WAVE PROPAGATION IN NON-UNIFORM PLASMAS

by

Declan Andrew Diver

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The Department of Physics and Astronomy,
University of Glasgow,
Glasgow G12 8QQ.

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Abstract

This thesis concentrates on phenomena associated with waves propagating in an inhomogeneous medium, in particular the transmission and reflection of wave motion in a non-uniform plasma, and the process of exciting secondary wave motion, termed mode conversion.

To this end, a thorough treatment of the phase integral, or WKBJ, method is given in Chapter II, together with a careful statement of under what circumstances it is applicable, and, if so, to what accuracy.

As a novel example of this powerful technique being applied in a specific physical context, WKBJ theory is used in Chapter III to solve for the transmission characteristics of an acoustic guide with varying cross-section.

Building on this experience, the phenomenon of mode conversion is defined in Chapter IV, and a critical review is undertaken of the various historical and contemporary approaches to quantifying this effect. Specific examples are cited as evidence of the inadequacy of the reverse Fourier transform technique, including a complete solution of the same non-uniform waveguide problem using this method. The

result is contrasted with the previous self-consistent analysis in support of the contention that such reverse transforms are not generally correct.

Since most mode conversion theories depend to some extent on the concept of a spatially dependent dispersion relation, Chapter V embarks on a self-consistent analysis of particular mode conversion events, deriving the coupling directly from the behaviour of the eigenvalues of the governing ordinary differential equation. Such analysis recovers some of the more desirable features of the other theories, but in a more rigorous mathematical setting.

Finally, wave propagation in a non-uniform mhd fluid plasma is studied, using only the appropriate fluid equations with the inhomogeneity present at the earliest possible stage. Computer algebra is used to perform the necessary algebraic manipulations, and full details are given in Chapter VI.

Extra physical effects of interest concerning the Alfven resonant layer are revealed.

Chapter VII summarises the conclusions, and suggests further work in the field.

Chapter I Introduction

Recent years have seen a dramatic escalation of interest and effort in the area of plasma physics, specifically on the feasibility of power generation by controlled thermonuclear fusion.

Although the basic properties of highly ionised gases have been studied in many devices over the years {1} (see table 1), this decade has seen the construction of very much larger and more sophisticated machines dedicated to solving the particular problems associated with plasmas of sufficient size and complexity as those considered necessary to contain in any prototype power station (see table 2). In fact, the Joint European Torus has the specific objective of an investigation of the physics of large volume plasmas close to fusion conditions {2}.

Motivation behind fusion

Nuclear fusion will be a very attractive means of generating electricity, not just for economic reasons, but also for environmental considerations {3}.

The economic argument in support of nuclear fusion is quite simple: the fuel resources are virtually unlimited, and are readily obtainable on a globally equitable basis. A power station of this type would use the following reactions to release the enormous stored energy by fusing together light nuclei:

$D + T \rightarrow {}^4He + n + 17.6MeV$	Fusion reaction
${}^6Li + n \rightarrow {}^4He + T + 4.8MeV$ ${}^7Li + n \rightarrow {}^4He + T + n - 2.5MeV$	Tritium producing reactions
$D + Li \rightarrow 2\,{}^4He + Energy$	Overall reaction

Deuterium, a hydrogen isotope, can be obtained from the oceans, and is consequently practically limitless in supply. The lithium can be extracted from the sea, but occurs in abundance in the form of natural ores distributed evenly across the planet.

As a guide to the fuel requirements, fusion engineers estimate that a typical fusion power plant with 1GW capacity, as envisaged by today's standards, would have an annual fuel requirement of 150kg of deuterium (extracted from $5000m^3$ of sea water) and around half a tonne of lithium {3}.

The environmental arguments are equally impressive. A fusion reactor would be inherently safe in operation, having only a few seconds worth of fuel at fusion conditions, thus eliminating the chance of an uncontrollable reaction. In addition, the reaction products, whilst still radioactive, have the great advantage over fission products in that the half-life of fusion waste is measured in tens of years, rather than thousands.

However, in common with current nuclear power plant, a fusion device will have to be shielded against fast neutron release into the environment, and the reactor vessel will need to be decommissioned in a similar fashion.

Problems

Great technical problems still stand in the way of practical fusion. Since the energies required to overcome the repulsion of the nuclei are so great, the fuel must be in the plasma state. This presents great difficulties in containment, since any gas at such temperatures would vapourise the material of a containment vessel on contact.

There are numerous ingenious devices designed to overcome this difficulty.

Inertial confinement contains the fuel when it is cold, in the form of small pellets. These micro-balloons are then heated rapidly by laser pulses, in order to achieve sufficient resulting compression that the fusion reaction can take place before the disintegration of the pellet.

The main emphasis of this thesis, however, will be on magnetic confinement. This uses the basic result that charged particles will spiral along magnetic field lines; if the magnetic field forms a closed loop, then the plasma may be contained by the field without coming into contact with the vessel walls. The simplest device based on this concept is the tokamak, first invented in the Soviet Union. The operational principles of a tokamak are shown in figure 1. Despite many variations on this theme, the tokamak is still the most promising machine for nuclear fusion, and the largest and most successful plasma device is of this type, namely the Joint European Torus at Culham laboratories in the United Kingdom (see figure 2).

In order to measure degrees of success in the performance of prototype fusion device, J D Lawson first proposed criteria {4} summarised in the famous inequalities

$$T \gtrsim 10^{8}\,K, \quad n\tau_{E} \gtrsim 10^{22}\,m^{-3}$$

These parameters have proved elusive; even today, devices such as JET have failed to achieve simultaneously all the required values (see figure 3). The fundamental problem is reaching and maintaining sufficiently high temperatures to exceed the Lawson criteria.

It was always planned that JET would require extra forms of energy input over and above the simplest ohmic heating, where the plasma is heated simply by passing a current through it. Despite the fact that JET recorded the highest ever temperatures by this method, ohmic heating alone cannot bring a plasma to fusion conditions. The reason for this is simple: as the temperature increases, the resistivity decreases, and then so too does the ohmic heat dissipated by the current. In fact, the resistivity for a stationary, source free plasma takes the form {5}

$$\eta \sim \frac{Z_{eff}}{T_e^{3/2}}$$

and consequently, the power dissipated saturates with temperature. This is a very well known and documented feature of magnetically confined plasmas. This being the case, most tokamaks have some form of additional heating to elevate the temperatures beyond those achievable by ohmic heating.

Additional Heating Schemes

neutral beam injection

By injecting a stream of highly energetic and electrically neutral particles into the plasma, energy from the beam can be dissipated throughout the background plasma by collisional processes.

The neutrality of the particles allow them to negotiate the containing magnetic fields, but as soon as the beam penetrates the plasma, it rapidly becomes ionised and is absorbed into the background gas of ions and electrons. Multiple coulomb collisions ensure beam energy is shared with the background plasma, and so heating takes place. The very recent performance of neutral beam injection in JET experiments show the promise of this technique as an effective heating mechanism.

radio frequency heating

Since a plasma can sustain a wide variety of complicated wave modes, excitation of these can provide a means of transferring energy from an external source into the plasma.

The basic idea is that an externally launched wave penetrates the plasma and couples to a natural internal plasma mode. This natural mode then propagates deep into the plasma, at some stage converting its energy into plasma thermal energy and thus raising the overall temperature.

It is this latter stage which is the most complicated and difficult to model. Since most fusion plasmas are non-uniform, any wave propagating in it will not retain a simple form, but will have a continuously varying wavelength and amplitude. This feature is exploited in rf heating schemes. Having penetrated the plasma, the wave may encounter a region in which it has a similar wavelength to another internal plasma mode, and so may lose some of its energy by exciting that other wave. If it is the case that the secondary, excited wave is more readily damped than the first, then this is potentially an efficient heating technique. This process is termed mode conversion, and its mathematical description forms the main content and motivation of this thesis.

The actual process by which any secondary wave loses energy to the plasma is not considered here, but possibilities are damping by collisionless or collisional processes, which include Landau damping and wave-particle interactions.

The candidate wave for rf heating is usually the fast magnetosonic wave, known also as the fast or compressional Alfven {6}. This is actually a cold plasma mode, which has its wavevector perpendicular to the magnetic field and causes bulk fluid motion parallel to $\underline{k}$. It is this latter feature which results in the term magnetosonic; in actual fact, the cold plasma is pressureless and so has no sound speed. Any resulting ambiguity is resolved by the fact that the fast magnetosonic mode of the warm plasma reduces to the cold compressional Alfven as $T \rightarrow 0$.

other methods

There are other wave heating schemes, such as ion hybrid resonance and electron cyclotron heating (the latter requiring microwave generators). In addition, plasmas may be heated by adiabatic compression, where the plasma is rapidly moved into a region of increased magnetic field.

Conclusion

Whilst impressive progress towards nuclear fusion has been made, there is still some considerable way to go. One of the most important phenomena in this context is the behaviour of waves in a non-uniform magnetised plasma, in view of the need for additional heating defined above. Given the technological importance and the physical significance of propagation and mode conversion in inhomogeneous media, it is this aspect of plasma physics which this thesis explores and hopefully makes some contribution.

For this reason, the next chapter gives detailed consideration to one of the most useful solution techniques in non-uniform wave propagation problems.

:-----------------------------:

TABLE 1

Representative Medium-Sized Tokamaks

Machine	Country	R(m)	a(m)	K	B(T)	I_p(MA)
T-3	USSR	1.0	0.17	1.0	2.5	0.1
PLT	USA	1.3	0.4	1.0	3.5	0.6
T-10	USSR	1.5	0.37	1.0	3.5	0.5
ASDEX	FRG	1.6	0.4	2.0	2.6	0.5
D III	USA	1.4	0.4	1.4 - 1.8	2.6	1.0
PDX	USA	1.4	0.4	1.0	2.4	0.5
FT	Italy	0.8	0.23	1.0	8.0	0.6
Alcator C	USA	0.64	0.16	1.0	12	0.8

R = Major radius of toroid

a = Minor "horizontal" radius of plasma

K = Elongation = vertical/horizontal radius of plasma

B = Toroidal field strength at plasma centre

I_p = Plasma current

TABLE 2

Large Tokamak Parameters

Machine	Country	R(m)	a(m)	K	B(T)	I_p (MA)	Operating Gas	Discharge Duration(s)	First Operation
JET	EEC	2.96	1.25	1.6	3.5	5	H/D/D-T	10 - 20	June '83
TFTR	USA	2.55	0.85	1.0	5.2	2.5	H/D/D-T	1 - 3	Dec '82
JT-60	Japan	3.0	0.95	1.0	4.5	2.7	H	5 - 10	April '85
T-15	USSR	2.4	0.70	1.0	4.0	2.0	H	5	'86
TORE-SUPRA	France	2.4	0.70	1.0	4.5	1.7	H/D	30	'87
D IIID	USA	1.67	0.67	1 - 2	2.2	2 - 3	H	2 - 5	'86
FT-U	Italy	0.92	0.31	1.0	8.0	1.6	H/D	1.5	'87

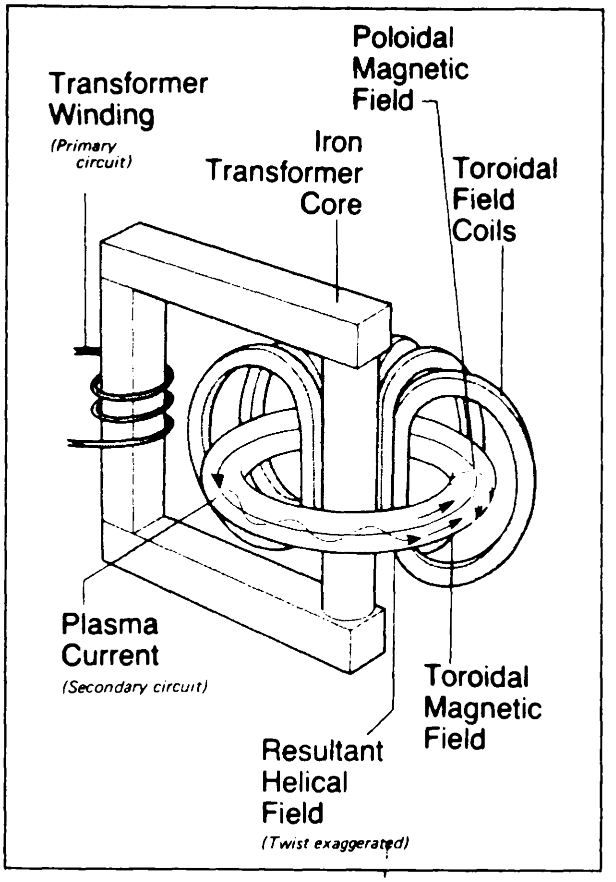


FIGURE 1.1

Schematic diagram of a tokamak.

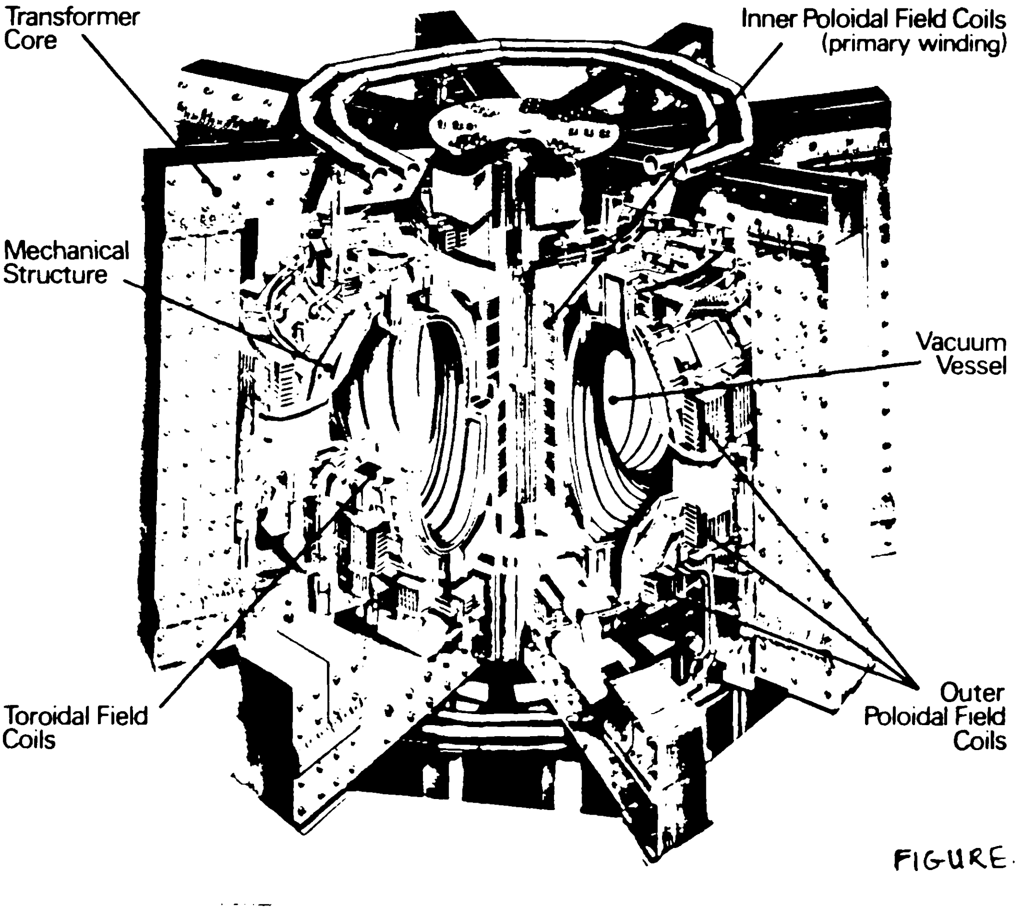
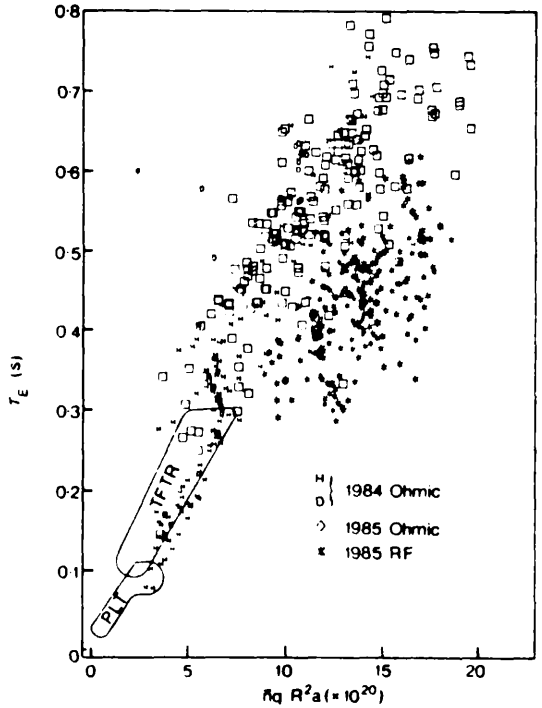
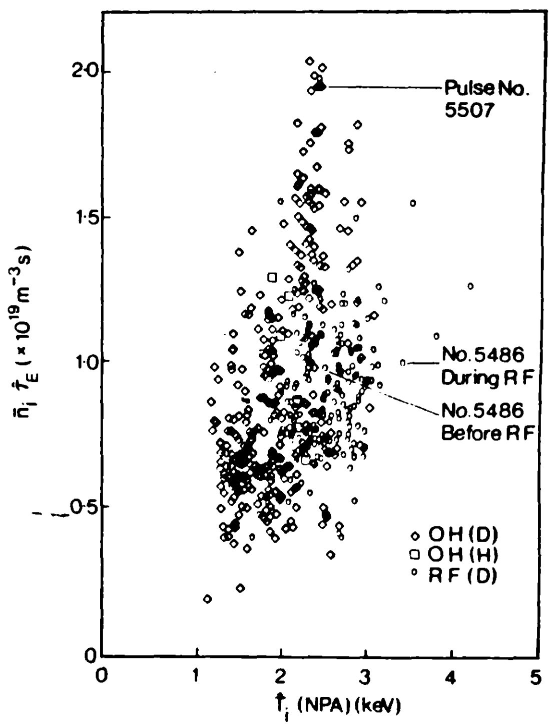


FIGURE 1.2

Diagram of JET



Global energy confinement time, τ_E, versus the scaling factor $\bar{n}qR^2a$ for both Ohmic and RF heated plasmas. The data from PLT and TFTR are shown for comparison;



Lawson diagram of $\hat{n}_i\hat{\tau}_E$ versus $\hat{T}_i$(NPA) for both hydrogen(H) and deuterium(D) plasmas during ohmic and RF Heating. Pulse No:5507 and Pulse No:5486 (both before and during RF heating) are highlighted;

FIGURE 1.3

Chapter 2 WKBJ Theory.

The technique for the asymptotic solution of certain ordinary differential equations, first proposed by Jeffrey in 1926 {7}, and later developed by Wentzel, Kramers and Brillouin, is still one of the most powerful analytic tools of its kind available. Unfortunately, the full theory is not well known: the phase integral method tends to be treated superficially and with some ambiguity in many texts (see for example reference {8}). For this reason, this chapter sets out the fundamental concepts of the theory, and illustrates how connection formulae generally arise.

WKBJ aims to solve ODEs of the type

$$\frac{d^2u}{dx^2} + h^2 q(x) u = 0, \quad h \text{ constant} \tag{2.1}$$

subject to the following conditions {9}:

(i) $q(x)$ is continuous for all x;

(ii) $h^2 q(x) \to$ constant as $h \to \infty$, with arbitrary, but fixed, x.

(iii) $|q'/q|^2, \ |q''/q| \ll 1 \quad \forall |x| > x_c$, for some x_c.

The approximate independent solutions of (2.1) for sufficiently large x take the well-known form

$$u_{1,2} \sim \left[q^{-1/4} \exp \pm ih \int^x q^{1/2} dx' \right] \left[1 + O(1/h^2) \right], \tag{2.2}$$

where

$$O(1/h^2) \sim \frac{5q'^2}{16q^2} - \frac{q''}{4q}.$$

The asymptotic solutions of (2.1) will be expressible in terms of these solutions (2.2) to an arbitrary accuracy, provided the conditions (i)-(iii) hold. It is this feature which makes WKBJ methods so attractive; its objective is to analytically continue these solutions into the complex plane, beyond their originally retricted domain of validity, in order that the asymptotic form of the solution for $x >> 0$ may be related self-consistently to that for $x << 0$. In doing so, a connection formula is established which enables the correct mixture of each asypmtotic solution of type (2.2) to be prescribed in a particular domain for a specific example (incorporating the relevant boundary conditions).

The great advantange of this technique is that the approximate solutions (2.2) can be readily evaluated, and that as a consequence of the connection formulae, the full ODE need not be solved explicitly in those regions in which the asymptotic form is not valid. However, there are certain complications involved in establishing the connection formulae, difficulties which are inherent in the very nature of the asymptotic solution procedure itself.

best estimate

One fundamental feature of an asymptotic series expansion is that it is unconditionally divergent {10}: in every case there is a finite number of terms which progressively decrease in magnitude, the remainder increasing continuously thereafter. It turns out that the best estimate of a quantity from an asymptotic expansion involves only those terms from the appropriate series which have magnitude less than unity.

stokes phenomenon

Another crucial aspect of asymptotic analysis is the phenomenon of Stokes' multipliers, and their role in the associated discontinuities in the expansion. An asymptotic series is one which tries to approximate to a function for large values of the function argument. The candidate function however possesses its own unique Taylor series representation (together with appropriate outer terms to accomodate branch points or poles). Thus the asymptotic series attempts to mimic this Taylor series, but using different coefficients and powers of the function argument. Since two different power series cannot indefinitely keep pace in magnitude and phase, the asymptotic one must have periodic corrections in order to ensure its accuracy. These jumps in value are referred to as **Stokes discontinuities,** and play a fundamental role in the evaluation of connection formulae.

However, any such behaviour must inevitably involve an error in the series. In order to minimise this inherent mismatch, any Stokes discontinuity is constrained to occur in the region of the complex plane where the relative error so introduced is minimal. These regions turn out to be lines in the complex plane, called **Stokes rays**, and the asymptotic series is then invalid on these rays, since it is ambiguously defined there.

In order to crystallise these concepts, an example involving all these effects is fully worked in the following section.

An example: the overdense potential barrier.

A fundamental example in wave propagation problems is that of waves penetrating a finite barrier, where there is a region of solution space in which all waves are evanescent in nature. This happens when the local wavenumber (or equivalently, the refractive index) passes through zero, remaining negative for a finite region of space before returning to positive values.

A specific example of this type of behaviour is the following ODE,

$$\frac{d^2u}{dx^2} + h^2(x^2-a^2)u = 0 \tag{2.3}$$

where, in the previous notation, $q(x)$ has zeros at $\pm a$. The function $q(x)$ is sometimes referred to as the wave potential, and its zeros are termed **transition points.**

Note that in this particular example, the transition points are of order unity, that is $q(x)$ goes linearly through zero as $x \to \pm a$. This simplifies the analysis considerably, since the order of the transition points has a significant effect on the nature of the barrier and any associated calculations. Higher order transition points can involve considerably more complicated analysis {11,12}.

We wish to use only the WKBJ solutions

$$u_{1,2} \sim q^{-1/4} \exp ih \int^{x} q^{1/2} dx' \tag{2.4}$$

but these are invalid at and near the transition points, since they are singular there, together with the error terms. As indicated earlier, we proceed by analytically continuing these expressions into the complex plane, and avoiding those regions where the approximations break down, viz the real axis close to the transition points. In order to do this, we must first find the Stokes rays and associated multipliers. In the interest of clarity, we adopt Heading's notation {9}, and define some rules necessary for progress.

<u>notation</u>

(i) $(\pm a, z) \overset{\partial}{=} q^{-1/4} \exp ih \int_{\pm a}^{z} q^{1/2} ds$.

(ii) $\{a, b\} \overset{\partial}{=} \exp ih \int_{a}^{b} q^{1/2} ds$.

(iii) A dominant solution is a WKBJ solution which possesses a positive real part in its exponential; the opposite behaviour is termed **subdominant.** Dominancy and subdominancy are designated by the respective suffices d and s.

Note however that, given the existence of transition points, a particular solution is dominant only in certain sectors of the complex plane, being subdominant in others. These sectors are delineated by lines of constant phase emanating from each transition point and defined by

$$\mathcal{I}m \left\{ \int_{z_0}^{z} q^{1/2} ds \right\} = 0.$$

Such lines are termed Anti-Stokes lines (ASL) or rays; along them, neither solution dominates. The complementary behaviour occurs along the Stokes lines (SL) or rays, defined by

$$\mathcal{R}e \left\{ \int_{z_0}^{z} q^{1/2} ds \right\} = 0 .$$

Here solutions achieve maximum dominancy; for this reason, the Stokes discontinuities are introduced on these rays in order to minimise the consequent relative error.

<u>rules</u>

Clearly a recipe is required for crossing Stokes lines in order that the Stokes discontinuity may be correctly evaluated and assigned. The required formula is summarised in the next two rules.

(iv) If an Anti-Stokes line is crossed in the complex plane, the dominancy of a solution is changed to sub-dominancy, and vice versa.

(v) If a Stokes line is crossed, a subdominant term changes its coefficient to the sum of its original plus S times that of the dominant, where S is the associated Stokes multiplier (to be evaluated).

A further 2 rules are needed before the barrier problem can be solved.

(vi) If the branch cut emanating from the transition point of order n is crossed, then the solutions are matched across it accordingly:

$$(a,z) \longmapsto i(-1)^{(n+1)/2}\,(z,a) \qquad n \text{ odd}$$

$$(a,z) \longmapsto (-1)^{n/2}\,(a,z) \qquad n \text{ even}$$

(vii) Heading's rule: if waves are taken to be of the form exp (-iwt), then along an Anti-Stokes line on the real axis, (x,a) represents a wave to the right if (z,a) is subdominant below the real axis (and vice versa).

We are now ready to solve the full problem by WKBJ. First of all, we must find the location of the Stokes and Anti-Stokes lines.

Consider a transition point of order unity. Then it possesses three ASLs, since

$$\mathcal{I}m\left\{\int q^{1/2}\,ds\right\} = \mathcal{I}m\,(z^{3/2}) = 0$$

$$\Rightarrow \arg(z) = 2n\pi/3 \ , \ n=0,1,2, \qquad (2.5)$$

and therefore also has three SLs.

The geometry of these lines in the complex plane is as given in figure 2.1.

In the case of the overdense barrier, which possesses two such points, the geometry must be similar in the neighbourhood of any one. Moreover, the number of such rays is n + 2, where n is the order of the transition point. At very large distances from the transition points the separation between them becomes negligible, and the barrier problem becomes one having a single transition point of order 2. We therefore expect only 4 Stokes rays to be present asymptotically {11}.

The arrangement of the rays for this problem is then as portrayed in figure 2.2. Note that ASLs may not cross, but can merge asymptotically. Also note that the ASLs align along the real axis for $|x|>a$, since there we expext oscillatory, not evanescent, solutions. The branch cuts, represented by wavy lines, may be inserted as shown without loss of generality.

To solve the problem, we impose only a transmitted wave on the far right, so that for $x >> a$, the solution is

$$u \sim (x,a)$$

Now trace this solution back through the complex plane, keeping $|z|>>a$, in order to discover what form the asymptotic solution must take for $x << -a$, and so establish a connection formula.

Using the notation of figure 2.2, and proceeding sector by sector following the rules defined earlier, we can write down the appropriate asymptotic solutions:

1. $(z,a)_s$ (by Heading's rule)

2. $(z,a)_s$ (since no dominant term is present, the subdominant coefficient is unchanged)

3. $(z,a)_d = \{-a,a\}(z,-a)_d$

4. $\{-a,a\}(z,-a)_d - S\{-a,a\}(-a,z)_s$ (S is the Stokes multiplier for an anti-clockwise crossing)

5. $i\{-a,a\}(-a,z)_d - iS\{-a,a\}(z,-a)_s$

Thus the solution for x << -a is

$$u \sim i\{-a,a\}(-a,x) - iS\{-a,a\}(x,-a)$$

where the second term represents the incident wave, since it is a solution travelling to the right. Consequently the reflection and transmission coefficients R and T may be expressed in the form

$$|R|^2 = \frac{1}{|S|^2}, \qquad |T|^2 = \frac{\{a,-a\}^2}{|S|^2} \tag{2.6}$$

It remains only to evaluate S. To do this, we note that $\forall x \in \mathbb{R},\ q(x) \in \mathbb{R}$. Returning to the original equation,

$$u'' + h^2 q(x)\, u = 0 \tag{2.3}$$

we may take the complex conjugate:

$$u^{*''} + h^2 q(x)\, u^* = 0 \tag{2.7}$$

Multiplying (2.3) by u^*, (2.7) by u and subtracting yields

$$u''u^* - u^{*''}u = 0$$

or
$$\frac{d}{dx}(u^*u' - u^{*'}u) = 0 \tag{2.8}$$

ie Im (u^*u') = constant, on the real axis. This extra relation, amounting to conservation of energy in a loss-free medium, allows S to be determined self consistently.

Thus for x >> a, we may write

$$u'u^* = (x,a)'(x,a)^* \approx -ihq^{1/2}(x,a)(a,x) = -ih, \tag{2.9}$$

where the term arising from differentiating $q^{1/2}$ has been ignored compared to that from the exponential; in examples where q $\rightarrow$ constant as $x \rightarrow \infty$, this approximation is excellent.

In the same way, for x << -a,

$$\begin{aligned} u'u^* &= \{-a,a\}^2\left[(-a,x) - S(x,-a)\right]'\left[(-a,x)^* - S^*(x,-a)^*\right] \\ &= ihq^{1/2}\{-a,a\}^2\left[(-a,x) + S(x,-a)\right]\left[(x,-a) - S^*(-a,x)\right] \\ &= ih\,\{-a,a\}^2\left[1 - |S|^2 + (S(x,-a)^2 - S^*(-a,x)^2\right] \end{aligned} \tag{2.10}$$

Now since

$$(x,-a)^2 = q^{-1/2}\exp -2ih\int_{-a}^{z} q^{1/2}ds = q^{-1/2}\exp 2ih\int_{z}^{-a} q^{1/2}ds = (-a,x)^{2*},$$

then

$$\mathrm{Re}\left[S(x,-a)^2 - S^*(a,x)^2\right] = 0.$$

Consequently, $\mathrm{Im}(u^*u')$ = constant implies

$$-h = h\{-a,a\}^2(1-|S|^2),$$

i.e. $$|S|^2 = 1 + \{a,-a\}^2, \tag{2.11}$$

and so the reflection and transmission coefficients take the form

$$|R|^2 = \frac{1}{1+\{a,-a\}^2}, \tag{2.12}$$

$$|T|^2 = \frac{\{a,-a\}^2}{1+\{a,-a\}^2}. \tag{2.13}$$

Summary

In this chapter, a technique of asymptotic analysis has been presented in considerable detail in order that the procedure by which connection formulae are constructed can be explicitly derived. Since asymptotic analysis is an important feature in mode conversion theory, this presentation of WKBJ analysis serves to illuminate the nature of the calculations involved.

In the next chapter, a particular example of a non-uniform wave propagation problem is solved using precisely the theory detailed in this chapter, so that the accuracy of WKBJ may be tested in a 'real' situation.

