APPLICATIONS OF STAR COMPLEXES

IN GROUP THEORY

by

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To the memory of my father

CONTENTS

STATEMENT

Chapter 1 covers basic material in geometric group theory and is based to some extent on notes of S.J.Pride. Chapters 2,3,5 are my own work. Chapter 4 involved some collaboration with S.J.Pride. I had used the method of triangulation to solve the word problem for T(6)-complexes, and had also investigated the conjugacy problem. However, to complete the solution of the conjugacy problem required a description of certain tesselations of the sphere. A rigorous proof that my descrption was correct was supplied by S.J.Pride. The suggestion to extend my formulae to try to solve the dependence problems DP(n) ($n \geq 3$) was due to S.J.Pride.

Chapter 2 except 2.4 will appear in [2].

An outline of Chapter 3, and most of chapter 4 will appear in a joint paper with S.J.Pride in [3].

After proving our result in Chapter 5 we discovered that Marcus [7] had obtained a similar result.

ACKNOWLEDGEMENTS

I would like to express my deepest gratitude to my supervisor Dr. S. J. Pride for suggesting the topics studied in this thesis, and for his kind and invaluable help and for his encouragement during this work.

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Finally I am very grateful to my wife for her patience and encouragement.

ABSTRACT

The main work of the thesis starts with Chapter 2.

Chapter 2 concerns free subgroups of C(4),T(4) groups. Collins has investigated the free subgroups of groups with presentations satisfying the C(4),T(4) conditions. He has shown that such groups contains a free subgroup of rank 2 except in some cases which he lists explicitly. The exceptions are all two generator groups. In Chapter 2 we give a simple proof, using star complexes, that if G is a C(4),T(4) group and if G can not be generated by fewer than three elements then G contains a free subgroup of rank 2. (In fact we prove a slightly stronger result.)

Much work has been done for C(4)-T(4), and C(6)-complexes. However T(6)-complexes have not so far been studied very much. For that reason our main work in the thesis is to study T(6)-complexes. This work is contained in Chapter 3 and Chapter 4.

In Chapter 3 we give some examples of T(6)-complexes and also examples of related complexes called hyperbolic complexes.

In Chapter 4 we obtain new solutions to the word and conjugacy problems for T(6)-complex, and we discuss the dependence

problems in general. (The word problem is DP(1), and the conjugacy problem is DP(2).)

In Chapter 5 we introduce the idea of the degree of a presentation, and also property-ST(m) and property-st(m). (These concepts are related to valences of vertices in star-complexes.) We ask whether having a presentation of degree m, property-ST(m) or property-st(m) (m>2) puts any restriction on the group defined by the presentation. We show that the answer is no.

NOTATIONS

Let G, H be groups:

$H \leqslant G$ means that H is a subgroup of G.

$H \trianglelefteq G$ means that H is normal subgroup of G.

$G \times H$ is the direct product.

$G * H$ is the free product.

$\mathrm{sgp}(x_1,\ldots,x_n)$ denotes the subgroup of G generated by $x_1,\ldots,x_n$ where the x_i belong to G.

We adopt the usual notation in set theory:

$R \cup S$ is the union of sets R, S.

$R \cap S$ is the intersection of sets R, S.

$R \subseteq S$ means that R is a subset of S.

$r \in R$ means that r is a member of R.

$|R|$ denotes the cardinality of R.

1 is the trivial group.

$\mathbb{Z}_n$ is the cyclic group of order n.

$\mathbb{Z}^n$ is the direct product of n copies of $\mathbb{Z}$.

F_n is the free group of rank n.

$\mathbb{Z}$ is the integers.

$\mathbb{Z}^+$ is the non-negative integers.

$\mathbb{Z}^-$ is the negative integers.

$\mathbb{Q}$ is the rationals.

The following notations are introduced in the text.

Let $\mathfrak{X}$ be a 1-complex.

$V(\mathfrak{X})$ set of vertices of 1-complex.

$E(\mathfrak{X})$ set of edges of 1-complex.

$P(\mathcal{L})$	set of paths of 1-complex.
$\iota(e)$	initial vertex of the edge e.
$\tau(e)$	terminal vertex of the edge e.
α^{-1}	the inverse of a path α.
$L(\alpha)$	length of path α.
1_v	the empty path for each v of $V(\mathcal{L})$.
$L_e(\alpha)$	number of times e, e^{-1} appear in a path α.
$\sigma_e(\alpha)$	exponent sum of e in α.
Star(v)	$\{ e: e \in E(\mathcal{L}),\ \iota(e)=v \}$

Let $\mathcal{K} = \langle \mathcal{L} ; \rho_\lambda (\lambda \in \Lambda) \rangle$ be a 2-complex.

$\mathcal{K}^{(1)}$	is the 1-skeleton $\mathcal{L}$.
ρ_λ	closed path in $\mathcal{K}$ called the defining path.
Λ	the set of elements called indices.
$R(\mathcal{K})$	the set of cyclic permutations of defining paths and their inverses.
r^*	the set of all cyclic permutation of elements of r and their inverses.
$\chi(\mathcal{K})$	Euler characterstic of $\mathcal{K}$.
$\alpha \underset{\mathcal{K}}{\sim} \beta$	α is equivalent to β in $\mathcal{K}$.
$\alpha \underset{\mathcal{K}}{\overset{1}{\sim}} \beta$	α is freely equivalent to β in $\mathcal{K}$.
$[\alpha]_{\mathcal{K}}$	the equivalences class of a path α with respect to $\underset{\mathcal{K}}{\sim}$.
$\pi_1(\mathcal{K}, v)$	the fundamental group of $\mathcal{K}$ at v.
$\alpha \underset{\mathcal{K}}{\sim} 1$	α is contractible in $\mathcal{K}$.
$\phi \approx \psi$	two mappings ϕ, ψ are homotopic.
Λ_ℓ	an element of Λ_ℓ is called of level ℓ.
$\mathcal{K}^{st}$	star complex of 2-complex .
$\iota^{st}(\gamma)$	first edge of γ.
$\tau^{st}(\gamma)$	inverse of last edge of γ.

$\mathcal{K}^{st}(v)$ full subcomplex of $\mathcal{K}^{st}$ on Star(v).

$\Gamma(\mathcal{K}^{st})$ star graph of $\mathcal{K}^{st}$.

$\mathcal{K}^{ST}$ extended star complex of 2-complex $\mathcal{K}$.

$\iota^{ST}(\lambda,\epsilon,\gamma)$ first edge of γ.

$\tau^{ST}(\lambda,\epsilon,\gamma)$ inverse of last edge of γ.

C(p) condition : no element of $R(\mathcal{K})$ is product of less than p pieces.

T(q) condition : there are no non-empty cyclically reduced closed path of length ℓ $(3\leq\ell<q)$ in star complexes.

$\tilde{T}(q)$ condition : there are no non-empty cyclically reduced closed path of length less than q in star complexes.

$(\omega_1,\ldots,\omega_n)\vdash_{\mathcal{K}}\omega_0$ ω_0 is dependent on $(\omega_1,\ldots,\omega_n)$ in $\mathcal{K}$.

DP(n) dependence problem.

DP(1) word problem.

DP(2) conjugancy problem.

$\mathcal{K}_\Delta$ a triangulation of 2-complex $\mathcal{K}$.

m weight function on $\mathcal{K}^{st}$.

deg($\mathcal{P}$) degree of $\mathcal{P}$.

CG($\mathcal{P}$) connectivty graph of $\mathcal{P}$.

CHAPTER 1

INTRODUCTION

1.1 Basic concepts and definitions.

Most of this section is collected from Pride's notes and also can be found in Pride [9], [10], [11].

1.1A.1. 1-complexes.

A *1-complex* $\mathcal{X}$ consists of two disjoint sets $V(\mathcal{X})$, $E(\mathcal{X})$ together with three functions ι: $E(\mathcal{X}) \to V(\mathcal{X})$, τ: $E(\mathcal{X}) \to V(\mathcal{X})$, $^{-1}$: $E(\mathcal{X}) \to E(\mathcal{X})$ satisfying $\iota(e^{-1})=\tau(e)$, $(e^{-1})^{-1}=e$, $e^{-1}\neq e$ for all $e\in E(\mathcal{X})$. The elements of $V(\mathcal{X})$ are called *vertices*, and the elements of $E(\mathcal{X})$ are called *edges*. We remark that 1-complexes are often called graphs in combinatorial group theory (see [14], [15]).

A non-empty *path* α in $\mathcal{X}$ is a finite sequence of edges $\alpha=e_1 \ldots e_n$ $(n\geq 1)$ such that $\iota(e_{i+1})=\tau(e_i)$ for $i=1,\ldots,n-1$. We define $\iota(\alpha)$, $\tau(\alpha)$ to be $\iota(e_1)$, $\tau(e_n)$ respectively. The *inverse path* α^{-1} of α is the path $e_n^{-1} \ldots e_1^{-1}$. The path α is said to be *closed* if $\iota(\alpha)=\tau(\alpha)$. The *length* $L(\alpha)$ of α is n. We say that α is *reduced* if $e_i \neq e_{i+1}^{-1}$ for all $i=1,\ldots,n-1$. Morever, if α is closed we say that α is *cyclically reduced* if all cyclic permutation are reduced. For each vertex v of $\mathcal{X}$, we introduce the *empty path* 1_v. This path has no edges ($L(\alpha)=o$). Morever, $\iota(1_v)=\tau(1_v)=v$ and $1_v^{-1}=1_v$. Sometimes we simply write 1 for an empty path if the particular vertex is clear.

We say that the product $\alpha\beta$ of two paths α, β *is defined* if $\tau(\alpha)=\iota(\beta)$. Then $\alpha\beta$ is the path consisting of the edges of α followed by the edges of β.

The set of all paths in $\mathcal{X}$ will be denoted by $P(\mathcal{X})$.

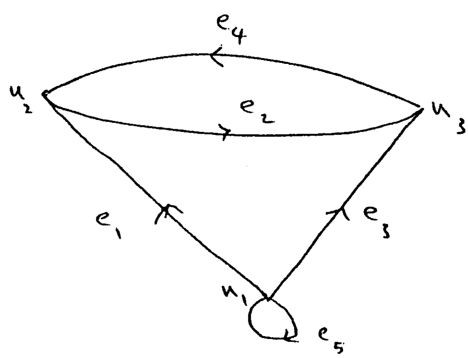
If α is a path in 1-complex and e is an edge then $L_e(\alpha)$ is the number of times e and e^{-1} *appear* in α. The *exponent sum* of e in α, denoted by $\sigma_e(\alpha)$, is the difference of the number of times e appears in α and the number of times e^{-1} appears in α. We say that e is *involved* in α if either e or e^{-1} appears in α.

We say that $\mathcal{X}$ is *connected* if, given any two vertices u, v of $\mathcal{X}$ there is a path γ such that $\iota(\gamma)=u$, $\tau(\gamma)=v$. A *subcomplex* of a 1-complex $\mathcal{X}$ is a subset of $V(\mathcal{X}) \cup E(\mathcal{X})$ which is closed under $\iota, \tau, {}^{-1}$. If $V_o \subseteq V(\mathcal{X})$ then the *full subcomplex* on V_o consists of V_o together with all edges e of $\mathcal{X}$ where both $\iota(e)$, $\tau(e)$ belongs to V_o. A maximal connected subcomplex of a 1-complex is called a *component.* If v is a vertex in a 1-complex $\mathcal{X}$ then

$$\mathrm{Star}(v)=\{\ e: e\in E(\mathcal{X}),\ \iota(e)=v\ \}.$$

A 1-complexes can be represented diagramatically as follows. A vertex is represented by a point. For each geometric edge $\{e, e^{-1}\}$, we select one of the pair say e and a draw segment (labelled e) joining the point corresponding to $\iota(e)$ to the point corresponding to $\tau(e)$.

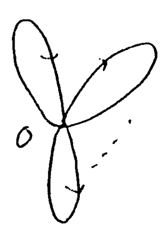
Example 1.1A.1.



This represents a 1-complex with 3-vertices u_1, u_2, u_3 and 10 edges e_i, e_i^{-1} $(1\leq i\leq 5)$. We have $\iota(e_1)=\iota(e_3)=\iota(e_5)=\tau(e_5)=u_1$, $\iota(e_2)=\tau(e_4)=\tau(e_1)=u_2$, $\iota(e_4)=\tau(e_2)=\tau(e_3)$.

A *tree* is a connected 1-complex in which no non-empty closed path is reduced.

A 1-complex with a single vertex is called a *bouquet*



1.1A.2. Mappings of 1-complexes.

Let $\mathcal{X}$, $\mathcal{Y}$ be 1-complexes. A *mapping of 1-complexes* from $\mathcal{X}$ to $\mathcal{Y}$ is a function

$$\phi\colon V(\mathcal{X})\cup E(\mathcal{X}) \to V(\mathcal{Y})\cup P(\mathcal{Y})$$

which sends vertices of $\mathcal{X}$ to vertices of $\mathcal{Y}$, and edges in $\mathcal{X}$ to paths in $\mathcal{Y}$ and satisfies for each $e\in\mathcal{X}$ $\phi(\iota(e))=\iota(\phi(e))$, $\phi(\tau(e))=\tau(\phi(e))$, $\phi(e^{-1})=\phi(e)^{-1}$. We can extend ϕ to a function (also denoted ϕ) from $V(\mathcal{X})\cup P(\mathcal{X})$ to $V(\mathcal{Y})\cup P(\mathcal{Y})$ as follows. Let α be a non-empty path in $\mathcal{X}$, say $\alpha=e_1e_2\ldots e_n$ $(n>0)$. We define $\phi(\alpha)$ to be the path $\phi(e_1)\phi(e_2)\ldots\phi(e_n)$. For an empty path 1_v we define $\phi(1_v)$ to be $1_{\phi(v)}$.

A mapping ϕ is said to be *rigid* if it preserve length, that is, $L(\phi(\alpha))=L(\alpha)$ for all paths α (ϕ takes edges to edges).

Suppose $\phi\colon \tilde{\mathcal{X}} \to \mathcal{X}$ is rigid, and let $\tilde{v}\in\tilde{\mathcal{X}}$. If $\tilde{e}\in\mathrm{Star}(\tilde{v})$ then $\phi(\tilde{e})\in\mathrm{Star}(\phi(\tilde{v}))$, thus $\phi\mathrm{Star}(\tilde{v})\subseteq \mathrm{Star}(\phi(\tilde{v}))$. We say that ϕ is *locally injective (resp. locally sutjective, locally bijective)* if $\phi\colon \mathrm{Star}(\tilde{v}) \to \mathrm{Star}(\phi(\tilde{v}))$ is injective (resp. surjective, bijective) for all $\tilde{v}\in\tilde{\mathcal{X}}$.

Let u be a vertex of $\mathcal{X}$ and let $\tilde{u}$ be a vertex of $\tilde{\mathcal{X}}$ such that $\phi(\tilde{u})=u$. We say that $\tilde{u}$ *lies over u*. If α is a path in $\mathcal{X}$ with $\iota(\alpha)=u$ then a path $\tilde{\alpha}$ in $\tilde{\mathcal{X}}$ such that $\iota(\tilde{\alpha})=\tilde{u}$ and $\phi(\tilde{\alpha})=\alpha$ is called a *lift* of α at $\tilde{u}$.

Lemma 1.1A.1. *Let ϕ be a rigid mapping of 1-complexes from $\tilde{\mathcal{X}}$ to $\mathcal{X}$. The following are equivalent.*

(i) For any path α in $\mathcal{X}$ and any vertex $\tilde{v}$ of $\tilde{\mathcal{X}}$ with $\phi(\tilde{v})=\iota(\alpha)$ there is at least one lift of α at $\tilde{v}$.

(ii) ϕ is locally surjective.

Proof.

(i) implies (ii). Let $\tilde{v}\in V(\tilde{\mathcal{X}})$ and let $e\in Star(\phi(\tilde{v}))$. Then by (i), the path consisting of e has a lift at $\tilde{v}$. This edge will be an edge in $Star(\tilde{v})$. So ϕ is locally surjective.

(ii) implies (i). We use induction on $L(\alpha)$. If $L(\alpha)=0$ there is nothing to prove. Suppose $L(\alpha)>0$ and write $\alpha=\beta e$ with e an edge. By induction hypothesis there is a lift $\tilde{\beta}$ of β at $\tilde{v}$. Suppose $\tau(\tilde{\beta})=\tilde{u}$. Since $e\in Star(\phi(\tilde{u}))$, and since ϕ is locally surjective, we have an edge $\tilde{e}\in Star(\tilde{u})$ with $\phi(\tilde{e})=e$. Then $\tilde{\beta}\tilde{e}$ is a lift of α at $\tilde{v}$.

Lemma 1.1A.2. *Let ϕ be a rigid mapping of 1-complexes from $\tilde{\mathcal{X}}$ to $\mathcal{X}$. The following are equivalent.*

(i) For any path α *in* $\mathcal{X}$ *and any vertex* $\tilde{v}$ *of* $\tilde{\mathcal{X}}$ *with* $\phi(\tilde{v})=\iota(\alpha)$ *there is at most one lift of* α *at* $\tilde{v}$.

(ii) ϕ *is locally injective.*

Proof. (i) implies (ii). Let $\tilde{v}\in V(\tilde{\mathcal{X}})$ and let $\tilde{e}_1,\tilde{e}_2\in \mathrm{Star}(\tilde{v})$. Suppose $\phi(\tilde{e}_1)=\phi(\tilde{e}_2)=e$. Then $\tilde{e}_1,\tilde{e}_2$ are lifts of the path in $\mathcal{X}$ consisting of e, so $\tilde{e}_1=\tilde{e}_2$ which implies that ϕ is locally injective.

(ii) implies (i). We use induction on $L(\alpha)$. If $L(\alpha)=0$ the result is obvious. Suppose $L(\alpha)>0$, and write $\alpha=\beta e$ with e an edge. Let $\tilde{\alpha}_1,\tilde{\alpha}_2$ be lifts of α at $\tilde{v}$. Then $\tilde{\alpha}_1=\tilde{\beta}_1\tilde{e}_1$, $\tilde{\alpha}_2=\tilde{\beta}_2\tilde{e}_2$ where $\phi(\tilde{\beta}_1)=\phi(\tilde{\beta}_2)=\beta$, $\phi(\tilde{e}_1)=\phi(\tilde{e}_2)=e$. Since $\tilde{\beta}_1,\tilde{\beta}_2$ are both lifts of β at $\tilde{v}$, $\tilde{\beta}_1=\tilde{\beta}_2$ by induction hypothesis. Let $\tilde{u}=\tau(\tilde{\beta}_1)$. Then $\tilde{e}_1,\tilde{e}_2\in \mathrm{Star}(\tilde{u})$ and since ϕ is locally injective we have $\tilde{e}_1=\tilde{e}_2$. Thus $\tilde{\alpha}_1=\tilde{\alpha}_2$.

Thus the locally bijectively of ϕ is equivalent to the condition that all possible lifts of paths exist and are unique.

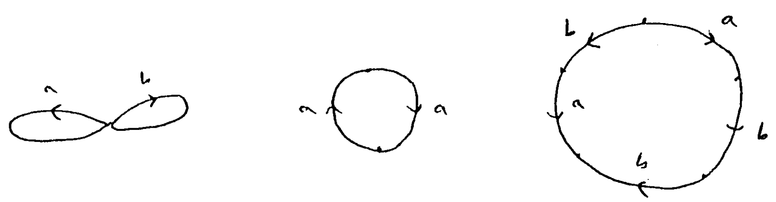
1.1B.1 2-complexes.

A *2-complex* $\mathcal{K}$ is an object $\langle\, \mathcal{X} : \rho_\lambda \ (\lambda\in\Lambda)\,\rangle$ where $\mathcal{X}$ is a 1-complex (called the *1-skeleton* of $\mathcal{K}$, and often denoted by $\mathcal{K}^{(1)}$) and the ρ_λ are a closed paths in $\mathcal{K}$. The ρ_λ are called ***defining paths***. The elements of Λ are called *indices*.

For a 2-complex K, we define $R(K)$ to be the set of cyclic permutation of non-empty defining paths and their inverses. We say that a 2-complex K is *finite* if $V(K)$, $E(K)$, Λ are all finite. We then define the *Euler characterstic* $\chi(K)$ to be

$$|V(K)|-1/2|E(K)|+|\Lambda|.$$

A 2-complex with a single vertex is called a *presentation*. If $\langle \mathcal{G} ; \rho_\lambda(\lambda\in\Lambda) \rangle$ is a presentation and if the edges of $\mathcal{G}$ are $y_1, y_1^{-1}, y_2, y_2^{-1}, \ldots$ then we will often use the more standard notation $\langle y_1, y_2, \ldots ; \rho_\lambda(\lambda\in\Lambda) \rangle$ for the presentation. Also we use $\langle y ; r \rangle$ if $\rho_\lambda(\lambda\in\Lambda)$ is a set and where y is the set of edges $y_1, y_2, y_3, \ldots$. Instead of $R(P)$ we sometimes use r^*. As an example, consider the presentation $\langle a, b ; a^2, b^2a^{-1}b^{-1}a \rangle$. The 2-complex associated with this presentation is



1.1B.2 Fundamental group of 2-complexes.

Let K be a 2-complex. There are two elementary transformations of paths in K.

(I) Deletion of an inverse pair ee^{-1} of succesive edges;

(II) If we have a path $\alpha=\alpha_1\gamma\alpha_2$ and $\rho=\gamma\delta$ where $\rho\in R(K)$, then replace

γ by δ^{-1} in α.

Two paths α,β are *equivalent* written $\alpha \underset{K}{\sim} \beta$ (or simply $\alpha \sim \beta$ if is understood) if there is a sequence of paths $\alpha=\alpha_0,\alpha_1,\ldots,\alpha_n=\beta$ where one of α_i,α_{i+1} is obtained from the other by operation (I) or (II). The equivalence class of a given path α with respect to this equivalence relation is denoted by $[\alpha]_K$. (or simply $[\alpha]$)

Let v be a fixed vertex of K. We define the *fundamental group* $\pi_1(K,v)$ *at* v to have underlying set

$$\{[\alpha]: \alpha \text{ a closed path with } \iota(\alpha)=v\}.$$

The vertex v is called the *base vertex*. The multiplication is defined by $[\alpha][\beta]=[\alpha\beta]$, the inverse $[\alpha]^{-1}$ is defined to be $[\alpha^{-1}]$, and the identity 1 is defined to be $[1_v]$. The multiplication is readily checked to be well defined. If K is connected then the fundamental group obtained is independent of our choice of the base vertex v, that is, if v_1,v_2 are two vertices of K then, it can be shown that $\pi_1(K,v_1)\cong\pi_1(K,v_2)$. (Consequently, we sometimes talk about *the* fundamental group of a connected 2-complex, that is, if K is connected we refer to the fundamental group of K and write $\pi_1(K)$.)

Two paths α, α' are freely equivalent, wtitten $\alpha \overset{I}{\sim} \alpha'$ if there is a sequence of paths $\alpha=\alpha_0, \alpha_1, \ldots, \alpha_n=\alpha'$ where one of α_i, α_{i+1} is obtained from the other by an operation of type (I).

A path which is equivalent to an empty path is said to be *contractible*. If α is contractible we define the *degree* of α, denoted by $\deg(\alpha)$, to be the smallest number of operations of type (II) used in any transformation of α to 1.

Lemma 1.1B.1. *If α is contractible with $\deg(\alpha)=n$ then $\alpha \overset{I}{\sim} (\gamma_1\beta_1\gamma_1^{-1})(\gamma_2\beta_2\gamma_2^{-1})\ldots(\gamma_n\beta_n\gamma_n^{-1})$ where the γ_i are suitable paths, and the $\beta_i \in R(K)$.*

Proof. We use induction on $\deg(\alpha)$. If n=0, then α is freely equivalent to 1. Now let n>o. Then we have a finite sequence of paths $\alpha=\alpha_0, \alpha_1, \ldots, \alpha_p=1$ where for $i=0,1,\ldots,p-1$ one of α_i, α_{i+1} is obtained from the other by operation of type (I) or (II). Let k be the first value of i for which an operation of type (II) is used. Then $\deg(\alpha_{k+1})=n-1$. Then by induction hypothesis we have

$$\alpha_{k+1} \overset{I}{\sim} (\gamma_1\beta_1\gamma_1^{-1})\ldots(\gamma_{n-1}\beta_{n-1}\gamma^{-1}{}_{n-1})$$ for suitable paths γ_j and $\beta_j \in R(K)$, $j=1,\ldots,n-1$. Now let $\alpha_k=\lambda_1\mu\lambda_2$ and $\alpha_{k+1}=\lambda_1\nu\lambda_2$ where $\mu\nu^{-1} \in R(K)$

We have

$\alpha \overset{!}{\sim} \alpha_k$ (since all operation of type (i))

$=\lambda_1\mu\lambda_2 \overset{!}{\sim} \lambda_1\nu\nu^{-1}\mu\lambda_2 \overset{!}{\sim} \lambda_1\nu\lambda_2[\lambda_2^{-1}(\nu^{-1}\mu)\lambda_2]$

$\overset{!}{\sim} (\gamma_1\beta_1\gamma_1^{-1})\ldots(\gamma_{n-1}\beta_{n-1}\gamma^{-1}{}_{n-1})[\lambda_2^{-1}(\nu^{-1}\mu)\lambda_2]$

$=(\gamma_1\beta_1\gamma_1^{-1})\ldots(\gamma_{n-1}\beta_{n-1}\gamma^{-1}{}_{n-1})(\gamma_n\beta_n\gamma_n^{-1})$

where $\lambda_2^{-1}=\gamma_n$ and $\nu^{-1}\mu=\beta_n$.

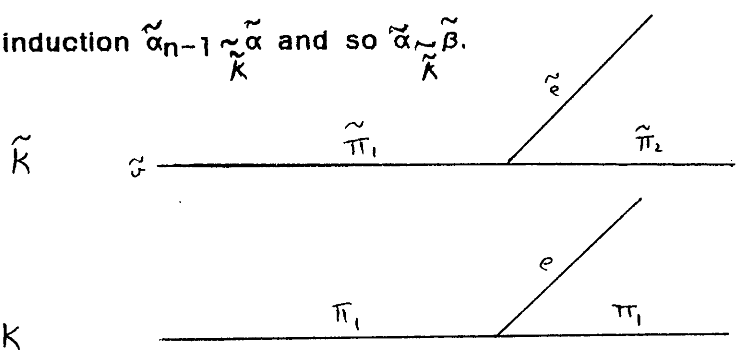
1.1B.3 Mappings of 2-complexes.

Let $\mathcal{K}, \mathcal{L}$ be 2-complexes. A *mapping of 2-complexes* ϕ: $\mathcal{K} \to \mathcal{L}$ is a mapping of 1-complexes from $\mathcal{K}^{(1)}$ to $\mathcal{L}^{(1)}$ with the property that the image of each contractible path of $\mathcal{K}$ is a contractible path of $\mathcal{L}$. If ϕ: $\mathcal{K} \to \mathcal{L}$ is a mapping of 2-complexes then sometimes for emphasis we will denote the underlying mapping of 1-complexes from $\mathcal{K}^{(1)}$ to $\mathcal{L}^{(1)}$ by $\phi^{(1)}$.

We say that a mapping of 2-complexes ϕ: $\tilde{\mathcal{K}} \to \mathcal{K}$ is *strong* if it does not map any edge to an empty path, and if $\phi R(\tilde{\mathcal{K}}) \subseteq R(\mathcal{K})$. We say that ϕ is *locally bijective* if $\phi^{(1)}$ is also locally bijective and $\phi^{-1}R(\mathcal{K})=R(\tilde{\mathcal{K}})$. If, in addition, $\tilde{\mathcal{K}}$ and $\mathcal{K}$ are connected then ϕ is called a *covering*.

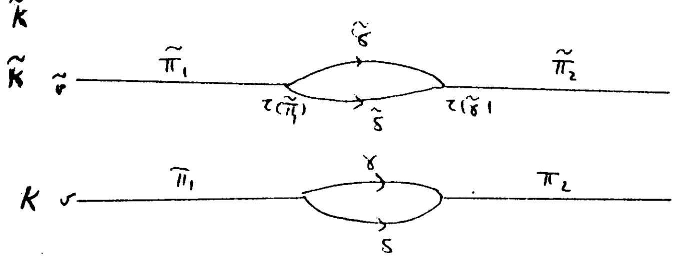
Theorem 1.1B.1. *Let $\phi: \tilde{K} \to K$ be locally bijective, let α, β be paths in K with $\iota(\alpha)=\iota(\beta)$, and let $\tilde{\alpha}, \tilde{\beta}$ be lifts of α, β with $\iota(\tilde{\alpha})=\iota(\tilde{\beta})$. If $\alpha \underset{K}{\sim} \beta$ then $\tilde{\alpha} \underset{\tilde{K}}{\sim} \tilde{\beta}$.*

Proof. Let α, β be two paths in K with $\iota(\alpha)=\iota(\beta)$, and $\tilde{\alpha}, \tilde{\beta}$ are lifts of α, β with $\iota(\tilde{\alpha})=\iota(\tilde{\beta})$. We will show that if $\alpha \underset{K}{\sim} \beta$ then $\tilde{\alpha} \underset{\tilde{K}}{\sim} \tilde{\beta}$. For let $\alpha \underset{K}{\sim} \beta$, that is, there is a sequence of paths $\alpha=\alpha_0, \alpha_1, \ldots, \alpha_n=\beta$ where one of α_i, α_{i+1} is obtained from the other by an operation of type (I) or (II). We use induction on n. If n=o there is nothing to prove. Now let n>o. Suppose first $\alpha_{n-1}=\pi_1\pi_2$, and $\beta=\pi_1 e e^{-1}\pi_2$ and let $\tilde{\pi}_1$ denote the unique lift of π_1 at $\tilde{v}$, and let $\tilde{\pi}_2$ be the unique lift of π_2 at the vertex $\tau(\tilde{\pi}_1)$, so that $\tilde{\alpha}_{n-1}=\tilde{\pi}_1\tilde{\pi}_2$. But the lift of β at $\tilde{v}$ is then $\tilde{\pi}_1\tilde{e}\tilde{e}^{-1}\tilde{\pi}_2$, where $\tilde{e}$ is the unique edge such that $\iota(\tilde{e})=\tau(\tilde{\pi}_1)$ and $\phi(\tilde{e})=e$ therefore $\tilde{\beta}=\tilde{\pi}_1\tilde{e}\tilde{e}^{-1}\tilde{\pi}_2 \underset{\tilde{K}}{\sim} \tilde{\pi}_1\tilde{\pi}_2=\tilde{\alpha}_{n-1}$ and by induction $\tilde{\alpha}_{n-1} \underset{\tilde{K}}{\sim} \tilde{\alpha}$ and so $\tilde{\alpha} \underset{\tilde{K}}{\sim} \tilde{\beta}$.



Now suppose that $\alpha_{n-1}=\pi_1\gamma\pi_2$ and $\beta=\pi_1\delta^{-1}\pi_2$ where $\rho=\gamma\delta\in R(K)$. Consider the lift $\tilde{\alpha}_{n-1}$ of α_{n-1} at $\tilde{v}$. We can write this as $\tilde{\pi}_1\tilde{\gamma}\tilde{\pi}_2$

where $\tilde{\pi}_1$ is the lift of π_1 at $\tilde{v}$, $\tilde{\gamma}$ is the lift of γ at $\tau(\tilde{\pi}_1)$, $\tilde{\pi}_2$ is the lift of π_2 at $\tau(\tilde{\gamma})$. Now by assumption the lift $\tilde{\rho}$ of ρ at $\tau(\tilde{\pi}_1)$ belongs to $R(\tilde{K})$. We have $\tilde{\rho}=\tilde{\gamma}\tilde{\delta}$ where $\tilde{\delta}$ is the lift of δ at $\tau(\tilde{\gamma})$. Now $\tilde{\pi}_1\tilde{\delta}^{-1}\tilde{\pi}_2$ starts at $\tilde{v}$ and is mapped onto β, so $\tilde{\beta}=\tilde{\pi}_1\tilde{\delta}^{-1}\tilde{\pi}_2$ by uniqueness. Since $\tilde{\pi}_1\tilde{\delta}^{-1}\tilde{\pi}_2$ is obtained from $\tilde{\alpha}_{n-1}$ by an operation of type (II), then we have $\tilde{\beta}=\tilde{\pi}_1\tilde{\delta}^{-1}\tilde{\pi}_2 \underset{\tilde{K}}{\sim} \tilde{\pi}_1\tilde{\pi}_2$ and by induction $\tilde{\alpha} \underset{\tilde{K}}{\sim} \tilde{\alpha}_{n-1}$ and so $\tilde{\alpha} \underset{\tilde{K}}{\sim} \tilde{\beta}$ as required.



Theorem 1.1B.2. *If $\phi: \tilde{K} \to K$ is locally bijective and $\tilde{v}$ is a vertex of $\tilde{K}$ then the induced homomorphism*

$$\phi_*: \pi_1(\tilde{K},\tilde{v}) \to \pi_1(K,v)$$

defined by $\phi_*[\tilde{\alpha}]=[\phi(\tilde{\alpha})]$ ($[\tilde{\alpha}]\in\pi_1(\tilde{K},\tilde{v})$) is injective.

Proof.

Suppose $[\tilde{\alpha}]\in \mathrm{Ker}\,\phi_*$. Then $[\phi(\tilde{\alpha})]=[1_v]$, that is, $\phi(\tilde{\alpha})\sim 1_v$. Now $\tilde{\alpha}$ and $1_{\tilde{v}}$ are the lifts of $\phi(\tilde{\alpha})$, 1_v respectively. Thus $\tilde{\alpha}\sim 1_{\tilde{v}}$ by Theorem 1.1B.1 above. Hence $[\tilde{\alpha}]=[1_{\tilde{v}}]$.

