The Low Energy Effective Action From The Heterotic Superstring

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Contents.

Abstract.

The heterotic superstring has been considered to be a candidate for a quantised unified theory of all physical interactions at high energies, including the unification of quantised gravity. The low energy phenomenology of such a theory is most easily described in terms of an effective quantum field theory, which describes the scattering of the massless modes of the string theory, as well as satisfying the anomaly freedom, supersymmetry, unitarity, etc. of the string theory. Previous attempts at the derivation of the low energy effective theory have concentrated on the bosonic sector of the action, and on the possible ambiguities in the dervivation of this action.

This thesis describes an attempt to derive the effective field theory including the fermionic sector of the theory, with the hope that the resulting action will demonstrate the supersymmetry of the heterotic superstring, as well as the anomaly freedom.

After an introduction to the construction of the free heterotic superstring, and the corresponding interacting string theory, two amplitudes are calculated. It is shown that these amplitudes naturally expand in a power series in the string parameter α'. The attempt at constructing a low energy theory which will reproduce these amplitudes order by order in this parameter will then be made.

The effective theory which has the same matter spectrum as the massless modes of the heterotic string is stated, and this free theory is quantised. The techniques of constrained quantisation are used to evaluate the propagators for the various fields of the theory. A comment will be made on the problems of quantisating higher derivative theories, and an ansatz will be made to avoid these problems.

The interacting field theory, which is uniquely determined by the Noether method is stated. It is noted that the required anomaly freedom of this action can only be included at the expense of supersymmetry, by the addition of a term of higher order in the parameter α'. The lowest order action thus derived is then used to construct the lowest order amplitudes in the α' parameter, (that is $O(\alpha'^0)$), which are compared with the corresponding string results. It is shown that the effective field theory gives the same amplitudes as the string theory.

The field theory is then extended in an attempt to match the next highest order amplitudes in the α' parameter, and also to retrieve the lost supersymmetry of the lowest order action. In this regard, known supersymmetric actions are used to construct the amplitudes at the appropriate level in the α' parameter, and these amplitudes are compared with the string results. These matching calculations are shown to fail implying that these actions do not correspond to the low energy

effective action for the heterotic string theory.

A more general action is constructed, subject to a guiding assumption that the action only contains the two-form form of the covariant derivative of the gravitino, and without regard to supersymmetry, which is then used to construct amplitudes which are compared to the string result. A match to the string amplitude is then found for specific values for the general coefficients introdced in this action. A comparison with the two actions previously used as trial low energy effective field theories is drawn, and the similarities and differences noted.

Finally some comments are made on the possibility of ambiguities which may be inherent in the procedure described in this thesis for deriving the low energy effective action. Some examples of field redefinitions which do not show the same properties as the field redefinitions used in the purely bosonic cases previously treated in the literature. The implications for the amplitude matching procedure are then noted. The conclusions of this work are then presented.

Declaration.

I would like to state that the work described in Chapters Three to Six of this thesis is the original work of the author, except where it is explicitly stated in the text where an appropriate reference will be provided.

R.J. Jenkins, March 1989.

"All the sanguine guesswork of youth is there, and the silliness; all the novelty of being alive and impressed by the urgency of tremendous trivialities."

Siegfried Sassoon,

Memoirs of a Fox-hunting Man.

Chapter Zero:
General Introduction.

Section 0.1 The general motivation for string theories.

The recent history of theoretical physics has been characterised by a search for a fully self consistent, completely unified quantum mechanical theory of the Universe. It is required that a complete description of all known interactions in the Universe should arise naturally from such a theory. Thus it is expected that any complete theory of the Universe should contain gravity. Already the three other known forces of nature, the strong interaction, the weak interaction and electromagnetism have been partially unified by gauge theories. The so called "standard model"[1,2,3] of elementary particle interactions has been extremely successful in describing particle interactions at the energies currently available in particle accelerators. However this is hardly a unification at all, as it does not answer any of the questions that a reasonable candidate for a unified theory should. Some of its more serious problems include its inability to explain charge quantisation, an unusually high number of arbitrary constants which have to be fixed by experiment, there are still three separate coupling constants in the theory whose values are independent of each other to one loop in the perturbation expansion, the hierarchy problem, the fact that left and right handed components of the same spinor field are in different representations of the gauge group and the sheer arbitrariness of the Higgs mechanism.

A unified theory ought to have several properties, these being:

1) That there should be a natural explanation for the observed gauge symmetries of nature, these being with respect to the gauge group $G=SU(3)\otimes SU(2)\otimes U(1)$. It might be that there is a group G' such that $G\supset G'$, and that some spontaneous symmetry breaking takes place at low energy, via the Higgs mechanism for example[4] breaking the group G' to the observed G.

2) There should be a natural explanation for the breaking scales involved in the theory, and that these should not introduce phenomenological defects of the form of proton decay etc.

3) That all matter fields should fall naturally into their correct representations of the observed gauge groups at a particular energy scale.

4) Gravity should be contained in the theory in a natural way, and be consistently quantised.

5) There should be as few arbitrary parameters in the theory as possible, the most desireable theory having no arbitrary parameters at all.

However gravity still does not easily find a place in such a scheme. Until quite recently the major approaches to a unified theory of gravity have been field theoretical, looking at gauge theories as described above for example, and simply coupling these to field theories of gravity. The field theoretical approaches to gravity have the problem of ultra violet divergences and renormalisability in the perturbation expansion, and so a recent trend has been the investigation of possible perturbatively finite gravitational field theories.

This trend has included the development of higher dimensional gravity theories incorporating local supersymmetry, for example ten and eleven dimensional supergravity theories[5,6,7]. It was hoped that these theories would lead to a peturbatively finite, completely consistent quantum theory of gravity, which would lead to a complete description of all known interactions. The attainment of this complete description would have to be achieved through some compactification scheme, whereby the four-dimensions of space-time are singled out from the higher number of dimensions. The remaining dimensions are "compactified" using a scheme which not only ensures that these dimensions should not be observable at realistically low energies but which also provide the low energy matter fields and their interactions. Several such schemes have been proposed[8,9,10], but are not successful in describing all known interactions, whilst also providing a perturbatively finite theory of gravity. The perturbatively divergent behaviour of these theories is due to the pointlike nature of the particles inolved, and the addition of finite-dimensional internal symmetries, though tending to reduce the problem, is not enough to cure it completely. It would therefore seem natural to define quanta with an internal linear degree of freedom which provides an infinite-dimensional internal symmetry[11]. In fact, the quanta of the theory can now be regarded as strings instead of point particles. As shall be discussed at length below, these string theories have higher dimensional gravity or supergravity theories as their low energy or massless sector limits, and can be seen to contain all of the advances mentioned above in a relatively simple formulation.

Most recently therefore, superstring theories have been regarded as being the most promising candidates for a complete, fundamental unified theory of the universe. In fact there are several string theories which can be regarded, to a greater or lesser extent, as possible unified theories of all known interactions, which are classified below.

Section 0.2 A Classification of String Theories.

The possible string theories can be classified as follows:

(1) The (open or closed) bosonic string, which is constructed in its critical space-time dimension of 26, but which only has bosonic space-time dimensions (i.e. there is no extension to superspace in the action for the string.) and so it does not contain spinor fields at the simple level. With the addition of Chan-Paton factors[12,13] it is possible to introduce a gauge group into the open bosonic string theory ,which has some constraints upon which groups are admissable[14].

(2) The open Neveu-Schwarz-Ramond or spinning string[15], which has such a superspace extension, is constructed in its critical space-time dimension of 10, and yields the open superstring when the Gliozzi-Scherk-Olive[16] projection conditions are imposed. Again it is possible to introduce a gauge group into this theory by the use of Chan-Paton factors. There are also constraints upon the gauge groups which are admitted in this theory. This theory has only one space-time supersymmetry when the projection conditions are imposed.

(3) The type I closed string which consists of the closed string sector of the open superstring[17], (which exists due to locality of string interactions and unitarity of the theory). This theory has one supersymmetry, but no gauge sector, and is therefore limited phenomenologically. This theory will not be discussed further in this thesis.

(4) The type IIa closed superstring, formulated in ten dimensions, which has only closed strings but which has different chirality spinors defined on left and right movers. This theory has an N=2 supersymmetry, but since it is a closed string theory, there is no gauge sector admitted. This theory will not be discussed further in this work.

(5) The type IIb closed superstring, again defined in ten dimensions, and which has the same chirality spinors defined on the left and right movers. The remarks made for the type IIa string apply here also.

(6) The heterotic string[18,19], which consists of the left moving sector of a bosonic string, and the right moving sector of a Neveu-Schwarz-Ramond string. This string theory is the most promising string for phenomenological studies as it has a gauge sector with tight constraints on the admissable gauge groups, as well as having only one space-time supersymmetry.

(7) Four dimensional string theories[20,21]. These strings are constructed by a generalisation of the fermionic construction of the heterotic string given in reference [18]. These string theories are particular examples of lower dimensional string theories that can be constructed in any dimension < 10. These theories give the possibility of constructing unified theories directly in four dimensions. The four dimensional theories suffer from a lack of uniqueness, since there are a vast number of possible theories, corresponding to the large number of possible fermionic boundary conditions that can be chosen.

At this point it would be wise to define some of the terms used in the above classification. By Neveu-Schwarz-Ramond string[15] it is meant that the spinors are on the world sheet. The spinning string spinor fields defined on the world sheet give rise to two distinct sectors in the theory. These correspond to different boundary conditions for the spinor fields on the world sheet, where periodicity of spinor fields yields the Ramond[22] sector and antiperiodicity the Neveu-Schwarz[23] sector. By selecting half the spectrum of states from each of these sectors it is possible to obtain a supersymmetric spectrum of states. This is what is termed the Gliozzi-Scherk-Olive projection. The nature of these fields is clear when the string theory is regarded as a conformal field theory on the world sheet. The existence of a spacetime spinor field on the world sheet becomes clear when this interpretation is made which corresponds to the space-time spinor of the Green-Schwarz string[24]. The conformal field theory approach to strings is outlined in chapter one below.

Section 0.3 The Phenomenological Implications of String Models.

Several of these theories are phenomenologically interesting. The theory which has most promise is the heterotic string. As stated above, this theory is a "cross" between the bosonic string and the superstring. It would appear that there is an immediate contradiction here, in that the bosonic string has critical dimension of 26, whereas the superstring has critical dimension 10. This difficulty is overcome by compactifying the extra 16 dimensions onto a 16-torus, and by considering the momenta in these dimensions to be the diagonal generators of a gauge group for the remaining 10 dimensional theory. An alternative way of looking at this is to consider the right moving spinors of the NSR string to be bosonised in some fashion and to give the Yang-Mills sector of the theory. The "conventional" gauge groups admissable in the heterotic string are, $E_8 \otimes E_8$ or Spin(32)/Z_2, though other gauge groups have been constructed, for example the O(16)⊗O(16) heterotic string.

At realistically low energies, string theories do not provide much in the way of useful physics, since string amplitudes contain contributions from massive modes which cannot be acheived at energies considerably lower than the Planck energy. Therefore it is postulated that interactions within the massless sector of the string theory form an effective field theory, which can be modelled by a suitable point particle field theory. Each of the string theories mentioned above has such a low energy effective field theory. In each case it is a gravity theory of some type, whose spectrum of fields corresponds to the massless sector of the string theory. This procedure involves matching string amplitudes for various processes involving massless external particles with those of a trial field theory. However the string amplitude generally involves terms which are of higher order in the string tension, T or the string parameter α'. The trial field theory must then be corrected to generate these terms calculated from the effective action. This procedure can be followed order by order in the parameter α', providing a complete low energy effective field theory. The effective field theory derived by this method can then be used in phenomenological studies, for example compactification schemes[8].

Several compactification schemes have been proposed to make the transition from a full ten dimensional effective field theory to a "four plus six" spacetime. The first attempts at this used standard Kaluza-Klein techniques[25] developed for the higher dimensional supergravity theories which were being explored in the late 1970's. These attempts failed due to the existence of various "ten into four won't go" no-go theorems[26]. The no-go theorems were circumvented by the addition of R^2 type terms in the action, which allowed the compactification schemes to proceed. The search for a supersymmetric four dimensional theory however which has only one supersymmetry which remains unbroken, culminated in the discovery of Calabi-Yau compactification schemes[8].

Other schemes have been developed which have had their individual successes and failures. These include coset space techniques, (symmetric and nonsymmetric[9]), and compactification of the string spacetime direct from its critical dimension to four dimensions plus a hypertorus in the manner of the original conception of the heterotic string[18]. Four dimensional string theories also exist, where the string is constructed in its critical dimension of four. This is achieved by the use of many extra fermionic fields on the string world sheet, which take up the place of the extra bosonic dimensions in the higher dimensional theories. These techniques in particular show a great deal of promise. However the most sophisticated way in which to proceed is to do string theory in a completely background independent formalism, and then solving an equation of motion for space-time. The major inspiration for this work coming from the techniques of statistical mechanics.

The next obvious step forward is to consider three dimensional field theories, with the intention being to consider the possibility of "membrane theories". These appear to have critical dimensions (bosonic and fermionic) which suggest that reduction of the dimension of the "world volume" leads to the equivalent reduction to the critical dimension of the space-time of the appropriate string theory. Although these theories show a great deal of promise, they and their consequences will not be discussed in this thesis.

Chapter One:
The Heterotic String:
An introduction to the calculation of amplitudes

Introduction.

This chapter will give an overview of superstring theory in general. The first section will outline the construction of the free heterotic string in ten dimensions as originally performed in reference [18]. The interacting string theory will be described in the following two sections. Section 1.2 will deal with the bosonic string theory, and Section 1.3 the NSR form[15] of the superstring. These will be dealt with using the conformal field theoretical approach to the quantisation of string theories, and where the techniques of BRST quantisation will be used extensively. The explicit details of these techniques can be found in references [11,27,28,29,30,31].

The final section will give an example of a string amplitude calculation using the techniques developed in the previous two sections. This amplitude will be used in subsequent chapters as the basis for amplitude matching calculations.

Section 1.1 The free heterotic string.

In this section the free heterotic superstring is constructed, the mass spectrum of the theory is obtained and the possible construction of other theories considered.

The heterotic string can be thought of in two distinct ways. The original formulation for this string theory, was in terms of the ten dimensional Green-Schwarz superstring[24], taking only the left moving sector, and "crossing" this with the right moving sector of the twenty-six dimensional bosonic string. The extra sixteen dimensions are compactified on a sixteen dimensional torus, on which the standard vertex operators transform as generators of a gauge group for the theory which is determined by the choice of normalisation for the lattice[18]. This approach allows the mass spectrum of the theory to be determined in a clear manner, however the theory is formulated in a specific space-time gauge, thus losing manifest covariance.

The more modern approach to string theory is by considering the standard NSR theory as a conformal field theory in two dimensions. The vertex operators can then be considered as conformal fields in the complete set of fields for the theory. (See Appendix One) This approach has the advantage of being manifestly space-time covariant, but the mass spectrum and supersymmetry become slightly obscure. To obtain the mass spectrum for the free theory the first approach will be used as in the original formulation of the heterotic string. The equivalence of this approach to the manifestly covariant approach can be seen in the formalism of Friedan, Martinec and Schenker where the ten dimensional spinor is introduced as an operator to interchange states between Neveu-Schwarz and Ramond sectors of the spinning string. This will be discussed more fully in the section dealing with the interacting heterotic string.

The free heterotic superstring is a theory of closed strings, and so the general solution of the wave equation in two dimensions (σ,τ) can be written in the form,

$$O^i(\tau,\sigma) = O^i_L(\tau\text{-}i\sigma) + O^i_R(\tau+i\sigma) \ , \qquad (1.1.1)$$

(where O^i is some field on the string), where the boundary conditions are periodic, and the left moving and right moving solutions are not related by the boundary conditions. This means that the left and right moving sectors can be thought of as separate closed string theories, which give the full theory when their "tensor product" is taken, where the two separate theories can be quite distinct in character. In the case of the heterotic string these two sectors can be taken to be two distinct theories, where the left moving sector is taken to be the left moving sector of the bosonic string, and the right moving sector that of the superstring. The mass spectrum of the combined theory is thus given by the tensor product of the two mass spectra for the two theories. This product only occurs between modes at the same mass level, due to the constraint of a rigid shift in σ, the world sheet space parameter.

The bosonic string is consistent in twenty-six dimensions and the superstring in ten-dimensions. The remaining sixteen dimensions of the left moving modes are compactified onto a sixteen dimensional manifold. The string action is given by,

$$S = \int d\tau d\sigma \sqrt{g}\, g_{\alpha\beta} \{\partial^\alpha X_i \partial^\beta X^i + \partial^\alpha X_I \partial^\beta X^I + i\, \bar{S}\gamma^- \sigma^\alpha \partial^\beta S \} \qquad (1.1.2)$$

which is invariant under the supersymmetry transformations,

$$\delta X^i = \frac{1}{\sqrt{p^+}} \bar{\varepsilon}\gamma^i S \;\; ; \;\; \delta S = \frac{i}{\sqrt{p^+}} \gamma_- \gamma_\mu \sigma_2 . \partial X^\mu \varepsilon \;\; ,$$

and applying the boundary conditions,

$$X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+\pi) \; ; \; S^\alpha(\tau,\sigma) = S^\alpha(\tau,\sigma+\pi) \;\; ,$$

(μ = i,I; α a spinor index in the space time dimension), and applying the constraint,

$$\Phi^I := (\partial_\tau - \partial_\sigma)\, X^I = 0 \, , \, I = 1, \ldots ,16 \;\; ,$$

(which merely eliminates the right moving modes of the bosonic string in the extra sixteen dimensions), the solutions can be seen to be of the general form,

$$X^i(z) = \frac{1}{2} \{ x^i + zp^i + i \sum_{n \neq 0} \frac{\bar{\alpha}^i_n}{n} e^{-2inz} \} \;\; ,$$

$$X^i(\bar{z}) = \frac{1}{2} \{ x^i + \bar{z}p^i + i \sum_{n \neq 0} \frac{\alpha^i_n}{n} e^{-2i\bar{z}} \} \;\; ,$$

$$S^\alpha(\bar{z}) = \sum_{n=-\infty}^{\infty} S^\alpha_n e^{-2i\bar{z}} \;\; , \qquad (1.1.3)$$

$$X^I(z) = x^I + zp^I + \frac{1}{2} i \sum_{n \neq 0} \frac{\bar{\alpha}^I_n}{n} e^{-2iz} \;\; , \qquad (1.1.4)$$

where the general mode expansion has been made in terms of the complex variables,

$$z = \sigma + i\tau \quad , \bar{z} = \sigma - i\tau \;\; ,$$

The quantisation of the X^I fields of this action is complicated by the appearance of a second class primary constraint[32], and so the action must be quantised using Dirac brackets instead of the usual Poisson brackets. The quantum commutators of the modes are given by,

$$[\, x^i, p^j] = i\, \delta^{ij} \;\; , \;\; [\, \alpha^i_n \, , \alpha^j_m \,] = [\, \bar{\alpha}^i_n \, , \bar{\alpha}^j_m \,] = n\delta_{n+m,0}\delta^{ij} \;\; ,$$

$$[\alpha^i_n , \bar{\alpha}^j_m] = 0 , \{S^\alpha_n , S^\beta_m\} = (\gamma^+ h)^{\alpha\beta}\delta_{n+m,0} \quad , \quad (1.1.5)$$

$$[\bar{\alpha}^I_n , \bar{\alpha}^J_m] = n\delta_{n+m,0} , [x^I , p^J] = \frac{1}{2} i\delta^{IJ} , (1.1.6)$$

where the extra factor of 1/2 in the $[x^I,p^J]$ commutator comes from the quantisation of the Dirac brackets.

The mass operator for the string is given by,

$$\frac{1}{4} m^2 = N + (\tilde{N}-1) + \frac{1}{2}\sum_{I=1}^{16} (p^I)^2 \qquad (1.1.7)$$

where the normal ordered number operators N, Ñ are given by the standard mode operators $\alpha_j^{\ i}$ and $\bar{\alpha}_j^{\ i}$. The condition that the zero point of the spacelike parametrisation of the string is arbitrary, gives another constraint on the number operators. The generator of a rigid displacement of the σ-coordinate is given by the unitary operator,

$$U(\Delta) \equiv e^{2i\Delta(N-\tilde{N}+1-\frac{1}{2}\sum_{I=1}^{16}(p^I)^2} \qquad (1.1.8)$$

which can be shown to satisfy $U(\Delta)O(\tau,\sigma)U(\Delta)^+ = O(\tau,\sigma+\Delta)$, where O can be any operator of the first quantised string. Since the theory must be unchanged by such a displacement, then the operator must be equal to the identity operator, and so the condition that the origin of the world sheet is not specified gives the constraint,

$$N = \tilde{N} - 1 + \frac{1}{2}\sum_{I=1}^{16} (p^I)^2 \qquad (1.1.9)$$

which means that only modes at the same mass level of the two separate string theories can be taken in a tensor product state.

As stated above, the string states are given by the tensor products of the Fock space states of the left and right moving theories, $|\phi\rangle_L \otimes |\psi\rangle_R$ where the states are constrained by equations (1.1.7-8) above. In the usual Fock space manner the right moving state is annihilated by the $\alpha_n^{\ i}$, $S_n^{\ a}$ where n > 0. Because of S_0, the zero mode of the fermion operator, the Fock space ground state must consist of a vector $|i\rangle_R$ and a fermion $|a\rangle_R$, which are defined by the S_0 operator to satisfy,

$$|a> = \frac{i}{8}(\gamma_i S_o)\,|i> \quad ,$$

$$|i> = \frac{i}{8}(\bar{S}_0 \gamma_{i+})^a\,|a> \qquad (1.1.10)$$

where the |a> and |i> states satisfy the normalisation conditions,

$$< i\,|\,j > = \delta^{ij} \quad ,$$

this being due to the fact that since the canonical commutation relations are,

$$\{\, S_0^a\,,\, \bar{S}_0^b \,\} = (\gamma^+ h)^{ab} \quad ,$$

where h is defined above, it follows that,

$$S_0^a\,\bar{S}_0^b = \frac{1}{2}(\gamma^+ h)^{ab} + \frac{1}{4}(\gamma^{ij-} h)^{ab} R_0^{ij} \qquad (1.1.11)$$

where $R_0^{ij} = {}^1/_8 \bar{S}_0 \gamma^{ij-} S_0$, which can be seen to satisfy the relations,

$$[\, R_0^{ij}\,,\, S_0^a \,] = -\frac{1}{2}(\gamma^{ij} S_0)^a \quad ,$$

$$[\, R_0^{ij}\,,\, R_0^{kl} \,] = \delta^{ik} R_0^{jl} + \delta^{jl} R_0^{ik} - \delta^{jk} R_0^{il} - \delta^{il} R_0^{jk} \quad ,$$

$$R_0^{ij}|\, k > = \delta^{ik}\,|\, j > - \delta^{jk}\,|\, i > \quad ,$$

$$R_0^{ij}\,|\, a > = -\frac{1}{2}(\,\gamma^{ij}\,)^{ab}\,|\, b > \qquad (1.1.12)$$

which are just the commutation relations of the SO(8) algebra, and where the |i> and |a> states form an irreducable representation of the S_0 subalgebra.

The fact that the algebra (1.1.12) is satisfied is due to the fact that the fermion and boson creation and annihilation operators commute with the S_0 operator, and by the fact that the ordinary vacuum $|0>_R$ which exist in the old superstring formalism is a singlet which is clearly not supersymmetric, and is projected out by the GSO projections[16].

The left moving spectrum is generated from the Fock space vacuum $|0>_L$ which is annihilated by the α_n^i, α_n^I where n > 0. Again the state is annihilated by the number operator $\tilde{N}$, and also by the operator p^I. This state in the bosonic string corresponds to a tachyon in the theory, but because of the condition given in equation (1.1.7) above, the first physical state must be $\tilde{N}=1$. So the theory is tachyon free. Furthermore the ground state of the heterotic string is massless and also the state transforms as either an $\underline{8}_v\otimes\underline{8}_v$ or $\underline{8}_v\otimes\underline{8}_s$ or $\underline{8}_v\otimes\underline{8}_c$. It should be noted that the restriction to the right moving modes α_n^i is made, thus ignoring the effects of the Yang-Mills sector of the string which is beyond the scope of this work. It can be seen by inspection of these tensor products that the spectrum of massless states correspond to those of N=1, D=10 supergravity.

Section 1.2 The interacting bosonic string as a conformal field theory.

The bosonic string action is given by,

$$S = \int d^2\xi \sqrt{g} \, . \, \{ g^{ab}\partial_a X^\mu \partial_b X_\mu + R^{(2)} + \lambda \} \tag{1.2.1}$$

where the $R^{(2)}$ and λ terms are added to give the most general Poincaré invariant, reparametrisation invariant, renormalisable action in flat space[31].

It is clear that this can be considered to be a conformal field theory in two dimensions. (See Appendix One for a summary of the techniques of conformal field theories in two dimensions.) The conformal symmetry must be gauged away before the functional integral can be evaluated, and this is done by imposing the conformal gauge,

$$g_{ab} = e^\sigma \eta_{ab} \tag{1.2.2}$$

where η_{ab} is a fixed reference metric on the world sheet. The remaining gauge fixed action is merely the action describing a set of d, (the space-time dimension), free fields. The solution to the equations of motion is as above for the free theory, and is given by the same mode expansion. The propagator is therefore of the form,

$$< X^\mu(z) \, X^\nu(w) > = -\eta^{\mu\nu} \ln(z-w) \tag{1.2.3}$$

This clearly demonstrates that $X^\mu(z,\bar{z})$ is not a conformal quantum field as the correlation function increases with separation. The stress energy tensor can be seen to be of the form,

$$T(z) = -\frac{1}{2} : \partial_z X . \partial_z X : \qquad (1.2.4)$$

and can be shown to satisfy the operator product,

$$T(z)\, T(w) \approx \frac{d}{2(z\text{-}w)^4} + \frac{2}{(z\text{-}w)^2} T(w) + \frac{1}{(z\text{-}w)} \partial T(w) \qquad (1.2.5)$$

which is equivalent to the algebra (A1.17) with c=d. The anomalous term, which corresponds to the Schwinger term in the central extension of the Virasoro algebra, can be seen as the term which is due to the fixing of the conformal gauge, that is, it describes the lack of invariance in the remaining fixed metric on the world sheet. However it should be noted that there is still an invariant subalgebra of Virasoro generators, i.e. L_{-1}, L_0 ,L_1 which correspond to a subalgebra SL(2), which corresponds to the remaining symmetry on the world sheet. This remaining symmetry is important in the evaluation of amplitudes, as it has to be gauged away before these amplitudes can be calculated. This can be done using Koba-Nielsen integrations, or more elegantly using the BRST invariance of vertex operators to eliminate complex contour integrations, as will be discussed below.

The conformal fields of the string theory are ∂X^μ and $e^{ik.X}$, which have conformal weights 1 and $\frac{1}{2}k^2$ respectively. The requirement that the second of these vertex terms must have conformal weight one implies that the momentum k^2 must be $1/\alpha'^2$. The construction of vertex operators for the massless sector of the theory merely corresponds to finding the correct combination of these conformal fields which transform in the correct manner to correspond with the conformal field required.

Because of the gauge fixing, a ghost action must be added to the theory to give the determinants for the gauge fixing,

$$\mathfrak{D}g = \mathfrak{D}\xi_z \mathfrak{D}\xi_{\bar{z}} \det \nabla_z \det \nabla_{\bar{z}}$$

where the ξ_z, and $\xi_{\bar{z}}$ are the variables which define the infinitesimal variations orthogonal to the gauge slice, that is,

$$\delta g_{zz} = \nabla_z \delta\xi_z \qquad \delta g_{\bar{z}\bar{z}} = \nabla_{\bar{z}} \delta\xi_{\bar{z}}$$

$$\delta g_{z\bar{z}} = (\nabla_z \delta\xi^z + \nabla_{\bar{z}} \delta\xi^{\bar{z}}) g_{z\bar{z}}$$

The ghost action is written in terms of the ghost field c, and its conjugate b, which have conformal weight 1 and -2 respectively, and is,

$$S_{gh} = \frac{1}{\pi} \int d^2z \, [\, b\, \partial\, c + \text{complex conjugate.}] \tag{1.2.6}$$

which gives equations of motion for the ghost fields,

$$\partial c = \partial b = 0 \tag{1.2.7}$$

and correlation function,

$$< b\, c > = \frac{1}{(\, z - w\,)} \tag{1.2.8}$$

The stress energy tensor for the ghost fields is given by considering the variation of the covariant derivative of rank n tensors, under a traceless deformation of the metric,

$$\delta\nabla^z = \frac{1}{2} \delta g^{zz} \nabla_z + \frac{n}{2} \nabla_z (\delta g^{zz}) \tag{1.2.9}$$

and applying this to the action (1.2.1), to yield,

$$T_{gh}(z) = c\partial b + 2\, (\partial c)\, b \tag{1.2.10}$$

Given the operator product (1.2.8) the operator product of the ghost field stress energy tensors is,

$$T_{gh}(z)\, T_{gh}(w) = \frac{-13}{(z\text{-}w)^4} + \frac{2}{(z\text{-}w)^2} T_{gh}(z) + \frac{1}{(z\text{-}w)} \partial T_{gh}(z) + \text{nonsingular terms.} \tag{1.2.11}$$

which demonstrates that the conformal anomaly vanishes in d=26. However it is more correct to use the gauge fixed form of the reparametrisation operator, T(z), in terms of the BRST algebra. The new operator is the BRST operator Q_{BRST}, constructed by the standard procedure of the Noether theorem. In terms of the conformal symmetry, the BRST current is given by,

$$j_{BRST}(z) = c\,(\,T(z) + \frac{1}{2}\,T_{gh}(z)\,) + \frac{3}{2}\,\partial^2 c \qquad (1.2.12)$$

and so the BRST charge is defined to be,

$$Q_{BRST} = \int \frac{dz}{2\pi i}\, j_{BRST} \qquad (1.2.13)$$

The BRST charge acts on the fields as follows;

$$[\,Q_{BRST}\,,\,O(z)\,] = \int \frac{d\bar{w}}{2\pi i}\, j_{BRST}(w)\, O(z),(z),$$

where the contour is about z, so that,

$$[\,Q_{BRST}\,,\,X^{\mu}\,] = c^{z}\,\partial_{z}\,X^{\mu} \quad ,$$

$$\{\,Q_{BRST}\,,\,c^{z}\,\} = c^{z}\,\partial_{z}\,c^{z} \quad ,$$

$$\{\,Q_{BRST}\,,\,b_{zz}\,\} = T_{zz} + T_{gh\;zz} \qquad (1.2.14)$$

which can be seen by the standard contour integral calculation and using the appropriate operator products given by the explicit form of the stress energy tensors, and the fundamental correlation functions for the fields.

Since the ghosts do not couple to the physical fields of the theory, as can be seen by simple inspection of the full action, then the Fock space of states can be written as a simple tensor product of the ordinary bosonic string Fock space states and the ghost Fock space states, for example,

$$|\,\{n\}\,;\,p\,>_{X} \otimes c_{1}\,|\,0\,>_{gh} \qquad (1.2.15)$$

where the extra ghost operator, c_1, is needed for BRST invariance of the state. This can be seen simply by considering the canonical choice for the vacuum: the ghost vacuum should be SL(2) invariant, that is annihilated by L_{-1} ,L_0, L_1. By noting the conformal weight of the ghost fields, it can be seen that their mode expansions must be,

$$c(z) = \sum_n c_n z^{-n+1} \quad , \ b(z) = \sum_n b_n z^{-n-2} \qquad (1.2.16)$$

and consequently, replacing these in the ghost stress-energy tensor, T_{gh}, it is possible to show that the vacuum must satisfy,

$$c_n \,|\, 0 >_{gh} = 0 \ , \quad n \geq 2$$

$$b_n \,|\, 0 >_{gh} = 0 \ , \quad n \geq -1 \qquad (1.2.17)$$

which has the interesting conclusion that,

$$c_1 \,|\, 0 >_{gh} \neq 0$$

which implies that the ghost vacuum is not a highest weight representation of the b,c algebra, although it is chosen to be a highest weight representation of the Virasoro algebra. So, taking the Hamiltonian to be the full $L_0^{tot.}$ operator, and noting the Q_{BRST} algebra, it is possible to show that the lowest energy state is the vacuum given above in equation (1.2.15), such that,

$$L_0^{tot.} (\,|\, 0 >_X \otimes c_1 \,|\, 0 >_{gh} \qquad (1.2.18)$$

The adjoint of this vacuum is clearly,

$$(c_1 \,|\, 0 >_{gh})^\dagger = {}_{gh}< 0 \,|\, c_{-1} c_0 \qquad (1.2.19)$$

These extra ghost modes play a crucial part in the evaluation of amplitudes, since if $\int d^2 z V_{phys.}(z)$ is a physical vertex then $c(z)c(z)V_{phys.}(z)$ is a Q_{BRST} combination, as can be seen by application of the BRST algebra. Hence the c operators of the ghost vacuum can be used in combinations with physical vertex operators, to soak up the background ghost charge, and to remove three of the integrations, since, because of the BRST invariance the location of the z's is unimportant. This allows three of the integrations to be removed from the amplitude calculation, and thus the remaining SL(2) invariance of the world sheet is gauge fixed.

The physical state condition on a general physical state |phys.>, is simply that it should satisfy,

$$Q_{BRST} \mid phys. > = 0 \qquad (1.2.20)$$

where the mode expansion of Q_{BRST} shows that this is simply the restatement of the standard decoupling of the negative norm states by the condition,

$$L_n \mid phys. > = 0 \quad , \quad n > 0 \qquad (1.2.21)$$

give by the older operator approach to covariant quantisation of the bosonic string. Given some vertex operator corresponding to the emission of a physical conformal field from the world sheet, the BRST equivalent of the old operator requirement is that,

$$[\, Q_{BRST} \, , V_{phys.} \,] = 0 \quad . \qquad (1.2.22)$$

As an example the three point calculation of three massless vectors, (in the left moving sector) will be performed. The appropriate vertex operator for (left moving) vector emission is,

$$V_B(k,\zeta,z) = c(z)\, \zeta.\partial X(z)\, e^{ik.X(z)} \qquad (1.2.23)$$

which then gives the correlation function,

$$< V_{B_1}(k_1,\zeta_1,z_1)\, V_{B_2}(k_2,\zeta_2,z_2)\, V_{B_3}(k_3,\zeta_3,z_3) > \qquad (1.2.24)$$

which can be evaluated using the appropriate two point functions given by,

$$<\partial X^\mu(z_i)\partial X^\nu(z_j)> = \frac{-\eta^{\mu\nu}}{(z_i-z_j)^2}$$

$$<\partial X^\mu(z_i)\, e^{ik.X(z_j)}> = \frac{-ik^\mu}{(z_i-z_j)}\, e^{ik.X(z_j)}$$

$$< e^{ik.X(z_i)} e^{ik.X(z_j)} > = (z_i - z_j)^{k_i.k_j} \qquad (1.2.25)$$

for the bosonic fields, and, (with reference to the bosonisation described below),

$$< c(z_1)\, c(z_2) > \sim (z_1 - z_2) \qquad (1.2.26)$$

for the fermionic ghosts, to yield the following result, (ignoring all external constants);

$$A_3 = \{ \zeta_1.k_2\zeta_2.\zeta_3 + \zeta_2.k_3\zeta_3.\zeta_1 + \zeta_3.k_1\zeta_1.\zeta_2 + 2\alpha'\zeta_1.k_2\zeta_2.k_3\zeta_3.k_1 \} \qquad (1.2.27)$$

up to an external constant, which is irrelevant in any of the considerations of this thesis. Furthermore the hitherto suppressed constant α' has now been explicitly shown by inserting a factor $2\alpha'$ in the last term of (1.2.27), in accord with the fact that $\alpha' = 1/2$ is the conventional *open* string value. Let us envisage for the moment keeping such explicit α' when we form the *closed* string amplitudes from the tensor product of two *open* string amplitudes as in reference [33]; at this time we must make the replacement $k^{\text{open string}} = 1/2\ k^{\text{closed string}}$ which is equivalent to multiplying each factor of α' by 1/4 in the final closed string amplitude, interpreting the momenta as closed string momenta, and α' as the *closed string* α' which takes the value α'=2.

The same result can be achieved without showing α' explicitly by the following procedure:- i) calculate the two open string amplitudes without showing the *open string* α' explicitly, (that is with α'=1/2); ii) form the closed string amplitude by taking the tensor product of the two open string amplitudes; iii) reinterpret the open string momenta in the resulting amplitude as closed string momenta *without rescaling as above*; iv) reinstate α' into the amplitude by inserting $\alpha'/2$ for each factor of k^2 equivalent to taking α'=2 for closed strings. Note that the powers of 1/4 that result from the rescaling the momenta as above are completely taken care of by the re-insertion of the α' parameter with the proper closed string value.

The procedure of the last paragraph will be followed in this thesis so that any re-insertion of the parameter α' will be done by inserting a factor $\alpha'/2$ for every factor of k^2 as on pages 29-31 below. This procedure is exactly as described in reference [33] and is completely consistent with the procedure carried out in the bosonic matching calculations of Gross and Sloan and Cai and Nunez[58].

This result has all the required symmetries. For example it is crossing symmetric, (i.e. it is symmetric with respect to 1↔2, 2↔3, etc..) This result can be "crossed" (i.e. the tensor product between this amplitude and some other) with itself to find the full bosonic string result, or it can be "crossed" with a corresponding superstring result to obtain the amplitude for a heterotic string process[33]. This second course is the one that will be followed below.

Section 1.3 The interacting closed superstring as conformal field theory.

In this section the conformal field theory techniques applied to the bosonic string will be applied to the spinning and, after applying the GSO projections, the superstring. The ghost algebra will be generalised and the calculation of amplitudes demonstrated. The section will end with the explicit calculation of a (right-moving) superstring amplitude. The next section will combine the techniques of this and the previous section to give the full four point heterotic superstring amplitude for the scattering of two massless fermionic states with two massless bosonic states.

The (right moving) spinning string action is given by,

$$S = \int d^2\xi \sqrt{g}\{ \frac{1}{2} g^{ab}\partial_a X^\mu \partial_b X_\mu + \frac{i}{2}\bar{\psi}^\mu \gamma^a \nabla_a \psi_\mu + \frac{i}{2}(\chi_a \gamma^b \gamma^a \psi^\mu).(\partial_b X_\mu - \frac{i}{4}\chi_b \psi_\mu) \} \tag{1.3.1}$$

which is invariant under the supersymmetry transformations,

$$\begin{aligned} \delta g_{ab} &= 2i\varepsilon\gamma_{(a}\chi_{b)} \quad , \\ \delta\chi_a &= 2\nabla_a\varepsilon \quad , \\ \delta X^\mu &= i\varepsilon\psi^\mu \quad , \\ \delta\psi^\mu &= \gamma^a(\partial_a X^\mu - \frac{i}{2}\chi_a\psi^\mu)\,\varepsilon \end{aligned} \tag{1.3.2}$$

and is also reparametrisation invariant. This action is superconformally invariant and so the superconformal symmetry must be gauged away before the functional integral can be performed. The appropriate conformal gauge fixing is given by the relations,

$$g_{ab} = \rho\, \delta_{ab} \quad , \quad \chi_a = \gamma_a \zeta$$

where the remaining gauge fixed theory is seen to be of the form,

$$S_{g.fixed.} = \int d^2z \{ \frac{1}{2}\partial_a X^\mu \partial^a X_\mu - \frac{i}{2}\bar{\psi}^\mu \gamma.\partial\psi_\mu \} \tag{1.3.3}$$

which is free in the remaining fields! The other fields of the theory can be seen to decouple from the action. The remaining free fields can be combined into one free superfield,

$$X^{\mu}(z,\theta) = X^{\mu}(z) + \theta.\psi^{\mu}(z) \qquad (1.3.4)$$

with the action,

$$S_{g.fixed} = \int d^2z\, d^2\theta\, \frac{1}{2} \bar{D}X^{\mu}DX_{\mu} \qquad (1.3.5)$$

where the D and $\bar{D}$ are the standard supercovariant derivatives.

Again, with reference to Appendix One, it can be seen that the methods of conformal field theory can be applied. The superconformal transformations are generated by the super-stress energy tensor,

$$T(z,\theta) = -\frac{1}{2} DX^{\mu}D^2X_{\mu} \qquad (1.3.6)$$

whose Laurent coefficents satisfy the super-Virasoro algebra,

$$\begin{aligned} \left[L_m,L_n\right] &= (m-n)\, L_{m+n} + \frac{1}{8}\hat{c}\,(m^3 - m)\delta_{m+n,0} \\ \left\{G_m,G_n\right\} &= 2L_{m+n} + \frac{1}{2}\hat{c}\,(m^2 - \frac{1}{4})\,\delta_{m+n,0} \\ \left[L_m,G_n\right] &= (\frac{m}{2} - n)\, G_{m+n} \qquad (1.3.7) \end{aligned}$$

because the super-stress-energy tensor satisfies the superconformal algebra, in the form of the operator product,

$$T(z_1,\theta_1)T(z_2,\theta_2) \approx \frac{\hat{c}}{4\, z_{12}^3} + \frac{3\,\theta_{12}}{2\, z_{12}^2} T(z_2,\theta_2) + \frac{D_2T(z_2,\theta_2)}{2\, z_{12}} + \frac{\theta_{12}}{z_{12}} \partial_2 T(z_2,\theta_2) + \ldots$$

$$(1.3.8)$$

where the $\hat{c}$ is defined to be 3/2 times c, and z_{12} , θ_{12} and the derivatives are defined in Appendix One.

The two point correlation function for the superfield $X^m(z,\theta)$ is given by the standard path integral procedure[31] and gives,

$$< X^{\mu}(z_1,\theta_1)\, X^{\nu}(z_2,\theta_2) > = -\eta^{\mu\nu}\log z_{12} \qquad (1.3.9)$$

(where z_{12} is defined in Appendix One.), which contains the seperate field operator products which will be needed in the evaluation of the super-amplitudes.

The correlations of the spin fields of the theory are also very important, as these fields occur in the definition of the fermionic vertices, due to the necessity for these vertices to transform under the symmetries of the string in a required fashion. It can be seen[30] that these correlations are completely specified by their transformation properties under the SO(9,1) current algebra generated by the global SO(9,1) symmetry of the string. (See Appendix One for a brief overview of this theory in general.) It is seen that the ψ^{μ} fields of the string transform as conformal fields in a conformal theory given that the stress energy tensor,

$$j^{\mu\nu}(z) = \psi^{\mu}\psi^{\nu}(z) \qquad (1.3.10)$$

defined in terms of the Noether currents of the symmetry, defines a conformal theory, the normal ordering of the expression being implicit. The spin field defined by the superconformal algebra must have non-zero operator product with this stress energy tensor, as can be seen by general conformal field theory techniques, (Appendix One), and so must also be a conformal field in this theory, and by inspection of the various operator products, it can be clearly seen that the spin field S must transform as a spinor under the Noether current. By the use of a trick typical of conformal field theories, it is possible to find the correlation functions for the various fields simply by their SO(9,1) properties. This is done fully in reference [30]. Summarising, the correlation functions for the ψ^{μ} and S fields are,

$$\psi^{\mu}(z)\, S_{\alpha}(w) = \frac{1}{(z-w)^{\frac{1}{2}}}\gamma^{\mu}_{\alpha\beta}\, S^{\beta}(w) + \ldots$$

$$S^{\alpha}(z)\, S_{\beta}(w) = \frac{1}{(z-w)^{\frac{5}{4}}}\delta^{\alpha}_{\beta} + \frac{1}{(z-w)^{\frac{1}{4}}}(\frac{1}{2}\gamma^{\mu}\gamma^{\nu})^{\alpha}_{\beta}\,\psi^{\mu}\psi^{\nu} + \ldots$$

$$S_{\alpha}(z)\, S_{\beta}(w) = \frac{1}{(z-w)^{\frac{3}{4}}}\gamma^{\mu}_{\alpha\beta}\psi_{\mu} + \ldots$$

which are used in the evaluation of string amplitudes. The correlation functions involving the symmetry current $j^{\mu\nu}(z)$ are given simply by the SO(9,1) transformation properties of the particular fields in question.

As in the bosonic case it is necessary to consider the ghost arising from the superconformal gauge fixing. As in the previous case these ghosts will give a ghost vacuum that is not self adjoint and so will give a means to eliminate the remaining SL(2) symmetry of the world sheet.

The arguments proceed in an exactly analogous fashion to the bosonic case, except for the fact that there is now a ghost action in terms of a ghost superfield, and that the BRST algebra must be extended to include the full super-Virasoro algebra. The ghost algebra of this system is most conveniently handled in the case where all ghost fields are bosonised. In this case it is convenient to write,

$$j(z) = \varepsilon\, \partial_z \phi \qquad (1.3.11a)$$

with the operator product,

$$\phi(z)\, \phi(w) = \varepsilon \ln (z-w) \qquad (1.3.11b)$$

where j(z) is the ghost current which generates the BRST transformation in the ghost sector of the theory, and ε is either +1 or -1 depending on whether fermionic or bosonic ghosts are being bosonised. In this case, the current generates transformations of the fields $e^{-q\phi}$, which can be represented in terms of the operator products,

$$T(z)\, e^{q\phi} = \left[\frac{\frac{1}{2}\varepsilon q(q+Q)}{(z-w)^2} + \frac{1}{(z-w)}\partial_w\right] e^{q\phi} \quad ,$$

$$j(z)\, e^{q\phi(w)} = \frac{q}{(z-w)}\, e^{q\phi(w)} \quad ,$$

and this can be seen to be described by the bosonic action,

$$S_{bos} = \int d^2z\, \{-2\varepsilon\, \partial_z\phi\, \partial_{\bar{z}}\, \phi - \frac{1}{2} Q\sqrt{g}\, R\phi\} \qquad (1.3.12)$$

which has equation of motion,

$$\partial_z \partial_{\bar{z}}\, \phi = \frac{1}{8}\varepsilon Q\sqrt{g}\, R \qquad (1.3.13)$$

The current anomaly which generates the ghost charge of the vacuum can be seen to be due to this term, when the stress-energy tensor is defined by the action in ϕ. With a specific choice of charges, the ghost system can be bosonised in this fashion, and for the superstring it is found that this is the easiest way to proceed.

The ghosts are bosonised as follows,

$$c(z) = e^{\sigma} \qquad b(z) = e^{-\sigma}$$

$$\gamma(z) = e^{\phi}\eta \qquad \beta(z) = e^{-\phi}\partial\xi \qquad (1.3.14)$$

where the extra η and ξ are fermionic fields introduced to retrieve the bosonic nature of the bosonic supersymmetric counterparts to the usual reparametrisation Fadeev-Popov ghosts. This is the case since the fields $e^{\pm\phi}$ are always fermionic. In this relation appears one of the most important points of the entire theory. It should be noted that the ξ_0 mode is projected out by a derivative. This is essential for the construction of irreducable representations of the conformal algebra, since the ξ_0 mode makes any representation reducible. That is, every state in the ghost system, $|\phi\rangle$ say, would have a degenerate partner in the representation given by $\xi_0|\phi\rangle$, which transforms in exactly the same way under the current j(z).

The conformal field theory approach to the string requires that "vertex" operators be constructed such that they correspond to the conformal fields of the theory, and that therefore they transform in all the correct ways, have the correct ghost numbers, etc.. Clearly $DX^{\mu}(z,\theta)$ is a conformal field, as is $e^{ik.X}$. So the obvious choice for the massless bosonic vertex is simply,

$$V(k,\zeta,z) = \zeta.DXe^{ik.X}(z) \qquad (1.3.15)$$

which can be rewritten in the bosonised ghost algebra by going into the enlarged ξ-algebra (by including the ξ_0 mode) and writing the bosonic vertex as,

$$V_B(k,\zeta,z) = [\, Q_{BRST} , \xi e^{-\phi}\, \zeta.\psi e^{ik.X} \,] \qquad (1.3.16)$$

This will be useful later when the background ghost charge of the vacuum is considered, and amplitudes are constructed.

The first choice for a covariant fermion vertex is determined by the consideration that the background charge of the vacuum should change by $^1/_2$ thus interpolating between Neveu-Schwarz and Ramond ground states. A first guess would be,

$$V_{-1/2}(k,u,z) = u^{\alpha}(k)\, e^{-\phi/2} S_{\alpha}(z)\, e^{ik.X(z)} \qquad (1.3.17)$$

where the subscript $^{-1}/_2$ denotes the ghost charge associated with the vertex and where the polarisation $u^{\alpha}(k)$ satisfies,

$$ku^{a}(k) = 0 \ , \ \text{where } k^2 = 0 \, .$$

By taking the operator product of this vertex with the various currents which contribute to the full superconformal Q_{BRST}, it is possible to show that the operator products are nonsingular, and hence the vertex commutes with Q_{BRST}. This is not quite enough to describe fermion scattering of the string theory, since the background charge of 2 can only be cancelled by four of these operators. The solution is to invoke the degeneracy involved with the ξ_0 mode.

In general all operators of the form $[Q_{BRST},O]$ are spurious, except when O is of the form ξV_{phys}. These operators are not spurious because of the fact that the ξ_0 mode is not part of the algebra of bosonic ghosts, γ, β. It is therefore possible to define a vertex with ghost charge $^1/_2$ by,

$$V_{1/2}(k,u,z) = \left[Q_{BRST} \, , 2\xi V_{-1/2}(k,u,z) \right] \qquad (1.3.18)$$

which has the explicit form,

$$V_{1/2}(k,u,z) = u^{\alpha}(k) \left\{ e^{\phi/2}\left[\partial X^{\mu} + \frac{1}{4}(k.\psi)\psi^{\mu}\right](\gamma_{\mu})_{\alpha\beta} S^{\beta} + \frac{1}{2} e^{3\phi/2} \eta b S_{\alpha} \right\} e^{ik.X(z)} \qquad (1.3.19)$$

when the operator product of $V_{-1/2}$ is taken with the full Q_{BRST} operator. In reference [27] this analysis is done fully and so the results will only be sketched here. This completes the list of all the vertex operators that will be needed to construct amplitudes in the final section.

Section 1.4 The construction of amplitudes.

In this section it is decribed how heterotic string amplitudes are constructed, and an example will be given. This particular example will be useful in a later chapter, when the amplitude matching procedure is considered.

A general string amplitude is given by a correlation function of the general form,

$$< V_1 V_2 .... V_n >$$

where each of the vertex operators corresponds to a conformal field in an underlying conformal field theory. The techniques of conformal field theory have already been developed above in some detail, and so it is merely a case of applying these general techniques to the case of the particular vertex operators of the theory being considered. At first sight this would seem to be merely a matter of replacing the appropriate combinations of fields with their operator products; performing some rearrangement of momenta to bring it into a simpler form; and finally integrating over the complex variables on the world sheet. However there is a level of subtlety in this analysis which has to be observed. The ghost charge on the vacuum has to be soaked up in a consistent manner, so as to remove three of the integrations mentioned above. Also the fermionic correlations appear to be quite complex, and difficult to evaluate. This will in fact be seen to pose no real problem in the evaluation of the amplitude considered, (or indeed any amplitude for that matter). The amplitude considered will be that for the scattering of two fermionic sates with two massless bosonic states. The procedure will be to calculate the appropriate bosonic left moving amplitude and the corresponding right moving superstring amplitude. These will be "crossed" with each other to find the complete heterotic string amplitude.

i) The bosonic amplitude.

The full bosonic amplitude is given by the correlation function, defined by the product of vertex operators,

$$< \zeta_4.\partial X(z_4)e^{ik_4.X}\zeta_3.\partial X(z_3)e^{ik_3.X}\zeta_2.\partial X(z_2)e^{ik_2.X}\zeta_1.\partial X(z_1)e^{ik_1.X}> \tag{1.4.1}$$

which then can be evaluated using the standard techniques described above. The amplitude contribution is clearly given by simply multiplying out the various terms in each of the possible Wick contractions, with regard to the operator product expansions given above. This is tedious. It is easier to perform the Koba-Nielsen integrations implicitly as the calculation is performed, by making the choice,

$$z_1 = 0 \ , \ z_2 = 1 \ , \ z_4 \to \infty \tag{1.4.2}$$

which allows the bosonic half of the amplitude to be written,

$$A = (\bar{z}_3 - 1)^{k_3.k_2} \bar{z}_3^{k_3.k_1} \left\{ \frac{1}{\bar{z}_3} A_1 + \frac{1}{(\bar{z}_3 - 1)} A_2 + \frac{1}{\bar{z}_3^2} A_3 + \frac{1}{(\bar{z}_3 - 1)^2} A_4 + A_5 \right\} \tag{1.4.3}$$

where the individual A_i terms are given in the form,

$$A_1 = [\, \zeta_4.\zeta_3 \zeta_2.k_4 \zeta_1.k_3 + \zeta_4.\zeta_2 \zeta_3.k_2 \zeta_1.k_3 - \zeta_4.\zeta_2 \zeta_3.k_1 \zeta_1.k_2$$

$$- \zeta_4.\zeta_1 \zeta_3.k_1 \zeta_2.k_4 - \zeta_3.\zeta_2 \zeta_1.k_3 \zeta_4.k_2 + \zeta_3.\zeta_1 \zeta_2.k_1 \zeta_4.k_3$$

$$- \zeta_3.\zeta_1 \zeta_2.k_3 \zeta_4.k_1 + \zeta_2.\zeta_1 \zeta_3.k_1 \zeta_4.k_2 + \zeta_3.k_2 \zeta_2.k_1 \zeta_1.k_3 \zeta_4.k_2$$

$$+ \zeta_3.k_1 \zeta_2.k_4 \zeta_1.k_2 \zeta_4.k_2 + \zeta_3.k_1 \zeta_2.k_4 \zeta_1.k_3 \zeta_4.k_3 \,]$$

$$A_3 = [\, -\zeta_4.\zeta_2 \zeta_3.k_1 \zeta_1.k_3 - \zeta_3.\zeta_1 \zeta_2.k_4 \zeta_4.k_2$$

$$+ \zeta_3.k_1 \zeta_2.k_4 \zeta_1.k_3 \zeta_4.k_2 + \zeta_4.\zeta_2 \zeta_3.\zeta_1 \,]$$

$$A_5 = [\, -\zeta_4.\zeta_3 \zeta_2.k_1 \zeta_1.k_2 - \zeta_2.\zeta_1 \zeta_3.k_4 \zeta_4.k_3$$

$$+ \zeta_3.k_4 \zeta_2.k_1 \zeta_1.k_2 \zeta_4.k_3 + \zeta_4.\zeta_3 \zeta_2.\zeta_1 \,]$$

and where A_2 and A_4 are related to A_1 and A_3 by the interchange of 1 and 2 indices respectively. The identity,

$$(k_1 + k_2 + k_3 + k_4)_\mu = 0 \tag{1.4.4}$$

has been used to eliminate the leading order terms in certain of the expansions, which would otherwise appear infinite when the Koba-Nielsen gauge fixing is performed. It only remains to calculate the fermionic half of the amplitude, and to take the tensor product with the bosonic term (1.4.3), and then perform the remaining complex integrals.

ii) The fermionic amplitude.

The calculation follows the example calculation given in reference [27]. (There are some errors in this calculation which have been corrected in what follows.) The appropriate correlation function for f-f-b-b scattering is given by,

$$<e^{-2\phi+3\sigma}\zeta_4.(\partial X(z_4) + ik_4.\psi\psi)e^{ik_4.X}\zeta_3.(\partial X(z_3) + ik_3.\psi\psi)e^{ik_3.X} \times$$

$$\times\ \bar{u}_2^{\alpha}S_{\alpha}e^{-\phi/2}e^{ik_2.X}(z_2)\ u_1^{\beta}\gamma^{\mu}_{\beta\gamma}S^{\gamma}[\partial X^{\mu}(z_1) + \frac{1}{4}ik_1.\psi\psi^{\mu}]e^{\phi/2}e^{ik_1.X}>$$

(1.4.5)

where the vertex operators are the $V_{1/2}$ and $V_{-1/2}$ fermionic vertex operators (1.3.17) and (1.3.18) and the bosonic vertex operator V_B (1.3.15) as given above. The fermionic vertices chosen have the correct neutral charge combination. The first ghost factor is introduced to cancel the vacuum charge. A ξ-algebra manipulation based on the definition of the $V_{1/2}$ vertex in terms of the Q_{BRST} operator can be used in addition to the conformal invariance of the theory to rewrite one of the bosonic vertices as the new vertex term,

$$\zeta_4.\psi\, e^{-\phi+\sigma}e^{ik_4.X}(z_4) \qquad (1.4.6)$$

which allows the direct evaluation of the amplitude by direct substitution of the appropriate operator product expansions. The individual operator products that are required are,

$$<e^{\sigma}(z_4)e^{\sigma}(z_2)e^{\sigma}(z_1)> = (z_4 - z_3)\,(z_2 - z_1)\,(z_4 - z_1)$$

(1.4.7)

$$<e^{-\phi}(z_4)e^{-\phi}(z_2)e^{-\phi}(z_1)> = (z_4 - z_2)^{-1/2}(z_4 - z_1)^{-1/2}(z_2 - z_1)^{-1/2}$$

(1.4.8)

$$<\psi^{\mu}(z_4)S_{\alpha}(z_2)S_{\beta}(z_1)> = (z_4 - z_2)^{-1/2}(z_4 - z_1)^{-1/2}(z_2 - z_1)^{-3/4}\gamma^{\mu}_{\alpha\beta}$$

(1.4.9)

$$<j^{\nu\lambda}(z_3)\psi^{\mu}(z_4)S_{\alpha}(z_2)S_{\beta}(z_1)> = \Big\{ \frac{1}{(z_3 - z_4)} g^{\mu[\nu}<\psi^{\lambda]}(z_4)S_{\alpha}(z_2)S_{\beta}(z_1)>$$

$$+ \frac{\frac{1}{2}}{(z_3 - z_2)}(\gamma^{\mu\nu})_{\alpha}^{\gamma}<\psi^{\mu}(z_4)S_{\gamma}(z_2)S_{\beta}(z_1)>$$

$$- \frac{\frac{1}{2}}{(z_3 - z_1)}(\gamma^{\nu\lambda})_{\beta}^{\gamma}<\psi^{\mu}(z_4)S_{\alpha}(z_2)S_{\gamma}(z_1)>\Big\}$$

(1.4.10)

and the remaining operator product expansions are just the bosonic type mentioned above. This implies that the amplitude may be written in the form,

$$A_F = (z_3 - 1)^{k_3.k_2} z_3^{k_3.k_1} . \Big\{ \frac{1}{z_3} \Big[\frac{1}{2} \bar{u}_2 \not\zeta_4 \not k_3 \not\zeta_3 u_1 + \frac{1}{2} \bar{u}_2 \not\zeta_4 \not k_1 \not\zeta_3 u_1 \Big]$$
$$+ \frac{1}{(z_3 - 1)} \Big[- \frac{1}{2} \bar{u}_2 \not\zeta_3 \not k_3 \not\zeta_4 u_1 + \frac{1}{2} \bar{u}_2 \not\zeta_3 \not k_2 \not\zeta_4 u_1 \Big] \Big\} \tag{1.4.11}$$

when the SL(2) gauge fixing is performed by the choice of z_1, z_2 and z_4 above. This must now be "crossed" with the bosonic term given above in equation (1.4.3) and the remaining complex world sheet variables integrated over. The appropriate integrals are of the form,

$$\int d^2z \, |z|^{-A} |1-z|^{-B} = -\pi \frac{\Gamma(1-\frac{A}{2})\,\Gamma(1-\frac{B}{2})\,\Gamma(\frac{A+B}{2}-1)}{\Gamma(\frac{A}{2})\,\Gamma(\frac{B}{2})\,\Gamma(2-\frac{A+B}{2})}$$

$$\int d^2z \, z \, |z|^{-A} |1-z|^{-B} = \int d^2z \, \bar{z} \, |z|^{-A} |1-z|^{-B}$$
$$= \pi \frac{\Gamma(1-\frac{A}{2})\,\Gamma(-\frac{B}{2})\,\Gamma(\frac{A+B}{2})}{\Gamma(\frac{A}{2})\,\Gamma(\frac{B}{2})\,\Gamma(1-\frac{A+B}{2})}$$

$$\int d^2z \, z^2 |z|^{-A} |1-z|^{-B} = \int d^2z \, \bar{z}^2 |z|^{-A} |1-z|^{-B}$$
$$= \pi \frac{\Gamma(3-\frac{A}{2})\,\Gamma(1-\frac{B}{2})\,\Gamma(\frac{A+B}{2}-1)}{\Gamma(\frac{A}{2})\,\Gamma(\frac{B}{2})\,\Gamma(4-\frac{A+B}{2})} \tag{1.4.12}$$

which can be proved by straightforward techniques, and which can be substituted in the amplitude given above to yield the final answer. The full amplitude is therefore,

$$A_{tot.} = \Gamma^{(3)} \Big\{ \bar{u}_2^{\mu} \gamma^{\alpha} \gamma^{\beta} \gamma^{\gamma} u_1^{\nu} \zeta_{3\alpha}^{\ \rho} \zeta_{4\gamma}^{\ \sigma} (k_{3\beta} + k_{1\beta}) .$$
$$. \Big[\frac{2u}{s(s-2)} A^1_{\mu\nu\rho\sigma} + \frac{2}{(u-2)} A^2_{\mu\nu\rho\sigma} + \frac{2u}{t(t-2)} A^3_{\mu\nu\rho\sigma} - \frac{2u}{ts} A^4_{\mu\nu\rho\sigma} + \frac{2}{s} A^5_{\mu\nu\rho\sigma} \Big]$$
$$+ \bar{u}_2^{\mu} \gamma^{\alpha} \gamma^{\beta} \gamma^{\gamma} u_1^{\nu} \zeta_{3\alpha}^{\ \rho} \zeta_{4\gamma}^{\ \sigma} (k_{3\beta} + k_{1\beta}) .$$

$$\cdot\Big[-\frac{2t}{s(s-2)}A^{1}_{\mu\nu\rho\sigma}-\frac{2t}{u(u-2)}\,A^{2}_{\mu\nu\rho\sigma}-\frac{2}{(t-2)}A^{3}_{\mu\nu\rho\sigma}+\frac{2}{s}A^{4}_{\mu\nu\rho\sigma}-\frac{2t}{us}A^{5}_{\mu\nu\rho\sigma}\Big]\Big\} \tag{1.4.13}$$

where the coefficients $A^1, A^2,\ldots,A^5$ are given by,

$$A^{1}_{\mu\nu\rho\sigma} = \{\, \eta_{\mu\nu}\eta_{\rho\sigma} + k_{1\mu}k_{2\nu}k_{3\rho}k_{4\sigma} - k_{1\mu}k_{4\nu}\eta_{\rho\sigma} - k_{2\rho}k_{3\sigma}\eta_{\mu\nu} \,\}$$

$$A^{2}_{\mu\nu\rho\sigma} = \{\, \eta_{\nu\rho}\eta_{\mu\sigma} + k_{1\rho}k_{2\sigma}k_{3\mu}k_{4\nu} - k_{1\rho}k_{4\nu}\eta_{\mu\sigma} - k_{2\sigma}k_{3\mu}\eta_{\nu\rho} \,\}$$

$$A^{3}_{\mu\nu\rho\sigma} = \{\, \eta_{\mu\rho}\eta_{\nu\sigma} + k_{1\sigma}k_{2\rho}k_{3\nu}k_{4\mu} - k_{2\rho}k_{4\mu}\eta_{\nu\sigma} - k_{1\sigma}k_{3\nu}\eta_{\mu\rho} \,\}$$

and where A^4 and A^5 are large complicated functions of momenta, which will not actually be used in calculation below. However, for the sake of reference, these are,

$$\begin{aligned} A^{4}_{\mu\nu\rho\sigma} = \{\, & k_{3\nu}k_{4\mu}\eta_{\rho\sigma} + k_{2\sigma}k_{3\nu}\eta_{\mu\rho} - k_{1\sigma}k_{2\nu}\eta_{\mu\rho} - k_{1\sigma}k_{4\mu}\eta_{\nu\rho} \\ & + k_{1\mu}k_{3\rho}\eta_{\nu\sigma} - k_{1\rho}k_{3\mu}\eta_{\nu\sigma} + k_{1\sigma}k_{2\rho}\eta_{\mu\nu} - k_{2\rho}k_{3\nu}\eta_{\mu\sigma} \\ & + k_{1\mu}k_{2\rho}k_{2\sigma}k_{3\nu} + k_{1\sigma}k_{2\rho}k_{2\nu}k_{4\mu} + k_{1\sigma}k_{3\rho}k_{3\nu}k_{4\mu} \,\} \end{aligned}$$

and where A^5 is minus A^4 with $(1\leftrightarrow 2)$ and $(3\leftrightarrow 4)$ indices reversed. The Mandlestam variables are defined to be $s = 2k_1.k_2$, $t = 2k_1.k_3$ and $u = 2k_1.k_4$ since the external string modes are taken to be massless, i.e. $k_i^2 = 0$. It should be noted that this definition is the same as that of Cai and Nunez, but differs by a sign to that of Gross and Sloan[58], and Schwarz[12]. The $\Gamma^{(3)}$ factor is given as a product of gamma functions, and contains the poles in the amplitude due to the massive modes of the theory,

$$\Gamma^{(3)} = \frac{\Gamma(\frac{s}{2})\,\Gamma(\frac{t}{2})\,\Gamma(\frac{u}{2})}{\Gamma(-\frac{s}{2})\,\Gamma(-\frac{t}{2})\,\Gamma(-\frac{u}{2})}$$

This is the form of the amplitude which will be used in the amplitude matching calculations in the following chapters. This amplitude can be expanded trivially in powers of the string parameter α'. This is done by re-inserting the α' parameter in the amplitude above, by noting its dimensionality, and the value chosen consistent with the *closed* string value of $\alpha'=2$, and then simply performing the appropriate

Taylor expansions. Explicitly this means that every occurence of s,t or u must be multiplied by the dimensionful term $\alpha'/2$, as in reference [33]. This yields a string amplitude completely consistent with the amplitudes listed in both of the papers in reference [58]. For example, the term,

$$\frac{1}{t-2} \rightarrow \frac{1}{2}\left\{ 1 + \frac{t}{2} + O(t^2) \right\} \qquad (1.4.14)$$

The $\Gamma^{(3)}$ term can be seen to expand in the form,

$$\Gamma^{(3)} = -\left\{ 1 + \frac{1}{4}\zeta(3)\, stu + \dots \right\}$$

where $\zeta(3)$ denotes the Riemann zeta function, and so the appropriate $O(\alpha'^0)$ amplitude is,

$$\begin{aligned} Amp_0 = &\ \bar{u}_2^{\nu}\gamma^{\gamma}(k_4 - k_3)\gamma^{\alpha}u_1^{\mu}\zeta_{3\gamma}{}^{\sigma}\zeta_{4\alpha}{}^{\rho}\left\{ \frac{t}{s}\eta_{\mu\nu}\eta_{\rho\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} + \frac{t}{u}\eta_{\mu\rho}\eta_{\nu\sigma} \right\} \\ & - \bar{u}_2^{\mu}\gamma^{\alpha}(k_4 - k_3)\gamma^{\gamma}u_1^{\nu}\zeta_{3\gamma}{}^{\sigma}\zeta_{4\alpha}{}^{\rho}\left\{ \frac{u}{s}\eta_{\mu\nu}\eta_{\rho\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} + \frac{u}{t}\eta_{\mu\rho}\eta_{\nu\sigma} \right\} \end{aligned} \qquad (1.4.15)$$

up to an irrelevant external constant, and the $O(\alpha')$ amplitude is,

$$\begin{aligned} Amp_1 = -\frac{\alpha'}{4}&\left\{ u\,\bar{u}_2^{\mu}\gamma^{\alpha}(k_4 - k_3)\gamma^{\gamma}u_1^{\nu} - t\,\bar{u}_2^{\nu}\gamma^{\gamma}(k_4 - k_3)\gamma^{\alpha}u_1^{\mu} \right\} \times \\ & \times \zeta_{3\gamma}{}^{\sigma}\zeta_{4\alpha}{}^{\rho}(\eta_{\mu\nu}\eta_{\rho\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} + \eta_{\mu\rho}\eta_{\nu\sigma}) \end{aligned} \qquad (1.4.16)$$

where α' has been replaced at the last stage. As a trivial example of the application of the conformal field theory techniques, it can be seen that by omitting the bosonic vertex at the complex z_3 point, then the amplitude immediately trivialises to the form,

$$A_3^f = \bar{u}_2\gamma^{\mu}u_1\zeta_{3\mu} \qquad (1.4.17)$$

which can be "crossed" with (1.2.27), using the techniques of reference [33], (with particular regard to equation (1.3) of that reference), to give the three point f-f-b scattering amplitude in the form,

$$A_3 = \bar{u}_2^{\mu} \gamma^{\sigma} u_1^{\nu} \zeta_{3\sigma}^{\ \ \rho} \left\{ k_{1\rho} \eta_{\mu\nu} + k_{2\nu} \eta_{\rho\mu} + k_{3\mu} \eta_{\nu\rho} + \frac{\alpha'}{2} k_{1\rho} k_{2\nu} k_{3\mu} \right\} \tag{1.4.18}$$

which will be used in conjunction with the four point amplitudes in the amplitude matching procedures carried out in the following chapters. It should be noted that the α' used in expression (1.4.18) is the complete closed string value $\alpha'=2$, as opposed to the open string value $\alpha'=1/2$ used in the expression (1.2.27). The reason for this is explained in depth in reference [33] where it is shown generally how to construct <u>any</u> closed string amplitude from a tensor product of two <u>open</u> string amplitudes. In all of what follows the value of α' will always be 2, since only closed string expressions will be used henceforth. In all of the remaining chapters of this thesis the dimensionful parameter α' will be suppressed, but when it is necessary to reintroduce it, for example to facilitate an amplitude matching to either the amplitude (1.4.18) or the four point amplitudes (1.4.15) and (1.4.16), then it will be reintroduced by introducing the factor $\alpha'/2$. This is the convention used from now on and will be adhered to throughout.

