# THE MAXIMUM PRINCIPLE *and* ITS APPLICATIONS

by

ATTA ABDUL KADER A. ABU HANY

A thesis submitted for the degree
of Master of Science at the
University of Glasgow

Department of Mathematics
University of Glasgow
September 1987

## ACKNOWLEDGMENTS

This thesis is submitted in accordance with the regulations for the degree of Master of Science (M.Sc.) in the Department of Mathematics in the University of Glasgow .

The work involved in preparing this thesis was done between October 1985 and September 1987, under the supervision of Dr. J. R. L. Webb , F.R.S.E., to whom I would like to express my deep gratitude for his guidance, constant help and encouragement.

I would also like to thank Professor W. D. Munn and Professor R. W. Ogden for providing me with every possible help in the department.

Finally, I am very grateful to my family for financing this work and to my Mother for her patience .

# CONTENTS

# SUMMARY

This dissertation is concerned with some of the many applications of the MAXIMUM PRINCIPLE .

In the first two chapters , we discuss and prove versions of the maximum principle first for Ordinary Differential equations, then for elliptic Partial Differential Equations , including some improvements due to Serrin .

In Chapter (III) , we study in detail symmetry properties of positive solutions of second order elliptic equations of the type

$$\Delta u + f(u) = 0$$

in a domain $\Omega$ with zero boundary conditions. This follows the important article of Gidas, Ni and Nirenberg and shows that the problem cited has radial solutions in a spherically symmetric domain, no matter what the function $f$ is.

We give extensions of these results to certain systems of second order elliptic equations in Chapter (IV) .

Chapters (V) and (VI) contain applications of different type. In Chapter (V), we study solutions of the equation

$$\Delta u + f(u) = 0$$

with either Dirichlet or Neumann boundary conditions, and obtain bounds for various quantities determined by a solution of $\Delta u + f(u) = 0$ .

We show that it is possible to find functions $g, f$ so that the function

$$P = g(u) \ |grad\ u|^2 + h(u)$$

satisfies an elliptic inequality and , by an application of the maximum principle, P either attains its maximum on the boundary of $\Omega$ or at a critical point of $u$.

We study particularly the case $h'(u) = c\, f(u)\, g(u)$ where $c$ is a constant. For $c \leq 1$ we show that, under suitable assumptions, the maximum of P occurs on $\partial\Omega$, whereas for $c \geq 2$ the maximum occurs at a critical point of $u$.

In the last chapter, we illustrate these results by giving some applications to the *torsion problem*, the *efficiency ratio* of a nuclear reactor and the *free membrane* problem.

# INTRODUCTION

The MAXIMUM PRINCIPLE is one of the most valuable tools in the study of second order Partial Differential Equations . This principle is a generalization of the elementary fact of calculus that any function $f(x)$ which satisfies the inequality $f'' > 0$ on an interval $[a, b]$ attains its maximum value at $a$ or $b$ .

In general, functions that satisfy elliptic inequalities on a domain $\Omega$ in n-dimensional Euclidean space take their maxima on the boundary of $\Omega$ . This is the simplest form of the maximum principle .

Maximum principles for solutions of second order elliptic equations (and inequalities) have been used in the mathematics literature since the early ninteenth century . These principles have been refined and extended by various authors (see *e.g.* references cited in the book of Protter and Weinberger [14]).

One of the more important refinements , known as the Hopf maximum principle , asserts that at a maximum on the boundary , the outward normal derivative is positive (unless the function is identically constant).

This dissertation is concerned with some of the many applications of the maximum principle . There are three main parts . The first one consists of two chapters where we discuss and prove versions of the maximum principle first for Ordinary Differential Equations , then for Elliptic Partial Differential Equations .

We follow fairly closely the book of Protter and Weinberger [14] but we also include some results due to Serrin [17], including a maximum principle for a domain with a corner. We give new proofs of some of the older results using those of Serrin .

The next part concerns symmetry properties of positive solutions of elliptic partial differential equations. This follows the paper by Gidas, Ni and Nirenberg [5] and an earlier one of Serrin [17].

This deals with equations of the type

$$\Delta u + f(u) = 0 \tag{1}$$

in a domain $\Omega$ with zero boundary conditions. For $f(u) = 1$, Serrin proved that if one has over-determined boundary conditions with also the normal derivative constant, then the domain $\Omega$ on which the solution of (1) is defined is necessarily a ball and the solution is radially symmetric. Later, Gidas, Ni and Nirenberg [5] showed that for a ball, positive solutions of the elliptic equation (1) are radially symmetric. This points out that on a symmetric domain, symmetric equations have symmetric solutions. The important point is that the results do not depend on $f$. We explain parts of this paper in much detail.

Chapter (IV) contains extensions of the results discussed in Chapter (III) to certain systems of second order elliptic equations, as given by Troy [23]. In some places we use a slightly different argument to deduce the same results of Troy in an easier way.

In the third part we consider solutions of the equation

$$\Delta u + f(u) = 0$$

and obtain bounds for various quantities associated with this problem. We show that (following work of Payne [9] and Sperb [21]) it is possible to find functions $g$, $h$ so that the function

$$P = g(u) \, |grad\ u|^2 + h(u)$$

satisfies an elliptic inequality and, by an application of the maximum principle , P either attains its maximum on the boundary of $\Omega$ or at a critical point of $u$ .

The cases $h' = 2\, f\, g$ and $h' = (2/n)\, f\, g$ (for $\Omega \subset \mathbb{R}^n$ ) have been of considerable use in obtaining bounds and are well covered in the book by Sperb [21].

We follow the procedure from Sperb's book but we study the more general case $h' = c\, f\, g$ . We see how $c = 2,\ 2/n$ arise in a natural way but that other choices may be possible. We show that for $c \leq 2/n$ , the maximum occurs on $\partial\Omega$ whereas for $c \geq 2$ , under conditions related to the curvature of $\partial\Omega$ , the maximum occurs at a point where $grad\ u = 0$ . Some of these results seem to be new for $c \neq 2,\ 2/n$ .

We illustrate these results by giving, in Chapter (VI), some applications to the torsion problem , the "efficiency ratio" of a nuclear reactor and the free membrane problem .

*Chapter* (I)

# MAXIMUM PRINCIPLES IN ORDINARY DIFFERENTIAL EQUATIONS

CHAPTER (I)

# MAXIMUM PRINCIPLES

## IN ORDINARY DIFFERENTIAL EQUATIONS (O.D.E.s)

SECTION 1

## THE ONE - DIMENSIONAL MAXIMUM PRINCIPLE

The maximum principle in Ordinary Differential Equations (O.D.E.s) is a generalization of the simple fact that any function $f$ which satisfies the inequality $f'' > 0$ on an interval $[a, b]$ attains its maximum at one of the endpoints of $[a, b]$. This is obvious from the fact that $f'' > 0$ is equivalent to convexity of $f$.

If $f'' \geq 0$ on $[a, b]$ the same conclusion may be drawn but now it is possible that $f$ is constant on $[a, b]$. We follow Protter and Weinberger [14]. The prime denotes differentiation with respect to $x$ . We shall always assume in this Chapter that the function $u$ is in the class $C^2(a, b) \cap C^0[a, b]$.

THEOREM 1.1

Let $u$ be $C^2$ function on the interval $(a, b)$, let $g(x)$ be a bounded function on $(a, b)$. Suppose $u$ satisfies the differential inequality

$$L[u] \equiv u'' + g(x)\, u' \geq 0, \quad x \in (a, b). \qquad ......(1.1)$$

Then $u$ attains its maximum $M$ at either $a$ or $b$. Moreover if $u(c) = M$ for some interior point $c$ of $(a, b)$ then

$$u \equiv M \quad \text{on } [a, b].$$

REMARK 1.1

If $u$ satisfies the strict inequality

$$u'' + g(x)\, u' > 0 \quad , x \in (a, b) \qquad ......(1.2)$$

Then $u$ cannot have an interior maximum. Because if $u$ has a maximum at an interior point $c$ in $(a, b)$, then by elementary calculus, we must have $u'(c) = 0$ and $u''(c) \leq 0$, which contradicts the strict inequality above. It is important for applications to consider the non - strict inequality.

PROOF OF THEOREM 1.1

The idea of the proof is to construct an auxiliary function $z$ such that

$$L[u + \varepsilon z] > 0 \qquad \text{for all } \varepsilon > 0 ,$$

so that Remark 1.1 applies to $u + \varepsilon z$. The proof is by contradiction.

Suppose that $u$ assumes its maximum $M$ at an interior point $c$ in $(a, b)$, but $u \not\equiv M$ in $(a, b)$. Then there is a point $d$ of $(a, b)$ such that $u(d) < M$. We suppose that $d > c$.

Define the auxiliary function $z$ by

$$z(x) = e^{\alpha(x - c)} - 1,$$

where $\alpha$ is a positive constant to be prescribed. By a simple calculation one gets :

$$L[z] = z'' + g(x)\ z = \alpha[\alpha + g(x)]\ e^{\alpha(x - c)} .$$

Choose $\alpha$ that $\alpha > - g(x)$ for $x \in [a, b]$; this can be done since $g$ is bounded. Then

$$L[z] > 0 \quad \text{on } (a, b). \qquad \ldots\ldots(1.3)$$

Therefore, for any $\varepsilon > 0$, by (1.1) and (1.3) we get $L[u + \varepsilon z] > 0$ on $(a, b)$ and a fortiori on $(a, d)$.

Now $z(x) > 0$ for $a \le x < c$, so $u + \varepsilon z < M$ for $a \le x < c$ and

$$u + \varepsilon z = M \quad \text{at } c,$$

$$u + \varepsilon z < M \quad \text{at } d,$$

for $\varepsilon < [M - u(d)]/z(d)$ .

Therefore $u + \varepsilon z$ must attain its maximum ( $\ge M$ ) on $[a, b]$ at an interior point of $[a, b]$. This contradicts Remark 1.1 above and therefore the assumption that $u(d) < M$ must be false. We conclude that $u \equiv M$ on $[a, b]$.

If $d < c$ an exactly similar argument applies taking the auxiliary function $z(x) = e^{-\alpha(x - c)} - 1$ with $\alpha > g(x)$ on $(a, b)$. □

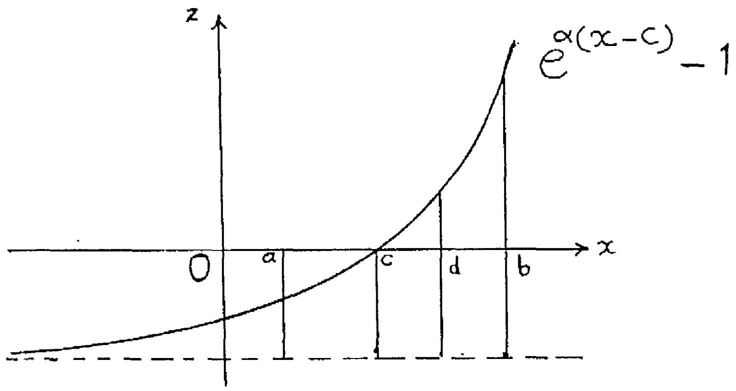


FIGURE [1.1]

REMARK 1.2

The boundedness assumption on the function $g(x)$ in Theorem 1.1 may be weakened. It suffices that $g(x)$ be bounded on every subinterval $[a', b']$ completely interior to $(a, b)$.

This observation is useful since it allows the coefficients of Differential Equations to become unbounded at the endpoints. This occurs in many of the equations arising in mathematical physics.

EXAMPLE 1.1 :

The differential equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = 0, \quad \text{for the disk} \quad 0 < r \leq 1,$$

is Laplace's equation in polars for radially symmetric solutions.

Theorem 1.1 tells us that a *non - constant* function which satisfies the inequality

$$u'' + g(x)\, u' \geq 0 \quad \text{in } (a, b)$$

attains its maximum at either $a$ or $b$. In fact $u$ decreases strictly as one moves into the interior of the interval $[a, b]$, that is the directional derivative of $u$ in the direction pointing interior to $[a, b]$ is *negative*.

More precisely we have the following result .

THEOREM 1.2

Let $u \in C^2(a, b)$ satisfy the inequality

$$u'' + g(x)\, u' \geq 0 \quad \text{in } (a, b)$$

with $g(x)$ bounded on every closed subinterval $[a', b']$ of $(a, b)$. Suppose that $u$ attains its maximum $M$ at one of the endpoints of $[a, b]$, $u \not\equiv M$ in $(a, b)$, and has one - sided derivatives at $a$ and $b$.

If $u(a) = M$ and $g$ is bounded from below at $x = a$, then $u'(a) < 0$. If $u(b) = M$ and $g$ is bounded from above at $x = b$, then $u'(b) > 0$.

PROOF:

Suppose that the function $u$ attains its maximum at the endpoint $b$ of $[a, b]$. Then $u(b) = M$, and $u(x) \leq M$ for $x \in [a, b]$.

Suppose that at an interior point $c \in (a, b)$, we have $u(c) < M$.

Consider the auxiliary function

$$z(x) = e^{-\alpha(x - b)} - 1 \quad \text{with } \alpha > 0 .$$

Note that $z(c) > 0$. By choosing $\alpha$ such that $\alpha > g(x)$ for $c \leq x \leq b$, we have

$$L(z) = \alpha^2 e^{-\alpha(x - b)} - \alpha\, g(x)\, e^{-\alpha(x - b)} > 0.$$

Now, we consider the function

$$w(x) = u(x) + \varepsilon\, z(x), \quad \text{where} \quad 0 < \varepsilon < [M - u(c)]/z(c).$$

Then we easily get

$$L(w) \equiv L(u) + \varepsilon\, L(z) > 0.$$

Hence $w$ attains its maximum at one of the endpoints $c$ or $b$ of the interval $[c, b]$. By the choice of $\varepsilon$, above, we have

$$\begin{aligned} w(c) &= u(c) + \varepsilon\, z(c) \\ &< u(c) + M - u(c) \\ &= M \end{aligned}$$

Therefore the maximum of $w$ occurs at $b$, and then $w$ has a nonnegative one - sided derivative at $b$

$$w'(b) = u'(b) + \varepsilon\, z'(b) \geq 0$$

but $z'(b) = -\alpha < 0$, so that $u'(b) > 0$. The result follows.□

<u>REMARKS 1.3</u>

(I) If $u$ attains its maximum at $x = a$, then the argument is similar. In this case, we choose the auxiliary function

$$z(x) = e^{\alpha(x - a)} - 1$$

with $\alpha > 0$ and we select $\alpha > -g(x)$ on an appropriate interval.

(II) The boundedness of $g$ is essential for the conclusion of Theorems 1.1 and 1.2 . To see this , we consider the O.D.E.

$$u'' + g(x)\,u' = 0 \quad \text{with } g(x) = \begin{cases} -3/x & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

We see that $u = 1 - x^4$ satisfies our problem . Now , if we take $x \in [-1, 1]$ then $u$ attains its maximum at the interior point $x = 0$ . Hence Theorem 1.1 is violated on $[-1, 1]$. Also if we take $x \in [0, 1]$ , then $u'(0) = 0$ . So that Theorem 1.2 is violated on $[0, 1]$.

## MORE GENERAL DIFFERENTIAL INEQUALITIES

We want to consider inequalities with zero order terms

$$(L + h)[u] = u'' + g(x)\,u' + h(x)\,u \geq 0$$

By virtue of an idea of Serrin's [17],we are able to prove the following :

**THEOREM 1.3**

Suppose that the function u satisfies

$$(L + h)[u] = u'' + g(x)\,u' + h(x)\,u \geq 0 \quad \text{on } (a, b), \qquad \ldots\ldots\ldots(1.4)$$

with $u \leq 0$ on $[a, b]$. Then

(i) if $u(c) = 0$ for some $c \in (a, b)$, $u \equiv 0$ on $[a, b]$.

(ii) if $u(a) = 0$, $u'(a) < 0$ $[\, u(b) = 0, \; u'(b) > 0 \,]$

where $h$ is bounded below , $g$ is bounded on every closed subinteravl of $(a, b)$.

PROOF:

Take $v(x) = e^{-\alpha x} u(x)$. Then

$$0 \le (L + h)[u] = e^{\alpha x} [v'' + (g + 2\alpha) v'] + e^{\alpha x} [\alpha^2 + \alpha g + h] v$$

$$= e^{\alpha x} [L_1 v + H v], \text{ say,}$$

where $L_1 v = v'' + (g + 2\alpha) v'$ contains no zero – order terms, and $H = \alpha^2 + \alpha g + h$.

For $\alpha$ sufficiently large, $H(x) > 0$ on $(a, b)$ so we have

$$L_1 v \ge - H(x) v \ge 0, \quad \text{since } v \le 0.$$

By Theorem 1.1, $v$ attains its maximum ($M = 0$) at $a$ or $b$ and (i) holds. Now, since

$$u'(x) = e^{\alpha x} (v' + \alpha v),$$

and if $u$ attains its maximum at $a$ we get

$$u'(a) = e^{\alpha a} v'(a)$$
$$< 0$$

Similarly, we apply the argument above to obtain $u'(b) > 0$ if $u(b) = 0$. This gives (ii).□

The following example illustrates that the hypotheses of Theorem 1.3, above, cannot be discarded .

EXAMPLE 1.2

The function $u(x) = \sin x$ is a solution of the equation

$$u'' + u = 0 \quad \text{on } (0, \pi).$$

But $u$ assumes its maximum at $x = \pi/2$, so there is no analogue of Theorem 1.1. Here $u$ is positive on $(0, \pi)$ and $h(x) > 0$. $u \leq 0$ does not hold.

THEOREM 1.4

Suppose that $g$ , $h$ are as in Theorem 1.3 , that

$$(L + h)[u] \geq 0 \quad \text{on } (a, b) \qquad \ldots\ldots(1.5)$$

and that $u$ attains *a positive maximum* $M > 0$ on $[a, b]$. Then if $h(x) \leq 0$ on $(a, b)$, we have $u$ attains its maximum $M$ at $a$ or at $b$.

Moreover

(i) if $u(c) = M$ for some $c \in (a, b)$, then $u \equiv M$.

(ii) if $u(a) = M$, then $u'(a) < 0$ unless $u \equiv M$

[ If $u(b) = M$, then $u'(b) > 0$ unless $u \equiv M$ ].

Thus maximum always occurs at an endpoint and either $u \equiv$ constant or (ii) holds.

PROOF:

Let $v(x) = u(x) - M$, then $v \leq 0$ and $v = 0$ at some point in $[a, b]$. Then

$$\begin{aligned}(L + h)[v] &= (L + h)[u] - h\,M \\ &\geq -h\,M \\ &\geq 0 \quad , \text{ since } h \leq 0.\end{aligned}$$

So Theorem 1.3 applies to $v$ and the proof is complete.□

EXAMPLE 1.3

The differential equation

$$u'' - u = 0 \quad \text{on the interval } (-1, 1)$$

has the solution

$$u(x) = - \{ e^{x} + e^{-x} \} = - 2 \cosh x.$$

Obvious, $u$ attains its maximum $M = - 2$ at $x = 0$. Here $h(x) \leq 0$, but $u$ has *a negative maximum*.

REMARK 1.4:

If $h(x)$ is negative somewhere in $(a, b)$ then part (i) of Theorem 1.4 can only occur if $M = 0$.

This is the well - known version of the Hopf Maximum **Principle**, as found, for example in the book of Protter & Weinberger [14]. We believe that our method of proof is new .

COROLLARY 1.5

Suppose that $(L + h) [u] \geq 0$ on $(a, b)$ with $h(x) \leq 0$. If $u$ is continuous on $[a, b]$, and $u(a) \leq 0$, $u(b) \leq 0$, then

$$u(x) < 0 \quad \text{in } (a, b) \text{ unless } u \equiv 0.$$

PROOF:

By hypothesis, $u$ attains its maximum $M$ on $[a, b]$. If $M \leq 0$, then by (i) of Theorem 1.3, either

$$u \equiv 0 \quad \text{or} \quad u(x) < 0 \quad \text{in } (a, b).$$

If $M > 0$, then the maximum occurs at an interior point. By Theorem 1.4 , this would imply

$$u \equiv M \text{ , impossible since } u(a) \leq 0, \ u(b) \leq 0. \square$$

SECTION 2

## THE GENERALIZED MAXIMUM PRINCIPLE

Consider the differential inequality

$$(L + h) \equiv u'' + g(x)\ u' + h(x)\ u \geq 0 \qquad \ldots\ldots(2.1)$$

with $h(x)$ not necessarily $\leq 0$. Assume that there exists a function $w \in C^2$ such that under some conditions, $w$ satisfies the following inequalities

$$w > 0 \qquad \text{on } [a,\ b] \qquad \ldots\ldots(2.2)$$

$$(L + h)[w] \leq 0 \qquad \text{in } (a,\ b) \qquad \ldots\ldots(2.3)$$

To see that such a function $w$ can exist, suppose $h(x)$ is bounded and the function $g(x)$ is bounded from below in $[a,\ b]$, with $[a,\ b]$ sufficiently short. Then take

$$w = 2 - e^{\alpha(x - a)} \qquad \ldots\ldots(2.4)$$

where $\alpha$ is a constant to be determined. We have by calculation:

$$(L + h)[w] = -\ e^{\alpha(x - a)}\ [\alpha^2 + \alpha\ g + h] + 2\ h \ \ldots.(2.5)$$

By assumption there are constants $G$ and $H$ such that $g \geq G$ and $h \geq H$. Then ,if $\alpha$ is sufficiently large, we have

$$\alpha^2 + \alpha\ g + h > 0$$

and

$$e^{\alpha(x - a)} \geq [2\ h\ /(\alpha^2 + \alpha\ g + h)].$$

This can be done since $h$ is also bounded above. From (2.5) we get

$$(L + h)[w] \leq 0 \quad \text{in } (a,\ b).$$

However (2.4) yields $w > 0$ on $[a,\ b]$, if $[a,\ b]$ is required to be small enough such that

$$e^{\alpha(x - a)} < 2.$$

NOTE:

One can also construct $w$ of the the form

$$w = 1 - \beta(x - a)^2 \quad , \text{ for suitable } \beta .$$

{see for example Protter and Weinberger [14]}.

When such a function $w$ exists, we define the new dependent variable $v = u/w$ , then one gets

$$(L + h)[u] \equiv v\, w'' + 2\, v'\, w' + v''\, w + g(v\, w' + v'\, w) + h(v\, w) \geq 0$$

Dividing by the positive quantity $w$ we get

$$v'' + [2\ (w'/w) + g]v' + (1/w)\ (L + h)[w]v \geq 0. \quad \ldots(2.6)$$

THEOREM 2.1

Let $u(x)$ satisfy the inequality

$$(L + h)[u] \equiv u'' + g(x)\ u' + h(x)\ u \geq 0$$

in a suitable domain $(a,\ b)$. Assume that there exists a function $w(x)$ which satisfies conditions (2.2),(2.3) in $[a,\ b]$. Then results of Theorem 1.4 hold for the dependent function $v = u/w$ .

REMARKS 2.1:

(I) In any interval $(a,\ b)$, where Theorem 2.1 holds, $u$ can have at most two zeros between which $u$ is negative. If we call these zeros $x = A$ and $x = B$ , if $u > 0$ at any point between A and B, $u/w$ would have a positive maximum between them which contradicts Theorem 2.1, unless the distance between A and B is so large that this theorem does not hold.

(II) If $u$ is a solution of the equation

$$u'' + g(x)\, u' + h(x) = 0,$$

the same reasoning can be applied to both $u$ and $(-u)$ to find that $u$ can have at most one zero in any interval $(a, b)$ where Theorem 2.1 holds.

Let $r(x)$ satisfy the equation

$$r'' + g(x)\, r' + h(x)\, r = 0 \ , \quad x \in (a, b), \quad \ldots.(2.7)$$

with $r(a) = 0$, $r(x) \not\equiv 0$ in $(a, b)$, and $h(x)$, $g(x)$ are bounded. If $r$ has any zeros to the right of $a$ we denote the first one by $a^*$, and we call $a^*$ the *conjugate point* of $a$. If $r$ has no zeros to the right of $a$ we set $a^* = \infty$ .

<u>NOTE</u>:

The function $r(x)$ does not change its sign in the interval $(a, a^*)$. For convenience we assume that $r > 0$ in the interval $(a, a^*)$.

Now, if $a^*$ is the conjugate point of $a$, we can find a function $w > 0$ such that Theorem 2.1 holds for $v = r/w$ on the interval $(a, b)$ if and only if $b < a^*$ . If $w$ exists, then $v = r/w$ is positive on $(a, a^*)$ and zero at $a$ and $a^*$, so $v$ has a positive maximum on $(a, a^*)$. Then $v$ would have to be identically constant on $[a, a^*]$ {i.e. $r = c\, w$, where $c$ is a constant} contradicting $w(a) > 0$.

If $b < a^*$, we take the function $w$ in the form

$$w = r + \varepsilon\, [2 - e^{\alpha(x - a)}]$$

for sufficiently small $\varepsilon > 0$. $w$ is positive on $[a, b]$ and we can have, for suitable choice of $\alpha$ ,

$$(L + h)[w] = (L + h)\left[2 - e^{\alpha(x - a)}\right] \leq 0 \quad \text{in } (a, b).$$

Hence we have constructed $w$ for which Theorem 2.1 holds.

(III) We remark that the boundedness of the functions $g$ and $h$ is essential.

EXAMPLE 2.1:

The function $u(x) = x \sin(1/x)$ satisfies the differential equation

$$u'' + x^{-4} u = 0 \quad \text{on } (0, \infty) .$$

Clearly $u$ vanishes at $x = 1/(n\pi)$ , $n = 1,2,\ldots,$ and so $a^*$ is not defined and no function $w > 0$ can exist. The problem here is that $h$ is unbounded at 0 . □

## SECTION 3

### UNIQUENESS RESULTS FOR INITIAL AND BOUNDARY VALUE PROBLEMS

One important application of the maximum principle is in the discussion of *uniqueness* of solutions to initial and boundary value problems.

### INITIAL VALUE PROBLEMS (I.V.P.s):

Consider the initial value problem

$$u'' + g(x)u' + h(x)\, u = f(x) \qquad \ldots\ldots(3.1)$$

with the conditions

$$u(a) = A, \quad u'(a) = B \qquad \ldots\ldots(3.2)$$

where the functions $h(x)$ and $g(x)$ are bounded in the interval $(a, b)$, and $A$ and $B$ are prescribed constants.

### THEOREM 3.1

Suppose $u_1(x)$ and $u_2(x)$ are solutions of (3.1) in $(a, b)$ and both of $u_1(x)$ and $u_2(x)$ satisfies the initial conditions (3.2). Then

$$u_1 \equiv u_2 \quad \text{in } (a, b).$$

NOTE : We do not require $h(x) \leq 0$.

### PROOF OF THEOREM 3.1 :

Let $u(x) = u_1(x) - u_2(x)$, $x \in (a, b)$. We want to show that $u(x) \equiv 0$ in $(a, b)$. We have that $u$ satisfies the equation

$$u'' + g(x)\, u' + h(x)\, u = 0$$

with the initial conditions

$$u(a) = u'(a) = 0.$$

Assume that $u \not\equiv 0$ in $(a, b)$. By Theorem 2.1 there exists $\varepsilon > 0$ and a function $w > 0$ on $[a, a + \varepsilon]$ such that $u/w$ attains its maximum at one of the endpoints of $[a, a + \varepsilon]$.

Since the same argument applies for $-u$, we observe that either the maximum or the minimum of $u/w$ must occur at $a$. But

$$(u/w)' = \frac{u' w - u w'}{w^2} = 0 \qquad \text{at } x = a.$$

Since Theorem 1.3 holds for the function $u/w$ we find that $u/w$ is constant, moreover $u/w = 0$ at $a$ since $u(a) = 0$. Contradiction. Therefore $u \equiv 0$ on $[a, a + \varepsilon]$, in particular

$$u(a + \varepsilon) = 0, \qquad u'(a + \varepsilon) = 0.$$

We may repeat the argument to conclude that $u \equiv 0$ in $(a + \varepsilon, a + 2\varepsilon)$, with $\varepsilon$ being unchanged since it depends only on bounds for $g$ and $h$ in $(a, b)$.

By employing the process above a finite number of times we deduce that $u \equiv 0$ in $(a, b)$.□

<u>BOUNDARY VALUE PROBLEMS</u> (B.V.P.s):

Consider the following B.V.P.

$$u'' + g(x)\, u' + h(x)\, u = f(x) \qquad \ldots\ldots(3.3)$$

with $x \in (a, b)$ and $g$ and $h$ bounded, subject to the boundary conditions

$$u(a) = S\ , \quad u(b) = R \qquad \ldots\ldots(3.4)$$

where S and R are prescribed constants.

THEOREM 3.2

Suppose that $u_1(x)$ and $u_2(x)$ are solutions of (3.3) and satisfy the boundary conditions (3.4) . If $h(x) \leq 0$ in $(a, b)$ then

$$u_1 \equiv u_2.$$

PROOF :

Let $u = u_1 - u_2$ , then $u$ satisfies

$$u'' + g(x)\, u' + h(x)\, u = 0 \qquad \text{......(3.5)}$$

with the boundary conditions

$$u(x) = 0\ , \quad u(b) = 0. \qquad \text{......(3.6)}$$

Assume that $u \not\equiv 0$. By Corollary 1.5 we know that $u(x) \leq 0$ in $(a, b)$. Moreover, since the function $-u$ satisfies (3.5) together with the conditions (3.6), then Corollary 1.5 also applies to $-u$ . Hence, $-u \leq 0$ in $(a, b)$. Therefore $u \equiv 0$ in $(a, b)$. □

In the following, we prove a uniqueness theorem for B.V.P. s without any restriction on the function $h(x)$. Meanwhile we put some condition on the required domain on which we wish to prove our theorem.

THEOREM 3.3

let $u_1(x)$, $u_2(x)$ be two solutions of (3.3) satisfying the same boundary conditions (3.4). If $b < a^*$, where $a^*$ is the conjugate point of $a$, then $u_1 \equiv u_2$.

PROOF :

Define a function $v(x)$ by

$$v(x) = u_1(x) - u_2(x).$$

Clearly $v(x)$ satisfies the differential equation

$$v'' + g(x)\, v' + h(x)\, v = 0$$

with the conditions

$$v(a) = 0, \quad v(b) = 0.$$

Since $b < a^*$, we can find a function $w(x) > 0$ such that $(L + h)[w] \leq 0$ in $(a, b)$. Applying Theorem 2.1 we get:

either

$$v(x) \equiv 0 \quad \text{or} \quad v(x) \neq 0 \ \text{ on } (a, a^*)$$

which is impossible, since $b < a^*$. □

SECTION 4

## NONLINEAR OPERATORS

We have seen how the maximum principle can be employed to give very important results in case of the linear operators. In this section we show how the maximum principle is applicable to some nonlinear operators.

Let $u(x)$ satisfy the nonlinear equation

$$u'' + H(x, u, u') = 0 \qquad \ldots\ldots(4.1)$$

on an interval $[a, b]$, where $H(x, u, p)$ is such that $\frac{\partial H}{\partial u}$ and $\frac{\partial H}{\partial p}$ are continuous functions from $[a, b] \times \mathbb{R}^2$ to $\mathbb{R}$. Let $w(x)$ satisfy the inequality

$$w'' + H(x, w, w') \geq 0 \qquad \ldots\ldots(4.2)$$

in $(a, b)$.

THEOREM 4.1

Suppose that the function $v(x) = w(x) - u(x)$ satisfies the inequality

$$v'' + H(x, w, w') - H(x, u, u') \geq 0$$

in $(a, b)$, where $\frac{\partial H}{\partial u}$, $\frac{\partial H}{\partial u'}$ are continuous and $\frac{\partial H}{\partial u} \leq 0$ .

If $v(x)$ attains a nonnegative maximum $M$ in $(a, b)$, then

$$v \equiv M.$$

PROOF :

By the Mean Value Theorem , for $0 < t < 1$

$H(x, w, w') - H(x, u, u') =$

$$\begin{bmatrix} \frac{\partial H}{\partial u} \\ \frac{\partial H}{\partial p} \end{bmatrix} (x, u + t(w - u), u' + t(w' - u')) \cdot \begin{bmatrix} w - u \\ w' - u' \end{bmatrix}$$

$$= \frac{\partial H}{\partial u} v + \frac{\partial H}{\partial u'} v'.$$

So $v$ satisfies the inequality:

$$v'' + \frac{\partial H}{\partial p} v' + \frac{\partial H}{\partial u} v \geq 0$$

which is linear, and hence the maximum principle as given in Theorem 1.4 applies.□

*Chapter* (II)

# MAXIMUM PRINCIPLES IN ELLIPTIC PROBLEMS

## CHAPTER (II)

## MAXIMUM PRINCIPLES IN ELLIPTIC PROBLEMS

### SECTION 1

### NOTATIONS AND SOME BASIC DEFINITIONS

Let $u(x) \in C^2(\Omega)$, where $\Omega$ is a bounded domain (open connected set) in the Euclidean space $\mathbb{R}^n$.

**DEFINITION 1.1**

We call the operator (we use summation convention)

$$P \equiv a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} \quad , \; a_{ij} = a_{ji} \quad , \qquad \ldots\ldots(1.1)$$

$(i, j = 1,2,\ldots,n)$ , *elliptic* at $x = (x_1,x_2,\ldots,x_n)$ if and only if there is a *positive* constant $\mu(x)$ such that

$$a_{ij}(x)\, \xi_i\, \xi_j \geq \mu(x)\, \xi_i\, \xi_i \qquad \ldots\ldots(1.2)$$

for any vector $\xi = (\xi_1,\xi_2,\ldots,\xi_n)$. The operator $P$ is said to be elliptic in a domain $\Omega$ if it is elliptic at each point of $\Omega$, and it is *uniformly elliptic* if (1.2) holds for each point of $\Omega$ and if there is a positive constant $\mu_0$ such that $\mu(x) \geq \mu_0$ for all x in $\Omega$.

**EXAMPLE** :

The Laplace operator defined by

$$\Delta \equiv \frac{\partial^2}{\partial x_1 \partial x_1} + \frac{\partial^2}{\partial x_2 \partial x_2} + \ldots + \frac{\partial^2}{\partial x_n \partial x_n} \, ,$$

is uniformly elliptic in any domain $\Omega$.

**DEFINITION 1.2**

We say that the operator

$$(L+h) \equiv a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + b_i(x) \frac{\partial}{\partial x_i} + h(x)$$

is (uniformly) elliptic in $\Omega$ if its principal part

$$P \equiv a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$$

is (uniformly) elliptic in $\Omega$.

## SECTION 2

## MAXIMUM PRINCIPLE FOR ELLIPTIC INEQUALITIES

We investigate maximum principles for inequalities satisfied by operators $L$ and $(L + h)$. We follow Protter and Weinberger [14]. We shall need the following lemmas.

