COSMOLOGICAL SOLUTIONS OF ELEVEN DIMENSIONAL SUPERGRAVITY

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CONTENTS

ABSTRACT

The first part of this thesis introduces, in succeeding chapters:

i) Kaluza-Klein theory, illustrated by a five dimensional example showing how the d=5 Lagrangian can give, up to a multiplicative constant, a d=4 Lagrangian. The equations of motion are derived and a Kasner type solution given for a time dependent version.

ii) Eleven dimensional supergravity. The Lagrangian for the bosonic sector is given together with the field equations derived from it.

iii) Dimensional reduction and spontaneous compactification. This explains how the fields present in a theory with greater than four dimensions may give classical solutions in which the metric describes a product space (the Freund-Rubin mechanism). One factor of the product representing ordinary space-time, the other the extra dimensions. Symmetry aspects are discussed.

iv) Cosmology and inflation. A description of standard hot cosmology is given, it being presumed that at 'late' times the extra dimensions have been frozen out and this description is approximately valid. The standard approach to inflation resulting from a phase transition is given to illustrate how different the d=11 supergravity approach given here is.

v) Kaluza-Klein cosmology, illustrating how spontaneous compactification can be formulated as a time dependent mechanism (dynamical compactification) for the early universe in a gravitational model with perfect fluid matter. Examples of how the scale factors, describing the size of the ordinary and extra dimensions, vary with time are given from my own calculations.

I then go on to show that for particular solutions of d=11 supergravity there exists a 4-form field F which has components on the internal space and that the existence of this F does not require or imply the existence of any Killing spinors. The internal space is not necessarily an Einstein space, an example being Q^{111} . The conditions which F must satisfy are derived.

Combining supergravity and dynamical compactification time dependent (cosmological) equations of the Robertson-Walker type are formulated for several models with different topologies. Suitable forms for F are chosen for each model. The Ricci tensor is in general

non-isotropic in the internal space, but must be diagonal.

The equations are then solved numerically with and without the new F field present. The solutions are studied to select those which give possible inflationary behaviour. It is shown that for the Freund-Rubin ansatz alone it is impossible to get sufficient expansion. A study of how the various components of the new type of F contribute to the energy-momentum tensor follows. This leads to necessary conditions to be imposed on F for inflation. The various models are studied to determine whether these conditions are fulfilled. In certain cases there is a class of solutions for which an arbitrarily large expansion of the ordinary dimensions can be obtained. In these cases analytic asymptotic solutions are given. Even with the necessary conditions satisfied other features of the model, the form of F and the topology of the internal space, may give solutions in which some of the extra dimensions expand.

The arbitrarily large expansion of the ordinary space can solve the horizon problem, but the associated rapid contraction of the extra dimensions is likely to be unphysical. To solve the flatness problem would require the present day density to be much larger than observed. If a 4-index tensor is similarly used to give the energy-momentum tensor in a non-supersymmetric gravity theory in greater than eleven dimensions the inflationary outlook is more hopefull because the contraction of the extra dimensions can be reduced by choosing more extra dimensions.

Next follows a digression to study chiral d=10 supergravity showing that inflationary type solutions with five dimensions expanding exist. In a static case it is possible to get one more dimension to be compact by the presence of a 1-form field. This method has been extended to a dynamical case.

Some discussion and comments are then given. Some technical details appear in appendices together with a note on supersymmetry and orientation of the internal space.

DECLARATION

This thesis has been composed by myself and is a record of work carried out by me in the Department of Natural Philosophy at the University of Glasgow. Part of the work described in chapters VII, VIII and IX was carried out in a collaboration with Professor R.G. Moorhouse. Some of the work has appeared in the following papers:

i) Cosmological Equations in Kaluza-Klein d=11 Supergravity Theory, R.G.Moorhouse and J.Nixon, Phys.Lett.145B(1984)39.

ii) Inflationary Cosmology and 4-Index Tensor Fields, R.G.Moorhouse and J.Nixon, Nucl.Phys.B261(1985)172.

This thesis, nor any part thereof, has not been submitted in any previous application for a degree.

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Chapter I INTRODUCTION

Recently, long after its original conception{1,2}, the Kaluza-Klein idea that gravity and other interactions can be unified in a theory with more than four space-time dimensions has enjoyed a revival. For a review see Salam and Strathdee{3}. There are many examples including the specific one of d=11 supergravity in which the symmetries of the extra dimensions can give the four dimensional effective theory a gauge group SU(3)xSU(2)xU(1) amongst others{4}. In static models the extra dimensions are assumed to be compact and of small enough size, of the order of the Planck length if gravity sets the scale, to be undetectable other than through their symmetry effects. Chodos and Detweiler{18} produced a time dependent model in which one extra dimension contracted with time whilst the ordinary dimensions expanded and this explained the non-observability of the extra dimension. This is the approach I take in considering the cosmology of various d=11 supergravity models. It is possible that by making the scale factors for the extra and ordinary dimensions time dependent these models may give an inflationary cosmology driven by a particular form of the 4-form field F, and hence solve some of the well known cosmological problems. In all the models considered here there is only one time-like dimension, thus ensuring that there are no closed time-like curves.

In chapter II Kaluza-Klein theory is introduced with a d=5 pure gravity theory which gives an effective d=4 theory of gravity coupled to electromagnetism. A simple time dependent treatment of the Chodos and Detweiler type{18} is given. Chapter III introduces supersymmetry and describes how local supersymmetry implies supergravity. The Lagrangian for the bosonic sector of d=11 supergravity is given together with the equations of motion for the two independent fields the vielbein and the 4-form F. The energy-momentum tensor is a quadratic in F.

Kaluza-Klein theory must have a four dimensional interpretation to be physically realistic and in chapter IV the concepts of dimensional reduction and spontaneous compactificatuion are discussed with particular reference to d=11 supergravity. Coset space compactification is introduced and the resulting symmetries described.

Spontaneous compactification is a result of the fields in the theory other than the vielbein, for example the 4-form F. In the bosonic sector of d=11 supergravity various solutions have been found in which the space compactifies to $M_4 \times M_7$ where M_4 is usually anti-de Sitter space and M_7 a compact coset space. The field F serves as a natural mechanism for splitting off four dimensions and gives spontaneous compactification in all the many ground state static solutions of d=11 supergravity - the Freund-Rubin mechanism{6}. Solutions also exist when F has components on M_7 constructed from a Killing spinor{7}.

It is assumed that at 'late' times the universe is well approximated by the four dimensional standard cosmology, the extra dimensions having been frozen out by some unknown process at some stage. The standard model{8} is described in chapter V together with the older phase transition forms of inflation{119} which I hope to show may be unnecessary. A list of some of the remaining cosmological problems is given.

In chapter VI the ideas of Kaluza-Klein and cosmology are brought together to show how spontaneous compactification can be formulated as a time dependent theory yielding dynamical compactification for a gravity model with perfect fluid matter. The Einstein equations are stretched to cover more dimensions and the energy-momentum tensor derived. The split between ordinary and extra dimensions is purely through assumptions about the space-time manifold. Examples of how the scale factors describing the size of the ordinary and extra dimensions vary with time are given from my own calculations. The usual question of why the extra dimensions are so small can now be answered by reversing the question and explaining why the ordinary dimensions are so large.

In chapter VII I return to d=11 supergravity to describe work carried out in collaboration with Professor R.G. Moorhouse. The Ricci tensor calculated for the space $M_4 \times M_7$ (as per the example in appendix A) is equated with that derived from the energy-momentum tensor, in terms of the bosonic fields in the theory, by the Einstein equations. I show that there exist new solutions with the 4-form field F having components on the internal space but not constructed from, or implying the existence of, a Killing spinor. This work was based on the observation by Professor Moorhouse that the field F must be of a particular form to satisfy its own equation of motion. Two static solutions are given - one of which has a non-Einstein space for M_7. The conditions which this new type of F must fulfill are derived. With this new F field equations in which the scale factors of the spaces and the coefficients of the field F are time dependent can be derived. Equations are formulated for several models with a field F suitably chosen for the different topologies of the various models.

Chapter VIII reviews previous time dependent solutions for d=11 supergravity and then proceeds to give my numerical solutions for the models of the previous chapter, the toroidal example being most extensively discussed. The new time dependent F acts to give dynamical compactification. There are several classes of solution, the various components of F contributing quite differently to the energy-momentum tensor and thus playing a different role in the Einstein equations. It is shown that for the Freund-Rubin ansatz alone it is impossible to get sufficient inflation. Study of the effects of the components of F in the equations of motion enable a class of solutions to be selected for which an arbitrarily large expansion of the ordinary dimensions occurs. This class may be suitable for inflation. The necessary condition on F for the solution to be in this class is derived. Even with this condition satisfied other features of the model, the choosing of F and the topology of M_7, can cause some of the directions in M_7 to expand. These features are discussed.

In chapter IX selected solutions are examined in relation to solving the horizon problem. A simple asymptotic analytic solution is given for the solutions which approach a singularity. A heuristic argument, extrapolating the solution to the present day by joining it to the standard model, shows that the expansion of the ordinary dimensions can solve the horizon problem. However the accompanying contraction of M_7 is likely to be unphysical and beset with quantum problems. A look at a higher dimensional theory containing a 4-form F which acts in a similar way causing dynamical compactification shows that the contraction of the extra dimensions can be reduced by having a greater number of extra dimensions. The equations for this model are given in appendix D.

In chapter X I digress to consider the d=10 chiral supergravity, a theory which has become more popular due to its chirality and connection with string theories. I show that a time dependent version can be derived in which five dimensions expand, whereas five contract. The field F, in this case a 5-form, acts in a different way to the F in the d=11 models. I show that the static method of Robb and Taylor{9} of 'compactifying' another dimension to S' can be generalised to the time dependent model, although the equations have yet to be studied in detail.

In the final chapter XI I summarise the results and give some discussion. Some technical details appear in the appendices together with a clarification of the effect of reversal of the orientation of M_7 on supersymmetry.

Most of the notation will be introduced in the text. I mainly consider tangent space indices (ie. 'flat' indices): M,N refering to the whole space-time; a,b to the d=4 space and m,n to the 7-dimensional space. If the time direction is treated separately it is denoted by 0 and the a,b then refer to the three spatial directions. In the models which are not 11-dimensional the obvious generalisations apply. The curvature 2-form is denoted R^{MN} , the Ricci tensor $R^M{}_N$, the Ricci scalar R and the connection 1-form ω^{MN}. If curved space indices occur, for example to label coordinates, the 4-space is covered by μ, ν and the 7-space by α, β. Some symbols have more than one meaning; the context makes it clear which is intended. The summing of repeated indices is implied except on the Ricci tensor.

Chapter II KALUZA-KLEIN THEORY

The aim of Kaluza-Klein theory is to construct a complete physical theory within a single geometrical structure. Newton's First Law for a three dimensional space states that a body has a uniform motion in a straight line unless acted upon by a force. Einstein's theory of general relativity absorbed the gravitational force into the geometry by introducing a curved metric into a 4-dimensional space-time so that Newton's law now states that a body follows a geodesic unless acted on by a non-gravitational force. The idea of Kaluza and Klein was to extend the idea by including the electromagnetic force in the geometry with a 5-dimensional curved metric, thus leaving only non-gravitational and non-electromagnetic forces. Kaluza and Klein suggested that electromagnetism and gravity could be unified by considering a pure gravity theory in 5-dimensions but looking at the theory from a 4-dimensional viewpoint. Both the Lagrangian and equations of motion are symmetric among all space-time components but this symmetry is broken by the solutions.

To include all forces in the geometry the idea has to be generalised to greater numbers of dimensions (D) and non-Abelian symmetries, where D depends on the forces to be included, and the higher dimensional theory need not be pure gravity but can be, for example, Einstein-Yang-Mills{10,11,12} higher derivative gravity {13,14,15} or, of most interest in this thesis, the maximal supergravity theories in 10 or 11 dimensions.

Kaluza-Klein theories contain extra dimensions which are not apparent to us today other than in the symmetry they create. The viewpoint taken here, and I think the only tenable one when considering cosmological theories, is that the extra dimensions are real with physical size. In some theories the extra dimensions are just assumed to form a compact manifold of a size comparable with the Planck length $(\hbar G/c)=1.6\times10^{-33}$ cm, but in time dependent theories it is hoped that this situation can be created dynamically. Some authors{16} have considered quantum effects to explain the size of the compact manifold. Here only classical theories are considered although quantum effects might be invoked to explain how initial conditions might have arisen or other features. However no detailed mechanisms will be given.

In this chapter I will describe a simple d=5 Kaluza-Klein model, and a time dependent solution, to introduce the ideas and concepts. These ideas will be extended in later chapters to cover non-Abelian symmetries and ground states of the form $M_4 \times M_7$.

d=5 Kaluza-Klein Theory

Since we do not "see" the fifth dimension it must somehow be treated differently from the other four in the choice of 5-dimensional metric and connection. Assume that there is no torsion and that the metric and connection are not independent; that is given one the other can be derived from the torsionfree condition. A basis of orthonormal one forms is $\{V^M\} = V^0, V^1, V^2, V^3, V^4$ and the metric tensor is

$$g_5 = \eta_{MN}\, V^M \otimes V^N , \tag{2.1}$$

where η_{MN}=diag(+1,-1,-1,-1,-1). The dual basis of orthonormal vectors is $X_M = X_0, X_1, X_2, X_3, X_4$. It is assumed that there exists a Killing vector field in the fifth direction and this causes the asymmetry splitting off the fifth direction. A choice for the metric, by no means unique, is

$$V^a = v^a(x^b) \quad , \quad V^4 = dy + A(x^a) \quad , \quad a, b = 0,1,2,3, \tag{2.2}$$

where $\{v^a\}$ is an orthonormal basis of 1-forms spanning 4-dimensional space-time, coordinates x^a, and y is a coordinate for the fifth dimension and A is a 1-form on 4-dimensional space-time only (ie. $A = A_a v^a$). The dual vectors X_A are

$$X_a = b_a - A_a \frac{\partial}{\partial y} \quad , \quad X_4 = \frac{\partial}{\partial y} , \tag{2.3}$$

where b_a are the 4-dimensional orthonormal vectors dual to v^a. The Killing vector in the fifth direction is

$$k = \frac{\partial}{\partial y} . \tag{2.4}$$

The 1-form A can be identified with the 1-form potential for a Maxwell 2-form F=dA. The choice of metric (2.2) will yield

i) 4-dimensional Einstein-Yang-Mills equations.

ii) Quantisation of charge if the fifth dimension is compact.

iii) The Lorentz force on a charged particle{17}.

Consider a pure gravity theory in 5-dimensions with the Lagrangian

$$\Lambda_5 = \mathbb{R}_{MN} \wedge *(V^{MN}) , \tag{2.5}$$

where $*$ is the Hodge duality operator, $V^{MN} = V^M \wedge V^N$ and $\mathbb{R}^{MN}$ is the curvature 2-form

$$\mathbb{R}_{MN} = d\omega_{MN} - \omega_M{}^P \wedge \omega_{PN} \tag{2.6}$$

and ω_{MN} the spin-connection. Assuming the torsion-free condition

$$DV^M \equiv dV^M - \omega^M{}_N \wedge V^N = 0 \tag{2.7}$$

variation of the vielbein V^M gives the field equation

$$\mathbb{R}_{MN} \wedge i_P *(V^M \wedge V^N) = \mathbb{R}_{MN} \wedge *(V^{MN}{}_P). \tag{2.8}$$

If I had not assumed (2.6) and V^M and ω^{MN} were independent the field equations would be (2.7) and

$$D(* V^{MN}) = 0 \tag{2.9}$$

which is equivalent to (2.6). Thus it does not matter which way I look at the theory, and this is also true if there is a 'matter' action which depends only on the metric. If, however, fermions are present torsion does exist.

The spin-connection 1-forms can be calculated by the method of appendix A and are

$$\omega_{4a} = \tfrac{1}{2}\, i_a\, dA , \tag{2.10a}$$

$$\omega_{ab} = \tilde{\omega}_{ab} - \tfrac{1}{2}\, V^4 i_a i_b\, dA , \tag{2.10b}$$

where terms with a tilde refer to the 4-dimensional space only. These give with (2.6)

$$\mathbb{R}_{a4} = -\frac{1}{2}\tilde{\mathcal{D}}(i_a F) \quad , \tag{2.12a}$$

$$\mathbb{R}_{ab} = \tilde{\mathbb{R}}_{ab} - \frac{1}{2} F \wedge i_a i_b F + \frac{1}{2} V^4 \wedge d(i_a i_b F) - \frac{1}{4}(i_a F) \wedge (i_b F). \tag{2.12b}$$

And so (2.5) becomes

$$\Lambda_5 = \left(\tilde{\mathbb{R}}_{ab} \wedge \underset{4}{*} V^{ab} - \frac{1}{2} F \wedge \underset{4}{*} F\right) \wedge V^4. \tag{2.13}$$

The 4-dimensional action density can be identified by integrating over the fifth dimension so

$$\Lambda_4 = \tilde{\mathbb{R}}_{ab} \wedge \underset{4}{*} V^{ab} - \frac{1}{2} F \wedge \underset{4}{*} F \tag{2.14}$$

which is obviously the 4-dimensional Einstein-Maxwell Lagrangian.

There are two approaches to finding the 4-dimensional equations of motion. Either take the 5-dimensional field equations and substitute (2.12) or take the 4-dimensional effective theory as given by (2.14) and derive the equations of motion from this.

The above theory has a U(1) gauge invariance which can be illustrated by setting

$$V^4 \to V^{4'} = V^4 + d\psi \tag{2.15}$$

for a scalar function $\psi(x)$.

The equations of motion are

$$d \underset{4}{*} F = 0 \quad , \tag{2.16a}$$

$$F \wedge \underset{4}{*} F = 0, \tag{2.16b}$$

$$\tilde{\mathbb{R}}_{ab} \wedge \underset{4}{*}(V^{ab}{}_c) = \frac{1}{2}\left(F \wedge i_c \underset{4}{*} F - (i_c F) \wedge \underset{4}{*} F\right). \tag{2.16c}$$

A cosmological term, proportional to $*1$, can be added to the action density (2.5) and sources would then appear in the Maxwell-Einstein equations. A Jordan-Thiry field can be introduced by setting

$$V^a = V^a(x^b) \quad , \quad V^4 = \phi(x)(dy + A) \tag{2.17}$$

instead of (2.2).

I have yet to describe the geometry of the 5-dimensional space. Suppose it is locally a product of a 4-dimensional space-time M_4 with a circle S^1. If S^1 has a radius a then the y coordinate has range $0 < y < 2\pi a$. A field $\phi(x,y)$ can be expanded in terms of eigenfunctions of the Killing vector X_4

$$\phi(x,y) = \sum_{m=-\infty}^{m=+\infty} e^{\frac{imy}{a}} \phi_m(x) , \tag{2.18}$$

where $\phi_m(x)$ are complex scalar fields on M_4. It is the choice that the extra dimension is compact which breaks the 5-dimensional general covariance.

The above theory has an extra dimension with the geometry S^1 and so the symmetry is U(1). This is fine for electromagnetism. To generalise this for a theory with a gauge group G consider a (4+D)-dimensional manifold which is locally $M_4 \times M_D$ where M_D is a manifold that admits a metric g_D with G as its group of Killing symmetries. The choice of G still allows some freedom in choosing M_D, especially with the coset spaces considered later.

For any cosmological treatment some, or all, of the fields must depend on time and the reason for our not seeing the extra dimension or dimensions must be some physical process such as their contraction with time although the extra dimensions might be of constant size even if some fields are time dependent. An example of a time dependent solution was given by Chodos and Detweiler{18}. Assuming isotropy in the normal spatial directions the metric (2.1) is

$$g = V^0 \otimes V^0 - V^a \otimes V^a - B^5 \otimes B^5 , \tag{2.19}$$

where $V^0 = dt$, $V^a = rv^a$, $B^5 = sb^5$, where r and s are functions of time. If no other fields are present the Einstein equations are

$$R^0{}_0 = \frac{3}{2}\frac{\ddot{r}}{r} + \frac{1}{2}\frac{\ddot{s}}{s} = 0 \tag{2.20a}$$

$$R^a{}_a = \frac{1}{2}\frac{\ddot{r}}{r} + \frac{k_3}{r^2} + \left(\frac{\dot{r}}{r}\right)^2 + \frac{1}{2}\frac{\dot{r}\dot{s}}{rs} = 0 , \tag{2.20b}$$

$$R^5{}_5 = \frac{1}{2}\frac{\ddot{s}}{s} + \frac{3}{2}\frac{\dot{r}\dot{s}}{rs} = 0 , \tag{2.20c}$$

where the dot represents the time derivative and k_3 is the 3-space curvature, which has the Kasner type solution (if $k_3 = 0$)

$$r = A t^{\alpha} \quad , \tag{2.21a}$$

$$s = B t^{\beta} \quad , \tag{2.21b}$$

where α and β must satisfy the relation

$$3\alpha^2 + \beta^2 = 1 = 3\alpha + \beta . \tag{2.22}$$

Either α or β must be negative and to make cosmological sense I take $\alpha > 0$, $\beta < 0$ so

$$\alpha = \frac{1}{2} \quad , \quad \beta = -\frac{1}{2} \quad . \tag{2.23}$$

Note that r = constant , $s = s_0 t$ is also a solution. The case $k_3 \neq 0$ has been studied by numerical computation and asymptotically approaches the above solution. Exact solutions have also been found when there is a cosmological constant in the d=5 theory{19}.

Chapter III d=11 SUPERGRAVITY

There is little, if any, evidence for supersymmetry in nature and even if it does exist in some form at high energies or in higher dimensions it is very badly broken in any effective four dimensional theory. However there are some good reasons for extending the Lie-Algebra of a theory to include a Majorana spin 1/2 generator Q_α, in the case of general relativity adding the relations

$$\left[Q_\alpha , M^{\mu\nu}\right] = i\left(\sigma^{\mu\nu}Q\right)_\alpha , \tag{3.1a}$$

$$\left[Q_\alpha , P^\mu\right] = 0 , \tag{3.1b}$$

$$\left\{Q_\alpha , \bar{Q}_\beta\right\} = -2\left(\gamma_\mu\right)_{\alpha\beta} P^\mu \tag{3.1c}$$

to those for space-time rotations ($M^{\mu\nu}$) and translations (P^μ) alone{20}. Any linear representation of supersymmetry describes a theory of equal numbers of boson and fermion states. It is hoped that a symmetry relating fermions to bosons will lead to cancellations between the divergences of both and that the resulting field theory will be finite. An example is the N=4 supersymmetric Yang-Mills theory which is probably finite to all orders{21}.

It is further hoped, although not realized, that local supersymmetry will lead to a finite theory of gravity. From (3.1) one can see that the product of two local supersymmetry transformations will lead to a position dependent space-time translation (ie. a general coordinate transformation). So local supersymmetry forces us to a theory of supergravity.

So why N=1 d=11 supergravity?

i) Supergravity is the supersymmetry model closest to Einstein gravity.

ii) It is the largest number of dimensions in which a consistent supergravity theory exists with spins not greater than 2, excluding quasi-Riemannian theories{22}.

iii) Hopefully all the space-time (geometric) symmetries in d>4 will lead to the correct set of space-time and internal symmetries in d=4. The d=11 theory is large enough to contain the SU(3)xSU(2)xU(1)

isometry group from the internal space. Higgs fields arise naturally.
iv) There are mechanisms for compactifying the seven extra dimensions to give an effectively four dimensional theory (see next chapter)-one of which is the maximal N=8 supergravity.

v) The field content and action density is fixed by the supersymmetry requirement to be the vielbein V^M, a three index antisymmetric tensor field A_{MNP} and a Rarita-Schwinger Majorana Spinor ψ_M . All fields are gauge fields. Possibly it is a candidate for a finite theory of gravity but it may be over specified in terms of the field content.

The number of bosonic and fermionic degrees of freedom match if instead of A_{MNP} a six index antisymmetric tensor A_{MNPQRS} appears, but so far attempts to construct a successfull theory this way have failed{23}.

The d=11 supergravity theory is symmetric under the following{24}

a) d=11 general coordinate transformations.

b) Local SO(1,10) Lorentz transformations.

c) N=1 local supersymmetry transformations.

d) Abelian gauge transformations given by

$$\delta A_{MNP} = \partial_M \zeta_{NP} + \partial_N \zeta_{PM} + \partial_P \zeta_{MN} \ .$$

N=1, d=11 supergravity is just one attempt to explain nature and there are still several problems for it to overcome. Despite hopes it is not obviously finite. There is the problem of chirality for solutions of the form M_4x M_7. As yet this theory produces an unacceptably large d=4 cosmological constant. In this thesis I show how supergravity might solve some of these problems in its use for explaining our universe.

Having said that supergravity has equal numbers of fermionic and bosonic degrees of freedom I shall now drop the fermions (ψ_M=0) and consider only the bosonic sector - assuming that for the period of the universe I am interested in only bosons are important. Supersymmetry still exists for the higher dimensional theory. The bosonic action density 11-form is{25,26}

$$\Lambda_{11} = \mathbb{R}_{MN} \wedge (* V^{MN}) - F \wedge * F + \frac{1}{6} F \wedge F \wedge A \ , \tag{3.2}$$

where $\mathbb{R}_{MN}$ is the curvature 2-form, F a 4-form derived, at least locally, from the 3-form potential A, F=dA. The Einstein equations are obtained by variation of V^M and are

$$\mathbb{R}_{MN} \wedge *(V^{MN}{}_P) = F \wedge i_M(* F) - (i_M F) \wedge * F \ . \tag{3.3}$$

The $F\wedge F\wedge A$ term does not contribute to the Einstein eqn.(3.3) because it depends on the volume not the metric. The field equation for F (in analogy to the Maxwell equations and hereafter referred to as the Maxwell equations) is

$$d * F = \frac{1}{4} F \wedge F . \tag{3.4}$$

The coefficient in (3.4) can be changed simply by rescaling A. There is also the identity

$$dF = 0 \tag{3.5}$$

required if there is no global potential A. The source term in (3.3) is fixed by the form F must take to satisfy its own equation of motion.

In calculations often the component version of (3.3) was easier to use and is

$$R^M{}_N - \frac{1}{2} \delta^M{}_N R = 6 F^{MP_1P_2P_3} F_{NP_1P_2P_3} - \frac{3}{4} \delta^M_N F^{P_1P_2P_3P_4} F_{P_1P_2P_3P_4} . \tag{3.6}$$

Chapter IV DIMENSIONAL REDUCTION and SPONTANEOUS COMPACTIFICATION

In the last chapter d=11 supergravity was described as a relatively simple geometrical theory. However, at low energies today space-time appears 4-dimensional, not 11-dimensional, and so any solution of the d=11 field equations with physical meaning must introduce an anisotropy which distinguishes between the ordinary dimensions (with coordinates $x^{\mu}, \mu=0,1,2,3$) and the extra dimensions (with coordinates y^{α}, $\alpha=1,\ldots,7$). The simplest procedure is ordinary dimensional reduction in which all fields and transformation parameters are quite arbitrarily taken to be independent of the y^{α} coordinates, so for a generic field $\phi(x,y)$, ignoring any indices it might carry,

$$\phi(x,y) = \phi(x,0) = \phi(x). \tag{4.1}$$

To obtain an effective 4-dimensional theory the 11-dimensional action density Λ_{11} is integrated with respect to the y coordinates

$$\int_{M_7} \Lambda_{11}(\phi(x,y))\, d\mu_y = V_{M_7}\, \Lambda_{11}(\phi(x)), \tag{4.2}$$

where μ_y is the measure over M_7 and V_{M_7} is the volume of the manifold M_7 (hopefully compact) and must be finite. The theory so obtained has exact gauge invariance ($U(1)^7$ from the geometry T^7) and retains any supersymmetries, for example the simple N=1 supergravity gives the extended N=8 supergravity in 4-dimensions (this being the consequence of a 32-component spinor in 11-dimensions giving eight 2-component spinors in 4-dimensions). Obviously extended supergravity is not the correct physical theory so we are led to more complicated reduction schemes which break some or all of the gauge and/or super-symmetries. Also, to ensure that the extra dimensions are not observable, we have to impose that their size is small.

The most usual generalisation of (4.2) is to expand the d=11 fields in terms of a complete set of functions $Y_n(y)$, the eigenfunctions of some operator on M_7,

$$\phi(x,y) = \sum_n \phi_n(x)\, Y_n(y), \tag{4.3}$$

where the expansion coefficients $\phi_n(x)$ are interpreted as the physical fields in d=4. It is the extra dimension part of an operator which gives masses in d=4 and so the $Y_n(y)$ are determined by the mass operator. If the extra dimensions form a compact manifold the mass spectrum will be discrete, the $m_n=0$ modes being massless. Integration over the y coordinates gives the effective d=4 theory describing a finite number of massless modes and an infinite tower of massive states. In this more general scheme a massless symmetric theory is reduced to a theory with massless and massive states and some or all of the symmetries broken. A simple example of (4.3) is the expansion in chapter II of a scalar field in (2.18).

There is another generalisation of (4.2) considered by Chaichion et al{27} in which the theory is reduced to a hypersurface $Y: x^a(z^i), y^m(z^i)$, where z^i (i=0,1,2,3) are some parameters which take on the role of position coordinates, by minimisation of the action

$$S = \int_Y \Lambda\left(\phi(x^a(z^i), y^m(z^i))\right) d\mu_z \, . \tag{4.4}$$

The claim of ref.{27} is that the reduction yields a theory invariant with respect to non-linear realisations of local transformations of some gauge group G. This opens up possibilities of supersymmetric theories with the numbers of bosonic and fermionic degrees of freedom not equal.

I now go on to describe the process called spontaneous compactification where I shall look for classical ('ground state') solutions of the d=11 field equations where the metric describes, at least locally, a product space $M_4 \times M_7$ for which the metric can be written in the form

$$g = g_{ab}(x)\, dx^a \otimes dx^b + g_{mn}(y)\, dy^m \otimes dy^n , \tag{4.5}$$

where M_4 is the usual space-time (+−−−) and M_7 is a compact manifold with Euclidian signature (−−−−−−−). This gives a compact gauge group, a discrete mass spectrum and prevents tachyons appearing in the reduced theory. Why $M_4 \times M_7$ should be the ground state solution rather than, say, Minkowski 11-space is not yet explained.

In one of the first approaches, Cremmer and Scherk{28}, the required structure was imposed on the space-time (in their case $R_{ab} = 0$ and that M_7 be maximally symmetric) and then a particular set of scalars chosen which complied with this. In ref.{28} it was shown that spontaneous compactification of extra dimensions was possible for arbitrary space-time dimensions and for general gauge groups which have an SO(N) subgroup. Scherk and Schwarz{29} extended this work to consider M_7 as a group space and they consider a set of 'flat' groups which give no d=4 cosmological constant. More recently numerous authors have considered the extra dimensions to be coset spaces.

If maximal symmetry is required for the d=4 space-time M_4 is restricted to being an Einstein space (ie. a space in which the Ricci tensor is proportional to its metric tensor)

$$R_{ab} = C_1 g_{ab} . \tag{4.6}$$

Some examples are given in table(4.1). When I refer to anti-de Sitter space-time I am really refering to the covering space in which the S^1 is unwrapped to R^1 {30}.

Table(4.1)

Examples for M_4.

C1	Space-time	Symmetry	Topology
>0	de Sitter	SO(1,4)	$R^1 \times S^3$
=0	Minkowski	Poincare	R^4
<0	Anti-de Sitter	SO(2,3)	$S^1 \times R^3$ ($R^1 \times R^3$)

It is desirable to have the extra dimensions compact to give a discrete mass spectrum and a compact gauge group. This is often achieved by demanding M_7 to be an Einstein space also

$$R_{mn} = C_2 g_{mn} , \tag{4.7}$$

$C_2>0$ ensures M_7 is compact if it is an Einstein space, the cases $C_2<0$ have no symmetries{31}. I shall show in a later chapter that (4.7) is not necessary, it is sufficient for the Ricci tensor to be diagonal. I shall assume (4.7) holds for the present discussion. All group spaces and coset spaces (excluding S^1 factors) admit Einstein metrics{32}.

In the case of pure gravity (see chapter VI with ρ =0)

$$C_1 = C_2 = 0 \tag{4.8}$$

which forces space-time to be Minkowski 4-space x T^7 which has only Abelian symmetries ($U(1)^7$) or Minkowski x K3 x T^3 with symmetry $U(1)^3$ {33}. With C_2 =0 the theory is consistent with compactification but does not imply it. Adding a cosmological constant (Λ) gives{31}

$$R_{MN} = \frac{2\Lambda}{d-2} g_{MN} \quad , \tag{4.9}$$

which implies either $C_1, C_2 > 0$ or $C_1, C_2 < 0$, neither of which is very satisfactory. A similar result could be derived for the perfect fluid model in chapter VI. To overcome this problem the theory must contain bose matter fields which will induce spontaneous compactification. There are numerous models with the matter fields being scalars{28,121}, Yang-Mills fields{10,12} or antisymmetric tensors{34,35} coupled to gravity. Here I am mainly concerned with d=11 supergravity in which the bose fields are determined by supersymmetry.

In Freund and Rubin{6} it is shown how, by what is now known as the Freund-Rubin mechanism, in a d-dimensional theory containing gravity and a rank (s-1) antisymmetric tensor field compactification occurs preferentially for (d-s) or s space-like dimensions. Ref.{6} looks first at a d-dimensional Einstein-Maxwell theory and then at a gravity theory with an arbitrary (s-1) rank antisymmetric tensor field A; but not the full bosonic supergravity which is a particular example of their mechanism. They found for the ansatz (where F = dA)

$$F_{a\dots b} = \frac{\oint}{s!} \varepsilon_{a\dots b} \tag{4.10}$$

the scalar curvatures of the (d-s) and s spaces to be

$$R_{d-s} = \frac{(s-1)(d-s)}{d-2} \lambda_s \quad , \tag{4.11a}$$

$$R_s = -\frac{s(d-s-1)}{d-2} \lambda_s \quad , \qquad \lambda_s = 8\pi G \oint^2 \frac{\det(g_s)}{|\det(g_s)|} \, . \tag{4.11b}$$

For d > s+1, since R_{d-s} and R_s have opposite signs, either (d-s) or s space-like dimensions compactify. Whichever one does compactify depending on the sign of $\det(g_s)$. Without supersymmetry the rank of the antisymmetric tensor is not fixed, nor is d, but in N=1 d=11 supergravity s=4 and so either 4 or 7 dimensions preferentially compactify. I will, of course, be considering the latter which includes the time-like direction in ordinary space-time.

In the d=11 supergravity I must set

$$g_{am} = 0 \tag{4.12}$$

as required in (4.5), and

$$F_{amnp} = F_{abmn} = F_{abcn} = 0 . \tag{4.13}$$

The requirement (4.12) that the gauge fields are zero is the Kaluza-Klein ansatz. There will be fluctuations in a full quantum theory, the classical solutions I am looking for are considered to be the ground state. For maximal symmetry of the ground state it is also required that

$$g_{ab} = g_{ab}(x) \quad , \quad F_{abcd} = F_{abcd}(x) , \tag{4.14a}$$

$$g_{mn} = g_{mn}(y) \quad , \quad F_{mnpq} = F_{mnpq}(y) . \tag{4.14b}$$

To give d=4 local Lorentz invariance F_{abcd} must be an invariant tensor, so a possible ansatz is

$$F_{abcd} = \frac{f}{4!}\,\varepsilon_{abcd} \; , \quad F_{mnpq} = 0 \; , \quad f = \text{constant} , \tag{4.15}$$

which gives for the Ricci tensor

$$R_{ab} = -\frac{f^2}{4!}\, g_{ab} \quad , \tag{4.16a}$$

$$R_{mn} = \frac{f^2}{2 \times 4!}\, g_{mn} \; . \tag{4.16b}$$

I shall refer to the ansatz (4.15) with the F field only on M_4 as a Freund-Rubin compactification, even in the more general cases when f is not constant. Thus an effective d=4 theory has been constructed with an AdS ground state and compact extra dimensions. The Einstein equations are

$$R^a{}_b - \frac{1}{2}\delta^a_b R = \frac{-3.3!}{(4!)^2}\delta^a_b f^2 , \qquad (4.17a)$$

$$R^m{}_n - \frac{1}{2}\delta^m_n R = \frac{3.3!}{(4!)^2}\delta^m_n f^2 \qquad (4.17b)$$

and the Maxwell equation is automatically satisfied by (4.15). There are many solutions with M_4 being AdS and M_7 some compact manifold. To obtain Minkowski 4-space requires f=0. Known M_7's are listed in table(4.2).

