Alfven Resonance Absorption in a Non-Uniform Magnetofluid

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by

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Contents

Abstract

One of the requirements for controlled nuclear fusion is the attainment of a high enough temperature so that nuclei can ovecome their mutual coulomb repulsion and fuse, releasing a great deal of energy in the process. To reach the required temperatures energy must be fed into the plasma. One method is high frequency heating of the plasma using electomagnetic waves, where the incoming wave is used to excite some of the many naturally occurring modes present in the plasma, which then decay giving up their energy to the plasma.

This project deals with Alfven resonance absorption in a plasma for supplementry heating of a magnetically confined plasma by extending the paper 'Alfven Resonance Absorption in a Magnetofluid' by Diver and Laing. The paper is unusual in that it treats not only the Alfven resonance but also the magnetosonic resonances which are then taken to be separated in the plasma by a small distance δ so that the absorption coefficient is dependent on both of these singularities. In this original paper some incorrect modelling assumptions about the magnetosonic singularities were made and dealt only with case of small θ, where θ is the angle between the x component of the incident wave vector and the magnetic field.

This thesis extends the work to finite θ and rederives the subsequent work in the paper. This work required the calculation of Bessel functions in which both the argument and the order, which were non integer varied as a continuous function of θ, this part of the project, the numerical simulation of the reflection formula, was a significant part of the whole project.

Chapter 1

1. Introduction

In recent years there has been considerable interest in the possibilty of controlled nuclear fusion, in particular in the deuterium, tritium reaction

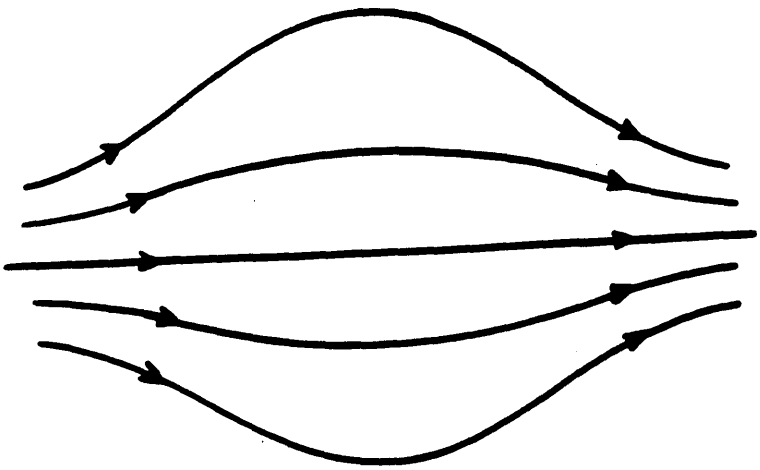
$$ {}_1D^2 + {}_1T^3 \rightarrow {}_2He^4 + n + 17.56\ MeV , $$

since this requires the lowest temperature for the reaction to become self sustaining. For fusion to occur in the first place the reacting nuclei must have enough kinetic energy to overcome their mutual Coulomb repulsion and approach sufficiently closely for there to be a reasonable probability of fusion . For the D-T reaction cross-section to be a maximum we require the particles' thermal energies to be of the order 10keV , this means for fusion we require to maintain a temperature of about $10^8 K$. The most promising method of achieving fusion is by magnetic confinement of the plasma ; this however has many problems .

In (1957) J. D. Lawson showed that in order to get the energy released to be greater than the energy supplied , $n\tau > 10^{14}\ cm^{-3} s$ where n is the plasma density and τ is the containment time for the plasma. A temperature greater than $10^7 K$ is also needed for fusion to be realised. At present JET has not achieved these requirements but is within an order of magnitude of these targets.

2. Containment

One of the major problems is in containing the plasma for long enough. There have been many methods proposed to contain the plasma , but due to the nature of the plasma there are a great many instabilities present in each of these methods . One of the first methods of containment was to use a cylindrical device where the magnetic field increased towards the ends of the device and the particles were trapped by a magnetic mirror (a sketch of the magnetic field in a mirror machine is shown in fig 1) .



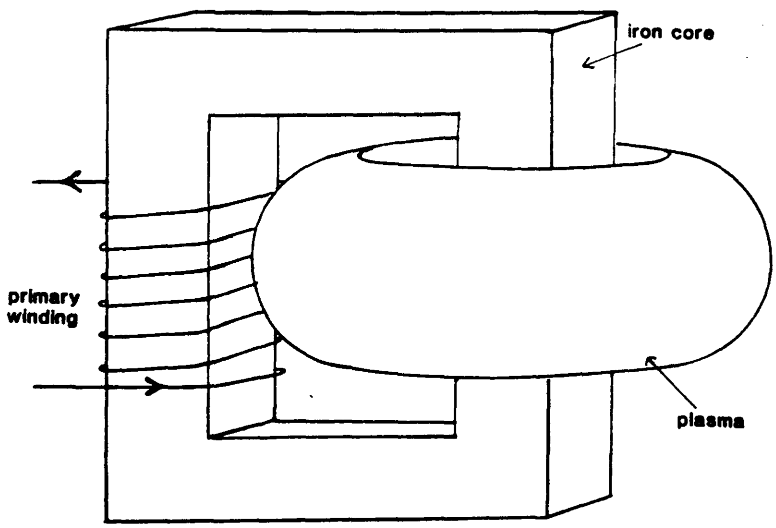
fig(1)

The problems with this method were that all particles were not trapped but that there was a 'loss cone' defined in velocity space where particles with these velocities were able to escape . This made the plasma velocities non-maxwellian after these particles had escaped . This caused enhanced electric field fluctuations causing more particles to be deflected into the loss cone which aggravated the situation more . All this meant that the simple mirror machine was not a useful containment device .

One solution to this problem is to join the ends of the cylinder to make a torus, this removes the need for magnetic mirrors as in the previous method . In fact this is the approach used by most workers in the field , the main torroidal containment devices are tokamaks and reverse field pinches . A torroidal device requires that the magnetic field have both toroidal and poloidal components with the poloidal component being provided by a current flowing in the plasma and the torroidal by external coils . The torus is susceptable to a great many instabilities , but these can be stabilised in a tokamak better than in other configurations .

3. Heating

The other requirement for fusion is a very high temperature . It was at first thought that resistive heating of the plasma would supply the necessary energy to the plasma . This current is induced in the plasma by making it the secondary winding in a large transformer (see fig. 2)



(fig 2)

The resistance of the plasma varies with temperature as $T^{-3/2}$ and with the current also being limited due to stability considerations the highest temperature that can be reached by ohmic heating is limited to 2-3 keV . So to reach the required higher temperatures needed for fusion some auxilliary heating system is required either neutral beam injection or radio-frequency heating .

3.1. Neutral beam injection

Neutral beam injection is the injection of high power beams of neutral hydrogen atoms , which can cross the containing field due to their neutrality and are then ionized and trapped within the target plasma , the energy of these atoms is then put into the plasma through collisional processes.

3.2. Radio-frequency heating

The alternative method of heating , that is radio-frequency heating can be split up into a further two sections which use either low frequency electromagnetic fields or high frequency electomagnetic waves . The low frequency method depends on the timescale of the induced field variations within the plasma being comparable to or shorter than the appropriate collisional relaxation time of each of the particle species . This causes a significant departure from thermal equilibrium of each species which lets energy be transferred irreversibly to the plasma from both the oscillatory and non-oscillatory fields .

The other form of heating using high-frequency waves relies on launching an externally generated wave into the plasma , this wave excites one of the many natural plasma modes . This natural mode propagates through the plasma losing its energy to the plasma thereby raising the overall temperature of the plasma . The processes by which the wave loses energy may be by exciting another mode in the plasma , that is one which is more easily absorbed by the plasma , this is called mode conversion . Another process is one in which the wave loses energy by collisional processes or collisionless proccesses such as Landau damping or wave-particle interactions.

The main reasons for a lowering of the plasma temperature during confinement are radiation losses due to Bremsstrhlung and impurity radiation caused by material from the walls entering the plasma . The second of these cases is the most important , to minimize impurity concentrations , the plasma must be kept away from the vessel wall . This is done by a limiter , which is an obstacle projecting into the plasma producing a plasma boundary away from the wall . An alternative is to use a divertor which takes the outer field lines into a chamber outside the main vessel , the plasma associated with these field lines is then pumped out of the divertor chamber .

4. Modelling of the plasma

There are a number of models to describe the behaviour of a plasma , the most accurate way is to describe the plasma using statistical mechanics . The behaviour of the plasma can be represented by a hierachy of equations determining the reduced distribution functions , this is called the BBGKY hierarchy . The problem with this system is that each reduced distribution function is defined in terms of a higher distribution function . To get any useful results we must close the system . The most useful one is the one particle distribution function and occasionally the two particle function . The equation describing the evolution of the one particle distibution function in time , space and velocity space is

$$\frac{\partial f}{\partial t}+\mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}}+\frac{\mathbf{F}}{m}\cdot\frac{\partial f}{\partial \mathbf{v}} = \left(\frac{\partial f}{\partial t}\right)_c ,$$

where $f = f(\mathbf{r},\mathbf{v},t)$. The term on the right was introduced to close the system of equations . The left hand side of the equation can be seen to be the total rate of change of the distribution function and the right hand side is the change in the distribution function due to collisions .