DEFINITION 2.1 :

An $n \times n$ matrix $A$ is called positive semidefinite,(or negative semidefinite ), if

$$\xi^T A \xi \geq 0 \ (\leq 0) \quad \text{for all } \xi \text{ in } \mathbb{R}^n.$$

LEMMA 2.1

Suppose that $A$ and $B$ are symmetric $n \times n$ matrices with $A \geq 0$ and $B \leq 0$. Then

$$trace\ (A\ B) \leq 0.$$

PROOF : (Smoller [18])

There exist orthogonal matrices $C$ and $D$ with $C A C^{-1} = \Delta_1$, $D B D^{-1} = \Delta_2$ where $\Delta_1$ and $\Delta_2$ are diagonal matrices and $\Delta_1$ has nonnegative elements and $\Delta_2$ has nonpositive elements.

By the fact that the *trace* of a product is independent of the order of the factors, we have

$$tr(A\ B) = tr(C\ A\ C^{-1}\ D\ B\ D^{-1}) = tr(\Delta_1\ \Delta_2) \leq 0. \ \square$$

DEFINITION 2.2

Let $n$ be the unit normal vector in an outward direction at a point $Q$ on the boundary $\partial\Omega$, and let $\nu$ be a vector pointing outward from $\Omega$ at $Q$, that is $\nu \cdot n > 0$ .

We define the *directional derivative* of $u$ at $Q$ in the direction $\nu$, if it exists, as

$$\frac{\partial u}{\partial \nu} = \left[ \lim_{t\to 0} \frac{u(Q - t\,\nu) - u(Q)}{t} \right].$$

If $u \in C^1$, then

$$\frac{\partial u}{\partial \nu} = \lim_{x\to Q} \left[ \nu_1 \frac{\partial u}{\partial x_1} + \dots + \nu_n \frac{\partial u}{\partial x_n} \right].$$

NOTE :

A well known outward directional derivative is the normal derivative.

LEMMA 2.2 (Elementary Calculus Lemma)

Suppose that the function $u \in C^2(\Omega \cup \partial\Omega)$ and that $u$ attains its maximum at a point $x \in \partial\Omega$ . Then the outward directional derivative

$$\frac{\partial u}{\partial \nu} \geq 0 \qquad \text{at } x . \qquad \dots\dots(2.1)$$

If $grad\ u(x) = 0$ , then

$$\frac{\partial^2 u}{\partial \nu^2} \leq 0 \qquad \text{at } x \qquad \dots\dots(2.2)$$

($\nu \equiv$ outward direction).

PROOF :

By virtue of the Mean Value Theorem in the form

$$u(x + h) - u(x) = grad\ u(x + s\,h)\ .\ h$$

we have

$$u(x - t\,\nu) - u(x) = -\,t\ grad\ u(\eta)\ .\ \nu$$

$$= -\,t\,\frac{\partial u}{\partial \nu}(\eta)\ . \qquad \dots\dots(2.3)$$

Therefore, as $u(x)$ is a maximum , for $t$ small ,

$$u(x - t\,\nu) - u(x) \leq 0 \quad .$$

If (2.1) is false we would have $\frac{\partial u}{\partial \nu} < 0$ at $x$, and therefore $\frac{\partial u}{\partial \nu} < 0$ on a neighbourhood of $x$ which gives a contradiction to (2.3) .

Suppose that *grad* $u(x) = 0$. We consider the second order directional derivative :

$$\frac{\partial}{\partial \nu}\left\{\frac{\partial u}{\partial \nu}\right\} = \frac{\partial}{\partial \nu}\left\{grad\ u\ .\ \nu\right\} = \frac{\partial}{\partial \nu}\left\{\sum_i \frac{\partial u}{\partial x_i}\nu_i\right\}$$

$$= \sum_j \frac{\partial}{\partial x_j}\left\{\sum_i \frac{\partial u}{\partial x_i}\nu_i\right\}\nu_j$$

$$= \sum_j \sum_i \frac{\partial^2 u}{\partial x_i \partial x_j}\nu_i\,\nu_j \qquad (i,\ j = 1,\ldots,n)$$

$$= \nu^T H\,\nu$$

where $H = \left[\frac{\partial^2 u}{\partial x_i \partial x_j}\right]$ is the Hessian matrix .

By Taylor expansion

$$u(x - t\,\nu) - u(x) = grad\ u(x).\ \nu + \frac{t^2}{2}\nu^T H\,\nu\,,$$

hence $\nu^T H\,\nu \leq 0$ , and the proof of Lemma 2.2 is complete.□

<u>REMARK 2.1</u>

If $x$ is an interior point, *where $u$ takes its maximum*, then *grad* $u(x) = 0$ and $H$ is negative semidefinite as the above applies to *all* directions $\nu$ .

<u>NOTE</u> : From elementary calculus we know that if a function $u(x)$ satisfies the strict inequality

$$L[u] \equiv a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + b_i\frac{\partial u}{\partial x_i} > 0 \qquad \ldots\ldots(2.4)$$

in $\Omega$, then $u$ cannot attain its maximum at any interior point of $\Omega$.

To see this, we assume that $u \in C^2(\Omega \cup \partial\Omega)$ has an interior maximum at some point $x \in \Omega$. Then

$$\frac{\partial u}{\partial x_i} = 0 \qquad \text{at } x$$

and the Hessian matrix

$$H = \left[\frac{\partial^2 u}{\partial x_i \partial x_j}\right]$$

is *negative semidefinite*.

Let $A$ denote the matrix $(a_{ij})$; $i,j = 1,2,\ldots,n$. Then by virtue of Lemma 2.1 we have :

$$tr(A\,H) \leq 0 \ ,$$

since $(a_{ij})$ is positive definite. Therefore we obtain a contradiction to inequality (2.4) .

We now wish to extend the maximum principle to allow the non - strict inequality. We will follow the proof given by Smoller [18] .

THEOREM 2.3

Suppose $u(x)$ satisfies the inequality

$$L[u] = a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + b_i \frac{\partial u}{\partial x_i} \geq 0 \qquad \ldots\ldots(2.5)$$

in $\Omega$, $L$ being uniformly elliptic with $a_{ij}$, $b_i$ uniformly bounded. Then $u$ cannot attain its maximum $M$ at an interior point of $\Omega$ unless $u \equiv M$ in $\Omega$.

PROOF :

Suppose that $u$ assumes its maximum $M$ at some point $x_0$ in $\Omega$, i.e. $u(x_0) = M$. Then we will show that $u \equiv M$.

Let $S$ be the set of all points $x$ in $\Omega$ for which $u(x) = M$, i.e. $S = \{ x \in \Omega : u(x) = M \}$. Clearly $S$ is not empty, since $x_0 \in S$. If $x_1 \in \Omega \backslash S$, we connect $x_1$ to $x_0$ by a curve $\gamma$ in $\Omega$. Since $\gamma$ is compact, we can find $\delta > 0$ such that if a point $Q \in \gamma$, dist $(Q, \partial\Omega) \geq \delta > 0$.

Since $u(x_1) < M$, $u(x) < M$ in some ball centred at $x_1$ of radius at most $\delta/2$. If $x_1$ moves along $\gamma$ towards $x_0$, the boundary of this ball eventually contains a point in $S$. Let $\bar{x}$ be the centre of the first ball whose boundary meets $S$. Thus there exists a ball $B$ whose closure is contained in $\Omega$ for which $\partial B \cap S \neq \varphi$, ($\varphi \equiv$ empty set), but $B \cap S = \varphi$. Let $y$ denote the point where $\partial B \cap S \neq \varphi$, see figure [2.1]

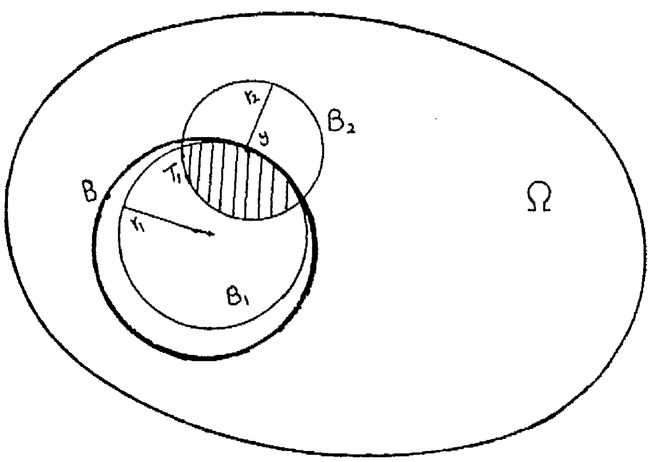


FIGURE [2.1]

Let $B_1 \subset \bar{B}$ be a smaller ball of radius $r_1$ such that $y \in \partial B_1$. Then $u < M$ in $\bar{B} \backslash \{y\}$. Let $B_2 \subset \Omega$ be a ball centred at $y$ and with radius $r_2 < r_1$. If $\partial B_2 = T_1 \cup T_2$ where $T_1 = \partial B_2 \cap \bar{B}_1$, then $T_1$ is compact, so since $u < M$ on $T_1$, $u \leq M - \varepsilon$ on $T_1$ for some $\varepsilon > 0$.

We choose the centre of $B_1$ to be the origin of our coordinate system, then we can define the auxiliary function $z$ by

$$z(x) = e^{-\alpha r^2} - e^{-\alpha r_1^2}$$

where $\alpha > 0$ is to be prescribed, $r^2 = |x|^2 = x_1^2 + x_2^2 + \ldots + x_n^2$.

By computation :

$$\frac{\partial z}{\partial x_i} = -2\alpha x_i e^{-\alpha r^2}$$

$$\frac{\partial^2 z}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_j}(-2\alpha x_i e^{-\alpha r^2})$$

$$= -2\alpha \delta_{ij} e^{-\alpha r^2} + 4\alpha^2 x_i x_j e^{-\alpha r^2}$$

where $\delta_{ij}$ is the Kronecker delta, $(i, j = 1,2,\ldots,n)$.

Thus

$$L[z] = a_{ij}(4\alpha^2 x_i x_j e^{-\alpha r^2}) + a_{ii}(-2\alpha e^{-\alpha r^2})$$

$$+ b_i(-2\alpha x_i e^{-\alpha r^2})$$

$$= 4\alpha^2 a_{ij} x_i x_j e^{-\alpha r^2} - 2\alpha(a_{ii} + b_i x_i) e^{-\alpha r^2}.$$

Dividing by the positive quantity $e^{-\alpha r^2}$ we get

$$e^{\alpha r^2} L[z] \equiv 4\alpha^2 a_{ij} x_i x_j - 2\alpha(a_{ii} + b_i x_i) \quad \ldots.(2.6)$$

Now since $r_2 < r_1$, the origin of our coordinate system $0 \notin \overline{B}_2$, and by the ellipticity condition we see that

$$a_{ij} x_i x_j \geq \sigma > 0 \quad \text{in } \overline{B}_2,$$

where $\sigma$ is a positive constant. Thus, for $\alpha$ sufficiently large, $L[z] > 0$ in $\overline{B}_2$.

We consider the new function :

$$w(x) = u(x) + \varepsilon_1 \; z(x)$$

and we take $k = \max \{z(x) : x \in T_1\}$. Then on $T_1$

$$\begin{aligned} w(x) &\leq M - \varepsilon + \varepsilon_1 \; z(x) \\ &\leq M - \varepsilon + \varepsilon_1 \; K \\ &< M \end{aligned}$$

if $\varepsilon_1 < \varepsilon/K$ .

Having chosen $\varepsilon_1 < \varepsilon/k$ , we see that on $T_2$

$$z(x) < 0 \quad \text{since } |x| > r_1 .$$

Therefore

$$w(x) = u(x) + \varepsilon_1 \; z(x) < u(x) \leq M.$$

Thus $w(x) < M$ on $T_1 \cup T_2 = \partial B_2$. Since $w(y) = M$, $w$ has a maximum at an interior point $x_2$ in $B_2$. But

$$L[w] = L[u] + \varepsilon_1 \; L[z] > 0 \quad \text{in } B_2$$

since by assumption $(Lu)(x) \geq 0$ in $\Omega$. We have obtained a contradiction to the previous comments.□

<u>REMARKS 2.2</u>

(I) Theorem 2.3 remains valid in case that $\Omega$ is not bounded.

(II) We can weaken the hypotheses in Theorem 2.3 by requiring only that the quantities

$$\frac{a_{ij}(x)}{\mu(x)} \quad \text{and} \quad \frac{b_i(x)}{\mu(x)} \quad ,$$

with $\mu(x) > 0$, are bounded on every ball contained entirely in the domain $\Omega$.

(III) A minimum principle applies to functions satisfying $L[u] \leq 0$ by applying the Theorem 2.3 to $(-u)$.

Let $u(x)$ be continuous and bounded function on $\overline{\Omega}$. If $u(x)$ attains its maximum at a point $Q \in \partial\Omega$, then the outward directional derivative of $u$ at $Q$ cannot be negative, by Lemma 2.2 .

In fact we shall see that the directional derivative must be positive unless u is constant.

<u>THEOREM 2.4</u> (<u>MAXIMUM PRINCIPLE OF E. HOPF</u>)

Let $u(x)$ satisfy the inequality

$$L[u] \equiv a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + b_i \frac{\partial u}{\partial x_i} \geq 0$$

in $\Omega$, and suppose that $u$ assumes its maximum $M$ at a point $Q \in \partial\Omega$. Assume that $u$ is continuous in $\Omega \cup \{Q\}$ and $\partial\Omega$ satisfies an interior sphere condition at $Q$. Then if $\nu$ points outward from $\Omega$ at $Q$

$$\frac{\partial u}{\partial \nu}(Q) > 0 \quad ;$$

if it exists, unless $u \equiv M$ .

<u>PROOF</u> :(Protter and Weinberger [14])

Since $\partial\Omega$ satisfies an interior sphere condition at $Q$, there exists a ball $B \subset \overline{\Omega}$ of radius $r_1$ with $\partial B \cap \overline{\Omega} = \{Q\}$. We construct another ball $B_1$ with radius $r_1/2$ and with $Q$ as a centre. (see figure 2.2 below)

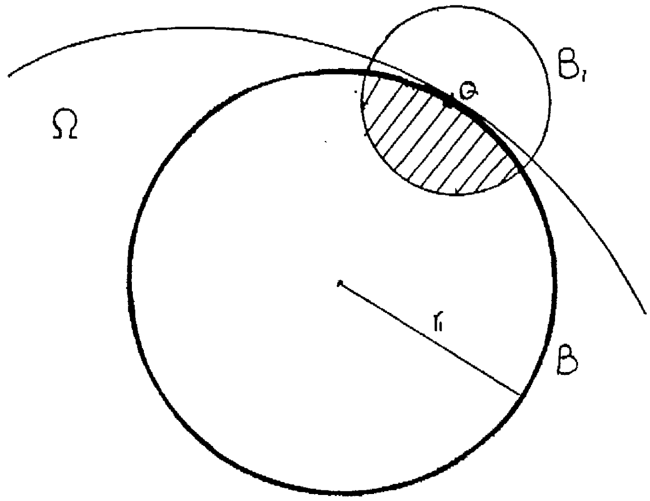


FIGURE [2.2]

We proceed as in proof of Theorem 2.3, and we choose the centre of the ball $B$ to be the origin of our coordinate system.

Now we introduce an auxiliary function

$$z(x) = e^{-\alpha r^2} - e^{-\alpha r_1{}^2} \quad , \; r^2 = |x| = x_1^2 + x_2^2 + \ldots + x_n^2$$

where $\alpha > 0$ is to be determined. We observe that

$$z > 0 \text{ in } B, \; z = 0 \text{ on } \partial B \text{ and } z < 0 \text{ outside } B.$$

By choosing $\alpha$ sufficiently large we obtain (see proof of Theorem 2.3.)

$$L[z] = a_{ij} \frac{\partial^2 z}{\partial x_i \partial x_j} + b_i \frac{\partial z}{\partial x_i} > 0 \quad \text{in } B_1.$$

Define the function

$$w(x) = u(x) + \varepsilon\, z(x) \quad , \; \varepsilon > 0.$$

If $u \not\equiv M$ in $\Omega$, then $u < M$ in $(B \cup \partial B)\setminus\{Q\}$ , by Theorem 2.3 . We choose $\varepsilon$ small enough to get $w \le M$ on $B \cap \partial B_1$. Then $w \le M$ on the entire boundary of the region $B \cap B_1$ (see proof of Theorem 2.3). In this region we have also

$$L[w] = L[u] + \varepsilon\, L[z] > 0$$

since $L[u] \geq 0$ and $L[z] > 0$. Thus $w(x)$ attains its maximum at $Q$, i.e. $w(Q) = M$, therefore

$$\frac{\partial w}{\partial \nu} = \frac{\partial u}{\partial \nu} + \varepsilon \frac{\partial z}{\partial \nu} \geq 0 \qquad \text{at } Q.$$

We shall show that $\dfrac{\partial z}{\partial \nu} < 0$ at $Q$, this will imply that

$$\frac{\partial u}{\partial \nu} > 0 \qquad \text{at } Q.$$

We compute

$$\frac{\partial z}{\partial x_i} = -\,2\,\alpha\, x_i\, e^{-\alpha r^2} \qquad , \ (i = 1,2,\ldots,n)$$

and we know that

$$\frac{\partial z}{\partial \nu} = \frac{\partial z}{\partial x_i} \cdot \nu_i \ ,$$

so if $\eta = (\eta_1, \eta_2, \ldots, \eta_n)$ is the unit outward pointing normal at $Q$, then $\eta_i = \dfrac{x_i}{r_1}$ . Hence at $Q$

$$\frac{\partial z}{\partial \nu} = -\,2\,\alpha\, r_1\, e^{-\alpha r^2} \sum_i \nu_i\, \eta_i < 0 \quad ,$$

since $\nu \cdot \eta > 0 \quad (i = 1, 2, \ldots, n)$ .

This completes the proof of Theorem 2.4 .□

<u>NOTE</u> :

The auxiliary function $z(x)$ can be chosen in different ways, for example as in Serrin [17], where it is defined by

$$z(x) = x_1 \left\{ e^{-\alpha|x|^2} - e^{-\alpha r^2} \right\} .$$

## MORE GENERAL DIFFERENTIAL INEQUALITIES

We now consider the linear partial differential inequalities in the domain $\Omega$ with zero order terms. The following results have been obtained by Serrin [17], see also Gidas, Ni, Nirenberg [5].

**THEOREM 2.5**

Suppose that $u \in C^2(\overline{\Omega})$ satisfies (we use summation convention)

$$(L + h)[u] \equiv a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + b_i \frac{\partial u}{\partial x_i} + hu \geq 0$$

in $\Omega$ and $u \leq 0$ in $\overline{\Omega}$. Then

(I) If $u$ vanishes at some interior point in $\Omega$, then $u \equiv 0$ in $\Omega$.

(II) If $\partial\Omega$ satisfies an interior sphere condition at a point $Q \in \partial\Omega$ with $u(Q) = 0$, then $\frac{\partial u}{\partial \nu}(Q) > 0$ unless $u \equiv 0$ in $\Omega$.

[ $\nu$ denotes a vector pointing outward from $\Omega$ at $Q$ ]

**PROOF** :

We define a function $v$ by

$$v(x) = e^{-\alpha x_1} u(x) \ ,$$

where $\alpha > 0$ is to be chosen.

Let $u(x) = e^{\alpha x_1} v = f(x) v$ (say). Then

$$\frac{\partial f}{\partial x_i} = \delta_{1i}\, \alpha\, e^{\alpha x_1}$$

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \alpha^2 e^{\alpha x_1} \delta_{1j}\, \delta_{1i} \quad \text{(zero unless } i = j = 1\text{)}.$$

Now

$$\frac{\partial u}{\partial x_i} = \frac{\partial f}{\partial x_i} v + f \frac{\partial v}{\partial x_i}$$

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_i \partial x_j} v + 2 \frac{\partial f}{\partial x_i} \frac{\partial v}{\partial x_j} + f \frac{\partial^2 v}{\partial x_i \partial x_j}$$

and hence,

$$0 \leq (L + h) = a_{ij}\left\{v\alpha^2 \; e^{\alpha x_1} \delta_{1j} \; \delta_{1i} + 2\delta_{1i} \; \alpha \; e^{\alpha x_1} \frac{\partial v}{\partial x_j} + e^{\alpha x_1} \frac{\partial^2 v}{\partial x_i \partial x_j}\right\}$$

$$+ \, b_i \left\{\delta_{1i} \; \alpha \; e^{\alpha x_1} \; v + e^{\alpha x_1} \; \frac{\partial v}{\partial x_i} \right\} + h \; e^{\alpha x_1} \; v \quad .$$

$$= e^{\alpha x_1} \left\{a_{ij} \frac{\partial^2 v}{\partial x_i \partial x_j} + b_i \frac{\partial v}{\partial x_i} + 2 \; \alpha \; a_{1i} \; \frac{\partial v}{\partial x_j} \right\}$$

$$+ \, e^{\alpha x_1} \; [\alpha^2 a_{11} \; v + \alpha \; b_1 \; v + h \; v \;]$$

$$= e^{\alpha x_1} \; L_1[v] + e^{\alpha x_1} \; [a_{11} \; \alpha^2 + b_1 \; \alpha + h] \; v$$

where $L_1$ is an elliptic operator containing no zero order terms. If we denote the term $[a_{11} \; \alpha^2 + b_1 \; \alpha + h]$ by $H$ and choose $\alpha$ sufficiently large such that $H \geq 0$ , we obtain

$$L_1[v] \geq 0 \quad \text{in } \Omega \quad ,$$

since $v \leq 0$ in $\Omega$.

(I) If $u = 0$ at some point in $\Omega$, $v = 0$ at that point and Theorem 2.3 applies to $v$ to get $v \equiv 0$ in $\Omega$. This completes (I).

(II) If $u = 0$ at a boundary point $Q$, $v = 0$ at $Q \in \partial\Omega$. Then by applying Theorem 2.4 to $v$ we get

$$\frac{\partial v}{\partial \nu} > 0 \qquad \text{at } Q.$$

Now

$$\frac{\partial u}{\partial \nu} = \frac{\partial u}{\partial x_i} \nu_i = \alpha\, e^{\alpha x_1}\, v\, \nu_i + e^{\alpha x_1} \frac{\partial v}{\partial x_i} \nu_i \ .$$

Therefore at $Q$ where $v = 0$,

$$\frac{\partial u}{\partial \nu} = e^{\alpha x_1} \frac{\partial v}{\partial \nu}$$

This proves (II). □

REMARK 2.3

It is interesting to note that Theorem 2.5 holds independently of the sign of $h(x)$. However the maximum of $u$ must be zero, otherwise the result may fail.

EXAMPLE 2.1 :

The function $u = (-\cosh x - \cosh y)$ satisfies the equation

$$\Delta u - u = 0$$

on a domain containing (0, 0). $u$ attains its maximum at the interior point (0,0) but is not constant. □

In the following, we study the well-known version of the Hopf Maximum Principle where it is required that $h \le 0$.

THEOREM 2.6

Let $u(x)$ satisfy the differential inequality

$$(L + h)[u] \equiv a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + b_i \frac{\partial u}{\partial x_i} + h\, u \ge 0$$

with $h(x) \le 0$, where $L$ is uniformly elliptic and the coefficients of $L$ and $h$ are bounded. We have the following:

(i) If $u$ assumes a positive maximum $M$ at an interior point of $\Omega$, then $u \equiv M$.

(ii) If $u$ assumes a positive maximum $M$ at a boundary point $Q$, and $\partial\Omega$ satisfies an interior sphere condition at $Q$, and $u$ is continuous in $\Omega \cup \{Q\}$, then if $\nu$ is the outward norml at $Q$,

$$\frac{\partial u}{\partial \nu}(Q) > 0$$

unless $u \equiv M$.

PROOF :

Let $w = u - M$, so that $w \leq 0$ in $\bar{\Omega}$.

Now

$$(L + h)[w] = (L + h)[u] - h\,M$$

$$\geq - h\,M \geq 0 \quad , \text{ since } \quad h \leq 0.$$

Therefore Theorem 2.5 applies to the function $w$ and hence (i),(ii) hold.□

REMARKS 2.4

(I) We believe that our method of proof Theorem 2.6 is new.

(II) Part (i) of Theorem 2.6 may fail if $h(x) > 0$.

EXAMPLE 2.2

Consider the differential equation

$$\Delta\, u + 2\, u = 0$$

in the domain

$$D = \{(x, y) : 0 \leq x, y \leq \pi\}$$

where $\Delta$ is the Laplacian. We find that the nonconstant solution

$$u(x, y) = \sin x \sin y$$

assumes its positive maximum (+ 1) at the interior point $(\pi/2\ , \pi/2)$ of D .

(III) In general whether or not the outer normal derivative exists, we get

$$\liminf_{x \to Q} \frac{u(Q) - u(x)}{|x - Q|} > 0$$

where the angle between the vector $Q - x$ and the normal at $Q$ is less than $\pi/2 - \delta$ for some fixed $\delta > 0$ (see Smoller [18] and Protter and Weinberger [14]).

SECTION 3

## (THE BOUNDARY POINT THEOREM AT A CORNER)

Our results, in Section 2, have required smoothness of the boundary $\partial\Omega$. Here we shall prove a boundary point Theorem at a special domain with a corner, which is suitable for applications to the non-linear elliptic equations.

We consider the results obtained by Serrin [17].

THEOREM 3.1

Consider the domain $\Omega$ with $C^2$ boundary and let $T$ be a plane containing the normal to $\partial\Omega$ at a boundary point $Q$. Let $\Omega^*$ denote the portion of $\Omega$ lying on some particular side of $T$.

Let $u$ be a function in $C^2(\overline{\Omega}^*)$ with $u \leq 0$ in $\Omega^*$, $u(Q)=0$, satisfying the differential inequality

$$L[u] \equiv a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + b_i \frac{\partial u}{\partial x_i} \geq 0 \qquad \text{......(3.1)}$$

in $\Omega^*$, with uniformly bounded coefficients. Assume that

$$|a_{ij}\,\zeta_i\,\eta_j| \leq K(|\zeta\,.\,\eta| + |\zeta|.|d|) \qquad \text{......(3.2)}$$

where $K$ is a positive constant, $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n)$ is an arbitrary real vector, $\eta = (\eta_1, \eta_2, \ldots, \eta_n)$ is the unit normal to the plane $T$, and $d$ is the distance from $T$. Then if $\nu$ is a vector pointing outward from $\Omega$ at $Q$,

either

$$\frac{\partial u}{\partial \nu}(Q) > 0 \quad \text{or} \quad \frac{\partial^2 u}{\partial \nu^2}(Q) < 0$$

unless $u \equiv 0$.

PROOF :

Since $\partial\Omega$ is of class $C^2$, there exists a ball $B_1 \subset \overline{\Omega}$ where $\overline{B}_1 \cap \overline{\Omega} = \{Q\}$ and with radius $r_1$ . Construct a ball $B_2$ centred at $Q$ and with radius $r_2 = \lambda\, r_1$ , where $\lambda$ is a constant to be prescribed. Let $H = B_1 \cap B_2 \cap \Omega^*$ noting that $B_1$, $B_2$ are open balls. (see figure 3.1)

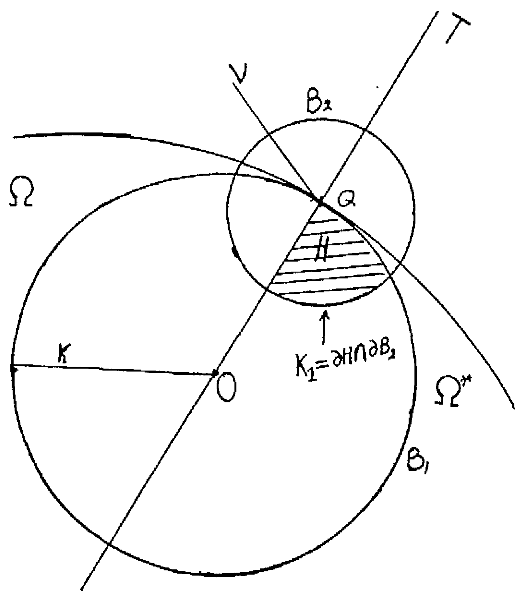


FIGURE [3.1]

Now we define the auxiliary function

$$z(x) = \left\{e^{-\alpha(x_1 - r_1)^2} - e^{-\alpha r_1^2}\right\} . \left\{e^{-\alpha r^2} - e^{-\alpha r_1^2}\right\} \qquad ......(3.3)$$

in $H$ , where $\alpha > 0$ is to be prescribed. Let us choose the centre of the ball $B_1$ to be the origin of our coordinate system and let $T$ be the plane $x_1 = 0$. For convenience assume that $\Omega^*$ is on the side of $T$ where $x_1 > 0$. We observe that :

$$\begin{cases} z > 0 & \text{in } H \\ z = 0 & \text{on } \partial B_1 \text{ and on } T \\ z < 0 & \text{outside } B_1 . \end{cases} \qquad .........(3.4)$$

To calculate $L[z]$, we first denote $e^{-\alpha(x_1 - r_1)^2}$ by $f$ and $e^{-\alpha r_1^2}$ by $\omega$. We compute

$$z = (f - \omega)(e^{-\alpha r^2} - \omega)$$

$$\frac{\partial f}{\partial x_i} = -2\alpha(x_1 - r_1) f \delta_{1i}$$

$$\frac{\partial z}{\partial x_i} = (f - \omega)(-2\alpha x_i e^{-\alpha r^2}) + \frac{\partial f}{\partial x_i}(e^{-\alpha r^2} - \omega)$$

$$\frac{\partial^2 z}{\partial x_i \partial x_j} = \frac{\partial f}{\partial x_j}(-2\alpha x_i e^{-\alpha r^2}) + (f - \omega)[-2\alpha e^{-\alpha r^2}\delta_{ij}$$

$$+ 4\alpha^2 x_i x_j e^{-\alpha r^2}] + \frac{\partial^2 f}{\partial x_i \partial x_j}(e^{-\alpha r^2} - \omega)$$

$$+ \frac{\partial f}{\partial x_i}(-2\alpha x_j e^{-\alpha r^2})$$

where

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_j}\left[\frac{\partial f}{\partial x_i}\right] = \begin{cases} 0 & \text{if } i \neq 1 \\ 0 & \text{if } i = 1,\ j \neq 1 \\ -2\alpha f + 4\alpha^2 (x_1 - r_1)^2 f, & \text{if } i = j = 1. \end{cases}$$

Then, from above we have

$$L[z] = e^{-\alpha r^2}[e^{-\alpha(x_1 - r_1)^2} - e^{-\alpha r_1^2}][4\alpha a_{ij} x_i x_j$$

$$- 2\alpha(a_{ij} + b_i x_i)] + e^{-\alpha(x_1 - r_1)^2}[e^{-\alpha r^2} - e^{-\alpha r_1^2}]$$

$$\cdot \left\{4\alpha^2 a_{11}(x_1 - r_1)^2 - 2\alpha[a_{11} + b_1(x_1 - r_1)]\right\}$$

$$+ 8\alpha^2 e^{-\alpha r^2} e^{-\alpha(x_1 - r_1)^2}(x_1 - r_1) a_{1j} x_j.$$

Since $r \geq (1/2) r_1$ in $H$, we have by ellipticity

$$a_{ij} x_i x_j \geq \mu r^2 \geq (1/4) \mu r_1^2 \quad \text{in } H, \quad \ldots\ldots(3.5)$$

and also

$$a_{11}\ (x_1 - r_1)^2 \geq\ (1/4)\ \mu\ r_1^2 \qquad \text{in } H\ .$$

From (3.2) we get

$$|a_{1j}\ x_j| = |a_{ij}\ \eta_i\ x_j| \leq\ K(|x_1| + |x_1|)$$

since in the present case $\eta = (1,\ 0, \ldots,\ 0)$. Thus

$$|(x_1 - r_1)\ a_{1j}\ x_j| \leq\ 2\ x_1\ r_1\ K \qquad \text{in } H.$$

Let $g(x) = e^{-\alpha x^2}$, by the Mean Value Theorem we get

$$g(r_1 - x_1) - g(r_1) = [(r_1 - x_1) - r_1]\ g'(\xi),$$

where $\xi = t(r_1 - x_1) + (1 - t)\ r_1$, $0 < t < 1$. Therefore

$$g(r_1 - x_1) - g(r_1) = (-\ x_1)(-\ 2\ \alpha\ \xi\ e^{-\alpha\xi^2})$$

$$= 2\ \alpha\ x_1\ \xi\ e^{-\alpha\xi^2}\ .$$

Since $\xi = r_1 - t\ x_1$, we get $\xi \geq r_1 - |x_1| \geq r_1(1 - \lambda)$ and $\xi \leq r_1$, since $|x_1| \leq \lambda\ r_1$ in $H$, and $x_1 > 0$ in $H$. Therefore

$$e^{-\alpha(x_1 - r_1)^2} - e^{-\alpha r_1^2} \geq 2\ \alpha\ x_1\ (1 - \lambda)\ r_1\ e^{-\alpha r_1^2} \ldots (3.6)$$

Using the fact that $e^{-\alpha(x_1 - r_1)^2} \leq 1$ and $\lambda \leq 1/2$ we get the inequality

$$2\ \alpha\ x_1\ (1 - \lambda)\ r_1\ e^{-\alpha r_1^2} \geq \alpha\ x_1\ r_1\ e^{-2\alpha\lambda r_1^2}\ e^{-\alpha(x_1 - r_1)^2}. \qquad \ldots\ldots(3.7)$$

From (3.6) and (3.7) the following inequality is achieved

$$e^{-(x_1-r_1)^2} - e^{-\alpha r_1^2} \geq 2\alpha x_1(1-\lambda)r_1 e^{-\alpha r_1^2} \geq \alpha x_1 r_1 e^{-2\alpha\lambda r_1^2 - \alpha(x_1-r_1)^2}.$$

Inserting the above inequalities into the expression for $L[z]$, by virtue of the fact that the terms $[a_{ii} + b_i x_i]$ and $[a_{11} + b_1 (x_1 - r_1)]$ are bounded, and for $\alpha$ sufficiently large we have :

$$L[z] \geq \alpha^2 x_1 r_1 e^{-\alpha[r^2 + (x_1 - x_1)^2]} \left[(\alpha\mu r_1^2 - S)\ e^{-2\alpha\lambda r_1^2} - 16K\right]$$

$$+ \alpha\ e^{-\alpha(x_1 - r_1)^2}[e^{-\alpha r^2} - e^{-\alpha r_1^2}]\ \left[\alpha\ \mu\ r_1^2 - S\right]$$

where $S$ is an appropriate constant being chosen as the following:

$$S = max\ \left[[a_{ii} + b_i\ x_i],\ [a_{11} + b_i\ (x_1 - r_1)]\right]\ .$$

We require $L[z]$ to be positive in $H$ . To see this, let $\lambda = 1/\alpha$ and choose $\alpha$ sufficiently large such that the quantities:

$$\left[(\alpha\ \mu\ r_1^2 - S)\ e^{-2r_1^2} - 16\ K\right] \text{ and } \left[\alpha\ \mu\ r_1^2 - S\right]$$

become *positive*. Hence we have constructed a function $z(x)$ with $L[z] > 0$ in $H$ .

