FOURIER INTEGRAL TRANSFORMS APPLIED TO BOUNDARY-VALUE PROBLEMS IN THE MATHEMATICAL THEORY OF ELASTICITY.

BY

JAMES FULTON.

A thesis submitted to the University of Glasgow for the degree of Doctor of Philosophy.

1 9 5 0.

PREFACE.

The problems which form the subject matter of this thesis are all concerned with the calculation of the stress set up in a two-dimensional elastic medium by the application of external applied forces, both statical and dynamical, considered as boundary value problems in the mathematical theory of elasticity. The solutions of the partial differential equations involved are obtained by using Fourier integral transform theory. This method of approach which has been used recently by Sneddon (32) to obtain the solutions of a number of problems in elasticity, is direct and yields formal solutions with general types of applied force from which the solutions to problems with special forms of loading may be deduced.

Part I is an account of those parts of the general mathematical theory of elasticity of which use is subsequently made, as well as a statement of some theorems and results from the theory of integral transforms. In Parts II and III formal solutions are derived for the problems of an infinite two-dimensional elastic medium with forces applied to its interior, and of the semi-infinite elastic solid and infinite elastic strip with boundary loading. While the results obtained are not new there are certain novel features in the method of solution. These general results are applied in Part IV to the case of forces applied to the interior of a semi-infinite elastic solid and the solution is obtained for a more general problem of this type than that considered previously by Sneddon (28).

The methods employed in Parts II and III are extended in Parts V and VI to problems in which the applied forces vary with time and among the special cases considered are some new solutions. In particular the problem of an impulsive force applied to the interior of an infinite two-dimensional elastic medium, and of applied forces moving with uniform velocity do not appear to have been dealt with previously. Numerical calculations have been made at various points to illustrate effects which appear to have some practical interest.

Fourier transforms can also be used in problems involving circular symmetry and while there are many interesting applications of this type of problem it has not been found possible at this stage to do more than give the analysis of a few problems and this forms the substance of Parts VII and VIII.

I should like to express my thanks to Professor I.N. Sneddon who suggested these problems to me as a subject of study and with whom I had many valuable conversations, and to Professor R.O. Street under whose supervision the work was carried out.

1950

CONTENTS.

PART VIII. THE APPLICATION OF DYNAMICAL FORCES TO THE BOUNDING SURFACES OF CIRCULARLY CYLINDRICAL ELASTIC SOLIDS.

PART I.

THE MATHEMATICAL THEORY OF ELASTICITY.

I. THE MATHEMATICAL THEORY OF ELASTICITY.

1.1 Historical résumé.

A knowledge of the behaviour of a solid body when acted upon by externally applied forces is of paramount importance in the design of engineering structures since the strength of the structure is dependent upon the resistance to rupture of the material comprising it. The realisation of this fact provided the impetus for an investigation into the strength of materials, one outcome of which was the formulation of the mathematical theory of elasticity. Technical advance was not however the sole stimulus for the subsequent development and much progress resulted from the academic urge shown during the golden age of mathematical physics to obtain an understanding of the material universe.

The first mathematician to concern himself with the problem of the strength of materials was Galileo when in 1638 he attempted to determine the effects of loading a beam one end of which was built into a wall. He did not consider the beam to be elastic since he had not the conception of displacement, but his work was the beginning of an investigation subsequently pursued by most of the great mathematical physicists. The experimental work of Hooke, circa 1678 and later by Young in 1807 and the theoretical work of Navier, Euler, Bernoulli and Cauchy culminated in the formulation by Cauchy in 1820 of the equations of elasticity. After this Cauchy, Clapeyron, Green and Kelvin, to mention only the greatest, applied the general theory to particular problems, greatest effort being directed towards obtaining solutions of the equations of equilibrium. Dynamical problems were not, however, entirely neglected and many problems relating to the free vibrations of elastic bodies were considered. Poisson deduced from the equations of motion that a disturbance would be transmitted through an isotropic elastic medium by two kinds of waves, waves of irrotational dilation, and waves of equivoluminal rotation, each type travelling with a different velocity. Dynamical problems however did not receive the same extensive treatment as statical problems, due no doubt to the fact that the additional independent variable in the equations of motion renders their solution more recondite.

Attention has also been given to problems in aeolotropic elasticity, but the complex nature of the equations involved makes the analysis extremely difficult.

In recent years attention has been focussed mainly on the development of methods of solving the equations of equilibrium and on the formulation of a mathematical theory of plasticity which will account of the behaviour of materials stressed beyond the elastic limit.

1.2 Specification of stress and displacement.

A solid body acted upon by externally applied forces may be considered to be rigid, elastic, or plastic. A rigid body is one in which there is no relative movement between the elements comprising the body. In an elastic body there is relative movement within the body but of a reversible nature so that when the forces producing the deformation are removed the body returns to its original configuration. This is not the case in a plastic medium; there the material does not return to its original state but is permanently deformed. In the problems considered here we assume perfect elasticity. This assumes that we neglect translational motion of the body as a whole, and that the applied forces are not great enough to produce plastic flow.

An elastic body when deformed by external forces is said to be in a state of stress, the stress being specified at every point in the body by the stress tensor S. If we refer a point P in the body to the rectangular Cartesian axes OX, OY, OZ by means of the co-ordinates (x,y,z) then the stress S may be denoted by

$$S = \begin{pmatrix} \sigma_x & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{pmatrix} \qquad (1.21)$$

where σ_x is the normal stress at P across a plane perpendicular to the axis OX, and τ_{xy} is the shearing stress across this plane in the direction of OY. Similar interpretations can be given to the other components. It may readily be shown that the tensor is symmetric.

Associated with this tensor is the displacement vector

$$\underset{\sim}{D} = (u, v, w)$$

where u, v and w are the components in the directions of the co-ordinate axes. It is sometimes convenient to specify the deformation by the strain tensor E given in this case by

$$E = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) & \frac{\partial w}{\partial z} \end{pmatrix} \tag{1.22}$$

where $\frac{\partial u}{\partial x}$ is the unit elongation in the direction OX and $\frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$ is a measure of the angular distortion about OZ of a plane perpendicular to OZ.

1·3 The equations of elasticity.

The experimental work of Hooke led to the enunciation in 1660 of Hooke's law which states that in an elastic body in a state of pure tension the stress is proportional to the strain within the elastic limits. This led to the assumption in the mathematical theory that, in general, the components of stress were linear functions of the components of strain. Subsequent work showed that for an isotropic medium only two **elastic** sonstants were involved, so that the stress-strain relations can be written in the form

$$\sigma_x = \lambda\Delta + 2\mu\frac{\partial u}{\partial x}, \quad \sigma_y = \lambda\Delta + 2\mu\frac{\partial v}{\partial y},$$

$$\sigma_z = \lambda\Delta + 2\mu\frac{\partial w}{\partial z},$$

$$\tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \quad \tau_{yz} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \qquad (1.31)$$

$$\tau_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

where Δ is the dilation defined by

$$\Delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

and λ and μ are the Lamé elastic constants. In practical applications it is more usual to use two other constants, namely E, Young's modulus and σ, Poisson's ration, related to the Lamé constants by means of the equations

$$\lambda = \frac{\sigma E}{(1+\sigma)(1-2\sigma)}, \quad \mu = \frac{E}{2(1+\sigma)} \qquad (1.32)$$

A further set of equations, the equations of motion, may be obtained by considering the motion of a small element of an elastic body in a state of stress due to applied forces which vary with time. If the forces are independent of time, so that the problem is statical, we consider the equilibrium of the element and obtain the equations of equilibrium. If we assume a body force (X,Y,Z) per unit mass acting throughout an elastic medium of density ρ it is readily shown*(21) that the equations of motion are

$$\begin{aligned}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + \rho X &= \rho \frac{\partial^2 u}{\partial t^2} \\
\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \rho Y &= \rho \frac{\partial^2 v}{\partial t^2} \qquad (1.33) \\
\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + \rho Z &= \rho \frac{\partial^2 w}{\partial t^2}
\end{aligned}$$

and the equations of equilibrium (1·34) are obtained by equating the left-hand side of these equations to zero.

If the elastic medium is bounded, then in order that equilibrium be maintained at the surface, the surface force per unit area at a point on the boundary must equal the internal stress at that point. This requirement gives rise to the boundary conditions.

*(21). p.83

1·4 Two-dimensional stress systems.

There are two types of two-dimensional problem in elasticity, plane strain and plane stress. If we assume that the body extends indefinitely in bothdirections perpendicular to the xy-plane, and that the applied forces are uniformly distributed through-out the body in this direction, then all planes parallel to the xy-plane before deformation remain parallel after deformation, so that $w = 0$, and u and v do not vary with z. The stress-strain relations then become

$$\sigma_x = \lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) + 2\mu\frac{\partial u}{\partial x}, \quad \sigma_y = \lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) + 2\mu\frac{\partial v}{\partial y},$$
$$\tau_{xy} = \mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right), \quad \sigma_z = \lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right), \quad \tau_{xz} = \tau_{yz} = 0 \qquad (1.41)$$

and if we assume furthermore that the component of body force Z is zero, then the equations of motion become

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \rho X = \rho\frac{\partial^2 u}{\partial t^2},$$
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \rho Y = \rho\frac{\partial^2 v}{\partial t^2}, \quad \frac{\partial \sigma_z}{\partial z} = 0 \qquad (1.42)$$

so that the stress and displacement are completely determined when σ_x, σ_y, τ_{xy}, u, and v are known. A similar set of equations (1·43) are obtained for the equations of equilibrium by equating the left-hand sides of (1·42) to zero.

If one the other hand we assume that the body is a plate parallel to the xy-plane, then we may assume that σ_z, τ_{xz}, and τ_{yz} are zero. From the stress-strain relations we have

$$0=\lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+2\mu\frac{\partial w}{\partial z} \tag{1.44}$$

so that when we substitute for $\frac{\partial w}{\partial z}$ from (1.44) into the remaining relations in (1.41) we get

$$\sigma_x=\lambda'\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+2\mu\frac{\partial u}{\partial x}$$

$$\sigma_y=\lambda'\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+2\mu\frac{\partial v}{\partial y} \tag{1.45}$$

$$\sigma_z=\lambda'\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

where $\lambda'=\frac{2\lambda\mu}{\lambda+2\mu}$. The equations of motion are the same as for plane strain.

We shall assume, therefore, that the problems considered here satisfy the conditions for plane strain. The solution for the corresponding problem in plane stress may readily be obtained by substituting λ' for λ in the solution.

1.5 Circular symmetry.

When dealing with a solid of revolution it is more convenient to employ the cylindrical polar co-ordinates (r, θ, z), to specify a point in the medium, the axis of symmetry being taken as the axis OZ. The stress tensor in this case is given by

$$S = \begin{pmatrix} \sigma_r & \tau_{\theta r} & \tau_{zr} \\ \tau_{r\theta} & \sigma_\theta & \tau_{z\theta} \\ \tau_{rz} & \tau_{\theta z} & \sigma_z \end{pmatrix} \tag{1.51}$$

where σ_r is the normal stress at a point across a plane perpendicular to the radius vector through the point, and $\tau_{r\theta}$ is the shearing stress at the point in the direction of the radius vector across a plane passing through the point and the axis OZ. Similar interpretations may be given to the other components.

If U, V, and w denote the components of the displacement vector in the directions of r, θ, and z then we have

$$\underset{\sim}{D} = (U, V, w)$$

and it may readily be shown*(21) that the strain tensor is given by

$$E = \begin{pmatrix} \frac{\partial U}{\partial r} & \frac{1}{r}\frac{\partial U}{\partial \theta} + \frac{\partial V}{\partial r} - \frac{V}{r} & \frac{\partial U}{\partial z} + \frac{\partial w}{\partial r} \\ \frac{\partial V}{\partial r} - \frac{V}{r} + \frac{1}{r}\frac{\partial U}{\partial \theta} & \frac{1}{r}\frac{\partial V}{\partial \theta} + \frac{U}{r} & \frac{\partial V}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta} \\ \frac{\partial w}{\partial r} + \frac{\partial U}{\partial z} & \frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{\partial V}{\partial z} & \frac{\partial w}{\partial z} \end{pmatrix} \tag{1.52}$$

*(21) p.56

The stress-strain relations in this case are

$$\sigma_r = \lambda\Delta + 2\mu\frac{\partial U}{\partial r}, \qquad \sigma_\theta = \lambda\Delta + 2\mu\left(\frac{1}{r}\frac{\partial V}{\partial \theta} + \frac{U}{r}\right),$$

$$\sigma_z = \lambda\Delta + 2\mu\frac{\partial w}{\partial z}$$

$$\tau_{r\theta} = \mu\left(\frac{\partial V}{\partial r} - \frac{V}{r} + \frac{1}{r}\frac{\partial U}{\partial \theta}\right), \quad \tau_{\theta z} = \mu\left(\frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{\partial V}{\partial z}\right), \tag{1.53}$$

$$\tau_{zr} = \mu\left(\frac{\partial w}{\partial r} + \frac{\partial U}{\partial z}\right)$$

where $\Delta = \frac{\partial U}{\partial r} + \frac{U}{r} + \frac{1}{r}\frac{\partial V}{\partial \theta} + \frac{\partial w}{\partial z}$

In the particular problems dealt with here, we shall be concerned with deformation produced by forces applied symmetrically with respect to the axis of symmetry, so that $V = 0$, and $\frac{\partial}{\partial \theta} = 0$. Equations (1.53) then reduce to

$$\sigma_r = \lambda\left(\frac{\partial U}{\partial r} + \frac{U}{r} + \frac{\partial w}{\partial z}\right) + 2\mu\frac{\partial U}{\partial r}, \quad \sigma_\theta = \lambda\left(\frac{\partial U}{\partial r} + \frac{U}{r} + \frac{\partial w}{\partial z}\right) + 2\mu\frac{U}{r},$$

$$\sigma_z = \lambda\left(\frac{\partial U}{\partial r} + \frac{U}{r} + \frac{\partial w}{\partial z}\right) + 2\mu\frac{\partial w}{\partial z}, \tag{1.54}$$

$$\tau_{rz} = \mu\left(\frac{\partial w}{\partial r} + \frac{\partial U}{\partial z}\right), \quad \tau_{r\theta} = \tau_{\theta z} = 0$$

so that the problem is a two-dimensional one in the sense that only two space co-ordinates are involved in the equations.

The equations of motion are given by*(21)

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} + \rho R = \rho \frac{\partial^2 U}{\partial t^2}$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} + \rho Z = \rho \frac{\partial^2 V}{\partial t^2} \tag{1.55}$$

where R and Z are the components of body-force per unit mass in the directions of r and z respectively and ρ is the density of the material. The equations of equilibrium in this co-ordinate system (1·56) are obtained as before, by equating the left-hand side of these equations to zero.

In the problems considered here the applied forces will be applied symmetrically to the surfaces of circular cylinders whose axis is the z-axis.

*(21) p.89

1.6 Methods of solution.

Fundamentally, the problem of determining the state of stress is an elastic body due to applied forces, is one of finding analytical functions for the components of stress and displacement which satisfy the equations of equilibrium in the case of statical problems and the equations of motion in the case of dynamical problems along with the boundary conditions, and which give the appropriate stress resultants around any force nuclei within the body. It may be shown that such a solution is unique.

The first solutions of special problems were obtained by taking particular solutions of the equations of equilibrium; solutions of more complicated problems were obtained by taking a series of solutions of the fundamental type, and determining the constants of the series so as to satisfy the prescribed boundary conditions.

A further development in two-dimensional problems came in the introduction by Airy (1) of a single function, in terms of which the stress components could be expressed as second derivatives and which was itself a solution of the biharmonic equation. The function is analagous to the potential function of classical hydrodynamical and electrical theory, but due to the fact that the biharmonic function is of a more complicated nature than Laplace's equation, the same extensive results as were obtained in other branches of mathematical physics were not forthcoming in elasticity. However, by choosing the appropriate form of stress function, many special problems were solved by this method: in particular, polynomial solutions of the biharmonic equation give certain stress distributions in rectangular plates. The stress distribution in rectangular beams under various forms of loading were obtained by choosing an elementary solution of trigonometric form and forming a series of such solutions.

The success which attended the application of complex variable theory to two-dimensional problems in hydrodynamics and allied subjects, led several writers to consider the possibility of a similar approach to statical problems in two-dimensional elasticity. The earliest methods developed were restricted in their application for various reasons but recently Stevenson (33) and Green (10) have produced methods which, although differing from one another in detail, have overcome many of the difficulties inherent in the original methods. In particular, by means of conformal transformations, a much wider class of problem becomes tractable. These methods are, however, restricted to two-dimensional problems, and while they constitute a great advance, their application calls for a great deal of mathematical intuition, particularly in the more complicated problems, solutions of which are constructed by a synthesis of elementary solutions. In fact, much of the difficulty with problems in elasticity is due to the lack of a direct approach: each special problem calling for a high degree of intuitive reasoning.

Sneddon (32) has succeeded in overcoming this difficulty to some extent in several papers on which he employs integral transforms to solve the partial differential equations involved and so obtains solutions which have a higher degree of generality, than those obtained previously and obtains them in a more direct fashion. The essence of the method is the well-known property of integral transforms which will reduce the number of independent variables in a partial differential equation by unity, provided the variable being removed is defined throughout the range $-\infty$ to $+\infty$. Thus if these conditions are satisfied, a partial differential equation in two variables may be reduced to an ordinary differential equation. For example, if we multiply $\frac{\partial^2}{\partial x^2} f(x,y)$ by $\exp i\xi x$ and integrate with respect to x over the range $-\infty$ to $+\infty$, assuming that $\frac{\partial f}{\partial x} e^{i\xi x}$, and $f \cdot e^{i\xi x}$ vanish at both limits of integration, we find that

$$\int_{-\infty}^{\infty} \frac{\partial^2}{\partial x^2} f(x,y) \cdot e^{i\xi x} dx = -\xi^2 \int_{-\infty}^{\infty} f(x,y) e^{i\xi x} dx$$

Thus if we multiply the partial differential equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f(x,y) = 0 \qquad (1.61)$$

by $\exp i\xi x$ and integrate over the entire range of the variable, we obtain

$$\left(\frac{d^2}{dy^2} - \xi^2\right)\bar{f}(\xi, y) = 0 \tag{1.62}$$

where we have written

$$\bar{f}(\xi, y) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} f(x, y)\, e^{i\xi x}\, dx$$

This is now an ordinary differential equation whose solution may be written down immediately as

$$\bar{f}(\xi, y) = A(\xi)\, e^{-\xi y} + B(\xi)\, e^{\xi y} \tag{1.63}$$

Where A and B are arbitrary functions of ξ, which may be chosen so as to satisfy certain prescribed conditions. The required solution of (1·61) then follows by inverting (1·63) using Fourier's inversion theorem

$$f(x, y) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} \bar{f}(\xi, y)\, e^{-i\xi x}\, d\xi$$

In some cases when more than one variable is defined over the range $-\infty$ to $+\infty$, a multiple transform may be used. For example, if we consider the equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f(x, y) = F(x, y) \tag{1.64}$$

and multiply both sides of the equation by $\exp i\,(\xi x + \eta y)$

and integrate over the entire xy-plane, the equation reduces to the ordinary algebraic equation

$$-(\xi^2+\eta^2)\,\bar{f}(\xi,\eta) = \bar{F}(\xi,\eta)$$

where we have written

$$\bar{f}(\xi,\eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{i(\xi x+\eta y)}\,dx\,dy$$

and

$$\bar{F}(\xi,\eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x,y)\, e^{i(\xi x+\eta y)}\,dx\,dy$$

Solving for $\bar{f}(\xi,\eta)$ and applying Fourier's inversion theorem for two-dimensional transforms

$$f(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{f}(\xi,\eta)\, e^{-i(\xi x+\eta y)}\,d\xi\,d\eta$$

we obtain a formal solution of (1.64) in the form of the double integral

$$f(x,y) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\bar{F}(\xi,\eta)}{\xi^2+\eta^2}\, e^{-i(\xi x+\eta y)}\,d\xi\,d\eta$$

which thus yields a solution provided the integral is convergent

Use has been made of Hankel (32) and Mellin (36) transforms as well as Fourier transforms to solve problems in elasticity, the nature of the problem and the co-ordinate system used determining the form of the kernel used. This approach is not restricted to two-dimensional elasticity but may with equal facility be applied to three dimensional problems. The method has also been applied to problems in which the applied forces vary with time and we shall consider some two-dimensional problems of this type in the later sections of this work.

1·7 Fourier integral transform theory.

It will be convenient at this stage to state some theorems associated with Fourier integral transforms of which frequent use will be made in the subsequent sections.

Theorem. If $\bar{f}(\xi)$ is defined by the integral

$$\bar{f}(\xi) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} f(x) e^{i\xi x} dx.$$

then

$$f(x) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \bar{f}(\xi) e^{-i\xi x} d\xi \qquad (I)$$

$\bar{f}(\xi)$ is the Fourier transform of $f(x)$, and the theorem as stated is the inversion theorem for Fourier transforms. In this form the theorem includes as special cases the Fourier cosine and Fourier sine transforms.* (35)

*(35) p.5

We shall also make use of the theorem in the following form

Theorem If $f(x) = 0$, for $x < 0$, and ξ is complex with a positive imaginary part, then

$$\bar{f}(\xi) = \frac{1}{(2\pi)^{1/2}} \int_0^\infty f(x) e^{i\xi x} dx$$

and

$$f(x) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty+i\gamma}^{\infty+i\gamma} \bar{f}(\xi) e^{-i\xi x} d\xi \qquad \text{(II)}$$

where γ is positive. (35)

Theorem. If $\bar{f}(\xi)$ and $\bar{g}(\xi)$ are the Fourier transforms of $f(x)$ and $g(x)$ then

$$\int_{-\infty}^{\infty} \bar{f}(\xi)\, \bar{g}(\xi) e^{-i\xi x} d\xi = \int_{-\infty}^{\infty} f(\alpha)\, g(x-\alpha)\, d\alpha. \qquad \text{(III)}$$

This is the faltung theorem for Fourier transforms (35)

Theorems I and III may be extended to functions of several variables* (35) in which case we have

Theorem If $\bar{f}(\xi_1, \xi_2, \ldots \xi_n)$ is defined by the relation

$$\bar{f}(\xi_1, \xi_2, \ldots \xi_n) = \left(\frac{1}{2\pi}\right)^{n/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, x_2, \ldots x_n) e^{i(\xi_1 x_1 + \cdots + \xi_n x_n)} dx_1 \cdots dx_n.$$

then

$$f(x_1, x_2, \ldots x_n) = \left(\frac{1}{2\pi}\right)^{n/2} \int_{-\infty} \cdots \int_{-\infty} \bar{f}(\xi_1, \xi_2, \ldots \xi_n) e^{-i(\xi_1 x_1 + \cdots + \xi_n x_n)} d\xi_1 \cdots d\xi_n \qquad \text{(IV)}$$

and the n-dimensional analogue of (III) is

* (35) p.50

<u>Theorem</u> If $\bar{f}(\xi_1, \xi_2, \cdots \xi_n)$ and $\bar{g}(\xi_1, \xi_2, \cdots \xi_n)$ are n-dimensional Fourier transforms of $f(x_1, x_2, \cdots x_n)$ and $g(x_1, x_2, \cdots x_n)$, then

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \bar{f}(\xi_1, \xi_2, \cdots \xi_n)\, \bar{g}(\xi_1, \xi_2, \cdots \xi_n)\, e^{i(\xi_1 x_1 + \cdots \xi_n x_n)}\, d\xi_1 d\xi_2 \cdots d\xi_n$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(\alpha_1, \alpha_2, \cdots \alpha_n)\, g(x_1 - \alpha_1, x_2 - \alpha_2, \cdots \cdots, x_n - \alpha_n)\, d\alpha_1 \cdots d\alpha_n \qquad (\text{V})$$

A function of which we shall make a great deal of use is the Dirac δ-function defined by

$$\begin{aligned} \delta(x) &= 0, \quad x \neq 0 \\ &= 1, \quad x = 0 \end{aligned} \qquad (1.71)$$

and we shall state here some results relating to $\delta(x)$

$$\int_{-\infty}^{\infty} f(x)\, \delta(x)\, dx = f(0) \qquad (\text{VI})$$

$$\int_{-\infty}^{\infty} f(x) \cdot \delta(x - a)\, dx = f(a) \qquad (\text{VII})$$

$$\int_{-\infty}^{\infty} e^{\pm i\xi x}\, dx = 2\pi\, \delta(\xi) \qquad (\text{VIII})$$

Two further results which will have application are:-

$$\int_{-\infty}^{\infty} f(x) e^{\pm i\xi x}\, dx = 2 \int_{0}^{\infty} f(x) \cos \xi x\, dx. \qquad (\text{IX})$$

if $f(x)$ is an even function of x, and

$$\int_{-\infty}^{\infty} f(x) e^{\pm i\xi x}\, dx = \pm 2i \int_{0}^{\infty} f(x) \sin \xi x\, dx \qquad (\text{X})$$

if $f(x)$ is an odd function of x.

PART II.

THE APPLICATION OF STATICAL FORCES TO THE INTERIOR OF AN INFINITE TWO-DIMENSIONAL ELASTIC MEDIUM.

II. THE APPLICATION OF STATICAL FORCES TO THE INTERIOR OF AN INFINITE TWO-DIMENSIONAL ELASTIC MEDIUM.

2·1 Introduction.

The determination of the stress in an infinite elastic medium due to a point force applied to the interior is a classical problem in the theory of elasticity, and the solution of the two-dimensional analogue of this problem has been obtained by various methods. Complex variable theory readily yields the solution through the correspondence of the point force to a singularity of the potential function from which the stress distribution is derived.

In this section we shall employ the two-dimensional Fourier transform to obtain a general solution of the equations of equilibrium, applicable to an infinite two-dimensional medium with forces applied to its interior and as special cases of the general solution we shall consider the stress distribution due to a point force acting at the origin and to a force uniformly distributed along a line.

2.2 Solution of the equations of equilibrium

Consider an elastic medium extending to infinity in all directions and in a state of stress due to external forces applied to its interior. If we assume that the forces are uniformly applied throughout the thickness of the medium, the problem is two-dimensional and is one of plane strain.

Let us set up the rectangular Cartesian axes OX, OY so that the component of the displacement vector in the direction perpendicular to the co-ordinate plane is zero, and the applied forces are in the directions parallel to the co-ordinate plane. The components of the stress tensor must satisfy the equations of equilibrium (1.43) while the components of the displacement vector are related to the components of the stress tensor by means of the equations (1.41)

We introduce now two functions $\phi(x,y)$, and $\psi(x,y)$ such that

$$u = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y} \quad , \quad v = \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x} \qquad (2.201)$$

and substitute for u and v in equations (1.41) to obtain the components of stress in terms of ϕ and ψ. It is easily shown that these are

$$\left(\frac{1+\sigma}{E}\right)\sigma_x = \frac{\sigma}{1-2\sigma}\nabla_1^2\phi + \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\psi}{\partial x\partial y} \quad , \quad \left(\frac{1+\sigma}{E}\right)\sigma_y = \frac{\sigma}{1-2\sigma}\nabla_1^2\phi + \frac{\partial^2\phi}{\partial y^2} - \frac{\partial^2\psi}{\partial x\partial y} \, ,$$

$$\left(\frac{1+\sigma}{E}\right)\tau_{xy} = \frac{\partial^2\phi}{\partial x\partial y} - \frac{1}{2}\left(\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2}\right) \qquad (2.202)$$

where we have expressed λ and μ in terms of σ and E, and we have written $\nabla_1^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ for the two-dimensional Laplacian operator.

We now write

$$X = \frac{\partial \Phi}{\partial x} + \frac{\partial \Psi}{\partial y} \quad , \quad Y = \frac{\partial \Phi}{\partial y} - \frac{\partial \Psi}{\partial x} \tag{2.203}$$

and substitute from equations (2·202) and (2·203) into the equations of equilibrium (1·43) which then may be written in the form

$$\begin{aligned} &\frac{\partial}{\partial x}\left\{\frac{E(1-\sigma)}{(1+\sigma)(1-2\sigma)}\nabla_1^2\phi + \rho\Phi\right\} + \frac{\partial}{\partial y}\left\{\frac{E}{2(1+\sigma)}\nabla_1^2\psi + \rho\Psi\right\} = 0 \\ &\frac{\partial}{\partial y}\left\{\frac{E(1-\sigma)}{(1+\sigma)(1-2\sigma)}\nabla_1^2\phi + \rho\Phi\right\} - \frac{\partial}{\partial x}\left\{\frac{E}{2(1+\sigma)}\nabla_1^2\psi + \rho\Psi\right\} = 0 \end{aligned} \tag{2.204}$$

These equations will be satisfied by solutions of the two non-homogeneous Laplacian equations

$$\begin{aligned} &\nabla_1^2\phi + \frac{\rho(1+\sigma)(1-2\sigma)}{E(1-\sigma)}\Phi = 0 \\ &\nabla_1^2\psi + 2\rho\frac{(1+\sigma)}{E}\Psi = 0 \end{aligned} \tag{2.205}$$

To obtain solutions of (2·205) we introduce the two-dimensional Fourier transforms of the functions ϕ and ψ, defined by the integrals

$$\begin{aligned} &\bar{\Phi}(\xi,\eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(x,y)\, e^{i(\xi x+\eta y)}\,dx\,dy \\ &\bar{\Psi}(\xi,\eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi(x,y)\, e^{i(\xi x+\eta y)}\,dx\,dy \end{aligned} \tag{2.206}$$

Multiplying equations (2·205) throughout by exp. $i(\xi x + \eta y)$ and integrating over the entire xy-plane, the partial differential equations (2·205) reduce to the ordinary algebraic equations

$$-(\xi^2+\eta^2)\bar{\Phi} + \rho\frac{(1+\sigma)(1-2\sigma)}{E(1-\sigma)}\bar{\bar{\Phi}} = 0$$

$$-(\xi^2+\eta^2)\bar{\Psi} + 2\rho\frac{(1+\sigma)}{E}\bar{\bar{\Psi}} = 0 \tag{2.207}$$

where we have assumed that $\emptyset$ and Ψ and their derivatives vanish at both limits of integration and $\bar{\bar{\emptyset}}$ and $\bar{\bar{\Psi}}$ are defined by integrals similar to (2·206). Transforming equations (2·203) in a similar manner we obtain the relations

$$\bar{X} = -i\xi\bar{\bar{\Phi}} - i\eta\bar{\bar{\Psi}} \quad , \bar{Y} = -i\eta\bar{\bar{\Phi}} + i\xi\bar{\bar{\Psi}} \tag{2.208}$$

Now if we solve equations (2·208) for $\bar{\bar{\emptyset}}$ and $\bar{\bar{\Psi}}$ and substitute into equations (2·207) we get $\bar{\emptyset}$ and $\bar{\Psi}$ in terms of $\bar{X}$ and $\bar{Y}$ as follows

$$\bar{\Phi} = \rho\frac{(1+\sigma)(1-2\sigma)}{E(1-\sigma)}\cdot\frac{i\xi\bar{X} + i\eta\bar{Y}}{(\xi^2+\eta^2)^2}$$

$$\bar{\Psi} = 2\rho\frac{(1+\sigma)}{E}\cdot\frac{i\eta\bar{X} - i\xi\bar{Y}}{(\xi^2+\eta^2)^2} \tag{2.209}$$

The transformed components of stress and displacement may be obtained by multiplying equations (2·201) and (2·202) throughout by exp. $i(\xi x + \eta y)$ and integrating over the entire xy-plane, assuming as before that all quantities and their derivatives vanish at the limits of integration.

Using equations (2.209) to eliminate $\bar{\emptyset}$ and $\bar{\Psi}$ we find the transformed components of stress and displacement to be given by the equations

$$\bar{\sigma}_x + \bar{\sigma}_y = -\frac{1}{1-\sigma}\rho\left\{\frac{i\xi\bar{X} + i\eta\bar{Y}}{\xi^2+\eta^2}\right\} \tag{2.210a}$$

$$\bar{\sigma}_x - \bar{\sigma}_y = -\left(\frac{1-2\sigma}{1-\sigma}\right)\rho\left\{\frac{(\xi^2-\eta^2)(i\xi\bar{X}+i\eta\bar{Y})}{(\xi^2+\eta^2)^2}\right\} - 4\rho\left\{\frac{\xi\eta(i\eta\bar{X}-i\xi\bar{Y})}{(\xi^2+\eta^2)^2}\right\}$$

$$\bar{\tau}_{xy} = -\left(\frac{1-2\sigma}{1-\sigma}\right)\rho\left\{\frac{\xi\eta(i\xi\bar{X}+i\eta\bar{Y})}{(\xi^2+\eta^2)^2}\right\} + \rho\left\{\frac{(\xi^2-\eta^2)(i\eta\bar{X}-i\xi\bar{Y})}{(\xi^2+\eta^2)^2}\right\}$$

(2.210)

$$\bar{u} = \frac{(1+\sigma)(1-2\sigma)}{E(1-\sigma)}\rho\left\{\frac{\xi(\xi\bar{X}+\eta\bar{Y})}{(\xi^2+\eta^2)^2}\right\} + \frac{2(1+\sigma)}{E}\rho\left\{\frac{\eta(\eta\bar{X}-\xi\bar{Y})}{(\xi^2+\eta^2)^2}\right\}$$

$$\bar{v} = \frac{(1+\sigma)(1-2\sigma)}{E(1-\sigma)}\rho\left\{\frac{\eta(\xi\bar{X}+\eta\bar{Y})}{(\xi^2+\eta^2)^2}\right\} - \frac{2(1+\sigma)}{E}\rho\left\{\frac{\xi(\eta\bar{X}-\xi\bar{Y})}{(\xi^2+\eta^2)^2}\right\}$$

The components of the stress tensor and the displacement vector may be obtained now by applying Fourier's inversion theorem (III) for two-dimensional transforms. Performing this operation on equation (2·210a) we obtain

$$\sigma_x + \sigma_y = -\frac{1}{2\pi(1-\sigma)}\rho\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{i\xi\bar{X}+i\eta\bar{Y}}{\xi^2+\eta^2}e^{-i(\xi x+\eta y)}\,d\xi\,d\eta.$$

Now

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{i\xi}{\xi^2+\eta^2}e^{-i(\xi x+\eta y)}\,d\xi\,d\eta = 4\int_0^{\infty}\xi\sin\xi x\,d\xi\int_0^{\infty}\frac{\cos\eta y}{\xi^2+\eta^2}\,d\eta$$

$$= \frac{2\pi x}{x^2+y^2}$$

using a result given in [*](7) so that $\frac{i\xi}{\xi^2+\eta^2}$ is the two-dimensional Fourier transform of $\frac{x}{x^2+y^2}$. Similarly it may be shown that $\frac{i\eta}{\xi^2+\eta^2}$ is the Fourier transform of $\frac{y}{x^2+y^2}$ so that when we apply the faltung theorem for two-dimensional transforms (IV) equation (2·211) becomes

$$\sigma_x + \sigma_y = -\frac{1}{2\pi(1-\sigma)}\rho \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{(x-\alpha)X(\alpha,\beta) + (y-\beta)Y(\alpha,\beta)}{(x-\alpha)^2 + (y-\beta)^2}\, d\alpha\, d\beta \qquad (2.212a)$$

The list of Fourier transforms at the end of the section may be proved in a similar manner and may be used in conjunction with (2·210) to yield the following expressions for the other components of stress

$$\sigma_x - \sigma_y = -\frac{(1-2\sigma)}{\pi(1-\sigma)}\rho \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{(x-\alpha)(y-\beta)\{(y-\beta)X(\alpha,\beta) - (x-\alpha)Y(\alpha,\beta)\}}{\{(x-\alpha)^2+(y-\beta)^2\}^2}\, d\alpha\, d\beta$$

$$-\frac{1}{\pi}\rho \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\{(x-\alpha)^2-(y-\beta)^2\}\{(x-\alpha)X(\alpha,\beta) + (y-\beta)Y(\alpha,\beta)\}}{\{(x-\alpha)^2+(y-\beta)^2\}^2}\, d\alpha\, d\beta$$

$$(2.212)$$

$$\tau_{xy} = \frac{(1-2\sigma)}{4\pi(1-\sigma)}\rho \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\{(x-\alpha)^2-(y-\beta)^2\}\{(y-\beta)X(\alpha,\beta) - (x-\alpha)Y(\alpha,\beta)\}}{\{(x-\alpha)^2+(y-\beta)^2\}^2}\, d\alpha\, d\beta$$

$$-\frac{1}{\pi}\rho \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{(x-\alpha)(y-\beta)\{(x-\alpha)X(\alpha,\beta) + (y-\beta)Y(\alpha,\beta)\}}{\{(x-\alpha)^2+(y-\beta)^2\}^2}\, d\alpha\, d\beta$$

In the expressions for the components of the displacement vector we make the assumption that $\frac{2\xi^2}{(\xi^2+\eta^2)^2}$ is the two-dimensional Fourier transform of $\frac{y^2}{x^2+y^2} - \frac{1}{2}\log(x^2+y^2)$.

[*](7) p. 480

This is difficult to prove rigorously, but it would seem to be the case, except for a constant which disappears on differentiation, and therefore will yield the same values for the components of strain. The difficulty in obtaining the correct components of displacement using transform theory would seem to be inherent in two-dimensional elasticityand has been encountered by others. The justification for the assumption in this case is merely that it gives the correct result in the special problems considered. We find the components of displacement to be given by

$$\frac{E}{1+\sigma}u = \frac{(1-2\sigma)}{4\pi(1-\sigma)}\rho\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big[X(\alpha,\beta)\Big\{\frac{(y-\beta)^2}{(x-\alpha)^2+(y-\beta)^2} - \tfrac{1}{2}\log\big((x-\alpha)^2+(y-\beta)^2\big)\Big\}$$
$$-Y(\alpha,\beta)\frac{(x-\alpha)(y-\beta)}{(x-\alpha)^2+(y-\beta)^2}\Big]\,d\alpha\,d\beta$$
$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big[X(\alpha,\beta)\Big\{\frac{(x-\alpha)^2}{(x-\alpha)^2+(y-\beta)^2} - \tfrac{1}{2}\log\big((x-\alpha)^2+(y-\beta)^2\big)\Big\}$$
$$+Y(\alpha,\beta)\frac{(x-\alpha)(y-\beta)}{(x-\alpha)^2+(y-\beta)^2}\Big]\,d\alpha\,d\beta$$

$$\frac{E}{1+\sigma}v = -\frac{(1-2\sigma)}{4\pi(1-\sigma)}\rho\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big[X(\alpha,\beta)\frac{(x-\alpha)(y-\beta)}{(x-\alpha)^2+(y-\beta)^2} \tag{2.212}$$
$$-Y(\alpha,\beta)\Big\{\frac{(x-\alpha)^2}{(x-\alpha)^2+(y-\beta)^2} - \tfrac{1}{2}\log\big((x-\alpha)^2+(y-\beta)^2\big)\Big\}\Big]\,d\alpha\,d\beta$$
$$+\frac{1}{2\pi}\rho\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big[X(\alpha,\beta)\frac{(x-\alpha)(y-\beta)}{(x-\alpha)^2+(y-\beta)^2}$$
$$+Y(\alpha,\beta)\Big\{\frac{(y-\beta)^2}{(x-\alpha)^2+(y-\beta)^2} - \tfrac{1}{2}\log\big((x-\alpha)^2+(y-\beta)^2\big)\Big\}\Big]\,d\alpha\,d\beta$$

Equations (2.212) represent a formal solution to the problems of an infinite two-dimensional medium with external forces applied to its interior. We shall now consider some special cases of the general formulae.

2·3 Point force.

Consider in the first instance the force applied to the interior to be defined by the equations

$$X(x,y) = \frac{1}{\rho} P\delta(x)\delta(y) \quad , Y(x,y) = 0 \tag{2.31}$$

This represents a force concentrated at the origin acting in the direction of the x-axis. $\delta(x)$ is the Dirac δ-function defined by (1·71)

Substituting these values for X and Y in equations (2·212), and making use of the result (VI), we obtain immediately the following expressions for the components of the stress tensor

$$\sigma_x + \sigma_y = -\frac{P}{2\pi(1-\sigma)} \frac{x}{x^2+y^2}$$

$$\sigma_x - \sigma_y = -\frac{(1-2\sigma)P}{\pi(1-\sigma)} \frac{xy^2}{(x^2+y^2)^2} - \frac{P}{\pi} \frac{x(x^2-y^2)}{(x^2+y^2)^2} \tag{2.32}$$

$$\tau_{xy} = \frac{(1-2\sigma)P}{4\pi(1-\sigma)} \frac{y(x^2-y^2)}{(x^2+y^2)^2} - \frac{P}{\pi} \frac{x^2y}{(x^2+y^2)^2}$$

If now we put $x^2 + y^2 = r^2$ and re-arrange the terms we obtain the expressions

$$\sigma_x + \sigma_y = -\frac{P}{2\pi(1-\sigma)} \frac{x}{r^2}$$

$$\sigma_x - \sigma_y = -\frac{P}{\pi(1-\sigma)} \frac{x}{r^2}\left\{(1-\sigma) - \frac{y^2}{r^2}\right\} \tag{2.33}$$

$$\tau_{xy} = -\frac{P}{4\pi(1-\sigma)} \frac{y}{r^2}\left\{(1-2\sigma) + \frac{2x^2}{r^2}\right\}$$

which are in agreement with those obtained by Love (21). The components of displacement obtained in this way differ from Love's result by a constant which of course will disappear on differentiation to yield the same components of strain, so it does not constitute an essential difference.

$$\frac{E}{1+\sigma}u = -\frac{P}{4\pi(1-\sigma)}\left\{(3-4\sigma)\log r + \frac{y^2}{r^2} - 2(1-\sigma)\right\}$$

$$\frac{E}{1+\sigma}v = +\frac{P}{4\pi(1-\sigma)} \frac{xy}{r^2} \tag{2.33}$$

It will be observed that these expressions lead to infinite stresses at the origin but this difficulty is removed if we assume that the origin is enclosed in a cavity and the applied force is distributed over the surface of the cavity in such a way that its resultant is of magnitude P and acts in the direction of the x-axis. This of course is the physical interpretation of the problem, the use of the Dirac δ-function being a mathematical idealisation which it is convenient to use because of the simplicity of the result (V).

(21) p.207

Most interest in problems in elasticity is in the lines of constant maximum shearing stress, since these are the lines along which rupture of the material is most likely to take place. These are also the lines produced by photo-elastic investigation and from this they are often referred to as "isochromatic lines." Mathematically they are the family of curves τ = constant, where τ is defined by the relation

$$\tau = \left\{ \left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2 \right\}^{1/2}$$

Using equation (2·33) these have been calculated for this problem and are shown in fig.(i) from which it can be seen that greatest stress concentration occurs immediately in front of and behind the point of application of the force.

We shall required in a subsequent problem to know the components of stress and displacement due to a force concentrated at the origin and ~~in~~ acting in a direction making angle α with the positive direction of the x-axis. If the force is of magnitude F, then it may be resolved into the two components F cos α, and F sin α in the directions of the x and y-axes respectively. Thus we have

$$X(x,y) = \frac{1}{\rho} F \cos\alpha \, \delta(x)\, \delta(y)$$

$$Y(x,y) = \frac{1}{\rho} F \sin\alpha \, \delta(x)\, \delta(y) \qquad (2.34)$$

and proceeding as in the previous case from equations (2·212), using the result (VI) we obtain for the components of stress and displacement

$$\sigma_x + \sigma_y = -\frac{F}{2\pi(1-\sigma)} \frac{x\cos\alpha + y\sin\alpha}{r^2}$$

$$\sigma_x - \sigma_y = -\frac{F}{\pi(1-\sigma)} \frac{1}{r^2}\left[x\cos\alpha\left\{-\sigma + \frac{x^2}{r^2}\right\} - y\sin\alpha\left\{(1-\sigma) - \frac{x^2}{r^2}\right\}\right]$$

$$\tau_{xy} = -\frac{F}{4\pi(1-\sigma)} \frac{1}{r^2}\left[x\sin\alpha\left\{(3-2\sigma) - \frac{2x^2}{r^2}\right\} + y\cos\alpha\left\{(1-2\sigma) + \frac{2x^2}{r^2}\right\}\right] \quad (2.35)$$

$$\frac{E}{1+\sigma} u = -\frac{F}{4\pi(1-\sigma)}\left[\left\{(3-4\sigma)\log r - \frac{x^2}{r^2} - (1-2\sigma)\right\}\cos\alpha - xy\sin\alpha\right]$$

$$\frac{E}{1+\sigma} v = +\frac{F}{4\pi(1-\sigma)}\left[xy\cos\alpha - \left\{(3-4\sigma)\log r + \frac{x^2}{r^2} - 2(1-\sigma)\right\}\sin\alpha\right]$$

2.4 Force distributed over a line.

As a further example of the general formulae (2.12) we shall obtain the components of stress and displacement in an infinite two-dimensional elastic medium due to a force F distributed uniformly over a line of length 2a, co-incident with the y-axis and having its centre at the origin. In this case we have

$$X(x,y) = \frac{1}{\rho}\frac{F}{2a}\delta(x) \quad , \quad |y| \leq a \; : \quad Y(x,y) = 0$$
$$= 0 \quad , \quad |y| > a$$

Substituting for X and Y in equation (2·212a) we have

$$\sigma_x+\sigma_y = -\frac{Fx}{4\pi a(1-\sigma)}\int_{-a}^{a}\frac{d\beta}{x^2+(y-\beta)^2}$$

Performing the integration which is elementary we obtain

$$\sigma_x+\sigma_y = \frac{F}{4\pi a(1-\sigma)}(\theta_2-\theta_1)$$

where θ_1 and θ_2 are defined by the equations

$$x = \lambda_1\cos\theta_1 \quad , \quad y+a = \lambda_1\sin\theta_1$$

$$x = \lambda_2\cos\theta_2 \quad , \quad y-a = \lambda_2\sin\theta_2$$

In a similar manner the following expressions for the other components of stress are obtained

$$\sigma_x-\sigma_y = \frac{F}{4\pi a(1-\sigma)}\left\{(1-2\sigma)(\theta_2-\theta_1)+\sigma(\sin 2\theta_2-\sin 2\theta_1)\right\}$$

$$\tau_{xy} = -\frac{F}{4\pi a(1-\sigma)}\left\{2(1-2\sigma)\log\frac{\lambda_2}{\lambda_1}+\cos 2\theta_2-\cos 2\theta_1\right\}$$

It may be readily shown by a similar proceedure that the components of the displacement vector are given in this case by the equations

$$\frac{E}{1+\sigma}u = \frac{F}{8\pi a(1-\sigma)}\Big[2x(1-2\sigma)(\theta_2-\theta_1)+(3-4\sigma)\left\{(y-a)\log\lambda_2-(y+a)\log\lambda_1\right\} + a(5-8\sigma)\Big]$$

$$\frac{E}{1+\sigma}v = -\frac{Fx}{8\pi a(1-\sigma)}\log\frac{\lambda_2}{\lambda_1}$$

Table of Two-dimensional Fourier Transforms.

