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FLUX-BREAKING OF GAUGE AND CENTRAL SYMMETRIES IN THE HOSOTANI MODEL

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Ph.D Thesis

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"...in every age there come forth things that are new and have no
foretelling, for they do not proceed from the past."

J. R. R. Tolkien

AINULINDALĒ

Declaration

The derivation of the potential in chapter 4 is based on techniques used in reference [39]. The generalisation to general gauge group and spacetime dimension and the introduction of the polylogarithm form of the potential is entirely original, as is the derivation of the potential directly in terms of this function. Chapters 5, 6, 7 and the appendices contain original work, except where indicated otherwise, some of which has been outlined in the following papers. The discussion of the congruency classes of $\text{spin}(4n)$ in chapter 3 is also claimed as original, and corrects a mistake in reference [22].

The numerical approach to the problem on $\mathbb{R}^3 \times S^1$ and the dependence on congruency classes is outlined in the paper

"Gauge Group Breaking By Wilson Loops"

Davies & McLachlan, Physics Letters B200, (1988), 305

The more analytic approach to the model, with a more detailed discussion of congruency classes, can be found in the paper

"Congruency Class Effects In The Hosotani Model"

Davies & McLachlan, Glasgow University preprint GUTPA/88/7-1

a shortened version of which shall appear in

"Proceedings of the XVII International Colloquium on Group Theoretical Methods in Physics", World Scientific Press (1988).

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Finally, as is my duty in such cases, I would like to thank the Science and Engineering Research Council, regular donations from whom have ensured peace of mind for both my bank manager and myself over the last three years. I would also like to thank the University of Glasgow for joining with SERC and the Institute of Physics in financing excursions both domestic and foreign in my not always successful quests for enlightenment.

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Abstract

The purpose of this work is to examine the flux-breaking mechanism for breaking gauge symmetries in the context of the generalised Hosotani model. This model consists of either fermions or scalars (in one arbitrary representation of the gauge group) minimally coupled to Yang-Mills fields on a background spacetime of the form $R^m \times S^1$. In this model, the one loop effective potential can be explicitly calculated in terms of the non-trivial background field components satisfying $F_{\mu\nu} = 0$ which exist in the circular dimension and therefore the particular breaking direction for any given case can be determined by minimising this potential.

We begin in Chapter 1 by describing the flux-breaking mechanism. The group theory required for this analysis is then reviewed in chapter 2 before we go on to discuss the effective potential and its calculation in the generalised Hosotani model. We show in Chapters 5 and 6 that, for the case of either antiperiodic fermions or periodic scalars, the zero background is preferred as the global minimum of the potential and hence no symmetry breaking occurs. For periodic fermions or antiperiodic scalars, the zero background is destabilised by the matter field contribution to the potential. If such fields are in a representation whose congruency class does not contain the adjoint representation, then there exist non-zero backgrounds which preserve the gauge symmetry and are preferred as the global minima of the potential. Such non-zero backgrounds, however, lead to the matter fields having no zero modes on compactification of the theory to R^m and hence there is an apparent breaking of the central symmetry of the gauge group.

If such destabilising fields are included in an adjoint class representation, then the zero background is found to be the only symmetry preserving background and, as this is eventually becomes a local maximum of the potential, the gauge symmetry must break. Examples illustrating this fact are given in numerical and graphical form. It is also found that the critical fermion number required to destabilise the zero

background decreases as the spacetime dimension is increased, whereas the corresponding critical scalar number increases. This then leads to the conjecture that fermions will be more conducive to symmetry breaking than scalar fields in higher dimensional theories. An additional feature is found whereby scalars given a phase $\delta = \pi/2$ can break the gauge symmetry providing the representation containing them generates a group with Z_2 centre. Examples of this type of breaking are also given, along with a discussion of permissible phases for the matter fields.

Finally, Lagrangians containing more than one matter representation are examined in Chapter 7 in order to determine if the single representation features persist. It is found that the situation becomes more complicated, but symmetry breaking is again only found if destabilising fields in the appropriate representations are included. This leads to a general conjecture about the possibilities for flux breaking of gauge symmetries in any model, a conjecture which seems to be validated in a brief review of other toy models. An original example of an E_6 model is also given, in which a realistic subgroup is obtained by flux breaking, provided that an upper limit is imposed on the number of fundamental representation fermion generations. We conclude by summarising the contributions made to the field in this thesis and we review the work of other authors on flux-breaking toy models.

CHAPTER 1

Gauge theories and symmetry breaking methods

i) Gauge theories

The purpose of this introductory chapter is to briefly outline the ideas behind gauge theories and to describe some methods for breaking gauge symmetries, including the flux-breaking mechanism. A detailed discussion of the Weinberg-Salam and standard models is not really necessary for the purpose of this thesis (which is a substantially numerical group theoretical analysis of a simple toy model not intended to be a serious candidate as a theory of nature) and hence such a discussion would either tend to detract from, or give a false impression of, the purpose of this work. The presentation of gauge theories in this section therefore, although seeming to be fairly naïve, shall contain the relevant group theoretical concepts required to permit an appreciation of the philosophy behind our examination of the Hosotani model. The reader is referred to the literature if detailed material on the standard model is required[1,2].

The simplest way to introduce the concept of a gauge theory is to examine a simple Lagrangian containing, for example, a fermion field with a global U(1) symmetry. We can write the Lagrangian as

$$L = \bar{\psi} \gamma^\mu \partial_\mu \psi \tag{1.1}$$

which is invariant under the following transformation of the fields

$$\begin{aligned} \psi &\rightarrow e^{i\alpha} \psi \\ \bar{\psi} &\rightarrow \bar{\psi} e^{-i\alpha} \end{aligned} \tag{1.2}$$

where α is some arbitrary real parameter. The group consisting of all the elements $e^{i\alpha}$ is isomorphic to U(1) and is called the symmetry group of the Lagrangian. By Noether's theorem[2], such a symmetry should lead to a conserved current and a

corresponding conserved charge, but as the symmetry here is global (the parameters α are not functions of the spacetime coordinates) then such conserved charges will also be global. If this model is interpreted as being a theory of electromagnetism then, due to the fact that the electromagnetic charge is conserved locally, it is clear that the transformation parameters must be taken to be functions of the coordinates. This immediately leads to a problem when we apply such a transformation to (1.1) as the derivative acting on the group elements will lead to $\partial_\mu \alpha(x)$ terms appearing in the transformed Lagrangian. In order to restore the invariance of the Lagrangian, an extra field A_μ can be introduced in the following manner

$$L = \bar{\Psi} \gamma^\mu D_\mu \Psi = \bar{\Psi} \gamma^\mu (\partial_\mu + A_\mu) \Psi \quad (1.3)$$

provided it transforms as

$$A_\mu \rightarrow e^{i\alpha(x)} A_\mu e^{-i\alpha(x)} + e^{i\alpha(x)} \partial_\mu e^{-i\alpha(x)} \quad (1.4)$$

Such local transformations are known as gauge transformations and the fields A_μ are known as gauge fields. The Lagrangian (1.3) can now be interpreted as a Lagrangian for electromagnetism, with the A_μ field manifesting itself as the photon.

In order to incorporate the other non-gravitational interactions into this formalism, a more complicated gauge group is required and it turns out that the smallest gauge group which describes quantum chromodynamics and the Weinberg-Salam electro-weak theory is $SU(3)^C \times [SU(2) \times U(1)]^{EW}$. Note that $U(1)$ electromagnetism does not appear explicitly in this group, as the $SU(2) \times U(1)$ factor must be broken down first.

ii) Higgs mechanism for symmetry breaking

Although the 'unified' symmetry group of the non-gravitational interactions is $SU(3)^C \times [SU(2) \times U(1)]^{EW}$, the observable symmetry group is actually $SU(3)^C \times U(1)^{EM}$, where the $SU(2) \times U(1)$ electro-weak group has been 'broken' in some manner in order to give the observed $U(1)$ electromagnetic interaction described earlier, and the most common symmetry breaking mechanism which is used to model this phenomenon is the Higgs mechanism[1-4]. This involves the introduction of scalar particles ϕ into the Lagrangian of the system, along with a quartic polynomial potential whose minima are not located at $\langle \phi \rangle = 0$. The Lagrangian will still possess the full symmetry, but when the scalar field is expanded about one of its minima, the resulting Lagrangian does not possess the full symmetry, with the residual symmetry group being that which leaves the vev $\langle \phi \rangle$ invariant. The introduction of this non-zero vev also leads to mass terms for the various matter fields as well as for the gauge boson components corresponding to the broken generators of the original symmetry, but will also result in the appearance of massive scalar(s) which should be observable at low energies. The main disadvantage with this mechanism (aside from the fact that no fundamental scalar fields have ever been observed) is that the various coefficients in the scalar potential have to be carefully selected in order to give the required residual symmetry along with the correct masses for the various fields, a degree of arbitrariness which is not particularly satisfactory.

This arbitrariness also persists in unified theories where the breaking of the unification group proceeds by the Higgs mechanism. An added complication arises when radiative corrections are taken into account, as the various parameters have to be fine tuned to an unreasonable degree at each order of perturbation theory in order to achieve agreement with the observed particle spectrum (this is known as the 'Hierarchy Problem'). In such theories, it would be desirable if some more fundamental and less arbitrary mechanism was responsible for the symmetry

breaking, perhaps leading to the electro-weak Higgs sector in some natural and completely specified manner. In theories which involve the introduction of extra compact dimensions, new mechanisms do appear, usually as a result of the non-trivial nature of the compact space. Two such mechanisms will now be described.

iii) Symmetry breaking in higher dimensions

For theories formulated in spacetimes with dimension greater than 4, possibilities for breaking symmetries and generating new symmetries arise, based on the geometry of the compact space dimensions. The prototype for such models was first examined by Kaluza[5] and Klein[6] and consisted of pure gravity in a spacetime of the form $M^4 \times S^1$. Using Greek letters for M^4 indices, the metric could be written as

$$g_{ij} = \begin{pmatrix} g_{\mu\nu} + A_\mu A_\nu \phi^2 & A_\mu \phi^2 \\ A_\mu \phi^2 & \phi^2 \end{pmatrix} \quad (1.5)$$

where A_μ is independent of the circular coordinate. On taking the scalar ϕ to be constant, the five dimensional Einstein's equations give four dimensional gravity and also U(1) electromagnetism. The reason for the appearance of the U(1) symmetry is that U(1) is the isometry group of the circle and hence translations of the circular coordinate manifest themselves as U(1) gauge transformations in the compactified theory. A detailed of this model can be found in the literature, but the phenomenon of components of the metric manifesting themselves as Yang-Mills fields on compactification persists in more complicated cases.

If we have a Yang-Mills theory to begin with, then the original symmetry group can be broken topologically on compactification, along with 'extra' Yang-Mills

symmetries being generated. If the compact space is actually a coset space S/R (where S is some compact Lie group), then the gauge symmetry G is broken to $S \times G'$ where G' is the centraliser of R in G (the subgroup of G whose elements commute with all the elements in R). The gauge group of the original Yang-Mills Lagrangian will now be G' , with the extra contribution from the metric having symmetry group S [7-11]. (The extra dimensional components of the original Yang-Mills fields will now appear as scalars in the dimensionally reduced theory.) While this is a desirable feature in the Kaluza-Klein approach, it is a distinct disadvantage when Yang-Mills terms are initially present as well as the gravitational fields. In order to avoid extra symmetries, it is necessary to work with manifolds for which the isometry group is trivial, thus rendering this particular method of symmetry breaking unworkable. Even if this extra Yang-Mills sector was acceptable, the simplest and most natural compact spaces are not of coset form (e.g. the tori T^n are just the group manifolds $[U(1)]^n$) and hence no breaking of the original symmetry group would be possible.

iv) The flux-breaking mechanism

This symmetry breaking mechanism is applicable in higher dimensional theories where the compact space is not simply connected. The method was first introduced by Hosotani[12] and was formulated in detail in order to apply it to superstring models[13-15] where models have been constructed for both Calabi-Yau[16-18] and orbifold[19,20] compactifications. We can write such a multiply connected space in the following form

$$K = \frac{K_0}{\Gamma} \tag{1.6}$$

where K_0 is a simply connected manifold and Γ is some freely acting discrete group

the elements of which identify points on K_0 , i.e. maps them to the same point on K . An ordinary field ψ on K is equivalent to a field on the covering space K_0 which takes the same value at identified points, i.e.

$$\psi(\gamma x) = \psi(x) , \quad \forall \gamma \in \Gamma \quad (1.7)$$

For fields which transform as some representation of the gauge group, the equivalent condition is that the fields on the covering space be equal up to a gauge transformation at identified points

$$\psi(\gamma x) = U_\gamma \psi(x) , \quad \forall \gamma \in \Gamma \quad (1.8)$$

An immediate consequence of this equation is

$$U_\gamma U_\delta \psi(x) = U_\gamma \psi(\delta x) = \psi(\gamma \delta x) = U_{\gamma \delta} \psi(x) \quad (1.9)$$

which implies that the matrices U form a homomorphism of the discrete group Γ into the gauge group. It is also clear that in order for (1.8) to be compatible with normal gauge transformations V , we must have

$$[V, U_\gamma] = 0 , \quad \forall \gamma \in \Gamma \quad (1.10)$$

The result of all this is that, on a multiply connected manifold, we may break an initial gauge group G to the subgroup G' which commutes with $\tilde{\Gamma} = \{U\}$, the image of the homomorphism of Γ into the gauge group. This is similar to the breaking described in the last section, as the residual group is the centraliser of the divisor group in both cases. All that is now required is some explicit expression for the matrices U which will permit their evaluation.

As is well known[2], parallel transport of a field (in some representation of the gauge group) along a path connecting two points x and x' in the presence of a gauge

field B_μ yields the following transformation

$$\psi(x') = \exp \left\{ \int_x^{x'} B_\mu dx^\mu \right\} \psi(x) \quad (1.11)$$

therefore we may write the U matrices as

$$U_\gamma = \exp \left\{ \int_\gamma B_\mu dx^\mu \right\} \quad (1.12)$$

where γ labels the path between two points identified by the action of the element $\gamma \in \Gamma$. These matrices are generally referred to as Wilson loops or Wilson lines, as analogous quantities were first defined in the context of lattice gauge theory. They are gauge covariant and therefore, if $U_\gamma \neq 1$, then no gauge transformation will set its value to unity[13].

We may adopt two equivalent viewpoints of this procedure, depending on whether we work with functions on the multiply connected space K or on its covering space K_0 . If we let the gauge field have a non-vanishing expectation value then, working in terms of fields on the multiply connected space K , there is no single valued gauge transformation which will gauge this vev to zero. We therefore can regard (1.12) as being constructed by integrating this non-zero vev around the closed loops on the manifold. The symmetry breaking can therefore be viewed as the non-zero vevs for the gauge fields behaving in a similar manner to adjoint representation Higgs fields. If we work with fields on the covering space, then such vevs can be gauged away at the expense of introducing the Wilson operators as given in equation (1.8).

We have therefore outlined the flux-breaking mechanism. One main advantage as regards superstring theory (the only serious model which employs this technique at present) is that the background field B_μ used to break the symmetry can be taken to satisfy $F_{\mu\nu} = 0$, i.e. can be taken locally as a pure gauge on K . This therefore

implies that these fields will not break any supersymmetries which may be present in the various superstring models and makes it extremely useful in breaking the residual E_6 symmetry obtained by embedding the spin connection of manifolds with $SU(3)$ holonomy into the original gauge group E_8 . Such manifolds can be made multiply connected by the action of some combination of discrete group, a technique which has the added beneficial feature of reducing the large number of matter generations specified by various index theorems. An added advantage is that the breaking is due to a non-zero vev for the original gauge fields and hence there is no need to introduce spurious scalar fields. By ensuring that the compact manifolds have no continuous symmetries (Calabi-Yau), the appearance of extra symmetries due to the symmetries of the compact space will be avoided. This mechanism can also be applied to the orbifold limit of Calabi-Yau manifolds[19,20], in which case it is the E_8 group which is to be broken.

As there is no added arbitrariness in the form of the Lagrangian when a non-zero vev for the gauge field is introduced, then the symmetry breaking direction should, in principle, be uniquely determined by minimisation of the potential terms in the action (including radiative corrections). In practice, such calculations tend to be difficult if not impossible and hence the superstring models tend to adopt the breaking directions which yield the most promising low energy groups and massless representations. Therein lies the motivation for this project : by examining simple models where the breaking directions can be determined, some general principles may be found which, while by no means determining the directions to be taken in more realistic cases, may give guidelines as to the features which may reasonably be expected in more sophisticated cases.

CHAPTER 2

Representation theory of Lie groups

i) Congruency classes of algebra representations

The aim of this chapter is not to provide a full description of the representation theory of Lie groups, but rather to concentrate on one main feature, i.e. the identification of the true symmetry group of a representation obtained by exponentiation of a representation of the corresponding Lie algebra. Some familiarity with the theory of Lie groups and their algebras will therefore be assumed. More general discussions of representation theory as applied to physics can be found in the literature[1,21,22]. This chapter will be loosely based on the discussion given in reference[22] and the terminology used therein will be adopted for the remainder of this thesis.

The first concept to be discussed is that of the congruency class of a Lie algebra representation[22-24]. A d -dimensional representation of a semi-simple complex Lie algebra L contains d weight vectors with $\text{rank}(L)$ components (the eigenvalues of the simultaneously diagonalisable generators when acting on an appropriate state). Each representation can be specified by a highest weight from which the other weights can be obtained by the action of the non-diagonal generators, which behave as ladder operators. All the weights of all representations can be considered as forming a lattice in a $\text{rank}(L)$ dimensional Euclidean space, with the root lattice (weights of the adjoint representation) forming a sublattice. The congruency classes can be defined as the cosets of the weight lattice modulo the root lattice. Put more simply, representations whose weights differ by an integer combination of roots will lie in the same congruency class. For the algebras G_2 , F_4 and E_8 , all representations are found to lie in the same class, but the concept is less trivial for the classical and remaining exceptional algebras. The importance of this classification of representations of the algebra is that representations in the same congruency class, when exponentiated, will generate the same Lie group. The fact that different groups can be generated from the same algebra is rarely emphasised in the physics literature, therefore we shall discuss this feature (without proofs) in the following section.

ii) Representations of locally isomorphic groups

For each semisimple complex Lie algebra, we can generate several different Lie groups. These groups will be locally isomorphic (isomorphic in a neighbourhood of the origin in the parameter space), and can all be written as homomorphic images of a simply connected covering group. For each group, the kernel of the mapping from the covering group (the set of elements mapping to the identity) will be a discrete central subgroup of the covering group. We can therefore write each locally isomorphic group as a factor group of the covering group modulo some discrete central subgroup. (If the covering group is not connected, then the above argument is to be taken as being relevant for the connected component containing the identity.) Exponentiation of a representation of the algebra will yield a single valued representation of the covering group, but this representation will not necessarily be faithful. A given representation will be both faithful and single-valued for only one of the locally isomorphic groups, and, following O'Raifeartaigh[22], we shall refer to this group as the true group of the representation. (A well known example is the case of the double-valued spinor representations of $SO(N)$ which should really be considered as single valued representations of the covering group $Spin(N)$). The relevance to flux-breaking is now clear : as the symmetry is preserved if the Wilson operators map to the centre, then it is necessary to identify which of the locally isomorphic group is the true symmetry group in order to determine and analyse the possible symmetry preserving configurations. The identification of the faithful single-valued representations of a particular group as being generated by representations of the algebra which lie in the same congruency class can be illustrated in the following manner for the simply-laced groups (groups generated from algebras whose simple roots all have the same length i.e. A_n , D_n and E_n). This relationship was explicitly derived by Andrew Davies in our second paper, although we expect that a more rigorous version can be found in the mathematics literature.

The argument proceeds as follows : the condition for $\underline{\lambda}$ to be a weight is

$2(\lambda, \alpha)/(\alpha, \alpha) \in \mathbb{Z}$ for all simple roots α , which reduces to $(\lambda, \alpha) \in \mathbb{Z}$ in the simply laced case, where we can normalise all (α, α) to 2.

The scalar product (λ, μ) is defined by

$$(\lambda, \mu) = \lambda^a G_{ab} \mu^b, \quad a, b = 1 \text{ to } N \quad (2.1)$$

G is a symmetric matrix, which in the simply laced case is simply A^{-1} where A is the Cartan matrix of the algebra. Our notation is similar to reference[1], where these matrices are listed.

The matrix A is integral. If $\det A = 1$, G is also integral, but in general its elements are rational fractions with denominator $\det A$ or a divisor thereof. λ^a are also integers, being components in the Dynkin basis. Defining $\lambda_b = \lambda^a G_{ab}$ with lowered indices as components in a dual basis, we can write

$$(\lambda, \mu) = \lambda_b \mu^b = \mu_b \lambda^b \quad (2.2)$$

As the simple root α^a is just row a of A , its dual has components

$$\alpha_b^a = \alpha^{ac} G_{cb} = (A A^{-1})_b^a = \delta_b^a \quad (2.3)$$

Since $\underline{\lambda}$ is in the same class as $\underline{\lambda} + \underline{\alpha}$, each congruency class, L , may be represented by a vector

$$\underline{\lambda}^L = r^{L,a} \underline{\alpha}^a, \quad 0 \leq r^{L,a} < 1 \quad (2.4)$$

and the dual components of $\underline{\lambda}^L$ are complete $r^{L,a}$.

The key observation is that a complete basis set of such fractional vectors is given by the duals of basic weights having one component unity and the others zero. The duals of such $\underline{\lambda}^L$ are therefore the rows of the matrix $G \pmod{1}$. They are not all

distinct, and form an additive abelian group isomorphic to Z_n (except in the case of $\text{Spin}(4n)$, where it is $Z_2 \times Z_2$).

If any non zero $\underline{\lambda}^L$ exist ,they satisfy $(\lambda, \alpha) = 0 \pmod{1}$ for all α , i.e. $\lambda_a \alpha^a = 0 \pmod{1}$, due to the fact that the adjoint group has unit centre. Conversely, any solution c_a of $c_a \alpha^a = 0$ must be a linear combination of duals of class vectors $\underline{\lambda}^L$. The additive group of class vectors is isomorphic to the centre of the group G , as we now show. Let G be the true group of a representation R , so that its elements are

$$g = \exp(-2\pi i \epsilon_a T_R^a) \quad (2.5)$$

where ϵ_a are the group parameters, and T_R^a ($a=1, \dim G$) are generators in representation R . The elements with $\epsilon_a T_R^a = \lambda_a^L H^a$ for different L form a discrete abelian subgroup. In the representation R , these are diagonal matrices, and furthermore are multiples of the identity. The i 'th diagonal element is $\exp\{-2\pi i (\lambda^L, \lambda^i)\}$, where $\underline{\lambda}^i$ is the i 'th weight of R . Due to the condition $(\lambda, \alpha) \in Z$ the inner products depend only on class, and are $q^{LM} = (\lambda^L, \lambda^M)$ where M labels the class of R . Thus for fixed M and varying L , $\exp(-2\pi i q^{LM})$ form a central subgroup of G . This is in fact the full centre, because any element of the centre will have the form $I_R \exp(-2\pi i q)$ where $q = (\epsilon, \lambda^R)$ in the representation R ; but $q = 0$ in the adjoint, so $(\epsilon, \alpha) = 0 \pmod{1}$. Hence ϵ must be a linear combination of $\underline{\lambda}^L$, the complete set of solutions of this equation.

To sum up: any representation in class M exponentiates to a true group which is unique to that M . Its discrete centre is isomorphic to the additive group of its congruency classes $\underline{\lambda}^L$.

We illustrate the foregoing procedure for determining the symmetry preserving backgrounds (= class vectors), and show how the class of the exponentiated representation determines the true group, in the example of A_5 , the Lie algebra of $SU(6)$, $SU(6)/Z_2$, $SU(6)/Z_3$, or $SU(6)/Z_6$. The Cartan matrix is

$$A(A_5) = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix} \quad (2.6)$$

with inverse

$$G(A_5) = \frac{1}{6} \begin{pmatrix} 5 & 4 & 3 & 2 & 1 \\ 4 & 8 & 6 & 4 & 2 \\ 3 & 6 & 9 & 6 & 3 \\ 2 & 4 & 6 & 8 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \quad (2.7)$$

The rows of $G \pmod{1}$ are the duals of the congruency class vectors. The first row is the dual of $\underline{\lambda}^{(1)} = (1 \ 0 \ 0 \ 0 \ 0)$ in the Dynkin basis. The other rows are $\underline{\lambda}^L = L\underline{\lambda}^{(1)} \pmod{1}$ for $L = 2, 3, 4, 5$. $\underline{\lambda}^{(6)} = 6\underline{\lambda}^{(1)} \pmod{1}$ represents the adjoint class.

The matrix of scalar products is

$$q^{LM} = (\lambda^L, \lambda^M) = \frac{1}{6} \begin{pmatrix} 5 & 4 & 3 & 2 & 1 & 0 \\ 4 & 2 & 0 & 4 & 2 & 0 \\ 3 & 0 & 3 & 0 & 3 & 0 \\ 2 & 4 & 0 & 2 & 4 & 0 \\ 1 & 2 & 3 & 4 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (2.8)$$

There are 6 $\lambda^{(L)}$ which all correspond to $q^{LM} = 0$ ($U_{ad} = I_{ad}$), but which generally differ in their q^{LM} values for other classes M . For the case of representations with sextality $M = 1$, the entries q^{L1} in column 1 of the q matrix give the values corresponding to the $\underline{\lambda}^L$. With an $M=1$ representation, the true group has a central Z_6 subgroup generated by q^{11} . The true group is therefore $SU(6)$. If we have sextality $M = 2$, the largest central subgroup generated by any q^{L2} is Z_3 , so the true group in

this case is $SU(6)/Z_2$.

For the nonsimply-laced groups with non-trivial centres (algebras B_n, C_n), the relationship between the algebra congruency classes and the locally isomorphic groups still holds and, as most of the classical groups and all of the exceptional groups have centres of the form Z_n , the identification of the true group centre is not particularly difficult. Exceptions to this, however, are the groups $Spin(4n)$ whose centres are of the form $Z_2 \times Z_2$, for which two index congruency class labels are required. (As was shown by Lemire and Patera[25], the congruency classes of the D_n algebras can all be labelled by two indices, but it is easy to see that for the cases of D_{4n+2} the two index label can be replaced by a one index label for which the q parameters as defined in [25] generate the Z_4 centre of $Spin(4n + 2)$. For the remaining case, it is necessary to examine the four classes in a little more detail in order to identify the corresponding true symmetry groups. In this brief discussion, we shall correct an error in O'Raifeartaigh[22], and shall independently obtain the results for this case which are given in Cornwell[21].

For the $Spin(4n)$ groups, it is necessary to distinguish between the two Z_2 factors which appear in the group centre. As no definite notation has been found in the literature, we shall label the factors in the centre by $Z_2(t) \times Z_2(s)$, with the individual classes for each Z_2 being denoted by M_t and M_s respectively. The reason for the notation above is that the spinor representations have $M_t = 1$ and $M_s = 1$ or 2 and the tensor representations have $M_t = 2$ and $M_s = 1$ (odd rank) or 2 (even rank), hence in a sense the two factors can be viewed as labelling tensors and spinors. As the centre has a product form, it turns out that none of the irreducible representations provides a faithful representation of $Spin(4n)$ and hence all the representations must have one of the locally isomorphic groups as a true group[21]. This is in direct contradiction with the presentation of this example in O'Raifeartaigh. The following table gives the true groups generated from the various congruency classes, a result which shall be explicitly verified for the case of $Spin(8)$.

Class	Kemel	Group
(2,2) even tensor	$Z_2(t) \times Z_2(s)$	$\text{Spin}(4n) / Z_2(t) \times Z_2(s)$
(1,2) odd tensor	$Z_2(s)$	$\text{Spin}(4n) / Z_2(s)$
(2,1) spinor	$Z_2(t)$	$\text{Spin}(4n) / Z_2(t)$
(1,1) spinor	$Z_2(t+s)$	$\text{Spin}(4n) / Z_2(t+s)$

As before, we see that the adjoint class representations generate a group with unit centre, and that the various other classes generate different groups, all of which have a Z_2 centre. The odd rank tensors have $\text{SO}(4n)$ as their true group and therefore the well known feature of the spinor representations being double-valued representations of $\text{SO}(4n)$ still holds. Unlike the $\text{SO}(4n+2)$ case, however, the spinor representations are not faithful representations of the covering group, but rather form representations of groups with smaller centres. The result of this is that, for either of the spinor true groups, the odd rank tensors and the other spinor representations will be double valued representations. In order to get a faithful representation of $\text{Spin}(4n)$, we would have to consider group representations generated from reducible algebra representations consisting if direct sums of representations in different congruency classes.

In order to apply this to our model, we must define the q^{LM} matrix as before. As we are dealing with matrix representations, then the 'physical' q parameters for the various groups will be

$$q_{ts}^{\text{LM}} = M_t q_t^{\text{L1}} + M_s q_s^{\text{L1}} \pmod{2} \quad (2.9)$$

where we have explicitly written q^{LM} in the form Mq^{L1} for the two Z_2 factors and the q parameters on the right hand side of (2.9) can take the values 0 or 1. This can be

seen to be valid if we write out the q^{LM} matrix in the form of a table as follows

(M_t, M_s) (q_t^1, q_s^1)	(1,1)	(1,2)	(2,1)	(2,2)
(0,0)	0	0	0	0
(1,0)	1	1	0	0
(0,1)	1	0	1	0
(1,1)	0	1	1	0

where we can see that this contains the relevant information derivable from the previous table. The double valuedness of the various representations discussed earlier can be seen in this table. If, for example, we take the $SO(4n)$ case, then it can be seen that the two choices of L giving the central element represented by $q^{LM} = 1$ in this class will give different values of q^{LM} in either of the spinor classes. To illustrate this explicitly we shall examine $Spin(8)$, as $Spin(4)$ is isomorphic to the product group $SU(2) \times SU(2)$. The algebra has three distinct eight dimensional representations, each of which lies in a different congruency class. In the following table, we explicitly indicate the vectors L which generate the various centres and, for each class, we list the vectors which generate the centre element $q^{LM} = 1$.

highest weight	(1 0 0 0)	(0 0 0 1)	(0 0 1 0)
class	(1,2)	(2,1)	(1,1)
$q = 1$ configurations	(1 0 1 0) (1,0)	(0 0 1 1) (0,1)	(1 0 1 0) (1,0)
with (q_t^1, q_s^1) values	(1 0 0 1) (1,1)	(1 0 0 1) (1,1)	(0 0 1 1) (0,1)

This clearly illustrates the points made previously, i.e. that none of the classes generate the covering group and vectors L which are 'equivalent' in one non-adjoint class will not be equivalent in another.

Having now given a fairly general discussion of the relevant group theoretical features which will be relevant later, we now give an alternative description of the flux-breaking mechanism which has a more group theoretical slant and which illustrates the importance of the last two sections.

iii) Flux-breaking re-examined

In our previous discussion of the flux-breaking mechanism, we concentrated mainly on the topological features. In this section, we shall examine the group theory of the mechanism, where the previous discussion will be seen to be relevant. As was stated in chapter 1, the set of Wilson loops provides a homomorphism of the discrete group Γ into the gauge group, leaving the residual group $G' \times \tilde{\Gamma}$, where G' commutes with $\tilde{\Gamma}$, the image of Γ in the gauge group. On compactification, however, it is necessary to expand the fields in terms of the harmonics on the compact space and integrate over their co-ordinates to leave the fields as functions of the macroscopic space co-ordinates only. In this procedure, various terms will appear which correspond to mass terms for the modes of the original fields. As the only scale parameter in this process is the 'size' of the compact space (around the Planck length) then these mass terms will be of the order of the inverse Planck length and hence only the zero modes of the various fields will be observable at low energies. The introduction of a non-vanishing background field also contributes to these mass terms and therefore may result in certain fields having no massless modes at low energy.

If the Wilson loops map to the identity matrix of a given representation in the Lagrangian, then the symmetry will be unbroken and fields in such representations

will retain massless modes on compactification. If the Wilson loops map to other elements of the centre, then the fields contained in such a representation will not retain massless modes and therefore, even though all the group elements commute with the Wilson loops, the apparent low energy group will not reflect the full symmetry. As will become clear when we discuss the Hosotani model in detail, the apparent low energy group in such a case will appear to be the original group modulo the set of Wilson loops. This will appear to be a breaking of the centre of the group, hence the title of this thesis.

For the cases where the symmetry breaks, only the representations of G' in the branching in which the background field vanishes will appear as massless fields at low energies. The true residual group (determined by the representations which appear in the branching) is then replaced by the effective residual group (determined by the representations containing massless fields) at low energies. It becomes cumbersome at this stage to continue specifying exact symmetry groups and the standard procedure in the literature is to quote the covering group of the residual group along with the massless field content of the residual action.

One important advantage of this feature is that the massless representations of the broken symmetry must come from different representations of the original symmetry group. This then avoids any awkward relationships between the masses of particles which, although in different representations of the residual symmetry group G' , came from the same representation of the original symmetry group G and therefore obey certain G relationships. Such problems are therefore avoided if the surviving representations of the subgroup come from distinct representations of the original group.

CHAPTER 3

Effective potentials and the background field

i) Definition of the path integral

The path integral approach to quantum theory was first conceived by Feynman[26], where he defined the path integral and used it to derive quantum mechanics. We shall follow the discussion given in this reference, although it is more common to find the alternative approach of deriving the path integral formalism from quantum mechanics[27].

In classical mechanics, the usual approach for determining the path of a particle is the principle of least action, i.e. there exists a quantity called the action which can be evaluated for all possible paths, and whose minimum selects the preferred trajectory. The action can be defined by

$$S = \int_{t_1}^{t_2} L(x, \dot{x}, t) dt \quad (3.1)$$

where L is the Lagrangian of the system. By using the techniques of the calculus of variations, the equations describing the classical path can be derived

$$\frac{d}{dt} \left[\frac{\partial L}{\partial \dot{x}} \right] - \frac{\partial L}{\partial x} = 0 \quad (3.2)$$

which are the usual Lagrange's equations of motion.

In quantum mechanics we are interested in calculating various amplitudes, but we cannot immediately apply the above procedure as each possible path from the initial to the final state contributes to the amplitude. Each path will contribute an equal amount to the amplitude, but they will contribute at different phases with the phases being proportional to the action S , with the proportionality constant being ih^{-1} (h is Planck's constant). In order to make contact with the previous classical situation we note that, if the parameters in the action are large in comparison to h , then a small move from the classical path will result in an enormous change in the

action (when measured in terms of \hbar) and therefore the large oscillations which take place in the phase factor $e^{iS/\hbar}$ will result in cancellation of contributions from paths other than the classical one (as any contribution from a given non-classical path will be cancelled by that from another path situated an infinitesimal distance away). The contribution in the classical path will not be cancelled in this manner as it corresponds to an extremum of the action and hence will not be cancelled by infinitesimally close paths. It therefore only remains to define some procedure for performing the weighted sum over paths.

To show how this may be done, we note that the usual Riemann integral I of a function $f(x)$ is just the sum of f over all the values x . We may take a set of equispaced x values x_i with a separation a , where we let the number of points x_i go to infinity, i.e. let the spacing a go to zero. In this case, the integration becomes

$$I = \lim_{a \rightarrow 0} \left[a \sum_i f(x_i) \right] \quad (3.3)$$

It is necessary to introduce the spacing a as a normalisation factor, as the summation in (3.3) has no limit as $a \rightarrow 0$. (Viewing the integration as representing the area under a curve, we see that (3.3) is effectively summing the areas of a series of rectangles as the rectangle width goes to zero, in which case the reason for the appearance of the factor a is clear.) In an analogous manner, we may choose a set of N paths by choosing a set of space points x_i at equispaced times t_i between the initial and final times, and joining these points with straight lines. We may then define the sum over paths as a multiple integral over the x_i , remembering that no integration is to take place with respect to the x coordinates which label the fixed end points. With an appropriate normalising factor such as that in (3.3), we can write the path integral P as

$$P = \lim_{a \rightarrow 0} N(a) \int \dots \int \prod_i dx_i \exp \left\{ \frac{i}{\hbar} \int_{t_1}^{t_2} L(x, \dot{x}, t) dt \right\} \quad (3.4)$$

This technique may break down if parameters in the theory turn out to be discontinuous at the points x_i , which would then result in some alternative definition of the summation over paths, The basic idea, however, is still valid and we can write down a formal expression for the path integral

$$P = \int d[x] \exp \left\{ \frac{i}{\hbar} S \right\} \quad (3.5)$$

where the measure $d[x]$ is to be defined according to the nature of the application.

ii) Generating functionals for Green's functions

Having outlined the ideas behind the path integral, we shall now discuss the concept of the generating functionals for various types of Green's functions[28]. There are many conventions in the literature which may be adopted for this analysis, and the one we shall adopt here is that where the background metric of the spacetime is taken to be Euclidean. With this in mind, we define the first of our generating functionals

$$Z[J] = \int d[Q] \exp \{ - S(Q) + J.Q \} \quad (3.6)$$

In this expression, Q is a field with classical action S coupled to an external source J , and the quantity $d[Q]$ is an appropriate functional measure for the integral. We have suppressed all indices and spacetime coordinates, and in general the fields and sources may have Lorentz, spinor or group indices or any combination thereof. To illustrate the coordinate dependence, the source term in (3.6) actually represents

$$J.Q = \int d^d x J Q \quad (3.7)$$

where $d^d x$ is an appropriate invariant measure on the spacetime manifold. The functional $Z[J]$ is actually the vacuum transition amplitude in the presence of the source J and can be written

$$Z[J] = \langle 0|0 \rangle_J \quad (3.8)$$

The various n point Green's functions of the theory can be obtained by repeated functional differentiation of the generating functional $Z[J]$ with respect to the source parameter J

$$\begin{aligned} \langle 0|T \{Q \dots Q\} |0 \rangle &= \int d[Q] (Q \dots Q) \exp \{ -S(Q) \} \\ &= \frac{\delta^n}{\delta J^n} Z[J] \Big|_{J=0} \end{aligned} \quad (3.9)$$

It can be shown that $Z[J]$ generates disconnected Green's functions and therefore we require another functional which will only generate the connected Feynman graphs of the theory. Such a functional can be obtained by taking the functional logarithm of $Z[J]$

$$W[J] = \ln Z[J] \quad (3.10)$$

To see that this does indeed generate the connected components, we again functionally differentiate with respect to the source J

$$\begin{aligned} \frac{\delta W}{\delta J} &= \frac{\langle 0|Q|0 \rangle_J}{\langle 0|0 \rangle_J} \\ \frac{\delta^2 W}{\delta J^2} &= \left[\frac{\langle 0|T \{QQ\} |0 \rangle_J}{\langle 0|0 \rangle_J} - \left(\frac{\langle 0|Q|0 \rangle_J}{\langle 0|0 \rangle_J} \right)^2 \right] \end{aligned} \quad (3.11)$$

The first term in the second line is the full two point Green's function as would be obtained directly from $Z[J]$. The second term, however, removes the disconnected part and thus leaves us with the connected two point Green's function. Subsequent differentiation will lead to the higher Green' functions with the disconnected part being removed at each stage.

iii) The effective action and the background field

We can also define another generating functional called the effective action[28-30]. This functional will generate what are known as one particle irreducible (1PI) Feynman diagrams, i.e. diagrams which cannot be split into two distinct diagrams by the severing of an internal line. The connected Green's functions generated by $W[J]$ are not 1PI but can be constructed from the 1PI Green's functions which the effective action will generate. The effective action can be obtained from $W[J]$ by performing a functional Legendre transformation

$$\Gamma[\bar{Q}] = W[J] - J\bar{Q} \quad (3.12)$$

The quantity \bar{Q} is the vacuum expectation value of the field Q in the presence of the source J

$$\bar{Q} = \frac{\langle 0|Q|0\rangle_J}{\langle 0|0\rangle_J} = \frac{\delta W}{\delta J} \quad (3.13)$$

and therefore the effective action Γ is a functional of \bar{Q} only

$$\frac{\delta \Gamma}{\delta \bar{Q}} = -J \quad , \quad \frac{\delta \Gamma}{\delta J} = 0 \quad (3.14)$$

This equation is just the quantum analogue of the classical field equation

$$\frac{\delta S}{\delta Q} = -J \quad (3.15)$$

If we differentiate Γ twice with respect to the vev Q , we obtain

$$\frac{\delta^2 \Gamma}{\delta \bar{Q}^2} = -\frac{\delta J}{\delta \bar{Q}} = -\left(\frac{\delta^2 W}{\delta J^2}\right)^{-1} = -D^{-1} \quad (3.16)$$

where D^{-1} is the inverse propagator of the field Q . We may expand the effective action in powers of \bar{Q}

$$\Gamma[\bar{Q}] = \sum_n \frac{1}{n!} \int d^d x_1 \dots d^d x_n \Gamma^{(n)}(x_1, \dots, x_n) \bar{Q}_1 \dots \bar{Q}_n \quad (3.17)$$

where the $\Gamma^{(n)}$ are the n point 1PI Green's functions. Similar expansions can be made for the previous generating functionals, but they will be of no real interest to us in this work. Whereas differentiating W with respect to J effectively added an extra external line to the Green's functions, differentiating Γ with respect to \bar{Q} adds an extra external line and removes a propagator from that line. We can also see that the two point 1PI Green's function is just the inverse propagator for the field Q . The functional Γ therefore generates all the vertices of the theory from which all other Green's functions may be constructed, hence the reasoning behind the name 'effective action'.