A simpler but still very accurate description of a plasma can be obtained by treating the plsama as a fluid , the equations describing the fluid model can be obtained by taking velocity moments of the kinetic equation above . We define the particle density to be

$$n(\mathbf{r},t) = \int f(\mathbf{r},\mathbf{v},t)d\mathbf{v} .$$

So the average of any arbitary funtion $\phi(\mathbf{r},\mathbf{v},t)$ is

$$\phi_{ave} = \frac{1}{n(\mathbf{r},t)} \int \phi(\mathbf{r},\mathbf{v},t) f(\mathbf{r},\mathbf{v},t) d\mathbf{v} \quad .$$

To get the moment equations from the kinetic equation , we multiply by ϕ and integrate over velocity. Taking $\phi = m$ in the kinetic equation we get the equation of continuity

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial \mathbf{r}}(n\mathbf{v}) = 0 \quad .$$

Lettinig ϕ equal $m\mathbf{v}$, $\frac{1}{2}mv^2$ we obtain a momentum equation and an energy equation respectively.

$$\rho \left[\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial \mathbf{r}} \right] = -\frac{\partial}{\partial \mathbf{r}} p + (\nabla \times \mathbf{B}) \times \frac{\mathbf{B}}{\mu_0}$$

where ρ and p are the density and scalar pressure respectively. Maxwell's equations were used in deriving the above momentum equation. We must also describe the process of energy exchange in the plasma , one of the most obvious is to take the plasma to be adiabatic, this gives us the following expression from thermodynamics

$$\frac{\partial}{\partial t}(p\rho^{-\gamma}) + v \cdot \nabla (p\rho^{-\gamma}) = 0 \quad .$$

We can get this equation from the energy equation

$$\frac{D}{Dt}(p\rho^{-\gamma}) = \frac{2}{3}\rho^{-\gamma}(\mathbf{J} - q\mathbf{v}) \cdot (\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

where { D / Dt } denotes the advective derivative, if we use the simple Ohm's law

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0.$$

Since the plasma is subject to magnetic fields we must include Maxwell's equations in describing the behaviour of the plasma .

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B}) \quad , \quad \nabla\cdot\mathbf{B} = 0$$

These equations form a closed system and so we can completely describe the plasma's physical proccesses within the applicability of the model .

Chapter 2

1. Setting up the MHD equations

$$\frac{D\rho}{Dt}+\rho\nabla\cdot\mathbf{u}=0 \tag{1}$$

$$\rho\frac{D\mathbf{u}}{Dt}=-\nabla p+(\nabla\times\mathbf{B})\times\frac{\mathbf{B}}{\mu_0} \tag{2}$$

$$\mathbf{E}+\mathbf{u}\times\mathbf{B}=0 \tag{3}$$

$$\frac{D}{Dt}(p\,\rho^{-\gamma})=0 \tag{4}$$

$$\frac{\partial\mathbf{B}}{\partial t}=\nabla\times(\mathbf{u}\times\mathbf{B}) \tag{5}$$

$$\nabla\cdot\mathbf{B}=0 \tag{6}$$

where $\frac{D}{Dt}=\frac{\partial}{\partial t}+\mathbf{u}\cdot\nabla$ denotes the advective derivative, and the dependent variables are

$\mathbf{u}$: *fluid bulk velocity*

p : *scalar pressure of the single fluid*

ρ: *mass density*

$\mathbf{B}$: *magnetic induction*

$\mathbf{E}$: *electric field*

γ=5/3, *the adiabatic constant for a simple gas.*

The list (1) - (6) forms a closed system, and so provides a complete mathematical description of all the physical processes of the model.

Since we are taking the plasma to be adiabatic , we are fixing the entropy of the plasma . The plasma fluid is initially taken to be at rest with uniform entropy S_0 on all of the stream lines so it must remain constant when in motion . So for all later times $S=S_0$ on all the stream lines , hence throughout the whole plasma .

This is described more fully in (WITHAM G. B. (1974)) , equation (4) is also quoted in more recent publications (CAIRNS R. A. (1985)) , with equation (4) being derived as a general energy equation but neglecting thermal transport processes .

We are going to restrict attention to small amplitude waves, so all the physical quantities can be written in the form $f=f_0+f_1$ where f_0 is the equilibrium value of the variable, and f_1 is the perturbation, which is assumed small compared with the equilibrium value so that we have only to keep up to first order quantities.

The equilibrium is taken as one in which there is no streaming velocity, no external electric field and the adiabatic law for simple gases holds,so

$$\mathbf{u}_0=0 \tag{7}$$

$$\mathbf{E}_0=0 \tag{8}$$

$$p_0\rho_0^{-\gamma}=constant \tag{9}$$

$$\nabla p_0=(\nabla\times\mathbf{B}_0)\times\frac{\mathbf{B}_0}{\mu_0} \tag{10}$$

equation (10) is just equation (2) with $\mathbf{u}=0$ and p and ρ set to their equilibrium values.Note that by saying that the plasma is adiabatic we cannot now take the density ρ_0 to be constant since this would contradict $p_0+B^2/2\mu_0=constant$

Setting

$$\mathbf{u}=\mathbf{u}_0+\mathbf{u}_1=\mathbf{u}_1$$

$$\mathbf{E}=\mathbf{E}_0+\mathbf{E}_1=\mathbf{E}_1$$

$$\mathbf{B}=\mathbf{B}_0+\mathbf{B}_1$$

$$p=p_0+p_1$$

$$\rho=\rho_0+\rho_1 \quad .$$

We are taking a static equilibrium, that is $\frac{\partial f_0}{\partial t}$ is always equal to zero since f_0 is the equilibrium value of the variable. Linearising equations (1)-(6) we find the following equations for the first order quantities.

Equation (1) becomes

$$\frac{\partial\rho_1}{\partial t}+\nabla\cdot(\rho_0\mathbf{u}_1)=0 \quad , \tag{11}$$

eqation (2) becomes

$$\rho_0\frac{\partial\mathbf{u}_1}{\partial t}=-\nabla p_1+(\nabla\times\mathbf{B}_1)\times\frac{\mathbf{B}_0}{\mu_0}+(\nabla\times\mathbf{B}_0)\times\frac{\mathbf{B}_1}{\mu_0} \quad , \tag{12}$$

equation (3) becomes

$$\mathbf{E}_1+\mathbf{u}_1\times\mathbf{B}_0=0 \quad , \tag{13}$$

equation (4) becomes

$$\frac{\partial(p_0+p_1)(\rho_0+\rho_1)^{-\gamma}}{\partial t}+\mathbf{u}_1\cdot\nabla((p_0+p_1)(\rho_0+\rho_1)^{-\gamma})=0 \quad ,$$

this becomes to first order

$$\frac{\partial(p_0\rho_1^{-\gamma}+p_1\rho_0^{-\gamma})}{\partial t}+\mathbf{u}_1\cdot\nabla(p_0\rho_0^{-}\gamma)$$

and using equation (9) we get

$$p_0\frac{\partial\rho_1^{-\gamma}}{\partial t}+\rho_0^{-\gamma}\frac{\partial p_1}{\partial t}=0$$

which finally gives

$$\frac{\partial p_1}{\partial t}=\frac{\gamma p_0}{\rho_0}\frac{\partial \rho_1}{\partial t} \quad , \tag{14}$$

equation (5) gives

$$\frac{\partial \mathbf{B}_1}{\partial t}=\nabla\times(\mathbf{u}_1\times\mathbf{B}_0) \tag{15}$$

equation (6) gives

$$\nabla\cdot\mathbf{B}_1=0 \; . \tag{16}$$

We need one more relationship before we can proceed; we obtain this by first taking the gradient of equation (9) and rearranging to get

$$\nabla\rho_0=\frac{\rho_0}{\gamma p_0}\nabla p_0 \, . \tag{17}$$

We now combine these equations self consistently by first differentiating (12) with respect to time, this gives

$$\rho_0\frac{\partial^2\mathbf{u}_1}{\partial t^2}=-\nabla\frac{\partial p_1}{\partial t}+(\nabla\times\frac{\partial \mathbf{B}_1}{\partial t})\times\frac{\mathbf{B}_0}{\mu_0}+(\nabla\times\mathbf{B}_0)\times\frac{\partial \mathbf{B}_1}{\partial t}\frac{1}{\mu_0} \; ,$$

now substitute in equations (11),(13),(14),(15),(16) and (17) to get

$$\rho_0\frac{\partial^2\mathbf{u}_1}{\partial t^2}=\nabla[\gamma p_0\nabla\cdot\mathbf{u}_1+\mathbf{u}_1\cdot\nabla p_0]+[\nabla\times(\nabla\times(\mathbf{u}_1\times\mathbf{B}_0))]\times\frac{\mathbf{B}_0}{\mu_0}+(\nabla\times\mathbf{B}_0)\times[\nabla\times(\mathbf{u}_1\times\mathbf{B}_0)]\frac{1}{\mu_0} \; . \tag{18}$$

