MODULES, LATTICES AND THEIR DIRECT SUMMANDS

BY

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TO

MY MOTHER

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STATEMENT

The first two sections of chapter I of this thesis are based on two papers [SHD] and [Sm2]. All the remaining sections of this thesis are original. Chapter II together with section 3.2 has appeared in [AS]. Chapters III and IV contain the duals of the results in chapter I. Chapter V generalizes results of chapter I.

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LIST OF SYMBOLS

$\neq$: does not equal.

$\nmid$: dose not divide

$\leq$: submodule.

$\leq_e$: essential submodule.

$\ll$: small submodule.

$\subseteq$: contained in.

$\in$: belongs to.

$\notin$: does not belong to.

$\oplus$: direct sum.

soc M: the socle of M.

soc_X M: $\sum \{N \leq M : N \in X\}$.

Rad M: the radical of M.

u-dim: the uniform dimension.

u^*-dim: the dual uniform dimension.

k-dim: the Krull dimension.

k^*-dim: the dual Krull dimension.

$\underline{X}$: a class of modules.

$\underline{Z}$: the class of zero modules.

$\underline{S}$: the class of simple modules.

$\underline{C}$: the class of semisimple modules.

$\underline{N}$: the class of Noetherian modules.

$\underline{A}$: the class of Artinian modules.

$\underline{I}$: the class of injective modules.

$\underline{P}$: the class of projective modules.

$\underline{T}$: the class of singular modules.

$\underline{T}_n$: the class of non-singular modules.

$\underline{G}$: the class of finitely generated modules.

$\underline{G}_c$: the class of finitely co-generated modules.

$\underline{U}$: the class of modules with finite uniform dimension.

$\underline{U}^*$: the class of modules with finite dual uniform dimension.

$\underline{K}$: the class of modules with Krull dimension.

$\underline{K}_\alpha$: the class of modules with Krull dimension at most α, for some ordinal $\alpha > 0$.

$\underline{K}^*$: the class of modules with dual Krull dimension.

$\underline{K}^*_\alpha$: the class of modules with dual Krull dimension at most α, for some ordinal $\alpha > 0$.

ACC: the ascending chain condition.

$(ACC)_e$: the ACC on essential submodules.

$(ACC)_s$: the ACC on small submodules.

DCC: the descending chain condition.

$(DCC)_e$: the DCC on essential submodules.

$(DCC)_s$: the DCC on small submodules.

$\vee$: the joint of two elements in a lattice.

$\wedge$: the meet of two elements in a lattice.

$\underline{X}_1$: a class of lattices.

$\underline{Z}_1$: the class of singleton lattices.

$\underline{C}_1$: the class of complemented lattices.

$\underline{N}_1$: the class of lattices satisfying the ACC.

$\underline{U}_1$: the class of lattices with finite uniform dimension.

$\underline{P}^c$: the class of pseudo-complemented lattices.

c(L): the set of complements in the lattice L.

e(L): the set of essential elements the lattice L.

s(L): $\sum \{a \in L : a \text{ is an atom}\}$.

$s_1(L)$: $\wedge \{a \in L : a \text{ is essential in } L\}$.

$s_2(L)$: $\vee \{a \in L : [0,a] \text{ is complemented}\}$.

f(L): $\{a \in L : \vee b_\lambda < a$ for every chain $\{b_\lambda\}_\Lambda$ in L with

$b_\lambda < a$ for all $\lambda \in \Lambda\}$.

L^0: the opposite lattice of the lattice L.

$f^0(L)$: $\{a \in L : \wedge b_\lambda > a$ for every chain $\{b_\lambda\}_\Lambda$ in L with

$b_\lambda > a$ for all $\lambda \in \Lambda\}$.

L(M): the lattice of all submodules of a module M.

$L^0(M)$: the opposite lattice of L(M) for some module M.

ABSTRACT

It is well known that any finitely generated $\mathbb{Z}$-module is a direct sum of a projective (in fact a free) module and a Noetherian module (in fact a module of finite length) (for example see [Fu]). More generally, [Sm1] proved that if R is a right Noetherian ring with maximal Artinian right ideal A, then every finitely generated right R-module is the direct sum of a projective module and a module of finite length if and only if the ideal A = eR for some idempotent e in R and the ring R/A is a left and right hereditary left and right Noetherian semiprime ring (see [Sm1, Theorem 3.3]). It was left open in [Sm1] whether the assumption that R be right Noetherian is necessary. In fact, it is not, as Chatters [Ch] showed, by proving that if R is a ring such that every cyclic right R-module is the direct sum of a projective module and a Noetherian module, then R is a right Noetherian ring (see [Ch, Theorem 3.1]).

Chatters [Ch, Theorem 4.1] also proved that if α is an ordinal and R a ring such that every cyclic right R-module is the direct sum of a projective module and a module of

Krull dimension at most α, then the right R-module R has Krull dimension at most $\alpha + 1$. Van Huynh and Dan [HD] have considered rings with the property that every cyclic right module is the direct sum of a projective module and an Artinian module, or the property that every cyclic right module is the direct sum of a projective module and a semisimple module. This led to the investigations in [SHD] and [Sm2]. The following terminology was introduced.

Let $\underline{X}$ be a class of modules. Then $h\underline{X}$ is defined to be the class of modules M such that for each submodule N of M, M/N belongs to $\underline{X}$. Moreover, $d\underline{X}$ is defined to be the class of modules M such that for each submodule N of M, there exists a direct summand K of M such that $N \subseteq K$ and K/N belongs to $\underline{X}$. Finally, $e\underline{X}$ is defined to be the class of modules M such that for each essential submodule E of M, M/E belongs to $\underline{X}$.

It is proved in [Sm2] that when $\underline{X}$ is the class $\underline{U}$: the class of modules with finite uniform dimension, then a module M belongs to $e\underline{U}$ if and only if M/N belongs to $h\underline{U}$ for some semisimple submodule N of M (Theorem 1.2.1). This fact led [Sm2] to prove that a module M belongs to $d\underline{U}$ if and only if $M = M_1 \oplus M_2$ where M_1 is a semisimple module and M_2 belongs to h$\underline{U}$ (Theorem 1.2.3). Moreover, [Sm2] proved that when $\underline{X}$ is the class $\underline{N}$: the class of Noetherian modules, or when $\underline{X}$ is the class $\underline{K}$: the class of modules which have Krull dimension then a module M belongs to d$\underline{N}$ (respectively, d$\underline{K}$) if and only if $M = M_1 \oplus M_2$ where M_1 is semisimple and M_2 belongs to $\underline{N}$ (respectively, $\underline{K}$) (Theorem 1.2.4).

In the first two sections of chapter I of this thesis, we present all of the background material from [SHD] and [Sm2] and, for completeness, we include the proofs. In the third section, we prove a generalization of Theorem 1.2.4. i.e. we prove that when $\underline{X}$ is the class of modules with dual Krull dimension at most α, for some ordinal $\alpha \geq 0$, then a module M belongs to d$\underline{X}$ if and only if $M = M_1 \oplus M_2$ where M_1 is a semisimple module and M_2 belongs to $\underline{X}$ (Theorem 1.3.11).

In section 2.1, we define the property (P): a module M satisfies (P) provided that for any submodule N of M, there exists a direct summand K of M such that Soc $K \subseteq N \subseteq K$. We prove that a module M is the direct sum of modules with (P) and M is eventually semisimple if and only if $M = M_1 \oplus M_2 \oplus M_3$

where M_1 is a semisimple module, M_2 a finite direct sum of uniform modules and M_3 has finite uniform dimension and zero socle (Theorem 2.1.5).

In section 2.2, we define the property (P^*): a module M satisfies (P^*) provided that for any submodule N of M, there exists a direct summand K of M with $K \subseteq N$ and $N/K \subseteq \text{Rad } M/K$. We prove that a module M is a direct sum of modules satisfying (P^*) and the radical of M has finite uniform dimension if and only if $M = M_1 \oplus M_2 \oplus M_3$ where M_1 is a semisimple module M_2 is a radical module with finite uniform dimension and M_3 is a finite direct sum of local submodules and has finite uniform dimension (Theorem 2.2.8).

In chapter III, we define $h^*\underline{X}$ (respectively, $e^*\underline{X}$) to be the class of modules M such that every (small) submodule of M belongs to $\underline{X}$. Moreover, we define $d^*\underline{X}$ to be the class of modules M such that for each submodule N of M, N contains a direct summand K of M such that N/K belongs to $\underline{X}$. We prove that when $\underline{X}$ is the class of Artinian modules, then for a module M, Rad M is artinian if and only if M belongs to $e^*\underline{X}$ if and only if M belongs to $(DCC)_s$ (Theorem 3.2.4).

In Chapter IV, we characterize $d^*\underline{X}$. We prove that a module M belongs to $d^*(h\underline{U})$ if and only if $M = M_1 \oplus M_2$ where M_1 is a semisimple module and M_2 belongs to $h\underline{U}$ (Theorem 4.2.2). We also prove that when $\underline{X}$ is the class of modules which have Krull dimension at most α for some ordinal $\alpha > 0$, then a module M belongs to $d^*\underline{X}$ if and only if $M = M_1 \oplus M_2$ where M_1 is a semisimple module and M_2 belongs to $\underline{X}$ (Theorem 4.2.3). Moreover, over an FBN-ring we prove that a module M belongs to $d^*\underline{U}$ if and only if $M = M_1 \oplus M_2$ where M_1 is a semisimple module and M_2 belongs to $\underline{U}$. Finally, over a non-local Dededind domain, we prove that a module M belongs to $d^*\underline{N}$ if and only if $M = M_1 \oplus M_2 \oplus M_3$ for some semisimple module M_1, Artinian module M_2 and a Noetherian module M_3 (Theorem 4.4.7).

In the last chapter, we prove general properties of the class $h\underline{X}$, $d\underline{X}$ and $e\underline{X}$ when $\underline{X}$ is a class of complete modular lattices, and hence, give one proof to the results in sections 1.1 and 3.1. In section 5.3, we prove Goodearl's Theorem for complete modular weak upper continuous lattices (Theorem 5.3.6).

Chapter 1.

The classes hX, dX and eX.

§ 1.1 Module classes.

Let R be a ring with 1. A module always means a right R-module. By a class $\underline{X}$ of modules we mean any collection of modules which contains a zero module and is closed under isomorphisims, i.e. any module which is isomorphic to some module in $\underline{X}$ also belongs to $\underline{X}$. Let M be a module and N be any submodule of M. We call M an *$\underline{X}$-module* if M belongs to the class $\underline{X}$. We call N an *$\underline{X}$-submodule* of M if N is an $\underline{X}$-module.

Let N be any submodule of a module M. Following the same terminology as in [Sm2] we call a class $\underline{X}$

s-closed if N is an $\underline{X}$-module whenever M is an $\underline{X}$-module,

q-closed if M/N is an $\underline{X}$-module whenever M is an $\underline{X}$-module and

p-closed if M is an $\underline{X}$-module when N and M/N are $\underline{X}$-modules.

Moreover, if g_1 and g_2 are any of these three properties then we say $\underline{X}$ is *(g_1, g_2)-closed* if $\underline{X}$ is g_1-closed and g_2-closed. We say that $\underline{X}$ is {s, p, q}-closed if $\underline{X}$ is s-closed, p-closed and q-closed.

Let $\underline{X}$ and $\underline{Y}$ be classes of modules over the same ring R. Then $\underline{XY}$ is defined to be the class of modules M which contains an $\underline{X}$-submodule N such that M/N is a $\underline{Y}$-module. In particular, $\underline{X}^2$ will denote the class $\underline{XX}$. On the other hand, a module M is an $(\underline{X} \oplus \underline{Y})$-module if M is the direct sum of an $\underline{X}$-submodule and a $\underline{Y}$-submodule. Note that any $\underline{X}$-module is an $(\underline{X} \oplus \underline{Y})$-module, and any $(\underline{X} \oplus \underline{Y})$-module is an $\underline{XY}$-module. More generally, if n is a positive integer and $\underline{X}_1$, $\underline{X}_2$, ..., $\underline{X}_n$ classes of modules over the same ring, then $\underline{X}_1 \oplus \ldots \oplus \underline{X}_n$ will denote the class of modules of the form $M_1 \oplus \ldots \oplus M_n$, where M_i is an $\underline{X}_i$-module for all $1 \leq i \leq n$.

A non-zero submodule N of a module M is called *essential* in M, written $N \leq_e M$, if it has a non-zero intersection with any non-zero submodule of M.

Let $\underline{X}$ be any class of modules. The class $d\underline{X}$ consists of all modules M such that, for every submodule N of M there exists a direct summand K of M such that $N \subseteq K$ and the factor module K/N is an $\underline{X}$-module. In [Sm2], classes $h\underline{X}$ and $e\underline{X}$ are defined as follows. The class $h\underline{X}$ (respectively, $e\underline{X}$) consists of all modules M such that, for every (essential) submodule N of M, the factor module M/N belongs to $\underline{X}$.

Gordon and Robson [GR] defined the *Krull dimension, K-dim M*, of M by transfinite induction as follows:

K-dim M = -1 if and only if M = 0 ; if $\alpha \geq 0$ is an ordinal and K-dim M $\not< \alpha$, then K-dim M = α provided for every descending chain $N_1 \geq N_2 \geq \ldots$ of submodules of M there exists a positive integer n such that K-dim $(N_i/N_{i+1}) < \alpha$ for all $i \geq n$.

Dualizing the above terminology we get the so called *dual Krull dimension*, K^**-dim M*, of M (i.e. K^*-dim M = -1 if and only if M = 0. If M ≠ 0, then for an ordinal $\alpha > 0$, K^*-dim M = α if K^*-dim M $\not< \alpha$ and for any ascending chain $N_1 \leq N_2 \leq \ldots$ of submodules of M there exists a positive integer n such that K^*-dim $(N_{i+1}/N_i) < \alpha$ for all $i \geq n$. Note that for a non-zero module M, K-dim M (respectively, K^*-dim M) = 0 if and only if M is Artinian (respectively, Noetherian).

Let S be any set of submodules of a module M. Then S is called *independent* (respectively, *coindependent*) provided that every finite subset $\{N_1, N_2, \ldots, N_k\}$ of S where $N_i \neq 0$ (respectively, $N_i \neq M$) for $i = 1, 2, \ldots, k$ has the following property: for every $1 \leq i \leq k-1$, $N_i \cap (N_1 + \ldots + N_{i-1}) = 0$ (respectively, $N_i + (N_1 \cap \ldots \cap N_{i-1}) = M$). We say that M has *finite uniform dimension* if and only if every independent set of submodules of M is finite. Equivalently, M has finite uniform dimension if and only if every direct sum of non-zero submodules of M has only a finite number of summands (see [Go, chapter 1]). The uniform dimension, u-dim M, of M is the largest cardinality of all the independent sets of submodules of M. Note that a module M has finite uniform dimension if and only if there exists a positive integer k such that the cardinality of each independent set of submodules of M is at most k (see [Goo2]). The smallest such integer is called the uniform dimension of M.

Similarly, a module M has *finite dual uniform dimension* provided every coindependent set of submodules of M is finite. The dual uniform dimension, u^*-dim M, of M is again the largest cardinality of all such coindependent sets (see [Ta],and [SV]).

Let N and N' be submodules of a module M. Then N is called a *relative complement* of N' in M if N is maximal with respect to the property $N \cap N' = 0$. We call N a *complement* in M if it is a relative complement for some submodule of M. Equivalently, N is a complement if it has no proper essential extension in M, that is, if $N \leq_e K \leq M$ then $K = N$ (see [Goo2, Proposition 1.4]. On the other hand, if N is a complement of N' in M, then $N \oplus N'$ is essential in M (see [Goo2, Proposition 1.3]).

For any ring R, we denote the classes of zero modules, simple modules, injective modules, projective modules, semisimple modules, singular modules, non-singular modules, finitely generated modules, finitely co-generated modules, Artinian modules, Noetherian modules, modules with Krull dimension, modules with Krull dimension at most α, for some ordinal $\alpha \geq 0$, modules with dual Krull dimension, modules with dual Krull dimension at most α, for some ordinal $\alpha \geq 0$, modules of finite uniform dimension, and modules of finite dual uniform dimension by:

$\underline{Z}$, $\underline{S}$, $\underline{I}$, $\underline{P}$, $\underline{C}$, $\underline{T}$, $\underline{T}_n$, $\underline{G}$, $\underline{G}_c$, $\underline{A}$, $\underline{N}$, $\underline{K}$, $\underline{K}_\alpha$, $\underline{K}^*$, $\underline{K}^*_\alpha$, $\underline{U}$, $\underline{U}^*$,

respectively. Unless it is mentioned, the ring R will be always an arbitrary ring.

In the next two sections, we recall results from [SHD] and [Sm2] and, for completeness, include the proofs. First, we recall some general properties of the classes $h\underline{X}$, $d\underline{X}$ and $e\underline{X}$.

Proposition 1.1.1. Let R be any ring. Let $\underline{X}$ and $\underline{Y}$ be any classes of right R-modules. Then

(i) $h\underline{X} \subseteq d\underline{X} \subseteq e\underline{X}$,

(ii) when $\underline{X} \subseteq \underline{Y}$, then $h\underline{X} \subseteq h\underline{Y}$, $d\underline{X} \subseteq d\underline{Y}$ and $e\underline{X} \subseteq e\underline{Y}$,

(iii) $\underline{C} \subseteq d\underline{X}$,

(iv) $h\underline{X} = h(h\underline{X}) \subseteq \underline{X}$,

(v) $d(\underline{P} \oplus \underline{X}) = d\underline{X}$ and

(vi) $\underline{P} \cap d\underline{X} \subseteq h(\underline{P} \oplus \underline{X})$.

Proof. (i), (ii) and (iii) are clear from definitions.

(iv) It is clear that $h\underline{X} \subseteq \underline{X}$, and hence, $h(h\underline{X}) \subseteq h\underline{X}$. Suppose that $M \in h\underline{X}$ and N be a submodule of M. Since any homomorphic image of M/N is also a homomorphic image of M, then $M/N \in h\underline{X}$. Therefore $M \in h(h\underline{X})$.

(v) By (ii), $d\underline{X} \subseteq d(\underline{P} \oplus \underline{X})$. Suppose that $M \in d(\underline{P} \oplus \underline{X})$ and N a submodule of M. Then there exists a direct summand K of M with $N \subseteq K$ and $K/N \in \underline{P} \oplus \underline{X}$. Therefore there exist L and L' submodules of K and containing N such that $K/N = L/N \oplus L'/N$ where $L/N \in \underline{P}$ and $L'/N \in \underline{X}$. Thus $K/L' \cong L/N$ is projective. So L' is a direct summand of K, and hence, a direct summand of M. Therefore $M \in d\underline{X}$.

(vi) Let $M \in \underline{P} \cap d\underline{X}$ and N a submodule of M. Then there exist submodules K and K' of M such that $M = K \oplus K'$, $N \subseteq K$ and K/N belongs to $\underline{X}$. Therefore K' is projective and $M/N \cong K' \oplus K/N$. Hence $M \in h(\underline{P} \oplus \underline{X})$.

<u>Proposition 1.1.2</u>. Let R be any ring. Let $\underline{X}$ be any class of right R-modules. Then

(i) $h\underline{X}$, $d\underline{X}$ and $e\underline{X}$ are q-closed and

(ii) $h\underline{X}$ and $e\underline{X}$ are s-closed if $\underline{X}$ is s-closed.

Proof. (i) Let $M \in e\underline{X}$ and N any submodule of M. Let K be any essential submodule of M/N. Then $K = L/N$ for some essential submodule L of M which contains N. Therefore $M/L \in \underline{X}$. Thus $(M/N)/K \cong M/L \in \underline{X}$. Therefore $M/N \in e\underline{X}$. So $e\underline{X}$ is q-closed. Similarly, $h\underline{X}$ and $d\underline{X}$ are q-closed.

(ii) Suppose that $\underline{X}$ is s-closed. Let $M \in h\underline{X}$ and N be any submodule of M. Let K be any submodule of N. Since $N/K \leq M/K$

and $\underline{X}$ is s-closed, then $N/K \in \underline{X}$. Thus $N \in h\underline{X}$. Now suppose that $M \in e\underline{X}$ and $K \leq_e N$. Suppose that L is a complement of K in M. Then $L \cap K = 0$. Therefore $N/K \cong (N \oplus L)/(K \oplus L)$. Note that

$[(N \oplus L)/(K \oplus L)] \leq [M/(K \oplus L)] \in \underline{X}$, because $K \oplus L$ is essential in M. Thus $(N \oplus L)/(K \oplus L) \in \underline{X}$, since $\underline{X}$ is s-closed. Therefore $N/K \in \underline{X}$, and hence, $N \in e\underline{X}$.

Proposition 1.1.3. Let R be any ring. Let $\underline{X}$ be any class of right R-modules. Then

(i) $\underline{C} \oplus e\underline{X} = e\underline{X}$ and

(ii) $\underline{C} \oplus d\underline{X} = d\underline{X}$.

Proof. (i) It is clear that $e\underline{X} \subseteq \underline{C} \oplus e\underline{X}$. Let $M \in \underline{C} \oplus e\underline{X}$. Then there exist submodules K, K' of M such that $M = K \oplus K'$, $K \in \underline{C}$ and $K' \in e\underline{X}$. Let N be any essential submodule of M. By [AF, Propositions 9.6 and 9.7], $K \subseteq N$. Therefore $N = K \oplus (N \cap K')$ and $M/N = (K \oplus K')/[K \oplus (N \cap K')] \cong K'/(N \cap K') \in \underline{X}$, because $N \cap K'$ is essential in K'. Thus $M/N \in \underline{X}$, and hence, $M \in e\underline{X}$.

(ii) It is clear that $d\underline{X} \subseteq \underline{C} \oplus d\underline{X}$. Let $M \in \underline{C} \oplus d\underline{X}$. Then there exist submodules K, K' of M such that $M = K \oplus K'$, $K \in \underline{C}$ and $K' \in d\underline{X}$. Let N be any submodule of M. Note that

$N + K' = [(N + K') \cap K] \oplus K'$. Now $K = [(N + K') \cap K] \oplus F$ for some submodule F of K. Thus $N + K'$ is a direct summand of M. Since $K' \in d\underline{X}$, then there exist submodules L, L' of K' such that $K' = L \oplus L'$, $(N \cap K') \subseteq L$ and $L/(N \cap K') \in \underline{X}$. But

$$(L + N)/N \cong L/(L \cap N) \text{ and } L \cap N = L \cap N \cap K' = N \cap K'.$$

Thus $(L + N)/N \in \underline{X}$. On the other hand,

$L' \cap (L + N) = L' \cap K' \cap (L + N) = L' \cap [L + (N \cap K')] = L' \cap L = 0$.

Thus $N + K' = L' \oplus (L + N)$, and hence, $L + N$ is a direct summand of M. Therefore $M \in d\underline{X}$.

Since $\underline{C} \oplus h\underline{X} \subseteq h\underline{X}$ implies $\underline{C} \subseteq h\underline{X}$, and hence, $\underline{C} \subseteq \underline{X}$, we conclude that $\underline{C} \oplus h\underline{X} \neq h\underline{X}$ in general. On the other hand, $\underline{C} \oplus h\underline{X} \subseteq d\underline{X}$ (see Proposition 1.1.3 (ii)).

<u>Proposition 1.1.4</u>. Let R be any ring. Let $\underline{X}$ be a p-closed class of right R-modules. Then

(i) $h\underline{X} \oplus h\underline{X} = h\underline{X}$,

(ii) $e\underline{X} \oplus e\underline{X} = e\underline{X}$ and

(iii) $h\underline{X} \oplus d\underline{X} = d\underline{X}$.

Proof. (i) It is clear that $h\underline{X} \subseteq h\underline{X} \oplus h\underline{X}$. Suppose that

$M = M_1 \oplus M_2$ where both M_1 and M_2 belongs to $h\underline{X}$. Let N be any submodule of M. Then $(M_1 + N)/N \cong M_1/(N \cap M_1) \in \underline{X}$. Since $M/M_1 \cong M_2 \in h\underline{X}$, then $M/(M_1 + N) \in \underline{X}$. But $\underline{X}$ is p-closed. Thus M/N belongs to $\underline{X}$, and hence, $M \in h\underline{X}$.

(ii) It is clear that $e\underline{X} \subseteq e\underline{X} \oplus e\underline{X}$. With the same notation in (i), suppose that $M_1 \in e\underline{X}$, $M_2 \in e\underline{X}$ and $N \leq_e M$. Then $N \cap M_1$ is also essential in M_1. Thus $(M_1 + N)/N \cong M_1/(N \cap M_1) \in \underline{X}$. But $M_1 + N = M_1 \oplus ((M_1 + N) \cap M_2)$ and $(M_1 + N) \cap M_2 \leq_e M_2$. Therefore $M/(M_1 + N) \cong M_2/[(M_1 + N) \cap M_2] \in \underline{X}$. But $\underline{X}$ is p-closed. Therefore $M/N \in \underline{X}$, and hence, $M \in e\underline{X}$.

(iii) It is clear that $d\underline{X} \subseteq h\underline{X} \oplus d\underline{X}$. If $M_1 \in h\underline{X}$ and $M_2 \in d\underline{X}$ in part (i), then $(M_1 + N)/N \cong M_1/(M_1 \cap N) \in \underline{X}$. Moreover, $M_1 + N = M_1 \oplus [(M_1 + N) \cap M_2]$. Therefore there exists a direct summand K of M_2 such that $(M_1 + N) \cap M_2$ is contained in K and $K/[(M_1 + N) \cap M_2] \in \underline{X}$. Thus $M_1 \oplus K$ is a direct summand of M and $(M_1 \oplus K)/(M_1 + N) \cong K/[(M_1 + N) \cap M_2] \in \underline{X}$. But $\underline{X}$ is p-closed. Therefore $(M_1 \oplus K)/N \in \underline{X}$, and hence, $M \in d\underline{X}$.

A module M is called a ***CS-module*** provided every submodule of M is essential in a direct summand of M.

The following example was given in [Sm2] to show that there exists a class $\underline{X}$ of modules such that $\underline{X}$ is (s, q, p)-closed but $d\underline{X}$ is not s-closed and $d\underline{X} \oplus d\underline{X} \neq d\underline{X}$.

Example 1.1.5. Let $R = \mathbb{Z}[x]$, the ring of polynomials in one indeterminate x over the integers. Let M be the R-module R_R. Then $M \in d\underline{T}$, $M \oplus M \notin d\underline{T}$ and $E(M) \oplus E(M) \in d\underline{T}$, where E(M) is the injective hull of M.

Proof. We know that the class $\underline{T}$ is (s, p, q)-closed. By [CK, Example 2.4], M is a CS-module but $M \oplus M$ is not a CS-module. Note that $M \in d\underline{T}$. Suppose that $M' = M \oplus M \in d\underline{T}$ and let N be a submodule of M'. Then there exists a direct summand K of M' such that $N \subseteq K$ and $K/N \in \underline{T}$. Let L be a submodule of K such that $N \cap L = 0$. Then L embeds in K/N, and hence, $L \in \underline{T}$. But it is clear that $M' \in \underline{T}_n$. Thus L = 0, and hence, N is essential in K. Therefore M' is a CS-module which is a contradiction. Hence $M' \notin d\underline{T}$. Since $E(M') = E(M) \oplus E(M)$ is injective, then E(M) is a CS-module. Therefore $E(M') \in d\underline{T}$.

§ 1.2 Some Characterizations of $d\underline{X}$ when $\underline{X} \subseteq \underline{U}$.

Smith [Sm2] generalized Goodearl's theorem, [Goo1], by characterizing the class $e\underline{U}$. In [SHD] and [Sm2], the classes $d\underline{N}$, $d\underline{K}$ and $d\underline{U}$ are completely characterized. In this section we recall some theorems from [SHD], [Sm2] and include the proofs, for completeness.