Suppose that $u \not\equiv 0$ in $\Omega^*$, then by virtue of Theorem 2.3, $u < 0$ in the domain $\Omega^*$ and hence $u < 0$ in $H$.

We consider the part of the boundary of $H$ lying on $\partial B_2$. This set $K_1 = \partial H \cap \partial B_2$, say, intersects the boundary of $\Omega^*$ only on the plane $T$. Furthermore the intersection set $K_1 \cap T$ lies at a finite distance from the corner of $\Omega^*$.

Moreover

$$u \leq 0 \qquad \text{on } \partial H \cap \partial B_1 \quad \text{and} \quad \text{on } \partial H \cap T.$$

Now let $g(x) = e^{-\alpha x^2}$ , then by the Mean Value Theorem, exactly as above

$$g(r_1 - x_1) - g(r_1) \leq 2\,\alpha\, x_1\, \xi \leq 2\,\alpha\, x_1 r_1$$

and since $\left[ e^{-\alpha r^2} - e^{-\alpha r_1^2} \right] \leq 1$ , we have

$$z \leq 2\,\alpha\, r_1 x_1 \qquad \text{on } K_1.$$

We define a function $v(x)$ by

$$v(x) = u(x) + [\varepsilon/(2\alpha r_1)]\, z(x).$$

Then $v \leq 0$ on $\partial H$ and $v = 0$ at $Q$ , since $z > 0$ in $H$ and $z = 0$ at $Q$ . We observe that

$$\begin{aligned} L[v] &= L[u + (\varepsilon/(2\alpha r_1))\, z] \\ &= L[u] + (\varepsilon/(2\alpha r_1))\, L[z] \\ &> 0 \qquad \text{in } H. \end{aligned}$$

Therefore (by the remarks preceding Theorem 2.3) we get

$$v < 0 \qquad \text{in } H ,$$

and at $Q$ , where $v = 0$ ,

$$\frac{\partial v}{\partial \nu} \geq 0 \qquad \text{and} \qquad \frac{\partial^2 v}{\partial \nu^2} \leq 0 \, , \quad \text{by Lemma 2.2 .}$$

We need to calculate $\frac{\partial z}{\partial \nu}$ explicitly at $Q$ , at which $x_1 = 0$

$$\begin{aligned} \frac{\partial z}{\partial \nu} &= grad\ z \,.\, \nu \\ &= \sum_i \frac{\partial z}{\partial x_i} \,.\, \nu_i \end{aligned}$$

$$= \frac{\partial f}{\partial x_1} (e^{-\alpha r^2} - \omega)\ \nu_1 + \sum_i^n (f - \omega)(- 2\alpha\ e^{-\alpha r^2})\ x_i\ \nu_i .$$

But at $Q$ , $\omega = e^{-\alpha r^2}$ and $f = \omega$ , so we get

$$\frac{\partial z}{\partial \nu} = 0 \qquad \text{at } Q .$$

If $\frac{\partial v}{\partial \nu} = 0$ we should move to the second derivative with respect to $\nu$, therefore we need to calculate $\frac{\partial}{\partial \nu}\left(\frac{\partial z}{\partial \nu}\right)$ at $Q$.

As earlier,

$$\frac{\partial^2 z}{\partial \nu^2} = \sum_{i,j} \frac{\partial^2 z}{\partial x_i \partial x_j}\ \nu_i\ \nu_j .$$

From earlier calculation

$$\frac{\partial^2 z}{\partial x_i \partial x_j} = (f - \omega)\left[- 2\alpha\ e^{-\alpha r^2}\ \delta_{ij} + 4\ \alpha^2\ x_i\ x_j\ e^{-\alpha r^2}\ \right]$$

$$+ \frac{\partial^2 f}{\partial x_i \partial x_j} (e^{-\alpha r^2} - \omega) + \frac{\partial f}{\partial x_j} (- 2\ \alpha\ x_i\ e^{-\alpha r^2})$$

$$+ \frac{\partial f}{\partial x_i} (- 2\ \alpha\ x_j\ e^{-\alpha r^2}) .$$

At Q , only the last two terms are non-zero, so

$$\frac{\partial^2 z}{\partial \nu^2} = 2 \sum_i^n \frac{\partial f}{\partial x_1}\ \nu_1 (- 2\ \alpha\ e^{-\alpha r^2}\ x_i\ \nu_i) .$$

[<u>NOTE:</u> The term i = 1 is zero, but this does not matter. ]

Now at the point $Q$ we have the following:

$\frac{\partial f}{\partial x_i} = 2\ \alpha\ r_1\ f$ is positive, $\nu_1 < 0$ , $\sum_i x_i\ \nu_i > 0$

where $\nu$ is outward pointing vector to $H$ at $Q$ . Hence we have

$$\frac{\partial^2 z}{\partial \nu^2} > 0 \qquad \text{at } Q.$$

Then, since

$$0 \geq \frac{\partial^2 v}{\partial \nu^2} - \frac{\partial^2 u}{\partial \nu^2} + \frac{\varepsilon}{2\alpha r_1} \frac{\partial^2 z}{\partial \nu^2} > \frac{\partial^2 u}{\partial \nu^2}$$

we deduce that

$$\frac{\partial^2 u}{\partial \nu^2} < 0 \qquad \text{at } Q$$

and the Theorem has been proved.□

REMARK 3.1 :

We remark that the same result holds if $(L + h)\, u \geq 0$ , by using the same procedure as in Theorem 2.5 .

The following example illustrates the Theorem .

EXAMPLE 3.1

The function $u(x, y) = -\cos x \cos y$ satisfies the elliptic equation

$$\Delta u - 2u = 0$$

on the square

$$D = \left\{ (x, y) : -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} , \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \right\}$$

and $u$ attains its maximum (zero) at *all* points of the boundary.

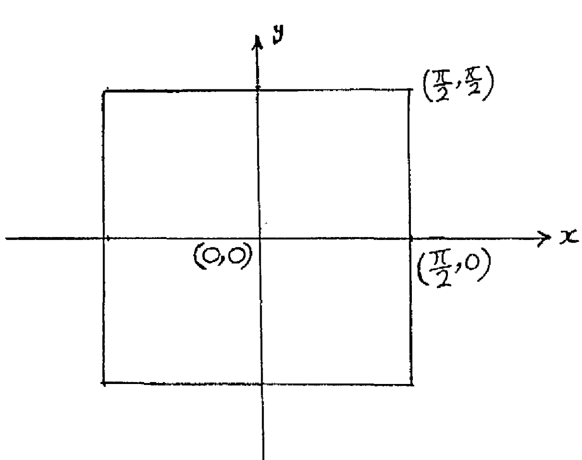


FIGURE [3.2]

It is sufficient to calculate $\frac{\partial u}{\partial \nu}$ at the right hand side of the square:

$$u_x = \sin x \cos y \ , \qquad u_{xx} = \cos x \cos y$$

$$u_y = \cos x \sin y \ , \qquad u_{yy} = \cos x \cos y$$

$$\therefore \Delta u = 2 \cos x \cos y \geq 0 \quad \text{in the given square .}$$

At the two right hand corners $grad\ u = 0$ , but at other points $u_x > 0$ and $u_y = 0$ . Therefore $\frac{\partial u}{\partial \nu} = 0$ at the corners and $\frac{\partial u}{\partial \nu} > 0$ at other points.

Now we shall show that $\frac{\partial^2 u}{\partial \nu^2} < 0$ at the corners.

It is enough to show this at the top R.H. corner $(\pi/2, \pi/2)$:

$$\frac{\partial u}{\partial \nu} = u_x \ \nu_1 + u_y \ \nu_2$$

$$\frac{\partial^2 u}{\partial \nu^2} = u_{xx} \ \nu_1^2 + 2 \ u_{xy} \ \nu_1 \ \nu_2 + u_{yy} \ \nu_2^2$$

$$= - 2 \ \nu_1 \ \nu_2 < 0$$

since $\nu_1, \nu_2 > 0$ at the top R.H. corner. Hence

$$\frac{\partial^2 u}{\partial \nu^2} < 0$$

at the top R.H. corner .□

<u>EXAMPLE 3.2</u>

Let $\Omega$ be the first quadrant in $\mathbb{R}^2$ . Let $u(x, y) = - xy$ in $\Omega$ . Then $u < 0$ in $\Omega$ , $u(0,0) = 0$ and $\Delta u = 0$ but

$$\frac{\partial u(\underline{0})}{\partial \nu} = \lim_{x \to 0} \frac{u(x)}{|x|} = 0$$

Also we find that :

$$\frac{\partial^2 u}{\partial \nu^2} = u_{xx}\, \nu_1^2 + 2\, u_{xy}\, \nu_1 \nu_2 + u_{yy}\, \nu_2^2$$

$$= -\,2\, \nu_1\, \nu_2 < 0 \quad ,$$

since

$$u_x = -\,y \qquad u_{xx} = 0 \quad ,$$

$$u_y = -\,x \qquad u_{yy} = 0 \quad ,$$

$$u_{xy} = -1 \quad .\square$$

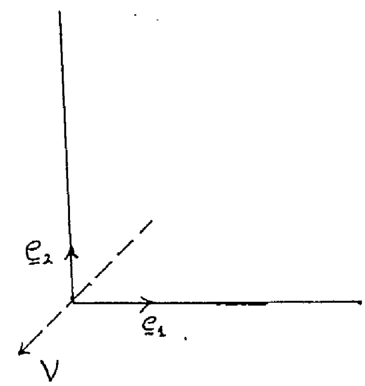


FIGURE [3.2]

$$\nu_1 = \underline{\nu} \cdot \underline{e}_1 < 0 \ ,$$
$$\nu_2 = \underline{\nu} \cdot \underline{e}_2 < 0$$

*Chapter* (III)

# SYMMETRY PROPERTIES VIA MAXIMUM PRINCIPLES

CHAPTER (III)

## SYMMETRY PROPERTIES VIA MAXIMUM PRINCIPLES

SECTION 1

### INTRODUCTION

We investigate symmetry of domains and symmetry of solutions of second order elliptic equations, in particular the symmetry of positive solutions of elliptic equations. The results are based on work of Serrin[17], Gidas,Ni,Nirenberg[5], Gidas[4] and use certain forms of the maximum principle (from Chapter(II)) together with a device of A.D.Alexandroff ( *Procedure of moving parallel planes to a critical point*). These techniques were employed before by Serrin[17] who treated solutions of elliptic equations with over - determined boundary conditions.

### PROCEDURE OF MOVING UP PARALLEL PLANES

This consists of moving up parallel planes perpendicular to a fixed direction, and then showing that the solution is symmetric about a limiting plane.

We assume that $\Omega$ is a bounded domain in $\mathbb{R}^n$ with smooth boundary. Let $\gamma$ be a unit vector in $\mathbb{R}^n$ and let $T_\lambda$ denote the hyperplane $\gamma \cdot x = \lambda$ . For sufficiently large $\bar{\lambda} > 0$ the plane $T_{\bar{\lambda}}$ does not intersect $\bar{\Omega}$ since $\Omega$ is bounded. Suppose that we decrease $\lambda$ , ( *i.e. we suppose this plane to be continuously moved towards* $\Omega$ , *normal to itself, to new positions*), until ultimately $T_\lambda$ begins to intersect $\bar{\Omega}$ . We denote by $\lambda_0$ the first value of $\lambda$ for which $T_\lambda$ intersects $\bar{\Omega}$ .

From that value of $\lambda$ on, the plane $T_\lambda$ cuts off from $\Omega$ an *open* cap $\Sigma(\lambda)$ ; that is, $\Sigma(\lambda)$ will be that portion of $\Omega$ which lies on the same side of $T_\lambda$ as $T_{\bar{\lambda}}$ .

Let $\Sigma'(\lambda)$ denote the reflection of $\Sigma(\lambda)$ in the plane $T_\lambda$. Clearly $\Sigma'(\lambda)$ will be contained in $\Omega$ at the beginning as $\lambda$ decreases, at least until one of the following occurs :

(I) $\Sigma'(\lambda)$ becomes internally tangent to $\partial\Omega$ at some point $P \notin T_\lambda$

or

(II) $T_\lambda$ reaches a position at which it is orthogonal to $\partial\Omega$ at some point $Q$ .

We denote by $T_{\lambda_1} : \gamma \cdot x = \lambda_1$ the plane $T_\lambda$ when it reaches either one of these positions. Evidently, $\Sigma'(\lambda_1) \subseteq \Omega$. It may happen that if we decrease $\lambda$ below $\lambda_1$, the reflected cap $\Sigma'(\lambda)$ of $\Sigma(\lambda)$ in $T_\lambda$ continues to be contained in $\Omega$. In that case $\Sigma'(\lambda) \subseteq \Omega$ for $\lambda \in [\lambda_2, \lambda_0]$, where

$$\lambda_2 = \inf \{ \tilde{\lambda} < \lambda_0 \mid \Sigma'(\lambda) \subset \Omega \text{ for } \tilde{\lambda} < \lambda < \lambda_0 \}.$$

Then, $\Sigma(\lambda_2)$ is called the optimal cap corresponding to the direction $\gamma$, and at $\lambda_2$ either (I) or (II), above, must occur. { see figure (1.1) below }

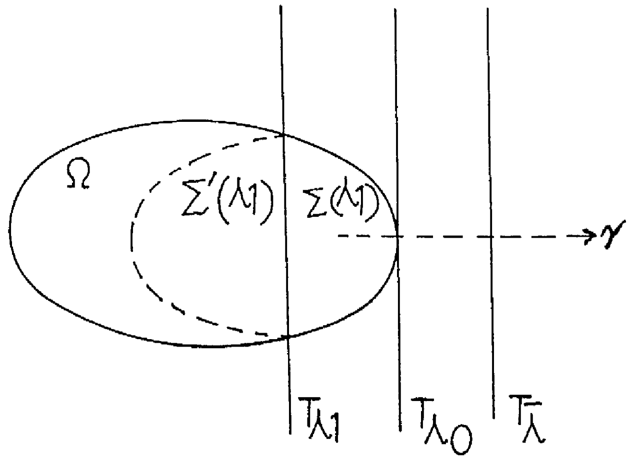


FIGURE [1.1]

SECTION 2

## MAIN THEOREMS

In this Section we will be concerned with positive functions $u(x_1, x_2, \ldots, x_n)$ that satisfy the semilinear elliptic equation:

$$\Delta u + f(u) = 0 \quad \text{in } \Omega \qquad (u > 0 \text{ in } \Omega) \tag{2.1}$$

with boundary condition

$$u = 0 \quad \text{on } \partial\Omega , \tag{2.2}$$

where $\Omega \subseteq \mathbb{R}^n$, $f$ is assumed to belong to the space $C^1(\Omega)$, is bounded.

We first give a result of Serrin [17] which shows that for an over - determined problem the domain $\Omega$ must be a ball in $\mathbb{R}^n$. We shall then give general results concerning symmetries of solutions. In particular, if $\Omega$ is a ball then the solution is radially symmetric .

THEOREM 2.1 (Serrin [17])

Let $\Omega$ be a domain whose boundary is of class $C^2$ in $\mathbb{R}^n$. Let $u \in C^2(\bar{\Omega})$ be a solution of the Poisson differential equation

$$\Delta u = -1 \quad \text{in } \Omega \tag{2.3}$$

together with the boundary conditions

$$u = 0, \quad \frac{\partial u}{\partial \nu} = \text{constant} \quad \text{on } \partial\Omega \tag{2.4}$$

Then $\Omega$ is a ball.

REMARKS 2.1

(I) By the maximum principle, we have $u > 0$ in $\Omega$.

(II) It will follow that $u$ is radially symmetric (see Theorem 2.2), and in fact $u$ must have the form $(R^2 - r^2)/2n$ , where $R$ is the radius of the ball and $r$ denotes distance from its centre. (see Serrin [17])

PROOF OF THEOREM 2.1 :

We first follow, exactly as in Section 1, the procedure of moving up parallel planes to a critical point. This will lead to the assertion that $\Omega$ must be symmetric about the plane $T_{\lambda_1}$ .

To see this, we observe that for any given direction in $\mathbb{R}^n$ there would then be a plane $T_{\lambda_1}$ with normal in that direction such that $\Omega$ is symmetric about $T_{\lambda_1}$ and $\Omega$ would have to be simply connected.

Assuming that the assertion holds, and since this is true for an arbitrarily chosen direction and since $\Omega$ is simply connected, then $\Omega$ must be a ball.

We choose an arbitrary direction, which we may assume to be $x_1$ and move the hyperplane towards $\Omega$ along the $x_1$- axis.

In order to show that $\Omega$ is symmetric about $T_{\lambda_1}$ , we recall, from Section 1, the definitions of $\lambda_0$, $\lambda_1$, $T_\lambda$, $\Sigma(\lambda)$ and $\Sigma'(\lambda)$ for $\lambda \in [\lambda_2, \lambda_0]$. Now we define a new function $v(x)$ in $\Sigma'(\lambda_1)$ by :

$$v(x) = u(x^{\lambda_1}) \qquad \text{for } x \in \Sigma'(\lambda_1) ,$$

where $x^{\lambda_1}$ is the reflected value of $x$ across $T_{\lambda_1}$ . Evidently $v$ satisfies the differential equation:

$$\Delta v = -1 \qquad \text{in } \Sigma'(\lambda_1)$$

and the boundary conditions

$$v = u \qquad \text{on } \partial\Sigma'(\lambda_1)\cap T_{\lambda_1}$$

$$v = 0, \quad \frac{\partial v}{\partial \nu} = \text{constant} = c \quad \text{on } \partial\Sigma'(\lambda_1)\cap(T_{\lambda_1})^C ,$$

where the constant being the same as in (2.4) and $(T_{\lambda_1})^C$ denotes the complement of $(T_{\lambda_1})$ .

We now wish to consider a new function

$$w = u - v \quad \text{in } \Sigma' = \Sigma'(\lambda_1) ,$$

since $\Sigma'(\lambda_1)$ is contained in $\Omega$ by construction. The following holds :

$$\Delta w = 0 \qquad \text{in } \Sigma',$$

$$w = 0 \qquad \text{on } \partial\Sigma'\cap T_{\lambda_1}$$

and

$$w \geq 0 \qquad \text{on } \partial\Sigma'\cap(T_{\lambda_1})^C ,$$

where the latter condition is a consequence of $u > 0$ in $\Omega$.

Applying the strong maximum principle as in Theorem 2.3 of Chapter (II) to the function $w$ we get : either

$$w > 0 \quad \text{in } \Sigma' , \tag{2.5}$$

or

$$w \equiv 0 \quad \text{in } \Sigma'. \tag{2.6}$$

Therefore if (2.6) holds, we get

$$u(x) \equiv u(x^{\lambda_1}) \quad \text{for } x \in \Sigma(\lambda_1) .$$

Since $u > 0$ in $\Omega$ and $u = 0$ on $\partial\Omega$, then the reflection of any point $x \in \partial\Omega$ can not lie *inside* $\Omega$ but along $\partial\Omega$. i.e. the reflected cap $\Sigma'$ must coincide with that part of $\Omega$ on the same side of $T_{\lambda_1}$ as $\Sigma'$ , that is $\Omega$ must be symmetric about $T_{\lambda_1}$ .

To complete the proof of the theorem we must show that (2.5) is impossible.

Recall that for $\lambda = \lambda_1$ either :

(a) $\Sigma'$ is internally tangent to the boundary of $\Omega$ at some point $P \notin T_{\lambda_1}$ , or

(b) $T_{\lambda_1}$ is orthogonal to the boundary of $\Omega$ at some point $Q$ .

Suppose that we are in case (a) , then $w = 0$ at $P$ . By virtue of Theorem 2.5 of Chapter (II) we have :

$$\frac{\partial w}{\partial \nu} = \frac{\partial (u - v)}{\partial \nu} > 0 \quad \text{at } P .$$

This contradicts the fact that

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \text{constant} = c \quad \text{at } P.$$

Hence in case (a), (2.5) is impossible.

Thus we assume that there is a point $Q \in \partial\Omega$ where $T_{\lambda_1}$ is orthogonal to $\partial\Omega$ [i.e. case(b)]. That means $Q$ is a right angled corner of $\Sigma'$ . Now we shall show that:

(i) $u - v$ (= 0 at $Q$) has a zero of order two ,

and

(ii) apply Theorem 3.1 of Chapter (II) to reach a contradiction.

For (i) , let $Q$ be the origin of our coordinate system , with $x_1$-axis being normal to $T_{\lambda_1}$ and $x_n$-axis being directed along the inward normal to $\partial\Omega$ at $Q$ . Since, by hypotheses , the boundary of $\Omega$ is of class $C^2$ , $u \in C^2(\bar{\Omega})$ and $u = 0$ on $\partial\Omega$, then we have the following representations:

$$x_n = \psi(x_1, x_2, \ldots, x_{n-1}) \ , \qquad (2.7)$$

$$u(x_1, \ldots, x_{n-1}, \psi) \equiv 0 \qquad (2.8)$$

We also wish to establish a representation for the boundary condition $\partial u/\partial \nu = c$ on $\partial\Omega$ in the light of the above coordinate system and new representations (2.7), (2.8). To do that we form, from (2.7), the function :

$$F(x_1, x_2, \ldots, x_n) = \psi(x_1, x_2, \ldots, x_{n-1}) - x_n = 0.$$

The direction numbers of the normal to the boundary can be given by :

$$grad\ F \equiv \left\{\frac{\partial\psi}{\partial x_1}, \ldots, \frac{\partial\psi}{\partial x_{n-1}}, -1\right\} \ldots\ldots\ldots\ldots\ldots (*)$$

and the direction cosines $\nu_i$ of the normal by :

$$\underline{\nu} = \frac{grad\ F}{|grad\ F|}$$

Therefore

$$\frac{\partial u}{\partial \nu} = grad\ u.\ \underline{\nu} = \frac{\frac{\partial u}{\partial x_1}\frac{\partial\psi}{\partial x_1} + \ldots + \frac{\partial u}{\partial x_{n-1}}\frac{\partial\psi}{\partial x_{n-1}} - \frac{\partial u}{\partial x_n}}{\left[1 + \sum_i \left[\frac{\partial\psi}{\partial x_i}\right]^2\right]^{\frac{1}{2}}}$$

where $i = 1, 2, \ldots, n-1$. Thus $\partial u/\partial\nu$ = constant $= c$ can be written on $\partial\Omega$ as :

$$\sum_i \frac{\partial u}{\partial x_i}\frac{\partial\psi}{\partial x_i} - \frac{\partial u}{\partial x_n} = c\left[1 + \sum_i \left[\frac{\partial\psi}{\partial x_i}\right]^2\right]^{\frac{1}{2}} \qquad (2.9)$$

$(i = 1, 2, \ldots, n-1)$ .

Differentiating (2.8) with respect to $x_i$ , $i = 1, 2, \ldots, n-1$, we obtain

$$\frac{\partial u}{\partial x_i} + \frac{\partial u}{\partial x_n}\frac{\partial\psi}{\partial x_i} = 0\ . \qquad (2.10)$$

From (*) and the fact that $\nu = (0,\ldots, 0, -1)$ at $Q$, we have $\partial\psi/\partial x_i = 0$, $i = 1, 2,\ldots, n-1$. Therefore at $Q$ we have: from (2.10),

$$\frac{\partial u}{\partial x_i} = 0\ ,\ i = 1, 2,\ldots, n-1, \quad \text{and from (2.9)} \quad \frac{\partial u}{\partial x_n} = -c\ .$$

Next we differentiate (2.10) with respect to $x_j$, $j = 1,2,\ldots n-1$. This gives :

$$\frac{\partial^2 u}{\partial x_i \partial x_j} + \frac{\partial^2 u}{\partial x_n \partial x_j}\frac{\partial \psi}{\partial xi} + \frac{\partial u}{\partial x_n}\frac{\partial^2 \psi}{\partial x_i \partial x_j} = 0$$

and evaluating at $Q$ we get:

$$\frac{\partial^2 u}{\partial x_i \partial x_j} - c\,\frac{\partial^2 \psi}{\partial x_i \partial x_j} = 0 \quad \text{at } Q \qquad (2.11)$$

Lastly differentiating (2.9) with respect to $x_k$, $k = 1, 2,\ldots, n-1$, we get :

$$\sum_{i,j}\frac{\partial^2 u}{\partial x_i \partial x_k}\frac{\partial \psi}{\partial x_i} + \frac{\partial u}{\partial x_i}\frac{\partial^2 \psi}{\partial x_i \partial x_k} - \frac{\partial^2 u}{\partial x_n \partial x_k} = c\left[\frac{\left\{\dfrac{\partial \psi}{\partial x_i}\right\}\displaystyle\sum_{i,j}\frac{\partial^2 \psi}{\partial x_i \partial x_k}}{\left\{1 + \displaystyle\sum_i \left\{\dfrac{\partial \psi}{\partial x_i}\right\}^2\right\}^{\frac{1}{2}}}\right]$$

and evaluating at $Q$ gives :

$$\frac{\partial^2 u}{\partial x_n \partial x_k} = 0 \qquad (\text{at } Q)$$

From (2.3) and (2.11) we obtain :

$$\frac{\partial^2 u}{\partial x_n \partial x_n} = c\,\Delta\,\psi - 1\ .$$

We have now determined all the first and second order derivatives of $u$ at $Q$.

Since, by the definition, $v(x_1, \bar{x}) = u(-x_1, \bar{x})$, $(\bar{x} = x_2, \ldots, x_n)$, in $\Sigma'(\lambda_1)$ we find that the first and second derivatives of $u$ and $v$ agree at $Q$. This completes the proof of (i).

Now for (ii) we apply the boundary point maximum principle to the function $w = u - v$ in $\Sigma'(\lambda_1)$. Since $w > 0$ in $\Sigma'(\lambda_1)$, and $w = 0$ at $Q$, we get :

$$\frac{\partial(u-v)}{\partial\nu} > 0 \quad \text{or} \quad \frac{\partial^2(u-v)}{\partial\nu^2} < 0 \quad (\text{at } Q).$$

This contradicts the fact that both $u$ and $v$ have the same first and second partial derivatives at $Q$, and the proof of the theorem is complete. □

REMARK 2.2

Serrin also gives a similar result for general elliptic equations and also for over-determined boundary conditions where $\partial u/\partial\nu$ = constant is replaced by $\partial u/\partial\nu = c(K)$, where $c$ is a $C^1$ non-decreasing function of the mean curvature K.

NOTES :

In the proof of Theorem 2.1 we have applied the following properties of the Poisson equation:

(I) The Poisson equation is invariant under the reflection $x \longrightarrow x^\lambda$.

(II) The difference of two solutions obeys the strong maximum principle.

The following theorem says that : if $\Omega$ is *a ball* in $\mathbb{R}^n$, then the positive solution of the elliptic equation $\Delta u + f(u) = 0$ is *radially symmetric* about the origin of $\Omega$. Moreover $\partial u/\partial r < 0$ for $0 < r < R$.

THEOREM 2.2

Let $\Omega$ be a ball of radius $R$ in $\mathbb{R}^n$. let $u > 0$ be a positive solution in $C^2(\bar{\Omega})$ of the differential equation

$$\Delta u + f(u) = 0.$$

Suppose that

$$u = 0 \quad \text{on } \partial\Omega = \{x \in \mathbb{R}^n : |x| = R\}$$

and the function $f$ is of the form $f_1 + f_2$ where $f_1 \in C^1$ and $f_2$ is monotonically increasing. Then $u$ is radially symmetric and $\partial u/\partial r < 0$, for $0 < r < R$.

PROOF:

Here we use the maximum principle forms as in Theorem 2.5 (Chapter (II)) together with the procedure of moving up parallel planes. We require, in addition, two technical lemmas to finish the proof of Theorem 2.2.

We pick an arbitrarily chosen direction, as in the proof of Theorem 2.1, which we may assume to be $x_1$; see figure (2.1) below. We move a hyperplane $T_\lambda$ along the $x_1$- axis, normal to itself, from the right towards the origin with $x_1$ positive. Let $\gamma = (1,0,\ldots,0)$ and recall from Section 1 the definitions of $\lambda_0$, $\lambda_1$, $\lambda_2$, $T_\lambda$, $\Sigma(\lambda)$ and $\Sigma'(\lambda)$ for $\lambda \in [\lambda_2, \lambda_0]$.

Let $T_{\lambda_1}$ be the hyperplane $x_1 = 0$. We define $x^{\lambda_1}$ to be the reflection of $x$ in the plane $T_{\lambda_1}$, where $x \in \Sigma(\lambda_1)$. We will show that

$$u(x) = u(x^{\lambda_1}) \quad x \in \Sigma(\lambda_1). \tag{2.12}$$

Since the $x_1$- direction is arbitrarily chosen, (2.12) proves the symmetry property.

By employing the device of A.D.Alexandroff we will be able to prove that, for $x \in \Sigma(\lambda)$

$$u(x) < u(x^{\lambda}) \tag{2.13}$$

where $x^{\lambda}$ is the reflection of $x$ in the plane $T_{\lambda}$, for $\lambda \in (\lambda_1, \lambda_0)$.

Now for a given $\varepsilon > 0$ and $x_0 \in \partial\Omega$ we define :

$$\Omega_{\varepsilon} = \Omega \cap \{|x - x_0| < \varepsilon\}$$

and

$$S_{\varepsilon} = \partial\Omega \cap \{|x - x_0| < \varepsilon\}.$$

LEMMA 2.3

Let $x_0 \in \partial\Omega$ with $\nu_1(x_0) > 0$. For a sufficiently small $\varepsilon > 0$, assume that $u \in C^2(\overline{\Omega}_{\varepsilon})$, $u > 0$ in $\Omega$ and $u = 0$ on $S_{\varepsilon}$. Then there exists $\delta > 0$ such that

$$\frac{\partial u}{\partial x_1} < 0 \quad \text{in } \Omega_{\delta}.$$

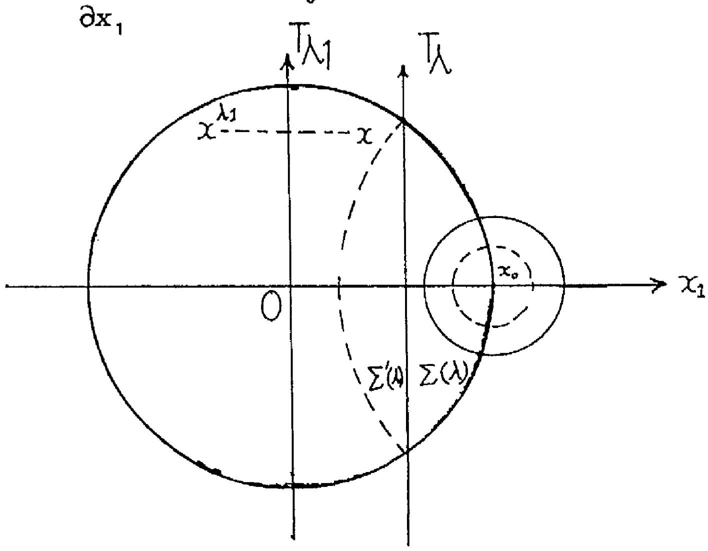


FIGURE [2.1]

LEMMA 2.4

Assume that for $\lambda \in [\lambda_1, \lambda_0)$, the function $u$ satisfies

$$\frac{\partial u}{\partial x_1} \leq 0$$

and

$$u(x) \leq u(x^{\lambda}) \quad \text{but} \quad u(x) \not\equiv u(x^{\lambda}) \quad \text{in } \Sigma(\lambda).$$

Then

$$u(x) < u(x^{\lambda}) \quad \text{in } \Sigma(\lambda) \tag{2.13}$$

and

$$\frac{\partial u}{\partial x_1} < 0 \quad \text{on } \Omega \cap T_{\lambda}. \tag{2.14}$$

(Recall that $x^{\lambda}$ is the reflection of $x$ in $T_{\lambda}$ )

REMARK 2.3

The set of positive $\lambda$ for which (2.13) and (2.14) hold is open .

Now we complete the proof of Theorem 2.2 . By Lemma 2.3 we remark that the set of positive $\lambda$ for which (2.13) and (2.14) hold is non empty . In light of Remark 2.3 these properties, (i.e. (2.13) and (2.14)), hold in a maximal interval $(\mu, R)$ , $R$ denotes the radius of the ball $\Omega$ .

We claim that $\mu = \lambda_1$ . To see this , assume that $\mu > \lambda_1$ . Let $x_0 \in \partial\Sigma(\lambda)\setminus T_{\lambda}$. Then $x_0^{\mu} \in \Omega$ . Since $0 = u(x_0) < u(x_0^{\mu})$ , $u(x) \not\equiv u(x^{\lambda})$ in $\Sigma(\mu)$. Therefore , Lemma 2.4 holds for $\mu = \lambda$ that is $\partial u/\partial x_1 < 0$ on $\Omega \cap T_{\mu}$ and $u(x) < u(x^{\mu})$ in $\Sigma(\mu)$ . This and the continuity of $\partial u/\partial x_1$ imply that there is an $\varepsilon > 0$ such that : $\partial u/\partial x_1 < 0$ on a neighbourhood in the region between $T_{\mu}$ and $T_{\mu-\varepsilon}$ . By compactness there exists a strip $\Omega \cap \{x_1 > \mu-\varepsilon\}$ on which

$$\frac{\partial u}{\partial x_1} < 0 . \qquad (\text{in } \Omega \cap \{x_1 > \mu-\varepsilon\}) \tag{2.15}$$

{Lemma 2.3 can be used to get neighbourhoods on the boundary}.