Theorem 1.1B.3. *Let $\phi: \tilde{K} \to K$ be locally bijective and let α be a closed path at v and suppose $\tilde{\alpha}$ is the lift of α at $\tilde{v}$. Then $\tilde{\alpha}$ is closed if and only if $[\alpha]\in\phi_*\pi_1(\tilde{K},\tilde{v})$.*

Proof.

If $\tilde{\alpha}$ is closed then $[\alpha]=[\phi(\tilde{\alpha})]=\phi_*[\tilde{\alpha}]\epsilon\phi_*\pi_1(\tilde{K},\tilde{v})$. Conversely if $[\alpha]\epsilon\phi_*\pi_1(\tilde{K},\tilde{v})$, then $[\alpha]=[\phi(\tilde{\beta})]$ for some closed path $\tilde{\beta}$ at $\tilde{v}$. Thus $\alpha\sim\phi(\tilde{\beta})$, and so $\tilde{\alpha}\sim\tilde{\beta}$ by Theorem 1.1B.1. In particular $\tau(\tilde{\alpha})=\tau(\tilde{\beta})=\tilde{v}$.

<u>1.1B.4 Representing subgroups by coverings</u>.

Consider a *connected 2-complex*, $K=\langle X;\ \rho_\lambda(\lambda\epsilon\Lambda)\rangle$ with basepoint v. *Let H be a subgroup of* $\pi_1(K,v)$. We will construct a covering

$$\phi_H : (K_H, v_H) \to (K, v)$$

such that $\phi_{H*}\ \pi_1(K_H,v_H)=H$.

We first construct the 1-skeleton X_H of K_H. Let

$$X = \{ [\alpha] : \iota(\alpha)=v \}.$$

We say that two elements $[\alpha],[\beta]$ of X are *equivalent mod H* if $\tau(\alpha)=\tau(\beta)$ and $[\alpha\beta^{-1}]\epsilon H$. We will show that "equivalent mod H" is an equivalence relation. For since $[\alpha\alpha^{-1}]=[1_v]\epsilon H$ then $[\alpha]=[\alpha]$ mod H, that is, it is *reflexive*. Let $[\alpha]=[\beta]$ mod H, then by the definition $\tau(\alpha)=\tau(\beta)$, $[\alpha\beta^{-1}]\epsilon H$. Since H is closed under inverses then $[\beta^{-1}\alpha]\epsilon H$. Thus $[\beta]=[\alpha]$ mod H. Then it is *symmetric*. Also it is *transitive* for, if $[\alpha]=[\beta]$ mod H, $[\beta]=[\gamma]$ mod H. We have $\tau(\alpha)=\tau(\beta)$, $[\alpha\beta^{-1}]\in H$, $\tau(\beta)=\tau(\gamma)$, $[\beta\gamma^{-1}]\in H$ which implies that $\tau(\alpha)=\tau(\gamma)$, $[\alpha\beta^{-1}\beta\gamma^{-1}]=[\alpha\gamma^{-1}]\in H$. Then "equivalent mod H" is an equivalence relation.

The equivalence class of $[\alpha]$ is the set of all elements $[\beta]\in X$ such that $[\alpha]=[\beta]$ mod H, that is, $\{[\beta]: [\alpha\beta^{-1}]\in H\}$ we will call this set E_1. We show that E_1 is equal to $E_2=\{ [\gamma][\alpha] : [\gamma]\in H \}$. For let $[\beta]\in E_1$ then $[\alpha\beta^{-1}]\in H$, that is, $[\beta\alpha^{-1}]\in H$ then $[\beta]=[\beta\alpha^{-1}][\alpha]$ which implies that $[\beta]\in E_2$. We have $E_1\subseteq E_2$. Now let $[\gamma][\alpha]\in E_2$, then $[\gamma]\in H$. We have $[\gamma][\alpha]\in E_1$ since $[\gamma]=[\gamma][\alpha][\alpha^{-1}]\in H$. Then $E_2\subseteq E_1$. Thus $E_1=E_2$. We denoted the set $\{[\gamma][\alpha]: [\gamma]\in H\}$ by $H[\alpha]$.

We define the 1-skeleton $\mathcal{X}_H$ of $\mathcal{K}_H$ as follows:

vertices $H[\alpha]$, $[\alpha]\in X$.

edges $(H[\alpha],e)$, $e\in E(\mathcal{X})$, $[\alpha]\in X$, $\tau(\alpha)=\iota(e)$.

For an edge $(H[\alpha],e)$, we set $\iota(H[\alpha],e)=H[\alpha]$, $\tau(H[\alpha],e)=H[\alpha e]$, $(H[\alpha],e)^{-1}=(H[\alpha e],e^{-1})$.

$\mathcal{X}_H$ is a 1-complex. For since $\tau(H[\alpha],e)=H[\alpha e]$, $\iota(H[\alpha],e)^{-1}=\iota(H[\alpha e],e^{-1})=H[\alpha e]$, that is, $\tau(H[\alpha],e)=\iota(H[\alpha],e)^{-1}$. Now $((H[\alpha],e)^{-1})^{-1}=(H[\alpha e],e^{-1}))^{-1}$

$=H[\alpha e e^{-1}], e) = (H[\alpha], e)$, that is, $((H[\alpha], e)^{-1})^{-1} = (H[\alpha], e)$.

Also $(H[\alpha], e)^{-1} = (H[\alpha e], e^{-1}) \neq (H[\alpha], e)$ since $e \neq e^{-1}$.

We take v_H to be the vertex $H[1_v]$. Note that $\mathfrak{X}_H$ is connected. For if $H[\alpha]$ is a vertex of $\mathfrak{X}_H$, let $\alpha = e_1 e_2 \ldots e_n$. Then $(H[1_v], e_1)(H[e_1], e_2)(H[e_1 e_2], e_3) \ldots (H[e_1 \ldots e_{n-1}], e_n)$ is a path in $\mathfrak{X}_H$ from v_H to $H[\alpha]$.

There is a locally bijective mapping of 1-complexes

$$\phi_H^{(1)} : \mathfrak{X}_H \to \mathfrak{X}$$

which takes $H[\alpha]$ to $\tau(\alpha)$ $(H[\alpha] \in V(\mathfrak{X}_H))$ and $(H[\alpha], e)$ to e ($(H[\alpha], e)$ an edge of $\mathfrak{X}_H$). For first we show that $\phi_H^{(1)}$ is a mapping of 1-complexes. We have $\phi_H^{(1)}(\iota(H[\alpha], e) = \phi_H^{(1)}(H[\alpha]) = \tau(\alpha)$, $\iota(\phi_H^{(1)}(H[\alpha], e)) = \iota(e)$. Since $\iota(e) = \tau(\alpha)$ we have that $\phi_H^{(1)}(\iota(H[\alpha], e) = \iota(\phi_H^{(1)}(H[\alpha], e))$. Also since $\tau(\phi_H^{(1)}(H[\alpha], e)) = \tau(e)$,
$\phi_H^{(1)}(\tau(H[\alpha], e)) = \phi_H^{(1)}(H[\alpha e]) = \tau(\alpha e) = \tau(e)$. Then we have $\phi_H^{(1)}(\tau(H[\alpha], e) = \tau(\phi_H^{(1)}(H[\alpha], e)$.

Now $\phi_H^{(1)}(H[\alpha], e)^{-1}) = \phi_H^{(1)}(H[\alpha e], e^{-1}) = e^{-1}$, and $(\phi_H^{(1)}(H[\alpha], e))^{-1} = (e)^{-1} = e^{-1}$. Then $\phi_H^{(1)}(H[\alpha], e)^{-1}) = (\phi_H^{(1)}(H[\alpha], e))^{-1}$. Now we show that $\phi_H^{(1)}$ is locally bijective. For let $H[\alpha]$ be a vertex of $\mathcal{X}_H$, and let $e \in Star(\tau(\alpha))$. Then $(H[\alpha], e) \in Star(H[\alpha])$, that is, $\phi_H^{(1)}$ is locally surjective. Also $\phi_H^{(1)}$ is locally injective for, let $H[\alpha]$ be a vertex of $\mathcal{X}_H$ and let $(H[\alpha], e_1), (H[\alpha], e_2) \in Star(H[\alpha])$. Suppose $\phi_H^{(1)}(H[\alpha], e_1) = \phi_H^{(1)}(H[\alpha], e_2)$. Then we have that $e_1 = e_2$ by the definition. We have $(H[\alpha], e_1) = (H[\alpha], e_2)$, that is, $\phi_H^{(1)}$ is locally injective. Thus $\phi_H^{(1)}$ is locally bijective.

If $\beta = f_1 f_2 \ldots f_n$ is a path in $\mathcal{X}$ and if $H[\alpha]$ lies over $\iota(\beta)$, then the lift of β at $H[\alpha]$ is

$$(H[\alpha], f_1)(H[\alpha f_1], f_2)(H[\alpha f_1 f_2], f_3) \ldots (H[\alpha f_1 \ldots f_{n-1}], f_n).$$

Note that this lift is closed if and only if $H[\alpha] = H[\alpha\beta]$, that is, if and only if $[\alpha][\beta][\alpha^{-1}] \in H$.

Let

$$\Lambda_H = \{ (\lambda, H[\alpha]) : \lambda \in \Lambda, [\alpha] \in X, \tau(\alpha) = \iota(\rho_\lambda) \}.$$

For $(\lambda, H[\alpha]) \in \Lambda_H$ let $\rho(\lambda, H[\alpha])$ be the lift of ρ_λ at $H[\alpha]$. This lift is closed. By the perivous paragraph the lift ends at $H[\alpha\rho_\lambda]$. Then this lift will be closed since $H[\alpha\rho_\lambda] = H[\alpha]$.

We let

$$\mathcal{K}_H = \langle \mathcal{X}_H ; \rho(\lambda, H[\alpha]) \ ((\lambda, H[\alpha]) \in \Lambda_H) \rangle$$

Then we have a mapping of 2-complexes

$$\phi_H : K_H \to K$$

which is locally bijective since $\phi_H^{(1)}$ is locally bijective and

$\phi_H^{-1}(R(K))=R(K_H)$ by the construction.

To show that $\phi_*\pi_1(K_H, v_H)=H$. We have by Theorem 1.1B.3 it sufficies to show that if α is closed path at v then the lift $\tilde{\alpha}$ of α at v_H is closed if and only if $[\alpha]\epsilon H$. Suppose $\alpha=e_1e_2...e_n$ then $\tilde{\alpha}=(H[\alpha],e_1)(H[\alpha e_1],e_2)...(H[\alpha e_1...e_{n-1}],e_n)$. Since $\tau(\alpha)=H[\alpha]$, so $\tilde{\alpha}$ is closed if and only if $[\alpha]\epsilon H$.

As an example let K=< 

; $ab^3a^{-1}b^3$ >.

Consider the homomorphism of $\pi_1(K,o)$ onto S_3 defined by a→(12), b→(123). Let H be the kernel of this homomorphisn. A transversal for H in $\pi_1(K,o)$ is [1], [a], [b], [b^2], [ab], [ab^2].

Then K_H has vertices

u_1=H[1], u_2=H[a], u_3=H[b], u_4=H[b^2], u_5=H[ab], u_6=H[ab^2], and

the edges are

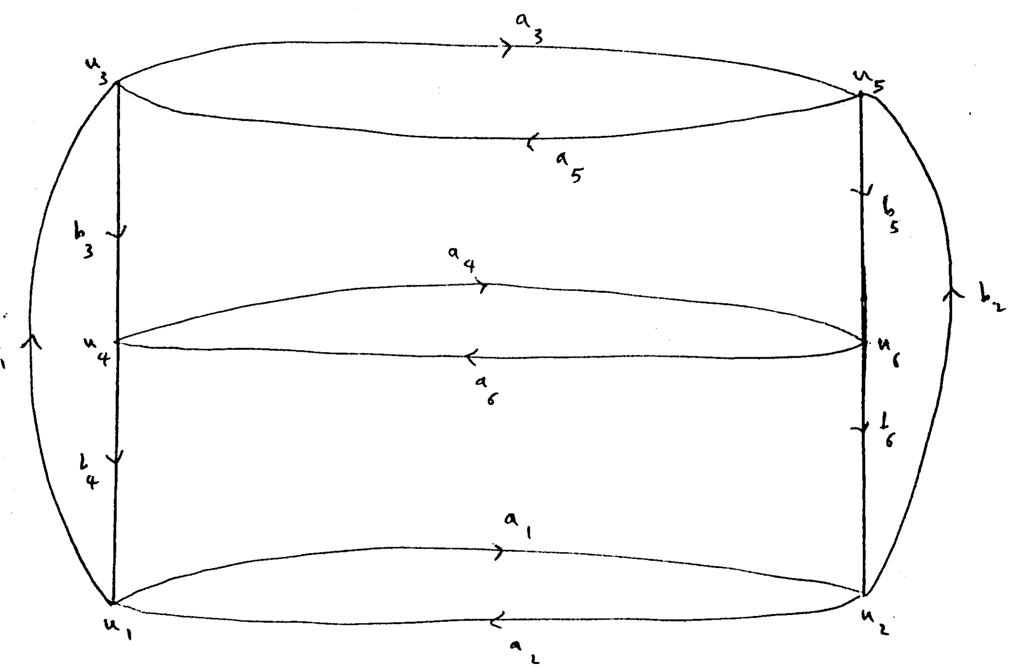
a_1=(H[1],a), a_2(H[a],a), a_3=(H[b],a), a_4=(H[b^2],a),

a_5=(H[ab],a), a_6=(H[ab^2],a),

b_1=(H[1],b), b_2=(H[a],b), b_3=(H[b],b), b_4=(H[b^2],b),

$b_5=(H[ab], b)$, $b_6=(H[ab^2], b)$.

Then K_H has 1-skeleton



The lifts of defining path $\rho=ab^3a^{-1}b^3$ are

$a_1b_2b_5b_6a_1^{-1}b_1b_3b_4$,

$a_2b_1b_3b_4a_2^{-1}b_2b_5b_6$,

$a_3b_5b_6b_2a_3^{-1}b_3b_4b_1$,

$a_4b_6b_2b_5a_4^{-1}b_4b_1b_3$,

$a_5b_3b_4b_1a_5^{-1}b_5b_6b_2$,

$a_6b_4b_1b_3a_6^{-1}b_6b_2b_5$.

1.1B.5. Equivalence of complexes.

Two mappings ϕ,ψ: $\mathcal{K}\to\mathcal{L}$ are said to be *homotopic* (written $\phi\approx\psi$) if $\phi(\alpha)\sim\psi(\alpha)$ for all paths α in $\mathcal{K}$.

A mapping ϕ: $\mathcal{K}\to\mathcal{L}$ is called an *equivalence* if there is a mapping θ: $\mathcal{L}\to\mathcal{K}$ such that $\phi\theta\approx id_{\mathcal{L}}$, $\theta\phi\approx id_{\mathcal{K}}$.

Two complexes are said to be *equivalent* if there is an equivalence between them.

The Level Theorem. [10], [11]

Let $\mathcal{K}=\langle\ \mathfrak{X}:\ \rho_\lambda(\lambda\in\Lambda)\ \rangle$. Consider a partition of Λ

$$\Lambda = \bigcup_{\ell=o}^{\infty} \Lambda_\ell\ .$$

An element of Λ_ℓ will be said of *level* ℓ. Assume that the following condition is satisfied: if λ has level $>o$, then some cyclic permutation of ρ_λ has the form $e_\lambda\alpha_\lambda^{-1}$, where e_λ is an edge, $L_{e_\lambda}(\alpha_\lambda)=0$, and $L_{e_\lambda}(\rho_\mu)=0$ $(\mu\neq\lambda)$ with μ of level k, $o<k\leqslant\ell$. We call e_λ *the edge associated with* ρ_λ. Let

$$\mathfrak{X}_o = \mathfrak{X} - \{\ e_\lambda^{\pm 1}\ :\ \lambda \text{ has level } >o\ \}$$

and for $\ell>o$ let

$$\mathfrak{X}_\ell = \mathfrak{X}_{\ell-1} \cup \{\ e_\lambda^{\pm 1}\ :\ \lambda\in\Lambda_\ell\ \}.$$

We note that if λ is of level $\ell>o$, then α_λ is a path in $\mathcal{K}_{\ell-1}$. For α_λ is a path in $\mathcal{K}_{\ell-1}$ if none of its edges are in $\{e_\mu^{\pm 1}: \mu \text{ of level} \geq \ell\}$, so we must show that there is no edge of this set occurs in α_λ. We have by the definition that any edge of this set does not appear in any other defining paths of level $k \leq \ell$ with $k>o$. Thus α_λ is a path in $\mathcal{K}_{\ell-1}$.

Define ϕ: $\mathcal{K} \to \mathcal{K}_o$ as follows. First define ϕ on $\mathcal{K}_o$ to be the identity. Suppose ϕ has defined on $\mathcal{K}_{\ell-1}$ $(\ell>o)$. Extend ϕ to $\mathcal{K}_\ell$ by setting $\phi(e_\lambda)=\phi(\alpha_\lambda)$ $(\lambda\in\Lambda_\ell)$. We note that

(i) $\phi(\rho_\mu)$ is freely equal to an empty path if μ has level>o.

(ii) for any path α in $\mathcal{K}$, $\phi(\alpha)\underset{K}{\sim}\alpha$.

To prove (i). We have $\phi(\rho_\mu)=\phi(e_\mu\alpha_\mu^{-1})\overset{1}{\sim}1$ (since $\phi(e_\mu)=\phi(\alpha_\mu)$).

To prove (ii). We prove by induction. Suppose α is a path in $\mathcal{K}_o$ then $\phi(\alpha)\sim\alpha$ since ϕ is the identity. Now suppose the result hold for all paths in $\mathcal{K}_{\ell-1}$ $(\ell>o)$. Let α be a path in $\mathcal{K}_\ell$. For each edge e_λ in $\mathcal{K}_\ell$ (λ of level ℓ) replace e_λ by α_λ. This gives a path α' in $\mathcal{K}_{\ell-1}$. Since $e_\lambda\alpha_\lambda^{-1}$ is a cyclic permutation of ρ_λ then $e_\lambda\underset{K}{\sim}\alpha_\lambda$ which implies $\alpha\sim\alpha'$. Now by definition of ϕ we have $\phi(\alpha)=\phi(\alpha')$, and by induction hypothesis $\phi(\alpha')\underset{K}{\sim}\alpha'$. We have $\phi(\alpha)\underset{K}{\sim}\alpha$.

Let $\mathcal{K}_0 = \langle \mathcal{X}_0 : \phi(\rho_\lambda)\ (\lambda \in \Lambda_0) \rangle$. We have a mapping of 2-complexes $\phi: \mathcal{K} \to \mathcal{K}_0$. For if $\lambda \in \Lambda_\ell\ (\ell > 0)$ then $\phi(\rho_\lambda) \underset{\mathcal{K}_0}{\sim} 1$ by (i) (that is, contractible paths in $\mathcal{K}$ maps into contractible paths in $\mathcal{K}_0$).

Also the inclusion of $\mathcal{X}_0$ in $\mathcal{X}$ gives rise to a mapping of 2-complexes $\theta: \mathcal{K}_0 \to \mathcal{K}$, since $\theta(\phi(\rho_\lambda)) = \phi(\rho_\lambda) \underset{\mathcal{K}}{\sim} 1$ where $\lambda \in \Lambda_0$ (by (ii)). It is clear that $\theta\phi \approx \mathrm{id}_{\mathcal{K}}$ and $\phi\theta = \mathrm{id}_{\mathcal{K}_0}$. For if α a path in $\mathcal{K}$ we have $\theta\phi(\alpha) = \phi(\alpha) \underset{\mathcal{K}}{\sim} \alpha$ (by (ii)). Also $\phi\theta = \mathrm{id}_{\mathcal{K}_0}$. Thus ϕ is equivalence.

We mention some special cases of the above.

(1) If there are no defining paths of level o then $\mathcal{K}$ will be equivalent to a 1-complex. In particular, $\pi_1(\mathcal{K}, v)$ will be free for any vertex v.

(2) Suppose

$$\mathcal{K} = \langle \mathcal{X} : \beta_i\ (i \in I),\ e_j \alpha_j^{-1}\ (j \in J) \rangle$$

where

$$L_{e_k}(e_j \alpha_j^{-1}) = \begin{cases} 1 & k = j \\ 0 & k \neq j. \end{cases}$$

Let $\mathcal{X}_0$ be the 1-complex obtained from $\mathcal{X}$ by removing all the edges $e_j^{\pm 1}$ $(j \in J)$. For $i \in I$, let β'_i be the path in $\mathcal{X}_0$ obtained from β_i by replacing each edge $e_j^{\pm 1}$ $(j \in J)$ by the path $\alpha_j^{\pm 1}$ $(j \in J)$. Let

$$\mathcal{K}_0 = \langle \mathcal{X}_0 \; ; \; \beta'_i \; (i \in I) \; \rangle.$$

Then the inclusion of $\mathcal{X}_0$ into $\mathcal{X}$ induces an equivalence between $\mathcal{K}_0$ and $\mathcal{K}$.

level 1 $\quad \boxed{e_j} \alpha_j^{-1}$ $(j \in J)$

level o $\quad \beta_i$ $(i \in I)$

$\phi : \mathcal{X} \to \mathcal{X}_0$ is the identity on $\mathcal{X}_0$ and maps e_j to α_j $(j \in J)$. Then $\phi(\beta_i) = \beta'_i$ $(i \in I)$.

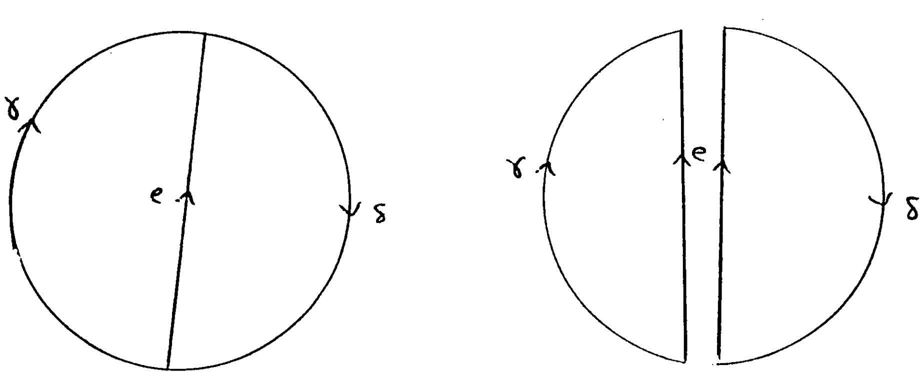
(3) Let

$$\mathcal{K}_0 = \langle \mathcal{X}_0 \; ; \; \beta_i \; (i \in I) \; \rangle.$$

Let j be an element of I and suppose that $\gamma\delta$ is a cyclic permutation of β_j. Adjoin to $\mathcal{X}_0$ a new edge pair e, e^{-1} with $\iota(e) = \iota(\gamma)$, $\tau(e) = \tau(\gamma)$, giving a new 1-complex $\mathcal{X}$, and let

$$\mathcal{K} = \langle \mathcal{X} : \beta_i \; (i \in I, \; i \neq j), \; \gamma e^{-1}, \; e\delta \; \rangle.$$

We say that $\mathcal{K}$ is obtained from $\mathcal{K}_0$ by *subdividing the defining path* β_j; dually, $\mathcal{K}_0$ is obtained from $\mathcal{K}$ by *coalation of defining paths.*



The inclusion of $\mathcal{X}_o$ into $\mathcal{X}$ induces an equivalence between $\mathcal{K}_o$ and $\mathcal{K}$.

level 1 $\boxed{e}$ δ

level o β_i $(i\in I,\ i\neq j)$, γe^{-1}

ϕ: $\mathcal{X} \to \mathcal{X}_o$ is the identity on $\mathcal{X}_o$ and maps e to δ^{-1}. Then $\phi(\gamma e^{-1}) = \gamma\delta$.

(Note that this is more or less a special case of (2)).

1.1C.1 Star complexes of 2-complexes.

Let K be 2-complex. We associate with K a 1-complex K^{st}, called the *star complex* of K as follows:

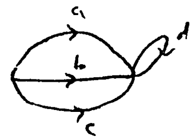
vertices : $E(K)$;

edges : $R(K)$.

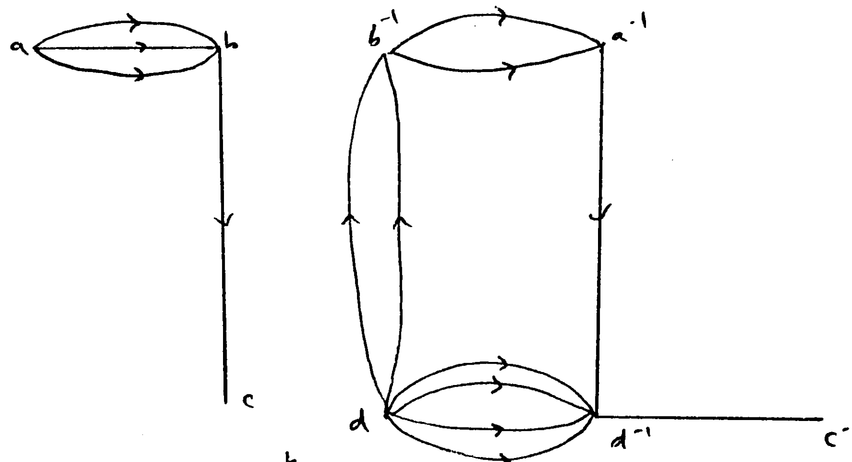
If γ is an edge of K^{st}, then we define the inverse edge to be the inverse path γ^{-1}. Also we need to define initial and terminal points of γ which usually would be denoted by $\iota(\gamma)$, $\tau(\gamma)$ but this has another meaning (see 1.1A.1), so we use $\iota^{st}(\gamma)$, $\tau^{st}(\gamma)$. We define $\iota^{st}(\gamma)$ to be the first edge of γ and τ^{st} to be the inverse of the last edge of γ.

We have that K^{st} is a 1-complex provided *no element of $R(K)$ is equal to its inverse*. For let $\gamma \in R(K)$ be an edge of K^{st} and suppose that $\gamma = e_1 e_2 \ldots e_n$. We have $\iota^{st}(\gamma^{-1}) = \tau^{st}(\gamma) = e_n{}^{-1}$. Also $(\gamma^{-1})^{-1} = \gamma$. Since no element of $R(K)$ is equal to its inverse then $\gamma^{-1} \neq \gamma$. So K^{st} is a 1-complex.

NOTE THAT WHENEVER WE WRITE K^{st} IT IS ASSUMED THAT NO ELEMENT OF $R(K)$ IS EQUAL TO ITS INVERSE.

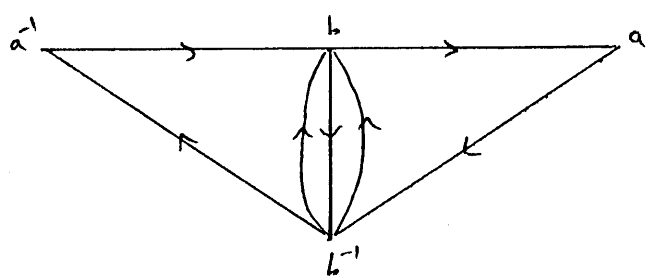
Example 1.1C.1. Let K=<  ; $(ab^{-1})^2$, d^3, $a^{-1}bdc^{-1}bd^2$ >

K^{st}:

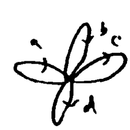


Example 1.1C.2. Let K=<  ; $a^{-1}b^2ab^{-3}$ >

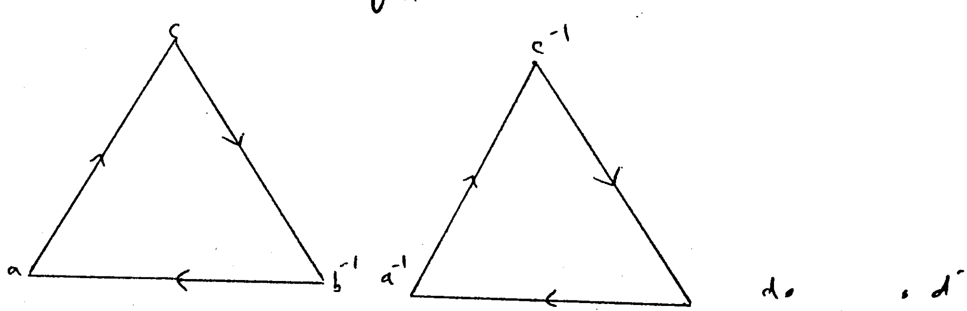
K^{st}:



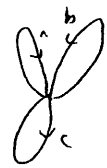
Example 1.1C.3.

Let K= <  ; $abca^{-1}b^{-1}c^{-1}$ >

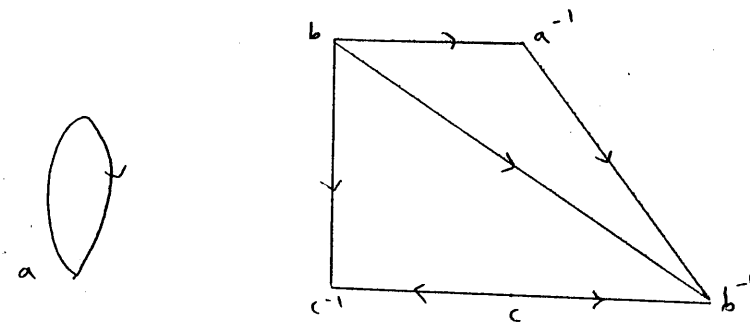
K^{st}:



Example 1.1C.4.

Let K= < 

; $abc^2b^2a^{-1}$ >

K^{st}: 

Note that the vertices e, e^{-1} of star complex has the same valence, that is, there is a bijection from Star(e) → Star(e^{-1}). This bijection is defined as follows. Let $\gamma\epsilon$Star(e) so that $\gamma = e\alpha$ for some path α. Then $e^{-1}\alpha^{-1}\epsilon$Star($e^{-1}$). The mapping Star(e) → Star(e^{-1}) defined by $e\alpha \rightarrow e^{-1}\alpha^{-1}$ is surjective for, let $\gamma\epsilon$Star(e^{-1}) then $\gamma = e^{-1}\beta$ for some path β then $e\beta^{-1}\epsilon$Star(e). Also it is injective for, let $e\alpha_1, e\alpha_2\epsilon$Star(e) for some paths α_1, α_2. Suppose $e^{-1}\alpha_1{}^{-1} = e^{-1}\alpha_2{}^{-1}$. We have $\alpha_1{}^{-1} = \alpha_2{}^{-1}$, that is, $\alpha_1 = \alpha_2$ then $e\alpha_1 = e\alpha_2$. Then it is injective. Thus it is bijective.

Note also that if two vertices e, f of K^{st} lie in the same component then $\iota(e) = \iota(f)$, that is, there a vertex v of K such that $e, f\epsilon$Star(v). For since e, f in the same component then there is a path π in K^{st} such that $\iota^{st}(\pi)=e$, $\tau^{st}(\pi)=f$. Let $\pi=\gamma_1\gamma_2\ldots\gamma_n$. We use induction on n. If n=0 there is nothing to prove. Suppose n>o. We have that $\iota^{st}(\gamma_1\ldots\gamma_{n-1})=e$, $\tau^{st}(\gamma_1\ldots\gamma_{n-1})=e_{n-1}$, $\iota^{st}(\gamma_n)=e_{n-1}$, $\tau^{st}(\gamma_n)=f$. Since γ_n is closed path in $R(K)$ then $\gamma_n=e_{n-1}\ldots f^{-1}$, that is, there is a vertex v of K with $\iota(e_{n-1})=\iota(f)$. By induction hypothesis we have $\iota(e)=\iota(e_{n-1})$. Then $\iota(e)=\iota(f)$ as required.

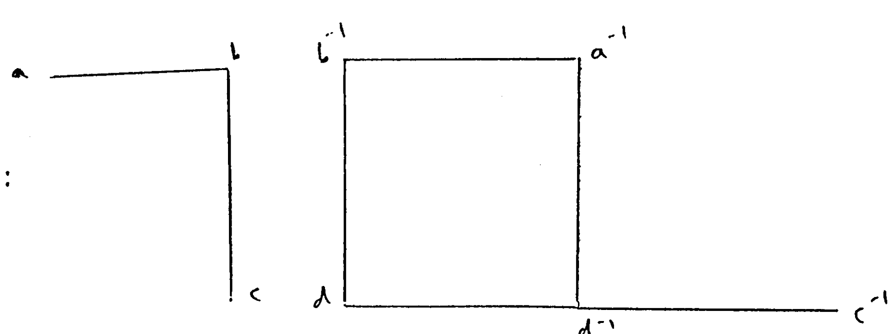
For a vertex v of K we denote the full subcomplex of K^{st} on Star(v) by $K^{st}(v)$. Then $K^{st}(v)$ is a union of components of K^{st}.

1.1C.2 Star graphs.

Associated with a star complex K^{st} we have the *star graph* $\Gamma(K^{st})$. This is obtained from K^{st} by identifying all edges with the same end points, that is, if γ_1,γ_2 are edges of K^{st} then we identify γ_1,γ_2 if either $\iota^{st}(\gamma_1)=\iota^{st}(\gamma_2)$ and $\tau^{st}(\gamma_1)=\tau^{st}(\gamma_2)$ or $\iota^{st}(\gamma_1)=\tau^{st}(\gamma_2)$ and $\tau^{st}(\gamma_1)=\iota^{st}(\gamma_2)$. Thus $\Gamma(K^{st})$ is undirected. $\Gamma(K^{st})$ is a graph in the sense of graph theory.