Theorem 1.2.1. For any ring R,

(i) $e\underline{U} = \underline{C}(h\underline{U})$.

(ii) $e\underline{N} = \underline{CN}$.

Proof. (i) Suppose that $M \in e\underline{U}$ and K any submodule of M which contains the socle of M, soc M. Let K' be a complement of K in M. Then $M/(K \oplus K') \in \underline{U}$. Suppose that $K' \notin \underline{U}$. Then there exists a submodule L of K' such that $L = L_1 \oplus L_2 \oplus L_3 \oplus \dots$ where each L_i is a non-zero submodule of K' with zero socle. Thus $L_i \notin \underline{C}$ for all i. Hence each L_i has a proper essential submodule H_i. Let $H = H_1 \oplus H_2 \oplus H_3 \oplus \dots$. Then H is essential in L. Moreover $L/H \cong (L_1/H_1) \oplus (L_2/H_2) \oplus (L_3/H_3) \oplus \dots$ is an infinite direct sum of non-zero submodules of L/H. But, by Proposition 1.1.2, $L \in e\underline{U}$, a contradiction. Hence K' has finite uniform dimension. Thus $K' \cong (K \oplus K')/K \in \underline{U}$. But $\underline{U}$ is p-closed. Thus $M/K \in \underline{U}$, and hence, $M/\text{soc } M \in h\underline{U}$. Therefore $M \in \underline{C}(h\underline{U})$.

Conversely, suppose that $M \in \underline{C}(h\underline{U})$ and $K \leq_e M$. Then there exists a submodule N of M such that $N \in \underline{C}$ and $M/N \in h\underline{U}$. But, by [AF, Theorem 9.6 and Proposition 9.7], $N \subseteq K$. Since $M/N \in h\underline{U}$, then $M/K \in \underline{U}$, and hence, $M \in e\underline{U}$.

(ii) It is clear that $\underline{NC} \subseteq e\underline{N}$. Suppose that $M \in e\underline{N}$. Then, by [PY, Corollary 2.6], $M/\text{Soc } M \in \underline{N}$. Hence $M \in \underline{CN}$.

<u>Lemma 1.2.2</u>. Let R be a ring and M a right R-module. Then

(i) $M \in h\underline{U}$ if and only if $M \in \underline{U} \cap e\underline{U}$ and

(ii) If $M \in d\underline{U}$, then $M \in \underline{U}$ if and only if the socle of M is contained in a finitely generated submodule of M.

Proof. (i) By the definition of the class $h\underline{U}$ and Proposition 1.1.1, $h\underline{U} \subseteq \underline{U} \cap e\underline{U}$. Suppose that $M \in \underline{U} \cap e\underline{U}$ and N a submodule of M. Let N' be a complement of N in M. Then $M/(N \oplus N') \in \underline{U}$. Since $\underline{U}$ is s-closed, then $N' \in \underline{U}$. But $(N + N')/N \cong N' \in \underline{U}$. Because $\underline{U}$ is p-closed, $M/N \in \underline{U}$, and hence, $M \in h\underline{U}$.

(ii) Let $M \in d\underline{U} \cap \underline{U}$. Let S be the socle of M. Since $\underline{U}$ is s-closed, then $S \in \underline{U}$. Thus $S \in \underline{G}$. Conversely, let $S \subsetneq N$ for some $\underline{G}$-submodule N of M. By Proposition 1.1.1 and the proof of Theorem 1.2.1, $M/S \in \underline{U}$. Therefore u-dim $M/N = n$ for some integer $n \geq 0$. Now we use the induction on n. If $n = 0$, then $M = S$, and hence, $M \in \underline{G}$. Thus $M \in \underline{U}$. Let $n > 0$. Suppose that M does not have finite uniform dimension. Then $S \notin \underline{G}$. Therefore there exist submodules S_1, S_2 of S such that $S = S_1 \oplus S_2$ and both $S_1, S_2 \notin \underline{G}$. By hypothesis, there exist submodules M_1 and M_2 of M such that $M = M_1 \oplus M_2$, $S_1 \subseteq M_1$, and $M_1/S_1 \in \underline{U}$. Note that $S_1 \subseteq \mathrm{soc}\, M_1$. Therefore $\mathrm{soc}\, M_1 = S_1 \oplus S'$ for some submodule S' of M_1. Since S' embeds in M_1/S_1, then $S' \in \underline{U}$, and hence, S' is finitely generated. Note that

$$S_1 \oplus S_2 = \mathrm{soc}\, M = \mathrm{soc}\, M_1 \oplus \mathrm{soc}\, M_2 = S_1 \oplus S' \oplus \mathrm{soc}\, M_2.$$

Thus $S_2 \cong S' \oplus \mathrm{soc}\, M_2 \notin \underline{G}$. Hence $\mathrm{soc}\, M_2 \notin \underline{G}$. But $\mathrm{soc}\, M_1 \notin \underline{G}$ and $M/S \cong (M_1/\ \mathrm{soc}\, M_1) \oplus (M_2/\ \mathrm{soc}\, M_2)$. If $M_1 = \mathrm{soc}\, M_1$, then $M_1 \subsetneq N$, and hence, $N = M_1 \oplus (N \cap M_2)$. Thus $M_1 \in \underline{G}$. Hence $\mathrm{soc}\, M_1 \in \underline{G}$, a contradiction. Therefore $M_1 \neq \mathrm{soc}\, M_1$. Similarly, $M_2 \neq \mathrm{soc}\, M_2$. Therefore $M_1/\ \mathrm{soc}\, M_1$ and $M_2/\ \mathrm{soc}\, M_2$ have uniform dimensions smaller than n. By induction on n, both M_1 and $M_2 \in \underline{U}$. Hence M has finite uniform dimension, a contradiction. Thus $M \in \underline{U}$.

Theorem 1.2.3. For any ring R, $d\underline{U} = d(h\underline{U}) = \underline{C} \oplus h\underline{U}$.

Proof. By Proposition 1.1.1, $d(h\underline{U}) \subseteq d\underline{U}$. Suppose that $M \in d\underline{U}$ and N be a submodule of M. Then there exists a direct summand L of M such that $N \subseteq L$ and $L/N \in \underline{U}$. By Proposition 1.1.2, $L \in d\underline{U}$, and hence, by the same Proposition $L/N \in d\underline{U}$. So, by Proposition 1.1.1, $L/N \in e\underline{U}$. So, by Lemma 1.2.2, $L/N \in h\underline{U}$. Thus $M \in d(h\underline{U})$. By Propositions 1.1.1 and 1.1.3, $\underline{C} \oplus h\underline{U} \subseteq d\underline{U}$.

Now suppose that $M \in d\underline{U}$. Then $M \in e\underline{U}$. By the proof of Theorem 1.2.1, $M/S \in h\underline{U}$ where S is the socle of M. Thus M/S has finite uniform dimension, say $n \geq 0$. Now we use the induction on n. If $n = 0$, then $M = S$, and hence, $M \in \underline{C} \subseteq \underline{C} \oplus h\underline{U}$. Suppose that $n > 0$ and $M \notin \underline{C} \oplus h\underline{U}$. Let $M = M_1 \oplus M_2$ for some submodules M_1, M_2 of M. Then $S = S_1 \oplus S_2$ where S_i is the socle of soc M_i $(i = 1, 2)$. Thus $M/S \cong (M_1/S_1) \oplus (M_2/S_2)$. Suppose that $M_1 \neq S_1$ and $M_2 \neq S_2$. Then u-dim $(M_1/S_1) < n$ and u-dim $(M_2/S_2) < n$. Thus $M_1 \in \underline{C} \oplus h\underline{U}$ and $M_2 \in \underline{C} \oplus h\underline{U}$. Hence $M \in \underline{C} \oplus h\underline{U}$, a contradiction. Therefore $M_1 = S_1$ or $M_2 = S_2$. Since $n > 0$, then $M/S \neq 0$. Thus there exists $m \in M$ such that $m \notin S$. Therefore there exist submodules K, K' of M such that $M = K \oplus K'$, $mR \subseteq K$ and K/mR has finite uniform dimension. Note that, by the previous argument, soc K = K $\in$ $\underline{C}$ or soc K' = K' $\in$ $\underline{C}$. But mR $\notin$ $\underline{C}$, since m $\notin$ S. Therefore K' $\in$ $\underline{C}$. There exists a submodule F of soc K such that soc K = (soc K $\cap$ mR) $\oplus$ F. Therefore

$$F \cong \text{soc } K/(\text{soc } K \cap mR) \cong (\text{soc } K + mR)/mR \leq K/mR.$$

But K/mR $\in$ $\underline{U}$. Therefore F $\in$ $\underline{U}$. Hence F is finitely generated. Thus soc K $\subseteq$ mR + F which is again finitely generated. Since K $\in$ d$\underline{U}$, then, by Lemma 1.2.2, K $\in$ $\underline{U}$. Therefore K $\in$ (e$\underline{U}$ $\cap$ $\underline{U}$). By Lemma 1.2.2, K $\in$ h$\underline{U}$, and hence, M $\in$ $\underline{C}$ $\oplus$ h$\underline{U}$.

<u>Theorem 1.2.4</u>. Let R be any ring, then

(i) d$\underline{N}$ = $\underline{C}$ $\oplus$ $\underline{N}$ and

(ii) d$\underline{K}$ = $\underline{C}$ $\oplus$ $\underline{K}$.

Proof. (i) By Propositions 1.1.1 and 1.1.3, $\underline{C}$ $\oplus$ $\underline{N}$ $\subseteq$ d$\underline{N}$. For the converse, Let M $\in$ d$\underline{N}$. Then, by Proposition 1.1.1, M $\in$ d$\underline{U}$. Hence, by Theorem 1.2.3, M = M_1 $\oplus$ M_2 where M_1 $\in$ $\underline{C}$ and M_2 $\in$ h$\underline{U}$. By Proposition 1.1.1, M $\in$ e$\underline{N}$. Hence, by [PY, Corollary 2.6], M/Soc M is Noetherian. Since M_2 $\in$ $\underline{U}$, then Soc M/M_1 $\in$ $\underline{U}$. Thus Soc M/M_1 is Noetherian. Thus M/M_1 $\in$ $\underline{N}$, and hence, M_2 $\in$ $\underline{N}$.

(ii) By [GR, Lemma 1.1], h$\underline{K}$ = $\underline{K}$. Therefore, by Propositions 1.1.1 and 1.1.3, $\underline{C}$ $\oplus$ $\underline{K}$ = $\underline{C}$ $\oplus$ h$\underline{K}$ $\subseteq$ $\underline{C}$ $\oplus$ d$\underline{K}$ = d$\underline{K}$. Let M $\in$ d$\underline{K}$.

Then, by [GR, Proposition 1.4] and Proposition 1.1.1, $M \in d\underline{U}$. Hence, by Theorem 1.2.3, $M = M_1 \oplus M_2$ where $M_1 \in \underline{C}$ and $M_2 \in h\underline{U}$. Note that $M_2 \in \underline{U} \cap d\underline{K}$. We shall prove that $M_2 \in \underline{K}$ by induction on the uniform dimension of M_2. Suppose that M_2 contains a non-zero submodule N such that $N \in \underline{K}$. Then $M_2 = L \oplus L'$, $N \subseteq L$, and $L/N \in \underline{K}$. Since $\underline{K}$ is p-closed (see [GR, Lemma 1]), then $L \in \underline{K}$. Moreover, L' has a smaller uniform dimension than M_2. Thus, by induction, $L' \in \underline{K}$. Therefore $M_2 \in \underline{K}$. On the other hand, if M_2 does not contain a non-zero submodule N with $N \in \underline{K}$, then $M_2 \in \underline{C}$, because $M_2 \in d\underline{K}$. Therefore $M_2 \in \underline{N}$. Hence, by [GR, Proposition 1.3], M_2 has Krull dimension.

§ 1.3 More results and examples.

The results in this section are not in [SHD] or [Sm2]. In this section we will prove more results about the class $d\underline{X}$, for a different class $\underline{X}$, and present some examples.

Lemma 1.3.1. For any ring R, if the class $\underline{X}$ is s-closed then $\underline{C} \oplus \underline{X}$ is also s-closed.

Proof. Suppose that $\underline{X}$ is s-closed. Let $M = M_1 \oplus M_2$ where M_1 is a semisimple module and $M_2 \in \underline{X}$. Let K be a submodule of M. Then, by [AF, Theorem 9.6], there exists a submodule M' of M_1 such that $M_1 = (K \cap M_1) \oplus M'$. Therefore $M = (K \cap M_1) \oplus M' \oplus M_2$. Hence $K = (K \cap M_1) \oplus L$ where $L = K \cap (M' \oplus M_2)$. Note that $L \leq M' \oplus M_2$ and $L \cap M' \subseteq K \cap M' = K \cap (M_1 \cap M') = (K \cap M_1) \cap M'$ which is contained in $(K \cap M_1) \cap (M' \oplus M_2) = 0$. Therefore L embeds in M_2. Thus $L \in \underline{X}$, because $\underline{X}$ is s-closed. It is clear that $K \cap M_1 \in \underline{C}$. Therefore $K \in \underline{C} \oplus \underline{X}$.

Corollary 1.3.2. For any ring R, the classes $d\underline{N}$, $d\underline{K}$ and $d\underline{U}$ are all s-closed.

Proof. By Lemma 1.3.1, Theorem 1.2.3 and Theorem 1.2.4.

Note that the converse of Lemma 1.3.1 is not true as we will show in the following example.

Example 1.3.3. Let R be any ring. Let $\underline{X}$ be the class of all semisimple right R-modules with composition series of even length. Then $\underline{C} \oplus \underline{X}$ is s-closed but $\underline{X}$ is not s-closed.

Proof. Since $\underline{C} \oplus \underline{X} = \underline{C}$, then $\underline{C} \oplus \underline{X}$ is s-closed. Now let M be the right R-module $N_1 \oplus N_2$ where N_1 and N_2 are non-zero simple modules. Then M is a semisimple of length 2, and hence, $M \in \underline{X}$. But the submodule N_1 is of length 1. Thus $N_1 \notin \underline{X}$.

Camillo and Yousif [CY] called a module M ***eventually semisimple*** if, for every direct sum $M_1 \oplus M_2 \oplus M_3 \oplus \ldots$ of submodules M_i $(i \geq 1)$ of M, there exists a positive integer n such that M_i is semisimple for all $i \geq n$. In [CY, Lemma 1], they prove that any module M, such that M/(Soc M) has finite uniform dimension, is eventually semisimple. Now we prove

Lemma 1.3.4. Let R be any ring. Let M_1 be a right R-module with finite uniform dimension and M_2 an eventually semisimple right R-module. Then $M = M_1 \oplus M_2$ is eventually semisimple.

Proof. Let $N = N_1 \oplus N_2 \oplus N_3 \oplus \ldots$ be a direct sum of submodules of M. Suppose that $N \cap M_1 \neq 0$. Then there exists $k(1) \geq 1$ such that $(N_1 \oplus \ldots \oplus N_{k(1)}) \cap M_1 \neq 0$. Let

$$N' = N_{k(1)+1} \oplus N_{k(1)+2} \oplus N_{k(1)+3} \oplus \ldots .$$

If $N' \cap M_1 \neq 0$, then there exists $k(2) \geq k(1)+1$ such that

$$(N_{k(1)+1} \oplus \dots \oplus N_{k(2)}) \cap M_1 \neq 0.$$

Repeating this process and noting that M_1 has finite uniform dimension, we conclude that there exists a positive integer k such that $(N_k \oplus N_{k+1} \oplus N_{k+2} \oplus \dots) \cap M_1 = 0$.

Let $\pi : M \to M_2$ denote the canonical projection. Then

$$\pi(N_k) + \pi(N_{k+1}) + \pi(N_{k+2}) + \dots$$

is a direct sum. Since M_2 is eventually semisimple, it follows that there exists $m > k$ such that $\pi(N_i)$ is semisimple for all $i > m$. But $N_i \cong \pi(N_i)$ for all $i > m$. Thus N_i is semisimple for all $i > m$. It follows that M is eventually semisimple.

We shall call a class of modules *eventually semisimple* if all of it's members are eventually semisimples.

Corollary 1.3.5. Let R be any ring. Then, the classes d$\underline{N}$, d$\underline{K}$, and d$\underline{U}$ are eventually semisimple.

Proof. By Lemma 1.3.4, Theorem 1.2.3 and Theorem 1.2.4.

Lemma 1.3.6. Let R be any ring and $\underline{X}$ be any s-closed class of right R-modules. Let M be a right R-module such that M does not have a non-zero $\underline{X}$-submodule. Then M is a $d\underline{X}$-module if and only if M is both a CS-module and an $e\underline{X}$-module.

Proof. Suppose that M does not have a non-zero $\underline{X}$-submodule. Let $M \in d\underline{X}$ and N a submodule of M. Then $M = K \oplus K'$ such that $N \subseteq K$ and $K/N \in \underline{X}$. Let L be a submodule of K. If $L \cap N = 0$, then L embeds in K/N which belongs to $\underline{X}$. Since $\underline{X}$ is s-closed, then L belongs to $\underline{X}$, and hence, $L = 0$. Therefore N is essential in K. Thus M is a CS-module and, by Proposition 1.1.1, $M \in e\underline{X}$.

Conversely, suppose that M is a CS-module and an $e\underline{X}$-module. Let N be a submodule of M. Then $M = K \oplus K'$ where N is essential in K. But $K \cong M/K' \in e\underline{X}$, because $e\underline{X}$ is q-closed. Therefore K/N belongs to $\underline{X}$, and hence, $M \in d\underline{X}$.

We know that the classes $d\underline{N}$, $d\underline{K}$ and $d\underline{U}$ are closed under finite direct sums (see Theorem 1.2.3 and Theorem 1.2.4). In the following example, we give a class of modules $\underline{X}$ such that the class $d\underline{X}$ is not closed under (finite) direct sums and $d\underline{X}$ is not s-closed.

Example 1.3.7. Let R be the ring of integers Z. For any prime number p, let $M = \{m/p^n : m, n \in Z \text{ and } n \geq 0\}$ and $M^* = M \oplus M$. Then M^* contains a submodule M' which belongs to $d\underline{A} \oplus d\underline{A}$ and does not belong to $d\underline{A}$.

Proof. It is clear that M is torsion free and $Z \subseteq M$. Let N be a non-zero submodule of M. If $N \cap Z = 0$, then N embedds in M/Z which is isomorphic to the singular artinian module $Z(p^\infty)$. Thus N is singular, and hence, N = 0, because $M \notin \underline{T}_n$. If $N \cap Z \neq 0$, then $Z/(N \cap Z) \in \underline{A}$. But $M/Z \in \underline{A}$. Therefore $M/(N \cap Z) \in \underline{A}$. Thus $M/N \in \underline{A}$, because $N \cap Z \subseteq N$. Thus $M \in d\underline{A}$. Thus, by Propositions 1.1.1 and 1.1.4, $M^* \in e\underline{A}$. But $Soc(M^*) = Soc(M) \oplus Soc(M) = 0$. Hence M^* does not contain a non-zero artinian submodule. Note that M is an additive subgroup of $\mathbb{Q}$. Therefore, by [MM, p.19], M^* is a CS-module. Hence, by Lemma 1.3.6, M^* belongs to $d\underline{A}$. Now let $M' = Z \oplus M$. Then M' is a submodule of M^*. By [MM, p.19], M' is not CS-module. Thus, by Lemma 1.3.6, $M' \notin d\underline{A}$. Since every proper homomorphic image of Z is artinian, then $Z \in d\underline{A}$. Therefore $M' \in d\underline{A} \oplus d\underline{A}$.

Note that, in particular, Example 1.3.7 shows that, for some ordinal $\alpha > 0$, $d\underline{K}_\alpha \oplus d\underline{K}_\alpha \neq d\underline{K}_\alpha$. On the other hand, by Example 1.3.7 compounded with Lemma 1.3.1, we conclude that $d\underline{A} \neq \underline{C} \oplus \underline{A}$.

In the next example we will show that, in general,

$$d\underline{U}^* \neq \underline{C} \oplus \underline{U}^*.$$

Example 1.3.8. Over itself the ring of integers Z belongs to $d\underline{U}^*$ but it does not have finite dual uniform dimension.

Proof. Let K be any submodule of Z. If $K = 0$, then K is a direct summand of Z. If $K \neq 0$, then Z/K is finite. Therefore $Z/K \in \underline{U}^*$. Hence Z is a $d\underline{U}^*$-module. Let $p_1, p_2, \ldots, p_n, p_{n+1}$ be distinct prime numbers. Since $\cap_{i=1}^{n} Zp_i = Z(p_1p_2\cdots p_n)$ and

$$Z(p_1p_2\cdots p_n) + Zp_{n+1} = Z$$

then, the set $\{Zp_1, Zp_2, \ldots, Zp_{n+1}\}$ is a coindependent set of submodules of Z for any $n > 0$. Therefore Z does not have finite dual uniform dimension.

Let R be a ring and M be a right R-module. By a *subquotient* of M we shall mean a right R-module N/K for some submodules K, N of M such that $K \subseteq N$. Next, we characterize the class h$\underline{U}$.

<u>Lemma 1.3.9</u>. Let R be any ring and M a right R-module. Then M belongs to h$\underline{U}$ if and only if every semisimple subquotient of M is finitely generated.

Proof. The necessity is clear. Conversely, suppose that M does not belong to h$\underline{U}$. Then there exists a submodule K of M such that M/K does not have finite uniform dimension. So M/K has an infinite direct sum of non-zero submodules. Therefore there exist elements $x_1, x_2, \ldots$ of the module M\K such that the sum $[(x_1R + K)/K] + [(x_2R + K)/K] + \ldots$ is direct. For each $n \geq 1$, K is contained in a maximal submodule P_n of $(x_nR + K)$. Now let $P = P_1 + P_2 + \ldots$ and $N = K + x_1R + x_2R + \ldots$ Therefore P is a submodule of N and $N/P \cong [(x_1R + K)/P_1] \oplus [(x_2R + K)/P_2] \oplus \ldots$ So N/P is a non-finitely generated semisimple subquotient of M.

In Proposition 1.1.2, it was shown that the class h$\underline{U}$ is {s,q}-closed. Now we use lemma 1.3.9 to show that the class h$\underline{U}$ is p-closed.

Proposition 1.3.10. For any ring R, the class $h\underline{U}$ is p-closed.

Proof. Suppose that N and M/N are both ($h\underline{U}$)-modules. Let $P \subseteq Q$ be submodules of M such that Q/P is semisimple. We shall prove that Q/P is finitely generated. Note that

$$Q/[P + (Q \cap N)] \cong (Q + N)/(P + N),$$

so that $Q/[P + (Q \cap N)]$ is finitely generated. Moreover,

$$[P + (Q \cap N)]/P \cong (Q \cap N)/(P \cap N),$$

so that $[P + (Q \cap N)]/P$ is finitely generated. It follows that Q/P is finitely generated. Therefore, by Lemma 1.3.9, $M \in h\underline{U}$.

Note that a module is Noetherian if and only if it has dual Krull dimension zero. Thus the next theorem is a generalization of Theorem 1.2.4.

Theorem 1.3.11. Let R be any ring and $\alpha > 0$ be an ordinal. Then $d\underline{K}^*_\alpha = \underline{C} \oplus \underline{K}^*_\alpha$.

Proof. Suppose that M belongs to $d\underline{K}^*_\alpha$. By [Le], $\underline{K}^*_\alpha \subseteq \underline{K}^* = \underline{K}$. Therefore $\underline{K}^*_\alpha \subseteq \underline{U}$ (see [GR, Proposition 1.4]). Hence M belongs to

$d\underline{U}$. Thus, by Theorem 1.2.3, $M = M_1 \oplus M_2$ where M_1 is semisimple module and M_2 belongs to $h\underline{U}$. Therefore M_2 has finite uniform dimension and, by Proposition 1.1.2, M_2 belongs to $d\underline{K}^*_\alpha$. First we claim that M_2/N belongs to $\underline{K}^*_\alpha$ for any non-zero submodule N of M_2. Suppose not. Then M_2/L does not belong to $\underline{K}^*_\alpha$ for some non-zero submodule L of M_2. So, by Proposition 1.1.2, there exist submodules K, K' of M_2 such that $M_2 = K \oplus K'$, $L \subseteq K$ and K/L belongs to $\underline{K}^*_\alpha$. Therefore, by the hypothesis, K is a proper submodule of M_2. Hence K and K' have smaller uniform dimension than M_2. By induction on the uniform dimension of M_2, K and K' belong to $\underline{K}^*_\alpha$. Therefore M_2 belongs to $\underline{K}^*_\alpha$. By [Le, Proposition 3], $\underline{K}^*_\alpha$ is q-closed. Thus M_2/L belongs to $\underline{K}^*_\alpha$, a contradiction. Therefore M_2/N belongs to $\underline{K}^*_\alpha$ for every non-zero submodule N of M_2. Now we show that M_2 belongs to $\underline{K}^*_\alpha$. Let $N_1 \leq N_2 \leq \dots$ be an ascending chain of submodules of M_2. If N_i is zero for all i, then $N_{i+1}/N_i \in \underline{K}^*_{-1}$ $(i \geq 1)$. Suppose that there exists $k \geq 1$ such that N_k is not zero. Then we have the ascending chain

$$0 \neq N_k \leq N_{k+1} \leq N_{k+2} \leq \dots \leq M_2.$$

Therefore $(N_{k+1} / N_k) \leq (N_{k+2} / N_k) \leq \dots \leq (M_2 / N_k)$. But we know that (M_2/N_k) belongs to $\underline{K}^*_\alpha$. Thus there exists $t \geq k+1$ such that for all $n \geq t$, $[(N_{n+1} / N_k) / (N_n / N_k)] \in \cup_{\beta<\alpha} \underline{K}^*_\beta$, and hence, we have $(N_{n+1} / N_n) \in \cup_{\beta<\alpha} \underline{K}^*_\beta$ for all $n > t$. Thus M_2 belongs to $\underline{K}^*_\alpha$. Therefore $M \in \underline{C} \oplus \underline{K}^*_\alpha$.

Conversely, suppose that $M = M_1 \oplus M_2$ for some semisimple module M_1 and a module M_2 belongs to $\underline{K}^*_\alpha$. Since $\underline{K}^*_\alpha$ is q-closed (see [Le, Proposition 3]), then M_2 belongs to $h\underline{K}^*_\alpha$. Therefore, by Proposition 1.1.1 and Proposition 1.1.3, M belong to $d\underline{K}^*_\alpha$.

<u>Corollary 1.3.12</u>. For any ring R and an ordinal $\alpha > 0$, the class $d\underline{K}^*_\alpha$ is s-closed and eventually semisimple.

Proof. By Theorem 1.3.11 and Lemma 1.3.1.

Chapter 2.

Modules with the properties (P) and (P^*).

§ 2.1. Modules satisfying the property (P).

Let R be a ring and M a right R-module. We shall say that the module M *has property (P)* if for any submodule N of M there exists a direct summand K of M such that Soc K $\subseteq$ N $\subseteq$ K.

In the next lemma, we give an alternative characterization of modules which have the property (P).

Lemma 2.1.1. Let R be any ring and M a right R-module. Then M satisfies the property (P) if and only if for every complement submodule K of M there exists a direct summand L of M such that Soc L $\subseteq$ K $\subseteq$ L.