Chapter Two: Quantisation of The Free Effective Field Theory: Evaluation of the Propagators

Introduction.

In this chapter the general principles of free Majorana-Weyl field theory will be developed, with special regard being paid to the extra constraints inherent in the quantisation of the effective field theory representing the heterotic string. Some care will be taken in the quantisation of the symmetric tensor field, and then the quantisation of both spin-1/2 and spin-3/2 fermion fields will be undertaken using the general Dirac approach to constrained quantisation. The final section will apply the same Dirac quantisation techniques to the quantisation of a higher derivative theory, and will be shown to be consistent with the path integral approach. The implication for the higher derivative propagator correcting terms will be discussed.

Section 2.1 The effective field theory for the heterotic string.

The free effective field theory for the string is determined by the spectrum of massless states determined by the equations of motion and symmetries of the string action. It can be shown that these states are (for the Spin32/Z_2 case):-

State	SO(8) Rep.	Gauge Rep.	Deg. of Freedom
$h_{\mu\nu}$	$\underline{35}_v$	$\underline{1}$	35
$a_{\mu\nu}$	$\underline{28}$	$\underline{1}$	28
ϕ	$\underline{1}$	$\underline{1}$	1
ψ_μ	$\underline{56}_s$	$\underline{1}$	56
λ	$\underline{8}_s$	$\underline{1}$	8
A_μ	$\underline{8}_v$	adj.	8x496
χ	$\underline{8}_s$	adj.	8x496

In all of what follows, only the 'gravitational' sector of the string will be considered. An action must now be constructed which is consistent with the global symmetries built into the string theory a priori. The procedure used is similar to that of Veltman[34] and will provide the unique free field theory effective action for the string, up to a possible overall constant. Essentially the procedure is as follows: a general action is written down with some arbitrary coeficients, then these coefficients are fixed by the stipulation that the action must be gauge invariant under transformations of the form,

$$\delta h_{\mu\nu} = \frac{1}{2} \left\{ \bar{\varepsilon}\gamma_\mu \psi_\nu + \bar{\varepsilon}\gamma_\nu \psi_\mu \right\}$$

$$\delta \psi_\mu = \gamma^{\alpha\beta} \partial_\alpha h_{\beta\mu} \varepsilon \qquad (2.1.1)$$

for example. It is extremely important to note that the restriction to the symmetric tensor $h_{\mu\nu}$ and the gravitino ψ_μ has been made, since these are the only fields that are considered in the amplitude calculations performed later in this thesis. It is possible to extend the discussion below to include all the fields from the massless spectrum of the string, but this only includes extra unnecessary detail and destroys what little clarity exists in the description of the quantisation of both the free and interacting theories, and the construction, eventually, of amplitudes. The action, in terms of these fields only, is then seen to be uniquely determined. Since the string contains several different states, both bosonic and fermionic, and since these are related by string supersymmetry, it is then possible to write down the unique effective field theory for the string, (again restricting to the same subset of fields). This action is,

$$\mathcal{L} = -\frac{1}{2} \left\{ -\frac{1}{4} h^{\alpha\mu} \partial^\sigma \partial_\sigma h_{\mu\alpha} + \frac{1}{2} h_{\mu\rho} \partial^\rho \partial_\sigma h^{\sigma\mu} \right.$$
$$\left. - \frac{1}{2} h_{\mu\rho} \partial^\rho \partial^\mu h^\sigma_\sigma + \frac{1}{4} h_\rho^{\ \rho} \partial_\sigma \partial^\sigma h_\mu^{\ \mu} + \bar{\psi}_\mu \gamma^{\mu\nu\rho} \overleftrightarrow{\partial}_\nu \psi_\rho \right\} \qquad (2.1.2)$$

where the consequences of the supersymmetry algebra and the necessity of its closure have been accounted for.

This free classical action can now be quantised. As mentioned in the introduction the fermionic part of the action will be quantised using the methods of constrained quantisation due to Dirac, and the actual quantisation and evaluation of propagators will be performed using standard, familiar, canonical techniques. In Appendix Two a description of the functional integral approach to the quantisation of the field theories with second class constraints will be outlined and the propagators evaluated below will be rederived using these methods to allow the calculation of the covariant form of the spin-3/2 propagator which is inaccessable to the methods used below, and to provide a useful check on the functional integral quantisation of theories with second class constraints[32].

The higher derivative extensions to the free action described schematically in the final section and in the following chapters will also be taken into account in

Appendix Two. The separate fields in the action (2.1.2) will be quantised in turn below. Only the symmetric tensor field will be quantised in the bosonic sector of the full action, since the remaining bosonic quantisations are either trivial, in the case of the ϕ field, or use exactly similar techniques to those used in the quantisation of the graviton. Both fermionic fields will be quantised in turn: the spin-1/2 field as a simple introduction to the application of the Dirac constrained quantisation to Grassman variables, which will be needed in the final section, and the spin-3/2 as a demonstration of the full complexity of this technique when gauge degrees of freedom are included in the action.

Section 2.2: The graviton and dilatino propagators.

In this section, each of the propagators for the graviton, spin-1/2 and spin-3/2 fields will be derived in turn using the usual canonical path to quantisation of defining the mapping from the classical theory to the quantum theory by[35],

$$i\hbar \left\{ f , g \right\}_D \rightarrow \hat{f}\,\hat{g} - (-1)^{n_f n_g}\,\hat{g}\,\hat{f} \qquad \text{(A3.2.11)}$$

and by finding a solution to the classical equations of motion which allows the mapping to the quantum algebra given by this equation. A well defined path then leads to the propagator for the free theory, which is what is required in the perturbative solution of scattering problems. However this standard definition of the quantisation condition must be modified when the theory is constrained. It is possible to solve the constraints and then to quantise in the remaining phase space which might be curved, and therefore requiring special geometrical techniques, but this is possible for only the simplest systems in general[36]. The alternative is to modify the definition of quantisation to include the constraints in the full flat phase space. This technique is the one originally developed by Dirac[32] and Bergmann[37], and is the approach followed here.

The first quantisation performed here will be that of the graviton. The quantisation procedure adopted will be a simple generalisation of the standard Fermi quantisation of the photon. The graviton action alone is,

$$\mathcal{L} = \frac{1}{2} \Big\{ -\frac{1}{4} h^{\alpha\mu}\, \partial^{\sigma}\partial_{\sigma}\, h_{\mu\alpha} + \frac{1}{2} h_{\mu\rho}\, \partial^{\rho}\partial_{\sigma}\, h^{\sigma\mu} - \frac{1}{2} h_{\mu\rho}\, \partial^{\rho}\partial^{\mu}\, h^{\sigma}_{\sigma} + \frac{1}{4} h^{\rho}_{\rho}\, \partial_{\sigma}\partial^{\sigma}\, h^{\mu}_{\mu} \Big\} \qquad (2.2.1)$$

and the appropriate covariant gauge fixing term is,

$$\mathcal{L}_{g.f.} = \frac{1}{2\xi}\{ h_{\mu\rho}\,\partial^{\rho}\partial^{\sigma} h^{\mu\sigma} - h_{\mu\rho}\,\partial^{\rho}\partial^{\mu} h^{\sigma}_{\sigma} + \frac{1}{4} h^{\rho}_{\rho}\,\partial_{\sigma}\partial^{\sigma} h^{\mu}_{\mu} \} \qquad (2.2.2)$$

yielding the full gauge fixed action in the form,

$$\mathcal{L}_{tot.} = -\frac{1}{2}\{ \frac{1}{4}\partial^{\sigma}h^{\alpha\mu}\partial_{\sigma}h_{\mu\alpha} - \frac{1}{8}\partial^{\mu}h^{\rho}_{\rho}\partial_{\mu}h^{\sigma}_{\sigma} \} \qquad (2.2.3)$$

where the gauge parameter has been chosen to be ξ=2. This choice will be used throughout. (It should be noted that the covariant gauge choice made here cannot be simply generalised to the case of the spin-3/2 vector spinor field in the case of the gravitino; it is possible to show that such a quantisation procedure is inconsistent with the Dirac quantisation procedure used in the quantisation of the spin-1/2 dilatino field.) Following the path of canonical quantisation, the canonically conjugate momentum is defined to be,

$$\pi^{\rho\sigma}_{h} = \frac{\delta\mathcal{L}}{\delta(\partial_0 h_{\rho\sigma})} = \frac{1}{4}\{ \dot{h}^{\rho\sigma} - \frac{1}{2}\eta^{\rho\sigma}\dot{h} \} \qquad (2.2.4)$$

(where $\dot{h}_{\mu\nu} = \partial_0 h_{\mu\nu} = -\partial^0 h_{\mu\nu}$). It can be seen that this can be written in the form,

$$\pi^{\rho\sigma}_{h} = \frac{1}{8}\{\eta^{\rho\alpha}\eta^{\sigma\beta} + \eta^{\rho\beta}\eta^{\sigma\alpha} - \eta^{\rho\sigma}\eta^{\alpha\beta}\}\dot{h}_{\alpha\beta} \qquad (2.2.5)$$

which will be the most useful form for the momentum in what follows. The canonical quantisation condition is defined to be,

$$[h_{\mu\nu}(\underline{x}), \pi^{\rho\sigma}(\underline{y})] = -\frac{i}{2}\{\eta^{\rho}_{\mu}\eta^{\sigma}_{\nu} - \eta^{\rho}_{\nu}\eta^{\sigma}_{\mu}\}\,\delta^{3}(\underline{x}-\underline{y}) \qquad (2.2.6)$$

at equal times. Equation (2.1.9) is used to get the quantisation in the somewhat more useable form,

$$G^{\rho\sigma,\alpha\beta}[h_{\mu\nu}, \dot{h}_{\alpha\beta}] = -i\,8\,1^{\rho\sigma}_{\mu\nu}\,\delta^{3}(\underline{x}-\underline{y}) \qquad (2.2.7)$$

where,

$$G^{\rho\sigma,\alpha\beta} = \{\eta^{\rho\alpha}\eta^{\sigma\beta} + \eta^{\rho\beta}\eta^{\sigma\alpha} - \eta^{\rho\sigma}\eta^{\alpha\beta}\}$$

$$1^{\rho\sigma}_{\mu\nu} = \frac{1}{2}\{\eta^{\rho}_{\mu}\eta^{\sigma}_{\nu} + \eta^{\rho}_{\nu}\eta^{\sigma}_{\mu}\} \qquad (2.2.8)$$

The standard quantisation can proceed when this operator is inverted. The inversion is carried out using the standard set of projection operators for rank two symmetric tensors. These are listed in Appendix Two. Noting that $\eta_{\mu\nu} = \theta_{\mu\nu} + \omega_{\mu\nu}$ (in the standard notation of Appendix Two) and the definitions for the projection operators it is possible to write (suppressing unnecessary indices),

$$G = 2\,P^{(1)} + 2\,P^{(2)} - (\theta\text{-}2)\,P^{(s)} + P^{(\omega)} - \sqrt{\theta}\,(P^{(s\omega)} + P^{(\omega s)})$$

which can be inverted using the standard multiplication algebra for these operators. (See Appendix Two for details of these calculations.) Defining an inverse,

$$P = a\,P^{(1)} + b\,P^{(2)} + c\,P^{(s)} + d\,P^{(\omega)} + e\,P^{(s\omega)} + f\,P^{(\omega s)}$$

it can be shown that $G^{\rho\sigma,\alpha\beta}.P_{\alpha\beta,\mu\nu} = 1^{\rho\sigma}{}_{\mu\nu}$, if the coefficients have the values, $a = b = {}^{1}/_{2}$ and,

$$c = \frac{-1}{2\,(\theta\text{-}1)} \qquad d = \frac{(\theta\text{-}2)}{2\,(\theta\text{-}1)}$$

$$e = \frac{-\sqrt{\theta}}{2\,(\theta\text{-}1)} \qquad f = \frac{-\sqrt{\theta}}{2\,(\theta\text{-}1)}$$

Consequently equations (2.2.6-7) are equivalent to,

$$[\,h_{\mu\nu}\,,\dot{h}_{\alpha\beta}\,] = -\,i\,8\,P_{\mu\nu,\alpha\beta}\,\delta^{9}(\underline{x} - \underline{y}) \qquad (2.2.9)$$

where,

$$P_{\mu\nu,\rho\sigma} = \frac{1}{2}\,1_{\mu\nu,\rho\sigma} - \frac{1}{2\,(\theta\text{-}1)}\,\eta_{\mu\nu}\,\eta_{\rho\sigma} \qquad (2.2.10)$$

It is now a simple but tedious path to the propagator. This begins with the mode expansion solution to the equation of motion of the theory. This is explicitly of the form,

$$h_{\mu\nu}(x) = \int d\tilde{k} \; \{ \; a_{(\lambda)}(k) \, \zeta^{(\lambda)}_{\mu\nu}(k) \, e^{ik.x} + \; a^{+}_{(\lambda)}(k) \, \zeta^{*(\lambda)}_{\mu\nu}(k) \, e^{-ik.x} \; \} \tag{2.2.11}$$

where the polarisation tensors have been introduced, and which are chosen to satisfy the normalisation conditions,

$$\sum_{(\lambda)} (\zeta^{(\lambda)}_{\mu\nu} \, \zeta^{(\lambda)}_{\alpha\beta}) = 0$$

$$\sum_{(\lambda)} (\zeta^{(\lambda)}_{\mu\nu} \, \zeta^{*(\lambda)}_{\alpha\beta}) = \{ \; \eta_{\mu\alpha} \, \eta_{\nu\beta} + \eta_{\mu\beta} \, \eta_{\nu\alpha} - \frac{1}{4} \eta_{\mu\nu} \, \eta_{\alpha\beta} \; \} \tag{2.2.12}$$

so that it can be seen that the mode coefficient algebra is of the form,

$$[\; a_{(\lambda)}(k) \, , \, a_{(\lambda')}(k') \;] = 0$$

$$[\; a_{(\lambda)}(k) \, , \, a^{+}_{(\lambda')}(k') \;] = - \, (2\pi)^3 \, k^0 \, \delta_{\lambda\lambda'} \, \delta^3(\underline{k} - \underline{k}') \tag{2.2.13}$$

The propagator now follows immediately from the definition,

$$G_h(x\text{-}x') = < 0 \mid T(\, h_{\mu\nu}(x) \, h_{\rho\sigma}(x') \,) \mid 0 > \tag{2.2.14}$$

which yields, after some standard manipulations[38],

$$G_h(x\text{-}x') = 2i \; \{ \; \eta_{\mu\rho} \, \eta_{\nu\sigma} + \eta_{\mu\sigma} \, \eta_{\nu\rho} - \frac{1}{4} \eta_{\mu\nu} \, \eta_{\rho\sigma} \; \} \, G_F(x\text{-}x') \tag{2.2.15}$$

where $G_F(x\text{-}x')$ is the standard Feynman propagator[39],

$$G_F(x) = \frac{1}{(2\pi)^{10}} \int d^{10}k \, \frac{e^{ik.x}}{p^2 + i\varepsilon} \tag{2.2.16}$$

This yields the form of the propagator that will be used in the amplitude calculations in the following chapters.

The derivation of the graviton propagator above is a tedious but well known procedure. The derivations of the propagators for the two fermion fields in the gravitational sector of the theory are equally tedious and require more novel techniques than the graviton case given above. These propagators require the use of the sophisticated constrained quantisation techniques of Dirac[32] for their evaluation. The simplest case, and the one considered first, is that of the dilatino, or spin-1/2 field. The use of these techniques will be useful in the discussion of the higher derivative Lagrangians in the final section. As an example of these techniques a new form of the quantisation of the spin-1/2 field with the Weyl constraint will be presented. This is trivial, since the Weyl constraint is linear and can be solved before the quantisation, but nonetheless interesting in that the application of the technique shows all the features that will be required in the final section.

In this case the action is given by the Lagranian,

$$\mathfrak{L} = \frac{1}{2}\bar{\lambda}\gamma^{\mu}\overleftrightarrow{\partial}_{\mu}\lambda \qquad (2.2.17)$$

where the fields are Majorana-Weyl fermions. The definition of the Majorana conjugate spinor is as given in Appendix Three,

$$\bar{\lambda} = \lambda^{T}\gamma^{0} \qquad (A3.1.12)$$

and so the Lagrangian can be explicitlty written in the form,

$$\mathfrak{L} = \frac{1}{2}\lambda^{a}(\gamma^{0}\gamma^{\mu})_{ab}\overleftrightarrow{\partial}_{\mu}\lambda^{b} \qquad (2.2.18)$$

so that the canonically conjugate momentum can be defined in the standard way,

$$\pi_{a} = \frac{\delta\mathfrak{L}}{\delta(\partial^{0}\lambda^{a})} = -\lambda_{a} \qquad (2.2.19)$$

where the rules of Grassman differentiation are used. (Note the all derivatives with respect to Grassman variables will be taken from the left. For explicit details of the definition of Poisson brackets over a Grassman algebra see Appendix Three.) While relation (2.2.19) is a primary constraint in the notation of Dirac[32], it is also second class, which means that the standard procedure of mapping from the classical Poisson brackets to the quantum commutators has to be modified. The fields are also defined to be Weyl spinors, that is they satisfy the constraint,

$$\gamma^{11}\lambda = \lambda$$

which is another primary second class constraint. Using the γ-matrix representation of Appendix Three, the γ^{11} matrix can be written in the block form,

$$\gamma^{11} = \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix} \qquad \text{(A3.1.11)}$$

where $A^T = -A$. The obvious, and standard procedure is to use a dotted spinor index notation so in what follows the spinor will be partitioned into two parts,

$$\lambda_a = \begin{pmatrix} \lambda_a^{(1)} \\ \lambda_{\dot{a}}^{(2)} \end{pmatrix} \qquad \text{(2.2.20)}$$

where the $\lambda^{(1)}$ corresponds to the undotted part of the spinor, λ_a, and $\lambda^{(2)}$ carries the dotted spinor index a, $\lambda_{\dot{a}}$.

Using the standard dotted index notation described above it is possible to summarise the constraints as follows;

$$\begin{aligned} \varphi_{ia} &= \pi_a^{(i)} + \lambda_a^{(i)} \quad , \quad i = 1,2 \quad a = a,\dot{a} \\ \varphi_{3a} &= \lambda_a^{(1)} - A_{a\dot{a}}\,\lambda_{\dot{a}}^{(2)} \end{aligned} \qquad \text{(2.2.21)}$$

where these are taken to be weakly zero, and can be shown to satisfy the Poisson bracket algebra,

$$\begin{aligned} \{\varphi_{1a}, \varphi_{1b}\} &= 2\,\delta_{ab} & \{\varphi_{1a}, \varphi_{3b}\} &= \delta_{ab} \\ \{\varphi_{2\dot{a}}, \varphi_{2\dot{b}}\} &= 2\,\delta_{\dot{a}\dot{b}} & \{\varphi_{2\dot{a}}, \varphi_{3b}\} &= -A_{b\dot{a}} \\ \{\varphi_{1a}, \varphi_{2\dot{b}}\} &= 0 & \{\varphi_{3a}, \varphi_{3b}\} &= 0 \end{aligned} \qquad \text{(2.2.22)}$$

It is trivial to show that there are no secondary constraints arising from these primary constraints since they are all second class constraints. It can be seen that the generalised Hamiltonian cannot generate any new constraints, but it will provide constraints on the general parameters introduced into the Hamiltonian by hand[32], which are introduced to include the effect of the constraints on the variational procedure of deriving the equations of motion. The Hamiltonian is never explicitly required in this work and so this aspect of the quantisation is never addressed. The matrix whose elements are the values of this algebra must be inverted and the Dirac bracket defined. The Dirac bracket is defined, (in terms of general constraints ϕ_m and general functions of the dynamical variables f and g), to be the bracket given by the sum of the canonical Poisson bracket with an extra term defined by the inverse of the matrix of constraints, as follows,

$$\{ f , g \}_D = \{ f , g \} - \{ f , \phi_m \} . \{ \phi_m , \phi_n \}^{-1} . \{ \phi_n , g \} \tag{2.2.23}$$

which in the case of the spin-1/2 field becomes,

$$\{ \lambda_a^{(i)} , \lambda_b^{(j)} \} = - \{ \lambda_a^{(i)} , \varphi_{kc} \} \{ \varphi_{kc} , \varphi_{ld} \}^{-1} \{ \varphi_{ld} , \lambda_b^{(j)} \} \tag{2.2.24}$$

which can be shown to be explicitly,

$$\{ \lambda_a^{(i)} , \lambda_b^{(j)} \}_D = -\frac{1}{4} [(\delta_1^i \delta_{ac}) (\delta_2^i \delta_{a\dot{c}}) \, 0] \; x$$

$$\begin{bmatrix} \delta_{cd} & A_{c\dot{d}} & 2\delta_{cd} \\ -A_{\dot{c}d} & \delta_{\dot{c}\dot{d}} & 2A_{\dot{c}d} \\ 2\delta_{cd} & -2A_{c\dot{d}} & -\delta_{cd} \end{bmatrix} x \begin{bmatrix} (\delta_1^j \delta_{db}) \\ (\delta_2^j \delta_{\dot{d}b}) \\ 0 \end{bmatrix}$$

by evaluating the appropriate general Poisson brackets. Returning to the full index notation and multiplying out the matrices, it is possible to write,

$$\{ \lambda_a , \lambda_b \}_D = \frac{i}{4} (\delta_{ab} + \gamma_{ab}^{11}) \tag{2.2.25}$$

which enables the canonical quantisation to proceed. To do this the standard path is trodden as in the graviton case above. First the mode expansion solution of the Dirac equation of motion is found, subject to the constraints of the theory, then the mode coefficients are quantised. The normalisation of the c-number solutions is noted, from which the propagator will be seen to follow directly. Given that the equation of motion of the spin-1/2 field is,

$$\gamma^{\mu}\partial_{\mu}\lambda = 0 \qquad (2.2.26)$$

which can be seen by direct calculation from equation (2.2.17) above, then the mode expansion solution is,

$$\lambda_a = \int d\tilde{k}\ \{\ a_{(\lambda)}(k)\, u_a^{(\lambda)}(k)\, e^{ik.x} + a^{+}_{(\lambda)}(k)\, u_a^{*(\lambda)}(k)\, e^{-ik.x}\ \}$$
$$(2.2.27)$$

and the conjugate expansion is obviously,

$$\bar{\lambda}_a = \int d\tilde{k}\ \{\ a_{(\lambda)}(k)\, \bar{u}_a^{*(\lambda)}(k)\, e^{ik.x} + a^{+}_{(\lambda)}(k)\, \bar{u}_a^{(\lambda)}(k)\, e^{-ik.x}\ \}$$
$$(2.2.28)$$

where the standard measure (up to the definition of the metric sign) of reference [38] is used, and where a consistent choice for the normalisation of the c-number polarisation states is,

$$\bar{u}^{(\lambda)}(k)\,\gamma^{0}\,u^{(\lambda')}(k) = \frac{1}{2}\,\delta^{\lambda\lambda'}\,k^{0}$$
$$u_a^{(\lambda)}(k)\,\bar{u}_b^{(\lambda)}(k) = \frac{1}{8}\,\{(\,1+\gamma^{11})\,\not{k}\,\}_{ab} \qquad (2.2.29)$$

The canonical quantisation condition is defined to be,

$$\{\ \lambda_a, \bar{\lambda}_b\ \} = \frac{i}{4}\,[(\,1+\gamma^{11}\,)\,\gamma^{0}\,]_{ab}\,\delta^{9}(\underline{x}-\underline{x}')$$
$$(2.2.30)$$

at equal times, (where here and following the standard curled bracket is used to denote anticommutators), which forces the mode coefficient algebra to be,

$$\{\,a_{(\lambda)}(k)\,,\,a^{+}_{(\lambda')}(k')\,\} = i\,k^{0}\,(2\pi)^{9}\,\delta_{\lambda\lambda'}\,\delta^{9}(\underline{k}-\underline{k}')$$
$$\{\,a_{(\lambda)}(k)\,,\,a_{(\lambda')}(k')\,\} = 0 \qquad (2.2.31)$$

The propagator is defined to be[38],

$$G_{\lambda}(x-x') = \langle\, 0 \mid T(\,\lambda_a(x)\,\bar{\lambda}_b(x')\,) \mid 0\,\rangle \qquad (2.2.32)$$

although it should be noted that the $\bar{\lambda}_b(x')$ is just a constant matrix times the λ field itself, which is different to the ordinary Dirac fermion case where the conjugate spinor is 'proportional' to the hermitian conjugate of the spinor. This has interesting consequences for the interacting field theory discussed in the next chapter. The propagator calculation is straightforward but tedious. The procedure is to expand (2.1.32) in terms of the step function,

$$\theta(t - t') = \begin{cases} 1 & t > t' \\ 0 & t < t' \end{cases}$$

and to work out each of the terms seperately. The first term can be written out in full, and gives,

$$< 0 \mid \lambda_a(x)\, \bar{\lambda}_b(x') \mid 0 > \; = \; < 0 \mid \lambda_a^{(-)}(x)\, \bar{\lambda}_b^{(+)}(x') \mid 0 >$$
$$= \int d\tilde{k} \int d\tilde{k}' \left\{ <0| \{ a_{(\lambda)}(k), a_{(\lambda')}(k') \} |0> \, u_a^{(\lambda)}(k)\, \bar{u}_b^{(\lambda')}(k')\, e^{i(k.x-k'.x')} \right\}$$
(2.2.33)

and using standard manipulations this can be written in terms of general invariant functions[39]. Explicitly,

$$< 0 \mid \lambda_a(x)\, \bar{\lambda}_b(x') \mid 0 > \; = -\frac{i}{4} [(1 + \gamma^{11})\, \gamma^{\mu}]_{ab}\, \partial_{\mu}\, G_+(x - x')$$
(2.2.34)

and so, noting that the second term also gives,

$$< 0 \mid \bar{\lambda}_b(x')\, \lambda_a(x) \mid 0 > \; = -\frac{i}{4} [(1 + \gamma^{11})\, \gamma^{\mu}]_{ab}\, \partial_{\mu}\, G_-(x - x')$$
(2.2.35)

Summing these two contributions yields a propagator of the form,

$$G_{\lambda}(x-x') = -\frac{i}{4} ((1 + \gamma^{11})\, \partial\!\!\!/ \,)_{ab}\, G_F(x-x') \qquad (2.2.36)$$

where $G_F(x-x')$ is the standard Feynman propagator, given by the definition,

$$G_F(x - x') = \theta(t - t')\, G_+(x - x') - \theta(t' - t)\, G_-(x - x')$$

The Fourier transformations of these propagators can now be calculated for use in the momenum representation Feynman rules. As stated above, this result is completly trivial in the sense that it is possible to solve the Weyl constraint at the beginning, before the definition of the Hamiltonian and the subsequent quantisation. This however does give the same result as the more sophisticated technique used here.

Section 2.2: The gravitino propagator.

The general procedure given in some depth for the graviton and dilatino must now be repeated for the case of the spin-3/2 field since it is also a Majorana-Weyl fermion field. This particular case is somewhat more complicated since the constraint algebra not only contains the Majorana and Weyl constraints as above, but also the gauge fixing conditions which do not trivially decouple from these constraints. As stated above the Weyl constraint can be solved at the Lagrangian stage simply adding a trivial $(1+\gamma^{11})$ factor to the Lagrangian. This will only serve to increase the complexity of the algebra and will be left implicit from now on. The γ^{11}'s can be reintroduced by replacing ψ by $(1+\gamma^{11})\psi/2$ throughout.

The action for the free spin-3/2 field in ten dimensions is[40],

$$\mathcal{L}_\psi = \overline{\psi}_\mu \gamma^{\mu\nu\rho} \overleftrightarrow{\partial}_\nu \psi_\rho \qquad (2.3.1)$$

from which it can be seen that the canonically conjugate momenta are defined to be,

$$\begin{aligned} \pi^0_a &= 0 \\ \pi^m_a - (\gamma^{mn})_{ab}\, \psi^b_n &= 0 \end{aligned} \qquad (2.3.2)$$

and thus the equation of motion is,

$$\gamma^{\mu\nu\rho}\partial_\nu \psi_\rho = 0 \qquad (2.3.3)$$

The Majorana constraint comes out implicitly in the normal set of fermionic primary, second class constraints. The gauge symmetries of this action arise form primary first class constraints, which yield primary second class constraints when the gauge fixing conditions,

$$\begin{aligned} \psi^a_0 &= 0 \\ \gamma^i_{ab}\, \psi^b_i &= 0 \end{aligned} \qquad (2.3.4)$$

are applied. It is also possible to use the transverse gauge fixing,

$$\psi_0^a = 0$$
$$\partial^m \psi_m^a = 0 \qquad (2.3.5)$$

It should be noted that explicit covariance has been abandoned temporarily, though it can be shown that the final result will be fully covariant. The functional integral calculation carried out in Appendix Two is fully covariant, but only implicitly observes the full constraint algebra[41]. The full constrained quantisation procedure is extremely similar to the spin-1/2 cae given above, and so will only be sketched here, since the propagator will not be evaluated in a form which will ever be used in this thesis. The complete set of primary constraints is given by (2.3.2) above. These generate secondary constraints;

$$\dot{\pi}_a^0 = \{\pi_a^0 , \mathcal{H} \}$$
$$= \{\pi_a^0 , - \overline{\psi}_\mu \gamma^{\mu n \rho} \overleftrightarrow{\partial}_n \psi_\rho \}$$
$$= 0$$

which can be seen to yield the secondary constraint,

$$(\gamma^{nr})_{ad} \partial_n \psi_r^d = 0 \qquad (2.3.6)$$

which is the only secondary constraint. The complete set of constraints must be separated into the first and second classes. The complete set of first class constraints is given by,

$$\varphi^0 = \pi^0$$
$$\varphi_a = \partial_k \pi_a^k - (\gamma^{nr})_{ab} \partial_n \psi_r^b \qquad (2.3.7)$$

and the second class constraints are,

$$\theta_a^k = \pi_a^k + (\gamma^{km})_{ab} \psi_m^b \qquad (2.3.8)$$

The chosen gauge fixing conditions eliminate the gauge degrees of freedom defined by the first class constraints. The construction of the Dirac brackets as in the spin-1/2 case is very complicated, and not terribly illuminating at this stage, and so only the result will be stated. Choosing the transverse gauge of equation (2.3.5), the evaluation of the Dirac bracket of ψ with itself is extremely tedious but straightforward, and yields[41],

$$\left\{ \psi_i^a(x), \psi_j^b(x') \right\}_D \Big|_{t=t'} = \Big\{ - \frac{(\gamma_{ij}+2\eta_{ij})_{ab}}{d(d-1)} + \frac{\delta_{ab}}{d} \frac{\partial_i \partial_j}{\partial^2} - \frac{\partial_i \partial^k (\gamma_{kj}+2\eta_{kj})_{ab}}{d(d-1)} - \frac{\partial_j \partial^k (\gamma_{ik}+2\eta_{ik})_{ab}}{d(d-1)} \Big\} \delta(\underline{x}-\underline{x}') \tag{2.3.9}$$

which will now be used to give the mode coefficient quantisation. Again the mode expansion must be determined subject to the constraints given by the equations of motion and the gauge conditions. The mode expansion is easily seen to be of the general form,

$$\psi_\mu(x) = \int d\tilde{k} \left\{ a_\lambda(k)\, u_\mu^\lambda(k)\, e^{ik.x} + a_\lambda^+(k)\, u_\mu^{*\lambda}(k)\, e^{-ik.x} \right\} \tag{2.3.10}$$

where the polarisation spinors must satisy the conditions,

$$\not{k}\, u_\mu^\lambda(k) = k^m u_m^\lambda(k) = \gamma^m u_m^\lambda(k) = u_0^\lambda(k) = 0 \tag{2.3.11}$$

and the polarisation states satisfy the normalisation conditions[42],

$$\bar{u}_\mu^\lambda(k)\gamma^\nu u_\rho^{\lambda'}(k)\eta^{\mu\rho} = -\,2\,\delta^{\lambda\lambda'} k^\nu \tag{2.3.12}$$

where the projection operator is defined,

$$P_{ij}(\underline{k}) = u_i^\lambda(k)\bar{u}_j^\lambda(k)$$

which satisfies the usual projection operator rules. The solution for this in the four dimensional case is given in reference [42]. The mode coefficient algebra when quantisation is performed, is,

$$\left\{ a_\lambda(k)\,,\, a_{\lambda'}(k') \right\} = \left\{ a_\lambda^+(k)\,,\, a_{\lambda'}^+(k') \right\} = 0$$

$$\left\{ a_\lambda(k)\,,\, a_{\lambda'}^+(k') \right\} = \delta_{\lambda\lambda'}\delta(\underline{k} - \underline{k}') \tag{2.3.13}$$

The propagator can now be found by straightforward manipulations, as in the case of the spin-1/2 field given above. The propagator is defined to be,

$$G_{\psi}(x - x') = \langle 0 | T (\psi^{a}_{\mu}(x) \bar{\psi}^{b}_{\nu}(x')) | 0 \rangle \quad (2.3.14)$$

which can be evaluated by the standard methods used above. The mode expansion is inserted in the definition of the time ordered product, and the normalisation conditions used to obtain the standard invariant functions, similar to the integrals found in the definition of the spin-1/2 propagator above. Finally the propagator is found to be,

$$G_{\psi}(x) = i \int \frac{d^d k}{(2\pi)^d} \frac{P_{ij}(\underline{k})}{k^2 + i\varepsilon} e^{ik.x} \qquad (2.3.15)$$

This form of the propagator can be seen to be very similar to the form derived using a covariant gauge choice in Appendix Two, when the projection operator of Das and Freedman[42]is used.

The covariant gauge choice is the most convenient for amplitude calculations, and so the propagator used in these calculations is,

$$Y^{(0)}_{\mu,\nu}(k) = \frac{i}{8k^2} \{ \gamma_{\nu} \not{k} \gamma_{\mu} - 6 (\eta_{\mu\nu} \not{k} - \frac{2}{k^2} k_{\mu} k_{\nu} \not{k}) \}$$
(A2.3.14)

which agrees with reference [43]. The higher derivative corrections to this are included also in Appendix Two. The subject of higher derivative quantisation will now be addressed.

Section 2.4: The higher derivative case.

This problem has been discussed in some detail by several authors[44]. Many of these attempts are not made in a language suitable for the work contained in the rest of this thesis. The problem of higher derivative quantisation will not be clarified completely by any work presented here, but the approach to the derivation of the propagators for higher derivative theories in general, which is used throughout this work, will be justified on the grounds that this approach is asymptotically correct in the limit when the parameter controlling the higher derivative parts of the action is chosen to makes these terms vanishingly small, and provides some means of getting amplitudes for these theories. A simple example will be sufficient to outline all the problems of higher derivative quantisation, as well as giving some idea of how the problem is solved. Only a sketch of the calculation will be provided as a detailed treatment lies outside the scope of the work of this thesis.

The action that will be considered as the toy example will be the higher derivative Lagrangian for a massless scalar field, where the higher derivative correction is introduced with the dimensionful parameter η. The action is,

$$\mathcal{L} = (\partial_\mu \phi)^2 + \eta (\partial_\mu \partial_\nu \phi)^2 \qquad (2.4.1)$$

and now the problem is how to construct an Hamiltonian and Poisson brackets with which to go over to a quantum theory. There are two well known approaches. The first approach, and the one which will be followed here, is due to reference [45]. The other approach is to separate the single variable ϕ into two new variables, whose equations of motion are the separate differential operators in the product (2.4.1) and to quantise these. The problem with this method is to define the reverse change of variables from these two new variables to the original physical variable. The problems of this approach will not be addressed here.

The first approach deals with the introduction of a new, dummy, variable which allows the definition of the canonically conjugate momenta for the theory, and thus the simple definition of the Poisson brackets. Explicitly the new variable introduced is,

$$\theta = \partial_0 \phi = - \partial^0 \phi \qquad (2.4.2)$$

which allows the canonically conjugate momenta to be defined,

$$\pi = \frac{\delta \mathcal{L}}{\delta \dot{\phi}} = -2\, \dot{\phi} = -2\theta$$

$$\xi = \frac{\delta \mathcal{L}}{\delta \dot{\theta}} = -2\eta \dot{\theta} \qquad (2.4.3)$$

and thus the Hamiltonian is seen to be,

$$\mathcal{H} = - \pi^2 - (\partial_m \phi)^2 + \frac{\xi^2}{4\eta} + 2\eta\, (\partial_m \theta)^2 - \eta\, (\partial_m \partial_n \phi)^2 \qquad (2.4.4)$$

The definition of the θ variable can be rewritten as a primary constraint,

$$\Phi_1 = \pi + 2\theta \approx 0 \tag{2.4.5}$$

which is first class with itself, implying that there must be a secondary constraint generated by the relation,

$$\Phi_2 = \{ \Phi_1 , \mathcal{H} \}$$

$$= \frac{1}{\eta}\xi + 2\eta\, \partial_m\partial_n\partial^m\partial^n\phi - 2\,\partial_m\partial^m\phi \approx 0 \tag{2.4.6}$$

which is second class with the primary constraint, and so there are no more secondary constraints. The constraint algebra is,

$$\{ \Phi_1 , \Phi_1 \} = \{ \Phi_2 , \Phi_2 \} = 0$$

$$\{ \Phi_1 , \Phi_2 \} = \frac{1}{\eta} - 2\,\partial_m\partial^m + 2\eta\, \partial_m\partial^m\partial_n\partial^n \tag{2.4.7}$$

where the Dirac delta functions are left implicit. (They may be reintroduced trivially). This algebra is trivial to invert, and is not demonstrated here. Thus the Dirac brackets for ϕ and π are,

$$\{ \phi(x) , \pi(x') \}_D\big|_{t=t'} = \int d^{d-1}\underline{k}\; \frac{1}{(2\eta\, \underline{k}^2 + 2\eta\, (\underline{k}^2)^2 + 1)}\, e^{i\underline{k}.(\underline{x}-\underline{x}')} \tag{2.4.8}$$

which now allow the quantisation to be performed with respect to the quantisation condition of equation (A3.2.11). The standard procedure for field theories will be followed here. The general solution of the equation of motion for the ϕ field can be seen to be,

$$\phi(x) = \int \frac{d^{d-1}\underline{k}}{2(2\pi)^{d-1}} \{ [\frac{a(k)e^{i\omega_k t}}{\omega_k} - \eta\, \frac{b(k)e^{i\bar{\omega}_k t}}{\bar{\omega}_k}]\, e^{i\underline{k}.\underline{x}} + (\text{complex conjugate}) \} \tag{2.4.9}$$

where the ω_k and ω_k coefficients are defined to be,

$$\omega_k = |\underline{k}| \quad , \qquad \overline{\omega}_k = (|\underline{k}|^2 + \frac{1}{\eta})^{1/2}$$

where some obvious Dirac delta function manipulations have been performed. The quantisation of the mode coefficients can be attempted now. Immediately it is obvious that the time dependence of the commutator will not completely cancel out, as would be necessary to be consistent with the Dirac bracket definiton in (2.4.8) above, unless the quantum algebra separates completely. Unless this is the case, there will always be residual mixing terms which contain a factor of the form,

$$\frac{1 + e^{i(\omega_k - \overline{\omega}_k)t}}{\omega_k} + \frac{1 + e^{i(\overline{\omega}_k - \omega_k)t}}{\overline{\omega}_k}$$

in this approach to the quantisation, indicating an inconsistency in the oscillator interpretation of the quantum field theory. Assuming the seperation of the two sectors of the algebra, i.e. the a and b oscillators commute with each other, allows the commutators to be solved and the propagator found. This is trivial to perform, and will not be made explicit here. It is shown in reference [44], (although they take no account of the Hamiltonian constraints in the quantisation) how this may be done.

How can the tedium of this procedure be avoided? The only reasonable way to get around this problem is simply to ignore it. The solution would seem to be to ignore the problems of the quantisation, and to simply equate the (Feynman) Green's function for the equation of motion, defined to be,

$$G_\phi(x) = \int d^d k \frac{e^{ik.x}}{\eta\,(k^2)^2 - k^2 + i\varepsilon} \qquad (2.4.10)$$

with the propagator, defined by the standard operator product which stems from the Wick expansion. This is the solution which is implicit in the functional integral approach of Appendix Two. This is also the approach adopted by most of the workers in the field at the present time, as is seen in references [58-61]. It would be reasonable to assume that the procedure followed above will generalise to the fermionic cases mentioned above, as well as to the graviton.

Chapter Three.
The Interacting Field Theory I:
The Lowest Order.

Introduction.

In this chapter the first aproximation to the full interacting field theory is stated and the corresponding interacting quantum theory is constructed. The unusual features of Majorana field theory are noted and their consequences derived. The procedure used will be that of standard canonical field theory. The interacting field theory can be constructed using the Noether method and can be seen to be simply the ten dimensional N=1 supergravity theory coupled to N=1 super Yang-Mills. This is just the modified non-abelian form of the action derived by dimensional reduction from N=1, d=11 supergravity in reference [6] to give the Einstein-Maxwell theory, derived in reference [7], where the addition of an extra Chern-Simons three form term ω^{3Y} is used as the non-abelian generalisation of the abelian coupling of the Maxwell field A_μ to the antisymmetric field $a_{\mu\nu}$ of reference [6], which maintains full gauge invariance and supersymmetry of the action. From now on this form of the action will be referred to as the Chapline-Manton action. This action is,

$$\mathcal{L} = -\frac{1}{2}eR - \frac{1}{2}e\,\overline{\psi}_\mu\gamma^{\mu\nu\rho}\mathcal{D}_\nu\psi_\rho - \frac{3}{4}e\,\phi^{-3/2}f_{\alpha\beta\gamma}f^{\alpha\beta\gamma} - \frac{1}{2}\overline{\lambda}\gamma^\mu\mathcal{D}_\mu\lambda$$
$$-\frac{9}{16}e\,(\partial_\mu\phi/\phi)^2 - \frac{3\sqrt{2}}{8}e\,\overline{\psi}_\mu\gamma^\nu\gamma^\mu\lambda\,(\partial_\mu\phi/\phi)$$
$$+\frac{\sqrt{2}}{16}e\,\phi^{-3/4}f_{\alpha\beta\gamma}\left\{\overline{\psi}_\mu\gamma^{\mu\alpha\beta\gamma\nu}\psi_\nu + 6\overline{\psi}^\alpha\gamma^\beta\psi^\gamma - \sqrt{2}\,\overline{\psi}_\mu\gamma^{\alpha\beta\gamma}\gamma^\mu\lambda\right\}$$

where,

$$f_{\alpha\beta\gamma} = \partial_{[\alpha}a_{\beta\gamma]} + \omega^{3Y}_{\alpha\beta\gamma}$$

(ignoring both the Yang-Mills sector and the four fermi terms) where it should be noted that the action is supersymmetric with respect to the transformations, (again ignoring the Yang-Mills sector of the theory),

$$\delta e^m{}_\mu = \frac{1}{2}\bar{\varepsilon}\gamma^m\psi_\mu\ ,\quad \delta\phi = -\frac{\sqrt{2}}{3}\bar{\varepsilon}\lambda f\phi$$

$$\delta a_{\mu\nu} = \frac{\sqrt{2}}{4}\phi^{3/4}\left\{\bar{\varepsilon}\gamma_\mu\psi_\nu - \bar{\varepsilon}\gamma_\nu\psi_\mu - \frac{\sqrt{2}}{2}\bar{\varepsilon}\gamma_{\mu\nu}\lambda\right\}$$

$$\delta\lambda = -\frac{3\sqrt{2}}{8}\gamma^{\mu}\varepsilon\,(\hat{\mathfrak{D}}_{\mu}\phi/\phi)$$

$$\delta\psi_{\mu} = \hat{\mathfrak{D}}_{\mu}\varepsilon - \frac{1}{16x32}(\gamma_{\mu}{}^{\alpha\beta\gamma} - 5\,\delta_{\mu}^{\alpha}\gamma^{\beta\gamma})\varepsilon\bar{\lambda}\gamma_{\alpha\beta\gamma}\lambda$$

$$+\frac{\sqrt{2}}{96}\left\{(\bar{\psi}_{\mu}\gamma_{mn}\lambda)\gamma^{mn}\varepsilon + (\bar{\lambda}\gamma_{mn}\varepsilon)\gamma^{mn}\psi_{\mu} + 2\,(\bar{\psi}_{\mu}\lambda)\varepsilon - 2\,(\bar{\lambda}\varepsilon)\psi_{\mu} + 4\,(\bar{\psi}_{\mu}\gamma_{m}\varepsilon)\gamma^{m}\lambda\right\}$$

Unfortunately this action is anomalous. The Green-Schwarz anomaly cancellation mechanism[46] is further necessitated by the addition of the Lorentz Chern-Simons three form term of the form,

$$\omega^{3L}_{\alpha\beta\gamma} = \sqrt{2}\,\left\{R^{mn}_{[\alpha\beta}\,\omega^{nm}_{\gamma]} - \frac{2}{3}\,\omega^{ma}_{[\alpha}\omega^{ab}_{\beta}\omega^{bm}_{\gamma]}\right\}$$

to the definition of the three form field strength,

$$G_{\alpha\beta\gamma} = \partial_{[\alpha}\,a_{\beta\gamma]} + \omega^{3Y}_{\alpha\beta\gamma} - \gamma\omega^{3L}_{\alpha\beta\gamma}$$

which unfortunately breaks the supersymmetry of this action. The form of this Chern-Simons term in the effective action can be explicitly checked by an amplitude matching calculation as carried out in Appendix Five, and the amplitudes calculated from the string can be seen to agree with those calculated from the form of the action stated here when the γ factor takes the value $\gamma = -1/4$. The discussion of supersymmetry and how it can be retrieved will be discussed in the next chapter, when extensions to this Lagrangian will be discussed. This chapter will deal only with the lowest order action given above and will ignore any of the more sophisticated aspects of the symmetries of this Lagrangian. The quantum theory will be set up using a neat formalism for the Feynman rules which is facilitated by the choice of Majorana fermions, which greatly reduces the amount of calculating normally required with Majorana field theories[47]. Throughout the restriction will be made to the graviton and gravitino fields, though the generalisation of what follows to the remaining fields in the theory is similar in procedure to that used below, due to the tedious nature of the calculations.

Section 3.1: First Steps: The Canonical Approach.

The construction of the interacting field theory described below shall follow (slightly pedantically) the standard canonical approach[38]. The basis of all perturbative field theory is the S-matrix and the LSZ reduction procedure[48] dealing with the perturbative solution of scattering problems. This standard perturbative method of solution is well known, and consequently need not be described in depth here. It will only be necessary to state some fundamental things which will be needed throughout.

The whole limiting approach of the LSZ procedure is given by the formal limit,

$$| \text{out} > = S^{-1} | \text{in} > = S^{\dagger} | \text{in} >$$

$$| \text{in} > = S \, | \text{out} > \qquad (3.1.1)$$

where S is the scattering operator which can be shown to be solved by,

$$S = \lim_{t \to +\infty} U(t)$$

where in terms of the incoming free fields. The time evolution operator U(t,t') is defined to be,

$$U(t',t) = T \exp \left\{ -i \int_{t}^{t'} dt \int d^{9}\underline{x} \ \mathcal{H}(\underline{x},t) \right\} \qquad (3.1.2)$$

where $\mathcal{H}(\underline{x},t)$ is defined in terms of the free incoming fields,

$$\mathcal{H}(\underline{x},t) = \mathcal{H}(\phi_{in}(\underline{x},t), \pi_{in}(\underline{x},t), t)$$

and where,

$$U(t) = \lim_{t' \to -\infty} U(t,t')$$

From the standard form of the LSZ reduction formula it can be seen that a scattering amplitude is defined by,

$$<p_1,\ldots,p_n \text{ out} \mid q_1,\ldots,q_m \text{ in}>$$

$$= (i\, Z^{-1/2})^{m+n} \int d^{10}x_1 \ldots d^{10}x_m d^{10}y_1 \ldots d^{10}y_n \times$$

$$\times \exp\{\, i \sum_1^m q_i.x_i - i \sum_1^n p_j.y_j \,\} \times$$

$$\times K_{x_1} \ldots K_{x_m} K_{y_1} \ldots K_{y_n} <0|T(\phi(y_1)\ldots\phi(x_m))|0>$$

(3.1.3)

for some general bosonic field ϕ, where the K's are the free field equations of motion and where the factor Z is the standard normalisation factor relating the interpolating interacting Heisenberg fields ϕ with the incoming free fields ϕ_{in} in the weak relation,

$$\phi(x) \rightarrow Z^{1/2}\phi_{in}(x) \quad \text{as} \quad x^0 \rightarrow -\infty \qquad (3.1.4)$$

There also exists the analogous result for fermions,

$$<p'_1,\ldots,p'_n,p_1,\ldots,p_m \mid q'_1,\ldots,q'_a,q_1,\ldots,q_b>$$

$$= (iZ_2^{-1/2})^{n+a}(-iZ_2^{-1/2})^{m+b} \int d^{10}x_1 \ldots d^{10}y_b \exp(-i \sum q.y + q'.y' - p.x - p'.x') \times$$

$$\times\, \bar{u}(p_1)\overrightarrow{K} \ldots <0|T(\{\bar{\psi}(x'_i)\psi(x_j)\bar{\psi}(y'_1)\psi(y_l)\})|0> \ldots \overleftarrow{K}v(q_b)$$

(3.1.5)

for some general fermionic field ψ, and where the K's denote the fermionic kinetic operators, and primes denote anti particles in the incoming or outgoing states. These can be recast by substitution of the expansion,

$$G(x_1,\ldots,x_{2n}) = \sum_{p=0}^{\infty} \frac{i^p}{p!} \int d^{10}y_1 \ldots d^{10}y_p \times$$

$$\times < 0 \mid T\{\, \phi_{in}(x_1)\ldots \phi_{in}(x_{2n})\, \mathcal{L}_{int}(y_1)\ldots\mathcal{L}_{int}(y_p) \,\} \mid 0 >^{(1)}$$

(3.1.6)

where $G(x_1,\ldots x_{2n}) = < 0 \mid T(\phi(x_1)\ldots\phi(x_{2n}) \mid 0 >$, and where the superscript (1) denotes the fact that when Wicks theorem is applied, only the connected terms are kept.

It should be noted at this point that the extended theory described in the following chapters will have higher derivative correction terms added to it, which will modify the propagators derived in the previous chapter. However it will be shown in the next chapter that these propagator corrections do not alter the fundamental structure of the LSZ procedure discussed above. It will be shown later how this pedantic approach can help in the discussion of possible field redefinitions in Chapter Six.

Using Wick's theorem allows the expansion of the various contributions to the amplitude into products of two point functions corresponding to connected Feynman diagrams. However at this point it should be noted that the Majorana definition of the fermions means that two point functions of the form,

$$G' = \langle 0 | T(\lambda_a \lambda_b) | 0 \rangle$$

are related by constant matrices to the standard propagator,

$$G = \langle 0 | T(\lambda_a \bar{\lambda}_b) | 0 \rangle$$

$$= -\frac{i}{4}((1+\gamma^{11})\partial)_{ab} G_F(x - x') \qquad (2.2.36)$$

This means that there appears to be a considerable increase in the number of contributions to the amplitude since many of the fermion contractions which normally do not contribute in the Wick expansion, now do so. However if these are examined more closely, it can be seen that a remarkable simplification can be made. It is easiest to see this using a simplified form for a general Majorana fermionic action. Consider a toy interaction Lagrangian, defined in terms of the free 'in' fields,

$$\mathcal{L}_{toy} = \bar{\psi}_{in} f \psi_{in} h_{in} + \bar{\psi}_{in} g \psi_{in} h_{in} h_{in} + q h_{in} h_{in} h_{in} \qquad (3.1.7)$$

where all unnecessary indices have been dropped, and all gamma matrices and partial derivatives have been taken up into the internal f, g and q factors. This means that the Green's function for $\bar{\psi}$-ψ-h-h scattering in terms of these generic fields can be written in the form,

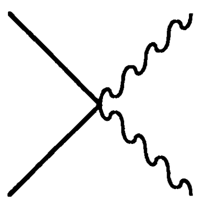
$$G(x_1,\ldots,x_4) = \langle 0 | T(\bar{\psi}_1 \psi_2 h_3 h_4 \exp\left\{ i\int dy\, \mathcal{L}_{toy}(y) \right\}) | 0 \rangle \qquad (3.1.8)$$

which can be written in the form (suppressing the y-integrations for clarity),

$$<0|T(\bar{\psi}_1\psi_2h_3h_4\sum_{p=0}^{\infty}\frac{i^p}{p!}\mathcal{L}_{toy}(y_1)\ldots\mathcal{L}_{toy}(y_p))|0>^{(1)}$$

(3.1.9)

which can be written out explicitly to lowest order in the expansion parameters, where a subscript i=1,...,4 denotes one of the external fields where the variable x_i will be integrated over in the LSZ formulae (3.1.3) and (3.1.5) above, and where a superscript j=1,... denotes the internal vertex point which will be integrated over in the evaluation of the Green's function. Because of the (1) only connected terms will be kept. The derivation which will now be given will be for one of the simpler cases, but it can easily be modified for one of the more complex terms in the expansion, (the Born terms for example). The case discussed will be for the simple point diagram,



which can be seen to originate from the g factor in the interaction Lagrangian above. The contribution to the Green's function due to this term can be written explicitly as (again suppressing the y-integrations for clarity),

$$<0|T(\bar{\psi}_1\psi_2h_3h_4\bar{\psi}^1g\psi^1h^1h^1)|0> \qquad (3.1.10)$$

and now Wick's theorem can be applied. The Majorana nature of the Fermions means that there are now four distinct terms, as opposed to the normal two terms which would occur in a normal field theory containing Dirac fermions. These are explicitly (ignoring for the moment the somewhat superfluous graviton terms),

$$<0|T(\bar{\psi}_1\psi_2\bar{\psi}^1g\,\psi^1)|0>$$

$$= <0|T(\bar{\psi}_1\bar{\psi}^1)|0>g<0|T(\psi^1\psi_2)|0>$$

$$- <0|T(\psi_2\bar{\psi}^1)|0>g<0|T(\psi^1\bar{\psi}_1)|0> \qquad (3.1.11)$$

where the two new factors are the contractions $<0|T(\psi^1\psi_2)|0>$ and $<0|T(\bar{\psi}_1\bar{\psi}^1)|0>$ which are related to the standard propagators by the relations such as,

$$<0|T(\psi^1\psi_2)|0> = -\ \gamma^0 <0|T(\psi_2\overline{\psi}^1)|0>$$
$$= -\ \gamma^0\ G_\psi(2,1) \qquad (3.1.12)$$

The extra γ^0 matrices can be taken through the Green's function calculation and combined with the definition of the g factor to define a new g factor, $\tilde{g}$ defined by the relation,

$$\tilde{g} = \gamma^0 g^T \gamma^0 \qquad (3.1.13)$$

where the transpose refers to the transpose of gamma matrices with respect to the spinor indices, and from which it can be seen that the two extra new product terms can be included into the two standard terms by noting that,

$$G_\psi(1,1)\gamma^0 g \gamma^0\ G_\psi(2,1) = G_\psi(2,1)\ \tilde{g}\ G_\psi(1,1)$$

and so the Green's function can be written in the form,

$$-\ G_\psi(2,1)\ \{\ \tilde{g} + g\ \}\ G_\psi(1,1) \qquad (3.1.14)$$

By redefining the vertex in the symmetrised form,

$$\overline{\psi}^1_a g \psi^1_b\, h^1 h^1 = \frac{1}{2}\ \{\ \overline{\psi}^1_a g \psi^1_b + \overline{\psi}^1_b \gamma^0 g^T \gamma^0 \psi^1_a\ \}\ h^1\ h^1$$
$$= \frac{1}{2}\ \{\ \overline{\psi}^1_a g \psi^1_b + \overline{\psi}^1_b \tilde{g} \psi^1_a\ \}\ h^1\ h^1 \qquad (3.1.15)$$

where it is noted that the a and b indices are introduced to keep track of the spinors in the calculation, it can be seen that the Green's function can be evaluated in terms of this new vertex factor by treating the fermions as ordinary Dirac fermions. So it is possible to see that the number of Feynman diagrams that have to be evaluated for Majorana fermions can be drastically reduced. In general it can be seen that pairs of Feynman diagrams combine to form 'symmetrised' vertices in this manner. In fact it is easiest to 'symmetrise' the Feynman vertices derived from the Lagrangian before calculating the amplitudes and then proceeding as in a normal field theory, which contains only Dirac fermions, but taking care to modify the Feynman combinatorial factors associated with each of the separate Feynman diagrams. This clearly depends on the weight convention chosen for the 'symmetrisation' of vertices.This is the procedure that will be adopted throughout this chapter, where the weight one convention has been adopted.