$f(x,y)$	$\bar{f}(\xi,\eta)$
$\frac{x}{x^2+y^2}$	$\frac{i\xi}{\xi^2+\eta^2}$
$\frac{y}{x^2+y^2}$	$\frac{i\eta}{\xi^2+\eta^2}$
$\frac{2xy^2}{(x^2+y^2)^2}$	$\frac{i\xi(\xi^2-\eta^2)}{(\xi^2+\eta^2)^2}$
$\frac{2x^2y}{(x^2+y^2)^2}$	$-\frac{i\eta(\xi^2-\eta^2)}{(\xi^2+\eta^2)^2}$
$\frac{x(x^2-y^2)}{(x^2+y^2)^2}$	$\frac{2i\xi\eta^2}{(\xi^2+\eta^2)^2}$
$\frac{y(x^2-y^2)}{(x^2+y^2)^2}$	$-\frac{2i\xi^2\eta}{(\xi^2+\eta^2)^2}$
$\frac{xy}{x^2+y^2}$	$-\frac{2\xi\eta}{(\xi^2+\eta^2)^2}$
$\frac{y^2}{x^2+y^2}-\frac{1}{2}\log(x^2+y^2)$	$\frac{2\xi^2}{(\xi^2+\eta^2)^2}$
$\frac{x^2}{x^2+y^2}-\frac{1}{2}\log(x^2+y^2)$	$\frac{2\eta^2}{(\xi^2+\eta^2)^2}$

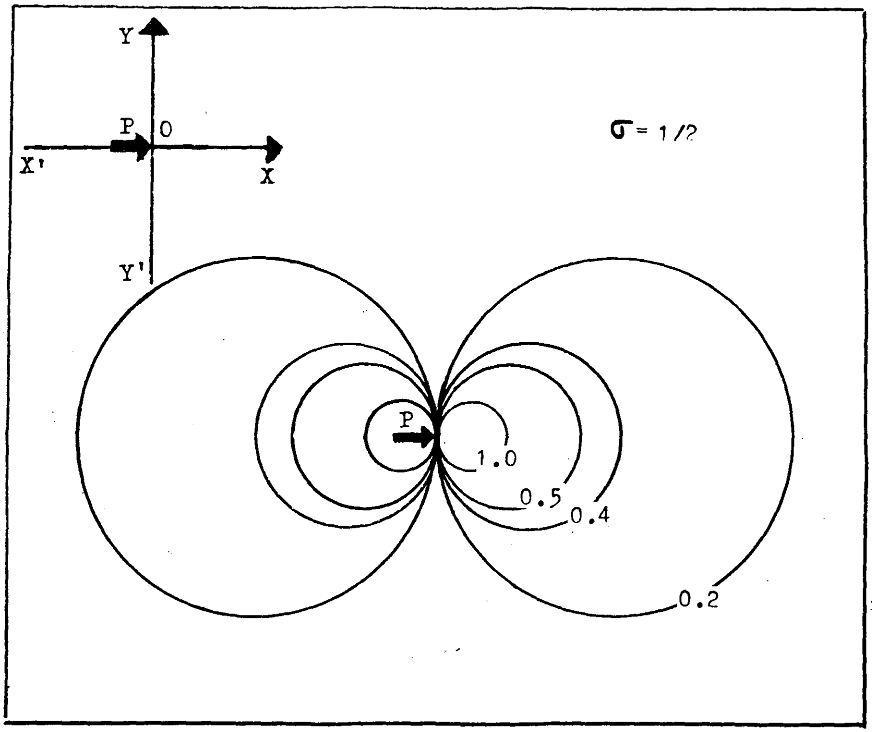

Fig.(i)a.

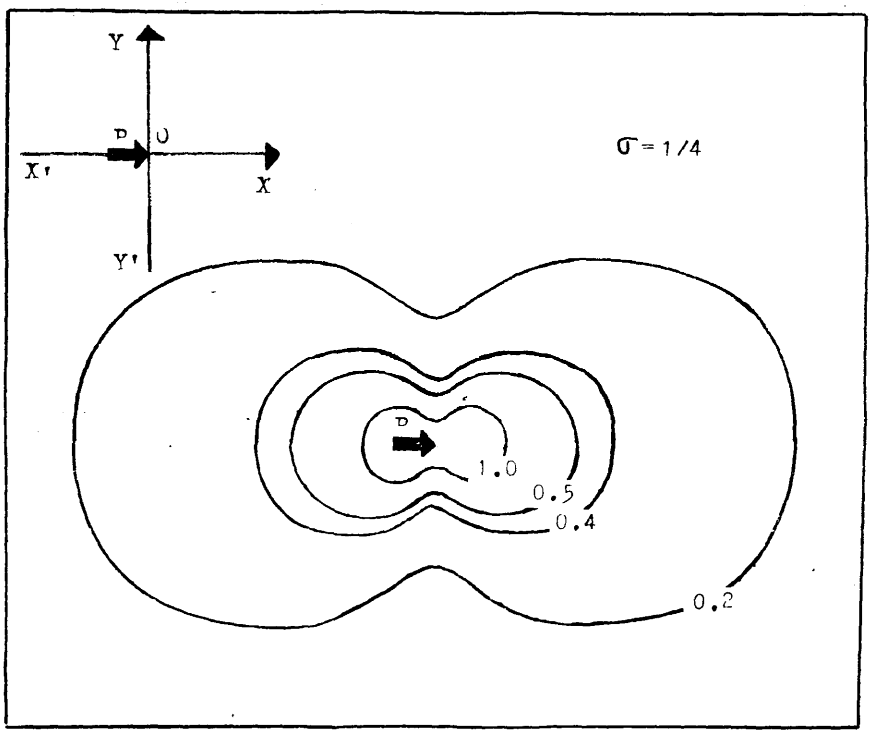

Fig.(i)b.

Lines of constant maximum shearing stress in an infinite two-dimensional elastic medium due to a point force acting in the interior.

PART III.

THE APPLICATION OF STATICAL FORCES TO THE SURFACES OF A TWO-DIMENSIONAL ELASTIC SOLID.

III. THE APPLICATION OF STATICAL FORCES TO THE SURFACES OF A TWO-DIMENSIONAL ELASTIC SOLID.

3·1 Introduction

The calculation of the stress set up in a two-dimensional semi-infinite elastic medium due to forces applied to the surface, is also a classical problem in mathematical elasticity, being the two-dimensional analogue of "Boussinesq's problem", and Love (21) records solutions to several problems of this type. Recently Sneddon (32) has used the Fourier cosine transform to obtain the solution with more general forms of loading on the boundary.

Filon (4) considered a two-dimensional strip of finite length with forces applied to the surface, and obtained the solution in the form of a Fourier series which he showed could be expressed as an infinite integral, when the length was great compared with the thickness. He also devised a method (5) for evaluating the type of integral obtained and an account of this is given at the end of this section. Sneddon (32) obtained the solution of the problem of the infinite strip using Fourier cosine transforms.

In this section we shall give a completely general solution for any form of loading of the boundary of a two-dimensional semi-infinite elastic medium, and the solutions of several particular problems will be deduced from the general solution, which will also be used in Part IV. These results were derived independently of those given by Sneddon (32) and are in a more general form, the faltung theorem (III) being employed here.

The general solution for the infinite strip is also given.

3·2. Solution of the equations of equilibrium.

We shall consider an elastic solid bounded by plane surfaces with statical foces applied to the boundaries, assuming that there are no forces applied to the interior. We choose the y-axis parallel to the bounding surfaces and x-axis perpendicular to this direction and in the sense making a right-handed set. The component of displacement in the direction perpendicular to the co-ordinate plane is assumed to be zero and the applied forces are uniform throughout the thickness to the solid so that the problem is one of plane strain. The stress tensor is, therefore, uniquely determined by the components σ_x, σ_y, and τ_{xy}. The solid is assumed to extend indefinitely in both directions of the y-axis.

If we eliminate u and v from the stress strain relation (1·41) we obtain another equation, namely the compatibility equation, which must be satisfied. This is

$$2\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = (1-\sigma)\left(\frac{\partial^2 \sigma_x}{\partial y^2} + \frac{\partial^2 \sigma_y}{\partial x^2}\right) - \sigma\left(\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2}\right) \tag{3.21}$$

We assume now that the components of the stress tensor may be derived from a single potential function $\chi(x,y)$, the Airy stress function, (1) such that

$$\sigma_x = \frac{\partial^2 \chi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 \chi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 \chi}{\partial x \partial y} \tag{3.22}$$

Now it is easily seen that solutions of this type satisfy the equations of equilibrium (1·43) when in addition X = Y = 0. Substituting for the stress components in terms of χ in equation (3·21) we see that χ must satisfy the biharmonic equation

$$\nabla_1^4 \chi = 0 \tag{3.23}$$

where we have written ∇_1^2 for the two-dimensional Laplacian operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. In order to solve this partial differential equation we introduce the Fourier transform of the function defined by the integral

$$\bar{\chi}(x,\eta) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} \chi(x,y)\, e^{i\eta y}\, dy \tag{3.24}$$

Multiplying equation (3·23) by exp.$i\eta y$ and integrating with respect to y over the range $-\infty$ to $+\infty$, assuming that χ and its first three derivatives with respect to x vanish at both limits of integration, the biharmonic equation reduces to the ordinary differential equation

$$\left(\frac{d^2}{dx^2} - \eta^2\right)^2 \bar{\chi}(x,\eta) = 0 \tag{3.25}$$

The solution of this equation may be written down immediately as

$$\bar{\chi}(x,\eta) = (A+Bx)e^{-\eta x} + (C+Dx)e^{\eta x} \tag{3.26}$$

or in hyperbolic form

$$\bar{\chi}(x,\eta) = (A'+B'x)\cosh\eta x + (C'+D'x)\sinh\eta x \tag{3.27}$$

where A,B,C,D are functions of η but are independent of x, and may be determined so as to satisfy conditions at the bounding surfaces.

Transforming the stress components as given by equations (3·22) by multiplying each by exp.$i\eta y$ and integrating with respect to y over the entire range of the variable, making the same assumptions as before, we obtain

$$\bar{\sigma}_x = -\eta^2\bar{\chi} \quad , \quad \bar{\sigma}_y = \frac{d^2\bar{\chi}}{dx^2} \quad , \quad \bar{\tau}_{xy} = i\eta\frac{d\bar{\chi}}{dx} \tag{3.28}$$

Similarly we obtain for the transformed components of the displacement vector

$$\frac{E}{1+\sigma}\,\bar{u} = \frac{1-\sigma}{\eta^2}\,\frac{d^3\bar{\chi}}{dx^3} - (2+\sigma)\,\frac{d\bar{\chi}}{dx}$$

$$\frac{E}{1+\sigma}\,\bar{v} = -\frac{(1-\sigma)}{i\eta}\,\frac{d^2\bar{\chi}}{dx^2} + \sigma i\eta\,\bar{\chi} \qquad (3.29)$$

Now having determined the arbitrary functions A, B, C, and D so as to satisfy the conditions prescribed on the bounding surfaces, we may substitute from equations (3•26) or (3•27) into equations (3•28) and (3•29) and so obtain the transformed components of stress and displacement appropriate to the prescribed conditions at the boundary. Applying Fourier's inversion theorem (I) yields the components of stress and displacement in the form of infinite integrals.

3•3 Semi-infinite medium with normal surface loading

Consider in the first instance the case of a semi-infinite solid with forces applied normal to the boundary. We shall take the bounding surface to be co-incident with y-axis and x-axis directed into the elastic medium. If we denote the forces applied to the boundary by p(y), then the conditions to be satisfied there, are

$$\sigma_x = -p(y)\;, \quad \tau_{xy} = 0\;, \quad x = 0 \qquad (3.31)$$

Transforming the boundary stresses and writing

$$\bar{p}(\eta) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} p(y)\, e^{i\eta y}\, dy \tag{3.32}$$

we have

$$\bar{\sigma}_x = -\bar{p}(\eta) \;, \quad \bar{\tau}_{xy} = 0 \quad , \quad x = 0 \tag{3.33}$$

Now according to St. Venant's principle the stresses at great distances from the point of application of the load are negligibly small, so that the form of Airy stress function chosen must be such as to vanish when x tends to infinity. It follows, therefore, that in the first of the solution (3•26) of the transformed biharmonic equations, C = D = 0.

Using equations (3•33) along with (3•26) and (3•28) we find that for these conditions at the boundary

$$A = \frac{\bar{p}(\eta)}{\eta^2} \;, \quad B = \frac{\bar{p}(\eta)}{\eta}$$

so that the transformed components of stress and displacement are given by

$$\begin{aligned}
\bar{\sigma}_x + \bar{\sigma}_y &= -2\bar{p}(\eta)\, e^{-\eta x} \\
\bar{\sigma}_x - \bar{\sigma}_y &= -2x\, \bar{p}(\eta) \cdot \eta\, e^{-\eta x} \\
\bar{\tau}_{xy} &= -x\, \bar{p}(\eta) \cdot i\eta\, e^{-\eta x}
\end{aligned} \tag{3.34}$$

$$\frac{E}{1+\sigma}\, \bar{u} = \frac{\bar{p}(\eta)}{\eta} (2 - 2\sigma + \eta x)\, e^{-\eta x}$$

$$\frac{E}{1+\sigma}\, \bar{v} = \frac{\bar{p}(\eta)}{i\eta} (1 - 2\sigma - \eta x)\, e^{-\eta x}$$

Now if we apply Fourier's inversion theorem (I) to the first of equations (3·34) we get

$$\sigma_x + \sigma_y = -\left(\frac{2}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} \bar{p}(\eta)\, e^{-\eta x} \cdot e^{-i\eta y}\, d\eta$$

which is in agreement with the result obtained by Sneddon (32) when p(y) is an even function. It may readily be shown that $e^{-|\eta|x}$ is the Fourier transform of the function $\left(\frac{2}{\pi}\right)^{1/2} \frac{x}{x^2+y^2}$ so that applying the Faltung theorem in the form (III) we obtain

$$\sigma_x + \sigma_y = -\frac{2x}{\pi} \int_{-\infty}^{\infty} \frac{p(\beta)\, d\beta}{x^2 + (y-\beta)^2} \tag{3.35a}$$

The other two equations for the components of the stress tensor may be dealt with in the same way, making use of the list of Fourier transforms at the end of the section to give the equations

$$\begin{aligned} \sigma_x - \sigma_y &= -\frac{2x}{\pi} \int_{-\infty}^{\infty} p(\beta) \frac{x^2 - (y-\beta)^2}{\{x^2 + (y-\beta)^2\}^2}\, d\beta \\ \tau_{xy} &= -\frac{2x^2}{\pi} \int_{-\infty}^{\infty} p(\beta) \frac{y-\beta}{\{x^2 + (y-\beta)^2\}^2}\, d\beta \end{aligned} \tag{3.35}$$

The components of the displacement vector are found in a similar manner to be

$$\begin{aligned} \frac{E}{1+\sigma} u &= -\frac{(1-\sigma)}{\pi} \int_{-\infty}^{\infty} p(\beta) \log\{x^2 + (y-\beta)^2\}\, d\beta + \frac{x^2}{\pi} \int_{-\infty}^{\infty} \frac{p(\beta)\, d\beta}{x^2 + (y-\beta)^2} \\ \frac{E}{1+\sigma} v &= -\frac{(1-2\sigma)}{\pi} \int_{-\infty}^{\infty} p(\beta) \tan^{-1}\left(\frac{y-\beta}{x}\right) d\beta + \frac{x}{\pi} \int_{-\infty}^{\infty} \frac{p(\beta)(y-\beta)\, d\beta}{x^2 + (y-\beta)^2} \end{aligned} \tag{3.35}$$

These formulae are quite general and represent the formal solution of the problem of a two-dimensional semi-infinite medium with any form of normal load applied to the surface.

A particular case of some interest is that of a point force applied at the origin. We have then $p(y) = P\delta(y)$, so that when we insert this value in equations (3·35) and apply the result (VI), we obtain the following expressions for the components of stress and displacement

$$
\left.
\begin{aligned}
\sigma_x + \sigma_y &= -\frac{2Px}{\pi r^2} \\
\sigma_x - \sigma_y &= -\frac{2Px}{\pi r^4}(x^2 - y^2) \\
\tau_{xy} &= -\frac{2Px^2y}{\pi r^4} \\
\frac{E}{1+\sigma}\cdot u &= \frac{P}{\pi}\left\{-2(1-\sigma)\log r + \frac{x^2}{r^2}\right\} \\
\frac{E}{1+\sigma}\, v &= \frac{P}{\pi}\left\{-(1-2\sigma)\tan^{-1}\frac{y}{x} + \frac{xy}{r^2}\right\}
\end{aligned}
\right\} \qquad (3{\cdot}36)
$$

where we have written $x^2 + y^2 = r^2$. This is in agreement with the result given by Sneddon (32), and others.

The maximum shearing stress τ defined by equation (2·34) is given in this case by

$$\tau = \frac{Px}{\pi r^2} = \frac{P}{\pi}\frac{\cos\theta}{r}$$

and so the 'isochromatics' or lines of constant maximum shearing stress are the family of coaxial circles shown in fig.(ii)

Another special case which is readily obtained from the general solution is that of uniform pressure over a segment of the boundary. We have

$$p(y) = P/2a \qquad |y| \leq a$$
$$= 0 \qquad |y| > a$$

so that when we insert this value in the first of equations (3·35) we get

$$\sigma_x + \sigma_y = -\frac{Px}{\pi a}\int_{-a}^{a}\frac{d\beta}{x^2+(y-\beta)^2}$$

Performing the integration with respect to β yields the result

$$\sigma_x + \sigma_y = \frac{P}{\pi a}\left\{\tan^{-1}\left(\frac{y-a}{x}\right) - \tan^{-1}\left(\frac{y+a}{x}\right)\right\} = \frac{P}{\pi a}(\theta_2 - \theta_1) \tag{3.38}$$

where θ_1, and θ_2, are defined by the equations

$$x = r_1 \cos\theta_1 \quad , \quad y + a = r_1 \sin\theta_1 \quad ,$$
$$x = r_2 \cos\theta_2 \quad , \quad y - a = r_2 \sin\theta_2$$

Similarly by substituting from equation (3·77) into the remaining equations (3·35) and performing the integrations, which are elementary, we obtain the following expressions for the components of stress

$$\sigma_x - \sigma_y = \frac{P}{2\pi a}\left(\sin 2\theta_2 - \sin 2\theta_1\right)$$
$$\tau_{xy} = -\frac{P}{4\pi a}\left(\cos 2\theta_2 - \cos 2\theta_1\right) \tag{3.38}$$

It is readily seen that the maximum shearing stress at any point is given in this case by

$$\tau = \frac{P}{2\pi a}\sin(\theta_2 - \theta_1)$$

so that the 'isochromatics' are of the form shown in fig.(iii).

3·4 Semi-infinite solid with shearing load applied to the surface.

If the loading is applied tangentially to the bounding surface and is denoted by q(y) the boundary conditions become

$$\sigma_x = 0 \quad , \quad \tau_{xy} = -q(y) \quad , \quad x = 0 \tag{3.41}$$

so that if we write

$$\bar{q}(\eta) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} q(y)\, e^{i\eta y}\, dy \tag{3.42}$$

we have

$$\bar{\sigma}_x = 0 \quad , \quad \bar{\tau}_{xy} = -\bar{q}(\eta) \quad , \quad x = 0 \tag{3.43}$$

Proceeding as before we find that the arbitrary functions take the values

$$B = -\frac{\bar{q}(\eta)}{i\eta} \quad , \quad A = C = D = 0$$

These lead to the following expressions for the components of the stress tensor and displacement vector in the general case

$$\sigma_x + \sigma_y = -\frac{2}{\pi}\int_{-\infty}^{\infty} \frac{q(\beta)(y-\beta)}{x^2+(y-\beta)^2}\,d\beta$$

$$\sigma_x - \sigma_y = -\frac{2}{\pi}\int_{-\infty}^{\infty} \frac{q(\beta)(y-\beta)\{x^2-(y-\beta)^2\}}{\{x^2+(y-\beta)^2\}^2}\,d\beta$$

$$\tau_{xy} = -\frac{2x}{\pi}\int_{-\infty}^{\infty} \frac{q(\beta)(y-\beta)^2}{\{x^2+(y-\beta)^2\}^2}\,d\beta \qquad (3.46)$$

$$\frac{E}{1+\sigma}u = \frac{(1-2\sigma)}{\pi}\int_{-\infty}^{\infty} q(\beta)\tan^{-1}\left(\frac{y-\beta}{x}\right)d\beta + \frac{x}{\pi}\int_{-\infty}^{\infty}\frac{q(\beta)(y-\beta)}{x^2+(y-\beta)^2}\,d\beta$$

$$\frac{E}{1+\sigma}v = -\frac{(1-\sigma)}{\pi}\int_{-\infty}^{\infty} q(\beta)\log\{x^2+(y-\beta)^2\}\,d\beta - \frac{x^2}{\pi}\int_{-\infty}^{\infty}\frac{q(\beta)}{x^2+(y-\beta)^2}\,d\beta$$

In the particular case where the load in concentrated at the origin and is of magnitude Q we have $q(y) = Q\delta(y)$, so that these expressions become

$$\sigma_x + \sigma_y = -\frac{2Qy}{\pi r^2}$$

$$\sigma_x - \sigma_y = -\frac{2Qy(x^2-y^2)}{\pi r^4}$$

$$\tau_{xy} = -\frac{2Qxy^2}{\pi r^4} \qquad (3.47)$$

$$\frac{E}{1+\sigma}u = \frac{Q}{\pi}\left\{(1-2\sigma)\tan^{-1}\frac{y}{x} + \frac{xy}{r^2}\right\}$$

$$\frac{E}{1+\sigma}v = \frac{Q}{\pi}\left\{-2(1-\sigma)\log r - \frac{x^2}{r^2}\right\}$$

The maximum shearing stress at any point is given by

$$\tau = \frac{Q \sin\theta}{\pi r}$$

and the 'isochromatics' are shown in fig.(iv).

When the shearing force is distributed over a segment of the boundary we have

$$\begin{aligned} q(y) &= \frac{Q}{2a}, \quad |y| \leq a. \\ &= 0, \quad |y| > a \end{aligned}$$

and the general expressions for the stress components reduce to integrals which are readily evaluated to give

$$\begin{aligned} \sigma_x + \sigma_y &= \frac{Q}{\pi a} \log \frac{\lambda_2}{\lambda_1} \\ \sigma_x - \sigma_y &= -\frac{Q}{\pi a} \left\{ \log \frac{\lambda_2}{\lambda_1} + \cos 2\theta_2 - \cos 2\theta_1 \right\} \\ \tau_{xy} &= -\frac{Q}{\pi a} \left\{ \theta_2 - \theta_1 + \frac{1}{4}(\sin 2\theta_2 - \sin 2\theta_1) \right\} \end{aligned}$$

3·5 Semi-infinite medium with general loading of the boundary.

From the results derived in the previous two paragraphs we can, from the principle of superposition, deduce the expressions for the components of stress and displacement in a semi-infinite elastic medium due to any form of loading of the boundary. The most general form of load can always be resolved into a normal load p(y) and a shearing load q(y) so that by adding the stresses and displacements in the elementary cases considered we obtain the solution for the more general case. This is possible because the equations of equilibrium are linear. We obtain then the equations

$$\sigma_x + \sigma_y = -\frac{2}{\pi}\int_{-\infty}^{\infty} \frac{x\,p(\beta) + (y-\beta)\,q(\beta)}{x^2+(y-\beta)^2}\,d\beta$$

$$\sigma_x - \sigma_y = -\frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\{x\,p(\beta)+(y-\beta)\,q(\beta)\}\{x^2-(y-\beta)^2\}}{\{x^2+(y-\beta)^2\}^2}\,d\beta$$

$$\tau_{xy} = -\frac{2x}{\pi}\int_{-\infty}^{\infty} \frac{\{x\,p(\beta)+(y-\beta)\,q(\beta)\}(y-\beta)}{\{x^2+(y-\beta)^2\}^2}\,d\beta$$

$$\frac{E}{1+\sigma}u = \frac{1}{\pi}\int_{-\infty}^{\infty}\Big[\frac{x\{x\,p(\beta)+(y-\beta)\,q(\beta)\}}{x^2+(y-\beta)^2} - (1-\sigma)\,p(\beta)\log\{x^2+(y-\beta)^2\} + (1-2\sigma)\,q(\beta)\tan^{-1}\Big(\frac{y-\beta}{x}\Big)\Big]\,d\beta \qquad (3.51)$$

$$\frac{E}{1+\sigma}v = \frac{1}{\pi}\int_{-\infty}^{\infty}\Big[\frac{x\{(y-\beta)\,p(\beta) - x\,q(\beta)\}}{x^2+(y-\beta)^2} - (1-\sigma)\,q(\beta)\log\{x^2+(y-\beta)^2\} - (1-2\sigma)\,p(\beta)\tan^{-1}\Big(\frac{y-\beta}{x}\Big)\Big]\,d\beta$$

A particular case of some interest is the point force acting at the origin and inclined at angle α to the normal to the bounding surface. If the magnitude of the force is F then we may resolve it into a normal component $F \cos\alpha$, and a tangential component $F \sin\alpha$ so that

$$p(y) = F \cos\alpha \; \delta(y)$$

$$q(y) = F \sin\alpha \; \delta(y)$$

Inserting these values in equations (3·51) and using the result (VI) gives the equations

$$\sigma_x + \sigma_y = -\frac{2F}{\pi} \frac{x\cos\alpha + y\sin\alpha}{r^2}$$

$$\sigma_x - \sigma_y = -\frac{2F}{\pi} \frac{(x\cos\alpha + y\sin\alpha)(x^2 - y^2)}{r^4}$$

$$\tau_{xy} = -\frac{2F}{\pi} \frac{(x\cos\alpha + y\sin\alpha)\, xy}{r^4} \qquad (3.46)$$

$$\frac{E}{1+\sigma} u = \frac{F}{\pi}\left[\frac{x(x\cos\alpha + y\sin\alpha)}{r^2} - 2(1-\sigma)\cos\alpha \log r + (1-2\sigma)\sin\alpha \tan^{-1}\frac{y}{x} \right]$$

$$\frac{E}{1+\sigma} v = \frac{F}{\pi}\left[\frac{x(x\sin\alpha - y\cos\alpha)}{r^2} - 2(1-\sigma)\sin\alpha \log r - (1-2\sigma)\cos\alpha \tan^{-1}\frac{y}{x} \right]$$

The maximum shearing stress at any point is given by

$$\tau = \frac{F}{\pi} \cos(\theta - \alpha)$$

and the 'isochromatics' are the family of curves shown in fig.(v)

TABLE OF ONE-DIMENSIONAL FOURIER TRANSFORMS.

$f(y)$	$\bar{F}(\eta)$
$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{x}{x^2 + y^2}$	$e^{-\eta x}$
$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{x^2 - y^2}{(x^2 + y^2)^2}$	$\eta e^{-\eta x}$
$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{2xy}{(x^2 + y^2)^2}$	$i\eta e^{-\eta x}$
$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{y}{x^2 + y^2}$	$ie^{-\eta x}$
$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{y(x^2 - y^2)}{(x^2 + y^2)^2}$	$-i(1 - \eta x)e^{-\eta x}$
$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{2xy^2}{(x^2 + y^2)^2}$	$(1 - \eta x)e^{-\eta x}$
$\frac{1}{(2\pi)^{\frac{1}{2}}} \log(x^2 + y^2)$	$-\frac{e^{-\eta x}}{\eta}$
$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \tan^{-1} \frac{y}{x}$	$-\frac{e^{-\eta x}}{i\eta}$

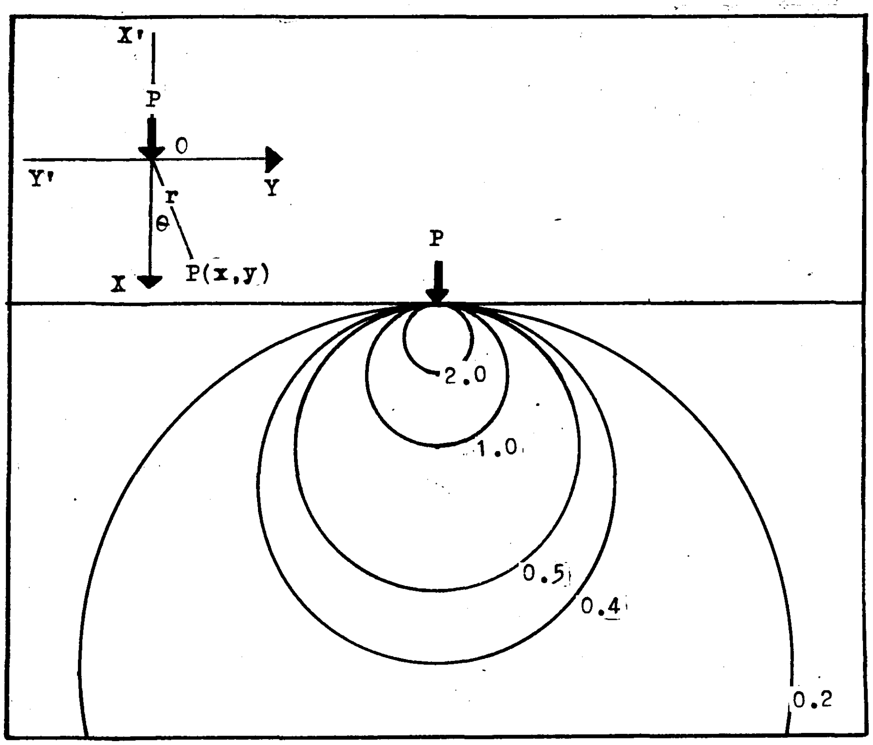

Fig.(ii)

Lines of constant maxium shearing stress in a semi-infinite elastic medium due to a point force acting normal to the boundary.

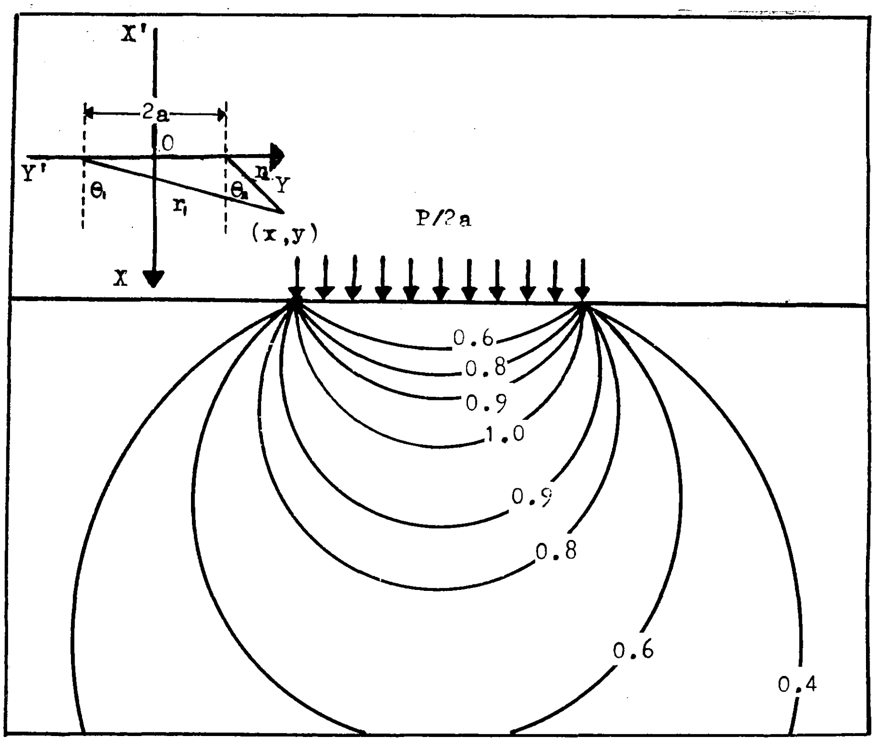

Fig.(iii)

Lines of constant maximum shearing stress in a semi-infinite two-dimensional elastic medium due to a normal force distributed uniformly over a segment of the boundary.

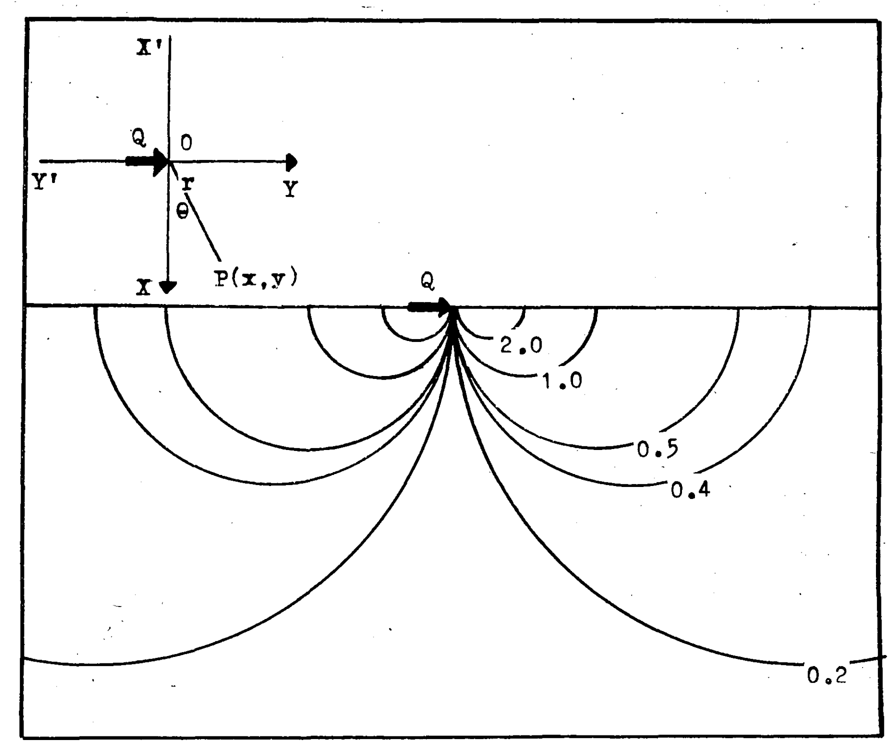

Fig. (iv)

Lines of constant maximum shearing stress in a semi-infinite two-dimensional elastic medium due to a point force acting tangentially to the boundary.

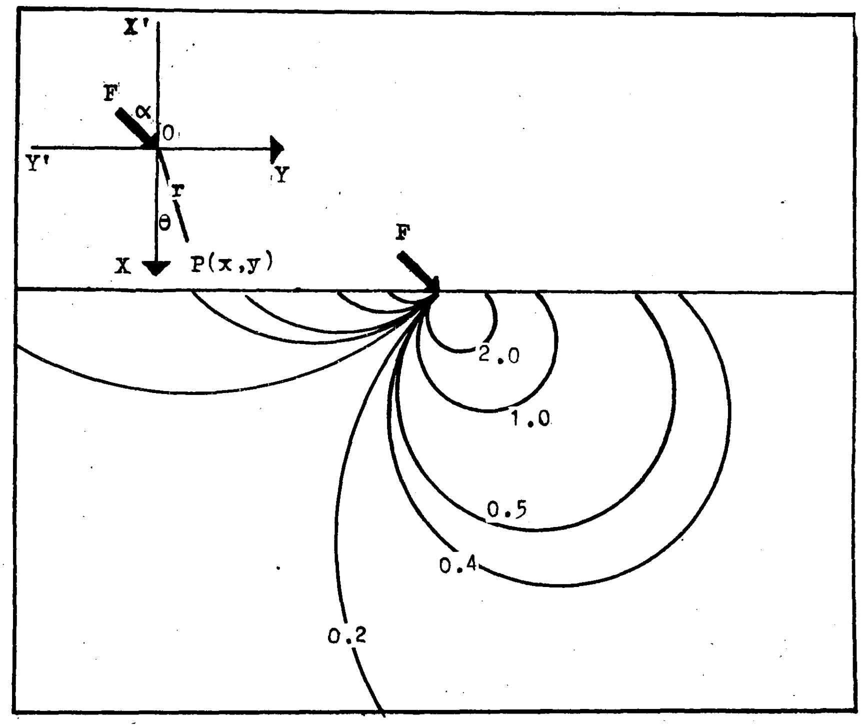

Fig. (v)

Lines of constant maximum shearing stress in a semi-infinite two-dimensional elastic medium due to a point force acting in the direction making angle α with the normal to the boundary.

3.6 Infinite strip of finite thickness with normal loading on one bounding surface.

Consider now an infinite strip of finite thickness with loads applied normal to one bounding surface, the other surface being free from stress. This system is not in statical equilibrium so it will be necessary at a later stage to modify the formal solution obtained by introducing infinite couples. We shall take the bounding surfaces to be $x = \pm d$, and the x- and y- axes to form a right-handed set. The applied forces are assumed uniform in the direction perpendicular to the co-ordinate plane, and the component of displacement in this direction is assumed zero, so that the stress system is two-dimensional.

If we take the applied force to be $p(y)$ acting on the surface $x = d$ then the boundary conditions are

$$\begin{aligned} \sigma_x &= -p(y), & \tau_{xy} &= 0, & x &= d \\ &= 0 & &= 0, & x &= -d \end{aligned} \tag{3.61}$$

Writing

$$\bar{p}(\eta) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} p(y)\, e^{i\eta y}\, dy \tag{3.62}$$

the transformed components of stress become

$$\begin{aligned} \bar{\sigma}_x &= -\bar{p}(\eta), & \bar{\tau}_{xy} &= 0, & x &= d \\ &= 0 & &= 0, & x &= -d \end{aligned} \tag{3.63}$$

at the boundaries. Now if we substitute from the hyperbolic form (3·27) of the solution of the transformed biharmonic equation (3·25) into the equations (3·28) and use equations (3·63), we obtain four equations from which we can determine the four arbitrary functions A', B', C', and D'. We find these to be

$$
\begin{aligned}
A' &= \frac{\bar{p}(\eta)}{2\eta^2}\,\frac{\sinh\eta d + \eta d\cosh\eta d}{\eta d + \sinh\eta d\cosh\eta d} \\[2ex]
B' &= \frac{\bar{p}(\eta)}{2\eta^2}\,\frac{\eta\cosh\eta d}{\eta d - \sinh\eta d\cosh\eta d} \\[2ex]
C' &= -\frac{\bar{p}(\eta)}{2\eta^2}\,\frac{\cosh\eta d + \eta d\sinh\eta d}{\eta d - \sinh\eta d\cosh\eta d} \\[2ex]
D' &= -\frac{\bar{p}(\eta)}{2\eta^2}\,\frac{\eta d\sinh\eta d}{\eta d + \sinh\eta d\cosh\eta d}
\end{aligned}
\qquad (3{\cdot}64)
$$

Substituting these values in the equations for the transformed components of stress and inverting according to Fourier's inversion theorem III we obtain the following integral expressions for the components of the stress tensor

$$
\sigma_x + \sigma_y = -\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\bar{p}(\eta)\left\{\frac{\cosh\eta x\sinh\eta d}{\eta d + \sinh\eta d\cosh\eta d} - \frac{\sinh\eta x\cosh\eta d}{\eta d - \sinh\eta d\cosh\eta d}\right\}e^{-i\eta y}\,d\eta
$$

$$
\sigma_x - \sigma_y = -\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\bar{p}(\eta)\left\{\frac{\eta x\sinh\eta x\sinh\eta d - \eta d\cosh\eta x\cosh\eta d}{\eta d + \sinh\eta d\cosh\eta d}\right.
$$

$$
\left. - \frac{\eta x\cosh\eta x\cosh\eta d - \eta d\sinh\eta x\sinh\eta d}{\eta d - \sinh\eta d\cosh\eta d}\right\}e^{-i\eta y}\,d\eta
$$

$$\tau_{xy} = -\frac{i}{2^{3/2}\pi^{1/2}}\int_{-\infty}^{\infty}\bar{p}(\eta)\left\{\frac{\eta x\cosh\eta x\sinh\eta d - \eta d\sinh\eta x\cosh\eta d}{\eta d + \sinh\eta d\cosh\eta d}\right.$$

$$\left. - \frac{\eta x\sinh\eta x\cosh\eta d - \eta d\cosh\eta x\sinh\eta d}{\eta d - \sinh\eta d\cosh\eta d}\right\} e^{-i\eta y}\, d\eta.$$

Now it will be observed that the first two of these integrals are divergent. The reason for this can be readily seen if we calculate the bending moment $\int_{-d}^{d} x\,\sigma_y\,dx$ across the strip at distance y from the origin. This contains a finite part due to the applied force p(y) and an infinite part which can be removed adding the term $-\frac{3x}{2^{3/2}\pi^{1/2}d^3}\int_{-\infty}^{\infty}\frac{\bar{p}(\eta)}{\eta^2}\,d\eta$. This is the mathematical equivalence of applying an infinite couple at infinity to maintain the system in statical equilibrium. With this modification the solution becomes

$$\sigma_x + \sigma_y = -\frac{1}{(2\pi)^{1/2}d}\int_{-\infty}^{\infty}\bar{p}\left(\frac{u}{d}\right)\left\{\left(\frac{\cosh\alpha u\sinh u}{u+\sinh u\cosh u} - \frac{\sinh\alpha u\cosh u}{u-\sinh u\cosh u}\right)e^{-i\beta u} - \frac{3\alpha}{2u^2}\right\}du$$

$$\sigma_x - \sigma_y = \frac{1}{(2\pi)^{1/2}d}\int_{-\infty}^{\infty}\bar{p}\left(\frac{u}{d}\right)\left\{\left(\frac{\alpha u\sinh\alpha u\sinh u - u\cosh\alpha u\cosh u}{u+\sinh u\cosh u}\right.\right.$$

$$\left.\left. - \frac{\alpha u\cosh\alpha u\cosh u - u\sinh\alpha u\sinh u}{u-\sinh u\cosh u}\right)e^{-i\beta u} - \frac{3\alpha}{2u^2}\right\}du$$

$$\tau_{xy} = -\frac{i}{2^{3/2}\pi^{1/2}d}\int_{-\infty}^{\infty}\bar{p}\left(\frac{u}{d}\right)\left\{\frac{\alpha u\cosh\alpha u\sinh u - u\sinh\alpha u\cosh u}{u+\sinh u\cosh u}\right. \tag{3.66}$$

$$\left. - \frac{\alpha u\sinh\alpha u\cosh u - u\cosh\alpha u\sinh u}{u-\sinh u\cosh u}\right\}e^{-i\beta u}\,du$$

where for convenience we have replaced ηd by u and written

$x/d = \alpha$, $y/d = \beta$. In a similar manner we find the components of the displacement vector to be

$$\frac{E}{1+\sigma}u = \frac{1}{2^{3/2}\pi^{1/2}}\int_{-\infty}^{\infty}\bar{p}\left(\frac{u}{d}\right)\Big\{\Big(\frac{\alpha u\cosh\alpha u\sinh u-2(1-\sigma)\sinh\alpha u\sinh u-u\sinh\alpha u\cosh u}{u+\sinh u\cosh u.}$$

$$-\frac{\alpha u\sinh\alpha u\cosh u-2(1-\sigma)\cosh\alpha u\cosh u-u\cosh\alpha u\sinh u}{u-\sinh u\cosh u.}\Big)e^{-i\beta u}$$

$$+\frac{3(1-\sigma)}{u^3}+\frac{3}{2u}\left(2-\sigma-\sigma\alpha^2-(1-\sigma)\gamma^2\right)\Big\}\frac{du}{u} \qquad (3.66)$$

$$\frac{E}{1+\sigma}v = -\frac{i}{2^{3/2}\pi^{1/2}}\int_{-\infty}^{\infty}\bar{p}\left(\frac{u}{d}\right)\Big\{\Big(\frac{\alpha u\sinh\alpha u\sinh u+(1-2\sigma)\cosh\alpha u\sinh u-u\cosh\alpha u\cosh u}{u+\sinh u\cosh u.}$$

$$-\frac{\alpha u\cosh\alpha u\cosh u+(1-2\sigma)\sinh\alpha u\cosh u-u\sinh\alpha u\sinh u}{u-\sinh u\cosh u.}\Big)e^{-i\beta u}$$

$$-\frac{3\alpha\gamma}{2u}-\frac{3}{2u^2}(1-2\sigma)\alpha\gamma\Big\}\frac{du}{u}$$

If we put $p(y) = P\delta(y)$ so that $\bar{p}(\eta) = P/(2\pi)^{\frac{1}{2}}$ and use the results (IX) and (X), we obtain expressions which are in agreement with those obtained otherwise by Filon (4). Integrals of this type do not appear to have been evaluated exactly, but various approximate methods have been suggested and these will be discussed later. The problem has, however, been discussed in detail by Filon.

3·7 Infinite strip of finite thickness with normal loading on both surfaces

We shall obtain now the solution to the problem of normal loading on both sides of the strip. The solution corresponding to the boundary conditions

$$\begin{aligned} \sigma_x &= 0, & \tau_{xy} &= 0, & x &= d. \\ &= -p(y) & &= 0 & x &= -d. \end{aligned} \qquad (3{\cdot}71)$$

is readily deduced from equations (3·66) by changing the sign of x, τ_{xy}, and u, and we can superimpose the two solutions to get the solution for normal loading on both boundaries.