Proof. The necessity is clear. Conversely, suppose that for every complement submodule K there exists a direct summand L of M such that Soc L $\subseteq$ K $\subseteq$ L. Let N be any submodule of M. First we show that N is essential in a complement submodule. Any zero submodule is a complement. Suppose that N is not zero. Then the set S of all submodules K of M such that $N \leq_e K$ is non-empty, since $N \in S$. Let $\{K_\lambda : \lambda \in \Lambda\}$ be any chain of submodules in S. Suppose that $K^* = \cup_{\lambda \in \Lambda} K_\lambda \notin S$. Then there exists $0 \neq L \leq K^*$ such that $N \cap L = 0$. Therefore $(K_\lambda \cap L) = 0$ for all $\lambda \in \Lambda$. Thus $L = 0$, a contradiction. Therefore $K^* \in S$, and hence, by Zorn's Lemma, S has a maximal member K'. Now suppose that $K' \leq_e F \leq M$. Then, by [Goo2, Prop.1.4], $N \leq_e F$. Therefore, by the maximality of K', $K' = F$. Hence K' is a complement submodule with $N \leq_e K'$. By hypothesis, there exists a direct summand L of M such that $\mathrm{Soc}\, L \subseteq K' \subseteq L$. Therefore $\mathrm{Soc}\, L \subseteq \mathrm{Soc}\, K'$. But N is essential in K'. Hence $\mathrm{Soc}\, K' \subseteq N$ (see [AF, Proposition 9.7]). It follows that $\mathrm{Soc}\, L \subseteq N \subseteq L$, and hence, M satisfies the property (P).

Note that, by the above Lemma, every CS-module satisfies (P). Our first aim in this section is to show that modules with the property (P) are built up from CS-modules and modules with zero socle. We first show that the property (P) is inherited by direct summands.

<u>Lemma 2.1.2</u>. Let R be any ring. Let M be a right R-module which satisfies the property (P). Then any direct summand of M also satisfies the property (P).

Proof. Suppose that M satisfies the property (P). Let K be a direct summand of M. Then $M = K \oplus K'$ for some submodule K' of M. Let N be any submodule of K. Consider the submodule $N \oplus K'$ of M. By hypothesis, there exist submodules L, L' of M such that $M = L \oplus L'$ and $\text{Soc } L \subseteq N \oplus K' \subseteq L$. Thus $L = (L \cap K) \oplus K'$, and hence, $K = (L \cap K) \oplus [K \cap (K' \oplus L')]$. Note that $N \subseteq L \cap K$. Moreover, $\text{Soc } (L \cap K) \subseteq (\text{Soc } L) \cap K \subseteq (N \oplus K') \cap K$. By modular law, $(N \oplus K') \cap K = N \oplus (K \cap K') = N$. Thus $\text{Soc } (L \cap K) \subseteq N$. It follows that K satisfies the property (P).

<u>Proposition 2.1.3</u>. Let R be any ring and M a right R-module which satisfies the property (P). Then $M = M_1 \oplus M_2$ where M_1 is a CS-module with essential socle and M_2 is a right R-module with zero socle.

Proof. Suppose that M satisfies the property (P). Let S be the socle of M. Then, by Zorn's Lemma there exists a submodule N of M which is maximal with respect to the property $S \cap N = 0$. By the hypothesis, there exists a direct summand K of M such that $\text{Soc } K \subseteq N \subseteq K$. Therefore $\text{Soc } K \subseteq S \cap N$, and hence, $\text{Soc } K = 0$. It follows that $K \cap S = 0$. By the maximality of N, $N = K$, and hence, N is a direct summand of M. Therefore $M = N \oplus N'$ for some submodule N' of M. By [AF, Corollary 9.9 and Proposition 9.19], $S = \text{Soc } M = (N \cap S) \oplus (N' \cap S) = N' \cap S \subseteq N'$. Let L be any submodule of N' such that $S \cap L = 0$. Then $S \cap (N \oplus L) = 0$, and hence, $N = N \oplus L$, by the choice of N. Therefore $L = 0$, and hence, S is an essential submodule of N'.

Let P be any submodule of N'. By Lemma 2.1.2, there exists a direct summand Q of N' such that $\text{Soc } Q \subseteq P \subseteq Q$. It is clear that $S = \text{Soc } N'$. By [AF, Corollary 9.9], $\text{Soc } Q = (\text{Soc } N') \cap Q$. Thus Soc Q is essential in Q, and hence, P is essential in Q. Hence every submodule of N' is essential in a direct summand. Thus N' is a CS-module.

In the next example, we shall show that the converse of Proposition 2.1.3 is false and, we also show that, a direct sum of modules with the property (P) does not need to have (P).

<u>Example 2.1.4</u>. Let R be the ring of integers $\mathbb{Z}$. For any prime p, let M_1 be the simple right R-module $\mathbb{Z}/\mathbb{Z}p$. Moreover, let M_2 be the right R-module $\mathbb{Z}_{\mathbb{Z}}$. Then M_1 and M_2 satisfy the property (P). On the other hand, $M = M_1 \oplus M_2$ does not satisfy (P).

Proof. Since M_1 is a semisimple module, then by [AF, Theorem 9.6], M_1 satisfies the property (P). Since Soc $M_2 = 0$, then by definition, M_2 satisfies the property (P). Let K be the cyclic submodule $Z(1+Zp,p)$ of M. Since, as an Abelian group, K is infinite cyclic, it follows that K is a uniform Z-module. Suppose that L is a submodule of M and K is essential in L. Then L is uniform. Now L is finitely generated. Thus, by the fundamental Theorem of Abelian groups, L is a finite direct sum of cyclic submodules, and hence, L is cyclic. Thus there exist elements a, b $\in$ Z such that $L = Z(a+Zp,b)$. Hence there exists an element n $\in$ Z such that $(1+Zp,p) = n(a+Zp,b)$. Therefore we have $1-na \in Zp$ and $p = nb$. It follows that $n = 1$ or -1, and hence, K = L. Thus K is a complement submodule of M.

Now suppose that M satisfies the property (P). Then there exists a direct summand N of M such that Soc N $\subseteq$ K $\subseteq$ N. Since Soc K = 0, then Soc N = 0, and hence, $N \cap M_1 = 0$. Therefore N embeds in M/M_1, and hence, N is uniform. But this implies that K is essential in N. Hence, by [Goo2, Proposition 1.4], K = N. Thus K is a direct summand of M. So there exists a submodule K' of M such that $M = K \oplus K'$. Thus K' is uniform. Since Soc K = 0, then by [AF, Proposition 9.19] M_1 = Soc M = Soc K' $\subseteq$ K', and hence, $M_2 \cap K' = 0$. Thus $K' = M_1$. But $K \oplus M_1 = M_1 \oplus pM_2 \neq M$, a contradiction. Therefore M does not satisfy the property (P).

It is easy to prove that any CS-module with finite uniform dimension is a finite direct sum of uniform modules. Let R be a ring and M a right R-module. For any $m \in M$ let $\underline{r}(m)$ denote the right ideal $\{r \in R : mr = 0\}$. Okado [Ok, Lemma 3] (see also [MM, Proposition 2.18]) proved that if R is a ring and M a right R-module such that M is a CS-module and R has the ascending chain condition on right ideals of the form $\underline{r}(m)$, where $m \in M$, then M is a direct sum of uniform submodules. Now we prove:

<u>Theorem 2.1.5</u>. Let R be any ring and M a right R-module. Then the following statements are equivalent.

(i) M is a direct sum of modules with (P) and M is eventually semisimple.

(ii) $M = M_1 \oplus M_2 \oplus M_3$ where M_1 is a semisimple module, M_2 a finite direct sum of uniform modules and M_3 a module with finite uniform dimension and zero socle.

Proof. Suppose that (ii) is true. Then, by Lemma 1.3.4, M is eventually semisimple. But it is clear that semisimple modules, uniform modules and modules with zero socles satisfy (P).

Suppose that M is eventually semisimple and there exist a non-empty index set Λ and submodules M_λ $(\lambda \in \Lambda)$ of M, each having (P), such that $M = \oplus_\Lambda M_\lambda$. By Proposition 2.1.3, for each $\lambda \in \Lambda$, M_λ is a direct sum of a CS-module and a module with zero socle. Thus, without loss of generality, we can suppose that M_λ is a CS-module or a module with zero socle for each $\lambda \in \Lambda$. Now let $\Lambda' = \{\lambda \in \Lambda : M_\lambda \text{ is not semisimple}\}$. Because M is eventually semisimple it follows that Λ' is a finite set. Thus there exist a positive integer k and submodules N_i $(1 \leq i \leq k)$ of M such that $M = N_1 \oplus \dots \oplus N_k$, N_1 is semisimple and, for each $2 \leq i \leq k$, N_i is a CS-module or a module with zero socle.

Let K be a direct summand of M such that K has zero socle. Since M is eventually semisimple it follows that K has finite uniform dimension. Now let N be a CS-module such that N is a direct summand of M. Because N is eventually semisimple, we can apply [CY, Lemma 2] to obtain submodules N_1 and N_2 of N such that $N = N_1 \oplus N_2$, N_1 is semisimple and N_2 has finite uniform dimension. But direct summands of CS-modules are CS-modules (see [MM, Proposition 2.7]). Therefore N_2 is a finite direct sum of uniform modules. Therefore (ii) follows.

We shall write ACC (respectively, DCC) to denote the ascending (respectively, descending) chain condition. Moreover, for any ring R, we say that a module M satisfies the $(ACC)_e$ (respectively, $(DCC)_e$) when M satisfies the ACC (respectively, DCC) on essential submodules.

Armendariz [Ar, Proposition 1.2] proved that the class of modules which satisfy the $(DCC)_e$ is {s,q}-closed. In the next lemma, we give a different proof and show that the class of modules which satisfy the $(ACC)_e$ is also {s,q}-closed.

<u>Lemma 2.1.6</u>. Let R be a ring. Then the class of modules which satisfy the $(ACC)_e$ ($(DCC)_e$) is {s,q}-closed.

Proof. Let M be a right R-module. Let N and L be submodules of M such that $N \subseteq L \subseteq M$ and L/N is an essential submodule of M/N. Then L is an essential submodule of M. In this way ACC (or DCC) on essential submodules passes from M to M/N. Thus the class of modules which satisfy the $(ACC)_e$ ($(DCC)_e$) is q-closed.

Let N' be a complement of N in M. Then $N \oplus N' \leq_e M$. Let K be an essential submodule of N. Then, by [AF, Propositions 5.16 and 5.20], $K \oplus N' \leq_e M$. Using this fact, it is easy to show that ACC (or DCC) on essential submodules passes from M to its submodule N. Thus the class of modules which satisfy the $(ACC)_e$ ($(DCC)_e$) is s-closed.

<u>Corollary 2.1.7.</u> Let R be any ring. Let M be a module which satisfies the $(ACC)_e$ (respectively, $(DCC)_e$). Moreover, let N be any uniform submodule of M. Then N is Noetherian (respectively, Artinian).

<u>Proof</u>. By Lemma 2.1.6, N satisfies the $(ACC)_e$ (respectively, $(DCC)_e$), and hence, N is Noetherian (respectively, Artinian).

<u>Corollary 2.1.8.</u> Let R be any ring. Let M be a direct sum of modules which satisfy the property (P). Then

(i) M satisfies the $(ACC)_e$ if and only if M belongs to $\underline{C} \oplus \underline{N}$.

(ii) M satisfies the $(DCC)_e$ if and only if M belongs to $\underline{C} \oplus \underline{A}$.

Proof. (i) Suppose that M satisfies the $(ACC)_e$. By the proof of

[Gool, Proposition 3.6] (see also [DHW, Lemma 2]), M/(Soc M) is Noetherian. Hence, by [CY, Lemma 1], M is eventually semisimple. Thus, by Theorem 2.1.5, $M = M_1 \oplus M_2 \oplus M_3$ where M_1 is a semisimple module, M_2 a finite direct sum of uniform modules and M_3 a submodule of M with zero socle. Therefore M_3 is Noetherian, because it embeds in M/(Soc M). On the other hand, by Corollary 2.1.7, M_2 is Noetherian. Thus $M_2 \oplus M_3$ is Noetherian. Therefore $M \in \underline{C} \oplus \underline{N}$. Conversely, let $M \in \underline{C} \oplus \underline{N}$. Then (M/Soc M) is Noetherian. Thus, by [AF, Proposition 9.7], M satisfies the $(ACC)_e$.

(ii) Suppose that M satisfies the $(DCC)_e$. Then, by [Ar, Proposition 1.1], M/(Soc M) is Artinian. By the proof of part (i), $M \in \underline{C} \oplus \underline{A}$. Conversely, suppose that M belongs to $\underline{C} \oplus \underline{A}$. Then M/(Soc M) is Artinian. Thus, by [AF, Proposition 9.7], M satisfies the $(DCC)_e$.

§ 2.2. Modules satisfying the property (P^*).

Let R be a ring and M a right R-module. The module M will be called a *radical module* if M = Rad M , the Jacobson radical of M. Also, the module M is called a *local module* provided that

Rad M is a maximal submodule of M and contains every proper submodule of M. A submodule N of M is called *small in M*, and written $N \ll M$, provided $M \neq N + K$ for any proper submodule K of M. We shall call a module M a *dual CS-module* provided, for any submodule N of M, there exist submodules K and K' of M such that $M = K \oplus K'$, $K \subseteq N$ and $N \cap K'$ is small in K'. Note that dual CS-modules are called "modules with (D_1)" in [MM] and "modules with (C_1)" in [Os].

We shall say that a module M *satisfies the property* (P^*) provided, for any submodule N of M, there exists a direct summand K of M such that $K \subseteq N$ and $N/K \subseteq \text{Rad}\,(M/K)$. Clearly every radical module satisfies the property (P^*)

Before dealing with the first result in this section, we recall a useful lemma about some properties of small submodules.

<u>Lemma 2.2.1.</u> Let R be any ring and M be a right R-module. Let $\{N_i : i \in I\}$ be the set of all small submodules of M. Then, for all $i \in I$,

(i) K_i is small in M for any submodule K_i of N_i,

(ii) $(N_i + L)/L$ is small in M/L for any submodule L of M,

(iii) $(N_i \cap K)$ is small in K for any direct summand K of M,

(iv) $\sum \{N_i : i \in I'\}$ is a submodule of M and small in M for any finite subset I' of I.

(v) Rad M = $\sum \{N_i : i \in I\}$.

Proof. See [AF, Proposition 5.17 and Proposition 9.13].

An alternative characterization of modules which satisfy the property (P^*) is given in the following lemma.

<u>Lemma 2.2.2.</u> Let R be any ring and M a right R-module. Then M satisfies the property (P^*) if and only if for any submodule N of M there exist submodules K and K' of M such that $M = K \oplus K'$, $K \subseteq N$ and $N \cap K' \subseteq \text{Rad } K'$.

Proof. Suppose that the module M satisfies the property (P^*). Let N be any submodule of M. Then there exist submodules K, K' of M such that $M = K \oplus K'$, $K \subseteq N$ and $N/K \subseteq \text{Rad } (M/K)$. Suppose that $K' = \text{Rad } K'$. Then it is clear that $N \cap K' \subseteq \text{Rad } K'$. Now suppose that $K' \neq \text{Rad } K'$. Then there exists a maximal submodule P of K', and hence, $(K \oplus P)/K$ is a maximal submodule of M/K. Consequently, $N/K \subseteq (K \oplus P)/K$. It follows that $N \subseteq K \oplus P$, and hence, $N \cap K'$ is contained in P. Thus every maximal submodule of K' contains $N \cap K'$. Therefore $N \cap K' \subseteq \text{Rad } K'$.

Conversely, suppose that M has the stated condition. Let N be any submodule of M. Then there exist submodules K, K' of M such that $M = K \oplus K'$, $K \subseteq N$ and $N \cap K' \subseteq \text{Rad } K'$. If M/K is a radical module then $N/K \subseteq \text{Rad } (M/K)$. Suppose that $\text{Rad } (M/K)$ is not equal to M/K. Then there exists a submodule Q of M such that $K \subseteq Q$ and Q/K is a maximal submodule of M/K. Therefore $Q = K \oplus (Q \cap K')$ and $Q \cap K'$ is a maximal submodule of K'. Thus $N \cap K' \subseteq Q \cap K'$. But this implies $N = K \oplus (N \cap K') \subseteq Q$. Hence $N/K \subseteq Q/K$. Therefore every maximal submodule of M/K contains N/K, and hence, $N/K \subseteq \text{Rad } (M/K)$ and M has (P^*).

Note that, by Lemma 2.2.2, local modules and dual CS-modules satisfy the property (P^*).

We know that the property (P) is inherited by direct summands (see Lemma 2.1.2). In the next lemma, we show that the property (P^*) is also inherited by direct summands.

<u>Lemma 2.2.3.</u> Let R be a ring. Let M be a right R-module which satisfies the propert (P^*). Let K be any direct summand of M. Then K satisfies the property (P^*).

Proof. Suppose that M satisfies the property (P^*). There exists a submodule K' of M such that $M = K \oplus K'$. Let L be a submodule of K. Then, by Lemma 2.2.2, there exist submodules N, N' of M such that $M = N \oplus N'$, $N \subseteq L$ and $L \cap N' \subseteq \text{Rad } N'$. By the modular law, $K = N \oplus (K \cap N')$. Now we have to show that $L \cap K \cap N'$ is contained in $\text{Rad } (K \cap N')$. Note that $L \cap K \cap N' = L \cap N'$. If $\text{Rad } (K \cap N') = K \cap N'$, then $L \cap N' \subseteq \text{Rad } (K \cap N')$. Otherwise, there exists a maximal submodule P of $K \cap N'$. Thus $N \oplus P \oplus K'$ is a maximal submodule of M. But, by [AF, Proposition 9.19], $\text{Rad } N' \subseteq \text{Rad } M$, and hence, $L \cap N' \subseteq N \oplus P \oplus K'$. Thus $L \cap N'$ is contained in P. Hence $L \cap N' \subseteq \text{Rad } (K \cap N')$. So K has (P^*).

Let N be a submodule of any module M. A submodule K of M is called a *supplement of N in M* provided $M = N + K$ and $M \neq N + L$ for any proper submodule L of K. It is easy to check that K is a supplement of N in M if and only if $M = N + K$ and $N \cap K$ is small in K. A submodule K of M is called a *supplement submodule* of M provided there exists a submodule N of M such that K is a supplement of N in M. Note that in any module M zero and M are supplement submodules of M. A module M is called *supplemented* provided that every submodule of M has a supplement in M. It is clear that semisimple modules are supplemented. Moreover, any Artinian module is supplemented (see [Mi, Prop. 1.6]). On the other hand, as a module over itself, the ring of integers is not supplemented (see Example 3.3.2).

In Section 2.1, we showed that modules which satisfy the property (P) are built up from CS-modules and modules with zero socle (see Proposition 2.1.3). We will prove the analogue of this result in the following Proposition.

Proposition 2.2.4. Let R be any ring and M a right R-module which satisfies the property (P^*). Moreover, suppose that the radical of M has a supplement in M. Then $M = M_1 \oplus M_2$ such that M_1 is a dual CS-module which has a small radical and M_2 is a radical module.

Proof. Suppose that M satisfies (P^*) and Rad M has a supplement in M. There exists a submodule N of M such that $M = N + \text{Rad } M$ and $M \neq L + \text{Rad } M$ for any proper submodule L of N. This means that $N \cap \text{Rad } M$ is small in N. There exist submodules K, K' of M such that $M = K \oplus K'$, $K \subseteq N$ and $N \cap K' \subseteq \text{Rad } K'$ (see Lemma 2.2.2). By [AF, Proposition 9.19], $\text{Rad } K' \subseteq \text{Rad } M$, and hence, $N \cap K' \subseteq \text{Rad } M$. But $N = K \oplus (N \cap K')$. Therefore $N = K + \text{Rad } M$, and hence, $M = K + \text{Rad } M$. Thus $N = K$, and hence, $M = N \oplus K'$. By [AF, Proposition 9.19], $\text{Rad } N \subseteq (N \cap \text{Rad } M) \ll N$. Therefore N has a small radical. Moreover, by Lemma 2.2.3, N satisfies (P^*). Let L be any submodule of N. Then there exist submodules P, P' of N such that $N = P \oplus P'$, $P \subseteq L$ and $L \cap P' \subseteq \text{Rad } P'$. But we know that $\text{Rad } P' \subseteq \text{Rad } N \ll N$. Thus $L \cap P'$ is small in P'. Hence N is a dual CS-module. We now consider the submodule K'. We have $M = N + \text{Rad } M = N + \text{Rad } N + \text{Rad } K' = N \oplus \text{Rad } K'$. But we know that $M = N \oplus K'$. Therefore $K' = \text{Rad } K'$.

Recall that $(P) \oplus (P) \neq (P)$. The analogue result is also true as we will prove in the following example.

Example 2.2.5. Let R be the ring of integers $\mathbb{Z}$. For any prime number p, Let $M_1 = \mathbb{Z}/\mathbb{Z}p$. Moreover, let $M_2 = \mathbb{Q}_{\mathbb{Z}}$ the rational numbers. Then M_1 and M_2 satisfy the property (P^*). On the other hand $M = M_1 \oplus M_2$ does not satisfy the property (P^*).

Proof. Since M_1 is a semisimple module, then by [AF, Theorem 9.6] and Lemma 2.2.2, M_1 satisfies (P^*). Since M_2 is a radical module, then M_2 satisfies (P^*). Let N be the submodule $M_1 \oplus \mathbb{Z}$. Since $\mathbb{Z}$ is essential in M_2, then N is essential in M. But N is a proper submodule of M. Therefore N is not a direct summand of M. On the other hand $N \not\subseteq \text{Rad } M = M_2$. Thus M does not satisfies the property (P^*).

Note that, in Example 2.2.5, M_1 is a dual CS-module and the radical of M has a supplement M_1 in M. Therefore the converse of Proposition 2.2.4 is not true.

Lemma 2.2.6. Let R be any ring and M a right R-module which satisfies the property (P^*). Let L be a submodule of M such that $L \cap \text{Rad } M = 0$. Then L is a semisimple module.

Proof. Suppose that M satisfies the property (P^*). Let L be a submodule of M such that $L \cap \text{Rad } M = 0$. Let L' be any submodule of L. By Lemma 2.2.2, there exist submodules K, K' of M such that $M = K \oplus K'$, $K \subseteq L'$ and $L' \cap K' \subseteq \text{Rad } K'$. But, by [AF, Proposition 9.19], $\text{Rad } K' \subseteq \text{Rad } M$. Thus $L' \cap K' \subseteq L \cap \text{Rad } M$, and hence, $L' \cap K' = 0$. On the other hand, $L' = K \oplus (L' \cap K')$. Thus $L' = K$, and L' is a direct summand of M. Thus L' is a direct summand of L. Therefore, by [AF, Theorem 9.6], L is semisimple.

<u>Corollary 2.2.7.</u> Let R be any ring and M a right R-module which satisfies the property (P^*). Then $M = M_1 \oplus M_2$ where M_1 is a semisimple module and M_2 is a module with essential radical.

Proof. Suppose that M satisfies the property (P^*). By Zorn's Lemma there exists a complement submodule N of Rad M in M. Thus $N \cap \text{Rad } M = 0$. There exist submodules K and K' of M such that $M = K \oplus K'$, $K \subseteq N$ and $N \cap K' \subseteq \text{Rad } K'$. But, by [AF, Proposition 9.19], $\text{Rad } K' \subseteq \text{Rad } M$. Hence $N \cap K' = 0$. By the maximality of K with respect to $K \cap K' = 0$, $N = K$. It is clear that $\text{Rad } N = 0$. Moreover, by Lemma 2.2.3, N satisfies (P^*). Thus by Lemma 2.2.6 N is a semisimple module, and hence, K is a semisimple module.

Therefore Rad M = Rad K'. Since N is a complement of Rad M in M, then N ⊕ Rad M is essential in M, and hence, K ⊕ Rad K' is essential in M. Therefore Rad K' is essential in K'.

It is proved in [Os, Theorem 3.5] (see also [MM, Theorem 4.15]) that if M is any dual CS module such that for every epimorphism from M onto a direct summand of M splits, then M is a direct sum of local modules and radical modules. In particular, for such a module M, Rad M has finite uniform dimension if and only if M is a direct sum of a semisimple module and a module with finite uniform dimension. We generalise this fact in our next result.

Theorem 2.2.8. Let R be any ring and M a right R-module. Then the following statements are equivalent.

(i) Rad M has finite uniform dimension and M is a direct sum of modules satisfying the property (P^*).

(ii) $M = M_1 \oplus M_2 \oplus M_3$ where M_1 is a semisimple module, M_2 is a radical module which has finite uniform dimension and M_3 is a module which has finite uniform dimension and which is a finite direct sum of local submodules of M.

Proof. Suppose that (ii) is true. Then it is clear that M is a direct sum of modules satisfying the property (P^*). Moreover, by [AF, Proposition 9.19], Rad M has finite uniform dimension.

Conversely, suppose that Rad M has finite uniform dimension and $M = \oplus_\Lambda M_\lambda$ such that , for each $\lambda \in \Lambda$, M_λ satisfies (P^*). By [AF, Proposition 9.19], Rad $M = \oplus_\Lambda$ (Rad M_λ). Since the radical of M has finite uniform dimension, then there exists a positive integer k such that Rad $M_\lambda = 0$ for all $\lambda > k$. Thus, by Lemma 2.2.6, M_λ is semisimple for all $\lambda > k$. Therefore, it remains to prove that if N is a module which satisfies the property (P^*) and its radical has finite uniform dimension, then N is a direct sum of N_1, N_2 and N_3 where N_1 is a semisimple module, N_2 is a radical module with finite uniform dimension and N_3 is a module with finite uniform dimension and a direct sum of local submodules.

Suppose that N has the stated conditions. Then, by Lemma 2.2.3 and Corollary 2.2.7, we can suppose without loss of generality that N has finite uniform dimension. Suppose that N is uniform. Suppose that N is not a radical module. Then there exists an element $m \in N \setminus$ Rad N. Therefore, by Lemma 2.2.2, there exist submodules K, K' of N such that $N = K \oplus K'$, $K \subseteq mR$

and $mR \cap K' \subseteq \text{Rad } K'$. Suppose that $K = 0$. Then $K' = N$, and hence, $m \in \text{Rad } N$, a contradiction. Therefore $K \neq 0$, and hence, $K' = 0$, by the hypothesis. In this case, $N = K \subseteq mR$. It follows that N is a local module.

Now suppose that u-dim $N > 2$. Again we shall suppose that N is not a radical module. Then there exists an element x such that $x \in N \setminus \text{Rad } N$. By hypothesis, there exist submodules K_x, L_x of N such that $N = K_x \oplus L_x$, $K_x \subseteq xR$ and $xR \cap L_x \subseteq \text{Rad } L_x$. But $x \notin \text{Rad } N$. Therefore $K_x \neq 0$. If $L_x \neq 0$ then K_x and L_x both have smaller uniform dimensions than N. Moreover, by Lemma 2.2.3, both K_x and L_x satisfy the property (P^*). By induction on the uniform dimension of N, K_x and L_x are both direct sums of semisimple modules, radical modules and local modules, and hence so also is N.

Now suppose that $L_x = 0$ for all $x \in N \setminus \text{Rad } N$. Then $N = xR$ for all $x \in N \setminus \text{Rad } N$. It follows that N is a local module. Therefore, in any case, N has the required structure.

Chapter 3.

The classes $h^*\underline{X}$, $d^*\underline{X}$ and $e^*\underline{X}$.

Let R be any ring. For any class $\underline{X}$ of right R-modules, we define classes $h^*\underline{X}$, $d^*\underline{X}$ and $e^*\underline{X}$ dual to the classes $h\underline{X}$, $d\underline{X}$ and $e\underline{X}$ respectively. First of all, the class $h^*\underline{X}$ (respectively $e^*\underline{X}$) consists of all modules M such that every (small) submodule of M belongs to $\underline{X}$. Note that $\underline{X} = h^*\underline{X}$ if and only if $\underline{X}$ is s-closed. Next, we define $d^*\underline{X}$ to be the class of modules M such that for each submodule N of M, N contains a direct summand K of M such that the factor module N/K belongs to $\underline{X}$.