Then the definition of $\mu$ implies that there is an increasing sequence $(\lambda_j)_{j\in\mathbb{N}} \subseteq (\mu-\varepsilon, \mu)$, with $\lim\limits_{j\to\infty} \lambda_j = \mu$, such that for each $j$ there is a point $x_j \in \Sigma(\lambda_j)$ for which :

$$u(x_j) \geq u(x_j^{\lambda_j}). \tag{2.16}$$

A subsequence which we still call $(x_j)$ converges to a point $x \in \overline{\Sigma(\mu)}$ as $j \to \infty$, then $x_j^{\lambda_j} \to x^{\mu}$ and $u(x) \geq u(x^{\mu})$. Since Lemma 2.4 holds for $\mu = \lambda$ we must have $x \in \partial\overline{\Sigma(\mu)}$. If x is not on $T_\mu$, then $x^\mu$ lies in $\Omega$, hence $0 = u(x) < u(x^\mu)$ a contradiction. Therefore $x \in T_\mu$ and $x^\mu = x$. On the other hand, for $j$ large, the straight line segment joining $x_j^{\lambda_j}$ to $x_j$ is contained in $\Omega$. Therefore, from (2.16) and the Mean Value Theorem it follows that there is a point $y_j$ in this straight line such that :

$$u(x_j^{\lambda_j}) - u(x_j) = \frac{\partial u}{\partial x_1}(y_j) \cdot tx_1 \qquad (t < 0)$$

Since the left hand side is $\leq 0$, this implies

$$\frac{\partial u}{\partial x_1}(y_j) \geq 0.$$

Since $\lim\limits_{j\to\infty} y_j = x$, we get $\frac{\partial u}{\partial x_1}(x) \geq 0$, a contradiction with (2.15). Thus we have proved that $\mu = \lambda_1$ and so far Remark 2.3. This completes the proof of Theorem 2.2 .□

To prove Lemma 2.3 we need a preliminary result.

LEMMA 2.5

INVARIANCE OF THE LAPLACIAN UNDER ORTHOGONAL TRANSFORMATION

If $\underline{y} = A\,\underline{x}$ where $A$ is an orthogonal matrix $[\alpha_{ij}]$ then,

$$\frac{\partial u}{\partial x_i} = \sum_{j=1}^{n} \alpha_{ij} \frac{\partial u}{\partial y_j} \quad , i = 1,2,\ldots,n,$$

and

$$\Delta_x u = \Delta_y u \ .$$

PROOF :

By the chain rule we get

$$\frac{\partial u}{\partial x_i} = \sum_j \frac{\partial u}{\partial y_j} \frac{\partial y_j}{\partial x_i} \quad , i = 1,2,\ldots,n,$$

$$= \sum_j \alpha_{ij} \frac{\partial u}{\partial y_j} \quad , j = 1,2,\ldots,n \ \ldots\ldots(2.17)$$

Differentiating (2.17) with respect to $x_l$ we get :

$$\frac{\partial^2 u}{\partial x_i \partial x_{\ell}} = \sum_j \alpha_{\ell j} \frac{\partial}{\partial y_j} \left[ \sum_k \alpha_{ik} \frac{\partial u}{\partial y_k} \right] \quad \ell = 1,2,\ldots,n,$$

$$= \sum_j \sum_k \alpha_{\ell j}\, \alpha_{ik} \frac{\partial^2 u}{\partial y_j \partial y_k} \ .$$

Thus

$$\Delta_x u = \sum_i \left[ \sum_j \sum_k \alpha_{ij}\, \alpha_{ik} \frac{\partial^2 u}{\partial y_j \partial y_k} \right]$$

$$= \sum_j \sum_k \frac{\partial^2 u}{\partial y_j \partial y_k} \delta_{jk} = \Delta_y u$$

since $\sum \alpha_{ij}\, \alpha_{ik} = \delta_{jk}$ where $\delta_{jk}$ is the Kronecker delta.□

PROOF OF LEMMA 2.3 :

Since $\Omega$ is an open connected subset of $\mathbb{R}^n$ and $u > 0$ in $\Omega$, then $\partial u/\partial \nu \leq 0$ on $\partial\Omega_\varepsilon$. By hypothesis $\nu_1(x_0) > 0$ implies that $\nu_1 > 0$ on $S_\varepsilon$ for small $\varepsilon > 0$. Therefore $\partial u/\partial x_1 \leq 0$ on $S_\varepsilon$.

If the Lemma was not true, there would be a sequence $\{x^j\} \subseteq \Omega_\varepsilon$ such that $x^j \longrightarrow x_0$, with $\partial u(x^j)/\partial x_1 \geq 0$. For $j$ large the interval in the positive $x_1$- direction from $x^j$ intersects $S_\varepsilon$ at a point $z^j$ where $\partial u/\partial x_1 \leq 0$. Thus, since $\lim\limits_{j\to\infty} z^j = x_0$, we conclude that

$$\frac{\partial u}{\partial x_1}(x_0) = 0 \quad \text{and} \quad \frac{\partial^2 u}{\partial x_1 \partial x_1}(x_0) \leq 0. \tag{2.18}$$

case: (i)

Suppose $f(0) \geq 0$ (we mean by $f(0)$ the function $f(u(x_0))$, where $u(x_0) = 0$ ). Then

$$\Delta u + f_1(u) - f_1(0) \leq f_2(0) - f_2(u) \leq 0$$

since $f_2$ is increasing. By the mean value theorem there exists a function $h_1(x)$ [whose sign is undetermined], such that

$$\Delta u + h_1(x) u \leq 0.$$

Applying the boundary point theorem (Theorem 2.5, Chapter (II)) to the function $-u$ we find

$$\frac{\partial u}{\partial \nu}(x_0) < 0, \quad \text{and therefore} \quad \frac{\partial u}{\partial x_1}(x_0) < 0$$

which contradicts (2.18) .

case: (ii)

Suppose $f(0) < 0$. Then at $x_0 \in \partial\Omega$ we find

$$\Delta u = -f(0) > 0 \qquad (\text{at } x_0) \tag{2.19}$$

Since $u > 0$ in $\Omega$, and $u = 0$ on $\partial\Omega$, we always have $\partial u/\partial\nu(x_0) \leq 0$ at $x_0 \in \partial\Omega$. Now if $\partial u/\partial\nu(x_0) < 0$, we are finished.

Suppose $\partial u/\partial\nu(x_0) = 0$. This implies *grad* $u = 0$ at $x_0$. We take a rotation of axes, that is a transformation from $x$ to $y$ given by :

$$y = A\,x$$

where $A$ is an orthogonal matrix.

Choose $A$ so that $y_1$- axis is along $\underline{\nu}$ at $x_0 \in \partial\Omega$, *i.e.*

$$A\,\underline{e}_1 = \underline{\nu}$$

so

$$\nu_i = \sum_j \alpha_{ij}\,\delta_{1j} = \alpha_{i1} \quad ,i = 1,2,\ldots,n.$$

Then $y_i = \Sigma_j\,\alpha_{ij}\,x_j$ and by Lemma 2.5 (setting $j = 1$) we get

$$\frac{\partial u}{\partial x_1} = \sum_i \frac{\partial u}{\partial y_i}\,\alpha_{i1}.$$

Since $u = 0$ on $\partial\Omega$, all tangential derivatives of $u$ are zero on $\partial\Omega$ [*c.f.* argument in Serrin's proof (Theorem 2.1)]. In particular $\partial u/\partial y_j = 0$ , for $j = 2,3,\ldots,n$, and therefore

$$\frac{\partial u}{\partial x_1} = \frac{\partial u}{\partial y_1}\,\alpha_{i1} = \nu_1\,\frac{\partial u}{\partial y_1} = \nu_1\,\frac{\partial u}{\partial \underline{\nu}} \quad (= 0 \text{ at } x_0).$$

Next we have :

$$\frac{\partial^2 u}{\partial x_1 \partial x_1} = \sum_{i,j} \frac{\partial^2 u}{\partial y_i \partial y_j}\,\alpha_{i1}\,\alpha_{j1}\ , \quad \text{as in Lemma 2.5,}$$

$$= \frac{\partial^2 u}{\partial y_1 \partial y_1}\,\alpha_{11}\,\alpha_{11} \quad \text{at } x_0 \in \partial\Omega\ .$$

since all other derivatives of $u$ with respect to $y$ are zero at $x_0$. We get at $x_0$ :

$$\frac{\partial^2 u}{\partial x_1 \partial x_1} = \frac{\partial^2 u}{\partial y_1 \partial y_1}\,\nu_1\,\nu_1.$$

Since the Laplacian is invariant under rotation of axes,

$$\Delta\, u(x_0) = \frac{\partial^2 u}{\partial y_1 \partial y_1} .$$

From (2.19) we have

$$0 < \Delta\, u = \frac{\partial^2 u}{\partial y_1 \partial y_1} \qquad (\text{at } x_0)$$

Thus we conclude

$$\frac{\partial^2 u}{\partial x_1 \partial x_1} > 0 \qquad (\text{at } x_0)$$

which contradicts (2.18). The Lemma is proved. □

<u>PROOF OF LEMMA 2.4</u>

Let $x \in \Sigma'(\lambda)$. Define the function $v(x) = u(x^\lambda)$. Then

$$v(x) \leq u(x) \qquad (v(x) \not\equiv u(x)) \tag{2.20}$$

and v satisfies

$$\Delta v + f(v) = 0. \tag{2.21}$$

Let $w = v - u$. Then

$$w \leq 0 \tag{2.22}$$

and satisfies

$$\Delta\, w + f_1(v) - f_1(u) \geq f_2(u) - f_2(v)$$
$$\geq 0$$

since $f_2$ is an increasing function. By the mean value theorem, there exists a function $c_1(x)$, whose sign is undetermined, such that

$$\Delta\, w + c_1(x)\, w \geq 0 \quad \text{for } x \in \Sigma'(\lambda). \tag{2.23}$$

Applying Theorem 2.5 in Chapter (II) to inequalities (2.22) and (2.23) and recalling that $w = 0$ on $\Omega \cap T_\lambda$ we get :

$$w < 0 \quad \text{in } \Sigma'(\lambda) \quad (\text{this gives } u(x) < u(x^\lambda),\ x \in \Sigma(\lambda))$$

and

$$\frac{\partial w}{\partial x_1} > 0 \quad \text{on } \Omega \cap T_\lambda .$$

But on $T_\lambda$

$$\frac{\partial w}{\partial x_1} = \frac{\partial v}{\partial x_1} - \frac{\partial u}{\partial x_1} > -2\frac{\partial u}{\partial x_1} .$$

Thus , $\partial u/\partial x_1 < 0$ on $\Omega \cap T_\lambda$ and the proof of Lemma 2.4 is complete . □

REMARK 2.3

The function $f(u)$ is of the form

$$f(u) = f_1(u) + f_2(u)$$

where $f_1 \in C^1(\Omega)$ and $f_2$ is *monotonically increasing* if $f$ is *locally Lipschitz continuous* , as stated in Gidas, Ni, Nirenberg [5] . This follows from the following facts.

DEFINITION :

The function $f(u)$ is called *locally Lipschitz* on $\mathbb{R}^n$ if for all $M$ there exists a constant $K = K(M)$ such that

$$|f(u) - f(v)| \leq K|u - v|, \text{if } |u| \leq M \text{ and } |v| \leq M ,$$

where $K$ is called a Lipschitz constant.

LEMMA 2.6

If $u \in C^0(\overline{\Omega})$ then there exists a positive constant $M$ such that :

$$|u(x)| \leq M \quad \text{for each } x \in \overline{\Omega}.$$

LEMMA 2.7

If on a neighbourhood $B_M \subseteq \mathbb{R}^n$

$$|f(u) - f(v)| \leq K|u - v|$$

then

$$f(u) = f_1(u) + f_2(u)$$

where $f_1$ is $C^1$ and $f_2$ is *monotonically increasing* function.

PROOF:

Take $\ell > K$ and let

$$f_2(u) = f(u) + \ell\, u + m. \ (m = \text{an arbitrary constant})$$

Then for $u > v$

$$\begin{aligned} f_2(u) - f_2(v) &= f(u) - f(v) + \ell\,(u - v) \\ &\geq -K\,(u - v) + \ell\,(u - v) \\ &= (\ell - K)(u - v) > 0. \end{aligned}$$

So $f_2$ is monotonically increasing. Also $f(u) = f_2(u) + (-\ell u - m)$ and $f_1(u) = -\ell u - m$ is certainly $C^1$. □

In the following theorem we investigate the symmetry of positive solution of the elliptic equation

$$\Delta u + f(u) = 0$$

in a domain $\Omega$, not necessarily a ball. We require $\Omega$ to be bounded and with smooth boundary $\partial\Omega$.

THEOREM 2.8

Let $u$ satisfy the differential equation

$$\Delta u + f(u) = 0 \quad \text{in } \Omega$$

with the conditions

$$u > 0 \quad \text{in } \Omega \quad \text{and} \quad u = 0 \quad \text{on } \partial\Omega .$$

Let $\lambda \in (\lambda_1, \lambda_0)$. Then

$$\frac{\partial u}{\partial x_1} < 0 \quad \text{and} \quad u(x) < u(x^{\lambda}) \tag{2.24}$$

for $x \in \Sigma(\lambda)$. Moreover, if $\partial u/\partial x_1 = 0$ at some point in $\Omega \cap T_{\lambda_1}$, then $u$ is symmetric relative to the plane $T_{\lambda_1}$ and $\Omega = \Sigma \cup \Sigma' \cup (T_{\lambda_1} \cap \Omega)$ .

<u>NOTE</u>:

The definitions of $\lambda$, $\lambda_0$, $\lambda_1$, $T_\lambda$, $T_{\lambda_1}$, $\Sigma(\lambda)$, $\Sigma'(\lambda)$, $\Sigma(\lambda_1)$ and $\Sigma'(\lambda_1)$ are as before. We define $\Sigma = \Sigma(\lambda_1)$ and $\Sigma' = \Sigma'(\lambda_1)$ .

<u>PROOF OF THEOREM 2.8</u>

Take $x_0 \in \partial\Omega$, such that $\nu_1(x_0) > 0$. Then by Lemmas 2.3 and 2.4 we have, for sufficiently small $\lambda_0 - \lambda > 0$,

$$\frac{\partial u}{\partial x_1} < 0 \quad \text{and} \quad u(x) < u(x^{\lambda}) \quad \forall\, x \in \Sigma(\lambda). \tag{2.25}$$

Decrease $\lambda$ until a critical value $\mu \geq \lambda_1$ is reached, beyond which (2.25) no longer holds . Then (2.25) holds for $\lambda > \mu$, while for $\lambda = \mu$ , by continuity, the following happens :

$$\frac{\partial u}{\partial x_1} \leq 0 \quad \text{and} \quad u(x) \leq u(x^{\mu}) \quad \text{for } x \in \Sigma(\mu) \tag{2.26}$$

The same argument as in the proof of Theorem 2.2 applies to show that $\mu = \lambda_1$ .

Now, since $\mu = \lambda_1$, it follows that (2.24) holds for $\lambda > \lambda_1$ . By continuity,

$$\frac{\partial u}{\partial x_1}(x) \leq 0 \quad \text{and} \quad u(x) \leq u(x^{\lambda_1}) \quad \text{in } \Sigma(\lambda_1).$$

Next suppose that there is a point $x \in \Omega \cap T_{\lambda_1}$ at which $\partial u/\partial x_1 = 0$. Then Lemma 2.4 implies that

$$u(x) \equiv u(x^{\lambda_1}) \quad \text{in } \Sigma(\lambda_1).$$

Therefore $u$ is symmetric in $T_{\lambda_1}$. Since $u > 0$ in $\Sigma(\lambda_1)$ and $u = 0$ on $\partial\Omega$, we conclude that

$$\Omega = \Sigma \cup \Sigma' \cup (T_{\lambda_1} \cap \Omega).$$

This completes the proof of Theorem 2.8. □

REMARK 2.5

A positive solution of $\Delta u + f(u) = 0$, $u = 0$ on $\partial\Omega$, satisfies $grad\ u \neq 0$ on the maximal cap $\Sigma(\lambda_1)$.

SECTION 3

## MORE GENERAL RESULTS

We show how Theorem 2.2 of Section 2 can be proved as a consequence of Theorem 2.8 (Section 2) .

PROOF OF THEOREM 2.2 USING THEOREM 2.8 :

For an arbitrarily chosen $x_1$- axis, we apply Theorem 2.8 to the function $u$ on the positive side of $x_1$- axis. Then we see that :

$$\frac{\partial u}{\partial x_1} < 0 \qquad (\text{for } x_1 > 0).$$

Similarly, applying Theorem 2.8 to $u$ in $x_1 < 0$ gives :

$$\frac{\partial u}{\partial x_1} > 0 \qquad (\text{for } x_1 < 0).$$

Hence , $\partial u/\partial x_1 = 0$ on $x_1 = 0$ . By the last assertion of Theorem 2.8 we infer that $u$ is symmetric in $x_1$ . Since the direction of $x_1$- axis is arbitrarily chosen, it follows that $u$ is radially symmetric and $\partial u/\partial r < 0$ for $0 < r < R$ . □

Theorem 2.2 suggests the following theorem. We shall use Theorem 2.8 in its proof.

THEOREM 3.1 (Gidas, Ni, Nirenberg [5])

Suppose that $u$ satisfies the equation :

$$\Delta u + f(u) = 0$$

in a ring-shaped domain $R' < |x| \leq R$ , with

$$u > 0 \qquad \text{in } R' < |x| \leq R\ ,$$

$$u = 0 \qquad \text{on } |x| = R\ ,$$

$$u \in C^2(R' < |x| \leq R)\ .$$

Then

$$\frac{\partial u}{\partial r} < 0 \quad \text{for} \quad \frac{R' + R}{2} \le |x| < R.$$

This means that $u$ has no *critical points* in the larger half of the ring .

PROOF:

Take the direction $\gamma$ , arbitrarily chosen, as positive $x_1$- axis. Let $\Sigma_\gamma$ denote the maximal open cap corresponding to $\gamma$ , and $\Sigma'_\gamma$ the reflected cap of $\Sigma_\gamma$ (see figure 3.1). It follows from Theorem 2.8 that

$$\gamma \,.\, \mathit{grad}\ u < 0 \quad \text{in } \Sigma_\gamma \,.$$

Since $\gamma$ is arbitrarily chosen , then the union of the maximal caps is the region $(R' + R)/2 < |x| < R$ .

Suppose that there is a point $y$ with $|y| = (R' + R)/2$, at which $\partial u/\partial r = 0$ . Then with $\gamma = y/|y|$ , the last assertion of Theorem 2.8 implies that

$$\Omega = \Sigma_\gamma \cup \Sigma'_\gamma$$

which is impossible . □

REMARK 3.1 :

If in addition to the hypotheses of the theorem above we assume that $u = 0$ on $|x| = R'$ and $u \in C^2(R' \le |x| \le R)$, then one might think that $u$ is radially symmetric . Using an example by Schaeffer, Gidas [4] shows that this is not true in general.

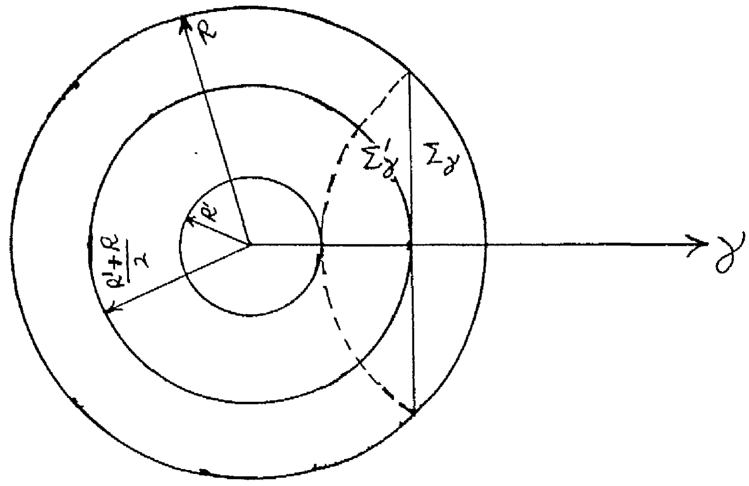


FIGURE [3.1]

COROLLARY 3.2 (Gidas [4])

Let $\Omega$ be a convex domain in $R^n$. If a function $u$ satisfies the hypotheses in Theorem 2.2 of Section 2, then there exists a neighbourhood of $\partial\Omega$ in $\Omega$ where $u(x)$ cannot have critical points .

EXAMPLE 3.1

We take an ellipse as an example of a convex domain. We find that the critical points of u ( if any ) lie in the shaded region in Figure (3.2) (the origin alone in this example).

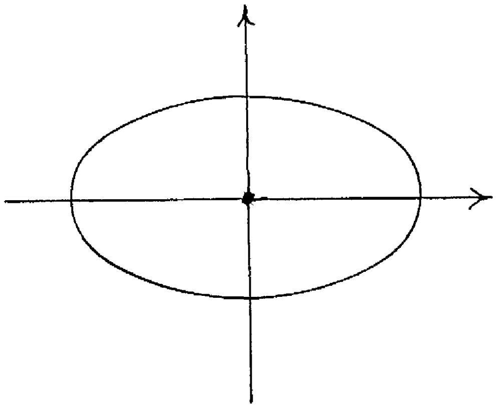

FIGURE [3.2]

<u>PROOF OF COROLLARY 3.2</u> :

Applying Theorem 2.8 implies that $u$ has no critical point in any maximal cap . The union of the maximal caps covers all of $\Omega$ except for a small region about the origin , see Example 3.1 above . □

*Chapter* (IV)

# SYMMETRY PROPERTIES OF SOLUTIONS OF SYSTEMS OF ELLIPTIC EQUATIONS

CHAPTER (IV)

# SYMMETRY PROPERTIES OF SOLUTIONS OF SYSTEMS OF ELLIPTIC EQUATIONS

## SECTION 1

### INTRODUCTION

We shall see in this chapter how the previous results can be extended to certain systems. This was done by Troy [23].

We shall be concerned in this chapter with solutions of systems of the form

$$\Delta u_i + f_i(u_1, u_2, \ldots, u_m) = 0 \quad \text{in } \Omega, \; i = 1,2,\ldots,m \qquad (1.1)$$

where $\Omega$ is a domain in $\mathbb{R}^n$, with the condition

$$u_i > 0 \quad \text{in } \Omega \quad \text{and} \quad u_i = 0 \quad \text{on } \partial\Omega \quad \forall\, i. \qquad (1.2)$$

The functions $f_i$ are assumed to be $C^1$ and satisfy the condition

$$\frac{\partial f_i}{\partial u_j} \geq 0 \qquad \text{for } i \neq j, \; 1 \leq i,j \leq m. \qquad (1.3)$$

We wish to determine a class of domains $\Omega$ for which the solution of the problem (1.1), (1.2) is symmetric about a point in $\Omega$. In particular, we shall show that the solution is radially symmetric in case $\Omega$ is a ball. However, if $\Omega$ is not given but we add the condition $\partial u_i/\partial\nu = C_i$ on $\partial\Omega$, $i = 1,2,\ldots,m$, where $C_i$ is a constant and $\nu$ is the outer normal to $\partial\Omega$, then $\Omega$ must be a ball in $\mathbb{R}^n$ and the solution is radially symmetric.

## MAXIMUM PRINCIPLE AND THE RELATED BOUNDARY POINT THEOREMS

Here we discuss the extension of Theorems 2.5 and 3.1 of chapter(II) from the scalar versions to systems.

We define the operators :

$$L_i \equiv \sum_{j,k} a^i_{jk}(x) \frac{\partial^2}{\partial x_j \partial x_k} + \sum_j b^i_j(x) \frac{\partial}{\partial x_j} \qquad (1.4)$$

$i = 1,2,\ldots,m$ and $1 \le j,\ k \le n$, where each $L_i$ is uniformly elliptic .

The following theorem is an extension of Theorem 2.5 of Chapter (II) to systems.

THEOREM 1.1

Let $u_i(x) \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfy the system of differential inequalties

$$L_i[u_i] + \sum_j h_{ij}(x)u_j \ge 0 \qquad \ldots\ldots.(1.5)$$

in a domain $\Omega \subseteq \mathbb{R}^n$ with $u_i \le 0$ in $\Omega$ for all $i = 1,2,\ldots,m$. Suppose that the coefficients $a^i_{jk}$, $b^i_j$, $h_{ij}$ are *uniformly bounded* in $\bar{\Omega}$, and that for $x \in \Omega$,

$$h_{ij}(x) \ge 0, \quad i \ne j,\ 1 \le i,\ j \le m. \quad \ldots\ldots(1.6)$$

(i) If for some $k$, $u_k$ vanishes at an interior point of $\Omega$, then

$$u_k \equiv 0 \quad \text{in } \Omega.$$

(ii) If $\partial\Omega$ satisfies an interior sphere condition at a point $Q \in \partial\Omega$ with $u_k(Q) = 0$ , for some $k$, then we have

$$\frac{\partial u_k(Q)}{\partial \nu} > 0$$

unless $u_k \equiv 0$ in $\Omega$, where $\nu$ denotes the outward normal to $\partial\Omega$ at $Q$.

PROOF :

From (1.5) we have

$$L_k[u_k] + h_{kk}\, u_k = -\sum_{j \neq k} h_{kj}\, u_j \geq 0,$$

$1 \leq k,j \leq m$, since $u_j \leq 0$ and $h_{kj} \geq 0$ for $k \neq j$. Therefore Theorem 2.5 of Chapter (II) applies to $u_k$ , and the proof is complete .□

REMARK 1.1 :

This is a shorter proof than the one given by Troy[23].

THEOREM 1.2

Let $\Omega \subseteq \mathbb{R}^n$ be a domain with $C^2$ boundary and let $T$ be a plane containing the normal to $\partial\Omega$ at a point $Q \in \partial\Omega$. Let $\Omega^*$ denote the portion of $\Omega$ lying on some particular side of $T$.

Let $u_i \in C^2(\bar{\Omega}^*)$, $i = 1,2,\ldots,m$, satisfy the system of differential inequalities (1.5). Assume that $u_i \leq 0$ in $\Omega^*$ for all $i$ and that there is $j$ such that $u_j < 0$ in $\Omega^*$ with $u_j(Q) = 0$. Then either :

$$\frac{\partial u_j(Q)}{\partial \nu} > 0 \quad \text{or} \quad \frac{\partial^2 u_j(Q)}{\partial \nu^2} < 0, \qquad \ldots\ldots(1.7)$$

unless $u_j \equiv 0$.

Here $\nu$ denotes the outward directional normal at $Q \in \partial\Omega$, also the coefficients of $L_i$ with $h_{ij}(x)$ in (1.5) are assumed to be uniformly bounded.

PROOF :

We proceed as in Serrin's proof of the Hopf boundary point theorem at a corner (Theorem 3.1, Chapter (II)).

We define the region $H$ by :

$$H = B_1 \cap B_2 \cap \Omega^*,$$

where $B_1 \subset \bar{\Omega}$ is an open ball with radius $r_1$ and internally tangent to $\bar{\Omega}$ at $Q$, $B_2$ is an open ball centred at $Q$ and with radius $r_2$. We take $r_2 \leq \frac{1}{2} r_1$.

Define

$$v_i = e^{-\alpha x_1} u_i \quad , \quad 1 \leq i \leq m.$$

Then $v_i$ satisfies

$$0 \leq L_i[u_i] + \sum_j^m h_{ij} u_j = L_i^1 [v_i] + (\alpha^2 a_{11} + \alpha b_1 + h_{ii}) v_i + \sum_j^m h_{ij} v_j , \qquad \dots\dots\dots(1.8)$$

where $L_i^1$ is an elliptic operator containing no zero – order terms . For large $\alpha$ and all $i$ ,

$$\alpha^2 a_{11} + \alpha b_1 + h_{ii} > 0$$

since $a_{11}$ , $b_1$ and $h_{ii}$ are bounded , and $a_{11}$ is positive . By (1.6) , it follows from (1.8)

$$L_i^1 [v_i] \geq 0 \quad \text{in } \Omega \quad , \quad i = 1,\dots,m$$

and $v_i \leq 0$ in $\Omega$ .

Applying Theorem 3.1 of Chapter (II) to $v_j$ for some $j$, the result follows .□

SECTION 2

## MAIN RESULTS

The following is a generalization to systems of the result in Section 2 of Chapter (III) . Our work is based on Troy [23].

THEOREM 2.1

Let $\Omega \subset \mathbb{R}^n$ be a ball of radius $R$. Let $u_i \in C^2(\bar{\Omega})$ satisfy the differential equations:

$$\Delta u_i + f_i(u_1, u_2, \ldots, u_m) = 0 \quad , i = 1,2,\ldots,m, \quad (2.1)$$

where $f_i$ is $C^1$ and satisfies the condition

$$\frac{\partial f_i}{\partial u_j} \geq 0 \ , \quad i \neq j, \quad 1 \leq i,j \leq m. \quad (2.2)$$

Suppose that

$$u_i > 0 \quad \text{in } \Omega \quad \text{and} \quad u_i \equiv 0 \quad \text{on } \partial\Omega \text{ for all } i. \quad (2.3)$$

Then for each $i$, $u_i$ is radially symmetric and $\partial u_i/\partial r < 0$ for $0 < r < R$.

PROOF :

We require three technical lemmas which are extensions of Lemmas 2.3, 2.4 and Theorem 2.8 of Chapter (III) for the scalar problem to systems.

We pick an arbitrarily chosen direction which we may assume to be the $x_1$- axis and move a hyperplane $T_\lambda$ from infinity towards $\Omega$ retaining its normal in the positive $x_1$- direction. In our construction of the caps, let $\gamma$ be the unit vector $(1,0,\ldots,0)$ and recall from Chapter (III) the definitions of $\lambda_0$, $\lambda_1$, $\lambda_2$, $T_\lambda$, $\Sigma(\lambda)$, $\Sigma'(\lambda)$, $\Sigma(\lambda_1)$ and $\Sigma'(\lambda_1)$ for $\lambda \in [\lambda_2, \lambda_0]$.

Now for a given $\varepsilon > 0$ and $x_0 \in \partial\Omega$ we define :

$$\Omega_\varepsilon = \Omega \cap \{|x - x_0| < \varepsilon\} \quad , x \in R^n,$$

and

$$S_\varepsilon = \partial\Omega \cap \{|x - x_0| < \varepsilon\} \quad , x \in R^n.$$

LEMMA 2.2

Let $x_0 \in \partial\Omega$ such that $\nu_1(x_0) > 0$ . Choose $\varepsilon > 0$ sufficiently small so that $\nu_1(x) > 0$ , for each $x \in S_\varepsilon$. Assume that for each $i$, $1 \leq i \leq m$ , $u_i \in C^2(\overline{\Omega}_\varepsilon)$, $u_i > 0$ in $\Omega_\varepsilon$ and $u_i = 0$ on $\partial\Omega_\varepsilon$. Then there exists $\delta > 0$ (independent of $i$) such that

$$\frac{\partial u_i}{\partial x_1} < 0 \quad \text{in } \Omega_\delta.$$

LEMMA 2.3

For $\lambda \in [\lambda_1, \lambda_0)$ and some $i, (1 \leq i \leq m)$, assume that the function $u_i$ satisfies :

$$\frac{\partial u_i}{\partial x_1} \leq 0 \quad , x \in \Sigma(\lambda), \tag{2.4}$$

and

$$u_i(x) \leq u_i(x^\lambda) \quad \text{but} \quad u_i(x) \not\equiv u_i(x^\lambda) \quad \text{in } \Sigma(\lambda). \tag{2.5}$$

Then

$$u_i(x) < u_i(x^\lambda) \quad \text{in } \Sigma(\lambda),$$

and

$$\frac{\partial u_i}{\partial x_1} < 0 \quad \text{on } \Omega \cap T_\lambda.$$

LEMMA 2.4

Let $u = (u_1, u_2, \ldots, u_m)$ satisfy the differential equations

$$\Delta u_i + f_i(u_1, u_2, \ldots, u_m) = 0 \quad \text{in } \Omega.$$

Suppose that

$$u_i > 0 \quad \text{in } \Omega \quad \text{and} \quad u_i \equiv 0 \quad \text{on } \partial\Omega.$$

Then for $\lambda \in (\lambda_1, \lambda_0)$,

$$\frac{\partial u_i}{\partial x_1} < 0 \quad \text{and} \quad u_i(x) < u_i(x^\lambda) \quad \text{for } x \in \Sigma(\lambda). \qquad \ldots\ldots(2.6)$$

If for some i, $\partial u_i/\partial x_1 = 0$ at some point in $\Omega \cap T_{\lambda_1}$, then $u_i$ is symmetric in the plane $T_{\lambda_1}$ and $\Omega = \Sigma(\lambda_1) \cup \Sigma'(\lambda_1) \cup (T_{\lambda_1} \cap \Omega)$.

We now complete the proof of Theorem 2.1 assuming Lemmas 2.2, 2.3 and 2.4. We may assume without loss of generality that the ball $\Omega$ is centred at $x = 0$ in $\mathbb{R}^n$.

By Lemma 2.4 we have, for any choice of the $x_1$- axis and each $i$,

$$\frac{\partial u_i}{\partial x_1} < 0 \qquad \text{for} \quad x_1 > 0.$$

Also, by the same lemma it is easy to see that

$$\frac{\partial u_i}{\partial x_1} > 0 \qquad \text{for } x_1 < 0.$$

Furthermore we get

$$\frac{\partial u_i}{\partial x_1} = 0 \qquad \text{at } x_1 = 0,$$

since $u_i \in C^2(\bar{\Omega})$. Therefore Lemma 2.4 implies that $u_i$ is symmetric in the $x_1$- axis. Since the $x_1$- axis is arbitrarily chosen, it follows that $u_i$ is radially symmetric for each $i$ and $\partial u_i/\partial x_1 < 0$ for $0 < r < R$ .□

We now turn to the proofs of Lemmas 2.2, 2.3 and 2.4 .

PROOF OF LEMMA 2.2

Assume that $i$ has been chosen and is held fixed. Since $\Omega$ is an open connected subset of $\mathbb{R}^n$ and $u_i > 0$ in $\Omega_\varepsilon$, then

$$\frac{\partial u_i}{\partial \nu} \le 0 \qquad \text{on} \quad \partial\Omega_\varepsilon .$$

Also, $\nu_1(x) > 0$ implies that $\nu_1 > 0$ on $S_\varepsilon$ for small $\varepsilon > 0$. Therefore

$$\frac{\partial u_i}{\partial x_1} \le 0 \qquad \text{on} \quad S_\varepsilon .$$

If the Lemma were false, then there would be a sequence $\{x^j\}_{j\in\mathbb{N}} \subseteq \Omega_\varepsilon$ such that $x^j \longrightarrow x_0$ as $j \longrightarrow \infty$ and $\partial u_i(x^j)/\partial x_1 \ge 0$.

On the other hand, for each $j$, the interval from $x^j$ in the positive $x_1$- direction intersects $S_\varepsilon$ at a point $z^j$ such that $z^j \longrightarrow x_0$ as $j \longrightarrow \infty$.