Section 3.2: The Feynman vertex rules.

The Feynman rules for the Chapline-Manton Lagrangian can now be derived, noting that in general a 'weight one' convention will be chosen for the vertices, in that given an unsymmetrised (generalised) vertex term,

$$\overline{\psi}_1 f\, \psi_2\, (h_3 + h_3.h_4 + \dots) \qquad (3.2.1)$$

the symmetrised form of the vertex will be,

$$\frac{1}{2}(\overline{\psi}_1\, f\, \psi_2 - \overline{\psi}_2\, \tilde{f}\, \psi_1).(h_3 + h_3.h_4 + \dots) \quad (3.2.2)$$

where the f has the obvious spinor conjugate meaning as defined in Appendix Three and as shown above. The weight one convention is not universal throughout this work but the cases which do not observe this convention will be explicitly stated when they arise. The action written above is the full gauge covariant form of the action, derived using the Noether method on the linear action stated in the previous chapter. It is necessary to consider the symmetric tensor field used in the previous chapter as the quantum field, propagating in some background which is the same as the string. To proceed it is necessary to determine the form of the background field expansion used in the construction of the Feynman vertices involving gravitons. The correct form of this expansion is given by the definitions,

$$\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} + h_{\mu\nu} \qquad \text{(exact)}, \\ g^{\mu\nu} &= \eta^{\mu\nu} - h^{\mu\nu} + h^{\mu\rho} h_\rho{}^{\nu} + \dots \end{aligned} \qquad (3.2.3)$$

and for the vielbein,

$$\begin{aligned} e^m_\mu &= \delta^m_\mu + \frac{1}{2} h^m_\mu - \frac{1}{8} h^m{}_n h^n{}_\mu + \dots \\ e^\mu{}_m &= \delta^\mu{}_m - \frac{1}{2} h^\mu{}_m + \frac{3}{8} h^\mu{}_n h^n{}_m + \dots \end{aligned} \qquad (3.2.4)$$

which allow the expansion of all the terms in the action. The definition of the expansion of the vielbein is Lorentz symmetric, that is it is symmetric if it is transposed as a matrix with two Lorentz indices in the sense of equation (3.2.5b) below. This is due to a symmetric gauge choice for the vielbein, that is, a choice of tangent frame in which the vielbein is a symmetric matrix. It is then necesary to prove that there exists a transformation of the vielbein given by the Lorentz transformation L,

$$e'^m{}_\mu = (Le)^m{}_\mu \qquad (3.2.5a)$$

where the e' matrix is Lorentz symmetric. (By Lorentz symmetry it is meant,

$$M^T = \eta M \eta^{-1} \qquad (3.2.5b)$$

where M is a matrix with two Lorentz indices.) If so, it is trivial to show that there must exist a Lorentz transformation matrix which satisfies the condition,

$$e^T = \eta L e L \eta^{-1} \qquad (3.2.6)$$

which is trivially possible in two dimensions. It is possible to show that this is also true in any dimension. The proof consists of the construction of a Lorentz transformation matrix L which satisfies (3.2.6). This proof begins by noting that the metric for the theory is defined by the relation,

$$g = e^T \eta e \qquad (3.2.7)$$

and there must exist an Lorentz matrix L_1 which satisfies the relation,

$$L_1^T g L_1 = \mathrm{diag}(d_1,\ldots,d_d)\,\eta \qquad (3.2.8)$$

and so defining the diagonal matrices,

$$\begin{aligned} d &= \mathrm{diag}(d_1,\ldots,d_d) \\ \sqrt{d} &= \mathrm{diag}(\sqrt{d_1},\ldots,\sqrt{d_d}) \\ \frac{1}{\sqrt{d}} &= \mathrm{diag}(\frac{1}{\sqrt{d_1}},\ldots,\frac{1}{\sqrt{d_d}}) \end{aligned} \qquad (3.2.9)$$

the expression (3.2.8) can be rewritten,

$$\frac{1}{\sqrt{d}} L_1^T e^T \eta e L_1 \frac{1}{\sqrt{d}} = \eta \qquad (3.2.10)$$

which implies by definition that the combination $eL_1(\sqrt{d})^{-1}$ must be a Lorentz transformation matrix. There must therefore be a Lorentz matrix L_2 which is equal to this combination above. This implies obviously,

$$e = L_2\sqrt{d}\,\eta^{-1}L_1^T\eta \qquad (3.2.11)$$

which can be rewritten in the new form,

$$e = L_2\eta^{-1}L_1^T\eta L_1\sqrt{d}\,\eta^{-1}L_1^T\eta \qquad (3.2.12)$$

by exploiting the invertibility condition, and the Lorentz definition of the matrix L_1,

$$L_1^{-1} = \eta^{-1}L_1^T\eta$$

The matrix L_2L_1 is clearly Lorentz since both L_1 and L_2 are. Therefore it is now possible to write,

$$\begin{aligned} e^T &= (L_2\eta^{-1}L_1^T\eta L_1\sqrt{d}\,\eta^{-1}L_1^T\eta)^T \\ &= \eta(L_1\eta^{-1}L_2^T\eta)L_2\sqrt{d}\,\eta^{-1}L_1^T\eta(L_1\eta^{-1}L_2^T\eta)\eta^{-1} \end{aligned} \qquad (3.2.13)$$

which is exactly what was required to be proved, by defining $L = L_1\eta^{-1}L_2{}^T\eta$ and comparing (3.2.13) with the definition (3.2.6).

It can be seen that the expansion of some of the terms in the nonlinear action to two or more gravitons will be extremely complicated, and some form of simplification is desireable in these higher order cases. To facilitate this a truncation procedure is used. For bosonic vertices this is based on the observation that any four point amplitude, in both string theory and field theory, can be expanded as a polynomial in the Mandlestam variables[51,58,59] s, t and u, which are defined in Appendix Three. It is possible to choose the highest order term in these variables and use this term only in matching procedures between the string theory and field theory. This is clearly necessary. It is also believed that the choice of matching to the highest order term is sufficient due to reasons of gauge invariance of the amplitude and unitarity of the amplitudes. No explicit proof of the sufficiency will be provided here, as no proof is known to the author. So only terms in the Feynman vertices which can contribute to the highest order terms in this polynomial expansion are kept, and all other terms are discarded. Essentially the procedure is as follows: since it can be seen that all momentum terms in the truncated amplitudes are taken up into terms of the form $k_i.k_j$, (with the possible exception of one momentum left over in the case of fermionic variables: this case will be dealt with in detail later), then it can be deduced that the terms which give contributions to terms which contain contractions of the form $k_i.\zeta_j$ can be omitted when it comes to the construction of amplitudes for the matching procedure, since only the highest order terms are being considered. This reduces most of the vertices to a more manageable scale. It is now possible to derive only the parts of the Feynman vertices needed in the construction of these highest order terms in the amplitudes.

Calculation of Feynman Vertices.

Since only $\overline{\psi}$-ψ-h and $\overline{\psi}$-ψ-h-h scattering processes will be examined in any detail in what follows, the restriction will be made to the vertices which can possibly contribute to these. It can be seen that the only contributions come from the sub-action,

$$\mathfrak{L} = -\frac{1}{2} e\, R - \frac{1}{2} e\, \overline{\psi}_\mu \gamma^{\mu\nu\rho} \overleftrightarrow{\mathfrak{D}}_\nu \psi_\rho \qquad (3.2.14)$$

each of whose terms will be treated in turn below. The bosonic vertex and fermionic vertices will be treated seperately. The fact that no other fields, for example the antisymmetric tensor, do not contribute to the scattering amplitude will be discussed in detail in the section dealing with the calculation of the amplitudes.

i) Bosonic Sector.

It can be noted that the two standard definitions of the Riemann tensor, given in terms of the Riemann-Christoffel conection[49,50],

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\eta} \left\{ \partial_\nu g_{\eta\mu} + \partial_\mu g_{\nu\eta} - \partial_\eta g_{\mu\nu} \right\} \quad (A3.4.1)$$

and the spin connection[50],

$$\omega_\mu{}^{mn} = 2\, e^{\nu[m}\, \partial_{[\mu}\, e^{n]}{}_{\nu]} + \partial_{[\rho} e^{p}{}_{\sigma]}\, e_{p\mu}\, e^{\rho n}\, e^{\sigma m} \quad (A3.4.3)$$

can be shown to satisfy,

$$R_{\mu\nu}^{\ \ \rho\sigma}(\Gamma) = e^{\rho}_{\ m}\, e^{\sigma}_{\ n}\, R_{\mu\nu}^{\ \ mn}(\omega) \qquad (3.2.15)$$

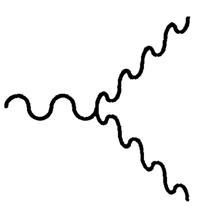
by a straightforward but tedious calculation. This implies that it is adequate to consider the simpler form of the Einstein-Hilbert action in terms of the Riemann-Christoffel connection when performing the background field expansion. Noting that the Riemann-Christoffel connection can be written,

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}\left(\eta^{\rho\eta} - h^{\rho\eta} + h^{\rho\xi}h_{\xi}^{\ \eta}\right).\left\{\partial_{\nu}h_{\eta\mu} + \partial_{\mu}h_{\nu\eta} - \partial_{\eta}h_{\mu\nu}\right\} \qquad (3.2.16)$$

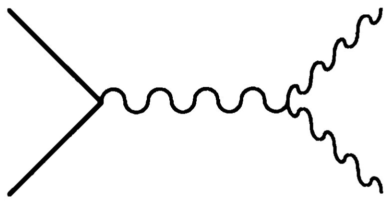
and that,

$$\sqrt{-g} = 1 + \frac{1}{2}\mathrm{tr}(h) - \frac{1}{4}\mathrm{tr}(h^2) + \frac{1}{8}(\mathrm{tr}(h))^2 + O(h^3) \qquad (3.2.17)$$

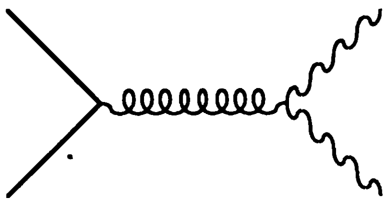
it is possible to expand the Einstein-Hilbert lagrangian to three gravitons. This is a tedious procedure in full generality, and will be easier if the vertex is derived with some regard to the amplitude it will be used to calculate. By this it is meant that if the amplitude is that of three graviton scattering, for example,



then it is easier to evaluate the appropriate vertex using the equations of motion to simplify continuously as the vertex is expanded. It is also useful to invoke the amplitude truncation procedure as the vertex is being expanded, in conjunction with the simplifications mentioned above. It is easiest to see this procedure in action, and so the two three graviton vertices will be derived and the simplifying conditions which are applied will be done so explicitly. The amplitudes that will be calculated are h-h-h all on-shell (as an introductory excercise) and h-h-h with one graviton off-shell, which contributes to the $\overline{\psi}$-ψ-h-h amplitude through diagrams of the form,



This is the only place where the pure graviton coupling is needed. It can also be seen by inspection of the action that no other bosonic field couplings of the diagrammatic form,



need be considered. There are several reasons for this:

i) $\underline{a_{\mu\nu}}$ field: There is no $O(\alpha'^0)$ a-h-h coupling as can be seen by examining the Chapline-Manton action. There does exist a $\overline{\psi}$-ψ-a coupling at the $O(\alpha'^0)$ level, which might give a contribution in $O(\alpha')$ diagrams. This point will be discussed in the next chapter, where it will be shown that there are again no $a_{\mu\nu}$ exchange diagrams.

ii) $\underline{\phi}$ field: From an inspection of the Einstein-Hilbert and Rarita-Schwinger parts of the full Chapline-Manton action, and by a consideration of the global scale covariance of the action, as detailed in reference [52], it can be seen that there are no h-h-ϕ or $\overline{\psi}$-ψ-ϕ couplings at $O(\alpha'^0)$.There are therefore no dilaton exchange diagrams in the evaluation of $O(\alpha'^0)$ or $O(\alpha')$ amplitudes.

The first term to consider is the three graviton coupling where all the fields are on shell, and where truncation is inappropriate. In this case the free field equation of motion can be applied to all of the fields in the Feynman vertex; that is if the equation of motion should arise naturally acting on one of the fields of the vertex, then that term in the expansion can be discarded. The justification for this is simple to see by considering a vertex of the form,

$$\overline{\psi}\overleftarrow{\partial} f(\gamma,h)\,\psi \qquad (3.2.18)$$

which can be replaced in the Green's function in equation (3.2.14) and where this form can be replaced in the LSZ formula,

$$\int dx \int dy\ e^{ik.x}\ \bar{u}(k) \overrightarrow{\partial}_x \langle 0|T(\psi(x)\ \bar{\psi}(y) \overleftarrow{\partial}_y\ )|0\rangle$$

where the Wick expansion has been implicitly performed with respect to an external fermion field, and the restriction to only the necessary fields for this discussion has been made. The propagator can be replaced in this expression to give the equation,

$$\int dx \int dy\ e^{ik.x}\ \bar{u}(k) \overrightarrow{\partial}_x G_\psi(x,y) \overleftarrow{\partial}_y \qquad (3.2.19)$$

and where the left derivative acting on the Green's function $G_\psi(x,y)$ gives,

$$\overrightarrow{\partial}_x G_\psi(x,y) = \delta(x-y) \qquad (3.2.20)$$

the right derivative can then be applied in turn to give,

$$\int dy\ e^{ik.y} \bar{u}(k) \overleftarrow{\partial}_y$$

$$= \int dy\ e^{ik.y} \bar{u}(k)\, k\!\!\!/ \ = 0 \qquad (3.2.21)$$

It can be seen from this derivation, which has been performed for the fermionic as opposed to bosonic case, which is completely analagous, that it is possible to ignore terms in the vertex which contain fields that are acted on by their appropriate kinetic operator, and which are connected to 'external' fields. This allows the final vertex to be written in the form,

$$\frac{1}{4}\ \{\ h^{\mu\alpha} h^{\sigma\rho}\ \partial_\sigma \partial_\alpha\, h_{\rho\mu} - \frac{1}{2} h^{\mu\alpha} h^{\sigma\rho}\ \partial_\mu \partial_\alpha\, h_{\rho\sigma}\ \} \qquad (3.2.22)$$

which enables the final answer for the amplitude to be calculated, which is,

$$\frac{i}{4}\{\ (\zeta_2^{\mu\alpha}\zeta_3^{\sigma\rho} + \zeta_3^{\mu\alpha}\zeta_2^{\sigma\rho})\zeta_{1\rho\mu}k_{1\sigma}k_{1\alpha} + \text{(cyclic perms of 1,2,3)}$$

$$- \frac{1}{2}(\zeta_2^{\mu\alpha}\zeta_3^{\sigma\rho} + \zeta_3^{\mu\alpha}\zeta_2^{\sigma\rho})\zeta_{1\rho\sigma}k_{1\mu}k_{1\alpha} + \text{(cyclic perms of 1,2,3)}\ \}$$

$$(3.2.23a)$$

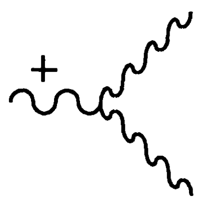
It can be seen that this corresponds to the string amplitude up to an external constant. Using the techniques described in Chapter One, the string amplitude can be shown to be,

$$\zeta_1^{\mu\nu}\zeta_2^{\rho\sigma}\zeta_3^{\alpha\beta}.(k_{1\sigma}\eta_{\beta\nu} + k_{2\beta}\eta_{\nu\sigma} + k_{3\nu}\eta_{\sigma\beta}) \times$$

$$\times (k_{1\rho}\eta_{\alpha\mu} + k_{2\alpha}\eta_{\mu\rho} + k_{3\mu}\eta_{\rho\alpha}) \qquad (3.2.23b)$$

These amplitudes will be used in the calculation of the γ factor mentioned above described in some detail in Appendix Five. They will be used to determine the normalisation of the string amplitudes used in that calculation.

The first example of the use of the truncation procedure will be the calculation of the vertex,



which will be necessary in the amplitude calculations later in this chapter and in the next chapter also. The procedure is similar to that used above. The same expansion of the Lagrangian to three gravitons as used in the derivation of equation (3.2.22) is used, and in each term of the expansion each graviton is taken off shell in turn, (this term being denoted by the underlined $\underline{h}$), and the truncation procedure applied to the remaining gravitons. This explicitly means eliminating all terms that contain an on-shell graviton contracted with <u>any</u> derivative as well as using the equation of motion when it occurs, for example,

$$\underline{h}_{\mu\nu}\partial^{\mu}h_{\alpha\beta}\partial^{\alpha}h^{\beta\nu}$$

would be eliminated, which can be shown to bring the vertex into the form,

$$-\frac{1}{4}\Big\{ -\frac{3}{4}\underline{h}^{\mu}{}_{\mu}\,\partial_{\rho}h^{\eta\xi}\partial^{\rho}h_{\eta\xi} + \underline{h}_{\rho}^{\sigma}\,\partial_{\mu}h^{\rho}_{\alpha}\,\partial^{\mu}h^{\alpha}_{\sigma}$$

$$+\frac{1}{2}\underline{h}_{\rho}^{\sigma}\,\partial_{\sigma}h^{\eta\xi}\,\partial^{\rho}h_{\eta\xi} + \underline{h}_{\rho}^{\sigma}\,h^{\eta\xi}\,\partial_{\sigma}\partial^{\rho}h_{\eta\xi}\Big\}$$

$$(3.2.24)$$

which is the form that will be used in the amplitude calculations below. The standard weight one convention has been broken here so as to agree with the work of reference [51]. It is now possible to consider the fermionic terms. It should be noted that this violation of the weight one convention is an isolated occurence.

ii) Fermionic Vertices.

In this subsection the same procedures used in the bosonic vertices will be applied to the case of the Rarita-Schwinger action, when the background field expansion is applied to it, and the three point and four point vertices calculated.

The massless Rarita-Schwinger action is given by considerations of local gauge invariance as stated in Chapter Two and explicitly in references [40,43], and can be written in the general covariant form,

$$\mathcal{L}_\psi = -\frac{1}{2}\overline{\psi}_\mu \gamma^{\mu\nu\rho} \overleftrightarrow{\mathcal{D}}_\nu \psi_\rho$$

where it is assumed that the ψ's are Majorana-Weyl and are quantised as in the previous chapter, and where the covariant derivative is given by the definition[50],

$$\mathcal{D}_\mu \psi_\rho = \partial_\mu \psi_\rho + \Gamma_\mu \psi_\rho - \Gamma^\lambda_{\mu\rho} \psi_\lambda \qquad (3.2.25)$$

and when replaced in the action term reduces to the form,

$$\mathcal{D}_\mu \psi_\rho = \partial_\mu \psi_\rho + \frac{1}{4}\gamma_m \gamma_n \psi_\rho \omega_\mu{}^{mn} \qquad (3.2.26)$$

by the skewsymmetry of the gamma matrix factor, and where the definition of the spinor connection has been used. It should be noted that the gamma matrices are defined to satisfy the Clifford algebra,

$$\{ \gamma_\mu , \gamma_\nu \} = 2\, g_{\mu\nu} \qquad (A3.1.8)$$

which will be important in the background field expansion discussed below. As can be seen above, the spin connection Γ_ρ is defined to be,

$$\Gamma_\rho = \frac{1}{4}\gamma_m\gamma_n\, \omega_\rho{}^{mn} \qquad (A3.4.14)$$

which is fully consistent with the use of the Riemann-Christoffel connection in the definition of the full covariant derivative defined above, and where the gamma matrices γ_m and γ_n are flat space gamma matrices which satisfy the flat space algebra,

$$\{ \gamma_m , \gamma_n \} = 2 \eta_{mn}$$

It should be noted that this definition of the covariant derivative of a gravitino field loses the rather nice 'formlike' geometric feel to the action, which has become increasingly popular recently. The notation of introducing an x-dependence to the gamma matrices will be used as a method of keeping track of the graviton expansion of the gamma matrices as in equations (3.2.28-29) below. The convention will be that a gamma matrix which has an explicit x-dependence will contain gravitons implicitly, as implied by the curved space form of the Clifford algebra, (A3.1.8), and no explicit x-dependence will imply that the background field expansion will have been performed on that particular gamma matrix. Thus it is possible to write out the full form of the action,

$$\mathfrak{L}_\psi = -\frac{1}{2} \{ \overline{\psi}_\mu \gamma^\mu(x)\gamma^\nu(x)\gamma^\rho(x) \overleftrightarrow{\mathfrak{D}}_\nu \psi_\rho - \overline{\psi}_\mu \gamma^\mu(x) \overleftrightarrow{\mathfrak{D}}_\nu \psi_\rho \, g^{\nu\rho}$$
$$+ \overline{\psi}_\mu \gamma^\nu(x) \overleftrightarrow{\mathfrak{D}}_\nu \psi_\rho \, g^{\mu\rho} - \overline{\psi}_\mu \gamma^\rho(x) \overleftrightarrow{\mathfrak{D}}_\nu \psi_\rho \, g^{\mu\nu} \} \quad (3.2.27)$$

As stated above, the introduction of the x-dependence of the gamma matrices is used to record the status of the gamma matrices with respect to the background field expansion, since by the right hand side of equation (A3.1.8) there must be gravitons associated with the gamma matrices. The symmetry of the Clifford algebra implies that the background field expansion of the gamma matrices is,

$$\gamma_\mu(x) = \gamma_\mu + \frac{1}{2}\gamma_m h^m_\mu - \frac{1}{8}\gamma_m h^m_n h^n_\mu + O(h^3) \quad (3.2.28)$$

and the obvious dual definition is therefore,

$$\gamma^\mu(x) = \gamma^\mu - \frac{1}{2}\gamma^m h^\mu_m + \frac{3}{8}\gamma^m h^n_m h^\mu_n + O(h^3) \quad (3.2.29)$$

which will be the most useful definition in what follows.

It is now possible to perform the background field expansion on the action as given in equation (3.2.24). All the necessary expansion definitions are given above.

As in the bosonic case it will be easier to apply a cumulative simplification procedure as a derivation proceeds. This will allow the selective 'pruning' of terms which vanish because of the gauge conditions or the equations of motion, or even the application of some truncation procedure. The truncation condition which can be seen to be the most convenient, and which is consistent with the bosonic truncation, will be a generalisation of the one used in the bosonic case above. This will be discussed in the section dealing with the construction of Feynman vertices for the evaluation of Born type graphs. For the moment, however, it will be adequate to consider the equations of motion and gauge conditions. The gauge conditions used here are defined by the gauge fixing term,

$$\mathcal{L}_{g.f.} = \frac{1}{4}\bar{\psi}_\mu \gamma^\mu \overleftrightarrow{\partial\!\!\!/} \gamma^\rho \psi_\rho \qquad (3.2.30)$$

which can be seen to be different to the conditions used in the canonical quantisation procedure. This is not a cause for worry due to the manifest gauge invariance of the theory, allowing an arbitrary choice of gauge to be made and the propagator used being modified accordingly as in reference [42]. The gauge choices used above will be the easiest for the calculation of Feynman diagrams later in this chapter.

The first vertex which will be calculated is the three point vertex which will be used in the calculation of the three point $\bar{\psi}$-ψ-h scattering. In the case where all the fields in the vertex "connect" directly to external fields, in the manner of the LSZ formula, then it can be seen (by a tedious application of the LSZ formalism, as in the case of the graviton above) that all the fields in the vertex which are acted upon by their equations of motion, or by their gauge conditions, can be neglected. This allows great simplifications to be made. It is possible to eliminate all terms which contain contributions of the form,

$$\bar{\psi}\overleftarrow{\partial\!\!\!/} \text{ and } \partial\!\!\!/\psi$$

as well as all terms containing the terms,

$$\gamma.\psi\ ,\ \bar{\psi}.\gamma\ ,\ \partial.\psi \text{ and } \bar{\psi}.\overleftarrow{\partial}$$

It can be seen that the final form for the vertex is,

$$\frac{1}{4}\{\, \bar{\psi}_\mu \gamma^\sigma \partial_\rho \psi^\mu h^\rho{}_\sigma - \bar{\psi}^\rho \gamma^\sigma \partial_\rho \psi_\mu h^\mu{}_\sigma - \bar{\psi}^\rho \gamma^\sigma \psi_\mu \partial^\mu h_{\sigma\rho} \,\}$$

(3.2.31a)

which can be recast in a more cyclically symmetric form,

$$\frac{1}{4}\{\bar{\psi}_\mu \gamma^\sigma \partial_\rho \psi^\mu h^\rho{}_\sigma + \bar{\psi}^\rho \gamma^\sigma \psi_\mu \partial_\rho h^\mu{}_\sigma + \partial^\mu \bar{\psi}^\rho \gamma^\sigma \psi_\mu h_{\sigma\rho} \,\}$$

(3.2.31b)

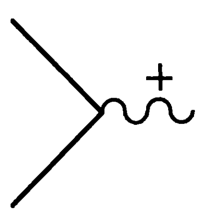
by some partial integrations. The remaining vertices all require a discussion of the truncation procedure. The procedure used in the fermionic case is very similar to the procedure in the calculation of bosonic amplitudes. It should be noted that the spinor fields have dimension $^1/_2$ and consequently there must always be one fewer momentum contribution to any fermionic amplitude containing two fermions as compared to a similar amplitude where the two fermions are replaced by bosonic fields. The $\bar{\psi}$-ψ-h-h scattering amplitude will only have one free momentum contribution, as compared to h-h-h-h having two. So the expansion of the amplitude as a polynomial (of at best rational functions) of the Mandlestam variables is complicated. However it can always be seen that one momentum in any amplitude always occurs in a contraction of the form,

$$\bar{u} \ldots \not{k} \ldots u\, \zeta\, \zeta$$

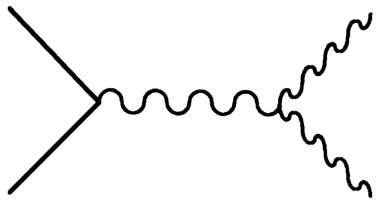
which enables the truncation procedure to be modified. The convention is that all contributions to the amplitude that contain terms where a polarisation tensor or spinor contracts with *either* an *associated* or *non-associated* momentum are discarded. For example, the matching of the four point amplitude contribution,

$$\bar{u}_2^\mu \gamma^\alpha (\not{k}_4 - \not{k}_3) \gamma^\gamma u_1^\nu \zeta_3^{\alpha\rho} \zeta_{4\rho}{}^{\gamma} k_{1\mu} k_{2\nu}$$

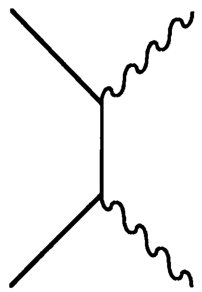
is discarded, since it does not contribute any more information than the highest order terms in the Mandlestam expansion. This truncation procedure will now be applied in the calculation of the remaining three point vertices. These are i) the graviton off-shell vertex, denoted diagrammatically by,



which is used in the construction of diagrams of the form,



and ii) the $\overline{\psi}$-ψ off-shell vertices, where the $\overline{\psi}$ and ψ fields are taken off-shell separately, and where the technique of vertex symmetrisation is employed. These vertices are used in the construction of diagrams of the form,



Each of these vertices will be considered in turn. i) The graviton off-shell vertex is simplified greatly by the observation that both the $\overline{\psi}$ and ψ fields are on-shell and their equations of motion can be invoked to provide great simplifications. In fact it can be seen that the only remaining contributions from equation (3.2.27) are,

$$-\frac{1}{2}\left\{ \overline{\psi}_\mu\gamma^\nu(x)\overleftrightarrow{\partial}_\nu\psi_\rho\, g^{\mu\rho} + \frac{1}{4}\overline{\psi}_\mu\,\gamma^\nu\gamma_m\gamma_n\psi^\mu\,\partial^{[n}\,h_\nu{}^{m]} \right\} \tag{3.2.32}$$

which can be simplified further by employing truncation on the $\overline{\psi}$ and ψ fields. The remaining contribution is,

$$\frac{1}{8}\left\{ \overline{\psi}_\mu\gamma^n\partial_\nu\,\psi^\mu - \partial_\nu\overline{\psi}_\mu\,\gamma^n\,\psi^\mu \right\} \underline{h}_n{}^\nu \tag{3.2.33}$$

which has been skewsymmetrised for the calculation of amplitudes in the next section. ii) The remaining term to consider is the $\overline{\psi}$-ψ off-shell vertex, which is the most difficult so far. The vertex contribution can be simplified most by first invoking the truncation procedure on the graviton contribution, as well as the equation of motion and gauge conditions. This gives a 'pre-vertex' of the form,

$$\frac{1}{8}\{ (2\,\overline{\psi}_\mu\gamma^\mu\gamma^\rho\gamma_n\psi_m - \overline{\psi}_\mu\gamma^\mu\gamma_m\gamma_n\psi^\rho - \overline{\psi}^\rho\gamma_m\gamma_n\gamma^\sigma\psi_\sigma$$

$$- 4\,\overline{\psi}^\rho\gamma_{[n}\psi_{m]})\,\partial^n h_\rho{}^m$$

$$+ 2\,\overline{\psi}_\mu\gamma^m\gamma^\rho\gamma^\sigma\partial_\rho\psi_\sigma h_m{}^\mu + 2\,\overline{\psi}_\mu\gamma^\mu\gamma^\rho\gamma^s\partial_\rho\psi_\sigma h^\sigma{}_s$$

$$- 2\,\overline{\psi}_\mu\gamma^m\partial_\rho\psi^\rho h_m{}^\mu - 2\,\overline{\psi}^\rho\gamma^s\partial_\rho\psi_\sigma h^\sigma{}_s + 4\,\overline{\psi}_\mu\gamma^\rho\partial_\rho\psi_\sigma h^{\mu\sigma} \}$$

(3.2.34)

which can now be used to construct actual vertices by first symmetrising the vertex, and then choosing which of $\overline{\psi}$ or ψ will be on-shell and applying the equations of motion, gauge conditions and truncation condition on it. When this is done the vertex takes the form,

$$\frac{1}{4}\{ \frac{1}{2}\overline{\psi}_\mu\gamma^\alpha\gamma^\rho\gamma^\sigma\partial_\rho\psi_\sigma + \partial^\sigma\overline{\psi}_\mu\,\gamma^\alpha\psi_\sigma + \overline{\psi}_\mu\,\gamma^\rho\partial_\rho\psi^\alpha \}.h_\alpha{}^\mu$$

(3.2.35)

when ψ is taken to be off-shell, and where the other vertex for $\overline{\psi}$ off-shell is simply the conjugate,

$$-\frac{1}{4}\{ \frac{1}{2}\partial_\rho\overline{\psi}_\sigma\,\gamma^\sigma\gamma^\rho\gamma^\alpha\psi_\mu + \overline{\psi}_\sigma\,\gamma^\alpha\partial^\sigma\,\psi_\mu + \partial_\rho\overline{\psi}_\alpha\,\gamma^\rho\,\psi_\mu \}h_\alpha{}^\mu$$

(3.2.36)

There remains only one more vertex to consider. This is the four point vertex $\overline{\psi}$-ψ-h-h, where all fields are on-shell, and where all reduction conditions can be freely applied to all fields. The Rarita-Schwinger action term can be expanded to two gravitons in the following manner,

$$\overline{\psi}_\mu\,(\gamma^{\mu\nu\rho}(x))|_{2h}\overleftrightarrow{\partial}_\nu\psi_\rho + \overline{\psi}_\mu\,\gamma^{\mu\nu\rho}(\overleftrightarrow{\mathfrak{D}}_\nu)|_{2h}\psi_\rho$$

$$+ \overline{\psi}_\mu\,(\gamma^{\mu\nu\rho}(x))|_h\,(\overleftrightarrow{\mathfrak{D}}_\nu)|_h\psi_\rho \qquad (3.2.37)$$

where it can be seen that each of these terms simplifies greatly after consideration of the equations of motion, gauge conditions and partial integration. Each term shall be considered in turn; each term will be examined in a little detail. The first term can be expanded to the form,

$$\overline{\psi}_\mu \gamma^\mu(x)\gamma^\rho(x)\gamma^\sigma(x)\partial_\rho\psi_\sigma - \overline{\psi}_\mu\gamma^\mu(x)\partial_\rho\psi_\sigma g^{\rho\sigma}$$

$$+ \overline{\psi}_\mu\gamma^\rho(x)\partial_\rho\psi_\sigma g^{\mu\sigma} - \overline{\psi}_\mu\gamma^\sigma(x)\partial_\rho\psi_\sigma g^{\mu\rho} \qquad (3.2.38)$$

where the two graviton fields must be taken out of the gamma matrices, which gives no contribution to the vertex, due to the truncation condition and the equations of motion. The second term can be expanded,

$$\frac{1}{4}\overline{\psi}^\sigma\gamma^\rho\gamma_m\gamma_n\psi_\sigma\,(-\frac{1}{4}h^{\eta[m}\partial_\rho h^{n]}{}_\eta + \frac{1}{2}h^{\eta[n}\partial^{m]}h_{\eta\rho})$$

$$-\overline{\psi}^\rho\gamma_{[n}\psi_{m]}\cdot\frac{1}{2}h^{\eta n}\partial^m h_{\eta\rho} \qquad (3.2.39)$$

which gives the vertex contribution,

$$\{\frac{1}{16}\overline{\psi}^\sigma\gamma^\rho\gamma_m\gamma_n\psi_\sigma h^{\eta m}\partial_\rho h_\eta{}^n + \frac{1}{4}\overline{\psi}^\rho\gamma_m\psi_n h^{n\eta}\partial^m h_{\eta\rho}\} \qquad (3.2.40)$$

It only remains to analyse the third, the most difficult and final term,

$$\frac{1}{4}\{\overline{\psi}_\mu\gamma^\mu(x)\gamma^\rho(x)\gamma^\sigma(x)\gamma_m\gamma_n\psi_\sigma - \overline{\psi}_\mu\gamma^\mu(x)\gamma_m\gamma_n\psi_\sigma g^{\rho\sigma}$$

$$+ \overline{\psi}_\mu\gamma^\rho(x)\gamma_m\gamma_n\psi_\sigma g^{\mu\sigma} - \overline{\psi}_\mu\gamma^\sigma(x)\gamma_m\gamma_n\psi_\sigma g^{\mu\rho}\}\,\partial^n h_\rho{}^m \qquad (3.2.41)$$

which reduces to the form,

$$\{-\frac{1}{2}\overline{\psi}_\mu\gamma^\rho\gamma^\rho\gamma_n\psi_m h_\rho{}^\mu + \frac{1}{8}\overline{\psi}_\mu\gamma^\rho\gamma_m\gamma_n\psi^\rho h_\rho{}^\mu - \frac{1}{8}\overline{\psi}_\mu\gamma^r\gamma_m\gamma_n\psi^\mu h_r{}^\rho$$

$$+ \frac{1}{8}\overline{\psi}^\rho\gamma^s\gamma_m\gamma_n\psi_\sigma h^\sigma{}_s + \frac{1}{2}\overline{\psi}_\mu\gamma_n\psi_m h^{\mu\rho}\}\,\partial^n h_\rho{}^m \qquad (3.2.42)$$

which is the contribution to the vertex. The total vertex contribution for the $\overline{\psi}$-ψ-h-h vertex, obtained by summing all the contributions above is therefore,

$$-\frac{1}{16}\overline{\psi}^\sigma\gamma^\rho\gamma_m\gamma_n\psi_\sigma\, h^{m\eta}\partial_\rho h_\eta{}^n \qquad (3.2.43)$$

which now must be symmetrised, and the overall action constant of -1/2 taken into account, to enable the construction of the $\bar{\psi}$-ψ-h and $\bar{\psi}$-ψ-h-h amplitudes. This will carried out in the next section.

Section 3.3 The calculation of the amplitudes

The Feynman rules evaluated rather tediously in the previous two sections will now be used in conjunction with the propagators described in the previous chapter to construct the amplitudes described in the introduction above. This will be done using the techniques outlined in the general discussion of Majorana field theory given above in Section 3.1. The 'corrected' combinatorial factors for the individual Feynman diagrams will be used, and the diagrams calculated. These amplitudes can then be compared with the string theory answers calculated in Chapter One. The first amplitude is for the $\bar{\psi}$-ψ-h three point amplitude, and which can be seen to be given by the Green's function,

$$< 0 \,|\, T(\bar{\psi}_1 \psi_2 h_3 \bar{\psi}^1 f^1 \psi^1 h^1) \,|\, 0 > \qquad (3.3.1)$$

which yields the amplitude,

$$- \frac{i}{2} . \{ \bar{u}_2^{\mu} \gamma^{\sigma} u_1^{\nu} \zeta_{3\sigma}^{\ \ \rho} (k_{1\rho} \eta_{\mu\nu} + k_{3\mu} \eta_{\nu\rho} + k_{2\nu} \eta_{\rho\mu}) \} \qquad (3.3.2)$$

after the application of Wick's theorem. It can be seen that this is identical to the string answer derived in Chapter One up to an external constant. This amplitude will be used in the normalisation of the extended O(α') amplitudes calculated in the next few chapters.

The $\bar{\psi}$-ψ-h-h four point amplitude is a little trickier to describe. It is given by the Green's function,

$$<0 |\, T(\bar{\psi}_1 \psi_2 h_3 h_4 \exp (i \int dx\, \mathcal{L}_{int})) \,|0> \qquad (3.3.3)$$

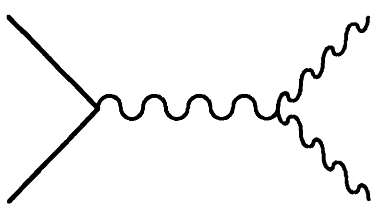
which expands to the sum,

$$
\begin{aligned}
& i \langle 0| T(\bar{\psi}_1 \psi_2 h_3 h_4 \bar{\psi}^1 g^1 \psi^1 h^1 h^1) |0\rangle \\
& - \frac{1}{2} \langle 0| T(\bar{\psi}_1 \psi_2 h_3 h_4 \{ \bar{\psi}^1 f^1 \psi^1 h^1 + h^1 h^1 h^1 p^1 \} \times \\
& \qquad \times \{ \bar{\psi}^1 f^1 \psi^1 h^1 + h^1 h^1 h^1 p^1 \} |0\rangle
\end{aligned}
\qquad (3.3.4)
$$

where integrations etc. have been left implicit for clarity. Each of these terms expands to give a set of diagrams, which obey the rules described in Section 3.1 above. The use of the rules yields the amplitude as a sum of contributions, denoted diagrammatically in Figure 3.1, each diagram of which can be constructed out of the Feynman rules given above. These amplitude contributions will be listed in turn. These individual contributions repeatedly use identities to be found in Appendix Three.

i) graviton exchange diagrams.

There is only one such diagram that need be calculated,



which gives amplitude contribution,

$$\frac{i}{32} \left(\frac{u-t}{s}\right) \bar{u}_2^{\rho} \not{k}_4 u_{1\rho} \zeta_3^{\eta\xi} \zeta_{4\eta\xi} \qquad (3.3.5)$$

All other diagrams of the same form come from interchange of indices.

ii) gravitino exchange diagrams.

Here again there is only one such term that needs to be considered,

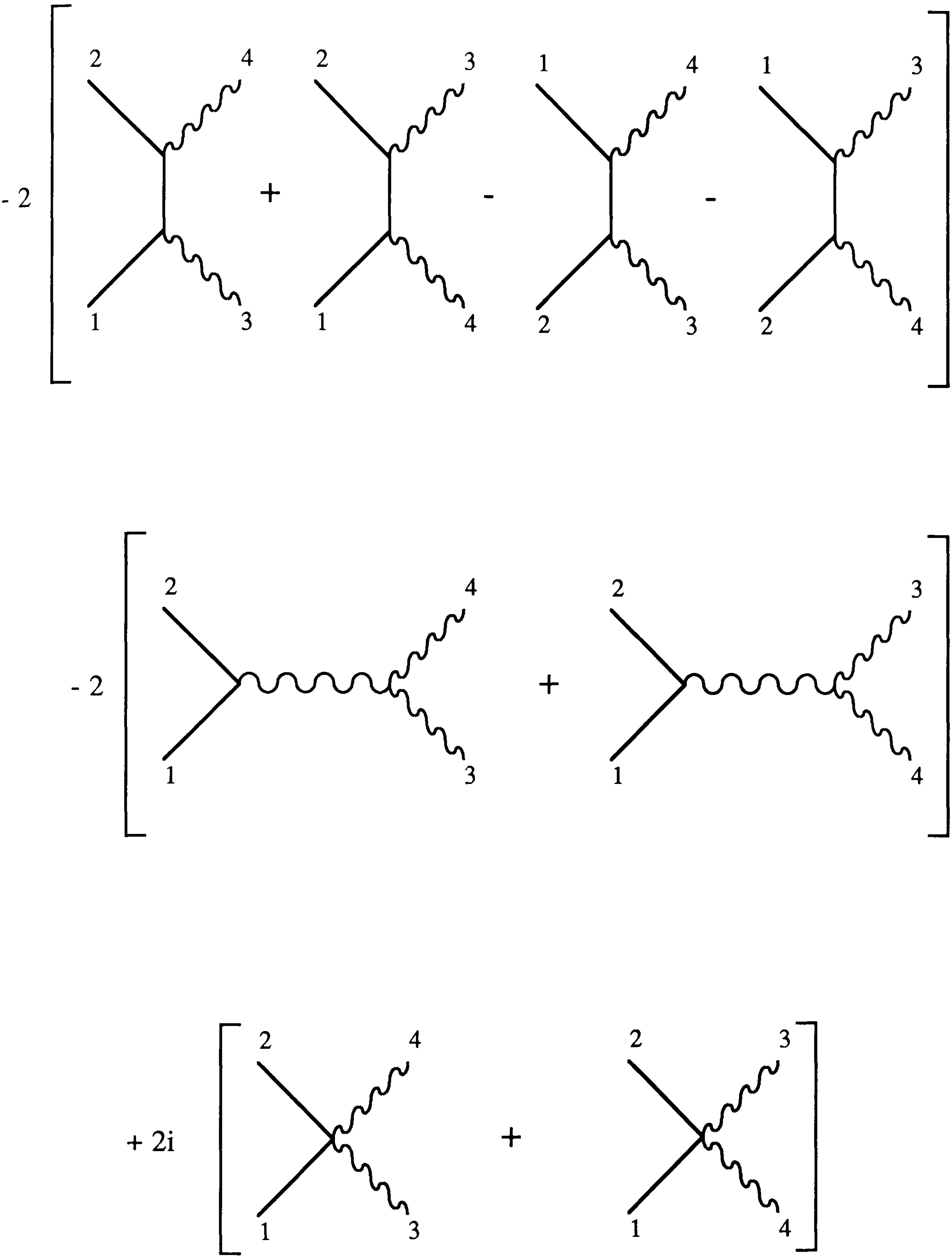
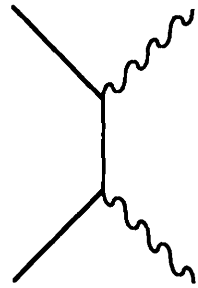


Figure 3.1: The Feynman graph scheme for the calculation of the $\overline{\Psi}$-ψ-h-h $O(\alpha'^0)$ amplitude.



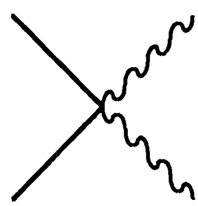
which gives the amplitude contribution,

$$-\frac{i}{32}\frac{s}{t}\bar{u}_{2\mu}\gamma^{\alpha}k\!\!\!/_{4}\gamma^{a}u_{1m}\zeta_{3a}{}^{m}\zeta_{4\alpha}{}^{\mu}+\frac{i}{16}\bar{u}_{2\mu}k\!\!\!/_{4}u_{1m}\zeta_{3}^{ms}\zeta_{4s}{}^{\mu} \qquad (3.3.6)$$

where again all the other diagrams of the same form come simply from interchange of indices and the application of the spinor conjugation rules.

iii) point diagrams.

Again there is only one contribution that need be considered. This is obviously,



which gives the amplitude contribution,

$$-\frac{1}{32}\bar{u}_{2}^{\sigma}\gamma_{m}k\!\!\!/_{4}\gamma_{n}u_{1\sigma}\zeta_{3}^{m\eta}\zeta_{4\eta}{}^{n} \qquad (3.3.7)$$

These separate contributions sum, with respect to the scheme given above, to give the amplitude,

$$- \frac{i}{16} \{ [\frac{u}{s} \bar{u}_2^{\sigma} \gamma_n (k_4 - k_3) \gamma_m u_{1\sigma} - \frac{t}{s} \bar{u}_2^{\sigma} \gamma_n (k_4 - k_3) \gamma_m u_{1\sigma}] \zeta_3^{m\eta} \zeta_{4\eta}^{\ \ n}$$

$$+ [\frac{u}{t} \bar{u}_{2\mu} \gamma^{\alpha} (k_4 - k_3) \gamma^{a} u_{1m} - \frac{t}{u} \bar{u}_{2m} \gamma^{a} (k_4 - k_3) \gamma^{\alpha} u_{1\mu}] \zeta_{3a}^{\ \ m} \zeta_{4\alpha}^{\ \ \mu}$$

$$+ [\bar{u}_{2m} \gamma^{\alpha} (k_4 - k_3) \gamma^{a} u_{1\mu} - \bar{u}_{2\mu} \gamma^{a} (k_4 - k_3) \gamma^{\alpha} u_{1m}] \zeta_{3a}^{\ \ m} \zeta_{4\alpha}^{\ \ \mu} \} \tag{3.3.8}$$

where the insertion of the gamma matrix identity,

$$\eta_{mn} = \frac{1}{2} \{ \gamma_m \gamma_n + \gamma_n \gamma_m \}$$

has been used. This is clearly allowed within the constraint of the truncation procedure. It can be seen furthermore that this amplitude is identical to the truncated string amplitude up to another overall constant. This amplitude will also be used to normalise the corresponding higher derivative extended amplitude, in the amplitude matching conditions used in the next two chapters.

Chapter Four.
The Interacting Field Theory II: The Extended Chapline-Manton Action and Supersymmetry.

Introduction.

In this chapter the supersymmetry of the Chapline-Manton action will be partially regained by the addition of various higher derivative terms to the action and then the corresponding $O(\alpha')$ contributions to the amplitudes will be calculated and these compared to the string $O(\alpha')$ amplitude terms. This will hopefully be able to single out one of the two candidate actions discussed below. There are two well known supersymmetric extensions to the Chapline-Manton action. These are the actions of reference [52], (hereafter known as the Romans and Warner action), and the action of reference [53], (henceforth known as the Han et al. action). Each of these will be treated in turn and their amplitudes compared with the string to try to determine which, if either, can be regarded as the low energy effective field theory limit of the string.

Each of these actions has its distinguishing characteristics, which will be discussed at some length. It is in fact postulated by the authors of reference [52] that there exists some form of singular field redefinition which relates the two actions. This matter will not be discussed at any length in this chapter, though the subject of field redefinitions in the context of these actions will be discussed in a later chapter.

The procedure of Romans and Warner[52] was to apply a Noether technique to the problem of deriving what the extra terms should be to correct the superymmetry of the Lagrangian. It is first noted that the addition of the Lorentz Chern-Simons three form term to the Chapline-Manton Lagrangian in the form given above in Chapter Three gives an anomaly when the supersymmetry transformations are applied to the action. The anomaly comes from two distinct areas: the addition of the Lorentz Chern-Simons three form to the supersymmetry transformations of the ψ_μ field and λ field, and by the variation of the Lorentz Chern-Simons three form in the $G_{\alpha\beta\gamma}$ terms in the action itself. It is necessary to introduce a small correction to the $a_{\mu\nu}$ supersymmetry transformation,

$$\delta a_{\mu\nu} = -2\sqrt{2}\,\gamma \left\{ (\delta\omega_{[\mu}^{\ mn})\, \omega_{\nu]}^{\ mn} \right\}$$

to restore the Lorentz covariance of the supersymmetry variation of $G_{\alpha\beta\gamma}$. Up to quadratic order in fermion fields the supersymmetry anomaly is easy to calculate and can be seen to be,

$$\delta\mathfrak{L} = - e\,\gamma \left\{ \phi^{-3/2} G^{\alpha\beta\gamma} \delta\omega_\alpha^{\ ab} R_{\beta\gamma\eta\xi} e^\eta_{\ a} e^\xi_{\ b} \right.$$

$$+ \frac{1}{16}\, \phi^{-3/4} \bar{\varepsilon}\gamma^{\mu\nu\eta\xi\rho} \psi_\rho R_{\mu\nu}^{\ \ \alpha\beta} R_{\eta\xi\alpha\beta}$$

$$\left. + \frac{1}{16\sqrt{2}}\, \phi^{-3/4}\, \bar{\varepsilon}\gamma^{\mu\nu\eta\xi}\lambda\, R_{\mu\nu}^{\ \ \alpha\beta} R_{\rho\sigma\alpha\beta} \right\}$$

It is now necessary to postulate a set of corrections to the action, and corrections to the supersymmetry transformations which will eliminate this anomaly. Romans and Warner attempt to achieve the much more modest goal of cancelling the second two of these anomalous terms. They do this by adding general terms to the action of the form, (only some illustrative examples are given here),

$$\mathfrak{L}' = \frac{1}{2} e\, \phi^{-3/4} \left\{ x_1 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + x_2 R_{\mu\nu} R^{\mu\nu} + x_3 R^2 \right\}$$

$$\mathfrak{L}'' = - \frac{3}{2} e\, \phi^{-3/4}\, \bar{\psi}_{[\mu} \gamma^{\rho\sigma\alpha} \mathfrak{D}_\nu \psi_{\alpha]} \left\{ b_1 R^{\mu\nu}_{\ \ \rho\sigma} + b_2\, R^{[\sigma}_{\ [\nu}\, \delta^{\rho]}_{\mu]} + b_3\, R\, \delta^{\rho}_{[\mu} \delta^{\sigma}_{\nu]} \right\}$$

and by correcting the supersymmetry transformations, by the addition of terms of the form, (again only an example is shown),

$$\delta\psi_\mu = \phi^{-3/4} \left\{ c_1 \gamma^{\rho\sigma} \varepsilon \mathfrak{D}_\rho R_{\sigma\mu} + c_2 \gamma_{\rho\mu} \varepsilon\, \mathfrak{D}^\rho R + c_3 \varepsilon\, \mathfrak{D}_\mu R \right\}$$

and then by varying the corrected action under the new transformations to yield a set of constraints on the general coefficients introduced. (Note that the conversion to the notation of the rest of the thesis has been made in the expressions above). The supersymmetric action is thus found up to the aforementioned higher order fermion terms, (which, it must be stressed, <u>cannot</u> affect any of the amplitudes calculated below), and up to terms linear in the $G_{\alpha\beta\gamma}$ tensor, (which also do not affect the amplitude matching calculations below). It is found when the cancellation is performed that there need not be any corrections to the supersymmetry transformations except for the additions mentioned above. It is important to note that the Romans and Warner fermionic corrections to the action as shown above, are chosen to maintain manifest supercovariance of the variations of the action, where the complete skewsymmetrisation of the lower indices of the example term is enforced. As will be noted below, when the calculation of amplitudes is performed, it is seen that the Romans and Warner action has a particularly nice form, where the quadratics of Riemann tensors, Ricci tensors and curvature scalars enter in the Gauss Bonnet combination, and it is also found that there is no need to introduce any pure higher derivative gravitino or dilatino terms, which would violate the manifest unitarity of the action. This is not so for the Han et al. solution to the same problem: they are forced to include such terms to cancel $\mathcal{D}_\mu\varepsilon$ type terms which arise in their higher derivative variation.

The Han et al solution to the cancellation of the supersymmetry anomaly given above is very similar in outline, but very different in structure to the Romans and Warner attempt given above. The Han et al attempt is similar to that of Romans and Warner in that the same Noether technique is used with the same constraints placed on the proposed cancellation, that is only the second two terms of the anomaly will be cancelled, and that the higher order fermion terms will be ignored. There is, however, a major difference in the Han et al procedure, in the respect that instead of using a general set of correction terms in both the action and the supersymmetry transformations, (up to a level of generality specified in advance), a restricted set of additional terms are used in the Lagrangian, and an analogy is drawn to the better defined Yang-Mills case, where the action and supersymmetry transformations look quite similar to the supergravity transformations of Chapline and Manton under the identification,

$$\begin{aligned} A_\mu{}^I &\leftrightarrow \omega_\mu{}^{ab} \\ F_{\mu\nu}{}^I &\leftrightarrow R_{\mu\nu}{}^{ab} \\ \chi^I &\leftrightarrow \psi_{ab} := 2\, e^\mu{}_a e^\nu{}_b\, \mathcal{D}_{[\mu}\psi_{\nu]} \end{aligned}$$

The restriction of choosing only fermionic corrections to the action which maintain the manifest supercovariance in the same manner as Romans and Warner is now abandoned. This however means that a pure higher derivative gravitino term of the form,

$$\mathcal{L}' = e\,\phi^{-3/4}\,\overline{\psi}^{ab}\gamma^\mu\mathcal{D}_\mu\psi_{ab}$$

has to be introduced to cancel off the $\mathcal{D}_\mu\varepsilon$ terms which now arise due to the variation of the term of the form,

$$\mathcal{L}'' = -\,e\,\phi^{-3/4}\,\overline{\psi}_\mu\gamma^{\rho\sigma}\gamma^\mu\psi_{ab}\,R_{\rho\sigma}{}^{ab}$$

which is introduced to cancel off the first of the last two terms in the supersymmetry anomaly above. The same procedure of varying the action and solving the resulting linear equations in the general coefficients can now be followed. Doing this yields an action, (see (4.2.1) below for a version of this action which is equivalent up to some four fermion terms due to the torsion induced by the fermion fields of the theory, and which does not contain the dilatino term which is necessary to cancel thethird term of the supersymmetry anomaly), which has not only a Riemann tensor squared without the Ricci tensor and curvature scalar combinations that make the graviton propagator manifestly unitary[54], but also a pure higher derivative, *propagator correcting* term for the gravitino in the Lagrangian. This must raise doubts about the validity of considering this as a candidate for the low energy action for the heterotic string. This point will be considered again when the subject of field redefinitions are raised in Chapter Six. The question of whether these two actions are in fact identical up to field redefinitions will be considered at the same time. Each of these two actions will be examined in turn using the amplitude matching technique to determine whether they are possible candidates for the low energy effective action of the heterotic superstring.

Section 4.1: The Romans and Warner Supersymmetric Action.

As can be seen above the Romans and Warner action is considerably more complicated than the Chapline-Manton action that it supercedes: it is clear that there are a considerable number of extra terms which have to been added to even partially retrieve the supersymmetry of the action. It must be emphasised that the action which results after the addition of these terms is not completely supersymmetric under the original supersymmetry transformations of Chapline and Manton. The Romans and Warner corrections to the gravitational sector of the corrected Chapline-Manton action are, (ignoring both the dilatino terms and quartic fermion terms, which will never contribute to any amplitudes which will be calculated below);

$$\mathcal{L}' = \gamma\Big[-\frac{3}{2} e\, \phi^{-\frac{3}{4}}\, \overline{\psi}_{[\rho}\, \gamma^{\mu\nu\alpha}\, \mathcal{D}_\sigma \psi_{\alpha]} \left\{ R_{\mu\nu}{}^{\rho\sigma} - 2\, R_{[\nu}{}^{[\sigma}\delta_{\mu]}{}^{\rho]} + \frac{1}{3} R\, \delta_{[\nu}^{\sigma}\, \delta_{\mu]}^{\rho} \right\}$$
$$+ \frac{1}{2} e\, \phi^{-\frac{3}{4}} \left\{ R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4\, R_{\mu\nu} R^{\mu\nu} + R^2 \right\}\Big] \qquad (4.1.1)$$

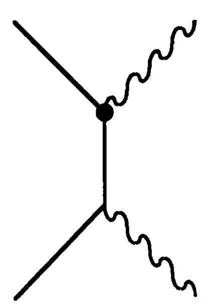
and the full action is supersymmetric under the old Chapline-Manton supersymmetry transformations, except for the $a_{\mu\nu}$ field whose supersymmetry transformation must be modified by the addition of a term of the form,

$$\delta a_{\mu\nu} = -2\sqrt{2}\, \gamma \left\{ (\delta\omega_{[\mu}{}^{mn})\, \omega_{\nu]}{}^{mn} \right\} \qquad (4.1.2)$$

where $\delta\omega_\mu{}^{mn}$ is the supersymmetry variation of the spin connection. The γ factor is the same γ factor as defined in the $G_{\alpha\beta\gamma}$ term in the corrected Chapline-Manton action in the introduction to Chapter Three. This constant will be suppressed in all the calculations below, particularly those of the Feynman vertex factors. It will be reintroduced at the amplitude matching stage, as will the dimensionful α' constant from the string theory, which will be reintroduced with the closed string value of $\alpha' = 2$ to be consistent with the notation of Chapter One.

It is important to stress that only the correction terms to the action and supersymmetry transformations that are pertinent to the amplitude calculations which will be considered in the remainder of this chapter have been kept, the remaining terms having been neglected as they cannot contribute to the conclusions of the amplitude matching analysis. These amplitudes are as stated in Chapter Three above: the three point $\overline{\psi}$-ψ-h amplitude, and the four point $\overline{\psi}$-ψ-h-h amplitude, where the amplitudes calculated will be of higher order in momenta than the amplitudes calculated in Chapter Three. The introduction of the higher derivative corrections allows the calculation of amplitudes which are higher order in momenta, and which suggest a classification scheme which is consistent with the string amplitude expansion discussed in chapter one. The dimensionful parameter α' is introduced, and the amplitudes are denoted by their order in this parameter. For a suitable choice of value for α' it can be seen that it can be identified with the string parameter with the same name. The amplitudes which will be evaluated in this chapter will simply be the $O(\alpha')$ counterparts to the fermionic amplitudes calculated in Chapter Three.

The perturbation theory arguments used in Chapter Three can be directly modified to account for the extra higher order (in α') terms in the background field expanded interaction Lagrangian. The only difference will be in the treatment of diagrams of the form,



where there will be extra diagrams due to the new vertices, which will modify the Feynman combinatoric factors in an amplitude calculation. The only thing that has to be done is the evaluation of the Feynman vertices required for the calculation of the amplitudes discussed above. This means that the O(α') vertices should be classified and evaluated. There is also the problem of the introduction of higher derivative terms of the form,

$$\mathcal{L}_{g.b.} = \frac{1}{2} e \, \phi^{-\frac{3}{4}} \left\{ R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \right\}$$

(4.1.3)

which naïvely might be thought to give higher derivative quadratic corrections to the graviton action that would introduce propagator correction terms. These extra poles in the propagator might be thought to violate the unitarity of the theory, but it has been shown that the combination given above does not correct the propagator[54], and consequently does not break the manifest unitarity of the theory. The possibility of general coefficients in the term (4.1.3) is discussed in Appendix Two, and will be required in the discussion of the Han et al. action in the next section. There are no pure higher derivative propagator correcting terms in the spin-3/2 sector of this action. Terms of higher derviative, propagator correcting, form exist however in the Han et al action. The procedure that has to be followed when these propagator correcting terms exist will be discussed in the section dealing with the Han et al. action.

The Feynman vertex factors should now be evaluated, and as in Chapter Three, this will be done with respect to the eventual amplitude calculations that will be performed. The first vertex that must be considered is the $\bar{\psi}$-ψ-h vertex used in the calculation of the $\bar{\psi}$-ψ-h amplitude. Here, as in Chapter Three, all the fields are on-shell and so the equations of motion and gauge conditions for the graviton and gravitino fields can be employed where appropriate. This will allow considerable simplifications in this case. The only term which can contribute to this vertex can be seen to be, setting the dilaton field to zero, (which will be done implicitly from now on),

$$-\frac{3}{2} \bar{\psi}_{[\rho} \gamma^{\mu\nu\alpha} \mathcal{D}_{\sigma} \psi_{\alpha]} R_{\mu\nu}{}^{\rho\sigma} \qquad (4.1.4)$$

since the other terms are functions of the Ricci tensor or curvature scalar, and cannot give any contribution due to the equations of motion of the graviton or the graviton gauge conditions. Explicitly the Riemann tensor expands to one graviton in the form,

$$R_{\mu\nu}{}^{\rho\sigma}|_h = -2\,\partial_{[\nu}\partial^{[\sigma}\,h_{\mu]}{}^{\rho]} + O(h^2) \qquad (4.1.5)$$

which can be shown to mean that the Ricci tensor can be written to one graviton in the form,

$$R_\nu{}^\rho|_h = R_{\mu\nu}{}^{\rho\mu}|_h = -\frac{1}{2}\left\{\partial_\nu\partial^\mu h_\mu{}^\rho - \partial_\nu\partial^\rho h_\mu{}^\mu - \partial_\mu\partial^\mu h_\nu{}^\rho + \partial_\mu\partial^\rho h_\nu{}^\mu\right\}$$

$$(4.1.6)$$

and the curvature scalar,

$$R|_h = R_\nu{}^\nu|_h = \left\{\partial_\nu\partial^\nu h_\mu{}^\mu - \partial_\nu\partial^\mu h_\mu{}^\nu\right\} \qquad (4.1.7)$$

Both of the terms above vanish when the graviton is on-shell because of the gauge condition and equation of motion of the graviton, and so the terms in the Lagrangian which depend on the Ricci tensor and the curvature scalar cannot contribute to any single graviton vertex where the graviton is on-shell. This observation will also be useful in the evaluation of four point $\overline{\psi}$-ψ-h-h vertices below. Thus the $\overline{\psi}$-ψ-h vertex is wholly from (4.1.4), leading to,

$$-\frac{3}{2}\overline{\psi}_{[\rho}\gamma^{\mu\nu\alpha}\partial_\sigma\psi_{\alpha]}\,.\,-2\,\partial_\nu\partial^\sigma h_\mu{}^\rho \qquad (4.1.8)$$

and the expansion of the fermionic term and subsequent application of gauge conditions, equations of motion and so on lead to the vertex in the form, where the γ dependence from (4.1.1) has been reintroduced, and the α' dependance has been introduced by inserting a factor of $\alpha'/2$ to agree with the conventional (implicit) closed string value of $\alpha'=2$, as discussed in Chapter One, and which agrees with the procedure developed in reference [33], (and agrees with the implicit convention of references [52,53,58]),

$$\frac{\gamma\alpha'}{2}\overline{\psi}_\sigma\gamma^\mu\partial_\rho\psi_\alpha\partial^\alpha\partial^\sigma h_\mu{}^\rho \qquad (4.1.9)$$

from which the amplitude can be directly calculated to be,

$$i\,\gamma\,\alpha'\,\bar{u}_{2\sigma}\gamma^\mu u_{1\alpha}\zeta_{3\mu\rho}k_1^\rho k_2^\alpha k_3^\sigma \qquad (4.1.10)$$

This can now be compared to the string answer, where the normalisation is taken care of by the relative weight of the $O(\alpha'^0)$ and $O(\alpha')$ subamplitudes from the string calculation, and the absolute normalisation between the $O(\alpha'^0)$ string and field theory amplitudes. That is, the relative weight of the $O(\alpha'^0)$ and $O(\alpha')$ subamplitudes of the field theory and string theory must be the same. This means that the Romans and Warner action receives its first check as a possible effective field theory of the string. The string amplitude after being normalised with respect to the $O(\alpha'^0)$ field theoretical amplitudes is, (see equations (1.4.18) and (3.3.2)),

$$-i\frac{\alpha'}{4}\bar{u}_2^{\mu}\gamma^{\sigma}u_1^{\nu}\zeta_{3\sigma}^{\ \ \rho}\, k_{1\rho}k_{2\nu}k_{3\mu}$$

It can be seen that the amplitudes do indeed match, if $\gamma = -1/4$. The results of Appendix Five show that $\gamma = \pm 1/4$, and so the three point matching is completely consistent with one of these values. The choice of the Riemann tensor chosen here agrees with references [6,7,52,53] and indeed with reference[68] and the Ricci tensor agrees with the definition used in references [6,7,43] which is consistent with the choice of sign of the Einstein-Hilbert action and supersymmetry. However it is possible that the ROmans and Warner and Han et al actions both define the Ricci tensor with a relative sign to the above. This negative sign problem will be discussed further in the section dealing with the Han et al action and the amplitude matching calculations carried out at that point. It will then be found that a three point match will exist, but for the value for the coefficient $\gamma = 1/4$. The problem of the Ricci tensor redefinition will be dealt with in the final section of this chapter, and will be shown not to affect any of the conclusions drawn in the preceding sections.