2. Homogeneous Plasma

First we will deal with the case of a uniform plasma ,that is a plasma with a constant density and a constant external magnetic field given by

$$\mathbf{B}=B_0(\cos\theta,\sin\theta,0) \, .$$

Now Fourier transform equation (18) in all dimensions with

$$\mathbf{k}=(k_x,0,k_z)\ .$$

The secular determinant of the resulting system of equations gives the familiar dispersion relation (VAN KAMPEN and FELDERHOF,(1967))

$$(\omega^2-k_x^2c_a^2\cos^2\theta)(\omega^4-k^2V^2\omega^2+k_x^2k^2c_s^2c_a^2\cos^2\theta)=0\ , \qquad (19)$$

$$c_a^2=\frac{B_0^2}{\mu_0\rho_0}\ ,$$

$$c_s^2=\frac{\gamma p_0}{\rho_0}\ ,$$

$$V^2=c_a^2+c_s^2\ ,$$

The normal modes of this system are the roots of equation (18),

$$\frac{\omega^2}{k_x^2}=c_a^2\cos^2\theta\ ,$$

$$\frac{\omega^2}{k^2}=½\left[V^2\pm\left[V^4-4\frac{k_x^2}{k^2}c_a^2c_s^2\cos^2\theta\right]^{½}\right],$$

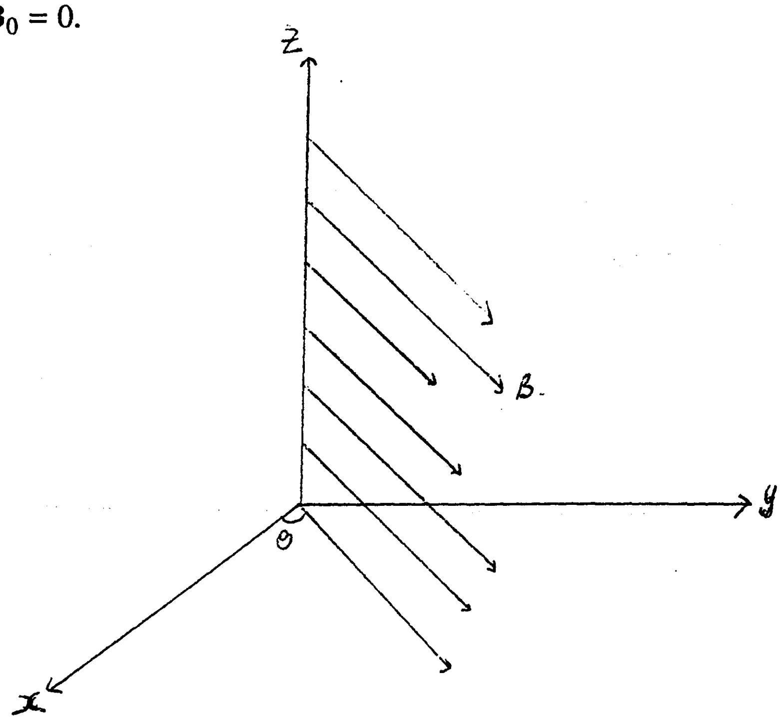
where the first root describes the shear alfven wave, and the other two are the fast and slow magnetosonic modes.

3. Inhomogeneous Plasma

A more realistic model is one in which the background state is not uniform in space. From now on, we will consider a plasma in which the equilibrium $\mathbf{B}_0$ field is plane stratified in the z direction :

$$\mathbf{B}_0=B_0(z)(\ \cos\theta\ ,\ \sin\theta\ ,\ 0\ )\ .$$

The angle θ is a constant, this is the simplest non-uniform field satisfying $\nabla\cdot\mathbf{B}_0 = 0$.



fig(3)

From this field we can determine the structure of the pressure and density functions.

Looking at equation (10) we see that

$$\left[\frac{\partial p_0}{\partial x},\frac{\partial p_0}{\partial y},\frac{\partial p_0}{\partial z}\right] = \left[0,0,-\frac{1}{2\mu_0}\frac{\partial B_0^2}{\partial z}\right] ,$$

and equating the z-compts

$$\frac{\partial p_0}{\partial z}+\frac{\partial B_0^2}{\partial z} = 0 ,$$

integrating with respect to z we finally get

$$p_0+\frac{B_0^2}{2\mu_0}=\chi_1,$$

where χ_1 is just the integration constant, now integrate (9) with respect to z to get

$$p_0\rho_0^{-\gamma}=\chi_2.$$

The structure of the plasma now is determined by the magnetic field structure $\mathbf{B}_0(z)$ and the two constants χ_1 and χ_2.

So we can see that the velocity functions now become functions of position, in fact all spatial information is now contained in these velocity functions.

$$c_a^2(z) = \frac{B_0^2(z)}{\mu_0\rho_0}$$

i.e

$$c_a^2(z) = \frac{B_0^2(z)}{\mu_0\left(\frac{p_0}{\chi_2}\right)^{1/\gamma}}$$

finally this gives

$$c_a^2(z)=\frac{\chi_2^{1/\gamma}B_0^2(z)}{\mu_0\left[\chi_1-\frac{B_0^2(z)}{2\mu_0}\right]^{1/\gamma}} , \qquad (20)$$

similarly for $c_s^2(z)$

$$c_s^2(z)=\gamma\chi_2^{1/\gamma}\left[\chi_1-B_0^2\frac{(z)}{2\mu_0}\right]^{1-1/\gamma} . \qquad (21)$$

The velocity functions now contain all the information about the spatial variation of the plasma.

4. Uncoupling the equations

Since equation (18) is a vector equation,we first Fourier transform in time and x and then write out the three resulting component equations below.

These equations were generated using REDUCE 3.3 to perform the simple but tedious algebraic manipulations,

The components of equation (18) can then be written as

$$[\omega^2 - kx^2(c_s^2 + c_a^2\nu^2)u_x + [k_x^2 c_a^2 \mu\nu]u_y + ik_x[c_s^2 + c_a^2\nu^2]u_z' = 0 \quad ,$$

$$[k_x^2 c_a^2 \nu\mu]u_x + [\omega^2 - k_x^2 c_a^2 \mu^2]u_y - [ik_x c_a^2 \mu\nu]u_z' = 0 \quad ,$$

$$ik_x[\hat{c}_s^2 + \hat{c}_a^2\nu^2]u_x - [ik_x c_a \mu\nu]u_y + ik_x[c_s^2 + c_a^2\nu^2]u_x'$$

$$-[ik_x c_a^2 \mu\nu]u_y' + [\omega^2 - k_x^2 c_a^2 \mu^2]u_z + [\hat{c}_s^2 + \hat{c}_a^2]u_z' + [c_s^2 + c_a^2]u_z'' = 0 \quad ,$$

where

$'$ *denotes* d/dz , $c_s^2 = \gamma p_0/\rho_0$, $c_a^2 = B_0^2/(\mu_0\rho_0)$, $\hat{c}_a^2 = B_0^2{}'/(\rho_0\mu_0)$, $\hat{c}_s^2 = \gamma p_0/\rho_0$, $\mu = \cos\theta$

$\nu = \sin\theta$.

By manipulating these equations using REDUCE we can reduce these component equations to the following ordinary differential equations

$$u_x = -g u_z' \quad ,$$

$$u_y = f u_x \quad ,$$

$$\xi\psi u_z'' + (\phi + \lambda\frac{q'}{q})u_z' + \psi q\, u_z = 0 \quad , \qquad (22)$$

where f , g , q , ξ , ψ , ϕ and λ are given as follows

$$f = \frac{\omega^2 c_a^2 \mu\nu}{\omega^2(c_s^2 + c_a^2\nu^2) - k_x^2 c_s^2 c_a^2 \mu^2} \quad ,$$

$$g = ik_x \frac{[\omega^2(c_s^2 + c_a^2\nu^2) - k_x^2 c_s^2 c_a^2 \mu^2]}{q} \quad ,$$

$$q = \omega^4 - k_x^2 V^2 \omega^2 + k_x^4 c_s^2 c_a^2 \mu^2 \quad , \tag{23}$$

$$\xi = \omega^2 V^2 - k_x^2 c_s^2 c_a^2 \mu^2 \quad ,$$

$$\psi = \omega^2 - k_x^2 c_a^2 \mu^2 \quad , \tag{24}$$

$$\phi = \tilde{c_a^2}[(2-\gamma)\omega^4 + k_x^2\omega^2(c_a^2(\gamma-2) - 2c_s^2)\mu^2 + 2k_x^4 c_a^2 c_s^2 \mu^4]$$

$$+ k_x^2 c_s^{2\prime} (\omega^2 - k_x^2 c_a^2 \mu^2)(c_a^2 \nu^2 + c_s^2)$$

$$+ k_x^2 c_a^{2\prime} [\omega^2 V^2 \nu^2 - k_x^2 c_s^2 (c_a^2 \nu^2 + c_s^2)\mu^2] \quad ,$$

$$\lambda = k_x^2[k_x^2 c_a^2 c_s^2 \mu^2 (c_a^2 \nu^2 + c_s^2) - \omega^2 (c_a^4 \nu^2 + c_s^4 + 2c_a^2 c_s^2 \nu^2)] \quad .$$

and

$$\tilde{c_a^2} = \frac{(\rho_0 c_a^2)'}{(2\rho_0)} \quad .$$

This completes the setting up of the equations required to give a description of the problem.

