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NOETHERIAN MODULES OVER HYPERFINITE GROUPS

by

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To

my wife and son
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SUMMARY

Let G be a group and A a \( \mathbb{Z}G \)-module. If \( A = \bar{A} \oplus A^f \), where \( A^f \) is a \( \mathbb{Z}G \)-submodule of A such that each irreducible \( \mathbb{Z}G \)-factor of \( A^f \) is finite and the \( \mathbb{Z}G \)-submodule \( \bar{A} \) of A has no nonzero finite \( \mathbb{Z}G \)-factors, then A is said to have an f-decomposition. If G is a hyperfinite locally soluble group, then it is known that any artinian \( \mathbb{Z}G \)-module A has an f-decomposition. In this thesis, especially by investigating the properties of the torsion-free noetherian \( \mathbb{Z}G \)-modules, we prove that any noetherian \( \mathbb{Z}G \)-module A over a hyperfinite locally soluble group G has an f-decomposition, too. Further, the structure of the noetherian \( \mathbb{Z}G \)-submodule \( \bar{A} \) is well described and the structure of the noetherian \( \mathbb{Z}G \)-submodule \( A^f \) is discussed in detail.

If G is a Černikov group (not necessarily locally soluble) or, more generally, if G is a finite extension of a periodic abelian group with \( |\pi(G)| < \infty \), where \( \pi(G) = \{\text{prime } p; G \text{ has an element of order } p\} \), then, for any noetherian \( \mathbb{Z}G \)-module A, we have that:

1. A has an f-decomposition;
2. \( A^f \) is finitely generated as an abelian group and \( G/C_G(A^f) \) is finite; and
3. \( \bar{A} \) is torsion as a group and has a finite \( \mathbb{Z}G \)-composition series as well as a finite exponent.

Moreover, we have generalized Zaïcev's results about modules over hyperfinite locally soluble groups to modules over hyper-(cyclic or finite) groups. In fact, we have got the following results:

Theorem C: Any periodic artinian \( \mathbb{Z}G \)-module A over a hyper-(cyclic or finite)
locally soluble group $G$ has an $f$-decomposition.

**Theorem D:** Let $E$ be an extension of a periodic abelian group $A$ by a hyper-(cyclic or finite) locally soluble group $G$. If $A$ is an artinian $\mathbb{Z}G$-module, then $E$ splits conjugately over $A$ modulo $A^f$. And

**Theorem E:** Let $E$ be an extension of an abelian group $A$ by a hyper-(cyclic or finite) locally soluble group $G$. If $A$ is a noetherian $\mathbb{Z}G$-module with $A = A^\mathcal{F}$, then $E$ splits conjugately over $A$.

A number of questions are given at the end of the work.
INTRODUCTION

A group $G$ is a hyperfinite group if $G$ has an ascending normal series $1 = G_0 \leq G_1 \leq \cdots \leq G_\alpha = G$ in which each factor $G_{\beta+1}/G_\beta$ is finite, where $\beta < \alpha$. The class of hyperfinite groups forms a subclass of the class of locally finite groups. In our work, we mainly consider a hyperfinite group $G$ acting on an abelian group $A$ and, by the action of $G$ on $A$, we consider $A$ as a $\mathbb{Z}G$-module.

In 1986, D. I. Zaicev proved that: if $G$ is a hyperfinite locally soluble group, then any artinian $\mathbb{Z}G$-module $A$ has an $f$-decomposition. That is, $A = A^f \oplus \overline{A}^f$, where $A^f$ is a $\mathbb{Z}G$-submodule of $A$ such that the irreducible $\mathbb{Z}G$-factors of $A^f$ are all finite and the $\mathbb{Z}G$-submodule $\overline{A}^f$ of $A$ has no nonzero finite $\mathbb{Z}G$-factors. Using this result, he proved a splitting theorem that: let $E$ be an extension of an abelian group $A$ by a hyperfinite locally soluble group $G$ and assume that $A$ is an artinian $\mathbb{Z}E$-module, then $E$ splits conjugately over $A$ modulo $A^f$ (we will explain this result later). In 1988, he used a strong condition for proving a splitting theorem dual to the above. That is, he proved the result that: let $E$ be an extension of an abelian group $A$ by a hyperfinite locally soluble group $G$ and assume that $A$ is a noetherian $\mathbb{Z}G$-module with $A = A^f$, then $E$ splits conjugately over $A$. Can we remove the condition $A = A^f$ and get exactly the form that $E$ splits conjugately over $A$ modulo $A^f$? This leads us to consider whether any noetherian $\mathbb{Z}G$-module $A$ over a hyperfinite locally soluble group $G$ has an $f$-decomposition. After investigating the properties of the torsion-free noetherian $\mathbb{Z}G$-modules, we have now successfully proved the required result. That is, we have:

Theorem A: Any noetherian $\mathbb{Z}G$-module $A$ over a hyperfinite locally soluble
group G has an f-decomposition, too.

In our proof, we proceed in the following steps.

Step 1: The important lemmas.

Lemma 1.2.5: (Wilson, [17]) Let G be a group, H a normal subgroup of finite index in G, and A a $\mathbb{Z}G$-module. Then A is a noetherian (resp. artinian) $\mathbb{Z}G$-module if and only if A is a noetherian (resp. artinian) $\mathbb{Z}H$-module.

Lemma 1.2.14: (Zalcve, [22]) Let H be a hyperfinitely embedded subgroup of a group G and A a noetherian $\mathbb{Z}G$-module. If $C_A(H) = 0$, then H contains a subgroup K and A contains a nonzero $\mathbb{Z}G$-submodule B such that K is normal in G, $C_B(K) = 0$, and $|K/C_A(B)| < \infty$.

Lemma 2.1.4: Let G be a locally finite group and A a torsion-free noetherian $\mathbb{Z}G$-module. Then $pA < A$ and $\bigotimes_{i=1}^{n} p_i A = 0$ for any prime $p$.

Lemma 2.4.5: Let G be a hyperfinite locally soluble group and A a noetherian $\mathbb{Z}G$-module with $pA = 0$ for some prime $p$. If all irreducible $\mathbb{Z}G$-factors of A are finite, then A is finite.

Lemma 2.4.7: Let G be a group, A a $\mathbb{Z}G$-module, and M a $\mathbb{Z}G$-submodule of A such that the factor module A/M is a p-group for some prime $p$. If $H = C_G(A/M)$ contains a nontrivial finite subgroup K being a q-group for some prime $q \neq p$, then $A = C_A(x)+M$ for any $x \in K$. Further, $A = C_A(K)+M$.

Step 2: Reducing A to be either torsion-free or an elementary abelian $p$-group for some prime $p$.

This step is very important in our proof and it much depends on the
following result.

**Corollary 3.3:** Let $G$ be a hyperfinite locally soluble group, $A$ a noetherian $\mathbb{Z}G$-module, and $B$ a $\mathbb{Z}G$-submodule of $A$ such that each irreducible $\mathbb{Z}G$-factor of $B$ is finite (resp. infinite) and $A/B$ contains no finite (resp. infinite) irreducible $\mathbb{Z}G$-factors. Then $B$ has a complement in $A$, i.e., $A = B \oplus C$ for some $\mathbb{Z}G$-submodule $C$ of $A$.

Step 3: Reducing $A$ to be torsion-free with all finite irreducible $\mathbb{Z}G$-factors being $p$-groups for some fixed prime $p$.

This has been achieved in Proposition 3.10.

**Proposition 3.10:** Let $G$ be a hyperfinite locally soluble group and $A$ a noetherian $\mathbb{Z}G$-module. If $A$ has no $f$-decomposition, then $A$ has a nonzero $\mathbb{Z}G$-image $\overline{A}$ satisfying:

(a) $\overline{A}$ has no $f$-decomposition;

(b) for every nonzero $\mathbb{Z}G$-submodule $C$ of $\overline{A}$, $\overline{A}/C$ has an $f$-decomposition;

(c) $\overline{A}$ has no nonzero $\mathbb{Z}G$-submodules with an $f$-decomposition;

(d) $\overline{A}$ is torsion-free; and

(e) the finite irreducible $\mathbb{Z}G$-factors of $\overline{A}$ are all $p$-groups for some fixed prime $p$.

Step 4: Discussing the properties of the torsion-free noetherian $\mathbb{Z}G$-modules.

Specially, for a fixed prime $p$, we have got a descending series of $\mathbb{Z}G$-submodules

$$A_{00} > A_{01} > A_{02} > \cdots > \bigcap_{i} A_{0i} = A_{0\infty},$$

in which, for any $\mathbb{Z}G$-submodule $A_{0i}$, $pA_{0i} < A_{0,i+1}$, the irreducible $\mathbb{Z}G$-factors of $A_{0i}/A_{0,i+1}$ are all finite, and the $\mathbb{Z}G$-submodule $A_{0\infty}$ has no
Step 5: Completing the proof.

The key result in completing our proof is the following:

**Proposition 3.14:** Let \( G \) be a hyperfinite locally soluble group and \( A \) a noetherian \( \mathbb{Z}G \)-module. If all finite irreducible \( \mathbb{Z}G \)-factors of \( A \) are \( p \)-groups for some fixed prime \( p \), then \( A \) has an \( f \)-decomposition.

Furthermore, in Chapter 4, we have well described the structure of the noetherian \( \mathbb{Z}G \)-submodule \( A^f \) of \( A \) and, under some conditions, that of the noetherian \( \mathbb{Z}G \)-submodule \( \hat{A}^f \) of \( A \). More exactly, we have:

**Theorem B:** Let \( G \) be a hyperfinite locally soluble group and \( A \) a noetherian \( \mathbb{Z}G \)-module. Then \( A^f \) is finitely generated as an abelian group and \( G/C(A^f) \) is finite.

**Proposition 4.4.6:** Let \( G \) be a periodic abelian group with \( |\pi(G)| < \infty \), where \( \pi(G) = \{ \text{prime } p; \ G \text{ has an element of order } p \} \), and let \( A \) be a noetherian \( \mathbb{Z}G \)-module. Then \( A^f \) is torsion and \( A^\hat{f} \) has a finite \( \mathbb{Z}G \)-composition series as well as a finite exponent.

In all the above results, \( G \) has been assumed to be locally soluble. However, it is not a necessary condition as we can take \( G \) to be a Černikov group (not necessarily locally soluble). More generally, we have:

**Theorem:** If \( G \) is a finite extension of a periodic abelian group with \( \pi(G) \) finite, then

- any noetherian \( \mathbb{Z}G \)-module \( A \) has an \( f \)-decomposition:
where $A^f$ is a $\mathbb{Z}G$-submodule of $A$ such that each irreducible $\mathbb{Z}G$-factor of $A^f$ is finite and the $\mathbb{Z}G$-submodule $\tilde{A}^f$ has no nonzero finite $\mathbb{Z}G$-factors;

(2) $A^f$ as a group is finitely generated and $G/C_G(A^f)$ is finite; and

(3) $\tilde{A}^f$ is torsion and has a finite $\mathbb{Z}G$-composition series as well as a finite exponent.

In Chapter 5, we have generalized Zaicev's results about modules over hyperfinite locally soluble groups to modules over hyper-(cyclic or finite) locally soluble groups. Especially, the obtained splitting theorems are new in the splitting theory, for which we will give a short review below.

A group $E$ is said to split over its subgroup $A$ modulo $C$ for some subgroup $C$ of $E$ if there is a subgroup $B$ of $E$ such that $E = AB$ and $A \cap B = C$. The subgroup $B$ is called a supplement to $A$ in $E$ or a complement to $A$ in $E$ modulo $C$. If all complements to $A$ in $E$ modulo $C$ are conjugate in $E$ modulo $C$, then $E$ is said to split conjugately over $A$ modulo $C$. If $C = 1$, then we naturally have the concepts of a complement to $A$ in $E$ and $E$ splitting conjugately over $A$.

It is well-known that if $A$ is a subgroup of a finite group $E$ such that $(|A|, |E/A|) = 1$ then $E$ splits conjugately over $A$ (The Schur–Zassenhaus Theorem, [15]). After this splitting theorem, many splitting results sprang up in the later years. Among these, we quote a few to stand as a background for our results.

a (M. L. Newell, 1975, [10]): Let $A$ be an abelian normal subgroup of a group $E$ such that $E/A$ is locally supersoluble and $[E', A] = 1$. If $A$ is noetherian as $\mathbb{Z}E$-module and has no nonzero cyclic $\mathbb{Z}E$-images, then $E$ splits
conjugately over A.

b (M. L. Newell, 1975, [10]): Let A be an abelian minimal normal subgroup of a group E such that E/A is hypercyclic. If A is not cyclic, then E splits conjugately over A.

c (M. J. Tomkinson, 1978, [16]): Let G be a hypercyclic group and A a finite \( \mathbb{Z}G \)-module. If A has no nonzero cyclic \( \mathbb{Z}G \)-images, then any extension E of A by G splits conjugately over A.

d (D. I. Zaicev, 1979 and 1980, [19] and [20]): Let A be an abelian normal subgroup of a group E such that E/A is hypercyclic. If A is artinian (resp. noetherian) as a \( \mathbb{Z}E \)-module and has no nonzero cyclic \( \mathbb{Z}E \)-submodules (resp. \( \mathbb{Z}E \)-images), then E splits conjugately over A.

e (D. I. Zaicev, 1986 and 1988, [21] and [22]): Let A be an abelian normal subgroup of a group E such that E/A is a hyperfinite locally soluble group. If A is artinian (resp. noetherian) as a \( \mathbb{Z}E \)-module and has no nonzero finite \( \mathbb{Z}E \)-submodules (resp. \( \mathbb{Z}E \)-images), then E splits conjugately over A.

Now we have proved the following results.

**Theorem D:** Let A be a periodic abelian normal subgroup of a group E such that E/A is a hyper-(cyclic or finite) locally soluble group. If A is artinian as a \( \mathbb{Z}E \)-module and has no nonzero finite \( \mathbb{Z}E \)-submodules, then E splits conjugately over A.

**Theorem E:** Let A be an abelian normal subgroup of a group E such that E/A is a hyper-(cyclic or finite) locally soluble group. If A is noetherian as a \( \mathbb{Z}E \)-module and has no nonzero finite \( \mathbb{Z}E \)-images, then E splits conjugately over A.
A.

There are lots of questions still remaining open. We list out some of these in Chapter 6.
In this chapter, as a beginning, we use two sections to recall some definitions and some well-known results, which will be quoted at least once in the later work. The indicated source of most of the results does not mean the original one. Some of the results have been literally rewritten to make them easily quoted. For the sake of convenience, we have also given a proof for some evident results. The terminology used in the work is standard as used in [15].

§1.1 NOTATION AND TERMINOLOGY

Throughout, we let \( G \) denote a group, \( \mathbb{Z} \) the ring of integers, \( \mathbb{Z}_p \) the prime field of characteristic \( p \), and \( \mathbb{Z}G \) (resp. \( \mathbb{Z}_p G \)) the group ring with \( G \) as a basis and the coefficients in \( \mathbb{Z} \) (resp. \( \mathbb{Z}_p \)).

Definition 1.1.1: An ascending series of a group \( G \) is a set of subgroups \( \{ G^\beta : \beta \leq \alpha \} \) indexed by ordinals less than or equal to an ordinal \( \alpha \) such that

(a) \( H_1 \leq H_2 \) if \( \beta_1 \leq \beta_2 \),

(b) \( H_0 = 1 \) and \( H_\alpha = G \),

(c) \( H_\beta \) is normal in \( H_{\beta + 1} \) and

(d) \( H_\lambda = \bigcup_{\beta \leq \lambda} H_\beta \) if \( \lambda \) is a limit ordinal.

It is convenient to write the ascending series in the form

\[
1 = G_0 \leq G_1 \leq \cdots \leq G_\alpha = G.
\]

If each \( G_\beta \) (\( \beta \leq \alpha \)) is normal in \( G \), then we say the ascending series \( \{ G^\beta : \beta \leq \alpha \} \) of \( G \) is an ascending normal series of \( G \).
Definition 1.1.2: Let \( \mathcal{P} \) denote a group property. A group is said to be a hyper-\( \mathcal{P} \) group if \( G \) has an ascending series

\[
1 = G_0 \leq G_1 \leq \cdots \leq G_\alpha = G,
\]

in which each factor \( G_{\beta+1}/G_\beta \) has \( \mathcal{P} \) and each \( G_\beta \) is normal in \( G \), where \( \beta < \alpha \). In particularly, if \( \mathcal{P} \) is the group property of finiteness, then we call \( G \) a hyperfinite group. If \( \alpha \) is finite and \( G_\beta \) need not to be normal in \( G \), then we call \( G \) a poly-\( \mathcal{P} \) group. Thus a poly-cyclic group \( G \) is that \( G \) has an ascending series \( 1 = G_0 \leq G_1 \leq \cdots \leq G_\alpha = G \), in which each factor \( G_{\beta+1}/G_\beta \) is cyclic, \( \beta < \alpha \), and \( \alpha < \infty \).

Definition 1.1.3: A normal subgroup \( H \) of \( G \) is said to be hyper-\( \mathcal{P} \) embedded in \( G \) if \( H \) itself is a hyper-\( \mathcal{P} \) group and in the corresponding ascending normal series \( 1 = H_0 \leq H_1 \leq \cdots \leq H_\alpha = H \) each \( H_\beta \) (\( \beta \leq \alpha \)) is normal in \( G \).

For a \( \mathbb{Z}G \)-module \( A \), being similar with the ascending series of groups, we define the ascending (resp. descending) \( \mathbb{Z}G \)-composition series of \( A \) as a set of \( \mathbb{Z}G \)-submodules \( \{ A_\gamma ; \gamma \leq \delta \} \) indexed by ordinals less than or equal to an ordinal \( \delta \) such that

(a) \( A_\gamma \leq A_\gamma \) (resp. \( A_\gamma \geq A_\gamma \)) if \( \gamma_1 \leq \gamma_2 \),

(b) \( A_0 = 0 \) and \( A_\delta = A \) (resp. \( A_0 = A \) and \( A_\delta = 0 \)),

(c) the \( \mathbb{Z}G \)-module \( A_{\gamma+1}/A_\gamma \) (resp. \( A_\gamma /A_{\gamma+1} \)) is irreducible for \( \gamma < \delta \),

(d) \( A_\lambda = \bigcup_{\gamma < \lambda} A_\gamma \) (resp. \( A_\lambda = \bigcap_{\gamma < \lambda} A_\gamma \)) if \( \lambda \) is a limit ordinal.

If \( \delta \) is finite, then we say that \( A \) has a finite \( \mathbb{Z}G \)-composition series and usually write in the form

\[
0 = A_0 < A_1 < \cdots < A_\delta = A. \quad \text{(resp.} \quad A = A_0 > A_1 > \cdots > A_\delta = 0). \]
**Definition 1.1.4:** A $\mathbb{Z}G$-module $A$ is said to be completely reducible if $A$ is a direct sum of some irreducible $\mathbb{Z}G$-submodules. Furthermore, $A$ is said to be semisimple if $A$ is a direct sum of finitely many irreducible $\mathbb{Z}G$-submodules.

The important concept is that:

**Definition 1.1.5:** A $\mathbb{Z}G$-module $A$ is said to have an $f$-decomposition if

$$A = A^f \oplus \bar{A}^f,$$

where $A^f$ is a $\mathbb{Z}G$-submodule of $A$ such that each irreducible $\mathbb{Z}G$-factor of $A^f$ is finite and the $\mathbb{Z}G$-submodule $\bar{A}^f$ has no nonzero finite $\mathbb{Z}G$-factors. Sometimes, we call $A^f$ the $f$-component of $A$ and $\bar{A}^f$ the $\bar{f}$-component of $A$.

**Definition 1.1.6:** Let $G$ be a group, $H$ a normal subgroup of $G$, and $A$ a $\mathbb{Z}G$-module. Then $A$ is said to be $H$-perfect if $A = [A, H]$.

Some other definitions are:

**Definition 1.1.7:** Let $E$, $A$, and $G$ be groups. If $A$ is normal in $E$ and the factor group $E/A \cong G$, then $E$ is called an extension of $A$ by $G$.

**Definition 1.1.8:** A group $E$ is said to split over its subgroup $A$ modulo $C$ for some subgroup $C$ of $E$ if there is a subgroup $B$ of $E$ such that $E = AB$ and $A \cap B = C$. The subgroup $B$ is called a supplement to $A$ in $E$ or a complement to $A$ in $E$ modulo $C$. If all complements to $A$ in $E$ modulo $C$ are conjugate in $E$ modulo $C$, then $E$ is said to split conjugately over $A$ modulo $C$. If $C = 1$, then we naturally have the concepts of a complement to $A$ in $E$ and $E$ splitting conjugately over $A$. 
A ring \( R \) is called (right) semisimple if each right \( R \)-module is semisimple or equivalently \( R \) is a direct sum of finitely many simple (right) artinian rings. Also, a ring \( R \) is called regular if each finitely generated right ideal of \( R \) is generated by a single element \( e \) with \( e^2 = e \) (such \( e \) is called an idempotent of \( R \)).

Other notations are: \( \pi(G) = \{ \text{prime } p; \ G \text{ has an element of order } p \} \), \( A \wr K \) denotes the semidirect product of \( A \) by \( K \), and \( A > B \) means \( B \) is properly contained in \( A \).

\[ \text{§ 1.2 THE WELL-KNOWN RESULTS} \]

The following two results are the module version of the according results in [15].

**Lemma 1.2.1:** (Thm 3.3.11 in [15]) If a \( \mathbb{Z}G \)-module \( A \) is a sum of a set of its irreducible \( \mathbb{Z}G \)-submodules, then it is the direct sum of certain of these \( \mathbb{Z}G \)-submodules. Thus \( A \) is a completely reducible \( \mathbb{Z}G \)-module.

**Lemma 1.2.2:** (Remark, Thm 3.3.12 in [15]) Let \( A = \bigoplus_{\lambda \in \Lambda} A^\lambda \), where \( A^\lambda \) is an irreducible \( \mathbb{Z}G \)-submodule of \( A \). Suppose that \( B \) is a \( \mathbb{Z}G \)-submodule of \( A \). Then \( A = B \oplus \bigoplus_{\mu \in \mathcal{M}} A^\mu \) for some \( \mathcal{M} \leq \Lambda \). Also, \( B \) is completely reducible.

If \( G \) is a locally finite group, then the group structure of an irreducible \( \mathbb{Z}G \)-module has been well described. That is, we have:

**Lemma 1.2.3:** (Baer, Lemma 5.26 in [14]) If \( G \) is a locally finite group, then any irreducible \( \mathbb{Z}G \)-module \( A \) (as a group) is an elementary abelian \( p \)-group for \( p \)-adic integers.
some prime $p$.

Two useful fundamental lemmas are:

**Lemma 1.2.4**: (Fitting's Lemma, Thm 5.2.3 in [3]) Let $p$ be a prime, $A$ an abelian $p$-group, and $H \leq \text{Aut}(A)$. If $H$ is a finite $p'$-group, then

$$A = C_A(H) \oplus [A, H].$$

Here $A$ is written additively.

**Lemma 1.2.5**: (Wilson, [17]) Let $G$ be a group, $H$ a normal subgroup of finite index in $G$, and $A$ a $\mathbb{Z}G$-module. Then $A$ is a noetherian (resp. artinian) $\mathbb{Z}G$-module if and only if $A$ is a noetherian (resp. artinian) $\mathbb{Z}H$-module.

We mention that:

**Lemma 1.2.6**: Let $G$ be a group, $A$ a $\mathbb{Z}G$-module, and $H$ a normal subgroup of $G$. Then both $C_A(H)$ and $[A, H]$ are $\mathbb{Z}G$-submodules of $A$.

For convenience, we prove:

**Lemma 1.2.7**: Let $G$ be a group, $A$ a $\mathbb{Z}G$-module, and $K$ a finite $p$-subgroup of $G$, where $p$ is a prime. If $A$ (as a group) is also a $p$-group, then $C_A(K) \neq 0$.

**Proof**: We note firstly that $A_1 = \langle a \rangle K$ is a finite $\mathbb{Z}K$-module for any fixed $0 \neq a \in A$. Since $A$ is a $p$-group, so $A_1$ is a finite $p$-group and then the semidirect product $L = A_1 \rtimes K$ is a finite $p$-group. Thus the normal subgroup $A_1$ of $L$ contains a nontrivial central element of $L$, say $a_0$. Therefore $a_0 \neq 0$ and $a_0 \in C_A(K)$. That is, $C_A(K) \neq 0$. The lemma is true.
Corollary 1.2.8: Under the hypotheses of Lemma 1.2.7, if $A$ is further irreducible, then $A = \mathcal{C}_A(K)$.

About hyperfinite groups, it is worth mentioning some results here.

Lemma 1.2.9: If $G$ is a hyperfinite group, then the subgroups and the homomorphic images of $G$ are all hyperfinite.

Proof: Suppose $G$ has an ascending normal series
\[ 1 = G_0 \leq G_1 \leq \cdots \leq G_\alpha = G \]
in which each $G_\beta$ is normal in $G$ and $G_{\beta+1}/G_\beta$ is finite, $\beta < \alpha$.

For a subgroup $H$ of $G$, it is clear that
\[ 1 = H \cap G_0 \leq H \cap G_1 \leq \cdots \leq H \cap G_\alpha = H \]
is an ascending normal series of $H$, and also each factor $\langle H \cap G_{\beta+1} \rangle / \langle H \cap G_\beta \rangle$ is finite, $\beta < \alpha$. So $H$ is hyperfinite.

Now, for a homomorphic image $\overline{G}$ of $G$, we may assume $\overline{G} = G/N$, where $N$ is a normal subgroup of $G$. Then
\[ 1 = NG_0/N \leq NG_1/N \leq \cdots \leq NG_\alpha/N = \overline{G} \]
is an ascending normal series of $\overline{G}$, and the factor $\langle NG_{\beta+1}/N \rangle / \langle NG_\beta/N \rangle$ is finite, $\beta < \alpha$. So $\overline{G}$ is hyperfinite.

Lemma 1.2.10: If $G$ is a hyperfinite group, then any nontrivial normal subgroup of $G$ contains nontrivial finite subgroups being (minimal) normal in $G$.

Proof: Since $G$ is hyperfinite, there is an ascending normal series
\[ 1 = G_0 \leq G_1 \leq \cdots \leq G_\alpha = G \]
in which each factor \( G_{\beta+1}/G_\beta \) is finite for all \( \beta < \alpha \).

Suppose \( H \) is a nontrivial normal subgroup of \( G \), then \( H \cap G_\beta \) is normal in \( G \), \( \forall \beta < \alpha \). Since \( H \cap G_{\alpha} \neq H \neq 1 \), there exists \( \alpha_0 \leq \alpha \) such that \( H \cap G_{\alpha_0} \neq 1 \) but \( H \cap G_\beta = 1 \) for all \( \beta < \alpha_0 \). If \( \alpha_0 - 1 \) does not exist, then \( G_{\alpha_0} = \bigcup_{\beta<\alpha_0} G_{\beta} \).

By \( H \cap G_{\alpha_0} \neq 1 \), there is \( 1 \not\in H \cap G_{\alpha_0} \) and then \( 1 \not\in g \in H \cap G_\beta \) for some \( \beta < \alpha_0 \), contrary to \( H \cap G_\beta = 1 \). So \( \alpha_0 - 1 \) exists. By \( H \cap G_{\alpha_0-1} = 1 \) we have

\[ 1 \not\in H \cap G_{\alpha_0} = \{G_{\alpha_0-1}(H \cap G_{\alpha_0})\}/G_{\alpha_0-1} \leq G_{\alpha_0}/G_{\alpha_0-1}, \]

and then \( |H \cap G_{\alpha_0}| \leq |G_{\alpha_0}/G_{\alpha_0-1}| < \infty \). That is, \( H \cap G_{\alpha_0} \) is a nontrivial finite subgroup of \( H \) and is normal in \( G \). Furthermore, by \( H \cap G_{\alpha_0} \) being finite, we can find a nontrivial finite subgroup \( N \) of \( H \) such that \( N \) is minimal with respect to \( N \) being normal in \( G \).

**Lemma 1.2.11:** A hyperfinite group is locally finite.

**Proof:** Suppose \( G \) is a hyperfinite group, then \( G \) has an ascending normal series \( 1 = G_0 \leq G_1 \leq \cdots \leq G_\alpha = G \), in which each \( G_\beta \) is normal in \( G \) and \( G_{\beta+1}/G_\beta \) is finite, \( \beta < \alpha \).

Let \( \gamma \leq \alpha \), and suppose that \( G_\beta \) is a locally finite group for all \( \beta < \gamma \), we prove that \( G_\gamma \) is also a locally finite group.

If \( \gamma - 1 \) exists, then \( G_\gamma \) is a (locally finite)–by–finite group and then is a locally finite group [15]; and on the other hand, if \( \gamma - 1 \) does not exist, then for any \( x_1, \ldots, x_\gamma \in G_\gamma = \bigcup_{\beta<\gamma} G_{\beta} \) there exists \( \beta_0 \) such that \( x_1, \ldots, x_\gamma \in G_{\beta_0} \).
and then \( < x_1, \ldots, x_n > \) is a finite subgroup of \( G_\beta \) (and of \( G_\gamma \)). So \( G_\gamma \) is a locally finite group. That is, from \( G_\beta \) being locally finite groups for all \( \beta < \gamma \) we have proved that \( G_\gamma \) is also a locally finite group. The lemma holds.

**Lemma 1.2.12:** A hyperfinite \( p \)-group \( G \) is a locally nilpotent group. Therefore, \( G \) has a nontrivial centre and then is a hypercentral group.

Before giving the proof for Lemma 1.2.12, we point out another well-known result:

**Lemma 1.2.13:** (Mal'cev, McLain, Thm 12.1.6 in [15]) A principal factor of a locally nilpotent group \( G \) is central.

**Proof of Lemma 1.2.12:** By Lemma 1.2.10, \( G \) is locally finite, then since \( G \) is a \( p \)-group we have \( G \) is a locally nilpotent group. By Lemma 1.2.13 and \( G \) having a minimal normal subgroup, \( G \) has a nontrivial centre. Let \( h(G) \) be the maximal hypercentrally embedded subgroup of \( G \), i.e., \( h(G) \) is the hypercentre of \( G \). If \( h(G) < G \), then \( G/h(G) \) as a nontrivial hyperfinite \( p \)-group has a nontrivial centre, say \( H/h(G) \). It is clear that \( H \) is a hypercentrally embedded subgroup of \( G \) and \( h(G) < H \) contrary to the maximality of \( h(G) \). So \( h(G) = G \) and then \( G \) is hypercentral.

In our work, we will quote a lot of results proved by D. I. Zaicev in the series of papers [19—22]. Among these, the most important one is:

**Lemma 1.2.14:** (Zaicev, [22]) Let \( H \) be a hyperfinitely embedded subgroup of \( G \), and \( A \) a nonzero noetherian \( ZG \)-module. If \( C_A(H) = 0 \), then \( H \) contains a subgroup \( K \) and \( A \) contains a nonzero \( ZG \)-submodule \( B \) such that \( K \) is normal in \( G \), \( C_B(K) = 0 \), and \( |K/C_K(B)| < \infty \).
The above lemma has a generalization for $H$ being a hyper-(cyclically or finitely) embedded subgroup of $G$. That is, it is a special case of our result—Proposition 5.1.2 (see p.100).

The other results proved by D. I. Zaicev or appearing in D. I. Zaicev's papers are:

**Lemma 1.2.15:** (Zaicev, [19]) If $G$ is a hypercentral group, then any artinian $\mathbb{Z}G$-module $A$ has a $\mathbb{Z}$-decomposition, i.e., $A = A^Z \oplus A^Z_G$, in which $A^Z$ is a $\mathbb{Z}G$-submodule of $A$ such that each irreducible $\mathbb{Z}G$-factor of $A^Z$ has $G$ as its centralizer in $G$ and the $\mathbb{Z}G$-submodule $A^Z_G$ of $A$ has no such irreducible $\mathbb{Z}G$-factors.

**Lemma 1.2.16:** (Frattini argument, [19]) Let $A$ be an abelian normal subgroup of the group $G$. Let $x \in G$ such that $A = [A, x]$ and $A < x >$ is normal in $G$. Then $G = AN_G(<x>)$.

**Lemma 1.2.17:** ([19]) If $A$ is a $\mathbb{Z}G$-module and $<x>$ is a normal cyclic subgroup of $G$, then $A(x-1)$ and $C_A(x)$ are $\mathbb{Z}G$-submodules of $A$.

**Lemma 1.2.18:** ([20]) Let $A$ be a $\mathbb{Z}G$-module and $<x>$ a normal cyclic subgroup of $G$. The map $a \mapsto a(x-1)$ induces an isomorphism of the groups $A/C_A(x)$ and $A(x-1)$ under which $\mathbb{Z}G$-submodules of $A/C_A(x)$ correspond to $\mathbb{Z}G$-submodules of $A(x-1)$. (If, further, $x$ is in the centre of $G$, then the map $a \mapsto a(x-1)$ induces a $\mathbb{Z}G$-isomorphism between the $\mathbb{Z}G$-modules $A/C_A(x)$ and $A(x-1)$.)

**Lemma 1.2.19:** ([20]) Let $A$ be a noetherian $\mathbb{Z}G$-module and $<x>$ a normal cyclic subgroup of $G$. Then there is an integer $n$ such that $A(x-1)^n \cap C_A(x) = 0$. 

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Lemma 1.2.20: ([21]) Let $A$ be an abelian normal $p$-subgroup of $E$ and $K/A$ a finite normal $p$-subgroup of the factor group $E/A$ such that $A = [A, K]$. Then $A$ contains a proper subgroup $D$ such that $D$ is normal in $E$ and (1) $A$ has a complement in $E$ modulo $D$; (2) if $A$ has a complement in $E$, then any two complements are conjugate modulo $D$.

Lemma 1.2.21: ([22]) Let $A$ be an abelian normal subgroup of $E$ and $K/A$ a finite normal subgroup of $E/A$ such that $|K/A| = p^k$, $C_A(K) = 1$. Then (1) if $A$ has a complement in $E$ modulo $A^D$, then $A$ has a complement in $E$; (2) if $A$ has a complement in $E$ and the complements are conjugate modulo $A^D$, then they are conjugate in $E$.

Lemma 1.2.22: (Zaicev, [22]) Let $G$ be a hyperfinite group, $A$ a noetherian $\mathbb{Z}G$-module, and $B$ a $\mathbb{Z}G$-submodule of $A$ such that $B$ (resp. $A/B$) is finite. If each irreducible $\mathbb{Z}G$-factor of $A/B$ (resp. that of $B$) is infinite, then $B$ has a complement in $A$, i.e., $A = B \oplus C$ for some $\mathbb{Z}G$-submodule $C$ of $A$.

There are some simple results we should mention:

Lemma 1.2.23: Let $G$ be a group, and $A$ a $\mathbb{Z}G$-module. If $A$ has an $f$-decomposition, then each $\mathbb{Z}G$-submodule of $A$ and each $\mathbb{Z}G$-image of $A$ has an $f$-decomposition, too. Furthermore, if $A = A^f \oplus A^\tilde{f}$, where $A^f$ is the $f$-component of $A$ and $A^\tilde{f}$ is the $\tilde{f}$-component of $A$, then (1) for any $\mathbb{Z}G$-submodule $B$ of $A$, $B = (B \cap A^f) \oplus (B \cap A^\tilde{f})$, in which $(B \cap A^f) = B^f$ and $(B \cap A^\tilde{f}) = B^\tilde{f}$; and (2) for any $\mathbb{Z}G$-image $A/D$ of $A$, $A/D = (A^f + D)/D \oplus (A^\tilde{f} + D)/D$, in which $(A^f + D)/D = (A/D)^f$ and $(A^\tilde{f} + D)/D = (A/D)^{\tilde{f}}$.

Proof: Let $A = A^f \oplus A^\tilde{f}$, where $A^f$ is a $\mathbb{Z}G$-submodule of $A$ such that each
irreducible $\mathbb{Z}G$-factor of $A^f$ is finite and the $\mathbb{Z}G$-submodule $A^\tilde{f}$ of $A$ has no nonzero finite $\mathbb{Z}G$-factors. Let $B$ be a $\mathbb{Z}G$-submodule of $A$. It is clear that $B \cong (B \cap A^f) \oplus (B \cap A^\tilde{f})$, each irreducible $\mathbb{Z}G$-factor of $B \cap A^f$ is finite and $B \cap A^\tilde{f}$ contains no nonzero finite $\mathbb{Z}G$-factors. Also, by $B/B \cap A^f \cong \mathbb{Z}G (B+A^f)/A^f$ and $B/B \cap A^\tilde{f} \cong \mathbb{Z}G (B+A^\tilde{f})/A^\tilde{f}$, we have each nonzero irreducible $\mathbb{Z}G$-factor of $B/B \cap A^f$ is infinite and all irreducible $\mathbb{Z}G$-factors of $B/B \cap A^\tilde{f}$ are finite. Thus $B/(B \cap A^f) \oplus (B \cap A^\tilde{f})$ has no nonzero irreducible $\mathbb{Z}G$-factors and then $B = (B \cap A^f) \oplus (B \cap A^\tilde{f})$, where $(B \cap A^f) = B^f$ and $(B \cap A^\tilde{f}) = B^\tilde{f}$. That is, $B$ has an $f$-decomposition.

For the $\mathbb{Z}G$-image, say $A/D$, it is clear that $A/D = (A^f + D)/D + (A^\tilde{f} + D)/D$. Since $(A^f + D)/D \cong \mathbb{Z}G A^f/(A^f \cap D)$ and $(A^\tilde{f} + D)/D \cong \mathbb{Z}G A^\tilde{f}/(A^\tilde{f} \cap D)$, we have each irreducible $\mathbb{Z}G$-factor of $(A^f + D)/D$ is finite and the $\mathbb{Z}G$-submodule $(A^\tilde{f} + D)/D$ has no nonzero finite irreducible $\mathbb{Z}G$-factors. Thus $(A^f + D)/D \cap (A^\tilde{f} + D)/D = 0$. It follows that $A/D = (A^f + D)/D \oplus (A^\tilde{f} + D)/D$, $(A^f + D)/D = (A/D)^f$ and $(A^\tilde{f} + D)/D = (A/D)^\tilde{f}$. That is, $A/D$ has an $f$-decomposition. The lemma is proved.

**Lemma 1.2.24:** Let $A = \sum_{i \in I} A_i$, where $A$ is a $\mathbb{Z}G$-module and $A_i$ are $\mathbb{Z}G$-submodules of $A$. If each $A_i$ has a $f$-decomposition: $A_i = A_i^f \oplus A_i^\tilde{f}$, then $A$ has a $f$-decomposition with $A^f = \sum_{i \in I} A_i^f$ and $A^\tilde{f} = \sum_{i \in I} A_i^\tilde{f}$.

**Proof:** It is clear that $A = \sum_{i \in I} A_i = \sum_{i \in I} (A_i^f \oplus A_i^\tilde{f}) = (\sum_{i \in I} A_i^f) + (\sum_{i \in I} A_i^\tilde{f})$.

For $\sum_{i \in I} A_i^f$, if it contains an infinite irreducible $\mathbb{Z}G$-factor, say $B_0/C_0$, then for $x \in B_0 \setminus C_0$, we have $x = x_1 + \cdots + x_r$ for some $x_j \in A_{i_j}$, $j = 1, \ldots, r$. Thus $B_0 \cap \sum_{j=1}^r A_{i_j}^f > C_0 \cap \sum_{j=1}^r A_{i_j}^f$. By $B_0/C_0$ being irreducible and

$$0 \neq (B_0 \cap \sum_{j=1}^r A_{i_j}^f)/(C_0 \cap \sum_{j=1}^r A_{i_j}^f) \cong \mathbb{Z}G [C_0 + (B_0 \cap \sum_{j=1}^r A_{i_j}^f)]/C_0.$$

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we have $\sum_{j=1}^{r} A_j^f$ contains infinite irreducible $\mathbb{Z}G$-factors.

Let $r_0$ be minimal such that $\sum_{j=1}^{r_0} A_j^f$ contains infinite irreducible $\mathbb{Z}G$-factors. It is clear that $r_0 > 1$. Let $B/C$ be an infinite irreducible $\mathbb{Z}G$-factor of $\sum_{j=1}^{r_0} A_j^f$, then by the minimality of $r_0$ we must have

$$B \cap \sum_{j=1}^{r_0-1} A_j^f = C \cap \sum_{j=1}^{r_0-1} A_j^f$$

and then $B + \sum_{j=1}^{r_0-1} A_j^f > C + \sum_{j=1}^{r_0-1} A_j^f$.

By $B/C$ being irreducible and

$$0 \neq (B + \sum_{j=1}^{r_0-1} A_j^f)/(C + \sum_{j=1}^{r_0-1} A_j^f) \equiv \mathbb{Z}G [B \cap (C + \sum_{j=1}^{r_0-1} A_j^f)]$$

we have $(B + \sum_{j=1}^{r_0-1} A_j^f)/(C + \sum_{j=1}^{r_0-1} A_j^f)$ is infinite and irreducible. Also

$$(B + \sum_{j=1}^{r_0-1} A_j^f)/(C + \sum_{j=1}^{r_0-1} A_j^f) \leq \sum_{j=1}^{r_0} A_j^f/(C + \sum_{j=1}^{r_0} A_j^f) \equiv \mathbb{Z}G [A_{r_0}^f/(C + \sum_{j=1}^{r_0} A_j^f)]$$

which shows that $A_{r_0}^f$ has an infinite irreducible $\mathbb{Z}G$-factor, a contradiction.

Therefore $\sum_{i \in I} A_i^f$ contains no infinite irreducible $\mathbb{Z}G$-factors.

Similarly, $\sum_{i \in I} \bar{A}_i^f$ contains no nonzero finite irreducible $\mathbb{Z}G$-factors.

Thus $(\sum_{i \in I} A_i^f) \cap (\sum_{i \in I} \bar{A}_i^f) = 0$ and then $A = (\sum_{i \in I} A_i^f) \oplus (\sum_{i \in I} \bar{A}_i^f)$ with $A_i^f = \sum_{i \in I} A_i^f$ and $\bar{A}_i^f = \sum_{i \in I} \bar{A}_i^f$.