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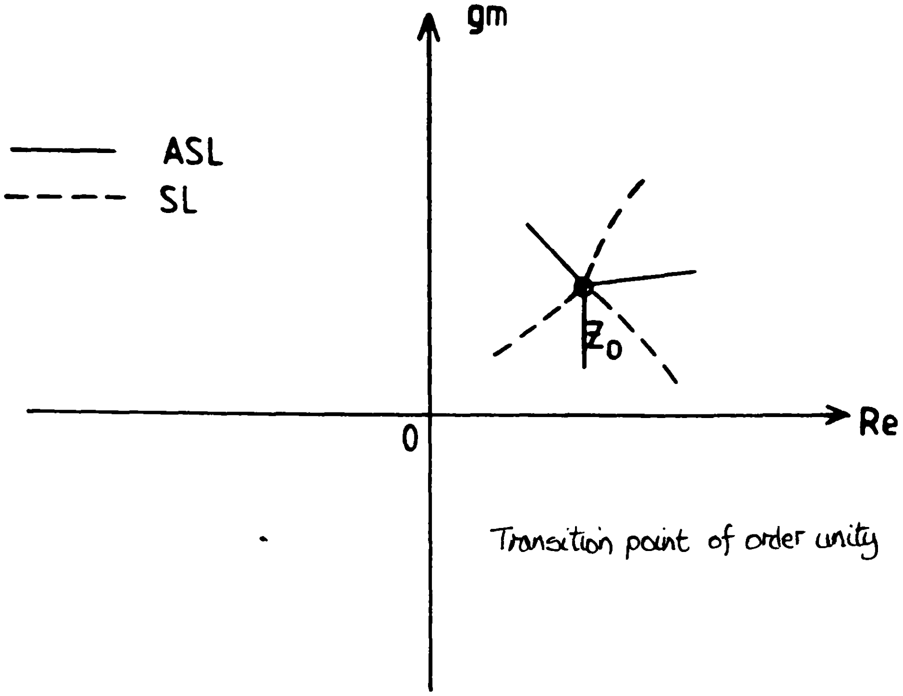


FIGURE 2·1

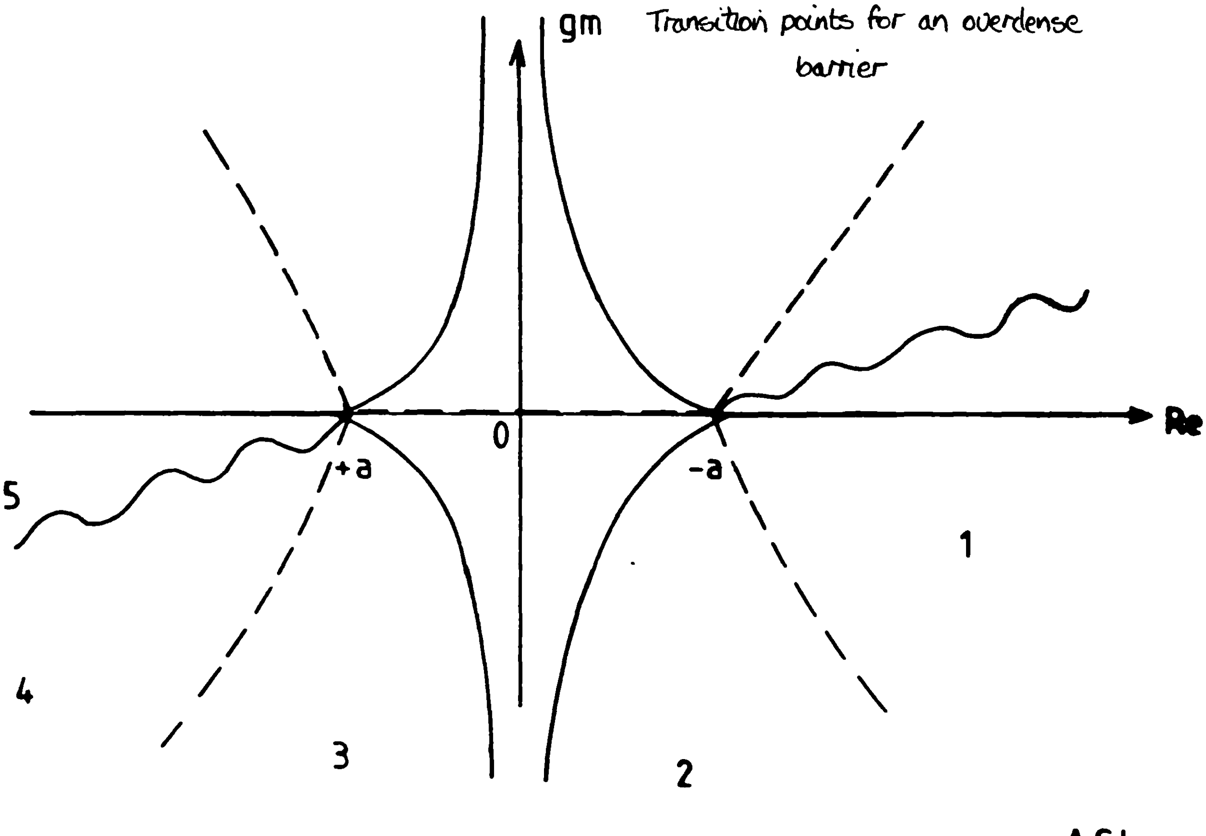


FIGURE 2·2

Chapter III Non-Uniform Waveguide

One of the simplest problems in which to study non-uniform wave propagation, with its consequent mode conversion in the form of reflected waves, is the waveguide with varying cross-section.

In this chapter, such a system will be solved using WKBJ methods to discover the transmission properties of a guide with asymptotically uniform cross section. This problem is relatively simple because only one mode of vibration is involved, the characteristics of which vary by virtue of the effect of the non-constant boundary on the transmitting medium.

Non-uniform guides have been widely studied in the past. Most analysis concentrates on piece-wise constant variations in width {13} or on slowly changing cross-sections {14}. More recently, technological considerations have stimulated interest in sinusoidally varying waveguides as possible mode-convertors {16-18}. However, in this chapter, we shall concentrate on a straightforward illustrative example.

acoustic guide

Consider a simple acoustic guide, and take the wave equation with the longitudinal displacement ξ of the gas as the scalar independent variable.

Consider then the relevant differential equation {18}:

$$\frac{d}{dx}\left[\frac{1}{A}\frac{d}{dx}\left(A\xi\right)\right] + k_0^2\xi = 0 \,,$$
$$k_0 = \omega/c \,, \tag{3.1}$$

where A is the cross-sectional area.

Now consider the case where the width varies with axial position, ie,

$$A = A(x) \,.$$

The full ODE then is

$$\xi'' + \rho\,\xi' + \left[k_0^2 + \rho'\right]\xi = 0 \,, \quad \rho = A'/A \,. \tag{3.2}$$

The first derivative term may be eliminated by changing the dependent variable to u, where

$$u = \xi \exp\tfrac{1}{2}\int^{x} \rho\, ds \,,$$

so that the resultant ODE for u is

$$u'' + \left[k_0^2 + \tfrac{1}{2}\rho' - \tfrac{1}{4}\rho^2\right]u = 0 \,. \tag{3.3}$$

In order that we might apply the WKBJ analysis of the previous chapter, we choose a simple cross sectional behaviour which involves a necking of the guide at the origin, returning to uniform cross section asymptotically. To this end, let

$$A(x) = A_0\left[1 - \theta e^{-|x|/L}\right] \,,$$

where L is a typical length over which the guide is non-uniform, and θ is the maximum depth of the constriction ($\theta < 1$). Figure 3.1 shows a typical example of such a guide. Then

$$A'(x) = \frac{\theta}{L} A_0 e^{-|x|/L} = (A_0 - A)/L,$$

so that

$$\rho = \frac{A'}{A} = \frac{1}{L}\left[\frac{A_0}{A} - 1\right],$$

and

$$\rho' = -\frac{1}{L}\frac{A_0}{A^2}A' = -\frac{1}{L}\frac{A_0}{A}\rho.$$

Thus

$$\frac{1}{2}\rho' - \frac{1}{4}\rho^2 = -\frac{\rho}{4L}\left[\frac{2A_0}{A} + L\rho\right] = -\frac{1}{4L^2}\left(\frac{A_0}{A} - 1\right)\left(\frac{3A_0}{A} - 1\right)$$

$$= -\frac{1}{4A^2L^2}\left(3A_0^2 - 4A_0A + A^2\right). \qquad (3.4)$$

So we may write (3.3) in the form

$$u'' + \Psi(x)\,u = 0, \qquad (3.5)$$

where the wave potential Ψ is defined by

$$\Psi = k_0^2 + \frac{1}{2}\rho' - \frac{1}{4}\rho^2,$$

which can be written, using (3.4), as

$$\Psi = \frac{1}{4L^2}\left[K_0^2 - \frac{3A_0^2 - 4A_0A + A^2}{A^2}\right], \quad K_0 = 2k_0L. \qquad (3.6)$$

Note that asymptotically, $A(x) \to A_0$, so that the guide attains uniform cross-section. Moreover, $u \to \xi$ and

$$\Psi \approx \frac{K_0^2}{4L^2} = k_0^2.$$

Thus for large $|x|$, we expect waves of the form

$$\xi \sim e^{\pm i k_0 x}.$$

asymptotic solution

The independent approximate WKBJ solution, valid far from the transition point, is

$$\psi^{-1/4} \exp \pm i \int^{x} \psi^{1/2} dx' . \tag{3.7}$$

Thus we are concerned with evaluating the integral

$$I = \int^{x} \psi^{1/2} dx' . \tag{3.8}$$

A change of independent variable ensures that this integral can be solved in closed form. Considering x>0 without loss of generality, let

$$y = e^{x/L} ,$$

so that

$$A(y) = A_0 (y-\theta)/y .$$

Then

$$\psi(y) = \frac{1}{4L^2}\left[\kappa^2 - \left(3y^2 - 4y(y-\theta) + (y-\theta)^2\right)/(y-\theta)^2\right]$$

$$= \frac{R}{4L^2(y-\theta)^2} ,$$

where R is the quadratic

$$R = \kappa^2 y^2 - 2\theta(1+\kappa^2)y + (\kappa^2-1)\theta^2 .$$

Then

$$I = \int \frac{\sqrt{R}}{2L(y-\theta)} \cdot \frac{L\,dy}{y} ,$$

$$\text{ie} \qquad I = \frac{1}{2}\int \frac{\sqrt{R}}{y(y-\theta)}\,dy \,. \tag{3.9}$$

Under this transformation, I becomes a standard integral {19}. In the interest of clarity, we detail the steps in the evaluation of I in the comprehensive Appendix A, rather than in the main text. Instead, we merely quote the result:

$$\begin{aligned} I = {} & \tfrac{1}{2}\kappa \ln\left\{2\left(\kappa\sqrt{R} + 2\kappa y - \theta(1+\kappa^2)\right)\right\} \\ & + \tfrac{1}{2}(\kappa^2-1)^{1/2} \ln\left\{\frac{2\theta}{y}\left(\kappa^2-1-(1+\kappa^2)y + \sqrt{(\kappa^2-1)R}\right)\right\} \\ & - \frac{\sqrt{3}}{2}\sin^{-1}\left\{-\frac{1+3\theta t}{\sqrt{1+3\kappa^2}}\right\}, \quad t = \frac{1}{y-\theta} \,. \end{aligned} \tag{3.10}$$

(NB we have assumed $\kappa > 1$: this is not a limitation at this stage, but the size of κ will be discussed later, when it is important.)

Note that as $x \to \infty$, $y \to \infty$, $t \to 0$ and $R \to \kappa^2 y^2$. Then

$$2I \to \kappa \ln(4\kappa^2 y) + \text{const} = \kappa \ln y + \text{const},$$

ie

$$I \to k_0 x + \text{const} \quad , \quad \text{as } x \to \infty \,.$$

Thus the expected wave solution is recovered, viz.

$$\xi \simeq u \sim e^{\pm i k_0 x} \,.$$

transition points

It now remains to specify the coupling between the eigenstates (3.7). In order to do this, we require the connection formula for this example. This involves the nature and location of the transition points, which are the zeros of Ψ, or equivalently, the roots of R. These are given by

$$y_{1,2} = \frac{\theta}{\kappa^2}\left[1+\kappa^2 \pm \Gamma\right], \quad \Gamma = \sqrt{1+3\kappa^2}. \tag{3.11}$$

In order that we can use the analysis of Chapter 2, we wish to encounter only 1 root of R for y > 1 (ie x > 0). This ensures that there are precisely two transition points in the range $x \in]-\infty, \infty[$. Clearly then we require

$$\frac{\theta}{\kappa^2}(1+\kappa^2+\Gamma) > 1 , \tag{3.12}$$

$$\frac{\theta}{\kappa^2}(1+\kappa^2-\Gamma) < 1 . \tag{3.13}$$

These equations impose a range of theta values, namely

$$\frac{\kappa^2}{1+\kappa^2+\Gamma} < \theta < \frac{\kappa^2}{1+\kappa^2-\Gamma} .$$

Now, since

$$\Gamma = \sqrt{1+3\kappa^2} > 1,$$

$$\kappa^2/[1+\kappa^2-\Gamma] > 1,$$

then (3.12) and (3.13) are satisfied simultaneously if, given κ,

$$\theta \in]\theta_{min}, 1[, \tag{3.14}$$

where

$$\theta_{min} = \kappa^2/[1+\kappa^2+\Gamma] .$$

Note that a restriction on the smallest value of the constriction is to be expected, since the analysis demands that the constriction be overdense for <u>all</u> K of interest.

Hence the guide has transition points $\pm a$ defined by

$$|x| = a = L \ln y_1 ,$$

$$y_1 = \theta / \theta_{min} . \tag{3.15}$$

From the previous working in Chapter II, we must calculate the integral

$$\mathcal{J} = \frac{1}{2L} \int (-\psi)^{1/2} dx . \tag{3.16}$$

Making the same change of variable as before, this reduces to

$$\mathcal{J} = \frac{1}{2} \int \frac{\sqrt{-R}}{y(y-\theta)} dy . \tag{3.17}$$

Whilst the procedure is similar to that adopted for the solution of I, $\mathcal{J}$ has solutions which are different because of the reversal in sign of the coefficients of R. Because of this, it is necessary to state whether $\kappa_0 > 1$. Now

$$\kappa_0 > 1 \Leftrightarrow 2k_0 L > 1 \Leftrightarrow k_0 > \frac{1}{2L} ,$$

ie, the asymptotic wavenumber is greater than half the reciprocal of the scalelength describing cross sectional variation. In fact, this may be too severe: at distances where the asymptotic wavelength is meaningful, the guide is almost uniform. There seems to be no prima facie case for assuming $\kappa > 1$; therefore the calculation will proceed for both cases of $\kappa > 1$ and $\kappa < 1$.

Again, the lengthy details involved in the explicit calculation of $\mathcal{J}$ are laid out in the appendix. Instead, we merely quote the result, which is

$$\mathcal{J} = \kappa \sin^{-1}\left[\frac{\kappa^2}{\Gamma}\left(\frac{1}{\theta_{min}} - \frac{y}{\theta}\right) - 1\right]$$

$$- \sqrt{3}\, \ln\left[2\sqrt{3\theta^2 \mathcal{D}} + 6\theta^2 t + 2\theta\right] \tag{3.18}$$

$$+ \begin{cases} (1-\kappa^2)^{1/2} \ln\left[\frac{2\theta}{y}\left((1+\kappa^2)y + \sqrt{(1-\kappa^2)\mathcal{R}} + (1-\kappa^2)\theta\right]\right. , & \kappa < 1 \\ (\kappa^2-1)^{1/2} \sin^{-1}\left[1 - \frac{\kappa^2-1}{\Gamma}\left(\frac{\theta}{y} - \theta_{min}\right)\right] , & \kappa > 1 \end{cases}$$

where

$$\mathcal{R} = -R,$$

$$\mathcal{D} = -\kappa^2 + 2\theta t + 3\theta^2 t^2 .$$

However, the evaluation of the indefinite integral is only the first stage: noting the formula (2.12) of chapter II, the desired term is

$$\{a, -a\} = \exp \int_a^{-a} (-\psi)^{1/2} ds$$

$$= \exp\left[\mathcal{J}(x=-a) - \mathcal{J}(x=a)\right]$$

$$= \exp 2\left[\mathcal{J}(x=0) - \mathcal{J}(x=a)\right],$$

where we have used the symmetry of the guide and the integral about the origin. The integral is then evaluated between the limits given by the half-width of the barrier, that is at the points

$$y = 1,$$
$$y = \theta/\theta_{min} .$$

On substitution of these values into (3.18), the result is

$$2\left[\mathcal{J}(x=0) - \mathcal{J}(x=a)\right]$$
$$= \kappa\left\{ \sin^{-1}\left[\frac{\kappa^2}{\Gamma}\left(\frac{1}{\theta_{min}} - \frac{1}{\theta}\right) - 1\right] + \frac{\pi}{2}\right\}$$
$$- \sqrt{3}\,\ln\left\{\frac{1-\theta_{min}}{1-\theta}\left[\frac{\sqrt{3(\theta^2+2\theta-\kappa^2(1-\theta)^2)}+2\theta+1}{2\theta_{min}+1}\right]\right\} \qquad (3.19)$$

$$\begin{cases} \kappa<1: & +(1-\kappa^2)^{1/2}\ln\left\{\left[(1-\kappa^2)\theta+1+\kappa^2+(1-\kappa^2)^{1/2}\sqrt{2\theta-((\kappa+1)\theta-\kappa)((\kappa-1)\theta-\kappa)}\right]/\Gamma\right\} \\ \kappa>1: & +(\kappa^2-1)^{1/2}\left\{\sin^{-1}\left(1-\frac{\kappa^2-1}{\Gamma}(\theta-\theta_{min})\right)-\frac{\pi}{2}\right\} \end{cases},$$

and so the term {a,-a} is given by

$$\kappa<1: \quad \{a,-a\} = A^{\sqrt{3}}\,B^{\sqrt{1-\kappa^2}}\,e^{\lambda}, \qquad (3.20)$$

$$\kappa>1: \quad \{a,-a\} = A^{\sqrt{3}}\,e^{\lambda+\mu}, \qquad (3.21)$$

where

$$A = \frac{1-\theta}{1-\theta_{min}}\left[\frac{2\theta_{min}+1}{\sqrt{3(\theta^2+2\theta-\kappa^2(1-\theta)^2)}+2\theta+1}\right],$$

$$B = \left[(1-\kappa^2)\theta+1+\kappa^2+(1-\kappa^2)^{1/2}\sqrt{2\theta-[(\kappa+1)\theta-\kappa][(\kappa-1)\theta-\kappa]}\right]/\Gamma,$$

$$\lambda = \kappa \left\{ \sin^{-1}\left[\frac{\kappa^2}{\Gamma}\left(\frac{1}{\theta_{min}} - \frac{1}{\theta} \right) - 1 \right] + \frac{\pi}{2} \right\} ,$$

$$\mu = (\kappa^2-1)^{1/2} \left\{ \sin^{-1}\left[1 - \frac{\kappa^2-1}{\Gamma}(\theta - \theta_{min}) \right] - \frac{\pi}{2} \right\} .$$

The reflection and transmission coefficients R and T follow using (2.12) and (2.13):

$$R^2 = 1/\left[1 + A^{2\beta} B^{2(1-\kappa^2)^{1/2}} e^{2\lambda}\right] \text{ or } 1/\left[1 + A^{2\beta} e^{2(\lambda+\mu)}\right] , \qquad (3.22)$$

$$T^2 = A^{2\beta} B^{2\sqrt{1-\kappa^2}} e^{2\lambda} . R^2 \text{ or } A^{2\beta} e^{2(\lambda+\mu)} R^2 . \qquad (3.23)$$

Discussion

Note that as $\theta \to 1$, $R \to 1$ and $T \to 0$, as expected. This shows that as the constriction becomes narrower, the transmission falls (for a given wavelength). Moreover, as $\theta \to \theta_{min}$, $R^2, T^2 \to 1/2$: this is a standard feature of all such barriers, being the result of the merger of the transition points (see equation (3.15)). In fact, this barrier problem is unusual in this respect, in that the width of the barrier depends on its depth, as well as on other parameters. This is then a feature of the reflection and transmission coefficients. The barrier width is the distance between transition points, and so is defined by

$$w = 2L \ln\left(\frac{\theta}{\theta_{min}} \right)$$

This expression contains all the key parameters, θ, κ and L. In order to compare reflection and transmission coefficients for varying depths of constriction, care must be taken to ensure a constant barrier width, so that valid comparisons may be made. The resulting analysis is best displayed graphically in figures 3.2 and 3.3.

Summary

This chapter has presented a fully worked example illustrating the value of the WKBJ technique in a real, physical problem. The transmission properties of the guide are deduced by constructing a connection formula which prescribes the correct mixture of asymptotic solutions far from the interaction region.

However, note that the approximations involved depended on knowing the _exact_ form of the wave potential in the interaction region, even though wave propagation is not solved in that area of solution space. The consequent estimation involved in the form of the asymptotic solutions themselves, and their relative mixtures via the connection formula, is only as accurate as the supplied wave potential.

In the next chapter, mode conversion proper is defined and the various current theories are briefly reviewed, with comments on their theoretical bases and underlying assumptions. The concept of a parametrised dispersion relation is introduced, and its relevance and accuracy is critically examined using two explicit examples.

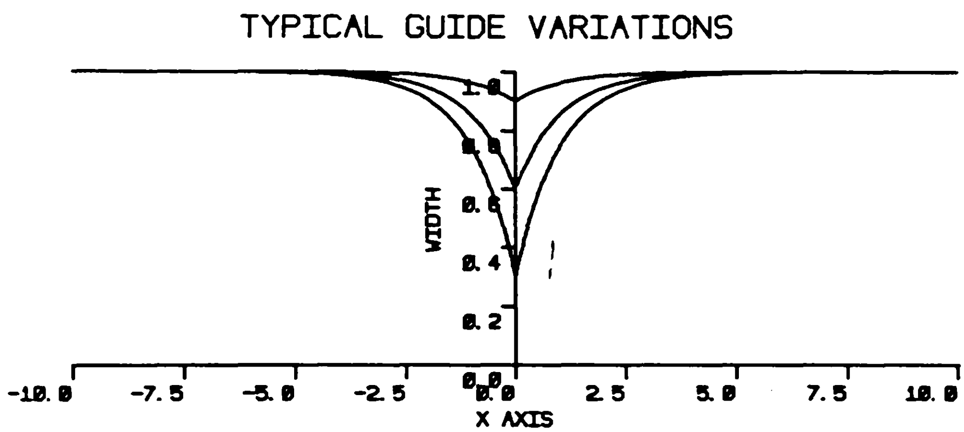


FIGURE 3.1

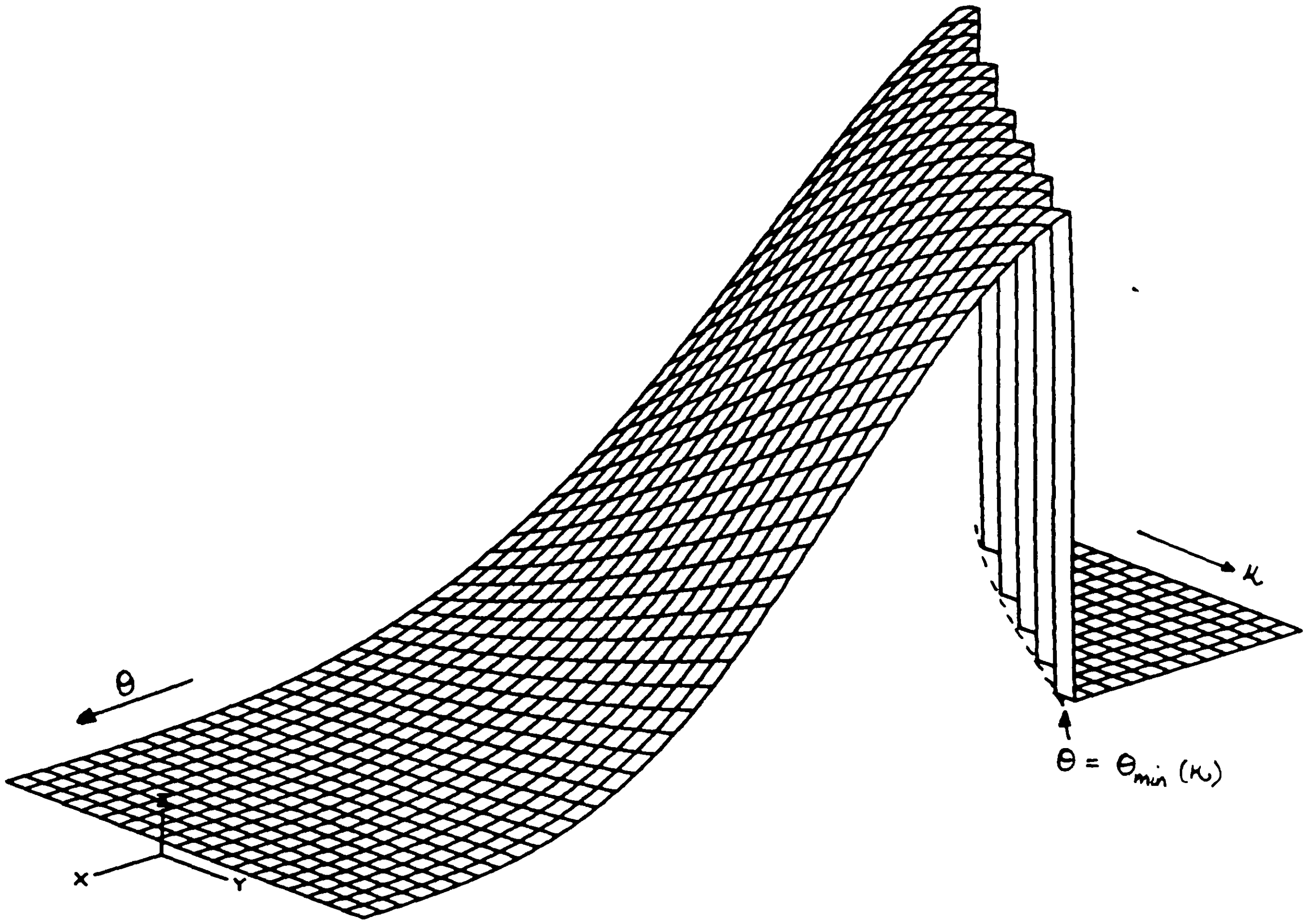


FIGURE 3.2 : $T^2 = T^2(\theta, \kappa)$

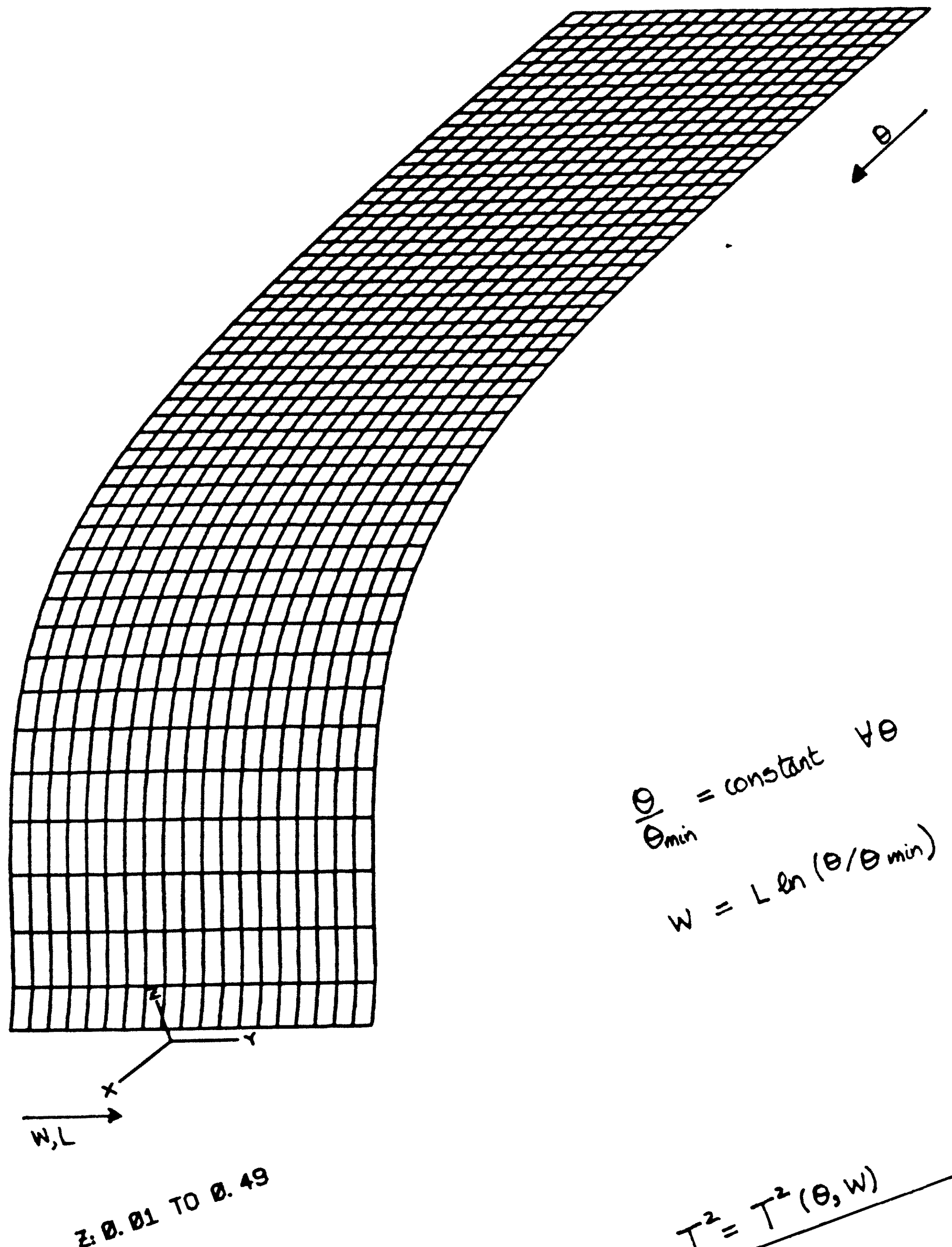


FIGURE 3.3 : $T^2 = T^2(\theta, W)$

Chapter IV Review of Mode Conversion

The study of waves in inhomogeneous media is particularly relevant to the fusion community, now that the need for additional heating schemes in fusion devices has been clearly established.

In this chapter, a review and critique of existing theoretical effort in this field is undertaken. The fundamental assumptions embodied in parametrised dispersion relations are examined critically, and their inadequacies exposed in two simple examples. In addition, each of the leading theories is outlined briefly, and their theoretical foundations commented upon.

Definition of mode conversion

In a plasma, many different oscillatory states are possible, and where the plasma is spatially non-uniform, some of these different modes may have a similar wavelength in a particular region of the solution space, due to the variation in wavelength of each mode as an inevitable consequence of the inhomogeneities. This being the case, it should be possible to propagate one such mode into an appropriate region of the non-uniform plasma, and in doing so, excite at least one other form of wave motion in the medium.

This phenomenon is termed mode conversion.

(Note that the transmission and reflection of waves through a barrier is a special case of mode conversion, so that there was already an historical background to the theory.)

dispersion relation

Early attempts to describe mode conversion quantitatively quickly found that the greatest difficulty lay in deriving suitable, appropriate differential equations to govern the process in a consistent way.

The first work in the field attempted to overcome this by using the homogeneous model's dispersion relation as a starting point.

In such a model, all the parameters p_i (for example $\underline{B}$, n etc) are constants. Thus the equations defining the model may be Fourier transformed in order to yield the algebraic quantity $D(\omega,\underline{k};p_i)$ - the dispersion relation. $D(\omega,\underline{k};p_i)=0$ defines all possible wave motions exp $i(\underline{k}.\underline{r}-\omega t)$ permitted as oscillatory solutions of the equations, by establishing a relationship between the allowed frequencies and wavelengths. For different values of the constants

p_i, the permitted waves have different, but constant, characteristic ω and $\underline{k}$ throughout the entire solution space.

In an attempt to generalise this notion to non-uniform media, the concept of a 'local' dispersion relation is created, in which the parameters p_i now vary with position in the desired locality. This new, restricted quantity $D(\omega,\underline{k};p_i(x))$ is used in an attempt to account for the local behaviour of waves whose characteristic features now vary continuously with position.

In this way, the parametric spatial dependencies are prescribed after the event, and an ODE with varying coefficients is then constructed by identifying powers of the wavenumber with a differential operator via the mapping

$$ik \longmapsto \frac{d}{dx} . \tag{4.1}$$

That this type of approach is desirable is without doubt; complicated systems of partial differential equations are at once reduced to a simple algebraic relation, which then generates an ordinary differential equation with minimal effort.

However, whilst it is possible to describe inhomogeneous media using very simple assumptions and ignoring parameter gradients, these only apply in those regions of solution space where such approximations are valid. Since these regions necessarily exclude (in general) the particular interaction areas where phenomena such as reflection or mode conversion occur by virtue of those very gradients, it would be naive to use the same techniques in such contexts and expect accurate results. In his basic text book, Stix {20} acknowledges the particular difficulties associated with this procedure (pp 240 - 241, p 258), given that the parameter gradients are inadequately treated, either by omission or by ambiguous assignment.

As elementary examples of how the mapping (4.1) may be inadequate in real systems, consider the following two problems.

waveguide revisited

We return to the case of an acoustic waveguide of varying cross-section, but this time we solve the problem using the mapping (4.1) in conjunction with a parametrised dispersion relation.

As a starting point, we require the dispersion relation for a simple uniform cross-section acoustic guide. The required expression is {18,21}

$$k_x^2 = \frac{\omega^2}{c^2} - \left(\frac{m\pi}{b_0}\right)^2 - \left(\frac{n\pi}{d_0}\right)^2 \quad , \quad m,n \in \mathbb{N}$$

where the guide has uniform cross-section $A_0 = b_0 d_0$. In a more convenient notation, this can be written as

$$k_x^2 = \alpha^2 \left[1 - \beta^2 \right], \tag{4.2}$$

where

$$\alpha^2 = k_0^2 - \left(\frac{m\pi}{b_0}\right)^2, \quad k_0 = \omega/c,$$

$$\beta = \frac{n\pi}{d_0 \alpha}. \tag{4.3}$$

At this stage, we demand a variable cross-section. In keeping with the previous treatment, let

$$d = d(x) = d_0\left[1 - \theta e^{-|x|/L}\right],$$

so that the parametrised dispersion relation now takes the form

$$k_x^2(x) = \alpha^2 \left[1 - \frac{\beta^2}{(1 - \theta e^{-|x|/L})^2} \right].$$

Note that asymptotically, we have

$$k_x^2 \longrightarrow \alpha^2 [1 - \beta^2], \quad x \to \infty,$$

and so in order that the asymptotic solutions are purely oscillatory, we require

$$\beta < 1 .$$

Now use the mapping (4.1) to construct an ODE to govern propagation down the guide:

$$\frac{d^2 u}{dx^2} + \psi(x)\, u = 0, \tag{4.4}$$

$$\psi(x) = \alpha^2 \left[1 - \frac{\beta^2}{(1 - \theta e^{-|x|/L})^2} \right].$$

Note that this is not the same as (3.5): although superficially of the same form, the consequent transmission characteristics are fundamentally different because of the altered wave potential Ψ.