Table(4.2)
Known compactifications of d=11 supergravity

	M_7	G	H	Killing spinors	Holonomy	Reference
1)	T^7	U(1)	1	8	1	25
2)	S^7	SO(8)	SO(7)	8	1	41,49
3)	J^7	SO(5)xSO(3)	SO(3)xSO(3)	1	G_2	42,43,44,50
4)	M^{pqr}	SU(3)xSU(2)xU(1)	SU(2)xU(1)^2	0	SO(7)	4,45,49
5)	N^{pqr}	SU(3)xU(1)	U(1)^2	1	G_2	5,51
6)	Q^{pqr}	SU(2)^4	SU(2)xU(1)	0	SO(7)	49
7)	S^4xS^3	SO(5)xSU(2)^2	SU(2)^3	0	SO(7)	49
8)	$\frac{SU(3)xS}{SO(3)}$	SU(3)xSU(2)	SO(3)xU(1)	0	SO(7)	48
9)	?	SO(5)	SO(3)$_{MAX}$	1	G_2	48
10)	$V_{5,2}$	SO(5)xU(1)	SO(3)xU(1)	2	SU(3)	48
11)	$K3xT^3$	not a coset space		4	SU(2)	33,46

Notes:

i) In cases 4,5 and 6 the holonomy group may be larger for specific values of the embedding parameters.

ii) There also exists a squashed version of solution 5 {116}.

There are also solutions in which F has components on M_7, F_{mnpq} being constructed from a 'Killing spinor' $\eta(y)$

$$F^{mnpq} = g\,\bar{\eta}(y)\,\tilde{\Gamma}^{mnpq}\,\eta(y) \ , \tag{4.18}$$

where $\tilde{\Gamma}^m$ are the 8x8 Dirac matrices and $\eta(y)$ satisfies the equation

$$D_m\,\eta(y) = \frac{m}{2}\,\tilde{\Gamma}_m\,\eta(y) \quad , \quad m = \text{constant} \ , \tag{4.19}$$

on M_7. Obviously not all M_7 allow this construction. The sign of m is important here. If m > 0 the solution is $M_4 \times M_7$, if m < 0 the solution is $M_4 \times \tilde{M}_7$ where $\tilde{M}_7$ is M_7 with the orientation reversed. This point is discussed in appendix C.

There exist other solutions with F components on M_7 not constructed as in (4.18). These are due to Moorhouse and Nixon{66} and are given in detail in chapter VII.

There is a problem. If the extra dimensions are of the order of the Planck length in size there appears in the d=4 theory an unacceptably large cosmological constant. To remove this it is possible to add a higher dimensional cosmological constant and fine tune it to cancel the d=4 one. Apart from this being against the spirit of Kaluza-Klein theory it would break the higher dimensional supersymmetry and the solution would be unstable{36}. In the cases where Killing spinors exist it may be possible to cancel the d=4 cosmological constant with fermi bilinears in the full theory{36}.

There are many spontaneously compactified solutions of d=11 supergravity listed above. Some of them will be described in greater detail later on as an introduction to their time dependent versions. First, though, I will discuss how to look at the symmetry of the solutions. Are there any remaining supersymmetries in the reduced theory ? Is it possible to obtain the phenomenological gauge symmetry SU(3)xSU(2)xU(1) (perhaps spontaneously broken) ?

A surviving supersymmetry would ensure a stable vacuum solution{37,38}, although this symmetry would have to be broken later for phenomenology. In the bosonic sector for the vacuum to have N supersymmetries requires the fermi field ψ to remain zero under supersymmetry transformations, that is

$$\delta\psi = \left(d - \tfrac{1}{4}\omega^{ab}\gamma_{ab} - \tfrac{1}{4}\omega^{mn}\tilde{\Gamma}_{mn} + m\,\gamma_5\,\gamma_a\,V^a - \tfrac{m}{2}\,\tilde{\Gamma}_m B^m\right)\varepsilon \tag{4.20a}$$

$$\equiv \bar{D}\,\varepsilon = 0 \tag{4.20b}$$

must have N linearly independent solutions, with the decomposition of the d=11 Γ matrices

$$\Gamma_M = \left(\gamma_a \otimes 1 \quad , \quad \gamma_5 \otimes \tilde{\tau}_m\right). \tag{4.21}$$

If the supersymmetry transformation parameter ε can be written

$$\varepsilon(x,y) = \sum_{I=1}^{N} \varepsilon(x)^I \eta(y)^I \tag{4.22}$$

(4.20) divides into two parts

$$\left(d - \tfrac{1}{4}\omega^{ab}\gamma_{ab} + m\gamma_5\gamma_a V^a\right)\varepsilon^I(x) = 0 \quad , \tag{4.23a}$$

$$\left(d - \tfrac{1}{4}\omega^{mn}\tilde{\tau}_{mn} - \tfrac{m}{2}\tilde{\tau}_m B^m\right)\eta^I(y) = 0 \quad . \tag{4.23b}$$

Eqn.(4.23b) reproduces the Killing spinor equation (4.19). The expression (4.22) is only true if M_4 x M_7 is a global product and not a non-trivial fibre bundle and so is presumably not necessary, although helpfull, in finding d=4 supersymmetries{39}. The integrability condition for (4.20) is

$$\bar{D}^2\varepsilon = 0 \; . \tag{4.24}$$

The operator $\bar{D}^2$ on M_7 takes the following form for spaces of the type AdS x M_7

$$\bar{D}^2 = -\tfrac{1}{4}\left(R^{mn}{}_{pq} - m^2\delta^{mn}{}_{pq}\right)\tilde{\tau}_{mn}\, B^{pq} \tag{4.25a}$$

$$\equiv -\tfrac{1}{4}\, C_{mn}{}^{pq}\,\Gamma_{pq}\, B^{mn} = -\tfrac{1}{4}\, C_{mn}\, B^{mn}. \tag{4.25b}$$

The operator C_{mn} contains linear combinations of SO(7) generators. If $\bar{D}^2$ has N zero eigenspinors η^I on M_7 C_{mn} will generate a subgroup of SO(7) - the holonomy group of M_7 which leaves the spinors invariant. N can take the values 0,1,2,4 and 8{40}. Since the eigenspinors must also satisfy (4.23b) the number of supersymmetries may be less than that allowed by holonomy. There are two cases for which there are N=8 solutions of (4.23b):

a) $m^2 = 0$, M_4 x M_7 =Minkowski x T^7. This corresponds to the original Cremmer-Julia theory{25} with only the massive sector retained.

b) $m^2 \neq 0$, M_4 x M_7 = AdS x S^7. Duff and Pope{41}.

In the cases where non-zero F_{mnpq} constructed by (4.18) are included all supersymmetries are broken{117}, some authors claiming{24,42} this is a spontaneously broken version of the supersymmetric $F_{mnpq}=0$ theory. F_{mnpq} might have a geometrical interpretation as a Ricci flattening torsion (see Englert{117}) and this type of field has been constructed for some of the examples in table(4.2){42,43,44,45}.

Now to consider the gauge symmetry. The gauge group on M_4 will contain the isometry group of the internal space M_7, thus the choice of M_7 is important. For the moment let me ignore the constraints imposed by (4.5) and consider a more general choice of metric

$$g_{4+D} = \eta_{MN}\, E^M \otimes E^N \quad , \tag{4.26}$$

where M,N=0,...,10 and

$$E^a = V^a \quad , \tag{4.27a}$$

$$E^m = B^m + B^m(K_i)\, A^i(x) \; . \tag{4.27b}$$

E^a are orthonormal 1-forms on M_4 (by the choice of V^a) and B^m are orthonormal 1-forms on M_7 such that

$$g_7 = \eta_{mn}\, B^m \otimes B^n \quad , \tag{4.28}$$

where η_{mn} is the Euclidean signature 7-dimensional flat metric diag(-------). Assume that the 7-dimensional space M_7 has a p-dimensional symmetry group G. Then there exist p Killing vectors for the metric g_7, denoted by $K_i(y)$,i=1,...,p ie.

$$\mathcal{L}_{K_i} g_7 = \eta_{mn}\left[(\mathcal{L}_{K_i} B^m)\otimes B^n + B^m \otimes (\mathcal{L}_{K_i} B^n)\right] = 0 \quad , \tag{4.29}$$

where the Lie derivative $\mathcal{L}$ with respect to K_i is most easily used in the form

$$\mathcal{L}_{K_i} = i_{K_i}\, d + d\, i_{K_i} \, . \tag{4.30}$$

The group structure is illustrated by the relation

$$\mathcal{L}_{K_i} K_j = [K_i \, , K_j] = f_{ij}{}^{k}\, K_k \quad , \tag{4.31}$$

where $f_{ij}{}^{k}$ are the structure constants of G.

The p 1-forms $A^i(x)$ can be expanded in terms of the basis of orthonormal 1-forms V^a on M_4

$$A^i(x) = A^i{}_a(x)\, V^a. \tag{4.32}$$

The $A^i{}_a(x)$ are identified with the potentials of the gauge group. Recall that in the spontaneously compactified ground state these are set to zero. Consider the restricted general coordinate transformation

$$x^{a'} = x^a , \tag{4.33a}$$

$$y^{m'} = y^m + \sum_i \varepsilon_i(x) K_i{}^m(y) , \tag{4.33b}$$

where $K_i = K_i{}^m dy_m$ generates an isometry of g_7. Since $\varepsilon_i(x)$ is ordinary space-time dependent the resulting d=4 theory will inherit the isometry group of M_7 as a local gauge group. It can be shown{3,25} that under (4.33)

$$\delta A_a{}^i(x) = \partial_a \varepsilon^i(x) - f^i{}_{jk}\, A_a{}^j(x)\, \varepsilon^k(x) , \tag{4.34}$$

the usual transformation law for a Yang-Mills field with gauge group G. In the case $F_{mnpq} \neq 0$ the isometry group is reduced to a subgroup of G which leaves F_{mnpq} invariant, this being equivalent to matter fields in non-supersymmetric theories which transform non-trivially under G.

Witten{4} pointed out that 7 was the least number of extra dimensions required to give the phenomenological gauge group SU(3)xSU(2)xU(1). Cremmer and Schwarz{28} had to assign a specific dependence of the fields on the extra dimensions and were thus restricted to $U(1)^7$. This would require some type of preon model to build a greater symmetry with composite particles. Scherk and Schwarz{29} gave a model in which spontaneous symmetry breaking resulted from the fields having non-trivial y dependence. Duff et al{46} introduced another form of spontaneous symmetry breaking by squashing the seven sphere (ie. keeping the topology but changing the geometry) which gives scalar fields in the reduced theory non-zero vacuum expection values (a Higgs mechanism).

The most economical way to gain a large symmetry from the extra dimensions is to make M_7 a coset space (a group space is a special case of a coset space). Since most M_7's are coset spaces I will now describe how a G invariant vielbein is constructed.

Let H be a compact Lie subgroup of a compact Lie group G. The coset space G/H , to be identified with M_7, is the set of cosets (equivalence classes) of elements of G defined, for left cosets of H in G by

$$gH = \{ gH \; ; \; h \in H \} . \tag{4.35}$$

If

$$\dim G = p \quad , \quad \dim H = q \tag{4.36}$$

then

$$\dim\left(\tfrac{G}{H}\right) = \dim G - \dim H . \tag{4.37}$$

Essentially the coset space G/H is the group manifold of G with the identification of points corresponding to equivalent group elements given by H. The advantage is that larger symmetry groups can be obtained for a given number of extra dimensions. The coset space is invariant under the action of G. The actual isometry group of G/H is often larger than G; being in fact GxN(H)/H where N(H) is the normaliser of H in G. See Castellani et al{47} for a proof of this and some exceptions. A list of possible G and H for d=11 supergravity is given in Castellani et al{48}.

It is necessary to find vielbein 1-forms B^m(m=1,...,7) for the 7-dimensional coset space which are G invariant and also give a space which is Ricci diagonal. It will be shown later that there exist solutions where M_7 is Ricci diagonal but not an Einstein space as demanded by other authors. First the torsion free condition

$$R^m = dB^m - \omega^m{}_n \wedge B^n = 0 \tag{4.38}$$

is solved for the SO(7) spin conection $\omega^{mn} = -\omega^{nm}$ and then the curvature calculated

$$\mathbb{R}^{mn} = d\omega^{mn} - \omega^{mp} \wedge \omega_p{}^n = R^{mn}{}_{pq} \, B^p \wedge B^q . \tag{4.39}$$

As stated it is required that the Ricci tensor

$$R^m{}_n = \sum_p R^{mp}{}_{np} \tag{4.40}$$

is diagonal. Note that in the time dependent cases to be considered later (4.38) must be replaced by

$$R^m = d B^m - \omega^m{}_n \wedge B^n - \omega^m{}_0 \wedge V^0 . \tag{4.41}$$

If G is a Lie group the coset space can be parametrised by (p-q)=7 coordinates y^α. On M_7 there exist p independent Killing vector fields $K_i(y)$, i=1,...,7 which form a basis for the tangent vectors on M_7 and obey the Lie algebra $\mathfrak{g}$ of the group G as in (4.31) and for vielbein G invariance the Lie derivative of B^m along K_i must satisfy

$$\mathcal{L}_{K_i} B^m \left(\equiv \mathcal{L}_i B^m \right) = W_i{}^{mn} \; B_n \quad , \tag{4.42}$$

where $W^{MN} = -W^{NM} \in SO(7)$ Lie algebra. Consistency of (4.42) yields{118}

$$\mathcal{L}_i W_j - \mathcal{L}_j W_i = \left[W_i , W_j \right] + f_{ij}{}^k W_k . \tag{4.43}$$

In each coset {y} one can choose a representative element L(y) which under left multiplication by $g \in G$ is in general carried into another coset with representative element L(y′). Thus

$$g L(y) = L(y') h , \quad h \in H , \tag{4.44}$$

where y′ and h are functions of y and g and depend on how coset representatives are chosen. Eqn.(4.44) defines the left translation of G/H by $g \in G$. To find the vielbein the 1-form

$$\Omega(y) = L(y)^{-1} d L(y) = \Omega^i T_i \quad , \tag{4.45}$$

is constructed where T_i are the generators of G (ie. $L^{-1}(y) dL(y) \in \mathfrak{g}$). The Ω^i are a set of left invariant 1-forms on G/H which satisfy the Maurer-Cartan equations for G

$$d \Omega^i = f^i{}_{jk} \Omega^j \wedge \Omega^k . \tag{4.46}$$

The T_i can be separated into a subset which spans the Lie algebra $\mathcal{H}$, $\{T_{\hat{m}}\}$, and a subset lying in G/H, $\{T_m\}$. It is the latter subset which is identified with the 7 G-invariant vielbeins on M_7. Thus

$$\Omega(y) = \Omega^{\hat{m}} T_{\hat{m}} + \Omega^{m} T_{m} , \tag{4.47}$$

where $\hat{m}$ take the q values complementary to the m. The vielbein is defined to be

$$B^{m} = s_{m} \Omega^{m} , \tag{4.48}$$

where s_m is an arbitrary real parameter which, in general, can be different for each m. It can be checked that B^m satifies (4.42). For a good description of how B^m transforms under gauge transformations see Strathdee{122}.

When G is a product of simple factors one is free to independently rescale the Killing metric on each simple factor. This does not change the topology of the coset space but alters the geometry. This fact will be important in time dependent theories in later chapters. In certain special cases one can rescale within a simple factor(Castellani{47}). Relaxing the condition requiring an Einstein space to that for a Ricci diagonal space increases the number of possible spaces, for instance including the space M^{001} of Witten{4} which will be of interest later.

The particular embedding of H in G will be labelled by integers, for example p_1, p_2 and p_3 in the case of G=SU(3)xSU(2)xU(1), H=SU(2)xU(1)xU(1) (for extensive discussion of this example see Castellani et al{45,48}), which label the topology and can be of importance in determining the type of time dependent behaviour.

Chapter V COSMOLOGY and INFLATION

In this chapter I shall briefly describe the 'standard' cosmological model and some of the problems it leaves unsolved. In my own models I assume that the standard model is at least substantially correct for some era of the evolution of the universe. I shall then go on to describe some attempts to solve these problems using the concept of inflation. The inflation described in this chapter is caused by a phase transition, this is not the case in the Kaluza-Klein approach although many of the benefits are the same.

According to the standard model the universe is assumed to have started in a singularity of infinite temperature and density and to be expanding and cooling. Present day observations of homogeneity and isotropy suggest the use of the Robertson-Walker metric

$$g = V^0 \otimes V^0 - \sum_{a=1}^{3} V^a \otimes V^a \quad , \tag{5.1}$$

where $\{V^A\}$ $A = 0,1,2,3$ is an orthonormal basis of 1-forms for the space-time manifold. Minkowski space, de Sitter space and anti-de Sitter space are all special cases of the general Robertson-Walker space{30}. In this case the 1-forms are

$$V^0 = dt \quad , \tag{5.2a}$$

$$V^1 = R(t)\, dr \quad , \tag{5.2b}$$

$$V^2 = R(t)\, C(r)\, d\theta \quad , \tag{5.2c}$$

$$V^3 = R(t)\, C(r) \sin\theta \, d\phi \quad , \tag{5.2d}$$

for coordinates (t,r,θ,ϕ). $R(t)$ is the scale factor of the universe, a measure of the radius of the universe, the precise relationship depending in general on the geometry considered. Note that r is the radial coordinate here and not the scale factor. There should be no confusion between the scale factor and the Ricci tensor. The function $C(r)$ is given by

$$C(r) = \frac{\sinh(\sqrt{k}\, r)}{\sqrt{k}} \tag{5.3}$$

where k = -1(+1) for a closed (open) universe (C(r)=sin(r),(sinh(r))) and C(r)=0 for a flat universe.

The spin-connection 1-forms are

$$\omega_{0a} = -\frac{\dot{R}}{R} V^a \quad , \quad \omega_{13} = -\frac{C'}{RC} V^3 \quad , \tag{5.4a}$$

$$\omega_{12} = -\frac{C'}{RC} V^2 \quad , \quad \omega_{23} = -\frac{\cot\theta}{RC} V^3 \quad , \tag{5.4b}$$

where $\dot{R} = \frac{\partial R}{\partial t}$ and $C' = \frac{\partial C}{\partial r}$. The curvature 2-forms can now be calculated and are

$$\mathbb{R}_{0a} = -\frac{\ddot{R}}{R} V^0 \wedge V^a \quad , \tag{5.5a}$$

$$\mathbb{R}_{ab} = \left[\left(\frac{\dot{R}}{R}\right)^2 - \frac{k}{R^2} \right] V^a \wedge V^b \ . \tag{5.5b}$$

The Ricci tensor is diagonal and is

$$R^0{}_0 = \frac{3}{2} \frac{\ddot{R}}{R} \quad , \tag{5.6a}$$

$$R^1{}_1 = R^2{}_2 = R^3{}_3 = \frac{1}{2}\frac{\ddot{R}}{R} + \left(\frac{\dot{R}}{R}\right)^2 - \frac{k}{R^2} \ . \tag{5.6b}$$

The derivation of these equations is a simple case of that described in chapter VII. The Einstein equations, without a cosmological constant, are{52}

$$R_{AB} - \frac{1}{2} g_{AB} R = -8\pi G\, T_{AB} \tag{5.7a}$$

or

$$R_{AB} = -8\pi G \left(T_{AB} - \frac{T^c{}_c}{2} g_{AB} \right) \tag{5.7b}$$

where G is the gravitational constant and T_{AB} the energy-momentum tensor. Assuming the universe to be radiation dominated and neglecting particle interactions the energy-momentum tensor is that of a perfect fluid

$$T_{AB} = -p\, g_{AB} + (p + \rho)\, u_A u_B \quad , \tag{5.8}$$

where $u_0 = 1$, $u_\alpha = 0$ is the comoving velocity, P the pressure and ρ the energy density. The pressure and density, functions of time only, are related by the equation of state $\rho = 3P$.

Substituting (5.6) and (5.8) into (5.7) gives

$$\frac{3}{2}\frac{\ddot{R}}{R} = -4\pi G(\rho + P) \tag{5.9a}$$

$$\frac{1}{2}\frac{\ddot{R}}{R} + \left(\frac{\dot{R}}{R}\right)^2 - \frac{k}{R^2} = 4\pi G(\rho - P) . \tag{5.9b}$$

Conservation of energy implies (as can be deduced from (5.9)) the Bianchi condition

$$\frac{\dot{R}}{R} = -\frac{\dot{\rho}}{3(\rho + P)} . \tag{5.10}$$

It is also assumed that the expansion is adiabatic so

$$S R^3 = constant \tag{5.11}$$

where S is the entropy.

For a radiation dominated era{53}

$$\rho \sim N(T)\, T^4 \tag{5.12}$$

$$S \sim N(T)\, T^3 \tag{5.13}$$

where N(T) is the effective number of particle species at temperature T. Taking N(T) independent of T the scale factor is approximately given by

$$R \sim t^{1/2} . \tag{5.14}$$

In a matter dominated era this would be modified to

$$R \sim t^{2/3} . \tag{5.15}$$

This model contains many approximations discussed in ref.{8}.

I shall now mention some of the problems of the standard scenario briefly described above. These and others are discussed in ref.{8} and references therein.

1 The Singularity

As $t \to 0$ $R(t) \to 0$ yet $\rho \to \infty$ so the standard model starts in a singularity. This raises the question of how the singularity arose out of whatever existed before. Was it a quantum fluctuation on a scale of 10^{-33} cm which just expanded ? If so it should contract to zero again sometime.

2 The Flatness Problem

From (5.9)

$$\frac{|\rho - \rho_c|}{\rho_c} = \frac{1}{\dot{R}^2} \tag{5.16}$$

where ρ_c is the energy density for a flat universe (k = 0) and ρ is the energy density for a closed or open universe(k = $\pm$ 1) with the same $(\dot{R}/R)^2$. The present energy density is not well known exactly $(0.03 < \frac{|\rho-\rho_c|}{\rho_c} < 2)$. For the early universe $\dot{R}^2 \sim t^{-1}$ so that $\frac{|\rho-\rho_c|}{\rho_c}$ was small, in fact near the Planck time($t_p \sim M_p^{-1}$)

$$\frac{|\rho - \rho_c|}{\rho_c} \lesssim 10^{-59} \tag{5.17}$$

If ρ was slightly greater than ρ_c the universe would be closed and have collapsed years ago, or if ρ was slightly less than ρ_c the universe would be open and the present ρ negligibly small. Why is ρ today so close to ρ_c ?

3 The Homogeneity, Isotropy and Horizon Problem.

It is difficult to understand why the universe is so isotropic and homogeneous today. Was this always true ? We can see everything within a proper distance

$$H(t) = R(t) \int_0^t \frac{dt'}{R(t')} . \tag{5.18}$$

A particle horizon will exist if this integral is finite. The visible universe might have expanded from a causally connected region and this could solve this problem. If the universe expanded in separate domains of different symmetry then our universe must be totally contained within one domain with a horizon. The size of the observable universe is $\sim 10^{28}$ cm and no domain walls are visible.

4 The Symmetry Problem.

Why is the universe now in a state of symmetry $SU(3) \times U(1)$? Kaluza-Klein theories attempt to explain this by the symmetry of the extra space.

5 Cosmological Constant Problem.

Any cosmological constant (ignored in the model above) has to be very small. Higher dimensional theories often give an effective four dimensional constant which is too big.

6 Dimensionality of Space-time

Space-time appears four dimensional and this is built into the standard model. In higher dimensional theories the goal is to produce an effective four-dimensional theory naturally but this is only achieved to a limited extent (ie. by giving possible mechanisms but not reasons). It is interesting to note that gravitational theories require greater than three space-time dimensions and that only theories with three effective spatial dimensions can have stable atomic structure and planetary systems because, in higher dimensions, the forces decrease too rapidly with distance between interacting objects{54}. Therefore only effectively four dimensional space times can support life as we know it (obviously). The dimensionality in different domains of the universe may be different but this is of only theoretical interest.

Attempts to solve problems (2) and (3) have been made by trying to produce a huge inflationary expansion. If it is possible to generate a huge amount of entropy (a factor of 10^{88}) in a causally connected volume early in the expansion of the universe by inflation these problems may be solved. These attempts will be described briefly as a comparison for the models in later chapters.

INFLATION

In the standard inflationary approach due to Guth{119} there is a field ϕ with a potential $V(\phi)$ which drives the inflation by a symmetry breaking cosmological phase transition. If the potential takes the form of fig.(5.1) the initial value of ϕ is zero due to the high temperature form of $V(\phi)$ but as the universe expands and cools the potential evolves to its low temperature form with a minimum

$\phi = \phi_c$. As ϕ moves to the low temperature minimum a first order phase transition will occur at temperature T_c. As the universe continues to expand and cool bubbles of the low temperature phase will form but, if the nucleation rate is low enough, it will supercool to , say, T_s in the high temperature phase. In the supercooled phase $\rho \sim V(0)$ which is constant so (5.10) gives an exponential expansion for R(t) before the phase transition. When the phase transition does take place much latent heat will be released and the universe will reheat to a temperature comparable to T_c. This temperature will not depend on the time of the exponential expansion. The horizon problem is thus solved by pushing the horizon back beyond the visible universe. Reheating of the universe only occurs after bubble wall collisions and leads to an inhomogeneous and anisotropic universe today.

To overcome these problems the New Inflationary Universe was proposed and has been applied to other models{8}. For potentials of a particular type the phase transition can be a slow rollover. This occurs if $\partial V/\partial \phi$ is small. If the effective potential $V(\phi, T)$ has only one minimum (at $\phi = 0$) for $T \gg T_c$ and this remains a local minimum for all $T \neq 0$ the phase transition from $\phi = 0$ to the global minimum $\phi = \phi_c$ proceeds by the formation and expansion of bubbles of the field ϕ . ϕ goes to ϕ_c in the bubble but $\phi < \phi_c$ whilst this is happening and for cases where $\phi \ll \phi_c$, $\rho \sim V(0)$ and is approximately constant so the inside of the bubble expands exponentially during the phase transition. It is suggested that the whole observable universe is contained inside one bubble and so any inhomogeneities caused by bubble wall collision are irrelevant. The reheating process occurs by decay of Higgs bosons produced from oscillation in the field ϕ . Just as in the Guth scenario an expansion of 10^{30} solves the horizon and flatness problems. Note that in the new inflationary universe inflation takes place during the phase transition.

The skill in looking at inflationary theories of this sort is in choosing a form of effective potential that gives the desired results. In supergravity the effective potential is fixed. New inflation has been applied to supersymmetric theories{55}.

Another method of inflation, called chaotic inflation, has been proposed which does not require a high temperature phase transition{8}. This scheme assumes that the initial distribution of ϕ was chaotic and that in some domain ϕ is quite large and decreases slowly so that the domain behaves as an exponentially expanding universe. The size of the domain must be large enough to explain the size of the homogeneous universe today.

Another benefit of inflation is that the few monopoles present in the original small volume of the universe that has expanded to 10^{28} cm across become insignificant and so their rarity today is explained.

The above models are purely classical. Quantum gravity effects become important at the Planck length ($L_p \sim 10^{-33}$ cm) and densities $>10^{94}$ g/cm^3. The standard model tells us this density was reached when the universe was 10^{-4} cm (ie. 10^{29} L_p) across.

Chapter VI KALUZA-KLEIN COSMOLOGY

Kaluza-Klein theories, as described in chapter II, contain extra dimensions which I consider to be physical and of such small size that they are not detectable and are hence ineffective other than through symmetry considerations. In Kaluza-Klein cosmology this is assumed to not always have been the case and the extra dimensions have a time dependence such that their scale could have been comparable to the ordinary dimensions at some point. The extra dimensions might thus have had an important effect on the evolution of our universe and might provide an answer to some of the problems of conventional cosmological models. When we consider Kaluza-Klein cosmology we usually assume that the effect of the extra dimensions was important during the very early universe and that the extra dimensions are somehow frozen-out at a later stage and that the universe becomes effectively 4-dimensional from then on. One of the first Kaluza-Klein approaches to cosmology was the 5-dimensional model of Chodos and Detweiler{18}, described in chapter II, who showed that solving the vacuum equations of general relativity in 4+1-dimensions with a Kasner type metric leads to a cosmology at the present epoch which has 3+1 observable dimensions in which the Einstein-Maxwell equations are obeyed. Guth and others{119} inflationary cosmology 'solves' the horizon and flatness problems but involves a non adiabatic period with a huge increase in entropy. Higher dimensional cosmologies hope to achieve inflation adiabatically by some generalisation of the Chodos and Detweiler model. Whether, when expansion is very rapid, the adiabatic approximation holds is an open question. Entropy in the extra dimensions is converted into entropy in the ordinary dimensions due to different time evolution of the scale factors of the extra and ordinary dimensions.

In this chapter I consider one simple model described by a generalised 1+3+D-dimensional Robertson-Walker metric for a space with energy density ρ behaving as a perfect fluid. There are many more complicated models (see eg. Lorenz-Petzold{56}) but my aim here is to describe the main features of the model for comparison with the specific case of supergravity considered later. It is assumed that the 4+D-dimensional space is governed by the normal Einstein equations stretched to cover the extra dimensions symmetrically

$$R_{MN} - \frac{1}{2} g_{MN} R = 8\pi \bar{G} T_{MN} - \Lambda_0 g_{MN} \,, \tag{6.1}$$

where T_{MN} is the 4+D-dimensional energy-momentum tensor and Λ_0 a 4+D-dimensional cosmological constant (from now taken to be zero) and subject to the adiabatic condition for the entropy S in a comoving volume

$$S \sim r^3 s^D T^{D+3} = constant, \tag{6.2}$$

where r(t),s(t) is the scale factor for the ordinary,extra dimensions and T the temperature. I assume all the extra dimensions have the same scale factor. The adiabatic assumption assumes interactions occur at a rate adequate to maintain thermal equilibrium (ie. reaction times are small compared to expansion and contraction times.) and is likely to be a reasonable approximation far away from the final singularity. $\bar{G}$ is the 4+D-dimensional gravitational constant and is related to G the present day gravitational constant by

$$\bar{G} = G V_{M_D} \tag{6.3}$$

where V_{M_D} is the volume of the D-dimensional space today. Note that G will vary today if the scale factor of the extra dimensions is not now constant.

A perfect fluid in 4+D-dimensions obeys a generalised form of (5.8) assuming isotropy between the 3 and D dimensional directions

$$T_{MN} = - p g_{MN} + (p + \rho) u_M u_N \tag{6.4}$$

and the equation of state for radiation is

$$\rho = (D + 3) p \,. \tag{6.5}$$

This equation is the integrated form of the Bianchi identity. All fields are massless in the higher dimensions in Kaluza-Klein theory. See Yoshimura{57} and Tosa{58} for anisotropic ρ models and eg. Sahdev{59,60} for a model similar to that considered here. For a radiation dominated universe the density is given by

$$\rho = N_{pol}\, a\, T^{D+4} \equiv C\,(r^3 s^D)^{-\left(\frac{D+4}{D+3}\right)} , \tag{6.6}$$

where N_{pol} is the number of polarisation states, a a function (given in Abbott et al{61}) and C a constant. The second part of (6.6) is a result of the Bianchi identity $T^{MN}{}_{;N} = 0$. The generalised Robertson-Walker metric is

$$g = V^0 \otimes V^0 - \sum_{a=1}^{3} V^a \otimes V^a - \sum_{m=1}^{D} B^m \otimes B^m , \tag{6.7}$$

where $\{L^M\} = \{V^0, V^a, B^m\}$ is an orthonormal basis for the space-time manifold which is, at least locally, a direct product$(R^1 \times M_3 \times M_7)$. This choice of metric determines the number of ordinary and extra compactifying dimensions. The split does not arise naturally from the equations of motion but once the split has been chosen then the behaviour of the two sets of dimensions is described by the equations of motion. It is not possible to say how three ordinary spatial directions were selected but it is assumed, by the choice of metric, that they are and that the anisotropy persisted all the way back to the original singularity. The 3 and D dimensional spaces are both taken to be maximally symmetric although there is no a priori reason why the extra dimensions possess this symmetry. The ordinary space-time can be taken to be anti-de Sitter.

To exhibit the time dependence the full vielbein is written

$$V^0 = dt , \tag{6.8a}$$

$$V^a = r(t)\, v^a , \tag{6.8b}$$

$$B^m = s(t)\, b^m , \tag{6.8c}$$

where v^a and b^m are the time independent vielbeins for the 3 and D dimensional spaces respectively.

The Ricci tensor is calculated to be

$$R^0{}_0 = \frac{3}{2}\frac{\ddot{r}}{r} + \frac{D}{2}\frac{\ddot{s}}{s} , \tag{6.9a}$$

$$R^a{}_a = \frac{1}{2}\frac{\ddot{r}}{r} + \frac{k_3}{r^2} + \left(\frac{\dot{r}}{r}\right)^2 + \frac{D}{2}\frac{\dot{r}\dot{s}}{rs} , \tag{6.9b}$$

$$R^m{}_m = \frac{1}{2}\frac{\ddot{s}}{s} + \frac{(D-1)}{2}\left(\frac{k_D}{s^2} + \left(\frac{\dot{s}}{s}\right)^2\right) + \frac{3}{2}\frac{\dot{r}\dot{s}}{rs} , \tag{6.9c}$$

where $k_3(k_D)$ is the curvature constant for the 3(D) dimensional space, flatness requires $k_3 \lesssim 0$ and to have compact extra dimensions $k_D > 0$; and the energy-momentum tensor substituted into the Einstein equations gives

$$R^0{}_0 = -8\pi\bar{G}\rho \quad , \tag{6.10a}$$

$$R^a{}_a = \frac{8\pi\bar{G}}{D+3}\rho \quad , \tag{6.10b}$$

$$R^m{}_m = \frac{8\pi\bar{G}}{D+3}\rho \quad . \tag{6.10c}$$

Note that my conventions differ from those of Abbott et al{61}. Equating (6.9) and (6.10) and substituting for $\ddot{r}$ and $\ddot{s}$ into the time direction equation one finds (equivalent to the Bianchi condition) the condition

$$\frac{3k_3}{r^2} + 3\left(\frac{\dot{r}}{r}\right)^2 + 3D\frac{\dot{r}\dot{s}}{rs} + D(D-1)\left(\frac{\dot{s}}{s}\right)^2 + \frac{D(D-1)}{2}\frac{k_D}{s^2} = 16\pi\bar{G}\rho . \tag{6.11}$$

Now the constant C in (6.6) can be calculated for a given set of initial values.

I have computed solutions to (6.9) and (6.10) subject to the condition (6.11) which, if satisfied initially, is maintained by (6.9) and (6.10). I chose $k_3 = -1$, $k_D = 1$ and D = 7 (for comparisons with d=11 supergravity later). The general behaviour of the scale factors r(t) and s(t) is shown in fig.(6.1) for the initial values r = s = 1, $\dot{r} = \dot{s} = 1$. The length scales are not set by the theory. This is also true of the supergravity models and a possible means of setting the scale is given in chapter IX. It can be seen that r expands, slowly at first and then goes through an 'exponential' expansion phase whilst s expands, reaches a maximum and then decreases to a singularity at time t_1, the same time r goes to infinity. The time dependence forces the volume of M_D to change with time. The results for a variety of initial values, given in table(6.1), are shown in figs.(6.1-4) and it is clear that although the initial behaviour varies depending on the initial conditions the behaviour near $t = t_1$ is in each case very similar and is in fact unavoidable (if changing the sign of k_3 or k_D is excluded for physical reasons). There are also thermodynamic arguments for the behaviour of r and s (see Abbott et al{61}). To solve the horizon problem requires

$$\frac{r}{s} \gtrsim \frac{10^{28}\,cm}{10^{-33}\,cm} = 10^{61} \tag{6.12}$$

which is possible since $r \to \infty$ and $s \to 0$. However other problems arise, some of which are considered later in the context of supergravity. If D increases the evolution of r and s is similar but $\dot{r}$ and $\dot{s}$ change more slowly at first but later change much quicker as $t \to t_1$.

TABLE(6.1)

Initial values for the perfect fluid gravity model.

r	s	$\dot{r}$	$\dot{s}$	Graph
1	1	1	1	fig.(6.1)
1	1	1	-1	fig.(6.2)
1	1	-1	-1	fig.(6.3)
1	1	-1	1	fig.(6.4)

For the case of fig.(6.1) the mean volume

$$V_M \sim r^3 s^D , \tag{6.13}$$

the temperature, assuming that it is the same for M_4 and M_D (as in Barr and Brown{120})

$$T \sim \left(\frac{C}{r^3 s^D}\right)^{\frac{1}{3+D}} = \left(\frac{C}{V_M}\right)^{\frac{1}{3+D}} \tag{6.14}$$

and the three dimensional entropy

$$S_3 \sim r^3 T^3 \tag{6.15}$$

were calculated as functions of the scale factors and are plotted, again with arbitrary scales, together with ρ in figs.(6.5-8). See Kolb{62} or Sahdev{59,60} for similar but more detailed calculations including a matter dominated universe. The effect of greater numbers of extra dimensions, which are not allowed in supergravity, will not be discussed here other than to say they can be used to improve behaviour with respect to solving the horizon problem. A specific model with a 4-form F will be considered in chapter X.