Chapter 3

When the equilibrium magnetic field is constant, then Fourier transforming equation (23) in z gives

$$k_z^2\,\xi\,\psi - \psi\,q = 0$$

i.e

$$\psi\,(\,k_z^2\,\xi - q\,) = 0$$

which is the expected dispersion relation equation (19) only in different notation.

We now go on to the full problem, with non-uniform $\mathbf{B}_0$ and oblique wavevector ($k_x \neq 0$), we must now solve equation (23) in its full complexity.

There are four possible singularities in equation (22),

$$\xi=0 \;:\; \omega^2 = k_x^2 \frac{c_a^2 c_s^2}{V^2} \mu^2 \tag{26}$$

$$\psi=0 \;:\; \omega^2 = k_x^2 c_a^2 \mu^2 \tag{27}$$

$$q=0 \;:\; \omega^2 = \tfrac{1}{2} k_x^2 V^2 \pm \tfrac{1}{2} k_x^2 [V^4 - 4 c_a^2 c_s^2 \mu^2]^{1/2} . \tag{28}$$

If we let $k_x=0$ then none of these roots would exist since the plasma flow would be only one-dimensional; we can see this from the component equations (22) where $g=0$ if $k_x=0$, this forces u_x and u_y to be zero. However by choosing $k_x \neq 0$, the bulk velocity of the plasma is three dimensional , and it is the flow of plasma in directions perpendicular to the inhomogeneity which causes these singular points to appear.

The root defined by (26) is the cusp singularity, well known in astrophysics literature, and is the dispersion relation for a strongly localised surface wave. $\psi=0$

gives the usual shear alfven wave dispersion function. $q=0$ however does not give the usual fast and slow magnetosonic modes of MHD since they do not involve the whole wavevector, only the k_x component.

When $\theta=0$, the root of q become

$$\omega^2 = k_x^2 c_a^2$$

or

$$\omega^2 = k_x^2 c_s^2 \quad ,$$

we can see there now exists a singularity of twice the expected order, one from $\psi=0$ and $q=0$.

At $\theta=\frac{1}{2}\pi$ however, only one singularity remains,

$$\omega^2 = k_x^2 V^2 \, ,$$

since neither the cusp nor the Alfven root exist at this angle.

1. Approximate modelling equations

In its present form equation (22) is very unlikely to have an analytic solution.So following the lead of past authors we concentrate on the behaviour of the equation near the Alfven resonance , but will extend previous treatments to cover the case of a finite θ so that there are two singularities close to one another, the Alfven resonance and one from $q=0$ corresponding to the fast magnetosonic mode of this model. We will assume these singularities are separated by a distance δ. The simplest possible way to model such behaviour is to take the following form for ψ and q :

$$q = q_0 z \, , \tag{29}$$

$$\psi = \psi_0 (z - \delta) \tag{30}$$

where

$$q_0 = \left[\frac{\partial q}{\partial z}\right]_{\delta/2} \,, \tag{31}$$

$$\psi_0 = \left[\frac{\partial \psi}{\partial z}\right]_{\delta/2} \,. \tag{32}$$

Note that λ may be written in the form

$$\lambda = -(a+b\,\psi)$$

where

$$a = 2k_x^2\omega^2 c_a^2 c_s^2 v^2 \,,$$

$$b = k_x^2(c_s^4 + c_a^4 v^2) \,,$$

and so we incorporate this feature into our modelling, which will concentrate primarily on the effects of the singularities on the wave properties of the model. Thus, although the other coefficients in the equation may also depend on z, they are taken to be constants since they do not affect the intrinsic nature of the singularities.

Thus (22) becomes

$$\psi_0\xi_0(z-\delta)u'' + (\phi_0 - (a+b\,\psi_0(z-\delta))/z\,)u' + \psi_0 q_0 z\,(z-\delta)u = 0, \tag{33}$$

where we evaluate ψ_0, ϕ_0, a, b at $z=\delta/2$. For reasons of convenience we now drop the subscript z on the velocity.

It is most convenient in the later solution of equation (33) if we rescale the ODE so that the singularities are situated at the origin and the point $z=1$. But we must treat the special case of $\theta=0$ separately, since in this case, $\delta=0$ and there is only one singularity.

2. Coincident roots : $\theta=0$

Setting $\delta=0$ in equation (33) yields the equation

$$\psi_0\xi_0 z u''+(\phi_0-b\psi_0)u'+\psi_0 q_0 z^2 = 0$$

i.e

$$zu''+\left(\frac{\phi_0-b\psi_0}{\psi_0\xi_0}\right)u'+\frac{q_0}{\xi_0}u=0$$

i.e

$$z\,u'' + A_0 u' + C_0 z^2 u = 0 \tag{34}$$

where

$$A_0=\frac{\phi_0-b\psi_0}{\psi_0\xi_0}$$

and

$$C_0=\frac{q_0}{\xi_0} \quad .$$

This equation is of standard form (MURPHY (1960)) and has solution

$$u = z^{\frac{1}{2}(1-A_0)} Z_{1/p}(\kappa z^{3/2}), \tag{35}$$

where

$$\kappa = \frac{4(1-2s)}{3(A_0-1)}\sqrt{C_0}\,, \quad p = \frac{3}{1-2s}, \quad 1-2s = \pm(1-A_0),$$

and where $Z_{1/p}$ is a Bessel function of order $1/p$. We have to take care here , since when $z<0$ the argument of the Bessel function becomes imaginary, so we must analytically continue this function for negative values of z. Later on for theta equal to zero we will see that A_0 is approximately zero so equation (33) becomes

$$u''+C_0 zu=0$$

this has Airy function solutions.We can see this in another way by noting that

$$p=\frac{3}{1-2s}=\pm\frac{3}{1-A_0}=\pm 3\ ,$$

so we see that solutions of equation (30) are bessel functions of order a third which can be represented in terms of Airy functions.

For z less than zero, u is imaginary due to the $z^{1/2}$ factor in the formula so the wave is completely reflected for theta equal to zero. We will use this as a check later on when we do the case for a finite theta, that the reflection coefficient tends to unity as theta tends to zero.

3. Separated Roots : $\theta\neq 0$

Again for later convenience we will rescale the singularities in order that they occur at the origin and unity.Thus we make a change of independent variable,

$$\eta=z/\delta,\quad \delta\neq 0\,,$$

so that we may write equation (29) in the form

$$\eta(\eta-1)u_{\eta\eta}+(A\eta+B)u_\eta+C\eta^2(\eta-1)u=0 \tag{36}$$

where

$$A=\frac{\phi_0-b\psi_0}{\xi_0\psi_0}\ , \tag{37}$$

$$B=\frac{b\psi_0\delta-a}{\delta\xi_0\psi_0}\ , \tag{38}$$

$$C=\frac{\delta q_0}{\xi_0}\sigma^2\ . \tag{39}$$

We now non-dimensionalise A,B and C so that the the parameters which now describe the problem are

$$\theta,\ \beta = c_s^2/c_a^2,\ \nu = \frac{k_x^2 c_a^2}{\omega^2},\ \sigma = k_x\delta,\ G = \frac{1}{B_0}\frac{\partial B_0}{\partial z}.$$

The non-dimensionalised forms and their derivation are shown in appendix A. Note that ν now no longer stands for $\sin\theta$.

4. Wave Matching

To constuct the form of the wave solution for the plasma, we are first going to find the solution in the region of the Alven resonance at η=1, the solution around the fast q=0 resonance at η=0 and the match them at some point between zero and delta to arrive at an overall description of the wave in the region of these two singularities. Such a process involves an element of analytic continuation which enables singular solutions to be connected in the complex plane, avoiding the troublesome points on the real line. In this way the wave absorption is encompassed in the overall process, and manifest itself in a loss of energy from the waves, which we will see when we plot the reflection coefficient for finite valu of θ.

5. Local Solutions

We will construct solutions in different regions by retaining the most significant terms in that region, and solving the modified version of (36) by referring to (MURPHY (1960)).