Since $\partial u_i(z^j)/\partial x_1 \le 0$, we conclude that

$$\frac{\partial u_i(x_0)}{\partial x_1} = 0 \quad \text{and} \quad \frac{\partial^2 u_i}{\partial x_1 \partial x_1}(x_0) \le 0 . \tag{2.7}$$

case(i):

Assume that $f_i(\bar{0}) \ge 0$ $[f_i(\bar{0}) = f_i(u_1(x_0),\ldots,u_m(x_0))$ with $u_i(x_0) = 0]$. Then in $\Omega_\varepsilon$, $u_i$ satisfies

$$\Delta\, u_i + f_i(u_1,\ldots,u_m) - f_i(\bar{0}) \le 0 .$$

hence, by the mean value theorem there exist functions $\eta_1(x),\ldots,\eta_n(x)$ defined for $x \in \Omega$ and with values in $\mathbb{R}^n$ such that

$$\Delta u_i + \sum_j \frac{\partial f_i}{\partial u_j} [\eta_j(x)]\ u_j \le 0. \tag{2.8}$$

Applying Theorem 1.1 (Section 1) to $-u_i$ implies

$$\frac{\partial u_i(x_0)}{\partial \nu} < 0, \quad \text{hence} \quad \frac{\partial u_i(x_0)}{\partial x_1} < 0$$

contradicting (2.7).

case(ii):

Assume that $f_i(\overline{0}) < 0$. Then at $x_0$ we get

$$\Delta u_i = - f_i(\overline{0}) > 0.$$

Applying Lemma 2.3 , Chapter (III) , the result follows.□

PROOF OF LEMMA 2.3

Let $\lambda \in [\lambda_1, \lambda_0)$. For each $i = 1,\ldots,m$, we define the function

$$v_i(x) = u_i(x^\lambda) \qquad \text{for } x \in \Sigma'(\lambda).$$

Then $v_i(x)$ satisfies

$$\Delta v_i + f_i(v_1,\ldots,v_m) = 0 \qquad \text{in } \Sigma'(\lambda).$$

Define the function

$$w_i(x) = v_i(x) - u_i(x) \qquad \text{in } \Sigma'(\lambda),\ 1 \le i \le n.$$

Applying Lemma 2.4 , Chapter(III), the result follows.□

PROOF OF LEMMA 2.4

Since $\nu_1(x) > 0$ for $x \in \partial\Omega \cap \partial(\Sigma(\lambda))$, then by Lemma 2.2 we have

$$\frac{\partial u_i}{\partial x_1} < 0 \quad \text{and} \quad u_i(x) < u_i(x^\lambda) \quad \text{at all } x \in \Sigma(\lambda), \tag{2.9}$$

for $\lambda_0 - \lambda > 0$ sufficiently small.

Decrease $\lambda$ below $\lambda_0$ until a critical value $\mu \geq \lambda_1$ is reached beyond which (2.9) no longer holds for some $u_j$. Then (2.9) holds for $u_j$ for $\lambda > \mu$, while for $\lambda = \mu$

$$\frac{\partial u_j}{\partial x_1} \leq 0 \quad \text{and} \quad u_j(x) \leq u_j(x^j) \quad \text{for } x \in \Sigma(\lambda).$$

Theorem 2.8 of Chapter (III) applies to $u_j$ and the result follows.□

THEOREM 2.5

Let $\Omega \subseteq \mathbb{R}$ be a domain whose boundary is of class $C^2$. Suppose that $u_i$, $i = 1,2,\ldots,m$, satisfies the system of differential equations :

$$\Delta\, u_i + f_i(u_1, u_2, \ldots, u_m) = 0 \quad \text{in } \Omega \tag{2.10}$$

with the condition

$$u_i \equiv 0 \quad \text{on } \partial\Omega \text{ at all } 1 \leq i \leq m, \tag{2.11}$$

where $f_i$ is assumed to be of class $C^1$ and satisfy the condition

$$\frac{\partial f_i}{\partial u_j} \geq 0\ , \quad i \neq j,\ 1 \leq i,j \leq m \tag{2.12}$$

Further we assume that

$$\frac{\partial u_i}{\partial \nu} = C_i \quad \text{on } \partial\Omega \tag{2.13}$$

where $C_i$ is a constant and $\nu$ denotes the outer normal to $\partial\Omega$. Then $\Omega$ must be a ball.

PROOF :

We use the same device of moving parallel planes as in Section 1, Chapter (III), and adopt the same notations.

Define the function $v_i$ by

$$v_i(x) = u_i(x^{\lambda_1}) \quad \text{for } x \in \Sigma'(\lambda_1),\ 1 \leq i \leq m \quad \ldots(2.14)$$

where $x^{\lambda_1}$ is the reflected value of $x$ in the plane $T_{\lambda_1}$. For each $i$ the function $v_i$ satisfies the differential equation

$$\Delta v_i + f_i(v_1, v_2, \ldots, v_m) = 0 \quad \text{in } \Omega \quad \ldots\ldots(2.15)$$

with the boundary conditions

$$v_i = u_i \qquad \text{On } \partial\Sigma'(\lambda_1) \cap T_{\lambda_1},$$

$$v_i = 0 \quad \text{and} \quad \frac{\partial v_i}{\partial \nu} = C_i \quad \text{on } \partial\Sigma'(\lambda_1) \cap (T_{\lambda_1})^C,$$

where the constant $C_i$ being the same as in (2.13) and $(T_{\lambda_1})^C$ denotes the complement of $(T_{\lambda_1})$. Further we define the functions

$$w_i = v_i - u_i \quad \text{in } \Sigma'(\lambda_1). \quad \ldots\ldots(2.16)$$

Then, by the mean value theorem, there exist functions $\xi_1(x), \ldots, \xi_n(x)$ defined for $x \in \Omega$ and with values in $\mathbb{R}^n$ such that

$$\Delta w_i + \sum_j \frac{\partial f_i}{\partial w_j}(\xi_i(x))\, w_j = 0.$$

Therefore ,

$$\Delta w_i + \frac{\partial f_i}{\partial w_i}(\xi_i(x))\, w_i = -\sum_j \frac{\partial f_i}{\partial w_j}(\xi_j(x)) w_j.$$

Since $\partial f_i/\partial w_j \geq 0$ for $i \neq j$ and by virtue of Lemma 2.3 , (2.16) implies $w_j \geq 0$ in $\Sigma'(\lambda_1)$. It follows that

$$\Delta w_i + h_{ii}\, w_i \leq 0 \quad \text{in } \Sigma'(\lambda_1) \quad \ldots(2.17)$$

$$w_i = 0 \quad \text{on } \partial\Sigma'(\lambda_1) \cap T_{\lambda_1} \quad \ldots(2.18)$$

$$w_i \geq 0 \quad \text{on } \partial\Sigma'(\lambda_1) \cap (T_{\lambda_1})^C \ , \ldots(2.19)$$

where $h_{ij} = \partial f_i(\xi_j(x))/\partial w_j$.

Applying Theorem 2.3 of Chapter (II) (*as in* Theorem 2.1 *of Chapter*(III) ) the result follows.□

REMARK 2.1

The proof above is similar to Troy's one .

EXAMPLE 2.1 :

STEADY STATE SOLUTIONS OF SOME REACTION-DIFFUSION EQUATIONS

The following model represents a system which satisfies the essential condition (1.3) of Section 1.

The Belousov - Zhabotinskii reaction in a capillary tube leads to a system of equations, given by Field and Noyes [2],

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x \partial x} + F(u, v), \qquad \text{.......(2.20)}$$

$$\frac{\partial v}{\partial t} = G(u, v), \qquad \text{.......(2.21)}$$

where

$$F(u, v) = s\,(v - u\,v + u - q\,u^2) \quad ,$$

$$G(u, v) = (1/s)(-\,v - u\,v + f\,u_0) \quad ,$$

$s$ is a given constant ($s = 77.27$ in the application), q is a small constant ($q = 8.375 \times 10^{-6}$) and $f$ is a numerical parameter taken in the range $(1+\sqrt{2}, \infty)$ (Field and Troy[3]), $(u_0, v_0)$ are the unique positive constant solutions of (2.20) and (2.21) given by

$$u_0 = \{1 - f - q + [(1 - f - q)^2 + 4\,q\,(1 + f)]^{\frac{1}{2}}\}/2\,q \qquad \text{......(2.22)}$$

and

$$v_0 = f\, u_0/(1 + u_0). \qquad \ldots\ldots(2.23)$$

Here we show that $u_0 > 1$. Since $(1 - f - q) < 0$ for $f > 1$, then

$$u_0 = \frac{1}{2q}\left\{(1 - f - q) + |1 - f - q|\left[1 + \frac{4q(1 + f)}{(1 - f - q)^2}\right]^{\frac{1}{2}}\right\}$$

$$\simeq \frac{1}{2q}\left\{(1 - f - q) + |1 - f - q|\left[1 + \frac{2q(1 + f)}{(1 - f - q)^2}\right]\right\}$$

(using $(1 + x)^{\frac{1}{2}} \simeq 1 + \frac{1}{2}x$ for small x)

$$= \frac{1}{2q}\left\{(1 - f - q) + |1 - f - q| + \frac{2q(1 + f)}{|1 - f - q|}\right\}$$

$$= \frac{1 + f}{|1 - f - q|}, \text{ since } |1 - f - q| = -(1 - f - q)$$

$$\simeq \frac{1 + f}{f - 1} = 1 + 2/(f - 1)$$

so $u_0 > 1$ for $f > 1$ (and q small). Note that this gives

$$u_0 \simeq 2.414 \quad \text{for} \quad f = 1 + \sqrt{2}\ .$$

REMARK 2.2

One can, of course, calculate $u_0$ using a calculator for the given values of $f$ and q. This gives the answer very close to that above (= 2.41433).

Now, It is reasonable to consider diffusion in v also. Then (2.21) gives

$$\frac{\partial v}{\partial t} = \Delta\, v + G(u,\, v).$$

Then steady state solutions satisfy the second order elliptic system:

$$\Delta u + F(u, v) = 0, \qquad ......(2.24)$$

$$\Delta v + G(u, v) = 0. \qquad ......(2.25)$$

By changing variables to

$$u_1 = u - u_0, \quad v_1 = v_0 - v,$$

one gets:

$$\Delta u_1 + F(u_1 + u_0, v_0 - v_1) = 0,$$

$$-\Delta v_1 + G(u_1 + u_0, v_0 - v_1) = 0,$$

or

$$\Delta u_1 + F_1(u_1, v_1) = 0, \qquad ......(2.26)$$

$$\Delta v_1 - G_1(u_1, v_1) = 0, \qquad ......(2.27)$$

where

$$F_1 = s[v_0 - v_1 - (u_1 v_0 - u_1 v_1 + u_0 v_0 - u_0 v_1) + (u_1 + u_0)$$

$$- q(u_1^2 + 2 u_1 u_0 + u_0^2)], \qquad ......(2.28)$$

$$G_1 = 1/s\,[-(v_0 - v_1) - (u_1 v_0 - u_1 v_1 + u_0 v_0 - u_0 v_1)].$$

$$......(2.29)$$

Note that for physical reasons, (Troy [23]), $(u, v)$ are constrained to satisfy the inequalities

$$u_0 \le u \le 1/q, \quad 0 \le v \le v_0, \qquad ......(2.30)$$

so $(u_1, v_1)$ satisfy the inequalities

$$u_0 \le u_1 + u_0 \le 1/q, \quad 0 \le v_1 \le v_0. \qquad ......(2.31)$$

Differentiating $F_1$ and $G_1$ with respect to $v_1$ and $u_1$, respectiv ly, we get

$$\frac{\partial F_1}{\partial v_1} = s\,[(u_1 + u_0) - 1],$$

$$\frac{\partial(-G_1)}{\partial u_1} = 1/s\,(v_0 - v_1).$$

From (2.22),(2.23) and (2.31) we have :

$$\frac{\partial F_1}{\partial v_1} > 0 \quad \text{and} \quad \frac{\partial(-G_1)}{\partial u_1} > 0.$$

Therefore, we conclude that Theorems 2.1 and 2.5 apply to equations (2.26),(2.27) and hence to the Field - Noyes model (2.24),(2.25).□

REMARK 2.3

Example 2.1 is a complete version of the one given by Troy [23]

NOTE :

A thorough treatment of system (2.20), (2.21) can be found in Field and Troy [3].

*Chapter* (V)

# THE P-FUNCTION FOR SOLUTIONS OF $\Delta u + f(u) = 0$

CHAPTER (V)

## THE P - FUNCTION FOR SOLUTIONS OF

## $\Delta u + f(u) = 0$

### INTRODUCTION

The elliptic partial differential equation $\Delta u + f(u) = 0$ has been of much interest because of its many applications. The maximum principle is an excellent tool for the study of properties of its solutions .

The papers by Stakgold and Payne [22], Payne, Sperb and Stakgold [12], Schaefer and Sperb [16], Payne [9] and the recent book by Sperb [21] show applications of different kinds of maximum principles to solutions of $\Delta u + f(u) = 0$.

In Chapter(III) we have seen how maximum principles can be used to show some symmetry properties of positive solutions of $\Delta u + f(u) = 0$. In the present chapter we follow Sperb [21] and we study the function P defined by

$$P(x) = g(u)|grad\ u|^2 + h(u),$$

where $u(x)$ is a solution of $\Delta u + f(u) = 0$ in $\Omega$.

We shall show that P satisfies a maximum principle if the functions $g(u)$ and $h(u)$ are chosen appropriately. This will lead to a derivation of useful bounds for all kinds of quantities that are of interest in problems governed by this equation as will be illustrated by some examples.

## SECTION 1

### THE ONE - DIMENSIONAL PROBLEM

It is convenient to start with the simplest cases and then proceed to more complicated situations.

we consider the differential equation

$$u'' + f(u) = 0 \quad \text{in } (a, b) \tag{1.1}$$

where $f > 0$ and $u$ is a function of one variable $x \in (a, b)$ and the primes denote differentiation with respect to x. For practical reasons we introduce a numerical parameter $\lambda$ which we may assume to be positive, since we are iterested in positive solutions, and take $\lambda\ f(u)$ in the place of $f(u)$ in (1.1). Also for convenience we assume that the interval under consideration is finite and take the interval (0, 1).

Adding the boundary conditions $u(0) = u(1) = 0$ to (1.1) brings us to the problem :

$$u'' + \lambda\ f(u) = 0\ , \quad u(0) = u(1) = 0. \tag{1.2}$$

REMARKS 1.1 :

(I) The nonlinear problem (1.2) arises as one of the physical problems involving the steady state temperature distribution in a material bounded by two finite parallel planes which lead to the problem of determining those positive numbers $\lambda$ for which (1.2) has a positive solution $u(x)$ in the interval (0, 1).

(II) All nonzero solutions of (1.2) for $\lambda > 0$ are strictly positive and have exactly one maximum on (0, 1) {Laetsch [6]}.

Now we let $F(u) = \int_0^u f(s)\, ds$ and multiply (1.2) by $u'$:

$$u'' u' + \lambda f(u) u' = 0, \tag{1.3}$$

which, on integration, gives

$$\frac{1}{2} (u')^2 + \lambda F(u) = \text{constant.} \tag{1.4}$$

Therefore, in the case of (1.1), the function

$$P = (u')^2 + 2 F(u) \tag{1.5}$$

is just a constant.

From (1.3) one can derive an implicit representation of the solution as follows :

Let $x_0 \in (0, 1)$ be the point at which some solution of (1.2) assumes its maximum $u_M = u(x_0)$, then $u'(x) \geq 0$ on $[0, x_0]$ and $u'(x) \leq 0$ on $[x_0, 1]$. From (1.4) we get

$$\frac{1}{2} (u')^2 + \lambda F(u) = \lambda F(u_M), \tag{1.6}$$

and integration gives :

$$\int_0^u [F(u_M) - F(s)]^{-\frac{1}{2}} ds = x_0 (2\lambda)^{\frac{1}{2}}, \quad x \in [0, x_0], \tag{1.7$_a$}$$

$$\int_0^u [F(u_M) - F(s)]^{-\frac{1}{2}} ds = (1 - x_0)(2\lambda)^{\frac{1}{2}}, \quad x \in [x_0, 1]. \tag{1.7$_b$}$$

Setting $x = x_0$ and $u(x) = u_M$, we see that $x_0 = \frac{1}{2}$ and $u(x) = u(1 - x)$; that is any solution of (1.2) is symmetric about $x = \frac{1}{2}$ .{This was shown by Laetsch [6]}

Therefore, equations (1.7) may be used to construct the solutions of (1.2). Similarly, different boundary conditions for

$u(x)$ can be treated provided that condition (1.7) is modified (see Sperb [21]).

For equations more general than (1.1) we may take, for example, the equation

$$h(u'^2)\, u'' + g(u) = 0, \tag{1.8}$$

where $h$ is a function of $(u'^2)$. Introducing $\zeta = u'^2$ and using the fact that

$$\frac{d\zeta}{dx} = \frac{d\zeta}{du'}\,\frac{du'}{dx} = 2\,\frac{du}{dx}\,\frac{d^2u}{dx^2},$$

we find that

$$\frac{1}{2}\,H(u'^2) + G(u) = \text{constant}, \tag{1.9}$$

where $dH/ds = h(s)$ and $dG/ds = g(s)$. From (1.9) one can construct an implicit representation of the solution of (1.8) following the above procedure.

Considering problems such as (1.1) and (1.8) gives an idea of what type of P - functions one has to look for in the n - dimensional case.

## SECTION 2

### DETERMINATION OF P - FUNCTIONS FOR SOLUTIONS OF $\Delta u + f(u) = 0$

Let $u \in C^2(\overline{\Omega}) \cap C^3(\Omega)$ satisfy the elliptic equation

$$\Delta u + f(u) = 0 \qquad \text{in } \Omega \tag{2.1}$$

where $\Omega$ is an open connected subset of $\mathbb{R}^n$. We shall be concerned with positive solutions of (2.1).

We aim to find conditions under which the function P, defined by :

$$P := g(u)|grad\ u|^2 + h(u),$$

satisfies a maximum principle, $u(x)$ being a solution of (2.1).

According to Section 1, the function

$$P = |grad\ u|^2 + 2\ F(u)$$

is a possible candidate. Recall, from Chapter (II), that if a function $u$ satisfies an elliptic inequality :

$$L[u] = \sum_{i,j} a_{ij}(x)\ \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_i b_i(x)\ \frac{\partial u}{\partial x_i} \geq 0 \qquad \text{in } \Omega$$

$(i,j = 1,\ldots,n)$, then the following holds :

(I) If $u$ assumes its maximum value M in $\Omega$, then $u \equiv M$ throughout $\overline{\Omega}$.

(II) If $u$ assumes its maximum value M at a point $Q \in \partial\Omega$, then either

$$u \equiv M \quad \text{in } \overline{\Omega} \quad \text{or} \quad \frac{\partial u}{\partial \nu}(Q) > 0$$

where $\nu$ is the outward normal to $\partial\Omega$ at Q.

Going back to our function $P = |grad\ u|^2 + 2F$, it will follow that P satisfies a maximum principle if P satisfies an inequality of the form $L[P] \geq 0$ in $\Omega$.

LEMMA 2.1

For any sufficiently smooth function $u : \mathbb{R}^n \longrightarrow \mathbb{R}^1$ and for $n > 1$, the following inequalities hold :

$$n \sum_{i,j} \left[\frac{\partial^2 u}{\partial x_i \partial x_j}\right]^2 \geq n \sum_k \left[\frac{\partial^2 u}{\partial x_k \partial x_k}\right]^2 \geq (\Delta u)^2.$$

PROOF :

By Schwarz's inequality

$$(\Delta u)^2 = \left[\sum_i \frac{\partial^2 u}{\partial x_i \partial x_i}\right]^2 \leq n \left[\sum_i \left[\frac{\partial^2 u}{\partial x_i \partial x_i}\right]^2\right] \leq n \sum_{i,j} \left[\frac{\partial^2 u}{\partial x_i \partial x_j}\right]^2$$

*(the last term contains more positive terms)*.□

At this stage, we would like to split Section 2 into three subsections.

## 2.1 UPPER AND LOWER SOLUTIONS AND EXISTENCE OF SOLUTIONS OF $\Delta u + f(u) = 0$

The formalism of upper and lower solutions is of importance in the following sections.

DEFINITION 2.1:

An upper solution to the boundary value problem

$$\begin{aligned} \Delta u + f(u) &= 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \qquad \ldots\ldots\ldots\ldots\ldots(2.2)$$

where $f \in C^1(\Omega)$, is a function $u_M(x)$ satisfying

$$\Delta u_M + f(u_M) \le 0 \qquad \text{in } \Omega$$

$$u_M \ge 0 \qquad \text{on } \partial\Omega.$$

A lower solution $u_m(x)$ is a function that satisfies

$$\Delta u_m + f(u_m) \ge 0 \qquad \text{in } \Omega$$

$$u_m \le 0 \qquad \text{on } \partial\Omega.$$

LEMMA 2.2

Let $u_m(x)$, $u_M(x)$ be lower and upper solutions, respectively, and suppose that

$$u_m(x) \le u_M(x) \qquad , \; x \in \Omega.$$

Then there exists at least one solution $u(x)$ of (2.2) satisfying the inequality

$$u_m(x) \le u\ (x) \le u_M(x) \quad , \; x \in \Omega.$$

NOTE : The proof of Lemma 2.2 can be found in Smoller [18], or Sattinger [15]. Stakgold and Payne [22] use this result to discuss the above equation in the special case that

$$f(s) = \lambda\, s - h(s), \quad \lambda > 0 \ ,$$

where $h(s) \in C^2(-\infty, \infty)$ and $h(0) = 0$, $h(s) > 0$ for $s > 0$.

## 2.2 REMARKS ON CURVATURE

Let $\Omega$ be a domain in $\mathbb{R}^n$ having nonempty boundary $\partial\Omega$. Let $\partial\Omega \in C^2$. For a point $Q \in \partial\Omega$, let $\nu(Q)$ and $T(Q)$ denote respectively the unit *outer normal* to $\partial\Omega$ at $Q$ and the tangent hyperplane to $\partial\Omega$ at $Q$.

The curvatures of $\partial\Omega$ at a fixed point $Q_0 \in \partial\Omega$ are determined as follows. By a rotation of coordinates we can assume that the $x_n$ - coordinate axis lies in the direction of the *inner normal* at $Q_0$. In this frame we can represent the boundary of $\partial\Omega$ *locally* by the equation :

$$x_n = \psi(x') \;, \qquad \psi \in C^2, \tag{2.3}$$

where

$$x' = (x_1, \ldots, x_{n-1}) \quad \text{and} \quad D\,\psi(Q_0') = 0, \tag{2.4}$$

where $D$ denotes derivative.

The curvature of $\partial\Omega$ at $Q_0$ is then described by the orthogonal invariants of the Hessian matrix $[D^2\psi]$ evaluated at $Q_0'$. The eigenvalues, $k_1, \ldots, k_{n-1}$, of $[D^2\psi(Q_0')]$ are called the *principal curvatures* of $\partial\Omega$ at $Q_0$ and the corresponding eigenvectors $x_1$, $x_2, \ldots, x_{n-1}$ are called the *principal directions* of $\partial\Omega$ at $Q_0$.

<u>DEFINITION</u> :

We define the *mean curvature* (or the average curvature) of $\partial\Omega$ at $Q_0$ by :

$$K(Q_0) = \frac{1}{n-1} \sum_{i=1}^{n-1} k_i = \frac{1}{n-1}\, \Delta\, \psi(Q_0'). \quad \ldots\ldots.(2.5)$$

<u>IMPLICIT REPRESENTATION OF THE BOUNDARY</u>

If the surface $\partial\Omega$ is given by an equation

$F(x_1, \ldots, x_n) = 0$ with $D\,F \neq 0$ on $\partial\Omega$, then

$\dfrac{\partial F/\partial x_i}{|D\,F|}$ is the unit normal to $\partial\Omega$ directed towards positive $F$.

It can be shown that the matrix $\dfrac{\partial}{\partial x_j}\left[\dfrac{\partial F/\partial x_i}{|D\ F|}\right]$ evaluated at a point $p \in \partial\Omega$ has eigenvalues $-k_1, -k_2, \ldots, -k_{n-1}, 0$.

So the mean curvature is given by

$$K(p) = \frac{-1}{n-1}\left[\frac{\partial}{\partial x_j}\left[\frac{\partial F/\partial x_i}{|D\ F|}\right](p)\right].$$

<u>EXAMPLE 2.1</u>

We compute the curvature at the "north pole" of the sphere of radius $R$ as an example. First we use the implicit representation. Let $F(x_1,\ldots,x_n) = R^2 - x_1^2 - x_2^2 - \ldots - x_n^2$ where the $x_n$ - axis lies along the *inner normal*, so that $\dfrac{D\ F}{|D\ F|}$ is the inner normal.

Then,

$$\frac{\partial F}{\partial x_i} = -2\ x_i \quad , i = 1,\ldots,n.$$

and

$$\frac{\partial F/\partial x_i}{|D\ F|} = -\frac{x_i}{\left[\sum_k x_k^2\right]^{\frac{1}{2}}}$$

so that

$$\frac{\partial}{\partial x_j}\left[\frac{\partial F/\partial x_i}{|D\ F|}\right] = -\frac{\delta_{ij}}{\left[\sum_k x_k^2\right]^{\frac{1}{2}}} + \frac{x_i\ x_j}{\left[\sum_k x_k^2\right]^{\frac{3}{2}}}.$$

At the north pole, $p = (0,\ 0, \ldots, -R)$, then

$$\frac{\partial}{\partial x_j}\left[\frac{\partial F/\partial x_i}{|D\ F|}\right] = -\frac{\delta_{ij}}{R} + \frac{R^2\ \delta_{in}\ \delta_{jn}}{R^3} = \begin{bmatrix} -\frac{1}{R} & 0 & \ldots & 0 \\ 0 & -\frac{1}{R} & 0\ldots & \\ \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & -\frac{1}{R} & \cdot \\ 0 & \cdot\ \cdot & \cdot & 0 \end{bmatrix}.$$

So

$$K = \frac{-1}{n-1}\left[\frac{-1}{R} - \dots \frac{-1}{R}\right]$$

$$= -\frac{1}{n-1}\quad(n-1)\left[-\frac{1}{R}\right] = \frac{1}{R}$$

EXPLICIT REPRESENTATION :

From the equation of the sphere we have

$$x_1^2 + x_2^2 + \dots + x_n^2 = R^2.$$

Then, near the north pole, we can represent the boundary $\partial\Omega$ as

$$x_n = -\sqrt{R^2 - x'^2} \qquad ; \; x'^2 = x_1^2 + \dots + x_{n-1}^2 .$$

Thus

$$\psi(x_1,\dots,x_{n-1}) = -\sqrt{R^2 - x_i x_i} \; .$$

Then

$$\frac{\partial\psi}{\partial x_i} = \frac{x_i}{\sqrt{R^2 - x'^2}}$$

and

$$\frac{\partial^2\psi}{\partial x_i \partial x_j} = \frac{\delta_{ij}}{\sqrt{R^2 - x'^2}} + \frac{x_i\, x_j}{\left[R^2 - x'^2\right]^{\frac{3}{2}}}$$

At the north pole $x' = 0$, and we get

$$[D^2\,\psi(p)] = \left[\frac{\delta_{ij}}{R}\right] = \begin{bmatrix} \frac{1}{R} & 0 & . & \dots \\ 0 & \frac{1}{R} & 0 & \dots \\ . & & . & \\ \vdots & & & \\ . & . & 0 & \frac{1}{R} \end{bmatrix}$$

$$\therefore K = \frac{1}{R} \; .$$

REMARKS 2.1

(I) If $x_n$ was chosen along the *outer normal* one gets the opposite signs in above calculations.

(II) The implicit and explicit representations are related. If we have

$$x_n = \psi\,(x_1,\ldots,x_{n-1}) \quad \text{on } \partial\Omega \quad \text{where} \quad grad\ \psi = 0,$$

one can take

$$F(x_1,\ldots,x_n) = x_n - \psi\,(x_1,\ldots,x_{n-1}).$$

In the following lemma we shall use the term "normal coordinates", by which we mean that we take the $x_1,\ldots,x_{n-1}$ axes along the principal directions corresponding to $k_1,\ldots,k_{n-1}$ at a point $Q \in \partial\Omega$.

<u>LEMMA 2.3</u>

Let $u \in C^2(\overline{\Omega})$ be a function vanishing on $\partial\Omega$ where $\partial\Omega$ is to be of class $C^2$. Then $\Delta u$ can be represented at $Q \in \partial\Omega$ by the identity

$$\Delta u = \frac{\partial^2 u}{\partial \nu^2} + (n-1)K\frac{\partial u}{\partial \nu} \tag{2.6}$$

where $\nu$ is the outward normal to $\partial\Omega$ at $Q \in \partial\Omega$, and K denotes the mean curvature of $\partial\Omega$ at $Q$.

<u>PROOF</u> : *(we use summation convention)*

Following the argument above, we get

$$x_n = \psi\,(x') \ ; \qquad \psi \in C^2,\ x' = (x_1,\ldots,x_{n-1}),$$

where we use normal coordinates at Q . Since $u$ is in $C^2(\overline{\Omega})$ the condition $u = 0$ on $\partial\Omega$ can be expressed as a twice differentiable identity

$$u\,(x', \psi) \equiv 0, \qquad x' = (x_1,\ldots,x_{n-1}). \tag{2.7}$$

Differentiating (2.7) with respect to $x_i$, $i = 1,\ldots,n-1$, we get

$$\frac{\partial u}{\partial x_i} + \frac{\partial u}{\partial x_n}\frac{\partial \psi}{\partial x_i} = 0. \tag{2.8}$$

Differentiating (2.8) with respect to $x_j$, $j = 1,\ldots,n-1$, and evaluating at $Q$, where $\partial\psi/\partial x_i = 0$ (by (2.4)) and hence $\partial u/\partial x_i = 0$ (by (2.8)), yields :

$$\frac{\partial^2 u}{\partial x_i \partial x_j} + \frac{\partial u}{\partial x_n}\,\frac{\partial^2 \psi}{\partial x_i \partial x_j} = 0 \qquad \text{(at } Q\text{)}. \qquad (2.9)$$

Considering the coordinate frame above at $Q \in \partial\Omega$, we get

$$\frac{\partial u}{\partial x_n} = -\frac{\partial u}{\partial \nu} \quad \text{and} \quad \frac{\partial^2 u}{\partial x_n \partial x_n} = \frac{\partial^2 u}{\partial \nu^2}\,. \qquad (2.10)$$

Since we have, from (2.9),

$$\frac{\partial^2 u}{\partial x_i \partial x_i} = -\frac{\partial u}{\partial x_n}\,\frac{\partial^2 \psi}{\partial x_i \partial x_i}\,, \quad i = 1,\ldots,n-1$$

then

$$\Delta\, u = \frac{\partial u}{\partial \nu}\,\frac{\partial^2 \psi}{\partial x_i \partial x_i} + \frac{\partial^2 u}{\partial \nu^2}\,.$$

Finally, by (2.5), we get

$$\Delta\, u = \frac{\partial^2 u}{\partial \nu^2} + (n-1)K\,\frac{\partial u}{\partial \nu}\,.\square$$

## 2.3 PRELIMINARY CALCULATIONS

In many calculations the case $n = 2$ (*i.e.* the two - dimensional case) allows a somewhat different treatment than that of $n \geq 3$. Also the difficulties that one encounters depend for a good part on the boundary conditions imposed on $u$.

As we mentioned before, the function

$$P = |grad\ u|^2 + 2\,F(u)$$

is a possible candidate, therefore we consider the more general form

$$P = g(u)\,|grad\ u|^2 + h(u) \qquad (2.11)$$

where $u$ is a solution of (2.1).

Differentiating (2.11) with respect to $x_j$ gives :

$$\frac{\partial P}{\partial x_j} = g'(u)\ |grad\ u|^2 \frac{\partial u}{\partial x_j} + 2\ g(u) \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial u}{\partial x_i} + h'(u) \frac{\partial u}{\partial x_j} \qquad \ldots\ldots.(2.12)$$

where the prime denotes differentiation with respect to u, and we use summation convention.

At this stage some simple notation will be convenient; thus

$$u_i = \frac{\partial u}{\partial x_i}\ ; \quad u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}\ ; \ u_{iij} = \frac{\partial^3 u}{\partial x_i \partial x_i \partial x_j}\ ,$$

$i,j = 1,\ldots,n.$

Now differentiating (2.12) with respect to $x_j$ gives :

$$\begin{aligned}\Delta\ P = P_{jj} = g''\ &|grad\ u|^4 + 2\ g'\ u_i\ u_j\ u_{ij} \\ &+ g'\ |grad\ u|^2\ u_{jj} + 2\ g'\ u_i\ u_j\ u_{ij} \\ &+ 2\ g\ u_{ijj}\ u_i + 2\ g\ u_{ij}\ u_{ij} + h''|grad\ u|^2 \\ &+ h'\ u_{jj}.\end{aligned}$$

$$\begin{aligned}\therefore \Delta\ P = g''\ &|grad\ u|^4 + |grad\ u|^2\ (g'\ u_{jj} + h'') \\ &+ 4\ g'\ u_i\ u_j\ u_{ij} + 2\ g\ u_i\ u_{ijj} + 2\ g\ u_{ij}\ u_{ij} \\ &+ h'\ u_{jj}. \qquad \ldots\ldots\ldots(2.13)\end{aligned}$$

Using (2.1), it is easy to see that

$$u_{iij} = u_{jii} = -\ f'\ u_j. \qquad \ldots\ldots\ldots(2.14)$$

The third term on the right in (2.13) can be expressed, using equation (2.12), as

$$4\ g'\ u_i\ u_j\ u_{ij} = 2\ \frac{g'}{g}\ u_j\ [P_j - g'|grad\ u|^2\ u_j - h'\ u_j] \qquad \ldots\ldots\ldots(2.15)$$

Thus we can rewrite (2.13) as

$$\Delta P = |grad\ u|^4 \left(g'' - 2\,\frac{g'^2}{g}\right)$$

$$+ |grad\ u|^2 \left(h'' - fg' - 2\,f'g - 2\,\frac{h'g'}{g}\right)$$

$$+ 2\,g\,u_{ij}\,u_{ij} - h'f + 2\,\frac{g'}{g}\,u_j\,P_j\ . \qquad \ldots\ldots\ldots(2.16)$$

<u>NOTE</u> :

Up to equation (2.16) the calculation is the same in any number of dimensions.

SECTION 3

## MAXIMUM PRINCIPLE FOR THE P – FUNCTION OF THE FORM $P(x) = g(u)|grad\ u|^2 + h(u)$

Equation (2.12) of Section 2 shows that P may assume its maximum at a point at which $grad\ u = 0$ (*i.e.* at a critical point of $u$). A second possibility is that P assumes its maximum somewhere on the boundary $\partial\Omega$. The third possibility is that P assumes its maximum at an interior point of $\Omega$, not a critical point of $u$, but a point where the determinant of the matrix

$$C_{ij} = 2\ g\ u_{ij} + (h' + g'|grad\ u|^2)\ \delta_{ij}$$

vanishes.

The latter possibility does not help us to achieve our aim, so we shall try to choose P in such a way that one of the first two possibilities mentioned above occurs.