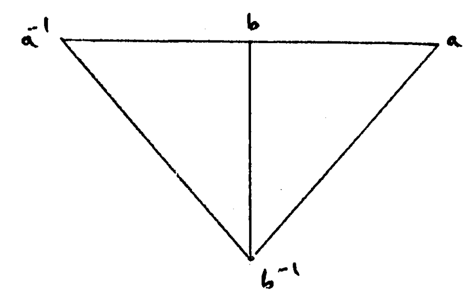
In Example 1.1C.1, we obtain

$\Gamma(K^{st})$:



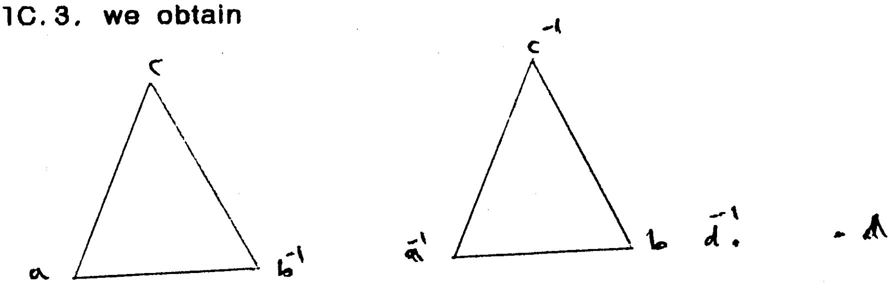
In Example 1.1C.2, we obtain

$\Gamma(K^{st})$:



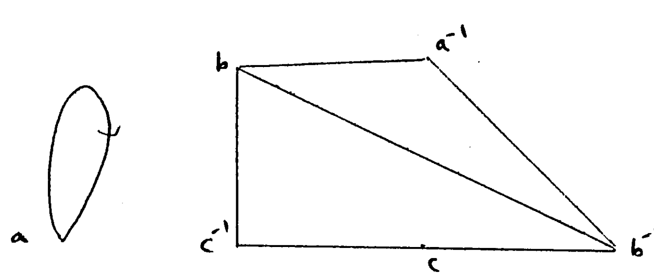
In Example 1.1C.3, we obtain

$\Gamma(K^{st})$:



In Example 1.1C.4, we obtain

$\Gamma(K^{st})$:



In passing from K^{st} to $\Gamma(K^{st})$ some information is lost. Information which is not lost is which edges of K can be predecesors of a given edge in a defining path of K. If we have an edge

in $\Gamma(K^{st})$ this tells us that in some defining paths of K, c is preceded by f^{-1}.

1.1C.3. Extended star complexes.

For some purposes it is necessary to consider the *extended star complex*.

Let $\mathcal{K}$=< $\mathcal{E}$: $\rho_\lambda(\lambda\in\Lambda)$ > be a 2-complex. We define the *extended star complexes* $\mathcal{K}^{ST}$ as follows:

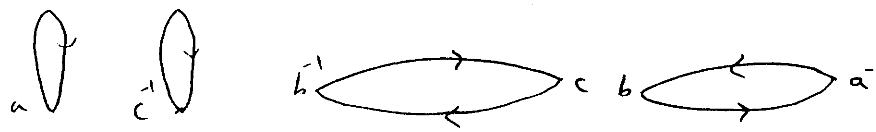
vertices : E($\mathcal{K}$) ;

edges : $(\lambda,\epsilon,\gamma)$ ($\lambda\in\Lambda$, $\epsilon=\pm1$, γ is a cyclic permutation of ρ_λ^{ϵ}).

We define $\iota^{ST}(\lambda,\epsilon,\gamma)$ to be a first edge of γ and $\tau^{ST}(\lambda,\epsilon,\gamma)$ to be the inverse of the last edge of γ, and $(\lambda,\epsilon,\gamma)^{-1}=(\lambda,-\epsilon,\gamma^{-1})$. We have that $\mathcal{K}^{ST}$ is a 1-complex. For let $(\lambda,\epsilon,\gamma)$ ($\lambda\in\Lambda$, $\epsilon=\pm1$, γ is a cyclic permutation of ρ_λ^{ϵ}) be an edge of $\mathcal{K}^{ST}$. Suppose $\gamma=e_1e_2...e_n$. We have $\iota^{ST}(\lambda,-\epsilon,\gamma^{-1})=e_n^{-1}$, $\tau^{ST}(\lambda,\epsilon,\gamma)=e_n^{-1}$. Then $\iota^{ST}(\lambda,-\epsilon,\gamma^{-1})=\tau^{ST}(\lambda,\epsilon,\gamma)$. Since by the definition $(\lambda,\epsilon,\gamma)^{-1}=(\lambda,-\epsilon,-\gamma)$ then $((\lambda,\epsilon,\gamma)^{-1})^{-1}=(\lambda,-\epsilon,-\gamma)^{-1}=(\lambda,\epsilon,\gamma)$. Then $((\lambda,\epsilon,\gamma)^{-1})^{-1}=(\lambda,\epsilon,\gamma)$. Also we have $(\lambda,\epsilon,\gamma)^{-1}=(\lambda,-\epsilon,-\gamma)\neq(\lambda,\epsilon,\gamma)$ since they have distinct second coodinates. Then $\mathcal{K}^{ST}$ is a 1-complex.

Example 1.1C.5. Let $\mathcal{P}$=< a,b,c ; $abcc^{-1}b^{-1}a^{-1}$ >. We have

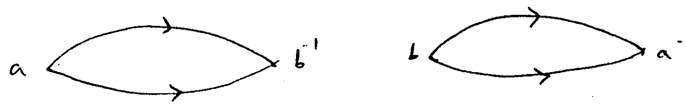
$\mathcal{P}^{ST}$:



Note that $\mathcal{P}^{st}$ is not defined.

Example 1.1C.6. Let $\mathcal{P}$=< a, b ; ab, ab >. We have

$\mathcal{P}^{ST}$:



Note that in this case $\mathcal{P}^{st}$ looks as follows.

$\mathcal{P}^{st}$:

If $\mathcal{K}^{st}$ is defined then there is a mapping of 1–complexes

$$\eta : \mathcal{K}^{ST} \to \mathcal{K}^{st}$$

which is defined to be the identity on vertices, and which takes an edge $(\lambda, \epsilon, \gamma)$ of $\mathcal{K}^{ST}$ to the edge γ of $\mathcal{K}^{st}$. This mapping is obviously surjective. Thus it is an isomorphism if and only if it is injective on edges. We say that $\mathcal{K}$ is *slender* if $\mathcal{K}^{st}$ is defined and η is an isomorphism.

Theorem 1.1C.1. *The following are equivalent.*

(1) K is slender.

(2) K satisfies the two conditions (i), (ii) below:

(i) if $\lambda,\mu\in\Lambda$ with $\lambda\neq\mu$, then ρ_μ is not a cyclic permutation of ρ_λ^ϵ.

(ii) no ρ_λ has a cyclic permutation of the form $\alpha\alpha^{-1}$.

Proof. (1) implies (2). Suppose K is slender. To show that (i) holds let λ,μ be distinct elements of Λ and let γ be a cyclic permutation of ρ_λ^ϵ ($\epsilon=\pm 1$). We want to show that $\gamma\neq\rho_\mu$. Now $(\mu,1,\rho_\mu)$ and $(\lambda,\epsilon,\gamma)$ are edges of K^{ST} and are distinct since they have distinct first coodinates. Thus their images ρ_μ,γ under η must be distinct. (ii) holds, for otherwise K^{st} would not be defined.

(2) implies (1). Suppose (i), (ii) hold. Then no element of $R(K)$ is equal to its inverse (by (ii)), so K^{st} is defined. Let $(\lambda,\epsilon,\gamma)$, $(\lambda',\epsilon',\gamma')$ be distinct edges of K^{ST}. We must show that their images under η are distinct. Now γ is a cyclic permutation of ρ_λ^ϵ, and γ' is a cyclic permutation of $\rho_{\lambda'}^{\epsilon'}$. Thus if $\lambda\neq\lambda'$then $\gamma\neq\gamma'$ by (i). Suppose $\lambda=\lambda'$. If $\epsilon=\epsilon'$ then $\gamma\neq\gamma'$ since $(\lambda,\epsilon,\gamma)\neq(\lambda',\epsilon',\gamma')$. Suppose $\epsilon'=-\epsilon$. Now if $\gamma=\gamma'$ then we would have that ρ_λ would be a cyclic permutation of ρ_λ^{-1}. The only way this

can happen is if some cyclic permutation of ρ_λ has the form $\alpha\alpha^{-1}$ which is excluded by (ii).

1.1C.4 Induced mappings of star complexes.

Let $\phi: \mathcal{K} \to \mathcal{L}$ be a strong mapping of 2-complexes. Then we have an *induced (rigid) mapping* of 1-complexes

$$\phi^{st}: \mathcal{K}^{st} \to \mathcal{L}^{st}$$

defined as follows:

on *vertices* of $\mathcal{K}^{st}$ $\quad \phi^{st}(e)$ is the first edge of $\phi(e)$

on *edges* of $\mathcal{K}^{st}$ $\quad \phi^{st}(\gamma)=\phi(\gamma)$.

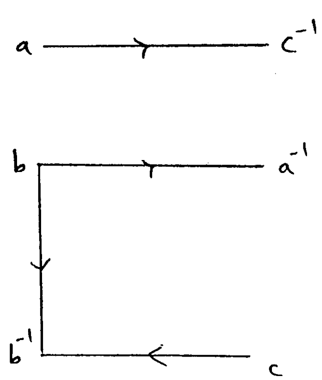
Now $\phi^{st}: \mathcal{K}^{st} \to \mathcal{L}^{st}$ is a mapping of 1-complexes. For, let γ be an edge of $\mathcal{K}^{st}$, suppose that $\gamma=e_1e_2\ldots e_n$, $\phi(e_i)=f_{i1}\ldots f_{ir(i)}$ $(i=1,\ldots,n)$. We have $\phi^{st}(\iota^{st}(\gamma))=\phi^{st}(e_1)=$first edge of $\phi(e_1)=f_{11}$, $\iota^{st}(\phi^{st}(\gamma))=\iota^{st}(\phi(e_1)\ldots\phi(e_n))=f_{11}$, that is, $\iota^{st}(\phi^{st}(\gamma))=\phi^{st}(\iota^{st}(\gamma))$. Also $\phi^{st}(\iota^{st}(\gamma))=\phi^{st}(e_n^{-1})=f_{nr(n)}^{-1}$, $\tau^{st}(\phi^{st}(\gamma))=\tau^{st}(\phi(e_1)\ldots\phi(e_n))=f_{nr(n)}^{-1}$, that is, $\tau^{st}(\phi^{st}(\gamma))=\phi^{st}(\tau^{st}(\gamma))$. Now we have $\phi^{st}(\gamma^{-1})=\phi(\gamma^{-1})=\phi(\gamma)^{-1}=\phi^{st}(\gamma)^{-1}$. Then ϕ^{st} is a mapping of 1-complexes.

Example 1.1C.7.

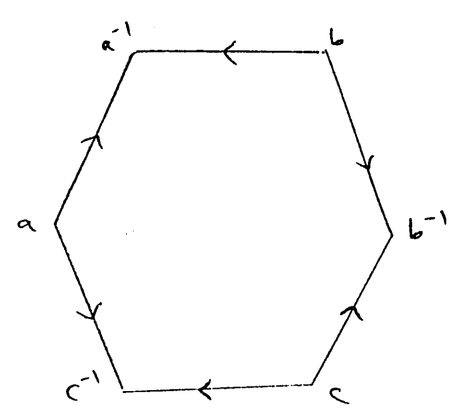
Let ϕ: K=< a, b, c ; ab^2c > $\to \mathcal{L}$=< a, b, c ; $a^2b^2c^2$ >

be a strong mapping of 2-complexes given by $a \to a^2$, $b \to b$, $c \to c^2$. Then we have a induced mapping of 1-complexes from K^{st} to $\mathcal{L}^{st}$ which takes the vertices of K^{st} into vertices of $\mathcal{L}^{st}$ as follows $a \to a$, $b \to b$, $c \to c$, $a^{-1} \to a^{-1}$, $b^{-1} \to b^{-1}$, $c^{-1} \to c^{-1}$; for edges takes $ab^2c \to a^2b^2c^2$, $b^2ca \to b^2c^2a^2$, $bcab \to bc^2a^2b$, $cab^2 \to c^2a^2b^2$, where

K^{st}:



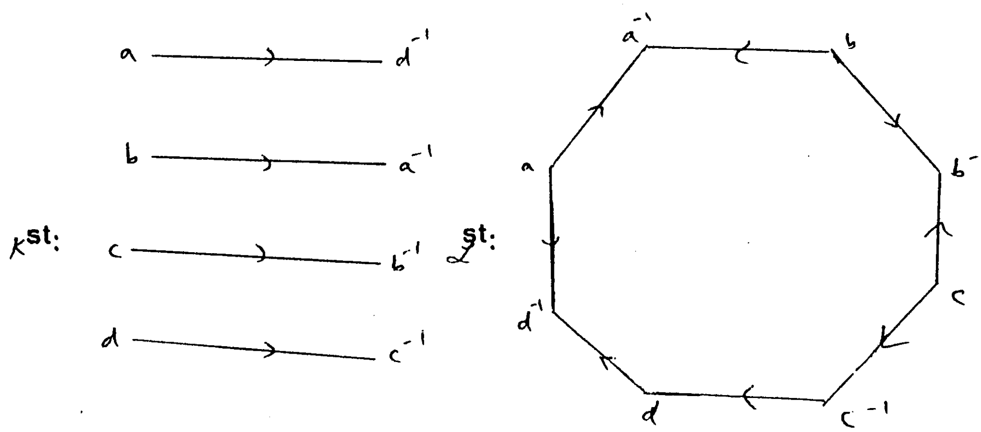
$\mathcal{L}^{st}$:



Example 1.1C.8. Let ϕ: K=<a, b, c, d; abcd> $\to$ $\mathcal{L}$=<a, b, c, d; $a^2b^2c^2d^2$>

be a strong mapping of 2-complexes given by $a \to ab$, $b \to bc$, $c \to cd$, $d \to da$. Then we have induced mapping of 1-complexes from $K^{st} \to \mathcal{L}^{st}$ which takes the vertices of K^{st} into vertices of $\mathcal{L}^{st}$ as follow

$a \to a$, $b \to b$, $c \to c$, $d \to d$, $a^{-1} \to b^{-1}$, $b^{-1} \to c^{-1}$, $c^{-1} \to d^{-1}$, $d^{-1} \to a^{-1}$; for the edges $abcd \to a^2b^2c^2d^2$, $bcda \to b^2c^2d^2a^2$, $cdab \to c^2d^2a^2b^2$, $dabc \to d^2a^2b^2c^2$ where



In (1.1B.3) we defined a mapping ϕ: $\tilde{K} \to K$ to be locally bijective if $\phi^{(1)}$ is locally bijective, and $\phi^{-1}R(K)=R(\tilde{K})$.

The next Theorem show that we could equally well have defined ϕ to be locally bijective if both $\phi^{(1)}$ and ϕ^{st} are locally bijective.

Theorem 1.1C.2. *Let ϕ: $\tilde{K} \to K$ be a strong mapping, and suppose that $\phi^{(1)}$ is locally bijective. Then the following are equivalent.*

(1) ϕ^{st} is locally bijective;

(2) $\phi^{-1}R(K)=R(\tilde{K})$

(3) for each vertex $\tilde{v}$ of $\tilde{K}$, ϕ^{st} maps $\tilde{K}^{st}(v)$ isomorphically onto $K^{st}(\phi(\tilde{v}))$.

Proof.

(1) implies (2). Let $\gamma \in R(\mathcal{K})$ and suppose $\phi(\tilde{\gamma})=\gamma$. Let $\iota^{st}(\gamma)=e$, and let $\tilde{e}$ be the unique edge of Star$(\iota(\tilde{\gamma}))$ lying over e. By the local surjectivity of ϕ^{st} there is an edge $\tilde{\delta} \in R(\tilde{\mathcal{K}})$ with $\iota^{st}(\tilde{\delta})=\tilde{e}$ and $\phi^{st}(\tilde{\delta})=\gamma$. Thus $\iota(\tilde{\gamma})=\iota(\tilde{\delta})$ and $\phi(\tilde{\gamma})=\phi(\tilde{\delta})=\gamma$. Hence $\tilde{\gamma}=\tilde{\delta}$ by uniqueness of lifts, so $\tilde{\gamma} \in R(\tilde{\mathcal{K}})$. We have $\phi^{-1}R(\mathcal{K}) \subseteq R(\tilde{\mathcal{K}})$. Since ϕ is strong mapping then $\phi^{-1}R(\mathcal{K}) \supseteq R(\tilde{\mathcal{K}})$. So we have $\phi^{-1}R(\mathcal{K})=R(\tilde{\mathcal{K}})$.

(2) implies (3). Since $\phi^{(1)}$ is locally bijective, ϕ^{st} maps the vertex set Star$(\tilde{v})$ of $\tilde{\mathcal{K}}^{st}(\tilde{v})$ bijectively onto the vertex set Star$(\phi(\tilde{v}))$ of $\mathcal{K}^{st}(\phi(\tilde{v}))$. For edges, let $\tilde{\gamma}_1, \tilde{\gamma}_2$ be edges of $\tilde{\mathcal{K}}^{st}(\tilde{v})$ such that $\phi^{st}(\tilde{\gamma}_1)=\phi^{st}(\tilde{\gamma}_2)$. Then $\iota(\tilde{\gamma}_1)=\iota(\tilde{\gamma}_2)=\tilde{v}$ and $\phi(\tilde{\gamma}_1)=\phi(\tilde{\gamma}_2)$ so $\tilde{\gamma}_1=\tilde{\gamma}_2$ by uniqueness of lifts. Thus ϕ^{st} is injective on edges of $\tilde{\mathcal{K}}^{st}(\tilde{v})$. To see that ϕ^{st}: $\tilde{\mathcal{K}}^{st}(\tilde{v}) \to \mathcal{K}^{st}(\phi(\tilde{v}))$ is surjective on edges, let γ be an edge of $\mathcal{K}^{st}(\phi(\tilde{v}))$. By (2) the (unique) lift $\tilde{\gamma}$ of γ at $\tilde{v}$ belongs to $R(\tilde{\mathcal{K}})$ and is thus an edge of $\mathcal{K}^{st}(\tilde{v})$. Moreover $\phi^{st}(\tilde{\gamma})=\gamma$.

(3) implies (1). This is a consequence of the following.

Let $\mathcal{A}$ and $\mathcal{B}$ be 1-complexes, each expressed as disjoint union of subcomplexes: $\mathcal{A}=\bigcup_{i \in I} \mathcal{A}_i$, $\mathcal{B}=\bigcup_{j \in J} \mathcal{B}_j$.

Let θ: $\mathcal{A} \to \mathcal{B}$ be a rigid mapping such that for each $i \in I$ θ maps

$\mathcal{A}_i$ isomorphically onto some $\mathcal{B}_{j(i)}$. Then e is locally bijective.

This complete the proof of Theorem 1.1C.2.

Let ϕ: $\mathcal{K} \to \mathcal{L}$, ψ: $\mathcal{L} \to \mathcal{M}$ be two strong mappings. We have that $\psi\phi$: $\mathcal{K} \to \mathcal{M}$ is strong mapping. Consider the induced mappings of 1-complexes. ϕ^{st}: $\mathcal{K}^{st} \to \mathcal{L}^{st}$, ψ^{st}: $\mathcal{L}^{st} \to \mathcal{M}^{st}$, $(\psi\phi)^{st}$: $\mathcal{K}^{st} \to \mathcal{M}^{st}$. It can be shown that $(\psi\phi)^{st}=\psi^{st}\phi^{st}$. Thus st is a covariant functor from the category of 2-complexes where the morphisms are strong mappings to the category of 1-complexes.

Let ϕ: $\mathcal{K} \to \mathcal{L}$ be a strong mapping of 2-complexes. Then we have an *induced extended (rigid) mapping* of 1-complexes

$$\phi^{ST}: \mathcal{K}^{ST} \to \mathcal{L}^{st}$$

defined as follows.

On vertices of $\mathcal{K}^{ST}$: $\phi^{ST}(e)$ is the first edge of $\phi(e)$.

On edges of $\mathcal{K}^{ST}$: $\phi^{ST}(\lambda,\epsilon,\gamma)=\phi(\gamma)$.

We will say that a strong mapping ϕ is *reduced* if ϕ^{ST} is locally injective.

1.2. Survey of thesis.

Putting restrictions on the star complex of a 2-complex will effect the structure of the 2-complex and hence will effect the fundamental group of the 2-complex.

We will consider three structural restrictions.

(1) Assigning numbers ("*weights*") to the edges, where these numbers satisfy certain conditions.

(2) Imposing restrictions on lengths of paths.

(3) Imposing restrictions on valence of vertices.

(1) If we consider the first restriction we are led to the concept of a *hyperbolic complex* [10] which we now define.

A *weight function* m on a 1-complex is a mapping from the edge set into $\mathbb{R}$ such that $m(x^{-1})=m(x)$ for all edges x. If $x_1x_2...x_n$ is a path in the 1-complex (where the x_i are edges) then the *weight* of the path is defined to be $\sum_{i=1}^{n} m(x_i)$

The situation we will be intersested in is when we have a 2-complex and a weight function m on K^{st}. We will use the notation *(K, m)* to denote this situation.

Let $\mathcal{K}$ be 2-complex and let m be a weight function on $\mathcal{K}^{st}$. Associated with m we have another weight function m^* on $\mathcal{K}^{st}$ defined as follows. Let $\gamma\in R(\mathcal{K})$ and write $\gamma=e_1e_2\ldots e_n$ where the e_i are edges of $\mathcal{K}$. Then

$$m^*(\gamma)=m(e_1\ldots e_n)+m(e_2\ldots e_1)+\ldots+m(e_n\ldots e_{n-1})$$

We say that $(\mathcal{K},m)$ is *hyperbolic* [10] if

(HI) $m^*(\gamma) < L(\gamma)-2$ for all $\gamma\in R(\mathcal{K})$.

(HII) The weight of any non-empty cyclically reduced closed path in $\mathcal{K}^{st}$ is at least 2.

(HIII) There exists a non-negative number N such that every reduced path in $\mathcal{K}^{st}$ has weight greater than or equal to $-N$.

We will say that a 2-complex $\mathcal{K}$ is hyperbolic if there is an m such that $(\mathcal{K},m)$ is hyperbolic.

In [10] it was mentioned that the surface presentations

$$\langle\, x_1,y_1,x_2,y_2,\ldots,x_n,y_n \;:\; \prod_{i=1}^{n} [x_i,y_i] \,\rangle \quad (n\geq 2)$$

$$\langle\, x_1,x_2,\ldots,x_n \;:\; \prod_{i=1}^{n} x_i^2 \,\rangle \quad (n\geq 3)$$

and the presentations of triangle groups

$$\langle\, a,b,c \;:\; a^p,\ b^q,\ c^r,\ abc \,\rangle \text{ where } 1/p+1/q+1/r<1$$

are hyperbolic.

In 3.3 we consider arbitary F-presentations [6, pp. 126–133].

An *orientable F-presentation* is a presentation

$$\mathcal{O}=\langle a_1, b_1, a_2, b_2, \ldots, a_g, b_g, e_1, \ldots, e_r \;:\; a_1b_1a_1^{-1}b_1^{-1}\ldots a_gb_ga_g^{-1}b_g^{-1}e_1\ldots e_r, e_1^{n_1}, \ldots, e_r^{n_r}\rangle$$

where $g, r \geq 0$ and $n_1, \ldots, n_r \geq 2$.

A *non-orientable F-presentation is a* presentation

$$\mathcal{N}=\langle a_1, \ldots, a_g, e_1, \ldots, e_r \;:\; a_1^2a_2^2\ldots a_g^2e_1\ldots e_r,\ e_1^{n_1}, \ldots, e_r^{n_r}\rangle$$

where $g, r \geq 0$ and $n_1, \ldots, n_r \geq 2$.

There is a standard parameter associated with these presentations

$$\mu(\mathcal{O})=2g-2+\sum_{i=1}^{r}(1-1/n_i)$$

$$\mu(\mathcal{N})=g-2+\sum_{i=1}^{r}(1-1/n_i)$$

In Theorem 3.3.1 we show that a necessary and sufficient condition for an F-presentation $\mathcal{P}$ to be hyperbolic is that $\mu(\mathcal{P})>0$.

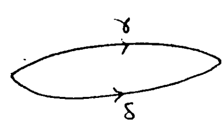
(2) Now if we consider the second restriction we are led to the definition of small cancellation condition $T(q)$ ($\widetilde{T}(q)$). A 2-complex K where each defining path is cyclically reduced satisfies the *T(q)* condition if and only if there are no

cyclically reduced closed path in $\mathcal{K}^{st}$ of length ℓ, $3 \leq \ell < q$. A 2-complex $\mathcal{K}$ where each defining path is cyclically reduced satisfies $\widetilde{T}(q)$ condition if there are no non-empty cyclically reduced closed path in $\mathcal{K}^{st}$ of length less than q.

WE EMPHASISE THAT WE ONLY DEFINE THE CONDITIONS T(q), $\widetilde{T}$(q) FOR COMPLEXES IN WHICH EACH DEFINING PATH IS CYCLICALLY REDUCED.

Now if $\mathcal{K}$ satisfies T(q) then, it will be also satisfy $\widetilde{T}(q)$ provided there are not distinct edges γ, δ in $\mathcal{K}^{st}$ with

$\iota^{st}(\gamma) = \iota^{st}(\delta)$, $\tau^{st}(\gamma) = \tau^{st}(\delta)$



Note that if $q \geq 5$ then T(q) and $\widetilde{T}(q)$ are the same property. For suppose $\mathcal{K}$ satisfied T(q) but not $\widetilde{T}(q)$ $(q \geq 5)$. Then in $\mathcal{K}^{st}$ there would be a cyclically reduced closed path of length 2. The square of this would then be a cyclically reduced closed path of length 4, contradicting T(q).

The condition T(q) ($\widetilde{T}(q)$) is only of use when considered in conjunction with small condition $C(p)$ where $1/p+1/q \leq 1/2$. To define C(p) condition we first give the following definition.

A non-empty path π in $\mathcal{K}$ is called a *piece* if there are distinct elements $\pi\alpha, \pi\beta \in R(\mathcal{K})$. $\mathcal{K}$ satisfies the small cancellation condition *C(p)* (p a positive integer) if no element of $R(\mathcal{K})$ is the product of less than p pieces.

For a $\tilde{T}(q)$-complex the only pieces are of length 1, that is, there are no pieces of length 2. For if we suppose that we have a piece of length 2 say $f^{-1}e$, that is $f^{-1}e\alpha, f^{-1}e\beta \in R(\mathcal{K})$ where α, β are distinct paths in $\mathcal{K}$. We have edges $e\alpha f^{-1}$, $e\beta f^{-1}$ in $\mathcal{K}^{st}$ with $\iota^{st}(e\alpha f^{-1}) = \iota^{st}(e\beta f^{-1}) = e$, $\tau^{st}(e\alpha f^{-1}) = \tau^{st}(e\beta f^{-1}) = f$; that is, we have a reduced closed path of length 2 which is a contradiction to $\tilde{T}(q)$.

We note from above that T(6)-complexes have no pieces of length 2, that is, the only pieces are of length 1.

FOR THE T(6)-COMPLEXES WE ALWAYS ASSUME THAT DEFINING PATHS ARE OF LENGTH >3.

Now we mention some results concerning T(q)-complexes in the thesis.

Collins in [1] investigated the free subgroups of groups with presentations satisfying C(4),T(4) conditions. He has shown that such a group contains a free subgroup of rank 2 except in some cases which he lists explicitly. The exceptions are all two generator groups.

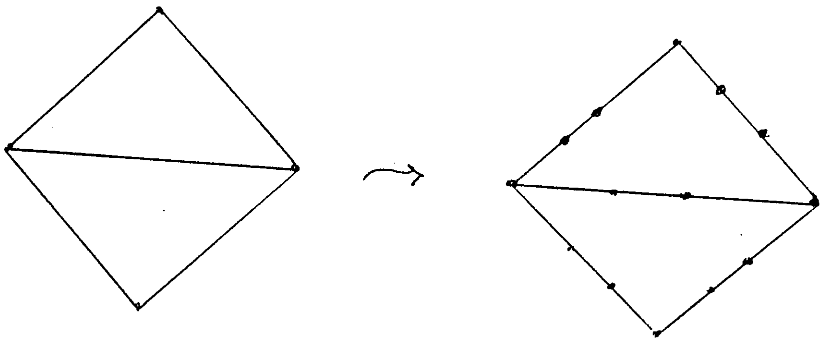
In [4] the T(4) condition was investigated graphically. In chapter 2 we give a simple proof, in the spirit of [4], that if G is the fundamental group of a C(4),T(4)-presentation and if G can not be generated by fewer than three elements then G contains a free subgroup of rank 2. In fact our proof actually shows that the group defined by a slender T(4)-presentation $\mathcal{P}=\langle x_1,\ldots,x_n ; r\rangle$ has a free subgroup of rank 2 provided there is a subset {a, b, c} of {$x_1,\ldots,x_n$} with the property that any non-empty freely reduced contractible word in a, b, c has a subword of length 2 contained in an element of r^*.

Recently M. Edjvet, and J. Howie have investigated free subgroups of T(6)-groups.

Much work has been done for C(4)-T(4), and C(6)-complexes. However T(6)-complexes have not so far been studied very much. For that reason our main work in the thesis is to study T(6)-complexes.

Now we give some results concerning T(6)-complexes.

In chapter 3 we give some examples of T(6)-complexes. In 3.1 we consider certain complexes whose star complexes are "trisected". Roughly speaking a trisected 1-complex is obtained by taking a 1-complex and trisecting each edge (making each edge of length 3).



More formally we define a *trisected 1-complex* as follow:

(1) The set of vertices is partitioned into two non-empty disjoint sets called type(1) and type(2), each of type(2) has valence 2.

(2) Call an edge a (1,2)-edge if one of its end point is of type(1) and the other of type(2), and call an edge a (2,2)-edge if its end points are of type(2). We require that if v is of type(1) then all edges in Star(v) are (1,2)-edges and if v is of type(2) then one edge in Star(v) is a (1,2)-edge and the other is a (2,2)-edge.

(3) There is no reduced closed path of length 3.

It follows from (2) that every cyclically reduced closed path has length a multiple of 3, and so by (3), every non-empty cyclically reduced closed path has length at least 6.

We say that a 2-complex $\mathcal{K}$ is an X-complex if

(i) $\mathcal{K}^{st}$ is a trisected 1-complex,

(ii) e, e^{-1} are the same type ($e \in E(\mathcal{K})$),

(iii) each defining path of $\mathcal{K}$ has length 3.

It follows from the definition of X-complex that X-complexes are T(6)-complexes. We will describe the structure of X-complexes (Theorem 3.1.2). The class of X-complexes is rather interesting and we will obtain other results concerning these complexes (Theorem 3.1.1, Theorem 3.1.3).

In 3.2 we give further examples of T(6)-presentations which we call *positive* T(6)-presentations. A presentation $\langle\, \mathbf{x}\ ;\ \mathbf{r}\,\rangle$ is said to be *positive* if each defining path $\rho \in \mathbf{r}$ is a positive path in $\mathbf{x}$ (that is, no element of $\mathbf{x}^{-1}$ occurs in ρ). In Theorem 3.2.1 we give a way of combining positive T(6)-presentations to get new ones, that is, if we have at least three positive T(6)-presentations $\mathcal{P} = \langle x_1, \ldots, x_n ; \mathbf{r} \rangle$, $\mathcal{Q} = \langle y_1, \ldots, y_n ; \mathbf{s} \rangle, \ldots, \mathcal{S} = \langle z_1, \ldots, z_n ; \mathbf{t} \rangle$ then

$\mathcal{P} = \langle x_1, \ldots, x_n, y_1, \ldots, y_n, \ldots, z_1, \ldots, z_n ; r, s, \ldots, t, z_i \ldots y_i x_i (1 \leq i \leq n) \rangle$

is a positive T(6)-presentation.

In [10] a sequence of decision problems (*the dependence problems*) were investigated, which we define.

Let $(\omega_0, \omega_1, \ldots, \omega_k)$ be a sequence of closed paths in a 2-complex K. We say that ω_0 is *dependent* on $(\omega_1, \ldots, \omega_k)$ in K written $(\omega_1, \ldots, \omega_k) \vdash_K \omega_0$ (or simply $(\omega_1, \ldots, \omega_k) \vdash \omega_0$) if there is a subset $(i_1, \ldots, i_\ell)$ of $(1, 2, \ldots, k)$ and paths $\eta_1, \ldots, \eta_\ell$ such that

$$\omega_0 (\eta_1 \omega_{i_1} \eta_1^{-1}) \ldots (\eta_\ell \omega_{i_\ell} \eta_\ell^{-1})$$

is contractible in K.

If n is a positive integer or ∞, then the *dependence problem* DP(n) asks for an algorithm to decide for any sequence $(\omega_0, \omega_1, \ldots, \omega_k)$ $(o \leq k < n)$ whether or not $(\omega_1, \ldots, \omega_k) \vdash_K \omega_0$. The problems DP(1), DP(2) are usually called the *word problem* and *conjugancy problem* for K respectively.

It was shown in [10] that DP(∞) is solvable for hyperbolic complexes ([10, Theorem 3]).

If K satisfies the small cancellation condition T(7), then is hyperbolic ([10, Theorem 4, Corollary]). On the other hand there are T(6)-complexes which are not hyperbolic (for example $\langle a, x_1, \ldots, x_n ; a x_1^2, \ldots, a x_n^2 \rangle$, $\langle a, b, c ; a^3, b^3, c^3, abc \rangle$ (see 4.1)). Then the natural question to ask is whether or not DP(∞) is solvable for T(6)-complexes. In chapter 4 we try to solve the above question.

There is an intimate connection between the dpendence problem DP(n) and diagrams on sphere a S with n distinguished regions (Theorem 4.1.1). In 4.3 we examine the geometry of spheres. We find in 4.3 a formula (*) concering pairs (S,Θ), where S is a finitely tesselated sphere and Θ is a subset of the set of regions of S. One of the terms in this formula is $2-|\Theta|$. Since this term is required to be non-negative gives $|\Theta|\leq 2$ which will allow us to give solutions to DP(1), DP(2) for T(6)-complexes. Solutions of these problems are already known [6,§§V.6,V.7], but our treatment is new and simpler. The fact that $2-|\Theta|$ becomes negative when $|\Theta|>2$ means that we are not able to extend our argument to solve DP(m), $m\geq 3$.