Let us consider then the boundary conditions

$$\begin{aligned} \sigma_x &= -p_1(y), & \tau_{xy} &= 0, & x &= d. \\ &= -p_2(y) & &= 0 & &= -d. \end{aligned} \qquad (3{\cdot}72)$$

This system will be in equilibrium provided

$$\int_{-\infty}^{\infty} p_1(y)\,dy = \int_{-\infty}^{\infty} p_2(y)\,dy$$

$$\int_{-\infty}^{\infty} y\,p_1(y)\,dy = \int_{-\infty}^{\infty} y\,p_2(y)\,dy$$

otherwise external forces must be applied as in the problem considered in the previous paragraph. It may readily be shown then that the formal solution of this problem is given by the expressions

$$\sigma_x+\sigma_y=-\frac{1}{(2\pi)^{1/2}\alpha}\int_{-\infty}^{\infty}\Big[\Big\{\bar{p}_1\big(\tfrac{y}{\alpha}\big)+\bar{p}_2\big(\tfrac{y}{\alpha}\big)\Big\}\Big\{\frac{\cosh\alpha u\sinh u}{u+\sinh u\cosh u}\Big\}$$

$$-\Big\{\bar{p}_1\big(\tfrac{y}{\alpha}\big)-\bar{p}_2\big(\tfrac{y}{\alpha}\big)\Big\}\Big\{\frac{\sinh\alpha u\cosh u}{u-\sinh u\cosh u}\Big\}\Big]e^{-i\beta u}\,du.$$

$$\sigma_x-\sigma_y=\frac{1}{(2\pi)^{1/2}\alpha}\int_{-\infty}^{\infty}\Big[\Big\{\bar{p}_1\big(\tfrac{y}{\alpha}\big)+\bar{p}_2\big(\tfrac{y}{\alpha}\big)\Big\}\Big\{\frac{\alpha u\sinh\alpha u\sinh u-u\cosh\alpha u\cosh u}{u+\sinh u\cosh u}\Big\}$$

$$-\Big\{\bar{p}_1\big(\tfrac{y}{\alpha}\big)-\bar{p}_2\big(\tfrac{y}{\alpha}\big)\Big\}\Big\{\frac{\alpha u\cosh\alpha u\cosh u-u\sinh\alpha u\sinh u}{u-\sinh u\cosh u}\Big\}\Big]e^{-i\beta u}\,du$$

$$\tau_{xy}=-\frac{i}{2^{3/2}\pi^{1/2}\alpha}\int_{-\infty}^{\infty}\Big[\Big\{\bar{p}_1\big(\tfrac{y}{\alpha}\big)+\bar{p}_2\big(\tfrac{y}{\alpha}\big)\Big\}\Big\{\frac{\alpha u\sinh\alpha u\cosh u-u\sinh\alpha u\cosh u}{u+\sinh u\cosh u}\Big\} \qquad (3.74)$$

$$-\Big\{\bar{p}_1\big(\tfrac{y}{\alpha}\big)-\bar{p}_2\big(\tfrac{y}{\alpha}\big)\Big\}\Big\{\frac{\alpha u\sinh\alpha u\cosh u-u\cosh\alpha u\sinh u}{u-\sinh u\cosh u}\Big\}\Big]e^{-i\beta u}\,du$$

$$\frac{E}{1+\sigma}u=\frac{1}{2^{3/2}\pi^{1/2}}\int_{-\infty}^{\infty}\Big[\Big\{\bar{p}_1\big(\tfrac{y}{\alpha}\big)+\bar{p}_2\big(\tfrac{y}{\alpha}\big)\Big\}\Big\{\frac{\alpha u\cosh\alpha u\sinh u-2(1-\sigma)\sinh\alpha u\sinh u-u\sinh\alpha u\cosh u}{u+\sinh u\cosh u}\Big\}$$

$$-\Big\{\bar{p}_1\big(\tfrac{y}{\alpha}\big)-\bar{p}_2\big(\tfrac{y}{\alpha}\big)\Big\}\Big\{\frac{\alpha u\sinh\alpha u\cosh u-2(1-\sigma)\cosh\alpha u\cosh u-u\cosh\alpha u\sinh u}{u-\sinh u\cosh u}\Big\}\Big]e^{-i\beta u}\,\frac{du}{u}$$

$$\frac{E}{1+\sigma}v=-\frac{i}{2^{3/2}\pi^{1/2}}\int_{-\infty}^{\infty}\Big[\Big\{\bar{p}_1\big(\tfrac{y}{\alpha}\big)+\bar{p}_2\big(\tfrac{y}{\alpha}\big)\Big\}\Big\{\frac{\alpha u\sinh\alpha u\sinh u+(1-2\sigma)\cosh\alpha u\sinh u-u\cosh\alpha u\cosh u}{u+\sinh u\cosh u}\Big\}$$

$$-\Big\{\bar{p}_1\big(\tfrac{y}{\alpha}\big)-\bar{p}_2\big(\tfrac{y}{\alpha}\big)\Big\}\Big\{\frac{\alpha u\cosh\alpha u\cosh u+(1-2\sigma)\sinh\alpha u\cosh u-u\sinh\alpha u\cosh u}{u-\sinh u\cosh u}\Big\}\Big]e^{-i\beta u}\,\frac{du}{u}$$

It is readily seen that the integrals are convergent at u = 0, if the condition for static equilibrium (3·73) is satisfied. The results are in agreement with those obtained by Sneddon (32).

A particular case of some interest is that where point forces are applied to both sides of the strip. We have then $p_1(y) = p_2(y) = P\delta(y)$, so that $\overline{p_1}(\eta) = \overline{p_2}(\eta) = P/(2\pi)^{1/2}$. The expressions for the components of the stress vector then become

$$\sigma_x+\sigma_y = -\frac{2P}{\pi d}\int_0^\infty \frac{\cosh\alpha u \sinh u \cos\beta u}{u+\sinh u\cosh u}\,du.$$

$$\sigma_x-\sigma_y = \frac{2P}{\pi d}\int_0^\infty \frac{(\alpha u\sinh\alpha u\sinh u - u\cosh\alpha u\cosh u)\cos\beta u}{u+\sinh u\cosh u}\,du \tag{3.75}$$

$$\tau_{xy} = -\frac{P}{\pi d}\int_0^\infty \frac{(\alpha u\cosh\alpha u\sinh u - u\sinh\alpha u\cosh u)\sin\beta u}{u+\sinh u\cosh u}\,du$$

and for the components of the displacement vector we have

$$\frac{E}{1+\sigma}u = \frac{P}{\pi}\int_0^\infty \frac{(\alpha u\cosh\alpha u\sinh u - 2(1-\sigma)\sinh\alpha u\sinh u - u\sinh\alpha u\cosh u)\cos\beta u}{u+\sinh u\cosh u}\,du$$

$$\frac{E}{1+\sigma}v = -\frac{P}{\pi}\int_0^\infty \frac{(\alpha u\sinh\alpha u\sinh u + (1-2\sigma)\cosh\alpha u\sinh u - u\cosh\alpha u\cosh u)\sin\beta u}{u+\sinh u\cosh u}\,du$$

We see from the expression for v that when x = 0, so that $\alpha = 0$, then v = 0. The solution, therefore, corresponds to that of an elastic strip lying on a rigid plane co-incident with the plane x = 0 acted upon by a force P at the point x = d, y = 0. We see from the equations for the stress components when x = 0 that the pressure of the strip on the plane is given by

$$(\sigma_x)_{x=0} = -\frac{P}{\pi d}\int_0^\infty \frac{(\sinh u + u\cosh u)\cos\beta u}{u+\sinh u\cosh u}\,du \tag{3.76}$$

Equation (3.76) was evaluated numerically using the method devised by Filon and which is discussed in 3·8. The pressure on the rigid plane is shown graphically in fig.(vi)

3·8 Evaluation of integrals.

The integral expressions obtained for the components of stress and displacement in (3·7) are of a type which are of frequent occurrence in mathematical physics. It has not been found possible to evaluate these exactly so resort must be made to numerical methods. The presence of the trigonometric factor in the integrand however makes the use of ordinary quadrature formulae, such as Simpson's Rule, very inaccurate, unless a great many ordinates are used, and this of course makes the calculation extremely laborious. A method employed by Sneddon (29) in which he replaces the integrand by another function which can be integrated exactly, and which "fits" the original approximately, might well be applied to this type of integral. Filon devised a new quadrature formula for the numerical evaluation of integrals containing a trigonometric factor in the integrand and it is this formula which has been used for the calculations made here. Recently, an account was given by Fürth and Pringle (6) of an electrical machine which would seem to be admirably suited for the approximate evaluation of integrals of this type, and it is hoped to investigate this possibility.

The formula which we have used here, and which is due to Filon is

$$\int_a^b \psi(x)\cos kx\,dx = h\left[\alpha\left\{\psi(b)\sin kb - \psi(a)\sin ka\right\} + \beta S_{2r} + \gamma S_{2r-1}\right] \qquad (3.81)$$

in which the range of integration is divided into an even number of intervals of length h, $2S_{2r}$ is the sum of all the even ordinates of the curve $y = \psi(x)\cos kx$, less the first and last, and S_{2r-1} is the sum of all the odd ordinates of this curve. The constants α, β, and γ are defined by the expression

$$\alpha = \frac{1}{\theta} + \frac{\cos\theta\sin\theta}{\theta^2} - \frac{2\sin^2\theta}{\theta^3} \doteq \frac{2\theta^3}{45} - \frac{2\theta^5}{315} - \frac{2\theta^7}{4725}$$

$$\beta = 2\left[\frac{1+\cos^2\theta}{\theta^2} - \frac{2\sin\theta\cos\theta}{\theta^3}\right] \doteq \frac{2}{3} + \frac{2\theta^2}{15} - \frac{4\theta^4}{105} + \frac{2\theta^6}{567} - \frac{4\theta^8}{22275}$$

$$\gamma = 4\left(\frac{\sin\theta}{\theta^3} - \frac{\cos\theta}{\theta^2}\right) \doteq \frac{4}{3} - \frac{2\theta^2}{15} + \frac{\theta^4}{270} - \frac{\theta^6}{11340} + \frac{\theta^8}{997920}$$

where $\theta = kh$.

In the numerical evaluation of (3·76) the value of the integral over the range 0 to 6 was obtained using (3·81) and from 6 to ∞ the integrand was replaced by the approximate expression ~~by~~ $e^{-u}(u + 1)\cos\beta u$ which can be evaluated exactly. In this way the ordinates of the graph shown in fig.(vi) were calculated.

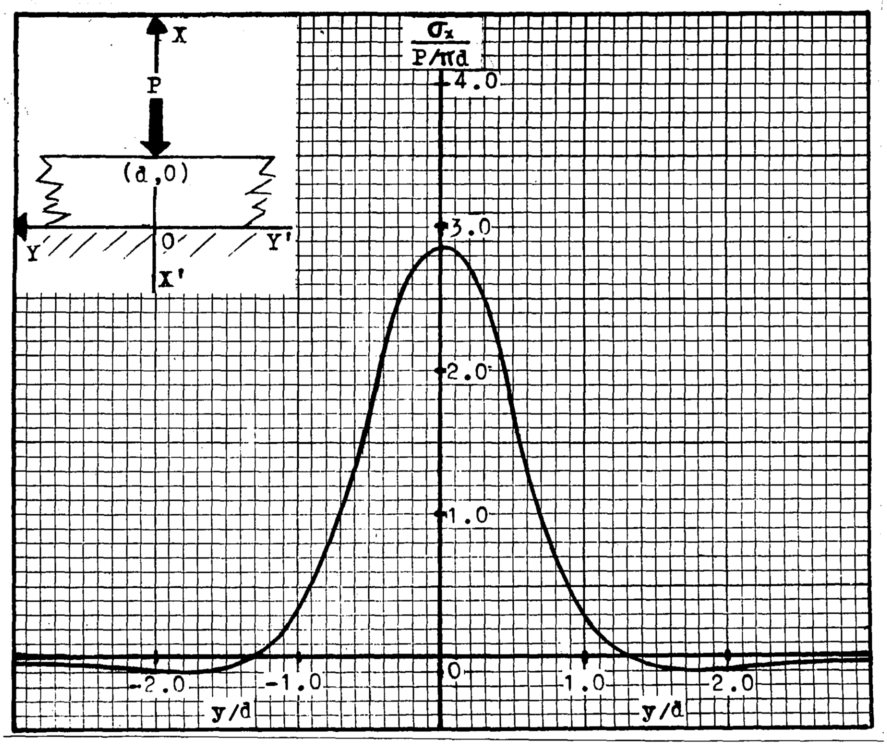

Fig. (vi).

Pressure of an elastic strip on a rigid plane due to a point force acting on the upper surface.

PART IV.

THE APPLICATION OF STATICAL FORCES TO THE INTERIOR OF A SEMI-INFINITE ELASTIC SOLID.

IV. THE APPLICATION OF STATICAL FORCES TO THE INTERIOR OF A SEMI-INFINITE ELASTIC SOLID.

4·1 Introduction

In a recent paper (28) Sneddon employed the Fourier cosine transform to solve the equations of equilibrium for a two dimensional elastic solid in a state of stress due to a force applied to its interior. He considered the case of a force concentrated at a point acting in the direction perpendicular to the boundary of the solid, and extended the analysis to the case of a force distributed uniformly over a line parallel to the boundary and in a direction normal to it.

The problem was treated in the first instance by Melan (24) who dealt with the point force acting (i) in the direction perpendicular to the boundary and (ii) int he direction parallel to the boundary. Both Green (10) and Stevenson (34) have given solutions of this problem employing the complex variable.

In this section we shall employ the results derived in the previous section to obtain first of all the stress distribution in a semi-infinite medium due to a point force acting in any direction. The results obtained are in agreement with the special cases mentioned above. The surface stress being of some importance was evaluated numerically and isochromatics for various directions of applied force were drawn. The analysis is extended to a force distributed over any curve lying within the medium and in some cases the surface stress was calculated.

4·2. Point force acting in any direction within the elastic solid.

Consider first of all an infinite two-dimensional elastic medium in which there has been set up the rectangular Cartesian co-ordinate axes OX, OY. Then if a force of magnitude F acts at the origin, in the direction making angle α with the negative direction of the x-axis, the components of the resulting stress tensor are obtained by replacing α by $\pi - \alpha$ in equation (2·35). These are

$$\sigma_x + \sigma_y = \frac{F}{2\pi(1-\sigma)} \cdot \frac{x\cos\alpha - y\sin\alpha}{r^2}$$

$$\sigma_x - \sigma_y = \frac{F}{\pi(1-\sigma)} \frac{1}{r^2}\left\{x\cos\alpha\left(-\sigma + \frac{x^2}{r^2}\right) + y\sin\alpha\left(1-\sigma-\frac{x^2}{r^2}\right)\right\} \qquad (4.21)$$

$$\tau_{xy} = \frac{F}{4\pi(1-\sigma)} \frac{1}{r^2}\left\{x\sin\alpha\left(-3+2\sigma+\frac{2x^2}{r^2}\right) + y\cos\alpha\left(1-2\sigma+\frac{2x^2}{r^2}\right)\right\}$$

where σ denotes Poisson's ration for the material comprising the solid. The components of the displacement vector are then

$$\frac{E}{1+\sigma}u = \frac{F}{4\pi(1-\sigma)}\left[\left\{(3-4\sigma)\log r - \frac{x^2}{r^2} - (1-2\sigma)\right\}\cos\alpha + xy\sin\alpha\right]$$

$$\frac{E}{1+\sigma}v = -\frac{F}{4\pi(1-\sigma)}\left[xy\cos\alpha + \left\{(3-4\sigma)\log r + \frac{x^2}{r^2} - 2(1-\sigma)\right\}\sin\alpha\right]$$

where E denotes Young's modulus for the material.

It is assumed as before that the origin is enclosed in a small cavity within the medium, and that the external force is applied over the surface of the cavity. The equilibrium of the solid as a whole is maintained by the application at a great distance from the origin of co-ordinates of a second external force system.

We consider now the equilibrium of the semi-infinite elastic medium $x \geqslant 0$ when a force of magnitude F acts at the point (h,0) in a direction making angle α with the negative direction of the x-axis.

The stress components must satisfy the equations of equilibrium (1.43) with $X = Y = 0$ and must have the same singularities at the point (h,0) as right hand sides of equations (4.21) have at the origin. Furthermore at the boundary $x = 0$ we have

$$\sigma_x = \tau_{xy} = 0 \tag{4.22}$$

We may take as before, solutions of the form

$$\sigma_x = \frac{\partial^2 \chi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 \chi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 \chi}{\partial x \partial y}$$

where $\chi(x,y)$ must be a solution of the biharmonic equation (3•23).

We can obtain a solution satisfying these conditions if we assume the existence of an image force at the point $(-h,0)$ acting in the opposite direction and then nullifying the stresses on the boundary so as to satisfy the boundary conditions. We write down a solution of the form

$$\sigma_x + \sigma_y = \frac{F\cos\alpha}{2\pi(1-\sigma)}\left[\frac{x-h}{\rho_1^2} - \frac{x+h}{\rho_2^2}\right] + \frac{F\sin\alpha}{2\pi(1-\sigma)}\,y\left[-\frac{1}{\rho_1^2} + \frac{1}{\rho_2^2}\right] + \frac{\partial^2\chi'}{\partial y^2} + \frac{\partial^2\chi'}{\partial x^2}$$

$$\sigma_x - \sigma_y = \frac{F\cos\alpha}{\pi(1-\sigma)}\left[-\frac{\sigma(x-h)}{\rho_1^2} + \frac{\sigma(x+h)}{\rho_2^2} + \frac{(x-h)^3}{\rho_1^4} - \frac{(x+h)^3}{\rho_2^4}\right]$$

$$+ \frac{F\sin\alpha}{\pi(1-\sigma)}\,y\left[\frac{1-\sigma}{\rho_1^2} - \frac{1-\sigma}{\rho_2^2} - \frac{(x-h)^2}{\rho_1^4} + \frac{(x+h)^2}{\rho_2^4}\right] + \frac{\partial^2\chi'}{\partial y^2} - \frac{\partial^2\chi'}{\partial x^2}$$

$$\tau_{xy} = \frac{F\sin\alpha}{4\pi(1-\sigma)}\left[-\frac{(3-2\sigma)(x-h)}{\rho_1^2} + \frac{(3-2\sigma)(x+h)}{\rho_2^2} + \frac{2(x-h)^3}{\rho_1^4} - \frac{2(x+h)^3}{\rho_2^4}\right]$$

$$+ \frac{F\cos\alpha}{4\pi(1-\sigma)}\,y\left[\frac{1-2\sigma}{\rho_1^2} - \frac{1-2\sigma}{\rho_2^2} + \frac{2(x-h)^2}{\rho_1^4} - \frac{2(x+h)^2}{\rho_2^4}\right] - \frac{\partial^2\chi'}{\partial x\partial y} \qquad (4{\cdot}23)$$

where $\rho_1^2 = (x-h)^2 + y^2$, and $\rho_2^2 = (x+h)^2 + y^2$

Then since these stress components satisfy the equations of equilibrium and have the correct singularities at the point (h,0) at which the force is applied it only remains to determine $\chi'(x,y)$ such that

$$\frac{\partial^2 \chi'}{\partial y^2} = p(y) \quad , \quad \frac{\partial^2 \chi'}{\partial x^2} = q(y) \tag{4.24}$$

where the functions p(y) and q(y) are the values obtained for σ_x and τ_{xy} by putting x = 0 in equations (4.23) giving

$$p(y) = -\frac{Fh\cos\alpha}{2\pi(1-\sigma)}\left\{\frac{1-2\sigma}{h^2+y^2} + \frac{2h^2}{(h^2+y^2)^2}\right\}$$

$$q(y) = \frac{Fh\sin\alpha}{2\pi(1-\sigma)}\left\{\frac{3-2\sigma}{h^2+y^2} - \frac{2h^2}{(h^2+y^2)^2}\right\}$$

Now if we insert these values in equations (3.51) and perform the integrations which are elementary we obtain the following expressions

$$\frac{\partial^2 \chi'}{\partial y^2} + \frac{\partial^2 \chi'}{\partial x^2} = \frac{2F\cos\alpha}{\pi(1-\sigma)}\left\{\frac{2(1-\sigma)x+(1-2\sigma)h}{\rho_2^2} - \frac{2h(x+h)^2}{\rho_2^4}\right\}$$

$$-\frac{2F\sin\alpha\, y}{\pi(1-\sigma)}\left\{\frac{1-\sigma}{\rho_2^2} - \frac{h(x+h)}{\rho_2^4}\right\}$$

$$\frac{\partial^2 \chi'}{\partial y^2} - \frac{\partial^2 \chi'}{\partial x^2} = \frac{2F\cos\alpha\, x}{\pi(1-\sigma)}\left\{-\frac{1-\sigma}{\rho_2^2} - \frac{\left(2(1-\sigma)x-(1+2\sigma)h\right)(x+h)}{\rho_2^4} + \frac{4h(x+h)^3}{\rho_2^6}\right.$$

$$\left.+\frac{2F\sin\alpha\, y}{\pi(1-\sigma)}\left\{\frac{1-\sigma}{\rho_2^2} - \frac{2(1-\sigma)x^2+2(2-\sigma)xh+h^2}{\rho_2^4} + \frac{4xh(x+h)^2}{\rho_2^6}\right\}\right.$$

$$\frac{\partial^2 \chi'}{\partial x \partial y} = -\frac{F\cos\alpha}{\pi(1-\sigma)}\, xy\left\{\frac{2(1-\sigma)x+(1-2\sigma)h}{\rho_2^4} + \frac{4h(x+h)^2}{\rho_2^6}\right\} \tag{4.26}$$

$$+\frac{F\sin\alpha}{2\pi(1-\sigma)}\left\{\frac{4(1-\sigma)x+(3-2\sigma)h}{\rho_2^2} - \frac{2(x+h)\left(2(1-\sigma)x^2+2(3-\sigma)xh+h^2\right)}{\rho_2^4} + \frac{8xh(x+h)^2}{\rho_2^6}\right\}$$

Combining equations (4·23) and (4·26) we find the components of the stress tensor for this system to be

$$\sigma_x+\sigma_y = \frac{F\cos\alpha}{2\pi(1-\sigma)}\left[\frac{x-h}{\rho_1^2} + \frac{(3-4\sigma)x+(1-4\sigma)h}{\rho_2^2} + \frac{4h(x+h)^2}{\rho_2^4}\right]$$
$$+\frac{F\sin\alpha}{2\pi(1-\sigma)}y\left[-\frac{1}{\rho_1^2} - \frac{3-4\sigma}{\rho_2^2} + \frac{4h(x+h)}{\rho_2^4}\right]$$

$$\sigma_x-\sigma_y = \frac{F\cos\alpha}{\pi(1-\sigma)}\left[-\frac{\sigma(x-h)}{\rho_1^2} - \frac{(2-3\sigma)x-\sigma h}{\rho_2^2} + \frac{(x-h)^3}{\rho_1^4} + \frac{(x+h)\{(3-4\sigma)x^2-4(1-\sigma)xh-h^2\}}{\rho_2^4} + \frac{8xh(x+h)^3}{\rho_2^6}\right]$$
$$+\frac{F\sin\alpha}{\pi(1-\sigma)}y\left[\frac{1-\sigma}{\rho_1^2} + \frac{1-\sigma}{\rho_2^2} - \frac{(x-h)^2}{\rho_1^4} - \frac{(3-4\sigma)x^2+2(3-2\sigma)xh+h^2}{\rho_2^4} + \frac{8xh(x+h)^2}{\rho_2^6}\right]$$

$$\tau_{xy} = \frac{F\sin\alpha}{4\pi(1-\sigma)}\left[-\frac{(3-2\sigma)(x-h)}{\rho_1^2} - \frac{(5-6\sigma)x+(3-2\sigma)h}{\rho_2^2} + \frac{2(x-h)^3}{\rho_1^4} + \frac{2(x+h)\{(3-4\sigma)x^2+2(5-2\sigma)xh+h^2\}}{\rho_2^4} - \frac{16xh(x+h)^3}{\rho_2^6}\right] \qquad (4{\cdot}27)$$
$$+\frac{F\cos\alpha}{4\pi(1-\sigma)}y\left[\frac{1-2\sigma}{\rho_1^2} - \frac{1-2\sigma}{\rho_2^2} + \frac{2(x-h)^2}{\rho_1^4} + \frac{2\{(3-4\sigma)x^2-4\sigma xh-h^2\}}{\rho_2^4} + \frac{16xh(x+h)^2}{\rho_2^6}\right]$$

The components of the displacement vector could be obtained in the same way but it was not considered worth while doing the analysis since most interest in this type of problem is in the stresses set up in the material.

At the boundary x = 0 there is only one non-vanishing components of stress and this is given by the equation

$$\sigma_y = \frac{2F}{\pi}\left[\frac{h\left(h^2-\frac{\sigma}{1-\sigma}y^2\right)\cos\alpha}{(h^2+y^2)^2} + \frac{y\left(\frac{\sigma}{1-\sigma}h^2-y^2\right)\sin\alpha}{(h^2+y^2)^2}\right] \tag{4.28}$$

The curves in fig.(vii). were computed from this expression.

The maximum shearing stress as defined by equation (2·34) was also evaluated for a number of points in the medium, taking $\sigma = 1/2$, for values of $\alpha = 0, \frac{\pi}{4}, \frac{\pi}{2}$, and the isochromatic lines shown in fig.(viii) were drawn.

4·3 Uniform pressure over a curve lying within the solid.

We consider now the stress produced in a semi-infinite elastic medium by a uniform pressure applied over a curve lying within the solid. We assume that at any point of the curve the applied force is acting in the direction of the normal to the curve at the point. Consider an element of the curve ds at the point $P(\xi,\eta)$ cf. fig (ix). Let the tangent PT to the curve make an angle β with the positive direction of the x-axis, and let the normal at P make angle a with the negative direction of the x-axis. If the parametric equations of the curve are $\xi = \xi(\theta)$, $\eta = \eta(\theta)$, and the extremities A and B of the curve being given by the values θ_1 and θ_2 of the parameter θ, then

$$\sin a = \cos\beta = d\xi/ds, \quad \cos a = \sin\beta = d\eta/ds$$

If the applied pressure is $p(\theta)$ then the force on the element is $F = p(\theta)\,ds$. It follows from the equation (4·28) that the surface stress due to the element of force at the point P is given by the equation

$$\sigma_y = \frac{2p(\theta)\,ds}{\pi}\left[\left\{\frac{\xi^2 - \frac{\sigma}{1-\sigma}(y-\eta)^2}{(\xi^2+(y-\eta)^2)^2}\right\}\xi\frac{d\eta}{ds} - \left\{\frac{(y-\eta)^2 - \frac{\sigma}{1-\sigma}\xi^2}{(\xi^2+(y-\eta)^2)^2}\right\}(y-\eta)\frac{d\xi}{ds}\right] \tag{4.301}$$

Now if we integrate with respect to θ from θ_1, to θ_2 we obtain an expression for the surface stress due to pressure $p(\theta)$ acting along the curve AB. This integral may be written in the form

$$\sigma_y = \frac{2}{\pi}\left\{I_1 + \frac{1}{1-\sigma}I_2\right\} \tag{4.302}$$

where

$$I_1 = \int_{\theta_1}^{\theta_2} p(\theta)\,\frac{\xi\eta' - (y-\eta)\xi'}{\xi^2+(y-\eta)^2}\,d\theta$$

and

$$I_2 = \int_{\theta_1}^{\theta_2} p(\theta)\,\frac{\{\xi\xi' - (y-\eta)\eta'\}\xi(y-\eta)}{\{\xi^2+(y-\eta)^2\}^2}\,d\theta$$

and we have written $\xi' = \dfrac{d\xi}{d\theta}$, and $\eta' = \dfrac{d\eta}{d\theta}$

By means of integration by parts it may be shown that

$$I_2 = \frac{1}{2}\left[-p(\theta)\frac{\xi(y-\eta)}{\xi^2+(y-\eta)^2}\right]_{\theta_1}^{\theta_2} - \frac{1}{2}I_1 + \frac{1}{2}\int_{\theta_1}^{\theta_2}\frac{p'(\theta)\,\xi(y-\eta)}{\xi^2+(y-\eta)^2}\,d\theta$$

so that in the special case in which $p(\theta)$ is constants, that is, the distribution of pressure is uniform, we have

$$I_2 = F(\theta_2,\theta_1) - \tfrac{1}{2}I_1$$

where

$$F(\theta_2,\theta_1) = \frac{p}{2}\left[\frac{\xi(y-\eta)}{\xi^2+(y-\eta)^2}\right]_{\theta_2}^{\theta_1} \tag{4.303}$$

and

$$I_1 = p\int_{\theta_1}^{\theta_2} \frac{\xi\eta' - (y-\eta)\xi'}{\xi^2+(y-\eta)^2}\,d\theta \tag{4.304}$$

substituting into equation (4·302) from (4·303) and (4·304) we have

$$\sigma_y = \frac{1}{\pi(1-\sigma)}\left[(1-2\sigma)I_1 + 2F(\theta_2,\theta_1)\right] \tag{4.305}$$

for the surface stress due to a pressure p applied uniformly along the curve.

In the special case in which the medium is incompressible so that $\sigma = 1/2$ this reduces to

$$\sigma_y = \frac{2p}{\pi}\left[\frac{\xi_1(y-\eta_1)}{\xi_1^2+(y-\eta_1)^2} - \frac{\xi_2(y-\eta_2)}{\xi_2^2+(y-\eta_2)^2}\right] \tag{4.306}$$

where (ξ_1, η_1) and (ξ_2, η_2) are the co-ordinates of the points A and B. (cf. fig.(x)). From fig.(x) it may be seen that

$$\sigma_y = \frac{p}{\pi}\left\{\sin 2\theta_1 - \sin 2\theta_2\right\}$$

and is thus independent of the shape of the curve, but is the same as would be produced by uniform pressure over the straight line AB.

If the curve is symmetrical about the x-axis (cf.fig.(xi) equation (4.36) may be put in the form

$$\sigma_y = \frac{2ph}{\pi}\left[\frac{y+a}{h^2+(y+a)^2} - \frac{y-a}{h^2+(y+a)^2}\right]$$

$$= \frac{2p}{\pi}\left[\frac{\beta+\alpha}{1+(\beta+\alpha)^2} - \frac{\beta-\alpha}{1+(\beta-\alpha)^2}\right] \qquad (4.307)$$

where we have written $\beta = y/h$, and $\alpha = a/h$. If we expand in ascending powers of α and neglect terms containing α^2 and higher powers we obtain the equation

$$\sigma_y = \frac{4p}{\pi}\,\frac{\alpha(1-\beta^2)}{(1+\beta^2)^2}$$

which is in agreement with equation (4.28) obtained for the surface stress due to a point force if we replace p by F/2a. We may deduce from equation (4.308) that the maximum surface stress occurs at the origin, while $y = \sqrt{3}h$ gives minimum values; the stress is zero at $y = \pm h$.

If we examine the function

$$f(\beta) = \frac{\beta+\alpha}{1+(\beta+\alpha)^2} - \frac{\beta-\alpha}{1+(\beta-\alpha)^2}$$

we find that if $\alpha \leq \sqrt{3}$ the function has one maximum turning value at $\beta = 0$ and two minimum turning values at $\beta = \pm\{\alpha^2 + 1 + 2(\alpha^2 + 1)^{\frac{1}{2}}\}^{\frac{1}{2}}$. On the other hand if $\alpha > \sqrt{3}$, $\beta = 0$ becomes a minimum turning value and two maxima appear at $\beta = \pm\{\alpha^2 + 1 + 2(\alpha^2 + 1)^{\frac{1}{2}}\}^{\frac{1}{2}}$.

We can obtain expressions for the components of stress in the interior of the medium by proceeding in the same way as for the surface stress. The expressions although more complicated are of the same form and are given by the equations

$$\sigma_x + \sigma_y = \frac{1}{2\pi(1-\sigma)}\left[F_1(\theta_2,\theta_1) + (1-2\sigma)\bar{\Phi}_1\right]$$

$$\sigma_x - \sigma_y = \frac{1}{2\pi(1-\sigma)}\left[F_2(\theta_2,\theta_1) + (1-2\sigma)\bar{\Phi}_2\right] \qquad (4.308)$$

$$\tau_{xy} = \frac{1}{2\pi(1-\sigma)}\left[F_3(\theta_2,\theta_1) + (1-2\sigma)\bar{\Phi}_3\right]$$

where F_1, F_2, F_3, $\bar{\Phi}_1$, $\bar{\Phi}_2$, and $\bar{\Phi}_3$ are defined by the equations

$$F_1 = p\int_{\theta_1}^{\theta_2}\left[\left\{\frac{x-\xi}{\rho_1^2} + \frac{x-\xi}{\rho_2^2} + \frac{4\xi(x+\xi)^2}{\rho_2^4}\right\}\eta' - \left\{\frac{1}{\rho_1^2} + \frac{1}{\rho_2^2} - \frac{4\xi(x+\xi)}{\rho_2^4}\right\}(y-\eta)\xi'\right]d\theta$$

$$F_2 = p\int_{\theta_1}^{\theta_2}\left[\left\{-\frac{x-\xi}{\rho_1^2} - \frac{x-\xi}{\rho_2^2} + \frac{2(x-\xi)^3}{\rho_1^4} + \frac{2(x-\xi)(x+\xi)^2}{\rho_2^4} - \frac{12x\xi(x+\xi)}{\rho_2^4} + \frac{16x\xi(x+\xi)^3}{\rho_2^6}\right\}\eta'\right.$$
$$\left. + \left\{\frac{1}{\rho_1^2} + \frac{1}{\rho_2^2} - \frac{2(x-\xi)^2}{\rho_1^4} - \frac{2(x+\xi)^2}{\rho_2^4} - \frac{4x\xi}{\rho_2^4} + \frac{16x\xi(x+\xi)^2}{\rho_2^6}\right\}(y-\eta)\xi'\right]d\theta.$$

$$F_3 = p\int_{\theta_1}^{\theta_2}\left[\left\{-\frac{x-\xi}{\rho_1^2} - \frac{x+\xi}{\rho_2^2} + \frac{(x-\xi)^3}{\rho_1^4} + \frac{(x+\xi)^3}{\rho_2^4} + \frac{6x\xi(x+\xi)}{\rho_2^4} - \frac{8x\xi(x+\xi)}{\rho_2^6}\right\}\xi'\right.$$
$$\left. + \left\{\frac{(x-\xi)^2}{\rho_1^4} + \frac{(x-\xi)^2}{\rho_2^4} - \frac{2\xi^2}{\rho_2^4} + \frac{8x\xi(x+\xi)^2}{\rho_2^6}\right\}(y-\eta)\eta'\right]d\theta$$

$$\bar{\Phi}_1 = 2p\int_{\theta_1}^{\theta_2}\left[\frac{(x+\xi)\eta' - (y-\eta)\xi'}{\rho_2^2}\right]d\theta \tag{4.310}$$

$$\bar{\Phi}_2 = p\int_{\theta_1}^{\theta_2}\left[\left\{\frac{x-\xi}{\rho_1^2} - \frac{3x+\xi}{\rho_2^2} + \frac{4x(x+\xi)^2}{\rho_2^4}\right\}\eta' + \left\{\frac{1}{\rho_1^2} + \frac{1}{\rho_2^2} - \frac{4x(x+\xi)}{\rho_2^4}\right\}(y-\eta)\xi'\right]d\theta$$

$$\bar{\Phi}_3 = \frac{p}{2}\int_{\theta_1}^{\theta_2}\left[\left\{-\frac{x-\xi}{\rho_1^2} - \frac{3x+\xi}{\rho_2^2} + \frac{4x(x+\xi)^2}{\rho_2^4}\right\}\xi' + \left\{\frac{1}{\rho_1^2} - \frac{1}{\rho_2^2} + \frac{4x(x+\xi)^2}{\rho_2^4}\right\}(y-\eta)\eta'\right]d\theta$$

If we take $\sigma = \frac{1}{2}$ then the expressions for the stress components reduce to

$$\sigma_x + \sigma_y = \frac{1}{\pi}F_1(\theta_2,\theta_1)$$

$$\sigma_x - \sigma_y = \frac{1}{\pi}F_2(\theta_2,\theta_1) \tag{4.311}$$

$$\tau_{xy} = \frac{1}{\pi}F_3(\theta_2,\theta_1)$$

These expressions are immediately inttegrable so that as in the case of the surface stress the components of stress in the interior are independent of the shape of the curve along which the pressure is applied, but depends only on the co-ordinates of the end points of the curve. In the notation of fig.(xii) these integrals are given by

$$\sigma_x+\sigma_y = \frac{p}{\pi}\left[\psi_2 - \psi_1 + \phi_2 - \phi_1 + \frac{h_2}{x+h_2}\sin 2\phi_2 - \frac{h_1}{x+h_1}\sin 2\phi_1\right]$$

$$\sigma_x - \sigma_y = \frac{p}{2\pi}\left[\sin 2\psi_2 - \sin 2\psi_1 + \frac{x^2+2xh_2-h_2^2}{(x+h_2)^2}\sin 2\phi_2 - \frac{x^2+2xh_1-h_1^2}{(x+h_1)^2}\sin 2\phi_1 \right.$$

$$\left. + \frac{xh_2}{(x+h_2)^2}\sin 4\phi_2 - \frac{xh_1}{(x+h_1)^2}\sin 4\phi_1\right]$$

$$\tau_{xy} = -\frac{p}{2\pi}\left[\frac{(x-h_2)^2}{d_2^2} - \frac{(x-h_1)^2}{d_1^2} + \frac{4xh_2(x+h_2)^2}{R_2^4} - \frac{4xh_1(x+h_1)}{R_1^4}\right.$$

$$\left. + \frac{x^2-2xh_2-h_2^2}{R_2^2} - \frac{x^2-2xh_1-h_1^2}{R_1^2}\right]$$

(4.312)

When AB is parallel to the boundary so that $h_1 = h_2$, these expressions agree with those given by Sneddon (28) for the case of $\sigma = \frac{1}{2}$, the case which he has evaluated numerically. If the elastic medium is compressible so that $\sigma \neq \frac{1}{2}$ the integrals Φ_1, Φ_2, Φ_3 must be taken into account, and these are not independent of the shape of the curve

4.4 Force uniformly distributed over a line.

We shall consider now the particular case of a force P distributed uniformly over a line of length 2a, inclined at an angle α to the y-axis and bisected by the x-axis at the point (h,0). The freedom equations of the line are

$$\eta = \theta \cos\alpha$$
$$\xi = h + \theta\sin\alpha \qquad -a \leq \theta \leq a \tag{4.41}$$

where for the analysis to be valid we must assume that $a \sin\alpha \leq h$.

In the first instance we shall consider the expressions for the surface stress. Substituting from equations (4.41) into equations (4.303) and putting $p = P/2a$ we have

$$F(\theta_2,\theta_1) = \frac{P}{4a}\left[\frac{(h+\theta\sin\alpha)(y-\theta\cos\alpha)}{(h+\theta\sin\alpha)^2+(y-\theta\cos\alpha)^2}\right]_a^{-a}$$

After a little reduction this becomes

$$F(\theta_2,\theta_1) = \frac{P}{2}\left[\frac{(y\sin\alpha + h\cos\alpha)(h^2+a^2\cos 2\alpha - y^2)}{(h^2+y^2+a^2) - 4a^2(h\sin\alpha - y\cos\alpha)^2}\right] \tag{4.42}$$

Also substituting from (4.41) into (4.304) and performing the integration we obtain

$$\sigma_y = \frac{P}{2\pi a(1-\sigma)}\left[\left(\frac{1-2\sigma}{2}\right)\sin 2\alpha \log\left\{\frac{(h+a\sin\alpha)^2+(y-a\cos\alpha)^2}{(h-a\sin\alpha)^2+(y+a\cos\alpha)^2}\right\}\right.$$
$$+ (1-2\sigma)\cos 2\alpha \tan^{-1}\left\{\frac{2a(h\cos\alpha + y\sin\alpha)}{h^2+y^2-a^2}\right\}$$
$$\left. - \frac{2a(y\sin\alpha + h\cos\alpha)(y^2-h^2-a^2\cos 2\alpha)}{(h^2+y^2+a^2)^2 - 4a^2(h\sin\alpha - y\cos\alpha)^2}\right] \tag{4.43}$$

which reduces to

$$\sigma_y = \frac{Ph}{\pi(1-\sigma)}\left[\frac{1-2\sigma}{2ah}\tan^{-1}\left(\frac{2ah}{h^2+y^2-a^2}\right) + \frac{a^2+h^2-y^2}{(h^2+y^2+a^2)^2-4a^2y^2}\right]$$

in the case when $\alpha = 0$, in agreement with the result obtained by Sneddon (28).

If we put $\alpha = \pi/2$ we obtain

$$\sigma_y = -\frac{Py}{\pi(1-\sigma)}\left[\frac{1-2\sigma}{2ay}\tan^{-1}\left(\frac{2ay}{h^2+y^2-a^2}\right)+\frac{a^2+y^2-h^2}{(h^2+y^2+a^2)^2-4a^2h^2}\right]$$

These expressions along with the one obtained by putting $\alpha = \pi/4$ in equation (4·43) are shown graphically in fig.(xiii) for the case $\sigma = 1/2$.

The stress components in the interior of the elastic medium may be obtained by substituting from equation (4·41) into the equations (4·310) and performing the integrations. These are found to be

$$\Phi_1 = \frac{P}{a}\left[\sin 2\alpha \log\frac{R_1}{R_2} + \cos 2\alpha\,(\phi_2-\phi_1)\right]$$

$$\Phi_2 = \frac{P}{2a}\left[-\sin 2\alpha \log\frac{R_1 r_1}{R_2 r_2} - \cos 2\alpha\,(\phi_2-\phi_1-\psi_2+\psi_1) + 2ax\cos 3\alpha\left(\frac{1}{R_1^2}+\frac{1}{R_2^2}\right)\right.$$

$$\left. -x\left[\frac{\{(x+h)^2+y^2\}\{(x+h)\sin 2\alpha\,(1+\cos 2\alpha)+y\cos 2\alpha(1-\cos 2\alpha)\}+2(x+h)^2 y}{\{(x+h)\cos\alpha+y\sin\alpha\}^2}\right]\left(\frac{1}{R_1^2}-\frac{1}{R_2^2}\right)\right]$$

$$\Phi_3 = \frac{P}{4a}\left[\cos 2\alpha \log\frac{R_1 r_2}{R_2 r_1} - \sin 2\alpha\,(\phi_2-\phi_1+\psi_2-\psi_1) + 2ax\sin 3\alpha\left(\frac{1}{R_1^2}+\frac{1}{R_2^2}\right)\right.$$

$$\left. +x\left[\frac{\{(x+h)^2+y^2\}\{(x+h)\cos 2\alpha\,(1+\cos 2\alpha)-y\sin 2\alpha(1-\cos 2\alpha)\}-2(x+h)^2 y}{\{(x+h)\cos\alpha+y\sin\alpha\}^2}\right]\left(\frac{1}{R_1^2}-\frac{1}{R_2^2}\right)\right] \tag{4.44}$$

where R_1, R_2, r_1, r_2, ϕ_1, ϕ_2, ψ_1, ψ_2, are defined in fig.(xii)

Combining these expressions with (4·311) we obtain the components of stress in the interior of the elastic medium due to a force P distributed over a line of length 2a inclined at the angle α to the surface. Putting $\alpha = 0$ we get agreement with Sneddon (28).

The integrals have also been evaluated for various other shapes of curve but the expressions are long and unwieldy and were not considered of sufficient importance to warrant inclusion here.

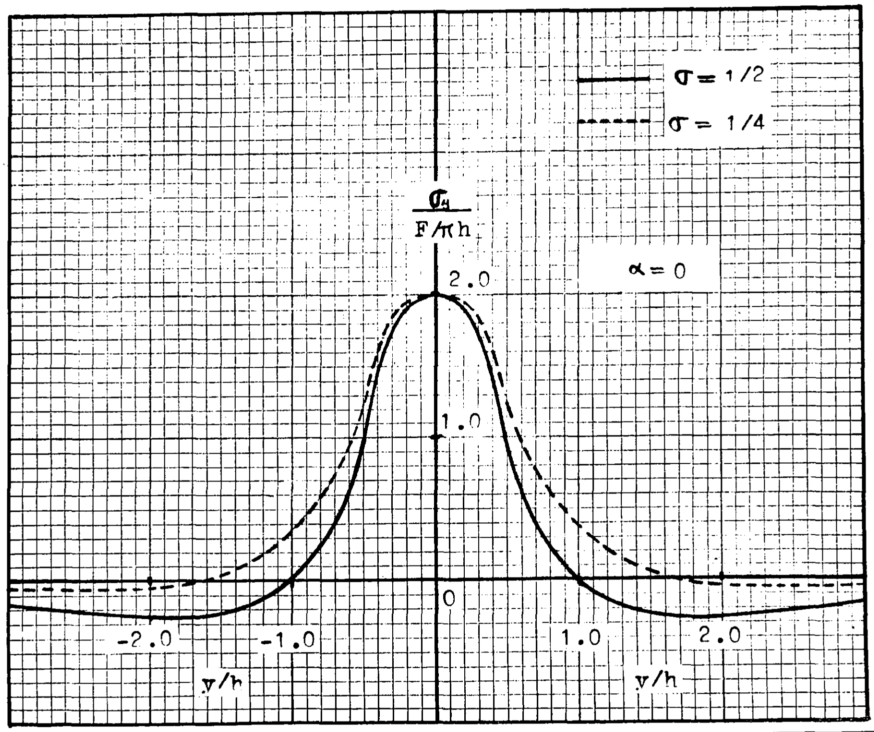

Fig.(vii)a.