The following illustrates a maximum principle for the P – function defined by :

$$P(x) = g(u)\ |grad\ u|^2 + h(u)$$

where $u$ is a solution of the differential equation

$$\Delta\ u + f(u) = 0 \qquad \text{in } \Omega. \tag{3.1}$$

Our work is based on Schaefer and Sperb [16], Sperb [21] and Payne [9].

## THE TWO - DIMENSIONAL CASE

We start with the case n = 2. We recall from Section 2 equations (2.12) and (2.16) which are just

$$P_i = g' \, |grad\ u|^2 \, u_i + 2\, g\, u_{ji}\, u_j + h'\, u_i \quad \ldots .(3.2)$$

and

$$\Delta\, P = |grad\ u|^4 \, (g'' - 2\, \frac{g'^2}{g})$$

$$+ \, |grad\ u|^2 \, (h'' - f\, g' - 2\, f'g - 2\, h'\frac{g'}{g})$$

$$+ \, 2\, g\, u_{ij}\, u_{ij} - h'\, f + 2\, \frac{g'}{g}\, u_i\, P_i\, . \quad \ldots ..(3.3)$$

In order to eliminate the term ( $2\, g\, u_{ij}\, u_{ij}$), we use the *following identity which only holds in two dimensions*. For any sufficiently smooth function *u*, we have (*for grad u* $\neq$ 0)

$$u_{ij}\, u_{ij} = (\Delta\, u)^2 + \frac{2}{|grad\ u|^2}\, u_i\, u_{ik}\, u_j\, u_{jk} - \frac{2\, \Delta\, u}{|grad\ u|^2}\, u_i\, u_j\, u_{ij}$$

$$\ldots \ldots (3.4)$$

while the simple form of (3.4) is :

$$u_{ij}\, u_{ij} = (\Delta\, u)^2 + 2(u_{xy}^2 - u_{xx}\, u_{yy})$$

which is easily seen. From (3.1), we may rewrite (3.4) in the form

$$u_{ij}\, u_{ij} = f^2 + \frac{2}{|grad\ u|^2}\, u_i\, u_{ik}\, u_j\, u_{jk} + \frac{2\, f}{|grad\ u|^2}\, u_i\, u_j\, u_{ij}$$

$$\ldots ..(3.5)$$

and we see that

$$\Delta P = |grad\ u|^4 \left(g'' - 2\,\frac{g'^2}{g}\right)$$

$$+ |grad\ u|^2 \left(h'' - fg' - 2\,f'g - 2\,\frac{h'g'}{g}\right)$$

$$+ 2\,g\,f^2 + \frac{4\,g}{|grad\ u|^2}\ u_i\,u_{ik}\,u_j\,u_{jk}$$

$$+ \frac{4\,g\,f}{|grad\ u|^2}\ u_i\,u_j\,u_{ij} - h'f + 2\,\frac{g'}{g}\,u_i\,P_i. \qquad .......(3.6)$$

Using (3.2), we can write

$$2\,g\,u_j\,u_{ij} = P_i - g'|grad\ u|^2\,u_i - h'\,u_i \qquad .......(3.7)$$

The combination of (3.6) and (3.7) gives :

$$\Delta P = |grad\ u|^4 \left(g'' - 2\,\frac{g'^2}{g}\right)$$

$$+ |grad\ u|^2 \left(h'' - f\,g' - 2\,f'g - 2\,\frac{h'g'}{g}\right) + 2\,g\,f^2$$

$$+ \frac{1}{g|grad\ u|^2}\Big[(P_k - g'\,u_k\,|grad\ u|^2 - h'\,u_k)$$

$$\times\ (P_k - g'\,u_k\,|grad\ u|^2 - h'\,u_k)\Big]$$

$$+ \frac{2\,f}{|grad\ u|^2}\ u_i\,(P_i - g'\,u_i\,|grad\ u|^2 - h'\,u_i)$$

$$- h'\,f + 2\,\frac{g'}{g}\,u_i\,P_i. \qquad ........(3.8)$$

The product term

$$\frac{1}{g|grad\ u|^2}\,(P_k - g'\,u_k\,|grad\ u|^2 - h'\,u_k)$$

$$\times\ (P_k - g'u_k\,|grad\ u|^2 - h'\,u_k)$$

can be estimated as follows

$$\frac{2}{g|grad\ u|^2} P_k(-\ g'\ u_k\ |grad\ u|^2 - h'\ u_k)$$

$$+ \frac{1}{g|grad\ u|^2} P_k\ P_k + \frac{g'^2}{g}|grad\ u|^4 + \frac{h'^2}{g} + 2\ \frac{g'\ h'}{g}\ |grad\ u|^2.$$

$$\geq \frac{2}{g|grad\ u|^2} P_k(-\ g'\ u_k\ |grad\ u|^2 - h'\ u_k) + \frac{g'^2}{g}\ |grad\ u|^4$$

$$+ \frac{h'^2}{g} + 2\ \frac{g'h'}{g}\ |grad\ u|^2.$$

The fifth term in (3.8) is

$$\frac{2\ f}{|grad\ u|^2}\quad u_i\ (P_i - g'\ u_i\ |grad\ u|^2 - h'\ u_i)$$

$$= \frac{2\ f}{|grad\ u|^2}\quad u_i\ P_i - 2\ f\ g'\ |grad\ u|^2 - 2\ f\ h'.$$

$$\therefore \Delta\ P \geq |grad\ u|^4\ (g'' - \frac{g'^2}{g}) + |grad\ u|^2\ (h'' - 2\ f'\ g - 3\ f\ g')$$

$$+ u_k\ P_k \left[\frac{2\ f}{|grad\ u|^2} - \frac{2\ h'}{g|grad\ u|^2}\right] + 2\ g\ f^2 + \frac{h'^2}{g}$$

$$- 3\ f\ h'. \qquad \ldots\ldots\ldots(3.9)$$

Inequality (3.9) may be written in the form

$$\Delta\ P + \frac{1}{|grad\ u|^2}\quad \psi_i\ P_i \geq g\ (log\ g)''\ |grad\ u|^4$$

$$+ ((h' - 2\ f\ g)' - f\ g')|grad\ u|^2$$

$$+ \frac{1}{g}\ (h' - f\ g)(h' - 2\ f\ g)$$

$$\ldots\ldots\ldots(3.10)$$

where $\psi_i = \frac{1}{g}\left[\ 2\ u_i\ (h' - f\ g)\right].$

Since we want to apply the maximum principle, we shall restrict our choice to functions $g(u)$ and $h(u)$ such that the right side in (3.10) becomes nonnegative. This will lead to the following result.

LEMMA 3.1

Let $u \in C^3(\Omega)$ satisfy (3.1) in a plane domain $\Omega \subset \mathbb{R}^2$. If $g(u)$ and $h(u)$ are chosen such that the coefficients of $|grad\ u|^4$ and $|grad\ u|^2$ and the constant term in (3.10) are nonnegative, then the corresponding function

$$P(x) = g(u)\ |grad\ u|^2 + h(u)$$

attains its maximum either on $\partial\Omega$ or at a critical point of $u$.

PROOF:

Suppose that P attains its maximum at an interior point $Q$ of $\Omega$, where $grad\ u(Q) \neq 0$. Let

$$\Omega' = \{x \in \Omega : grad\ u(x) \neq 0\}$$

an open subset of $\Omega$ and $Q$ is an interior point of $\Omega'$. Then, by the above calculation, P satisfies

$$\Delta P + \frac{1}{|grad\ u|^2} \psi_i P_i \geq 0 \quad \text{on } \Omega'.$$

By the maximum principle, either $P \equiv$ constant on $\Omega'$ or P attains its maximum on the boundary of $\Omega'$. The second possibility cannot occur since P attains its maximum at $Q$. Therefore, P is constant on $\Omega'$ and therefore also on $\bar{\Omega}'$, so P attains its maximum at a point where $grad\ u = 0$. This completes the proof of Lemma 3.1 .□

In the applications, important choices of $h$ are $h' = f\, g$ and $h' = 2\, f\, g$ . We shall study here the more general case

$$h' = c\, f\, g, \qquad \text{for} \quad c \in \mathbb{R}.$$

Hence $h'' = c\, [f'g + f\, g']$, and by substituting into (3.9) we get

$$\Delta\, P \geq |grad\ u\,|^4\, (g'' - \frac{g'^2}{g})$$

$$+ |grad\ u|^2 (c\, f'\, g + c\, f\, g' - 2\, f'\, g - 3\, f\, g')$$

$$+ u_k\, P_k \left[\frac{2\, f}{|grad\ u|^2} - \frac{2\, c\, f}{|grad\ u|^2}\right]$$

$$+ (c - 2)(c - 1)\, g\, f^2\ . \qquad \text{..........(3.11)}$$

Inequality (3.11) may be written in the form :

$$\Delta\, P + \frac{[2(c - 1)\ f]}{|grad\ u|^2}\, u_k\, P_k \geq |grad\ u|^4\, (g'' - \frac{g'^2}{g})$$

$$+ |grad\ u|^2\, [(c - 2)\, f'\, g + (c - 3)\, f\, g']$$

$$+ [(c - 1)(c - 2)]\, g\, f^2\ . \qquad \text{.........(3.12)}$$

As the constant term in (3.12) must be nonnegative, we must have either

$$c \leq 1 \quad \text{or} \quad c \geq 2.$$

We consider the special cases $c = 1$, $c = \frac{1}{2}$, $c = 2$, $c = 3$.

<u>CASE $c = 1$</u> :

If $c = 1$ ; $h' = f\, g$ then from (3.12) we get

$$\Delta\, P \geq |grad\ u|^4\, (g'' - \frac{g'^2}{g})$$

$$+ |grad\ u|^2\, (- f'\, g - 2\, f\, g'\, ).$$

Therefore with the assumptions :

(i) $(\log g)'' \geq 0$ , $g > 0$,

(ii) $f' g + 2 f g' \leq 0$

*(or in other words* $(\log f)' + 2 (\log g)' \leq 0$ ),

we get

$$\Delta P \geq 0$$

hence Lemma 3.1 is applicable to the function

$$P(x) = g(u) |grad\ u|^2 + \int_0^u f(s)\ g(s)\ ds \quad \text{in } \Omega.$$

<u>CASE $c < 1$</u>:

As an example of the case $c < 1$, we take $c = \frac{1}{2}$ . From (3.12) we get :

$$\Delta P - \frac{f}{|grad\ u|^2} u_k P_k \geq |grad\ u|^4 \left(g'' - \frac{g'^2}{g}\right)$$

$$+ |grad\ u|^2 \left[\tfrac{1}{2}(- 3 f'g - 5 f g')\right]$$

$$+ \frac{3}{4} g f^2 .$$

Therefore, with the assumptions

(i) as above

(ii) $3 f' g + 5 f g' \leq 0$ ,

we get

$$\Delta P - \frac{f}{|grad\ u|^2} u_k P_k \geq 0 .$$

So Lemma 3.1 applies to the function

$$P(x) = g(u) |grad\ u|^2 + \tfrac{1}{2} \int_0^u f(s)\ g(s)\ ds.$$

CASE $c = 2$ :

If $c = 2$ ; $h' = 2 f g$ then from (3.12) we get

$$\Delta P + \frac{2 f}{|grad\ u|^2} u_k P_k \geq |grad\ u|^4 \left(g'' - \frac{g'^2}{g}\right)$$

$$+ |grad\ u|^2(- f g') .$$

Therefore with the assumptions

(i) as before

(ii) $f g' \leq 0$ ,

we get

$$\Delta P + \frac{2 f}{|grad\ u|^2} u_k P_k \geq 0$$

and Lemma 3.1 applies to the function

$$P(x) = g(u)\ |grad\ u|^2 + 2 \int_0^u f(s)\ g(s)\ ds \quad \text{in } \Omega.$$

An interesting consequence of this case (*when* $c = 2$) holds if we take $g = 1$ ; $h' = 2 f$, then we get the inequality

$$\Delta P + \frac{2 f}{|grad\ u|^2} u_k P_k \geq 0$$

directly without putting conditions on $f(u)$. Clearly we get the function $P(x)$ in the form

$$P = |grad\ u|^2 + 2 F(u)$$

which we have conjectured on the basis of Section 1.

CASE $c > 2$ :

As an example of the case $c > 2$, we take $c = 3$ ; $h' = 3 f g$. From (3.12) we get

$$\Delta P + \frac{4f}{|grad\ u|^2} u_k P_k \geq |grad\ u|^4 \left(g'' - \frac{g'^2}{g}\right)$$

$$+ |grad\ u|^2 (f'\ g) + 2\ g\ f^2 .$$

With the assumptions :

(i) as before

(ii) $f' \geq 0$ (*i.e.* $f(x)$ monotonically increasing)

we get

$$\Delta P + \frac{4f}{|grad\ u|^2} u_k P_k \geq 0 .$$

Therefore, Lemma 3.1 applies to the function

$$P(x) = g(u)\ |grad\ u|^2 + 3 \int_0^u f(s)\ g(s)\ ds .$$

EXAMPLE 3.1

For the case $c = 1$, one can take $g(u) = e^{-\alpha u}$, $\alpha > 0$. Then Lemma 3.1 applies to the function

$$P(x) = |grad\ u|^2\ e^{-\alpha u} + \int_0^u f(s)\ e^{-\alpha s}\ ds$$

provided $f' \leq 2\ \alpha\ f$ (so that $f(u) \leq f(0)\ e^{\alpha u}$ ). Note that the equality sign in assumption (i) is admissible since

$$(log\ e^{-\alpha u})'' = 0 .\square$$

EXAMPLE 3.2 : (Sperb [21])

The following example is given in the case $c = 2$.

Take $g(u) = e^{-\alpha u}$, $\alpha > 0$,

and

$$h(u) = 2 \int_0^u e^{-\alpha s}\ f(s)\ ds\ ;\ \ h' = 2\ f\ g$$

so that

$$P(x) = e^{-\alpha u} \; |grad\ u|^2 + 2 \int_0^u e^{-\alpha s} \; f(s)\ ds \ .$$

Assumptions (i), (ii) in the case $c = 2$ above are satisfied and again the equality sign in assumption (i), for $c = 2$, is admitted.□

EXAMPLE 3.3

For the case $c = 3$ , one can take $g(u) = e^{\alpha u}$ , $\alpha > 0$. Then Lemma 3.1 applies to the function

$$P(x) = |grad\ u|^2 \ e^{\alpha u} + 3 \int_0^u f(s)\ e^{\alpha s} \ ds$$

provided $f' \geq 0$ . Note again that for $c = 3$, the equality sign in assumption (i) holds since

$$(log\ e^{\alpha u})'' = 0 \ . \ \square$$

THE N - DIMENSIONAL CASE

The main problem which one encounters is the elimination of the term $u_{ij}\ u_{ij}$ in equation (3.3) as we have seen in the Two - dimensional case above. Since the identity (3.4) is only valid in two dimensions, we use Schwarz's inequality.

Now, from (3.2) we have

$$(P_j - g' \ |grad\ u|^2 u_j - h'\ u_j)\ (P_j - g' \ |grad\ u|^2\ u_j - h'\ u_j)$$

$$= 4\ g^2\ u_{ij}\ u_i\ u_{kj}\ u_k \leq 4\ g^2\ u_{ij}\ u_{ij}\ |grad\ u|^2 \qquad \ldots\ldots.(3.14)$$

$$\therefore \Delta P \geq |grad\ u|^4 \left(g'' - 2\frac{g'^2}{g}\right) + |grad\ u|^2 \left(h'' - fg' - 2f'g - 2\frac{h'g'}{g}\right) - h'f + 2\frac{g'}{g} u_i P_i + \left[\frac{1}{2g|grad\ u|^2} \times (P_i - g'|grad\ u|^2 u_i - h'u_i)(P_i - g'|grad\ u|^2 u_i - h'u_i)\right].$$

.......(3.15)

The last term on the right side of (3.15) can be estimated as follows :

$$\frac{1}{g|grad\ u|^2} P_i(-g'u_i|grad\ u|^2 - h'u_i) + \frac{1}{2g|grad\ u|^2} P_i P_i$$

$$+ \frac{1}{2g|grad\ u|^2}(g'u_i|grad\ u|^2 + h'u_i)(g'u_i|grad\ u|^2 + h'u_i)$$

$$\geq \frac{1}{g|grad\ u|^2} P_i(-g'u_i|grad\ u|^2 - h'u_i) + \frac{g'^2}{2g}|grad\ u|^4$$

$$+ \frac{h'^2}{2g} + \frac{g'h'}{g}|grad\ u|^2 .$$

$$\therefore \Delta P \geq |grad\ u|^4 \left(g'' - \frac{3}{2}\frac{g'^2}{g}\right)$$

$$+ |grad\ u|^2 \left(h'' - fg' - 2f'g - \frac{h'g'}{g}\right)$$

$$+ \frac{h'^2}{2g} + \frac{g'}{g}u_i P_i - h'f - \frac{h'}{g|grad\ u|^2} u_i P_i .$$

.......(3.16)

So (3.15) can be written in the form

$$\Delta P + \frac{1}{|grad\ u|^2}\psi_i P_i \geq |grad\ u|^4\left(g'' - \frac{3}{2}\frac{g'^2}{g}\right)$$

$$+ |grad\ u|^2\left[(h' - 2fg)' + \frac{g'}{g}(fg - h')\right]$$

$$+ \frac{h'}{2g}(h' - 2fg) \qquad \text{...........(3.17)}$$

where $\psi_i = \frac{1}{g} (h' u_i - g' u_i |grad\ u|^2)$.

The analogue of Lemma 3.1 in the n - dimensional case is:

LEMMA 3.2

Let $u \in C^3(\Omega)$ be a solution of (3.1), $\Omega \subset \mathbb{R}^n$, $n \geq 2$. If $g(u)$, $h(u)$ are chosen such that the coefficients of $|grad\ u|^4$ and $|grad\ u|^2$ and the constant term in (3.17) are nonnegative, then the corresponding function

$$P = g(u)\ |grad\ u|^2 + h(u)$$

must assume its maximum value either on $\partial\Omega$ or at a critical point of $u$.

PROOF :

The proof is the same as for Lemma 3.1 .□

NOTE : This calculation allows somewhat different results from those of Lemma 3.1 when n = 2.

At this stage we consider $h'(u)$ in the form

$$h' = c\ f\ g\ ; \qquad c \in \mathbb{R}\ (real\ number).$$

Therefore, from (3.17) we get :

$$\Delta P + \frac{1}{|grad\ u|^2} \psi_i P_i \geq |grad\ u|^4 \left(g'' - \frac{3}{2} \frac{g'^2}{g}\right) + |grad\ u|^2 [(c - 2)f'g - fg'] + \frac{c}{2} (c - 2)\ g\ f^2 \qquad \text{...........(3.18)}$$

where $\psi_i = \frac{1}{g} [c\ g\ f\ u_i - g'\ u_i\ |grad\ u|^2]$.

Recalling Lemma 3.2 , we shall discuss the required hypotheses for different values of $c$, for which Lemma 3.2 is applicable to the function

$$P(x) = g(u)\ |grad\ u|^2 + h(u)\ ; \quad h' = c\ f\ g.$$

REMARK 3.1

(I) From (3.18) it is clear that the admissible choices of $c$ are :

$$c \geq 2 \quad \text{or} \quad c \leq 0.$$

(II) Since the coefficient of $|grad\ u|^4$ , $(g'' - \frac{3}{2}\frac{g'^2}{g})$, does not depend on $c$, then the variety of the hypotheses will rest in the coefficient of $|grad\ u|^2$ and in the constant term only.

CASE $c = 2$ :

For $c = 2$; $h' = 2\ f\ g$ , we get , from (3.18),

$$\Delta\ P + \frac{1}{|grad\ u|^2}\ \psi_i\ P_i \geq |grad\ u|^4\ (g'' - \frac{3}{2}\frac{g'^2}{g})$$

$$- |grad\ u|^2\ f\ g' ,$$

where $\psi_i = \frac{1}{g}\,[\ 2\ f\ g\ u_i - g'\ u_i\ |grad\ u|^2].$

Therefore, with the assumptions

(i) $(g'' - \frac{3}{2}\frac{g'^2}{g}) \geq 0$ , $g > 0$,

(ii) $f\ g' \leq 0$

we get

$$\Delta\ P + \frac{1}{|grad\ u|^2}\ \psi_i\ P_i \geq 0 \quad .$$

Then Lemma 3.2 applies to the function

$$P(x) = g(u)\ |grad\ u|^2 + 2 \int_0^u f(s)\ g(s)\ ds \quad \text{in } \Omega.$$

REMARK 3.2

For n = 2 , the previous result is less restrictive .

CASE $c > 2$ :

As an example, we take $c = 3$, $h' = 3\ f\ g$ . From (3.18), we get

$$\Delta\ P + \frac{1}{|grad\ u|^2}\ \psi_i\ P_i \geq |grad\ u|^4\ (g'' - \frac{3}{2}\frac{g'^2}{g})$$

$$+ |grad\ u|^2\ (f'\ g - f\ g') + \frac{3}{2}\ g\ f^2.$$

where $\psi_i = \frac{1}{g}\left[3\ g\ f\ u_i - g'\ u_i\ |grad\ u|^2\right]$

Therefore with the assumptions

(i) as above

(ii) $f'\ g - f\ g' \geq 0$

we get

$$\Delta\ P + \frac{1}{|grad\ u|^2}\ \psi_i\ P_i \geq 0\ .$$

Therefore Lemma 3.2 applies to the function

$$P(x) = g(u)\ |grad\ u|^2 + 3 \int_0^u f(s)\ g(s)\ ds \quad \text{in } \Omega.$$

CASE $c = 0$ :

For $c = 0$ ; $h' = 0$ , we get from (3.18)

$$\Delta\ P + \frac{1}{|grad\ u|^2}\ \psi_i\ P_i \geq |grad\ u|^4\ (g'' - \frac{3}{2}\frac{g'^2}{g})$$

$$+ |grad\ u|^2\ (-\ 2\ f'g - f\ g')$$

where $\psi_i = \frac{1}{g} (- g' u_i \, |grad\ u|^2)$ .

With the assumptions :

(i) as before

(ii) $2 f' g + f g' \leq 0$

we get

$$\Delta P + \frac{1}{|grad\ u|^2} \psi_i P_i \geq 0 ,$$

therefore Lemmma 3.2 applies to the function

$$P(x) = g(u) \, |grad\ u|^2 \quad \text{in } \Omega.$$

<u>CASE $c < 0$</u> :

As an example we take $c = -1$. From (3.18), we get

$$\Delta P + \frac{1}{|grad\ u|^2} \psi_i P_i \geq |grad\ u|^4 \left(g'' - \frac{3}{2} \frac{g'^2}{g}\right)$$

$$+ |grad\ u|^2 (- 3 f' g - f g') + \frac{3}{2} g f^2,$$

where $\psi_i = \frac{1}{g} [- g f u_i - g' u_i \, |grad\ u|^2]$.

Therefore with the assumptions

(i) as before

(ii) $3 f' g + f g' \leq 0$ ,

we get

$$\Delta P + \frac{1}{|grad\ u|^2} \psi_i P_i \geq 0 .$$

Therefore, Lemma 3.2 applies to the function

$$P(x) = g(u) \, |grad\ u|^2 - \int_0^u f(s) \, g(s) \, ds \quad \text{in } \Omega.$$

REMARK 3.3

The case $c = 2$ is useful because no hypotheses are needed on $f$, only on $g$.

The following theorem is based on arguments given by Payne [9] for the case $c = 2$ only. It gives somewhat different results to the ones above.

We shall, first, introduce the following calculations. Consider the function $P(x)$ in $\Omega \subset \mathbb{R}^n$, $n \geq 2$, in the form :

$$P(x) = g(u)\ |grad\ u|^2 + c \int_0^u f(s)\ g(s)\ ds$$

(*thus* $h' = c\ f\ g$ ) and $c \in \mathbb{R}$. From (2.12) and (2.16) of Section 2 we get :

$$P_k = 2\ g\ u_i\ u_{ik} + g'\ |grad\ u|^2\ u_k + c\ f\ g\ u_k \qquad (3.19)$$

and

$$\therefore \Delta P = 2\ g\ u_{ij}\ u_{ij} + |grad\ u|^4\ g''$$
$$+ |grad\ u|^2 [-2\ g\ f' - f\ g' + c\ f'g + c\ f\ g']$$
$$+ 4\ g'\ u_i\ u_j\ u_{ij} - c\ g\ f^2 . \qquad .......(3.20)$$

Suppose that $P(x)$ takes its maximum at an interior point Q where $grad\ u \neq 0$. At Q we can orient our axes such that $u_j(Q) = 0$, for $j \neq 1$ and $u_1(Q) \neq 0$. Since, by assumption, $P_k = 0$ at Q, it follows from (3.19) that :

$$2\ g\ u_{11} + g'\ u_1^2 + c\ g\ f = 0 \qquad .......(3.21)$$

and

$$u_{1i} = 0\ , \quad i = 2,\dots,n. \qquad .......(3.22)$$

Therefore

$$4\ g'\ u_i\ u_j\ u_{ij} = 4\ g'\ u_1^2\ u_{11}\ .$$

We will use the inequality

$$u_{ij}\, u_{ij} \geq u_{kk}\, u_{kk} = u_{11}^2 + \sum_{k=2}^{n} u_{kk}^2$$

$$\geq u_{11}^2 + \frac{1}{n-1} (\Delta u - u_{11})^2$$

by Schwarz's inequality. .......(3.23)

So

$$2\, g\, u_{ij}\, u_{ij} \geq 2\, g \left[ u_{11}^2 + \frac{1}{(n-1)} (f + u_{11})^2 \right] \quad \text{....(3.24)}$$

From (3.21) we have

$$u_{11} = -\frac{c}{2} f - \frac{g'}{2\, g} u_1^2$$

and

$$f + u_{11} = (1 - \frac{c}{2})\, f - \frac{g'}{2g} u_1^2 . \quad \text{.......(3.25)}$$

Inserting from above into (3.20) we get :

$$\Delta\, P \geq 2\, g \left[ (\frac{c}{2} f + \frac{g'}{2\, g} u_1^2) + \frac{1}{n-1} \{(1 - \frac{c}{2})f - \frac{g'}{2\, g} u_1^2 \}^2 \right]$$

$$+ |grad\ u|^4 g'' + |grad\ u|^2 \left[-2\, g\, f' + c\, f' g - f g' + c\, f g'\right]$$

$$+ 4\, g'\, u_1^2 (-\frac{c}{2} f - \frac{g'}{2g} u_1^2) - c\, f^2\, g .$$

$$= |grad\ u|^4 \left[g'' + \frac{g'^2}{2g} + \frac{g'^2}{2g(n-1)} - 4 \frac{g'^2}{2\, g} \right]$$

$$+ |grad\ u|^2 \left[c\, f\, g' - 2 \frac{\left[1 - \frac{c}{2}\right]}{(n-1)} g'\, f - 2\, g\, f' + c\, f' g \right.$$

$$\left. - f\, g' + c\, f\, g' - 2\, c\, g' f\right] - c f^2 g + \frac{c}{2}^2 g\, f^2 + \frac{2\, g}{n-1} (1 - \frac{c}{2})^2 f^2 .$$

$$\therefore \Delta P \geq |grad\ u|^4 \left[ g'' - \frac{(3n-4)}{2(n-1)} \frac{g'^2}{g} \right]$$

$$+ |grad\ u|^2 \left[ (c-2) g f' - \left[ 1 + \frac{(2-c)}{(n-1)} \right] g' f \right]$$

$$+ \frac{2\ g\ f^2}{n-1} \left[ - \frac{c}{2} (n-1) + (n-1) \frac{c^2}{4} + (1 - \frac{c}{2})^2 \right]. \quad \ldots\ldots.(3.26)$$

We shall restrict our choice to functions $g(u)$ and $f(u)$ such that the coefficients of $|grad\ u|^4$ and $|grad\ u|^2$ *and the constant term* are *nonnegative* and one of these is strictly positive to get a contradiction.

We note that the coefficient of $|grad\ u|^4$ does not depend on $c$ . So we want

$$g'' - \frac{(3n-4)}{2(n-1)} \frac{g'^2}{g} \geq 0 \quad .$$

Also we want the coefficient of $|grad\ u|^2$

$$(c-2)\ g\ f' - \left[ 1 + \frac{(2-c)}{(n-1)} \right] g'\ f \geq 0.$$

<u>THEOREM 3.3</u>

Let $u \in C^3(\Omega)$ satisfy (3.1) in $\Omega \subset \mathbb{R}^n$, $n \geq 2$. If $g(u)$ and $f(u)$ are chosen such that the coefficients of $|grad\ u|^4$ and $|grad\ u|^2$ and the constant term in (3.26) are nonnegative and one of these is strictly positive, then the corresponding function

$$P(x) = g(u)\ |grad\ u|^2 + c \int_0^u f(s)\ g(s)\ ds$$

attains its maximum value either on the boundary of $\Omega$ or at a critical point of $u$.

PROOF :

If the hypotheses above are satisfied, then by the above calculation, if P attained its maximum at an interior point Q where $grad\ u \neq 0$ , we would have

$$\Delta P > 0 \quad \text{at } Q$$

which is impossible. This completes the proof of the Theorem 3.3.□

REMARK 3.4

The constant term is, in fact,

$$\frac{2\,g\,f^2}{(n-1)}\left[(n-1)\frac{c}{2}\left(\frac{c}{2}-1\right)+\left(\frac{c}{2}-1\right)^2\right]$$

$$=\frac{2\,g\,f^2}{(n-1)}\left[\left(\frac{c}{2}-1\right)\left(\frac{c}{2}n-1\right)\right].$$

Therefore the constant term is *zero* when

$$c = 2 \quad \text{and when} \quad c = 2/n ,$$

and *positive* when

$$c > 2 \quad \text{and when} \quad c < 2/n$$

(*negative in between*).

Therefore the admissible values of $c$ are :

$$c \geq 2 \quad \text{and} \quad c \leq 2/n \quad , \quad n \geq 2.$$

SPECIAL CASES OF THEOREM 3.3

CASE $c = 2$ :

For $c = 2$ ; $h' = 2\,f\,g$ we get from (3.26)

$$\Delta P \geq |grad\ u|^4 \left(g'' - \frac{(3n-4)}{2(n-1)}\frac{g'^2}{g}\right)$$

$$+ |grad\ u|^2 (-\,f\,g').$$

Therefore, with the assumptions :

(i) $\quad [g'' - \frac{(3n-4)}{2(n-1)} \frac{g'^2}{g}] \geq 0 \ , \quad g > 0 \ ,$

(ii) $\quad g' f \leq 0 \ ,$

and if one of these inequalities is strict, Theorem 3.3 applies to the function

$$P(x) = g(u) \ |grad\ u|^2 + 2 \int_0^u f(s)\ g(s)\ ds.$$

<u>CASE $c > 2$</u> :

As an example we take $c = 3$. From (3.26), we get

$$\Delta P \geq |grad\ u|^4 \ [g'' - \frac{(3n-4)}{2(n-1)} \frac{g'^2}{g}]$$

$$+ \ |grad\ u|^2 \ [g\ f' - \frac{(n-2)}{(n-1)} g' f] + \frac{g\ f^2}{(n-1)} [\frac{3}{2} n - 1].$$

Therefore, with the assumptions :

(i) as above

(ii) $\quad g\ f' \geq \frac{(n-2)}{(n-1)} g'\ f \ ,$

Theorem 3.3 applies to the function

$$P(x) = g(u) \ |grad\ u|^2 + 3 \int_0^u f(s)\ g(s)\ ds.$$

<u>CASE $c = 2/n$</u> :

For $c = 2/n$ $\quad (h' = (2/n)\ f\ g)$ we have from (3.26)

$$\Delta P \geq |grad\ u|^4 \ [g'' - \frac{(3n-4)}{2(n-1)} \frac{g'^2}{g}]$$

$$+ \ |grad\ u|^2 \ [\frac{2}{n} (1-n)\ g\ f' - \frac{(n+2)}{n} g'\ f\ ].$$

Therefore, with the assumptions :

(i) as before

(ii) $\frac{2(1-n)}{(n+2)}\, g\, f' \geq f\, g'$ ,

and if one of the inequalities in (i) and (ii) is strict , Theorem 3.3 is applies to the function

$$P(x) = g(u)\ |grad\ u|^2 + \frac{2}{n}\int_0^u f(s)\ g(s)\ ds\ .$$

REMARK 3.5 :

Since we must have $c \leq 2/n$ or $c \geq 2$ we remark that the case c = 1 ($h' = f\, g$) is admissible *only* for n = 2 *i.e. for a plane domain* $\Omega \subset \mathbb{R}^2$ .

CASE c < 2/n :

As an example we shall take $c = 1/n$ ($h' = (1/n)\, f\, g$). From (3.26) one gets :

$$\Delta\ P \geq |grad\ u|^4 \left[g'' - \frac{(3n-4)}{2(n-1)}\frac{g'^2}{g}\right]$$

$$+ |grad\ u|^2 \left[(\frac{1}{n} - 2)\ g\ f' - \frac{(1+n-(1/n))}{(n-1)}\ g'\ f\right]$$

$$+ \frac{g\ f^2}{(n-1)}\ (1 - \frac{1}{2n}\ )\ .$$

Therefore, with the assumptions :

(i) as before

(ii) $\frac{(3-(1/n)-2\,n)}{(1-(1/n)+n)}\, g\, f' \geq g'\, f$ ,

Theorem 3.3 applies to the function

$$P(x) = g(u)\ |grad\ u|^2 + (1/n)\int_0^u f(s)\ g(s)\ ds\ .$$

**NOTE** : We note again that for $c = 2$ , no hypotheses are needed on $f$.

EXAMPLE 3.4

If $\Omega \subset \mathbb{R}^n$ , $n > 2$, we consider the function

$$P(x) = g(u)\ |grad\ u|^2 + h(u)$$

where $h' = 2\,f\,g$ and $g(u) = (u + \beta)^{-\alpha}$ , $\beta > 0$.

To apply Lemma 3.2 we need :

$$\text{(i)} \qquad (g'' - \frac{3}{2}\frac{g'^2}{g}) = (u + \beta)^{-\alpha-2}\ (\alpha^2 + \alpha - \frac{3}{2}\alpha^2) \geq 0$$

*i.e.* $\alpha^2 - 2\,\alpha \leq 0$ ,

and

$$\text{(ii)} \qquad g' = -\,\alpha(u + \beta)^{-\alpha-1} \leq 0 ,$$

that is $0 \leq \alpha \leq 2$ .