In order to quantify this difference, (4.4) can be solved in a similar fashion. Following the same change of variable as before,

$$\Psi(y) = \alpha^2\left[1 - \frac{\beta^2 y^2}{(y-\theta)^2}\right] , \quad y = e^{x/L} \quad (x>0).$$

Moving straight to the reflection and transmission calculations, the required integral takes the form

$$\mathcal{J} = \alpha L \int \frac{\sqrt{\mathcal{R}}\, dy}{y(y-\theta)} \qquad (4.5)$$

where

$$\mathcal{R} = -(1-\beta^2)y^2 + 2\theta y - \theta^2 .$$

The transition points are given by the roots of R. Following the previous calculation, we choose the larger root to define the transition point on either side of the origin, and use the fact that only one root must be accessible to place a limit on the minimum constriction. Thus the transition point at x=a is defined by

$$y_1 \stackrel{?}{=} e^{a/L} \stackrel{?}{=} \frac{\theta}{1-\beta} , \qquad (4.6)$$

$$\frac{\theta}{1-\beta} > 1 \quad \Rightarrow \quad \theta \in \,]1-\beta, 1[\; .$$

Now $\mathcal{J}$ can be written as

$$\frac{1}{\alpha L}\mathcal{J} = -(1-\beta^2)\int\frac{dy}{\sqrt{R}} + \beta^2\theta\int\frac{dy}{(y-\theta)\sqrt{R}} + \theta\int\frac{dy}{y\sqrt{R}} , \qquad (4.7)$$

and so using the same standard integrals, we have

$$\frac{1}{\alpha L}\mathcal{J} = (1-\beta^2)^{1/2}\sin^{-1}A_1 + \sin^{-1}A_2 - \beta\ln A_3 , \qquad (4.8)$$

$$A_1 = \frac{(\beta^2-1)y+\theta}{\beta\theta}$$

$$A_2 = (y-\theta)/\beta y,$$

$$A_3 = 2\beta\theta\left[\sqrt{\delta} + \beta\theta t + \beta\right],$$

$$\delta = \beta^2-1 + 2\beta^2\theta t + \beta^2\theta^2 t^2, \quad t = 1/(y-\theta) .$$

Evaluating the arguments at the origin and at the transition point yields

$y = 1$:

$$A_1 = (\beta^2-1+\theta)/\beta\theta ,$$

$$A_2 = (1-\theta)/\beta ,$$

$$A_3 = 2\beta\theta\left[\sqrt{\beta^2-(1-\theta)^2}+\beta\right]/(1-\theta) .$$

$y = \theta/(1-\beta)$:

$$A_1 = -1 ,$$

$$A_2 = 1 ,$$

$$A_3 = 2\beta\theta .$$

Consequently

$$\frac{1}{\alpha L}\left[\mathcal{J}(1) - \mathcal{J}(\theta/(1-\beta))\right] = (1-\beta^2)^{1/2}\left[\sin^{-1}\left(\frac{\beta^2-1}{\beta\theta}\right) + \frac{\pi}{2}\right]$$

$$+\sin^{-1}\left[\frac{1-\theta}{\beta}\right] - \frac{\pi}{2} - \beta\ln\left[\frac{\sqrt{\beta^2-(1-\theta)^2}+\beta}{1-\theta}\right], \qquad (4.9)$$

and so

$$\{a,-a\} = D^{p} e^{\nu}, \qquad (4.10)$$

where

$$D = \frac{1-\theta}{\sqrt{\beta^2-(1-\theta)^2}+\beta},$$

$$p = 2\alpha\beta L = \frac{2n\pi}{d}L,$$

$$\nu = 2\alpha L(1-\beta^2)^{1/2}\left[\sin^{-1}\left(\frac{\beta^2-1}{\beta\theta}\right) + \frac{\pi}{2}\right]$$

$$+ 2\alpha L\left[\sin^{-1}\left(\frac{1-\theta}{\beta}\right) - \frac{\pi}{2}\right].$$

comparison of results

Note that the expected behaviour is present, in that transmission falls with increasing depth of constriction, ie $T \to 0$ as $\theta \to 1$.

However, the most significant differences are the definition of the minimum possible overdense constriction (that is, the expression for θ_{min}) and the exponent involving the scalelength L. The departure of (4.10) from the results of Chapter III can be very significant in these respects. As before, the results are best displayed in graphical form (see figure 4.1), because of the complicated nature of the formulae involved.

perpendicular propagation in a cold plasma

Consider the propagation of electromagnetic waves orthogonal to the magnetic field in a cold, inhomogeneous plasma. Let $\underline{B} = \hat{z}B(x)$. Then the starting equation is the usual one {20,22},

$$\nabla\times\nabla\times\underline{E} - \frac{\omega^2}{c^2}K\cdot\underline{E} = 0, \qquad (4.11)$$

where K is the standard cold plasma dielectric tensor, defined by

$$K = \begin{bmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{bmatrix},$$

$$S = \tfrac{1}{2}(R+L), \quad D = \tfrac{1}{2}(R-L), \qquad (4.12)$$

$$R = 1 - \sum_j \frac{\omega_{pj}^2}{\omega^2}\left(\frac{\omega}{\omega+\epsilon_j\Omega_j}\right), \qquad \omega_{pj}^2 = \frac{B^2}{\mu_0\rho_j}$$

$$L = 1 - \sum_j \frac{\omega_{pj}^2}{\omega^2}\left(\frac{\omega}{\omega-\epsilon_j\Omega_j}\right), \qquad \Omega_j = \frac{q_jB}{m_j}$$

$$P = 1 - \sum_j \frac{\omega_{pj}^2}{\omega^2}. \qquad \text{for species } j.$$

Note that in constructing K, no assumption is made as to the uniformity or otherwise of the plasma. Thus K may have spatially dependent elements. Setting $k_0 = \omega/c$ and choosing $\nabla = (\partial_x, ik_y, 0)$, the components of (4.11) are

$$\hat{\underline{x}}: \ \partial_x(\partial_x E_x + ik_y E_y) - (\partial_x^2 - k_y^2)E_x - k_0^2(SE_x - iDE_y) = 0,$$

ie. $$ik_y E_y' + ik_0^2 D E_y + (k_y^2 - k_0^2 S)E_x = 0\,; \qquad (4.13)$$

$\underline{\hat{y}}$: $$ik_y(\partial_x E_x + ik_y E_y) - (\partial_x^2 - k_y^2)E_y - k_0^2(iDE_x + SE_y) = 0,$$

ie. $$-E_y'' - k_0^2 S E_y + ik_y E_x' - ik_0^2 D E_x = 0\,. \qquad (4.14)$$

From (4.13),

$$E_x = i\,(k_y E_y' + k_0^2 D E_y)/\Delta\,, \qquad (4.15)$$

where

$$\Delta \hat{=} k_0^2 S - k_y^2\,.$$

Consequently

$$E_x' = \frac{i}{\Delta^2}\left\{\Delta k_y E_y'' + (k_0^2 D\Delta - k_y\Delta')E_y' + k_0^2(D'\Delta - D\Delta')E_y\right\}, \qquad (4.16)$$

and this may be substituted into (4.14) to yield an equation for E_y alone:

$$-E_y'' - k_0^2 S E_y + ik_y\cdot\frac{i}{\Delta^2}\left[k_y\Delta E_y'' + (k_0^2 D\Delta - k_y\Delta')E_y' + k_0^2(D'\Delta - \Delta' D)E_y\right]$$

$$- ik_0^2\cdot D\cdot\frac{i}{\Delta}(k_y E_y' + k_0^2 D E_y) = 0,$$

that is,

$$\left(1+\frac{k_y^2}{\Delta}\right)E_y'' - \frac{k_y^2\Delta'}{\Delta^2}E_y' + k_0^2\left[S + \frac{k_0^2 k_y}{\Delta^2}(D'\Delta - D\Delta') - \frac{k_0^2 D^2}{\Delta}\right]E_y = 0. \qquad (4.17)$$

Multiplying through by Δ, and noting that $\Delta + k_y^2 = k_0^2 S$, we have

$$k_0^2 S E_y'' - k_y^2\frac{\Delta'}{\Delta}E_y' + k_0^2\left[S\Delta - k_0^2 D^2 + k_0^2 k_y\left(D' - D\frac{\Delta'}{\Delta}\right)\right]E_y = 0. \qquad (4.18)$$

Now $$S^2 - D^2 = RL$$

Thus (4.18) may be written

$$k_0^2 S E_y'' - k_y^2 \frac{\Delta'}{\Delta} E_y' + k_0^2 \left[k_0^2 RL - k_y^2 S + k_y \left(D' - D\frac{\Delta'}{\Delta} \right) \right] E_y = 0. \tag{4.19}$$

Note that in the inhomogeneous plasma, (4.19) recovers the correct dispersion relation{20}:

$$- k_x^2 k_0^2 S + k_0^2 \left[k_0^2 RL - k_y^2 S \right] = 0,$$

$$\text{i.e.} \quad (k_x^2 + k_y^2) k_0^2 S = k_0^4 RL,$$

or, writing $k_\perp^2 = k_x^2 + k_y^2$,

$$k_\perp^2 = k_0^2 \frac{RL}{S}. \tag{4.20}$$

However the technique of associating $d/dx \leftrightarrow ik_x$ to arrive at the equation

$$\left[\frac{d^2}{dx^2} + K^2 \frac{R(x)L(x)}{S(x)} \right] E_y = 0 \tag{4.21}$$

holds only for the special case of perpendicular propagation with zero wavevector in the y-direction. For general orthogonal propagation, (4.19) is the correct equation, not (4.21) as implied by Swanson.{23} Note also that while (4.21) is the correct equation for a special case, it nevertheless describes the electric field in the y direction, which is orthogonal to the direction of inhomogeneity. The expression for E_x, when substituted for E_y, contains parameter gradients regardless of wavevector components.

more advanced approaches

The examples above illustrate the inadequacies inherent in the simple mapping (4.1) when constructing ODEs to describe complicated wave phenomena in non-uniform media. In order to address these problems directly, more sophisticated techniques of constructing the equations were recently developed. In the following sections, the leading theories are briefly reviewed, and examined in the light of experiences with the dispersion relation.

FKB - a dispersion relation approach

The theory of Fuchs, Ko and Bers, (24-26) places its main emphasis on the dispersion relation derived from the homogeneous model, taking a lead from the very early attempts.

Again, the homogeneous dispersion relation is made position dependent by direct substitution of the required spatial variations of the parameters. Therefore, the starting point is the expression $D(\omega,\underline{k};p_i(z))$.

However, this theory attempts to avoid the pitfalls of the simple mapping (4.1) by trying to assign parameter gradients in a consistent manner.

Since in the context of fusion plasma heating, the frequency ω is a constant determined by the driving source (an rf antenna), the local dispersion relation is here written as D(k,z), the implication being that

$$\mathcal{D}(k,z) = 0 \tag{4.22}$$

yields solutions for waves having spatially dependent wavelength.

With this information, solutions $k_i(z)$ given by (4.22) are examined to reveal those in particular which are almost equal over some finite region, that is,

$$k_i(z) \simeq k_j(z) \;, \quad z \in \mathcal{R} \;, \quad j \neq i$$

where $\mathcal{R}$ is a finite locus lying within the region of solution space over which the approximating dispersion relation D(k,z) is deemed to be valid.

Mode conversion in the FKB theory is defined to be the redistribution of energy flow amongst the possible branches $k_i(z)$ of (4.22).

Restricting attention to pairwise events (in order that only a two-wave interaction need be considered) means extracting an appropriate embedded dispersion relation

$$\hat{\mathcal{D}}(k,z)$$

from the full D(k,z), the former restricting attention to only those two branches for which mode conversion is deemed desirable. The FKB theory then prescribes a recipe for constructing a suitable ODE with which to mathematically describe the resulting redistribution of energy between the branches.

recipe

In order that $\hat{D}(k,z)$ actually describes a mode-conversion event, FKB proposes that the following equations describing the locations z_s of saddle points of $\hat{D}(k,z)$ and the consequent critical wavenumbers k_c be satisfied simultaneously:

$$\left.\begin{array}{l}\text{(i) } \hat{D}(k_c, z_s) = 0 \\ \\ \text{(ii) } \hat{D}_k(k_c, z_s) = 0\end{array}\right\} \quad \text{subject to } \hat{D}_{kk}(k_c, z_s) \neq 0 . \qquad (4.23)$$

In addition, the theory requires

$$\partial \hat{D}(k_c, z_s)/\partial z \neq 0 . \qquad (4.24)$$

The crucial ODE is then given to be

$$\frac{d^2 y}{dz^2} + Q(z)y = 0 , \qquad (4.25)$$

where

$$Q(z) = -\frac{2\hat{D}(k_c(z), z)}{\hat{D}_{kk}(k_c(z), z)} , \qquad (4.26)$$

and where y is postulated to be the power flow in the branches (modes).

This summary states the essentials of the FKB theory; much greater detail concerning the motivation behind the above concepts is given in the relevant publications, but the equations (4.23) and (4.25) are its fundamental basis.

comment

The most fundamental objection to any theory of this type must be the importance attached to the function $D(\omega, k; p_i(z)) = D(k,z)$. Since the dispersion relation exists precisely because the parameters are assumed constant throughout the domain of interest, it is a contradiction in terms to postulate the existence of a dispersion relation incorporating spatially dependent parameters. Even if this proposed function were considered to be a good approximation, the recipe for constructing the differential equation fails to account properly for the parameter gradients which must be present, since any such gradients by the definition of the Fourier transform have been omitted, and cannot be unambiguously restored after the event (in the general case).

It is no argument to state that the gradients are small in this limit, and so are negligible: the crucial role they play will be explicitly shown in the next chapter.

Another possible flaw in this kind of theory is the parametrisation by only one independent variable; such dependencies may in fact be forbidden by the original model equations, but since this information is lost in the dispersion relation, the omission of curvature in the construction of an ODE may hide a serious inconsistency. (Chapter VI gives an explicit example of such a mistake arising in MHD publications.)

Finally, in common with most of these theories, the power flow ODE will be solved asymptotically, calling into question the validity of the resulting asymptotic solutions, given the context of a local model and a restricted dispersion relation. No attempt is made to clarify the inherent relative error of the assumed approximations.

CLD - a coupled equation theory

The theoretical approach developed by Cairns and Lashmore-Davies {27-30} builds on the basic premise that if two modes are to be involved in a mode-conversion event, then there must exist distinct differential equations which separately govern the independent propagation of each, except in a certain region of solution space where the equations are coupled. This local coupling provides the mechanism by which power can flow from one mode to the other in the mode conversion region.

To this end, the dispersion relation for the homogeneous model is again the starting point, and the candidate modes for conversion are factored out. However, the difference in this theory is that the factorisation is not exact; rather the modes are only approximated locally, leaving a remainder term which multiplies the rest of the full dispersion relation, and which plays a central role in the conversion process.

The CLD theory proceeds as follows. In the neighbourhood of a designated mode conversion point, the dispersion relation is assumed to take the form

$$(\omega-\omega_1)(\omega-\omega_2) \simeq \eta \tag{4.27}$$

where $\omega_1(k,x)$, $\omega_2(k,x)$ are the frequencies of the candidates, and η is a 'small' quantity, arising from the approximate factorisation of the full dispersion relation and containing the remaining modes. In CLD, η 'small' implies that it is only relatively significant in the conversion region. As in FKB, this local form (4.27) is expanded about x_0 and k_0, the mode conversion point and its associated communal wavenumber, defined by $\omega_1(k_0,x_0) \simeq \omega_2(k_0,x_0)$. Writing

$$k = k_0 + \delta \ , \quad x = x_0 + \xi ,$$

we have, on a Taylor expansion,

$$\begin{aligned} \omega_1 &\simeq \omega_0 + \left.\frac{\partial\omega_1}{\partial k}\right|_{k_0}\delta + \left.\frac{\partial\omega_1}{\partial x}\right|_{x_0}\xi \triangleq \omega_0 + a\delta + b\xi, \\ \omega_2 &\simeq \omega_0 + \left.\frac{\partial\omega_2}{\partial k}\right|_{k_0}\delta + \left.\frac{\partial\omega_2}{\partial x}\right|_{x_0}\xi \triangleq \omega_0 + f\delta + g\xi. \end{aligned} \tag{4.28}$$

Then in the locality of x_0, k has the spatial dependence defined by

$$(k - k_o + \frac{b}{a}\xi)(k - k_o + \frac{g}{f}\xi) = \frac{\eta_o}{af} , \qquad (4.29)$$

where $\eta_o = \eta(x_o, k_o)$.

The coupling inherent in (4.29) is then equally divided between an appropriate pair of ODEs, each governing the propagation of one of the candidates:

$$\frac{d\phi_1}{d\xi} - i(k_o - \frac{b}{a}\xi)\phi_1 = i\lambda\phi_2, \qquad (4.30)$$

$$\frac{d\phi_2}{d\xi} - i(k_o - \frac{g}{f}\xi)\phi_2 = i\lambda\phi_1 , \qquad (4.31)$$

ϕ_i representing the ith wave amplitude, and

$$\lambda = (\eta_o / af)^{1/2} \qquad (4.32)$$

The pair (4.30,4.31), on elimination of one of the ϕ_i (say ϕ_2), yielding

$$\frac{d^2\phi_1}{d\xi^2} - 2i\left[k_o - \frac{1}{2}\left(\frac{b}{a} + \frac{g}{f}\right)\xi\right]\frac{d\phi_1}{d\xi} + \left[i\frac{b}{a} + \lambda^2 - (k_o - \frac{b}{a}\xi)(k_o - \frac{g}{f}\xi)\right]\phi_1 = 0.$$

This expression may be transformed to the Weber equation,

$$\frac{d^2\psi}{d\varsigma^2} + \left[\frac{i\eta_o}{|ag - bf|} + \frac{1}{2} - \frac{1}{4}\varsigma^2\right]\psi = 0, \qquad (4.33)$$

where

$$\psi(\xi) = \phi_1(\xi).\exp\left\{\frac{i}{4}\left(\frac{b}{a} + \frac{g}{f}\right)\xi^2 - ik_o\xi\right\},$$

$$\xi = \varsigma\left[\frac{af}{|ag - bf|}\right]^{1/2} e^{-i3\pi/4}$$

The asymptotic analysis of the Weber equation, after some algebraic manipulation, can then be interpreted as showing the resultant redistribution of energy between the modes after a mode conversion event.

comment

Again the concept of a local, position dependent dispersion relation has been invoked, but this time with a different emphasis from FKB theory. In CLD, the dispersion relation is imperfectly factored using only a local approximation for the candidate modes, and the remainder, viz. the term involving higher order corrections to the candidates, together with the other non-participant modes, becomes the crucial factor in determining the degree of coupling. This term η is then set to a constant evaluated at the appropriate mode conversion point.

The consequent splitting of λ between the ODEs to produce a constant symmetric coupling is rather arbitrary; there is no reason why the inhomogeneous term in each ODE should be equal and independent of position.

In fact, the principle that the coupling take place in a 'small', finite region, implicit in the approximation (4.27) and the asymptotic expansion of (4.33), should suggest a strongly localised interaction term, rather than a constant.

Moreover, the concept of 'locally significant' as expressed in (4.27) is not related to the wave amplitudes of the candidates, and so the inference that λ is the coupling term seems unjustified. (The mechanism by which coupling self-consistently arises will be shown in chapter V.)

Furthermore, this coupling must clearly apply throughout the region of solution space in which both the local approximation and the asymptotic expansions are deemed to hold.

Finally, in common with FKB, the parameter gradients are inadequately treated; the presence of a term $\partial\omega_i/\partial x$ implies that parameter gradients may be extracted from the dispersion relation after prescribing the latter's spatial dependence. That this is incorrect is clear; whether it is a reasonable estimate will be discussed in chapter V.

Note that in CLD, energy conservation is identified with the constancy of the sums of the squares of the wave amplitudes, ie

$$|\phi_1|^2 + |\phi_2|^2 = \text{const.}$$

Contrast this with the usual conservation law for a loss-free medium:

$$\mathrm{Im}\,(\varphi'\varphi^*) = \text{const} .$$

The above quantity is the one used in WKBJ theory to extract connection formulae. That CLD opts for another form has been noted by Swanson {31} who comments on the divergence of the CLD solutions from those expected from WKBJ.

Swanson and Stix

This theoretical work {23,31-35} tackles one dimensional local approximations where the coefficients of the governing ODE are no worse than linear. The basic premise here is that mode conversion arises from fourth or higher order differential equations, which describe two or more wave solutions.

By solving certain types of these equations analytically, a basis set of mode-conversion solutions is constructed. These known solutions are then used to solve other ODEs which arise in specific models.

In this way, an analytic method of solving higher-order mode conversion equations is constructed, and a systematic way of coupling the relevant asymptotic solutions is proposed. Thus fourth order ODEs are studied analytically using Laplace integration techniques and matched asymptotic expansions.

Mode conversion here is concerned with the splitting of energy flow between the various branches of the local dispersion relation. Without detailing extensively the intricacies of this theory, a basic description is that it is mainly concerned with reducing the equation governing the mode conversion to one of two basic types, viz.

$$\psi'''' + \lambda^2 z \psi'' + \gamma \psi = 0, \tag{4.22}$$

$$\psi'''' + \lambda^2 z \psi'' + \alpha\lambda^2 \psi' + (\lambda^2 z + \gamma)\psi = 0,$$
$$\lambda, \gamma, \alpha \text{ constant.} \tag{4.23}$$

The authors treat these equations by matching their asymptotic solutions across the complex plane (and so deriving appropriate connection formulae). These equations then become the kernels for an analytic attack on any mode conversion equation, for if any such equation is not already of the form (4.22) or (4.23), one of the kernel equations is extracted from it, and any remaining terms are written on the right-hand side. The resulting equation is then treated as though it were an inhomogeneous equation, with the non-conforming terms appearing as a driving term on the right, modifying the already known solutions of the kernel on the left.

comment

The technical treatment of an equation in this way is very complex, treating the same dependent variable in two fundamentally and logically different ways.

However, the principle that the equations must first be put into the kernel form is very restrictive, and is hard to justify given modern numerical methods. Despite Swanson's claim that such systems are numerically unstable due to the possible presence of evanescent solutions {33}, Appert {36} has overcome these difficulties and presents numerical solutions for the same mode-conversion scenarios.

The moulding of model equations into one of the types (4.22), (4.23) can be particulary difficult in certain contexts, as will be evident in the MHD analysis of chapter VI.

These authors are not really concerned with the construction of the candidate ODE, rather they present an analytic technique for any equation so arising. However, they show clearly in their publications (eg {34}) that they are content with the early notions of Stix, in that appropriate ODEs are generated from the dispersion relation directly, using (4.1).

Summary

In this chapter, a very brief review was given of some of the the existing theories which claim to describe mathematically wave phenomenon in non-uniform media. These methods, together with other theories, are detailed extensively in the recent review article by Swanson {31}. The extent of the effort in this field underlines the importance of mode conversion to the fusion community.

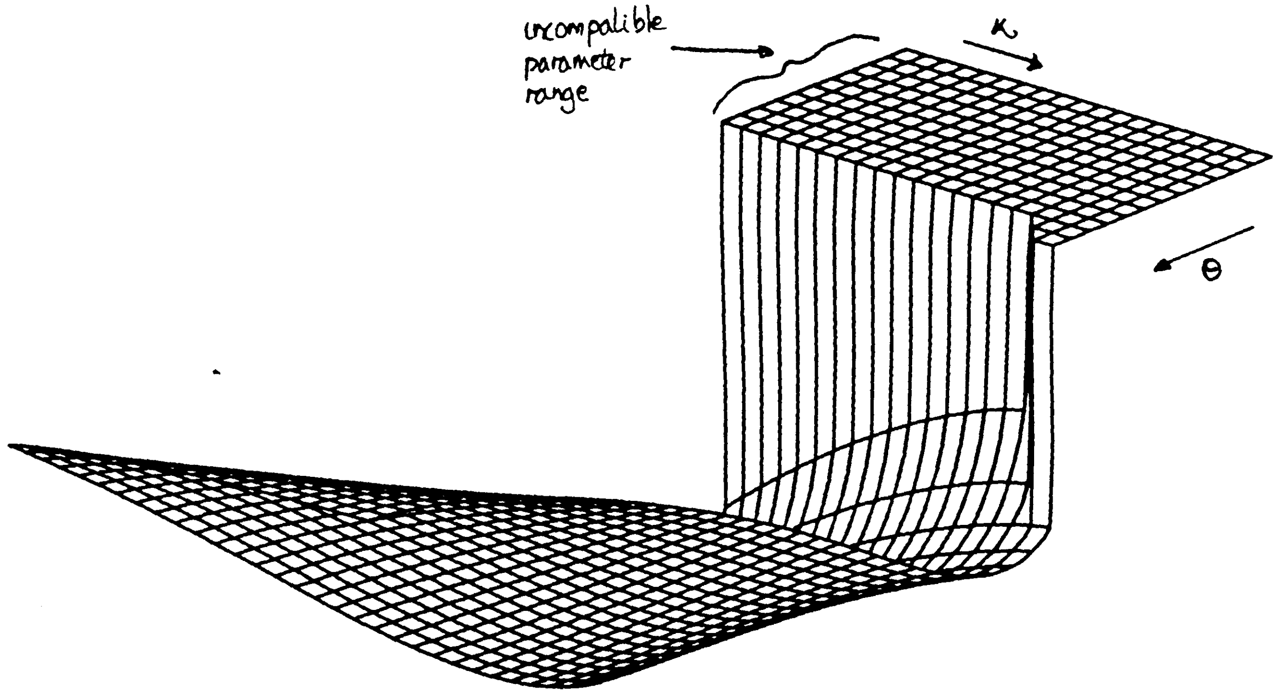
However, the adequacy of the work in this field is open to question. That some of the various theories reach agreement in particular problems {26} reflects more their common theoretical basis, rather than serving as an independent test of their veracity.

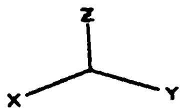
It should be noted that some disquiet concerning the nature of the approximations and their relevance is being expressed in the publications, for example references {37-39} question the lack of parameter gradients despite their demonstrable significance.

In contrast, the straightforward approach of 4th order ODEs directly from dispersion relations, using only the mapping (4.1), is still in use (eg {40}) without any apparent reservations.

Since the solution of these types of problems would be useful in a wide range of scientific applications, further efforts should be made to study these effects in a self-consistent manner, and point out the pitfalls in making too many simplifying assumptions.

The latter aim has been the main goal of this chapter; the next will tackle the self-consistent problem.





$$\Delta T^2 = T^2(\text{CHAPTER III}) - T^2(\text{CHAPTER IV})$$

over common θ, κ.

Z: -0.8 TO 0.0

FIGURE 4·1 : $\Delta T^2 = \Delta T^2(\theta, \kappa)$

Chapter V Mode Conversion Theory

Linear mode conversion in inhomogeneous media describes the process whereby energy is redistributed amongst the various possible eigenstates (or 'modes') of the system as a direct result of the spatial variation of the model parameters.

As was explained in the previous chapter, much effort has been expended in extracting desirable differential equations from dispersion relations which can be used to quantify this phenomenon. In this chapter, a more straightforward and consistent theoretical approach is developed, building on earlier work of Heading {41} and incorporating some of the more satisfactory features of the less rigorous approaches.

governing equation

The most fundamental aspect of mode conversion must be the existence of coupled equations describing the interaction of the modes in question. In the simplest possible case, where only two modes are involved, such a binary interaction in its most elementary form will be governed by a second order ordinary differential equation. In other words, the simplest case will be a restriction to precisely two modes in the absence of any curvature.

Clearly an elemental ODE of this type will possess coefficients which vary with position, since the medium of propagation is inhomogeneous. Let such an equation be the following:

$$a(x)\,y'' + b(x)\,y' + c(x)\,y = 0. \tag{5.1}$$

So far, we have used the vague term 'mode' in the discussion. It is apparent from the equation above that we cannot mean mode in the sense of a Fourier component, since Fourier transformation of this equation is not meaningful. How then are the 'modes' of the system to be identified? This difficulty is resolved unambiguously in the following analysis.

coupled equations

We can write (5.1) as a pair of coupled equations by expressing the ODE in matrix form:

$$\underset{\sim}{y}' = \begin{bmatrix} y \\ y' \end{bmatrix}' = M\underline{y}\,, \quad M \doteq \begin{bmatrix} 0 & 1 \\ -c/a & -b/a \end{bmatrix}.$$

We may now extract the eigenvalues of the characteristic matrix:

$$\lambda_{1,2} = -\frac{b}{2a} \pm \left[\left(\frac{b}{2a}\right)^2 - \frac{c}{a}\right]^{1/2}.$$

We shall see that these eigenvalues, the roots of the characteristic equation, play the same role in inhomogeneous media as do the wavenumbers extracted from the roots of the dispersion relation (or secular determinant) for problems with uniform media.

To show this generalisation, and to reveal the coupling inherent in such systems, make the transformation

$$\underset{\sim}{y} = A\underline{u}, \quad A \triangleq \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}, \quad \underline{u} \triangleq \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

where the A is actually a diagonalising matrix for M. Thus substituting for $\underline{y}$ produces the matrix equation

$$\underset{\sim}{u}' = A^{-1}MA\underline{u} - A^{-1}A'\underline{u},$$

that is,

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}' = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \frac{1}{\Delta}\begin{bmatrix} -\lambda_1' & -\lambda_2' \\ \lambda_1' & \lambda_2' \end{bmatrix}, \tag{5.2}$$

where

$$\Delta(x) = \lambda_1 - \lambda_2 .$$

To emphasise the main features, we may write (5.2) in the form

$$\begin{aligned} u_1' - k_1(x)\,u_1 &= c_{12}(x)\,u_2 , \\ u_2' - k_2(x)\,u_2 &= c_{21}(x)\,u_1 , \end{aligned} \tag{5.3}$$

where

$$\begin{aligned} k_1 &= \lambda_1 - \lambda_1'/\Delta, \qquad & c_{12} &= -\lambda_2'/\Delta, \\ k_2 &= \lambda_2 + \lambda_2'/\Delta, \qquad & c_{21} &= \lambda_1'/\Delta . \end{aligned} \tag{5.4}$$

At this stage, we can compare these equations with those of Cairns and Lashmore - Davies (see (4.30), (4.31)). The latter description has much the same form as (5.3) but with two important differences :

(i) the 'wavenumbers' $k_{1,2}(x)$ are derived from the eigenvalues, <u>not</u> from a 'dispersion relation', and contain an extra term dependent on the gradient of and difference between those same eigenvalues ;

(ii) the coupling terms are self-consistent functions of position, not externally supplied constants.

It is interesting to note the role of the parameter gradients at this early stage. Where the eigenvalues are well separated, or have negligible gradients, the coupling terms are small, and the system then has the quasi - uniform states

$$u_i(x) \sim \exp \int^x k_i(s)\, ds \,. \qquad (5.5)$$

These recover the expected behaviour for distinct modes (cf equations (2.2)). However, recalling the procedure involved in WKBJ analysis, in order to describe the relative mixture of these asymptotic states, the full system must be studied.

<u>single mode equation</u>

Note that u_2 can be eliminated from (5.3) in order to yield a single second order ODE for u_1 alone. Since

$$\frac{d}{dx}\left[(u_1'-k_1u_1)/c_{12}\right] = u_2' ,$$

$$\frac{k_2}{c_{12}}\left[u_1'-k_1u_1\right] = k_2u_2 ,$$

subtracting these equations produces

$$\left[\frac{u_1'-k_1u_1}{c_{12}}\right]' - \frac{k_2}{c_{12}}\left[u_1'-k_1u_1\right] = c_{21}u_1 ,$$

ie

$$u_1'' - (k_1u_1)' - \frac{c_{12}'}{c_{12}}(u_1'-k_1u_1) - k_2(u_1'-k_1u_1) - c_{12}c_{21}u_1 = 0,$$

or

$$u_1'' - (\rho+\sigma)u_1' + (k_1k_2+k_1\rho-k_1'-c_{12}c_{21})u_1 = 0,$$

where

$$\sigma = k_1+k_2 , \quad \rho = c_{12}'/c_{12} .$$

The change of variable

$$v = u_1 \exp\left\{-\tfrac{1}{2}\int(\rho+\sigma)ds\right\}$$

will eliminate the first derivative term, finally producing the equation

$$v'' = \left[-k_1k_2 - k_1\rho + k_1' + c_{12}c_{21} - \tfrac{1}{2}(\rho+\sigma)' + \tfrac{1}{4}(\rho+\sigma)^2\right]v. \quad (5.6)$$

Now

$$\tfrac{1}{4}(\rho+\sigma)^2 - \tfrac{1}{2}(\rho+\sigma)' - k_1k_2 - k_1\rho + k_1'$$

$$= \tfrac{1}{4}(\rho^2+\sigma^2) - k_1k_2 + \tfrac{1}{2}\rho\sigma - k_1\rho - \tfrac{1}{2}\rho' - \tfrac{1}{2}\sigma' + k_1'$$

$$= \tfrac{1}{4}(\rho-\delta)^2 - \tfrac{1}{2}(\rho-\delta)' , \quad \text{where} \quad \delta = k_1-k_2 .$$

Thus setting

$$\kappa = \tfrac{1}{2}(\rho-\delta)$$

we can write (5.6) in the form

$$v'' = \left[\kappa^2 + \kappa' + c_{12}c_{21}\right] v \quad . \tag{5.7}$$

This elimination technique is the same as that used by CLD (see p.56, and ref.{28}), and so a comparison of the two equations can be made to see the differences arising from the inclusion of all parameter gradients.

However, recall that the basic starting equation was already a second-order ODE; the procedure detailed above is lengthy and complicated, producing an equation which may be even more complicated than the original. In fact there is a simpler transformation which retains the fundamental features and interpretations of the preceding one, but is simpler and more symmetric in its treatment of the two states.

Noting that the original equation (5.1) may be written as

$$y'' - \Sigma(x) y' + \Pi(x) y = 0,$$

where

$$\Sigma = \lambda_1 + \lambda_2 = 2\alpha \ , \quad \Pi = \lambda_1 \lambda_2 = \alpha^2 - \beta^2,$$

the dependent variable may be changed from y to

$$W = y \exp \int^x -\tfrac{1}{2} \Sigma(s)\, ds,$$

yielding the equation

$$W'' = \left[-\pi - \tfrac{1}{2}\Sigma' + \tfrac{1}{4}\Sigma^2\right]W,$$

ie

$$W'' = \left[\tfrac{1}{4}\Delta^2 - \tfrac{1}{2}\Sigma'\right]W. \tag{5.8}$$

This equation serves as an easily derivable 'master' equation for describing binary mode conversion. Note again the importance of the parameter gradients, and also that the difference between the eigenvalues will be an important feature.

