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SOME FORMULAE FOR THE E-FUNCTIONS AND RELATED FUNCTIONS

— by —
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Preface

This thesis, as is seen from its name, is based mainly upon MacRoberts' E-functions. For the definitions and properties of these functions the reader is referred to MacRobert, Functions of a Complex Variable, third edition. This work will be denoted by the letters C.V.

The thesis is divided into four chapters. In chapter I. linear relations between E-functions are established. In chapter II. some multiple integrals involving Bessel-functions are evaluated, while in chapter III. many integrals, involving E-functions are obtained. The last chapter is devoted to formulae for the associated Legendre functions, where the sum of the degree and the order is a positive integer. These formulae were proved by Gegenbauer, Wien, Sitzungsberichte, using his function $C_n^{\nu}(z)$. My proofs are very much simpler than his, though I make considerable use of one of his methods, involving the extended Rodrigues' formula. Many of the formulae used are given in MacRobert, Spherical Harmonics, second edition, which will be referred to as S.H. Also some known formulae will be shown to be particular cases of E-functions formulae in an appendix.

I wish to record my indebtedness to Professor T. M. MacRobert, my supervisor during my three years in Glasgow, under whose direction this work has been carried out, for the interest he has taken and the encouragement and advice he has given during the course of the work.

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CHAPTER I.

LINEAR RELATIONS BETWEEN E-FUNCTIONS

§1. First formula. The formula to be proved is

$$\sum_{r=0}^{n} \frac{(-)^r \, {}^nC_r \, (\alpha;r)(1+\tfrac{1}{2}\alpha;r)(\beta;r)}{(\tfrac{1}{2}\alpha;r)(1+\alpha-\beta;r)(1+\alpha+n;r)} x^{-r}$$

$$\times E(\alpha_1+r,\ldots,\alpha_p+r,1+\alpha+n+r,1+\alpha-\beta+r:\rho_1+r,\ldots,\rho_q+r,1+\alpha-\beta+n+r,1+\alpha+2r:x)$$

$$=(1+\alpha;n)\{(1+\alpha-\beta;n)\}^{-1}E(p;\alpha_r:q;\rho_s:x). \qquad (1)$$

The following formulae are required in the proof: [MacRobert, Phil. Mag., Ser. 7, XXXI., p. 260, 1941]

$$\alpha_1 E(p;\alpha_r:q;\rho_s:x)=E(\alpha_1+1,\alpha_2,\ldots,\alpha_p:q;\rho_s:x)+\frac{1}{x}E(p;\alpha_r+1:q;\rho_s+1:x), \quad (2)$$

$$(\rho_1-1)E(p;\alpha_r:q;\rho_s:x)=E(p;\alpha_r:\rho_1-1,\rho_2,\ldots,\rho_q:x)+\frac{1}{x}E(p;\alpha_r+1:q;\rho_s+1:x). \quad (3)$$

Now multiply (2) by $(\rho_1 - 1)$ and (3) by α_1, substract and then replace $\underline{\rho_1}$ by (ρ_1+1); this gives

$$E(p;\alpha_r:q;\rho_s:x) = \frac{\rho_1}{\alpha_1} E(\alpha_1+1, \alpha_2, \dots, \alpha_p : \rho_1+1, \rho_2, \dots, \rho_q : x)$$

$$\frac{\rho_1-\alpha_1}{\alpha_1 x} E(p;\alpha_r+1 : \rho_1+2, \rho_2+1, \dots, \rho_q+1 : x). \qquad (4)$$

It follows that if, in the following formula, the parameters $(1+\alpha-\beta)$ and $(1+\alpha)$ play the role that $\underline{\alpha_1}$ and $\underline{\rho_1}$ play in (4),

$$E(\alpha_1, \alpha_2, \dots, \alpha_p, 2+\alpha, 1+\alpha-\beta : \rho_1, \rho_2, \dots, \rho_q, 2+\alpha-\beta, 1+\alpha : x)$$

$$- \frac{\beta}{(1+\alpha-\beta)x} E(\alpha_1+1, \dots, \alpha_p+1, 2+\alpha-\beta : \rho_1+1, \dots, \rho_q+1, 3+\alpha-\beta : x)$$

$$= (1+\alpha)\{(1+\alpha-\beta)\}^{-1} E(p;\alpha_r : q; \rho_s : x). \qquad (5)$$

This is formula (1) with $n=1$. When $n=0$ the identity is obvious.

Now assume that (1) holds for a particular value of $\underline{n}$ and multiply by $(1+\alpha+n)/(1+\alpha-\beta+n)$: then the R.H.S. becomes the R.H.S. of (1) with $(n+1)$ in place of $\underline{n}$, while the L.H.S., on applying (4) with $(1+\alpha+n+r)$ and $(1+\alpha-\beta+n+r)$ in the roles of $\underline{\alpha_1}$ and $\underline{\rho_1}$, becomes

$$\sum_{r=0}^{n} \frac{(-1)^r\, {}^nC_r (\alpha;r)(1+\tfrac{1}{2}\alpha;r)(\beta;r)(1+\alpha-\beta+n+r)}{(1+\alpha-\beta+n)(\tfrac{1}{2}\alpha;r)(1+\alpha-\beta;r)(2+\alpha+n;r)} I_r$$
$$+\sum_{r=0}^{n} \frac{(-1)^r\, {}^nC_r (\alpha;r)(1+\tfrac{1}{2}\alpha;r)(\beta;r)(-\beta)}{(1+\alpha-\beta+n)(\tfrac{1}{2}\alpha;r)(1+\alpha-\beta;r)(2+\alpha+n;r)} J_r$$

where

$$I_r = x^{-r}E(2+\alpha+n+r, \alpha_1+r, \ldots, \alpha_p+r, 1+\alpha-\beta+r : 2+\alpha-\beta+n+r, \rho_1+r, \ldots, \rho_q+r, 1+\alpha+2r : x),$$

$$J_r = x^{-r-1}E(2+\alpha+n+r, \alpha_1+r+1, \ldots, \alpha_p+r+1, 2+\alpha-\beta+r : 3+\alpha-\beta+n+r, \rho_1+r+1, \ldots, \rho_q+r+1, 2+\alpha+2r : x).$$

If in the second of these series, $\underline{r}$ is replaced by $(r-1)$, the series becomes

$$\sum_{r=1}^{n+1} \frac{(-1)^{r-1}\,{}^{n}C_{r-1}\,(\alpha;r-1)\,(1+\tfrac{1}{2}\alpha;r-1)\,(\beta;r-1)\,(-\beta)}{(1+\alpha-\beta+n)\,(\tfrac{1}{2}\alpha;r-1)\,(1+\alpha-\beta;r-1)\,(2+\alpha+n;r-1)}\,x^{-r}$$

$$\times E\left(1+\alpha+n+r,\alpha_1+r,\dots,\alpha_p+r,1+\alpha-\beta+r:\alpha+2r,\rho_1+r,\dots,\rho_q+r,2+\alpha-\beta+n+r:x\right)$$

and on applying (4), this is found to be equal to

$$\sum_{r=1}^{n+1} \frac{(-1)^{r-1}\,{}^{n}C_{r-1}(\alpha;r-1)(1+\tfrac{1}{2}\alpha;r-1)(\beta;r-1)(-\beta)}{(1+\alpha-\beta+n)(\tfrac{1}{2}\alpha;r-1)(1+\alpha-\beta;r-1)(2+\alpha+n;r)}(\alpha+2r)\,I_r$$

$$+\sum_{r=1}^{n} \frac{(-1)^{r-1}\,{}^{n}C_{r-1}(\alpha;r-1)(1+\tfrac{1}{2}\alpha;r-1)(\beta;r-1)(-\beta)}{(1+\alpha-\beta+n)(\tfrac{1}{2}\alpha;r-1)(1+\alpha-\beta;r-1)(2+\alpha+n;r)}(r-n-1)\,J_r.$$

Note - The $(n+1)^{th}$ term in the last series vanishes because of the factor $(r-n-1)$.

Similarly, on replacing r by $(r-1)$ in the last series and then applying (4) the series is found to be equal to

$$\sum_{r=2}^{n+1} \frac{(-1)^{r-2}\, {}^{n}C_{r-2}\,(\alpha;r-2)(1+\tfrac{1}{2}\alpha;r-2)(\beta;r-2)(-\beta)(\alpha+2r)}{(1+\alpha-\beta+n)(\tfrac{1}{2}\alpha;r-2)(1+\alpha-\beta;r-2)(2+\alpha+n;r)}(r-n-2)I_r$$

$$+\sum_{r=2}^{n} \frac{(-1)^{r-2}\, {}^{n}C_{r-2}\,(\alpha;r-2)(1+\tfrac{1}{2}\alpha;r-2)(\beta;r-2)(-\beta)}{(1+\alpha-\beta+n)(\tfrac{1}{2}\alpha;r-2)(1+\alpha-\beta;r-2)(2+\alpha+n;r)}(r-n-1)(r-n-2)J_r.$$

Proceeding thus we find that the coefficient of I_r, where $r=1,2,\dots,(n+1)$, is

$$\frac{(-n;r)(\alpha;r)(1+\tfrac{1}{2}\alpha;r)(\beta;r)(1+\alpha-\beta+n+r)}{r!\,(1+\alpha-\beta+n)(\tfrac{1}{2}\alpha;r)(1+\alpha-\beta;r)(2+\alpha+n;r)}$$

$$+\frac{(-n;r-1)(\alpha+2r)(-\beta)}{(1+\alpha-\beta+n)\alpha(2+\alpha+n;r)}\sum_{m=1}^{r}\frac{(\alpha;r-m)(\beta;r-m)(\alpha+2r-2m)}{(r-m)!\,(1+\alpha-\beta;r-m)},$$

since
$$\frac{(1+\tfrac{1}{2}\alpha;r)}{(\tfrac{1}{2}\alpha;r)}=\frac{\alpha+2r}{\alpha}.$$

Now the last sum is equal to

$$\sum_{m=1}^{r} \frac{(\alpha; r-m)(\beta+1; r-1-m)}{(r-m)!\,(1+\alpha-\beta; r-m)}\,\beta(\alpha+2r-2m),$$

and $\beta(\alpha+2r-2m) = (\alpha+r-m)(\beta+r-m) - (r-m)(\alpha-\beta+r-m)$,

so that the sum is equal to

$$\sum_{m=1}^{r} \frac{(\alpha; r+1-m)(\beta+1; r-m)}{(r-m)!\,(1+\alpha-\beta; r-m)} - \sum_{m=1}^{r-1} \frac{(\alpha; r-m)(\beta+1; r-1-m)}{(r-m-1)!\,(1+\alpha-\beta; r-1-m)}$$

$$= \frac{(\alpha; r)(\beta+1; r-1)}{(r-1)!\,(1+\alpha-\beta; r-1)}.$$

Thus the coefficient of I_r is

$$\frac{(-1)^r\, {}^nC_r\,(\alpha; r)(1+\tfrac{1}{2}\alpha; r)(\beta; r)}{(\tfrac{1}{2}\alpha; r)(1+\alpha-\beta; r)(2+\alpha+n; r)}\left\{\frac{1+\alpha-\beta+n+r}{1+\alpha-\beta+n} + \frac{r(\alpha-\beta+r)}{(n-r+1)(1+\alpha-\beta+n)}\right\}.$$

But

$$(n-r+1)(1+\alpha-\beta+n+r) + r(\alpha-\beta+r) = (n+1)(1+\alpha-\beta+n).$$

Therefore the coefficient of I_r is

$$\frac{(-1)^r\, {}^{n+1}C_r\,(\alpha;r)\,(1+\tfrac{1}{2}\alpha;r)\,(\beta;r)}{(\tfrac{1}{2}\alpha;r)\,(1+\alpha-\beta;r)\,(2+\alpha+n;r)},$$

and from this it follows that if the formula holds for a particular value of n, it holds when n is replaced by (n+1). But it holds for $n=0$, $n=1$; hence it holds for all positive integral values of n.

§2. Alternative proof. Formula (1) may also be established by using the integral of Barnes's type

$$E(p;\alpha_r:q;\rho_s:z)=\frac{1}{2\pi i}\int\frac{\Gamma(\xi)\prod_{r=1}^{p}\Gamma(\alpha_r-\xi)}{\prod_{s=1}^{q}\Gamma(\rho_s-\xi)}z^{\xi}\,d\xi, \qquad (6)$$

where $p>q+1$, and Dougall's formula [Proc. Edin. Math. Soc., XXV., 1906, p. 10]

$$F\left(\begin{matrix} a, & 1+\frac{1}{2}a, & c, & d, & e & : 1 \\ & \frac{1}{2}a, & 1+a-c, & 1+a-d, & 1+a-e & \end{matrix}\right)$$

$$= \frac{\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-e)\Gamma(1+a-c-d-e)}{\Gamma(1+a)\Gamma(1+a-d-e)\Gamma(1+a-c-d)\Gamma(1+a-c-e)}, \tag{7}$$

where one of the parameters c, d, e is a negative integer and $R(a-c-d-e) > -1$.

For by (6), the L.H.S. of (1) is equal to if $p > q+1$,

$$\sum_{r=0}^{n} \frac{(-1)^r\, {}^nC_r (\alpha;r)(1+\frac{1}{2}\alpha;r)(\beta;r)}{(\frac{1}{2}\alpha;r)(1+\alpha-\beta;r)(1+\alpha+n;r)} x^{-r}$$

$$\times \frac{1}{2\pi i}\int \frac{\Gamma(\xi)\Gamma(1+\alpha+n+r-\xi)\Gamma(1+\alpha-\beta+r-\xi)\prod_{t=1}^{p}\Gamma(\alpha_t+r-\xi)}{\Gamma(1+\alpha-\beta+n+r-\xi)\Gamma(1+\alpha+2r-\xi)\prod_{s=1}^{q}\Gamma(\rho_s+r-\xi)} x^{\xi}\, d\xi.$$

Here replace $\underline{\xi}$ by $(\xi+r)$ and get

$$\frac{1}{2\pi i}\int \frac{\Gamma(\xi)\Gamma(1+\alpha+n-\xi)\Gamma(1+\alpha-\beta-\xi)\prod_{t=1}^{p}\Gamma(\alpha_t-\xi)}{\Gamma(1+\alpha-\beta+n-\xi)\Gamma(1+\alpha-\xi)\prod_{s=1}^{q}\Gamma(\rho_s-\xi)} x^{\xi} F\left(\begin{matrix} -n, \alpha, 1+\frac{1}{2}\alpha, \beta, \xi & : 1 \\ \frac{1}{2}\alpha, 1+\alpha-\beta, 1+\alpha+n, 1+\alpha-\xi & \end{matrix}\right) d\xi.$$

where $R(\alpha-\beta+n-\xi)>-1$.

Then by (7), the L.H.S. of (1) is equal to

$$\frac{(1+\alpha;n)}{(1+\alpha-\beta;n)}\frac{1}{2\pi i}\int\frac{\Gamma(\xi)\prod_{t=1}^{p}\Gamma(\alpha_t-\xi)}{\prod_{s=1}^{q}\Gamma(\rho_s-\xi)}x^{\xi}d\xi=\frac{(1+\alpha;n)}{(1+\alpha-\beta;n)}E(p;\alpha_r:q;\rho_s:x),$$

provided that $p>q+1$. The restriction $R(\alpha-\beta+n-\xi)>-1$ can be removed by analytical continuation. Also the condition $p>q+1$ may now be removed by writing $x\xi$ for ξ in (1), multiplying by $e^{\xi}\xi^{-\rho_{q+1}}$, and then integrating both sides, applying the formula

$$\frac{1}{2\pi i}\int_C e^{\xi}\xi^{-\rho_{q+1}}E(p;\alpha_r:q;\rho_s:\xi x)=E(p;\alpha_r:q+1;\rho_s:x), \tag{8}$$

[MacRobert, Phil. Mag., Ser. 7, $\overline{\text{XXXI}}$., p. 255]

where C is a contour starting at $-\infty$ on the real axis, passing positively round the origin and returning to $-\infty$ on the real axis, and amp $\xi=0$ when $\underline{\xi}$

is on the right of the origin

§3. Second formula. The formula to be established is

$$\sum_{r=0}^{n} \frac{{}^{n}C_r(\alpha;r)(1+\tfrac{1}{2}\alpha;r)}{(\tfrac{1}{2}\alpha;r)(1+\alpha+n;r)} x^{-r} E(\alpha_1+r,\dots,\alpha_p+r,1+\alpha+n+r:\rho_1+r,\dots,\rho_q+r,1+\alpha+2r:x)$$
$$= (1+\alpha;n)E(p;\alpha_r:q;\rho_s:x). \tag{9}$$

From (2) it follows that

$$(\alpha+1)E(\alpha+1,\alpha_1,\dots,\alpha_p:\alpha+1,\rho_1,\dots,\rho_q:x) = E(\alpha+2,\alpha_1,\dots,\alpha_p:\alpha+1,\rho_1,\dots,\rho_q:x)$$
$$+\frac{1}{x}E(\alpha+2,\alpha_1+1,\dots,\alpha_p+1:\alpha+2,\rho_1+1,\dots,\rho_q+1:x),$$

or

$$E(\alpha_1,\alpha_2,\dots,\alpha_p,\alpha+2:\rho_1,\dots,\rho_q,\alpha+1:x)+\frac{1}{x}E(p;\alpha_r+1:q;\rho_s+1:x)$$
$$=(\alpha+1)E(p;\alpha_r:q;\rho_s:x), \tag{10}$$

which is (9) with $n=1$. Formula (9) is obvious when $n=0$.

Now assume (9) for a particular value of $\underline{n}$ and multiply by $(1+\alpha+n)$: then the R.H.S. becomes the R.H.S of (9) with $(n+1)$ in place of $\underline{n}$, while the L.H.S. becomes

$$\sum_{r=0}^{n} \frac{{}^nC_r\,(\alpha;r)\,(1+\tfrac{1}{2}\alpha;r)}{(\tfrac{1}{2}\alpha;r)(2+\alpha+n;r-1)} x^{-r} E(\alpha_1+r,\ldots,\alpha_p+r,1+\alpha+n+r:\rho_1+r,\ldots,\rho_q+r,1+\alpha+2r:x).$$

Here apply formula (2) with $(1+\alpha+n+r)$ in the role of $\underline{\alpha_1}$, and the series becomes

$$\sum_{r=0}^{n} \frac{{}^nC_r\,(\alpha;r)\,(1+\tfrac{1}{2}\alpha;r)}{(\tfrac{1}{2}\alpha;r)(2+\alpha+n;r)} K_r + \sum_{r=0}^{n} \frac{{}^nC_r\,(\alpha;r)\,(1+\tfrac{1}{2}\alpha;r)}{(\tfrac{1}{2}\alpha;r)\,(2+\alpha+n;r)} L_r, \qquad (11)$$

where
$$K_r = x^{-r} E(\alpha_1+r,\ldots,\alpha_p+r,2+\alpha+n+r:\rho_1+r,\ldots,\rho_q+r,1+\alpha+2r:x)$$
and
$$L_r = x^{-r-1} E(\alpha_1+r+1,\ldots,\alpha_p+r+1,2+\alpha+n+r:\rho_1+r+1,\ldots,\rho_q+r+1,2+\alpha+2r:x)$$

In the second series in (11) replace $\underline{r}$ by $(r-1)$ and then apply formula (4) with $(1+\alpha+n+r)$ and $(\alpha+2r)$ in the roles of $\underline{\alpha_1}$ and $\underline{\rho_1}$ respectively; then it becomes

$$\sum_{r=1}^{n+1} \frac{{}^nC_{r-1}(\alpha;r-1)\,(1+\tfrac{1}{2}\alpha;r-1)}{(\tfrac{1}{2}\alpha;r-1)(2+\alpha+n;r)} (\alpha+2r) K_r + \sum_{r=1}^{n} \frac{{}^nC_{r-1}\,(\alpha;r-1)(1+\tfrac{1}{2}\alpha;r-1)}{(\tfrac{1}{2}\alpha;r-1)\,(2+\alpha+n;r)} (r-n-1) L_r$$

<u>Note</u>. The $(n+1)^{th}$ term in the second series vanishes because of the factor $(r-n-1)$.