**Lemma 1.2.25:** Let $A = B + A_i$ with $B \cap A_i = B_i$, where $A$ is a $\mathbb{Z}G$-module, $B$, $A_i$, and $B_i$ are $\mathbb{Z}G$-submodules. If $A_i = B_i \oplus C_i$ for some $\mathbb{Z}G$-submodule $C_i$, then $A = B \oplus C_i$.

**Proof:** Since $A = B + A_i = B + (B_i + C_i) = (B + B_i) + C_i = B + C_i$. 

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and \( B \cap C_1 = B \cap (A \cap C_1) = (B \cap A) \cap C_1 = B_1 \cap C_1 = 0 \), so \( A = B \oplus C_1 \).

The following form of Maschke's Theorem will be used in a later proof.

**Lemma 1.2.26**: (Maschke's Theorem) Let \( V \) be a torsion-free \( \mathbb{Z}G \)-module with \( G \) being finite, and \( W \) a \( \mathbb{Z}G \)-submodule of \( V \). If \( V = W \oplus V_1 \) for some subgroup \( V_1 \) of \( V \), then there exists a \( \mathbb{Z}G \)-submodule \( U \) such that \( |G| V \leq W \oplus U \).

**Proof**: It is a special case of Theorem 4.1 in [12].

Finally, we end this chapter with three results related to semisimple rings.

**Lemma 1.2.27**: (Corollary 2.16, [2, p.21]) Any noetherian regular ring is semisimple.

**Lemma 1.2.28**: If \( F \) is a finite \( p' \)-group, then the group ring \( \mathbb{Z}_p F \) is semisimple.

**Proof**: Since \( \mathbb{Z}_p F \) is finite, so the result is a consequence of Theorem 4.2 in [11] and Theorem 0.1.11 in [9].

**Lemma 1.2.29**: Any right ideal of a semisimple ring is generated by a single idempotent.

**Proof**: Using Lemma 1.2.2 in this section and Lemma 6 in [13], we get the result.
This chapter consists of basic lemmas, of which most will be necessarily used in the later proof of our main results. We deal with them in four sections.

§2.1 ON TORSION-FREE MODULES

Among the \( \mathbb{Z}G \)-factors of a torsion-free \( \mathbb{Z}G \)-module \( A \), there are some factors which have a very nice relation between them. We list some of these as

**Lemma 2.1.1:** Let \( G \) be a group, \( A \) a torsion-free \( \mathbb{Z}G \)-module, and \( B \) a \( \mathbb{Z}G \)-submodule of \( A \). Then \( p^iA/p^iB \cong _{\mathbb{Z}G} A/B \) for any integer \( p > 0 \) and any integer \( i \geq 0 \).

**Proof:** Let \( \varphi: a \mapsto p^i a + p^i B \), where \( a \in A \). Since

\[
\begin{align*}
a. & \quad \varphi(a + b) = p^i(a + b) + p^i B \\
& \quad = (p^i a + p^i B) + (p^i b + p^i B) = \varphi(a) + \varphi(b), \text{ and} \\
b. & \quad \varphi(\alpha g) = p^i(\alpha g) + p^i B \\
& \quad = (p^i \alpha)g + p^i B = (p^i a + p^i B)g = [\varphi(a)]g,
\end{align*}
\]

where \( a, b \in A \) and \( g \in G \). So \( \varphi \) is a \( \mathbb{Z}G \)-homomorphism from \( A \) to \( p^iA/p^iB \), thus \( A/\ker \varphi \cong _{\mathbb{Z}G} p^iA/p^iB \).

Since \( B \trianglelefteq \ker \varphi \) is clear; and, on the other hand, \( \alpha \in \ker \varphi \) implies that \( p^i \alpha \in p^i B \). By \( A \) being torsion-free, \( \alpha \in B \) and so \( \ker \varphi \trianglelefteq B \). Thus \( \ker \varphi = B \) and then \( p^iA/p^iB \cong _{\mathbb{Z}G} A/B \) for any integer \( p > 0 \) and any integer \( i \geq 0 \).

From Lemma 2.1.1, we have:
Corollary 2.1.2: Let $G$ be a group, $A$ a torsion-free $\mathbb{Z}G$-module, and $B$ a $\mathbb{Z}G$-submodule of $A$. Then $p^iA/p^jB \cong_{\mathbb{Z}G} p^jA/p^jB$ for any integer $p > 0$ and any integers $i, j \geq 0$.

If we let $B = p^tA$, where $t$ is an integer not less than zero, then we have:

Corollary 2.1.3: Let $G$ be a group and $A$ a torsion-free $\mathbb{Z}G$-module. Then

$$\frac{k^iA}{k^{i+t}A} \cong_{\mathbb{Z}G} \frac{k^jA}{k^{j+t}A}$$

for any integer $k > 0$ and any integers $i, j, t \geq 0$.

In the torsion-free case, we often need $A$ contains some nonzero $\mathbb{Z}G$-factors being $p$-groups for some fixed prime $p$. The following lemma indicates that conditions for the purpose.

Lemma 2.1.4: Let $G$ be a locally finite group and $A$ a torsion-free noetherian $\mathbb{Z}G$-module. Then $pA < A$ and $\bigcap_{i=1}^{\infty} p^iA = 0$ for any integer $p > 0$.

Proof: Firstly, we prove $pA < A$ for any integer $p > 0$.

Since $A$ is noetherian, there exist $a_1, \ldots, a_n$ such that

$$A = \langle a_1, \ldots, a_n \rangle^G.$$

Suppose that $pA = A$ for some integer $p > 0$, then

$$a_i = p^t(a_i m_{i_1} g_{i_1} + \cdots + a_i m_{i_t} g_{i_t})$$

for some integer $t$, where $m_{i_j} \in \mathbb{Z}$, $g_{i_j} \in G$, $j = 1, 2, \ldots, t$, and $i = 1, 2, \ldots, n$.

Let $F = \langle g_{i_j} \in G; j = 1, \ldots, t, i = 1, \ldots, n \rangle$. By $G$ being locally
finite, $F$ is a finite group. Also, $a_i \in p^i <a_1, \ldots, a_n >^F$ for all $i$. Thus $<a_1, \ldots, a_n > \leq p^i <a_1, \ldots, a_n >^F$ and then $<a_1, \ldots, a_n >^F = p^i <a_1, \ldots, a_n >^F$.

But $<a_1, \ldots, a_n >^F$ is a finitely generated torsion-free abelian group and so $p^i <a_1, \ldots, a_n >^F$ is a proper subgroup of $<a_1, \ldots, a_n >^F$ for $p > 0$, a contradiction. Thus we have proved that $pA < A$ for any integer $p > 0$.

Secondly, if $B = \bigoplus_{i=1}^{\infty} p^i A \neq 0$, then since $B$ is also a torsion-free noetherian $\mathbb{Z}G$-module, thus $pB < B$ for the integer $p > 0$. But by $A$ being torsion-free we can easily get $B = pB$, a contradiction. Thus $\bigoplus_{i=1}^{\infty} p^i A = 0$. The lemma holds.

Related to Lemma 2.1.4, we have:

**Lemma 2.1.5**: For any noetherian $\mathbb{Z}G$-module $A$, $pA < A$ if and only if $A$ has a nonzero $\mathbb{Z}G$-factor being a $p$-group, where $p$ is a prime.

**Proof**: The necessity is evident.

We prove the sufficiency. Let $U/V$ be a nonzero $\mathbb{Z}G$-factor of $A$ such that $U/V$ is a $p$-group. By the noetherian condition, we may assume that $U/V$ is an elementary abelian $p$-group and $A/U$ has no nonzero $\mathbb{Z}G$-factors being $p$-groups.

If $pA = A$, then $A/V = (pA + V)/V = p(A/V)$. For $(V \neq 0)$ $u + V \in U/V \leq p(A/V)$, there is $a_0 \in A$ such that $u + V = p(a_0 + V)$. Thus $a_0 \not\in U$. But $p^2(a_0 + V) = p(u + V) = V$, which implies that $(A/U)[p] = \{a + U \in A/U; p(a + U) = U\}$ is a nonzero $\mathbb{Z}G$-submodule, contrary to $A/U$ having no nonzero $\mathbb{Z}G$-factors being $p$-groups. So $pA < A$, the result holds.

The following lemmas are useful and interesting.
Lemma 2.1.6: Let $G$ be a locally finite group, $A$ a torsion-free noetherian $ZG$-module, and $C_G(A) = 1$. Then

1. $H = C_G(A/pA) = 1$, if $p$ is an odd prime; and
2. $H = C_G(A/pA)$ is an elementary abelian 2-group, if $p = 2$.

Proof: We prove first that $H = C_G(A/pA)$ is a $p$-group for the prime $p$.

Suppose $x \in H$ and $x$ is of order $q$ for some prime $q$ other than $p$. Since $x \notin C_H(A) \leq C_G(A) = 1$, there exists $a_0 \in A$ such that $a_0 x \neq a_0$. Let $A_i = p^i A$, then, by applying Lemma 2.1.4, we have $\bigcap A_i = 0$. Therefore, there exists $i_0$ such that $a_0 \in A_i \setminus A_{i+1}$. By $A_i/A_{i+1} \cong ZG_i/A_i$, for any integer $i$, we have $x \in H = C_G(A_i/A_{i+1})$ for any integer $i$. So $a_0 x = a_0 + a_1$, where $a_1 \neq 0$ and $a_1 \in A_i \setminus A_{i+1}$ for some integer $i > i_0$. If $a_1 x = a_1$, then $a_0 = a_0 x^q = a_0 + qa_1$.

That is, $qa_1 = 0$ and then $a_1 = 0$, a contradiction. So $a_1 x \neq a_1$.

Suppose $a_0 x^r = \sum_{j=0}^{r} C_j a_j$, where for any $j$, $a_j \in A_i \setminus A_{i+1}$, $a_j x \neq a_j$, and $a_1 x = a_1 + a_1'$ for some $i > i_1 > \cdots > i_r > i_0 \geq 0$. By $a_1 x \neq a_1$ and $x \in H = C_G(A_i/A_{i+1})$, we have $a_i x = a_i + a_i r$, where $a_i r \in A_i \setminus A_{i+1}$, and $i + r > i_0$. Also we easily obtain $a_{i+r} x \neq a_{i+r}$. Now we have

$$a_0 x^{r+1} = (a_0 x^r) x = \left[\sum_{j=0}^{r} C_j a_j\right] x = \sum_{j=0}^{r} C_j x a_j$$

$$= \sum_{j=0}^{r} C_j x + a_{j+1} = a_0 + \sum_{j=1}^{r+1} C_{j-1} x a_j a_{j+1}$$

$$= a_0 + \sum_{j=1}^{r+1} C_{j-1} x a_j a_{j+1} + a_{r+1} = \sum_{j=0}^{r+1} C_{j+1} x a_j.$$  

Hence $a_0 = a_0 x^q = \sum_{j=0}^{q} C_j a_j = a_0 + \sum_{j=1}^{q} C_{j-1} x a_j$. That is,

$$0 = \sum_{j=1}^{q} C_{j-1} x a_j = qa_1 + \sum_{j=2}^{q-1} \frac{1}{q} a_{j+1} + a_q.$$
Since $q \neq p$, so $q = kp + t$ for some $t$ with $0 < t < p$. Thus we have $ta_1 = -p(ka_1) - \sum_{j=2}^{q} (\binom{q}{j})a_j \in (A_{i+1} + A_{i}) \leq A_{i+1}$ and then $a_1 \in A_{i+1}$, a contradiction. Hence we have in fact proved that $H = C_{G}(A/pA)$ contains only $p$-elements.

Now we begin to prove the required results.

(1) If $p > 2$ and $H \neq 1$, then, as above, we have the equation

$$0 = p(a_1 + \sum_{j=2}^{p-1} \binom{p}{j}a_j) + a_p,$$

where $a_1 x = a_1 x + a_1$ for some $x \in H$, $a_j \in A_{i+1}$ and $i > i_1 > \cdots > i_{p-1} > i > 0$.

Since $i \geq 1$ and $i \geq 2$, therefore $a_1 + \sum_{j=2}^{p-1} \binom{p}{j}a_j \in A_{i+1} \leq A_i$ and then $a_1 \in A_{i+1}$, contrary to $a_1 \in A_i$. So $p > 2$ implies that $H = C_{G}(A/pA) = 1$.

(2) For the 2-group $H = C_{G}(A/2A)$, if $H$ has an element, say $x$, with order 4, then, since $x^2 \neq 1$ and $x^2 \notin C_{G}(A) = 1$, there exists $0 \neq a_0 \in A$ such that $a_0 x^2 \neq a_0$. Let $A_i = 2^i A$, then, by Lemma 2.1.4, $A_i \subset A_{i+1}$ and $\cap A_i = 0$. Thus $a_0 \in A_i \setminus A_{i+1}$ for some $i \geq 0$. Since $x \in H = C_{G}(A_i/A_{i+1}) = C_{G}(A_i/A_{i+1})$ for any integer $i \geq 0$, so $a_0 x = a_0 x^4 = a_0 + 4a_1$, which implies that $4a_1 = 0$ and then $a_1 = 0$, contrary to $a_0 x^2 \neq a_0$. So we have $a_1 x \neq a_1$. Let $a_1 \in A_i \setminus A_{i+1}$, where $i_1 > i_0$, then, similarly, we have $a_1 x = a_1 + a_2$ for some $a_2 \in A_{i+1}$, where $i_2 > i_1$ and $a_2 x \neq a_2$. Generally, we have
\[ a_{j-1} x = a_{j-1} + a_j \quad \text{with} \quad a_j \in A_{j-1} \setminus A_{j+1}, \quad a x \neq a_j, \quad j > j-1, \quad j = 1, 2, 3, 4. \]

Therefore \[ a_0 = a_0 x^4 = a_0 + 4a_1 + 6a_2 + 4a_3 + a_4. \] That is, \[ 4a_1 + 6a_2 + 4a_3 + a_4 = 0. \]

Since \[ i_4 \geq i_3 + 1 \geq i_2 + 2 > 2, \] so we have \[ a_4 = 2a_4^*, \] where \[ a_4^* \in A_{i_4-1} \setminus A_{i_4}. \]

Thus \[ 2a_1 + 3a_2 + 2a_3 + a_4^* = 0 \] and then \[ a_2 = -2a_1 - (2a_2 + 2a_3 + a_4^*) = -2a_1 + b_2, \]

where \[ b_2 = -(2a_2 + 2a_3 + a_4^*) \in A_{i_2+1}. \] By \[ a_0 x^2 \neq a_0 \] and \[ a_0 x^2 = (a_0 + a_1)x \]

\[ a_0 + 2a_1 + a_2 = a_0 + b_2, \] we have \[ b_2 \neq 0, \] thus \[ b_2 \in A_{i_2} \setminus A_{i_2+1} \] for some \[ j_2 > i_2. \]

If \[ b_2 x = b_2, \] then \[ a_0 = a_0 x^4 = (a_0 x^2)^2 = (a_0 + b_2)^2 = a_0 + 2b_2, \]

which implies that \[ 2b_2 = 0 \] contrary to \[ b_2 \neq 0. \] So \[ b_2 x \neq b_2. \]

As above, we may have \[ b_{k-1} x = b_{k-1} + b_k, \] where \[ b_k \in A_{i_k} \setminus A_{i_k+1}, \] \[ b_k x \neq b_k, \] \[ j_k > j_{k-1}, \] and \[ k = 3, 4. \] Therefore,

\[
\begin{align*}
    a_0 &= a_0 x^4 = (a_0 x^2)^2 = (a_0 + b_2)^2 \\
    &= a_2^2 + b_2^2 = (a_0 + b_2^2) = (a_0 + b_2) + (b_2 + b_4) \\
    &= a_0 + (2b_2 + 2b_3 + b_4).
\end{align*}
\]

That is, \[ 2b_2 + 2b_3 + b_4 = 0 \] and then \[ b_2 + b_3 \in A_{i_4-1} \leq A_{i_3}. \] Hence we get the contradiction that \[ b_2 \in A_{i_3} \leq A_{i_2+1}. \]

By the above contradiction, we have \( H \) contains no elements of order 4 and then, since \( H \) is a 2-group, we must have \( H \) is an elementary abelian 2-group.

The result is proved.

A consequence of Lemma 2.1.6 is that:
**Corollary 2.1.7**: Let $G$ be a locally finite group and $A$ a torsion-free noetherian $\mathbb{Z}G$-module. Then $C_G(A/pA) = C_G(A)$ for any prime $p > 2$.

The following possibly well-known result follows immediately from Corollary 2.1.7.

**Corollary 2.1.8**: If a locally finite group $G$ is contained in the group $GL_n(\mathbb{Z})$, then $G$ is isomorphic with some subgroup of $GL_n(\mathbb{Z}/p)$ for any prime $p > 2$.

Further, $G$ is finite with order at most $3^{2n} \prod_{i=1}^{n} (3^i-1)$.

As an interesting aside, we prove a lemma, even though it has not been used anywhere in the later discussion, but it has a very close relationship with Lemma 2.1.6.

**Lemma 2.1.9**: Let $G$ be a locally finite group, $A$ a torsion-free noetherian $\mathbb{Z}G$-module, and $C_G(A) = 1$. Then $H = C_G(A/4A) = 1$.

**Proof**: Since $H = C_G(A/4A) \leq C_G(A/2A)$, so, by Lemma 2.1.6, $H$ is an elementary abelian 2-group. If $H \neq 1$, then $H$ contains an element $x$, say, of order 2. By $C_G(A) = 1$, there exists $a_0 \in A$ such that $a_0 x \neq a_0$. Let $A_i = 4^iA$, then, by Lemma 2.1.4, $A_i < A$ and $\cap A_i = 0$. Thus there exists $i_0 \geq 0$ such that $a_0 \in A_{i_0} \setminus A_{i_0+1}$. Using Corollary 2.1.3, $A_0/A_i \equiv \mathbb{Z}/4A_i/A_{i+1}$, so $H = C_G(A_i/A_{i+1})$ for any $i \geq 0$. Since $x \in H = C_G(A_i/A_{i+1})$ and $a_0 \in A_i$, so $a_0 x = a_0 + a_1$ for some $0 \neq a_1 \in A_{i+1}$. Let $a_1 \in A_{i+1} \setminus A_{i+2}$, where $i > i_0$, then by $x \in H = C_G(A_i/A_{i+1})$ we have $a_1 x = a_1 + a_2$ for
some $a_2 \in A_{i_1+1}$, On the other hand, $a_0 = a_0 x^2 = (a_0 + a_1) x = a_0 + a_1 + a_1 x$. That is, $a_1 x = -a_1$. So $a_2 = -2a_1$ and then $2a_1 = -a_2$. Similarly, if we let $a_2 \in A_{i_2} \setminus A_{i_2+1}$, where $i_2 > i_1$, then $a_2 x = a_2 + a_3$ for some $a_3 \in A_{i_2+1}$ and $2a_2 = -a_3$. Thus $4a_1 = -2a_2 = a_3 \in A_{i_2+1}$. Since $A$ is torsion-free, $a_1 \in A_{i_2} \leq A_{i_1+1}$ a contradiction. Hence we must have $H = C_G(A/4A)$ is a trivial group. That is, $H = 1$. The result is proved.

§2.2 RELATED TO THE NORMAL SUBGROUPS OF FINITE INDEX

From Wilson's lemma (Lemma 1.2.5), we know that some important properties are inherited by the normal subgroups of finite index. In this section, we give some more properties of this kind.

Lemma 2.2.1: Let $H$ be a normal subgroup of finite index in a group $G$, and $A$ a $\mathbb{Z}G$-module. Then $A$ has a finite $\mathbb{Z}G$-composition series if and only if $A$ has a finite $\mathbb{Z}H$-composition series.

Proof: Since $A$ has a finite composition series if and only if $A$ is both artinian and noetherian, thus the lemma follows from Lemma 1.2.5.

Corollary 2.2.2: If $H$ is a normal subgroup of finite index in a group $G$, then any finite (resp. infinite) irreducible $\mathbb{Z}G$-module $A$ contains a finite (resp. infinite) irreducible $\mathbb{Z}H$-submodule.

Proof: By Lemma 2.2.1, $A$ contains an irreducible $\mathbb{Z}H$-submodule, say $V$. Let $T$
be a left transversal to \( H \) in \( G \), then \( \sum_{t \in T} V_t \) is a nonzero \( \mathbb{Z}G \)-submodule of \( A \) and then \( A = \sum_{t \in T} V_t \) by the irreducibility of \( A \). Since \( T \) is finite, so \( A \) is finite if and only if \( V \) is finite. Thus \( V \) is finite (resp. infinite) if \( A \) is finite (resp. infinite).

**Corollary 2.2.3:** If \( H \) is a normal subgroup of finite index in a group \( G \), and if a \( \mathbb{Z}G \)-module \( A \) contains an irreducible \( \mathbb{Z}G \)-factor \( B/C \), then \( A \) contains an irreducible \( \mathbb{Z}H \)-factor \( U/V \) such that: if \( B/C \) is finite, infinite, or a \( p \)-group for some prime \( p \), then so is \( U/V \).

**Proof:** Consider the \( \mathbb{Z}G \)-module \( B/C \), then the result follows from Corollary 2.2.2 and its proof.

Let \( G \) be a group, \( H \) a normal subgroup of \( G \), and \( A \) a \( RG \)-module, where \( R \) is a ring with 1. Let \( \mathbb{A}^H = \{ U/V ; \ U/V \text{ is a } RH \text{-factor of } A \} \), then \( U_g \) and \( V_g \) are \( RH \)-submodules of \( A \) and so \( G \) acts naturally on \( \mathbb{A}^H \) by the action

\[
(U/V)_g = U_g/V_g \quad \text{for every } g \in G.
\]

**Lemma 2.2.4:** If \( H \) is a normal subgroup of finite index in a group \( G \), and if a \( \mathbb{Z}G \)-module \( A \) contains a nonzero (irreducible) \( \mathbb{Z}H \)-factor \( U/V \), then

1. \( U_g/V_g \) is also a nonzero (irreducible) \( \mathbb{Z}H \)-factor of \( A \), \( \forall \ g \in G \);
2. as groups, \( U/V \cong U_g/V_g \), \( \forall \ g \in G \);
3. \( C_H(U_g/V_g) = g^{-1} C_H(U/V)_g \), \( \forall \ g \in G \);
4. if \( V = 0 \) and \( U \) is irreducible, then \( A \) contains an irreducible \( \mathbb{Z}G \)-submodule \( B \) such that \( B \) has an irreducible \( \mathbb{Z}H \)-submodule \( W \) with \( W \cong \mathbb{Z}G \ U_g \) for some \( g \in G \);
5. for \( B \) and \( W \) in the above (4), \( B = \bigcup_{s \in S} W_s \), where \( S \) is a subset of...
a transversal $T$ to $H$ in $G$; and

(6) if $U/V$ is irreducible, then $A$ contains an irreducible $ZG$-factor $B/C$ such that: if $U/V$ is finite, infinite, or a $p$-group for some prime $p$, then so is $B/C$.

Proof: (1) For any $g \in G$, we show firstly that: (a) $U$ is a $ZH$-submodule of $A$ if and only if $Ug$ is also, and (b) for any $ZH$-submodules $U$ and $V$ of $A$, $U \simeq V$ if and only if $Ug \simeq Vg$.

(a) "\rightleftharpoons": For any $a_1g, a_2g \in Ug$ and any $h \in H$,

$$a_1g - a_2g = (a_1 - a_2)g \in Ug,$$

and

$$(a_1gh = a_1(gh) = a_1(h'g) = (a_1h')g \in Ug$$

since $U$ is a $ZH$-submodule and $H$ is normal in $G$. So $Ug$ is a $ZH$-submodule.

"\rightleftharpoons": Since $Ug$ is a $ZH$-submodule of $A$, by the arbitrariness of $g$ and the above proof, we have $U = (Ug)g^{-1}$ is also a $ZH$-submodule of $A$.

(b) "\rightleftharpoons": For any $ag \in Vg$, since $a \in V \leq U$ so $ag \in Ug$, thus $Vg \leq Ug$;

"\rightleftharpoons": By the arbitrariness of $g$ and the above proof, we have

$$V = (Vg)g^{-1} \leq (Ug)g^{-1} = U.$$

We secondly note that: (c) if $W$ is a $ZH$-submodule of $Ug$ for some $ZH$-submodule $U$ of $A$, then there exists a $ZH$-submodule $V$ of $U$ such that $W = Vg$, namely $V = Wg^{-1}$.

By (a), (b) and (c), it is clear that $Ug/Vg$ is also a nonzero (irreducible) $ZH$-factor of $A$ for any nonzero (irreducible) $ZH$-factor $U/V$ of $A$, where $g \in G$.

(2) Let $\varphi: u \mapsto ug + Vg$, where $u \in U$. Then it is clear that $\varphi$ is a group homomorphism from $U$ to $Ug/Vg$, thus $U/\ker\varphi \simeq Ug/Vg$. Obviously,
$V \leq \ker \varphi$. On the other hand, $u \in \ker \varphi$ implies that $ug \in Vg$, thus there exists $v \in V$ such that $ug = vg$ and then $u = (ug)g^{-1} = (vg)g^{-1} = v$, so $\ker \varphi \leq V$ and so $V = \ker \varphi$. That is, as groups, $U/V \cong Ug/Vg$.

(3) Let $h \in C_H(Ug/Vg)$; then $ghg^{-1} \in H$ and

$$(u + V)ghg^{-1} = (ug + Vg)gh^{-1} = (ug + Vg)g^{-1} = u + V,$$

so $ghg^{-1} \in C_H(U/V)$ and then $h \in g^{-1}C_H(U/V)g$. That is,

$$C_H(Ug/Vg) \leq g^{-1}C_H(U/V)g.$$

On the other hand, $h \in g^{-1}C_H(U/V)g$ implies that $h = g^{-1}h^*g \in H$, where $h^* \in C_H(U/V)$. Since

$$(ug + Vg)h = (ug + Vg)g^{-1}h^*g = (u + V)h^*g = (u + V)g = ug + Vg,$$

so $h \in C_H(Ug/Vg)$ and thus $g^{-1}C_H(U/V)g \leq C_H(Ug/Vg)$.

Therefore $C_H(Ug/Vg) = g^{-1}C_H(U/V)g$.

(4) Let $T$ be a left transversal to $H$ in $G$, then $T$ is a finite set. Let $D = \sum_{t \in T} U_t$, then $D$ is a $\mathbb{Z}G$-submodule of $A$ and, by (1), $D$ is a sum of finitely many irreducible $\mathbb{Z}H$-submodules thus $D$ is completely reducible (Lemma 1.2.1). Since $D$ has a finite $\mathbb{Z}H$-composition series, so $D$ has a finite $\mathbb{Z}G$-composition series (Lemma 2.2.1); thus $D$ (and then $A$) contains an irreducible $\mathbb{Z}G$-submodule, say $B$. By Lemma 1.2.2, $D = B \oplus D_r \in \mathcal{S}U_s$, $B \equiv \mathbb{Z}H D_r \in \mathcal{S}U_s$, and $S$ is a subset of $T$ and $S' = T \backslash S$. Thus, it is clear that $B$ contains an irreducible $\mathbb{Z}H$-submodule $W$ such that $W \equiv \mathbb{Z}U_g$ for some $g \in S' \subseteq G$.

(5) For the above $B$ and $W$, $\sum_{t \in T} W_t$ is a nonzero $\mathbb{Z}G$-submodule of $B$ and then, by the irreducibility of $B$, $B = \sum_{t \in T} W_t$. Thus by Lemma 1.2.1, $B = D_r \in \mathcal{S}W_s$ for some subset $S \subseteq T$.

(6) Let $U/V$ be an irreducible $\mathbb{Z}H$-factor of $A$, then by Zorn's lemma,
there is a $\mathbb{Z}H$-submodule $M$ of $A$ maximal with respect to $U \cap M = V$. This implies that $V \leq M$ and so $(U+M)/M \cong \mathbb{Z}H U/V$ is irreducible. If $M < M_1$ then $U+M \leq M_1$ otherwise $(U+M) \cap M_1 = M$ so $U \cap M_1 = U \cap (U+M) \cap M_1 = U \cap M = V$, contrary to the choice of $M$. If $M$ is not a $\mathbb{Z}G$-submodule of $A$, then there is a $\mathbb{Z}H$-submodule $W$ of $A$ of the form $W = \bigcap_{i=1}^{n} Mg_i$ such that $M \cap W$ is a $\mathbb{Z}G$-submodule properly contained in $W$; thus $M$ is properly contained in the $\mathbb{Z}H$-submodule $M+W$ and $U+M = (U+M) \cap (M+W) = M + [(U+M) \cap W]$, so $(U+M)/M \cong \mathbb{Z}H [(U+M) \cap W]/(M \cap W)$. That is $[(U+M) \cap W]/(M \cap W)$ is irreducible and $M \cap W$ is a $\mathbb{Z}G$-submodule of $A$. Also, by

\[
[(U+M) \cap W]/(M \cap W) \cong \mathbb{Z}H (U+M)/M \cong \mathbb{Z}H U/V
\]

we have that: if $U/V$ is finite, infinite, or a $p$-group, then so is $[(U+M) \cap W]/(M \cap W)$. Now by (4) and (5) above, we have the required $\mathbb{Z}G$-factor $B/C$.

The lemma is proved.

From Lemma 2.2.4, we have the following consequence:

**Lemma 2.2.5:** Let $H$ be a normal subgroup of finite index in a group $G$, and let the $\mathbb{Z}G$-module $A$ have a nonzero $\mathbb{Z}H$-submodule $W$. Then there is a one-to-one correspondence between the $\mathbb{Z}H$-factors of $W$ and those of $Wg$ for any $g \in G$, and this correspondence preserves the finiteness and the irreducibility of these $\mathbb{Z}H$-factors.

**Proof:** From the proof of (1) in Lemma 2.2.4, it is clear that the mapping $\varphi: U/V \mapsto Ug/Vg$ is a one-to-one correspondence between the $\mathbb{Z}H$-factors of $W$ and those of $Wg$, and from (1) and (2) in Lemma 2.2.4, $\varphi$ preserves the finiteness and the irreducibility of the $\mathbb{Z}H$-factors. So the lemma is true.
Another consequence of Lemma 2.2.4 is:

**Lemma 2.2.6:** Let $H$ be a normal subgroup of finite index in a group $G$ and let $A$ be a $\mathbb{Z}G$-module. Then $A$ contains an irreducible $\mathbb{Z}G$-factor being finite, infinite, or a $p$-group for some prime $p$ if and only if $A$ as a $\mathbb{Z}H$-module contains an irreducible $\mathbb{Z}H$-factor being finite, infinite, or a $p$-group for the prime $p$, respectively.

**Proof:** It follows from Corollary 2.2.3 and (6) in Lemma 2.2.4.

Furthermore, we have:

**Lemma 2.2.7:** If $H$ is a normal subgroup of finite index in a group $G$, and if $D$ is a $\mathbb{Z}H$-submodule of a $\mathbb{Z}G$-module $A$, then the $\mathbb{Z}G$-submodule $D_G^G = \sum_{g \in G} D_g$ of $A$ has a finite (resp. infinite) irreducible $\mathbb{Z}G$-factor if and only if $D$ has a finite (resp. infinite) irreducible $\mathbb{Z}H$-factor.

**Proof:** By (6) in Lemma 2.2.4, the sufficiency is evident. So we prove the necessity.

Suppose $T = \{t_1, t_2, \ldots, t_n\}$ is a transversal to $H$ in $G$. If $D_G^G$ has a finite (resp. infinite) irreducible $\mathbb{Z}G$-factor then, by Corollary 2.2.3, $D_G^G$ has a finite (resp. infinite) irreducible $\mathbb{Z}H$-factor, say $B_0/C_0$. Since $D_G^G = \sum_{t_i \in T} D_t$, we may choose an integer $n_0$ such that $n_0$ is minimal with respect to $\sum_{t_i = 1}^{n_0} D_t$ has a finite (resp. infinite) irreducible $\mathbb{Z}H$-factor $B/C$. We show that $n_0 = 1$. If not, we should have $(B + \sum_{t_i = 1}^{n_0-1} D_t) > (C + \sum_{t_i = 1}^{n_0-1} D_t)$ for otherwise by $B/C$ being irreducible we have

$$B/C = \mathbb{Z}H \left( B \cap \sum_{t_i = 1}^{n_0-1} D_t \right) / \left( C \cap \sum_{t_i = 1}^{n_0-1} D_t \right)$$

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contrary to the minimality of $n_0$. So

$$B/C \cong_{ZH} \left( B + \sum_{i=1}^{n_0} D_{t_i} \right) / \left( C + \sum_{i=1}^{n_0} D_{t_i} \right)$$

$$\leq \sum_{i=1}^{n_0} D_{t_i} / \left( C + \sum_{i=1}^{n_0} D_{t_i} \right)$$

$$\cong_{ZH} \left[ D_{t_0} / \left( D_{t_0} \cap \left( C + \sum_{i=1}^{n_0} D_{t_i} \right) \right) \right].$$

That is, $D_{t_0}$ has a finite (resp. infinite) irreducible $ZH$-factor. By Lemma 2.2.5, $D_{t_0}$ has a finite (resp. infinite) irreducible $ZH$-factor, contrary to the minimality of $n_0$ again. So $n_0 = 1$. That is, $B/C$ is a finite (resp. infinite) irreducible $ZH$-factor of $D$. The lemma is true.

Using Lemma 2.2.7, we have:

**Corollary 2.2.8:** If $H$ is a normal subgroup of finite index in a group $G$, then the $ZG$-module $A$ contains no nonzero $ZG$-submodules with an $f$-($ZG$)-decomposition if and only if $A$ as a $ZH$-module contains no nonzero $ZH$-submodules with an $f$-($ZH$)-decomposition.

**Proof:** For a $ZG$-submodule $C$ of $A$, if $C$ has an $f$-($ZG$)-decomposition

$$C = C^f \oplus C^\bar{f},$$

then it follows from Lemma 2.2.6 that this is also an $f$-($ZH$)-decomposition.

Suppose $A$ as a $ZH$-module contains a nonzero $ZH$-submodule $D$ with an $f$-($ZH$)-decomposition, i.e., $D = D^f \oplus D^\bar{f}$, where $D^f$ is the $f$-component of $D$ and $D^\bar{f}$ is the $\bar{f}$-component of $D$. Since $D^G = (D^f)^G + (D^\bar{f})^G$, and by Lemma 2.2.7 $(D^f)^G$ has only finite irreducible $ZG$-factors and $(D^\bar{f})^G$ has only infinite irreducible $ZG$-factors, it follows that $D^G = (D^f)^G \oplus (D^\bar{f})^G$ is the
f-(ZG)-decomposition of $D^G$. That is, the nonzero $ZG$-submodule $D^G$ of $A$ has an f-(ZG)-decomposition, contrary to $A$ having no such $ZG$-submodules. So the corollary is true.

From the proof of Corollary 2.2.8, we have the following two corollaries.

**Corollary 2.2.9:** Let $H$ be a normal subgroup of finite index in a group $G$ and let $D$ be a $ZH$-submodule of the $ZG$-module $A$. Then $D$ has an $f-(ZH)$-decomposition if and only if $D^G$ has an $f-(ZG)$-decomposition.

**Corollary 2.2.10:** If $H$ is a normal subgroup of finite index in a group $G$, then a $ZG$-module $A$ has an $f-(ZG)$-decomposition if and only if $A$ has an $f-(ZH)$-decomposition.

§2.3 RELATED TO $f$-DECOMPOSITION

At the end of the last section, we have noted some results related to the $f$-decomposition of the $ZG$-modules. Here, we prove some more results, which will play an important role in the proof of our main results.

**Lemma 2.3.1:** Let $G$ be a group, $F$ a nontrivial finite normal subgroup of $G$, $A$ a $ZG$-module, and $B$ a $ZG$-submodule of $A$. If $A/B$ has an $f$-decomposition and $F \trianglelefteq C_G(B)$ but $F$ is not contained in $C_G(A)$, then $A$ has a nonzero $ZG$-submodule $D$ with an $f$-decomposition, too. Furthermore, $D$ can be chosen such that:

1. if $(A/B)^f = 0$ then $D^f = 0$, and
2. if $(A/B)^f = 0$ then $D^f = 0$. 

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Proof: Let $H = C(G)(F)$, then $|G/H| < \infty$. By Corollary 2.2.10, $A/B$ as a $ZH$-module has an $f-(ZH)$-decomposition, and then each $ZH$-image of $A/B$ has an $f-(ZH)$-decomposition (Lemma 1.2.23). For $x \in F$, we have $B \leq C_A(x) \leq A$. Since $F$ is not contained in $C_G(A)$, there exists $x_0 \in F$ such that $B \leq C_A(x_0) < A$. Thus the nonzero $ZH$-submodule $C = A(x_0 - 1) (\equiv ZH A/C_A(x_0))$ has an $f-(ZH)$-decomposition. By (6) in Lemma 2.2.4, the $ZH$-submodule $C$ has the properties that: (i) if $(A/B)f = 0$, then $Cf = 0$; and (ii) if $(A/B)f = 0$, then $Cf = 0$. Let $D = C_G = \sum_{g \in G} C_g$, then $D$ is a nonzero $ZG$-submodule of $A$, and, by Corollary 2.2.9, $D$ has an $f-(ZG)$-decomposition and, further, satisfies (1) if $(A/B)f = 0$, then $Df = 0$; and (2) if $(A/B)f = 0$, then $Df = 0$. The lemma is proved.

Corollary 2.3.2: Let $G$ be a hyperfinite group, $A$ a $ZG$-module, and $B$ a nonzero $ZG$-submodule of $A$ such that each irreducible $ZG$-factor of $A/B$ is finite (resp. infinite). If $A$ has no nonzero $ZG$-submodules with all irreducible $ZG$-factors being finite (resp. infinite), then $C_G(B) = C_G(A)$.

Proof: We assume that $G$ acts faithfully on $A$, i.e., $C_G(A) = 1$. If $C_G(B) \neq 1$, then since $G$ is hyperfinite, $C_G(B)$ contains a nontrivial finite subgroup $F$ being normal in $G$. By Lemma 2.3.1, $A$ has a nonzero $ZG$-submodule with all irreducible $ZG$-factors being finite (resp. infinite), a contradiction. So $C_G(B) = 1$. That is, $C_G(B) = C_G(A)$ as required.

The following lemma will be important in our proof for the main decomposition theorem in Chapter 3 as it allows us to assume that $G$ acts faithfully when we pass to certain submodules.

Lemma 2.3.3: Let $G$ be a hyperfinite group which contains a normal locally
soluble subgroup $H$, $A$ a noetherian $\mathbb{Z}G$-module, and $B$ a nonzero $\mathbb{Z}G$-submodule of $A$ satisfying $B = B^f$ and $A/B = (A/B)^f$ (resp. $B = B^f$ and $(A/B)^f = A/B$).

If $C_B(H) = 0$, $A/C$ has an $f$-decomposition for any nonzero $\mathbb{Z}G$-submodule $C$ of $B$, and $A$ has no nonzero $\mathbb{Z}G$-submodule $D$ with $D = D^f$ (resp. $D = D^f$), then there is a $K \leq H$ and a nonzero $\mathbb{Z}G$-submodule $B^* \leq B$ such that $K$ is normal in $G$, $A/B^* = B/B^* \oplus A^*/B^*$ for some $\mathbb{Z}G$-submodule $A^*$ of $A$, $C_{B^*}(KC_G(A^*)/C_G(A^*)) = 0$, and $KC_G(A^*)/C_G(A^*)$ is a finite elementary abelian $q$-subgroup of $G/C_G(A^*)$ for some prime $q$.

**Proof:** By Lemma 1.2.14, there is a $K \leq H$ and a nonzero $\mathbb{Z}G$-submodule $B^*_1 \leq B$ such that $K$ is normal in $G$, $C_{B^*_1}(K) = 0$, and $|K/C_K(B^*_1)| < \infty$.

Let $A^*_1$ be the $\mathbb{Z}G$-submodule of $A$ such that $A/B^*_1 = B/B^*_1 \oplus A^*_1/B^*_1$, then by Corollary 2.3.2, $C_G(B^*_1) = C_G(A^*_1)$. Since $|K/C_K(B^*_1)| < \infty$ and $KC_G(A^*_1)/C_G(A^*_1) = KC_G(B^*_1)/C_G(B^*_1) \cong K/C_K(B^*_1)$, $KC_G(A^*_1)/C_G(A^*_1)$ is a finite subgroup of $G/C_G(A^*_1)$.

Choose $K$ such that $KC_G(A^*_1)/C_G(A^*_1)$ is minimal with respect to $C_{B^*_1}(K) = 0$.

Let $K_0$ be a normal subgroup of $G$ such that $KC_G(A^*_1) > K_0 \supseteq C_G(A^*_1)$ and $KC_G(A^*_1)/K_0$ is a minimal normal subgroup of $G/K_0$. By the minimality of $KC_G(A^*_1)/C_G(A^*_1)$, we have $C_{B^*_1}(K_0) \neq 0$. Let $B^*_2 = C_{B^*_1}(K_0)$, then $A^*_1/B^*_2 = B^*_1/B^*_2 \oplus A^*_2/B^*_2$. Since $C_{B^*_2}(K) \subseteq C_{B^*_1}(K) = 0$ and $C_G(A^*_1) \subseteq C_G(A^*_2)$, so $K$ is not contained in $C_G(A^*_2)$ and $KC_G(A^*_1)C_G(A^*_2) = KC_G(A^*_2)$. That is, $KC_G(A^*_2)/C_G(A^*_2)$ is a nontrivial subgroup of $G/C_G(A^*_2)$. Also, by Corollary 2.3.2 again, $C_G(B^*_2) = C_G(A^*_2)$. Since $K_0 \leq C_G(B^*_2)$, so $K_0 \leq C_G(A^*_2)$, and then $KC_G(A^*_1) \geq C_G(A^*_2) \cap (KC_G(A^*_1)) \geq K_0$. By
\[ \frac{K(G(A_2))}{C_0(A_2)} = K(G(A_1))C_0(A_2) \equiv \frac{K(G(A_1))}{[C_0(A_2) \cap (K(G(A_1)))]} \]

and \( \frac{K(G(A_1))}{K_0} \) being a finite characteristically simple group, we must have \( \frac{K(G(A_2))}{C_0(A_2)} \) is a finite characteristically simple group. Since \( H \) is locally soluble and \( K \leq H \), \( \frac{K(G(A_2))}{C_0(A_2)} \) is a finite characteristically simple subgroup of the locally soluble group \( HC_0(A_2)/C_0(A_2) \). So \( \frac{K(G(A_2))}{C_0(A_2)} \) is a finite elementary abelian \( q \)-group for some prime \( q \). Since \( C_{B_{2}}(K) = 0 \), we have \( C_{B_{2}}(K(G(A_2))/C_0(A_2)) = 0 \).