§ 3.1. General Properties.

Throughout this section, we shall consider classes of modules over any ring R.

We start this section with an elementary result which gives an alternative characterization of the class $d^*\underline{X}$.

Lemma 3.1.1. Let R be any ring. Let $\underline{X}$ be any class of right R-modules. Then a module M belongs to $d^*\underline{X}$ if and only if, for every submodule N of M, there exist a direct summand K of M and an $\underline{X}$-submodule L of M such that $N = K \oplus L$.

Proof. The sufficiency is clear. Conversely, suppose that M belongs to $d^*\underline{X}$ and N is a submodule of M. By hypothesis, there exist submodules K, K' of M such that $M = K \oplus K'$, $K \subseteq N$ and $N/K \in \underline{X}$. By the Modular Law, $N = K \oplus (N \cap K')$. Note that $N \cap K' \cong N/K \in \underline{X}$. Therefore M has the stated condition.

<u>Proposition 3.1.2.</u> Let R be any ring. Then

(i) $\underline{Z} = h^*\underline{Z}$,

(ii) $e^*\underline{Z} = e^*\underline{I} = \{M : \text{Rad } M = 0\}$,

(iii) $\underline{C} = d^*\underline{Z} = d^*\underline{I}$,

(iv) $h^*\underline{I} = \underline{I} \cap \underline{C}$ and

(v) $\underline{P} \cap d^*\underline{P} = h^*\underline{P}$.

Proof. (i), (ii) and (iv) are elementary (see [AF, Theorem 9.6 and Proposition 9.13]).

(iii) By [AF, Theorem 9.6], $\underline{C} = d^*\underline{Z} \subseteq d^*\underline{I}$. Suppose that M belongs to $d^*\underline{I}$ and N a submodule of M. There exist submodules K and K' of M such that $M = K \oplus K'$, $K \subseteq N$ and N/K is injective.

Thus $N \cap K'$ is injective, and hence, $N \cap K'$ is a direct summand of K'. Hence N is a direct summand of M. By [AF, Theorem 9.6], M is a semisimple module.

(v) Suppose that M belongs to $\underline{P} \cap d^*\underline{P}$ and N is a submodule of M. By Lemma 3.1.1, $N = K \oplus L$ for some direct summand K of M and a projective submodule L. Since M is projective, then K is projective, and hence, N is projective. Therefore M belongs to $h^*\underline{P}$. It follows that $\underline{P} \cap d^*\underline{P} \subseteq h^*\underline{P}$. On the other hand, it is clear that $h^*\underline{P} \subseteq \underline{P}$ and $h^*\underline{P} \subseteq d^*\underline{P}$.

The next two propositions contain some basic information about the classes $h^*\underline{X}$, $e^*\underline{X}$ and $d^*\underline{X}$. The first one is the analogue of Proposition 1.1.1 and the second one contains the analogue of Proposition 1.1.2.

<u>Proposition 3.1.3.</u> Let R be any ring. Let $\underline{X}$ and $\underline{Y}$ be any classes of right R-modules. Then

(i) $h^*\underline{X} \subseteq d^*\underline{X} \subseteq e^*\underline{X}$,

(ii) If $\underline{X} \subseteq \underline{Y}$, then $h^*\underline{X} \subseteq h^*\underline{Y}$, $d^*\underline{X} \subseteq d^*\underline{Y}$ and $e^*\underline{X} \subseteq e^*\underline{Y}$,

(iii) $\underline{C} \subseteq d^*\underline{X}$,

(iv) $h^*\underline{X} = h^*(h^*\underline{X}) \subseteq \underline{X}$,

(v) $d^*(\underline{I} \oplus \underline{X}) = d^*\underline{X}$,

(vi) $\underline{I} \cap d^*\underline{X} \subseteq h^*(\underline{I} \oplus \underline{X})$ and

(vii) $e^*(\underline{I} \oplus \underline{X}) = e^*\underline{X}$.

Proof. (i), (ii) and (iv) are elementary. (iii) is an immediate consequence of (ii) and Proposition 3.1.2 (iii).

(v) Let $M \in d^*(\underline{I} \oplus \underline{X})$. Let N be any submodule of M. Then there exist submodules K, K' of M such that $M = K \oplus K'$, $K \subseteq N$ and N/K belongs to $\underline{I} \oplus \underline{X}$. Thus $N = K \oplus (N \cap K')$ and $N \cap K'$ has the form $I \oplus L$ for some injective submodule I and $\underline{X}$-submodule L, since $N \cap K' \cong N/K$. Thus I is a direct summand of K', and hence, $K \oplus I$ is a direct summand of M. But $N/(K \oplus I) \cong L \in \underline{X}$. It follows that $M \in d^*\underline{X}$. Thus $d^*(\underline{I} \oplus \underline{X}) \subseteq d^*\underline{X}$. On the other hand, by (ii), $d^*\underline{X} \subseteq d^*(\underline{I} \oplus \underline{X})$.

(vi) Suppose that $M \in \underline{I} \cap d^*\underline{X}$. Let N be a submodule of M. Then $M = K \oplus K'$, $K \subseteq N$ and $N/K \in \underline{X}$. Since M is injective then K is injective. But $N = K \oplus (N \cap K')$ and $(N \cap K') \cong N/K \in \underline{X}$. Thus N belongs to $\underline{I} \oplus \underline{X}$.

(vii) By (ii), $e^*\underline{X} \subseteq e^*(\underline{I} \oplus \underline{X})$. Moreover, $e^*(\underline{I} \oplus \underline{X}) \subseteq e^*\underline{X}$ is a consequence of the fact that injective small submodules of any module are zero.

<u>Proposition 3.1.4.</u> Let R be any ring. Let $\underline{X}$ be any class of right R-modules. Then

(i) $h^*\underline{X}$, $e^*\underline{X}$ and $d^*\underline{X}$ are all s-closed,

(ii) $h^*\underline{X}$ is q-closed, if $\underline{X}$ is q-closed,

(iii) $h^*\underline{X}$ is p-closed, if $\underline{X}$ is p-closed,

(iv) $e^*\underline{X} = (h^*\underline{X})(e^*\underline{X})$, if $\underline{X}$ is p-closed, and

(v) $h^*\underline{X} = \underline{X} \cap d^*\underline{X}$, if $\underline{X}$ is (p,q)-closed.

Proof. (i) The classes $h^*\underline{X}$ and $e^*\underline{X}$ are trivially s-closed. Suppose that M belongs to $d^*\underline{X}$. Let N be a submodule of M and H be a submodule of N. By Lemma 3.1.1, $H = K \oplus L$ for some direct summand K of M and $\underline{X}$-submodule L of M. But K is also a direct summand of N. Therefore, by Lemma 3.1.1, N belongs to $d^*\underline{X}$. It follows that $d^*\underline{X}$ is s-closed.

(ii) Suppose that $\underline{X}$ is q-closed. Let M be an $h^*\underline{X}$-module and N be a submodule of M. Any submodule of M/N has the form K/N, where K is a submodule of M containing N. Thus $K \in \underline{X}$, and hence, $K/N \in \underline{X}$. Therefore $M/N \in h^*\underline{X}$.

(iii) Suppose that $\underline{X}$ is p-closed. Let K be a submodule of a module M such that K and M/K are both $h^*\underline{X}$-modules. Let N be a submodule of M. Then $N/(N \cap K) \cong (N + K)/K \in \underline{X}$. Hence $N/(N \cap K)$

is an $\underline{X}$-module. Since $K \in h^*\underline{X}$ then $N \cap K \in \underline{X}$. Therefore $N \in \underline{X}$, and hence, $M \in h^*\underline{X}$.

(iv) It is clear that $e^*\underline{X} \subseteq (h^*\underline{X})(e^*\underline{X})$. Conversely, suppose that $M \in (h^*\underline{X})(e^*\underline{X})$. Then there exists a submodule K of M such that $K \in h^*\underline{X}$ and $M/K \in e^*\underline{X}$. Let N be any small submodule of M. Then, by Lemma 2.2.1, $(N + K)/K \ll M/K$. Hence $(N + K)/K \in \underline{X}$. But $(N + K)/K \cong N/(N \cap K)$ and $N \cap K \in \underline{X}$. Thus $N \in \underline{X}$. So $M \in e^*\underline{X}$.

(v) Clearly $h^*\underline{X} \subseteq \underline{X} \cap d^*\underline{X}$, for any class $\underline{X}$. Suppose that $\underline{X}$ is {p,q}-closed. Let $M \in \underline{X} \cap d^*\underline{X}$ and N be any submodule of M. By Lemma 3.1.1, there exists a direct summand K of M and an $\underline{X}$-submodule L such that $N = K \oplus L$. Since $\underline{X}$ is q-closed then K belongs to $\underline{X}$, and hence, N/L belongs to $\underline{X}$. Therefore $N \in \underline{X}$. It follows that $M \in h^*\underline{X}$.

In the following example we show that the class $e^*\underline{X}$ is neither q-closed nor p-closed, even when $\underline{X}$ is {s,q,p}-closed. Thus the analogue of Proposition 1.1.2 (ii) is not true

<u>Example 3.1.5</u>. Let R be the ring of integers. Then the class $e^*\underline{Z}$ is neither q-closed nor p-closed.

Proof. Let F be a free right R-module of non-zero rank. Then, by [AF, Proposition 17.10], Rad F = F J(R) = 0. Hence, by Lemma 2.2.1, F belongs to $e^*\underline{Z}$. Let p be any prime number. Therefore

$$\mathrm{Rad}(F/Fp^2) = Fp/Fp^2 \neq 0.$$

Therefore $F/Fp^2 \notin e^*\underline{Z}$ (see Lemma 2.2.1), and hence, $e^*\underline{Z}$ is not q-closed. On the other hand, F/Fp and Fp/Fp^2 are semisimple modules. Therefore $F/Fp \in e^*\underline{Z}$ and $Fp/Fp^2 \in e^*\underline{Z}$. Hence $e^*\underline{Z}$ is not p-closed.

Note that $\underline{C} \oplus h^*\underline{X} \subseteq h^*\underline{X}$ implies $\underline{C} \subseteq \underline{X}$, (see Proposition 3.1.4), which is not true in general. Hence $\underline{C} \oplus h^*\underline{X} \neq h^*\underline{X}$. But the classes $d^*\underline{X}$ and $e^*\underline{X}$ behave nicer as we will show in the next proposition which contains the analogue of Proposition 1.1.3.

__Proposition 3.1.6.__ Let R be any ring. Let $\underline{X}$ be any class of right R-modules. Then

(i) $\underline{C} \oplus h^*\underline{X} \subseteq h^*(\underline{C} \oplus \underline{X}) \subsetneq \underline{C} \oplus \underline{X}$,

(ii) $\underline{C} \oplus d^*\underline{X} = d^*\underline{X}$ and

(iii) $\underline{C} \oplus e^*\underline{X} = e^*\underline{X}$.

Proof. (i) Let $M = M_1 \oplus M_2$ where M_1 is a semisimple module and M_2 belongs to $h^*\underline{X}$. Let N be a submodule of M. By [AF, Theorem 9.6], $M_1 = K \oplus (N \cap M_1)$ for some submodule K of M_1. Therefore $M = K \oplus (N \cap M_1) \oplus M_2$. Thus, by modular law, $N = (N \cap M_1) \oplus L$ where $L = N \cap (K \oplus M_2)$. Note that $(K \oplus M_2)/K \cong M_2 \in h^*\underline{X}$. Thus $(L + K)/K \in \underline{X}$. But $L \cap K \subseteq (N \cap M_1) \cap K = 0$. So $L \cong (L + K)/K$ which belongs to $\underline{X}$. Thus $N \in \underline{C} \oplus \underline{X}$. Therefore $M \in h^*(\underline{C} \oplus \underline{X})$. It is clear that $h^*(\underline{C} \oplus \underline{X}) \subseteq \underline{C} \oplus \underline{X}$.

(ii) It is clear that $d^*\underline{X} \subseteq \underline{C} \oplus d^*\underline{X}$. Suppose that a module M belongs to $\underline{C} \oplus d^*\underline{X}$. Then $M = M_1 \oplus M_2$ where M_1 is a semisimple module and M_2 belongs to $d^*\underline{X}$. Let N be a submodule of M. Then, by [AF, Theorem 9.6], $M_1 = (N \cap M_1) \oplus M'$ for some submodule M' of M_1. Thus $M = (N \cap M_1) \oplus M' \oplus M_2$, and hence, $N = (N \cap M_1) \oplus H$ where $H = N \cap (M' \oplus M_2)$. Since $(M_2 \oplus M')/M' \in d^*\underline{X}$, then

$$(H + M')/M' = (K/M') \oplus (L/M')$$

for some submodules K and L containing M' such that K/M' is a direct summand of $(M_2 \oplus M')/M'$ and $L/M' \in \underline{X}$ (see Lemma 3.1.1). Thus K is a direct summand of M. But $K = M' \oplus (K \cap H)$, so that $(K \cap H)$ is also a direct summand of M. Thus $(N \cap M_1) \oplus (K \cap H)$ is a direct summand of M. On the other hand,

$N/[(N \cap M_1) \oplus (K \cap H)] \cong H/(K \cap H) \cong (H + K)/K$ which is equal to $(H + M')/K \cong L/M' \in \underline{X}$. Thus $M \in d^*\underline{X}$.

(iii) It is clear that $e^*\underline{X} \subseteq \underline{C} \oplus e^*\underline{X}$. Suppose that a module M belongs to $\underline{C} \oplus e^*\underline{X}$. Then $M = M_1 \oplus M_2$ where M_1 is semisimple and $M_2 \in e^*\underline{X}$. Let N be a small submodule of M. By Lemma 2.2.1, $N \cap M_1$ is small in M_1. Hence $N \cap M_1 = 0$. Thus $N \cong (N + M_1)/M_1$ which is small in $M/M_1 \cong M_2$ (see Lemma 2.2.1). Thus $N \in \underline{X}$. We conclude that $M \in e^*\underline{X}$.

<u>Corollary 3.1.7.</u> Let R be any ring. Let $\underline{X}$ be any s-closed class of right R-modules. Then $\underline{C} \oplus \underline{X} \subseteq d^*\underline{X}$.

Proof. By Proposition 3.1.3 (i) and Proposition 3.1.6 (ii),

$$\underline{C} \oplus \underline{X} = \underline{C} \oplus h^*\underline{X} \subseteq \underline{C} \oplus d^*\underline{X} = d^*\underline{X}.$$

The next two propositions contain the analogue of Proposition 1.1.4. In these two propositions, we examine how different classes behave under direct sums.

Proposition 3.1.8. Let R be any ring. Let $\underline{X}$ be a p-closed class of right R-modules. Then

(i) $h^*\underline{X} \oplus h^*\underline{X} = h^*\underline{X}$ and

(ii) $e^*\underline{X} \oplus e^*\underline{X} = e^*\underline{X}$.

Proof. (i) It is clear that $h^*\underline{X} \subseteq h^*\underline{X} \oplus h^*\underline{X}$. By Proposition 3.1.4 (iii), the converse is true.

(ii) It is clear that $e^*\underline{X} \subseteq e^*\underline{X} \oplus e^*\underline{X}$. Let $M = M_1 \oplus M_2$ where both M_1 and M_2 belong to $e^*\underline{X}$. Let N be a small submodule of M. By Lemma 2.2.1, $N \cap M_1$ is small in M_1. Hence $N \cap M_1 \in \underline{X}$. But, by Lemma 2.2.1, $(N + M_1)/M_1$ is small in $M/M_1 \cong M_2$. Hence $(N + M_1)/M_1 \in \underline{X}$. It follows that $N \in \underline{X}$. Thus $M \in e^*\underline{X}$.

In the next chapter (see Example 4.1.9) we shall give a class of modules $\underline{X}$ which is (s,q,p)-closed but,

$$d^*\underline{X} \oplus d^*\underline{X} \neq d^*\underline{X}.$$

This fact is interesting in view of the following proposition.

Proposition 3.1.9. Let R be any ring. Let $\underline{X}$ be a p-closed class of right R-modules. Then

(i) $d^*\underline{X} = d^*\underline{X} \oplus h^*\underline{X}$ and

(ii) $d^*\underline{X} = d^*\underline{X} \oplus (\underline{P} \cap d^*\underline{X})$.

Proof. (i) It is clear that $d^*\underline{X} \subseteq d^*\underline{X} \oplus h^*\underline{X}$. Suppose that M belongs to $d^*\underline{X} \oplus h^*\underline{X}$. Then $M = M_1 \oplus M_2$ where M_1 belongs to $d^*\underline{X}$ and M_2 belongs to $h^*\underline{X}$. Let N be any submodule of M. Then

$$N + M_1 = M_1 \oplus [(N + M_1) \cap M_2].$$

Therefore $(N + M_1)/M_1 \in \underline{X}$, and hence, $N/(N \cap M_1) \in \underline{X}$. By Lemma 3.1.1, there exist a direct summand K and an $\underline{X}$-submodule L of M_1 such that $N \cap M_1 = K \oplus L$. Note that K is a direct summand of M and $(N \cap M_1)/K \in \underline{X}$. Thus $N/K \in \underline{X}$. Therefore $M \in d^*\underline{X}$.

(ii) It is clear that $d^*\underline{X} \subseteq d^*\underline{X} \oplus (\underline{P} \cap d^*\underline{X})$. Conversely, suppose that M belongs to $d^*\underline{X} \oplus (\underline{P} \cap d^*\underline{X})$. Then $M = M_1 \oplus M_2$ where M_1 belongs to $\underline{P} \cap d^*\underline{X}$ and M_2 belongs to $d^*\underline{X}$. Let N be a submodule of M. Note that $M/M_2 \in \underline{P} \cap d^*\underline{X}$. Therefore there exist submodules K and K' of M containing M_2 such that

$$M/M_2 = (K/M_2) \oplus (K'/M_2),$$

$K \subseteq N + M_2$ and $(N + M_2)/K \in \underline{X}$. Note that $K = (K \cap N) + M_2$ and

$$K/M_2 = [(K \cap N) + M_2]/M_2 \cong (K \cap N)/(M_2 \cap N).$$

Therefore $(K \cap N)/(M_2 \cap N)$ is a projective module. Therefore

$$(K \cap N) = H \oplus (M_2 \cap N)$$

for some submodule H of M. Therefore $M = H \oplus K'$. But $M_2 \in d^*\underline{X}$. Hence $M_2 = F \oplus F'$ for some submodules F and F' of M_2 such that $F \subseteq (N \cap M_2)$ and $(N \cap M_2)/F$ belongs to $\underline{X}$. Note that

$$M = H \oplus K' = H \oplus M_2 \oplus (K' \cap M_1) = H \oplus F \oplus F' \oplus (K' \cap M_1).$$

Since $N/(K \cap N) \cong (N + M_2)/K \in \underline{X}$, then $N/(K \cap N) \in \underline{X}$. On the other hand $(K \cap N)/(H \oplus F) = [H \oplus (N \cap M_2)]/(H \oplus F)$ which is isomorphic to the $\underline{X}$-module $(N \cap M_2)/F$. Thus $(K \cap N)/(H \oplus F)$ belongs to $\underline{X}$. Since $\underline{X}$ is p-closed we conclude that $N/(H \oplus F)$ belongs to $\underline{X}$. Therefore $M \in d^*\underline{X}$.

<u>Corollary 3.1.10.</u> Let R be any ring. Let $\underline{X}$ be any p-closed class of right R-modules. Then

$$d^*\underline{X} = d^*\underline{X} \oplus h^*\underline{X} \oplus (\underline{P} \cap d^*\underline{X}) \oplus \underline{C}.$$

Proof. By Proposition 3.1.6 and Proposition 3.1.9.

<u>§ 3.2. Modules with Noetherian or Artinian radical</u>.

Let R be a ring and M a right R-module. We shall say that M satisfies the $(ACC)_s$ (respectively, $(DCC)_s$) when M satisfies the ACC (respectively, DCC) on small submodules of M.

Let R be a ring and M a right R-module. It is known that M satisfies the $(ACC)_e$ if and only if M/(Soc M) is Noetherian (for example, see [DHW, Lemma 2]). Dually, M satisfies the $(DCC)_e$ if and only if M/(Soc M) is Artinian (see [Ar, Prop. 1.1]). We prove analogues of these results for the radical of M. Specifically, the module M satisfies the $(ACC)_s$ ($(DCC)_s$) if and only if Rad M is Noetherian (Artinian).

The next result was proved by Varadarajan [Va, Lemma 2.1]. We shall give a proof for completeness.

<u>Proposition 3.2.1.</u> Let R be any ring and M a right R-module. Then Rad M is Noetherian if and only if M satisfies the $(ACC)_s$.

Proof. The necessity is an immediate consequence of Lemma 2.2.1

Conversely, suppose that M satisfies the $(ACC)_s$. Then M contains a maximal small submodule K. Therefore, by Lemma 2.2.1 Rad M = K. Finally, by Lemma 2.2.1, Rad M is Noetherian.

A companion result of Proposition 3.2.1 is the following.

<u>Proposition 3.2.2</u>. Let R be any ring and M any right R-module. Then the following statements are equivalent.

(i) Rad M has finite uniform dimension.

(ii) M belongs to $e^{*}\underline{U}$ and there exists a positive integer k such that u-dim $N \leq k$ for every small submodule N of M.

(iii) M does not contain an infinite direct sum of non-zero small submodules.

Proof. Suppose that Rad M has finite uniform dimension. Suppose that N is a small submodule of M. Therefore, by Lemma 2.2.1, N has finite uniform dimension which smaller than u-dim Rad M.

Now assume statement (ii). Let $N_1 \oplus N_2 \oplus \dots$ be an infinite direct sum of non-zero small submodules of M. Then, by Lemma 2.2.1, $N_1 \oplus \dots \oplus N_{k+1}$ is small in M. On the other hand we have u-dim $(N_1 \oplus \dots \oplus N_{k+1}) \geq k + 1$, a contradiction. So M does not contain an infinite direct sum of non-zero small submodules.

Assume that statement (iii) is true. Suppose that Rad M does not have finite uniform dimension. Therefore there exist non-zero submodules K_i $(i = 1, 2, \dots)$ of Rad M such that $K_1 \oplus K_2 \oplus \dots \subseteq$ Rad M. For each $i \geq 1$, let k_i be a non-zero element of K_i. By Lemma 2.2.1, k_iR is small in M for all i.

Thus $k_1R + k_2R + \dots$ is an infinite direct sum of non-zero small submodules of M, a contradiction. Therefore Rad M has finite uniform dimension.

Recall the following result (see Theorem 1.2.1).

<u>Proposition 3.2.3</u>. Let R be any ring and M a right R-module. Then the following statements are equivalent.

(i) M/(Soc M) is Noetherian.

(ii) M belongs to $e\underline{N}$.

(iii) M satisfies the $(ACC)_e$.

In the following theorem, we prove that the dual of Proposition 3.2.3 is true.

<u>Theorem 3.2.4</u>. Let R be any ring and M a right R-module. Then the following statements are equivalent

(i) Rad M is Artinian.

(ii) M belongs to $e^*\underline{A}$.

(iii) M satisfies the $(DCC)_s$.

Proof. Suppose that Rad M is Artinian. Then, by Lemma 2.2.1, M belongs to $e^*\underline{A}$.

Now suppose that M belongs to $e^*\underline{A}$. Let $N_1 > N_2 > \ldots$ be a descending chain of small submodules N_i of M. Since N_1 is Artinian, then the chain is finite. So M satisfies the $(DCC)_S$.

Finally, suppose that M satisfies the $(DCC)_S$. Let N be a finitely generated submodule of Rad M. Then, by Lemma 2.2.1, N is small in M. Thus, by Lemma 2.2.1, N is Artinian. Therefore Rad M is locally Artinian. Now let K be any proper submodule of Rad M. Let $x \in (\text{Rad } M)\backslash K$. Then xR is artinian. On the other hand $(xR + K)/K \cong xR/(xR \cap K)$. Thus $(xR + K)/K$ is a non-zero Artinian module. It follows that (Rad M)/K has essential socle. Suppose that Rad M is not Artinian. By [AF, Proposition 10.10], the set Ω of submodules L of Rad M such that (Rad M)/L is not finitely cogenerated, is non-empty. Let $\{L_\lambda : \lambda \in \Lambda\}$ be any chain of submodules in Ω. Let $L = \cap_{\lambda \in \Lambda} L_\lambda$. Suppose that $L \notin \Omega$, then (Rad M)/L is finitely cogenerated, and hence, $L = L_\lambda$ for some $\lambda \in \Lambda$. Thus $L \in \Omega$, a contradiction. Therefore, by Zorn's Lemma, Ω has a minimal member, say P.

Let S denote the submodule of Rad M, containing P, such that S/P is the socle of (Rad M)/P. We have seen already that

S/P is an essential submodule of $(\text{Rad } M)/P$. Therefore, by [AF, Proposition 10.7], S/P is not finitely generated.

We next prove that P is small in M. Suppose that $M = P + Q$ for some submodule Q of M. So, by modular law, $S = P + (S \cap Q)$. Suppose that $P \cap Q \neq P$. Therefore $(\text{Rad } M)/(P \cap Q)$ is finitely cogenerated, by the choice of P. On the other hand,

$S/P = [P + (S \cap Q)]/P \cong (S \cap Q)/(P \cap Q) \subseteq \text{Soc}[(\text{Rad } M)/(P \cap Q)]$,

and hence S/P is finitely generated, a contradiction. Therefore $P = (P \cap Q) \subseteq Q$, and hence, $M = P + Q = Q$. So P is small in M.

Finally, we prove that S is small in M. Let $M = S + V$ for some submodule V of M. Then

$$M/(P + V) = (S + V)/(P + V) \cong S/[P + (S \cap V)].$$

Thus $M/(P + V)$ is semisimple. If $M \neq P + V$, then there exists a maximal submodule W of M such that $(P + V) \subseteq W$. On the other hand, $S \subseteq \text{Rad } M \subseteq W$. So $S + (P + V) \subseteq W$, and hence, $S + V \subseteq W$. Thus $M \subseteq W$, a contradiction, because $M \neq W$. Thus $M = P + V$, and hence, $M = V$, because P is small in M. Therefore S is small in M. Since M belongs to $e^*\underline{A}$, then S is Artinian. It follows that S/P is Artinian, and hence, S/P is finitely co-generated. But S/P is a semisimple module. Therefore S/P is finitely generated which is a contradiction. Therefore $\text{Rad } M$ is Artinian.

We shall call a submodule N of a module M *semimaximal* provided N is an intersection of M and a finite number of maximal submodules of M.

<u>Corollary 3.2.5.</u> Let R be a ring and M a right R-module. Then the following statements are equivalent for M.

(i) M is Artinian.

(ii) M satisfies both the $(DCC)_s$ and the DCC on semimaximal submodules.

(iii) M satisfies the $(DCC)_s$ and Rad M is semimaximal.

Proof. If M is Artinian, then clearly statement (ii) is true.

Suppose that M satisfies the DCC on semimaximal submodules. Let N be a minimal semimaximal submodule of M. It is clear that Rad M $\subseteq$ N. If M is a radical module, then Rad M = N. Suppose that M $\neq$ Rad M. If P is a maximal submodule of M, then N $\cap$ P is semimaximal, and hence, N = N $\cap$ P, so that N $\subseteq$ P. It follows that N $\subseteq$ Rad M. Hence N = Rad M. Thus, in any case, Rad M is semimaximal.