To apply Theorem 3.3 to $P(x)$ as above, we want (the coefficient of $|grad\ u|^4$),

$$\text{(i)} \qquad (\alpha^2 + \alpha - \frac{3n - 4}{2n - 2}\,\alpha^2\ )\ (u + \beta)^{-\alpha-2} \geq 0 \quad .$$

Hence ,

$$\alpha\ (\alpha - \frac{2n - 2}{n - 2}) \leq 0$$

Also we want (the coefficient of $|grad\ u|^2$),

$$\text{(ii)} \qquad -\,\alpha\ (u + \beta)^{-\alpha-1} \geq 0$$

with one of (i) and (ii) strict .

Therefore, we need

$$0 < \alpha \leq \frac{2n - 2}{n - 2} = 2 + \frac{2}{n - 2} ,$$

so Theorem 3.3 can be better .

However, for $g(u) \equiv 1$, if $c = 2$, Lemma 3.2 can be applied but Theorem 3.3 cannot.□

## SECTION 4

### THE MAXIMUM OF P ON $\partial\Omega$

We study the second possibility of the maximum of the function P in which P assumes its maximum value on the boundary of $\Omega$. For these, the calculations as given by Sperb are appropriate.

### THE TWO - DIMENSIONAL CASE

We start with the plane domain case.

**THEOREM 4.1**

Let $u$ be a solution of the elliptic equation

$$\Delta u + f(u) = 0 \tag{4.1}$$

in a plane domain $\Omega$, with $u_m \leq u \leq u_M$, where $u_m$ and $u_M$ are lower and upper bounds respectively. Suppose that for $u_m \leq s \leq u_M$ the following conditions are satisfied :

(i) $(\log g(s))'' \geq 0$ , $g(s) > 0$ ,

(ii) $(c - 2) f'g + (c - 3) f g' \geq 0$ , $c \leq 1$ .

Then the function

$$P = g(u)|grad\, u|^2 + c\int_0^u g(s)\, f(s)\, ds$$

assumes its maximum on $\partial\Omega$.

**PROOF** :

First suppose that $c < 1$ . By Lemma 3.1 , Section 3, P assumes its maximum either on $\partial\Omega$ or at a critical point of $u$ . At an interior point of $u$ , from (3.3) of Section 3, we get :

$$\Delta P = 2\, g\, u_{ij}\, u_{ij} - c\, g\, f^2 \, .$$

Using Lemma 2.1, Section 2, we find that

$$\Delta P \geq (1 - c)\, g\, f^2 > 0 \,, \quad \text{since} \quad c < 1 \,.$$

Therefore P cannot take its maximum at an interior critical point of $u$ when $c < 1$.

For $c = 1$, from (3.9) of Section 3 we have :

$$\Delta P \geq |grad\ u|^4 \left(g'' - \frac{g'^2}{g}\right)$$

$$+ |grad\ u|^2 \left(-2 f g' - f' g\right) . \qquad \text{.....(4.2)}$$

Note that inequality (3.9) is derived assuming that $grad\ u \neq 0$. So at a point Q where $grad\ u \neq 0$, there is a neighbourhood of Q on which $grad\ u \neq 0$, and the above calculation gives

$$\Delta P \geq 0 \quad \text{at } Q.$$

Also, if $grad\ u = 0$ at a point $Q'$, then as above

$$\Delta P \geq 0 \quad \text{at } Q' .$$

Therefore

$$\Delta P \geq 0 \quad \text{in } \Omega,$$

and hence P attains its maximum on $\partial\Omega$ and

$$\frac{\partial P}{\partial \nu} > 0 \quad \text{there} ,$$

unless $P \equiv$ constant in $\Omega$ .□

REMARK 4.1

Sperb [21] only considers the case $c = 1$ ; moreover we believe that his proof is incomplete .

COROLLARY 4.2

For $c < 1$ , if $u$ is a positive solution of (4.1) with $u = 0$ on $\partial\Omega$ and P attains its maximum at a point Q on $\partial\Omega$ , then

$$\frac{\partial P}{\partial \nu} > 0 \quad \text{at } Q$$

unless $P \equiv$ constant near Q .

PROOF :

From (3.12) of Section 3,

$$\Delta P + \psi_k P_k > 0$$

at a point in $\Omega$ where $grad\ u \neq 0$ , where $\psi_k = \dfrac{2(c-1) f u_k}{|grad\ u|^2}$ .

From Remark 2.5 of Chapter (III), $u$ has no critical point in any maximal cap . Therefore $grad\ u$ is bounded away from zero in a neighbourhood of Q and the maximum principle in Theorem 2.4 of Chapter (II) applies .□

REMARK 4.2

If $\Omega$ is convex , Corollary 3.2 of Chapter (III) gives $grad\ u \neq 0$ on a neighbourhood of $\partial\Omega$ .

EXAMPLES 4.1 : (Sperb [21])

(a) Take $g(u) = (f(u))^{-\frac{1}{2}}$ if $f(s) > 0$ and

$$(log\ f(s))'' \leq 0 \quad \text{for } u_m \leq s \leq u_M ,$$

Then

$$P = (f(u))^{-\frac{1}{2}} \left[ |grad\ u|^2 + \frac{2}{3} f^2 (u) \right] .$$

(b) Choose $g(u) = e^{-\alpha u}$ , $\alpha > 0$, then the equality sign in assumption (i) is admitted , and (ii) is satisfied provided

$$(\log f(s))' \leq 2\,\alpha \qquad \text{for } u_m \leq s \leq u_M .$$

Then

$$P = e^{-\alpha u}\, |grad\ u|^2 + \int_0^u e^{-\alpha s} f(s)\, ds$$

assumes its maximum on $\partial\Omega$.

## THE N - DIMENSIONAL CASE

Now we give a theorem which applies to $\Omega \subset \mathbb{R}^n$, with $n \geq 2$ . Note that for $n = 2$ the hypothesis (i) is changed from that of Theorem 4.1 .

THEOREM 4.3

Let $u$ be a sufficiently smooth solution of (4.1) with $u_m \leq u \leq u_M$ . Suppose that for $u_m \leq s \leq u_M$ the following assumptions are satisfied

(i) $g > 0$ , $(1/g)'' \leq 0$ ,

(ii) $\dfrac{(c - 2)}{(c + 1)} f' g \geq f g'$ .

Then the function

$$P = g(u)\ |grad\ u|^2 + c \int_0^u f(s)\ g(s)\ ds\ , \qquad c \leq 2/n$$

assumes its maximum on $\partial\Omega$.

PROOF :

We have taken $h$ so that $h' = c\ f\ g$. By substituting into (3.2) and (3.3) of Section 3, we get

$$P_k = g' |grad\ u|^2 u_k + 2\ g\ u_{ik}\ u_i + c\ f\ g\ u_k$$

and

$$\Delta P = |grad\ u|^4 \left[g'' - 2\frac{g'^2}{g}\right]$$

$$+ |grad\ u|^2 \left[c f' g + c f g' - f g' - 2 f' g - 2c f g'\right]$$

$$+ 2 g u_{ik} u_{ik} - c f^2 g + 2\frac{g'}{g} u_k P_k .$$

Exploiting Lemma 2.1 of Section 2 for the n - dimensional case we get

$$2 g u_{ik} u_{ik} \geq \frac{2}{n} g (\Delta u)^2 = \frac{2}{n} g f^2$$

Thus

$$\Delta P - 2\frac{g'}{g} u_k P_k \geq |grad\ u|^4 \left[g'' - 2\frac{g'^2}{g}\right]$$

$$+ |grad\ u|^2\left[(c - 2) f' g - (c + 1) f g'\right]$$

$$+ \left(\frac{2}{n} - c\right) g f^2 . \qquad \ldots\ldots(4.4)$$

Now, assumptions (i) and (ii) state that the coefficients of $|grad\ u|^4$ and $|grad\ u|^2$ in (4.4) are nonnegative, for $c \leq 2/n$ , noting that :

$$g'' - 2\left(\frac{g'^2}{g}\right) = -\left(\frac{1}{g^2}\right)\left(\frac{1}{g}\right)'' .$$

Hence P satisfies

$$\Delta P - 2 (\log g)' u_k P_k \geq 0 \quad \text{in } \Omega,$$

and the result follows.□

<u>REMARK 4.3</u>

Sperb[21] only considers the case $c = 2/n$ .

For $c < 2/n$ a different argument yields the following result which appears to be *new* .

THEOREM 4.4

Let $u$ be a sufficiently smooth solution of (4.1) with $u_m \leq u \leq u_M$ . If for $u_m \leq u \leq u_M$ the following assumptions are satisfied

(i) $$g'' - \left[\frac{3n-4}{2n-2}\right] \frac{g'^2}{g} \geq 0$$

(ii) $$(c-2)\, g\, f' + \left[\frac{c-2}{n-1} - 1\right] g' f \geq 0 \ , \quad c < 2/n$$

then the function

$$P(x) = g(u)\ |grad\ u|^2 + c \int_0^u f(s)\ g(s)\ ds$$

attains its maximum on $\partial\Omega$ .

PROOF :

By Theorem 3.3 of Section 3, either the maximum of P occurs on the boundary or at a critical point of $u$ .

At an interior critical point of $u$ , from (3.3) of Section 3,

$$\Delta P = 2\ g\ u_{ij}\ u_{ij} - c\ g\ f^2 \ .$$

Using Lemma 2.1 , Section 2 , we find that

$$\Delta P \geq ((2/n) - c)\ f^2\ g > 0 \ , \text{ since } \ c < 2/n \ ,$$

impossible . □

REMARK 4.4

A similar result to Corollary 4.2 holds .

Also for $c < 0$ another variant is possible .

THEOREM 4.5

Let $u$ be a sufficiently smooth solution of (4.1) . If, for $u_m \le u \le u_M$ the following assumptions are satisfied

(i) $g'' - \dfrac{3}{2}\dfrac{g'^2}{g} \ge 0$ ,

(ii) $(c-2)\, f'\, g - f\, g' \ge 0$ , $c < 0$ ,

then the function

$$P(x) = g(u)\ |grad\ u|^2 + c\int_0^u f(s)\ g(s)\ ds$$

takes its maximum on $\partial\Omega$ .

PROOF :

By Lemma 3.2 , Section 3, either the maximum of P occurs on $\partial\Omega$ or at a critical point of $u$ .

At an interior critical point of $u$ , from (3.3) of Section 3 ,

$$\Delta\ P = 2\ g\ u_{ij}\ u_{ij} - c\ f^2\ g\ .$$

Using Lemma 2.1 of Section 2, we find that

$$\Delta\ P \ge ((2/n) - c)\ f^2\ g > 0\ ,\quad \text{since } c < 0\ ,$$

impossible .□

REMARK 4.5

A similar result to Corollary 4.2, again, holds .

EXAMPLE 4.2 : (Sperb [21])

Take $g(u) = 1/(\alpha\ u + \beta)$, $\alpha > 0$ ,$\beta > 0$. Then $(1/g)'' = 0$. Assumption (ii) of Theorem 4.3 for $c = 2/n$ requires that for $u_m \le u \le u_M$ we have :

$$\alpha\, u + \beta > 0 \quad \text{and} \quad f(u) \leq C\,(\alpha\, u + \beta)^{\frac{n+2}{2n-2}} \quad , \text{ where}$$

$$C = \frac{f(u_m)}{(\alpha u_m + \beta)^{\frac{n+2}{2n-2}}} \quad .$$

REMARK 4.6 :

In Sperb [21] , the inequality for $f(u)$ in the example above seems to be incorrect.

SPECIAL CASE OF THEOREM 4.3

With somewhat more restrictive assumptions on $f(u)$ it is possible to obtain upper bounds for $|grad\ u|^2$ as we shall see later. Choosing $g(u) = 1$, the following demonstrates our aim.

COROLLARY 4.6

Let $u \in C^3(\bar{\Omega})$ satisfy equation (4.1). If $f'(u) \leq 0$ in $\Omega$, then the function

$$P(x) = |grad\ u|^2 + c \int_0^u f(s) \quad ds$$

where $c \leq 2/n$ , attains its maximum on $\partial\Omega$.

## SECTION 5

## THE MAXIMUM OF P AT A POINT WHERE *grad* $u = 0$

We study the maximum of P at a critical point of $u$ under some conditions on $\partial\Omega$. Recall that we choose our functions $g(u)$ and $h(u)$ such that the corresponding function $P(x)$ assumes its maximum either on $\partial\Omega$ or at a point where *grad* $u = 0$.

To achieve our goal, mentioned above, we must select the functions $g(u)$ and $h(u)$ with appropriate conditions such that the normal derivative $\partial P/\partial\nu$, where $\nu$ is *an outward normal* to $\partial\Omega$, is *nonpositive* at a point on $\partial\Omega$ which contradicts the strong maximum principle. In such a case P must take its maximum at a point where *grad* $u = 0$.

We shall be concerned with positive solutions of the elliptic equation

$$\Delta u + f(u) = 0, \quad \text{in } \Omega \tag{5.1}$$

where $f \in C^1$ is positive in $\Omega$. Also we shall confine ourself to convex domains for which the *mean curvature* K is nonnegative at each point of $\partial\Omega$. We shall denote by $K_0$ the nonnegative lower bound of K ; often we shall assume $K_0 > 0$. In addition to the convexity of $\partial\Omega$ we require $\partial\Omega$ to be $C^2$. we shall use the symbol $\tau$ defined as

$$\tau = \max_{\partial\Omega} |grad\ u| . \tag{5.2}$$

Now we shall seek conditions so that the maximum of the function P defined by

$$P(x) = g(u)\ |grad\ u|^2 + h(u)$$

cannot occur on $\partial\Omega$.

To accomplish this, we assume that u satisfies

$$\Delta u + f(u) = 0 \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \partial\Omega$$

which is known as Dirichlet Problem (D P).

LEMMA 5.1

Let $u(x)$ be a sufficiently smooth solution of (5.1) vanishing on the boundary of $\Omega$. The normal derivative of the function

$$P(x) = g(u)\ |grad\ u|^2 + h(u)$$

can be represented at a point $Q \in \partial\Omega$ by the identity

$$\frac{\partial P}{\partial \nu} = -\ |grad\ u| \Big[ g'(0)\ |grad\ u|^2 - 2\ f(0)\ g(0) + 2\ K(n-1)\ g(0)\ |grad\ u| + h'(0) \Big] \quad \ldots\ldots..(5.3)$$

where $\nu$ is an outward normal to $\partial\Omega$ .

PROOF :

The boundary $\partial\Omega$ can be represented locally by the identity

$$x_n = \psi(x_1,\ x_2,\ \ldots\ ,\ x_{n-1})\ ,\quad \psi \in C^2$$

where $grad\ \psi = 0$ at $Q \in \partial\Omega$.

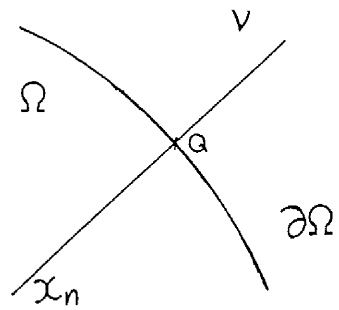


FIGURE [5.1]

By a rotation of coordinates, we assume that the $x_n$ - coordinate axis lies in the direction of the *inner* normal at Q. The outward normal $\underline{\nu}(x)$ is given by

$$\underline{\nu}(x) = \frac{\left(\frac{\partial\psi}{\partial x_i}, -1\right)}{(1 + |grad\ \psi|^2)^{\frac{1}{2}}} . \qquad \ldots\ldots\ldots(5.4)$$

The condition $u = 0$ on $\partial\Omega$ can be expressed as a twice differentiable identity

$$u(x_1, x_2, \ldots, x_{n-1}, \psi) \equiv 0 \qquad \text{on } \partial\Omega. \quad \ldots\ldots(5.5)$$

So

$$u_i + u_n \psi_i = 0 \qquad \text{on } \partial\Omega, \quad i = 1,\ldots,n-1 .$$

Then

$$u_{ij} + u_{nj} \psi_i + u_n \psi_{ij} = 0 \qquad \text{on } \partial\Omega , \quad j = 1,\ldots, n .$$

$\therefore$ At Q

$$u_{ii} = - u_n \psi_{ii} \qquad \textit{since} \quad grad\ \psi = 0 \quad \text{at Q}$$

and

$$\Delta u = u_{nn} - u_n \psi_{ii} . \qquad \ldots\ldots\ldots(5.6)$$

Note that $grad\ u$ points into $\Omega$, so

$$|grad\ u| = \frac{\partial u}{\partial x_n} \quad \text{(at Q)} \quad \text{and} \quad |grad\ u| = - \frac{\partial u}{\partial \nu} .$$

Therefore, from (5.6) we have

$$\Delta u = u_{nn} + \frac{\partial u}{\partial \nu} (n - 1)K$$

where K is the mean curvature ; see Section 2. So

$$u_{nn} = - f - \frac{\partial u}{\partial \nu} (n - 1) K \qquad \text{(at Q)} \qquad \ldots\ldots(5.7)$$

Now

$$\frac{\partial P}{\partial x_i} = g' u_i |grad\ u|^2 + 2 g u_{ij} u_j + h' u_i .$$

So

$$\frac{\partial P}{\partial \nu} = P_i \, \nu_i = g' \, |grad\ u|^2 \, \frac{\partial u}{\partial \nu} + h' \, \frac{\partial u}{\partial \nu} + 2\, g\, u_{ij}\, u_j\, \nu_i$$

$$= g' \, |grad\ u|^2 \, \frac{\partial u}{\partial \nu} + h' \, \frac{\partial u}{\partial \nu} + 2\, g\, u_{nn} \, \frac{\partial u}{\partial \nu} \qquad \text{(at Q)}$$

since at Q , $u_j = 0$ for $j \neq n$ and $\nu_i = 0$ for $i \neq n$.

From (5.7) we get (at Q)

$$\frac{\partial P}{\partial \nu} = g' \, |grad\ u|^2 \, \frac{\partial u}{\partial \nu} + h' \, \frac{\partial u}{\partial \nu} + 2\, g \left[ - f - \frac{\partial u}{\partial \nu} (n-1)K \right] \frac{\partial u}{\partial \nu}$$

$$= - |grad\ u| \left[ g'(0) \, |grad\ u|^2 - 2\, g(0)\, f(0) \right.$$

$$\left. + \, 2\, g(0)\, (n-1)K \, |grad\ u| + h'(0) \right] ,$$

since $u = 0$ on $\partial\Omega$. □

REMARK 5.1

This direct proof is not the one given by Sperb [21] who uses some tensor analysis .

Our work is based on Sperb [21], Payne, Sperb and Stakgold [12], Stakgold and Payne [22], Schaefer and Sperb [16], Sperb [20] and Payne [8]. We discuss the Two - dimensional and the N - dimensional cases separately.

THE TWO - DIMENSIONAL CASE

In (D P), if by appropriate choice of the functions $f(u)$ and $g(u)$ on $\partial\Omega$ , the term between braces in the right side of (5.3) becomes nonnegative, for $n = 2$, we get

$$\frac{\partial P}{\partial \nu} \leq 0 \qquad \text{at } Q \in \partial\Omega.$$

We know by the maximum principle that if P has a maximum at Q on $\partial\Omega$, then $\partial P/\partial\nu > 0$ at Q unless P is a constant in $\Omega$.

Consequently we arrive at the following result

THEOREM 5.2

Let $u \in C^3(\Omega)$ be a solution of (5.1) in a convex plane domain $\Omega$ with $u = 0$ on $\partial\Omega$ and $u_m \leq u \leq u_M$. Suppose that for this range of $u$, we have :

(i) $(\log g(u))'' \geq 0$, $c \geq 2$ and $(c - 2)f'g + (c - 3)fg' \geq 0$, $g(u) > 0$,

(ii) $g'(0)\,\tau + 2\,K_0\,g(0) > 0$, $K \geq K_0 > 0$.

Then the function

$$P(x) = g(u)\ |grad\ u|^2 + c \int_0^u f(s)\ g(s)\ ds \quad \ldots(5.8)$$

assumes its maximum where $grad\ u = 0$.

PROOF

From inequality (3.12) of Section 3 we have, for $h' = c\ f\ g$

$$\Delta\ P + \frac{2(c - 1)\ f}{|grad\ u|^2}\ u_k\ P_k \geq |grad\ u|^4 \left[g'' - \frac{g'^2}{g}\right]$$

$$+ |grad\ u|^2[(c - 2)\ f'g + (c - 3)\ fg')]$$

$$+ (c - 1)(c - 2)\ g\ f^2 . \quad \ldots\ldots.(5.9)$$

We know that P attains its maximum either at a point where $grad\ u = 0$ or at an interior point where $grad\ u \neq 0$ or somewhere on $\partial\Omega$ where $grad\ u \neq 0$.

Let $\Omega'$ be the subdomain of $\Omega$ defined by :

$$\Omega' = \{x \in \Omega : grad\ u \neq 0\} .$$

If P attains its maximum at Q in $\Omega'$ then by the maximum principle, $P \equiv$ constant in $\Omega'$. Therefore P is constant on $\bar{\Omega}'$

and also attains its maximum at a point on the boundary of $\Omega'$ where *grad* $u = 0$ .

Suppose $P$ attains its maximum at $Q_1 \in \partial\Omega'$ where *grad* $u(Q_1) \neq 0$ . Then $Q_1 \in \partial\Omega$ and by the maximum principle either $P \equiv$ constant or $\partial P/\partial\nu\ (Q_1) > 0$ , where $\nu$ is the outward normal at $Q_1$ .

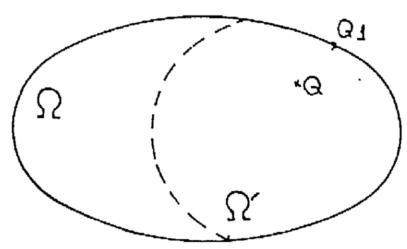


FIGURE [5.2]

If $P \equiv$ constant on $\Omega'$ , as before $P$ attains its maximum where *grad* $u = 0$. Suppose that $P \not\equiv$ constant on $\Omega'$ . Now, for $h' = c\,f\,g$ and $n = 2$ equation (5.3) can be written in the form :

$$\frac{\partial P}{\partial\nu} = -\ |grad\ u|\ \left[2\ g(0)\,|grad\ u|\ K + g'(0)\,|grad\ u|^2 + (c-2)\ f(0)\ g(0)\right] .$$

We note that if $P$ takes its maximum on $\partial\Omega$ it must be where

$$|grad\ u| = \tau := \max_{\partial\Omega}\ |grad\ u| .$$

Therefore, at $Q_1$ we get

$$\frac{\partial P}{\partial \nu} \leq - \ \tau^2 \left[ 2\ g(0)\ K_0 + g'(0)\ \tau \ \right],$$

since $(c - 2)\ f(0)\ g(0) \geq 0$ . By assumption (ii) then

$$\frac{\partial P}{\partial \nu} \leq \ 0 \qquad \text{at } Q_1 \in \partial\Omega,$$

a contradiction. Hence P attains its maximum at a critical point where $grad\ u = 0$.□

REMARK 5.2

Sperb [21] only gives the case $c = 2$ when the conditions are independent of $f$ .

REMARK 5.3

If $c \leq 1$ , the function $P(x)$ as defined in (5.8) is unlikely to assume its maximum where $grad\ u = 0$ , but to take its maximum on the boundary of $\Omega$ ; *c.f.* Theorem 4.2 .

EXAMPLE 5.1

Let $u(x)$ be a sufficiently smooth solution of Poisson equation

$$\Delta\ u + 1 = 0 \quad \text{in } \Omega$$

and

$$u = 0 \quad \text{on } \partial\Omega$$

with $\Omega$ as above .

We find that the function $P(x)$ , with $g(u) \equiv 1$ , defined as

$$P(x) = |grad\ u|^2 + c\ u \quad , \quad c \leq 1$$

takes its maximum on $\partial\Omega$ , by Theorem 4.2 .

In Payne [10], the author states that for $c < 1$, there is no region $\Omega$ on which $P(x) \equiv$ constant .

REMARKS 5.4

(I) For a convex $\Omega$, Sperb [20] shows that for a class of functions $f(u)$ the convexity of $\Omega$ implies that the solution of (5.1) with zero - boundary condition has only one critical point in $\Omega$. If $f(u) \geq 0$ and $u \geq 0$ then $u$ has exactly one maximum in $\Omega$.

(II) Let $g(u) = 1$, $h' = 2f$. The function

$$P = |grad\ u|^2 + 2 \int_0^u f(s)\ ds,$$

where $u = 0$ on $\partial\Omega$ and $u$ satisfies (5.1), takes its maximum at a point where $grad\ u = 0$ if $\Omega$ is convex. This was first found by Stakgold and Payne [22], and it marked the beginning of a series of papers that were concerned with various generalizations and applications of maximum principles for such a function associated with the solution of some boundary value problems.

EXAMPLE 5.2

Let $\Omega$ be a simply connected cross section of a cylindrical bar that is twisted by terminal couples. If the angle of twist per unit length is sufficiently small, one is led to the Saint Venant torsion problem. It can be formulated mathematically as follows :

We seek a solution $u(x)$ of

$$\Delta u = -2 \qquad \text{in } \Omega \subset \mathbb{R}^2 ,$$

$$u = 0 \qquad \text{on } \partial\Omega.$$

The components of the resulting stress are then given by :

$$\tau_1 = \mu \theta \frac{\partial u}{\partial y} \quad \text{and} \quad \tau_2 = -\mu \theta \frac{\partial u}{\partial x}$$

where $\mu$ is the shear modulus and $\theta$ is the angle of twist per unit length. The function $u(x)$ is called the stress function. The magnitude $\tau$ of the *shearing stress* is given by :

$$\tau = \mu \theta \, |grad\ u| \ ,$$

where the *torsional regidity* of $\Omega$ defined as :

$$S := \int_{\Omega} |grad\ u|^2 \, ds \ ,$$

that is the Dirichlet integral of $u$ . For more details one is referred to Payne [8] , Weinberger [24] and Sokolnikoff [19].

In the Saint Venant torsion problem the assumptions (i) and (ii) of Theorem 5.2 are satisfied with

$$g(u) = 1 \ , \quad h(u) = 4\, u \ .$$

We shall come back to the torsion problem in Chapter (VI), where we seek bounds for the maximum stress $\tau$ .

REMARK 5.5

There is some thing interesting about Payne [8], that is the author makes use of the maximum principle for elliptic equations to compute upper and lower bounds for the maximum stress $\tau$ in the Saint Venant torsion problem in terms of geometric properties of the cross section of the beam. That is the cross section $\Omega$ is assumed to be a bounded two - dimensional simply connected region.

These bounds are claimed to be better than those obtained by making use of "sub" and "super" solutions, which are often too crude to be of practical value.

EXAMPLE 5.3

The assumptions (i) and (ii) of Theorem 5.2 are satisfied with

$$g(u) = \frac{1}{(u+\alpha)^2}, \quad h(u) = 2\int_0^u \frac{f(s)}{(s+\alpha)^2}\, ds,$$

in the nonlinear Dirichlet Problem (Payne, Sperb and Stakgold [12]), where $\alpha \geq \tau / K_0$ .

We may also take (Schaefer and Sperb [16])

$$g(u) = e^{-\beta u}, \quad h'(u) = 2\, e^{-\beta u} f(u),$$

where $\beta = 2K_0/\tau$ .

THE N - DIMENSIONAL CASE

Let us now consider a domain in n - dimensions with $n \geq 2$. The following is an extension to the result in the Two - dimensional case to the n - dimensional case.

From Lemma 5.1, we recall that at $Q \in \partial\Omega$ :

$$\frac{\partial P}{\partial \nu} = -\,|grad\ u|\,\big[2\, g(0)\,(n-1)\, K\, |grad\ u|^2 - 2\, g(0)\, f(0)$$

$$+\,|grad\ u|^2\, g'(0) + h'(0)\,\big] \qquad \text{..........(5.12)}$$

since $u = 0$ at $Q \in \partial\Omega$.

As in the two-dimensional case we shall take P(x) to be

$$P(x) = g(u)\,|grad\ |^2 + c\int_0^u f(s)\, g(s)\, ds$$

Therefore, equation (5.12) can be written in the form :

$$\frac{\partial P}{\partial \nu} = - |grad\ u| \left[\ 2\ g(0)|grad\ u|\ (n-1)\ K + |grad\ u|^2\ g'(0)\right.$$

$$\left. + (c-2)\ f(0)\ g(0)\ \right]. \qquad .........(5.13)$$

THEOREM 5.3

Let $u \in C^3(\Omega)$ be a solution of (5.1) in a convex domain $\Omega$ with zero boundary condition and let $u_m \leq u \leq u_M$ . Suppose that, for this range of $u$, we have

(i) $g(u) > 0,\ \left[g^{-\frac{1}{2}}(u)\right]'' \leq 0,\ c \geq 2$ and $(c-2)f'g - g'f \geq 0$ ,

(ii) $2\ g(0)\ (n-1)\ K_0 + \tau\ g'(0) \geq 0$ , $K \geq K_0 > 0$.

Then the function

$$P(x) = g(u)\ |grad\ u|^2 + c \int_0^u f(s)\ g(s)\ ds$$

assumes its maximum where $grad\ u = 0$.

PROOF :

We proceed as in Theorem 5.2 . For $h' = c\ f\ g$ , inequality (3.17) of Section 3 can be written in the form

$$\Delta P + \frac{1}{|grad\ u|^2}\ \psi_i\ P_i \geq |grad\ u|^4 \left[g'' - \frac{3}{2}\frac{g'^2}{g}\right]$$

$$+ |grad\ u|^2 \left[(c-2)f\ g' - f\ g')\right]$$

$$+ \frac{1}{2}\ c\ (c-2)\ g\ f^2 , \qquad ......(5.14)$$

where $\psi_i = \frac{1}{g}\ (c\ f\ g\ u_i - g'\ u_i\ |grad\ u|^2)$ .

Since $\left[g^{-\frac{1}{2}}\ \right]'' = \frac{1}{2}\ g^{-\frac{3}{2}} \left[\frac{3}{2}\frac{g'^2}{g} - g''\right]$, the coefficients of $|grad\ u|^4$ and $|grad\ u|^2$ in the right side of (5.14) are nonnegative by assumption (i). We know that P takes its maximum either at a point where $grad\ u = 0$ or at an interior point where $grad\ u \neq 0$ or somewhere on the boundary where $grad\ u \neq 0$.

If (i) is satisfied, then by the maximum principle P can not attain its maximum at an interior point where $grad\ u \neq 0$. Suppose that P attains its maximum at $Q \in \partial\Omega$ where $grad\ u \neq 0$, and let $\Omega'$ be a subdomain of $\Omega$ defined by :

$$\Omega' = \{x \in \Omega : grad\ u(x) \neq 0\}.$$

So there exists $Q_1 \in \partial\Omega'$ such that $grad\ u(Q_1) = 0$. On $\Omega'$ P satisfies (5.14), so by the maximum principle either

$$P \equiv \text{constant} \quad \text{or} \quad \partial P/\partial\nu\ (Q\ ) > 0\ .$$

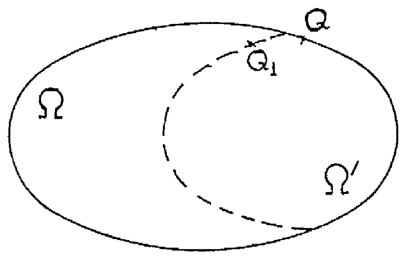


FIGURE [5.3]

If $P \equiv$ constant on $\Omega'$, then $P(Q) = P(Q_1)$ and therefore P attains its maximum where $grad\ u = 0$. Suppose that $P \not\equiv$ constant on $\Omega'$. From (5.13) we get at Q :

$$\frac{\partial P}{\partial \nu} \leq - |grad\, u|^2 \, [\, 2\, g(0)\, (n-1)\, K_0 + \tau\, g'(0)\,]\,,$$

......(5.15)

since $(c - 2)\, f(0)\, g(0) \geq 0$ (for $c \geq 2$). By assumption (ii) then we have

$$\frac{\partial P}{\partial \nu} \leq 0 \quad \text{at } Q \in \partial\Omega,$$

contradiction with the strong maximum principle. We deduce that $P(x)$ assumes its maximum value at a critical point of $u$ where $u$ is a maximum.□

REMARKS 5.6

(I) Again Sperb [21] only gives the case $c = 2$ when the hypotheses do not depend on $f$.

(II) The function

$$P(x) = |grad\, u|^2 + 2 \int_0^u f(s)\, ds$$

where $g(u) = 1$, $h' = 2f$, satisfies the assumptions of Theorem 5.3. This proves the fact noted in (II) of Remarks 5.4.

COROLLARY 5.4 (Payne, Sperb and Stakgold [12])

Let $u(x)$ be a positive solution of (5.1) vanishing on $\partial\Omega$. For $0 \leq \alpha \leq 2$ and $\beta \geq \dfrac{\alpha\, \tau}{2(n-1)\, K_0}$, where $K_0 > 0$ is the lower bound of the mean curvature $K$ of $\partial\Omega$, the function

$$P(x) = \frac{|grad\, u|^2}{(u+\beta)^\alpha} + 2 \int_0^u \frac{f(s)}{(s+\beta)^\alpha}\, ds \qquad .....(5.16)$$

assumes its maximum value where $grad\, u = 0$.

PROOF :

For $0 \leq \alpha \leq 2$ , $g(u) = (u+\beta)^{-\alpha}$ satisfies (i) of Theorem 5.3 and (ii) is satisfied by the choice of $\beta$ .□

REMARK 5.7

If $\alpha = 0$, in (5.16) then we are left with the function

$$P(x) = |grad\ u|^2 + 2\int_0^u f(s)\ ds$$

which satisfies the required result.

P(x) as defined in (5.16) is claimed to be more powerful than that used by Stakgold and Payne [22], in particular to get bounds to the gradient of u via the maximum principle (see Payne, Sperb and Stakgold [12]).

NEUMANN BOUNDARY CONDITION

We study the maximum of P where u satisfies (5.1) and

$$\frac{\partial u}{\partial \nu} = 0 \qquad on \qquad \partial\Omega,$$

which is known as Neumann boundary condition.

LEMMA 5.5

Let $u(x)$ be a sufficiently smooth solution of (5.1), and let P(x) be defined as

$$P(x) = g(u)\ |grad\ u|^2 + h(u)\ .$$

The normal derivative of P , $\partial P/\partial\nu$ , can be represented at a point $Q \in \partial\Omega$ by the identity

$$\frac{\partial P}{\partial \nu} = -2\ g \sum_{i=1}^{n-1} k_i\ u_i^2 \qquad \text{.......(5.17)}$$

provided that $\partial u/\partial\nu = 0$ at $Q \in \partial\Omega$, $\nu$ is the outward normal at Q and $k_i$ are the principal curvatures of $\partial\Omega$ at Q.

PROOF :

It is possible to represent the boundary $\partial\Omega$ locally by

$$x_n = \psi(x_1, \ldots, x_{n-1}) \ , \quad \psi \in C^2$$

where $grad\ \psi = 0$ at $Q \in \partial\Omega$. We choose the $x_n$ - coordinate axis to be inner normal and the $x_1, \ldots, x_{n-1}$ axes to be along the principal directions corresponding to the principal curvatures $k_1, \ldots, k_{n-1}$ .