Having gone this far into the functional approach, it is now appropriate to revise the above analysis for the case where the field Q has a non-vanishing classical background B and see where the connections can be made with the previous quantities[31-33]. We can define a generating functional corresponding to the Z functional given earlier, but this time having the background B as an argument. This gives

$$\tilde{Z}[J, B] = \int d[Q] \exp \{ -S(Q+B) + J \cdot Q \} \quad (3.18)$$

where the background field B is not coupled to the source J (therefore it does not appear as a propagating field) and can itself be regarded as an external source. The physical interpretation of this quantity is exactly as before, and we can therefore define the generating functional for connected Feynman diagrams

$$\tilde{W}[J, B] = \ln \tilde{Z}[J, B] \quad (3.19)$$

If we now define the vev \tilde{Q} of the quantum field Q by

$$\tilde{Q} = \frac{\delta \tilde{W}}{\delta J} \quad (3.20)$$

then we can define the background field effective action by the following functional Legendre transformation

$$\tilde{\Gamma}[\tilde{Q}, B] = \tilde{W}[J, B] - J \cdot \tilde{Q} \quad (3.21)$$

To relate these to the previous expressions, we can shift the \tilde{Q} field by $\tilde{Q} \rightarrow \tilde{Q} - B$, which then gives us

$$\begin{aligned} \tilde{Z}[J, B] &= Z[J] e^{-J \cdot B} \\ \tilde{W}[J, B] &= W[J] - J \cdot B \end{aligned} \quad (3.22)$$

which further illustrates the point that B is to be considered as an external source term. The implication of this, which may have been intuitively expected, is that the vacuum expectation values of the Q fields in the two cases are simply related

$$\bar{Q} = \tilde{Q} + B \quad (3.23)$$

We also have

$$\begin{aligned} \bar{\Gamma}[\tilde{Q}, B] &= W[J] - J.B - J.\bar{Q} + J.B \\ &= \Gamma[\bar{Q}] \end{aligned} \quad (3.24)$$

Combining the two previous equations gives us the relationship between the two effective actions

$$\bar{\Gamma}[\tilde{Q}, B] = \Gamma[\tilde{Q} + B] \quad (3.25)$$

Taking the vanishing source case, which implies that $\tilde{Q} = 0$, this becomes

$$\bar{\Gamma}[0, B] = \Gamma[B] \quad (3.26)$$

which tells us that the effective action for a theory with a quantum field expanded about a classical background is the same as that for a theory where the field is taken as having non-vanishing expectation value. This may seem a trivial result, but the two quantities in (3.26) are calculated in very different manners and in cases where the calculation of one of these effective actions is very difficult, the other effective action may be obtainable with much greater ease.

If we consider the background effective action Γ then we can obviously see that $\Gamma[0, B]$ is independent of \tilde{Q} and therefore generates legless diagrams (vacuum graphs). The propagators of the fields in these vacuum graphs are obtained from the shifted action $S(Q+B)$ and therefore have a dependence on the background field B . To calculate the effective action in this case, we evaluate Feynman diagrams which have no external lines and use the corrected propagators which can be derived from

the shifted action. If we consider the original effective action, then we do not have a classical and a quantum field, but only a single field with a non-vanishing expectation value. Γ therefore generates Feynman diagrams whose internal lines are represented by the usual propagators derived from the action $S(Q)$, but which now have external lines corresponding to the non-vanishing vev B .

In general, the calculation of $\Gamma[B]$ can usually be performed in cases where the calculation of $\Gamma[0,B]$ is far too difficult but, in simple cases, it occasionally proves to be more convenient to determine the corrected propagators and hence calculate $\Gamma[0,B]$ and this is the approach we shall adopt in chapter 4. We shall now go on to discuss the concept of the effective potential.

iv) The effective potential and the loop expansion

In (3.17), we effectively expanded the effective action $\Gamma[B]$ in terms of B . We may also expand Γ in powers of the derivatives of B , giving

$$\Gamma[B] = \int d^d x [- V(B) + \text{higher derivative terms}] \quad (3.27)$$

This is just the configuration space equivalent of the expansion of 1PI Green's functions around zero external momenta[2]. The quantity $V(B)$ is the effective potential density, which is just the sum of all 1PI diagrams with zero external momenta. If, as will be the case in the Hosotani model, the background field B is a constant, then the higher derivative terms in (3.27) will vanish and the effective action will therefore reduce to the effective potential. This quantity is analogous to the statistical mechanical free energy F (and can for all intents and purposes be identified with the free energy of systems in which gravitational effects are being neglected[34]), therefore the preferred background field B should be that which minimises the effective potential. (Note that in all subsequent discussions, the source

J is being set to zero.)

We shall not be dealing with the regularisation of the theory in any great detail, but we should point out that we can calculate the regularised (mass)² of the quantum field by functional differentiation of the effective potential. From equation (3.16) we see that the two point 1PI Green's function $\Gamma^{(2)}$ is effectively the inverse propagator for the field. As we can also expand the effective potential in terms of the 1PI Green's functions with zero external momenta

$$V[B] = - \sum_n \frac{1}{n!} \Gamma^{(n)}(0, \dots, 0) B^n \quad (3.28)$$

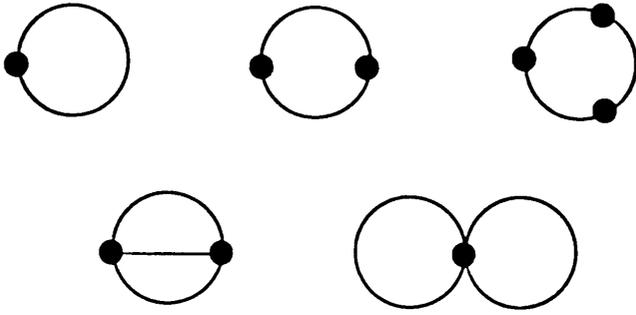
then we have

$$\left. \frac{\delta^2 V}{\delta B^2} \right|_{B=0} = - \Gamma^{(2)}(0,0) = m^2 \quad (3.29)$$

This relationship will turn out to be very important in subsequent chapters.

Although we have now outlined the method by which the background field B should be selected, it turns out that some perturbative scheme is usually required to calculate the effective potential. We shall now outline such an expansion scheme [2,31], and shall explain why it is preferable to the usual perturbation expansion in terms of the coupling constant. To begin, we introduce a parameter c^{-1} as a coefficient of the entire action of the system. By equations (3.16) and (3.17) we see that each vertex will be associated with a factor c^{-1} whereas each propagator will be associated with a factor c . Irreducible tree level diagrams (diagrams with no loops) will have one vertex and therefore will be of order c^{-1} . Diagrams with one internal loop will have the same number of vertices as internal lines and therefore will be of order $c^{-1}c = 1$. For two loops, there will be one more internal line than the number of vertices (in order to give an extra loop) giving a contribution of order $c^{-1}c^2 = c$. This is illustrated below for some possible vacuum diagrams in some

hypothetical theory with quadratic, cubic and quartic interactions.



In general, the number of vertices V in a diagram with L loops and P propagators is $V = P - L + 1$ hence giving a factor $(c^{-1})^{L - P + 1} c^L = c^{L - 1}$. It can therefore be seen that an expansion of the action in terms of this parameter c is equivalent to an expansion in terms of the number of loops of the various 1PI diagrams, hence the reason why this is referred to as the loop expansion. There is no requirement at any stage that the expansion parameter c be small, but it is common to find it being identified with Planck's constant \hbar .

It should be noted that the loop expansion is in general distinct from the usual expansion in powers of the coupling constant. The reason for this is that the coupling constant is only associated with the interaction terms in the Lagrangian whereas the loop parameter is associated with the entire Lagrangian. The coupling constant is therefore associated with the vertices and not the number of loops. (This can easily be seen in the previous diagrams.) Any given term in the loop expansion can therefore be expanded perturbatively in terms of the coupling constant. The reasoning behind the difference between the two expansions is precisely the reason why the loop expansion is to be preferred. As the loop parameter c is a coefficient of the entire Lagrangian, then the shifting of the fields as given in (3.23) leaves each term in the loop expansion unaffected - graphs which contain n loops before the shift will also contain n loops after the shift thus making this approach much less awkward. Another advantage is that, when adopting the corrected propagator approach as detailed earlier, the one loop contribution may be expressed in a simple

manner, unlike the higher loop contributions[35]. To determine this form we first of all note that the action $S(Q+B)$ can be expanded in powers of the quantum field

$$S(Q+B) = S(B) + \frac{1}{2} Q D^{-1} Q + S^{(3)}(Q,B) \quad (3.30)$$

where $S^{(3)}$ contains terms which are at least cubic in the quantum field Q . This then gives us the partition function

$$Z[J, B] = \exp \{-c^{-1} S(B)\} \int d[Q] \exp \{-c^{-1} [\frac{1}{2} Q D^{-1} Q + S^{(3)}(Q,B) - J \cdot Q]\} \quad (3.31)$$

where we have explicitly introduced the loop parameter c . For notational convenience, we shall denote the quadratic functional integral by

$$\int d[Q] \exp \{-\frac{1}{2c} Q D^{-1} Q\} = F(D^{-1}) \quad (3.32)$$

and shall return to its evaluation once we have discussed the application of this technique to gauge theories. We therefore rewrite (3.31) as

$$Z[J, B] = \exp \{-c^{-1} S(B)\} F(D^{-1}) Z_2[J, B] \quad (3.33)$$

where

$$Z_2[J, B] = \frac{\int d[Q] \exp \{-c^{-1} [Q D^{-1} Q + S^{(3)}(Q,B) - J \cdot Q]\}}{F(D^{-1})} \quad (3.34)$$

If we now rescale the quantum field by $Q \rightarrow c^{1/2} Q$, then we can easily see that $F(D^{-1})$ will become independent of c , and that $S^{(3)}$ will become at least of order c^2

...g the field

(as fractional powers of the loop parameter do not have any physical meaning). We therefore remove the global c^{-1} factor from the exponents in the partition function and, as before, define the generating functional for connected Green's functions

$$W[J, B] = -S(B) + c \ln F(D^{-1}) - W_2[J, B]$$

$$W_2[J, B] = \ln Z_2[J, B] \quad (3.35)$$

where $F(D^{-1})$ is as defined in (3.32) but with the loop parameter removed. $W_2[J, B]$ generates connected graphs from the action $1/2(QD^{-1}Q) + S^{(3)}(Q, B)$ and, as was mentioned above, is at least quadratic in the loop parameter. This therefore implies that in (3.35) we have explicitly separated out the tree level term $S(B)$ and the one loop term $\ln F(D^{-1})$ from the generating functional. By Legendre transforming (3.35) we find the tree level effective action

$$\Gamma_0 = -S(B) \quad (3.36)$$

and the one loop effective action

$$\Gamma_1 = c \ln F(D^{-1}) \quad (3.37)$$

The zero and one loop effective potential terms for the field Q with classical background B are therefore

$$V_0 = V(B)$$

$$V_1 = -c \ln F(D^{-1}) \quad (3.38)$$

where $V(B)$ is the classical potential term. The renormalised (mass)² parameter can be found using the identity

$$\frac{\delta^2 V_1}{\delta B^2} = m^2 \quad (3.39)$$

Before dealing with the evaluation of F , we shall now apply the above formalism to the case of relevance to the remainder of this work, i.e. to gauge theories.

v) Application to gauge theories

The discussion up to now has been completely general, i.e. we have not indicated which type of field we are giving a non-zero background. As the flux breaking mechanism relies on a non-trivial background gauge field, we shall now give a more specific discussion of the background field effective action, paying particular attention to the effects of gauge transformations on both the quantum and background fields[31]. The partition function for a pure gauge theory can be written as

$$Z[J] = \int d[Q] \det \left[\frac{\delta G^a}{\delta \omega^b} \right] \exp \left\{ - S[Q] - \frac{1}{2\alpha} G \cdot G + J \cdot Q \right\} \quad (3.40)$$

where Q represents the gauge field Q^a_μ and G^a is the gauge fixing term. (Note that we have again suppressed all indices and coordinates in this equation.) The various expressions in (3.40) are

$$S(Q) = \frac{1}{4} \int d^d x F_{\mu\nu}^a F^{a\mu\nu}$$

$$F_{\mu\nu}^a = \partial_\mu Q_\nu^a - \partial_\nu Q_\mu^a + g f^{abc} Q_\mu^b Q_\nu^c$$

$$J.Q + \int d^d x J_\mu^a \frac{\delta Q^a}{\delta x} G^a$$

$$G.G = \int d^d x G^a G^a \quad (3.41)$$

The Faddeev-Popov determinant removes a volume factor from the measure, i.e. removes the contribution from non-physical degrees of freedom of the gauge fields (longitudinal and scalar components). The derivative in the determinant is the functional derivative of the gauge fixing term with respect to the infinitesimal gauge transformation

$$Q_\mu^a \rightarrow Q_\mu^a + f^{abc} \omega^b Q_\mu^c - \frac{1}{g} \partial_\mu \omega^a \quad (3.42)$$

We also have the background field form of the partition function

$$\tilde{Z}[J, B] = \int d[Q] \det \left[\frac{\delta \tilde{G}^a}{\delta \omega^b} \right] \exp \left\{ -S(Q+B) - \frac{1}{2\alpha} \tilde{G} \cdot \tilde{G} + J.Q \right\} \quad (3.43)$$

where the relevant gauge transformation on the integration variable Q for the Faddeev-Popov determinant can now mix between Q and B

$$Q_\mu^a \rightarrow Q_\mu^a + f^{abc} \omega^b (Q+B)_\mu^c - \frac{1}{g} \partial_\mu \omega^a \quad (3.44)$$

We can proceed as before to derive the effective action via the W generating functionals and make the usual identification

$$\tilde{\Gamma}[0, B] = \Gamma[B] \quad (3.45)$$

provided

$$G^a(Q) = \tilde{G}^a(Q - B, B) \Big|_{\bar{Q} = B} \quad (3.46)$$

i.e. $\exists G^a$ for which $\Gamma[0, B]$ is a gauge invariant functional of B , a condition which imposes a severe restriction on the possible choices of gauge fixing. The most usual choice, which is commonly known as the background gauge, is

$$\tilde{G}^a = \partial_\mu Q_\mu^a + g f^{abc} B_\mu^b Q_\mu^c \quad (3.47)$$

\tilde{G}^a is just the background field covariant derivative acting on the quantum field. This choice of gauge fixing results in the partition function (3.43) being invariant under the transformations

$$B_\mu^a \rightarrow B_\mu^a + f^{abc} \omega^b B_\mu^c - \frac{1}{g} \partial_\mu \omega^a \quad (3.48)$$

$$J_\mu^a \rightarrow J_\mu^a + f^{abc} \omega^b J_\mu^c \quad (3.49)$$

If we simultaneously perform a change of variables for the quantum fields

$$Q_\mu^a \rightarrow Q_\mu^a + f^{abc} \omega^b Q_\mu^c \quad (3.50)$$

then we see that equations (3.49) and (3.50) represent adjoint rotations and hence leave the term $J \cdot Q$ invariant. Adding (3.48) and (3.50) gives us

$$(Q+B)_\mu^a \rightarrow (Q+B)_\mu^a + f^{abc} \omega^b (Q+B)_\mu^c - \frac{1}{g} \partial_\mu \omega^a \quad (3.51)$$

which is just a gauge transformation on the field variable $Q+B$ and as such leaves the action $S(Q+B)$ invariant. As the gauge fixing term is just the background field

covariant derivative of Q then it is invariant under the adjoint rotation (3.50) and the gauge transformation (3.48). Likewise, the Faddeev-Popov determinant is also invariant. As the quantum vev \tilde{Q} is conjugate to J then the effective action $\Gamma[\tilde{Q},B]$ must be invariant under (3.48) and (3.50). As (3.50) is a homogeneous transformation of the quantum field Q , it is clear that $\Gamma[0,B]$ is invariant under (3.48) alone, i.e. it is a gauge invariant functional of the background field.

Having attained our goal of finding a suitable background field formalism for gauge theories, we can now proceed with the determination of the function F defined earlier, and also find the one loop contributions to the effective potential from matter fields which couple to the gauge field and hence to the background B . In order to calculate F , we require the corrected gauge field propagator, i.e. we require the terms in the exponent of Z which are quadratic in the quantum field. The expansion in terms of Q is extremely tedious, resulting in the appropriate expression for F [36]

$$F = \int d[Q] \exp \left\{ \frac{1}{2} Q^{a\mu} \left[-g_{\mu\nu} D^2(B) + \left(1 - \frac{1}{\alpha}\right) D_\mu(B) D_\nu(B) \right. \right. \\ \left. \left. + R_{\mu\nu} - 2F_{\mu\nu}(B) \right] Q^{a\nu} \right\} \quad (3.52)$$

where $g_{\mu\nu}$ is the metric, $R_{\mu\nu}$ is the Ricci tensor and α is the gauge fixing parameter. The term in brackets is the corrected inverse propagator $(\Delta_{\mu\nu})^{-1}$. By analogy with the standard result for Gaussian integrals

$$\int \frac{dx}{2\pi} \exp \{ -a x^2 \} = \frac{1}{\sqrt{4\pi a}} \quad (3.53)$$

and its matrix equivalent

$$\int \frac{dx}{2\pi} \exp (x, Ax) = \frac{1}{\sqrt{4\pi \det (A)}} \quad (3.54)$$

where (x, Ax) represents the matrix product, we have the path integral equivalent for bosonic fields

$$\int d[Q] \exp \left\{ -\frac{1}{2} Q M Q \right\} = \frac{1}{\sqrt{\det M}} \quad (3.55)$$

where the determinant is taken with respect to all suppressed indices. Using this formula, equations (3.38) and (3.52) then give us the one loop contribution to the effective potential from the gauge fields alone

$$(V_1)_{\text{gauge}} = -\frac{1}{2} \ln \det (\Delta_{\mu\nu}^{-1}) \quad (3.56)$$

where it should be remembered that the inverse propagator is a matrix in the adjoint representation. While we are dealing with Bose fields, we shall discuss the contribution to the potential from scalar fields which are minimally coupled to the gauge fields. As before, we can separate out the background and quantum gauge fields and expand the scalar term in the action in powers of the loop parameter c . Rescaling all quantum fields by a factor $c^{1/2}$ shows that, as before, the only term contributing to the one loop potential is the term which is quadratic in the quantum fields. This therefore implies that we need only find the corrected propagator for the scalars. If we have real scalars, then we may use (3.54) as it stands, whereas the appropriate identity for complex fields is

$$\int d[\bar{Q} Q] \exp \left\{ -\bar{Q} M Q \right\} = \frac{1}{\det M} \quad (3.57)$$

As the inverted scalar propagator is just the background field covariant derivative, we have the complex scalar contribution to the one loop effective potential

$$(V_1)_{\text{scalar}} = -\ln \det (-D^2(B) + m^2) \quad (3.58)$$

This term must be multiplied by a factor of 1/2 if the scalars are taken to be real.

For the cases where there are fermionic fields, the above functional integration identities must be modified. It can be shown that the identities corresponding to (3.54) and (3.57) for Grassmann variables are

$$\int d[\psi] \exp \left\{ -\frac{1}{2} \psi M \psi \right\} = \sqrt{\det M} \quad (3.59)$$

if ψ real, and

$$\int d[\bar{\psi} \psi] \exp \left\{ -\bar{\psi} M \psi \right\} = \det M \quad (3.60)$$

if ψ complex. If we have Dirac fermions minimally coupled to the gauge fields then we make use of (3.60) for the quadratic term in the fermion action, giving us

$$\begin{aligned} (V_1)_{\text{Dirac}} &= \ln. \det (S_F^{-1}) = \ln. \det (\gamma^\mu \nabla_\mu + m^2) \\ &= \frac{1}{2} \ln. \det (-D^2(B) + \frac{R}{4} + m^2 - \frac{1}{2} \gamma^\mu \gamma^\nu F_{\mu\nu}(B)) \end{aligned} \quad (3.61)$$

where ∇ is the full covariant derivative, including the spin connection. Note that the Fermi contribution has a different sign from the Bose terms. For all matter contributions, the propagators are to be taken as matrices in the relevant representations. The only remaining contribution we require is that from the Faddeev-Popov determinant. This determinant can, by using identities similar to those given above, be written in terms of the contribution from spinless complex fermionic fields known as Faddeev-Popov ghosts. These fields only appear as internal lines in Feynman diagrams and remove the contribution from the unphysical degrees of freedom in the gauge fields which is present in (3.40). With the choice of gauge fixing term (3.47), we find that the inverse propagator is

$$\frac{\delta G}{\delta \omega} \Big|_{Q=0} = -D^2(B) \quad (3.62)$$

and therefore the ghost contribution to the one loop potential is

$$(V_1)_{\text{ghost}} = \text{ln.det} (-D^2(B)) \quad (3.63)$$

where the covariant derivative is obviously in the adjoint representation.

We have therefore derived all the necessary contributions to the one loop potential which will be relevant in our analysis of the Hosotani model. It is important to remember that the determinants apply to all suppressed indices, and that for notational convenience we have suppressed the configuration and momentum space coordinates throughout. In the calculation of the effective potential for the Hosotani model in chapter 4, we shall express the various propagators in momentum space, remembering to integrate over all possible loop momenta.

CHAPTER 4

Effective potential for the generalised Hosotani model

i) Description of the model

After the preparatory work in the previous chapters, we are now in a position to examine a particular model. The simplest and most obvious choice for the multiply connected compact space is the circle S^1 , which is isomorphic to R/Z , where R is the real line and Z is the additive group of integers. This quotient just implies periodic identification of points on R , and is multiply connected with first homotopy group $\pi_1(S^1) = Z$. Putting this more simply, any closed curve on S^1 will wind round the circle a certain number of times and cannot be continuously deformed to give a curve with a different winding number. In terms of the covering space R , this corresponds to sections of a line with end points identified under the action of an element z_1 of Z not being deformable into a line whose endpoints are related by the action of a different element z_2 . In this case, the set Γ of Wilson loop operators provides a mapping from the divisor group Z into the gauge group G . (Note that the circle is isomorphic to $U(1)$ and hence is not a coset space of the form described in chapter 1. Any symmetry breaking will therefore be due to the Wilson loops alone.)

As was stated in chapter 1, it would be preferable to employ some dynamical principle to determine the background field and hence Γ and, in this simple case, we can achieve this by minimising the effective potential V expanded to some chosen order in the coupling constant. The very nature of the loop expansion implies that we have a small loop parameter ($\ll 1$) therefore we shall calculate the effective potential to first order and shall assume that this dominant term determines the correct background field. This calculation appears in many forms in the literature, usually for the case $R^3 \times S^1$ where the techniques have been developed for finite temperature gauge theories where the circular dimension represents Euclidean time (temperature)[37-45]. Although the interpretation of the extra dimension is markedly different here, the procedure used is still perfectly valid. In the version to be presented here, I have expanded and modified the calculation as presented in reference [39] in order to generalise the spacetime dimension from 3 to $m \geq 3$. This

has been done independently by Actor for the case of scalar fields[46]. An alternative and original derivation is also given which derives the potential in terms of the polylogarithm function[47,48]. The resulting one loop potential is therefore presented in an unconventional form, but one which is considerably more conducive to the detailed analytic examination which is to be presented in subsequent chapters. We shall derive the contributions from gauge fields, Faddeev-Popov ghosts, Dirac fermions and complex scalars, though we shall not commit ourselves to a particular Lagrangian at this stage.

ii) Parametrisation of the background field

It is necessary to find a form for the gauge inequivalent configurations which will allow a convenient parametrisation. The analysis here will be performed in a similar manner to that done in reference [43] for the simple case of SU(2). The first thing to note is that, as R^m is simply connected, we are able to gauge away the background field components in R^m leaving a non-vanishing component on the circle.

We first perform a gauge transformation on B_μ with the group element

$$g(x) = P \exp \left\{ \int_{|x_j|}^0 dy_i B_i(y) \right\} \quad (4.1)$$

The integral is path ordered along any path in R^m connecting the point (x_j) with the origin. The transformed field is

$$B_\mu \rightarrow g^{-1} B_\mu g + g^{-1} \partial_\mu g \quad (4.2)$$

If we examine the R^m components of the transformed field, we find

$$B_j(x) \rightarrow g^{-1} B_j(x) + g^{-1} (-B_j(x)) = 0 \quad (4.3)$$

i.e. the transformed field only has components in the circular dimension. Furthermore, as we are dealing with vacuum fields, we must satisfy $F_{\mu\nu} = 0$. This therefore gives

$$\partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu] = 0 \quad (4.4)$$

this is trivially satisfied by the macroscopic space components and, when we take into account the fact that $B_\mu = B_S \delta_{\mu S}$, we find

$$F_{iS}(B) = \partial_i B_S = 0 \quad (4.5)$$

which implies that the remaining component of the background field is a function of the circular co-ordinate only. We have therefore transformed a general background field $B_\mu(x_\mu)$ into the form $B_S(x_S)$. We can simplify this further by eliminating the residual gauge degrees of freedom which depend on x_S .

As we are on a circle, the gauge fields obey periodic boundary conditions (as there are terms in the action which are linear in the gauge fields, periodic boundary conditions are necessary for the action to be single-valued) and hence any valid gauge transformation should also be periodic. This is also necessary in order to preserve the boundary conditions on any matter fields which may be present. (A discussion of an alternative viewpoint will be presented in the next chapter.) We may therefore define the Θ matrix, which is an element of the gauge algebra, by

$$\exp \{ i\Theta \} = P \exp \left\{ \int_{\beta}^0 dx_S B_S(x_S) \right\} \quad (4.6)$$

where β is the circumference of the circle. This is actually an element of the set of Wilson operators, and all other Wilson loops can be generated from this element

simply by raising it to the n 'th power ($n \in \mathbb{Z}$ is the winding number of the loop).
Acting on B_S with the transformation matrix

$$g(x_S) = P \exp \left\{ \int_{x_S}^0 dy_S B_S(y_S) - \frac{i \Theta x_S}{\beta} \right\} \quad (4.7)$$

gives the transformed field

$$B_S \rightarrow g^{-1} B_S(x_S) g - g^{-1} \left\{ - B_S(x_S) - \frac{i \Theta}{\beta} \right\} g = - \frac{i \Theta}{\beta} \quad (4.8)$$

i.e. we have gauge transformed the vacuum field into a constant component in the circular direction. We may now diagonalise this matrix by the similarity transformations, i.e. by the action of constant elements of the gauge group, thus gauging the background field into the Cartan subalgebra. We can therefore write B_S (or more conveniently, Θ) in terms of the Cartan subalgebra generators

$$\Theta = i \beta B_S^a H^a \quad (4.9)$$

These are still not gauge invariant. If we now consider further gauge transformations which preserve the form (4.9)

$$g = \exp \{ c^a(x_S) H^a \} \quad (4.10)$$

then we have

$$\begin{aligned} g^{-1} \partial_S g &= B_S - \hat{B}_S \\ \Rightarrow g &= g_0 \exp \{ x_S (B_S - \hat{B}_S) \} \end{aligned} \quad (4.11)$$

imposing the periodicity condition, and writing the resulting equation in terms of the

Θ matrix gives

$$\exp \{ -i (\Theta - \hat{\Theta}) \} = 1 \quad (4.12)$$

This implies that two background fields with Θ matrix elements equal modulo 2π are gauge equivalent, i.e. both give the same set of Wilson loops.

We shall be interested in the eigenvalues of the Θ matrix as it acts on various states, and this obviously leads us to write Θ in terms of the weights of the appropriate representation as detailed in chapter 2. As the generators are anti-Hermitian, the eigenvalues of the H^a will be $-i \lambda_j^a$, where j runs from 1 to the dimension of the representation. If we change to the Dynkin basis, these λ will be integer, with any normalisation factors incurred being absorbed into the arbitrary B_S^a coefficients. The products $B_S^a \lambda_j^a$, and hence the Θ matrix, will be unaffected by this manipulation. We have therefore obtained the final form of the background vacuum field, i.e. as a constant circular component of the field lying in the Cartan subalgebra. The Θ matrix elements are now

$$\Theta_{jk} = \beta B_S^a \lambda_j^a \delta_{jk} \quad (4.13)$$

where $\Theta_{jk} \in [0, 2\pi]$, $\forall j,k$.

We have now achieved our objective, i.e. a parametrisation of the background field in terms of the Θ matrix, and may now proceed with the derivation of the one loop effective potential in terms of the Θ matrices of the various representations.

iii) Calculation of the one loop potential

As was described in chapter 3, the one loop effective potential can be expressed in terms of functional determinants of the various inverse propagators with

coefficients depending on the nature of the field in question. Referring to equation (), we have to evaluate expressions of the form

$$\int \frac{d^{m+1}k}{(2\pi)^{m+1}} \frac{1}{2} \ln.\det (- \eta_{\mu\nu} D^2 (B)^{ab}) \quad \text{for gauge fields}$$

$$\int \frac{d^{m+1}k}{(2\pi)^{m+1}} - \ln.\det (- D^2(B)^{ab}) \quad \text{for F-P ghosts}$$

$$\int \frac{d^{m+1}k}{(2\pi)^{m+1}} - N_F \ln.\det (\gamma_{\mu}^{\alpha\beta} D_{\mu} (B)^{ab}) \quad \text{for Dirac fermions}$$

and

$$\int \frac{d^{m+1}k}{(2\pi)^{m+1}} N_S \ln.\det (- D^2 (B)^{ab}) \quad \text{for complex scalars} \quad (4.14)$$

The a,b indices are group indices for the matrices in the appropriate representation, i.e. adjoint for the gauge and ghost terms and in a general representation for the matter fields. The μ, ν indices are the usual Lorentz indices, and α, β are spinor indices. These indices have been explicitly indicated in these expressions as the determinant is taken over all indices, not just the group matrices.

The evaluation of these contributions is performed using the same technique in each case, therefore we shall illustrate the procedure for one case only, the Dirac fermion contribution. As it stands, it is the only one not to have D^2 terms in the determinant, so we start by transforming it into the same basic form as the others. One added complication is that, while the gauge and ghost fields obey periodic boundary conditions, we may impose more general boundary conditions on the matter fields. We shall therefore incorporate a general phase factor δ via

$$\Phi(x_S + \beta) = e^{+i\delta} \Phi(x_S) \quad (4.15)$$

and we shall delay a discussion of the possible values of δ until later.

We wish to evaluate

$$V_F = -N_F \int \frac{d^{m+1}k}{(2\pi)^{m+1}} \ln.\det (\gamma_\mu^{\alpha\beta} D_\mu (B)^{ab}) \quad (4.16)$$

The eigenvalues of the normal derivative ∂_μ can be obtained by expanding ψ in terms of the harmonics on the circle, i.e. as a Fourier series in x_S with the summation over discrete momenta $k_S = 2\pi n/\beta$, $n \in \mathbb{Z}$.

$$\psi(x_i, x_S) = \sum_{k_S} \psi_{k_S}(x_i) \exp \left\{ -i x_S \left(k_S + \frac{\delta}{\beta} \right) \right\} \quad (4.17)$$

This satisfies (4.15) as $\exp(-i\beta k_S) = 1$. We therefore have

$$\partial_S \rightarrow -i k_S - \frac{i\delta}{\beta} \quad , \quad \partial_\mu \rightarrow k_\mu \quad (4.18)$$

Coupled with our previous discussion of the background field and the Θ matrix, equation (4.16) becomes

$$V_F = -N_F \int \frac{d^{m+1}k}{(2\pi)^{m+1}} \ln.\det \left[-i \gamma_\mu \left\{ k_\mu + \frac{1}{\beta} \delta_{\mu S} (\Theta_R + \delta) \right\} \right] \quad (4.19)$$

where Θ_R is the Θ matrix in the fermion representation R and where we have now suppressed group and spinor indices. We may now manipulate this expression as follows

$$\begin{aligned}
& \ln.\det \left[-i \gamma_\mu \left\{ k_\mu + \frac{1}{\beta} \delta_{\mu S} (\Theta_R + \delta) \right\} \right] \\
&= \text{Tr}.\ln \left[-i \gamma_\mu \left\{ k_\mu + \frac{1}{\beta} \delta_{\mu S} (\Theta_R + \delta) \right\} \right] \\
&= \frac{1}{2} \text{Tr}.\ln \left[-\gamma_\mu \gamma_\nu \left\{ k_\mu + \frac{1}{\beta} \delta_{\mu S} (\Theta_R + \delta) \right\} \left\{ k_\nu + \frac{1}{\beta} \delta_{\nu S} (\Theta_R + \delta) \right\} \right]
\end{aligned} \tag{4.20}$$

where the trace is over group and spinor indices. Using the fact that

$$\{ \gamma_\mu, \gamma_\nu \} = 2\eta_{\mu\nu} \tag{4.21}$$

we have

$$\text{Tr} (\gamma_\mu \gamma_\nu) = \text{Tr} (I \eta_{\mu\nu}) \tag{4.22}$$

where I is the unit matrix in the spinor space. As the dimension of spinors in d dimensions is $2^{\lfloor d/2 \rfloor}$ where the square brackets refer to the integer part, we can take the trace over spinor indices in (4.20) to give

$$\frac{1}{2} 2^{\lfloor (m+1)/2 \rfloor} \text{Tr}.\ln \left[\mathbf{k}^2 + \left\{ k_S + \frac{1}{\beta} (\Theta_R + \delta) \right\}^2 \right] \tag{4.23}$$

The quantity \mathbf{k} is the magnitude of the momentum m -vector in the macroscopic space.

We should now note that the use of an integration sign in equation (4.16) is purely symbolic, as we should actually be summing over the discrete momenta on the circle, i.e.

$$\int \frac{d^{m+1} \mathbf{k}}{(2\pi)^{m+1}} f(\mathbf{k}) \rightarrow \int \frac{d^m \mathbf{k}}{(2\pi)^m} \frac{1}{\beta} \sum_{k_S} f(k_1, k_S) \tag{4.24}$$

However the summation in this case is divergent, and we must adopt some regularisation procedure in order to proceed with the evaluation of V_F . To do this we make use of a summation formula akin to that of Plana, which re-expresses the divergent sum as a contour integral in the complex k_S plane. We can rewrite the summation in (4.24) as

$$\frac{1}{\beta} \sum_{k_S} f(k_S) = \frac{1}{4\pi} \int_C dz f(z) \cot\left(\frac{\beta z}{2}\right) \quad (4.25)$$

The contour C encloses the real axis in an anti-clockwise. This expression is valid for any function f which is analytic in a neighbourhood of the real axis.

[Proof :

$$\begin{aligned} \frac{1}{\beta} \sum_{k_S} f(k_S) &= \frac{1}{4\pi} \int_C dz f(z) \cot\left(\frac{\beta z}{2}\right) \\ &= \frac{1}{4\pi i} \int_C dz f(z) \frac{\cos\left(\frac{\beta z}{2}\right)}{\sin\left(\frac{\beta z}{2}\right)} \end{aligned}$$

which has simple poles at $z = 2\pi n / \beta = k_S$ along the real axis. The residues are

$$\begin{aligned} \lim_{z \rightarrow k_S} \frac{f(z) (z - k_S) \cos\left(\frac{\beta z}{2}\right)}{\sin\left(\frac{\beta z}{2}\right)} \\ = \lim_{z \rightarrow k_S} \frac{f(z) \cos\left(\frac{\beta z}{2}\right)}{\frac{\beta}{2} \cos\left(\frac{\beta z}{2}\right)} \quad \text{by L'Hôpital's rule} \end{aligned}$$

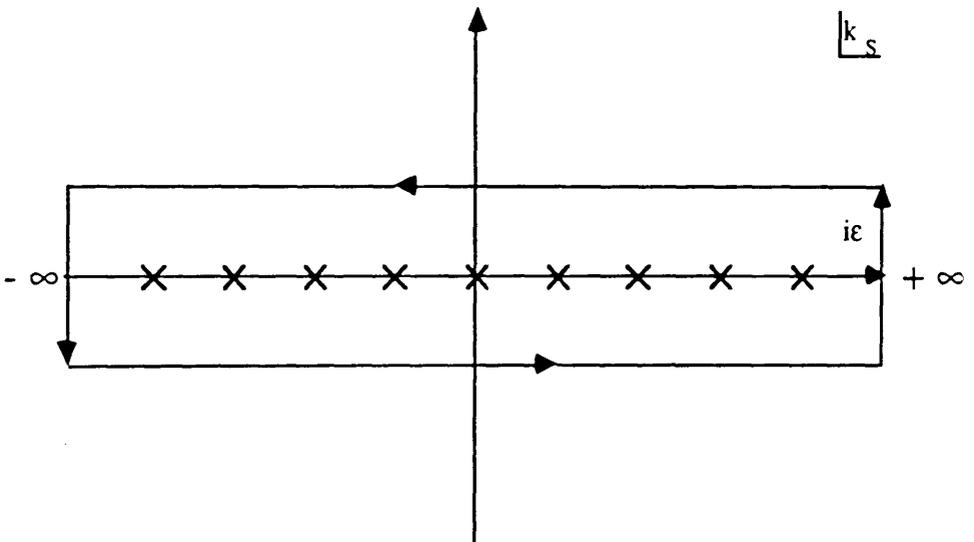
$$= \frac{2}{\beta} f(k_S)$$

The integral therefore becomes

$$2\pi i \sum_{k_S} \frac{1}{4\pi i} \frac{2}{\beta} f(k_S) = \frac{1}{\beta} \sum_{k_S} f(k_S)$$

as required.]

The next step is to deform the contour round the poles, for which the following diagram is helpful.



We have deformed the curve such that the integration can be conveniently split into four constituent parts.

$$\int_C \rightarrow \int_{-\infty - i\epsilon}^{\infty - i\epsilon} + \int_{\infty - i\epsilon}^{\infty + i\epsilon} + \int_{\infty + i\epsilon}^{-\infty + i\epsilon} + \int_{-\infty + i\epsilon}^{-\infty - i\epsilon} \quad (4.26)$$

The integrations from $+\infty$ to $-\infty$ at real infinity do not contribute in the limit $\epsilon \rightarrow 0$.