Finally, suppose that M satisfies the $(DCC)_s$ and Rad M is semimaximal. Then, by Theorem 3.2.4, Rad M is Artinian. If M is a radical module, then M is Artinian. Suppose that $M \neq \text{Rad } M$. Then $\text{Rad } M = P_1 \cap \ldots \cap P_n$ for some positive integer n and maximal submodules P_i $(1 \leq i \leq n)$ of M. Therefore M/(Rad M) embeds in $(M/P_1) \oplus \ldots \oplus (M/P_n)$ which is a finitely generated semisimple module. Therefore M/(Rad M) is Artinian. Hence M is Artinian, because the class $\underline{A}$ is p-closed.

<u>Corollary 3.2.6.</u> Let R be any ring and M a right R-module which is a direct sum of modules, each having (P^*). Suppose that R satisfies the $(ACC)_s$ (respectively, $(DCC)_s$). Then, $M = M_1 \oplus M_2$ for some semisimple module M_1 and a Noetherian (respectively, Artinian) module M_2.

Proof. By Proposition 3.2.1, Theorem 3.2.4 and Theorem 2.2.8.

Note that the module M is Noetherian if and only if M satisfies the $(ACC)_e$ and the ACC on complement submodules. To see why this is so, recall that a module M satisfies the ACC on complement submodules if and only if M has finite uniform dimension ([Go, Lemma 1.3]). Thus, if a module M satisfies the ACC on complement submodules, then Soc M is Noetherian, and if, in addition, it satisfies the $(ACC)_e$, then M is Noetherian (see Proposition 3.2.3).

The analogue of the fact that $(ACC)_e$ and ACC on complement submodules gives ACC on all submodules will be proved in the next lemma.

Lemma 3.2.7. Let R be any ring. Let M be a right R-module which is a supplemented module and satisfies the DCC on supplement submodules. Then M/(Rad M) is a finitely generated semisimple module.

Proof. Suppose that M satisfies the stated conditions. Let N be any submodule of M containing Rad M. There exists a supplement K of N in M. Since $N \cap K$ is small in K, then $N \cap K$ is small in M, and hence, $N \cap K \subseteq \text{Rad } M$. Thus M/(Rad M) can be decomposed as follow $M/(\text{Rad } M) = \{N/(\text{Rad } M)\} \oplus \{(K + \text{Rad } M)/(\text{Rad } M)\}$. Thus every submodule of M/(Rad M) is a direct summand, and hence, by [AF, Theorem 9.6], M/(Rad M) is semisimple.

Now suppose that $\text{Rad } M \subseteq N_1 \subseteq N_2 \subseteq \ldots$ is an ascending chain of submodules of M. Because M is supplemented, there exists a descending chain of submodules $K_1 \supseteq K_2 \supseteq \dots$ such that K_i is a supplement of N_i in M for each $i \geq 1$. Therefore there exists a positive integer t such that $K_t = K_{t+1} = \dots$. Because $M/(\text{Rad } M) = \{N_i/(\text{Rad } M)\} \oplus \{(K_i + \text{Rad } M)/(\text{Rad } M)\}$, for all $i \geq t$, it follows that $N_t = N_{t+1} = N_{t+2} = \dots$. Therefore M/(Rad M) is Noetherian, and hence, finitely generated.

<u>Theorem 3.2.8.</u> Let R be any ring and M a right R-module. Then M is Artinian if and only if M is supplemented and M satisfies the $(DCC)_s$ and the DCC on supplement submodules.

Proof. The necessity is clear. Conversely, suppose that M is a supplemented module which satisfies the $(DCC)_s$ and the DCC on supplement submodules. By Theorem 3.2.4, Rad M is Artinian. By Lemma 3.2.7, M/(Rad M) belongs to $\underline{G} \cap \underline{C}$. Therefore M /(Rad M) is Artinian, and hence, M is Artinian.

§ 3.3. Counterexamples.

Let R be any ring. We know from Theorem 1.2.1 that $e\underline{U} = \underline{C}(h\underline{U})$. Moreover, in Theorem 3.2.4, we proved that $e^*\underline{A} = \{M : \text{Rad } M \text{ is Artinian}\}$. In view of these facts, the following question arises: if M is a module such that every small submodule has finite uniform dimension, does Rad M have finite uniform dimension? i.e. does $e^*\underline{U} = \{M : \text{Rad } M \in \underline{U}\}$?

In this section, we will give a negative answer to the above question (see Example 3.3.3).

We start this section with an example to show that Theorem 3.2.4 is not true for the class of Noetherian modules.

Example 3.3.1. Let R be the ring of integers $\mathbb{Z}$ and p is a prime number. Let M be the right R-module $\mathbb{Z}(p^\infty)$, the Prufer p-group. Then M belongs to $e^*\underline{N}$, but the radical of M is not Noetherian.

Proof. Since every proper submodule of M is cyclic and spanned by $1/p^n$ for some $n > 0$, then every proper submodule of M is finite ,and hence, Noetherian. For $i = 1, 2, \ldots$, let N_i be the cyclic submodule spanned by $1/p^i$. Then

$$0 < N_1 < N_2 < \ldots < \cup N_i = M$$

is an infinite ascending chain of submodules of M. Hence M is not Noetherian. Since $N_i < N_{i+1}$ for all $i \geq 1$, then M has no maximal submodules. Therefore Rad M = M. Since M is infinite and the sum of finite submodules is finite then every proper submodule of M is small. Thus $M \in e^*\underline{N}$ and Rad $M \notin \underline{N}$.

Note that the ring of integers Z is not Artinian. In the next example, we show that Theorem 3.2.8 is not true for the class of non-supplemented modules.

<u>Example 3.3.2.</u> Let M be the right Z-module Z_Z. Then M has no proper supplements and no non-zero small submodules.

Proof. Let $n \in Z$. If $n = 0$, 1 or -1, then nZ is a supplement submodule of M. Suppose that nZ is a supplement submodule where $n \neq 0$, 1 and -1. Then there exists $m \in Z$ which is relatively prime to n and mZ is minimal with respect to $nZ + mZ = Z$. Let

$k > 1$ be any integer which is relatively prime to n. Then mk is also relatively prime to n. Thus $n\mathbb{Z} + mk\mathbb{Z} = \mathbb{Z}$, a contradiction, because $mk\mathbb{Z} < m\mathbb{Z}$. Thus M has no proper supplement submodule. In the other hand, $n\mathbb{Z} + f\mathbb{Z} = \mathbb{Z}$ for any $f \in \mathbb{Z}$ which is relatively prime to n. Hence zero is the only small submodule of M.

In the next Example, we show that Theorem 3.2.4 can not be generalized to the class of modules which has finite uniform dimension. In fact, it can not be even generalized to the class of modules with Krull dimension one. We start with an example which was constructed by E. R. Puczylowski.

<u>Example 3.3.3.</u> Let $R = \{s/t\colon s, t \in \mathbb{Z} \text{ and } p \nmid t\}$ where p is a prime number. Let M be the right R-module $R \oplus R \oplus \ldots$. Then every small submodule of M is Noetherian of Krull dimension one but the radical of M does not have Krull dimension.

Proof. It is clear that $\text{Rad } R = Rp = \{sp/t\colon s, t \in \mathbb{Z} \text{ and } p \nmid t\}$. Therefore $\text{Rad } M = Rp \oplus Rp \oplus \ldots$. Thus the radical of M does not have finite uniform dimension. Let N be any small submodule of M. Without loss of generality, we can suppose that N is a non-Artinian submodule. Suppose that N does not have finite uniform dimension. Then N contains an infinite direct sum of non-zero submodules, say F. Thus $F = Rx_1 \oplus Rx_2 \oplus \ldots \leq N$ where $x_i \neq 0$ for all $i \geq 1$. Hence F is a free R-module of infinite rank. The field of fractions of R is the rational numbers $\mathbb{Q}$ which is countable. But $\mathbb{Q}$ is countably generated over R. Thus there exists an R-epimorphisim Φ from F onto $\mathbb{Q}$. Let K be the kernel of Φ. Then $F/K \cong \mathbb{Q}$ which is an injective R-module. Therefore $M/K = F/K \oplus L/K$ for some submodule L of M which contains K. Hence $M = F + L$. Thus $M = N + L$, and hence, $M = L$. Therefore $F = K$, a contradiction. Thus N has finite uniform dimension. By Goldie's Theorem, there exists $k \geq 1$ such that $N' = Rx_1 \oplus Rx_2 \oplus \ldots \oplus Rx_k$ is an essential submodule of N where $x_i \in N$ for all $1 \leq i \leq k$. Therefore $N' \subseteq G = R \oplus R \oplus \ldots \oplus R$ (t-direct summands and $t \geq k$). Hence N/N' is torsion. Therefore $(N+G)/G \cong N/(N \cap G)$ is torsion. But $M/G \cong M$ which is torsion free. Therefore M/G is torsion free, and hence, $(N+G)/G = 0$. So $N \subseteq G$. But G is Noetherian. Therefore N is Noetherian, and hence, N has Krull dimension one (see [MR, 1.8 ch.6]). Thus M belongs to $e^*\underline{K}$, and hence, belongs to $e^*\underline{U}$. Since Rad M does not have finite uniform dimension, then Rad $M \notin \underline{K}$.

Chapter 4.

Some Characterizations of d^*X

§ 4.1. Further properties of d^*X and examples.

Let R be a ring. Let $\underline{X}$ be a class of right R-modules. For any right R-module M, we define the *$\underline{X}$-socle $Soc_X(M)$* to be the sum of all $\underline{X}$-submodules of M. It is clear that $M = Soc_X(M)$ if M belongs to $\underline{X}$. Moreover, $Soc\ M = Soc_X(M)$ when $\underline{X}$ is the class of semisimple right R-modules. In the first result of this section we investigate the internal structure of $d^*\underline{X}$-modules.

Proposition 4.1.1. Let R be any ring. Let $\underline{X}$ be a class of right R-modules. Let M be a $d^*\underline{X}$-module. Then $M/Soc_X(M)$ is semisimple.

Proof. Suppose that M belongs to $d^*\underline{X}$. Let $S = Soc_X(M)$. Any submodule of M/S has the form N/S for some submodule N of M containing S. There exist submodules K and K' of M such that $M = K \oplus K'$, $K \subseteq N$ and N/K belongs to $\underline{X}$. Thus, by modular law, $N = K \oplus (N \cap K')$ and $N \cap K' \in \underline{X}$. Thus $N \cap K' \subseteq S$, and hence, $M/S = (N/S) \oplus [(K'+ S)/S]$. Therefore, by [AF, Theorem 9.6], M/S is semisimple module.

Corollary 4.1.2. Let R be any ring. Let $\underline{X}$ be a class of right R-modules such that $\underline{S} \subseteq \underline{X}$. Let M be a $d^*\underline{X}$-module. Then $Soc_X(M)$ is an essential submodule of M.

Proof. Suppose that M belongs to $d^*\underline{X}$. Let N be any submodule of M such that $N \cap Soc_X(M) = 0$. Then N embeds in $M/Soc_X(M)$. Thus, by Proposition 4.1.1, N is semisimple. Thus $N \subseteq Soc_X(M)$, and hence, $N = 0$. Therefore $Soc_X(M)$ is essential in M.

Lemma 4.1.3. Let R be any ring. Let $\underline{X}$ be any class of right R-modules. Let M be a $d^*\underline{X}$-module and N be any submodule of M. Then N contains a non-zero $\underline{X}$-submodule or N is a semisimple direct summand of M.

Proof. Suppose that M belongs to $d^*\underline{X}$ and N be a submodule of M. Suppose that N does not contain a non-zero $\underline{X}$-submodule. Let P be any submodule of N. By Lemma 3.1.1, $P = K \oplus L$ where K is a direct summand of M and L is an $\underline{X}$-submodule of M. Thus $L = 0$, and hence, $P = K$. Therefore, by [AF, Theorem 9.6], N is a semisimple. Moreover, by [Lemma 3.1.3], N is a direct summand of M.

Proposition 4.1.4. Let R be any ring. Let $\underline{X}$ be any s-closed class of right R-modules and M be any $d^*\underline{X}$-module. Then there exist a semisimple submodule M_1 of M and a submodule M_2 of M such that $M = M_1 \oplus M_2$ and every non-zero submodule of M_2 contains a non-zero $\underline{X}$-submodule.

Proof. By Zorn's Lemma M contains a submodule M_1 maximal with respect to the property that it does not contain a non-zero $\underline{X}$-submodule. By Lemma 4.1.3, M_1 is a semisimple direct summand of M. There exists a submodule M_2 such that $M = M_1 \oplus M_2$. Let N be a non-zero submodule of M_2. Then $M_1 \oplus N$ contains a non-zero $\underline{X}$-submodule K, by the choice of M_1. Note that $K \cap M_1 \in \underline{X}$, and hence, $K \cap M_1 = 0$. Thus K embeds in N, and hence, N contains a non-zero $\underline{X}$-submodule.

Corollary 4.1.5. Let R be any ring. Then

(i) $\underline{C} \oplus \underline{T} \subseteq d^*\underline{T} \subseteq \underline{C} \oplus [\underline{T}(\underline{C} \cap \underline{T})] \subseteq \underline{TC}$ and

(ii) $d^*\underline{T}_n = \underline{C} \oplus \underline{T}_n$.

Proof. (i) By Corollary 3.1.7, $\underline{C} \oplus \underline{T} \subseteq d^*\underline{T}$. Moreover, by Proposition 4.1.4, any $d^*\underline{T}$-module M can be written in the form $M = M_1 \oplus M_2$ where M_1 is a semisimple module and M_2 a module with essential singular submodule. Thus M_2 belongs to $\underline{T}(\underline{C} \cap \underline{T})$ (see Proposition 4.1.1). Therefore $d^*\underline{T} \subseteq \underline{C} \oplus [\underline{T}(\underline{C} \cap \underline{T})]$ and it is clear that $\underline{C} \oplus [\underline{T}(\underline{C} \cap \underline{T})] \subseteq \underline{TC}$.

(ii) By Corollary 3.1.7, we get the first inclusion and ,by Proposition 4.1.4, we get the second one.

<u>Corollary 4.1.6</u>. Let R be any ring. Then

(i) $d^*\underline{T} = \underline{C} \oplus \underline{T}$, if R is right non-singular,

(ii) $\underline{T} \subseteq d^*\underline{T} \subseteq \underline{T}^2$, if R has zero right socle, and

(iii) $d^*\underline{T} = \underline{T}$, if R is right nonsingular and has zero right socle.

Proof. (i) Suppose that R is right non-singular. Then $\underline{T}^2 = \underline{T}$. Thus, by Corollary 4.1.5, $d^*\underline{T} = \underline{C} \oplus \underline{T}$.

(ii) Suppose that R has zero right socle. Then $\underline{C} \subseteq \underline{T}$, and hence, (ii) follows by Corollary 4.1.5.

(iii) Follows from part (i) and part (ii).

Note that, if R is a commutative domain which is not a field, then Corollary 4.1.6 gives that $d^*\underline{T} = h^*\underline{T} = \underline{T}$. Moreover, Note that, for any ring R, $e^*\underline{T}$ is the class of right R-modules which have a singular radical (see Lemma 2.2.1).

Corollary 4.1.6 suggests that $d^*\underline{T} = \underline{C} \oplus \underline{T}$, but this is not true in general. This fact will be proved in the next example.

<u>Example 4.1.7</u>. Let K be a field. Let $R = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} : a, b \in K \right\}$. Then over the commutative ring R, $d^*\underline{T} \neq \underline{C} \oplus \underline{T}$.

Proof. Let M be the right R-module R_R. Then the singular submodule, Z(M), of M is the subring of R consisting of all elements of R with a = 0. It is clear that 0, Z(M), and M are the only submodules of M. Moreover, Z(M) is the unique maximal submodule of M, and hence, Z(M) = Rad(M). Thus Z(M) is small in M (see Lemma 2.2.1). Therefore M is indecomposable, and hence, M is not a semisimple module (see [AF, Theorem 9.6]). But M belongs to $d^*\underline{T}$. Moreover, M is not singular.

In Example 1.1.5, It was shown that the class $d\underline{X}$ is not s-closed, in general, and $d\underline{X} \oplus d\underline{X} \neq d\underline{X}$. The analogue of these results is true as we will show in the next two examples.

Example 4.1.8. Let R be the ring of rational integers. Then $d^{*}\underline{T}_n$ is not q-closed.

Proof. Let M be the additive group of rational numbers considered as an R-module. Then M is non-singular and M has zero socle. Thus, by Corollary 4.1.5, M belongs to $d^{*}\underline{T}_n$. On the other hand, the ring of integers Z is a submodule of M such that M/Z is singular and M/Z is not semisimple. By Corollary 4.1.5, M/Z does not belong to $d^{*}\underline{T}_n$.

A module M is called ***small*** provided there exists an extension module M' of M such that M is small in M'. Leonard, in [Leo, Theorem 1], proved that M is small if and only if M is small in its injective hull E(M). In [Pa, 3.2], Pareigis showed that the small modules form a class which is {q,s}-closed and is also closed under finite direct sums. Moreover, if $\underline{X}$ is the class of small modules, then the dual CS-modules belongs to $d^{*}\underline{X}$.

Example 4.1.9. Let R be a ring which is local commutative and principal ideal domain. Let P be the unique maximal ideal of R. Suppose further that R is not complete in the P-adic topology. Then $d^*\underline{X} \neq d^*\underline{X} \oplus d^*\underline{X}$ where $\underline{X}$ is the class of small R-modules.

Proof. Let K be the field of fractions of R. Let $M = K \oplus K$. Then, by [MM, Proposition A.7], M is not a dual CS-module. On the other hand, by [MM, Proposition A.7], K is a dual CS-module. Therefore K belongs to $d^*\underline{X}$. Note that, by [Ha, Lemma 2.1], M is injective. If M belongs to $d^*\underline{X}$, then, by Lemma 4.4.1 M is a dual CS-module, a contradiction. Therefore M does not belong to $d^*\underline{X}$.

§ 4.2. The classes $d^*(hU)$ and d^*K_α.

In this section, for an arbitrary ring, we shall completely characterize the classes $d^*(h\underline{U})$ and the classes $d^*\underline{K}_\alpha$, for an ordinal $\alpha > 0$.

Let R be a ring and let $\underline{X}$ be any class of right R-modules.

We shall denote $a\underline{X}$ to the class of right R-modules M such that every proper submodule of M belongs to $\underline{X}$. It is clear that, for any class $\underline{X}$, $a\underline{X} \subseteq d^*\underline{X}$. We also define, for any positive integer n, $\underline{X}^{(n)} = \underline{X} \oplus \underline{X} \oplus \ldots \oplus \underline{X}$ (n summands). Finally we define

$$\underline{X}^{(\omega)} = \cup_{n\geq 1} \underline{X}^{(n)}.$$

Lemma 4.2.1. Let R be any ring. let $\underline{X}$ be any class of right R-modules. Then $\underline{U} \cap d^*\underline{X} \subseteq (a\underline{X})^{(\omega)}$.

Proof. Let M be a $(\underline{U} \cap d^*\underline{X})$-module. We prove that M is a (finite) direct sum of $a\underline{X}$-modules by induction on the uniform dimension of M. Suppose first that M is uniform. Let N be a proper submodule of M. By hypothesis, there exist submodules K and K' of M such that $M = K \oplus K'$, $K \subseteq N$ and N/K is an $\underline{X}$-module. Because $K \neq M$, it follows that $K = 0$, and hence, N belongs to $\underline{X}$. Thus M is an $a\underline{X}$-module. Now suppose that M has uniform dimension $n \geq 2$. Suppose that $M \notin (a\underline{X})^{(\omega)}$. Then $M \notin a\underline{X}$. There exists a proper submodule L of M such that $L \notin \underline{X}$. Since $M \in d^*\underline{X}$ then there exist submodules P and P' of M such that $M = P \oplus P'$,

$P \subsetneq L$ and L/P is an $\underline{X}$-module. Note that $P \neq 0$ and $P' \neq 0$. Thus P and P' both have uniform dimension at most $n - 1$. Therefore, by Proposition 3.1.4, and induction on n, P and P' belong to $(a\underline{X})^{(\omega)}$, and hence, M belongs to $(a\underline{X})^{(\omega)}$, a contradiction. Thus M belongs to $(a\underline{X})^{(\omega)}$.

It is proved, in Theorem 1.2.3, that $d\underline{U} = d(h\underline{U}) = \underline{C} \oplus (h\underline{U})$. the analogue of this fact will be proved in the next Theorem.

<u>Theorem 4.2.2</u>. For any ring R, $d^*(h\underline{U}) = \underline{C} \oplus (h\underline{U})$.

Proof. By Corollary 3.1.7, $\underline{C} \oplus (h\underline{U}) \subseteq d^*(h\underline{U})$. Conversely, let $M \in d^*(h\underline{U})$. Then there exist submodules K and K' of M such that $M = K \oplus K'$, $K \subseteq \text{Soc } M$ and $(\text{Soc } M)/K \in h\underline{U}$. Thus K is semisimple. But $\text{Soc } K' \cong (\text{Soc } M)/K$. Hence Soc K' is finitely generated. By Proposition 3.1.4, we can suppose without loss of generality, that Soc M is finitely generated and prove that $M \in h\underline{U}$.

Let $P \subseteq Q$ be submodules of M such that Q/P is semisimple. Proposition 3.1.4 allows us to assume $M = Q$. By hypothesis, there exist submodules L, L' of M such that $M = L \oplus L'$, $L \subseteq P$ and $P/L \in h\underline{U}$. By modular law, $P = L \oplus (P \cap L')$. Hence $P \cap L'$ has finite uniform dimension. But, by Zorn's Lemma, there exists a submodule L'' of L' maximal with respect to $L'' \cap (P \cap L') = 0$. Since $L'' \cap P = 0$, then L'' is semisimple, and hence, finitely generated, because Soc M is finitely generated. The submodule L' contains the essential submodule $L'' \oplus (P \cap L')$ which has finite uniform dimension. Thus L' has finite uniform dimension.

By Lemma 4.2.1, $L' \in (a(h\underline{U}))^{(\omega)}$. Let N be a submodule of L' such that $N \in a(h\underline{U})$. If $N \subseteq P \cap L' \cong P/L$, then $N \in h\underline{U}$. Otherwise $P \cap N \subset N$, and hence, $N/(P \cap N)$ has a maximal submodule. By Proposition 1.3.10, $N \in h\underline{U}$. Therefore, in any case, $N \in h\underline{U}$. Since $L' \in (a(h\underline{U}))^{(\omega)}$, then, by Proposition 1.3.10, $L' \in h\underline{U}$. But $M/L \cong L'$. Therefore M/P is finitely generated. Thus, by Lemma 1.3.9, $M \in h\underline{U}$.

In Theorem 1.3.11, for an ordinal $\alpha > 0$, we proved that

$$d\underline{K}^*_\alpha = \underline{C} \oplus \underline{K}^*_\alpha.$$

We prove the dual of this fact in the next theorem.

<u>Theorem 4.2.3</u>. Let R be any ring R. Then, for an ordinal $\alpha > 0$, $d^*\underline{K}_\alpha = \underline{C} \oplus \underline{K}_\alpha$.

Proof. By Corollary 3.1.7, $\underline{C} \oplus \underline{K}_\alpha \subseteq d^*\underline{K}_\alpha$. Conversely, suppose that $M \in d^*\underline{K}_\alpha$. Since $\underline{K}_\alpha \subseteq h\underline{U}$ (see, for example, [GR, Lemma 1.1 and Proposition 1.4]), then by Proposition 3.1.3 and Theorem 4.2.2, $M = M_1 \oplus M_2$ where M_1 is semisimple module and M_2 has finite uniform dimension. By Proposition 3.1.4, $M_2 \in d^*\underline{K}_\alpha$. On the other hand, by Lemma 4.2.1, $M_2 \in (a\underline{K}_\alpha)^{(\omega)}$. But it is clear that $a\underline{K}_\alpha = \underline{K}_\alpha$ (see [GR,Lemma 1.1]). Therefore $M_2 \in \underline{K}_\alpha$. Thus M belongs to $\underline{C} \oplus \underline{K}_\alpha$.

In Theorem 1.2.3, it was proved that $d\underline{N} = \underline{C} \oplus \underline{N}$. The dual of this fact will be proved in the following Corollary.

<u>Corollary 4.2.4.</u> For any ring R, $d^*\underline{A} = \underline{C} \oplus \underline{A}$.

Proof. Since $\underline{K}_0 = \underline{A}$, then the proof follows by Theorem 4.2.3.

§ 4.3. FBN-rings.

Let R be a ring. We know from Theorem 1.2.3 that

$$d\underline{U} = d(h\underline{U}) = \underline{C} \oplus (h\underline{U}).$$

Clearly $d^*(h\underline{U}) \subseteq d^*\underline{U}$. However, if $R = K[X_1, X_2, \ldots]$ denotes the polynomial ring over a field K in a countably infinite number of commuting indeterminates X_1, X_2, ..., then R_R is uniform. Thus $R_R \in d^*\underline{U}$. On the other hand, $R_R \notin \underline{C} \oplus h\underline{U} = d^*(h\underline{U})$, because R_R has zero socle and the ring R has a homomorphic image with non-finitely generated socle. Therefore $d^*(h\underline{U}) \neq d^*\underline{U}$.

This raises the question: given a ring R, what is $d^*\underline{U}$ for the ring R? We shall show that $d^*\underline{U} = \underline{C} \oplus \underline{U}$ for the class of right FBN-rings.

Proposition 4.3.1. For any ring R, $d^*\underline{U} \cap \underline{CG} \subseteq \underline{C} \oplus \underline{U}$.

Proof. Let $M \in d^*\underline{U} \cap \underline{CG}$. Let S denote the socle, Soc M, of M. There exists a $\underline{C}$-submodule N of M such that M/N is finitely generated. Thus (M/S) is finitely generated. Hence there exists a finitely generated submodule L of M such that $M = L + S$. But S is semisimple. Therefore $S = (L \cap S) \oplus S_1$ for some submodule

S_1 of S. Hence S_1 is semisimple and $M = L \oplus S_1$.

Suppose that L is not a $\underline{U}$-module. Then there exists a submodule $N' = \oplus_{\alpha \in \Lambda} N_\alpha$ where N_α is a non-zero submodule of L for all $\alpha \in \Lambda$. By Proposition 3.1.4, $L \in d^*\underline{U}$. Thus, by Lemma 3.1.1, $N' = K \oplus P$ for some direct summand K of L and a submodule P of L which belongs to $\underline{U}$. Note that L is finitely generated. Therefore K is finitely generated. But P has finite uniform dimension. Therefore there exists a finitely generated submodule Q of P which is essential in P. Therefore $K \oplus Q$ is essential in $K \oplus P$. But $K \oplus Q$ is finitely generated. Therefore $K \oplus Q \subseteq \oplus_{\alpha \in \Lambda'} N_\alpha$ for some finite subset Λ' of Λ. Therefore $(K \oplus Q) \cap N_\alpha = 0$ for all $\alpha \notin \Lambda'$. Hence $N_\alpha = 0$ for all $\alpha \notin \Lambda'$, a contradiction. Therefore L has finite uniform dimension, and hence, $M \in \underline{C} \oplus \underline{U}$.

<u>Corollary 4.3.2</u>. Let R be any ring. Let M be a right R-module which belongs to $d^*\underline{U}$. Then any finitely generated submodule of M has finite uniform dimension.

Proof. By Proposition 3.1.4 and Proposition 4.3.1.

<u>Lemma 4.3.3</u>. Let R be any ring. Let M be a nonsingular right R-module which belongs to $d^*\underline{U}$ and has zero socle. Then M has finite uniform dimension.