Proceeding thus, we find that the co-factor of K_r is

$$\frac{{}^nC_r(\alpha;r)(1+\tfrac{1}{2}\alpha;r)}{(\tfrac{1}{2}\alpha;r)(2+\alpha+n;r)}+\sum_{m=1}^{r}\frac{{}^nC_{r-m}\,(\alpha;r-m)\,(1+\tfrac{1}{2}\alpha;\,r-m)}{(\tfrac{1}{2}\alpha;r-m)\,(2+\alpha+n;r)}(\alpha+2r)(r-n-2)\cdots(r-n-m).$$

Now the latter series can be written

$$\sum_{m=1}^{r}\frac{(-1)^{r-m}(-n;r-1)(\alpha;r-m)}{(r-m)!\,(2+\alpha+n;r)}(\alpha+2r)\,\frac{\alpha+2r-2m}{\alpha}$$

$$=\frac{(-1)^r(-n;r-1)(\alpha+2r)}{(2+\alpha+n;r)\,r!}\sum_{m=1}^{r}\frac{(-r;m)(\alpha;r-m)}{\alpha}\left\{(\alpha+r-m)+(r-m)\right\}$$

$$=\frac{(-1)^r(-n;r-1)\,(\alpha+2r)}{(2+\alpha+n;r)\,r!}\left\{\sum_{m=1}^{r}(-r;m)(\alpha+1;r-m)-\sum_{m=1}^{r-1}(-r;m+1)(\alpha+1;r-m-1)\right\}$$

$$=\frac{(-1)^{r-1}(-n;r-1)(1+\tfrac{1}{2}\alpha;r)\;(\alpha;r)}{(\tfrac{1}{2}\alpha;r)(2+\alpha+n;r)\;\;(r-1)!}$$

$$=\frac{{}^nC_{r-1}\,(\alpha;r)(1+\tfrac{1}{2}\alpha;r)}{(\tfrac{1}{2}\alpha;r)(2+\alpha+n;r)}.$$

Thus the coefficient of K_r is

$$\frac{{}^{n+1}C_r\,(\alpha;r)\,(1+\tfrac{1}{2}\alpha;r)}{(\tfrac{1}{2}\alpha;r)\,(2+\alpha+n;r)},$$

which is the coefficient of K_r in (9) with $(n+1)$ in place of $\underline{n}$. Thus the formula holds for $(n+1)$ if it holds for $\underline{n}$; but it holds for $n=0,1$; therefore it holds for all values of $\underline{n}$.

<u>Note</u>. Formula (9) can be verified by substituting from formula (6) in the L.H.S., replacing $\underline{\xi}$ by $(\xi+r)$ and then summing by means of Whipple's formula [<u>Proc. Lond. Math. Soc.</u>, Ser. 2, $\overline{\underline{XXIV}}$, 251]

$$F\left(\begin{matrix} a, & 1+\tfrac{1}{2}a, & c, & d, & -1 \\ & \tfrac{1}{2}a, & 1+a-c, & 1+a-d \end{matrix}\right) = \frac{\Gamma(1+a-c)\,\Gamma(1+a-d)}{\Gamma(1+a)\,\Gamma(1+a-c-d)}. \qquad (12)$$

The condition $p>q+1$ of (6) can be removed as in §2. by applying (8).

§4. Third Formula. The formula to be established is

$$\sum_{s=0}^{n} \frac{{}^{n}C_s(n+1;s)\,(4x)^{-s}}{\Gamma(\alpha-\frac{1}{2}n+\frac{1}{2}s-\frac{1}{2})\Gamma(\alpha-\frac{1}{2}n+\frac{1}{2}s)} E(\alpha-\tfrac{1}{2}n+\tfrac{1}{2}s-\tfrac{1}{2}, \alpha-\tfrac{1}{2}n+\tfrac{1}{2}s, \alpha_1+s, \ldots, \alpha_p+s : q ; \rho_t+s : x)$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(\alpha-n-\frac{1}{2})} E(\alpha-n-\tfrac{1}{2}, \alpha, \alpha_1, \ldots, \alpha_p : q ; \rho_t : x) \quad (13)$$

The identity [MacRobert, Phil., Mag., Ser. 7, XXXIX., 466]

$$(\alpha_1-1)E(\alpha_1-1, \alpha_2+1, \alpha_3, \ldots, \alpha_p : q ; \rho_t : x)$$

$$+(\alpha_1-\alpha_2-1)x^{-1}E(\alpha_1, \alpha_2+1, \alpha_3+1, \ldots, \alpha_p+1 : q ; \rho_t+1 : x) = \alpha_2 E(p ; \alpha_r : q ; \rho_s : x), \quad (14)$$

will be required; as will also be the formula

$${}^{n+1}C_s(n+2;s) - 4(n+\tfrac{1}{2})\,{}^{n}C_{s-1}(n+1;s-1) = {}^{n-1}C_s(n;s), \quad (15)$$

where $s = 1, 2, \ldots, n-1$.

Formula (13) obviously holds for $n=0$. The case $n=1$ can be verified by taking (14) in the form

$$(\alpha-1)E(\alpha-1, \alpha-\tfrac{1}{2}, \alpha_1, \ldots, \alpha_p : q ; \rho_t : x)$$

$$+\tfrac{1}{2}x^{-1}E(\alpha, \alpha-\tfrac{1}{2}, \alpha_1+1, \ldots, \alpha_p+1 : q ; \rho_t+1 : x) = (\alpha-\tfrac{3}{2})E(\alpha, \alpha-\tfrac{3}{2}, \alpha_1, \ldots, \alpha_p : q ; \rho_t : x).$$

For the general case assume that (13) holds for $(n-1)$ and $\underline{n}$; it will be shown that in consequence it holds for $(n+1)$; thus, by induction, since it holds for $n=0$ and $n=1$, it holds for all values of $\underline{n}$.

In (13), with $(n-1)$ in place of $\underline{n}$, replace $\underline{\alpha}$ by $(\alpha-1)$: then

$$\sum_{s=0}^{n-1} {}^{n-1}C_s(n;s)F_s = \frac{1}{\Gamma(\alpha-1)\Gamma(\alpha-n-\tfrac{1}{2})}E(\alpha-n-\tfrac{1}{2}, \alpha-1, \alpha_1, \ldots, \alpha_p : q ; \rho_t : x) \qquad (16)$$

where

$$F_s = \frac{(4x)^{-s}}{\Gamma(\alpha-\tfrac{1}{2}n+\tfrac{1}{2}s-1)\Gamma(\alpha-\tfrac{1}{2}n+\tfrac{1}{2}s-\tfrac{1}{2})}E(\alpha-\tfrac{1}{2}n+\tfrac{1}{2}s-1, \alpha-\tfrac{1}{2}n+\tfrac{1}{2}s-\tfrac{1}{2}, \alpha_1+s, \ldots, \alpha_p+s : q ; \rho_t+s : x)$$

Now the L.H.S. of (13) with $(n+1)$ in place of $\underline{n}$ is equal to

$$\sum_{s=0}^{n} {}^{n+1}C_s(n+2;s)F_s + (n+2;n+1)F_{n+1} \qquad (w)$$

In (13) replace α_r by (α_r+1), $r=1,2,\dots,p$ and ρ_t by (ρ_t+1), $t=1,2,\dots,q$: then

$$\sum_{s=0}^{n-1} \frac{{}^nC_s\,(n+1;s)\,(4x)^{-s}}{\Gamma(\alpha-\frac{1}{2}n+\frac{1}{2}s-\frac{1}{2})\,\Gamma(\alpha-\frac{1}{2}n+\frac{1}{2}s)}\,E\left(\alpha-\tfrac{1}{2}n+\tfrac{1}{2}s-\tfrac{1}{2},\alpha-\tfrac{1}{2}n+\tfrac{1}{2}s,\alpha_1+s+1,\dots,\alpha_p+s+1:q;\rho_t+s+1:x\right)$$

$$+\frac{(n+1;n)\,(4x)^{-n}}{\Gamma(\alpha-\frac{1}{2})\,\Gamma(\alpha)}\,E\left(\alpha-\tfrac{1}{2},\alpha,\alpha_1+n+1,\dots,\alpha_p+n+1:q;\rho_t+n+1:x\right)$$

$$=\frac{1}{\Gamma(\alpha)\,\Gamma(\alpha-n-\frac{1}{2})}\,E\left(\alpha-n-\tfrac{1}{2},\alpha,\alpha_1+1,\dots,\alpha_p+1:q;\rho_t+1:x\right). \qquad (14)$$

Now substitute from the last on the left of (14) in (w), and the latter formula becomes

$$\sum_{s=0}^{n} {}^{n+1}C_s\,(n+2;s)\,F_s$$

$$-\frac{n+\frac{1}{2}}{x}\sum_{s=0}^{n-1}\frac{{}^{n}C_s\,(n+1;s)\,(4x)^{-s}}{\Gamma(\alpha-\frac{1}{2}n+\frac{1}{2}s-\frac{1}{2})\Gamma(\alpha-\frac{1}{2}n+\frac{1}{2}s)}E\left(\alpha-\tfrac{1}{2}n+\tfrac{1}{2}s-\tfrac{1}{2},\alpha-\tfrac{1}{2}n+\tfrac{1}{2}s,\alpha_1+s+1,\ldots,\alpha_p+s+1:q;\rho_t+s+1:x\right)$$

$$+\frac{n+\frac{1}{2}}{x}\;\frac{1}{\Gamma(\alpha)\Gamma(\alpha-n-\frac{1}{2})}E\left(\alpha-n-\tfrac{1}{2},\alpha,\alpha_1+1,\ldots,\alpha_p+1:q;\rho_t+1:x\right)$$

$$=\sum_{s=1}^{n}\left\{{}^{n+1}C_s\,(n+2;s)-4(n+\tfrac{1}{2})\,{}^{n}C_{s-1}(n+1;s-1)\right\}F_s+F_0+A$$

$$=\sum_{s=0}^{n-1}{}^{n-1}C_s\,(n;s)\,F_s+A,$$

where A is the last term on the left.

It should be noted that

$$^{n+1}C_n(n+2;n)-4(n+\tfrac{1}{2})\,{}^{n}C_{n-1}(n+1;n-1)=0$$

so that the $(n+1)^{th}$ term in the series vanishes.

But, by (16), this is equal to

$$\frac{1}{\Gamma(\alpha-1)\Gamma(\alpha-n-\frac{1}{2})}E(\alpha-n-\tfrac{1}{2},\alpha-1,\alpha_1,\ldots,\alpha_p:q;\rho_t:x)$$

$$+\frac{n+\frac{1}{2}}{\Gamma(\alpha)\Gamma(\alpha-n-\frac{1}{2})x}E(\alpha-n-\tfrac{1}{2},\alpha,\alpha_1+1,\ldots,\alpha_p+1:q;\rho_t+1:x)$$

$$=\frac{1}{\Gamma(\alpha)\Gamma(\alpha-n-\frac{3}{2})}E(\alpha,\alpha-n-\tfrac{3}{2},\alpha_1,\ldots,\alpha_p:q;\rho_t:x),$$

by (14).

Now this is the R.H.S. of (13) with $(n+1)$ in place of $\underline{n}$. Hence the formula holds for all values of $\underline{n}$.

<u>Note</u>. Formula (13) can be verified by substituting from (6) in the L.H.S., replacing $\underline{\xi}$ by $(\xi+s)$ and then summing by means of the formula

[Whipple, Proc. Lond. Math. Soc. Ser. 2. Vol. XXIII., p. 114],

if

$$a+b=1$$
$$e+f=2c+1 \quad , \text{ then}$$

$$F\begin{bmatrix} a, & b, & c \\ & e & f \end{bmatrix} = \frac{\pi\Gamma(e)\Gamma(f)}{2^{2c-1}\Gamma(\frac{1}{2}a+\frac{1}{2}e)\Gamma(\frac{1}{2}a+\frac{1}{2}f)\Gamma(\frac{1}{2}b+\frac{1}{2}e)\Gamma(\frac{1}{2}b+\frac{1}{2}f)} . \quad (18)$$

The condition $p>q+1$ of (16) can now be removed as in § 2. by applying (8).

§5. Two expansions Theorems: The two theorems to be proved are

$$\lambda^{-\alpha_t} E(p;\alpha_r:q;\rho_s:\lambda z) = \sum_{m=0}^{\infty} \frac{(1-\lambda)^m}{m!} E(\alpha_1,\alpha_2,\ldots,\alpha_t+m,\ldots,\alpha_p:q;\rho_s:z), \tag{19}$$

where $|\lambda-1|<1$, $t=1,2,\ldots,p$;

$$\lambda^{1-\rho_t} E(p;\alpha_r:q;\rho_s:\lambda z) = \sum_{m=0}^{\infty} \frac{1}{m!}(1/\lambda-1)^m E(p;\alpha_r:\rho_1,\rho_2,\ldots,\rho_t-m,\ldots,\rho_q:z), \tag{20}$$

where $R(\lambda)>\frac{1}{2}$, $t=1,2,\ldots,q$.

In the course of the proof, use will be made of the formula

$$\frac{d}{dz}E(p;\alpha_r:q;\rho_s:z) = 1/z^2\, E(p;\alpha_r+1:q;\rho_s+1:z), \tag{21}$$

[MacRobert, Phil. Mag., Ser. 7, $\overline{\text{XXXI}}$., p. 260]

Now differentiating the function $z^{-\alpha_t}E(p;\alpha_r:q;\rho_s:z)$ $(t=1,2,\dots,p.)$ w.r.t. z, using (20) and (2), and noting that the E-function is symmetrical in the $\underline{\alpha's}$ and $\underline{\rho's}$, we have

$$\frac{d}{dz}\left\{z^{-\alpha_t}E(p;\alpha_r:q;\rho_s:z)\right\}=-z^{-\alpha_t-1}E(\alpha_1,\alpha_2,\dots,\alpha_t+1,\dots,\alpha_p:q;\rho_s:z);$$

and on differentiating m times we have

$$\frac{d^m}{dz^m}\left\{z^{-\alpha_t}E(p;\alpha_r:q;\rho_s:z)\right\}=(-)^m z^{-\alpha_t-m}E(\alpha_1,\dots,\alpha_t+m,\dots,\alpha_p:q;\rho_s:z), \quad (21)$$

where $t=1,2,\dots,p$.

Similarly, differentiating the function $z^{\rho_t-1}E(p;\alpha_r:q;\rho_s:1/z)$ $(t=1,2,\dots,q)$ w.r.t. z and using (20) and (3), we have

$$\frac{d}{dz}\left\{z^{\rho_t-1}E(p;\alpha_r:q;\rho_s:1/z)\right\}=-z^{\rho_t-2}E(p;\alpha_r:\rho_1,\dots,\rho_t-1,\dots,\rho_q:z);$$

and on differentiating m times, we have

$$\frac{d^m}{dz^m}\left[z^{\rho_t-1}E(p;\alpha_r:q;\rho_s:1/z)\right]=(-)^m z^{\rho_t-m-1}E(p;\alpha_r:\rho_1,\ldots,\rho_t-m,\ldots,\rho_q:z), \qquad (22)$$

where $t=1,2,\ldots,q$.

To obtain (19), note that the function $z^{-\alpha_t}E(p;\alpha_r:q;\rho_s:z)$ is for all values of z except $z=0$, is an analytic function of z: so we can use if $|\xi|<|z|$, Taylors series for the function $(z+\xi)^{-\alpha_t}E(p;\alpha_r:q;\rho_s:z+\xi)$ $[t=1,2,\ldots,p]$. Thus we get

$$(z+\xi)^{-\alpha_t}E(p;\alpha_r:q;\rho_s:z+\xi)=\sum_{m=0}^{\infty}\frac{\xi^m}{m!}\frac{d^m}{dz^m}\left\{z^{-\alpha_t}E(p;\alpha_r:q;\rho_s:z)\right\}.$$

Using (21) and substituting $\lambda z-z$ for ξ (19) is obtained.

Also $z^{\rho_t - 1} E(p; \alpha_r : q; \rho_s : 1/z)$ is an analytic function of z in a closed region not including the origin. Thus Taylor's series can be applied, giving

$$(z+\xi)^{\rho_t - 1} E\left(p; \alpha_r : q; \rho_s : \frac{1}{z+\xi}\right) = \sum_{m=0}^{\infty} \frac{\xi^m}{m!} \frac{d^m}{dz^m} \left\{ z^{\rho_t - 1} E(p; \alpha_r : q; \rho_s : 1/z) \right\}$$

in the region $|\xi| < |z|$, $t = 1, 2, \dots, q$.

Using (22) and writing $1/z$ for z, $(1-\lambda)/\lambda z$ for ξ, (20) is obtained. The condition $|\xi| < |z|$ becomes $|\lambda - 1| < |\lambda|$, or $R(\lambda) > \frac{1}{2}$.

<u>Note</u>: (19) and (20) can also be obtained by using (6) if $p > q+1$ and changing the order of integration and summation. After that the restriction $p > q+1$ can be removed by applying (8).

CHAPTER II.

SOME BESSEL FUNCTIONS FORMULAE

§1. An Alternative proof of an integral due to Hardy: The integral

$$\int_0^\infty K_n(x) K_n\left(\frac{b}{x}\right) dx = \pi K_{2n}(2\sqrt{b}), \tag{1}$$

where $R(b) > 0$ was proved by Hardy [Mess. of Math., LVI, (1927), p. 190] by an application of Mellin's inversion formula. An alternative proof is given here and some related formulae are deduced.

To prove it, denote the integral on the left of (1) by $F(b)$; then

$$F'(b) = \int_0^\infty K_n(x) K_n'\left(\frac{b}{x}\right) \frac{1}{x} dx,$$

and

$$F''(b) = \int_0^\infty K_n(x) K_n''\left(\frac{b}{x}\right) \frac{1}{x^2} dx,$$

so that

$$b^2F''(b) = \int_0^\infty K_n(x)\left\{\left(\frac{b^2}{x^2}+n^2\right)K_n\left(\frac{b}{x}\right) - \frac{b}{x}K_n'\left(\frac{b}{x}\right)\right\}dx$$

$$= n^2F(b) - bF'(b) + b^2\int_0^\infty K_n(x)K_n\left(\frac{b}{x}\right)\frac{dx}{x^2}.$$

But on replacing x by b/x in $F(b)$, it is seen that

$$F(b) = b\int_0^\infty K_n(x)K_n\left(\frac{b}{x}\right)\frac{1}{x^2}\,dx.$$

Hence

$$b^2F''(b) + bF'(b) - (b+n^2)F(b) = 0 \tag{2}$$

Now in the equation

$$x^2y'' + xy' - (x^2+4n^2)y = 0, \tag{3}$$

with solutions $K_{2n}(x)$ and $I_{2n}(x)$, put $x = 2\sqrt{b}$ and it reduces to (2). Therefore

$$\int_0^\infty K_n(x)K_n\left(\frac{b}{x}\right)dx = AK_{2n}(2\sqrt{b}) + BI_{2n}(2\sqrt{b}).$$

Here let $b \to \infty$, and it is seen that B must be zero. Thus

$$\int_0^\infty K_n(x)\left\{I_{-n}\left(\frac{b}{x}\right) - I_n\left(\frac{b}{x}\right)\right\}dx = \frac{A}{2\cos n\pi}\left\{I_{-2n}(2\sqrt{b}) - I_{2n}(2\sqrt{b})\right\}.$$

Now assume that $R(n) > 0$, multiply by b^n and let $b \to 0$; then

$$\frac{2^n}{\Gamma(1-n)}\int_0^\infty K_n(x)\, x^n\, dx = \frac{A}{2\cos n\pi\ \Gamma(1-2n)}$$

But, if $R(m \pm n) > 0$,

$$\int_0^\infty K_n(x)\, x^{m-1}\, dx = 2^{m-2}\,\Gamma\left(\frac{m+n}{2}\right)\Gamma\left(\frac{m-n}{2}\right), \qquad (4)$$

[Bessel functions, by Gray, Mathews and MacRobert, p. 66].