It will be necessary to carry out a full four point amplitude matching calculation for the Romans and Warner action to see if the action can continue to match the four point $O(\alpha')$ amplitude (1.4.16) subject to the correct normalisation, and this is now done. As before the γ and α' coefficients will be suppressed throughout, since they only enter linearly in the final $O(\alpha')$ amplitudes calculated at the end of this section. The first thing to do is to calculate the apropriate Feynman rules. The vertices that have to be evaluated are;

i) the h-h-h vertex, $\underline{h}$ off-shell,
ii) the $\bar{\psi}$-ψ-h vertex, $\underline{h}$ off-shell,
iii) the $\bar{\psi}$-ψ-h vertex, $\bar{\psi}$ or ψ off-shell,
iv) the $\bar{\psi}$-ψ-h-h vertex, all on-shell,

where the truncation procedure adopted in Chapter Three will be adopted in this chapter also. This will not prove useful at the moment, but will be seen to provide a great simplification in the case of the Han et al. action. It will be discussed later that these are the only vertices that need to be calculated, due to the truncation procedure. Each of the vertices above will be considered in turn. The first term to consider is the three graviton vertex.

<u>i) the h-h-h vertex, h off-shell.</u>

In this case it is necessary to expand the bosonic higher derivative term in (4.1.1) to three gravitons. It is much simpler to do this in the higher derivative case than the lowest order action case given in Chapter Three above. This is because only the one and two graviton expansions of the Riemann tensor are required, as opposed to the full expansion to three gravitons in the case of the standard Einstein-Hilbert action. The appropriate expansions are as those in Chapter Three, particularly in equations (3.2.3-4), (3.2.16-17) and the definition of the Riemann tensor as in Appendix Three, and so it only remains to multiply these together in the appropriate combinations. The vertex is derived by taking each of the graviton fields off-shell in turn, and applying the standard simplification procedure. The standard truncation procedure can be applied, as well as applying the equations of motion and gauge conditions. Doing this gives a vertex of the form,

$$\frac{1}{2}\left\{ \frac{1}{2}\underline{h}_{\mu}{}^{\mu}\,\partial^{\sigma}\partial^{\nu}h^{\rho\alpha}\partial_{\sigma}\partial_{\nu}h_{\rho\alpha} - 2\,\underline{h}^{\sigma\nu}\partial_{\alpha}h^{\rho\mu}\partial^{\alpha}\partial_{\sigma}\partial_{\nu}h_{\rho\mu} \right.$$
$$\left. - 9\,\underline{h}_{\alpha}{}^{\mu}\,\partial^{\sigma}\partial^{\nu}h^{\rho\alpha}\partial_{\sigma}\partial_{\nu}h_{\rho\mu} + \underline{h}_{\mu\nu}\,\partial^{\mu}\partial_{\alpha}h_{\sigma}{}^{\rho}\partial^{\alpha}\partial^{\nu}h_{\rho}{}^{\sigma} \right\} \qquad (4.1.11)$$

which will be the form used in the amplitude calculations below. The fermionic vertices can now be dealt with.

<u>ii) the $\overline{\psi}$-ψ-h vertex, h off-shell.</u>

In this case each of the correction terms must be expanded to one graviton. Since the curvature scalar term is multiplied by the free gravitino Lagrangian it can be shown, by a simple application of the gravitino equations of motion, that this term can give no contribution. It is left only to consider the Riemann and Ricci tensor terms. The Riemann tensor term can be expanded to one graviton in the background field expansion,

$$\{ \overline{\psi}_\rho\gamma^{\mu\nu\alpha}\partial_{[\sigma}\psi_{\alpha]} + \overline{\psi}_\sigma\gamma^{\mu\nu\alpha}\partial_{[\alpha}\psi_{\rho]} + \overline{\psi}_\alpha\gamma^{\alpha\mu\nu}\partial_{[\rho}\psi_{\sigma]} \}\partial_\nu\partial^\sigma\underline{h}_\mu{}^\rho \tag{4.1.12}$$

where the only graviton can possibly come from the Riemann tensor, from the expansion (4.1.5). The equations of motion and gauge conditions can be applied, to the fermionic terms, leaving a term of the form,

$$\overline{\psi}^\nu\gamma^\mu\partial_{[\rho}\psi_{\sigma]}\partial_\nu\partial^\sigma\underline{h}_\mu{}^\rho \tag{4.1.13}$$

at which point the truncation procedure may also be applied, eliminating the entire contribution. Hence the Riemann term does not contribute to this vertex. The Ricci tensor term is the only term which can possibly contribute to the vertex. Again the expansion of the term to one graviton is of the form,

$$3\,\{ \overline{\psi}_{[\rho}\gamma^{\mu\nu\alpha}\,\partial_\sigma\psi_{\alpha]}\,R_\nu{}^\sigma\,\delta_\mu{}^\rho \}\,|_{\underline{h}} \tag{4.1.14}$$

where the skewsymmetrised fermion term can be expanded to the form,

$$\{ \overline{\psi}_\rho\gamma^{\rho\nu\alpha}\partial_{[\sigma}\psi_{\alpha]} + \overline{\psi}_\alpha\gamma^{\rho\nu\alpha}\partial_{[\rho}\psi_{\sigma]} \}\;R_\nu{}^\rho\,|_{\underline{h}} \tag{4.1.15}$$

thence substituting the expansion for the Ricci tensor given above yields,

$$\{ \overline{\psi}_\rho\gamma^{\rho\nu\alpha}\partial_{[\sigma}\psi_{\alpha]} + \overline{\psi}_\alpha\gamma^{\rho\nu\alpha}\partial_{[\rho}\psi_{\sigma]} \}\;\times$$
$$\times\,-\frac{1}{2}\,\{ \partial^\mu\partial_\nu\underline{h}^\sigma{}_\mu - \partial^\mu\partial_\mu\underline{h}^\sigma{}_\nu - \partial^\sigma\partial_\nu\underline{h}^\mu{}_\mu + \partial^\sigma\partial_\mu\underline{h}^\mu{}_\nu \} \tag{4.1.16}$$

and the application of partial integration, the equations of motion of the gravitino field, the gravitino gauge conditions and the truncation procedure gives the vertex in the form,

$$\frac{-1}{2}\,\{ \overline{\psi}_\alpha\gamma^\nu\partial_\sigma\psi^\alpha \}.\{ \partial^\sigma\partial_\mu\underline{h}^\mu{}_\nu - \partial^\mu\partial_\mu\underline{h}^\sigma{}_\nu \} \tag{4.1.17}$$

which can be symmetrised to comply with the formalism developed in Chapter Three for the perturbative evaluation of amplitudes in a Majorana fermion field theory. The final vertex is therefore,

$$\frac{1}{4}\{\partial_\sigma\overline{\psi}_\alpha\gamma^\nu\psi^\alpha - \overline{\psi}_\alpha\gamma^\nu\partial_\sigma\psi^\alpha\}.\{\partial^\sigma\partial_\mu\underline{h}^\mu{}_\nu - \partial^\mu\partial_\mu\underline{h}^\sigma{}_\nu\}$$

(4.1.18)

The remaining three point vertex can be calculated.

<u>iii) the $\overline{\psi}$-ψ-h vertex, $\overline{\psi}$ or ψ off-shell.</u>

The only term which can possibly contribute to this vertex is the Riemann tensor term, because of the on-shell property of the graviton. The equations of motion of the gravitino cannot be applied in this case, but truncation <u>is</u> allowed on the graviton, and some simplification is facilitated by the observation that $\overline{\psi}$ <u>and</u> ψ cannot both be off-shell simultaneously, which means that certain terms can be neglected. The full term is as in equation (4.1.8) above, and each of these three terms can be expanded in some detail. The first term is,

$$\overline{\psi}_\rho\gamma^{\mu\nu\alpha}\partial_{[\sigma}\psi_{\alpha]}\,\partial_\nu\partial^\sigma h_\mu{}^\rho$$

$$= \frac{1}{2}\overline{\psi}_\rho\gamma^\mu\gamma^\nu\gamma^\alpha\partial_\sigma\psi_\alpha\partial_\nu\partial^\sigma h_\mu{}^\rho - \frac{1}{2}\overline{\psi}_\rho\gamma^\mu\gamma^\nu\gamma^\alpha\partial_\alpha\psi_\sigma\,\partial_\nu\partial^\sigma h_\mu{}^\rho$$

$$+ \frac{1}{2}\overline{\psi}_\rho\gamma^\nu\partial_\sigma\psi_\alpha\,\partial_\nu\partial^\sigma h^{\alpha\rho} \qquad (4.1.19)$$

The second term gives the contribution,

$$\overline{\psi}_\sigma\gamma^{\mu\nu\alpha}\partial_{[\alpha}\psi_{\rho]}\,\partial_\nu\partial^\sigma h_\mu{}^\rho$$

$$= -\frac{1}{2}\overline{\psi}_\sigma\gamma^\mu\partial^\nu\psi_\rho\,\partial_\nu\partial^\sigma h_\mu{}^\rho \qquad (4.1.20)$$

and finally the third term gives,

$$\overline{\psi}_\alpha\gamma^{\mu\nu\alpha}\partial_{[\rho}\psi_{\sigma]}\,\partial_\nu\partial^\sigma h_\mu{}^\rho$$

$$= -\frac{1}{2}\overline{\psi}_\alpha\gamma^\alpha\gamma^\mu\gamma^\nu\partial_\sigma\psi_\rho\,\partial_\nu\partial^\sigma h_\mu{}^\rho + \frac{1}{2}\overline{\psi}^\mu\gamma^\nu\partial_\sigma\psi_\rho\,\partial_\nu\partial^\sigma h_\mu{}^\rho$$

$$-\frac{1}{2}\,\overline{\psi}^\nu\gamma^\mu\partial_\sigma\psi_\rho\,\partial_\nu\partial^\sigma h_\mu{}^\rho\,\} \qquad (4.1.21)$$

where each of these terms has been simplified as much as possible using the conditions mentioned above. The vertex must now be symmetrised subject to the formalism developed in Chapter Three, and $\overline{\psi}$ and ψ taken off-shell in turn, to give the vertices,

$$\{ \frac{1}{2} \overline{\psi}_\rho \gamma^\mu \gamma^\nu \gamma^\alpha \partial_\sigma \psi_\alpha + \frac{1}{2} \partial_\sigma \overline{\psi}_\rho \gamma^\mu \psi^\nu - \frac{1}{4} \overline{\psi}_\rho \gamma^\mu \gamma^\nu \gamma^\alpha \partial_\alpha \psi_\sigma$$

$$+ \frac{1}{2} \overline{\psi}^\mu \gamma^\nu \partial_\sigma \psi_\rho - \frac{1}{2} \partial_\sigma \overline{\psi}_\rho \gamma^\nu \psi^\mu \} \, \partial_\nu \partial^\sigma h_\mu{}^\rho \qquad (4.1.22a)$$

for the ψ field off-shell, and,

$$- \{ \frac{1}{2} \partial_\sigma \overline{\psi}_\alpha \gamma^\alpha \gamma^\nu \gamma^\mu \psi_\rho + \frac{1}{2} \overline{\psi}^\nu \gamma^\mu \partial_\sigma \psi_\rho - \frac{1}{4} \partial_\alpha \overline{\psi}_\sigma \gamma^\alpha \gamma^\nu \gamma^\mu \psi_\rho$$

$$+ \frac{1}{2} \partial_\sigma \overline{\psi}_\rho \gamma^\nu \psi^\mu - \frac{1}{2} \overline{\psi}^\mu \gamma^\nu \partial_\sigma \psi_\rho \} \partial_\nu \partial^\sigma h_\mu{}^\rho \qquad (4.1.22b)$$

for the $\overline{\psi}$ field, which is simply the conjugate of the ψ off-shell vertex. The final vertex is a little more complicated than either of the cases demonstrated above.

<u>iv) $\overline{\psi}$-ψ-h-h vertex, all on-shell.</u>

In this case all the fields are on-shell, and can be used in the iterative simplification procedure used above. It should be noted, in this case, that there is also the extra contribution from the Chern-Simons term which occurs in the Green-Schwarz modified form of the Chapline-Manton action, in the term,

$$\frac{\sqrt{2}}{16} \cdot G_{\alpha\beta\gamma} \{ \overline{\psi}_\mu \gamma^{\mu\alpha\beta\gamma\nu} \psi_\nu + 6 \, \overline{\psi}^\alpha \gamma^\beta \psi^\gamma \} \qquad (4.1.23)$$

where the $G_{\alpha\beta\gamma}$ term is defined to be, (continuing the suppresion of the γ and α' parameters),

$$G_{\alpha\beta\gamma} = \{ \partial_{[\alpha} a_{\beta\gamma]} + c \, \omega^{Y}_{3\alpha\beta\gamma} - \gamma \, \omega^{L}_{3\alpha\beta\gamma} \} \qquad (4.1.24)$$

and where the Lorentz Chern-Simons three form term is defined to be,

$$\omega^{L}_{3\alpha\beta\gamma} = \sqrt{2} \; \{ R_{[\alpha\beta}{}^{mn} \omega_{\gamma]}{}^{nm} - \frac{2}{3} \, \omega_{[\alpha}{}^{mn} \, \omega_{\beta}{}^{na} \, \omega_{\gamma]}{}^{am} \}$$

which gives a contribution to the $\overline{\psi}$-ψ-h-h vertex from the term,

$$-\frac{1}{8}\left\{R_{[\alpha\beta}{}^{mn}\,\omega_{\gamma]}{}^{nm}\right\}.\left\{\overline{\psi}_\mu\gamma^{\mu\alpha\beta\gamma\nu}\psi_\nu+6\,\overline{\psi}^\alpha\gamma^\beta\psi^\gamma\right\}$$

which is of the correct order in the α' expansion, and has the correct terms in the background field expansion. The case of the two graviton fermionic vertex is a little more complicated than the previous vertices discussed above, since the background field expansion of the gravitino part of some of the terms must be considered.However the fact that all fields are on-shell in this vertex provides a great simplifying observation: the Ricci tensor and curvature scalar can never contribute only a single graviton. The Ricci tensor and curvature scalar must contribute two gravitons in the expansion, if they contribute anything at all. Noting this and the observation used above that the curvature scalar term multiplies the free gravitino action, it can be seen easily that the curvature scalar term does not contribute to this vertex either. However the Ricci tensor term does contribute the term,

$$-\frac{3}{2}\overline{\psi}_{[\rho}\gamma^{\mu\nu\alpha}\,\mathfrak{D}_\sigma\psi_{\alpha]}\left\{-2\,R_\nu{}^\sigma\,\delta_\mu{}^\rho\right\}\Big|_{2h} \qquad (4.1.25)$$

where the Ricci tensor is expanded to two gravitons,

$$R_\beta{}^\alpha\Big|^T_{2h}=\Big\{-\frac{1}{2}h^{\sigma\eta}\partial_\beta\partial^\alpha h_{\eta\sigma}-\frac{1}{4}\partial^\alpha h_\sigma{}^\rho\partial_\beta h_\rho{}^\sigma$$
$$-\frac{1}{2}\partial_\sigma h^{\rho\alpha}\partial^\sigma h_{\rho\beta}\Big\} \qquad (4.1.26)$$

where the superscript T denotes that as much simplification has been made using the observation that in any case where this expansion is used both fields will be on-shell and so the truncation convention, equations of motion and gauge conditions can be applied freely. Using the standard expansion of the fermionic part of the term, the contribution from (4.1.25) can be seen to be of the form,

$$\overline{\psi}^\alpha\gamma^\nu\partial_\sigma\psi_\alpha\left\{-\frac{1}{2}h^{\eta\xi}\partial_\nu\partial^\sigma h_{\eta\xi}-\frac{1}{4}\partial^\sigma h^{\eta\xi}\partial_\nu h_{\eta\xi}\right\}$$
$$(4.1.27)$$

The Riemann tensor term will be expanded according to the scheme,

$$-\frac{3}{2}\overline{\psi}_{[\rho}\gamma^{\mu\nu\alpha}\mathfrak{D}_\sigma\psi_{\alpha]}\Big|_h R_{\mu\nu}{}^{\rho\sigma}\Big|_h-\frac{3}{2}\overline{\psi}_{[\rho}\gamma^{\mu\nu\alpha}\partial_\sigma\psi_{\alpha]}\,R_{\mu\nu}{}^{\rho\sigma}\Big|_{2h}$$
$$(4.1.28)$$

Each of these terms will be expanded in turn. The second term is the easiest to analyse, and so will be treated first. When the fermion term is expanded, this term gives,

$$- \{ \overline{\psi}_\rho \gamma^{[\nu} \partial_{[\sigma} \psi_{\alpha]} \eta^{\mu]\alpha} + \overline{\psi}_\sigma \gamma^{[\nu} \partial_{[\alpha} \psi_{\rho]} \eta^{\mu]\alpha} + \overline{\psi}^{[\nu} \gamma^{\mu]} \partial_{[\rho} \psi_{\sigma]} \} . R_{\mu\nu}{}^{\rho\sigma} |_{2h}$$

(4.1.29)

will require that the appropriate form of the Riemann tensor be expanded to two graviton fields. This is,

$$R^{\sigma\alpha}{}_{\beta\gamma} \Big|^{T}_{2h} = - \frac{1}{2} \{ [h^{\sigma\eta} \partial_\beta \partial^\alpha h_{\gamma\eta} - h^{\sigma\eta} \partial_\gamma \partial^\alpha h_{\beta\eta} - h^{\alpha\eta} \partial_\beta \partial^\sigma h_{\gamma\eta} + h^{\alpha\eta} \partial_\gamma \partial^\sigma h_{\beta\gamma}]$$

$$+ \frac{1}{2} [\partial^\alpha h_\beta{}^\rho + \partial_\beta h^{\rho\alpha} - \partial^\rho h^\alpha{}_\beta] . [\partial_\rho h_\gamma{}^\sigma - \partial_\gamma h^\sigma{}_\rho - \partial^\sigma h_{\rho\gamma}]$$

$$- \frac{1}{2} [\partial^\alpha h_\gamma{}^\rho + \partial_\gamma h^{\rho\alpha} - \partial^\rho h^\alpha{}_\gamma] . [\partial_\rho h_\beta{}^\sigma - \partial_\beta h^\sigma{}_\rho - \partial^\sigma h_{\rho\beta}] \}$$

(4.1.30)

which can be seen to give a contribution to the vertex in the form,

$$- \{ \overline{\psi}^\nu \gamma^\mu \partial_\rho \psi_\sigma \, h^{\sigma\eta} \partial_\mu \partial^\rho h_{\nu\eta} + \frac{1}{2} \overline{\psi}_\rho \gamma^{[\nu} \partial^{\mu]} \psi_\sigma \, \partial_\nu h^{\sigma\eta} \partial_\mu h_\eta{}^\rho$$

$$+ \frac{1}{4} \overline{\psi}_\rho \gamma^\mu \partial_\sigma \psi^\nu \, \partial^\rho h_\nu{}^\eta \partial_\mu h^\rho{}_\eta + \frac{1}{4} \overline{\psi}^\mu \gamma^\nu \partial_\rho \psi_\sigma \, \partial^\rho h_{\eta\mu} \partial_\nu h^{\eta\sigma} \}$$

after a somewhat tedious calculation.

The remaining term requires that the fermionic part of the term be expanded subject to the rules developed in Chapter Three above, and where the standard simplifications are invoked. The first thing to calculate is the background field expansion of the fermionic term,

$$\overline{\psi}_{[\rho} \gamma^{\mu\nu\alpha}(x) \mathfrak{D}_\sigma \psi_{\alpha]}$$

$$= \frac{1}{3} \{ \overline{\psi}_\rho \gamma^{\mu\nu\alpha}(x) \partial_{[\sigma} \psi_{\alpha]} + \frac{1}{4} \overline{\psi}_\rho \gamma^{\mu\nu\alpha} \gamma_m \gamma_n \psi_{[\alpha} \partial^n h_{\sigma]}{}^m$$

$$(+ \text{cyclic perms. of } \rho, \sigma, \text{and } \alpha) \}$$

(4.1.31)

where each of these terms can be expanded in turn, with special regard to the expansion of the Riemann tensor to one graviton. This is again a tedious business which will not be written out in detail, but an example of this calculation will be given before the full final contribution is stated. It can be noted that all the terms that contain five gamma matrices vanish by truncation with respect to the derivatives apparent in the one graviton background field expansion of the Riemann tensor. This leaves only three terms to evaluate, all of which use similar expansion techniques. The sample term chosen is,

$$\overline{\psi}_\rho \gamma^{\mu\nu\alpha}(x)\partial_{[\sigma}\psi_{\alpha]}\,\partial_\nu\partial^\sigma h_\mu{}^\rho$$

$$= \{\, \overline{\psi}_\rho \gamma^\mu(x)\gamma^\nu(x)\gamma^\alpha(x)\partial_{[\sigma}\psi_{\alpha]} - \overline{\psi}_\rho \gamma^\mu(x)\partial_{[\sigma}\psi_{\alpha]} g^{\nu\alpha}$$

$$+ \overline{\psi}_\rho \gamma^\nu(x)\partial_{[\sigma}\psi_{\alpha]} g^{\mu\alpha} \,\}\, \partial_\nu\partial^\sigma h_\mu{}^\rho \qquad (4.1.32)$$

where the gamma matrices have been written using the explicit spacetime dependence notation used in the previous chapter. Using the expansion stated there, and the usual truncation conditions this can immediately be contracted down to the form,

$$\{\, -\frac{1}{4}\overline{\psi}_\rho \gamma^\mu\gamma^\nu\gamma^a\partial_\sigma\psi_\alpha h^\alpha{}_a - \frac{1}{2}\overline{\psi}_\rho\gamma^\nu\partial_\sigma\psi_\alpha h^{\mu\alpha} \,\}\, \partial_\nu\partial^\sigma h_\mu{}^\rho \qquad (4.1.33)$$

Calculating the other terms, and summing gives the contribution to the $\overline{\psi}$-ψ-h-h vertex in the form from the mixed term,

$$\{\, -\frac{1}{4}\overline{\psi}_\rho \gamma^\mu\gamma^\nu\gamma^a\partial_\sigma\psi_\alpha h^\alpha{}_a + \frac{1}{4}\overline{\psi}_\alpha\gamma^a\gamma^\mu\gamma^\nu\partial_\sigma\psi_\rho\, h^\alpha{}_a$$

$$- \frac{1}{2}\overline{\psi}_\rho\gamma^\nu\partial_\sigma\psi_\alpha h^{\mu\alpha} - \frac{1}{2}\overline{\psi}_\alpha\gamma^\nu\partial_\sigma\psi_\rho h_\mu{}^\rho \,\}\, \partial_\nu\partial^\sigma h_\mu{}^\rho \qquad (4.1.34)$$

so the total vertex contribution from the Romans and Warner additions to the action is, after the standard simplifications,

$$\{\, -\frac{1}{4}\overline{\psi}_\rho \gamma^\mu\gamma^\nu\gamma^a\partial_\sigma\psi_\alpha h^\alpha{}_a + \frac{1}{4}\overline{\psi}_\alpha\gamma^a\gamma^\mu\gamma^\nu\partial_\sigma\psi_\rho h^\alpha{}_a - \frac{1}{2}\overline{\psi}_\alpha\gamma^\nu\partial_\sigma\psi_\rho h^{\mu\alpha}$$

$$- \overline{\psi}^\mu\gamma^\nu\partial_\sigma\psi_\alpha h_\rho{}^\alpha - \frac{1}{4}\overline{\psi}^\alpha\gamma^\nu\partial_\sigma\psi_\alpha h_\rho{}^\mu \,\}\, \partial_\nu\partial^\sigma h_\mu{}^\rho \qquad (4.1.35)$$

This vertex excludes any contribution from the Lorentz Chern-Simons term, which will be considered shortly. Thus the proper symmetrised vertex is clearly seen to be,

$$\{ -\frac{1}{8}\overline{\psi}_\rho\gamma^\mu\gamma^\nu\gamma^a\partial_\sigma\psi_\alpha h^\alpha{}_a + \frac{1}{8}\partial_\sigma\overline{\psi}_\alpha\gamma^a\gamma^\nu\gamma^\mu\psi_\rho h^\alpha{}_a$$

$$-\frac{1}{8}\overline{\psi}_\alpha\gamma^a\gamma^\nu\gamma^\mu\partial_\sigma\psi_\rho h^\alpha{}_a + \frac{1}{8}\partial_\sigma\overline{\psi}_\rho\gamma^\mu\gamma^\nu\gamma^a\psi_\alpha h^\alpha{}_a$$

$$-\frac{1}{4}\overline{\psi}_\alpha\gamma^\nu\partial_\sigma\psi_\rho h^{\mu\alpha} + \frac{1}{4}\partial_\sigma\overline{\psi}_\rho\gamma^\nu\psi_\alpha h^{\mu\alpha}$$

$$-\frac{1}{2}\overline{\psi}^\mu\gamma^\nu\partial_\sigma\psi_\alpha h^\alpha{}_\rho + \frac{1}{2}\partial_\sigma\overline{\psi}_\alpha\gamma^\nu\psi^\mu h^\alpha{}_\rho$$

$$-\frac{1}{8}\overline{\psi}^\alpha\gamma^\nu\partial_\sigma\psi_\alpha h_\rho{}^\mu + \frac{1}{8}\partial_\sigma\overline{\psi}_\alpha\gamma^\nu\psi^\alpha h_\rho{}^\mu \}\partial_\nu\partial^\sigma h_\mu{}^\rho \tag{4.1.36}$$

This only leaves the derivation of the $\overline{\psi}$-ψ-h-h vertex contribution due to the Lorentz Chern-Simons term. This is simple to evaluate using the identity for <u>on-shell</u> gravitino fields which satisfy the standard $\gamma.\psi$ gauge condition,

$$\overline{\psi}_\mu\gamma^{\mu\alpha\beta\gamma\nu}\psi_\nu = 6\,\overline{\psi}^{[\alpha}\gamma^\beta\psi^{\gamma]} + \overline{\psi}_\mu\gamma^{\alpha\beta\gamma}\psi^\mu \tag{4.1.37}$$

where an implicit external factor completely antisymmetric in the α,β,and γ indices is assumed. This allows the vertex contribution to be written in the form,

$$\frac{-1}{8}\ \{\overline{\psi}_\mu\gamma^\alpha\gamma^\beta\gamma^\gamma\psi^\mu + \overline{\psi}_\mu\gamma^\beta\psi^\mu\eta^{\alpha\gamma} + 4\,\overline{\psi}^\alpha\gamma^\beta\psi^\gamma\,\}\partial^n\partial_\beta h^m{}_\alpha\partial_n h_{\gamma m} \tag{4.1.38}$$

which may be skewsymmetrised and added to the $\overline{\psi}$-ψ-h-h vertex derived above, and can be seen to be of the form,

$$\frac{-1}{16}\ \{\overline{\psi}_\mu\gamma^\alpha\gamma^\beta\gamma^\gamma\psi^\mu - \overline{\psi}_\mu\gamma^\gamma\gamma^\beta\gamma^\alpha\psi^\mu$$

$$+ 4\,\overline{\psi}^\alpha\gamma^\beta\psi^\gamma - 4\,\overline{\psi}^\gamma\gamma^\beta\psi^\alpha\ \}\ \partial^n\partial_\beta h^m{}_\alpha\partial_n h_{\gamma n}$$

when symmetrised. It can be seen that the middle term of (4.1.38) has skewsymmetrised out completely.

It is possible to see that the $G_{\alpha\beta\gamma}$ term in (4.1.23) contains an $O(\alpha'^0)$ $\overline{\psi}$-ψ-a vertex with the antisymmetric tensor field a taken to be off-shell, *raising the* *possibility of an antisymmetric tensor exchange diagram contribution to $\bar{\psi}$-ψ-h-h scattering.* This *cannot* be the case, since symmetry considerations immediately imply that such a diagram can only contribute to terms which are neglected when the truncation procedure is applied. Explicitly this is very easy to see: it can be seen that the antisymmetric tensor propagator is of the form,

$$A_{\mu\nu,\alpha\beta}(k^2) = \frac{i}{k^2}\left(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha} \right)$$

which means that it is possible to rewrite all the $\bar{\psi}$-ψ parts of the diagram as a simple function which contains all the polarisation spinor dependence and k dependence, for example, $f_{\alpha\beta}(u_2,u_1,k)$ for example, where it can be seen that,

$$f_{\alpha\beta}(u_2,u_1,k) = - f_{\beta\alpha}(u_2,u_1,k)$$

because of the form of the antisymmetric tensor propagator above. Now because of the truncation convention there are only two possible combinations which the a-h-h vertex can possibly take. (Only one of these arises from the $G_{\alpha\beta\gamma}G^{\alpha\beta\gamma}$ term in the Lorentz Chern-Simons modified Chapline-Manton action, but the argument given below does not depend on the explicit form of the vertex.) These are,

$$\underline{a}_{\mu\nu}\partial^{\mu}\partial^{\alpha}h^{\rho\sigma}\partial^{\nu}\partial_{\alpha}h_{\rho\sigma}$$

and,

$$\underline{a}_{\mu\nu}\partial^{\beta}\partial^{\alpha}h^{\mu\sigma}\partial_{\beta}\partial_{\alpha}h_{\sigma}{}^{\nu}$$

When these are replaced in the Green's function for the antisymmetric tensor exchange diagram, it can be shown that the Wick expansion yields either the term,

$$\frac{s}{2}\left(f_{\alpha\beta} + f_{\beta\alpha}\right) k_3^{\alpha} k_4^{\beta}\, \zeta_3^{\eta\xi}\, \zeta_{4\xi\eta}$$

or the term,

$$\frac{s^2}{4}\left(f_{\alpha\beta} + f_{\beta\alpha}\right) \zeta_3^{\alpha\eta}\, \zeta_{4\eta}{}^{\beta}$$

both of which vanish by the symmetry properties of the function $f_{\alpha\beta}$.

By similar arguments to those given in Chapter Three it can be seen that there can be no spin-1/2 exchange or dilatino exchange diagrams, due to the absence of the appropriate $O(\alpha'^0)$ Feynman rules from the original Chapline-Manton action.

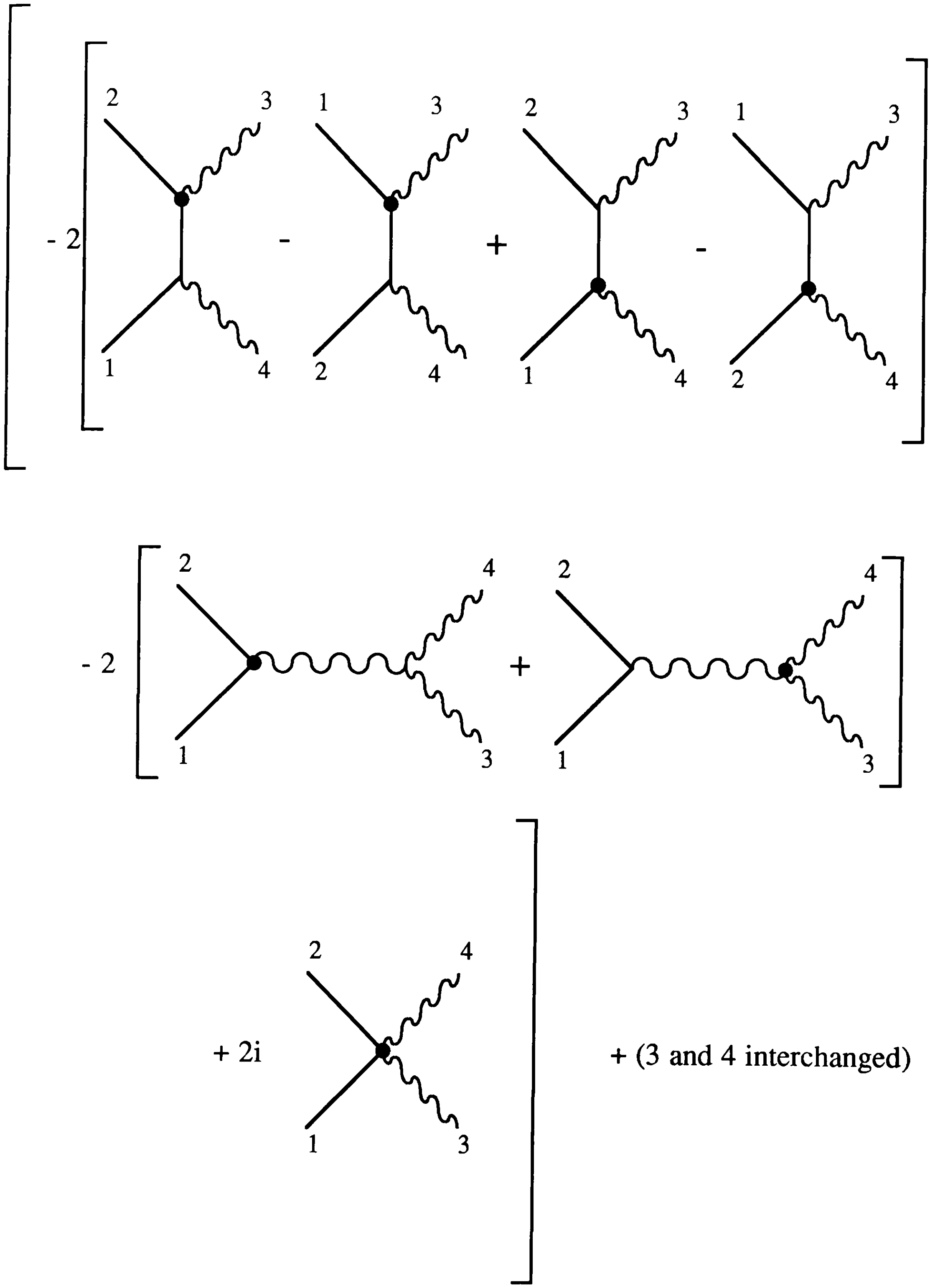
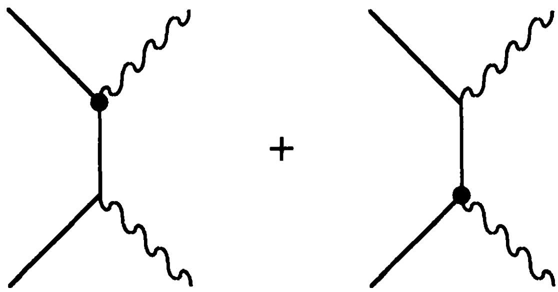


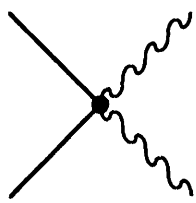
Figure 4.1: The Feynman diagram scheme for the calculation of the $\overline{\psi}$-ψ-h-h $O(\alpha')$ amplitude, Romans and Warner case.

This completes the list of vertices necessary for the calculation of the $\overline{\psi}$-ψ-h-h, O(α') amplitude. This will now be performed. The amplitude will be calculated using similar techniques to those developed for the lowest order Lagrangian in Chapter Three. The only modification that needs to be incorporated is the observation that there are now two $\overline{\psi}$-ψ-h vertices, with $\overline{\psi}$ or ψ off-shell, which will slightly modify the Feynman combinatorial factors. Diagrammatically it can be seen that there are now combinations of diagrams of the form,



which have to be included in the perturbative solution, where the dotted vertex denotes the O(α') vertex. The amplitude is therefore evaluated according to the scheme given in Figure 4.1, which means that each of these diagrams has to be evaluated seperately. Each diagram will be treated in turn below.

The first diagram will be the point diagram,



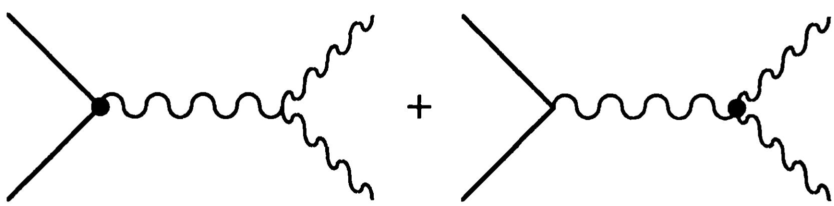
which can simply be evaluated by inspection of the appropriate vertex due to (4.1.36) and (4.1.38). The amplitude contribution is,

$$\Big\{ \frac{1}{8}\frac{(t-u)}{2}\bar{u}_{2\rho}\gamma^{\mu}\not{k}_4\gamma^{a}u_{1\alpha}\zeta_{3a}^{\alpha}\zeta_{4\mu}^{\rho} + \frac{1}{8}\frac{(t-u)}{2}\bar{u}_{2\alpha}\gamma^{a}\not{k}_4\gamma^{\mu}u_{1\rho}\zeta_{3a}^{\alpha}\zeta_{4\mu}^{\rho}$$
$$+\frac{1}{8}\frac{(t-u)}{2}\bar{u}_{2\alpha}\not{k}_4 u_1^{\alpha}\zeta_3^{\eta\xi}\zeta_{4\eta\xi}$$
$$+\frac{1}{4}\Big[\frac{t}{2}\bar{u}_{2\rho}\not{k}_4 u_{1\mu} - \frac{u}{2}\bar{u}_{2\mu}\not{k}_4 u_{1\rho}\Big]\zeta_3^{\mu\alpha}\zeta_{4\alpha}{}^{\rho}$$
$$+\frac{1}{4}\Big[t\,\bar{u}_{2\sigma}\not{k}_4 u_1^{\nu} - u\,\bar{u}_2^{\nu}\not{k}_4 u_{1\sigma}\Big]\zeta_3^{\sigma\eta}\zeta_{4\eta\nu}\Big\} \tag{4.1.39}$$

due to the Romans and Warner terms, and

$$-\frac{1}{8}\left\{ \frac{s}{2}\bar{u}_{2\mu}\gamma^{\alpha}k\!\!\!/_3\gamma^{\gamma}u_1^{\mu}\,\zeta_{3\alpha}{}^{m}\zeta_{4m\gamma} + s\left[\bar{u}_2^{\alpha}k\!\!\!/_3u_1^{\gamma} - \bar{u}_2^{\gamma}k\!\!\!/_3u_1^{\alpha}\right]\zeta_{3\alpha}{}^{m}\zeta_{4m\gamma}\right\} \qquad (4.1.40)$$

due to the Lorentz Chern-Simons three form term. These will be symmetrised with respect to the 3 and 4 indices according to the scheme given above when the full amplitude is evaluated. At this stage it is possible to see that the amplitude can be split into the two distinct types of subamplitude: terms where the $\bar{u}$ and u polarisation spinor indices contract together, which will be called 's'-channel terms, and the remaining terms, which will be given the name 't-u'-terms. It will be convenient to separate the amplitude into these two types of subamplitude when the comparison with the string amplitude for the same process is considered. The next easiest set of diagrams to consider is the set of graviton exchange diagrams,



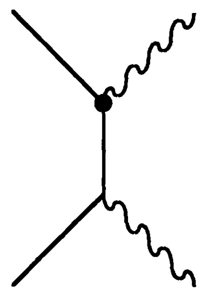
which can be simply evaluated, and yield the amplitude contribution,

$$\frac{i}{4}(u-t)\,\bar{u}_2^{\sigma}k\!\!\!/_4u_{1\sigma}\,\zeta_3^{\eta\xi}\zeta_{4\eta\xi} \qquad (4.1.41)$$

which can be seen to give entirely 's'-channel contributions in the notation used above. The gamma matrix algebra, equation (A3.1.8) can be used, in conjunction with the truncation procedure, to rewrite this amplitude contribution in the form,

$$\frac{i}{8}(t-u)\left\{ \bar{u}_2^{\sigma}\gamma^{\alpha}k\!\!\!/_4\gamma^{\gamma}u_{1\sigma} + \bar{u}_2^{\sigma}\gamma^{\gamma}k\!\!\!/_4\gamma^{\alpha}u_{1\sigma}\right\}\zeta_{3\alpha}{}^{\eta}\zeta_{4\eta\gamma} \qquad (4.1.42)$$

which corresponds more closely to the form in which the string amplitude occurs. The only diagrams that are left to calculate are the gravitino exchange diagrams, which are the most complicated to evaluate. It can be seen that as a consequence of the symmetrisation of vertices it is only necessary to consider one of the complete set of diagrams. The rest can be found simply by the spinor conjugation of the amplitude and relabelling of the polarisation states. The diagram calculated will be,



which requires the use of several gamma matrix identities. A list of these useful identities is presented in Appendix Three. The amplitude contribution can be split into four separate diagrams where each of these may be split into five sets of sums of three terms, each of which will be analysed in a little detail below. It is necessary only to consider one of the four diagrams, since the remaining diagrams can be derived by relabelling polarisation states, and using the spinor conjugation rules.

The analysis of this one diagram is done in a little detail since it will be shown below and in the next chapter, where more general action terms will be discussed, that the same types of amplitude contributions will arise on several occasions. It will therefore be useful to discuss these calculations once in a little detail for reference. The individual terms contained within this diagram are given by the multiplication of the five terms in the $O(\alpha')$ action with each of the three terms in the $O(\alpha'^0)$ vertex. Each of the $O(\alpha')$ terms will be treated in turn, where it will be multiplied into the full $O(\alpha'^0)$ vertex. The first of these terms is,

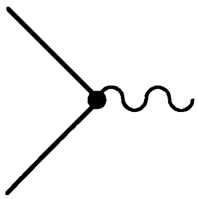
$$\frac{1}{2}\bar{\psi}_\rho\gamma^\mu\gamma^\nu\gamma^\alpha\partial_\sigma\psi_\alpha\partial_\nu\partial^\sigma h_\mu{}^\rho \qquad (4.1.43)$$

which can be seen to give zero contribution. (As an example of this type of calculation, this particular derivation will be included as Appendix Four) The next contribution is the term,

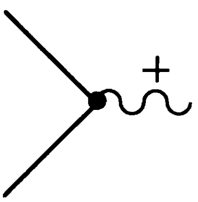
$$\frac{1}{2}\partial_\sigma\bar{\psi}_\rho\gamma^\mu\psi^\nu\partial_\nu\partial^\sigma h_\mu{}^\rho \qquad (4.1.44)$$

which by similar manipulations to the first term gives the contribution,

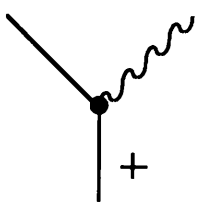
$$-\frac{i}{32}u\,\bar{u}_{2\rho}\gamma^\mu \not{k}_4\gamma^a u_{1m}\zeta_{3a}{}^m\zeta_{4\mu}{}^\rho \qquad (4.1.45)$$



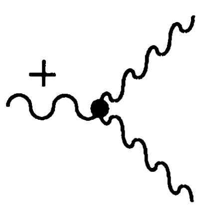
$$\frac{\gamma\alpha'}{2}\bar{\psi}_\sigma\gamma^\mu\partial_\rho\psi_\alpha\partial^\alpha\partial^\sigma h_\mu{}^\rho$$



$$\frac{\gamma\alpha'}{2}\ \Big[\ \frac{1}{4}\{\partial_\sigma\bar{\psi}_\alpha\gamma^\nu\psi^\alpha - \bar{\psi}_\alpha\gamma^\nu\partial_\sigma\psi^\alpha\ \}.\{\ \partial^\sigma\partial_\mu\underline{h}^\mu{}_\nu - \partial^\mu\partial_\mu\underline{h}^\sigma{}_\nu\ \}\ \Big]$$

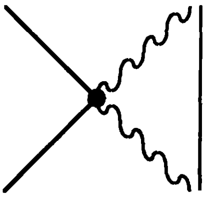


$$\frac{\gamma\alpha'}{2}\ \Big\{\ \frac{1}{2}\bar{\psi}_\rho\gamma^\mu\gamma^\nu\gamma^\alpha\partial_\sigma\psi_\alpha + \frac{1}{2}\partial_\sigma\bar{\psi}_\rho\gamma^\mu\psi^\nu - \frac{1}{4}\bar{\psi}_\rho\gamma^\mu\gamma^\nu\gamma^\alpha\partial_\alpha\psi_\sigma$$
$$+ \frac{1}{2}\bar{\psi}^\mu\gamma^\nu\partial_\sigma\psi_\rho - \frac{1}{2}\partial_\sigma\bar{\psi}_\rho\gamma^\nu\psi^\mu\ \Big\}\ \partial_\nu\partial^\sigma h_\mu{}^\rho$$



$$\frac{\gamma\alpha'}{2}\ \Big[\frac{1}{2}\Big\{\ \frac{1}{2}\underline{h}_\mu{}^\mu\,\partial^\sigma\partial^\nu h^{\rho\alpha}\partial_\sigma\partial_\nu h_{\rho\alpha} - 2\,\underline{h}^{\sigma\nu}\partial_\alpha h^{\rho\mu}\partial^\alpha\partial_\sigma\partial_\nu h_{\rho\mu}$$
$$- 9\,\underline{h}_\alpha{}^\mu\,\partial^\sigma\partial^\nu h^{\rho\alpha}\partial_\sigma\partial_\nu h_{\rho\mu} + \underline{h}_{\mu\nu}\,\partial^\mu\partial_\alpha h_\sigma{}^\rho\partial^\alpha\partial^\nu h_\rho{}^\sigma\ \Big\}\Big]$$

Fig. 4.2a The three point O(α') vertices from the Romans and Warner action.



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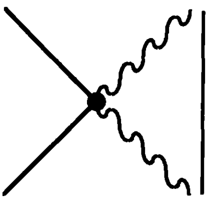
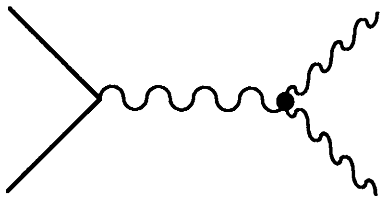
$$\begin{aligned}\frac{\gamma\alpha'}{2}\Big\{ &-\frac{1}{8}\overline{\psi}_\rho\gamma^\mu\gamma^\nu\gamma^a\partial_\sigma\psi_\alpha h^\alpha{}_a + \frac{1}{8}\partial_\sigma\overline{\psi}_\alpha\gamma^a\gamma^\nu\gamma^\mu\psi_\rho h^\alpha{}_a \\ &-\frac{1}{8}\overline{\psi}_\alpha\gamma^a\gamma^\nu\gamma^\mu\partial_\sigma\psi_\rho h^\alpha{}_a + \frac{1}{8}\partial_\sigma\overline{\psi}_\rho\gamma^\mu\gamma^\nu\gamma^a\psi_\alpha h^\alpha{}_a \\ &-\frac{1}{4}\overline{\psi}_\alpha\gamma^\nu\partial_\sigma\psi_\rho h^{\mu\alpha} + \frac{1}{4}\partial_\sigma\overline{\psi}_\rho\gamma^\nu\psi_\alpha h^{\mu\alpha} \\ &-\frac{1}{2}\overline{\psi}^\mu\gamma^\nu\partial_\sigma\psi_\alpha h^\alpha{}_\rho + \frac{1}{2}\partial_\sigma\overline{\psi}_\alpha\gamma^\nu\psi^\mu h^\alpha{}_\rho \\ &-\frac{1}{8}\overline{\psi}^\alpha\gamma^\nu\partial_\sigma\psi_\alpha h_\rho{}^\mu + \frac{1}{8}\partial_\sigma\overline{\psi}_\alpha\gamma^\nu\psi^\alpha h_\rho{}^\mu \Big\}\partial_\nu\partial^\sigma h_\mu{}^\rho\end{aligned}$$



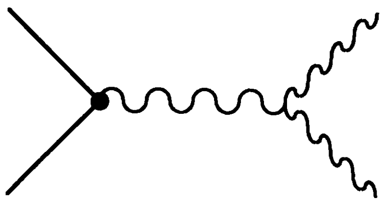
Lorentz Chern-Simons.

$$\begin{aligned}\frac{\gamma\alpha'}{32}\Big\{&\overline{\psi}_\mu\gamma^\alpha\gamma^\beta\gamma^\gamma\psi^\mu - \overline{\psi}_\mu\gamma^\gamma\gamma^\beta\gamma^\alpha\psi^\mu \\ &+ 4\,\overline{\psi}^\alpha\gamma^\beta\psi^\gamma - 4\,\overline{\psi}^\gamma\gamma^\beta\psi^\alpha\Big\}\,\partial^n\partial_\beta h^m{}_\alpha\partial_n h_{\gamma m}\end{aligned}$$

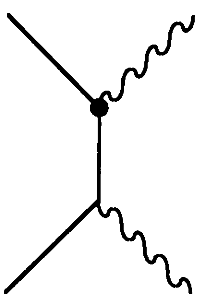
Fig. 4.2b The four point O(α') vertex from the Romans and Warner action, and the four point O(α) vertex due to the Lorentz Chern-Simons term.



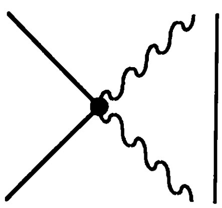
$$\frac{i\gamma\alpha'3}{16}(t-u)\,\bar{u}_2^{\alpha}(\not{k}_4-\not{k}_3)u_{1\alpha}\zeta_3^{\eta\xi}\zeta_{4\xi\eta}$$



$$\frac{i\gamma\alpha'}{16}(t-u)\,\bar{u}_2^{\alpha}(\not{k}_4-\not{k}_3)u_{1\alpha}\zeta_3^{\eta\xi}\zeta_{4\xi\eta}$$

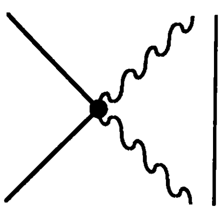


$$\frac{i\gamma\alpha'}{8}\Big\{\big(u\,\bar{u}_{2\rho}\gamma^{\mu}(\not{k}_4-\not{k}_3)\gamma^{a}u_{1m}-t\,\bar{u}_{2m}\gamma^{a}(\not{k}_4-\not{k}_3)\gamma^{\mu}u_{1\rho}\big)\zeta_{3a}{}^{m}\zeta_{4\mu}{}^{\rho}$$
$$-2\big(t\,\bar{u}_{2\beta}(\not{k}_4-\not{k}_3)u_{1m}-u\,\bar{u}_{2m}(\not{k}_4-\not{k}_3)u_{1\beta}\big)\zeta_3^{ms}\zeta_{4s}{}^{\beta}\Big\}$$



Pure R.W.

$$\frac{i\gamma\alpha'}{16}\Big\{(t-u)\big(\bar{u}_{2\rho}\gamma^{\mu}(\not{k}_4-\not{k}_3)\gamma^{a}u_{1\alpha}+\bar{u}_{2\alpha}\gamma^{a}(\not{k}_4-\not{k}_3)\gamma^{\mu}u_{1\rho}\big)\zeta_{3a}{}^{\alpha}\zeta_{4\mu}{}^{\rho}$$
$$+2\big(t\,\bar{u}_{2\rho}(\not{k}_4-\not{k}_3)u_{1\mu}-u\,\bar{u}_{2\mu}(\not{k}_4-\not{k}_3)u_{1\rho}\big)\zeta_3^{\mu\alpha}\zeta_{4\alpha}{}^{\rho}$$
$$+4\big(t\,\bar{u}_{2\sigma}(\not{k}_4-\not{k}_3)u_1^{\nu}-u\,\bar{u}_2^{\nu}(\not{k}_4-\not{k}_3)u_{1\sigma}\big)\zeta_3^{\sigma\eta}\zeta_{4\eta\nu}$$
$$+(t-u)\,\bar{u}_{2\alpha}(\not{k}_4-\not{k}_3)u_1^{\alpha}\zeta_3^{\eta\xi}\zeta_{4\eta\xi}\Big\}$$



Lorentz Chern-Simons

$$\frac{i\gamma\alpha's}{32}\Big\{\big(\bar{u}_{2\mu}\gamma^{\alpha}(\not{k}_4-\not{k}_3)\gamma^{\gamma}u_1^{\mu}-\bar{u}_{2\mu}\gamma^{\gamma}(\not{k}_4-\not{k}_3)\gamma^{\alpha}u_1^{\mu}\big)$$
$$+4\big(\bar{u}_2^{\alpha}(\not{k}_4-\not{k}_3)u_1^{\gamma}-\bar{u}_2^{\gamma}(\not{k}_4-\not{k}_3)u_1^{\alpha}\big)\Big\}\zeta_{3\alpha}{}^{m}\zeta_{4m\gamma}$$

Fig. 4.3 The summary of O(α') four point contributions to $\bar{\psi}$-ψ-h-h scattering from the Romans and Warner action. The full amplitude is merely the sum of these diagrams: *all* factors have been absorbed in the subamplitude contributions.

The next term is the second three gamma matrix term,

$$-\frac{1}{4}\bar{\psi}_\rho\gamma^\mu\gamma^\nu\gamma^\alpha\partial_\alpha\psi_\sigma\partial_\nu\partial^\sigma h_\mu{}^\rho \qquad (4.1.46)$$

which gives the contribution,

$$-\frac{i}{32}u\,\bar{u}_{2\rho}\gamma^\mu k\!\!\!/_4\gamma^a u_{1m}\zeta_{3a}{}^m\zeta_{4\mu}{}^\rho \qquad (4.1.47)$$

which is the same as the second term. The remaining two terms can be seen to be the same by partially integrating, and using the symmetry of the graviton, therefore it is necessary only to consider,

$$\bar{\psi}^\mu\gamma^\nu\partial_\sigma\psi_\rho\partial_\nu\partial^\sigma h_\mu{}^\rho \qquad (4.1.48)$$

which gives the final amplitude contribution,

$$\frac{i}{8}t\,\bar{u}_{2\beta}k\!\!\!/_4 u_{1m}\zeta_3^{ms}\zeta_{4s}{}^\beta \qquad (4.1.49)$$

Which completes the gravitino exchange diagram contributions.

All of these contributions sum to give the full contribution from the diagram in the form,

$$\left\{-\frac{i}{16}u\,\bar{u}_{2\rho}\gamma^\mu k\!\!\!/_4\gamma^a u_{1m}\zeta_{3a}{}^m\zeta_{4\mu}{}^\rho+\frac{i}{8}t\,\bar{u}_{2\beta}k\!\!\!/_4 u_{1m}\zeta_3^{ms}\zeta_{4s}{}^\beta\right\} \qquad (4.1.50)$$

The full amplitude can now be evaluated. It is useful at this point to summarise both the Feynman rules used, and the complete list of contributions listed diagrammatically with the correct combinatorial factors as used in Fig 4.1. These summaries are given in Figs. 4.2a,4.2b and 4.3 respectively. In Figs. 4.2a and 4.2b a complete summary of the Feynman rules used in the calculation of the diagrams given in the scheme of Fig. 4.1. These have been skewsymmetrised as described in Chapter Three to take account of the Majorana constraint on the fermions in the derivation of the scattering amplitude. Fig. 4.3 gives a complete list of the diagrams calculated using these Feynman rules. It should be noted that the contributions listed next to each diagram correspond to the complete sum of all diagrams of the same generic form, i.e. the amplitude contribution listed next to a diagram consists of the sum of all sub-amplitude contributions coming from diagrams which have the same form. In the case of the gravitino exchange diagram this includes the sum of the diagrams where the 'dot' is on the lower vertex. The amplitude contribution listed in this figure also contain the combinatorial factors arising from the Wick expansion. The full amplitude is merely the sum of the terms listed in this figure, and where it can be seen that the γ and α' factors have been reintroduced.

Having evaluated the complete amplitude, it will be convenient to split the amplitude into the 's' and 't-u' parts described above for the purposes of comparison with the string. It is at this point that the γ constant will be reintroduced, and the dimensionful parameter α' introduced also, as in the three point case discussed above. The separate contributions are,

$$\frac{\gamma\alpha'}{2}\Big\{\bar{u}_{2\mu}\gamma^{\alpha}(\not{k}_4-\not{k}_3)\gamma^{\gamma}u_1^{\mu}\Big(\frac{is}{16}-\frac{i5}{16}(t-u)\Big) - \bar{u}_{2\mu}\gamma^{\gamma}(\not{k}_4-\not{k}_3)\gamma^{\alpha}u_1^{\mu}\Big(\frac{is}{16}-\frac{i5}{16}(u-t)\Big)\Big\}\zeta_{3\alpha}{}^{m}\zeta_{4m\gamma} \tag{4.1.51}$$

for the 's'-channel terms, and,

$$\Big\{\Big[\bar{u}_{2\rho}\gamma^{\mu}(\not{k}_4-\not{k}_3)\gamma^{a}u_{1m}\Big(\frac{i}{8}(u-s)\Big) - \bar{u}_{2m}\gamma^{a}(\not{k}_4-\not{k}_3)\gamma^{\mu}u_{1\rho}\Big(\frac{i}{8}(t-s)\Big)\Big]\zeta_{3a}{}^{m}\zeta_{4\mu} + \Big[\bar{u}_{2\rho}\gamma^{a}(\not{k}_4-\not{k}_3)\gamma^{\mu}u_{1m}\Big(\frac{i}{8}u\Big) - \bar{u}_{2m}\gamma^{a}(\not{k}_4-\not{k}_3)\gamma^{\mu}u_{1\rho}\Big(\frac{i}{8}t\Big)\Big]\zeta_{3a}{}^{m}\zeta_{4\mu}{}^{\rho}\Big\}\frac{\gamma\alpha'}{2} \tag{4.1.52}$$

for the 't-u'-channel terms. The comparison with the string requires the normalisation with respect to the $O(\alpha'^0)$ amplitudes for string and field theory. Noting that the field theory $O(\alpha'^0)$ amplitude has an overall normalisation constant of i/16, then the relative normalisation of field theory amplitudes to string amplitudes is fixed. This implies that the field theory amplitude must equal the string amplitude in the normalised form,

$$-\frac{i\alpha'}{64}\Big\{\Big[u\,\bar{u}_2^{\mu}\gamma^{\alpha}(\not{k}_4-\not{k}_3)\gamma^{\gamma}u_{1\mu} - t\,\bar{u}_2^{\mu}\gamma^{\gamma}(\not{k}_4-\not{k}_3)\gamma^{\alpha}u_{1\mu}\Big]\zeta_{3\gamma\sigma}\zeta_4^{\sigma}{}_{\alpha} + \Big[u\,\bar{u}_2^{\rho}\gamma^{\alpha}(\not{k}_4-\not{k}_3)\gamma^{\gamma}u_1^{\sigma} - t\,\bar{u}_2^{\sigma}\gamma^{\gamma}(\not{k}_4-\not{k}_3)\gamma^{\alpha}u_1^{\rho}\Big]\zeta_{3\gamma\sigma}\zeta_{4\alpha\rho} + \Big[u\,\bar{u}_2^{\sigma}\gamma^{\alpha}(\not{k}_4-\not{k}_3)\gamma^{\gamma}u_1^{\rho} - t\,\bar{u}_2^{\rho}\gamma^{\gamma}(\not{k}_4-\not{k}_3)\gamma^{\alpha}u_1^{\sigma}\Big]\zeta_{3\gamma\sigma}\zeta_{4\alpha\rho}\Big\} \tag{4.1.53}$$

taking into account the normalisation factor. The string amplitude also breaks up into the separate 's'-channel and 't-u'-channel contributions mentioned above. The match for each of these sectors will be made separately.

i) The 's'-channel.

In this section equations (4.1.51) and appropriate sub terms of equation (4.1.53) will be compared. This will be done by a simple subtraction. When this is done it is found that no match can occur for these terms. Explicitly the comparison therefore takes the form,

$$\gamma \left\{ \frac{is}{16} + \frac{5i}{16}(u - t) \right\} \equiv \frac{it}{32} \qquad (4.1.54)$$

which can be seen clearly not to be consistent. The application of the identity,

$$s + t + u = 0$$

which derives from the kinematics of the theory, allows many of the terms which are derived to be transformed. This identity will be useful in the more general amplitude matching calculations carried out in the next chapter. For completeness the remaining 't-u'-channel matching calculation will be carried out, and also shown to fail.

ii) The 't-u'-channel.