5.1. solution local to origin : η=0

When η is approximatly zero , equation (36) becomes

$$\eta u_{\eta\eta} - Bu_\eta + C\eta^2 u = 0, \tag{40}$$

which has the solutions

$$u_0 = \eta^{\frac{1}{2}(1+B)} Z_{\pm(1+B)/3}[2/3\sqrt{C}\,\eta^{3/2}] \,. \tag{41}$$

5.2. solution local to δ : η=1

Here we make the change of variable ζ=$\eta-1$ and use the local equation

$$\zeta u_{\zeta\zeta} + (A+B)u_\zeta + C\zeta u = 0 \,, \tag{42}$$

which possesses solutions of the form

$$u_1 = \zeta^{\frac{1}{2}(1-A-B)} \, Z_{\pm\frac{1}{2}(1-A-B)}[\sqrt{C}\,\zeta],$$

i.e

$$u_1 = (\eta-1)^{\frac{1}{2}(1-A-B)} Z_{\pm\frac{1}{2}(1-A-B)}[\sqrt{C}\,(\eta-1)] \,. \tag{43}$$

5.3. asymptotic solution :

For the overall picture in this analysis, we require the behaviour of u for large η. Taking the leading terms in each coefficient for large η, we have

$$\eta u_{\eta\eta} + A u_\eta + C\eta^2 u = 0, \tag{44}$$

the solutions of which are

$$u_\infty = \eta^{\frac{1}{2}(1-A)} \, Z_{\pm(1-A)/3}[2/3\sqrt{C}\,\eta^{3/2}]. \tag{45}$$

In the above solutions Z can stand for any of the essel function pairs J_ν and $J_{-\nu}$ or J_ν and Y_ν or finally $H_\nu^{(1)}$ and $H_\nu^{(2)}$;we may also need to use modified Bessel functions if the arguments become imaginary. So we now have a complete set of solutions that we require in order to construct an approximate picture of the behaviour of waves in this model. Before we can interpret these solutions we have to calculate A,B and C in terms of the parameters given in (34)

The equilibrium plasma is assumed to be capable of sustaining waves of the required frequency, in the presence of a non-uniform magnetic field that is consistent with $\nabla\cdot\mathbf{B}=0$, in such a way that these waves encounter two singularities separated by a finite distance δ taken to be small in comparison with the gradient scale-lengths of the plasma.Not all of the parameters $\theta,\delta,\nu,\beta,\sigma$ and G can be independent.

6. Determination of ν and δ

The relationship between the parameters come from the definition of ψ,q .

$$\psi = 1-\nu\mu^2 \tag{46}$$

$$q = 1-\nu(1+\beta)+\beta\nu^2\mu^2. \tag{47}$$

From now on when using ψ , q etc, we will use the non dimensionalised forms found in Appendix A

Since this model has been designed with tokamak plasmas in mind,we will henceforth take the ratio $\beta = c_s^2/c_a^2$ to be β_{plasma}, and so is a small quantity (typically less than 10%). So in the following analysis we will use β as an expansion parameter and approximate factors of the form $(1+\beta)$ to one.

First step is to find an expression for delta. Returning to the modelling equations (29) and (30)

$$\psi = \psi_0(z-\delta) ,$$

$$q = q_0 z ,$$

we can Taylor expand (46) and (47) around the point $\frac{\delta}{2}$ and equate these expansions with the modelling equations (29) and (30)

$$\psi_0(z-\delta) = \psi = \psi(\tfrac{1}{2}\delta) + (z-\tfrac{1}{2}\delta)\left[\frac{\partial\psi}{\partial z}\right]_{\frac{1}{2}\delta} ,$$

$$q_0 z = q = q(\tfrac{1}{2}\delta) + (z-\tfrac{1}{2}\delta)\left[\frac{\partial q}{\partial z}\right]_{\frac{1}{2}\delta} ,$$

evaluating these at the point $z = \frac{1}{2}\delta$,we can now see

$$\delta/2 = -\left[\frac{\psi}{\psi_0}\right]_{\frac{1}{2}\delta} = \left[\frac{q}{q_0}\right]_{\frac{1}{2}\delta} . \tag{48}$$

Equation (48) gives us δ , and the left hand side of equation (48) allows us to arrive at an expression for ν .

However before we can proceed to calculate ν we must first find expressions for $\partial\nu/\partial z$ and $\partial\beta/\partial z$; these are are derived in appendix A.

In fact

$$\frac{\partial\nu}{\partial z}=\frac{\nu}{\beta}[1+2\beta]G,$$

$$\frac{\partial\beta}{\partial z}=-[\gamma+2\beta]G.$$

Now writing equation (48) out explicitly

$$\frac{1-\mu^2\nu}{\nu(1+2\beta)\mu^2} = \frac{1-\nu(1+\beta)+\beta\nu^2\mu^2}{2\beta^2\mu^2\nu(1+\beta)+\beta\gamma(1-\nu\mu^2)-(1+3\beta)} . \tag{49}$$

This equation allows us to find ν from the resulting quadratic equation, which was done using REDUCE .

As θ goes to zero only the Alfven resonance exists and since

$$\psi = 1-\nu\mu^2 ,$$

when θ=0 then $\psi = 0$, so

$$0 = 1-\nu ,$$

so we can see that when $\theta = 0$ then $\nu = 1$, this fact allows us to select the correct root from the resulting quadratic equation (49). The full expression for ν is

$$\nu = \frac{[2\beta^2+\beta(4-\gamma)+1]+(\beta+½)(\beta-1)\left[1+\frac{2\beta(1-\mu^2)(\beta+(3-\gamma)/4)}{(\beta+½)(\beta-1)^2}\right]^{½}}{4\mu^2\beta(\beta+(3-\gamma)/4)} \quad . \qquad (50)$$

The factor inside the square root is much less than one (with β less than ten percent) so we can Taylor expand the square root term truncating the series after the first term to give

$$\nu = \frac{1}{\mu^2}\,\frac{(2\beta^2+\beta(4-\gamma)+1)+2(\beta+½)(\beta-1)\left[1+\frac{\beta(1-\mu^2)(\beta-(3-\gamma)/4)}{(\beta+½)(\beta-1)^2} - \ldots\right]}{4\beta(\beta+(3-\gamma)/4)} \quad .$$

Multiplying through and cancelling the many common factors we get

$$\nu = \frac{1}{\mu^2}\,\frac{1-½(1-\mu^2)}{(1-\beta)} \quad ,$$

and finally using the approximation $(1-\beta) \approx 1$, the final form of ν is

$$\nu = \frac{(1+\mu^2)}{2\mu^2} \quad . \qquad (51)$$

This gives the behavior we expect , that is $\nu = 1$ when $\theta = 0$.

There were two ways of finding δ from equation (48) these were using q and q_0 or using ψ and ψ_0 ; both ways gave the same answer , but using ψ was the simplest.

$$\delta = -2\left[\frac{\psi}{\psi_0}\right]_{\delta/2}$$

this becomes

$$\delta = -2\frac{1-\nu\mu^2}{-\frac{\nu}{\beta}(1+2\beta)G} \quad ,$$

substituting for ν ,equation (51) we get

$$\delta = \frac{2\beta}{G}\frac{(1-\mu^2)}{(1+\mu^2)} \quad . \tag{52}$$

The coefficients A , B and C were then derived using the above formulas for δ and ν these were

$$A = -\frac{(1-\mu^2)(3+\mu^2)}{2\mu^2} \;, \tag{53}$$

$$B = \frac{(1+\mu^2)(1-\mu^2)}{2\mu^2} \;, \tag{54}$$

$$C = -2\frac{(1-\mu^2)}{(1+\mu^2)}\sigma^2 \;. \tag{55}$$

The REDUCE programs deriving these coefficients are in appendix B

Note that $\sigma = k_x\delta$ where δ has a β/G factor so we can write σ as

$$\sigma = \frac{k_x\beta}{G}\,\frac{2(1-\mu^2)}{(1+\mu^2)} \;. \tag{56}$$

As you can see all the external parameters k_x , β and G all appear only here and we can take these as the one parameter to be put in .

7. Discussion and extensions of solutions obtained so far

Henceforth we will assume that $G>0$ since this is the behaviour most relevant to tokamak plasmas since this is similar to propagation from the low field side .

Looking at the asymptotic formula equation (45) , then u_∞ gives oscillatory solutions for $\eta \ll 0$, and evanescent solutions for $\eta \gg 0$. This is clear since

$$C<0 \quad so \;\; C^{1/2}=i\,|C\,|^{1/2} ,$$

so for $\eta < 0$ the argument of the Bessel function is real. This means that we have an oscillitory solution for $\eta \ll 0$ which is what we were wanting , since this is the side the incident wave would be launched from. We can see that u_0 agrees with u_∞ for $\eta \rightarrow -\infty$. For $\eta>1$ we can see that u_0 , u_1 and u_∞ all become evanescent, which is again what we expect since the incident wave must be absorbed or reflected after encountering the two singularities; there is no oscillatory solutions for $\eta>0$.

We want to match the u_0 and the u_1 solutions at a point midway between the two singularities to give an overall solution.To do this we must first find the correct form of Bessel function for u_1 for $\eta \gg 0$ then analytically continue u_0 and u_1 into the region where they will be matched.

For $\eta<0$ we will take the following convenient form for u_0

$$u_0 = \eta^{3l/2}\left[D_1 H_l^{(1)}(i\kappa_0\eta^{3/2}) + D_2 H_l^{(2)}(i\kappa_0\eta^{3/2})\right] \quad \text{for } \eta < 0$$

where

$$l = (1+B)/3, \quad \kappa_0 = 2|C|^{1/2}/3.$$

D_1 and D_2 are just constants and $H_l^{(1)}$ and $H_l^{(2)}$ are Hankel functions which were chosen because they have plane waves as asymptotic solutions . Note this formula is only applicable for $\eta<0$ since then $\eta^{3/2} = i\,|\eta|^{3/2}$ and the argument of the Bessel function is real.