Therefore

$$\frac{2^{2n-1}}{\Gamma(1-n)}\Gamma\left(n+\tfrac{1}{2}\right)\Gamma\left(\tfrac{1}{2}\right) = \frac{A}{2\cos n\pi\,\Gamma(1-2n)},$$

and from this it follows that $A = \pi$.

From (1) other formulae of the same type can be derived as follows.

In (1) let amp b decreases by $\frac{1}{2}\pi$, finally writing b/i in place of b; then, since

$$K_n(t) = i^n \mathcal{G}_n(it), \tag{5}$$

the formula becomes

$$\int_0^\infty K_n(x)\, \mathcal{G}_n\left(\frac{b}{x}\right) dx = \pi i^{-n} K_{2n}\left(2\sqrt{b}\cdot e^{-i\pi/4}\right), \tag{6}$$

provided that $-\frac{5}{2} < R(n) < \frac{5}{2}$.

Similarly, on replacing b by ib, it is seen that

$$\int_0^\infty K_n(x)\, \mathcal{G}_n\left(\frac{b}{x} e^{i\pi}\right) dx = \pi i^{-n} K_{2n}\left(2\sqrt{b}\cdot e^{i\pi/4}\right), \tag{7}$$

where $-\frac{5}{2} < R(n) < \frac{5}{2}$.

Hence, using the formula

$$\pi i\, J_n(t) = \mathcal{G}_n(t) - i^{2n}\mathcal{G}_n(te^{i\pi}) \tag{8}$$

it follows that

$$i\int_0^\infty K_n(x)\, J_n\left(\frac{b}{x}\right) dx = i^{-n} K_{2n}(2\sqrt{b}\cdot e^{-i\pi/4}) - i^n K_{2n}(2\sqrt{b}\cdot e^{i\pi/4}), \tag{9}$$

where $-\frac{5}{2} < R(n) < \frac{5}{2}$. This formula also is given in Hardy's paper.

Again let amp x and amp b increase simultaneously by $\frac{1}{2}\pi$, so that $\underline{x}$ becomes $\underline{ix}$ and $\underline{b}$ becomes $\underline{ib}$; then

$$\int_0^\infty \mathcal{G}_n(xe^{i\pi})\, J_n\left(\frac{b}{x}\right) dx = K_{2n}(2\sqrt{b}\, e^{i\pi/2}) - i^{-2n} K_{2n}(2\sqrt{b}), \tag{10}$$

where $-\frac{1}{2} < R(n) < \frac{5}{2}$.

Similarly if amp x and amp b decrease simultaneously by $\frac{1}{2}\pi$ (9) becomes

$$\int_0^\infty \mathcal{G}_n(x)\, J_n\left(\frac{b}{x}\right) dx = i^{-2n} K_{2n}(2\sqrt{b}\cdot e^{-i\pi/2}) - K_{2n}(2\sqrt{b}), \tag{11}$$

where $\quad -\frac{1}{2} < R(n) < \frac{5}{2}.$

Finally from (8), (10) and (11) it follows that

$$\pi i \int_0^\infty J_n(x)\, J_n\left(\frac{b}{x}\right) dx = i^{-2n} K_{2n}(2\sqrt{b}\cdot e^{-i\pi/2}) - i^{2n} K_{2n}(2\sqrt{b}\cdot e^{i\pi/2})$$

$$= \mathcal{G}_{2n}(2\sqrt{b}) - i^{4n}\mathcal{G}_{2n}(2\sqrt{b}\cdot e^{i\pi})$$

$$= \pi i\, J_{2n}(2\sqrt{b}),$$

so that

$$\int_0^\infty J_n(x)\, J_n\left(\frac{b}{x}\right) dx = J_{2n}(2\sqrt{b}), \tag{12}$$

where $R(n) > -\frac{1}{2}.$

This formula was given by Bateman [Proc. Camb. Phil. Soc., XXI., (1908) p. 186].

Another proof of (1). In this proof, the following formulae are required:

$$\int_0^\infty \exp\{-x - y^2/(8x)\} \frac{K_n(x)}{x^{3/2}}\,dx = \frac{4\sqrt{(2\pi)}\;K_{2n}(y)}{y}, \tag{13}$$

[CH. III. equation (34)],

$$\frac{1}{2}\int_0^\infty \exp\{-\tfrac{1}{2}\xi - (x^2+y^2)/2\xi\} K_n(xy/\xi)\, 1/\xi \,d\xi = K_n(x) K_n(y), \tag{14}$$

where $R(x+y)^2 > 0$

[MacRobert, C.V., p. 383 ex 123],

and

$$\int_0^\infty e^{-(\alpha + k^2/\alpha)} \alpha^{-\frac{1}{2}}\, d\alpha = \sqrt{\pi}\, e^{-2k}, \tag{15}$$

where $R(k^2) > 0$ [Gibson, An Elementary Treatize on the Calculus second edition, p. 440, ex. 6].

To prove (1), substitute from (14) in the L.H.S. of (1); then it becomes

$$\frac{1}{2}\int_0^\infty dx \int_0^\infty \exp\left\{-\tfrac{1}{2}\xi - (x^2 + b^2/x^2)/(2\xi)\right\} K_n(b/\xi)\ 1/\xi\ d\xi.$$

Here change the order of integration and write $\sqrt{(2\alpha\xi)}$ for x, then the L.H.S. becomes

$$\frac{1}{2\sqrt{2}}\int_0^\infty e^{-\xi/2} K_n(b/\xi)\ 1/\sqrt{\xi}\ d\xi \int_0^\infty e^{-\left(\alpha + \frac{b^2/4\xi^2}{\alpha}\right)} \frac{1}{\sqrt{\alpha}}\, d\alpha$$

$$= \tfrac{1}{2}\sqrt{\pi/2}\int_0^\infty \exp\left\{-\tfrac{1}{2}\xi - b/\xi\right\} K_n(b/\xi)\ 1/\sqrt{\xi}\ d\xi \qquad \text{by (15)}.$$

In this last integral write t for b/ξ and (1) is easily obtained by means of (13).

§2. Generalization of Hardy's Integral. It is proposed here to establish a generalization of (1); namely

$$\prod_{s=1}^{p-1}\int_0^\infty K_n(t_s)\, t_s^{\,2s/p-1}\, dt_s\, K_n\left(\frac{b}{t_1 t_2 \cdots t_{p-1}}\right) = \pi^{p-1} K_{pn}\left(p b^{1/p}\right), \qquad (16)$$

where $b>0$, $p=2,3,4,\ldots$.

The proof is based on the differential equation

$$x^2y'' + xy' - (x^2+n^2)\,y = 0 \qquad (17)$$

satisfied by $K_n(x)$ and $I_n(x)$.

The following formula will be required in the proof:

$$\Gamma(z)\Gamma\left(z+\tfrac{1}{m}\right)\cdots\Gamma\left(z+\tfrac{m-1}{m}\right) = (2\pi)^{\frac{1}{2}m-\frac{1}{2}}\, m^{\frac{1}{2}-mz}\,\Gamma(mz), \qquad (18)$$

where m is a positive integer [MacRobert, C.V., p. 154].

<u>Proof of the formula</u>. If the L.H.S. of (16) is denoted by $F(b)$, then

$$F'(b) = \prod_{s=1}^{p-1} \int_0^\infty K_n(t_s)\, t_s^{2s/p-1}\, dt_s\, K_n'\left(\frac{b}{t_1 t_2 \cdots t_{p-1}}\right) \frac{1}{t_1 t_2 \cdots t_{p-1}},$$

and

$$F''(b) = \prod_{s=1}^{p-1} \int_0^\infty K_n(t_s)\, t_s^{2s/p-1}\, dt_s\, K_n''\left(\frac{b}{t_1 t_2 \cdots t_{p-1}}\right) \frac{1}{(t_1 t_2 \cdots t_{p-1})^2}.$$

Then, from (17)

$$b^2 F''(b) = \prod_{s=1}^{p-1} \int_0^\infty K_n(t_s)\, t_s^{2s/p-1}\, dt_s$$

$$\left[-\frac{b}{t_1 t_2 \cdots t_{p-1}} K_n'\left(\frac{b}{t_1 t_2 \cdots t_{p-1}}\right) + \left\{ \left(\frac{b}{t_1 t_2 \cdots t_{p-1}}\right)^2 + n^2 \right\} K_n\left(\frac{b}{t_1 t_2 \cdots t_{p-1}}\right) \right]$$

$$= -bF'(b) + n^2 F(b) + L,$$

where

$$L \equiv b^2 \prod_{s=1}^{p-1} \int_0^\infty K_n(t_s)\, t_s^{2s/p-3}\, dt_s\, K_n\left(\frac{b}{t_1 t_2 \cdots t_{p-1}}\right).$$

In this multiple integral change the order of integration so that the first integral becomes the last and replace t_1 by $b/(\lambda t_2 t_3 \cdots t_{p-1})$, where $\underline{\lambda}$ is the new variable; then

$$L = b^2 \prod_{s=2}^{p-1} \int_0^\infty K_n(t_s)\, t_s^{2s/p-3}\, dt_s \int_0^\infty K_n(\lambda) \left(\frac{b}{\lambda t_2 \cdots t_{p-1}}\right)^{2/p-3} K_n\left(\frac{b}{\lambda t_2 \cdots t_{p-1}}\right) \frac{b\, d\lambda}{\lambda^2 t_2 \cdots t_{p-1}}$$

$$= b^{2/p} \prod_{s=2}^{p-1} \int_0^\infty K_n(t_s)\, t_s^{2(s-1)/p-1}\, dt_s \int_0^\infty K_n(\lambda)\, \lambda^{2(p-1)/p-1} K_n\left(\frac{b}{\lambda t_2 t_3 \cdots t_{p-1}}\right) d\lambda.$$

Here write $\underline{t_{s-1}}$ for $\underline{t_s}$ and t_{p-1} for $\underline{\lambda}$; then

$$L = b^{2/p} F(b);$$

so that

$$b^2F''(b)+bF'(b)-(b^{2/p}+n^2)F(b)=0.$$

Now, in (14) put $b=(x/p)^p$, and it becomes

$$b^2\frac{d^2y}{db^2}+b\frac{dy}{db}-(b^{2/p}+n^2/p^2)\,y=0$$

and therefore

$$F(b)=A\,K_{pn}(pb^{1/p})+B\,I_{pn}(pb^{1/p}).$$

Here let $b\longrightarrow\infty$ and it is seen that B must be zero. (For the purpose of the proof, it may be assumed for the time being that $n\geqq 0$).

In order to determine A the equation may be put in the form

$$\prod_{s=1}^{p-1}\int_0^\infty K_n(t_s)\, t_s^{\,2s/p-1}\, dt_s \;\frac{\pi}{2\sin n\pi}\left\{I_{-n}\left(\frac{b}{t_1\cdots t_{p-1}}\right)-I_n\left(\frac{b}{t_1\cdots t_{p-1}}\right)\right\}$$

$$= A\,\frac{\pi}{2\sin(pn\pi)}\left\{I_{-pn}(pb^{1/p})-I_{pn}(pb^{1/p})\right\}.$$

Now multiply by b^n and let $b\to 0$; then

$$\prod_{s=1}^{p-1}\int_0^\infty K_n(t_s)\, t_s^{\,2s/p-1}\, dt_s \;\frac{2^n\pi}{2\sin n\pi}\;\frac{(t_1t_2\cdots t_{p-1})^n}{\Gamma(1-n)} = A\,\frac{\pi\,(2/p)^{pn}}{2\sin(pn\pi)}\;\frac{1}{\Gamma(1-pn)},$$

or from (4)

$$\prod_{s=1}^{p-1} 2^{\,n+2s/p-2}\,\Gamma(n+s/p)\,\Gamma(s/p)\,2^{n-1}\Gamma(n) = A\,2^{pn-1}p^{-pn}\,\Gamma(pn).$$

Hence from (18) and from (18) with $1/m$ in place of z,

$$2^{(p-1)n-(p-1)+n-1}(2\pi)^{\frac{1}{2}p-\frac{1}{2}}p^{\frac{1}{2}-pn}\,\Gamma(pn)\,(2\pi)^{\frac{1}{2}p-\frac{1}{2}}p^{-\frac{1}{2}} = A\,2^{pn-1}p^{-pn}\,\Gamma(pn).$$

Therefore $A = \pi^{p-1}$

Thus formula (16) has been obtained

§3. A multiple Integral involving Bessel Function of the First Kind. The formula to be proved is

$$\prod_{s=1}^{p-1} \int_0^\infty J_n(t_s)\, t_s^{2s/p-1}\, dt_s\, J_n\left(\frac{b}{t_1 t_2 \cdots t_{p-1}}\right) = J_{pn}(p b^{1/p}), \qquad (19)$$

where $b>0$, $R(n) > \frac{1}{2} - \frac{2}{p}$. For the particular case, when $p=2$,

$$\int_0^\infty J_n(t) J_n\left(\frac{b}{t}\right) dt = J_{2n}(2\sqrt{b}),$$

where $b>0$, $R(n) > -\frac{1}{2}$, see Watson's Bessel Functions, p. 437.

The proof depends on the differential equation

$$x^2 y'' + x y' + (x^2 - n^2) y = 0 \qquad (20)$$

satisfied by $J_n(x)$ and $J_{-n}(x)$.

The formula

$$\int_0^\infty J_n(\lambda)\,\lambda^{m-1}\,d\lambda = 2^{m-1}\Gamma\left(\frac{m+n}{2}\right)\Big/\Gamma\left(1+\frac{n-m}{2}\right), \tag{21}$$

where $R(m+n)>0$, $R(m)<\frac{3}{2}$, [MacRobert C.V., p. 372], is required

Denoting the L.H.S. of (19) by $\phi(b)$, we have as in §2., if $R(n)>\frac{3}{2}-\frac{2}{p}$,

$$b^2\phi''(b) = -b\phi'(b) + n^2\phi(b) - L,$$

where

$$L = b^2\prod_{s=1}^{p-1}\int_0^\infty J_n(t_s)\,t_s^{2s/p-3}\,dt_s\; J_n\left(\frac{b}{t_1t_2\cdots t_{p-1}}\right).$$

On proceeding as in §2., it is found that

$$L = b^{2/p}\phi(b),$$

so that

$$b^2\phi''(b) + b\phi'(b) + (b^{2/p} - n^2)\phi(b) = 0.$$

Now in (20) put $b = (x/p)^p$ and it becomes

$$b^2\frac{d^2y}{db^2} + b\frac{dy}{db} + (b^{2/p} - n^2/p^2)\,y = 0$$

Therefore

$$\phi(b) = A J_{pn}(pb^{1/p}) + B J_{-pn}(p\,b^{1/p}).$$

Here multiply by $\underline{b^n}$ and let $\underline{b} \to 0$, then clearly B must be zero.

Again to determine A multiply by $\underline{b^{-n}}$ and let $b \to 0$; then

$$\prod_{s=1}^{p-1}\int_0^\infty J_n(t_s)\,t_s^{\,2s/p-n-1}\,dt_s\,\frac{1}{2^n\Gamma(n+1)} = A\,\frac{p^{pn}}{2^{pn}\Gamma(pn+1)}.$$

But from (21), the L.H.S. is equal to

$$\prod_{s=1}^{p-1} 2^{2s/p-n-1}\,\Gamma(s/p)\,\Gamma\left(n+\frac{p-s}{p}\right)\frac{1}{2^n\,\Gamma(n+1)} = \frac{A.\,2^{-n(p-1)}\,(2\pi)^{\frac{1}{2}p-\frac{1}{2}}\,p^{-\frac{1}{2}}}{2^n\,n\,(2\pi)^{\frac{1}{2}p-\frac{1}{2}}\,p^{\frac{1}{2}-pn}\,\Gamma(pn)}.$$

Hence $A=1$, so that (19) has been proved. By applying analytical continuation the restriction $R(n)>\frac{3}{2}-\frac{2}{p}$ can be altered to $R(n)>\frac{1}{2}-\frac{2}{p}$.

CHAPTER III.

INTEGRALS INVOLVING E-FUNCTIONS

§1. An Integral involving the product of a Bessel function and an E-function. The formula

$$4\int_0^\infty \lambda^{m-1} K_n(2\lambda) E(p;\alpha_r:q;\rho_s:x\lambda^{-2})\,d\lambda = E(p+2;\alpha_r:q;\rho_s:x), \qquad (1)$$

where $\alpha_{p+1} = \frac{1}{2}(m+n)$, $\alpha_{p+2} = \frac{1}{2}(m-n)$, $R(m\pm n) > 0$ and $\underline{x}$ is real and positive, was given by MacRobert [Phil., Mag., Ser. 7, XXXI. p. 258]. From it formula (4) below will be deduced.

In (1) let it be assumed that $R(m\pm n)>0$, $R(\frac{3}{2}-m+2\alpha_r)>0$, $r=1,2,\ldots,p$, and let amp λ decrease by $\frac{1}{2}\pi$ and amp x decreasing simultaneously by π, finally writing λ/i in place of $\underline{\lambda}$ and $xe^{-i\pi}$ in place of $\underline{x}$, then applying (5) CH. II.

we have

$$4i^{n-m}\int_0^\infty \lambda^{m-1} G_n(2\lambda)E(p;\alpha_r:q;\rho_s:x\lambda^{-2})\,d\lambda = E(p+2;\alpha_r:q;\rho_s:xe^{-i\pi}). \quad (2)$$

Similarly, on increasing amp λ by $\frac{1}{2}\pi$ and amp x by π, we have

$$4i^{n+m}\int_0^\infty \lambda^{m-1} G_n(2\lambda e^{i\pi})E(p;\alpha_r:q;\rho_s:x\lambda^{-2})\,d\lambda = E(p+2;\alpha_r:q;\rho_s:xe^{i\pi}). \quad (3)$$

Hence applying (8) CH. II. it is found that

$$4i\pi\int_0^\infty \lambda^{m-1} J_n(2\lambda)E(p;\alpha_r:q;\rho_s:x\lambda^{-2})\,d\lambda$$

$$= i^{m-n}E(p+2;\alpha_r:q;\rho_s:xe^{-i\pi}) - i^{-\,m+n}E(p+2;\alpha_r:q;\rho_s:xe^{i\pi}), \quad (4)$$

where $R(m+n)>0$, $R(\frac{3}{2}-m+2\alpha_r)>0$ $r=1,2,\dots,p$ and $\underline{x}$ is real and positive.

In particular, if $p \geqq q-1$, formula (4) can be written

$$2\pi\int_0^\infty \lambda^{m-1} J_n(2\lambda) E(p;\alpha_r:q;\rho_s:x\lambda^{-2})\,d\lambda$$

$$= \sum_{r=1}^{p+1} \frac{\prod_{s=1}^{p+2}{}' \Gamma(\alpha_s-\alpha_r)}{\prod_{t=1}^{q} \Gamma(\rho_t-\alpha_r)} \Gamma(\alpha_r) \sin(\tfrac{1}{2}m-\tfrac{1}{2}n-\alpha_r)\pi\, x^{\alpha_r}$$

$$\times F\left(\alpha_r, \alpha_r-\rho_1+1, \ldots, \alpha_r-\rho_q+1; \alpha_r-\alpha_1+1, \ldots * \ldots, \alpha_r-\alpha_{p+2}+1; (-1)^{p-q+1}x\right), \quad (5)$$

where $R(m+n)>0$, $R(\frac{3}{2}-m+2\alpha_r)>0$, $r=1,2,\ldots,p$, and $\underline{x}$ is real and positive. It should be noted that the $(p+2)^{th}$ term on the right of (5) does not appear because $\alpha_{p+2}=\frac{1}{2}(m-n)$.