This matching can be subdivided into two smaller matching conditions, by noticing that there are two distinct contraction schemes of indices which are not 's'-channel, which take the generic forms,

$$\bar{u}_{2\mu}\gamma^{\rho}(\not{k}_4 - \not{k}_3)\gamma^{a}u_{1m}\zeta_{3\ a}^{m}\zeta_{4\ \rho}^{\mu} \qquad (4.1.55)$$

and the corresponding 'crossed' term,

$$\bar{u}_{2\mu}\gamma^{a}(\not{k}_4 - \not{k}_3)\gamma^{\rho}u_{1m}\zeta_{3\ a}^{m}\zeta_{4\ \rho}^{\mu} \qquad (4.1.56)$$

The same subtraction technique will be applied to the two subterms listed above. It must be noted that this is only one of two possible bases of comparison which are completely equivalent up to the truncation procedure. The alternative basis involves rewriting the amplitudes all in the form where the gamma matrices occur either as a completely skewsymmetrised product of three matrices, or as a single matrix. This basis will not be used or discussed further here. Doing the subtraction explicitly yields,

$$\frac{i\gamma}{8}(u - s) \equiv - \frac{iu}{32} \qquad (4.1.57)$$

for the first term, and,

$$\frac{i\gamma u}{8} \equiv \frac{it}{32} \qquad (4.1.58)$$

for the second 'crossed' term. It can be seen immediately that the subtraction does not give a null result: the Romans and Warner action can not correspond to the low energy effective action for the string. This is quite a remarkable result. It would not have been unreasonable to expect such a symmetric, 'geometrical' action to correspond to the low energy effective action for the string in some sense. This hope is shown to be groundless by the amplitude matching. There is the alternative supersymmetric action which may be a better choice for the low energy effective action for the string. This will now be considered.

Section 4.2: The Han, Kim, Koh and Tanii action.

This action is an alternative supersymmetrisation of the modified Chapline-Manton action of given in the introduction of Chapter Three. In this case the supersymmetry transformations of Chapline and Manton are kept without modification, and only the action is modified by the addition of higher derivative corrections. These are explicitly,

$$\mathfrak{L}' = \gamma \Big[e\,\phi^{-\frac{3}{4}} \overline{\psi}_\eta \gamma^{\rho\sigma} \gamma^\eta \mathfrak{D}_{[\mu} \psi_{\nu]} R_{\rho\sigma}{}^{\mu\nu} - \frac{1}{2} e\,\phi^{-\frac{3}{4}} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4\, e\,\phi^{-\frac{3}{4}} \mathfrak{D}^{[\mu} \psi^{\nu]} \not{\mathfrak{D}}\, \mathfrak{D}_{[\mu} \psi_{\nu]} \Big] \qquad (4.2.1)$$

in the notation of Chapline and Manton, where special attention should be drawn to the second and third terms, (and where it should be noted that the third term has been rewritten by excluding the contribution of some of the torsion terms inherent in the definition of the spin connection by its equation of motion, since these only give four fermion couplings which are ignored throughout). (The γ factor will be suppressed throughout this section as it was above, and will only be replaced, with the α' factor at the very end, when the amplitude matching is performed. The α'

factor will of course be reintroduced with the implicit value $\alpha' = 2$). These have the unique property of being propagator modifying higher derivative terms, (in the meaning of Chapter Two): that is, after the background expansion has been performed, these action terms give rise to the higher derivative terms,

$$\mathcal{L}'_{2h} = -\frac{1}{2}\Big\{ h_\mu{}^\rho \partial_\nu \partial^\nu \partial_\sigma \partial^\sigma h_\rho{}^\mu - h_\mu{}^\sigma \partial_\nu \partial^\nu \partial_\sigma \partial^\rho h_\rho{}^\mu$$

$$- h_\nu{}^\rho \partial_\mu \partial^\nu \partial_\sigma \partial^\sigma h_\rho{}^\mu + h_\nu{}^\sigma \partial_\mu \partial^\nu \partial_\sigma \partial^\rho h_\rho{}^\mu \Big\} + O(h^3) \qquad (4.2.2)$$

in the case of the graviton, and,

$$\mathcal{L}'_{\bar{\psi}\psi} = -2\Big\{ \bar{\psi}^\nu \gamma^\rho \partial_\rho \partial^\mu \partial_\mu \psi_\nu - \bar{\psi}^\mu \gamma^\rho \partial_\rho \partial^\nu \partial_\mu \psi_\nu \Big\} + O(h) \qquad (4.2.3)$$

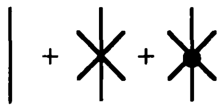
in the case of the gravitino. Both of these can be seen to be quadratic in the fields, and hence contribute to the linear, free, part of the actions for the respective fields, and therefore modify the equations of motion and thus the quantisation and subsequent derivation of the propagators. This is extremely worrying, for these extra higher derivative corrections give extra unphysical poles in the propagators which may violate unitarity of the quantum theory. The cure for this worry is related to the field redefinition analysis of general field theories[55]. In particular the case of the first term is treated in more depth in references [51,56] which restricts the discussion to the case of the graviton field only. In this work field redefinition analysis is used to show that no matter what choice of generalised Gauss-Bonnet combination is chosen the theory will remain unitary. This conclusion is justified by direct calculation. The subject of field redefinition analysis, and its application to the low energy effective action of the heterotic superstring in particular will be taken up again later. For the moment only the two actions discussed above will be compared with the string and with each other.

The problem of the propagator corrections must be faced in the calculation of Feynman diagrams. The procedure is to note that in the momentum representation the propagators can be written in the generic form, (see Appendix Two),

$$P \sim \frac{f(k)}{k^2\,(\,1 + \varepsilon k^2)} \qquad (4.2.4)$$

where the extra term in the denominator is directly due to the higher derivative corrections to the action, using the ansatz of Chapter Two. Since the k^2 term in this factor is of order $O(\alpha')$, then it makes sense to Taylor expand this factor in powers of α' and to truncate at the appropriate point in the expansion to keep only terms up to the power in α' for which amplitudes will be calculated. This is done to $O(\alpha'^2)$ in Appendix Two. Graphically this can be represented in the form,

for the graviton propagator, and the corresponding expansion for the gravitino propagator is,



where each of the terms in the expansions are given in Appendix Two, and the appropriate terms for the propagators will be taken from there with the appropriate values taken for the various variables. The first order in α' correction to the spin-3/2 propagator is of the form,

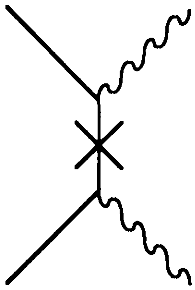
$$Y^{(1)}_{\mu,\nu}(k) = 4i \left\{ -\frac{50}{64}\eta_{\mu\nu}\not{k} + \frac{7}{64}\gamma_\nu \not{k}\, \gamma_\mu - \frac{1}{8}(\gamma_\mu k_\nu + \gamma_\nu k_\mu) + \frac{68}{64}\frac{k_\mu k_\nu \not{k}}{k^2} \right\}$$

and the corresponding correction to the graviton propagator is of the form,

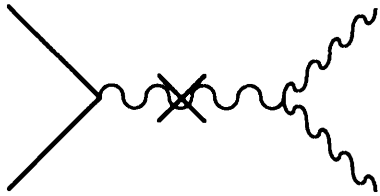
$$\begin{aligned} D^{(1)}_{\mu\nu,\alpha\beta}(k) = i \Big\{ & -8\,(\delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\mu\beta}\delta_{\nu\alpha}) \\ & + 8\,(\delta_{\mu\alpha}\omega_{\nu\beta} + \delta_{\mu\beta}\omega_{\nu\alpha} + \delta_{\nu\beta}\omega_{\mu\alpha} + \delta_{\nu\alpha}\omega_{\mu\beta}) \\ & + \frac{32}{18}(\delta_{\mu\nu} - \omega_{\mu\nu}).(\delta_{\alpha\beta} - \omega_{\alpha\beta}) \\ & - \frac{1}{4}\Big[\frac{1}{9}(\delta_{\mu\nu} - \omega_{\mu\nu}).(\delta_{\alpha\beta} - \omega_{\alpha\beta}) + 9\,\omega_{\mu\nu}\omega_{\alpha\beta} \\ & + 3\,(\delta_{\mu\nu}\omega_{\alpha\beta} + \delta_{\alpha\beta}\omega_{\mu\nu} - 2\,\omega_{\mu\nu}\omega_{\alpha\beta})\Big]\Big\} \end{aligned}$$

This means that the Wick expansion is modified by the inclusion of extra

O(α') diagrams of the form,



&



which give extra contributions to the amplitude. These amplitude contributions will be calculated below.

The only remaining parts needed for the amplitude calculation are the Feynman vertices. The procedure is completely similar to that presented in some detail in the case of the Romans and Warner action. The vertices required for calculation are the same type as the vertices derived above. Each of the vertices is calculated below. The only difference in technique needed in the evaluation of these terms to the techniques used in the Romans and Warner case, is in the background field expansion of the pure higher derivative gravitino action term,

$$\mathfrak{L}'_{\bar{\psi}\psi} = -4\, e\, \phi^{-\frac{3}{4}} \{ \mathfrak{D}^{[\mu}\bar{\psi}^{\nu]}\gamma^{\rho}(x)\mathfrak{D}_{\rho}\mathfrak{D}_{[\mu}\psi_{\nu]} \} \quad (4.2.5)$$

which calls for the expansion of the space dependent gamma matrices, defined by the Clifford algebra (A3.1.8), and the spin connection to two gravitons. Each of the vertices will be treated in turn as in the Romans and Warner case above. The first term to be treated will be the three point $\bar{\psi}$-ψ-h vertex with all fields on-shell. In this case, as above, all fields are on-shell. This allows the same simplifications as used above. This vertex has contributions only from the first and last terms in equation (4.2.1), where the Riemann tensor term gives the contribution,

$$-2\,\bar{\psi}^{\sigma}\gamma^{\rho}\partial_{\mu}\psi_{\nu}\partial_{\sigma}\partial^{\nu}h_{\rho}{}^{\mu} \qquad (4.2.6)$$

which is trivial to derive using the simplification techniques as developed above. It will be useful for the purposes of the more general terms that will be invoked in the next chapter to treat the second term in some detail, in each of the calculations below. The third term in (4.2.1) above gives a contribution to the $\bar{\psi}$-ψ-h all on-shell vertex in the form,

$$-4\{ \mathfrak{D}^{[\mu}\overline{\psi}^{\nu]}\gamma^{\rho}\mathfrak{D}_{\rho}\mathfrak{D}_{[\mu}\psi_{\nu]} \}|_h = -4\ \{ [\mathfrak{D}^{[\mu}\overline{\psi}^{\nu]}]|_h\gamma^{\rho}\partial_{\rho}\partial_{[\mu}\psi_{\nu]}$$

$$+ \partial^{[\mu}\overline{\psi}^{\nu]}[\gamma^{\rho}(x)\mathfrak{D}_{\rho}]|_h\partial_{[\mu}\psi_{\nu]} + \partial^{[\mu}\overline{\psi}^{\nu]}\gamma^{\rho}\partial_{\rho}[\mathfrak{D}_{[\mu}\psi_{\nu]}]|_h \} \tag{4.2.7}$$

where partial integration and the equations of motion immediately imply that the first and third terms must vanish. This simplification is extremely useful on several occasions. Each of the remaining terms can be expanded using the definitions given in equations (3.2.3-4). Doing this term by term yields;

$$\partial_{\sigma}\overline{\psi}_{\rho}\gamma^{\mu}\partial_{\rho}\psi_{\lambda}\partial^{\lambda}h_{\mu}{}^{\sigma} \tag{4.2.8}$$

Thus the full $\overline{\psi}$-ψ-h all on-shell vertex is,

$$-\ \overline{\psi}^{\sigma}\gamma^{\rho}\partial_{\mu}\psi_{\nu}\partial_{\sigma}\partial^{\nu}h_{\rho}{}^{\mu} \tag{4.2.9}$$

which allows an immediate calculation of the three point amplitude. As in the Romans and Warner case this is quite trivial to perform, and yields the amplitude,

$$-\ i\ \gamma\ \alpha'\ \overline{u}_2^{\sigma}\gamma^{\rho}u_1^{\nu}\zeta_{3\rho}{}^{\mu}k_{1\mu}k_{2\nu}k_{3\sigma} \tag{4.2.10}$$

which can be compared to the O(α') three point string amplitude with due respect given to the normalisation of the amplitudes. In this case it can be seen that the string amplitude matches the field theory amplitude (4.2.10) if the parameter γ takes the value γ = 1/4. This is quite an interesting conclusion: the Romans and Warner matching required that the γ parameter should take the value γ = -1/4! This will have some important implications for any possibilities for a field redefinition which links the Romans and Warner and Han et al actions, since any field redefinition which links the two actions must change the dynamics of the theory.

The corresponding four point generalisation of this matching will be performed as in the Romans and Warner case, to see if this form of the supersymmetric Lagrangian can match the four point string amplitude at the O(α') level. An interesting result will be seen to result from the 's'-channel amplitude matching conditions from this action. This will also provide a useful starting point for the more general matching attempted in Chapter Five.

i) The h-h-h vertex, h off-shell.

In this case the vertex is merely a subset of the terms evaluated for the Romans and Warner case. Since this is the case it is necessary only to state the required form for the vertex, which is,

$$-\frac{1}{2}\left\{\frac{1}{2}\underline{h}_{\mu}{}^{\mu}\partial^{\sigma}\partial^{\nu}h^{\rho\alpha}\partial_{\sigma}\partial_{\nu}h_{\alpha\rho} - \underline{h}^{\sigma\nu}\partial_{\alpha}h^{\rho\mu}\partial^{\alpha}\partial_{\sigma}\partial_{\nu}h_{\rho\mu}\right.$$

$$\left. - 5\,\underline{h}_{\alpha}{}^{\mu}\partial^{\sigma}\partial^{\nu}h^{\rho\alpha}\partial_{\sigma}\partial_{\nu}h_{\rho\mu}\right\} \qquad (4.2.11)$$

which will be used in the amplitudes calculated below.

ii) The $\overline{\psi}$-ψ-h vertex, h off-shell.

There are only two terms in the Han et al. action which can possibly contribute to this vertex. These are the first and third terms in equation (4.2.1). The Riemann tensor term can be shown to give no contribution to this vertex, by a similar argument to the one employed in the Romans and Warner case. It only remains to discuss the pure gravitino higher derivative term. As in the case of the $\overline{\psi}$-ψ-h vertex above, both $\overline{\psi}$ and ψ are on-shell, and so the first and third terms in the general expansion do not contribute in this case either. Expanding the remaining second term with regard to the fact that the graviton field is off-shell, and employing the usual simplification methods and also the truncation procedure, it is possible to determine the vertex in the form,

$$\frac{-1}{2}\,\overline{\psi}^{\sigma}\gamma^{\nu}\partial_{\mu}\psi_{\sigma}\partial^{\rho}\partial_{\rho}\underline{h}^{\mu}{}_{\nu} \qquad (4.2.12)$$

which can be symmetrised to give the total vertex,

$$\frac{-1}{4}\left\{\overline{\psi}^{\sigma}\gamma^{\nu}\partial_{\mu}\psi_{\sigma} - \partial_{\mu}\overline{\psi}_{\sigma}\gamma^{\nu}\psi^{\sigma}\right\}\partial^{\rho}\partial_{\rho}\underline{h}^{\mu}{}_{\nu} \qquad (4.2.13)$$

It can be noted that this term can only give 's'-channel amplitude contributions.

iii) The $\overline{\psi}$-ψ-h vertex, $\overline{\psi}$ or ψ off-shell.

In this case the calculation of the vertex contribution is very similar to the one carried out above, and so will only be stated here. Due to the symmetrisation procedure in the derivation of vertices where one of the gravitino fields is off-shell, it is only necessary to state one of the two possible vertices, since they are conjugate to each other with respect to the transposition of spinor indices. The convention chosen is that the ψ off-shell vertex is stated and the conjugate vertex is left implicit. The contribution to the vertex from the first of the terms added to the action in equation (4.2.1) above is,

$$2\left\{ -\frac{1}{4}\partial_\mu\overline{\psi}_\nu\gamma^\sigma\gamma^\rho\gamma^\eta\psi_\eta\partial_\sigma\partial^\mu h_\rho{}^\nu - \frac{1}{2}\overline{\psi}^\sigma\gamma^\rho\partial_\mu\psi_\nu\partial_\rho\partial^\mu h_\sigma{}^\nu \right.$$

$$\left. - \partial_\mu\overline{\psi}_\nu\gamma^\rho\psi^\sigma\partial_{[\sigma}\partial^\mu h_{\rho]}{}^\nu \right\} \qquad (4.2.14)$$

where the symmetrisation of the vertex has already been performed. It remains to discuss the background field expansion of the new term with respect to the truncation convention and the equations of motion and gauge conditions of the appropriate fields. The argument used in the cases where $\overline{\psi}$ and ψ are on-shell cannot be applied in this case and so the first and third terms in the expansion (4.2.7) above must be considered. As before each subterm will be considered in turn. This vertex expansion requires the 'building block' terms,

$$\left\{ \mathfrak{D}_{[\rho}\psi_{\sigma]} \right\}\Big|_h = \frac{1}{4}\gamma_m\gamma_n\psi_{[\sigma}\partial^{[n}h_{\rho]}{}^{m]} \qquad (4.2.15)$$

and,

$$\left\{ \mathfrak{D}^{[\rho}\overline{\psi}^{\sigma]} \right\}\Big|_h = \frac{1}{2}\left\{ \frac{1}{2}\partial_{[n}h^{[\rho}{}_{m]}\overline{\psi}^{\sigma]}\gamma^n\gamma^m - 2\partial^{[\alpha}\overline{\psi}^{\sigma]}h_\alpha{}^\rho - 2\,\partial^{[\rho}\overline{\psi}^{\beta]}h_\beta{}^\sigma \right\}$$

$$(4.2.16)$$

which will also be used in the evaluation of the graviton expansions of the pure gravitino term in (4.2.1) to two gravitons. Although there exists an expansion of these terms to two gravitons, these are never needed in the work described below. Continuing with the $\overline{\psi}$-ψ-h vertex, taking the expansion (4.2.7) term by term;

a) The first term.

The first term can be expanded and simultaneously simplified by noting that the presence of the equation of motion implies that the ψ field must be considered to be off-shell, allowing the usual simplification procedure to be applied to the $\overline{\psi}$ field. This allows the contribution to the vertex to be written,

$$2\left\{-\frac{1}{2}\partial_n h^{[\rho}{}_m \overline{\psi}^{\sigma]}\gamma^n\gamma^m \partial\!\!\!/\, \partial_{[\rho}\psi_{\sigma]} + 2\,\partial^{[\alpha}\overline{\psi}^{\sigma]}\partial\!\!\!/\,\partial_{[\rho}\psi_{\sigma]} + 2\,\partial^{[\rho}\overline{\psi}^{\beta]}\partial\!\!\!/\,\partial_{[\rho}\psi_{\sigma]}\right\} \tag{4.2.17}$$

which can immediately be simplified to the form,

$$2\,\partial^{\sigma}\overline{\psi}^{\alpha}\gamma^{\mu}\partial_\mu\partial_\sigma\psi_\rho h^{\rho}{}_{\alpha} \tag{4.2.18}$$

b) The second term.

The second term is a little more complicated because it will involve the Riemann-Christoffel connection in the definition of the covariant derivative of the two form term. The condition used above for the first term, where the ψ field is taken necessarily off-shell, cannot be applied. This term can be expanded to the form,

$$\begin{aligned} -4\,\partial^{[\rho}\overline{\psi}^{\sigma]} \Big\{ & \frac{1}{4}\gamma^{\mu}\gamma_m\gamma_n\partial_{[\rho}\psi_{\sigma]}\partial^n h_\mu{}^m \\ & -\frac{1}{2}\gamma^m\partial_\mu\partial_{[\rho}\psi_{\sigma]}h^{\mu}{}_m \\ & -(\partial_\mu h^{\lambda}{}_\sigma + \partial_\sigma h_\mu{}^{\lambda} - \partial^{\lambda}h_{\sigma\mu})\gamma^{\mu}\partial_{[\rho}\psi_{\lambda]} \Big\} \end{aligned} \tag{4.2.19}$$

which can be shown to simplify through the gauge condition of the graviton, or the truncation procedure to the term,

$$\left\{ 2\,\partial^{[\rho}\overline{\psi}^{\sigma]}\gamma^{\mu}\partial_\rho\psi_\lambda\partial_\sigma h_\mu{}^{\lambda} - 2\,\partial^{\rho}\overline{\psi}^{\sigma}\gamma^{\mu}\partial_{[\rho}\psi_{\lambda]}\partial^{\lambda}h_{\sigma\mu} \right\} \tag{4.2.20}$$

which is interestingly symmetric.

c) The third term.

The third term is the easiest to analyse, where the $\overline{\psi}$ must be taken off-shell by the same argument used in a) above, and which can expanded using partial integration and the building block term (4.2.16). This is,

$$\partial_\mu\partial^{[\rho}\overline{\psi}^{\sigma]}\gamma^{\mu}\gamma_m\gamma_n\psi_\sigma\partial^n h_\rho{}^m \tag{4.2.21}$$

which truncates immediately to zero.

The total vertex is now derived by first symmetrising the vertex, then applying the equation of motion, as in the Romans and Warner case above. The total vertex contribution from the pure gravitino term is therefore,

$$\{ \ \partial^{\sigma}\overline{\psi}^{\alpha}\gamma^{\mu}\partial_{\mu}\partial_{\sigma}\psi_{\rho}h^{\rho}{}_{\alpha} - \partial^{\rho}\overline{\psi}^{\sigma}\gamma^{\mu}\partial_{\rho}\psi_{\lambda}\partial^{\lambda}h_{\sigma\mu}$$

$$+ \partial^{\rho}\overline{\psi}^{\sigma}\gamma^{\mu}\partial_{\lambda}\psi_{\rho}\partial^{\lambda}h_{\sigma\mu} \} \qquad (4.2.22)$$

which can be added to the vertex derived from the Riemann tensor term, (the first term in the O(α') action correction (4.2.1)), to give the full vertex. This is finally,

$$\{ -\frac{1}{2}\partial_{\mu}\overline{\psi}_{\nu}\gamma^{\sigma}\gamma^{\rho}\gamma^{\eta}\psi_{\eta}\partial_{\sigma}\partial^{\mu}h_{\rho}{}^{\nu} - 3\,\overline{\psi}^{\sigma}\gamma^{\rho}\partial_{\mu}\psi_{\nu}\partial_{\rho}\partial^{\mu}h_{\sigma}{}^{\nu}$$

$$+ \partial^{\rho}\overline{\psi}^{\sigma}\gamma^{\mu}\partial_{\lambda}\psi_{\rho}\partial^{\lambda}h_{\sigma\mu} \} \qquad (4.2.23)$$

from which the conjugate vertex with $\overline{\psi}$ off-shell can be found, and the amplitude contributions calculated.

iv) The $\overline{\psi}$-ψ-h-h vertex, all on-shell.

In this case the Riemann tensor term can be seen to give the contribution,

$$\{ -\frac{1}{2}\overline{\psi}_{\eta}\gamma^{\lambda}\gamma^{\rho}\gamma^{\sigma}\partial_{\mu}\psi_{\nu}h_{\lambda}{}^{\eta}\partial_{\sigma}\partial^{\mu}h_{\rho}{}^{\nu} + 2\,\overline{\psi}_{s}\gamma^{\rho}\partial_{\mu}\psi_{\nu}h^{s\sigma}\partial_{\rho}\partial^{\mu}h_{\sigma}{}^{\nu}$$

$$+ \ \overline{\psi}^{\sigma}\gamma^{\rho}\partial_{\mu}\psi_{\nu}h^{\nu\alpha}\partial_{\rho}\partial^{\mu}h_{\nu\sigma} \} \qquad (4.2.24)$$

by applying the standard arguments used above. The contribution from the Lorentz Chern-Simons anomaly cancelling term is identical to the contribution derived above. The new term gives a contribution according to the scheme,

$$
\begin{aligned}
&-4\,\{\, \mathfrak{D}^{[\rho}\overline{\psi}^{\sigma]}\gamma^{\mu}\mathfrak{D}_{\mu}\mathfrak{D}_{[\rho}\psi_{\sigma]}\,\}|_{2h} \\
&= -4\,\{\, [\mathfrak{D}^{[\rho}\overline{\psi}^{\sigma]}]|_{2h}\gamma^{\mu}\partial_{\mu}\partial_{[\rho}\psi_{\sigma]} + \partial^{[\rho}\overline{\psi}^{\sigma]}\gamma^{\mu}\partial_{\mu}[\mathfrak{D}_{[\rho}\psi_{\sigma]}]|_{2h} \\
&\quad + \partial^{[\rho}\overline{\psi}^{\sigma]}[\gamma^{\mu}\mathfrak{D}_{\mu}]|_{2h}\partial_{[\rho}\psi_{\sigma]} + [\mathfrak{D}^{[\rho}\overline{\psi}^{\sigma]}]|_{h}\gamma^{\mu}\partial_{\mu}[\mathfrak{D}_{[\rho}\psi_{\sigma]}]|_{h} \\
&\quad + [\mathfrak{D}^{[\rho}\overline{\psi}^{\sigma]}]|_{h}[\gamma^{\mu}\mathfrak{D}_{\mu}]|_{h}\partial_{[\rho}\psi_{\sigma]} + \partial^{[\rho}\overline{\psi}^{\sigma]}[\gamma^{\mu}\mathfrak{D}_{\mu}]|_{h}[\mathfrak{D}_{[\rho}\psi_{\sigma]}]|_{h}\,\}
\end{aligned} \tag{4.2.25}
$$

The argument used in the three point case can be applied to show that the first and second terms vanish. Each of the remaining terms will give a contribution when the appropriate expansions from above are replaced in the nonzero contributions. This is a long and somewat tedious task. Only the result need be stated here. The full vertex derived from (4.2.25) before symmetrisation, and after a little simplification, is,

$$
\begin{aligned}
-\{\, &-\tfrac{1}{2}\overline{\psi}^{\sigma}\gamma^{n}\gamma^{m}\gamma_{\mu}\partial_{\nu}\psi_{[\sigma}\partial^{\nu}h_{\rho]}{}^{\mu}\partial_{n}h^{\rho}{}_{m} + \partial_{\lambda}\overline{\psi}_{\rho}\gamma^{\mu}\gamma^{\eta}\gamma^{p}\psi^{[\sigma}\partial_{p}h^{\rho]}{}_{q}\partial^{\lambda}h_{\sigma\mu} \\
&-\tfrac{1}{8}\partial^{\rho}\overline{\psi}^{\sigma}\gamma^{\mu}\gamma_{m}\gamma_{n}\partial_{\rho}\psi_{\sigma}h^{m\nu}\partial_{\mu}h_{\nu}{}^{n}\,\}
\end{aligned} \tag{4.2.26}
$$

which can now be symmetrised and used in the calculation of the four point amplitude. All the necessary components for the calculation have been assembled. The amplitude can now be evaluated.

The amplitude calculation proceeds according to the sum of Feynman diagrams, which is found by a simple application of the modified form of the Wick expansion developed in Chapter Three, which isgiven in Figure 4.2. Each of these diagrams can be evaluated using the Feynman rules as above. As in Chapter Three and in the Romans and Warner case, the individual classes of diagrams will be dealt with in turn, and the full amplitude given at the end. The contribution from each class of diagrams is as follows;

<u>The point diagram:</u>

The point diagram contribution due only to the terms in the Han et al action, is given by considering the vertices (4.2.24) and (4.2.26). The vertex contribution (4.2.24) gives the amplitude contribution,

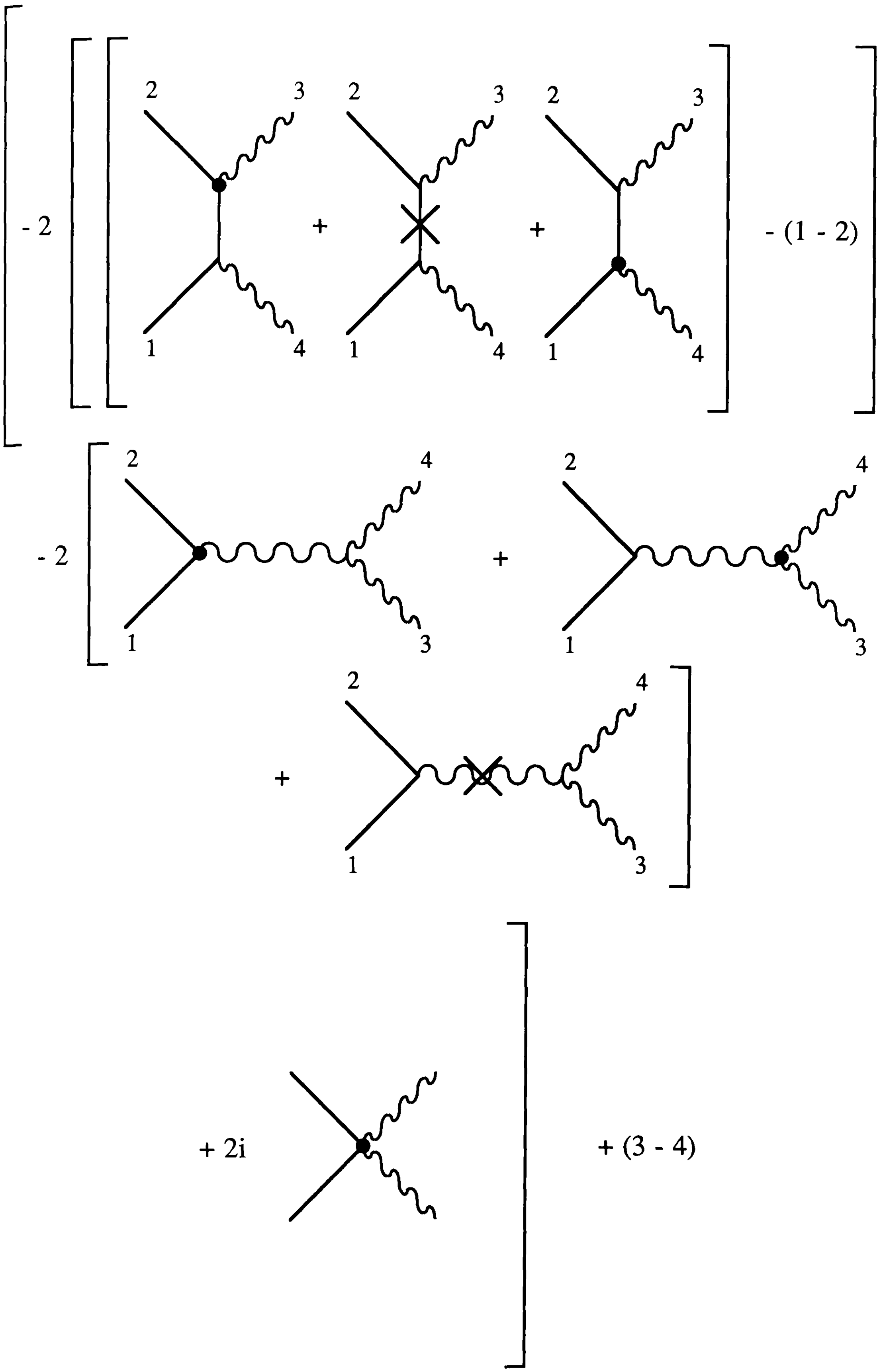


Figure 4.4: The Feynman diagram scheme in the case of the calculation of the $\overline{\psi}$-ψ-h-h O(α') amplitude with propagator corrections

$$\left\{ \frac{1}{8}\left(u\, \bar{u}_{2\eta}\gamma^{l} k\!\!\!/_{4}\gamma^{\rho} u_{1\nu} - t\, \bar{u}_{2\nu}\gamma^{\rho} k\!\!\!/_{4}\gamma^{l} u_{1\eta}\right) \zeta_{3l}{}^{\eta}\zeta_{4\rho}{}^{\nu} \right.$$

$$\left. - \left[\left(\frac{2t-u}{4}\right) \bar{u}_{2}^{\sigma} k\!\!\!/_{4} u_{1\nu} - \left(\frac{2u-t}{4}\right) \bar{u}_{2\nu} k\!\!\!/_{4} u_{1}^{\sigma}\right]\zeta_{3}^{\nu\alpha}\zeta_{4\alpha\sigma} \right\}$$

(4.2.27)

and the vertex contribution (4.2.26) gives the amplitude contribution,

$$\left\{ \left[\frac{(2t+s)}{32} \bar{u}_{2}^{\sigma}\gamma_{n} k\!\!\!/_{4}\gamma_{m} u_{1\sigma} - \frac{(2u+s)}{32} \bar{u}_{2}^{\sigma}\gamma_{m} k\!\!\!/_{4}\gamma_{n} u_{1\sigma}\right] \zeta_{3}^{m\nu}\zeta_{4\nu}{}^{n} \right.$$

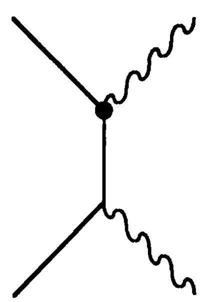
$$\left. - \frac{1}{16}\left(t\, \bar{u}_{2}^{\rho}\gamma^{\mu} k\!\!\!/_{4}\gamma^{q} u_{1}^{\sigma} - u\, \bar{u}_{2}^{\sigma}\gamma^{q} k\!\!\!/_{4}\gamma^{\mu} u_{1}^{\rho}\right) \zeta_{3q\rho}\zeta_{4\mu\sigma} \right\}$$

(4.2.28)

These should be added to the Lorentz Chern-Simons contribution, to give the full point diagram contribution. The addition will not be performed explicitly here, and will be left to the calculation of the full amplitude.

<u>The gravitino exchange diagrams:</u>

The pure gravitino exchange diagram,

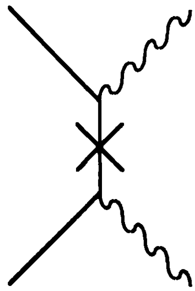


gives the amplitude contribution,

$$\left\{ \frac{is}{16} \bar{u}_{2\lambda}\gamma^{\mu} k\!\!\!/_{4}\gamma^{a} u_{1m}\zeta_{3}^{m}{}_{a}\zeta_{4}^{\lambda}{}_{\mu} - \frac{i3t}{8} \bar{u}_{2\beta} k\!\!\!/_{4} u_{1m}\zeta_{3}^{ms}\zeta_{4s}{}^{\beta} \right\}$$

(4.2.29)

and the propagator correction diagram,



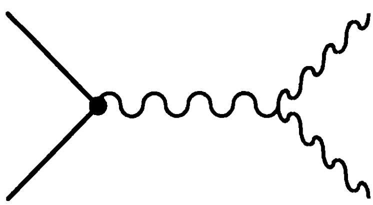
gives the amplitude contribution,

$$\frac{i}{8}\left\{ \frac{2s+t}{4}\,\bar{u}_{2\mu}\gamma^{\alpha}\not{k}_{4}\gamma^{a}u_{1m}\zeta_{3a}{}^{m}\zeta_{4\alpha}{}^{\mu} - t\,\bar{u}_{2\mu}\not{k}_{4}u_{1m}\zeta_{3}^{ms}\zeta_{4s}{}^{\mu} \right\} \tag{4.2.30}$$

which add together as in the diagram scheme, Fig. 4.4.

<u>The graviton exchange diagram:</u>

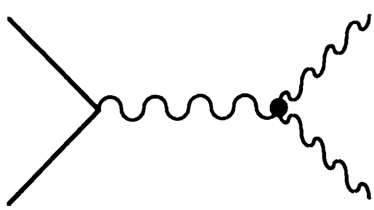
There are three diagrams which contribute in this sector. The first diagram has the O(α') fermionic vertex given by equation (4.2.13),



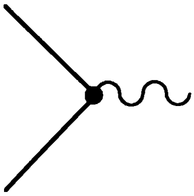
and gives the amplitude contribution,

$$-\frac{i}{16}\,\bar{u}_{2}^{\sigma}\not{k}_{4}u_{1\sigma}\zeta_{3}^{\eta\xi}\zeta_{4\eta\xi}(u-t) \tag{4.2.31}$$

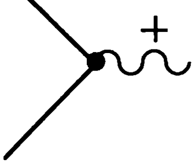
The second diagram to consider has the O(α') <u>h</u>-h-h vertex of equation (4.2.11),



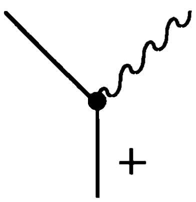
and gives contribution,



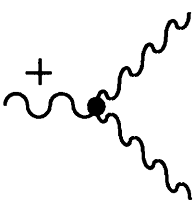
$$-\frac{\gamma\alpha'}{2}\overline{\psi}^{\sigma}\gamma^{\rho}\partial_{\mu}\psi_{\nu}\partial_{\sigma}\partial^{\nu}h_{\rho}{}^{\mu}$$



$$\frac{\gamma\alpha'}{2}\left[\ \frac{-1}{4}\left\{\ \overline{\psi}^{\sigma}\gamma^{\nu}\partial_{\mu}\psi_{\sigma} - \partial_{\mu}\overline{\psi}_{\sigma}\gamma^{\nu}\psi^{\sigma}\ \right\}\partial^{\rho}\partial_{\rho}\underline{h}^{\mu}{}_{\nu}\right]$$

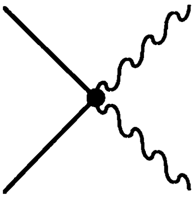


$$\frac{\gamma\alpha'}{2}\left\{\ -\frac{1}{2}\partial_{\mu}\overline{\psi}_{\nu}\gamma^{\sigma}\gamma^{\rho}\gamma^{\eta}\psi_{\eta}\partial_{\sigma}\partial^{\mu}h_{\rho}{}^{\nu} - 3\ \overline{\psi}^{\sigma}\gamma^{\rho}\partial_{\mu}\psi_{\nu}\partial_{\rho}\partial^{\mu}h_{\sigma}{}^{\nu} + \partial^{\rho}\overline{\psi}^{\sigma}\gamma^{\mu}\partial_{\lambda}\psi_{\rho}\partial^{\lambda}h_{\sigma\mu}\ \right\}$$



$$\frac{\gamma\alpha'}{2}\left[\ -\frac{1}{2}\left\{\frac{1}{2}\underline{h}_{\mu}{}^{\mu}\partial^{\sigma}\partial^{\nu}h^{\rho\alpha}\partial_{\sigma}\partial_{\nu}h_{\alpha\rho} - \underline{h}^{\sigma\nu}\partial_{\alpha}h^{\rho\mu}\partial^{\alpha}\partial_{\sigma}\partial_{\nu}h_{\rho\mu} - 5\ \underline{h}_{\alpha}{}^{\mu}\partial^{\sigma}\partial^{\nu}h^{\rho\alpha}\partial_{\sigma}\partial_{\nu}h_{\rho\mu}\ \right\}\right]$$

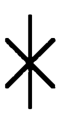
Fig. 4.5a The three point $O(\alpha')$ Feynman rules from the Han et al. action.



$$\frac{\gamma\alpha'}{2}\Big[\Big\{-\frac{1}{2}\overline{\psi}_\eta\gamma^\lambda\gamma^\rho\gamma^\sigma\partial_\mu\psi_\nu h_\lambda{}^\eta\partial_\sigma\partial^\mu h_\rho{}^\nu + 2\,\overline{\psi}_s\gamma^\rho\partial_\mu\psi_\nu h^{s\sigma}\partial_\rho\partial^\mu h_\sigma{}^\nu + 4\,\overline{\psi}^\sigma\gamma^\rho\partial_\mu\psi_\nu h^{\nu\alpha}\partial_\rho\partial^\mu h_{\nu\sigma}\Big\}$$

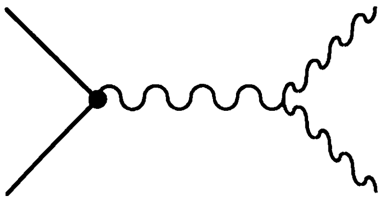
$$-\Big\{-\frac{1}{2}\overline{\psi}^\sigma\gamma^n\gamma^m\gamma_\mu\partial_\nu\psi_{[\sigma}\partial^\nu h_{\rho]}{}^\mu\partial_n h^\rho{}_m + \partial_\lambda\overline{\psi}_\rho\gamma^\mu\gamma^n\gamma^p\psi^{[\sigma}\partial_p h^{\rho]}{}_q\partial^\lambda h_{\sigma\mu} - \frac{1}{8}\partial^\rho\overline{\psi}^\sigma\gamma^\mu\gamma_m\gamma_n\partial_\rho\psi_\sigma h^{m\nu}\partial_\mu h_\nu{}^n\Big\}\Big]$$

$$D^{(1)}_{\mu\nu,\alpha\beta}(k) = \frac{\gamma\alpha' i}{2}\Big\{-8(\delta_{\mu\alpha}\delta_{\nu\beta}+\delta_{\mu\beta}\delta_{\nu\alpha})$$
$$+8(\delta_{\mu\alpha}\omega_{\nu\beta}+\delta_{\mu\beta}\omega_{\nu\alpha}+\delta_{\nu\beta}\omega_{\mu\alpha}+\delta_{\nu\alpha}\omega_{\mu\beta})$$
$$+\frac{32}{18}(\delta_{\mu\nu}-\omega_{\mu\nu}).(\delta_{\alpha\beta}-\omega_{\alpha\beta})$$
$$-\frac{1}{4}\Big[\frac{1}{9}(\delta_{\mu\nu}-\omega_{\mu\nu}).(\delta_{\alpha\beta}-\omega_{\alpha\beta})+9\,\omega_{\mu\nu}\omega_{\alpha\beta}$$
$$+3(\delta_{\mu\nu}\omega_{\alpha\beta}+\delta_{\alpha\beta}\omega_{\mu\nu}-2\,\omega_{\mu\nu}\omega_{\alpha\beta})\Big]\Big\}$$

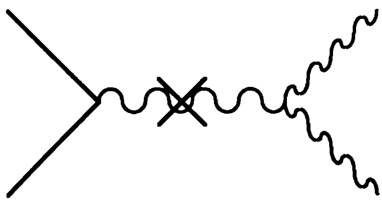


$$Y^{(1)}_{\mu,\nu}(k) = \frac{\gamma\alpha' i}{2}\Big[4\Big\{-\frac{50}{64}\eta_{\mu\nu}\not{k} + \frac{7}{64}\gamma_\nu\not{k}\,\gamma_\mu - \frac{1}{8}(\gamma_\mu k_\nu + \gamma_\nu k_\mu) + \frac{68}{64}\frac{k_\mu k_\nu \not{k}}{k^2}\Big\}\Big]$$

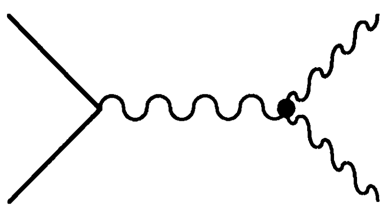
Fig. 4.5b The four point O(α') Feynman rule and O(α') propagator corrections from the Han et al action.



$$\frac{\gamma\alpha'}{2}\frac{i}{8}\{\, \bar{u}_2^{\sigma}(\not{k}_4 - \not{k}_3)\, u_{1\sigma}\, \zeta_3^{\eta\xi}\zeta_{4\xi\eta}(u-t) \,\}$$

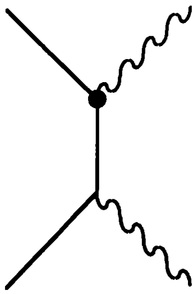


$$-\frac{\gamma\alpha'}{2}\frac{i}{4}\{\, \bar{u}_2^{\sigma}(\not{k}_4 - \not{k}_3)\, u_{1\sigma}\, \zeta_3^{\eta\xi}\zeta_{4\xi\eta}(u-t) \,\}$$



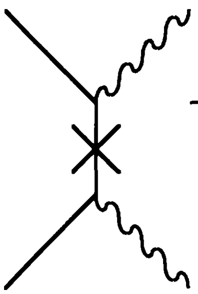
$$-\frac{\gamma\alpha'}{2}\frac{i}{8}\{\, \bar{u}_2^{\sigma}(\not{k}_4 - \not{k}_3)\, u_{1\sigma}\, \zeta_3^{\eta\xi}\zeta_{4\xi\eta}(u-t) \,\}$$

Fig. 4.6a The O(α') graviton exchange diagrams from the Han et al action.



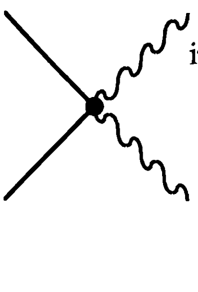
$$-\frac{i\gamma\alpha'}{8}\Big\{s\big(\bar{u}_{2\lambda}\gamma^{\mu}(\not k_4-\not k_3)\gamma^{a}u_{1m}-\bar{u}_{2m}\gamma^{a}(\not k_4-\not k_3)\gamma^{\mu}u_{1\lambda}\big)\zeta_3{}^{m}{}_{a}\zeta_4{}^{\lambda}{}_{\mu}$$

$$-6\big(t\,\bar{u}_{2\beta}(\not k_4-\not k_3)u_{1m}-u\,\bar{u}_{2m}(\not k_4-\not k_3)u_{1\beta}\big)\zeta_3^{ms}\zeta_{4s}{}^{\beta}\Big\}$$



$$\frac{i\gamma\alpha'}{4}\Big\{\Big[\frac{(2s+t)}{4}\bar{u}_{2\mu}\gamma^{\alpha}(\not k_4-\not k_3)\gamma^{a}u_{1m}-\frac{(2s+u)}{4}\bar{u}_{2m}\gamma^{a}(\not k_4-\not k_3)\gamma^{\alpha}u_{1\mu}\Big]\zeta_{3a}{}^{m}\zeta_{4\alpha}{}^{\mu}$$

$$-\big(t\,\bar{u}_{2\mu}(\not k_4-\not k_3)u_{1m}-u\,\bar{u}_{2m}(\not k_4-\not k_3)u_{1\mu}\big)\zeta_3^{ms}\zeta_{4s}{}^{\mu}\Big\}$$



$$i\gamma\alpha'\Big\{\frac{1}{8}\big(u\,\bar{u}_{2\eta}\gamma^{l}(\not k_4-\not k_3)\gamma^{\rho}u_{1\nu}-t\,\bar{u}_{2\nu}\gamma^{\rho}(\not k_4-\not k_3)\gamma^{l}u_{1\eta}\big)\zeta_{3l}{}^{\eta}\zeta_{4\rho}{}^{\nu}$$

$$-\Big[\frac{(2t-u)}{4}\bar{u}_2^{\sigma}(\not k_4-\not k_3)u_{1\nu}-\frac{(2u-t)}{4}\bar{u}_{2\nu}(\not k_4-\not k_3)u_1^{\sigma}\Big]\zeta_3^{\nu\alpha}\zeta_{4\alpha\sigma}$$

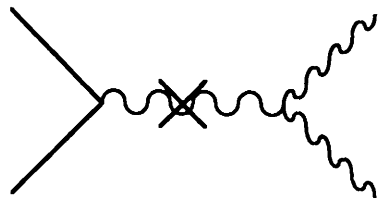
$$+\Big[\frac{(2t+s)}{32}\bar{u}_2^{\sigma}\gamma_n(\not k_4-\not k_3)\gamma_m u_{1\sigma}-\frac{(2u+s)}{32}\bar{u}_2^{\sigma}\gamma_m(\not k_4-\not k_3)\gamma_n u_{1\sigma}\Big]\zeta_3^{m\nu}\zeta_{4\nu}{}^{n}$$

$$-\frac{1}{16}\big(t\,\bar{u}_2^{\rho}\gamma^{\mu}(\not k_4-\not k_3)\gamma^{q}u_1^{\sigma}-u\,\bar{u}_2^{\sigma}\gamma^{q}(\not k_4-\not k_3)\gamma^{\mu}u_1^{\rho}\big)\zeta_{3q\rho}\zeta_{4\mu\sigma}\Big\}$$

Fig. 4.6b The O(α') gravitino exchange diagrams, and point diagram from the Han et al action. (No Lorentz Chern-Simons contribution note.)

$$\frac{i}{16}\,\bar{u}_2^{\sigma}\mathbf{k}_4 u_{1\sigma}\zeta_3^{\eta\xi}\zeta_{4\eta\xi}(u-t) \qquad (4.2.32)$$

and the final diagram is,



which gives the contribution,

$$\frac{i}{8}\,\bar{u}_2^{\sigma}\mathbf{k}_4 u_{1\sigma}\zeta_3^{\eta\xi}\zeta_{4\eta\xi}(u-t) \qquad (4.2.33)$$

The full amplitude can now be derived. Several of these calculations can be considerably simplified where appropriate reference has been made to the similar calculations performed in the Romans and Warner case. As in the Romans and Warner case given above a summary of the Feynman rules derived above will be given. The complete list of three point Feynman rules is given in Fig. 4.5a and the four point Feynman rules and $O(\alpha')$ propagator corrections are listed in Fig. 4.5b. As in the Romans and Warner case the factors of γ and α' have been reintroduced in these. The Feynman graphs are listed in Figs. 4.6a-b, where the same procedure used in the Romans and Warner case is used here: all amplitude contributions listed next to a diagram include the sum of all diagrams of that form, and the appropriate combinatorial factor from Fig. 4.4. The full amplitude will be given in the 's'-channel and 't-u'-channel form used above. This will aid the comparison with the string and the Romans and Warner calculation. The γ constant and the α' parameter will now be replaced in the amplitudes. The 's'-channel matching can clearly be seen to be,

$$\frac{i\gamma\alpha'}{16}\left[\,(u-t)+\frac{1}{2}(2t+s)-\frac{s}{2}\,\right]\equiv\frac{-\,i\alpha'}{64}\,u \qquad (4.2.34)$$

which can be rewritten in a slighlty neater form. When this is done the matching condition takes the form,

$$(2\gamma+1)\,u-2\gamma t-2\gamma s=0 \qquad (4.2.35)$$

which can be seen to be true if and only if γ = -1/4! So it can be seen that it is possible to have either a three point amplitude match, *or*, it is possible to get a four point 's'-channel match to the string from the Han et al action. It is now important to see if such a result extends to the 't-u'-channel also. Unfortunately it will not do so. It is quite interesting to note that the Hanet al amplitude matching result is somewhat 'closer' than that of the Romans and Warner result! This raises doubts about the possibility of field redefinitions linking the two supersymmetric actions. This will be discussed more fully in a later chapter. The 't-u'-channel matchings will now be derived for completeness. The matching conditions can be seen to be simply,

$$\frac{-i\gamma\alpha'}{8}\left[\frac{t}{2}-\frac{s}{2}+u\right] \equiv \frac{-i\alpha'}{64}u \qquad (4.2.36)$$

for the standard generic term (4.1.56),

$$\frac{-i\gamma\alpha'}{8}\left[u-\frac{s}{2}\right] \equiv \frac{i\alpha'}{64}t \qquad (4.2.37)$$

for the crossed generic term (4.1.57). As mentioned above, it can be seen that the amplitude derived from the Han et al action cannot match the string amplitude for any value of the parameter γ. The amplitude matching is however very close, which tends to suggest that the low energy effective action for the heterotic string must be quite similar to the Han et al action.

The question that must now be asked is what form the low energy effective action for the string can possibly take, and what supersymmetry transformation rules make it supersymmetric? The 's'-channel result above is quite tantalising and suggests that the final action may be quite similar to that of Han et al. This question will be partially answered in the next chapter: the derivation of the effective action will be attempted. The subject of supersymmetry of the effective action is left until a discussion of field redefinitions, and redundancy and ambiguities in the effective action.

Section 4.3: The Ricci tensor: Possible Ambiguities.

The Ricci tensor used in the work given above is defined as in Appendix Three by the contraction of the Riemann tensor in the form,

$$R_{\mu\nu} = R^{\sigma}_{\ \mu\nu\sigma} \qquad (A3.4.6)$$

or equivalently in the form,

$$R_{\mu\nu} = R_{\alpha\mu}{}^{mn} e^{\alpha}_{n} e_{m\nu} \qquad (4.3.1)$$

However it is possible to use the definition,

$$R_{\mu\nu} = R^{\sigma}{}_{\mu\sigma\nu} \qquad (4.3.2)$$

which differs from the previous definition only by an overall negative sign. The action of Romans and Warner lists the curvature scalar as having a negative sign difference from that of Chapline-Manton when the conversion table as listed in their paper is used to convert between the two actions. (Han et al have the same problem as they use exactly the same notation of Romans and Warner). This means that their are two possible conclusions,

i) The negative sign multiplying the Einstein-Hilbert action in Romans and Warner is a typographical error, in which case the results given above for both the Romas and Warner and Han et al amplitude matchings are correct.

ii) Romans and Warner use the second definition of the Ricci tensor given above, and omit to mention this in their conversion table. In this case the Ricci tensor and curvature scalar terms in the $O(\alpha')$ part of the Romans and Warner action enter with the opposite sign, though the Riemann tensor terms stay the same. This implies that the amplitude matching for the Romans and Warner action will differ slightly from that given above. The Han et al results will be unaffected by a redefinition of the Ricci tensor, since the $O(\alpha')$ sector of the Han et al action does not depend on the Ricci tensor or curvature scalar. The changes to the Romans and Warner amplitude matching will be discussed now. It will be shown that the conclusions stand even though the details of the matching change slightly.

If the Ricci tensor is assumed to have the second form defined above in the action of Romans and Warner, then the only terms that vary under this redefinition are,

$$\mathfrak{L}'' = \gamma\left\{ -\frac{3}{2} e\, \phi^{-3/4}\, \overline{\psi}_{[\rho} \gamma^{\mu\nu\alpha} \mathfrak{D}_{\sigma} \psi_{\alpha]} \left[2 R_{[\nu}{}^{[\sigma} \delta_{\mu]}^{\rho]} - \frac{1}{3} R\, \delta^{\sigma}_{[\nu} \delta^{\rho}_{\mu]} \right] \right\}$$

(4.3.3)

It is therefore only necessary to examine the Feynman diagrams that are affected by these terms in the action. It is immediately clear that the second of these two terms never contributes to the amplitude matching in any way, and so it only remains to examine the effect of the change of sign of the first term. The Ricci tensor term only contributes to the four point $\bar{\psi}$-ψ-h-h Feynman vertex in the term,

$$-\bar{\psi}^{\alpha}\gamma^{\nu}\partial_{\sigma}\psi_{\alpha}\left\{-\frac{1}{2}h^{\eta\xi}\partial_{\nu}\partial^{\sigma}h_{\eta\xi}-\frac{1}{4}\partial^{\sigma}h^{\eta\xi}\partial_{\nu}h_{\eta\xi}\right\}$$

(4.3.4)

and to the three point $\bar{\psi}$-ψ-h vertex where the graviton h is taken off-shell,

$$-\frac{1}{4}\left\{\partial_{\sigma}\bar{\psi}_{\alpha}\gamma^{\nu}\psi^{\alpha}-\bar{\psi}_{\alpha}\gamma^{\nu}\partial_{\sigma}\psi^{\alpha}\right\}.\left\{\partial^{\sigma}\partial_{\mu}\underline{h}^{\mu}{}_{\nu}-\partial^{\mu}\partial_{\mu}\underline{h}^{\sigma}{}_{\nu}\right\}$$

(4.3.5)

where the Ricci tensor redefinition has been taken into account in the two expressions above. It is important to stress that no other vertices are affected in any way, (so the three point matching is not altered by a convenient sign for example) so the diagrams which do not use these vertices are unchanged. It can be seen that both these vertices contribute only to 's'-channel amplitude terms. This implies that the 's'-channel matching condition must be modified, but that the 't-u'-channel matching conditions are unaffected. Since these are enough to demonstrate that the Romans and Warner action is unable to match the string amplitude even after taking into account the ambiguity of sign in the definition of the Ricci tensor, the changes due the redefinition will not be considered explicitly. The conclusion that the Romans and Warner action cannot be considered as the low energy effective action for the herterotic string is unchanged. It must be stressed most emphatically that the Han et al amplitude considered above will not change in any way under the Ricci tensor redefinition implying that the conclusion that the Han et al action cannot be considered as the low energy action for the string continues to hold. In the more general amplitude matching calculation that follows in the next chapter it is obvious by the way in which the action is constructed that the consideration of the particular definition of the Ricci tensor is irrelevant.

Chapter Five.
The Interacting Field Theory III:
The General Case.

Introduction.

This chapter extends the attempts of Chapter Four to derive the low energy effective action for the heterotic superstring to the O(α') order. This is done by constructing the most general action possible subject to the restriction that skewsymmetrisation of the terms of the form,

$$\mathfrak{D}_{[\rho}\psi_{\sigma]} = \partial_{[\rho}\psi_{\sigma]} + \frac{1}{4}\gamma_m\gamma_n\psi_{[\sigma}\partial^{[n}h_{\rho]}{}^{m]}$$

will be assumed. This restriction will obviously limit the absolute generality of the action used to the standard types of supergravity type terms of references [43,52,53]. There will be no discussion of supersymmetry in this chapter due to the extreme complexity of the action used, though a general discussion of supersymmetry of such extended actions will be attempted in the context of field redefinition analysis. The general action developed below will be shown also to be inadequate as a low energy effective description of the string by the same argument of amplitude matching as in the previous two chapters.

Section 5.1: The most general Lagrangian and its Feynman Rules.

In this section the most general possible action will be constructed, and the corresponding Feynman vertices evaluated. It should be noted that the approach that will be followed here is completely different in spirit to the Noether method technique which was used in references [52,53] in the two trial actions tested above by the amplitude matching procedure. In this chapter the action is found by a direct method completely ignoring *any* symmetries that the final action must have, particularly supersymmetry. It is assumed that the symmetry transformations that correspond to the supersymmetry, local Lorentz symmetry, general coordinate invariance, and so on, that leave the action invariant can be found later and shown to satisfy the appropriate algebraic conditions. The symmetry of the action is assumed to be assured by the symmetries of the string which generate the amplitudes that are matched. In the Noether methods used in the Romans and Warner and Han et al actions the symmetries are postulated a priori, and the actions derived from these by consistency arguments. Clearly the string may not generate these particular forms of the symmetry transformations in its low energy limit. It is in fact quite possible that the supersymmetry transformations, for example, that arise from the string in the low energy limit, (although these are more or less forced to be the same as those of the Chapline and Manton action at the $O(\alpha'^0)$ level due to the Noether method applied as in reference[70]), may be very complicated and non-geometrical at the $O(\alpha')$ level and above, the full string supersymmetry only being retrieved at the $n \to \infty$ limit of the $O(\alpha'^n)$ expansion. It is conceivable, though unpalatable, that the supersymmetry transformations may take a very un-geometrical form, including complicated higher derivative terms which maintain supercovariance only through pathological cancellations, which nevertheless close the supersymmetry algebra <u>order by order</u> in the α' expansion. Such a supposition may appear somewhat surprising, but it does not seem so when considered in the light of the comments of the section dealing with supersymmetry of the interacting superstring in reference [27], where it is noted that the interacting string supersymmetry transformations do not form a closing algebra, unless one works in all the string pictures simultaneously. The most that is claimed for the current algebra of supersymmetry transformations developed there is that they are suggestive of a full supersymmetry algebra. It is not clear how these transformations relate to the transformations of the low energy effective action.

The construction of the general action, its corresponding Feynman rules and the contruction of amplitudes in this chapter will proceed by observing the similarities of the action to the less general Romans and Warner or Han et al. actions. The first stage is to classify the various types of terms that can be added to the action subject to the constraint given above. This is simple to do. The terms that are chosen for the most general action are further restricted by the amplitude matchings that will be performed to fix the coefficients in the action, that is the general terms chosen will only be those that contribute to the amplitudes considered for the matching procedure. These amplitudes will be as in Chapter four above, that is the $\overline{\psi}$-ψ-h amplitude and $\overline{\psi}$-ψ-h-h amplitude at $O(\alpha')$. This means that only certain types of terms need be considered. For example, since there is <u>no</u> $O(\alpha'^0)$ $\overline{\psi}$-λ-h vertex with the λ field off-shell, it will be unnecessary to consider any higher derivative terms of the generic form,

$$\bar{\lambda}\,\gamma\,\mathfrak{D}\psi\,R \qquad (5.1.1)$$

or of the form,

$$\mathfrak{D}\bar{\lambda}\,\gamma\,\mathfrak{D}\,\mathfrak{D}\psi \qquad (5.1.2)$$

since these can never contribute to the amplitude matchings. The arguments presented in Chapter Four to show that no antisymmetric propagator diagrams need be considered continue to hold in this case, and so no generalised antisymmetric tensor vertices need be considered, and so no action terms will be constructed.

The only types of terms which need to be considered for the amplitude matching calculations performed below are therefore easily classified into the sets of terms,

a) Fermionic Terms.

i) $\bar{\psi}\gamma\mathfrak{D}\psi\, R_{\mu\nu}{}^{\rho\sigma}$: γ = antisymmetric product of three gamma matrices.
ii) $\bar{\psi}\gamma\mathfrak{D}\psi\, R_{\mu\nu}{}^{\rho\sigma}$: γ = single gamma matrix.
iii) $\bar{\psi}\gamma\mathfrak{D}\psi\, R_{\mu}{}^{\sigma}$: γ = any product of gamma matrices.
iv) $\mathfrak{D}\bar{\psi}\gamma\mathfrak{D}\mathfrak{D}\psi$: γ = single gamma matrix.
v) $\mathfrak{D}\bar{\psi}\gamma\mathfrak{D}\mathfrak{D}\psi$: γ = antisymmetric product of three gamma matrices.

b) Bosonic Terms.

i) $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$
ii) $R_{\mu\nu}R^{\mu\nu}$
iii) R^2

most of which have been seen in the previous chapter in a less general way. Each of these classes of terms can be further subdivided into the most general set of possible terms. This will be done class by class. The bosonic class is already the most general. It remains only to consider the fermionic terms.

i) The most general set of terms containing a completely antisymmetrised product of three gamma matrices, and a Riemann tensor is highly restricted by the symmetries of the Riemann tensor, and although many terms can be written, only two of these terms are independent. These are;

$\underline{K_1}$:-

$$\bar{\psi}_\mu\gamma^{\rho\sigma\tau}\mathfrak{D}_{[\nu}\psi_{\tau]}R_{\rho\sigma}{}^{\mu\nu} \qquad (5.1.3)$$

$\underline{K_3}$:-

$$\bar{\psi}_\tau\gamma^{\tau\rho\sigma}\mathfrak{D}_{[\mu}\psi_{\nu]}R_{\rho\sigma}{}^{\mu\nu} \qquad (5.1.4)$$

where the underlined letter before the colon denotes the arbitrary coefficient that will be associated with the term throughout the calculation of the Feynman rules, calculation of the amplitudes, and finally the amplitude matching process. This notation will be used throughout. Each of these terms will give contributions to the

Feynman rules described below.

ii) The same arguments using symmetry can be applied to the most general set of terms containing a single gamma matrix and a Riemann tensor. In this case there is only one independent term,

$\underline{K}_2$:-

$$\bar{\psi}_\mu \gamma^\sigma \mathfrak{D}_{[\nu}\psi_{\rho]} R_\sigma{}^{\mu\nu\rho} \qquad (5.1.5)$$

which completes the terms containing the Riemann tensor.

iii) There are only two terms containing Ricci tensors and any number of gamma matrices which are independent, and which contribute to the amplitudes calculated later. These are of the form,

$\underline{l}_1$:-

$$\bar{\psi}^\nu \gamma^\sigma \mathfrak{D}_{[\nu}\psi_{\rho]} R_\sigma{}^\rho \qquad (5.1.6)$$

$\underline{l}_2$:-

$$\bar{\psi}_\sigma \gamma^{\rho\sigma\tau} \mathfrak{D}_{[\nu}\psi_{\tau]} R_\rho{}^\nu \qquad (5.1.7)$$

which will give Feynman rules which will be seen to contribute to the 's'-channel of the matching.