(3) Now if we consider the third restriction we are led to *property-ST*, and *property-st* which we define. Consider a

presentation $\mathcal{P}$=< x ; ρ_λ ($\lambda\in\Lambda$) >. We say that $\mathcal{P}$ has *property*-ST(m) if each vertex in $\mathcal{P}^{ST}$ has valence m. We say that $\mathcal{P}$ has property-st(m) if each vertex in $\mathcal{P}^{st}$ has valence m. We say that $\mathcal{P}$ has *degree* m, denoted by deg($\mathcal{P}$)=m if each $x\in \mathbf{x}\cup \mathbf{x}^{-1}$ appears exactly m times in the totality of paths ρ_λ, ρ_λ^{-1} ($\lambda\in\Lambda$). We say that $\mathcal{P}$ is *rootless* if no defining path is a proper power. We note that if $\mathcal{P}$ is rootless with degree m then $\mathcal{P}$ has property-ST(m). We note also that property-ST(m), property-st(m) coincide if and only if $\mathcal{P}^{ST}$ is isomorphic to $\mathcal{P}^{st}$, that is, if and only if $\mathcal{P}$ is slender.

Let $\mathcal{P}$=< x ; ρ_λ($\lambda\in\Lambda$) > be a presentation of degree 2. As is well-known the structure of the group G defined by $\mathcal{P}$ is rather special. (see 5.2)

The natural question to ask is whether having a presentation of degree m, property-ST(m) or property-st(m) (m>2) puts any restriction on the group defined by $\mathcal{P}$?. The answer is no by our result in Theorem 5.3.1. After we proved the result we found that Marcus [7] had also obtained a similar result, though his treatment is more complicated than ours.

CHAPTER 2

FREE SUBGROUPS OF SMALL CANCELLATION GROUPS

2.1 Introduction.

Collins [1] investigated the free subgroups of groups with presentations satisfying C(4),T(4) conditions (see also Johnson [5]). He has shown that such a group contains a free subgroups of rank 2 except in some cases which he lists explicity (see 2.4 below). The exceptions are all two generators groups.

In [4] the T(4) condition was investigated graphically. Here we give a simple proof, in the spirit of [4], that is if G is the fundamental group of a C(4),T(4)-presentation and if G can not be generated by fewer than three elements then G contains a free subgroup of rank 2. In fact we prove a slightly stronger result which will be stated in Theorem 2.1.1 .

Let $\mathcal{P}, \mathcal{Q}$ be two presentations with the same 1-skelton $\mathcal{X}$. An equivalence ϕ: $\mathcal{P} \to \mathcal{Q}$ will be called a *permutational equivalence* if it maps $R(\mathcal{P})$ to $R(\mathcal{Q})$ and if it permutes the edge set of $\mathcal{X}$.

We have the following.

Theorem 2.1.1. *Let* $\langle x_1, x_2, \ldots, x_n;\ r\rangle$ $(n\geq 3)$ *be a slender C(4), T(4)-presentation not permutational equivalent to a presentation*

$$(*)\qquad \langle x_1, x_2, \ldots, x_n; x_1\alpha_1, x_2\alpha_2, \ldots, x_{n-2}\alpha_{n-2}, s\rangle$$

where $s \subseteq r$ *and* $L_{x_i}(W)=o$ $(W\in\{\alpha_1, \alpha_2, \ldots, \alpha_{n-2}\}\cup s,\ i=1,2,\ldots,n-2)$. *Then the group defined by the presentation has a free subgroup of rank 2.*

Remark: Our proof actually shows that the group defined by a slender T(4)-presentation $\mathcal{P}=\langle x_1, x_2, \ldots, x_n; r\rangle$ has a free subgroup of rank 2 provided there is a subset $\{a, b, c\}$ of $\{x_1, x_2, \ldots, x_n\}$ with the property that any non-empty freely reduced contractible word in a, b, c has a subword of length 2 contained in an element of r^*. It will be seen (see 2.2) that a presentation satisfying the hypotheses of Theorem 2.1.1. has such a subset.

2.2 Preliminaries.

Let $\mathcal{P}=\langle x_1, x_2, \ldots, x_n; r\rangle$ be a slender presentation satisfying the assumption of Theorem 2.1.1 . We want to show that there is a 3-element subset $\{a, b, c\}$ of $\{x_1, x_2, \ldots, x_n\}$ with the property:

(†) *Any non-empty freely reduced contractible word on a, b, c has a subword of length 2 contained in an element of* r^*.

Lemma 2.2.1. *A 3-element subset $\{a,b,c\}$ of $\{x_1,x_2,\ldots,x_n\}$ satisfies (†) except possibly if there is an element $\rho_o \in r$ where for some $t \in \{a,b,c\}$, $L_t(\rho_o)=1$, $L_t(\rho)=o$ for $\rho_o \neq \rho$.*

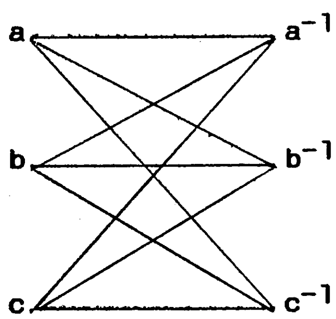
Proof. Suppose there is a non-empty freely reduced word W on a, b, c which is contractible but which does not have a subword of length 2 contained in an element of r^*. Let M be a reduced Van Kampen diagram with boundary label W. By standard small cancellation theory ([6, ChapterV] M has a boundary region Δ (labelled by ρ_o say) such that Δ has at most two interior edges, $\partial\Delta \cap \partial M$ is a consective part of M, the label u on $\partial\Delta \cap \partial M$ is a subword of W. By assumption u has length 1. Since the label on the interior edges of Δ are pieces, the C(4)-condition implies that u is not a piece. Then u cannot occur in any defining path except ρ_o. Also it occurs once in ρ_o because it cannot be part of a label of an interior edge of Δ.

Now suppose $\{x_1, x_2, x_3\}$ does not satisfy (†). Then by Lemma 2.1.1, we can assume (relabelling if necessary) that for some $\rho_1 \in r$, $L_{x_1}(\rho_1)=1$ and $L_{x_1}(\rho)=o$ for $\rho \neq \rho_1$. Some cyclic permutation of $\rho_1^{\pm 1}$ will then have the form $x_1\alpha_1$ where $L_{x_1}(\alpha_1)=o$. Now consider

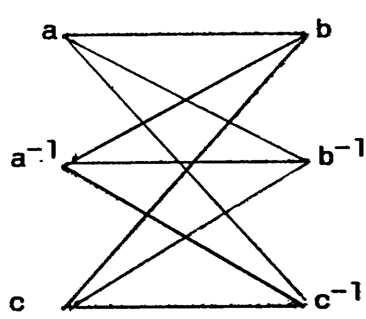
$\{x_2, x_3, x_4\}$ and so on. If we do not eventually find a subset satisfying (†) then we will have that $\langle x_1, x_2, \ldots, x_n; r\rangle$ is permutational equivalent to a presentation of the form (*) which contradicts the assumption.

2.3 Proof of Theorem 2.1.1.

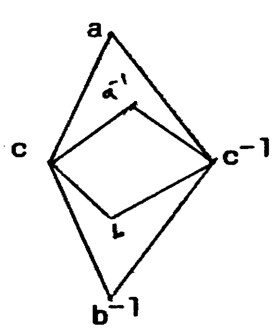
Let $\{a, b, c\}$ be a subset of $\{x_1, x_2, \ldots, x_n\}$ satisfying (†) and consider the full subgraph $\Gamma(a, b, c)$ of the star graph of the presentation $\langle x_1, x_2, \ldots, x_n; r\rangle$ on the vertices $a, a^{-1}, b, b^{-1}, c, c^{-1}$. Let Ω_3 be the subgroup of the symmetric group on $\{a, b, c, a^{-1}, b^{-1}, c^{-1}\}$ generated by the elements $(xy)(x^{-1}y^{-1}), (xx^{-1})$ $(x, y \in \{a, b, c\})$. Then it is shown in [4] that, up to permuting the vertices by an element of Ω_3, $\Gamma(a, b, c)$ is a subgraph of one of the following.



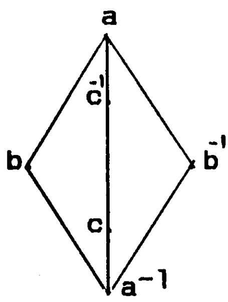
Fig(1)



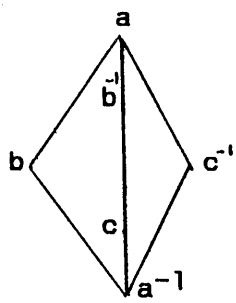
Fig(2)



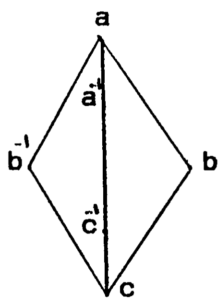
Fig(3)



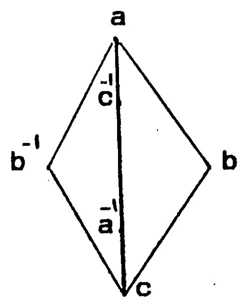
Fig(4)



Fig(5)



Fig(6)



Fig(7)

It therefore sufficies to assume that $\Gamma(a,b,c)$ is a subgraph of one of the above.

Note that a word xy of length 2, $(x,y \in \{a,a^{-1},b,b^{-1},c,c^{-1}\})$ is a subset of an element of r^* if and only if (x^{-1},y) is an edge of $\Gamma(a,b,c)$.

Suppose $\Gamma(a,b,c)$ is a subgraph of the first graph (Fig(1)). Then the words $c^{-1}b, b^{-1}a, c^{-1}a, ba^{-1}, ca^{-1}, cb^{-1}$ do not occur as subwords of elements of r^*. It follows that $[a^{-1}b], [a^{-1}c]$ freely generate a subgroup of the group G defined by the presentation, for no non-empty freely reduced word in $a^{-1}b, a^{-1}c$ has (after freely reducing in terms of a, b, c) a subword of length 2 contained in an element of r^*. For the remaining cases similar argument apply. For each case we list two words U, V in a, b, c and we leave it to the reader to verify that if W(U, V) is a non-empty freely reduced word in U,V then after freely reducing W(U,V) in terms of a,b,c, we obtain a non-empty word with no subword of length 2 contained in an element of r^*.

Fig( )	U	V
2	cbc^{-1}	a
3	bab^{-1}	a
4	cbc^{-1}	b
5	$ab^{-1}cba^{-1}$	cbc^{-1}
6	cbc^{-1}	aba^{-1}
7	aba^{-1}	cbc^{-1}

2.4 2-generator groups.

A presentation as in (*) is obviously equivalent to

$$\langle x_{n-1}, x_n \; ; \; s \rangle$$

(by taking $x_i \to \alpha_i^{-1}$ $i=1,2,\ldots,n-2$ see p.20).

For convenience, we write a,b for x_{n-1}, x_n respectively. Let G be the group defined by the presentation.

If one of a,b (say a) is not a piece then

$$s^* = \{(ab^p)^m, b^n\}^*$$

and G is a free product $Z_m * Z_n$ which has no free subgroup if $m=n=2$ or one of m,n equal to 1.

Now suppose that a and b are both pieces. Then Collins [1] has shown that G has a free subgroup of rank 2 unless $\langle a, b; s \rangle$ is permutational equivalent to one of the following.

(i) $\langle a, b; a^{-1}b^{-1}ab \rangle$

(ii) $\langle a, b; a^{-1}bab \rangle$

(iii) $\langle a, b; a^2b^2 \rangle$

(iv) $\langle a, b; a^4, b^4, (ab)^2 \rangle$

(v) $\langle a, b; (ab)^2, (ab^{-1})^2 \rangle$

The exceptions are as follows.

(i) This is a free abelan group on two generators.

(ii), (iii) These are presentations of the same group. Working with the second presentation (iii), we see that sgp([a^2]) is central and the corresponding factor group is the infinite dihedral group $C_2 * C_2$.

(iv) The group is a Euclidean triangle group $(1/4+1/4+1/2=1)$ and so has an abelian normal subgroup of finite index.

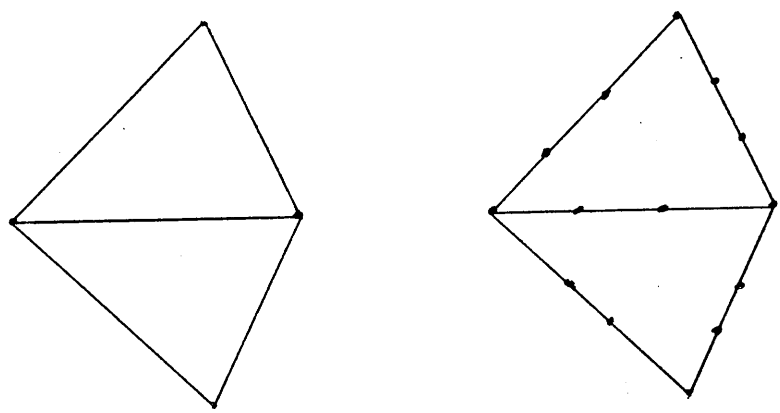
(v) As a consequence of the two given relations $[a^{-1}]=[aba]=[b^{-1}ab^{-1}]$ so that $[b^2ab^2]=[a]$, and $[b^{-1}]=[a^{-1}ba^{-1}]$ so that $[a^2ba^2]=[b]$. It follows that sgp([a^2], [b^2]) is an abelian normal subgroup with the Klein-four group as corresponding factor group.

CHAPTER 3

EXAMPLES OF T(6)-COMPLEXES AND HYPERBOLIC COMPLEXES.

3.1 X-complexes and Y-complexes.

In this section we consider certain complexes whose star complexes are "trisected". Roughly speaking, a trisected 1-complex is obtained by taking a 1-complex and trisecting each edge (making each edge of length 3). For example



Note that if the original 1-complex has no closed path of length 1, then the trisected 1-complex will have no reduced closed path of length less than 6.

More formally we define a *trisected 1-complex* as follow:

(1) The set of vertices is partitioned into two non-empty disjoint sets called type(1) and type(2), each of type(2) has valence 2.

(2) Call an edge a (1,2)-edge if one of its end point is of type(1) and the other of type(2), and call an edge a (2,2)-edge

if its end points are of type(2). We require that if v is of type(1) then all edges in Star(v) are (1,2)-edges and if v is of type(2) then one edge in Star(v) is a (1,2)-edge and the other is a (2,2)-edge.

(3) There is no reduced closed path of length 3.

It follows from (2) that every reduced closed path has length a multiple of 3, and so by (3), every non-empty reduced closed path has length at least 6.

Class-X

A 2-complex $\mathcal{K}$ is in the class X, that is, $\mathcal{K}$ is an ***X-complex***, if

(i) $\mathcal{K}^{st}$ is a trisected 1-complex,

(ii) e, e^{-1} are the same type ($e \in E(\mathcal{K})$),

(iii) each defining path of $\mathcal{K}$ has length 3.

A group G is said to be an ***X-group*** if $G \cong \pi_1(\mathcal{K})$, where $\mathcal{K}$ is a connected X-complex.

It follows from the definition of X-complex that X-complexes are T(6)-complexes. We will describe the structure of X-complexes in Theorem 3.1.2 below. First, however, we prove

Theorem 3.1.1. *If $\Phi: \tilde{K} \to K$ is a locally bijective mapping of 2-complexes and if K is an X-complex then so is $\tilde{K}$.*

From this it follows that the class of X-groups is closed under taking subgroups (see 1.1B.4).

In order to give a proof of Theorem 3.1.1. we give the following:

Lemma 3.1.1. *Let $\psi: \tilde{X} \to X$ be a locally bijective mapping of 1-complexes. If X is a trisected 1-complex then so is $\tilde{X}$.*

Proof. (1) We define a vertex $\tilde{v} \in \tilde{X}$ to be of type(1) or type(2) if $\psi(\tilde{v})$ is a vertex in X of type(1) or type(2). To show that a vertex $\tilde{v} \in \tilde{X}$ of type(2) has valence 2, we have by the definition that $\psi(\tilde{v}) \in X$ of type(2) has valence 2. Since ψ is locally bijective there is a bijection from the set of edges in Star($\tilde{v}$) to the set of edges in Star($\psi(\tilde{v})$). Thus we have that $\tilde{v}$ has valence 2.

(2) Let $\tilde{u}$ be a vertex of type (1) in $\tilde{X}$. We must show that all edges in Star($\tilde{u}$) are (1,2)-edges. Let $\tilde{e} \in$Star($\tilde{u}$) then $\psi(\tilde{e}) \in$Star($\psi(\tilde{u})$). Since $\psi(\tilde{u})$ is of type(1), $\tau\psi(\tilde{e})$ is of type(2). But $\tau\psi(\tilde{e}) = \psi(\tau\tilde{e})$, so $\tau\tilde{e}$ is of type(2) by definition. Similarly

if $\tilde{w}$ of type(2) then one edge in Star($\tilde{w}$) is a (1,2)-edge and the other is a (2,2)-edge.

(3) Let $\tilde{\beta}$ be reduced closed path in $\tilde{\mathcal{X}}$. Since ψ is locally injective then $\psi(\tilde{\beta})$ is a reduced closed path in $\mathcal{X}$, and since in $\mathcal{X}$ there is not such a path of length 3, $L(\tilde{\beta})\neq 3$.

Proof of Theorem 3.1.1.

Let ϕ: $\tilde{K} \to K$ be locally bijective. Since the defining paths of $\tilde{K}$ are lifts of defining paths in K, then each defining path of $\tilde{K}$ has length 3. We want to show that $\tilde{K}^{st}$ is a trisected. For that we use the fact that ϕ^{st}: $\tilde{K}^{st} \to K^{st}$ is a locally bijective mapping of 1-complexes (see 1.1C.3) and then by Lemma 3.1.1 above $\tilde{K}^{st}$ is a trisected. Now we show that for each $\tilde{e}\in\tilde{K}$ that $\tilde{e},\tilde{e}^{-1}$ have the same type. By definition of ϕ^{st}: $\tilde{K}^{st} \to K^{st}$, we have that $\phi^{st}(\tilde{e})=\phi(\tilde{e})=e$, $\phi^{st}(\tilde{e}^{-1})=\phi(\tilde{e}^{-1})=e^{-1}$, and since e,e^{-1} have the same type in K^{st}, then $\tilde{e},\tilde{e}^{-1}$ have the same type in $\tilde{K}^{st}$. Thus we have that $\tilde{K}$ is an X-complex.

Now we state our main Theorem in this section, which gives a charcterization of X-complexes.

Theorem 3.1.2. *Let $K = \langle X; r \rangle$ be a 2-complex where each defining path has length 3. Then the following are equivalent*

(I) K is an **X**-*complex.*

(II) K has the properties:

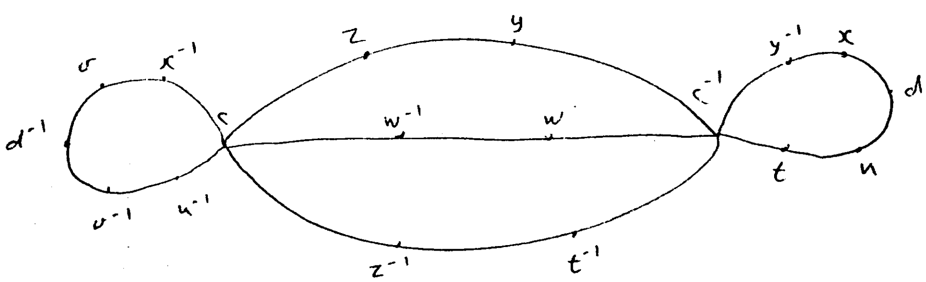
(i) The edge set of K is partitioned into two non-empty disjoint subsets A, B, each closed under inversion.

(ii) Replacing the defining paths by cyclic permutations if necessary, we have that each defining path has the form aba' *$(a, a' \in A, b \in B)$.*

(iii) Each $a \in A$ occurs exactly twice in the totality of paths $r \cup r^{-1}$, once on the left, and once on the right (that is one path in the set $r \cup r^{-1}$ has the form $a\alpha$ *and another* (possibly *the same) has the form βa, and no other element of $r \cup r^{-1}$ involves a).*

(iv) If $a_1 b_1 a$, aba', $a' b_2 a_2 \in r \cup r^{-1}$ then $b_1 \neq b_2^{-1}$.

Example 3.2.1. Let $\mathcal{P}$ be an **X**-presentation which has the star complex



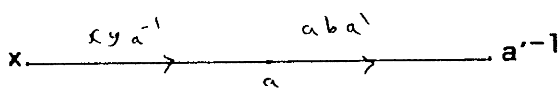
$\mathcal{P} = \langle c, d, x, y, z, t, u, v, w; xcy, yc^{-1}z^{-1}, z^{-1}ct, tc^{-1}u^{-1}, u^{-1}dv, vd^{-1}x, wcw \rangle$

where A={x, y, z, t, u, v, w}, B={c, d}. Note that the left occurence and right occurence of w are both in the same defining path.

Proof of Theorem 3.1.2. (I) implies (II)

(i) Since K is an X-complex, then K^{st} is a trisected 1-complex and by definition of trisected 1-complex K^{st} is partitioned into two disjoint subsets called type(1) and type(2). We call such sets B, A and by (ii) of definition of X-complex A, B are closed under inversion.

(ii) To show that each defining path has the form aba′ (a, a′∈A, b∈B), we first show that each defining path has at most one b∈B. Let ρ=aba′ and assume a, b∈B. Then in K^{st} there is an edge ba′a. This edge belongs to Star(b), but is not a (1,2)-edge, since a, b are of type(1), a contradiction. Then each defining path has at most one b∈B. Secondly we prove that each defining path has at least b∈B. Let a, a′∈A we show that b∈B. In K^{st} there are edges



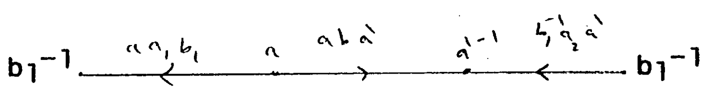
We have that x is of type(1), since a, a'^{-1} are of type(2), and then by the first part we have that y is of type(2). Also we have in K^{st} the edges

Since y^{-1}, a^{-1} are of type(2) by (ii) of the definition of X-complex, then b is of type(1), which implies $b \in B$.

(iii) To show that each $a \in A$ occurs exactly twice in the totality of paths rur^{-1} once on the right and once on the left. Let $\rho = aba'$ so $\rho^{-1} = a'^{-1}b^{-1}a^{-1}$. In K^{st} a has valence 2 and so we have uniquely determined edges

Since a, a'^{-1} are of type(2) then b_1^{-1} is of type(1) and hence a_1 is of type(2). Thus we have a_1b_1a, aba'.

(iv) We suppose that a_1b_1a, aba', $a'b_2a_2 \in rur^{-1}$ with $b_1 = b_2^{-1}$. Then in K^{st} we have edges



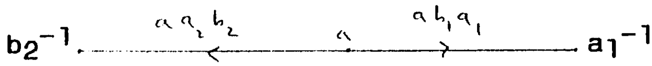
which is a reduced closed path of length 3, a contradiction to the fact that K^{st} is a trisected 1-complex. Then we have $b_1 \neq b_2^{-1}$.

(II) implies (I)

(i) First we show that K^{st} is a trisected 1-complex.

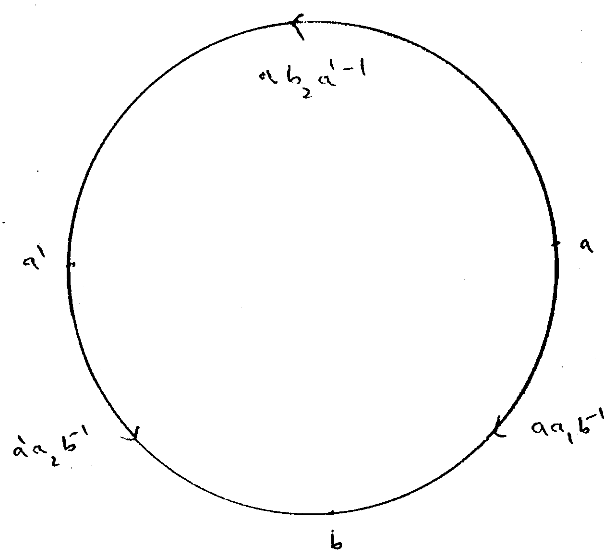
(1) We define an element $b \in B$ to be of type(1) and an element $a \in A$ to be of type(2). Then each vertex of type(2) has valence 2 by II(iii).

(2) Let $b \in \mathcal{K}^{st}$ be of type(1), and consider an edge $b \xrightarrow{\delta}$ in Star(b). Then $\gamma = ba_1a_2$ where $a_1, a_2 \in A$ by II(ii), so $\tau^{st}(\gamma) \in A$ that is $\tau^{st}(\gamma)$ is of type(2). Then all edges in Star(b) are (1,2)-edges. Now let $a \in \mathcal{K}^{st}$ be of type(2). By II(iii), we have the elements $ab_1a_1, a_2b_2a \in r \cup r^{-1}$. Thus the two edges of $\mathcal{K}^{st}$ in Star(a) are ab_1a_1, aa_2b_2



Then one of these is a (1,2)-edge and the other is a (2,2)-edge.

(3) To show there is no reduced closed path in $\mathcal{K}^{st}$ of length 3. Suppose there was such a path. Then it would have to pass through one vertex of type(1) and two vertices of type(2).



Then we get $a_1b^{-1}a$, $ab_2a'^{-1}$, $a'^{-1}ba_2^{-1}$ which a contradiction to II(iv). Then there is no reduced closed path of length 3.

(ii) From II(i), since A, B are closed under inversion, we have that for all e in $\mathcal{K}^{st}$, e, e^{-1} are of the same type.

(iii) Each defining path of $\mathcal{K}$ has length 3, by hypothesis.

This complete the proof of the Theorem.

We use the Structure Theorem above to arrange the defining paths of an **X**-complex in *chains*.

Let $\mathcal{K}$ be an **X**-complex. We consider subcomplexes $\mathcal{E}$ of $\mathcal{K}^{st}$ satisfying :

(A) $\mathcal{E}$ consists of a collection of (2,2)-edges together with their end points.

(B) If e is a vertex of $\mathcal{E}$ then so is e^{-1}.

Now suppose we have a subcomplex $\mathcal{A}$ of $\mathcal{K}^{st}$.

$$
\begin{array}{ccc}
e_0 & \longrightarrow & e_1^{-1} \\
e_1 & \longrightarrow & e_2^{-1} \\
 & \vdots & \\
e_k & \longrightarrow & e_{k+1}^{-1}
\end{array}
$$

where the e's are of type(2). Then we have the following.

Lemma 3.1.2. *The following are equivalent.*

(i) $\mathcal{A}$ *satisfies (B).*

(ii) $e_0 = e_{k+1}$.

(iii) $e_{k+1} \in \mathcal{A}$

Proof. (i) implies (ii) for, let satisfies (B). Then the set of vertices of $\mathcal{A}$ is closed under inversion. Also $\{e_1, e_1^{-1}, \ldots, e_k^{-1}, e_k\}$ is closed under inversion. Then $V(\mathcal{A}) - \{e_1, e_1^{-1}, \ldots, e_k^{-1}, e_k\}$ is closed under inversion, that is $\{a_0, a_{k+1}^{-1}\}$ is closed under inversion. We have $e_0 = e_{k+1}$ as required. (ii) implies (iii) is obvious. (iii) implies (ii) (and (ii) obviously implies (i)), for if we assume that $e_{k+1} \in \mathcal{A}$, but $e_{k+1} \neq e_0$ then we have $\{e_1^{-1}, e_1, \ldots, e_k^{-1}, e_k, e_{k+1}^{-1}\}$ is closed under inversion which is a contradiction since the cardinality of the set is odd.

Now suppose we have a (2,2)-edge ρ_0 of K^{st}. We can construct a subcomplex

$$\mathscr{E}_0(\rho_0): \quad a_0 \xrightarrow{\ \rho_0\ } a_1^{-1}$$

If $\mathscr{E}_0(\rho_0)$ satisfies (B), we have $a_0 = a_1$ by Lemma 3.1.2 and we stop. If $\mathscr{E}_0(\rho_0)$ does not satisfy (B) then by Lemma 3.1.2 $a_1 \notin \mathscr{E}_0(\rho_0)$. Consider the unique (2,2)-edge ρ_1 of K^{st} which starts with a_1. Then we have the subcomplex

$$\mathscr{E}_1(\rho_0): \quad \begin{array}{l} a_0 \xrightarrow{\ \rho_0\ } a_1^{-1} \\ a_1 \xrightarrow{\ \rho_1\ } a_2^{-1} \end{array}$$

If $\mathscr{E}_1(\rho_0)$ satisfies (B) we have $a_0 = a_2$ by Lemma 3.1.2 and we stop.

If $\mathcal{C}_1(\rho_0)$ does not satisfy (B), then $a_2 \notin \mathcal{C}_1(\rho_0)$ by Lemma 3.1.2.

Consider the unique (2,2)-edge ρ_2 of K^{st} which starts with a_2.

We then have the subcomplex

$$\mathcal{C}_2(\rho_0): \quad \begin{array}{lcl} a_0 & \xrightarrow{\rho_0} & a_1^{-1} \\ a_1 & \xrightarrow{\rho_1} & a_2^{-1} \\ a_2 & \xrightarrow{\rho_2} & a_3^{-1} \end{array}$$

Now continuing with the above argument, we will have two possible

outcomes. We either end up with a finite subcomplex

$$\mathcal{C}(\rho_0): \quad \begin{array}{lcl} a_0 & \xrightarrow{\rho_0} & a_1^{-1} \\ a_1 & \xrightarrow{\rho_1} & a_2^{-1} \\ & \vdots & \\ a_k & \xrightarrow{\rho_k} & a_0^{-1} \end{array}$$

satisfying (A) and (B) or the process continues indefinitely

giving

$$\mathcal{D}_0(\rho_0): \quad \begin{array}{lcl} a_0 & \xrightarrow{\rho_0} & a_1^{-1} \\ a_1 & \xrightarrow{\rho_1} & a_2^{-1} \\ & \vdots & \end{array}$$

In the later case $a_0^{-1} \notin \mathcal{D}_0(\rho_0)$, for if $a_0^{-1} \in \mathcal{D}_0$ then $V(\mathcal{D}_0)$ would be closed under inversion. But the set $\{a_1^{-1}, a_1, a_2^{-1}, a_2, \ldots\}$ is closed under inversion so $V(\mathcal{D}_0) - \{a_1^{-1}, a_1, a_2^{-1}, a_2, \ldots\} = \{a_0\}$ would be closed under inversion, a contradiction. Then $a_0^{-1} \notin \mathcal{D}_0(\rho_0)$. Let ρ_{-1} be the unique (2,2)-edge ending with a_0^{-1}. Then we have the subcomplex

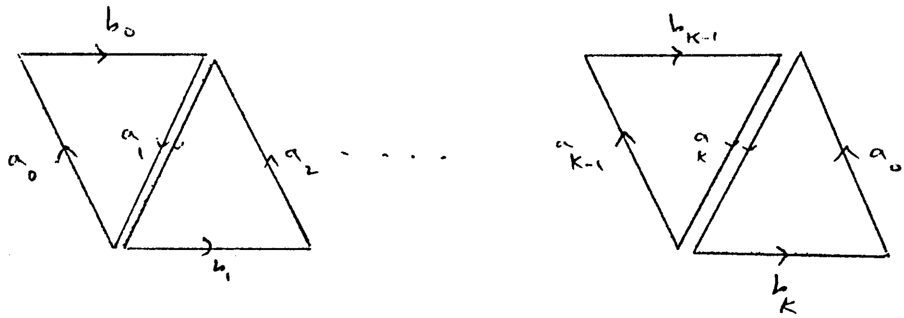
$$\begin{array}{llcl}
 & a_{-1} & \xrightarrow{\rho_{-1}} & a_0^{-1} \\
\mathcal{D}_1(\rho_0): & a_0 & \xrightarrow{\rho_0} & a_1^{-1} \\
 & a_1 & \xrightarrow{\rho_1} & a_2^{-1}
\end{array}$$

then $a^{-1}_{-1} \notin \mathcal{D}_1(\rho_0)$ (by the same reasoning as above), so we consider the unique (2,2)-edge ρ_{-2} ending with a^{-1}_{-1}, and so on. Eventually we will get a subcomplex

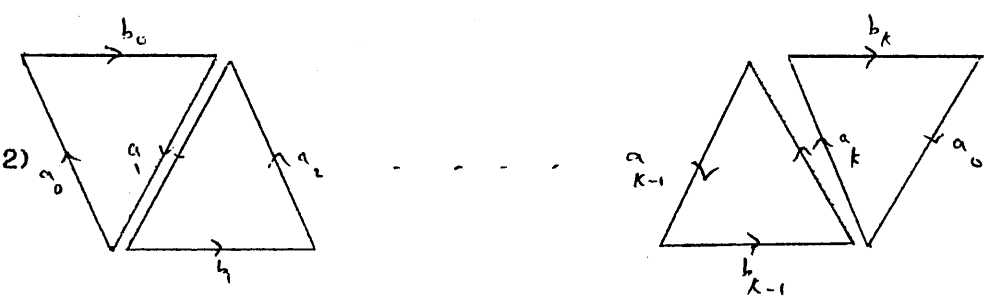
$$\begin{array}{llcl}
 & & \vdots & \\
 & a_{-2} & \xrightarrow{\rho_{-2}} & a^{-1}_{-1} \\
 & a_{-1} & \xrightarrow{\rho_{-1}} & a_0^{-1} \\
\mathcal{D}(\rho_0): & a_0 & \xrightarrow{\rho_0} & a_1^{-1} \\
 & a_1 & \xrightarrow{\rho_1} & a_2^{-1} \\
 & a_2 & \xrightarrow{\rho_2} & a_3^{-1} \\
 & & \vdots &
\end{array}$$

Now each ρ_i belong to $r \cup r^{-1}$. If ρ_i belongs to r^{-1} then $\rho_i^{-1} \in r$ and we modify r by replacing the elements $\rho_i^{-1} \in r$ by ρ_i. Write $\rho_i = a_i b_i a_{i+1}$ where b_i is of type(1). Then the ρ's fit together as follows:

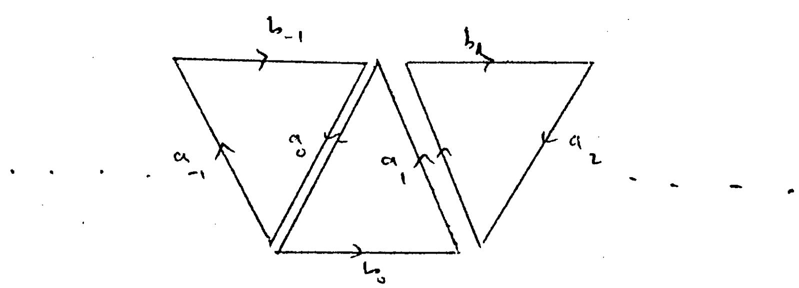
(1)



or (2)



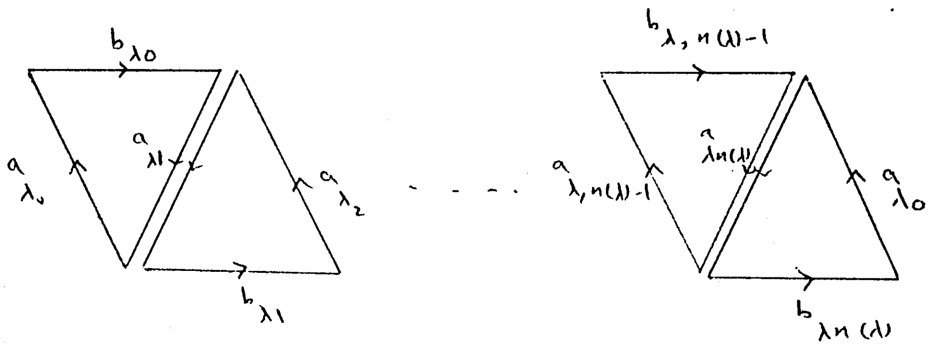
or (3)



We call a chain of form (1) a finite chain of even length, a chain of form (2) a finite chain of odd length, and a chain of form (3) a chain of infinite length.