The remaining two integrals can be manipulated in the following manner

$$\begin{aligned}
& \int_{-\infty - i\epsilon}^{\infty - i\epsilon} \frac{dz}{4\pi} f(z) \cot\left(\frac{\beta z}{2}\right) + \int_{\infty + i\epsilon}^{-\infty - i\epsilon} \frac{dz}{4\pi} f(z) \cot\left(\frac{\beta z}{2}\right) \\
&= - \int_{-\infty + i\epsilon}^{\infty + i\epsilon} \frac{dz}{4\pi} [f(z) + f(-z)] \cot\left(\frac{\beta z}{2}\right) \\
&= + \int_{-\infty + i\epsilon}^{\infty + i\epsilon} \frac{dz}{4\pi} [f(z) + f(-z)] \left[\frac{e^{-i\beta z} + 1}{e^{-i\beta z} - 1} \right] \\
&= + \int_{-\infty + i\epsilon}^{\infty + i\epsilon} \frac{dz}{4\pi} [f(z) + f(-z)] \left[1 + \frac{2}{e^{-i\beta z} - 1} \right]
\end{aligned}$$

as the first term is free of any singularities in the neighbourhood of the real axis, we are justified in taking the limit $\epsilon \rightarrow 0$. In the second term, the integration contour lies an infinitesimal distance above the real axis, so there are no divergences due to the poles at $z = 2\pi n / \beta$. On this understanding, it is customary to write the integration in terms of a real variable as follows

$$\frac{1}{\beta} \sum_{k_S} f(k_S) = \int_{-\infty}^{\infty} \frac{dx}{2\pi} f(x) + \int_{-\infty}^{\infty} \frac{dx}{2\pi} \frac{f(x) + f(-x)}{e^{-i\beta x} - 1} \quad (4.27)$$

where the function f in our case is

$$f(k_S) = \text{Tr} \cdot \ln \left[k^2 + \left\{ k_S + \frac{1}{\beta} (\Theta_R + \delta) \right\}^2 \right] \quad (4.28)$$

The divergent first term corresponds to an infinite vacuum energy, present because we did not deal with the problem of normal ordering at the outset. The second term is the finite correction due to the circular nature of one of the

dimensions. As we are interested in minimising the potential then the difference between the potentials at any two background configurations will be finite, hence the divergent term can be neglected for our purposes.

The remaining term may be evaluated in the following manner

$$\begin{aligned} & \int \frac{d^{m+1}k}{(2\pi)^{m+1}} \frac{1}{e^{-i\beta k_S} - 1} \left\{ \ln \left[k^2 + \left\{ k_S + \frac{1}{\beta} (\Theta_R + \delta) \right\}^2 \right] \right. \\ & \qquad \qquad \qquad \left. + \ln \left[k^2 + \left\{ k_S - \frac{1}{\beta} (\Theta_R + \delta) \right\}^2 \right] \right\} \\ &= \int \frac{d^{m+1}k}{(2\pi)^{m+1}} \ln(k^2) \left\{ \frac{1}{e^{-i\beta k_S + i(\Theta_R + \delta)} - 1} \right. \\ & \qquad \qquad \qquad \left. + \frac{1}{e^{-i\beta k_S - i(\Theta_R + \delta)} - 1} \right\} \end{aligned}$$

where we have performed a translation in the k_S direction in each of the two logarithm terms. If we now make the change $k_S \rightarrow -k_S$ in the first term above, then we see that it is the complex conjugate of the second term, hence we can write the expression as

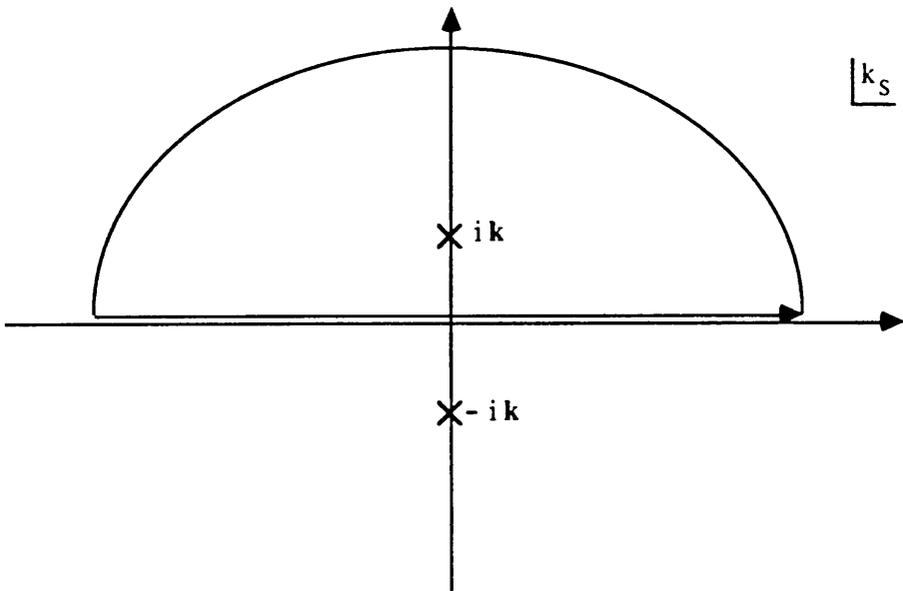
$$2 \operatorname{Re} \int \frac{d^{m+1}k}{(2\pi)^{m+1}} \frac{\ln(k^2)}{e^{-i\beta k_S - i(\Theta_R + \delta)} - 1}$$

Integrating by parts gives

$$\frac{2}{\beta} \operatorname{Re} \int \frac{d^{m+1}k}{(2\pi)^{m+1}} \frac{-2ik_S}{k^2} \ln \left[1 - e^{i\beta k_S + i(\Theta_R + \delta)} \right]$$

where the surface term vanishes. The remaining k_S integration can be performed

using complex integration techniques once again. The integrand has simple poles at $k_S = \pm i k$ but, due to the existence of an essential singularity at $k_S = -i \infty$ (where the exponential 'blows up'), we must ensure that our contour lies in the upper half of the complex k_S plane. We therefore consider the contour shown in the following diagram.



The contribution from the semi-circle vanishes in the limit as $k_S \rightarrow \infty$, leaving only the contribution from the pole at $k_S = +i k$. At this pole, the residue is

$$\begin{aligned} \lim_{k_S \rightarrow +i k} \frac{-2 i k_S (k_S - i k)}{k^2} \ln [1 - e^{i \beta k_S + i (\Theta_R + \delta)}] \\ = -i \ln [1 - e^{-\beta k + i (\Theta_R + \delta)}] \end{aligned}$$

This leaves

$$\frac{2}{\beta} \text{Re} \int \frac{d^m k}{(2\pi)^m} \ln [1 - e^{-\beta k + i (\Theta_R + \delta)}]$$

We therefore have only to deal with the integration over the remaining momenta. To evaluate this we use spherical integration techniques, where we write the measure $d^m k$ as the product of the line element, dk , with the surface area of the sphere in m dimensions to give the m dimensional volume element. The area of the $(m-1)$ sphere is given by the formula

$$A_{m-1} = \frac{2\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2})} k^{m-1} \quad (4.29)$$

hence the integral now becomes

$$\frac{2}{\beta} \text{Re} \int_0^\infty dk \frac{k^{m-1}}{(2\pi)^m} \ln [1 - e^{-\beta k + i(\Theta_R + \delta)}] \quad (4.30)$$

There are now two ways to proceed from this point. The more sophisticated approach will be given in the next section, while we give the more common approach here[39,46]. If we expand the logarithm, then (4.30) becomes

$$\begin{aligned} & -\frac{2}{\beta} \text{Re} \frac{2\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2})} \int_0^\infty dk \frac{k^{m-1}}{(2\pi)^m} \sum_{n=1}^\infty \frac{e^{-n\beta k + i n(\Theta_R + \delta)}}{n} \\ & = -\frac{2}{\beta} \frac{2\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2})} \frac{1}{n\beta(2\pi)^m} \cos[n(\Theta_R + \delta)] \int_0^\infty dk k^{m-1} e^{-n\beta k} \\ & = -\frac{2\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2})} \Gamma(m) \sum_{n=1}^\infty \frac{\cos[n(\Theta_R + \delta)]}{(n\beta)^{m+1}} \end{aligned}$$

where the surface terms vanish at each partial integration. To simplify this expression, we shall define the quantity

$$C(m) = \frac{2 \pi^{\frac{m}{2}} \Gamma(m)}{\Gamma(\frac{m}{2}) \beta^{m+1}}$$

as this appears in all terms in the one loop potential as an overall positive definite multiplicative coefficient.

Incorporating this expression back into the fermion contribution term, and performing a similar analysis for the other types of field that we will be considering, we obtain the following terms which may be incorporated into our model

$$\begin{aligned} V_F &= 2^{[(m+1)/2]} N_F C(m) \text{Tr} \sum_{n=1}^{\infty} \frac{\cos [n(\Theta_R + \delta)]}{n^{m+1}} \\ V_S &= -2 N_S C(m) \text{Tr} \sum_{n=1}^{\infty} \frac{\cos [n(\Theta_R + \delta)]}{n^{m+1}} \\ V_G &= (1 - m) C(m) \text{Tr} \sum_{n=1}^{\infty} \frac{\cos [n(\Theta_{ad} + \delta)]}{n^{m+1}} \end{aligned} \quad (4.31)$$

The subscripts F, S and G refer to fermions, complex scalars, and gauge + ghost fields respectively, and the traces are over group indices. The coefficient $(1 - m)$ in V_G comes from a contribution of $+2$ from the ghosts and a contribution of $-(m + 1)$ which comes from taking the trace over Lorentz indices in the gauge contribution. The Fourier series form of the potential is the most convenient form for numerical examination, and is the form used by Hosotani[12] in his analysis of $SU(N)$ fundamental representation on $R^3 \times S^1$, but there is an alternative form which lends itself more readily to analytical investigations, and we shall describe this in the next section.

iv) Polylogarithm form of the one loop potential

The polylogarithm function[47] was first examined in its power series form

$$\text{Li}_p(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^p}, \quad |z| \leq 1 \quad (4.32)$$

for integer order $p \geq 2$. Its definition can be extended to allow for general argument z by repeated integration of the normal logarithm function

$$\text{Li}_2(z) = - \int dz \frac{\ln(1-z)}{z} \quad (4.33)$$

and

$$\text{Li}_p(z) = \int dz \frac{\text{Li}_{p-1}(z)}{z} \quad (4.34)$$

Further refinements can extend the definition to negative and non-integral order, but such developments are of no relevance to this work.

As it happens, the effective potential as calculated in section iii is exactly of the form (4.32) and, if we write the polylogarithm as a function of e^{ix} , then we have

$$\text{Li}_p(e^{ix}) = \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^p} + i \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^p} \quad (4.35)$$

and hence

$$\sum_{n=1}^{\infty} \frac{\cos[n(\Theta_R + \delta)]}{n^{m+1}} = \text{Re Li}_{m+1}(e^{i\delta} U_R) \quad (4.36)$$

where U_R is the fundamental Wilson loop operator which was encountered earlier.

The potential could have been evaluated directly in terms of the polylogarithm function by noting that the expression (4.30) has a similar form to (4.33) and can be treated without expanding the logarithm. If we integrate (4.33) by parts, then we obtain the following integral

$$-(m-1) \int_0^{\infty} dk k^{m-2} \int dx \ln [1 - e^{-\beta x + i(\Theta_R + \delta)}] \Big|_{x=k} \quad (4.37)$$

making the change of variables

$$w = e^{-\beta x + i(\Theta_R + \delta)} \quad , \quad dw = -\beta w dx$$

in (4.37) leads to the expression

$$\begin{aligned} & -\frac{(m-1)}{\beta} \int_0^{\infty} dk k^{m-2} \int dw \frac{-\ln(1-w)}{w} \Big|_{w=e^{-\beta k + i(\Theta_R + \delta)}} \\ & = -\frac{(m-1)}{\beta} \int_0^{\infty} dk k^{m-2} \text{Li}_2(e^{-\beta k + i(\Theta_R + \delta)}) \end{aligned} \quad (4.38)$$

subsequent partial integrations of (4.38) can be performed until we come to the final integration

$$-\frac{\Gamma(m)}{\beta^m} \int_0^{\infty} dk \text{Li}_m(e^{-\beta k + i(\Theta_R + \delta)})$$

$$\begin{aligned}
&= -\frac{\Gamma(m)}{\beta^m} \int_{e^{i(\theta_R + \delta)}}^0 dw \frac{-\text{Li}_m(w)}{\beta w} \\
&= -\frac{\Gamma(m)}{\beta^{m+1}} \text{Li}_{m+1}(e^{i(\theta_R + \delta)}) \tag{4.39}
\end{aligned}$$

as $\text{Li}_p(0) = 0$. Taking the real part of (4.39) gives the Fourier series form as given in (4.31) in the previous section.

The two Fourier series in (4.35) fall into two independent series, the generalised Clausen functions and the associated Clausen functions. The associated Clausen functions are given by (for $p \geq 1$)

$$\begin{aligned}
\text{Gl}_{2p}(x) &= \sum_{n=1}^{\infty} \frac{\cos(nx)}{x^{2p}} \\
\text{Gl}_{2p+1}(x) &= \sum_{n=1}^{\infty} \frac{\sin(nx)}{x^{2p+1}} \tag{4.40}
\end{aligned}$$

These functions are the Fourier series of polynomial expressions of order $2p, 2p+1$ respectively, and are also known as Glaisher polynomials (hence the Gl notation). They are closely related to the Bernoulli polynomials

$$\begin{aligned}
\text{Gl}_{2p}(x) &= (-1)^{p-1} \frac{1}{2} \frac{(2\pi)^{2p}}{(2p)!} B_{2p}\left(\frac{x}{2\pi}\right) \\
\text{Gl}_{2p+1}(x) &= (-1)^{p-1} \frac{1}{2} \frac{(2\pi)^{2p+1}}{(2p+1)!} B_{2p+1}\left(\frac{x}{2\pi}\right) \tag{4.41}
\end{aligned}$$

and in some papers on theories in $\mathbb{R}^3 \times S^1$, the potential is given in terms of the Bernoulli polynomials, which unfortunately leads to extremely cumbersome coefficients.

The generalised Clausen functions are (again for $p \geq 1$)

$$\begin{aligned} Cl_{2p}(x) &= \sum_{n=1}^{\infty} \frac{\sin(nx)}{x^{2p}} \\ Cl_{2p+1}(x) &= \sum_{n=1}^{\infty} \frac{\cos(nx)}{x^{2p+1}} \end{aligned} \quad (4.42)$$

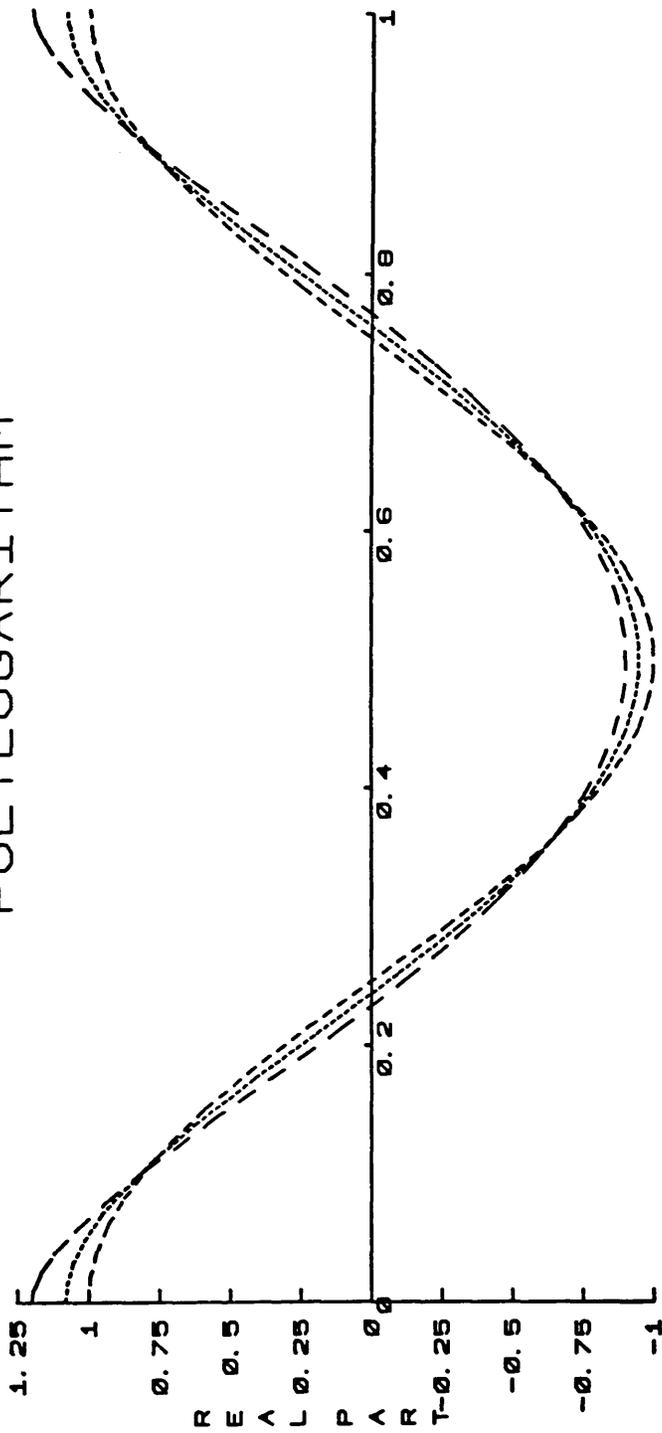
but, unlike their counterparts are transcendental in nature, being the Fourier series of multiple integrals of logarithms of trigonometric expressions, e.g.

$$Cl_2(x) = - \int_0^x dy \ln \left[2 \sin \left(\frac{y}{2} \right) \right] \quad (4.43)$$

It therefore appears that we may have a problem with the analysis of the potential in a general $(m + 1)$ dimensional space, as it is polynomial for m even and transcendental for m odd but, as will be explained, the relevant properties of these functions are independent of the dimension of the macroscopic space.

If we plot the cosine Fourier series of a single variable x as in the graph, we see that these functions have only one minimum which is located at $x = 1/2$, one maximum located at $x = 0$ and have reflection symmetry about the minimum. The zeros of these functions lie in the ranges $[0, 1/4)$ and $(3/4, 1]$ and tend to the values $1/4$ and $3/4$ as the order of the function tends to infinity. This is simply because the leading terms in the series are just cosines, and these terms will dominate as the order increases. We could therefore expect many of our results to be independent of the dimension of the spacetime chosen for our examination of the model and this indeed turns out to be the case for much of the one matter representation analysis. As can be seen from the graph, the low order polylogarithm is much 'flatter' and hence we may expect different results to occur in some lower dimensional cases (Appendix C) and, in models with several matter representations, the spacetime dimension does

POLYLOGARITHM



- - - ORDER = 3
 ORDER = 4
 - . - . ORDER = 10

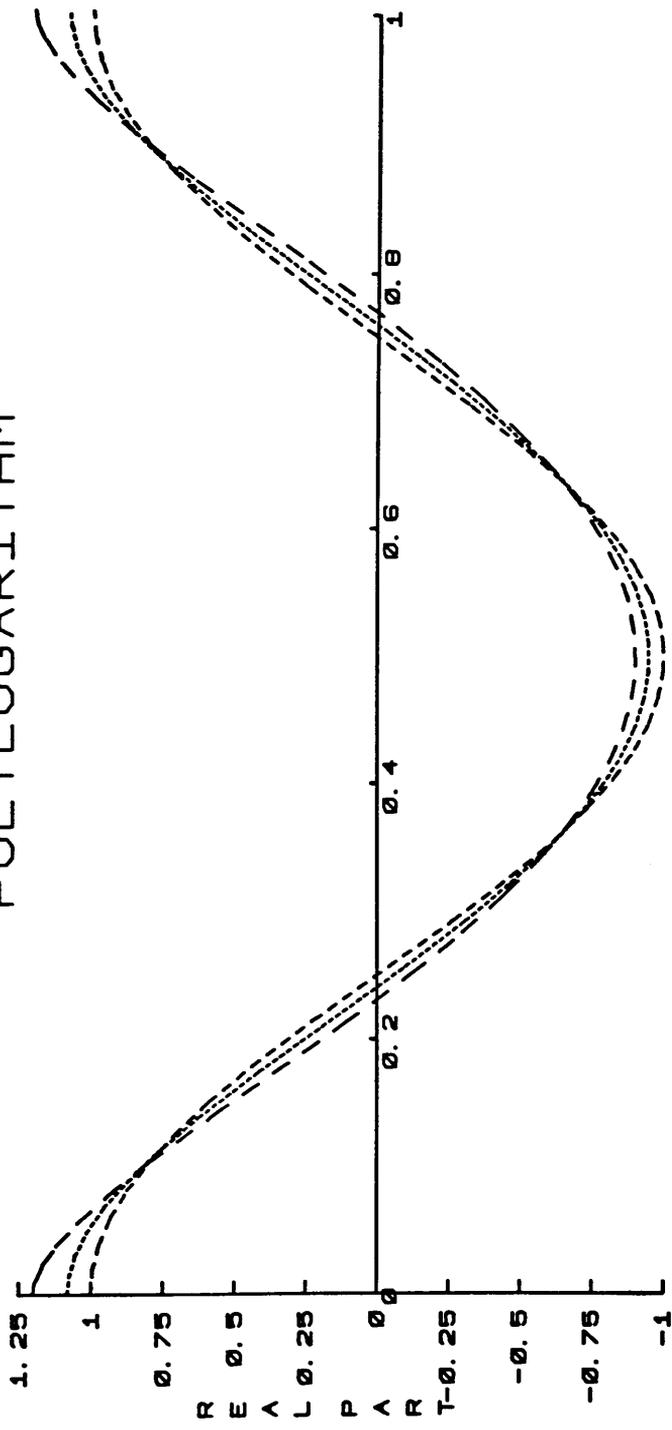
seem to play a more important role. This will be illustrated in Chapter 7.

Another feature of these graphs is that the real part of the polylogarithm function will always be negative for at least half of the range of the variable x . This seemingly unimportant feature will lead to a subtle difference between the Dirac fermion case and the complex scalar case.

v) Graph

The graph is of the function $\operatorname{Re} \operatorname{Li}_m(e^{-2\pi i x})$ for the sample values $m = 3, 4$ and 10 . The main features are that the positions of the minimum at $x = 1/2$ and the maxima at $x = 0$ and 1 are independent of the order of the polylogarithm. The zeros are always in the range $[0, 1/4) \cup (3/4, 1]$ and move towards the points $1/4$ and $3/4$ as the order m tends to infinity.

POLYLOGARITHM



i) Symmetry preserving minima

In this chapter, we shall give an analysis of a system consisting of fermions in one irreducible representation of the gauge group minimally coupled to the gauge fields. We shall adopt an analytic approach, resorting to numerical methods only when the complexity of the minimisation procedure justifies it.

In order to determine when breaking of the gauge group occurs, it is logical to determine which configurations preserve the gauge algebra and then analyse such cases in detail. This relates back to the group theoretical discussion given in chapter 2, where we discussed the concepts of the true group of a representation congruency class and its true centre. This can be extended to the concept of the true group of a Lagrangian, which will be the locally isomorphic group with largest centre generated by the gauge algebra for which all the representations present in the Lagrangian are single valued and at least one is faithful. In this and the next chapter, the true symmetry groups of the Lagrangians will just be the true groups of the matter representations. We shall touch on more complicated cases, with more than one matter representation present, in chapter 7.

For a Lagrangian with a particular true group G/D , where D is some discrete central subgroup of the covering group G ($C(G) \supseteq D$) as before, the only symmetry preserving configurations are those whose Wilson operators map Z homomorphically into the true centre $C(G)/D$, i.e. $\text{Hom}(Z, C(G)/D) \supseteq \Gamma$. For such configurations, the fundamental Wilson loop operator is a multiple of the unit matrix and can be written in terms of the q_{LM} parameter as defined in chapter 3, where $e^{-2\pi i q}$, $\forall L$, form a (not necessarily faithful representation) of the centre of the true group of the class labelled by M . Such backgrounds (labelled by L) will be symmetric vacua for any choice of matter representation in the Lagrangian, with the q parameters for such a field in two different classes being related by

$$M_2 q_{LM_1} = M_1 q_{LM_2} \quad (5.1)$$

It should be noted that, as the true group depends on the congruency class, then two vacua which are gauge equivalent for matter fields in one class will not necessarily be gauge equivalent in another. In general, for the L symmetric vacua of a particular covering group, those which are represented by the same q parameter in the class L will be gauge equivalent, i.e. the vacua will be related by a periodic gauge transformation matrix in that class. An obvious example is the case of the adjoint class, where the true group has unit centre and hence all symmetric vacua are equivalent to the zero background ($q_{L0} = 0, \forall L$).

In order to determine whether or not the symmetry breaks, we have to find out if any of the symmetric vacua forms the global minimum. At the moment, it is not even clear whether they are stationary points of the one loop potential. The most obvious way to proceed, therefore, is to evaluate the derivatives of V at the symmetric backgrounds, but, in order to achieve this, we must know how the polylogarithm of an exponential argument behaves under differentiation. From the defining relationship (4.32), we see that

$$\frac{d}{dz} \text{Li}_p(z) = \frac{1}{z} \text{Li}_{p-1}(z) \quad (5.2)$$

therefore, making the substitution $z = e^{cx}$ yields

$$\frac{d}{dx} \text{Li}_p(e^{cx}) = c \text{Li}_{p-1}(e^{cx}) \quad (5.3)$$

This can be applied directly to the case where c is a matrix in the Cartan subalgebra, as all such matrices commute with each other.

Constructing our potential from (4.31) we have

$$V = C(m) \text{Re Tr} \left[(1-m) \text{Li}_{m+1}(U_{ad}) + 2^{\lfloor (m+1)/2 \rfloor} N_F \text{Li}_{m+1}(e^{i\delta} U_R) \right] \quad (5.4)$$

where all the quantities in (5.4) were defined in the previous chapter. Using the relationship (5.3), we find on differentiation of the potential

$$\begin{aligned} \frac{\partial V}{\partial B^a} = & \beta C(m) \operatorname{Re} \operatorname{Tr} \left[(1-m) \operatorname{Li}_m (U_{ad}) H_{ad}^a \right. \\ & \left. + 2^{[(m+1)/2]} N_F \operatorname{Li}_m (e^{i\delta} U_R) H_R^a \right] \end{aligned} \quad (5.5)$$

At the symmetric backgrounds we have

$$U_{ad} = I_{ad} \quad , \quad U_R = e^{-2\pi i q} I_R \quad (5.6)$$

giving

$$\begin{aligned} \frac{\partial V}{\partial B^a} = & \beta C(m) \operatorname{Re} \left[(1-m) \operatorname{Li}_m (1) \operatorname{Tr} H_{ad}^a \right. \\ & \left. + 2^{[(m+1)/2]} N_F \operatorname{Li}_m (e^{-2\pi i q + i\delta}) \operatorname{Tr} H_R^a \right] = 0 \end{aligned} \quad (5.7)$$

as the generators of all representations of a semi-simple complex Lie algebra are traceless.

We therefore find that all symmetry preserving configurations are stationary points of the one loop potential. In order to determine the nature of these stationary points, we must perform yet another differentiation of V . This then gives

$$\begin{aligned} \frac{\partial^2 V}{\partial B^a \partial B^b} = & \beta^2 C(m) \operatorname{Re} \left[(1-m) \operatorname{Li}_{m-1} (1) \operatorname{Tr} (H_{ad}^a H_{ad}^b) \right. \\ & \left. + 2^{[(m+1)/2]} N_F \operatorname{Li}_{m-1} (e^{-2\pi i q + i\delta}) \operatorname{Tr} (H_R^a H_R^b) \right] \end{aligned} \quad (5.8)$$

The traces of the generator products in this expression can be written in terms of the Dynkin indices of the representations[24], giving

$$\frac{\partial^2 V}{\partial B^a \partial B^b} = -\beta^2 C(m) \text{Re} \left[(1-m) \text{Li}_{m-1}(1) L_2(\text{ad}) \right. \\ \left. + 2^{[(m+1)/2]} N_F \text{Li}_{m-1}(e^{-2\pi i q + i \delta}) L_2(\mathbb{R}) \right] \delta^{ab} \quad (5.9)$$

In order for any background labelled by q_{LM} to represent a minimum of V , the second derivative (5.9) must be positive, corresponding to a positive curvature of the function at q_{LM} . (5.9) also has a more physical interpretation : by (3.29) it corresponds to a mass term for the quantum field associated with the classical background term, i.e. the gauge field components on the circle. On compactification of the theory to give its low energy limit on R^m , these gauge field components manifest themselves as scalar fields. The condition that the second derivative be positive is then ensuring that these scalar fields do not acquire an imaginary mass. (An alternative approach to the calculation of these masses via the vacuum polarisation tensor can be adopted[39] which agrees with the form given in (5.9). This technique is unnecessarily complicated when the simple method of differentiating the potential can be adopted.) It should also be noted that, as the second derivatives of the polylogarithm are just polylogarithms with the same sign, then the configuration which minimises the potential will also give the largest value for the curvature at that minimum, i.e. will maximise the (mass)² term for these scalar fields.

We can therefore write a general stability condition for symmetric backgrounds

$$N_F \frac{\text{Re Li}_{m-1}(e^{-2\pi i q + i \delta})}{\zeta(m-1)} < (m-1) 2^{-[(m+1)/2]} \frac{L_2(\text{ad})}{L_2(\mathbb{R})} \quad (5.10)$$

where we have used the fact that

$$\text{Re Li}_p(1) = \text{Li}_p(1) = \sum_{n=1}^{\infty} \frac{1}{n^p} = \zeta(p) \quad (5.11)$$

ζ is the usual Riemann zeta function. Setting $q = 0$ and $m = 3$ gives the stability condition for the zero background as derived by Toms in reference [50].

Any vacuum whose q parameter in the fermion representation congruency class satisfies (5.10) will be at least a local minimum of the one loop potential. Any such vacuum whose q parameter gives a negative value for the left hand side of this inequality will be a local minimum for all choices of the fermion generation number N_F . We shall refer to such configurations as being stable minima. Any other configuration will either be a local maximum or else will be a minimum only for a limited range of N_F and we shall call such vacua unstable.

If we consider the case of gauge fields without any matter present, then we see from the graph of the polylogarithm given in the previous chapter that the maximum was at $x = 0$ hence, when a negative coefficient $(1-m)$ is introduced, this configuration becomes the minimum. This therefore implies that the minimum of the one loop potential without matter fields corresponds to the zero background, which is the only symmetric configuration for the adjoint group. We therefore require the presence of matter fields if we are to break the gauge symmetry.

We are now ready to examine the case where fermion matter fields are introduced, but first we must discuss which values of the fermion phase δ will be considered.

ii) Spin structures on $R^m \times S^1$

It is not our intention to prove any of the following facts and the reader is referred to the literature for the detailed mathematics[51-53]. They are merely stated to give some idea of why different fermion phases may be permissible and, if so, what the restrictions are on the possible values.

The conventional geometrical view of a spinor is as a single valued section of some principal fibre bundle with the relevant spacetime as the base space and a $\text{Spin}(d)$

structure group, where d is the spacetime dimension. Such a bundle is known as a spinor structure, S , and is defined as a double covering of an $SO(d)$ principal bundle, B , which gives the non-trivial $Spin(d)$ covering of $SO(d)$ when restricted to the $SO(d)$ fibres. Spinor structures may only exist if the second Stiefel-Whitney class of the base space vanishes, i.e. the cohomology group $H^2(Z, Z_2) = 0$, where M is the spacetime manifold in question. The number of inequivalent spinor structures on this manifold is given by the group of homomorphisms of the first homotopy (or fundamental) group $\pi_1(M)$ into the two element cyclic group Z_2 , as we may also consider S to be a Z_2 bundle over B . On a simply connected manifold, the first homotopy group vanishes and hence there is only one permissible spinor structure. On multiply connected manifolds, however, the group $\text{Hom}(\pi_1(M), Z_2)$ may not be trivial, corresponding to the bundle S not being isomorphic to a product bundle, and as a result there may exist inequivalent types of spinor which, in our case, would manifest themselves by obeying different boundary conditions on translation around the circle. It is not our intention to go into detail, but it turns out that the group $\text{Hom}(\pi_1(M), Z_2)$ is isomorphic to the first cohomology group $H^1(M, Z_2)$ and for the case where $M = R^m \times S^1$ we have $H^1(M, Z_2) = Z_2$, implying the existence of two distinct types of spinor which correspond to the phase choices $\delta = 0$ and π corresponding to the trivial product bundle over the circle and the twisted bundle (cf. Möbius strip, Klein bottle etc.). Hence we have a choice of including either periodic or antiperiodic spinors in our Lagrangian. (A more detailed discussion of the definition of spinors on general manifolds can be found in the literature.)

There is, however, an alternative point of view in the literature which regards the fields as multi-valued sections of the principal bundle over the base manifold, which is equivalent to having the fields as single valued sections of the bundle whose base space is the covering space of the multiply-connected spacetime manifold. The argument for this viewpoint is that the Lagrangian is still single-valued and hence physical amplitudes are still independent of the particular

coordinate patch chosen, therefore there is no reason to be as restrictive in the definitions of the various fields. The implications for our model is that the fermions would be permitted to have a general phase change on translation around S^1 as was the case in the original paper by Hosotani.

(A similar argument can be presented for the case of scalar matter fields, in which case the previously permissible phases of 0 and π are again replaced by general phase factors.)

In sections iii to vi, we shall give the detailed analysis for the conventional periodic or antiperiodic fermions (although it is postulated in the literature that the spinor generating functionals are not generally Lorentz invariant unless both types of spinor are present simultaneously[53]), and shall discuss the effect of a general phase factor in the final section.

iii) Antiperiodic fermions and the low energy theory

In this case, we are choosing the fermion phase factor to be $\delta = \pi$, i.e. the fermion field changes sign on one complete translation round the circle. The stable symmetric backgrounds are now those which satisfy

$$\text{Re Li}_{m-1}(-e^{-2\pi i q}) < 0 \quad (5.12)$$

Looking back to the graph of the polylogarithm given in the last chapter, we see that in this case the variable x corresponds to $(q - \delta/2\pi)$ with only certain values of q being permissible of course, i.e. the minimum of the graph at $x = 1/2$ signifies that the global minimum lies at $q = 0$. The physical significance of this is as follows. In the full Lagrangian, the non-zero background field is represented by the gauge invariant Θ matrices in the appropriate representations. The number of Θ matrices which are multiples of the identity in a given representation is related to the order of

the centre of the group which that representation generates. For all representations in all classes, the group centre will always contain the identity matrix, i.e. there will always be background fields which give $\Theta = 0$. (There obviously may be other symmetric vacua, depending on the class of the representation.) Equation (5.10) shows that this is the global minimum of the one loop potential, i.e. that for anti-periodic fermions, there should be no classical background ($U_R = 1, \forall R$). This is illustrated in the graphs for fermions in the $\underline{2}$ of $SU(2)$, the $\underline{3}$ of $SU(3)$, the $\underline{7}$ of G_2 and the $\underline{14}$ of G_2 , where the zero background minimum deepens as the fermion generation number increases. In the $SU(2)$ graph, we see that there is a symmetric configuration at $q = 1/2$, present as the true centre is Z_2 (although such backgrounds were gauge equivalent to zero under $SU(2)/Z_2$ i.e. they have $q_{10} = 0$, they are not equivalent to zero under $SU(2)$, i.e. $q_{11} \neq 0$, where $M = 0,1$ labels respectively the adjoint and non-adjoint classes of $\mathfrak{su}(2)$). Such a configuration corresponds to the global maximum of the antiperiodic fermion contribution.

The obvious result of this is that we have left the gauge symmetry completely unaffected. Whatever true group was originally present is preserved in the low energy theory on R^m . On integrating out the circular coordinate from the Fourier expanded fields (4.17) we find that, with no background field

$$\begin{aligned} \int_0^\beta dx_S \bar{\psi} \gamma_S D_S \psi &= \int_0^\beta dx_S \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \bar{\psi}_n \gamma_S \frac{2\pi m}{\beta} \psi_m e^{-2\pi i x_S (m-n)} \\ &= \bar{\psi}_n \gamma_S \frac{2\pi n}{\beta} \psi_n \end{aligned} \quad (5.13)$$

as the integration of the exponential just yields a delta function. This corresponds to mass terms of the order of the inverse scale length of the compact space for the Fourier modes of the fermion field. As β will be small (of the order of the Planck

length), these mass terms will be extremely large, with only the $n = 0$ mode surviving. We therefore have massless fermions after compactification. Similarly, the gauge field components on R^m will also have massless modes (as $\Theta_{\text{ad}} = 0$), corresponding to the gauge bosons of the dimensionally reduced theory. (It should be noted that only the full Fourier expansion will be gauge invariant, and just taking the zero modes will not be permissible. A general gauge transformation may interpolate between the different Fourier modes, but the final spectrum will be the same.)

In short, taking anti-periodic fermions in any representation of the gauge group will lead to the zero background being preferred and hence give a low energy theory with the same symmetry group.

iv) Periodic fermions in a non-adjoint class

The next example to be considered is the case of periodic fermions in a congruency class other than that containing the adjoint representation. The reason for this restriction will become apparent when we consider adjoint class fields in a later section. For this case, the stable minima satisfy

$$\text{Re Li}_{m-1}(e^{-2\pi i q}) < 0 \quad (5.14)$$

As the gauge group is not the adjoint group (the groups G_2 , F_4 , and E_8 have unit centre and are technically their own adjoint groups), then we have a non-trivial centre and hence non-zero q values. If the true centre is Z_2 , then there exists a $q = 1/2$ symmetric configuration. If the centre is Z_3 then there are symmetric configurations labelled by $q = 1/3$ and $2/3$. In general, for Z_n true centre, there are symmetric configurations at $q = p/n$ for $p = 0, 1, \dots, n-1$. This may seem trivial, but the important point becomes clear on referring back to the polylogarithm graph and

substituting $x = q$. As was pointed out in chapter 4, section iv, the polylogarithm takes negative values for q in the range $1/4$ to $3/4$. As can be seen from (5.14), q values within this range will correspond to stable symmetric minima and hence the point is that, for Z_n true centre, there will always be stable symmetric vacua. As the global minimum of the matter contribution corresponds to $x = 0$, then the vacua with q parameters nearest the value $1/2$ will be the deepest of the symmetric minima. In order to clarify the situation, we now discuss the cases of even and odd order centres separately.

1/ Even order true centres.

In the case of the true centre having even order, the central group Z_{2n} will contain the element -1 corresponding to $q = p/2n = 1/2$ for $p = n$. For such a system, this symmetric configuration will trivially minimise the fermionic contribution to the potential and, as it is also one of the degenerate global minima of the gauge term, it will form the global minimum of the one loop potential, and should therefore be the preferred background field for the theory. This is illustrated in the graph of the potential for periodic fermions in the $\underline{2}$ of $SU(2)$ when $m = 3$. Note that, from (5.9), we can determine the fermion number at which any particular symmetric background will become a local maximum and, for the case shown in the graph, the zero background becomes unstable at $N_F = 2$. (For this choice of boundary condition, the zero background actually maximises the fermion contribution.) In general, the critical number decreases with the dimension of the spacetime due to the relative increase of the fermion contribution in relation to the gauge field term. This therefore implies that the symmetry algebra remains unbroken (as was the case with the antiperiodic boundary conditions), but there is a slight subtlety involved when determining the low energy group.

If we look back at (5.13), then we see that the covariant derivative was replaced by the eigenvalues of the usual spatial derivative only because the background field vanished. Here, however, we have a non-zero background field and, as a result, the

quantity n in (5.13) should be replaced by $(n - \Theta/2\pi)$, i.e. by $(n - 1/2)$. The immediate consequence of this is that there will be no massless Fourier modes on compactification, and hence we must see a different low energy gauge group. As the algebra is unaffected, and therefore there are massless gauge boson modes surviving compactification, and as the full Fourier series expansion is still invariant under the full true group, then the true low energy group must be one of the locally isomorphic groups which can be obtained from the same algebra. In order to identify the low energy group, we should refer back to the example tables of q_{LM} values in chapter 2. By finding the configuration L which gives $q = 1/2$ in the class M , and then examining the q parameters of the remaining symmetric vacua, we see that they generate a discrete group Z_n/Z_2 which forms the centre of our low energy group. We have therefore identified our effective symmetry as being the original true group modulo Z_2 , the set of Wilson loops. As our fermion representation can never be a representation of this group (as the group is determined by this very fact!), then no matter fields contained in this representation will be observed at low energies. If we added other matter fields but still retained this configuration as the vacuum (by, for example, having an abnormally large number of fields in the original class), then only those fields in classes whose true group is identical to the low energy group will retain massless modes on compactification as the background field will not appear in their covariant derivatives. This could in principle determine the low energy group as, in our case, the effective group is indistinguishable from the adjoint group. due to the fact that the only surviving massless fields are the gauge bosons.