Because of the collapsing extra dimensions and expanding ordinary ones the mean volume V_M increases and then decreases eventually approaching zero (see fig.(6.5)), this causing the eventual reheating and increasing ρ of figs.(6.6) and (6.8). As $t \to t_1$ the quantities T, S_3 and ρ approach a singularity. However as s decreases there must come a point when quantum effects become important, even if the approximations are still valid up to this point - which is unlikely. Let me assume that a miracle called 'quantum gravity stabilisation' occurs at t_c just less than t_1 when s is of the order of the Planck length

$$s = s_{KK} \sim L_P \tag{6.16}$$

and s becomes constant, or very nearly so, to prevent time variation in the gravitational constant today. Even if a miracle does not occur the perfect fluid approximation would break down because of local anisotropies in the expansion rates, however the qualitative behaviour of the solutions will remain the same (see Kolb et al{63}). Assuming $s = s_{KK}$ at and after t_c it is insufficient simply to set $\ddot{s} = \dot{s} = 0$ in (6.9) as this would require some finite jumps in other quantities such as the d=4 cosmological constant. There must be some process acting over a short period of time which has the effect of maintaining a constant energy density in the compact space as the two spaces effectively decouple. During this time the equations of motion are unknown. The effective 3-dimensional density at decoupling is

$$\rho_3 = \int_{M_D} \rho \, d\mu_{M_D} = V_{M_D} \rho \tag{6.17}$$

where μ_{M_D} is the measure over the compact space M_D. The equations describing the 3-dimensional universe become

$$\frac{3}{2}\frac{\ddot{r}}{r} = 8\pi G \rho_3 + \Lambda_4 \quad , \tag{6.18a}$$

$$\frac{1}{2}\frac{\ddot{r}}{r} + \frac{k_3}{r^2} + \left(\frac{\dot{r}}{r}\right)^2 = \frac{8\pi G}{D+3}\rho_3 + \Lambda_4 \tag{6.18b}$$

which correspond to standard cosmology, with a cosmological constant

Λ_4 a remnant of the extra dimensions, with behaviour dominated by

$$r \sim t^{1/2} \quad . \tag{6.19}$$

See Bailin et al{64} for a discussion of behaviour after t_c. The d=4 cosmological constant would be modified by a non-zero curvature in the extra dimensions. There are numerous other models with arbitrary matter fields coupled to gravity (for example Randjbar-Daemi et al{65} find a solution $r \sim (t(t_1 - t))^{1/2}$, $s \sim$ constant) but I am interested in N=1 d=11 supergravity in which the matter content is specified.

Chapter VII COSMOLOGICAL EQUATIONS OF d=11 SUPERGRAVITY

In the last chapter I described a scheme for dynamical compactification of gravity with a perfect fluid, the geometry of the extra dimensions being maximally symmetric. Now I want to apply this idea to the Kaluza-Klein form of d=11 supergravity to obtain a model which is hopefully more realistic in terms of field content and symmetries as well as having the pleasing aspect of being a unified geometric theory. I shall show that there exist time dependent scale factors for the various spontaneously compactified d=11 solutions given in chapter IV.

First I will show, as was demonstrated by Moorhouse and Nixon{66}, that there exist many spontaneous compactifying solutions of the bosonic sector of d=11 supergravity in which the field A_{MNP} (and hence F_{MNPQ}) has components on the 7-space but does not require the existence of a Killing spinor. The form of F, however, is severely restricted. I shall show that the Ricci tensor is in general non-isotropic in the internal space (ie. M_7 is not an Einstein space). I then go on to describe the derivation of time-dependent cosmological equations. The first time dependent solutions for d=11 supergravity were given by Freund{67} with M_7 a seven torus. Solutions for the different manifolds given here will be given in chapter VIII, together with the constraints imposed by cosmology. The symmetry of solutions with $F_{mnpq} \neq 0$ and not constructed from an η has yet to be studied.

Firstly two cases, that of the seven-sphere and seven-torus, will be described in some detail and then equations for other models will be stated with some comments. The form of solution is still $M_4 \times M_7$ but now M_4 can be considered as $R^1 \times M_3$, recalling that for anti-de Sitter space I mean its covering space.

As stated before spontaneously compactified solutions of d=11 supergravity have been found in which the 4-form field lives on M_4 creating the split $M_4 \times M_7${41}. Solutions have also been found in which F has components on M_7, the first such solution was the round seven-sphere{7} and subsequently a particular squashed seven-sphere{24,42,117} and then other manifolds{45}. In all these cases the field F = dA has to be constructed using a covariantly constant (Killing) spinor $\eta(y)$ satisfying (4.19) which can be written

$$D\eta = \frac{m}{2}\tilde{\Gamma}_m B^m \eta \tag{7.1}$$

with the potential (A_7) on M_7

$$A_7 = \frac{g}{3!}\bar{\eta}\tilde{\Gamma}_{mnp}\eta B^{mnp} \quad , \tag{7.2}$$

where B^m and $\tilde{\Gamma}_m$ are the vielbein and gamma matrices on M_7, and g is a constant. Using (7.1) and (7.2) the 4-form F_7 on M_7 is

$$F_7 = dA_7 = \mathcal{D}A_7 \tag{7.3a}$$

$$= -\frac{gm}{3!}\bar{\eta}\tilde{\Gamma}_{mnpq}\eta B^{mnpq} \quad , \tag{7.3b}$$

where it can be noticed that

$$\underset{7}{*}F = -m A_7 . \tag{7.4}$$

In these particular Kaluza-Klein theories the 4-form F is completed by adding the Freund-Rubin ansatz for M_4, thus the complete F is

$$F = f\Omega + F_7 \tag{7.5}$$

where f is a constant and Ω the volume element for M_4

$$\Omega = \frac{\varepsilon_{abcd}}{4!} V^{abcd} = V^{0123} \tag{7.6}$$

where the V^a are vielbeins on M_4. If

$$f = -4m \tag{7.7}$$

then F also satisfies the 11-dimensional Maxwell equation

$$d * F = \frac{1}{4} F \wedge F . \tag{7.8}$$

The fact that F satisfies (7.8) is a consequence of (7.4) and thus follows naturally for an A_7 constructed from η as in (7.3). The condition (7.8) is not compatible with the condition for supersymmetry given in appendix C. As stated F given by (7.5) can also solve the Einstein equations in certain cases.

The construction (7.2) leading to (7.4) is not available when no Killing spinors exist on M_7 (eg. on $M^{p_1p_2p_3}$ solutions for arbitrary p_1, p_2, p_3). In the case of the squashed seven-sphere, with squashing parameter λ, an η exists for $\lambda^2 = 1$ (the round seven-sphere) and for $\lambda^2 = 1/5$. In both these cases one can put on M_7 a field F which, by its construction will satisfy (7.8). In each of these cases there is a solution of the Einstein equations with M_7 an Einstein space. This is, in part, due to the properties of S^7 which has two Einstein metrics on it; one for $\lambda^2 = 1$, the other for $\lambda^2 = 1/5${24}.

However for $\lambda^2 \neq 1, 1/5$, and for other manifolds in table(4.2), no Killing spinors exist. This implies, as described in chapter IV, that no supersymmetries exist. I shall show that a 4-form F_7 exists on the seven-sphere for arbitrary squashing such that F given by (7.5) satisfies both the Maxwell and Einstein equations. Afterwards I shall show how the properties inherent in this construction of F allow for the introduction of time dependence, in a way not otherwise available, which will be used to formulate time dependent cosmological equations of the Robertson-Walker type which will allow the squashing parameter to vary with time. I then look at other models. The properties of this construction also make possible a more complete formulation of one approach to stability for the static solution on the $\lambda^2 = 1/5$ squashed seven-sphere{68}.

Now for the construction of the 4-form F on the squashed seven-sphere. Although the round seven-sphere S^7 and the squashed seven-sphere J^7 can both be regarded as coset spaces I do not here follow the procedure of chapter IV but use the vielbeins of Awada et al{50} with a slight change in notation,

$$b^m = \lambda\, e^m \quad , \quad m = 4, 5, 6, \tag{7.9a}$$

$$b^m = e^m \quad , \quad m = 7, 1, 2, 3, \tag{7.9b}$$

where λ is the squashing parameter and the e^m, given by

$$e^7 = d\mu\,, \quad e^1 = \tfrac{1}{2}\sin\mu\ \omega_1\,, \quad e^2 = \ldots, \quad e^3 = \ldots, \tag{7.10a}$$

$$e^4 = \tfrac{1}{2}(\nu_1 + \cos\mu\ \omega_1), \quad e^5 = \ldots, \quad e^6 = \ldots, \tag{7.10b}$$

form an orthonormal basis for the round SO(8) invariant seven-sphere. The 1-forms ν_i and ω_i are linear combinations of σ_i and Σ_i, which are left invariant 1-forms satisfying the SU(2) algebra

$$d\sigma_1 = -\sigma_2 \wedge \sigma_3 \quad , \quad d\Sigma_1 = -\Sigma_2 \wedge \Sigma_3 \quad , \tag{7.11}$$

plus cyclic permutations, and are given by

$$\nu_i = \sigma_i + \Sigma_i \quad , \quad \omega_i = \sigma_i - \Sigma_i \quad . \tag{7.12}$$

From (7.11) and (7.12)

$$d\nu_1 = -\tfrac{1}{2}\left[\nu_2 \wedge \nu_3 + \omega_2 \wedge \omega_3\right] \quad , \tag{7.13a}$$

$$d\omega_1 = -\tfrac{1}{2}\left[\nu_2 \wedge \omega_3 + \omega_2 \wedge \nu_3\right] \quad , \tag{7.13b}$$

plus cyclic permutations, which gives for the vielbein equations

$$de^7 = 0 \quad , \tag{7.14a}$$

$$de^1 = -e^5 \wedge e^3 - e^2 \wedge e^6 + 2\cot\mu \, e^2 \wedge e^3 + \cot\mu \, e^7 \wedge e^1 \quad , \tag{7.14b}$$

$$de^4 = -e^5 \wedge e^6 - e^2 \wedge e^3 - e^7 \wedge e^1 \quad , \tag{7.14c}$$

together with those obtained by simultaneous cyclic permutations of 1,2,3 and 4,5,6. I now define the 3-forms

$$E_1 \equiv e^{351} + e^{324} + e^{621} + e^{417} + e^{527} + e^{637} \quad , \tag{7.15a}$$

$$E_2 \equiv e^{456} \quad , \tag{7.15b}$$

and the 4-forms

$$G_1 \equiv e^{7426} + e^{7156} + e^{7453} + e^{2356} + e^{3164} + e^{1245} \quad , \tag{7.16a}$$

$$G_2 \equiv e^{1237} \quad . \tag{7.16b}$$

G_1 and G_2 are dual, with respect to the metric $-\sum_m b^m \otimes b^m$ on M_7, to E_1 and E_2 when $\lambda = 1$ and satisfy

$$G_1 = \frac{1}{\lambda} \, {*_7} E_1 \quad , \quad G_2 = \lambda^3 \, {*_7} E_2 \tag{7.17a,b}$$

for the orientation defined on M_7 to be

$$\underset{7}{*}1 = b^{1\dots 7} = \lambda^3 e^{1\dots 7} . \qquad (7.18)$$

It follows from (7.14) that

$$dE_1 = -G_1 - 6G_2 , \quad dE_2 = -G_1 , \qquad (7.19a,b)$$

and consequently the important result

$$dG_1 = 0 , \quad dG_2 = 0 . \qquad (7.20a,b)$$

I now construct F by (7.5) using the ansatz

$$A_7 = g_1 E_1 + g_2 E_2 , \quad F = f\Omega + dA_7 , \qquad (7.21)$$

so the Maxwell equation (7.8) may be solved for g_1 and g_2. The 11-dimensional metric is taken to be

$$g = V^0 \otimes V^0 - \sum_{a=1}^{3} V^a \otimes V^a - s^2 \sum_{m=1}^{7} b^m \otimes b^m , \qquad (7.22)$$

where $V^M = \{V^0, V^a, sb^m\}$ is the full vielbein on $M_4 \times M_7$. The scale factor s governs the scale of the seven-sphere with respect to M_4 and is, for the static solution being considered here, constant. The Maxwell equation (7.8) yields for F in (7.21)

$$\frac{1}{s\lambda}\Omega\wedge[(g_1+g_2+6\lambda^4 g_1)G_1 + 6(g_1+g_2)G_2] = \tfrac{1}{2}f\ \Omega\wedge[(g_1+g_2)G_1 + 6g_1G_2] \qquad (7.23)$$

which is solved by

$$x \equiv \frac{g_1+g_2}{g_1} = \tfrac{1}{2}\left(1 \pm (1+24\lambda^4)^{\frac{1}{2}}\right) \qquad (7.24a)$$

$$f = \frac{2x}{\lambda s} . \qquad (7.24b)$$

This procedure is equivalent to finding the ratio g_1/g_2 which satisfies the duality condition (7.4); however in this formulation the existence of a Killing spinor is not implied. Eqn.(7.4) can be satisfied because there exist independent 3-forms E_1 and E_2 such that dE_1 and dE_2 are linear combinations of $\underset{7}{*}E_1$ and $\underset{7}{*}E_2$.

In the special case $\lambda^2 = 1/5$, $x = 6/5$ the field constructed from an η satisfying (7.1) is reproduced, which is constant in the basis (7.9) so that

$$\frac{1}{3!}\,\bar{\eta}\,\Gamma_{mnp}\,\eta\; b^{mnp} = E_1 + \frac{1}{5} E_2 \qquad (7.25)$$

in a suitable representation of the Γ_m. Although a covariantly constant spinor exists for $\lambda^2 = 1/5$, with $|m| = \frac{3S}{\sqrt{20}}$ there is no supersymmetry because not only is the existence of an η required for some m but also that $m = \frac{\sqrt{2}}{4!} f$ in my conventions and this cannot be compatible with f given by (7.21). A similar equation to (7.25) does not exist for $\lambda^2 = 1$ because constant η have not been found on the seven-sphere.

There is a trivial interchange between left and right squashing{42,117} caused by putting $e^m \to -e^m$ in equations (7.14) and (7.15). This would change the sign of f in (7.24b) but does not effect either the existence of a solution or the lack of supersymmetry. Since there is some confusion in the literature on this point I have included some notes on the matter in Appendix C.

I now look, unsuccesfully, for a static solution of all the equations for the squashed seven-sphere for an arbitrary λ which will necessarily not be an Einstein space. The Einstein equation is given in (3.6). Summing over M in (3.6) yields

$$R = \frac{1}{2} F^{MNPQ} F_{MNPQ} \equiv \frac{F^2}{2} \qquad (7.26)$$

and reinserting this in (3.6) gives

$$R^M{}_N = 6 F^{MN_1N_2N_3} F_{NN_1N_2N_3} - \frac{F^2}{2} \delta^M{}_N \,. \qquad (7.27)$$

For a given degree of squashing the ansatz for F in (7.21) has three constants occuring (f,g_1 and g_2). Two of these are fixed by the the solution to the Maxwell equation (7.24). The Einstein equations (7.27) become, after substituting from (7.24)

$$R^a{}_b = -\frac{\delta^a_b}{4!}\left(\frac{4x^2}{\lambda^2 S^2} + \frac{g_1^2}{S^8\lambda^4}\left(18\lambda^4 + 3x^2\right)\right), \qquad (7.28a)$$

$$R^m{}_n = -\frac{\delta^m_n}{4!}\left(\frac{-2x^2}{\lambda^2 S^2} + \frac{g_1^2}{S^8\lambda^4}\left(-36\lambda^4 - \frac{3}{2}x^2\right)\right), \qquad (7.28b)$$

$$R^u{}_v = -\frac{\delta^u_v}{4!}\left(\frac{-2x^2}{\lambda^2 S^2} + \frac{g_1^2}{S^8\lambda^4}\left(18\lambda^4 - 3x^2\right)\right), \qquad (7.28c)$$

where a,b are tangent space indices on M_4, m,n = 1,2,3,7 and u,v = 4,5,6 all on M_7. Anti-de Sitter space will satisfy (7.28a) for any values of g_1, s and λ ; so I can take M_4 to be anti-de Sitter space-time.

The calculation of the Ricci tensor from the vielbein 1-forms (7.10) proceeds as in appendix A and gives

$$R^m{}_n = \delta^m_n \frac{3}{2s^2}\left(1 - \frac{\lambda^2}{2}\right), \qquad (7.29a)$$

$$R^u{}_v = \delta^u_v \frac{1}{4s^2}\left(2\lambda^2 + \frac{1}{\lambda^2}\right). \qquad (7.29b)$$

Equating (7.28) and (7.29) gives

$$\frac{4!}{s^2}\left[\frac{3}{2}\left(1-\frac{\lambda^2}{2}\right) - \frac{x^2}{12\lambda^2}\right] + \frac{g_1^2}{s^8\lambda^4}\left(-36\lambda^4 - \frac{3}{2}x^2\right) = 0 , \qquad (7.30a)$$

$$\frac{4!}{s^2}\left[\frac{1}{4}\left(2\lambda^2+\frac{1}{\lambda^2}\right) - \frac{x^2}{12\lambda^2}\right] + \frac{g_1^2}{s^8\lambda^4}\left(18\lambda^4 - 3x^2\right) = 0 . \qquad (7.30b)$$

For $\lambda^2 = 1$ (7.30) gives $g_1^2 = 0$ implying $g_1 = g_2 = 0$. This might be expected as another consequence of constant η not having been found on the round seven-sphere. The solution for $\lambda^2 = 1/5$ is $g_2 = g_1/5$ (hence the form of (7.25)) and $g_1^2/s^6 = 1/5$. In both these cases M_7 is an Einstein space. For $\lambda^2 > 0$ there are no other solutions to (7.30); so there are no static non-Einstein space solutions for the squashed seven-sphere. Although nothing seems to specifically prevent them they do not exist.

I now consider non-static equations, which will hopefully give a more realistic cosmological model, with a Robertson-Walker type metric of the form

$$g = V^0 \otimes V^0 - r^2(t)\sum_{a=1}^{3} v^a \otimes v^a - s^2(t)\sum_{m=1}^{7} b^m \otimes b^m , \qquad (7.31)$$

where $V^0 = dt$, $V^a = rv^a$ and the squashing parameter is now also time dependent: $\lambda = \lambda(t)$. The 3-form potential and 4-form field must also be made time dependent by making the coefficients g_1, g_2 and f time dependent:

$$A_7 = g_1(t)\, E_1 + g_2(t)\, E_2 , \qquad (7.32a)$$

$$F = f(t)\,\Omega + dA_7 , \qquad (7.32b)$$

where Ω is the M_4 volume element $\Omega = V^{0123} = r^3 V^0 \wedge V^{123} \equiv V^0 \wedge \Omega^3$. From (7.32) I develop the Maxwell equation (7.8)

$$dA_7 = -(q_1+q_2)\,G_1 - 6q_1 G_2 + \dot{q}_1 V^0 \wedge E_1 + \dot{q}_2 V^0 \wedge E_2 \quad , \tag{7.33a}$$

$$F \wedge F = -2f\left[(q_1+q_2)\,\Omega \wedge G_1 + 6q_1\,\Omega \wedge G_2\right] + 12 V^0 \wedge \left[\dot{q}_1(q_1+q_2) + q_1\dot{q}_2\right]\bar{\Omega} \,, \tag{7.33b}$$

where $\bar{\Omega} = e^{1\ldots7}$,

$$*F = \Omega \wedge \left[\left(\frac{q_1+q_2}{\lambda s}\right)E_1 + \left(\frac{6\lambda^3 q_1}{s}\right)E_2\right] - \Omega^3 \wedge \left(s\lambda\dot{q}_1 G_1 + \frac{\dot{q}_2 s}{\lambda^3} G_2\right) - f\,\bar{\Omega}\, s^7\lambda^3 . \tag{7.34}$$

The expressions (7.33) and (7.34) have to satisfy the Maxwell equation (7.8) and for this it is necessary that $dG_1 = dG_2 = 0$ because otherwise the 8-form $d{*F}$ would contain 5-forms on M_7 wedged with 3-forms on M_4 and such forms are not contained in $F \wedge F$. In fact dG_1 and dG_2 are zero from the property that dE_1 and dE_2 are linearly expressible in terms of their duals G_1 and G_2. This property, as shown by (7.19), which allows solutions of the static Maxwell equations thus also leads to the existence of a time dependent F with components on M_4 and M_7 which satisfies the identity $dF = 0$ (by construction) and has no M_7 5-forms in $d{*F}$.

Inserting (7.33) and (7.34) into (7.8) the Maxwell equations are satisfied if

$$\frac{d}{dt}\left(f s^7\lambda^3\right) + 3\dot{q}_1(q_1+q_2) + 3q_1\dot{q}_2 = 0 \quad , \tag{7.35a}$$

$$\frac{1}{r^3}\frac{d}{dt}\left(r^3\dot{q}_1\lambda s\right) + \frac{1}{\lambda s}(q_1+q_2) + \frac{6q_1\lambda^3}{s} - \frac{f}{2}(q_1+q_2) = 0 \quad , \tag{7.35b}$$

$$\frac{1}{r^3}\frac{d}{dt}\left(r^3\frac{\dot{q}_2 s}{\lambda^3}\right) + \frac{6}{\lambda s}(q_1+q_2) - 3fq_1 = 0 \, . \tag{7.35c}$$

The first of these is easily integrated with respect to time to give

$$f s^7\lambda^3 + \frac{3}{2}q_1^2 + 3q_1 q_2 = \text{constant}. \tag{7.35a'}$$

From the Einstein equations the following conditions arise where, as well as $\lambda(t)$, the variable $u(t) = \lambda(t)\,s(t)$ is introduced

$$\frac{1}{2}\left(3\frac{\ddot{r}}{r} + 4\frac{\ddot{s}}{s} + \frac{3\ddot{u}}{u}\right) + \frac{1}{4!}\left(\frac{9\dot{q}_1^2}{s^6\lambda^2} + \frac{3}{2}\frac{\dot{q}_2^2}{s^6\lambda^6} + \frac{3}{2}f^2\right) + \frac{F^2}{2} = 0 \, , \tag{7.36a}$$

$$\frac{k_3}{r^2} + \frac{1}{2}\frac{\ddot{r}}{r} + \frac{1}{2}\frac{\dot{r}}{r}\left(2\frac{\dot{r}}{r} + 4\frac{\dot{s}}{s} + 3\frac{\dot{u}}{u}\right) + \frac{3}{2.4!}f^2 + \frac{F^2}{2} = 0 \quad , \tag{7.36b}$$

$$\frac{3}{2s^2}\left(1 - \frac{\lambda^2}{2}\right) + \frac{1}{2}\frac{\ddot{s}}{s} + \frac{1}{2}\frac{\dot{s}}{s}\left(3\frac{\dot{r}}{r} + 3\frac{\dot{s}}{s} + 3\frac{\dot{u}}{u}\right) - \frac{1}{4!}\left(\frac{9(g_1+g_2)^2}{2s^8\lambda^4} + \frac{54g_1^2}{s^8} - \frac{9}{2}\frac{\dot{g}_1^2}{s^6\lambda^2}\right) + \frac{F^2}{2} = 0, \tag{7.36c}$$

$$\frac{1}{4s^2}\left(2\lambda^2 + \frac{1}{\lambda^2}\right) + \frac{1}{2}\frac{\ddot{u}}{u} + \frac{1}{2}\frac{\dot{u}}{u}\left(3\frac{\dot{r}}{r} + 4\frac{\dot{s}}{s} + 2\frac{\dot{u}}{u}\right) - \frac{1}{4!}\left(\frac{6(g_1+g_2)^2}{s^8\lambda^4} - \frac{3\dot{g}_1^2}{s^6\lambda^2} - \frac{3}{2}\frac{\dot{g}_2^2}{s^6\lambda^6}\right) + \frac{F^2}{2} = 0 \quad , \tag{7.36d}$$

where

$$F^2 = \frac{1}{4!}\left(\frac{6(g_1+g_2)^2}{\lambda^4 s^8} + \frac{36\,g_1^2}{s^8} - f^2 - \frac{6\,\dot{g}_1^2}{\lambda^2 s^6} - \frac{\dot{g}_2^2}{s^6\lambda^2}\right). \tag{7.37}$$

Calculation of the Ricci tensor to construct these equations is given, as an example, in Appendix A. Eqn.(7.36a) comes from the time index Einstein equation, (7.36b) from the ordinary space index equation with k_3 the curvature for that space, (7.36c) comes from the seven-sphere indices m = 1,2,3,7 and (7.36d) from the seven-sphere indices u = 4,5,6; note that $u(t) = \lambda\, s(t)$ sets the scale of these vielbeins and s(t) sets the scale of the 1,2,3,7 vielbeins.

Eqns.(7.36) are of diagonal form because $F^{MP_1P_2P_3}F_{NP_1P_2P_3} = 0$ if $M \neq N$. This arises from the properties of the forms E_1, E_2, G_1 and G_2 which appear in the F_7 part of F. These properties can be related to those of the structure constants of the octonion algebra{69,117} and are essential for this formulation.

I have discussed in some detail the squashed seven-sphere model with the forms E_1 and E_2. I will later consider several other models and suitable forms E_i on other compact manifolds (where i might take values from 1 to 7 and still preserve diagonality of the Ricci tensor). However the search for these E_i has not been exhaustive, although extensive, and there may be other forms which satisfy the requirements.

One approach to cosmological solutions, not that considered in this thesis, is to consider perturbation from special solutions of a simple type such as non-static seven-sphere solutions or even static solutions. Page{68} has investigated the stability of static solutions both without and with a component F_7 of F on M_7. With an F_7 from η he is unable to consider changes in the squashing parameter λ because of a lack of F except when $\lambda^2 = 1, 1/5$. With the F of (7.32) now available for arbitrary λ a more general consideration of stability by Page's method is possible. One can use a metric

$$g = V^0 \otimes V^0 - \sum_{a=1}^{3} V^a \otimes V^a - s^2(x)\left(\sum_{\substack{m=1,\\2,3,7.}} e^m \otimes e^m + \lambda^2(x) \sum_{\substack{n=4,\\5,6.}} e^n \otimes e^n\right), \quad (7.38)$$

where x denotes the coordinates of ordinary space-time. (It is not implied in (7.38) that $V^0 = dt$). Using the ansatz for F of the type (7.21) or (7.32) with f, g_1, g_2 now functions of x, all the good properties of E_1 and E_2 play again, as in the cosmological case, and stability conditions can be formulated. This tests stability against dilations only. A more general method of testing stability, but which is harder, is to look for dangerous eigenmodes of differential operators on M_7. See Duff et al{37} and D'Auria et al{70}.

I can find 3-forms E_i for constructing time dependent equations, analogous to those for the squashed seven-sphere, for other models with anisotropic internal spaces. Some of these will be given later, but first I shall look at the static version of one of these which has the interesting feature, unlike the squashed seven-sphere, of giving a static non-Einstein space solution.

Consider the 7-dimensional coset space SU(2)xSU(2)xSU(2)/U(1)xU(1) of D'Auria et al{70}. The manifold is characterised by three integers (p_1, p_2, p_3) which determine the topology by labelling the embedding of U(1)xU(1) in SU(2)xSU(2)xSU(2) and internal parameters(a,b,c) which determine the geometry and take on the role of inverse scale factors for the internal space. Adopting the notation of ref.{70} I consider the special case $p_1 = p_2 = p_3 = 1$ and a = b = c = 1/s. The metric is

$$g = V^0 \otimes V^0 - \sum_{a=1}^{3} V^a \otimes V^a - \sum_{m=1}^{7} B^m \otimes B^m \quad (7.39)$$

and the siebenbein B^m obey the equations, with Ω^i the left-invariant 1-forms on G/H (see(4.45),

$$dB^1 = B^2 \wedge \Omega^0, \quad dB^2 = -B^1 \wedge \Omega^0, \quad d\Omega^0 = \frac{1}{s^2} B^1 \wedge B^2, \quad (7.40a)$$

$$dB^4 = B^5 \wedge \Omega^{0'}, \quad dB^5 = -B^4 \wedge \Omega^{0'}, \quad d\Omega^{0'} = \frac{1}{s^2} B^4 \wedge B^5, \quad (7.40b)$$

$$dB^6 = B^7 \wedge \Omega^{0''}, \quad dB^7 = -B^6 \wedge \Omega^{0''}, \quad d\Omega^{0''} = \frac{1}{s^2} B^6 \wedge B^7, \quad (7.40c)$$

$$B^3 = w\left(\Omega^0 + \Omega^{0'} + \Omega^{0''}\right), \quad B^m = s\,\Omega^m. \quad (7.40d)$$

Here w and s are the remaining parameters of the manifold, which for static solutions are constant and for cosmological equations of the

Robertson-Walker type are functions of time. Defining 3-forms

$$E_1 \equiv B^{417} + B^{426} + B^{516} + B^{257} , \tag{7.41a}$$

$$E_2 \equiv B^{123} + B^{453} + B^{673} , \tag{7.41b}$$

$$-G_1 \equiv \overset{*}{_7} E_1 = B^{2563} + B^{1573} + B^{2473} + B^{4163} , \tag{7.42a}$$

$$-G_2 \equiv \overset{*}{_7} E_2 = -B^{4567} - B^{6712} - B^{1245} \tag{7.42b}$$

which, using (7.40), yield

$$dE_1 = \frac{1}{w} G_1 \quad , \quad dE_2 = \left(\frac{2w}{s^2}\right) G_2 , \tag{7.43}$$

so that E_1 and E_2 fullfill the necessary condition to have solutions for the Maxwell equations and also to permit a cosmological development, but in a simpler way than in the case of the squashed seven-sphere. Looking for static solutions I set

$$F = f\,\Omega + g_1 G_1 + g_2 G_2 , \tag{7.44}$$

where f, g_1 and g_2 are constants. Using this ansatz, rather than $A_7 = g_1 E_1 + g_2 E_2$, only amounts to a rescaling of the constants g_1 and g_2 by factors of s and w when dE_1 and dE_2 are as in (7.43). The Maxwell equation (7.8) requires

$$\frac{1}{2} g_1 f = \frac{g_1}{w} \quad , \quad \frac{1}{2} g_2 f = \frac{2 g_2 w}{s^2} \tag{7.45}$$

and if $f \neq 0$, $g_1 \neq 0$, $g_2 \neq 0$ (7.45) requires $2w^2/s^2 = 1$, which is the condition in ref.{70} for an internal Einstein space. However if either g_1 or g_2 is zero this conclusion is avoided. Using (7.27) I find the Ricci tensor

$$R^a{}_b = \frac{\delta^a_b}{4!}\left(-f^2 - 2g_1^2 - \frac{3}{2} g_2^2\right) , \tag{7.46a}$$

$$R^m{}_n = \frac{\delta^m_n}{4!}\left(\frac{1}{2} f^2 + g_1^2 + \frac{9}{2} g_2^2\right) , \quad m,n = 1,2,4,5,6,7, \tag{7.46b}$$

$$R^m{}_3 = \frac{\delta^m_3}{4!}\left(\frac{1}{2} f^2 + 4 g_1^2 - \frac{3}{2} g_2^2\right) \tag{7.46c}$$

which can be equated with the results in ref.{70} for M_7

$$R^m{}_n = \frac{S^m{}_n}{2S^2}\left(1 - \frac{1}{2}\frac{W^2}{S^2}\right) , \tag{7.47a}$$

$$R^m{}_3 = S^m{}_3 \; \frac{3W^2}{4S^4} . \tag{7.47b}$$

If $g_1 \neq 0$, $g_2 = 0$ (7.45) only requires $f = 2/w$ and w^2/s^2 is not fixed. Equating corresponding terms in (7.46) and (7.47) for the Ricci tensor on M_7 yields

$$\frac{1}{2S^2}\left(1 - \frac{1}{2}\frac{W^2}{S^2}\right) = \frac{1}{24}\left(\frac{2}{W^2} + g_1{}^2\right) , \tag{7.48a}$$

$$\frac{3}{4}\frac{W^2}{S^4} = \frac{1}{24}\left(\frac{2}{W^2} + 4g_1{}^2\right) , \tag{7.48b}$$

which can both be satisfied by $w^2 = s^2$, $g_1{}^2 = 4/w^2$ or $7w^2 = s^2$, $g_1{}^2 = -20/49w^2$; either of which is a static non-Einstein internal space singling out the $m = 3$ tangent space direction, the ratio w/s giving its size relative to the other directions. Although the space is not Einstein it is still Ricci diagonal. Obviously the latter solution is unphysical. Ordinary space-time is again anti-de Sitter.

For the other case $g_1 = 0$, $g_2 \neq 0$ (7.45) requires $f = 4w/s^2$ and w is not fixed; the conditions corresponding to (7.48) are solved by $w^2/s^2 = -3/4$, $g_2{}^2 = 5/s^2$ - which is also unphysical.

The cosmological equations can be developed, as for the squashed seven sphere with $A_7 = g_1(t)E_1 + g_2(t)E_2$. These equations will be given later. There are non-Einstein space solutions with both $g_1(t)$ and $g_2(t)$ non-zero.

I have looked at two cases where the E_i have been chosen to satisfy the Maxwell equations and then proceeded to try and solve the Einstein equations. I now derive the conditions on E_i for this to be so. For simplicity I assume that the scale factors for each direction in M_7 are equal, although the result is more generally valid. Let the E_i be linearly independent 3-forms on M_7 which can be written

$$E_i = \sum_{p,q,r} a^i_{pqr} \; b^p \wedge b^q \wedge b^r , \tag{7.49}$$

where a^i_{pqr} are constants, or functions of y only, and b^p, $p=1,\dots,7$ an orthonormal basis of 1-forms on M_7. The ansatz for the potential A_7 and for F is

$$A_7 = \sum_i g_i(t) \; E_i , \tag{7.50a}$$

$$F = f(t)\,\Omega + d A_7 \,, \tag{7.50b}$$

so that

$$F = f\,\Omega + \sum_i \dot{q}_i V^0 \wedge E_i + \sum_i q_i \, dE_i \,, \tag{7.51}$$

$$F \wedge F = \sum_i 2 f q_i \, \Omega \wedge dE_i + 2 \sum_{i,j} \dot{q}_i q_j \, V^0 \wedge E_i \wedge dE_j \,. \tag{7.52}$$

Using the results

$$*(V^0 \wedge E_i) = s\, V^{123} \wedge \underset{7}{*} E_i \,, \tag{7.53a}$$

$$* dE_i = \frac{1}{s}\, \Omega \wedge \underset{7}{*} dE_i \,, \tag{7.53b}$$

$$d * dE_i = \frac{1}{s}\, \Omega \wedge \underset{7}{*} E_i \tag{7.53c}$$

I find

$$*F = -f\,\Omega + \sum_i s \dot{q}_i V^{123} \wedge \underset{7}{*} E_i + \sum_i q_i * dE_i \,, \tag{7.54}$$

$$d*F = -\frac{1}{s^7}\frac{d}{dt}(f s^7)\, V^0 \wedge \bar{\Omega} + \sum_i \frac{1}{r^3}\frac{d}{dt}(s \dot{q}_i r^3)\, \Omega \wedge \underset{7}{*} E_i - \sum_i s \dot{q}_i V^{123} \wedge d \underset{7}{*} E_i + \sum_i \frac{q_i}{s}\, \Omega \wedge \underset{7}{*} E_i \,. \tag{7.55}$$

The Maxwell equation (7.8) is satisfied by (7.52) and (7.55) if

$$dE_i = \sum_j c_{ij} \underset{7}{*} E_j \tag{7.56}$$

where c_{ij} are constant coefficients, $\det(c) \neq 0$. This is true even if the a^i_{pqr} are functions of y, however the c_{ij} will be different. A consequence of (7.56) is

$$d(\underset{7}{*} E_i) = 0 \,, \qquad \forall i \,. \tag{7.57}$$

This is equivalent to $dG_i = 0$.

The conditions arising from the Maxwell equation are

$$\frac{1}{r^3}\frac{d}{dt}(s \dot{q}_j r^3) - \frac{q_j}{s} - \sum_i 2 f q_i c_{ij} = 0 \,, \qquad \forall j \,, \tag{7.58a}$$

$$\frac{d}{dt}(f s^7) + 2 \sum_{i,j} c_{ij} \dot{q}_i q_j = 0 \,. \tag{7.58b}$$

When the 7-space is anisotropic these expressions become much more complicated; for instance g_1/s in (7.58a) is replaced by $\sum_{ij} \frac{d_{ij} s}{\sigma(s_m)}$ where d_{ij} is some constant and $\sigma(s_m)$ a function of the scale factors s_m of weight 1. Eqns.(7.35) are a specific example of the more general formulae in the case of the squashed seven-sphere. In practice the c_{ij} were not calculated explicitly but results obtained directly from (7.52) and (7.55). Note that (7.58b) can be straightforwardly integrated with respect to time whereas, in general, (7.58a) cannot.