If $m=\beta+1$, $n=\beta-1$ and $\rho_q=\beta$ then

$$\alpha_{p+1}=\beta, \quad \alpha_{p+2}=1$$

and $\qquad \alpha_r-\alpha_{p+1}+1 = \alpha_r-\beta+1,$

which cancels $\alpha_r-\rho_q+1$ on the right of (5).

Also $\alpha_r - \alpha_{p+2} + 1 = \alpha_r$
which cancels α_r on the right of (5).

Again $\rho_q - \alpha_{p+1} = 0$ so that $\dfrac{1}{\Gamma(\rho_q - \alpha_{p+1})} = 0$,
and therefore the last term on the right of (5) disappears.

Finally, noting that
$$\Gamma(\alpha_{p+2} - \alpha_r)\Gamma(\alpha_r) = \frac{\pi}{\sin(\alpha_r \pi)},$$
that
$$\sin(\tfrac{1}{2}m - \tfrac{1}{2}n - \alpha_r)\pi = \sin \alpha_r \pi,$$
and that
$$\frac{\Gamma(\alpha_{p+1} - \alpha_r)}{\Gamma(\rho_q - \alpha_r)} = 1,$$
we have if $p \geq q-1$
$$2\int_0^\infty \lambda^{\beta} J_{\beta-1}(2\lambda) E(p; \alpha_r : q; \rho_s : x\lambda^{-2})\, d\lambda$$
$$= \sum_{r=1}^{p} \frac{\prod_{s=1}^{p}{}' \Gamma(\alpha_s - \alpha_r)}{\prod_{t=1}^{q-1} \Gamma(\rho_t - \alpha_r)} x^{\alpha_r} F\left(\begin{matrix}\alpha_r - \rho_1 + 1, \ldots, \alpha_r - \rho_{q-1} + 1 \\ \alpha_r - \alpha_1 + 1, \ldots * \ldots, \alpha_r - \alpha_p + 1\end{matrix}; (-1)^{p-q+1} x\right), \quad (6)$$

where $p_q = \beta$, $R(\beta) > 0$, $R(\frac{1}{2} - \beta + 2\alpha_r) > 0$ $r = 1, 2, \ldots, p$.
It should be noted that (β) does not appear on the right of (6).

This result was given for the case $p = q+1$, by Meijer [Proc. Akad. te Amsterdam, XXXIX., 1936, p. 394].

§2. A generalization of some integrals involving Bessel functions and E-functions.

It is here proposed to establish a generalization of formula (1). Some related formulae will also be deduced. The formula to be proved is

$$2^{2^r+r+1}\,\pi^{2^r-1}\int_0^\infty \lambda^{2^r m-1} K_{2^r n}(2^{r+1}\lambda)\,E(p;\alpha_s:q;\rho_t:x\lambda^{-2^{r+1}})\,d\lambda$$

$$= E(p+2^{r+1};\alpha_s:q;\rho_t:x), \tag{7}$$

where $r = 0, 1, 2, \ldots,$ and

$$\left.\begin{aligned}\alpha_{p+2k+1} &= \tfrac{1}{2}m + \tfrac{1}{2}n + \frac{k}{2^r}\\ \alpha_{p+2k+2} &= \tfrac{1}{2}m - \tfrac{1}{2}n + \frac{k}{2^r}\end{aligned}\right\} k = 0, 1, 2, \ldots, 2^r - 1, \qquad (\gamma')$$

$R(m \pm n) > 0$ and $\underline{x}$ is real and positive

It can be proved by induction; for, assuming that it is valid, it follows that

$$E(p+2^{r+2}; \alpha_s : q; \rho_t : x)$$

$$= 2^{2^r + r + 1}\, \pi^{2^r - 1} \int_0^\infty \lambda^{2^r l - 1} K_{2^r n}(2^{r+1}\lambda) E(p+2^{r+1}; \alpha_s : q; \rho_t : x\lambda^{-2^{r+1}})\, d\lambda$$

where

$$\left.\begin{aligned}\alpha_{p+2^{r+1}+2k+1} &= \tfrac{1}{2}l + \tfrac{1}{2}n + \frac{k}{2^r}\\ \alpha_{p+2^{r+1}+2k+2} &= \tfrac{1}{2}l - \tfrac{1}{2}n + \frac{k}{2^r}\end{aligned}\right\} k = 0, 1, 2, \ldots, 2^r - 1,$$

$$=2^{2^{r+1}+2r+2}\pi^{2^{r+1}-2}\int_0^\infty \lambda^{2^r l-1}K_{2^r n}(2^{r+1}\lambda)\,d\lambda$$

$$\times\int_0^\infty \mu^{2^r m-1}K_{2^r n}(2^{r+1}\mu)E\left\{p;\alpha_s:q;\rho_t:x(\lambda\mu)^{-2^{r+1}}\right\}d\mu.$$

Here replace $\underline{\mu}$ by μ/λ and get

$$2^{2^{r+1}+2r+2}\pi^{2^{r+1}-2}\int_0^\infty \lambda^{2^r(l-m)-1}K_{2^r n}(2^{r+1}\lambda)\,d\lambda$$

$$\times\int_0^\infty \mu^{2^r m-1}K_{2^r n}\left(2^{r+1}\frac{\mu}{\lambda}\right)E(p;\alpha_s:q;\rho_t:x\mu^{-2^{r+1}})\,d\mu.$$

Next put $l=m+2^{-r}$ and change the order of integration, so getting

$$2^{2^{r+1}+2r+2}\pi^{2^{r+1}-2}\int_0^\infty \mu^{2^r m-1}E(p;\alpha_s:q;\rho_t:x\mu^{-2^{r+1}})\,d\mu\int_0^\infty K_{2^r n}(2^{r+1}\lambda)K_{2^r n}\left(2^{r+1}\frac{\mu}{\lambda}\right)d\lambda,$$

where

$$\left.\begin{aligned}\alpha_{p+2^{r+1}+2k+1} &= \tfrac{1}{2}m+\tfrac{1}{2}n+\frac{2k+1}{2^{r+1}}\\ \alpha_{p+2^{r+1}+2k+2} &= \tfrac{1}{2}m-\tfrac{1}{2}n+\frac{2k+1}{2^{r+1}}\end{aligned}\right\}\begin{aligned}&k=0,1,2,\dots,2^r-1,\\ &\text{or}\\ &2k+1=1,3,5,\dots,2^{r+1}-1.\end{aligned}$$

But from (γ'),

$$\left.\begin{aligned}\alpha_{p+2k+1} &= \tfrac{1}{2}m+\tfrac{1}{2}n+\frac{2k}{2^{r+1}}\\ \alpha_{p+2k+2} &= \tfrac{1}{2}m-\tfrac{1}{2}n+\frac{2k}{2^{r+1}}\end{aligned}\right\}\begin{aligned}&k=0,1,2,\dots,2^r-1,\\ &\text{or}\\ &2k=0,2,4,\dots,2^{r+1}-2.\end{aligned}$$

Therefore

$$\left.\begin{aligned}\alpha_{p+2k+1} &= \tfrac{1}{2}m+\tfrac{1}{2}n+\frac{k}{2^{r+1}}\\ \alpha_{p+2k+2} &= \tfrac{1}{2}m-\tfrac{1}{2}n+\frac{k}{2^{r+1}}\end{aligned}\right\}k=0,1,2,\dots,2^{r+1}-1.$$

Now from (1) CH. II., the last integral is equal to

$$\frac{1}{2^{r+1}}\pi K_{2^{r+1}n}(2^{r+2}\sqrt{u}).$$

Hence on replacing u by λ^2 we have

$$E(p+2^{r+2};\alpha_s:q;\rho_t:x) = 2^{2^{r+1}+r+2}\,\pi^{2^{r+1}-1}$$

$$\int_0^\infty \lambda^{2^{r+1}m-1} K_{2^{r+1}n}(2^{r+2}\lambda)\,E(p;\alpha_s:q;\rho_t:x\lambda^{-2^{r+2}})\,d\lambda,$$

which is (7) with $(r+1)$ in place of r

But the formula holds when $r=0$: hence it holds for all positive integral values of $\underline{r}$.

If in (7) amp λ is decreased by $\frac{1}{2}\pi$ and amp x by $2^r\pi$, it becomes by (5) CH. II.

$$2^{2^r+r+1}\,\pi^{2^r-1}\,i^{2^r(n-m)}\int_0^\infty \lambda^{2^r m-1} \mathcal{G}_{2^r n}(2^{r+1}\lambda)\,E(p;\alpha_s:q;\rho_t:x\lambda^{-2^{r+1}})\,d\lambda$$

$$= E(p+2^{r+1};\alpha_s:q;\rho_t:xe^{-i\pi 2^r}), \qquad (8)$$

where $R(m \pm n) > 0$, $R(\frac{3}{2} - 2^r m + 2^{r+1}\alpha_s) > 0$, $s = 1, 2, \dots, p$ and $\underline{x}$ is real and positive.

Similarly and subject to the same conditions,

$$2^{2^r+r+1}\pi^{2^r-1} i^{2^r(n+m)} \int_0^\infty \lambda^{2^r m-1} \mathcal{Y}_{2^r n}(2^{r+1}\lambda e^{i\pi}) E(p; \alpha_s : q; \rho_t : x\lambda^{-2^{r+1}}) d\lambda$$

$$= E(p+2^{r+1}; \alpha_s : q; \rho_t : x e^{i\pi 2^r}), \tag{9}$$

and from (8) CH. II.

$$2^{2^r+r+1}\pi^{2^r} i \int_0^\infty \lambda^{2^r m-1} J_{2^r n}(2^{r+1}\lambda) E(p; \alpha_s : q; \rho_t : x\lambda^{-2^{r+1}}) d\lambda$$

$$= i^{2^r(m-n)} E(p+2^{r+1}; \alpha_s : q; \rho_t : x e^{-i\pi 2^r})$$

$$- i^{-2^r(m-n)} E(p+2^{r+1}; \alpha_s : q; \rho_t : x e^{i\pi 2^r}). \tag{10}$$

§3. An E-function formula. It is here proposed to establish the formula

$$\int_0^\infty e^{-n\lambda}\lambda^{nk-1}E\left(p;\alpha_r;q;\rho_s:\frac{x}{\lambda^n}\right)d\lambda = \frac{1}{(2\pi)^{\frac{1}{2}n-\frac{1}{2}}\sqrt{n}}E(p+n;\alpha_r;q;\rho_s:x), \tag{11}$$

where $R(k)>0$, n is a positive integer, and

$$\alpha_{p+\nu+1} = k+\frac{\nu}{n}, \quad \nu=0,1,2,\ldots,n-1.$$

This formula will be derived by means of the following subsidiary formula

$$\int_0^\infty\int_0^\infty\cdots\int_0^\infty e^{-F}G\,dx_1\,dx_2\ldots dx_{n-1} = (2\pi)^{\frac{1}{2}n-\frac{1}{2}}n^{-\frac{1}{2}}e^{-nb^{1/n}}, \tag{12}$$

where $b>0$ and

$$F = x_1+x_2+\cdots+x_{n-1}+b/(x_1x_2\cdots x_{n-1}),$$

$$G = x_1^{1/n-1}x_2^{2/n-1}\cdots x_{n-1}^{(n-1)/n-1}.$$

To prove (12) let the L.H.S. of it be denoted by $F(b)$: then:

$$F'(b) = \int_0^\infty \int_0^\infty \cdots \int_0^\infty e^{-t} g\left(\frac{-1}{x_1 x_2 \cdots x_{n-1}}\right) dx_1\, dx_2 \ldots dx_{n-1}$$

Here change the order of integration so that the first integral becomes the last and put

$$x_1 = \frac{b}{\mu x_2 x_3 \cdots x_{n-1}}$$

in the innermost integral: then

$$F'(b) = -\int_0^\infty \int_0^\infty \cdots \int_0^\infty e^{-t_1} g_1\, \mu\, dx_2 \ldots dx_{n-1} \frac{d\mu}{\mu^2 x_2 \cdots x_{n-1}},$$

where

$$t_1 = x_2 + x_3 + \cdots + x_{n-1} + \mu + b/(\mu x_2 \cdots x_{n-1})$$

and

$$g_1 = \left(\frac{b}{\mu x_2 \cdots x_{n-1}}\right)^{1/n-1} x_2^{2/n-1} \cdots x_{n-1}^{(n-1)/n-1}$$

$$= b^{1/n-1} x_2^{1/n} x_3^{2/n} \cdots x_{n-1}^{(n-2)/n} \mu^{-1/n+1}.$$

Hence, on replacing $x_2, x_3, \ldots, x_{n-1}, \mu$ by $x_1, x_2, \ldots, x_{n-1}$ respectively we have

$$F'(b) = -b^{1/n-1} F(b).$$

Thus $$F(b) = A e^{-n b^{1/n}}.$$

To determine A let $b \to 0$: then

$$\Gamma(\tfrac{1}{n})\Gamma(\tfrac{2}{n}) \cdots \Gamma(\tfrac{n-1}{n}) = A, \quad \text{so that}$$

$$A = (2\pi)^{\frac{1}{2}n - \frac{1}{2}} n^{-\frac{1}{2}},$$

from which (12) follows.

<u>Proof of (11)</u>: If the formula

$$\int_0^\infty e^{-\lambda} \lambda^{k-1} E(p; \alpha_r : q; \varrho_s : \tfrac{x}{\lambda})\, d\lambda = E(p+1; \alpha_r : q; \varrho_s ; x), \quad (13)$$

where $\alpha_{p+1}=k$, $R(k)>0$ [MacRobert, Phil. Mag., Ser. 7, XXXI., p. 255], is applied repeatedly to itself, it becomes

$$\int_0^\infty\int_0^\infty\cdots\int_0^\infty e^{-(x_0+x_1+\cdots+x_{n-1})}x_0^{k-1}x_1^{k+1/n-1}\cdots x_{n-1}^{k+(n-1)/n-1}E\left(p;\alpha_r:q;\rho_s:\frac{x}{x_0\cdots x_{n-1}}\right)dx_0\cdots dx_{n-1}$$

$$=E(p+n;\alpha_r:q;\rho_s:x),$$

where the α's are those given in (11). Now change the order of integration so that the first integral becomes the last and put

$$x_0=\frac{\sigma}{x_1x_2\cdots x_{n-1}}.$$

Then the L.H.S. becomes

$$\int_0^\infty\int_0^\infty\cdots\int_0^\infty e^{-(x_1+x_2+\cdots+x_{n-1})}x_1^{1/n-1}x_2^{2/n-1}\cdots x_{n-1}^{(n-1)/n-1}dx_1\cdots dx_{n-1}$$

$$\times\int_0^\infty e^{-\sigma/(x_1x_2\cdots x_{n-1})}\sigma^{k-1}E\left(p;\alpha_r:q;\rho_s:\frac{x}{\sigma}\right)d\sigma.$$

Here change the order of integration so that the last integral becomes the first, apply (12) and get

$$(2\pi)^{\frac{1}{2}n-\frac{1}{2}}\, n^{-\frac{1}{2}} \int_0^\infty e^{-n\sigma^{1/n}}\, \sigma^{k-1} E(p;\alpha_r;q;\rho_s : \tfrac{x}{\sigma})\, d\sigma .$$

Finally, put $\sigma = \lambda^n$ and so obtain (11).

§4. An Integral involving a Product of two E-functions. The formula to be established is

$$\int_0^\infty e^{-t}\, t^{k-1} E(\gamma, \delta :: t) E(p; \alpha_r : q; \rho_s : x/t)\, dt$$
$$= \Gamma(\gamma)\Gamma(\delta) E(p+2; \alpha_r : q+1; \rho_s : x), \qquad (14)$$

where $\alpha_{p+1} = \gamma + k$, $\alpha_{p+2} = \delta + k$, $\rho_{q+1} = \gamma + \delta + k$ and $R(\gamma + k) > 0$, $R(\delta + k) > 0$.

The following formulae are required in the proof:

$$\int_0^\infty \mu^{\rho_{q+1} - \alpha_{p+1} - 1} (1+\mu)^{-\rho_{q+1}} E\{p; \alpha_r : q; \rho_s : (1+\mu)x\}\, d\mu$$
$$= \Gamma(\rho_{q+1} - \alpha_{p+1}) E(p+1; \alpha_r : q+1; \rho_s : x), \qquad (15)$$

where $R(\alpha_{p+1}) > 0$, $R(\rho_{q+1} - \alpha_{p+1}) > 0$,

[MacRobert, Phil. Mag., Ser. 7, XXXI., p. 256, (8)];

$$E(\alpha, \beta :: x) = \Gamma(\alpha) \int_0^\infty e^{-\lambda} \lambda^{\beta-1} (1+\lambda/x)^{-\alpha} d\lambda, \tag{16}$$

where $R(\beta) > 0$ [MacRobert C.V., p. 348, (10)].

It is assumed that $\underline{x}$ is real and positive.

<u>Proof of the formula</u>: on applying (15) to (13) it is seen that

$$\int_0^\infty \mu^{\delta-1}(1+\mu)^{-\gamma-\delta-k} d\mu \int_0^\infty e^{-\lambda} \lambda^{\delta+k-1} E\{p; \alpha_r : q; \rho_s : (1+\mu)x/\lambda\} d\lambda$$

$$= \Gamma(\delta) E(p+2; \alpha_r : q+1; \rho_s : x),$$

where $R(\gamma+k) > 0$, $R(\delta+k) > 0$, $R(\delta) > 0$.

Now put $\lambda = (1+\mu)t$ and change the order of integration: then the double integral becomes

$$\int_0^\infty e^{-t} t^{\delta+k-1} E(p; \alpha_r : q; \rho_s : x/t)\, dt \int_0^\infty e^{-\mu t} \mu^{\delta-1} (1+\mu)^{-\gamma} d\mu.$$

In the inner integral put $\mu = \xi/t$ and it becomes

$$t^{-\delta}\int_0^\infty e^{-\xi}\xi^{\delta-1}(1+\xi/t)^{-\gamma}d\xi = \{1/\Gamma(\gamma)\}t^{-\delta}E(\gamma,\delta::t), \text{ by } (16).$$

From this (14) follows.

§5. Integral involving a product of three E-functions. The formula to be proved is

$$\int_0^\infty e^{-t}t^{k-1}E(\alpha,\beta::t)E(\gamma,\delta::t)E(p;\alpha_r:q;\rho_s:x/t)\,dt$$
$$=\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(\delta)\sum_{m=0}^{\infty}\frac{(\alpha;m)(\delta;m)}{m!}$$
$$\times E(\alpha_1,\dots,\alpha_p,\alpha+\gamma+k,\beta+\delta+k,\alpha+\delta+k+m:\rho_1,\dots,\rho_q,\alpha+\beta+\delta+k+m,\alpha+\gamma+\delta+k+m:x), \quad (17)$$

where $p \geq q+1$ and the constants are such that the integral converges.

The following formula is required in the proof:

$$\int_0^\infty e^{-t} t^{k-1} E(\alpha, \beta :: t) E(\gamma, \delta :: t)\, dt$$

$$= \Gamma(\beta)\Gamma(\gamma)\Gamma(\alpha+\gamma+k)\Gamma(\beta+\delta+k) \sum_{m=0}^{\infty} \frac{\Gamma(\alpha+m)\Gamma(\delta+m)\Gamma(\alpha+\delta+k+m)}{m!\,\Gamma(\alpha+\beta+\delta+k+m)\Gamma(\alpha+\gamma+\delta+k+m)}, \quad (18)$$

provided that $R(\alpha+\gamma+k)>0$, $R(\beta+\gamma+k)>0$, $R(\alpha+\delta+k)>0$, $R(\beta+\delta+k)>0$ [MacRobert, Quart. Journ. of Math. Oxford, XIII., 1942, 68].