There are several other terms which are independent of these two, but which do not contribute to any of the amplitudes considered below.

iv) There is only one term which explicitly contains only higher covariant derivative combinations of the gravitino fields and a single gamma matrix. This is,

L:-

$$\mathfrak{D}^{[\mu}\bar{\psi}^{\nu]}\gamma^\rho \mathfrak{D}_\rho \mathfrak{D}_{[\mu}\psi_{\nu]} \qquad (5.1.8)$$

which provides a propagator correcting term in the background field expansion zero graviton limit.

v) There is only one term which explicitly contains only higher covariant derivative combinations of the gravitino fields and an antisymmetrised product of three gamma matrices. This is,

L':-

$$\mathfrak{D}^{[\mu}\overline{\psi}^{\nu]}\gamma^{\rho\sigma\tau}\mathfrak{D}_{\tau}\mathfrak{D}_{[\mu}\psi_{\sigma]}g_{\nu\rho} \qquad (5.1.9)$$

which is also propagator correcting in the background field expansion limit. The first of these last two terms has appeared in the Han et al. action, but the second is an entirely new term. This completes the list of the terms which comprise the general higher derivative action which may be compared with the string. This means that the most general action, subject to the constraint mentioned above, is of the form,

$$\begin{aligned}\mathfrak{L}' = e\phi^{-3/4}\{ & x\,[R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4y\,R_{\mu\nu}R^{\mu\nu} + z\,R^2] \\ & + \kappa_1\overline{\psi}_\mu\gamma^{\rho\sigma\tau}\mathfrak{D}_{[\nu}\psi_{\tau]}R^{\mu\nu}{}_{\rho\sigma} + \kappa_2\overline{\psi}_\mu\gamma^{\sigma}\mathfrak{D}_{[\nu}\psi_{\rho]}R_{\sigma}{}^{\mu\nu\rho} \\ & + \kappa_3\overline{\psi}_\tau\gamma^{\tau\rho\sigma}\mathfrak{D}_{[\mu}\psi_{\nu]}R^{\mu\nu}{}_{\rho\sigma} + l_1\overline{\psi}^{\mu}\gamma^{\sigma}\mathfrak{D}_{[\mu}\psi_{\rho]}R_{\sigma}{}^{\rho} \\ & + l_2\overline{\psi}_\mu\gamma^{\rho\mu\tau}\mathfrak{D}_{[\nu}\psi_{\tau]}R_{\rho}{}^{\nu} + 4L\mathfrak{D}^{[\mu}\overline{\psi}^{\nu]}\gamma^{\rho}\mathfrak{D}_{\rho}\mathfrak{D}_{[\mu}\psi_{\nu]} \\ & + 4L'\mathfrak{D}^{[\mu}\overline{\psi}^{\nu]}\gamma^{\rho\sigma\tau}\mathfrak{D}_{\tau}\mathfrak{D}_{[\mu}\psi_{\sigma]}g_{\nu\rho}\,\}\end{aligned} \qquad (5.1.10)$$

where it should be emphasised that <u>only</u> the terms which contribute to the amplitudes considered below are kept. The procedure is now as follows: the Feynman rules will be calculated, then the appropriate amplitudes will be evaluated using them. The standard comparison to the string will then be made.

The Feynman rules are calculated by the same tedious and laborious methods used in the previous two chapters. First the propagators need to be calculated, then the Feynman vertices need to be derived subject to the rules developed in Chapter Three.

<u>i) The Propagators.</u>

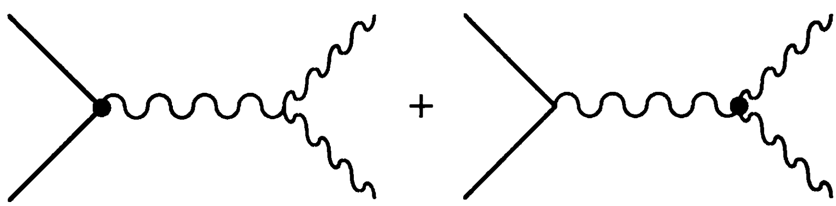
The propagators are simply the generalisation of the form of the propagators derived subject to the assumption introduced in Chapter Two and applied in Chapter Four. The general propagators are derived in Appendix Two. The graviton propagator is of the general form,

$$D_{\mu\nu,\alpha\beta} = \frac{2i}{k^2}.\Big\{ -2P^{(1)}_{\mu\nu,\alpha\beta} - \frac{2}{(1+8k^2x(1-y))} P^{(2)}_{\mu\nu,\alpha\beta}$$
$$+ \frac{2}{((\theta-1) - 8k^2x((1-y)+\theta(y-z)))} P^{(s)}_{\mu\nu,\alpha\beta}$$
$$- 2\frac{((\theta-2) - 16k^2x((1-y)+\theta(y-z)))}{((\theta-1) - 8k^2x((1-y)+\theta(y-z)))} P^{(\omega)}_{\mu\nu,\alpha\beta}$$
$$+ 2\frac{\sqrt{\theta}}{((\theta-1) - 8k^2x((1-y)+\theta(y-z)))} (P^{(s\omega)}_{\mu\nu,\alpha\beta} + P^{(\omega s)}_{\mu\nu,\alpha\beta}) \Big\}$$
(5.1.11)

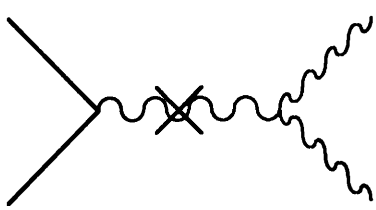
using the standard projection operators used in Appendix Two), which can be Taylor expanded to give the $O(\alpha'^0)$ and $O(\alpha')$ terms, which will be used in the calculation of amplitudes. The $O(\alpha'^0)$ must be the same as the propagator derived by pure canonical means in Chapter Two. This is, (in the notation of Appendix Two),

$$D^{(0)}_{\mu\nu,\alpha\beta} = \frac{2i}{k^2} \left\{ \delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\mu\beta}\delta_{\nu\alpha} - \frac{1}{4}\delta_{\mu\nu}\delta_{\alpha\beta} \right\}$$
(2.2.15)

which can be seen to occur in the standard $O(\alpha')$ diagrams,



The $O(\alpha')$ propagator correction for the graviton gives rise to extra diagrams which must be considered, for example,



The $O(\alpha')$ propagator correction is given by the standard Taylor expansion and is,

$$D^{(1)}_{\mu\nu,\alpha\beta} = i \{ 16x(1-y)(\delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\mu\beta}\delta_{\nu\alpha})$$

$$- 16x(1-y)(\delta_{\mu\alpha}\omega_{\nu\beta} + \delta_{\mu\beta}\omega_{\nu\alpha} + \delta_{\nu\beta}\omega_{\mu\alpha} + \delta_{\nu\alpha}\omega_{\mu\beta})$$

$$- \frac{32}{9} x(1-y)(\delta_{\mu\nu} - \omega_{\mu\nu})(\delta_{\alpha\beta} - \omega_{\alpha\beta})$$

$$+ \frac{1}{2} x((1-y)+9(y-z)) [\frac{1}{9} (\delta_{\mu\nu} - \omega_{\mu\nu})(\delta_{\alpha\beta} - \omega_{\alpha\beta})$$

$$+ 9\, \omega_{\mu\nu}\omega_{\alpha\beta} + 3\, (\delta_{\mu\nu}\omega_{\alpha\beta} + \delta_{\alpha\beta}\omega_{\mu\nu} - 2\omega_{\mu\nu}\omega_{\alpha\beta})] \}$$

(5.1.12)

which can be seen to be extremely complicated. The next higher order correction, the $O(\alpha'^2)$ one is very much more complicated, and is given in Appendix Two. The great complexity of these terms is negated somewhat by the truncation convention chosen for the calculation of four point diagrams. It can be seen that most of the terms in the propagator corrections vanish trivially by the truncation procedure, as will be demonstrated later.

The complete gravitino propagator derived from the generalised action of equation (5.1.10) is of the form,

$$Y_{\mu,\nu} = - \frac{2i\not{k}}{k^2} \{ \frac{1}{(2 + (8L-4L')k^2)} P^{3/2}_{\mu,\nu} - \frac{1}{(2(\theta-1) - (8L+ 4L'(\theta-1))k^2)} P^{1/2}_{11\mu,\nu}$$

$$+ \frac{(2 - \theta + (8L+ 4L'(\theta-1))k^2)}{(2(\theta-1) - (8L+ 4L'(\theta-1))k^2)} P^{1/2}_{22\mu,\nu}$$

$$+ \frac{\sqrt{\theta}}{(2(\theta-1) - (8L+ 4L'(\theta-1))k^2)} (P^{1/2}_{12\mu,\nu} - P^{1/2}_{21\mu,\nu}) \}$$

(5.1.13)

which is derived explicitly in Appendix Two. This can be expanded as in the graviton case by a Taylor expansion. The Taylor expansion of the gravitino case is similar to that of the graviton. As in the case of graviton, the $O(\alpha'^0)$ uncorrected propagator is simply the standard propagator, calculated by standard canonical manipulations, which in the covariant gauge is,

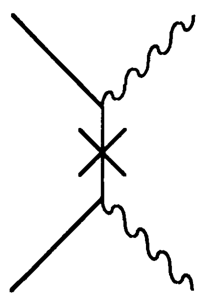
$$Y^{(0)}_{\mu,\nu} = \frac{i}{8k^2} \{ \gamma_\nu \not{k} \gamma_\mu - 6\, (\eta_{\mu\nu}\not{k} - \frac{2k_\mu k_\nu \not{k}}{k^2}) \} \quad (5.1.14)$$

and where the $O(\alpha')$ correction to this is,

$$
\begin{aligned}
Y^{(1)}_{\mu,\nu} = -2i\,(2L-L')\,\Big\{ &-\frac{50}{64}\,\eta_{\mu\nu}\not{k} + \frac{7}{64}\,\gamma_\nu \not{k}\, \gamma_\mu - \frac{1}{8}\,(\gamma_\mu k_\nu + \gamma_\nu k_\mu) \\
&+ \frac{68}{64}\,\frac{k_\mu k_\nu \not{k}}{k^2} \Big\} \\
- 2iL' \Big\{ &-\frac{1}{32}\,\eta_{\mu\nu}\not{k} - \frac{1}{64}\,\gamma_\nu \not{k}\, \gamma_\mu + \frac{100}{64}\,\frac{k_\mu k_\nu \not{k}}{k^2} \\
&- \frac{1}{8}\,(k_\mu \gamma_\nu + \gamma_\mu k_\nu) \Big\}
\end{aligned}
\qquad (5.1.15)
$$

which can be seen to explicitly vanish only for the trivial choice of parameters, i.e. $L = L' = 0$. That is the action is only manifestly unitary when neither of the propagator correcting terms exist in the action. This is quite different to the case of the graviton, where the choice of the Gauss-Bonnet combination produces a manifestly unitary theory[51,54].

Again as in the graviton case, this new propagator generates extra diagrams from the Wick expansion, which must be evaluated. These are of the form,



The complete set of diagrams that have to be considered is now, as for the Han et al action given in Figure 4.4. All that remains to be derived before these diagrams can be evaluated are the Feynman vertices. The techniques used in these calculations have all been developed in the previous chapters, and so a recapitulation is unnecessary. The appropriate vertices will therefore be stated rather than derived. The calculation of the various diagrams can then be performed.

The Feynman vertices will be listed in a systematic way. First the three point vertex where all fields are on shell, which allows the three point calculation to be performed will be stated. For the four point amplitude a set of vertices are needed, which are;

i) the $\underline{h}$-h-h vertex, $\underline{h}$ off-shell,

ii) the $\overline{\psi}$-ψ-$\underline{h}$ vertex, $\underline{h}$ off-shell,

iii) the $\overline{\psi}$-ψ-h vertex, $\overline{\psi}$ or ψ off-shell,

iv) the $\overline{\psi}$-ψ-h-h vertex, all on-shell.

which will be simply stated below. Some of the terms in the extended action are extremely difficult to expand. Most of these have been examined previously for the individual cases of the Romans and Warner, and Han et al. Lagrangians. The arbitrary coefficients will keep account of the individual terms in the action. The vertices will all be symmetrised according to the rules developed in Chapter Three, and will thus be ready simply to construct the Feynman diagrams and finally the amplitude. It may be the case that it will be simpler to write the vertices in parts, corresponding to individual terms in the extended action. It will be obvious when this occurs.

i) h-h-h vertex, h off-shell.

This vertex is given by identical arguments as used in the previous chapters. The vertex is,

$$x\left\{\left(\frac{1}{2}-3y+3z\right)\underline{h}_{\mu}{}^{\mu}\partial^{\sigma}\partial^{\nu}h^{\rho\alpha}\partial_{\sigma}\partial_{\nu}h_{\rho\alpha}+(-1+2y-3z)\underline{h}^{\sigma\nu}\partial_{\alpha}h^{\rho\mu}\partial^{\alpha}\partial_{\sigma}\partial_{\nu}h_{\rho\mu}\right.$$
$$\left.-(5+4y)\underline{h}_{\alpha}{}^{\mu}\partial^{\sigma}\partial^{\nu}h^{\rho\alpha}\partial_{\sigma}\partial_{\nu}h_{\rho\mu}+(4y-3z)\underline{h}_{\mu\nu}\partial^{\mu}\partial_{\alpha}h_{\sigma}{}^{\rho}\partial^{\alpha}\partial^{\nu}h_{\rho}{}^{\sigma}\right\} \tag{5.1.16}$$

which can be seen to correspond to the vertices derived from the Han et al. and Romans and Warner Lagrangians when suitable values for the coefficients x,y and z are chosen. For example the Han et al form of the vertex clearly corresponds to the choice $x = -\gamma/2$, $y = z = 0$. (Note the factor γ has been included here even though it was suppressed for the sake of clarity in the derivations of Chapter Four).

ii) $\overline{\psi}$-ψ-h vertex, h off-shell.

There are several contributions to this vertex from different sectors of the action. It will be desirable in this instance to separate the various contributions for clarity. The vertex due to the terms given by the Ricci tensor terms given by the coefficients l_1 and l_2 is,

$$-\frac{1}{8}(l_1+l_2)\left\{\overline{\psi}^{\nu}\gamma^{\rho}\partial_{\rho}\psi_{\nu}-\partial_{\rho}\overline{\psi}_{\nu}\gamma^{\rho}\psi^{\nu}\right\}\partial_{\mu}\partial^{\mu}\underline{h}_{\sigma}{}^{\rho} \tag{5.1.17}$$

and the similar contribution due to the pure gravitino terms denoted by the coefficients L and L' is,

$$\frac{(L-L')}{2}\{\ \partial^{\rho}\overline{\psi}^{\sigma}\gamma^{\nu}\partial_{\mu}\partial_{\rho}\psi_{\sigma}-\partial_{\mu}\partial_{\rho}\overline{\psi}_{\sigma}\gamma^{\nu}\partial^{\rho}\psi^{\sigma}\ \}\underline{h}_{\nu}{}^{\mu} \tag{5.1.18}$$

which sum to give the full vertex, since these are the only contributions to the vertex.

<u>iii) $\overline{\psi}$-ψ-h vertex, $\overline{\psi}$ or ψ off-shell.</u>

This vertex comes only from the Lagrangian terms denoted by the coefficients κ_1, κ_2, κ_3, L and L'. The vertex due to the κ coefficients is,

$$\{\ -\frac{1}{4}(\kappa_1+2\kappa_3)\,\overline{\psi}_{\mu}\gamma^{\rho}\gamma^{\sigma}\gamma^{\tau}\partial_{\nu}\psi_{\tau}\partial_{\sigma}\partial^{\nu}h_{\rho}{}^{\mu}+\frac{1}{4}\kappa_1\overline{\psi}_{\mu}\gamma^{\rho}\gamma^{\sigma}\gamma^{\tau}\partial_{\tau}\psi_{\nu}\partial_{\sigma}\partial^{\nu}h_{\rho}{}^{\mu}$$
$$-\frac{1}{2}(\kappa_1+\kappa_2+2\kappa_3)\,\overline{\psi}_{\mu}\gamma^{\sigma}\partial_{\nu}\psi_{\tau}\partial_{\sigma}\partial^{\nu}h^{\mu\tau}+\frac{1}{4}(\kappa_1+\kappa_2+2\kappa_3)\,\overline{\psi}_{\mu}\gamma^{\rho}\partial_{\nu}\psi_{\tau}\partial^{\tau}\partial^{\nu}h_{\rho}{}^{\mu}\ \} \tag{5.1.19}$$

and that due to the L and L' terms is,

<u>L</u>:-

$$L\ \{\ -\partial^{\sigma}\overline{\psi}^{\alpha}\gamma^{\mu}\partial_{\mu}\partial_{\sigma}\psi_{\rho}h^{\rho}{}_{\alpha}+\partial^{\rho}\overline{\psi}^{\sigma}\gamma^{\mu}\partial_{\rho}\psi_{\lambda}\partial^{\lambda}h_{\sigma\mu}$$
$$-\partial^{\rho}\overline{\psi}^{\sigma}\gamma^{\mu}\partial_{\lambda}\psi_{\rho}\partial^{\lambda}h_{\sigma\mu}\ \} \tag{5.1.20}$$

<u>L'</u>:-

$$L'\ \{\ -\frac{1}{4}\partial^{\rho}\overline{\psi}^{\mu}\gamma^{m}\gamma^{\nu}\gamma^{\tau}\partial_{\tau}\partial_{\rho}\psi_{\nu}h_{\mu m}+\partial^{\nu}\overline{\psi}^{\mu}\gamma^{m}\psi_{\rho}\partial_{\nu}\partial^{\rho}h_{\mu m}$$
$$+\frac{1}{2}\partial^{\tau}\overline{\psi}_{\lambda}\gamma_{\mu}\partial^{\mu}\psi^{\rho}\partial_{\tau}h^{\lambda}{}_{\rho}\ \} \tag{5.1.21}$$

which give the full $\overline{\psi}$-ψ-h vertex for ψ off-shell on addition.

<u>iv) $\overline{\psi}$-ψ-h-h vertex, all on-shell.</u>

This vertex has contributions from most sectors of the theory. It will be desirable to split them up for the sake of clarity. The κ_1, κ_2, κ_3 contribution is of the form,

$\underline{\kappa_1}$:-

$$\frac{\kappa_1}{2}\ \{\ \bar{\psi}_\mu\gamma^\rho\partial_\nu\psi^\rho[\ h^{\mu\eta}\partial_\sigma\partial^\nu h_{\eta\rho}+h_{\rho\tau}\partial_\sigma\partial^\nu h^{\tau\mu}+\frac{1}{2}\partial^\nu h_\rho{}^\lambda\partial_\sigma h_\lambda{}^\mu\ ]$$

$$-\ \bar{\psi}_\mu\gamma^\sigma\partial^\rho\psi_\nu\ [\ -\frac{1}{2}\partial_\sigma h^{\lambda\nu}\partial_\rho h_\lambda{}^\mu+\frac{1}{2}\partial_\rho h^{\nu\lambda}\partial_\sigma h_\lambda{}^\mu\ ]$$

$$+\frac{1}{2}\bar{\psi}_\mu\gamma^\rho\gamma^\sigma\gamma^t\partial_\nu\psi_\tau h^\tau{}_t\partial_\sigma\partial^\nu h_\rho{}^\mu\ \}\qquad(5.1.22)$$

$\underline{\kappa_2}$:-

$$\frac{\kappa_2}{2}\{\ \bar{\psi}_s\gamma^\rho\partial_\nu\psi_\mu h^{s\sigma}\partial_\rho\partial^\nu h_\sigma{}^\mu+\frac{1}{2}\bar{\psi}^\sigma\gamma^\rho\partial_\mu\psi_\nu h^{\nu\eta}\partial_\rho\partial^\mu h_{\eta\sigma}\ \}$$

(5.1.23)

$\underline{\kappa_3}$:-

$$\kappa_3\ \{\ \frac{1}{2}\ \bar{\psi}^\sigma\gamma^\rho\partial_\mu\psi_\nu\ [\ h^{\nu\eta}\partial_\rho\partial^\mu h_{\eta\sigma}+2\ h_{\sigma\eta}\partial_\rho\partial^\mu h^{\eta\nu}\ ]$$

$$+\frac{1}{2}\bar{\psi}_\tau\gamma^t\gamma^\sigma\gamma^\rho\partial_\nu\psi_\mu h_t{}^\tau\partial_\sigma\partial^\nu h_\rho{}^\mu\ \}\qquad(5.1.24)$$

The L and L' contribution is,

$\underline{L}$:-

$$-\frac{L}{4}\{\ \partial^\sigma\bar{\psi}^\rho\gamma^\mu\gamma_m\gamma_n\psi_\rho\partial^n h_\lambda{}^m\partial_\sigma h^\lambda{}_\mu-\partial^\sigma\bar{\psi}^\rho\gamma^\mu\gamma_m\gamma_n\psi_\lambda\partial^n h_\rho{}^m\partial_\sigma h^\lambda{}_\mu$$

$$+\frac{1}{2}\partial^\sigma\bar{\psi}^\rho\gamma^\mu\gamma_m\gamma_n\partial_\sigma\psi_\rho h_{\mu\eta}\partial^n h^{\eta m}\ \}$$

(5.1.25)

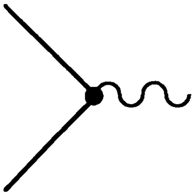
$\underline{L'}$:-

$$\frac{L'}{2}\ \{\bar{\psi}_m\gamma_n\partial_\nu\psi_\lambda\partial^n h_\alpha{}^m\partial^\nu h^{\alpha\lambda}+\frac{1}{4}\partial^\rho\bar{\psi}^\nu\gamma^\tau\gamma_m\gamma_n\partial_\rho\psi_\nu h^{m\eta}\partial_\tau h_\eta{}^n$$

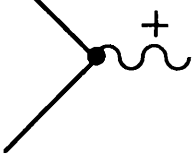
$$+\frac{3}{2}\ \partial^\tau\bar{\psi}^\rho\gamma^n\gamma_p\gamma_q\psi_{[\rho}\partial^q h_{\nu]}{}^p\partial_\tau h_n{}^\nu\ \}$$

(5.1.26)

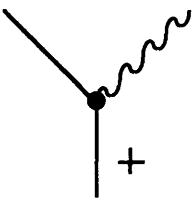
where the vertex contributions have not been explicitly symmetrised for the sake of



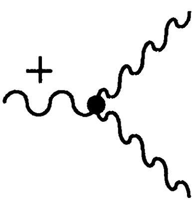
$$\frac{\alpha'}{2}\left\{ -\frac{1}{2}(\kappa_1+\kappa_2+2\kappa_3) - L + L' \right\}\partial^{\nu}\overline{\psi}^{\rho}\gamma^{\tau}\partial_{\rho}\psi_{\lambda}\partial^{\lambda}h_{\nu\tau}$$



$$\frac{\alpha'}{2}\left\{ \frac{(L-L')}{4} - \frac{(l_1+l_2)}{8} \right\}\left(\overline{\psi}^{\nu}\gamma^{\sigma}\partial_{\rho}\psi_{\nu} - \partial_{\rho}\overline{\psi}^{\nu}\gamma^{\sigma}\psi_{\nu} \right)\partial_{\mu}\partial^{\mu}\,\underline{h}_{\sigma}{}^{\rho}$$

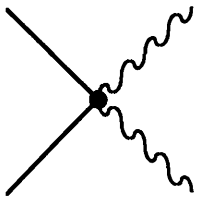


$$\frac{\alpha'}{2}\left\{ \frac{-(\kappa_1+2\kappa_3+L')}{4}\overline{\psi}_{\mu}\gamma^{\rho}\gamma^{\sigma}\gamma^{\tau}\partial_{\nu}\psi_{\tau}\partial_{\sigma}\partial^{\nu}h_{\rho}{}^{\mu} + \frac{\kappa_1}{4}\overline{\psi}_{\mu}\gamma^{\rho}\gamma^{\sigma}\gamma^{\tau}\partial_{\tau}\psi_{\nu}\partial_{\sigma}\partial^{\nu}h_{\rho}{}^{\mu} \right.$$
$$+ \frac{(2L-L')}{2}\overline{\psi}_{\mu}\gamma^{\rho}\partial_{\nu}\partial^{\tau}\psi_{\tau}\partial^{\nu}h_{\rho}{}^{\mu} + \frac{(\kappa_1+\kappa_2+2\kappa_3-4L'+8L)}{4}\overline{\psi}_{\mu}\gamma^{\rho}\partial_{\nu}\psi_{\tau}\partial^{\tau}\partial^{\nu}h_{\rho}{}^{\mu}$$
$$\left. + \frac{(\kappa_1+\kappa_2+2\kappa_3-2L-L')}{2}\overline{\psi}_{\mu}\gamma^{\rho}\partial_{\rho}\partial_{\nu}\psi_{\tau}\partial^{\nu}h^{\mu\tau} \right\}$$



$$\frac{\alpha' x}{2}\left\{ \left(\frac{1}{2}-3y+3z\right)\underline{h}_{\mu}{}^{\mu}\partial^{\sigma}\partial^{\nu}h^{\rho\alpha}\partial_{\sigma}\partial_{\nu}h_{\rho\alpha} + (2y-3z-1)\underline{h}^{\sigma\nu}\partial_{\alpha}h^{\rho\mu}\partial^{\alpha}\partial_{\sigma}\partial_{\nu}h_{\rho\mu} \right.$$
$$\left. - (5+4y)\,\underline{h}_{\alpha}{}^{\mu}\partial^{\sigma}\partial^{\nu}h^{\rho\alpha}\partial_{\sigma}\partial_{\nu}h_{\rho\alpha} + (4y-3z)\,\underline{h}_{\mu\nu}\partial^{\mu}\partial_{\alpha}h_{\sigma}{}^{\rho}\partial^{\nu}\partial^{\alpha}h_{\rho}{}^{\sigma} \right\}$$

Fig. 5.1a The O(α') three point Feynman vertices used in the calculation of O(α') amplitudes.



$$\frac{\alpha'}{2}\Big\{\overline{\psi}_\mu\gamma^\sigma\partial_\nu\psi^\rho\Big[\Big(\frac{\kappa_1}{4}+\frac{\kappa_2}{2}+\kappa_3-\frac{L'}{2}\Big)h^{\mu\eta}\partial_\sigma\partial^\nu h_{\eta\rho}$$

$$+\frac{(2\kappa_1+\kappa_2+2\kappa_3)}{4}h_{\rho\eta}\partial_\sigma\partial^\nu h^{\eta\mu}\Big]$$

$$-\frac{\kappa_1}{2}\overline{\psi}_\mu\gamma^\sigma\partial^\rho\psi_\nu\,\partial_{[\rho}h^{\nu\eta}\partial_{\sigma]}h_\lambda^{\ \mu}$$

$$+\frac{(\kappa_1+2\kappa_3)}{4}\overline{\psi}_\mu\gamma^\rho\gamma^\sigma\gamma^t\partial_\nu\psi_\tau h^\tau_{\ t}\partial_\sigma\partial^\nu h_\rho^{\ \mu}$$

$$+\frac{\kappa_3}{2}\overline{\psi}_\mu\gamma^\rho\gamma^\sigma\gamma^t\psi_\tau\partial_\nu h^\tau_{\ t}\partial_\sigma\partial^\nu h_\rho^{\ \mu}$$

$$+\frac{(3L'-2L)}{8}\overline{\psi}_\rho\gamma_m\gamma_n\gamma^\mu\partial^\sigma\psi^\rho\partial^n h_\lambda^{\ m}\partial_\sigma h^\lambda_{\ \mu}$$

$$+\frac{(L-L')}{8}\partial^\sigma\overline{\psi}^\rho\gamma^\mu\gamma_n\gamma_m\partial_\sigma\psi_\rho h_{\mu\eta}\partial^n h^{\eta m}$$

$$+\frac{(2L-3L')}{8}\partial^\sigma\overline{\psi}^\rho\gamma^\mu\gamma_m\gamma_n\psi_\lambda\partial^n h_\rho^{\ m}\partial_\sigma h^\lambda_{\ \mu}$$

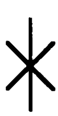
$$+\frac{(l_1+l_2)}{4}\overline{\psi}^\mu\gamma^\sigma\partial_\rho\psi_\mu h_{\eta\xi}\partial_\sigma\partial^\rho h^{\xi\eta}\Big\}$$

$$\frac{\alpha' i}{2}\Big\{16x(1-y)[(\delta_{\mu\alpha}\delta_{\nu\beta}+\delta_{\mu\beta}\delta_{\nu\alpha})-(\delta_{\mu\alpha}\omega_{\nu\beta}+\delta_{\mu\beta}\omega_{\nu\alpha}+\delta_{\nu\alpha}\omega_{\mu\beta}+\delta_{\nu\beta}\omega_{\mu\alpha})]$$

$$-\frac{32}{9}x(1-y).(\delta_{\mu\nu}-\omega_{\mu\nu}).(\delta_{\alpha\beta}-\omega_{\alpha\beta})$$

$$+\frac{1}{2}(x((1-y)+9(y-z)).\Big[\frac{1}{9}(\delta_{\mu\nu}-\omega_{\mu\nu}).(\delta_{\alpha\beta}-\omega_{\alpha\beta})$$

$$+9\,\omega_{\alpha\beta}\omega_{\mu\nu}+3(\delta_{\mu\nu}\omega_{\alpha\beta}+\delta_{\alpha\beta}\omega_{\mu\nu}-2\,\omega_{\mu\nu}\omega_{\alpha\beta})\Big]\Big\}$$



$$-\alpha' i(2L-L')\Big\{-\frac{50}{64}\eta_{\mu\nu}\not{k}+\frac{7}{64}\gamma_\nu\not{k}\gamma_\mu-\frac{1}{8}(\gamma_\mu k_\nu+\gamma_\nu k_\nu)+\frac{68}{64}\frac{k_\mu k_\nu\not{k}}{k^2}\Big\}$$

$$-\alpha' iL'\Big\{-\frac{1}{32}\eta_{\mu\nu}\not{k}-\frac{1}{64}\gamma_\nu\not{k}\gamma_\mu+\frac{100}{64}\frac{k_\mu k_\nu\not{k}}{k^2}-\frac{1}{8}(k_\mu\gamma_\nu+\gamma_\mu k_\nu)\Big\}$$

Fig. 5.1b The four point O(α') Feynman rule and propagator corrections. (Note the four point Feynman rule has not been skewsymmetrisedfor simplicity).

simplicity and clarity. Finally the l_1 and l_2 contribution is,

$$\frac{(l_1 + l_2)}{4}\bar{\psi}^{\mu}\gamma^{\sigma}\partial_{\rho}\psi_{\mu}h_{\eta\xi}\partial_{\sigma}\partial^{\rho}h^{\eta\xi} \qquad (5.1.27)$$

All of the contributions given above add to the contribution from the Lorentz Chern-Simons three form term, equation (4.1.38) to give the full vertex factor. The complete set of Feynman vertices used are summarised in Figure 5.1a and b. These Feynman factors include the α' factors omitted throughout in the discussion above. The figures also include the Feynman propagator corrections. The calculation of the amplitude is now merely a matter of multiplying these various factors together in the momentum representation in the manner of the formalism developed in Chapter Three. This is carried out in the next section.

Section 5.2: The amplitude calculation and amplitude matching.

In this section the full, general, amplitude for the ψ-ψ-h and ψ-ψ-h-h scattering processes are calculated. The amplitudes thus derived are then used in the amplitude matching procedure, to try to determine values for the general coefficients introduce into the general action. The three point matching will give a single relation in five of the arbitrary coefficients introduced in the action above. Clearly this is not enough to determine these coefficients on its own. The same coefficients arise in the four point matching, and so the three and four point matchings together may fix some of the coefficients in the general action. A consideration of the four point action is very similar to the specific cases dealt with in Chapter Four above. This will again be seen to separate into the 's'-channel and 't-u'-channel subterms used in the matching conditions in Chapter Four. The 's'-channel will be seen to give a possible match to the string for a particular set of values for the L' and γ coefficients in the general action given above. This can then be used in the 't-u' matching conditions. The 't-u'-channel matching conditions will be seen to reduce to a set of three linear simultaneous equations which can be solved. When the three point matching condition is reintroduced, however, there is an *extra* constraint on the general coefficients of the action. This means that the variables are overconstrained, and it would therefore require a remarkable coincidence for this system to be soluble. It can in fact be noted as an aside that two of the general cofficients always occur in the combination $\kappa_1 + \kappa_2$ and consequently only count as one degree of freedom in the matching. It transpires that a solution *is* in fact possible, and *still* leaves some ambiguity in the leading order terms in the effective action! This statement will now be proved using the standard amplitude matching procedure. The conclusions of this analysis will be discussed at the end of the chapter and in a little more detail in Chapter Six.

The first amplitude to consider is obviously the three point amplitude $\overline{\psi}$-ψ-h. The appropriate contributions come from the terms whose coefficients are κ_1, κ_2, κ_3, L and L'. All other terms have vanishing contribution to this amplitude. The amplitude is,

$$- \alpha' i \left\{ -\frac{1}{2}(\kappa_1 + \kappa_2 + 2\kappa_3) - L + L' \right\} \bar{u}_2^{\rho} \gamma^{\tau} u_1^{\lambda} \zeta_{3\tau}{}^{\nu} k_{1\rho} k_{2\nu} k_{3\lambda} \tag{5.2.1}$$

which can now be compared to the string subject to the normalisation condition used in Chapter Four. This implies that the matching condition is,

$$-\frac{1}{2}(\kappa_1 + \kappa_2 + 2\kappa_3) - L + L' = -\frac{1}{4} \tag{5.2.2}$$

which will be the first of five constraints in these variables. The four point amplitude will contribute the remaining matching conditions discussed above. The four point $\overline{\psi}$-ψ-h-h amplitude will be evaluated according to the Feynman diagram scheme, given in Figure 4.4. Each of the classes of diagrams will be evaluated in turn. Obviously all of the terms considered above in the general action contribute to this amplitude. It will be shown that the coefficients discussed in the three point case will be the overconstrained coefficients, and all the others will be completely underconstrained. Each of the diagrams will be treated in turn,

<u>i) The 'point' diagram.</u>

The amplitude contribution due to this diagram is most easily analysed by splitting the amplitude contributions into the separate contributions due to the individual subamplitudes quoted above. The amplitude contribution due to the κ terms is,

$$\left\{ \frac{(\kappa_1+\kappa_2+2\kappa_3)}{16} \left[(2u-t)\bar{u}_{2\mu} k\!\!\!/_4 u_{1\rho} - (2t-u)\bar{u}_{2\rho} k\!\!\!/_4 u_{1\mu}\right] \zeta_3^{\mu\eta} \zeta_{4\eta}{}^{\rho} \right.$$
$$\left. + \frac{1}{16}\left[(\kappa_1 u - 2\kappa_3 t)\bar{u}_{2\mu}\gamma^{\rho} k\!\!\!/_4 \gamma^{r} u_{1m} - (\kappa_1 t - 2\kappa_3 u)\bar{u}_{2m}\gamma^{r} k\!\!\!/_4 \gamma^{\rho} u_{1\mu}\right] \zeta_{3r}{}^{m} \zeta_{4\rho}{}^{\mu} \right\} \tag{5.2.3}$$

The contribution due to the L and L' terms is of the form, L:-

$$\Big\{ -\frac{1}{8}\Big[\frac{t}{2}\bar{u}_2^{\rho}\gamma^{\mu}\not{k}_4\gamma^{q}u_{1\rho} - \frac{u}{2}\bar{u}_2^{\rho}\gamma^{q}\not{k}_4\gamma^{\mu}u_{1\rho}\Big]\zeta_{3q}{}^{\sigma}\zeta_{4\sigma\mu}$$

$$+\frac{1}{8}\Big[\frac{t}{2}\bar{u}_2^{\rho}\gamma^{\mu}\not{k}_4\gamma^{q}u_1^{\sigma} - \frac{u}{2}\bar{u}_2^{\sigma}\gamma^{q}\not{k}_4\gamma^{\mu}u_1^{\rho}\Big]\zeta_{3\rho q}\zeta_{4\mu\sigma}$$

$$+\frac{s}{32}\Big[\bar{u}_2^{\sigma}\gamma_m\not{k}_4\gamma_n u_{1\sigma} - \bar{u}_{2\sigma}\gamma_n\not{k}_4\gamma_m u_1^{\sigma}\Big]\zeta_3^{m\nu}\zeta_{4\nu}{}^{n}\Big\} \tag{5.2.4}$$

L':-

$$\Big\{ \frac{1}{4}\Big[\frac{u}{2}\bar{u}_{2m}\not{k}_3u_{1\lambda} - \frac{t}{2}\bar{u}_{2\lambda}\not{k}_3u_{1m}\Big]\zeta_{3\alpha}{}^{m}\zeta_4^{\alpha\lambda}$$

$$+\frac{s}{32}\Big[\bar{u}_2^{\nu}\gamma_m\not{k}_3\gamma_n u_{1\nu} - \bar{u}_2^{\nu}\gamma_n\not{k}_3\gamma_m u_{1\nu}\Big]\zeta_3^{\eta m}\zeta_{4\eta}{}^{n}$$

$$+\frac{3}{16}\Big[\frac{t}{2}\bar{u}_2^{\nu}\gamma^{n}\not{k}_4\gamma_p u_{1\nu} - \frac{t}{2}\bar{u}_2^{\nu}\gamma_p\not{k}_4\gamma^{n}u_{1\nu}\Big]\zeta_{3}^{p}{}_{\rho}\zeta_4^{\rho}{}_{n}$$

$$-\frac{3}{16}\Big[\frac{t}{2}\bar{u}_2^{\rho}\gamma^{n}\not{k}_4\gamma_p u_{1\nu} - \frac{u}{2}\bar{u}_{2\nu}\gamma_p\not{k}_4\gamma^{n}u_1^{\rho}\Big]\zeta_{3}^{p}{}_{\rho}\zeta_4^{\nu}{}_{n}\Big\} \tag{5.2.5}$$

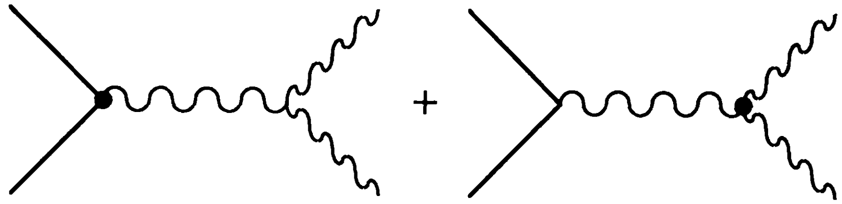
and the remaining contribution from the l_1 and l_2 terms is,

$$-\frac{(l_1+l_2)}{32}\bar{u}_2^{\tau}\not{k}_4u_{1\tau}\zeta_3^{\eta\xi}\zeta_{4\eta\xi}(u-t) \tag{5.2.6}$$

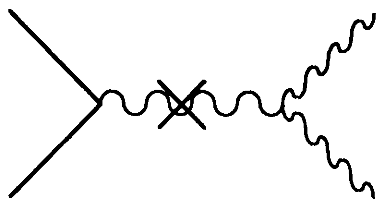
These contributions add to the standard Lorentz Chern-Simons contribution of equation (4.1.40), to give the full contribution. This will not be stated for the sake of simplicity. The contribution to the amplitude from this diagram can be seen to split up in an interesting way. It can be seen that the κ_1, κ_2, κ_3 coefficients which appear in the three point matching only give contributions to the 't-u'-channel subamplitudes, where the l_1 and l_2 coefficients only contribute to the 's'-channel. The L and L' give contributions to both sectors of the amplitude. This will be seen to be a common feature of the amplitude contributions in general. When considering the relative number of coefficients which contribute to each of these sectors, this would normally signify a distinction between the leading order terms and the nonleading order terms in the action, in the limited sense of the field redefinition analyses carried out in the bosonic matchings of references [51,56-61].

ii) The graviton exchange diagrams.

The diagrams that have to be considered in this sector are,



and the higher order propagator term,



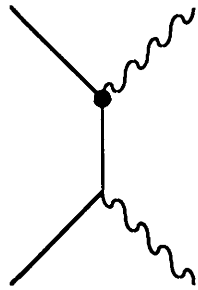
which include the higher order graviton propagator correction discussed above. Inspection of the various fermionic vertices required reveals that all of these diagrams are 's'-channel. The contribution from these diagrams is especially simple to calculate, given the vertices above and the truncation convention, and can be shown to be,

$$\frac{i}{32}\left\{2\,(L - L') - 12x - (l_1 - l_2)\right\}\bar{u}_2^{\rho}\not{k}_4 u_{1\rho}\,\zeta_3^{\eta\xi}\,\zeta_{4\eta\xi}(u - t) \tag{5.2.7}$$

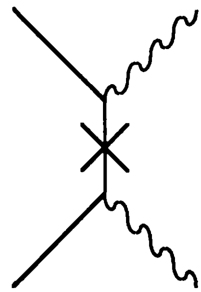
which completes the 's'-channel contributions to the amplitude. It is interesting to note that all y and z dependence has vanished from the amplitude. This may be constued to be due to invariance of the amplitude under field redefinitions. This point will be discusssed further in Chapter Six. All remaining diagrams will contribute only to the 't-u'-channel.

iii) Gravitino exchange diagrams.

The only diagrams that need to be considered in the case of the gravitino exchange diagrams are,



and



where the appropriate symmetrisations can be derived as in Chapters Three and Four by relabelling polarisation states, and using the Majorana spinor conjugation. These diagrams are most easily evaluated by using the toolbox of terms developed in Chapter Four. Doing this gives the first diagrammatic contribution in the form,

$\underline{\kappa}$:-

$$\Big\{ -\frac{i}{16}(\kappa_1+\kappa_2+2\kappa_3)\left[t\,\bar{u}_{2\beta}\not{k}_4 u_{1m} - u\,\bar{u}_{2m}\not{k}_4 u_{1\beta}\right]\zeta_3^{ms}\zeta_{4s}^{\ \ \beta} + \frac{i}{64}(3\kappa_1+\kappa_2+2\kappa_3)\left[u\,\bar{u}_{2\mu}\gamma^{\rho}\not{k}_4\gamma^{r}u_{1m} - t\,\bar{u}_{2m}\gamma^{r}\not{k}_4\gamma^{\rho}u_{1\mu}\right]\zeta_{3r}^{\ \ m}\zeta_{4\rho}^{\ \ \mu}\Big\} \tag{5.2.8}$$

$\underline{L}$:-

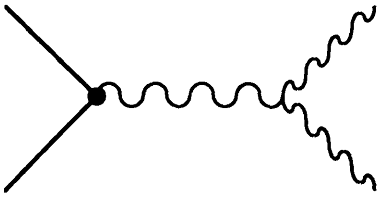
$$\frac{iL}{8}\Big\{u\,\bar{u}_{2\nu}\gamma^{\rho}\not{k}_4\gamma^{a}u_{1m}\zeta_{3a}^{\ \ m}\zeta_{4\rho}^{\ \ \nu} - \frac{t}{2}\bar{u}_{2\nu}\gamma^{a}\not{k}_4\gamma^{\rho}u_{1m}\zeta_{3a}^{\ \ m}\zeta_{4\rho}^{\ \ \nu}\Big\} \tag{5.2.9}$$

$\underline{L'}$:-

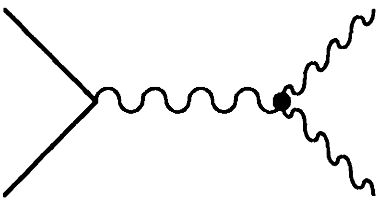
$$\frac{iL'}{16}\Big\{\frac{(s-u)}{2}\bar{u}_{2\nu}\gamma^{\rho}\not{k}_4\gamma^{a}u_{1m}\zeta_{3a}^{\ \ m}\zeta_{4\rho}^{\ \ \nu} + t\,\bar{u}_{2\beta}\not{k}_4 u_{1m}\zeta_3^{ms}\zeta_{4s}^{\ \ \beta}\Big\} \tag{5.2.10}$$

all of whose contributions are dependent only on the κ_1, κ_2, κ_3, L and L' coefficients as expected. The second diagram yields the two terms,

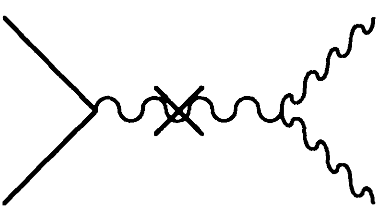
$$\frac{i\,(2L-L')}{8}\Big\{\frac{(t+2s)}{4}\bar{u}_{2\mu}\gamma^{\alpha}\not{k}_4\gamma^{a}u_{1m}\zeta_{3a}^{\ \ m}\zeta_{4\alpha}^{\ \ \mu} - t\,\bar{u}_{2\mu}\not{k}_4 u_{1m}\zeta_3^{ms}\zeta_{4s}^{\ \ \mu}\Big\} + \frac{iL't}{32}\bar{u}_{2\mu}\gamma^{\alpha}\not{k}_4\gamma^{a}u_{1m}\zeta_{3a}^{\ \ m}\zeta_{4\alpha}^{\ \ \mu} \tag{5.2.11}$$



$$-\frac{i\alpha'}{32}\left\{2(L-L') - (l_1+l_2)\right\}(u-t)\,\bar{u}_2^{\rho}(\not{k}_4 - \not{k}_3)\,u_{1\rho}\,\zeta_3^{\eta\xi}\,\zeta_{4\eta\xi}$$

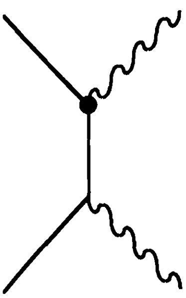


$$\frac{i\alpha' x(1+2y)}{8}(u-t)\,\bar{u}_2^{\rho}(\not{k}_4 - \not{k}_3)u_{1\rho}\,\zeta_3^{\eta\xi}\,\zeta_{4\eta\xi}$$



$$\frac{i\alpha' x(1-y)}{4}(u-t)\,\bar{u}_2^{\rho}(\not{k}_4 - \not{k}_3)u_{1\rho}\,\zeta_3^{\eta\xi}\,\zeta_{4\eta\xi}$$

Fig. 5.2a Summary of the graviton exchange Feynman $O(\alpha')$ diagrams for the general case.

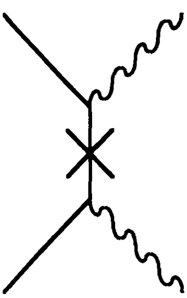


$$\frac{i\alpha'}{32}\Big\{4(\kappa_1+\kappa_2+2\kappa_3-L')\big[t\,\bar{u}_{2\beta}(\not k_4-\not k_3)u_{1m}-u\,\bar{u}_{2m}(\not k_4-\not k_3)u_{1\beta}\big]\zeta_3^{ms}\zeta_{4s}{}^{\beta}$$

$$-\big[[(3\kappa_1+\kappa_2+2\kappa_3+8L-2L')u+2L's]\,.\,\bar{u}_{2\nu}\gamma^{\rho}(\not k_4-\not k_3)\gamma^{a}u_{1m}$$

$$-[(3\kappa_1+\kappa_2+2\kappa_3+8L-2L')t+2L's]\,.\,\bar{u}_{2m}\gamma^{a}(\not k_4-\not k_3)\gamma^{\rho}u_{1\nu}\big]\zeta_{3a}{}^{m}\zeta_{4\rho}{}^{\nu}$$

$$+4L\big[t\,\bar{u}_{2\nu}\gamma^{a}(\not k_4-\not k_3)\gamma^{\rho}u_{1m}-u\,\bar{u}_{2m}\gamma^{\rho}(\not k_4-\not k_3)\gamma^{a}u_{1\nu}\big]\zeta_{3a}{}^{m}\zeta_{4\rho}{}^{\nu}\Big\}$$



$$\frac{i\alpha'(L'-2L)}{8}\Big\{\Big[\frac{(t+2s)}{4}\bar{u}_{2\mu}\gamma^{\alpha}(\not k_4-\not k_3)\gamma^{a}u_{1m}-\frac{(u+2s)}{4}\bar{u}_{2m}\gamma^{a}(\not k_4-\not k_3)\gamma^{\alpha}u_{1\mu}\Big]\zeta_{3a}{}^{m}\zeta_{4\alpha}{}^{\mu}$$

$$-\big[t\,\bar{u}_{2\mu}(\not k_4-\not k_3)u_{1m}-u\,\bar{u}_{2m}(\not k_4-\not k_3)u_{1\mu}\big]\zeta_3^{ms}\zeta_{4s}{}^{\mu}\Big\}$$

$$-\frac{iL'\alpha'}{32}\Big\{t\,\bar{u}_{2\mu}\gamma^{\alpha}(\not k_4-\not k_3)\gamma^{a}u_{1m}-u\,\bar{u}_{2m}\gamma^{a}(\not k_4-\not k_3)\gamma^{\alpha}u_{1\mu}\Big\}\zeta_{3a}{}^{m}\zeta_{4\alpha}{}^{\mu}$$

Fig. 5.2b The gravitino exchange diagrams used in the evaluation of the $O(\alpha')$ $\bar{\psi}$-ψ-h-h amplitude in the most general case considered.

which depends only on L and L'.

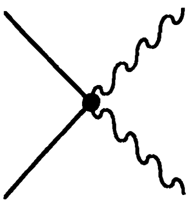
These diagrams, in addition to the others described above, can be summed according to the symmetrisation and combinatorics of the scheme given in Figure 4.4. As in Chapter four the various contributions from individual types of diagrams will be summarised. The graviton exchange diagrams are listed in Figure 5.2a, the gravitino exchange diagrams are listed in Figure 5.3b and the point diagrams (excluding the Lorentz Chern-Simons contribution from Chapter Four) are listed in Figure 5.2c. As in the summaries given in Chapter Four the complete amplitude is merely the sum of these individual terms listed in these figures, and the contribution from the Lorentz Chern-Simons term given in Figure 4.3, since all the various factors have been taken into account, including the trivial reintroduction of the α' factor as in the previous chapters. This gives the final amplitude which can be matched with the string to attempt to fix the arbitrary coefficients. This is done in the following section. Due to the vast complexity of the amplitude and the lack of necessity of seeing it complete, the full field theory amplitude will not be listed: rather the individual classes of sub terms within the amplitude will be matched separately. This is now done.

Section 5.3: The amplitude matching and its consequences.

As mentioned above in this section the amplitude matching is performed and an attempt will be made to solve the resulting linear simultaneous equations. The string answer is given by equation (4.1.53), which has already been normalised subject to the $O(\alpha'^0)$ matching condition. Splitting the full trial effective field theory amplitude into the 's' and 't-u'-channels, allows a direct matching to be made as done for the Romans and Warner, and Han et al. actions in Chapter Four. This will be done below, where the sectors will be examined in turn.

a) The 's'-channel.

This is the easiest of the matching conditions to analyse. There will turn out to be two conditions in the matching process as usual. The matching condition is simply determined by comparing the sum of the diagrams given in Figure 5.2a, the 's'-channel parts of the point diagram contribution of Figure 5.2c and the Lorentz Chern-Simons term contribution given in Figure 4.3, with the 's'-channel sector of the string amplitude given in equation (1.4.16), and can be seen to give the condition,



$$\frac{i\alpha'}{32}\Big\{\Big[\,(2(\kappa_1+\kappa_2+2\kappa_3)(2u-t)+4L'u)\,\bar{u}_{2\mu}(\not k_4-\not k_3)u_{1\rho}$$

$$-\,(2(\kappa_1+\kappa_2+2\kappa_3)(2t-u)+4L't\,)\,\bar{u}_{2\rho}(\not k_4-\not k_3)u_{1\mu}\Big]\zeta_3^{\mu\eta}\zeta_{4\eta}{}^{\rho}$$

$$+\,2\Big[\,(\kappa_1 u-2\kappa_3 t\,)\,\bar{u}_{2\mu}\gamma^{\rho}(\not k_4-\not k_3)\gamma^{r}u_{1m}-(\kappa_1 t-2\kappa_3 u)\,\bar{u}_{2m}\gamma^{r}(\not k_4-\not k_3)\gamma^{\rho}u_{1\mu}\Big]\zeta_{3r}{}^{m}\zeta_{4\rho}{}^{\mu}$$

$$+\,(2L-3L')\Big[\,t\,\bar{u}_2^{\rho}\gamma^{\mu}(\not k_4-\not k_3)\gamma^{q}u_1^{\sigma}-u\,\bar{u}_2^{\sigma}\gamma^{q}(\not k_4-\not k_3)\gamma^{\mu}u_1^{\rho}\Big]\zeta_{3q\rho}\zeta_{4\mu\sigma}$$

$$-\,\Big[((s+2t)L+(s-3t)L')\,\bar{u}_2^{\rho}\gamma^{\mu}(\not k_4-\not k_3)\gamma^{q}u_{1\rho}$$

$$-\,((s+2u)L+(s-3u)L')\,\bar{u}_2^{\rho}\gamma^{q}(\not k_4-\not k_3)\gamma^{\mu}u_{1\rho}\Big]\zeta_{3q\sigma}\zeta_{4\mu}^{\sigma}$$

$$-\,(l_1+l_2)\,(u-t\,)\,\bar{u}_2^{\rho}(\not k_4-\not k_3)u_{1\rho}\zeta_3^{\eta\xi}\zeta_{4\eta\xi}\Big\}$$

Fig. 5.2c The point diagram for the calculation of the O(α') $\bar{\psi}$-ψ-h-h amplitude in the most general case.

$$-\frac{i\alpha'}{32}\Big\{ -\frac{1}{2}\big[2(L-L') - 12x\big](u-t) + \gamma s$$

$$+ (s+2t)\,L + (s-3t)\,L' \Big\} \equiv -\frac{i\alpha'}{64} u \qquad (5.3.1a)$$

which can be rewritten in the somewhat more illuminating form,

$$\Big\{ \big(-\frac{1}{2} - (2L - 6x)\big)u + (2L - 6x)t + \gamma s \Big\} = 0 \qquad (5.3.1b)$$

which can be seen to be soluble by the choice $\gamma = -1/4$ *and* $L' = 0$, which is the same as the Han et al case. However, the possibility of a full match to the string amplitude remains in this case, due to the increased generality of the action used, and so the 't-u'-channel matching must now be addressed. It can be seen that the combination $L - 3x$ occurs in the matching condition, leaving the 't-u'-channel matching conditions to solve for both L and x. This is now carried out. It will be hoped that the choice $L' = 0$ will be consistent in these matching conditions. In fact it will be shown that this *is* the case.

b) The 't-u'-channel matching.

As in Chapter Four, it will be convenient to subdivide these amplitude contribution into two further subclasses. Using the standard trick of introducing gamma matrices into the amplitude produces the "standard" term matching condition in the form,

$$\Big\{ \frac{i\alpha'}{32}\big[-2(\kappa_1 + \kappa_2 + 2\kappa_3 - L')t - ((3\kappa_1 + \kappa_2 + 2\kappa_3 + 8L - 2L')u + 2L's)$$

$$+ \frac{i\alpha'(L' - 2L)}{8}\big[\frac{(2s+t)}{4} + \frac{t}{2}\big] - \frac{iL'\alpha' t}{32} + \frac{i\gamma\alpha' s}{16}$$

$$+ \frac{i\alpha'}{32}\big[-\frac{1}{2}(2(\kappa_1 + \kappa_2 + 2\kappa_3).(2u-t) + 4L'u) + 2(\kappa_1 u - 2\kappa_3 t)\big]\Big\} \equiv -\frac{i\alpha'}{64} u \qquad (5.3.2a)$$

which is interesting, in that the combination $(\kappa_1 + \kappa_2 + 2\kappa_3)$ appears, which is the same as in the three point matching. This matching can be split into the usual two Mandlestam independent matching conditions,

$$5\kappa_{12} + 6\kappa_3 + 6L + 4L' = -2\,(-\frac{1}{2} + \gamma)$$

and,

$$\kappa_{12} + 6\kappa_3 + 2L - 4L' = -2\,\gamma \qquad (5.3.2b)$$

where the combination $\kappa_{12} = \kappa_1 + \kappa_2$. The same must now be done for the "crossed" term denoted genericaly by amplitude terms of the form (4.1.56). The amplitude matching condition for this term is,

$$\Big\{ \frac{i\alpha'}{32}\Big[-2(\kappa_1 + \kappa_2 + 2\kappa_3 - L')t + 4Lt\Big] + \frac{i\alpha'(L' - 2L)t}{16}$$

$$+ \frac{i\gamma\alpha' s}{16} - \frac{i\alpha'}{64}\Big[2(\kappa_1 + \kappa_2 + 2\kappa_3).(2u-t) + 4L'u\Big]$$

$$- \frac{i\alpha'(2L - 3L')u}{32} \Big\} \equiv \frac{i\alpha'}{64}t \qquad (5.3.3a)$$

which again can be rewritten in the Mandlestam variable independent form,

$$2L + 7L' = 2(\gamma + \frac{1}{2})$$

and,

$$2\kappa_{12} + 4\kappa_3 + 2L - L' = -2\gamma \qquad (5.3.3b)$$

which gives the final two matching conditions. This set of linear simultaneous equations can now be analysed. Immediately it is clear that the combination $(\kappa_1 + \kappa_2)$ is the only combination of these two variables which appears in any of the matching conditions. Henceforth this comination will be replaced by the coefficient κ_{12} in all the matching conditions. Immediately it is clear that the first of the last pair of matching conditions can be used to eliminate L from the remaining matching conditions. When this is done the matching conditions can be seen to be of the form;

$$5\kappa_{12} + 6\kappa_3 - 17L' = 4\,(-\frac{1}{2} - 2\gamma)$$

$$\kappa_{12} + 6\kappa_3 - 11L' = 2\,(-\frac{1}{2} - 2\gamma)$$

$$2\kappa_{12} + 4\kappa_3 - 8L' = 2\,(-\frac{1}{2} - 2\gamma) \qquad (5.3.4)$$

and,

$$-\frac{1}{2}\kappa_{12} - \kappa_3 - L + L' = -\frac{1}{4} \qquad (5.2.2)$$

where (5.2.2) is the three point amplitude matching condition. For consistency with the 's'-channel matching, γ must take the value $\gamma = -1/4$. When this value is substituted into the matching conditions above, it can be seen that the equations may be solved to give the values, $\kappa_{12} = \kappa_3 = L' = 0$, which implies $L = 1/4$, which is consistent with (5.2.2) the three point matching result. This is remarkable in that a solution exists for the complete set of equations which overconstrain the variables in question.

Finally it can be seen from the's'-channel matching condition, (5.3.1b), that,

$$2L - 6x = \gamma \qquad (5.3.5)$$

which implies that the coefficient x takes the value $x = 1/8$, which is completly consistent with the analyses of Cai and Nunez, and also Gross and Sloan[58]. This implies that the $O(\alpha')$ part of the effective action for the heterotic string takes the form,

$$\begin{aligned}\mathfrak{L}' = e\phi^{-3/4} \{ & \frac{1}{8}[R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4yR_{\mu\nu}R^{\mu\nu} + zR^2] \\ & + \delta (\overline{\psi}_\mu \gamma^{\rho\sigma\tau} \mathfrak{D}_{[\nu}\psi_{\tau]} + \overline{\psi}_\mu \gamma_\nu \mathfrak{D}^{[\rho}\psi^{\sigma]}) R^{\mu\nu}{}_{\rho\sigma} \\ & + l_1 \overline{\psi}^\mu \gamma^\sigma \mathfrak{D}_{[\mu}\psi_{\rho]} R_\sigma{}^\rho + l_2 \overline{\psi}_\mu \gamma^{\rho\mu\tau} \mathfrak{D}_{[\nu}\psi_{\tau]} R_\rho{}^\nu \\ & + \mathfrak{D}^{[\mu}\overline{\psi}^{\nu]} \gamma^\rho \mathfrak{D}_\rho \mathfrak{D}_{[\mu}\psi_{\nu]} \ \}\end{aligned} \qquad (5.3.6)$$

which can be seen to be very similar to the Han et al action when the γ factor in that action takes the value -1/4, and when the l_1, l_2, y, and z coefficients are set to zero. It should be noted that there is a remaining ambiguity in the *leading order* terms denoted by the κ_1 and κ_2 coefficients, where this is given by the δ coefficient. The Han et al action would have required κ_2=-1/2 and κ_3=-1/4, which is impossible.

It is also interesting that the choice of the Gauss-Bonnet form is left open by the amplitude matching procedure, which suggests that possibly the field redefinition analysis used previously in a purely bosonic context might continue to be a useful tool in fermionic actions. This is examined in the next and final chapter.

Chapter Six: Field Redefinitions, Supersymmetry and Conclusions.

Introduction.

In this chapter, some of the topics mentioned, but not covered, in previous chapters will be addressed. These topics include the discussion of field redefinition analysis and a brief discussion of supersymmetry in this context. The final section will give a summary of the work in this thesis, and the conclusions that can be drawn from it.

The field redefinition analysis will be covered in some depth, although the practicality of certain calculations will preclude a complete description. Section 6.1 will deal with field redefinitions and supersymmetry. Section 6.2 will contain the conclusions of the work contained in this thesis.

The subject of field redefinition analysis as applied to the bosonic sector of the low energy effective action for the heterotic string has been extensively examined in the literature[59,60,63,64,65,66] but the subject of the extension of these arguments to the fermionic sector of the low energy effective action has not been addressed as fully. Some comments are presented here to suggest that the arguments used do not extend simply to the general fermionic action. The consideration of supersymmetry must play some part in the generalisation of the field redefinition analysis, since there already exist two quite separate actions which are supersymmetric up to the same level of generality, (though neither action is completely supersymmetric, they are each supersymmetric up to terms dependent on the antisymmetric tensor field as discussed in Chapter Four), but which have been demonstrated in Chapter Four to give different scattering amplitudes for the same scattering process. They are quite distinct dynamically, and clearly cannot be related by the naïve types of field redefinition used in the purely bosonic case. However the rôle of supersymmetry will not be discussed in any depth below. Some simple examples of field redefinitions will be presented to demonstrate that a more subtle approach must be adopted in any field redefiniton analysis.

The results of this analysis in conjunction with the results of Chapters Four and Five will be tied together in the final section when the conclusions of the work described in this thesis will be presented.

Section 6.1: Field Redefinitions: some comments.