Now, since \( A = B + A_1 = B + A_2 + A_2 = B + A_2 \) and \( B \cap A_2 = B \cap (A_1 \cap A_2) = (B \cap A_1) \cap A_2 = B_1 \cap A_2 = B_2 \), so \( A/B_2 = B/B_2 \oplus A_2/B_2 \). Let \( B^{*} = B_2 \) and \( A^{*} = A_2 \). The required results follow.

§2.4 SOME OTHERS

This section comprises generalizations of one of Zaicev's results as well as other important results.

**Lemma 2.4.1:** Let \( G \) be a hyperfinite group, \( A \) a noetherian \( ZG \)-module, and \( B \) a \( ZG \)-submodule of \( A \) such that each irreducible \( ZG \)-factor of \( B \) is not a finite \( p \)-group for some fixed prime \( p \). If \( A/B \) is a finite \( p \)-group, then \( B \) has a complement in \( A \), i.e., \( A = B \oplus C \) for some \( ZG \)-submodule \( C \) of \( A \).

**Proof:** Suppose \( B \) does not have a complement in \( A \), then by the noetherian condition we may assume that for every nonzero \( ZG \)-submodule \( C \) of \( B \), \( B/C \) has a complement in \( A/C \). Let \( D_{0} \) be a \( ZG \)-submodule of \( A \) maximal subject to \( B \cap D_{0} = 0 \).
Since $A \neq B \oplus D_0$, by replacing $A$ by $A/D_0$ we may assume that for each nonzero \(ZG\)-submodule $D$ of $A$, $B \cap D \neq 0$.

We show first that $A$ is not torsion-free. For otherwise, since $A/B$ is a finite $p$-group, there is a $ZG$-submodule $A^*$ of $A$ such that $A^*/B$ is a nontrivial elementary abelian $p$-group and, by $A$ being torsion-free, $A^*$ is also torsion-free, thus $pA^* \neq 0$ and then $A^* \cong ZG pA^* \leq B$ contrary to $B$ containing no irreducible $ZG$-factors being finite $p$-groups. So $A$ is not torsion-free. Let $T(A)$ be the torsion part of $A$, then $T(A)$ is a nonzero $ZG$-submodule of $A$. Thus $T(B) = T(A) \cap B \neq 0$. Let $B_1$ be the nonzero $ZG$-submodule of $B$ generated by all the elements of order $q$ for some prime $q$, then $A/B_1 = B/B_1 \oplus A_1/B_1$ for some $ZG$-submodule $A_1$ of $A$. Since $A_1/B_1 \cong ZG A/B$, $A_1/B_1$ is also a finite $p$-group. If $q \neq p$, then it is clear that $A_1 = B_1 \oplus A_2$ for some $ZG$-submodule $A_2$ of $A_1$ (and hence of $A$), thus $A = B \oplus A_2$ contrary to $B$ having no complement in $A$. Therefore, $q = p$ and then $A_1$ is a $p$-group. Thus $B_1$ is a $ZG$-submodule of $A_1$ such that $A_1/B_1$ is finite while $B_1$ has no nonzero finite irreducible $ZG$-factors. By Lemma 1.2.22, $B_1$ has a complement in $A_1$, and then $B$ has a complement in $A$, a contradiction.

**Lemma 2.4.2:** Let $G$ be a hyperfinite group, $A$ a noetherian $ZG$-module, and $B$ a $ZG$-submodule of $A$ such that $B$ is a finite $p$-group for some prime $p$. If $A/B$ has no nonzero finite $ZG$-factors being $p$-groups, then $B$ has a complement in $A$, i.e., $A = B \oplus C$ for some $ZG$-submodule $C$ of $A$.

**Proof:** Suppose $B$ does not have a complement in $A$, then by the noetherian condition we may assume that for every nonzero $ZG$-submodule $C$ of $B$, $B/C$ has a complement in $A/C$. Let $D_0$ be a $ZG$-submodule of $A$ maximal with respect to $B \cap D_0 = 0$. Since $A \neq B \oplus D_0$, by replacing $A$ by $A/D_0$ we may assume that for
each nonzero $\mathbb{Z}G$-submodule $D$ of $A$, $B \cap D \neq 0$, i.e., each nonzero $\mathbb{Z}G$-submodule of $A$ contains nonzero $\mathbb{Z}G$-submodules being finite $p$-groups.

Let $B_1$ be the nonzero $\mathbb{Z}G$-submodule of $B$ generated by all the elements of order $p$, then $A/B_1 = B/B_1 \oplus A_1/B_1$ for some $\mathbb{Z}G$-submodule $A_1$ of $A$. Let $\phi: A_1 \rightarrow pA_1$ defined by $\phi: a \mapsto pa$ ($\forall a \in A_1$), then $\phi$ is a $\mathbb{Z}G$-homomorphism from $A_1$ to $pA_1$. It is clear that $B_1 \leq \ker \phi$. Since $pA_1 \cong_\mathbb{Z}G A_1/\ker \phi$ and $A_1/B_1 (\cong_\mathbb{Z}G A/B)$ has no nonzero $\mathbb{Z}G$-factors being finite $p$-groups, $pA_1$ has no nonzero $\mathbb{Z}G$-factors being finite $p$-groups. But each nonzero $\mathbb{Z}G$-submodule of $A$ (and hence of $A_1$) contains nonzero $\mathbb{Z}G$-submodules being finite $p$-groups, therefore we must have $pA_1 = 0$ and then $A_1$ is a $\mathbb{Z}G$-module with the $\mathbb{Z}G$-submodule $B_1$ such that $B_1$ is finite and the factor-module $A_1/B_1$ has no nonzero finite $\mathbb{Z}G$-factors. By Lemma 1.2.22, $B_1$ has a complement in $A_1$ and then $B$ has a complement in $A$, a contradiction.

Combining the above two lemmas, we have:

**Corollary 2.4.3:** Let $G$ be a hyperfinite group, $A$ a noetherian $\mathbb{Z}G$-module, and $B$ a $\mathbb{Z}G$-submodule of $A$ such that $B$ (resp. $A/B$) is a finite $p$-group for some prime $p$. If $A/B$ (resp. $B$) contains no nonzero finite $\mathbb{Z}G$-factors being $p$-groups, then $B$ has a complement in $A$, i.e., $A = B \oplus C$ for some $\mathbb{Z}G$-submodule $C$ of $A$.

From Corollary 2.4.3, we have:

**Corollary 2.4.4:** Let $G$ be a hyperfinite group and $A$ a noetherian $\mathbb{Z}G$-module. Then some irreducible $\mathbb{Z}G$-image of $A$ is a finite $p$-group for some prime $p$ if and only if so is some irreducible $\mathbb{Z}G$-factor of $A$.

**Proof:** If $A$ has an irreducible $\mathbb{Z}G$-factor being a finite $p$-group for the
prime $p$, then by the noetherian condition $A$ has an irreducible $ZG$-factor, say
$B/C$, such that the $ZG$-submodule $B/C$ of $A/C$ is a finite $p$-group and the
factor-module $A/C/B/C$ ($\cong ZG A/B$) contains no irreducible $ZG$-factors being
finite $p$-groups. By Corollary 2.4.3, $A/C$ and then $A$ has an irreducible
$ZG$-image being a finite $p$-group.

The above are the generalizations of Zaicev's result (see Lemma 1.2.22).
Now we prove some other important lemmas.

**Lemma 2.4.5:** Let $G$ be a hyperfinite locally soluble group and $A$ a noetherian
$ZG$-module with $pA = 0$ for some prime $p$. If all irreducible $ZG$-factors of $A$
are finite, then $A$ is finite.

**Proof:** Suppose $A$ is not finite, then by the noetherian condition we may
assume that for every nonzero $ZG$-submodule $C$ of $A$, $A/C$ is finite.

We have $A = \langle a_1, \ldots, a_n \rangle^G$ with $n$ being an integer and the order of $a_i$
is $p$ for each $i$. Also we may assume that $G$ acts faithfully on $A$ and, since $A$
is infinite, $G$ is infinite.

Let $M$ be a maximal $ZG$-submodule of $A$, then $A/M$ is finite and hence for
$H = C_G(A/M)$ we have $|G/H| < \infty$. Since $H$ is nontrivial, so $H$ contains a
nontrivial finite subgroup, say $K$, being minimal normal in $G$ (Lemma 1.2.10).

By $G$ being a locally soluble group, $K$ is an elementary abelian subgroup of $G$.
Let $G_1 = C_G(K)$, then $K < G_1$ and $|G/G_1| < \infty$. By Lemma 1.2.5 and (6) in
Lemma 2.2.4, $A$ is an infinite noetherian $ZG_1$-module and all irreducible
$ZG_1$-factors of $A$ are finite. Since the $ZG_1$-image $A/M$ is finite, using the
noetherian condition, we may have a $ZG_1$-image $A^* (= A/D)$ of $A$ such that
$D < M$ and $A^*$ is infinite but for every nonzero $ZG_1$-submodule $C^*$ of $A^*$, $A^*/C^*$
is finite. By $G$ being faithful on $A$, we have the $ZG$-submodule $[A, K] \neq 0$ and
then the $\mathcal{Z}G$-image (and so $\mathcal{Z}G_1$-image) $A/[A, K]$ is finite. Thus $[A, K]$ is not contained in $D$ and then $K$ is not contained in $C_{G_1}(A/D) = C_{G_1}(A^*)$.

Replacing $G_1$ by $G_1/C_{G_1}(A^*)$ and $K$ by $(KC_{G_1}(A^*)/C_{G_1}(A^*)$ we may assume that $G_1$ acts faithfully on $A^*$, then $K$ is a nontrivial finite central subgroup of $G_1$. Let $1 \neq x \in K$ such that $x$ is of order $q$ for some prime $q$. If $q \neq p$, then, by Lemma 1.2.4, $A^* = C_{A^*}(\langle x \rangle) \oplus [A^*, \langle x \rangle]$. Since $C_{G_1}(A^*) = 1$, so the $\mathcal{Z}G_1$-submodule $A^*(x-1) = [A^*, x] = [A^*, \langle x \rangle] \neq 0$. Also, since $M^* = M/D$ and $A^*/M^* \cong \mathcal{Z}G_1 A/M$, so $\langle x \rangle \leq K \leq C_{G_1}(A^*/M^*)$, therefore $A^*(x-1) \leq M^* < A^*$. Thus $C_{A^*}(x) = C_{A^*}(\langle x \rangle) \neq 0$. But $C_{A^*}(x) (\cong \mathcal{Z}G_1 A^*/A^*(x-1))$ and $A^*(x-1)$ are both finite and then $A^*$ is finite, a contradiction. So $q = p$. Consider the finite $\mathcal{Z}<x>$-module $A^*_1 = \langle a \rangle^{\langle x \rangle}$, where $0 \neq a \in A^*$. Since $A^*_1$ is a finite $p$-group, there exists $0 \neq a_0 \in A^*_1$ such that $a_0 \in C_{A^*_1}(x)$. By $C_{G_1}(A^*) = 1$ we have $A^* \neq C_{A^*_1}(x)$ and then $A^*(x-1)$ is a nonzero finite $\mathcal{Z}G_1$-module. Therefore $A^*/A^*(x-1)$ is finite and then $A^*$ is finite, a contradiction again. The result holds.

For a general hyperfinite group $G$ (that is, $G$ need not to be locally soluble), we have:

**Lemma 2.4.6:** Let $G$ be a hyperfinite group and $A$ a noetherian $\mathcal{Z}G$-module with $pA = 0$ for some prime $p$. If $G$ is a $p'$-group and all irreducible $\mathcal{Z}G$-factors of $A$ are finite, then $A$ is finite.

**Proof:** Suppose $A$ is infinite, then using the noetherian condition we may
assume that for any nonzero $\mathbb{Z}G$-submodule $C$ of $A$, $A/C$ is finite.

We have $A = \langle a_1, \ldots, a_n \rangle^G$ with $n$ being an integer and the order of $a_i$ is $p$ for each $i$. Also we may assume that $G$ acts faithfully on $A$ and, since $A$ is infinite, $G$ is infinite.

Let $M$ be a maximal $\mathbb{Z}G$-submodule of $A$, then $A/M$ is finite and hence for $H = C_G(A/M)$ we have $|G/H| < \infty$. Since $H$ contains nontrivial finite subgroups being normal in $G$ (Lemma 1.2.10), so we may let $K \leq H$ and $K$ is a finite normal subgroup of $G$. By $G$ being a $p'$-group, we have $K$ is a finite $p'$-group, thus, by Lemma 1.2.4, $A = C_A(K) \oplus [A, K]$. Since $K \leq H = C_G(A/M)$, so $[A, K] \leq M$ and then $C_A(K) \neq 0$. By $G$ being faithful on $A$, we have $C_A(K) \neq A$ and then $[A, K] \neq 0$. Thus $C_A(K) (\cong \mathbb{Z}G A/[A, K])$ and $[A, K] (\cong \mathbb{Z}G A/C_A(K))$ are both finite and then $A$ is finite, a contradiction. Hence the result holds.

The final result of this chapter has a very special but simple proof. We mention that it will play an important role in our work similar with that of Fitting's lemma.

**Lemma 2.4.7**: Let $G$ be a group, $A$ a $\mathbb{Z}G$-module, and $M$ a $\mathbb{Z}G$-submodule of $A$ such that $A/M$ is a $p$-group for some prime $p$. If $H = C_G(A/M)$ contains a nontrivial subgroup $K$ such that $K$ is a finite $q$-group for some prime $q$ other than $p$, then $A = C_A(x) + M$ for any $x \in K$. Further, $A = C_A(K) + M$. (We note that the subgroups $C_A(x)$ and $C_A(K)$ may not be $\mathbb{Z}G$-submodules of $A$.)

**Proof**: Let $x \in K$, then $x^q = 1$ for some integer $n$. Since

$$(x^{q_n-1} + x^{q_n-2} + \cdots + x + 1)(x-1) = x^{q_n} - 1 = 0,$$

and

$$q^n = (x^{q_n-1} + x^{q_n-2} + \cdots + x + 1)$$

$$-[x^{q_n-2} + 2x^{q_n-3} + \cdots + (q_n-2)x + (q_n-1)](x-1).$$

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So $q^n A \leq A(x^{q^n-1} + x^{q^n-2} + \cdots + x + 1) + A(x-1) \leq C_A(x) + M$. That is, $A/(C_A(x) + M)$ is a $q$-group. But $A/M$ is a $p$-group and $p \neq q$, so we must have $A = C_A(x) + M$.

Let $K = \{x_1 = 1, x_2, \cdots, x_t\}$ and let $C_m = C_A(x_1, \cdots, x_m)$, where $m = 1, 2, \cdots, t$. Suppose that $A = C_m + M$, we prove that $A = C_{m+1} + M$. By the equations in (*) above, we have

$$q^n C_m \leq C_m(x^{q^n-1} + x^{q^n-2} + \cdots + x_{m+1} + 1) + C_m(x_{m+1} - 1)$$

$$\leq C_m(x_{m+1}) + (M \cap C_m) \leq C_{m+1} + (M \cap C_m).$$

Since $A = C_m + M$ so, as groups, $A/M \equiv C_m/(M \cap C_m)$. That is, $C_{m+1}/(M \cap C_m)$ is also a $p$-group. By $q^n C_m \leq C_{m+1} + (M \cap C_m)$ and $q \neq p$, we have $C_m = C_{m+1} + (M \cap C_m)$, and then

$$A = C_m + M = C_{m+1} + (M \cap C_m) + M = C_{m+1} + M$$

as required. So $A = C_m + M$ for all $m$. In particular, put $m = t$, then $C_t = C_A(K)$ and $A = C_A(K) + M$. The lemma is proved.
In this chapter, we aim to prove the main result—Theorem A.

If a noetherian $\mathbb{Z}G$-module $A$ over a hyperfinite group $G$ contains a $\mathbb{Z}G$-submodule $B$ such that $B$ (resp. $A/B$) is finite and $A/B$ (resp. $B$) contains no nonzero finite irreducible $\mathbb{Z}G$-factors, then $B$ has a complement in $A$ (Lemma 1.2.22). In §2.4, we have given a generalization of this result. Now we will prove a number of other generalizations and corollaries under the condition that $G$ is hyperfinite and locally soluble.

The following is in fact the beginning of the main proof for Theorem A.

**Proposition 3.1:** Let $G$ be a hyperfinite locally soluble group, $A$ a noetherian $\mathbb{Z}G$-module, and $B$ a $\mathbb{Z}G$-submodule of $A$ such that each irreducible $\mathbb{Z}G$-factor of $B$ is finite while $A/B$ has no finite irreducible $\mathbb{Z}G$-factors. Then $B$ has a complement in $A$, i.e., $A = B \oplus C$ for some $\mathbb{Z}G$-submodule $C$ of $A$.

**Proof:** Suppose that $B$ does not have a complement in $A$. By considering an appropriate factor-module of $A$ we may assume that for every nonzero $\mathbb{Z}G$-submodule $B$ of $B$, $B/B'$ has a complement in $A/B'$. Also, let $C$ be a $\mathbb{Z}G$-submodule of $A$ maximal with respect to $B \cap C = 0$. Then $A \neq B \oplus C$. By replacing $A$ by $A/C$ we may assume that for every nonzero $\mathbb{Z}G$-submodule $D$ of $A$, $B \cap D \neq 0$. That is, $A$ contains no nonzero $\mathbb{Z}G$-submodules with all irreducible $\mathbb{Z}G$-factors being infinite.

If $A$ is not torsion-free, let $T(A)$ be the torsion part of $A$, then $T(A)$ is a nonzero $\mathbb{Z}G$-submodule of $A$ and then $T(B) = B \cap T(A) \neq 0$. Let $B^*$ be the nonzero $\mathbb{Z}G$-submodule of $B$ generated by all the elements of order $p$ for some prime $p$, then $A/B^* = B/B^* \oplus A^*/B^*$ for some $\mathbb{Z}G$-submodule $A^*$ of $A$. Let $\varphi: A^* \to pA^*$
be defined by φ: a → pa for all a ∈ A*, then φ is a ZG-homomorphism from A* to pA* and B* ≤ ker φ. If pA* ≠ 0, then since pA* ≅ G A*/kerφ and A*/B* has no finite irreducible ZG-factors (as A*/B* ≅ G A/B), the irreducible ZG-factors of pA* are all infinite, a contradiction. So pA* = 0. That is, A* is an elementary abelian p-group for the prime p. If B* has a complement in A*, i.e., A* = B* ⊕ C* for some ZG-submodule C*, then A = B ⊕ C* (Lemma 1.2.25), a contradiction. So B* has no complement in A*, and A* satisfies all the conditions that are satisfied by A. Hence we may replace A by A* (when A is not torsion-free) and then we may assume that A is either torsion-free or an elementary abelian p-group for some prime p.

Let L = C_G(A/B), where we have B is a nonzero proper ZG-submodule of A such that all irreducible ZG-factors of B are finite and A/B has no finite irreducible ZG-factors. We consider the following two cases: (a) L = 1, or (b) L ≠ 1.

(a) L = C_G(A/B) = 1. In this case, let N_1 and N_2 be two maximal ZG-submodules of B such that B/N_1 is a finite p-group and B/N_2 is a finite q-group, where p and q are primes and, if A is torsion-free, we can assume p ≠ q (by using Lemma 2.1.4). Let M_1 and M_2 be two ZG-submodules of A such that A/M_1 ≅ G B/N_1 and A/M_2 ≅ G B/N_2 (such M_1 and M_2 exist as we can take A/N_1 = B/N_1 ⊕ M_1/N_1 and A/N_2 = B/N_2 ⊕ M_2/N_2). Let M = M_1 ∩ M_2 and let H = C_G(A/M), then |G/H| < ∞.

Let W = C_B(H). If W ≠ 0, then let 0 ≠ a ∈ W, we have U = <a>^G is a nonzero ZG-submodule of B and then A/U = B/U ⊕ V/U for some ZG-submodule V of A. By Corollary 2.3.2, we have C_G(U) = C_G(V). But H ≤ C_G(W) ≤ C_G(U) and C_G(V) ≤ C_G(V/U) = C_G(A/B) = L = 1, then it follows from |G/H| < ∞ that G is a finite group, contrary to A having infinite irreducible ZG-factors.
\( W = C_B(H) = 0 \). Now, by Lemma 2.3.3, there exists a \( K \leq H \) and a nonzero \( \mathbb{Z}G \)-submodule \( B_1 \) of \( B \) such that \( K \) is normal in \( G \), \( A/B_1 = B/B_1 \oplus A_1/B_1 \) for some \( \mathbb{Z}G \)-submodule \( A_1 \) of \( A \), \( C_{B_1}(K_1/C_{G}(A_1)) = 0 \), and \( KC_{G}(A_1)/C_{G}(A) \) is a finite elementary abelian \( r \)-subgroup of \( G/C_{G}(A) \) for some prime \( r \). Since \( C_{G}(A_1) \leq C_{G}(A_1/B_1) = C_{G}(A/B) = L = 1 \), so \( C_{G}(A_1) = 1 \) and then \( K \) is a nontrivial finite elementary abelian \( r \)-subgroup of \( G \) for some prime \( r \).

If \( C_A(K) \neq 0 \), then since \( B_1 \cap C_A(K) = C_{B_1}(K) = 0 \), we have \( C_A(K) = C_A(K) \oplus B_1 \subseteq A/B_1 = B/B_1 \oplus A_1/B_1 \). Using Lemma 1.2.23, we get \( C_A(K) \) has an \( f \)-decomposition and, by the fact that \( A \) has no nonzero \( \mathbb{Z}G \)-submodules with all irreducible \( \mathbb{Z}G \)-factors being infinite, we have \( C_A(K) < B \). Let \( B_3 = C_A(K) \), then \( A/B_3 = B/B_3 \oplus A_3/B_3 \) for some \( \mathbb{Z}G \)-submodule \( A_3 \) and \( K \leq C_{G}(B_3) = C_{G}(A_3) \) (Corollary 2.3.2). But

\[
C_{G}(A_3) \leq C_{G}(A_3/B_3) = C_{G}(A/B) = L = 1,
\]

thus \( K = 1 \), a contradiction. So \( C_A(K) = 0 \).

If \( A \) is an elementary abelian \( p \)-group, then by \( C_A(K) = 0 \) we have \( r \neq p \);

also, on the other hand, if \( A \) is torsion-free, then we have \( r \neq p \) or \( r \neq q \). By Lemma 2.4.7 and \( C_A(K) = 0 \) we have the contradiction that

\[ A = C_A(K) + M_i = M_i < A, \]

where \( i = 1 \) or \( 2 \) if \( r \neq p \) or \( r \neq q \).

Case (a) is proved.

(b) \( L = C_{G}(A/B) \neq 1 \). In this case, let \( B^* = C_B(L) \), and we consider the following two subcases: (i) \( B^* = 0 \), or (ii) \( B^* \neq 0 \).

(i) \( B^* = C_B(L) = 0 \). By Lemma 2.3.3, there exists a \( K \leq L \) and a nonzero \( \mathbb{Z}G \)-submodule \( B_1 \) of \( B \) such that \( K \) is normal in \( G \), \( A/B_1 = B/B_1 \oplus A_1/B_1 \),
for some $ZG$-submodule $A_1$ of $A$, $C_{B_1}(KC_G(A_1)/C_G(A_1)) = 0$, and $KC_G(A_1)/C_G(A_1)$ is a finite elementary abelian $q$-subgroup of $G/C_G(A_1)$ for some prime $q$.

Consider $A_1$ as a $Z\overline{G}$-module, where $\overline{G} = G/C_G(A_1)$. Then it is evident that $\overline{\mathcal{K}} = \overline{\mathcal{L}} = C_{\overline{G}}(A_1/B_1)$ and $C_{B_1}(\overline{\mathcal{K}}) = 0$. Also it is clear that all the irreducible $Z\overline{G}$-factors of $B_1$ are finite, the factor-module $A_1/B_1$ has no finite irreducible $Z\overline{G}$-factors, and $A_1$ has no nonzero $Z\overline{G}$-submodules with all irreducible $Z\overline{G}$-factors being infinite. Thus, since $B_1 \cap C_{A_1}(\overline{\mathcal{K}}) = C_{B_1}(\overline{\mathcal{K}}) = 0$, we have $C_{A_1}(\overline{\mathcal{K}}) = 0$.

If $A_1$ is an elementary abelian $p$-group, then by $C_{A_1}(\overline{\mathcal{K}}) = 0$ we have $q \neq p$. Also, if $A_1$ is torsion-free but $A_1/B_1$ is not torsion-free, then we may assume that $A_1$ has a $Z\overline{G}$-submodule $A_2$ such that $A_2/B_1$ ($\leq A_1/B_1$) is a nontrivial elementary abelian $r$-group for some prime $r$, and then $A_2 \equiv Z\overline{G} rA_2 \leq B_1$ (by $A_2$ being torsion-free), contrary to $B_1$ having no infinite irreducible $Z\overline{G}$-factors.

So $A_1$ is torsion-free implies that $A_1/B_1$ is torsion-free, too. By Lemma 2.1.4, we have $A_1/B_1 > p(A_1/B_1)$ for any prime $p$. Therefore $A_1$ always contains a proper $Z\overline{G}$-submodule $M$ such that $B_1 \leq M$ and $A_1/M$ is a $p$-group for some prime $p$ other than $q$. Since $\overline{K} \leq L = C_{G}(A_1/B_1) \leq C_{G}(A_1/M)$, by Lemma 2.4.7, $A_1 = C_{A_1}(\overline{\mathcal{K}})+M = M < A_1$, a contradiction.

(ii) $B^* = C_B(L) \neq 0$. We write $A$ as a sum $A = B+A^*$ with $B \cap A^* = B^*$.

For any $C \leq B^*$ ($C \neq 0$), since $A^*/C = B^*/C \oplus \overline{A}/C$ for some $ZG$-submodule $\overline{A}$ of $A$, and since $L$ centralizes both $B^*/C$ and $\overline{A}/C$ (as $L \leq C_{G}(B^*)$ and $L = C_{G}(A/B)$
\[ C_G(A^*/B^*) = C_G(A^*/C^*) \], then \( L \subseteq C_G(A^*/C^*) \). Let \( C^* = k \cap C \), where \( 0 \neq C \leq B^* \), then either \( C^* = 0 \) or \( C^* \) is a finite irreducible \( \mathbb{Z}G \)-submodule of \( A \). If \( C^* \neq 0 \), then \( A/C^* = B/C^* \oplus \tilde{A}/C^* \) for some \( \mathbb{Z}G \)-submodule \( \tilde{A} \) of \( A \) and then, by Lemma 1.2.22, \( \tilde{A} = C^* \oplus \tilde{A} \) for some \( \mathbb{Z}G \)-submodule \( \tilde{A} \). It is clear that \( \tilde{A} \) is a nonzero \( \mathbb{Z}G \)-submodule with all irreducible \( \mathbb{Z}G \)-factors being infinite, contrary to \( A \) having no such \( \mathbb{Z}G \)-submodules. So we must have \( C^* = 0 \). As \( [A^*, L] \subseteq C \) for all \( 0 \neq C \leq B^* \), so \( [A^*, L] \subseteq \cap C = C^* = 0 \); that is, \( L \subseteq C_G(A^*) \). Also since \( C_G(A^*) \subseteq C_G(A^*/B^*) = L \), so \( L = C_G(A^*) \). Now consider \( A^* \) as a \( \tilde{G} \)-module, where \( \tilde{G} = G/C_G(A^*) \), and let \( \bar{L} = C_{\tilde{G}}(A^*/B^*) \), then it is clear that \( \bar{L} = \bar{I} \).

Thus, by the proof of the above for the case \( (a) \), we get a contradiction. So we have in fact finished the proof.

Dual to Proposition 3.1, we have

**Proposition 3.2:** Let \( G \) be a hyperfinite locally soluble group, \( A \) a noetherian \( \mathbb{Z}G \)-module, and \( B \) a \( \mathbb{Z}G \)-submodule of \( A \) such that each irreducible \( \mathbb{Z}G \)-factor of \( A/B \) is finite while \( B \) has no finite irreducible \( \mathbb{Z}G \)-factors. Then \( B \) has a complement in \( A \), i.e., \( A = B \oplus C \) for some \( \mathbb{Z}G \)-submodule \( C \) of \( A \).

**Proof:** Suppose \( B \) does not have a complement in \( A \). By considering an appropriate factor-module of \( A \) we may assume that for every nonzero \( \mathbb{Z}G \)-submodule \( \bar{B} \) of \( B \), \( B/\bar{B} \) has a complement in \( A/\bar{B} \). Also, let \( C \) be a \( \mathbb{Z}G \)-submodule of \( A \) maximal subject to \( B \cap C = 0 \). Then \( A \neq B \oplus C \). By replacing \( A \) by \( A/C \) we may assume that for every nonzero \( \mathbb{Z}G \)-submodule \( D \) of \( A \), \( B \cap D \neq 0 \). That is, \( A \) contains no nonzero \( \mathbb{Z}G \)-submodules with all irreducible \( \mathbb{Z}G \)-factors
being finite.

If $A$ is torsion-free but $A/B$ is not, then let $A^*$ be a $\mathbb{Z}G$-submodule of $A$ such that $A^*/B$ is a nontrivial elementary abelian $p$-group for some prime $p$. Thus $A^* \cong \mathbb{Z}G pA^* \leq B$ by $A^*$ being torsion-free, but this is contrary to $B$ containing no finite irreducible $\mathbb{Z}G$-factors. So $A$ being torsion-free implies that $A/B$ is also torsion-free.

If $A$ is not torsion-free, let $T(A)$ be the torsion part of $A$, then $T(A)$ is a nonzero $\mathbb{Z}G$-submodule of $A$ and then $T(B) = T(A) \cap B \neq 0$. Let $B_1$ be the nonzero $\mathbb{Z}G$-submodule of $B$ generated by all the elements of order $p$ for some prime $p$, then $A/B_1 = B/B_1 \oplus A_1/B_1$ for some $\mathbb{Z}G$-submodule $A_1$ of $A$. Let $\varphi: A_1 \rightarrow pA_1$ be defined by $\varphi: a \mapsto pa$ for all $a \in A_1$; then $\varphi$ is a $\mathbb{Z}G$-homomorphism from $A_1$ to $pA_1$ and $B_1 \leq \ker \varphi$. If $pA_1 \neq 0$, then since $pA_1 \cong \mathbb{Z}G A_1/\ker \varphi$ and $A_1/B_1$ has no infinite irreducible $\mathbb{Z}G$-factors, $pA_1$ is a nonzero $\mathbb{Z}G$-submodule with all irreducible $\mathbb{Z}G$-factors being finite, contrary to $A$ having no such $\mathbb{Z}G$-submodules. So $pA_1 = 0$. That is, $A$ contains a nonzero $\mathbb{Z}G$-submodule $A_1$, which is an elementary abelian $p$-group for some prime $p$, such that $A = B + A_1$.

As before, we can replace $A$ by $A_1$ (if necessary), so we may assume that $A$ is either torsion-free or an elementary abelian $p$-group for some prime $p$. For the nonzero $\mathbb{Z}G$-submodule $B$ of $A$ with $B = B^f$ and $A/B = (A/B)^f$, since $A/B$ is accordingly either torsion-free or an elementary abelian $p$-group, we can assume that $A$ contains two maximal $\mathbb{Z}G$-submodules $M_1$ and $M_2$, both containing $B$, such that $A/M_1$ is a finite $p$-group and $A/M_2$ is a finite $q$-group, where $p$ and $q$ are primes, and in the case that $A/B$ is torsion-free, by Lemma 2.1.4, we may assume that $p \neq q$. Let $M = M_1 \cap M_2$ and let $H = C_G(A/M)$, then $|G/H| < \infty$.

If $a \in C_B(H)$, then the $\mathbb{Z}G$-submodule $<a>^G$ as a group is finitely generated and so each irreducible $\mathbb{Z}G$-factor of $<a>^G$ is finite. Since $B$ has no

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nonzero finite irreducible \( \mathbb{Z}G \)-factors, we must have \( \langle a \rangle^G = 0 \) and so \( a = 0 \); that is, \( C_B(H) = 0 \). Now, by Lemma 2.3.3, there is a \( K \leq H \) and a nonzero \( \mathbb{Z}G \)-submodule \( B^* \leq B \) such that \( K \) is normal in \( G \), \( A/B^* = B/B^* \oplus A^*/B^* \) for some \( \mathbb{Z}G \)-submodule \( A^* \) of \( A \), \( C_B(K)(A^*/C_G(A^*)) = 0 \), and \( KC_G(A^*)/C_G(A^*) \) is a finite elementary abelian \( r \)-subgroup of \( G/C_G(A^*) \) for some prime \( r \).

Since \( A^*/B^* \cong \mathbb{Z}G A/B \), there exist \( \mathbb{Z}G \)-submodules \( M^*, M_1^*, \) and \( M_2^* \) such that \( M^* = M_1^* \cap M_2^* \), \( A^*/M_1^* \cong \mathbb{Z}G A/M_1 \), \( A^*/M_2^* \cong \mathbb{Z}G A/M_2 \), \( A^*/M^* \cong \mathbb{Z}G A/M \), and \( B^* \leq M^* \). Let \( \mathbb{G} = G/C_G(A^*) \), then, since \( C_G(A^*) \leq C_G(A^*/M^*) = C_G(A/M) = H \), \( \mathbb{H} = H/C_G(A^*) \). It is clear that \( \mathbb{H} = C_G(A^*/M^*) \). Also \( \mathbb{I} \neq \mathbb{K} \leq \mathbb{H} \) and \( \mathbb{K} \) is a finite elementary abelian \( r \)-group for the prime \( r \). Since \( C_{B^*}(\mathbb{K}) = 0 \), if \( A^* \) is a \( p \)-group, then \( r \neq p \), and if \( A^*/B^* \) is torsion-free, then \( r \neq p \) or \( r \neq q \). By Lemma 2.4.7, we have \( A^* = C_{A^*}(\mathbb{K}) + M^* \), where \( M^* = M_1^* \) or \( M_2^* \) if \( r \neq p \) or \( r \neq q \). Since \( B^* \cap C_{A^*}(\mathbb{K}) = C_{B^*}(\mathbb{K}) = 0 \) and \( A^* \) has no nonzero \( \mathbb{Z}G \)-submodules with all irreducible \( \mathbb{Z}G \)-factors being finite, so \( C_{A^*}(\mathbb{K}) = 0 \) and then \( A^* = M^* < A^* \), a contradiction.

The result is proved.

Joining these two propositions, we get the following corollary, which generalizes Lemma 1.2.22 in the case that \( G \) is hyperfinite and locally soluble.

**Corollary 3.3:** Let \( G \) be a hyperfinite locally soluble group, \( A \) a noetherian \( \mathbb{Z}G \)-module, and \( B \) a \( \mathbb{Z}G \)-submodule of \( A \) such that each irreducible \( \mathbb{Z}G \)-factor of \( B \) is finite (resp. infinite) and \( A/B \) contains no finite (resp. infinite) irreducible \( \mathbb{Z}G \)-factors. Then \( B \) has a complement in \( A \), i.e., \( A = B \oplus C \) for some \( \mathbb{Z}G \)-submodule \( C \) of \( A \).
From Corollary 3.3, we have

**Corollary 3.4:** Let $G$ be a hyperfinite locally soluble group and $A$ a noetherian $\mathbb{Z}G$-module. Then $A$ has a finite (resp. infinite) irreducible $\mathbb{Z}G$-image if and only if $A$ has a finite (resp. infinite) irreducible $\mathbb{Z}G$-factor.

**Proof:** Suppose $A$ has a finite (resp. infinite) irreducible $\mathbb{Z}G$-factor, then by the noetherian condition we have $A$ contains an irreducible $\mathbb{Z}G$-factor, say $B/C$, such that the irreducible $\mathbb{Z}G$-submodule $B/C$ of $A/C$ is finite (resp. infinite) and the factor-module $A/C/B/C (\equiv_{\mathbb{Z}G} A/B)$ contains no finite (resp. infinite) irreducible $\mathbb{Z}G$-factors. By Corollary 3.3, we have $A/C$ and hence $A$ has a finite (resp. infinite) irreducible $\mathbb{Z}G$-image. The corollary is true.

Another consequence of Corollary 3.3 is that

**Corollary 3.5:** Let $G$ be a hyperfinite locally soluble group and $A$ a noetherian $\mathbb{Z}G$-module. If $A$ has a $\mathbb{Z}G$-composition series in which the finite (resp. infinite) irreducible $\mathbb{Z}G$-factors of $A$ are only finitely many, then $A$ has an $f$-decomposition.

**Proof:** It follows from Corollary 3.3 and induction on the finite number of the finite (resp. infinite) irreducible $\mathbb{Z}G$-factors in a $\mathbb{Z}G$-composition series.

As in the hyperfinite case, we can generalize Corollary 3.3 in the following forms, and these results will be seen to be useful in the later discussions.

**Proposition 3.6:** Let $G$ be a hyperfinite locally soluble group, $A$ a noetherian $\mathbb{Z}G$-module, and $B$ a $\mathbb{Z}G$-submodule of $A$ such that each irreducible $\mathbb{Z}G$-factor of $A/B$ is a finite (resp. infinite) $p$-group and $B$ has no irreducible $\mathbb{Z}G$-factors
being finite (resp. infinite) $p$-groups, where $p$ is a fixed prime. Then $B$ has a complement in $A$, i.e., $A = B \oplus C$ for some $\mathbb{Z}G$-submodule $C$ of $A$.

**Proof:** Suppose $B$ does not have a complement in $A$, then by the noetherian condition we may assume that for every nonzero $\mathbb{Z}G$-submodule $C$ of $B$, $B/C$ has a complement in $A/C$. Let $D_0$ be a $\mathbb{Z}G$-submodule of $A$ maximal with respect to $B \cap D_0 = 0$. Since $A \not= B \oplus D_0$, by replacing $A$ by $A/D_0$ we may assume that for any nonzero $\mathbb{Z}G$-submodule $D$ of $A$, $B \cap D \not= 0$.

Since each irreducible $\mathbb{Z}G$-factor of $A/B$ is a finite (resp. infinite) $p$-group for the fixed prime $p$, by Lemma 2.1.4, $A/B$ is not torsion-free, and further $A/B$ is a $p$-group. We claim that $A$ is also not torsion-free. For otherwise, let $A^*$ be a $\mathbb{Z}G$-submodule of $A$ such that $A^*/B$ is a nontrivial elementary abelian $p$-group, then $pA^* \not= 0$ and then $A^* \cong \mathbb{Z}G pA^* \not= B$, contrary to $B$ having no irreducible $\mathbb{Z}G$-factors being finite (resp. infinite) $p$-groups. So $A$ is not torsion-free. Let $T(A)$ be the torsion part of $A$, then $T(A)$ is a nonzero $\mathbb{Z}G$-submodule of $A$ and then $T(B) = B \cap T(A) \not= 0$. Let $B_1$ be the nonzero $\mathbb{Z}G$-submodule of $B$ generated by all the elements of order $q$ for some prime $q$, then $A/B_1 = B/B_1 \oplus A_1/B_1$ for some $\mathbb{Z}G$-submodule $A_1$ of $A$. By $A_1/B_1 \cong \mathbb{Z}G A/B$, we have $A_1/B_1$ is a $p$-group. If $q \not= p$, then it is clear that $A_1 = B_1 \oplus A_2$ for some $\mathbb{Z}G$-submodule $A_2$ and then $A = B \oplus A_2$, a contradiction. Thus $q = p$, and then $A_1$ is a $p$-group. Now each irreducible $\mathbb{Z}G$-factor of $A_1/B_1$ is finite (resp. infinite) and the $\mathbb{Z}G$-submodule $B_1$ has no finite (resp. infinite) irreducible $\mathbb{Z}G$-factors, by Corollary 3.3, we have $A_1 = B_1 \oplus C_1$ and then $A = B \oplus C_1$, a contradiction again.

**Corollary 3.7:** Let $G$ be a hyperfinite locally soluble group, $A$ a noetherian $\mathbb{Z}G$-module and $p$ a prime. Then $A$ has an irreducible $\mathbb{Z}G$-image being not a
finite (resp. infinite) $p$-group if and only if the same is true for some irreducible $\mathbb{Z}G$-factor of $A$.

**Proof:** If $A$ has an irreducible $\mathbb{Z}G$-factor being not a finite (resp. infinite) $p$-group, then by the noetherian condition we have $A$ contains an irreducible $\mathbb{Z}G$-factor, say $B/C$, such that the irreducible $\mathbb{Z}G$-submodule $B/C$ of $A/C$ is not a finite (resp. infinite) $p$-group but the irreducible $\mathbb{Z}G$-factors of the factor-module $A/C/B/C \cong \mathbb{Z}G A/B$ are all finite (resp. infinite) $p$-groups.

By Proposition 3.6, we have $A/C$ and then $A$ has a nonzero irreducible $\mathbb{Z}G$-image being not a finite (resp. infinite) $p$-group.

The dual of proposition 3.6 is:

**Proposition 3.8:** Let $G$ be a hyperfinite locally soluble group, $A$ a noetherian $\mathbb{Z}G$-module, and $B$ a $\mathbb{Z}G$-submodule of $A$ such that each irreducible $\mathbb{Z}G$-factor of $B$ is a finite (resp. infinite) $p$-group and $A/B$ has no irreducible $\mathbb{Z}G$-factors being finite (resp. infinite) $p$-groups, where $p$ is a fixed prime. Then $B$ has a complement in $A$, i.e., $A = B \oplus C$ for some $\mathbb{Z}G$-submodule $C$ of $A$.

**Proof:** Suppose $B$ does not have a complement in $A$, then by the noetherian condition we may assume that for every nonzero $\mathbb{Z}G$-submodule $C$ of $B$, $B/C$ has a complement in $A/C$. Let $D_0$ be a $\mathbb{Z}G$-submodule of $A$ maximal subject to $B \cap D_0 = 0$. Since $A \neq B \oplus D_0$, by replacing $A$ by $A/D_0$ we may assume that for each nonzero $\mathbb{Z}G$-submodule $D$ of $A$, $B \cap D \neq 0$, i.e., each nonzero $\mathbb{Z}G$-submodule of $A$ contains irreducible $\mathbb{Z}G$-factors being finite (resp. infinite) $p$-groups for the fixed prime $p$.