Now substitute from (6) CH. I. in the L.H.S. of (17), and change the order of integration; then if $p \geq q+1$,

$$\frac{1}{2\pi i}\int \frac{\Gamma(\xi)\prod_{r=1}^{p}\Gamma(\alpha_r-\xi)}{\prod_{s=1}^{q}\Gamma(\rho_s-\xi)} x^\xi d\xi \int_0^\infty e^{-t} t^{k-\xi-1} E(\alpha,\beta::t) E(\gamma,\delta::t)\, dt$$

$$= \frac{1}{2\pi i}\int \frac{\Gamma(\xi)\prod_{r=1}^{p}\Gamma(\alpha_r-\xi)}{\prod_{s=1}^{q}\Gamma(\rho_s-\xi)} x^\xi d\xi\, \Gamma(\beta)\Gamma(\gamma)\Gamma(\alpha+\gamma+k-\xi)\Gamma(\beta+\delta+k-\xi)$$

$$\sum_{m=0}^{\infty} \frac{\Gamma(\alpha+m)\Gamma(\delta+m)\Gamma(\alpha+\delta+k+m-\xi)}{m!\,\Gamma(\alpha+\beta+\delta+k+m-\xi)\Gamma(\alpha+\gamma+\delta+k+m-\xi)},$$

by (18).

On changing the order of integration and summation and applying (6) CH. I., formula (17) is obtained.

<u>Alternative proof.</u> In this proof the following formulae are required

$$\sum_{\gamma,\delta} \Gamma(\delta-\gamma)\Gamma(\gamma)t^{\gamma}F(\gamma;\gamma-\delta+1;t) = E(\gamma,\delta::t), \tag{19}$$

[E.V. p. 348, (11)];

$$\frac{\Gamma(a)\Gamma(b)\Gamma(f-c)\Gamma(e-a-b)}{\Gamma(f)}\,{}_3F_2\left[\begin{matrix}a, & b, & f-c\\ a+b-e+1, & f\end{matrix}\right]$$

$$+\frac{\Gamma(e-a)\Gamma(e-b)\Gamma(a+b-e)\Gamma(e+f-a-b-c)}{\Gamma(e+f-a-b)}\,{}_3F_2\left[\begin{matrix}e-a, e-b, e+f-a-b-c\\ e-a+b+1, e+f-a-b\end{matrix}\right]$$

$$=\frac{\Gamma(a)\Gamma(b)\Gamma(e-a)\Gamma(e-b)\Gamma(f-c)}{\Gamma(e)\Gamma(f)}\,{}_3F_2\left[\begin{matrix}a, & b, & c\\ & e, & f\end{matrix}\right], \tag{20}$$

where $R(a+b+c-e-f)<0$, $R(c-e+1)>0$,

[Hardy, <u>Proc. Camb. Phil. Soc.</u>, p. 498; (5.2)].

To prove (17), substitute from (19) in the L.H.S. of (17) then if $p \geq q+1$ it becomes

$$\int_0^\infty e^{-t} t^{k-1} E(\alpha, \beta :: t) E(p; \alpha_r : q; \rho_s : x/t) \sum_{\gamma, \delta} \Gamma(\delta-\gamma)\Gamma(\gamma) t^{\gamma} F(\gamma; \gamma-\delta+1; t)\, dt$$

$$= \int_0^\infty e^{-t} t^{k-1} E(\alpha, \beta :: t) E(p; \alpha_r : q; \rho_s : x/t) \sum_{\gamma, \delta} \Gamma(\delta-\gamma)\Gamma(\gamma) t^{\gamma} \sum_{m=0}^{\infty} \frac{(\gamma; m)\, t^m}{m!\,(\gamma-\delta+1; m)}\, dt$$

$$= \sum_{\gamma, \delta} \Gamma(\delta-\gamma)\Gamma(\gamma) \sum_{m=0}^{\infty} \frac{(\gamma; m)}{m!\ (\gamma-\delta+1; m)}$$

$$\times \int_0^\infty e^{-t}\, t^{k+\gamma+m-1} E(\alpha, \beta :: t) E(p; \alpha_r : q; \rho_s : x/t)\, dt$$

by expanding the hypergeometric function and changing the order of integration and summation.

Now apply (14) to evaluate the integral in the last expansion; then it becomes

$$\sum_{\gamma,\delta}\Gamma(\delta-\gamma)\Gamma(\gamma)\sum_{m=0}^{\infty}\frac{(\gamma;m)}{m!\,(\gamma-\delta+1;m)}\Gamma(\alpha)\Gamma(\beta)$$

$$\times E(\alpha_1,\dots,\alpha_p,\alpha+k+\gamma+m,\beta+k+\gamma+m:\rho_1,\dots,\rho_q,\alpha+\beta+\gamma+k+m:x)$$

Next, substitute from (6) CH. I. for the E-function, and the L.H.S. of (17) becomes, if $p \geq q+1$,

$$\sum_{\gamma,\delta}\Gamma(\delta-\gamma)\Gamma(\gamma)\sum_{m=0}^{\infty}\frac{(\gamma;m)}{m!\,(\gamma-\delta+1;m)}\Gamma(\alpha)\Gamma(\beta)$$

$$\times\frac{1}{2\pi i}\int\frac{\Gamma(\xi)\prod_{r=1}^{p}\Gamma(\alpha_r-\xi)\,\Gamma(\alpha+\gamma+k+m-\xi)\,\Gamma(\beta+\gamma+k+m-\xi)}{\prod_{s=1}^{q}\Gamma(\rho_s-\xi)\,\Gamma(\alpha+\beta+\gamma+k+m-\xi)}x^{\xi}d\xi$$

which can be written in the form

$$\sum_{\gamma,\delta}\Gamma(\delta-\gamma)\Gamma(\gamma)\Gamma(\alpha)\Gamma(\beta)\,\frac{1}{2\pi i}\int\frac{\Gamma(\xi)\prod_{r=1}^{p}\Gamma(\alpha_r-\xi)\Gamma(\alpha+\gamma+k-\xi)\Gamma(\beta+\gamma+k-\xi)}{\prod_{s=1}^{q}\Gamma(\rho_s-\xi)\Gamma(\alpha+\beta+\gamma+k-\xi)}$$

$${}_3F_2\left[\begin{matrix}\gamma,\ \alpha+\gamma+k-\xi,\ \beta+\gamma+k-\xi\\ \gamma-\delta+1,\ \alpha+\beta+\gamma+k-\xi\end{matrix}\right]x^{\xi}d\xi$$

$$= \Gamma(\alpha)\Gamma(\beta)\frac{1}{2\pi i}\int \frac{\Gamma(\xi)\prod_{r=1}^{p}\Gamma(\alpha_r-\xi)}{\prod_{s=1}^{q}\Gamma(\rho_s-\xi)} x^{\xi}$$

$$\times\left\{\frac{\Gamma(\gamma)\Gamma(\delta-\gamma)\Gamma(\alpha+\gamma+k-\xi)\Gamma(\beta+\gamma+k-\xi)}{\Gamma(\alpha+\beta+\gamma+k-\xi)}\,{}_3F_2\left[\begin{matrix}\gamma, \alpha+\gamma+k-\xi, \beta+\gamma+k-\xi\\ \gamma-\delta+1, \alpha+\beta+\gamma+k-\xi\end{matrix}\right]\right.$$

$$\left.+\frac{\Gamma(\delta)\Gamma(\gamma-\delta)\Gamma(\alpha+\delta+k-\xi)\Gamma(\beta+\delta+k-\xi)}{\Gamma(\alpha+\beta+\delta+k-\xi)}\,{}_3F_2\left[\begin{matrix}\delta, \alpha+\delta+k-\xi, \beta+\delta+k-\xi\\ \delta-\gamma+1, \alpha+\beta+\delta+k-\xi\end{matrix}\right]\right\}d\xi.$$

But the L.H.S. of formula (20) with the following substitutions

$a=\delta$, $b=\alpha+\delta+k-\xi$, $c=\alpha$

$e=\alpha+\gamma+\delta+k-\xi$, $f=\alpha+\beta+\delta+k-\xi$,

gives the quantity between the brackets { }.

Thus the L.H.S. of formula (17) becomes, if

$p \geq q+1$, $R(-k+\xi-\beta-\gamma)<0$, $R(1-\gamma-\delta-k+\xi)>0$

$$\Gamma(\alpha)\Gamma(\beta)\times\frac{1}{2\pi i}\int\frac{\Gamma(\xi)\prod_{r=1}^{p}\Gamma(\alpha_r-\xi)}{\prod_{s=1}^{q}\Gamma(\rho_s-\xi)}x^{\xi}$$

$$\times\frac{\Gamma(\gamma)\Gamma(\delta)\Gamma(\alpha+\delta+k-\xi)\Gamma(\alpha+\gamma+k-\xi)\Gamma(\beta+\delta+k-\xi)}{\Gamma(\alpha+\gamma+\delta+k-\xi)\Gamma(\alpha+\beta+\delta+k-\xi)}$$

$${}_3F_2\left[\begin{matrix}\delta, & \alpha+\delta+k-\xi, & \alpha \\ & \alpha+\gamma+\delta+k-\xi, & \alpha+\beta+\delta+k-\xi\end{matrix}\right]d\xi.$$

$$=\Gamma(\alpha)\Gamma(\beta)\frac{1}{2\pi i}\int\frac{\Gamma(\xi)\prod_{r=1}^{p}\Gamma(\alpha_r-\xi)}{\prod_{s=1}^{q}\Gamma(\rho_s-\xi)}x^{\xi}$$

$$\times\frac{\Gamma(\gamma)\Gamma(\delta)\Gamma(\alpha+\delta+k-\xi)\Gamma(\alpha+\gamma+k-\xi)\Gamma(\beta+\delta+k-\xi)}{\Gamma(\alpha+\gamma+\delta+k-\xi)\Gamma(\alpha+\beta+\delta+k-\xi)}$$

$$\times\sum_{m=0}^{\infty}\frac{(\delta;m)(\alpha+\delta+k-\xi;m)(\alpha;m)}{m!\,(\alpha+\gamma+\delta+k-\xi;m)(\alpha+\beta+\delta+k-\xi;m)}.$$

Again change the order of integration and summation, apply (6) CH-I., then formula (17) is obtained. The restrictions $R(-k+\xi-\beta-\gamma)<0$ and $R(1-\gamma-\delta-k+\xi)>0$ can be removed by analytical continuation.

§6. Further Generalization of an integral involving a product of a Bessel function and an E-function. The formula to be established is

$$2^m m \pi^{m-1} \int_0^\infty K_{mn}(2m\lambda)\, \lambda^{mk-1} E\left(p ; \alpha_r : q ; \rho_s : \frac{x}{\lambda^{2m}}\right) d\lambda$$

$$= E(p+2m ; \alpha_r : q ; \rho_s : x), \qquad (21)$$

where

$$\left.\begin{aligned} \alpha_{p+2\nu+1} &= \tfrac{1}{2}m + \tfrac{1}{2}n + \frac{\nu}{m} \\ \alpha_{p+2\nu+2} &= \tfrac{1}{2}m - \tfrac{1}{2}n + \frac{\nu}{m} \end{aligned}\right\} \nu = 0, 1, 2, \ldots, m-1, \qquad (21')$$

and the constants are such that the integral converges.

This formula was proved by induction in §2. for the case $m = 2^l$, $l = 1, 2, 3, \ldots$

To prove it, apply formula (1) repeatedly to itself taking in place of $\underline{m}$ there the values $k + 2r/m$, $r = 1, 2, \dots, m-1$: then

$$4^m \int_0^\infty K_n(2\lambda)\,\lambda^{k-1}\,d\lambda \prod_{r=1}^{m-1} \int_0^\infty K_n(2t_r)\, t_r^{k+2r/m-1}\, dt_r\, E\left(p; \alpha_r : q; \rho_s : \frac{x}{\lambda^2 t_1^2 \cdots t_{m-1}^2}\right)$$

$$= E(p+2m; \alpha_r : q; \rho_s : x)$$

where the $\underline{\alpha\text{'s}}$ are given by (21')

Here, on the left, put the first integral last and then put $\lambda = \mu/(t_1 t_2 \cdots t_{m-1})$; the expression becomes

$$4^m \prod_{r=1}^{m-1} \int_0^\infty K_n(2t_r)\, t_r^{2r/m-1}\, dt_r \int_0^\infty K_n\left(\frac{2\mu}{t_1 t_2 \cdots t_{m-1}}\right) \mu^{k-1} E\left(p; \alpha_r : q; \rho_s : \frac{x}{\mu^2}\right) d\mu$$

Now change the order of integration so that the last integral becomes the first integral and apply (16) CH. II., this gives

$$2^{m+1}\pi^{m-1}\int_0^\infty K_{mn}(2m\,\mu^{1/m})\,\mu^{k-1}E\left(p;\alpha_r:q;\rho_s:\frac{x}{\mu^2}\right)d\mu,$$

on putting $\mu=\lambda^n$, formula (21) is obtained.

<u>Corollary</u>. On replacing $\underline{\lambda}$ by λ/i and $\underline{x}$ by $xe^{-im\pi}$ in (21) and making use of the formula (5) CH. II., it is found that

$$2^{m+1}m\,\pi^{m-1}\int_0^\infty G_{mn}(2m\lambda)\,\lambda^{mk-1}E\left(p;\alpha_r:q;\rho_s:\frac{x}{\lambda^{2m}}\right)d\lambda$$

$$=i^{m(k-n)}E\left(p+2m;\alpha_r:q,\rho_s:xe^{-im\pi}\right), \qquad (22)$$

where $(k\pm n)>0$, $m(2\alpha_r-k)>-\frac{3}{2}$, $r=1,2,\ldots,p$.

Similarly, on replacing $\underline{\lambda}$ by $\underline{\lambda i}$ and $\underline{x}$ by $xe^{im\pi}$, it is found that

$$2^{m+1} m \pi^{m-1} \int_0^\infty \mathcal{Y}_{mn}(2m\lambda e^{i\pi}) \lambda^{mk-1} E(p; \alpha_r : q; \rho_s : \frac{x}{\lambda^{2m}}) d\lambda$$

$$= i^{-m(k+n)} E(p+2m; \alpha_r : q; \rho_s : x e^{im\pi}), \quad (23)$$

where $(k \pm n) > 0$, $m(2\alpha_r - k) > -\frac{3}{2}$, $r = 1, 2, \ldots, p$.

From (22) and (23) and making use of (8) CH. II., it follows that

$$2^{m+1} m \pi^m i \int_0^\infty J_{mn}(2m\lambda) \lambda^{mk-1} E(p; \alpha_r : q; \rho_s : \frac{x}{\lambda^{2m}}) d\lambda$$

$$= i^{m(k-n)} E(p+2m; \alpha_r : q; \rho_s : x e^{-im\pi})$$

$$- i^{-m(k-n)} E(p+2m; \alpha_r : q; \rho_s : x e^{im\pi}), \quad (24)$$

where the $\underline{\alpha's}$ are given by (21′) and $(k+n) > 0$, $m(2\alpha_r - k) > -\frac{3}{2}$, $r = 1, 2, \ldots, p$.

§7. A multiple Integral involving E-functions. The formula to be established is

$$\prod_{r=1}^{m-1}\int_0^\infty e^{-t_r}E(\alpha,\beta::t_r)\,t_r^{r/m-1}\,dt_r\; e^{-\frac{b}{t_1t_2\cdots t_{m-1}}}E\left(\alpha,\beta::\frac{b}{t_1t_2\cdots t_{m-1}}\right)$$

$$=\frac{\{\Gamma(\alpha)\Gamma(\beta)\}^m}{\Gamma(m\alpha)\Gamma(m\beta)}(2\pi)^{\frac{1}{2}m-\frac{1}{2}}m^{-\frac{1}{2}}e^{-mb^{1/m}}E(m\alpha,m\beta::mb^{1/m}), \quad (25)$$

where $\underline{b}>0$.

It will be shown that the function

$$w(z)\equiv e^{-z}E(\alpha,\beta::z)$$

satisfies the differential equation

$$z^2w''+z(z-\alpha-\beta+1)w'+(\alpha\beta-\alpha z-\beta z+z)w=0 \quad (26)$$

and then (25) will be deduced by means of (26).

The following formulae are also required

$$\int_0^\infty e^{-t} t^{\gamma-1} E(\alpha, \beta :: t)\, dt = \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\gamma)\Gamma(\beta+\gamma)}{\Gamma(\alpha+\beta+\gamma)}, \tag{27}$$

where $R(\alpha+\gamma)>0$, $R(\beta+\gamma)>0$. [C. V., p. 381, ex 113.] and

$$F(\alpha; \rho; z) = e^{z} F(\rho-\alpha; \rho; -z). \tag{28}$$

<u>The differential equation</u>. If $y \equiv E(\alpha, \beta :: z)$, it satisfies the equation

$$z^2 y'' = z(z+\alpha+\beta-1)y' - \alpha\beta y. \tag{29}$$

[C. V., p. 349]

Now let $w = e^{-z} y$, so that $y = e^{z} w$;

then

$$z^2(w''+2w'+w)=z(z+\alpha+\beta-1)(w'+w)-\alpha\beta w,$$

from which (26) follows. Other solutions of (26) are

$$e^{-z}z^{\alpha}F(\alpha;\alpha-\beta+1;z)=z^{\alpha}F(1-\beta;\alpha-\beta+1;-z),$$

and $e^{-z}z^{\beta}F(\beta;\beta-\alpha+1;z)=z^{\beta}F(1-\alpha;\beta-\alpha+1;-z)$.