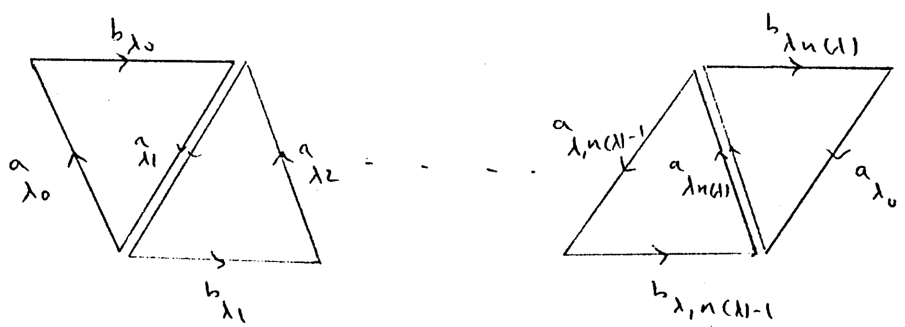
Now we index the edges and defining paths of K to coincides with their arrangement into chains.

Finite chains of even length



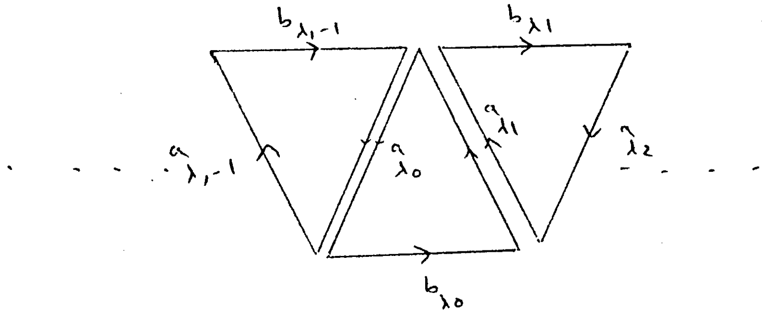
$(\lambda \in \Lambda_1)$

Finite chains of odd length



$(\lambda \in \Lambda_2)$

Chains of infinite length



$(\lambda \in \Lambda_3)$

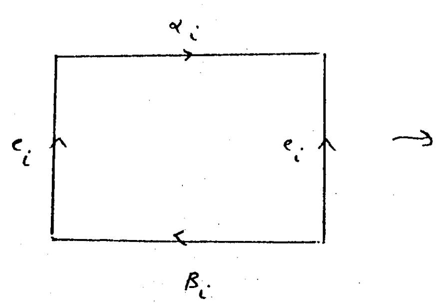
Note that for a chain of *even* length $n(\lambda)$ is *odd*, and for a chain of *odd* length $n(\lambda)$ is *even*.

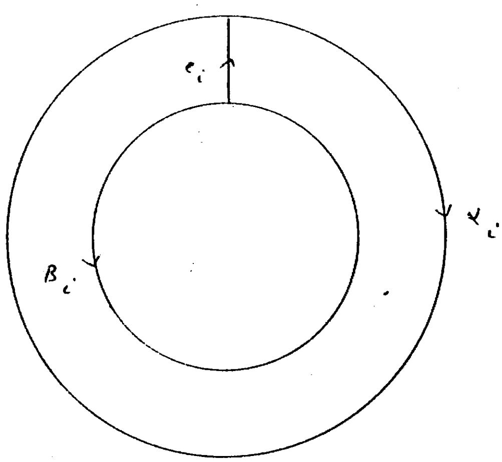
Class–Y

A 2–complex K is in the class Y, that is, K is a *Y–complex* if, the edge set of K is partitioned into two non–empty disjoint subsets $\{e_i, e_i^{-1}(i \in I)\} \cup A$, where A is closed under inversion, and for $i \in I$ there is a defining path $\rho_i = e_i \alpha_i e_i^{\epsilon_i} \beta_i$ $(\epsilon_i = \pm 1)$ where α_i, β_i are non–empty paths whose edges belong to A. We further require that if $\epsilon_i = -1$ then α_i and β_i are cyclically reduced and have the same length, while if $\epsilon_i = 1$ then $\alpha_i \beta_i^{-1}$ is cyclically reduced and the lengths of α_i and β_i differ by 1.

We say that a Y–complex K is *orientable* if each $\epsilon_i = -1$ otherwise it is *non–orientable*.

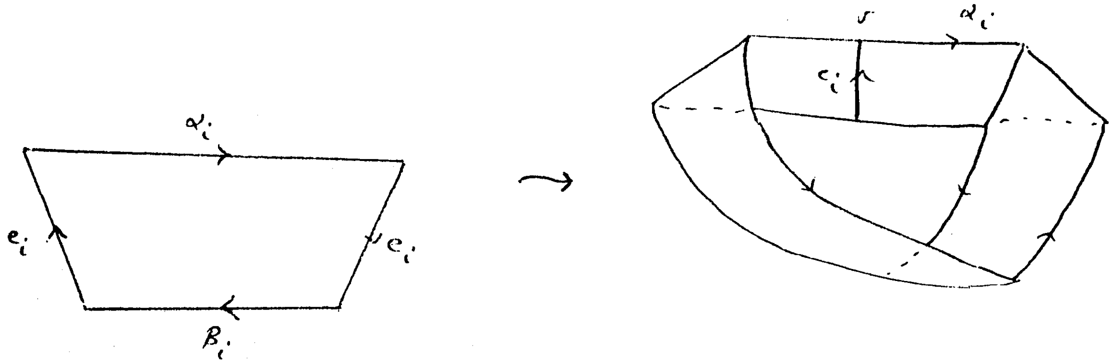
For orientable, we have





Orientable surface with boundary

For non-orientable, we have



Mobius strip, and the label reading

around the boundary starting at v is $\alpha_i\beta_i^{-1}$

A group G is said to be a *Y-group* if $G \cong \pi_1(\mathcal{K})$, where $\mathcal{K}$ is a connected Y-complex.

Theorem 3.1.3. *Every X-complex is equivalent to a Y-complex. Conversely, every Y-complex is equivalent to an X- complex.*

Proof.

Let $\mathcal{K}$ be an X-complex. We index the defining paths of $\mathcal{K}$ to coincide with their arrangement into chains (see above). We have

$\mathcal{K} = \langle \mathfrak{X};\ \rho_{\lambda l}\ (\lambda\in\Lambda_1\cup\Lambda_2\cup\Lambda_3,\ 0\leq l\leq n(\lambda)$ for $\lambda\in\Lambda_1\cup\Lambda_2$, $l\in\mathbf{Z}$ for $\lambda\in\Lambda_3)\rangle$.

We arrange the defining paths into levels, to use Level Theorem (see 1.1B.5). In the following the edge associated with a defining path of a given level is marked ⌐¬.

	defining paths in finite chains	defining paths in infinite chains	
level $n(\lambda)$	$\rho_{\lambda n(\lambda)-1}=a_{\lambda,n(\lambda)-1}\,b_{\lambda,n(\lambda)-1}\,\boxed{a_{\lambda n(\lambda)}}$	.	.
.	.	.	.
level 4	$\rho_{\lambda 3}=a_{\lambda 3}\,b_{\lambda 3}\,\boxed{a_{\lambda 4}}$	$\rho_{\lambda,-3}=\boxed{a_{\lambda,-3}}\,b_{\lambda,-3}\,a_{\lambda,-2}$	$\rho_{\lambda 3}=a_{\lambda 3}\,b_{\lambda 3}\,\boxed{a_{\lambda 4}}$
level 3	$\rho_{\lambda 2}=a_{\lambda 2}\,b_{\lambda 2}\,\boxed{a_{\lambda 3}}$	$\rho_{\lambda,-2}=\boxed{a_{\lambda,-2}}\,b_{\lambda,-2}\,a_{\lambda,-1}$	$\rho_{\lambda 2}=a_{\lambda 2}\,b_{\lambda 2}\,\boxed{a_{\lambda 3}}$
level 2	$\rho_{\lambda 1}=a_{\lambda 1}\,b_{\lambda 1}\,\boxed{a_{\lambda 2}}$	$\rho_{\lambda,-1}=\boxed{a_{\lambda,-1}}\,b_{\lambda,-1}\,a_{\lambda 0}$	$\rho_{\lambda 1}=a_{\lambda 1}\,b_{\lambda 1}\,\boxed{a_{\lambda 2}}$
level 1	$\rho_{\lambda 0}=a_{\lambda 0}\,b_{\lambda 0}\,\boxed{a_{\lambda 1}}$	$\rho_{\lambda 0}=a_{\lambda 0}\,b_{\lambda 0}\,\boxed{a_{\lambda 1}}$	
level o	$\rho_{n(\lambda)}=a_{\lambda n(\lambda)}\,b_{\lambda n(\lambda)}\,a_{\lambda 0}$		

Let $\mathcal{X}_0$ be the 1-complex obtained from $\mathcal{X}$ by removing all these edges marked $\square$ above and their inverses and for $\ell>0$, let

$\mathcal{X}_\ell = \mathcal{X}_{\ell-1}\cup\{$all edges marked $\square$ above and their inverses of level $\ell\}$.

We define a mapping $\phi: \mathcal{X} \to \mathcal{X}_0$ as follows. we define ϕ on $\mathcal{X}_0$ to be the identity. Suppose ϕ has defined on $\mathcal{X}_{\ell-1}$ $(\ell>0)$. Extend ϕ to $\mathcal{X}_\ell$ by setting $\phi(a_{\lambda\ell})=\phi(b^{-1}_{\lambda,\ell-1}a^{-1}_{\lambda,\ell-1})$ $(\lambda\in\Lambda_1\cup\Lambda_2\cup\Lambda_3,\ \ell\leq n(\lambda)$ if $\lambda\in\Lambda_1\cup\Lambda_2)$ and $\phi(a_{\lambda,-\ell+1})=\phi(a^{-1}_{\lambda,-\ell+2}b^{-1}_{\lambda,-\ell+1})$ $(\lambda\in\Lambda_3)$.

We have

$\phi(a_{\lambda 1})=\phi(b_{\lambda 0}{}^{-1}a_{\lambda 0}{}^{-1})=b_{\lambda 0}{}^{-1}a_{\lambda 0}{}^{-1}$,

$\phi(a_{\lambda 2})=\phi(b_{\lambda 1}^{-1}a_{\lambda 1}^{-1})=b_{\lambda 1}^{-1}\phi(a_{\lambda 1})^{-1}$

$=b_{\lambda 1}^{-1}a_{\lambda o}b_{\lambda o}.$

$\phi(a_{\lambda 3})=\phi(b_{\lambda 2}^{-1}a_{\lambda 2}^{-1})=b_{\lambda 2}^{-1}\phi(a_{\lambda 2})^{-1}$

$=b_{\lambda 2}^{-1}(b_{\lambda 1}^{-1}a_{\lambda o}b_{\lambda o})^{-1}$

$=b_{\lambda 2}^{-1}b_{\lambda o}^{-1}a_{\lambda o}^{-1}b_{\lambda 1}.$

$\phi(a_{\lambda 4})=\phi(b_{\lambda 3}^{-1}a_{\lambda 3}^{-1})=b_{\lambda 3}^{-1}\phi(a_{\lambda 3})^{-1}$

$=b_{\lambda 3}^{-1}(b_{\lambda 2}^{-1}b_{\lambda o}^{-1}a_{\lambda o}^{-1}b_{\lambda 1})^{-1}$

$=b_{\lambda 3}^{-1}b_{\lambda 1}^{-1}a_{\lambda o}b_{\lambda o}b_{\lambda 2}.$

In general we have for $\lambda\in\Lambda_1\cup\Lambda_2$ and $\ell<n(\lambda)$

$$\phi(a_{\lambda\ell})=\begin{cases} b^{-1}{}_{\lambda,\ell-1}\cdots b_{\lambda 3}^{-1}b_{\lambda 1}^{-1}a_{\lambda o}b_{\lambda o}b_{\lambda 2}\cdots b_{\lambda,\ell-2} & \text{if } \ell \text{ is even} \\ b^{-1}{}_{\lambda,\ell-1}\cdots b_{\lambda 2}^{-1}b_{\lambda o}^{-1}a_{\lambda o}^{-1}b_{\lambda 1}b_{\lambda 3}\cdots b_{\lambda,\ell-2} & \text{if } \ell \text{ is odd.} \end{cases}$$

We prove this by induction. It is certainly true for $\ell=1$. Now

suppose $\ell>1$.

$\phi(a_{\lambda\ell})=\phi(b^{-1}{}_{\lambda,\ell-1}a^{-1}{}_{\lambda,\ell-1})$

$=b^{-1}{}_{\lambda,\ell-1}\phi(a_{\lambda,\ell-1})^{-1}$ and by induction hypothesis

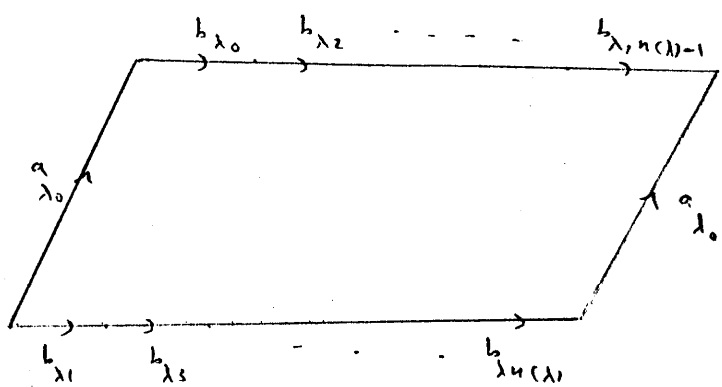
$$=\begin{cases} b^{-1}{}_{\lambda,\ell-1}(b^{-1}{}_{\lambda,\ell-2}\cdots b_{\lambda 3}^{-1}b_{\lambda 1}^{-1}a_{\lambda o}b_{\lambda o}b_{\lambda 2}\cdots b_{\lambda,\ell-3})^{-1} & \text{if } \ell-1 \text{ is even} \\ b^{-1}{}_{\lambda,\ell-1}(b^{-1}{}_{\lambda,\ell-2}\cdots b_{\lambda 2}^{-1}b_{\lambda o}^{-1}a_{\lambda o}^{-1}b_{\lambda 1}b_{\lambda 3}\cdots b_{\lambda,\ell-3})^{-1} & \text{if } \ell-1 \text{ is odd.} \end{cases}$$

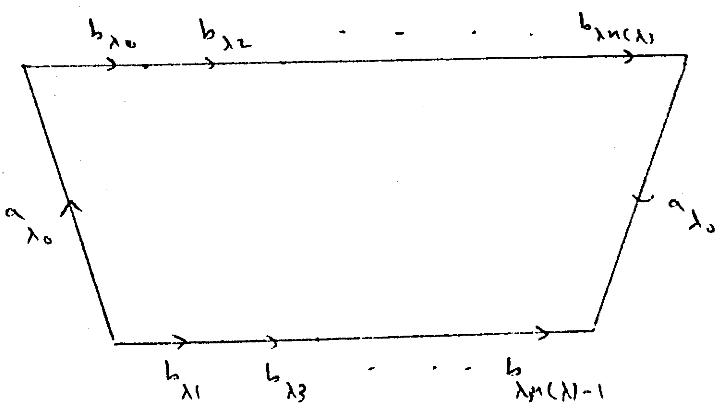
$$= \begin{cases} b^{-1}_{\lambda,\ell-1}b^{-1}_{\lambda,\ell-3}\cdots b_{\lambda 2}^{-1}b_{\lambda o}^{-1}a_{\lambda o}^{-1}b_{\lambda 1}b_{\lambda 3}\cdots b_{\lambda,\ell-2} & \text{if } \ell \text{ is odd} \\ b^{-1}_{\lambda,\ell-1}b^{-1}_{\lambda,\ell-3}\cdots b_{\lambda 3}^{-1}b_{\lambda 1}^{-1}a_{\lambda o}b_{\lambda o}b_{\lambda 2}\cdots b_{\lambda,\ell-2} & \text{if } \ell \text{ is even} \end{cases}$$

as required.

Now we have

$$\phi(\rho_{\lambda n(\lambda)}) = \begin{cases} b^{-1}_{\lambda,n(\lambda)-1}\cdots b_{\lambda o}^{-1}a_{\lambda o}^{-1}b_{\lambda 1}\cdots b_{\lambda n(\lambda)-2}b_{\lambda n(\lambda)}a_{\lambda o} & \text{if } \lambda\in\Lambda_1 \\ b^{-1}_{\lambda,n(\lambda)-1}\cdots b_{\lambda 1}^{-1}a_{\lambda o}b_{\lambda o}\cdots b_{\lambda,n(\lambda)-2}b_{\lambda,n(\lambda)}a_{\lambda o} & \text{if } \lambda\in\Lambda_2 \end{cases}$$





To show that $\langle \mathcal{K}_0 : \phi(\rho_{\lambda n(\lambda)}) \ (\lambda\in\Lambda_1\cup\Lambda_2) \rangle$ is a Y-complex. Since

$L(b_{\lambda o}b_{\lambda 2}\ldots b_{\lambda,n(\lambda)-1})=L(b_{\lambda 1}b_{\lambda 3}\ldots b_{\lambda n(\lambda)})$ $(\lambda\in\Lambda_1)$, and

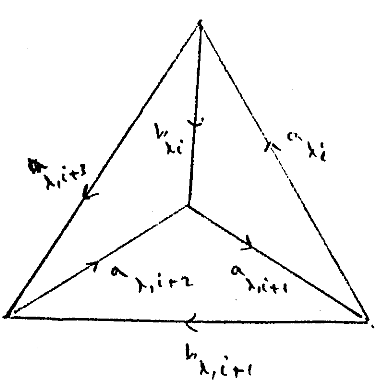
$L(b_{\lambda o}b_{\lambda 2}\ldots b_{\lambda n(\lambda)})$, $L(b_{\lambda 1}b_{\lambda 3}\ldots b_{\lambda,n(\lambda)-1})$ differ by 1 $(\lambda\in\Lambda_2)$,

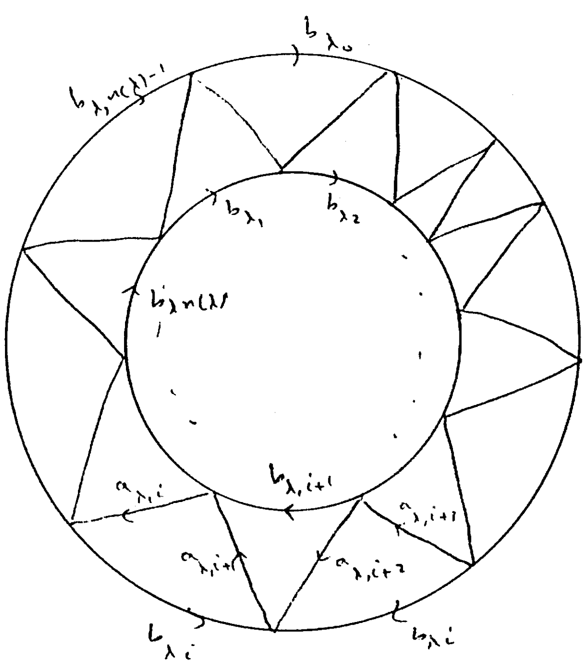
it remains to show

(1) if $\lambda\in\Lambda_1$ then $b_{\lambda o}b_{\lambda 2}\ldots b_{\lambda,n(\lambda)-1}$ and $b_{\lambda 1}b_{\lambda 3}\ldots b_{\lambda n(\lambda)}$ are cyclically reduced and

(2) if $\lambda\in\Lambda_2$ then $b_{\lambda o}b_{\lambda 2}\ldots b_{\lambda n(\lambda)}b_{\lambda 1}b_{\lambda 3}\ldots b_{\lambda,n(\lambda)-1}$ is cyclically reduced.

For (1) to prove that $b_{\lambda o}b_{\lambda 2}\ldots b_{\lambda,n(\lambda)-1}$, $b_{\lambda 1}b_{\lambda 3}\ldots b_{\lambda n(\lambda)}$ are cyclically reduced. Assume that $b_{\lambda o}b_{\lambda 2}\ldots b_{\lambda,n(\lambda)-1}$ not cyclically reduced, then we have two consective edges say the edges $b_{\lambda i}, b_{\lambda i+1}$ are inverses. We have

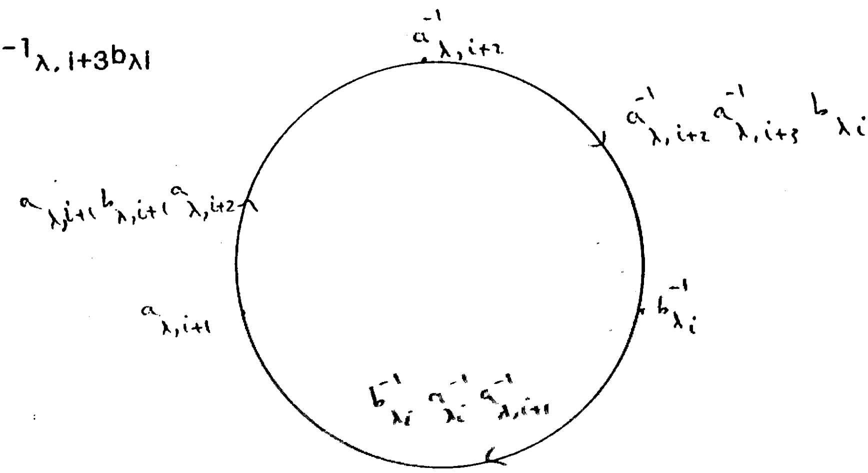




which gives the reduced closed path of length 3 in K^{st} namely

the path $b_{\lambda i}^{-1}a_{\lambda i}^{-1}a^{-1}_{\lambda,i+1}$, $a_{\lambda,i+1}b_{\lambda,i+1}a_{\lambda,i+2}$,

$a^{-1}_{\lambda,i+2}a^{-1}_{\lambda,i+3}b_{\lambda i}$



which is a contradiction since K is an X-complex. Then we have

that $b_{\lambda o}b_{\lambda 2}\ldots b_{\lambda n(\lambda)-1}$, $b_{\lambda 1}b_{\lambda 3}\ldots b_{\lambda n(\lambda)}$ are cyclically reduced.

Similarly for (2). Then by (1), (2), we have that

$\langle K_o ;\ \Phi(\rho_{\lambda n(\lambda)}),\ \lambda\in\Lambda_1\cup\Lambda_2 \rangle$ is a Y-complex.

Conversely. Let L be a Y-complex. Then we have

(1) A set of defining paths of the forms $\rho_\lambda = e_{\lambda o}\alpha_\lambda e_{\lambda o}^{-1}\beta_\lambda^{-1}$ $(\lambda\in\Lambda_1)$,

where $L(\alpha_\lambda)=L(\beta_\lambda)$ and $\alpha_\lambda, \beta_\lambda$ are cyclically reduced.

(2) A set of defining paths of the forms $\rho_\lambda = e_{\lambda o}\alpha_\lambda e_{\lambda o}\beta_\lambda^{-1}$ $(\lambda\in\Lambda_2)$,

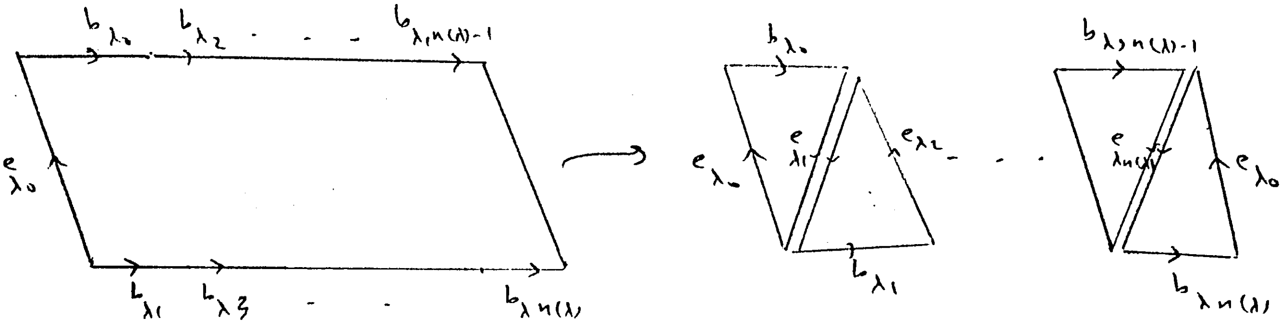
where $L(\alpha_\lambda)-L(\beta_\lambda)=1$ and $\alpha_\lambda\beta_\lambda$ is cyclically reduced.

Suppose $\lambda\in\Lambda_1$ and let $\alpha_\lambda=b_{\lambda o}b_{\lambda 2}\ldots b_{\lambda n(\lambda)-1}$, $\beta_\lambda=b_{\lambda 1}b_{\lambda 3}\ldots b_{\lambda n(\lambda)}$.

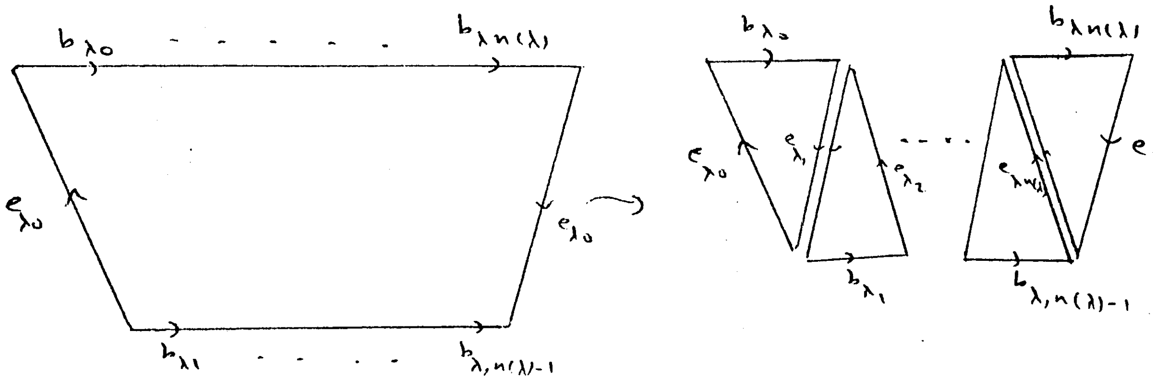
Adjoin edges $e_{\lambda 1}, e_{\lambda 2}, \ldots, e_{\lambda n(\lambda)}$ to $L^{(1)}$ and replace ρ_λ by the

defining paths $e_{\lambda o}b_{\lambda o}e_{\lambda 1}$, $e_{\lambda 1}b_{\lambda 1}e_{\lambda 2}$, $e_{\lambda 2}b_{\lambda 2}e_{\lambda 3}$, ...,

$e_{\lambda n(\lambda)}b_{\lambda n(\lambda)}e_{\lambda o}$



Now suppose $\lambda \in \Lambda_2$ and let $\alpha_\lambda = b_{\lambda o} b_{\lambda 2} \dots b_{\lambda n(\lambda)}$, $\beta_\lambda = b_{\lambda 1} b_{\lambda 3} \dots b_{\lambda, n(\lambda)-1}$. Adjoin edges $e_{\lambda 1}, e_{\lambda 2}, \dots, e_{\lambda n(\lambda)}$ to $\mathcal{L}^{(1)}$ and replace ρ_λ by the defining paths $e_{\lambda o} b_{\lambda o} e_{\lambda 1}$, $e_{\lambda 1} b_{\lambda 1} e_{\lambda 2}$, $e_{\lambda 2} b_{\lambda 2} e_{\lambda 3}$, $\dots$, $e_{\lambda n(\lambda)} b_{\lambda n(\lambda)} e_{\lambda o}$



Then we obtain a 2-complex satisfying (i), (ii), (iii), (iv) of (II) of Theorem 3.1.2. Then we have that $\mathcal{L}$ is an X-complex.

This complete the proof of Theorem 3.1.3.

Corollary 3.1.1. *If K is connected X-complex in which all chains are infinite then $\pi_1(K)$ is free.*

There is a conjecture that finite T(6)-complex have residually finite fundamental group ([12], [16, Problem F.5]). It is therefore of interest to consider the residual finiteness of $\pi_1(\mathcal{K})$ where $\mathcal{K}$ is a finite X-complex. Now $\mathcal{K}$ is equivalent to a Y-complex $\mathcal{L}$ by Theorem 3.1.3, so $\pi_1(\mathcal{K}) \cong \pi_1(\mathcal{L})$. Thus it sufficies to consider the residual finiteness of Y-complex. Of particular interest is the fact that many one relator group with a presentation of the form

$$\langle t, x_1, \ldots, x_n \ ; \ t^{-1}\alpha t\beta \rangle$$

(α, β are paths not involving t) are not residually finite [8], but can this happen if the presentation is a Y-presentation, that is, if α, β are cyclically reduced and $L(\alpha)=L(\beta)$?. We have not been able to make any progress to prove that.

Theorem 3.1.4. *Every non-orientable Y-complex has a 2-fold covering which is equivalent to an orientable Y-complex.*

Proof.

Let $\mathcal{K}$ be a non-orientable Y-complex. We construct a 2-fold covering $\tilde{\mathcal{K}}$ of $\mathcal{K}$. We define the 1-skeleton $\tilde{\mathcal{K}}^{(1)}$ of $\tilde{\mathcal{K}}$ as follows. For each vertex v of $\mathcal{K}$ there are two vertices v_0, v_1, and for each $a \in A$ there are two edges a_0, a_1 where

$\iota(a_o) = \iota(a)_o$, $\tau(a_o) = \tau(a)_o$,

$\iota(a_1) = \iota(a)_1$, $\tau(a_1) = \tau(a)_1$

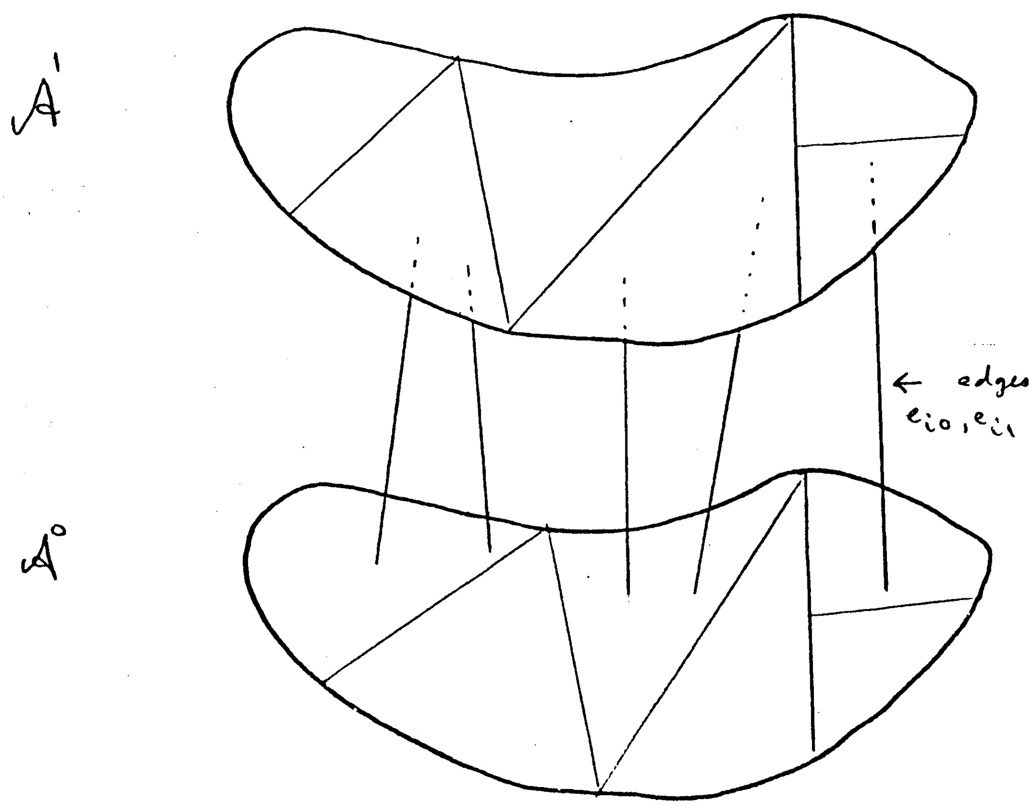
and $a_o^{-1} = (a^{-1})_o$, $a_1^{-1} = (a^{-1})_1$

For each e_i there are e_{io} , e_{i1} such that

$\iota(e_{io}) = (\iota e_i)_o$, $\tau(e_{io}) = (\tau e_i)_1$,

$\iota(e_{i1}) = (\iota e_i)_1$, $\tau(e_{i1}) = (\tau e_i)_o$

Note that $\tilde{K}^{(1)}$ has the following structure. Let $\mathcal{A}$ be the sub-complex of $K^{(1)}$ obtained by deleting all the edges e_i, e_i^{-1} $(i \in I)$. Then $\tilde{K}^{(1)}$ consists of two copies of $\mathcal{A}$ denoted by $\mathcal{A}^o$, $\mathcal{A}^1$ ($\mathcal{A}^o$ consists of all vertices of $\tilde{K}^{(1)}$ with subscript o and all edges a_o $(a \in A)$; $\mathcal{A}^1$ is defined analogously) and $\mathcal{A}^o$, $\mathcal{A}^1$ are linked by the edges e_{io}, e_{i1}.