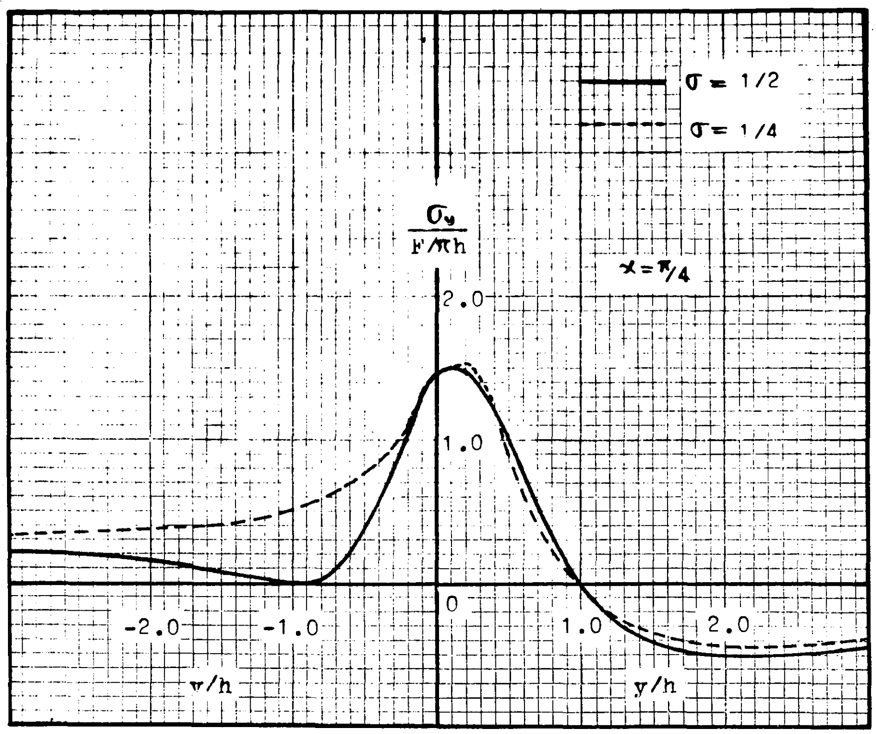

Fig.(vii)b.

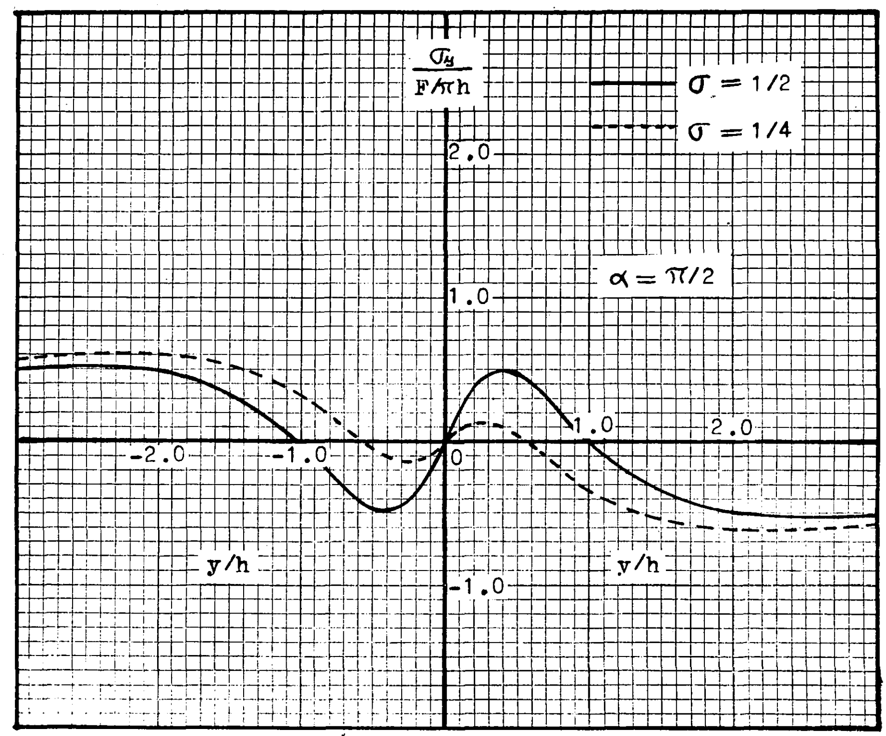

Fig.(vii) c.

Surface stress due to a point force acting in the interior of a semi-infinite two-dimensional elastic medium.

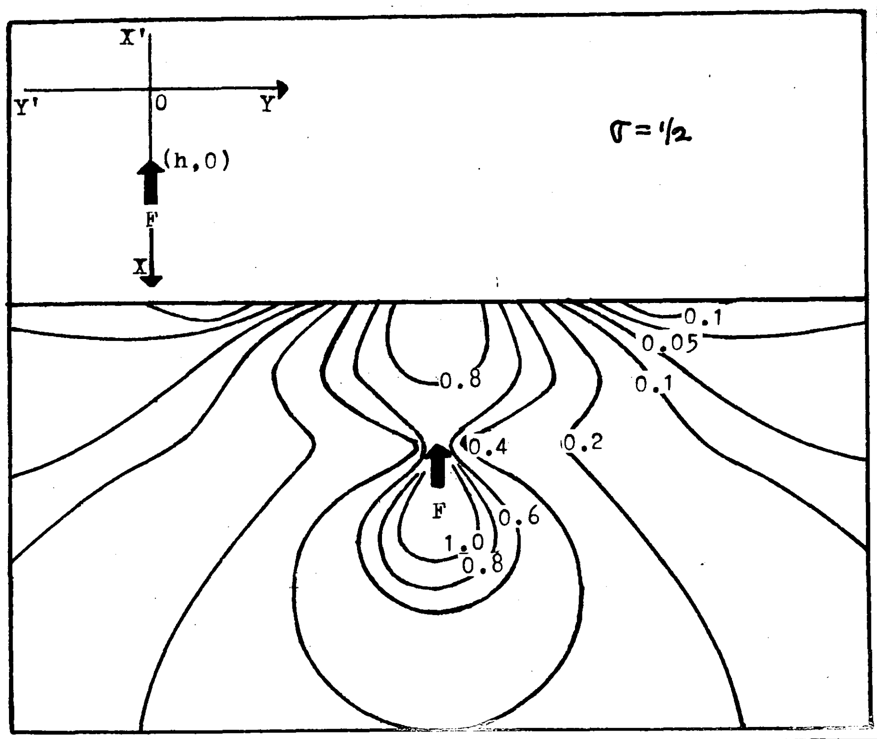

Fig. (viii) a

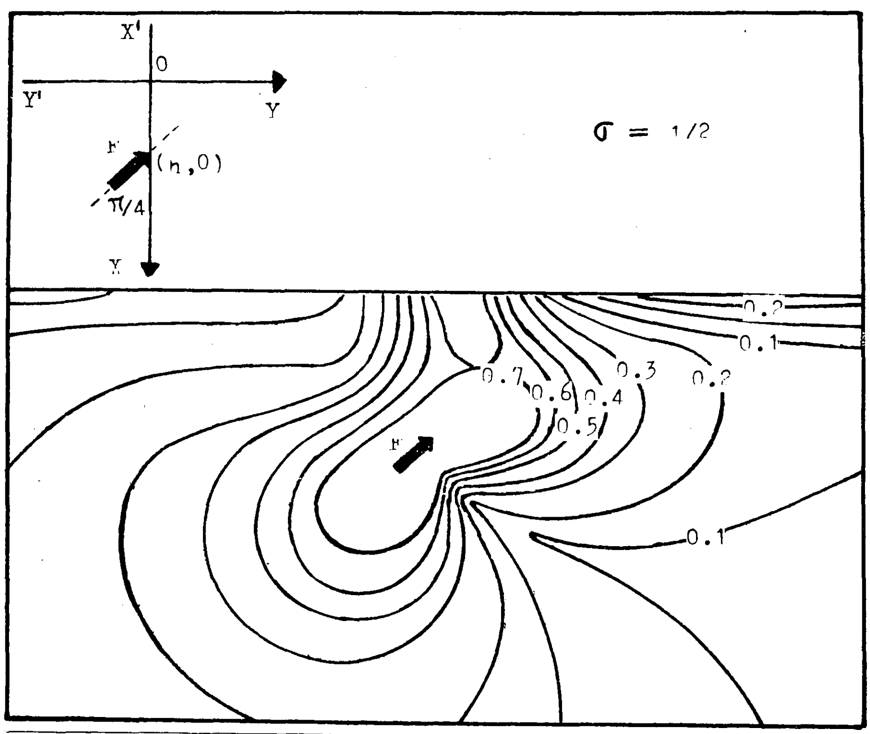

Fig. (viii) b.

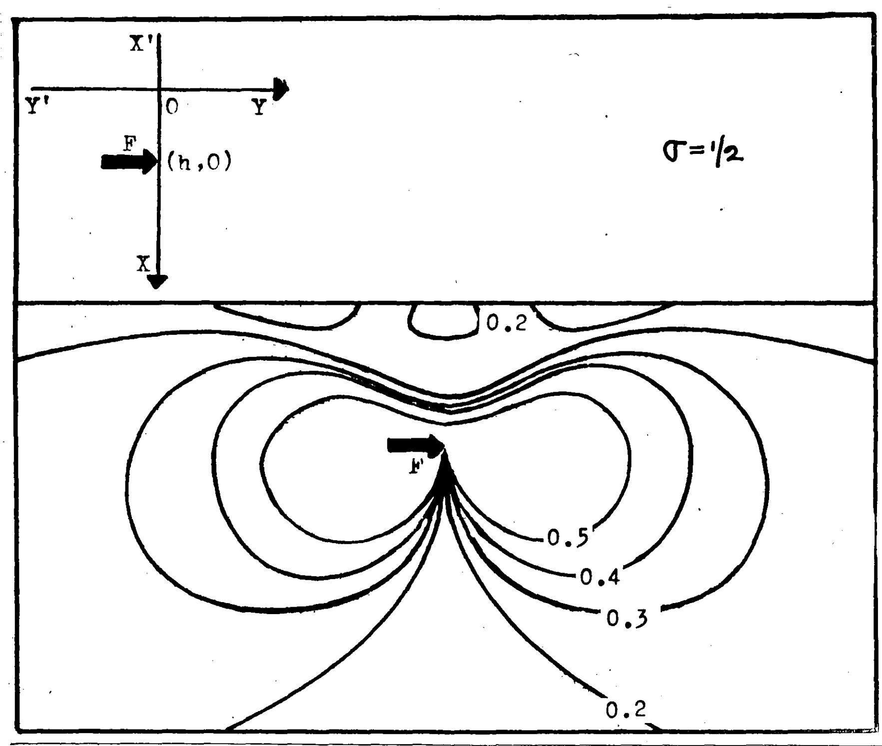

Fig.(viii)c

Lines of constant maximum shearing stress due to a point force acting in the interior of a semi-infinite two-dimensional elastic medium.

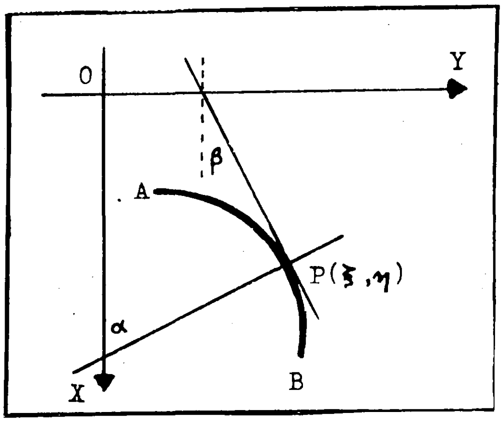

Fig. (ix)

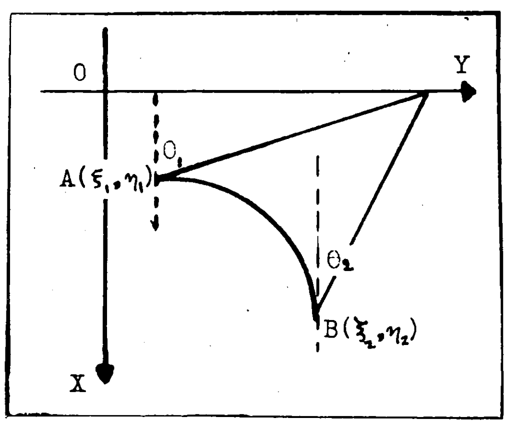

Fig. (x)

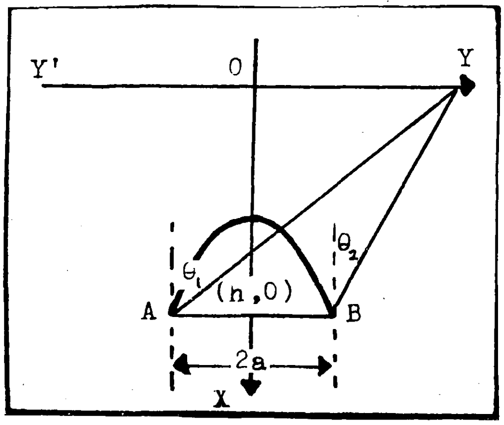

Fig.(xi).

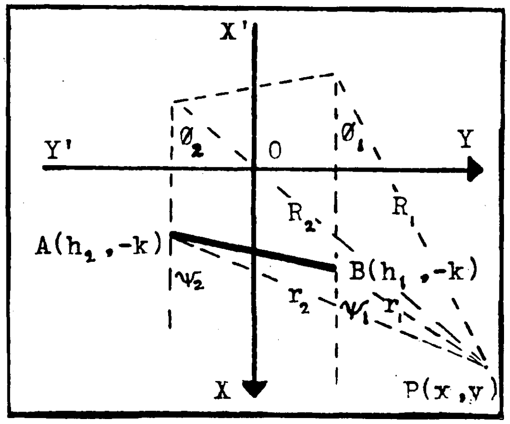

Fig.(xii).

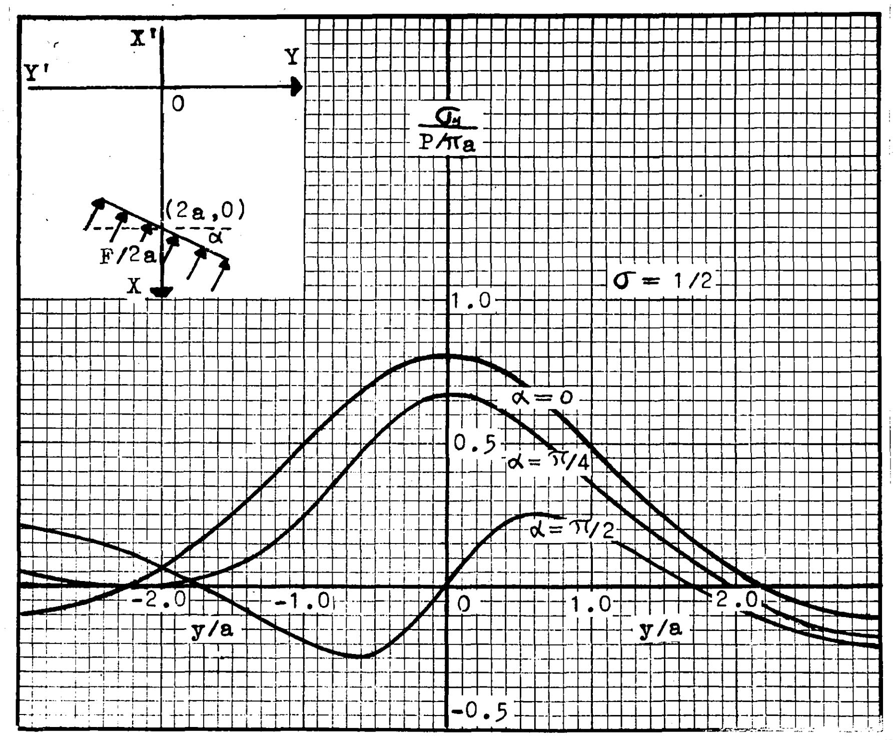

Fig.(xiii).

Surface stress due to a force uniformly distributed over a line in the interior of a semi-infinite two-dimensional elastic medium.

PART V.

THE APPLICATION OF DYNAMICAL FORCES TO THE INTERIOR OF AN INFINITE TWO-DIMENSIONAL ELASTIC MEDIUM.

V. THE APPLICATION OF DYNAMICAL FORCES TO THE INTERIOR OF AN INFINITE TWO-DIMENSIONAL ELASTIC MEDIUM.

5·1 Introduction.

In this section we shall be concerned with the solution of the equations of motion applicable to an infinite two-dimensional isotropic elastic medium with arbitrary forces which may vary with time applied to its interior.

The fact that a disturbance in an elastic medium is propagated by two different types of waves, travelling with different velocities, is evident from the general theory, and was pointed out in the first instance by Poisson. The character of the two types of waves is indicated by the names assigned to them, namely waves of irrotational distortion and waves of equivoluminal rotation but more recently they have been referred to as P-waves, and S-waves respectively. It is readily seen from the subsequent analysis that the velocities of propagation of the P-waves and S-waves are in the ratio $\frac{\lambda+2\mu}{\mu}$, λ and μ being the Lamé elastic constants.

The particular case in which the applied force is periodic was considered by Rayleigh (27) who showed that at a considerable distance from the point of application of the force the resulting disturbance consisted of two trains of waves moving with the velocities of P- and S-waves. Subsequently Lamb in the course of an investigation into the effect of a periodic force applied to the bounding surface of a semi-infinite elastic medium, also derived the solution of this problem.

The problem of an arbitrary force applied to the interior of an infinite three dimensional elastic medium has been considered by Love (20) who used a property of the Poisson integral formula to obtain the solution.

Here we shall use the three dimensional Fourier transform to obtain a formal solution of the two-dimensional problem with arbitrary forces applied to the interior. Rayleigh's problem will be considered briefly and the complete solution will be given when the applied forces have the form of the Dirac δ-function, and the Heavieside unit function since these problems appear to be some interest in geophysics. The problem of a moving pulse of pressure is also considered and it will be seen that the form of solution is similar to that obtained by Eshelby (2) when discussing moving dislocations.

5·2 Solution of the equations of motion.

We shall consider the distribution of stress, and the associated displacement in an infinite two-dimensional isotropic elastic medium when forces which vary with time are applied to certain regions of the medium. If we describe the position of a point in the elastic solid by means of the rectangular Cartesian co-ordinates x and y, then the state of stress will be uniquely determined by the three components σ_x, σ_y, and τ_{xy} of the stress tensor. Taking the components of the displacement vector to be u and v and denoting by X and Y the components of the applied force per unit mass in the directions of the x- and y-axes respectively, the equations of motion are equations (1·42) ρ denoting the density of the material comprising the solid and t being the time variable. Furthermore the components of stress are expressed in terms of the components of displacement by the relations (1·41) in which λ and μ are the Lamé elastic constants.

We introduce now the two functions $\phi(x,y,t)$, and $\psi(x,y,t)$ such that

$$u = \frac{\partial\phi}{\partial x} + \frac{\partial\psi}{\partial y}, \quad v = \frac{\partial\phi}{\partial y} - \frac{\partial\psi}{\partial x} \qquad (5.201)$$

and the expressions for the components of stress are then given in terms of ϕ and ψ by the equations

$$\sigma_x = \lambda\nabla_1^2\phi + 2\mu\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\psi}{\partial x\partial y}\right), \quad \sigma_y = \lambda\nabla_1^2\phi + 2\mu\left(\frac{\partial^2\phi}{\partial y^2} - \frac{\partial^2\psi}{\partial x\partial y}\right),$$

$$\tau_{xy} = \mu\left(2\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) \qquad (5.202)$$

in which we have written $\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ for the two-dimensional Laplacian operator. Substituting from equations (5•202) into the equations of motion and writing

$$X = \frac{\partial \Phi}{\partial x} + \frac{\partial \Psi}{\partial y} \quad , \quad Y = \frac{\partial \Phi}{\partial y} - \frac{\partial \Psi}{\partial x} \tag{5.203}$$

we find that the equations of motion may be written in the form

$$\begin{aligned} &\frac{\partial}{\partial x}\left\{(\lambda+2\mu)\nabla_1^2\phi - \rho\frac{\partial^2\phi}{\partial t^2} + \rho\Phi\right\} + \frac{\partial}{\partial y}\left\{\mu\nabla_1^2\psi - \rho\frac{\partial^2\psi}{\partial t^2} + \rho\Psi\right\} = 0 \\ &\frac{\partial}{\partial y}\left\{(\lambda+2\mu)\nabla_1^2\phi - \rho\frac{\partial^2\phi}{\partial t^2} + \rho\Phi\right\} - \frac{\partial}{\partial x}\left\{\mu\nabla_1^2\psi - \rho\frac{\partial^2\psi}{\partial t^2} + \rho\Psi\right\} = 0 \end{aligned} \tag{5.204}$$

It follows immediately from these equations that the expressions (5•201) and (5•202) yield a solution of the equations of motion provided ϕ and ψ satisfy the wave equations

$$\begin{aligned} \nabla_1^2\phi + \frac{1}{c_1^2}\Phi &= \frac{1}{c_1^2}\frac{\partial^2\phi}{\partial t^2} \\ \nabla_1^2\psi + \frac{1}{c_2^2}\Psi &= \frac{1}{c_2^2}\frac{\partial^2\psi}{\partial t^2} \end{aligned} \tag{5.205}$$

in which we have written $c_1^2 = \frac{\lambda+2\mu}{\rho}$, and $c_2^2 = \frac{\mu}{\rho}$. If for convenience we replace the time variable by the space-like variable τ defined by $\tau = c_1 t$, then we may write equations (5•205) in the form

$$\begin{aligned} \nabla_1^2\phi + \frac{1}{c_1^2}\Phi &= \frac{\partial^2\phi}{\partial \tau^2} \\ \nabla_1^2\psi + \frac{1}{c_2^2}\Psi &= \kappa^2\frac{\partial^2\psi}{\partial \tau^2} \end{aligned} \tag{5.206}$$

where $\kappa^2 = \frac{c_1^2}{c_2^2} = \frac{\lambda+2\mu}{\mu}$.

To solve the non-homogeneous wave equations (5·206) we introduce the three-dimensional Fourier transforms

$$\bar{\Phi}(\xi,\eta,\varsigma) = \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(x,y,\tau)\, e^{i(\xi x+\eta y+\varsigma\tau)}\,dx\,dy\,d\tau.$$

$$\bar{\Psi}(\xi,\eta,\varsigma) = \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi(x,y,\tau)\, e^{i(\xi x+\eta y+\varsigma\tau)}\,dx\,dy\,d\tau \qquad (5{\cdot}207)$$

of the functions $\phi(x,y,\tau)$, and $\psi(x,y,\tau)$. Multiplying both sides of equations (5·206) by exp. i $(\xi x+\eta y+\varsigma\tau)$, and integrating throughout the entire (x,y,τ) space we find that the functions $\bar{\phi}(\xi,\eta,\varsigma)$ and $\bar{\Psi}(\xi,\eta,\varsigma)$ must satisfy the simple relations

$$(\xi^2+\eta^2-\varsigma^2)\,\bar{\Phi} = \frac{1}{c_1^2}\,\bar{\bar{\Phi}}$$

$$(\xi^2+\eta^2-\kappa^2\varsigma^2)\,\bar{\Psi} = \frac{1}{c_2^2}\,\bar{\bar{\Psi}} \qquad (5{\cdot}208)$$

where $\bar{\bar{\phi}}$ and $\bar{\bar{\Psi}}$ are the Fourier transforms of $\bar{\phi}$ and $\bar{\Psi}$ defined by equations of the type (5·207). If we define similarly the Fourier transforms $\bar{X}$, $\bar{Y}$ of X and Y, it follows by multiplying equations (5·203) throughout by exp. i $(\xi x+\eta y+\varsigma\tau)$, and integrating throughout the entire (x, y, τ)-space, that

$$\bar{X} = -i\xi\bar{\bar{\Phi}} - i\eta\bar{\bar{\Psi}}$$

$$\bar{Y} = -i\eta\bar{\bar{\Phi}} + i\xi\bar{\bar{\Psi}} \qquad (5{\cdot}209)$$

Solving these equation for $\bar{\bar{\Phi}}$ and $\bar{\bar{\Psi}}$ and inserting in equations (5•208), we obtain the expressions

$$\bar{\Phi}(\xi,\eta,s) = \frac{1}{c_1^2}\,\frac{i\xi\bar{X} + i\eta\bar{Y}}{(\xi^2+\eta^2)(\xi^2+\eta^2-s^2)}$$

$$\bar{\Psi}(\xi,\eta,s) = \frac{1}{c_2^2}\,\frac{i\eta\bar{X} - i\xi\bar{Y}}{(\xi^2+\eta^2)(\xi^2+\eta^2-\kappa^2s^2)} \qquad (5.210)$$

determining $\bar{\Phi}$ and $\bar{\Psi}$ once the form of the applied force $X(x,y,t)$ $Y(x,y,t)$ is known.

Applying the appropriate inversion theorem (IV) to (5•210) we find that

$$\phi = \frac{1}{(2\pi)^{3/2}c_1^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{i\xi\bar{X} + i\eta\bar{Y}}{(\xi^2+\eta^2)(\xi^2+\eta^2-s^2)}\,e^{-i(\xi x+\eta y+s\tau)}\,d\xi\,d\eta\,ds$$

$$\psi = \frac{1}{(2\pi)^{3/2}c_2^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{i\eta\bar{X} - i\xi\bar{Y}}{(\xi^2+\eta^2)(\xi^2+\eta^2-\kappa^2s^2)}\,e^{-i(\xi x+\eta y+s\tau)}\,d\xi\,d\eta\,ds \qquad (5.211)$$

and these results inserted in equations (5•201) and (5•202) constitute a formal solution of the general problem.

It may be necessary to ensure convergence of the integrals with some forms of applied force to take ξ,η,s complex. In one of the examples considered later we shall take ξ,η real and s complex with a positive imaginary part. The appropriate form of the inversion theorem then gives

$$\phi = \frac{1}{(2\pi)^{3/2}c_1^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty+i\gamma}^{\infty+i\gamma}\frac{i\xi\bar{X} + i\eta\bar{Y}}{(\xi^2+\eta^2)(\xi^2+\eta^2-s^2)}\,e^{-i(\xi x+\eta y+s\tau)}\,d\xi\,d\eta\,ds$$

where γ is positive and real. A similar modification is required in the integral giving ψ.

Finally we substitute for ϕ and ψ from (5·211) into (5·201) and (5·202) to obtain the following formal expressions for the components of stress and displacement which constitute a solution of the general problem

$$\sigma_x + \sigma_y = -\frac{2(\kappa^2-1)\rho}{(2\pi)^{3/2}\kappa^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{i\xi\bar{\bar{X}} + i\eta\bar{\bar{Y}}}{\xi^2+\eta^2-\varsigma^2}\, e^{-i(\xi x+\eta y+\varsigma\tau)}\, d\xi\, d\eta\, d\varsigma$$

$$\sigma_x - \sigma_y = -\frac{2\rho}{(2\pi)^{3/2}\kappa^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{(\xi^2-\eta^2)(i\xi\bar{\bar{X}} + i\eta\bar{\bar{Y}})}{(\xi^2+\eta^2)(\xi^2+\eta^2-\varsigma^2)}\, e^{-i(\xi x+\eta y+\varsigma\tau)}\, d\xi\, d\eta\, d\varsigma$$

$$-\frac{4\rho}{(2\pi)^{3/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\xi\eta(i\eta\bar{\bar{X}} - i\xi\bar{\bar{Y}})}{(\xi^2+\eta^2)(\xi^2+\eta^2-\kappa^2\varsigma^2)}\, e^{-i(\xi x+\eta y+\varsigma\tau)}\, d\xi\, d\eta\, d\varsigma$$

$$\tau_{xy} = -\frac{2\rho}{(2\pi)^{3/2}\kappa^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\xi\eta(i\xi\bar{\bar{X}} + i\eta\bar{\bar{Y}})}{(\xi^2+\eta^2)(\xi^2+\eta^2-\varsigma^2)}\, e^{-i(\xi x+\eta y+\varsigma\tau)}\, d\xi\, d\eta\, d\varsigma$$

$$+\frac{\rho}{(2\pi)^{3/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{(\xi^2-\eta^2)(i\eta\bar{\bar{X}} - i\xi\bar{\bar{Y}})}{(\xi^2+\eta^2)(\xi^2+\eta^2-\kappa^2\varsigma^2)}\, e^{-i(\xi x+\eta y+\varsigma\tau)}\, d\xi\, d\eta\, d\varsigma$$

$$u = \frac{\rho}{(2\pi)^{3/2}\kappa^2\mu}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\xi^2\bar{\bar{X}} + \xi\eta\bar{\bar{Y}}}{(\xi^2+\eta^2)(\xi^2+\eta^2-\varsigma^2)}\, e^{-i(\xi x+\eta y+\varsigma\tau)}\, d\xi\, d\eta\, d\varsigma$$

$$+\frac{\rho}{(2\pi)^{3/2}\mu}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\eta^2\bar{\bar{X}} - \xi\eta\bar{\bar{Y}}}{(\xi^2+\eta^2)(\xi^2+\eta^2-\kappa^2\varsigma^2)}\, e^{-i(\xi x+\eta y+\varsigma\tau)}\, d\xi\, d\eta\, d\varsigma$$

(5.212)

$$v = \frac{\rho}{(2\pi)^{3/2}\kappa^2\mu}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(\xi\eta\overline{X}+\eta^2\overline{Y})e^{-i(\xi x+\eta y+\varsigma t)}}{(\xi^2+\eta^2)(\xi^2+\eta^2-\varsigma^2)}d\xi d\eta d\varsigma$$

$$-\frac{\rho}{(2\pi)^{3/2}\mu}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(\xi\eta\overline{X}-\xi^2\overline{Y})e^{-i(\xi x+\eta y+\varsigma t)}}{(\xi^2+\eta^2)(\xi^2+\eta^2-\kappa^2\varsigma^2)}d\xi d\eta d\varsigma$$

5·3 Periodic force.

In the first instance we shall consider briefly the form of the solution when the applied force is periodic. Assume that the force acts in the direction of the x-axis so that

$$\overline{X} = \frac{1}{\rho}f(x,y)e^{ipt} = \frac{1}{\rho}f(x,y)e^{i\lambda\tau} \quad ; \quad \overline{Y} = 0 \tag{5.301}$$

where $\frac{1}{\rho}f(x,y)$ is the amplitude of the force, $\frac{2\pi}{p}$ is its period and $\lambda = \frac{p}{c_1}$ has the dimensions of wave-length. In order to satisfy convergency conditions in the evaluation of the subsequent integrals we shall replace λ in the meantime by $\lambda' = \lambda + i\alpha$ where α is positive for positive values of t, and negative for negative values of t.

Substituting from (5·301) into equations of the form of (5·207) we obtain

$$\overline{X} = \frac{1}{(2\pi)^{3/2}\rho}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)e^{i(\xi x+\eta y)}dx\,dy\int_{-\infty}^{\infty}e^{i(\varsigma+\lambda')\tau}d\tau \quad ; \quad \overline{Y}=0$$

$$= \frac{1}{(2\pi)^{3/2}\rho}\overline{f}(\xi,\eta).\,\delta(\varsigma+\lambda') \tag{5.302}$$

where we have made use of the result (VIII) and we have written

$$\overline{f}(\xi,\eta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)e^{i(\xi x+\eta y)}dx\,dy \tag{5.303}$$

Substituting from (5·302) into (5·211a) we get

$$\phi = \frac{1}{4\pi k^2\mu}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{i\xi \bar{f}(\xi,\eta)}{\xi^2+\eta^2} e^{-i(\xi x+\eta y)} d\xi d\eta \int_{-\infty}^{\infty} \frac{e^{-i\varsigma\tau}\delta(\varsigma+\lambda')}{\xi^2+\eta^2-\varsigma^2} d\varsigma$$

$$= \frac{e^{i\lambda'\tau}}{4\pi^2 k^2\mu}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{i\xi \bar{f}(\xi,\eta) e^{-i(\xi x+\eta y)}}{(\xi^2+\eta^2)(\xi^2+\eta^2-\lambda'^2)} d\xi d\eta. \qquad (5{\cdot}304a)$$

where we have applied the result (VI). In a similar manner we obtain from (5·211b)

$$\psi = \frac{e^{i\lambda'\tau}}{4\pi^2\mu}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{i\eta \bar{f}(\xi,\eta) e^{-i(\xi x+\eta y)}}{(\xi^2+\eta^2)(\xi^2+\eta^2-k^2\lambda'^2)} d\xi d\eta. \qquad (5{\cdot}304b)$$

Consider now the integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{i\xi e^{-i(\xi x+\eta y)}}{(\xi^2+\eta^2)(\xi^2+\eta^2-\lambda'^2)} d\xi d\eta = \frac{1}{\lambda'^2}\left[I_1 - I_2\right] \qquad (5{\cdot}305)$$

where

$$I_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{i\xi e^{-i(\xi x+\eta y)}}{\xi^2+\eta^2-\lambda'^2} d\xi d\eta. \qquad (5{\cdot}306)$$

and

$$I_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{i\xi e^{-i(\xi x+\eta y)}}{\xi^2+\eta^2} d\xi d\eta \qquad (5{\cdot}307)$$

Making the substitution $\xi = \rho\cos\phi$, $\eta = \rho\sin\phi$, $x = r\cos\theta$, $y = r\sin\theta$ in (5·307) we have

$$I_2 = i\int_0^{\infty}\int_0^{2\pi} e^{-i\rho r\cos(\phi-\theta)}\cos\phi \, d\rho\, d\phi$$

$$= 2\pi\cos\theta\int_0^{\infty} J_1(\rho r)\, d\rho$$

$$= \frac{2\pi x}{r^2} \qquad (5{\cdot}308)$$

where we have used some well known results relating to Bessel functions. (39).p.124 ,(8) p.65.

Now if we denote by I the integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-i(\xi x+\eta y)}}{\xi^2+\eta^2-\lambda'^2}\,d\xi\,d\eta \qquad (5{\cdot}309)$$

we see that

$$I_2 = -\frac{\partial I}{\partial x} \qquad (5{\cdot}310)$$

Changing to cylindrical polar co-ordinates in (5•309) gives

$$I = \int_0^{\infty}\int_0^{2\pi} \frac{e^{-i\rho r\cos(\phi-\theta)}}{\rho^2-\lambda'^2}\,\rho\,d\rho\,d\phi$$

$$= 2\pi \int_0^{\infty} \frac{\rho J_0(\rho r)}{\rho^2-\lambda'^2}\,d\rho$$

Now it can be shown, making use of a result in (39) that

$$I = 2\pi G_0(\lambda' r) \qquad (5{\cdot}311)$$

where $G_0(z)$ is defined as in (8) by expressions

$$G_0(z) = -Y_0(z) + (\log 2-\gamma)J_0(z) + \frac{\pi i}{2}J_0(z)$$

$J_0(z)$ is the Bessel function of zero order of the first kind and $Y_0(z)$ is Neumann's Bessel function of zero order of the second kind. γ is the Euler constant.

Thus, using (5•311) (5•310) and (5•308) we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{i\xi e^{-i(\xi x+\eta y)}}{(\xi^2+\eta^2)(\xi^2+\eta^2-\lambda'^2)}\,d\xi\,d\eta = \frac{2\pi}{\lambda'^2}\left\{\frac{\lambda' x}{r}G_1(\lambda' r) - \frac{x}{r^2}\right\} \qquad (5{\cdot}312)$$

in which use has been made of the recurrence formula for this type of Bessel function. We can now let α tend to zero so that λ' tends to λ. Applying the faltung theorem (V) for two-dimensional Fourier transforms to (5•304) using (5.312)

(39). p.424 ; (8). p.23.

gives

$$\phi = \frac{e^{ipt}}{2\pi\mu\kappa^2\lambda^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{\lambda G_1\{\lambda((x-\alpha)^2+(y-\beta)^2)^{1/2}\}}{\{(x-\alpha)^2+(y-\beta)^2\}^{1/2}} - \frac{1}{(x-\alpha)^2+(y-\beta)^2}\right](x-\alpha)\,f(\alpha,\beta)\,d\alpha\,d\beta \tag{5.313}$$

Similarly we find from (5·304b) that

$$\psi = \frac{e^{ipt}}{2\pi\mu\kappa^2\lambda^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{\kappa\lambda G_1\{\kappa\lambda((x-\alpha)^2+(y-\beta)^2)^{1/2}\}}{\{(x-\alpha)^2+(y-\beta)^2\}^{1/2}} - \frac{1}{(x-\alpha)^2+(y-\beta)^2}\right](y-\beta)\,f(\alpha,\beta)\,d\alpha\,d\beta \tag{5.313}$$

From these the components of stress and displacement may readily be obtained.

If the applied force is concentrated at the origin and has amplitude P so that

$$f(x,y) = P\delta(x)\delta(y) \tag{5.314}$$

we find on inserting (5·314) in (5·313) and using the result (VI) that

$$\phi = \frac{Pe^{ipt}x}{2\pi\mu\kappa^2\lambda^2}\left\{\frac{\lambda G_1(\lambda r)}{r} - \frac{1}{r^2}\right\} \tag{5.315}$$

and

$$\psi = \frac{Pe^{ipt}y}{2\pi\mu\kappa^2\lambda^2}\left\{\frac{\kappa\lambda G_1(\lambda r)}{r} - \frac{1}{r^2}\right\} \tag{5.315}$$

The expressions for ϕ and ψ given by (5·315) differ slightly from those obtained by Lamb (17). His solution omits the part due to I_2 which however contributes nothing to the components of stress and displacement. Apart from this there is a difference of sign which does not constitute an essential difference.

5·4 Impulsive force.

We will now consider the effect of an impulsive force applied to the interior of an infinite two-dimensional elastic medium. We may represent such a force idealistically by the Dirac δ-function, and assume that it acts in the positive direction of the x-axis so that we have

$$X = \frac{1}{\rho} f(x,y)\,\delta(t)\ , \quad Y = 0 \tag{5.401}$$

Substituting X and Y in formulae of the type (5.207) gives

$$X = \frac{1}{(2\pi)^{3/2}\rho}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{i(\xi x+\eta y)}\,dx\,dy \int_{-\infty}^{\infty}\delta(\tau)\,e^{is\tau}\,d\tau\ , \quad \overline{Y} = 0$$

$$= \frac{1}{(2\pi)^{3/2}\rho}\,\bar{f}(\xi,\eta) \tag{5.402}$$

where we have made use of a property of the δ-function and we have written

$$\bar{f}(\xi,\eta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{i(\xi x+\eta y)}\,dx\,dy\ . \tag{5.403}$$

Now if we substitute these values for $\overline{X}$ and $\overline{Y}$ in equations (5.211) we obtain

$$\phi = \frac{1}{8\pi^3\mu K^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{i\xi\,\bar{f}(\xi,\eta)}{(\xi^2+\eta^2)(\xi^2+\eta^2-s^2)}\,e^{-i(\xi x+\eta y+s\tau)}\,d\xi\,d\eta\,ds$$

$$\psi = \frac{1}{8\pi^3\mu}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{i\eta\,\bar{f}(\xi,\eta)}{(\xi^2+\eta^2)(\xi^2+\eta^2-K^2s^2)}\,e^{-i(\xi x+\eta y+s\tau)}\,d\xi\,d\eta\,ds \tag{5.404}$$

Now let us consider the integral

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{i\xi}{(\xi^2+\eta^2)(\xi^2+\eta^2-s^2)}\,e^{-i(\xi x+\eta y+s\tau)}\,d\xi\,d\eta\,ds$$

If we put $\xi = \rho \cos \phi, \eta = \rho \sin \phi, x = r \cos \theta$, $y = r\sin\theta$ this integral becomes

$$I = i\int_0^\infty d\rho \int_0^{2\pi} e^{-i\rho r\cos(\phi-\theta)} \cos\phi \, d\phi \int_{-\infty}^{\infty} \frac{e^{-i\varsigma\tau}}{\rho^2-\varsigma^2} d\varsigma$$

The integration with respect to ϕ may be performed immediately to give

$$I = 2\pi\cos\theta \int_0^\infty J_1(\rho r)\, d\rho \int_{-\infty}^{\infty} \frac{e^{-i\varsigma\tau}}{\rho^2-\varsigma^2} d\varsigma \qquad (5.405)$$

where J_1 is the Bessel function of the first kind of order one.

We consider now the integration with respect to ς. The integrand has discontinuities at $\varsigma = \pm\rho$ so we will take ς to be complex and integrate along the equivalent path consisting of the real axis from $-\infty$ to $+\infty$ with semi-circular indentations at the poles $\varsigma = \pm\rho$. We then have, making use of a well known theorem in*(23),

$$\int_{-\infty}^{\infty} \frac{e^{-i\varsigma\tau}}{\rho^2-\varsigma^2} d\varsigma = \mathcal{P}\int_{-\infty}^{\infty} \frac{e^{-iu\tau}}{\rho^2-u^2} du + \frac{\pi\sin\tau\rho}{\rho}$$

$\mathcal{P}$ denoting the principal value of the integral. Furthermore, if we integrate the function $\frac{e^{i\tau\rho}}{\varsigma-\rho}$ around a contour consisting of the real axis from $-\infty$ to $+\infty$, with a semi-circular indentation at $\varsigma = \rho$, and a semi-circle with infinite radius lying in the upper plane, and apply Cauchy's theorem we find that

$$\mathcal{P}\int_{-\infty}^{\infty} \frac{e^{-iu\tau}}{\rho^2-u^2} du = \frac{\pi\sin\tau\rho}{\rho}$$

hence

$$I = 4\pi^2\cos\theta \int_0^\infty \frac{J_1(\rho r)\sin\tau\rho}{\rho} d\rho \qquad (5.406)$$

* (23) p.57.

Finally making use of a further result in*(39), we have

$$I = 4\pi^2 \begin{cases} \dfrac{\tau x}{\lambda^2} & , \lambda \geq \tau \\ \dfrac{x}{\lambda^2}\left(\tau - \sqrt{\tau^2 - \lambda^2}\right) & , \lambda < \tau \end{cases} \tag{5.407}$$

so that, if we apply the Faltung theorem for three dimensional Fourier transforms to (5.404) we have

$$\phi = \frac{1}{2\pi\mu\kappa^2} \begin{cases} \tau \displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{f(\alpha,\beta)(x-\alpha)\,d\alpha\,d\beta}{(x-\alpha)^2+(y-\beta)^2} & , x^2+y^2 \geq \tau^2 \\ \displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{f(\alpha,\beta)(x-\alpha)\{\tau - (\tau^2-(x-\alpha)^2-(y-\beta)^2)^{1/2}\}\,d\alpha\,d\beta}{(x-\alpha)^2+(y-\beta)^2} & , x^2+y^2 < \tau^2 \end{cases} \tag{5.408}$$

In a similar manner we can show that

$$\psi = \frac{1}{2\pi\mu\kappa} \begin{cases} \tau' \displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{f(\alpha,\beta)(y-\beta)\,d\alpha\,d\beta}{(x-\alpha)^2+(y-\beta)^2} & , x^2+y^2 \geq \tau^2 \\ \displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{f(\alpha,\beta)(y-\beta)\{\tau' - (\tau'^2-(x-\alpha)^2-(y-\beta)^2)^{1/2}\}\,d\alpha\,d\beta}{(x-\alpha)^2+(y-\beta)^2} & , x^2+y^2 < \tau'^2 \end{cases} \tag{5.409}$$

where $\tau' = \dfrac{\tau}{\kappa}$.

These expressions along with (5.201) and (5.202) constitute a solution of the problem.