If however $\eta>0$ then $\eta^{3/2}$ is real,so the argument of the essel function is imaginary and we need to analytically continue u_0 to get a solution valid for $\eta>0$. The formulas for analytically continuing Bessel functions were taken from (ABRAMOWITZ M. and STEGUN I. A.(1968)).

The form of this equation is

$$u_0 = \frac{2}{\pi}\eta^{3l/2}\left[(D_2 - D_1)e^{i\frac{\pi}{2}(1-l)} K_l(\kappa_0\eta^{3/2}) + \pi D_2 I_l(\kappa_0\eta^{3l/2}) e^{il\frac{\pi}{2}}\right] \quad \eta>0 \ . \quad (57)$$

For u_1 we have the general solution

$$u_1 = (\eta-1)^m Z_{\pm m}[\kappa_1(\eta-1)] \quad \eta-1>0 \ .$$

Again because C is negative there is an imaginary term in the argument so the solution has to be a modified Bessel function . The asymptotic solution requires evcanescent behaviour so we only require one solution for u_1 namely a modified Bessel function of the second kind . So we can write it as

$$u_1 = (\eta-1)^m \ K_m(\kappa_1(\eta-1)), \quad (\eta-1) > 0 \ , \tag{58}$$

where

$$m = \tfrac{1}{2}[1-A-B], \quad \kappa_1 = |C|^{\frac{1}{2}} \ ,$$

Similarly for u_1 we have to analytically continue equation (56) to find a solution valid in the region $\eta-1<0$; this is

$$u_1 = (1-\eta)^m \left[K_m(\kappa_1(1-\eta)) - \pi e^{i\frac{\pi}{2}(1+2m)} I_m(\kappa_1(1-\eta)) \right] \quad \eta-1<0 \ . \tag{59}$$

8. Full Reflection Coefficient

To calculate D_1 and D_2 we must now match u_1 and u_0 at some common point . The point $\delta/2$ was chosen since it favours neither singularity and this is where all the parameters and gradients were defined in the modelling . This corresponds to $\eta=\frac{1}{2}$.

Setting $\eta=\frac{1}{2}$ we must match both the functions and their gradients at this point , i.e

$$u_0(\tfrac{1}{2}) = u_1(\tfrac{1}{2}) \qquad \left[\frac{\partial u_0}{\partial \eta}\right]_{\eta=\frac{1}{2}} = \left[\frac{\partial u_1}{\partial \eta}\right]_{\eta=\frac{1}{2}} . \tag{60}$$

To use (60) we must use the analytically continued versions of u_0 and u_1 namely equations (55) and (57) . Then solving the resulting set of simultaneous equations (60) for D_1 and D_2 we may arrive at an expression for the reflection coefficient that is

$$|R|^2 = \frac{D_2 D_2^*}{D_1 D_1^*}.$$

The square of the reflection coefficient was obtained using REDUCE (appendix-C) to do the algebra. The final formula for the reflection coefficient is an extremely large and unwieldy equation giving little information easily on the behaviour of the function as θ varies . Below is given the correct but simplified forms of D_1 and D_2 .

$$D_1 = -2[\sqrt{2}\tau(e^{i(m+1/2)\pi}\pi I_m - K_m)(e^{i\pi l}\pi I_l + e^{i\pi/2}K_l)$$

$$+\kappa_1(e^{i(m+1/2)\pi}\pi I_m - K_m)(e^{i\pi l}\pi I_l' + e^{i\pi/2}K_l')$$

$$-\sqrt{2}\kappa_1(e^{i(m+1/2)\pi}\pi I_m' - K_m')(e^{i\pi l}\pi I_l + e^{i\pi/2}K_l)] \ ,$$

$$D_2 = -(\sqrt{2}\tau K_l - \kappa_1 K_l')(e^{i(m+\frac{1}{2})\pi} I_m \pi - K_m) + \sqrt{2}\kappa_1 K_l (e^{i(m+\frac{1}{2})\pi} I_m' \pi - K_m')$$

where

$$\tau = 3l + 2m$$

and the $'$ denotes the derivative with respect to the whole argument. The arguments were left out mainly to make these formulae more readable; the arguments for the modified Bessel functions of order l and m were $\kappa_0/2\sqrt{2}$ and $\kappa_1/2$ respectively . In the above forms of D_1 and D_2 , κ_0 has been set equal to $2\kappa_1/3$.

9. Approximate reflection formula

Since the full formula for the reflection coefficient is so unwieldy we will now derive an approximate formula for the reflection coefficient suitable for small angles .

First a suitable approximation for the Bessel functions has to be decided on, so we begin by examining the argument of the Bessel functions . For the Bessel functions of order 1, the arguments are given by

$$x = \frac{\kappa_0}{2\sqrt{2}} = \frac{|C|^{1/2}}{3\sqrt{2}} ,$$

and for order m

$$x = \frac{\kappa_1}{2} = \frac{|C|^{1/2}}{2} .$$

If we now limit θ , so that we can use the following approximation for $(\cos\theta)^2$ that is

$$\mu^2 = 1-\theta^2 ,$$

then we can then take C to leading order as

$$C \approx -\theta^2\sigma^2, \quad \sigma \approx \frac{k_x\beta}{G}\theta^2 ,$$

so the Bessel function arguments now become

$$\frac{k_x\beta}{3\sqrt{2}G}\theta^3 \quad \text{and} \quad \frac{k_x\beta}{2G}\theta^3 .$$

To get some idea of the magnitude of these quantities we take

$$k_x \approx \omega / c_a \quad \textit{where } \omega \textit{ is the wave frequency}$$

$$c_a = \frac{B_0}{\mu_0\rho_0} \approx B_0\frac{10^{17}}{\sqrt{n}} ,$$

where n is the number density of the plasma. $1/G$ is effectively the gradient scale length so we may equate this with the tokamak minor diameter $2a$. So $|C|^{1/2}$ is given by

$$|C|^{1/2} \approx \left[2\times10^{-17}\frac{n^{1/2}}{B_0} a\,\omega\beta \right]\theta^3 \approx 2\theta^3 ,$$

so the arguments for the Bessel functions become θ^3 and $8\theta^3$, which for small θ are clearly very small.

So clearly a small argument expansion of the Bessel functions is appropriate in this case ;the expansions are (AMBRAMOWITZ and STEGUN (1968)):

$$K_\nu(z) \approx \tfrac{1}{2}\Gamma(\nu)\left(\frac{z}{2}\right)^{-\nu}, \quad I_\nu(z) \approx \frac{\left(\frac{z}{2}\right)^{\nu}}{\Gamma(1+\nu)} .$$

The small argument expansions for A and B are to leading order in θ are

$$A \approx -2\theta^2 \quad B \approx \theta^2$$

so the orders of the Bessel functions become

$$l \approx \frac{(1+\theta^2)}{2} \approx \frac{1}{2} \quad \text{and} \quad m \approx \frac{(1+\theta^2)}{3} \approx \frac{1}{3}.$$

So again using REDUCE to make the above substitutions in the full formula for the reflection coefficient and arrive at the following approximate formula

$$|R|^2 = \frac{1}{1+b_0\theta^2} , \tag{59}$$

where the b_0 is a positive constant.

This approximate formula clearly gives the behaviour we expect of the reflection coefficient , that is $|R|^2 \to 1$ *as* $\theta \to 0$.

Chapter 4

1. Presentation of Results

Now that the full reflection coefficient has been derived, and has been checked that it gives the correct behaviour for small θ (See appproximate reflection formula, equation (59)). We must now plot the reflection coefficient to see its behaviour because of the complicated form of the equation.

The orders of the modified Bessel functions l , m are dependent only on θ through their dependence on A and B. The arguments of these Bessel functions are however dependent on $|C|$ which not only depends on θ but on k_x , G and β . C is given as

$$|C| = 8\left[\frac{k_x\beta}{G}\right]\left[\frac{1-\mu^2}{1+\mu^2}\right]^3 .$$

Henceforth we will take

$$P = \frac{k_x\beta}{G} ,$$

where P stands for the plasma parameters to be put into the model. P is a non dimensional parameter since β is non dimensional and k_x and G both have dimensions of m^{-1}. The simplest way to plot the reflection coefficient is by plotting the reflection coefficient $|R|^2$ as a function of θ for different values of P ; this gives a set of nested curves along which P is constant (see graph). On the graph the value of P increases in jumps of eight staring on the uppermost curve with $P = 0.01$ and finishing on the last curve with $P = 56.01$.