If γ is a path in $\mathcal{A}$ then the lifts in $\mathcal{A}^o$, $\mathcal{A}^1$ are denoted by γ^o, γ^1 respectively.

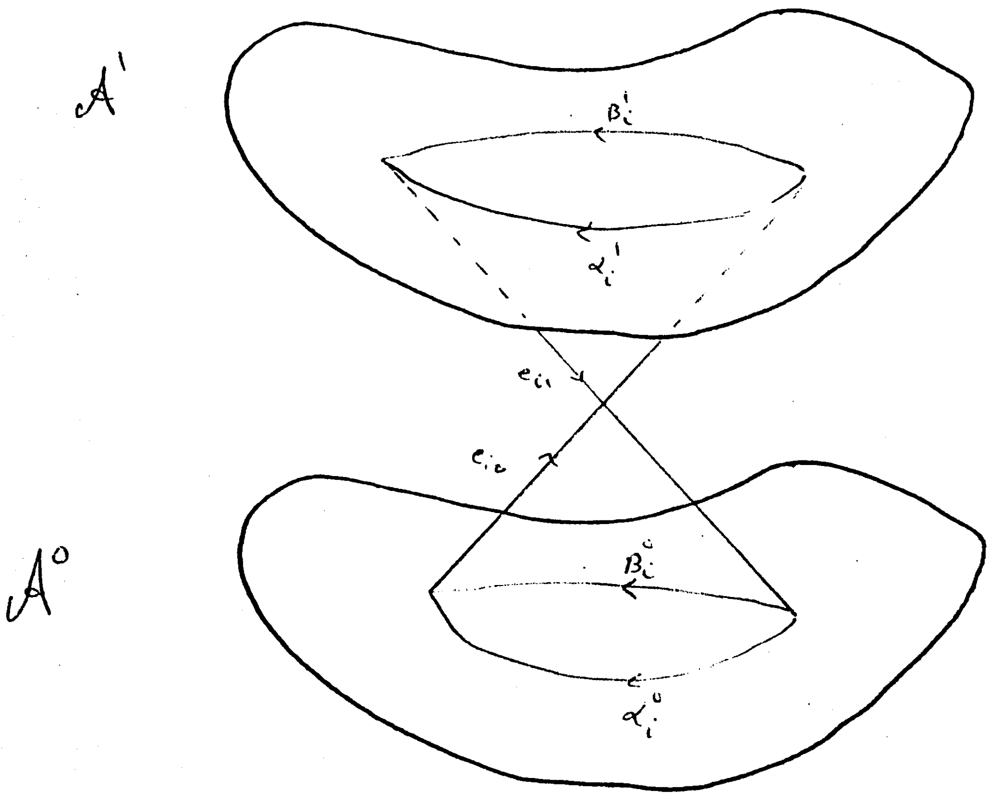
We now consider the defining paths of $\widetilde{K}$. Let $\rho_i = e_i \alpha_i e_i^{\epsilon_i} \beta_i$ be a defining path of K. This gives rise to two defining paths ρ_{io}, ρ_{i1} of $\widetilde{K}$ as follows.

<u>Case 1</u> <u>$\epsilon_i = 1$</u>

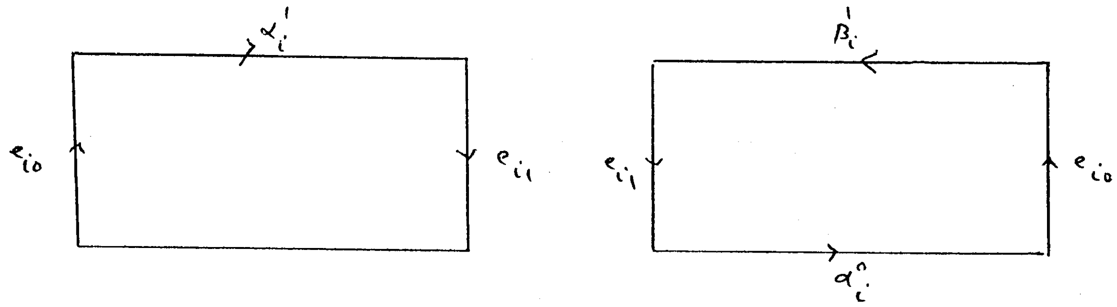
Then the lifts of ρ_i in $\widetilde{K}$ are

$\rho_{io} = e_{io} \alpha_i^1 e_{i1} \beta_i^o$,

$\rho_{i1} = e_{i1} \alpha_i^o e_{io} \beta_i^1$



We have

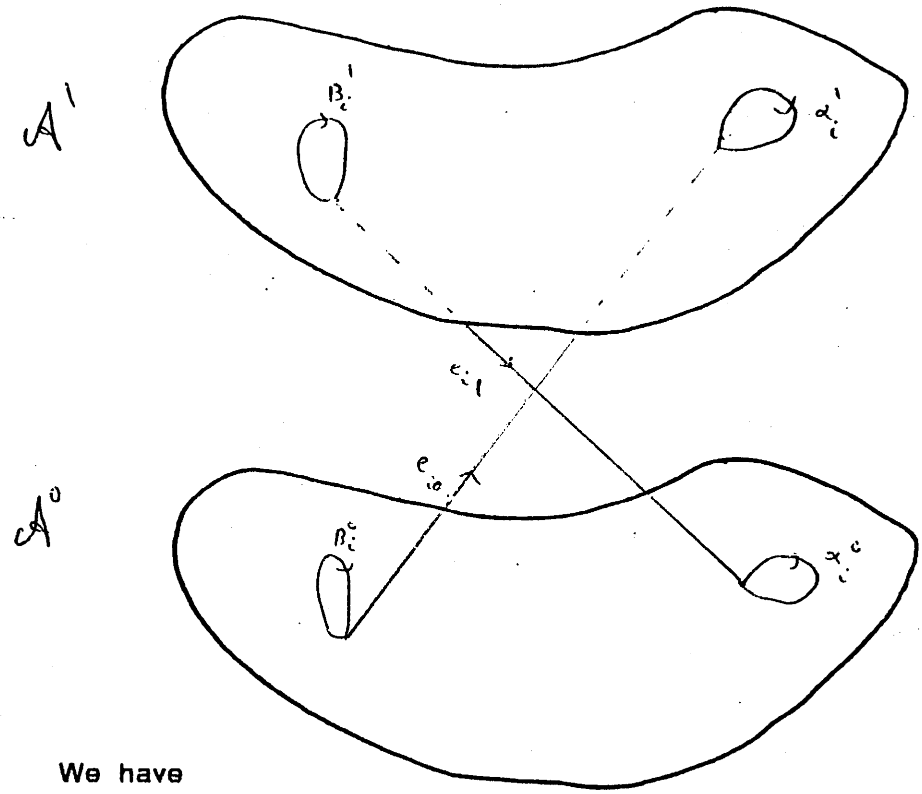


<u>Case 2</u> <u>$\epsilon_i=-1$</u>

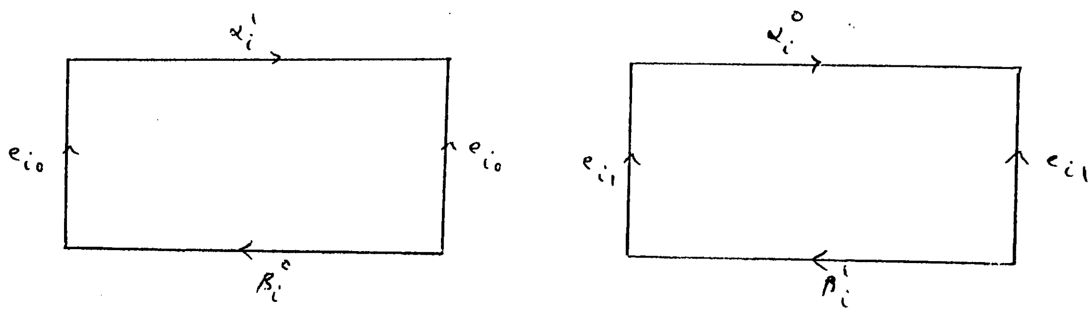
Then the lifts of ρ_i in $\tilde{K}$ are

$\rho_{i0}=e_{i0}\alpha_i{}^1 e^{-1}{}_{i0}\beta_i{}^0$

$\rho_{i1}=e_{i1}\alpha_i{}^0 e_{i1}{}^{-1}\beta_i{}^1$

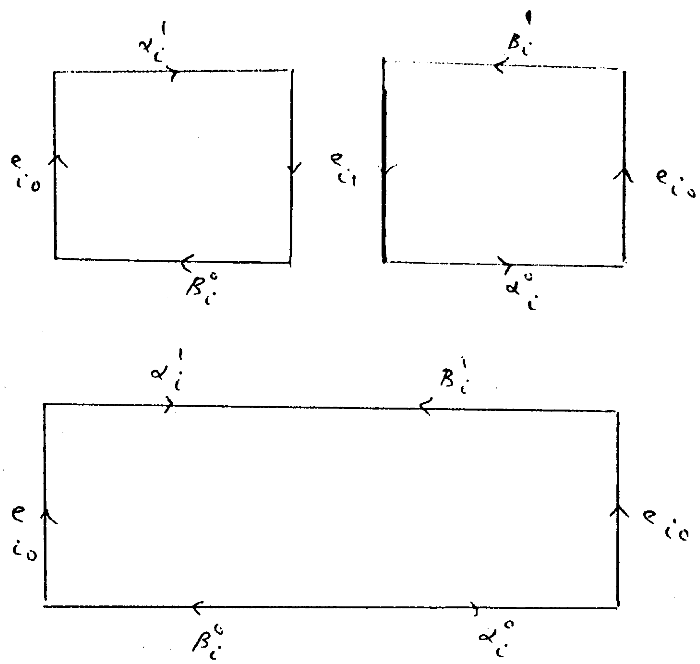


We have



Let $\mathcal{L}$ be the 2-complex obtained from $\tilde{K}$ as follows. For each $i \in I$ for which $\epsilon_i=1$ remove the edges e_{i1}, e^{-1}_{i1} from $\tilde{K}^{(1)}$, remove the two defining paths ρ_{i0}, ρ_{i1}, and replace them by a single defining path

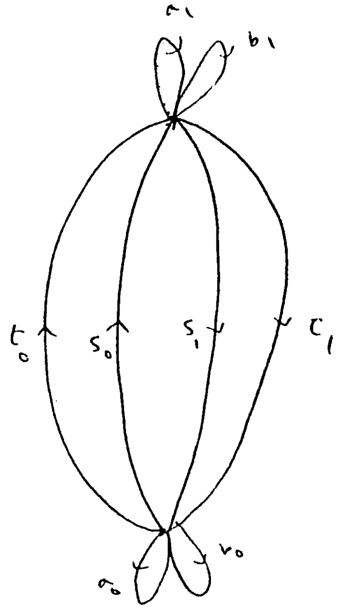
$$e_{i0}\alpha^1{}_i\beta^1{}_i{}^{-1}e_{i0}\alpha^0{}_i{}^{-1}\beta^0{}_i$$



Then $\mathcal{L}$ is an oriented Y-complex and is equivalent to $\tilde{K}$ by Level Theorem (see 1.1B.5)

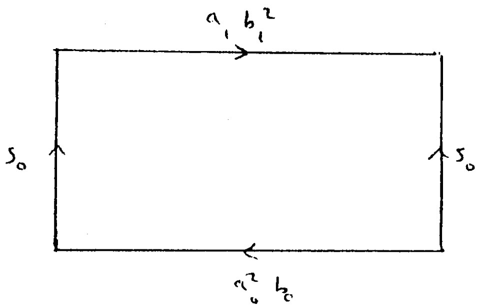
Example 3.1.2. Consider the non-orientable Y-presentation

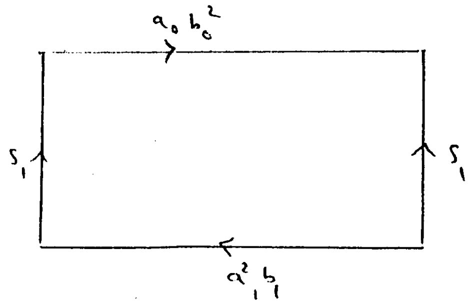
$K = \langle\, a, b, s, t\ ;\ tabab^2tb^2a^2,\ sab^2s^{-1}a^2b\,\rangle$. We have



The lift of $sab^2s^{-1}a^2b$ are

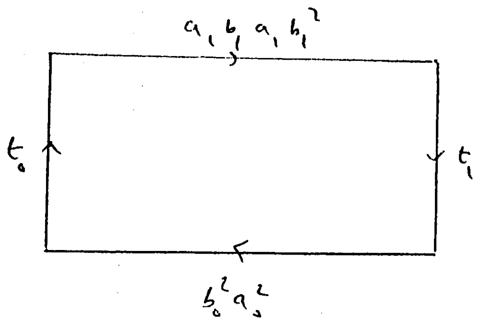
$s_0a_1b_1^2s_0^{-1}a_0^2b_0$, $s_1a_0b_0^2s_1^{-1}a_1^2b_1$

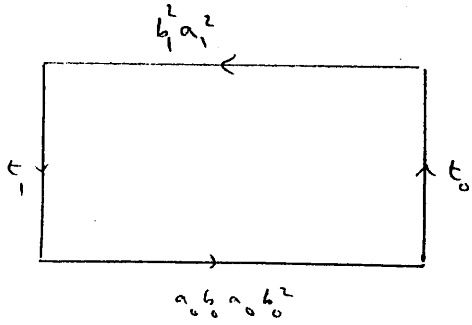




The lifts of $tabab^2tb^2a^2$ are

$t_0a_1b_1a_1b_1^2t_1b_0^2a_0^2$, $t_1a_0b_0a_0b_0^2t_0b_1^2a_1^2$





Then $\tilde{K}$ is equivalent to the orienntable Y-complex with the above 1-skeleton and with the defining paths

$s_0 a_1 b_1^2 s_0^{-1} a_0^2 b_0$,

$s_1 a_0 b_0^2 s_1^{-1} a_1^2 b_1$,

$t_0 a_1 b_1 a_1 b_1^2 a_1^{-2} b_1^{-2} t_0^{-1} b_0^{-2} a_0^{-1} b_0^{-1} a_0^{-1} b_0^2 a_0^2$.

3.2 Positive T(6)-presentations.

A presentation $\langle x; r\rangle$ is said to be *positive* if each defining path $\rho \in r$ is a positive path in x (that is, only positive powers of elements of x occurs in ρ).

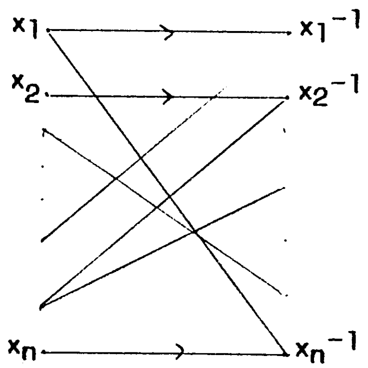
The following gives a way of combining positive T(6)-presentation. to get new ones.

Theorem 3.2.1. *Let* $\mathcal{P}=\langle x_1, x_2, \ldots, x_n; p\rangle$, $\mathcal{R}=\langle y_1, y_2, \ldots, y_n; r\rangle$ $\mathcal{S}=\langle z_1, z_2, \ldots, z_n; s\rangle$ *be positive T(6)-presentations. Then*

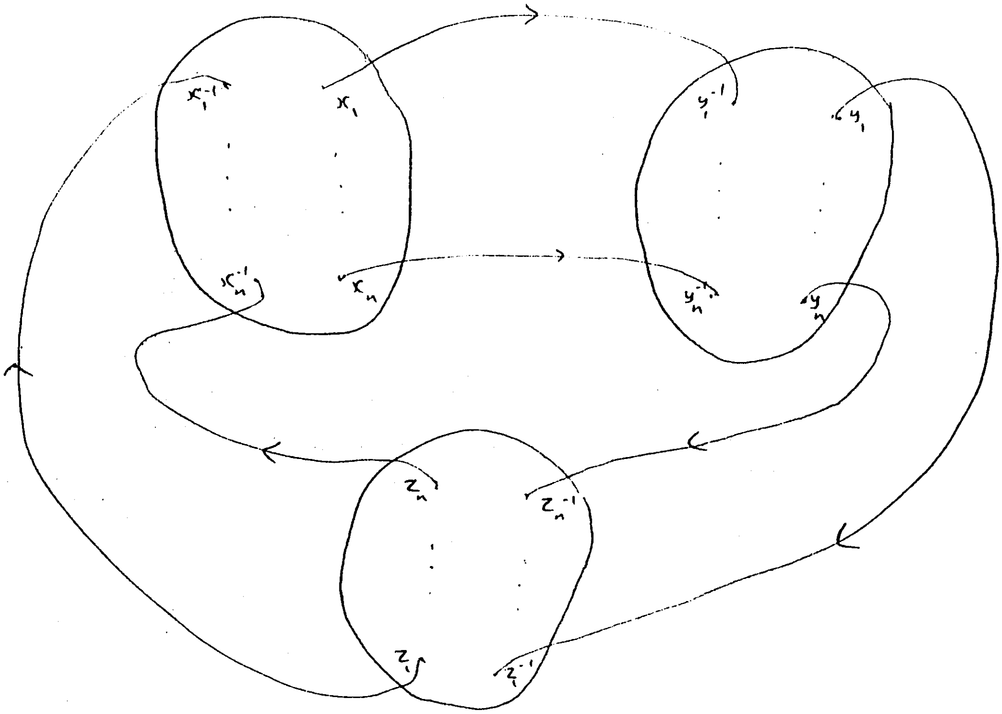
$$\mathcal{B} = \langle x_i, y_i, z_i (1 \leq i \leq n); p, r, s, z_i y_i x_i \ (1 \leq i \leq n) \rangle$$

is a positive T(6)-presentation.

Proof. Since $\mathcal{P}$ is positive then each edge of $\mathcal{P}^{st}$ has one endpoint in $\{x_1, x_2, \ldots, x_n\}$ and the other end point in $\{x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}\}$



Similaraly for $\mathcal{R}^{st}$, $\mathcal{S}^{st}$. Now $\mathcal{B}^{st}$ is obtained by taking $\mathcal{P}^{st}$, $\mathcal{R}^{st}$ $\mathcal{S}^{st}$ and adding new edges $x_i \longrightarrow y_i^{-1}$, $y_i \longrightarrow z_i^{-1}$, $z_i \longrightarrow x_i^{-1}$ $(1 \leq i \leq n)$ and their inverses. We have



From this it is obvious that $\mathcal{B}^{st}$ has no reduced closed path of length less than 6.

Example 3.2.1. Consider the T(6)-presentations $\langle x; x^k \rangle$, $\langle y; y^\ell \rangle$, $\langle z; z^m \rangle$ $(k, \ell, m \geq 3)$. Then we obtain the positive T(6)-presentation $\langle x, y, z; x^k, y^\ell, z^m, zyx \rangle$ which is a presentation of triangle group.

Remarks:

(1) We note that the presentation $\langle x_1, \ldots, x_n; x_1^2, \ldots, x_n^2 \rangle$ $(n \geq 3)$ is positive T(6)-presentation.

(2) We note also that in Theorem 3.2.1 we have combined three positive T(6)-presentations together. Obviously we can combine more than three positive presentations together in an analogous way.

3.3. Example of hyperbolic complexes.

A *weight function* on a 1-complex is a mapping m from the edge set into $\mathbb{R}$ such that $m(x^{-1})=m(x)$ for all edges x. If $x_1x_2...x_n$ is a path in the 1-complex (where the x_i are edges) then the *weight* $m(x_1x_2...x_n)$ of the path is defined to be $\sum_{i=1}^{n} m(x_i)$.

The situation we will be interested in is when we have a 2-complex $\mathcal{K}$ and a weight function m on $\mathcal{K}^{st}$. We will use the notation $(\mathcal{K}, m)$ to denote this situation.

Let K be 2-complex and let m be a weight function on $\mathcal{K}^{st}$. Associated with m we have another weight function m^* on $\mathcal{K}^{st}$ defined as follows. Let $\gamma \in R(K)$ and write $\gamma = e_1e_2...e_n$ where the e_i are edges of $\mathcal{K}$. Then

$m^*(\gamma)=m(e_1...e_n)+m(e_2...e_1)+...+m(e_n...e_{n-1})$.

We say that $(\mathcal{K}, m)$ is *hyperbolic* if

(HI) $m^*(\gamma) < L(\gamma)-2$ for all $\gamma \in R(\mathcal{K})$

(HII) The weight of any non-empty cyclically reduced closed path in $\mathcal{K}^{st}$ is at least 2.

(HIII) There exist a non-negative number N such that every reduced path in $\mathcal{K}^{st}$ has weight greater than or equal to −N.

We will say that a complex K is *hyperbolic* if there is an m such that (K, m) is hyperbolic.

In [10] it was mentioned that the surface presentations

$$< x_1, y_1, x_2, y_2, \ldots, x_n, y_n \; ; \; \prod_{i=1}^{n} [x_i, y_i] > \quad (n \geq 2)$$

$$< x_1, x_2, \ldots, x_n \; ; \; \prod_{i=1}^{n} x_i^2 > \quad (n \geq 3)$$

and the presentations of triangle groups

$$< a, b, c \; ; \; a^p, \; b^q, \; c^r, \; abc > \text{ where } 1/p+1/q+1/r < 1$$

are hyperbolic. Here we consider arbitary F-presentations [6, pp. 126–133].

An *orientable F-presentation* is a presentation

$$(*) \mathcal{O} = < a_1, b_1, a_2, b_2, \ldots, a_g, b_g, e_1, \ldots, e_r \; :$$

$$a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} e_1 e_2 \ldots e_r, \; e_1^{n_1}, \ldots, e_r^{n_r} >$$

where $g, r \geq 0$, and $n_1, n_2, \ldots, n_r$ are integers ≥ 2.

A *non-orientable F-presentation* is a presentation

$$\mathcal{N} = < a_1, \ldots, a_g, e_1, \ldots, e_r ; a_1^2 a_2^2 \ldots a_g^2 e_1 \ldots e_r, e_1^{n_1}, e_2^{n_2}, \ldots, e_r^{n_r} >$$

where $g, r \geq 0$ and, $n_1, n_2, \ldots, n_r$ are integers ≥ 2 .

There is a standard prameter associated with these presentations

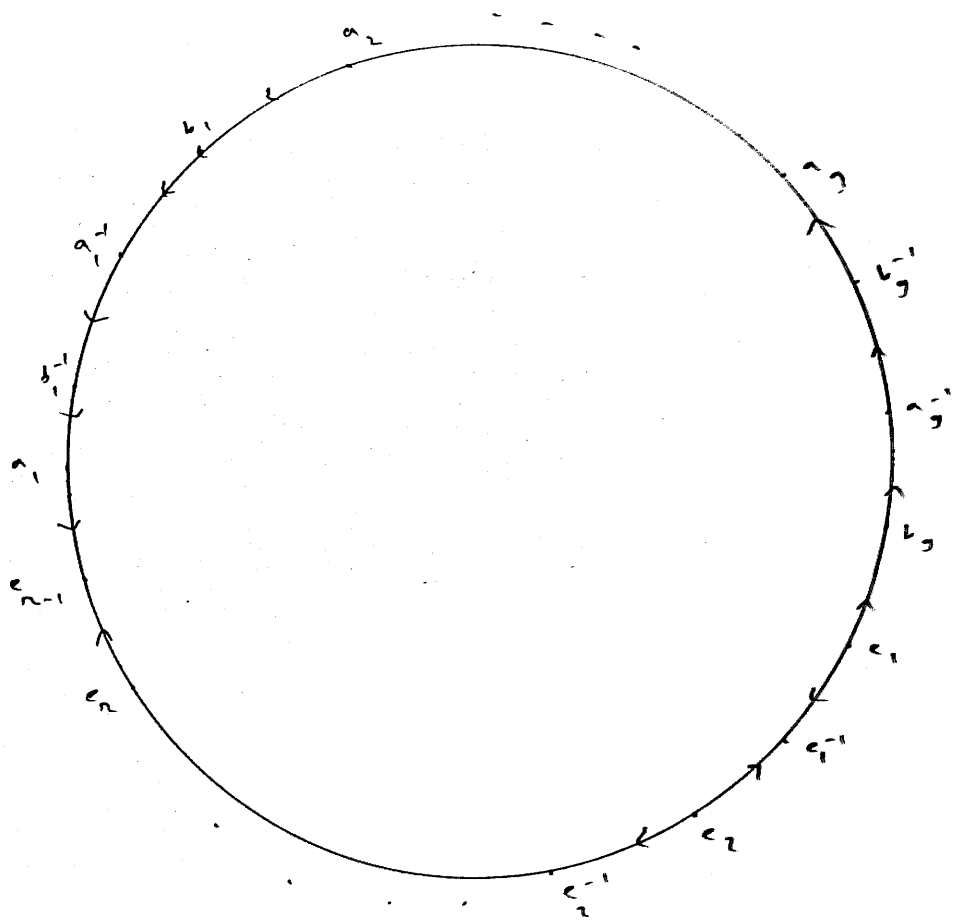
$$\mu(\mathcal{O}) = 2g-2+ \sum_{i=1}^{r} (1-1/n_i)$$

$$\mu(\mathcal{N}) = g-2+\sum_{i=1}^{n}(1-1/n_i)$$

Theorem 3.3.1. *An F-presentation $\mathcal{P}$ is hyperbolic if and only if $\mu(\mathcal{P})>0$.*

Proof.

We prove this in the case when $\mathcal{P}$ is orientable. The other case is similar. Then suppose $\mathcal{P}=\mathcal{O}$, as in (*). Note that $\mathcal{P}^{st}$ looks as follows.



Let m be a weight function on $\mathcal{P}^{st}$. Then in order for $(\mathcal{P}, m)$ to be hyperbolic we must have by (HI) that

$$n_1\, m(e_1^{n_1}) < n_1-2$$

$$n_2\, m(e_2^{n_2}) < n_2-2$$

$$\vdots$$

$$n_r\, m(e_r^{n_r}) < n_r-2$$

and

$$m(a_1b_1a_1^{-1}b_1^{-1}\ldots e_1\ldots e_r)+m(b_1\ldots a_1)+\ldots+m(e_r\ldots e_{r-1}) < 4g+r-2.$$

Thus

$$m(e_1^{n_1})+m(e_2^{n_2})+\ldots+m(e_r^{n_r})+m(a_1b_1\ldots e_r)+\ldots+m(e_r\ldots e_{r-1})$$

$$< \sum_{i=1}^{r}(1-2/n_i)+4g+r-2$$

and by (HII) we have

$$m(e_1^{n_1})+m(e_2^{n_2})+\ldots+m(e_r^{n_r})+m(a_1b_1\ldots e_r)+\ldots+m(e_r\ldots e_{r-1}) > 2.$$

So we have that $\mathcal{P}$ is hyperbolic only if

$$\sum_{i=1}^{r}(1-2/n_i)+4g+r-2 > 2$$

then $$2\left(\sum_{i=1}^{r}(1-1/n_i)\right)+4g-2 > 2$$

which implies that $\sum_{i=1}^{r}(1-1/n_i)+2g-2 > 0$ as required.

For "if" put $m(e_1^{n_1})=1-2/n_1-\epsilon$

$$m(e_2^{n_2})=1-2/n_2-\epsilon$$

$$\vdots$$

$$m(e_r^{n_r})=1-2/n_r-\epsilon$$

$$m(a_1b_1a_1^{-1}b_1^{-1}\ldots e_r)=m(b_1\ldots a_1)=\ldots=m(e_r\ldots e_{r-1})=\frac{4g+r-2}{4g+r}-\epsilon$$

where $\epsilon>0$. Then (HI) is satisfied.

To satisfy (HII) we must have

$$(1-2/n_1-\epsilon)+(1-2/n_2-\epsilon)+\ldots+(1-2/n_r-\epsilon)+4g+r-2-\epsilon(4g+r) > 2$$

ie $\sum_{i=1}^{r}(1-2/n_i)-\epsilon r+4g+r-2-\epsilon(4g+r) > 2$

ie $2(\sum_{i=1}^{r}(1-1/n_i))+4g-2-2\epsilon(4g+r) > 2$

ie $\sum_{i=1}^{r}(1-1/n_i)+2g-2 > \epsilon(2g+r)$

So a choice of ϵ is possible since $\dfrac{\sum_{i=1}(1-1/n_i)+2g-2}{2g+r}$ is positive.

To satisfy (HIII). Consider all simple non-closed paths in $\wp^{st}$.

Let $-N$ denote the minimum value of the weights of all simple non-closed paths. Since empty paths are simple non-closed paths then $-N<0$, which implies that $N>o$. If π is any reduced path in $\wp^{st}$ consisting of a closed path π_1 going around a circle ℓ times and simple non-closed path π_2, that is, $\pi=\pi_1\pi_2$. Then

$$m(\pi)=m(\pi_1)+m(\pi_2)$$
$$>2\ell-N > -N.$$

Remark.

Note in the above that $\mathcal{P}^{st}$ is a circle with $4g+2r$ geomtric edges. Thus if $4g+2r \geq 7$ and if each defining relator has length ≥ 3 then $\mathcal{P}$ is a T(7)-presentation, and so the fact that it is hyperbolic also follows from [10, Theorem 4, Corollary].

CHAPTER 4

WORD AND CONJUGACY PROBLEMS FOR T(6)-COMPLEXES

THROUGOUT THIS CHAPTER WE WILL ASSUME THAT ALL 2-COMPLEXES ARE FINITE.

4.1. The dependence problems and diagrams over 2-complexes.

4.1.1. The dependence problems.

Let $(\omega_0, \omega_1, \ldots, \omega_k)$ be a sequence of closed paths in $\mathcal{K}$. We say that ω_0 is *dependent* on $(\omega_1, \ldots, \omega_k)$ in $\mathcal{K}$ written $(\omega_1, \ldots, \omega_k) \vdash_{\mathcal{K}} \omega_0$ (or simply $(\omega_1, \ldots, \omega_k) \vdash \omega_0$) if there is a subset $\{i_1, \ldots, i_\ell\}$ of $\{1, 2, \ldots, k\}$ and paths $\eta_1, \ldots, \eta_\ell$ such that

$$\omega_0(\eta_1 \omega_{i_1} \eta_1^{-1}) \ldots (\eta_\ell \omega_{i_\ell} \eta_\ell^{-1})$$

is contractible in $\mathcal{K}$.

If n is a positive integer or ∞, then the *dependence problem* DP(n) asks for an algorithm to decide for any sequence $(\omega_0, \omega_1, \ldots, \omega_k)$ $(o \leq k < n)$ whether or not $(\omega_1, \omega_2, \ldots, \omega_k) \vdash_{\mathcal{K}} \omega_0$. The problems DP(1), DP(2) are usually called the ***word problem*** and ***conjugacy problem*** for $\mathcal{K}$ respectively. Note that if DP(n) is solvable then DP(m) is solvable for all $1 \leq m < n$.

Lemma 4.1.1. *Let ϕ: $\mathcal{K} \to \mathcal{L}$ be an equivalence of 2-complexes, and let $\alpha_0, \alpha_1, \ldots, \alpha_m$ be closed paths in $\mathcal{K}$. Then there are paths $\gamma_1, \ldots, \gamma_m$ in $\mathcal{K}$ such that $\alpha_0(\gamma_1\alpha_1\gamma_1^{-1})\ldots(\gamma_m\alpha_m\gamma_m^{-1})$ is contractible in $\mathcal{K}$ if and only if there are paths $\delta_1, \ldots, \delta_m$ in $\mathcal{L}$ such that $\phi(\alpha_0)(\delta_1\phi(\alpha_1)\delta_1^{-1})\ldots(\delta_m\phi(\alpha_m)\delta_m^{-1})$ is contractible in $\mathcal{L}$.*

Proof.