By Lemma 2.1.4, $B$ is not torsion-free, and further $B$ is a $p$-group for the prime $p$. Let $B_1$ be the nonzero $\mathbb{Z}G$-submodule of $B$ generated by all the elements
of order \( p \), then \( A/B_1 = B/B_1 \oplus A_1/B_1 \) for some \( \mathbb{Z}G \)-submodule \( A_1 \) of \( A \). Let 
\[ \varphi: A_1 \rightarrow pA_1 \] 
be defined by \( \varphi: a \rightarrow pa \) for all \( a \in A \); then \( \varphi \) is a \( \mathbb{Z}G \)-homomorphism from \( A_1 \) to \( pA_1 \), and it is clear that \( B_1 \leq \ker \varphi \). Since 
\[ pA_1 \cong \mathbb{Z}G A_1/\ker \varphi \] 
and \( A_1/B_1 \cong \mathbb{Z}G A/B \) has no irreducible \( \mathbb{Z}G \)-factors being finite (resp. infinite) \( p \)-groups, so \( pA_1 \) has no irreducible \( \mathbb{Z}G \)-factors being finite (resp. infinite) \( p \)-groups. But each nonzero \( \mathbb{Z}G \)-submodule of \( A \) (and hence of \( A_1 \)) contains irreducible \( \mathbb{Z}G \)-factors being finite (resp. infinite) \( p \)-groups; therefore we must have \( pA_1 = 0 \) and then \( A_1 \) is a \( \mathbb{Z}G \)-module with the \( \mathbb{Z}G \)-submodule \( B_1 \) such that each irreducible \( \mathbb{Z}G \)-factor of \( B_1 \) is finite (resp. infinite) and the factor-module \( A_1/B_1 \) has no finite (resp. infinite) irreducible \( \mathbb{Z}G \)-factors. Thus, by Corollary 3.3, we have \( A_1 = B_1 \oplus C_1 \) and then 
\( A = B \oplus C_1 \), a contradiction.

**Corollary 3.9:** Let \( G \) be a hyperfinite locally soluble group, \( A \) a noetherian \( \mathbb{Z}G \)-module and \( p \) a prime. Then \( A \) has an irreducible \( \mathbb{Z}G \)-image being a finite (resp. infinite) \( p \)-group if and only if the same is true for some irreducible \( \mathbb{Z}G \)-factor of \( A \).

**Proof:** If \( A \) has an irreducible \( \mathbb{Z}G \)-factor being a finite (resp. infinite) \( p \)-group, then by the noetherian condition we have \( A \) contains an irreducible \( \mathbb{Z}G \)-factor, say \( B/C \), such that the irreducible \( \mathbb{Z}G \)-submodule \( B/C \) of \( A/C \) is a finite (resp. infinite) \( p \)-group and the factor-module \( A/C/B/C \cong \mathbb{Z}G A/B \) has no irreducible \( \mathbb{Z}G \)-factors being finite (resp. infinite) \( p \)-groups. By Proposition 3.8, we have \( A/C \) and hence \( A \) contains the required \( \mathbb{Z}G \)-images.

Comparing with Corollary 2.4.3 and Corollary 2.4.4, we see that Proposition 3.8 and Corollary 3.9 are generalizations of these results in the
locally soluble case.

An important step in proving Theorem A is the following reduction result.

**Proposition 3.10:** Let $G$ be a hyperfinite locally soluble group and $A$ a noetherian $\mathbb{Z}G$-module. If $A$ has no $f$-decomposition, then $A$ has a nonzero $\mathbb{Z}G$-image $\overline{A}$ satisfying:

(a) $\overline{A}$ has no $f$-decomposition;

(b) for every nonzero $\mathbb{Z}G$-submodule $\overline{C}$ of $\overline{A}$, $\overline{A}/\overline{C}$ has an $f$-decomposition;

(c) $\overline{A}$ has no nonzero $\mathbb{Z}G$-submodules with an $f$-decomposition;

(d) $\overline{A}$ is torsion-free; and

(e) the finite irreducible $\mathbb{Z}G$-factors of $\overline{A}$ are all $p$-groups for some fixed prime $p$.

**Proof:** Since $A$ has no $f$-decomposition, then by the noetherian condition there is a nonzero $\mathbb{Z}G$-image $\overline{A}$ satisfying the conditions (a) and (b).

For $\overline{A}$, suppose $\overline{B} \leq \overline{A}$ and $\overline{B}$ has an $f$-decomposition, i.e., $\overline{B} = \overline{B}^f \oplus \overline{B}^\perp$. If $\overline{B}^f \neq 0$, then since $\overline{B}^f$ is a nonzero $\mathbb{Z}G$-submodule of $\overline{A}$, by (b), $\overline{A}/\overline{B}^f$ has an $f$-decomposition. Let $\overline{A}/\overline{B}^f = \overline{A}_1/\overline{B}^f \oplus \overline{A}_2/\overline{B}^f$, where $\overline{A}_1$ is a $\mathbb{Z}G$-submodule of $\overline{A}$ such that $\overline{A}_1 \simeq \overline{B}^f$ and $\overline{A}_1/\overline{B}^f = (\overline{A}/\overline{B})^f$ and the $\mathbb{Z}G$-submodule $\overline{A}_2 (\simeq \overline{B}^\perp)$ is such that $\overline{A}_2/\overline{B}^f = (\overline{A}/\overline{B})^{\perp}$. By Proposition 3.1, $\overline{A}_2 = \overline{B}^{\perp} \oplus \overline{C}$ for some $\mathbb{Z}G$-submodule $\overline{C}$ whose irreducible $\mathbb{Z}G$-factors are all infinite. And then it is clear that $\overline{A} = \overline{A}_1 \oplus \overline{C}$, where evidently each irreducible $\mathbb{Z}G$-factor of $\overline{A}_1$ is finite. That is, $\overline{A}$ has an $f$-decomposition, contrary to (a). So $\overline{B}^f = 0$. Similarly, by applying Proposition 3.2, we can prove $\overline{B}^\perp = 0$. So $\overline{B} = 0$. Therefore the condition (e) is satisfied by $\overline{A}$.
If $\overline{A}$ does not satisfy the condition (d) or the condition (e), then by (c), we may assume that $\overline{A}$ is either an elementary abelian $p$-group for some prime $p$ or a torsion-free group which contains at least two nonzero finite irreducible $\mathbb{Z}G$-factors, one being a $p$-group and the other a $q$-group, where $p$ and $q$ are two distinct primes.

We may also assume that $G$ acts faithfully on $\overline{A}$, i.e., $C_G(\overline{A}) = 1$. By Corollary 3.9, we may assume that $\overline{A}$ contains two maximal $\mathbb{Z}G$-submodules $\overline{M}_1$ and $\overline{M}_2$ such that $\overline{A}/\overline{M}_1$ is a nontrivial finite $p$-group and $\overline{A}/\overline{M}_2$ is a nontrivial finite $q$-group, where $r = p$ or $q$ according to $\overline{A}$ being torsion or torsion-free.

Let $\overline{M} = \overline{M}_1 \cap \overline{M}_2$ and let $H = C_G(\overline{A}/\overline{M})$, then $|G/H| < \infty$. For $C = C_{\overline{M}}(H)$, if $C \neq 0$, let $0 \neq a \in C$, then the $\mathbb{Z}G$-submodule $\langle a \rangle^G$ is nonzero and each of its irreducible $\mathbb{Z}G$-factors is finitely generated as an abelian group and hence is finite, contrary to the condition (c). So $C = 0$. By Lemma 1.2.14, there exists a $K \subseteq H$ and a nonzero $\mathbb{Z}G$-submodule $\overline{M}^* \leq \overline{M}$ such that $K$ is normal in $G$, $C_{\overline{M}^*}(K) = 0$, and $|K/C_{\overline{M}^*}(\overline{M}^*)| < \infty$.

We show that $C_K(\overline{M}^*) = 1$. If not, then $C_K(\overline{M}^*)$ contains a nontrivial finite subgroup $F$ being normal in $G$. Since $F$ is not contained in $C_G(\overline{A}) = 1$ and $\overline{A}/\overline{M}^*$ has an $f$-decomposition, by Lemma 2.3.1, $\overline{A}$ has a nonzero $\mathbb{Z}G$-submodule with an $f$-decomposition, a contradiction. So $C_K(\overline{M}^*) = 1$ and then $K$ is finite.

Choose $K$ to be minimal with respect to $C_{\overline{M}^*}(K) = 0$. If $K$ is not a minimal normal subgroup of $G$, then $K$ contains a nontrivial proper subgroup $K_1$ being normal in $G$. By the minimality of $K$, we have $C_{\overline{M}^*}(K_1) \neq 0$. Let $\overline{M}^{**} = C_{\overline{M}^*}(K_1)$, then $1 \neq K_1 \leq C_K(\overline{M}^{**})$. As in the last paragraph, we obtain a contradiction. So $K$ must be a minimal normal subgroup of $G$. Since $G$ is locally soluble, $K$ is an elementary abelian $k$-group for some prime $k$.

In order to apply Lemma 2.4.7, we should consider that $k \neq p$ or $k \neq q$. 

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in the case that $\overline{A}$ is torsion-free; or we have $k \neq p$ by using $C_{M^*}(K) = 0$ in the case that $\overline{A}$ is a $p$-group (Lemma 1.2.7). Now from Lemma 2.4.7 we have $\overline{A} = C_{\overline{A}}(K) + M$, where $M_i = \overline{M}_i$ or $\overline{M}_2$ according to $k \neq p$ or $k \neq q$. Since $\overline{M}_i < \overline{A}$, so $C_{\overline{A}}(K) \neq 0$. But, since $\overline{M} \cap C_{\overline{A}}(K) = C_{\overline{M}}(K) = 0$, we have $\overline{C}_{\overline{A}}(K) \equiv_{ZG}(C_{\overline{A}}(K) \oplus \overline{M})/\overline{M} \leq \overline{A}/\overline{M}$ and then, since $\overline{A}/\overline{M}$ has an $f$-decomposition, each $ZG$-submodule of $\overline{A}/\overline{M}$ has an $f$-decomposition (Lemma 1.2.23) and hence so does $C_{\overline{A}}(K)$, a contradiction. So we have, in fact, proved that $\overline{A}$ satisfies the conditions (d) and (e).

The result is proved.

From Proposition 3.10 and its proof, we can get a number of results which assert that the noetherian $ZG$-module $A$ has an $f$-decomposition under certain conditions. Among these, the simplest and most useful one is:

Corollary 3.11: If $G$ is a hyperfinite locally soluble group, then any periodic noetherian $ZG$-module $A$ has an $f$-decomposition.

By Proposition 3.10, the task for proving Theorem A is now reduced to considering torsion-free noetherian $ZG$-modules. For such modules, we need to prove the following

Proposition 3.12: Let $G$ be a hyperfinite locally soluble group, $A$ a torsion-free noetherian $ZG$-module, and $A_i = p^iA$, where $p$ is a prime and $i = 0, 1, 2, \cdots$. Then

\begin{enumerate}
\item for any $0 \leq j < i$, $A_j/A_i$ has an $f$-decomposition
\[ A_j/A_i = A_j/A_i \oplus A_i/A_i \]
\end{enumerate}

where $A_j$ is the $ZG$-submodule of $A_j$ such that $A_j \equiv A_i$ and $A_j/A_i = (A_j/A_i)^f$.
and the ZG-submodule $A_{ji}$ is such that $A_{ji}/A_{ii} = (A_{jj}/A_{ii})^f$.

(2) $A_{ij} \leq A_{ik}$ and $A_{ij} \leq A_{sj}$, $\forall k \leq j$ and $\forall s \leq i$;

(3) $A_{ii} = A_{ik} \cap A_{s}$, where $k \leq i$, $s \leq i$, and $i = 0, 1, 2, \ldots$;

(4) $A_{ij} = A_{ik} \cap A_{s}$, where $k \leq j$, $s \leq i$, and $i, j = 0, 1, 2, \ldots$;

(5) $A_{ij}/A_{kk} = A_{kj}/A_{kk} \oplus A_{ik}/A_{kk}$, $A_{kj}/A_{kk} = (A_{ij}/A_{kk})^f$, and $A_{ik}/A_{kk} = (A_{ij}/A_{kk})^f$, where $k \geq i, j$;

(6) $A_{ij}/A_{sk} = A_{sj}/A_{sk} \oplus A_{ik}/A_{sk}$, $A_{sj}/A_{sk} = (A_{ij}/A_{sk})^f$, and $A_{ik}/A_{sk} = (A_{ij}/A_{sk})^f$, where $k \geq j$, $s \geq i$, and $i, j = 0, 1, 2, \ldots$;

(7) $A_{ij}/A_{i,j+s+t} \equiv \mathbb{Z}/A_{ks}/A_{k,s+t}$ and $A_{ij}/A_{i,j+s+t} \equiv \mathbb{Z}/A_{k,s}/A_{k,s+t}$ where $i, j, k, s, t = 0, 1, 2, \ldots$;

(8) $p^k A_{ij} = A_{i+k,j+k}$, $i, j = 0, 1, 2, \ldots$.

**Proof:** (1) By Lemma 2.1.4, $A_{ii} = pA < A$, and then $A_{ii} < A_{jj}$ for any $0 \leq j < i$. Since
A is noetherian, so $A_{ij}/A_{ii}$ is noetherian and is also periodic. Thus, by Corollary 3.11, $A_{ij}/A_{ii}$ has an $f$-decomposition, i.e.,

$$A_{ij}/A_{ii} = A_{ij}/A_{ii} \oplus A_{ij}/A_{ii},$$

where $A_{ij}$ is the $ZG$-submodule of $A_{jj}$ such that $A_{ij} \geq A_{ii}$ and $A_{ij}/A_{ii} = (A_{jj}/A_{jj})^f$ and the $ZG$-submodule $A_{ij}(\geq A_{ii})$ is such that $A_{ij}/A_{ii} = (A_{jj}/A_{jj})^f$.

(2) We prove $A_{ij} \subseteq A_{ik}$ for any $k \leq j$ and $i, j = 0, 1, 2, \ldots$.

If $j \leq i$, then $A_{ii} \leq A_{ij} \leq A_{kk}$ for any $k \leq j$. Thus $A_{jj}/A_{ii} \leq A_{kk}/A_{ii}$ and then $A_{ij}/A_{ii} \leq A_{ik}/A_{ii}$. That is, $A_{ij} \leq A_{ik}$.

If $j > i$, then if $k \leq i$, we have $A_{ik} \geq A_{ii} \geq A_{ij}$; so we may assume that $i < k \leq j$. Thus $A_{ii} \geq A_{ik} \geq A_{jj}$. Since $A_{ii}/A_{jj}$ has an $f$-decomposition, we have $A_{ik}/A_{jj}$ has an $f$-decomposition (Lemma 1.2.23). Let $B$ be the $ZG$-submodule of $A_{ik}$ such that $B \geq A_{ij}$ and $B/A_{jj} = (A_{ik}/A_{jj})^f$, then $B \leq A_{ii}$ and every irreducible $ZG$-factor of $A_{ij}/B$ is isomorphic either with one of the irreducible $ZG$-factors of $A_{ii}/A_{ik}$ or with one of those of $A_{ij}/B$ and so is finite. So we will have $B = A_{ij}$ and then $A_{ik} \geq A_{ij}$. Thus, we have proved that $A_{ij} \leq A_{ik}$ for any $k \leq j$.

Similarly, we have $A_{ij} \leq A_{si}$, $\forall s \leq i$.

(3) By (2), $A_{ii} \leq A_{ik} \cap A_{si}$ for any $k \leq i$ and any $s \leq i$.

Since $(A_{ik} \cap A_{si})/A_{ii} \leq A_{ik}/A_{ii} = (A_{kk}/A_{ii})^f$ and $(A_{ik} \cap A_{si})/A_{ii} \leq A_{si}/A_{ii} = (A_{ss}/A_{ii})^f$, so $(A_{ik} \cap A_{si})/A_{ii}$ is trivial. That is, $A_{ii} = A_{ik} \cap A_{si}$.
for any \(k, s \leq i\), and \(i = 0, 1, 2, \ldots\).

(4) By (2), \(A_{ij} \leq A_{ik} \cap A_{sj}\) for any \(k \leq j\) and any \(s \leq i\).

On the other hand, we let \(j = s\) i, then by
\[
(A_{ik} \cap A_{sj})/A_{ii} \leq A_{ik}/A_{ii} \cap A_{jj}/A_{ii} \leq (A_{jj}/A_{ii})^f = A_{ij}/A_{ii},
\]
we have \(A_{ik} \cap A_{jj} \leq A_{ij}\). For \(s \leq i\), if \(s < j\), then
\[
A_{ik} \cap A_{sj} = (A_{ik} \cap A_{jk}) \cap A_{sj} = A_{ik} \cap A_{jk} \cap A_{sj} = A_{ik} \cap A_{jj} \leq A_{ij},
\]
and if \(j \leq s \leq i\), then \(A_{ik} \cap A_{sj} \leq A_{ik} \cap A_{jj} \leq A_{ij}\). That is, \(A_{ij} \geq A_{ik} \cap A_{sj}\) for \(j \leq i\), \(k \leq j\), and \(s \leq i\). Similarly, we have \(A_{ij} \geq A_{ik} \cap A_{sj}\) for \(j > i\), \(k \leq j\), and \(s \leq i\). Thus \(A_{ij} = A_{ik} \cap A_{sj}\) for any \(k \leq j\) and any \(s \leq i\).

(5) Suppose \(i \geq j\), then \(A_{ij} \geq A_{ij} \geq A_{ii} \geq A_{kk}\), where \(k \geq i\). Since \(A_{jj}/A_{kk}\) has an \(f\)-decomposition, we have \(A_{ij}/A_{kk}\) has an \(f\)-decomposition. Let
\[
A_{ij}/A_{kk} = B/A_{kk} \oplus C/A_{kk},
\]
in which \(B\) is the \(\mathbb{Z}G\)-submodule of \(A_{ij}\) such that \(B \geq A_{kk}\) and \(B/A_{kk} = (A_{ij}/A_{kk})^f\) and the \(\mathbb{Z}G\)-submodule \(C\) \((\geq A_{kk})\) is such that \(C/A_{kk} = (A_{ij}/A_{kk})^f\). Then, by Lemma 1.2.23, \(B \leq A_{kj}\) and \(C \leq A_{ik}\). Since \(A_{ij}/A_{kk} \geq (A_{ij}/A_{kk})^f\) and \(A_{ij}/B\) has no finite irreducible \(\mathbb{Z}G\)-factors, so \(B = A_{kj}\). Meanwhile, by \(A_{ij}/A_{kk} \geq A_{ik}/A_{kk} = (A_{ii}/A_{kk})^f\) and \(A_{ij}/C\) having no infinite irreducible \(\mathbb{Z}G\)-factors, we have \(C = A_{ik}\). Thus
\[
A_{ij}/A_{kk} = A_{kj}/A_{kk} \oplus A_{ik}/A_{kk}
\]
for any \(k \geq i \geq j\).

Similarly, the result is true for \(k \geq j > i\).
(6) Suppose \(i \leq j\), then, by (2), \(A_{ij} \geq A_{sj}\) and \(A_{ij} \geq A_{ik}\) for any \(s \geq i\) and any \(k \geq j\). Thus \(A_{ij} \geq A_{sj} + A_{ik}\). By (5), if \(s \geq k\), then \(A_{ij} = A_{sj} + A_{is} \leq A_{sj} + A_{ik}\), and if \(s < k\), then \(A_{ij} = A_{kj} + A_{ik} \leq A_{sj} + A_{ik}\). So \(A_{ij} = A_{sj} + A_{ik}\).

By (4), \(A_{sk} = A_{sj} \cap A_{ik}\), so \(A_{sk} / A_{sj} = A_{ik} / A_{sk}\), where \(k \geq j\), \(s \geq i\), and \(j \geq i = 0, 1, 2, \ldots\).

For \(A_{sk} / A_{ij}\): (i) if \(s \geq k(\geq j)\), then \(A_{sk} \geq A_{sj} \geq A_{ss}\) and, since each irreducible \(\mathbb{Z}G\)-factor of \(A_{sj} / A_{ss}\) is finite, we have each irreducible \(\mathbb{Z}G\)-factor of \(A_{sk} / A_{sj}\) is finite; (ii) if \(k \geq s \geq j\), then \(A_{sk} \geq A_{ss} \geq A_{sj}\) and, by each irreducible \(\mathbb{Z}G\)-factor of \(A_{sk} / A_{sj}\) is isomorphic to one of the irreducible \(\mathbb{Z}G\)-factors of \(A_{ss} / A_{sk}\) or one of that of \(A_{sj} / A_{ss}\), we have \(A_{sk} / A_{sj}\) contains only finite irreducible \(\mathbb{Z}G\)-factors; and (iii) if \((k \geq j) \geq s\), then \(A_{sk} \geq A_{ss} \geq A_{sj}\) and, since each irreducible \(\mathbb{Z}G\)-factor of \(A_{sk} / A_{ss}\) is finite, we have any irreducible \(\mathbb{Z}G\)-factor of \(A_{sk} / A_{ss}\) is finite. Thus \(A_{sk} / A_{ss} \leq (A_{sj} / A_{sk})^{t}\). Similarly, we have \(A_{ik} / A_{sk} \leq (A_{ij} / A_{sk})^{t}\). Therefore, \(A_{sk} / A_{sj} = (A_{sk} / A_{js})^{t}\) and \(A_{ik} / A_{sk} = (A_{ik} / A_{sk})^{t}\).

For \(i > j\), the proof is similar.

(7) We only consider the case in which \(i \leq j\), \(k \leq s\), and \(k \leq i\) (as we can similarly prove the other cases).

By (5), we have
\[
A_{ij} / A_{j+t,j+t} = A_{j+t,j+t} / A_{j+t,j+t} \oplus A_{j+t,j+t} / A_{j+t,j+t},
\]
and
\[
A_{ks} / A_{s+t,s+t} = A_{s+t,s+t} / A_{s+t,s+t} \oplus A_{s+t,s+t} / A_{s+t,s+t}.
\]
Thus \( A_{i,j}/A_{i,j+t} \cong_{\mathbb{Z}G} A_{j+1,i}/A_{j+1,i+t} \) and \( A_{k,s}/A_{k,s+t} \cong_{\mathbb{Z}G} A_{s+t,s}/A_{s+t,s+t} \).

By \( A_{i,j}/A_{i,j+t} \cong_{\mathbb{Z}G} A_{i,i}/A_{i,i+s} \) (Corollary 2.1.3), we have

\[
A_{j+1,i}/A_{j+1,i+t} = (A_{i,j}/A_{i,j+t})^f \cong_{\mathbb{Z}G} (A_{i,i}/A_{i,i+S})^f = A_{s+t,s}/A_{s+t,s+t}
\]

Thus \( A_{i,j}/A_{i,j+t} \cong_{\mathbb{Z}G} A_{i,i}/A_{i,i+s} \).

Similarly, we have \( A_{i,j}/A_{i+1,j} \cong_{\mathbb{Z}G} A_{i,i}/A_{i+1,i+s} \).

(8) By induction, we only need to prove \( pA_{i,j} = A_{i+1,j+1} \) for any \( i, j \geq 0 \).

Let \( i \geq j \); by (6), \( A_{i,j}/A_{i+1,j+1} = A_{i+1,j}/A_{i+1,j+1} \oplus A_{i,j+1}/A_{i+1,j+1}' \)

and by (7), \( A_{i+1,j}/A_{i+1,j+1} \cong_{\mathbb{Z}G} A_{00}/A_{01} \) and \( A_{i,j+1}/A_{i+1,j+1} \cong_{\mathbb{Z}G} A_{00}/A_{10} \).

So

\[
p(A_{i,j}/A_{i+1,j+1}) = p(A_{i+1,j}/A_{i+1,j+1}) \oplus p(A_{i,j+1}/A_{i+1,j+1})
\]

\[
\cong_{\mathbb{Z}G} p(A_{00}/A_{01}) \oplus p(A_{00}/A_{10}) = 0.
\]

That is, \( pA_{i,j} \leq A_{i+1,j+1} \).

On the other hand, let \( a \in A_{i+1,j+1}' \) then \( a \in A_{i+1,j+1} \setminus A_{i+1,j+1}' \). Thus \( a = pb \) for some \( b \in A_{i,j} \). If \( b \in A_{i,j} \), then \( (b^G + A_{i,j})/A_{i,j} \) is not contained in \( A_{i,j}/A_{i,j} \) and then \( (b^G + A_{i,j})/A_{i,j} \) contains infinite irreducible \( \mathbb{Z}G \)-factors. Since

\[
A_{i+1,j+1}/A_{i+1,i+1} \cong (a^G + A_{i+1,i+1}) / A_{i+1,i+1}
\]

\[
= (p < b^G + A_{i+1,i+1}) / A_{i+1,i+1}
\]

\[
\cong_{\mathbb{Z}G} p < b^G / (p < b^G \cap A_{i+1,i+1}) \]

\[
= p < b^G / p( < b^G \cap A_{i,i}) (as A is torsion-free)
\]

\[
\cong_{\mathbb{Z}G} < b^G / ( < b^G \cap A_{i,i}) (Lemma 2.1.1)
\]

\[
\cong_{\mathbb{Z}G} (b^G + A_{i,i}) / A_{i,i}.
\]
we have $A_{i+1,j+1}/A_{i+1,i+1}$ contains infinite irreducible $\mathbb{Z}G$-factors, a contradiction. So $b \in A_{ij}$ and then $a \in pA_{ij}$. Thus $A_{i+1,j+1} \leq pA_{ij}$. Therefore, $pA_{ij} = A_{i+1,j+1}$ for $i \geq j$.

Similarly, we have $pA_{ij} = A_{i+1,j+1}$ for $i < j$. Thus $pA_{ij} = A_{i+1,j+1}$ for any $i, j \geq 0$.

Furthermore, let $A_{\infty,1} = \bigcap_{j \neq j} A_{ij}$ and $A_{1,\infty} = \bigcap_{j \neq j} A_{ij}$ for $i = 0, 1, 2, \ldots$, then by applying Proposition 3.12, we can prove the result which will be very important in the following critical proof for Theorem A.

**Proposition 3.13:** Under the hypothesis of Proposition 3.12 and the notation above, one has:

(a) $p^k A_{\infty,1} = A_{\infty,i+k}$ and $p^k A_{1,\infty} = A_{i+k,\infty}$, $i, k = 0, 1, 2, \ldots$;

(b) $A_{\infty,k} = A_{\infty,j} \cap A_{ij}$ and $A_{k,\infty} = A_{j,\infty} \cap A_{ij}$, $k \geq j$, and $i = 0, 1, 2, \ldots$;

(c) $A_{i,\infty}/A_{k,\infty} \leq \mathbb{Z}G (A_{i,j} + A_{kk})/A_{kk} \leq A_{k,i}/A_{kk}$ and $A_{j,\infty}/A_{k,\infty} \leq \mathbb{Z}G (A_{j,k} + A_{kk})/A_{kk} \leq A_{k,j}/A_{kk}$, $k \geq j = 0, 1, 2, \ldots$; and

(d) $A_{i,\infty}$ (resp. $A_{1,\infty}$) has no finite (resp. infinite) irreducible $\mathbb{Z}G$-factors being $p$-groups, $i = 0, 1, 2, \ldots$.

**Proof:** (a) We only prove $pA_{\infty,i} = A_{\infty,i+1}$ for $i = 0, 1, \ldots$.

By (8) in Proposition 3.12, $pA_{ji} = A_{j+1,i+1}$, so

$$pA_{\infty,i} = p(\cap A_{ji}) = \cap (pA_{ji}) \quad \text{(as $A$ is torsion-free)}$$

$$= \cap A_{j+1,i+1} = \cap A_{j,i+1} = A_{\infty,i+1}.$$

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(b) For \( k > j \geq 0 \), using (6) in Proposition 3.12, we have

\[
A_{\infty} \cap A_{ik} = \left( \bigcap_{s > j} A_{sj} \right) \cap A_{ik} = \bigcap_{s > i} \left( A_{sj} \cap A_{ik} \right) = \bigcap_{s > i} A_{sk} = A_{\infty k}.
\]

Similarly, we have \( A_{k\infty} = A_{j\infty} \cap A_{k_k} \) for \( k \geq j \geq 0 \), \( i = 0, 1, \ldots \).

(c) By (b), we have \( A_{\infty k} = A_{\infty j} \cap A_{k_k} \) for any \( k \geq j \geq 0 \). Also, it is clear that

\[
A_{\infty j} + A_{kk} \leq A_{kj},
\]

thus

\[
A_{\infty j}/A_{\infty k} = A_{\infty j}/(A_{\infty j} \cap A_{kk}) \equiv_{ZG} (A_{\infty j} + A_{kk})/A_{kk} \leq A_{kj}/A_{kk}, \quad k \geq j \geq 0.
\]

Similarly, \( A_{j\infty} / A_{k\infty} \equiv_{ZG} (A_{j\infty} + A_{kk})/A_{kk} \leq A_{jk}/A_{kk}, \quad k \geq j \geq 0.\)

(d) By (a) and (c), we have

\[
A_{i\infty}/pA_{i\infty} = A_{i\infty}/A_{i+1,\infty} \equiv_{ZG} (A_{i\infty} + A_{i+1,i+1})/A_{i+1,i+1} \leq A_{i+1,i+1}/A_{i+1,i+1},
\]

Since \( A_{i+1,i+1}/A_{i+1,i+1} \) has no finite irreducible \( ZG \)-factors, we have \( A_{i\infty}/pA_{i\infty} \) has no finite irreducible \( ZG \)-factors. Since \( A_{i\infty} \) is also a noetherian \( ZG \)-module, by Corollary 3.9, \( A_{i\infty} \) has no finite irreducible \( ZG \)-factors being \( p \)-groups \( (i=0,1,2,\ldots) \).

Similarly, we have \( A_{\infty i} \) has no infinite irreducible \( ZG \)-factors being \( p \)-groups, \( i=0,1,2,\ldots \).

The Proposition 3.13 is proved.

Now the critical proof for Theorem A is coming. It enables us to deal with those modules which remain after Proposition 3.10.

**Proposition 3.14:** Let \( G \) be a hyperfinite locally soluble group and \( A \) a
noetherian \( \mathbb{Z}G \)-module. If all finite irreducible \( \mathbb{Z}G \)-factors of \( A \) are \( p \)-groups for some fixed prime \( p \), then \( A \) has an \( f \)-decomposition.

**Proof:** Suppose that \( A \) does not have an \( f \)-decomposition, then by Proposition 3.10, we may further assume that \( A \) satisfies the following conditions:

(a) for every nonzero \( \mathbb{Z}G \)-submodule \( C \) of \( A \), \( A/C \) has an \( f \)-decomposition;

(b) \( A \) is torsion-free; and

(c) \( A \) has no nonzero \( \mathbb{Z}G \)-submodules with an \( f \)-decomposition.

Furthermore, we assume that \( G \) acts faithfully on \( A \), i.e., \( C_G(A) = 1 \).

For the prime \( p \), by Lemma 2.1.4, \( pA < A \) (and then \( p^{i+1}A < p^iA \) for any integer \( i \)) and \( \bigcap_i p^iA = 0 \). Applying Corollary 3.11, we have \( p^jA/p^jA \) has an \( f \)-decomposition for any integers \( 0 \leq j < i \). Let \( A_{kk} = p^kA \) for any integer \( k \geq 0 \) and, for any \( 0 \leq j < i \), let \( A_{ij}/A_{ii} = A_{ij}/A_{ii} \oplus A_{ji}/A_{ii} \), where \( A_{ij} \) is the \( \mathbb{Z}G \)-submodule of \( A_{ii} \) such that \( A_{ij}/A_{ii} = (A_{ij}/A_{ii})^f \) and the \( \mathbb{Z}G \)-submodule \( A_{ji}(\geq A_{ii}) \) is such that \( A_{ji}/A_{ii} = (A_{ji}/A_{ii})^f \). Since \( A \) does not have an \( f \)-decomposition, it does have finite irreducible \( \mathbb{Z}G \)-factors. Together with the hypothesis of the proposition, this shows that \( A \) contains finite irreducible \( \mathbb{Z}G \)-factors being \( p \)-groups for the prime \( p \); then by Corollary 3.9, \( A \) contains irreducible \( \mathbb{Z}G \)-images being finite \( p \)-groups. Thus \( A_{00}/A_{01} \) (\( \equiv \mathbb{Z}G \) \( A_{10}/A_{11} \)) is nonzero and then, by Lemma 2.4.5, \( |A_{00}/A_{01}| < \infty \).

Let \( H = C_G(A_{00}/A_{01}) \), then \( |G/H| < \infty \). Consider \( A \) as a \( \mathbb{Z}H \)-module and then,
by Lemma 1.2.5, \(A\) is noetherian and, by Corollary 2.2.10, \(A\) has no \(f-(ZH)\)-decomposition. By Proposition 3.10, there is a \(ZH\)-image \(A^* (=A/C,\) where \(C\) is a \(ZH\)-submodule \) of \(A\) such that

(a) \(A^*\) has no \(f-(ZH)\)-decomposition;

(b) for every nonzero \(ZH\)-submodule \(D^*\) of \(A^*\), \(A^*/D^*\) has an \(f-(ZH)\)-decomposition;

(c) \(A^*\) has no nonzero \(ZH\)-submodules with an \(f-(ZH)\)-decomposition; and

(d) \(A^*\) is torsion-free.

Since the finite irreducible \(ZG\)-factors of \(A\) are all \(p\)-groups for the prime \(p\), applying (6) in Lemma 2.2.4, the finite irreducible \(ZH\)-factors of \(A\), and hence those of \(A^*\) too, are all \(p\)-groups for the prime \(p\).

As above, for \(A^*\), we have \(pA^* < A^*\) (and then \(p^{i+1}A^* < p^iA^*\) for any integer \(i\) ) and \(\bigcap_{i} p^iA^* = 0\). For integers \(k \geq 0\) and \(0 \leq j < i\), let \(A^*_{ij} = p^kA^*\) and let

\[
A^*/A^*_{ij} = A^*/A^*_{ij} \oplus A^*/A^*_{ij},
\]

where \(A^*_{ij}\) is the \(ZH\)-submodule of \(A^*\) such that

\[
A^*_{ij} = (A^*/A^*_{ij})^f
\]

and the \(ZH\)-submodule \(A^*_{ij} (\cong A^*_{ij})\) is such that \(A^*_{ij}/A^*_{ij} = (A^*_{ij}/A^*_{ij})^f\). Then \(A^*/A^*_{ij} (\cong_{ZH} A^*/A^*_{ij})\) is nonzero and \(|A^*/A^*_{ij}| < \infty\).

Since

\[
A^*_{00} = A^* = A/C = A_{00}/C,
\]

\[
A^*_{11} = pA^*/A^* = (pA^*/A^*)/(A^*/A^*) = (A^*/A^*_{11})/(A^*/A^*_{11}) = A^*/A^*_{11},
\]

\[
(A^*_{10} + C)/(A^*_{11} + C) \cong_{ZH} A^*/(A^*/A^*_{11} + (A^*/A^*_{11} \cap C)),
\]

and

\[
(A^*_{01} + C)/(A^*_{11} + C) \cong_{ZH} A^*/(A^*/A^*_{11} + (A^*/A^*_{11} \cap C)),
\]

so

\[
A^*/A^*_{11} \cong_{ZH} A^*/(A^*/A^*_{11} + C) = (A^*/A^*_{11} + C)/(A^*/A^*_{11} + C) = (A^*/A^*_{11} + C)/(A^*/A^*_{11} + C).
\]

And also
\( A_{00}^* = A/C = (A + C)/C = (A_{10} + C)/C + (A_{01} + C)/C, \) and

\[(A_{10} + C)/C \cap (A_{01} + C)/C = ((A_{10} + C) \cap (A_{01} + C))/C = (A_{11} + C)/C = A_{11}^*.
\]

Thus \( A_{01}^* = (A_{01} + C)/C, \) and then

\[ H = C_G(A_{00}/A_{01}) = C_H(A_{00}/A_{01}) \leq C_H(A_{00}/A_{01} + C) = C_H(A_{00}/A_{01}), \]

That is, \( H = C_H(A_{00}/A_{01}^*). \) Also, by Proposition 3.12,

\[ A_{00}^*/A_{01}^* \cong \text{ZH} A_{ij}/A_{i,j+1}, \]

so \( H = C_H(A_{00}/A_{01}) = C_H(A_{00}/A_{01}) \) for any \( i, j \geq 0. \) By Proposition 3.13, the ZH-submodule \( A_{0\infty}^* \) has no finite irreducible ZH-factors being \( p \)-groups and then has no finite irreducible ZH-factors. Since \( A^* \) has no nonzero ZH-submodules with an \( f \)-decomposition, we must have \( A_{0\infty}^* = 0. \)

Replacing \( H \) by \( H/C_H(A^*) \) we may assume that \( H \) acts faithfully on \( A^* \). Now we will obtain a contradiction by proceeding in the following four steps:

(i) \( H = C_H(A_{00}/A_{01}) \) is a \( p \)-group for the prime \( p \).

Suppose \( x \notin H \) and \( x \) has order \( q \) for some prime \( q \). Since \( x \notin C_H(A_{00}^*) = 1, \) there exists \( a_0 \in A_{00}^* \) such that \( a_0 x \neq a_0 \). By \( \cap_i A_{01}^* = A_{0\infty}^* = 0, \) we have

\[ a_0 \in A_{0,i_0}^* \setminus A_{0,i_0+1}^* \]

for some integer \( i_0 \geq 0. \) Since \( A_{00}^*/A_{01}^* \cong \text{ZH} A_{0,i_0}^*/A_{0,i_0+1}^* \), \( x \in H = C_H(A_{00}^*/A_{01}^*) \) for any \( i_0 \), so \( a_0 x = a_0 + a_{i_0} = \sum_{j=0}^{i_0} \binom{i_0}{j} a_j, \) where \( a_1 \neq 0 \) and \( a_{i_0} \in A_{0,i_0+1}^* \) for some integer \( i_0 \). If \( a_1 x = a_1 \), then \( a_0 = a_0 x^q = a_0 + qa_1. \) That is, \( qa_1 = 0 \) and then \( a_1 = 0, \) a contradiction. So \( a_1 x \neq a_1 \).

Suppose \( a_0 x^r = \sum_{j=0}^{i_0} \binom{i_0}{j} a_j, \) where for any \( j, a_j \in A_{0,j}^* \setminus A_{0,j+1}^* \), \( a_j x \neq a_j \) and \( a_{j-1} x = a_{j-1} + a_j, i \geq j \geq 0. \) By \( a_j \neq a_j \) and
x \in H = C_H\left( A^*_{0,i}/A^*_{0,i+1} \right), \quad \text{we have } a_r x = a_r + a_{r+1}, \quad \text{where}

a_{r+1} \in A^*_{0,i+1}/A^*_{0,i+1} \text{ and } i_{r+1} > i_r. \quad \text{As above we also have } a_{r+1} x \neq a_{r+1}.

Now we have:

\[
a_0 x^{r+1} = \left( a_0 x^r \right) = \left[ \sum_{j=0}^{r} \binom{r}{j} a_j \right] x = \sum_{j=0}^{r} \binom{r}{j} (a_j x)
\]

\[
= \sum_{j=0}^{r} \binom{r}{j} (a_j + a_{j+1}) = a_0 + \sum_{j=1}^{r} \left[ \binom{r}{j} + \binom{r}{j-1} \right] a_j + a_{r+1}
\]

\[
= a_0 + \sum_{j=1}^{r} \binom{r+1}{j} a_j + a_{r+1} = \sum_{j=0}^{r+1} \binom{r+1}{j} a_j.
\]

Hence \( a_0 = a_0 x^q = \sum_{j=0}^{q} \binom{q}{j} a_j = a_0 + \sum_{j=1}^{q} \binom{q}{j} a_j \). That is,

\[
0 = \sum_{j=1}^{q} \binom{q}{j} a_j = q \left[ a_1 + \sum_{j=2}^{q-1} \frac{1}{q} \binom{q}{j} a_j \right] + a_q. \quad (\ast)
\]

If \( q \neq p \), by \( q = kp + t \) for some \( t \) with \( 0 < t < p \), we have

\[
ta_1 = -P(ka_1) - \sum_{j=2}^{q} \binom{q}{j} a_j \in (A^*_{0,i+1} + A^*_{0,i+1}) \leq A^*_{0,i+1}. \quad \text{contrary to the}
\]

abelian \( p \)-group \( A^*_{0,i+1}/A^*_{0,i+1} \) being elementary. So \( p = q \) and then \( H \) must be a \( p \)-group.

(ii) \( p = 2 \).

Now from (\ast), we have

\[
p[a_1 + \sum_{j=2}^{p-1} \frac{1}{p} \binom{p}{j} a_j] \in pA^*_{0,i} \cap A^*_{0,i} = A^*_1 \cap A^*_1 = A^*_1 \leq pA^*_{0,i+1}.
\]

Since \( A^* \) is torsion-free, \( a_1 + \sum_{j=2}^{p-1} \frac{1}{p} \binom{p}{j} a_j \in A^*_{0,i+1} \). If \( p > 2 \), then

\[
i_{p-1} \geq i_3 > i_2 \geq i_1 + 1.
\]

Thus \( a_1 + \sum_{j=2}^{p-1} \frac{1}{p} \binom{p}{j} a_j \in A^*_{0,i+1} \) and then

\[
a_1 \in A^*_{0,i+1} \leq A^*_{0,i+1}. \quad \text{contrary to } a_1 \in A^*_{0,i+1} \cap A^*_{0,i+1} \text{. Thus we must have}
\]

\( p = 2 \).

(iii) \( Z(H) \) contains only one element with order 2.
Since $H$ is a hyperfinite 2-group, by Lemma 1.2.12, we have $Z(H)$ is nontrivial and so contains at least one element $x_0$, say, with the order of $x_0$ being 2.