Let the 'wave potential' Ψ be defined by

$$\Psi(x) = \tfrac{1}{4}\Delta^2 - \tfrac{1}{2}\Sigma'$$

Then the nature of the coupling between eigenstates is dependent both on the zeros of Ψ and on its turning points, since these define the barrier width and depth.

the Weber equation

At this stage, recall that most of the contemporary mode conversion theories rely on the Weber equation {42} as the typical comparison equation for this binary interaction. This confluent hypergeometric equation takes the form

$$\varphi'' + \left[k + \tfrac{1}{2} - \tfrac{1}{4}x^2\right]\varphi = 0. \tag{5.9}$$

It is favoured in mode conversion because it contains two transition points

$$x_{1/2} = \pm (2k+1)^{1/2}$$

and because its asymptotic behaviour is well known {43} (see Appendix B).

If the Weber equation is to be used in this analysis, a polynomial representation of the eigenvalues must be taken, in the form

$$\tfrac{1}{2}\Sigma = s_0 + s_1 x + s_2 x^2 + s_3 x^3, \qquad (= \alpha)$$

$$\tfrac{1}{4}\Delta^2 = d_0 + d_1 x + d_2 x^2, \qquad (= \beta^2)$$

in order that ψ may be written as

$$\psi(x) = p_0 + p_1 x + p_2 x^2$$

where

$$p_0 = d_0 - s_1, \quad p_1 = d_1 - 2s_2, \quad p_3 = d_2 - 3s_3 .$$

Now ψ can be written in the required standard form by employing a change of independent variable. Noting that

$$\psi'(x_0) = 0 \quad \Rightarrow \quad x_0 = -p_1/2p_2,$$

and that

$$\psi(x-x_0) = p_0 + p_1(x-x_0) + p_2(x-x_0)^2$$

$$= \psi_0 + p_2 x^2,$$

$$\psi_0 = \psi(x_0),$$

changing from x to η, where η is defined by

$$\eta = \pm(4p_2)^{1/4}(x-x_0)$$

allows (5.8) to be written as

$$\frac{d^2W}{d\eta^2} = \left[\frac{\psi_0}{2p_2^{1/2}} + \tfrac{1}{4}\eta^2\right]W, \qquad (5.10)$$

which is of the required form (5.9). Note that the Weber equation describes a barrier problem with transition points defined as

$$\eta_{1,2}^{2} = - \frac{2\psi_0}{p_2^{1/2}} .$$

In general, η_i will be complex. In terms of the original coordinate x, these barrier points are given as

$$x_{1,2} = x_0 \pm \left(- \frac{\psi_0}{p_2}\right)^{1/2}$$

Clearly, the spatial dependence of ψ is a crucial factor in determining the consequent barrier and thus the transmission and conversion coefficients. With this in mind, we take particular examples of the wave potential deemed to be relevant in rf heating analysis and examine the resulting behaviour.

polynomial representations

Note that the Weber equation applies when the eigenvalues take the form

$$\lambda_{1,2} = \text{CUBIC} \pm [\text{QUADRATIC}]^{1/2} \stackrel{\partial}{=} \alpha \pm \beta ,$$

or lower order variations.

linear eigenvalues

Consider the simplest case of α and β linear, in particular

$$\alpha(x) = a_0 + a_1 x \,, \quad \beta(x) = b_1 x \qquad a_1, a_2, b_1 \text{ const.} \tag{5.11}$$

Thus the eigenvalues are equal at x=0. Now

$$\Psi(0) = a_0 \,,$$

and so we are dealing with the case

$$\psi(x) = -a_1 + b_1^2 x^2 \,.$$

Thus choosing

$$\eta = (4b_1^2)^{1/4} x$$

yields

$$\frac{d^2 W}{d\eta^2} = \left[\frac{\psi(0)}{2b_1} + \frac{1}{4}\eta^2 \right] W = \left[-\frac{a_1}{2b_1} + \frac{1}{4}\eta^2 \right] W. \tag{5.12}$$

If we apply the asymptotic expansion, with full detailed working in the appendix, we find the following result:

$$x \gg 0 : \quad y(x) \sim I_+ \exp\left\{ a_0 x + \tfrac{1}{2}(a_1 - b_1) x^2 \right\}$$

$$x \ll 0 : \quad y(x) \sim I_- \exp\left\{ a_0 x + \tfrac{1}{2}(a_1 - b_1) x^2 \right\} \tag{5.13}$$

$$+ B \exp\left\{ a_0 x + \tfrac{1}{2}(a_1 + b_1) x^2 \right\} ,$$

where the coefficients I_+, I_- and B are defined as

$$I_+ = (2b_1)^{\frac{1}{2}k} x^{k-1}$$

$$I_- = -(2b_1)^{\frac{1}{2}k} e^{ik\pi} |x|^k$$

$$B = \frac{\sqrt{2\pi}}{\Gamma(-k)} \cdot (2b_1)^{-\frac{1}{2}(k+1)} |x|^{-k-1}$$

with

$$k = -\tfrac{1}{2}\left(\frac{a_1}{b_1} + 1 \right) .$$

Since we are really interested in oscillatory eigenstates (at least asymptotically), the simplest such case would require a_1, a_2 and b_1 purely imaginary; this would result in η^2 being purely imaginary, but k real. In such a situation,

$$\left|\frac{I_+}{I_-}\right| = |\exp(ik\pi)| = 1.$$

In this example, mode conversion is negligible. This has a direct analogy in the underdense potential barrier, where to a first approximation, mode conversion is zero. This conclusion is in qualitative agreement with the ideas of Fuchs et al {25} in that 'only saddle points of the dispersion relation can lead to mode conversion'. However, this contradicts the prediction of CLD, whose theory is centred on a 'local' linear approximation for the 'wavenumbers'. That CLD has non-negligible conversion is due entirely to the fact that the coupling is externally supplied, and is not a consequence of the self-consisent variation of the eigenvalues.

nonlinear eigenvalues

The next most complicated case of interest is that where β^2 depends quadratically on the independent variable x. Thus we consider eigenvalues of the form

$$\lambda_{1/2} = \alpha \pm \beta$$

where

$$\alpha = a_0 + a_1 x ,$$

$$\beta = (b_0 + b_1 x + b_2 x^2)^{1/2} . \tag{5.14}$$

For this case,

$$\psi(z) = -a_1 + (b_0 + b_1 x + b_2 x^2) ,$$

$$\psi(x_0) = \psi(-b_1/2b_2) = -a_1 + b_0 - b_1^2/4b_2 .$$

Consequently, the 'k' in the Weber equation must be

$$k = -\frac{\psi(x_0)}{2 b_2^{1/2}} - \frac{1}{2}$$

$$= \frac{a_1 - \beta_0^2}{2\sqrt{b_2}} - \frac{1}{2} ,$$

where

$$\beta_0^2 = \beta^2(x_0) = b_0 - b_1^2/4b_2 .$$

Note that β_0 is actually the minimum separation of the eigenvalues, and x_0 is the point of closest approach. This is similar to the local behaviour ansatz of CLD, in equation (4.27).

The transition points for this problem are given as

$$x_{1,2} = x_0 \pm \left(\frac{a_1 - \beta_0^2}{b_2}\right)^{1/2} \stackrel{\text{a}}{=} x_0 \pm x_w , \tag{5.15}$$

and these are clearly roots of the potential :

$$\psi(x_{1,2}) = \beta^2(x_{1,2}) - a_1$$

$$= \beta_0^2 \pm (b_1 + 2b_2 x_0) x_w + b_2 x_w^2 - a_1$$

$$= 0 .$$

Moreover, should the eigenvalues be purely imaginary, for example, $0 > b_2, \beta_0^2 \in \mathbb{R}, a_j = i\alpha_j$, then the barrier is indeed complex.

A graph of typical eigenvalues for this example is shown in figure 5.1. Since they show the kind of behaviour most often considered to be relevant to mode conversion problems in plasma physics (for example see figure 1 of {28}), it is worth examining in detail the structure of this example.

detailed analysis

Returning to the coupled-mode form of the equation, (see equations (5.3) and (5.4)), the coupling functions $C_{12}(x)$ and $C_{21}(x)$ which control the interdependence of the eigenstates, take the form

$$C_{12}(x) = -\lambda_2'/\Delta \quad ; \quad C_{21}(x) = \lambda_1'/\Delta \quad .$$

Since

$$\beta' = \tfrac{1}{2}(b_1 + 2b_2 x)/\beta \;\stackrel{\partial}{=}\; \gamma(x)/\beta ,$$

we can write

$$C_{12}(x) = \frac{-a_1\beta + \gamma}{\beta^2} \quad , \quad C_{21}(x) = \frac{a_1\beta + \gamma}{\beta^2} \quad .$$

Thus as $|x| \to \infty$, $C_{ij}(x) \to 0$ and so the coupling becomes negligible far from the interaction region, implying that the asymptotic solution is expressible in terms of the approximate independent eigenstates.

extrema

It is useful to know where the extrema of the coupling functions occur, in relation to the transition points.

Consider first $C_{12}(x)$. Then

$$
\begin{aligned}
2C_{12}' &= (a_1\beta+\gamma)'/\beta^2 - 2(a_1\beta+\gamma)\beta'/\beta^3 \\
&= \frac{1}{\beta^3}\left[a_1\gamma + \gamma'\beta - 2\beta'(a_1\beta+\gamma)\right] \\
&= \frac{1}{\beta^4}\left[a_1\beta\gamma + b_2\beta^2 - 2\gamma(a_1\beta+\gamma)\right] \\
&= \left[\beta(b_2\beta - a_1\gamma) - 2\gamma^2\right]/\beta^4 \;.
\end{aligned}
$$

Thus the extrema of C_{12} occur at the roots of the polynomial

$$b_2\beta^2 - a_1\gamma\beta - 2\gamma^2 = 0. \tag{5.16}$$

Similarly, C_{21} possesses extrema a positions defined by the roots of

$$b_2\beta^2 + a_1\gamma\beta - 2\gamma^2 = 0. \tag{5.17}$$

We can solve for all four roots simultaneously by multiplying (5.16) and (5.17) together, and solving the quartic equation in x,

$$(b_2\beta^2 - 2\gamma^2)^2 - (a_1\gamma\beta)^2 = 0. \tag{5.18}$$

Substituting $x_c = x_0 + \epsilon$ into (5.18) yields a biquadratic in ϵ . The full details of this calculation are not essential to the main text, and so are laid out in full in Appendix B. Stating only the result in the interest of clarity, we have

$$x_c = x_0 + \epsilon \tag{5.19}$$

where

$$\epsilon^2 = \frac{\beta_0^2}{b_2 - a_1^2}\left[1 + \frac{a_1^2}{2b_2}\left(1 \pm \left(1 + \frac{8b_2}{a_1^2}\right)^{1/2}\right)\right] \tag{5.20}$$

Therefore the coupling is at a maximum on either side of the conversion point. In addition, note that

$$C_{12} + C_{21} = \beta'/\beta = \frac{\gamma}{\beta^2} .$$

Hence

$$(C_{12} + C_{21})' = \frac{\gamma'}{\beta^2} - \frac{2\gamma\beta'}{\beta^3} = \frac{b_2\beta^2 - 2\gamma^2}{\beta^4} .$$

Now

$$b_2\beta^2 - \gamma^2 = (b_0 b_2 - \tfrac{1}{2}b_1^2) - b_1 b_2 x - b_2 x^2 ,$$

and so

$$C_{12} + C_{21} \text{ has extrema at } -\frac{b_1}{2b_2} \pm \frac{1}{2b_2}\left(4b_0 b_2 - b_1^2\right)^{1/2} .$$

Furthermore,

$$C_{12} - C_{21} = \frac{\alpha'}{\beta} = \frac{a_1}{\beta} ,$$

thus

$$(C_{12} - C_{21})' = -\frac{a_1\gamma}{\beta^3}$$

Hence

$$C_{12} - C_{21} \text{ has extremum at } x_0 .$$

If $a_1 = 0$, then $C_{21} = C_{12}$, and the various extrema would coalesce; non-zero a_1 introduces the evident fine structure.

Graphs of $C_{12}(x)$ and $C_{21}(x)$ for the typical eigenvalues of figure 5.1 are shown in figure 5.2. It can be seen that they exhibit the desired behaviour for a mode conversion problem, in that the coupling is negligible far from the conversion point, but peaking around the critical region.

Since these functions are the 'driving' terms in the coupled system, they can be viewed as the mechanism which 'switches on' the complementary mode when the original propagates through a certain region of solution space.

It must be emphasised that this behaviour derives consistently from the intrinsic variation of the eigenvalues, and does not appear as a consequence of artificial coupling induced from a 'dispersion relation'.

asymptotic solution

With these points in mind, we return to the form of the Weber equation appropriate to this example, and write down its asymptotic solution. The appropriate ODE is

$$\frac{d^2W}{d\eta^2} + \left[\frac{a_1 - \beta_0^2}{2\sqrt{b_2}} - \frac{1}{4}\eta^2\right]W = 0 .$$

The usual asymptotic expansion, when written in terms of the original variables y=y(x), takes the form

$x \gg x_0,\ x > 0$:

$$y \sim I_+(x-x_0)^k \exp\left\{\int^x \alpha\, ds - \tfrac{1}{2}\sqrt{b_2}\,(x-x_0)^2\right\}$$

$x \ll x_0,\ x < 0$:

$$y \sim I_-(x-x_0)^k \exp\left\{\int^x \alpha\, ds - \tfrac{1}{2}\sqrt{b_2}\,(x-x_0)^2\right\}$$
$$+ B\,|x-x_0|^{-k-1} \exp\left\{\int^x \alpha\, ds + \tfrac{1}{2}\sqrt{b_2}\,(x-x_0)^2\right\},$$

where ...

$$I_+ = (4b_2)^{k/4}, \quad I_- = (4b_2)^{k/4} e^{ik\pi}, \quad B = \frac{\sqrt{2\pi}}{\Gamma(-k)} e^{-2ik\pi} (4b_2)^{-\frac{1}{4}(k+1)} . \tag{5.21}$$

In order to see that these expressions are meaningful, we need to consider the appropriate eigenstates for large $|x|$.

approximate eigenstates

Consider the form of the eigenvalues far from the interaction region. Now

$$\begin{aligned}\beta &= \sqrt{b_0 + b_1 x + b_2 x^2} \\ &= b_2^{1/2} x \left[1 + \frac{b_1}{b_2}\cdot\frac{1}{x} + \frac{b_0}{b_2}\cdot\frac{1}{x^2}\right]^{1/2} \\ &\simeq b_2^{1/2} x \left[1 + \frac{b_1}{2b_2}\frac{1}{x} + \frac{b_0}{2b_2}\frac{1}{x^2} - \frac{b_1^2}{8b_2}\cdot\frac{1}{x^2} + O(1/x^3)\right]\end{aligned}$$

that is, for large $|x|$,

$$\beta \sim b_2^{1/2}(x - x_0) + \frac{\beta_0^2}{2b_2^{1/2}}\frac{1}{x} + O(1/x^2),$$

where $\quad \beta_0 = \beta(x_0)$.

Consequently,

$$\int^x \beta\, ds \sim \frac{1}{2} b_2^{1/2}(x^2 - 2x x_0) + \frac{\beta_0^2}{2\sqrt{b_2}} \ln x, \qquad x \gg 0.$$

Moreover,

$$\int^x \frac{\alpha'}{2\beta} ds = \frac{a_1}{2}\int^x \frac{ds}{\beta} \simeq \frac{a_1}{2b_2^{1/2}}\int \frac{ds}{s} = \frac{a_1}{2b_2^{1/2}} \ln x, \quad x \gg 0.$$

Combining these results, we may evaluate the approximate uncoupled solutions given by

$$\int^x k_{1,2}(s)ds \simeq \int^x (\alpha \pm \beta)ds - \ln\sqrt{\beta} \mp \frac{1}{2} a_1 \int \frac{ds}{\beta}$$

$$\simeq x^{p_\pm} \exp\left[a_0 x + \frac{1}{2} a_1 x^2 \pm \frac{1}{2} b_2^{1/2}(x^2 - 2x x_0)\right]$$

where $\quad p_\pm = -\frac{1}{2} - (k + \frac{1}{2})$ or $-\frac{1}{2} + (k + \frac{1}{2}) = -k - 1$ or k.

By comparison with (5.21), it is clear that these approximate eigenstates are precisely the important components of the asymptotic expansion, enabling the latter to be interpreted as a superposition of the two independent eigenstates existing far from the interaction region. This is a satisfactory result, given the nature of the problem and the behaviour of the coupling terms.

With this interpretation, we can identify a transmission factor of

$$\frac{I_-}{I_+} = \exp(ik\pi) \, . \tag{5.22}$$

In a realistic problem, we would be concerned with oscillatory eigenstates. Thus assuming complex eigenvalues here implies that k is complex. Consequently, we are dealing with a complex potential barrier, in which there is significant mode conversion.

accuracy of uncoupled eigenstates

Since much of the theory so far has centred on reducing the solution to a superposition of the uncoupled approximate eigenstates it is appropriate to comment upon the accuracy of such solutions.

In order to assess how well these forms estimate the correct solution, consider the relevant eigenstates of the Bessel equation,

$$y' + \frac{1}{x}y'' + \left(1-\frac{n^2}{x^2}\right)y = 0. \tag{5.23}$$

The eigenvalues in this case are

$$\lambda_{1,2} = \alpha \pm \beta$$

where

$$\alpha = -\frac{1}{2x}, \quad \beta^2 = \frac{1}{4x^2} - \left(1-\frac{n^2}{x^2}\right) = \frac{d^2-x^2}{x^2}, \quad d^2 = n^2+\frac{1}{4}.$$

The desired expressions are the uncoupled approximate eigenstates of (5.5), namely

$$\exp\int k_i(s)\,ds = \beta^{-1/2}\exp\int^x \left(\alpha+\beta-\frac{\alpha'}{2\beta}\right)ds. \quad (i=1)$$

Now

$$\beta - \frac{\alpha'}{2\beta} = \frac{\sqrt{d^2-x^2}}{x} - \frac{x}{4x^2\sqrt{d^2-x^2}} = \frac{n^2-x^2}{x\sqrt{d^2-x^2}}.$$

Choosing $x>d$, we have

$$\int^x \left(\beta-\frac{\alpha'}{2\beta}\right)dx' = in^2\int^x \frac{dx'}{x'u} - i\int^x \frac{x'dx'}{u}, \quad u \stackrel{2}{=} x'^2-d^2. \tag{5.24}$$

These integrals are of a standard form {19}, thus

$$\int^x \left(\beta-\frac{\alpha'}{2\beta}\right)dx' = i\frac{n^2}{d}\sec^{-1}(x/d) - iu. \tag{5.25}$$

Hence the solution is

$$y \sim (x^2-d^2)^{-1/4} \cos\left[\sqrt{x^2-d^2} - \frac{\pi}{4} - \frac{n^2}{d}\sec^{-1}(x/d)\right]. \qquad (5.26)$$

Compare this with the usual asymptotic formula {44}

$$J_n(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left[x - \frac{\pi}{4} - \frac{n\pi}{2}\right]. \qquad (5.27)$$

For a more quantitative comparison, see Figure 5.3, which graphs the full eigenstate approximation against Bessel functions generated by a standard numerical package {45}.

Summary

In this section, the self-consistent form of the mode conversion equations has been considered, without relying on ad-hoc dispersion relations or arbitrary, externally imposed coupling coefficients.

In this context, two simple examples were studied: one in which the eigenvalues varied linearly with position, sharing a common value at the origin, and a more complex case where the eigenvalues converged, then diverged without necessarily crossing. In the former example, mode conversion was found to be negligible, in contrast to the local linear theory of Cairns and LashmoreDavies. This discrepancy arises wholly from the lack of self-consistent coupling in the CLD model: there, mode conversion occurs only by virtue of a purely imaginary constant supplied independently of the spatial dependence of the modes in question.

In the second example, however, mode conversion is not negligible, and arises naturally as a consequence of localised coupling functions derived self-consistently from the candidate eigenvalues. The resulting solution can then be expressed as the superposition of the asymptotic uncoupled quasi-uniform states, as in WKBJ.

Note that the analysis is not restricted to considering these special cases: variations such as

$$\text{CUBIC} \pm \sqrt{\text{QUADRATIC}}$$

can be studied within the same framework and using the Weber equation. Such eigenvalue behaviour may prove of interest in special applications. Moreover, mode conversion need not (and should not) be restricted to the solutions afforded by only the Weber equation. Higher order ODEs can be studied using the same techniques, the only difference being that the matrix yielding the pair of equations (5.3) will produce instead n ODEs. By considering each of the coupling functions $C_{ij}(x)$ in turn, a quantitative judgement can be made on whether the mode conversion involves two or more waves, and on the validity of the ensuing asymptotic solutions.

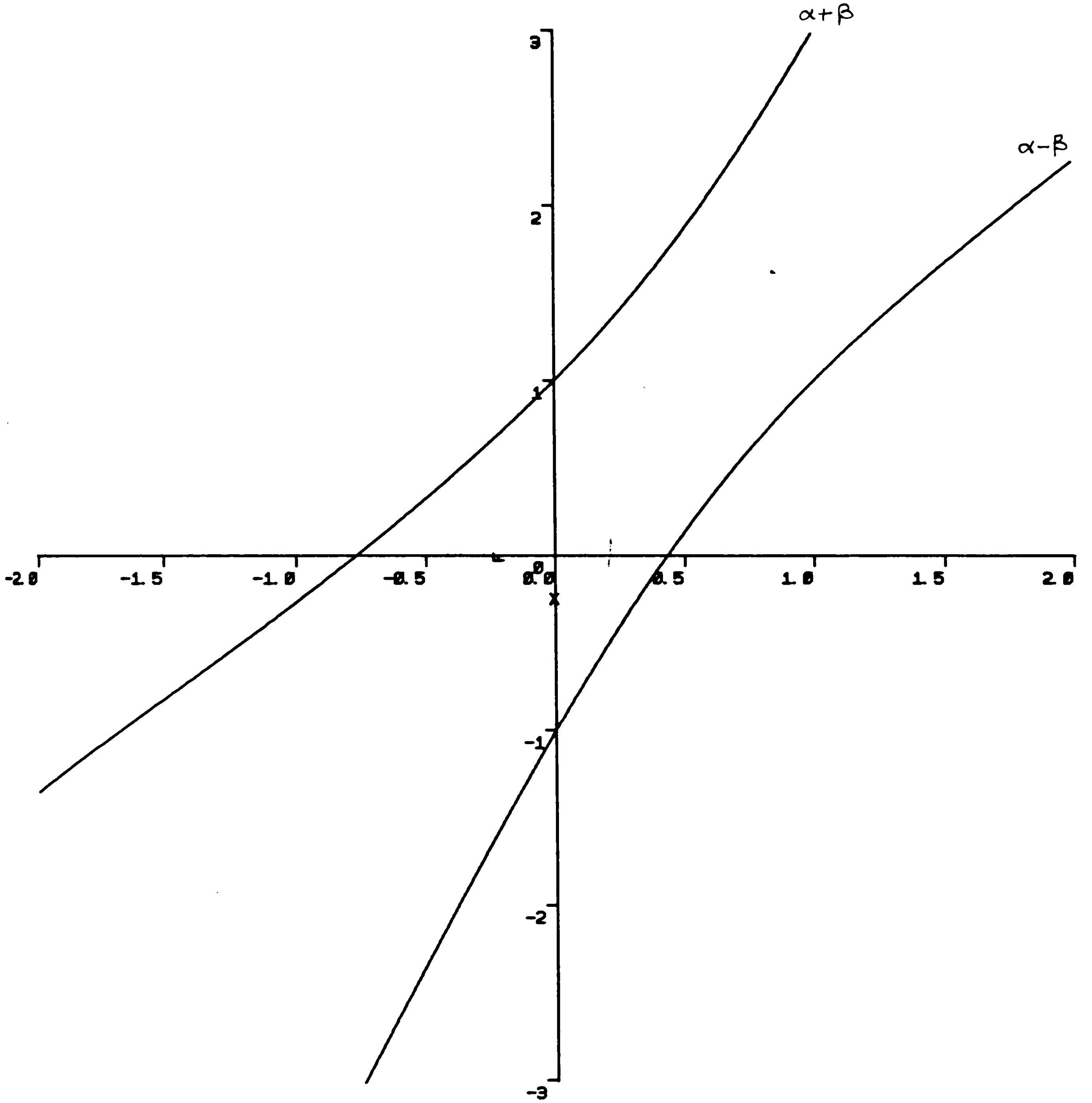
Indeed, such systems lend themselves readily to computational solution, given that the coupling is localised.

Finally, note that only ODEs were studied here: real physical systems are described by partial differential equations, and the underlying problem of tackling mode conversion in this context has not been addressed here.

In the next chapter, a simple mhd plasma is studied self-consistently as an example of the dangerous over-simplifications involved in analysing inhomogeneous media by any means other than the correct differential equations.

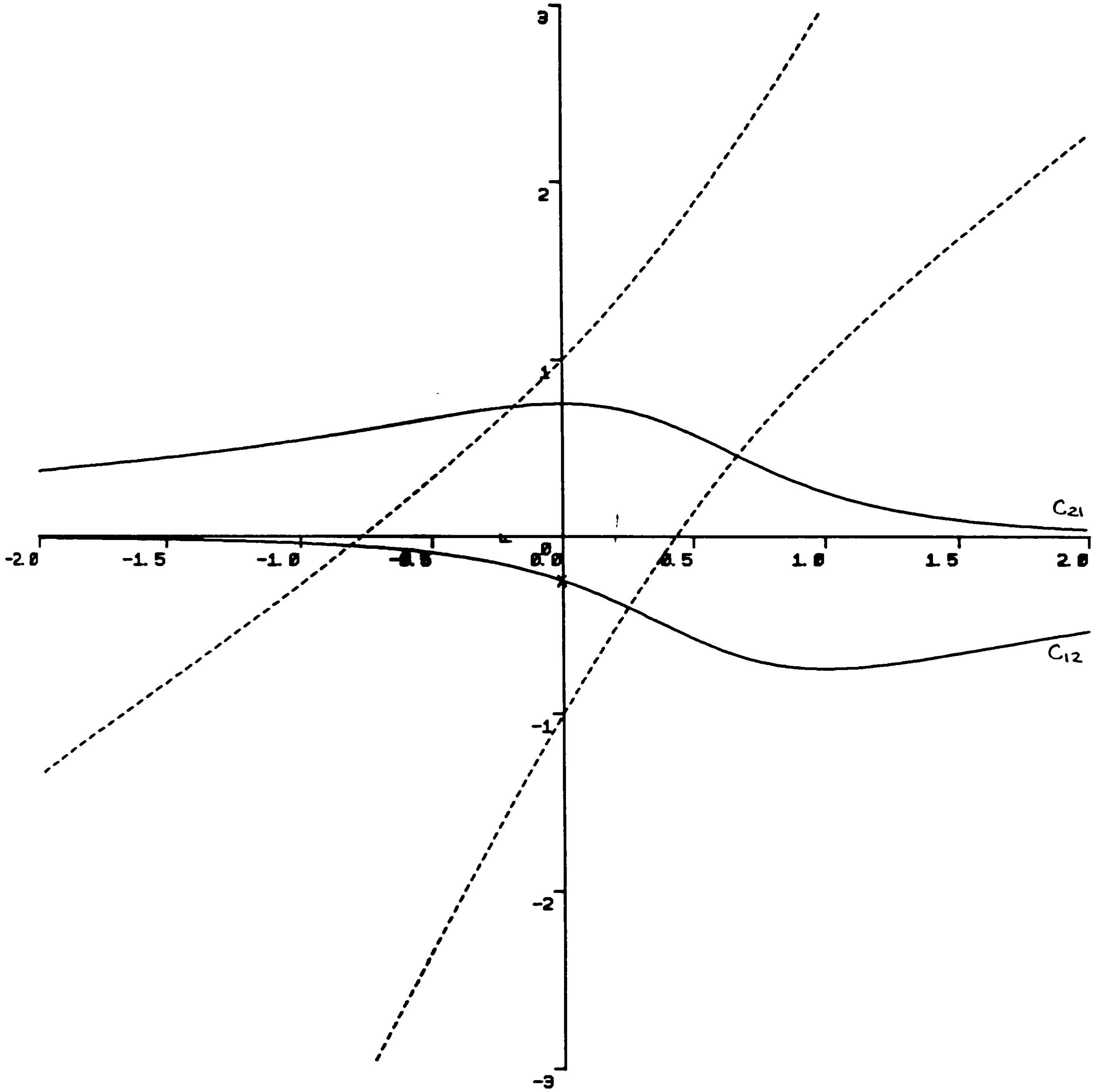
:----------------------------:

FIGURE 5.1



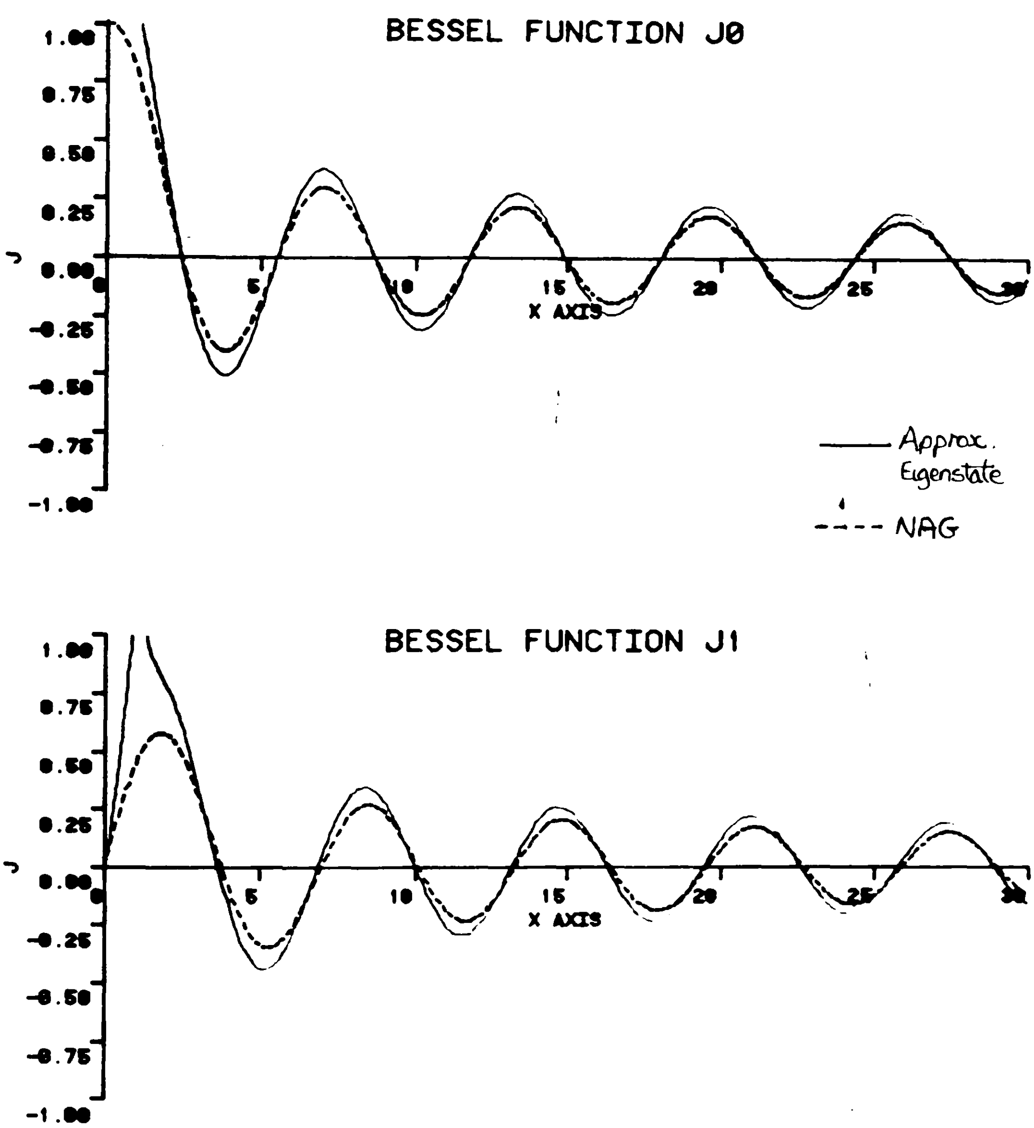
EIGENVALUES AS FUNCTION OF POSITION

FIGURE 5.2



EIGENVALUES AND COUPLING FUNCTIONS

FIGURE 5.3



Approximate eigenstate and NAG comparison for Bessel functions

Chapter VI Warm Plasma Model

The ideal mhd plasma model has a long history in plasma physics, and enjoys wide ranging popularity and applicability.

The particular version under consideration in this chapter is the one-fluid model with zero resistivity and viscosity, and a scalar pressure.

We will analyse the case of a plane stratified non-uniform magnetic field in a warm fluid plasma, by including the inhomogeneity from the very beginning, and self consistently generating the resulting equations governing the behaviour of the perturbed plasma.

In so doing, we reveal extra physical effects of interest, and show that dispersion relation approaches compare poorly with consistent analysis.

The fluid equations

The fluid equations defining the model are derived from an appropriate kinetic equation for each species, by integrating out the velocity dependence in a series of moment equations.

The resulting fluid equations are then combined in a consistent manner to yield a set of bulk fluid equations for the plasma. This procedure is detailed extensively in well known texts, eg {46-48}.

The relevant model equations under these conditions are

$$\frac{D\rho}{Dt} = -\rho\nabla\cdot\underline{u}\ , \tag{6.1}$$

$$\rho\frac{D\underline{u}}{Dt} = -\nabla p + \underline{J}\times\underline{B}\ , \tag{6.2}$$

$$\frac{D}{Dt}(p\rho^{-\gamma}) = 0\ , \tag{6.3}$$

$$\underline{E} + \underline{u}\times\underline{B} = 0, \tag{6.4}$$

$$\nabla\times\underline{B} = \mu_0\underline{J}\ , \tag{6.5}$$

$$\nabla\cdot\underline{B} = 0, \tag{6.6}$$

$$\frac{\partial\underline{B}}{\partial t} = -\nabla\times\underline{E}\ , \tag{6.7}$$

where $D/Dt = \partial/\partial t + \underline{u}\cdot\nabla$ denotes the advective derivative, and the dependent variables are

$\underline{u}$: the fluid bulk velocity

p : the scalar pressure of the single fluid

ρ : mass density

$\underline{J}$: the total current in the fluid

$\underline{B}$: the magnetic induction

$\underline{E}$: the electric field

$\gamma = 5/3$ - the adiabatic constant for a simple gas with 3 degrees of freedom.