The outward normal at x is given by

$$\underline{\nu}(x) = \frac{\left(\dfrac{\partial\psi}{\partial x_i} \ , \ -1\right)}{(1 + |grad\ \psi|^2)^{\frac{1}{2}}} \quad , \ i = 1,\ 2,\ \ldots,\ n-1,$$

and the components are

$$\nu_i(x) = \frac{\partial\psi/\partial x_i}{(1 + |grad\ \psi|^2)^{\frac{1}{2}}} \quad , \ i = 1, \ldots, n-1 \ ,$$

and

$$\nu_n(x) = \frac{-1}{(1 + |grad\ \psi|^2)^{\frac{1}{2}}} \ .$$

Now

$$\frac{\partial u}{\partial \nu} = grad\ u(x) \ . \ \nu(x) = \frac{\left[- \dfrac{\partial u}{\partial x_n} + \dfrac{\partial u}{\partial x_i} \dfrac{\partial \psi}{\partial x_i}\right]}{(1 + |grad\ \psi|^2)^{\frac{1}{2}}} \ .$$

On $\partial\Omega$, where $\partial u/\partial\nu = 0$ , we get :

$$u_n(x', \psi) = u_i(x', \psi)\ \psi_i \ , \quad (x' = x_1, x_2, \ldots, x_{n-1}),$$

$(i = 1,2,\ldots,n-1)$ .

Note that this gives $u_n = 0$ at Q (as it must since $u_n = -\partial u/\partial\nu$ clearly) .

Differentiating with respect to $x_j$ , $j = 1,2,\ldots,n-1$ , gives

$$u_{nj}(x', \psi) + u_{nn}(x', \psi)\ \psi_j = [u_{ij}(x', \psi) + u_{in}(x', \psi)\psi_j]\ \psi_i$$
$$+ u_i(x', \psi)\ \psi_{ij} \ . \quad \ldots\ldots(5.18)$$

At $Q$, where $grad\,\psi = 0$, we find

$$u_{nj} = u_i\,\psi_{ij} \quad .$$

Now

$$\frac{\partial P}{\partial \nu} = P_i\,\nu_i = 2\,g\,u_{ij}\,u_j\,\nu_i + |grad\,u|^2\,g' \frac{\partial u}{\partial \nu} + h' \frac{\partial u}{\partial \nu}\,,$$

$$(i,j = 1,2,\ldots,n)\ .$$

At $Q$, we get

$$\frac{\partial P}{\partial \nu} = 2\,g\,u_{nj}\,u_j \qquad (j = 1,2,\ldots,n)$$

$$= -\,2\,g\,u_{nn}\,u_n - 2\,g\,u_{nj}\,u_j \qquad (j \neq n)$$

$$= -\,2\,g\,u_i\,\psi_{ij}\,u_j \quad , \textit{since} \quad u_n = 0 \quad \text{at } Q\ .$$

This last is the quadratic form in $(n-1)$ variables $u_i$ relative to the matrix $\psi_{ij}$. Since we have chosen coordinates so that the matrix $\psi_{ij}$ is diagonal, and its eigenvalues are $k_1,\ldots,k_{n-1}$ we get

$$\frac{\partial P}{\partial \nu} = -\,2\,g \sum_{i=1}^{n-1} k_i\,u_i^2 \quad .\square$$

REMARK 5.8

Our proof differs from that of Sperb [21] .

The following theorem illustrates a maximum priciple for the function

$$P(x) = g(u)\,|grad\,u|^2 + c\int_0^u f(s)\,g(s)\,ds\ , \quad c \in \mathbb{R},$$

where $u$ satisfies (5.1) and the Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega$$

with $\nu$ is the outward normal at $Q \in \partial\Omega$.

THEOREM 5.6 :

Let $u(x)$ be a sufficiently smooth solution of (5.1) in a convex domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, with $\partial u/\partial\nu = 0$ on $\partial\Omega$. Suppose that the hypotheses of Lemma 3.2 , Section 3, are satisfied, then the function

$$P(x) = g(u)\ |grad\ |^2 + c \int_0^u f(s)\ g(s)\ ds\ ,$$

where $c \geq 2$ or $c \leq 0$, takes its maximum at a critical point of $u$ .

PROOF :

The proof is obvious (*c.f.* arguments in Lemma 3.2 of Section 3). Only we need to show that

$$\frac{\partial P}{\partial \nu} \leq 0 \quad \text{at } Q \in \partial\Omega\ .$$

Since for a convex domain , the curvatures $k_i$ are nonnegative , then by Lemma 5.5 we get at Q

$$\frac{\partial P}{\partial \nu} = -\ 2\ g \sum_{i=1}^{n-1} k_i\ u_i \quad \leq 0 \quad .$$

Therefore , P cannot assume its maximum on $\partial\Omega$, and the proof is complete .□

REMARK 5.9

Results in Theorem 5.6 *extend* those of Sperb [21] who took $c = 2$.

REMARK 5.10

Take $g(u) = 1$ ; $h' = 2f$ , then we arrive at an analogue to Remark 5.6 , that is : if the function

$$P(x) = |grad\ u|^2 + 2 \int_0^u f(s)\ ds$$

satisfies the hypotheses in Theorem 5.6 , then P takes its maximum where $grad\ u = 0$ .

Using Lemma 5.5 we can also prove the following result given by Payne [9] for n = 2,3 .

PROPOSITION 5.7

Let $v(x)$ be any $C^2(\bar{\Omega})$ function satisfying $\partial v/\partial \nu = 0$ on a strictly convex portion $\Gamma$ of $\partial\Omega$. Then , if $|grad\ v|^2$ attains its maximum on $\Gamma$ , it follows that $v \equiv$ constant .

PROOF

Suppose that $u = |grad\ v|^2$ takes its maximum at a point Q on $\Gamma$ . Then $\partial u/\partial \nu \geq 0$ by elementary calculus lemma. Taking $g = 1$ and $h = 0$ in Lemma 5.5 we have

$$\frac{\partial u}{\partial \nu} = -2 \sum_{i=1}^{n-1} k_i v_i^2 < 0 \quad , \text{ if } v_i \neq 0, \text{ for any } i \ .$$

Therefore we must have

$$v_i = 0 \quad \text{for all } i \ ,$$

i.e.

$$grad\ v = 0 \quad \text{at Q.}$$

Thus $|grad\ v|^2 = 0$ on $\Omega$ and $v \equiv$ constant .□

NOTE : For $n = 2$ , $k_i$ is to be replaced by the ordinary curvature of $\partial\Omega$.

*Chapter* (VI)

# APPLICATIONS

## CHAPTER (VI)

# APPLICATIONS

## SECTION 1

### TORSION PROBLEM

We consider the classical torsion problem, that is

$$\Delta u = -2 \quad \text{in } \Omega \qquad \text{......(1.1)}$$

with zero boundary condition

$$u = 0 \quad \text{on } \partial\Omega, \qquad \text{......(1.2)}$$

where $\Omega$ is a convex *plane domain*.

We are mainly interested in obtaining information about the maximum stress $\tau$ defined by

$$\tau = max \ |grad\ u| \qquad \text{......(1.3)}$$

which is known to occur on the boundary.

Here we wish to employ the results in Theorem 4.1, Section 4 and Theorem 5.2 , Chapter (V) .

First we shall make use of Theorem 4.1 which, for

$$P(x) = |grad\ u|^2 + 2\ u\ , \qquad \text{......(1.4)}$$

states that P takes its maximum on $\partial\Omega$ , at Q say. That is,

$$|grad\ u|^2 + 2\ u \leq \max_{\partial\Omega} \ |grad\ u|^2 = \tau^2 \qquad \text{......(1.5)}$$

where $u$ satisfies (1.1) and (1.2) .

Then we have, by the maximum principle ,

$$\frac{\partial P}{\partial \nu} = \frac{\partial}{\partial \nu} (|grad\ u|^2 + 2\ u) > 0 \quad \text{at Q ,}$$

where $\nu$ denotes the outward normal at Q , unless $P \equiv$ constant.

Exploiting Lemma 5.1 , Chapter (V) , we get :

$$\frac{\partial}{\partial \nu} (|grad\ u|^2 + 2\ u) = -\ |grad\ u| [2K\ |grad\ u| - 2] > 0 \qquad \ldots\ldots (1.6)$$

where K denotes the curvature of $\partial\Omega$ . From (1.6) then one has

$$K\ |grad\ u| < 1 \ , \quad \text{at } Q$$

that is ,

$$K\ \tau < 1 \qquad (\text{at } Q)\ . \qquad \ldots\ldots (1.7)$$

Note that K is the curvature of $\partial\Omega$ at Q where $|grad\ u| = \tau$ .

Now , if $K \geq K_0 > 0$ , we get

$$\tau \leq 1/K_0 \quad . \qquad \ldots\ldots (1.8)$$

<u>REMARK 1.1</u> :

If $\Omega$ is a disk , then the equality sign holds in (1.8), and P(x) becomes a constant . To show this , we proceed as follows :

On the basis of Chapter (III) , $u$ is radially symmetric in our disk , so that $u$ is independent of the angle in $\Omega$ , and then the polar form for the Laplacian of $u$ is just

$$\Delta\ u = u_{rr} + (1/r)\ u_r = -\ 2\ .$$

By elementary calculus , one gets

$$u = \frac{R^2 - r^2}{2} \quad ,$$

$R$ denotes the radius of the disk, since $u(R) = 0$ (by (1.2)). We have then

$$grad\ u = u_r = -\ r$$

and therefore

$$\tau = \max_{\partial\Omega} |grad\ u| = R\ .$$

Thus $\tau = 1/K_0$ , since for a disk $K = 1/R$ .□

Starting with (1.5), one can obtain other information on $\tau$ . We define

$$S := \int_\Omega |grad\ u|^2 \ ds$$

as the torsional rigidity of $\Omega$ . Then , integrating (1.5) over $\Omega$ gives

$$S + 2 \int_\Omega u \ dx \leq \tau^2 A , \qquad \text{......(1.9)}$$

where A denotes the area of $\Omega$ .

Using Green's identity , yields :

$$2 \int_\Omega u \ dx = \int_\Omega |grad\ u|^2 \ dx = S .$$

Thus (1.9) gives

$$\tau^2 \geq \frac{2\ S}{A} . \qquad \text{......(1.10)}$$

A combination of (1.7) and (1.10) then gives

$$K(Q) < \sqrt{A/2S} \qquad \text{......(1.11)}$$

where K is the curvature at a point Q where $|grad\ u| = \tau$ . Also a combination of (1.8) and (1.10) gives

$$S \leq \frac{A}{2\ K_0^2} . \qquad \text{......(1.12)}$$

Finally , evaluating (1.5) at a point where $u$ takes its maximum $u_M$ gives :

$$\tau^2 \geq 2\ u_M . \qquad \text{......(1.13)}$$

We want to find a lower bound for $u_M$. The following application of the maximum principle achieves this aim.

LEMMA 1.1

Let $\Omega_1$, $\Omega_2$ be two domains with $\Omega_1 \subset \Omega_2$ and let $u_i$ satisfy

$$\Delta u_i + f(u_i) = 0 \quad \text{in } \Omega_i \qquad (i = 1, 2),$$

$$u_i = 0 \quad \text{on } \partial\Omega_i \qquad (i = 1, 2),$$

where $f(u) \geq 0$ and $f'(u) \leq 0$ for $u \geq 0$. Then $u_1 \leq u_2$ in $\Omega_1$.

PROOF :

By the maximum principle, $u_i > 0$ in $\Omega_i$ (*unless* $f(0) = 0$).

Let $w = u_1 - u_2$. Then $w \leq 0$ on $\partial\Omega_1$ and

$$\Delta w = f(u_2) - f(u_1)$$

$$= - f'(\xi)\, w \quad ,$$

by the mean value theorem, so

$$\Delta w + f'(\xi)\, w = 0 \quad \text{in } \Omega_1 .$$

If $w$ has a positive maximum in $\Omega_1$, by Theorem 2.6, Chapter (II), $w$ would be identically constant. Therefore

$$w \leq 0 \quad \text{in } \Omega_1 . \ \square$$

For the torsion problem, $f(u) \equiv 2$ so the above applies. Let $D_\rho$ be the largest disk inside $\Omega$ and let $v$ be the solution on $D_\rho$. Then $v \leq u$ in $D_\rho$. If $v$ attains its maximum at M we have

$$v_{max} = v(M) \leq u(M)$$

so $$u_{max} \geq v_{max} .$$

By the calculation of Remark 1.1 ; $v_{max} = \rho^2/2$ .

Therefore

$$u_M \geq \rho^2/2 \quad . \qquad \ldots\ldots(1.14)$$

A combination of (1.13) and (1.14) gives

$$\tau \geq \rho \ . \qquad \ldots\ldots(1.15)$$

Similarly , if $D_R$ is the smallest disk containing $\Omega$, we have

$$u_M \leq R^2/2 \quad . \qquad \ldots\ldots(1.16)$$

We now take $c = 2$ and using Theorem 5.2, Chapter (V), we see that the function

$$P(x) = |grad\ u|^2 + 4\ u$$

attains its maximum at a point where $grad\ u = 0$ that is ,

$$|grad\ u|^2 + 4\ u \leq 4\ u_M \ . \qquad \ldots\ldots(1.17)$$

On $\partial\Omega$ this gives

$$\tau^2 \leq 4\ u_M \ . \qquad \ldots\ldots(1.18)$$

So a combination of (1.18) and (1.13) yields

$$2\ u_M \leq \tau^2 \leq 4\ u_M \quad , \qquad \ldots\ldots(1.19)$$

which gives an upper and lower bound for $\tau$ .

One can also use the above inequality (1.17) to get an upper bound for $u_M$ in the following way :

Let M be the point where $u = u_M$ , Q a point on $\partial\Omega$ nearest to M and $r$ measure the distance from M along the ray connecting M and Q . Since $-\dfrac{du}{dr} \leq |grad\ u|$ we have :

$$\int_0^{u_M} \frac{du}{2\sqrt{u_M - u}} \leq \int_M^Q dr =: \overline{MQ} \quad .$$

Taking $\overline{MQ} = \rho$ , where $\rho$ denotes the radius of the largest inscribed circle we get

$$u_M \leq \rho^2 \quad . \qquad \ldots\ldots(1.20)$$

This can be a better inequality than $u_M \leq R^2/2$ , for example , when $\Omega$ is an equilateral triangle .

Now , we wish to employ Theorem 5.2 of Chapter (V) which, in an inequality form, states that :

$$g(u)\ |grad\ u|^2 + h(u) \leq h(u)_{max} \qquad (h' = 2\ f\ g). \ \ldots\ldots(1.21)$$

Now, our aim is to choose $g(u)$ and $h(u)$ optimal in the sense that:

(I) (1.21) is as sharp as possible at every point of $\Omega \cup \partial\Omega$ and for any $f(u) > 0$ .

(II) (1.21) becomes an equality in the limit as the domain shrinks to a narrow strip.

Inequality (1.21) , for $h' = 2\ f\ g$ , can be written as

$$g(u)\ |grad\ u|^2 + 2\int_0^u f(s)\ g(s)\ ds \leq 2\int_0^{u_M} f(s)\ g(s)\ ds \qquad \ldots\ldots.(1.22)$$

where $u_M$ is the maximum of $u$ . According to Schaefer and Sperb [16] the optimal choice of P(x) in the sense of (I) and (II) is

$$P(x) = |grad\ u|^2\ e^{-\beta u} + 4\int_0^u e^{-\beta s}\ ds \ , \ \beta = 2K_0/\gamma \qquad \ldots\ldots(1.23)$$

(*where in the torsion problem* $f = 2$), where $K_0 \geq 0$ is the lower bound to the mean curvature $K$ to $\partial\Omega$.

REMARK 1.2

$P(x)$ as defined in (1.23) satisfies the assumptions of Theorem 5.2 of Chapter (V).

Therefore, from (1.22) and (1.23) we get

$$|grad\ u|^2 \leq 4\, e^{\beta u} \int_u^{u_M} e^{-\beta s}\, ds = \frac{4}{\beta}\,(1 - e^{-\beta(u_M - u)})$$

$(\beta = 2K_0/\tau)$. .......(1.24)

Thus, on $\partial\Omega$ where $u = 0$ and $\tau = max\ |grad\ u|$, we get :

$$\tau^2 \leq \frac{4\tau}{2K_0}\,[1 - e^{(-2K_0 u_M/\tau)}]$$

$$\therefore \quad \tau \leq \frac{2}{K_0}\,[1 - e^{(-2K_0 u_M/\tau)}] . \qquad .......(1.25)$$

From (1.25) we have :

$$\tau + \frac{2}{K_0}\, e^{(-2K_0 u_M/\tau)} \leq \frac{2}{K_0} .$$

Setting $x = \dfrac{2\ K_0\ u_M}{\tau}$ , *i.e.* $\tau = \dfrac{2\ K_0\ u_M}{x}$ .......(1.26)

we get

$$2\ K_0\ u_M + \frac{2\ x}{K_0}\ e^{-x} \leq \frac{2\ x}{K_0} .$$

$$\therefore \quad x - x\, e^{-x} \geq K_0^2\ u_M . \qquad ......(1.27)$$

Taking $\bar{x}$ such that

$$\bar{x}\ (1 - e^{-\bar{x}}) = K_0^2\ u_M$$

then

$$x \geq \bar{x}$$

*(since the left side in* (1.27) *is increasing in x)*.

Therefore , from (1.26) , we arrive at

$$\tau \leq \frac{2\, K_0\, u_M}{\bar{x}} \quad . \qquad \ldots\ldots.(1.28)$$

Inequality (1.28) gives upper bounds for the maximum stress $\tau$ when $u_M$ and $K_0$ are given explicitly.

Employing a similar technique as described in Payne [8], we can also obtain an upper bound for $\tau$ . We proceed as follows :

Let M be the point where $u = u_M$ , and $Q$ a point on $\partial\Omega$ nearest to M . Let $\rho$ measure the distance between M and $Q$ and let $r$ be the distance between M and a variable point in $\Omega$. Certainly $-\dfrac{du}{dr} \leq |grad\, u|$ , and therefore from (1.24) we get :

$$-\frac{du}{dr} \leq \frac{2}{\sqrt{\beta}}\,[1 - e^{-\beta(u_M-u)}\,]^{\frac{1}{2}}, \quad \beta = 2K_0/\tau \qquad \ldots\ldots(1.29)$$

Integrating along the ray from $Q$ to M gives :

$$\frac{2}{\sqrt{\beta}} \int_Q^M dr \geq \int_0^{u_M} [1 - e^{-\beta(u_M-u)}\,]^{-\frac{1}{2}}\, du \quad ,$$

$$\int_0^{u_M} [1 - e^{-\beta(u_M-u)}\,]^{-\frac{1}{2}}\, du \leq \frac{2}{\sqrt{\beta}}\,\rho \quad . \qquad \ldots\ldots(1.30)$$

By change of variables, the left hand side of (1.30) can be integrated as follows :

Write $e^{-\beta(u_M-u)} = \sin^2 x$ , so that we have

$$du = \frac{2 \sin x \cos x}{\beta \sin^2 x} \, dx = \frac{2}{\beta} \frac{\cos x}{\sin x} \, dx .$$

Substituting into (1.30), we have :

$$\frac{2}{\beta} \int_A^B \frac{1}{\sqrt{1 - \sin^2 x}} \cdot \frac{\cos x}{\sin x} \, dx \le \frac{2}{\sqrt{\beta}} \rho ,$$

$$\int_A^B \frac{1}{\sin x} \, dx \le \rho \sqrt{\beta} \quad i.e. \quad \int_A^B \csc x \, dx \le \rho \sqrt{\beta} .$$

$$\therefore \log (\csc x - \cot x)\Big|_A^B \le \rho \sqrt{\beta} \quad i.e. \quad \log \left[\frac{1 - \cos x}{\sin x}\right]_A^B \le \rho \sqrt{\beta} . \qquad \ldots\ldots.(1.31)$$

We have :

$$\sin^2 B = 1 , \qquad \sin B = 1 , \quad \cos B = 0$$

and

$$\sin^2 A = e^{-\beta u_M} , \quad \cos A = (1 - \sin^2 A)^{\frac{1}{2}} .$$

Then (1.31) becomes :

$$\log \left[\frac{\sin x_0}{\{1 - (1-\sin^2 x_0)^{\frac{1}{2}}\}}\right] \le \rho \sqrt{\beta} , \text{ setting } \sin^2 x_0 = e^{-\beta u_M} .$$

This implies , after some steps :

$$\frac{\sin^2 x_0}{[1 + (1-\sin^2 x_0)^{\frac{1}{2}}]^2} \ge e^{-2\rho\sqrt{\beta}} , \ \beta = 2K_0/\tau . \quad \ldots.(1.32)$$

Now , we let $y = (1 - \sin^2 x_0)^{\frac{1}{2}}$ . Then (1.32) can be written as

$$\frac{1 - y^2}{(1 + y)^2} \geq A \quad (= e^{-2\rho\sqrt{\beta}}) .$$

So

$$\frac{1 - y}{1 + y} \geq A \quad i.e. \quad y \leq \frac{1 - A}{1 + A} = tanh(\rho\sqrt{\beta}) .$$

Using (1.25) we get

$$\tau \leq \frac{2}{K_0} (1 - sin^2 x_0)^2 \leq \frac{2}{K_0} tanh^2(\rho\sqrt{\beta})$$

*i.e.*

$$\tau \leq \frac{2}{K_0} tanh^2(\rho\sqrt{(2K_0/\tau)}) .$$

Let $v = \sqrt{(\tau K_0/2)}$ . Then $v^2 \leq tanh^2(\rho K_0/v)$

and so

$$v \leq tanh(\rho K_0/v) . \qquad \ldots\ldots\ldots(1.33)$$

Since, $v - tanh(\rho K_0/v)$ is an increasing function of $v$ , taking $\bar{v}$ such that $\bar{v} = tanh(\rho K_0/\bar{v})$ , we finally get

$$\tau \leq \frac{2}{K_0} \bar{v}^2 \qquad \ldots\ldots\ldots(1.34)$$

where $\bar{v}$ is the positive solution of

$$\bar{v}\ artanh\ \bar{v} = \rho K_0 . \qquad \ldots\ldots\ldots(1.35)$$

A series expansion in (1.35) leads to the following :

First we arrange the identity in (1.35) to the form :

$$\bar{v}\ (1 + e^{-2z/\bar{v}}) = 1 - e^{-2z/\bar{v}} , \quad z = \rho K_0 ,$$

which can be written as :

$$e^{-2z/\bar{v}} = \frac{1 - \bar{v}}{1 + \bar{v}}$$

*i.e.*

$$\frac{z}{\bar{v}} = \frac{1}{2} \; log \; \left(\frac{1 + \bar{v}}{1 - \bar{v}}\right) \; . \qquad \ldots\ldots\ldots(1.36)$$

The right side in (1.30) is just :

$$\bar{v} + \frac{\bar{v}^3}{3} + \frac{\bar{v}^5}{5} \ldots \quad \text{for} \quad -1 < \bar{v} < 1 \; .$$

Therefore (1.36) gives for $\bar{v}$ small ,

$$\bar{v}\;(\bar{v} + \frac{\bar{v}^3}{3}) \cong z \quad .$$

Set $u = \bar{v}^2$ , then approximately we have

$$u + \frac{u^2}{3} = z \qquad i.e. \; u^2 + 3\,u - 3\,z = 0 \; .$$

Then

$$u = \frac{-\,3 \pm \sqrt{9 + 12\,z}}{2} \qquad , \; z = \rho K_0$$

*i.e.*

$$\bar{v}^2 = \frac{\sqrt{(9 + 12\;\rho K_0)} \;\; - \; 3}{2} \quad .$$

Substituting into (1.34) one gets :

$$\tau \leq \frac{(\sqrt{9 + 12\;\rho K_0} \;\; - \;\; 3)}{K_0} \qquad \ldots\ldots\ldots(1.37)$$

which is an improvement of the result given by Payne [8].

SECTION 2

## A BOUND FOR THE "EFFICIENCY RATIO"

In the steady - state operation of a bare, homogeneous, monoenergetic nuclear reactor, the neutron density $w(x)$ satisfies the boundary value problem

$$\Delta w + \eta w = 0 , \quad x \in \Omega ; \quad w = 0 , \quad x \in \partial\Omega \quad \ldots(2.1)$$

where $\Omega$ is the domain occupied by the reactor , $\partial\Omega$ its boundary, and $\eta$ is a positive parameter. In the linear problem , where $\eta$ does not depend on $w$, $\eta = \lambda_1$ , where $\lambda_1$ is the first (positive) eigenvalue of

$$\Delta u + \lambda u = 0 , \quad x \in \Omega ; \quad u = 0 \quad \text{on } \partial\Omega. \qquad (2.2)$$

In (2.2), $\lambda_1$ is simple and positive with an associated positive eigenfunction $u$ (see Stakgold and Payne [22]).

The Efficiency of the reactor is given by :

$$E := \frac{\int_\Omega u \, dx}{u_M \ A} \quad , \quad u_M = \max_{x\in\Omega} u(x) \ , \quad \ldots..(2.3)$$

where $u$ is the first eigenfunction in (2.2) and $A$ is the area of $\Omega \subset \mathbb{R}^2$ .

We investigate an upper bound for $E$ . To achieve this, we use Theorem 5.2 , Chapter (V) , with P(x) defined by :

$$P(x) = |grad\ u|^2 \ e^{-\beta u} + 2 \lambda_1 \int_0^u e^{-\beta s} . s \, ds \ , \quad \ldots...(2.4)$$

where $\beta = 2K_0/\tau$ ($K_0 \geq 0$ is the lower bound of the mean curvature K), $\tau = \max_{\partial\Omega} |grad\ u|$ and $u > 0$ satisfies (2.2).

The function P assumes its maximum where $grad\ u = 0$. Therefore

$$|grad\ u|^2 \quad e^{-\beta u} + 2\ \lambda_1 \int_0^u e^{-\beta s} .\ s\ ds \leq 2\ \lambda_1 \int_0^{u_M} e^{-\beta s} .\ s\ ds\ .$$

Evaluating on $\partial\Omega$ , where $u = 0$ , one gets

$$|grad\ u|^2 \leq 2\ \lambda_1 \int_0^{u_M} e^{-\beta s} .\ s\ ds \qquad \ldots\ldots(2.5)$$

By integrating the right side of (2.5) by parts one gets :

$$\tau^2 \leq \frac{2\ \lambda_1}{\beta^2} \left[1 - e^{-\beta u_M} (1 + \beta\ u_M)\right] . \qquad \ldots\ldots(2.6)$$

Now, we let

$$x = \beta\ u_M = \frac{2\ K_0\ u_M}{\tau} \quad , \quad \tau = \frac{2\ K_0\ u_M}{x} \qquad \ldots\ldots(2.7)$$

Then, from (2.6), we get

$$\frac{2\ K_0^2}{\lambda_1} \leq 1 - (1 + x)\ e^{-x} \quad ,$$

which , in passing shows that $\lambda_1 \geq 2\ K_0^2$ ,

and hence

$$(1 + x)\ e^{-x} \leq 1 - \frac{2\ K_0^2}{\lambda_1} . \qquad \ldots\ldots(2.8)$$

Since , $(1 + x)\ e^{-x}$ is a decreasing function of $x$,

taking $\bar{x}$ such that $(1 + \bar{x})\ e^{-\bar{x}} = 1 - \frac{2\ K_0^2}{\lambda_1}$ , then $x \geq \bar{x}$ , and

therefore ,

$$\frac{\tau}{u_M} \leq \frac{2\ K_0}{\bar{x}} \qquad \ldots\ldots.(2.9)$$

where $\bar{x}$ is the positive solution of

$$(1+\bar{x})\, e^{-\bar{x}} = 1 - \frac{2\,K_0^2}{\lambda_1} .$$

Using Green's first identity (with $\psi = 1$) :

$$\int_\Omega \psi\, \Delta u\, dx + \int_\Omega \mathit{grad}\, \psi \cdot \mathit{grad}\, u\, dx = \int_{\partial\Omega} \psi \frac{\partial u}{\partial \nu}\, ds ,$$

we get (since $\Delta u = -\lambda_1 u$) :

$$-\lambda_1 \int_\Omega u\, dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu}\, ds .$$

Therefore

$$\lambda_1 \int_\Omega u\, dx = \int_{\partial\Omega} -\frac{\partial u}{\partial \nu}\, ds = \int_{\partial\Omega} |\mathit{grad}\, u|\, ds \leq \tau L$$

where $L$ is the arc length of $\partial\Omega$ .

$$\therefore \int_\Omega u\, dx \leq \frac{\tau L}{\lambda_1} . \qquad \text{.......(2.10)}$$

Substituting into (2.3) one finds :

$$E \leq \frac{\tau L}{\lambda_1\, u_M\, A} ,$$

and using (2.9) , we arrive at :

$$E \leq \frac{2\, K_0\, L}{\lambda_1\, A\, \bar{x}} . \qquad \text{.......(2.11)}$$

For $\Omega$ a disk , the inequality (2.11) gives (Schaefer and Sperb [16]) :

$$E \leq 0.565$$

which is an improvement of a result of Payne and Stakgold [11] who obtained

$$E \leq 2/\pi = 0.6366 .$$

## SECTION 3

### THE FREE MEMBRANE PROBLEM

The following is to give an improved inequality to the one by Payne and Weinberger [13] which gives an upper bound to the first *nonzero* eigenvalue of the *"free membrane problem"* in the *plane*,

$$\Delta u + \mu u = 0 \quad \text{in } \Omega \subset \mathbb{R}^2 , \qquad \ldots\ldots.(3.1)$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega , \qquad \ldots\ldots.(3.2)$$

where $\nu$ is the outward normal to $\partial\Omega$ , $u \in C^2$ and $\Omega$ is assumed to be *convex* .

It is possible to reflect $\Omega$ across a line-segment $\Gamma$ of the boundary $\partial\Omega$ , (see Courant and Hilbert [1]), obtaining a new domain $\Omega'$ , and continue the function $u$ into $\Omega'$ in the following way : If $y'$ is the mirror-image of the point $y$ of $\Omega$ under reflection , let $\bar{u}(y') = u(y)$ when $\partial u/\partial\nu = 0$ on $\Gamma$ . Then $\bar{u}$ is a continuous solution of $\Delta u + \mu u = 0$ in the combined domain $\Omega + \Omega'$ with $C^2$ derivatives .

REMARK 3.1

Under the boundary condition $\partial u/\partial\nu = 0$ , the first eigenvalue of (3.1) is *zero* and the associated eigenfunction is *constant*. The second eigenfunction changes its sign in $\Omega$ since

$$-\mu \int_\Omega u \, dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \, ds = 0 \quad (\textit{by Green's identity}).$$

DEFINITION : NODAL POINTS *and* NODAL LINES

In the case of a string or a rod, the points at which an eigenfunction $u$ vanishes are of practical interest :

these points are called the *"nodal points"* of the associated eigenvibration $u\ e^{i\omega t}$ , where $\omega$ is the frequency of the string or the rod .

In the case of eigenvibrations of a membrane , we consider *"nodal lines"* *i.e.* the curves $u = 0$ . These nodal lines are the curves along which the membrane remains at rest during eigenvibrations.

THEOREM 3.1 (Courant and Hilbert [1], pp. 395)

If several branches of the curve $u = 0$ intersect in the interior of a plane domain in which $u$ is regular , then the set of "nodal lines" which meet at the point of intersection forms an equiangular system of rays .

Now, by the theorem of Courant and Hilbert , it follows that $\Omega$ is divided into two subdomains $\Omega^+$ and $\Omega^-$ such that the second eigenfunction $u > 0$ in $\Omega^+$ and $u < 0$ in $\Omega^-$ . In Payne [7], the author shows that $u$ cannot have a closed nodal line in $\Omega$. On the other hand , if $\Omega$ has two axes of symmetry , the same is true for the corresponding eigenfunction $u$ . In this case the nodal line of $u$ must contain one of the axes .

Now, we consider the following function :

$$P(x) = g(u)\ |grad\ u|^2 + h(u) \quad ,$$

($h' = c\ f\ g$ , and for convenience we take $c = 2$) on a domain $\Omega$ with two axes of symmetry .

In the light of the above , we consider the second eigenfunction $u$ of (3.1) in $\Omega$, which also satisfies (3.2) on $\partial\Omega$.

Take $g(u) \equiv 1$ and $h'(u) = 2\,\mu\,u$ , and therefore, the function

$$P(x) = |grad\ u|^2 + \mu\,u^2 \qquad .......(3.3)$$

attains its maximum at a critical point of $u$ (*c.f.* argument of Theorem 5.6 of Chapter (V)), where $u$ is a maximum (= $u_M$ , say).

Let M be the point where $u = u_M$ and let Q be the centre of symmetry , *i.e.* , the point of intersection of the two axes . Let $r$ measure the distance from M along the ray connecting M and Q. Note that the point M must lie on one of the axes .

Now , since (3.3) takes its maximum where $grad\ u = 0$ , then

$$|grad\ u|^2 + \mu\,u^2 \le \mu\,u_M^2 . \qquad .......(3.4)$$

We proceed as in Section 1 . Certainly $-\dfrac{du}{dr} \le |grad\ u|$ ,

therefore from (3.4) one gets

$$-\frac{du}{dr} \le \sqrt{\mu}\ \sqrt{u_M^2 - u^2} \qquad ......(3.5)$$

and hence

$$\int_0^{u_M} \frac{du}{\sqrt{u_M^2 - u^2}} \le \sqrt{\mu}\ \overline{MQ} \qquad .......(3.6)$$

The best value for $\overline{MQ}$ (Sperb [21]) is $\overline{MQ} = |a|/2$ where $|a|$ is the length of longer axis of $\Omega$. Therefore, from (3.6) we get :

$$\int_0^{u_M} \frac{du}{\sqrt{u_M^2 - u^2}} \le \sqrt{\mu}\ \frac{|a|}{2} . \qquad ......(3.7)$$

Integrating the left side of (3.7) gives

$$\frac{\pi}{2} \leq \sqrt{\mu} \; \frac{|a|}{2} \quad .$$

Hence

$$\mu \geq \frac{\pi^2}{|a|^2} \qquad \ldots\ldots\ldots(3.8)$$

if $\Omega$ is convex and symmetric (Sperb [21]), which is an improvement of the inequality given by Payne and Weinberger [13] who showed by entirely different methods that for a convex plane domain one has

$$\mu \geq \frac{\pi^2}{|b|^2}$$

where $|b|$ is the diameter of $\Omega$, but no symmetry assumption is needed for the validity of their inequality .

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