The other requirement demanded of the E_i is that they give an F which yields a Ricci diagonal space in (7.27). There are $7!/4!.3! = 35$ possible E_i which might satisfy (7.56) on some manifold M_7 but these are restricted to a choice from 7 when I require $F^{MP_1P_2P_3} F_{NP_1P_2P_3} = \delta^M_N$, which can be written

$$b^{351}, b^{324}, b^{621}, b^{417}, b^{527}, b^{637}, b^{456}. \tag{7.59}$$

These correspond to the indices for non-zero octonion structure constants and are thus related to the A fields giving parallelising torsion on the seven-sphere{42,71}. The choice (7.59) together with (7.56) ensures a Ricci diagonal space. The indices in (7.59) can of course be relabelled in many ways by cyclic permutations.

The time dependent equations for the squashed seven-sphere are the Maxwell equations (7.35) and the Einstein equations (7.36). Note that not all the Einstein equations are independent. What now follows is a catalogue of the models I studied with their time dependent equations. To recap, I am dealing with the metric

$$g = V^0 \otimes V^0 - \sum_a V^a \otimes V^a - \sum_m B^m \otimes B^m , \tag{7.60}$$

where

$$V^0 = dt , \quad V^a = r(t)\, v^a , \quad B^m = S_m(t)\, b^m . \tag{7.61}$$

The 4-space is anti-de Sitter or de Sitter in all models. The procedure is as follows. Firstly a basis was chosen from the left-invariant 1-forms Ω^i on G, from which the b^m are taken (see chapter IV). The Ricci tensor was then calculated by Cartan's moving frame method (see Appendix A) and for the time and ordinary space directions is

$$R^{0}{}_{0} = \frac{3}{2}\frac{\ddot{r}}{r} + \sum_{m} \frac{1}{2}\frac{\ddot{s}_m}{s_m} \quad , \tag{7.62a}$$

$$R^{a}{}_{a} = \frac{1}{2}\frac{\ddot{r}}{r} + \frac{k_3}{r^2} + \left(\frac{\dot{r}}{r}\right)^2 + \sum_{m} \frac{1}{2}\frac{\dot{r}\,\dot{s}_m}{r\,s_m} \quad , \tag{7.62b}$$

where some, or all, of the s_m may be equal. The constant k_3 is the curvature of the 3-space. The equation for the M_7 directions varies more with the topology of M_7 and is

$$R^{m}{}_{m} = \frac{1}{2}\frac{\ddot{s}_m}{s_m} + \sum_{n} H_n(s_p) + \frac{3}{2}\frac{\dot{r}\,\dot{s}_m}{r\,s_m} + \sum_{\substack{n \\ n\neq m}} \frac{1}{2}\frac{\dot{s}_m\,\dot{s}_n}{s_m\,s_n} \quad , \tag{7.62c}$$

where H_n is a function of the s_p of weight -2. The E_i, 3-forms on M_7, are then chosen subject to the conditions of this chapter, leading to the construction of A_7 and F by (7.50). The Ricci tensor can then be calculated from the Einstein equation (7.27) and equated with (7.62). F can also be substituted in the Maxwell equation (7.8) and the conditions arising from this found, as for the seven-sphere.

MODEL 1: Squashed Seven-Sphere J^7.

The equations for this model are given in (7.35) and (7.36).

MODEL 2: Seven-Torus T^7.

$\{\Omega^i\}$, i=1,...7 is a basis of left-invariant 1-forms for T^7 and the Ω^i obey

$$d\,\Omega^i = 0 \tag{7.63}$$

the space having only abelian $U(1)^7$ symmetry. Eqn.(7.63) leads to special features for this model. The Ricci tensor is (7.62a,b) and

$$R^{m}{}_{m} = \frac{1}{2}\frac{\ddot{s}_m}{s_m} + \frac{3}{2}\frac{\dot{r}\,\dot{s}_m}{r\,s_m} + \sum_{\substack{n \\ n\neq m}} \frac{\dot{s}_m\,\dot{s}_n}{s_m\,s_n} \quad . \tag{7.64}$$

The $H_n(s_p)$ terms being zero due to the flatness of T^7. The B^m are chosen

$$B^m = s_m\, b^m = s_m\, \Omega^i\, \delta^m{}_i \tag{7.65}$$

and the E_i chosen

$$E_1 = b^{351},\ E_2 = b^{324},\ E_3 = b^{621},\ E_4 = b^{417},\ E_5 = b^{527},\ E_6 = b^{637},\ E_7 = b^{456}, \tag{7.66}$$

$$G_1 = b^{7426},\ G_2 = b^{7156},\ G_3 = b^{7453},\ G_4 = b^{2356},\ G_5 = b^{3164},\ G_6 = b^{1245},\ G_7 = b^{1237}. \quad (7.67)$$

Because of (7.63)

$$dE_i = 0 = dG_i \qquad \forall i\ , \qquad (7.68)$$

which gives a special form for dA_7

$$dA_7 = \sum_i \dot{q}_i\, V^\circ \wedge E_i\ . \qquad (7.69)$$

F_7 can be constructed from dA_7 in the usual manner but, because $db^m = 0$ there can also be added a term $\sum_i l_i G_i$ so

$$F = \oint \Omega + dA_7 + \sum_i l_i\, G_i\ . \qquad (7.70)$$

This satisfies the identity $dF = 0$ if the l_i are constant. Note that

$$E_i \wedge E_j = -\delta_{ij}\, \Omega\ ,\quad \sum_i \sum_j E_i \wedge G_j = \sum_i E_i \wedge G_i = -7\, \Omega. \qquad (7.71)$$

Introducing the notation

$$s_i^3 = s_m s_n s_p \quad \text{where } m,n,p \text{ are the indices in } E_i, \qquad (7.72a)$$

$$s_i^4 = s_m s_n s_p s_q \quad \text{where } m,n,p,q \text{ are the indices in } G_i, \qquad (7.72b)$$

$$s^7 = \prod_{m=1}^{7} s_m, \quad \text{Eg.} \quad s_1^3 = s_3 s_5 s_4, \quad s_1^4 = s_7 s_4 s_2 s_6, \qquad (7.72c)$$

the Einstein equations are

$$R^0_M = \frac{\delta^0_M}{4!}\left(-\oint^2 - \sum_i \frac{\dot{q}_i^2}{(s_i^3)^2} - \frac{1}{2}\sum_i \frac{l_i^2}{(s_i^4)^2}\right), \qquad (7.73a)$$

$$R^a_M = \frac{\delta^a_M}{4!}\left(-\oint^2 + \frac{1}{2}\sum_i \frac{\dot{q}_i^2}{(s_i^3)^2} - \frac{1}{2}\sum_i \frac{l_i^2}{(s_i^4)^2}\right), \qquad (7.73b)$$

$$R^m_M = \frac{\delta^m_M}{4!}\left(\frac{1}{2}\oint^2 - \frac{3}{2}\sum_{i_s} \frac{\dot{q}_i^2}{(s_i^3)^2} + \frac{3}{2}\sum_{j_s} \frac{l_j^2}{(s_j^4)^2} + \frac{1}{2}\sum_i \frac{\dot{q}_i^2}{(s_i^3)^2} - \frac{1}{2}\sum_i \frac{l_i^2}{(s_i^4)^2}\right), \qquad (7.73c)$$

where $\sum_{i_s}$ ($\sum_{j_s}$) are sums over the indices m for which the $E_i(G_i)$ contains b^m. The Maxwell equations are

$$\frac{d}{dt}\left(\oint s^7\right) - \sum_i \frac{\dot{q}_i l_i}{2} = 0\ , \qquad (7.74a)$$

$$\ddot{g}_i = -\dot{g}_i\left(3\frac{\dot{r}}{r} + \frac{s_i^3}{s_i^4}\frac{d}{dt}\left(\frac{s_i^4}{s_i^3}\right)\right) - f\frac{\lambda_i}{2}\frac{s_i^3}{s_i^4} \, . \tag{7.74b}$$

MODEL 3: The Coset Space SU(2)xSU(2)xSU(2)/U(1)xU(1).

$\{\Omega^i\}$, $i=1,2,0,4,5,0',6,7,0''$ are a basis of left-invariant 1-forms and obey the Maurer-Cartan equations (7.40) with the G invariant vielbein given by

$$B^i = s\,\Omega^i \, , \qquad i = 1, 2, \tag{7.75a}$$

$$B^{i'} = u\,\Omega^{i'} \, , \qquad i' = 4, 5, \tag{7.75b}$$

$$B^{i''} = v\,\Omega^{i''} \, , \qquad i'' = 6, 7, \tag{7.75c}$$

$$B^3 = w\left(p_1\Omega^0 + p_2\Omega^{0'} + p_3\Omega^{0''}\right) = w\,\Omega^3, \tag{7.75d}$$

where the integers p_1, p_2, p_3 define the topology{70}. The Ricci tensor is (7.62a,b) with s_m taking the appropriate values from (7.75) and on M_7

$$R^3{}_3 = \frac{1}{2}\frac{\ddot{w}}{w} + \frac{w^2}{4}\left(\frac{p_1^2}{s^4} + \frac{p_2^2}{u^4} + \frac{p_3^2}{v^4}\right) + \frac{3}{2}\frac{\dot{r}\dot{w}}{rw} + \frac{\dot{w}}{w}\left(\frac{\dot{s}}{s} + \frac{\dot{u}}{u} + \frac{\dot{v}}{v}\right) , \tag{7.76a}$$

$$R^1{}_1 = R^2{}_2 = \frac{1}{2}\frac{\ddot{s}}{s} + \frac{1}{2s^2} - \frac{w^2 p_1^2}{4s^4} + \frac{3}{2}\frac{\dot{r}\dot{s}}{rs} + \frac{\dot{s}}{s}\left(\frac{1}{2}\frac{\dot{w}}{w} + \frac{\dot{u}}{u} + \frac{\dot{v}}{v} + \frac{1}{2}\frac{\dot{s}}{s}\right). \tag{7.76b}$$

$R^4{}_4 = R^5{}_5$ and $R^6{}_6 = R^7{}_7$ are obtained by the transformation $(s,u,v,p_1) \rightarrow (u,v,s,p_2) \rightarrow (v,s,u,p_3)$. A choice of E_i is

$$E_1 = b^{417} + b^{426} + b^{516} + b^{257} \, , \tag{7.77a}$$

$$E_2 = b^{123} \, , \quad E_3 = b^{453} \, , \quad E_4 = b^{673} \tag{7.77b}$$

with $G_i = {}^*_7E_i$. E_1 will only satisfy $dE_1 = \frac{1}{7}{}^*_7E_1$ if $p_1 = p_2 = p_3 = c$ or $g_1 = \dot{g}_1 = 0$. If I take the former case notice that the $p_2 = p_3 = 0$ coset manifold $S^2 \times S^2 \times S^3$ does not permit a time dependent formalism. The Einstein equations are

$$R^o{}_M = \frac{\delta^o_M}{4!}\left[-f^2 - 4A - B - C - D - 2H - \frac{I}{2} - \frac{J}{2} - \frac{K}{2}\right] , \tag{7.78a}$$

$$R^a{}_M = \frac{\delta^a_M}{4!}\left[-f^2 + 2A + \frac{B}{2} + \frac{C}{2} + \frac{D}{2} - 2H - \frac{I}{2} - \frac{J}{2} - \frac{K}{2}\right] , \tag{7.78b}$$

$$R^3{}_M = \frac{\delta^3_M}{4!}\left[\frac{1}{2}f^2 + 2A - B - C - D + 4H - \frac{I}{2} - \frac{J}{2} - \frac{K}{2}\right], \tag{7.78c}$$

$$R^1{}_M = \frac{\delta^1_M}{4!}\left[\frac{1}{2}f^2 - A - B + \frac{C}{2} + \frac{D}{2} + H - \frac{I}{2} + J + K\right], \tag{7.78d}$$

$$R^4{}_M = \frac{\delta^4_M}{4!}\left[\frac{1}{2}f^2 - A + \frac{B}{2} - C + \frac{D}{2} + H + I - \frac{J}{2} + K\right], \tag{7.78e}$$

$$R^6{}_M = \frac{\delta^6_M}{4!}\left[\frac{1}{2}f^2 - A + \frac{B}{2} + \frac{C}{2} - D + H + I + J - \frac{K}{2}\right], \tag{7.78f}$$

where

$$A = \frac{\dot{g}_1^2}{s^2u^2v^2}, \quad B = \frac{\dot{g}_2^2}{s^4w^2}, \quad C = \frac{\dot{g}_3^2}{u^4w^2}, \quad D = \frac{\dot{g}_4^2}{v^4w^2}, \quad E = \frac{g_1^2}{s^2u^2v^2w^2}, \tag{7.79a}$$

$$I = \frac{(p_3 g_3 + p_2 g_4)^2}{u^4 v^4}, \quad J = \frac{(p_3 g_2 + p_1 g_4)^2}{s^4 v^4}, \quad K = \frac{(p_2 g_2 + p_1 g_3)^2}{s^4 u^4}. \tag{7.79b}$$

The Maxwell equations are

$$-\frac{d}{dt}(f s^2u^2v^2w) = 2\dot{g}_1 g_1 + \frac{\dot{g}_2}{2}(p_3 g_3 + p_2 g_4) + \frac{\dot{g}_3}{2}(p_3 g_2 + p_1 g_4) + \frac{\dot{g}_4}{2}(p_2 g_2 + p_1 g_3), \tag{7.80a}$$

$$\frac{1}{r^3}\frac{d}{dt}(\dot{g}_1 w r^3) = \frac{f g_1}{2} - \frac{c g_1}{w}, \tag{7.80b}$$

$$\frac{1}{r^3}\frac{d}{dt}\left(\frac{\dot{g}_2 u^2v^2r^3}{s^2w}\right) = \frac{f}{2}(p_3 g_3 + p_2 g_4) - (p_3 g_2 + p_1 g_4)p_3\frac{u^2w}{s^2v^2} - (p_2 g_2 + p_1 g_3)\frac{p_2 v^2 w}{s^2u^2}, \tag{7.80c}$$

$$\frac{1}{r^3}\frac{d}{dt}\left(\frac{\dot{g}_3 v^2s^2r^3}{u^2w}\right) = \frac{f}{2}(p_3 g_2 + p_1 g_4) - (p_3 g_3 + p_2 g_4)\frac{p_3 s^2w}{u^2v^2} - (p_2 g_2 + p_1 g_3)\frac{p_1 v^2 w}{s^2u^2}, \tag{7.80d}$$

$$\frac{1}{r^3}\frac{d}{dt}\left(\frac{\dot{g}_4 u^2s^2r^3}{v^2w}\right) = \frac{f}{2}(p_2 g_2 + p_1 g_3) - (p_3 g_3 + p_2 g_4)\frac{p_2 s^2w}{u^2v^2} - (p_3 g_2 + p_1 g_4)\frac{p_1 u^2 w}{s^2v^2}. \tag{7.80e}$$

Note that (7.80b) only applies if $c = p_1 = p_2 = p_3$. Eqn.(7.80a) can be integrated

$$f s^2u^2v^2w + g_1^2 + \frac{p_3}{2}g_2 g_3 + \frac{p_1}{2}g_3 g_4 + \frac{p_2}{2}g_2 g_4 = \text{constant}. \tag{7.80a'}$$

MODEL 4: The Coset Space SU(3)xSU(2)xU(1)/SU(2)xU(1)xU(1).

$\{\Omega^i\}$, $i=\dot{1},\dot{2},\dot{3},8,4,5,6,7,\ddot{3},1,2,\ddot{3}$ are a basis of left-invariant 1-forms for the cotangent space of G. Setting

$$\gamma^0 = dt, \quad \gamma^a = r\,\nu^a, \tag{7.81a}$$

$$B^A = s\,\Omega^A = s\,b^A, \quad A = 4,5,6,7, \tag{7.81b}$$

$$B^m = u\,\Omega^m = u\,b^m\ , \quad m = 1, 2, \tag{7.81c}$$

$$B^3 = v\left(\sqrt{3}\,p_1\Omega^8 + p_2\Omega^{\ddot{3}} + 2p_3\Omega^{\dddot{3}}\right) = v\,\Omega^3 = v\,b^3\ , \tag{7.81d}$$

where again p_1, p_2, p_3 define the topology of particular embeddings. The Ricci tensor is (7.62a,b) and on M_7

$$R^3{}_3 = \frac{1}{2}\frac{\ddot{v}}{v} + \frac{p_2^2 v^2}{4u^4} + \frac{9p_1^2}{8}\frac{v^2}{s^4} + \frac{3}{2}\frac{\dot{r}\dot{v}}{rv} + \frac{\dot{v}}{v}\left(\frac{\dot{u}}{u} + 2\frac{\dot{s}}{s}\right), \tag{7.82a}$$

$$R^1{}_1 = R^2{}_2 = \frac{1}{2}\frac{\ddot{u}}{u} - \frac{p_2^2 v^2}{4u^4} + \frac{1}{2u^2} + \frac{3}{2}\frac{\dot{r}\dot{u}}{ru} + \frac{\dot{u}}{u}\left(\frac{1}{2}\frac{\dot{v}}{v} + 2\frac{\dot{s}}{s} + \frac{1}{2}\frac{\dot{u}}{u}\right), \tag{7.82b}$$

$$R^4{}_4 = R^5{}_5 = R^6{}_6 = R^7{}_7 = \frac{1}{2}\frac{\ddot{s}}{s} - \frac{9p_1^2}{16}\frac{v^2}{s^4} + \frac{3}{4s^2} + \frac{3}{2}\frac{\dot{r}\dot{s}}{rs} + \frac{\dot{s}}{s}\left(\frac{1}{2}\frac{\dot{v}}{v} + \frac{\dot{u}}{u} + \frac{3}{2}\frac{\dot{s}}{s}\right). \tag{7.82c}$$

A choice of E_i, G_i is

$$E_1 = b^{123}\ , \quad E_2 = \left(b^{45} + b^{67}\right)\wedge b^3\ , \tag{7.83a}$$

$$G_1 = b^{4567}\ , \quad G_2 = b^{12}\wedge\left(b^{45} + b^{67}\right) \tag{7.83b}$$

related by

$$dE_1 = \frac{3p_1}{2}G_2\ , \quad dE_2 = 3p_1G_1 - p_2G_2\ . \tag{7.84}$$

The Einstein equations give

$$R^0{}_M = \frac{\delta^0_M}{4!}\left(-f^2 - \frac{\dot{q}_1^2}{u^4v^2} - \frac{2\dot{q}_2^2}{s^4v^2} - \frac{9}{2}\frac{p_1^2q_2^2}{s^8} - \frac{1}{4u^4s^4}\left(3p_1q_1 - 2p_2q_2\right)^2\right), \tag{7.85a}$$

$$R^a{}_M = \frac{\delta^a_M}{4!}\left(-f^2 + \frac{\dot{q}_1^2}{2u^4v^2} + \frac{\dot{q}_2^2}{s^4v^2} - \frac{9}{2}\frac{p_1^2q_2^2}{s^8} - \frac{1}{4u^4s^4}\left(3p_1q_1 - 2p_2q_2\right)^2\right), \tag{7.85b}$$

$$R^3{}_M = \frac{\delta^3_M}{4!}\left(\frac{1}{2}f^2 - \frac{\dot{q}_1^2}{u^4v^2} - \frac{2\dot{q}_2^2}{s^4v^2} - \frac{9}{2}\frac{p_1^2q_2^2}{s^8} - \frac{1}{4u^4s^4}\left(3p_1q_1 - 2p_2q_2\right)^2\right), \tag{7.85c}$$

$$R^m{}_M = \frac{\delta^m_M}{4!}\left(\frac{1}{2}f^2 - \frac{\dot{q}_1^2}{u^4v^2} + \frac{3\dot{q}_2^2}{s^4v^2} - \frac{9}{2}\frac{p_1^2q_2^2}{s^8} + \frac{1}{2u^4s^4}\left(3p_1q_1 - 2p_2q_2\right)^2\right), \tag{7.85d}$$

$$R^A{}_M = \frac{\delta^A_M}{4!}\left(\frac{1}{2}f^2 + \frac{\dot{q}_1^2}{2u^4v^2} - \frac{\dot{q}_2^2}{2s^4v^2} + \frac{9p_1^2q_2^2}{s^8} + \frac{1}{8u^4s^4}\left(3p_1q_1 - 2p_2q_2\right)^2\right). \tag{7.85e}$$

The Maxwell equations are

$$\frac{d}{dt}\left(f s^4u^2v\right) + \frac{3}{2}p_1\dot{q}_1q_2 + \frac{3}{2}p_1q_1\dot{q}_2 - p_2q_2\dot{q}_2 = 0 \tag{7.86a}$$

$$\frac{1}{r^3}\frac{d}{dt}\left(\frac{\dot{q}_1 s^4 r^3}{u^2 v}\right) + \frac{3p_1}{2}\frac{v}{u^2}\left(3p_1 q_1 - 2p_2 q_2\right) - \frac{3}{2} f\, p_1 q_2 = 0 , \tag{7.86b}$$

$$\frac{1}{r^3}\frac{d}{dt}\left(\dot{q}_2 \frac{u^2 r^3}{v}\right) - \frac{p_2 v}{2u^2}\left(3p_1 q_1 - 2p_2 q_2\right) + \frac{9}{2} p_1^2 q_2 \frac{u^2 v}{s^4} - \frac{f}{4}\left(3p_1 q_1 - 2p_2 q_2\right) = 0. \tag{7.86c}$$

Eqn.(7.86a) can be integrated

$$f s^4 u^2 v + \frac{3}{2} p_1 q_1 q_2 - \frac{p_2}{4} q_2^2 = \text{constant}. \tag{7.86a'}$$

I shall now consider two models in which M_7 is a coset space only in a trivial sense (ie. H = 1). The reason for considering these models will become clear later. Because H = 1 the left-invariant 1-forms on G can be taken as the vielbein 1-forms.

MODEL 5: $S^1 \times S^2_{(1)} \times S^2_{(2)} \times S^2_{(3)}$.

If $\{\Omega^i\}$, i=1,...,7 are left-invariant 1-forms on $S^1 \times S^2 \times S^2 \times S^2$ the vielbeins on M_7 can be taken to be

$$B^7 = s\,\Omega^7 , \quad B^m = u\,\Omega^m , \quad m = 1, 2, \tag{7.87a}$$

$$B^{m'} = v\,\Omega^{m'} , \quad m' = 3, 4, \qquad B^{m''} = w\,\Omega^{m''} , \quad m'' = 5, 6. \tag{7.87b}$$

The Maurer-Cartan equations are

$$db^7 = 0 , \quad db^1 = 0 , \quad db^2 = \frac{1-\rho^2}{\rho}\, b^1 \wedge b^2 , \tag{7.88a,b,c}$$

where ρ is a coordinate on $S^2_{(1)}$. The pairs of indices (3,4),(4,5) reproduce (7.87b,c) with appropriate changes of indices. The Ricci tensor is as (7.62a,b) and on M_7

$$R^7{}_7 = \frac{1}{2}\frac{\ddot{s}}{s} + \frac{3}{2}\frac{\dot{r}\dot{s}}{rs} + \frac{\dot{s}}{s}\left(\frac{\dot{u}}{u} + \frac{\dot{v}}{v} + \frac{\dot{w}}{w}\right) , \tag{7.89a}$$

$$R^m{}_m = \frac{1}{2}\frac{\ddot{u}}{u} + \frac{2}{u^2} + \frac{3}{2}\frac{\dot{r}\dot{u}}{ru} + \frac{\dot{u}}{u}\left(\frac{\dot{v}}{v} + \frac{\dot{w}}{w} + \frac{1}{2}\frac{\dot{s}}{s}\right) . \tag{7.89b}$$

$R^{m'}{}_{m'}$, $R^{m''}{}_{m''}$ are given by (7.89b) with the permutations $(u,v,w) \rightarrow (v,w,u) \rightarrow (w,u,v)$. A choice of E_i, G_i is

$$E_1 = b^{712} , \quad E_2 = b^{734} , \quad E_3 = b^{756} , \tag{7.90a}$$

$$G_1 = b^{3456} , \quad G_2 = b^{1256} , \quad G_3 = b^{1234} , \tag{7.90b}$$

which satisfy

$$dE_i = 0 = dG_i \qquad \forall i . \tag{7.91}$$

The Einstein equations are

$$R^0{}_0 = \frac{\delta^0_0}{4!}\left(-f^2 - \frac{\dot{g}_1^2}{s^2u^4} - \frac{\dot{g}_2^2}{s^2v^4} - \frac{\dot{g}_3^2}{s^2w^4}\right) , \tag{7.92a}$$

$$R^a{}_M = \frac{\delta^a_M}{4!}\left(-f^2 + \frac{\dot{g}_1^2}{2s^2u^4} + \frac{\dot{g}_2^2}{2s^2v^4} + \frac{\dot{g}_3^2}{2s^2w^4}\right) , \tag{7.92b}$$

$$R^m{}_M = \frac{\delta^m_M}{4!}\left(\frac{1}{2}f^2 - \frac{\dot{g}_1^2}{s^2u^4} + \frac{\dot{g}_2^2}{2s^2v^4} + \frac{\dot{g}_3^2}{2s^2w^4}\right), \tag{7.92c}$$

$$R^{m'}{}_M = \frac{\delta^{m'}_M}{4!}\left(\frac{1}{2}f^2 + \frac{\dot{g}_1^2}{2s^2u^4} - \frac{\dot{g}_2^2}{s^2v^4} + \frac{\dot{g}_3^2}{2s^2w^4}\right), \tag{7.92d}$$

$$R^{m''}{}_M = \frac{\delta^{m''}_M}{4!}\left(\frac{1}{2}f^2 + \frac{\dot{g}_1^2}{2s^2u^4} + \frac{\dot{g}_2^2}{2s^2v^4} - \frac{\dot{g}_3^2}{s^2w^4}\right) , \tag{7.92e}$$

$$R^7{}_M = \frac{\delta^7_M}{4!}\left(\frac{1}{2}f^2 - \frac{\dot{g}_1^2}{s^2u^4} - \frac{\dot{g}_2^2}{s^2v^4} - \frac{\dot{g}_3^2}{s^2w^4}\right) . \tag{7.92f}$$

The Maxwell equations can all be integrated because of (7.91) and are

$$f\, s u^2 v^2 w^2 = c1 , \tag{7.93a}$$

$$\dot{g}_1 \frac{v^2 w^2 r^3}{s u^2} = c2 , \tag{7.93b}$$

$$\dot{g}_2 \frac{u^2 w^2 r^3}{s v^2} = c3 , \tag{7.93c}$$

$$\dot{g}_3 \frac{u^2 v^2 r^3}{s w^2} = c4 , \tag{7.93d}$$

where c1, c2, c3 and c4 are constants.

MODEL 6: $S^1 \times S^3_{(1)} \times S^3_{(2)}$.

If $\{\Omega^i\}$, i=1,...,7 are left invariant 1-forms on $S^1 \times S^3 \times S^3$ the vielbeins can be taken as

$$B^7 = s\,\Omega^7 , \tag{7.94a}$$

$$B^m = u\,\Omega^m , \quad m = 1, 2, 3 , \tag{7.94b}$$

$$B^{m'} = v\,\Omega^{m'} , \quad m' = 4, 5, 6 . \tag{7.94c}$$

The Maurer-Cartan equations are those of U(1)xSU(2)xSU(2)

$$db^7 = 0 \,, \tag{7.95a}$$

$$db^1 = b^2 \wedge b^3 \,, \quad db^2 = b^3 \wedge b^1 \,, \quad db^3 = b^1 \wedge b^2 \,, \tag{7.95b}$$

$$db^4 = b^5 \wedge b^6 \,, \quad db^5 = b^6 \wedge b^4 \,, \quad db^6 = b^4 \wedge b^5 . \tag{7.95c}$$

The Ricci tensor is as (7.62a,b) and on M_7

$$R^7{}_7 = \frac{1}{2}\frac{\ddot{s}}{s} + \frac{3}{2}\frac{\dot{r}\dot{s}}{rs} + \frac{3}{2}\frac{\dot{s}}{s}\left(\frac{\dot{u}}{u} + \frac{\dot{v}}{v}\right) \,, \tag{7.96a}$$

$$R^m{}_m = \frac{1}{2}\frac{\ddot{u}}{u} + \frac{3}{4u^2} + \frac{3}{2}\frac{\dot{r}\dot{u}}{ru} + \frac{\dot{u}}{u}\left(\frac{\dot{u}}{u} + \frac{3}{2}\frac{\dot{v}}{v} + \frac{1}{2}\frac{\dot{s}}{s}\right) \,, \tag{7.96b}$$

the equation for $R^{m'}{}_{m'}$ is obtained from (7.96b) by interchanging u and v. A choice of E_i, G_i is

$$E_1 = b^{123} \,, \quad E_2 = b^{456} \,, \tag{7.97a}$$

$$G_1 = b^{4567} \,, \quad G_2 = b^{1237} \,, \tag{7.97b}$$

satisfying

$$dE_i = 0 = dG_i \,, \quad i = 1, 2 . \tag{7.98}$$

The Einstein equations are

$$R^0{}_M = \frac{\delta^0_M}{4!}\left(-f^2 - \frac{\dot{g}_1^2}{u^6} - \frac{\dot{g}_2^2}{v^6}\right) \,, \tag{7.99a}$$

$$R^a{}_M = \frac{\delta^a_M}{4!}\left(-f^2 + \frac{\dot{g}_1^2}{u^6} + \frac{\dot{g}_2^2}{v^6}\right) \,, \tag{7.99b}$$

$$R^7{}_M = \frac{\delta^7_M}{4!}\left(\frac{1}{2}f^2 + \frac{\dot{g}_1^2}{2u^6} + \frac{\dot{g}_2^2}{2v^6}\right) \,, \tag{7.99c}$$

$$R^m{}_M = \frac{\delta^m_M}{4!}\left(\frac{1}{2}f^2 - \frac{\dot{g}_1^2}{u^6} + \frac{\dot{g}_2^2}{2v^6}\right) \,, \tag{7.99d}$$

$$R^{m'}{}_M = \frac{\delta^{m'}_M}{4!}\left(\frac{1}{2}f^2 + \frac{\dot{g}_1^2}{2u^6} - \frac{\dot{g}_2^2}{v^6}\right) . \tag{7.99e}$$

The Maxwell equations can be integrated because of (7.98) and are

$$\phi \, s \, u^3 v^3 = c1, \tag{7.100a}$$

$$\dot{q}_1 \, s \, v^3/u^3 = c2, \tag{7.100b}$$

$$\dot{q}_2 \, s \, u^3/v^3 = c3, \tag{7.100c}$$

where c1, c2 and c3 are constants.

Chapter VIII TIME-DEPENDENT SOLUTIONS OF d=11 SUPERGRAVITY

In the last chapter I derived time dependent equations for several models of d=11 supergravity with components of the 4-form F on both M_4 and M_7 with coefficients functions of time. I now give solutions. In most cases these have to be numerical or approximate but there are some exact solutions for special cases. I will first give some exact solutions, then some discussion of other solutions from refs.{72,73,74,75,76,77,78} and my own work, and then solutions of the models described in chapter VII. The originality of the approach here is the ability to put a time varying F_7 on M_7, and by this not require M_7 to be an Einstein space, even when no covariantly constant spinors exist. I then select solutions that give possible cosmological behaviour for the early universe in terms of an inflationary period with a large expansion of the scale parameter r(t) for the ordinary dimensions and a constant or decreasing scale factor s(t) for the extra dimensions. I showed in chapter VI that such a dynamical compactification can occur for non-supergravity models. Here I show that it can occur in more realistic models with the matter fields specified by supergravity and perhaps with a more realistic gauge group. I identify the conditions required for inflationary behaviour.

The first cosmological solution of d=11 supergravity was by Freund{67}. Consider a metric

$$g = V^0 \otimes V^0 - r^2(t) \sum_{a=1}^{3} V^a \otimes V^a - s^2(t) \sum_{m=1}^{7} b^m \otimes b^m \tag{8.1}$$

for a product space $R^1 \times M_3 \times M_7$ with curvatures k_3, k_7 respectively for the isotropic spaces M_3, M_7. The Ricci tensor for such a space is given by (7.62) with $s_m = s$ for all m and $H_n = k_7/s^2$. For F given by the Freund-Rubin ansatz

$$F = f(t)\, \Omega \tag{8.2}$$

the Einstein equation (7.27) gives

$$R^0{}_0 = -\frac{1}{4!} f^2 \delta^0_0, \tag{8.3a}$$

$$R^{a}{}_{M} = -\frac{1}{4!} \oint^{2} \delta^{a}{}_{M} \quad , \tag{8.3b}$$

$$R^{m}{}_{M} = \frac{1}{2.4!} \oint^{2} \delta^{m}{}_{M} \quad , \tag{8.3c}$$

and the Maxwell equation (7.8), which can be integrated, gives

$$\frac{d}{dt}\left(\oint s^{7}\right) = 0 \quad ; \quad \oint s^{7} = constant = \oint_{1} \quad . \tag{8.4}$$

Note that all the models in chapter VII reproduce (8.3) and (8.4) in the special case $g_i = \dot{g}_i = 0$, $s_m = s$, $\forall$ m. Freund{67} gave two solutions to (8.3-4):

1) $r = r_0 t$, $s = s_0 t^{\frac{1}{7}}$, $f = \sqrt{\frac{12}{7}}\, t^{-1}$, $k_3 = -\frac{27}{14} r_0^{\,2}$, $k_7 = 0$. In this case M_7 is flat (ie. locally, though not necessarilly globally, a maximally symmetric torus) and M_3 a pseudosphere.

2) $r = r_0 \cos(\alpha t)$ (or $\sin(\alpha t)$), s = constant, f = constant, $f^2 = 36\alpha^2$, $k_7 = \frac{3}{4}\alpha^2 s^2$, $k_3 = -\alpha^2 r_0^{\,2}$. M_7 is static and compact.

There are no power law solutions other than (1) except $r = r_0 t$, $s = s_0 t$ which forces $f = 0$ and $k_3 = -r_0^{\,2}$, $k_7 = -9s^2/2$ which is unacceptable. More disappointing is the absence of solutions of the form $r = r_0 e^{\alpha t}$, $s = s_0 e^{-\beta t}$ where α and β are constants. Gleiser et al{77} point out that solutions of the type (2) exist when an Englert type F_7 field (7.3) is included, the relations between α, f etc. having different coefficients. However their generalisation of solutions of type (1) is inconsistent, as was pointed out by Lorenz-Petzold{75}.

Alvarez{73} takes F of the form (8.2) and reproduces (7.62) and (8.3-4) and takes the particular values $k_3 = -1$, $k_7 = 1$ (ie. open M_4 and closed M_7). Introducing the variables $\theta_1 \equiv \dot{r}/r$, $\theta_2 \equiv \dot{s}/s$ he notes some limits (if θ_1 and θ_2 are bounded):

1) $r \to \infty$, $s \to \infty$. The system can be completely analysed in terms of θ_1 and θ_2. However this limit is unphysical.

2) $r \to 0$, $s \to \infty$; $\dot{\theta}_1 > 0$, $\dot{\theta}_2 < 0$.

3) $r \to \infty$, $s \to 0$; $\dot{\theta}_1 < 0$, $\dot{\theta}_2 > 0$.

I later confirm findings (2) and (3) for specific cases of asymptotic solutions. If $r \sim t^{n}$, $s \sim t^{-m}$, n,m > 0 then $\dot{\theta}_1 < 0$, $\dot{\theta}_2 > 0$; if n,m < 0 then $\dot{\theta}_1 > 0$, $\dot{\theta}_2 < 0$. Alvarez{73} performs some numerical calculations, which are a special case of my own work, and concludes that for F as in (8.2) there is not a non-singular solution with dynamical compactification. This is not entirely correct. Ref.{73} considers, without physical justification, initial values for the scale factors and their time derivatives which differ by several orders of magnitude.