Note that (6.4) is the Ohm's Law for the model, and is derived from the charge flow analogue of the momentum equation (6.2). Each of (6.1) to (6.4) are derived from moments in velocity space of the Vlasov equation. The full form of (6.4) is in fact {47}

$$\frac{D}{Dt}\left(p\rho^{-\gamma}\right) = \frac{2}{3}\rho^{-\gamma}\left(\underline{J}-q\underline{u}\right)\cdot\left(\underline{E}+\underline{u}\times\underline{B}\right),$$

where q is the electronic charge appropriate for the model species. Thus if the simple Ohm's Law (6.4) holds, we see that the energy equaiton in the form of the adiabatic law (6.3) must hold to the same approximation. Thus the system is closed and self-consistent.

Perturbation analysis

For any fruitful analytic study of the model properties, the full set of equations (6.1-6.7) are linearised, so that small amplitude behaviour may be analysed.

linearised equations

We adopt the static equilibrium quantities $p_0, \rho_0, E_0, \underline{B}_0, \underline{u}_0$ given by

$$\underline{u}_0 = 0, \tag{6.9}$$

$$\underline{E}_0 = 0, \tag{6.10}$$

$$\nabla p_0 = \frac{1}{\mu_0}(\nabla\times \underline{B}_0)\times \underline{B}_0 \,, \tag{6.11}$$

$$p_0 \rho_0^{-\gamma} = \text{constant} \,, \tag{6.12}$$

$$\nabla\cdot \underline{B}_0 = 0. \tag{6.13}$$

Note that (6.9) and (6.12) are postulated: the remaining equations then are the consequences of these assumptions as dictated by the model equations. Given these, the first order, perturbed quantities then obey

$$\dot{\rho}_1 + \underline{u}_1\cdot\nabla\rho_0 + \rho_0\nabla\cdot\underline{u}_1 = 0, \tag{6.14}$$

$$\rho_0\dot{\underline{u}}_1 = -\nabla p_1 + \frac{1}{\mu_0}\left\{(\nabla\times\underline{B}_1)\times\underline{B}_0 + (\nabla\times\underline{B}_0)\times\underline{B}_1\right\}, \tag{6.15}$$

$$\underline{E}_1 + \underline{u}_1\times\underline{B}_0 = 0, \tag{6.16}$$

$$\dot{p}_1\rho_0^{-\gamma} - \gamma p_0\rho_0^{\gamma-1}\dot{\rho}_1 = 0,$$

i.e. $$\dot{p}_1 = \frac{\gamma p_0}{\rho_0}\dot{\rho}_1 \,, \tag{6.17}$$

$$\dot{\underline{B}}_1 = \nabla\times(\underline{u}_1\times\underline{B}_0), \tag{6.18}$$

$$\nabla\cdot\underline{B}_1 = 0, \tag{6.19}$$

where dot denotes $\partial/\partial t$.

magnetic field geometry

Throughout this chapter, the magnetic field is taken as lying in the x,y-plane, making angle Θ with the x axis (similar to Boyd and Sanderson {47}). Thus

$$\underset{\sim}{B} = \hat{\underline{x}}\, B\cos\Theta + \hat{\underline{y}}\, B\sin\Theta .$$

Homogeneous plasma

In the uniform plasma, where the equilibrium quantities are constant in space (and time), the system of equations may be Fourier transformed in all dimensions. The secular determinant of the resulting system of algebraic equations then yields the familiar dispersion relation {49,50},

$$(\omega^2 - k_x^2 c_a^2 \cos^2\theta)(\omega^4 - k^2 V^2 \omega^2 + k_x^2 k^2 c_s^2 c_a^2 \cos^2\theta) = 0, \tag{6.20}$$

in which the wavevector $\underline{k}$ is taken to lie in the x,z plane, and

$$\underset{\sim}{k} = \hat{\underline{x}}\, k_x + \hat{\underline{z}}\, k_z,$$

$$c_s^2 = \frac{\gamma p_0}{\rho_0},$$

$$c_a^2 = \frac{B_0^2}{\mu_0 \rho_0},$$

$$V^2 = c_s^2 + c_a^2 .$$

The normal modes of the system, that is, the possible oscillatory solutions exp $i(\underline{k}.\underline{r} - \omega t)$, are the roots of (6.20), viz.,

$$\omega^2/k^2 = C_a^2 \cos^2\theta, \tag{6.21}$$

$$\omega^2/k^2 = \tfrac{1}{2}V^2 \pm \tfrac{1}{2}\left[V^4 - 4\frac{k_x^2}{k^2} C_s^2 C_a^2 \cos^2\theta\right]^{1/2}. \tag{6.22}$$

Equation (6.21) gives the shear Alfven wave solution; (6.22) represents the fast and slow magnetosonic modes.

Inhomogeneous plasma

A more realistic model is one in which the magnetic field is not spatially uniform. The simplest possible case is where $\underline{B}$ is plane stratified in the z-direction, ie

$$\underline{B}_0(z) = \hat{\underline{x}}\, B_{0x}(z) + \hat{\underline{y}}\, B_{0y}(z) = B_0(z)\left[\hat{\underline{x}}\cos\theta + \hat{\underline{y}}\sin\theta\right]. \tag{6.23}$$

Note that this is the simplest possible variation that satisfies $\nabla\cdot\underline{B}_0 = 0$. This imposed magnetic field now determines the permitted pressure variation by virtue of equation (6.11):

$$\hat{\underline{z}}: \quad p_0' = \frac{1}{\mu_0}\left[\hat{\underline{x}}(-B_{0y}') + \hat{\underline{y}}\, B_{0x}'\right] \times \left[\hat{\underline{x}}\, B_{0x} + \hat{\underline{y}}\, B_{0y}\right]$$

$$= \hat{\underline{z}}\, \frac{1}{\mu_0}\left[-B_{0x}' B_{0x} - B_{0y}' B_{0y}\right],$$

ie. $$p_0' + \frac{B_0 B_0'}{\mu_0} = 0,$$

ie. $$p_0(z) + \frac{|\underline{B}_0(z)|^2}{2\mu_0} = \chi_1, \quad \text{constant}. \tag{6.24}$$

Moreover, the adiabatic law (6.12) now fixes the density behaviour:

$$\rho_0(z) = \left[\frac{p_0(z)}{\chi_2}\right]^{1/\gamma}, \quad \chi_2 \text{ constant}. \tag{6.25}$$

Consequently the entire equilibrium is characterised by the field variation $\underline{B}_0(z)$ and the two constants χ_1 and χ_2.

Following on from this, the Alfven and sound speeds have the functional dependences determined by

$$c_a^2(z) = \frac{B_0^2(z)}{\mu_0 \rho_0(z)} = \frac{\chi_2^{1/\gamma}}{\mu_0} \cdot \frac{B_0^2(z)}{\left[\chi_1 - B_0^2(z)/2\mu_0\right]^{1/\gamma}}, \tag{6.26}$$

and

$$c_s^2(z) = \frac{\gamma p_0(z)}{\rho_0(z)} = \gamma \chi_2^{1/\gamma}\left[\chi_1 - B_0^2(z)/2\mu_0\right]^{1-1/\gamma}. \tag{6.27}$$

an equation for $\underline{u}_1$

Returning to the master system governing the perturbed quantities, we may eliminate all first-order terms except $\underline{u}_1$ by differentiating (6.15) with respect to time, and using (6.14), (6.17) and (6.18) to eliminate $\dot{\rho}_1$, $\dot{p}_1$ and $\dot{\underline{B}}_1$:

$$\begin{aligned} \rho_0 \ddot{\underline{u}}_1 &= -\nabla \dot{p}_1 + \frac{1}{\mu_0}\left[(\nabla\times\dot{\underline{B}}_1)\times\underline{B}_0 + (\nabla\times\underline{B}_0)\times\dot{\underline{B}}_1\right] \\ &= -\nabla(c_s^2\dot{\rho}_1) + \frac{1}{\mu_0}\left[(\nabla\times(\nabla\times(\underline{u}_1\times\underline{B}_0)))\times\underline{B}_0 + (\nabla\times\underline{B}_0)\times(\nabla\times(u_1\times B_0))\right] \\ &= \nabla\left[c_s^2(\underline{u}_1\cdot\nabla\rho_0 + \rho_0\nabla\cdot\underline{u}_1)\right] + \frac{1}{\mu_0}\left[(\nabla\times(\nabla\times(\underline{u}_1\times\underline{B}_0)))\times\underline{B}_0\right. \\ &\qquad \left.+ (\nabla\times\underline{B}_0)\times(\nabla\times(\underline{u}_1\times\underline{B}_0))\right]. \end{aligned} \tag{6.28}$$

To simplify this expression, first note that

$$0 = \nabla(p_0 \rho_0^{-\gamma}) = \rho_0^{-\gamma} \nabla p_0 - \frac{\gamma p_0}{\rho_0^{\gamma+1}} \nabla \rho_0 ,$$

so that

$$\nabla \rho_0 = \frac{1}{c_s^2} \nabla p_0 . \tag{6.29}$$

Thus the first term in (6.28) may be written

$$\nabla \left[\underline{u}_1 \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \underline{u}_1 \right] . \tag{6.30}$$

Also, using the vector relation {51},

$$\underline{a} \times (\nabla \times \underline{b}) + \underline{b} \times (\nabla \times \underline{a}) = \nabla \cdot (\underline{a} \cdot \underline{b}) - (\underline{a} \cdot \nabla)\underline{b} - (\underline{b} \cdot \nabla)\underline{a},$$

we have

$$\underline{B}_0 \times \left[\nabla \times (\nabla \times (\underline{u}_1 \times \underline{B}_0))\right] + \left[\nabla \times (\underline{u}_1 \times \underline{B}_0)\right] \times (\nabla \times \underline{B}_0)$$

$$= \nabla(\underline{B}_0 \cdot \underline{\zeta}) - (\underline{\zeta} \cdot \nabla)\underline{B}_0 - (\underline{B}_0 \cdot \nabla)\underline{\zeta} , \tag{6.31}$$

where

$$\begin{aligned} \underline{\zeta} &= \nabla \times (\underline{u}_1 \times \underline{B}_0) \\ &= \underline{u}_1 \cdot (\nabla \cdot \underline{B}_0) - \underline{B}_0 (\nabla \cdot \underline{u}_1) + (\underline{B}_0 \cdot \nabla)\underline{u}_1 - (\underline{u}_1 \cdot \nabla)\underline{B}_0 \\ &= (\underline{B}_0 \cdot \nabla)\underline{u}_1 - \underline{B}_0(\nabla \cdot \underline{u}_1) - (\underline{u}_1 \cdot \nabla)\underline{B}_0 . \end{aligned} \tag{6.32}$$

Thus combining (6.30), (6.31) and (6.32), (6.28) may be written in the form

$$\rho_0 \ddot{\underline{u}}_1 = \nabla\left[\underline{u}_1 \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \underline{u}_1 - \frac{1}{\mu_0} \underline{B}_0 \cdot \underline{\zeta}\right] + (\underline{\zeta} \cdot \nabla)\underline{B}_0 + (\underline{B}_0 \cdot \nabla)\underline{\zeta},$$

ie

$$-\omega^2 \rho_0 \underline{u}_1 = \nabla\left[\underline{u}_1 \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \underline{u}_1 - \frac{1}{\mu_0} \underline{B}_0 \cdot \underline{\zeta}\right] + (\underline{\zeta} \cdot \nabla)\underline{B}_0 + (\underline{B}_0 \cdot \nabla)\underline{\zeta} . \tag{6.33}$$

Note that since the magnetic field depends only on z, we may Fourier transform in all other dimensions.

At this stage the algebraic manipulation system REDUCE {52} is employed, in order to simplify the algebra and minimise both the effort involved, and the chance of error.

The full REDUCE output is detailed in Appendix C. However, in the main text, we confine ourselves to outlining the method of calculation, and the intermediate answers at each stage. This allows the heavy algebraic manipulations to be followed without unnecessary detail.

Equation (6.33) is first split into its components, resulting in the system of equations

$$\hat{\underline{x}}: \quad a_0 u_x + a_1 u_y + a_2 u_z' = 0, \tag{6.34}$$

$$\hat{\underline{y}}: \quad b_0 u_x + b_1 u_y + b_2 u_z' = 0, \tag{6.35}$$

$$\hat{\underline{z}}: \quad d_0 u_z + d_1 u_z' + d_2 u_z'' + d_3 u_x + d_4 u_x' + d_5 u_y + d_6 u_y' = 0, \tag{6.36}$$

where the coefficients are as follows:

$$a_0 = \omega^2 - k_x^2 c_s^2 - k^2 c_a^2 \sin^2\theta ,$$

$$a_1 = -k_x k_y c_s^2 + k^2 c_a^2 \cos\theta \sin\theta ,$$

$$a_2 = ik_x\left[c_s^2 + c_a^2 \sin^2\theta\right] - ik_y c_a^2 \cos\theta \sin\theta ,$$

$$b_0 = a_1 ,$$

$$b_1 = \omega^2 - k_y^2 c_s^2 - k^2 c_a^2 \cos^2\theta ,$$

$$b_2 = i k_y \left[c_s^2 + c_a^2 \cos^2\theta \right] - i k_x c_a^2 \cos\theta \sin\theta ,$$

$$d_0 = \omega^2 - (k_x^2 \cos^2\theta + k_y^2 \sin^2\theta + 2 k_x k_y \cos\theta \sin\theta) c_a^2 = \omega^2 - (k_x \cos\theta + k_y \sin\theta)^2 c_a^2 ,$$

$$d_1 = (2-\gamma) \tilde{c_a^2} , \quad \text{where} \quad \tilde{c_a^2} = \frac{1}{2\mu_0 \rho_0} (B_0^2)' = \frac{1}{2\rho_0} \frac{d}{dx} (\rho_0 c_a^2) ,$$

$$d_2 = c_a^2 + c_s^2 = V^2 ,$$

$$d_3 = i\, 2k_x \left[\sin^2\theta - \gamma \right] \tilde{c_a^2} - i k_y \tilde{c_a^2} \cos\theta \sin\theta ,$$

$$d_4 = a_2 ,$$

$$d_5 = i\, 2k_y \left[\cos^2\theta - \gamma \right] \tilde{c_a^2} - i k_x \tilde{c_a^2} \cos\theta \sin\theta ,$$

$$d_6 = b_2 .$$

<u>uncoupling the equations</u>

In order to arrive at independent ODEs for each velocity component, (6.34)-(6.36) may be manipulated in a straightforward manner to achieve the necessary eliminations.

Firstly, u_y can be given in terms of u_x using the procedure

$b_2 \times (6.34) - a_2 \times (6.35)$

$$= (b_2 a_0 - a_2 b_0) u_x + (a_1 b_2 - a_2 b_1) u_y = 0,$$

which yields

$$u_y = f u_x \,, \quad f = \frac{a_2 b_0 - a_0 b_2}{a_1 b_2 - a_2 b_1} \,. \tag{6.37}$$

Next, if (6.37) is substituted into (6.35), u_x can be expressed as a function of u_z' alone:

$$u_x = -\frac{b_2}{b_0 + b_1 f} u_z' \stackrel{\text{a}}{=} -g u_z' \,. \tag{6.38}$$

Finally, using both (6.37) and (6.38) in the z-component equation (6.36), a single second order equation in u_z can be found:

$$[d_2 - d_4 g - d_6 f g] u_z'' + [d_1 - d_3 g - d_4 g' - d_5 f g - d_6 (fg)'] u_z' + d_0 u_z = 0,$$

ie

$$[d_2 - (d_4 + d_6 f) g] u_z'' + [d_1 - d_3 g - d_4 g' - d_5 f g - d_6 (fg)'] u_z' + d_0 u_z = 0. \tag{6.39}$$

Note that we now have three ODEs governing the perturbed bulk velocity of the plasma in three dimensions. Equation (6.39) governs the velocity in the direction of the inhomogeneity; ODEs for the x and y components are calculated using (6.38) and (6.37) in combination with (6.39).

evaluating the coefficients

Since (6.39) is the key ODE to solve, REDUCE is again enlisted in order to calculate the coefficients of this equation. As a simplification, we set $k_y=0$, and refer to Appendix C, from which the algebra steps can be read directly.

Consider each of the coefficients in turn.

u" coefficient

From (6.39), and the Appendix, we have

$$d_2-(d_4+d_6 f)g = \{k_x^2 C_a^4 (k_x^2 C_s^2 \cos^4\theta - \omega^2\cos^2\theta) - 2k_x^2\omega^2 C_s^2 C_a^2\cos^2\theta + (C_a^2\sin^2\theta + C_s^2)\omega^4\} / \{k_x^4 C_a^2 C_s^2 \cos^4\theta - k_x^2 V^2\omega^2 + \omega^4\}, \tag{6.40}$$

which can be written as

$$\frac{1}{q}\left(\omega^2 V^2 - k_x^2 C_s^2 C_a^2 \cos^2\theta\right)\left(\omega^2 - k_x^2 C_a^2\cos^2\theta\right), \tag{6.41}$$

where

$$q = \omega^4 - k_x^2\omega^2 V^2 + k_x^4 C_s^2 C_a^2 \cos^2\theta. \tag{6.42}$$

Thus defining

$$\psi = \omega^2 - k_x^2 C_a^2 \cos^2\theta, \tag{6.43}$$

$$\xi = \omega^2 V^2 - k_x^2 C_s^2 C_a^2 \cos^2\theta, \tag{6.44}$$

we may write

$$d_2 - (d_4 + d_6 f) g = \frac{\xi \psi}{q} . \qquad (6.45)$$

u' coefficient

In order to simplify the calculation, we use the notation

$$g = \frac{1}{q}\left[i k_x \left[\omega^2 (c_s^2 + c_a^2 \sin^2\theta) - k_x^2 c_s^2 c_a^2 \cos^2\theta \right]\right] \overset{\partial}{=} \frac{h}{q} ,$$

and also

$$fg \overset{\partial}{=} j/q ,$$

where

$$j = - i k_x \omega^2 c_a^2 \sin\theta \cos\theta .$$

Then the coefficient can easily be split into two parts, as follows:

$$d_1 - d_3 g - d_5 fg - d_4 g' - d_6 (fg)'$$

$$= \frac{1}{q}\left[d_1 q - (d_3 h + d_4 h' + d_5 j + d_6 j') \right] + \frac{q'}{q^2}\left[d_4 h + d_6 j \right] .$$

We proceed by evaluating separately the coefficients of $1/q$ and q'/q^2.

From the REDUCE output,

$$d_1 q - (d_3 h + d_4 h' + d_5 j + d_6 j')$$

$$= -\tilde{c}_a^2 \Big\{ \big[2(k_x^2 c_s^2 - \omega^2)\sin^2\theta - k_x^2 c_s^2 \big] k_x^2 c_a^2 \cos^2\theta + k_x^2 \omega^2 (\gamma c_a^2 - 2 c_s^2)\sin^2\theta - (\gamma - 2)\omega^2 (k_x^2 c_a^2 - \omega^2) - 2 k_x^2 \omega^2 (c_a^2 \sin^4\theta - c_s^2) \Big\}$$

$$- 2 c_s c_s' \Big\{ (k_x c_a \cos\theta + \omega)(k_x c_a \cos\theta - \omega)(c_a^2 \sin^2\theta + c_s^2) \Big\}$$

$$- 2 c_a c_a' \Big\{ \big[(k_x c_s + \omega)(k_x c_s - \omega)\sin^2\theta \, c_a^2 + k_x^2 c_s^4 \big] \cos^2\theta - (c_a^2 \sin^4\theta + c_s^2 \sin^2\theta)\omega^2 \Big\},$$

$$= \tilde{c}_a^2 \Big\{ (2-\gamma)\omega^4 + k_x^2 \omega^2 \big[c_a^2 (\gamma - 2)\cos^2\theta - 2 c_s^2 \cos^2\theta \big] + 2 k_x^4 c_s^2 c_a^2 \cos^4\theta \Big\}$$

$$+ c_s^{2\,\prime} \Big\{ (\omega^2 - k_x^2 c_a^2 \cos^2\theta)(c_a^2 \sin^2\theta + c_s^2) \Big\}$$

$$+ c_a^{2\,\prime} \Big\{ \omega^2 V^2 \sin^2\theta - k_x^2 c_s^2 (c_a^2 \sin^2\theta + c_s^2)\cos^2\theta \Big\}.$$

For convenience, we use the notation

$$d_1 q - (d_3 h + d_4 h' + d_5 j + d_6 j') \;\hat{=}\; \varphi. \tag{6.46}$$

In a similar fashion, we use REDUCE to calculate the other part of the coefficient of u'. Thus

$$
\begin{aligned}
d_4 h + d_6 j &= k_x^2 \Big\{ (k_x^2 c_s^2 c_a^2 \cos^2\theta - c_a^2 \sin^2\theta\, \omega^2 - c_s^2 \omega^2)(c_a^2 \sin^2\theta + c_s^2) \\
&\qquad - \omega^2 c_a^4 \sin\theta \cos^2\theta \Big\} \\
&= k_x^2 \Big\{ k_x^2 c_s^2 c_a^2 \cos^2\theta\, (c_a^2 \sin^2\theta + c_s^2) \\
&\qquad - \omega^2 \left[(c_a^2 \sin^2\theta + c_s^2)(c_a^2 \sin^2\theta + c_s^2) + c_a^4 \sin^2\theta \cos^2\theta \right] \Big\} \\
&= k_x^2 \Big\{ k_x^2 c_a^2 c_s^2 \cos^2\theta\, (c_a^2 \sin^2\theta + c_s^2) \\
&\qquad - \omega^2 (c_a^4 \sin^2\theta + c_s^4 + 2 c_s^2 c_a^2 \sin^2\theta) \Big\}, \qquad (6.47)
\end{aligned}
$$

which we denote by λ.

Finally, we can write the coefficient of the first derivative as

$$\frac{\varphi}{q} + \frac{\lambda q'}{q^2} . \qquad (6.48)$$

u coefficient

The last remaining term to be calculated is the coefficient of the zeroth derivative. Clearly, from the definition of d_0 on page (98), and equation (6.43), we identify

$$d_0 = \Psi .$$

the full equation

Using equations (6.43) to (6.48), we may write the full ODE for the perturbed velocity in the direction of the inhomogeneity as

$$\frac{\xi\psi}{q} u_z'' + \left(\frac{\varphi}{q} + \frac{\lambda q'}{q^2}\right) u_z' + \psi u_z = 0,$$

or

$$\xi\psi u_z'' + \left(\varphi + \frac{\lambda q'}{q}\right) u_z' + \psi q u_z = 0. \tag{6.49}$$

Note that this equation has been generated by other authors - see for example {49,53-57}. However, few of these publications give the warm plasma ODE in as general a form as (6.49). Moreover, most interest centres on the Alfven resonance effect as a plasma heating mechanism; we will avoid specialising to that degree, and embark on as comprehensive as possible a study of the qualities of (6.49).

Special Cases

Now that the full ODE is known, special cases of the parameters may be considered in detail.

Homogeneous plasma

As a check on the accuracy of (6.49), take the magnetic field to be constant, and Fourier transform (6.49) in the z direction. This yields

$$(-k_z^2 \xi\Psi + \Psi q)\, u_z = 0,$$

Substituting for q, Ψ and ξ then gives

$$(\omega^2 - k_x^2 c_a^2 \cos^2\theta)(\omega^4 - k_x^2 V^2\omega^2 + k_x^4 c_s^2 c_a^2 \cos^2\theta - k_z^2(\omega^2 V^2 - k_x^2 c_s^2 c_a^2 \cos^2\theta))$$

$$= (\omega^2 - k_x^2 c_a^2 \cos^2\theta)(\omega^4 - k^2 V^2 \omega^2 + k_x^2 k^2 c_s^2 c_a^2 \cos^2\theta)$$

$$= 0,$$

which is the familiar dispersion relation for the homogeneous warm plasma model (see equation (6.20)).

Homogeneous, k_x=0

Note that in the particular case of k_x=0, the ODE reduces to

$$V^2 u_z'' + \omega^2 u_z = 0,$$

which on Fourier transforming, yields the dispersion relation for the propagation of the Fast Magnetosonic mode normal to the x,y-plane:

$$k_z^2 = \frac{\omega^2}{V^2}.$$

This agrees with k_x=0 in (6.20).

Inhomogeneous, k_x=0

Now consider the same scenario as above, but this time with a non-uniform magnetic field. Then (6.49) yields

$$V^2 u_z'' + (2-\gamma)\tilde{c}_0^2 u_z' + \omega^2 u_z = 0. \tag{6.50}$$

By changing the dependent variable to v, where

$$v = u_z \exp \frac{1}{2}\int^z (2-\gamma)\,\tilde{c}_0^2/V^2\, dz',$$

equation (6.50) can be written in normal form as

$$v'' + \left[\frac{\omega^2}{V^2} - \left(1-\frac{\gamma}{2}\right)\left\{\left(\frac{\tilde{c}_0^2}{V^2}\right)' + \left(\frac{\tilde{c}_0^2}{V^2}\right)^2\left(1-\gamma/2\right)\right\}\right]v = 0. \tag{6.51}$$

By so arranging the magnetic field variation that the wave potential can go to zero, (6.51) may describe a barrier problem involving at least partial reflection of the Fast Magnetosonic mode. Depending on the nature and order of the transition points, WKBJ methods may be useful in determining the transmission characteristics without the relevant analysis becoming prohibitively complicated (see comment on page 16).

In the previous two sections, we have considered uniform and non-uniform plasmas with zero wavevector in the x-direction. The consequence of k_x=0 is that u_x, u_y = 0 (from equations (6.37),(6.38) and (6.41)).

For the most general behaviour, we take $k_x \neq 0$ for the remaining analysis.

The General Case

In this section, the full ODE (6.49) is studied, with nonzero k_x. In the following analysis, there will be no asssumption of very low beta, as in {55,56}, neither will there be an arbitrary density variation independent of the pressure profile, as in {54}, since we assume the adiabatic law in equilibrium as well as in the perturbed state.

Instead, we will analyse the most general behaviour, in Cartesian coordinates, for all θ of interest.

Recall that the equation takes the form

$$\xi \psi u_z'' + \left(\varphi + \frac{\lambda q'}{q}\right) u_z' + \psi q u_z = 0.$$

Thus there are four possible singularities in this ODE, namely

$$\xi = 0 : \quad \omega^2 = k_x^2 \frac{c_s^2 c_a^2}{V^2} \cos^2\theta \tag{6.52}$$

$$\psi = 0 : \quad \omega^2 = k_x^2 c_a^2 \cos^2\theta \tag{6.53}$$

$$q = 0 : \quad \omega^2 = \tfrac{1}{2} k_x^2 V^2 \pm \tfrac{1}{2} k_x^2 \left[V^4 - \tfrac{1}{4} c_s^2 c_a^2 \cos^2\theta\right]^{1/2} \tag{6.54}$$

These 4 roots have some physical significance. In choosing $k_x \neq 0$, the bulk velocity of the plasma is now truly three dimensional, and it is the flow of plasma in directions perpendicular to the inhomogeneity which causes these singular points to occur in the differential equation.

The root defined by (6.52) is often referred to as the cusp singularity in astrophysics literature (see {57}); it is the dispersion relation for a strongly localised surface wave.

The shear Alfven wave dispersion relation is given by (6.53), again a well-known singularity in the literature {49,53-56}.

Equation (6.54) defines two singularities, corresponding to the roots of the dispersion relation for the Fast and Slow Magnetosonic modes in the plane perpendicular to the inhomogeneity (and so in the plane of the equilibrium magnetic field). Note that they are not the <u>actual</u> magnetosonic modes, since they do not involve the total wavevector.

For $\theta = 0$, the roots of q become

$$\omega^2 = k_x^2 C_a^2 \quad \text{or} \quad k_x^2 C_s^2,$$

thus sharing a root with Ψ, so that there is a singularity at the Alfven resonance of twice the previous order.

However, at $\theta = \pi/2$, only one singularity remains, viz.

$$\omega^2 = k_x^2 V^2,$$

since neither the cusp nor the Alfven root exist at this angle.

local solutions

Consider the neighbourhood $\mathcal{O}$ of a zero of q.

In this immediate locality, we may make the following approximations to arrive at a local solution for the velocity in $\mathcal{O}$.

From an initial suggestion by Cairns {58}, we assume $q \simeq z$ near each of its zeros. Then equation (6.49) reads

$$\xi \psi u_z'' + (\varphi + \tfrac{\lambda}{z}) u_z' + \psi z u_z = 0.$$

Making the further assumption that ξ, ψ, φ and λ are constant in $\mathcal{O}$, we may write (6.49) as

$$u_z'' + (a_0 + \tfrac{b_0}{z}) u_z' + c_0 z u_z = 0,$$
$$a_0 = \varphi_0/\xi_0\psi_0 \;, \quad b_0 = \lambda_0/\xi_0\psi_0 \;, \quad c_0 = 1/\xi_0 \;. \tag{6.55}$$

Eliminate the first derivative by the usual change of variable

$$v = u_z \exp \tfrac{1}{2}\int^z (a_0 + \tfrac{b_0}{z'})\, dz' ,$$

to arrive at the normal form

$$v'' = \left[- c_0 z + \tfrac{1}{4}(a_0 + \tfrac{b_0}{z})^2 - \tfrac{b_0}{2z^2} \right] v,$$

or

$$z^2 v'' + \left[A + Bz + Cz^2 + Dz^3 \right] v = 0, \tag{6.56}$$

where

$$A = \frac{b_0}{2}\left(\frac{b_0}{2}-1\right), \quad B = -\tfrac{1}{2}a_0 b_0, \quad C = -\tfrac{1}{4}a_0^2, \quad D = -c_0.$$

further approximations

(i) Discard terms higher than linear in the potential of (6.56), and solve the equation

$$z^2 v'' + [A + Bz]\,v = 0.$$

This can be transformed to a standard form (see example 251 of {42}) using changes of dependent and independent variable. Setting

$$v = z^k W, \quad k^2 - k + A = 0,$$

produces the equation

$$z^2\left[z^k W'' + 2k z^{k-1} W' + k(k-1) z^{k-2} W\right] + [A + Bz]\,z^k W = 0,$$

ie

$$z W'' + 2kW' + BW = 0.$$

This can be transformed (using example 198 of {42}) by the mapping

$$\zeta = 2\sqrt{Bz}$$

to the equation

$$\zeta \frac{d^2W}{d\zeta^2} + (4k-1)\frac{dW}{d\zeta} + \zeta W = 0,$$

which has solution

$$W = z^{1-2k} I_p(2\sqrt{Bz}), \qquad p = 1-2k$$

where $I_p(s) = i^{-p} J_p(is)$ is the modified Bessel function of the third kind, of order p.

Thus under these assumptions, the velocity behaves like

$$u_z \simeq z^{1-k-\frac{1}{2}b_0} e^{-\frac{1}{2}a_0 z} I_p(2\sqrt{Bz})$$

in the vicinity of a zero of q.

(ii) Retain the quadratic term in (6.55), and solve the equation

$$z^2 v'' + [A + Bz + Cz^2] v = 0.$$

Using the same initial change of variable as above results in the equation

$$z w'' + 2k w' + (B + Cz) w = 0.$$

Now form the new dependent variable W via

$$W = e^{-qz} w, \quad q^2 = -C = \tfrac{1}{4} a_0^2$$

to arrive at an equation for W of the form

$$z W'' + (2k + a_0 z) W' + a_0(k - \tfrac{1}{2} b_0) W = 0.$$

This is in fact one form of the confluent hypergeometric equation, with confluent hypergeometric or Pochammer-Barnes functions as (series) solutions.

Note that so far, all the approximations have depended on q going linearly to zero, and have depended intricately on the local values of ξ, ψ, φ and λ.

The approximation of a zero of order unity, while convenient, may also be poor, given the parameter dependencies defined in equations (6.26) and (6.27).

Given that q is small in $\mathscr{D}$, it should be possible to approximate (6.49) by

$$\xi_0 \psi_0 u_z'' + \left(\varphi_0 + \frac{\lambda_0 q'}{q}\right) u' \simeq 0,$$

assuming u_z remains finite. Then defining v by

$$v = u_z'$$

we can solve the equation by integrating directly:

$$v' + \left(a_0 + \frac{b_0 q'}{q}\right) v = 0,$$

ie. $$v \sim \exp - \int_{\mathscr{D}} (a_0 + b_0\, q'/q)\, ds$$

ie. $$v \sim |q|^{-b_0} \exp(-a_0 z) . \cos(b_0 \arg(q)) , \quad z \in \mathscr{D} .$$

Note that the sign of b_0 is now crucial: where $b_0 < 0$, $u_z' \to 0$ as $z \to z_{1,2}$. However, $b_0 > 0$ implies $u_z' \to \infty$, though u_z may still remain finite over $\mathscr{D}$ depending on the order of the transition point.

In all of the local solutions, the behaviour of ξ, ψ, φ and λ has been ignored. This is not a good approximation for small θ, since as $\theta \to 0$, the Alfven root approaches one of the roots of q, violating the approximation of relatively constant a_b and b_o.

Nevertheless, all of the local estimates could be of some use in enabling a numerical solution to negotiate these singularities for moderate θ .

The Alfven singularity has been studied in considerable detail by Tataronis and Grossman {56} and Chen and Hasegawa {55}. These authors found that the resonant layer at the Alfven singularity is modified by the inclusion of physical effects external to the fluid model, for instance finite ion Larmor radius and non-zero electron mass. However, these additional effects raise the order of the differential equation, and analysis close to the singularity shows that the energy absorbed at the resonant layer excites these extra wave solutions in a mode conversion process {56,23}.