For "if", let $\phi(\alpha_0)(\delta_1\phi(\alpha_1)\delta_1^{-1})\ldots(\delta_m\phi(\alpha_m)\delta_m^{-1})$ be a contractible path in $\mathcal{L}$. Since ϕ is an equivalence there is a map θ: $\mathcal{L} \to \mathcal{K}$ such that $\theta\phi(\alpha) \sim \alpha$ for all paths in $\mathcal{K}$. We have

$$1 \underset{\mathcal{K}}{\sim} \theta(\phi(\alpha_0)(\delta_1\phi(\alpha_1)\delta_1^{-1})\ldots(\delta_m\phi(\alpha_m)\delta_m-1))$$

$$\underset{\mathcal{K}}{\sim} (\alpha_0(\theta(\delta_1)\alpha_1\theta(\delta_1^{-1}))\ldots(\theta(\delta_m)\alpha_m\theta(\delta_m^{-1}))$$

$$= (\alpha_0(\gamma_1\alpha_1\gamma_1^{-1})\ldots(\gamma_m\alpha_m\gamma_m^{-1}))$$

where $\theta(\delta_j) = \gamma_j$, $j=1,\ldots,m$.

For "only if" it follows directly since ϕ is a mapping of 2-complexes.

There is a connection between the dependence problems and diagrams on sphere which we now explain.

4.1.2 Diagrams.

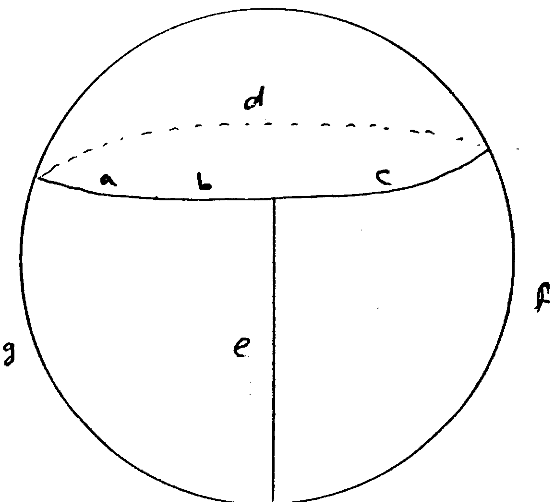
Let S be a finitely tesselated sphere. We associate with S a 2-complex $\mathcal{C}_S$ as follows.

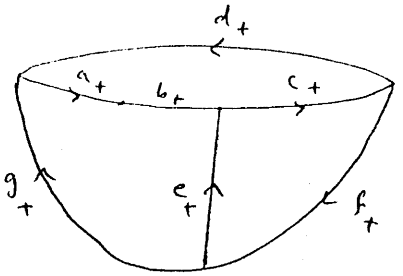
The 1-skeleton $\mathcal{C}_S^{(1)}$ of $\mathcal{C}_S$ has vertices the vertices of S. Let e be an edge of S with endpoints u, v and choose an orienation of e, say the orientation running from u to v. Then e gives rise to an inverse pair e_+, e_- of edges of $\mathcal{C}_S^{(1)}$ with $\iota(e_+)=u$, $\tau(e_+)=v$. For the defining paths of $\mathcal{C}_S$, let D be a region of S and choose a fixed but arbitary vertex O on ∂D. Read around ∂D in the clockwise direction starting at O. If we traverse an edge e in the direction of the orientation then we write down the symbol e_+, and if we traverse e against its orientation then we write down e_-. In this way we obtain a closed path β_D in $\mathcal{C}_S^{(1)}$. The defining paths of $\mathcal{C}_S$ are then all paths β_D as D ranges over the regions of S.

Note that $\mathcal{C}_S$ is not uniquely defined since it depends on the choice of orientation of the edges of S and on which vertices we choose to start reading around the boundaries of regions.

Example 4.1.1.

S:



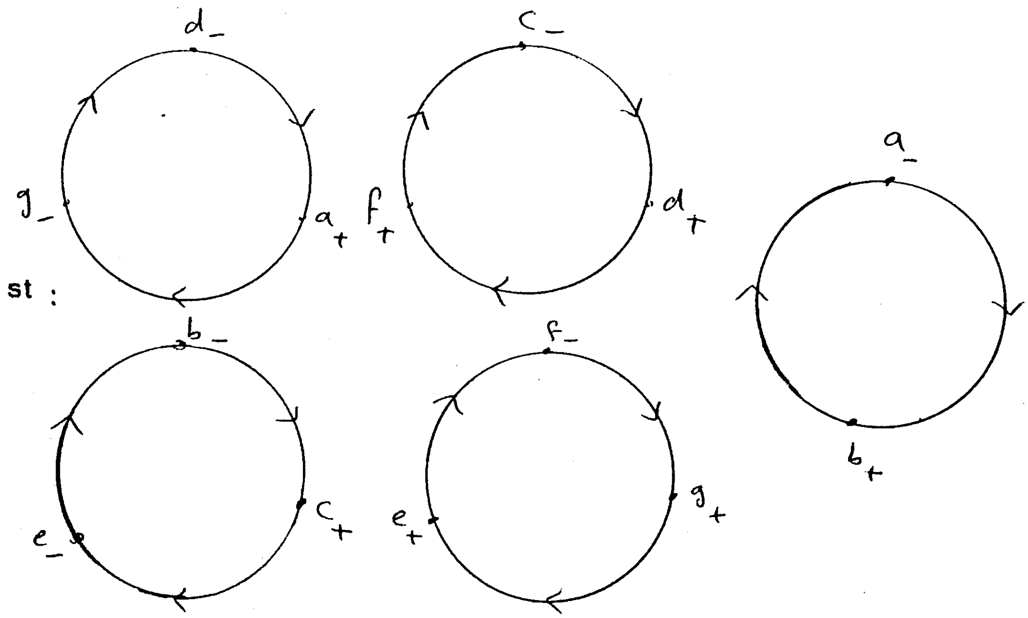


1-skeleton

$d_- c_- b_- a_-$
$g_+ a_+ b_+ e_-$
$e_+ c_+ f_+$
$f_- d_+ g_-$

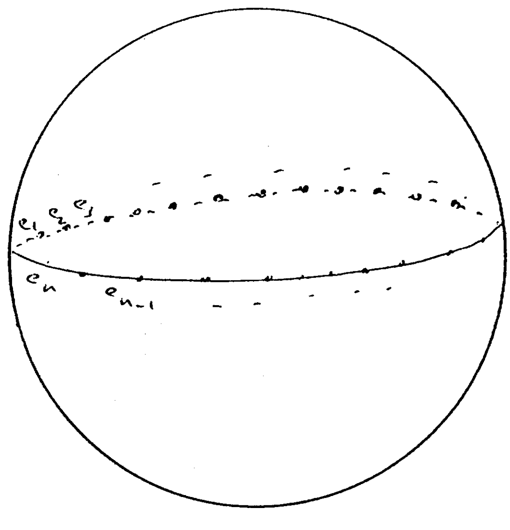
defining paths

$\mathcal{C}_S^{st}$:

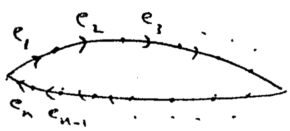


Example 4. 1. 2. ("Exceptional" tesselations.)

S:



$\mathcal{C}_S^{(1)}$:



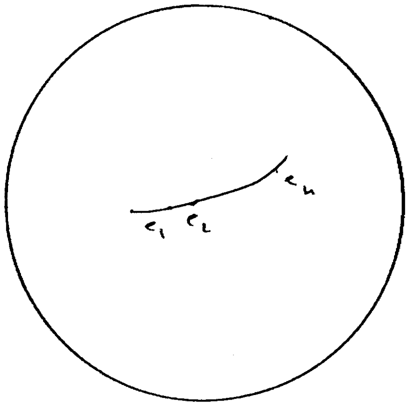
$\beta D_1 = e1_+ e2_+ \cdots en_+ .$

$\beta D_2 = en_- en{-}1_- \cdots e1_- .$

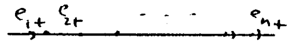
$\mathcal{C}_S^{st}$:

$\mathcal{C}_S^{ST}$:

T:



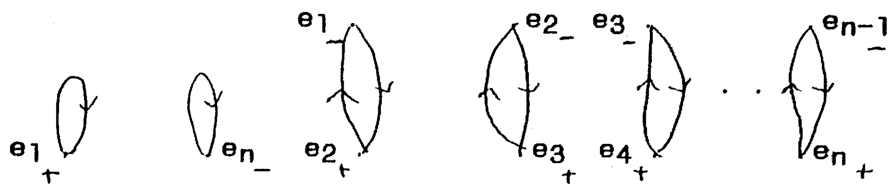
$\mathcal{C}_T(1)$:



$\beta_D = e1_+ e2_+ \dots en_+ en_- en\text{-}1_- \dots e1_-$.

Note that $\mathcal{C}_S^{st}$ is not defined. However $\mathcal{C}_S^{ST}$ looks as follow.

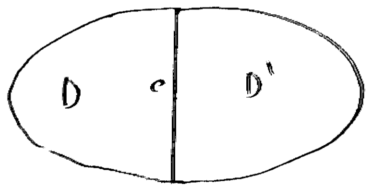
$\mathcal{C}_S^{ST}$:



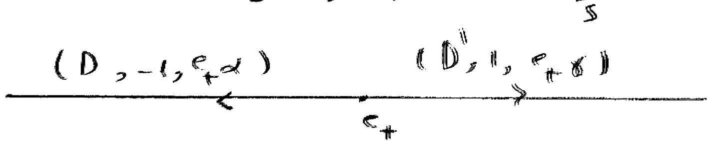
A *diagram over* $\mathcal{K}$ is a triple (S, e, φ) where S is a finitely tesselated sphere, φ a strong rigid mapping from $\overset{\circ}{\mathcal{C}}_S = \langle \overset{(1)}{\mathcal{C}}_S : \beta_D (D \in e) \rangle$ to $\mathcal{K}$. The diagram is said to be *reduced* if $\phi^{ST} : \check{\mathcal{C}}_S^{ST} \to \mathcal{K}^{st}$ is locally injective.

We sometimes describe the diagram informally. For this we will combine S and $\mathcal{C}_S$ into a single picture and write against the oriented edges of S the edges of $\mathcal{K}$ by which they are labelled.

The condition that ϕ be reduced translates into the assertion that wherever we have any two regions D, D′ not in e, which have an edge in common



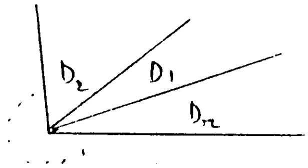
the label on the anticlockwise boundary cycle of D starting with e is not the label on the clockwise boundary cycle of D′ starting with e. For reading around D in the anticlockwise direction starting at v will give a cyclic permutation of $\beta_D{}^{-1}$ say $e_+\alpha$, reading around D′ in the clockwise direction will give a cyclic permutation of $\beta_{D'}$ say $e_+\gamma$. Then in $\mathcal{E}_S^{ST}$ we have



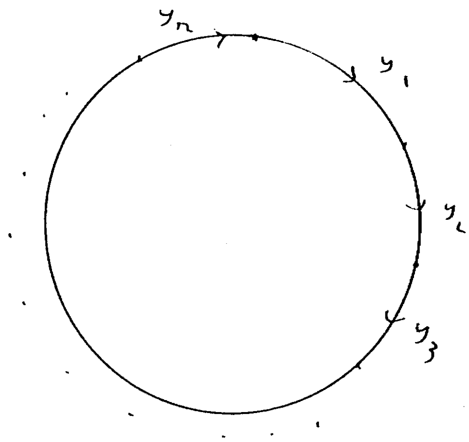
Now suppose that $\phi(e_+\alpha)=\phi(e_+\gamma)$ and so by definition of ϕ^{ST}, we have $\phi^{ST}(D,-1,e_+\alpha)=\phi^{ST}(D',1,e_+\gamma)$ which a contradiction since $(D,-1,e_+\alpha)\neq(D',1,e_+\gamma)$.

If S is any tesselated sphere then $\mathcal{E}_S^{ST}$ consists of a collection of circles $\mathcal{E}_S^{ST}(v)$ for each vertex v. For let v be a vertex in a tesselation, looking at the regions incident with v say

$D_1, D_2, \ldots, D_r$. (which may not all be distinct)

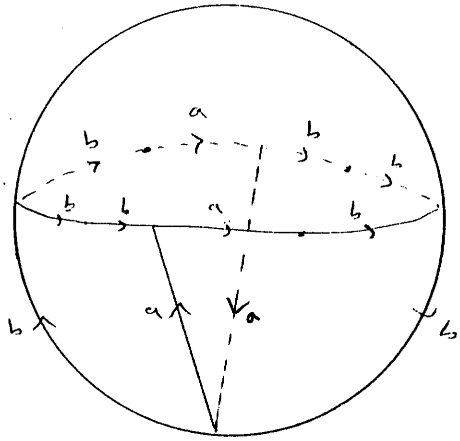


Reading around ∂D_i $(i=1,2,\ldots,r)$ in the clockwise direction starting at v will give a cyclic permutation γ_i of β_D . Then the γ_i and their inverses will gives the edges $y_i^{\pm 1}$ of $\mathcal{G}_S^{ST}(v)$ where $y_i=(D_i, 1, \gamma_i)$ $(i=1,2,\ldots,r)$. We have $\mathcal{G}_S^{ST}(v)$ will be a circle since $\tau^{ST}(y_i)=\iota^{ST}(y_{i+1})$ $(i=1,2,\ldots,r$, subscripts mod $r)$.



We call the regions in Θ the *distinguished regions*. A vertex on the boundary of at least one distinguished region will be called a *distinguished vertex*.

Example 4.1.3. Let $\mathcal{P}=\langle a, b; a^2b^2, ab^{-3}\rangle$. Then a diagram over $\mathcal{P}$ is illustrated below, where the northern hemisphere is the single distinguished region.



The following result gives a relation between the dependence problems and diagrams.

Theorem 4.1.1. *$(\omega_1, \omega_2, \ldots, \omega_k) \vdash_K \omega_0$ if and only if there is a reduced diagram $(S, \{D_1, \ldots, D_\ell\}, \phi)$ over K with $\phi(\partial D_0) = \omega_0$ and $(\phi(\beta_{D_1}), \ldots, \phi(\beta_{D_\ell}))$ is a subsequence of $(\omega_1, \ldots, \omega_k)$.*

Proof.

For "if". We have that

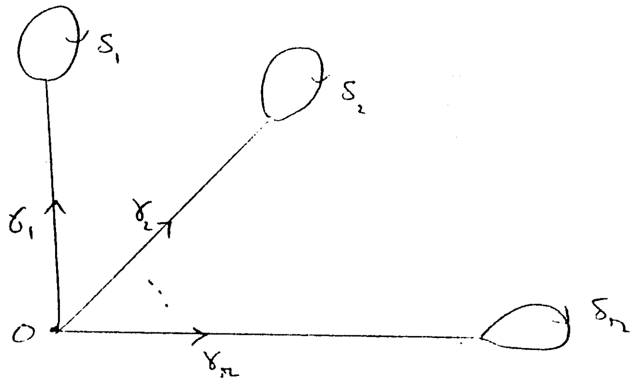
$$[\omega_0^{-1}] = [\eta_1 \omega_{i_1} \eta_1^{-1}] \ldots [\eta_\ell \omega_{i_\ell} \eta_\ell^{-1}]$$

(by [6, Lemma V.1.2]), that is, $(\omega_1, \ldots, \omega_k) \vdash_k \omega_0$.

For "only if". Suppose $(\omega_1, \ldots, \omega_k) \vdash_K \omega_0$. By cancelling inverses pairs, obtain cyclically reduced paths $\bar{\omega}_0, \ldots, \bar{\omega}_k$ from $\omega_0, \ldots, \omega_k$. Then

$$\bar{\omega}_0^{-1} \sim \prod_{i=1}^{n} \gamma_i \delta_i \gamma_i^{-1}$$

where for some subset $\{j_1, \ldots, j_\ell\}$ of $\{1, \ldots, k\}$, $(\delta_{j_1}, \ldots, \delta_{j_\ell})$ is a subsequence of $(\bar{\omega}_1, \ldots, \bar{\omega}_k)$ and for $i \neq j_1, \ldots, j_\ell$, $\delta_i \in R(K)$. Among all products of the above form which are freely equal to $\bar{\omega}_0^{-1}$, choose one with r minimal. Form a diagram by taking each factor $\gamma_1, \ldots, \gamma_r$ and arrange these in order around common base point O.



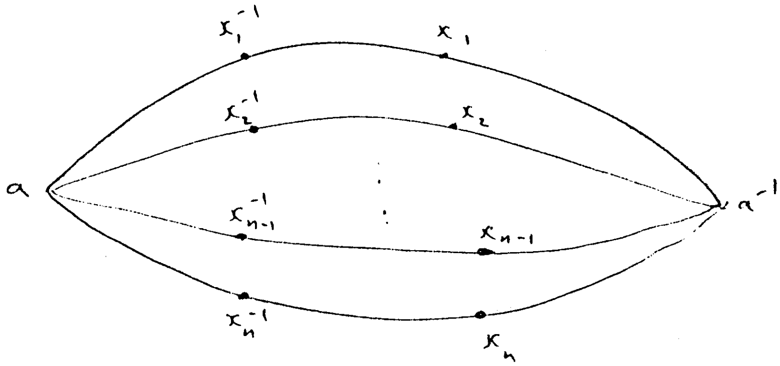
Now we identify the succesive edges whose labels are inverses, we obtain a diagram $(S', \{D_0', D_1', \ldots, D_\ell'\}, \Phi')$ with $(\Phi'(\beta_{D_0'}), \Phi'(\beta_{D_1'}), \ldots, \Phi'(\beta_{D_\ell'}))$ a subsequence of $(\bar{\omega}_0, \bar{\omega}_1, \ldots, \bar{\omega}_k)$ containing $\bar{\omega}_0$. This diagram is reduced since the product is minimal (see [6, Lemma V. 2. 1]). Now by inserting labelled trees to the boundaries $D_0', D_1', \ldots, D_\ell'$ we then obtain a diagram $(S, \{D_0, D_1, \ldots, D_\ell\}, \Phi)$ with $(\Phi(\beta_{D_0}), \ldots, \Phi(\beta_{D_\ell}))$ a subsequence of $(\omega_0, \omega_1, \ldots, \omega_k)$ containing ω_0.

In [10] some result are obtained about the dependence problems for *hyperbolic complexes*.

In [10] it was shown that DP(∞) is solvable for hyperbolic complexes ([10], Theorm 3). If $\mathcal{K}$ satisfies the small cancellation condition T(7) then $\mathcal{K}$ is hyperbolic ([10], Theorem 4, Corollary).

On the other hand there are T(6) complexes which are not hyperbolic. For example

(1) Let $\mathcal{P}=\langle a, x_1, x_2, \ldots, x_n; ax_1^2, ax_2^2, \ldots, ax_n^2\rangle$. We will show that $\mathcal{P}$ is not hyperbolic. Note that $\mathcal{P}^{st}$ looks as follows.



Suppose by way of contradiction that we have a weight function m on $\mathcal{P}^{st}$ such that $(\mathcal{P}, m)$ is hyperbolic. Then by (HI) of the definition we have that

$$m^*(ax_1^2)=m(ax_1^2)+m(x_1ax_1)+m(x_1^2a) < 1$$

$$\vdots$$

$$m^*(ax_n^2)=m(ax_n^2)+m(x_nax_n)+m(x_n^2a) < 1$$

we have

$$\sum_{i=1}^{n} m^*(ax_i^2) < n \qquad (1)$$

and by (HII) we have

<u>case 1</u> If n is even

$$m^*(ax_1^2)+m^*(ax_2^2) \geq 2$$

$$\vdots$$

$$m^*(ax_{n-1}^2)+m^*(ax_n^2) \geq 2$$

adding we have $\sum_{i=1}^{n} m^*(ax_i^2) \geq n$ (2)

from (1), (2) we have a contradiction to $\mathcal{G}$ is hyperbolic.

<u>case 2</u> If n is odd, we have

$$m^*(ax_1^2)+m^*(ax_2^2) \geq 2$$

$$\vdots$$

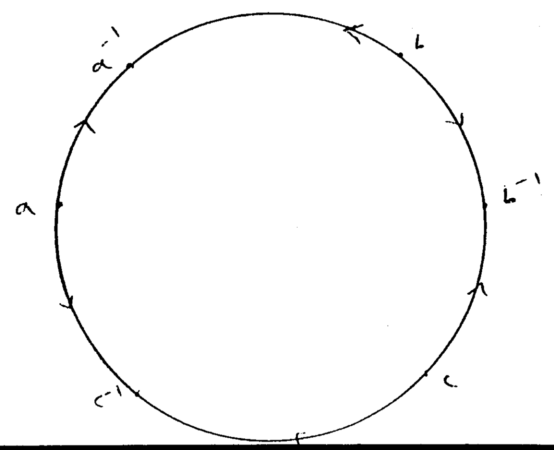
$$m^*(ax_n^2)+m^*(ax_1^2) \geq 2$$

adding together we have

$$\left(\sum_{i=1}^{n} m^*(ax_i^2)\right)+m^*(ax_1^2) \geq n+1$$

then $n + m^*(ax_1^2) > n+1$ and $m^*(ax_1^2) > 1$, which is a contradiction to (HI). Then $\mathcal{G}$ is not hyperbolic.

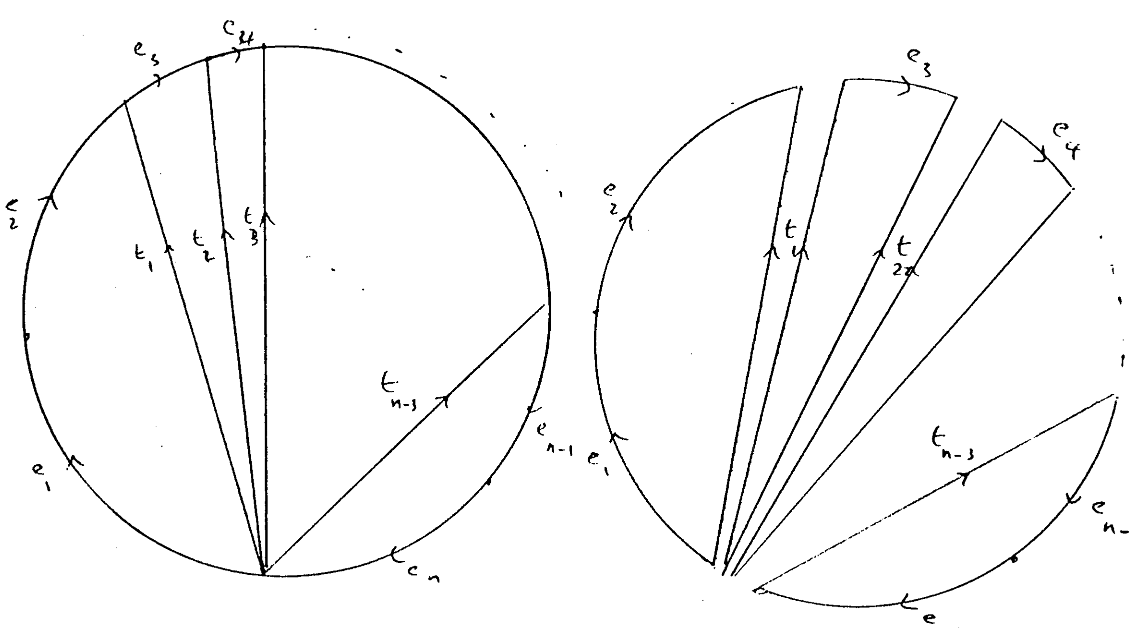
(2) Let $\mathcal{G}=\langle a,b,c; a^3, b^3, c^3, abc\rangle$. Then $\mathcal{G}$ is not hyperbolic since $\mu(\mathcal{G})=0$ (by Theorem 3.3.1). Note that $\mathcal{G}^{st}$ looks as follows.



4.2. Triangulated complexes.

A *triangulated 2-complex* is a 2-complex in which each defining path has length 3. Given a 2-complex in which each defining path has length at least 3, we can triangulate it as follows.

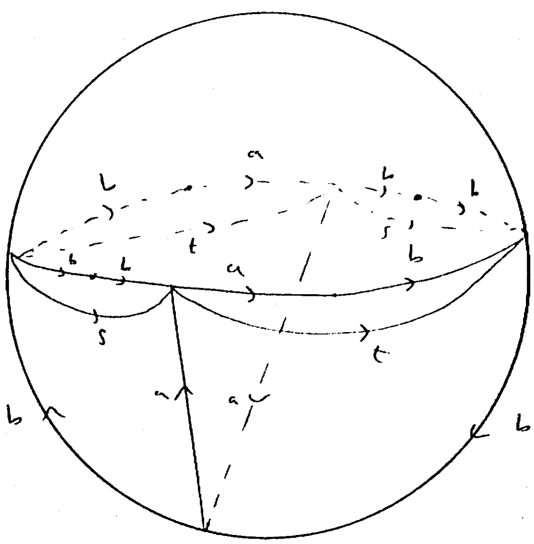
Let $\rho=e_1 \ldots e_n$ be a defining path of K of length greater than 3. Adjoin new edges $t_1, t_1^{-1}, \ldots, t_{n-3}, t_{n-3}^{-1}$ to the 1-skeleton of K, where $\iota(t_1)=\iota(t_2)=\ldots=\iota(t_{n-3})$, $\tau(t_1)=\tau(e_2)$, $\tau(t_2)=\tau(e_3)$, $\ldots$, $\tau(t_{n-3})=\tau(e_{n-2})$, and replace ρ by the new paths $e_1e_2t_1^{-1}$, $t_1e_3t_2^{-1}, \ldots, t_{n-3}e_{n-1}e_n$.



Carrying out the above procedure for each defining path of K of length greater than 3, we obtain a triangulated 2-complex K_Δ which we call the *triangulation* of K. Since K_Δ is obtained from K by subdividing of defining paths, the inclusion of $K^{(1)}$ into $K_\Delta^{(1)}$ induced an equivalence between K and K_Δ (see 1.1B.5). Thus if $(\omega_0, \omega_1, \ldots, \omega_n)$ is a sequence of closed paths in K then $(\omega_1, \ldots, \omega_k) \vdash_K \omega_0$ if and only if $(\omega_1, \ldots, \omega_k) \vdash_{K_\Delta} \omega_0$ (see Lemma 4.1.1).

If $\mathcal{D}$ is a diagram over K then we can triangulate the non-distinguished regions of $\mathcal{D}$ to obtain a diagram $\mathcal{D}_\Delta$ over K_Δ.

Example 4.2.1. Consider the presentation $\mathcal{P}=\langle a, b; a^2b^2, ab^{-3}\rangle$ then $\mathcal{P}_\Delta=\langle a, b, s, t; b^2s^{-1}, sa^{-1}b, abt^{-1}, tba\rangle$. If $\mathcal{D}$ is the diagram over $\mathcal{P}$ in Example 4.1.3 then $\mathcal{D}_\Delta$ is the diagram



Note that if each non-distinguished vertex of $\mathcal{D}$ has valence at least 6, then when we triangulate $\mathcal{D}$ to get $\mathcal{D}_\Delta$, we have that each non-distinguished vertex of $\mathcal{D}_{\Delta}$ has valence at least 6.

4.3 Tesselation of sphere.

In this section we consider a pair (S,Θ) where S is a non-trivial finitely tesselated sphere (A tesselation is trivial if it has one vertex.) and Θ a subset of the set of regions. We call the regions in Θ distinguished regions. We associate with S a 2-complex $\mathcal{E}_S$ (see 4.1.2), and we assume that $L(\beta_D)=3$ for all $D \notin \Theta$.

If S is any non-trivial tesselation then $\mathcal{E}_S^{ST}$ consists of a collection of circles $\mathcal{E}_S^{ST}(v)$ for each vertex v (see 4.1.2). An edge $(D,1,\gamma)$ of $\mathcal{E}_S^{ST}$ is said to be *distinguished* if $D \in \Theta$, and is said to be *non-distinguished* if $D \notin \Theta$. (Note that the term "distinguished" or "non-distinguished" is only used for "positive" edges, that is, edges whose second coordinate is 1).

We say that a component $\mathcal{E}_S^{ST}(v)$ is of *type (k,n)* if it has n positive edges of which k are distinguished.

We say that a vertex v is of *type (k,n)* if $\mathcal{E}_S^{ST}(v)$ is of type (k,n).

We say that the component $\mathcal{E}_S^{ST}(v)$ is *distinguished* if it has at least one distinguished edge, otherwise it is *non-distinguished.*

Let V^o denote the number of non-distinguished components of $\mathcal{C}_S^{ST}$. For $n>0$ and $1 \le k \le n$, let V^{kn} denote the number of distinguished components of $\mathcal{C}_S^{ST}$ of type (k,n). (Note that V^{kn} is not defined for $k=0$). So the total number of distinguished positive edges is $\sum_{k,n} kV^{kn}$ and the total number of all positive edges in all distinguished components is $\sum_{k,n} nV^{kn}$. The number of distinguished components is $\sum V^{kn}$.

Let V, E, F denote the number of vertices, edges, regions of the tesselation respectively. We suppose that the average of positive edges in non-distinguished components is $6+\alpha$. We have

$$2E = \sum_{D \notin \Theta} L(\beta_D) + \sum_{D \in \Theta} L(\beta_D)$$

$$= 3(\ F-|\Theta|) + \sum_{k,n} kV^{kn}$$

$$\frac{2E}{3} = (F-|\Theta|) + \sum_{k,n} \frac{kV^{kn}}{3} \qquad (1)$$

also we have

$$2E = (6+\alpha)V^o + \sum_{k,n} nV^{kn}$$

$$= \alpha V^o + 6(V - \sum_{k,n} V^{kn}) + \sum_{k,n} nV^{kn}$$

$$= \alpha V^o + 6V + \sum_{k,n} (n-6)V^{kn}$$

so $\frac{2E}{6} = \frac{\alpha}{6} V^o + V + \sum_{k,n} \frac{(n-6)}{6} V^{kn}$ (2)

adding (1), (2) we have

$$E = (F-|\Theta|) + V + \frac{\alpha}{6} V^o + \sum_{k,n} \frac{(n+2k-6)}{6} V^{kn}$$

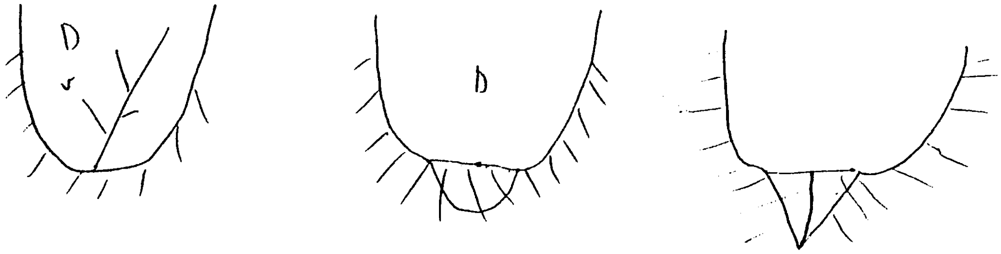
by using V−E+F=2, we have

$$(*) \qquad (2-|\Theta|) + \frac{\alpha}{6} V^o + \sum_{k,n} \frac{(n+2k-6)}{6} V^{kn} = o$$

We note from (*) that first term $(2-|\Theta|)$ is non-negative for $|\Theta|<2$ which allows to use (*) to give solution to the word and conjugancy problems for T(6)-complexes. When $|\Theta|>2$ we not able to extend our argument for solving DP(m), m>2.

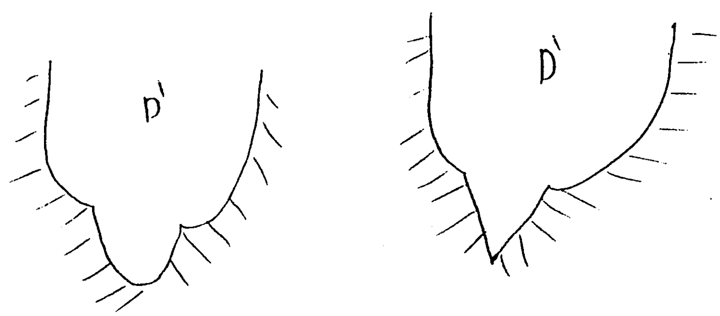
4.3.1 The case of one distinguished region.

Suppose $|\Theta|=1$, that is there is a single distinguished region D and assume that the number of positive edges in each non- distin guished component is at least 6. In (*) we have the first term is positive and second term is non-negative, so we have $(n+2k-6)V^{kn}<o$ for some k, n. Thus $V^{kn}>o$ (since V^{kn} is non-negative) and $(n+2k-6) < o$, so (k, n) must be either (1, 1), (1, 2), (1, 3). The cases are illustrated below.



(i) (ii) (iii)

Now in each case we modify the tesselation to obtain a new tesselation with a single distinguished region D′ obtained from D by eliminating the vertex v.



(i) (ii) (iii)

This new tesselation still has that each non-distinguished vertex has valence at least 6. We can therefore repeat the process for the new tesselation and so on, eventually we obtain the trivial tesselation with one vertex and a single distinguished region.

Lemma 4.3.1. *In a non-trivial tesselation with one distinguished region and in which each non-distinguished vertex has valence at least 6, there are at least two vertices of type (1,1), (1,2) or (1,3).*

Proof.

In (*) since $|\theta|=1$ we have that

$$1+\frac{\alpha}{6}V^{o}+\sum_{k,n}\frac{n+2k-6}{6}V^{kn}=0$$

Since $\frac{\alpha}{6}V^{o}\geq 0$ by hypothesis, so $\sum_{k,n}\frac{n+2k-6}{6}V^{kn}\leq -1$.