The universe is isotropic on a large scale today but need not have been near some initial singularity. I assume it was and consider only Robertson-Walker type cosmologies. Lorenz-Petzold{75,76} and Demaret et al{72} have more recently looked at more general solutions for M_4 which are products of a Bianchi type space and a seven-space described by the metric

$$ds^2 = -dt^2 + a^2(t)\, V^1 \otimes V^1 + b^2(t)\, V^2 \otimes V^2 + c^2(t)\, V^3 \otimes V^3 + \sum_m s^2(t)\, b^m \otimes b^m . \quad (8.5)$$

Demaret et al{72} reproduce the Freund{67} power law solution ($a=b=c \sim t$, $s \sim t^{\frac{1}{7}}$), and find a generalisation of it ($a \sim t$, $b \sim t^{\alpha}$, $c \sim t^{\beta}$, $s \sim t^{\gamma}$) in which the indices are constrained by algebraic relations, for $F = f\Omega$ and a seven-torus for the extra dimensions. The method of solution of ref.{72} is different to my own. They use what they call 'Bianchi techniques'{79} which require spatial sections (ie. $M_3 \times M_7$ at a fixed time) to be 10-dimensional group spaces. This obviously excludes coset spaces for the extra dimensions, however they express some hope of extending their methods to include them. Due to the actual form of their metric my results and theirs are not always easily comparable, sometimes requiring non-trivial transformations of the time coordinates to make $V^0 = dt$; however they give no inflationary solutions.

The Lorenz-Petzold{75,76} approach is similar to my own but only considers the seven-sphere(S^7) or seven-torus(T^7) for the extra dimensions and the field F given by the Englert ansatz (7.5). However refs.{75,76} do give new analytic solutions, available due to the extra freedom of (8.5), including a special case of a Bianchi solution the Robertson-Walker solution of Freund{67}. The concentrating on new M_4 does not significantly improve the cosmological behaviour.

Fujii and Okada{74} consider time dependent equations when M_7 is a round or squashed seven-sphere without torsion and a round seven-sphere with Englert torsion{117}. For the case of the seven-sphere without torsion they consider a squashing parameter λ which depends on time. The approach of ref.{74} is very much to look at the theory from the 4-dimensional viewpoint, performing conformal transformations (Weyl rescalings) to give the standard d=4 Einstein-Hilbert action as the first term in the action density.

I now describe a numerical method for solving the equations of motion, illustrating it with the particular example of the seven-sphere described in chapter VII before going on to discuss particular features of the various models. The 4-form F is constructed as in (7.21) from the 3-forms E_1, E_2 given in (7.15) giving, after substitution in (7.8) the Maxwell conditions (7.35). Equating the Ricci tensor calculated from the curvature with that given by the Einstein equations gives (7.36). This gives a system of seven equations with six unknowns. However all of (7.36) are not independent due to the Bianchi identity ($D^M T_{MN} = 0$) and so I can substitute for $\ddot{r}$, $\ddot{s}$ and $\ddot{u}$ from (7.36b,c,d) into (7.36a) to obtain the condition

$$\frac{3k_3}{r^2} + \frac{1}{s^2}\left(6 - \frac{3}{2}\lambda^2 + \frac{3}{4\lambda^2}\right) + 3\left(\frac{\dot{r}}{r}\right)^2 + 6\left(\frac{\dot{s}}{s}\right)^2 + 3\left(\frac{\dot{u}}{u}\right)^2 + 12\frac{\dot{r}\dot{s}}{rs} \tag{8.6}$$

$$+ 9\frac{\dot{r}\dot{u}}{ru} + 12\frac{\dot{s}\dot{u}}{su} - \frac{1}{4!}\left[\frac{3}{2}f^2 + \frac{9(g_1+g_2)^2}{s^8\lambda^4} + 54\frac{g_1^2}{s^8} + \frac{9\dot{g}_1^2}{s^6\lambda^2} + \frac{3}{2}\frac{\dot{g}_2^2}{s^6\lambda^6}\right] = 0.$$

If this is satisfied at any time the equations of motion will ensure that it is satisfied at all times. Eqn.(8.6) obviously imposes restrictions on the values all the parameters can take in a numerical calculation; it was imposed as a condition on the initial values.

Eqns.(7.36b,c,d) and (7.35b,c) were rearranged to give second order differential equations for $\ddot{r}$, $\ddot{s}$, $\ddot{u}$, $\ddot{g}_1$, $\ddot{g}_2$ respectively in terms of r, s, u, g_1, g_2 and their first time derivatives and f. Initial values were chosen for r, s, u, g_1, g_2 and their first time derivatives. Eqn.(8.6) was then used to calculate f, for a specific value of k_3, usually $k_3 = -1$; thus giving the constant in (7.35a'). The system was then numerically integrated (see Appendix B) with respect to time. Since there is no equation describing the time evolution of f once (7.35a) has been integrated it was recalculated at each step using (7.35a'). In some models the equivalent equations to (7.35b,c) can be integrated exactly and in these cases they are treated in the same way as (7.35a).

One of the difficulties in studying these models is the large number of initial conditions which have to be chosen, ten in the case of the squashed seven-sphere. After preliminary studies I decided to set the modulus values of the scale factors r and s and their time derivatives equal to one. Choosing, say, $\dot{r} \gg r$ etc. changed the initial shape of the curve of the scale factors with time but did not qualitatively change the behaviour. The g_i, $\dot{g}_i$ and f were chosen in various ways suggested by the forms of the equations to show the effect of the different terms. Innumerable results could have been produced, many were; what follows is an illustrative selection. I do not address the problem of how the initial values arose out of whatever existed before.

Table(8.1)
Initial values for solutions of the seven-sphere.

r	s	$\dot{r}$	$\dot{s}$	λ^2	f	g_1	g_2	$\dot{g}_1$	$\dot{g}_2$	$\dot{g}_2/\dot{g}_1$	Graph
1	1	1	1	1/5	28.69	0	0	0	0	-	fig.(8.1)
1	1	1	-1	1/5	12.30	0	0	0	0	-	fig.(8.2)
1	1	-1	1	1/5	12.30	0	0	0	0	-	fig.(8.3)
1	1	1	-1	1/5	0	0	0	2.08	0.42	1/5	fig.(8.4)
1	1	-1	1	1/5	0	0	0	2.08	0.42	1/5	fig.(8.5)
1	1	1	1	1/10	0	0	0	3.56	3.56	1	fig.(8.6)

Firstly I will look at solutions for Model 1. Table(8.1) contains the initial values for the scale factors and field coefficients for the graphs plotted in figs.(8.1-6). A study of these and other solutions together with (7.35,36) reveals certain features of the system. All the graphs show that at some critical time t_1 there is a singularity in the solution with $r \to 0$, $s \to \infty$, although the behaviour before the singularity is very varied. If $g_1, g_2, \dot{g}_1, \dot{g}_2 = 0$ initially then this is so for all time by (7.35b); this being the case in figs.(8.1-3). Figs.(8.1-5) illustrate the solutions with $\lambda^2 = 1/5$. The solutions with $\lambda^2 = 1$ are qualitatively the same with regard to r and s. In fig.(8.4) there is a break in the line for r, this is because the scale is insufficient to clearly show r dropping rapidly from 1.97 to 0.14 and then rising sharply to 2.83 before the final descent to the singularity. For $\lambda^2 = 1/5$ if $g_2/g_1 = \dot{g}_2/\dot{g}_1 = \lambda^2$ (even if $g_2 = g_1 = 0$ initially) the value of λ^2 is maintained for all t. This is to be expected because when $\lambda^2 = 1/5$ eqns.(7.35b,c) are the same, as are (7.36c,d). In the case of $\lambda^2 = 1$ this value will be maintained only if $g_1, g_2, \dot{g}_1, \dot{g}_2 = 0$. In all other cases λ^2 is not constant, how it varies depending on all the other parameters involved. In the case of fig.(8.6) λ^2 = 1/10 initially, this value decreasing until near the final singularity and then increasing again. Often λ^2 changes slowly until near the singularity and then changes rapidly.

Table(8.2)

Initial values for solutions of the seven torus.

r	s	$\dot{r}$	$\dot{s}$	k_3	f	$\dot{g}$	Graph
1	1	1	1	-1/2	26.38	0	fig.(8.7)
1	1	1	-1	-1/2	4.9	0	fig.(8.8)
1	1	-1	1	-1/2	4.9	0	fig.(8.9)
1	1	1	0	-1/2	4.9	0	fig.(8.10)
1	1	0	1	-1/2	17.66	0	fig.(8.11)
1	1	-1	-1	-1/2	26.38	0	fig.(8.12)
1	1	1	1	-1/2	18.65	7.05	fig.(8.13)
1	1	1	-1	-1/2	3.47	1.31	fig.(8.14)
1	1	-1	1	-1/2	3.47	1.31	fig.(8.15)
1	1	1	0	-1/2	3.47	1.31	fig.(8.16)
1	1	0	1	-1/2	12.49	4.72	fig.(8.17)
1	1	-1	-1	-1/2	18.65	7.05	fig.(8.18)
1	1	1	1	-1/2	0	9.97	fig.(8.19)
1	1	1	-1	-1/2	0	1.85	fig.(8.20)
1	1	-1	1	-1/2	0	2.22	fig.(8.21)
1	1	1	0	-1/2	0	1.85	fig.(8.22)
1	1	0	1	-1/2	0	7.24	fig.(8.23)
1	1	-1	-1	-1/2	0	9.97	fig.(8.24)
1	1	1	-1	0	6.93	0	fig.(8.25)
1	1	1	0	0	6.93	0	fig.(8.26)
1	1	1	1	0	0.06	10.14	fig.(8.27)
1	1	1	-1	0	0.15	2.62	fig.(8.28)
1	1	1	-1	-1	0	0	fig.(8.29)
1	1	1/ 2	1/ 2	-1/2	0	0	fig.(8.30)

If $f = 0$ initially it will not remain at this value as time progresses unless $g_1, g_2, \dot{g}_1, \dot{g}_2 = 0$ (by (7.35a')) which is incompatible with my choice of initial values for the scale factors and their time derivatives. Comparison of figs.(8.2) and (8.4), where $f \neq 0$ from $t = 0$ in fig.(8.2) whereas $f = 0$ initially in fig.(8.4), illustrates the effect of the f term. The curves are different but the ultimate fate of r and s is the same, in each case a study of the relative effects of the various terms in the equations of motion show that this is due to the f term. As $t \to t_1$ the f term becomes smaller and approaches zero as $s \to \infty$. The final behaviour for this model seems unavoidable, even for cases such as fig.(8.5) where the behaviour looks promising at first due to the particular choice of initial values.

The solutions of the seven-sphere are obviously not compatible with inflationary cosmology. I now consider Model 2, where M_7 is a seven-torus, in the special case when all the extra dimensions are treated equally (ie. $s_m = s \; \forall m$, $g_i = g \; \forall i$) and $l_i = 0$. In this case the 4-form F is given by

$$F = f \, \Omega + \dot{g} \, V^0 \wedge E \tag{8.7}$$

where $E = \sum_j E_j$, and the condition which the initial values must fulfill is

$$\frac{f^2}{16} = \frac{3k_3}{r^2} + 3\left(\frac{\dot{r}}{r}\right)^2 + 21\frac{\dot{r}\dot{s}}{rs} + 21\left(\frac{\dot{s}}{s}\right)^2 - \frac{7}{16}\frac{\dot{g}^2}{s^6}. \tag{8.8}$$

The Einstein and Maxwell equations are given by (7.73-74). Notice that g does not appear anywhere in the equations of motion; this will lead to special effects.

The initial values for several computations are shown in table(8.2) and the solutions graphed in figs.(8.7-30). Again the solutions shown are a representative selection. They can be divided into categories. The first category, figs.(8.7-12), exhibit cases where $g = 0$ initially and by (7.74b) for all time. For these solutions $F = f\Omega$ is entirely on M_4 as in the case of Freund{67}. Despite the variety of initial conditions, as t becomes larger there are only two types of behaviour. There is that of figs.(8.7,9,10,11) in which $r \sim t$ and s approaches a constant value ($f \to 0$) and that of figs.(8.8,12) in which the solution approaches a singularity in which $r \to 0$, $s \to \infty$. The second type of solution is, as in the case of the seven-sphere, due to the f term in (7.62,64,73) which dominates the equations for $\ddot{r}$ and $\ddot{s}$ for an important period leading to the final behaviour. It is clear that these solutions do not give satisfactory inflationary behaviour.

The second category contains those with $f \neq 0$, $\dot{g} \neq 0$ as illustrated in figs.(8.13-18) which show many of the features of the first category; this being due to the fact, as explicit calculation of the relevant terms shows, that the f term is dominant compared to the $\dot{g}$ term. Rewriting the two Maxwell equations (7.74)

$$\dot{f}/f = -7\,\frac{\dot{s}}{s}\,, \tag{8.9a}$$

$$\ddot{g}/\dot{g} = -\left(\frac{\dot{s}}{s} + 3\frac{\dot{r}}{r}\right) \tag{8.9b}$$

note that $\dot{f}$ is positive whenever $\dot{s}$ is negative; so if s is sharply decreasing f will increase and again dominate in the Einstein equations for $\ddot{r}$ and $\ddot{s}$. When $\dot{s}$ and $\dot{r}$ are negative $\dot{g}$ will increase, but as $\dot{r}$ becomes positive $\ddot{g}$ becomes less positive and eventually negative and so the effect of the $\dot{g}$ term becomes smaller. This is the struggle that has taken place in the complicated behaviour in fig.(8.18); however the final behaviour approaches that of fig.(8.14).

The third category of solutions (figs.(8.19-24)) is when $f = 0$, $\dot{g} \neq 0$. That is the 4-form F is of the form $V^0 \wedge B^{mnp}$, partly in the time direction and partly on the seven-space. The form of F_7 is maintained by the equations of motion. This is a very important difference between this model and that of the seven-sphere and leads to some interesting features. Again there are two types of behaviour; the $r \sim t$, s = constant type and a more promising one in which $r \to \infty$, $s \to 0$ at a singularity (in contrast to the first two categories). The cosmological possibilities of this second type of behaviour will be investigated later.

The interplay between the different terms in the equations of motion can be complicated. Consider the expression for $\ddot{r}$ (7.62b,73b). In fig.(8.20) it is the $(\dot{r}\dot{s}/rs)$ term which dominates; although the $\dot{g}^2$ term helps keep $\ddot{r}$ positive it becomes increasingly insignificant. This explains why fig.(8.29) with $\dot{g} = 0$ throughout is similar to fig.(8.20). For fig.(8.14) the $(\dot{r}\dot{s}/rs)$ term is initially dominant, thus the first part of the curve is similar to fig.(8.20), however this causes an increase in the f^2 term which assumes dominance for a while making $\ddot{r}$ large and negative and so $\dot{r}$ large and negative. Although the f^2 term now fades it has so increased $\dot{r}$ that the $-(\dot{r}/r)^2$ term now dominates and this brings r crashing down.

For the solutions ending in a singularity the change in behaviour due to the f term can be traced by considering the series of graphs figs.(8.8,14,20). Figs.(8.8) and (8.14) are similar, $\dot{g} = 0$ in fig.(8.8) and $f^2 \approx 7\dot{g}^2$ in fig.(8.14). The dotted and dot-dashed graphs in fig.(8.14) shows the result with the same initial values for the scale factors and their derivatives as fig.(8.14) but with $\dot{g} \gg f$ initially. The behaviour is much more like that of fig.(8.20) until near the end, however the f term eventually causes the about turn in $\dot{r}$ and $\dot{s}$. In fig.(8.20) with f = 0 for all the time this turn over does not occur. The same progression exists in the series figs.(8.12,18,24).

Figs.(8.25-28) show solutions with $k_3 = 0$, thus showing for this model there are other types of solution. Figs.(8.29-30) give, by special choice of the initial values, cases where $f = \dot{g} = 0$ for all time. Note that these results are similar to figs.(8.20,24) thus illustrating the relative unimportance of the $\dot{g}$ term in some cases.

It can be seen from the two models considered so far in this chapter that there are families of solutions, each solution being selected by a particular set of initial values which has to obey the condition arising from the Bianchi identity. I have no explanation for how any of these different values arose out of whatever existed before and so, a priori, each is a possibility. I now consider the form of the equations in more detail to see which terms contribute to an inflationary type behaviour (eg. perhaps as in figs.(8.20.24)) with a large expansion of the ordinary dimensions and with the extra dimensions contracting or constant, and which act against it. I can then impose the conditions required to give this type of behaviour and note how it restricts the possible spaces for M_7 for my construction of the 4-form F. In the next chapter I will discuss whether the solutions can solve some of the remaining cosmological problems and compare inflation in d=11 supergravity with the stretched ordinary gravity of chapter VI and also a non-supersymmetric gravity model in greater than 11-dimensions which contains a 4-form F.

The various components of F given by

$$F = f\,\Omega + \sum_i \dot{g}_i\, V^0 \wedge E_i + \sum_i g_i\, dE_i \qquad (8.10)$$

contribute quite differently to the energy-momentum tensor and thus play different roles in the Einstein equations. In chapter VII I derived

$$dE_i = \sum_j c_{ij} * E_j \tag{8.11}$$

as a necessary condition for the Maxwell equations to be satisfied. The choice of E_i satisfying (8.11) is important.

I first return to the model described at the start of this chapter given by (8.1-4). I equate the Ricci tensor in (8.3) with (7.62) and, because of the Bianchi identity, I need here only consider two of the three equations, say the (b) and (c) parts. These equations can be divided into kinetic terms with accelerations and velocities, ($\ddot{r}/2r+\ldots$) and ($\ddot{s}/2s+\ldots$) respectively on the LHS and 'potential' terms on the RHS which are ($-k_3/r^2 - f_1^2/4!\,s^{14}$) and ($-k_7/s^2 + f_1^2/2.4!\,s^{14}$) respectively, where $k_3 < 0$, $k_7 > 0$ and I have used (8.4) to replace f with f_1. In both equations the f_1^2 term is of opposite sign to, and thus acts against, the curvature term. The f_1^2 term would on balance, despite the complicated nature of the kinetic terms, be expected to act against the positive acceleration of the scale factor r and for that of the extra dimensions s. This is just the opposite of what is required for a universe in which the ordinary dimensions expand and the extra dimensions contract. The results of the seven-sphere and seven-torus with $F = f\Omega$ confirm that it is not possible to have sufficient expansion of r(t). Some results (eg. figs.(8.4,8)) show an initial expansion of r and contraction of s, however this phase is brought to an end with $\dot{r}$ becoming negative and $\dot{s}$ positive. Just to reiterate this is because, according to (8.4), as s becomes large and negative f becomes large and positive so the $f^2 = f_1^2/s^{14}$ term, with the wrong sign, dominates. This s dependence of F is a dominant feature of the equations and persists as such even when F takes the more general form as in (8.10), which I shall now suppose.

With F given by (8.10) the Einstein equations replacing (8.3) are more complicated, as I have shown in chapter VII. Consider the contribution of the dA_7 term to the Ricci tensor derived from the Einstein equations by taking specimen terms from the dE_i and from the $V^o \wedge E_i$ parts of (8.10). I shall ignore the inessential complication arising from the a^i_{pqr} in (7.49) being a function of y. I will clarify and extend the conclusions given by Moorhouse and Nixon{80}.

The two important specimen terms can be written

$$\tilde{g}(t)\, b^{m_1 m_2 m_3 m_4} \quad ; \quad \dot{\hat{g}}\; V^0 \wedge b^{n_1 n_2 n_3} \tag{8.12}$$

and using the metric in (7.60) with the s_m distinct the components of F are

$$F_{m_1 m_2 m_3 m_4} = \frac{\tilde{g}(t)}{s_{m_1} s_{m_2} s_{m_3} s_{m_4}} \equiv \frac{\tilde{g}(t)}{s^4} \; ; \; F_{0 n_1 n_2 n_3} = \frac{\dot{\hat{g}}}{s_{n_1} s_{n_2} s_{n_3}} \equiv \frac{\dot{\hat{g}}}{\hat{s}^3} \,. \tag{8.13}$$

So, assuming that the E_i are chosen to satisfy (8.11) and give a diagonal Ricci tensor the three space components of the Ricci tensor are

$$R^a{}_b = \frac{\delta^a_b}{4!}\Big[-f^2 + \big(-\tfrac{1}{2}\tilde{g}^2 s^{-8} + \tfrac{1}{2}\dot{\hat{g}}^2 \hat{s}^{-6} + \text{similar terms}\;\big)\Big]. \tag{8.14}$$

Since the 'similar terms' containing g_i or $\dot{g}_i$ in (8.14) all arise from the $-F^2\delta^a_b/2$ part of the Einstein equation (7.27), and not the $F^{a p_1 p_2 p_3} F_{b p_1 p_2 p_3}$ part, the g_i^2 terms will always give negative contributions and the $\dot{g}_i^2$ terms will always give positive contributions whatever the form of the E_i. Thus (8.14) shows that for inflation of r the f and g_i components of F in (8.10) are bad and the $\dot{g}_i$ components good. A study of the models in chapter VII will show this to be true in each case.

Now I examine the effect of the $g_i, \dot{g}_i$ components on the extra space, where the Ricci tensor from the curvature gives seven separate equations

$$R^m{}_m = \frac{1}{2}\frac{\ddot{s}_m}{s_m} + \frac{\dot{s}_m}{s_m}\Big(\frac{3}{2}\frac{\dot{r}}{r} + \frac{1}{2}\sum_{n\neq m}\frac{\dot{s}_n}{s_n}\Big) + \frac{1}{s_m^2}\tilde{R}^m{}_m \,, \tag{8.15}$$

where $\tilde{R}^m{}_m$ is the Ricci tensor calculated for the 7-space alone with metric $-\sum_m b^m \otimes b^m$; and from the Einstein equations

$$R^m{}_m = \frac{1}{4!}\Big[\frac{1}{2}f^2 + \frac{3}{2}\tilde{g}^2 s^{-8}\big(\delta^{m_1}_m + \delta^{m_2}_m + \delta^{m_3}_m + \delta^{m_4}_m - \tfrac{1}{3}\big) \tag{8.16}$$

$$- \frac{3}{2}\dot{\hat{g}}^2 \hat{s}^6\big(\delta^{n_1}_m + \delta^{n_2}_m + \delta^{n_3}_m - \tfrac{1}{3}\big) + \begin{matrix}\text{similar terms with various}\\ \text{sets of } m_i \text{ and } n_i\end{matrix}\;\Big]$$

As before the f^2 term tends to accelerate s, acting against diminution of the extra space. As for the other terms the details depend on the E_i and the form of the dE_i, so the signs of the $g_i, \dot{g}_i$ terms depend on whether E_i and/or dE_i contains b^m. This must be studied for individual models.

The f term continues to be important because with F as in (8.10) the general Maxwell equation equivalent to (8.4) is also exactly integrable and gives

$$f\Big(\prod_{m=1}^{7} s_m\Big) + \sum_{i,j} b_{ij}\, g_i\, g_j = \text{constant}\ , \tag{8.17}$$

where b_{ij} are constants independent of y if the a^i_{pqr} of (7.49) are independent of y. Consequently even if f starts off zero, in the course of time, for a shrinking internal space, it can be expected to dominate the potential and bring any expansion of r to a sharp end. This has been shown for the seven-sphere and seven-torus models, and computations show it to be true for the other models also.

There are two aspects of the solution to the problem of finding inflationary behaviour. The first is to eliminate all terms in the expression (8.10) for F which cause difficulties (ie. the f and g_i terms); the second is to choose the E_i and M_7 such that the $\dot{g}_i$ terms give negative contributions to $R^m{}_m$. The first step requires f(t) = 0 and $dE_i = 0$ to remove the g_i terms. Note that to remove all 'bad' terms requires all dE_i to be zero not just the sum $\sum_i dE_i$. Thus F has the form

$$F = \sum_i \dot{g}_i\, V^0 \wedge E_i\ . \tag{8.18}$$

Subject to the further condition

$$d \underset{7}{*}\, E_i = 0 \tag{8.19}$$

the Maxwell equations (7.8) are now soluble and can be integrated with respect to time to give

$$\dot{g}_i(t)\, r^3\, \sigma_i = \text{constant} = h\ , \tag{8.20}$$

where σ_i is a product of s_m of overall degree 1. In the case of the torus $\sigma_i = s$. The new condition (8.19) was already satisfied in the more general case when $dE_i \neq 0$ since the matrix c_{ij} in (7.56) is non-singular.

The less stringent condition of just setting f = 0, which removes the worst inflationary term, is insufficient because equation (8.17) has still to be satisfied. This takes the form $\sum_{i,j} b_{ij}\, g_i\, g_j = 0$ which, along with (8.20) over determines the g_i unless the $b_{ij} = 0$. This is the case for the special case of the torus, and some other models in chapter VII.

Now I can study the models of chapter VII to see if they fulfill the conditions

$$dE_i = 0 \quad , \quad d * E_i = 0 \tag{8.21}$$

as well as giving $\dot{g}_i$ terms of the correct sign in (8.16). The E_i I consider are those of chapter VII, I have not yet studied the cohomology of the M_7's sufficiently to completely exclude the existence of other forms which may satisfy (8.16).

It follows directly from (7.19) that Model 1, the squashed seven-sphere, cannot give inflationary behaviour.

Returning to Model 2, the seven-torus, for the special case with all the extra dimensions treated equally notice that (8.21) are satisfied and (8.20) gives

$$\dot{g}\, s\, r^3 = h \tag{8.22}$$

and the Einstein equations are

$$\frac{3}{2}\frac{\ddot{r}}{r} + \frac{7}{2}\frac{\ddot{s}}{s} = -\frac{7}{24}\frac{h^2}{r^6 s^8} , \tag{8.23a}$$

$$\frac{1}{2}\frac{\ddot{r}}{r} + \left(\frac{\dot{r}}{r}\right)^2 + \frac{7}{2}\frac{\dot{r}\dot{s}}{rs} = -\frac{k_3}{r^2} + \frac{7}{48}\frac{h^2}{r^6 s^8} , \tag{8.23b}$$

$$\frac{1}{2}\frac{\ddot{s}}{s} + 3\left(\frac{\dot{s}}{s}\right)^2 + \frac{3}{2}\frac{\dot{r}\dot{s}}{rs} = -\frac{1}{24}\frac{h^2}{r^6 s^8} \tag{8.23c}$$

which yield the condition the initial values must satisfy (replacing (8.8)

$$3\left(\frac{\dot{r}}{r}\right)^2 + 21\frac{\dot{r}\dot{s}}{rs} + 21\left(\frac{\dot{s}}{s}\right)^2 = \frac{7}{16}\frac{h^2}{r^6 s^8} - \frac{3k_3}{r^2} . \tag{8.24}$$

Note that the $\dot{g}_i$ (ie. h) term does act the correct way in (8.23c). This is not necessarily true in the more general case with the s_m distinct and some $\dot{g}_i = 0$ (eg. $R'_M < 0$ if $\dot{g}_3 = \dot{g}_5 = \dot{g}_1 = 0$).

An exact solution $r = at$, $s = bt^{-\frac{1}{4}}$, corresponding to $r = at$, $s = bt^{1/7}$ for the torus with the Freund-Rubin ansatz, does not exist because it would require $h^2 < 0$ which is not allowed.

The solutions given for the torus show that even with the conditions (8.21) satisfied not all initial values give rise to a huge expansion of r, but there is at least a class which does; see figs.(8.20,24). The numerical solution for these cases, as $t \to t_1$, is characterised by

$$\frac{\ddot{r}}{r}\ ,\ \left|\frac{\ddot{s}}{s}\right|\ ,\ \frac{\dot{r}}{r}\ ,\ \left|\frac{\dot{s}}{s}\right| \gg s\ ,\ \frac{k_3}{r^2}\ ,\ \frac{h^2}{r^6 s^8} \tag{8.25}$$

and consequently approaches a Kasner type solution

$$r = a(t_1 - t)^{\alpha}\ ,\quad s = b(t_1 - t)^{\beta}\ , \tag{8.26}$$

with the right hand sides of (8.23) and (8.24) becoming negligible and consequently the left hand sides give the Kasner conditions

$$3\alpha + 7\beta = 1 = 3\alpha^2 + 7\beta^2\ , \tag{8.27}$$

which have solutions

$$\alpha = -0.358\ ,\quad \beta = 0.296\ , \tag{8.28a}$$

$$\alpha = 0.558\ ,\quad \beta = -0.096\ . \tag{8.28b}$$

Eqn.(8.28a) corresponds to the desired case of r expanding and s contracting and is selected by f = 0 and (8.21). Eqn.(8.28b) corresponds to the solutions I do not want with $r \to 0$, $s \to \infty$ and which are selected by $f \neq 0$. The constants a,b depend on the precise initial conditions and t_1 is, as yet, only given in arbitrary time units. The constants will be given values in the next chapter when I look at the cosmology of the solutions. The solutions (8.28) satisfy the Alvarez{73} conditions (2) and (3) described earlier in the chapter.

Another asymptotic solution is

$$r = at + c\ ,\quad s = b = \text{constant}\ . \tag{8.29}$$

where $a = \sqrt{k_3}$ and b,c depend on the initial values. This applies for eg. the solutions of figs.(8.21,22). This does not fall into the Alvarez categories.

For the torus with the $\dot{g}_i$ not all equal, and other models, solutions with the s_m not all equal can approach

$$r = a(t_1 - t)^{\alpha} \quad , \quad s_m = b(t_1 - t)^{\beta_m} \quad , m = 1, \dots, 7, \tag{8.30}$$

instead of (8.26), with the Kasner condition

$$3\alpha + \sum_m \beta_m = 1 = 3\alpha^2 + \sum_m \beta_m{}^2 . \tag{8.31}$$

However the solution of (8.31) giving the maximum rate of expansion of r occurs for $\beta_m = \beta$ (m=1,...,7) and this solution is given by (8.28a). Note that in some models it is not possible to have all the β_m equal due to the geometry, and even when the geometry permits it the choice of the g_i is important.

The effect of the $\sum_i l_i G_i$ terms in (7.70) has not been studied in detail but their presence seems to act against good cosmological solutions.

I now return to a study of the other models with regard to inflationary possibilities.

Model 3.

Since $dE_1 \neq 0$ I must set $g_1 = 0$. The derivatives of the other E_i's are

$$dE_2 = -(p_2 G_4 + p_3 G_3) , \tag{8.32a}$$

$$dE_3 = -(p_1 G_4 + p_3 G_2) , \tag{8.32b}$$

$$dE_4 = -(p_1 G_3 + p_2 G_2) . \tag{8.32c}$$

Eqns.(8.32) give $d *_7 E_i = 0$ but imposing $dE_i = 0$ requires only one of g_2, g_3, g_4 non-zero (say g_2) and only one of p_1, p_2, p_3 non-zero (p_1 in the case of $g_2 \neq 0$). Thus M_7 is restricted to the manifold $Q^{p_1 0 0}$ (which is in fact $S^2 \times S^2 \times S^3$ when $p_1 = 1$ which has symmetry ($SU(2)^4$){70}.

However, with only $p_1, g_2 \neq 0$ a study of (7.78) shows that although g_2^2 acts for increasing $\ddot{r}$ and decreasing $\ddot{w}$ and $\ddot{s}$ it also acts for increasing $\ddot{u}$ and $\ddot{v}$. Thus the model is unlikely to give suitable inflationary behaviour. Lack of time prevented a full numerical study of this model.

Model 4.

Initial values are given in table(8.3). The solution shown in fig.(8.31) shows the type of behaviour expected when $f \neq 0$ for the topology $p_1 = p_2 = 0$, $p_3 = 1$. Note that this topology greatly simplifies the equations of motion (7.82,85,86). From now on I consider only $f = 0$ solutions.

Table(8.3)
Initial values for solutions for Model 4

r	s	$\dot{r}$	$\dot{s}$	f	$\dot{g}_1$	$\dot{g}_2$	Topology	Graph
1	1	1	1	8	0	0	$p_1 = p_2 = 0$ $p_3 = 1$	fig.(8.31)
1	1	1	1	0	15.66	15.66	$p_1 = p_2 = 0$ $p_3 = 1$	fig.(8.32)
1	1	1	1	0	27.13	0	$p_1 = p_2 = 0$ $p_3 = 1$	fig.(8.33)
1	1	1	1	0	15.43	15.43	$p_1 = p_2 = p_3 = 1$	fig.(8.34)

For possible inflation it is required that $d^*E_i = 0$ which follows directly from (7.84) and $dE_i = 0$ which requires $p_1 = p_2 = 0$ thus restricting the topology to M^{00p_3}, an example of which is M^{001} the space $CP^2 x S^2 x S^1$ with symmetry SU(3)xSU(2)xU(1){4}. Thus this model is appealing from the symmetry aspect. Does it also give inflation ? A study of (7.85) with $p_1 = p_2 = 0$ reveals that in general the $\dot{g}_i$ terms act for increasing $\ddot{r}$ and decreasing $\ddot{v}$, which is good, but that $\dot{g}_1$ acts for decreasing $\ddot{u}$ and increasing $\ddot{s}$ whereas $\dot{g}_2$ acts for increasing $\ddot{u}$ and decreasing $\ddot{s}$. In fig.(8.32) is shown a solution which does give inflationary type behaviour starting with the initial values of $\dot{g}_1$ and $\dot{g}_2$ equal. What has happened is that in (7.85d,e) the $\dot{g}_1$ and $\dot{g}_2$ terms cancel each other out sufficiently to prevent either s or u expanding much beyond their initial values before the final Kasner type behaviour takes over. This is not the case in fig.(8.33) which starts with $\dot{g}_1 \neq 0$, $\dot{g}_2 = 0$; the $\dot{g}_1$ term here causing the expansion of s and contraction of u. With the initial values $\dot{g}_1 = 0$, $\dot{g}_2 \neq 0$ u can be made to expand and s to contract. Further tuning of the initial values of $\dot{g}_1$ and $\dot{g}_2$ might improve the inflationary behaviour beyond that of fig.(8.32).

In fig.(8.34) is shown a solution with $\dot{g}_1 = \dot{g}_2$ initially but for a different topology. Comparing this graph with fig.(8.32) shows how the solutions are crucially dependent on the topology with even the asymptotic behaviour of the scale factors markedly different.

Model 5.

For this model $dE_i = d{*}E_i = 0$ by construction. Again the $\dot{g}_i$ terms in the Einstein equations (7.92) act both for and against decreasing accelerations of the scale factors for the extra dimensions in various combinations. Initial values are given in table(8.4). The solution shown in fig.(8.35) gives inflationary behaviour for r together with u,v,w,s contracting and is dependent on the initial choice of the $\dot{g}_i$'s, here all equal. The solution in Fig.(8.36) with initial values $\dot{g}_1 \neq 0$, $\dot{g}_2 = \dot{g}_3 = 0$ has s as well as r expanding.

Table(8.4)

Initial values for solutions for Models 5 and 6

	r	s	$\dot{r}$	$\dot{s}$	$\dot{g}_1$	$\dot{g}_2$	$\dot{g}_3$	Graph
Model 5								
	1	1	1	1	29.39	29.39	29.39	fig.(8.35)
	1	1	1	1	50.91	0	0	fig.(8.36)
Model 6								
	1	1	1	1	13.36	13.36	-	fig.(8.37)
	1	1	1	1	19.29	0	-	fig.(8.38)

Model 6.

Again $dE_i = d{*}E_i = 0$ by construction. However for this model both $\dot{g}_1$ and $\dot{g}_2$ act for the increase of $\ddot{s}$ and consequently there appears to be no solution in which s does not expand. Initial values are given in table(8.4). Fig.(8.37) shows a solution in which u = v by the initial choice $\dot{g}_1 = \dot{g}_2$ and they both contract. By choosing $\dot{g}_1 \neq 0$, $\dot{g}_2 = 0$ v can be made to expand as shown in fig.(8.38).

Chapter IX COSMOLOGY OF THE SUPERGRAVITY MODEL

I have shown that there is a class of models which, under certain conditions, will give solutions for the scale factors which become Kasner like as the time t approaches some critical time t_1 as illustrated by the solution for the torus in (8.27). For manifolds, including tori, with s_m not all equal solutions may, if the necessary conditions for the previous chapter are satisfied, approach

$$r = a(t_1 - t)^{\alpha}, \quad s_m = b(t_1 - t)^{\beta_m}, \quad m = 1, \dots, 7, \tag{9.1}$$

with α and β_m satisfying

$$3\alpha + \sum_m \beta_m = 1 = 3\alpha^2 + \sum_m \beta_m^2 . \tag{9.2}$$

Using Lagrange's method of undetermined multipliers I find that for maximum $|\alpha|$ then $\beta_m = \beta$ for all m; $\alpha < 0$ corresponding to the inflationary behaviour of r, $\alpha > 0$ corresponding to the crash of r and expansion of s. I will now consider whether this type of solution can solve the horizon and flatness problems. The parameters a, b, t_1 in (9.1) depend on the model and the precise initial values and for the case of the torus with the initial values $r = s = 1$, $\dot{r} = 1$, $\dot{s} = -1$, $k_3 = -1/2$, $h = \sqrt{\frac{24}{7}}$ are $a = 0.605$, $b = 1.481$ and $t_1 = 0.27$ in an as yet arbitrary time scale. Any horizon and flatness considerations are insensitive to these values but the values of the dimensionless characteristic constants α and β, which are fixed by (9.2) for any d=11 supergravity model, are crucial. I shall now present a heuristic argument to test whether these solutions can solve the flatness and horizon problems. It should be noted that $t = 0$ does not necessarily correspond to the origin of the universe.