Let us consider now the case in which the impulsive force is a point force applied at the origin of co-ordinates. That is,

$$f(x,y) = P\,\delta(x)\,\delta(y).$$

where P is the magnitude of the force, so that if we insert this value for $f(\alpha,\beta)$ in equations (5.408) and (5.409) and use

* (39) p.405

the property of the δ-function ($\overline{\text{VI}}$) we have

$$\phi = \frac{P}{2\pi\mu K^2}\begin{cases} \dfrac{\tau x}{r^2} & , r \geq \tau \\ \dfrac{x}{r^2}\left(\tau - \sqrt{\tau^2 - r^2}\right) & , r < \tau \end{cases}$$

and (5.410)

$$\psi = \frac{P}{2\pi\mu K}\begin{cases} \dfrac{\tau' y}{r^2} & , r \geq \tau' \\ \dfrac{y}{r^2}\left(\tau' - \sqrt{\tau'^2 - r^2}\right) & , r < \tau \end{cases}$$

Now substituting from (5·410) into (5·201) and (5·202) we find that the functions giving the stress and displacement are discontinuous at $r = \tau'$, and $r = \tau$. The expressions obtained are

$$\sigma_x + \sigma_y = \begin{cases} 0 & , r \geq \tau \\ \dfrac{P(K^2-1)}{\pi K^2}\cdot x\left(\tau^2 - r^2\right)^{-3/2} & , r < \tau \end{cases} \tag{5.411}$$

$$\sigma_x - \sigma_y = \begin{cases} 0 \quad , r \geq \tau \\ \dfrac{Px}{\pi K^2}\left\{\left(1 - \dfrac{2y^2}{r^2}\right)\left(\tau^2 - r^2\right)^{-3/2} - \dfrac{2}{r^2}\left(1 - \dfrac{4y^2}{r^2}\right)\left(\tau^2 - r^2\right)^{-1/2} - \dfrac{4}{r^4}\left(1 - \dfrac{4y^2}{r^2}\right)\left(\tau^2 - r^2\right)^{1/2}\right\} , \tau' \leq r < \tau \\ \dfrac{Px}{\pi K^2}\left\{\left(\tau^2 - r^2\right)^{-3/2} - \dfrac{2y^2}{r^2}\left(\left(\tau^2 - r^2\right)^{-3/2} - K\left(\tau'^2 - r^2\right)^{-3/2}\right) - \dfrac{2}{r^2}\left(1 - \dfrac{4y^2}{r^2}\right)\left(\left(\tau^2 - r^2\right)^{-1/2} - K\left(\tau'^2 - r^2\right)^{-1/2}\right) \right. \\ \qquad \left. - \dfrac{4}{r^4}\left(1 - \dfrac{4y^2}{r^2}\right)\left(\left(\tau^2 - r^2\right)^{1/2} - K\left(\tau'^2 - r^2\right)^{1/2}\right)\right\} \quad , r < \tau' \end{cases} \tag{5.412}$$

$$\tau_{xy} = \begin{cases} 0 \quad , r \geq \tau \\ \dfrac{Py}{2\pi K^2}\left\{\dfrac{2x^2}{r^2}\left(\tau^2 - r^2\right)^{-3/2} + \dfrac{2}{r^2}\left(1 - \dfrac{4x^2}{r^2}\right)\left(\tau^2 - r^2\right)^{-1/2} + \dfrac{4}{r^4}\left(1 - \dfrac{4x^2}{r^2}\right)\left(\tau^2 - r^2\right)^{1/2}\right\} , \tau' \leq r < \tau \\ \dfrac{Py}{2\pi K^2}\left\{K\left(\tau'^2 - r^2\right)^{-3/2} - \dfrac{2x^2}{r^2}\left(K\left(\tau'^2 - r^2\right)^{-3/2} - \left(\tau^2 - r^2\right)^{-3/2}\right) - \dfrac{2}{r^2}\left(1 - \dfrac{4x^2}{r^2}\right)\left(K\left(\tau'^2 - r^2\right)^{-1/2} - \left(\tau^2 - r^2\right)^{-1/2}\right) \right. \\ \qquad \left. - \dfrac{4}{r^4}\left(1 - \dfrac{4x^2}{r^2}\right)\left(K\left(\tau'^2 - r^2\right)^{1/2} - \left(\tau^2 - r^2\right)^{1/2}\right)\right\} \quad , r < \tau' \end{cases} \tag{5.413}$$

$$u = \begin{cases} 0 \ , & \lambda \geq \tau \\ \dfrac{P}{2\pi\mu\kappa^2}\left\{ \dfrac{x^2}{\lambda^2}(\tau^2-\lambda^2)^{-1/2} + \dfrac{1}{\lambda^2}\left(1-\dfrac{2y^2}{\lambda^2}\right)(\tau^2-\lambda^2)^{1/2} \right\} \ , & \tau' \leq \lambda < \tau \\ \dfrac{P}{2\pi\mu\kappa^2}\left\{ (\tau^2-\lambda^2)^{-1/2} - \dfrac{y^2}{\lambda^2}\left((\tau^2-\lambda^2)^{-1/2} - \kappa(\tau'^2-\lambda^2)^{-1/2}\right) + \dfrac{1}{\lambda^2}\left(1-\dfrac{2y^2}{\lambda^2}\right)\left((\tau^2-\lambda^2)^{1/2} - \kappa(\tau'^2-\lambda^2)^{1/2}\right) \right\} & \lambda < \tau' \end{cases} \tag{5.414}$$

$$v = \begin{cases} 0 \ , & \lambda \geq \tau \\ \dfrac{P}{2\pi\mu\kappa^2} \cdot \dfrac{xy}{\lambda^2}\left\{ (\tau^2-\lambda^2)^{-1/2} + \dfrac{2}{\lambda^2}(\tau^2-\lambda^2)^{1/2} \right\} \ , & \tau' \leq \lambda < \tau \\ \dfrac{P}{2\pi\mu\kappa^2} \cdot \dfrac{xy}{\lambda^2}\left\{ (\tau^2-\lambda^2)^{-1/2} - \kappa(\tau'^2-\lambda^2)^{-1/2} + \dfrac{2}{\lambda^2}\left((\tau^2-\lambda^2)^{1/2} - \kappa(\tau'^2-\lambda^2)^{1/2}\right) \right\} \ , & \lambda < \tau' \end{cases} \tag{5.415}$$

It is readily seen from these expressions that the disturbance is propagated outwards from the centre with velocities c_1 and c_2 and that the wave-fronts are circles, centre the origin, with radii c_1t and c_2t. At the wavefront the stress and displacement have infinite discontinuities. This is of course a physical impossibility in a perfectly elastic medium and is probably due to the adoption of the idealistic Dirac δ-function to represent a physical impulse.

An interesting fact emerges from the evaluation of the expressions for the displacement shown graphically in fig. (xiv). In the direction in which the force is applied the wave-front of the P-wave is an infinite discontinuity, but not so the wave-front of the S-wave, while in a direction perpendicular to this, the wave-front of the S-wave is an infinite discontinuity but not the wave-front of the P-wave. In directions intermediate to these both wave-fronts are infinite discontinuities. This fact may explain some of discrepancies existing in geophysical problems, since in the first case the arrival of the S-wave and in the second the arrival of the P-wave would not be apparent.

5·5. Force suddenly applied.

Another example which can be treated by the same method as the previous one, and which can be reduced to a problem whose solution is known, thus providing a check on the method, is that of an applied force which can be represented by the Heavieside unit function. We again assume that the force acts in the direction of the x-axis so that

$$X = \frac{1}{\rho} f(x,y) \quad , t \geq 0 \quad ; \quad Y = 0$$
$$= 0 \quad , t < 0 \tag{5.501}$$

Substitution in formulae of the type (5·207) yields

$$\bar{X} = \frac{1}{(2\pi)^{3/2}\rho} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{i(\xi x+\eta y)}\, dx\, dy \int_{0}^{\infty} e^{i\varsigma\tau}\, d\tau \quad , \bar{Y} = 0$$

$$= \frac{1}{(2\pi)^{3/2}\rho} \bar{f}(\xi,\eta)\left(-\frac{1}{i\varsigma}\right) \tag{5.502}$$

where to ensure convergence at $\tau = \infty$ it is necessary to take ς complex with a positive imaginary part, and we have written

$$\bar{f}(\xi,\eta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{i(\xi x+\eta y)}\, dx\, dy \tag{5.503}$$

Substituting in the alternative form of (5.211) gives

$$\phi = -\frac{1}{8\pi^3\mu\kappa^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty+i\gamma}^{\infty+i\gamma}\frac{\xi\bar{f}(\xi,\eta)\,e^{-i(\xi x+\eta y)}}{s(\xi^2+\eta^2)(\xi^2+\eta^2-s^2)}\,d\xi\,d\eta\,ds$$

$$\psi = -\frac{1}{8\pi^3\mu}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty+i\gamma}^{\infty+i\gamma}\frac{\eta\bar{f}(\xi,\eta)\,e^{-i(\xi x+\eta y+s\tau)}}{s(\xi^2+\eta^2)(\xi^2+\eta^2-\kappa^2s^2)}\,d\xi\,d\eta\,ds \tag{5.504}$$

Denoting by I the integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty+i\gamma}^{\infty+i\gamma}\frac{\xi\,e^{-i(\xi x+\eta y+s\tau)}}{s(\xi^2+\eta^2)(\xi^2+\eta^2-s^2)}\,d\xi\,d\eta\,ds$$

and proceeding as in the previous example we get

$$I = -2\pi i\cos\theta\int_0^{\infty}J_1(\rho\lambda)\,d\rho\int_{-\infty+i\gamma}^{\infty+i\gamma}\frac{e^{-is\tau}}{s(\rho^2-s^2)}\,ds \tag{5.505}$$

Now the path of integration with respect to s may be deformed into the real axis from $-\infty$ to $+\infty$ with semi-circular detours around the poles $-\rho, 0, +\rho$. This gives

$$\int_{-\infty+i\gamma}^{\infty+i\gamma}\frac{e^{-is\tau}}{s(\rho^2-s^2)}\,ds = \mathcal{P}\int_{-\infty}^{\infty}\frac{e^{-iu\tau}}{u(\rho^2-u^2)}\,du - \frac{i\pi}{\rho^2}(1-\cos\rho\tau)$$

and if we integrate the function $\frac{e^{is\tau}}{s(s-\rho)}$ around the same contour as in the previous section with the addition of a semi-circular indentation at $s = 0$ we find that

$$\mathcal{P}\int_{-\infty}^{\infty}\frac{e^{-iu\tau}}{u(\rho^2-u^2)}\,du = -\frac{i\pi}{\rho^2}(1-\cos\rho\tau)$$

so that

$$\int_{-\infty+i\gamma}^{\infty+i\gamma}\frac{e^{-is\tau}}{s(\rho^2-s^2)}\,ds = -\frac{2i\pi}{\rho^2}(1-\cos\rho\tau)$$

and

$$I = -4\pi^2\cos\theta\int_0^{\infty}\frac{J_1(\rho\lambda)(1-\cos\rho\tau)}{\rho^2}\,d\rho$$

Now it is readily seen that

$$\int_0^\infty \frac{J_1(\rho r)(1-\cos\rho\tau)}{\rho^2}\,d\rho = \int_0^\infty\int_0^\tau \frac{J_1(\rho r)\sin\tau\rho}{\rho}\,d\rho$$

Changing the order of integration and using the result in*(39) we find that

$$I = -2\pi^2 \begin{cases} \dfrac{\tau^2 x}{r^2}, & r \geq \tau \\[2ex] \dfrac{x}{r^2}\left(\tau^2 - \tau\sqrt{\tau^2 - r^2} + r^2 \log\left(\dfrac{\tau + \sqrt{\tau^2 - r^2}}{r}\right)\right), & r < \tau \end{cases} \qquad 5.506$$

so that using (5.506) and applying the Faltung theorem to (5.504a) we get

$$\phi = \frac{1}{4\pi\mu K^2}\begin{cases} \tau^2 \displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{f(\alpha,\beta)(x-\alpha)\,d\alpha\,d\beta}{(x-\alpha)^2+(y-\beta)^2} \\[3ex] \displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{f(\alpha,\beta)(x-\alpha)}{(x-\alpha)^2+(y-\beta)^2}\left\{\tau^2 - \tau\left(\tau^2-(x-\alpha)^2-(y-\beta)^2\right)^{1/2} + \left((x-\alpha)^2+(y-\beta)^2\right)\log\frac{\tau+\left(\tau^2-(x-\alpha)^2-(y-\beta)^2\right)^{1/2}}{\left((x-\alpha)^2+(y-\beta)^2\right)^{1/2}}\right\}d\alpha\,d\beta \end{cases} \qquad (5.507)$$

In a similar manner we get from (5.504b)

$$\psi = \frac{1}{4\pi\mu}\begin{cases} \tau'^2 \displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{f(\alpha,\beta)(y-\beta)\,d\alpha\,d\beta}{(x-\alpha)^2+(y-\beta)^2} \\[3ex] \displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{f(\alpha,\beta)(y-\beta)}{(x-\alpha)^2+(y-\beta)^2}\left\{\tau'^2 - \tau'\left(\tau'^2-(x-\alpha)^2-(y-\beta)^2\right)^{1/2} + \left((x-\alpha)^2+(y-\beta)^2\right)\log\frac{\tau'+\left(\tau'^2-(x-\alpha)^2-(y-\beta)^2\right)^{1/2}}{\left((x-\alpha)^2+(y-\beta)^2\right)^{1/2}}\right\}d\alpha\,d\beta \end{cases} \qquad (5.507)$$

where $\tau' = \frac{\tau}{K}$.

If the applied force is concentrated at the origin of coordinates and is of magnitude P we have

$$f(x,y) = P\delta(x)\delta(y)$$

* (39) p.405

and (5·507) reduce to

$$\Phi = \frac{P}{4\pi\mu\kappa^2}\begin{cases} \dfrac{\tau^2 x}{\lambda^2} & , \lambda \geq \tau \\ \dfrac{x}{\lambda^2}\left\{\tau^2 - \tau\sqrt{\tau^2-\lambda^2} + \lambda^2 \log\left(\dfrac{\tau+\sqrt{\tau^2-\lambda^2}}{\lambda}\right)\right\} & , \lambda < \tau . \end{cases} \tag{5.508}$$

$$\psi = \frac{P}{4\pi\mu}\begin{cases} \dfrac{\tau'^2 y}{\lambda^2} & , \lambda \geq \tau' \\ \dfrac{y}{\lambda^2}\left\{\tau'^2 - \tau'\sqrt{\tau'^2-\lambda^2} + \lambda^2 \log\left(\dfrac{\tau'+\sqrt{\tau'^2-\lambda^2}}{\lambda}\right)\right\} & , \lambda < \tau' \end{cases}$$

and the expressions for the stress and displacement components are

$$\sigma_x + \sigma_y = \begin{cases} 0 & , \lambda \geq \tau \\ -\dfrac{P(\kappa^2-1)\tau x}{\pi\kappa^2\lambda^2}\cdot(\tau^2-\lambda^2)^{-1/2} & , \lambda < \tau \end{cases} \tag{5.509}$$

$$\sigma_x - \sigma_y = \begin{cases} 0 & , \lambda \geq \tau \\ -\dfrac{P\tau x}{\pi\kappa^2\lambda^4}\left\{\left(1-\dfrac{2y^2}{\lambda^2}\right)(\tau^2-\lambda^2)^{-1/2} + \dfrac{2}{\lambda^2}(\tau^2-\lambda^2)^{1/2}\right\} & , \tau' \geq \lambda > \tau . \\ -\dfrac{P\tau x}{\pi\kappa^2\lambda^4}\left\{(\tau^2-\lambda^2)^{-1/2} - \dfrac{2y^2}{\lambda^2}\left((\tau^2-\lambda^2)^{-1/2} - \kappa(\tau'^2-\lambda^2)^{-1/2}\right)\right. & \\ \qquad\left. + \dfrac{2}{\lambda^2}\left(1-\dfrac{4y^2}{\lambda^2}\right)\left((\tau^2-\lambda^2)^{1/2} - \kappa(\tau'^2-\lambda^2)^{1/2}\right)\right\} & , \lambda < \tau' \end{cases}$$

$$\tau_{xy} = \begin{cases} 0 & , \lambda \geq \tau \\ -\dfrac{P\tau y}{2\pi\kappa^2\lambda^4}\left\{\dfrac{2x^2}{\lambda^2}(\tau^2-\lambda^2)^{-1/2} - \dfrac{2}{\lambda^2}\left(1-\dfrac{4x^2}{\lambda^2}\right)(\tau^2-\lambda^2)^{1/2}\right\} & , \tau' \geq \lambda > \tau \\ -\dfrac{P\tau y}{2\pi\kappa^2\lambda^4}\left\{\kappa(\tau'^2-\lambda^2)^{-1/2} - \dfrac{2x^2}{\lambda^2}\left(\kappa(\tau'^2-\lambda^2)^{-1/2} - (\tau^2-\lambda^2)^{-1/2}\right)\right. & \\ \qquad\left. + \dfrac{2}{\lambda^2}\left(1-\dfrac{4x^2}{\lambda^2}\right)\left(\kappa(\tau'^2-\lambda^2)^{1/2} - (\tau^2-\lambda^2)^{1/2}\right)\right\} & , \lambda < \tau' \end{cases}$$

$$
u = \begin{cases} 0 \quad , \quad \lambda \geqslant \tau \\[2ex] \dfrac{P}{4\pi\mu K^2\lambda^2}\left\{\left(1-\dfrac{2y^2}{\lambda^2}\right)\tau\left(\tau^2-\lambda^2\right)^{1/2} + \lambda^2 \log\left(\dfrac{\tau+\sqrt{\tau^2-\lambda^2}}{\lambda}\right)\right\} \quad , \tau' \geqslant \lambda > \tau \\[2ex] \dfrac{P}{4\pi\mu K^2\lambda^2}\left\{\left(1-\dfrac{2y^2}{\lambda^2}\right)\tau\left(\left(\tau^2-\lambda^2\right)^{1/2} - K\left(\tau'^2-\lambda^2\right)^{\frac{1}{2}}\right)\right. \\ \qquad\qquad \left. + \lambda^2\left(\log\dfrac{\tau+\sqrt{\tau^2-\lambda^2}}{\lambda} + K^2 \log\dfrac{\tau'+\sqrt{\tau'^2-\lambda^2}}{\lambda}\right)\right\}, \\ \qquad\qquad\qquad\qquad\qquad\qquad \lambda < \tau' \end{cases}
$$

$$
v = \begin{cases} 0 \quad , \quad \lambda \geqslant \tau \\[2ex] \dfrac{P\tau x y}{2\pi\mu K^2\lambda^4}\left(\tau^2-\lambda^2\right)^{1/2} \quad , \quad \tau' \geqslant \lambda > \tau \\[2ex] \dfrac{P\tau x y}{2\pi\mu K^2\lambda^4}\left\{\left(\tau^2-\lambda^2\right)^{1/2} - K\left(\tau'^2-\lambda^2\right)^{1/2}\right\} \quad , \quad \lambda < \tau. \end{cases}
$$

If we let τ tend to infinity in these expressions we find that (5·509) reduce to

$$
\begin{aligned}
\sigma_x + \sigma_y &= -\frac{P(K^2-1)x}{\pi K^2\lambda^2} \\
\sigma_x - \sigma_y &= -\frac{Px}{\pi\lambda^2}\left(1 - \frac{K^2-1}{K^2}\,\frac{2y^2}{\lambda^2}\right) \\
\tau_{xy} &= -\frac{Py}{2\pi\lambda^2}\left(\frac{1}{K^2} + \frac{K^2-1}{K^2}\,\frac{2x^2}{\lambda^2}\right) \\
v &= \frac{P(K^2-1)xy}{4\pi\mu K^2\lambda^2}
\end{aligned}
\qquad (5.510)
$$

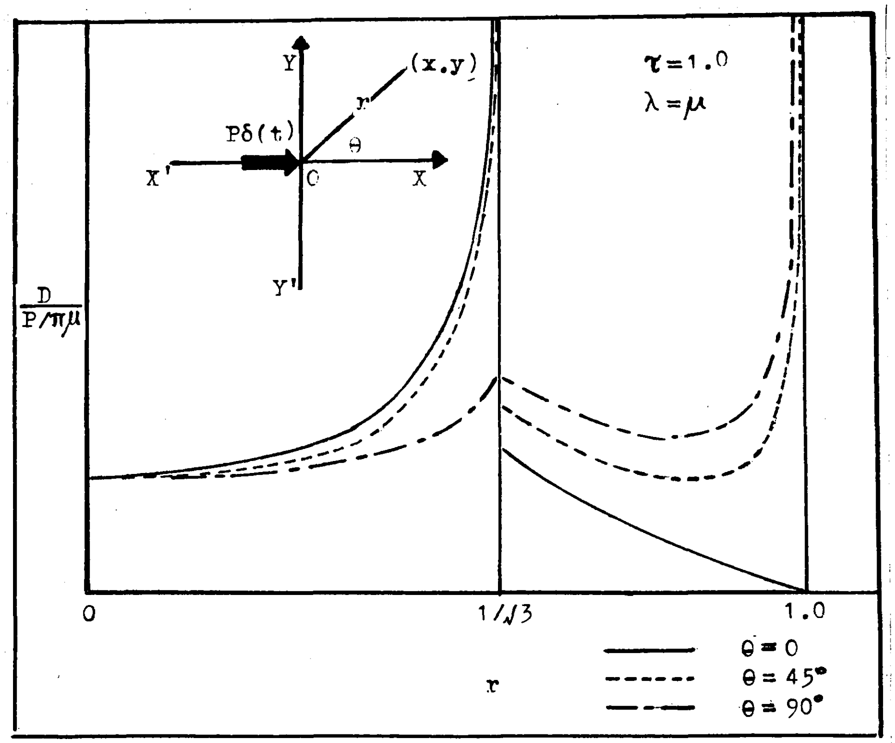

Fig. (xiv)

Displacement due to an impulsive force in the interior of an infinite two-dimensional elastic medium.

which are in agreement with the expressions (2·33) obtained for the statical problem. The x-component of displacement is indeterminate in the limit, a characteristic revealed in the analysis of the statical problem

5·6 Pulse of pressure moving with uniform velocity.

As an example of the general formulae (5·212) we shall now consider the stresses set up in an infinite two-dimensional elastic medium when a pulse of pressure moves uniformly along the line y = a. If we suppose that the applied force acts in the positive direction of the y-axis, then we may take

$$X = 0 \quad , \quad Y = \frac{1}{\rho} f(x - vt) . \delta(y-a)$$

it being assumed that the pulse has a shape "$\frac{1}{\rho} f(x)$" and that it moves with uniform velocity v along the line y = a. With these assumptions we have

$$\bar{X} = 0 \quad , \quad \bar{Y} = \frac{1}{(2\pi)^{3/2}\rho} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x - vt)\, e^{i(\xi x + \eta y)} dx\, d\tau \int_{-\infty}^{\infty} e^{i\eta y} \delta(y-a)\, dy$$

$$= \frac{e^{i\eta a}}{(2\pi)^{3/2}\rho} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x - \kappa_1 \tau)\, e^{i(\xi x + \eta y)} dx\, d\tau \qquad (5.61)$$

where we have written $\kappa_1 = \frac{v}{c_1}$, and we have used the property (VII) of the Dirac δ-function. By a trivial change of variable we find that (5·61) reduces to

$$\bar{Y} = \frac{e^{i\eta a}}{(2\pi)^{3/2}\rho}\int_{-\infty}^{\infty} f(\lambda)\, e^{i\xi\lambda}\, d\lambda \int_{-\infty}^{\infty} e^{i\tau(\zeta+\kappa_1\xi)}\, d\tau.$$

$$= \frac{e^{i\eta a}}{\rho}\,\bar{f}(\xi)\cdot\delta(\zeta+\kappa_1\xi) \qquad (5.62)$$

where we have written

$$\bar{f}(\xi) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} f(\lambda)\, e^{i\xi\lambda}\, d\lambda \qquad (5.63)$$

and we have applied the result (VIII).

Substituting for $\bar{X}$ and $\bar{Y}$ in equation (5.212a) we obtain

$$\sigma_x+\sigma_y = -\frac{2(\kappa^2-1)}{(2\pi)^{3/2}\kappa^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{i\eta\,\bar{f}(\xi)\cdot\delta(\zeta+\kappa_1\xi)\, e^{-i\{\xi x+\eta(y-a)+\zeta\tau\}}}{\xi^2+\eta^2-\zeta^2}\, d\xi\, d\eta\, d\zeta,$$

$$= -\frac{2(\kappa^2-1)}{(2\pi)^{3/2}\kappa^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{i\eta\,\bar{f}(\xi)\, e^{-i\{\xi(x-vt)+\eta(y-a)\}}}{(1-\kappa_1^2)\xi^2+\eta^2}\, d\xi\, d\eta \qquad (5.64)$$

where we have applied result (VII). Now making use of the result

$$\int_{-\infty}^{\infty}\frac{i\eta\, e^{-i\eta(y-a)}}{(1-\kappa_1^2)\xi^2+\eta^2}\, d\eta = \pi\, e^{-|\xi|(y-a)(1-\kappa_1^2)^{1/2}}$$

we find that on performing the integration with respec to η that

$$\sigma_x+\sigma_y = -\frac{\kappa^2-1}{(2\pi)^{1/2}\kappa^2}\int_{-\infty}^{\infty}\bar{f}(\xi)\, e^{-|\xi|(y-a)(1-\kappa_1^2)^{1/2}-i\xi(x-vt)}\, d\xi$$

Now the function

$$e^{-|\xi|(y-a)(1-\kappa_1^2)^{1/2}}$$

is the Fourier transform of the function

$$\left(\frac{2}{\pi}\right)^{1/2}\frac{(1-\kappa_1^2)^{1/2}(y-a)}{(1-\kappa_1^2)(y-a)^2+x^2}$$

so that by the Faltung theorem (III) we have

$$\sigma_x+\sigma_y=-\frac{(\kappa^2-1)}{\pi\kappa^2}(1-\kappa_1^2)^{1/2}(y-a)\int_{-\infty}^{\infty}\frac{f(\alpha)\,d\alpha}{(1-\kappa_1^2)(y-a)^2+(x-vt-\alpha)^2} \qquad (5\cdot65)$$

Substituting from (5·62) into equations (5·212b) and (5·212c) proceeding as before we find that

$$\sigma_x-\sigma_y=-\frac{2(y-a)}{\pi\kappa_2^2}\int_{-\infty}^{\infty}\left\{\frac{(1-\frac{1}{2}\kappa_1^2)(1-\kappa_1^2)^{1/2}}{(1-\kappa_1^2)(y-a)^2+(x-vt-\alpha)^2}-\frac{(1-\kappa_2^2)^{1/2}}{(1-\kappa_2^2)(y-a)^2+(x-vt-\alpha)^2}\right\}f(\alpha)\,d\alpha.$$

and

$$\tau_{xy}=\frac{1}{\pi\kappa_2^2}\int_{-\infty}^{\infty}\left\{\frac{(1-\kappa_1^2)^{1/2}}{(1-\kappa_1^2)(y-a)^2+(x-vt-\alpha)^2}-\frac{1-\frac{1}{2}\kappa_2^2}{(1-\kappa_2^2)^{1/2}}\frac{1}{(1-\kappa_2^2)(y-a)^2+(x-vt-\alpha)^2}\right\}(x-vt-\alpha)f(\alpha)\,d\alpha. \qquad (5\cdot65)$$

where we have written $\kappa_2=\frac{v}{c_2}=\kappa\kappa_1$. It is apparent that these formulae hold only if $\kappa_2<1$, that is, provided the velocity with which the pulse moves is less than the velocity of S-waves. Most practical interest in a calculation of this kind is however when κ_1 and κ_2 are small, and in this case we can obtain approximate expressions for the stress components by expanding the integrands in powers of κ_1^2 and neglecting terms containing κ_1^4 and higher powers. Equations (5·65) in their approximate form are

$$\sigma_x+\sigma_y=-\frac{(\kappa^2-1)}{\pi\kappa^2}(y-a)\int_{-\infty}^{\infty}\left[\frac{1}{(y-a)^2+(x-vt-\alpha)^2}+\frac{\kappa_1^2}{2}\frac{(y-a)^2-(x-vt-\alpha)^2}{\{(y-a)^2+(x-vt-\alpha)^2\}^2}+\cdots\right]f(\alpha)\,d\alpha$$

$$\sigma_x-\sigma_y=\frac{(y-a)}{\pi\kappa^2}\int_{-\infty}^{\infty}\left[\frac{\kappa^2(y-a)^2-(\kappa^2-2)(x-vt-\alpha)^2}{\{(y-a)^2+(x-vt-\alpha)^2\}^2}\right. \qquad (5\cdot66)$$

$$\left.+\frac{\kappa_1^2}{4}\frac{(3\kappa^4-1)(y-a)^4-6(\kappa^4-1)(y-a)^2(x-vt-\alpha)^2-(\kappa^4+1)(x-vt-\alpha)^4}{\{(y-a)^2+(x-vt-\alpha)^2\}^3}+\cdots\right]f(\alpha)\,d\alpha$$

$$\tau_{xy} = -\frac{1}{2\pi K^2}\int_{-\infty}^{\infty}\Bigg[\frac{(2K^2-1)(y-a)^2+(x-vt-\alpha)^2}{\{(y-a)^2+(x-vt-\alpha)^2\}^2}$$

$$-\frac{K_1^2}{4}\frac{3(3K^4-1)(y-a)^4+2(K^4+3)(y-a)^2(x-vt-\alpha)^2+(K^4+1)(x-vt-\alpha)^4}{\{(y-a)^2+(x-vt-\alpha)^2\}^3}+\cdots\Bigg](x-vt-\alpha)f(\alpha)d\alpha$$

when v = 0, so that $K_1 = K_2 = 0$ the pulse is stationary and (5·66) become

$$\sigma_x+\sigma_y = -\frac{(K^2-1)}{\pi K^2}(y-a)\int_{-\infty}^{\infty}\frac{f(\alpha)\,d\alpha.}{(x-\alpha)^2+(y-a)^2}$$

$$\sigma_x-\sigma_y = \frac{(y-a)}{\pi K^2}\int_{-\infty}^{\infty}\frac{\{K^2(y-a)^2-(K^2-2)(x-\alpha)^2\}f(\alpha)\,d\alpha.}{\{(x-\alpha)^2+(y-a)^2\}^2} \tag{5.67}$$

$$\tau_{xy} = -\frac{1}{2\pi K^2}\int_{-\infty}^{\infty}\frac{\{(2K^2-1)(y-a)^2+(x-\alpha)^2\}(x-\alpha)f(\alpha)\,d\alpha.}{\{(x-\alpha)^2+(y-a)^2\}^2}$$

These are readily seen to be identical with the expressions obtained for the components of stress by putting

$$X = 0\ ,\quad Y = \frac{1}{\rho}f(x).\delta(y-a)$$

in equations (2·212).

A special case of this solution which is of some interest is that of a point force acting in the negative direction of the y-axis whose point of application is moving along the line y = a with a uniform velocity v. In this case

$$f(x) = -P\,\delta(x)$$

where P denotes the magnitude of the force. Making this substitution in equations (5·65) we obtain on using (VI) the simple expressions

$$
\begin{aligned}
\sigma_x+\sigma_y &= \frac{P(k^2-1)(1-k_1^2)^{1/2}(y-a)}{\pi k^2 r_1^2} \\
\sigma_x-\sigma_y &= \frac{2P}{\pi k_2^2}(y-a)\left\{\frac{(1-\tfrac{1}{2}k_1^2)(1-k_1^2)^{1/2}}{r_1^2} - \frac{(1-k_2^2)^{1/2}}{r_2^2}\right\} \qquad (5{\cdot}68) \\
\tau_{xy} &= -\frac{P}{\pi k_2^2}(x-vt)\left\{\frac{(1-k_1^2)^{1/2}}{r_1^2} - \frac{(1-\tfrac{1}{2}k_2^2)}{(1-k_2^2)^{1/2} r_2^2}\right\}
\end{aligned}
$$

where we have written

$$r_1^2 = (1 - k_1^2)(y - a)^2 + (x - vt)^2,$$

$$r_2^2 = (1 - k_2^2)(y - a)^2 + (x - vt)^2.$$

If the velocity v is small we have on neglecting terms of the order k_1^4 and higher powers

$$
\begin{aligned}
\sigma_x+\sigma_y &= \frac{P(k^2-1)\cancel{(1-k_1^2)^{1/2}}\sin\phi}{\pi k^2 r_0}\left\{1-\tfrac{1}{2}k_1^2\cos 2\phi + \cdots\cdots\right\} \qquad (5{\cdot}69) \\
\sigma_x-\sigma_y &= -\frac{P\sin\phi}{\pi k^2 r_0}\left\{1-(k^2-1)\cos 2\phi - \tfrac{1}{4}k_1^2\left(2\cos 2\phi-(k^4-1)\cos 4\phi\right)+\cdots\right\} \\
\tau_{xy} &= \frac{P\cos\phi}{2\pi k^2 r_0}\left\{k^2-(k^2-1)\cos 2\phi+\tfrac{1}{2}k_1^2\left(4k^4-(2k^4-1)\cos 2\phi+(k^4-1)\cos 4\phi\right)+\cdots\right\}
\end{aligned}
$$

in which the angle ϕ is defined by the equations

$$x - vt = r_0 \cos\phi \,, \qquad y - a = r_0 \sin\phi$$

Putting $v = 0$ yields the solution of the statical problem of a point force in the interior of an infinite two-dimensional elastic medium.

Using formulae (5·68) the maximum shearing stress was calculated at various points in the medium and the "isochromatics" shown in fig.(xv) where drawn. Comparing with fig.(i)a it is seen that the stress concentration is less localised and at any given point in the medium is greater when the force is moving than when it is static.

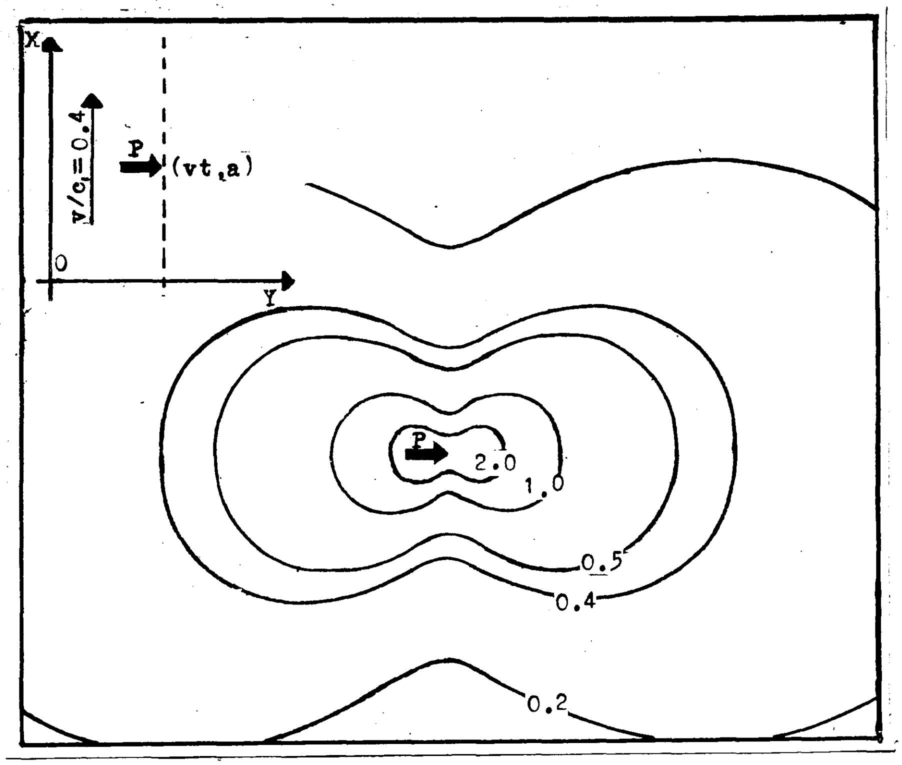

Fig. (xv)

Lines of constant maximum shearing stress due to a point force moving with uniform velocity in the interior of an infinite two-dimensional elastic medium. ($v/c_1 = 0{\cdot}4$).

PART VI.

THE APPLICATION OF DYNAMICAL FORCES TO THE SURFACES OF A TWO-DIMENSIONAL ELASTIC SOLID.

VI. THE APPLICATION OF DYNAMICAL FORCES TO THE SURFACES OF A TWO-DIMENSIONAL ELASTIC SOLID.

6·1 Introduction.

The problem of forces which vary with time acting on the bounding surfaces of a semi-infinite two-dimensional isotropic elastic solid, is one which has been the subject of several researches, considered mainly from the point of view of its importance ingeophysics. The first of these was a classical contribution by Lamb (17) in which he first of all considered the effect of a periodic force applied to the bounding surface, and then generalised his results to give the case of an impulsive force. The result which he obtained depended upon the evaluation of a particularly recondite type of integral which defied exact solution, and so adopting the device of integrating certain functions of a complex variable around various contours in the complex plane, he endeavoured to extract the important features of the solution even although he could not evaluate it exactly. Subsequently, various attempts were made on this and similar types of problem, the most important being due to Lapwood (18) who considered a "jerk" applied near the surface of a semi-infinite medium; a jerk being a displacement which was represented by the Heavieside unit function. The integral expressions obtained were similar to those encountered by Lamb, and he applied a method first introduced by Sommerfeld in which he distorted the path of integration on a Riemann surface so as to concentrate the important contributions on certain sections of the plane.

, Thus despite the attention the problem has received it has only been possible to discuss the results qualitatively, the formidable integrals involved, rendering quantitative analysis impossible.

One important feature of this type of problem which was discovered in the first instance by Rayleigh, is that, in addition to the P-waves and S-waves encountered in the previous section, a further type of wave may exist, which is essentially a surfacewave. These waves travel with a velocity different from both that of the P-wave and the S-wave and are referred to as Rayleigh waves.

The corresponding problem of an elastic solid bounded by two parallel planes, with dynamical forces applied to the bounding surfaces, is even more retractable as is apparent from a discussion by Lamb (93) on the possible modes of vibration of such a solid.

Not all forms of dynamical loading, however, lead to these recondite integrals and in the subsequent paragraphs of the section we shall consider a form of loading for which the problem can be solved exactly. We shall obtain in the first instance a formal solution of the equations of motion applicable to a semi-infinite two-dimensional elastic medium with general dynamical loading of the surface, from which a formal solution of Lamb's problem can be deduced; and then in some detail we shall consider the case of forces moving with uniform velocity along the bounding surface. The problem of the elastic solid bounded by two parallel planes with this form of loading will also be considered briefly.

6·2. Solution of the equations of motion.

Consider an elastic solid bounded by plane surfaces, with forces which vary with time applied to the bounding planes, assuming that there are no forces applied to the interior. We choose the y-axis parallel to the bounding plane, and the x-axis in a direction perpendicular to this, so as to form a right-handed set. The component of displacement in the direction perpendicular to the co-ordinate plane is assumed to be zero, and the applied forces uniform throughout the thickness of the solid so that the problem is a two-dimensional one of plane strain. The stress tensor is, therefore, uniquely determined by the components σ_x, σ_y, τ_{xy}, and the displacement vector by the components u and v. The equations of motion are then obtained from equation (1·42) by putting X and Y equal to zero.

If as before we introduce the functions $\phi(x,y,t)$, and $\psi(x,y,t)$ defined by equations (5·201), the components of stress are given in terms of ϕ and ψ by equations (5·202), and it follows that the equations of motion are satisfied provided ϕ and ψ are solutions of the wave equations

$$\nabla_1^2\phi = \frac{1}{c_1^2}\frac{\partial^2\phi}{\partial t^2}, \quad \nabla_1^2\psi = \frac{1}{c_2^2}\frac{\partial^2\psi}{\partial t^2}$$

where c_1 and c_2 are defined by the relations $c_1^2 = \frac{\lambda+2\mu}{\rho}$, $c_2^2 = \mu/\rho$. Replacing t by $\tau = c_1 t$ and writing $\kappa = c_1/c_2$ we may write these equations in the form

$$\nabla_1^2\phi = \frac{\partial^2\phi}{\partial\tau^2} \quad , \quad \nabla_1^2\psi = \kappa^2\frac{\partial^2\psi}{\partial\tau^2} \tag{6.21}$$

To find solutions of these equations we introduce now the two-dimensional Fourier transforms, defined by

$$\begin{aligned}\bar{\Phi}(x,\eta,\varsigma) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(x,y,\tau)\,e^{i(\eta y+\varsigma\tau)}\,dy\,d\tau. \\ \bar{\Psi}(x,\eta,\varsigma) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi(x,y,\tau)\,e^{i(\eta y+\varsigma\tau)}\,dy\,d\tau.\end{aligned} \tag{6.22}$$

of the functions ϕ and ψ. If we multiply both sides of equation (6.21) by exp. i (ηy + $\varsigma\tau$) and integrate over the entire $y\tau$ -plane, assuming that both ϕ and ψ and their partial derivatives with respect to y and τ vanish at both limits of integration, we find that the functions ϕ and ψ must satisfy the equations

$$\begin{aligned}\left\{\frac{d^2}{dx^2} - (\eta^2-\varsigma^2)\right\}\bar{\Phi} &= 0 \\ \left\{\frac{d^2}{dx^2} - (\eta^2-\kappa^2\varsigma^2)\right\}\bar{\Psi} &= 0\end{aligned} \tag{6.23}$$

The solutions of these equations may be written down immediately as

$$\begin{aligned}\bar{\Phi}(x,\eta,\varsigma) &= A\,e^{-(\eta^2-\varsigma^2)^{1/2}x} + B\,e^{(\eta^2-\varsigma^2)^{1/2}x}. \\ \bar{\Psi}(x,\eta,\varsigma) &= C\,e^{-(\eta^2-\kappa^2\varsigma^2)^{1/2}x} + D\,e^{(\eta^2-\kappa^2\varsigma^2)^{1/2}x}\end{aligned} \tag{6.24}$$

where A,B,C and D are arbitrary functions of η and ς which must be chosen so as to satisfy the conditions prescribed at the boundaries. Alternatively the solutions may be written in the form

$$\begin{aligned}\overline{\Phi}(x,\eta,\varsigma) &= A'\cosh(\eta^2-\varsigma^2)^{1/2}x + B'\sinh(\eta^2-\varsigma^2)^{1/2}x \\ \overline{\Psi}(x,\eta,\varsigma) &= C'\cosh(\eta^2-\kappa^2\varsigma^2)^{1/2}x + D'\sinh(\eta^2-\kappa^2\varsigma^2)^{1/2}x\end{aligned} \tag{6.25}$$

the latter form being more convenient for the problems considered in paragraph 6·8 et seq.

Transforming equations (5·202) in the same manner we find that the transformed components of stress are given in terms of $\overline{\phi}$ and $\overline{\Psi}$ by the equations

$$\begin{aligned}\overline{\sigma}_x &= \lambda\overline{\nabla}_1^2\overline{\Phi} + 2\mu\left(\frac{d^2\overline{\Phi}}{dx^2} - i\eta\frac{d\overline{\Psi}}{dx}\right) \\ \overline{\sigma}_y &= \lambda\overline{\nabla}_1^2\overline{\Phi} + 2\mu\left(-\eta^2\overline{\Phi} + i\eta\frac{d\overline{\Psi}}{dx}\right) \\ \overline{\tau}_{xy} &= \mu\left(-2i\eta\frac{d\overline{\Phi}}{dx} - \frac{d^2\overline{\Psi}}{dx^2} - \eta^2\overline{\Psi}\right)\end{aligned} \tag{6.26}$$

where $\overline{\nabla}_1^2 = \dfrac{d^2}{dx^2} - \eta^2$. The transformed components of displacement are found in a similar manner from equations (5·201) These are

$$\begin{aligned}\overline{u} &= \frac{d\overline{\Phi}}{dx} - i\eta\overline{\Psi} \\ \overline{v} &= -i\eta\overline{\Phi} - \frac{d\overline{\Psi}}{dx}\end{aligned} \tag{6.27}$$

When the arbitrary constants have been determined in accordance with the boundary conditions we can substitute from (6•24) or (6•25) intwo equations (6•26) and (6•27) and using the inversion theorem (**IV**) for two-dimensional Fourier transforms we obtain the require.. components of stress and displacement.

6•3 Free vibrations.

In the case of a semi-infinite solid the appropriate solution of equations (6•21) may be taken as (6•24) with B = D = 0 since the stress and displacement must be finite as x tends to infinity. Hence we have

$$\bar{\Phi} = Ae^{-(\eta^2-s^2)^{1/2}x}, \quad \bar{\Psi} = Ce^{-(\eta^2-\kappa^2 s^2)^{1/2}x} \tag{6.31}$$

Substituting from (6•31) into equations (6•26) gives

$$\begin{aligned} \frac{\bar{\sigma}_x}{2\mu} &= (\eta^2-\tfrac{1}{2}\kappa^2 s^2)Ae^{-(\eta^2-s^2)^{1/2}x} + i\eta(\eta^2-\kappa^2 s^2)^{1/2}Ce^{-(\eta^2-\kappa^2 s^2)^{1/2}x} \\ \frac{\bar{\tau}_{xy}}{2\mu} &= i\eta(\eta^2-s^2)Ae^{-(\eta^2-s^2)^{1/2}x} - (\eta^2-\tfrac{1}{2}\kappa^2 s^2)Ce^{-(\eta^2-\kappa^2 s^2)^{1/2}x} \end{aligned} \tag{6.32}$$

Now the conditi on for free vibrations is that the normal and shearing stress on the boundary x = 0 should be zero. Substituting these conditions in the equations (6•32) and and eliminating A and C gives

$$(\eta^2-\tfrac{1}{2}\kappa^2 s^2)^2 - \eta^2(\eta^2-s^2)^{1/2}(\eta^2-\kappa^2 s^2)^{1/2} = 0 \tag{6.33}$$

The solution of this equation is discussed by Lamb (17). It is found to have only one real root, and this corresponds to the velocity of Rayleigh waves, which are thus a form of free vibration and may therefore be superimposed on any forced vibration without altering the conditions at the boundary.

In the same way using the form of solution (6·25) and equation can be set up to give the modes of free vibration of an elastic plate. The equation has been obtained, and its roots discussed by Lamb (16).

6·4. Semi-infinite medium with dynamical forces applied normal to the bounding surface.

We shall now obtain the formal solution applicable to a semi-infinite medium x = 0 when normal forces which vary with time are applied to the bounding surface x = 0, it being assumed that the shearing force is zero on this surface. If we denote the applied force by p(y,t) then the conditions at the boundary are

$$\sigma_x = -p(y,t) \quad , \quad \tau_{xy} = 0 \quad , \quad x = 0$$

Transforming, these become

where $p(\eta,s)$ is defined by the integral, $x = 0$ (6.41)

where $\overline{p}(\eta,s)$ is defined by the integral

$$\overline{p}(\eta,s) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(y,\tau)\, e^{i(\eta y + s\tau)}\, dy\, d\tau \tag{6.42}$$

Furthermore we may take B = D = 0 since the stress must be finite when x tends to infinity so that when we substitute from (6·24) into (6·26) and use (6·41) we obtain two equations from which A and C can be determined. Performing the sequence of operations we find that

$$A = -\frac{\bar{p}(\eta,\varsigma)}{2\mu}\,\frac{(\eta^2-\frac{1}{2}\kappa^2\varsigma^2)}{f(\eta^2,\varsigma^2)}$$

$$B = -\frac{\bar{p}(\eta,\varsigma)}{2\mu}\,\frac{i\eta(\eta^2-\varsigma^2)^{1/2}}{f(\eta^2,\varsigma^2)} \tag{6.43}$$

where $f(\eta^2, \varsigma^2)$ is given by the equation

$$f(\eta^2,\varsigma^2) = (\eta^2-\tfrac{1}{2}\kappa^2\varsigma^2)^2 - \eta^2(\eta^2-\varsigma^2)^{1/2}(\eta^2-\kappa^2\varsigma^2)^{1/2}$$

Substituting from (6·24) into (6·26) using (6·43) and applying the inversion theorem (**IV**) for two dimensional Fourier transforms, we find that the components of stress are given by

$$\sigma_x+\sigma_y = \frac{\kappa^2-1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\bar{p}(\eta,\varsigma)}{f(\eta^2,\varsigma^2)}\,\varsigma^2(\eta^2-\tfrac{1}{2}\kappa^2\varsigma^2)\,e^{-(\eta^2-\varsigma^2)^{1/2}x-i(\eta y+\varsigma\tau)}\,d\eta\, d\varsigma$$

$$\sigma_x-\sigma_y = -\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\bar{p}(\eta,\varsigma)}{f(\eta^2,\varsigma^2)}\Big\{(\eta^2-\tfrac{1}{2}\varsigma^2)(\eta^2-\tfrac{1}{2}\kappa^2\varsigma^2)\,e^{-(\eta^2-\varsigma^2)^{1/2}x}$$
$$-\eta^2(\eta^2-\varsigma^2)^{1/2}(\eta^2-\kappa^2\varsigma^2)^{1/2}\,e^{-(\eta^2-\kappa^2\varsigma^2)^{1/2}x}\Big\}\,e^{-i(\eta y+\varsigma\tau)}\,d\eta\, d\varsigma$$

$$\tau_{xy} = -\frac{i}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\bar{p}(\eta,\varsigma)}{f(\eta^2,\varsigma^2)}\,\eta(\eta^2-\varsigma^2)^{1/2}(\eta^2-\tfrac{1}{2}\kappa^2\varsigma^2)\Big\{e^{-(\eta^2-\varsigma^2)^{1/2}x}-e^{-(\eta^2-\kappa^2\varsigma^2)^{1/2}x}\Big\}\,e^{-i(\eta y+\varsigma\tau)}\,d\eta\, d\varsigma$$

$$\tag{6.44}$$

Again, using (6.24), (6.27) and (6.43) we obtain the components of the displacement vector as the following integrals

$$u = \frac{1}{4\pi\mu}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\bar{p}(\eta,\varsigma)}{f(\eta^2,\varsigma^2)}(\eta^2-\varsigma^2)^{1/2}\left\{(\eta^2-\tfrac{1}{2}\kappa^2\varsigma^2)e^{-(\eta^2-\varsigma^2)^{1/2}x} - \eta^2 e^{-(\eta^2-\kappa^2\varsigma^2)^{1/2}x}\right\}e^{-i(\eta y+\varsigma\tau)}\,d\eta\,d\varsigma$$

$$v = \frac{i}{4\pi\mu}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\bar{p}(\eta,\varsigma)}{f(\eta^2,\varsigma^2)}\eta\left\{(\eta^2-\tfrac{1}{2}\kappa^2\varsigma^2)e^{-(\eta^2-\varsigma^2)^{1/2}x} - (\eta^2-\varsigma^2)^{1/2}(\eta^2-\kappa^2\varsigma^2)^{1/2}e^{-(\eta^2-\kappa^2\varsigma^2)^{1/2}x}\right\}e^{-i(\eta y+\varsigma\tau)}\,d\eta\,d\varsigma$$

It may readily be verified that these expressions satisfy the equations of motion and give the prescribed stresses on the boundary.