<u>Proof of the Multiple Integral</u>. Let $F(b)$ denote the L.H.S. of (25), if $w(z)\equiv e^{-z}E(\alpha,\beta::z)$,

$$F(b)=\prod_{r=1}^{m-1}\int_0^{\infty}e^{-t_r}E(\alpha,\beta::t_r)\,t_r^{r/m-1}\,dt_r\;w\left(\frac{b}{t_1t_2\cdots t_{m-1}}\right),$$

$$F'(b)=\prod_{r=1}^{m-1}\int_0^{\infty}e^{-t_r}E(\alpha,\beta::t_r)\,t_r^{r/m-2}\,dt_r\;w'\left(\frac{b}{t_1t_2\cdots t_{m-1}}\right),$$

and

$$F''(b) = \prod_{r=1}^{m-1} \int_0^\infty e^{-t_r} E(\alpha, \beta :: t_r)\, t_r^{r/m-3}\, dt_r \; w''\left(\frac{b}{t_1 t_2 \cdots t_{m-1}}\right).$$

Hence from (26)

$$b^2 F''(b) = \prod_{r=1}^{m-1} \int_0^\infty e^{-t_r} E(\alpha, \beta :: t_r)\, t_r^{r/m-1}\, dt_r$$

$$\times \left\{ \frac{-b}{t_1 \cdots t_{m-1}} \left(\frac{b}{t_1 \cdots t_{m-1}} - \alpha - \beta + 1 \right) w' - \alpha\beta w + \frac{b}{t_1 \cdots t_{m-1}} (\alpha+\beta-1) w \right\}.$$

Therefore

$$b^2 F''(b) - (\alpha+\beta-1) b F'(b) + \alpha\beta F(b) = L + M, \tag{30}$$

where

$$L = -b^2 \prod_{r=1}^{m-1} \int_0^\infty e^{-t_r} E(\alpha, \beta :: t_r)\, t_r^{r/m-3}\, dt_r \; w',$$

and

$$M = (\alpha+\beta-1)\, b \prod_{r=1}^{m-1} \int_0^\infty e^{-t_r} E(\alpha, \beta :: t_r)\, t_r^{r/m-2}\, dt_r \; w.$$

Now in L change the order of integration so that the first integral becomes the last; then

$$L = b\prod_{r=2}^{m-1}\int_0^\infty e^{-t_r}E(\alpha,\beta::t_r)\,t_r^{r/m-2}\,dt_r\int_0^\infty e^{-t_1}E(\alpha,\beta::t_1)\,t_1^{1/m-1}\frac{\partial w}{\partial t_1}\,dt_1.$$

Here integrate by parts, noting that $\underline{w}$ vanishes at both limits, and get

$$L = b\prod_{r=2}^{m-1}\int_0^\infty e^{-t_r}E(\alpha,\beta::t_r)\,t_r^{r/m-2}\,dt_r$$

$$\times -\int_0^\infty e^{\frac{-b}{t_1\cdots t_{m-1}}}E\left(\alpha,\beta::\frac{b}{t_1\cdots t_{m-1}}\right)\left[\left(\tfrac{1}{m}-1\right)e^{-t_1}E(\alpha,\beta::t_1)\,t_1^{1/m-2}+t_1^{1/m-1}\frac{d}{dt_1}\left\{e^{-t_1}E(\alpha,\beta::t_1)\right\}\right]dt_1$$

On substituting $b/(x t_2\cdots t_{m-1})$ for t_1, this becomes

$$L = -\left(\frac{1}{m}-1\right) b^{1/m} \prod_{r=2}^{m-1} \int_0^\infty e^{-t_r} E(\alpha,\beta::t_r)\, t_r^{(r-1)/m-1}\, dt_r$$

$$\times \int_0^\infty e^{-\lambda} E(\alpha,\beta::\lambda)\, \lambda^{(m-1)/m-1}\, w\left(\frac{b}{\lambda t_2 \cdots t_{m-1}}\right) d\lambda$$

$$- b^{1/m+1} \prod_{r=2}^{m-1} \int_0^\infty e^{-t_r} E(\alpha,\beta::t_r)\, t_r^{(r-1)/m-2}\, dt_r$$

$$\times \int_0^\infty e^{-\lambda} E(\alpha,\beta::\lambda)\, \lambda^{(m-1)/m-2}\, w'\left(\frac{b}{\lambda t_2 \cdots t_{m-1}}\right) d\lambda$$

$$= \left(1-\frac{1}{m}\right) b^{1/m} F(b) - b^{1/m+1} F'(b).$$

Similarly

$$M = (\alpha+\beta-1)\, b^{1/m} F(b).$$

Hence (30) can be written

$$b^2 F''(b) - (\alpha+\beta-1-b^{1/m})\, b F'(b) + \left\{\alpha\beta - (\alpha+\beta-1/m)\, b^{1/m}\right\} F(b) = 0 \tag{31}$$

Next in (26) put $z = m\lambda^{1/m}$, replace α and β by $m\alpha$ and $m\beta$, and get

$$\lambda^2 \frac{d^2w}{d\lambda^2} - (\alpha+\beta-1-\lambda^{1/m})\lambda\frac{dw}{d\lambda} + \left\{\alpha\beta - (\alpha+\beta-\tfrac{1}{m})\lambda^{1/m}\right\} w = 0,$$

which is (31) with w in place of $F(b)$ and λ in place of b. Thus

$$F(b) = Ae^{-mb^{1/m}}(mb^{1/m})^{m\alpha}F(m\alpha; m\alpha - m\beta + 1; mb^{1/m})$$
$$+ Be^{-mb^{1/m}}(mb^{1/m})^{m\beta}F(m\beta; m\beta - m\alpha + 1; mb^{1/m}).$$

Now $F(b)$ is symmetrical in α and β, and so are the coefficients A and B. Therefore, if $A = f(\alpha,\beta)$ it follows that $B = f(\beta,\alpha)$.

Let it be assumed that $\alpha<\beta$, multiply by $b^{-\alpha}$ and let $b\to 0$; then, if $1/m+\beta-\alpha>0$,

$$\prod_{r=1}^{m-1}\int_0^\infty e^{-t_r}E(\alpha,\beta::t_r)\,t_r^{r/m-\alpha-1}\,dt_r\,\Gamma(\alpha)\Gamma(\beta-\alpha)=A\,m^{m\alpha},$$

or by (24)

$$\prod_{r=1}^{m-1}\frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(r/m)\Gamma(\beta-\alpha+r/m)}{\Gamma(\beta+r/m)}\Gamma(\alpha)\Gamma(\beta-\alpha)=A\,m^{m\alpha}.$$

Hence by (18) CH. II.

$$A=\frac{\{\Gamma(\alpha)\Gamma(\beta)\}^m}{\Gamma(m\alpha)\Gamma(m\beta)}(2\pi)^{\frac{1}{2}m-\frac{1}{2}}m^{-\frac{1}{2}}\Gamma(m\alpha)\Gamma\{m(\beta-\alpha)\},$$

from which, and (19); (25) follows

If, on the other hand, $\beta<\alpha$, multiply by $b^{-\beta}$, let $b \to 0$ and the same results are obtained.

The case $\underline{\alpha = \beta}$ can be derived by continuity.

§8. Second Integral involving the product of two E-functions.

It is proposed to establish the formula

$$\int_0^\infty e^{-nt} t^{nk-1} E(n\gamma, n\delta :: nt) E(p; \alpha_r : q; \rho_s : \frac{x}{t^n}) dt$$

$$= \frac{\Gamma(n\gamma)\Gamma(n\delta)}{(2\pi)^{\frac{1}{2}n-\frac{1}{2}}\sqrt{n}} E(p+2n; \alpha_r : q+n; \rho_s : x), \qquad (32)$$

where $R(k+\gamma) > 0$, $R(\delta+k) > 0$, $\underline{n}$ is a positive integer and

$$\left.\begin{aligned} \alpha_{p+2\nu+1} &= \gamma + k + \frac{\nu}{n} \\ \alpha_{p+2\nu+2} &= \delta + k + \frac{\nu}{n} \\ \rho_{q+\nu+1} &= \gamma + \delta + k + \frac{\nu}{n} \end{aligned}\right\} \nu = 0, 1, 2, \ldots, n-1. \qquad (32')$$

To prove this apply (14) repeatedly to itself $(n-1)$ times with $k+1/n$, $k+2/n$, $\ldots$, $k+(n-1)/n$

in turn in place of k: it is found that

$$\prod_{r=0}^{n-1}\int_0^\infty e^{-t_r}\, t_r^{\,k+r/n-1}\, E(\gamma,\delta::t_r)\,dt_r\, E\left(p;\alpha_r:q;\rho_s:\frac{x}{t_0t_1\cdots t_{n-1}}\right)$$

$$=\{\Gamma(\gamma)\Gamma(\delta)\}^n E(p+2n;\alpha_r:q+n;\rho_s:x)$$

where the $\underline{\alpha's}$ and $\underline{\rho's}$ are given by (3i)

Now change the order of integration so that the first integral becomes the last, put

$$t_0=\frac{\lambda}{t_1t_2\cdots t_{n-1}},$$

and change the order of integration so that the last integral becomes the first. Then the L.H.S. becomes

$$\int_0^\infty \lambda^{k-1} E\left(p; \alpha_r : q; \rho_s : \frac{x}{\lambda}\right) d\lambda \prod_{r=1}^{n-1} \int_0^\infty e^{-t_r}\, t_r^{r/n-1}\, E(\gamma, \delta :: t_r)\, dt_r \; e^{\frac{-\lambda}{t_1 \cdots t_{n-1}}} E\left(\gamma, \delta :: \frac{\lambda}{t_1 \cdots t_{n-1}}\right)$$

$$= \frac{\{\Gamma(\gamma)\Gamma(\delta)\}^n (2\pi)^{\frac{1}{2}n-\frac{1}{2}}}{\Gamma(n\gamma)\Gamma(n\delta)\sqrt{n}} \int_0^\infty e^{-n\lambda^{1/n}} \lambda^{k-1} E(n\gamma, n\delta :: n\lambda^{1/n})\, E\left(p; \alpha_r : q; \rho_s : \frac{x}{\lambda}\right) d\lambda,$$

by (25). On replacing $\underline{\lambda}$ by $\underline{t}^n$ formula (32) is obtained.

§4. An Integral involving a simple type of E-function.

The formula to be proved is

$$\int_0^\infty \exp\{-x - y^2/(4x)\} x^{-\alpha-\beta-1} E(\alpha,\beta::x)\,dx = 2\Gamma(\alpha)\Gamma(\beta)\left(\tfrac{1}{2}y\right)^{-\alpha-\beta} K_{\alpha-\beta}(y), \quad (33)$$

where $R(y^2) > 0$.

The following formulae will be required in the proof:

$$K_n(z) = \tfrac{1}{2}\left(\tfrac{1}{2}z\right)^n \int_0^\infty \exp\left\{-\tau - \frac{z^2}{4\tau}\right\} \tau^{-n-1}\,d\tau, \quad (34)$$

where $R(z^2) > 0$ [Gray, Mathews and MacRobert, <u>Bessel Functions</u>, p. (51), (33)];

$$\int_0^\infty \frac{K_n\{a\sqrt{(t^2+z^2)}\}}{(t^2+z^2)^{\frac{1}{2}n}} t^{2m+1}\,dt = \frac{2^m\Gamma(m+1)}{a^{m+1}z^{n-m-1}} K_{n-m-1}(az). \quad (35)$$

[This integral, due to Sonine and Gegenbauer, may be found in Watson's <u>Bessel Functions</u>, p. 417, (6)].

To prove the formula, substitute for the E-function on the L.H.S. of (33) from (16), and it becomes if $R(\beta)>0$

$$\Gamma(\alpha)\int_0^\infty \exp\{-x-y^2/(4x)\}x^{-\alpha-\beta-1}dx\int_0^\infty e^{-\lambda}\lambda^{\beta-1}\left(1+\frac{\lambda}{x}\right)d\lambda.$$

Here put $\lambda=x\xi$ and change the order of integration, so getting

$$\Gamma(\alpha)\int_0^\infty \xi^{\beta-1}(1+\xi)^{-\alpha}d\xi\int_0^\infty \exp\left\{-\frac{y^2}{4x}-x(1+\xi)\right\}x^{-\alpha-1}dx.$$

Next put $x(1+\xi)=\tau$ and the expression becomes

$$\Gamma(\alpha)\int_0^\infty \xi^{\beta-1}d\xi\int_0^\infty \exp\left\{-\frac{y^2(1+\xi)}{4\tau}-\tau\right\}\frac{d\tau}{\tau^{\alpha+1}}$$

$$= \frac{\Gamma(\alpha)\, 2^{\alpha+1}}{y^\alpha} \int_0^\infty \xi^{\beta-1} \frac{K_\alpha\{y\sqrt{(1+\xi)}\}}{(1+\xi)^{\frac{1}{2}\alpha}}\, d\xi, \text{ by (34).}$$

Now replace $\underline{\xi}$ by t^2; then, on applying (35), formula (33) is obtained. The condition $R(\beta)>0$ may be removed by analytical continuation.

<u>A new integral involving Bessel Function</u>: As a special case of (33), put $\alpha = \frac{1}{2} - n$, $\beta = \frac{1}{2} + n$ and write $\underline{2x}$ for $\underline{x}$; then, applying the formula

$$\cos n\pi\, E\left(\tfrac{1}{2}+n, \tfrac{1}{2}-n :: 2x\right) = \sqrt{(2\pi x)}\, e^x K_n(x), \qquad (36)$$

[C. V., p. 351, (14)], we have if $R(y^2)>0$,

$$\int_0^\infty \exp\{-x - y^2/(8x)\} \frac{K_n(x)}{x^{3/2}}\, dx = 4\sqrt{(2\pi)}\, \frac{K_{2n}(y)}{y}. \qquad (37)$$

§10. An Integral involving a Product of two E-functions:

The formula to be established is

$$\int_0^\infty x^{-\lambda-1} E(p;\alpha_r:q;\rho_s:\mu x)\,E(l;\beta_t:m;\sigma_u:\nu x)\,dx =$$

$$\mu^\lambda G^{l+1,\,p+1}_{p+m+1,\,q+l+1}\left(\frac{\nu}{\mu}\middle|\begin{matrix}-\alpha_1+\lambda+1,\dots,-\alpha_p+\lambda+1,1,\sigma_1,\dots,\sigma_m\\ \lambda,\beta_1,\dots,\beta_l,1-\rho_1+\lambda,\dots,1-\rho_q+\lambda\end{matrix}\right), \qquad (38)$$

where

$$q+1 \leqq p \leqq q+l-m,$$

$$l > m+1,$$

$$R(\beta_t+\alpha_r-\lambda)>0\ [t=1,\dots,l\ ;\ r=1,\dots,p],$$

$$R(\lambda)>0$$

and μ,ν are supposed for simplicity to be real and positive.

The constants are assumed such that all the functions included in the formula (38) exist.

The function on the R.H.S. of (38) is Meijer's G-function, defined by him in the Proceedings of the Akademie te Amsterdam [XLIV, p. 82, 1941] by the following two formulae. Firstly

$$G_{p,q}^{m,n}\left(w\,\middle|\,\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right)=\sum_{h=1}^{m}\frac{\prod_{j=1}^{m}{}'\Gamma(b_j-b_h)\prod_{j=1}^{n}\Gamma(1+b_h-a_j)}{\prod_{j=m+1}^{q}\Gamma(1+b_h-b_j)\prod_{j=n+1}^{p}\Gamma(a_j-b_h)}\,w^{b_h}F\left(\begin{matrix}1+b_h-a_1,\dots,1+b_h-a_p\\ 1+b_h-b_1,\dots *\dots,1+b_h-b_q\end{matrix};(-)^{m+n+p}w\right), \tag{39}$$

where

$$0\leq n\leq p\leq q; \quad , \quad 1\leq m\leq q.$$

If $p=q$, then the formula holds provided that $|w|<1$.

Secondly*,

$$G^{m,n}_{p,q}\left(w\middle|\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right)=\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}w^s\,ds. \quad (40)$$

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* The contour L is of Barnes's type [Proc. Lond. Math. Soc. Ser. 2, 5 pp 65-71]. It starts from $-\infty i+b$ to $\infty i+b$ (b is any real number) and is curved if necessary so that the points $b_j, b_j+1, \dots$ $(j=1,\dots,m)$ lie on the right of it and the points $-a_j-1, -a_j-2, \dots$ $(j=1,\dots,n)$ lie on the left of the contour. This is possible since $a_j-b_h \neq 1,2,3,\dots$ $(j=1,\dots,n$ and $h=1,\dots,m)$.

The formulae

$$E(p; \alpha_r : q ; \rho_s : \mu x) = \Gamma(\alpha_p)\left\{\prod_{n=1}^{q} \Gamma(\rho_n - \alpha_n)\right\}^{-1} \prod_{n=1}^{q} \int_0^1 \lambda_n^{\alpha_n - 1} (1-\lambda_n)^{\rho_n - \alpha_n - 1} d\lambda_n$$

$$\times \prod_{n=q+1}^{p-2} \int_0^\infty e^{-\lambda_n} \lambda_n^{\alpha_n - 1} d\lambda_n \int_0^\infty e^{-\lambda_{p-1}} \lambda_{p-1}^{\alpha_{p-1}-1} \left(1 + \lambda_1 \lambda_2 \cdots \lambda_{p-1}/(\mu x)\right)^{-\alpha_p} d\lambda_{p-1}, \quad (41)$$

where

$R(\alpha_n) > 0 \quad [n = 1, 2, \ldots, p-1]$

$R(\rho_n - \alpha_n) > 0 \quad [n = 1, 2, \ldots, q]$

$p \geq q+1,$

[MacRobert, Proc. Roy. Soc. Edin., LVIII., p. 3, 1937];

and *

$$E(l;\beta_t:m;\sigma_u:\nu x)=\frac{1}{2\pi i}\int_{\Gamma}\frac{\Gamma(\xi)\prod_{t=1}^{l}\Gamma(\beta_t-\xi)}{\prod_{u=1}^{m}\Gamma(\sigma_u-\xi)}(\nu x)^{\xi}\,d\xi, \tag{42}$$

provided $l>m+1$, are also required.

Formula (38) will be derived from the following subsidiary formula which involves one E-function

$$\int_0^{\infty}x^{-k-1}E(p;\alpha_r:q;\rho_s:\mu x)\,dx=\frac{\Gamma(k)\prod_{r=1}^{p}\Gamma(\alpha_r-k)}{\prod_{s=1}^{q}\Gamma(\rho_s-k)}\mu^{k}, \tag{43}$$

where $R(\alpha_r-k)>0$ $(r=1,\dots,p)$, $R(k)>0$.

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* The contour Γ is equivalent to Barnes's contour (loc. cit). The points β_t, β_t+1, ... $(t=1,\dots,l)$ lie on the right and the points $0, 1, \dots$ lie on the left of it.

Proof of the subsidiary formula. Substitute from the R.H.S. of (41) in the L.H.S. of (43); then change the order of integration; put $\mu x = y$, and get

$$\text{L.H.S. of (43)} = \mu^{k}\Gamma(\alpha_p)\left\{\prod_{n=1}^{q}\Gamma(\rho_n-\alpha_n)\right\}^{-1}\prod_{n=1}^{q}\int_0^1 \lambda_n^{\alpha_n-1}(1-\lambda_n)^{\rho_n-\alpha_n-1}d\lambda_n$$

$$\times\prod_{n=q+1}^{p-1}\int_0^\infty e^{-\lambda_n}\lambda_n^{\alpha_n-1}d\lambda_n\int_0^\infty y^{-k-1}(1+\lambda_1\lambda_2\cdots\lambda_{p-1}/y)^{-\alpha_p}dy.$$

[Note. The integral $\int_0^\infty e^{-\lambda_{p-1}}\lambda_{p-1}^{\alpha_{p-1}-1}d\lambda_{p-1}$, appearing in the R.H.S. of (41), is added to the product $\prod_{n=q+1}^{p-2}$ after changing the order of integration.]

To evaluate the integral $\int_0^\infty y^{-k-1}(1+\lambda_1\lambda_2\cdots\lambda_{p-1}/y)^{-\alpha_p}dy$ put $y=\lambda_1\lambda_2\cdots\lambda_{p-1}z$ and get

$$\int_0^\infty y^{-k-1}(1+\lambda_1\lambda_2\cdots\lambda_{p-1}/y)^{-\alpha_p}dy = (\lambda_1\lambda_2\cdots\lambda_{p-1})^{-k}\int_0^\infty \frac{z^{\alpha_p-k-1}}{(1+z)^{\alpha_p}}dz$$

$$= (\lambda_1\lambda_2\cdots\lambda_{p-1})^{-k}B(\alpha_p-k, k),$$

where $R(\alpha_p-k)>0$, $R(k)>0$.

Now, putting each $\underline{\lambda}$ in its proper integral and evaluating each integral seperately by the Beta function and Gamma function formulae, formula (43) is obtained.

In (43) change $\underline{k}$ to $(\lambda-\xi)$, then, if

$$R(\lambda-\alpha_r)<R(\xi)<R(\lambda) \quad [r=1,\dots,p], \tag{44}$$

$$\int_0^\infty x^{-\lambda+\xi-1}E(p;\alpha_r:q;\rho_s:\mu x)dx = \mu^{\lambda-\xi}\frac{\Gamma(\lambda-\xi)\prod_{r=1}^{p}\Gamma(\alpha_r-\lambda+\xi)}{\prod_{s=1}^{q}\Gamma(\rho_s-\lambda+\xi)}. \tag{45}$$

To obtain formula (38), note that, since, for convergence of the integral in (38), $R(\beta_t+\alpha_r-\lambda)>0$, ($t=1,\dots,l$ and $r=1,\dots,p$) and $R(\lambda)>0$, it is always possible to select the contour Γ of (42) so that the condition (44) is satisfied at all points on Γ.

Now substitute (42) for the second E-function occuring in the L.H.S. of (38), change the order of integration, use (45) to evaluate the inner integral, and get

$$\text{L.H.S. of (38)} = \frac{\mu^\lambda}{2\pi i}\int_\Gamma \frac{\Gamma(\xi)\prod_{t=1}^{l}\Gamma(\beta_t-\xi)\Gamma(\lambda-\xi)\prod_{r=1}^{p}\Gamma(\alpha_r-\lambda+\xi)}{\prod_{u=1}^{m}\Gamma(\sigma_u-\xi)\prod_{s=1}^{q}\Gamma(\rho_s-\lambda+\xi)}\left(\frac{\nu}{\mu}\right)^\xi d\xi$$

$$=\mu^\lambda G^{l+1,\,p+1}_{p+m+1,\,l+q+1}\left(\frac{\nu}{\mu}\,\middle|\,\begin{matrix}-\alpha_1+\lambda+1,\dots,-\alpha_p+\lambda+1,\ 1,\sigma_1,\dots,\sigma_m\\ \lambda,\beta_1,\beta_2,\dots,\beta_l,\ 1-\rho_1+\lambda,\dots,1-\rho_q+\lambda\end{matrix}\right),$$

by using (40).