So far the length and time scales have been arbitrary. To fix a physical length scale I must determine L, where 1 unit of the length scale in fig.(8.20) is L metres. Since the solution (9.1) is approaching a singularity at $t = t_1$ I have to postulate an exit from this classical solution, say at time $t_1 = t - \tau$ where τ is a small time interval. I follow the idea of an exit{61,63} through stabilisation of s when it has collapsed to the Kaluza-Klein scale near the Planck length, perhaps due to the onset of quantum effects or some other miracle. This requires that

$$L b \tau^{\beta} \approx L_p \tag{9.3}$$

where L_p is the Planck length; for a given τ this fixes the approximate scale of the numbers. However τ remains to be determined by the required expansion of r to solve the horizon problem, that is $r = a\tau^{\alpha}$ has to be large enough.

Introducing capital letters for the physical quantities (and small letters for quantities in my scale) I have

$$cT = Lt \tag{9.4}$$

where c is the speed of light. I require the causally connected region at time $\bar{T}(\bar{t})$ when the 11-dimensional phase comes to an end(ie. the extra dimensions are frozen out) to include the region which we now, at the present time $T_o(t_o)$, observe in the particle horizon of 2×10^{20} light years = 1.8×10^{26} metres. The distance of causal connection at time $\bar{T}$ is

$$H(\bar{T}) = R(\bar{T}) \int_0^{\bar{T}} \frac{c\,dT'}{R(T')} \quad , \tag{9.5}$$

where R(T) = Lr(t), which becomes a distance

$$H(\bar{T}) \frac{R(T_o)}{R(\bar{T})} = R(T_o) \int_0^{\bar{T}} \frac{c\,dT'}{R(T')} \quad , \tag{9.6}$$

at time T_o today. Consequently I require

$$R(T_o) \int_0^{\bar{T}} \frac{c\,dT'}{R(T')} \geqslant 1\cdot 8 \times 10^{26} \text{ metres.} \tag{9.7}$$

It is necessary to make some assumptions about the evolution of the 4-dimensional universe after time $\bar{T}$ so that the solution before $\bar{T}$ can be continuously connected to the present time. Let me assume that there is an immediate transition to the usual scenario, with first a radiation dominated universe. This could be so at a point in time if s and g_i immediately become constant. It would however require adjustment of the d=4 cosmological constant and radiation pressure. If it is so there is a continuous join at $\bar{T}$ to the solution

$$R(T) = A\, R(T_0)\, T^{1/2} \tag{9.8}$$

where A can be approximately evaluated by the standard formulae{52} to give

$$A = 1\cdot 8 \times 10^{-10} \ \text{seconds}^{-1/2}. \tag{9.9}$$

The power of T in (9.8) will change as T goes from $\bar{T}$ to T_0, however the change will not effect this calculation. From (9.8)

$$R(\bar{T}) = A\, R(T_0)\, \bar{T}^{1/2} = L\, r(\bar{t}). \tag{9.10}$$

Eqn.(9.1) gives a good approximation to the solution with M_7 a seven-torus so it is reasonable to take

$$R(T') = L\, r(t') = L\, a\, (t_1 - t)^{\alpha}, \quad t_1 > t, \tag{9.11}$$

and on matching (9.10) and (9.11) at $T = \bar{T}$ $(t = \bar{t})$ I obtain

$$R(T_0) = \frac{L\, a\, \tilde{t}^{\beta}}{A\, \bar{T}^{1/2}} \tag{9.12}$$

and so the requirement (9.7) becomes, taking the equality

$$\frac{L\, \tilde{t}^{\beta}}{A\, \bar{T}^{\alpha}} \int_0^{\bar{T}} \frac{dt'}{(t_1 - t')^{\alpha}} = 1\cdot 8 \times 10^{26}\ m. \tag{9.13}$$

Performing the integration and substituting the numerical values for the torus the following approximate results are obtained:

$$L \approx 4 \times 10^{17} L_p \approx 6\cdot 4 \times 10^{-18}\ m, \tag{9.14a}$$

$$\bar{T} \approx 5\cdot 8 \times 10^{-27}\ s, \tag{9.14b}$$

$$R(0) = S(0) = L \approx 6\cdot 4 \times 10^{-18}\ m, \tag{9.14c}$$

$$R(\bar{T}) = 1\cdot 2 \times 10^{4}\ m. \tag{9.14d}$$

This result cannot be regarded as satisfactory. Initial values as in (9.14c) of many orders of magnitude times the Planck length have no natural explanation yet their size is a consequence of the values α

and β are required to have by the Kasner equation. For models with M_3 expanding and all directions in M_7 contracting α and β_m do not vary enough among different models to give significantly different results. The torus model gives the 'best' behaviour. Another assumption to avoid large initial length scales could be to set $R(0) \sim S(0) \sim L_P$. This would however suffer seriously from quantum effects as the extra dimensions contracted. Notice that the d=11 phase is over in less than 10^{-26} seconds.

Evaluating the flatness parameter $\frac{|\rho - \rho_c|}{\rho_c}$ at $\bar{t}$ using $r = ar(t_o)t^{\frac{1}{2}}$ gives

$$\left|\frac{k_3}{r^2}\right| \Big/ \left(\frac{\dot{r}}{r}\right)^2 \approx 10^{-33} \tag{9.15}$$

to be compared with (5.17). This seems too large and supposing $T^o{}_o = -8\pi G\rho$ at $t = \bar{t}$ the value of ρ extrapolated to present times is about a factor of 10^{10} larger than commonly accepted values.

Could the other asymptotic torus solution $r = at$, $s = b$ where $a = \frac{1}{\sqrt{2}}$, $b = 1.413$ solve the horizon problem ? Let me assume that s attains the Planck length and take

$$L = \frac{L_P}{b} \; . \tag{9.16}$$

Performing a similar calculation to that above gives

$$L = 1 \cdot 1 \times 10^{-35} \, m \, . \tag{9.17a}$$

$$\bar{T} = 7 \times 10^{11} \, s. \tag{9.17b}$$

$$R(o) = S(o) = L \, , \tag{9.17c}$$

$$R(\bar{T}) = 1 \cdot 5 \times 10^{20} \, m \, . \tag{9.17d}$$

Note that in this case it is not possible to use $r = at$ all the way back to $t = 0$ in (9.7) due to the logarithmic nature of the integral and an approximation is made. This would appear to solve the horizon problem but expansion has to continue for a long time (over 20,000 years and 10^{38} times the expansion time in (9.14)) and by this time other effects, such as particle creation, would have to be included and the model would be insufficient. Some comments on this point are given in chapter XI. Maybe this mildly inflationary solution would work in a more complicated model in which these effects are included.

I have looked at time dependent solutions in d=11 supergravity and have identified rather restrictive conditions for the 4-form F so that expansion of r is not brought to an untimely end and I have found that there exist a class of solutions for certain compact manifolds M_7 which give huge expansions for r. In taking the 4-form

$$F = \sum_i \dot{\gamma}_i \, V^0 \wedge E_i \, . \tag{9.18}$$

I derived an energy momentum tensor which has important differences to that of the stretched ordinary gravity of various authors{59,61,63,81} and described in chapter VI where the energy-momentum tensor comes, typically, from the higher dimensional radiation pressure and density. Comparing the Ricci tensor (6.8-9) for a perfect fluid cosmology with that for supergravity, (7.64) in the case of the torus, one can see that the energy-momentum tensor arising from the density ρ in one case and from F as in (9.18) in the other act in the same direction for the time and ordinary space components, but for the compact dimension equations the signs are different; supergravity has F acting for s decreasing, stretched gravity for it increasing. Another difference, which will be discussed for arbitrary dimensions, is that the energy-momentum tensor given by (9.18) is proportional to $r^{-6} s^{-8}$ for supergravity whereas the corresponding term for stretched gravity in 11 dimensions is $(r^{-3} s^{-7})^{\frac{11}{10}}$. In cases where there is rapid expansion of r(t) these two factors will explain initial differences in the two models (compare fig.(6.2) and fig.(8.20)) but, since the energy-momentum terms become of smaller magnitude than the kinetic terms which are the same in each case, the solutions tended to are the same and the problem of obtaining satisfactory inflation the same. Thus the supergravity model does not improve inflationary prospects. Although the form of F does not change the asymptotic behaviour it is important in selecting which type of asymptotic behaviour results, that is whether r expands and s contracts or vice versa. In stretched ordinary gravity for a perfect fluid only one type of asymptotic behaviour is possible with r expanding and s contracting. If the energy-momentum tensor arose from different fields and/or a cosmological constant this could be changed. To improve the inflationary behaviour one might consider a larger number of extra dimensions in a stretched cosmology as the equations described in chapter VI allow. Some authors have speculated on the use of the

4-form field F in d>11 dimensions{34,82}. I shall consider this possibility in a cosmological context, since, as I have shown, F gives different behaviour to ρ in the perfect fluid theory. Since supergravity cannot exist in dimensions greater than 11 I will consider a minimal theory in which F only appears in the Lagrangian as $F \wedge {}^*F$, leading to the equation of motion $d^*F = 0$ together with the constraint $dF = 0$. The term $F \wedge F \wedge A$ is only appropriate in 11 dimensions for a 3-form A.

The Einstein equations are just the same as (7.62) except now the sums run from 1 to m and not 7. There is no restriction on the number of extra dimensions m but if, for simplicity, I want to consider only one scale factor for them all I must be able to choose $E = \sum_i E_i$,where E_i are 3-forms on the m-dimensional manifold, such that the indices are equally spread over all directions (and only give diagonal contributions to the Ricci tensor). For simplicity, and because it will give inflationary behaviour for the same reasons as with supergravity, consider the case of an m torus where $m = 2^M-1$, M an integer and consider E constructed from E_i using combinations allowed by a generalisation of the octonion structure constants (multonions?). The number of E_i is $(2^M-1)(2^M-2)/6$. With

$$A_7 = \sum_i g_i E_i \quad , \quad F = f\,\Omega + dA_7 \tag{9.19}$$

the Einstein and Maxwell equations can be derived in a similar manner to the method of chapter VII and they are given for completeness in Appendix D. First, just to note in passing, that when F is given by the Freund-Rubin ansatz $F = f(t)\Omega$ the equation $d^*F = 0$ implies that the energy-momentum tensor contributes terms of the form f_1^2/s^{2m}, where f_1 is a constant, and I have found a generalisation of the Freund power law solution (1) in chapter VIII of the form

$$r = a t \quad , \quad s = b t^{\frac{1}{m}} . \tag{9.20}$$

Although for m large the increase in s is negligible compared to r the increase of r itself is not sufficiently fast to overcome the problems described before for power law inflation. Using the cosmological form of F in (9.18) the energy-momentum tensor contributes terms on the right hand side of the Einstein equations proportional to $r^{-6}s^{-2m+6}$. In the case of the stretched cosmology the corresponding terms are proportional to $(r^{-3}s^{-m})^{\frac{m+4}{m+3}}$. In both cases the solution approaches, as

$t \to t_1$, the asymptotic Kasner form

$$r = a\,(t_1 - t)^{\alpha}\ ,\quad s = b\,(t_1 - t)^{\beta}\ , \tag{9.21}$$

where the indices are required by the Einstein equations to satisfy

$$3\alpha + m\beta = 1 = 3\alpha^2 + m\beta^2 . \tag{9.22}$$

For inflationary behaviour (9.22) is solved by

$$\alpha = \frac{\frac{1}{m} - \sqrt{\frac{1+2/m}{3}}}{1 + 3/m}\ ,\quad \beta = \frac{1 + \sqrt{3 + 3/m}}{m+3} \tag{9.23}$$

and for m large

$$\alpha \sim \frac{-1}{\sqrt{3}} = -0.577\ ,\quad \beta \sim \frac{1+\sqrt{3}}{m} . \tag{9.24}$$

Thus, as was previously noted in the stretched cosmology{61,63} and applying equally to the m large F-field cosmology, r now increases much faster than s decreases and this gives a much more promising scenario for inflation. Note that the improvement is more due to the change in β slowing the decrease in s rather than an increase in $|\alpha|$ causing faster expansion of r. There is also an asymptotic solution with r decreasing and s increasing ($\alpha > 0$). Again it is the form of F and M_7 which selects one type of solution or the other. For the m-sphere it may not be possible to obtain the required expansion of r just as for S^7 in d=11 supergravity.

There are differences in how the asymptotic form is approached in the two models since firstly, the sign of the compact dimension energy-momentum tensor components is different and secondly, for m large using (9.21) and (9.24) the contributions are proportional to

$$r^{-6} s^{-2m+6} \sim (t_1 - t)^{-2}\ ,\quad \text{F-field}\ , \tag{9.25a}$$

$$\left(r^{-3} s^{-m}\right)^{\frac{m+4}{m+3}} \sim (t_1 - t)^{-1}\ ,\quad \text{Stretched cosmology,} \tag{9.25b}$$

so that in the case of the 4-form model the energy-momentum tensor is, while still less than the kinetic terms, much nearer their magnitude (which is proportional to $(t_1 - t)^2$) than in the stretched cosmology. Specimen results are shown in fig.(9.1) for the initial values

$r = s = 1$, $\dot{r} = 1$, $\dot{s} = -1$, $k_3 = -600$ when a) $m = 15$,b) $m = 63$. Although the increase in r is largely a direct result of more extra dimensions appearing in the kinetic terms the full role of the extra dimensions is more complicated and they contribute, for instance, to increasing the $\dot{g}^2/s^6$ term since

$$\frac{d}{dt}\left(\frac{\dot{g}^2}{s^6}\right) = \frac{\dot{g}^2}{s^6}\left[(6-2m)\frac{\dot{s}}{s} - 3\frac{\dot{r}}{r}\right] \tag{9.26}$$

and this increases $\ddot{r}$ and decreases $\ddot{s}$.

To fix the physical scales I use the same procedure as for the supergravity model in requiring an expansion of r large enough to solve the horizon problem. For the initial values above and m = 63 the asymptotic forms are

$$r = 0.079\,(0.028 - t)^{-0.545} \tag{9.27a}$$

$$s = 1.19\,(0.028 - t)^{0.042}. \tag{9.27b}$$

Performing the analysis gives

$$L = 185\,L_p\,, \tag{9.28a}$$

$$\bar{T} = 2.76 \times 10^{-43}\,s. \tag{9.28b}$$

$$R(0) = S(0) = L = 2.96 \times 10^{-33}\,m. \tag{9.28c}$$

$$R(\bar{T}) = 9.71 \times 10^{-4}\,m. \tag{9.28d}$$

$$S(\bar{T}) = L_p. \tag{9.28e}$$

This gives a much more attractive inflationary prospect, and L can be brought down to L_p by further increasing m. The disadvantage of this scheme is the very ad hoc appearance of the F field whereas in supergravity its presence is required. It would be interesting to consider what non trivial manifolds in the m extra dimensions could satisfy $dE_i = 0$, $d^*_m E_i = 0$ other than the m torus considered here. This might lead to a theory with a more realistic symmetry after accounting for the symmetry reduction due to a field F_m on M_m.

Chapter X TEN-DIMENSIONAL SUPERGRAVITY

One of the shortcomings of N=1 d=11 supergravity is that it is a non-chiral theory and this leads to problems introducing realistic fermions into the theory. The other maximal supergravity, the N=2 d=10 theory, does not have this problem and it is free of chiral anomalies{83}.

The d=10 chiral supergravity cannot be obtained from a manifestly covariant action principle and so it is necessary to deal directly with the equations of motion which are derived, up to quadratic terms in the fermi fields, in Howe and West{84} by requiring closure of the local supersymmetry algebra on shell. I looked briefly at this theory to see whether the methods of dynamical compactification could be applied. The fields in the bosonic sector, other than the vielbein, are a 4-form A, a complex 2-form C and a complex 0-form B which give the derived fields

$$F = 5\, dA - \frac{5}{4}\, \mathrm{Im}(C \wedge H^*) , \tag{10.1a}$$

$$P = \frac{dB}{(1 - BB^*)} , \tag{10.1b}$$

$$G = \frac{(H - BH^*)}{(1 - BB^*)^{1/2}} , \tag{10.1c}$$

$$H = dC. \tag{10.1d}$$

The full bosonic sector equations of motion are given in refs.{9,83,85,86} but first I shall consider a much simpler theory in which I set C = B = 0 leaving, after some rescaling of the fields for convenience

$$F = * F , \tag{10.2a}$$

$$R_{MN} = \frac{1}{6} F_{M_1 M_2 M_3 M_4 M} F^{M_1 M_2 M_3 M_4}{}_{N} , \tag{10.2b}$$

together with the constraint

$$dF = 0 . \tag{10.3}$$

I first look for compactifications which are the local direct product of two 5-dimensional spaces, one non-compact (M5') and the other compact and Einstein (M5) described by a metric

$$g = V^0 \otimes V^0 - \sum_{a=1}^{4} V^a \otimes V^a - \sum_{m=5}^{10} V^m \otimes V^m , \tag{10.4a}$$

$$V^0 = dt , \quad V^a = r(t)\, v^a , \quad V^m = s(t)\, v^m . \tag{10.4b}$$

Choosing the simplest Freund-Rubin type ansatz for F

$$F = \sqrt{5 \times 6!}\, f \left(V^{01234} + V^{56789} \right) \tag{10.5}$$

which, because of (10.2a), must have components on both M5' and M5. The Ricci tensor is calculated from the Einstein equation to be

$$R^0{}_0 = f^2 \delta^0{}_0 , \tag{10.6a}$$

$$R^a{}_b = f^2 \delta^a{}_b , \tag{10.6b}$$

$$R^m{}_n = -f^2 \delta^m{}_n \tag{10.6c}$$

so that the space-time can be taken to be the direct product of 5-dimensional anti-de Sitter space and a 5-dimensional Einstein space. Note that in the trivial case f = 0 the Ricci flat Minkowski d-space is allowed with the extra dimensions a torus T^{10-d} (d<10). Calculating the Ricci tensor from the curvature also and equating with (10.6) yields

$$2\frac{\ddot{r}}{r} + \frac{5}{2}\frac{\ddot{s}}{s} = f^2 , \tag{10.7a}$$

$$\frac{1}{2}\frac{\ddot{r}}{r} + \frac{3}{2}\frac{k_4}{r^2} + \frac{3}{2}\left(\frac{\dot{r}}{r}\right)^2 + \frac{5}{2}\frac{\dot{r}\dot{s}}{rs} = f^2 , \tag{10.7b}$$

$$\frac{1}{2}\frac{\ddot{s}}{s} + 2\frac{k_5}{s^2} + 2\frac{\dot{r}\dot{s}}{rs} + 2\left(\frac{\dot{s}}{s}\right)^2 = -f^2 . \tag{10.7c}$$

Since F is not constructed as the exterior derivative of a 4-form it must also satisfy (10.3) which implies the constraint

$$\oint S^5 = \text{constant}. \tag{10.8}$$

Treating these equations as for d=11 supergravity numerical solutions were computed and the results for $k_4 = -1$, $k_5 = 0$ and the two sets of initial values $r = s = 1$, $\dot{r} = \dot{s} = 1$, $f^2 = 5$ and $r = s = 1$, $\dot{r} = -1$, $\dot{s} = 1$, $f^2 = 5$ are shown in figs.(10.1-2) respectively, clearly showing rapid expansion of the scale factor r(t) for "ordinary" 5-dimensional space-time and diminution of s(t). Although (10.7-8) resemble those of d=11 supergravity in form there is an important difference in the time evolution in that the f term eventually dominates as time proceeds, especially since the $(\dot{r}\dot{s}/rs)$ term is largely cancelled by the $(\dot{r}/r)^2$ term in (10.7b) and by the $(\dot{s}/s)^2$ term in (10.7c). Note that in (10.5) F is constructed from the full vielbein V^M for the 10-dimensional space and not the v^M. The f term acts for increasing $\ddot{r}$ and decreasing $\ddot{s}$. A survey of known compactifications to M5' X M5 is given in Romans{85} and their symmetries discussed therein. I have not derived time dependent equations for them all but presume it is possible. Their study might give some interesting models.

The above compactification leaves a d=5 theory. Robb and Taylor{9} present a process of spontaneous compactification to AdS4 x S^1 x M5 first by the step above and then 'compactifying' the M5' to AdS4 x S^1 by giving a non-zero value to P with P only having a component in the 4-direction (ie. $P = P_4 V^4$). The equations of motion are now

$$F = * F, \tag{10.9a}$$

$$d * P = 0, \tag{10.9b}$$

$$R_{MN} = P_M P_N^* + P_M^* P_N + \frac{1}{6} F_{M_1M_2M_3M_4M} F^{M_1M_2M_3M_4}{}_N. \tag{10.9c}$$

P_4 must be chosen such that (10.9c) gives $R^4{}_4 = 0$ yet maintains (10.9b). In the static case this can be done by setting $P_4 = f/\sqrt{2}$. I have extended this to the time dependent case. The generalisation is straightforward because f is now time dependent and so P_4 is. If P still only has a component in the 4-direction (10.9b) is automatically satisfied. Letting u be the scale factor in the fourth direction the Ricci tensor is calculated and is the same as (10.6) except $R^4{}_4 = 0$ and the indices a,b only run over the values 1,2,3. The Einstein equations become

$$\frac{3}{2}\frac{\ddot{r}}{r} + \frac{1}{2}\frac{\ddot{u}}{u} + \frac{5}{2}\frac{\ddot{s}}{s} = f^2 , \tag{10.10a}$$

$$\frac{1}{2}\frac{\ddot{r}}{r} + \frac{k_3}{r^2} + \left(\frac{\dot{r}}{r}\right)^2 + 2\frac{\dot{r}\dot{s}}{rs} + \frac{1}{2}\frac{\dot{r}\dot{u}}{ru} = f^2 , \tag{10.10b}$$

$$\frac{1}{2}\frac{\ddot{u}}{u} + 2\frac{\dot{u}\dot{s}}{us} + \frac{3}{2}\frac{\dot{u}\dot{r}}{ur} = 0 , \tag{10.10c}$$

$$\frac{1}{2}\frac{\ddot{s}}{s} + \frac{2k_5}{s^2} + \frac{3}{2}\frac{\dot{r}\dot{s}}{rs} + \frac{1}{2}\frac{\dot{s}\dot{u}}{su} + 2\left(\frac{\dot{s}}{s}\right)^2 = -f^2 , \tag{10.10d}$$

which must be satisfied together with (10.8). The Bianchi condition gives

$$\frac{3k_3}{r^2} + \frac{10k_5}{s^2} + 3\left(\frac{\dot{r}}{r}\right)^2 + 10\left(\frac{\dot{s}}{s}\right)^2 + 21\frac{\dot{r}\dot{s}}{rs} + 3\frac{\dot{r}\dot{u}}{ru} + \frac{9}{2}\frac{\dot{s}\dot{u}}{su} + 3f^2 = 0 . \tag{10.11}$$

Unfortunately lack of time prevented a numerical study of these equations, however I think it would be interesting to pursue this possibility further. It appears that f still acts for $\ddot{r}$ increasing and $\ddot{s}$ decreasing. A first choice would be to keep u constant. The theory could be further extended by making C non-zero.

There is another method of obtaining an effective d=4 theory by compactifying six dimensions by only giving the derived field H a non-zero value{87}.

Since the d=10 theory is a limiting case of a superstring theory it might be possible that this 'hierarchy' approach to spontaneous compactification is applicable to superstring theories. I have not considered the non-chiral d=10 supergravities, all of which can be obtained from the d=11 theory by dimensional reduction, although there has been some work on these, see eg. Gleiser et al{88}, Chapline and Manton{89}. The two types of d=10 supergravity are equivalent for dimensional reduction to d<10 if the extra dimensions are tori{90}.

Chapter XI CONCLUSIONS and DISCUSSION

Assuming that the classical picture has some validity for the early universe I have presented some new results which have shown the possibility of obtaining inflation in d=11 supergravity despite some disappointing features. I have also shown that going to higher dimensions in a non-supersymmetric gravity theory coupled to a 4-form F may improve some of these. In both types of theory the presence of the 4-form F is important in determining the dimensionality of ordinary space-time as well as, in time dependent theories, the behaviour of the various scale factors.

The first important result is that it is possible to extend the F-field in d=11 supergravity from the Freund-Rubin ansatz by including components on M_7 which do not require the existence of a Killing spinor as per Englert{7}. This observation allows F_7 to be constructed when no supersymmetry exists. The choice of F_7 depends on the properties of M_7 and the properties of F can be very different from those of, when it exists, the Englert form. Two new static solutions are given; the first showing that for the seven-sphere this new F only exists when $\lambda^2 = 1$ or 1/5 but for another case, the coset space Q^{111}, the new F exists when the internal space is not an Einstein space. It is relaxing the condition that M_7 be an Einstein space but still requiring it to be Ricci diagonal that allows for the more general time dependent equations to be derived with, for example, time dependent squashing in the case of the seven-sphere.

The importance of the new F is that in certain cases it permits a class of time dependent solutions for the scale factors which give arbitrarily large expansion of the ordinary space dimensions. The components of F which produce this have been identified and conditions derived for this expansion to occur. Other features of the model may still effect the results and these have been discussed for the various models considered. It is clear that the Englert F_7 arising from

$$A_7 = g \bar{\eta} \tilde{\Gamma}_{mnp} \eta \, b^{mnp} \qquad (11.1)$$

which gives

$$F_7 = \dot{q}\,\eta\,\tilde{\Gamma}_{mnp}\,\eta\,V^0 \wedge b^{mnp} - m\,q\,\bar{\eta}\,\tilde{\Gamma}_{mnpq}\,\eta\,b^{mnpq} \tag{11.2}$$

fails the inflationary requirement unless $D\eta = 0$ (ie. $m = 0$). The only cases I know with η existing and $m = 0$ are the torus T^7 and the $K3 \times T^3$ space{24}. In the former case both prescriptions are equivalent if the 7-torus is isotropic. Neither the Englert nor new F_7 on S^7 or $\lambda^2 = 1/5\ J^7$ fulfill the inflationary requirement. Although the new F allows freedom to look at arbitrary squashing in the time dependent model it gives no significant result other than a squashing which, in general, varies in time. Calculations I performed with the Englert F_7 for $\lambda^2 = 1,\ 1/5$ gave results which are very similar to those of the new model for the specific cases discussed when λ^2 is maintained.

I have shown that if F obeys certain conditions it is possible to obtain arbitrarily large expansion of the ordinary dimensions and, if other features of the model permit, all the extra dimensions will contract. The cases where some of the extra dimensions expand are rejected for this reason. The expansion can solve the horizon problem but unfortunately sufficient expansion is accompanied by a rapid contraction of the extra dimensions. It should be stressed that the scale is not set by the theory. If the scale of the extra dimensions is stabilised at the Planck length the enormous initial value of $\sim 10^{17}\ L_P$ seems unphysical. If the initial values are of the order of L_P the contraction would be beset by quantum problems. Both these are unsatisfactory. For the numerical solutions initial values were chosen but no physical reasoning given. If they arose out of a quantum fluctuation one would expect the initial scale to be of the order of L_P. In the higher dimensional gravity coupled to F the contraction of the extra dimensions can be reduced simply by increasing the number of extra dimensions. Although this latter theory is not as attractive as supergravity it perhaps deserves more study.

The solutions with r expanding and s contracting approach a singularity and if this is to be avoided the extra dimensions must become stabilised at some fixed value. Is this a quantum effect ? It has been suggested that quantum effects may set the scale of the extra dimensions in static spontaneous compactification{16,91} or even that quantum fluctuations of matter fields may induce spontaneous compactification{92}. Although I have ignored quantum effects they will modify the results somehow. In a non supersymmetric model

Yoshimura{57} has calculated some solutions including a gravitational Casimir energy-momentum tensor.

Expansion must occur in the very early universe before particle creation. This may be so if the d=11 phase only lasts 10^{-26} seconds. As far as I am aware the effects of particle creation in supergravity Kaluza-Klein cosmologies have not been studied but preliminary work in other Kaluza-Klein cosmologies suggests that the energy densities of created particles may prevent dynamical compactification{58,93,94,95}.

For static solutions one criterion is that they be stable, a general method of testing against all instabilities being given by Duff et al.{37} for static Freund-Rubin type compactifications. This would have to be extended to incorporate F_7 and spaces which are not Einstein before being applied to a time dependent model. The particle mass spectrum could then be studied to see which were the dangerous modes. This extension of the method of ref.{37} may not be easy.

In the d=10 supergravity model the 5-form F can be chosen to give five expanding and five contracting dimensions. In static cases there is a method of 'compactifying' one more dimension to S^1 using a 1-form present in the theory{9}. This model has been generalised to give time dependent equations and it is likely that there exist solutions with the S^1 radius constant, 3 spatial dimensions expanding and 5-dimensions contracting. The idea does suggest the possibility of a field fluctuation in the d=11 theory causing the ultimate contraction of the rogue extra dimensions which expand initially in some of the solutions. This could lead to complicated hierarchies of compactification.

The d=10 supergravity is a limiting case of a string theory. The study of the cosmology of string theories requires dynamical compactification as I have described as well as dynamical 'localisation' to remove the non-local degrees of freedom in a d=4 effective theory{96}.

Now to discuss the symmetry of the solutions. The presence of F_7 will reduce the bosonic symmetry of the theory to a subgroup of the symmetry group of M_7. The particular symmetries have yet to be studied. Coupling constants can be associated with the radii of the compact manifold{92} and these results have been modified for the presence of F{97}. Attempts to identify the coupling constants of Freund-Rubin compactifications with symmetry SU(3)xSU(2)xU(1) with the physical $SU(3)_c \times SU(2)_W \times U(1)_Y$ coupling constants extrapolated by the renormalisation group equations to the compactification mass scale have been largely unsuccessful{98,99}. In the solutions given here the coupling constants will be time dependent until the stabilisation of the extra dimensions, the relations between the coupling constants of different interactions varying due to the non-Einstein nature of M_7. It is also possible to obtain SU(3)xSU(2)xU(1) from d=10 theories but these do not extrapolate correctly either{98}.

Since the presence of components of F on the extra dimensions reduces the symmetry, but is necessary for inflation, to obtain the ultimately required SU(3)xSU(2)xU(1) symmetry requires more than seven extra dimensions. Thus it would be interesting to consider non-trivial manifolds for the extra dimensions for the higher dimensional gravity theory coupled to F and explicitly study the remaining symmetries for particular M_m and F. The geometry of 8-dimensional compact manifolds has been studied with a view to solving the fermion representation problem{100}. To include a 4-form F in this case would certainly require M_m to be anisotropic.

If d=11 supergravity is to explain low energy physics by dynamical compactification it must be some kind of preon model so that the gauge fields need not arise from the Killing symmetries of the extra space but are composites produced by some unknown dynamics. This could perhaps also overcome the problems with spinors arising from the non-chirality of the theory{101}. Duff et al.{102} discuss the possibility of obtaining the standard model from the squashed seven-sphere.

It is possible to generalise the ansatzen made for the fields in the models in several ways, such as letting the functions $g_i(t)$ in (7.50a) be functions of y^m, the coordinates on M_7, also. These generalisations lead to more complicated conditions on the E_i for solving the Maxwell equations together with more complicated Einstein equations. With no systematic method for selecting the E_i the way forward appears neither clear nor obviously profitable. One generalisation which might be useful is to let the $a^i{}_{pqr}$ of (7.49) be functions of y^m. The requirement (7.56) to satisfy the Maxwell equations remains but with different c_{ij}. It may be possible to pick $a^i_{pqr}(y)$ such that $dE_i = 0$ for inflation and yet still retain $d{}^*_7E_i = 0$, which is necessary if the $c_{ij} = 0$. Letting $r = r(x)$, $s = s(x)$ etc. leads to local compactification which has been studied in d=11 supergravity in a static case{103}. Unless the dependence on the ordinary dimensions is specialised to one radial coordinate the equations are very complicated and the spaces no longer Ricci diagonal. This would vastly complicate the choice of F. I think this approach is unlikely to lead to an improved cosmology.

There is also the problem of the cosmological constant appearing in the effective d=4 theory from the scalar curvature of the extra dimensions. This 'constant' would be time varying until fixed when the extra dimensions are frozen out, its exact value depending on when the process occurs and by what mechanism. For parallalisable M_7 (eg. S^7 from the models here) it is possible that fermi bilinears may flatten the extra dimensions to remove this problem{36} but in general the problem remains unresolved. Based on an idea by Rubakov and Shapashnikov{104} a new type of static solution has been found which is anti-de Sitter coupled to a seven space, the AdS metric being scaled by a function of the coordinates of M_7{105,106,107}. It is hoped that it is possible to choose this function so that the d=4 cosmological constant is eliminated. Introducing time dependence would proceed as in the models described but the new function would become time dependent too. This opens up new possibilities.

There are also solutions of d=11 supergravity, which I have not considered, where F is constructed from Kahler 2-forms when M_7 is a product of a Kahler manifold and S^1 factors{108}.

Since this work was carried out another static AdS x S^7 solution of d=11 supergravity has been found by Pope and Warner{109} with F non-zero on both the ordinary and extra dimensions for which the metric on S^7 is not Einstein and there are no Killing Spinors. The method used, the inverse Kaluza-Klein method, allows the remaining symmetries to be more easily identified. The method could presumably be extended to higher dimensional gravity and might thus lead to the phenomenological gauge group. Hopefully the method could be formulated to include time dependence.

I have presented some results which have illustrated a mechanism for producing inflation. There are still many problems to be overcome in making it realistic and many variants of the theories discussed to be studied.

Appendix A CALCULATION OF THE RICCI TENSOR

The Ricci tensor is calculated by Cartan's Moving Frame Method which was found to be the easiest and quickest method. In this thesis I consider mainly 11-dimensional space-times which are pseudo-Riemanian manifolds and are either Lie group or coset spaces. The modifications for other dimensional manifolds are straightforward. For any definitions and details not included here see refs.{110,111,112}. Let $\{L^M\}$, $M=0,1,\ldots,10$ be an orthonormal basis of left invariant 1-forms. All indices are in the tangent space to the manifold and are raised and lowered by the flat metric $\eta_{MN} = \mathrm{diag}(+-\ldots-)$. The Cartan structural equations give the torsion 1-form

$$R^M = dL^M - \omega^M{}_N \wedge L^N , \tag{A.1}$$

which by the torsionless condition is set equal to zero, and the curvature 2-form

$$\mathbb{R}^{MN} = d\omega^{MN} - \omega^M{}_P \wedge \omega^{PN} , \tag{A.2}$$

where ω^{MN} is the connection 1-form given by

$$\omega_{MN} = -\tfrac{1}{2}\left(L^P i_M i_N dL_P - i_M dL_N + i_N dL_M \right) , \tag{A.3}$$

where i_M is the interior derivative taking (p+1)-forms to p-forms and is defined by

$$i_M L^N = \delta_M{}^N , \tag{A.4a}$$

$$i_M (L^N \wedge L^P) = \delta_M{}^N L^P - L^N \delta_M{}^N . \tag{A.4b}$$

The Riemann tensor is given by

$$R^{MN}{}_{PQ} = \tfrac{1}{2}\, i_P i_Q \,\mathbb{R}^{MN} \tag{A.5}$$

the Ricci tensor and the Riemann curvature scalar follow

$$R^{M}{}_{N} = \sum_{P=0}^{10} R^{MP}{}_{NP} \quad , \tag{A.6}$$

$$R = \sum_{P=0}^{10} R^{P}{}_{P} \quad . \tag{A.7}$$

Although in general the summation convention is assumed for repeated indices an exception is for the Ricci tensor - $R^{M}{}_{M}$ does not imply a sum over M.