Now $\frac{n+2k-6}{6}V^{kn}$ is negative if and only if $V^{kn}>0$ (since V^{kn} is non-negative) and $\frac{(n+2k-6)}{6}<0$, so (k,n) must be either $(1,1)$, $(1,2)$, $(1,3)$. Now if there is only one vertex of any of the types $(1,1)$, $(1,2)$, $(1,3)$ then only one term of the sum will be negative. This term will be either $-\frac{1}{2}$, $-\frac{1}{3}$ or $-\frac{1}{6}$ according to (k,n) of type $(1,1)$, $(1,2)$ or $(1,3)$. Thus the sum will be $-\frac{1}{2}+m$, $-\frac{1}{3}+m$, $-\frac{1}{6}+m$ for some non-negative number m according to each case. But this contradicts the fact that the sum is ≤ -1.

Thus we have at least two vertices of type $(1,1)$, $(1,2)$ or $(1,3)$.

4.3.2 The case of two distinguished regions.

Suppose $|\Theta|=2$, that is we have a tesselation with two distinguished regions, and we assume that each non-distinguished vertex has valence at least 6. If there are any distinguished vertices of type (1,1), (1,2), or (1,3), then we eliminate them as in 4.3.1. We then obtain a tesselation with two regions D_0, D_1 which still has each non-distinguished vertex having valence at least 6. In (*) we have all terms are non-negative and so all terms are zeros. So we have that each non-distinguished vertex has valence exactly 6, and we have for each (k,n) ($1\leq k\leq n$) that

$$\frac{(n+2k-6)}{6}\, V^{kn}=o,$$

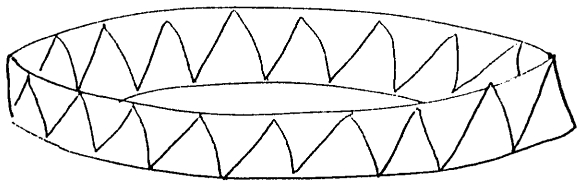
so $V^{kn} = o$ or $\frac{(n+2k-6)}{6}=o$.

If $\frac{(n+2k-6)}{6}=o$

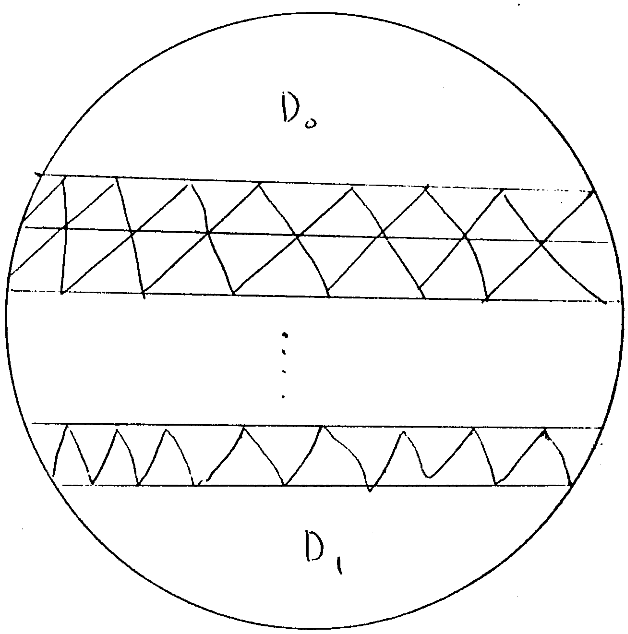
then the only possibilities for (k,n) are (1,4) or (2,2).

Now consider a tesselation with two distinguished regions D_0,D_1 and suppose that each non-distinguish vertex has valence at least 6 and there are no vertices of type (1,1), (1,2), (1,3). We have each vertex of the tesselation is of type (0,6), (2,2) or (1,4).

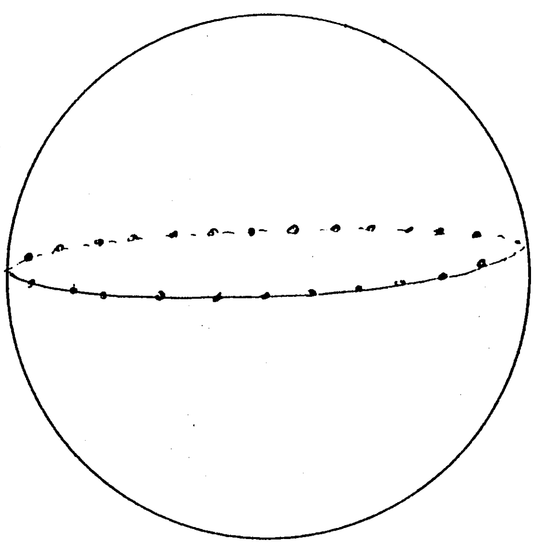
We show that the tesselation is made up of a sequence of layers



fitted together as follows



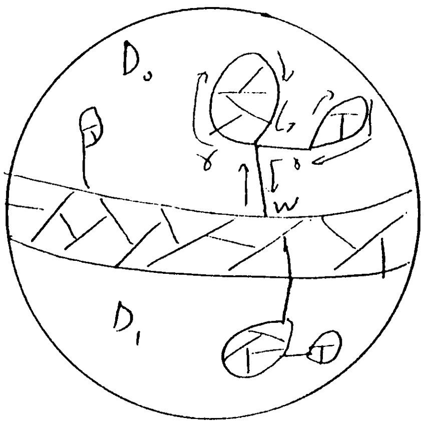
(The number of layers may be zero in which case we have simply



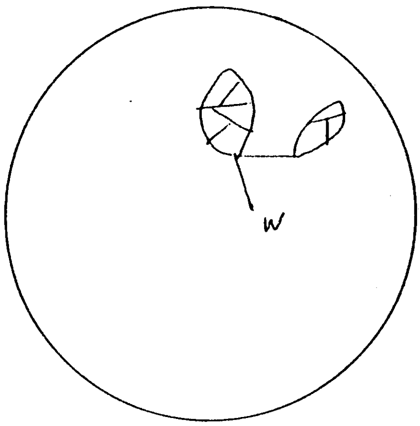
.)

The following until the end of this section given by S. Pride in our joint paper [3].

First we show that $\partial D_0, \partial D_1$ are simple closed paths. Suppose ∂D_0 is not simple closed path so that some closed subpath γ of ∂D_0 will bound part of the tesselation excluding D_0 and D_1.



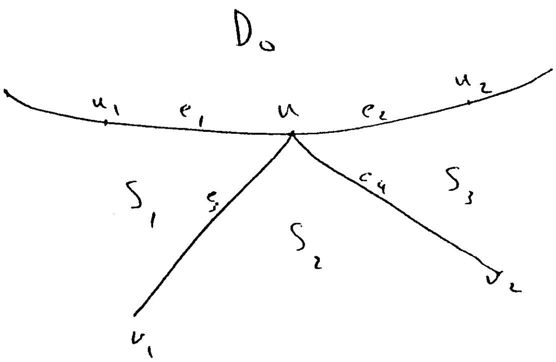
Remove all the regions of the tesselation except those in the part bounded by γ.



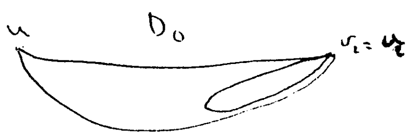
Then by Lemma 4.3.1, some vertex of this tesselation other than w is of type (1,1), (1,2) or (1,3). But this vertex will be a vertex of type (1,1), (1,2) or (1,3) in our orignal tesselation, which is a contradiction. Thus we have that $\partial D_0, \partial D_1$ are simple closed paths.

Consider the vertices $u_1, u_2, \ldots u_r$ on ∂D_0. If all these vertices are of type (2,2) then we have the tesselation with no layers. Thus can suppose that one of the vertices is of type (1,4).

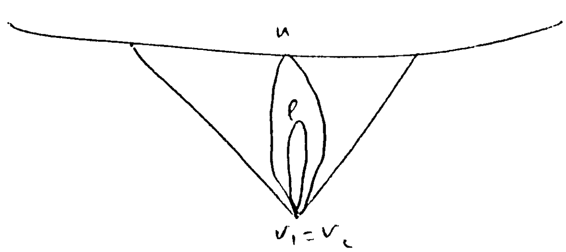
Assume first that r>1. Let u be a vertex on ∂D_0 of type (1,4). There are three non-distinguished regions S_1, S_2, S_3 incident with u. We show that $\partial S_1, \partial S_2, \partial S_3$ are simple closed paths.



We have that $e_3 \neq e_4$ for otherwise S_2 would have a boundary cycle of length 1. Thus $v_1 \neq u$, $v_2 \neq u$. Now $v_2 \neq u_2$ for otherwise the third edge of ∂S_3 would be a loop at v_2 which means that v_2 would be a distinguished vertex with valence at least 5 a contradiction.

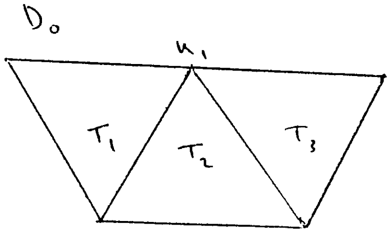


Similary $v_1 \neq u_1$. Now it is left to show that $v_1 \neq v_2$. If $v_1 = v_2$ then the third edge of ∂S_2 would be a loop ℓ at v_1.

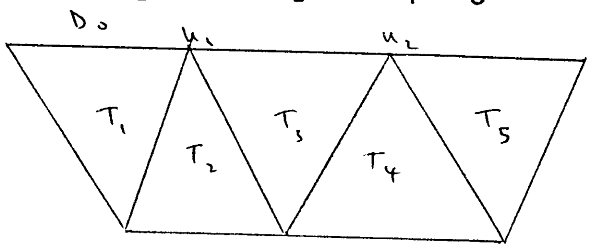


But then v_1 has valence at least 6, and so must have valence exactly 6 (since v_1 is non-distinguished). But there can be no non-distinguished region having boundary cycle of length 1 (region labelled by ℓ) a contradiction. Thus $\partial S_1, \partial S_2, \partial S_3$ are simple closed paths.

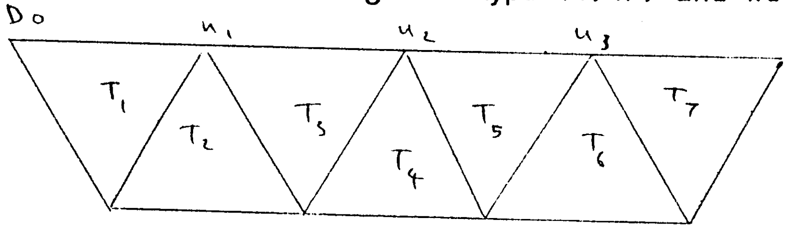
Now consider the vertices $u_1, u_2, \ldots, u_r$ on ∂D_0 and suppose that u_1 is of type (1,4). We will show all the vertices are of type (1,4) and we will determine the arrangement of the regions incident with $u_1, \ldots, u_r$. There are four regions incident with u_1, D_0 and three non-distinguished regions T_1, T_2, T_3.



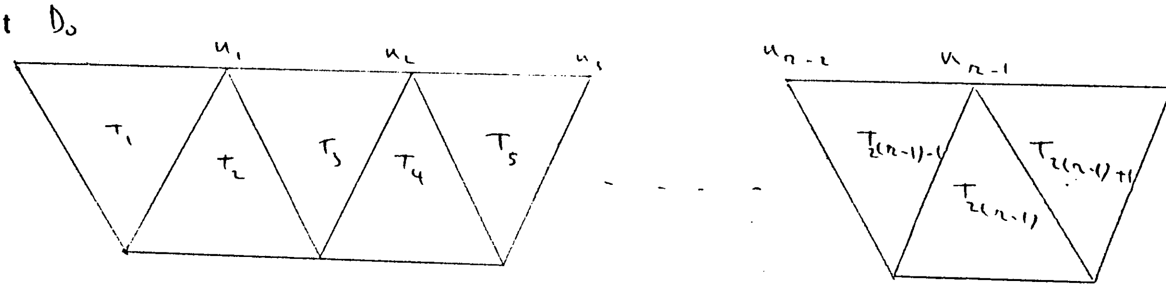
Suppose u_2 is the vertex to the right of u_1. Since D_0 is incident with u_2, u_2 is either of type (1,4) or (2,2). However, it cannot be of type (2,2) because of the region T_3. Hence it is of type (1,4). Thus the regions incident with u_2 are D_0, T_3 and two further non-distinguished regions T_4, T_5.



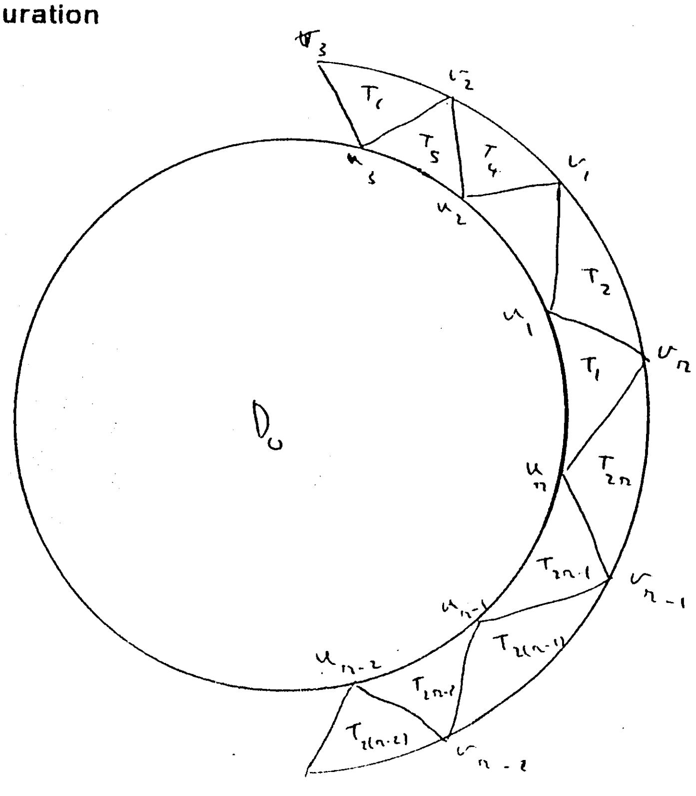
Now suppose u_3 is the vertex to the right of u_2. Then arguing as above, we have that u_3 is of type (1,4), and we obtain



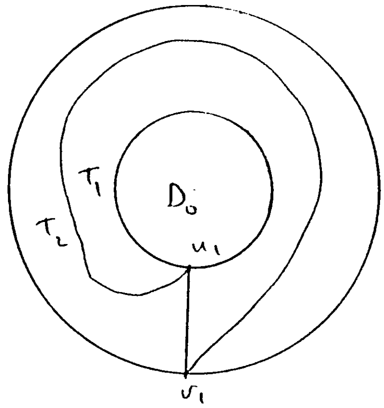
Continuing this way we will eventually arrive (after r−1 steps) at



The vertex to the right of u_{r-1} is the same as the vertex to the left of u_1, namely u_r. The regions D_o, $T_{2(r-1)+1}$, T_1 are incident with u_r so the vertex is of type (1,4) and there is one further non-distinguished region T_{2r} incident with u_r giving the final configuration

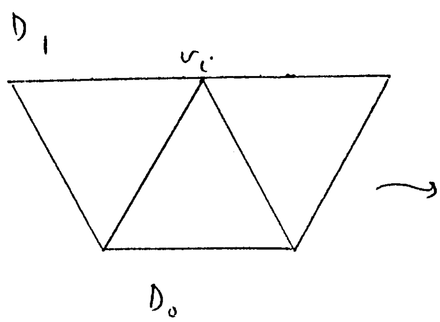


If r=1 then we simply have



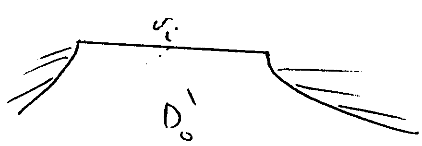
Now obtain a new tesselation by removing the above "layer", that is, remove all the vertices $u_1, \ldots, u_r$ and all the edges and regions incident with these vertices. The distinguished region D_0 then becomes a new distinguished region D_0'. We claim that in this new tesselation each vertex is of type (1,4), (2,2), (0,6). This is clear except for the vertices $v_1, \ldots, v_r$. Now in the old tesselation v_i is either of type (1,4) or type (0,6) according to whether or not D_1 is incident with v_i. Thus in the new tesselation v_i is either of type (2,2) or (1,4).

Old tesselation

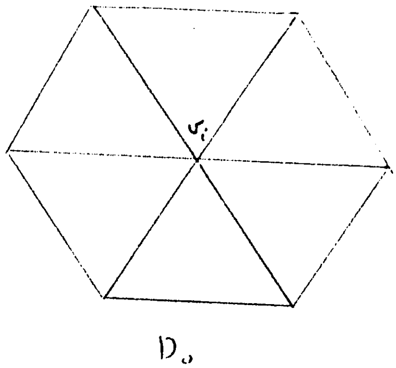


v_i of type (1,4)

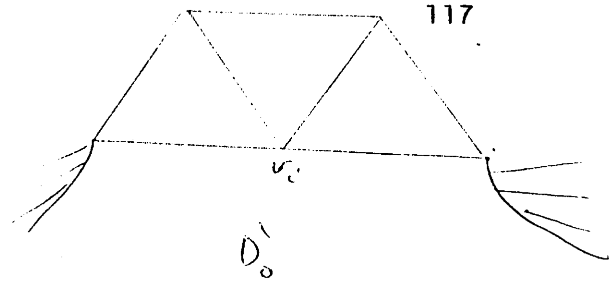
New tesselation



v_i of type (2,2)



v_i of type (0, 6)



v_i of type (1, 4)

By induction our new tesselation is made up of layers fitted together in the required way. Hence our orginial tesselation also has this form.

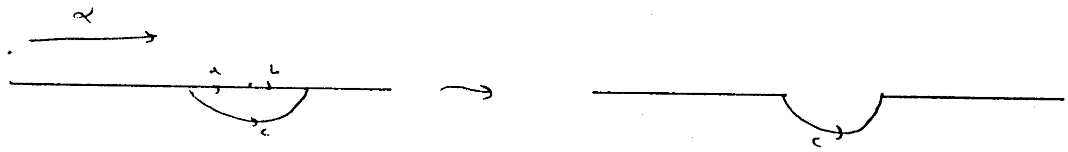
4.4 The word and conjugacy problem for T(6)-complexes.

Let $K = \langle \mathcal{X} : \rho_\lambda (\lambda \in \Lambda) \rangle$ be a triangulated 2-complex. We define certain operations on closed paths α in K.

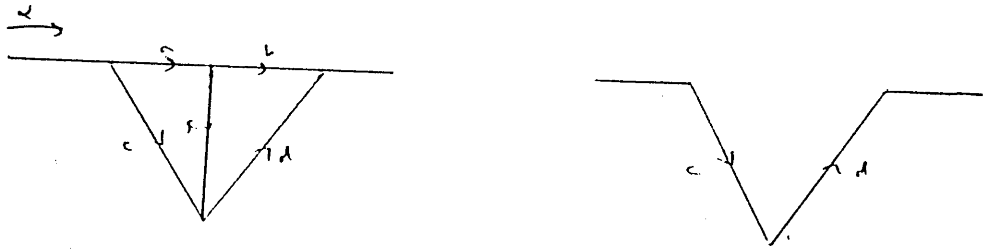
(I) Cyclically permute α.

(II) Delete an adjacent pair of inverse edges aa^{-1} (free reduction).

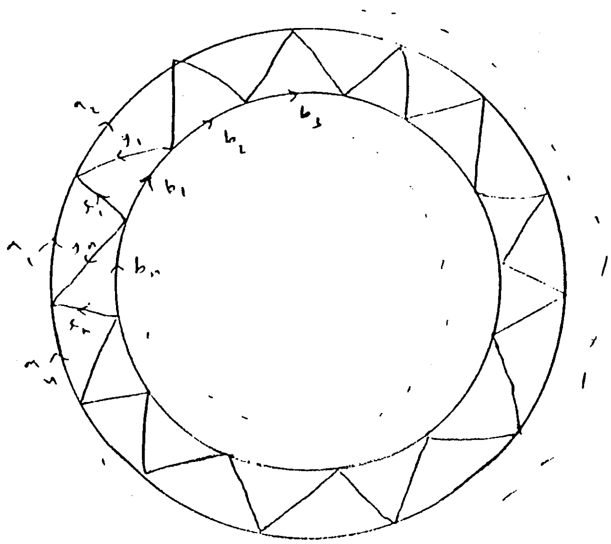
(III) Replace a subpath ab of α by c if $abc^{-1} \in R(K)$.



(IV) Replace a subpath ab of α by cd if there are elements axc^{-1}, $bd^{-1}x^{-1} \in R(K)$.



(V) If $\alpha = a_1a_2...a_n$ then change α to $b_1b_2...b_n$ if there are elements $a_1x_1^{-1}y_n, x_1y_1^{-1}b_1^{-1}, a_2x_2^{-1}y_1, ..., a_nx_n^{-1}y_{n-1}, x_ny_n^{-1}b_n^{-1}$ in $R(K)$.



Note that the operations from (I) to (V) do not increase the length of a path α.

Now we state our result.

Theorem 4.4.1. *Let $\mathcal{K}$ be a T(6)-complex.*

(i) A closed path in $\mathcal{K}$ is contractible if and only if it can be converted to an empty path by a finite sequence of operations (I) to (IV) in $\mathcal{K}_\Delta$.

(ii) Two closed paths in $\mathcal{K}$ are conjugate in $\mathcal{K}$ if and only if they can be converted to the same paths by a finite sequence of operations (I) to (V) in $\mathcal{K}_\Delta$.

Corollary 4.4.1.

DP(1) and DP(2) are solvable for T(6)-complexes.

Proof of Theorem 4.4.1.

Let $\mathcal{K}$ be T(6)-complex. Let ω be contractible path in $\mathcal{K}$. Then there is a reduced diagram $\mathcal{D}$ over $\mathcal{K}$ with single distinguished region whose boundary label is ω. Since $\mathcal{D}$ is reduced each non-distinguished vertex has valence at least 6. Consider $\mathcal{D}_\Delta$ (a triangulation of $\mathcal{D}$ see 4.2). The underlying sphere of this diagram satisfies the assumption (4.3.1) and we deduce from 4.3.1 that ω can be converted to the empty path by a finite sequence of operations (I) to (IV) in K_Δ. Conversely, if ω is a closed path in K which can be converted to an empty path by a finite sequence of operations (I) to (IV) in K_Δ then ω is contractible in K_Δ and hence is contractible in K, since the inclusion of $K^{(1)}$ into $K_\Delta^{(1)}$ induces an equivalence between K and K_Δ (see 1.1B.6) and by using Lemma 4.1.1 .

Now suppose ω_0, ω_1 are two closed paths in K which are conjugate in K, that is there is a path η such that $\omega_0 \underset{K}{\sim} \eta\omega_1\eta^{-1}$. If one of ω_0, ω_1 is contractible, then so is the other, so we may assume that neither is contractible. Then there is a reduced diagram $\mathcal{E}$ with two distinguished regions labelled by ω_0, ω_1^{-1}. Now the underlying sphere of the diagram $\mathcal{E}_\Delta$ satisfies the assumption (4.3.2) and it follows that ω_0, ω_1 can be converted to the same path by a finite sequence of operations (I) to (V) in K_Δ. Conversly if ω_0 and ω_1 are closed paths in K which can be converted to the same path by a finite sequence of operations (I) to (V) in K_Δ then ω_0 and ω_1 are conjugate in K_Δ and hence are conjugate in K (by using Lemma 4.1.1).

<u>Algorithm for word problem for a triangulated T(6)-complex</u>.

1) Let ω be a closed path in $\mathcal{K}$ of length n.

2) Draw a directed graph Γ_n whose vertices are all closed paths in $\mathcal{K}_\Delta$ of length $\leqslant$ n and (α, β) be an edge if and only if β is obtained from α by (I), (II), (III), (IV) moves.

3) Check if there is a path from ω to empty path in Γ_n.

4) If <u>yes</u> then ω is contractible. If <u>no</u> then ω is not contractible.

Algorithm for conjugacy problem for a triangulated T(6)-complex.

1) Let ω_0, ω_1 be closed paths in K of lengths n, m.

2) Draw a directed graphs Γ_n, Γ_m whose vertices are all closed paths in K_Δ of lengths $\leq$ n, m respectively. (α, β) be an edge if and only if β is obtained from α by (I), (II), (III), (IV) (V) moves.

3) Check if there exist a vertex γ in $\Gamma_n \cap \Gamma_m$ and a path in Γ_n from ω_0 to γ and a path in Γ_m from ω_1 to γ.

4) If yes then ω_0, ω_1 are conjugate in K_Δ. If no then ω_0, ω_1 are not conjugate in K_Δ.

CHAPTER 5

VALENCE OF VERTICES IN STAR COMPLEXES

5.1 Degree m, property-ST(m), property-st(m).

Consider a presentation $\mathcal{P}=\langle \mathbf{x}; \rho_\lambda (\lambda\in\Lambda)\rangle$. We say that $\mathcal{P}$ has *degree m*, denoted by deg($\mathcal{P}$)=m, if each $x\in\mathbf{x}\cup\mathbf{x}^{-1}$ appears exactly m times in the totality of paths $\rho_\lambda, \rho_\lambda^{-1} (\lambda\in\Lambda)$. We say that $\mathcal{P}$ has *property-ST(m)* if each vertex in $\mathcal{P}^{ST}$ has valence m, and we say that $\mathcal{P}$ has *property-st(m)* if each vertex in $\mathcal{P}^{st}$ has valence m.

We say a presentation $\mathcal{P}$ is *rootless* if each defining path of $\mathcal{P}$ is not a proper power.

We note that a rootless presentation with degree m has property-ST(m).

We note also that property-ST(m) and property-st(m) coincide if and only if $\mathcal{P}^{ST}$ is isomorphic to $\mathcal{P}^{st}$, that is, if and only if $\mathcal{P}$ is slender (see 1.1C.3).

Example 5.1.1. Let $\mathcal{P}=\langle a, b; a^2b^3a^{-1}\rangle$. Then deg($\mathcal{P}$)=3, and $\mathcal{P}$ has property-ST(3), property-st(3).

Example 5.1.2. Let $\mathcal{P}=\langle a, b, c; abcc^{-1}b^{-1}a^{-1}\rangle$. Then deg($\mathcal{P}$)=2 and has property-ST(2). Note that $\mathcal{P}$ has not property-st since $\mathcal{P}^{st}$ is not defined.

Example 5.1.3. Let $\mathcal{P}$=<a,b;(ab)^3>. Then deg($\mathcal{P}$)=3, and $\mathcal{P}$ has property-ST(1), property-st(1).

Example 5.1.4. Let $\mathcal{P}$=<a,b;a^{2}b^2,a^{2}b^2>. Then deg($\mathcal{P}$)=4, and $\mathcal{P}$ has property-ST(4), property-st(2).

Example 5.1.4. Let $\mathcal{P}$=<a,b,c;(aba^{-1}cb^{-1}c)^3>. Then deg($\mathcal{P}$)=6, and $\mathcal{P}$ has property-ST(2), property-st(2).

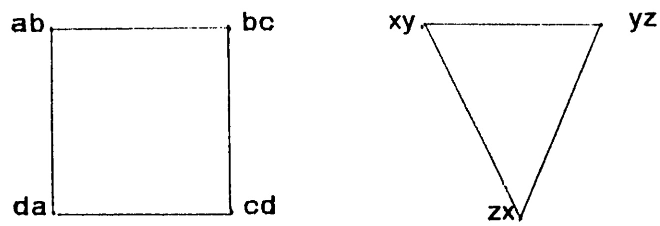
5.2 The case m=2.

Let $\mathcal{P}$=<x ;ρ_λ($\lambda\in\Lambda$)> be a presentation with degree 2. As is well-known the structure of the group G defined by $\mathcal{P}$ is rather special. We briefly discuss this.

The *connectivity graph* of $\mathcal{P}$, denoted by CG($\mathcal{P}$), is the graph whose vertex set is Λ and {λ,μ} is an edge if $L_x(\rho_\lambda)$, $L_x(\rho_\mu)\neq o$ for some x$\in$x.

Example 5.2.1. Let

$\mathcal{P}$=<a,b,c,d,x,y,z,p,q;ab,bc,cd,da,xy,yz,zx,p^{2}q^2>. Then CG($\mathcal{P}$) is



. p^2q^2

Now consider a presentation $\mathcal{P}= \langle \mathbf{x}; \rho_\lambda(\lambda\in\Lambda)\rangle$ of degree 2. Let $C_i(i\in I)$ be the components of $CG(\mathcal{P})$ with vertex sets $\Lambda_i(i\in I)$. Let $\mathbf{x}_i$ be the set of all elements of $\mathbf{x}$ involved in the relators $\rho_\lambda(\lambda\in\Lambda_i)$. We have that $\mathbf{x}= \bigcup_{i\in I} \mathbf{x}_i$. Let $\mathcal{P}_i=\langle \mathbf{x}_i; \rho_\lambda(\lambda\in\Lambda_i)\rangle$. If G_i is the group defined by $\mathcal{P}_i$, then we have that $G= \ast_{i\in I} G_i$. Now if Λ_i is infinite then G_i is free [6, Prop. I. 7. 4]. If Λ_i is finite then G_i is the free product of a surface group and a finitely generated free group [6, Prop. I. 7. 6].

5.3 The case m>2 .

The natural question to ask is whether having a presentation with degree m, property-ST(m) or property st(m) (m>2) puts any restriction on the group defined by $\mathcal{P}$?. The answer is no by proving the following.

Theorem 5.3.1. *Given m≥3 and a countable presentation, then the presentation is equivalent to a presentation of degree m which is rootless and slender (and hence has property-ST(m) and property-st(m)).*

Proof.

The proof is divided into two parts. In the first part we show how to convert a countable presentation into a presentation of degree m. In the second part we show how to convert a countable presentation with degree m into a rootless slender presentation of degree m. But first we give the following concept.

If y is a set of symbols and y_1 is a set in obvious 1-1 correspondence with y then we write yy_1^{-1} to mean all words of the form $yy_1^{-1}(y \in \mathbf{y})$.

<u>Part 1</u>. Let $\mathcal{P} = \langle\, \mathbf{a}, \mathbf{x}, \mathbf{d} \;;\; \rho_1, \rho_2, \ldots \rangle$ be a countable presentation, where for each $a \in \mathbf{a}$, $L_a(\rho_1) + L_a(\rho_2) + \ldots = 0$, for each $x \in \mathbf{x}$ $0 < L_x(\rho_1) + L_x(\rho_2) + \ldots < \infty$, and for each $d \in \mathbf{d}$ $L_d(\rho_1) + L_d(\rho_2) + \ldots = \infty$.

Now $\mathcal{P}$ is equivalent to

$$\langle \mathbf{a}, \mathbf{a}_1, \mathbf{x}, \mathbf{d}; \rho_1, \rho_2, \ldots, \mathbf{a}\mathbf{a}_1^{-1}, \mathbf{a}_1\mathbf{a}_1^{-1}, \mathbf{a}\mathbf{a}^{-1} \rangle.$$

Since each ρ_i can be considered as a finite sequence of generators and their inverses, so we can write $\rho = \rho_1\rho_2 \ldots$ as a (possibly infinite) sequence of generators and their inverses. Now let $x \in \mathbf{x}$ and suppose $L_x(\rho_1) + L_x(\rho_2) + \ldots = p_x$. Look at the terms of $\boldsymbol{\rho}$ which are $x^{\pm 1}$. There are p_x such terms. Replace the j^{th} term x^{ϵ_j} $(j = 1, 2, \ldots, p_x)$ by $x_j^{\epsilon_j}$. Now consider $d \in \mathbf{d}$ and look at the terms of ρ

which are $d^{\pm 1}$. There are infinitely many such terms. Replace the

i^{th} term d^{ϵ_i} $(i=1,2,\ldots)$ by $d_{\frac{i}{2}}^{\epsilon_i}$ if i is even or $d_{-\frac{i-1}{2}}^{\epsilon_i}$ if i is odd.

This gives a new sequence $\bar{\rho}=\bar{\rho}_1\bar{\rho}_2\ldots$. Then $\mathcal{P}$ is equivalent to

$$\langle a, a_1, x_1, x_2, \ldots, x_{p_x} (x\in X),\ d_0, d_{-1}, d_1, \ldots ; \bar{\rho}_1, \bar{\rho}_2, \ldots, aa_1^{-1}, a_1a_1^{-1}, aa^{-1}, x_1x_2^{-1}, x_2x_3^{-1}, \ldots, x_{p_x}x_1^{-1}, \ldots, d_{-1}d_0^{-1}, d_0d_1^{-1}, d_1d_2^{-1}, \ldots\rangle$$

which is a presentation of degree 3. Now if we repeat

$\bar{\rho}_1, \bar{\rho}_2, \ldots, aa_1^{-1}$ $(m-2)$-times, we have a presentation of degree m.

Such a presentation is not slender because of the repetition of

the relators $\bar{\rho}_1, \bar{\rho}_2, \ldots, aa_1^{-1}$ and because of the relators

$aa^{-1}, a_1a_1^{-1}$.

Part 2. Let $\mathcal{P}=\langle X; \rho_1, \rho_2, \ldots\rangle$ be a presentation of degree m.

Consider $\rho=\rho_1\rho_2\ldots$. For each $x\in X$, we have $L_x(\rho_1)+L_x(\rho_2)+\ldots=m$.

Look at the terms of ρ which are $x^{\pm 1}$. There are m such terms.

Replace the i^{th} term x^{ϵ_i} $(i=i,2,\ldots,m)$ by $x_i^{\epsilon_i}$. This gives a new

sequence $\bar{\rho}=\bar{\rho}_1\bar{\rho}_2\ldots$ Then $\mathcal{P}$ is equivalent to

$$\langle x_1, x_2, \ldots, x_m; \bar{\rho}_1, \bar{\rho}_2, \ldots, x_tx_q^{-1} (1\leq t<q\leq m)$$

which is a presentation of degree m since each $x_j\in x_j (j=1,2,\ldots,m)$

is involved once in $\bar{\rho}_1, \bar{\rho}_2, \ldots$, and $(m-1)$-times in the relators

$x_tx_q^{-1}$ $(1\leq q\leq m)$. This presentation is rootless and slender.

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