For $1 \neq x \in Z(H)$, if $C_{A^*}(x) \neq 0$, then, by $A^* \neq C_{A^*}(x)$, we have the $ZH$-submodule $A^*(x-1)$ $(\equiv_{ZH} A^*/C_{A^*}(x))$ is nonzero and has an $f$–$(ZH)$–decomposition, contrary to $A^*$ having no such $ZH$–submodules. So $C_{A^*}(x) = 0$ for any $x \in Z(H)$ with $x \neq 1$. In particular, $C_{A^*}(x_0) = 0$, where $x_0 \in Z(H)$ and the order of $x_0$ is 2. By $A^*(x_0+1) \leq C_{A^*}(x_0)$, we have $A^*(x_0+1) = 0$ and then $ax_0 = -a$ for any $a \in A^*$.

If $x \in H$ with the order of $x$ being 2 and $C_{A^*}(x) = 0$, then, since $A^*(x+1) \leq C_{A^*}(x) = 0$ we have $ax = -a$ for any $a \in A^*$ and then $a(xx_0) = (ax)x_0 = (-a)x_0 = -ax_0 = a,$

for any $a \in A^*$. Thus $xx_0 \in C_H(A^*) = 1$ and so $xx_0 = 1$, i.e., $x = x_0$. So it follows that $Z(H)$ contains only one element with order 2.

(iv) $H$ has no elements of order 4.

In fact, from the proof of the above in (iii), we have: if $x \in H$ with $x^2 = 1$ and $x \neq x_0'$, then $C_{A^*}(x) \neq 0$.

Let $y \in H$ and the order of $y$ be 4, then $(y^2x_0')^2 = 1$ and $y^2x_0 \neq x_0'$. Thus $C_{A^*}(y^2x_0') \neq 0$. Let $0 \neq a \in C_{A^*}(y^2x_0')$, since $a \in A^*$ and $\cap A^*_{0i} = A^*_{0\infty} = 0$, there exists $j_1$ such that $a \in A^*_{0,j_1}$. By $y \in H = C_H(A^*/A^*_{0,i+1})$ for any integer $i$, we have $ay = a + b$, where $b \in A^*_{0,j_1+1}$. Let
b ∈ A^* \setminus A^*_{0,j_2+1} for some integer j_2 > j_1, then by = b + c, where
c ∈ A^*_{0,j_2+1}. Thus, -a = ax_0 = a(y^2x_0)y^2 = ay^2 = (a+b)y = a+2b+c. Therefore,
2(a+b) = -c ∈ (2A^*_{0,j_2} ∩ A^*_{0,j_2+1}) = (A^*_{11} ∩ A^*_{0,j_2+1}) = A^*_{i,j_2+1} = 2A^*_{i,j_2}. Since A^* is
torsion-free, we have a+b ∈ A^*_{0,j_2} and then a ∈ A^*_{0,j_2} ≤ A^*_{0,j_2+1}, contrary to
a ∈ A^* \setminus A^*_{0,j_2+1}. So H has no elements of order 4.

Now by (i), (ii) and (iv), H is an elementary abelian 2-group and, by
(iii), |H| = 2. But, G is infinite and |G/H| < ∞, we must have H is infinite,
a contradiction. So the result is true.

Now Theorem A is followed.

Theorem A: If G is a hyperfinite locally soluble group, then any noetherian
ZG-module A has an f—decomposition.

Proof: Suppose A does not have an f—decomposition, then, by applying
Propositions 3.10. and 3.14, we will get a contradiction, So the theorem is
ture.

In our proof of Theorem A, the locally soluble condition is necessary.
However, it is not a necessary condition for the result as we can see from the
following results:

Corollary A1: If G is a hyperfinite almost locally soluble group, then any
noetherian ZG-module A has an f—decomposition. (Here almost locally soluble
means (locally soluble)—by—finite.)

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Proof: It follows from Lemma 1.2.5, Corollary 2.2.10 and Theorem A.

A special and very important case of Corollary A1 is that:

**Corollary A2:** If $G$ is a Černikov group, then any noetherian $\mathbb{Z}G$-module $A$ has an $f$-decomposition.

Another special case worthy of mention is:

**Corollary A3:** If $G$ is a locally finite group satisfying the minimal condition on subgroups, then any noetherian $\mathbb{Z}G$-module $A$ has an $f$-decomposition.

**Proof:** Since a locally finite group $G$ satisfying the minimal condition on subgroups is almost abelian [8] and therefore is a Černikov group, so the result follows from Corollary A2.
From Theorem A, we know that any noetherian $\mathbb{Z}G$-module $A$ over a hyperfinite locally soluble group $G$ has an $f$-decomposition: $A = A^f \oplus A^\ell$. In this chapter, we are going to discuss the details of the structure of the submodules $A^f$ and $A^\ell$.

Because of the complicated structure of $A^\ell$, we need first in §4.1 to recall some knowledge of injective hull and this yields, in §4.2, examples of $A^\ell$ with exponent $n$ for any integer $n > 0$. §4.3 contains the complete results about the structure of $A^f$. In §4.4, we focus our attention on $A^\ell$ again and have proved some results which look interesting. Especially, in some important cases we can prove that $A^\ell$ must be torsion and so have finite exponent. The general question of whether $A^\ell$ must be torsion remains open.

§4.1 INJECTIVE HULL

We follow the treatment given by B. Hartley and D. McDougall in their paper [6].

Let $R$ be a ring with 1. An $R$-module $X$ is called injective if whenever $U \subseteq W$ are $R$-submodules then every $R$-homomorphism of $U$ into $X$ can be extended to $W$. This is equivalent (but not immediately) to the requirement that $X$ be a direct summand of every $R$-module which contains it. A well-known result is that:

**Proposition 4.1.1:** (Hartley, [5]) Let $K$ be a field of characteristic $p \geq 0$ and $H$ a countable group. Every irreducible $KH$-module is injective if and only if $H$ is a periodic almost abelian $p'$-group.
If $V$ is an arbitrary $R$-module then an injective hull of $V$ (in the category of $R$-modules) is an $R$-module $\overline{V}$ satisfying:

(i) $\overline{V}$ is injective, and either

(ii) no proper $R$-submodule of $\overline{V}$ containing $V$ is injective, or

(ii)' $\overline{V}$ is an essential extension of $V$.

Here an $R$-module $W$ is said to be an essential extension of an $R$-submodule $U$ if every nonzero $R$-submodule of $W$ has a nonzero intersection with $U$. It was shown by Eckmann and Schopf [1] that every $R$-module $V$ has an injective hull $\overline{V}$ which is unique in the sense that if $V^*$ is another injective hull of $V$ then there is an isomorphism from $\overline{V}$ to $V^*$ extending the identity map on $V$.

The following simple fact was proved by B. Hartley and D. McDougall.

**Proposition 4.1.2:** Let $R$ be a ring with 1, let $V$ be an $R$-module and let $\overline{V}$ be an injective hull of $V$. Suppose $V = \bigoplus_{\lambda \in \Lambda} V_\lambda$, where each $V_\lambda$ is an $R$-submodule of $V$. If either (i) $\Lambda$ is finite, or (ii) $R$ satisfies the maximal condition on right ideals, then $\overline{V} = \bigoplus_{\lambda \in \Lambda} \overline{V}_\lambda$, where $\overline{V}_\lambda$ is an injective hull of $V_\lambda$.

B. Hartley and D. McDougall had pointed out that every injective $R$-module $U$ is divisible in the sense that $Ud = U$ for every element $d$ of $R$ which is not a zero-divisor, and they call an $R$-module $V$ $Z$-divisible if the additive group $V^+$ of $V$ is a divisible group. Then, immediately, they have

**Proposition 4.1.3:** Every injective $ZG$-module is $Z$-divisible.

For a prime $p$ and an abelian group $V$, let $V[p^k]$ denote the set of elements $v \in V$ satisfying $p^k v = 0$ (where $k \geq 0$ is an integer). If $V$ is in
addition an $R$-module then evidently $V[p^k]$ is an $R$-submodule of $V$. B.Hartley and D.McDougall have proved that:

**Proposition 4.1.4:** Let $G$ be a centre-by-finite $p'$-group and $V$ a $\mathbb{Z}G$-module such that, as an additive group, $V$ is a $p$-group (where $p$ is a prime). Let $\overline{V}$ be an injective hull of $V$. Suppose that either (i) $G$ is finite, or (ii) $V$ is an artinian $\mathbb{Z}G$-module. Then

(a) $\overline{V}$ (as an additive group) is a $p$-group and $\overline{V}[p] = V[p]$,

(b) $V$ is injective if and only if $V$ is $\mathbb{Z}$-divisible.

§4.2 **EXAMPLES OF $\mathbb{F}$**

First of all, from Carin's group (cf. [14] p.152), there follows a construction of an infinite irreducible $\mathbb{Z}G$-module, which as a group is a $p$-group, over the group $G = C_\infty$ for any two distinct primes $p$ and $q$. By applying Proposition 4.1.4, we have that:

**Proposition 4.2.1:** For any finite integer $n>0$, there exists a noetherian $\mathbb{Z}G$-module $A$ over a periodic abelian group $G$ such that $A^\mathbb{F}$ is of exponent $n$.

**Proof:** Suppose $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, where $p_1, \ldots, p_r$ are different primes and $\alpha_1, \ldots, \alpha_r$ are positive integers. Let $q$ be a prime satisfying $q \nmid n$. Let $G$ be the quasicyclic group $C_\infty$ and let $V_i$, which is an infinite elementary abelian $p_i$-group, be the irreducible $\mathbb{Z}G$-module arising from the Carin group $V_i[G]$, where $i = 1, 2, \ldots, r$. Let $\overline{V_i}$ be an injective hull of $V_i$ then, since $V_i$ is not $\mathbb{Z}$-divisible, $\overline{V_i} > V_i$ and $\overline{V_i}[p_i] = V_i[p_i] = V_i$ (Proposition 4.1.4).
Let $V_{i,j} = V_i^p V_j$. Since $V_{i,j}/V_{i,j-1} \cong \mathbb{Z}G V_{i,j} = V_i$, so the $\mathbb{Z}G$-submodule $V_{i,j}$ (and then $V_{i\alpha_1}$) has a finite $\mathbb{Z}G$-composition series with all nonzero $\mathbb{Z}G$-factors being infinite. Put $A_{\alpha_1} = V_{i,\alpha_1}$, then the noetherian $\mathbb{Z}G$-submodule $A_{\alpha_1}$ is of exponent $p_{\alpha_1}$ and $A_{\alpha_1} = A_{\alpha_1}^{\infty}$. 

Let $A = A_{\alpha_1} \oplus A_{\alpha_2} \oplus \cdots \oplus A_{\alpha_r}$, then the noetherian $\mathbb{Z}G$-module $A$ is of exponent $n$ and $A = A^n$. That is, $A$ is the required $\mathbb{Z}G$-module.

In order to get some more general examples, we investigate the relations between the $RG$-modules and the $R(G/N)$-modules, where $N$ is some normal subgroup of $G$ and $R$ is a ring with 1.

As B. Hartley and D. McDougall [6] have noted: if $G$ is a periodic abelian group, then all irreducible $\mathbb{Z}_p^G$-modules can be obtained (up to isomorphism) by the following:

Let $G$ be a periodic abelian group and $K$ an algebraic closure of $\mathbb{Z}_p^r$. Suppose $\delta$ is a homomorphism of $G$ into the multiplicative group $K^*$ of nonzero elements of $K$. Then since the elements of $\delta(G)$ are all roots of unity, it follows that the additive group $L_\delta$ generated by $\delta(G)$ is in fact a field. Let $K_\delta$ be the $\mathbb{Z}_p^r$-module whose underlying vector space is $L_\delta$ with the $G$-action given by

$$v g = v \cdot \delta(g) \quad (v \in K_\delta, \; g \in G).$$

Since $\delta(G)$ generates $L_\delta$ additively any $G$-submodule of $K_\delta$ is invariant under multiplication by any element of $L_\delta$; consequently $K_\delta$ is irreducible.

**Lemma 4.2.2:** (B. Hartley and D. McDougall) With the above notation

(i) every irreducible $\mathbb{Z}_p^r$-$G$-module is isomorphic to some $K_\delta$;
For a general (irreducible) RG-module V, using a natural method, we can always view V as an (irreducible) R(G/N)-module for some normal subgroup \( N \leq C_G(V) \) of G and in this case we denote the (irreducible) R(G/N)-module V by \( V^* \). That is, for any (irreducible) RG-module V, there exists \( N \leq C_G(V) \), being normal in G, and \( \theta \in \text{Hom}(G, G/N) \) with \( \text{Im}\theta = G/N \) and \( \text{Ker}\theta = N \) such that on the set V an (irreducible) R(G/N)-module structure (denote this R(G/N)-module by \( V^\theta \)) can be given by

\[ v \cdot (rg) = v(rg), \]

where \( v \in V, r \in R, \text{ and } g \in G \) such that \( \theta(g) = g \in G/N \). If \( N = C_G(V) \), then we denote the faithful (irreducible) R(G/N)-module \( V^\theta \) by \( V^{\theta^*} \).

On the other hand, if W is an (irreducible) R(G/N)-module for some normal subgroup N of G, then for any \( \theta \in \text{Hom}(G, G/N) \) satisfying \( \text{Im}\theta = G/N \) and \( \text{Ker}\theta = N \) we have an (irreducible) RG-module (denoted by \( W_\theta \)) defined by the following:

1. the underlying vector space of \( W_\theta \) is W, and
2. the RG-action \( \circ \) is given by

\[ w \circ (rg) = w(rg), \quad (w \in W, r \in R, \text{ and } g \in G). \]

It is clear that the above is well-defined and then \( W_\theta \) is an (irreducible) RG-module with \( C_G(W_\theta) \geq N \). Evidently, \( C_G(W_\theta) = N \) if and only if W is faithful on G/N, and in this case we denote \( W_\theta \) by \( W^{\theta^*} \). Since there exists \( \theta \in \text{Hom}(G, G/N) \) satisfying \( \text{Im}\theta = G/N \) and \( \text{Ker}\theta = N \), so for any (irreducible) R(G/N)-module W there is at least one (irreducible) RG-module V.
satisfying $C_G(V) \cong N$.

From the above definitions, we immediately have:

**Lemma 4.2.3:** (a) If $V$ is an (irreducible) RG-module with $C_G(V) = N$, then for any $\theta \in \text{Hom}(G, G/N)$ with $\text{Im}\theta = G/N$ and $\text{Ker}\theta = N$,

$$V = \left( V^{\theta} \right)^{\overline{\theta}};$$

(b) If $W$ is an (irreducible) R(G/N)-module and $\theta \in \text{Hom}(G, G/N)$ with $\text{Im}\theta = G/N$ and $\text{Ker}\theta = N$, then

$$W = \left( W^{\theta} \right)^{\overline{\theta}}.$$

**Lemma 4.2.4:** (a) Let $V_1$ and $V_2$ be two RG-modules with $N$ being their centralizer in $G$ for some normal subgroup $N$ of $G$, and let $\theta \in \text{Hom}(G, G/N)$ with $\text{Im}\theta = G/N$ and $\text{Ker}\theta = N$. If $V_1$ is RG-isomorphic with $V_2$, i.e., $\varnothing_{1} = \text{R}G \; V_{1} \text{R}G \; V_{2}$, then $V_{1} \psi = \text{R}(G/N) \; V_{2}$ for some $R(G/N)$-isomorphism $\psi$.

(b) Let $W_1$ and $W_2$ be two $R(G/N)$-isomorphic $R(G/N)$-modules, i.e., $W_{1} \psi = \text{R}(G/N) \; W_{2}$. Let $\theta \in \text{Hom}(G, G/N)$ with $\text{Im}\theta = G/N$ and $\text{Ker}\theta = N$. Then $W_{1} \psi = \text{R}G \; W_{2}$ for some RG-isomorphism $\psi$.

**Proof:** (a) Let $\psi: a \mapsto \varphi(a)$ for any $a \in V_{1}^{\theta}$, then $\psi$ is a group-isomorphism from $V_{1}^{\theta}$ to $V_{2}^{\theta}$. Now for any $a \in V_{1}^{\theta}$, any $r \in R$, and any $\tilde{g} \in G/N$, since $g = \theta(\tilde{g})$ for some $\tilde{g} \in G$, and since

$$\psi[a \cdot (rg)] = \psi[a(rg)] = \varphi[a(rg)]$$

$$= [\varphi(a)](rg) = [\psi(a)](r \tilde{g})$$

$$= \psi(a \cdot (rg)),$$

so $\psi$ is a $R(G/N)$-isomorphism from $V_{1}^{\theta}$ to $V_{2}^{\theta}$. That is, $V_{1}^{\theta} \cong _{R(G/N)} V_{2}^{\theta}$. 

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(b) The proof is almost as same as that of (a).

In fact, let \( \varphi: a \mapsto \varphi(a) \) for any \( a \in W_1 \), then \( \psi \) is a group-isomorphism from \( W_1 \) to \( W_2 \). Now for any \( a \in W_1 \), any \( r \in R \), and any \( g \in G \), since

\[
\psi[a \circ (rg)] = \psi[a(rg\theta)] = \varphi[a(rg\theta)]
\]

\[
= [\varphi(a)](rg\theta) = [\varphi(a)] \circ (rg\theta)
\]

\[
= [\psi(a)] \circ (rg\theta),
\]

so \( \psi \) is an RG-isomorphism from \( W_1 \) to \( W_2 \). That is, \( W_1 \cong_{RG} W_2 \).

Also, from the definition, we immediately have:

Lemma 4.2.5: (a) Let \( V_1 \) and \( V_2 \) be two RG-modules with \( N \) being their centralizer in \( G \) for some normal subgroup \( N \) of \( G \), and let \( \theta \in \text{Hom}(G, G/N) \) satisfying \( \text{Im}\theta = G/N \) and \( \text{Ker}\theta = N \). Then

\[
(V_1 \cap V_2) \cong_{RG} V_1 \cap V_2.
\]

(b) Let \( W_1 \) and \( W_2 \) be two \( R(G/N) \)-modules and let \( \theta \in \text{Hom}(G, G/N) \) with \( \text{Im}\theta = G/N \) and \( \text{Ker}\theta = N \). Then

\[
(W_1 \cap W_2) \cong_{RG} W_1 \cap W_2.
\]

Lemma 4.2.6: Let \( G \) be a periodic abelian \( p' \)-group for some prime \( p \) and \( R \) a ring with 1.

(a) Let \( V \) be an irreducible RG-module such that \( pV = 0 \), \( \overline{V} \) an injective hull of \( V \), and \( N = C_G(\overline{V}) \). Let \( \theta \in \text{Hom}(G, G/N) \) satisfying \( \text{Im}\theta = G/N \) and \( \text{Ker}\theta = N \). If the \( R(G/N) \)-module \( (V) \) is an injective hull of \( \overline{V} \), then

\[
(V) \cong_{R(G/N)} \overline{V}.
\]
(b) Let $N$ be a normal subgroup of $G$, $W$ an irreducible $R(G/N)$-module such that $pW = 0$, and let $\theta \in \text{Hom}(G, G/N)$ with $\text{Im} \theta = G/N$ and $\text{Ker} \theta = N$. If $\overline{W}$ is an injective hull of $W$ and if the $RG$-module $(W^{\overline{\theta}})$ is an injective hull of $W^{\overline{\theta}}$, then

$$(W^{\overline{\theta}}) \cong_{RG} (\overline{W})^{\overline{\theta}}.$$ 

**Proof:** (a) By Proposition 4.1.4, it is easy to know that $(V^{\overline{\theta}})$ is injective.

Since $V \leq \overline{V}$, so $V^{\overline{\theta}} \leq (\overline{V})^{\overline{\theta}}$; also for any nonzero $R(G/N)$-submodule $U$ of $(\overline{V})^{\overline{\theta}}$, since $U^{\overline{\theta}} \leq ((\overline{V})^{\overline{\theta}})^{\overline{\theta}} = \overline{V}$, so $V \cap U^{\overline{\theta}} \neq 0$ and then

$$V^{\overline{\theta}} \cap U = V^{\overline{\theta}} \cap (U^{\overline{\theta}})^{\overline{\theta}} = (V \cap U^{\overline{\theta}})^{\overline{\theta}} \neq 0.$$ 

That is, $(\overline{V})^{\overline{\theta}}$ is an injective hull of $V^{\overline{\theta}}$ and then $(V^{\overline{\theta}}) \cong_{R(G/N)} (\overline{V})^{\overline{\theta}}$.

(b) By Proposition 4.1.4, we may easily have $(\overline{W})^{\overline{\theta}}$ is injective.

Since $W \leq \overline{W}$, so $W^{\overline{\theta}} \leq (\overline{W})^{\overline{\theta}}$; also for any nonzero $RG$-submodule $U$ of $(\overline{W})^{\overline{\theta}}$, since $U^{\overline{\theta}} \leq ((\overline{W})^{\overline{\theta}})^{\overline{\theta}} = \overline{W}$, so $W \cap U^{\overline{\theta}} \neq 0$ and then

$$W^{\overline{\theta}} \cap U = W^{\overline{\theta}} \cap (U^{\overline{\theta}})^{\overline{\theta}} = (W \cap U^{\overline{\theta}})^{\overline{\theta}} \neq 0.$$ 

That is, $(\overline{W})^{\overline{\theta}}$ is an injective hull of $W^{\overline{\theta}}$ and then $(W^{\overline{\theta}}) \cong_{RG} (\overline{W})^{\overline{\theta}}$.

A special case is $N = 1$, thus the homomorphism $\theta$ is actually an automorphism of $G$ and then we use $V^{\theta}$ to denote either $V^{\overline{\theta}}$ or $V^{\overline{\theta}}$. Thus we have:

**Lemma 4.2.7:** Let $V$ be an $RG$-module and let $\varphi \in \text{Aut}(G)$. If $\overline{V}$ and $(V^{\varphi})$ are the injective hull of $V$ and $V^{\varphi}$, respectively, then

$$(V^{\varphi}) \cong_{RG} (\overline{V})^{\varphi}.$$
Now we continue to consider the examples of a noetherian $\mathbb{Z}G$-module $A$ with $A^r$ having a finite $\mathbb{Z}G$-composition series and being of finite exponent $n$. In fact, we will get a complete description for the $\mathbb{Z}G$-submodule $A^r$ of $A$ in the case that (1) $G \cong \mathbb{C}_q \wr \mathbb{Z}_n$; or (2) $G$ is Černikov such that its finite residual $H$ satisfies that: if $q \in \pi(H)$, then $q \not| n$.

As B. Hartley and D. McDougall pointed out: if $G$ is a nontrivial locally cyclic $p'$-group then there is, up to automorphism conjugacy, exactly one faithful irreducible $\mathbb{Z}G$-module. Here automorphism conjugacy is defined as: for a ring $R$ with 1, two $RG$-modules $V_1$ and $V_2$ are automorphism conjugate iff there is an automorphism $\varphi \in \text{Aut}(G)$ such that $V_1 \cong_{RG} V_2^\varphi$. Using the above discussion, we have: if $G$ is a nontrivial periodic abelian group then there is, up to quotient-automorphism conjugacy, exactly one irreducible $\mathbb{Z}G$-module such that $N$ is its centralizer in $G$ for some normal subgroup $N$ of $G$. Here quotient-automorphism conjugacy is defined as: for a ring $R$ with 1, if $W_1$ and $W_2$ are two $RG$-modules with $N$ being their centralizer in $G$ for some normal subgroup $N$ of $G$, then the $RG$-modules $W_1$ and $W_2$ are said to be quotient-automorphism conjugate iff there is an automorphism $\varphi \in \text{Aut}(G/N)$ and a homomorphism $\theta \in \text{Hom}(G, G/N)$ with $\text{Im} \theta = G/N$ and $\text{Ker} \theta = N$ such that

$$W_1 \cong_{RG} [(W_2^\varphi)^\theta]^\theta.$$

If $G = \mathbb{C}_q \wr \mathbb{Z}_n = \langle x_0, x_1, x_2, \cdots; x_0 = 1, x_i^q = x_i, i = 0, 1, 2, \cdots \rangle$ and if $A$ is a noetherian $\mathbb{Z}G$-module such that $A^r$ has a finite $\mathbb{Z}G$-composition series and is of exponent $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ with $q \not| n$, for each $i \leq r$, using Proposition 4.1.1, we may suppose

$$A[p_i] = D_{k=1}^{t_i} [(V_{i,j}^{\theta_1,j,k})_{i,j,k}^{\theta_1,j,k}].$$

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where $V_{i_jk}$ is the "unique" irreducible $\mathbb{Z}_G$-module with $C_{\alpha_{i_jk}}(V_{i_jk}) = < x_{i_jk} >$, $\phi_{i_jk} \in \text{Aut}(G/\langle x_{i_jk} \rangle)$, $\theta_{i_jk} \in \text{Hom}(G, G/\langle x_{i_jk} \rangle)$ satisfying $\text{Im} \theta_{i_jk} = G/\langle x_{i_jk} \rangle$ and $\text{Ker} \theta_{i_jk} = \langle x_{i_jk} \rangle$, $j_k \geq 0$, $k = 1, 2, \ldots, t_i$, and $t_i \geq 1$.

We claim that: up to isomorphism, $\mathbb{F}$ is a $\mathbb{Z}_G$-submodule of

$$D_r \otimes_{t=1} \bigotimes_{k=1}^{t} \left[ (A_{\alpha_{i_jk}}^{\theta_{i_jk}}) \right]^{ijk}$$

where $A_{\alpha_{i_jk}}$ is obtained from $V_{i_jk}$ over the group ring $\mathbb{Z}(G/\langle x_{i_jk} \rangle)$ as in the proof of Proposition 4.2.1. In fact, let $\mathbb{A}$ be an injective hull of $\mathbb{F}$ and let $\mathbb{A}^{[p_i]}$ be an injective hull of $\mathbb{A}^{[p_i]}$, $i=1,2,\ldots,t$. Since $\mathbb{D}_{r}^{\otimes_{i=1}^{r}} \mathbb{A}^{[p_i]} \leq \mathbb{F} \leq \mathbb{A}$, so the injective module $\mathbb{A}$ is an extension of $\mathbb{D}_{r}^{\otimes_{i=1}^{r}} \mathbb{A}^{[p_i]}$. For any nonzero $\mathbb{Z}_G$-submodule $B$ of $\mathbb{A}$, since $B \cap \mathbb{F} \neq 0$, so it is clear that $B \cap (\mathbb{D}_{r}^{\otimes_{i=1}^{r}} \mathbb{A}^{[p_i]}) \neq 0$. That is, $\mathbb{A}$ is an essential extension of $\mathbb{D}_{r}^{\otimes_{i=1}^{r}} \mathbb{A}^{[p_i]}$ and then, by definition, $\mathbb{A}$ is an injective hull of $\mathbb{D}_{r}^{\otimes_{i=1}^{r}} \mathbb{A}^{[p_i]}$.

Thus $\mathbb{A} \cong \mathbb{Z}_G \mathbb{D}_{r}^{\otimes_{i=1}^{r}} \mathbb{A}^{[p_i]} = \mathbb{D}_{r}^{\otimes_{i=1}^{r}} \mathbb{A}^{[p_i]}$. If $B$ is a $\mathbb{Z}_G$-submodule of $\mathbb{D}_{r}^{\otimes_{i=1}^{r}} \mathbb{A}^{[p_i]}$ such that $B$ is of exponent $n (=p_{i_1}^{\alpha_1} \cdots p_{i_t}^{\alpha_t})$, then, by using Proposition 4.1.2, Lemma 4.2.6, Lemma 4.2.7 and Lemma 4.2.4, we have:

$$\mathbb{D}_{r}^{\otimes_{i=1}^{r}} \mathbb{A}^{[p_i]} = \mathbb{D}_{r}^{\otimes_{i=1}^{r}} \mathbb{D}_{k=1}^{\otimes_{k=1}^{t}} \left[ (V_{i_jk})^{ijk} \right]^{\theta_{i_jk}}$$

$$\equiv \mathbb{Z}_G \mathbb{D}_{r}^{\otimes_{i=1}^{r}} \mathbb{D}_{k=1}^{\otimes_{k=1}^{t}} \left[ (V_{i_jk})^{ijk} \right]^{\theta_{i_jk}}$$

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Thus, up to isomorphism, \( B \) is a \( \mathbb{Z}G \)-submodule of \( \mathbb{Z}G \) \( \bigoplus_{i=1}^{t} \left\{ \left( v_{i,j} \right)_{j,k} \theta \right\} \).

So, up to isomorphism, \( \tilde{A} \) is a \( \mathbb{Z}G \)-submodule of \( \mathbb{Z}G \) \( \bigoplus_{i=1}^{t} \left\{ \left( a_{i,j,k} \right) \theta \right\} \).

A similar result is also true for any periodic abelian group \( G \) (especially, for an abelian Černikov group). That is, we could give a similar description of the \( \mathbb{Z}G \)-submodule \( \tilde{A} \) of a noetherian \( \mathbb{Z}G \)-module \( A \) satisfying that \( \tilde{A} \) is of finite exponent and has a finite \( \mathbb{Z}G \)-composition series, where \( G \) is a periodic abelian group satisfying that the intersection of the sets \( \pi(G) \) and \( \{ p; p \mid \exp \tilde{A} \} \) is empty.

Furthermore, if \( G \) is a Černikov group, then the finite residual \( H \) of \( G \) is a direct product of finitely many quasicyclic groups. If \( A \) is an irreducible \( \mathbb{Z}G \)-module then, as a \( \mathbb{Z}H \)-module, \( A \) is a direct sum of finitely many irreducible \( \mathbb{Z}H \)-submodules (Lemma 2.2.4). On the other hand, if \( V \) is an (infinite) irreducible \( \mathbb{Z}H \)-module (it is clear that such a \( V \) always exists), then the
following method guarantees the existence of the (infinite) irreducible $\mathbb{Z}G$-modules. And, from the "uniqueness" of the infinite irreducible $\mathbb{Z}H$-module $V$ (up to quotient-automorphism conjugacy), it follows that the infinite irreducible $\mathbb{Z}G$-modules are "almost" unique. Here we mean that: if $B$ is an infinite irreducible $\mathbb{Z}G$-module over a Černikov group $G$, which has $H$ as its finite residual, then

$$B = \bigoplus_{i=1}^{n} \left( (V_i^*)^{\phi_i^*} \right)^{\theta_i^*},$$

where $V_i$ is the "unique" infinite irreducible $\mathbb{Z}H$-module with $N_i$ being its centralizer in $H$ for some normal subgroup $N_i$ of $H$, $\phi_i \in \text{Aut}(H/N_i)$, and $\theta_i \in \text{Hom}(H, H/N_i)$ with $\text{Im}\theta_i = H/N_i$ and $\text{Ker}\theta_i = N_i$.

Let $G$ be a Černikov group, $H$ the finite residual of $G$, and $V$ an (infinite) irreducible $\mathbb{Z}H$-module. Let $T = \{t_1, t_2, \ldots, t_n\}$ be a transversal to $H$ in $G$. Consider the induced $\mathbb{Z}G$-module $V \otimes_{\mathbb{Z}H} \mathbb{Z}G = (V \otimes_{\mathbb{Z}H} t_1) \otimes \cdots \otimes (V \otimes_{\mathbb{Z}H} t_n)$ defined by

$$(v \otimes t_i)_{t_j} = v_{t_{ij}} \otimes t_{ij},$$

where $t_{ij} = ht_i$ with $h \in H$, $t_i^{-1} = v_{t_i} \otimes t_i$.

Here $\otimes$ is the direct sum of $\mathbb{Z}H$-submodules. It is easy to show that the above $\mathbb{Z}G$-module $V \otimes_{\mathbb{Z}H} \mathbb{Z}G$ is well-defined. Now, since $V \otimes_{\mathbb{Z}H} t_i \equiv_{\mathbb{Z}H} V^\phi$ for the automorphism $\phi$ of $H$ induced by $t_i^{-1}$ acting on $H$ by conjugation, and since $V$ is irreducible, so $V \otimes_{\mathbb{Z}H} \mathbb{Z}G$ as a $\mathbb{Z}H$-module has a finite $\mathbb{Z}H$-composition series and then, by Lemma 2.2.1, $V \otimes_{\mathbb{Z}H} \mathbb{Z}G$ has a finite $\mathbb{Z}G$-composition series. Therefore $V \otimes_{\mathbb{Z}H} \mathbb{Z}G$ contains an irreducible $\mathbb{Z}G$-submodule, say $B$. As in the proof of (4) and (5) in Lemma 2.2.4, $B$ is a direct sum of finitely many
irreducible $ZH$-submodules. Using the "uniqueness" of the infinite irreducible $ZH$-modules (up to quotient-automorphism conjugacy), we have the fact that $B$ is "almost" uniquely determined.

For a given Černikov group $G$, let $H$ be its finite residual and let

$$B = \bigoplus_{i=1}^{\infty} \left[ (V_i^1)^* \right]^i.$$

where $B$ is an infinite irreducible $ZG$-module, $V_i$ is a fixed infinite irreducible $ZH$-module with $N_i$ as its centralizer in $H$ for some normal subgroup $N_i$ of $H$, $\varphi_i \in \text{Aut}(H/N_i)$, and $\theta_i \in \text{Hom}(H, H/N_i)$ satisfying $\text{Im} \theta_i = H/N_i$ and $\text{Ker} \theta_i = N_i$. It is clear that $pB = 0$ for some prime $p \notin \pi(H)$. Consider $B$ as a $ZH$-module and let $\overline{B}$ be an injective hull of $B$ (here we note that it can be shown

$$\overline{B} = \bigoplus_{i=1}^{\infty} \left[ (V_i^1)^* \right]^i.$$

where $V_i^1$ is an injective hull of $V_i$. By Proposition 4.1.4, $B < \overline{B}$ and $\overline{B}[p] = B[p] = B$. Let $B_j = \overline{B}[p^j]$, then $B_j/B_{j-1} \cong ZH B_{j-1} = B$, so $B_j$ has a finite $ZH$-composition series in which each factor is infinite. Evidently, $B_j$ is of exponent $p^j$. From $B_j$, we consider the induced $ZG$-module $A_j = B_j \otimes ZG$ defined as above, then as $A_j$ is a direct sum of finitely many $ZH$-submodules, $A_j$ has a finite $ZH$-composition series with all factors being infinite. Thus $A_j$ has a finite $ZG$-composition series in which each factor is infinite (Lemma 2.2.1 and Corollary 2.2.3) and also $A_j$ is of exponent $p^j$.

We claim that:

$$A_j = \bigoplus_{i=1}^{\infty} U_{ij},$$

where $U_{ij} = \left[ \left[ \left[ V_i^1 \right]^j \right]^1 \right]^i \otimes ZH ZG.$

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In order to find a $\mathbb{Z}G$-isomorphism from $A_j$ to $\bigoplus_{i=1}^{r_j} U_i$, we must first show that (which has been mentioned in the above)

$$\overline{B} \cong_{\mathbb{Z}H} \bigoplus_{i=1}^{r_j} [(V_i^i)^{\theta^*_i \phi^+_i}].$$

Since $V_i^i \leq V_i$, so $[(V_i^i)^{\theta^*_i \phi^+_i}]_i \leq [(V_i^i)^{\theta^*_i \phi^+_i}]_i$ and then

$$B \leq \bigoplus_{i=1}^{r_j} [(V_i^i)^{\theta^*_i \phi^+_i}];$$

also for any nonzero $\mathbb{Z}H$-submodule $C$ of $\bigoplus_{i=1}^{r_j} [(V_i^i)^{\theta^*_i \phi^+_i}]_i$.

using Proposition 4.1.4, we can easily get $C$ is also a $p$-group, so

$$0 \neq C[p] \leq \left\{ \bigoplus_{i=1}^{r_j} [(V_i^i)^{\theta^*_i \phi^+_i}]_i \right\}[p]$$

$$= \bigoplus_{i=1}^{r_j} [(V_i^i)^{\theta^*_i \phi^+_i}]_i$$

$$= \bigoplus_{i=1}^{r_j} [(V_i^i)^{\theta^*_i \phi^+_i}]_i$$

$$= \bigoplus_{i=1}^{r_j} [(V_i^i)^{\theta^*_i \phi^+_i}]_i[p]$$

$$= B[p] = B,$$

and then $B \cap C \neq 0$. Therefore, $\bigoplus_{i=1}^{r_j} [(V_i^i)^{\theta^*_i \phi^+_i}]_i$ is an injective hull of $B$ and

so

$$\overline{B} \cong_{\mathbb{Z}H} \bigoplus_{i=1}^{r_j} [(V_i^i)^{\theta^*_i \phi^+_i}]_i.$$

Thus $B_j = \overline{B}[p^j] \geq_{\mathbb{Z}H} \left\{ \bigoplus_{i=1}^{r_j} [(V_i^i)^{\theta^*_i \phi^+_i}]_i[p^j] \right\} = \bigoplus_{i=1}^{r_j} \left\{ [(V_i^i)^{p^j}]_i \right\} = \bigoplus_{i=1}^{r_j} W_{ij}$

for some $\mathbb{Z}H$-isomorphism $\alpha$, where $W_{ij} = \left\{ [(V_i^i)^{p^j}]_i \right\}$. Since

$$A_j = B_j \otimes_{\mathbb{Z}H} \mathbb{Z}G = (B_j \otimes_{\mathbb{Z}H} \mathbb{Z}T_1) \otimes \cdots \otimes (B_j \otimes_{\mathbb{Z}H} \mathbb{Z}T_n)$$

and

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so for \( a \in A_j \), let \( a = (b_1 \otimes t_1) + \cdots + (b_n \otimes t_n) \) with \( b_1, \ldots, b_n \in B_j \), and let \( \beta: a \mapsto (b_1^\alpha \otimes t_1) + \cdots + (b_n^\alpha \otimes t_n) \in \bigoplus_{i=1}^n U_{ij} \), then it is routine to check that \( \beta \) is a \( \mathbb{Z}G \)-isomorphism from \( A_j \) to \( \bigoplus_{i=1}^n U_{ij} \). So we get the required isomorphism.

For any integer \( n > 0 \), let \( n = p_1^{\alpha_1} \cdots p_r^{\alpha_r} \), where \( p_1, \ldots, p_r \) are distinct primes and \( \alpha_1, \ldots, \alpha_r \) are positive integers. Suppose \( G \) is a Černikov group with \( p_i \neq \pi(H) \) for all \( 1 \leq i \leq r \), where \( H \) is the finite residual of \( G \). As above, there exists a \( \mathbb{Z}G \)-module \( A_i \) such that \( A_i \) has a finite \( \mathbb{Z}G \)-composition series in which each factor is infinite and \( A_i \) is of exponent \( p_i^{\alpha_i} \), where \( i = 1, 2, \ldots, r \). Let \( A = A_1 \oplus A_2 \oplus \cdots \oplus A_r \), then \( A \) has a finite \( \mathbb{Z}G \)-composition series (and so is noetherian), \( A = A_i \), and \( A \) is of exponent \( n \).

Let the noetherian \( \mathbb{Z}G \)-module \( A \) over a Černikov group \( G \) satisfy the condition that \( A_i \) has a finite \( \mathbb{Z}G \)-composition series and is of exponent \( n \) \((-p_1^{\alpha_1} \cdots p_r^{\alpha_r})\) with \( q \mid n \) for any \( q \in \pi(H) \), where \( H \) is the finite residual of \( G \). Suppose

\[
A_i[p_i] = \bigoplus_{j=1}^{s_j} \bigoplus_{k=1}^{t_j} [(V_{ij})^{p_i}]_{i j k} \theta_{i j k}^+,
\]

where \( \bigoplus_{j=1}^{s_j} \bigoplus_{k=1}^{t_j} [(V_{ij})^{p_i}]_{i j k} \theta_{i j k}^+ \) is an infinite irreducible \( \mathbb{Z}G \)-module, \( V_{ij} \) is the "unique" infinite irreducible \( \mathbb{Z}H \)-module (viewed as a \( \mathbb{Z}H \)-module) with \( N_{ijk} \) being its centralizer in \( H \) for some normal subgroup \( N_{ijk} \) of \( H \), \( \varphi_{ijk} \in \text{Aut}(H/N_{ijk}) \), \( \theta_{ijk} \in \text{Hom}(H, H/N_{ijk}) \) satisfying \( \text{Im} \theta_{ijk} = H/N_{ijk} \) and

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Kerθ_{ijk} = N_{ijk} and t_i ≥ 1. Then, up to a \(\mathbb{Z}G\)-isomorphism, \(A^f\) is a \(\mathbb{Z}G\)-submodule of the \(\mathbb{Z}H\)-module

\[
M = D_{r}^{t} D_{t}^{i} D_{t}^{s} \left( \left( V_{i,j,k}^{i} \left[ \left( V_{i,j,k}^{i} \right) \left( V_{i,j,k}^{i} \right) \right] \right) \theta_{ijk}^{*} \right).
\]

where the \(\mathbb{Z}H\)-module \(V_{i,j,k}^{i}\) is an injective hull of the \(\mathbb{Z}H\)-module \(V_{i,j,k}^{i}\) In fact, consider \(A^f\) as a \(\mathbb{Z}H\)-module, then, up to a \(\mathbb{Z}H\)-isomorphism, \(A^f\) is a \(\mathbb{Z}H\)-submodule of \(M\) (this claim can be proved by almost just quoting that of quasicyclic case). Since \(A^f\) is a \(\mathbb{Z}G\)-module, let \(U = \psi(A^f) \leq M\), where \(\psi\) is a \(\mathbb{Z}H\)-isomorphism from \(A^f\) to \(U\), then we may define a \(\mathbb{Z}G\)-module \(U^\psi\), which as a \(\mathbb{Z}H\)-module is contained in \(M\) and is \(\mathbb{Z}G\)-isomorphic with \(A^f\).

Let the underlying vector space of \(U^\psi\) be \(U\), and let the \(G\)-action \(\circ\) on \(U^\psi\) be given by

\[
u \circ g = \psi([\nu^{-1}(\nu)]g) \quad (u \in U^\psi, \ g \in G).
\]

It is clear that the above is well-defined and, as a \(\mathbb{Z}H\)-module, \(U^\psi\) is contained in \(M\). Now we prove \(U^\psi\) is \(\mathbb{Z}G\)-isomorphic with \(A^f\). For \(\psi: a \mapsto \psi(a)\), where \(a \in A^f\), it is evident that \(\psi\) is a group-isomorphism from \(A^f\) to \(U^\psi\); also since \(\varphi(a) \circ g = \varphi(a) \circ g = \varphi(a) \circ g\), for any \(g \in G\) and any \(a \in A^f\), so we have \(\varphi\) is actually a \(\mathbb{Z}G\)-isomorphism from \(A^f\) to \(M\). Thus, up to a \(\mathbb{Z}G\)-isomorphism, \(A^f\) is a \(\mathbb{Z}G\)-submodule of the \(\mathbb{Z}H\)-module \(M\).