It must be noted that this latter function exists since $p \leq q + l - m$, the constants being such that all the Gamma functions and the hypergeometric functions included in the G-function exist.

§11. An Integral involving a product of an E-function and a hypergeometric function. The formula to be proved is

$$\frac{\eta}{2\pi i}\int_C (\eta z)^{-\lambda} {}_pF_p\left(\begin{matrix}1-\lambda+a_1,\dots,1-\lambda+a_p\\1-\lambda+b_1,\dots,1-\lambda+b_p\end{matrix};\eta z\right)E(l;\beta_s:m;\sigma_u:\nu z)\,dz =$$

$$\prod_{r=1}^{p}\left\{\frac{\Gamma(1-\lambda+b_r)}{\Gamma(1-\lambda+a_r)}\right\}E\left(\beta_1,\dots,\beta_l,a_1,\dots,a_p:\lambda,\sigma_1,\dots,\sigma_m,b_1,\dots,b_p:\frac{\nu}{\eta}\right), \quad (46)$$

where $R(a_r)>0$ $[r=1,\dots,p]$, $l>m+1$, η and ν are assumed to be real and positive and C is a contour starting at $-\infty$ on the real axis, passing positively round the origin and returning to $-\infty$ on the real axis, and amp $z=0$ to the right of the origin.

The following formulae are required in the proof:

$$\prod_{r=1}^{p}\int_0^1 t_r^{a_r-\lambda}(1-t_r)^{b_r-a_r-1}dt_r\, e^{t_1t_2\cdots t_p \eta z} =$$

$$\prod_{r=1}^{p}\left\{\frac{\Gamma(b_r-a_r)\Gamma(1-\lambda+a_r)}{\Gamma(1-\lambda+b_r)}\right\}\cdot F\left(\begin{matrix}1-\lambda+a_1,\dots,1-\lambda+a_p\\1-\lambda+b_1,\dots,1-\lambda+b_p\end{matrix};\eta z\right), \qquad (47)$$

where $R(a_r-\lambda+1)>0$, $R(b_r-a_r)>0$, $[r=1,2,\dots,p]$,

[Erdelyi, Quart. Journ. of Math. Oxford series Volume 8, 1937, p. 241, equation (3.6)].

and

$$\frac{1}{2\pi i}\int_C e^{\xi}\,\xi^{-z}\,d\xi = 1/\Gamma(z), \qquad (48)$$

[MacRobert, C.V., p. 143].

To prove (46), substitute from (47) and (42) for the hypergeometric function and the E-function respectively in the L.H.S. of (46), then it becomes

$$\frac{\eta}{(2\pi i)^2}\prod_{r=1}^{p}\left\{\frac{\Gamma(1-\lambda+b_r)}{\Gamma(b_r-a_r)\Gamma(1-\lambda+a_r)}\right\}$$

$$\times\int_C (\eta z)^{-\lambda}\prod_{r=1}^{p}\int_0^1 t_r^{a_r-\lambda}(1-t_r)^{b_r-a_r-1}dt_r\, e^{t_1\cdots t_p\eta z}\int_\Gamma \frac{\Gamma(\xi)\prod_{s=1}^{l}\Gamma(\beta_s-\xi)}{\prod_{u=1}^{m}\Gamma(\sigma_u-\xi)}(\nu z)^{\xi}d\xi,$$

where $l>m+1$, $R(a_r-\lambda+1)$ and $R(b_r-a_r)>0$ $[r=1,2,\ldots,p]$.

Now change the order of integration so that the last integral becomes the first, then the L.H.S. of (46) becomes

$$\frac{\eta^{1-\lambda}}{(2\pi i)^2}\prod_{r=1}^{p}\left\{\frac{\Gamma(1-\lambda+b_r)}{\Gamma(b_r-a_r)\Gamma(1-\lambda+a_r)}\right\}$$

$$\times\int_\Gamma \frac{\Gamma(\xi)\prod_{s=1}^{l}\Gamma(\beta_s-\xi)}{\prod_{u=1}^{m}\Gamma(\sigma_u-\xi)}\nu^{\xi}d\xi\prod_{r=1}^{p}\int_0^1 t_r^{a_r-\lambda}(1-t_r)^{b_r-a_r-1}dt_r\int_C e^{t_1\cdots t_p\eta z}z^{\xi-\lambda}dz,$$

where $l>m+1$, $R(a_r-\lambda+1)>0$ and $R(b_r-a_r)>0$ $[r=1,2,\ldots,p]$.

But using (48) it is seen that

$$\frac{1}{2\pi i}\int_C e^{t_1\cdots t_p \eta z}\, z^{\xi-\lambda}\, dz = (t_1\cdots t_p\eta)^{\lambda-\xi-1}\left\{1/\Gamma(\lambda-\xi)\right\}.$$

Substituting in the last expression this value, then the variables $t_1,\ldots,t_p$ become seperable and their integrals can be evaluated by the Beta function and Gamma function formulae. Then again using (47) formula (46) is obtained. The restrictions $R(a_r-\lambda)>-1$ and $R(b_r-a_r)>0$ $[r=1,2,\ldots,p]$ can now be removed by analytical continuation.

§12. Another E-function formula. The formula to be proved is

$$\int_0^1 x^{a_2-1}(1-x)^{\alpha+\beta-a_1-a_2-1} F\left[\begin{matrix}\alpha-a_1, \beta-a_1\\ \alpha+\beta-a_1-a_2\end{matrix}; (1-x)\right] E\left(\alpha, \beta, a_3, \ldots, a_p : q ; \rho_s : \frac{\xi}{x}\right) dx$$

$$= \Gamma(\alpha+\beta-a_1-a_2) E(a_1, a_2, \ldots, a_p : q ; \rho_s : \xi), \qquad (49)$$

where $R(\alpha+\beta-a_1-a_2)>0$, $R(a_1)>0$, $R(a_2)>0$.

It may be noted that $\underline{\alpha}$ and $\underline{\beta}$ do not appear in the E-function on the right of (49).

The following formulae are required in the proof:

$$\int_0^1 \lambda^{\alpha_{p+1}-1}(1-\lambda)^{\rho_{q+1}-\alpha_{p+1}-1} E(p ; \alpha_r : q ; \rho_s : x/\lambda)\, d\lambda =$$

$$\Gamma(\rho_{q+1}-\alpha_{p+1}) E(p+1 ; \alpha_r : q+1 ; \rho_s : x), \qquad (50)$$

where $R(\alpha_{p+1})>0$, $R(\rho_{q+1}-\alpha_{p+1})>0$.

[This is really (15) after putting $1/\lambda$ for $(1+\mu)$].

and

$$E(p;\alpha_r:q;\rho_s:z) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_p)}{\Gamma(\rho_1)\Gamma(\rho_2)\cdots\Gamma(\rho_q)} F(p;\alpha_r:q;\rho_s:-1/z), \qquad (51)$$

where $p \leqslant q$ [MacRobert, C.V., p. 352.(21)].

Now to obtain (49) expand the hypergeometric function in the L.H.S., and integrate term by term applying (50) with $\alpha_{p+1} = a_2$ and $\rho_{q+1} = \alpha+\beta-a_1+m$; then the L.H.S. of (49) becomes

$$\Gamma(\alpha+\beta-a_1-a_2) \sum_{m=0}^{\infty} \frac{(\alpha-a_1;m)(\beta-a_1;m)}{m!}$$

$$\times E(\alpha,\beta,a_2,a_3,\ldots,a_p:\rho_1,\rho_2,\ldots,\rho_q,\alpha+\beta-a_1+m:\xi). \qquad (A)$$

Firstly, if $p \geqslant q+1$, substitute from (41) for the E-function in (A), then it becomes

$$\Gamma(\alpha+\beta-a_1-a_2)\sum_{m=0}^{\infty}\frac{(\alpha-a_1;m)\,(\beta-a_1;m)\,\Gamma(\alpha_{p+1})}{\prod_{n=1}^{q+1}\Gamma(\rho_n-\alpha_n)}$$

$$\times\prod_{n=1}^{q+1}\int_0^1\lambda_n^{\alpha_n-1}(1-\lambda_n)^{\rho_n-\alpha_n-1}d\lambda_n\prod_{n=q+2}^{p-1}\int_0^{\infty}e^{-\lambda_n}\lambda_n^{\alpha_n-1}d\lambda_n$$

$$\times\int_0^{\infty}e^{-\lambda_p}\lambda_p^{\alpha_p-1}(1+\lambda_1\lambda_2\cdots\lambda_p/\xi)^{-\alpha_{p+1}}d\lambda_p,$$

where in this expression

$\alpha_1,\ \alpha_2,\ \alpha_3,\ \ldots,\ \alpha_q,\ \alpha_{q+1},\ \ldots,\ \alpha_p,\ \alpha_{p+1}$ are replaced by $\beta,\ a_q,\ a_2,\ \ldots,\ a_{q-1},\ \alpha,\ \ldots,\ a_{p-1},\ a_p$ respectively. (A′)

[Note. This is possible since the E-function is symmetrical in the α's and ρ's.].

Then since $\rho_{q+1}=\alpha+\beta-a_1+m$ and $\alpha_{q+1}=\alpha$, (A) becomes

$$\frac{\Gamma(\alpha+\beta-a_1-a_2)\Gamma(\alpha_{p+1})}{\prod_{n=1}^{q}\Gamma(\rho_n-\alpha_n)}\sum_{m=0}^{\infty}\frac{(\alpha-a_1;m)(\beta-a_1;m)}{m!\,\Gamma(\beta-a_1+m)}$$

$$\times\prod_{n=1}^{q}\int_0^1\lambda_n^{\alpha_n-1}(1-\lambda_n)^{\rho_n-\alpha_n-1}d\lambda_n\int_0^1\lambda_{q+1}^{\alpha-1}(1-\lambda_{q+1})^{\beta-a_1+m-1}d\lambda_{q+1}$$

$$\times\prod_{n=q+2}^{p-1}\int_0^\infty e^{-\lambda_n}\lambda_n^{\alpha_n-1}d\lambda_n\int_0^\infty e^{-\lambda_p}\lambda_p^{\alpha_p-1}(1+\lambda_1\lambda_2\cdots\lambda_p/\xi)^{-\alpha_{p+1}}d\lambda_p,$$

where the $\underline{\alpha's}$ are given by (A').

Changing the order of integration and summation, the last expression can be written in the form

$$\frac{\Gamma(\alpha+\beta-a_1-a_2)\Gamma(\alpha_{p+1})}{\prod_{n=1}^{q}\Gamma(\rho_n-\alpha_n)\Gamma(\beta-a_1)}\prod_{n=1}^{q}\int_0^1\lambda_n^{\alpha_n-1}(1-\lambda_n)^{\rho_n-\alpha_n-1}d\lambda_n$$

$$\times\left\{\int_0^1\lambda_{q+1}^{\alpha-1}(1-\lambda_{q+1})^{\beta-a_1-1}\sum_{m=0}^{\infty}\frac{(\alpha-a_1;m)}{m!}(1-\lambda_{q+1})^m\,d\lambda_{q+1}\right\}$$

$$\times\prod_{n=q+2}^{p-1}\int_0^\infty e^{-\lambda_n}\lambda_n^{\alpha_n-1}d\lambda_n\int_0^\infty e^{-\lambda_p}\lambda_p^{\alpha_p-1}(1+\lambda_1\lambda_2\cdots\lambda_p/\xi)^{-\alpha_{p+1}}d\lambda_p,$$

where the $\underline{\alpha^{m}}$ are given by (A').

But
$$\sum_{m=0}^{\infty} \frac{(\alpha-a_1;m)}{m!}(1-\lambda_{q+1})^m = \{1-(1-\lambda_{q+1})\}^{a_1-\alpha} = \lambda_{q+1}^{a_1-\alpha}.$$

Now substitute and the quantity between brackets $\{\quad\}$ becomes

$$\int_0^1 \lambda_{q+1}^{a_1-1}(1-\lambda_{q+1})^{\beta-a_1-1}\,d\lambda_{q+1}$$

From this it is seen that α_{q+1} and ρ_{q+1} in the E-function are replaced after summation by $\underline{a_1}$ and $\underline{\beta}$ instead of $\underline{\alpha}$ and $\alpha+\beta-a_1+m$. Hence by (41), the L.H.S. of (49) is equal to

$$\Gamma(\alpha+\beta-a_1-a_2)E(\beta, a_1, a_2, \dots, a_{q-1}, a_q, \dots, a_p : \rho_1, \dots, \rho_q, \beta : \xi)$$

$$=\Gamma(\alpha+\beta-a_1-a_2)E(a_1, a_2, \dots, a_p : \rho_1, \rho_2, \dots, \rho_q : \xi),$$

since $\underline{\beta}$ cancels. Therefore (49) is proved for the case $p \geqslant q+1$.

Secondly, if $p \leqq q$, substitute from (51) for the E-function in (A), and it becomes

$$\frac{\Gamma(\alpha+\beta-a_1-a_2)\Gamma(\alpha)\Gamma(\beta)\Gamma(a_3)\cdots\Gamma(a_p)\Gamma(a_2)}{\Gamma(\rho_1)\Gamma(\rho_2)\cdots\Gamma(\rho_q)}$$

$$\times\sum_{m=0}^{\infty}\frac{(\alpha-a_1; m)(\beta-a_1; m)}{m!\,\Gamma(\alpha+\beta-a_1+m)}\sum_{t=0}^{\infty}\frac{(\alpha; t)(\beta; t)(a_2; t)(a_3; t)\cdots(a_p; t)}{t!(\rho_1; t)\cdots(\rho_q; t)(\alpha+\beta-a_1+m; t)}(-1/\xi)^t.$$

Here change the order of summation, sum the last series by Gauss's theorem, and (49) is obtained for $p \leqq q$.

§13. <u>An Integral involving a product of a Bessel function and a simple type of an E-function.</u> The formula to be proved is

$$\int_0^\infty \lambda^{m-1} J_n(p\lambda) E\left(\alpha, \tfrac{1}{2}n-\tfrac{1}{2}m+1 :: \frac{\lambda^2}{a}\right) d\lambda = \frac{a^{-\alpha}}{2}\left(\frac{p}{2}\right)^{-m-2\alpha} E\left(\alpha, \frac{n}{2}+\frac{m}{2}+\alpha :: \frac{ap^2}{4}\right), \quad (52)$$

where $R(n+m+2\alpha)>0$, $R(n)>-1$ and $\underline{p}$ is an unrestricted complex number.

The formula [Watson, Bessel Functions, p. 394 (4)]

$$\int_0^\infty J_n(p\lambda) e^{-\mu\lambda^2} \lambda^{n+1} d\lambda = \frac{p^n}{(2\mu)^{n+1}} e^{-\frac{p^2}{4\mu}}, \quad (53)$$

where $R(n)>-1$, is required

To prove this, we have from (16)

$$E\left(\alpha, \tfrac{1}{2}n-\tfrac{1}{2}m+1 :: \frac{\lambda^2}{a}\right) = \Gamma(\alpha)\int_0^\infty e^{-\xi} \xi^{\frac{1}{2}n-\frac{1}{2}m}\left(1+\frac{a\xi}{\lambda^2}\right) d\xi,$$

Now substitute from this in the L.H.S. of (52), then it becomes after writing $\lambda^2\mu$ for $\underline{\xi}$ and then changing the order of integration

$$\Gamma(\alpha)\int_0^\infty \mu^{\frac{1}{2}n-\frac{1}{2}m}(1+a\mu)^{-\alpha}d\mu\int_0^\infty e^{-\lambda^2\mu}\lambda^{n+1}J_n(p\lambda)\,d\lambda.$$

Now substitute from (53) and this becomes

$$\frac{\Gamma(\alpha)\,p^n}{2^{n+1}}\int_0^\infty e^{-\frac{p^2}{4\mu}}\mu^{-\frac{1}{2}m-\frac{1}{2}n-1}(1+a\mu)^{-\alpha}d\mu$$

$$=\frac{\Gamma(\alpha)\,p^n}{2^{n+1}\,a^\alpha}\int_0^\infty e^{-\frac{p^2}{4\mu}}\mu^{-\frac{1}{2}m-\frac{1}{2}n-\alpha-1}\left(1+\frac{1}{a\mu}\right)^{-\alpha}d\mu.$$

Using (16) again (52) is obtained.

CHAPTER IV.

SOME FORMULAE FOR THE ASSOCIATED LEGENDRE FUNCTIONS WHEN THE SUM OF THE DEGREE AND ORDER IS A POSITIVE INTEGER

§1. Introductory. The function

$$(\mu^2-1)^{-\frac{1}{2}m} P_{m+n}^{-m}(\mu) \equiv (1-\mu^2)^{-\frac{1}{2}m} T_{m+n}^{-m}(\mu)$$

where n is any positive integer, are generalizations of the Legendre Polynomials and are closely related to Gegenbauer's Functions [Phil. Mag., Ser. 7, XXI., 698, 1936; Quart. Journ. of Math., XIV., 1, 1943]. They possess many properties analogous to those of the Legendre polynomials.

For instance, they can be expressed in the polynomial form

$$(1-\mu^2)^{-\frac{1}{2}m}\, T_{m+n}^{-m}(\mu) = \frac{2^{m+n}\,\Gamma(m+n+\frac{1}{2})}{\Gamma(\frac{1}{2})\,\Gamma(2m+n+1)}\,\mu^n F\left(-\tfrac{1}{2}n, \tfrac{1}{2}-\tfrac{1}{2}n; \tfrac{1}{2}-m-n; \mu^{-2}\right), \quad (1)$$

or by the extended Rodrigues' Formula

$$(1-\mu^2)^{\frac{1}{2}m}\, T_{m+n}^{-m}(\mu) = \frac{(-1)^n}{2^{m+n}\,\Gamma(m+n+1)}\,\frac{d^n}{d\mu^n}\left\{(1-\mu^2)^{m+n}\right\}. \quad (2)$$

An account of these functions will be found in S.H. pp. 332–336. A study of Gegenbauer's papers has brought other formulae to light. They are given below, sometimes in forms more general than Gegenbauer's, and, in many cases, with simpler proofs.

The following formulae are required in the proofs:

$$P_\nu^{-\mu}(\cosh\psi) = \frac{2^{-\mu}(\sinh\psi)^\mu}{\Gamma(\frac{1}{2})\Gamma(\mu+\frac{1}{2})}\int_0^\pi (\cosh\psi + \cos\theta\sinh\psi)^{\nu-\mu}(\sin\theta)^{2\mu}\,d\theta, \qquad (3)$$

where $\mu > -\frac{1}{2}$, $\psi > 0$ [S.H., p. 325, eq. 8]; and

$$T_\nu^{-\mu}(x) = \frac{2^{-\mu}(1-x^2)^{\frac{1}{2}\mu}}{\Gamma(\frac{1}{2})\Gamma(\mu+\frac{1}{2})}\int_0^\pi \left\{x + i\sqrt{(1-x^2)}\cos\phi\right\}^{\nu-\mu}(\sin\phi)^{2\mu}\,d\phi, \qquad (4)$$

where $\mu > -\frac{1}{2}$, $-1 < x < 1$, [S.H., p. 325, eq. 9].