2/ Odd order true centres

When the true centre has odd order, then there are no symmetric configurations which can be labelled by $q = 1/2$, i.e. there are no integers p satisfying $2p = 2n + 1$. In such a case, the minima labelled by $q = n/(2n + 1)$ and $(n + 1)/(2n + 1)$ are the (degenerate) deepest symmetric minimum, as shown in the graph for fermions in the 3 of $SU(3)$. Such configurations still globally minimise the gauge field contribution,

but it is conceivable that there may exist broken symmetry vacua which may form the global minima of the potential. For this to happen, the diagonal blocks in the corresponding fermion Θ matrix would have to give multiples of the identity with 'q' parameters nearer to $1/2$ on average than the symmetric minima. As the 'q' parameters of each block could not be equal to the q parameters in another block, and as such vacua would immediately increase the contribution from the gauge field term, we would intuitively expect the symmetry to remain unbroken as before.

It was not clear how the general problem could be approached analytically, therefore we resorted to a numerical examination of several cases. (This was achieved using a Newton-Raphson algorithm on an IBM 43-61 computer. The algorithm can be found in the Numerical ALgorithm (NAG) library under the code E04 JAF). In all the cases examined, it was found that the symmetric minima nearest the value $1/2$ did indeed form the global minima of the potential. As happens in the even order case, this non-zero background results in the fermions having no zero modes on compactification, and again the residual group is the true group modulo the set of Wilson loop $\tilde{\Gamma}$. This time, however, $\tilde{\Gamma}$ forms a faithful representation of the true centre as $pn/(2n + 1) \pmod{1}$ generates all the centre elements as p runs over the homotopically distinct loops. The effective low energy symmetry group is therefore the adjoint group.

In either of these cases, the algebra remains unbroken, with only the effective low energy group being changed. This is not a breaking as such, as the low energy group is not a subgroup of the original group, and in fact the original group is still the true symmetry group of the dimensionally reduced Lagrangian.

(In the last chapter, we insisted on gauge transformations preserving the phases of all the fields in the Lagrangian. In the Higuchi-Parker paper, as they have allowed the fermion phase to be a variable in the theory, they have introduced gauge transformation matrices $e^{iC(x)}$ which preserve the gauge field boundary conditions but change the fermion phase. They then state that if the group has a central element -1 , i.e. the true centre has even order, then an appropriate choice of C in the

transformation matrix can gauge away the background field at the expense of altering the fermion phase from 0 to π . They therefore have the full symmetry group as the effective low energy group as they have effectively 'gauge transformed' the massive periodic spinors into massless antiperiodic spinors. They did not give much detail in the case where the true centre has odd order, as this requires a numerical examination of several cases, but it is not clear how their approach could be applied to this case as there are degenerate symmetric minima which could not both be gauged away simultaneously. It initially seemed that they were adopting the covering space approach as outlined in chapter 1, where the background field could be gauged away at the expense of introducing non-trivial Wilson loop matrices $U = e^{iC(x)}$ which would then change the apparent phase of the fermion fields on the covering space (in which case any degenerate minima would just correspond to degenerate choices of the gauge transformation matrices U). This however does not bear any relation to the genuine fermion phase on the circle which is related to the choice of either of the spinor structures on S^1 and where the non-trivial Wilson loops are represented by non-vanishing background gauge fields. It seems that they confused the two cases in their analysis which resulted in an incorrect identification of the true residual groups generated from the unbroken algebras.

With our philosophy, the spinor phase is not variable and therefore the symmetric minima may not be gauged away. As we are only examining a toy model to see what the features as regards flux-breaking patterns are concerned, we shall not be investigating the low energy theory in greater detail than is necessary in order to examine symmetry breaking patterns.)

We therefore proceed to the final case, that of adjoint class periodic fermions, having had no success in finding an energetically preferred broken symmetry vacuum. Before we move on to this case, we will give a detailed analysis of the original Hosotani model, in which the reason for its failure to break the symmetry will become apparent.

v) The original Hosotani model

The original paper by Hosotani discussed the model for fermions in the fundamental representations of the $SU(N)$ groups and in particular, fermions in the $\underline{5}$ of $SU(5)$. He gave the Fourier series form of the potential given in (4.31), and seems to have adopted a numerical approach. He found that no breaking took place, but gave no conjecture as to why this was the case. In this section, we shall now re-examine this model where, in the light of the previous sections, it will become clear why Hosotani obtained these results.

The q_{LM} matrix for $SU(5)$ is

$$q_{LM} = \frac{1}{5} \begin{pmatrix} 4 & 3 & 2 & 1 & 0 \\ 3 & 1 & 4 & 2 & 0 \\ 2 & 4 & 1 & 3 & 0 \\ 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (5.15)$$

where, as in chapter 3, the rows represent the q parameters of a particular vacuum L in the various congruency classes M . If we take periodic fermions in the $\underline{5}$ (class 1), then from the last section, we have the global minima of the potential at the symmetric configurations labelled by $q_{L1} = 2/5$ and $3/5$, corresponding to the vacua $L = 2$ and 3 . We also find that the zero background becomes unstable at $N_F = 5$ for this representation. (In general, for fermions in the defining representation of a general $SU(N)$ group, the critical number for the zero background is $N_F = N$.) Hosotani then proceeded to examine the case when the fermions were put into the $\underline{10}$ representation (class 2). In his work, however, he dealt with only the Θ matrix elements in the $\underline{5}$, writing the elements of the other Θ matrices in terms of $\Theta_{\underline{5}}$. He then found that the backgrounds which formed the minima for the $\underline{5}$ case were no longer stable, and that the vacua labelled in our notation by $L = 3$ and 4 were the new preferred backgrounds. This becomes trivial in our formalism. Either by referring to

the matrix (5.15) or by using the relationship (5.1), we see that the vacua $L = 2$ and 3 , which gave $q = 2/5$ and $3/5$ in the class $M = 1$ now give $q = 1/5$ and $4/5$ in the class $M = 2$, and hence are no longer stable minima of the potential. The preferred vacua are $L = 1$ and 4 , which give $q_{L2} = 3/5$ and $2/5$ respectively. In terms of the class 1 parameters, the new vacua correspond to $q_{L1} = 4/5$ and $1/5$, as was found by Hosotani.

vi) Periodic adjoint class fermions

The final case we have to consider is that of periodic fermions in a representation which lies in the same congruency class as the adjoint representation. As has been stated earlier, all symmetric vacua are represented by $q = 0$ in the adjoint class, as the true group has unit centre only. We have already seen, however, that the $q = 0$ configuration is the global maximum of the fermion contribution and hence becomes a maximum of the full one loop potential at some critical value of the fermion number N_F . (The calculations of the critical numbers for some of our graphical examples are detailed in appendix A.) This obviously implies that the global minimum of the potential must lie elsewhere but, as $q = 0$ is the only symmetric configuration in this case, the new vacuum must correspond to some broken symmetry state. We have therefore found a case where the original symmetry group must be broken by the addition of fermion matter fields.

(Adopting the Higuchi-Parker philosophy, this case corresponds to the fact that no central transformation can gauge away the new broken minimum, and any transformation which gauges away some of the background field components will lead to a change in the boundary conditions of the gauge fields, hence losing the single valued nature of the Lagrangian. The only example they give is that of the adjoint breaking of $SU(2)$ or, to be more precise, $SO(3) = SU(2)/\mathbb{Z}_2$, the result of which confirmed our independently derived conclusions, modulo the different

philosophy on spinor phases.)

The identification of the new minima is best performed numerically. We know that the Θ matrix elements will tend to prefer values near π , but they are constrained by the algebraic structure of the original Θ matrices. We may anticipate that these matrices must lie in the centre of the new gauge group, but the situation is complicated by the fact that the residual group need not be semi-simple, i.e. there may exist $U(1)$ factors in the product. It may be possible to do a detailed group theoretical analysis given a particular initial gauge group, but such a procedure would be extremely complicated, whereas a numerical minimisation would be considerably easier and quicker. We therefore proceed with the examination of several cases, including some which may be of interest in superstring theory or in some subsequent unified model theory. Before any such examples are given, it is necessary to explain how the residual group may be identified by the Θ matrix elements at a particular minimum, and how we may determine the effective low energy group by examination of the massless content of the dimensionally reduced theory.

Recalling that the Θ matrices are written in terms of the Cartan subalgebra generators of a given representation of the algebra, we would therefore expect the Θ matrices to adopt some block form corresponding to the branching of the particular representation into representations of the subgroup in question. As was illustrated before, non-zero matrix elements correspond to representations which do not have massless modes and therefore do not appear in the low energy theory. It is important to note that the group may be misidentified if only the matter representation Θ matrix is examined in this manner and it is best to look at both the adjoint and fermion matrices at the new minimum. In particular, the subgroup may be identified by examination of the adjoint matrix, as this gives the root system of the original algebra and will yield the necessary information about the root system of the appropriate subalgebra. As was detailed in chapter 2, the roots of the symmetry algebra are those which are orthogonal to the background field components. Note that this also

includes roots which give scalar product 2π , as such products are gauge equivalent to zero by the implementation of some appropriate transformation matrix. As a consequence, if we find the Θ_{ad} elements which are zero mod 2π at the minimum, then this will give us the root system of the residual subalgebra and hence the massless gauge bosons of the symmetry in the low energy theory. The non-zero elements will correspond to the broken adjoint generators of the original symmetry, and will not appear in the low energy spectrum due to the inverse scale length order of any masses acquired in this type of breaking. As the matter fields are also in the adjoint class, then they will share some weight vectors with the adjoint representation, and therefore we are guaranteed some surviving massless fermion modes in the low energy theory, the nature of which will be indicative of the particular low energy symmetry group generated from the algebra. Note that, as in previous cases, the residual low energy group is not the true residual group, as the true group will be determined by the representations appearing in the branchings of the original matrices, both massless and massive.

We may now proceed with our numerical examination of several cases, using the same computer routine as was used in the examination of the odd order centre cases as described in section iv. (Originally all cases were examined in this manner using the Fourier series form of the potential, but the implementation of the analytic procedure using the polylogarithm made much of the numerical work redundant. Suffice to say that the analytic results obtained earlier verified the previous numerical work and hence we may be confident in the reliability of the chosen minimisation procedure.)

The simplest example to illustrate this is the only one to have been found elsewhere in the literature, i.e. the $SU(2)$ example obtained independently by Higuchi and Parker and Hosotani. In this case, we can easily illustrate the results graphically as well as analytically, as the background field, and hence the potential are one parameter functions. The fermions in this case are in the adjoint representation, hence the true symmetry group is $SO(3)$. As the generator of the

three dimensional representation of SU(2) is $\text{diag}(1, 0, -1)$, we can write the general Θ matrix as $\Theta = \text{diag}(\theta, 0, -\theta)$. The θ dependent part of the function to be minimised is

$$\begin{aligned} V &= C(m) \left[1 - m + 2^{[(m+1)/2]} N_F \right] \left[\text{Li}_{m+1}(e^{i\theta}) + \text{Li}_{m+1}(e^{-i\theta}) \right] \\ &= 2 C(m) \left[1 - m + 2^{[(m+1)/2]} N_F \right] \text{Re Li}_{m+1}(e^{i\theta}) \end{aligned} \quad (5.16)$$

where the sum of polylogarithms in the square brackets is obviously real. As the coefficient in square brackets is positive for any reasonable choice of m with $N_F = 0$, then the minimum of the function can be trivially seen to lie at $\theta = \pi$. This yields the fundamental Wilson loop

$$U_{\text{ad}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (5.17)$$

As only one Θ_{ad} element remains at 0, (as it corresponds to the SU(2) zero weight), then the residual true group can be easily identified as being U(1), with the relevant branching being

$$\underline{\mathfrak{3}} \rightarrow \underline{\mathfrak{1}}(0) + \underline{\mathfrak{1}}(1) + \underline{\mathfrak{1}}(-1) \quad (5.18)$$

where the numbers in brackets are the U(1) representation numbers. As the Wilson loops generated from (5.17) will form a group isomorphic to Z_2 , then the effective low energy group will be $U(1)/Z_2$ which is trivially isomorphic to U(1) itself. The graph illustrates clearly that this broken symmetry minimum is the global minimum for all fermion numbers, and this turns out to be a general feature of periodic adjoint class breaking in all the cases which have been examined. It seems that, as the destabilising effect on the zero background is maximised by fermions satisfying

periodic boundary conditions, the new preferred minimum is lowered by a large enough margin with respect to the zero background for the zero vacuum to be deposited as the global minimum when even one generation of fermions is added. This is trivially true in the graph as the zero background becomes a maximum even for $N_F = 1$, but the statement appears to be true for all cases, even when N_F is below its critical value. This is demonstrated in the next case.

A slightly more complicated example is that where the symmetry group is the exceptional group generated from the algebra G_2 . As this group has unit centre, periodic fermions in any representation will break the symmetry, and we shall give the details here for fermions in the $\underline{7}$ and then in the $\underline{14}$ representations. It turns out that, as G_2 is a rank 2 algebra, the effective potential becomes a function of two variables and therefore, by choosing an appropriate section through weight space, we may illustrate the example graphically as before. (It is also possible to depict this information in the form of contour plots, giving equipotential lines as a function of the two background field components but, as can be seen from the example graphs, it is much easier to use the other method. The first method also has the benefit of allowing us to plot the function for several fermion numbers on the same graph, thus showing the development of the various turning points as N_F increases.) When we minimise the potential for fermions in the $\underline{7}$ representation we find that the seven dimensional Wilson loops form a representation of Z_3 at the global minima (see graph)

$$U_7 = \text{diag} (\omega, \omega, \omega, \omega^2, \omega^2, \omega^2, 1) \quad (5.19)$$

where ω is a cube root of unity. If we examine the branching rules for the $\underline{7}$ of G_2 into its regular subgroups (i.e. subgroups whose algebras have the same rank as the original algebra and whose Cartan subalgebras are contained in the original Cartan subalgebra), we find that (5.19) corresponds to

$$G_2 \rightarrow SU(3)$$

$$\underline{7} \rightarrow \underline{3} + \underline{3}^* + \underline{1} \quad (5.20)$$

where the Wilson loops can now be seen to lie in the centre of $SU(3)$. In order to confirm this, we should examine the branching of the adjoint (14) representation as this will give us the root system of the residual group without any ambiguity. At the minima, this gives

$$U_{ad} = \omega I_{6 \times 6} \oplus \omega^2 I_{6 \times 6} \oplus I_{8 \times 8} \quad (5.21)$$

corresponding to the branching

$$\underline{14} \rightarrow \underline{6} \oplus \underline{6}^* \oplus \underline{8} \quad (5.22)$$

The 8 zero elements of Θ_{ad} at the minima correspond to the roots of $SU(3)$, giving us 8 massless gauge bosons of the residual symmetry as required. It should be noted that, in one of the degenerate vacua, the $\underline{3}$ and $\underline{6}^*$ representations correspond to, say, the Wilson loop element ω , with the $\underline{3}^*$ and $\underline{6}$ representations corresponding to the ω^2 term. This indicates the congruency class nature of the subgroup representations. Another important point which should be noted is that, as the Θ matrices must contain non-zero blocks at any broken symmetry minimum, then many of the representations which appear in the branching will not have massless modes and therefore will not be seen at low energy. In this case, although the true residual group is $SU(3)$ due to the presence of the triplets, the effective low energy group is actually $SU(3)/Z_3$ and hence only the singlet and octet will contain massless fields.

If we now put fermions in the 14 representation, then, as can be seen from the graphs, the turning point at $\beta B^2 = 1$ deepens to become the global minimum, with the previous $SU(3)$ minima becoming only local. At this minimum, the fundamental Wilson loops become

$$U_7 = \text{diag} (-1, -1, -1, -1, 1, 1, 1) \quad (5.23)$$

and

$$U_{\text{ad}} = -I_{8 \times 8} \oplus I_{6 \times 6} \quad (5.24)$$

which correspond to the breaking

$$G_2 \rightarrow SU(2) \times SU(2)$$

$$\underline{7} \rightarrow (\underline{3}, \underline{1}) + (\underline{2}, \underline{2})$$

$$\underline{14} \rightarrow (\underline{3}, \underline{1}) + (\underline{1}, \underline{3}) + (\underline{4}, \underline{2}) \quad (5.25)$$

where again the adjoint representations are the only surviving representations in the low energy theory.

In such cases, it becomes cumbersome to determine the effective residual group, and it is customary in the superstring models where this breaking mechanism is used to quote only the true residual group and the representations which will contain massless modes on compactification. In the above case, although the true residual group is clearly $SU(2) \times SU(2)$ as both factors have doublet representations appearing in the branchings, it is unclear whether one or both of the $SU(2)$ factors has had its centre 'broken' at low energy, due to the absence of any $(\underline{1}, \underline{2})$ and $(\underline{2}, \underline{1})$ representations appearing in either of the branchings. The low energy situation would only become clear if another G_2 representation with such representations in its branching had been present in the original Lagrangian, but this would almost inevitably have changed the global minimum of the potential unless a large number of $\underline{14}$ fermions had been present. This type of ambiguity will also occur in more realistic cases and as, in general, we would prefer to have a small number of matter fields, it is far more reasonable just to quote the residual covering

group as was stated earlier.

In both these cases, the broken symmetry minima were shown graphically to be turning points of the potential for the cases of no fermion fields, 7 fermion fields and 14 fermion fields. The graphs for the antiperiodic case also show that the $SU(3)$ and $SU(2) \times SU(2)$ backgrounds are always turning points. In general, a procedure akin to that implemented for the symmetric minima case can be used to show that backgrounds corresponding to Wilson loops which represent the centre of any semi-simple subgroup of a given initial group are always turning points of the one loop potential. This proof is given in appendix B, along with a stability analysis of the $SU(3)$ minima as the number of antiperiodic fermions increases. It seems to be the case that the backgrounds corresponding to non semi-simple subgroups (groups with $U(1)$ factors) whose Wilson loops lie in the centre of the semi-simple component of the subgroup are also turning points, but we have not as yet proved this to be the case. The reason for this is the abelian nature of the $U(1)$ factors which effectively gives a one dimensional centre as the target space for the Wilson loops. This is to be contrasted with the much simpler zero dimensional finite order centres of the non-abelian groups.

We are now in a position to give several examples, the interpretation of the numerical results being carried out in exactly the manner detailed above for the $SU(2)$ and G_2 cases. In the table, we quote the results ($m \geq 3$) for selected representations of classical groups which could be of possible interest in model building as well for low dimensional representations of all the exceptional groups. The branchings of the original representations in each case are also given, with only the adjoint class representations of the residual groups containing massless fermions in the compactified theory. The significance of these results will be discussed in the final chapter.

group	rep	residual group	branching to irreducible reps
SO(2N+1)	2N+1	SO(2N)	(2N+1) \rightarrow 2N + 1
SU(N+1)	adjoint	[U(1)] ^N	singlets
G2	7	SU(3)	7 \rightarrow 3 + 3* + 1
G2	14	SU(2)xSU(2)	14 \rightarrow (1,3) + (3,1) + (2,4)
F4	26	SO(9)	26 \rightarrow 16 + 9 + 1
F4	52	SU(3)xSU(2)xU(1)	52 \rightarrow (8,1) + (1,3) + 2(1,2) + (1,1) + (6,2) + (6,1) + (6*,2) + (6*,1)
E6	78	SU(3)xSU(3)xSU(3)	78 \rightarrow (8,1,1) + (1,8,1) + (1,1,8) + (3,3,3*) + (3*,3*,3)
E8	248	SU(6)xSU(3)xU(1)	248 \rightarrow (15*,3*) + (15,3) + 2(6,3*) + 2(6*,3) + 2(20,1) + 3(1,1) + (35,1) + (1,8)

vii) General fermion phases

In this section, we shall discuss the effect of giving the fermions a general phase change on translation around the circle, i.e. by defining the fields as multi-valued sections of the principal fibre bundle. Our philosophy again will be that

the choice of a particular phase for the fermions in a particular model will be fixed for that particular model, i.e. that gauge transformations will leave the fermion phase invariant.

The first case we shall examine is that of fermions in a non-adjoint class. We have already seen that, for the cases of $\delta = 0$ or π , symmetric configurations form the global minima of the one loop potential irrespective of the order of the true centre. As the stability condition (5.10) is satisfied for more than half the range of a continuous parameter $x = q - \delta/2\pi$, then there will be at least one stable symmetric minimum for any choice of the phase δ . Numerical examinations of several cases have verified Hosotani's SU(5) result that the symmetric minima always form the global minima of the potential. As the true centre always has order Z_n , then it can easily be seen from the graph of the polylogarithm that for phases $\delta = 2\pi p/n$, $p \in Z$, there will be degenerate symmetric minima forming the global minima of the potential, whereas for any other phase there will be a unique symmetric configuration as the preferred background.

For the adjoint class case, numerical investigations reveal that the phases can be split into two subsets for a given spacetime dimension. As the only symmetric configuration is at $q = 0$, then there will be a range of phases centred on $\delta = \pi$, which will stabilise this configuration and, by the previously stated trend for a stable symmetric configuration to form the global minimum, the zero background will be preferred as the background for the theory. For phases outside this range, the minimum corresponding to the 'dominant subgroup' for that representation is found to be the global minimum. There will be two choices of phase for which the zero background and the broken minimum will be degenerate, and it would require some detailed analysis of a particular model to decide which configuration would be the preferred vacuum. If we adhere to the single valued section definition of the fields, then this problem will not arise.

viii) Graphs

1) Potential for antiperiodic fermions in the doublet of SU(2)

The x coordinate corresponds to $B^1/2\pi$ where B^1 is the non vanishing background field component. The symmetric configurations $q^{L1} = 0, 1/2$ are represented by $B^1/2\pi = 0, 1$ and are, as expected, degenerate when $N_F = 0$ ($q^{L0} = 0, \forall L$). This degeneracy is broken when fermions are introduced and the zero background becomes the global minimum for all non-zero N_F , thus leaving the gauge symmetry unbroken.

2) Potential for antiperiodic fermions in the triplet of SU(2) / Z_2

In this case, the only symmetric configuration is at $B = 0$ and, as before, this configuration deepens as N_F increases.

3) Potential for antiperiodic fermions in the triplet of SU(3)

For the SU(3) case, the Cartan subalgebra is two-dimensional and hence there are two non-trivial components to the background field. This graph shows a section through weight space corresponding to $(B^1 + B^2) / 2\pi = 1$, the reason for which is that this line intersects the three symmetric configurations $q^{L1} = 0, 1/3, 2/3$. The graph again possesses reflection symmetry, hence the two degenerate minima at $N_F = 0$ correspond to the zero background and one of the symmetric configurations. This degeneracy is again broken as N_F increases, thus leaving the symmetry unbroken.

4) Potential for antiperiodic fermions in the \mathbb{Z} of G_2

As G_2 also has a two dimensional Cartan subalgebra, then we again have two non-trivial background field components. As the group has unit centre, however, the only symmetric minimum will be the zero minimum for any representation. The graph shows the potential through the section $B^1 = 0$ (the reason for choosing this

section will be explained later) and, as before, the zero background deepens as N_F increases.

5) Potential for antiperiodic fermions in the 14 of G_2

For this case, the x coordinate is again $B^2/2\pi$, and again the zero background is the preferred global minimum.

6) Potential for periodic fermions in the doublet of $SU(2)$

When periodic fermions are added to the original Yang-Mills term in the Lagrangian, the zero background is destabilised. The graph verifies that the critical fermion number for destabilising the zero background is $N_F = 2$ (as derived in appendix A), and that the symmetric minimum at $q = 1/2$ is now the global minimum of the potential.

7) Potential for periodic fermions in the triplet of $SU(2) / Z_2$

In this graph, we see that, as predicted, the zero background is destabilised even for $N_F = 1$. As this is the only symmetric configuration for $SO(3)$, then the new minimum must correspond to a broken symmetry state. In this case the residual symmetry is $U(1)$.

8) Potential for periodic fermions in the triplet of $SU(3)$

For the $SU(3)$ triplet case, the critical number for the zero background is $N_F = 3$ as illustrated in the graph. Although there is no $q = 1/2$ configuration for this group, the symmetric minima at $q = 1/3$ and $2/3$ can be seen to be the new global minima.

9) Contour plot for periodic fermions in the triplet of $SU(3)$

In this contour plot, the potential is shown as a function of both B^1 and B^2 , clearly showing that the $q = 1/3$ and $2/3$ configurations are the global minima.

10) Potential for periodic fermions in the \mathbb{Z} of G_2

As in the previous examples, the zero background is again destabilised by periodic fermions with the critical number in this case being $N_F = 2$. As with the adjoint $SO(3)$ case, the new global minima must correspond to broken symmetry states and in this case, the residual symmetry is $SU(3)$.

11) Contour plot for periodic fermions in the \mathbb{Z} of G_2

As with the $SU(3)$ example, we can give a contour plot of the G_2 potential. This shows that the section $B^1 = 0$ does indeed intersect the global $SU(3)$ minima of the potential.

12) Potential for periodic fermions in the $\underline{14}$ of G_2

In this case, the broken symmetry minimum at $B^2 = \pi$ corresponds to residual group $SU(2) \times SU(2)$. The zero background is again destabilised at $N_F = 1$.

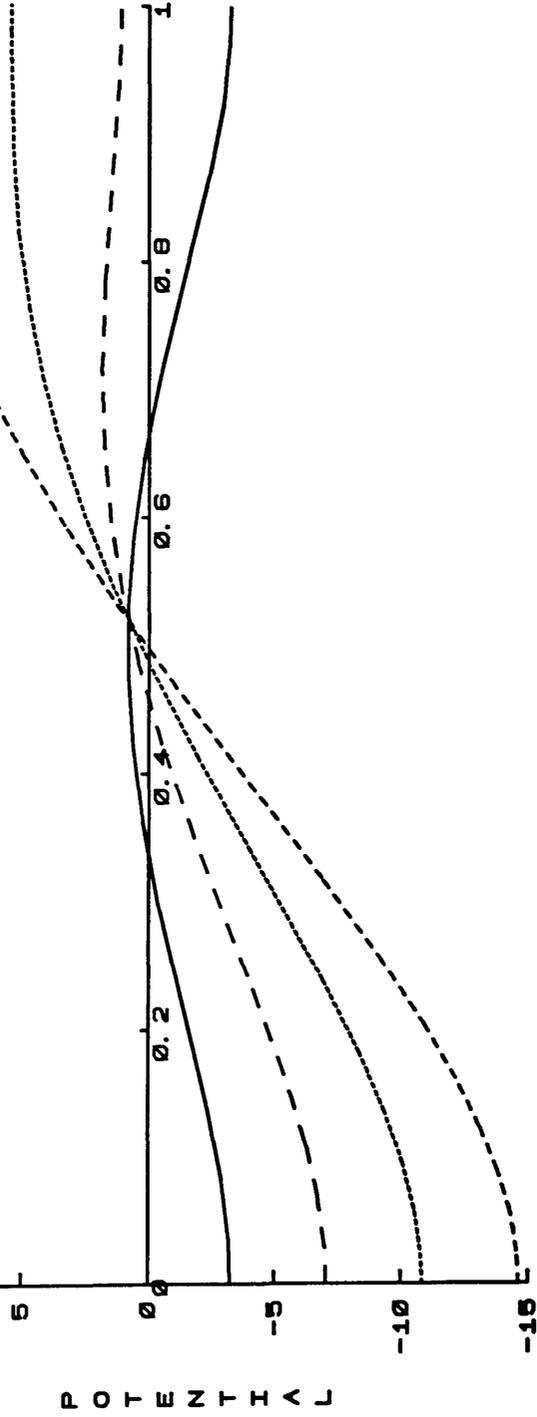
13) Contour plot for periodic fermions in the $\underline{14}$ of G_2

This graph shows that there are degenerate global minima at $(1,0)$, $(0,1)$ and $(1,1)$. All these minima correspond to residual true group $SU(2) \times SU(2)$ with the branchings as given in the text, but the three minima correspond to the three different effective low energy groups $SU(2) / \mathbb{Z}_2 \times SU(2)$, $SU(2) \times SU(2) / \mathbb{Z}_2$ and $SU(2) / \mathbb{Z}_2 \times SU(2) / \mathbb{Z}_2$. These groups are degenerate and indistinguishable as there are no $(\underline{1}, \underline{2})$ or $(\underline{2}, \underline{1})$ representations present in the branching of the $\underline{14}$ of G_2 .

14) Dimensional dependence of stability coefficient for fermion fields

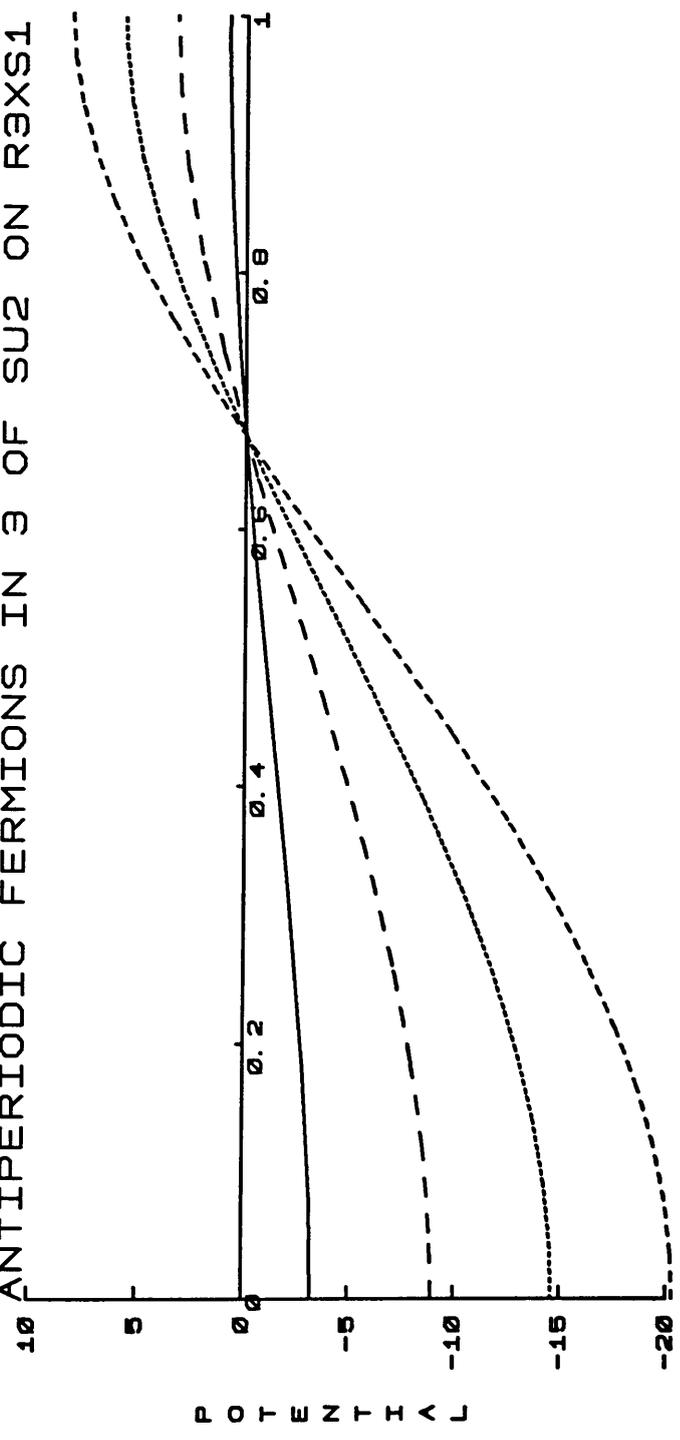
In the stability equation (5.9), we can ascertain the dimensional dependence of the critical number for the zero background (see appendix A). The graph shows the stability coefficient $(m - 1) 2^{[(m+1)/2]}$ as a function of m , illustrating the fact that the symmetric backgrounds are more readily destabilised by adjoint class periodic fermions in theories formulated in higher dimensions.

10 ANTIPERIODIC FERMIONS IN 2 OF SU2 ON R3XS1



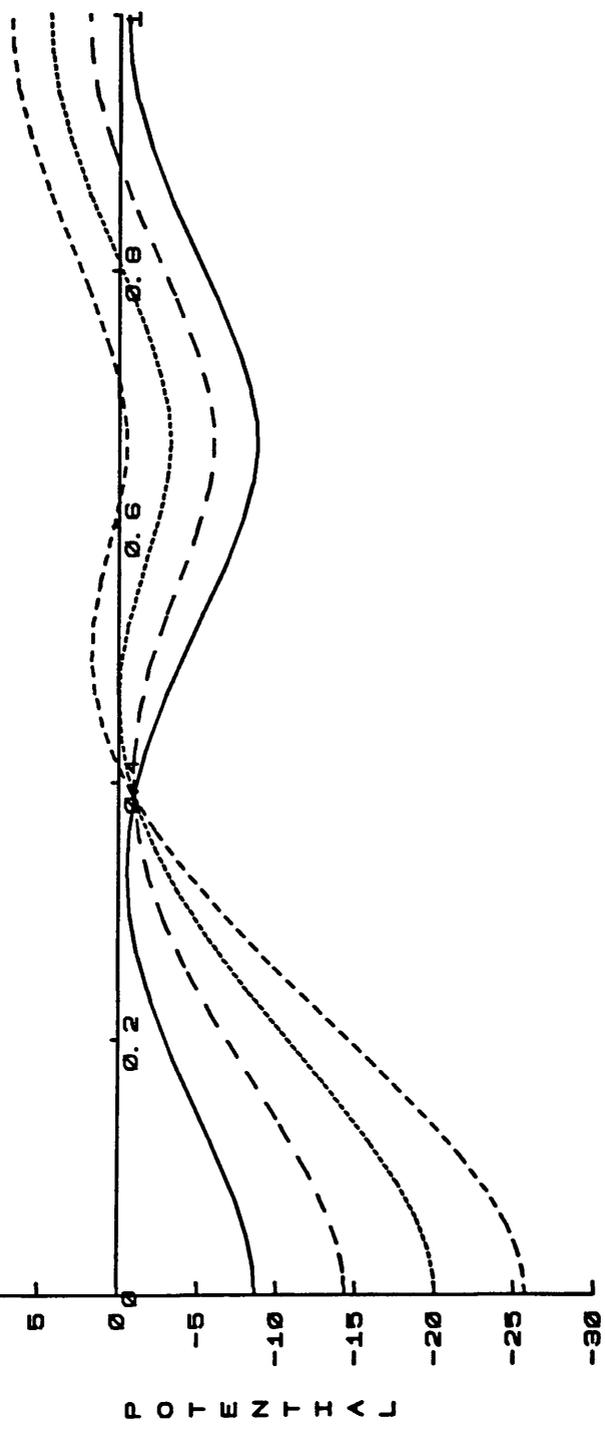
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- - - NF = 1
- NF = 2
- . - . NF = 3

ANTIPERIODIC FERMIONS IN 3 OF SU2 ON R3XS1



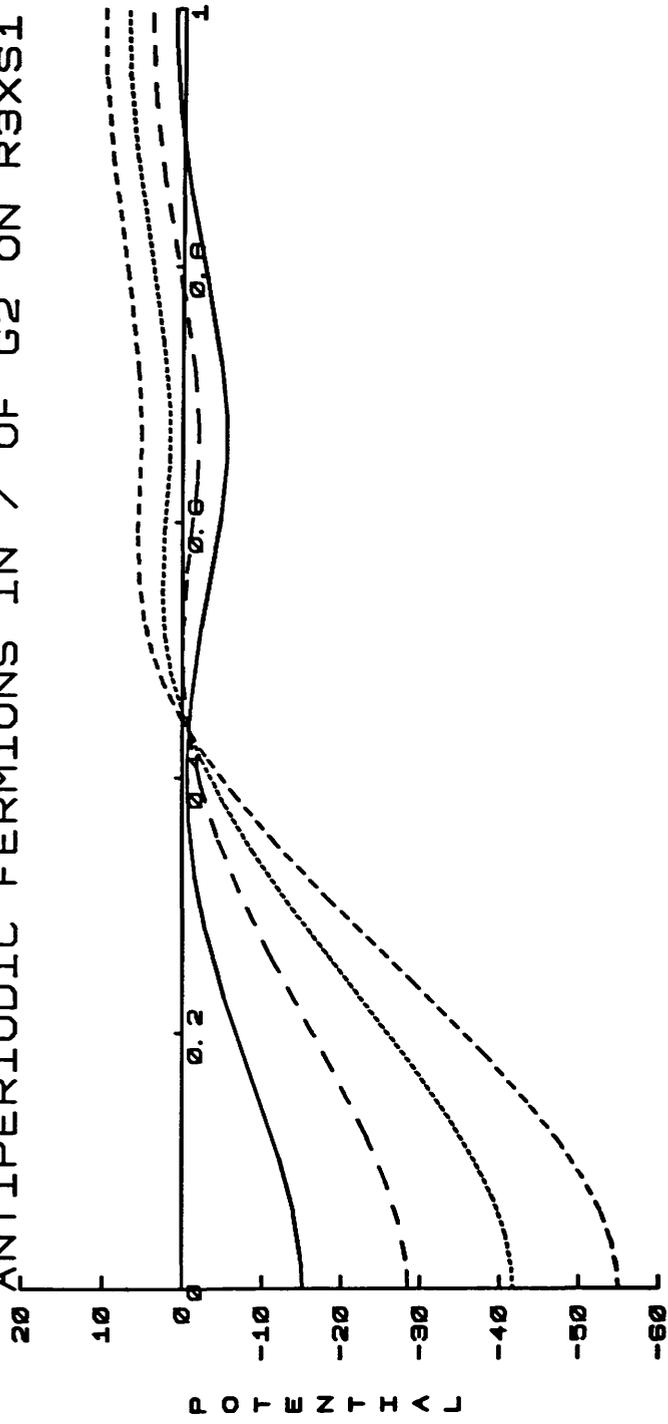
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- - - NF = 1
- NF = 2
- . - . NF = 3

10 ANTIPERIODIC FERMIONS IN 3 OF SU3 ON R3XS1



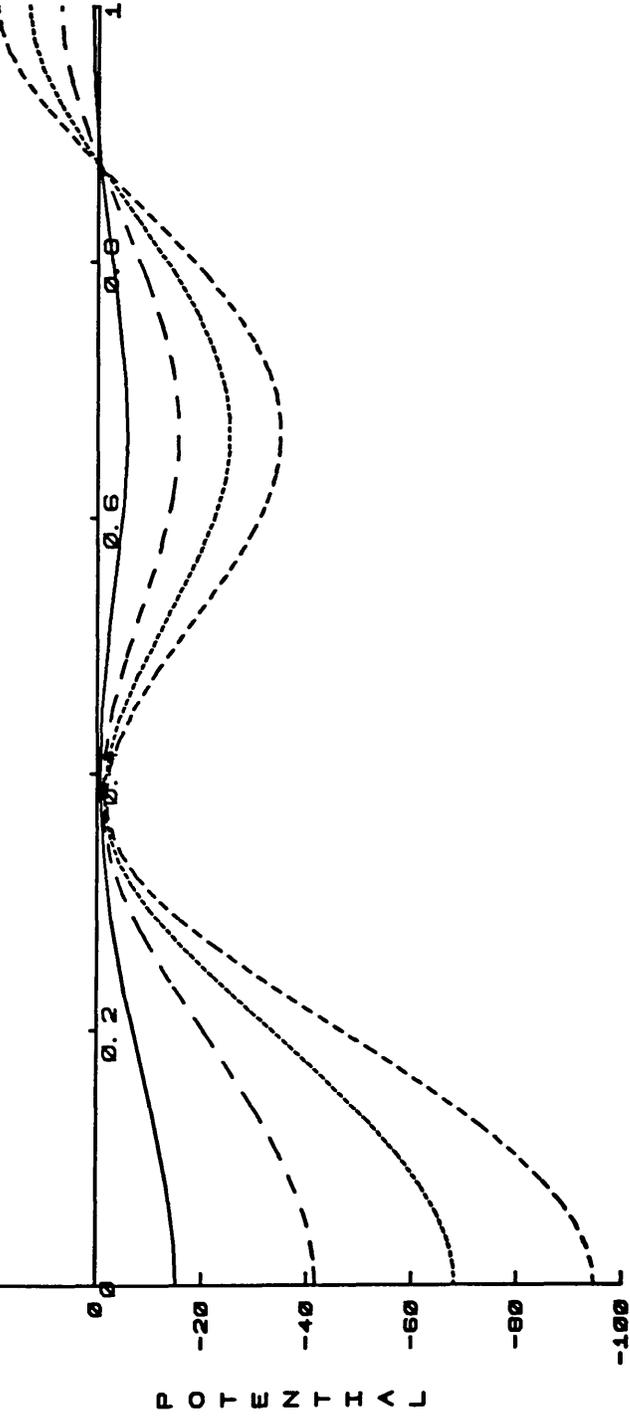
- NF = 0
- - - NF = 1
- NF = 2
- . - . NF = 3