§4.3 THE STRUCTURE OF \(A^f\)

This section is short, however, the results have completely shown the structure of \(A^f\) without further restriction on \(G\).

**Proposition 4.3.1:** Let \(G\) be a hyperfinite locally soluble group and \(A\) a
torsion-free noetherian \( \mathbb{Z}G \)-module with all irreducible \( \mathbb{Z}G \)-factors being finite. Then \( A \) is finitely generated as an abelian group and \( G/C_G(A) \) is finite.

**Proof:** We may assume that \( G \) acts faithfully on \( A \), i.e., \( C_G(A) = 1 \). In order to apply Corollary 2.1.7, we let \( H = C_G(A/pA) \) for some prime \( p > 2 \). By Lemma 2.4.5, \( A/pA \) is finite, and then \( |G/H| < \infty \). Applying Corollary 2.1.7, we get \( H = C_G(A) = 1 \). Thus \( G \) is finite. By the noetherian condition, we have \( A = \langle a_1, \ldots, a_n \rangle^G \) for some elements \( a_1, \ldots, a_n \). Hence, since \( G \) is finite, \( A \) is finitely generated as an abelian group. The result holds.

A generalization of Proposition 4.3.1 is that:

**Corollary 4.3.2:** Let \( G \) be a hyperfinite almost locally soluble group and \( A \) a torsion-free noetherian \( \mathbb{Z}G \)-module with all irreducible \( \mathbb{Z}G \)-factors being finite. Then \( A \) is finitely generated as an abelian group and \( G/C_G(A) \) is finite.

**Proof:** Let \( H \) be a normal subgroup of \( G \) such that \( H \) is locally soluble and \( G/H \) is finite. Consider \( A \) as a \( \mathbb{Z}H \)-module then, by Lemma 1.2.5 and Lemma 2.2.6, the torsion-free \( \mathbb{Z}H \)-module \( A \) is also noetherian and has all irreducible \( \mathbb{Z}H \)-factors being finite. Since \( H \) is also hyperfinite thus, by Proposition 4.3.1, \( A \) is finitely generated as an abelian group and \( H/C_H(A) \) is finite. For \( G/C_O(A) \), since \( |G/C_O(A)| = |G/H| \cdot |H/C_H(A)| < \infty \) and \( C_H(A) \leq C_G(A) \), so \( |G/C_G(A)| < \infty \), the result is proved.

An important consequence is that:

**Corollary 4.3.3:** Let \( G \) be a Černikov group and \( A \) a torsion-free noetherian \( \mathbb{Z}G \)-module with all irreducible \( \mathbb{Z}G \)-factors being finite. Then \( A \) is finitely
generated as an abelian group and \( G/C_G(A) \) is finite.

In particular, we have:

**Corollary 4.3.4:** If \( G \) is a locally finite group satisfying the minimal condition on subgroups, and if \( A \) is a torsion-free noetherian \( \mathbb{Z} \)G-module with all irreducible \( \mathbb{Z} \)G-factors being finite. Then \( A \) is finitely generated as an abelian group and \( G/C_G(A) \) is finite.

For a general \( \mathbb{Z} \)G-module \( A \) over a hyperfinite locally soluble group \( G \), we have:

**Theorem B:** Let \( G \) be a hyperfinite locally soluble group and \( A \) a noetherian \( \mathbb{Z} \)G-module with all irreducible \( \mathbb{Z} \)G-factors being finite. Then \( A \) is finitely generated as an abelian group and \( G/C_G(A) \) is finite.

**Proof:** Let \( T(A) \) be the torsion part of \( A \), then, by Proposition 4.3.1, \( A/T(A) \) is finitely generated as an abelian group. Since \( T(A) \) is also a noetherian \( \mathbb{Z} \)G-module, so \( T(A) \) has a finite exponent, say \( n \). Let \( n = p_1 p_2 \cdots p_m \), where \( p_i \) are primes, \( i = 1, 2, \cdots, m \). By Lemma 2.4.5, \( p_1 \cdots p_j T(A)/p_1 \cdots p_j T(A) \) is finite for any \( j \in \{1, 2, \cdots, m\} \), where \( p_0 = 1 \). Thus

\[
|T(A)| = |T(A)/p_1 T(A)| \cdot |p_1 T(A)/p_2 T(A)| \cdots |p_1 \cdots p_{m-1} T(A)| < \infty.
\]

Therefore the group \( A \) is finite-by-(finitely generated) and then is finitely generated as an abelian group. The other conclusion that \( G/C_G(A) \) is finite follows immediately from the following two simple results.

**Proposition 4.3.5:** Let \( G \) be a group, \( A \) a \( \mathbb{Z} \)G-module, and \( B \) a finite \( \mathbb{Z} \)G-submodule of \( A \) such that the \( \mathbb{Z} \)G-module \( A/B \) as a group is finitely
generated. If \( C_G(B) = G = C_G(A/B) \), and if \( C_G(A) = 1 \), then \( G \) is finite.

Proof: Let \( A/B = \langle a_1 + B, a_2 + B, \ldots, a_n + B \rangle \). Since \( G = C_G(A/B) \), so for \( g \in G \), we have \( a_i g + B = (a_i + B)g = a_i + B \), where \( 1 \leq i \leq n \). Thus \( a_i g = a_i + b_i \) for some \( b_i \in B \). Since \( B \) is finite and \( n \) is finite, there are only finitely many maps \( g^* : a_i \mapsto a_i + b_i \), where \( b_i \in B \) and \( 1 \leq i \leq n \). If \( G \) is infinite, then there exist two elements \( g_1, g_2 \in G \) such that \( a_i g_1 = a_i g_2 \) for all \( i \). Therefore \( a_i (g_1 g_2^{-1}) = a_i \), \( i = 1, 2, \ldots, n \). Also \( g_1 g_2^{-1} \in G = C_G(B) \), so it is clear that \( g_1 g_2^{-1} \in C_G(A) = 1 \) and then \( g_1 = g_2 \), a contradiction. So \( G \) is finite.

**Proposition 4.3.6:** If all irreducible \( \mathbb{Z}G \)-factors of a noetherian \( \mathbb{Z}G \)-module \( A \) over a hyperfinite locally soluble group \( G \) are finite, and if \( C_G(A) = 1 \), then \( G \) is a finite group.

Proof: Let \( T(A) \) be the torsion part of \( A \), then from the above proof of Theorem B we know that the \( \mathbb{Z} \)-submodule \( T(A) \) is finite, so \( |G/C_G(T(A))| < \infty \). Also, using Proposition 4.3.1, we have \( |G/C_G(A/T(A))| < \infty \). Let \( H = C_G(T(A)) \cap C_G(A/T(A)) \), then \( |G/H| < \infty \). Consider \( A \) as a \( \mathbb{Z}H \)-module, then \( T(A) \) is a finite \( \mathbb{Z}H \)-submodule of \( A \) and the \( \mathbb{Z}H \)-module \( A/T(A) \) as a group is finitely generated. It is clear that \( C_H(T(A)) = H = C_H(A/T(A)) \). Also, since \( C_G(A) = 1 \), so \( C_H(A) = 1 \). Thus, by Proposition 4.3.5, \( H \) is finite and then \( G \) is finite. The result holds.

As before, from Theorem B, we have

**Corollary B1:** Let \( G \) be a hyperfinite almost locally soluble group and \( A \) a noetherian \( \mathbb{Z}G \)-module with all irreducible \( \mathbb{Z}G \)-factors being finite. Then \( A \) is finitely generated as an abelian group and \( G/C_G(A) \) is finite.
Corollary B2: Let $G$ be a Černikov group and $A$ a noetherian $\mathbb{Z}G$-module with all irreducible $\mathbb{Z}G$-factors being finite. Then $A$ is finitely generated as an abelian group and $G/C_G(A)$ is finite.

Corollary B3: Let $G$ be a locally finite group satisfying the minimal condition on subgroups, and let $A$ be a noetherian $\mathbb{Z}G$-module with all irreducible $\mathbb{Z}G$-factors being finite. Then $A$ is finitely generated as an abelian group and $G/C_G(A)$ is finite.

§4.4 THE STRUCTURE OF $A^*$

In §4.2, we have saw that for any integer $n > 0$, there exists a noetherian $\mathbb{Z}G$-module $A$ over a periodic abelian (and hence hyperfinite and locally soluble) group $G$ such that $A^*$ is of exponent $n$. Must the $\mathbb{Z}G$-submodule $A^*$ of any noetherian $\mathbb{Z}G$-module $A$ over a hyperfinite locally soluble group $G$ necessarily be torsion? Further, if $pA = 0$ for some prime $p$, does $A$ always have a finite $\mathbb{Z}G$-composition series? Should these two questions both have a positive answer, the structure of the $\mathbb{Z}G$-submodule $A^*$ of a noetherian $\mathbb{Z}G$-module $A$ over a hyperfinite locally soluble group $G$ would become much clearer, and the examples given in §4.2 would be the typical models for other modules.

Conjecture A: If $G$ is a hyperfinite locally soluble group and if $A$ is a noetherian $\mathbb{Z}G$-module with $pA = 0$ for some prime $p$, then $A$ has a finite $\mathbb{Z}G$-composition series.

Conjecture B: If $G$ is a hyperfinite locally soluble group, then any
noetherian \( \mathbb{Z}G \)-module \( A \) with all irreducible \( \mathbb{Z}G \)-factors being infinite is torsion and so has finite exponent.

We now prove that these two conjectures are positive if \( G \) satisfies some further condition, and then the conjectures are also true even for Černikov groups (which need not be locally soluble).

First, we consider Conjecture A.

**Proposition 4.4.1:** Let \( G \) be a periodic abelian group and \( A \) a noetherian \( \mathbb{Z}G \)-module with \( pA = 0 \) for some prime \( p \). If \( G \) is a \( p' \)-group, then \( A \) has a finite \( \mathbb{Z}G \)-composition series.

**Proof:** Suppose \( A \) does not have a finite \( \mathbb{Z}G \)-composition series, then by the noetherian condition we may assume that for any nonzero \( \mathbb{Z}G \)-submodule \( C \) of \( A \), \( A/C \) has a finite \( \mathbb{Z}G \)-composition series. It is clear that every nonzero \( \mathbb{Z}G \)-submodule of \( A \) does not have a finite \( \mathbb{Z}G \)-composition series but any proper \( \mathbb{Z}G \)-image of the nonzero \( \mathbb{Z}G \)-submodules of \( A \) has one, so we may assume that \( A = \langle a \rangle^G \). Also we may assume \( G \) acts faithfully on \( A \).

Since \( pA = 0 \), so we may consider \( A \) as a \( \mathbb{Z}^G \)-module instead of \( \mathbb{Z}G \)-module. Let \( L \) denote the annihilator ideal \( \text{Ann}_{\mathbb{Z}^G} a = \{ r \in \mathbb{Z}^G; ra = 0 \} \), then \( \mathbb{Z}^G / L \cong \mathbb{Z}^G \langle a \rangle^G = A \). Thus the ring \( \mathbb{Z}^G / L \) is noetherian by \( A \) being a noetherian \( \mathbb{Z}G \)-module. If the ring \( \mathbb{Z}^G \) is regular, i.e., every finitely generated (right) ideal is generated by a single idempotent, then so is \( \mathbb{Z}^G / L \) a regular ring. And then, by Lemma 1.2.27, \( \mathbb{Z}^G / L \) is semisimple and so has
only finitely many (right) ideals. But this is not true as \( \mathbb{Z}_p G/L \cong \mathbb{Z}_p G \) and A has no finite \( \mathbb{Z}_p G \)-composition series.

The remainder is to prove \( \mathbb{Z}_p G \) is regular. In fact, let I be a finitely generated ideal of \( \mathbb{Z}_p G \), then \( I = \sum_{i=1}^n a_i \mathbb{Z}_p G \). Since G is a periodic abelian group, so G is locally finite, and then there is a finite subgroup F such that \( a_i \in \mathbb{Z}_p F \) for all \( i = 1, 2, \ldots, n \). Since G is a \( p' \)-group, so is F, and then \( \mathbb{Z}_p F \) is semisimple (Lemma 1.2.28). Thus any right ideal of \( \mathbb{Z}_p F \) is generated by a single idempotent (Lemma 1.2.29). Therefore \( \sum_{i=1}^n a_i \mathbb{Z}_p F = v \mathbb{Z}_p F \) for some idempotent \( v \in \mathbb{Z}_p F \). Hence

\[
\mathbb{Z}_p G = (v \mathbb{Z}_p F) \mathbb{Z}_p G = (\sum_{i=1}^n a_i \mathbb{Z}_p F) \mathbb{Z}_p G \leq \sum_{i=1}^n a_i \mathbb{Z}_p G
\]

\[
= I = \sum_{i=1}^n a_i \mathbb{Z}_p G \leq \sum_{i=1}^n (v \mathbb{Z}_p F) \mathbb{Z}_p G
\]

\[
= v \sum_{i=1}^n (\mathbb{Z}_p F) \mathbb{Z}_p G \leq v \mathbb{Z}_p G.
\]

That is, \( I = v \mathbb{Z}_p G \) with \( v = v^2 \). So \( \mathbb{Z}_p G \) is regular, the result is proved.

**Proposition 4.4.2:** Let G be a periodic abelian group and let A be a noetherian \( \mathbb{Z}G \)-module with \( pA = 0 \) for some prime \( p \). Then A has a finite \( \mathbb{Z}G \)-composition series.

**Proof:** Suppose A does not have a finite \( \mathbb{Z}G \)-composition series, then by the noetherian condition we may assume that for every nonzero \( \mathbb{Z}G \)-submodule C of A, \( A/C \) has a finite \( \mathbb{Z}G \)-composition series. Also we may assume that G acts faithfully on A, i.e., \( C_G(A) = 1 \).

Since G is abelian, every subgroup is normal in G. If G contains an
element, say $x$, with the order of $x$ being $p$ for the prime $p$, then since $A$ is a $p$-group, we have $C_A(x) \neq 0$. Also by $C_G(A) = 1$, we have $C_A(x) \neq A$. So $A(x-1) \equiv_{ZG} A/C_A(x)$ is nonzero and has a finite $ZG$-composition series. Also $A/A(x-1)$ has a finite $ZG$-composition series and hence so does $A$. This is contrary to the choice of $A$. Therefore $G$ contains no elements with order being the prime $p$, i.e., $G$ is a $p'$-group. Thus, by Proposition 4.4.1, $A$ has a finite $ZG$-composition series, a contradiction again. Hence we have proved the result.

From Proposition 4.4.2, using Lemma 1.2.5 and Lemma 2.2.1, we immediately have:

**Corollary 4.4.3:** Let $G$ be a periodic almost abelian group and let $A$ be a noetherian $ZG$-module with $pA = 0$ for some prime $p$. Then $A$ has a finite $ZG$-composition series.

As before, we have the following results:

**Corollary 4.4.4:** If $G$ is a Černikov group and if $A$ is a noetherian $ZG$-module with $pA = 0$ for some prime $p$, then $A$ has a finite $ZG$-composition series.

**Corollary 4.4.5:** If $G$ is a locally finite group satisfying the minimal condition on subgroups and if $A$ is a noetherian $ZG$-module with $pA = 0$ for some prime $p$, then $A$ has a finite $ZG$-composition series.

Now, for Conjecture B, we have:

**Proposition 4.4.6:** Let $G$ be a periodic abelian group with $|\pi(G)| < \infty$, where $\pi(G) = \{\text{prime } p; \ G \text{ has an element of order } p\}$, and let $A$ be a noetherian $ZG$-module with all irreducible $ZG$-factors being infinite. Then $A$ is torsion
and so has a finite $\mathbb{Z}G$-composition series as well as a finite exponent.

**Proof:** We only need to prove $A$ is torsion. Suppose $A$ is not torsion, then by the noetherian condition we may assume that for every nonzero $\mathbb{Z}G$-submodule $C$ of $A$, $A/C$ is torsion. Certainly, $A$ is a torsion-free $\mathbb{Z}G$-module. Also, we may assume that $G$ acts faithfully on $A$, i.e., $C^G(A) = 1$.

For any element $1 \neq x \in G$, since $G$ is abelian, so $A(x-1)$ and $C^A(x)$ are both $\mathbb{Z}G$-submodules of $A$ and $A(x-1) \cong \mathbb{Z}G A/C^A(x)$. If $C^A(x) \neq 0$, then $A(x-1) \neq 0$ by $C^G(A) = 1$, thus $A/A(x-1)$ and $A(x-1)$ being torsion implies that $A$ is torsion, a contradiction. So we must have $C^A(x) = 0$ for any $1 \neq x \in G$. If $G$ has two elements $x$ and $y$ satisfying $\langle x \rangle \cap \langle y \rangle = 1$ and both $x$ and $y$ are of the same order $p$ for some prime $p$, by

$$a[1 + (x^y) + (x^y)^2 + \cdots + (x^y)^{p-1}(x^y - 1) = a[(x^y)^p - 1] = a0 = 0$$

for any $a \in A$, we have

$$a[1 + (x^y) + (x^y)^2 + \cdots + (x^y)^{p-1}] \in C^A(x^y) = 0.$$  

Thus

$$pa = a[(1 + y + y^2 + \cdots + y^{p-1})$$
$$+(1 + xy + (xy)^2 + \cdots + (xy)^{p-1})$$
$$+(1 + x^2y + (x^2y)^2 + \cdots + (x^2y)^{p-1})$$
$$\vdots$$
$$+(1 + x^{p-1}y + (x^{p-1}y)^2 + \cdots + (x^{p-1}y)^{p-1})] = 0,$$  

where $0 \neq a \in A$. 

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This is contrary to \( A \) being torsion-free. Hence, if \( p \in \pi(G) \), then \( G \) has only one subgroup of order \( p \), and then the Sylow \( p \)-subgroup of \( G \) is locally cyclic. Thus the Sylow \( p \)-subgroup \( S_p \) of \( G \) is either finite cyclic or is isomorphic to \( C_p^\infty \). By \( A \) having infinite irreducible \( \mathbb{Z}G \)-factors, we have \( G \) is infinite. Since \( \pi(G) \) is a finite set and \( G \) is abelian, at least one Sylow subgroup of \( G \) is infinite, so there is at least one \( p \in \pi(G) \) such that the Sylow \( p \)-subgroup \( S_p \) is a quasicyclic group. Let

\[
S_p = \langle x_1, x_2, x_3, \ldots; x_1^p = 1, x_i^{p+1} = x_i \rangle, \quad i = 1, 2, \ldots >.
\]

For the prime \( p \), using Lemma 2.1.4, we have \( A/pA \) is nonzero and \( \bigcap_i^p A = 0 \). By Proposition 4.4.2, \( A \) has a descending series of \( \mathbb{Z}G \)-submodules:

\[
A = M_0 > M_1 > M_2 > \cdots > \bigcap M_i = 0,
\]

in which each \( \mathbb{Z}G \)-factor \( M_i/M_{i+1} \) is irreducible and, as a group, is an infinite elementary abelian \( p \)-group, \( i = 0, 1, 2, \ldots \). For any \( x \in S_p \), since \( x \) is of order a power of \( p \), by Lemma 1.2.8, we have \( x \in C_G(M_i/M_{i+1}) \), and then \( S_p \) is contained in \( C_G(M_i/M_{i+1}) \) for any \( i \geq 0 \).

Now for any \( a \in M_0 \setminus M_{i+1} \) and any \( x \in S_p \), \( ax = a + b \) for some \( b \in M_{i+1} \), thus \( a(1 + x + \cdots + x^{p-1}) = pa + b[(p-1) + (p-2)x + \cdots + x^{p-2}] \in M_{i+1} \) and then \( M_i(1 + x + \cdots + x^{p-1}) \subseteq M_{i+1} \). For \( 0 \neq a \in A \), by \( \bigcap_i M_i = 0 \), there exists \( i_0 \) such that \( a \in M_{i_0} \setminus M_{i_0+1} \). For \( x_j \in S_p \), let \( ax_j = a + b_j \), where \( b_j \in M_{i_0+1} \), \( j = 1, 2, \ldots \). Since

\[
a + b_j = ax_j = ax_{j+1}^p = (a + b_{j+1})x_{j+1}^{p-1}
\]

\[
= a + b_{j+1}(1 + x_{j+1} + \cdots + x_{j+1}^{p-2}),
\]

then \( b_j = b_{j+1}(1 + x_{j+1} + \cdots + x_{j+1}^{p-2}) \in M_{i_0+2} \), \( j = 1, 2, \ldots \). We suppose that
b_j \in M_\alpha, j = 1, 2, \ldots. Then by \( a + b_j = a + b_{j+1}(1 + x_{j+1} + \cdots + x_{j+1}^{p-1}) \) again, we have \( b_j = b_{j+1}(1 + x_{j+1} + \cdots + x_{j+1}^{p-1}) \in M_{\alpha+1} \). So, by induction, we have \( b_j \in M_i \) for all \( i = 1, 2, \ldots \). Then \( b_j \in \cap_i M_i = 0 \), i.e., \( b_j = 0 \) for all \( j \). That is, \( a = ax_j \) for all \( j \). Therefore \( a \in C_A(S_p) \) and then, by the arbitrariness of \( a \), we have \( 1 \neq S_p \leq C_G(A) \) contrary to \( G \) being faithful on \( A \).

By this contradiction, we have proved the result.

Consequently, by using Lemma 1.2.5 and Lemma 2.2.6, we have:

**Corollary 4.4.7:** Let \( G \) be a periodic almost abelian group with \( \pi(G) \) being finite and let \( A \) be a noetherian \( \mathbb{Z}G \)-module with all irreducible \( \mathbb{Z}G \)-factors being infinite. Then \( A \) is torsion and so has finite exponent.

**Corollary 4.4.8:** If \( G \) is a Černikov group and if \( A \) is a noetherian \( \mathbb{Z}G \)-module with all irreducible \( \mathbb{Z}G \)-factors being infinite, then \( A \) is torsion and so has finite exponent.

**Corollary 4.4.9:** If \( G \) is a locally finite group satisfying the minimal condition on subgroups, and if \( A \) is a noetherian \( \mathbb{Z}G \)-module with all irreducible \( \mathbb{Z}G \)-factors being infinite, then \( A \) is torsion and so has finite exponent.

For the general case, we have finally proved neither Conjecture A nor Conjecture B. But the following results are worth mentioning here.

**Proposition 4.4.10:** Let \( G \) be a hyperfinite \( p \)-group and \( A \) a noetherian
$\mathbb{Z}G$-module with $pA = 0$, where $p$ is a prime. Then $A$ is finite.

**Proof:** Suppose $A$ is not finite, then by the noetherian condition we may assume that for any nonzero $\mathbb{Z}G$-submodule $C$ of $A$, $A/C$ is finite. Also, we may assume that $A = \langle a_1, \cdots, a_n \rangle^G$, where $a_i$ is of order $p$ for all $i$. So it follows that $G \neq C_G(A)$ and then, by replacing $G$ by $G/C_G(A)$, we may assume that $G$ acts faithfully on $A$.

Since $G$ is a hyperfinite $p$-group, so $Z(G) \neq 1$ (Lemma 1.2.12). Let $x \in Z(G)$ with $x$ being of order $p$, then $A_1 = \langle a_i \rangle^{<x>}$ is a finite $\mathbb{Z}<x>$-module. Thus there exists $0 \neq a_0 \in A_1$ such that $a_0 \in C_A(x)$. By $G$ being faithful on $A$, we have $A \neq C_A(x)$, and so $A(x-1) (\equiv ZG A/C_A(x))$ is a nonzero finite $\mathbb{Z}G$-module. Also $A/A(x-1)$ is finite, which implies that $A$ is finite, a contradiction. So the result is true.

**Proposition 4.4.11:** Let $G$ be a hyperfinite $p$-group and $A$ a noetherian $\mathbb{Z}G$-module with all irreducible $\mathbb{Z}G$-factors being infinite. Then $A$ is a torsion $p'$-group of finite exponent.

**Proof:** Let $T(A)$ be the torsion part of $A$. If $T(A) < A$, then $A/T(A)$ is a torsion-free noetherian $\mathbb{Z}G$-module. By Lemma 2.1.4, $p(A/T(A)) < A/T(A)$, and by Proposition 4.4.10, $\langle A/T(A) \rangle / p(A/T(A))$ is finite. But $A$ and hence $A/T(A)$ has no nonzero finite $\mathbb{Z}G$-factors, a contradiction. Thus $T(A) = A$. Since $A$ is noetherian, so is of finite exponent and then, by Proposition 4.4.10 again, we have $A$ must be a $p'$-group. So the result holds.

Combining all the main results in the two chapters above, we have:

**Theorem:** If $G$ is a finite extension of a periodic abelian group with $\pi(G)$
finite, then

(1) any noetherian \( \mathbb{Z}G \)-module \( A \) has an \( f \)-decomposition

\[
A = A^f \oplus \tilde{A}^f
\]

where \( A^f \) is a \( \mathbb{Z}G \)-submodule of \( A \) such that each nonzero irreducible \( \mathbb{Z}G \)-factor of \( A^f \) is finite while the \( \mathbb{Z}G \)-submodule \( \tilde{A}^f \) has no nonzero finite \( \mathbb{Z}G \)-factors;

(2) \( A^f \) as a group is finitely generated and \( G/C_G(A^f) \) is finite; and

(3) \( \tilde{A}^f \) is torsion and has a finite exponent as well as a finite \( \mathbb{Z}G \)-composition series.
D.I. Zaïcev has proved a number of results about modules over hypercyclic groups [19, 20] and modules over hyperfinite groups [21, 22]. In this chapter, we consider modules over hyper-(cyclic or finite) groups and get a lot of results which generalize all the Zaïcev’s results about modules over hyperfinite groups.

As we shall mention in §6.1, there exist torsion-free irreducible \( \mathbb{Z}G \)-modules over hypercyclic abelian groups (such irreducible \( \mathbb{Z}G \)-modules do not occur in the previous discussion). Due to the existence of such a module, we meet some difficulties in the research for the structure of the \( \mathbb{Z}G \)-modules over hyper-(cyclic or finite) groups. However, if we restrict ourselves only to the periodic case for artinian \( \mathbb{Z}G \)-modules and to the generalization of the Zaïcev’s results for noetherian \( \mathbb{Z}G \)-modules, we successfully get the required results. But we have not been able to generalize completely our decomposition theorem for hyperfinite groups.

§5.1 THE I–DECOMPOSITION

We have seen that: if \( A \) is an artinian (or noetherian) \( \mathbb{Z}G \)-module over a hyperfinite locally soluble group \( G \), then \( A \) always has an \( i \)-decomposition

\[
A = A^i \oplus A^f,
\]

where \( A^f \) is a \( \mathbb{Z}G \)-submodule of \( A \) such that each irreducible \( \mathbb{Z}G \)-factor of \( A^f \) is finite and the \( \mathbb{Z}G \)-submodule \( A^f \) contains no nonzero finite \( \mathbb{Z}G \)-factors. Now we consider \( G \) to be hyper-(cyclic or finite) instead of \( G \) being just hyperfinite.
and prove the following results (we note that these results are essentially
generalizations of those of Zaïcev).

Artinian Case:

**Theorem C:** If G is a hyper-(cyclic or finite) locally soluble group, then any
periodic artinian \(\mathbb{Z}G\)-module \(A\) has an \(f\)-decomposition.

**Proof:** We may assume that \(G\) acts faithfully on \(A\).

Suppose that the \(\mathbb{Z}G\)-module \(A\) does not have an \(f\)-decomposition, then one
can find a \(\mathbb{Z}G\)-submodule not having an \(f\)-decomposition but each of its proper
\(\mathbb{Z}G\)-submodules does have. We may suppose that \(A\) satisfies this condition. It
follows that \(A\) is not a sum of proper \(\mathbb{Z}G\)-submodules (Lemma 1.2.24) and so \(A\)
has a unique maximal \(\mathbb{Z}G\)-submodule \(M\), containing every proper \(\mathbb{Z}G\)-submodule of
\(A\). For each \(a \in A\backslash M\), certainly \(<a>^G = A\). If \(G\) were finite, then \(A\) would be
finitely generated as an abelian group and therefore finite, contrary to the
choice of \(A\). So \(G\) is infinite.

It is clear that \(A\) is a \(p\)-group for some prime \(p\) (since a periodic
abelian group is the direct sum of its components). Let \(M = M^f \oplus M^I\) be the
\(f\)-decomposition of \(M\), we consider the following two cases: (1) \(A/M\) is finite,
in this case, we may suppose that \(M^f = 0\) by considering \(A/M^f\); (2) \(A/M\) is
infinite, similarly we suppose that \(M^I = 0\).

(1) \(A/M\) is finite and \(M = M^I\).

Now, for \(H = C_G(A/M)\), since \(G\) is infinite, \(H \neq 1\). Thus \(H\) contains a
nontrivial normal subgroup of \(G\) being infinite cyclic or finite.

Suppose firstly that \(1 \neq <x> \leq H\) and \(<x>\) is normal in \(G\), where \(x\) is of
infinite order, then the \(\mathbb{Z}G\)-submodule \(A(x-1) \leq M\). Since
\[ \varphi: \ a + M \mapsto a(x-1) + M(x-1) \]

is clearly a homomorphism from the group \( A/M \) to the group \( A(x-1)/M(x-1) \), so the \( \mathbb{Z}G \)-factor \( A(x-1)/M(x-1) \) is finite. By \( M (= M') \) having no nonzero finite \( \mathbb{Z}G \)-factors, we have \( A(x-1) = M(x-1) \). Thus, for \( a \in A \), there exists \( m \in M \) such that \( a(x-1) = m(x-1) \), i.e., \( (a-m)(x-1) = 0 \). Then \( A = M + C_A(x) \). But this is contrary to \( G \) acting faithfully on \( A \) and all proper \( \mathbb{Z}G \)-submodules being contained in \( M \). So \( H \) contains a nontrivial finite minimal normal subgroup, say \( N \), of \( G \). Since \( G \) is locally soluble, \( N \) is an elementary abelian \( q \)-group for some prime \( q \). We show that \( q \neq p \) by showing that \( Q_p(G) = 1 \).

In fact, if \( Q_p(G) \neq 1 \), then \( G \) has a finite normal \( p \)-subgroup \( K \), \( K \neq 1 \), and we put \( L = C_G(K) \). Since \( A \) is an artinian \( \mathbb{Z}G \)-module and \( |G/L| < \infty \), \( A \) is also an artinian \( \mathbb{Z}L \)-module (Lemma 1.2.5). Thus \( A \) has a least \( \mathbb{Z}L \)-submodule \( A_1 \) such that \( A_1 \) is not contained in \( M \). Then \( A_1/(A_1 \cap M) \isom \mathbb{Z}L(A_1 + M)/M \) is a finite irreducible \( \mathbb{Z}L \)-module. Since \( A/M \) is an irreducible \( \mathbb{Z}G \)-module (and is a \( p \)-group), \( K \) acts trivially on \( A/M \). Therefore \( A_1(x-1) \subseteq M \) for each \( x \in K \).

Since \( |G/L| < \infty \), \( M \) has no nonzero finite \( \mathbb{Z}L \)-factors (Lemma 2.2.6) and so \( A_1(x-1) \) is a \( \mathbb{Z}L \)-module with no nonzero finite \( \mathbb{Z}L \)-factors. Thus \( A_1 \cap C_{A_1}(x) \isom \mathbb{Z}L(A_1(x-1)) \) has no nonzero finite \( \mathbb{Z}L \)-factors. But \( A_1/(A_1 \cap M) \) is finite and so \( C_{A_1}(x) \) is not contained in \( M \). By the choice of \( A_1 \) we have \( C_{A_1}(x) = A_1 \) for each \( x \in K \). Thus \( A_1 \leq C_A(K) \) and so \( C_A(K) \) is not contained in \( M \). Since \( M \) contains all proper \( \mathbb{Z}G \)-submodules of \( A \), we must have \( C_A(K) = A \), contrary to \( G \) being faithful on \( A \). Thus \( Q_p(G) = 1 \) and then \( q \neq p \).

By Lemma 1.2.4, \( A = [A, N] \leq C_A(N) \). Since \( N \leq H = C_G(A/M) \), \( [A, N] \leq M \) and then \( C_A(N) \neq 0 \). Since \( M \) contains all proper \( \mathbb{Z}G \)-submodules of \( A \), so we have \( C_A(N) = A \) and so \( [A, N] = 0 \) contrary to \( G \) acting faithfully on \( A \). We
have proved case (1).

(2) \( A/M \) is infinite and \( M^f = 0 \).

In this case, we choose a finite \( \mathbb{Z}G \)-submodule \( D \), say, of \( M \) and let \( H = C_D(D) \), then \( |G/H| < \infty \) and so \( H \) contains either a nontrivial finite normal subgroup of \( G \) or an infinite cyclic subgroup being normal in \( G \). If \( H \) contains a nontrivial finite subgroup \( N \) being minimal normal in \( G \), then it is easy to know that \( N \) is a \( p' \)-group (by almost using the method used in case (1) to show that \( O_{p'}(G) = 1 \)). Hence, by Lemma 1.2.4, \( A = [A, N] \oplus C_A(N) \). The \( \mathbb{Z}G \)-submodule \( M \) does not contain both factors of this decomposition and so one of them is \( A \) (and the other is zero). But \( [A, N] \neq 0 \) by \( G \) being faithful on \( A \). On the other hand, \( D \triangleleft C_A(N) \) and so \( C_A(N) \neq 0 \). This contradiction shows that \( H \) does not contain any nontrivial finite subgroups being normal in \( G \). So, we may suppose that \( 1 \neq <x> \leq H \), \( <x> \) is normal in \( G \) and \( x \) is of infinite order. Let \( G_1 = C_G(x) \), then \( |G/G_1| \leq 2 \) and \( x \in Z(G_1) \). Since \( A/M \) is infinite and irreducible, \( A \) has a least \( \mathbb{Z}G_1 \)-submodule \( A_1 \) such that \( A_1/M \) is an infinite irreducible \( \mathbb{Z}G_1 \)-module. If \( A_1 \) has an \( f-(\mathbb{Z}G_1) \)-decomposition, i.e., \( A_1 = B \oplus M \), where \( B \) is an infinite irreducible \( \mathbb{Z}G_1 \)-submodule of \( A_1 \), then the nonzero \( \mathbb{Z}G \)-submodule \( B^G \) of \( A \) has no nonzero finite \( \mathbb{Z}G \)-factors (Lemma 2.2.7). Thus \( B^G \cap M = 0 \) and then \( A = B^G \oplus M \). That is, \( A \) has an \( f-(\mathbb{Z}G) \)-decomposition with \( A^f = M \) and \( A^f = B^G \), a contradiction. So \( A_1 \) has no \( f-(\mathbb{Z}G_1) \)-decomposition. By passing from \( G \) to \( G_1 \) and \( A \) to \( A_1 \), we may assume that \( 1 \neq x \in H \cap Z(G) \).

(a) If \( A(x-1) \neq A \), then \( A(x-1) \leq M \). For \( \varphi: a+M \mapsto a(x-1)+M(x-1) \) \( (a \in A) \), we have \( A/M \mathop{\rightarrow}\limits^{p} \mathbb{Z}G A(x-1)/M(x-1) \) and \( \ker \varphi = 0 \) or \( A/M \). If \( \ker \varphi = 0 \), then \( A(x-1)/M(x-1) \) is an infinite irreducible factor of \( M \), a contradiction. So \( \ker \varphi = A/M \). That is, \( A(x-1) = M(x-1) \), and then \( A = M + C_A(x) \), a
contradiction again.

(b) \( A(x-1) = A \). Then, for \( a \in A \setminus M \), there exists \( a_0 \in A \) such that \( a = a_0(x-1) \) and \( A = \langle a \rangle^G \). Choose a finitely generated subgroup \( K \) of \( G \) such that \( a_0 \in \langle a \rangle^K \), \( D \leq \langle a \rangle^K \), and \( x \in K \). Let \( A_1 = \langle a \rangle^K \), then \( A_1 \) is a finitely generated \( ZK \)-module and \( K \) is a finitely generated hyper-(cyclic or finite) soluble group. If \( K \) is a supersoluble–by–finite group, then \( K \) is a polycyclic group (since supersoluble groups and finite soluble groups are both polycyclic). Thus \( A_1 \) has a \( ZK \)-submodule \( B_1 \) of finite index such that \( D \cap B_1 \neq D \) by the residual finiteness of finitely generated abelian–by-polycyclic groups [7]. Consider the finite \( ZK \)-module \( A_1/B_1 \). Since \( x \in Z(K) \), \( A_1/B_1 \) can be viewed as a \( Z \langle x \rangle \)-module. Then, by [19], we can get \( A_1/B_1 = B/B_1 \oplus C/B_1 \), where the \( Z \langle x \rangle \)-submodule \( B/B_1 \) has a \( Z \langle x \rangle \)-composition series in which each \( Z \langle x \rangle \)-factor is \( < x \rangle \)-trivial and the \( Z \langle x \rangle \)-submodule \( C/B_1 \) has no nonzero \( Z \langle x \rangle \)-factors which are \( < x \rangle \)-trivial. Since \( (D+B_1)/B_1 \) is an \( < x \rangle \)-trivial \( Z \langle x \rangle \)-submodule of \( A_1/B_1 \), so \( B/B_1 \neq 0 \). Thus \( A_1/C \) is a nonzero finite \( Z \langle x \rangle \)-module and \( A_1/C \cong (A_1/B_1)/(C/B_1) \cong Z \langle x \rangle \) shows that \( A_1/C \) has a finite \( Z \langle x \rangle \)-composition series in which each \( Z \langle x \rangle \)-factor is \( < x \rangle \)-trivial. Hence \( (A_1(x-1)+C)/C = \overline{A}_1(x-1) < \overline{A}_1 \), where \( \overline{A}_1 = A_1/C \). That is, \( A_1(x-1) < A_1 \).

But, on the other hand, since \( A_1(x-1) \) is a \( ZK \)-module and \( a_0 \in A_1 \), so
\[
A_1 = \langle a \rangle^K = \langle a_0(x-1) \rangle^K = \left( \langle a_0 \rangle(x-1) \right)^K \leq \left( A_1(x-1) \right)^K = A_1(x-1),
\]
a contradiction.

The remainder is to prove that \( K \) is a supersoluble–by–finite group. However, it follows from the following result.

**Lemma 5.1.1**: Any finitely generated hyper–(cyclic or finite) soluble group is a supersoluble–by–finite group.
Proof: In fact, let

\[ G = G_\alpha \triangleright \cdots \triangleright G_1 \triangleright G_0 = 1 \]

be an ascending normal series of subgroups of a finitely generated soluble group \( G \) in which each factor \( G_{\beta+1}/G_\beta \) is cyclic or finite. Since \( G/G_\alpha = 1 \) is clearly a supersoluble–by-finite group, we may assume that there exists \( \beta \leq \alpha \) such that \( G/G_\beta \) is supersoluble–by-finite but \( G/G_\gamma \) is not for all \( \gamma < \beta \).

We claim that \( \beta = 0 \). Otherwise, if \( \beta - 1 \) exists, then, (i) \( G_\beta /G_{\beta-1} \) is cyclic would imply that \( G/G_{\beta-1} \) is supersoluble–by-finite, a contradiction; and (ii) \( G_\beta /G_{\beta-1} \) is finite implies that \( G/G_{\beta-1} \) is polycyclic and so is residually finite, therefore there is an \( N \) with \( G/N \) finite and \( N \cap G_\beta = G_{\beta-1} \), thus \( N/G_{\beta-1} \cong NG_\beta /G_\beta \) is supersoluble–by-finite and so is \( G/G_{\beta-1} \), a contradiction.

Thus, \( \beta - 1 \) does not exist, i.e., \( \beta \) is a limit ordinal. Since \( G/G_\beta \) is finitely generated, by \([15, \text{p.} 403]\), \( G_\beta \) is finitely generated as a \( G \)-operator group.

Thus, let \( G_\beta = \langle x_1, \ldots, x_n \rangle^G \), since \( G_\beta = \bigcup_{\gamma < \beta} G_\gamma \) so there exist \( \gamma_1, \ldots, \gamma_n \) such that \( x_i \in G_{\gamma_i} \). Let \( \gamma_0 < \beta \) such that \( \gamma_i < \gamma_0 \) for all \( i = 1, \ldots, n \), then \( x_i \in G_{\gamma_0} \) for all \( i \). Since \( G \triangleright G_{\gamma_0} \), so

\[ G_\beta = \langle x_1, \ldots, x_n \rangle^G \leq (G_{\gamma_0})^G = G_{\gamma_0}. \]

Thus \( G/G_\beta = G/G_{\gamma_0} \), contrary to the hypothesis for \( \beta \). Hence \( \beta = 0 \) and then the result is proved.

**Noetherian Case:**

We have not yet got the complete \( f \)-decomposition theory for a noetherian \( \mathbb{Z}G \)-module over a hyper–(cyclic or finite) group, however, the following
results look like a good start.

**Proposition 5.1.2:** Let $H$ be a normal hyper-(cyclically or finitely) embedded subgroup of a group $G$, and let $A$ be a nonzero noetherian $\mathbb{Z}G$-module. If $C_A(H) = 0$, then there is a subgroup $K$ of $H$ and a nonzero $\mathbb{Z}G$-submodule $B$ of $A$ such that $K$ is normal in $G$, $C_B(K) = 0$, and $K$ induces in $B$ a cyclic or finite group of automorphisms.

**Proof:** Suppose the lemma is false. Using the noetherian condition we may assume that the lemma is true in all proper $\mathbb{Z}G$-images of the $\mathbb{Z}G$-module $A$. We may also assume that $G$ acts faithfully on $A$.

There is a cyclic or finite subgroup $F \leq H$ with $F$ being normal in $G$. If $C_A(F) = 0$ then the lemma is true taking $F, A$ for $K, B$.

Consider the second possibility $C_A(F) \neq 0$. We let $A_1$ be the $\mathbb{Z}G$-submodule $C_A(F)$ and let $H_1 = C_H(F)$. Then $H_1$ is normal in $G$ and $|H/H_1| < \infty$.

(1) Suppose that the centralizer $A_2/A_1 = C_{A_2}(H_1)$ is nonzero, i.e., $A_2 \neq A_1$. Consider the $\mathbb{Z}H$-isomorphism $A_2/C_{A_2}(f) \cong \mathbb{Z}H_1 A_2(f-1)$, where $f \in F$.