Proof. Suppose that M does not have finite uniform dimension. Then there exists $N = N_1 \oplus N_2 \oplus \ldots$ which is an infinite direct sum of non-zero submodules of M. Note that, for each $i \geq 1$, N_i has zero socle, and hence, N_i has a proper essential submodule K_i (see [AF, Proposition 9.7]). Let $K = K_1 \oplus K_2 \oplus \ldots$. Then K is essential in N (see [AF, Proposion 5.20]). Note that, by Proposition 3.1.4, $N \in d^*\underline{U}$. Thus there exist submodules L, L' of N such that $N = L \oplus L'$, $L \subseteq K$ and K/L has finite uniform dimension. By modular law $K = L \oplus (K \cap L')$. Thus $K \cap L'$ has finite uniform dimension. Since K essential in N, then $K \cap L'$ is essential in L', and hence, L' has finite uniform dimension. Therefore L' contains a finitely generated essential submodule H, and hence, there exists $t \geq 1$ such that $H \subseteq N_1 \oplus \ldots \oplus N_t$.

Let $x \in N_{t+1}$ and $x \notin K_{t+1}$. Then $x = y - y'$ for some $y \in L$ and $y' \in L'$. Since $y \in K$, then $y = k_1 + \ldots + k_n$ for some $n \geq 1$ and $k_i \in K_i$ $(1 \leq i \leq n)$. There exists an essential right ideal E of R such that $y'E \subseteq H \subseteq N_1 \oplus \ldots \oplus N_t$. Note that

$y' = y - x = k_1 + \ldots + k_t + (k_{t+1} - x) + k_{t+2} + \ldots + k_n$. But $y'E \subseteq (N_1 \oplus \ldots \oplus N_t)$. Thus $(k_{t+1} - x)E = 0$. Since N_{t+1} is nonsingular, then $k_{t+1} - x = 0$. Therefore $x = k_{t+1} \in K_{t+1}$, a contradiction. It follows that M has finite uniform dimension.

<u>Proposition 4.3.4</u>. Let R be any ring. Then

$\underline{C} \oplus \underline{U} \subseteq d^*\underline{U} \subseteq \underline{T} \oplus \underline{C} \oplus \underline{U}$.

Proof. By Corollary 3.1.7, $\underline{C} \oplus \underline{U} \subseteq d^*\underline{U}$. Suppose that $M \in d^*\underline{U}$. Let S be the socle of M. Then there exist submodules K and K' of M such that $M = K \oplus K'$, $K \subseteq S$ and $S \cap K' \in \underline{U}$. Therefore K is semisimple and K' has finitely generated socle, $S \cap K'$. Let Z be the singular submodule of K'. But, by Proposition 3.1.4, K' belongs to $d^*\underline{U}$, and hence, there exist submodules L and L' of K' such that $K' = L \oplus L'$, $L \subseteq Z$ and $Z \cap L' \in \underline{U}$. Therefore L is singular and the singular submodule Z' of L' has finite uniform dimension.

We claim that L' has finite uniform dimension. Suppose not. Then there exists an infinite direct sum $P = P_1 \oplus P_2 \oplus \ldots$ of non-zero submodules of L'. Since Soc K' is finitely generated, then Soc $K' \in \underline{U}$. Moreover, $Z' \in \underline{U}$. So without loss of generality suppose that P_i is nonsingular with zero socle for all $i > 1$. Thus P is a nonsingular $d^*\underline{U}$-module with zero socle. Thus, by Lemma 4.3.3, P has finite uniform dimension, a contradiction. Therefore L' has finite uniform dimension. Thus any $d^*\underline{U}$-module belongs to $\underline{T} \oplus \underline{C} \oplus \underline{U}$.

Using the last proposition we shall prove that for any commutative Noetherian ring R, $d^*\underline{U} = \underline{C} \oplus \underline{U}$. In fact we can do rather better. Recall that a ring R is a ***right FBN-ring*** if R is right Noetherian and, for every prime homomorphic image S of R, every essential right ideal of S contains a non-zero two-sided ideal. Examples of right FBN-rings are commutative Noetherian rings and right Noetherian rings which satisfy a polynomial identity (i.e. right Noetherian PI-rings).

Let R be a right Noetherian ring and U a uniform right R-module. Recall that $P = \{r \in R : Ar = 0$ for some non-zero submodule A of $U\}$ is a prime ideal of R and $P = ann(W)$, the annihilator of some non-zero submodule W of U. We call P the ***assassinator of U*** and write $P = ass(U)$.

Lemma 4.3.5. For a prime right FBN-ring, $d^*\underline{U} = \underline{C} \oplus \underline{U}$.

Proof. Let R be a prime right FBN-ring. Then, by Corollary 3.1.7, $\underline{C} \oplus \underline{U} \subseteq d^*\underline{U}$. Suppose that $d^*\underline{U} \not\subseteq \underline{C} \oplus \underline{U}$. Because R is right Noetherian, we can suppose that $d^*\underline{U} = \underline{C} \oplus \underline{U}$ for any proper prime homomorphic image of R. Let M be a $d^*\underline{U}$-module such that $M \notin \underline{C} \oplus \underline{U}$. Then, by Proposition 4.3.4, $M \in \underline{T} \oplus \underline{C} \oplus \underline{U}$. Thus there exists an infinite direct sum $N = N_1 \oplus N_2 \oplus \ldots$ of non-zero singular submodules N_i $(i \geq 1)$ of M such that N_i has zero socle (see the proof of Proposition 4.3.4) and is uniform for each $i \geq 1$.

Let $i \geq 1$ and let $N' = N_i$. Let $0 \neq x \in N'$. Then $xE = 0$ for some essential right ideal E of R. Because R is a prime right FBN-ring, there exists a non-zero ideal I of R such that $I \subseteq E$. Then $xI = 0$, and hence, $xRa = 0$ for each $a \in I$. Thus $0 \neq I \subseteq P$ where $P = \operatorname{ass}(N')$. Let $P_i = \operatorname{ass}(N_i)$, for $i \geq 1$. By the above argument, $P_i \neq 0$, and we can suppose without loss of generality that $N_iP_i = 0$.

For each $i \geq 1$, let K_i be a proper essential submodule of N_i (N_i has zero socle, so we use [AF, Proposition 9.7]). Let $K = K_1 \oplus K_2 \oplus \ldots$. Then $K \leq_e N = N_1 \oplus N_2 \oplus N_3 \oplus \ldots$. Thus, by

Proposition 3.1.4, there exist submodules L and L' of N such that $N = L \oplus L'$, $L \subseteq K$ and $K \cap L' \in \underline{U}$. By the argument used in the proof of Lemma 4.3.3, $L' \in \underline{U}$. Therefore there exists a finitely generated essential submodule H of L'. Thus there exists a positive integer t such that $H \subseteq N_1 \oplus \ldots \oplus N_t$, (see the proof of Lemma 4.3.3). Let $x \in L'$. Then $x = x_1 + \ldots + x_n$ for some $n \geq 1$ and $x_i \in N_i$ $(1 \leq i \leq n)$. On the other hand,

$$x(P_1 \cap \ldots \cap P_t) \subseteq N_{t+1} \oplus \ldots \oplus N_n.$$

Hence $x(P_1 \cap \ldots \cap P_t) \cap H = 0$. Therefore $x(P_1 \cap \ldots \cap P_t) = 0$. It follows that $L'(P_1 \cap \ldots \cap P_t) = 0$.

Now we adapt the last part of the proof of Lemma 4.3.3. Let $x \in N_{t+1}$ and $x \notin K_{t+1}$. Adopting the same notations of Lemma 4.3.3, $y' \in L'$, so that $y'(P_1 \cap \ldots \cap P_t) = 0$, and hence,

$$(k_{t+1} - x)(P_1 \cap \ldots \cap P_t) = 0.$$

It follows that $(P_1 \cap \ldots \cap P_t) \subseteq P_{t+1}$, because $k_{t+1} - x \neq 0$. Hence $(P_1 \ldots P_t) \subseteq P_{t+1}$ and $P_i \subseteq P_{t+1}$ for some $1 \leq i \leq t$. Similarly, for each $j \geq t+1$, there exists $1 \leq i \leq t$ such that $P_i \subseteq P_j$. There exist an infinite subset Ω of $\{t+1, t+2, \ldots\}$ and $1 \leq i \leq t$ such that $P_i \subseteq P_j$ for all $j \in \Omega$. Let $M' = \oplus N_i$, where the sum is taken over all $i \in \Omega$. Then $M'P_i = 0$. Hence M' is a right module over the prime right FBN-ring R/P_i. Note that

$P_i \neq 0$ and the (R/P_i)-module M' is a $d^*\underline{U}$-module. Therefore, by Proposition 3.1.4, it follows that M' belongs to $\underline{C} \oplus \underline{U}$. But M' has zero socle (see [AF, Proposition 9.19]). Therefore M' has finite uniform dimension, a contradiction. Thus $d^*\underline{U} = \underline{C} \oplus \underline{U}$.

<u>Theorem 4.3.6</u>. Let R be a right FBN-ring. Then $d^*\underline{U} = \underline{C} \oplus \underline{U}$.

Proof. Let R be a right FBN-ring. By Corollary 3.1.7, it is always true that $\underline{C} \oplus \underline{U} \subseteq d^*\underline{U}$. Suppose that $d^*\underline{U} \not\subseteq \underline{C} \oplus \underline{U}$. Then there exists a $d^*\underline{U}$-module M which does not belong to $\underline{C} \oplus \underline{U}$. But, by Proposition 4.3.4, $M \in \underline{C} \oplus \underline{U} \oplus \underline{T}$. Thus there exists an infinite direct sum $N = N_1 \oplus N_2 \oplus \dots$ of non-zero singular submodules N_i $(i \geq 1)$ of M, each with zero socle. As in the proof of Lemma 4.3.5, we can suppose that N_i is uniform and $N_iP_i = 0$ for some prime ideal P_i, for each $i \geq 1$. Because R is right Noetherian, R contains only a finite number of minimal prime ideals. Let $Q_1, Q_2, \dots, Q_k$ denote the minimal prime ideals of R. For each $j \geq 1$, there exists $1 \leq i \leq k$ such that $Q_i \subseteq P_j$. Hence there exists $Q \in \{Q_1, \dots, Q_k\}$ and an infinite set Ω of positive integers such that $Q \subseteq P_j$ $(j \in \Omega)$. In this case, let $K = \oplus N_j$, where the sum is taken over all $j \in \Omega$.

Note that $KQ = 0$. Thus K is a $d^*\underline{U}$-module with zero socle over the prime right FBN-ring (R/Q). Therefore, by Lemma 4.3.5, K has finite uniform dimension, a contradiction. It follows that over a right FBN-ring $d^*\underline{U} = \underline{C} \oplus \underline{U}$.

§ 4.4. The class $d^*\underline{N}$.

We know that, by Corollary 3.1.7, that $\underline{C} \oplus \underline{N} \subsetneq d^*\underline{N}$. On the other hand if M is the Prufer p-group $Z(p^\infty)$, then $M \in a\underline{N} \subseteq d^*\underline{N}$ (see Example 3.3.1). Moreover M is not Noetherian and the socle of M is zero. It follows that $d^*\underline{N} \neq \underline{C} \oplus \underline{N}$ in general.

This raises the question: given a ring R, what is $d^*\underline{N}$ for the ring R?

In this section, we shall show that, if R is a non-local Dedekind domain, then $d^*\underline{N} = \underline{C} \oplus \underline{A} \oplus \underline{N}$.

<u>Lemma 4.4.1</u>. For any ring R,

(i) $\underline{G} \cap d^*\underline{G} = \underline{N} = h^*\underline{G}$,

(ii) $d^*\underline{G} = d^*\underline{N}$ and

(iii) $d^*\underline{N} \cap \underline{CG} = \underline{C} \oplus \underline{N}$.

Proof. (i) Since the class $\underline{G}$ is (p,q)-closed, then by Proposition 3.1.4 (v), $\underline{G} \cap d^*\underline{G} = h^*\underline{G}$. Clearly $h^*\underline{G} = \underline{N}$.

(ii) First, note that $\underline{N} \subseteq \underline{G}$ implies $d^*\underline{N} \subseteq d^*\underline{G}$. Let M be a $d^*\underline{G}$-module. Let L be any submodule of M. Then, by Lemma 3.1.1, $L = K \oplus P$ for some direct summand K of M and $\underline{G}$-submodule P. By Proposition 3.1.4 (i), $P \in \underline{G} \cap d^*\underline{G} \subseteq \underline{N}$. Again applying Lemma 3.1.1, we have $M \in d^*\underline{N}$.

(iii) By Corollary 3.1.7, $\underline{C} \oplus \underline{N} \subseteq d^*\underline{N}$. It is clear that $\underline{C} \oplus \underline{N} \subseteq \underline{CG}$. Conversely, let $M \in d^*\underline{N} \cap \underline{CG}$. Then $M \in d^*\underline{U}$. Thus, by Proposition 4.3.1, $M = M_1 \oplus M_2$ where $M_1 \in \underline{C}$ and $M_2 \in \underline{U}$. Therefore M_1 is contained in the socle of M and $M/M_1 \in \underline{G}$. Thus $M_2 \in \underline{G}$. But M_2 is $d^*\underline{N}$-module, whence a $d^*\underline{G}$-module. Thus, by (i) $M_2 \in \underline{N}$. Therefore $M \in \underline{C} \oplus \underline{N}$.

<u>Lemma 4.4.2.</u> Let R be a non-local Dedekind domain. Suppose that the R-module $M \in a\underline{N}$. Then M is an Artinian injective module or M is Noetherian.

Proof. Let M be an injective R-module. Then M is indecomposable and hence, $M \cong K$, the field of fractions of R or $M \cong E(U)$, the injective hull of a simple R-module U (see [SVa, Theorem 2.32

Corollary]). Let P and Q be distinct maximal ideals of R. Then $E(R/P) \oplus E(R/Q)$ embeds in K/R, so that $K \notin a\underline{N}$. Thus $M \cong E(U)$, and hence, M is Artinian (see [SVa, Theorem 4.30]).

Suppose that M is not injective. Then M is not divisible (see [SVa, Theorem 4.25]). Therefore there exists a maximal ideal P of R such that $M \neq MP$. On the other hand M/MP is a non-zero semisimple module. Therefore M has a maximal submodule N. But, by hypothesis, N is Noetherian. Thus M is Noetherian.

Note that, if R is a local Dedekind domain, then Lemma 4.4.2 is not true as we will show in the following example.

Example 4.4.3. Let R be a local Dedekind domain. Then there exists a right R-module M such that M belongs to $a\underline{N}$ but M is neither Noetherian nor Artinian.

Proof. Let R be a local Dedekind domain. Let $0 \neq P$ be the unique maximal ideal of R. Then P is a principal ideal Rp for some element $p \in P$. Let K be the field of fractions of R. Then the left R-submodules of K form a chain as follow:

$$0 = \cap_n Rp^n \subsetneq .. \subsetneq Rp^2 \subsetneq Rp \subsetneq R \subsetneq Rp^{-1} \subsetneq Rp^{-2} \subsetneq ... \subsetneq \cup_n Rp^{-n} = K.$$

Thus K belongs to $a\underline{N}$ and K is neither Artinian nor Noetherian.

As usual, we shall call a module *reduced* if it contains no non-zero injective submodules.

Lemma 4.4.4. Let R be a Dedekind domain. Suppose that M is an Artinian R-module. Then any reduced submodule of M is Noetherian.

Proof. Let M be an Artinian R-module. Suppose that there exists a reduced submodule of M which is not Noetherian. Let N be a reduced submodule of M which is minimal with respect to not being Noetherian. Then N is not divisible, and hence, $N \neq NP$ for some maximal ideal P. Note that N/NP is semisimple and Artinian. Therefore N/NP is Noetherian. But, by the choice of N, NP is Noetherian. Thus N is Noetherian, a contradiction. Therefore any reduced submodule of M must be Noetherian.

Corollary 4.4.5. Let R be a Dedekind domain. Let M be an R-module which belongs to $\underline{N} \oplus \underline{A}$. Then any reduced submodule of M is Noetherian.

Proof. Suppose that R is a Dedekind domain. Let M be the R-module $M_1 \oplus M_2$ where M_1 is a Noetherian R-module and M_2 is an Artinian R-module. Let N be any reduced submodule of M. Then $N \cap M_2$ is reduced. Hence, by Lemma 4.4.4, $N \cap M_2$ is Noetherian. Now $N/(N \cap M_2) \cong (N + M_2)/M_2 \subseteq M/M_2 \cong M_1$. Thus $N/(N \cap M_2)$ is Noetherian, and hence, N is Noetherian.

<u>Crollary 4.4.6</u>. Let R be a Dedekind domain. Then $\underline{N} \oplus \underline{A} \subseteq d^*\underline{N}$.

Proof. Let R be a Dedekind domain. Suppose that $M \in \underline{N} \oplus \underline{A}$. Let N be any submodule of M. We know that $N = N_1 \oplus N_2$ for some injective submodule N_1 and a reduced submodule N_2. By Corollary 4.4.5, N_2 is Noetherian. Hence $M \in h^*(\underline{I} \oplus \underline{N})$. Therefore, by Proposition 3.1.3 (v), M belongs to $d^*\underline{N}$.

<u>Theorem 4.4.7</u>. Let R be a non-local Dedekind domain. Then $d^*\underline{N} = \underline{C} \oplus \underline{A} \oplus \underline{N}$.

Proof. Let R be a non-local Dedekind domain and M an R-module. Suppose that $M \in d^*\underline{N}$. Then, by Proposition 3.1.3, M belongs to $d^*\underline{U}$. Hence, by Theorem 4.3.6, $M \in \underline{C} \oplus \underline{U}$. But, by Proposition

3.1.4 and Lemma 4.2.1, $M \in \underline{C} \oplus (a\underline{N})^{(\omega)}$. Therefore, by Lemma 4.4.2, M belongs to $\underline{C} \oplus \underline{A} \oplus \underline{N}$.

Conversely, suppose that $M \in \underline{C} \oplus \underline{A} \oplus \underline{N}$. Then, by Corollary 4.4.6 and Proposition 3.1.6, $\underline{C} \oplus \underline{A} \oplus \underline{N} \subseteq \underline{C} \oplus d^*\underline{N} \subseteq d^*\underline{N}$.

Lemma 4.4.8. Let R be a ring. Let M be a uniform right R-module such that $M \in d^*\underline{N}$. Then $M \in a\underline{N}$.

Proof. Suppose that $M \notin a\underline{N}$. Then there exists a non-Noetherian proper submodule N of M. Therefore there exist submodules K, K' of M such that $M = K \oplus K'$, $K \subseteq N$ and $N/K \in \underline{N}$. If $K = 0$, then N is Noetherian, a contradiction. Moreover, if $K' = 0$, then $N = M$, a contradiction. Thus K and K' are non-zero submodules of M such that $K \cap K' = 0$, a contradiction. Therefore $M \in a\underline{N}$.

Theorem 4.4.9. Let R be a ring. Let M be a right R-module such that $M \in d^*\underline{N}$. Then $M \in \underline{C} \oplus (a\underline{N})^{(k)}$ for some positive integer k.

Proof. Suppose that $M \in d^*\underline{N}$. By Proposition 3.1.3, $M \in d^*(h\underline{U})$. Thus, by Theorem 4.2.2, $M \in \underline{C} \oplus h\underline{U}$. So there exist submodules M_1, M_2 of M such that $M_1 \in \underline{C}$ and $M_2 \in h\underline{U}$. Thus M_2 has finite uniform dimension. Now we finish the proof by using induction on the uniform dimension of M_2. If u-dim $M_2 = 1$, then, by Lemma 4.4.8, $M_2 \in a\underline{N}$. Suppose that u-dim $M_2 > 1$. Moreover, suppose that $M_2 \notin a\underline{N}$. Then there exists a proper submodule N of M_2 such that N is not Noetherian. But, by Proposition 3.1.4, $M_2 \in d^*\underline{N}$. Thus there exist submodules K, K' of M_2 such that $M_2 = K \oplus K'$, $K \subseteq N$ and N/K is Noetherian. If $K = 0$, then N is Noetherian, a contradiction. On the other hand, if $K' = 0$, then $N = M_2$, a contradiction. Therefore K and K' are non-zero direct summands of M_2. But $M_2 \in h\underline{U}$. Hence u-dim K and u-dim K' are smaller than u-dim M_2. Therefore, by induction on the uniform dimension of M_2, $K \in (a\underline{N})^{(s)}$ and $K' \in (a\underline{N})^{(t)}$ for some positive integers s, t. Therefore $M_2 \in (a\underline{N})^{(s + t)}$. Thus $M \in \underline{C} \oplus (a\underline{N})^{(k)}$ for some positive integer k.

For some prime number p, let $M = Z(p^\infty) \oplus Z(p^\infty)$. Then it is clear that $M \in (a\underline{N})^{(2)}$. But $Z(p^\infty) \notin \underline{N}$. Hence $M \notin a\underline{N}$. Therefore, for an integer $k \geq 2$, $(a\underline{N})^{(k)} \nsubseteq a\underline{N}$ in general.

Chapter 5.

Complete Modular Lattices.

Let L be a lattice. Then L is said to be *modular* if for all elements a, b, c of L, $a \wedge (b \vee c) = (a \wedge b) \vee c$ when $c \leq a$. A lattice L is called *complete* if every non-empty subset S of L has a least upper bound $\vee S$ and a greatest lower bound $\wedge S$.

Throughout this chapter "a lattice" always means a complete modular lattice which has least element 0, greatest element 1 and symmetric relations $\wedge$ and $\vee$. Let L be a lattice. Then an element e of L is called *essential* if $e \wedge a \neq 0$ for all non-zero elements a of L. Denote the set of all essential elements of L by e(L).

Let L be a lattice and c be an element of L. We shall call c a *complement* in L if there exists $c' \in L$ such that $c \vee c' = 1$ and $c \wedge c' = 0$. Denote the set of complements of L by c(L).

For any element $a \leq b$ in a lattice L, we define [a,b] to be the set $\{x \in L : a \leq x \leq b\}$. A well known fact about a modular lattice L which we shall use repeatedly is that, for all a, b in L, $[a \wedge b, a]$ is lattice isomorphic to $[b, a \vee b]$(for example see [Gg, Theorem 4.2]).

By a *class* of lattices we mean any collection of complete modular lattices which contains all the singleton lattices and is closed under lattice isomorphisms.

In previous chapters we always used $\underline{X}$ to denote a class of modules. To avoid any confusion, throughout this chapter we use $\underline{X}_1$ to denote a class of lattices.

Let $\underline{X}_1$ be a class of lattices. By $h\underline{X}_1$ we mean the class of lattices L such that $[a,1] \in \underline{X}_1$ for all $a \in L$. Moreover, define $e\underline{X}_1$ to be the class of all lattices L such that $[a,1] \in \underline{X}_1$ for all $a \in e(L)$. Finally, let $d\underline{X}_1$ be the class of all lattices L such that for every element $a \in L$ there exists $c \in c(L)$ such that $a \leq c$ and $[a,c] \in \underline{X}_1$.

Let $\underline{X}_1$ be a class of lattices. Consider the dual classes $h^*\underline{X}_1$, $e^*\underline{X}_1$, $d^*\underline{X}_1$ of $h\underline{X}_1$, $e\underline{X}_1$, $d\underline{X}_1$, respectively. Moreover, let $\underline{X}^0 = \{L : L^0 \in \underline{X}\}$ where L^0 is the opposite lattice of the lattice L. Then $h^*\underline{X}_1 = h\underline{X}_1$, $e^*\underline{X}_1 = e\underline{X}_1$, $d^*\underline{X}_1 = d\underline{X}_1$. This duality motivates studying lattices as well as modules.

A lattice L is called *complemented* provided that $c(L) = L$. We denote the class of singleton lattices by $\underline{Z}_1$ and the class of complemented lattices by $\underline{C}_1$.

§ 5.1 General Properties.

In the first section of chapter one some general properties of the classes $h\underline{X}$, $d\underline{X}$, and $e\underline{X}$ were proved. In this section we prove the same properties for $h\underline{X}_1$, $d\underline{X}_1$, and $e\underline{X}_1$.

Proposition 5.1.1. Let $\underline{X}_1$ and $\underline{Y}_1$ be classes of lattices. Then

(i) $h\underline{X}_1 \subseteq d\underline{X}_1 \subseteq e\underline{X}_1$,

(ii) $h\underline{X}_1 \subseteq h\underline{Y}_1$, $d\underline{X}_1 \subseteq d\underline{Y}_1$, and $e\underline{X}_1 \subseteq e\underline{Y}_1$, when $\underline{X}_1 \subseteq \underline{Y}_1$,

(iii) $\underline{C}_1 = d\underline{Z}_1 \subseteq d\underline{X}_1$, and

(iv) $h\underline{X}_1 = h(h\underline{X}_1) \subseteq \underline{X}_1$.

Proof. (i) It is clear that $h\underline{X}_1 \subseteq d\underline{X}_1$. Suppose that $L \in d\underline{X}_1$ and $a \in e(L)$. Then there exist $b, b' \in L$ with $b \wedge b' = 0$, $b \vee b' = 1$, $a \leq b$, and $[a,b] \in \underline{X}_1$. Therefore, $a \wedge b' = 0$, and hence, $b = 1$. Therefore, $L \in e\underline{X}_1$.

(ii) Clear, because for any elements a, b of L $[a,b] \in \underline{Y}_1$ when $[a,b] \in \underline{X}_1$.

(iii) By (ii), $d\underline{Z}_1 \subseteq d\underline{X}_1$. If L is a complemented lattice, then $c(L) = L$, and hence, $L \in d\underline{Z}_1$. Thus $\underline{C}_1 \subseteq d\underline{Z}_1$. Suppose that $L \in d\underline{Z}_1$ and $a \in L$, then there exists $b \in c(L)$ such that $a \leq b$

and $[a,b] \in \underline{Z}_1$. Thus $a = b$, and hence, $L \in \underline{C}_1$.

(iv) Let $L \in h\underline{X}_1$. Then $[0,1] \in \underline{X}_1$, so $L \in \underline{X}_1$. Therefore, by (ii), $h(h\underline{X}_1) \subseteq h\underline{X}_1$. On the other hand, if $L \in h\underline{X}_1$ and $a, b \in L$ with $a \leq b$, then $[b,1] \in \underline{X}_1$. Thus $[a,1] \in h\underline{X}_1$. So $L \in h(h\underline{X}_1)$.

Let $\underline{X}_1$ be a class of lattices. Then $\underline{X}_1$ is called *q-closed* if $[a,1] \in \underline{X}_1$ for all lattices L in $\underline{X}_1$ and $a \in L$. On the other hand, $\underline{X}_1$ is called *s-closed* if $[0,a] \in \underline{X}_1$ for all lattices L in $\underline{X}_1$ and $a \in L$. Also, $\underline{X}_1$ is *p-closed* if $L \in \underline{X}_1$ for any lattice L such that $[0,b] \in \underline{X}_1$ and $[b,1] \in \underline{X}_1$, for some $b \in L$.

Lemma 5.1.2. Let L be a lattice. Let a, b be any two elements of L such that $a \leq b$ and $b \in e([a,1])$. Then $b \in e(L)$.

Proof. Let $x \in L$ such that $x \wedge b = 0$. Then, by modularity of L, $(x \vee a) \wedge b = (x \wedge b) \vee a = 0 \vee a = a$. Since $b \in e([a,1])$, then $(x \vee a) = a$. Thus $x \leq a \leq b$. Therefore, $x = 0$, and hence, b is an essential element of L.

Proposition 5.1.3. Let $\underline{X}_1$ be any class of lattices. Then

(i) $h\underline{X}_1$, $d\underline{X}_1$, and $e\underline{X}_1$ are all q-closed, and

(ii) $h\underline{X}_1$ is s-closed if $\underline{X}_1$ is s-closed.