ANTIPERIODIC FERMIONS IN 7 OF G2 ON R3XS1



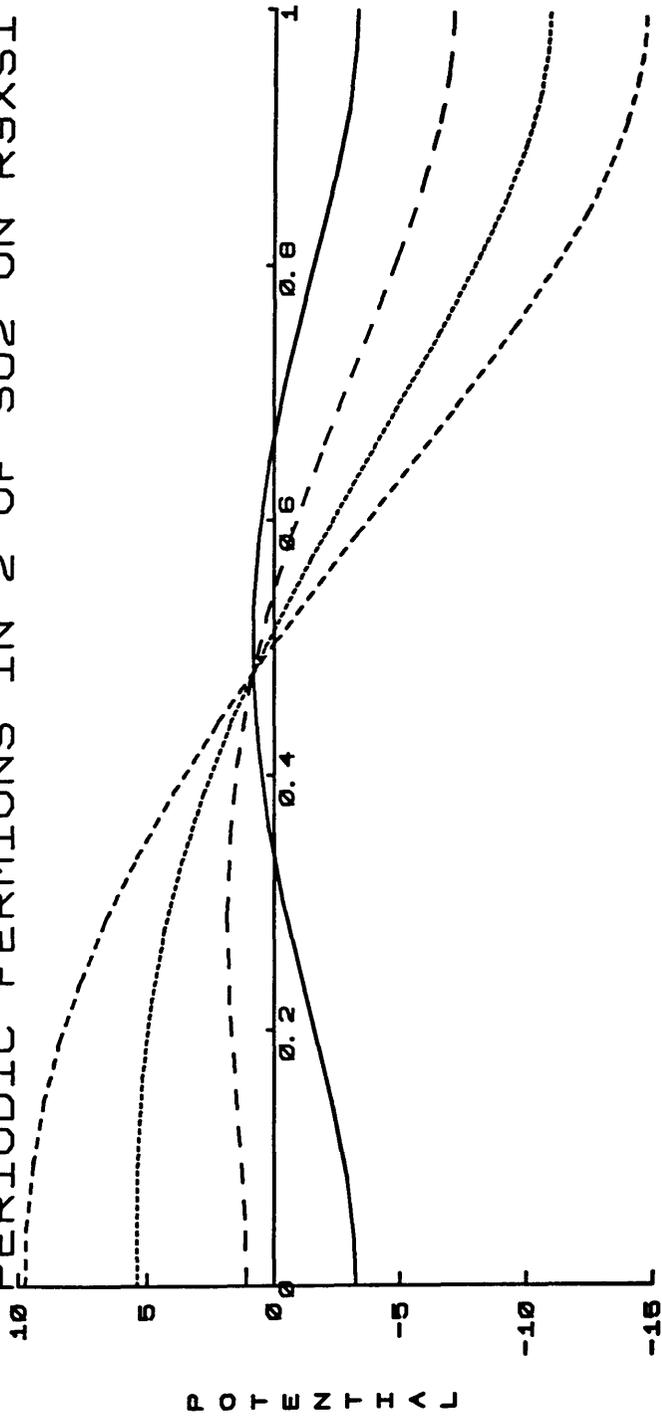
— NF = 0
 - - - NF = 1
 NF = 2
 - . - . NF = 3

ANTIPERIODIC FERMIONS IN 14 OF G2 ON R9XS1

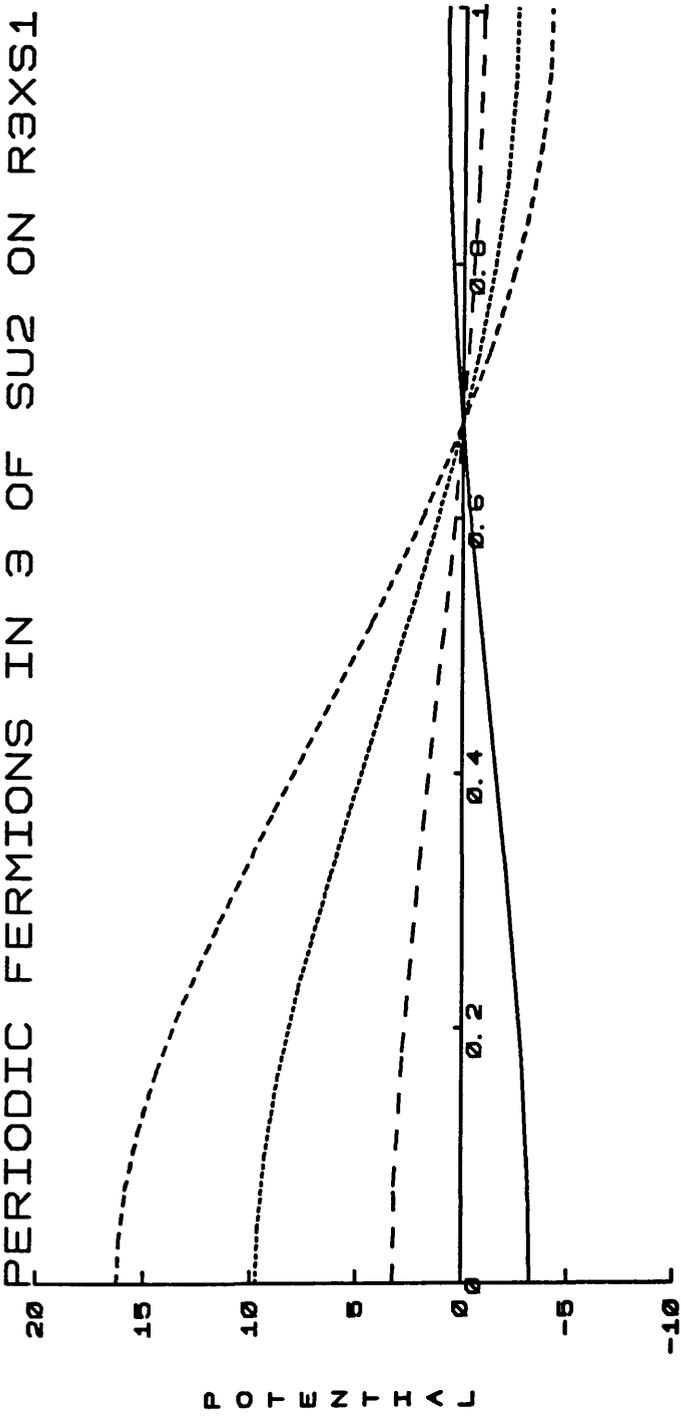


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N F = 1
N F = 2
N F = 3

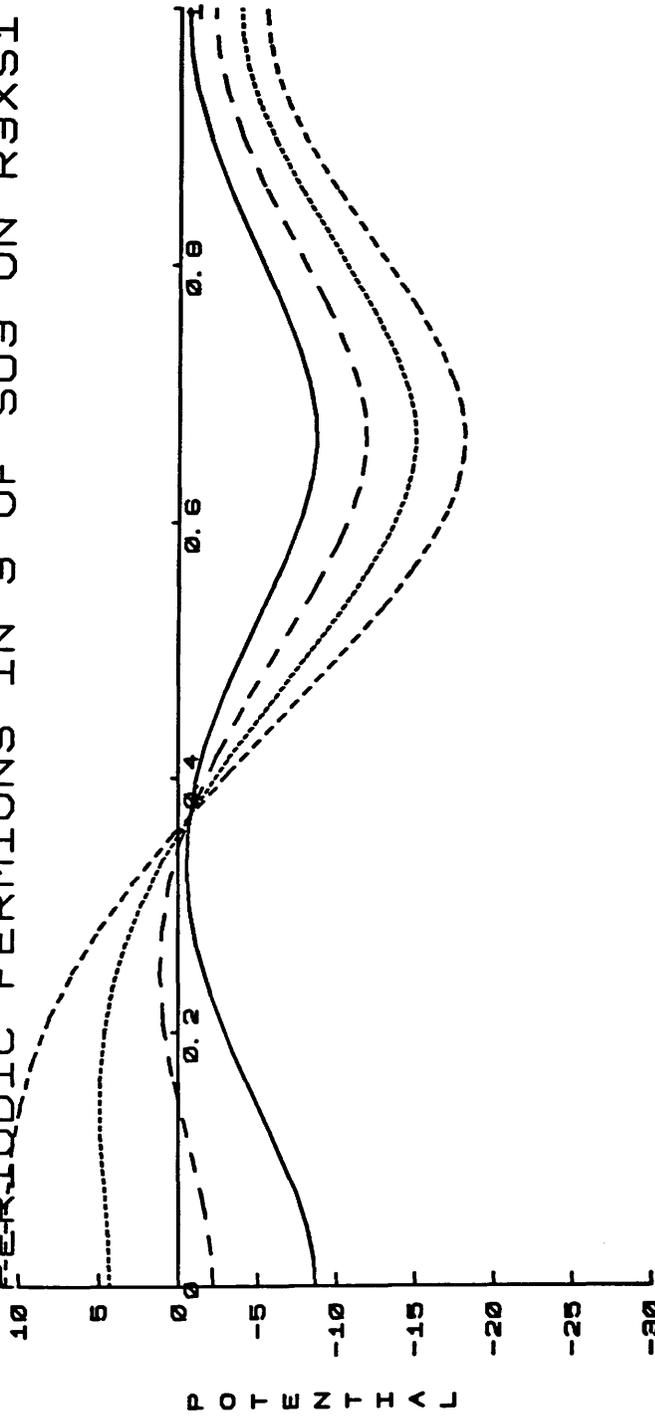
PERIODIC FERMIONS IN 2 OF SU2 ON R3XS1



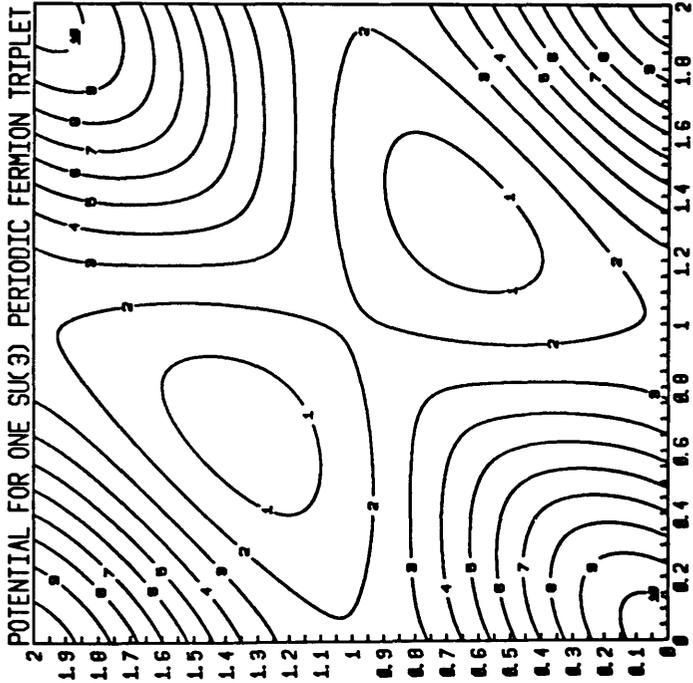
PERIODIC FERMIONS IN 3 OF SU2 ON R3XS1



PERIODIC FERMIONS IN 3 OF SU3 ON R3XS1

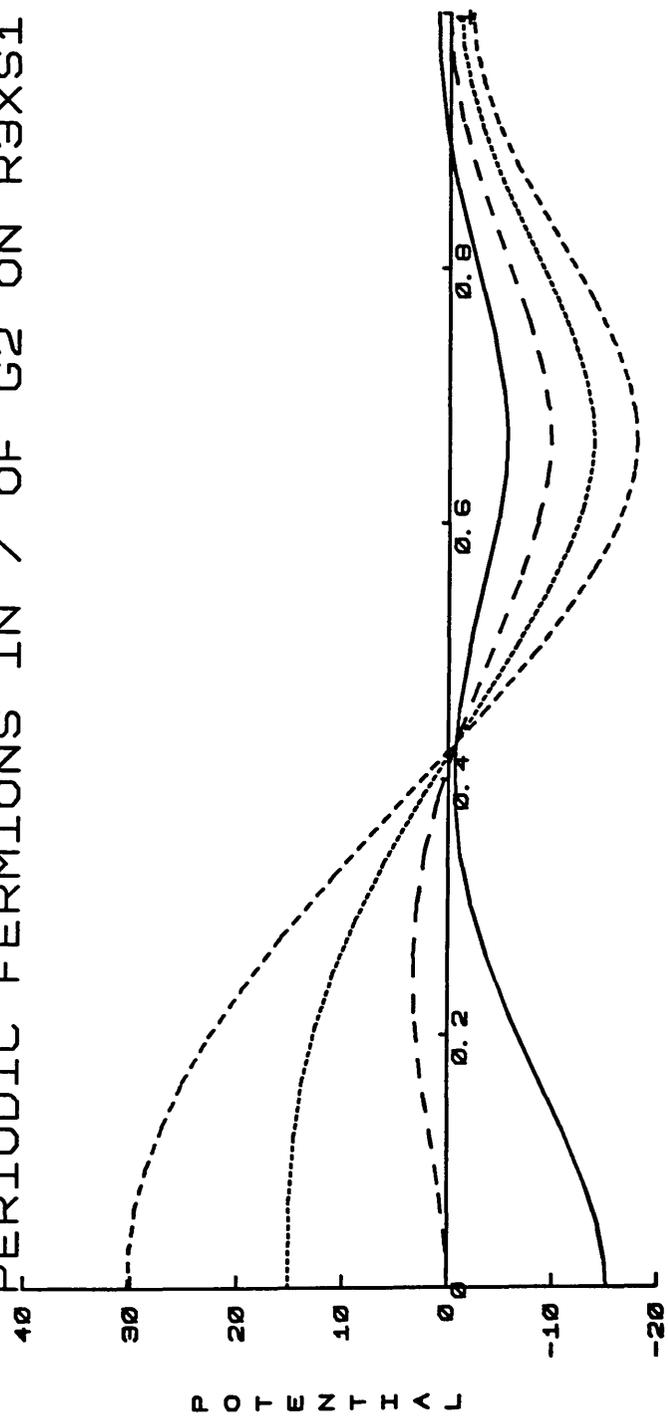


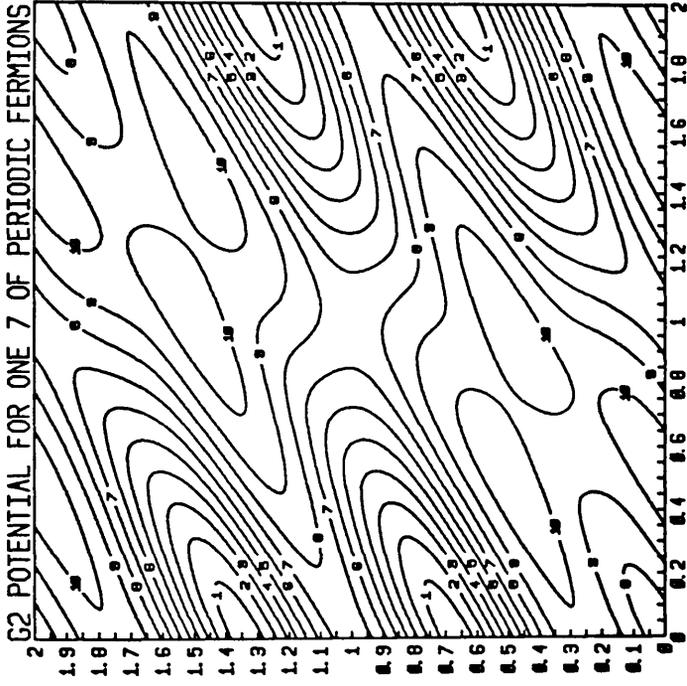
- N F = 0
- - - N F = 1
- N F = 2
- . - . N F = 3



CONTOUR KEY	
1	-11.303984
2	-10.341940
3	-9.379897
4	-8.417853
5	-7.455809
6	-6.493766
7	-5.531722
8	-4.569678
9	-3.607635
10	-2.645591

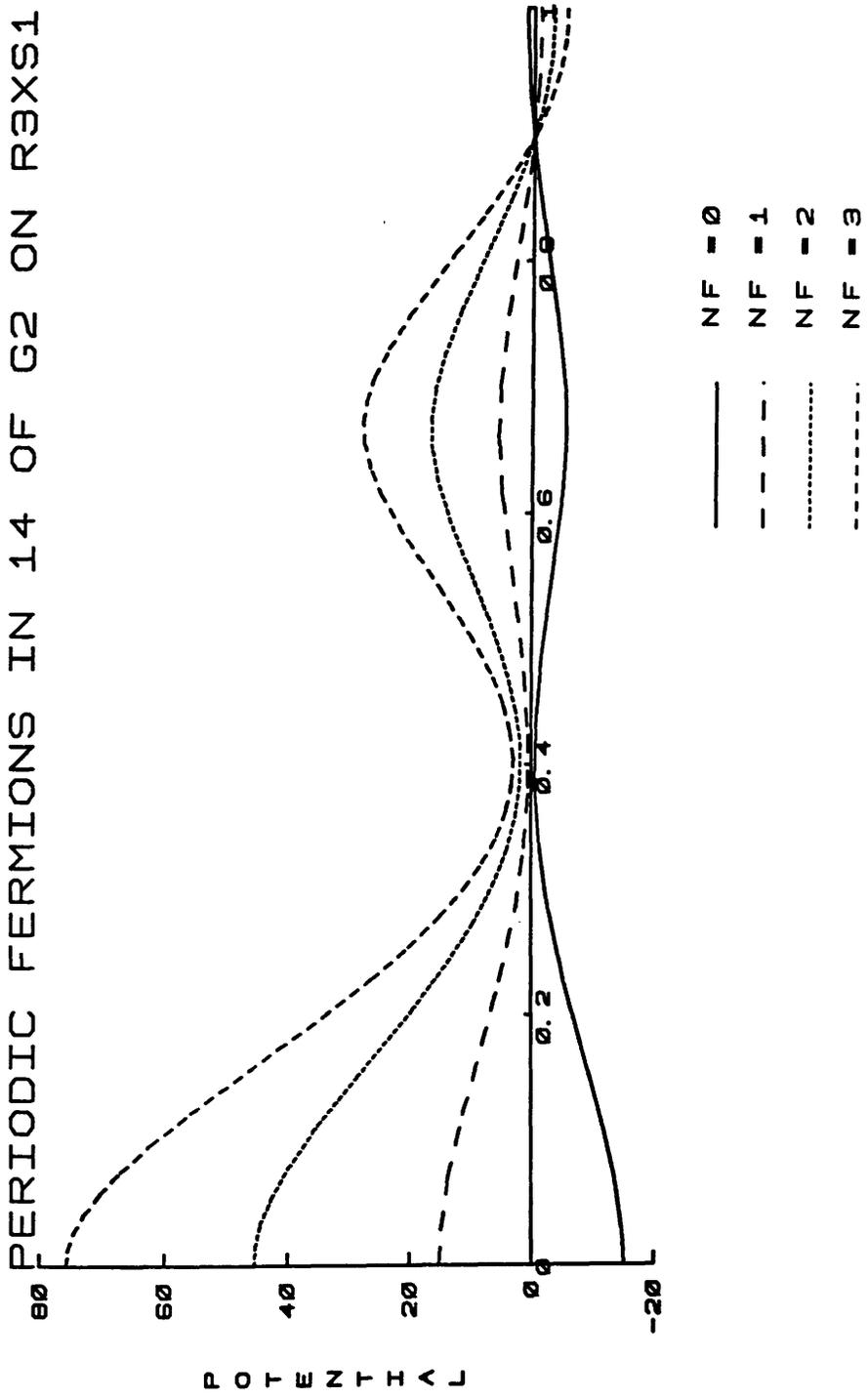
PERIODIC FERMIONS IN 7 OF G2 ON R3XS1

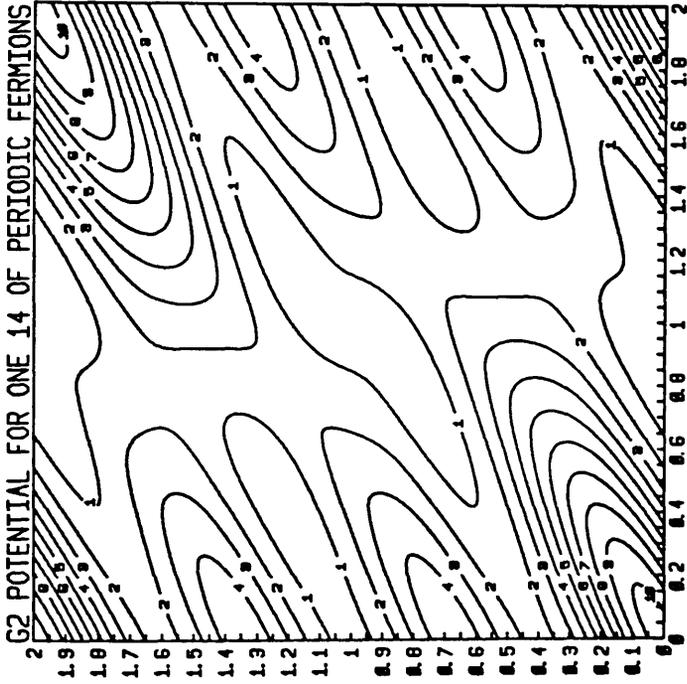




CONTOUR KEY	
1	-8.881441
2	-7.519743
3	-6.158044
4	-4.796346
5	-3.434648
6	-2.072950
7	-0.711252
8	0.650447
9	2.012145
10	3.373843

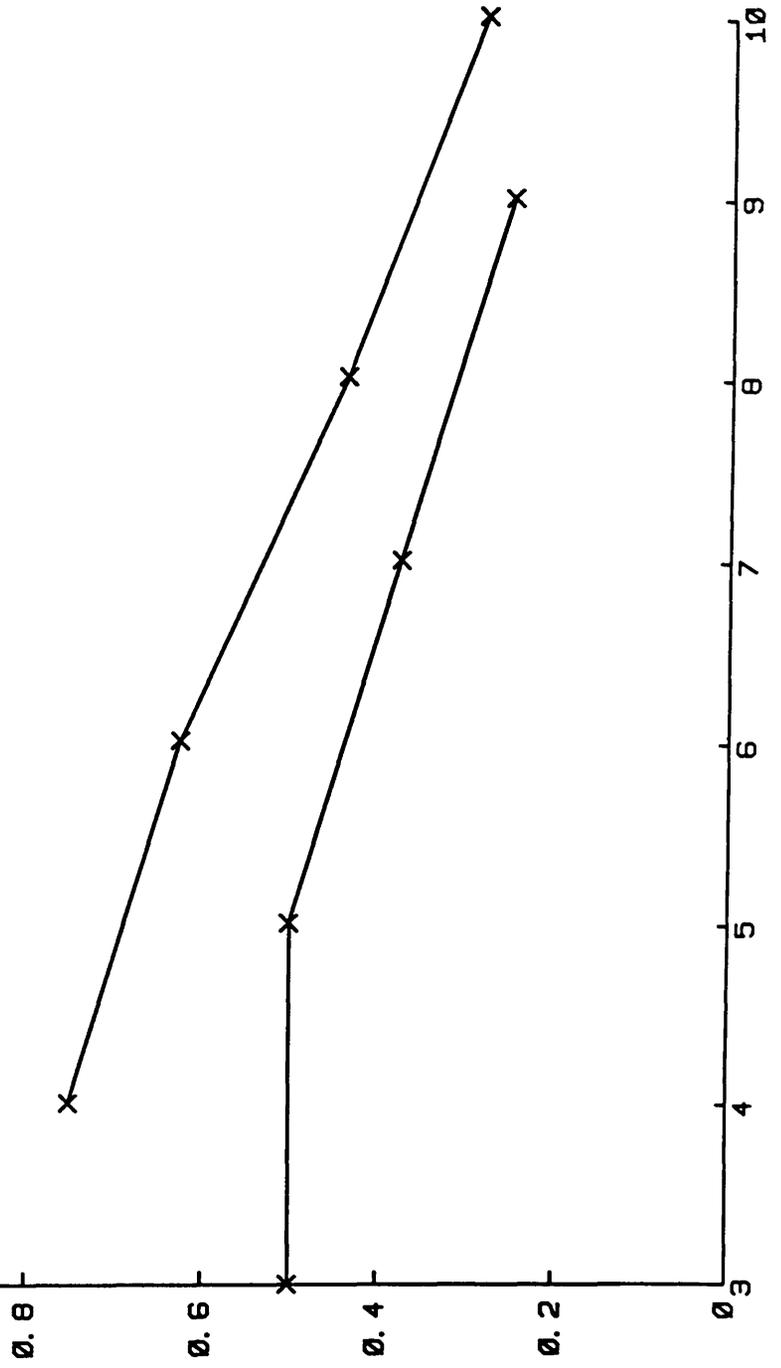
PERIODIC FERMIONS IN 14 OF G2 ON R3XS1





CONTOUR KEY	
1	-0.635497
2	1.025978
3	2.687452
4	4.348927
5	6.010402
6	7.671876
7	9.333351
8	10.994825
9	12.656300
10	14.317774

PERIODIC FERMION STABILITY COEFFICIENT



CHAPTER 6

Generalised Hosotani model - scalar fields

Consider the Lagrangian

$$\mathcal{L} = -\frac{1}{2} \text{Tr}(\partial_\mu \phi \partial^\mu \phi) - \frac{1}{2} \text{Tr}(\phi^2)$$

where ϕ is a scalar field in the adjoint representation of $U(N)$.

The equations of motion are

$$\partial_\mu \partial^\mu \phi + \phi = 0$$

which can be solved by taking the ansatz

$$\phi = \sum_{\mathbf{k}} \tilde{\phi}_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}}$$

where $\tilde{\phi}_{\mathbf{k}}$ is a function of the momentum \mathbf{k} .

Substituting this into the equations of motion

$$-\mathbf{k}^2 \tilde{\phi}_{\mathbf{k}} + \tilde{\phi}_{\mathbf{k}} = 0$$

we find that the solutions are

$$\tilde{\phi}_{\mathbf{k}} = \frac{1}{\mathbf{k}^2 - 1} \tilde{\phi}_{\mathbf{k}}(0)$$

where $\tilde{\phi}_{\mathbf{k}}(0)$ is a constant.

The general solution is therefore

$$\phi = \sum_{\mathbf{k}} \frac{1}{\mathbf{k}^2 - 1} \tilde{\phi}_{\mathbf{k}}(0) e^{i\mathbf{k} \cdot \mathbf{x}}$$

where the sum is over all momenta \mathbf{k} .

The energy of the system is given by

$$E = \int d^3x \mathcal{L}$$

which can be evaluated using the above solution.

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The equations of motion are

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$$\phi = \sum_{\mathbf{k}} \frac{1}{\mathbf{k}^2 - 1} \tilde{\phi}_{\mathbf{k}}(0) e^{i\mathbf{k} \cdot \mathbf{x}}$$

where the sum is over all momenta \mathbf{k} .

The energy of the system is given by

$$E = \int d^3x \mathcal{L}$$

which can be evaluated using the above solution.

i) Scalar equivalents of fermion results

In this chapter, we shall examine the model in the case where the Dirac fermion field have been replaced by complex scalars whose phase factors initially will be either $\delta = 0$ or π . we will show that there is an extra subtlety in this case which was not present with fermionic fields, and this will allow slightly more freedom in breaking symmetries for certain types of true group.

The fermion term in the one loop potential is now replaced by

$$V_S = -2 N_S C(m) \text{Re Tr Li}_{m+1} (e^{i\delta} U_R) \quad (6.1)$$

it is trivial to see that, at the symmetric minima

$$\frac{\partial V_S}{\partial B^a} = -2 N_S \beta C(m) \text{Re Li}_m (e^{2\pi i q + i\delta}) \text{Tr H}^a = 0 \quad (6.2)$$

and hence the symmetric backgrounds still correspond to stationary points of the one loop potential. We can therefore proceed with the derivation of the stability condition for these configurations by taking the second derivative of the full potential as before. The scalar version of (5.10) then becomes

$$N_S \frac{-\text{Re Li}_{m-1} (e^{-2\pi i q + i\delta})}{\zeta(m-1)} < \frac{m-1}{2} \frac{I_2(\text{ad})}{I_2(\text{R})} \quad (6.3)$$

The change in statistics has resulted in a minus sign appearing on the left hand side of this equation, the repercussions of which will be greater than first anticipated, due to the fact that the zeros now lie in the range $(1/4, 3/4)$. Although the polylogarithm graphs have already been given, it will turn out to be useful to refer to the the graph of $-\text{Li}_m(e^{-2\pi i x})$ given at the end of the chapter. As the global minimum of this graph is now at $x = 0$ and the maximum at $x = 1/2$, then we would expect scalars

with phase δ to mimic the effects of fermions with phase $(\delta - \pi)$.

It is easy to see that the periodic scalars give the potential minimum at zero background and therefore give exactly the same result as the antiperiodic fermion case. (With the Higuchi - Parker philosophy, the choice of the zero background will determine the scalar phase to be $\delta = 0$.) For the case of antiperiodic scalars, we must deal separately with the different representation congruency classes in order to determine what happens. As with the non-adjoint class periodic fermions, the non-adjoint class antiperiodic scalars prefer one of the non-zero symmetric backgrounds, hence leaving the true symmetry group unaffected and giving as an effective low energy group either the original group modulo Z_2 or the adjoint group, depending on whether the true centre has even or odd order respectively. What is not clear is what will happen when antiperiodic scalars are put into the adjoint class as, even though the minimum of the negative polylogarithm is at the same configuration as the positive polylogarithm for the adjoint class periodic fermions case, the shape of the potential either side of the minimum is markedly different.

Surprisingly enough, the results given for the periodic fermion case in the last chapter are reproduced exactly by antiperiodic scalars, although the critical numbers obviously change, as can be seen from the scalar $SU(2)$, $SU(3)$ and $G(2)$ graphs. The calculation of the critical numbers for the various graphical examples are again given in appendix A. The fact that the dominant subgroups seem to be the same for either fermion or scalars in a particular representation seems to imply that the connection between the breaking pattern and the matter representation is more important than the statistics of the matter fields in question. The fact that the minimum of $-\text{Li}_m(e^{ix})$ coincides with the minimum of $+\text{Li}_m(-e^{ix})$ seems to be enough to select a particular dominant turning point for the chosen representation, with the change in the potential shape around the minimum seeming to be relevant only for the (gauge) scalar mass calculation when the gauge fields are compactified.

We have therefore determined the correspondence between scalar and fermion breaking of the symmetry, but there is another possibility for symmetry breaking

which we shall now describe.

ii) Scalar breaking of true groups with Z_2 centre

In the previous section, we restricted ourselves to the case where the scalars were either periodic or antiperiodic, in which case the global features were the same as for the fermion case. As in the fermion case however, we can now consider cases where the phase takes a general value δ in the range 0 to 2π . For most cases, the effect of this is exactly as described for fermions, but there is one major exception. If we have adjoint class scalars, there will be a transition from the zero background to the broken background at some choice of the scalar phase, the exact position of this transition depending on the dimension of the spacetime. This is exactly as was found for the fermion case, and again it must be emphasised that, even though we are considering the development of the potential as the phase changes, the choice of a particular phase in any model will be treated as being fixed, and will not be gauge transformable to any other phase value.

The new feature appears when we consider non-adjoint class scalar fields with a general boundary condition. When we had fermions, the left hand side of the stability condition (5.10) had a plus sign in front of the polylogarithm, whereas for the scalar equivalent, there is a minus sign on the left hand side. The consequence of this is that, whereas the fermion stability function was satisfied for more than half the range of the continuous x parameter as defined earlier, the scalar function is stable for less than half the range. As the stable area for scalars increases as the dimension increases, then the smallest stable area will occur for the smallest spacetime dimension in which the stability conditions are valid, i.e. the $m = 3$ case as studied by Hosotani [12]. In such a case, we can easily work out the position of the zeros of the left hand side of (6.3) as the stability condition just becomes a quadratic in the continuous variable x . The zeros are found to be at

$$B_2(x) = x^2 - x + \frac{1}{6}$$

$$\Rightarrow x = \frac{1 \pm \sqrt{1 - \frac{2}{3}}}{2} \quad (6.4)$$

and therefore we see that the scalar stable region width for $m = 3$ is the range where the Bernoulli function in (6.4) is positive, i.e.

$$\Delta x_{\text{stable}} = \frac{1 - \sqrt{1 - \frac{2}{3}}}{2} + \left[1 - \frac{1 + \sqrt{1 - \frac{2}{3}}}{2} \right]$$

$$= \frac{\sqrt{3} - 1}{\sqrt{3}} \quad (6.5)$$

and therefore is never less than $1/3$ of the full range. This then implies that if we have scalars in a representation whose true symmetry group has centre Z_n of order 3 or greater then, as for the fermion case, there will always be a stable symmetric configuration which will form the global minimum of the potential, with two degenerate symmetric configurations will becoming degenerate global minima at $\delta = 2\pi p/n$ for integer p . This is again analogous to the fermion case. Where the difference comes in is when we have the true group of the scalar representation having a Z_2 centre. In this case, the only symmetric minima are at $q = 0$ and $q = 1/2$. In this case, if we give the scalars a phase $\delta = \pi/2$, then the scalar term will give a destabilising effect to both the symmetric configurations, and therefore we may expect to be able to break the symmetry at some stage. Unlike the adjoint breaking, where the destabilising effect on the symmetric fields was maximised and hence the symmetry broke even when one matter representation was present, the destabilising effect here is fairly small, which means that a large number of representations will be required in order for any breaking to occur. A critical number beyond which the symmetry must break is again given in appendix A.

An example of this was found numerically by Hosotani. In his paper, he found that $SU(2)$ doublet scalars on $R^3 \times S^1$ could break $SU(2)$ to $U(1)$ provided that the number N_S of such representations was greater than 16. He also gave the range of phase values around the value $\pi/2$ which would still give this destabilising effect, and this range can be derived from (6.4) and (6.5). To do this, we require the zeros found for $x = q + \delta/2\pi$ to lie between the symmetric values $q = 0$ and $q = 1/2$. This then gives

$$\delta_{\min} = 2\pi \frac{1 - \sqrt{\frac{1}{3}}}{2} = \frac{\sqrt{3} - 1}{\sqrt{3}} \pi$$

$$\delta_{\max} = 2\pi \left[\frac{1 + \sqrt{\frac{1}{3}}}{2} - \frac{1}{2} \right] = \frac{1}{\sqrt{3}} \pi \quad (6.6)$$

As the spacetime dimension increases, this range of phases will become smaller as the stable region increases, with the number of scalar representations also increasing. The table gives some example breaking patterns, again for $m \geq 3$.

group	rep	residual group	branching to irreducible reps
$SU(4)$	6	$SU(3) \times U(1)$	6 + 9 (massless)
$SU(6)$	15	$SU(5) \times U(1)$	10 + 15 (massless)
$SO(10)$	10	$[U(1)]^5$	singlets
E_7	56	$[U(1)]^7$	singlets

iii) Graphs

1) Polylogarithm

This graph shows the negative real part of $\text{Li}_m(e^{-2\pi ix})$ as a function of x for the cases $m = 3, 4$ and 10 . This emphasises the fact that the zeros now lie in the range $(1/4, 3/4)$, a fact which proves crucial if scalars are permitted to have general phases.

2) Potential for periodic scalars in doublet of $SU(2)$

3) Potential for periodic scalars in triplet of $SU(2)$

4) Potential for periodic scalars in $\underline{7}$ of G_2

5) Potential for periodic scalars in $\underline{14}$ of G_2

All these graphs illustrate the deepening of the zero background when periodic boundary conditions are imposed.

6) Potential for antiperiodic scalars in doublet of $SU(2)$

This shows the deepening of the $q = 1/2$ configuration. The critical number for the zero background is now $N_F = 8$, although this fact is not clear in the graph.

7) Potential for antiperiodic scalars in doublet of $SU(2)$

A magnified version of the previous graph near the $q = 0$ configuration shows that the critical number is indeed 8. The $N_F = 8$ and 9 curves have been translated in order to illustrate the nature of the turning points on the same graph.

8) Potential for antiperiodic scalars in triplet of $SU(2)$

In this case, we again obtain the $U(1)$ minimum as the new global minimum.

Note that, unlike the adjoint representation periodic fermion case, the $q = 0$ configuration is not destabilised at $N_F = 1$, due to the fact that the scalar contribution to the potential is not as great as the corresponding fermion term. In this case, the critical number is $N_F = 2$.

9) Potential for antiperiodic scalars in $\underline{7}$ of G_2

For this case, the $SU(3)$ minima are still the preferred global minima. The zero background destabilises at $N_F = 8$.

10) Potential for antiperiodic scalars in $\underline{14}$ of G_2

The $SU(2) \times SU(2)$ minimum is still preferred for the adjoint G_2 case, with the zero background destabilising at $N_F = 2$.

11) Dimensional dependence of the antiperiodic scalar stability coeff.

This stability coefficient is derived in appendix A, and it turns out that, unlike the fermion case, the critical scalar number increases as the dimension of the spacetime is increased. It therefore seems that, in cases with more complicated Lagrangians, many scalar fields may be required in order to achieve the results obtainable by considerably fewer fermion fields.

12) Potential for $\delta = \pi/2$ scalars in the doublet of $SU(2)$

This shows the destabilising effect of the scalars on both the symmetric configurations $q = 0$ and $1/2$. The critical number for both these configurations is calculated in appendix A to be 32, but this is not clear from this form of the graph. The new minimum corresponds to residual $U(1)$ symmetry.