6·5 Periodic force

We shall consider briefly now the form which the general solution takes when the applied force is periodic. We may take

$$p(y,t) = p(y)\,e^{i\omega t} = p(y)\,e^{i\lambda\tau}$$

where $\frac{\omega}{c_1} = \lambda$, λ having the dimensions of wave length. Substituting into equation (6·42) we obtain

$$\bar{p}(\eta,\varsigma) = \frac{1}{2\pi}\int_{-\infty}^{\infty}p(y)\,e^{i\eta y}\,dy\int_{-\infty}^{\infty}e^{i(\varsigma+\lambda)\tau}\,d\tau.$$

$$= \bar{p}(\eta)\,\delta(\varsigma+\lambda) \qquad (6.51)$$

where have made use of the result (VIII) and we have written

$$\bar{p}(\eta) = \int_{-\infty}^{\infty} p(y) e^{i\eta y} dy \tag{6.52}$$

Inserting this value for $\bar{p}(\eta, \varsigma)$ into equations (6·44) we get

$$\sigma_x + \sigma_y = \frac{K^2-1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\bar{p}(\eta) . \delta(\varsigma+\lambda)}{f(\eta^2, \varsigma^2)} \varsigma^2 (\eta^2 - \tfrac{1}{2}K^2\varsigma^2) e^{-(\eta^2-\varsigma^2)^{1/2}x - i(\eta y + \varsigma\tau)} d\eta d\varsigma$$

$$= \frac{K^2-1}{2\pi} e^{i\lambda\tau} \int_{-\infty}^{\infty} \frac{\bar{p}(\eta) e^{-i\eta y}}{f(\eta^2, \lambda^2)} \cdot \lambda^2 (\eta^2 - \tfrac{1}{2}K^2\lambda^2) e^{-(\eta^2-\lambda^2)^{1/2}x} d\eta$$

$$= \frac{(K^2-1)}{2\pi} \lambda e^{i\lambda\tau} \int_{-\infty}^{\infty} \frac{\bar{p}(\frac{u}{\lambda}) e^{i\lambda u y}}{f(u)} (u^2 - \tfrac{1}{2}K^2) e^{-(u^2-1)^{1/2}\lambda x} du \tag{6.53}$$

where we have made the slight change of variable $\eta = \lambda u$. so that now

$$f(u) = (u^2 - \tfrac{1}{2}K^2)^2 - u^2(u^2-1)^{1/2}(u^2-K^2)^{1/2}$$

In the case of a force of magnitude P concentrated at the origin we have $p(y) = P\delta(y)$ so that from (6·52) $\bar{p}(\frac{u}{\lambda}) = P$ and substituting in (6·53) gives

$$\sigma_x + \sigma_y = \frac{P e^{i\lambda\tau} \lambda (K^2-1)}{\pi} \int_0^{\infty} \frac{(u^2 - \tfrac{1}{2}K^2)}{f(u)} e^{-(u^2-1)^{1/2}\lambda x} \cos(\lambda y u)\, du$$

where we have made use of the result (**IX**).

In a like manner we find the other expressions for the stress components to be

$$\sigma_x - \sigma_y = -\frac{P\lambda e^{i\lambda\tau}}{2\pi}\int_0^\infty \frac{\{(u^2-\frac{1}{2})(u^2-\frac{1}{2}\kappa^2)e^{-(u^2-1)^{1/2}\lambda x} - u^2(u^2-1)^{1/2}(u^2-\kappa^2)^{1/2}e^{-(u^2-\kappa^2)^{1/2}\lambda x}\}}{f(u)}\cos(\lambda y u)\,du.$$

$$\tau_{xy} = -\frac{P\lambda e^{i\lambda\tau}}{\pi}\int_0^\infty \frac{u(u^2-1)^{1/2}(u^2-\frac{1}{2}\kappa^2)\{e^{-(u^2-1)^{1/2}\lambda x} - e^{-(u^2-\kappa^2)^{1/2}\lambda x}\}}{f(u)}\sin(\lambda y u)\,du$$

By the same process we find the components of the displacement vector to be

$$u = \frac{Pe^{i\lambda\tau}}{2\pi\mu}\int_0^\infty \frac{(u^2-1)^{1/2}\{(u^2-\frac{1}{2}\kappa^2)e^{-(u^2-1)^{1/2}\lambda x} - u^2 e^{-(u^2-\kappa^2)^{1/2}\lambda x}\}}{f(u)}\cos(\lambda y u)\,du.$$

$$v = \frac{Pe^{i\lambda\tau}}{2\pi\mu}\int_0^\infty \frac{u\{(u^2-\frac{1}{2}\kappa^2)e^{-(u^2-1)^{1/2}\lambda x} - (u^2-1)^{1/2}(u^2-\kappa^2)^{1/2}e^{-(u^2-\kappa^2)^{1/2}\lambda x}\}}{f(u)}\sin(\lambda y u)\,du.$$

At a point on the surface x = 0, the components of displacement become

$$u = \frac{Pe^{i\omega t}}{2\pi\mu}\int_0^\infty \frac{-\frac{1}{2}\kappa^2(u^2-1)^{1/2}}{f(u)}\cos(\lambda y u)\,du.$$

$$v = \frac{Pe^{i\omega t}}{2\pi\mu}\int_0^\infty \frac{u\{(u^2-\frac{1}{2}\kappa^2)-(u^2-1)^{1/2}(u^2-\kappa^2)^{1/2}\}}{f(u)}\sin(\lambda y u)\,du.$$

and these are in agreement with the expressions obtained by Lamb (17).

6·6 Pulse of pressure moving uniformly along the boundary.

The dynamical problems so far attempted have all been based on the solution for a periodic force and consequently have contained the integrals of the type obtained in section (6·5). We shall now consider a type of dynamical problem in which this is not so, namely, that of determining the stress distribution in a semi-infinite two-dimensional elastic medium due to a pulse of pressure which moves with uniform velocity along the boundary. This problem is of some practical interest and some particular forms of loading will be discussed numerically.

Let us assume in the first instance that a pulse of pressure of shape "p(y)" moves along the bounding surface $x = 0$ with uniform velocity v. We then have

$$p(y,t) = p(y - vt) = p(y - \kappa_1\tau)$$

where $\kappa_1 = v/c_1$, so that from (6·42) we have

$$\bar{p}(\eta,\varsigma) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(y-\kappa_1\tau)\, e^{i(\eta y+\varsigma\tau)}\, dy\, d\tau.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} p(\lambda)\, e^{i\eta\lambda}\, d\lambda \int_{-\infty}^{\infty} e^{i(\varsigma+\kappa_1\eta)\tau}\, d\tau.$$

where we have made the change of variable $y - \kappa_1\tau = \lambda$. Making use of result (VIII) and writing

$$\bar{p}(\eta) = \int_{-\infty}^{\infty} p(\lambda)\, e^{i\eta\lambda}\, d\lambda \qquad (6.601)$$

we obtain

$$\bar{p}(\eta, \zeta) = \bar{p}(\eta)\,\delta(\zeta + \kappa_1 \eta) \tag{6.602}$$

Substituting from (6•602) into (6•26) we get

$$\sigma_x + \sigma_y = \frac{\kappa^2 - 1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\bar{p}(\eta)\,\delta(\zeta + \kappa_1\eta)}{f(\eta^2, \zeta^2)} \zeta^2 (\eta^2 - \tfrac{1}{2}\kappa^2\zeta^2)\, e^{-(\eta^2 - \zeta^2)^{1/2} x - i(\eta y + \zeta\tau)}\, d\eta\, d\zeta$$

$$= \frac{(\kappa^2 - 1)\kappa_1^2(1 - \frac{1}{2}\kappa_2^2)}{2\pi g(\kappa_1^2, \kappa_2^2)} \int_{-\infty}^{\infty} \bar{p}(\eta)\, e^{-(1-\kappa_1^2)^{1/2}\eta x - i\eta(y - vt)}\, d\eta$$

where, in performing the integration with respect to ζ we have used result (VII). We have written $\kappa_2 = \frac{v}{c_2} = \kappa\kappa_1$ and

$$g(\kappa_1^2, \kappa_2^2) = (1 - \tfrac{1}{2}\kappa_2^2)^2 - (1 - \kappa_1^2)^{1/2}(1 - \kappa_2^2)^{1/2}$$

In a similar manner, the other expressions for the components of stress are found to be

$$\sigma_x - \sigma_y = -\frac{1}{\pi g(\kappa_1^2, \kappa_2^2)} \int_{-\infty}^{\infty} \bar{p}(\eta) \left\{ (1 - \tfrac{1}{2}\kappa_1^2)(1 - \tfrac{1}{2}\kappa_2^2)\, e^{-(1-\kappa_1^2)^{1/2}\eta x} - (1-\kappa_1^2)^{1/2}(1-\kappa_2^2)^{1/2} e^{-(1-\kappa_2^2)^{1/2}\eta x} \right\} e^{-i\eta(y - vt)}\, d\eta$$

$$\tau_{xy} = -\frac{i(1-\kappa_1^2)^{1/2}(1 - \frac{1}{2}\kappa_2^2)}{2\pi g(\kappa_1^2, \kappa_2^2)} \int_{-\infty}^{\infty} \bar{p}(\eta) \left\{ e^{-(1-\kappa_1^2)^{1/2}\eta x} - e^{-(1-\kappa_2^2)^{1/2}\eta x} \right\} e^{-i\eta(y - vt)}\, d\eta \tag{6.603}$$

The components of the displacement vector are found in the same way to be

$$u = \frac{(1-\kappa_1^2)^{1/2}}{4\pi\mu g(\kappa_1^2, \kappa_2^2)} \int_{-\infty}^{\infty} \frac{\bar{p}(\eta)}{\eta} \left\{ (1 - \tfrac{1}{2}\kappa_2^2)\, e^{-(1-\kappa_1^2)^{1/2}\eta x} - e^{-(1-\kappa_2^2)^{1/2}\eta x} \right\} e^{-i\eta(y - vt)}\, d\eta \tag{6.603}$$

$$v = \frac{i}{4\pi\mu g(\kappa_1^2, \kappa_2^2)} \int_{-\infty}^{\infty} \frac{\bar{p}(\eta)}{\eta} \left\{ (1 - \tfrac{1}{2}\kappa_2^2)\, e^{-(1-\kappa_1^2)^{1/2}\eta x} - (1-\kappa_1^2)^{1/2}(1-\kappa_2^2)^{1/2} e^{-(1-\kappa_2^2)^{1/2}\eta x} \right\} e^{-i\eta(y - vt)}\, d\eta$$

Now it can easily be shown that $\sqrt{\frac{\pi}{2}}\, e^{-(1-\kappa_1^2)^{1/2}\eta x}$ is the Fourier transform of $\frac{(1-\kappa_1^2)^{1/2} x}{(1-\kappa_1^2)x^2 + (y - vt)^2}$, while $\frac{1}{(2\pi)^{1/2}}\bar{p}(\eta)$.

is the Fourier transform of p(y) so that applying the Faltung theorem (III) to equation (6·603a) we obtain

$$\sigma_x+\sigma_y = \frac{(K^2-1)K_1^2(1-\frac{1}{2}K_2^2)(1-K_1^2)^{1/2}x}{\pi g(K_1^2,K_2^2)}\int_{-\infty}^{\infty}\frac{p(\beta)}{(1-K_1^2)x^2+(y-vt-\beta)^2}\,d\beta \qquad (6\cdot 604)$$

Similarly we obtain for the other components of stress and displacement

$$\sigma_x-\sigma_y = -\frac{2(1-K_1^2)^{1/2}x}{\pi g(K_1^2,K_2^2)}\int_{-\infty}^{\infty}p(\beta)\left\{\frac{(1-\frac{1}{2}K_1^2)(1-\frac{1}{2}K_2^2)}{(1-K_1^2)x^2+(y-vt-\beta)^2}-\frac{1-K_2^2}{(1-K_2^2)x^2+(y-vt-\beta)^2}\right\}d\beta$$

$$\tau_{xy} = -\frac{(1-K_1^2)^{1/2}(1-\frac{1}{2}K_2^2)}{\pi g(K_1^2,K_2^2)}\int_{-\infty}^{\infty}p(\beta)\left\{\frac{1}{(1-K_1^2)x^2+(y-vt-\beta)^2}-\frac{1}{(1-K_2^2)x^2+(y-vt-\beta)^2}\right\}(y-vt-\beta)\,d\beta$$

$$u = -\frac{(1-K_1^2)^{1/2}}{4\pi\mu g(K_1^2,K_2^2)}\int_{-\infty}^{\infty}p(\beta)\left\{(1-\tfrac{1}{2}K_2^2)\log\left[(1-K_1^2)x^2+(y-vt-\beta)^2\right]-\log\left[(1-K_2^2)x^2+(y-vt-\beta)^2\right]\right\}d\beta$$

$$v = \frac{1}{2\pi\mu g(K_1^2,K_2^2)}\int_{-\infty}^{\infty}p(\beta)\left\{(1-\tfrac{1}{2}K_2^2)\tan^{-1}\left[\frac{y-vt-\beta}{(1-K_1^2)x}\right]-(1-K_1^2)^{1/2}(1-K_2^2)^{1/2}\tan^{-1}\left[\frac{y-vt-\beta}{(1-K_2^2)x}\right]\right\}d\beta$$

It is readily seen that a value of v which makes $g(K_1^2,K_2^2)=0$ correspond to the velocity of Rayleigh waves and the expressions then become indeterminate. For small values of v approximate expressions for the components of stress and displacement can be obtained by expanding the integrands in powers of K_1^2, and neglecting terms of the order K_1^4 and higher powers. These are found to be

$$\sigma_x+\sigma_y=-\frac{2x}{\pi}\int_{-\infty}^{\infty}\frac{p(\beta)}{x^2+(y-vt-\beta)^2}\left\{1+\frac{K_1^2}{4}\left(\frac{4x^2}{x^2+(y-vt-\beta)^2}+\frac{K^4-2K^2+3}{K^2-1}\right)\right\}d\beta$$

$$\sigma_x-\sigma_y=-\frac{2x}{\pi}\int_{-\infty}^{\infty}\frac{p(\beta)}{x^2+(y-vt-\beta)^2}\left\{\frac{x^2-(y-vt-\beta)^2}{x^2+(y-vt-\beta)^2}+\frac{K_1^2}{4}\left(\frac{8(K^2+1)x^4}{\left(x^2+(y-vt-\beta)^2\right)^2}\right.\right.$$

$$\left.\left.-\frac{K^4+2K^2-6}{K^2-1}\,\frac{2x^2}{x^2+(y-vt-\beta)^2}-\frac{3K^4-2K^2+3}{K^2-1}\right)\right\}d\beta \qquad (6.605)$$

$$\tau_{xy}=-\frac{2x^2}{\pi}\int_{-\infty}^{\infty}\frac{p(\beta)}{\{x^2+(y-vt-\beta)^2\}^2}\left\{1+\frac{K_1^2}{4}\left(\frac{4(K^2+1)x^2}{x^2+(y-vt-\beta)^2}+\frac{K^4-2K^2+3}{K^2-1}\right)\right\}(y-vt-\beta)\,d\beta$$

Putting v = 0 in these expressions yields the results obtained in section 3·3 for the statical problem.

As a particular example of these formulae, let us consider the stress set up in a semi-infinite elastic medium due to a point force of magnitude P acting normal to the bounding surface and moving along it with uniform velocity v. The expression for the pressure pulse is then

$$p(y-vt) = P\,\delta(y-vt).$$

so that using result (VI) we obtain from equations (6·604)

$$\sigma_x+\sigma_y=\frac{(K^2-1)K_1^2(1-\tfrac{1}{2}K_2^2)(1-K_1^2)^{1/2}}{\pi g(K_1^2,K_2^2)}\,\frac{P\cos\psi}{r(1-K_1^2\cos^2\psi)}$$

$$\sigma_x - \sigma_y = -\frac{2(1-K_1^2)^{1/2}}{\pi g(K_1^2,K_2^2)}\frac{P\cos\psi}{r}\left\{\frac{(1-\frac{1}{2}K_1^2)(1-\frac{1}{2}K_2^2)}{1-K_1^2\cos^2\psi} - \frac{1-K_2^2}{1-K_2^2\cos^2\psi}\right\} \tag{6.605}$$

$$\tau_{xy} = -\frac{(1-K_1^2)^{1/2}(1-\frac{1}{2}K_2^2)}{\pi g(K_1^2,K_2^2)}\frac{P\sin\psi}{r}\left\{\frac{1}{1-K_1^2\cos^2\psi} - \frac{1}{1-K_2^2\cos^2\psi}\right\}$$

where r and ψ are defined by the equations

$$x = r\cos\psi \quad , \quad y - vt = r\sin\psi$$

The approximate expressions then become

$$\sigma_x + \sigma_y = -\frac{2P\cos\psi}{\pi r}\left\{1 + \frac{1}{4}K_1^2\left(2\cos 2\psi + \frac{K^4+1}{K^2-1}\right)\right\}$$

$$\sigma_x - \sigma_y = -\frac{2P\cos\psi}{\pi r}\left\{\cos 2\psi + \frac{1}{4}K_1^2\left((K^2+1)\cos 4\psi + \frac{3K^4-2K^2+1}{K^2-1}\cos 2\psi - \frac{K^4+1}{K^2-1}\right)\right\}$$

$$\tau_{xy} = -\frac{P\sin 2\psi\cos\psi}{\pi r}\left\{1 + \frac{1}{4}K_1^2\left(2(K^2+1)\cos 2\psi + \frac{3K^4-2K^2+1}{K^2-1}\right)\right\} \tag{6.606}$$

Most interest in the corresponding statical problem is centred on the lines of maximum shearing stress, or "isochromatics" which are the loci of constant τ where τ is defined by the equation

$$\tau = \left\{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2\right\}^{1/2}. \tag{6.607}$$

so to provide a comparison we have evaluated τ numerically for a range of values of K_1. An approximate relation can be obtained from equations (6.606) and (6.607), namely

$$\tau = \frac{P\cos\psi}{\pi r}\left\{1 + \frac{1}{4}K_1^2\left(\frac{3K^4-2K^2+1-2\cos 2\psi}{K^2-1}\right)\right\}$$

Taking the case of $\lambda = \mu$ or $K^2 = 3$, this becomes

$$\tau = \frac{P\cos\psi}{\pi r}\left\{1 + \frac{1}{2}K_1^2\left(6 - \cos^2\psi\right)\right\}$$

so that for small values of κ_1 the isochromatics do not differ much from the circles of the statical problem. Fig. (xvi) illustrates this point and it was found that the approximate formula was correct to within 2% for values of κ_1 up to 0·25, that is, up to about 4,000 ft/sec. on rock. The "isochromatics" shown in fig. (xvii) were calculated from the exact formulae (6·605) for a value of $\kappa_1 = 0{\cdot}4$.

It is clear from these calculations that when the force is moving the maximum shearing stress at any point in the medium is greater than in the corresponding statical problem and furthermore it increases as the speed with which the force moves increases. Thus structures designed on the basis of statical applied forces will have a greater tendency to rupture when the applied forces are moving.

Another example of some interest is that of a rectangular pulse of pressure moving with uniform velocity along the boundary. In this case we have

$$p(y) = P/2a, \qquad |y| \leq a.$$

$$= 0 \qquad |y| > a.$$

Substituting from (6·609) into equations (6·604) and performing the integrations, which are elementary, we obtain

$$\sigma_x + \sigma_y = \frac{(\kappa^2-1)\kappa_1^2(1-\frac{1}{2}\kappa_2^2)P}{2\pi a g(\kappa_1^2,\kappa_2^2)}\left\{\tan^{-1}\frac{\tan\psi_1}{(1-\kappa_1^2)^{1/2}} - \tan^{-1}\frac{\tan\psi_2}{(1-\kappa_1^2)^{1/2}}\right\}$$

$$\sigma_x - \sigma_y = -\frac{P}{\pi a g(\kappa_1^2,\kappa_2^2)}\Big\{(1-\tfrac{1}{2}\kappa_1^2)(1-\tfrac{1}{2}\kappa_2^2)\left(\tan^{-1}\frac{\tan\psi_1}{(1-\kappa_1^2)^{1/2}} - \tan^{-1}\frac{\tan\psi_2}{(1-\kappa_1^2)^{1/2}}\right)$$

$$-(1-\kappa_1^2)^{1/2}(1-\kappa_2^2)^{1/2}\left(\tan^{-1}\frac{\tan\psi_1}{(1-\kappa_2^2)^{1/2}} - \tan^{-1}\frac{\tan\psi_2}{(1-\kappa_2^2)^{1/2}}\right)\Big\} \qquad (6{\cdot}610)$$

$$\tau_{xy} = -\frac{(1-\kappa_1^2)^{1/2}(1-\frac{1}{2}\kappa_2^2)P}{4\pi a g(\kappa_1^2,\kappa_2^2)} \log\left\{\frac{(1-\kappa_1^2\cos^2\psi_1)(1-\kappa_2^2\cos^2\psi_2)}{(1-\kappa_1^2\cos^2\psi_2)(1-\kappa_2^2\cos^2\psi_1)}\right\}$$

where ψ_1 and ψ_2 are defined by the relations

$$\tan\psi_1 = \frac{y+a-vt}{x}, \quad \tan\psi_2 = \frac{y-a-vt}{x}$$

When $\lambda = \mu$ and κ_1 is small equations (6·610) may be replaced the approximate expressions

$$\sigma_x + \sigma_y = -\frac{P}{\pi a}(\psi_1-\psi_2)\left\{1+\tfrac{1}{4}\kappa_1^2\left(5+\frac{\sin 2\psi_1 - \sin 2\psi_2}{\psi_1-\psi_2}\right)\right\}$$

$$\sigma_x - \sigma_y = \frac{P}{2\pi a}\Big\{\sin 2\psi_1 - \sin 2\psi_2 - \tfrac{1}{4}\kappa_1^2\Big(10(\psi_1-\psi_2) - 11(\sin 2\psi_1 - \sin 2\psi_2) - (\sin 4\psi_1 - \sin 4\psi_2)\Big)\Big\}$$

$$\tau_{xy} = \frac{P}{4\pi a}(\cos 2\psi_2 - \cos 2\psi_1)\left\{1+\tfrac{1}{4}\kappa_1^2\left(19+8(\cos 2\psi_1+\cos 2\psi_2)\right)\right\} \qquad (6.11)$$

Using the approximate relations (6·611), the isochromatic lines shown in fig(xviii) were calculated for a value of $\kappa_1 = 0{\cdot}2$. The lines in fig.(xix) were obtained from the exact formulae (6·610) taking a value 0·4 for κ_1.

6·7 Shearing stress prescribed on the boundary.

We now consider briefly the solution when a shearing force which may vary with time, is applied to the boundary $x = 0$, it being assumed that the component normal to the boundary is zero. The conditions to be satisfied in this case are $\sigma_x = 0$, $\tau_{xy} = -q(y,t)$.

If we write

$$\bar{q}(\eta,\zeta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} q(y,\tau)\, e^{i(\eta y+\zeta\tau)}\, dy\, d\tau$$

it may be shown, using equations (6·24) and (6·26), that $B = D = 0$ and

$$A = -\frac{\bar{q}(\eta,\zeta)}{2\mu}\,\frac{i\eta(\eta^2-\kappa^2\zeta^2)^{1/2}}{f(\eta^2,\zeta^2)}$$

$$B = \frac{\bar{q}(\eta,\zeta)}{2\mu}\,\frac{(\eta^2-\frac{1}{2}\kappa^2\zeta^2)}{f(\eta^2,\zeta^2)}$$

Proceeding as in (6·6) we find that the components of stress and displacement are given by

$$\sigma_x+\sigma_y = \frac{i(\kappa^2-1)}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\bar{q}(\eta,\zeta)}{f(\eta^2,\zeta^2)}\,\eta\zeta^2(\eta^2-\kappa^2\zeta^2)^{1/2}\, e^{-(\eta^2-\zeta^2)^{1/2}x-i(\eta y+\zeta\tau)}\, d\eta\, d\zeta$$

$$\sigma_x-\sigma_y = -\frac{i}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\bar{q}(\eta,\zeta)}{f(\eta^2,\zeta^2)}\,\eta(\eta^2-\kappa^2\zeta^2)^{1/2}\left\{(\eta^2-\tfrac{1}{2}\zeta^2)e^{-(\eta^2-\zeta^2)^{1/2}x}-(\eta^2-\tfrac{1}{2}\kappa^2\zeta^2)e^{-(\eta^2-\kappa^2\zeta^2)^{1/2}x}\right\}e^{-i(\eta y+\zeta\tau)}\, d\eta\, d\zeta$$

$$\tau_{xy} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\bar{q}(\eta,\zeta)}{f(\eta^2,\zeta^2)}\left\{\eta^2(\eta^2-\zeta^2)^{1/2}(\eta^2-\kappa^2\zeta^2)^{1/2}e^{-(\eta^2-\zeta^2)^{1/2}x}-(\eta^2-\tfrac{1}{2}\kappa^2\zeta^2)^2e^{-(\eta^2-\kappa^2\zeta^2)^{1/2}x}\right\}e^{-i(\eta y+\zeta\tau)}\, d\eta\, d\zeta$$

$$u = \frac{i}{2\pi\mu}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\bar{q}(\eta,\zeta)}{f(\eta^2,\zeta^2)}\,\eta\left\{(\eta^2-\zeta^2)^{1/2}(\eta^2-\kappa^2\zeta^2)^{1/2}e^{-(\eta^2-\zeta^2)^{1/2}x}-(\eta^2-\tfrac{1}{2}\kappa^2\zeta^2)e^{-(\eta^2-\kappa^2\zeta^2)^{1/2}x}\right\}e^{-i(\eta y+\zeta\tau)}\, d\eta\, d\zeta$$

$$v = -\frac{1}{2\pi\mu}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\bar{q}(\eta,\zeta)}{f(\eta^2,\zeta^2)}(\eta^2-\kappa^2\zeta^2)^{1/2}\left\{\eta^2e^{-(\eta^2-\zeta^2)^{1/2}x}-(\eta^2-\tfrac{1}{2}\kappa^2\zeta^2)e^{-(\eta^2-\kappa^2\zeta^2)^{1/2}x}\right\}e^{-i(\eta y+\zeta\tau)}\, d\eta\, d\zeta \tag{6.71}$$

In the case of a shearing force moving uniformly along the boundary $x = 0$, we have

$$q(y,t) = q(y - vt)$$

and proceeding as before we obtain

$$\sigma_x+\sigma_y = \frac{(K^2-1)K_1^2(1-K_2^2)^{1/2}}{\pi g(K_1^2,K_2^2)} \int_{-\infty}^{\infty} \frac{q(\beta)(y-vt-\beta)\,d\beta}{(1-K_1^2)x^2+(y-vt-\beta)^2} \tag{6.72}$$

$$\sigma_x-\sigma_y = -\frac{2(1-K_2^2)^{1/2}}{\pi g(K_1^2,K_2^2)} \int_{-\infty}^{\infty} q(\beta)\left\{\frac{1-\frac{1}{2}K_1^2}{(1-K_1^2)x^2+(y-vt-\beta)^2} - \frac{1-\frac{1}{2}K_2^2}{(1-K_2^2)x^2+(y-vt-\beta)^2}\right\}(y-vt-\beta)\,d\beta$$

$$\tau_{xy} = \frac{(1-K_2^2)^{1/2}x}{\pi g(K_1^2,K_2^2)} \int_{-\infty}^{\infty} q(\beta)\left\{\frac{1-K_1^2}{(1-K_1^2)x^2+(y-vt-\beta)^2} - \frac{(1-\frac{1}{2}K_2^2)^2}{(1-K_2^2)x^2+(y-vt-\beta)^2}\right\}d\beta$$

In particular if $q(y) = Q\delta(y)$ these expressions become

$$\sigma_x+\sigma_y = \frac{(K^2-1)K_1^2(1-K_2^2)^{1/2}}{\pi g(K_1^2,K_2^2)} \frac{Q\sin\psi}{r(1-K_1^2\cos^2\psi)}$$

$$\sigma_x-\sigma_y = -\frac{2(1-K_2^2)^{1/2}}{\pi g(K_1^2,K_2^2)} \frac{Q\sin\psi}{r}\left\{\frac{1-\frac{1}{2}K_1^2}{1-K_1^2\cos^2\psi} - \frac{1-\frac{1}{2}K_2^2}{1-K_2^2\cos^2\psi}\right\} \tag{6.73}$$

$$\tau_{xy} = \frac{(1-K_2^2)^{1/2}}{\pi g(K_1^2,K_2^2)} \frac{Q\cos\psi}{r}\left\{\frac{1-K_1^2}{1-K_1^2\cos^2\psi} - \frac{(1-\frac{1}{2}K_2^2)^2}{1-K_2^2\cos^2\psi}\right\}$$

where $x = r\cos\psi$, and $y - vt = r\sin\psi$. The approximate form of these expressions are

$$\sigma_x+\sigma_y = -\frac{2Q\sin\psi}{\pi r}\left\{1+\frac{1}{4}K_1^2\left(\frac{K^4+2K^2-1}{K^2-1} + 2\cos2\psi\right)\right\}$$

$$\sigma_x-\sigma_y = -\frac{2Q\sin\psi\cos2\psi}{\pi r}\left\{1+\frac{1}{4}K_1^2\left(\frac{3K^4-1}{K^2-1} + 2(K^2+1)\cos2\psi\right)\right\} \tag{6.74}$$

$$\tau_{xy} = -\frac{Q\cos\psi}{\pi r}\left\{1-\cos2\psi-\frac{1}{4}K_1^2\left(\frac{K^4+1}{K^2-1}\cos2\psi + (K^2+1)\cos4\psi\right)\right\}$$

The maximum shearing stress to the same degree of approximation is given by the expression

$$\tau = \frac{Q\sin\psi}{\pi r}\left\{1+\frac{1}{4}K_1^2\left(K^2+1 - \frac{2\cos2\psi}{K^2-1}\right)\right\} \tag{6.75}$$

so that when $\lambda = \mu$ this becomes

$$\tau = \frac{Q \sin\psi}{\pi\lambda}\left\{1 + \frac{1}{4}K_1^2(5 - 2\cos^2\psi)\right\} \tag{6·76}$$

Calculations using (6·76) shows that for small values of K_1 the isochromatics do not differ appreciably from the circles shown in fig.(iv) for the corresponding statical problem.

The results contained in 6·6 and 6·7 can be combined to give the solution to the problem in which a force which can be resolved into a normal stress and shearing stress moves with uniform velocity along the boundary x = 0. For the particular case of a point force F inclined at angle α to the x - axis moving along the boundary x = 0 with uniform velocity v we may put P = F cos α, Q = F sin α and superimpose equations (6·605) and (6·73). The approximate expression for the maximum shearing stress is given by

$$\tau = \frac{F}{\pi\lambda}\left[\cos(\alpha-\psi)\left(1 - \frac{1}{2}K_1^2\cos 2\psi\right) + \frac{1}{4}K_1^2\left\{\frac{(K^4-K^2+1)\cos(\alpha+\psi) + (2K^4-K^2)\cos(\alpha-\psi)}{K^2-1}\right\}\right]$$

and when $K^2 = 3$ this becomes

$$\tau = \frac{F}{\pi\lambda}\left[\cos(\alpha-\psi)\left(1 - \frac{1}{2}K_1^2\cos 2\psi\right) + \frac{1}{8}K_1^2\left\{15\cos(\alpha-\psi) + 7\cos(\alpha+\psi)\right\}\right]$$

Again we see that for small values of K_1 the "isochromatics" do not differ much in shape from the circles of fig.(v) but that the maximum shearing stress at any point is increased.

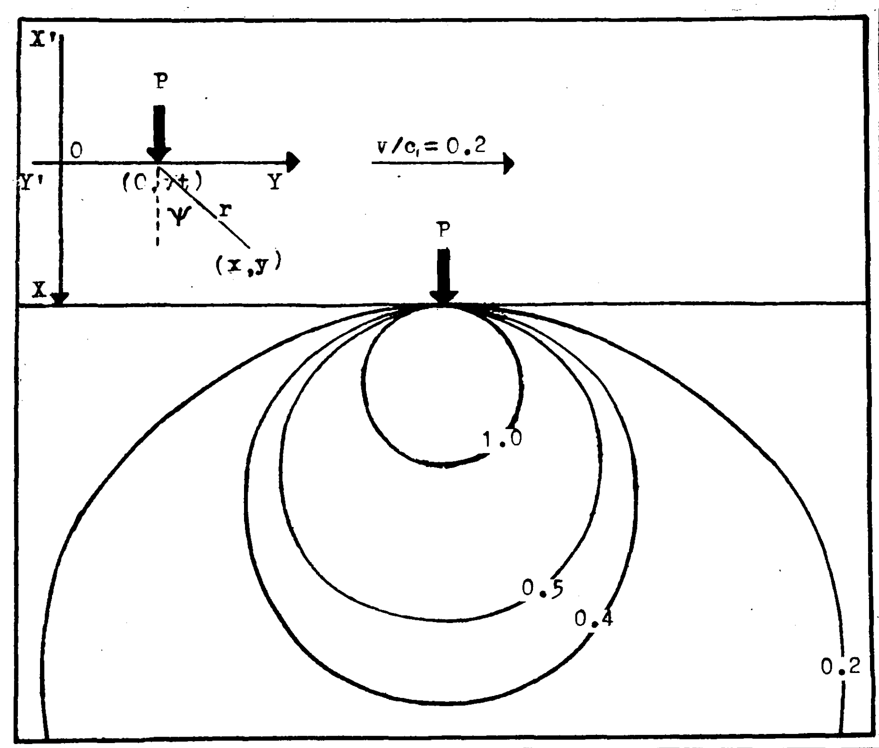

Fig. (xvi).

Lines of constant maximum shearing stress in a semi-infinite two-dimensional elastic medium due to a point force acting normal to and moving with uniform velocity along the boundary. ($v/c_1 = 0{\cdot}2$).

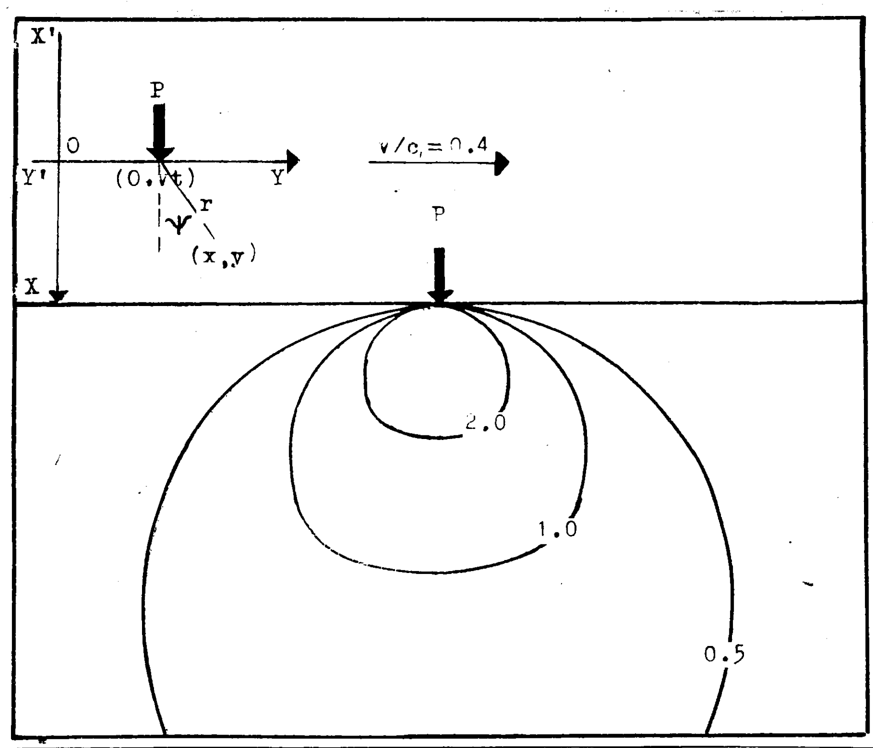

Fig.(xvii).

Lines of constant maximum shearing stress in a semi-infinite two-dimensional elastic medium due to a point force acting normal to and moving with uniform velocity along the boundary. ($v/c_1 = 0{\cdot}4$).

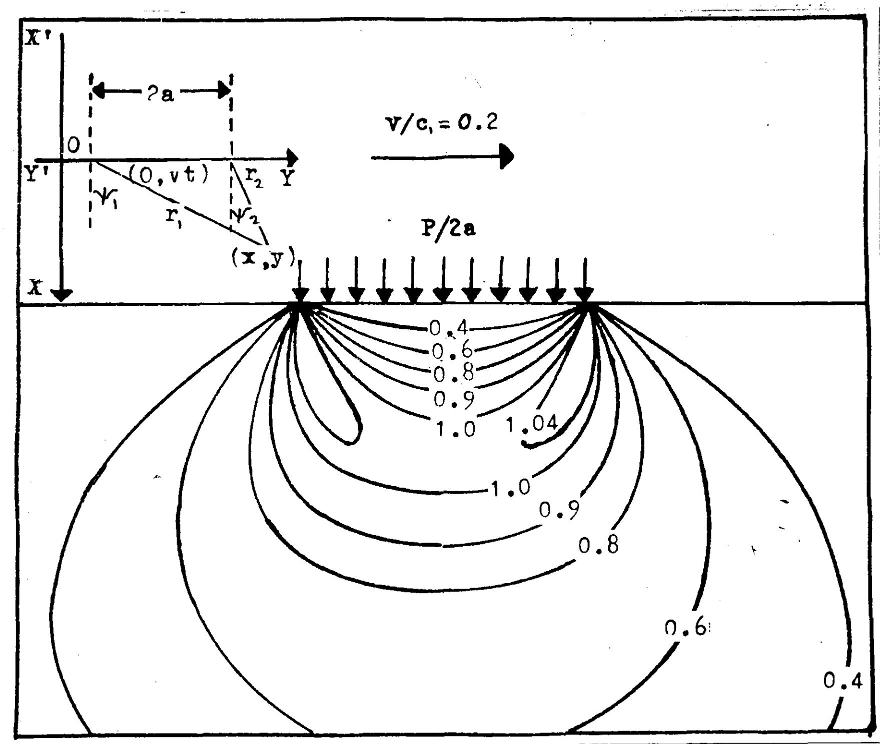

Fig. (xviii).

Lines of constant maximum shearing stress due to a rectangular pulse of pressure moving with uniform velocity along the boundary. ($v/c_1 = 0{\cdot}2$).

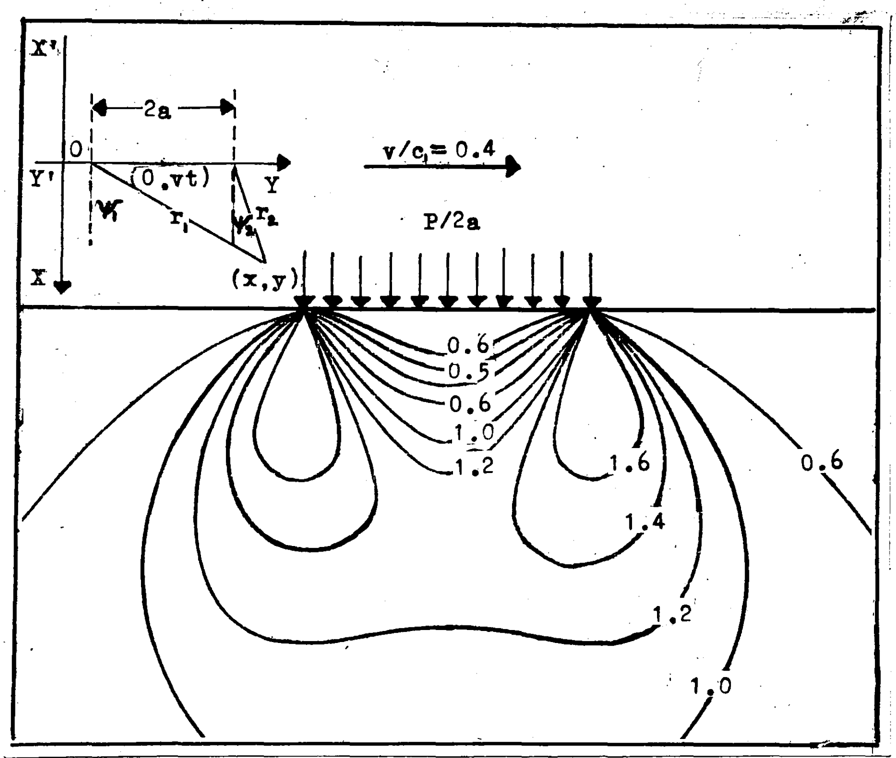

Fig.(xix).

Lines of constant maximum shearing stress due to a rectangular pulse of pressure moving with uniform velocity along the boundary. ($v/c_1 = 0{\cdot}4$).

6·8 Infinite strip of finite thickness with dynamical forces applied to the bounding surfaces.

Consider now an elastic body bounded by two parallel planes in a state of stress due to the application to the bounding planes, of normal forces which may vary with time. As in the previous paragraphs of this section we shall assume that there are no body forces present. If the y-axis is taken to lie in the medial plane, and the x-axis perpendicular to it so as to form a right-handed set the bounding planes may be taken as $x = \pm d$, where 2d is the thickness of the solid. The applied forces are assumed to be uniform in the direction perpendicular to the xy-plane, so that the components of the displacement vector in this direction is zero and the problem is one of plane strain.

We shall confine our attention to problems in which the applied forces are symmetrical with respect to the medial plane, and are in a direction normal to this plane so that the shearing stress vanishes everywhere on the boundaries. If we denote the applied force by $p(y,t)$, then the conditions at the bounding planes are

$$\sigma_x = -p(y,t), \qquad \tau_{xy} = 0, \qquad x = \pm d \tag{6.81}$$

Writing

$$\bar{p}(\eta,\varsigma) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(y,t)\, e^{i(\eta y + \varsigma\tau)}\, dy\, d\tau. \tag{6.82}$$

and using (6·25), (6·26), and (6·81) we can readily show that in this case, $B' = C' = 0$ and

$$A' = -\frac{\bar{p}(\eta,\varsigma)}{2\mu}\,\frac{(\eta^2 - \tfrac{1}{2}k^2\varsigma^2)\sinh(\eta^2 - k^2\varsigma^2)^{1/2} d.}{f(\eta^2,\varsigma^2)}$$

$$D' = \frac{\bar{p}(\eta,\varsigma)}{2\mu}\,\frac{i\eta(\eta^2-\varsigma^2)^{1/2}\sinh(\eta^2-\varsigma^2)^{1/2} d.}{f(\eta^2,\varsigma^2)} \tag{6.83}$$

where for compactness we have written

$$f(\eta^2,s^2) = (\eta^2-\tfrac{1}{2}K^2s^2)\cosh(\eta^2-s^2)^{1/2}\alpha\,\sinh(\eta^2-K^2s^2)^{1/2}\alpha - \eta^2(\eta^2-s^2)^{1/2}(\eta^2-K^2s^2)^{1/2}\sinh(\eta^2-s^2)^{1/2}\alpha.\cosh(\eta^2-K^2s^2)^{1/2}\alpha.$$

Using the values for A', B', C', and D' given by equations (6•83) and substituting in (6•25), (6•26) and (6•27) and inverting according to Fourier's inversion formula (IV) for two-dimensional Fourier transforms, we obtain the following expressions for the components of stress and displacement, consistent with the prescribed boundary conditions

$$\sigma_x+\sigma_y = \frac{K^2-1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\bar{p}(\eta,s)}{f(\eta^2,s^2)}\,s^2(\eta^2-\tfrac{1}{2}K^2s^2)\sinh(\eta^2-K^2s^2)^{1/2}\alpha\,\cosh(\eta^2-s^2)^{1/2}x\,e^{-i(\eta y+s\tau)}\,d\eta\,ds$$

$$\sigma_x-\sigma_y = \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\bar{p}(\eta,s)}{f(\eta^2,s^2)}\Big\{\eta^2(\eta^2-s^2)^{1/2}(\eta^2-K^2s^2)^{1/2}\sinh(\eta^2-s^2)^{1/2}\alpha.\cosh(\eta^2-K^2s^2)^{1/2}x$$
$$-(\eta^2-\tfrac{1}{2}s^2)(\eta^2-\tfrac{1}{2}K^2s^2)\sinh(\eta^2-K^2s^2)^{1/2}\alpha\cosh(\eta^2-s^2)^{1/2}x\Big\}e^{-i(\eta y+s\tau)}\,d\eta\,ds$$

$$\tau_{xy} = \frac{i}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\bar{p}(\eta,s)}{f(\eta^2,s^2)}\,\eta(\eta^2-s^2)^{1/2}(\eta^2-\tfrac{1}{2}K^2s^2)\Big\{\sinh(\eta^2-K^2s^2)^{1/2}\alpha.\sinh(\eta^2-s^2)^{1/2}x.$$
$$-\sinh(\eta^2-s^2)^{1/2}\alpha.\sinh(\eta^2-K^2s^2)^{1/2}x\Big\}e^{-i(\eta y+s\tau)}\,d\eta\,ds$$

$$u = -\frac{1}{4\pi\mu}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\bar{p}(\eta,s)}{f(\eta^2,s^2)}(\eta^2-s^2)^{1/2}\Big\{(\eta^2-\tfrac{1}{2}K^2s^2)\sinh(\eta^2-K^2s^2)^{1/2}\alpha.\sinh(\eta^2-s^2)^{1/2}x$$
$$+\eta^2\sinh(\eta^2-s^2)^{1/2}\alpha.\sinh(\eta^2-K^2s^2)^{1/2}x\Big\}e^{-i(\eta y+s\tau)}\,d\eta\,ds$$

$$v = \frac{i}{4\pi\mu}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\bar{p}(\eta,s)}{f(\eta^2,s^2)}\,\eta\Big\{(\eta^2-\tfrac{1}{2}K^2s^2)\sinh(\eta^2-K^2s^2)^{1/2}\alpha.\cosh(\eta^2-s^2)^{1/2}x$$
$$-(\eta^2-s^2)^{1/2}(\eta^2-K^2s^2)^{1/2}\sinh(\eta^2-s^2)^{1/2}\alpha\cosh(\eta^2-K^2s^2)^{1/2}x\Big\}e^{-i(\eta y+s\tau)}\,d\eta\,ds$$

(6.84)

6·9. Pulses of pressure moving uniformly along the boundaries.