Global view

In the general case, although individual local solutions for each singularity is an essential part of the analysis, the full solution must account for all singular points arising in the solution space in a global treatment.

Since most attention has focussed on the Alfven resonance in the low beta limit, the remainder of this chapter will concentrate on the preliminary analysis of the entire ODE (6.49).

eigenvalues

Note that the eigenvalues of (6.49) are given by

$$\lambda_{1,2} = \frac{\varphi + \lambda q'/q}{2\xi\psi} \pm \left[\frac{(\varphi+\lambda q'/q)^2 - 4q\psi^2\xi}{4\xi^2\psi^2}\right]^{1/2}$$

$$= [2\xi\psi]^{-1}\left\{\varphi + \frac{\lambda q'}{q} \pm \sqrt{(\varphi+\lambda q'/q)^2 - 4q\xi\psi^2}\right\}. \tag{6.57}$$

In order that there exists an oscillatory component in the solution, we must have

$$4q\xi\psi^2 > \varphi + \frac{\lambda q'}{q}.$$

Clearly this is not satisfied in the neighbourhood of a root of q. Thus we expect to see evanescent solutions in a region of solution space over which $q \lesssim 0$.

Moreover, note that q has its roots at positions z_1, z_2 such that

$$z_{1,2}: \quad \frac{\omega^2}{k_x^2} = \frac{1}{2}\left[V^2 \pm \sqrt{V^2 - 4C_s^2 C_a^2 \cos^2\theta}\right].$$

Compare these with the Alfven root z_3:

$$z_3: \quad \frac{\omega^2}{k_x^2} = C_a^2 \cos^2\theta.$$

Clearly the Alfven singularity lies between the two other roots. Thus we have the interesting case of an overdense potential barrier defined by the zeros of q, which are singular points, and containing in its interior a further singularity defining the Alfven resonance layer.

numerical work

In order to quantify this behaviour, consider the particular model with the following initial parameter values, typical of a current fusion device:

$$|\underline{B}_0| = 3{\cdot}5\,T$$

$$n_0 = 10^{22}\,m^{-3}$$

$$T = 10^6\,K$$

$$\omega/k_x = 10^6\,ms^{-1}$$

(Note that a relatively high number density has been taken in order to lessen the sensitivity of (6.24) to magnetic field variations.)

Allowing a linearly increasing magnetic field of the type

$$B_0(z) = B_0 + B_s\eta$$

where B_s = 0.1T, and $\eta = k_x z$, produces the graphs given in figures 6.1-6.5.

These plots show the eigenvalue behaviour in those regions of relevant solution space where they are purely real.

There are several points of interest to note.

1. As predicted above, the oscillations cease before reaching the first root of q (the 'fast magnetosonic'), and therefore no oscillatory solutions are present near the Alfven singularity.

2. The real part of each eigenvalue, viz

$$(\varphi + \lambda q'/q)/2\xi\psi$$

changes sign with θ, that is

$$\theta \in [0, 0\cdot 7],\ \mathcal{R}e(\lambda_{1,2}) \geqslant 0 \qquad \forall z \in [0, z_1]$$

$$\theta \in [0\cdot 8, 1\cdot 3],\ \mathcal{R}e(\lambda_{1,2}) \leqslant 0 \qquad \forall z \in [0, z_1].$$

The effect of this behaviour is shown in the graphs of the partial solutions, figure 6.6. For $\theta < 0.7$, the growth in envelope of the oscillation is greatest; for larger values of θ, the envelope grows much more slowly.

This is in keeping with the WKBJ solutions, which have an envelope factor of

$$\beta^{-1/2} \exp \int \alpha\, ds$$

if the eigenvalues take the form $\lambda_{1,2} = \alpha \pm i\beta$.

3. Note that between singularities, the eigenvalues converge then diverge. Recalling the theory of Chapter IV, and the fact that the Weber equation is a local form in the area, we might infer that mode conversion is present to some degree, with the consequence that the system inevitably finds itself in a singular eigenstate at each singular point.

4. Finally, we find that the second, smaller root of q is not present with these parameters, because of the restriction on the range of independent variable represented by (6.27): we cannot go far enough to pick up the second zero.

dispersion relation approaches

Since we have the full consistent treatment of this model, we should be able to contrast the implications of the dispersion relation approaches for the same model.

FKB

Recall the dispersion relation for the homogeneous model (6.20):

$$D = (\omega^2 - k_x^2 c_a^2 \cos^2\theta)(\omega^4 - k^2 V^2 \omega^2 + k_x^2 k^2 c_s^2 c_a^2 \cos^2\theta).$$

Extract from this an expression governing wave propagation in the direction of the proposed inhomogeneity:

$$\hat{D} = k_z^2 \left[k_x^2 c_s^2 c_a^2 \cos^2\theta - \omega^2 V^2\right] + \omega^4 + k_x^4 c_s^2 c_a^2 \cos^2\theta - k_x^2 V^2 \omega^2.$$

Now insert into the parameters the appropriate spatial dependencies, ie

$$c_s = c_s(z) \quad , \quad c_a = c_a(z) \; .$$

Now, for a mode conversion point, we have to satisfy the criteria

$$\hat{D}(k_c, z_c) = 0$$

$$\frac{\partial \hat{D}}{\partial k_z}(k_c, z_c) = 0 \quad \text{but} \quad \frac{\partial^2 \hat{D}}{\partial k_z^2}(k_c, z_c) \neq 0,$$

and

$$\frac{\partial \hat{D}}{\partial z}(k_c, z_c) \neq 0 .$$

Clearly, if

$$k_z^2 = -\frac{q}{k_x^2 c_s^2 c_a^2 \cos^2\theta - \omega^2 V^2} = \frac{q}{\xi},$$

then the roots z_1, z_2 of q satisfy all the relevant conditions, and have the appropriate critical wavenumber $k_c = 0$ (as expected for a reflection point).

Thus we have the operator (4.25):

$$\left[\frac{d^2}{dz^2} + \frac{q}{\xi}\right] \tag{6.58}$$

which acts on the energy flow.

Note that this fails to recover even the famous Alfven singularity, $\Psi = 0$, let alone the singular behaviour at q=0.

Figure 6.7 graphs the wave potential q/ξ of (6.58) as a function of position: contrast this graph with the earlier plots of eigenvalue behaviour. (Note that q/ξ is almost independent of θ).

Clearly, (6.58) is an inadequate approximation. Although FKB theory was designed to tackle wave problems involving ODEs of fourth order and above, by reducing the relevant equation to second order, in those cases where the entire problem is described by a second order ODE anyway, FKB should be able to recover the exact equation. In this instance, it has not.

CLD

In reference {28}, the authors cite a Russian paper on wave transformation {59} as an example of the excellent agreement of the coupled equation approach with the fully worked solution. The example chosen was mode conversion involving the fast and slow magnetosonic modes in a plane stratified magnetic field.

However, the fourth order ODE derived by the authors Moiseev and Smilyanskii is in fact wrong, since its construction depends on the assumption of an isothermal law,

$$p_0 \rho_0 = \text{constant}$$

together with the imposed equilibrium

$$\underset{\sim}{H}_0 = \hat{z}\, H_{0z}\ , \quad \text{constant}$$

$$\rho_0 = \rho_0(z).$$

That these conditions are mutually contradictory is clear from (6.10) and (6.11), since H_0 = constant $\Rightarrow p_0, \rho_0$ = constant.

That the coupled mode theory agrees with the conversion coefficients so derived is a warning that such approaches may infer mode conversion where it is not actually permitted by the governing master equations of the model.

In a final comment, note how difficult the actual model equations are compared to the ideal equations of the Swanson technique; given the parameter variations in this model (see (6.26) and (6.27)), it would be impractical to attempt to extract either of (4.22) and (4.23) from any 4th order (partial) differential equation arising.

Summary

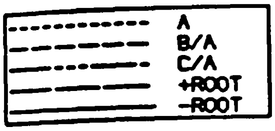
This chapter has set out the equations for an inhomogeneous warm plasma model with a plane stratified, unidirectional magnetic field.

Using only these equations, the most general case of wave propagation in the direction of the non-uniformity was constructed, without restriction to low beta plasmas or non-zero wavevector along the equilibrium magnetic field.

The resulting second order ODE describing the fluid velocity is found to have 4 singularities, two of which can have a profound effect on the nature of the Alfven resonant layer, in that their presence excludes the possibility of wave motion reaching the Alfven resonant layer.

Finally, the contrasting results of using parameterised dispersion relations to describe wave motion serve as a warning of the restricted validity of such theories.

:----------------------------:



(6·49) WRITTEN $Au'' + Bu' + Cu = 0$

$+\text{ROOT} = -\frac{B}{2A} + \sqrt{\left(\frac{B^2}{4A^2} - \frac{C}{A}\right)}$

$-\text{ROOT} = \quad '' \quad - \quad ''$

FIGURE 6·1

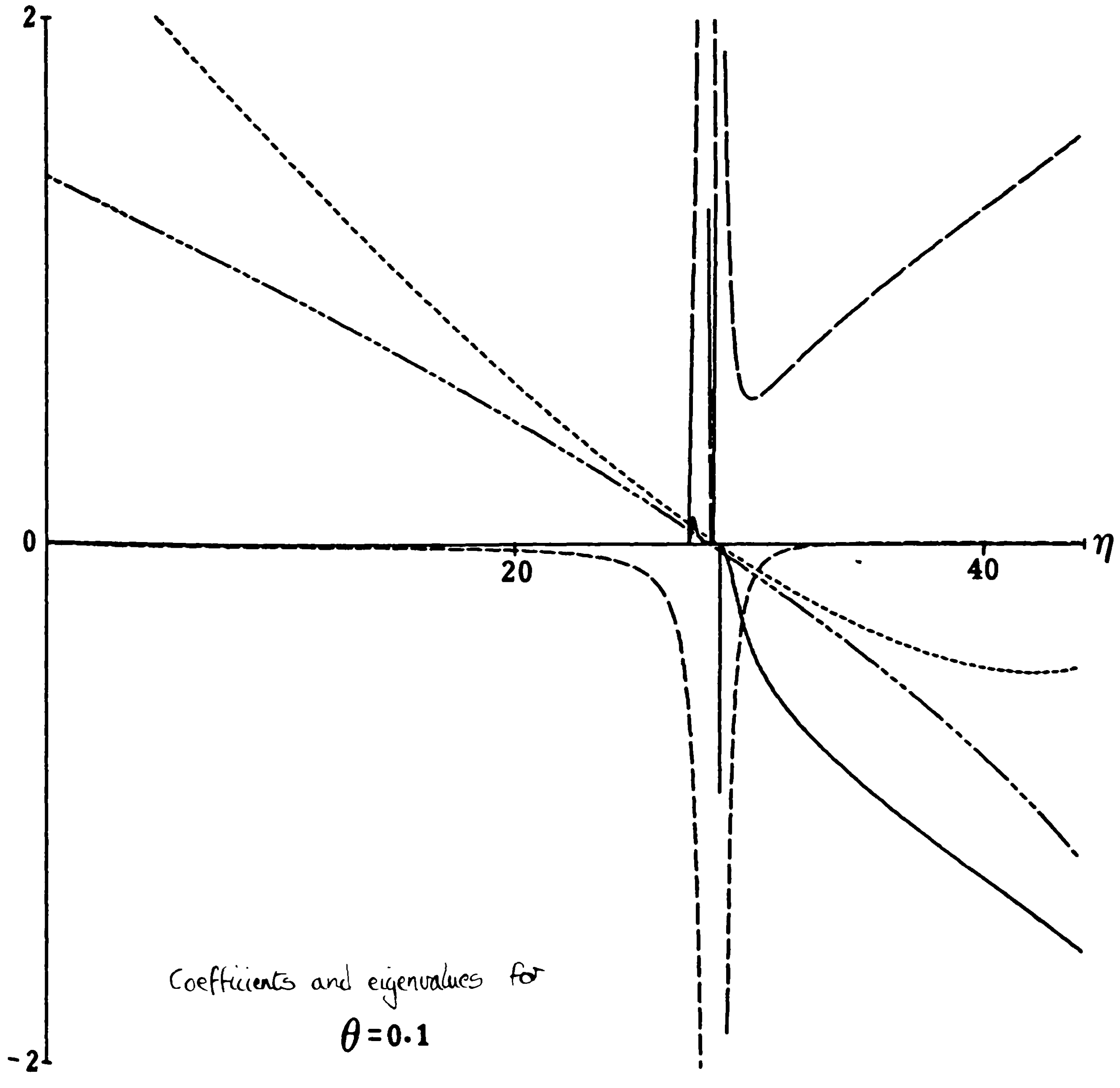
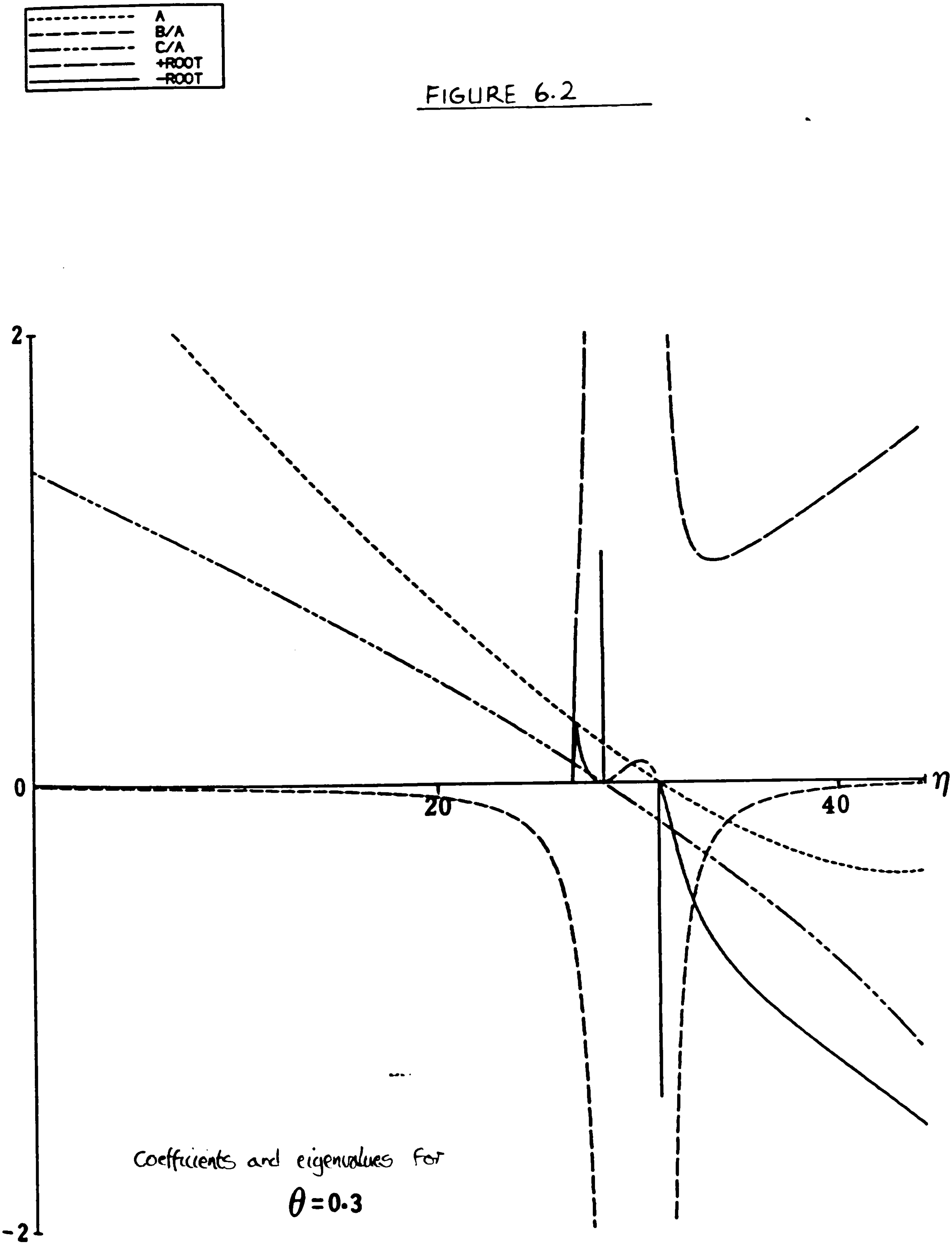


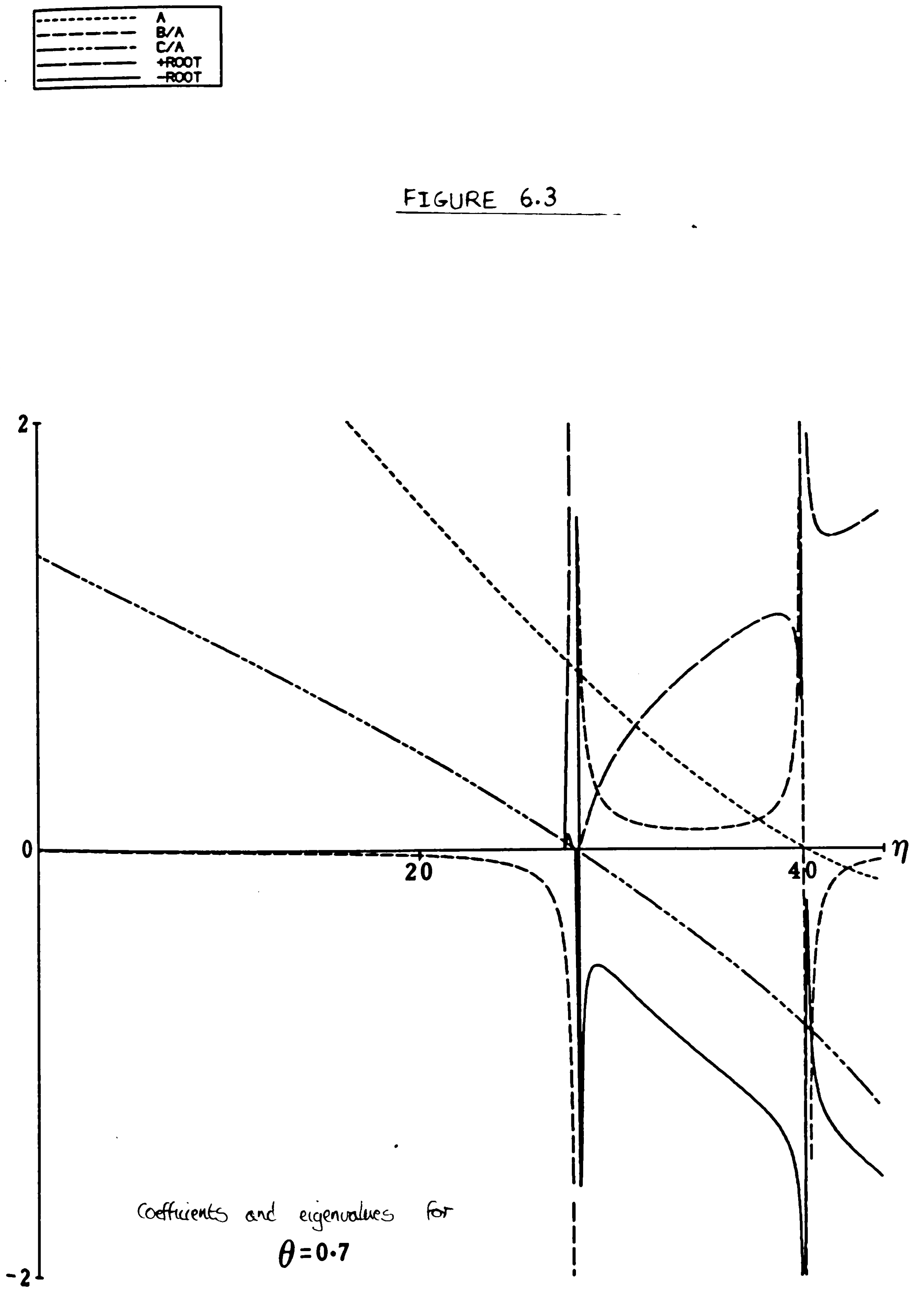
FIGURE 6.2



Coefficients and eigenvalues for

$\theta = 0{\cdot}3$

FIGURE 6.3



Coefficients and eigenvalues for

$\theta = 0\cdot7$

FIGURE 6.4

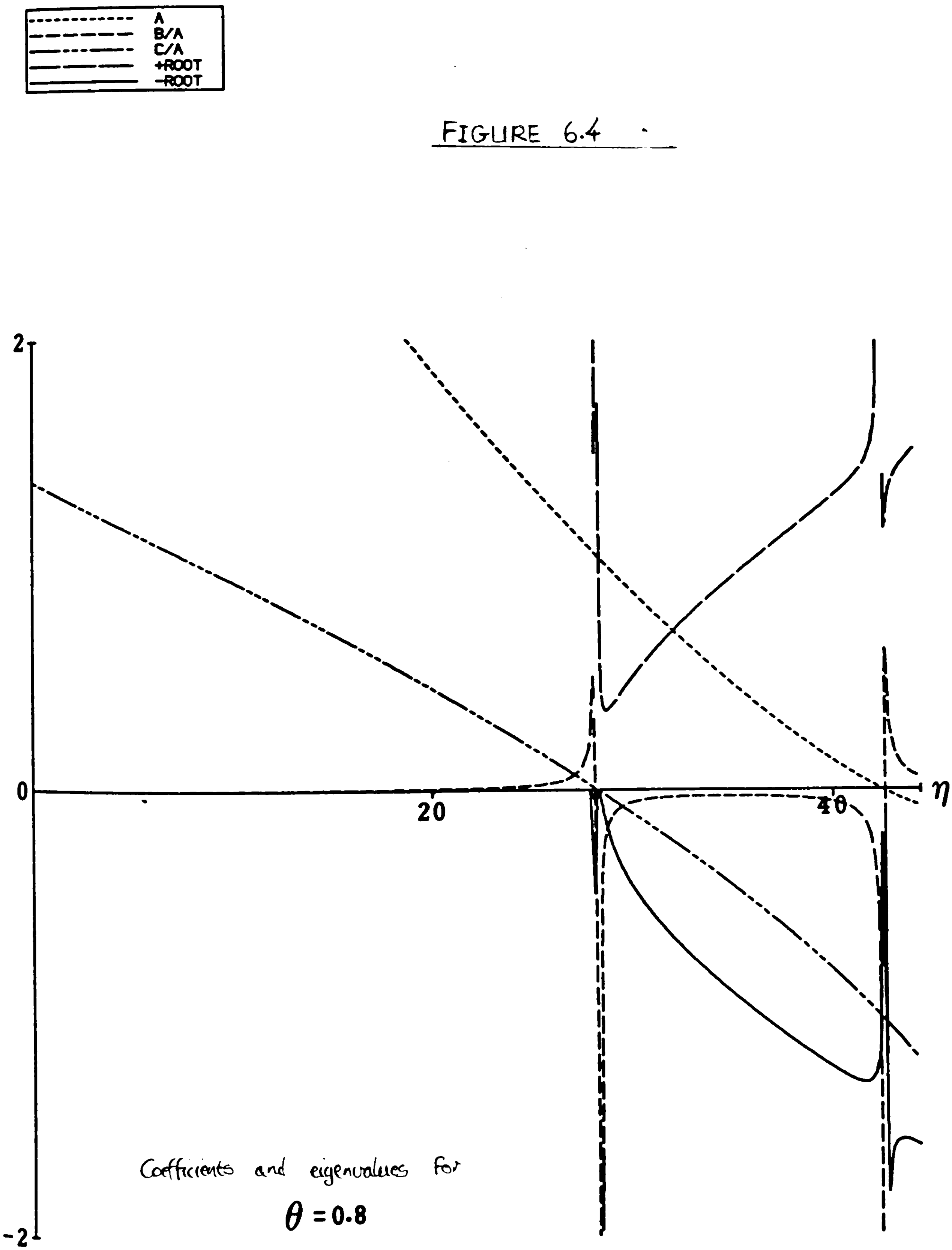
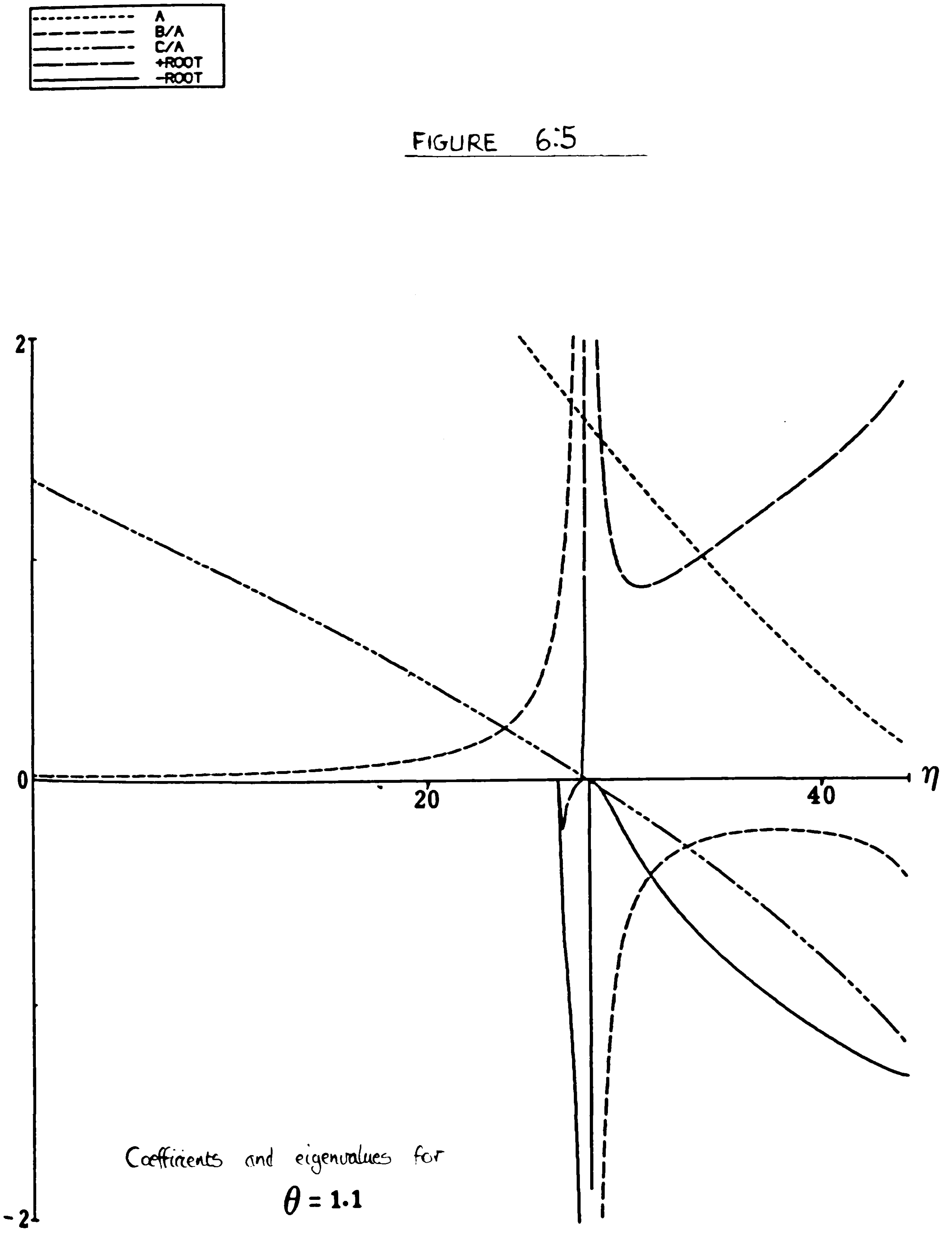

A
B/A
C/A
+ROOT
−ROOT
2
0
-2
20
40
η
Coefficients and eigenvalues for
θ = 0.8

FIGURE 6:5



Coefficients and eigenvalues for

$\theta = 1.1$

FIGURE 6.6

Solutions of (6.49) for various θ values

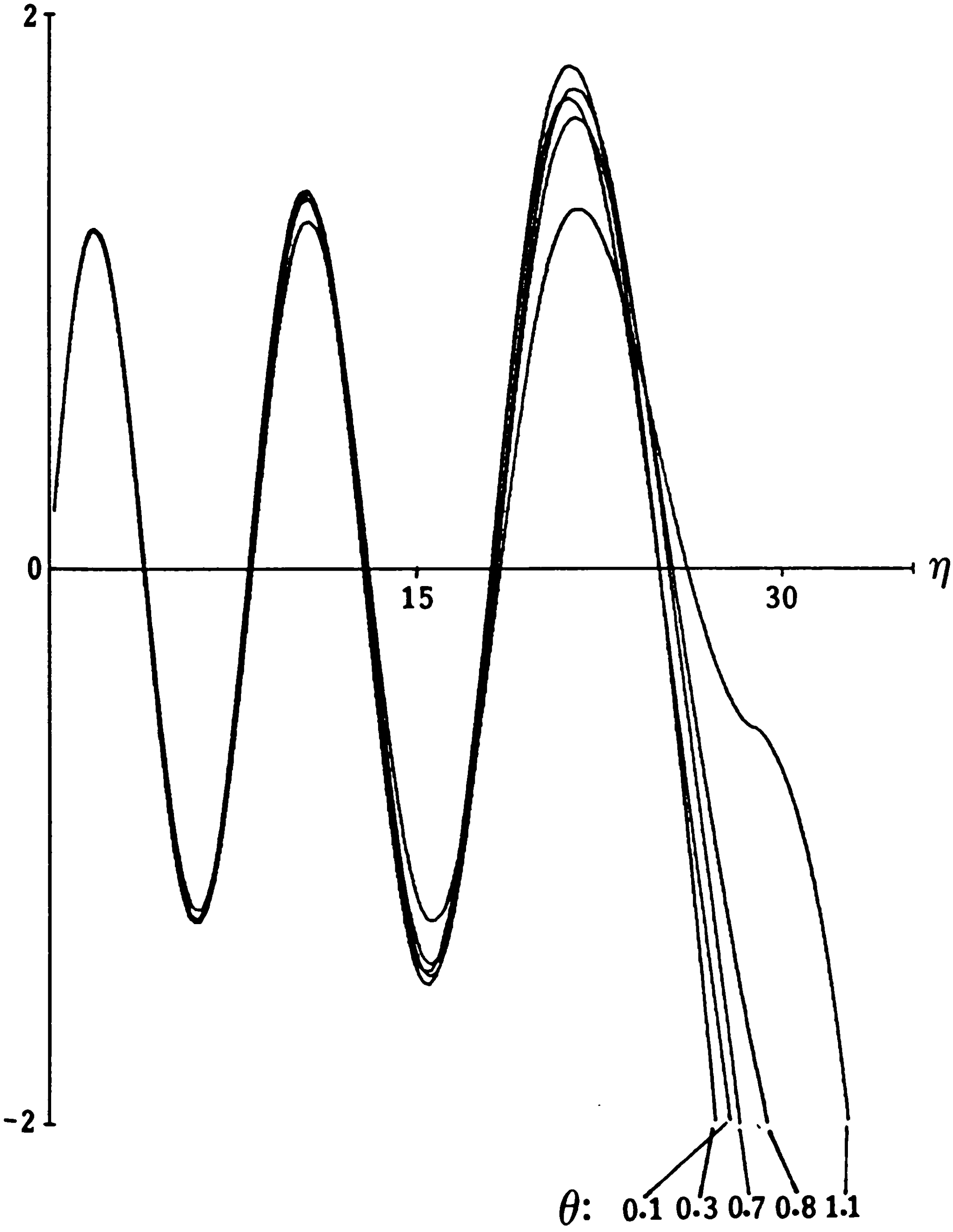
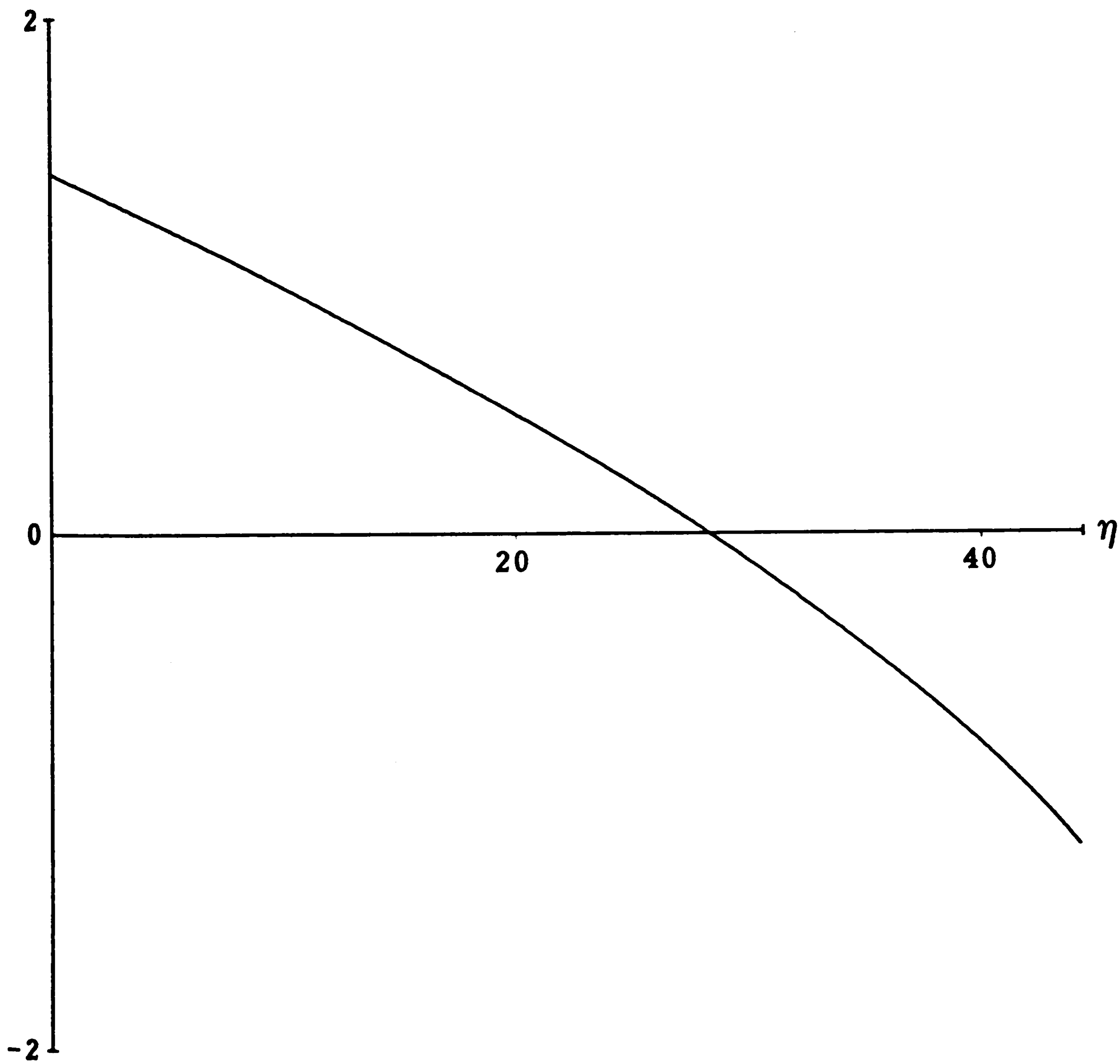


FIGURE 6.7

Wave potential derived from FKB



Chapter VII Conclusions and Possible Future Work

Conclusions

This thesis has been concerned mainly with the rigorous description of wave propagation in non-uniform media, in particular, with the analysis of linear mode conversion in an inhomogeneous plasma.