13) Potential for $\delta = \pi/2$ scalars in the doublet of $SU(2)$

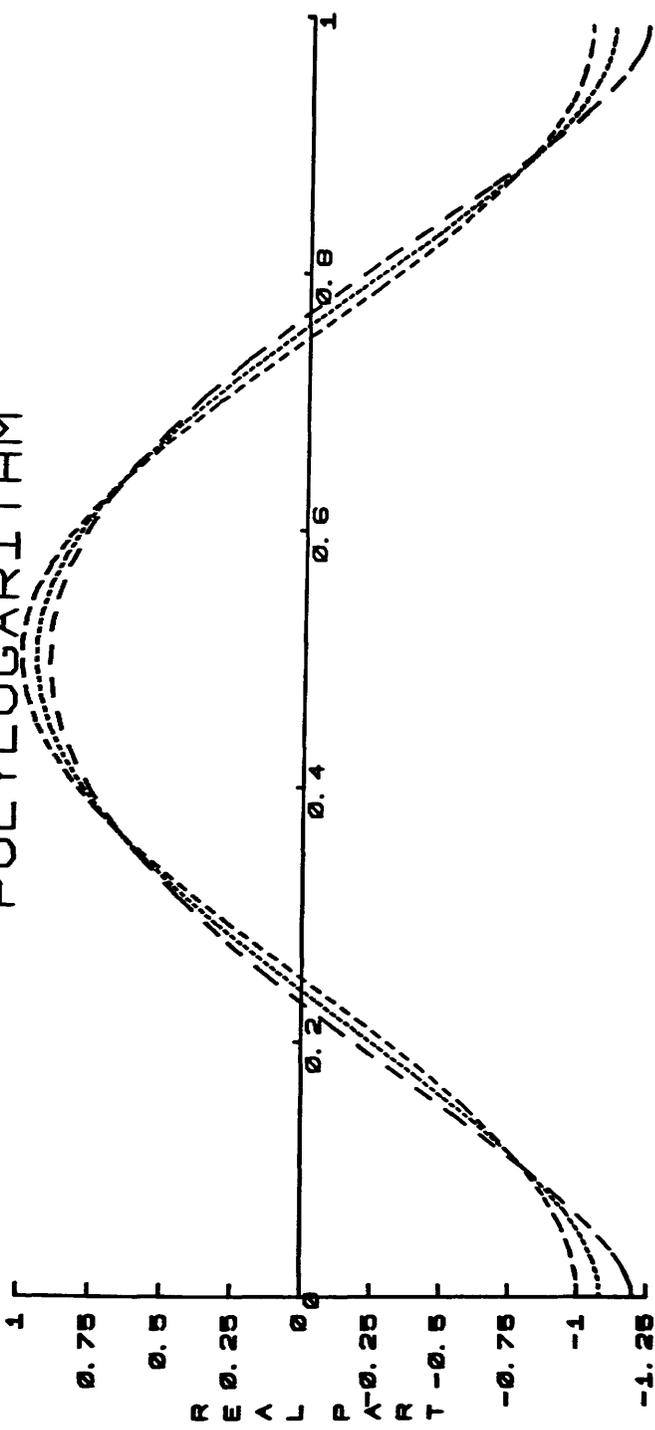
This enlarged area around the potential axis verifies the critical number as being

$N_S = 32$, where again we have translated the curves in order to fit them on the same graph.

14) Stability coefficient for $\pi/2$ scalars

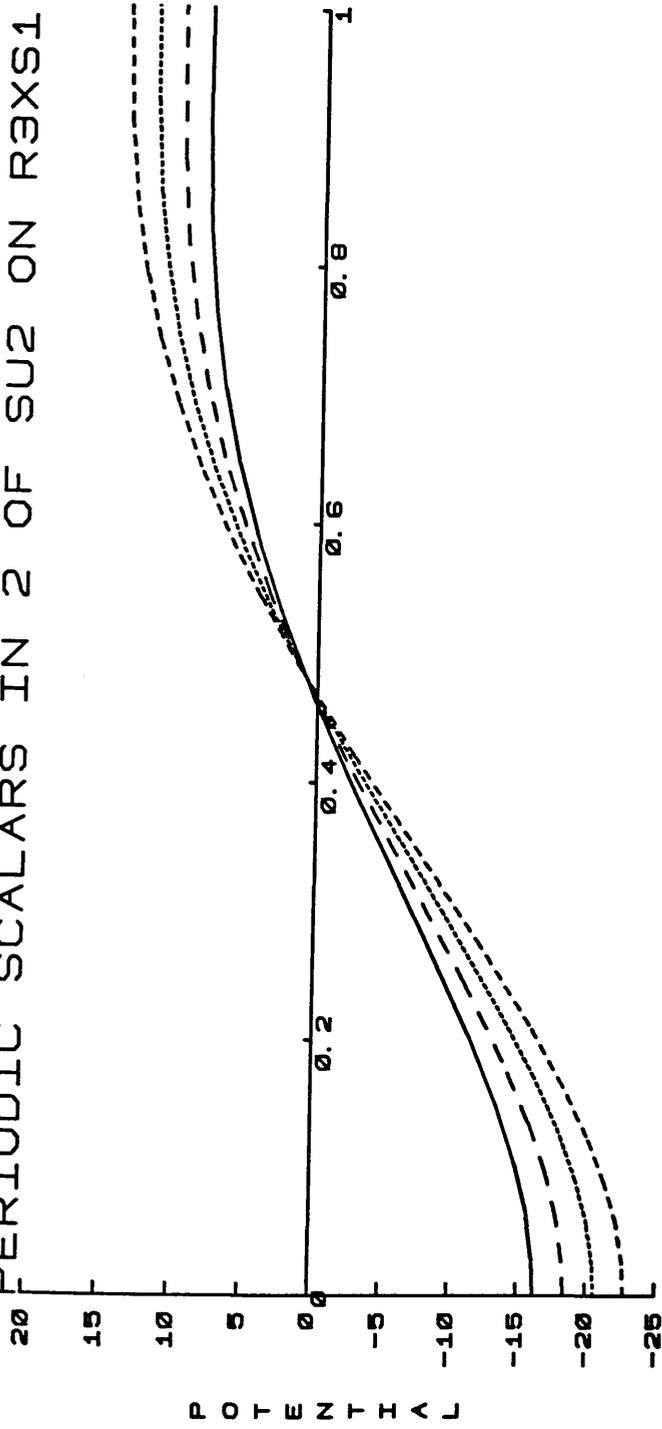
This stability coefficient is also calculated in appendix A and shows an even more dramatic increase with m . Such scalars are therefore unlikely to be of much use for breaking the symmetry in a more realistic Lagrangian.

POLYLOGARITHM



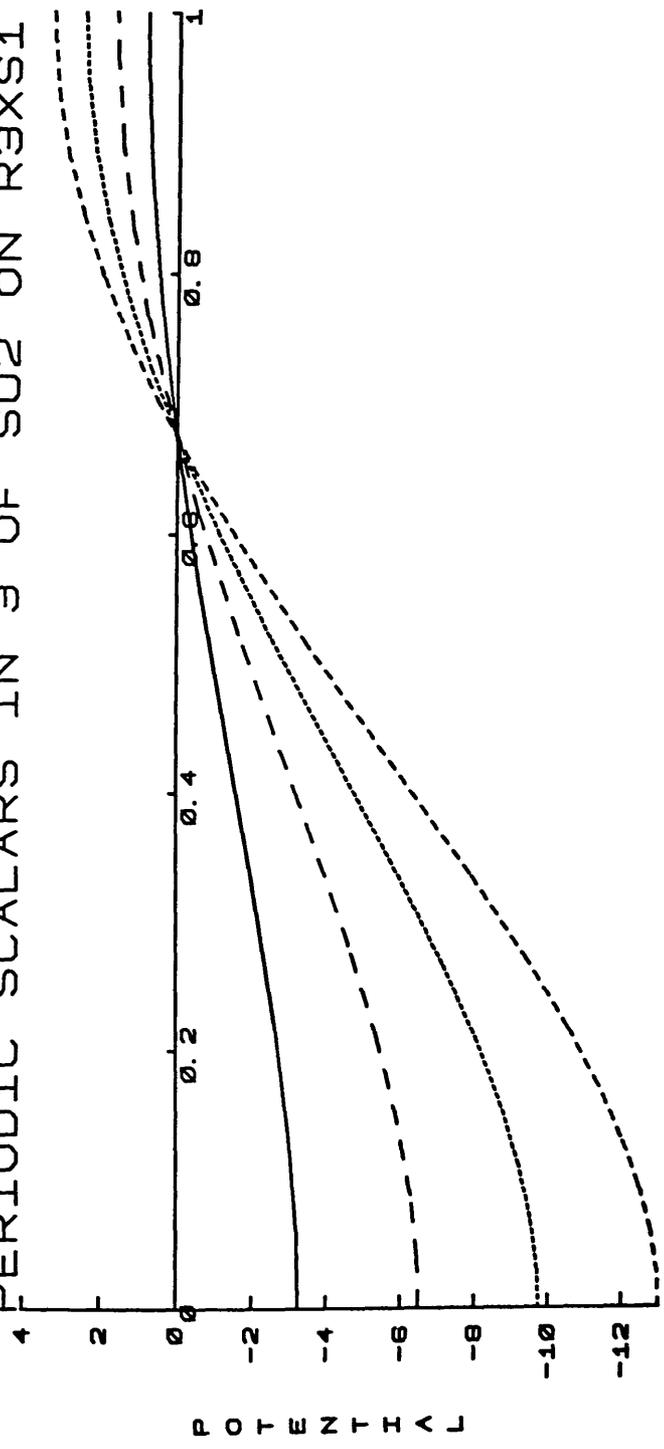
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 ORDER = 4
 - - - - - ORDER = 10

PERIODIC SCALARS IN 2 OF SU2 ON R3XS1



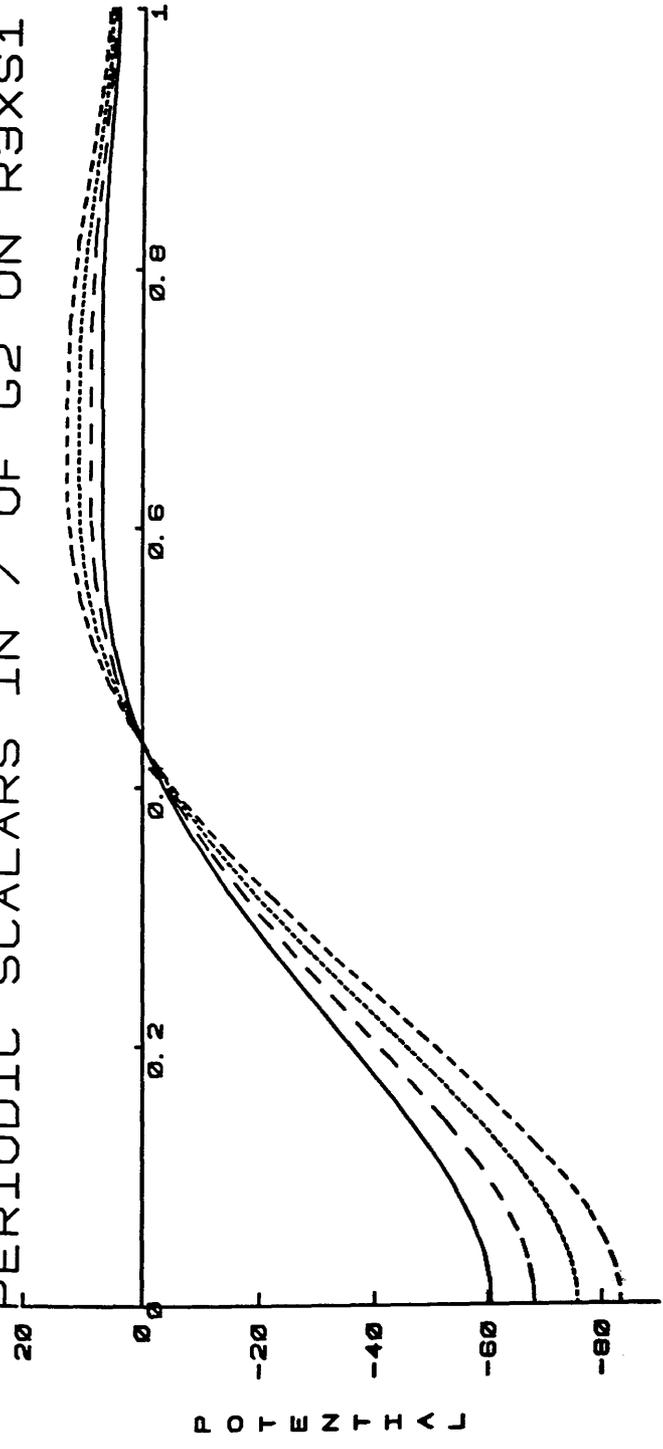
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- - - NF = 7
- NF = 8
- . - . NF = 9

PERIODIC SCALARS IN 3 OF SU2 ON R3XS1



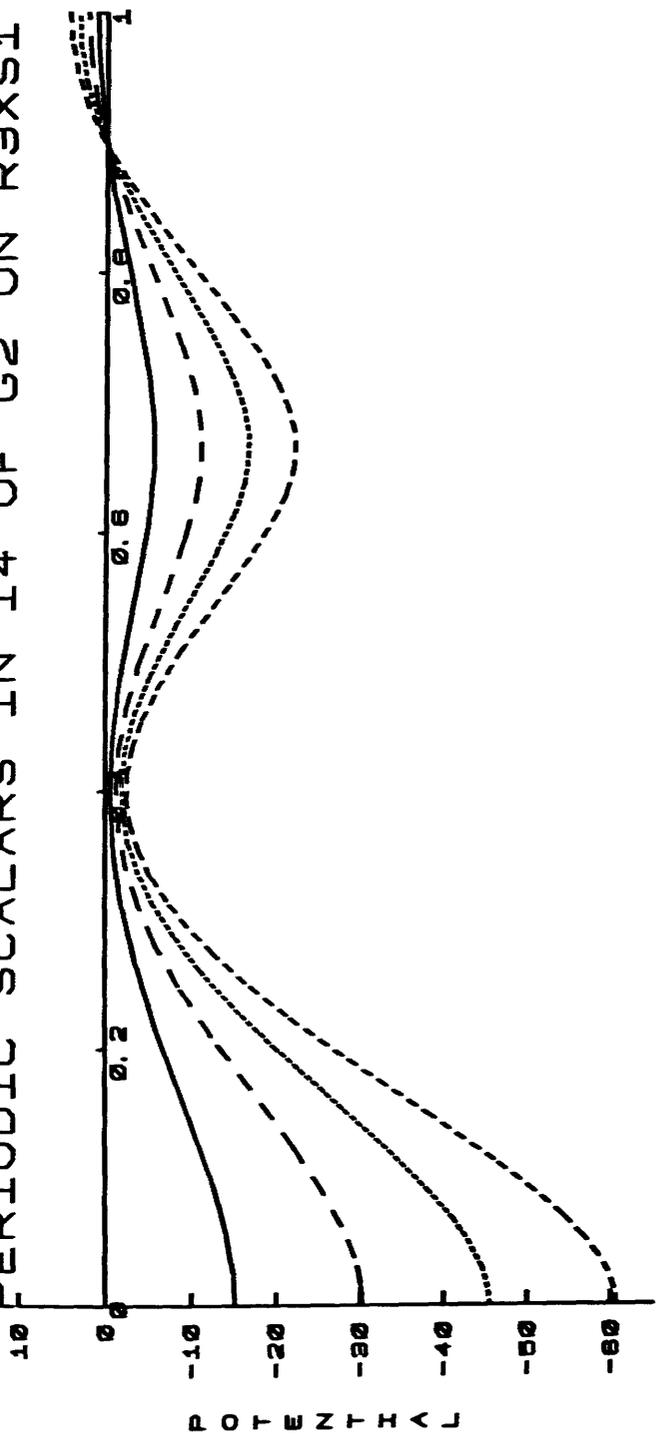
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- - - NF = 1
..... NF = 2
- . - . NF = 3

PERIODIC SCALARS IN 7 OF G2 ON R3XS1



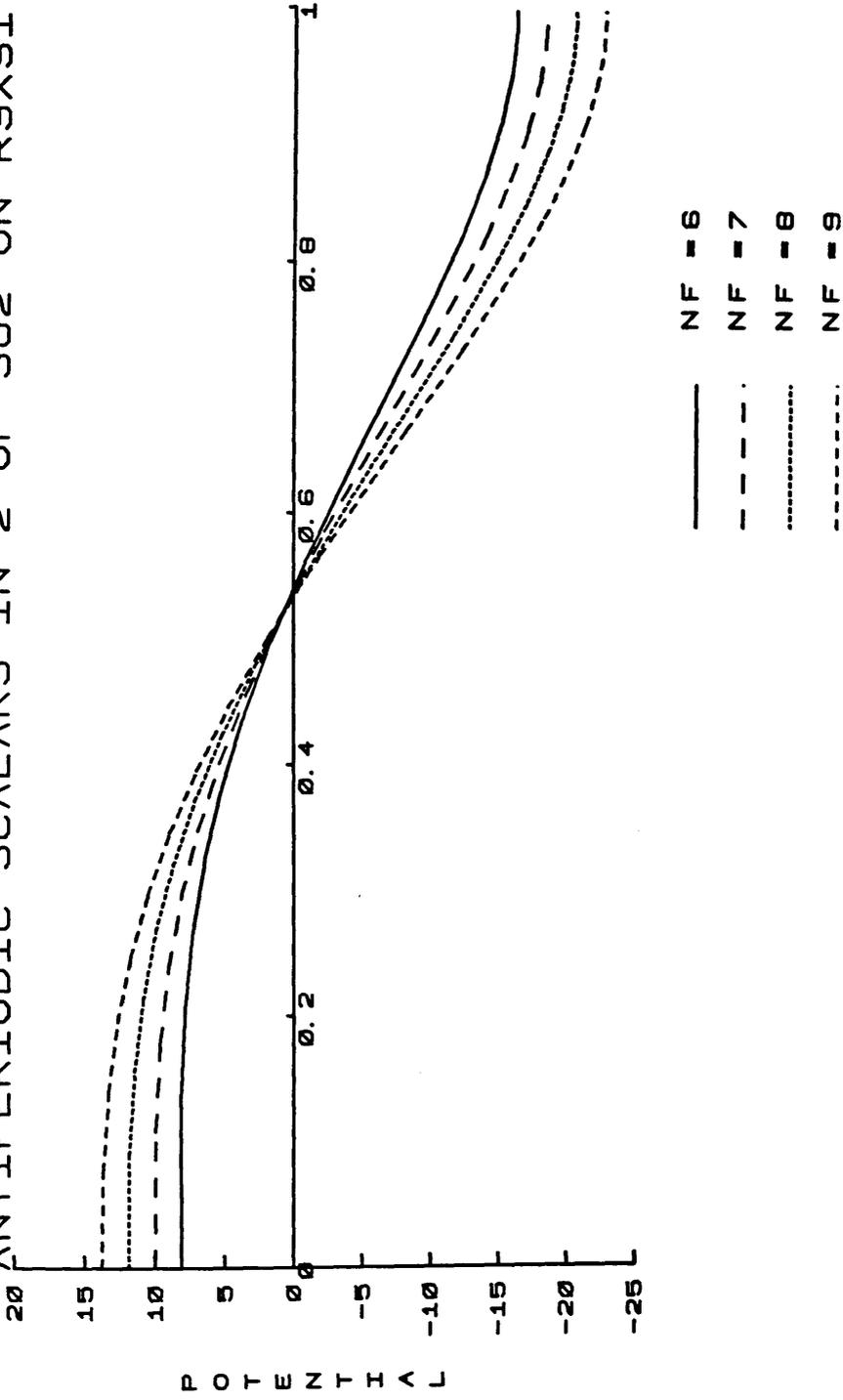
- NF — 6
- - - NF - 7
- NF - 8
- · - · NF - 9

PERIODIC SCALARS IN 14 OF G2 ON R3XS1

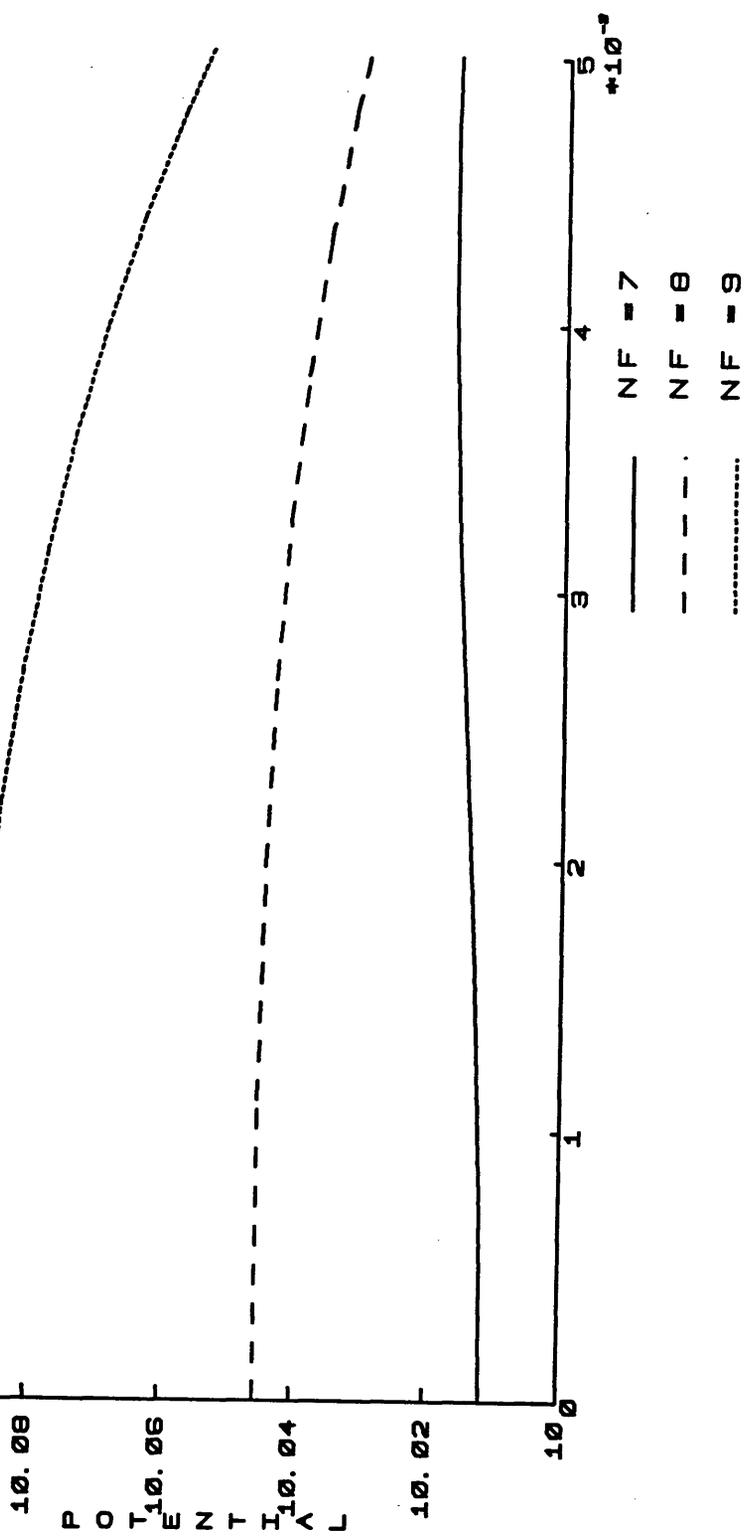


— NF = 0
- - NF = 1
..... NF = 2
- . - . NF = 3

ANTIPERIODIC SCALARS IN 2 OF SU2 ON R3XS1

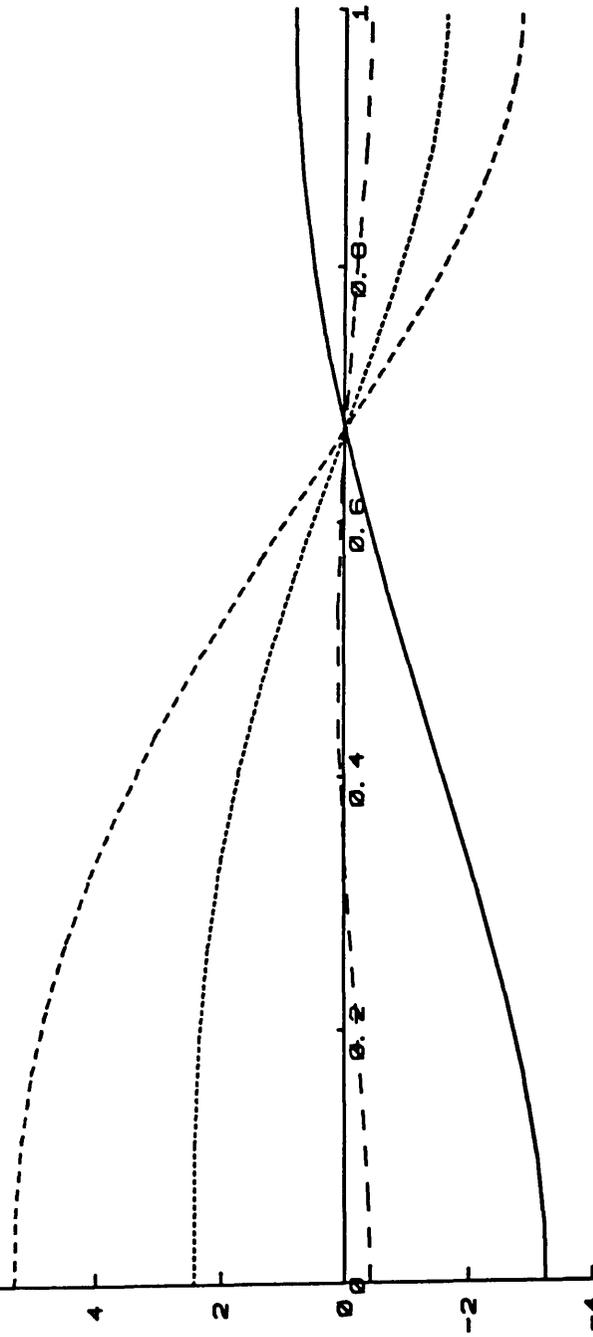


10.1 ANTIPERIODIC SCALARS IN 2 OF SU2 ON R3XS1



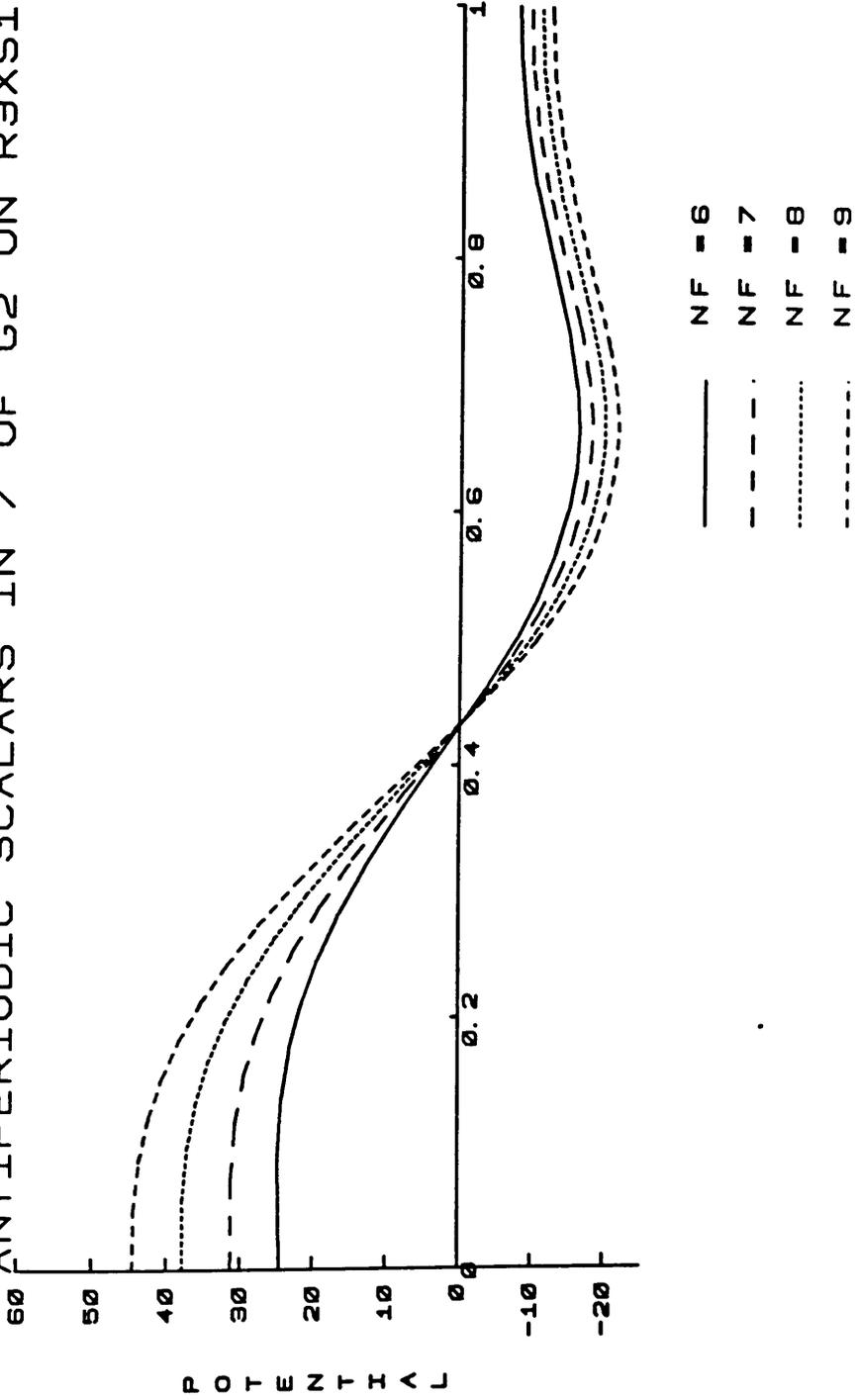
6 ANTIPERIODIC SCALARS IN 3 OF SU2 ON R3XS1