Since $A_1 \leq C_{A_2}(f)$ and $A_2/A_1$ is $H_1$-trivial, we have that $A_2(f-1)$ is $H_1$-trivial for any $f \in F$. It follows that $[A_2, F] = \sum_{f \in F} A_2(f-1)$ is $H_1$-trivial and so $H$ induces a finite group of automorphisms on $[A_2, F]$. Since $A_2 \neq A_1$ the $\mathbb{Z}G$-submodule $[A_2, F] \neq 0$ and $C_{[A_2, F]}(H) = 0$ since $C_A(H) = 0$. Therefore the lemma is true with $K = H$, $B = [A_2, F]$.

(2) Suppose now that $A_2 = A_1$, i.e., $C_{A/A_1}(H_1) = 0$. Then the $\mathbb{Z}G$-module $A/A_1$ and the normal subgroup $H_1$ satisfy the hypotheses of the lemma and so
there is a subgroup $K_1$ of $H_1$ and a nonzero $\mathbb{Z}G$-submodule $B_1/A_1$ of $A/A_1$ such that $K_1$ is normal in $G$, $C_{B_1/A_1}(K_1) = 0$, and $K_1$ induces in $B_1/A_1$ a cyclic or finite group of automorphisms.

Put $G_1 = C_G(F)$; clearly $H_1 = H \cap G_1$, $|G/G_1| < \infty$.

(a) We consider firstly the case that $K_1/C_{K_1}(B_1/A_1)$ is cyclic.

Let $B_2 = [B_1, F]$ and let $K_0 = C_{K_1}(B_1/A_1)$. Since $A_1 = C_A(F)$, so

$$[K_0, B_1, F] = [[K_0, B_1], F] \leq [A_1, F] = 0;$$

also by $K_0 \leq K_1 < H_1 = C_H(F)$, we have

$$[F, K_0, B_1] = [[F, K_0], B_1] = [1, B_1] = 0.$$

Thus, by three subgroup lemma,

$$[B_2, K_0] = [[B_1, F], K_0] = [B_1, F, K_0] = 0.$$

Therefore $B_2 \leq C_A(K_0)$ and we then can view the noetherian $\mathbb{Z}G$-module $B_2$ as a

noetherian $\mathbb{Z}(G/K_0)$-module. Applying Lemma 1.2.19 to the cyclic normal subgroup $K_1/K_0$ of $G/K_0$, there is an integer $m$ such that

$$B_2^{(k-1)^m} \cap C_{B_2}(k) = 0,$$

where $k$ is an element such that $K_1 = K_0 < k>$.  

If $B_2^{(k-1)^m} = 0$, then

$$0 = B_2^{(k-1)^m} = \left(\sum_{f \in F} B_1(f-1)\right)^{(k-1)^m}$$

$$= \sum_{f \in F} B_1((f-1)(k-1)^m) = \sum_{f \in F} B_1((k-1)^m(f-1))$$

$$= \sum_{f \in F} (B_1(k-1)^m)(f-1).$$

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That is, \( B^k - 1 \leq A_1 \). But this is contrary to

\[
C_{B_i/A_1}(k) = C_{B_i/A_1}(K_i) = 0.
\]

So we have \( B_2(k-1)^m \neq 0 \) and then the lemma is true by taking \( B = B_2(k-1)^m \) and \( K = K_i \).

(b) Secondly, we consider the case that \( K_i/C_{K_1}(B_i/A_1) \) is finite.

Choose in \( F \) a least set of elements \( \{x_1, \ldots, x_n\} \) satisfying

\[
A_i = C_{B_i}(F) = C_{B_i}(x_1) \cap \cdots \cap C_{B_i}(x_n)
\]

and put \( B_2 = C_{B_i}(x_1) \cap \cdots \cap C_{B_i}(x_n) \) if \( n > 1 \) and \( B_2 = B_1 \) if \( n = 1 \). Then

\[
B_2 \neq A_1
\]

and \( C_{B_2}(x_n) = C_{B_1}(x_1) \cap \cdots \cap C_{B_1}(x_n) = A_1 \). Consider the \( ZG_i \)-isomorphism

\[
B_2/A_1 = B_2/C_{B_2}(x_n) \equiv ZG_i B_2(x_n -1).
\]

Since \( K_i \leq G_i \), \( B_2 \leq B_1 \), and \( K_i \) induces a finite group of automorphisms on \( B_1/A_1 \), so \( K_i \) induces a finite group of automorphisms on \( B_2/A_1 \) and hence on \( B_2(x_n -1) \). Since \( C_{B_1/A_1}(K_i) = 0 \) we also have \( C_{B_2(x_n -1)}(K_i) = 0 \).

Let \( D = B_2(x_n -1) \). Then \( D \) is a \( ZG_i \)-submodule of \( B_1 \), \( C_{D_i}(K_i) = 0 \), and \( |K_i/C_{K_1}(D)| < \infty \). Let \( \overline{D} \) be the \( ZG \)-module generated by \( D \), then \( \overline{D} = \sum g \in T D_g \) is a finite sum of \( ZG_i \)-submodules \( D_g \), where \( T \) is a transversal to \( G_i \) in \( G \).

Note that since \( K_i \) is normal in \( G \), \( C_{D_i}(K_i) = C_D(K_i)g = 0 \), and \( C_{K_1}(Dg) = g^{-1} C_{K_1}(D)g \), it follows that \( |K_i/\cap_{g \in T} C_{K_1}(Dg)| < \infty \) and so \( K_i \) induces a finite group of automorphisms in \( \overline{D} \).
Now consider two cases.

(A) \(D\) contains an element of finite order.

Then \(D\) contains a maximal elementary abelian \(p\)-subgroup \(D_1 \neq 0\) and we let \(D_1 = \sum_{g \in T} D_{1g}\). Let \(S\) be the \(K_1\)-socle of the \(ZG_1\)-submodule \(D_1\), i.e., sum of all irreducible \(ZK_1\)-submodules (these irreducible \(ZK_1\)-submodules are all finite since \(K_1\) induces a finite group of automorphisms in \(D\)). Since \(D_1\) is a \(ZG_1\)-submodule and \(K_1\) is normal in \(G\) so \(S\) is a \(ZG_1\)-submodule and \(\bar{S} = \sum_{g \in T} S_g\) is a \(ZG\)-submodule. Now \(S_g\) is a sum of irreducible \(ZK_1\)-submodules and so \(\bar{S}\) is a sum of irreducible \(ZK_1\)-submodules each being contained in some \(S_g\). Since \(C_{Dg}(K_1) = 0\) it follows that \(C_{\bar{S}}(K_1) = 0\). Thus we can take \(K_1\) and \(\bar{S}\) satisfying the conclusion of the lemma.

(B) The group \(D\) is torsion-free.

Let \(T(\bar{D})\) be the torsion part of \(\bar{D}\). Since \(\bar{D}\) is a noetherian \(ZG\)-module, \(T(\bar{D})\) has a finite exponent. Therefore \(n\bar{D} \cap T(\bar{D}) = 0\) for some \(n\) and \(n\bar{D}\) is torsion-free.

We put \(m = |K_1/C_{K_1}(\bar{D})|\), \(C = C_{D}(K_1)\) and show that

\[
[mn\bar{D}, K_1] \cap C = 0
\]  

(3)

In fact, if \(a \in [mn\bar{D}, K_1] \cap C\), then \(a \in [mn\bar{B}, K_1]\) for some finitely generated \(K_1\)-admissible subgroup \(\bar{B}\) of \(\bar{D}\). Since \(n\bar{B} \cap C = C_{n\bar{D}}(K_1)\), \(\bar{B} \leq \bar{D}\), and \(n\bar{D}\) is torsion-free, so \(n\bar{B}/(n\bar{B} \cap C)\) is torsion-free and then \(n\bar{B} = (n\bar{B} \cap C) \oplus V\), where \(V\) is a free abelian subgroup. Using Lemma 1.2.26, there is in \(n\bar{B}\) a
$K_1$-admissible subgroup $W$ such that $(n \mathcal{B} \cap C) \cap W = 0$ and the factor group $n \mathcal{B} / [(n \mathcal{B} \cap C) \oplus W]$ has a finite exponent, dividing $m$. Thus $mn \mathcal{B} \leq (n \mathcal{B} \cap C) \oplus W$. It follows that $[mn \mathcal{B}, K_1] \leq W$ and so $[mn \mathcal{B}, K_1] \cap C = 0$. Hence $a = 0$ and (3) is proved.

Note now that $[mn \mathcal{D}, K_1] \neq 0$. In fact, if $[mn \mathcal{D}, K_1] = 0$, then $mn \mathcal{D} \leq C_D(K_1) = C$. Therefore $mn \mathcal{D} \leq C$ and since $D$ is torsion-free, $D \leq C$. This shows that $D$ is a $K_1$-trivial $\mathbb{Z}G_1$-module and since $D = B_2(x_n - 1)$ and is $G_1$-isomorphic to $B_2/A_1$ (2) we have $B_2/A_1$ is also $K_1$-trivial. But $C_{B_1/A_1}(K_1) = 0$ and so $B_2 = A_1$ contrary to (1). Thus $[mn \mathcal{D}, K_1] 
eq 0$.

Since $[mn \mathcal{D}, K_1]$ is a $\mathbb{Z}G$-submodule and $K_1$ induces in it (as in $D$) a finite group of automorphisms then it follows from (3) that the conditions of the lemma are satisfied by $K_1$ and $[mn \mathcal{D}, K_1]$. This completes the proof of the proposition.

**Proposition 5.1.3:** Let $G$ be a hyper-(cyclic or finite) group, $A$ a noetherian $\mathbb{Z}G$-module, and $B$ a $\mathbb{Z}G$-submodule of $A$ such that $B$ is of finite index in $A$ and $B$ has no nonzero finite $\mathbb{Z}G$-factors, then $B$ has a complement in $A$, i.e., $A = B \oplus C$ for some finite $\mathbb{Z}G$-submodule $C$ of $A$.

**Proof:** Suppose that $B$ does not have a complement in $A$. By considering an appropriate factor-module of $A$ we may assume that for every $\mathbb{Z}G$-submodule $D$ of $B$ with $D \neq 0$, $B/D$ has a complement in $A/D$.

Put $H = C_G(A/B)$, then, since $G/H$ is finite and the irreducible $\mathbb{Z}G$-factors of $B$ are all infinite, we have $C_B(H) = 0$ so we can apply Proposition 5.1.2 to the subgroup $H$ and the $\mathbb{Z}G$-module $B$. So there is a subgroup $K$ of $H$ and a nonzero $\mathbb{Z}G$-submodule $D$ of $B$ such that $K$ is normal in $G$,
\( C_D(K) = 0 \) and \( K \) induces on \( D \) a cyclic or finite group of automorphisms, i.e., 
\[ \frac{K}{C_K(D)} \] is cyclic or finite.

We write \( A \) as a sum \( A = B + A_1 \) with \( B \cap A_1 = D \) and we will consider the 
\( \mathbb{Z}G \)-submodule \( A_1 \) as a faithful \( \mathbb{Z}G_0 \)-module, where \( G_0 = G/C_G(A_1) \). It is clear 
that \( D \) is a \( \mathbb{Z}G_0 \)-submodule of \( A_1 \) such that \( D \) is of finite index in \( A_1 \) and \( D \) has 
no nonzero finite \( \mathbb{Z}G_0 \)-factors. Also \( D \) has no complements in \( A_1 \) for otherwise 
if \( A_1 = D \oplus C_1 \) for some \( \mathbb{Z}G_0 \)-submodule \( C_1 \) of \( A_1 \) then \( C_1 \) can be viewed as a 
\( \mathbb{Z}G \)-submodule of \( A \) by \( G_0 = G/C_G(A_1) \) and then \( A = B + A_1 = B \oplus C_1 \) (Lemma 1,2,25), 
a contradiction.

Since \( C_D(K) = 0 \) and \( D \leq A_1 \), so \( K \) is not contained in \( C_G(A_1) \). Let 
\( K_0 = (K \cap C_0(A_1))/C_0(A_1) \), then \( K_0 \neq 1 \). Also, it is clear that \( C_D(K_0') = 0 \) and 
\( K_0 \) induces on the \( \mathbb{Z}G_0 \)-submodule \( D \) of \( A_1 \) a cyclic or finite group of 
automorphisms. We prove that \( C_K(D) = 1 \). For suppose \( C_K(D) \neq 1 \) and let \( F_0 \) 
bef a nontrivial cyclic or finite normal subgroup of \( G_0 \) contained in \( C_K(D) \). If 
\( x \in F_0 \), then \( D \leq C_{A_1}(x) \). Since \( |A_1/D| = |A/B| < \infty \) and, as groups, 
\( A_1/C_{A_1}(x) \cong A_1(x-1) \), we see that \( A_1(x-1) \) is finite. Thus the \( \mathbb{Z}G_0 \)-submodule 
\( [A_1, F_0] \) is finite. Also \( F_0 \leq C_K(D) \leq K_0 = (K \cap C_0(A_1))/C_0(A_1) \leq (H \cap C_0(A_1))/C_0(A_1) = (C_G(A/B)C_G(A_1))/C_G(A_1) \), thus 
\( [A_1, F_0] \leq B \), and then \( [A_1, F_0] \leq D \). By \( D \) having no nonzero finite \( \mathbb{Z}G_0 \)-factors, we have 
\( [A_1, F_0] = 0 \) contrary to \( G_0 \) acting faithfully on \( A_1 \). So \( C_K(D) = 1 \) and 
hence \( K_0 \) is cyclic or finite.

Now put \( G_1 = C_G(K_0) \), \( K_0 = \langle x_1, x_2, \ldots, x_m \rangle \), \( C_n = C_{A_1}(\langle x_1, \ldots, x_n \rangle) \), 
\( n = 1, 2, \ldots, m \). We prove that \( A_1 = D + C_n \), \( n = 1, 2, \ldots, m \).
It is clear that $A_1 = D + C_1$. Suppose $A_1 = D + C_n$ we prove $A_1 = D + C_{n+1}$.

Consider the isomorphism of $\mathbb{Z}G$ modules

$$C_n / C_{n+1} = C_n / C_n (x_{n+1}^{-1}) \cong \mathbb{Z}G C_n (x_{n+1}^{-1}),$$

where $C_n (x_{n+1}^{-1})$ may not be contained in $C_n$ if $K_0$ is nonabelian.

Since $x_{n+1} \in K_0 = \frac{(KC_G(A_1))}{C_G(A_1)} \leq \frac{(HC_G(A_1))}{C_G(A_1)} = \frac{(C_G(A/B)C_G(A_1))}{C_G(A_1)}$, the $\mathbb{Z}G$-module $C_n (x_{n+1}^{-1})$ of $A_1$ is contained in $B$ and then in $D$. Since $|G_0 / G_1| < \infty$ it follows from Lemma 2.2.6 that the irreducible $\mathbb{Z}G$-factors of $D$ are all infinite, hence so are the factors of $C_n / C_{n+1}$. But $C_n / (C_n + D \cap C_n) \cong \mathbb{Z}G (C_n + D) / (C_n + D)$, a factor module of the finite module $A_1 / D$. Hence $C_n + D = C_{n+1} + D$ and so $A_1 = C_{n+1} + D$. Thus $A_1 = C_n + D$ for all $n = 1, 2, \cdots, m$. In particular, put $n = m$, $C_m = C_{A_1} (K_0)$ and $A_1 = D + C_{A_1} (K_0)$. By Lemma 1.2.6, $C_{A_1} (K_0)$ is a $\mathbb{Z}G$-submodule of $A_1$.

Since $D \cap C_{A_1} (K) = C_D (K) = 0$ we have $A_1 = D \oplus C_{A_1} (K)$, contrary to $D$ having no complements in $A_1$. The proof is completed.

Using almost the same proof as above, we immediately have:

**Proposition 5.1.4:** Let $G$ be a hyper-(cyclic or finite) group, $A$ a noetherian $\mathbb{Z}G$-module, and $B$ a $\mathbb{Z}G$-submodule of $A$ such that, as group, $A/B$ is a finite $p$-group for some prime $p$ and the $\mathbb{Z}G$-submodule $B$ contains no nonzero $\mathbb{Z}G$-factors being finite $p$-groups. Then $B$ has a complement in $A$, i.e., $A = B \oplus C$ for some $\mathbb{Z}G$-submodule $C$ of $A$. 


Dual to Proposition 5.1.3, we have:

**Proposition 5.1.5:** Let $G$ be a hyper-(cyclic or finite) group, $A$ a $\mathbb{Z}G$-module, and $B$ a finite $\mathbb{Z}G$-submodule of $A$ such that all irreducible $\mathbb{Z}G$-factors of $A/B$ are infinite. Then $B$ has a complement in $A$, i.e., $A = B \oplus C$ for some $\mathbb{Z}G$-submodule $C$ of $A$.

**Proof:** By Zorn's Lemma, $A$ has a $\mathbb{Z}G$-submodule $D$ maximal with respect to $B \cap D = 0$. We show that $A = B \oplus D$. Suppose not, then by replacing $A$ by $A/D$ we may assume that for any nonzero $\mathbb{Z}G$-submodule $C$ of $A$, $B \cap C = 0$. We also assume that $G$ acts faithfully on $A$.

Put $H = C^g(B), \quad |G/H| < \infty$, so there is a normal subgroup $K$ of $G$ contained in $H$ such that $K$ is either cyclic or finite. Put $H_1 = C^h(K)$. Since $H_1$ is normal in $G$ and $|G/H_1| < \infty$ it follows from Lemma 2.2.6 that the irreducible $\mathbb{Z}H_1$-factors of $A/B$ are infinite. If $x \in K$, then $B \leq C^A(x)$ and so the irreducible $\mathbb{Z}H_1$-factors of $A/C^A(x)$ and hence $A(x-1)$ are infinite.

We prove that $[A, K] \cap B = 0$. If not, then there is a minimal set of elements $x_1, \ldots, x_n$ such that $B_1 = B \cap \sum_{i=1}^n A(x_i - 1) \neq 0$. Then

$$B_1 \equiv \mathbb{Z}H_1 \left( B \oplus \sum_{i=1}^{n-1} A(x_i - 1) \right) / \left( \sum_{i=1}^{n-1} A(x_i - 1) \right)$$

$$\leq \left( \sum_{i=1}^{n} A(x_i - 1) \right) / \left( \sum_{i=1}^{n-1} A(x_i - 1) \right)$$

$$\equiv \mathbb{Z}H_1 A(x_n - 1) / \left( A(x_n - 1) \cap \sum_{i=1}^{n-1} A(x_i - 1) \right).$$

This shows that $A(x_n - 1)$ has a nonzero finite $\mathbb{Z}H_1$-factor contrary to the fact that the irreducible $\mathbb{Z}H_1$-factors of $A(x-1)$ are all infinite. Thus $[A, K] \cap B = 0$ and hence $[A, K] = 0$, contrary to $G$ acting faithfully on $A$. So the result is true.
From Proposition 5.1.5, we have:

**Corollary 5.1.6:** Let $G$ be a hyper-(cyclic or finite) group, and $A$ a noetherian $\mathbb{Z}G$-module. Then $A$ has a nonzero finite $\mathbb{Z}G$-factor if and only if $A$ has a nonzero finite $\mathbb{Z}G$-image.

**Proof:** We only need to suppose that $A$ has a finite $\mathbb{Z}G$-factor $B/C$, then using the noetherian condition we may assume that every irreducible $\mathbb{Z}G$-factor of $A/B$ is infinite. Then applying Proposition 5.1.5 to $A/C$ with the finite $\mathbb{Z}G$-submodule $B/C$ we obtain a finite $\mathbb{Z}G$-image of $A$.

Also, almost follow the proof of Proposition 5.1.5, we have:

**Proposition 5.1.7:** Let $G$ be a hyper-(cyclic or finite) group, $A$ a $\mathbb{Z}G$-module and $B$ a $\mathbb{Z}G$-submodule of $A$. If as a group $B$ is a finite $p$-group for some prime $p$, and if the factor module $A/B$ contains no nonzero finite $\mathbb{Z}G$-factors being $p$-groups, then $B$ has a complement in $A$, i.e., $A = B \oplus C$ for some $\mathbb{Z}G$-submodule $C$ of $A$.

By a similar proof to Corollary 5.1.6, we have:

**Corollary 5.1.8:** Let $G$ be a hyper-(cyclic or finite) group, and $A$ a noetherian $\mathbb{Z}G$-module. Then $A$ has a nonzero $\mathbb{Z}G$-image being a finite $p$-group for some prime $p$ if and only if $A$ has such a nonzero $\mathbb{Z}G$-factor.

§5.2 THE SPLITTING THEORY

Depending on the corresponding decomposition of modules, D. I. Zaïcev has
proved the splitting results for hypercyclic (hypercentral) extensions ([19], [20]) and hyperfinite extensions ([21], [22]) of an abelian normal subgroup. Similarly, we will prove splitting results for hyper-(cyclic or finite) extensions of abelian groups in this section. We divide the discussion in two parts: one is for the artinian case, and another for the noetherian case. The results proved in this section can all be viewed as a generalization of Zaicov's results.

\textbf{Artinian Case:}

Before proving the main result of this part, we prove some lemmas.

\textbf{Lemma 5.2.1:} Let $A$ be a nonzero artinian $\mathbb{Z}G$-module and $H$ a normal hyper-(cyclically or finitely) embedded subgroup of $G$. If $A$ is an $H$-perfect module then $H$ has a subgroup $K$ such that $K$ is normal in $G$ and $A$ has a nonzero $K$-perfect $\mathbb{Z}G$-image on which $K$ induces a cyclic or finite group of automorphisms.

\textbf{Proof:} (It is similar to the corresponding one in [21].)

We assume that $G$ acts faithfully on $A$ and take a nontrivial cyclic or finite normal subgroup $F$ of $G$ with $F \leq H$. If $A = [A, F]$, then we can take $F$ to be the required subgroup and $A$ the $F$-perfect $\mathbb{Z}G$-image.

So we may suppose that $A_1 = [A, F] \neq A$. Put $H_1 = C_H(F)$, then $H_1$ is normal in $G$ and $|H/H_1|$ is finite.

(1) Suppose $A_1 \neq [A_1, H_1]$. Consider the $\mathbb{Z}G$-module $\overline{A} = A/[A_1, H_1]$ and its $\mathbb{Z}G$-submodule $\overline{A}_1 = A_1/[A_1, H_1]$. Since $\overline{A}_1$ is an $H_1$-trivial $\mathbb{Z}H_1$-module and, for each $x \in F$, $\overline{A}(x^{-1}) \leq [\overline{A}, F] \leq \overline{A}_1$, so $\overline{A}(x^{-1})$ is an $H_1$-trivial $\mathbb{Z}H_1$-module. Therefore the $\mathbb{Z}H_1$-isomorphism $\overline{A}/C_{\overline{A}}(x) \cong \overline{A}(x^{-1})$ shows that

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\( \overline{\mathcal{A}}/\mathcal{C}_A(x) \) is an \( H_1 \)-trivial \( \mathcal{Z}H_1 \)-module. It follows that \( \overline{\mathcal{A}}/\mathcal{C}_A(F) = \overline{\mathcal{A}}/\bigcap_{x \in F} \mathcal{C}_A(x) \) (or \( \overline{\mathcal{A}}/\mathcal{C}_A(F) = \overline{\mathcal{A}}/\mathcal{C}_A(x_0) \), where \( F = \langle x_0 \rangle \) if \( F \) is cyclic) is an \( H_1 \)-trivial \( \mathcal{Z}H_1 \)-module.

The factor \( \overline{\mathcal{A}}/\mathcal{C}_A(F) \) is nonzero, for if \( \overline{\mathcal{A}} = \mathcal{C}_A(F) \), then \( [A, F] = [A_1, H_1] \) and hence \( A_1 = [A, F] = [A_1, H_1] \) contrary to assumption. Furthermore, the \( \mathcal{Z}G \)-module \( \overline{\mathcal{A}}/\mathcal{C}_A(F) \) is \( H \)-perfect, since \( A \) is \( H \)-perfect, and \( H \) induces in \( \overline{\mathcal{A}}/\mathcal{C}_A(F) \) a finite group of automorphisms (since it is \( H_1 \)-trivial and \( H/H_1 \) is finite). Thus we can take \( H \) to be the required subgroup and \( \overline{\mathcal{A}}/\mathcal{C}_A(F) \) as the \( H \)-perfect \( \mathcal{Z}G \)-image.

(2) Suppose \( A_1 = [A_1, H_1] \). Suppose that the lemma is true for the pair \( (A_1, H_1) \), i.e., \( H_1 \) has a subgroup \( K_1 \) with \( K_1 \) being normal in \( G \) and \( A_1 \) has a nonzero \( K_1 \)-perfect \( \mathcal{Z}G \)-image \( \hat{A}_1 \) on which \( K_1 \) induces a cyclic or finite group of automorphisms. We show that this implies the result for the pair \( (A, H) \).

Let \( \hat{B}_i = A(x_{i-1}) \), where \( x_i \) (\( 1 \leq i \leq m \)) are elements of the group \( F \) with \( m = 1 \) and \( F = \langle x_1 \rangle \) if \( F \) is infinite or \( m = |F| \) if \( F \) is finite. Then \( A_1 = [A, F] = \hat{B}_1 + \cdots + \hat{B}_m \) and \( \hat{A}_1 = \hat{B}_1 + \cdots + \hat{B}_m \), where the \( \hat{B}_i \) are \( \mathcal{Z}G_1 \)-modules, \( G_1 = C_G(F) \). Let \( k \) be the largest integer such that \( \overline{A}_1 = \hat{B}_{k-1} + \cdots + \hat{B}_m \). Then \( \overline{\hat{A}_1}/(\hat{B}_{k+1} + \cdots + \hat{B}_m) \cong \mathcal{Z}G_1 \hat{B}_k/(\hat{B}_k \cap (\hat{B}_{k+1} + \cdots + \hat{B}_m)) \) is a \( K_1 \)-perfect \( \mathcal{Z}G_1 \)-module, since \( \hat{A}_1 \) is \( K_1 \)-perfect, and is nonzero by the choice of \( k \). Thus \( \hat{B}_k \), and hence \( B_k \), has a nonzero \( K_1 \)-perfect \( \mathcal{Z}G \)-image. By the \( \mathcal{Z}G_1 \)-isomorphism \( \mathcal{A}/\mathcal{C}_A(x_k) \equiv A(x_{k-1}) = B_k \), \( A \) has a nonzero \( K_1 \)-perfect \( \mathcal{Z}G_1 \)-image \( A/D \), say.

Since \( D \) is a \( \mathcal{Z}G_1 \)-module and \( |G/G_1| < \infty \), there are only finitely many \( \mathcal{Z}G_1 \)-modules of the form \( Dg \) (\( g \in G \)). Put \( C = [A, K_1] \cap D \). Then \( C \) is a \( \mathcal{Z}G_1 \)-module and \( C_0 = \sum_{g \in G} \langle [A, K_1] \cap Dg \rangle \) is a \( \mathcal{Z}G \)-submodule of \( A \) generated by \( C \).

(a) Suppose \( A = C_0 \). Since \( C_0 \leq [A, K_1] \) we have \( A = [A, K_1] \), i.e., \( A \)
is a $K_1$-perfect $\mathbb{Z}G$-module. By hypothesis, $K_1$ induces a cyclic or finite group of automorphisms on the $\mathbb{Z}G$-image $\hat{A}_1$, $C_{K_1}(\hat{A}_1)$ is normal in $G$ and $K_1/C_{K_1}(\hat{A}_1)$ is cyclic or finite. Elements of $C_{K_1}(\hat{A}_1)$ act trivially on $\hat{A}_1$ and hence on $B_k$ and so on $A/D$. Therefore $[A, C_{K_1}(\hat{A}_1)] \leq D$ and $A/[A, C_{K_1}(\hat{A}_1)]$ is a nonzero $K_1$-perfect $\mathbb{Z}G$-image of $A$ on which $K_1$ induces a cyclic or finite group of automorphisms.

(b) Suppose $A \neq C_0$. Since $A/D$ is a $K_1$-perfect $\mathbb{Z}G_1$-image we have $A = [A, K_1] + D$ and, since $C = [A, K_1] \cap D \leq C_0$, $A/C_0 = ([A, K_1]/C_0) \oplus ((D + C_0)/C_0)$. By considering the $\mathbb{Z}G$-factor-module $A/C_0$, we may assume that $C_0 = 0$ and hence $A = [A, K_1] + D$.

Put $D_0 = \sum_{g \in G} D_g$. Since $|G/G_1| < \infty$ this is a sum of finitely many $\mathbb{Z}G_1$-modules $D_g$. Also $[D_g, K_1] \leq [A, K_1] \cap D_g \leq C_0 = 0$ so that each $D_g$ is $K_1$-trivial. It follows that $D_0$ has a finite series of $\mathbb{Z}G_1$-submodules in which the factors are $K_1$-trivial. Therefore, since $A/D$ is $K_1$-perfect, we must have $D_0 \neq A$. Thus $A/D_0$ is a nonzero $K_1$-perfect $\mathbb{Z}G$-image of $A$. Also $K_1$ induces on $A/D$ and hence on $A/D_0$ a cyclic or finite group of automorphisms. Thus we can take $K_1$ to be the required subgroup and $A/D_0$ the required $\mathbb{Z}G$-image.

In Case (2), we have now shown that if the lemma fails to hold for the pair $(A, H)$ where $A$ is an $H$-perfect $\mathbb{Z}G$-module and $H$ is a normal subgroup of $G$, then there is an $H_1$-perfect proper $\mathbb{Z}G$-submodule $A_1$ with $H_1$ being normal in $G$ and contained in $H$ such that the lemma is false for the pair $(A_1, H_1)$. Hence
A_1 has a proper H^-perfect ZG-submodule A_2 such that the lemma is false for (A_2, H_2). This process leads to an infinite properly descending chain of ZG-submodules A > A_1 > A_2 > ⋯, contrary to the artinian condition, and so the lemma is proved.

Lemma 5.2.2: Let G be a hyper-(cyclic or finite) locally soluble group, B a ZG-module, and A a periodic artinian ZG-submodule of B such that all irreducible ZG-factors of A are infinite. If B/A as an abelian group is finitely generated and is G-trivial, then B = A ⊕ B_1 for some ZG-submodule B_1 of B.

Proof: Since A is an artinian ZG-submodule of B, it is possible to choose a ZG-submodule B_1 such that B = A + B_1 and for each U ≤ B_1 with B = A + U, the intersections A ∩ U and A ∩ B_1 are equal. We prove that A ∩ B_1 = 0.

Suppose A_1 = A ∩ B_1 ≠ 0. Since B/A is finitely generated and B/A ≅ ZG B_1/A_1, we have B_1 = A_1 + < b_1, ⋯, b_n > for some b_1, ⋯, b_n ∈ B_1. Firstly, if Z(G) = 1, since G is a hyper-(cyclic or finite) locally soluble group, G has a nontrivial normal subgroup H, which is either cyclic or abelian and finite. Let G_1 = C_G(H) and consider B as a ZG-module. Since |G/G_1| < ∞, the conditions of the lemma are clearly satisfied by G_1, B and A. So we may assume that Z(G) ≠ 1. Secondly, if C_G(B_1) ≠ 1, then C_G(B_1) < G by the nonzero ZG-submodule A_1 being contained in B_1 and A having no nonzero finite ZG-factors. Let G = G/C_G(B_1), then G is a hyper-(cyclic or finite) locally soluble group. We regard B_1 as a ZG-module, and then, as above, we may assume that Z(G) ≠ 1. Let 1 ≠ g ∈ Z(G), then < g > is a central cyclic subgroup of G and B_1(g−1) = A_1(g−1) + < b_1(g−1), ⋯, b_n(g−1) >. Since B_1/A_1 (≈ ZG B/A) is G-trivial and therefore is G-trivial, we have B_1(g−1)/A_1(g−1) is G-trivial.
and $B_i^{(g-1)} \leq A_i$. Also, since $A_i$ is periodic and has no nonzero finite $\mathbb{Z}G$-factors (therefore has no nonzero finite $\mathbb{Z}\overline{G}$-factors), we have $B_i^{(g-1)} = A_i^{(g-1)}$. Thus, there exist $a_i \in A_i$ such that $b_i^{(g-1)} = a_i^{(g-1)}$ for all $1 \leq i \leq n$, that is, $b_i - a_i \in C_{B_1}(g)$. So $B_1 = A_1 + C_{B_1}(g)$, where $C_{B_1}(g)$ is certainly a proper $\mathbb{Z}\overline{G}$-submodule of $B_1$. Any $\mathbb{Z}\overline{G}$-submodule of $B_1$ is a $\mathbb{Z}G$-submodule and so $C_{B_1}(g)$ is a proper $\mathbb{Z}G$-submodule of $B_1$. But, since $A + C_{B_1}(g) = A + B_1 = B$, then $A \cap C_{B_1}(g) < A \cap B_1$, which is contrary to the choice of $B_1$. So $A \cap B_1 = 0$, the lemma is proved.

**Corollary 5.2.3:** Let $G$, $B$, and $A$ be as in Lemma 5.2.2 with one exception that $B/A$ is just cyclic, then the same is true for $B$, i.e., $B$ has a $\mathbb{Z}G$-submodule $B_1$ such that $B = A \oplus B_1$.

**Proof:** Let $G_1 = C_{G_i}(B/A)$, then $|G/G_1| \leq 2$. Regarding $B$ as a $\mathbb{Z}G_1$-module, then, by Lemma 5.2.2, $B = A \oplus B_1$ for some $\mathbb{Z}G_1$-submodule $B_1$ of $B$. For $g \in G$, if $B_1g \neq B_1$, then $0 \neq B_1g/(B_1 \cap B_1g) \cong \mathbb{Z}G_1(B_1 + B_1g)/B_1 \leq B/B_1 \cong \mathbb{Z}G_1 A$.

That is, $A$ has a cyclic (and hence finite by $A$ being periodic) $\mathbb{Z}G_1$-factor. By Lemma 2.2.6, $A$ has a nonzero finite $\mathbb{Z}G$-factor, a contradiction. So $B_1g = B_1$ for all $g \in G$. That is, $B_1$ is a $\mathbb{Z}G$-submodule of $B$. Thus the result holds.

**Lemma 5.2.4:** Let $E$ be an extension of a periodic abelian subgroup $A$ by a hyper-(cyclic or finite) locally soluble group $G$ such that $A$ is an artinian $\mathbb{Z}G$-module without any nonzero finite $\mathbb{Z}G$-factors. If $N/A$ is normal in $E/A$ and $N \leq C_{E_i}(A)$, then $N = A \times M$, where $M$ is normal in $E$ and is contained in any supplement to $A$ in $E$.

**Proof:** Let $M$ be a normal subgroup of $E$ contained in $N$ and maximal subject to
By considering the factor group $E/M$, we may assume that $M = 1$. Then $E$ satisfies the following property (*): if $S$ is normal in $E$, $S \leq N$, and $S \neq 1$, then $A \cap S \neq 1$. We show that in this situation $A = N$.

Suppose that $A \neq N$. Since the factor group $E/A$ is hyper-(cyclic or finite), there is a nontrivial finite subgroup $K/A \leq N/A$ such that $K$ is normal in $E$ or an infinite cyclic subgroup $L/A \leq N/A$ such that $L$ is normal in $E$.

For $K$, by the hypothesis of the lemma, $K \leq \mathbb{C}_E(A)$ and so $K$ is a finite extension of its central subgroup $A$. Hence $K'$ is finite (Schur, Thm 10.1.4 in [15]). It follows that $A \cap K'$ is finite and by $A$ having no nonzero finite $\mathbb{Z}G$-factors, $A \cap K' = 1$, and so, by (*), $K' = 1$. That is, $K$ is abelian. By Theorem C, $K = A \times K'$, where $K'$ is a finite normal subgroup of $E$. By (*) again, $K' = 1$, that is, $K = A$, contrary to $K/A$ being nontrivial.

For $L$, by the hypothesis of the lemma, $L \leq \mathbb{C}_E(A)$ and so $L$ is a cyclic extension of its central subgroup $A$. Thus $L$ is abelian. By Corollary 5.2.3, $L = A \times L_1$, where $L_1$ is a cyclic normal subgroup of $E$. By (*), $L_1 = 1$, that is, $L = A$, a contradiction.

Thus, we have got $N = A \times M$ with $M$ being normal in $E$.

If $E_1$ is a supplement to $A$ in $E$, then $AE_1 = E$ and so $N = A(N \cap E_1)$ with $N \cap E_1$ being normal in $AE_1 = E$. Since $M$ is hyper-(cyclically or finitely) embedded in $E$, and so is $M(N \cap E_1)/(N \cap E_1)$ in $E/(N \cap E_1)$. But, since $A$ is periodic and has no nontrivial finite $E$-factors (otherwise $A$ would have a nonzero finite $\mathbb{Z}G$-factor, a contradiction), by

$$M(N \cap E_1)/(N \cap E_1) \leq N/(N \cap E_1) = A(N \cap E_1)/(N \cap E_1),$$

we have $M(N \cap E_1) = N \cap E_1$, i.e., $M \leq E_1$. The result is proved.
Comparing with Lemma 5.2.1, we have the following partial dual result:

**Lemma 5.2.5:** Let $E$ be an extension of a periodic abelian group $A$ by a hyper-(cyclic or finite) locally soluble group $G$ and suppose that $A$ is an artinian $\mathbb{Z}G$-module without any nonzero finite $\mathbb{Z}G$-factors. If also $G$ has no nontrivial finite normal subgroups and $E$ has no nontrivial cyclic normal subgroups, then there is a normal subgroup $K$ of $E$ such that $[A \cap K, K] = A \cap K$ and $K/(A \cap K)$ is a nontrivial cyclic group. If, furthermore, $A$ has a complement $L$ in $E$, then $K$ can be chosen so that $K \leq (A \cap K)L$.

**Proof:** By $G$ being hyper-(cyclic or finite) and having no nontrivial finite normal subgroups, we have $E$ contains a normal subgroup $K_0$ such that $A < K_0$ and $K_0/A$ is cyclic. By Lemma 5.2.4 and $E$ having no nontrivial normal cyclic subgroups, we have $C^*_E(A) = A$. Thus, $K_0$ is nonabelian. Let $A_0 = [A, K_0]$. If $A = A_0$, then $K_0$ is the required subgroup. If $A \neq A_0$, consider the $\mathbb{Z}G$-module $K_0/A_0 = \overline{K}_0$. By Corollary 5.2.3, we have $\overline{K}_0 = \overline{A} \oplus K_1$ for some $\mathbb{Z}G$-submodule $\overline{K}_1$, where $\overline{A} = A/A_0$. Let $K_1$ be the preimage of $\overline{K}_1$, then $K_1$ is a normal subgroup of $E$, $K_1 \cap A = A_0$, and $K_1/A_0 \cong K_0/A$. Furthermore, if $A_1 = [A_0, K_1] \neq A_1$, then similarly we can find a subgroup $K_2$ being normal in $E$ such that $K_2 \leq K_1$, $K_2 \cap A = A_1$, and $K_2/A_1 \cong K_0/A$. If $A_2 = [A_1, K_2] \neq A_1$, then there exists a subgroup $K_3$. The chain of submodules $A > A_0 > A_1 > \cdots$ must terminate at $A_n$, say, and there is a normal subgroup $K_{n+1}$ such that $[A_n, K_{n+1}] = A_n$, $A \cap K_{n+1} = A_n$, and $K_{n+1}/A_n \cong K_0/A$. Thus $K_{n+1}$ has the required properties.

Now suppose that $A$ has a complement $L$ in $E$. Then $K_0 \leq E = AL = (A \cap K_0)L$ so that the second part is also proved if $K = K_0$. Since $E = A]L$, we have $K_0/A_0 = \overline{K}_0 = \overline{A} \times (K_0 \cap \overline{L})$. By $\overline{A}$ having no nonzero finite (and hence cyclic) $\mathbb{Z}G$-factors, we clearly have that the direct factor $\overline{A}$ of $\overline{K}_0$ has a unique complement in $\overline{K}_0$. Therefore, it follows that $\overline{K}_0 \cap \overline{L} = \overline{K}_1$ and then $K_1 \leq A_0 L$.  

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Using a similar argument and induction on \( n \), we immediately have
\[
K_{n+1} \leq A_n L = (A \cap K_{n+1}) L.
\]
The result holds.

Now we prove the main result of this part.

**Theorem D:** Let \( G \) be a hyper-(cyclic or finite) locally soluble group and \( A \) a periodic artinian \( \mathbb{Z}G \)-module. If \( A \) has no nonzero finite \( \mathbb{Z}G \)-submodules, then any extension \( E \) of \( A \) by \( G \) splits conjugately over \( A \) and \( A \) has no nonzero finite \( \mathbb{Z}G \)-factors. Also, every complement to \( A \) in \( E \) is self-normalizing.

**Proof:** By Theorem C, \( A \) has no nonzero finite \( \mathbb{Z}G \)-factors.

Existence of self-normalizing complements: Since \( A \) is artinian, \( E \) has a subgroup \( E_1 \) minimal with respect to \( E = AE_1 \). We prove that \( A \cap E_1 = 1 \).

Suppose that \( A_1 = A \cap E_1 \neq 1 \). The group \( E_1 \) and its subgroup \( A_1 \), considered as a \( \mathbb{Z}G \)-module, satisfy the conditions of the theorem (since \( E/A \cong E_1/A_1 \)). We may therefore assume that \( A = A_1 \) and \( E = E_1 \), then \( A \) has no proper supplement in \( E \) and \( A \neq 1 \).

Since \( A \) is periodic, by passing from \( E \) to its factor group \( E/A \), we may assume that \( A \) is a \( p \)-group for some prime \( p \). By \( A \) having no nonzero finite \( \mathbb{Z}G \)-factors, we have \( A = [A, G] \). So, by Lemma 5.2.1, there is a nonzero \( \mathbb{Z}G \)-image \( A/B \) and a nontrivial normal subgroup \( K/A \) of \( E/A \) such that \( A/B = [A/B, K] \) and \( K/C_K(A/B) \) is cyclic or finite. Passing to the factor group \( E/B \), we can take \( B = 1 \) so that \( A = [A, K] \) and \( K/C_K(A) \) is cyclic or finite.