This section deals with the extension of the traditional approach to field redefinitions in the context of the fermionic sector of the heterotic string effective action. This approach derived originally from the work discussed in reference [63] with respect to the analysis of the nonrenormalisability and divergences of the background field expansion approach to the quantisation of gravity theories. The technique has lately been extensively applied to the particular case of the full bosonic sector of the effective field theory of the heterotic string in references [60,64,65] amongst many others. The field redefinition analysis of the pure gravitational sector of the theory will used as an example of the technique. The validity of the techniques as applied in the references mentioned above when the quantum theory is considered[66] will not be discussed. The connection between the essentially classical field redefinitions performed in the example presented, and the extreme complexity of the corresponding quantum field redefinition is unclear. The naïve general principles outlined in the example presented, will then be applied in a completely similar fashion to the fermionic sector of the action. The method outlined will be shown to give somewhat ambiguous results which will seem to suggest that the standard approach to field redefinitions as used in bosonic field theories is not trivially extendable to the case of fermionc actions. It will be shown that the field redefinition analysis is probably *not* consistent with the results of Chapter Five in that certain field redefinitions will be shown to vary coefficients in the action which have been absolutely fixed by the amplitude matching, but it will not be shown whether this result can be strengthened to demonstrate that the field redefinition analysis is indeed not consistent with the amplitude calculations carried out above. It will require a great deal of work to decide whether the field redefinition analysis is a useful tool in the fermionic field theory case or whether the methods used succesfully in the bosonic case do not extend to the case of fermionic actions. This work is beyond the scope of this thesis.

The starting point for the field redefinition analysis is the consideration of the types of field redefinitions that need to be considered. Since the natural expansion parameter for string effective theories is the α' constant, the restriction to field redefinitions of the form,

$$\phi \rightarrow \phi + k\,\alpha'\partial^2\phi \qquad (6.1.1)$$

(where ϕ is some general field), can be made, which can be seen to highlight only the ambiguities in the $O(\alpha')$ parts of the action, as required. The field considered is the graviton, so the action that will be considered is the most general possible action up to the α' order, any terms higher in α' being ignored. (This analysis has been performed up to $O(\alpha'^3)$ level to the author's knowledge[60], and probably higher.) The action is,

$$\mathcal{L} = -\frac{1}{2}\sqrt{-g}\,R + x\sqrt{-g}\left\{R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4yR_{\mu\nu}R^{\mu\nu} + zR^2\right\}$$

The most general possible $O(\alpha')$ field redefinition of the metric tensor $g_{\mu\nu}$, with respect to the various symmetries of the theory, is the redefinition,

$$g_{\mu\nu} \rightarrow g_{\mu\nu} + b_1 R_{\mu\nu} + b_2 g_{\mu\nu} R \qquad (6.1.2)$$

This redefinition must be replaced in the action above, and the new action derived. This is trivial to do using Bianchi identities, tensorial symmetries etc., and yields the modified action,

$$\mathcal{L}' = -\frac{1}{2}\sqrt{-g}\,R + x\sqrt{-g}\left\{R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4y'R_{\mu\nu}R^{\mu\nu} + z'R^2\right\} \qquad (6.1.3a)$$

where the modified coefficients y' and z' can be written in the form,

$$y' = y - \frac{b_1}{4x} \quad , \quad z' = z - \frac{1}{2x}\left(b_1 + \frac{(d-2)}{2}b_2\right) \qquad (6.1.3b)$$

It is immediately obvious that only the Ricci tensor and curvature scalar terms vary under this redefinition. This implies that these two terms in the action are ambiguous with respect to field redefinitions, and consequently can never contribute to any physical amplitude, and therefore no amplitude matching procedure. This means that the amplitude matching procedure, as discussed in the previous chapters, can never determine the coefficients of these particular terms in the action.

The relation of this field redefinition analysis has been investigated, in references [51,56], in the context of the bosonic sector of the theory, by explicitly showing the exact, (and miraculous), cancellation of these ambiguous coefficients in an amplitude calculation. The generalisation of this technique to the case of fermions is obvious. The same techniques will be applied to the various fields in the complete action and the ambiguous coefficients determined.

The obvious starting point for any examination of this type is a toy model which has the practical advantage of demonstrating all the features that would be apparent in the full theory, without the concomitant complexity. The restriction chosen will be to the graviton field in the form of the vierbein, $e^m{}_\mu$, and the spin-3/2 field ψ_μ. Since the field variations are of the order $O(\alpha')$, and only the $O(\alpha')$ variations of the action are needed, then the only action terms that need to be considered in the variation are the ones of order $O(\alpha'^0)$, that is the order below the variations looked for. Any terms that arise from the variation procedure can then be compared to the $O(\alpha')$ terms used in Chapter Five.

The first problem of this work immediately presents itself when the most general field redefinitions must be found. The question of what guiding principles must be used in their selection? How does one ensure the supersymmetry of the lowest order action, and also the anomaly freedom, gauge invariance, etc.. These problems will be dealt with below.

If the field redefinition is written in the form,

$$e^m_\mu \rightarrow e'^m_\mu \quad , \quad \psi_\mu \rightarrow \psi'_\mu \qquad (6.1.4a)$$

and since the varied Lagrangian is clearly given by,

$$\mathcal{L}(\phi) \rightarrow \mathcal{L}(\phi') = \mathcal{L}'(\phi) \qquad (6.1.4b)$$

under such a transformation, then if the supersymmetry transformations can be written formally,

$$\delta e^m_\mu = f_1{}^m{}_\mu(e,\psi) \;,\quad \delta\psi_\mu = f_2{}_\mu(e,\psi) \qquad (6.1.5a)$$

then the transformed variations,

$$\delta e'^m_\mu = f_1{}^m{}_\mu(e',\psi') \;,\quad \delta\psi'_\mu = f_2{}_\mu(e',\psi') \qquad (6.1.5b)$$

can be seen to leave the new action invariant, because of the definition of the varied Lagrangian. The invariance of the action under these new supersymmetry transformations is manifest due to the supersymmetry of the "old" theory. The subject of $O(\alpha')$ corrections to the supersymmetry transformations has been extensively examined[67]. The closure of the supersymmetry algebra is also manifest, since although the equations of motion are used to enforce the closure of the algebra, the new equations of motion obtained after the field variation <u>must</u> ensure closure due to the form invariance of the action under the transformation (6.1.4a-b).

The only remaining problem is maintaining anomaly freedom of the quantum theory. This is not a problem by the nature of the field redefinition theorem, where quantum S-matrix amplitudes are unaffected by the variations of the fields that are chosen, and so the amplitudes which could display the anomalous behaviour must all reflect the anomaly freedom of the initial Lagrangian, and so must all be anomaly free. Thus if the initial Lagrangian is anomaly free, then the final varied Lagrangian must also be anomaly free.

The action that will be varied is simply the action quantised in Chapters Two and Three,

$$\mathcal{L} = -\frac{1}{2} eR - \frac{1}{2} e\, \overline{\psi}_\mu \gamma^{\mu\nu\rho} \overleftrightarrow{\mathcal{D}}_\nu \psi_\rho \qquad (3.2.14)$$

Rather than considering the most general set of possible field redefinitions of the fields in this action, which can be seen to be quite a large set of terms when the gamma matrices are taken into account, only certain redefinitions will be considered. The fermionic form of the field redefinition theorem will be shown to be more complicated than its bosonic counterpart: it will be shown that it is possible to vary what are termed leading order terms[60] in the action, and consequently vary the physical amplitudes in contradiction with the results found in the bosonic case. The starting point of the field redefinition will be to find how the variation of the field produces a variation of the Lagrangian. The standard way of doing this is to consider the variation of the Lagrangian,

$$\begin{aligned} \delta\mathcal{L}(\phi,\partial_\mu\phi) &= \frac{\delta\mathcal{L}}{\delta\phi}\delta\phi + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\delta\partial_\mu\phi \\ &= \left\{ \frac{\delta\mathcal{L}}{\delta\phi} - \partial_\mu \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi} \right\}\delta\phi \qquad (6.1.6) \end{aligned}$$

which allows the variation of the Lagrangian to be calculated most simply. All that need be done is to multiply the equation of motion into the field variation to get the variation of the action. The simplest way to proceed is to consider the simplest field equation for the action given above. This is the equation of motion for the gravitino field, which is, (noting the rules of Grassman Lagrangian dynamics stated in Appendix Three),

$$\overline{\psi}_\mu \gamma^{\mu\nu\rho} \overleftarrow{\mathcal{D}}_\nu = 0 \qquad (6.1.7)$$

which may now be multiplied into any of the field variations of the gravitino field. The example variations that will be considered are;

$$\psi_\rho \rightarrow \psi'_\rho = \psi_\rho + f_\rho(e,\psi)$$

where $f_\rho(e,\psi)$ takes the following three forms,

i) $$\delta\psi_\rho = \gamma^{\nu\lambda}\mathfrak{D}_{[\nu}\mathfrak{D}_{\lambda]}\psi_\rho \qquad (6.1.8a)$$

ii) $$\delta\psi_\rho = \gamma^{\nu\lambda}\mathfrak{D}_\rho\mathfrak{D}_{[\nu}\psi_{\lambda]} \qquad (6.1.8b)$$

iii) $$\delta\psi_\rho = \gamma_{\mu\nu}\psi^\sigma R^{\mu\nu}{}_{\rho\sigma} \qquad (6.1.8c)$$

iv) $$\delta\psi_\rho = \mathfrak{D}^\alpha\mathfrak{D}_{[\rho}\psi_{\alpha]} \qquad (6.1.8d)$$

which will be seen to give variations of some of those terms in the action which do not cancel completely from the amplitude, and will *not* vary some terms in the action which will demonstrate that the Romans and Warner action and Han et al action cannot be field redefined into each other since there exists a term, (the term denoted by the coefficient L in the most general action chosen in Chapter Five which exists in the Han et al action but not in the Romans and Warner action), which does not vary under these, or *any*, field redefinitions. Such a field redefinition would be necessary for the Noether method to be able to give a unique supersymmetric Lagrangian at the $O(\alpha')$ level.

i) The first variation.

The first variation is of the form,

$$\delta\psi_\rho = \gamma^{\nu\lambda}\mathfrak{D}_\nu\mathfrak{D}_\lambda\psi_\rho \qquad (6.1.8a)$$

which can be rewritten in the more interesting form,

$$\delta\psi_\rho = \gamma^{\nu\lambda}\gamma_{\alpha\beta}R_{\nu\lambda}{}^{\alpha\beta}\psi_\rho \qquad (6.1.9)$$

The gamma matrix products can be simplified using the identities stated in Appendix Three, to give,

$$\gamma^{\nu\lambda}\gamma^{\alpha\beta} = (\gamma^\nu\gamma^\lambda - g^{\nu\lambda})(\gamma^\alpha\gamma^\beta - g^{\alpha\beta})$$

which allows the field variation to be rewritten in the form,

$$\delta\psi_\rho = \gamma^\nu\gamma^\lambda\gamma^\alpha\gamma^\beta R_{\nu\lambda\alpha\beta}\psi_\rho$$

$$= \gamma^\nu\,(\,\gamma^{\lambda\alpha\beta} + \gamma^\lambda g^{\alpha\beta} - \gamma^\alpha g^{\lambda\beta} + \gamma^\beta g^{\alpha\eta}\,)R_{\nu\lambda\alpha\beta}\psi_\rho$$

$$= 2\,\gamma^\nu\gamma^\alpha R_{\nu\alpha}\psi_\rho$$

$$= 2R\psi_\rho \qquad (6.1.10)$$

where the symmetries of the Riemann tensor have been observed. This gives a variation in the Lagrangian of the form,

$$\delta\mathfrak{L} = 2\,\overline{\psi}_\mu\gamma^{\mu\nu\rho}\mathfrak{D}_\nu\psi_\rho R \qquad (6.1.11)$$

which is dependent only on the curvature scalar. The curvature scalar terms in the most general action can be seen to give no contributions to the amplitudes as calculated above, so it can be seen so far that the field redefinition theorem appears to be consistent with the bosonic field redefinitions: the physical amplitudes will remain unchanged with respect to such a variation. The remaining three redefinitions will give variations which do not agree at the naïve level with this conclusion.

ii) The second variation.

The second variation will be seen to be much less trivial, and consequently much more interesting than the first variation. The second variation is of the form,

ii) $$\delta\psi_\rho = \gamma^{\eta\lambda}\mathfrak{D}_\rho\mathfrak{D}_{[\eta}\psi_{\lambda]} \qquad (6.1.8b)$$

which gives a variation of the Lagrangian of the form,

$$\delta\mathfrak{L} = \mathfrak{D}_{[\nu}\overline{\psi}_{\mu]}\gamma^{\mu\nu\rho}\gamma^\lambda\gamma^\tau\mathfrak{D}_\rho\mathfrak{D}_{[\lambda}\psi_{\tau]} \qquad (6.1.12)$$

Again the gamma matrix identities can be applied to simplify this expression. The identity needed is quite complicated, and is,

$$\gamma^{\mu\nu\rho}\gamma^{\eta}\gamma^{\xi} = \gamma^{\mu\nu\rho\eta\xi} + \gamma^{\mu\nu\rho}g^{\eta\xi} - \gamma^{\mu\nu\eta}g^{\rho\xi} + \gamma^{\mu\rho\eta}g^{\nu\xi}$$

$$- \gamma^{\nu\rho\eta}g^{\mu\xi} + \gamma^{\nu\rho\xi}g^{\mu\eta} - \gamma^{\mu\eta\xi}g^{\nu\rho}$$

$$+ \gamma^{\mu}(g^{\nu\xi}g^{\rho\eta} - g^{\nu\eta}g^{\rho\xi})$$

$$+ \gamma^{\nu}(g^{\mu\eta}g^{\rho\xi} - g^{\mu\xi}g^{\rho\eta})$$

$$+ \gamma^{\rho}(g^{\nu\eta}g^{\mu\xi} - g^{\nu\xi}g^{\mu\eta}) \qquad (6.1.13)$$

which can be seen now to give terms in the Lagrangian variation which <u>are</u> dependent on Riemann tensors. The full variation due to this term can be seen to be of the form,

$$\delta\mathfrak{L} = 3\, g^{\nu\xi}\mathfrak{D}_{[\nu}\overline{\Psi}_{\mu]}\gamma^{\mu\rho\eta}\mathfrak{D}_{\rho}\mathfrak{D}_{[\eta}\Psi_{\xi]} + \overline{\Psi}_{\mu}\gamma^{\mu\alpha\beta}\mathfrak{D}_{[\eta}\Psi_{\xi]}R_{\alpha\beta}{}^{\eta\xi}$$

$$+ 2\,\overline{\Psi}^{\beta}\gamma^{\alpha}\mathfrak{D}_{[\eta}\Psi_{\xi]}R_{\alpha\beta}{}^{\eta\xi} + 4\,\overline{\Psi}^{\xi}\gamma_{\alpha}\mathfrak{D}_{[\eta}\Psi_{\xi]}R^{\alpha\eta}$$

which can be seen to contain variations of leading order terms denoted by the coefficients κ_2, κ_3 and L', though *not* the leading order terms denoted by κ_1 or L. This is clearly inconsistent with the amplitude matching results presented in Chapter Five above, since these coefficients are all completely determined by an amplitude matching and since these coefficients all appear explicitly in the amplitude: the physical amplitude varies under this variation.

There is also a Ricci tensor term denoted by l_1, though this is consistent with the amplitude matching result given in Chapter Five above, since this coefficient is left arbitrary under the amplitude matching.

<u>iii) The third variation.</u>

The third variation is quite trivial to analyse compared to the last example, and is of the form,

$$\delta\psi_{\rho} = \gamma_{\mu\nu}\psi^{\sigma}R^{\mu\nu}{}_{\rho\sigma} \qquad (6.1.8c)$$

which might be expected to produce terms dependent on Riemann tensors in a similar fashion to the second variation. The variation of the Lagrangian is clearly seen to be,

$$\delta\mathfrak{L} = \mathfrak{D}_\nu \overline{\psi}_\mu \gamma^{\mu\nu\rho} \gamma_{\alpha\beta} \psi^\sigma R^{\alpha\beta}{}_{\rho\sigma} \qquad (6.1.14)$$

which requires the gamma matrix identity of equation (6.1.13) again. As in the second case above, when the gamma matrix identity (6.1.13) is substituted it can be shown that the κ_1, κ_2 and κ_3 terms all vary. It is clear that neither L or L' enter in this case. This variation must therefore alter physical amplitudes. This particular case will not be discussed any further.

iv) The fourth variation.

The fourth variation takes the form,

$$\text{iv)} \qquad \delta\psi_\rho = \mathfrak{D}^\alpha \mathfrak{D}_{[\rho}\psi_{\alpha]} \qquad (6.1.8d)$$

which can be substituted into the usual equation (6.1.6) to give a variation of the Lagrangian of the form,

$$\delta\mathfrak{L} = - g^{\alpha\beta} \mathfrak{D}_{[\beta}\overline{\psi}_{\mu]} \gamma^{\mu\nu\rho} \mathfrak{D}_\nu \mathfrak{D}_{[\rho}\psi_{\alpha]} + \overline{\psi}^\xi \gamma^{\nu\rho\eta} \mathfrak{D}_{[\rho}\psi_{\alpha]} R^\alpha{}_{\nu\eta\xi}$$

$$- \frac{1}{2} \overline{\psi}_\mu \gamma^{\mu\nu\eta} \mathfrak{D}_{[\rho}\psi_{\alpha]} R^{\alpha\rho}{}_{\eta\nu} + \overline{\psi}_\xi \gamma_\nu \mathfrak{D}_{[\rho}\psi_{\alpha]} R^{\alpha\nu\rho\xi}$$

$$+ \text{curvature scalar and Ricci terms.} \qquad (6.1.15)$$

(where the curvature scalar and Ricci tensor terms have been suppressed since they are obviously not of leading order and so need not be considered for the purposes of this section). This variation can be seen to give a variation of the physical amplitude as in the case of the second variation above. In this case all the fermionic leading order coefficients *except one* vary. The exception is the coefficient L, which is quite an interesting observation, as will be discussed below.

This variation of the ψ_ρ field has been included since it can be seen that this is the only term which can 'generate' the L term from the Rarita-Schwinger action by the variation of the term,

$$\overline{\psi}^\mu \gamma^\rho \mathfrak{D}_\rho \psi_\mu$$

which would give exactly the $O(\alpha')$ action variation required. As is obvious from (6.1.15) above there is no L type term generated. (It can also be seen that the second variation might give a term of the required form. This cannot occur without the introduction of extra terms which do not occur in the most general basis of O(α') action correction terms chosen for the action in Chapter Five. When these spurious terms are eliminated by rewriting them in the basis chosen in Chapter Five they can be shown to completely eliminate all terms of the L type leaving only the terms shown above.) It can be seen that there are *no* variations of the ψ_ρ field which can generate an L type term in the variation of the action. This immediately implies that the Noether method does not give a unique O(α') corrected supersymmetric Lagrangian since there exists a term in the Han et al action which cannot be created by field redefinitions from the Romans and Warner action.

The field variations shown above can all be shown, with the exception of the first variation, to give variations of the physical amplitudes calculeted in Chapter Five. The question remains: what principle must be invoked to 'select' field variations so that they give a suitable generalisation of the bosonic field redefinition theorem discussed above? Is it possible to use a linear combination of *all* the variations listed above so as to provide miraculous cancellations of all the variations of the leading order type terms? If so, what principle underlies such a choice? Are the field variations which generate leading order variations somehow disallowed for some reason? These questions will not be discussed here, but it would appear that the best line of attack on these problems would be to find some parallel to the link between renormalisation and field redefinitions in the manner of reference [63]. This line of thought will not be pursued here.

Section 6.2: Summary of Conclusions.

This section presents a summary of the conclusions of the work performed in this thesis. The purpose of the amplitude matching procedure followed in Chapters Three to Five was to attempt to find some fermionic correction terms to the already well known O(α') correction terms[58,59,60,] to the Lorentz-Chern-Simons modified Chapline-Manton Lagrangian as low energy limit to the heterotic superstring. For each of the three actions used as trial actions in the amplitude matching procedure, *two* separate matchings are performed: a three point matching, and a four point matching. The three point matching for the Romans and Warner terms confirmed the result of references [68,69], and the Han et al. action is shown to fail at this level. When this calculation is extended to the four point matching, it is shown that neither the Romans and Warner, nor Han et al actions are completely capable of matching string amplitudes, though the Han et al action can be seen to give a much closer match than the Romans and Warner action. It is important to note that the two amplitudes are *different* in terms of the dynamics of the theories.

The work of Chapter Five used a general form of the O(α') action consisting of a complete set of O(α') terms with undetermined coefficients. The same amplitudes as calculated for the two actions found by the Noether method were derived, and the amplitude matching to the string performed. This enabled certain of the coefficients in the general action to be fixed absolutely, others to be fixed up to a linear dependence, and other coefficients were shown either not to appear in the ampltiude at all, or to completely cancel out. The terms which cannot be matched by such an amplitude matching procedure are called the non-leading order terms and in the case of the y and z coefficients these coefficients are the ones that vary under a bosonic field redefinition, and so are postulated not to ever occur in physical amplitudes by the field redefinition theorem. The l_1 and l_2 coefficients may also be related to some fermionic extension of the field redefinition theorem as discussed above. If this is the case then these coefficients also never contrtibute to physical amplitudes. More interestingly there are two terms in the effeitve action which are determined only up to a linear combination: i.e. the κ_1 and κ_2 terms. The restriction that arises from the amplitude matching is simply that $\kappa_1 = - \kappa_2$ and that neither κ_1 and κ_2 are determined separately. This is quite an unprecedented result since nothing like it appears in the bosonic sector of this theory. Naïvely it would appear that from the field redefinitions κ_1 and κ_2 can be varied independently suggesting that they are non-leading order terms, but the amplitude matching suggests that only the combination of these terms as given in Chapter Five is of non-leading order. It is postulated but not discussed that perhaps supersymmetry may play a rôle in the determination of some of the remaining ambiguities in the action. This question will now be addressed in a *somewhat speculative* manner, as the work described does not fit in with the main thrust work described in the previous chapters of this thesis, where the main goal was the construction of part of the fermionic sector of the low energy effective action for the heterotic string.

The previous section in this chapter demonstrated that there appears to be no field redefinition which links the Romans and Warner action to the Han et al action of the form which is usually considered, and to which the discussion was restricted above, implying that the two actions considered *cannot* be considered as two degenerate forms of the same action. It is clear that although the two actions discussed in Chapter Four were derived by the *same* Noether type of technique, and they are *both* supersymmetric with respect to the *same* supersymmetry transformations, (up to one small difference which cannot be detected in the amplitude matching calculations used above since it is dependent on the antisymmetric tensor field which is shown not to contribute to the amplitudes considered above) the actions are completely different dynamically as the amplitudes calculated in that chapter show. It must be concluded that the Noether method must give *nonunique* actions which depend on the trial terms used to cancel the supersymmetry anomaly generated by the inclusion of the Lorentz-Chern-Simons three form term. For example the Romans and Warner action and the Han et al action start with completely different sets of terms to try and cancel the same anomaly, each term given its own arbitrary coefficient which is fixed by the Noether method as applied to that action. It can be seen by examining the action used in Chapter Five, which is the most general action of its type, that the Noether method can be generalised further than either of these two references. It would therefore seem probable that the rather limited results of the two trial actions used in Chapter Four would generalise to give an action from the Noether method which would still contain some arbitrariness in the coefficients used in the action. It can be seen from the final result obtained from the amplitude matchings of Chapter Five still contains δ as an arbitrary parameter in what are usually regarded as leading order terms in the action. This arbitrariness would not affect the supersymmetry of the action since the Noether method would provide constraints on the variation of the general coefficients in the action, and the general coefficients introduced in the supersymmetry variations (as in the Romans and Warner action), which would ensure the preservation of supersymmetry. Moreover, each of these actions given by a supersymmetric choice in this parameter space can be expected to be dynamically distinct from any others given by a different choice of the parameters.

So summarising: it can be surmised that the Noether method may be used, as in the construction of the Romans and Warner and Han et al actions, to generate a set of at least two, (but probably more), *dynamically nondegenerate* actions, (obviously at least the Romans and Warner and Han et al actions), which are all forced to be supersymmetric by the definition of the method, but which all give quite distinct physical amplitudes for the same physical processes. It must be hoped that the low energy effective action for the heterotic superstring is given by one of these actions. This is quite hopeful because it can be seen that the final action found by amplitude matching is almost, but not quite identical to the Han et al action. It would be interesting, but exteremely difficult, to perform the most general Noether type calculation using an action of the form of (5.1.10). It would also be interesting, and somewhat more practicable, to try to show by the same techniques that the action found by amplitude matching in Chapter Five can be shown to be supersymmetric at least for some choice of the remaining arbitrary parameters. Neither of these calculations will be performed in this thesis.

Appendix One: Two Dimensional Conformal Field Theories.

This appendix contains a review of the conformal field theoretical techniques used in the calculation of string amplitudes, as given in Chapter One. It is a resumé of the work of references [11,27-31] which will be of use in the discussion of string theory amplitude calculations.

In discussing two dimensional conformal field theories, it is possible to consider a general action of the form;

$$S = \int d^D\xi \, L \quad , \qquad \text{(A1.1)}$$

where L is the Lagrangian, which is some functional of the physical fields in the theory. The theory is said to have a conformal symmetry if the action is invariant under transformations of the form;

$$\xi^a \to \lambda \, \xi^a \qquad a = 1, \ldots, D \quad , \qquad \text{(A1.2a)}$$

or more generally reparametrisations of the form,

$$\xi^a \to \eta^a(\xi) \quad , \qquad \text{(A1.2b)}$$

such that the metric transforms as,

$$g_{ab}(\xi) \to \frac{\partial\xi^{a'}}{\partial\eta^a} \cdot \frac{\partial\xi^{b'}}{\partial\eta^b} \, g_{a'b'}(\eta) = \rho(\xi) \, g_{ab} \quad , \qquad \text{(A1.2c)}$$

The condition for such a conformal symmetry is that the trace of the stress energy tensor should vanish,

$$T^a_a(\xi) = 0 \quad , \qquad \text{(A1.3)}$$

where it should be noted that this condition is true for any number of dimensions. In the case where we have two space-time dimensions, the coordinates may be parametrised using complex coordinates of the form,

$$z = \xi^1 + i\,\xi^2\,,\ \bar{z} = \xi^1 - i\,\xi^2\ , \qquad \text{(A1.4a)}$$

and where the complex derivatives may also be defined by,

$$\partial_z = \frac{1}{2}(\,\partial_1 - i\,\partial_2\,)\ ,\ \partial_{\bar{z}} = \frac{1}{2}(\,\partial_1 + i\,\partial_2\,)\quad . \qquad \text{(A1.4b)}$$

The fields of the theory define a <u>complete</u> set of fields $\{A_j(0)\}$, where the completeness axiom is defined to mean that the fields satisfy the operator product algebra, where any product of fields can be represented by a linear sum of other fields in the algebra; that is, it is possible to write the operator product expansions,

$$A_i(\xi)\,A_j(0) = \sum_k C^k_{ij}(\xi)\,A_k(0)\quad , \qquad \text{(A1.5)}$$

where this operator algebra satisfies the bootstrap conditions: in a general correlation function, the complete operator expansion is associative.

At this point it should be noted that the complete set of possible conformal transformations of the type (A1.2a-2c) form a group G. If a complex parametrisation is chosen, then G is the group of all transformations of the form,

$$z \rightarrow \zeta(z)\quad ,\quad \bar{z} \rightarrow \bar{\zeta}(\bar{z})\quad ,$$

such that z, $\bar{z}$, are analytic and antianalytic respectively. Furthermore, it can be shown that the stress energy tensor also satisfies the analyticity requirement, i.e.,

$$T(z) = T_{11} - T_{22} + 2iT_{12}\quad ,\quad \bar{T}(\bar{z}) = T_{11} - T_{22} - 2iT_{12}\quad ,$$

where,

$$\partial_{\bar{z}}\,T(z) = 0\quad ,\quad \partial_z\bar{T}(\bar{z}) = 0\quad . \qquad \text{(A1.6)}$$

Thus it is easily seen that the group may be naturally decomposed into two parts,

$$G = \Gamma \otimes \bar{\Gamma}\quad , \qquad \text{(A1.7)}$$

corresponding to the analytic and antianalytic parts respectively. By considering an infintesimal conformal transformation of the form, (noting that from now on, only the analytic half of the theory is considered),

$$z \rightarrow z + \varepsilon(z) \quad , \qquad \text{(A1.8)}$$

and expanding this in a Laurent expansion,

$$\varepsilon(z) = \sum_{n=-\infty}^{\infty} \varepsilon_n \, z^{n+1} \quad , \qquad \text{(A1.9)}$$

it is possible to see that the generators of Γ are given by,

$$l_n = z^{n+1} \frac{d}{dz} \quad , \quad n \in \mathbb{Z} \quad , \qquad \text{(A1.10)}$$

where these generators satisfy the commutation relations,

$$[l_n, l_m] = (n-m)\, l_{n+m} \quad . \qquad \text{(A1.11)}$$

The central extension of this algebra, (which will be discussed below), forms the Virasoro algebra. The full Virasoro algebra reappears when conformal reparametrisations are discussed in conjunction with the full quantum theory.

By generalising the usual argument found in the Lagrangian formulation of field theories, it can be seen that the stress energy tensor T(z) generates conformal reparametrisations of the fields of the theory. So it helps to define an infinitesimal operator to generate these reparametrisations by writing,

$$T_\varepsilon = \int_{C_o} dz \, \varepsilon(z)\, T(z) \quad , \qquad \text{(A1.12)}$$

(where C_0 is a contour about the origin), for the generator of the specific conformal reparametrisation ε, so that the conformal variation of the (now quantum) field $A_j(\zeta)$, is, (where the contour is now about the point ζ),

$$< \delta_\varepsilon A_j(\zeta) > = < [T_\varepsilon , A_j(\zeta)] >$$
$$= \int_{C_\zeta} dz\, \varepsilon(z) < T(z)\, A_j(\zeta) > \quad . \qquad \text{(A1.13)}$$

This generalises to the case of any correlation function of the fields in the complete set, where this can be written,

$$\sum_{k=1}^{N} < A_{i_1}(\xi_1) \,\ldots\ldots\, \delta_\varepsilon A_{i_k}(\xi_k) \,\ldots\ldots\, A_{i_N}(\xi_N) >$$
$$+ \int d^2\xi\, \partial^a \varepsilon^b(\xi) < T_{ab}(\xi)\, X > \; = 0 \;, \quad \text{where } X = < A_{i_1}(\xi_1) \ldots A_{i_N}(\xi_N) >$$
-----(A1.14)

The stress energy tensor also varies with respect to these conformal reparametrisations, and it is possible to write down the most general possible variation of the tensor, with respect to these reparametrisations,

$$\delta_\varepsilon T(z) = \varepsilon(z) \frac{\partial}{\partial z} T(z) + 2 \frac{\partial \varepsilon}{\partial z} T(z) + \frac{1}{12} c \frac{\partial^3 \varepsilon}{\partial z^3} \quad , \qquad \text{(A1.15)}$$

where c is an arbitrary parameter of the theory, and can be regarded as its parameter. Now, this relation may be rewritten,

$$[T_\varepsilon , T(z)] = \varepsilon(z)\, T'(z) + 2\varepsilon'(z)\, T(z) + \frac{1}{12} c\, \varepsilon'''(z) \quad , \qquad \text{(A1.16)}$$

where the dash denotes differentiation with respect to z, which gives the Virasoro algebra,

$$[L_n , L_m] = (n-m)\, L_{n+m} + \frac{1}{12} c\, (n^3-n)\, \delta_{n+m,0} \quad , \qquad \text{(A1.17)}$$

where the Laurent expansion of the stress energy tensor is assumed to be,

$$T(z) = \sum_{n=-\infty}^{\infty} L_n z^{-n-2} \quad . \qquad \text{(A1.18)}$$

It is useful to note at this point that the generators L_0, L_{-1}, and L_1 generate a subalgebra of SL(2,C), which will be used to factor out some of the integrations on the world sheet. A suitable definition of the Hamiltonian of the theory is given by the generators of local dilations,

$$H = L_0 + \overline{L_0} \quad . \qquad \text{(A1.19)}$$

The vacuum of the field theory is defined to be the ground state of this Hamiltonian, and since the stress energy tensor T(z) is required <u>not</u> to be singular at $z = 0$, and to be regular as $z \to \infty$ in the parametrisation,

$$z = e^{(\tau+i\sigma)} \ , \ \bar{z} = e^{(\tau-i\sigma)} \quad , \qquad \text{(A1.20)}$$

the vacuum satisfies the conditions,

$$\begin{aligned} L_n|0\rangle &= 0 \ , \ n \geq -1 \quad , \\ \langle 0| L_n &= 0 \ , \ n \leq 1 \quad . \end{aligned} \qquad \text{(A1.21)}$$

The most important structures in the algebra of fields that have been developed so far, are the generalised Ward identities. These identities give the relationships between the fields in the complete set $\{A_j(0)\}$ which are needed for the evaluaton of <u>any</u> correlation function of the specific theory under examination. Since the algebra of fields $\{A_j(0)\}$ is complete, it is possible to expand the variation of a particular field linearly in terms of derivatives of the infinitesimal conformal reparametrisation and other fields in the algebra. This can be explicitly written in the form,

$$\delta_\varepsilon A_j(z) = \sum_{k=0}^{v_j} B_j^{(k-1)}(z) \frac{\partial^k}{\partial z^k} \varepsilon(z) \quad , \qquad \text{(A1.22)}$$

where the fields B_j are members of the complete set. Now, noting in general that the $k^{th.}$ derivative of any function f(z) at the point ζ can be written,

$$f^{(k)}(\zeta) = k! \int_{C_\zeta} \frac{dz}{2\pi i} \frac{f(z)}{(\zeta - z)^{k+1}} \quad , \qquad (A1.23)$$

and that the Laurent expansion of the stress energy tensor is given by equation (A1.18), the expansion (A1.14) can be written,

$$< T(z)\, A_{i_1}(z_1) \,.....\, A_{i_N}(z_N) >$$

$$= \sum_{k=1}^{N} \sum_{l=0}^{\nu_k} l!\, (z-z_k)^{-l-1} < A_{i_1}(z_1) \ldots B_{i_k}^{(l-1)}(z_k) \ldots A_{i_N}(z_N) > , \quad (A1.24)$$

which is the generalised Ward identity for the complete set of fields chosen for the theory. By considering simple translations and scalings, it is possible to see that,

$$B_j^{(-1)}(z) = \frac{\partial}{\partial z} A_j(z) \ , \ B_j^{(0)}(z) = \Delta_j\, A_j(z) \quad , \qquad (A1.25)$$

where Δ_j is the anomalous scaling dimension of the field $A_j(z)$. Furthermore, by noting that the anomalous conformal dimension of $\varepsilon(z)$ is -1, it can be seen that the fields $B_j^{(l-1)}$ have dimensions given by,

$$\Delta_{j,(k-1)} = \Delta_j + 1 - k \quad , \quad k = 0,1, \ldots ,\nu_j \qquad (A1.26)$$

and since any physically realistic theory has fields which transform with anomalous conformal dimension $\Delta_j \geq 0$, then it can be seen that ν_j is finite and satisfies the inequality $\nu_j \leq \Delta_j + 1$.

It can also be seen that the spectrum of dimensions of fields in the theory considered, consists of an infinite set of series of integer spacing,

$$\Delta_n^{(k)} = \Delta_n + k \quad , \qquad (A1.27)$$

where Δ_n is the minimum dimension of each series. This is equivalent to saying that the decomposition (A1.22) does not continue to $B_j^{(k-1)}$ fields of anomalous dimension Δ_n' such that $\Delta_n' - 1 < 0$, for every possible field in the complete set $\{A_j(0)\}$. This implies the existence of a field ϕ_n for each field A_j in the complete set, such that the field ϕ_n has anomalous scaling dimension Δ_n. The field ϕ_n with dimension Δ_n, now called a primary field, can be seen to transform under a general infinitesimal conformal transformation with variation,

$$\delta_\varepsilon \phi_n(z) = \varepsilon(z) \frac{\partial}{\partial z} \phi_n(z) + \Delta_n \frac{\partial \varepsilon(z)}{\partial z} \phi_n(z) \quad . \qquad \text{(A1.28)}$$

Thus it can be seen that in the case of a primary correlation function, (that is a correlation function consiting of the primary fields defined above), the generalised Ward identity can be written,

$$< T(z)\, \phi_1(z_1) \, ... \, \phi_N(z_N) > \; = \sum_{i=1}^{N} \{ \frac{\Delta_i}{(z-z_i)^2} + \frac{1}{(z-z_i)} \frac{\partial}{\partial z_i} \} \; . \; < \phi_1(z_1) \, ... \, \phi_N(z_N) > \quad , \qquad \text{(A1.29)}$$

which can be applied in the simple case of N=1. Doing this and applying the operator product expansion it is possible to write,

$$T(z)\, \phi_n(\zeta) = \sum_{k=0}^{\infty} (z - \zeta)^{-2+k} \phi_n^{(-k)}(\zeta) \quad , \qquad \text{(A1.30)}$$

where the $\phi_n{}^{(-k)}(\zeta)$ are called the secondary fields of the theory. These secondary fields may themselves vary with respect to conformal reparametrisations and so it can be shown that,

$$\delta_\varepsilon \phi_n^{(-k)}(z) = \varepsilon(z) \frac{\partial}{\partial z} \phi_n^{(-k)}(z) + (\Delta_n + k)\, \varepsilon'(z)\, \phi_n^{(-k)}(z) + \sum_{l=1}^{k} \frac{k+1}{(l+1)!} [\frac{\partial^{l+1}}{\partial z^{l+1}} \varepsilon(z)]\, \phi_n^{(l-k)}(z) + \frac{1}{12} c \frac{1}{(k-2)!} [\frac{\partial^{k+1}}{\partial z^{k+1}} \varepsilon(z)]\, \phi_n(z) \quad , \qquad \text{(A1.31)}$$

where this can written in terms of the general Ward identity, as above in equation (A1.29),

$$T(\zeta)\, \phi_n{}^{(-k)}(z) = ... \qquad \text{etc.} \quad .$$

More secondary fields are obtained from this relation and the procedure can be repeated for these fields. This means that for any primary field ϕ_n there exists a conformal family denoted by $[\phi_n]$. In fact it is simple to see that the complete set of conformal fields $\{A_j(0)\}$ must decompose into a direct sum of all the conformal families,

$$\{A_j(0)\} = \bigoplus_n [\phi_n] \qquad . \qquad (A1.32)$$

Explicitly the derivation of the secondary fields, given a specific primary field can be summarised neatly by writing,

$$L_{-k}(z) = \int_{C_z} d\zeta \frac{T(\zeta)}{(z-\zeta)^{-k-2}} \qquad , \qquad (A1.33a)$$

and,

$$\phi_n^{(-k_1, \, \ldots\ldots, -k_N)}(z) = L_{-k_1}(z) \ldots\ldots L_{-k_N}(z) \, \phi_n(z) \qquad . \qquad (A1.33b)$$

So, it can be seen that any secondary field can be written as a function of the primary field which generates the conformal family of which it is a member. It is now possible to consider the evaluation of general correlation functions. From (A1.33) and (A1.30), it can be shown that any general correlation function can be written in the form,

$$< T(\zeta_1) \ldots\ldots T(\zeta_M) \, \phi_1(z_1) \ldots\ldots \phi_N(z_N) > \qquad , \qquad (A1.34)$$

which can be expanded by repeated application of the generalised Ward identity, giving a sum of differential operators acting on the 'primary' correlator.

For example consider the simple case,

$$< \phi_n^{(-k_1, \, \ldots, -k_m)}(z) \, \phi_1(z_1) \ldots\ldots \phi_N(z_N) >$$

$$= L_{-k_m}(z,z_i) \ldots L_{-k_1}(z,z_j) < \phi_n(z) \, \phi_1(z_1) \ldots\ldots \phi_N(z_N) > \quad , \quad (A1.35)$$

where the differential operators are given by,

$$L_{-k_p}(z,z_i) = \sum_{i=1}^{N} \left(\frac{(1-k_p).\Delta_i}{(z-z_{k_p})^{k_p}} - \frac{1}{(z-z_i)^{k_p-1}} \frac{\partial}{\partial z_i} \right) \quad . \qquad (A1.36)$$

All of the above is derived for a general action of indeterminate form. It is possible to consider actions which have an internal symmetry, either global or local. (The case of string theory is an example where the action has a global SO(1,D-1) symmetry.). Consider an action with an internal symmetry, of the form given in (A1.1) with Lagrangian of the form,

$$L = L(X^{\mu}, \partial_{\tau} X^{\nu}) \quad , \qquad (A1.37)$$

where μ,ν are the indices of the internal symmetry and ∂_τ is the world sheet derivative with respect to time. The action is defined to satisfy the condition of invariance under transformations of the form,

$$X(\xi) \rightarrow \Omega(z)\, X(\xi)\, \bar{\Omega}^{-1}(\bar{z}) \quad , \qquad (A1.38)$$

where $\Omega(z)$ and $\Omega(z)$ are arbitrary G valued matrices, G being the group of symmetry transformations. Noether's theorem implies that there are an infinite number of conserved currents on the world sheet. These are easily seen to be analytic (antianalytic) by similar considerations to those of equation (A1.6), and it is also true that these symmetry currents generate transformations of the fields with respect to the group G. So it is possible to write,

$$J = J^{\mu}\, t^{\mu} \quad , \quad J = J^{\nu}\, t^{\nu} \qquad (A1.39a)$$

$$\partial_{\bar{z}} J = 0 \quad , \quad \partial_z \bar{J} = 0 \quad , \qquad (A1.39b)$$

where the t^{μ} are the generators of the symmetry group G, and consequently satisfy the Lie algebra, $[t^{\mu},t^{\nu}] = f^{\mu\nu\rho}\, t^{\rho}$, $f^{\mu\nu\rho}$ being the structure constants of the group G, and where J^{μ} may be written,

$$J^{\mu} = J^{\mu}(z) \quad , \quad \bar{J}^{\nu} = \bar{J}^{\nu}(\bar{z}) \quad , \qquad (A1.39c)$$

due to equation (A1.39b). Under infinitesimal tranformations,

$$\Omega(z) = I + \omega(z) = I + \omega^{\mu}(z)\, t^{\mu} \quad , \qquad \text{(A1.40a)}$$

$$\bar{\Omega}(\bar{z}) = I + \bar{\omega}(\bar{z}) = I + \bar{\omega}^{\mu}(\bar{z})\, t^{\mu} \quad , \qquad \text{(A1.40b)}$$

it can be seen that the currents transform in the manner,

$$\delta_{\omega} J(z) = \left[\omega(z)\, ,J(z)\right] + \frac{1}{2} k\, \omega'(z) \quad ,$$

$$\delta_{\bar{\omega}} \bar{J}(\bar{z}) = \left[\bar{\omega}(\bar{z})\, ,\bar{J}(\bar{z})\right] + \frac{1}{2} k\, \bar{\omega}(\bar{z}) \quad , \qquad \text{(A1.41)}$$

which will give a Kac-Moody algebra, when the operator algebra is constructed later. (The restriction to the analytic part is now made, leaving the antianalytic part implicit. The properties of the antianalytic half of the theory are directly analagous to the analytic part.).

The generator of the symmetry transformations can be written in direct analogy to the generator for conformal reparametrisations, given by (A1.12) and (A1.13). Explicitly the transformation takes the form,

$$\delta_{\omega} A_j(\zeta) = \int_{C_\zeta} dz\, J^{\mu}(z)\, \omega^{\mu}(z)\, A_j(\zeta)\, , \qquad \text{(A1.42)}$$

where the quantum mechanical vacuum expectation is implied. The J fields transform under group transformations and conformal reparametrisations as follows,

$$\delta_{\varepsilon} J^{\mu}(z) = \varepsilon(z) \frac{\partial}{\partial z} J^{\mu}(z) + \frac{\partial}{\partial z} \varepsilon(z)\, J^{\mu}(z) \quad , \qquad \text{(A1.43a)}$$

$$\delta_{\omega} J^{\mu}(z) = f^{\mu\nu\rho}\, \omega^{\nu}(z)\, J^{\rho}(z) + \frac{1}{2} k \frac{\partial}{\partial z} \omega^{\mu}(z) \ , \qquad \text{(A1.43b)}$$

which may be rewritten in terms of the operator product expansions,

$$T(z)J^{\mu}(\zeta) = \frac{1}{(z-\zeta)^2} J^{\mu}(\zeta) + \frac{1}{(z-\zeta)} \frac{\partial}{\partial \zeta} J^{\mu}(\zeta) + \dots \quad , \qquad \text{(A1.44a)}$$

$$J^\mu(z)\, J^\nu(\zeta) = \frac{k\, \delta^{\mu\nu}}{(z-\zeta)^2} + \frac{f^{\mu\nu\rho}}{(z-\zeta)} J^\rho(\zeta) + \ldots \qquad , \qquad \text{(A1.44b)}$$

where all regular terms at $z \rightarrow \zeta$ are omitted. The equations (A1.44a-b) define the Kac-Moody algebra which was mentioned above and also the coupling of this to the Virasoro algebra given in equation (A1.17), as may be explicitly written,

$$\left[J^\mu_n , J^\nu_m\right] = f^{\mu\nu\rho}\, J^\rho_{m+n} + \frac{1}{2} k\, n\, \delta^{\mu\nu} \delta_{n,0} \qquad , \qquad \text{(A1.45a)}$$

$$\left[L_n , J^\mu_m\right] = -m\, J^\mu_{n+m} \qquad , \qquad \text{(A1.45b)}$$

and where the structure constants are those of the symmetry group. (The Laurent expansion of the $J^\mu(z)$ field is given by,

$$J^\mu(z) = \sum_{n=-\infty}^{\infty} J^\mu_n\, z^{-n+1} \qquad , \qquad \text{(A1.45c)}$$

the conformal dimension being -1). It can now be seen that the primary fields of the theory transform under the symmetry analogously to the way they transform under conformal reparametrisations, that is,

$$J^\mu(\zeta)\, \phi_l(z) = \frac{t^\mu_l}{(\zeta-z)} \phi_l(z) \qquad , \qquad \text{(A1.46)}$$

where t_l^μ is a G valued matrix which forms a representation of the group G. It can now be seen that the argument which led to the invocation of secondary fields in the algebra of conformal fields can be generalised to the case of the internal symmetry. The fields thus generated can be seen to decompose into families generated by the operator product algebra acting on some primary field of the theory. This means that now any general correlation function can be written in terms of the stress energy tensor, the symmetry current and the new primary fields. Thus it can be seen that it is possible to write any correlation function in the form,

$$< T(z_1) \ldots T(z_M)\, J^{\mu_1}(z'_1) \ldots J^{\mu_N}(z'_N)\, \phi_{l_1}(\zeta_1) \ldots \phi_{l_P}(\zeta_P) > \quad , \quad \text{(A1.47)}$$

and by repeated application of the operator product algebra, and the variation rules for the fields, as given in equations (A1.44a-b) and (A1.46) for the symmetry group, and by using the generalised Ward identity for the conformal reparametrisations repeatedly.

In fact the symmetry group gives rise to a 'generalisation' of the generalised Ward identities, of the form,

$$< J^{\mu}(\zeta)\,\phi_1(z_1)\,...\,\phi_N(z_N) > \;=\; \sum_{j=1}^{N} \frac{t_j^{\mu}}{(\zeta - z_j)} < \phi_1(z_1)\,...\,\phi_N(z_N) > \tag{A1.48}$$

so that any correlation function can be expressed as a sum of products of differential operators acting on the primary correlation function.

If the theory contains fermions, then there exists a specific representation of the symmetry currents, which generate the Kac-Moody algebra, that is, it is possible to write the symmetry current in terms of fermionic bilinears, that is, the normal fermionic currents of the theory. (In the case of world sheet supersymmetric string theory, there is a manifest SO(1,D-1) symmetry and then the symmetry currents can take the form,

$$J^{\mu\nu}(z) = \psi^{\mu}\psi^{\nu}(z) \quad , \tag{A1.49}$$

where the normal ordered product is assumed to be taken. (In the NSR string theory, the two point correlation function for the world sheet fermions is,

$$< \psi^{\mu}(z)\,\psi^{\nu}(\zeta) > \;=\; \eta^{\mu\nu} \frac{1}{(z-\zeta)} + \;.... \tag{A1.50}$$

where the finite corrections depend on the world sheet topology and include the symmetry current, which must be regular at all points. The normal ordering merely subtracts the singular term.). It can also be seen that the stress energy tensor constructed from the symmetry current by writing,

$$T(z) = \frac{1}{2\kappa} :J(z).J(z): \quad , \tag{A1.51}$$

where κ is a parameter of the theory, which is dependant on the previous two parameters c and k, as well as the normalisation of the structure constants of the symmetry group. It can be seen that this stress energy tensor gives the same operator product algebra when the relation (A1.44b) is used. This is known as the Sugawara construction. This will be used in the string theory to define the correct vacuum of the theory, and to show that the vacuum is not self adjoint, in the ghost sector . The same construction will be used in the evaluation of superstring amplitudes, since the symmetry current arising from the world sheet fermions appears in the bosonic superstring vertex. The techniques involved are described in the section dealing with string theory, as a particular example of conformal field theories in two dimensions.

The techniques of superconformal field theory in two dimensions are similar, but have an interesting generalisation of the Virasoro algebra. These techniques are very important in the construction of superstring theories, and so it will be productive to generalise the techniques outlined above to the superconformal case.

The easiest way to generalise to the superconformal case is to extend the world sheet into superspace, and to construct superconformal superfields. Many of the techniques can then simply be generalised from the bosonic case. The important point is to generalise the results of complex analysis to the complex superspace case. It can be seen that Cauchy's theorem, Taylor's theorem and contour integration have analogues in complex superspace. Defining Grassman differentiation and integration by the usual rules,

$$\frac{d}{d\psi}\psi = 1 - \psi\frac{d}{d\psi} \quad , \quad \frac{d}{d\psi}\bar{\psi} = - \bar{\psi}\frac{d}{d\psi}$$

$$\int (\dots)\, d\psi = \frac{d}{d\psi} = [\frac{d}{d\psi}]^{-1}$$

and supposing that a function f(z,ψ) can be defined, then it is possible to state that,

$$f(z,\psi) = f_o(z) + \psi f_1(z)$$

so that the definitions of the integration and differentiation can be applied consistently to this.Defining the complex superderivatives in the usual manner,

$$D = \partial_\theta + \theta\partial_z \quad , \quad \bar{D} = \partial_{\bar{\theta}} + \bar{\theta}\partial_{\bar{z}} \qquad \text{(A1.52)}$$

by which it can be seen that the supersymmetry algebra in two dimensions has the form,

$$D^2 = \partial_z \qquad \bar{D}^2 = \partial_{\bar{z}} \tag{A1.53}$$

means that the results of the bosonic theory above can be almost directly applied with only a little modification. A little care must be taken in the definition of the coordinates used in the Laurent expansion of the various fields of the theory. The most convenient way of writing this is to use the coordinate definitions to denote displacements of the coordinates of superspace,

$$z_{12} = z_1 - z_2 - \theta_1\theta_2 \ , \quad \theta_{12} = \theta_1 - \theta_2 \tag{A1.54}$$

so that the superspace derivatives can be written in the form,

$$D_1 z_{12} = D_2 z_{12} = \theta_{12}$$

$$D_1 \theta_{12} = D_2 \theta_{12} = 1 \tag{A1.55}$$

Infinitesimal reparametrisations of the superspace variables are most easily accomplished by generalising the infinitesimal vector field used to parametrise general field reparametrisations, as in standard differential geometry, to a superfield parameter $V(z,\theta) = v_0 + \theta\, v_1$, such that,

$$\delta z = V - \theta\delta\theta$$

$$\delta\theta = \frac{1}{2} D V$$

where these transformations act on conformal superfields $\phi(z,\theta)$ of superconformal weight h,

$$\delta\phi = \left\{ V\partial + \frac{1}{2}(D V)D + h\,(\partial V) \right\}\phi \tag{A1.56}$$

(which can be shown by general geometrical considerations). Again it can be shown that the generator of the superconformal transformations is the super stress-energy tensor, which generates superconformal transformations of the local superfields of the theory, and so the operator product which generates (A1.56) above is defined by the standard contour integral technique applied to the superspace case, in the form,

$$\delta\phi = \frac{1}{2\pi i}\oint dz\, d\theta\, VT\phi \qquad \text{(A1.57)}$$

which yields the appropriate operator product,

$$T(z_1,\theta_1)\,\phi(z_2,\theta_2) \sim h\,\frac{\theta_{12}}{z_{12}^2}\,\phi(z_2,\theta_2) + \frac{1}{2z_{12}}\,D_2\phi(z_2,\theta_2) + \frac{\theta_{12}}{z_{12}}\,\partial_2\phi(z_2,\theta_2) + \ldots$$

(A1.58)

and the operator product of the super stress-energy tensor is,

$$T(z_1,\theta_1)\,T(z_2,\theta_2) \sim \frac{\hat{c}}{4z_{12}^3} + \frac{3\theta_{12}}{2z_{12}^2}\,T(z_2,\theta_2) + \frac{1}{2z_{12}}\,D_2T(z_2,\theta_2) + \frac{\theta_{12}}{z_{12}}\,\partial_2T(z_2,\theta_2) + \ldots$$

(A1.59)

which generates the superconformal algebra with central extension given by the parameter $\hat{c}$. This is done in strict analogy to the derivation sketched above.

The Laurent coefficients given by expanding the super stress-energy tensor as above, (noting the stress energy operator product expansion above), must satisfy the generalisation of the Virasorso algebra given above. The Laurent expansion is of the form,

$$T(z,\theta) = \sum_n z^{-n-3/2}\,(\frac{1}{2})\,G_n + \theta\sum_m z^{-n-2}L_n$$

(A1.60)

and so the algebra becomes,

$$[\,L_n\,,L_m\,] = (n-m)\,L_{n+m} + \frac{1}{8}\,\hat{c}\,(n^3-n)\,\delta_{n+m,0}$$

$$\{\,G_n\,,G_m\,\} = 2L_{m+n} + \frac{1}{2}\,\hat{c}\,(\,n^2 - \frac{1}{4}\,)\,\delta_{m+n,0}$$

$$[\,L_n\,,G_m\,] = (\,\frac{1}{2}n - m\,)\,G_{n+m} \qquad \text{(A1.61)}$$

which is just the super-Virasoro algebra[27].

It can now be seen that there are two distinct sectors of the Fock space generated by the operators in this algebra, because of the fermionic sector of the conformal superfields. The fermions in such a superfield can be double valued; that is, it is possible to write,

$$\phi_f(z) = +\phi_f(e^{2\pi i}z) \qquad \text{(A1.62a)}$$

<u>or</u>,

$$\phi_f(z) = -\phi_f(e^{2\pi i}z) \qquad \text{(A1.62b)}$$

These sectors are the Neveu-Schwarz[23] and Ramond[22] sectors of the theory respectively. From inspection of the mode expansion it can be seen that the G_n in the algebra have integer n in the Ramond sector corresponding to periodic boundary condition, and half-integer n in the antiperiodic Neveu-Schwarz sector. Each of these sectors is seperately supersymmetric in two dimensions, and the expression of closure of the supersymmetry algebra can be seen by a generalisation of the argument which shows that L_{-1}, L_0, L_1forms an SL(2) subalgebra of the full Virasoro algebra, in the bosonic case, which corresponds to the local Lorentz symmetry of the two dimensional manifold on which the fields are defined. The condition is that, in the Neveu-Schwarz sector,

$$G^2_{-1/2} = L_{-1} \qquad \text{(A1.63)}$$

(which can be seen by the mode expansion corresponding to the super-reparametrisation of a general conformal field given the definition of the super-Virasoro algebra given above), and in the Ramond sector,

$$G_0^2 = L_0 - \frac{1}{16}\hat{c} \qquad \text{(A1.64)}$$

where the ground state preserves supersymmetry if there exists a vacuum state such that,

$$G^2_0 = 0 \qquad \text{(A1.65)}$$

The vacuum is the lowest energy state, which is also a conformal singlet, i.e. h=0. This is clearly in the Neveu-Schwarz sector. This must be the case as two-dimensional supersymmetry will be broken in the Ramond sector, unless $G_0{}^2=0$. Thus the Ramond ground state has $h={}^1/_{16}\hat{c}$, where $\hat{c}$ is nonzero. As before the conformal superfields create the set of Highest weight states in the Neveu-Schwarz sector, which are defined to be annihilated by all of the lowering operators in the algebra. However this procedure does not exhaust all the possible states in the theory, since this approach does not create fields in the Ramond sector, which is obvious due to the invariance of the boundary conditions of the fermion fields created by this process. The answer is to introduce a new field into the theory which is called a spin field, which has a double valued operator product with Neveu-Schwarz fields to 'simulate' the Ramond boundary conditions of the superfields, so that the Laurent expansion of fermions in the Ramond sector have the correct, double valued form. The operator product of such a spin field chosen to satisfy the constraint of supersymmetry of the theory[27], is of the form,

$$T(z,\theta)\, S(\zeta) \sim \frac{1}{2}\frac{1}{(z-\zeta)^{-3/2}}\, S(\zeta) \qquad \text{(A1.66)}$$

This completes the set of tools required for the transition to string theory proper. This is examined in some detail in Chapter One, where these techniques are applied to the specific case of the calculation of string amplitudes.

Appendix Two.
The propagators of the extended Chapline-Manton action.

Section A2.1 The graviton propagator.