We shall now consider the particular case of pulses of pressure applied to the bounding surfaces and moving along them with uniform velocity v. If the pulses have shape p(y), then we have

$$p(y,t) = p(y - vt) = p(y - \kappa_1\tau)$$

where $\kappa_1 = v/c$. Inserting this expression in formula (6·82) and performing the integration with respect to τ making use of (VIII) gives

$$\bar{p}(\eta,\varsigma) = \bar{p}(\eta)\cdot\delta(\varsigma+\kappa_1\eta). \tag{6.91}$$

where we have written

$$\bar{p}(\eta) = \int_{-\infty}^{\infty} p(\lambda)e^{i\lambda\eta}\,d\lambda \tag{6.92}$$

If we then substitute from (6·91) into equations (6·84) and perform the integration with respect to ς, making use of (VII) we obtain

$$\sigma_x+\sigma_y = \frac{(\kappa^2-1)\kappa_1^2(1-\frac{1}{2}\kappa_2^2)}{2\pi}\int_{-\infty}^{\infty}\frac{\bar{p}(\eta)}{f(\eta^2,\kappa_1^2\eta^2)}\sinh(1-\kappa_2^2)^{1/2}\eta d\cdot\cosh(1-\kappa_1^2)^{1/2}\eta x\, e^{-i\eta(y-vt)}\,d\eta$$

If further we assume that p(y) is an even function, and make the slight change of variable $\eta d = u$, this equation becomes

$$\sigma_x+\sigma_y = \frac{(\kappa^2-1)\kappa_1^2(1-\frac{1}{2}\kappa_2^2)}{\pi d}\int_0^{\infty}\frac{\bar{p}(\frac{u}{d})}{f(u)}\sinh u_2\cosh u_1\frac{x}{d}\cos u\left(\frac{y-vt}{d}\right).\,du.$$

where we have written $u_1 = (1 - K_1^2)^{\frac{1}{2}} u$, $u_2 = (1 - K_2^2)^{\frac{1}{2}} u$,

and

$$f(u) = (1-\tfrac{1}{2}K_2^2)^2 \cosh u_1 \sinh u_2 - (1-K_1^2)^{1/2}(1-K_2^2)^{1/2} \sinh u_1 \cosh u_2$$

Similarly the following expressions for the other components of the stress vector may be obtained

$$\sigma_x - \sigma_y = \frac{2}{\pi d}\int_0^\infty \frac{\bar{p}(\frac{u}{d})}{f(u)}\left\{(1-K_1^2)^{1/2}(1-K_2^2)^{1/2}\sinh u_1 \cosh u_2 \tfrac{x}{d} - (1-\tfrac{1}{2}K_1^2)(1-\tfrac{1}{2}K_2^2)\sinh u_2 \cosh u_1 \tfrac{x}{d}\right\} \times \cos u\left(\frac{y-vt}{d}\right) . du$$

$$\tau_{xy} = \frac{(1-K_1^2)^{1/2}(1-\frac{1}{2}K_2^2)}{\pi d}\int_0^\infty \frac{\bar{p}(\frac{u}{d})}{f(u)}\left\{\sinh u_2 \sinh u_1 \tfrac{x}{d} - \sinh u_1 \sinh u_2 \tfrac{x}{d}\right\} \sin u\left(\frac{y-vt}{d}\right) . du.$$

(6.93)

The corresponding components of the displacement vector are found to be

$$u = -\frac{(1-K_1^2)^{1/2}}{2\pi\mu}\int_0^\infty \frac{\bar{p}(\frac{u}{d})}{f(u)}\left\{(1-\tfrac{1}{2}K_2^2)\sinh u_2 \sinh u_1 \tfrac{x}{d} + \sinh u_1 \sinh u_2 \tfrac{x}{d}\right\}\cos u\left(\frac{y-vt}{d}\right) . \frac{du}{u}.$$

$$v = \frac{1}{2\pi\mu}\int_0^\infty \frac{\bar{p}(\frac{u}{d})}{f(u)}\left\{(1-\tfrac{1}{2}K_2^2)^2 \sinh u_2 \cosh u_1 \tfrac{x}{d} - (1-K_1^2)^{1/2}(1-K_2^2)^{1/2}\sinh u_1 \cosh u_2 \tfrac{x}{d}\right\} \sin u\left(\frac{y-vt}{d}\right)\frac{du}{u}.$$

(6.93)

It may readily be shown by expanding the integrands in powers K_1^2 and letting K_1 to zero that these results reduce to the solution of the statical problem obtained in 3·7.

Now it will be observed that the component of the displacement vector in the direction of the x-axis is zero everywhere on the medial plane $x = 0$, so that this solution also corresponds to an elastic strip of thickness d lying on a rigid plane co-incident with the plane $x = 0$, with a pulse of pressure $p(y)$ moving with velocity v along the surface $x = d$. The pressure of the elastic plane on the rigid plane is of some practical interest and is given by the expression

$$(\sigma_x)_{x=0} = -\frac{1}{\pi d}\int_0^\infty \frac{\bar{p}(\frac{u}{d})}{f(u)}\left\{(1-\tfrac{1}{2}K_2^2)^2\sinh u_2 - (1-K_1^2)^{1/2}(1-K_2^2)^{1/2}\sinh u_1\right\}\cos u\left(\frac{y-vt}{d}\right).du. \qquad (6.94)$$

In particular if the loading is a point force of magnitude P, then $p(y) = P\delta(y)$, and $\bar{p}(\frac{u}{d}) = P$, so that equation (6·94) reduces to

$$(\sigma_x)_{x=0} = -\frac{P}{\pi d}\int_0^\infty \frac{\left\{(1-\tfrac{1}{2}K_2^2)^2\sinh u_2 - (1-K_1^2)^{1/2}(1-K_2^2)^{1/2}\sinh u_1\right\}}{f(u)}\cos u\left(\frac{y-vt}{d}\right).du.$$

This integral may be evaluated by one of the methods discussed in section 3·8. Filon's method was used and the pressure is shown graphically in fig. (xx)

Comparing fig. (vi) and fig. (xx) we see that when the force is moving the pressure is greater immediately below the point of application of the force but falls off more rapidly on either side. A rough check on the calculations is provided by the fact that the area under the curve is the same in both cases. This problem has some bearing on the design of aircraft runways and the result of these calculations suggest that the effect of moving loads should be considered more carefully.

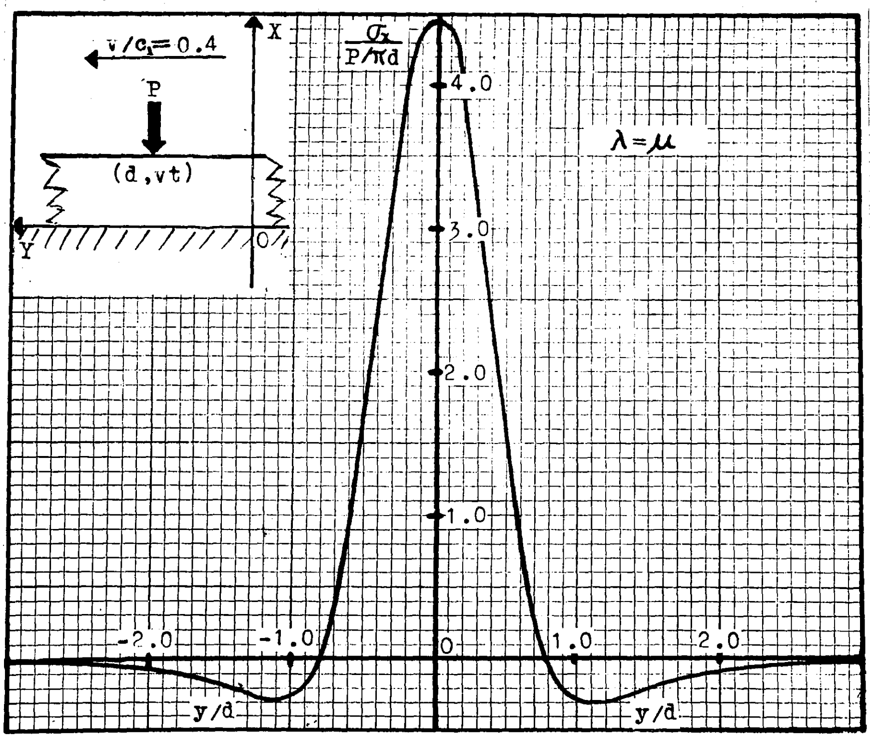

Fig.(xx)

Pressure of an elastic strip on a rigid plane due to a point force moving along the upper surface with uniform velocity.

PART VII

THE APPLICATION OF STATICAL FORCES TO THE BOUNDING SURFACES OF CIRCULARLY CYLINDRICAL ELASTIC SOLIDS.

VII. THE APPLICATION OF STATICAL FORCES TO THE BOUNDING SURFACES OF CIRCULARLY CYLINDRICAL ELASTIC SOLIDS.

7·1 Introduction.

Problems in which forces are applied to the bounding surfaces of circularly cylindrical elastic solids are of considerable importance in engineering work and consequently many specific problems of this type have been discussed in some detail. Filon (3) obtained the solution for a cylinder of finite length in the form of a Fourier series and Rankin (26) and Tranter and Claggs (38) gave the solutions to problems involving an infinite cylinder with certain types of loading in the form of infinite integrals. Tranter (36) also discussed a problem considered previously by Westergaard (40) in which pressure was applied to the surfaces of a cylindrical hole.

In this section we shall obtain using Fourier transforms, formal solutions to problems of this type with general forms of load applied to the surfaces. The components of stress and displacement are given in the form of infinite integrals which could be evaluated by the methods discussed in section 3·8

7·2 Solution of the equations of equilibrium

Consider an elastic solid of revolution and denote the position of a point in the body by the cylindrical polar co-ordinates (r, θ, z), where the z-axis is the axis of symmetry and $z = 0$ is the central cross-sectional plane. If the solid is in a state of stress due to statical forces acting symmetrically with respect to the axis, so that the deformation is symmetrical, then the components of the stress tensor at every point in the interior of the body must satisfy the equations of equilibrium (1.56).

We shall assume that the normal components of stress can be derived from a single function $\chi(r,z)$ according to the equations

$$\sigma_r = \frac{\partial}{\partial z}\left\{\sigma\nabla_1^2 - \frac{\partial^2}{\partial r^2}\right\}\chi \quad , \quad \sigma_\theta = \frac{\partial}{\partial z}\left\{\sigma\nabla_1^2 - \frac{1}{r}\frac{\partial}{\partial r}\right\}\chi$$

$$\sigma_z = \frac{\partial}{\partial z}\left\{(2-\sigma)\nabla_1^2 - \frac{\partial^2}{\partial z^2}\right\}\chi \qquad (7.21)$$

where σ denotes Poisson's ratio and

$$\nabla_1^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

Substituting in the first of the equations (1.56) we find that

$$\tau_{rz} = \frac{\partial}{\partial r}\left\{(1-\sigma)\nabla_1^2 - \frac{\partial^2}{\partial z^2}\right\}\chi \tag{7.22}$$

and when these expressions are substituted into the second of equations (1.56) we find that χ must satisfy the biharmonic equation

$$\nabla_1^4 \chi = 0 \tag{7.23}$$

From the stress-strain relations it may readily be shown that the components of the displacement vector are given in terms of χ by means of the equations

$$U = -\frac{1+\sigma}{E}\cdot\frac{\partial^2\chi}{\partial r\,\partial z}$$

$$w = \frac{1+\sigma}{E}\left\{(1-2\sigma)\nabla_1^2 + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right\} \tag{7.24}$$

In order to obtain a solution of (7·23) of the appropriate form we introduce the Fourier transform of the function χ (r,z) defined by

$$\bar{\chi}(r,\xi) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\chi(r,z)\,e^{i\xi z}\,dz \tag{7.25}$$

and we shall assume that the elastic solid extends indefinitely in both directions of the z-axis. Multiplying equation (7·23) by exp.$i\xi z$ and integrating from $-\infty$ to $+\infty$ assuming that χ and its derivatives vanish at both limits of integration we find that $\bar{\chi}$ (r,ξ) must satisfy the ordinary differential equation

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \xi^2\right)^2 \bar{\chi} = 0 \tag{7.26}$$

The general solution of equation (7·26) is

$$\bar{\chi}(r,\varsigma) = A I_0(\rho) + B\rho I_1(\rho) + C K_0(\rho) + D\rho K_1(\rho) \tag{7.27}$$

where we have written $\rho = \varsigma r$, and $I_n(\rho)$, and $K_n(\rho)$ are the modified Bessel functions of order n of the first and second kind respectively, A,B,C and D are arbitrary functions of ς and are independent of r. Multiplying equations (7·21) and (7·22) by exp.$i\varsigma z$ and integrating with respect to z, making the same assumptions as before, we obtain the transformed components of stress as

$$\bar{\sigma}_r = -i\varsigma\left\{\sigma\bar{\nabla}_1^2 - \frac{d^2}{dr^2}\right\}\bar{\chi}, \quad \bar{\sigma}_\theta = -i\varsigma\left\{\sigma\bar{\nabla}_1^2 - \frac{1}{r}\frac{d}{dr}\right\}\bar{\chi}$$
$$\bar{\sigma}_z = -i\varsigma\left\{(2-\sigma)\bar{\nabla}_1^2 + \varsigma^2\right\}\bar{\chi}, \quad \bar{\tau}_{rz} = \frac{d}{dr}\left\{(1-\sigma)\bar{\nabla}_1^2 + \varsigma^2\right\}\bar{\chi} \tag{7.28}$$

where

$$\bar{\nabla}_1^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \varsigma^2$$

In a similar manner the transformed components of the displacement vector are found to be

$$\bar{U} = \frac{1+\sigma}{E}.i\varsigma\frac{d\bar{\chi}}{dr}$$
$$\bar{w} = \frac{1+\sigma}{E}\left\{(1-2\sigma)\bar{\nabla}_1^2 + \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right\}\bar{\chi} \tag{7.29}$$

The conditions existing at the bounding surfaces may now be used to determine the functions A,B,C and D and hence from (7·28) and (7·29) we obtain the transformed components of stress appropriate to the problem. Inverting according to Fourier's inversion theorem (II)

$$\chi(r,z) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \bar{\chi}(r,\xi)\, e^{-i\xi z}\, d\xi$$

gives the components of the stress tensor. In a similar manner the components of the displacement vector may be obtained.

7·3. Infinite elastic cylinder with normal loading of the surface.

We shall consider in the first instance an elastic cylinder of radius a extending indefinitely in both directions of OZ, having normal forces applied symmetrically to the curved surface. If we denote the applied force by p(z) then the conditions existing at the boundary are given by

$$\sigma_r = -p(z) \quad , \quad \tau_{rz} = 0 \quad , \quad r = a.$$

Transforming, we obtain

$$\bar{\sigma}_r = -\bar{p}(\xi) \quad , \quad \bar{\tau}_{rz} = 0 \quad , \quad r = a \tag{7.31}$$

where $\bar{p}(\xi)$ is defined by the equation

$$\bar{p}(\xi) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} p(z)\, e^{i\xi z}\, dz \tag{7.32}$$

Now the required solution must give finite stresses along the axis r = 0, so that in equation (7·27) we must take C = D = 0, since $K_0(\rho)$ and $K_1(\rho)$ both become infinite when $\rho = 0$.

If we substitute now from equation (7·27) into equations (7·28) and use (7·31) we obtain two equations from which to determine A and B. Solving these equations we find that

$$A(s) = \frac{\alpha \bar{p}(s)}{i s^3} \, \frac{2(1-\sigma) I_1(\alpha) + \alpha I_0(\alpha)}{\Delta(\alpha)}$$

$$B(s) = -\frac{\alpha \bar{p}(s)}{i s^3} \, \frac{I_1(\alpha)}{\Delta(\alpha)} \tag{7.33}$$

where $\alpha = as$ and

$$\Delta(\alpha) = \{\alpha^2 + 2(1-\sigma)\} I_1^2(\alpha) - \alpha^2 I_0^2(\alpha)$$

Substituting these values into equations (7·28) and (7·29) and inverting according to (II) we find the following integral expressions for the components of the stress tensor and displacement vector where for convenience of notation we have replaced α by u so that $\rho = (r/a)\alpha$,

$$\sigma_r = \frac{1}{(2\pi)^{1/2} r} \int_{-\infty}^{\infty} \bar{p}\left(\frac{u}{a}\right) \bar{\Psi}_1(u) \, e^{-i\frac{z}{a}u} \, du$$

$$\sigma_\theta = \frac{1}{(2\pi)^{1/2} r} \int_{-\infty}^{\infty} \bar{p}\left(\frac{u}{a}\right) \bar{\Psi}_2(u) \, e^{-i\frac{z}{a}u} \, du$$

$$\sigma_z = \frac{1}{(2\pi)^{1/2} a} \int_{-\infty}^{\infty} \bar{p}\left(\frac{u}{a}\right) \bar{\Psi}_3(u) \, e^{-i\frac{z}{a}u} \, du \tag{7.34}$$

$$\tau_{rz} = \frac{i}{(2\pi)^{1/2} a} \int_{-\infty}^{\infty} \bar{p}\left(\frac{u}{a}\right) \bar{\Psi}_4(u) \, e^{-i\frac{z}{a}u} \, du.$$

$$\frac{E}{1+\sigma}U = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\bar{p}\left(\frac{u}{a}\right)\bar{\Psi}_5(u)\,e^{-i\frac{z}{a}u}\,du$$

$$\frac{E}{1+\sigma}w = \frac{i}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\bar{p}\left(\frac{u}{a}\right)\bar{\Psi}_6(u)\,e^{-i\frac{z}{a}u}\,du$$

in which we have written

$$\Delta(u).\bar{\Psi}_1(u) = \frac{d}{a}u\,I_1(u)I_0\left(\frac{d}{a}u\right) + \frac{d}{a}u^2\,I_0(u)I_0\left(\frac{d}{a}u\right) - u\,I_0(u)I_1\left(\frac{d}{a}u\right) - \left\{\frac{d^2}{a^2}u^2 + 2(1-\sigma)\right\}I_1(u)I_1\left(\frac{d}{a}u\right)$$

$$\Delta(u)\,\bar{\Psi}_2(u) = u\,I_0(u)I_1\left(\frac{d}{a}u\right) + 2(1-\sigma)I_1(u)I_1\left(\frac{d}{a}u\right) - (1-2\sigma)\frac{d}{a}u\,I_1(u)I_0\left(\frac{d}{a}u\right)$$

$$\Delta(u).\bar{\Psi}_3(u) = u\left\{2I_1(u)I_0\left(\frac{d}{a}u\right) - u\,I_0(u)I_0\left(\frac{d}{a}u\right) + \frac{d}{a}u\,I_1(u)I_1\left(\frac{d}{a}u\right)\right\}$$

$$\Delta(u).\bar{\Psi}_4(u) = u^2\left\{\frac{d}{a}I_1(u)I_0\left(\frac{d}{a}u\right) - I_0(u)I_1\left(\frac{d}{a}u\right)\right\}$$

$$\Delta(u).\bar{\Psi}_5(u) = 2(1-\sigma)I_1(u)I_1\left(\frac{d}{a}u\right) + u\,I_0(u)I_1\left(\frac{d}{a}u\right) - \frac{d}{a}u\,I_1(u)I_0\left(\frac{d}{a}u\right)$$

$$\Delta(u).\bar{\Psi}_6(u) = 2(1-\sigma)I_1(u)I_0\left(\frac{d}{a}u\right) - u\,I_0(u)I_0\left(\frac{d}{a}u\right) + \frac{d}{a}u\,I_1(u)I_1\left(\frac{d}{a}u\right)$$

In a recent paper (38) Tranter and Craggs considered the particular loading conditions

$$\sigma_r = -1, \quad \tau_{rz} = 0, \quad 0 \le r < \infty$$
$$= 0, \quad = 0, \quad -\infty < r < 0$$

so that making the assumption that ς is complex with a positive imaginary part and using form (III) of the inversion theorem we obtain

$$\bar{p}(\varsigma) = -\frac{1}{(2\pi)^{1/2} i\varsigma}$$

which when substituted in (7·34) gives expressions which are in agreement with (38)

As an example of these general formulae let us consider the case of uniform pressure P/2b applied over the part of the bounding surface contained between the planes r = b, and r = -b. The boundary conditions then become

$$\begin{aligned} \sigma_r &= -\frac{P}{2b}, & \tau_{rz} &= 0, & -b \leq z \leq b \\ &= 0, & &= 0, & -\infty < z < -b\,;\ b < z < \infty \end{aligned}$$

so that

$$\bar{p}(\varsigma) = \frac{P}{(2\pi)^{1/2}.2b}\int_{-b}^{b} e^{i\varsigma z}\,dz$$

$$= \frac{P}{(2\pi)^{1/2}}\frac{\sin\left(\frac{b}{a}u\right)}{\frac{b}{a}u}.$$

Inserting this value in equations (7·34) and using the results (IX) and (X) we find that the components of stress and displacement are given by

$$\sigma_r = \frac{Pa}{\pi b r}\int_0^\infty \overline{\mathcal{V}_1}(u)\sin\frac{b}{a}u\cos\frac{z}{a}u.\frac{du}{u}$$

$$\sigma_\theta = \frac{Pa}{\pi b r}\int_0^\infty \overline{\mathcal{V}_2}(u)\sin\frac{b}{a}u\cos\frac{z}{a}u.\frac{du}{u}. \qquad (7.35)$$

$$\sigma_z = \frac{P}{\pi b}\int_0^\infty \bar{\Psi}_3(u)\sin\frac{b}{a}u\cos\frac{z}{a}u\,.\frac{du}{u}.$$

$$\tau_{rz} = \frac{P}{\pi b}\int_0^\infty \bar{\Psi}_4(u)\sin\frac{b}{a}u\sin\frac{z}{a}u\,.\frac{du}{u}.$$

$$\frac{E}{1+\sigma}\frac{U}{a} = \frac{P}{\pi b}\int_0^\infty \bar{\Psi}_5(u)\sin\frac{b}{a}u\cos\frac{z}{a}u\,.\frac{du}{u}.$$

$$\frac{E}{1+\sigma}\frac{w}{a} = \frac{P}{\pi b}\int_0^\infty \bar{\Psi}_6(u)\sin\frac{b}{a}u\sin\frac{z}{a}u\,.\frac{du}{u}.$$

If the applied pressure is concentrated round the circumference of the circle r = a, z = 0, then

$$\bar{p}(s) = \frac{P}{(2\pi)^{1/2}}\lim_{b\to 0}\frac{\sin(\frac{b}{a}u)}{(\frac{b}{a}u)} = \frac{P}{(2\pi)^{1/2}}$$

and the expressions for the components of stress and displacement may readily be deduced from (7·35)

7·4. Infinite elastic cylinder with shearing forces applied to the boundary.

We now obtain expressions for the components of the stress tensor and the displacement vector in an infinite elastic cylinder due to shearing forces applied to the bounding surface. In this case the conditions at the boundary are expressed by the equations

$$\sigma_r = 0 \quad , \quad \tau_{rz} = q(z) \quad , \quad r = a$$

so that the transformed components of stress become

$$\bar{\sigma}_r = 0 \quad , \quad \bar{\tau}_{rz} = \bar{q}(\xi) \quad , \quad r = a \tag{7.41}$$

where $\bar{q}(\xi)$ is defined by the integral

$$\bar{q}(\xi) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} q(z)\, e^{i\xi z}\, dz \tag{7.42}$$

Proceeding as in the previous case it is found that the arbitrary functions A and B consistent with (7·41) are given by the expressions

$$A(\xi) = \frac{\bar{q}(\xi)}{\xi^3} \frac{\{(1-2\sigma)\alpha I_0(\alpha) + \alpha^2 I_1(\alpha)\}}{D(\alpha)}$$

$$B(\xi) = -\frac{\bar{q}(\xi)}{\xi^3} \frac{\{\alpha I_0(\alpha) - I_1(\alpha)\}}{D(\alpha)} \tag{7.43}$$

where $\alpha = \xi a$ and

$$D(\alpha) = \{\alpha^2 + 2(1-\sigma)\} I_1^2(\alpha) - \alpha^2 I_0^2(\alpha)$$

These lead to the following expressions for the components of stress and displacement

$$\sigma_r = \frac{i}{(2\pi)^{1/2} r} \int_{-\infty}^{\infty} \bar{q}\left(\frac{u}{a}\right) \Phi_1(u)\, e^{-i\frac{z}{a}u}\, du.$$

$$\sigma_\theta = \frac{i}{(2\pi)^{1/2} r} \int_{-\infty}^{\infty} \bar{q}\left(\frac{u}{a}\right) \Phi_2(u)\, e^{-i\frac{z}{a}u}\, du \tag{7.44}$$

$$\sigma_z = \frac{i}{(2\pi)^{1/2} a} \int_{-\infty}^{\infty} \bar{q}\left(\frac{u}{a}\right) \Phi_3(u)\, e^{-i\frac{z}{a}u}\, du$$

$$\tau_{rz} = \frac{1}{(2\pi)^{1/2} a} \int_{-\infty}^{\infty} \bar{q}\left(\frac{u}{a}\right) \Phi_4(u)\, e^{-i\frac{z}{a}u}\, du.$$

$$\frac{E}{1+\sigma} U = \frac{i}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \bar{q}\left(\frac{u}{a}\right) \Phi_5(u)\, e^{-i\frac{z}{a}u}\, du.$$

$$\frac{E}{1+\sigma} w = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \bar{q}\left(\frac{u}{a}\right) \Phi_6(u)\, e^{-i\frac{z}{a}u}\, du.$$

where $\Phi_i(u)$, $i = 1\cdots\cdots 6$, denotes the functions

$$D(u).\Phi_1(u) = \frac{d}{a}\{(1-2\sigma)+u^2\} I_1(u) I_0\left(\frac{d}{a}u\right) - \left\{(1-2\sigma)+\frac{d^2}{a^2}u^2\right\} I_0(u) I_1\left(\frac{d}{a}u\right) - \left(1-\frac{d^2}{a^2}\right) u\, I_1(u) I_1\left(\frac{d}{a}u\right)$$

$$D(u).\Phi_2(u) = (1-2\sigma) I_0(u) I_1\left(\frac{d}{a}u\right) + u\, I_1(u) I_1\left(\frac{d}{a}u\right) - \frac{d}{a}(1-2\sigma) u\, I_0(u) I_0\left(\frac{d}{a}u\right) + \frac{d}{a}(1-2\sigma) I_1(u) I_0\left(\frac{d}{a}u\right)$$

$$D(u).\Phi_3(u) = 3u\, I_0(u) I_0\left(\frac{d}{a}u\right) - \frac{d}{a}u\, I_1(u) I_1\left(\frac{d}{a}u\right) + \frac{d}{a}u^2 I_0(u) I_1\left(\frac{d}{a}u\right) - \{2(2-\sigma)+u^2\} I_1(u) I_0\left(\frac{d}{a}u\right)$$

$$D(u).\Phi_4(u) = \frac{d}{a}u\, I_1(u) I_0\left(\frac{d}{a}u\right) - u\, I_0(u) I_1\left(\frac{d}{a}u\right) - \frac{d}{a}u^2 I_0(u) I_0\left(\frac{d}{a}u\right) + \{2(1-\sigma)+u^2\} I_1(u) I_1\left(\frac{d}{a}u\right)$$

$$D(u).\Phi_5(u) = u\, I_1(u) I_1\left(\frac{d}{a}u\right) + \frac{d}{a} I_1(u) I_0\left(\frac{d}{a}u\right) - \frac{d}{a}u\, I_0(u) I_0\left(\frac{d}{a}u\right) + (1-2\sigma) I_0(u) I_1\left(\frac{d}{a}u\right)$$

$$D(u).\Phi_6(u) = \frac{d}{a}u\, I_1(u) I_1\left(\frac{d}{a}u\right) - \frac{d}{a}u^2 I_0(u) I_1\left(\frac{d}{a}u\right) - (3-2\sigma) u\, I_0(u) I_0\left(\frac{d}{a}u\right) + \{4(1-\sigma)+u^2\} I_1(u) I_0\left(\frac{d}{a}u\right)$$

As an example of these general formulae let us consider a shear force of intensity $S/2c$ acting on the part of the bounding surface contained between the planes $z = b - c$, and $z = -b + c$ and a similar force of equal intensity but in the opposite direction acting on the part contained between $z = -b - c$, and $z = -b + c$, that is

$$q(z) = -S/2c, \qquad -b - c \leqslant z \leqslant -b + c$$
$$\quad = S/2c, \qquad b - c \leqslant z \leqslant b + c$$

so that

$$\bar{q}(\xi) = \frac{S}{2^{3/2}\pi^{1/2}c}\left[-\int_{-b-c}^{-b+c} e^{i\xi z}\,dz + \int_{b-c}^{b+c} e^{i\xi z}\,dz\right]$$

$$= \left(\frac{2}{\pi}\right)^{1/2} \frac{iS\sin\frac{b}{a}u\sin\frac{c}{a}u.}{\frac{c}{a}u.}$$

Inserting this value in equations (7·44) and using the results (IX) and (X) the expressions for the components of stress and displacement become

$$\sigma_r = -\frac{Sa}{\pi rc}\int_0^\infty \Phi_1(u)\sin\frac{b}{a}u\sin\frac{c}{a}u\cos\frac{z}{a}u.\frac{du}{u}.$$

$$\sigma_\theta = -\frac{Sa}{\pi rc}\int_0^\infty \Phi_2(u)\sin\frac{b}{a}u\sin\frac{c}{a}u\cos\frac{z}{a}u.\frac{du}{u}$$

$$\sigma_z = -\frac{S}{\pi c}\int_0^\infty \Phi_3(u)\sin\frac{b}{a}u\sin\frac{c}{a}u\cos\frac{z}{a}u.\frac{du}{u}.$$

$$\tau_{rz} = \frac{S}{\pi c}\int_0^\infty \Phi_4(u)\sin\frac{b}{a}u\sin\frac{c}{a}u\sin\frac{z}{a}u.\frac{du}{u}.$$

$$\frac{E}{1+\sigma}\frac{U}{a} = -\frac{S}{\pi c}\int_0^\infty \bar{\Phi}_5(u)\sin\frac{b}{a}u\,\sin\frac{c}{a}u\,\cos\frac{z}{a}u\,.\frac{du}{u}.$$

$$\frac{E}{1+\sigma}\frac{w}{a} = \frac{S}{\pi c}\int_0^\infty \bar{\Phi}_6(u)\sin\frac{b}{a}u\,\sin\frac{c}{a}u\,\sin\frac{z}{a}u\,.\frac{du}{u}.$$

Filon (3) has considered the problem of an elastic cylinder of finite length with this type of loading which as he points out is of considerable importance due to the fact that in a standard tension test using cylindrical bars, the tension is applied by means of pressure applied to collars projecting from the cylinder so that the forces applied to the surface of the test piece are in fact shearing forces applied in the manner described above. Since the part of the speciman of most interest is confined between the collars there should be no serious error incurred in the assumption that the ends are some distance from the collars. Since Filon's solution is in the form of an infinite series and is consequently not amenable to numerical calculation it may be that the above solution has some advantages.

7·5. Infinitely long cylindrical cavity in an elastic solid of infinite extent with normal forces applied to the surface of the cavity.

We shall consider now the general solution to the problem of elastic medium of infinite extent with forces applied normally to the surface of a circularly cylindrical cavity in the medium.

We take the axis of symmetry to be the z-axis and employ cylindrical polar co-ordinates as before so that a suitable solution of the transformed biharmonic equation (7·26) must be obtained. The general solution was given previously by (7·27) and since the required solution must give finite stresses at infinity we take A = B = 0. The conditions at the surface of the cavity may then be used to determine the arbitrary functions C and D. Let us denote the normal force applied to the bounding surface r = a by p(z) so that we have

$$\sigma_r = -p(z) \quad , \quad \tau_{rz} = 0 \quad , \quad r = a.$$

then $$\qquad (7{\cdot}51)$$

$$\overline{\sigma}_r = -\overline{p}(\zeta) \quad , \quad \overline{\tau}_{rz} = 0 \quad , \quad r = a.$$

where $\overline{p}(\zeta)$ is defined by the integral (7·23). Using equations (7·28) along with (7·27) and (7·51) we find that C and D are given by

$$C = \frac{\overline{p}(\zeta)}{i\zeta^3} \, \frac{\alpha\{2(1-\sigma)K_1(\alpha) - \alpha K_0(\alpha)\}}{\omega(\alpha)}$$

$$\qquad (7{\cdot}52)$$

$$D = \frac{\overline{p}(\zeta)}{i\zeta^3} \, \frac{\alpha K_1(\alpha)}{\omega(\alpha)}$$

where

$$\omega(\alpha) = \alpha^2 K_0^2(\alpha) - \{2(1-\sigma) + \alpha^2\} K_1^2(\alpha)$$

Proceeding as in the previous examples we obtain the following integral expressions for the components of stress and displacement in the elastic medium

$$\sigma_r = \frac{1}{(2\pi)^{1/2} r}\int_{-\infty}^{\infty} \bar{p}\left(\frac{u}{a}\right) \Theta_1(u)\, e^{-i\frac{z}{a}u}\, du$$

$$\sigma_\theta = \frac{1}{(2\pi)^{1/2} r}\int_{-\infty}^{\infty} \bar{p}\left(\frac{u}{a}\right) \Theta_2(u)\, e^{-i\frac{z}{a}u}\, du.$$

$$\sigma_z = \frac{1}{(2\pi)^{1/2} a}\int_{-\infty}^{\infty} \bar{p}\left(\frac{u}{a}\right) \Theta_3(u)\, e^{-i\frac{z}{a}u}\, du.$$

$$\tau_{rz} = \frac{i}{(2\pi)^{1/2} a}\int_{-\infty}^{\infty} \bar{p}\left(\frac{u}{a}\right) \Theta_4(u)\, e^{-i\frac{z}{a}u}\, du$$

$$\frac{1+\sigma}{E}\, U = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} \bar{p}\left(\frac{u}{a}\right) \Theta_5(u)\, e^{-i\frac{z}{a}u}\, du$$

$$\frac{1+\sigma}{E}\, w = \frac{i}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} \bar{p}\left(\frac{u}{a}\right) \Theta_6(u)\, e^{-i\frac{z}{a}u}\, du$$

where $\Theta_i(u)$, $i = 1 \cdots\cdots 6$ denotes the following functions

$$\omega(u).\Theta_1(u) = \frac{r}{a}u\, K_1(u) K_0\left(\frac{r}{a}u\right) - u\, K_0(u) K_1\left(\frac{r}{a}u\right) - \frac{r}{a}u^2 K_0(u) K_0\left(\frac{r}{a}u\right)$$
$$+ \left\{2(1-\sigma) + \frac{r^2}{a^2}u^2\right\} K_1(u) K_1\left(\frac{r}{a}u\right)$$

$$\omega(u).\Theta_2(u) = u\, K_0(u) K_1\left(\frac{r}{a}u\right) - 2(1-\sigma) K_1(u) K_1\left(\frac{r}{a}u\right)$$
$$-(1-2\sigma)\frac{r}{a}u\, K_1(u)\, K_0\left(\frac{r}{a}u\right)$$

$$\omega(u).\Theta_3(u) = u\left\{2K_1(u) K_0\left(\frac{r}{a}u\right) + u\, K_0(u) K_0\left(\frac{r}{a}u\right) - \frac{r}{a} K_1(u) K_1\left(\frac{r}{a}u\right)\right\}$$

$$\omega(u)\cdot\Theta_4(u) = u^2\left\{\tfrac{d}{a}K_1(u)K_0(\tfrac{d}{a}u) - K_0(u)K_1(\tfrac{d}{a}u)\right\}$$

$$\omega(u)\cdot\Theta_5(u) = u\,K_0(u)K_1(\tfrac{d}{a}u) - \tfrac{d}{a}u\,K_1(u)K_0(\tfrac{d}{a}u) - 2(1-\sigma)K_1(u)K_1(\tfrac{d}{a}u)$$

$$\omega(u)\cdot\Theta_6(u) = -\tfrac{d}{a}u\,K_1(u)K_1(\tfrac{d}{a}u) + u\,K_0(u)K_0(\tfrac{d}{a}u) + 2(1-\sigma)K_1(u)K_0(\tfrac{d}{a}u)$$

The case of shearing forces applied to the bounding surface may be obtained in exactly the same way.

7·6 Infinitely long cylindrical tube with normal loading of the curved surfaces.

Finally in this section we shall consider the case of an infinitely long elastic tube of circular cross-section in a state of stress due to symmetrical loading of the curved surfaces. The axis of symmetry is assumed to the co-incident with the axis of the tube so that the problem is one of symmetrical deformation. Proceeding as before we see that it is necessary in this case to determine the four arbitrary functions A,B,C and D in the solution (7·27)of the transformed biharmonic equation (7·26) from the conditions prescribed on the two bounding surfaces of the tube. Let us denote the normal force applied to the bounding surface r = a by p(z), and assume that the other bounding surface r = b is free from stress then the conditions to be satisfied are

$$\sigma_r = -p(z) \quad , \quad \tau_{rz} = 0 \, , \quad r = a$$
$$= 0 \qquad\qquad = 0 \, , \quad r = b$$

Writing

$$\bar{p}(\varsigma) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} p(z) e^{i\varsigma z} dz \, .$$

the transformed components of stress at the boundaries become

$$\bar{\sigma}_r = -\bar{p}(\varsigma) \quad , \quad \bar{\tau}_{rz} = 0 \, , \quad r = a$$
$$= 0 \qquad\qquad = 0 \, , \quad r = b \tag{7.61}$$