P O T E N T I A L

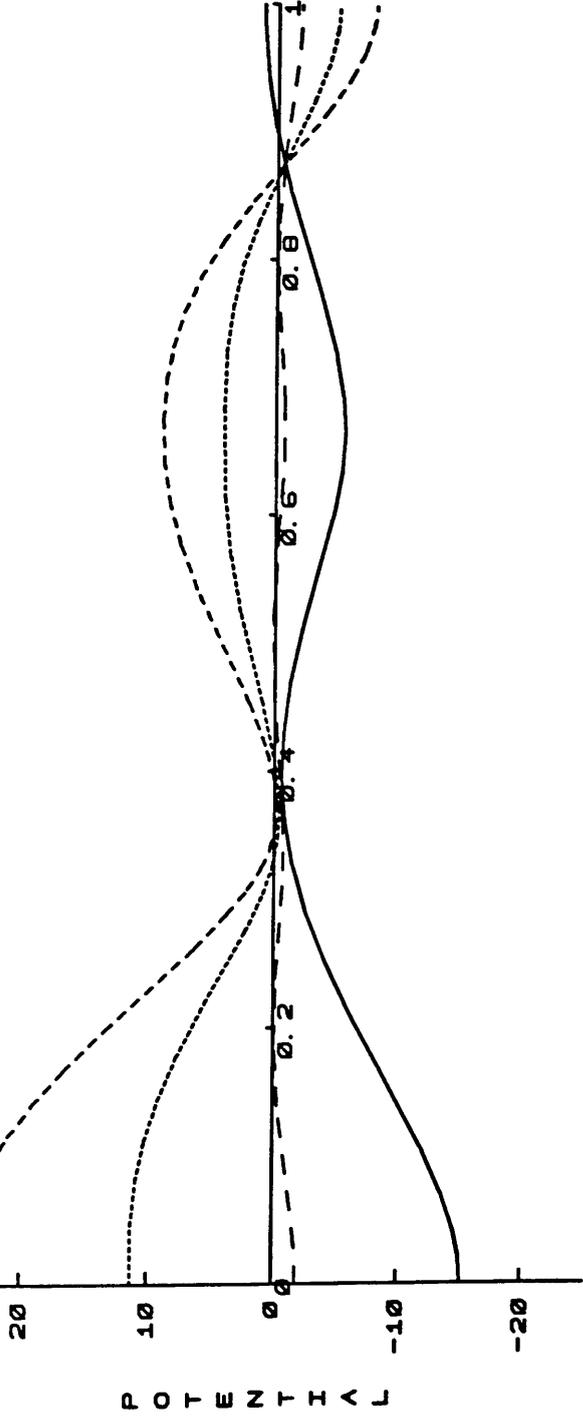


- NF = 0
- - - NF = 1
- NF = 2
- · - · NF = 3

ANTIPERIODIC SCALARS IN 7 OF G2 ON R3XS1

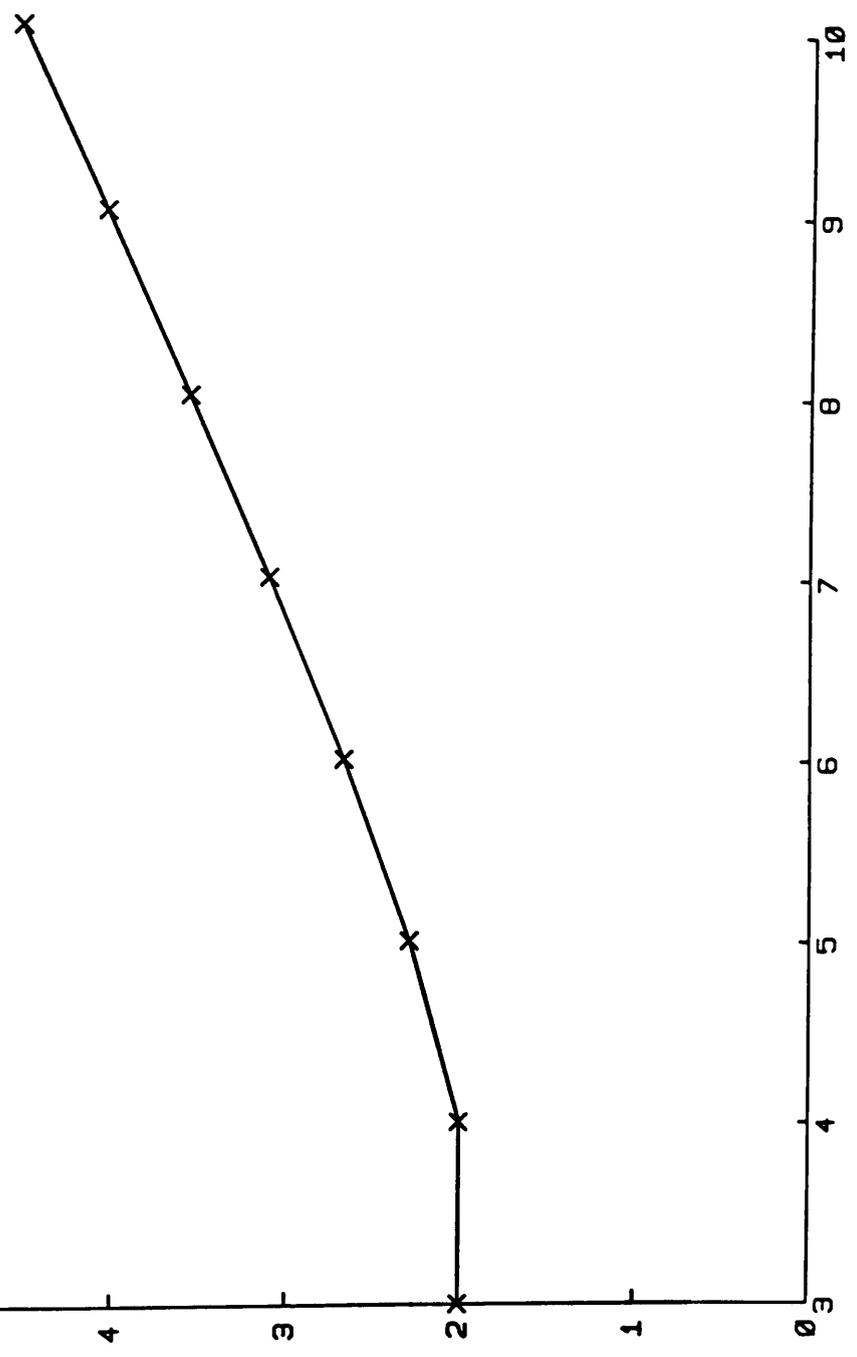


ANTIPERIODIC SCALARS IN 14 OF G2 ON R9XS1

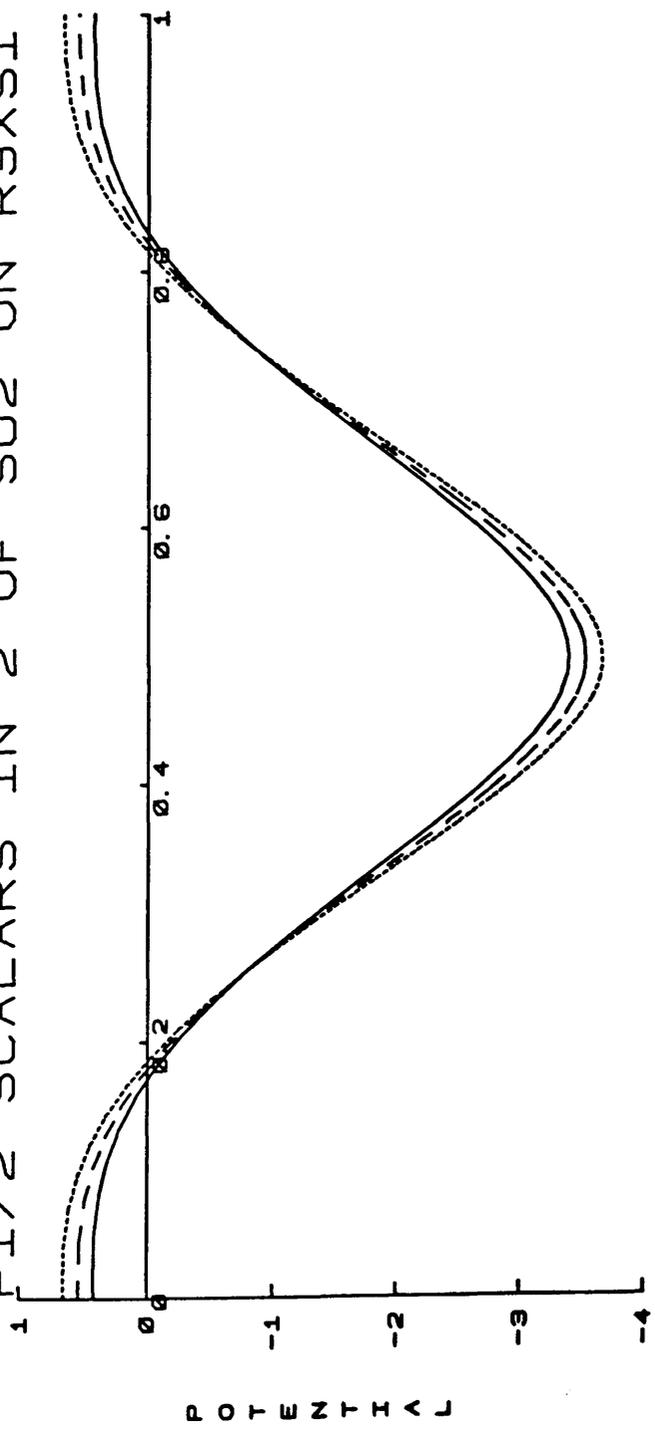


- NF = 0
- - - NF = 1
- NF = 2
- · - · - NF = 3

ANTIPERIODIC SCALAR STABILITY COEFFICIENT

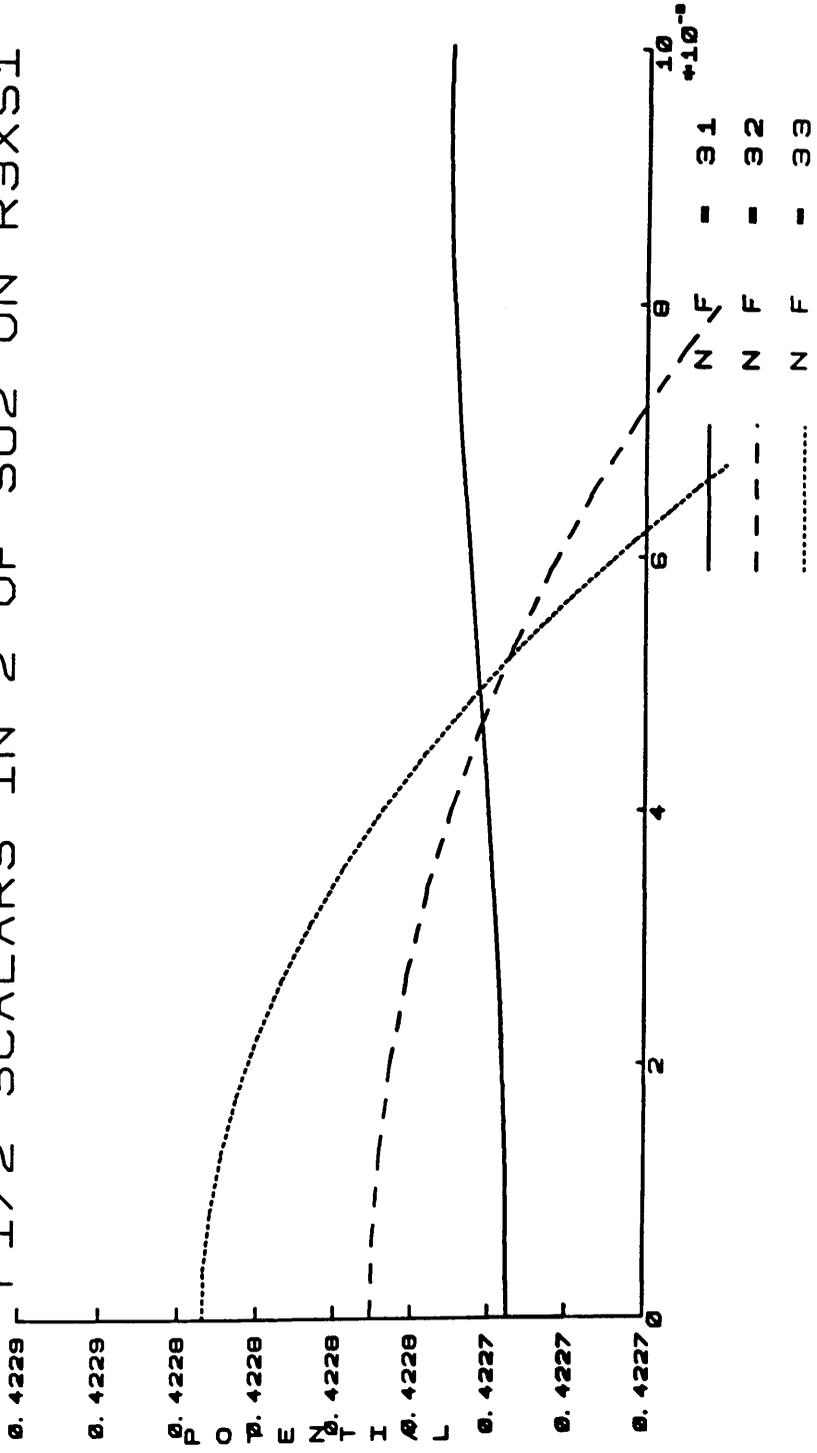


PI/2 SCALARS IN 2 OF SU2 ON R3XS1

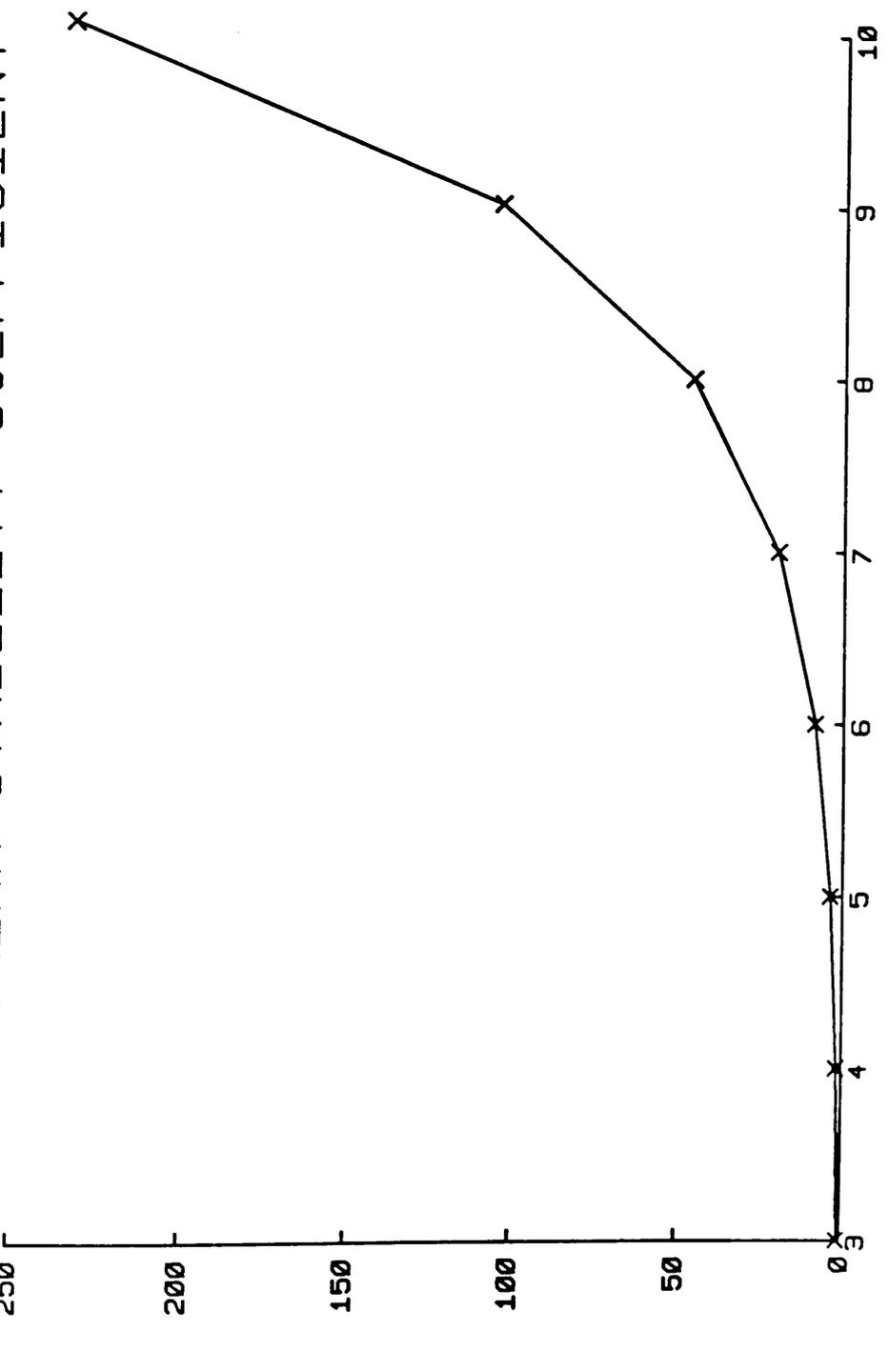


NF = 31
NF = 32
NF = 33

PI/2 SCALARS IN 2 OF SU2 ON R3XS1



*10¹ PI/2 SCALAR STABILITY COEFFICIENT



CHAPTER 7

Generalised Hosotani model - multiple representations

i) Preliminary discussion

In this chapter, we shall extend the model to include matter fields in more than one representation of the gauge group. In the previous two chapters, the true symmetry group of the Lagrangian was determined by the congruency class of the matter representation chosen, i.e. it was the group for which the representation was faithful and single-valued. In the more general case, the various representations may lie in different congruency classes and hence the determination of the true symmetry group is a little more complicated. The true group in such cases will be the 'smallest' of the locally isomorphic groups generated from the given algebra for which all the representations are single valued. (By smallest, we mean the group with the lowest order centre.) It may then be the case that none of the individual representations are faithful representations of the symmetry group (just as the adjoint representation in the previous cases was not a faithful representation of the symmetry group when the matter representation did not lie in the adjoint congruency class).

In order to determine whether the symmetry may be broken, it is necessary to perform some sort of stability analysis of the various symmetry preserving background configurations as in the previous cases. An added complication in this case is that any particular symmetric configuration will be represented by different q parameters in the different congruency classes and therefore, while one particular class may prefer a particular symmetric configuration as the global minimum, another representation in the Lagrangian which lies in a different class may provide a destabilising effect on that same configuration due to the different q parameter by which it is represented in that class. This problem will become more pronounced, as the number of different class representations which are included in the model is increased. In order to make some sort of headway with this problem, we shall first of all consider systems which contain matter representations lying in two different representations of the covering group.

ii) Models with two matter representations

The situation for such systems is complicated by the fact that, as well as having the option of introducing either fermion or scalars in either of the representations, we also have the freedom of choosing either periodic or anti-periodic boundary conditions. There is also no reason why, for example, if fermions are chosen for both representations, different boundary conditions cannot be adopted independently for each representation. We will therefore classify the matter fields into two categories, depending on how they affect the zero background (or any other configuration represented by $q = 0$ for the representation in question). Periodic scalars and antiperiodic fermions shall be referred to as stabilising fields ; antiperiodic scalars and periodic fermions shall be referred to as destabilising fields. (Note that we shall not be too concerned about the Z_2 scalar breaking as outlined in the last chapter as, unless both representations include $\delta = \pi/2$ scalars in the same congruency class, the destabilising effect on the symmetric backgrounds will be swamped by the contribution from the other field unless a considerable number of such scalar fields are introduced.)

The simplest model is one in which both representations contain stabilising fields. Such cases are, as in the one representation model, trivial, as the contribution from both fields to the one loop potential is such that the $q = 0$ configuration forms the global minimum. The true symmetry group of the Lagrangian is therefore preserved, and all fields retain massless modes on compactification to the macroscopic space.

The situation where either one or both of the fields is destabilising is for more complicated. For such cases, the most obvious way to proceed is to determine whether there are any symmetric configurations for which the q parameters in the two representations are in the appropriate stable regions of the polylogarithm function as described earlier. (If both representations are in the same congruency class, then we are effectively back to the situation already discussed in chapters 5

and 6.) To proceed with this analysis therefore, we shall first of all consider the simplest case for which two different congruency classes may exist, i.e. the case where the covering group has a Z_2 centre. Groups of this type include $SU(2)$, E_7 and $\text{spin}(2n+1)$. The orthogonal groups $SO(2n)$, while having Z_2 centres, are not simply connected and in fact have $\text{spin}(2n)$ covering groups. Such groups have larger centres, and will be discussed later. In order to clarify this discussion, it will be convenient to quote the q_{LM} matrix for Z_2 as defined in chapter 2, which gives the q parameters for the symmetric vacua L in the congruency class M .

$$q_{LM}(Z_2) = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (7.1)$$

where the rows are labelled by the symmetric background parameter L and the columns by the congruency class label M . Using the conventions of chapter 2, the vacuum labelled by $L = 2$ is the zero background and the congruency class labelled by $M = 2$ contains the adjoint representation. As representations from both congruency classes are present in this case, the true symmetry group of the Lagrangian will be the covering group, as the $M = 1$ column of the q_{LM} matrix contains q parameters which appear in the Lagrangian and which generate a faithful representation of the Z_2 group centre. We can now examine the three possibilities for this model.

1) Destabilising 'fundamental' class fields and stabilising adjoint class fields.

In this case, the stabilising fields deepen the $q_{L2} = 0$ symmetric configurations, i.e. they deepen both of the symmetric backgrounds $L = 1$ and $L = 2$. The contribution from the destabilising fields has its global minimum at $q_{L1} = 1/2$, i.e. it breaks the degeneracy of the two symmetric backgrounds and deepens the $q_{L1} = 1/2$ minimum, while providing a maximum destabilising effect on the $q_{L1} = 0$ background. This is not dissimilar to the cases discussed in chapters 5 and 6, as the stabilising adjoint

class fields are just enhancing the minima of the gauge field term in the potential. The only effect of this will be to increase the critical number of destabilising fields for the zero background, but this is not all that important as the other symmetric configuration becomes the global minimum.

2) Stabilising fundamental fields and destabilising adjoint fields.

For this combination, the stabilising fields will, by definition, deepen the $L = 2$ background and destabilise the $L = 1$ configuration ($q_{11} = 1$). The effect of the destabilising fields in the adjoint class will be to destabilise both symmetric backgrounds ($q_{L2} = 0$ for $L = 1$ and $L = 2$). The overall effect will therefore depend on the relative sizes of the different contributions, i.e. on the dimensions of the representations and the generation numbers of the two fields. If the stabilising term dominates, then the zero background will be the global minimum of the potential, whereas, if the destabilising term is the dominant one, the symmetry will be broken.

3) Destabilising fields in both classes.

For this final case, the situation is almost identical to the previous case, with the only real difference being that the $L = 2$ background will now be the preferred global minimum if the fundamental class fields dominate the potential.

To summarise the Z_2 case, the symmetry will only be broken if we have destabilising fields in the adjoint congruency class. This is much the same as before, although in this case, due to the fundamental class representation deepening one of the symmetric minima, the symmetry will in general only be broken if a sufficient number of the adjoint class fields as incorporated into the model. The extension of the adjoint breaking phenomenon to the multiple representation case was to be expected, and as we know that stabilising adjoint class fields in the two representation models effectively return us to the analysis presented in earlier chapters, we shall only examine combinations of non - adjoint representations in our

subsequent discussion of covering groups with higher order centres in order to see if there exists any possibility of symmetry breaking without adjoint class fields.

With this in mind, we move on to the cases of covering groups with Z_3 centres (e.g. $SU(3)$, E_6). The q_{LM} matrix for Z_3 is

$$q_{LM}(Z_3) = \frac{1}{3} \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (7.2)$$

and we will now be restricting our attention to the cases where the Lagrangian matter representations lie in the $M = 1$ and $M = 2$ congruency classes, i.e. to the first two columns of the matrix (7.2). As before, we shall now examine the various possibilities for the types of fields which may be put into the appropriate representations.

1) Destabilising fields in both representations.

This is the simplest of the possibilities, due to the reflection symmetry of the polylogarithm function. As the centre has odd order, there is no $q_{LM} = 1/2$ for any of the backgrounds L in any class M . Both matter representations will therefore prefer the backgrounds labelled by $q_{LM} = 1/3$ and $2/3$ for the appropriate values of M . As the backgrounds $L = 1$ and 2 are labelled by $q_{LM} = 1/3$ and $2/3$ for both the congruency classes in question, then the matter terms in the potential will combine to deepen these symmetric minima and destabilise the zero ($L = 3$) background. No symmetry breaking is then possible by this particular combination and the effective low energy group will be $SU(3)/Z_3$.

2) One field destabilising and the other field stabilising.

Due to the symmetry between the two classes we can, as in the Z_2 case, discuss the two cases simultaneously. If, for the sake of argument, we have the stabilising field in the $M = 1$ class, then the $L = 3$ background will obviously be preferred as the

minimum for that particular term in the potential. This term will also have a destabilising effect on the other symmetric minimum as the values of q_{11} and q_{12} lie outside the stable range of $[0, 1/4) \cup (3/4, 1]$. The other term, however, will tend to prefer the $L = 1$ and $L = 2$ backgrounds as before, and will have its maximum at the $L = 3$ configuration. It is therefore the case that none of the symmetric configurations are stable minima of the one loop potential with respect to changes in the matter field generation numbers. This then provides us with the hope that an appropriate choice of matter representations and generation numbers may result in some broken symmetry minimum of the potential being deeper than any of the symmetric minima. The problem with this is that the dominant matter term in the Lagrangian will ensure that either one or two of the symmetric configurations will definitely be minima of the potential, and the gauge field term in the potential will still have an enhancing effect on all the symmetric backgrounds. We have by no means conducted an exhaustive examination of the various possibilities for the $SU(3)$ and E_6 cases, but the few cases which have been done, involving the lower dimensional representations of $SU(3)$ seem to indicate that the dominant symmetric configuration is preferred as the global minimum, and any changes in the generation numbers can at most induce a transition to one of the other symmetric backgrounds. The effective symmetry group will be either $SU(3)$ or $SU(3)/Z_3$ depending on which of the two possibilities is chosen.

To summarise for this case, the by now well known adjoint breaking is still applicable, but a new possibility for symmetry breaking involving different categories of fields in different non-adjoint classes arises. The results of some preliminary numerical investigations seem to indicate that, in this latter case, a symmetric configuration is still preferred as the global minimum of the one loop potential, the particular backgrounds depending on the weighting of the contributions to the potential from the two fields. We have therefore still to provide an example of symmetry breaking which does not rely on the destabilising effect of appropriate

adjoint class fields.

The next major case of interest is the Z_4 analysis (groups with Z_4 centre are $SU(4)$ and $spin(4n+2)$). As usual, the analysis of the situation is facilitated by the introduction of the Z_4 q_{LM} matrix

$$q_{LM}(Z_4) = \frac{1}{4} \begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 0 & 2 & 0 \\ 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (7.3)$$

As before, we shall be looking for any interesting features concerning the non-adjoint class representations. A detailed account of the various features at this stage tends to be somewhat cumbersome, especially as we have an idea of what to expect from our examination of the Z_2 and Z_3 cases, therefore the analysis of this example shall be much more brief than before. From (7.3) we can see that destabilising fields in any two of the three non-adjoint classes share a common stable symmetric configuration (for example, the $M = 1$ and $M = 2$ classes have the $L = 1$ and $L = 3$ backgrounds as common stable minima). If we have one stabilising and one destabilising then again there is always at least one common stable symmetric minimum of the potential. The result of this is that two fields in representations of a covering group with Z_4 centre will always preserve one of the symmetric backgrounds as a stable minimum of the potential, and we have found no numerical example where such minima are not global. The effective low energy group will be determined by the representation whose q parameter is zero at the minimum, and will either be G , G/Z_2 or G/Z_4 , where G is the relevant covering group.

The next case is that of groups with Z_5 centre ($SU(5)$). The q_{LM} matrix is

$$q_{LM}(Z_5) = \frac{1}{5} \begin{pmatrix} 1 & 2 & 3 & 4 & 0 \\ 2 & 4 & 1 & 3 & 0 \\ 3 & 1 & 4 & 2 & 0 \\ 4 & 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (7.4)$$

For this case, the q parameters ($\times 5$) in the stable range for stabilising fields are 0,1 and 4, whereas the stable q values ($\times 5$) for the destabilising fields are 2 and 3. It is then clear from the matrix (7.4) that the following combinations do not lead to a stable symmetric minimum :

- a) 2 destabilising fields in the class combinations 1+2,1+3,4+2 or 4+3
- b) 1 stabilising and one destabilising in the class combinations 2+3 or 1+4

Be this as it may, the models we examined in this case still show that the dominant matter contribution to the potential ensures that its preferred symmetric minimum/minima become global. A model of type a consisting of periodic fermions in the $\underline{5} + \underline{10}$ of $SU(5)$ was discussed by Hosotani in his original paper, where he found the $\underline{10}$ fermions dominated the potential and determined the global minima to be the symmetric backgrounds $L = 1$ and $L = 4$ as discussed in chapter 5. We shall discuss this phenomenon further in the next section, in the context of an $SU(5)$ model put forward by Svetovoi and Khariton[54].

This sort of approach can be extended to higher order cyclic centres, but no new features seem to appear. As only the $SU(n)$ groups have centres of this form, there is no point in indulging in a repetitious discussion of these cases.

The only case of interest for which the covering group centre is not of the form Z_n is the case of $spin(4n)$, where the centre is $Z_2 \times Z_2$. A detailed discussion of the congruency classes for these groups was given in chapter 2, but we now shall outline the nature of the different symmetric backgrounds for the multiple representation case. The reason why no mention was made of any complication with such groups in the previous two chapters was due to the fact that, as outlined in chapter 2, these groups do not possess any faithful irreducible representations. In

order to obtain a realisation of $\text{spin}(4n)$ it is necessary to have a reducible representation constructed from representations lying in different congruency classes, which is effectively the situation here, hence it is only now that we may experience any difficulties which may arise due to the product form of the group centres.

As was detailed in chapter 2, the congruency classes of the $\text{spin}(4n)$ groups may be distinguished by means of a two index label (where each index is a label for the individual Z_2 factors comprising the group centres). The two spinor classes are labelled by $M = (1,2)$ and $M = (2,1)$, the odd rank tensors by $M = (1,1)$ and the adjoint class (even rank tensors) by $M = (2,2)$. We can, as before, define a q_{LM} matrix for $Z_2 \times Z_2$ by

$$q_{LM}(Z_2 \times Z_2) = \frac{1}{2} \begin{pmatrix} (1,0) & (0,0) & (1,0) & (0,0) \\ (0,0) & (0,1) & (0,1) & (0,0) \\ (1,0) & (0,1) & (1,1) & (0,0) \\ (0,0) & (0,0) & (0,0) & (0,0) \end{pmatrix} \quad (7.5)$$

where the rows are labelled by the background L parameter as before, and the columns are the $q_{LM}(Z_2)$ values in the various congruency classes (the first two columns are the spinor classes and the third column is the odd tensor class). As the Wilson loops representing the $Z_2 \times Z_2$ centre will be direct products of the Wilson loops constructed out of the individual $q_{LM}(Z_2)$ parameters, then we may construct the 'physical' q parameters by summing the individual $q_{LM}(Z_2)$ parameters in a given congruency class, in which case (7.5) becomes

$$q_{LM}(Z_2 \times Z_2) = \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (7.6)$$

From this form of the q_{LM} matrix, it can clearly be seen that for either category of matter field in any two combinations of classes will always preserve a stable symmetric minimum (any two of the first three columns have adjacent $(0\ 0)$, $(1/2\ 0)$ and $(1/2\ 1/2)$ entries). In fact, as the only entries in the matrix are 0 and $1/2$, whichever symmetric minimum is preferred will trivially be the global minimiser for all the representations present. The true symmetry group of the Lagrangian will be $\text{spin}(4n)$ and the effective symmetry group at low energies will be $\text{spin}(4n)$ modulo the Z_2 factor which contains non-zero q parameters for the particular background in the original form of the q_{LM} matrix (7.5). (The details of the locally isomorphic groups were given in chapter 2.).

To illustrate that the manoeuvre which obtained (7.6) from (7.5) is legitimate, we shall explicitly find the three non-zero symmetric backgrounds for the group $\text{spin}(8)$. The three locally isomorphic groups generated from the $\text{spin}(8)$ algebra (apart from the adjoint group) all have faithful 8 dimensional representations. By calculating the weights for each of these representations, the three non-zero symmetric background components are easily found and turn out to be (modulo factors of π)

$$\begin{aligned}
 (1\ 0\ 1\ 0) \quad L = 1 \\
 (1\ 0\ 0\ 1) \quad L = 2 \\
 (0\ 0\ 1\ 1) \quad L = 3
 \end{aligned}
 \tag{7.7}$$

The $L = 1$ and $L = 2$ backgrounds are labelled by $q_{LM} = 1/2$ in the 8 dimensional spinor representation with highest weight $(1\ 0\ 0\ 0)$. In this representation, the $L = 3$ background is labelled by $q_{LM} = 0$. For the other spinor representation with highest weight $(0\ 0\ 0\ 1)$ the $L = 2$ and $L = 3$ backgrounds have $q_{LM} = 1/2$, with the $L = 1$ background having $q_{LM} = 0$. The tensor \underline{g} has highest weight $(0\ 0\ 1\ 0)$ and therefore has the backgrounds $L = 1$ and $L = 3$ with $q_{LM} = 1/2$ and $L = 2$ with $q_{LM} = 0$. (All these backgrounds have $L = 0$ in the adjoint representation $(0\ 1\ 0\ 0)$ as required.) It

can therefore be seen that we have explicitly allocated a background to every L label in (7.6) for the spin(8) case, and that all the conclusions remain valid.

The points to be taken from this section are as follows :

- 1) For the cases of Lagrangians containing two matter representations, the symmetry can be broken by the introduction of destabilising fields in the adjoint congruency class.
- 2) If no adjoint class representations are present then, depending on the group and class of representations chosen, there may not exist symmetric configurations which are stable with respect to changes in all the matter field generation numbers.
- 3) If there exist stable symmetric backgrounds, then they contain among them the global minimum of the potential.
- 4) If there are no stable symmetric backgrounds, then it appears that the dominant term in the potential selects its preferred symmetric minimum as the global minimum of the potential.
- 5) It therefore seems as if destabilising fields in the adjoint class are a necessary feature of the model if flux breaking of the gauge symmetry is to occur.

iii) The Svetovoi-Khariton SU(5) model

In this section, we shall examine a model first outlined by Svetovoi and Khariton[54] in which they take a multiple representation version of the Hosotani model with SU(5) gauge group and attempt to incorporate multiple symmetry breakings in an attempt to produce the SU(3) x U(1) gauge group as the residual symmetry group. A brief review of their paper will be given here, with appropriate corrections, and we shall conclude with a numerical examination of both this model and similar SU(5) models.

The Svetovoi-Khariton Lagrangian contains periodic fermions in the $\underline{5} + \underline{10}$

reducible representation of $SU(5)$, periodic scalars in the $\underline{15}$, and one periodic scalar family in the $\underline{5}$. It should be noted that none of these representations lie in the adjoint (quintality 0) congruency class, and therefore any breaking found will indicate that adjoint class fields are not a necessity for flux breaking in the Hosotani model. They claim that their spacetime is $R^4 \times S^1$, but the effective potential which they quote has errors in the overall coefficient $C(m)$ and in the coefficient of the gauge term $(1-m)$ and in fact corresponds to $m = 3$ rather than $m = 4$. (In our numerical work at the end of this section, we shall also examine the model for the case $m = 3$.) The first step in their procedure is to find an appropriate background field configuration for which the $SU(5)$ symmetry breaks to the standard model group $SU(3) \times SU(2) \times U(1)$. An extra constraint to be imposed is due to the fact that a surviving scalar doublet field is required at low energies in order to break the $SU(2) \times U(1)$ factor to $U(1)$. The appropriate background field quoted gives $q = 1/3$ or $2/3$ for the $\underline{3}$ of $SU(3)$ produced in the branching, $q = 0$ for the $SU(2)$ doublet and $q = 0$ from the $U(1)$ contribution.

In order to break the $SU(2) \times U(1)$ group to $U(1)$, the possibility of incorporating $SU(5)$ invariant term in the Lagrangian is mentioned which will give the doublet a small vev and lead to further breaking of the symmetry. Simple models were first examined in order to find a potential whose minimum was at the appropriate configuration, and the simplest such model contains the matter representations detailed previously. It was found that one of the reducible fermion representations and 3 of the scalar $\underline{15}$'s were enough to give a minimum at an appropriate background.

As this was the first paper found which dealt with a case of symmetry breaking in the Hosotani model, we examined it numerically and allowed the generation numbers and the phases of all fields to vary. For all cases, it was found that one or more of the symmetric configurations were preferred as the global minima. The global minima in any given case were determined by the congruency classes of the dominant contributions. The quintality 1 representation (i.e. the fundamental

fermions and scalars) preferred the $q_1 = 2/5$ and $3/5$ minima when given periodic boundary conditions, and obviously preferred the $q_1 = 0$ configuration when allocated an antiperiodic boundary condition. The quintality 2 and 3 representations (i.e. the 10 and 15) preferred the $q_1 = 1/5$ and $4/5$ minima when given periodic conditions, and these were the preferred global minima of the potential when such representations dominated.

When we looked for local minima in order to see if the Svetovoi-Khariton background was a deep minimum, we found instead that, for their choice of periodic boundary conditions for all fields, the deepest minima which would give the standard model group had $q = 1/2$ for the SU(2) doublet and hence no massless doublets would survive compactification. (This was to be expected as the minimum of the polylogarithm function for this case is at $q = 1/2$.) The Svetovoi-Khariton background at best corresponds to a fairly shallow minimum of the potential and as such is unlikely to form the gauge background for any reasonable length of time, as quantum tunnelling effects should take the background field into a much deeper minimum of the potential. This rather disappointing feature was hinted at in their paper, where it was claimed that the model was intended for illustrative purposes only. In this vein, it would be preferable if a semi-realistic example could be found for which the broken symmetry configuration actually corresponded to the global minimum of the potential. Such an example will be given in the following section.

iv) A simple E_6 model

In the last section, we outlined a model where the fact that the matter representations lay in non-adjoint congruency classes resulted in symmetric global minima of the potential. In this section, we shall now see an example where the existence of an adjoint class representation allows the symmetry to be broken for an appropriate range of matter generation numbers. There are obviously no end of such

examples which could be given, so, as the main use of this mechanism at present is in the case of superstring inspired models, we shall give a very simple model with E_6 gauge group. Several such models were examined with varying degrees of success as regards the outcome, but one in particular yielded some interesting results, especially considering the simplicity of the model.

This model, which is initially formulated in our original $m = 3$ dimensions, consists of the usual E_6 Yang-Mills term minimally coupled to N_A generations of periodic Dirac fermions in the adjoint (78) representation, and N_F generations of antiperiodic Dirac fermions in the fundamental (27) representation. (The existence of the different boundary conditions is disappointing if the desire was to have originally included all the fermions fields in a single representation of E_6 or some larger symmetry group. Cases where the 27 fermions had periodic boundary conditions were examined but as such fields contribute to the destabilising of $q = 0$ configurations, the end result in all cases was that, no matter how interesting the residual group, few of the representations in the branching of the 27 corresponded to $q = 0$ and therefore the low energy spectrum of the compactified theory contained few massless fermions in any realistic representations.)

As the $N_F = 0$ case corresponds to the adjoint breaking of E_6 as given in chapter 5, then the residual group in this case is $[SU(3)]^3$. The antiperiodic fermions will tend to prefer the zero background as before and will inevitably determine this configuration to be the potential minimum as N_F increases. The problem is therefore to determine the symmetry group for N_F less than this critical number.

A stability analysis of the zero and $[SU(3)]^3$ configurations is in appendix C, where it can be seen that the $[SU(3)]^3$ backgrounds remain minima of the potential whereas the zero background is only a minimum when $N_F \geq 8N_A - 4$. In order to determine the residual groups for all values of N_F , we again resort to a numerical approach. The results are

1) The symmetry for $N_F = 0$ is $[SU(3)]^3$

2) The symmetry is $SU(3) \times SU(3) \times SU(2) \times U(1)$ for $1 \leq N_F \leq 3$, $N_A = 1$ and $1 \leq N_F \leq 12$, $N_A = 2$

3) The zero background is preferred for $N_F \geq 4$, $N_A = 1$ and $N_F \geq 12$, $N_A = 2$.

For both the $N_A = 1$ and $N_A = 2$ cases, the surviving fundamental fermions are in the representation $(\underline{3}^*, \underline{3}, \underline{1})$, with the surviving adjoint fermions being in the adjoint representation of the subgroup, although the background configurations are not the same. The effective low energy symmetry for the matter fields would then appear to be $[SU(3)]^2$, but it should be noted that there is a maximum number of 3 fermion $\underline{27}$'s giving this subgroup for $N_A = 1$. If a model along the lines of the Svetovoi-Khariton model was desired, then families of scalar fields could be added in the $\underline{27}$, the surviving modes of which could be used to break down one of the remaining $SU(3)$ factors in the residual subgroup to $SU(2) \times U(1)$ and then to $U(1)$. This would, however, detract from the fermion contribution and, as well as the $SU(3) \times SU(3) \times SU(2) \times U(1)$ background still being preferred as the global minimum even for $N_F = 1$, the maximum number of fermion generations giving this minimum would, as given by (C.10), rise by 1/2 for every $\underline{27}$ or $\underline{27}^*$ of scalars added. (The integer part is, of course, the only significant quantity.)

Countering the positive feature of having a realistic subgroup, there are the not so attractive features of having an unrealistic spacetime dimension, having massless gauge bosons corresponding to the extra $SU(2) \times U(1)$ factor in the original subgroup, having gauge bosons with 'observable' masses due to the breaking of one of the low energy $SU(3)$ factors and having both massless and massive adjoint fermions in correspondence with the gauge bosons of the original broken symmetry subgroup. As the $\underline{27}$ and $\underline{27}^*$ representations give identical contributions to the potential for this case, then it would not be possible to have an upper limit of three families of fermions and their corresponding antiparticles in the $\underline{27} \oplus \underline{27}^*$ reducible representation for $N_A = 1$ unless either 4 or 5 representations of scalars were added. (For $N_A = 2$, we can have six $\underline{27} \oplus \underline{27}^*$ fermion representations before

scalars are added). We can, however, get round some of these problems by formulating the model in a different number of spacetime dimensions.

If we examine this model in 4+1 dimensions then, as can be seen from appendix C, the critical number of $\underline{27}$'s for the $N_A = 1$ case is 2. This is due to the spinor 'size' remaining the same as for the $m = 3$ case hence increasing the weighting of the gauge and ghost contribution. If we add an extra $\underline{78}$ of periodic fermions, then the critical number of $\underline{27}$'s increases to 7. At this critical number, the zero background is again preferred as the global minimum hence, in order for the symmetry to break, we cannot have more than 3 generations of $\underline{27} \oplus \underline{27}^*$ fermions. The $m = 4$ results are

1) For $N_A = 1$, the symmetry is $SU(6) \times U(1)$ for $N_F = 1$, E_6 for $N_F > 1$.

In this case, the surviving fermions from the $\underline{27}$ lie in the $\underline{15}$ of $SU(6)$.

2) For $N_A = 2$ the symmetry is

a) $[SU(3)]^3$ for $N_F = 1$ ($\underline{27} \rightarrow$ massless $(\underline{3}^*, \underline{3}, \underline{1})$)

b) $SU(3) \times SU(3) \times SU(2) \times U(1)$ for $N_F = 2$ (massless $(\underline{3}^*, \underline{3}, \underline{1})$)

c) $SU(3) \times SU(3) \times U(1) \times U(1)$ for $N_F = 3, 4, 5, 6$ (massless $(\underline{3}^*, \underline{3})$)

d) E_6 for $N_F \geq 7$ (massless $\underline{27}$)

The main differences from the $m = 3$ case are that we now have a realistic number of non-compact dimensions and that the residual symmetry for a 3 generation model is now $SU(6) \times U(1)$. This then introduces the need for at least two further stages of symmetry breaking if a realistic low energy group is to be obtained.

If we consider a 9+1 dimensional theory (in order to get some vague idea of what might happen from some superstring inspired model) then we find that the critical numbers of $\underline{27}$ and $\underline{27}^*$ fields are $N_F = 4$ for $N_A = 1$ and $N_F = 8$ for $N_A = 2$. In such a case, the scalar contribution to the potential is considerably smaller than the fermion term (due to the larger dimension of the spinors) and therefore adding

scalars in small numbers does not affect these critical fermion numbers. We can therefore put a limit of $N_F = 3$ of the number of $\underline{27} \oplus \underline{27}^*$ fermion representations provided we take $N_A = 2$, and may have several representations of scalars which may be used for multi-stage symmetry breaking. The results in this case are

1) $N_A = 1$

- a) $SU(6) \times SU(2)$ for $N_F = 1$ (massless $(\underline{15}, \underline{1})$)
- b) $SU(6) \times U(1)$ for $N_F = 2,3$ (massless $\underline{15}$)
- c) unbroken for $N_F \geq 4$

2) $N_A = 2$

- a) $SU(6) \times SU(2)$ for $N_F = 1,2,3$ (massless $(\underline{15}, \underline{1})$)
- b) $SU(6) \times U(1)$ for $N_F = 4,5,6,7$ (massless $\underline{15}$)
- c) unbroken for $N_F \geq 8$

These are far more interesting examples of possible routes to reasonable low energy physics from some larger group by making use of the flux-breaking mechanism, due to the fact that the background configurations in these cases correspond to the global minima of the one loop potential rather than the local minimum model of Svetovoi and Khariton.

v) The centre conjecture

The findings of this chapter can be summarised as follows :

- 1) If the Lagrangian is such that the corresponding potential has a symmetric minimum which is stable with respect to changes in any of the matter field generation numbers, then this minimum is a global minimum.

2) If the Lagrangian is such that no such stable symmetric minimum exists then, the dominant term in the Lagrangian determines the global minimum, which is therefore symmetric unless the dominant field is a destabilising field in the adjoint congruency class.

We can combine these two points into the following conjecture :

The Wilson loop operators (constructed from the background field configuration which globally minimises the one loop effective potential) will map the discrete group Γ into the gauge group centre unless suitable fields are included in a congruency class for which a destabilising effect is produced on all the symmetry preserving backgrounds.

This 'centre conjecture' implies that the only way in which the symmetry can be broken in the Hosotani model is to have adjoint class periodic fermions or adjoint class antiperiodic scalars in the Lagrangian. If general phases are to be accepted, then non-adjoint class scalars can also break the symmetry, provided that they have $\delta \approx \pi/2$ phases and are in a class whose true group has Z_2 centre. No cases contradicting this conjecture have been found.

CHAPTER 8

Final conclusions and possible developments

The following table summarizes the key findings and potential future research directions:

Area	Key Findings	Possible Developments
1. Market Structure	Highly competitive market with low barriers to entry.	Investigate the impact of new entrants on existing players.
2. Consumer Behavior	Price sensitivity is high, and brand loyalty is low.	Study the effects of marketing strategies on consumer loyalty.
3. Supply Chain	Fragmented supply chain with significant inefficiencies.	Explore the benefits of vertical integration or strategic alliances.
4. Innovation	Low R&D spending relative to industry peers.	Assess the impact of increased R&D investment on product differentiation.
5. Regulatory Environment	Stringent regulations, particularly regarding environmental standards.	Monitor regulatory changes and their impact on business operations.
6. Global Expansion	Successful in domestic market, but limited international presence.	Identify key international markets for expansion and develop tailored strategies.
7. Digital Transformation	Slow adoption of digital technologies across the organization.	Develop a comprehensive digital strategy to improve efficiency and customer experience.
8. Sustainability	Increasing focus on ESG (Environmental, Social, Governance) factors.	Implement sustainable practices to enhance brand reputation and attract investors.

We shall now summarise the results of the work detailed in this thesis, and afterwards, we shall review the current status this and other flux-breaking toy models.

When we started work on the Hosotani model, the only facts known were those outlined in the original paper, i.e. that fermions in the 5 or 10 dimensional representations of SU(5) on $R^3 \times S^1$ could not break the gauge symmetry. If the antiperiodic fermion boundary condition was imposed, then the zero background was preferred and all fermions retained massless modes on compactification. If, however, the periodic boundary condition was chosen, then a non-zero background was preferred which still preserved the SU(5) symmetry but lead to masses of the order of the Planck mass for all fermion modes on compactification. An example of scalars with $\delta = \pi/2$ phase breaking SU(2) to U(1) was given as a numerical curiosity. Our work has resulted in

- 1) Identification of the Fourier series form of the potential with the polylogarithm function for $R^m \times S^1$.
- 2) Showing analytically that the zero background is trivially the global minimum of the one loop potential for either antiperiodic fermions or periodic scalars in any representation.
- 3) Showing that either periodic fermions or antiperiodic scalars in a representation congruency class such that the group generated from it has non-trivial centre, leads to a non-zero symmetry preserving background, the particular one being determined by the proximity of its q parameter in that class to the value 1/2 hence leading to no massless modes on compactification and therefore effective breaking of the group centre at low energies.
- 4) Showing that adjoint class periodic fermions or antiperiodic scalars can break the symmetry due to the fact that the symmetry group has unit centre and hence all backgrounds are equivalent to zero under adjoint group gauge transformations.

Destabilising this background leads to a broken symmetry global minimum of the potential.

- 5) Providing examples of such breakings and showing that periodic fermions lead to the same residual groups as those found for antiperiodic scalars.
- 6) Showing that the Hosotani $\pi/2$ scalar breaking of $SU(2)$ is only one example of breakings induced by such scalars in representations which generate groups with Z_2 centres.
- 7) Showing that the critical periodic fermion number required for destabilising the zero background decreases as the spacetime dimension decreases, whereas the critical antiperiodic or $\pi/2$ scalar number increases.
- 8) Showing a tendency for potentials containing contributions from matter fields in various representations to globally minimise at a symmetry preserving background unless, as for the single representation case, periodic fermions or antiperiodic scalars are included in the adjoint congruency class.
- 9) Providing more reasonable examples of 'physically' interesting toy models than previously existed.

As this thesis was being prepared, the Higuchi-Parker paper[49] was brought to our attention which found breaking of $SU(2)/Z_2$ ($SO(3)$) by triplet periodic fermions on $R^3 \times S^1$. No further examples or postulates along the lines of the current work were given. Another paper[55], produced by Hosotani, postulated that adjoint periodic fermions on $R^m \times S^1$ would break the gauge symmetry. The only example given as proof of this was again the adjoint $SU(2)$ case, which he showed to be broken on $R^3 \times S^1$ by expressing the potential explicitly in terms of a quartic polynomial (i.e. the Glaisher function Gl_4), and on $R^1 \times S^1$, where this time he minimised the quadratic polynomial potential. In chapter 4, we show analytically the breaking of $SU(2)$ to $U(1)$ for a general dimension $m \geq 3$. The results at lower dimensions were known to us, but in such cases the derivatives of the potentials, if

not the potentials themselves, misbehave quite badly. As such cases are of no physical interest anyway, we have not pursued this line of research in any great detail.