The modified Bessel functions were calculated using the confluent hypergeometric equation representation, in particular Kummer's function which converged quickly for the range of parameters used.

$$I_\nu(z) = \left[\frac{z}{2}\right]^\nu \frac{e^{-z}}{\Gamma(\nu+1)} M(\nu+\tfrac{1}{2}\ ,\ 2\nu+1\ ,\ ,2z)$$

where ν is the order, and

$$K_\nu(z) = \left[\frac{\pi}{2}\right] \frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)}$$

and Kummer's function $M(a,b,z)$ is

$$M(a,b,z) = 1 + \frac{az}{b} + \frac{(a)_2 z^2}{(b)_2 2!} + \cdots + \frac{(a)_n z^n}{(b)_n n!} + \cdots$$

where

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1) \quad , \quad (a)_0 = 1 \quad ,$$

2. Disscussion of results

To simplify the following analysis we will take $k_x \approx \omega / c_a$,where $c_a = B/\sqrt{\mu_0 \rho_0}$. If we now define ρ_0 in terms of the number density we arrive at the following expression for { P },

$$P \approx \sqrt{n}\ 10^{-19}\ \frac{\omega\beta}{B'} \quad ,$$

remembering that $G = B'/B$. We will not let n vary so the behaviour of the plasma model is now dependent only on ω , β and B'.Listed below are various wave heating modes and their associated frequency ranges.

mode	frequency
Shear Alfven	$1 < \omega < 2$ MHz
Ion cyclotron	$10 < \omega < 120$ MHz

Fast Alfven	$10 < \omega < 120$ MHz
Lower hybrid	$0.5 < \omega < 2.5$ Ghz
Electon cyclotron	$15 < \omega < 300$ GHz

Looking again to the graph and taking all the parameters fixed for the moment except for ω , we can see that as ω increases the reflection coefficient decreases hence the absorption coefficient increases. As can be seen from the above table there are a number of modes for the incoming wave to couple to, of which only a few are shown above except at very low frequencies where there are no modes and hence no absorption.

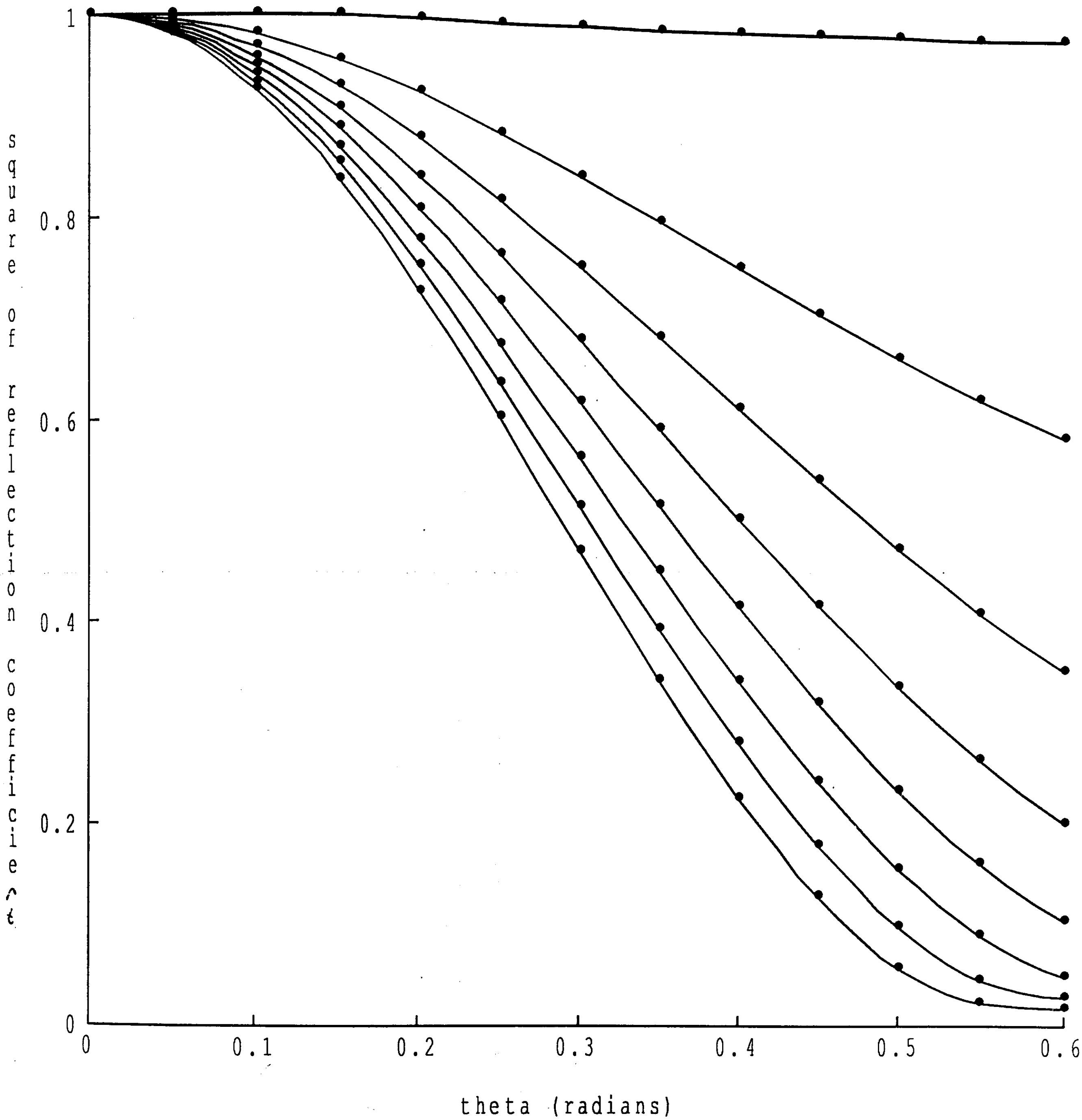
If we now fix all the parameters except for B' we see that for large B' , P is small hence the absorption coefficient is small, and for small B' P is large, hence a large absorption coefficient. If B' is large then the various plasma modes whose frequencies are dependent on the magnetic field, vary significantly across the plasma, so an incident wave of frequency ω will only be able to be resonantly absorbed in a small region of the plasma leading to a small absorption coefficient. We can use a similar argument to prove that a small value of B' would lead to a better absorption coefficient. We can see this behaviour clearly reflected in the graph.

Finally confining our attention to β, β can be related to the efficiency of a fusion plasma at producing energy with a high β (large P) corresponding to a high efficiency and a low β (small P) corresponding to a low efficiency; again we can see this behaviour reflected in the graph.

So in summary the behaviour of the reflection coefficient behaves generally as we would expect as we vary the various plasma parameters and we can see that the inclusion of the second singularity does indeed lead to a significantly increased absorpti on for large angles of θ.

3. Future Work

The next step in this work is to consider more complex magnetic fields; the next up would be to consider a plane stratified spatially rotating magnetic field. As can be easily guessed the mathematics of this problem would become much more involved with computer algebra being required to do the much of the algebra.


1
0.8
0.6
0.4
0.2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
theta (radians)
square of reflection coefficie

appendix A

We non-dimensionalise the coefficients from equation (36) ; we do this by dividing the equation by $\omega^4 c_a^2$ .Equation (36) is

$$\eta(\eta-1)u_{\eta\eta}+(A\eta+B)u_\eta+C\eta^2(\eta-1)u = 0 \quad , \tag{36}$$

where

$$A=\frac{\phi_0-b\psi_0}{\xi_0\psi_0} \quad B=\frac{b\psi_0\delta-a}{\delta\xi_0\psi_0} \quad C=\frac{\delta q_0}{\xi_0}\sigma^2 \ .$$

To non-dimensionalise we divide equation (36) by $\omega^4 c_a^2$; we do this term by term so

$$\frac{q}{w^4} = \frac{\omega^4-k_x^2(c_a^2+c_s^2)\omega^2+k_x^4c_s^2c_a^2\mu^2}{\omega^4} \quad ,$$

becomes after rearranging,

$$q = 1-\frac{k_x^2c_a^2}{\omega^2}(1+\frac{c_s^2}{c_a^2})+\frac{k_x^4c_a^4}{\omega^4}\frac{c_s^2}{c_a^2}\mu^2 \quad ,$$

this finally becomes

$$q = 1-\nu(1+\beta)+\beta\nu^2\mu^2 \quad ,$$

where ν and β are as defined in the main text.