Proof. (i) By 5.1.1 (iv), $h\underline{X}_1$ is q-closed. Let a, b ϵ L such that b ϵ [a,1]. Suppose that L ϵ $d\underline{X}_1$. Then there exist c, d ϵ L such that $c \wedge d = 0$, $c \vee d = 1$, $b \leq c$, and [b,c] ϵ $\underline{X}_1$. By the modular law, $c \wedge (a \vee d) = a \vee (c \wedge d) = a \vee 0 = a$. On the other hand $c \vee (a \vee d) = 1$. Therefore c ϵ c([a,1]), and hence, [a,1] ϵ $d\underline{X}_1$. Finally suppose that L ϵ $e\underline{X}_1$ and b ϵ e([a,1]). By Lemma 5.1.2, b is essential in L, and hence, [b,1] ϵ $\underline{X}_1$. Thus [a,1] ϵ $e\underline{X}_1$. So all the classes $h\underline{X}_1$, $d\underline{X}_1$ and $e\underline{X}_1$ are q-closed.

(ii) Suppose that $\underline{X}_1$ is s-closed and L ϵ $h\underline{X}_1$. Let a, b ϵ L such that $0 \leq b \leq a$. Then [b,1] ϵ $\underline{X}_1$. Since a ϵ [b,1] and $\underline{X}_1$ is s-closed, then [b,a] ϵ $\underline{X}_1$. Therefore, [0,a] ϵ $h\underline{X}_1$, and hence, $h\underline{X}_1$ is s-closed.

Note that in Proposition 1.1.2 it was shown that $e\underline{X}$ is s-closed when $\underline{X}$ is s-closed. But this fact can not be extended to lattices, because the class $\underline{Z}_1$ is s-closed but $e\underline{Z}_1$ is not s-closed (see Example 3.1.5).

Let $\underline{X}_1$ and $\underline{Y}_1$ be classes of lattices. Then $\underline{X}_1 \oplus \underline{Y}_1$ will denote the class of lattices L such that there exists $c \in c(L)$ with $[0,c] \in \underline{X}_1$ and $[c,1] \in \underline{Y}_1$. Note that a lattice $L \in \underline{X}_1 \oplus \underline{Y}_1$ if and only if there exist $c, c' \in L$ with $c \vee c' = 1$, $c \wedge c' = 0$, $[0,c] \in \underline{X}_1$, and $[0,c'] \in \underline{Y}_1$.

For any classes of lattices $\underline{X}_1$ and $\underline{Y}_1$ we define the class $\underline{X}_1\underline{Y}_1$ to be the class of lattices L such that there exists an element a of L with $[0,a] \in \underline{X}_1$ and $[a,1] \in \underline{Y}_1$. It is clear that, for any classes of lattices $\underline{X}_1$ and $\underline{Y}_1$, $\underline{X}_1 \oplus \underline{Y}_1 \subseteq \underline{X}_1\underline{Y}_1$.

<u>Lemma 5.1.4</u>. Let L be any lattice. let $a, b, c \in L$. then

(i) If $a \wedge (b \vee c) = 0$, and $b \wedge c = 0$, then $(a \vee b) \wedge c = 0$.

(ii) If b is essential in L, then $a \wedge b$ is essential in $[0,a]$.

Proof. (i) Since $(b \vee c) \wedge (a \vee b) = ((b \vee c) \wedge a) \vee b = b$, then $(a \vee b) \wedge c \leq (a \vee b) \wedge (b \vee c) = b$, by the modular law. But $(a \vee b) \wedge c \leq c$. Therefore, $(a \vee b) \wedge c \leq b \wedge c = 0$.

(ii) It is clear that $a \wedge b \in [0,a]$. Suppose that d is a non-zero element such that $d \in [0,a]$. Since $d \wedge (a \wedge b) = d \wedge b$ and $b \in e(L)$, then $d \wedge (a \wedge b) \neq 0$. Thus $a \wedge b \in e([0,a])$.

Proposition 5.1.5. Let $\underline{X}_1$ be any class of lattices. Then

(i) $\underline{C}_1 \oplus d\underline{X}_1 = d\underline{X}_1$, and

(ii) $\underline{C}_1 \oplus e\underline{X}_1 = e\underline{X}_1$.

Proof. (i) By the definition $d\underline{X}_1 \subseteq \underline{C}_1 \oplus d\underline{X}_1$. Let L be a lattice such that $L \in \underline{C}_1 \oplus d\underline{X}_1$. Then there exist two elements a, a' of L such that $a \wedge a' = 0$, $a \vee a' = 1$, $[0,a] \in \underline{C}_1$, and $[0,a'] \in d\underline{X}_1$. Let $b \in L$. Then $a' \vee b = (a' \vee b) \wedge (a' \vee a) = a' \vee ((a' \vee b) \wedge a)$, by the modular law. Let $c = (a' \vee b) \wedge a$. Then $c \leq a$. But $[0,a]$ is complemented. Thus there exists c' such that $a = c \vee c'$ and $c \wedge c' = 0$. Hence $(a' \vee b) \vee c' = (a' \vee c) \vee c' = a' \vee a = 1$. By Lemma 5.1.4, $(a' \vee c) \wedge c' = 0$. Therefore, $(a' \vee b) \wedge c' = 0$, and hence, $a' \vee b \in c(L)$. By Proposition 5.1.3, $[0,a'] \in d\underline{X}_1$. Thus there exist $d, d' \in L$ such that $d \vee d' = a'$, $d \wedge d' = 0$, $a' \wedge b \leq d$, and $[a' \wedge b, d] \in \underline{X}_1$. Therefore, $a' \wedge b \leq b \wedge d$. On the other hand, $b \wedge d = b \wedge a' \wedge d = b \wedge a'$. Thus $a' \wedge b = b \wedge d$. So $[b \wedge d, d] \in \underline{X}_1$. Therefore, $[b, b \vee d] \in \underline{X}_1$.

Note that $(b \vee d) \vee d' = b \vee (d \vee d') = b \vee a'$. Moreover, $(b \vee d) \wedge d' = ((b \vee d) \wedge a') \wedge d' = ((b \wedge a') \vee d) \wedge d' = d \wedge d'$. Therefore, $(b \vee d) \wedge d' = 0$. Hence, $b \vee d \in c([0, a' \vee b])$. Since $a' \vee b \in c(L)$, then $b \vee d \in c(L)$. On the other hand, $b \leq b \vee d$

and $[b, b \vee d] \in \underline{X}_1$. It follows that $L \in d\underline{X}_1$.

(ii) By definition, $e\underline{X}_1 \subseteq \underline{C}_1 \oplus e\underline{X}_1$. Let $L \in \underline{C}_1 \oplus e\underline{X}_1$. Then there exist a, $b \in L$ such that $a \wedge b = 0$, $a \vee b = 1$, $[0,a] \in \underline{C}_1$ and $[0,b] \in e\underline{X}_1$. Let x be any essential element of L. Then, by Lemma 5.1.4, $x \wedge a \in e([0,a])$. But $x \wedge a \in c([0,a])$. Therefore, $x \wedge a = a$. So $a \leq x$ and $x \vee b = 1$. Therefore, $[x,1] = [x, x \vee b]$ which is isomorphic to $[x \wedge b, b]$. On the other hand, by Lemma 5.1.4, $x \wedge b \in e([0,b])$. But $[0,b] \in e\underline{X}_1$. Thus $[x \wedge b, b] \in \underline{X}_1$, and hence, $[x,1] \in \underline{X}_1$. Therefore, $L \in e\underline{X}_1$.

<u>Proposition 5.1.6</u>. Let $\underline{X}_1$ be any p-closed class of lattices. Then,

(i) $h\underline{X}_1 \oplus h\underline{X}_1 = h(h\underline{X}_1) = h\underline{X}_1$,

(ii) $e\underline{X}_1 \oplus e\underline{X}_1 = (e\underline{X}_1)(h\underline{X}_1) = e\underline{X}_1$, and

(iii) $h\underline{X}_1 \oplus d\underline{X}_1 = d\underline{X}_1$.

Proof. (i) Using the definition and Proposition 5.1.1, we can see that $h(h\underline{X}_1) = h\underline{X}_1 \subseteq h\underline{X}_1 \oplus h\underline{X}_1$. Suppose that $L \in h\underline{X}_1 \oplus h\underline{X}_1$. Then there exists an element $b \in c(L)$ such that $[0,b] \in h\underline{X}_1$ and $[b,1] \in h\underline{X}_1$. Let $a \in L$. Since $a \wedge b \leq b$, then $[a \wedge b, b] \in \underline{X}_1$. Therefore, $[a, a \vee b] \in \underline{X}_1$. Since $[b,1] \in h\underline{X}_1$ and $b \leq a \vee b \leq 1$, then $[a \vee b,1] \in \underline{X}_1$. But $\underline{X}_1$ is p-closed. Therefore, $[a,1] \in \underline{X}_1$, and hence, $L \in h\underline{X}_1$.

(ii) By definition, $e\underline{X}_1 \subseteq e\underline{X}_1 \oplus e\underline{X}_1$. Let $L \in e\underline{X}_1 \oplus e\underline{X}_1$. Then there exist $b, b' \in L$ such that $[0,b] \in e\underline{X}_1$, $[0,b'] \in e\underline{X}_1$, $b \wedge b' = 0$, and $b \vee b' = 1$. Let $a \in e(L)$. Then, by Lemma 5.1.4, $a \wedge b \in e([0,b])$. Thus, $[a \wedge b,b] \in \underline{X}_1$. Hence $[a,a \vee b] \in \underline{X}_1$. Since $a \in e(L)$, then $a \vee b \in e(L)$. Therefore, by Lemma 5.1.4, $(a \vee b) \wedge b' \in e([0,b'])$. Thus, $[(a \vee b) \wedge b',b'] \in \underline{X}_1$. Since $[a \vee b,1] = [a \vee b,(a \vee b) \vee b'] \cong [(a \vee b) \wedge b',b']$, then $[a \vee b,1] \in \underline{X}_1$. But $\underline{X}_1$ is p-closed. Therefore, $[a,1] \in \underline{X}_1$. It follows that $L \in e\underline{X}_1$.

It is clear that $e\underline{X}_1 \subseteq (e\underline{X}_1)(h\underline{X}_1)$ for any class $\underline{X}_1$ of lattices. Suppose that $L \in (e\underline{X}_1)(h\underline{X}_1)$. Then, there exists $a \in L$ such that $[0,a] \in e\underline{X}_1$ and $[a,1] \in h\underline{X}_1$. Let $b \in e(L)$. Then, by Lemma 5.1.4, $a \wedge b \in e([0,a])$. Therefore, $[a \wedge b,a] \in \underline{X}_1$, and hence, $[b, a \vee b] \in \underline{X}_1$. Since $[a,1] \in h\underline{X}_1$, then $[a \vee b,1] \in \underline{X}_1$. Thus, $[b,1] \in \underline{X}_1$, because $\underline{X}_1$ is p-closed. Therefore, $L \in e\underline{X}_1$.

(iii) By the definition $d\underline{X}_1 \subseteq h\underline{X}_1 \oplus d\underline{X}_1$. Let $L \in h\underline{X}_1 \oplus d\underline{X}_1$. Then there exist $b, b' \in L$ such that $[0,b] \in h\underline{X}_1$, $[0,b'] \in d\underline{X}_1$, $b \wedge b' = 0$, and $b \vee b' = 1$. Let $a \in L$. Since $0 \leq a \wedge b \leq b$, then $[a \wedge b,b] \in \underline{X}_1$, and hence, $[a,a \vee b] \in \underline{X}_1$. Since $[0,b'] \in d\underline{X}_1$, then there exist $d, d' \in L$ such that $(a \vee b) \wedge b' \leq d$, $d \wedge d' = 0$, $d \vee d' = b'$ and $[(a \vee b) \wedge b', d] \in \underline{X}_1$. But $(a \vee b) \wedge b' \leq a \vee b$. So $(a \vee b) \wedge b' \leq (a \vee b) \wedge d$. Since $(a \vee b) \wedge d \leq (a \vee b) \wedge b'$, then $(a \vee b) \wedge d = (a \vee b) \wedge b'$. Therefore $[(a \vee b) \wedge b', d]$ is lattice isomorphic to $[a \vee b, b \vee d]$. So $[a \vee b, b \vee d] \in \underline{X}_1$. Since $\underline{X}_1$ is p-closed, then $[a, b \vee d] \in \underline{X}_1$. Finally, we need to show that $b \vee d \in c(L)$. Note that, by Lemma 5.1.4 (i), we have $(b \vee d) \wedge d' = 0$. But $(b \vee d) \vee d' = b \vee (d \vee d') = b \vee b' = 1$. Thus $b \vee d \in c(L)$. Therefore $L \in d\underline{X}_1$.

§ 5.2 Pseudo-complemented Lattices.

Let L be a complete lattice. A subset X of L is called *independent* if any finite subset $\{x_1, x_2, \ldots, x_n\}$ of non-zero elements of X satisfies the following: for every $1 \leq i \leq n$, $x_i \wedge (\vee x_j) = 0$ where $i \neq j \in \{1, 2, \ldots, n\}$. For any subset $S = \{x_i : i \in I\}$ of L we write $\vee_I x_i$ for $\vee S$, and $\wedge_I x_i$ for $\wedge S$.

Define a lattice L to be *essentially good* if the following condition hold: suppose that $A = \{a_1, a_2, \ldots\}$ is a countable independent subset of L and $B = \{b_1, b_2, \ldots\}$ is a subset of A such that for all i, b_i is essential in $[0,a_i]$. Then $\vee B$ is essential in $[0,\vee A]$.

Let L be a lattice and $a \in L$. By a ***pseudo-complement*** b of a (if it exists) we mean an element b which is maximal in the set $S = \{ c \in L : a \wedge c = 0\}$. We shall call L ***pseudo-complemented*** if every element of L has a pseudo-complement. We denote the class of pseudo-complemented lattices by $\underline{P}^c$.

Lemma 5.2.1. Let L be a lattice. For any elements a, $b \in L$, If a is a pseudo-complement of b, then $a \wedge b = 0$ and $a \vee b \in e(L)$.

Proof. Suppose that a is a pseudo-complement of b. Let $c \in L$ such that $(a \vee b) \wedge c = 0$. Then by the modular law we obtain

$$b \wedge (a \vee c) = b \wedge (b \vee a) \wedge (a \vee c) = b \wedge (a \vee ((b \vee a) \wedge c))$$

$$= b \wedge a = 0.$$ By the maximality of a, $a \vee c = a$.

Thus $c \leqslant a \leqslant a \vee b$. Therefore $c = (a \vee b) \wedge c = 0$, and hence, $a \vee b$ is essential in L.

Lemma 5.2.2. Let L be a lattice. Then L is complemented if and only if L is pseudo-complemented and L does not have an element b such that $b \neq 1$ and $b \in e(L)$.

Proof. Let $a \in L$. Suppse that $L \in \underline{C}_1$. Then there exists $b \in L$ such that $a \wedge b = 0$ and $a \vee b = 1$. Thus, $L \in \underline{P}^c$. Moreover, if $a \neq 1$, then $b \neq 0$. Therefore $a \notin e(L)$. Conversely, Suppose that there is no element b with $b \neq 1$ and $b \in e(L)$. By Lemma 5.2.1 and the hypothesis, there exist $c \in L$ such that $a \wedge c = 0$ and $a \vee c \in e(L)$. Therefore $a \vee c = 1$. Thus L is complemented.

A lattice L is called *upper continuous* if for every chain $\{a_\lambda\}$ of elements of L and every $b \in L$, $b \wedge (\vee a_\lambda) = \vee (b \wedge a_\lambda)$. We call L *weak upper continuous*, and denote it by WUC-lattice, if for every chain $\{a_\lambda\}_\Lambda$ of elements of L and each element b of L, the condition $(\vee a_\lambda) \wedge b \neq 0$ implies $a_\mu \wedge b \neq 0$ for some μ in Λ (i.e. if $a_\lambda \wedge b = 0$ for all $\lambda \in \Lambda$, then $(\vee a_\lambda) \wedge b = 0$). It clear that every upper continuous lattice is a WUC-lattice.

An element x of a lattice L is called an *atom* provided that $[0,x] = \{0, x\}$. We shall call L *atomic* if $[0,a]$ contains an atom for every non-zero element a of L. For any lattice L, Let $s(L) = \vee \{a : a \text{ is an atom}\}$. For an upper continuous lattice L, $[0,s(L)]$ is atomic (see [CD, 4.3]).

For any lattice L, let $s_1(L) = \wedge \{a : a \text{ is essential in } L\}$ and $s_2(L) = \vee \{b : [0,b] \text{ is complemented}\}$. Before we study the relation between s(L), $s_1(L)$ and $s_2(L)$ we introduce the following terminology.

For any lattice L, let $f(L) = \{ a \in L : a > \vee b_\lambda$ for every chain $\{b_\lambda\}_\Lambda$ of elements of L with $a > b_\lambda$ for all $\lambda \in \Lambda\}$. We define L to be *weak noetherian* if for all elements a, b $\in$ L with a < b, there exists c $\in$ f(L) such that $c \leq b$ and $c \nleq a$. Note that finite lattices are weak noetherian.

Lemma 5.2.3. Let L be any lattice. Then $s(L) \leq s_2(L) \leq s_1(L)$. If L is pseudo-complemented, then $s_1(L) = s_2(L)$. On the other hand $s_1(L) = s_2(L) = s(L)$ whenever L is pseudo-complemented and weak noetherian.

Proof. Let a be an atom. Then [0,a] is complemented.Therefore, $a \leq s_2(L)$. Thus $s(L) \leq s_2(L)$. Let c, b $\in$ L with c $\in$ e(L) and [0,b] is complemented. Then, by Lemma 5.1.4, $c \wedge b \in e([0,b])$. Therefore there exists d $\in$ [0,b] such that $b = (c \wedge b) \vee d$ and $c \wedge d = 0$. So d = 0, and hence, $b \leq c$. So $b \leq s_1(L)$. Therefore $s(L) \leq s_2(L) \leq s_1(L)$.

Suppose that L is pseudo-complemented. Now we show that $[0, s_1(L)]$ is complemented. Let $a \in [0, s_1(L)]$. Then there exists $b \in L$ such that $a \wedge b = 0$ and $a \vee b \in e(L)$. Thus $s_1(L) \leq a \vee b$. Hence $s_1(L) = s_1(L) \wedge (a \vee b) = a \vee (s_1(L) \wedge b)$, by the modular law. But $a \wedge (s_1(L) \wedge b) = a \wedge b = 0$. Therefore $[0, s_1(L)]$ is complemented. Thus $s_1(L) \leq s_2(L)$, and hence, $s_1(L) = s_2(L)$.

Let L be a pseudo-complemented and weak noetherian lattice. Suppose that $s(L) < s_1(L)$. Then there exists $c \in f(L)$ such that $c \nleq s(L)$ and $c \leq s_1(L)$. Since $[0,c]$ is complemented, then there exists $d \in [0,c]$ such that $c = d \vee (c \wedge s(L))$ and $d \wedge s(L) = 0$. Suppose that $d > 0$. Then there exists $c' \in f(L)$ with $c' > 0$ and $c' \leq d$. Let $X = \{x \in L : x < c'\}$. Then X is not empty, because $0 \in X$. Since $c' \in f(L)$, then by Zorn's Lemma X has a maximal member p. But $[0,c]$ is complemented and $c' \leq d \leq c$. Thus $[0,c']$ is also complemented. Therefore there exists $y \in [0,c']$ such that $c' = p \vee y$ and $p \wedge y = 0$. Since p is maximal and $[0,y]$ is lattice isomorphic to $[p,c']$, then y is an atom. Therefore, by definition, $y \in [0,s(L)]$. Thus $y \leq c' \wedge s(L) \leq d \wedge s(L) = 0$. Hence $y = 0$. Therefore $c' = p$, a contradiction. Thus $d = 0$, and hence, $c = c \wedge s(L)$. Therefore $c \leq s(L)$, a contradiction. Thus $s(L) = s_1(L) = s_2(L)$.

We shall call L ***amply pseudo-complemented*** if for every interval [a,b] of L, [a,b] is pseudo-complemented. It is clear that every complemented lattice is amply pseudo-complemented and every amply pseudo-complemented is pseudo-complemented.

<u>Lemma 5.2.4</u>. Let L be a WUC-lattice. Then for any $y \in L$, $[0,y]$ is pseudo-complemented and essentially good. Moreover, if L is upper continuous then $[x,y]$ is pseudo-complemented for any elements x and y of L.

Proof. Suppose that L is WUC-lattice. Let $a \in [0,y]$. Then the set $S = \{ b \in [0,y] : a \wedge b = 0 \}$ is not empty, because $0 \in S$. Let $(b_\lambda)_\Lambda$ be any chain in S. Then $a \wedge b_\lambda = 0$ for every $\lambda \in \Lambda$. Since L is WUC-lattice, then $a \wedge (\vee b_\lambda) = 0$. Thus $\vee b_\lambda \in S$, and hence, by Zorn's Lemma S has a maximal member p. Thus a has a pseudo-complement p in $[0,y]$, and hence, $[0,y] \in \underline{P}^c$.

Let $\{a_i, i \in I\}$ be a countable independent set of elements of $[0,y]$. Let $b_1, b_2, \ldots$ be elements of $[0,y]$ such that for each i, $b_i \in e([0,a_i])$. Then $\{b_i\}$ is an independent set of elements of $[0,y]$. Let $a = \vee a_i$, $b = \vee b_i$, and $0 \neq c \leq a$. Then $c = c \wedge a = c \wedge (\vee a_i) \neq 0$. Therefore, there exists $j \in I$ such that $c \wedge a_j \neq 0$. Thus $(c \wedge a_j) \wedge b_j = c \wedge b_j \neq 0$. Therefore, $c \wedge b \neq 0$. Hence $a \in e([0,b])$. Thus $[0,y]$ is essentially good.

Suppose that L is upper continuous. Let $a \in [x,y]$. Then the set $S = \{b \in [x,y] : a \wedge b = x\}$ is not empty, because $x \in S$. Let $\{b_\lambda\}_\Lambda$ be a chain in S. Then $a \wedge (\vee b_\lambda) = \vee (a \wedge b_\lambda) = x$. Thus $\vee b_\lambda \in S$. So, by Zorn's Lemma, S has a maximal element which is a pseudo-complement of a in $[x,y]$. Therefore $[x,y]$ is pseudo-complemented, and hence, L is amply pseudo-complemented.

Lemma 5.2.5. Let L be an upper continuous lattice. Then $[0,s(L)]$ is complemented and every element of $[0,s(L)]$ is a join of atoms.

Proof. Suppose that L is upper continuous. Let $x \in [0,s(L)]$. Then, by Lemma 5.2.4, $[0,s(L)] \in \underline{P}^c$. Thus, by Lemma 5.2.1, there exists an element $y \in [0,s(L)]$ such that $x \wedge y = 0$ and $x \vee y \in e([0,s(L)])$. Let $0 \neq a \in L$ be an atom. Then, by the definition of $s(L)$, $a \leq s(L)$. Thus $(x \vee y) \wedge a \neq 0$, and hence, $(x \vee y) \wedge a = a$. Hence $a \leq x \vee y$. Therefore $s(L) = x \vee y$. Thus $[0,s(L)]$ is complemented.

Now let $b = \vee \{a : a \leq x \text{ where a is an atom}\}$. Then $b \leq x$.

Suppose that $b \neq x$. Since $[0,s(L)]$ is complemented, then $[0,x]$ is complemented. Therefore there exists $0 \neq c \in [0,x]$ such that $b \wedge c = 0$ and $b \vee c = x$. By [CD, 4.3], $[0,s(L)]$ is atomic. But $c \neq 0$. Therefore there exists an atom $d \leq c$. But $d \wedge b = 0$. Thus $d \not\leq x$, a contradiction. Thus $b = x$.

Note that, by the above lemma , for any upper continuous lattice L and for any element $x \in L$, $s([0,x]) = x \wedge s(L)$.

Let L be a lattice. We say that L has *finite uniform dimension* if every independent subset of L is finite. In this case, there exists a least positive integer k, called the *uniform dimension* of L, such that every independent subset of L contains at most k elements (see [Pu, Theorem 2.10]).

We shall call a lattice L *uniform* when every non zero element of L is essential in L. It is known that a lattice L has finite uniform dimension k if and only if there exist elements $a_1, a_2, \ldots, a_k$ of L such that $(a_1 \vee a_2 \vee \ldots \vee a_k)$ is essential in L and $[0,a_i]$ is uniform for all $i = 1, 2, \ldots , k$ (see [Pu, Theorem 2.8]). We denote the class of lattices which have finite uniform dimension by $\underline{U}_1$.

Lemma 5.2.6. Suppose that L is pseudo-complemented and has finite uniform dimension. Moreover let $L \in e\underline{U}_1$. Then $L \in h\underline{U}_1$.

Proof. Suppose that $L \in \underline{U}_1 \cap e\underline{U}_1 \cap \underline{P}^c$. Let $a \in L$. Then there exists a pseudo-complement c of a. By Lemma 5.2.1, $a \vee c$ is essential in L. Thus $[a \vee c, 1]$ has finite uniform dimension. Since $[a \wedge c, c] = [0, c]$ and $\underline{U}_1$ is s-closed, then $[a, a \vee c]$ has finite uniform dimension. But, by [Pu, Corollary 2.9], $\underline{U}_1$ is p-closed. Therefore $[a,1]$ has finite uniform dimension, and hence, $L \in h\underline{U}_1$.

Lemma 5.2.7. Let L be an amply pseudo-complemented lattice. Then $L \in d\underline{U}_1$ if and only if $L \in d(h\underline{U}_1)$.

Proof. By Proposition 5.1.1, $d(h\underline{U}_1) \subseteq d\underline{U}_1$. Suppose that L is amply pseudo-complemented such that $L \in d\underline{U}_1$. Let $a \in L$. Then there exist $c, c' \in L$ such that $c \wedge c' = 0$, $c \vee c' = 1$, $a \leq c$, and $[a,c]$ has finite uniform dimension. By Proposition 5.1.3, $[c',1] \in d\underline{U}_1$. Therefore $[0,c] \in d\underline{U}_1$, and hence, by Proposition 5.1.3, $[a,c] \in d\underline{U}_1$. Thus, by Proposition 5.1.1, $[a,c] \in e\underline{U}_1$. By the hypothesis and Lemma 5.2.6, $[a,c] \in h\underline{U}_1$. Thus $L \in d(h\underline{U}_1)$.

§ 5.3 Lattices which satisfy the ACC.

Let L be a lattice. Then L is said to satisfy the ACC (or, be noetherian) if for every ascending chain $a_1 \leq a_2 \leq \ldots$ of elements of L there exists n such that $a_{n+i} = a_n$ for all $i \geq 1$. Let $\underline{N}_1$ denote the class of lattices which satisfy the ACC.

Lemma 5.3.1. Let L be a lattice and a, b ϵ f(L). Then

(i) L is noetherian if and only if f(L) = L.

(ii) If c is a complement in [0,a], then c ϵ f(L).

(iii) If L is upper continuous, then $a \vee b \ \epsilon$ f(L).

Proof. (i) If L is noetherian, then for any chain $(a_\lambda)_\Lambda$ of L, $\vee a_\lambda = a_\beta$ for some $\beta \ \epsilon \ \Lambda$. Hence f(L) = L. Conversely, suppose that f(L) = L. Let $a_1 < a_2 < \ldots$ be an infinite ascending chain of L. Then $a_i < \vee a_k$ $(k \geq i)$. But, by hypothesis, $\vee a_k \ \epsilon$ f(L). Thus $\vee a_i < \vee a_k$, a contradiction. Hence the chain is finite. Therefore L is noetherian.