Most theoretical approaches to mode conversion have relied on the accuracy of the spatially dependent dispersion relation as a means of deriving the behaviour of waves in a mode conversion region. We have shown that this is not an acceptable general technique, by supplying explicit examples where this method fails to recover the characteristics derived from rigorous analysis. The waveguide of varying cross-section and the warm plasma model are prime evidence against using reverse Fourier transform techniques in such circumstances.

The main difficulty in assessing the accuracy of the various mode conversion theories and their predictions has been that they are applied to wave propagation examples which are extremely complicated, and in which a rigorous analytic solution is impratical. Since numerical models are not yet sufficiently sophisticated to predict mode conversion coefficients for realistic tokamak simulations, we can only analyse the accuracy of such theoretical work by critical comparison with simpler, but more rigorous, examples.

This thesis has presented several such cases, and the conclusion must be that the three main theories of mode conversion depend too much on simple assumptions, and pay little regard to the actual effects of the inhomogeneities. By including parameter gradients, Chapter VI reworks special cases in mode conversion to show how the coupling potential arises naturally and self-consistently.

It cannot be emphasised too strongly that the accuracy of any asymptotic solution of a wave problem is crucially dependent on knowing the exact wave potential in the interaction region. Thus theories which postulate approximate scattering potentials and then embark on asymptotic expansions may yield seriously inaccurate results (consider the waveguide problem, for example).

Moving on to the warm plasma model, it is clear that including the inhomogeneities at the earliest possible stage (that is, in the master equations for the model) is the only way to guarantee rigorous analysis.

Moreover, computer algebra is clearly one of the most powerful computational tools available, allowing very complicated and tedious analysis to be tackled with maximum accuracy and confidence.

The fact that analysis using REDUCE allowed a full study of the warm plasma model with a plane stratified magnetic field is evidence of this, particularly when extra physical effects are discovered, such as the possibility of the Alfven resonance layer being inaccessible to oscillatory solutions of the equations.

The extra singularities in this model for general propagation may prove very significant in plasma heating schemes, particularly since most use the fast magnetosonic mode as the mode conversion candidate.

Future work

There are two main obstacles to progress in the field of mode conversion: the dependence on the Weber equation, and the restriction to one independent variable.

The former arises as a result of confining attention to binary conversion events, which reduces the order of the differential to two (in this context). The next step is to assume linear transition points and proceed with the asymptotic solution via the well-known parabolic expansions.

For more realistic cases, such as the mhd problem, the parabolic cylinder equation may be a poor comparison equation. Moreover, physical problems are likely to involve partial differential equations.

It may be possible to convert an approach of Chapter V to cope with more than one independent variable by extending the matrices to tensors, and incorporating many more cross-coupling terms.

Analysis like this may also make any numerical attack easier to implement and interpret, thus freeing further progress from impractical restrictions.

In addition, the q=0 singularities of the warm plasma model deserve much more detailed analysis. The concept of the Alfven resonance being trapped within a potential barrier is intriguing, and will require a numerical solution to fully exploit its consequences.

Again, it should be possible to exploit the power of computer algebra in all these endeavours, both in the analysis, and in the construction of computer codes for quantitative simulations.

Appendix A

Calculations for the non-uniform waveguide (Chapter III)

For convenience, we quote the following standard integrals from Gradshteyn and Ryzhik {19}:

using the notation

$$R = a + bx + cx^2 , \quad \Delta = 4ac - b^2 :$$

(2.261)
$$\int \frac{dx}{\sqrt{R}} = \frac{1}{\sqrt{c}} \ln\left[2\sqrt{cR} + 2cx + b\right] , \quad c>0$$

$$= \frac{-1}{\sqrt{-c}} \sin^{-1}\left[\frac{2cx+b}{\sqrt{-\Delta}}\right] , \quad c<0, \Delta<0.$$

(2.266)
$$\int \frac{dx}{x\sqrt{R}} = -\frac{1}{\sqrt{a}} \ln\left[\frac{1}{x}\left(2a+bx+2\sqrt{aR}\right)\right] , \quad a>0$$

$$= \frac{1}{\sqrt{-a}} \sin^{-1}\left[\frac{2a+bx}{x\sqrt{-\Delta}}\right] .$$

(2.281)
$$\int \frac{dx}{(x+p)^n\sqrt{R}} = -\int \frac{t^{n-1}\,dt}{\sqrt{c+(b-2pc)t+(a-bp+cp^2)t^2}} , \quad t = 1/(x+p)$$

(2.282(1))
$$\int \frac{\sqrt{R}\,dx}{x+p} = c\int \frac{x\,dx}{\sqrt{R}} + (b-cp)\int \frac{dx}{\sqrt{R}} + (a-bp+cp^2)\int \frac{dx}{(x+p)\sqrt{R}} .$$

(2.282(3)) $$\int \frac{\sqrt{R}\,dx}{(x+p)(x+q)} = \frac{1}{q-p}\int \frac{\sqrt{R}\,dx}{x+p} + \frac{1}{p-q}\int \frac{\sqrt{R}\,dx}{x+q}.$$

---

evaluation of indefinite integrals

We calculate the indefinite integral I defined in equation (3.9),

$$I = \frac{1}{2}\int \frac{\sqrt{R}}{y(y-\theta)}\,dy,$$

$$R = K^2y^2 - 2\theta(1+K^2)y + (K^2-1)\theta^2 \triangleq cy^2+by+a.$$

Now by 2.282(3),

$$2I = \frac{1}{\theta}\int \frac{\sqrt{R}\,dy}{(y-\theta)} - \frac{1}{\theta}\int \frac{\sqrt{R}\,dy}{y} \triangleq I_1 - I_2.$$

Using 2.282(1) we have

$$I_1 = c\int \frac{y\,dy}{\sqrt{R}} + (b+c\theta)\int \frac{dy}{\sqrt{R}} + (a+b\theta+c\theta^2)\int \frac{dy}{(y-\theta)\sqrt{R}}$$

$$I_2 = c\int \frac{y\,dy}{\sqrt{R}} + b\int \frac{dy}{\sqrt{R}} + a\int \frac{dy}{y\sqrt{R}}.$$

so that

$$2\theta I = c\theta\int \frac{dy}{\sqrt{R}} + (a+b\theta+c\theta^2)\int \frac{dy}{(y-\theta)\sqrt{R}} - a\int \frac{dy}{y\sqrt{R}}.$$

Now

$$a+b\theta+c\theta^2 = (K^2-1)\theta^2 - 2\theta^2(1+K^2) + K^2\theta^2 = -3\theta^2,$$

Thus

$$2I = K^2\int \frac{dy}{\sqrt{R}} - 3\theta\int \frac{dy}{(y-\theta)\sqrt{R}} - (K^2-1)\theta\int \frac{dy}{y\sqrt{R}}.$$

Taking each integral in turn,

$$\int \frac{dy}{\sqrt{R}} = \frac{1}{\kappa} \ln\left[2\kappa\sqrt{R} + 2\kappa^2 y - 2\theta(1+\kappa^2)\right].$$

$$\int \frac{dy}{(y-\theta)\sqrt{R}} = -\int \frac{dt}{\sqrt{S}}, \quad t = \frac{1}{y-\theta}, \; S = \kappa^2 - 2\theta t - 3\theta^2 t^2$$

$$= \frac{1}{\sqrt{3\theta^2}} \sin^{-1}\left[\left(\frac{1+3\theta t}{\Gamma}\right)\right], \quad \Gamma = \sqrt{1+3\kappa^2}.$$

$$\int \frac{dy}{y\sqrt{R}} = -\frac{1}{\theta\sqrt{\kappa^2-1}} \ln\left[\frac{2\theta}{y}\left(\kappa^2-1-(1+\kappa^2)y + \sqrt{(\kappa^2-1)R}\right)\right], \quad \kappa > 1.$$

Finally, combining these calculations yields

$$2I = \kappa \ln\left[2\left(\kappa\sqrt{R} + \kappa^2 y - \theta(1+\kappa^2)\right)\right]$$
$$- \sqrt{3}\sin^{-1}\left[-\frac{1+3\theta t}{\Gamma}\right]$$
$$+ (\kappa^2-1)^{1/2} \ln\left[\frac{2\theta}{y}\left(\kappa^2-1-(1+\kappa^2)y + \sqrt{(\kappa^2-1)R}\right)\right], \quad \kappa > 1.$$

reflection and transmission coefficients

The first step is to evaluate the integral of (3.17), viz.

$$\mathcal{J} = \frac{1}{2}\int \frac{\sqrt{\mathcal{R}}}{y(y-\theta)}\, dy, \quad \mathcal{R} = -R.$$

As before, we have

$$2\mathcal{J} = -\kappa^2 \int \frac{dy}{\sqrt{\mathcal{R}}} + 3\theta \int \frac{dy}{(y-\theta)\sqrt{\mathcal{R}}} + (\kappa^2-1)\theta \int \frac{dy}{y\sqrt{\mathcal{R}}}.$$

Again, following the procedure for I, each integral is evaluated in turn:

$$\int \frac{dy}{\sqrt{R}} = -\frac{1}{\kappa} \sin^{-1}\left[\frac{\theta(1+\kappa^2)-\kappa^2 y}{\theta\Gamma}\right]$$

$$\int \frac{dy}{(y-\theta)\sqrt{R}} = -\int \frac{dt}{\sqrt{\delta}} \quad , \quad \delta = -\kappa^2+2\theta t+3\theta^2 t^2$$

$$= -\frac{1}{\sqrt{3\theta^2}} \ln\left[2\sqrt{3\theta^2\delta}+6\theta^2 t+2\theta\right]$$

$$\int \frac{dy}{y\sqrt{R}} = -\frac{1}{\theta\sqrt{1-\kappa^2}} \ln\left[\frac{2(1-\kappa^2)\theta^2+2\theta(1+\kappa^2)y+2\theta\sqrt{(1-\kappa^2)R}}{y}\right] \quad , \kappa^2<1$$

$$= \frac{1}{\theta\sqrt{\kappa^2-1}} \sin^{-1}\left[\frac{(1+\kappa^2)y-\theta(\kappa^2-1)}{\Gamma y}\right] \quad , \kappa^2>1.$$

Thus we have that

$$2\mathcal{J} = \kappa\sin^{-1}(\alpha_1) - \sqrt{3}\ln(\alpha_2) + \begin{cases}(1-\kappa^2)^{1/2}\ln(\alpha_3) & \kappa<1 \\ (\kappa^2-1)^{1/2}\sin^{-1}(\alpha_4), & \kappa>1\end{cases}$$

where the arguments α_i are given as

$$\alpha_1 = [\theta(1+\kappa^2)-\kappa^2 y]/(\theta\Gamma)$$

$$\alpha_2 = 2\theta\left[\sqrt{3\delta}+3\theta t+1\right]$$

$$\alpha_3 = 2\theta\left[(1-\kappa^2)\theta+(1+\kappa^2)y+\sqrt{(1-\kappa^2)R}\right]/y$$

$$\alpha_4 = \left[(1+\kappa^2)y-\theta(\kappa^2-1)\right]/(\Gamma y).$$

The algebra may be simplified as follows. Noting that

$$\theta_{min} = \frac{\kappa^2}{1+\kappa^2+\Gamma} = \frac{1+\kappa^2-\Gamma}{\kappa^2-1} ,$$

then α_1 and α_4 can be written in terms of θ_{min}:

$$\alpha_1 = \frac{1+\kappa^2}{\Gamma} - \frac{\kappa^2 y}{\Gamma} \Rightarrow y = \frac{\theta(1+\kappa^2)}{\kappa^2} - \frac{\Gamma\theta}{\kappa^2}\alpha_1$$

$$= \theta\left[\frac{1}{\theta_{min}} - \frac{\Gamma}{\kappa^2}\right] - \frac{\Gamma\theta}{\kappa^2}\alpha_1 .$$

so

$$\underline{\alpha_1 = \frac{\kappa^2}{\Gamma}\left[\frac{1}{\theta_{min}} - \frac{y}{\theta}\right] - 1} .$$

$$\alpha_4 = \frac{1+\kappa^2}{\Gamma} - \frac{\theta(\kappa^2-1)}{\Gamma y} \Rightarrow y = \frac{\theta(\kappa^2-1)}{1+\kappa^2-\Gamma\alpha_4}$$

$$\Rightarrow \frac{1}{y} = \frac{1+\kappa^2-\Gamma+(1-\alpha_4)\Gamma}{\theta(\kappa^2-1)} = \frac{\theta_{min}}{\theta} + \frac{(1-\alpha_4)\Gamma}{\theta(\kappa^2-1)}$$

so

$$\alpha_4 = 1 - \frac{\kappa^2-1}{\Gamma}\left[\frac{\theta}{y} - \theta_{min}\right] .$$

The next stage is to evaluate $\mathcal{g}$ at $x=0$ and $x=x_c$.

x=0:

$$y = 1 \qquad t = \frac{1}{1-\theta}$$

$$\alpha_1 = \frac{\kappa^2}{\Gamma}\left[\frac{1}{\theta_{min}} - \frac{1}{\theta}\right] - 1$$

$$\mathcal{g} = -\kappa^2 + \frac{2\theta}{1-\theta} + \frac{3\theta^2}{(1-\theta)^2} = \frac{1}{(1-\theta)^2}\left[-\kappa^2(1-\theta)^2 + 2\theta(1-\theta) + 3\theta^2\right]$$

$$= -\frac{\kappa^2(1-\theta)^2 + 2\theta + \theta^2}{(1-\theta)^2}$$

So

$$\alpha_2 = \frac{2\theta}{1-\theta}\left[\sqrt{(3(\theta^2+2\theta-\kappa^2(1-\theta)^2)} + 2\theta + 1\right] .$$

$$\alpha_3 = 2\theta\left[(1-K^2)\theta + (1+K^2) + (1-K^2)^{1/2}\left[2\theta - ((K+1)\theta - K)((K-1)\theta - K)\right]^{1/2}\right].$$

$$\alpha_4 = 1 - \frac{K^2-1}{\gamma}\left[\theta - \theta_{min}\right].$$

—— // ——

$x = x_c$: $\quad y = \theta/\theta_{min} \qquad t = \dfrac{\theta_{min}}{\theta(1-\theta_{min})}$.

$$\alpha_1 = -1.$$

Note that $\mathcal{R}(\theta/\theta_{min}) = \mathcal{J}(\theta/\theta_{min}) = 0$, by definition. Hence

$$\alpha_2 = 2\theta\left[\frac{3\theta\,\theta_{min}}{\theta(1-\theta_{min})} + 1\right] = \frac{2\theta}{(1-\theta_{min})}\left[2\theta_{min} + 1\right].$$

$$\alpha_3 = 2\theta\left[(1-K^2)\,\theta + (1+K^2)\frac{\theta}{\theta_{min}}\right]\cdot\frac{\theta_{min}}{\theta}$$

$$= 2\theta\left[(1-K^2)\theta_{min} + (1+K^2)\right] = 2\theta\Gamma.$$

$$\alpha_4 = 1.$$

Consequently,

$$2\left[g(x=0) - g(x=x_c)\right] = K\left\{\sin^{-1}\left[\frac{K^2}{\gamma}\left[\frac{1}{\theta_{min}} - \frac{1}{\theta}\right] - 1\right] + \frac{\pi}{2}\right\}$$

$$- \sqrt{3}\ln\left\{\frac{1-\theta_{min}}{1-\theta}\left[\frac{\sqrt{(3(\theta^2+2\theta-K^2(1-\theta)^2)} + 2\theta + 1}{2\theta_{min}+1}\right]\right\}$$

$$\begin{cases} K<1 & + (1-K^2)^{1/2}\ln\left[\frac{1}{\Gamma}\left((1-K^2\theta) + 1 + K^2 + (1-K^2)^{1/2}\sqrt{2\theta - ((K+1)\theta - K)((K-1)\theta - K)}\right)\right] \\ K>1 & + (K^2-1)^{1/2}\left[\sin^{-1}\left(1 - \frac{K^2-1}{\gamma}(\theta - \theta_{min})\right) - \frac{\pi}{2}\right] \end{cases}$$

Thus

$$\{a,-a\} = \begin{cases} A^{\sqrt{3}}\, e^{\lambda+\mu} & \kappa > 1 \\ A^{\sqrt{3}}\, B^{\sqrt{1-\kappa^2}}\, e^{\mu} & \kappa < 1 \end{cases}$$

$$A = \frac{1-\theta}{1-\theta_{min}} \left[\frac{2\theta_{min}+1}{3\sqrt{\left(\theta^2+2\theta-\kappa^2(1-\theta)^2\right)}+2\theta+1} \right]$$

$$\lambda = \kappa\left[\sin^{-1}\left(\frac{\kappa^2}{\Gamma}\left(\frac{1}{\theta_{min}}-\frac{1}{\theta}\right)-1\right)+\frac{\pi}{2}\right]$$

$$\mu = (\kappa^2-1)^{1/2}\left[\sin^{-1}\left(1-\frac{\kappa^2-1}{\Gamma}(\theta-\theta_{min})\right)-\frac{\pi}{2}\right]$$

$$B = \frac{1}{\Gamma}\left\{(1-\kappa^2)\theta+1+\kappa^2+(1-\kappa^2)^{1/2}\sqrt{2\theta-\left[(\kappa+1)\theta-\kappa\right]\left[(\kappa-1)\theta-\kappa\right]}\right\}.$$

Appendix B

expansions of the Parabolic Cylinder Functions

The Weber equation

$$\frac{d^2 D_k}{dz^2} + \left[k+\tfrac{1}{2}-\tfrac{1}{4}z^2\right] D_k = 0$$

is satisfied by the parabolic cylinder functions D_k, which have the following asymptotic expansions {43}

$|\arg z| < \frac{3\pi}{4}$:

$$D_k(z) \sim e^{-\frac{1}{4}z^2} z^k \left\{1 - \frac{k(k-1)}{2z^2} + \frac{k(k-1)(k-2)(k-3)}{2\cdot 4\cdot z^4} - \dots\right\}$$

$\arg z \in \left]\frac{\pi}{4}, \frac{5\pi}{4}\right[$:

$$D_k(z) \sim e^{-\frac{1}{4}z^2} z^k \left\{1 - \frac{k(k-1)}{2z^2} + \frac{k(k-1)(k-2)(k-3)}{2\cdot 4\cdot z^4} - \dots\right\}$$

$$- \frac{\sqrt{2\pi}}{\Gamma(-k)} e^{i\pi k} e^{\frac{1}{4}z^2} z^{-k-1} \left\{1 + \frac{(k+1)(k+2)}{2z^2} + \frac{(k+1)(k+2)(k+3)(k+4)}{2\cdot 4\cdot z^4} \dots\right\}.$$

expansion for linear eigenvalues

Note that here,

$$k = -\tfrac{1}{2}\left(1+\tfrac{a_1}{b_1}\right) \qquad \eta = (4b_1^2)^{1/4} z$$

$b_1 \in \mathbb{R}$: $\arg(\eta) = \arg(z)$, since $\arg(\eta) = \arg\left((4b_1^2)^{1/4} z\right)$.

$|z| >> 0$:
$$W \sim \eta^{-1/2\,(1+\frac{a_1}{b_1})} e^{-\frac{1}{4}\eta^2} = \left(\sqrt{2b_1}\, z\right)^{-\frac{1}{2}(1+\frac{a_1}{b_1})} e^{-\frac{1}{2}b_1 z^2}$$

and so

$$y = We^{\frac{1}{2}\int^{z}\Sigma\,ds} = \{(4b_1^2)^{1/4}z\}^{-\frac{1}{2}(1+a_1/b_1)} \times e^{\frac{1}{2}(2a_0z+a_1z^2)}$$

i.e. $z \gg 0$: $y \sim [\sqrt{2b_1}\,z]^k\, e^{\int^z \lambda_2\, ds}$.

Similarly,

$|z| \ll 0$:

$$\frac{\sqrt{2\pi}}{\Gamma(-k)}\, e^{i\pi k} \cdot e^{\frac{1}{4}\eta^2} \cdot z^{-(k+1)} = \frac{\sqrt{2\pi}}{\Gamma(-k)} \cdot e^{i\pi k}\, e^{\frac{1}{4}\eta^2} \cdot |z|^{-k-1}\, e^{i\pi(-k-1)}$$

$$= -\frac{\sqrt{2\pi}}{\Gamma(-k)} \cdot e^{\frac{1}{4}\eta^2}\, |z|^{-k+1} .$$

So

$$D_k \sim e^{-\frac{1}{4}\eta^2}|\eta|^k e^{i\pi k} + \frac{\sqrt{2\pi}}{\Gamma(-k)}\, e^{\frac{1}{4}\eta^2}|\eta|^{-k+1)}$$

Thus finally,

$$y \sim |\sqrt{2b_1}\,z|^k e^{i\pi k} \exp\int^z \lambda_2(s)\,ds$$

$$+ \frac{\sqrt{2\pi}}{\Gamma(-k)} \cdot |\sqrt{2b_1}\,z|^{-(k+1)} \exp\int^z \lambda_1(s)\,ds .$$

expansion for linear $\pm\sqrt{(\text{quadratic})}$

Here,

$$k + \tfrac{1}{2} = \frac{a_1 - \beta_0^2}{2\sqrt{b_2}}$$

Assume $\beta^2(z) < 0 \ . \ \forall z \ : \ \rightarrow b_2 < 0, \quad b_i \in \mathbb{R}, \quad i = 0,1,2.$

Then $\qquad a_0 = i\alpha_0, \quad a_1 = i\alpha_1.$

$$\eta = (4b_2)^{1/4}(z - z_0) \quad , \quad z_0 = -\frac{b_1}{2b_2} \in \mathbb{R}.$$

$z > z_0 \ : \quad \arg(\eta) = \arg(b_2^{1/4}) = \arg(-1)^{1/4} = \pi/4$

$z < z_0 \ : \quad \arg(\eta) = \pi/4 - \pi = -3\pi/4.$

$|z| >> z_0$:

$$W \sim e^{-\frac{1}{4}\eta^2}\eta^k$$

i.e. $$W(z) \sim e^{-\frac{1}{4}(4b_2)^{1/2}(z-z_0)^2}\left[(4b_2)^{1/4}(z-z_0)\right]^k.$$

$|z| << z_0$: $\quad W(\eta) \sim e^{-\frac{1}{4}\eta^2}\eta^k - \frac{\sqrt{2\pi}}{\Gamma(-k)}e^{-ik\pi}e^{\frac{1}{4}\eta^2}\eta^{-k-1}$

i.e. $$W(z) \sim e^{-\frac{1}{4}(4b_2)^{1/2}(z-z_0)^2}\left[(4b_2)^{1/4}(z-z_0)\right]^k$$

$$- \frac{\sqrt{2\pi}}{\Gamma(-k)}e^{-ik\pi}e^{\frac{1}{4}(4b_2)^{1/2}(z-z_0)^2}\left((4b_2)^{1/4}(z-z_0)\right)^{-(k+1)}.$$

Thus we have

$\eta >> |k+\tfrac{1}{2}|$:

$$y(z) \sim I_+ (z-z_0)^k \exp\left(\int^z \alpha\, ds - \tfrac{1}{2}\sqrt{b_2}\,(z-z_0)^2\right),$$

$\eta \ll -|k+\tfrac{1}{2}|$

$$y(z) \sim \quad I_- \, |z-z_0|^k \exp\left(\int^z \alpha\, ds - \tfrac{1}{2}\sqrt{b_2}\,(z-z_0)^2 \right)$$

$$+ B \, |z-z_0|^{-(k+1)} \exp\left[\int^z \alpha\, ds + \tfrac{1}{2}\sqrt{b_2}\,(z-z_0)^2 \right] ,$$

with

$$I_+ = (4b_2)^{k/4}$$

$$I_- = (4b_2)^{k/4} e^{ik\pi}$$

$$B = \frac{\sqrt{2\pi}}{\Gamma(-k)} \cdot e^{-i\,2k\pi} \quad (4b_2)^{-\frac{1}{4}(k+1)} .$$

Appendix C

REDUCE listing and output

The first part of this Appendix lists a computer code written in REDUCE 2 for the calculation of coefficients in the warm plasma fluid equations.
The second part gives the (edited) output of this code, showing the calculation of the coefficients a_i, b_i and d_i.
The third part is an (edited) account of an interactive session of REDUCE 3.2, used to tidy up the algebra involved in the coefficient of the first derivative.

part one

```
//PL12MHDN JOB PL12,DIVER,CLASS=E,TIME=(1,59)
// EXEC FVLG,PRINT='SYSOUT=T'
//L.LIB DD DSN=GU1.PACKAGES,DISP=SHR
 INCLUDE LIB(LISP)
//G.REDUCE DD DSN=GU1.REDUCE,DISP=SHR
//G.LISPOUT DD SYSOUT=T
//G.LISPIN DD *
 RESTORE(REDUCE)
BEGIN NIL
COMMENT
THIS PROGRAM TACKLES THE RHS OF THE WARM PLASMA MODEL MHD EQUATION
IN CARTESIAN GEOMETRY. THE MATRIX W IS A GLOBAL ONE, WHERE THE FIRST
TWO ROWS ARE THE VELOCITY AND MAGNETIC FIELD, AND THE REMAINING ROWS
CONTAIN ALL THE INTERMEDIATE CALCULATIONS USED TO DERIVE THE FINAL ANSWER
AS STORED IN ROW 12.  EACH PROCEDURE IS SELF EXPLANATORY. IT IS INTENDED
THAT THE FIRST ARGUMENT AND THE SECOND ARGUMENT OF A VECTOR PROCEDURE
SHOULD BE IN STRICT ORDER, FOR INSTANCE DOTGRAD(U,B,N) MEANS U.GRADB WITH
THE ANSWER BEING WRITTEN IN THE NTH ROW OF W.     $
ON NERO;
OFF ECHO;
ARRAY UXCFT(3),UYCFT(3),UZCFT(3),
   DUXCFT(3),DUYCFT(3),DUZCFT(3),D2UZCFT(3);
MATRIX Q(13,3);
OPERATOR UX,UY,UZ,BO,P,RO,CA,CS,TERM,L,F,GG,QQ ;
INTEGER U1,B,PSI;
```

```
FACTOR UX,UY,UZ,DF(UX(X,Y,Z),Z),DF(UY(X,Y,Z),Z),DF(UZ(X,Y,Z),Z)$
FACTOR DF(UX(X,Y,Z),X),DF(UX(X,Y,Z),Y),DF(UX(X,Y,Z),X,Y),
DF(UX(X,Y,Z),X,Z),DF(UX(X,Y,Z),Y,Z)$
FACTOR DF(UY(X,Y,Z),X),DF(UY(X,Y,Z),Y),DF(UY(X,Y,Z),X,Y),
DF(UY(X,Y,Z),X,Z),DF(UY(X,Y,Z),Y,Z)$
FACTOR DF(UZ(X,Y,Z),X),DF(UZ(X,Y,Z),Y),DF(UZ(X,Y,Z),X,Y),
DF(UZ(X,Y,Z),X,Z),DF(UZ(X,Y,Z),Y,Z)$
U1:=1;B:=2;PSI := 3;
UX:=UX(X,Y,Z);
UY:=UY(X,Y,Z);
UZ:=UZ(X,Y,Z);
BO:=BO(Z);
P:=P(Z);
RO:=RO(Z);
DF(BO,X):=DF(BO,Y):=DF(BO,X,Y):=DF(BO,X,Z):=DF(BO,Y,Z):=0;
DF(P,X):=DF(P,Y):=DF(P,X,Y):=DF(P,X,Z):=DF(P,Y,Z):=0;
DF(CA(Z),X):=DF(CA(Z),Y):=DF(CA(Z),X,Y):=
    DF(CA(Z),X,Z):=DF(CA(Z),Y,Z):=0;
DF(CS(Z),X):=DF(CS(Z),Y):=DF(CS(Z),X,Y):=
    DF(CS(Z),X,Z):=DF(CS(Z),Y,Z):=0;
Q(1,1):=UX;Q(1,2):=UY;Q(1,3):=UZ;
Q(2,1):=BO*COS(T);Q(2,2):=BO*SIN(T);
PROCEDURE GRAD(F,J);
BEGIN
Q(J,1):=DF(F,X);
Q(J,2):=DF(F,Y);
Q(J,3):=DF(F,Z);
RETURN;
```

```
END;
PROCEDURE DIV(N);
BEGIN
DIV:=DF(Q(N,1),X) + DF(Q(N,2),Y) + DF(Q(N,3),Z);
RETURN DIV;
END;
PROCEDURE DOT(M,N);
BEGIN
DOT:= FOR J:=1:3  SUM ( Q(M,J)*Q(N,J) );
RETURN DOT;
END;
PROCEDURE DOTGRAD(M,N,J);
BEGIN
FOR K:=1:3 DO Q(J,K):=Q(M,1)*DF(Q(N,K),X) + Q(M,2)*DF(Q(N,K),Y)
                        + Q(M,3)*DF(Q(N,K),Z);
RETURN;
END;
%  WE NOW LOAD THE ENTIRE MHD EQUATION INTO THE CODE.     $
ON RAT,DIV;
BSQ := DOT (B,B) / 2   $
GRAD ( BSQ,4 ) ;   %  GRAD (- P) IN ROW    4    $
DOTGRAD ( B, U1, 5 ) ;    % B . GRAD U   IN ROW  5    $
DOTGRAD ( U1, B, 6 ) ;    % U . GRAD B  IN ROW   6    $
FOR J:=1:3 DO  Q( PSI, J ) := Q(5,J) - ( Q(6,J) + DIV (U1)*Q(B,J) ) $
%   QUANTITY PSI NOW IN ROW IDENTIFIED BY INTEGER PSI=7    $
%TERM2 := GA*P*DIV ( U1 ) - DOT (U1, 4) - DOT ( B, PSI ) $
TERM2 := GA*P*DIV(U1) - DOT( U1,4) - DOT ( B, PSI )  $
%TERM2 := ( GA*P + DOT(B,B) )*DIV(U1) - DOT (B,5) $
```

```
GRAD (TERM2, 8)   ;        % 1ST MAJOR TERM IN ROW 8   $
DOTGRAD ( B, PSI, 9 );   % B . GRAD PSI  IN ROW 9    $
DOTGRAD ( PSI, B, 10 );  % PSI . GRAD B IN ROW  10    $
% WE ARE READY TO CONSTRUCT THE ENTIRE MHD EQUATION & STORE IN ROW 13 $
FOR J:= 1 : 3 DO
    BEGIN
      Q(13,J) := Q(8,J)/RO + Q(9,J)/RO + Q(10,J)/RO
                                + Q(U1,J)*W**2   $
     WRITE Q(13,J):=Q(13,J);
    END $
%  NOW MAKE A FEW SIMPLIFICATIONS : DEFINE THE ALFVEN AND SOUND SPEEDS,
  AND FOURIER TRANSFORM IN X,Y...    $
%      FOR ALL Z LET BO(Z) = CA(Z)*(RO(Z)**(1/2));
%      FOR ALL Z LET P(Z) = RO(Z)*CS(Z)**2 / GA;
      DF(UX,X):= I*KX*UX  $
      DF(UY,X):= I*KX*UY$
      DF(UZ,X):= I*KX*UZ$
      DF(UX,Y):= I*KY*UX$
      DF(UY,Y):= I*KY*UY$
      DF(UZ,Y):= I*KY*UZ$
%-----------------------------------------------------------------;
FOR J:=1:3 DO WRITE Q(13,J) := Q(13,J)  $
%-----------------------------------------------------------------;
%   NOW SPLIT OFF THE VARIOUS COEFFICIENTS OF THE DERIVATIVES OF THE
VARIOUS COEFFICIENTS.;
FACTOR CS(Z), CA(Z)$       ON ALLFAC$
%   NOW INITIALISE ALL THE FOLLOWING VARIABLES TO ZERO, SO THAT THEY
    WILL NOT APPEAR IN THE OUTPUT UNLESS THEY ARE RESET:          $
```

```
UXD0:=UXD1:=UXD2:=UYD0:=UYD1:=UYD2:=UZD0:=UZD1:=UZD2:= 0$
DUXD0:=DUXD1:=DUXD2:=DUYD0:=DUYD1:=DUYD2:=DUZD0:=DUZD1:=DUZD2:= 0$
D2UXD0:=D2UXD1:=D2UXD2:=D2UYD0:=D2UYD1:=D2UYD2:=
D2UZD0:=D2UZD1:=D2UZD2:= 0$
MATCH KX**2 = K**2 - KY**2  $
FOR J := 1 : 3  DO  BEGIN
COEFF ( Q(13,J), UX(X,Y,Z), UXD )$   UXCFT(J) := UXD1;
COEFF ( Q(13,J), UY(X,Y,Z), UYD )$   UYCFT(J) := UYD1;
COEFF ( Q(13,J), UZ(X,Y,Z), UZD )$   UZCFT(J) := UZD1;
COEFF ( Q(13,J), DF(UX(X,Y,Z),Z),DUXD)$  DUXCFT(J):= DUXD1;
COEFF ( Q(13,J), DF(UY(X,Y,Z),Z),DUYD)$  DUYCFT(J):= DUYD1;
COEFF ( Q(13,J), DF(UZ(X,Y,Z),Z),DUZD)$  DUZCFT(J):= DUZD1;
COEFF ( Q(13,J), DF(UZ(X,Y,Z),Z,2),D2UXD)$  D2UZCFT(J):= D2UXD1;
WRITE UXCFT(J):=UXCFT(J);
WRITE UYCFT(J):=UYCFT(J);
WRITE UZCFT(J):=UZCFT(J);
WRITE DUXCFT(J):=DUXCFT(J);
WRITE DUYCFT(J):=DUYCFT(J);
WRITE DUZCFT(J):=DUZCFT(J);
WRITE D2UZCFT(J):=D2UZCFT(J);
END;
% ...........................................................$
%    NOW MAKE THE NECESSARY CANCELLATIONS IN ORDER TO GENERATE COMPLETELY
%    THE WARM PLASMA ODES
%............................................................$
      MATCH B0(Z)**2 = RO(Z)*CA(Z)**2;
      MATCH P(Z) = RO(Z)*CS(Z)**2 / GA;
      FOR ALL Z LET   DF(P(Z),Z) = -CT*RO(Z);
```

```
      FOR ALL Z LET   DF(BO(Z),Z) = CT*RO(Z)/BO(Z);
D0:= UZCFT(3)$  D1:= DUZCFT(3)$   D2:= D2UZCFT(3)$
D3:= UXCFT(3)$  D4:= DUXCFT(3)$
D5:= UYCFT(3)$  D6:= DUYCFT(3)$
OFF DIV, RAT $    ON ALLFAC, GCD $
KY:=0; K:=KX ;% <<<<<<<<<<<<<<<<<<    SPOT THE SIMPLIFICATION    $
%  CALCULATE FF, GG EXPLICITLY AND MAKE THE SUBSTITION:     $
A0:=`UXCFT(1)$     A1 := UYCFT(1) $   A2:= DUZCFT(1)  $
BB0 := UYCFT(2) $   BB1 := UXCFT(2) $   BB2 := DUZCFT(2) $
%=============================================================;
WRITE A0:=A0;   WRITE A1:=A1;  WRITE A2:=A2;
WRITE BB0:=BB0;   WRITE BB1:=BB1;  WRITE BB2:=BB2;
WRITE D0:=D0;   WRITE D1:=D1;  WRITE D2:=D2;  WRITE D3:=D3;
WRITE D4:=D4;   WRITE D5:=D5;  WRITE D6:=D6;
%=============================================================;
ON ECHO;
FF := ( A0*BB2 - A2*BB1) / ( A2*BB0 - A1*BB2 );
GG := A2 / ( A0 + A1*FF );
%  NOW SPECIFY THE SUBSTITUTIONS TO BE MADE BEFORE DEFINING THEM
      EXPLICITLY:          $
   OFF GCD;   FACTOR CT, CA(Z), CS(Z), DF(CA(Z),Z), DF(CS(Z),Z);
%     DF(RO(Z),Z) := -CT*RO(Z)/(CS(Z)**2);
SECOND := D2 - D4*GG - D6*FF*GG;
H:= NUM(GG);    QQ:= DEN(GG);     J := NUM( FF*GG);
FIRSTA := D1*QQ - (H*D3 + DF(H,Z)*D4 + J*D5 + DF(J,Z)*D6) ;
FIRSTB := H*D4 + J*D6 ;
ZEROTH := D0;
%  NOW MAKE THE ALTERNATIVE SUBSTITUTION WHICH ELIMINATES DF(CS**2,Z)
```

```
   WRT CT:      ;
/* END OF JOB
```