Using these equations along with (7·28) and (7·27) we obtain four equations which may be solved to give the following expression for A, B, C, and D

$$\Pi(\alpha,\beta).A(\varsigma) = -\frac{\bar{p}(\varsigma)}{i\varsigma^3}\alpha\Big[\{[2(1-\sigma)+\beta^2]I_1(\beta)K_1(\beta)+\beta^2 I_0(\beta)K_0(\beta)\}\{2(1-\sigma)K_1(\alpha)-\alpha K_0(\alpha)\}$$
$$-\{[2(1-\sigma)+\beta^2]K_1^2(\beta)-\beta^2K_0^2(\beta)\}\{2(1-\sigma)I_1(\alpha)+\alpha I_0(\alpha)\}$$
$$+\{2(1-\sigma)+\beta^2\}K_1(\alpha)-2(1-\sigma)\alpha K_0(\alpha)\Big]$$

$$\Pi(\alpha,\beta).B(\varsigma) = \frac{\bar{p}(\varsigma)}{i\varsigma^3}\alpha\Big[\{[2(1-\sigma)+\beta^2]I_1(\beta)K_1(\beta)+\beta^2 I_0(\beta)K_0(\beta)\}K_1(\alpha)$$
$$-\{[2(1-\sigma)+\beta^2]K_1^2(\beta)-\beta^2K_0^2(\beta)\}I_1(\alpha)-\alpha K_0(\alpha)\Big]$$

$$\Pi(\alpha\beta).C(s) = \frac{\bar{p}(s)}{is^3}\alpha\Big[\{[2(1-\sigma)+\beta^2]I_1(\beta)K_1(\beta) + \beta^2 I_0(\beta)K_0(\beta)\}\{2(1-\sigma)I_1(\alpha) + \alpha I_0(\alpha)\}$$

$$-\{[2(1-\sigma)+\beta^2]I_1^2(\beta) - \beta^2 I_0^2(\beta)\}\{2(1-\sigma)K_1(\alpha) - \alpha K_0(\alpha)\}$$

$$-\{2(1-\sigma)+\beta^2\}I_1(\alpha) - 2(1-\sigma)\alpha I_0(\alpha)$$

$$\Pi(\alpha,\beta).D(s) = \frac{\bar{p}(s)}{is^3}\alpha\Big[\{[2(1-\sigma)+\beta^2]I_1(\beta)K_1(\beta) + \beta^2 I_0(\beta)K_0(\beta)\}I_1(\alpha)$$

$$-\{[2(1-\sigma)+\beta^2]I_1^2(\beta) - \beta^2 I_0^2(\beta)\}K_1(\alpha) - \alpha I_0(\alpha)\Big]$$

where $\alpha = as$, $\beta = bs$ and

$$\Pi(\alpha,\beta) = \Big[\{2(1-\sigma)+\alpha^2\}\{2(1-\sigma)+\beta^2\}\{I_1(\alpha)K_1(\beta) - K_1(\alpha)I_1(\beta)\}^2$$

$$-\alpha^2\{2(1-\sigma)+\beta^2\}\{I_0(\alpha)K_1(\beta) + K_0(\alpha)I_1(\beta)\}^2$$

$$-\beta^2\{2(1-\sigma)+\alpha^2\}\{I_1(\alpha)K_0(\beta) + K_1(\alpha).I_0(\beta)\}^2$$

$$+\alpha^2\beta^2\{K_0(\alpha)I_0(\beta) - I_0(\alpha)K_0(\beta)\}^2$$

$$+ 4(1-\sigma) + \alpha^2 + \beta^2\Big]$$

If now we substitute thes values for A,B,C,and D in equations (7.28) and then obtain the transformsd components of stress and displacement we obtain on inverting

$$\sigma_r = \frac{1}{(2\pi)^{1/2} r}\int_{-\infty}^{\infty} \bar{p}\left(\tfrac{u}{a}\right)\Omega_1(u)\, e^{-i\frac{z}{a}u}\,du$$

$$\sigma_\theta = \frac{1}{(2\pi)^{1/2} r}\int_{-\infty}^{\infty} \bar{p}\left(\tfrac{u}{a}\right)\Omega_2(u)\, e^{-i\frac{z}{a}u}\,du.$$

$$\sigma_z = \frac{1}{(2\pi)^{1/2} a}\int_{-\infty}^{\infty} \bar{p}\left(\tfrac{u}{a}\right)\Omega_3(u)\, e^{-i\frac{z}{a}u}\,du$$

$$\tau_{rz} = \frac{i}{(2\pi)^{1/2} a}\int_{-\infty}^{\infty} \bar{p}\left(\tfrac{u}{a}\right)\Omega_4(u)\, e^{-i\frac{z}{a}u}\,du$$

$$\frac{E}{1+\sigma}U = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} \bar{p}\left(\tfrac{u}{a}\right)\Omega_5(u)\, e^{-i\frac{z}{a}u}\,du.$$

$$\frac{E}{1+\sigma}w = \frac{i}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} \bar{p}\left(\tfrac{u}{a}\right)\Omega_6(u)\, e^{-i\frac{z}{a}u}\,du.$$

where the Ω are defined by the expressions

$$\Pi\left(u,\tfrac{b}{a}u\right)\cdot\Omega_1(u) = \Big[\Theta\left(\tfrac{b}{a}u\right)\Big\{u\left[I_0(u)K_1\left(\tfrac{\lambda}{a}u\right) - K_0(u)I_1\left(\tfrac{\lambda}{a}u\right)\right] + \frac{\lambda}{a}u\left[I_1(u)K_0\left(\tfrac{\lambda}{a}u\right) - K_1(u)I_0\left(\tfrac{\lambda}{a}u\right)\right]$$

$$+\frac{\lambda}{a}u^2\left[I_0(u)K_0\left(\tfrac{\lambda}{a}u\right) + K_0(u)I_0\left(\tfrac{\lambda}{a}u\right)\right] + \left[\frac{\lambda^2}{a^2}u^2 + 2(1-\sigma)\right]\left[K_1(u)I_1\left(\tfrac{\lambda}{a}u\right) + I_1(u)K_1\left(\tfrac{\lambda}{a}u\right)\right]$$

$$+\Psi\left(\tfrac{b}{a}u\right)\Big\{uK_0(u)K_1\left(\tfrac{\lambda}{a}u\right) - \frac{\lambda}{a}uK_1(u)K_0\left(\tfrac{\lambda}{a}u\right) + \frac{\lambda}{a}u^2K_0(u)K_0\left(\tfrac{\lambda}{a}u\right)$$

$$-\left[\frac{\lambda^2}{a^2}u^2 + 2(1-\sigma)\right]K_1(u)K_1\left(\tfrac{\lambda}{a}u\right)\Big\}$$

$$-\Phi\left(\tfrac{b}{a}u\right)\Big\{uI_0(u)I_1\left(\tfrac{\lambda}{a}u\right) - \frac{\lambda}{a}uI_1(u)I_0\left(\tfrac{\lambda}{a}u\right) - \frac{\lambda}{a}u^2I_0(u)I_0\left(\tfrac{\lambda}{a}u\right)$$

$$+\left[\frac{\lambda^2}{a^2}u^2 + 2(1-\sigma)\right]I_1(u)I_1\left(\tfrac{\lambda}{a}u\right)\Big\}$$

$$-X\left(\tfrac{b}{a}u\right)\Big\{I_1(u)K_1\left(\tfrac{\lambda}{a}u\right) - K_1(u)I_1\left(\tfrac{\lambda}{a}u\right) + \frac{\lambda}{a}u\left[I_1(u)K_0\left(\tfrac{\lambda}{a}u\right) + K_1(u)I_0\left(\tfrac{\lambda}{a}u\right)\right]\Big\}$$

$$-\frac{\lambda}{a}u^2\left[I_0(u)K_0\left(\tfrac{\lambda}{a}u\right) - K_0(u)I_0\left(\tfrac{\lambda}{a}u\right)\right] - u\left[\frac{\lambda^2}{a^2}u^2 + 2(1-\sigma)\right]\left[I_0(u)K_1\left(\tfrac{\lambda}{a}u\right) + K_0(u)I_1\left(\tfrac{\lambda}{a}u\right)\right]$$

$$
\begin{aligned}
\Pi\left(u,\tfrac{b}{a}u\right)\cdot\Omega_2(u) = \Big[& -\Theta\left(\tfrac{b}{a}u\right)\Big\{2(1-\sigma)\left[I_1(u)K_1\left(\tfrac{d}{a}u\right)+K_1(u)I_1\left(\tfrac{d}{a}u\right)\right] + u\left[I_0(u)K_1\left(\tfrac{d}{a}u\right)-K_0(u)I_1\left(\tfrac{d}{a}u\right)\right] \\
& \qquad + (1-2\sigma)\tfrac{d}{a}u\left[I_1(u)K_0\left(\tfrac{d}{a}u\right)-K_1(u)I_0\left(\tfrac{d}{a}u\right)\right]\Big\} \\
& + \Psi\left(\tfrac{b}{a}u\right)\Big\{2(1-\sigma)K_1(u)K_1\left(\tfrac{d}{a}u\right) - uK_0(u)K_1\left(\tfrac{d}{a}u\right) + (1-2\sigma)\tfrac{d}{a}u\,K_1(u)K_0\left(\tfrac{d}{a}u\right)\Big\} \\
& + \Phi\left(\tfrac{b}{a}u\right)\Big\{2(1-\sigma)I_1(u)I_1\left(\tfrac{d}{a}u\right) + uI_0(u)I_1\left(\tfrac{d}{a}u\right) - (1-2\sigma)\tfrac{d}{a}u\,I_1(u)I_0\left(\tfrac{d}{a}u\right)\Big\} \\
& + \chi\left(\tfrac{b}{a}u\right)\Big\{I_1(u)K_1\left(\tfrac{d}{a}u\right) - K_1(u)I_1\left(\tfrac{d}{a}u\right)\Big\} \\
& + 2(1-\sigma)u\Big\{I_0(u)K_1\left(\tfrac{d}{a}u\right) + K_0(u)I_1\left(\tfrac{d}{a}u\right)\Big\} \\
& + (1-2\sigma)\tfrac{d}{a}u^2\Big\{I_0(u)K_0\left(\tfrac{d}{a}u\right) - K_0(u)I_0\left(\tfrac{d}{a}u\right)\Big\}\Big]
\end{aligned}
$$

$$
\begin{aligned}
\Pi\left(u,\tfrac{b}{a}u\right)\cdot\Omega_3(u) = u\Big[& \Theta\left(\tfrac{b}{a}u\right)\Big\{2\left[I_1(u)K_0\left(\tfrac{d}{a}u\right)-K_1(u)I_0\left(\tfrac{d}{a}u\right)\right] - u\left[I_0(u)K_0\left(\tfrac{d}{a}u\right)+K_0(u)I_0\left(\tfrac{d}{a}u\right)\right] \\
& \qquad - \tfrac{d}{a}u\left[I_1(u)K_1\left(\tfrac{d}{a}u\right)+K_1(u)I_1\left(\tfrac{d}{a}u\right)\right]\Big\} \\
& - \Psi\left(\tfrac{b}{a}u\right)\Big\{2K_1(u)K_0\left(\tfrac{d}{a}u\right) + uK_0(u)K_0\left(\tfrac{d}{a}u\right) - \tfrac{d}{a}u\,K_1(u)K_1\left(\tfrac{d}{a}u\right)\Big\} \\
& + \Phi\left(\tfrac{b}{a}u\right)\Big\{2I_1(u)I_0\left(\tfrac{d}{a}u\right) - uI_0(u)I_0\left(\tfrac{d}{a}u\right) + \tfrac{d}{a}u\,I_1(u)I_1\left(\tfrac{d}{a}u\right)\Big\} \\
& + \chi\left(\tfrac{b}{a}u\right)\Big\{I_1(u)K_0\left(\tfrac{d}{a}u\right) + K_1(u)I_0\left(\tfrac{d}{a}u\right)\Big\} \\
& - 2u\Big\{I_0(u)K_0\left(\tfrac{d}{a}u\right) - K_0(u)I_0\left(\tfrac{d}{a}u\right)\Big\} \\
& + \tfrac{d}{a}u^2\Big\{I_0(u)K_1\left(\tfrac{d}{a}u\right) + K_0(u)I_1\left(\tfrac{d}{a}u\right)\Big\}\Big]
\end{aligned}
$$

$$\Pi\left(u,\tfrac{b}{a}u\right).\Omega_4(u) = u\Big[\Theta\left(\tfrac{b}{a}u\right)\Big\{u\left[I_0(u)K_1\left(\tfrac{d}{a}u\right) - K_0(u)I_1\left(\tfrac{d}{a}u\right)\right] + \tfrac{d}{a}u\left[I_1(u)K_0\left(\tfrac{d}{a}u\right) - K_1(u)I_0\left(\tfrac{d}{a}u\right)\right]\Big\}$$

$$+\Psi\left(\tfrac{b}{a}u\right)\Big\{uK_0(u)K_1\left(\tfrac{d}{a}u\right) - \tfrac{d}{a}uK_1(u)K_0\left(\tfrac{d}{a}u\right)\Big\}$$

$$+\Phi\left(\tfrac{b}{a}u\right)\Big\{uI_0(u)I_1\left(\tfrac{d}{a}u\right) - \tfrac{d}{a}uI_1(u)I_0\left(\tfrac{d}{a}u\right)\Big\}$$

$$-\chi\left(\tfrac{b}{a}u\right)\Big\{I_1(u)K_1\left(\tfrac{d}{a}u\right) - K_1(u)I_1\left(\tfrac{d}{a}u\right)\Big\}$$

$$-\tfrac{d}{a}u^2\Big\{I_0(u)K_0\left(\tfrac{d}{a}u\right) - K_0(u)I_0\left(\tfrac{d}{a}u\right)\Big\}\Big]$$

$$\Pi\left(u,\tfrac{b}{a}u\right).\Omega_5(u) = \Big[-\Theta\left(\tfrac{b}{a}u\right)\Big\{2(1-\sigma)\left[I_1(u)K_1\left(\tfrac{d}{a}u\right) + K_1(u)I_1\left(\tfrac{d}{a}u\right)\right]$$

$$+u\left[I_0(u)K_1\left(\tfrac{d}{a}u\right) - K_0(u)I_1\left(\tfrac{d}{a}u\right)\right] + \tfrac{d}{a}u\left[I_1(u)K_0\left(\tfrac{d}{a}u\right) - K_1(u)I_0\left(\tfrac{d}{a}u\right)\right]\Big\}$$

$$+\Psi\left(\tfrac{b}{a}u\right)\Big\{2(1-\sigma)K_1(u)K_1\left(\tfrac{d}{a}u\right) - uK_0(u)K_1\left(\tfrac{d}{a}u\right) + \tfrac{d}{a}uK_1(u)K_0\left(\tfrac{d}{a}u\right)\Big\}$$

$$+\Phi\left(\tfrac{b}{a}u\right)\Big\{2(1-\sigma)I_1(u)I_1\left(\tfrac{d}{a}u\right) + uI_0(u)I_1\left(\tfrac{d}{a}u\right) - \tfrac{d}{a}uI_1(u)I_0\left(\tfrac{d}{a}u\right)\Big\}$$

$$+\chi\left(\tfrac{b}{a}u\right)\Big\{I_1(u)K_1\left(\tfrac{d}{a}u\right) - K_1(u)I_1\left(\tfrac{d}{a}u\right)\Big\}$$

$$+2(1-\sigma)u\Big\{I_0(u)K_1\left(\tfrac{d}{a}u\right) + K_0(u)I_1\left(\tfrac{d}{a}u\right)\Big\}$$

$$+\tfrac{d}{a}u^2\Big\{I_0(u)K_0\left(\tfrac{d}{a}u\right) - K_0(u)I_0\left(\tfrac{d}{a}u\right)\Big\}\Big]$$

$$\Pi\left(u,\tfrac{b}{a}u\right)\cdot\Omega_0(u) = \Big[\Theta\left(\tfrac{b}{a}u\right)\Big\{2(1-\sigma)\left[I_1(u)K_0\left(\tfrac{d}{a}u\right) - K_1(u)I_0\left(\tfrac{d}{a}u\right)\right]$$

$$- u\left[I_0(u)K_0\left(\tfrac{d}{a}u\right) + K_0(u)I_0\left(\tfrac{d}{a}u\right)\right] - \tfrac{d}{a}u\left[I_1(u)K_1\left(\tfrac{d}{a}u\right) + K_1(u)I_1\left(\tfrac{d}{a}u\right)\right]\Big\}$$

$$- \Psi\left(\tfrac{b}{a}u\right)\Big\{2(1-\sigma)K_1(u)K_0\left(\tfrac{d}{a}u\right) + uK_0(u)K_0\left(\tfrac{d}{a}u\right) + \tfrac{d}{a}uK_1(u)K_1\left(\tfrac{d}{a}u\right)\Big\}$$

$$+ \Phi\left(\tfrac{b}{a}u\right)\Big\{2(1-\sigma)I_1(u)I_0\left(\tfrac{d}{a}u\right) - uI_0(u)I_0\left(\tfrac{d}{a}u\right) + \tfrac{d}{a}uI_1(u)I_1\left(\tfrac{d}{a}u\right)\Big\}$$

$$+ \chi\left(\tfrac{b}{a}u\right)\Big\{I_1(u)K_0\left(\tfrac{d}{a}u\right) + K_1(u)I_0\left(\tfrac{d}{a}u\right)\Big\}$$

$$- 2(1-\sigma)u\Big\{I_0(u)K_0\left(\tfrac{d}{a}u\right) - K_0(u)I_0\left(\tfrac{d}{a}u\right)\Big\}$$

$$+ \tfrac{d}{a}u^2\Big\{I_0(u)K_1\left(\tfrac{d}{a}u\right) + K_0(u)I_1\left(\tfrac{d}{a}u\right)\Big\}\Big]$$

The factors dependent only on the thickness of the tube being defined by the expressions

$$\Theta\left(\tfrac{b}{a}u\right) = \left\{2(1-\sigma) + \tfrac{b^2}{a^2}u^2\right\} I_1\left(\tfrac{b}{a}u\right)K_1\left(\tfrac{b}{a}u\right) + \tfrac{b^2}{a^2}u^2 I_0\left(\tfrac{b}{a}u\right)K_0\left(\tfrac{b}{a}u\right)$$

$$\Psi\left(\tfrac{b}{a}u\right) = \left\{2(1-\sigma) + \tfrac{b^2}{a^2}u^2\right\} I_1^2\left(\tfrac{b}{a}u\right) - \tfrac{b^2}{a^2}u^2 I_0^2\left(\tfrac{b}{a}u\right)$$

$$\Phi\left(\tfrac{b}{a}u\right) = \left\{2(1-\sigma) + \tfrac{b^2}{a^2}u^2\right\} K_1^2\left(\tfrac{b}{a}u\right) - \tfrac{b^2}{a^2}u^2 K_0^2\left(\tfrac{b}{a}u\right)$$

$$\chi\left(\tfrac{b}{a}u\right) = \left\{2(1-\sigma) + \tfrac{b^2}{a^2}u^2\right\}$$

$$\Pi\left(\tfrac{b}{a}u\right) = \Big[\left\{2(1-\sigma)+u^2\right\}\left\{2(1-\sigma)+\tfrac{b^2}{a^2}u^2\right\}\left\{I_1(u)K_1\left(\tfrac{b}{a}u\right)-K_1(u)I_1\left(\tfrac{b}{a}u\right)\right\}^2$$

$$-u^2\left\{2(1-\sigma)+\tfrac{b^2}{a^2}u^2\right\}\left\{I_0(u)K_1\left(\tfrac{b}{a}u\right)+K_0(u)I_1\left(\tfrac{b}{a}u\right)\right\}^2$$

$$-\tfrac{b^2}{a^2}u^2\left\{2(1-\sigma)+u^2\right\}\left\{I_1(u)K_0\left(\tfrac{b}{a}u\right)+K_1(u)I_0\left(\tfrac{b}{a}u\right)\right\}^2.$$

$$-\tfrac{b^2}{a^2}u^4\left\{K_0(u)I_0\left(\tfrac{b}{a}u\right)-I_0(u)K_0\left(\tfrac{b}{a}u\right)\right\}^2+4(1-\sigma)+u^2\left(1+\tfrac{b^2}{a^2}\right)\Big]$$

It is readily shown that when b tends to zero these expressions reduce to (7.34) while if b tends to infinity we obtain the results (7.53).

The solution for a tube with normal loading on both bounding surfaces may readily be deduced from (7.63).

Numerical evaluation of the results obtained in this part has not yet been attempted but it is hoped that a computing device such as that referred to in (3.8) will reduce the labour involved in the use of quadrature formulae.

PART VIII.

THE APPLICATION OF DYNAMICAL FORCES TO THE BOUNDING SURFACES OF CIRCULARLY CYLINDRICAL ELASTIC SOLIDS.

VIII. THE APPLICATION OF DYNAMICAL FORCES TO THE BOUNDING SURFACES OF CIRCULARLY CYLINDRICAL ELASTIC SOLIDS.

8.1 Introduction.

The various modes of vibration of an infinitely long cylindrical tube of circular cross-section is discussed by Love (21) but problems in which forces which vary with time are applied to the bounding surface do not seem to have been considered to any great extent. It is clear however that the methods developed in the previous parts of this thesis can be used to give formal solutions of the problems. In general the components of stress and displacement are given as double integrals which are impossibly difficult to evaluate, but in the particular case in which the forces are moving with uniform velocity these integrals reduce to single integrals which can be evaluated by numerical methods.

This type of problem has many interesting applications particularly in armaments research but a complete treatment would involve a considerable amount of numerical computation which it was impractical to undertake at this stage, so we shall conclude by giving the analysis for a few special problems to illustrate the method.

Problems in which the forces are moving along the bounding surface of an infinitely long circular cylindrical cavity, or along the surfaces of a cylindrical tube of circular cross-section may also be solved by this method.

8·2 Solution of the equations of motion.

We shall consider now the solution of the equations of motion applicable to an elastic solid of revolution when forces which may vary with time are applied to the bounding surfaces. If we describe the position of a point on the elastic medium by means of the cylindrical polar co-ordinates (r, θ, z), taking OZ to be the axis of symmetry, and assume that the applied forces are also symmetrical so that the problem is one of symmetrical deformation, then the components of stress tensor at any point within the body must satisfy the equations of motion (1·55) and the components of the displacement vector at any point are related to the components of stress by means of the equations (1·54). We shall assume that there are no body forces present.

We introduce two functions $\phi(r, z, t)$, and $G(r, z, t)$ defined by the equations

$$
\begin{aligned}
U &= \frac{\partial \phi}{\partial r} - \frac{\partial G}{\partial z} \\
w &= \frac{\partial \phi}{\partial z} + \frac{\partial G}{\partial r} + \frac{G}{r}
\end{aligned}
\tag{8.21}
$$

Substituting from equations (8•21) into equations (1•54) we find the components of stress expressed in terms of ϕ and G, to be

$$
\begin{aligned}
\sigma_r &= \lambda \nabla_1^2 \phi + 2\mu\left(\frac{\partial^2 \phi}{\partial r^2} - \frac{\partial^2 G}{\partial r \partial z}\right) \\
\sigma_\theta &= \lambda \nabla_1^2 \phi + 2\mu\left(\frac{1}{r}\frac{\partial \phi}{\partial r} - \frac{1}{r}\frac{\partial G}{\partial z}\right) \\
\sigma_z &= \lambda \nabla_1^2 \phi + 2\mu\left(\frac{\partial^2 \phi}{\partial z^2} + \frac{\partial^2 G}{\partial r \partial z} + \frac{1}{r}\frac{\partial G}{\partial z}\right) \\
\tau_{rz} &= \mu\left(2\frac{\partial^2 \phi}{\partial r \partial z} - \frac{\partial^2 G}{\partial z^2} + \frac{\partial^2 G}{\partial r^2} + \frac{1}{r}\frac{\partial G}{\partial r} - \frac{G}{r^2}\right)
\end{aligned}
\tag{8.22}
$$

If now we substitute from (8•22) into equations of motion (1•55) we find that ϕ and G must satisfy the two partial differential equations

$$
\begin{aligned}
(\lambda + 2\mu)\nabla_1^2 \phi - \rho \frac{\partial^2 \phi}{\partial t^2} &= 0 \\
\mu\left(\nabla_1^2 G - \frac{1}{r^2} G\right) - \rho \frac{\partial^2 G}{\partial t^2} &= 0
\end{aligned}
\tag{8.23}
$$

Writing $\frac{\lambda + 2\mu}{\rho} = c_1^2$, $\frac{\mu}{\rho} = c_2^2$, $K^2 = \frac{c_1^2}{c_2^2}$, and $\tau = c_1 t$ these equations become

$$\nabla^2\phi - \frac{\partial^2\Phi}{\partial\tau^2} = 0$$
$$\nabla^2 G - \frac{1}{r^2}G - \kappa^2\frac{\partial^2 G}{\partial\tau^2} = 0 \qquad (8.24)$$

We assume now that the elastic body extends indefinitely in both directions of the z-axis, and introduce the two-dimensional Fourier transforms $\bar{\phi}(r,\eta,\zeta)$, $\bar{G}(r,\eta,\zeta)$ of the functions ϕ and G to obtain solutions of equations (8•24). The transforms in this case are defined by means of the equations

$$\bar{\Phi}(r,\eta,\zeta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi(r,z,\tau)\,e^{i(\eta z+\zeta\tau)}\,dz\,d\tau.$$
$$\bar{G}(r,\eta,\zeta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(r,z,\tau)\,e^{i(\eta z+\zeta\tau)}\,dz\,d\tau. \qquad (8.25)$$

Multiply equations (8•23) by exp.i $(\eta z + \zeta\tau)$ and integrate over the entire $z\tau$-plane and these equations reduce to the ordinary differential equations

$$r^2\frac{d^2\bar{\Phi}}{dr^2} + r\frac{d\bar{\Phi}}{dr} - r^2(\eta^2 - \zeta^2)\bar{\Phi} = 0$$
$$r^2\frac{d^2\bar{G}}{dr^2} + r\frac{d\bar{G}}{dr} - \{r^2(\eta^2 - \kappa^2\zeta^2) + 1\}\bar{G} = 0 \qquad (8.26)$$

which are readily seen to be forms of Bessel's equation. The solutions of (8•26) may be written down immediately as

$$\bar{\Phi}(r,\eta,\zeta) = A(\eta,\zeta)\cdot I_0(\alpha r) + B(\eta,\zeta).K_0(\alpha r)$$
$$\bar{G}(r,\eta,\zeta) = C(\eta,\zeta)\,I_1(\beta r) + D(\eta,\zeta).K_1(\beta r). \qquad (8.27)$$

where I_n , and K_n are the modified Bessel functions of order n, of the first and second kind respectively, and $(\eta^2 - \zeta^2)^{\frac{1}{2}}$ and $(\eta^2 - K^2\zeta^2)^{\frac{1}{2}}$ being denoted by α and β . A, B, C, and D are arbitrary functions of η and ζ . Transforming equations (8•22) we find that the transformed components of stress are given by

$$\bar{\sigma}_r = \lambda \bar{\nabla}_1^2 \bar{\Phi} + 2\mu \left(\frac{d^2\bar{\Phi}}{dr^2} + i\eta \frac{d\bar{G}}{dr} \right)$$

$$\bar{\sigma}_\theta = \lambda \bar{\nabla}_1^2 \bar{\Phi} + 2\mu \left(\frac{1}{r} \frac{d\bar{\Phi}}{dr} + \frac{i\eta}{r} \bar{G} \right)$$

$$\bar{\sigma}_z = \lambda \bar{\nabla}_1^2 \bar{\Phi} + 2\mu \left(-\eta^2 \bar{\Phi} - i\eta \frac{d\bar{G}}{dr} - \frac{i\eta}{r} \bar{G} \right) \tag{8.28}$$

$$\bar{\tau}_{rz} = \mu \left(-2i\eta \frac{d\bar{\Phi}}{dr} + \eta^2 \bar{G} + \frac{d^2\bar{G}}{dr^2} + \frac{1}{r} \frac{d\bar{G}}{dr} - \frac{1}{r^2} \bar{G} \right)$$

where $\bar{\nabla}_1^2 \equiv \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \eta^2$.

Similarly from equations (8•21) we obtain the transformed components of the displacement vector

$$\bar{U} = \frac{d\bar{\Phi}}{dr} + i\eta \bar{G} \ , \ \bar{w} = -i\eta \bar{\Phi} + \frac{d\bar{G}}{dr} + \frac{\bar{G}}{r} \tag{8.29}$$

As in the previous examples considered the functions A, B, C, and D may be determined from the conditions prescribed on the boundaries of the elastic medium. The inversion theorem may then be used to give the components of stress and displacement appropriate to the problem.

8·3 Infinite circular cylinder with dynamical forces applied normal to the curved surface.

We shall now consider an infinitely long elastic cylinder of circular cross-section in a state of stress due to forces, which may vary with time, applied to the bounding surface $r = a$. If we denote the applied force by $p(z,t)$ the conditions at the boundary are

$$\sigma_r = -p(z,t) \;,\quad \tau_{rz} = 0 \;,\quad r = a. \tag{8·31}$$

Writing

$$\bar{p}(\eta,s) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(z,t)\, e^{i(\eta z + s\tau)}\, dz\, d\tau \tag{8·32}$$

these conditions become when transformed

$$\bar{\sigma}_r = -\bar{p}(\eta,s) \;,\quad \bar{\tau}_{rz} = 0 \;,\quad r = a \tag{8·34}$$

Now on the axis $r = 0$ the stresses must be finite so that the required solution must not contain the functions K_0, or K_1, both of which tend to infinity as the value of the argument tends to zero. Thus we have two arbitrary functions A and C to determine from equations (8·28). Substituting for $\bar{\phi}$ and $\bar{G}$ from (8·27) into (8·28) and using (8·34) we obtain two equations which on solving give the following expressions for A and C

$$A(\eta,s) = -\frac{\bar{p}(\eta,s)}{2\mu}\,\frac{(\eta^2 - \tfrac{1}{2}k^2 s^2)\, I_1(\beta a)}{\Delta(\eta^2,s^2)}$$

$$B(\eta,s) = -\frac{\bar{p}(\eta,s)}{2\mu}\,\frac{i\eta\,(\eta^2 - s^2)^{1/2}\, I_1(\alpha a)}{\Delta(\eta^2,s^2)} \tag{8·35}$$

where

$$\Delta(\eta^2,\varsigma^2) = (\eta^2-\tfrac{1}{2}k^2\varsigma^2)^2 I_0(\alpha a) I_1(\beta a) + \tfrac{1}{2a}k^2\varsigma^2(\eta^2-\varsigma^2)^{1/2} I_1(\alpha a) I_1(\beta a) - \eta^2(\eta^2-\varsigma^2)^{1/2}(\eta^2-k^2\varsigma^2)^{1/2} I_1(\alpha a) I_0(\beta a)$$

Now if we insert these values of A, B, C, and D in the expressions for the transformed components of stress we obtain the following double integrals which are the formal solution of this problem

$$\sigma_r = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{p}(\eta,\varsigma)\,\bar{\Psi}_1(\eta,\varsigma)\,e^{-i(\eta z+\varsigma\tau)}\,d\eta\,d\varsigma$$

$$\sigma_\theta = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{p}(\eta,\varsigma)\,\bar{\Psi}_2(\eta,\varsigma)\,e^{-i(\eta z+\varsigma\tau)}\,d\eta\,d\varsigma \tag{8.36}$$

$$\sigma_z = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{p}(\eta,\varsigma)\,\bar{\Psi}_3(\eta,\varsigma)\,e^{-i(\eta z+\varsigma\tau)}\,d\eta\,d\varsigma$$

$$\tau_{rz} = -\frac{i}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{p}(\eta,\varsigma)\,\bar{\Psi}_4(\eta,\varsigma)\,e^{-i(\eta z+\varsigma\tau)}\,d\eta\,d\varsigma$$

where the $\bar{\Psi}_i$, i = 1,·····,4 are defined by

$$\Delta(\eta^2,\varsigma^2)\cdot\bar{\Psi}_1(\eta,\varsigma) = (\eta^2-\tfrac{1}{2}k^2\varsigma^2)^2 I_1(\beta a) I_0(\alpha r) - \tfrac{1}{r}(\eta^2-\tfrac{1}{2}k^2\varsigma^2)(\eta^2-\varsigma^2)^{1/2} I_1(\beta a) I_1(\alpha r)$$
$$- \eta^2(\eta^2-\varsigma^2)^{1/2}(\eta^2-k^2\varsigma^2)^{1/2} I_1(\alpha a) I_0(\beta r) + \tfrac{1}{r}\eta^2(\eta^2-\varsigma^2)^{1/2} I_1(\alpha a) I_1(\beta r)$$

$$\Delta(\eta^2,\varsigma^2)\cdot\bar{\Psi}_2(\eta,\varsigma) = (1-\tfrac{1}{2}k^2)\varsigma^2(\eta^2-\tfrac{1}{2}k^2\varsigma^2) I_1(\beta a) I_0(\alpha r) + \tfrac{1}{r}(\eta^2-\tfrac{1}{2}k^2\varsigma^2)(\eta^2-\varsigma^2)^{1/2} I_1(\beta a) I_1(\alpha r)$$
$$- \tfrac{1}{r}\eta^2(\eta^2-\varsigma^2)^{1/2} I_1(\alpha a) I_1(\beta r)$$

$$\Delta(\eta^2,\varsigma^2)\cdot\bar{\Psi}_3(\eta,\varsigma) = (\eta^2-\tfrac{1}{2}k^2\varsigma^2)\{(1-\tfrac{1}{2}k^2)\varsigma^2-\eta^2\} I_1(\beta a) I_0(\alpha r) + \eta^2(\eta^2-\varsigma^2)^{1/2}(\eta^2-k^2\varsigma^2)^{1/2} I_1(\alpha a) I_0(\beta r)$$

$$\Delta(\eta^2,\varsigma^2)\cdot\bar{\Psi}_4(\eta,\varsigma) = \eta(\eta^2-\varsigma^2)^{1/2}(\eta^2-\tfrac{1}{2}k^2\varsigma^2)\{I_1(\alpha a) I_1(\beta r) - I_1(\beta a) I_1(\alpha r)\}$$

In a similar manner we find the components of displacement to be

$$U = -\frac{1}{4\pi\mu}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\bar{p}(\eta,s)\,\Psi_5(\eta,s)\,e^{-i(\eta z+s\tau)}\,d\eta\,ds$$

$$w = \frac{i}{4\pi\mu}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\bar{p}(\eta,s)\,\Psi_6(\eta,s)\,e^{-i(\eta z+s\tau)}\,d\eta\,ds \qquad (8.36)$$

where

$$\Delta(\eta^2,s^2)\cdot\Psi_5(\eta,s) = (\eta^2-s^2)^{1/2}(\eta^2-\tfrac{1}{2}K^2s^2)\,I_1(\beta a)\,I_1(\alpha r) - \eta^2(\eta^2-s^2)^{1/2}\,I_1(\alpha a)\,I_1(\beta r)$$

$$\Delta(\eta^2,s^2)\cdot\Psi_6(\eta,s) = \eta(\eta^2-\tfrac{1}{2}K^2s^2)\,I_1(\beta a)\,I_0(\alpha r) - \eta(\eta^2-s^2)^{1/2}(\eta^2-K^2s^2)^{1/2}\,I_1(\alpha a)\,I_0(\beta r)$$

8·4 Pulse of pressure moving uniformly along the boundary.

The integrals in the general solution can be considerably simplified if the applied force is a pulse of pressure moving with uniform velocity along the surface of the cylinder. If the pulse is moving with velocity v then we may take the applied force to be represented by p(z - vt) so that

$$\bar{p}(\eta,s) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}p(z-K_1\tau)\,e^{i(\eta z+s\tau)}\,dz\,d\tau \qquad (8.41)$$

where $K_1 = v/c_1$. Now if we put $z - K_1\tau = m$, and use the result (VIII) we get

$$\bar{p}(\eta,s) = \bar{p}(\eta)\,\delta(s+K_1\eta) \qquad (8.42)$$

where $\bar{p}(\eta)$ is defined by the expression

$$\bar{p}(\eta) = \int_{-\infty}^{\infty} p(u) e^{i\eta u} du. \tag{8.43}$$

Inserting this value in the first of equations (8·36) we get

$$\sigma_r = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{p}(\eta) e^{-i\eta z} d\eta \int_{-\infty}^{\infty} \bar{\Psi}_1(\eta, s) e^{-is\tau} \delta(s + \kappa_1\eta)\, ds$$

$$= -\frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{p}(\eta)\, \bar{\Psi}_1(\eta, -\kappa_1\eta)\, e^{-i(z-vt)\eta} d\eta$$

Using the result (VII). Putting $\eta a = u$ this becomes

$$\sigma_r = -\frac{1}{2\pi r}\int_{-\infty}^{\infty} \bar{p}\left(\frac{u}{a}\right) \bar{\Psi}_1(u)\, e^{-i\left(\frac{z-vt}{a}\right) u} du. \tag{8.44}$$

where now

$$\Delta(u)\cdot\bar{\Psi}_1(u) = \left(1-\tfrac{1}{2}\kappa_2^2\right)^2 \frac{r}{a} u\, I_1(u_2)\, I_0\left(\frac{r}{a}u_1\right) - \left(1-\tfrac{1}{2}\kappa_2^2\right)\left(1-\kappa_1^2\right)^{1/2} I_1(u_2)\, I_1\left(\frac{r}{a}u_1\right)$$
$$- \frac{r}{a}u\left(1-\kappa_1^2\right)^{1/2}\left(1-\kappa_2^2\right)^{1/2} I_1(u_1)\, I_0\left(\frac{r}{a}u_2\right) + \left(1-\kappa_1^2\right)^{1/2} I_1(u_1)\, I_1\left(\frac{r}{a}u_2\right)$$

and

$$\Delta(u) = \left(1-\tfrac{1}{2}\kappa_2^2\right)^2 u\, I_0(u_1)\, I_1(u_2) + \tfrac{1}{2}\kappa_2^2\left(1-\kappa_1^2\right)^{1/2} I_1(u)\, I_1(u_2) - \left(1-\kappa_1^2\right)^{1/2}\left(1-\kappa_2^2\right)^{1/2} u\, I_1(u_1)\, I_0(u_2)$$

u_1, and u_2 denote the expressions $(1-\kappa_1^2)^{\frac{1}{2}}u$, and $(1-\kappa_2^2)^{\frac{1}{2}}u$ respectively. In the same way it may be shown that the other components of the stress tensor are given by the integrals

$$\sigma_\theta = -\frac{1}{2\pi r}\int_{-\infty}^{\infty} \bar{p}\left(\frac{u}{a}\right) \bar{\Psi}_2(u)\, e^{-i\left(\frac{z-vt}{a}\right) u} du.$$

$$\sigma_z = -\frac{1}{2\pi a}\int_{-\infty}^{\infty} \bar{p}\left(\frac{u}{a}\right) \bar{\Psi}_3(u)\, e^{-i\left(\frac{z-vt}{a}\right) u} du. \tag{8.44}$$

$$\tau_{rz} = -\frac{i}{2\pi a}\int_{-\infty}^{\infty} \bar{p}\left(\frac{u}{a}\right) \bar{\Psi}_4(u)\, e^{-i\left(\frac{z-vt}{a}\right)u} du$$

where

$$\Delta(u).\,\bar{\Psi}_2(u) = (1-\tfrac{1}{2}K^2)K_1^2(1-\tfrac{1}{2}K_2^2)\tfrac{d}{a}u\, I_1(u_2)\, I_0(\tfrac{d}{a}u_1)$$
$$+(1-\tfrac{1}{2}K_2^2)(1-K_1^2)^{1/2} I_1(u_2)\, I_1(\tfrac{d}{a}u_1) - (1-K_1^2)^{1/2} I_1(u_1)\, I_1(\tfrac{d}{a}u_2)$$

$$\Delta(u).\,\bar{\Psi}_3(u) = (1-\tfrac{1}{2}K_2^2)\{(1-\tfrac{1}{2}K^2)K_1^2-1\}u\, I_1(u_2)\, I_0(\tfrac{d}{a}u_1)$$
$$+(1-K_1^2)^{1/2}(1-K_2^2)^{1/2}u\, I_1(u_1)\, I_0(\tfrac{d}{a}u_2)$$

$$\Delta(u).\,\bar{\Psi}_4(u) = (1-K_1^2)^{1/2}(1-\tfrac{1}{2}K_2^2)u\{I_1(u_1)\, I_1(\tfrac{d}{a}u_2) - I_1(u_2)\, I_1(\tfrac{d}{a}u_1)\}$$

Similarly it may be shown that the components of displacement become

$$U = -\frac{1}{4\pi\mu}\int_{-\infty}^{\infty} \bar{p}\left(\frac{u}{a}\right) \bar{\Psi}_5(u)\, e^{-i\left(\frac{z-vt}{a}\right)u} du$$

(8.44)

$$w = \frac{i}{4\pi\mu}\int_{-\infty}^{\infty} \bar{p}\left(\frac{u}{a}\right) \bar{\Psi}_6(u)\, e^{-i\left(\frac{z-vt}{a}\right)u} du.$$

where

$$\Delta(u)\cdot\overline{\Psi}_5(u) = (1-\kappa_1^2)^{1/2}(1-\tfrac{1}{2}\kappa_2^2)\, I_1(u_2)\, I_1(\tfrac{d}{a}u_1) - (1-\kappa_1^2)^{1/2} I_1(u_1)\, I_1(\tfrac{d}{a}u_2)$$

$$\Delta(u)\cdot\overline{\Psi}_6(u) = (1-\tfrac{1}{2}\kappa_2^2)\, I_1(u_2)\, I_0(\tfrac{d}{a}u_1) - (1-\kappa_1^2)^{1/2}(1-\kappa_2^2)^{1/2} I_1(u_1)\, I_0(\tfrac{d}{a}u_2)$$

As an example of a particular distribution of pressure applied to the curved surface, let us consider the rectangular pulse moving uniformly along it with velocity v. We have then

$$\begin{aligned} p(z,t) &= P/2b \quad , \quad |z - vt| \leq b \\ &= 0 \quad , \quad |z - vt| > b \end{aligned}$$

so that

$$\bar{p}(\eta) = \frac{P}{2b}\int_{-b}^{b} e^{i\eta m}\, dm = P.\frac{\sin \frac{b}{a}u}{\frac{b}{a}u}$$

where we have put $\eta a = u$

Substituting for $\bar{p}(\frac{u}{a})$ in equations (8·44) and using the results IX and X, we find that the components of stress are given by the integrals

$$\begin{aligned} \sigma_r &= -\frac{Pa}{\pi b r}\int_{-\infty}^{\infty} \overline{\Psi}_1(u) \sin\frac{b}{a}u \cos\left(\frac{z-vt}{a}\right)u \cdot \frac{du}{u} \\ \sigma_\theta &= -\frac{Pa}{\pi b r}\int_0^{\infty} \overline{\Psi}_2(u) \sin\frac{b}{a}u \cos\left(\frac{z-vt}{a}\right)u \cdot \frac{du}{u} \\ \sigma_z &= -\frac{P}{\pi b}\int_0^{\infty} \overline{\Psi}_3(u) \sin\frac{b}{a}u \cos\left(\frac{z-vt}{a}\right)u \cdot \frac{du}{u} \\ \tau_{rz} &= -\frac{P}{\pi b}\int_0^{\infty} \overline{\Psi}_4(u) \sin\frac{b}{a}u \cdot \sin\left(\frac{z-vt}{a}\right)u \cdot \frac{du}{u} \end{aligned} \qquad (8{\cdot}45)$$

To avoid the difficulty of dealing with a divergent integrand in the expression for w we shall consider instead the component of strain in the direction of the z-axis which is given by $\frac{\partial w}{\partial z}$

$$U = -\frac{Pa}{2\pi\mu b}\int_0^\infty \overline{\Psi}_5(u)\sin\frac{b}{a}u\cos\left(\frac{z-vt}{a}\right)u\cdot\frac{du}{u}.$$

$$\frac{\partial w}{\partial z} = -\frac{P}{2\pi\mu b}\int_0^\infty \overline{\Psi}_6(u)\sin\frac{b}{a}u\cos\left(\frac{z-vt}{a}\right)\cdot du.$$

The corresponding expressions for a point force uniformly distributed over the circle r = a, z = vt may readily be deduced from these by letting b tend to zero.

If the $\overline{\Psi}_i$ are expanded in terms of $K_1^{\,2}$ and then v, and K_1 are put equal zero we obtain the expressions (7·35) for the components of the stress and displacement in the corresponding statical problem.

A further example which is of some interest and which must be treated in a somewhat different way is that of a single discontinuity of pressure moving uniformly along the surved surface. In this case the applied pressure takes the form

$$\begin{aligned} p(z,t) &= p \qquad && vt \leq z < \infty \\ &= 0 && -\infty < z < vt \end{aligned} \tag{8.46}$$

so that

$$\overline{p}(\eta) = p\int_0^\infty e^{i\eta\mu}\,d\mu \tag{8.47}$$

In order that $\bar{p}(\eta)$ will have a definite value as m tends to infinity it is necessary to assume that η is complex and has a positive imaginary part. We then obtain

$$\bar{p}(\eta) = -\frac{p}{i\eta} = -\frac{pa}{iu} \tag{8.48}$$

where u is also complex. We are dealing now with the generalised Fourier integral so that the inversion theorem to be used is (II). With this modification the expression for the radial stress becomes

$$\sigma_r = -\frac{1}{2\pi r}\int_{ic-\infty}^{ic+\infty} \bar{p}\left(\frac{u}{a}\right)\bar{\Psi}_1(u)\, e^{-i\left(\frac{z-vt}{a}\right)u}\, du$$

which on substituting from (8·48) gives

$$\sigma_r = -\frac{ipa}{2\pi r}\int_{ic-\infty}^{ic+\infty} \bar{\Psi}_1(u)\, e^{-i\left(\frac{z-vt}{a}\right)u}\, \frac{du}{u} \tag{8.49}$$

Now the integrand has a simple pole at u = 0 so that the path of integration may be deformed into the real axis from $-\infty$ to $+\infty$ with a semi-circular indentation at the origin. The value of the integrals round the indentations may be found by making use of a well known theorem*(23) in complex variable theory and the integrals along the negative and positive parts of the real axis may be combined making use of the results (IX) and (X) so that the components of the stress tensor are given by

$$\sigma_r = -\tfrac{1}{2}p - \frac{pa}{\pi r}\int_0^\infty \bar{\Psi}_1(u)\sin\left(\frac{z-vt}{a}\right)u \cdot \frac{du}{u}.$$

$$\sigma_\theta = -\tfrac{1}{2}p - \frac{pa}{\pi r}\int_0^\infty \bar{\Psi}_2(u)\sin\left(\frac{z-vt}{a}\right)u \cdot \frac{du}{u}.$$

*(23) p.57.

$$\sigma_z = -\frac{1}{4}p\,\frac{\left(1-\frac{1}{2}K^2\right)K_2^2}{\left\{1-\frac{3}{4}K^2-\frac{1}{4}(1-K^2)K_2^2\right\}} - \frac{p}{\pi}\int_0^\infty \overline{\Psi}_3(u)\sin\left(\frac{z-vt}{a}\right)u\cdot\frac{du}{u}.$$

$$\tau_{rz} = \frac{p}{\pi}\int_0^\infty \overline{\Psi}_4(u)\cos\left(\frac{z-vt}{a}\right)u\cdot\frac{du}{u}.$$

In a similar manner we find the displacement given by

$$\frac{U}{a} = \frac{pv}{16\mu a}\,\frac{K^2(1-K_1^2)^{1/2}}{1-\frac{3}{4}K^2-\frac{1}{4}(1-K^2)K_2^2} - \frac{p}{4\pi\mu}\int_0^\infty \overline{\Psi}_5(u)\sin\left(\frac{z-vt}{a}\right)u\cdot\frac{du}{u}.$$

$$\frac{\partial w}{\partial z} = \frac{p}{4\mu}\,\frac{1-\frac{1}{2}K^2}{1-\frac{3}{4}K^2-\frac{1}{4}(1-K^2)K_2^2} + \frac{p}{2\pi\mu}\int_0^\infty \overline{\Psi}_6(u)\sin\left(\frac{z-vt}{a}\right)u.\,du.$$

where the $\overline{\Psi}_i$ are as defined in (8.44).

The solution to the statical problem is readily obtained by expanding the above expressions as powers of K_1^2 and then letting v tend to zero. Proceeding in this way we obtain agreement with the solution given by Tranter and Craggs. to the statical problem.

The integrals which have been obtained in the problems involving circular symmetry do not appear to be capable of exact evaluation but resort may always be made to one of the numerical methods discussed in 3·8. It is hoped however that a mechanical of electrical device might prove satisfactory for this purpose and in particular that referred to in (3·8) seems to have distinct possibilities.

BIBLIOGRAPHY AND REFERENCES.

1. G. Airy British Association Report 1862, p.62.
2. J.D.Eshelby
3. L.N.G. Filon Phil. Trans.Roy.Soc.A.198,(1902), 241.
4. L.N.G. Filon Phil.Trans.Roy.Soc.A. 201,(1903), 63.
5. L.N.G. Filon Proc.Roy.Soc.Edin. 49,(1928),38.
6. R. Fürth and R.W. Pringle Phil.Mag.(7),(37,)(1946).
7. G.A. Gibson "Elem.Treat. on the Calculus." 2nd ed.1905
8. A.Gray, G.B. Mathews, and T.M. MacRobert "Treat. on Bessel Functions." 2nd. ed. 1922.
9. A.E. Green Proc.Roy.Soc.A. 173,(1939), 173.
10. A.E. Green Proc.Roy.Soc.A. 180,(1942), 173.
11. A.E. Green Proc.Camb.Phil.Soc.41,(1945).
12. A.E. Green Phil.Mag.(7),34,(1943),416.
13. A.E. Green Phil.Mag.(7),34,(1943),420.
14. A.E. Green and G.I. Taylor Proc.Roy.Soc.A.173,(1939)
15. J.W. Harding and I.N. Sneddon Proc.Camb.Phil.Soc.41,(1945).
16. H. Lamb Proc.Roy.Soc.A. 93,(1917),114.
17. H. Lamb Phil.Trans.Roy.Soc.A.203,(1904),1.
18. E.R. Lapwood Phil. Trans.Roy.Soc.A. 242,(1949), 63.
19. A.E.H. Love Proc.Lon.Math.Soc.1,(1903),37.
20. A.E.H. Love Proc.Lon.Math.Soc.1,(1903),291.
21. A.E.H. Love "The Math.Theory of Elast.", 3rd.ed.1920.

22. A.N. Lowan "Tables of Circular and Hyperbolic Sines and Cosines." New York, 1939.

23. T.M. MacRobert "Functions of a Complex Variable." 1st.ed. MacMillan, 1917.

24. E. Melan Z.Angew.Math.Mech., 12,(1932),343.

25. L.M. Milne-Thomson and L.J. Comrie "Standard Four-Figure Mathematical Tables."MacMillan,1948.

26. A.W. Rankin Jour.Appl.Mech., 11,A.77,(1944).

27. Lord Rayleigh. Proc.Lon.Math.Soc.17,(1885),4.

28. I.N. Sneddon Proc.Camb.Phil.Soc.40,(1944), 229.

29. I.N. Sneddon Proc.Camb.Phil.Soc.42,(1945), 260.

30. I.N. Sneddon Proc.Camb.Phil.Soc.41,(1945), 239.

31. I.N. Sneddon Proc.Camb.Phil, Soc.42,(1945), 29.

32. I.N. Sneddon D.Sc.Thesis,Glasgow, 1947.

33. A.C. Stevenson Proc.Roy.Soc.A. 184, (1945), 129.

34. A.C. Stevenson Phil.Mag.(7), 34,(1943), 766.

35. A.C.Titchmarch "An Introduction to the Theory of Fourier Integrals." 2nd.ed.Oxford 1948.

36. C.J. Tranter Quart. Appl. Math.4(1946), 298.

37. C.J. Tranter Quart Jour.Mech.and App. Maths..1(2)(1948)

38. C.J. Tranter and J.W. Craggs Phil.Mag.(7),36,(1945), 241.

39. G.N. Watson "The Theory of Bessel Functions." 1st. ed. Cambridge, 1922.

40. H.M. Westergaard Karman Ann, Vol, 1941, p.154.

41. T.J. Wilmore Quart.Jour. Mech.Appl.Maths.2,(1949)