(ii) Suppose that there exists d ϵ L such that $c \vee d = a$ and $c \wedge d = 0$. Let $(b_\lambda)_\Lambda$ be a chain of elements of L. Suppose that $c = \vee b_\lambda$. Then $a = c \vee d = (\vee b_\lambda) \vee d = \vee (b_\lambda \vee d)$. Thus, by the hypothesis, there exists $\mu \in \Lambda$ such that $a = b_\mu \vee d$. Hence $c = c \wedge a = c \wedge (b_\mu \vee d) = b_\mu \vee (c \wedge d) = b_\mu \vee 0 = b_\mu$. Therefore $c \in f(L)$.

(iii) Suppose that L is upper continuous and $a \vee b = \vee b_\lambda$ for some chain $\{b_\lambda\}_\Lambda$. Then $a = a \wedge (a \vee b) = a \wedge (\vee b_\lambda)$. Thus $a = \vee (a \wedge b_\lambda)$, and hence, $a = a \wedge b_\mu$ for some $\mu \in \Lambda$. Therefore $a \leqslant b_\mu$. Similarly, $b \leqslant b_\gamma$ for some $\gamma \in \Lambda$. Since $\{b_\lambda\}_\Lambda$ is a chain, assume that $b_\gamma \leqslant b_\mu$. Thus $a \vee b \leqslant b_\mu \vee b_\gamma = b_\mu \leqslant \vee b_\lambda$. Therefore $a \vee b = b_\mu$, and hence, $a \vee b \in f(L)$.

Note that, by the above lemma, every noetherian lattice is weak noetherian.

<u>Lemma 5.3.2.</u> Let L be an upper continuous and weak noetherian lattice. Suppose that for some elements a and b of L $[a,b]$ is noetherian. Then there exists $d \in f(L)$ such that $b = a \vee d$.

Proof. If $a = b$, then we can take $d = 0$. Suppose that $a < b$. Then, by hypothesis, there exists $c_1 \in f(L)$ such that $c_1 \leqslant b$ and $c_1 \not\leqslant a$. Suppose that $a \vee c_1 < b$. Then there exists $c_2 \in f(L)$ such that $c_2 \leqslant b$ and $c_2 \not\leqslant a \vee c_1$. By continuing these process we get the ascending chain $a < a \vee c_1 < a \vee c_1 \vee c_2 < \dots$. Since $[a,b] \in \underline{N}_1$, then $b = a \vee c_1 \vee c_2 \vee \dots \vee c_n$ such that $c_i \in f(L)$ for $i = 1,2,\dots,n$. Let $d = c_1 \vee c_2 \vee \dots \vee c_n$. Since $[a,b] \in \underline{N}_1$, then any chain in $[a,b]$ is finite, and hence, $[a,b]$ is upper continuous. Thus, by Lemma 5.3.1, $d \in f(L)$.

<u>Lemma 5.3.3.</u> Let L be a complemented upper continuous lattice. Suppose that $1 \notin f(L)$. Then there exist elements b and c of L such that both $b, c \notin f(L)$ with $b \vee c = 1$ and $b \wedge c = 0$.

Proof. If L has finite uniform dimension, then there exist elements $a_1, a_2, \dots, a_n$ such that $(a_1 \vee a_2 \vee \dots \vee a_n) \in e(L)$ and $[0,a_i]$ is a uniform lattice for all $i = 1, 2, \dots, n$ (see [Pu, Theorem 2.8]). Since L is complemented, then $[0,a_i]$ is complemented ($i = 1, 2, \dots, n$) and $a_1 \vee a_2 \vee \dots \vee a_n = 1$. Therefore a_i is an atom ($i = 1, 2, \dots, n$). Hence $a_i \in f(L)$ for all $i = 1, 2, \dots, n$. Thus, by Lemma 5.3.1 (iii), $1 \in f(L)$, a contradiction. Thus L does not have finite uniform dimension. Hence there exists an infinite independent set $\{b_1, b_2, \dots\}$ of elements of L. Let $a = \vee b_i$ where i is odd, and $b = \vee b_i$ where i is even. Now we show that $a \notin f(L)$. For $i = 1, 2, \dots$, Let

$d_i = b_1 \vee b_3 \vee \dots \vee b_{2i-1}$. Then $\{d_i\}$ is a chain with $d_i \leq a$ for all i But $\vee d_i = a$. Thus $a \notin f(L)$. Similarly, $b \notin f(L)$.

Suppose that $a \wedge b \neq 0$, then , by hypothesis, $b_j \wedge b_i \neq 0$ for some odd j, even i, a contradiction. Hence $a \wedge b = 0$. By the hypothesis, there exists $c \in L$ such that $(a \vee b) \vee c = 1$ and $(a \vee b) \wedge c = 0$. Hence, by Lemma 5.1.4, $a \wedge (b \vee c) = 0$. Note that $b \wedge c = 0$. Suppose that $b \vee c \in f(L)$. Then, by Lemma 5.3.1, $b \in f(L)$, a contradiction. Therefore, $b \vee c \notin f(L)$.

<u>Lemma 5.3.4</u>. Let L be a pseudo-complemented lattice such that $L \in e\underline{N}_1$. Then, for any element a of L, $[0,a] \in e\underline{N}_1$.

Proof. Let $b \in e([0,a])$. Let $b \leq a_1 \leq a_2 \leq \dots \leq a$ be an ascending chain of elements in [0,a]. By the hypothesis, there exists $c \in L$ such that $a \wedge c = 0$ and $a \vee c \in e(L)$. Therefore $b \vee c \in e(L)$. Thus $[b \vee c, 1] \in \underline{N}_1$. Now we have the ascending chain $b \vee c \leq a_1 \vee c \leq a_2 \vee c \leq \dots \leq 1$. Hence there exists a positive integer n such that $a_n \vee c = a_{n+1} \vee c = a_{n+2} \vee c = \dots$ Therefore $a_{n+1} = a_{n+1} \wedge (a_{n+1} \vee c) = a_{n+1} \wedge (a_n \vee c)$. Thus, by modular law, $a_{n+1} = a_n \vee (a_{n+1} \wedge c) = a_n \vee 0 = a_n$. Thus [b,a] is noetherian, and hence, $[0,a] \in e\underline{N}_1$.

<u>Lemma 5.3.5</u>. Let L be a WUC-lattice. Suppose that $s_1(L) = 0$ and $L \in e\underline{N}_1$. Then L has finite uniform dimension.

Proof. Suppose that L does not have finite uniform dimension. Then there exists an infinite countable set of independent elements $x_1, x_2, \ldots$ of L. By Lemma 5.2.4, L is essentially good and pseudo-complemented. Thus, by Lemma 5.2.3, $[0,x_i]$ is not complemented for all i. Therefore, by Lemma 5.2.2, for all i there exists $x_i \neq b_i \in e([0,x_i])$. Therefore $\vee b_i$ is essential in $[0,\vee x_i]$. Hence, by Lemma 5.3.4, $[0,\vee x_i] \in e\underline{N}_1$. Therefore $[\vee b_i,\vee x_i]$ is noetherian. On the other hand

$$\vee b_i \leq x_1 \vee b_2 \vee b_3 \vee \ldots \leq x_1 \vee x_2 \vee b_3 \vee \ldots \leq \ldots \leq \vee x_i$$

is an ascending chain in $[\vee b_i,\vee x_i]$. Therefore it must be finite, and hence, there exists a positive integer n such that $a \vee (\vee b_j) = a \vee x_{n+1} \vee (\vee b_k)$ where $a = x_1 \vee x_2 \vee \ldots \vee x_n$, $j \in \{n+1, n+2, \ldots\}$ and $k \in \{n+2, n+3, \ldots\}$. Therefore we have $x_{n+1} = x_{n+1} \wedge (a \vee x_{n+1} \vee (\vee b_k)) = x_{n+1} \wedge (a \vee (\vee b_j))$. Note that x_{n+1} can be written as $x_{n+1} = x_{n+1} \wedge (a \vee b_{n+1} \vee (\vee b_k))$. Thus, by modular law, $x_{n+1} = b_{n+1} \vee (x_{n+1} \wedge (a \vee (\vee b_k)))$.

But $x_1, x_2, \ldots$ are independent. Therefore $x_{n+1} \wedge a = 0$. Note that $a \vee b_{n+2} \leq a \vee b_{n+2} \vee b_{n+3} \leq \ldots$ is a chain such that

$x_{n+1} \wedge (a \vee b_{n+2} \vee \ldots \vee b_t) = 0 \quad (t \geq n + 2)$. But the lattice is WUC-lattice. Therefore $x_{n+1} \wedge (a \vee (\vee b_k)) = 0$, and hence, $x_{n+1} = b_{n+1}$, a contradiction. Therefore L has finite uniform dimension.

Note that for any lattice L, if $[s_1(L),1]$ is noetherian then $L \in e\underline{N}_1$. For WUC-lattices the converse is true as we will see in the next generalization of Theorem 1.2.1.

<u>Theorem 5.3.6</u>. Suppose that L is a WUC-lattice and $L \in e\underline{N}_1$. Then $[s_1(L),1]$ is noetherian.

Proof. Suppose that L is a WUC-lattice and $L \in e\underline{N}_1$. Then, by Lemma 5.2.4, L is pseudo-complemented. Thus there exists $k \in L$ which is a pseudo-complement of $s_1(L)$ in L. Let $s = s_1(L)$. Then, by Lemma 5.2.1, $s \vee k \in e(L)$. Thus, by the hypothesis, $[s \vee k,1] \in \underline{N}_1$. Moreover, by Lemma 5.3.4, $[0,k] \in e\underline{N}_1$. Note that, by Lemma 5.1.4, $x \wedge k \in e([0,k])$ for all $x \in e(L)$. Thus $\wedge \{y : y \in e([0,k])\} \leq \wedge \{x \wedge k : x \in e(L)\} \leq \wedge \{x : x \in e(L)\}$. Hence $s_1([0,k]) \leq s_1(L)$. But $s_1([0,k]) \leq k$ and $k \wedge s_1(L) = 0$. Therefore $s_1([0,k]) = 0$. Therefore, by Lemma 5.3.5, $[0,k]$ has finite uniform dimension. Thus there exist uniform elements b_1, b_2, ... , b_n of $[0,k]$ such that $(b_1 \vee b_2 \vee \ldots \vee b_n) \in e([0,k])$ (see [Pu, Theorem 2.8]). Let $g = b_1 \vee b_2 \vee \ldots \vee b_n$. Therefore $[g,k] \in \underline{N}_1$, because $[0,k] \in e\underline{N}_1$. On the other hand, by Lemma 5.3.4, $[0,b_i] \in e\underline{N}_1$ for all $1 \leq i \leq n$. Therefore $[0,b_i]$ is noetherian for all $1 \leq i \leq n$, because b_1, b_2, ..., b_n are uniforms. Thus $[0,g]$ is noetherian. Hence $[0,k]$ is noetherian. But $[s,s \vee k]$ is a lattice isomorphic to $[0,k]$. Thus $[s,s \vee k]$ is noetherian. But $\underline{N}_1$ is p-closed (see [Pu, Corollary 2.9]). Therefore $[s,1]$ is noetherian.

<u>§ 5.4 Examples and Applications</u>.

In this section we apply our terminologies to some examples and prove some facts about the lattice of all submodules of any module. Before we start this section note that for any lattice L the opposite set $f^{o}(L)$ of $f(L)$ is equal to the following set: $\{ a \in L : \wedge b_\lambda > a$ for every chain $(b_\lambda)_\Lambda$ in L such that $b_\lambda > a$ for all $\lambda \in \Lambda\}$. Now we prove the following lemma.

Lemma 5.4.1. Let L be a complete, modular, upper continuous and weak noetherian lattice. Then the opposite lattice L^{o} of L is weak noetherian.

Proof. Let u, v be any two elements of L such that $u < v$. Then we need to find $c \in L$ such that $c \geq u$, $c \not\geq v$, and $c \in f^0(L)$. By hypothesis, there exists $w \in f(L)$ such that $w \leq v$ and $w \not\leq u$. Let $S = \{x \in L : u \leq x \text{ and } w \not\leq x\}$. Then $u \in S$. Let $\{x_\lambda\}_\Lambda$ be any chain in S. Let $y = \vee \{x_\lambda\}$. Suppose that $y \notin S$. Therefore $w \leq y$ because $u \leq y$. Hence $w = w \wedge y = w \wedge (\vee x_\lambda) = \vee \ (w \wedge x_\lambda)$. But $w \in f(L)$. Therefore $w = w \wedge x_\mu$ for some $\mu \in \Lambda$. Hence $w \leq x_\mu$, a contradiction. Thus $y \in S$, and hence, by Zorn's Lemma S has a maximal member p. We claim that $p \in f^0(L)$. Let $\{a_\lambda\}_\Lambda$ be a chain in L such that $p < a_\lambda$ for all λ in Λ. Then $u \leq p < a_\lambda$ for all λ in Λ. Thus, by the maximality of p, $a_\lambda \notin S$ for all λ in Λ. Thus $w \leq a_\lambda$ for all λ in Λ, and hence, $w \leq \wedge \{a_\lambda : \lambda \in \Lambda\}$. We know that $\wedge \{a_\lambda : \lambda \in \Lambda\} \geq p$. Suppose that $\wedge \{a_\lambda : \lambda \in \Lambda\} = p$. Then $\wedge \{a_\lambda : \lambda \in \Lambda\} \not\geq w$, a contradiction. Hence $\wedge \{a_\lambda : \lambda \in \Lambda\} > p$. So $p \in f^0(L)$. But $p \geq u$ and $p \not\geq w$. Thus L^0 is weak noetherian.

Now we give an example to show that a WUC-lattice does not need to be upper continuous.

<u>Example 5.4.2</u>. Let $S = \{(x,y)\colon 0 < x,y < 1$, and x, y are real numbers$\}$. Let $L = S \cup \{(0,0), (1,1)\}$ with the following order

$(a,b) \leqslant (c,d)$ if and only if $a \leqslant c$ and $b \leqslant d$. Moreover, let $(a,b) \wedge (c,d) = (\min a,c , \min b,d)$ "min means minimum" and $(a,b) \vee (c,d) = (\max a,c , \max b,d)$ "max means maximum". Then L is a lattice with the following properties.

__Fact 1__. The lattice L and its opposite lattice L^0 are both complete modular lattices.

Proof. It is clear that L and L^0 are complete lattices. Now let $a = (x_1,y_1)$, $b = (x_2,y_2)$, and $c = (x_3,y_3)$ such that $c \leqslant a$. Then $x_1 \geqslant x_3$ and $y_1 \geqslant y_3$. Moreover,

$a \wedge (b \vee c) = a \wedge (\max x_2,x_3, \max y_2,y_3)$

$\qquad = (\min\{x_1,\max x_2,x_3\}, \min\{y_1,\max y_2,y_3\})$ and

$(a \wedge b) \vee c = (\min x_1,x_2, \min y_1,y_2) \vee c$

$\qquad = (\max\{x_3,\min x_1,x_2\}, \max\{y_3,\min y_1,y_2\})$.

Thus to calculate $a \wedge (b \vee c)$ and $(a \wedge b) \vee c$ we only have the following three cases:

Case I. $x_1 \geqslant x_2 \geqslant x_3$. So we have the following cases:

1. If $y_1 \geqslant y_2 \geqslant y_3$, then $a \wedge (b \vee c) = (x_2,y_2) = (a \wedge b) \vee c$.
2. If $y_2 > y_1 \geqslant y_3$, then $a \wedge (b \vee c) = (x_2,y_1) = (a \wedge b) \vee c$.
3. If $y_2 \leqslant y_3 \leqslant y_1$, then $a \wedge (b \vee c) = (x_2,y_3) = (a \wedge b) \vee c$

Case II. $x_2 \geq x_1 \geq x_3$. So we have the following cases:

1. If $y_1 \geq y_2 \geq y_3$, then $a \wedge (b \vee c) = (x_1, y_3) = (a \wedge b) \vee c$.

2. If $y_2 \geq y_1 \geq y_3$, then $a \wedge (b \vee c) = (x_1, y_1) = (a \wedge b) \vee c$.

3. if $y_1 \geq y_3 \geq y_2$, then $a \wedge (b \vee c) = (x_1, y_3) = (a \wedge b) \vee c$.

Case III. $x_1 \geq x_3 \geq x_2$. So we have the following cases:

1. If $y_1 \geq y_2 \geq y_3$, then $a \wedge (b \vee c) = (x_3, y_2) = (a \wedge b) \vee c$.

2. If $y_2 \geq y_1 \geq y_3$, then $a \wedge (b \vee c) = (x_3, y_1) = (a \wedge b) \vee c$.

3. If $y_1 \geq y_3 \geq y_2$, then $a \wedge (b \vee c) = (x_3, y_3) = (a \wedge b) \vee c$.

Therefore, L is a modular lattice. Hence L^0 is modular, because the opposite lattice of a modular lattice is a modular lattice.

<u>Fact 2</u>. It is clear that L satisfies the following properties: $s(L) = s_1(L) = s_2(L) = 0$ and $c(L) = \{(0,0), (1,1)\}$. Moreover, $e(L) = L \setminus (0,0)$. Thus L is uniform, and hence, L is essentially good and pseudo-complemented.

<u>Fact 3</u>. L and L^0 are both weak upper continuous but neither L nor L^0 is upper continuous.

Proof. Let $\{a_\lambda\}_\Lambda$ be a chain of elements of L and $b \in L$. Suppose that $(\vee a_\lambda) \wedge b \neq 0$. Then $b \neq 0 \neq \vee a_\lambda$. Therefore, there exists

$\mu \in \Lambda$ such that $a_\mu \neq 0$. But $b \in e(L)$. Thus $a_\mu \wedge b \neq 0$. Hence L is weak upper continuous. Note that L^0 is WUC-lattice if and only if for every chain $\{a_\lambda\}_\Lambda$ in L and every $b \in L$ such that $a_\lambda \vee b = (1,1)$ for all $\lambda \in \Lambda$, $(\wedge a_\lambda) \vee b = (1,1)$. But it is clear that, if $b \vee a_\lambda = (1,1)$ then $b = (1,1)$ or $a_\lambda = (1,1)$. Therefore, L^0 is WUC-lattice.

Now Let $D = \{(1/2,y) : 0 < y < 1\}$ and $a = (3/4,1/4)$. Then D is a chain with $\vee D = (1,1)$. Thus $(\vee D) \wedge a = a$. On the other hand $\vee \{(d \wedge a): d \in D\} = (1,1) \neq a$. Therefore L is not upper continuous. On the other hand, $a \vee (\wedge D) = a \vee (0,0) = a$ and $\wedge \{(a \vee d): d \in D)\} = (0,0) \neq a$. Therefore the opposite lattice L^0 is not upper continuous.

<u>Fact 4</u>. L and L^0 are both not weak noetherian.

Proof. Let $I = \{1, 2, \ldots \}$. Let $(0,0) \neq a = (x,y) \in L$. For $n \in I$, let $b_n = (1 - 1/n)(x,y)$. Then $\{b_n\}_I$ is a chain such that $b_n < a$ for all $n \in I$. But $\vee b_n = a$. Thus $a \notin f(L)$, and hence, $f(L) = (0,0)$. Thus, by definition, L is not weak noetherian. Now let $(1,1) \neq a' = (x',y') \in L$. Moreover, for each $n \in I$, let $c_n = 1/n\ (1 - x', 1 - y') + (x',y')$. Then $c_n > a'$ for all n. But $\wedge c_n = a'$. Therefore $f^0(L) = (1,1)$, and hence, L^0 is not weak noetherian.

__Example 5.4.3__. Let $S_1 = \{(1 - 1/n , 0) : n = 1, 2, \ldots\}$ and $S_2 = \{(0,1), (1,1)\}$. Let $L = S_1 \cup S_2$. Then with the same ordering in Example 5.4.1, L has the following properties:

__Fact 1__. L and L^0 are both complete complemented lattices. L is an essentially good lattice with $s(L) = s_1(L) = s_2(L) = (1,1)$.

Proof. It is clear that L and L^0 are complete lattices. Note that $(0,1) \vee (1/2,0) = (1,1)$ and $(0,1)$, $(1/2,0)$ are atoms. Therefore, $s(L) = (1,1)$. Let $a \in L$. It is clear that a has a complement if $a = (0,0)$ or $(1,1)$. Suppose that $a = (0,1)$, then any $b \in S_1$ is a complement of a. Finally, If $a \in S_1$, then $(0,1)$ is a complement of a. Therefore, $c(L) = L$, and hence, L is complemented. Thus L^0 is also complemented. Since $c(L) = L$, then $s_2(L) = (1,1)$ and $e(L) = \{(1,1)\}$. Thus $s_1(L) = (1,1)$. Hence $s(L) = s_1(L) = s_2(L)$ and L is essentially good..

__Fact 2__. L and L^0 are not modular. L is not WUC-lattice, and hence, L is not upper continuous. But L^0 is upper continuous, and hence, L^0 is WUC-lattice.

Proof. Let $a = (2/3,0)$, $b = (0,1)$ and $c = (1/2,0)$. Then $c \leq a$ and $a \wedge (b \vee c) = a \wedge (1,1) = a$. But $(a \wedge b) \vee c = c \neq a$. Thus

L is not modular, and hence, L^0 is also not modular. Note that S_1 is a chain such that $(0, 1) \wedge a = (0,0)$ for any element a in S_1 and $(0,1) \wedge (\vee S_1) = (0,1) \wedge (1,1) = (0,1) \neq (0,0)$. Thus L is not WUC-lattice, and hence, L is not upper continuous. It is clear that L is artinian. Thus L^0 is noetherian, and hence, L^0 is upper continuous.

Fact 3. L and L^0 are both weak noetherian lattices.

Proof. Let $a, b \in L$ such that $a < b$. We need to find $c \in f(L)$ such that $c \leq b$ and $c \not\leq a$. Note that S_1 is a chain in L such that $\vee S_1 = (1,1)$ and $a < (1,1)$ for every $a \in S_1$. Therefore $(1,1) \notin f(L)$. Moreover, note that $f(L) = L \setminus (1,1)$. Therefore, if $b \neq (1,1)$ then we take $c = b$. Suppose that $b = (1,1)$. Then either $a \in S_2$ or $a \in S_1$. In the first case take c to be any non- zero element of S_1. In the second case take c to be any $x \in S_1$ such that $a < x$. Thus L is weak noetherian. It is clear that L^0 is weak noetherian, because $f^0(L) = L$.

Remark 5.4.4. In this remark we use Example 5.4.3 to check the validity of some results in sections 5.2 and 5.3 when we drop some assumptions. It is clear that $(1/2,0), (0,1) \in f(L)$. But

$(1/2,0) \vee (0,1) = (1,1) \notin f(L)$. So Lemma 5.3.1 (iii) is not true if L were not upper continuous. Note that $(0,1) \in f(L)$. Thus there exist no a, b $\in$ L such that $a \wedge b = 0$, $a \vee b = 1$, and both a, b $\notin f(L)$. Therefore Lemma 5.3.3 is not true if L were not upper continuous. On the other hand if we take x to be (2/3,0), then $s([0,x]) = (1/2,0) \neq x \wedge s(L) = x$. Therefore the second part of Lemma 5.2.5 is not true for lattices which are not upper continuous.

5.4.5 The lattice of submodules.

Let R be any ring and M be a right R-module. Let S be the set of all submodules of M such that S is ordered by inclusion. For any K, $K' \in S$, define $\wedge$ and $\vee$ as follow: $K \wedge K' = K \cap K'$ and $K \vee K' = K + K'$. Then it is clear tha $(S, \cap, +)$ is a lattice. Let $L(M) = (S, \cap, +)$ and $L^0(M)$ be the opposite lattice of L(M). Then the lattices L(M) and $L^0(M)$ satisfy the following properties:

Fact 1. L(M) and $L^0(M)$ are complete modular lattices.

Proof. It is clear that L(M) and $L^0(M)$ are complete lattices with 0 and 1 to be the zero submodule and M respectively. Let

K, K', and N be submodules of M such that $K \subseteq N$. Then it is clear that $K + (N \cap K') \subseteq N \cap (K + K')$. Let $x \in N \cap (K + K')$. Then $x = k + k'$ where $k \in K$ and $k' \in K'$. So $k' = x - k \in K' \cap N$. Therefore $x \in K + (N \cap K')$. Hence $N \cap (K + K') = K + (N \cap K')$. Thus $L(M)$ is modular, and hence, $L^0(M)$ is also modular, because the opposite lattice of a modular lattice is again modular.

Fact 2. $L(M)$ is upper continuous, and hence, WUC-lattice. But in general $L^0(M)$ is not WUC-Lattice, and hence, not upper continuous.

Proof. Let $K \in L(M)$. Let B_1, B_2, ... be any chain of elements of $L(M)$. Then $\sum B_i = \cup B_i$ and $\sum (K \cap B_i) = \cup (K \cap B_i)$. But it is clear that $K \cap (\cup B_i) = \cup (K \cap B_i)$. Therefore $L(M)$ is upper continuous, and hence, $L(M)$ is WUC-lattice. To show that the lattice $L^0(M)$ is not WUC-lattice, let M be the $\mathbb{Z}$-module $\mathbb{Z}$. Then, by Example 3.3.2, M is not supplemented. Thus $L^0(M)$ is not pseudo-complemented. Thus, by Lemma 5.2.4, $L^0(M)$ is not WUC-lattice. Hence $L^0(M)$ is not upper continuous.

Fact 3. $L(M)$ and $L^0(M)$ are both weak noetherian lattices.

Proof. Let A, B be submodules of M such that $A \subset B$. Then there exists $x \in B$ such that $x \notin A$. Therefore, $xR \in f(L(M))$ such that $xR \subseteq B$ and $xR \nsubseteq A$. Thus L(M) is weak noetherian. Therefore, by Lemma 5.4.1, $L^0(M)$ is also weak noetherian.

5.5 Open Questions:

In Example 1.1.5 it was shown that d$\underline{T}$ is not s-closed. Thus, by Lemma 1.3.1, d$\underline{T} \neq \underline{C} \oplus \underline{T}$. Moreover, in Examples 1.3.7 and 1.3.8 we showed that d$\underline{A} \neq \underline{C} \oplus \underline{A}$ and d$\underline{U}^* \neq \underline{C} \oplus \underline{U}^*$. This raises the following question:

Question 1. Let $\underline{X}$ be any one of the classes of modules $\underline{T}$, $\underline{A}$, or $\underline{U}^*$. What is the structure of the class d$\underline{X}$?

We have shown that over a right FBN-ring $d^*\underline{U} = \underline{C} \oplus \underline{U}$ (see Theorem 4.3.6). We do not know what $d^*\underline{U}$ is over a general ring! Indeed The following question is also open:

Question 2. Over a right noetherian ring does $d^*\underline{U} = \underline{C} \oplus \underline{U}$?

In studying the class $d^*\underline{N}$ we showed that $d^*\underline{N} \subseteq \underline{C} \oplus (a\underline{N})^{(k)}$ for some positive integer k (see Theorem 4.4.9). Moreover, it is clear that $a\underline{N} \subseteq d^*\underline{N}$ and $(a\underline{N})^{(k)} \not\subseteq a\underline{N}$, in general (see page 99). A natural question is:

<u>Question 3</u>. For some positive integer k does (or when does) $(a\underline{N})^{(k)} \subseteq d^*\underline{N}$?

In Theorem 1.3.11 we showed that, for an ordinal $\alpha > 0$ and for any ring, $d\underline{K}^*_\alpha = \underline{C} \oplus \underline{K}^*_\alpha$. We also showed that the dual of this fact is true (see Theorem 4.2.3). Our question therefore is:

<u>Question 4</u>. Can we prove the above facts for lattices ?

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