Apart from the Svetovoi-Khariton model, the only other work done on generalising the Hosotani model has been performed by Shiraishi[56], who has examined the $SU(2)$ doublet model at finite temperature and density, where we only have an effective Z_2 centre breaking by periodic fermions. His examinations showed that the determination of the $q = 1/2$ symmetric configuration as the global minimum of the potential was unaffected by changes in the temperature of the system, thus showing that the Z_2 symmetry could not be restored at high temperature. He then performed a similar analysis with $\pi/2$ scalars and showed that increases in the temperature parameter could lead to breaking for any non-zero number of scalars. Moving on to the case of finite density[57] (effectively incorporating an external charge density) he found that an appropriate choice of the density parameter could lead to restoration of the Z_2 symmetry. It would be desirable to examine such effects in a model for which the symmetry was broken at zero temperature, e.g. our E_6 example as given in chapter 7.

Another toy model which has been examined is that for which the compact space is a three sphere on which a Z_2 translation group is allowed to act freely. In this case, which was first examined by Evans and Ovrut[58], the first homotopy group of the manifold is Z_2 and therefore the Wilson loops will map this group into the gauge group. An added feature of this model is that, unlike the Hosotani model, there is no continuum of background fields. (In the Hosotani model, the vacuum parameters B^a were continuous parameters whereas in the Evans-Ovrut model, the interpolating solutions between vacuum fields do not have vanishing field strength and therefore the distinct vacua are separated by energy barriers. A discussion of instanton tunnelling between such vacua was given by Kolb et al.[59]) The only case

examined on this manifold was that of periodic fermions and scalars in the triplet representation of $SU(3)$. In this case, the fermions were found not to contribute to the one loop potential and it was claimed that the gauge (and ghost) contributions lead to a background field corresponding to residual group $SU(2) \times U(1)$ having a lower value for the one loop potential than the only permissible symmetric minimum at zero. When scalar fields were included, it was found they tended to restore the $SU(3)$ symmetry by deepening the zero background minimum.

This model was examined further by Shiraishi[60], who generalised it to $M^4 \times S^N/Z_2$ (where S^N is the N -dimensional sphere) and who also examined temperature effects. By examining high and low temperature expansions of the potential for gauge and scalar fields, he found that the $SU(3)$ symmetry was unbroken at the extremes of zero and infinite temperature. He claimed, however, that the gauge and ghost contribution did not, as stated in the Evans and Ovrut paper, break the symmetry.

The next shot in this battle was provided by Freire, Romao and Barroso[61], who examined the $SU(3)$ symmetry on $M^4 \times S^3/Z_2$. They claimed that Shiraishi had dealt incorrectly with the Faddeev-Popov ghost term and as a consequence the results of Evans and Ovrut for the case S^3/Z_2 were correct and could be carried over to the more general case.

The situation was seemingly resolved in another paper by Shiraishi[62] in collaboration with Nakamura. In this work, the calculation for $M^4 \times S^3/Z_2$ was detailed explicitly. It was claimed to have been shown that the contribution from the flat space components of the gauge fields had been neglected in [61] and that the previous Shiraishi paper had been correct. Just As this thesis was being completed, a new paper was produced by Dowker and Jadhav[63] who, while pointing out an error in [62], still agree with the basic features of Shiraishi's work. (The results of Evans and Ovrut still stand but, as their manifold consists only of S^3/Z_2 , their results are for the most part irrelevant.)

Adopting the Dowker-Jadhav paper as providing the correct analysis of the generalised Evans-Ovrut model, we see that the features found are not inconsistent with our analysis of the Hosotani model. In both cases, the gauge and ghost fields prefer the symmetric configuration as the minimum of the potential and the scalar fields (periodic by necessity in the Evans-Ovrut model) tend to deepen this minimum. As the fermions do not contribute on this manifold at one loop, there seems no possibility of inducing flux-breaking of any gauge group with S^3/Z_2 as the compact space.

The only other toy model which has been examined (at the time of writing) was again introduced by Evans and Ovrut. For this case, they retained the three sphere but chose Z_3 rather than Z_2 as the point group. In such a case, there is a non-vanishing fermion contribution at one loop and it was found that, for the case of $SU(2)$ gauge group, the gauge+ghost and scalar contributions tended to prefer the zero background, but the fermions destabilised it and lead to a $U(1)$ residual group. In the context of our centre conjecture, the destabilising of the zero background was to be expected and, as the only permissible symmetric background on this manifold is the zero background, then the symmetry must break when enough fermions are present, irrespective of the choice of fermion representation. (This case has also been examined by Dowker and Jadhav. They also generalise the model to include all lens spaces of the form S^3/Z_m , as well as some prism spaces where the factor group is not of the simple form considered up to now. They have still to generalise the analysis to cases other than the fundamental representations of $SU(3)$ and $SU(5)$, where they claim that such calculations are pointless without further physical motivation.)

There are several possibilities for extending the toy model examinations of flux breaking. In the spirit of Shiraishi's work, a finite temperature and density examination of the generalised Hosotani model could be made, in particular for the

cases where the symmetry was found to break at zero temperature. An examination of a torus compactified model would also be desirable in order to see if our 1-torus results carry forward to the more general case. The effective potentials and partition functions for both the generalised Hosotani and Evans-Ovrut models could be evaluated to second order in the loop expansion in order to see if the features of the one loop contributions may be carried forward to all orders in the perturbation expansion and hence free the results from the perturbative regime. It would also be preferable to consider more complex Lagrangians on more realistic manifolds, perhaps trying to find some supersymmetric example in the process, and in such cases it would be difficult to give a general analysis along the lines of our chapters 5 and 6. A particular model would then have to be adopted, which would hopefully resemble some plausible unified model and thus remove some of the arbitrariness in the choice of the background gauge fields which unfortunately exists in current superstring models.

The main conclusions to be taken from our examination of the generalised Hosotani model and the examinations of the Evans-Ovrut model are as follows :

- 1) The use of the flux-breaking mechanism in model building is not unreasonable, although there may be non-trivial constraints on the possible choices of the matter representations, i.e. on the choice of true symmetry group for the system.
- 2) The mechanism avoids both the need for fundamental scalars (although such scalars may be needed for further stages of symmetry breaking in a realistic model) and the need for finely tuned parameters in the action. There is, however, no supersymmetric example of this mechanism at work.
- 3) Simple as the model is, the most popular choices of breaking directions preferred in superstring inspired models seem to be energetically favoured.

Promising though these results are, there is much more work to be done with more realistic models before the mechanism can be used with any degree of reliability in unified model building.

Appendix A

Critical numbers for the zero background

Table 1: Critical numbers for the zero background. The table is mostly illegible due to low contrast.

Table 2: Critical numbers for the zero background. The table is mostly illegible due to low contrast.

Table 3: Critical numbers for the zero background. The table is mostly illegible due to low contrast.

Table 4: Critical numbers for the zero background. The table is mostly illegible due to low contrast.

Table 5: Critical numbers for the zero background. The table is mostly illegible due to low contrast.

Table 6: Critical numbers for the zero background. The table is mostly illegible due to low contrast.

Table 7: Critical numbers for the zero background. The table is mostly illegible due to low contrast.

Table 8: Critical numbers for the zero background. The table is mostly illegible due to low contrast.

Table 9: Critical numbers for the zero background. The table is mostly illegible due to low contrast.

Table 10: Critical numbers for the zero background. The table is mostly illegible due to low contrast.

Table 11: Critical numbers for the zero background. The table is mostly illegible due to low contrast.

In this appendix, we shall give the explicit derivations of the various critical fermion and scalar numbers for stability of the $q = 0$ background, and shall find a dramatic difference between the fermion and scalar cases as regards their dependence on the dimension of the spacetime.

For the periodic fermion case, the stability criterion for the $q = 0$ background configuration can easily be obtained from equation (5.10), and turns out to be

$$N_F < (m - 1) 2^{-[(m + 1) / 2]} \frac{I_2(\text{ad})}{I_2(R)} \quad (\text{A.1})$$

If we restrict our attention to the case $m = 3$ for the moment, then this becomes

$$N_F < \frac{1}{2} \frac{I_2(\text{ad})}{I_2(R)} \quad (\text{A.2})$$

For any case where the fermions lie in the adjoint representation, any non-zero fermion number will result in the $q = 0$ configuration becoming a maximum of the one loop potential. This can easily be seen in the adjoint G_2 and adjoint $SU(2)$ graphs given in chapter 5. For the other graphical examples, we can derive the critical number simply by finding the various Dynkin indices.

Group	Representation	Index	Adjoint index	NFC
SU(2)	<u>2</u>	2	8	2
SU(3)	<u>3</u>	1	6	3
G2	<u>7</u>	2	8	2

The dimension dependence of the $q = 0$ critical number can be illustrated by plotting

a graph of the m dependent coefficient in equation A.2, in which case we see that the critical number drops as m increases. This is due to the increase in the spinor degrees of freedom in higher dimensions, and one would therefore hope that a similar phenomenon would occur when a more complicated higher dimensional compact space is chosen.

The $q = 0$ critical number for antiperiodic adjoint class scalars has a slightly different dependence on m . This is due to the fact that the stability condition derived from (6.3) has the form

$$-N_S \frac{\text{Li}_{m-1}(-1)}{\text{Li}_{m-1}(1)} < \frac{(m-1)}{2} \frac{I_2(\text{ad})}{I_2(\text{R})} \quad (\text{A.3})$$

where both polylogarithm terms are real. In this case, the polylogarithms have different arguments and therefore some manipulation is required before the explicit m dependence of the critical number is obtained. To achieve this, we may manipulate the series expansion for the polylogarithm in the numerator. By equation (4.32), we have the following expansion

$$\text{Li}_p(-1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^p} \quad (\text{A.4})$$

which can be rewritten as

$$\begin{aligned} \text{Li}_p(-1) &= -\sum_{n=1}^{\infty} \frac{1}{n^p} + 2\sum_{n=1}^{\infty} \frac{1}{(2n)^p} \\ &= -\text{Li}_p(1) + \frac{1}{2^{p-1}} \text{Li}_p(1) \\ &= \frac{1 - 2^{p-1}}{2^{p-1}} \text{Li}_p(1) \end{aligned} \quad (\text{A.5})$$

Incorporating this expression into (A.3) gives us the required form of the stability condition

$$N_S < \frac{(m-1)2^{m-3}}{2^{m-1}-1} \frac{I_2(\text{ad})}{I_2(R)} \quad (\text{A.6})$$

Unlike the fermion case, this critical number increases with the dimension of the spacetime. The reason for this is that the coefficient of the scalar term in the potential is independent of m , whereas the fermion coefficient increased with m . If this were the only difference between the cases, then the critical scalar number would be independent of the spacetime dimension. As can be seen from the above equations, however, the polylogarithm cancellation in the scalar case is not the same as for the fermions, due to the fact that the destabilising term depends on the ratio of the depth of the polylogarithm minimum to the height of the polylogarithm maximum (see graph at end of chapter 4). As, when m is increased, the minimum drops more quickly than the maximum rises. then more scalar fields are required to destabilise the $q = 0$ background at higher dimensions, as is shown in the graph. The critical values for the $m = 3$ graphs in chapter 5 can be found from (A.6) and are

Group	Representation	Index	Adjoint index	NSC
SU(2)	<u>2</u>	2	8	8
SU(2)	<u>3</u>	8	8	2
G2	<u>7</u>	2	8	8
G2	<u>14</u>	8	8	2

The remaining case to be considered is that of $\delta = \pi/2$ scalars in a representation of a group with Z_2 centre. In this case, the stability condition for the $q = 0$

configuration is exactly the same as that for the other symmetric configuration at $q = 1$, due to the reflection symmetry of the polylogarithm graph. This condition is

$$-N_S \frac{\operatorname{Re} \operatorname{Li}_{m-1}(i)}{\operatorname{Li}_{m-1}(1)} < \frac{(m-1)}{2} \frac{I_2(\text{ad})}{I_2(\text{R})} \quad (\text{A.7})$$

As for the previous scalar case, we wish to write the numerator of the left hand side in terms of $\operatorname{Li}_{m-1}(1)$. We therefore proceed as follows

$$\begin{aligned} \operatorname{Re} \operatorname{Li}_p(i) &= \operatorname{Re} \sum_{n=1}^{\infty} \frac{i^n}{n^p} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)^p} \end{aligned} \quad (\text{A.8})$$

We now re-express the series (A.8) in two different ways. As in the previous case, we can write this series as

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{(2n)^p} - 2 \sum_{n=1}^{\infty} \frac{1}{(4n-2)^p} \\ = \frac{1}{2^p} \operatorname{Li}_p(1) - 2 \sum_{n=1}^{\infty} \frac{1}{(4n-2)^p} \end{aligned} \quad (\text{A.9})$$

We can also rewrite (A.8) by separating out the positive and negative terms, giving

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{(4n)^p} - \sum_{n=1}^{\infty} \frac{1}{(4n-2)^p} \\ = \frac{1}{4^p} \operatorname{Li}_p(1) - \sum_{n=1}^{\infty} \frac{1}{(4n-2)^p} \end{aligned} \quad (\text{A.10})$$

By equating (A.9) and (A.10), we may express the remaining summation in terms of $\text{Li}_p(1)$

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{(4n-2)^p} &= \left(\frac{1}{2^p} - \frac{1}{4^p} \right) \text{Li}_p(1) \\ &= \frac{2^p - 1}{4^p} \text{Li}_p(1) \end{aligned} \tag{A.11}$$

and, by substituting back into (A.10), we obtain

$$\text{Re Li}_p(i) = \frac{2 - 2^p}{4^p} \text{Li}_p(1) \tag{A.12}$$

We can now substitute this expression into the stability condition (A.7) to obtain

$$N_S < \frac{(m-1) 4^{m-2}}{2^{m-2} - 1} \frac{I_2(\text{ad})}{I_2(\text{R})} \tag{A.13}$$

As in the previous scalar case, the critical number increases with m (see graph). This is due to the zeros moving towards their asymptotic positions at $1/4$ and $3/4$, thus decreasing the destabilising effect at higher dimensions. Looking at the $\text{SU}(2)$ example in chapter 6, the critical value can be found from (A.13) to be $N_S = 32$ but, as originally found by Hosotani, the symmetry will break at $N_S = 16$, well below this critical value. The fact remains, however, that more and more unrealistic numbers of scalars will be required to break the symmetry as the dimension increases.

The conclusion to be drawn from this is that the symmetry is more readily broken by fermion fields than by scalars, a result which, if mirrored in more sophisticated models, may reduce the need for the introduction of spurious scalar fields.

Appendix B

Turning points for semisimple subgroups

In chapter 4, we predicted the existence of symmetry preserving configurations as turning points of the one loop potential and, in appendix A, gave an explicit derivation of the general stability conditions for the $q = 0$ symmetric configuration. (This analysis can also be performed for any other symmetric configuration assuming, of course, that the true group of the matter representation has non-trivial centre).

It is also possible, however, to predict the existence of turning points at non-symmetric configurations and to give a corresponding stability analysis. The main problem with trying to determine all turning points of the potential this way is due to the existence of non-semisimple subgroups. For the symmetry to break to any particular subgroup, the set of Wilson operators created from the true vacuum must form a representation of the subgroup centre. For the semisimple subgroups of the original group, the centre will be a product of finite order cyclic groups, and hence the form of the Wilson loops is severely restricted, allowing the necessary analysis to proceed. If, however, we are interested in the subgroups which contain $U(1)$ factors then, due to the fact that these groups are abelian and therefore have a one dimensional centre, the analysis is considerably complicated.

As it is easier to adopt a numerical approach to this particular problem, not too much work has been done on this cumbersome analytic approach, but in this appendix, the proof of the existence of turning points for every semisimple subgroup of a general gauge group shall be given, along with a sample stability calculation for the $SU(3)$ turning points of the G_2 potential for $\underline{7}$ representation fermions. In the case where the fermions obey an antiperiodic boundary condition, these broken symmetry minima are destabilised, and the critical number will be derived and verified graphically.

To begin, we take the form of the one loop potential as given in (5.4). (In this appendix, we shall work with fermion fields only, noting that, as outlined in chapter 6, the generalisation to scalar fields is straightforward.) The fermion potential is

$$V = C(m) \text{Re Tr} \left[(1-m) \text{Li}_{m+1} (U_{\text{ad}}) + 2^{[(m+1)/2]} N_F \text{Li}_{m+1} (e^{i\delta} U_R) \right] \quad (\text{B.1})$$

Noting the usual differentiation rule

$$\frac{d}{dx} \text{Li}_p (e^{cx}) = c \text{Li}_{p-1} (e^{cx}) \quad (\text{B.2})$$

we obtain the general condition for a stationary point

$$\frac{\partial V}{\partial B^a} = \beta C(m) \text{Re Tr} \left[(1-m) \text{Li}_m (U_{\text{ad}}) H_{\text{ad}}^a + 2^{[(m+1)/2]} N_F \text{Li}_m (e^{i\delta} U_R) H_R^a \right] \quad (\text{B.3})$$

In chapter 5, we proceeded to examine the case where the fundamental Wilson loops U_R and U_{ad} were appropriate multiples of the group identity, i.e. lay in the true group centre. These quantities could then be moved outside the trace, leaving only a trace over the Cartan subalgebra generators and therefore giving a vanishing first derivative of the potential. In this case, the situation is a little more complicated, as the Wilson loops now form a representation of the centre of some subgroup of the original group, and therefore are no longer simply multiples of the identity matrix. In order to proceed, we note that, as the point group acting on the covering manifold is just the abelian discrete translational group Z then, as was pointed out in chapter 1, the flux breaking mechanism on this spacetime is rank preserving. This then implies that we are only interested in regular subgroups of the original gauge group and, in such cases, we can apply suitable inner automorphisms on the algebra to arrange for the Cartan subalgebra of the subgroup to be contained within the Cartan subalgebra of the original group, i.e. that the original Cartan subalgebra matrices become reducible Cartan subalgebra generators for the regular subalgebra in question.

As was detailed in Slansky's review article, we may define a projection matrix for the branching of the original representation R into the subgroup representations r_j by

$$R \rightarrow \bigoplus_j r_j, \quad H_R^a \rightarrow \bigoplus_j \sum_{b=1}^{\text{rank}} c^{ab} h_j^b \quad (\text{B.4})$$

where the a and b indices are for the original and residual algebras respectively and the h_j are the Cartan subalgebra generators of the representation r_j . (For the case where the subgroup is semisimple, the matrix elements c^{ab} are integer.) The Wilson loop operators also take block form

$$U_R \rightarrow \bigoplus_j \exp \{ \beta B_S^a c^{ab} h_j^b \} \quad (\text{B.5})$$

where summation is implied over the a and b indices.

If we now restrict our attention to the cases where the h_j generate a semisimple group, then the Wilson operators at the appropriate background field configuration must lie in the centre of this group, i.e. the blocks in equation (B.5) must lie in the centre of the true group of the particular representation r_j . (Note that the true residual group will in most cases be the covering group, due to the fact that representations in various residual group congruency classes will appear in the branchings of the fermion and gauge field representations.) This implies that the Wilson loops may be written in the form

$$U_R \rightarrow \bigoplus_j I_j \exp \{ -2\pi i q_j \} \quad (\text{B.6})$$

where the q_j are exactly analogous to the q parameters defined in chapter 5 for the symmetry preserving configurations, with $\exp\{-2\pi i q_j\}$ generating a representation of the centre of the true group for which r_j is a single valued faithful representation.

For convenience, we can also redefine the background field components B_S by

$$\hat{B}_S^b = B^a c^{ab} \quad (B.7)$$

and can therefore rewrite the condition for a stationary point as

$$\frac{\partial V}{\partial \hat{B}_S^b} = \beta C(m) \text{Re} \left\{ (1 - m) \oplus_j \text{Li}_m \left(e^{-2\pi i q_j} \right) \text{Tr} h_j^b \right. \\ \left. + 2^{[(m+1)/2]} N_F \oplus_k \text{Li}_m \left(e^{-2\pi i q_k + i \delta} \right) \text{Tr} h_k^b \right\} \quad (B.8)$$

As for the symmetric cases, this expression vanishes due to the tracelessness of the generators h_j , hence every semisimple subgroup of any original symmetry group will have a corresponding background field configuration which will always be a turning point of the one loop potential and whose Wilson loops will form a representation of the group centre.

The problem with the subgroups which contain $U(1)$ factors is that, while we can separate out the generators of the semisimple parts by inner automorphism as before and therefore constrain the Wilson loops constructed out of these generators with the appropriate rescaled fields in (B.7) to lie in the centre of the semisimple part, there is no such restriction on the abelian component as the target space for the Wilson loops is one dimensional.

Another factor contributing to the complexity of such cases is that the $U(1)$ generators of the representation r_j are $\text{dim}(r_j)$ dimensional reducible matrices whose trace is non-vanishing, unlike the semisimple case. These matrices are multiples of the $\text{dim}(r_j)$ identity matrix, with the multiplying factor just being the $U(1)$ number for the representation. In such cases, it is necessary to look for the rescaled background field components associated with the $U(1)$ generators which will give a vanishing first derivative of the potential, a task which can be done in particular cases, but which is not as satisfying as the general result obtained for the semisimple

cases.

Having dealt, albeit incompletely, with the determination of the turning points of the one loop potential, we shall now give an example for one of the semisimple cases, i.e. the SU(3) turning points of the G_2 potential with fundamental fermions. We shall explicitly determine the projection matrix and shall then determine the critical number of antiperiodic fermion families for which these configurations no longer form local minima of the potential. The generators of the seven dimensional representation of G_2 are

$$\begin{aligned} H^1 &= \text{diag} (1, -1, 0, 0, -1, 1, 0) \\ H^2 &= \text{diag} (-2, 1, 1, 0, 2, -1, -1) \end{aligned} \tag{B.9}$$

As the branching of the $\underline{7}$ into SU(3) representations is

$$\underline{7} \rightarrow \underline{3} + \underline{3}^* + \underline{1} \tag{B.10}$$

and as the generators of the $\underline{3}$ of SU(3) are

$$\begin{aligned} h^1 &= \text{diag} (1, -1, 0) \\ h^2 &= \text{diag} (1, 1, -2) \end{aligned} \tag{B.11}$$

then we can immediately see that the projection matrix elements as defined in equation (B.4) are

$$c = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \tag{B.12}$$

where the rows are labelled by the G_2 index and the columns by the SU(3) index.

Note that these matrix elements are integer, as was stated earlier for the general case, and this matrix is applicable to the Cartan subalgebra of any representation of G_2 .

In this simple case, it is easy to derive the background field components which give $SU(3)$ as the subgroup. As the transformed fields are

$$\hat{B}^1 = B^1, \quad \hat{B}^2 = 2B^1 + B^2 \quad (\text{B.13})$$

and as the triplet Wilson loop must form a representation of Z_3 , then we have

$$\exp \{ \hat{B}^1 h^1 + \hat{B}^2 h^2 \} = \exp \left\{ \frac{2n\pi}{3} I \right\} \quad (\text{B.14})$$

which has the solution

$$\begin{aligned} \hat{B}^1 = B^1 &= 0 \\ \hat{B}^2 = B^2 &= \frac{2\pi}{3}, \frac{4\pi}{3} \end{aligned} \quad (\text{B.14})$$

This background configuration will always be a turning point of the G_2 potential and the $B^1 = 0$ section through weight space used in the graphs in chapters 4 and 5 illustrates this point. Note that in equation (B.14), the value $n = 0$ would not be permissible as the Wilson loops would just be the G_2 identity matrices, thus leaving the symmetry unbroken. The effective low energy symmetry group at these backgrounds is therefore $SU(3)/Z_3$.

We can now proceed with a stability analysis of these turning points for the destabilising antiperiodic fermion case. As the projection matrix (B.12) is the same for all representations of G_2 , then we may, for simplicity, deal directly with the Dynkin indices of $SU(3)$ when performing the stability analysis, as the correction term due to the projection matrices will just appear as an overall multiplicative factor in the stability inequation, i.e.

$$\text{Tr} \{ H^a H^b \} = \text{Tr} \{ c^{ay} h^y c^{bz} h^z \} = c^{ay} c^{by} I_2 \quad (\text{B.15})$$

Noting that the branching of the $\underline{14}$ into $SU(3)$ representations is

$$\underline{14} \rightarrow \underline{8} + \underline{3} + \underline{3}^* \quad (\text{B.16})$$

the stability condition obtained by double differentiation of the potential can be written as

$$\begin{aligned} & \text{Re} \left\{ (1-m) \left[6 \text{Li}_{m-1}(1) + 2 \text{Li}_{m-1} \left(e^{2\pi i / 3} \right) \right] \right. \\ & \left. + 2^{\lfloor (m+1)/2 \rfloor} N_F \left[\text{Li}_{m-1} \left(e^{2\pi i / 3 + i\delta} \right) + \text{Li}_{m-1} \left(e^{4\pi i / 3 + i\delta} \right) \right] \right\} < 0 \end{aligned} \quad (\text{B.17})$$

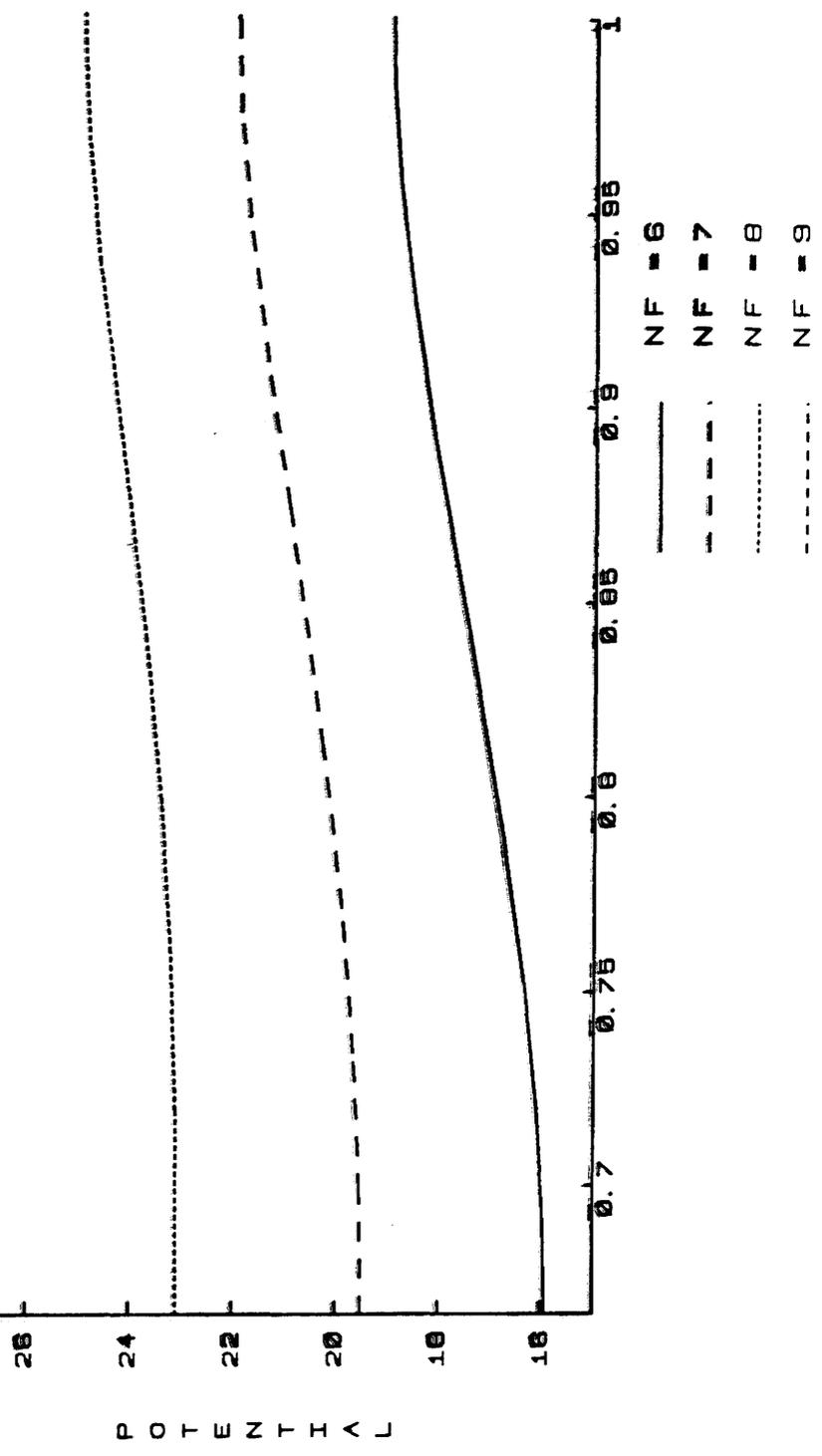
where we have used the reflection symmetry of the polylogarithm in the gauge term and have substituted the values for the relevant $SU(3)$ Dynkin indices, i.e. $I_2(\underline{8}) = 1$ and $I_2(\underline{3}) = I_2(\underline{3}^*) = 1$. If we now take the case $m = 3$ and choose $\delta = \pi$, we see that (B.17) becomes

$$\text{Re} \left\{ -3 \text{Li}_2(1) - \text{Li}_2 \left(e^{2\pi i / 3} \right) + 2 N_F \text{Li}_2 \left(e^{\pi i / 3} \right) \right\} < 0 \quad (\text{B.18})$$

where we can see that the argument $e^{\pi i / 3}$ in the fermion polylogarithm will provide a destabilising effect (see polylogarithm graph in chapter 5). Substitution of the appropriate polylogarithm values in (B.18) leads to the critical antiperiodic fermion number $N_F = 8$, a prediction verified by the graph at the end of this section.

A similar form of analysis can in principle be performed for any case but it is much quicker and easier to examine the complicated cases using a computer.

ANTIPERIODIC FERMIONS IN 7 OF G2 DN-RXSI-



Appendix C

Stability analysis for the E_6 model

The stability analysis for the E_6 model is performed by examining the eigenvalues of the linearized system around the equilibrium point. The system is defined by the following set of equations:

$$\begin{aligned} \dot{X} &= -\lambda X + \mu Y \\ \dot{Y} &= \lambda X - \mu Y \end{aligned}$$

where X and Y are the state variables, λ and μ are parameters, and the dot denotes the time derivative. The equilibrium point is found by setting the time derivatives to zero, which yields the system of equations:

$$\begin{aligned} -\lambda X + \mu Y &= 0 \\ \lambda X - \mu Y &= 0 \end{aligned}$$

The characteristic equation for the linearized system is given by the determinant of the Jacobian matrix:

$$\det \begin{pmatrix} -\lambda & \mu \\ \lambda & -\mu \end{pmatrix} = 0$$

which simplifies to:

$$\lambda^2 - \mu^2 = 0$$

The eigenvalues are $\lambda = \pm \mu$. The stability of the equilibrium point depends on the sign of μ . If $\mu > 0$, the equilibrium point is a saddle point. If $\mu < 0$, the equilibrium point is a center.

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In this appendix, we shall give a stability analysis similar to that given for the SU(3) minima of G_2 in the previous chapter, but we shall now move on to the more complicated (and more physically interesting) case where the covering group of the Lagrangian symmetry group is E_6 . To achieve this, it is necessary to introduce the Dynkin indices for product groups, so we shall first of all consider the adjoint breaking case touched on in chapter 4, in which case the new minima correspond to a residual group $[SU(3)]^3$. We shall then move on to the two representation case which was introduced in chapter 7 (N_A families of periodic 78 fermions and N_F families of anti periodic 27 fermions), and shall conduct a stability analysis for the $[SU(3)]^3$ minima for this more complicated case, showing that the fundamental antiperiodic fermions destabilise these minima and result in a different residual group being preferred.

The generalisation of the Dynkin indices to groups consisting of products of semi-simple factors is as given in reference[24]. If we define the zero'th order index I_0 to be simply the dimension of the representation, then the Dynkin (second order) index of the representation $R = (r_1, \dots, r_k)$ of a semi-simple Lie algebra L consisting of the direct sum of k semi-simple factors L_i is

$$I_2(R) = \prod_{i=1}^k I_0(r_i) \sum_{j=1}^k \frac{I_2(r_j)}{I_0(r_j)} \quad (C.1)$$

The sum is effectively the weighted sum of the Dynkin indices of the constituent representations r_i , and the product simply multiplies this sum by the dimension of the representation R . As the branching of the adjoint of E_6 to representations of $[SU(3)]^3$ is

$$\underline{78} \rightarrow (\underline{8}, \underline{1}, \underline{1}) + (\underline{1}, \underline{8}, \underline{1}) + (\underline{1}, \underline{1}, \underline{8}) + (\underline{3}, \underline{3}, \underline{3}^*) + (\underline{3}^*, \underline{3}^*, \underline{3}) \quad (C.2)$$

then we can use (C.1) to evaluate the Dynkin indices of the various representations

in this branching. (We should really adopt the normalisation of the $SU(3)$ generators as imbedded in E_6 but, as this will only lead to a rescaling of the stability condition by a positive factor, such subtleties can be ignored for the purposes of this calculation.) Adopting the $SU(3)$ Dynkin indices as given in [24], we find that

$$\begin{aligned}
 I_2(\underline{8}) &= 6 \\
 I_2(\underline{3}) &= 1 \\
 I_2(\underline{8}, \underline{1}, \underline{1}) &= I_2(\underline{1}, \underline{8}, \underline{1}) = I_2(\underline{1}, \underline{1}, \underline{8}) = 6 \\
 I_2(\underline{3}, \underline{3}, \underline{3}^*) &= I_2(\underline{3}^*, \underline{3}^*, \underline{3}) = 27
 \end{aligned} \tag{C.3}$$

When the function is numerically minimised for the periodic fermion case, it is found that the global minima corresponding to this breaking has 'q' values such that $q = 0$ for the adjoint representation of $[SU(3)]^3$, i.e. the sum of individual $SU(3)$ octets, has $q = 1/3$ or $2/3$ for one of the 'triplet' representations, and has $q = 2/3$ or $1/3$ for the other 'triplet' representation. As the polylogarithm function is symmetric about the value $q = 1/2$, we can combine these terms in the stability inequation for these configurations. This then gives

$$\begin{aligned}
 (1 - m) \left[18 \zeta_{(m-1)} + 54 \operatorname{Re} \operatorname{Li}_{m-1}(e^{2\pi i / 3}) \right] \\
 + 2^{\lfloor (m+1)/2 \rfloor} N_F \operatorname{Re} \left[18 \operatorname{Li}_{m-1}(e^{i\delta}) + 54 \operatorname{Li}_{m-1}(e^{2\pi i / 3 + i\delta}) \right] < 0
 \end{aligned} \tag{C.4}$$

If we now specialise to the example case $m = 3$ as usual, and for simplicity write the polylogarithms as Bernoulli polynomials, then the antiperiodic adjoint fermion case becomes

$$-2 \left[B_2(0) + 3 B_2(1/3) \right] + 4 N_F \left[B_2(1/2) + 3 B_2(1/6) \right] < 0$$

$$\Rightarrow N_F > 0 \tag{C.5}$$

This inequality is satisfied for any N_F , hence these turning points are local minima of the one loop potential in the presence of antiperiodic adjoint fermions. The feature also appears for $m > 3$.

For the periodic case, we can easily write the entire expression in terms of the zeta function by a similar approach to that used in appendix A. The left hand side of (C.4) becomes

$$(1 - m + 2^{\lfloor (m+1)/2 \rfloor} N_F) \left[1 - \frac{3}{2} \left(1 - \frac{1}{3^{m-2}} \right) \right] < 0 \tag{C.6}$$

We find that this is always negative for $m \geq 4$ and zero for $m = 3$. This implies that the turning point will always be a stable minimum except when $m = 3$, in which case we can extract no information from the second derivatives. Extensive numerical examinations have shown that this minimum is always stable for $m \geq 3$ and is the preferred global minimum for the adjoint periodic fermion case.

We are now in a position to analyse these turning points in the case of our simple E_6 example, where we have a matter Lagrangian consisting of N_A generations of periodic adjoint representation fermions and N_F generations of antiperiodic fundamental fermions. For this case, we shall analyse the stability of the $q = 0$ symmetric background as well as the $[SU(3)]^3$ backgrounds. The first thing to do, however, is to give the branching of the $\underline{27}$ into $[SU(3)]^3$ representations, and to find the Dynkin indices of these representations. We therefore have

$$\underline{27} \rightarrow (\underline{3}^*, \underline{3}^*, \underline{1}) + (\underline{3}^*, \underline{1}, \underline{3}) + (\underline{1}, \underline{3}, \underline{3})$$

$$I_2(\underline{3}^*, \underline{3}^*, \underline{1}) = I_2(\underline{3}^*, \underline{1}, \underline{3}) = I_2(\underline{1}, \underline{3}, \underline{3}) = 6 \quad (\text{C.7})$$

As these representations are labelled by $q = 1/3$ ($2/3$), 0 , $2/3$ ($1/3$) respectively at the $[\text{SU}(3)]^3$ configurations, then the stability condition for this example is

$$\begin{aligned} & (1 - m + 2^{[(m+1)/2]} N_A) [18 \zeta(m-1) + 54 \text{Re Li}_{m-1}(e^{2\pi i/3})] \\ & + 2^{[(m+1)/2]} N_F [6 \text{Li}_{m-1}(-1) + 12 \text{Re Li}_{m-1}(e^{\pi i/3})] < 0 \quad (\text{C.8}) \end{aligned}$$

In the $m = 3$ case, this reduces to

$$\begin{aligned} & 3(-1 + 2 N_A) [B_2(0) + 3 B_2(1/3)] + 2 N_F [B_2(1/2) + 2 B_2(1/6)] < 0 \\ \Rightarrow N_F > 0 \quad (\text{C.9}) \end{aligned}$$

The result of this is that, as before, the introduction of fundamental fermions obeying antiperiodic boundary conditions still stabilise the $[\text{SU}(3)]^3$ turning points. We may therefore have this group as the residual symmetry group for this model but, as the antiperiodic fermions will tend to deepen the $q = 0$ symmetric turning point, we shall work out the appropriate stability condition for this configuration in order to see whether this background may possibly be the global minimum of the potential for $N_F > 0$ as it was in the antiperiodic adjoint case. The stability condition for this configuration is ($m = 3$, $I_2(\underline{27}) = 4$, $I_2(\underline{78}) = 24$)

$$\begin{aligned} & 4(-2 + 4 N_A) B_2(0) + 4 N_F B_2(1/2) < 0 \\ \Rightarrow N_F > 8 N_A - 4 \quad (\text{C.10}) \end{aligned}$$

which implies that, for $N_A = 1$, the $q = 0$ symmetric configuration will only be a minimum of the potential when four or more generations of fundamental fermions are added. If we add one family of antiperiodic scalars in the 27 representation as in chapter 7, then we should replace N_F in (C.10) by $N_F - 1/2$, the result of which is to destabilise the zero background at $N_A = 1$, $N_F = 4$. In such a case, 5 or more fermion generations would be required in order to stabilise the zero background. If we take $N_A = 2$, then we find the critical number of fundamental fermions to be $N_F = 12$.

If we now change the spacetime dimension to 4+1 or 9+1, then using (A.5) we find

$$B_3(1/2) = -\frac{3}{4} B_3(0) \tag{C.11}$$

$$B_8(1/2) = -\frac{127}{128} B_8(0) \tag{C.12}$$

and hence, after making the appropriate changes in the m dependent coefficients, the critical numbers from (C.8) become :

- 1) $m = 4$; $N_F = 2$, ($N_A = 1$) and $N_F = 7$, ($N_A = 2$)
- 2) $m = 9$; $N_F = 4$, ($N_A = 1$) and $N_F = 8$, ($N_A = 2$),

results which are confirmed by the numerical examinations.

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