The extended Chapline-Manton action described in Chapter Two has various correction terms which can give corrections to the propagators derived in the background field expansion approach to the quantisation of the theory, as discussed in Chapter Two. For the graviton, these terms take the form,

$$xe\phi^{-3/4}(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4y\, R_{\mu\nu}R^{\mu\nu} + z\, R^2) \qquad \text{(A2.1.1)}$$

where the background field expansion is defined by,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} ,$$

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + h^{\mu\eta} h_{\eta}^{\ \nu} + \ldots \qquad (3.2.3)$$

$$e_{\mu}^{\ m} = \delta_{\mu}^{\ m} + \frac{1}{2} h_{\mu}^{\ m} - \frac{1}{8} h_{\mu}^{\eta} h_{\eta}^{\ m} + \ldots$$

$$e_{m}^{\mu} = \delta_{m}^{\mu} - \frac{1}{2} h_{m}^{\mu} + \frac{3}{8} h_{m}^{\eta} h_{\eta}^{\ \mu} + \ldots \qquad (3.2.4)$$

in which case the Riemann tensor takes the form,

$$\begin{aligned} R^{\sigma}_{\alpha\beta\gamma} = -\frac{1}{2} \{ & (\partial_{\gamma}\partial_{\alpha} h^{\sigma}_{\beta} - \partial_{\gamma}\partial^{\sigma} h_{\beta\alpha} - \partial_{\beta}\partial_{\alpha} h^{\sigma}_{\gamma} + \partial_{\beta}\partial^{\sigma} h_{\gamma\alpha}) \\ & + \partial_{\gamma} [-h^{\sigma\eta} (\partial_{\beta} h_{\eta\alpha} + \partial_{\alpha} h_{\beta\eta} - \partial_{\eta} h_{\alpha\beta})] \\ & - \partial_{\beta} [-h^{\sigma\eta} (\partial_{\gamma} h_{\eta\alpha} + \partial_{\alpha} h_{\gamma\eta} - \partial_{\eta} h_{\alpha\gamma})] \\ & -\frac{1}{2} (\partial_{\gamma} h^{\rho}_{\alpha} + \partial_{\alpha} h^{\rho}_{\gamma} - \partial^{\rho} h_{\alpha\gamma}) (\partial_{\beta} h^{\sigma}_{\rho} + \partial_{\rho} h^{\sigma}_{\beta} - \partial^{\sigma} h_{\rho\beta}) \\ & + \frac{1}{2} (\partial_{\beta} h^{\rho}_{\alpha} + \partial_{\alpha} h^{\rho}_{\beta} - \partial^{\rho} h_{\alpha\beta}) (\partial_{\gamma} h^{\sigma}_{\rho} + \partial_{\rho} h^{\sigma}_{\gamma} - \partial^{\sigma} h_{\rho\gamma}) \} \end{aligned}$$

(A2.1.2)

The lowest order action in the expansion parameter, α', defined to be,

$$-\frac{1}{2}eR$$

can be expanded, by noting that,

$$e = \det(e^{m}_{\mu}) = 1 + \frac{1}{2}\,tr(h) - \frac{1}{4}\,tr(h^2) + \frac{1}{8}\,(tr(h))^2 + O(h^3)$$

(3.2.17)

and, noting that the contraction of the curvature scalar is of the form,

$$R = R^{\mu\nu}{}_{\rho\sigma}g_{\mu}{}^{\sigma}g_{\nu}{}^{\rho} \qquad \text{(A3.4.7)}$$

it can be seen that the full $O(\alpha'^{0})$ term bilinear in gravitons is,

$$-\frac{1}{2}eR = -\frac{1}{2}\Big\{ -\frac{1}{4}h^{\alpha\mu}\partial^{\sigma}\partial_{\sigma}h_{\mu\alpha} + \frac{1}{2}h^{\alpha\mu}\partial^{\sigma}\partial_{\mu}h_{\alpha\sigma}$$
$$-\frac{1}{2}h^{\nu}_{\rho}\partial^{\rho}\partial_{\nu}h^{\sigma}_{\sigma} + \frac{1}{4}h^{\mu}_{\mu}\partial^{\rho}\partial_{\rho}h^{\sigma}_{\sigma}\Big\} \qquad (2.1.5)$$

This term must now be added to the correction term bilinear in gravitons derived from the correction term (A2.1.1) above. The appropriate term is given by the expression,

$$x\,(\, R^{\mu\nu\rho\sigma}|_h R_{\mu\nu\rho\sigma}|_h - 4y\, R^{\mu\nu}|_h R_{\mu\nu}|_h + z\, R|_h R|_h\,)$$

(A2.1.3)

which can be evaluated using the background expansion of the Riemann tensor, and thus yields,

$$x\,\Big\{\,(1-y)\,h_{\mu\nu}\Big[\frac{\partial^{\rho}\partial_{\rho}\partial^{\sigma}\partial_{\sigma}}{2}\,(\delta^{\mu\alpha}\delta^{\nu\beta} + \delta^{\mu\beta}\delta^{\nu\alpha})\Big]\,h_{\alpha\beta}$$

$$-\,2(1-y)\,h_{\mu\nu}\Big[\frac{\partial^{\rho}\partial_{\rho}}{4}\,(\,\delta^{\mu\alpha}\partial^{\nu}\partial^{\beta} + \delta^{\mu\beta}\partial^{\nu}\partial^{\alpha} + \delta^{\nu\beta}\partial^{\mu}\partial^{\alpha} + \delta^{\nu\alpha}\partial^{\mu}\partial^{\beta}\,)\Big]\,h_{\alpha\beta}$$

$$+\,(1-2y+z)\,h_{\mu\nu}\,[\,\partial^{\mu}\partial^{\nu}\partial^{\alpha}\partial^{\beta}\,]\,h_{\alpha\beta}$$

$$-\,(y-z)\,h_{\mu\nu}[\,\partial^{\rho}\partial_{\rho}\partial^{\sigma}\partial_{\sigma}(\delta^{\mu\nu}\delta^{\alpha\beta})]\,h_{\alpha\beta}$$

$$+ 2\,(y-z)\, h_{\mu\nu} \left[\frac{\partial^{\rho}\partial_{\rho}}{2} \left(\delta^{\mu\nu}\partial^{\alpha}\partial^{\beta} + \delta^{\alpha\beta}\partial^{\mu}\partial^{\nu} \right)\right] h_{\alpha\beta} \Big\}$$

(A2.1.4)

The full term bilinear in the gravitons is given by the sum of the terms (2.1.5) and (A2.1.4) above. To evaluate the propagator the terms bilinear in the appropriate field can be written in the momentum representation for maximum convenience, and suitably symmetrised with respect to the fields being considered, as has already been performed above. When this is done the kinetic matrix obtained must be inverted. This will yield the appropriate propagator for the field. In general it can be seen that a singular matrix must be inverted. However this is not possible without the usual gauge fixing term. The gauge chosen here is the standard transverse gauge given by the general gauge fixing term,

$$L_{g.f.} = -\frac{1}{2\xi}\left(\partial^{\rho} h_{\mu\rho} - \frac{1}{2}\partial_{\mu} h^{\rho}_{\ \rho} \right)^{2} - \frac{1}{2\zeta}\left(\partial^{\rho} a_{\mu\rho}\right)^{2} + \frac{1}{4\eta}\,\bar{\psi}^{\mu}\gamma_{\mu}\,\partial\!\!\!/\,(\gamma_{\nu}\psi^{\nu})$$

(A2.1.5)

where the gauge fixing term is the full term needed for the derivation of all the propagators described below. It should be noted that the gauge chosen is only meaningful in the "linearised" theory described throughout this work. The more general form of this gauge is important in the consideration of the complete general action used in the field redefinition analysis of Chapter Six. The discussion of this point is left to that chapter and will not affect the analysis given here. In what follows the gauge parameter ξ will be chosen to have the value $\xi = 2$, in which case the lowest order propagator given by first term in the Taylor expansion of the propagator about the origin with respect to k^2 has an extremely simple form. Adding the gauge fixing term can be seen to give the kinetic matrix term which should be inverted. To do this it is convenient to follow the standard procedure of defining a set of projection operators for rank two symmetric tensors. It is most convenient to work in the momentum representation of field theories in these calculations. The most convenient representation will be chosen for the context of the calculation being performed. A specific choice of projection operators is[51];

<u>Projection operator</u>

$$P^{(2)}_{\mu\nu,\alpha\beta}(k) = \frac{1}{2}\left(\theta_{\mu\alpha}\theta_{\nu\beta} + \theta_{\mu\beta}\theta_{\nu\alpha}\right) - \frac{1}{\theta}\,\theta_{\mu\nu}\theta_{\alpha\beta}$$

$$P^{(1)}_{\mu\nu,\alpha\beta}(k) = \frac{1}{2}(\theta_{\mu\alpha}\omega_{\nu\beta} + \theta_{\mu\beta}\omega_{\nu\alpha} + \theta_{\nu\alpha}\omega_{\mu\beta} + \theta_{\nu\beta}\omega_{\mu\alpha})$$

$$P^{(s)}_{\mu\nu,\alpha\beta}(k) = \frac{1}{\theta}\theta_{\mu\nu}\theta_{\alpha\beta}$$

$$P^{(\omega)}_{\mu\nu,\alpha\beta}(k) = \omega_{\mu\nu}\omega_{\alpha\beta}$$

$$P^{(s\omega)}_{\mu\nu,\alpha\beta}(k) = \frac{1}{\sqrt{\theta}}\theta_{\mu\nu}\omega_{\alpha\beta}$$

$$P^{(\omega s)}_{\mu\nu,\alpha\beta}(k) = \frac{1}{\sqrt{\theta}}\omega_{\mu\nu}\theta_{\alpha\beta} \qquad \text{(A2.1.7)}$$

where, $\theta_{\mu\nu} = \delta_{\mu\nu} - \omega_{\mu\nu}$ and $\omega_{\mu\nu} = k_\mu k_\nu/k^2$, and where the parameter θ denotes the trace of $\theta_{\mu\nu}$, and where θ = d-1, d being the space-time dimension, (in this case ten). The kinetic matrix must be rewritten in terrms of these projection operators, and is consequently of the form,

$$V^{(0)}_{\mu\nu,\alpha\beta} = V^{(0)}_{\mu\nu,\alpha\beta} + V^{(1)}_{\mu\nu,\alpha\beta} \qquad \text{(A2.1.8)}$$

where the separate terms are,

$$V^{(0)}_{\mu\nu,\alpha\beta}(k) = \frac{k^2}{2}\Big\{ -\frac{1}{\xi}P^{(1)}_{\mu\nu,\alpha\beta}(k) - \frac{1}{2}P^{(2)}_{\mu\nu,\alpha\beta}(k) + \frac{((\theta-1)\xi-\theta)}{2\xi}P^{(s)}_{\mu\nu,\alpha\beta}(k) - \frac{1}{2\xi}P^{(\omega)}_{\mu\nu,\alpha\beta}(k) + \frac{\sqrt{\theta}}{2\xi}\left(P^{(s\omega)}_{\mu\nu,\alpha\beta}(k) + P^{(\omega s)}_{\mu\nu,\alpha\beta}(k)\right)\Big\}$$

(A2.1.9)

and,

$$V^{(1)}_{\mu\nu,\alpha\beta}(k) = \frac{k^2}{2}\Big\{ -4k^2x((1-y)\,P^{(2)}_{\mu\nu,\alpha\beta}(k) + ((1-y)+\theta(y-z))\,P^{(s)}_{\mu\nu,\alpha\beta}(k)\,)\Big\}$$

(A2.1.10)

The projection operators satisfy the multiplication algebra given in Figure A2.1. The algebra is simple to check using the orthogonality relations,

$$\theta_{\mu\nu}\theta_{\nu\rho} = \theta_{\mu\rho} \ , \ \theta_{\mu\nu}\omega_{\nu\rho} = 0 \ , \ \omega_{\mu\nu}\theta_{\nu\rho} = 0 \ , \ \omega_{\mu\nu}\omega_{\nu\rho} = \omega_{\mu\rho} \tag{A2.1.11}$$

are satisfied. It is possible to define the identity,

$$1_{\mu\nu,\alpha\beta} = P^{(1)}_{\mu\nu,\alpha\beta}(k) + P^{(2)}_{\mu\nu,\alpha\beta}(k) + P^{(s)}_{\mu\nu,\alpha\beta}(k) + P^{(\omega)}_{\mu\nu,\alpha\beta}(k) \tag{A2.1.12}$$

and so a unique inverse can be found which satisfies the relation,

$$D_{\mu\nu,\alpha\beta}(k)\, V_{\alpha\beta,\rho\sigma}(k) = 1_{\mu\nu,\rho\sigma} = V_{\mu\nu,\alpha\beta}(k)\, D_{\alpha\beta,\rho\sigma}(k)$$

To do this a general term is written for the inverse,

$$D_{\mu\nu,\alpha\beta}(k) = \frac{2}{k^2}\big(aP^{(1)}_{\mu\nu,\alpha\beta}(k) + bP^{(2)}_{\mu\nu,\alpha\beta}(k) + cP^{(s)}_{\mu\nu,\alpha\beta}(k)$$
$$+ dP^{(\omega)}_{\mu\nu,\alpha\beta}(k) + eP^{(s\omega)}_{\mu\nu,\alpha\beta}(k) + fP^{(\omega s)}_{\mu\nu,\alpha\beta}(k) \big) \tag{A2.1.13}$$

and the product with the kinetic matrix taken. The identity can then be applied and the equation of coefficents used to determine the general coefficients for the propagator. The conditions are that the coefficients must satisfy;

$$a = -\xi$$

$$b = -\frac{1}{\frac{1}{2} + 4k^2x(1-y)} = \frac{-2}{1 + 8k^2x(1-y)}$$

$$\frac{f\sqrt{\theta}}{2\xi} - \frac{d}{2\xi} = 1$$

$$f\left\{ \frac{(\theta-1)\xi-\theta}{2\xi} - 4k^2x\,((1-y)+\theta(y-z)) \right\} + \frac{d\sqrt{\theta}}{2\xi} = 0$$

1 \ 2	$P^{3/2}$	$P^{1/2}_{11}$	$P^{1/2}_{22}$	$P^{1/2}_{12}$	$P^{1/2}_{21}$
$P^{3/2}$	$P^{3/2}$	0	0	0	0
$P^{1/2}_{11}$	0	$P^{1/2}_{11}$	0	$P^{1/2}_{12}$	0
$P^{1/2}_{22}$	0	0	$P^{1/2}_{22}$	0	$P^{1/2}_{21}$
$P^{1/2}_{12}$	0	0	$P^{1/2}_{12}$	0	$P^{1/2}_{11}$
$P^{1/2}_{21}$	0	$P^{1/2}_{21}$	0	$P^{1/2}_{22}$	0

Figure A2.1 The Projection Operator Algebra for Rank Two Symmetric Tensor Operators.

$$\frac{c\sqrt{\theta}}{2\xi} - \frac{e}{2\xi} = 0$$

$$c \left\{ \frac{(\theta-1)\xi-\theta}{2\xi} - 4k^2x\,((1-y)+\theta(y-z)) \right\} + \frac{e\sqrt{\theta}}{2\xi} = 1$$

(A2.1.14a)

which are uniquely soluable and yield; a, b as above, and,

$$c = \frac{2}{((\theta-1) - 8k^2x((1-y)+\theta(y-z)))}$$

$$d = -2\,\frac{((\theta-1)\xi-\theta - 8\xi k^2x((1-y)+\theta(y-z)))}{((\theta-1) - 8k^2x((1-y)+\theta(y-z)))}$$

$$e = f = \frac{2\sqrt{\theta}}{((\theta-1) - 8k^2x((1-y)+\theta(y-z)))}$$

(A2.1.14b)

The graviton propagator can now be written in terms of the projection operators, in the form of equation (A2.1.13) above. The propagator is not terribly useful in this form. In general since only the <u>low energy</u> behaviour of the effective theory is of interest, i.e.

$$k^2 << \frac{1}{\alpha'}$$

then the propagator can be expanded using the binomial theorem, (the Taylor expansion by any other name) in powers of k^2, which is equivalent to expanding in orders of α'. From now on the gauge parameter is chosen to have value ξ=2. Noting that the denominators can be expanded,

$$\frac{1}{(1 + 8k^2x(1-y))} = 1 - 8k^2x(1-y) + \dots$$

$$\frac{1}{((\theta-1) - 8k^2x((1-y)+\theta(y-z)))} = \frac{1}{(\theta-1)} + \frac{8k^2x((1-y)+\theta(y-z))}{(\theta-1)^2} + \dots$$

(A2.1.15)

the propagator takes the form,

$$D_{\mu\nu,\alpha\beta}(k) = D^{(0)}{}_{\mu\nu,\alpha\beta} + D^{(1)}{}_{\mu\nu,\alpha\beta} + \dots$$

(A2.1.16)

where the lowest order propagator term is just the usual spin-2 propagator, in the momentum representation,

$$D^{(0)}_{\mu\nu,\alpha\beta}(k) = \frac{2}{k^2}\left\{\delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\mu\beta}\delta_{\nu\alpha} - \frac{1}{4}\delta_{\mu\nu}\delta_{\alpha\beta}\right\}$$

(2.2.15)

and the $O(\alpha')$ correction is,

$$\begin{aligned} D^{(1)}_{\mu\nu,\alpha\beta} = \{\; & 16\, x(1-y).(\delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\mu\alpha}\delta_{\nu\beta}) \\ & - 16\, x(1-y)\,(\delta_{\mu\alpha}\omega_{\nu\beta} + \delta_{\mu\beta}\omega_{\nu\alpha} + \delta_{\nu\alpha}\omega_{\mu\beta} + \delta_{\nu\beta}\omega_{\mu\alpha}) \\ & - \frac{32}{9}\, x(1-y)\,(\delta_{\mu\nu} - \omega_{\mu\nu})(\delta_{\alpha\beta} - \omega_{\alpha\beta}) \\ & + \frac{1}{2}\, x\,((1-y) + 9(y-z))\,[\frac{1}{9}(\delta_{\mu\nu} - \omega_{\mu\nu})(\delta_{\alpha\beta} - \omega_{\alpha\beta}) \\ & + 9\,\omega_{\mu\nu}\omega_{\alpha\beta} + 3\,(\delta_{\mu\nu}\omega_{\alpha\beta} + \delta_{\alpha\beta}\omega_{\mu\nu} - 2\,\omega_{\mu\nu}\omega_{\alpha\beta})\,]\,\} \end{aligned}$$

(A2.1.17)

It is also possible to calculate the higher order corrections. This is not necessary for the work performed above. The projection operators have been expanded into their components, thus writing the propagators in a somewhat more useful form. These can now be used in the calculation of amplitudes. The propagators are left in their most general form here, and the truncation convention is only applied when explicit calculations are performed.

Section A2.2 The antisymmetric tensor propagator.

The procedure developed for the graviton can be generalised to the case of the antisymmetric tensor field, where the projection operators for the rank two antisymmetric tensor are defined with the same notation as above, to be, Projection operators.

$$P^{1}_{\mu\nu,\alpha\beta}(k) = \frac{1}{2}(\theta_{\mu\alpha}\theta_{\nu\beta} - \theta_{\mu\beta}\theta_{\nu\alpha})$$

$$P^{2}_{\mu\nu,\alpha\beta}(k) = \frac{1}{2}(\omega_{\mu\alpha}\omega_{\nu\beta} - \omega_{\mu\beta}\omega_{\nu\alpha})$$

$$P^{\theta\omega}_{\mu\nu,\alpha\beta}(k) = (\theta_{\mu\alpha}\omega_{\nu\beta} - \theta_{\mu\beta}\omega_{\nu\alpha})$$

$$P^{\omega\theta}_{\mu\nu,\alpha\beta}(k) = (\omega_{\mu\alpha}\theta_{\nu\beta} - \omega_{\mu\beta}\theta_{\nu\alpha}) \qquad \text{(A2.2.1)}$$

These projection operators span the four dimensional space of rank two antisymmetric tensors, and satisfy the multiplication algebra given by the table,

1 \ 2	P^1	P^2	$P^{\omega\theta}$	$P^{\theta\omega}$
P^1	P^1	0	0	0
P^2	0	P^2	0	0
$P^{\omega\theta}$	0	0	$P^{\omega\theta}$	$P^{\omega\theta}$
$P^{\theta\omega}$	0	0	$P^{\theta\omega}$	$P^{\theta\omega}$

The identity of the algebra is defined to be,

$$I_{\mu\nu,\alpha\beta} = P^{1}_{\mu\nu,\alpha\beta}(k) + \frac{1}{2}(P^{\omega\theta}_{\mu\nu,\alpha\beta}(k) + P^{\theta\omega}_{\mu\nu,\alpha\beta}(k)) + P^{2}_{\mu\nu,\alpha\beta}(k) \qquad \text{(A2.2.2)}$$

The kinetic term for the antisymmetric tensor field is given by the term,

$$-\frac{3}{4} e\, \phi^{-3/4}\, \partial_{[\alpha} a_{\beta\gamma]}\, \partial^{[\alpha} a^{\beta\gamma]} \qquad \text{(A2.2.3)}$$

in the extended action. (Note: there are no higher derivative additions to this term which are purely bilinear in the $a_{\mu\nu}$ field.) This expands to give the non gauge fixed kinetic term,

$$A = \frac{1}{4} \left\{ a_{\mu\nu} \partial^{\rho} \partial_{\rho} a^{\mu\nu} + 2\, a_{\mu\nu} \partial_{\alpha} \partial^{\nu} a^{\alpha\mu} \right\} \qquad \text{(A2.2.4)}$$

The corresponding gauge fixing term is as given above, and can be seen to give the gauge fixed kinetic term, (written in terms of the projection operators),

$$A_{g.f.} = \frac{k^2}{2} a_{\mu\nu} \, \mathit{1}_{\mu\nu,\alpha\beta} \, a_{\alpha\beta} \qquad \text{(A2.2.5)}$$

where the propagator has the simplest form when the gauge parameter has the value $\zeta = 1$, which is the choice implicitly made above.

The projection operators satisfy the multiplication algebra given above, and so the inverse can be found by application of the method used above in the graviton case. The inverse is trivially,

$$A_{\mu\nu,\alpha\beta}(k) = \frac{2}{k^2} \mathit{1}_{\mu\nu,\alpha\beta} \qquad \text{(A2.2.6)}$$

written in terms of the identity relation. This can then be expanded into the explicit form used in calculations,

$$A_{\mu\nu,\alpha\beta}(k) = \frac{1}{k^2} (\delta_{\mu\alpha} \delta_{\nu\beta} - \delta_{\mu\beta} \delta_{\nu\alpha}) \qquad \text{(A2.2.7)}$$

which is the form required in the amplitude matching procedure. In fact this propagator is only included for the sake of completeness as it has been demonstrated above that this propagator will not ever be needed in any of the calculations described elsewhere in this thesis.

Section A2.3 The spin-$^3/_2$ propagator.

It is now time to consider the trickiest of the propagators to derive. The extended Chapline-Manton action has two terms which can give corrections to the propagator for the spin-$^3/_2$ field;

$$4\, L\, e\, \phi^{-3/4} \, \mathfrak{D}^{[\mu} \overline{\psi}^{\nu]} \gamma^{\rho} \mathfrak{D}_{\rho} \mathfrak{D}_{[\mu} \psi_{\nu]} \qquad \text{(A2.3.1)}$$

and,

$$4\,L'\,e\,\phi^{-3/4}\,\mathfrak{D}^{[\mu}\overline{\psi}^{\rho]}\gamma^{\alpha\beta\tau}\mathfrak{D}_{\tau}\mathfrak{D}_{[\mu}\psi_{\beta]}g_{\rho\alpha} \qquad \text{(A2.3.2)}$$

It will be shown that the combination of these with 2L=L' gives no correcting contribution to the propagator. Doing the background field expansion as usual,

$$\psi_{ab} = 2\partial_{[a}\psi_{b]} + \psi_n\partial_{[a}h^n{}_{b]} + \frac{1}{2}\gamma_p\gamma_q\psi_{[b}\partial^{[q}h_{a]}{}^{p]} \qquad \text{(A2.3.3)}$$

and noting that to zeroth order in the graviton field the action terms take the form,

$$4L\,\partial^{[a}\overline{\psi}^{b]}\not{\partial}\,\partial_{[a}\psi_{b]} = -\,2L\,\overline{\psi}_a\not{\partial}\,(\,\delta^{ab}\partial^2 - \partial^a\partial^b)\psi_b \qquad \text{(A2.3.4)}$$

and,

$$4L'\,\partial^{[a}\overline{\psi}^{b]}\gamma_b{}^{c\tau}\partial_\tau\partial_{[a}\psi_{c]} = -\,\overline{\psi}_b\gamma^{bc\tau}\partial_\tau\partial^a\partial_a\psi_c \qquad \text{(A2.3.5)}$$

As usual, it is necessary to define the projection operators for the particular field under consideration. A suitable choice is[43];

$$P^{3/2}_{\mu\nu}(k) = \theta_{\mu\nu} - \frac{1}{\theta}\hat{\gamma}_\mu\hat{\gamma}_\nu$$

$$(P^{1/2}_{11}(k))_{\mu\nu} = \frac{1}{\theta}\hat{\gamma}_\mu\hat{\gamma}_\nu$$

$$(P^{1/2}_{12}(k))_{\mu\nu} = \frac{1}{\sqrt{\theta}}\hat{\gamma}_\mu\omega_\nu$$

$$(P^{1/2}_{21}(k))_{\mu\nu} = \frac{1}{\sqrt{\theta}}\omega_\mu\hat{\gamma}_\nu$$

$$(P^{1/2}_{22}(k))_{\mu\nu} = \omega_\mu\omega_\nu \qquad \text{(A2.3.6)}$$

which satisfy the multiplication algebra,

$$(P^{I}{}_{ij})_{\mu\nu}\,(P^{J}{}_{kl})_{\nu\rho} = \delta^{IJ}\delta_{jk}\,(P^{J}{}_{il})_{\mu\rho} \qquad \text{(A2.3.7)}$$

which can be summarised in the table given in Figure A2.2. The identity of the algebra is defined to be,

$$\mathrm{I}_{\mu\nu} = P^{3/2}_{\mu\nu}(k) + (P^{1/2}_{11}(k))_{\mu\nu} + (P^{1/2}_{22}(k))_{\mu\nu} \qquad \text{(A2.3.8)}$$

The gauge fixing is the usual Rarita-Schwinger gauge fixing,

$$\mathcal{L}_{g.f.} = \frac{1}{4\eta}\,\overline{\psi}_{\mu}\gamma^{\mu}\gamma^{\alpha}\partial_{\alpha}\gamma^{\nu}\psi_{\nu} \qquad \text{(A2.3.9)}$$

which gives the kinetic term when the gauge parameter is chosen to be $\eta=1$,

$$-\frac{1}{4}\overline{\psi}_{\mu}\,\Big\{\,(2 + (4L\text{-}2L')k^{2})\,P^{3/2}_{\mu\nu}(k) - (P^{1/2}_{22}(k))_{\mu\nu}$$
$$+\sqrt{\theta}\,\,(\,(P^{1/2}_{21}(k) - P^{1/2}_{12}(k)\,)_{\mu\nu} + ((2\text{-}\theta) + (4L\text{-}2L')k^{2})(P^{1/2}_{11}(k))_{\mu\nu}\,\Big\}\not{k}\,\psi_{\nu}$$
$$\text{(A2.3.10)}$$

where the action terms have been expanded using the identity I, and the derivatives and gamma metrices have been rewritten in terms of the projection operator terms γ_{μ} and ω_{μ}, and these have been collected into projection operators. It should be noted that the kinetic matrix has been written in a form that does not display the symmetries which one might expect. In fact the symmetries are still present, though in a much more convoluted manner than in either of the cases discussed above. It can be seen that the terms of order k^2 in the kinetic term, which are the terms which will give the unphysical poles in the full propagator are always multiplied by the coefficient 2L-L'. This implies that there is a manifestly unitary combination of these terms when 2L-L' = 0.

This kinetic term may now be inverted by using the projection operator algebra. The propagator is of the general form, (suppressing indices on the projection operators),

$$Y(k) = \frac{2\not{k}}{k^{2}}\,\Big\{\,a\,P^{3/2}(k) + bP^{1/2}_{11}(k) + cP^{1/2}_{22}(k) + dP^{1/2}_{12}(k) + eP^{1/2}_{21}(k)\,\Big\}$$
$$\text{(A2.3.11)}$$

1 \ 2	P^1	P^2	$P^{(s)}$	$P^{(\omega)}$	$P^{(s\omega)}$	$P^{(\omega s)}$
P^1	P^1	0	0	0	0	0
P^2	0	P^2	0	0	0	0
$P^{(s)}$	0	0	$P^{(s)}$	0	$P^{(s\omega)}$	0
$P^{(\omega)}$	0	0	0	$P^{(\omega)}$	0	$P^{(\omega s)}$
$P^{(s\omega)}$	0	0	0	$P^{(s\omega)}$	0	$P^{(s)}$
$P^{(\omega s)}$	0	0	$P^{(\omega s)}$	0	$P^{(\omega)}$	0

Figure A2.2 The Projection Operator Algebra For The Evaluation of The Gravitino Propagator.

where the symmetries are again obscured within the notation. The condition that $V \times Y = \mathrm{I}$ means that the general coefficients should satisfy the following set of linear equations;

$$a = \frac{1}{(2 + (8L+ 4L'(\theta-1)k^2)}$$

$$(2-\theta+(8L+4L'(\theta-1))k^2)b - \sqrt{\theta}\, e = 1$$

$$\sqrt{\theta}\, b - e = 0$$

$$(2-\theta+ (8L+4L'(\theta-1))k^2)\, d - \sqrt{\theta}\, c = 0$$

$$\sqrt{\theta}\, d - c = 1 \qquad \text{(A2.3.12a)}$$

which are uniquely soluble and can be seen to give, (where a is given above),

$$b = \frac{-1}{(2(\theta-1) - Pk^2)}$$

$$d = \frac{\sqrt{\theta}}{(2(\theta-1) - Pk^2)}$$

$$e = \frac{-\sqrt{\theta}}{(2(\theta-1) - Pk^2)}$$

$$c = \frac{(2-\theta + Pk^2)}{(2(\theta-1) - Pk^2)}$$

$$P = (8L + 4L'(\theta-1)) \qquad \text{(A2.3.12b)}$$

where the usual low energy expansion is given by the binomial expansion, as for the graviton above. This yields the propagator in the form (which manifestly displays all the required symmetries),

$$Y_{\mu,\nu}(k) = Y^{(0)}{}_{\mu,\nu}(k) + Y^{(1)}{}_{\mu,\nu}(k) + Y^{(2)}{}_{\mu,\nu}(k) + \ldots \qquad \text{(A2.3.13)}$$

where,

$$Y^{(0)}_{\mu,\nu}(k) = \frac{i}{8k^2}\{ \gamma_\nu \not{k} \gamma_\mu - 6 (\eta_{\mu\nu} \not{k} - \frac{2 k_\mu k_\nu \not{k}}{k^2}) \} \tag{A2.3.14}$$

which is just the usual spin-3/2 propagator in d-dimensions, and,

$$\begin{aligned} Y^{(1)}_{\mu,\nu}(k) = \Big[-2i\,(2L-L') \Big\{ & -\frac{50}{64}\eta_{\mu\nu}\not{k} + \frac{7}{64}\gamma_\nu \not{k}\, \gamma_\mu + \frac{68}{64}\frac{k_\mu k_\nu \not{k}}{k^2} \\ & -\frac{1}{8}(\gamma_\mu k_\nu + \gamma_\nu k_\mu) \Big\} \\ -2i\,L' \Big\{ & -\frac{1}{32}\eta_{\mu\nu} - \frac{1}{64}\gamma_\nu \not{k}\, \gamma_\mu + \frac{100}{64}\frac{k_\mu k_\nu \not{k}}{k^2} \\ & -\frac{1}{8}(\gamma_\mu k_\nu + \gamma_\nu k_\mu) \Big\} \Big] \end{aligned} \tag{A2.3.15}$$

is the $O(\alpha')$ correction term. The $O(\alpha'^2)$ propagator correction term is superfluous to the work carried out in this thesis and is not presented here.

Section A2.4 The dilatino propagator.

The dilatino propagator is corrected by the addition of the term,

$$M e \phi^{-3/4} D_a \lambda \gamma^\mu D_\mu D^a \lambda \tag{A2.4.1}$$

which can be seen to give a term of the form,

$$- M \bar{\lambda} \not{\partial} \partial^a \partial_a \lambda \tag{A2.4.2}$$

which can be seen to correct the kinetic matrix,

$$-\frac{1}{2} \bar{\lambda} \not{\partial} \,(1 + 2M\partial^2)\lambda \tag{A2.4.3}$$

and can thus be seen to lead trivially to the propagator in the form,

$$\Lambda(k) = \frac{-\not{k}}{k^2(1 + 2Mk^2)} \tag{A2.4.4}$$

This calculation, with that of the antisymmetric tensor field has been included merely for the sake of completeness, and has no bearing on any of the other work carried out in this thesis.

Appendix Three:
Notations and Conventions.

Section A3.1 The Gamma Matrix Algebra.

In this section the gamma matrix algebra inplicitly used throughout all of the work described in this thesis is constructed, and some gamma matrix identities are stated.

The flat space Minkowski metric used throughout is $\eta_{\mu\nu} = \mathrm{diag}(-1,1,\ldots,1)$. This leads neatly to the definition of the gamma matrix algebra. The gamma matrices in ten dimensions are 32x32 complex valued matrices. Given that there exists a consistent Majorana-Weyl spinor representation, then a completely real (or imaginary) representation for the gamma matrices must exist. These are constructed as follows. First the Pauli matrices are defined to be,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A3.1.1}$$

and the gamma matrices are constructed out of tensor products of these. First define the 4x4 gamma matrices,

$$\begin{aligned} \Gamma^0 &= I \otimes i\,\sigma_2 \quad , \Gamma^1 = I \otimes \sigma_3 \\ \Gamma^2 &= \sigma_1 \otimes \sigma_1 \quad , \Gamma^3 = \sigma_3 \otimes \sigma_1 \end{aligned} \tag{A3.1.2}$$

and where the Γ^5 matrix is defined to be,

$$\Gamma^5 = \Gamma^0\Gamma^1\Gamma^2\Gamma^3 \tag{A3.1.3}$$

which can then be used in the standard Gliozzi-Scherk-Olive gamma matrix definitions[16]. Supposing that the uppercase latin indices run from zero to three, the ten dimensional gamma matrix represention is defined as,

$$\begin{aligned} \gamma^M &= \Gamma^M \otimes I_4 \otimes \sigma_3 \quad , \\ \gamma^{3+i} &= \Gamma^5 \otimes \begin{pmatrix} \rho_i & 0 \\ 0 & \rho'_i \end{pmatrix} \end{aligned} \tag{A3.1.4}$$

where the ρ_i and ρ'_i (i=1,2,3) are defined to be,

$$\rho_1 = \rho'_1 = \Gamma^0 \quad , \quad \rho_2 = \rho'_2 = \Gamma^5$$

$$\rho_3 = - \rho'_3 = \Gamma^0\Gamma^5 \qquad \text{(A3.1.5)}$$

and where the remaining matrices are defined by,

$$\Gamma^{6+j} = I_4 \otimes \begin{pmatrix} 0 & \zeta_j \\ \zeta_j & 0 \end{pmatrix} \qquad \text{(A3.1.6)}$$

where ζ_j (j=1,2,3) is defined to be,

$$\zeta_j = \gamma^j \qquad \text{(A3.1.7)}$$

where this set of gamma matrices satisfies all the necessary properties of the Clifford algebra. Explicitly,

$$\{ \gamma_\mu , \gamma_\nu \} = 2g_{\mu\nu} \qquad \text{(A3.1.8)}$$

and each of the gamma matrices satisfies,

$$(\gamma^0)^2 = -1 \quad , \quad (\gamma^i)^2 = 1$$

$$(\gamma^0)^T = -\gamma^0 \quad , \quad (\gamma^i)^T = \gamma^i \qquad \text{(A3.1.9)}$$

Finally the γ^{11} matrix is defined to be,

$$\gamma^{11} = \gamma^0\gamma^1 \ldots \gamma^9 \qquad \text{(A3.1.10)}$$

which can be seen to be of the block form,

$$\gamma^{11} = \begin{pmatrix} 0 & I_4\otimes\rho_3 \\ -I_4\otimes\rho_3 & 0 \end{pmatrix} \qquad \text{(A3.1.11)}$$

This can be seen to be a completely real representation of the gamma matrices and hence is Majorana. The spinors are defined to be Majorana spinors, and therefore satisfy the Majorana condition,

$$\overline{\psi} = \psi^T C \ , \text{ where } \overline{\psi} = \psi^\dagger \gamma^0 \qquad \text{(A3.1.12)}$$

and where C is the charge conjugation matrix which is defined to satisfy,

$$(\gamma^\mu)^T = - C \gamma^\mu C^{-1}$$

which is solved by choosing $C = \gamma^0$, and where it can now be seen that the spinors are real[62]. This gamma matrix representation is the one that is used in the fermion quantisation of Chapter Two.

There are some useful gamma matrix identities that will be useful throughout this thesis. These are presented below;

$$\gamma^{\rho\sigma\mu} = \gamma^\rho\gamma^\sigma\gamma^\mu - \gamma^\rho g^{\sigma\mu} + \gamma^\sigma g^{\mu\rho} - \gamma^\mu g^{\rho\sigma}$$

$$= \frac{1}{2} [\gamma^\rho\gamma^\sigma\gamma^\mu - \gamma^\mu\gamma^\sigma\gamma^\rho]$$

$$[\, \gamma_\rho\gamma_\sigma , \gamma_\mu \,] = 4\, \gamma_{[\rho} g_{\sigma]\mu}$$

$$\gamma^\mu\gamma_\mu = d$$

$$\gamma^\mu\gamma_\rho\gamma_\mu = - (d-2)\gamma_\rho$$

$$\gamma^\mu\gamma^\nu\gamma_\rho\gamma_\mu\gamma_\nu = - (d^2 - 6d - 4)\gamma_\rho \qquad \text{(A3.1.13)}$$

These identities, as well as some others which occur less frequently, are used constantly throughout this work.

Section A3.2 Grassman Hamiltonian Mechanics.

The discussion of fermion quantisation necessitates the discussion of the definition of generalised Poisson brackets for Grassman variables and of Hamiltonian techniques for Grassman variables in general. The Grassman algebra can be separated into two subspaces, the even and odd subspaces. Elements from the even subspace are taken to be commuting and those from the odd subspace anticommuting. It is possible to formalise this to a great extent[35].

Consider a set of Grassmann variables, (from which the Lagrangian and Hamiltonian wil be constructed later), denoted by q_α. The corresponding velocities will be needed later, but these need not be considered at the moment. It will be asumed that any function $f(q_\alpha)$ can be shown to be in either the even or odd subspaces. It is then possible to associate a number n_f (= 0 or 1 depending on whether f is in the even or odd subspaces, respectively) to any such function, and where it is now possible to write,

$$f\,g = (-1)^{n_f n_g}\, g\,f \qquad \text{(A3.2.1)}$$

For a discussion of the Lagrangian and Hamiltonian dynamics of Grassmann variables it will be necessary to consider derivatives with respect to these variables. The problem of the anticommuting nature of the odd Grassmann variables is now apparent. Care will need to be taken in the definition of these derivatives. It is possible to define two different derivatives with respect to grassman variables. These are left differentiation, defined by considering the differential,

$$\delta f = \left.\frac{\partial f}{\partial q_\alpha}\right|_L \delta q_\alpha \qquad \text{(A3.2.2a)}$$

and right differentiation, given by,

$$\delta f = \delta q_\alpha \left.\frac{\partial f}{\partial q_\alpha}\right|_R \qquad \text{(A3.2.2b)}$$

It can be seen that these are related by the equation,

$$\left.\frac{\partial f}{\partial q_\alpha}\right|_L = -(-1)^{n_f} \left.\frac{\partial f}{\partial q_\alpha}\right|_R \qquad \text{(A3.2.3)}$$

In all the work described in this thesis the Grassman derivatives are always taken to be left derivatives.

The Lagrangian can now be constructed. Given a Lagrangian $L = L(q_\alpha, \dot{q}_\alpha)$, the Euler Lagrange equations of motion are as usual,

$$\frac{d}{dt}\left\{ \frac{\partial L}{\partial \dot{q}_\alpha} \right\} = \frac{\partial L}{\partial q_\alpha} \qquad \text{(A3.2.4)}$$

The canonically conjugate momentum is defined to be,

$$p^{\alpha} = \frac{\partial L}{\partial \dot{q}_{\alpha}} \qquad \text{(A3.2.5)}$$

and so it is possible to define the Hamiltonian,

$$H = p^{\alpha}\dot{q}_{\alpha} - L \qquad \text{(A3.2.6)}$$

The variation of the Hamiltonian can now be taken in analogy to the standard case,

$$\delta H = \delta p^{\alpha}\, \dot{q}_{\alpha} + p^{\alpha}\, \delta\dot{q}_{\alpha} - \frac{\partial L}{\partial q_{\alpha}}\, \delta q_{\alpha} - \frac{\partial L}{\partial \dot{q}_{\alpha}}\, \delta\dot{q}_{\alpha}$$

$$= - \dot{q}_{\alpha}\, \delta p^{\alpha} - \frac{\partial L}{\partial q_{\alpha}}\, \delta q_{\alpha} \qquad \text{(A3.2.7)}$$

revealing that the Hamiltonian equations of motion are,

$$\dot{q}_{\alpha} = - \frac{\partial H}{\partial p^{\alpha}} \quad , \quad \dot{p}^{\alpha} = - \frac{\partial H}{\partial q_{\alpha}} \qquad \text{(A3.2.8)}$$

where both of the Hamiltonian equations of motion have minus signs. It is now simple to see how the Poisson brackets should be constructed. The Poisson bracket is defined to be,

$$\{ f , g \} = - (-1)^{n_g} \frac{\partial f}{\partial q_{\alpha}} \frac{\partial g}{\partial p^{\alpha}} + (-1)^{n_f(n_g+1)} \frac{\partial g}{\partial q_{\alpha}} \frac{\partial f}{\partial p^{\alpha}}$$

(A3.2.9)

which clearly gives the Hamiltonian equations of motion in the form required by taking $\{q_{\alpha} , H\}$ and $\{p_{\alpha} , H\}$. This form of the Poisson brackets clearly satisfies the normal set of Poisson bracket identities, generalised to the Grassmann case;

$$\{ f , g \} = - (-1)^{n_f n_g} \{ g , f \}$$

$$\{ f , g + h \} = \{ f , g \} + \{ f , h \}$$

$$\{ f , gh \} = (-1)^{n_f n_g} g \{ f , h \} + \{ f , g \} h$$

(A3.2.10a)

as well as the Jacobi identity,

$$(-1)^{n_f n_h}\{f,\{g,h\}\} + (-1)^{n_g n_f}\{g,\{h,f\}\} + (-1)^{n_h n_g}\{h,\{f,g\}\} = 0$$

(A3.2.10b)

The quantisation of Fermionic theories now follows by following the usual quantisation prescription,

$$i\hbar\{f,g\} \rightarrow \hat{f}\hat{g} - (-1)^{n_f n_g}\hat{g}\hat{f} \qquad (A3.2.11)$$

which is the course followed in Chapter Two. The usual Dirac methods[32] can be generalised to the Grassman case also. These Grassmann Dirac brackets are used in Chapter Two.

Section A3.3 The Invariant Functions.

The work carried out in Chapter Two requires the use of the various definitions for the invariant functions, and the relations between them[39]. It is necessary to make some preliminary definitions. The step function is defined to be,

$$\theta(x) = \begin{cases} 1 & , \quad x^0 > 0 \\ \frac{1}{2} & , \quad x^0 = 0 \\ 0 & , \quad x^0 < 0 \end{cases} \qquad (A3.3.1a)$$

$$\varepsilon(x) = 2\theta(x) - 1 \qquad (A3.3.1b)$$

which can be seen to satisfy the relations,

$$\theta^2(x) = \theta(x) \quad , \qquad \theta(x)\theta(-x) = 0$$

$$\theta(x) + \theta(-x) = 1 \, , \qquad (2\theta(x) - 1)^2 = 1 \qquad (A3.3.2)$$

The Dirac delta function and its derivaives are defined to be,

$$\int dx\, f(x)\, \delta(x) = f(0)$$

$$\int dx\, f(x)\, \delta'(x) = -\, f'(0)$$

$$\int dx\, f(x)\, \delta^{(n)}(x) = (-1)^n f^{(n)}(0) \qquad \text{(A3.3.3)}$$

where f is some general function, chosen to be suitably smooth; that is, differentiable to the required degree.

The various invariant functions can all be represented by the general form,

$$G_i(x) = \int_{\gamma_i} \frac{d^p k}{(2\pi)^p} \frac{e^{ik.x}}{k^2 - m^2} \qquad \text{(A3.3.4)}$$

from which all the specific delta functions can be seen to follow by choosing specific contours γ_i for the evaluation of the integral. The Green's functions of some relevance to the work contained in this thesis will be treated in turn below. The identities relating these invariant functions will be stated at the end.

i) The Retarded and Advanced Feynman Green's Functions.

The retarded Feynman Green's function is evaluated using the C_{Fr} contour given in Figure A3.1. This Green's function can be identified with the propagator of the harmonic oscillator, and therefore with the propagator of most quantum systems. The integral can be written in the more usual form,

$$G_{Fr}(x) = \int \frac{d^p k}{(2\pi)^p} \frac{e^{ik.x}}{k^2 - m^2 + i\varepsilon} \qquad \text{(A3.3.5)}$$

from which it can be seen that there exists an analogous advanced propagator, evaluated using the C_{Fa} contour as in Figure A3.1, or the term $-i\varepsilon$ is added to the denominator of (A3.3.5) above, instead of $i\varepsilon$.

ii) The Positive and Negative Green's Functions.

These Green's functions only use a contour about the positive and negative valued poles respectively, as demonstrated by the C_+ and C_- contours in Figure A3.1. These are the most useful of the invariant functions required in the quantisation of invariant wave equations. They satisfy a useful identity stated below.

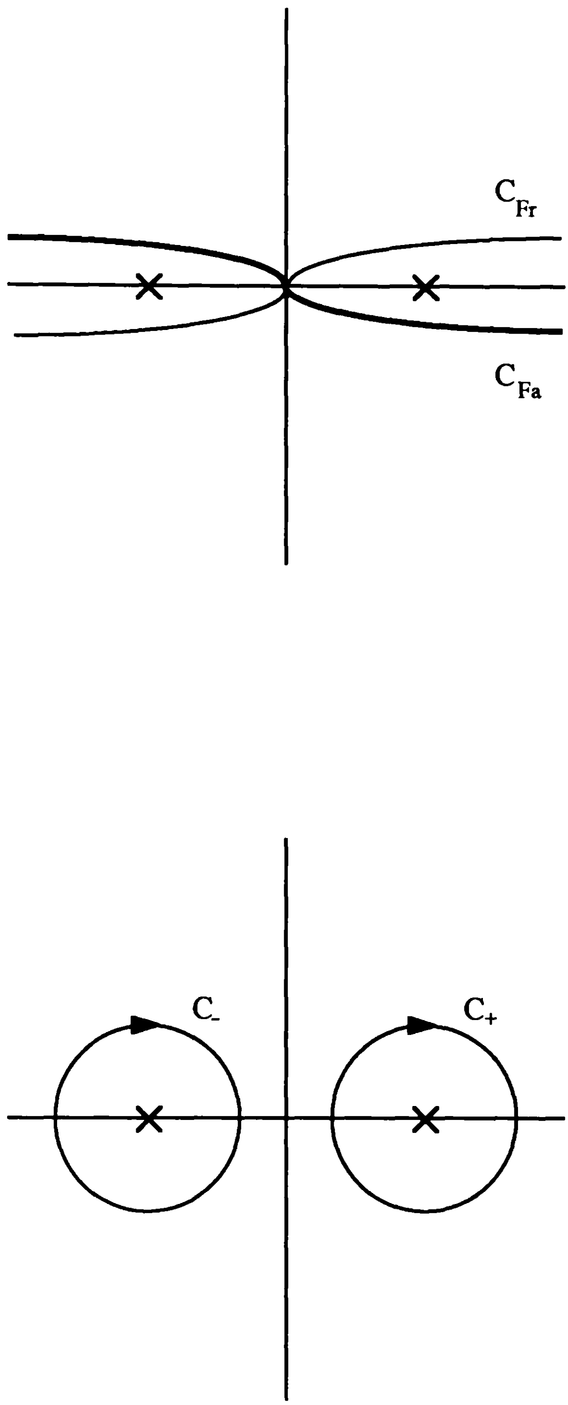


Figure A3.1. The contours necessary for the evaluation of the invariant functions, and the quantum propagators.

These Green's functions can be rewritten in rather more natural and useful forms. The most useful form of these integrals is,

$$G_{\pm}(x) = \pm i \int \frac{d^p k}{(2\pi)^{p-1}} \; (1 \pm \varepsilon(k)) \; \delta(k^2 - m^2) \; e^{ik.x} \tag{A3.3.6}$$

which will be seen to be the form which arises most naturally in the quantisations given in Chapter Two. There are other Green's functions which will not be needed, and consequently will not be discussed here.

These various Green's functions are related by the identities,

$$G_{F_r}(x) = \theta(x) \; G_+(x) - \theta(-x) \; G_-(x)$$

$$G_{F_a}(x) = \theta(x) \; G_-(x) - \theta(-x) \; G_+(x) \tag{A3.3.7}$$

which allow the transition from the quantised mode coefficients to the propagators.

Section A3.4: The Riemann Tensor.

The definition used throughout for the Riemann tensor can be written in two distinct, but physically equivalent ways. Both of the definitions given below can be shown to be equivalent in the background field expansion used throughout in this work. One definition will prove to be the most useful however; the definition given in terms of the Riemann Christoffel connection, defined to be[49,50],

$$\Gamma^{\mu}_{\rho\sigma} = \frac{1}{2} g^{\mu\eta} \{ \partial_{\sigma} g_{\eta\rho} + \partial_{\rho} g_{\sigma\eta} - \partial_{\eta} g_{\rho\sigma} \} \tag{A3.4.1}$$

The definition of the Riemann tensor in this case is[50],

$$R^{\sigma}{}_{\alpha\beta\gamma}(\Gamma) = \{ \partial_{\beta} \Gamma^{\sigma}_{\alpha\gamma} - \partial_{\gamma} \Gamma^{\sigma}_{\alpha\beta} + \Gamma^{\lambda}_{\alpha\gamma} \Gamma^{\sigma}_{\lambda\beta} - \Gamma^{\lambda}_{\alpha\beta} \Gamma^{\sigma}_{\lambda\gamma} \} \tag{A3.4.2}$$

This definition is adequate in the case of purely bosonic theories, but is inadeqate in the case where spinors are included in the theory. In this case the Riemann-Christoffel connection is inadequate to maintain the covariance of the theory. The spin connection $\omega_{\mu}{}^{mn}$ has to be defined. The appropriate definition is[50],

$$\omega_\mu{}^{mn} = 2\, e^{\nu[m}\partial_{[\mu}e^{n]}{}_{\nu]} + \partial_{[\rho}e^{p}{}_{\sigma]}e_{p\mu}e^{\rho n}e^{\sigma m} \qquad (A3.4.3)$$

The definition of the Riemann tensor is now defined to be,

$$R_{\mu\nu}{}^{mn} = \partial_\mu\omega_\nu{}^{mn} - \partial_\nu\omega_\mu{}^{mn} + \omega_\mu{}^{mt}\omega_{\nu t}{}^{n} - \omega_\nu{}^{mt}\omega_{\mu t}{}^{n}$$
$$(A3.4.4)$$

which can be shown to be completely equivalent to the definition (A3.4.2). This equivalence takes the form,

$$R_{\mu\nu}{}^{mn}(\omega)e^{\rho}{}_{m}e^{\sigma}{}_{n} = R_{\mu\nu}{}^{\rho\sigma}(\Gamma) \qquad (A3.4.5)$$

The choice of which definition to use clearly depends on the convenience of use in any partcular situation.

The Ricci tensor is defined to be,

$$R_{\mu\nu} = R^{\sigma}{}_{\mu\nu\sigma} \qquad (A3.4.6)$$

and the curvature scalar is defined to be,

$$R = g^{\mu\nu}R_{\mu\nu} \qquad (A3.4.7)$$

which can now be used in the definition of the Einstein-Hilbert action. The action for free gravitational fields is,

$$A = \int d^dx\,\sqrt{-g}\,R \qquad (A3.4.8)$$

from which the free field equations of motion immediately follow[49,50],

$$R_{\mu\nu} = 0 \qquad (A3.4.9)$$

This result generalises to the full Einstein equations of motion,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T_{\mu\nu} \qquad (A3.4.10)$$

Where $T_{\mu\nu}$is the stress energy tensor for any other fields that are coupled to gravity in the generalised form of action (A3.4.8).

The covariant derivatives of both covariant and contravariant vectors should now be defined. The covariant derivative of a covariant vector is,

$$\mathfrak{D}_\mu \varphi_\nu = \partial_\mu \varphi_\nu - \Gamma^\lambda_{\mu\nu} \varphi_\lambda \qquad \text{(A3.4.11)}$$

and the covariant derivative of a contravariant derivative is defined to be,

$$\mathfrak{D}_\mu \varphi^\nu = \partial_\mu \varphi^\nu + \Gamma^\nu_{\alpha\mu} \varphi^\alpha \qquad \text{(A3.4.12)}$$

The covariant derivative of a spinor now presents some problems which must be dealt with using some care. It is required that a special spinor connection Γ_ρ be defined such that the covariant derivative of a spinor can be defined[50],

$$\mathfrak{D}_\mu \psi = \partial_\mu \psi + \Gamma_\mu \psi \qquad \text{(A3.4.13)}$$

where the spinor connection can be shown to satisfy the definition,

$$\Gamma_\rho = \frac{1}{4} \gamma_m \gamma_n \, \omega_\rho^{\ mn} \qquad \text{(A3.4.14)}$$

The covariant derivative of a vector spinor, that is a spinor that transforms as a vector rather than as a scalar under local Lorentz transformations, can now be shown to be,

$$\mathfrak{D}_\mu \psi_\nu = \partial_\mu \psi_\nu + \frac{1}{4} \gamma_m \gamma_n \psi_\nu \omega_\mu^{\ mn} - \Gamma^\lambda_{\mu\nu} \psi_\lambda \qquad \text{(A3.4.15)}$$

with the corresponding definition for a contravariant vector spinor. Normally ψ_μ is treated as a spinor valued one form, and the two form $\mathfrak{D}\psi$ is given by the skewsymmetrisation of (A3.4.15) above, in which case the Riemann Christoffel connection does not contribute.

Appendix Four.
Sample Calculation.

This short appendix contains the sample calculation of the gravitino exchange diagram mentioned in Section 4.1 of Chapter Four. This involves only the first term in the $O(\alpha')$ vertex, contracted with the entire $O(\alpha^0)$ vertex. This will be explicitly shown to vanish. The $O(\alpha')$ term that will be used is,

$$-\frac{1}{2}\partial_\mu\overline{\psi}_\nu\gamma^\sigma\gamma^\rho\gamma^\eta\psi_\eta\partial_\sigma\partial^\mu h_\rho{}^\nu \qquad \text{(A4.1)}$$

and the $O(\alpha'^0)$ vertex is given by equation (3.2.36). This is,

$$-\frac{1}{4}\Big\{ \frac{1}{2}\partial_\rho\overline{\psi}_\sigma\gamma^\sigma\gamma^\rho\gamma^\alpha\psi_\mu h_\alpha{}^\mu + \overline{\psi}_\sigma\gamma^\alpha\partial^\sigma\psi_\mu h_\alpha{}^\mu$$

$$+ \partial_\rho\overline{\psi}_\sigma\gamma^\rho\psi_\mu h^{\mu\sigma} \Big\} \qquad \text{(3.2.36)}$$

The propagator is as given in Appendix Two, and is,

$$Y^{(0)}_{\mu,\nu}(k) = \frac{i}{8k^2}\Big\{ \gamma_\nu \not{k} \gamma_\mu - 6\,(\eta_{\mu\nu}\not{k} - \frac{2}{k^2}k_\mu k_\nu \not{k}) \Big\} \qquad \text{(A2.3.14)}$$

The contribution to the amplitude is thus given by the contraction of these three together;

$$\frac{i}{8t}\frac{1}{8}\Big\{ \frac{1}{2}\bar{u}_{2\nu}\gamma^\sigma\gamma^\rho\gamma^\eta\big[\gamma_s(\not{k}_1 + \not{k}_3)\gamma_\eta - 6(\eta_{s\eta} - \frac{2}{t}(k_{1s} + k_{3s})(k_{1\eta} + k_{3\eta})(\not{k}_1 + \not{k}_3))\big] \mathrm{x}$$

$$\mathrm{x}\Big\{ \gamma^s\gamma^r\gamma^a u_{1m}\, k_{2\mu}(k_{2r} + k_{4r})k_{4\sigma}k_4{}^\mu\zeta_{3a}{}^m\zeta_{4\rho}{}^\nu$$

$$+ \gamma^a u_{1m}\, k_{2\mu}k_{1s}k_{4\sigma}k_4{}^\mu\zeta_{3a}{}^m\zeta_{4\rho}{}^\nu$$

$$+ \gamma^r u_{1m}\, k_{2\mu}(k_{2r} + k_{4r})k_{4\sigma}k_4{}^\mu\zeta_3{}^{sm}\zeta_{4\rho}{}^\nu \Big\}\Big\} \qquad \text{(A4.2)}$$

which must now be simplified by the application of some of the identities listed in Appendix Three, and by the explicit use of the truncation procedure. The first term can be seen to simplify to the form,

$$\frac{i}{64t}\Big\{\frac{1}{2}\bar{u}_{2\nu}\not{k}_4\gamma^\rho\big[-44\,(\not{k}_1 + \not{k}_3) - 6\,(-10(\not{k}_1 + \not{k}_3)\,)\big](\not{k}_2 + \not{k}_4)\gamma^a u_{1m}\frac{t}{2}\zeta_{3a}{}^m\zeta_{4\rho}{}^\nu \qquad \text{(A4.3)}$$

the second term yields,

$$\frac{i}{64t}\left\{ \bar{u}_{2\nu}\not{k}_4\gamma^\rho\gamma^\eta\left[\not{k}_1\not{k}_3\gamma_\eta + 6k_{3\eta}(\not{k}_1+\not{k}_3)\right]\gamma^a u_{1m}\frac{t}{2}\zeta_{3a}{}^m\zeta_{4\rho}{}^\nu\right\} \tag{A4.4}$$

and finally the third term gives,

$$\frac{i}{64t}\left\{ \bar{u}_{2\nu}\not{k}_4\gamma^\rho\gamma^\eta\left[\gamma_s(\not{k}_1+\not{k}_3)\gamma_\eta - 6\,\eta_{s\eta}(\not{k}_1+\not{k}_3)\right](\not{k}_2+\not{k}_4)u_{1m}\frac{t}{2}\zeta_3^{ms}\zeta_{4\rho}{}^\nu\right\} \tag{A4.5}$$

which can be seen to vanish on the application of the identity,

$$\gamma^\eta\gamma_\mu\gamma_\nu\gamma_\eta = (d-4)\gamma_{\mu\nu} + 4\eta_{\mu\nu} \tag{A4.6}$$

The first and second terms must be added together. Doing this gives the final answer for the contribution to the amplitude,

$$\frac{i}{64}\left\{ -\frac{16t^2}{4}\bar{u}_{2\nu}\not{k}_4\gamma^\rho\gamma^a u_{1m}\zeta_{3a}{}^m\zeta_{4\rho}{}^\nu + \frac{8t^2}{2}\bar{u}_{2\nu}\not{k}_4\gamma^\rho\gamma^a u_{1m}\zeta_{3a}{}^m\zeta_{4\rho}{}^\nu \right\} \tag{A4.7}$$

(using the identities of equation (A3.1.13)), which can be seen to be zero. This is the result that was promised in Chapter Four. Although many of the intermediate detail steps have been left out, it can immediately be appreciated that the complexity of these calculations would be overwhelming if it were not for the truncation procedure discussed in Chapter Three.

Appendix Five
Amplitude Matching Determination
of the Coefficient γ

Introduction

In this Appendix the matching calculations for the parameter γ used in Chapters Three, Four and Five will be exhibited. This entails the construction of three separate matching calculations, none of which will be explained in any depth in this Appendix as it would not be too illuminating to include too detailed a summary. The parameter will be shown to take the form of $\gamma=\pm 1/4$, but preferably of the form $\gamma=1/4$. The procedure for calculating the value of γ from the string is quite simple and requires that only three quite simple three point amplitude matchings be performed.

The procedure is that first the bosonic string amplitude for the heterotic string is constructed, and the overall normalisation to the $O(\alpha'^0)$ bosonic amplitude for three graviton scattering is found. This serves as the absolute normalisation for the string amplitude. It is known that the string polarisation tensor obtained by the tensor product of two vectors, one from the bosonic string and one from the fermionic string can be expanded as follows,

$$\zeta_{\mu\nu} = \zeta^{h}_{\mu\nu} + \varepsilon\, \zeta^{a}_{\mu\nu} + \varphi\, \eta_{\mu\nu}\zeta^{\phi} \qquad \text{(A5.1)}$$

(where the superscripts refer to h - the graviton, a - the antisymmetric tensor field, and ϕ - the dilaton fied. In all that follows the h superscript will be omitted in consistency with what appears in the rest of this thesis: the appearance of a propagating antisymmetric tensor field occurs only in this appendix, and so no ambiguity will arise. When the *full* tensor ζ is used the fact will be noted, and when ζ is used to denote a graviton polarisation tensor *no* note will be made.) It can be seen that it still remains to find the relative normalisation constant ε from a second $O(\alpha'^0)$ amplitude matching, this time matching the a-a-h scattering amplitude from the field theory with the term from the string, before it is possible to calculate the possible values for the coefficient γ. This will enable the value of ε^2 to be found, unfortunately leaving the value of ε ambiguous with respect to a sign. This sign ambiguity will persist up to the amplitude matching which will allow the calculation of γ which will then be seen to be either $\gamma=1/4$ or $\gamma=-1/4$. However it has been shown in Chapter Five that only one of these values, that is $\gamma = -1/4$, can be seen to give a full four point amplitude match to the string theory. (If $\gamma=1/4$ is chosen it can be shown that no matching exists at all, and so this value must be discarded.)

It should be noted that the string constant α' is taken to have value $\alpha'=2$ throughout, which is completely consistent with the procedure of reference [33] and with the calculations of Chapter One. The first two matching calculations do not involve the α' parameter at all.

As discussed above the first amplitude matching calculaiton that must be performed will be used to determine the absolute overall normalisation for the bosonic string amplitude, which can be seen to be of the form,

$$\zeta_1^{\mu\nu}\zeta_2^{\rho\sigma}\zeta_3^{\alpha\beta}\left\{ k_{1\beta}\eta_{\nu\sigma} + k_{2\nu}\eta_{\sigma\beta} + k_{3\sigma}\eta_{\beta\nu} \frac{\alpha'}{2} k_{1\alpha}k_{2\mu}k_{3\rho} \right\}$$
$$.\left\{ k_{1\alpha}\eta_{\mu\rho} + k_{2\mu}\eta_{\rho\alpha} + k_{3\rho}\eta_{\alpha\mu} + \frac{\alpha'}{2} k_{1\alpha}k_{2\mu}k_{3\rho} \right\} \tag{A5.2}$$

by a trivial application of the techniques described in Chapter One and also in reference [33]. It should be noted tht the tensors in the above are the full tensors as in (A5.1) above. This amplitude can now be 'normalised' by deriving the three graviton scattering amplitude from the Chapline Manton action and comparing this with the three graviton term derived from (A5.2) using relation (A5.1). The string amplitude prediction for a-a-h scattering can now be derived from (A5.2) using (A5.1). This can be seen to contain a factor of ε^2 which can be fixed by comparison with the calculation of the same a-a-h amplitude from the Chapline-Manton action. This normalisation procedure will now be carried out.

The three graviton scattering amplitude due to the Chapline-Manton action can be derived immediately from the vertex factor (3.2.23a), which can be compared with the string answer, (3.2.23b) to yield the overall normalising factor of -i/4. This means that the *string* prediction for the $O(\alpha'^0)$ a-a-h scattering amplitude is therefore,

$$-\frac{i}{4}\left\{ \varepsilon^2 \zeta_1^{a\mu\nu}\zeta_2^{a\rho\sigma}\zeta_3^{\alpha\beta} (k_{1\sigma}\eta_{\beta\nu} + k_{2\beta}\eta_{\nu\sigma} + k_{3\nu}\eta_{\sigma\beta}).\right.$$
$$\left. .(k_{1\rho}\eta_{\alpha\mu} + k_{2\alpha}\eta_{\mu\rho} + k_{3\mu}\eta_{\rho\alpha}) \right\} \tag{A5.3}$$

The field theory amplitude for the same process is simply derived by expanding the action term, (using the notation of Chapter Three),

$$-\frac{3}{4} f_{\alpha\beta\gamma} f_{\mu\nu\rho}\, g^{\alpha\mu} g^{\beta\nu} g^{\gamma\rho} \tag{A5.4}$$

employing the methods explained in detail above, and by using the one graviton expansion of the metric. (*Note*: no truncation will be performed here for the same reasons that it is not used in the three point fermionic amplitude calculations). This implies that the amplitude is of the form,

$$\frac{i}{4}\{\, 2\,(\zeta^{a}_{1\nu\rho}\zeta_2^{a\gamma\mu}k_{1\mu}k_2^{\nu} + \zeta_1^{a\gamma\mu}\zeta^{a}_{2\nu\rho}k_{2\mu}k_1^{\nu})\,\zeta_{3\,\gamma}^{\rho}$$

$$+\,4\,(\zeta^{a}_{1\nu\rho}\zeta_2^{a\mu\nu}k_{1\mu}k_2^{\gamma} + \zeta_1^{a\mu\nu}\zeta_{2\nu\rho}k_{2\mu}k_1^{\gamma})\,\zeta_{3\,\gamma}^{\rho}$$

$$+\;(\zeta^{a}_{1\mu\nu}\zeta_2^{a\mu\nu}(\,k_{1\rho}k_2^{\gamma} + k_{2\rho}k_1^{\gamma}))\,\zeta_{3\,\gamma}^{\rho}\,\} \qquad \text{(A5.5)}$$

using the standard techniques developed in Chapter Three.

It can be seen obviously that the ε coefficient must satisfy $\varepsilon^2 = 2$, implying that $\varepsilon = \pm\sqrt{2}$. This ambiguity in the sign will persist to the final matching to determine γ, which can be derived from the $O(\alpha')$ a-h-h amplitude matching derived directly from the appearance of the Lorentz-Chern-Simons term in the G^2 term in the action. It should be noted that the α' parameter will be kept in explicitly throughout this calculation in contrast to the procedure adopted in the main text above. The appropriate vertex for the matching to determine γ comes from the term,

$$-\frac{3}{4}\{\, f_{\alpha\beta\gamma} - \frac{\gamma\alpha'}{2}\sqrt{2}\,R_{[\alpha\beta}{}^{mn}\omega_{\gamma]}{}^{nm}\,\}^2 \qquad \text{(A5.6)}$$

which eventually yields an amplitude of the form,

$$-\frac{\gamma\alpha'\sqrt{2}\,i}{2}\{\, \zeta^{a}_{1\nu\rho}\zeta_{2n\mu}\zeta_{3\;m}^{\rho}\; k_1^{\mu}k_2^{m}k_2^{\nu}k_3^{n}$$

$$+\;\zeta^{a}_{1\nu\rho}\zeta_{3n\mu}\zeta_{2\;m}^{\rho}\; k_1^{\mu}k_3^{m}k_3^{\nu}k_2^{n}\,\} \qquad \text{(A5.7)}$$

which must be compared with the string result. This is now done. From (A5.2) and (A5.1), and using the value for ε derived above, the string amplitude for a-h-h scattering can be seen to be,

$$\pm\frac{i\sqrt{2}\,\alpha'}{8}\{\, \zeta_1^{a\mu\nu}\zeta_2^{\rho\sigma}\zeta_3^{\alpha\beta}\,(\,k_{1\beta}\eta_{\nu\sigma} + k_{3\sigma}\eta_{\beta\nu}\,)\,k_{1\alpha}k_{2\mu}k_{3\rho}\,\} \qquad \text{(A5.8)}$$

(where the $\pm$ denotes the sign of ε) which immediately implies that the parameter γ

takes the values $\gamma = \pm 1/4$, when $\varepsilon = \pm\sqrt{2}$. As discussed above, only the $\gamma = -1/4$ can be seen to give an amplitude match to the string in Chapter Five. As discussed both there and in Chapter Six, this gives results completely consistent with both Gross and Sloan, and Cai and Nunez[58].

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