part two

```
COMMENT
THIS PROGRAM TACKLES THE RHS OF THE
WARM PLASMA MODEL MHD EQUATION
IN CARTESIAN GEOMETRY. THE MATRIX W
IS A GLOBAL ONE, WHERE THE FIRST
TWO ROWS ARE THE VELOCITY AND MAGNETIC FIELD,
AND THE REMAINING ROWS
CONTAIN ALL THE INTERMEDIATE CALCULATIONS
USED TO DERIVE THE FINAL ANSWERr
AS STORED IN ROW 12.  EACH PROCEDURE IS SELF EXPLANATORY.
IT IS INTENDED  THAT THE FIRST ARGUMENT AND THE
SECOND ARGUMENT OF A VECTOR PROCEDURE
SHOULD BE IN STRICT ORDER, FOR INSTANCE
DOTGRAD(U,B,N) MEANS U.GRADB WITH
THE ANSWER BEING WRITTEN IN THE NTH ROW OF W.     $
ON NERO;
OFF ECHO;
U1 := 1
B := 2
PSI := 3
UX := UX(X,Y,Z)
UY := UY(X,Y,Z)
UZ := UZ(X,Y,Z)
BO := BO(Z)
P := P(Z)
```

```
RO := RO(Z)

Q(1,1) := UX(X,Y,Z)

Q(1,2) := UY(X,Y,Z)

Q(1,3) := UZ(X,Y,Z)

Q(2,1) := COS(T)*BO(Z)

Q(2,2) := SIN(T)*BO(Z)

                        2                           (-1)
Q(13,1) := UX(X,Y,Z)*W  + DF(UY(X,Y,Z),X,Y)*GA*RO(Z)     *P(Z) + DF(

                                    (-1)              2      (-1)
           UZ(X,Y,Z),X,Z)*(GA*RO(Z)     *P(Z) + SIN(T) *RO(Z)     *BO

                2                                              (-1)
             (Z) ) - DF(UZ(X,Y,Z),Y,Z)*SIN(T)*COS(T)*RO(Z)     *BO(

            2                              2      (-1)     2
           Z)  + DF(UX(X,Y,Z),Y,2)*SIN(T) *RO(Z)     *BO(Z)  - DF(UY

                                                (-1)     2
           (X,Y,Z),Y,2)*SIN(T)*COS(T)*RO(Z)     *BO(Z)  - DF(UY(X,Y,

                                           (-1)     2
           Z),X,2)*SIN(T)*COS(T)*RO(Z)     *BO(Z)  + DF(UX(X,Y,Z),X,

                      (-1)                                  2      (
           2)*GA*RO(Z)     *P(Z) + DF(UX(X,Y,Z),X,2)*SIN(T) *RO(Z)

           -1)      2
              *BO(Z)

                        2                           (-1)
Q(13,2) := UY(X,Y,Z)*W  + DF(UX(X,Y,Z),X,Y)*GA*RO(Z)     *P(Z) - DF(

                                                  (-1)     2
           UZ(X,Y,Z),X,Z)*SIN(T)*COS(T)*RO(Z)     *BO(Z)  + DF(UZ(X,

                               (-1)              2      (-1)     2
           Y,Z),Y,Z)*(GA*RO(Z)     *P(Z) + COS(T) *RO(Z)     *BO(Z) )
```

```
                                                   (-1)      2
 - DF(UX(X,Y,Z),Y,2)*SIN(T)*COS(T)*RO(Z)      *BO(Z)   + DF
                               (-1)
(UY(X,Y,Z),Y,2)*GA*RO(Z)     *P(Z) + DF(UY(X,Y,Z),Y,2)*
      2       (-1)       2                              2
COS(T) *RO(Z)      *BO(Z)   + DF(UY(X,Y,Z),X,2)*COS(T) *RO(
  (-1)       2                                          (-1
Z)     *BO(Z)   - DF(UX(X,Y,Z),X,2)*SIN(T)*COS(T)*RO(Z)
)       2
 *BO(Z)

                  2                                        (-1)
Q(13,3) := UZ(X,Y,Z)*W  + DF(UZ(X,Y,Z),Z)*(DF(P(Z),Z)*GA*RO(Z)
                                     2       (-1)
   + 2*DF(BO(Z),Z)*SIN(T) *RO(Z)      *BO(Z) + 2*DF(BO(Z)
                     2       (-1)
   ,Z)*COS(T) *RO(Z)      *BO(Z)) + DF(UX(X,Y,Z),X)*(DF(P(
                  (-1)                           2       (-1)
   Z),Z)*GA*RO(Z)       + 2*DF(BO(Z),Z)*SIN(T) *RO(Z)      *
   BO(Z)) - 2*DF(UX(X,Y,Z),Y)*DF(BO(Z),Z)*SIN(T)*COS(T)*
    (-1)                                         (-1)
RO(Z)     *BO(Z) + DF(UX(X,Y,Z),X,Z)*(GA*RO(Z)      *P(Z)
                2       (-1)       2
   + SIN(T) *RO(Z)      *BO(Z) ) - DF(UX(X,Y,Z),Y,Z)*SIN(
                 (-1)       2
T)*COS(T)*RO(Z)      *BO(Z)   - 2*DF(UY(X,Y,Z),X)*DF(BO(Z),
                          (-1)
Z)*SIN(T)*COS(T)*RO(Z)      *BO(Z) + DF(UY(X,Y,Z),Y)*(DF(P
                  (-1)                             2      (-1)
```

```
     (Z),Z)*GA*RO(Z)      + 2*DF(BO(Z),Z)*COS(T) *RO(Z)

                                                    (-1)
     *BO(Z)) - DF(UY(X,Y,Z),X,Z)*SIN(T)*COS(T)*RO(Z)     *
       2                           (-1)            2
    BO(Z)  + DF(UY(X,Y,Z),Y,Z)*(GA*RO(Z)     *P(Z) + COS(T) *
           (-1)       2
     RO(Z)     *BO(Z) ) + 2*DF(UZ(X,Y,Z),X,Y)*SIN(T)*COS(T)
        (-1)       2                           2      (-1)
    *RO(Z)     *BO(Z)  + DF(UZ(X,Y,Z),Y,2)*SIN(T) *RO(Z)     *
       2                          2      (-1)      2
    BO(Z)  + DF(UZ(X,Y,Z),X,2)*COS(T) *RO(Z)     *BO(Z)  + DF
                               (-1)
    (UZ(X,Y,Z),Z,2)*GA*RO(Z)     *P(Z) + DF(UZ(X,Y,Z),Z,2)*
         2      (-1)      2                             2
    SIN(T) *RO(Z)     *BO(Z)  + DF(UZ(X,Y,Z),Z,2)*COS(T) *RO(
      (-1)      2
    Z)     *BO(Z)

                     2     2       2      (-1)       2     2
Q(13,1) := UX(X,Y,Z)*(W  - KY *SIN(T) *RO(Z)     *BO(Z)  - KX *GA*RO
            (-1)          2       2      (-1)       2
     (Z)     *P(Z) - KX *SIN(T) *RO(Z)     *BO(Z) ) + UY(X,Y
        2                      (-1)       2
    ,Z)*(KY *SIN(T)*COS(T)*RO(Z)     *BO(Z)  - KY*KX*GA*RO(Z)
          (-1)          2                    (-1)       2
            *P(Z) + KX *SIN(T)*COS(T)*RO(Z)     *BO(Z) ) +
                                                (-1)       2
    DF(UZ(X,Y,Z),Z)*( - I*KY*SIN(T)*COS(T)*RO(Z)     *BO(Z)
                          (-1)                    2      (-1)
```

```
      + I*KX*GA*RO(Z)    *P(Z) + I*KX*SIN(T) *RO(Z)    *BO
        2
     (Z) )

                 2                           (-1)       2
Q(13,2) := UX(X,Y,Z)*(KY *SIN(T)*COS(T)*RO(Z)    *BO(Z)  - KY*KX*GA
                 (-1)          2                       (-1)      2
       *RO(Z)    *P(Z) + KX *SIN(T)*COS(T)*RO(Z)    *BO(Z) )
                   2      2         (-1)           2        2
     + UY(X,Y,Z)*(W  - KY *GA*RO(Z)    *P(Z) - KY *COS(T) *
           (-1)      2      2        2      (-1)       2
       RO(Z)    *BO(Z)  - KX *COS(T) *RO(Z)    *BO(Z) ) + DF
                                  (-1)                   2
      (UZ(X,Y,Z),Z)*(I*KY*GA*RO(Z)    *P(Z) + I*KY*COS(T) *RO(
         (-1)      2                          (-1)      2
       Z)    *BO(Z)  - I*KX*SIN(T)*COS(T)*RO(Z)    *BO(Z) )
                                                              (-1)
Q(13,3) := UX(X,Y,Z)*( - 2*I*KY*DF(BO(Z),Z)*SIN(T)*COS(T)*RO(Z)
                                                  (-1)
       *BO(Z) + I*KX*DF(P(Z),Z)*GA*RO(Z)     + 2*I*KX*DF(BO(
                 2        (-1)
       Z),Z)*SIN(T) *RO(Z)    *BO(Z)) + UY(X,Y,Z)*(I*KY*DF(P
                    (-1)                             2
       (Z),Z)*GA*RO(Z)     + 2*I*KY*DF(BO(Z),Z)*COS(T) *RO(Z
        (-1)                                                 (
       )    *BO(Z) - 2*I*KX*DF(BO(Z),Z)*SIN(T)*COS(T)*RO(Z)
       -1)                     2     2        2      (-1)
         *BO(Z)) + UZ(X,Y,Z)*(W  - KY *SIN(T) *RO(Z)    *BO
         2                               (-1)      2     2
```

```
(Z)  - 2*KY*KX*SIN(T)*COS(T)*RO(Z)    *BO(Z)  - KX *

        2      (-1)      2
COS(T) *RO(Z)     *BO(Z) ) + DF(UX(X,Y,Z),Z)*( - I*KY*

                      (-1)      2                 (-1)
SIN(T)*COS(T)*RO(Z)     *BO(Z)  + I*KX*GA*RO(Z)     *P(

                    2      (-1)      2
Z) + I*KX*SIN(T) *RO(Z)     *BO(Z) ) + DF(UY(X,Y,Z),Z)

                 (-1)                  2      (-1)      2
*(I*KY*GA*RO(Z)     *P(Z) + I*KY*COS(T) *RO(Z)     *BO(Z)

                                  (-1)      2
 - I*KX*SIN(T)*COS(T)*RO(Z)     *BO(Z) ) + DF(UZ(X,Y,Z)

                              (-1)                      2
,Z)*(DF(P(Z),Z)*GA*RO(Z)      + 2*DF(BO(Z),Z)*SIN(T) *RO(

      (-1)                               2      (-1)
  Z)     *BO(Z) + 2*DF(BO(Z),Z)*COS(T) *RO(Z)     *BO(Z

                                             (-1)
  )) + DF(UZ(X,Y,Z),Z,2)*GA*RO(Z)     *P(Z) + DF(UZ(X,

                   2      (-1)      2
Y,Z),Z,2)*SIN(T) *RO(Z)     *BO(Z)  + DF(UZ(X,Y,Z),Z,2)*

       2      (-1)      2
COS(T) *RO(Z)     *BO(Z)

                 2           (-1)         2       2      (-1)       2
UXCFT(1) :=  - K *GA*RO(Z)     *P(Z) - K *SIN(T) *RO(Z)     *BO(Z)

               2     2          (-1)
             + W  + KY *GA*RO(Z)     *P(Z)

              2                          (-1)       2                (-1)
UYCFT(1) := K *SIN(T)*COS(T)*RO(Z)     *BO(Z)  - KY*KX*GA*RO(Z)     *

             P(Z)
```

```
                                         (-1)      2
DUZCFT(1) :=  - I*KY*SIN(T)*COS(T)*RO(Z)     *BO(Z)  + I*KX*GA*RO(Z)
             (-1)                          2     (-1)      2
                *P(Z) + I*KX*SIN(T) *RO(Z)     *BO(Z)

             2                      (-1)      2                   (-1)
UXCFT(2) := K *SIN(T)*COS(T)*RO(Z)     *BO(Z)  - KY*KX*GA*RO(Z)     *
            P(Z)

               2      2     (-1)      2   2    2           (-1)
UYCFT(2) :=  - K *COS(T) *RO(Z)     *BO(Z)  + W  - KY *GA*RO(Z)     *
            P(Z)

                        (-1)                  2    (-1)       2
DUZCFT(2) := I*KY*GA*RO(Z)     *P(Z) + I*KY*COS(T) *RO(Z)     *BO(Z)
                                         (-1)      2
              - I*KX*SIN(T)*COS(T)*RO(Z)     *BO(Z)

                                                           (-1)
UXCFT(3) :=  - 2*I*KY*DF(BO(Z),Z)*SIN(T)*COS(T)*RO(Z)     *BO(Z) + I
                                     (-1)
            *KX*DF(P(Z),Z)*GA*RO(Z)      + 2*I*KX*DF(BO(Z),Z)*SIN(T)
            2      (-1)
             *RO(Z)     *BO(Z)

                                      (-1)
UYCFT(3) := I*KY*DF(P(Z),Z)*GA*RO(Z)      + 2*I*KY*DF(BO(Z),Z)*COS(T
             2      (-1)
            ) *RO(Z)     *BO(Z) - 2*I*KX*DF(BO(Z),Z)*SIN(T)*COS(T)*
                (-1)
            RO(Z)     *BO(Z)

               2      2     (-1)      2   2    2      2
UZCFT(3) :=  - K *COS(T) *RO(Z)     *BO(Z)  + W  - KY *SIN(T) *RO(Z)
```

```
         (-1)      2     2       2     (-1)      2
             *BO(Z)  + KY *COS(T) *RO(Z)    *BO(Z)  - 2*KY*KX*

                                (-1)      2
         SIN(T)*COS(T)*RO(Z)    *BO(Z)

                                         (-1)      2
DUXCFT(3) :=  - I*KY*SIN(T)*COS(T)*RO(Z)    *BO(Z)  + I*KX*GA*RO(Z)

          (-1)                     2     (-1)      2
              *P(Z) + I*KX*SIN(T) *RO(Z)    *BO(Z)

                        (-1)                     2     (-1)      2
DUYCFT(3) := I*KY*GA*RO(Z)    *P(Z) + I*KY*COS(T) *RO(Z)    *BO(Z)

                                         (-1)      2
            - I*KX*SIN(T)*COS(T)*RO(Z)    *BO(Z)

                                 (-1)                        2      (
DUZCFT(3) := DF(P(Z),Z)*GA*RO(Z)     + 2*DF(BO(Z),Z)*SIN(T) *RO(Z)

          -1)                                2      (-1)
             *BO(Z) + 2*DF(BO(Z),Z)*COS(T) *RO(Z)    *BO(Z)

                       (-1)                2      (-1)      2
D2UZCFT(3) := GA*RO(Z)    *P(Z) + SIN(T) *RO(Z)    *BO(Z)  + COS(T)

             2      (-1)      2
              *RO(Z)    *BO(Z)

K := KX

              2  2        2  2       2  2
A0 :=  - CS(Z) *KX  - CA(Z) *KX *SIN(T)  + W

           2  2
A1 := CA(Z) *KX *SIN(T)*COS(T)

           2             2          2
A2 := CS(Z) *I*KX + CA(Z) *I*KX*SIN(T)

              2  2        2  2
```

```
BB0 :=  - CA(Z) *KX *COS(T)  + W

             2   2
BB1 := CA(Z) *KX *SIN(T)*COS(T)

                2
BB2 :=  - CA(Z) *I*KX*SIN(T)*COS(T)

               2    2       2    2
D0 :=  - CA(Z) *KX *COS(T)  + W

                           2           2
D1 := CT*( - GA + 2*SIN(T)  + 2*COS(T) )

            2         2        2         2
D2 := CS(Z)  + CA(Z) *(SIN(T)  + COS(T) )

                                 2
D3 := I*KX*CT*( - GA + 2*SIN(T) )

            2             2            2
D4 := CS(Z) *I*KX + CA(Z) *I*KX*SIN(T)

D5 :=  - 2*I*KX*CT*SIN(T)*COS(T)

               2
D6 :=  - CA(Z) *I*KX*SIN(T)*COS(T)

FF:=(A0*BB2-A2*BB1)/(A2*BB0-A1*BB2);

                2  2                         2      2   2       2
FF := ( - CA(Z) *W *SIN(T)*COS(T))/( - CS(Z) *CA(Z) *KX *COS(T)  +

               2  2         2  2       2
         CS(Z) *W  + CA(Z) *W *SIN(T) )

GG:=A2/(A0+A1*FF);

                2      2     3       2        2    2          2
GG := ( - CS(Z) *CA(Z) *I*KX *COS(T)  + CS(Z) *I*W *KX + CA(Z) *I*W

        2          2        2      2  4       2        2  2   2
        *KX*SIN(T) )/(CS(Z) *CA(Z) *KX *COS(T)  - CS(Z) *W *KX  - (
```

```
           2 2  2         2        2    4
        CA(Z) *W *KX )*(SIN(T)  + COS(T) ) + W )
% NOW SPECIFY THE SUBSTITUTIONS TO BE MADE BEFORE DEFINING THEM
        EXPLICITLY:           $
OFF GCD;
FACTOR CT,CA(Z),CS(Z),DF(CA(Z),Z),DF(CS(Z),Z);
```

==

NOW COME THE COEFFICIENTS OF THE NTH DERIVATIVE:

==

```
SECOND:=D2-D4*GG-D6*FF*GG;

                   4       2  4       4          4  2   2        2
SECOND := (CA(Z) *CS(Z) *KX *COS(T)  - (CA(Z) *W *KX *COS(T) )*(SIN

                 2          2           2       2  2   2        2
          (T)  + COS(T) ) - 2*CA(Z) *CS(Z) *W *KX *COS(T)  + CA

          2  4          2          2          2  4         2       2
        (Z) *W *(SIN(T)  + COS(T) ) + CS(Z) *W )/(CA(Z) *CS(Z) *

           4        2           2  2   2          2         2
         KX *COS(T)  - (CA(Z) *W *KX )*(SIN(T)  + COS(T) ) - CS

             2  2  2    4
          (Z) *W *KX  + W )

H:=NUM(GG);

                2       2     3       2       2   2           2
H :=  - CA(Z) *CS(Z) *I*KX *COS(T)  + CA(Z) *I*W *KX*SIN(T)  + CS(Z

      2   2
     ) *I*W *KX

QQ:=DEN(GG);

              2       2  4        2         2  2   2          2
QQ := CA(Z) *CS(Z) *KX *COS(T)  - (CA(Z) *W *KX )*(SIN(T)  + COS(T)
```

```
          2          2  2   2    4

           ) - CS(Z) *W *KX  + W

J:=NUM(FF*GG);

             2    2

J :=  - CA(Z) *I*W *KX*SIN(T)*COS(T)

FIRSTA:=D1*QQ-(H*D3+DF(H,Z)*D4+J*D5+DF(J,Z)*D6);

                    2      2   4       4           2  2   2       2

FIRSTA := 2*CT*CA(Z) *CS(Z) *KX *COS(T)  + CT*CA(Z) *W *KX *COS(T)

                         2           2              2  2   2

          *(GA - 2*SIN(T)  - 2*COS(T) ) - 2*CT*CS(Z) *W *KX *COS(T)

          2       4                  2           2           4

            + CT*W *( - GA + 2*SIN(T)  + 2*COS(T) ) - 2*CA(Z) *CS(Z

                          4       2       2          3      2

          )*DF(CS(Z),Z)*KX *SIN(T) *COS(T)  - 2*CA(Z) *CS(Z) *DF(CA

                   4       2       2          3              2   2

          (Z),Z)*KX *SIN(T) *COS(T)  + 2*CA(Z) *DF(CA(Z),Z)*W *KX *

                2        2         2           2      3

          SIN(T) *(SIN(T)  + COS(T) ) - 2*CA(Z) *CS(Z) *DF(CS(Z),Z)

             4       2          2                    2   2       2

          *KX *COS(T)  + 2*CA(Z) *CS(Z)*DF(CS(Z),Z)*W *KX *SIN(T)

                          4               4       2

           - 2*CA(Z)*CS(Z) *DF(CA(Z),Z)*KX *COS(T)  + 2*CA(Z)*CS(Z)

          2              2   2       2          3              2

           *DF(CA(Z),Z)*W *KX *SIN(T)  + 2*CS(Z) *DF(CS(Z),Z)*W *KX

          2

FIRSTB:=H*D4+J*D6;

               4      2   4       2       2         4  2   2

FIRSTB := CA(Z) *CS(Z) *KX *SIN(T) *COS(T)  - (CA(Z) *W *KX *SIN(T)
```

```
                    2           2           2           2      4   4       2
                     )*(SIN(T)   + COS(T) ) + CA(Z) *CS(Z) *KX *COS(T)   - 2

                       2        2  2   2        2        4  2   2
                 *CA(Z) *CS(Z) *W  *KX *SIN(T)  - CS(Z) *W  *KX

ZEROTH:=DO;

                        2   2       2    2
ZEROTH :=  - CA(Z) *KX *COS(T)  + W

%  NOW MAKE THE ALTERNATIVE SUBSTITUTION WHICH ELIMINATES DF(CS**2

,Z)      WRT CT:      ;

%DF(CS(Z),Z) := (1-GA)*CT/(2*CS(Z));

%------------------------------------------------------ $

*** END OF DATA
```

```
    REDUCE 3.2, 15-Apr-85 ...

    1:

d1:= ct*(2-ga);

    D1 := CT*( - GA + 2)

    2:

d2:= cs(z)**2 + ca(z)**2;

    Declare CS operator ?  (Y or N)

y

    Declare CA operator ?  (Y or N)

y

                  2          2
    D2 := CA(Z)   + CS(Z)

    3:

d3 :=i*kx*ct*(2*sin(t)**2 - ga);

                                 2
    D3 := I*CT*KX*(2*SIN(T)   - GA)

    4:

d4:= i*kx*(ca(z)**sin(t)**2 + cs(z)**2);

                         (2*SIN(T))        2
    D4 := I*KX*(CA(Z)              + CS(Z) )

    5:

clear d4;

    6:

d4:= i*kx*(ca(z)**2*sin(t)**2 + cs(z)**2);

                          2      2         2
```

```
    D4 := I*KX*(SIN(T) *CA(Z)  + CS(Z) )

    7:

d5:=-2*i*kx*ct*sin(t)*cos(t);

    D5 :=  - 2*COS(T)*SIN(T)*I*CT*KX

1

    8:

d6:= -cos(t)*sin(t)*ca(z)**2*i*kx;

                                2

    D6 :=  - COS(T)*SIN(T)*CA(Z) *I*KX

    9:

h:= i*kx*w**2(ca(z)**2*sin(t)**2 + cs(z)**2)

-i*kx**3*cs(z)**2*ca(z)**2*cos(t)**2;

    H:=I*KX*W**2(CA(Z)**2*SIN(T)**2+CS(Z)**2)-I*KX**3*CS(Z)**2*CA(Z)**2*

    COS(T)**2;

    ***** Missing Operator

    10:

-i*kx**3*cs(z)**2*ca(z)**2*cos(t)**2;

                2       2       2     3

      - COS(T) *CA(Z) *CS(Z) *I*KX

    11:

h:= i*kx*w**2*(ca(z)**2*sin(t)**2 + cs(z)**2)

-i*kx**3*cs(z)**2*ca(z)**2*cos(t)**2;

                          2       2       2   2           2       2  2

    H := I*KX*( - COS(T) *CA(Z) *CS(Z) *KX  + SIN(T) *CA(Z) *W  +

                      2  2

                 CS(Z) *W )

    12:

j:=-ca(z)**2*i*w**2*kx*sin(t)*cos(t);
```

```
                                2   2
J :=  - COS(T)*SIN(T)*CA(Z) *I*W *KX

13:

q:= w**4 - w**2*kx**2*(cs(z)**2+ca(z)**2)+(kx**2*cs(z)*ca(z)*cos(T))**2;

                 2      2      2   4         2  2   2          2  2   2   4
    Q := COS(T) *CA(Z) *CS(Z) *KX  - CA(Z) *W *KX  - CS(Z) *W *KX  + W

    14:

d1;

    CT*( - GA + 2)

    15:

d2;

          2         2
    CA(Z)  + CS(Z)

    16:

d3;

                     2
    I*CT*KX*(2*SIN(T)  - GA)

    17:

d4;

                   2       2        2
    I*KX*(SIN(T) *CA(Z)  + CS(Z) )

    18:

d5;

     - 2*COS(T)*SIN(T)*I*CT*KX

    19:

d6;

                               2
     - COS(T)*SIN(T)*CA(Z) *I*KX
```

```
   20:

h;

               2       2      2   2         2      2  2       2  2
   I*KX*( - COS(T) *CA(Z) *CS(Z) *KX  + SIN(T) *CA(Z) *W  + CS(Z) *W )

   21:

j;

                         2   2
   - COS(T)*SIN(T)*CA(Z) *I*W *KX

   22:

q;

          2      2      2   4        2  2   2       2  2   2   4
   COS(T) *CA(Z) *CS(Z) *KX  - CA(Z) *W *KX  - CS(Z) *W *KX  + W

   23:

firsta:= d1*q - (h*d3+df(h,z)*d4+j*d5+df(j,z)*d6);

                                       2       2      3      2   4
   FIRSTA :=  - 2*DF(CA(Z),Z)*COS(T) *SIN(T) *CA(Z) *CS(Z) *KX  + 2*DF

                           2       2      3  2   2
             (CA(Z),Z)*COS(T) *SIN(T) *CA(Z) *W *KX  - 2*DF(CA(Z),Z)*

                   2           4   4                     4     3
             COS(T) *CA(Z)*CS(Z) *KX  + 2*DF(CA(Z),Z)*SIN(T) *CA(Z) *

              2  2                      2           2  2   2
             W *KX  + 2*DF(CA(Z),Z)*SIN(T) *CA(Z)*CS(Z) *W *KX  - 2*DF

                           2       2      4        4
             (CS(Z),Z)*COS(T) *SIN(T) *CA(Z) *CS(Z)*KX  - 2*DF(CS(Z),Z

                     2      2      3   4                     2
             )*COS(T) *CA(Z) *CS(Z) *KX  + 2*DF(CS(Z),Z)*SIN(T) *

                  2        2  2                      3  2   2
             CA(Z) *CS(Z)*W *KX  + 2*DF(CS(Z),Z)*CS(Z) *W *KX  - 2*
```

```
            2        2        2        2       4             2         2
COS(T) *SIN(T) *CA(Z) *CS(Z) *CT*KX  + 2*COS(T) *SIN(T) *

       2  2       2            2        2        2       4
CA(Z) *W *CT*KX  + 2*COS(T) *CA(Z) *CS(Z) *CT*KX  + 2*

       4        2  2       2            2       2  2           2
SIN(T) *CA(Z) *W *CT*KX  - SIN(T) *CA(Z) *W *GA*CT*KX  +

         2        2  2       2         2  2           2
2*SIN(T) *CS(Z) *W *CT*KX  + CA(Z) *W *GA*CT*KX  - 2*

       2  2       2            2  2       2    4             4
CA(Z) *W *CT*KX  - 2*CS(Z) *W *CT*KX  - W *GA*CT + 2*W *

CT
```

24:

on factor;

25:

firsta;

```
                                                     2        2         4   2
- (2*(((CS(Z)*KX + W)*(CS(Z)*KX - W)*SIN(T) *CA(Z)  + CS(Z) *KX )*

          2          4       2  2           2       2  2
    COS(T)  - SIN(T) *CA(Z) *W  - SIN(T) *CS(Z) *W )*DF(CA(Z),Z)

                 2                                      2           2
  *CA(Z)*KX  + 2*((CS(Z)*KX + W)*(CS(Z)*KX - W)*SIN(T)  - CS(Z) *

      2          2        2       2
    KX )*COS(T) *CA(Z) *CT*KX  + 2*(COS(T)*CA(Z)*KX + W)*(COS(T)

                                2       2          2
    *CA(Z)*KX - W)*(SIN(T) *CA(Z)  + CS(Z) )*DF(CS(Z),Z)*CS(Z)*

    2           2               2         2  2        2
  KX  + (CA(Z) *GA - 2*CS(Z) )*SIN(T) *W *CT*KX  - (CA(Z)*KX + W)

                           2                 4      2  2       2
```

```
       *(CA(Z)*KX - W)*(GA - 2)*W *CT - 2*SIN(T) *CA(Z) *W *CT*KX  + 2

             2  2      2
       *CS(Z) *W *CT*KX )

   26:

coeff(firsta,ct,r);

   *** R1 R0 are non zero

   27:

r1;

                                                        2        2   2       2
    - (2*((CS(Z)*KX + W)*(CS(Z)*KX - W)*SIN(T)  - CS(Z) *KX )*COS(T) *

             2   2         2            2        2  2   2
       CA(Z) *KX  + (CA(Z) *GA - 2*CS(Z) )*SIN(T) *W *KX  - (CA(Z)*KX

                                          2          4       2  2   2
         + W)*(CA(Z)*KX - W)*(GA - 2)*W  - 2*SIN(T) *CA(Z) *W *KX

                  2  2   2
        + 2*CS(Z) *W *KX )

   28:

saveas poly;

   29:

coeff(poly,w,rw);

   *** RW0 is non zero

   0

   30:

clear poly;

   31:

firstb := h*d4+j*d6;
                          2      2     2  2         2      2  2      2
   FIRSTB := ((COS(T) *CA(Z) *CS(Z) *KX  - SIN(T) *CA(Z) *W  - CS(Z) *
```

```
                   2           2       2         2            2        2      4
                  W )*(SIN(T) *CA(Z)  + CS(Z) ) - COS(T) *SIN(T) *CA(Z) *

                  2    2
                 W )*KX

     32:

bye;

*** END OF RUN
```

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