Similar procedures are followed for the other coefficients which become,

$$\xi = 1+\beta-\nu\beta\mu^2 \ ,$$

$$\psi = 1-\nu\mu^2 \ ,$$

$$\lambda = -(a+b\psi)\ ,\ a = 2\nu\beta(1-\mu^2)\ ,\ b = \nu(\beta^2+1-mu^2)\ ,$$

$$\phi = \frac{\tilde{c}_a^2}{c_a^2}\left[(2-\gamma)+2\nu\mu^2(\gamma-2)-2\nu^2\beta\mu^2+2\nu^4\beta\mu^4\right]+\frac{c_s^{2\prime}}{c_a^2}\nu(1-2\nu\mu^2)(1-\mu^2\beta)+\frac{c_a^{2\prime}}{c_a^2}\left[\nu(1+\beta)(1-\mu^2)^2+\nu^4\beta\right] .$$

Now require to calculate

$$\frac{\tilde{c}_a^2}{c_a^2}\ ,\ \frac{c_s^{2\prime}}{c_a^2}\ ,\ \frac{c_a^{2\prime}}{c_a^2}\ .$$

So we begin with $\tilde{c}_a^2$

$$\tilde{c}_a^2 = \frac{(\rho_0 c_a^2)'}{2\rho_0}$$

$$= \frac{\rho_0' c_a^2}{2\rho_0} + \frac{c_a^{2\prime}}{2}$$

$$= \frac{\rho_0' B_0^2}{2\rho_0\mu_0\rho_0} + \frac{1}{2}\left[\frac{B_0^2}{\mu_0\rho_0}\right]$$

$$= \left[\frac{B_0^2}{\mu_0\rho_0}\right]\left[\frac{B_0'}{B_0}\right] .$$

We will let $B_0'/B_0 = G$, so

$$\frac{\tilde{c}_a^2}{c_a^2} = G\ .$$

The derivation for $c_s^{2\prime}$ follows the same lines

$$\frac{c_s^{2\prime}}{c_a^2} = \frac{\mu_0\rho_0}{B_0^2}\left[\gamma\frac{p_0}{\rho_0}\right]' ,$$

$$= \frac{\gamma\mu_0}{B_0^2}\left[p_0' - \frac{\rho_0'}{\rho_0}p_0\right] .$$

Now making use of the expression $\rho_0' = (\rho_0 p_0')/(\gamma p_0)$ and $p_0' = -B_0 B_0'/\mu_0$ we finally arrive at

$$\frac{c_s^{2\prime}}{c_a^2} = (1-\gamma)G \, .$$

Finally we calculate $c_a^{2\prime}$

$$c_a^{2\prime} = \frac{1}{\mu_0}\left[\frac{2B_0 B_0' \rho_0 - B_0^2 \rho_0'}{\rho_0^2}\right]$$

$$= c_a^2 \left[2G - \frac{\rho_0'}{\rho_0}\right] \, .$$

Differentiating $p_0 \rho_0^{-\gamma} = \chi_2$ and $p_0 + B_0^2/(2\mu_0) = \chi_1$ and combining these we can write

$$\frac{\rho_0'}{\rho_0} = \frac{G}{\beta} \, ,$$

so we finally get

$$\frac{c_a^{2\prime}}{c_a^2} = G\left(2 + \frac{1}{\beta}\right) \, .$$

This completes the non-dimensionalised forms for the coefficients.

1. Further proofs

We now go on to prove the following two equations

$$\frac{\partial \nu}{\partial z} = \frac{\nu}{\beta}[1+2\beta]G\,, \quad \frac{\partial \beta}{\partial z} = -[\gamma+2\beta]G\,.$$

Proofs

$$\frac{\partial \nu}{\partial z} = \frac{k_x}{\omega^2} c_a^{2\prime}$$

$$= \frac{k_x}{\omega^2} c_a^2 G \left[2+\frac{1}{\beta}\right]$$

$$= \frac{\nu}{\beta}[1+2\beta]G \quad ,$$

using equation (A3). The next equation is proved similarly

$$\frac{\partial \beta}{\partial z} = \frac{\partial}{\partial z}\frac{c_s^2}{c_a^2}$$

$$= \frac{c_s^{2\prime}}{c_a^2} - \beta\frac{c_a^{2\prime}}{c_a^2}$$

$$= (1-\gamma)G - \beta\frac{G}{\beta}(1+2\beta)$$

$$= [\gamma+2\beta]G\,.$$

Appendix-B

The following REDUCE programs were used to calculate the full coefficients A,B and C ; that is without approximating $\cos\theta$.

In these programs most of the expressions are obvious , but some are not these are ; v is used instead of ν and GG is used instead of G .

1. A-coefficient

```
phi:=-(GG*(2*Beta**3*MU**2*V**2 - Beta**2*GM*MU**2*V**2 + Beta**2*GM*V - 4*Beta**2*MU**4*V**2 + 4*Beta**2*MU**2*V**2 + 4*Beta**2*MU**2*V - 3*Beta**2*V + Beta*GM*MU**4*V**2 - Beta*GM*MU**2*V**2 - 2*Beta*GM*MU**2*V + Beta*GM*V + Beta*GM - 2*Beta*MU**4*V**2 + 2*Beta*MU**2*V**2 + 6*Beta*MU**2*V - 4*Beta*V - 2*Beta + MU**2*V - V))/Beta;

b:=v*(beta^2+1-mu^2);

psi:=1-v*mu^2;

df(v,z):=v*(1+2*beta)*gg/beta;

dpsi:=df(phi,z);

xi:=1+beta-v*beta*mu^2;

acoef:=(phi-b*dpsi)/(xi*dpsi);

v:=(1-(1-mu^2)/2)/mu^2;

acoef; % Now let beta =0 ;

let beta=0;
```

```
acoef;

end;
```

2. B-Coefficient

```
a:=v*beta*(1-v*mu^2);

b:=v*(beta^2+1-mu^2);

psi:=1-mu^2*v;

xi:=1+beta-v*beta*mu^2;

dd:=beta*(1-mu^2)/(gg*(1-(1-mu^2)/2));

df(v,z):=v*(1+2*beta)*gg/beta;

df(beta,z):=-(gamma+2*beta)*gg;

dpsi:=df(psi,z);

bcoef:=(b*dpsi*dd-a)/(dd*xi*dpsi);

v:=(1-(1-mu^2)/2)/mu^2;

ans;

let b^2=0;

ans;

end;
```

3. C-coefficient

```
q:=1-v*(1+beta)+beta*v^2*mu^2;

df(v,z):=v*(1+2*beta)*gg/beta;

df(beta,z):=-(gm+2*beta)*gg;

dq:=df(q,z);

xi:=1+beta-v*beta*mu^2;

dd:=beta*(1-mu^2)/(gg*(1-(1-mu^2)/2));

ccoef:=dd*dq/(xi);

v:=(1-(1-mu^2)/2)/mu^2;

let beta=0;

ccoef;

end;
```

4. Delta

```
%this calculates delta using psi ;

psi:=1-v*mu^2;

df(v,z):=v*(1+2*b)*GG/b;

df(b,z):=(gm+2*b)*gg;

dpsi:=df(phi,z);

dd:=-2*psi/dpsi;

end;
```

Appendix C

1. Calculation of modulus d1 depend kk(l),x;

```
depend kk(m),x;

depend ii(l),x;

depend ii(m),x;

u0:=(2/pi)*x^(3*l/2)*((d2-d1)*exp(i*a*(1-l))*kk(l)+pi*d2*exp(i*l*a)*ii(l));

u1:=(1-x)^m*(kk(m)-pi*exp(i*2*a*(m+1/2))*ii(m));

df(kk(l),x):=3*k0*x^(1/2)*dk(l)/2;

df(ii(l),x):=3*k0*x^(1/2)*di(l)/2;

df(kk(m),x):=k1*dk(m);

df(ii(m),x):=k1*di(m);

du0:=df(u0,x);

du1:=df(u1,x);

li:=solve ({u1=u0,du0=du1},{d1,d2});

d1:=rhs first first li;

factor ii(l),kk(l),ii(m),kk(m);

factor di(l),di(m),dk(m),dk(l);

c1:=(num d1)/(exp(i*a)*(1-x)^m);

3*l-(3*l-2*m)*x:=tau;

f1:=sub(i=cc,c1);

c1s:=sub(cc=-i,f1);
```

```
modc1:=c1*c1s;

operator f,fd;

let pi^2*di(m)^2+dk(m)^2=fd(m);

let pi^2*ii(m)^2+kk(m)^2=ff(m);

on div;

%now have to get rid of the cross terms ;

let 2*m*i*a+i*a=ss;

let 2*l*i*a-i*a=tt;

let e^(ss)=uu;

let e^(tt)=yy;

let yy^2+1=2*yy*cos(tt);

let uu^2+1=2*uu*cos(ss);

off exp;

let 2*cos(tt)*kk(l)*ii(l)*pi+pi^2*ii(l)^2+kk(l)^2=f(l);

modc1;

on exp;

off div;

let uu*yy=qq;

%this is to get rid of the 1/uu,1/yy terms;

part1:=coeffn(num modc1,qq,1);

part2:=coeffn(num modc1,qq,0);

off exp;

%substitute for wronskians;

operator w;

let di(l)*kk(l)-dk(l)*ii(l)=-w(l)/pi;
```

```
let di(m)*kk(m)-dk(m)*ii(m)=-w(m)/pi;

let w(l)=2/k1;

let w(m)=1/(2*sqrt(2)*k0);

let cos(ss)*uu+cos(tt)*yy-1=cos(m*pi)*cos(l*pi)-sin(m*pi)*sin(l*pi);

qw1:=part1;

qw2:=part2;

let x=1/2;

let cos(ss)=-sin(m*pi);

let cos(tt)=sin(l*pi);

let sin(ss)=cos(m*pi);

let sin(tt)=-cos(l*pi);

mc1:=qw1+qw2;

on factor;

on gcd;

mc1;

end;
```

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