As an example of the calculation of the Ricci tensor by this method I give, in detail, the derivation for the time dependent squashed seven-sphere introduced in chapter VII. All other models follow similarly. The basis chosen was $\{L^{M}\} = \{V^{0}, V^{a}, B^{m}\}$ and is

$$V^{0} = dt \; , \quad V^{a} = r(t)\, v^{a}(x) \; , \quad B^{7} = s(t)\, e^{7} \; , \tag{A.8a}$$

$$B^{m} = s(t)\, e^{m}(y) \; , \quad m = 1,2,3, \quad B^{m'} = u(t)\, e^{m'}(y), \; m' = 4,5,6 \; , \tag{A.8b}$$

where the e^{m}, m=1,...,7 are the basis of (7.10) obeying (7.14). Immediately the relations for the full vielbein can be derived

$$dV^{0} = dt \; , \tag{A.9a}$$

$$dV^{a} = \frac{\dot r}{r} V^{0}\wedge V^{a} + r\, dv^{a} = \frac{\dot r}{r} V^{0}\wedge V^{a} + \tilde\omega^{a}{}_{b}\wedge V^{b} \; , \tag{A.9b}$$

$$dB^{m} = \frac{\dot s}{s} V^{0}\wedge B^{m} + s\, de^{m} \; , \tag{A.9c}$$

$$dB^{m'} = \frac{\dot u}{u} V^{0}\wedge B^{m'} + u\, de^{m'} \; , \tag{A.9d}$$

$$dB^{7} = \frac{\dot s}{s} V^{0}\wedge B^{7} . \tag{A.9e}$$

The tilde in $\tilde\omega^{a}{}_{b}$ implies it refers only to the 4-space, that is in (A.3) the L^{M} are replaced by v^{a} and the i_{M} by $\tilde i_{a}$ which obey

$$\tilde i_{a} v^{b} = \delta_{a}{}^{b} \quad , \quad \tilde i_{a} B^{m} = 0 . \tag{A.10}$$

Substitution from (7.14) into (A.9d,e) gives

$$dB^{1} = \frac{\dot s}{s} V^{0}\wedge B^{1} - \frac{1}{u} B^{53} - \frac{1}{u} B^{26} + \frac{2}{s}\cot\mu\, B^{23} + \frac{1}{s}\cot\mu\, B^{71} \; , \tag{A.11a}$$

$$dB^{4} = \frac{\dot u}{u} V^{0}\wedge B^{4} - \frac{1}{u} B^{56} - \frac{u}{s^{2}} B^{23} - \frac{u}{s^{2}} B^{71} \; , \tag{A.11b}$$

plus simultaneous cyclic permutations of (1,2,3) and (4,5,6). Using (A.4), or directly from (A.9), the components of the spin connection are

$$\omega_{0a} = -\dot{r}\, V^a \,, \quad \omega_{07} = -\dot{s}\, e^7 \,, \quad \omega_{0m} = -\dot{s}\, e^m \,, \quad \omega_{0m'} = -\dot{u}\, e^{m'} \,, \tag{A.12a}$$

$$\omega_{ab} = \tilde{\omega}_{ab} \,, \quad \omega_{71} = -\cot\mu\, e^1 + \frac{u^2}{2s^2}\, e^4 \,, \tag{A.12b}$$

$$\omega_{74} = \frac{u}{2s}\, e^1 \,, \quad \omega_{12} = -\cot\mu\, e^3 + \left(\frac{u^2}{2s^2} - 1\right) e^6 \,, \tag{A.12c}$$

$$\omega_{14} = -\frac{u}{2s}\, e^7 \,, \quad \omega_{15} = -\frac{u}{2s}\, e^3 \,, \tag{A.12d}$$

$$\omega_{16} = \frac{u}{2s}\, e^2 \,, \quad \omega_{45} = -\frac{e^6}{2} \,, \tag{A.12e}$$

plus cyclic permutations. Note that the product form of the d=11 manifold ensures $\omega_{am} = 0$. This would not be so if the scale factors in (A.8) were functions of x or y. The curvature 2-form is calculated by the straightforward, but lengthy, substitution of (A.12) into (A.2) and gives the non-zero components

$$\mathbb{R}^{ab} = \tilde{\mathbb{R}}^{ab} + \left(\frac{\dot{r}}{r}\right)^2 V^a \wedge V^b = \left[\frac{k_3}{r^2} + \left(\frac{\dot{r}}{r}\right)^2\right] V^a \wedge V^b \,, \tag{A.13a}$$

$$\mathbb{R}^{0a} = \frac{\ddot{r}}{r}\, V^0 \wedge V^a \,, \tag{A.13b}$$

$$\mathbb{R}^{07} = \frac{\ddot{s}}{s}\, V^0 \wedge B^7 + \frac{u\dot{s}}{2s^2}\left(B^{14} + B^{25} + B^{36}\right) + \frac{\dot{u}}{2s}\, B^1 \wedge \left(B^4 + B^5 + B^6\right) , \tag{A.13c}$$

$$\mathbb{R}^{01} = \frac{\ddot{s}}{s}\, V^0 \wedge B^1 \,, \quad \mathbb{R}^{04} = \frac{\ddot{u}}{u}\, V^0 \wedge B^4 \,, \tag{A.13d}$$

$$\mathbb{R}^{1a} = \frac{\dot{r}\dot{s}}{rs}\, B^1 \wedge V^a \,, \quad \mathbb{R}^{4a} = \frac{\dot{r}\dot{u}}{ru}\, B^4 \wedge V^a \,, \tag{A.13e}$$

$$\mathbb{R}^{71} = \left(\frac{1}{s^2} - \frac{3}{4}\frac{u^2}{s^4} + \left(\frac{\dot{s}}{s}\right)^2\right) B^{71} + \frac{1}{2s^2}\left(1 - \frac{u^2}{s^2}\right) B^{56} + \left(\frac{\dot{u}}{s} - \frac{u\dot{s}}{s^2}\right) V^0 \wedge B^5 \,, \tag{A.13f}$$

$$\mathbb{R}^{74} = \left(\frac{\dot{s}\dot{u}}{su} + \frac{u^2}{4s^4}\right) B^{74} + \frac{1}{4s^2}\left(1 - \frac{u^2}{s^2}\right)\left(B^{35} - B^{26}\right) + \frac{1}{2}\left(\frac{\dot{u}}{s} - \frac{u\dot{s}}{s^2}\right) V^0 \wedge B^1 \,, \tag{A.13g}$$

$$\mathbb{R}^{12} = \left(\frac{1}{s^2} - \frac{3}{4}\frac{u^2}{s^4} + \left(\frac{\dot{s}}{s}\right)^2\right) B^{12} + \frac{1}{2s^2}\left(1 - \frac{u^2}{s^2}\right) B^4 \wedge B^5 + \left(\frac{\dot{u}}{s} - u\frac{\dot{s}}{s^2}\right) V^0 \wedge B^6 \,, \tag{A.13h}$$

$$\mathbb{R}^{14} = \left(\frac{\dot{s}\dot{u}}{su} + \frac{u^2}{4s^4}\right) B^{14} + \frac{1}{4s^2}\left(1 - \frac{u^2}{s^2}\right)\left(B^{36} + B^{25}\right) - \frac{1}{2}\left(\frac{\dot{u}}{s} - \frac{u\dot{s}}{s^2}\right) V^0 \wedge B^7 \,, \tag{A.13i}$$

$$\mathbb{R}^{15} = \left(\frac{\dot{s}\dot{u}}{su} + \frac{u^2}{4s^4}\right) B^{15} - \frac{1}{4s^2}\left(1 - \frac{u^2}{s^2}\right)\left(B^{24} + B^{76}\right) - \frac{1}{2}\left(\frac{\dot{u}}{s} - \frac{u\dot{s}}{s^2}\right) V^0 \wedge B^3 \,, \tag{A.13j}$$

$$\mathbb{R}^{16} = \left(\frac{\dot{s}\dot{u}}{su} + \frac{u^2}{4s^4}\right) B^{16} + \frac{1}{4s^2}\left(1-\frac{u^2}{s^2}\right)\left(B^{75} - B^{34}\right) + \frac{1}{2}\left(\frac{\dot{u}}{s} - \frac{u\dot{s}}{s^2}\right) V^{o}_{\wedge} B^2, \quad \text{(A.13k)}$$

$$\mathbb{R}^{45} = \left(\left(\frac{\dot{u}}{u}\right)^2 + \frac{1}{4u^2}\right) B^{45} + \frac{1}{2s^2}\left(1 - \frac{u^2}{s^2}\right)\left(B^{12} + B^{73}\right). \quad \text{(A.131)}$$

Eqns.(A.13f-1) are a time dependent version of eqns.(21) of Awada et al{50}. Now using (A.5,6) the equations (A.13) give

$$R^{o}{}_{o} = \frac{1}{3}\left(3\frac{\ddot{r}}{r} + 4\frac{\ddot{s}}{s} + 3\frac{\ddot{u}}{u}\right) \quad , \quad \text{(A.14a)}$$

$$R^{a}{}_{a} = \frac{k_3}{2r^2} + \frac{1}{2}\frac{\ddot{r}}{r} + \frac{1}{2}\frac{\dot{r}}{r}\left(2\frac{\dot{r}}{r} + 4\frac{\dot{s}}{s} + 3\frac{\dot{u}}{u}\right), \quad \text{(A.14b)}$$

$$R^{m}{}_{m} = \frac{3}{2s^2}\left(1 - \frac{\lambda^2}{2}\right) + \frac{1}{2}\frac{\ddot{s}}{s} + \frac{1}{2}\frac{\dot{s}}{s}\left(3\frac{\dot{r}}{r} + 3\frac{\dot{s}}{s} + 3\frac{\dot{u}}{u}\right), m = 7,1,2,3, \quad \text{(A.14c)}$$

$$R^{m'}{}_{m'} = \frac{1}{4s^2}\left(2\lambda^2 + \frac{1}{\lambda^2}\right) + \frac{1}{2}\frac{\ddot{u}}{u} + \frac{1}{2}\frac{\dot{u}}{u}\left(3\frac{\dot{r}}{r} + 4\frac{\dot{s}}{s} + 2\frac{\dot{u}}{u}\right), \ m' = 4,5,6, \quad \text{(A.14d)}$$

which gives (7.29) for the static case and leads, with the Einstein equations to (7.36) for the time dependent version.

APPENDIX B   NUMERICAL SOLUTIONS OF EQUATIONS

Due to the highly non-linear nature of the model dependent systems of coupled second and first order differential equations (eg. the Einstein equations (7.36) and the Maxwell equations (7.35) for Model 1) it was, in general, necessary to solve the Maxwell and Einstein equations numerically. I shall briefly describe the procedure with some reference to Model 1, the squashed seven-sphere. Let the system be represented by

$$F_I(\ddot{G}_J, \dot{G}_K, G_L, t) = 0 , \tag{B.1}$$

where I labels the equations, I = 1,...,5 for Model 1 after (7.35a) has been integrated and is imposed as a constraint on the system and one of (7.36) is also imposed as a Bianchi condition. The G_L are in the case of Model 1 r, s, u, g_1 and g_2. In this work the independent variable t does not appear explicitly. By setting

$$H_K = \dot{G}_K \tag{B.2}$$

(eg. $\rho = \dot{r}$, $\sigma = \dot{s}$ etc.), the system is converted to one of first order in H_K, G_K, namely

$$\dot{H}_K = L_K(H_I, G_J) , \tag{B.3a}$$

$$\dot{G}_L = H_K(H_I, G_J). \tag{B.3b}$$

Eqns.(B.3a) are the equations for $\dot{\rho}$, $\dot{\sigma}$ etc. (five of them) and (B.3b) the equations for $\dot{r}$, $\dot{s}$ etc. which are $\dot{r} = \rho$, $\dot{s} = \sigma$ etc. (and there are five of these). A computer program was written (for an IBM 43/31) which called a Harwell subroutine (DC01AD) to numerically integrate the system using a Runge-Kutta method.

Initial values had to be chosen for the G_I and $\dot{G}_I$ which would satisfy the condition arising from the Einstein equations due to the Bianchi identity; (8.6) in the case of Model 1. The constant in (7.35a') was calculated for the initial values and then in each step of the integration f was recalculated by (7.35') from this equation using the current values. Alternatively (7.35a) could have been used as a first order differential equation for f. Eqns.(B.3a) are obtained from (7.35b,c) and (7.36b-d). The program then gave values for $G_I, \dot{G}_I$ (eg. $r, \dot{r}$) at time intervals as required.

The accuracy of the solutions was set by allowing an error on J_K ($= H_K, G_K$) of $|10^{-10} + |J_K| \times 10^{-10}|$. The program uses Gear's predictor-corrector method{113} which automatically chooses step size and order of integration. For some cases, with particular initial values, the system of differential equations becomes unstable for certain values of t and this accuracy could not be achieved. However, when this occurred the equations were so unstable that setting a lesser accuracy only allowed the integration to continue for a short time longer. In most cases when this happened the behaviour of the equations was clear and approximate analytic solutions could be studied as time approached this critical value. The error could build up over several steps. To check that this was not important the permitted error was changed and the results were found to be very similar.

The Harwell subroutine was designed for up to ten equations. In the case of Model 4 up to twelve equations were needed and this required modification of the program. This modification did not give the same accuracy as before and to some extent limited the detail study of this model.

APPENDIX C ORIENTATION of M_7 and SUPERSYMMETRY

There is some confusion in the literature about the effect on any remaining supersymmetries of reversing the orientation of M_7, giving rise to the misleading concept of left and right squashing in the case of the seven-sphere{42,46,114} and other solutions{5,115}. I have chosen the orientation for M_7 as

$$\overset{*}{_7} 1 = B^{1234567} . \tag{C.1}$$

By sending $B^m \to -B^m$ the orientation is reversed because the space is of odd dimension but this should have no consequence in terms of supersymmetry etc.

As discussed in chapter IV a surviving supersymmetry requires that there exist a spinor η (y) which is a solution of

$$D\eta = \frac{m}{2} \eta \tilde{\Gamma}_m B^m . \tag{C.2}$$

Note that the sign of m is not fixed. The integrability condition for (C.2), given by (4.25), requires

$$R_{mn} = 6 m^2 g_{mn} . \tag{C.3}$$

Choosing the usual Freund-Rubin ansatz

$$F = f \Omega \tag{C.4}$$

gives, as in chapter IV

$$R_{mn} = \frac{1}{2.4!} f^2 g_{mn} . \tag{C.5}$$

Comparing (C.3) and (C.5) gives the result

$$f^2 = 12.4! \, m^2 . \tag{C.6}$$

Thus m and f are related but their relative sign is not fixed. This is the important point to note. If the relative sign is fixed a priori, as in refs.{5,42,46,47,45} then if there is an η satisfying (C.2) there is not necessarily one satisfying the equation with the orientation reversed

$$\mathcal{D}\eta = -\frac{m}{2}\,\eta\,\tilde{\Gamma}_m B^m . \tag{C.7}$$

However if the sign of m in (C.7) is changed, this being allowed by (C.6), the equation is restored to its original form and a solution exists. Thus left and right squashing are equivalent, whichever it is depending on whether $m > 0$ or $m < 0$ and this does not effect the existence of an η or not. If an η exists on M_7, an F can be constructed on M_7 (by the Englert method, see (4.18)) and also on $\tilde{M}_7$, that is M_7 with the orientation reversed.

APPENDIX D EQUATIONS for the HIGHER DIMENSIONAL THEORY

For completeness here is included the Einstein and Maxwell equations for the minimal theory of gravity coupled to a 4-form F in higher dimensions considered in chapter IX. If m is the number of extra dimensions with an isotropic geometry and M_4 is the usual anti-de Sitter or de Sitter 4-space the Ricci tensor calculated from the curvature is

$$R^0{}_0 = \frac{3}{2}\frac{\ddot{r}}{r} + \frac{m}{2}\frac{\ddot{s}}{s} \quad , \tag{D.1a}$$

$$R^a{}_a = \frac{1}{2}\frac{\ddot{r}}{r} + \frac{k_3}{r^2} + \left(\frac{\dot{r}}{r}\right)^2 \quad , \tag{D.1b}$$

$$R^n{}_n = \frac{1}{2}\frac{\ddot{s}}{s} + \frac{(m-1)}{2}\frac{k_m}{s^2} + \frac{3}{2}\frac{\dot{r}\dot{s}}{rs} + \frac{m(m-1)}{2}\left(\frac{\dot{s}}{s}\right)^2 , \tag{D.1c}$$

where $k_3(k_m)$ are the 3(m)-space curvatures. For the m-torus $k_m = 0$. Eqn.(D.1) is equated with the Ricci tensor from the Einstein equation, which is the same as the d=11 supergravity case,

$$R^M{}_N = 6\, F^{MP_1P_2P_3} F_{NP_1P_2P_3} - \frac{F^2}{2}\,\delta^M{}_N \tag{D.2}$$

and is for F given in (9.19) if M_m is a m-torus

$$R^0{}_M = \frac{\delta^0_M}{4!}\left[-f^2 - m\frac{\dot{g}^2}{s^6}\right] , \tag{D.3a}$$

$$R^a{}_M = \frac{\delta^a_M}{4!}\left[-f^2 - m\frac{\dot{g}^2}{s^6}\right] , \tag{D.3b}$$

$$R^m{}_M = \frac{\delta^m_M}{4!}\left[\frac{1}{2}f^2 - \frac{(m-3)}{4}\frac{\dot{g}^2}{s^6}\right] . \tag{D.3c}$$

The Maxwell equation

$$d * F = 0 \tag{D.4}$$

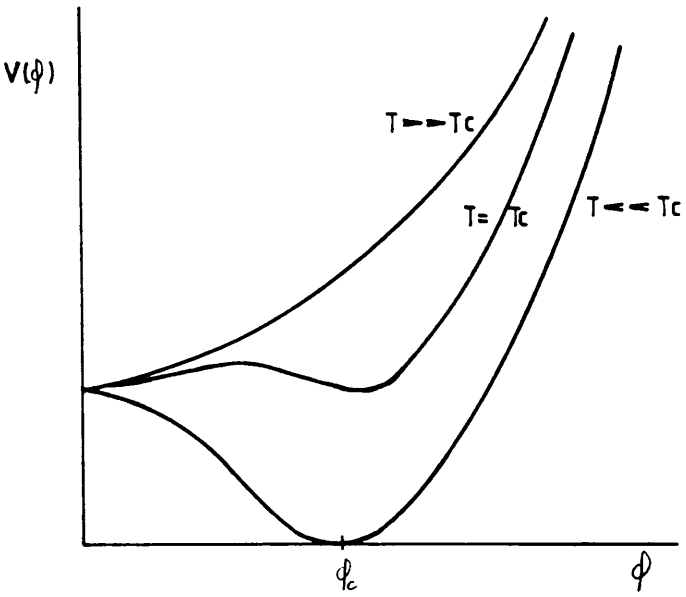
gives the conditions

$$f\, s^m = c_1 \quad , \tag{D.5a}$$

$$\dot{q}\, s^{m-6}\, r^{3} \;=\; c2 \quad , \tag{D.5b}$$

where c1, c2 are constants. Eqns.(D.3,5) would be more complicated for other m-dimensional manifolds and choices of the 3-forms E_i.

FIGURE CAPTIONS



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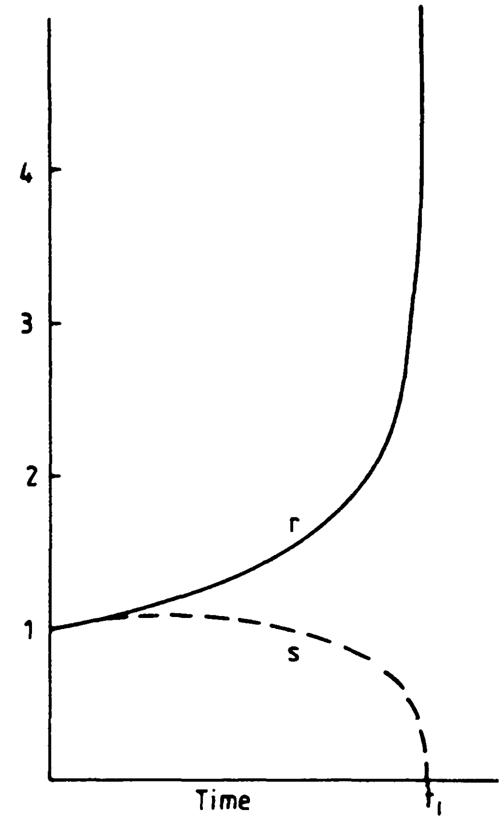

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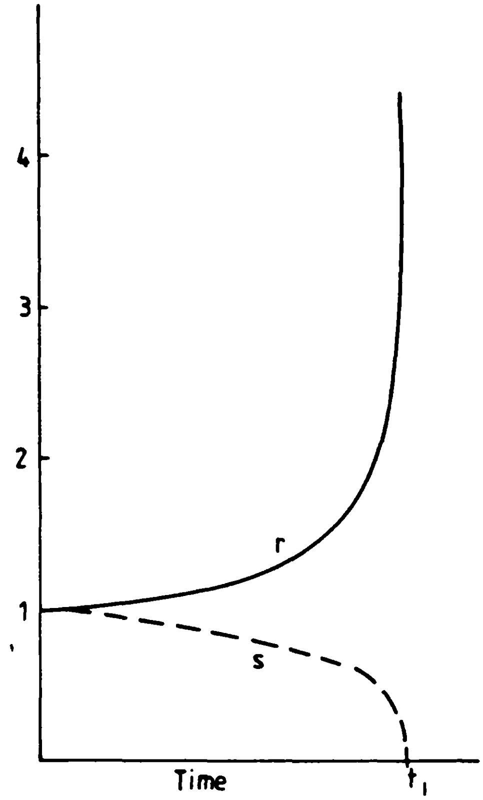

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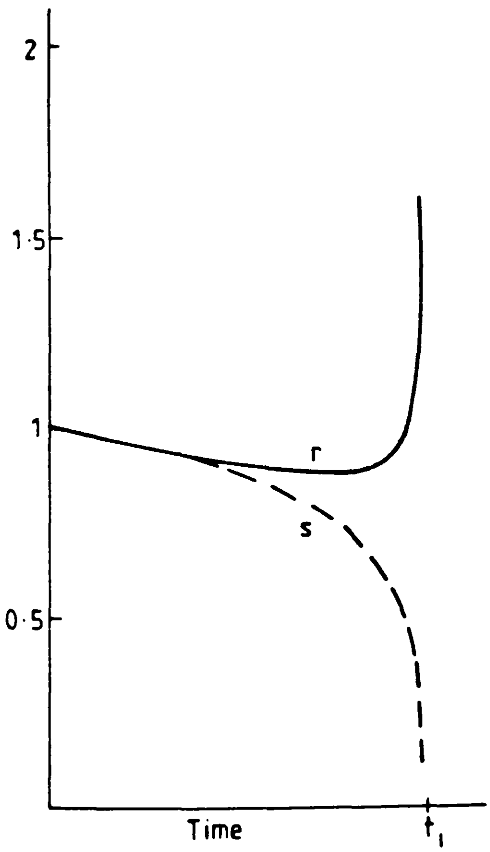

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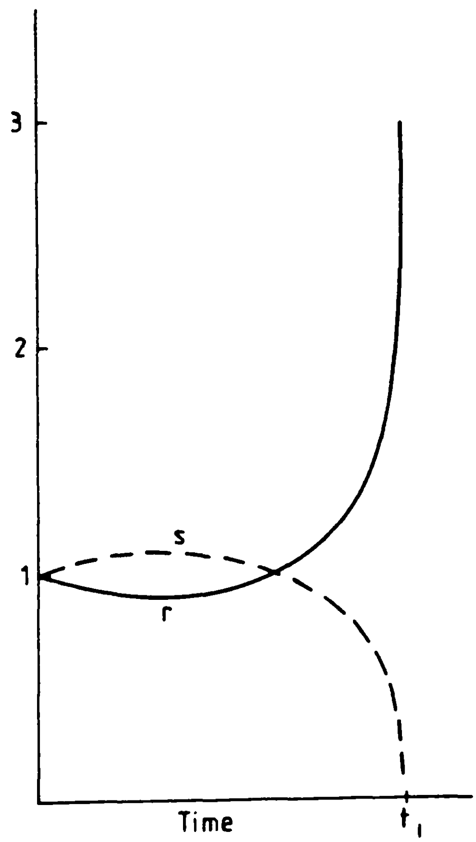

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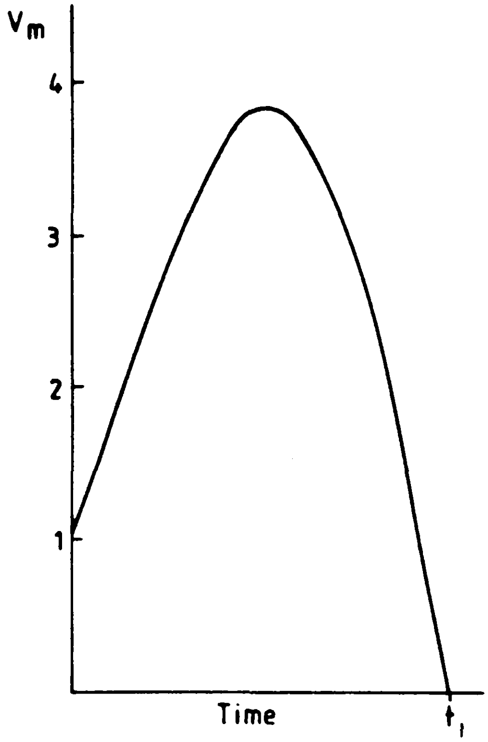

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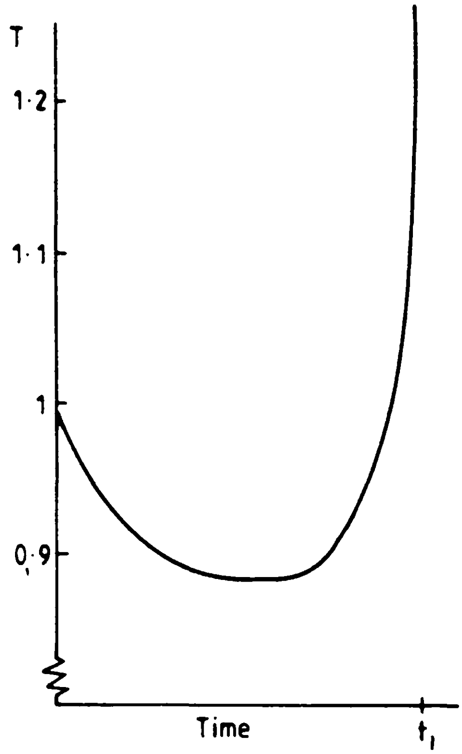

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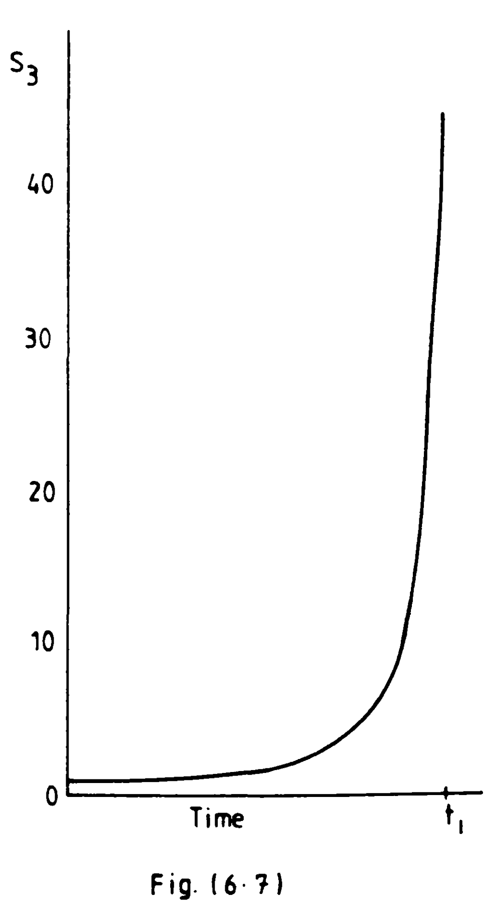

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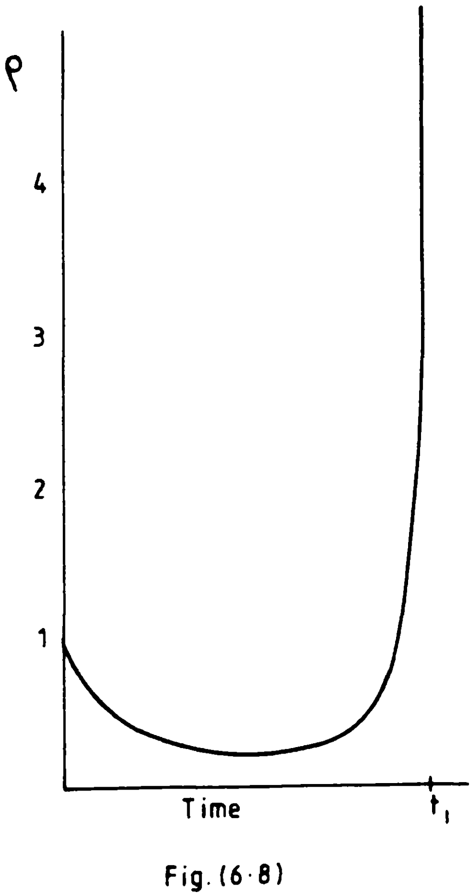

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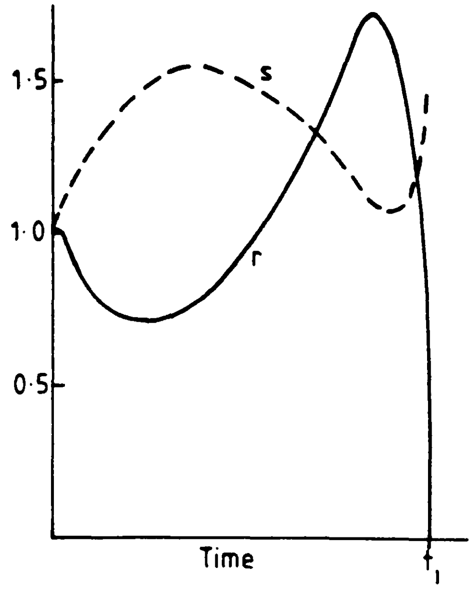

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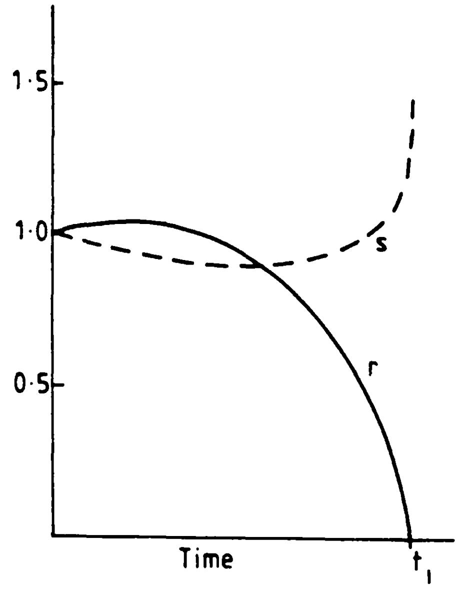

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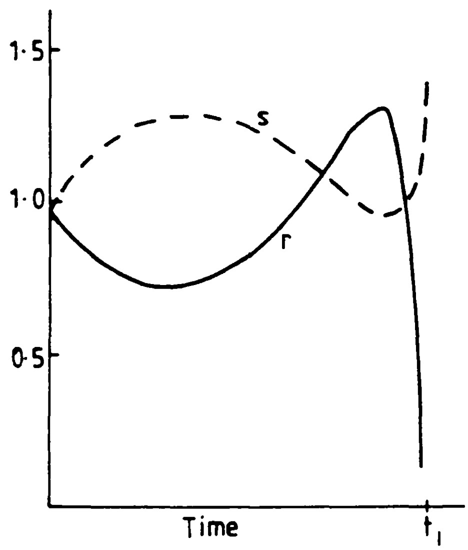

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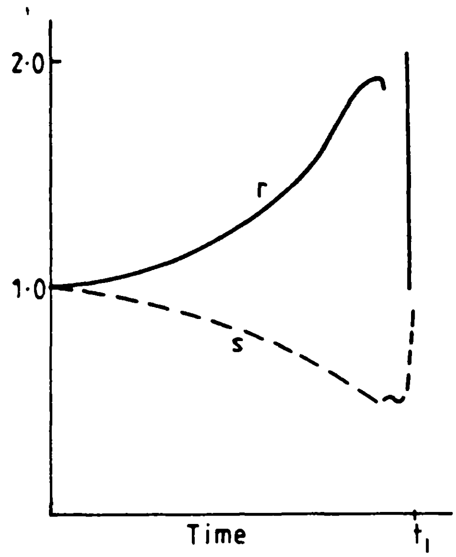

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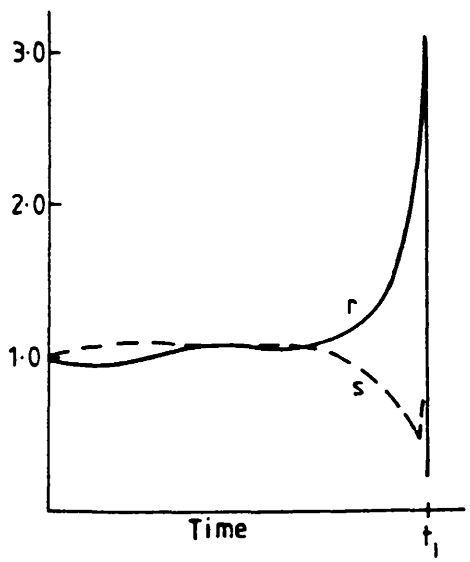

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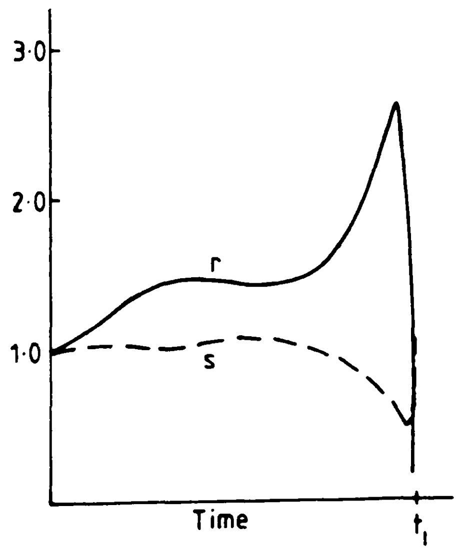

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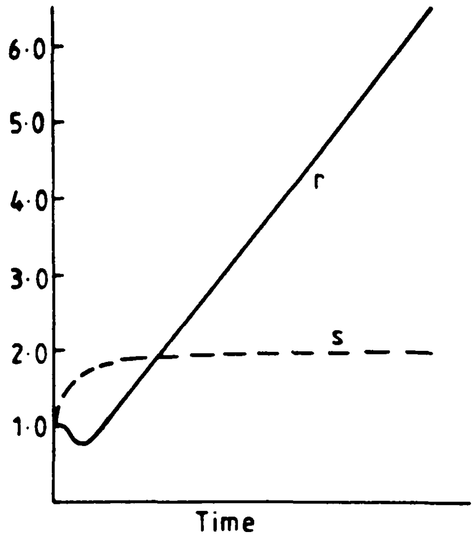

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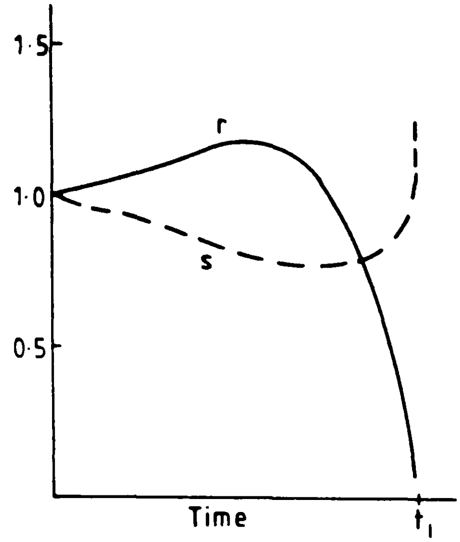

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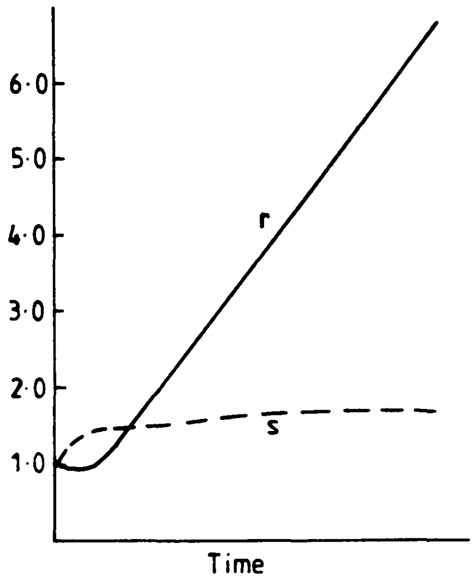

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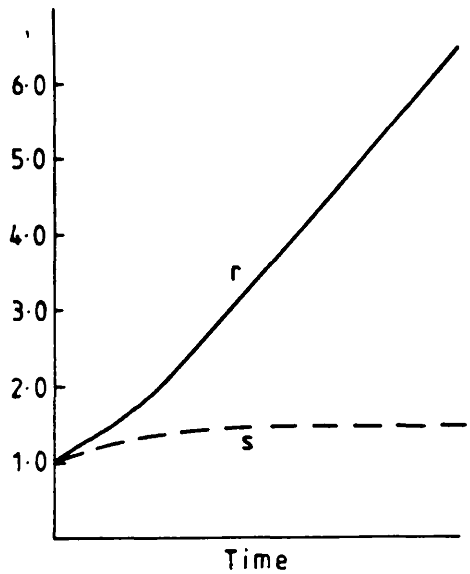