(1) Suppose \( K/C_K(A) \) is cyclic. By Lemma 5.2.4, \( C_E(A) = A \times M \), where \( M \) is a normal subgroup of \( E \). Passing to \( E/M \), we may assume that \( C_E(A) = A \). So \( C_K(A) = K \cap C_E(A) = A \), i.e., \( K/A \) is cyclic. Passing from \( E \) to \( E/C \), where \( C \) is
the maximal hypercyclically embedded normal subgroup of $E$, we may assume that $E$ has no nontrivial cyclic normal subgroups. Since $A = [A, K]$ and $K/A$ is a nontrivial cyclic normal subgroup of $E/A$, there is an element $x$ such that $K = A < x >$. Then, by Lemma 1.2.16, we have $E = AN_E(<x>)$. By $A$ having no proper supplement in $E$, we have $E = N_E(<x>)$, i.e., $<x>$ is a cyclic normal subgroup of $E$, a contradiction.

(2) Suppose $K/C_K(A)$ is finite. In this case, we need only consider the situation in which $K/C_K(A)$ is a $q$-group. For, let

$$K = K_0 > K_1 > \cdots > K_s = C_K(A)$$

be a series of normal subgroups of $E$, the factors of which are $p_i$-groups. Let $i$ be the largest such that $A = [A, K_i]$. Then $[A, K_{i+1}] < A$, $[A, K_{i+1}]$ is normal in $E$, and we can pass to the factor group $E/[A, K_{i+1}]$. So we may assume that $[A, K_{i+1}] = 1$, $K_{i+1} \leq C_{K_i}(A)$, and $K_i/C_{K_i}(A)$ is a $q$-group ($q = p_i$). We can replace $K$ by $K_i$ and so assume that $K/C_K(A)$ is a $q$-group.

By Lemma 5.2.4, we may further assume that $C_K(A) = A$, and so $C_K(A) = A$. Thus $K/A$ is a finite $q$-group for some prime $q$.

Consequently, we need only consider the following situation: $E$ is an extension of the $p$-group $A$, where $E/A$ has a nontrivial finite normal $q$-subgroup $K/A$ such that $A = [A, K]$.

Let $q = p$. By Lemma 1.2.20, $A$ has a proper supplement in $E$, a contradiction;

Let $q \neq p$. If $Q$ is a Sylow $q$-subgroup of $K$, then $K = AQ$ is abelian—by-finite and so is locally finite, which implies all Sylow $q$-subgroups of $K$ are conjugate in $K$ [15]. Therefore, by the Frattini argument, $E = AN_E(Q)$. Also $A \cap N_E(Q) < A$ for otherwise $[A, K] = [A, Q] = 1$. Hence
$N_E(Q)$ is a proper supplement to $A$ in $E$, a contradiction again.

So, we have proved the existence of the complements.

If, finally, a complement $K$ is properly contained in its normalizer $N_E(K)$, then consider $K < K < \leq N_E(K)$. Since $K < x > \cap A$ is normal in $AK = E$ and $K < x > \cap A \cong ((K < x > \cap A)K)/K \leq K < x > /K$, we see that $K < x > \cap A$ is a cyclic (and hence finite) $ZG$-submodule of $A$, contrary to $A$ having no nonzero finite $ZG$-factors. So all the complements to $A$ in $E$ are self-normalizing.

Conjugacy of complements: Let $S_1$ and $S_2$ be two complements to $A$ in $E$ and take an $E$-invariant subgroup $A_0$ of $A$ such that $S_1$ and $S_2$ are conjugate modulo $A_0$ but are not conjugate modulo any proper $E$-invariant subgroup of $A_0$. If $A_0 \neq 1$, we may clearly assume that $A = A_0$. By $A$ being artinian, $E$ has a subgroup $E_1$ minimal with respect to $E = AE_1$ and $E_1$ is generated by $S_1$ and $S_2$ — conjugates of $S_1$ and $S_2$. We prove that $A \cap E_1 = 1$ so that $E_1 = S_1 = S_2$, a contradiction.

Suppose that $A_1 = A \cap E_1 \neq 1$ and note that $E_1 = A_1 S_1$. Apply Lemma 5.2.1 to $E_1$, we obtain a series of normal subgroups

$$B < A_1 \leq C_K(A_1/B) \leq K \leq E_1$$

with $A_1/B = [A_1/B, K]$ and $K/C_K(A_1/B)$ is cyclic or finite. Passing to the factor group $E_1/B$, we can take $B = 1$ so that $A_1 = [A_1, K]$ and $K/C_K(A_1)$ is cyclic or finite.

(1) Suppose $K/C_K(A_1)$ is cyclic. In this case, we pass from $E_1$ to the factor group $E_1 = E_1/C$, where $C$ is the maximal hypercyclically embedded normal subgroup of $E_1$. Certainly, by $A_1$ having no nonzero finite $ZG$-factors, we have $K$ is not contained in $C$ and, since a complement to $A_1$ in $E_1$ is a maximal hyper-(cyclic or finite) subgroup of $E_1$, $C$ is contained in any
complement to $A_1$ in $E_1$. Also, by Lemma 5.2.4, we may assume that $C_{K}(A_1) = A_1$.

Hence, we have $E_1 = A_1 S_{1}$ and $K/A_1$ is cyclic. It is clear that $K = A_1 K_i$, where $K_i$ is the cyclic subgroup $K_i = K \cap S_{1}$. By $A_1 = [A_1, K]$, we have $K_i$ are conjugate in $K$, i.e., $K_i = K_{2}^{a}$ for some $a \in A_1$. By Lemma 1.2.16, $E_1 = A_1 N_{E_1}(K_i)$. By $E_1 = E_1/C$, we have $E_1 > N_{E_1}(K_i)$.

Let $E_2$ be the preimage of $N_{E_1}(K_i)$ in $E_1$; since $C \leq S_{1}$, $S_{1} \leq N_{E_1}(K_i)$, and $E_1 = A_1 N_{E_1}(K_i)$, we have $E_2 < E_1$ and $E_1 = A_1 E_{2}$. Thus, $E = A E_1 = A E_2$. But $K_i = K_{2}$ so $S_1$ and $S_2$ are both contained in $E_2$ and hence the subgroup of $E_1$ generated by $S_{1}^{g_1}$ and $S_{2}^{g_2}$ should be the group $E_1$ by the minimality of $E_1$. That is, $E_2 = E_1$ contrary to $E_2 < E_1$.

(2) Suppose $K/C_{K}(A_1)$ is finite. As before, we may assume that $K/C_{K}(A_1)$ is a $q$-group and, further, we may assume that $C_{K}(A_1) = A_1$, i.e., $K/A_1$ is a finite $q$-group.

If $p = q$, then by Lemma 1.2.20, $S_{1}^{g_1}$ and $S_{2}^{g_2}$ are conjugate modulo some proper $E_1$-invariant subgroup of $A_1$ and hence $S_{1}$ and $S_{2}$ are conjugate modulo some proper $E$-invariant subgroup of $A$, a contradiction;

If $p \neq q$, then by Lemma 1.2.4, $A_1 = C_{A_1}(K) \oplus [A_1, K]$, and it follows from $A_1 = [A_1, K]$ that $C_{A_1}(K) = 1$. The intersection $K \cap S_{1}^{g_1}$ is a Sylow $q$-subgroup of $K$ and $K \cap S_{1}^{g_1}$ is normal in $S_{1}^{g_1}$ so that $S_{1}^{g_1} \leq N_{E_1}(K \cap S_{1}^{g_1})$. But $N_{A_1}(K \cap S_{1}^{g_1}) = C_{A_1}(K \cap S_{1}^{g_1}) = C_{A_1}(K) = 1$ so that $S_{1}^{g_1} = N_{E_1}(K \cap S_{1}^{g_1})$.

Similarly, $S_{2}^{g_2} = N_{E_1}(K \cap S_{2}^{g_2})$ and so $S_{1}^{g_1}$ and $S_{2}^{g_2}$ are conjugate in $E_1$ (since
$K \cap S_1$ and $K \cap S_2$ are conjugate in $K$ and so in $E_1$. Thus, $S_1$ and $S_2$ are conjugate in $E$, a contradiction.

Theorem D is proved.

Noetherian Case:

Similar with the artinian case, we have the following lemmas:

Lemma 5.2.6: Let $G$ be a hyper-(cyclic or finite) group, $B$ a $\mathbb{Z}G$-module, and $A$ a noetherian $\mathbb{Z}G$-submodule of $B$ such that all irreducible $\mathbb{Z}G$-factors of $A$ are infinite. If $B/A$ is torsion-free and $G$-trivial, then $B = A \oplus B_1$ for some $\mathbb{Z}G$-submodule $B_1$ of $B$.

Proof: Suppose that $A$ has no complements in $B$. Since $A$ is noetherian, we may assume that for each nonzero $\mathbb{Z}G$-submodule $C$ of $A$, $A/C$ has a complement in $B/C$.

In $B$, we choose a $\mathbb{Z}G$-submodule $M$ maximal with respect to $A \cap M = 0$. We show that if $S$ is any $\mathbb{Z}G$-submodule such that $B = A + S$ then $M \leq S$.

Since $B/A \cong (A \oplus M)/A \cong \mathbb{Z}G M$, we have $M$ is a $G$-trivial $\mathbb{Z}G$-module and hence all of its irreducible $\mathbb{Z}G$-factors are finite. Also

$$A/(A \cap S) \cong \mathbb{Z}G (A + S)/S = B/S \geq (M + S)/S \cong \mathbb{Z}G M/(M \cap S),$$

by $A$ being noetherian and having no nonzero finite $\mathbb{Z}G$-factors, we must have $M = M \cap S$, i.e., $M \leq S$.

Consider the factor-module $B/M$. Every nonzero $\mathbb{Z}G$-submodule of $B/M$ has nonzero intersection with $(A \oplus M)/M$. In particular, $(A \oplus M)/M$ has no complements in $B/M$. If $V/M$ is a nonzero $\mathbb{Z}G$-submodule of $(A \oplus M)/M$ then $V = C \oplus M$, where $C = A \cap V$ is nonzero and so $B/C = A/C \oplus S_1/C$ for some
ZG-submodule $S_i$ of $B$. As above, $M \leq S_i$ and so $(A \oplus M) \cap S_i = (A \cap S_i) \oplus M = C \oplus M = V$. Thus $S_i/V$ is a complement to $(A \oplus M)/V$ in $B/V$.

By passing to the factor-module $B/M$ we may assume that $M = 1$ so that:

(a) $A$ has no complements in $B$ but for any nonzero $ZG$-submodule $C$ of $A$, $A/C$ has a complement in $B/C$; (b) if $N$ is a nonzero $ZG$-submodule of $B$ then $A \cap N \neq 0$.

We may assume that $A$ is torsion-free. For otherwise, we may let $A[p]$ be the nonzero $ZG$-submodule generated by all the elements of order $p$, where $p$ is a prime. By (a), $B/A[p] = A/A[p] \oplus B_1/A[p]$. Since $B_1/A[p] \cong ZG B/A$ is torsion-free, $pB_1 \neq 0$, then, by (b), $0 \neq A \cap pB_1 \leq A[p] \cap pB_1$. That is, $B_1$ has elements of order $p^2$, contrary to $B_1/A[p]$ being torsion-free. So $A$ is torsion-free and then $B$ is torsion-free. Since $A$ has no nonzero finite $ZG$-factors, we have $C_A(G) = 0$. By Proposition 5.1.2, $G$ has a normal subgroup $K$ and $A$ has a nonzero $ZG$-submodule $A_1$ such that $C_{A_1}(K) = 0$ and $K/C_{A_1}(A_1)$ is cyclic or finite. By (a), $B/A_1 = A/A_1 \oplus B_1/A_1$. Consider the $ZG$-module $B_1$ and we prove that $B_1 = A_1 \oplus B_2$ for some $ZG$-submodule $B_2$ (and hence we get $B = A \oplus B_2$ as required).

Suppose $B_1 \neq A_1 \oplus B_2$ for any $ZG$-submodule $B_2$ and suppose that $G$ acts faithfully on $B_1$, i.e., $C_G(B_1) = 1$. It is clear that we still have that $K$ is normal in $G$, $C_{A_1}(K) = 0$, and $K/C_{A_1}(A_1)$ is cyclic or finite. If $C_K(A_1) \neq 1$, then, since $C_K(A_1) = K \cap C_G(A_1)$ is a normal subgroup of $G$, $C_K(A_1)$ contains a nontrivial cyclic or finite subgroup $F$ being normal in $G$. Let $F = \langle f_1, \ldots, f_n \rangle$ and let $G_1 = C_G(F)$, then $|G/G_1| < \infty$. By Lemma 2.2.6, the irreducible $ZG_1$-factors of $A_1$ are infinite. Since $B_1/A_1$ is $G$-trivial, it is also $G_1$-trivial. By $B_1/C_{B_1}(f_i) \cong ZG_1 B_1(f_i-1) \leq A_1$ and $A_1 \leq C_{B_1}(f_i)$, we must have
$B_i(f_i-1) = 0$, for all $i$. That is, $1 \neq F \leq C_G(B_i)$, contrary to $G$ acting faithfully on $B_i$. So $C_K(A_i) = 1$ and so $K$ is a nontrivial cyclic or finite normal subgroup of $G$. Let $K = \langle k_i, \cdots, k_t \rangle$. Being similar with the above, we have $B_i/C_{B_i}(k_i) \equiv_{ZG_i} B_i(k_i-1) \leq A_i$ for all $i$, where $G_i = C_K(k_i)$. Thus $B_i/(A_i + C_{B_i}(k_i))$ must be zero for all $i$. That is, $B_i = A_i + C_{B_i}(k_i)$ for any $i$.

Let $C_m = C_{B_1}(\langle k_{i_1}, \cdots, k_m \rangle)$, $m = 1, \cdots, t$. Then we have $B_1 = A_1 + C_1$. Suppose that $B_1 = A_1 + C_2$; we prove that $B_1 = A_1 + C_{m+1}$.

Consider the $ZG_2$-modules $C_m/C_{m+1} = C_m/C_m(k_{m+1}) \equiv_{ZG_2} C_m(k_{m+1})$. Since $B_1/A_1$ is $G$-trivial, $C_{m+1}(k_{m+1}) \leq A_1$ and so $C_m(k_{m+1})$ has no nonzero finite $ZG_2$-factors; hence the irreducible $ZG_2$-factors of $C_m/C_{m+1}$ are all infinite. But $C_{m+1}/(A_1 \cap C_{m+1}) \equiv_{ZG_2} (C_{m+1}/A_1)/(C_{m+1}/A_1)$ a factor module of the $G_2$-trivial $ZG_2$-module $B_1/A_1$. Hence $A_1 + C_m = A_1 + C_{m+1}$. That is, $B_1 = A_1 + C_{m+1}$. Therefore $B_1 = A_1 + C_m$ for all $m$. Put $m = n$, then $C_n = C_{B_1}(K)$ and $B_1 = A_1 + C_{B_1}(K)$, which implies that $C_{B_1}(K) \neq 0$. Hence, by (b) and $B/A_1 = A/A_1 \oplus B_1/A_1$, we have $C_{A_1}(K) = A_1 \cap C_{B_1}(K) = A \cap C_{B_1}(K) = 0$, a contradiction. So $B_1 = A_1 \oplus B_2$ for some $ZG$-submodule $B_2$ and hence the lemma is proved.

**Corollary 5.2.7:** Let $G$ be a hyper-(cyclic or finite) group, $B$ a $ZG$-module, and $A$ a noetherian $ZG$-submodule of $B$ such that all irreducible $ZG$-factors of $A$ are infinite. If $B/A$ is an infinite cyclic group, then $B = A \oplus B_1$ for some $ZG$-submodule $B_1$ of $B$.

**Proof:** Let $G_1 = C_G(B/A)$, then $|G/G_1| \leq 2$ and $B/A$ is torsion-free and 122
G-trivial. By Lemma 5.2.6, \( B = A \oplus B_1 \) for some \( G \)-trivial \( ZG \)-submodule \( B_1 \) of \( B \). For \( g \in G \), if \( B_1 g \neq B_1 \), then \( B_1 g \) is \( G \)-trivial and

\[
0 \neq B_1 g / (B_1 \cap B_1 g) \cong ZG_1 (B_1 + B_1 g) / B_1 \leq B / B_1 \cong ZG_1 A.
\]

That is, \( A \) has a nonzero \( G \)-trivial \( ZG \)-factor and then a nonzero finite irreducible \( ZG \)-factor, which will imply that \( A \) has a nonzero finite irreducible \( ZG \)-factor, a contradiction. So \( B_1 g = B_1 \) for all \( g \in G \). That is, \( B_1 \) is a \( ZG \)-submodule of \( B \). The result is proved.

Lemma 5.2.8: Let \( E \) be an extension of the abelian group \( A \) by a hyper- (cyclic or finite) group \( G \) such that \( A \) is a noetherian \( ZG \)-module and all irreducible \( ZG \)-factors of \( A \) are infinite. Then if \( C / A \) is a normal subgroup of \( E / A \) and \( C \leq C_E(A) \), then \( C = A \times N \), where \( N \) is a normal subgroup of \( E \) and is contained in every supplement to \( A \) in \( E \).

Proof: Let \( N \) be a normal subgroup of \( E \) contained in \( C \) and maximal subject to \( N \cap A = 1 \). By considering the factor group \( E / N \) we may suppose that \( N = 1 \). Then \( E \) satisfies the following condition: if \( S \) is normal in \( E \), \( S \leq C \), and \( S \neq 1 \), then \( S \cap A \neq 1 \). We show that this implies that \( A = C \).

Suppose that \( A \neq C \). Since \( E / A \) is hyper- (cyclic or finite), there is a nontrivial finite subgroup \( K / A \leq C / A \) such that \( K \) is normal in \( E \) or an infinite cyclic subgroup \( L / A \leq C / A \) such that \( L \) is normal in \( E \).

For \( K \), by the hypothesis of the lemma, \( K \leq C_E(A) \) and so \( K \) is a finite extension of its central subgroup \( A \). Hence \( K' \) is finite. It follows that \( A \cap K' \) is finite and so \( A \cap K' = 1 \) by \( A \) having no nonzero finite \( ZG \)-factors. By the condition above, we have \( K' = 1 \) and so \( K \) is abelian. Apply Proposition 5.1.3 to the \( Z(E/K) \)-module \( K \) and its submodule \( A \), then \( A = A \times K_1 \) for some normal
subgroup $K_1$ of $E$, contrary to the condition above.

For $L$, by the hypothesis of the lemma, $L \leq C_E(A)$ and so $L$ is a cyclic extension of its central subgroup $A$. Thus $L$ is abelian. By Corollary 5.2.7, $L = A \times L_1$ for some normal subgroup $L_1$ of $E$, contrary to the condition above.

Thus we have proved that $C = A \times N$, where $N$ is normal in $E$.

Now let $E_1$ be a supplement to $A$ in $E$ so that $E = AE_1$, $C = A(C \cap E_1)$ and $C \cap E_1$ is normal in $AE_1 = E$. We have

$$N(C \cap E_1)/(C \cap E_1) \leq C/(C \cap E_1) = A(C \cap E_1)/(C \cap E_1).$$

Since $N$ is hyper-(cyclically or finitely) embedded in $E$ and the irreducible $ZG$-factors of $A$ are all infinite, we must have $N(C \cap E_1)/(C \cap E_1) = 1$, i.e., $N(C \cap E_1) = C = E_1$. Hence $N \leq E_1$ as required.

Now, we prove the last main result of this part.

**Theorem E:** Let $G$ be a hyper-(cyclic or finite) locally soluble group and $A$ a noetherian $ZG$-module. If $A$ has no nonzero finite $ZG$-images, then the extension $E$ of $A$ by $G$ splits conjugately over $A$ and $A$ has no nonzero finite $ZG$-factors.

**Proof:** By Corollary 5.1.6, $A$ has no nonzero finite $ZG$-factors.

Suppose the theorem is false, then using the fact that $A$ is a noetherian $ZG$-module we may assume that: $A$ has conjugate complements in $E$ modulo any nontrivial $E$-invariant subgroup of $A$.

Since $A$ has no nonzero finite $ZG$-factors, $C_A(E) = 1$. By Proposition 5.1.2, $E/A$ has a normal subgroup $K/A$ and $A$ has a nontrivial $E$-invariant subgroup $A_0$ such that $C_{A_0}(K) = 1$ and $K/C_K(A_0)$ is cyclic or finite.

(1) If $K/C_K(A_0)$ is finite, then we may choose $K$ and $A_0$ such that $K/C_K(A_0)$ is minimal and so $K/C_K(A_0)$ is a chief factor of $E$. (For if $L$ is
normal in $E$ and $C_k(A_0) < L < K$ then if $C_{A_0}(L) = 1$ we have $L$, $A_0$ contrary to minimality of $|K/C_k(A_0)|$ and if $C_{A_0}(L) \neq 1$ then $K$, $C_{A_0}(L)$ is contrary to minimality of $|K/C_k(A_0)|$. Hence $K/C_k(A_0)$ has order $p^k$ for some prime $p$ and integer $k \geq 1$. From $C_{A_0}(K) = 1$ it follows that $A_0[p] = 1$ and so $A_0^p \neq 1$.

By the assumption on $A$, we have $E$ splits conjugately over $A$ modulo $A_0^p$.

Let $E_1$ be a complement to $A$ in $E$ modulo $A_0^p$: $E = AE_1$, $A \cap E_1 = A_0^p$; put $E_0 = A_0E_1$, $K_0 = K \cap E_0$, and $C_0 = C_{K_0}(A_0)$. By Lemma 5.2.8, $C_0 = A_0 \times N$, where $N$ is normal in $E_0$ and is contained in $E_1$. Consider the factor group $E_0 = E_0/N$ and the subgroups $K_0$, $A_0$. Since $K_0/A_0 = K_0/C_0 \cong K_0/C_0 \equiv K/C_k(A_0)$, we have $|K_0/A_0| = p^k$. Corresponding to $C_{A_0}(K) = 1$ we have $C_{A_0}(K_0) = 1$ and also $A_0 \cap E_1 = A_0^p$. It follows, by applying Lemma 1.2.21 to $E_0$ and its subgroups $K_0$, $A_0$, that $E_0$ splits over $A_0$: $E_0 = A_0E_2$, $A_0 \cap E_2 = 1$. The complete preimage $E_2$ of $E_2$ in $E_0$ gives $E_0 = A_0E_2$ and $A_0 \cap E_2 = 1$. So that $E_2$ is a complement to $A$ in $E$. Let $S_1$, $S_2$ be any two complements to $A$ in $E$. Then, since $E$ splits conjugately over $A$ modulo $A_0^p$, we have $S_1$ and $S_2$ are conjugate modulo $A_0^p$ and we may assume that $A_0^p S_1 = A_0^p S_2$. Put $E_0 = A_0 S_1 = A_0 S_2$, $K_0 = K \cap E_0$, and $C_0 = C_{K_0}(A_0)$. By Lemma 5.2.8, $C_0 = A_0 \times N$, where $N$ is normal in $E_0$ and is contained in every supplement to $A_0$ in $E_0$: in particular, $N \leq S_1 \cap S_2$. Consider the factor group $E_0 = E_0/N$ and its
subgroups $\overline{K}_0$, $\overline{A}_0$. Since $\overline{K}_0/\overline{A}_0 \cong K/C_K(A_0)$, so $\overline{K}_0/\overline{A}_0$ is a group of order $p^k$, and also $C_{\overline{A}_0}(\overline{K}_0) = 1$ by $C_{\overline{A}_0}(K) = 1$. From $A_0^k S_1 = A_0^k S_2$ it follows that $S_1$ and $S_2$ are complements to $\overline{A}_0$ in $E_0$ which coincide modulo $A_0^p$. Apply Lemma 1.2.21 to the group $E_0$ and its subgroups $\overline{K}_0$, $\overline{A}_0$, we have the conjugacy of the complements: $\overline{S}_1^a = \overline{S}_2^a$, $a \in A_0$. Since $\overline{S}_1 = S_1/N$, $\overline{S}_2 = S_2/N$, and $N$ is normal in $E_0$ it follows that $S_1^a = S_2^a$, i.e., $E$ splits conjugately over $A$, a contradiction.

(2) Now we may suppose that $K/C_K(A_0)$ is cyclic.

In this case, we let $A_1 = [A_0, K] \leq A_0$, then, by $C_{\overline{A}_0}(K) = 1$, we have $A_1 \neq 1$. Thus $E$ splits conjugately over $A$ modulo $A_1$, i.e., $E = A E_1$, $A \cap E_1 = A_1$.

Let $K_1 = K \cap E_1$ and $C_1 = C_{K_1}(A_0)$. It is clear that $A_1 \leq C_1 \leq C_{K_1}(A_1)$ $\leq C_{E_1}(A_1)$. By Lemma 5.2.8, $C_1 = A_1 \times N$ for some normal subgroup $N$ of $E_1$.

Since $K_1/C_1 \cong K/C_K(A_0)$, we have $K_1 = C_1 < x >$ for some $x \in K_1$. Let $M = N < x >$, then $K_1 = C_1 < x > = A_1 M$. Since

$$[A_1 \cap M, K] = [A_1 \cap M, C_K(A_0) < x >]$$

$$= [A_1 \cap M, < x >] = [A_1 \cap M, x]$$

$$\leq [A_1, x] \cap [M, x]$$

$$\leq A_1 \cap N = 1,$$

we have $A_1 \cap M \leq C_{\overline{A}_0}(K) = 1$. Thus $K_1 = A_1 M$, i.e., $M$ is a complement to $A_1$ in $K_1$.

Suppose that $M_0$ is also a complement to $A_1$ in $K_1$ with $N \leq M_0$; we show
that $M$ and $M_0$ are conjugate by an element of $A_0$. We can write $x = a_1 x_0$ with $a_1 \in A_1$ and $x_0 \in M_0$. Since

$$A_1 = [A_0', K] = [A_0', C_{K_1}(A_0) <x>]$$

$$= [A_0', <x>] = [A_0', x^{-1}],$$

so $a_1 = [a_0^{-1}, x^{-1}]$ for some $a_0 \in A_0$, and therefore

$$x = a_1 x_0 = [a_0^{-1}, x^{-1}] x_0 = a_0 (a_0^{-1})^{-1} x_0$$

$$= (a_0^{-1}) x_0 = x(a_0^{-1}) x_0,$$

i.e., $x_0 = x^0$. Since $N \leq M_0$ and $N \leq C_1 = C_{K_1}(A_0)$, we have

$$M^0 = (N <x>)^0 = N <x^0> = N <x_0> \leq M_0.$$ 

As $C_K(A_0) = A C_{K_1}(A_0)$ and $K = K_1 C_{K_1}(A_0)$, so

$$AM_0 = A (A_1 M_0) = AK_1 = A C_{K_1}(A_0) K_1 = C_K(A_0) K_1$$

$$= K = K^0 = (AM)^0 = AM^0,$$

also $A \cap M_0 = A_1 \cap M_0 = 1$ and $A \cap M = 1$ implies that $A \cap M^0 = 1$. Thus $M_0 = M^0$.

We now prove that $A$ has conjugate complements in $E$ and that the complements are of the form $L = N_{E_0}(M)$, where $E_0 = A_0 E_1$ and $M$ is, as above, a complement to $A_1$ in $K_1$ containing $N$.

If $g \in E_1$, then since $N$ and $K_1$ are both normal in $E_1$ and the subgroup $M^g$ is a complement to $A_1$ in $K_1$ containing $N$, thus $M^g = M^0$ for some $a_0 \in A_0$ and so $g a_0^{-1} \in N_{E_0}(M) = L$, hence $E = AE_1 = AL$. We show that $L$ is a
complement to $A$ in $E$. That is, we need to prove that $A \cap L = 1$.

Since $L \leq E_0 = A_0E_1$ and $A \cap E_1 = A_1$, so

$$A \cap L = A \cap (E_0 \cap L) = (A \cap E_0) \cap L = (A \cap A_0E_1) \cap L$$

$$= A_0(A \cap E_1) \cap L = A_0A_1 \cap L = A \cap L;$$

also $A_0$ is normal in $E$ and $L = N_{E_0}(M)$, hence $[A_0 \cap L, M] \leq A_0 \cap M$.

Since $A_0 \cap M = A_0 \cap (E_1 \cap M) = (A_0 \cap E_1) \cap M = A_1 \cap M = 1$, so $A \cap L \leq C_{A_0}(M)$. Therefore, by $K = AM$ and $C_{A_0}(K) = 1$, we have $A \cap L \leq C_{A_0}(M) = C_{A_0}(K) = 1$.

That is, $A \cap L = 1$ and so $L$ is a complement to $A$ in $E$.

Now let $S$ be any complement to $A$ in $E$. Thus $S$ and $L$ are conjugate modulo $A_1$ and we may assume that $A_1L = A_1S$. Therefore, we have

$$E_0 = E_0 \cap E = E_0 \cap A_1L = (E_0 \cap A)L = (A_0E_1 \cap A)L$$

$$= A_0L = A_0A_1L = A_0A_1S = A_0S.$$

Since $K_1 = A_1M \leq A_1L = A_1S$, so $K_1 = A_1M_1$, $A_1 \cap M_1 = 1$, where $M_1 = K_1 \cap S$; thus $M$ and $M_1$ are complements to $A_1$ in $K_1$. We show that $N \leq M_1$. By $K_1 \leq A_1S$ and $C_{K_1}(A_0) \leq K_1$, we have $C_1 = C_1 \cap A_1S = A_1(C_1 \cap S)$, thus $C_1 = A_1 \times N_1$, where $N_1 = C_1 \cap S \leq M_1$ and $N_1$ is normal in $A_0S = E_0$ since $C_1 = C_{K_1}(A_0)$ is normal in $A_0E_1 = E_0$. In particular, $N_1$ is normal in $E_1 \leq E_0$ and, since $E_1/A_1$ is hyper-(cyclic or finite), $N_1$ is hyper-(cyclically or finitely) embedded in $E_1$. Consider the product $NN_1$, if $NN_1 \neq N_1$ then, by $C_1 = A_1 \times N_1 \times N_1 \neq N_1 \neq N_1$ and so $A_1$ contains a nontrivial cyclic or finite
subgroup normal in $E$. By $A \leq A$ and $E/A \cong E/A \cong G$, we have $A$ has a nonzero cyclic or finite $ZG$-submodule and hence contains a nonzero finite $ZG$-factor, a contradiction. Thus $NN_1 = N_1$, $N \leq N_1$ and so $N \leq M_1$.

This shows that $M$ and $M_1$ are conjugate by an element $a_0 \in A_0$, i.e.,

$M^{a_0} = M_1$, and hence $L^{a_0} = N_{E_0}(M)^{a_0} = N_{E_0}(M_1)^{a_0} = N_{E_0}(M_1)$. From $K_1 = A_1 M$ and $M$ is normal in $L$ it follows that $K_1$ is normal in $A_1 L$. Therefore, by $A_1 L = A_1 S$, we have $K_1$ is normal in $A_1 S$, and so $M_1 = K_1 \cap S$ is normal in $S$ and $S \leq N_{E_0}(M_1)$. By $L^{a_0} = N_{E_0}(M_1)$, we have $S \leq L^{a_0}$ and so

$L^{a_0} = AS \cap L^{a_0} = (A \cap L^{a_0})S = S$.

That is, $S$ and $L$ are conjugate in $E$, i.e., $E$ splits conjugately over $A$, a contradiction again.

Thus, we have finished the proof of the theorem.
After Theorem A is proved, the similar results about the modules over some special groups are expected. However, these expected results are not true in most cases, which can be seen from the examples given in the following §6.1. §6.2 contains questions arising from our work which are still open.

§6.1 EXAMPLES OF SPECIAL MODULES

a There exists a torsion-free irreducible $\mathbb{Z} G$-module $A$ over a hypercyclic group $G$.

As P. Hall has shown: there exists a 3-generator torsion-free soluble group $E$ with derived length 3 and having a minimal normal subgroup $A$ isomorphic with the direct product of a countable infinity of copies of the additive group of rational numbers $[4]$. That is, in the category of $\mathbb{Z} G$-modules, $A$ is a torsion-free irreducible $\mathbb{Z}(E/A)$-module over the soluble group $E/A$. From the example given by him, we can have a torsion-free irreducible $\mathbb{Z} G$-module $A$ over a hypercyclic group $G$. In order to get such a module, we recall firstly P. Hall's example.

Let $V$ be a vector space of dimension $\infty$ over the field of rational numbers $\mathbb{Q}$ and let $\{v_i; i = 0, \pm 1, \pm 2, \cdots\}$ be a basis for $V$. For each integer $i$ we select a prime number $p_i$ in such a way that $p_i \neq p_j$ if $i \neq j$, and every prime occurs among the $p_i$.

Let $\xi$ and $\eta$ be the linear transformations of $V$ defined by

$$v_i \xi = v_{i+1} \quad \text{and} \quad v_i \eta = p_i v_i, \quad (i = 0, \pm 1, \pm 2, \cdots).$$
Let $H = \langle \xi, \eta \rangle$, and writing $\eta^j = \eta_j$, then
\[ v_i \eta_j = p_{i,j} v_i. \]

Thus the $\eta_j$ commute with each other and $\eta^H$ is a normal abelian subgroup of $H$. Therefore $H$ is metabelian, and indeed, $H$ is isomorphic with the standard wreath product of two infinite cyclic groups ($H$ has $\eta^H$ as its base subgroup).

It was shown that $V$ contains no nonzero proper $H$-admissible additive subgroups. Let $A$ be the additive group of $V$ and let $E = A \rtimes H$, the semidirect product of $A$ by $H$, then $E$ is a 3-generator torsion-free soluble group with derived length 3 and has $A$ as its minimal normal subgroup. View $A$ as a $\mathbb{Z}H$-module, then $A$ is a torsion-free irreducible $\mathbb{Z}H$-module.

Now let $G = \eta^H$ and let $A$ be the additive group of the 1-dimensional vector subspace, say spanned by $v_0$ of $V$. Then $G$ is a torsion-free hypercyclic abelian group and $A$ is a torsion-free $\mathbb{Z}G$-module. Let $B$ be a nonzero $\mathbb{Z}G$-submodule of $A$ and let $0 \neq w \in B$, then $w = rv_0$ for some nonzero rational number $r \in \mathbb{Q}$. For any prime $p$ and any integer $n$, since $p = p_i$ for some $i$ and $v_0 \eta^{-i} = p_{0,(-i)} v_0 = p_{i} v_0 = p^n v_0$, so
\[ tp^n v_0 = tp^n v_0 = (rv_0) \eta^{-i} \in B. \]

Thus, it follows that $B$ contains each rational multiple of $v_0$ and then $B = A$. That is, $A$ is irreducible. (If we take $A$ to be the additive group of all rationals and $G$ the multiplicative group of all positive rationals and assume that $G$ acts on $A$ by the natural multiplication, then we get the same required example by taking $\eta_j = p^{-j} \in G$, where $p_i$ runs over all primes when $i$ runs over all integers.)

\[ b \text{ There is a noetherian } \mathbb{Z}G\text{-module } A \text{ over a hypercyclic group } G \text{ such that} \]
A has no C-decomposition.

A C-decomposition of a \(ZG\)-module \(A\) over a group \(G\) is that:

\[ A = A^c \oplus A^c, \]

where \(A^c\) is a \(ZG\)-submodule of \(A\) such that each irreducible \(ZG\)-factor of \(A^c\) is a cyclic group and the \(ZG\)-submodule \(A^c\) has no that kind of irreducible \(ZG\)-factors.

D. I. Zaicev proved that: any artinian \(ZG\)-module \(A\) over a hypercyclic group \(G\) has a C-decomposition [19]. For noetherian modules, we have the following counterexample.

Let \(A = \mathbb{Z} \oplus \mathbb{Z} (= \langle a, b; ab = ba \rangle)\), the free abelian group of rank 2. Let \(G (= \langle x \rangle)\) be a cyclic group of order 3. Define a G-action on \(A\) by

\[
\begin{align*}
    a &\mapsto b \quad ( \mapsto -(a+b) \mapsto a) \\
    b &\mapsto -(a+b) \quad ( \mapsto a \mapsto b).
\end{align*}
\]

Then we can see that \(A\) is a noetherian \(ZG\)-module over the hypercyclic group \(G\).

Since \(A/3A\) is finite and is of order \(3^2\), so the 3-group \(G\) acts trivially on the irreducible \(ZG\)-factors of \(A/3A\) (Corollary 1.2.8) and then the irreducible \(ZG\)-factors of \(A/3A\) are cyclic. Thus, the \(ZG\)-module \(A\) contains irreducible \(ZG\)-factors being cyclic groups. On the other hand, it is clear that \(A/2A\) is an irreducible \(ZG\)-factor of \(A\) and is an elementary abelian group of order 4. That is, \(A\) contains irreducible \(ZG\)-factors being not cyclic groups.

Suppose \(A\) has a C-decomposition, i.e., \(A = A^c \oplus A^c\), then \(A^c \neq 0\) and \(A^c \neq 0\). Since \(A\) is a free abelian group of rank 2, so both \(A^c\) and \(A^c\) must be infinite cyclic groups and then \(A^c/3A^c\) is irreducible and is of order 3, which contrary
to $A^c$ having no irreducible $ZG$-factors being cyclic groups. So $A$ has no $C$-decomposition.

c There is a noetherian $ZG$-module $A$ over a hypercentral group $G$ such that $A$ has no $Z$-decomposition.

A $Z$-decomposition of a $ZG$-module $A$ over a group $G$ is that:

$$A = A^Z \oplus A^Z,$$

where $A^Z$ is a $ZG$-submodule of $A$ such that each irreducible $ZG$-factor of $A^Z$ is $G$-trivial, i.e., the centralizer in $G$ is the whole group $G$, and the $ZG$-submodule $A^Z$ has no nonzero $G$-trivial $ZG$-factors.

In the same paper [19], D. I. Zaicev pointed out that: any artinian $ZG$-module $A$ over a hypercentral group $G$ has a $Z$-decomposition. But, the result does not hold again in the noetherian case as we can see from the following simple example.

Let $A = \langle a \rangle$, the infinite cyclic group, and let $G = \langle x \rangle$, the cyclic group of order 2. Define the $G$-action on $A$ by

$$a^x = a^{-1}.$$

It is clear that the above is, in fact, the definition of an infinite dihedral group, thus we have got a noetherian $ZG$-module $A = \langle a \rangle$ over the hypercentral group $G = \langle x \rangle$. Since $A$ is indecomposable as a group, so $A$ has no nontrivial $Z$-decomposition. But each irreducible $ZG$-factor $2^iA/2^{i+1}A$ is clearly $G$-trivial and each irreducible $ZG$-factor $p^jA/p^{j+1}A$ with $p \neq 2$ is clearly not $G$-trivial, so $A$ does not have a $Z$-decomposition.
§6.2 UNSOLVED QUESTIONS

Through our work, we often assume that $G$ is locally soluble. However, from the corollaries of our main results in Chapter 3 and Chapter 4, we have noted that this condition is not necessary. Therefore, the general question arises (as D. I. Zaïcev has mentioned for artinian case).

**Question 1:** Let $G$ be a hyperfinite group, does any noetherian $\mathbb{Z}G$-module $A$ have an $f$-decomposition?

From our proof for the main result — Theorem A, we see that the above question may have a positive answer if the following three questions all have a positive answer.

**Question 2:** Let $G$ be a hyperfinite group and $A$ a noetherian $\mathbb{Z}G$-module with $pA = 0$ for some prime $p$. If all irreducible $\mathbb{Z}G$-factors of $A$ are finite, should $A$ be finite? (It is almost true, see Lemma 2.4.6)

**Question 3:** Let $G$ be a hyperfinite group, $A$ a noetherian $\mathbb{Z}G$-module, and $B$ a $\mathbb{Z}G$-submodule of $A$. If all irreducible $\mathbb{Z}G$-factors of $B$ are finite and $A/B$ has no nonzero finite $\mathbb{Z}G$-factors, does $B$ have a complement in $A$?

**Question 4:** Let $G$ be a hyperfinite group, $A$ a noetherian $\mathbb{Z}G$-module, and $B$ a $\mathbb{Z}G$-submodule of $A$. If all irreducible $\mathbb{Z}G$-factors of $A/B$ are finite while $B$ has no nonzero finite $\mathbb{Z}G$-factors, does $B$ have a complement in $A$?

In §4.4, we have proved that: if $A$ is a noetherian $\mathbb{Z}G$-module over a periodic abelian group $G$ with $\pi(G)$ being finite, then $A^f$ is torsion and has a finite $\mathbb{Z}G$-composition series as well as a finite exponent. Now the general
question is:

**Question 5:** For any noetherian $\mathbb{Z}G$-module $A$ over a hyperfinite locally soluble group $G$, must $\overline{A}$ always be torsion?

Specially, we still have:

**Question 6:** If $A$ is a noetherian $\mathbb{Z}G$-module over a periodic abelian group $G$, must $\overline{A}$ be torsion?

The other challenges rising from Chapter 5 are the following:

**Question 7:** Let $G$ be a hyper-(cyclic or finite) locally soluble group, does any (torsion-free) artinian $\mathbb{Z}G$-module $A$ have an $f$-decomposition?

**Question 8:** Let $G$ be a hyper-(cyclic or finite) locally soluble group, does any noetherian $\mathbb{Z}G$-module have an $f$-decomposition?

Finally, we have:

**Question 9:** Let $G$ be a hyperfinite locally soluble group, $A$ a $\mathbb{Z}G$-module, and $B$ a $\mathbb{Z}G$-submodule of $A$. If $B$ is an artinian (resp. noetherian) $\mathbb{Z}G$-module and $A/B$ is a noetherian (resp. artinian) $\mathbb{Z}G$-module, does $A$ have an $f$-decomposition?
REFERENCES

Arch. Math., 4 (1953), 75 - 78.


[4] P. Hall, On the finiteness of certain soluble groups,

[5] B. Hartley, Injective Modules over Group Rings,

[6] B. Hartley and D. McDougall, Injective modules and soluble groups
satisfying the minimal condition for normal subgroups,

[7] A. V. Jategaonkar, Integral group rings of polycyclic-by-finite groups,

[8] O. H. Kegel and B. A. F. Wehrfritz, Strong finiteness conditions in

[9] J. C. McConnell and J. C. Robson, Noncommutative Noetherian Rings,

Math. Z., 144 (1975), 165 - 175.


[13] P. Ribenboim, Rings and Modules,