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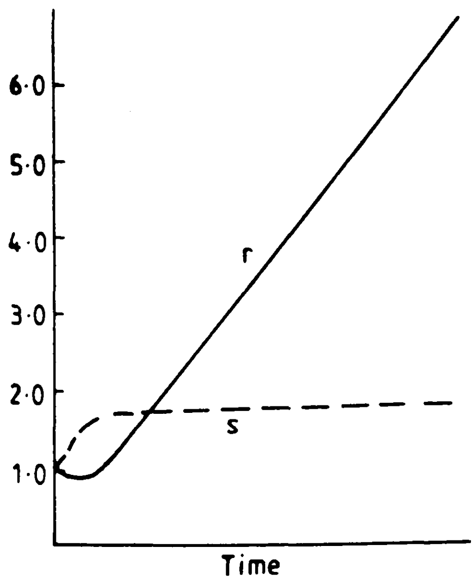

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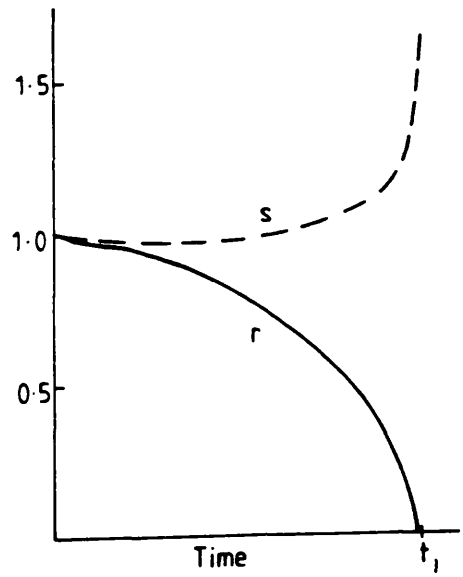

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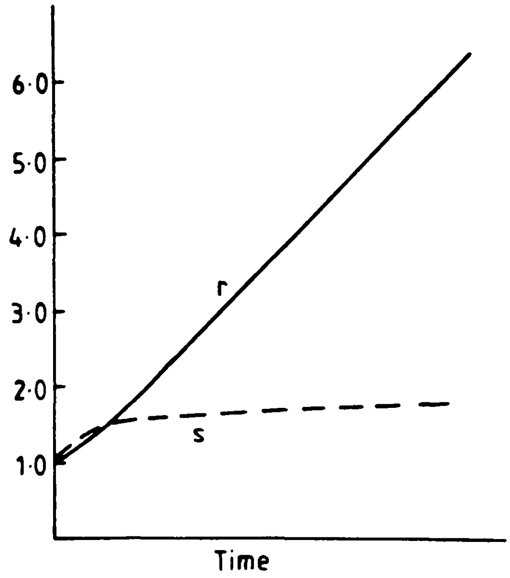

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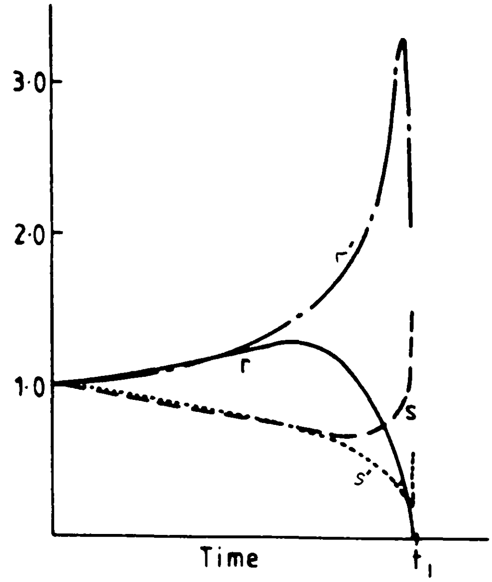

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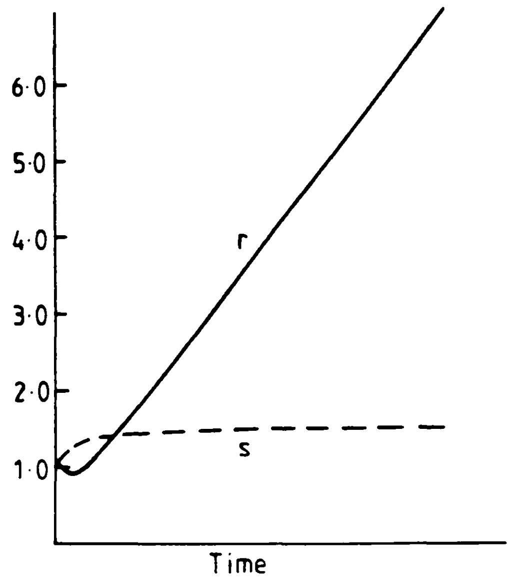

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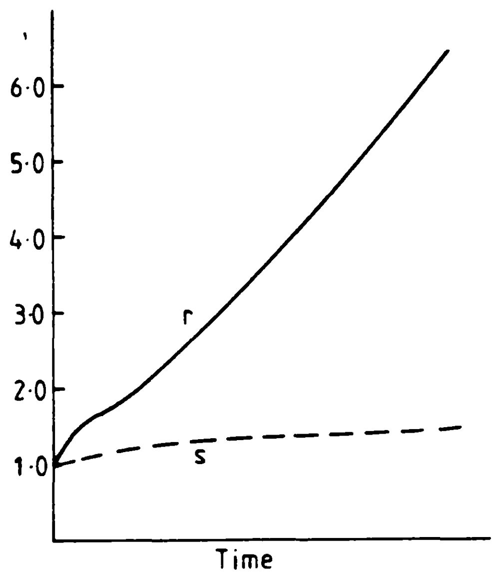

Fig. (8·16)

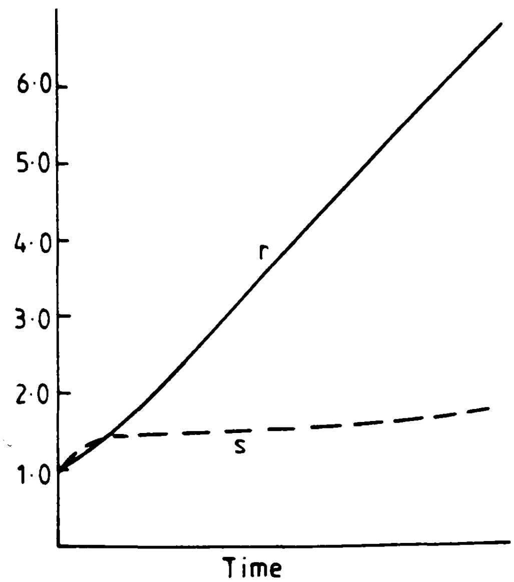

Fig. (8·17)

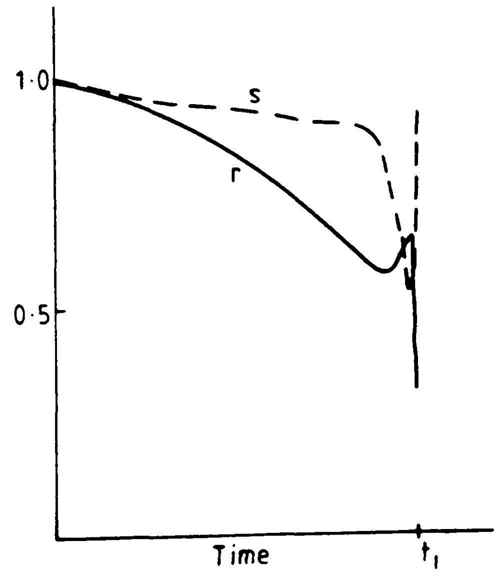

Fig. (8·18)

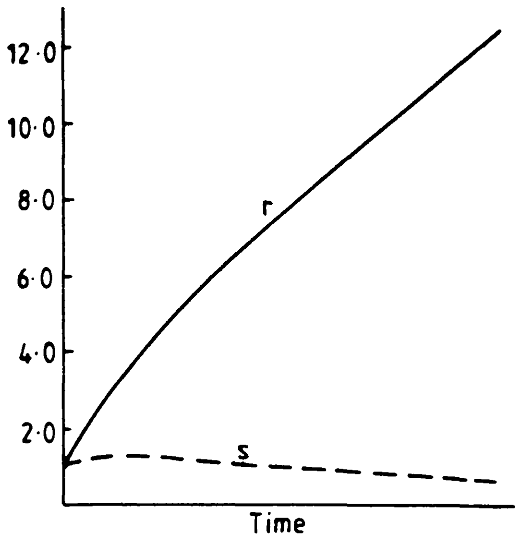

Fig. (8·19)

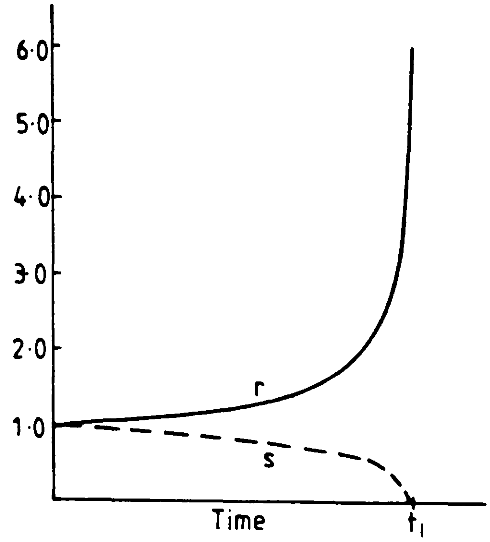

Fig. (8·20)

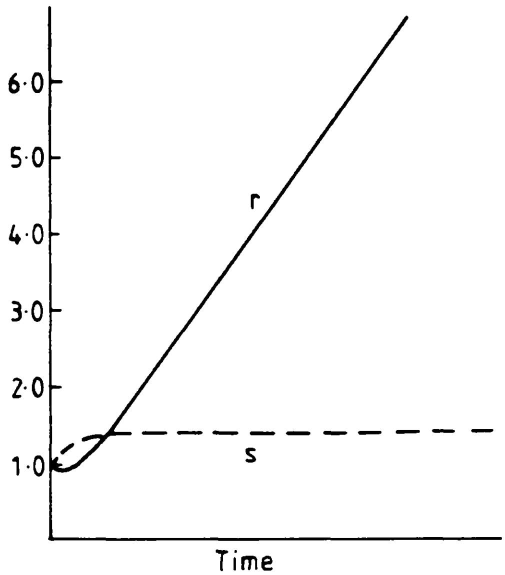

Fig. (8·21)

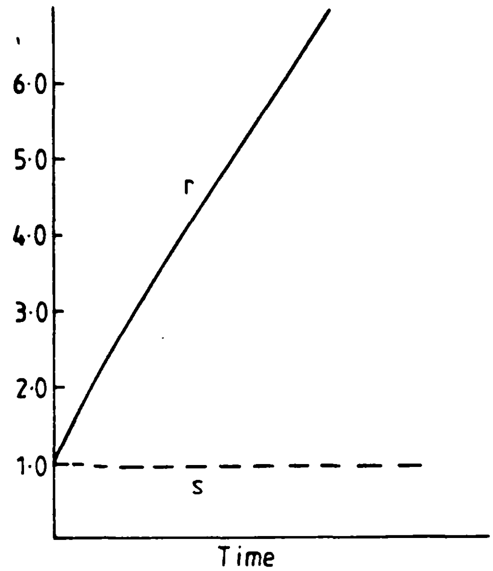

Fig. (8·22)

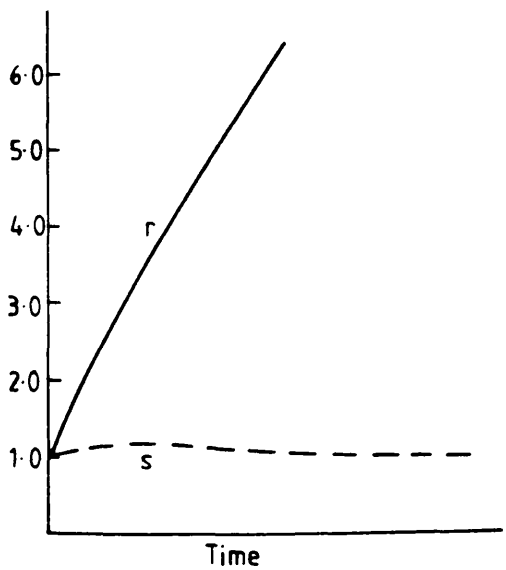

Fig. (8·23)

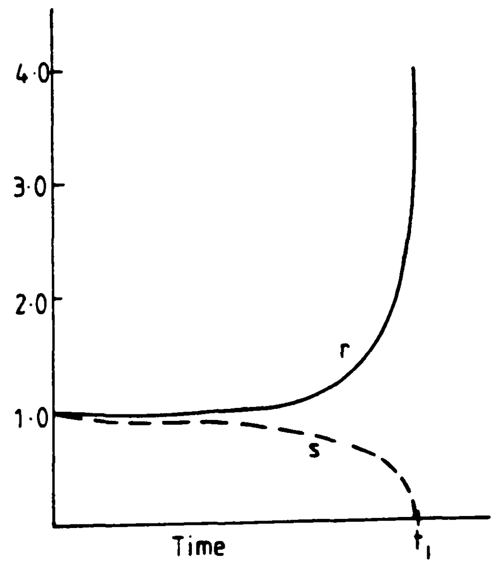

Fig. (8·24)

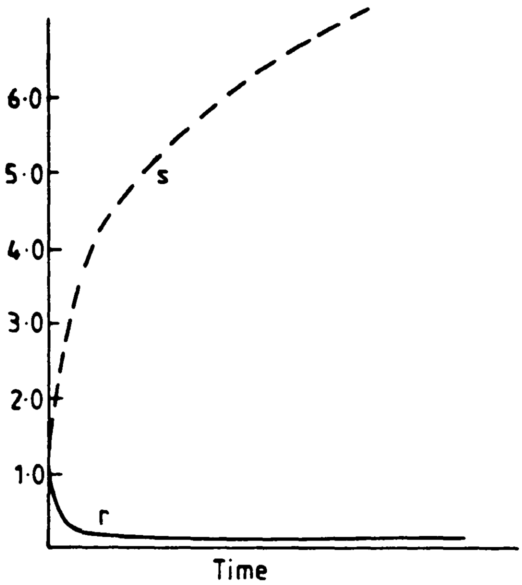

Fig. (8·25)

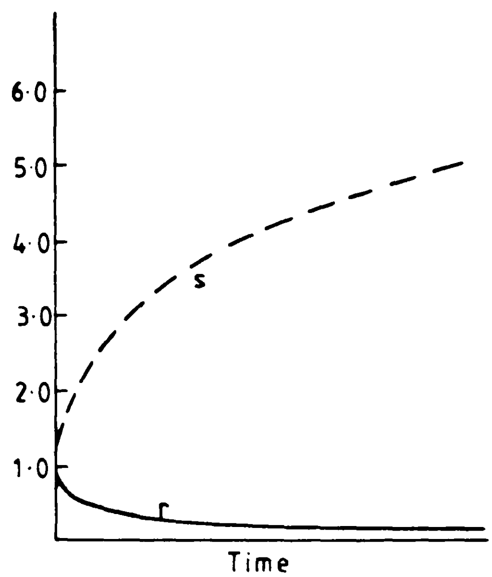

Fig. (8·26)

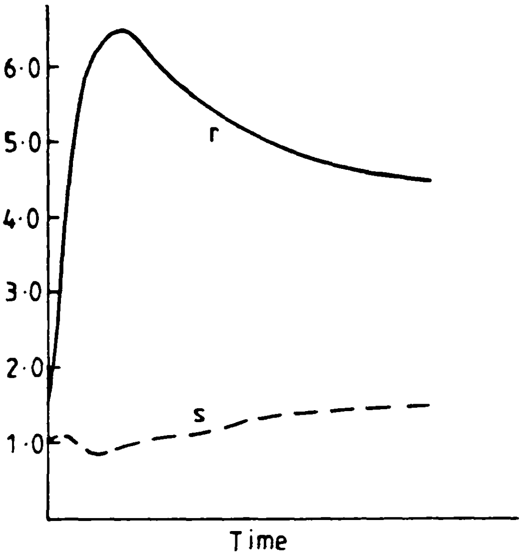

Fig. (8·27)

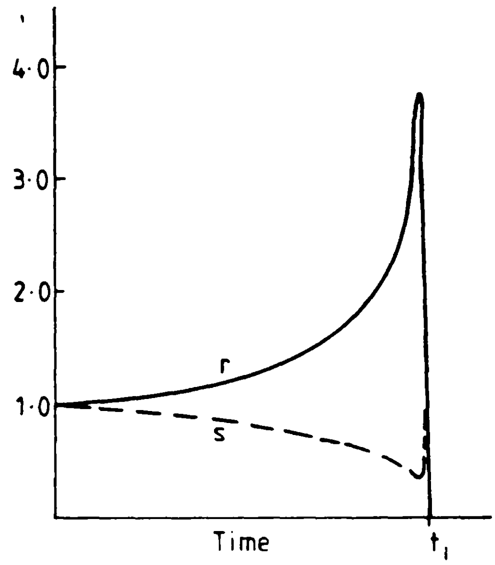

Fig. (8·28)

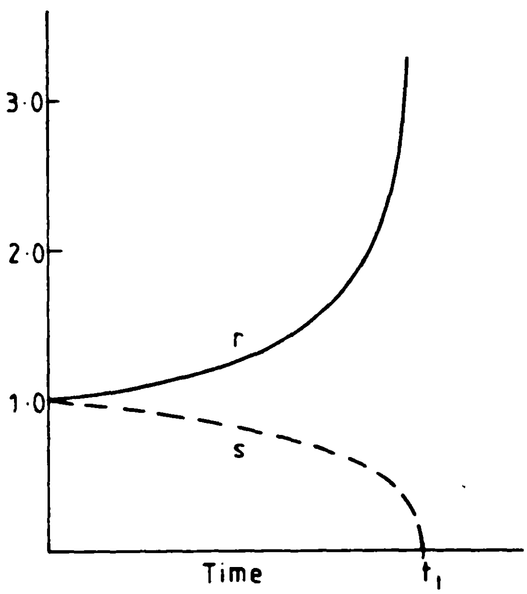

Fig. (8·29)

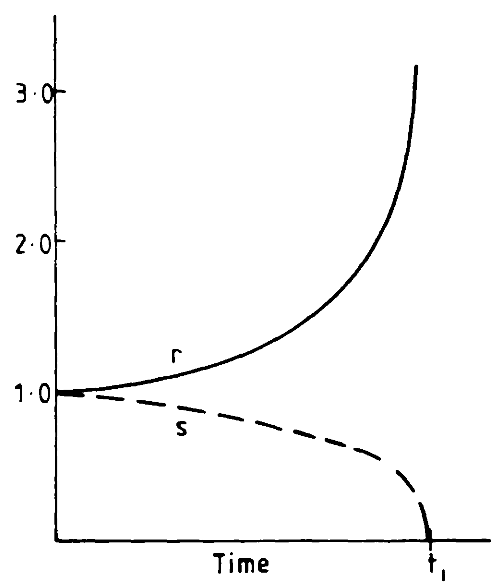

Fig. (8·30)

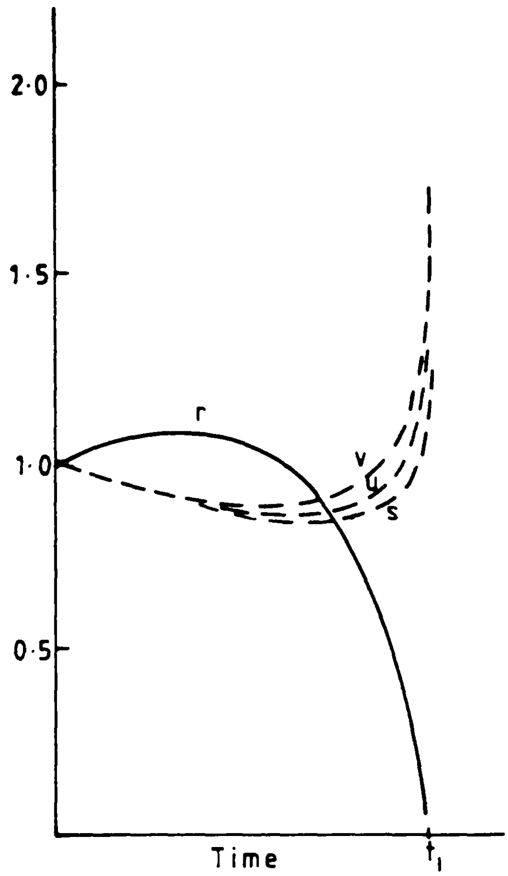

Fig. (8·31)

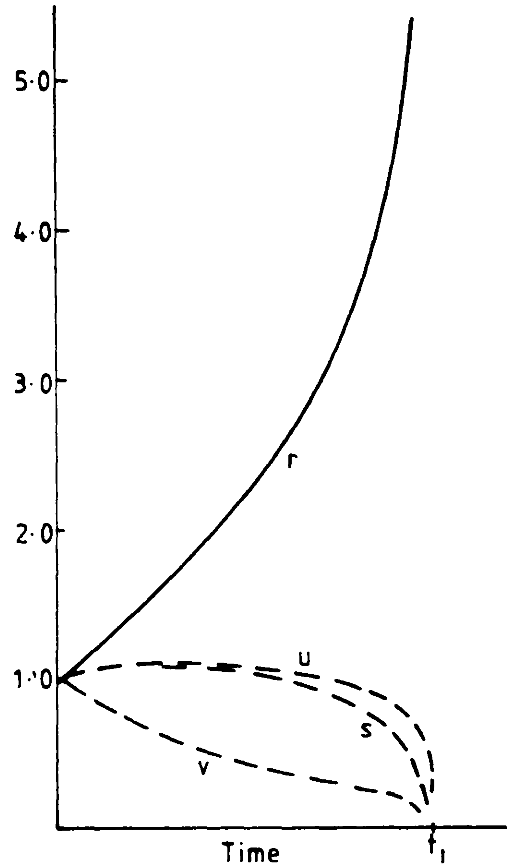

Fig. (8·32)

Fig. (8·33)

Fig. (8·34)

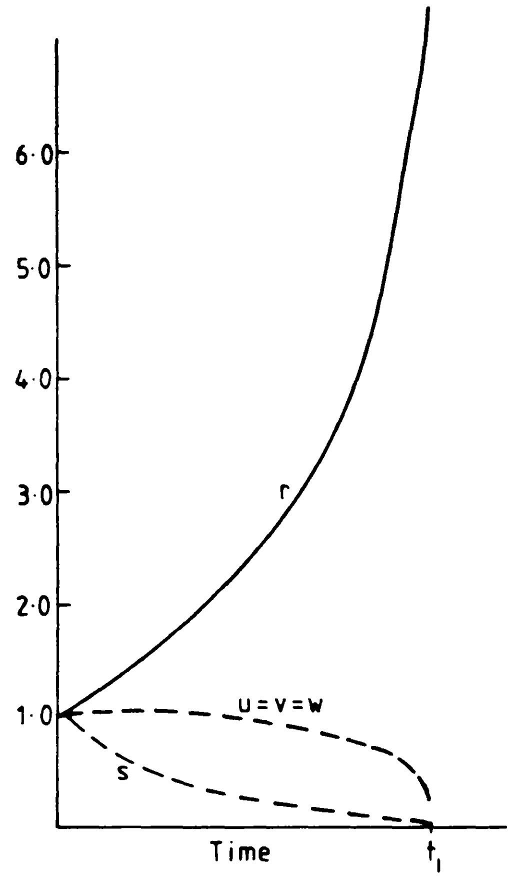

Fig. (8·35)

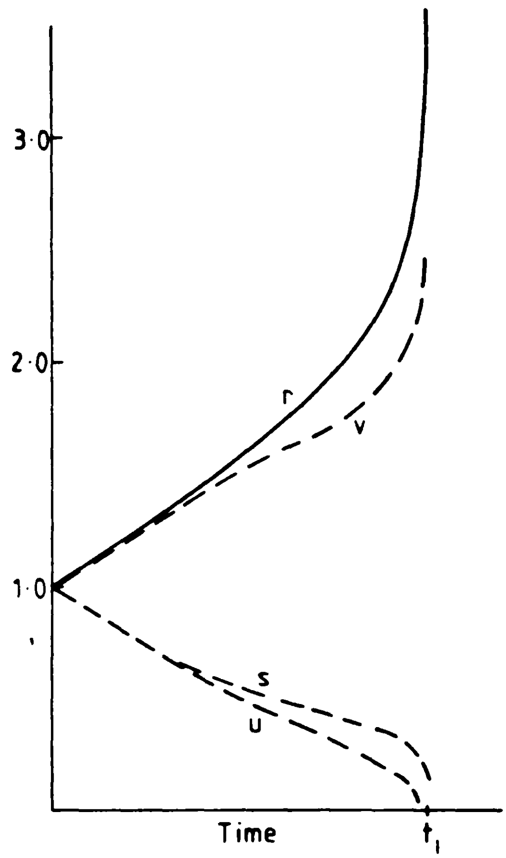

Fig. (8·36)

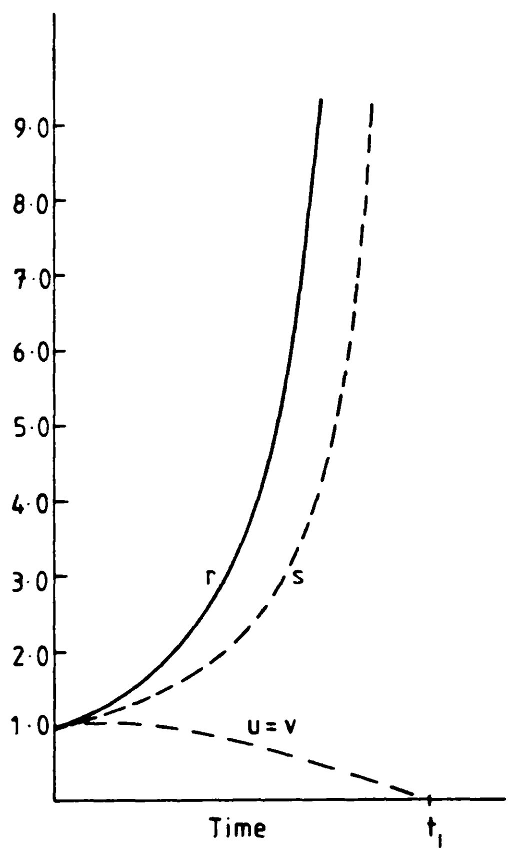

Fig. (8·37)

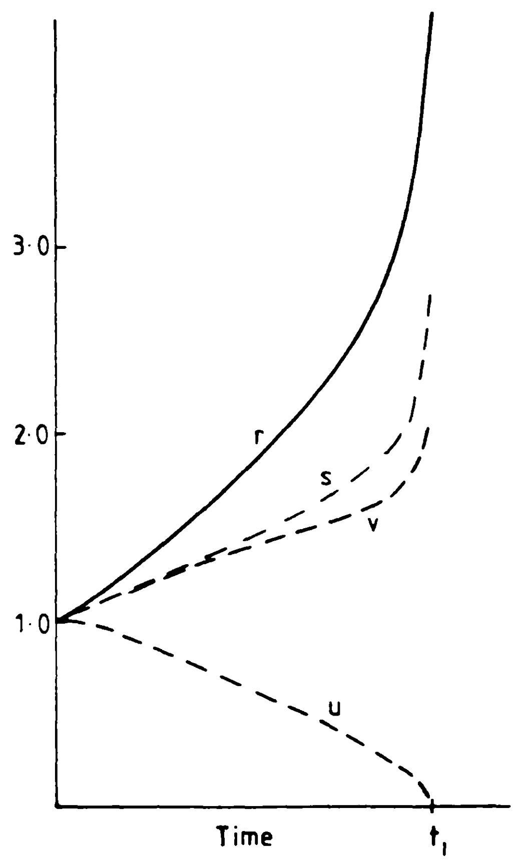

Fig. (8·38)

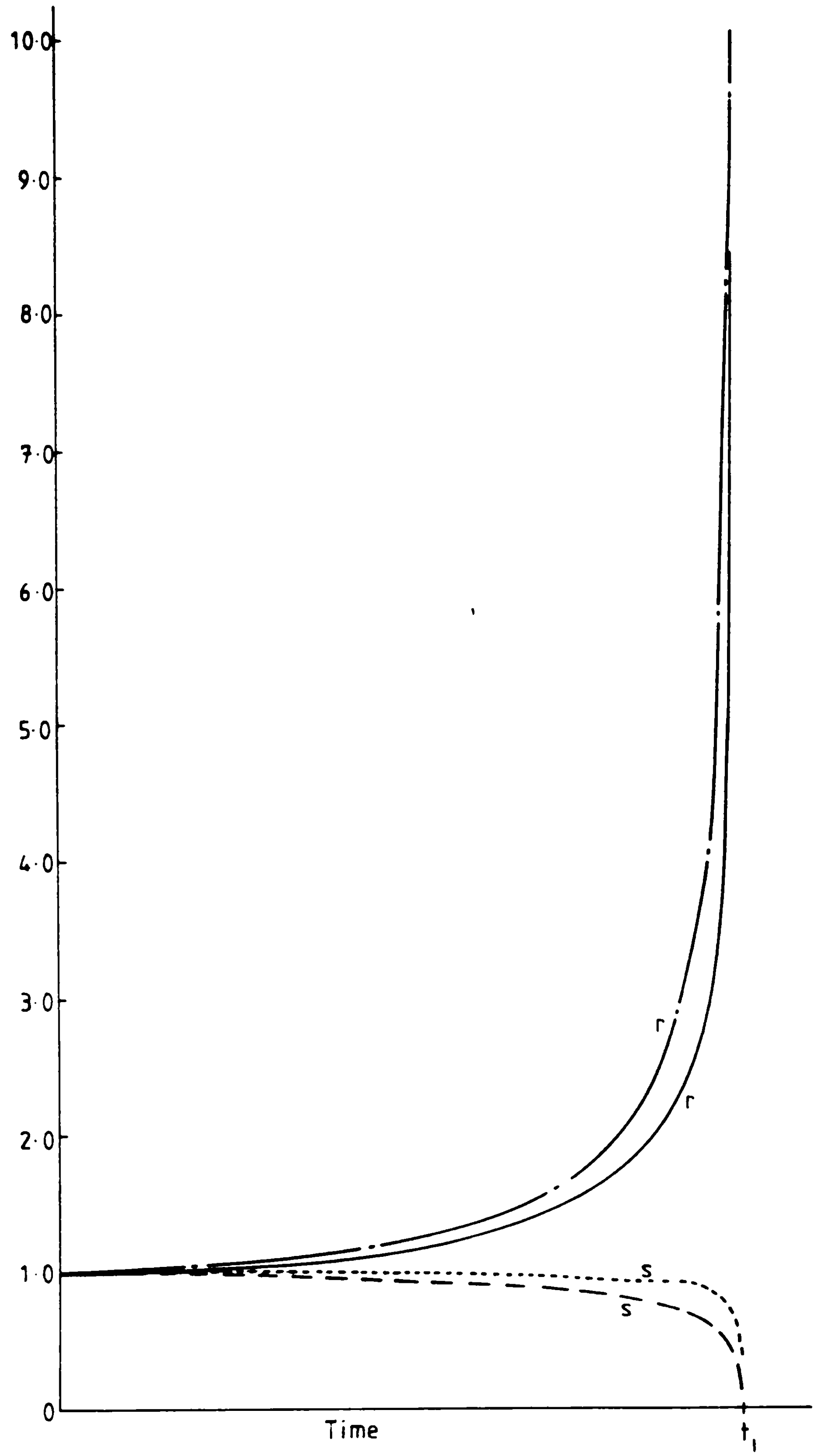

Fig. (9·1)

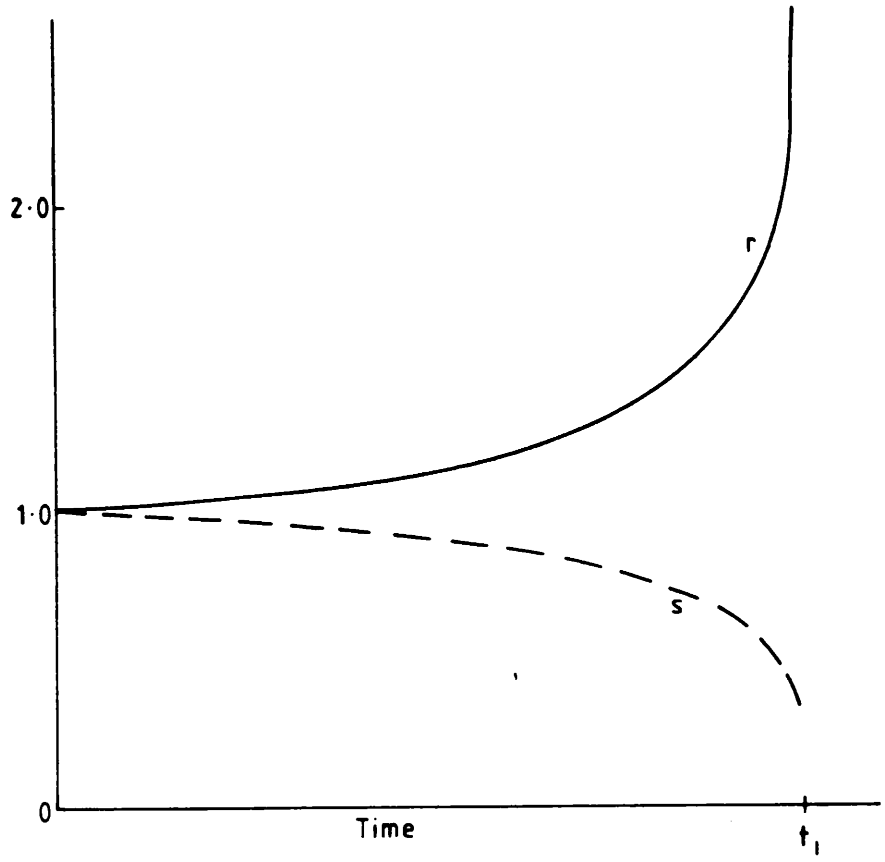

Fig.(10·1)

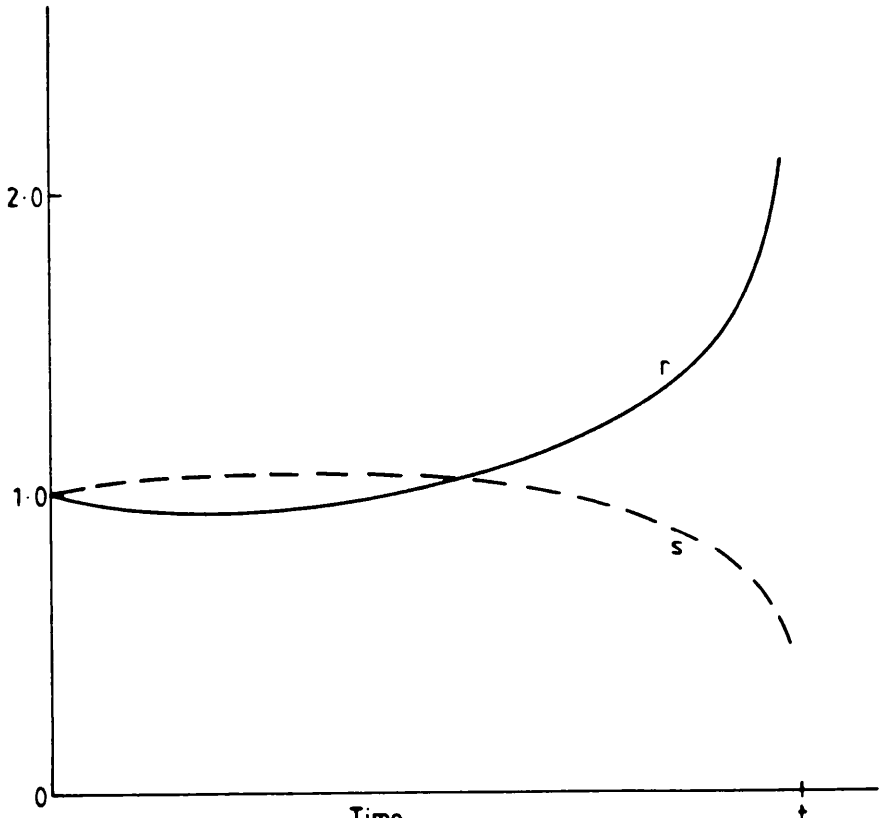

Fig.(10·2)

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