§2. Formulae and Proofs. If p and q are zero or positive integers and if $R(l+p) > -1$, the integral*

$$I \equiv \int_{-1}^{+1} T_{l+p}^{-l}(\mu)\, T_{m+q}^{-m}(\mu)\,(1-\mu^2)^{\frac{1}{2}l-\frac{1}{2}m}\,d\mu$$

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* Wiener Sitzungsberichte, LXX. (2), 434, 1874.

is zero if $q < p$ or if $q - p = 1, 3, 5, \ldots$, and, if $q - p = 0, 2, 4, \ldots$, has the value

$$(-1)^{\frac{q-p}{2}} \frac{q!\, 2^{m-l}\, \Gamma(1+l-m)\, \Gamma(\tfrac{1}{2}+m+\tfrac{1}{2}p+\tfrac{1}{2}q)}{(\tfrac{1}{2}q-\tfrac{1}{2}p)!\, \Gamma(l+\tfrac{1}{2}p+\tfrac{1}{2}q+\tfrac{3}{2})\, \Gamma(1+l-m+\tfrac{1}{2}p-\tfrac{1}{2}q)\, \Gamma(2m+q+1)} \cdot \quad (5)$$

For, from formula (1) and formula (2),

$$I = (-1)^p \frac{2^{m+q-l-p}\, \Gamma(m+q+\tfrac{1}{2})}{\Gamma(l+p+1)\, \Gamma(\tfrac{1}{2})\, \Gamma(2m+q+1)}$$

$$\times \int_{-1}^{1} \frac{d^p}{d\mu^p}(1-\mu^2)^{l+p} \left\{ \mu^q - \frac{q(q-1)}{2(2m+2q-1)} \mu^{q-2} + \cdots \right\} d\mu.$$

Here integrate by parts $\underline{p}$ times. If $q < p$, the integral vanishes. If $(q-p)$ is an odd positive integer, the resulting integrand is odd, and therefore the integral vanishes. If $q - p = 0, 2, 4, \ldots$, then

$$I = \frac{q!\, 2^{m+q-l-p}\, \Gamma(m+q+\frac{1}{2})}{(q-p)!\, \Gamma(l+p+1)\, \Gamma(\frac{1}{2})\, \Gamma(2m+q+1)}$$

$$\int_{-1}^{1} (1-\mu^2)^{l+p} \left\{ \mu^{q-p} - \frac{(q-p)(q-p-1)}{2(2m+2q-1)} \mu^{q-p-2} + \cdots \right\} d\mu.$$

Now the integral is equal to

$$B(l+p+1, \tfrac{1}{2}q-\tfrac{1}{2}p+\tfrac{1}{2}) F(\tfrac{1}{2}p-\tfrac{1}{2}q, -l-\tfrac{1}{2}p-\tfrac{1}{2}q-\tfrac{1}{2}; \tfrac{1}{2}-m-q; 1).$$

Hence, on applying Gauss's Theorem and simplifying, (5) is obtained.

Again*, if $\underline{p}$ is a positive integer, and $-1<x<1$,

$$\sum_{n=0}^{2p} \frac{(-1)^n x^{2p-n}\, T^{-m}_{m+n}(x)}{n!\,(2p-n)!} = (-1)^p \frac{(1-x^2)^{\frac{1}{2}m+p}}{2^{m+2p}\, p!\, \Gamma(m+p+1)}. \quad (6)$$

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* Wiener Sitzungsberichte, XCVII. (2), 270, 1888.

To prove this, substitute from (4) in the L·H·S· of (6) and get

$$\frac{2^{-m}(1-x^2)^{\frac{1}{2}m}}{\Gamma(\frac{1}{2})\Gamma(m+\frac{1}{2})(2p)!}\int_0^{\pi}\sum_{n=0}^{2p}(-1)^n\,{}^{2p}C_n\,x^{2p-n}\{x+i\sqrt{(1-x^2)}\cos\phi\}^n(\sin\phi)^{2m}\,d\phi$$

$$=\frac{2^{-m}(1-x^2)^{\frac{1}{2}m}}{\Gamma(\frac{1}{2})\Gamma(m+\frac{1}{2})(2p)!}\int_0^{\pi}\{-i\sqrt{(1-x^2)}\cos\phi\}^{2p}(\sin\phi)^{2m}\,d\phi,$$

from which, the result follows. The condition $m>-\frac{1}{2}$ may be removed by analytical continuation.

Next*

$$\int_{-1}^{1}\{\cosh\psi+\mu\sinh\psi\}^{l}(1-\mu^2)^{\frac{1}{2}m}\,T_{m+n}^{-m}(\mu)\,d\mu$$

$$=\frac{\sqrt{(2\pi)}\,\Gamma(l+1)}{\Gamma(l-n+1)}(\sinh\psi)^{-m-\frac{1}{2}}\,P_{l+m+\frac{1}{2}}^{-m-n-\frac{1}{2}}(\cosh\psi), \qquad (7)$$

where $R(m)>-1$, $\psi>0$ and n is zero or a positive integer.

* Wiener Sitzungsberichte, LXX., (2), 435, 1874

To prove this, substitute from (2) in the integral, integrate by parts $\underline{n}$ times, and get

$$\frac{\Gamma(l+1)\, 2^{-m-n}}{\Gamma(l-n+1)\Gamma(m+n+1)}(\sinh\psi)^n\int_{-1}^{1}(1-\mu^2)^{m+n}\left\{\cosh\psi+\mu\sinh\psi\right\}^{l-n}d\mu,$$

from which and (3) the formula is obtained.

Gegenbauer proved the formula for $\underline{l}$ integral; it holds for general values of $\underline{l}$.

On replacing $\underline{l}$ by $(-l-1)$ and noting that

$$P_n^{-m}(z)=P_{-n-1}^{-m}(z),$$

the formula may be written

$$\int_{-1}^{1} \frac{(1-\mu^2)^{\frac{1}{2}m}\, T_{m+n}^{-m}(\mu)}{\{\cosh\psi + \mu \sinh\psi\}^{l+1}} = (-1)^n \sqrt{(2\pi)}\, \frac{\Gamma(l+n+1)}{\Gamma(l+1)} (\sinh\psi)^{-m-\frac{1}{2}} P_{l-m-\frac{1}{2}}^{-m-n-\frac{1}{2}}(\cosh\psi), \quad (7')$$

where $R(m) > -1$ and $\underline{n}$ is zero or a positive integer.

The formula*

$$\int_{-1}^{1} (1-\mu^2)^{\frac{1}{2}m}\, T_{m+n}^{-m}(\mu)\, (a-b\mu)^{-l}\, d\mu$$

$$= \frac{\Gamma(\frac{1}{2})\Gamma(l+n)\, 2^{-m-n}\, b^n}{\Gamma(l)\Gamma(m+n+\frac{3}{2})\, a^{l+n}} F\left(\frac{l+n}{2}, \frac{l+n+1}{2}; m+n+\frac{3}{2}; \frac{b^2}{a^2}\right), \quad (8)$$

where $R(m) > -1$, $n = 0, 1, 2, 3, \ldots$, $|a| > |b|$,
is obtained by expanding the last factor in the integrand in powers of $\underline{\mu}$ and integrating by parts using (2). On replacing $\underline{l}$ by $(2m+n+2)$ and by

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* Wiener Sitzungsberichte, C. (2), 749, 1891.

and by $(2m+n+3)$ the formulae*

$$\int_{-1}^{1}(1-\mu^2)^{\frac{1}{2}m}\,T_{m+n}^{-m}(\mu)\,(a-b\mu)^{-2m-n-2}\,d\mu=\frac{2^{m+n+1}\,\Gamma(m+n+1)}{\Gamma(2m+n+2)}\,\frac{b^n}{(a^2-b^2)^{m+n+1}},\tag{9}$$

where $R(m)>-1$, $n=0,1,2,3,\ldots$, $|a|>|b|$, and

$$\int_{-1}^{1}(1-\mu^2)^{\frac{1}{2}m}\,T_{m+n}^{-m}(\mu)\,(a-b\mu)^{-2m-n-3}\,d\mu=\frac{2^{m+n+2}\,\Gamma(m+n+2)}{\Gamma(2m+n+3)}\,\frac{a\,b^n}{(a^2-b^2)^{m+n+2}},\tag{10}$$

where $R(m)>-1$, $n=0,1,2,3,\ldots$, $|a|>|b|$, are obtained.

If in (8), l is replaced by $(l+1)$, the formula can be written

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* Wiener Sitzungsberichte, LXX. (2), 443, 1874.

$$\int_{-1}^{1}(1-\mu^2)^{\frac{1}{2}m}\, T_{m+n}^{-m}(\mu)\,(a-b\mu)^{-l-1}\,d\mu$$

$$= \frac{2}{\Gamma(l+1)}\, b^{-m-1}(a^2-b^2)^{\frac{1}{2}m-\frac{1}{2}l}\, Q_{m+n}^{l-m}\left(\frac{a}{b}\right), \tag{11}$$

where $R(m)>-1$, $n=0,1,2,3,\ldots$, $|a|>|b|$.

Next*, if $\underline{n}$ and $\underline{p}$ are positive integers or zero and such that $n \geqq p$, if

$$Z = xy - \sqrt{(x^2-1)}\sqrt{(y^2-1)}\,\mu,$$

and if $R(m)>-\frac{1}{2}$,

$$\int_{-1}^{1}(Z^2-1)^{-\frac{1}{2}m}\, P_{m+n}^{-m}(Z)\,(1-\mu^2)^{\frac{1}{2}m-\frac{1}{4}}\, T_{m-\frac{1}{2}+p}^{\frac{1}{2}-m}(\mu)\,d\mu$$

$$=(-1)^p\sqrt{(2\pi)}\,\frac{\Gamma(n+p+2m+1)\,n!}{\Gamma(n+2m+1)\,(n-p)!}\,(x^2-1)^{-\frac{1}{2}m}(y^2-1)^{-\frac{1}{2}m}\, P_{m+n}^{-m-p}(x)\, P_{m+n}^{-m-p}(y). \tag{12}$$

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* Wiener Sitzungsberichte, LXX. (2), 443, 1874.

For, from (1) and (2), the integral is equal to

$$(-1)^p \frac{2^{n-p+\frac{1}{2}}\,\Gamma(m+n+\frac{1}{2})}{\Gamma(\frac{1}{2})\,\Gamma(2m+n+1)\,\Gamma(m+p+\frac{1}{2})}$$

$$\times\int_{-1}^{1}\left\{z^n-\frac{n(n-1)}{2(2m+2n-1)}z^{n-2}+\cdots\right\}\frac{d^p}{d\mu^p}\left\{(1-\mu^2)^{m+p-\frac{1}{2}}\right\}d\mu$$

$$=(-1)^p\frac{2^{n-p+\frac{1}{2}}\,\Gamma(m+n+\frac{1}{2})\,n!}{\Gamma(\frac{1}{2})\,\Gamma(2m+n+1)\,\Gamma(m+p+\frac{1}{2})\,(n-p)!}(x^2-1)^{\frac{1}{2}p}(y^2-1)^{\frac{1}{2}p}$$

$$\times\int_{-1}^{1}(1-\mu^2)^{m+p-\frac{1}{2}}\left\{z^{n-p}-\frac{(n-p)(n-p-1)}{2(2m+2n-1)}z^{n-p-2}+\cdots\right\}d\mu.$$

Now

$$\int_{-1}^{1}(1-\mu^2)^{m+p-\frac{1}{2}}\,z^{n-p-2r}\,d\mu$$

$$=\int_{-1}^{1}(1-\mu^2)^{m+p-\frac{1}{2}}\left\{(xy)^{n-p-2r}-\frac{(n-p-2r)(n-p-2r-1)}{2!}(xy)^{n-p-2r-2}(x^2-1)(y^2-1)\mu^2+\cdots\right\}d\mu$$

$$= B(m+p+\tfrac{1}{2}, \tfrac{1}{2})\,(xy)^{n-p-2r}$$

$$- \frac{(n-p-2r)(n-p-2r-1)}{2!} B(m+p+\tfrac{1}{2}, \tfrac{3}{2})(xy)^{n-p-2r-2}(x^2-1)(y^2-1) + \cdots$$

Thus the integral on the left of (12) is equal to

$$(-1)^p \frac{2^{n-p+\frac{1}{2}}\,\Gamma(m+n+\frac{1}{2})\,n!}{\Gamma(2m+n+1)\Gamma(m+p+1)(n-p)!}(xy)^{n-p}(x^2-1)^{\frac{1}{2}p}(y^2-1)^{\frac{1}{2}p}$$

$$\times \sum_{r=0} \frac{(\frac{p-n}{2};r)\,(\frac{p-n+1}{2};r)}{r!\,(\frac{1}{2}-m-n;r)}(xy)^{-2r} F\left\{\begin{matrix}\frac{p-n}{2}+r,\ \frac{p-n+1}{2}+r;\ \frac{(x^2-1)(y^2-1)}{(xy)^2}\\ m+p+1\end{matrix}\right\},$$

where $(\alpha;0)=1$, $(\alpha;r)=\alpha(\alpha+1)\cdots(\alpha+r-1)$.

But the last summation is equal to

$$\sum_{r=0}\sum_{s=0} \frac{(\frac{p-n}{2};r+s)(\frac{p-n+1}{2};r+s)}{r!\,s!\,(\frac{1}{2}-m-n;r)(m+p+1;s)}\,\frac{1}{(x^2y^2)^r}\left\{\frac{(x^2-1)(y^2-1)}{(xy)^2}\right\}^s$$

$$= F\left(\begin{matrix}\frac{p-n}{2}, \frac{p-n+1}{2}\\ \frac{1}{2}-m-n\end{matrix}; \frac{1}{x^2}\right) F\left(\begin{matrix}\frac{p-n}{2}, \frac{p-n+1}{2}\\ m+p+1\end{matrix}; 1-\frac{1}{y^2}\right),$$

[Bailey: Generalized hypergeometric Series, Cambridge Tracts in Math., p. 81, (2)].

From this [S.H., p. 319, ex 1, p. 321, (23)] formula (12) is obtained.

<u>Note</u>. Gegenbauer's addition formula*

$$(1-z^2)^{-\frac{1}{2}m}\,T_{m+n}^{-m}(z)=\frac{\sqrt{(2\pi)}}{\Gamma(2m+n+1)}(1-x^2)^{-\frac{1}{2}m}(1-y^2)^{-\frac{1}{2}m}$$

$$\times\sum_{p=0}^{n}\Gamma(2m+p)\,\Gamma(2m+n+p+1)\,(m+p)\;{}^{n}C_p\,(-1)^p$$

$$\times\,T_{m+n}^{-m-p}(x)\,T_{m+n}^{-m-p}(y)\,(1-\mu^2)^{\frac{1}{4}-\frac{1}{2}m}\,T_{m-\frac{1}{2}+p}^{\frac{1}{2}-m}(\mu),\quad(13)$$

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* <u>Wiener Sitzungsberichte</u>, LXX. (2), 443, 1874, and <u>Wiener Sitzungsberichte</u>, CII. (2), 949, 1893.

where $-1<\mu<1$, can now be deduced by using (12) to evaluate the integrals in the expansions of the L.H.S. in terms of $(1-\mu^2)^{\frac{1}{4}-\frac{1}{2}m}\,T^{\frac{1}{2}-m}_{m-\frac{1}{2}+p}(\mu)$, where $p=0,1,2,\ldots,n$.

[S.H., p. 336].

Again* if $\underline{p}$ and $\underline{n}$ are positive integers or zero such that $n \geqq p$, if

$$z = xy - \sqrt{(x^2-1)}\sqrt{(y^2-1)}\,\mu, \text{ and if } R(m) > -\tfrac{1}{2},$$

$$\int_{-1}^{1}(z^2-1)^{-\frac{1}{2}m}\,Q^{m}_{n+m}(z)\,(1-\mu^2)^{\frac{1}{2}m-\frac{1}{4}}\,T^{\frac{1}{2}-m}_{m-\frac{1}{2}+p}(\mu)\,d\mu =$$

$$\sqrt{2\pi}\,(x^2-1)^{-\frac{1}{2}m}\,(y^2-1)^{-\frac{1}{2}m}\,Q^{p+m}_{n+m}(x)\,P^{-p-m}_{n+m}(y), \text{ where } |z|>1. \tag{14}$$

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*Wiener Sitzungsberichte, LXX. (2), 443, 1874.

The proof of (14), runs on the same lines as the proof of (12) except that instead of using (1), the following formula is used for $(z^2-1)^{-\frac{1}{2}m} Q^{m}_{n+m}(z)$:

$$Q^{m}_{n+m}(z) = \frac{\Gamma(\frac{1}{2})\Gamma(n+2m+1)(z^2-1)^{\frac{1}{2}m}}{2^{m+n+1}\Gamma(n+m+\frac{3}{2})\, z^{n+2m+1}} F\left(\begin{matrix} m+\frac{n+1}{2}, m+1+\frac{n}{2} ; \\ n+m+\frac{3}{2} \end{matrix} \frac{1}{z^2}\right), \quad (15)$$

where $|z|>1$ [S.H., p. 316, (9)].

It will be found that the L.H.S. of (14) is equal to

$$\frac{\pi(xy)^{-(n+2m+p+1)}(x^2-1)^{\frac{p}{2}}(y^2-1)^{\frac{p}{2}}\Gamma(n+2m+p+1)}{2^{p+n+2m+\frac{1}{2}}\Gamma(n+m+\frac{3}{2})\Gamma(p+m+1)}$$

$$\times F\left(\begin{matrix} m+\frac{n+p+1}{2}, m+1+\frac{n+p}{2} ; \\ n+m+\frac{3}{2} \end{matrix} \frac{1}{x^2}\right) F\left(\begin{matrix} m+\frac{n+p+1}{2}, m+1+\frac{n+p}{2} ; \\ p+m+1 \end{matrix} 1-\frac{1}{y^2}\right)$$

$$= \sqrt{2\pi}\,(x^2-1)^{-\frac{1}{2}m}(y^2-1)^{-\frac{1}{2}m} Q^{p+m}_{n+m}(x)\, P^{-p-m}_{n+m}(y),$$

by (15) and S.H., p. 321, (23).

<u>Note</u>. Gegenbauer's second addition formula*

$$(z^2-1)^{-\frac{1}{2}m} Q^m_{n+m}(z) = \sqrt{(2\pi)}\,(x^2-1)^{-\frac{1}{2}m}(y^2-1)^{-\frac{1}{2}m}$$

$$\times \sum_{p=0}^{\infty} (p+m)\frac{\Gamma(p+2m)}{p!}\, Q^{p+m}_{n+m}(x)\, T^{-m-p}_{n+m}(y)\,(1-\mu^2)^{\frac{1}{4}-\frac{1}{2}m}\, T^{\frac{1}{2}-m}_{m-\frac{1}{2}+p}(\mu), \quad (16)$$

where $-1<\mu<1$, $|z|>1$, can now be deduced by using (15) to evaluate the integrals in the expansion of the L.H.S. in terms of $(1-\mu^2)^{\frac{1}{4}-\frac{1}{2}m}\, T^{\frac{1}{2}-m}_{m-\frac{1}{2}+p}(\mu)$, where $p=0,1,2,\ldots,\infty$. (S.H., p. 336).

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* Wiener Sitzungsberichte, LXX. (2) (1874), 443.

Next, the formula*

$$\sum_{m=0}^{\infty} \frac{T_{n+m}^{-m}(x)}{n!} = e^{x} J_{m}\{\sqrt{(1-x^2)}\}, \tag{17}$$

now can easily be proved by substituting from (4), and changing the order of integration and summation.

Thus, if $R(m) > -\frac{1}{2}$,

$$\sum_{m=0}^{\infty} \frac{T_{n+m}^{-m}(x)}{n!} = \frac{(1-x^2)^{\frac{1}{2}m} e^{x}}{2^{m}\Gamma(\frac{1}{2})\Gamma(m+\frac{1}{2})} \int_0^{\infty} e^{i\sqrt{(1-x^2)}\cos\phi} (\sin\phi)^{2m}\, d\phi$$

$$= e^{x} J_{m}\{\sqrt{(1-x^2)}\},$$

by [G.V., p. 267]. The condition $R(m) > -\frac{1}{2}$ can be removed by analytical continuation.

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* Denkschriften der Kaiserlichen Akademie Der Wissenschaften, LVII., p. 471, 1890.

Finally* the determinant

$$\begin{vmatrix} P^{-m}_{n+m}(x) & P^{-m}_{n+m-1}(x) \\ Q^{-m}_{n+m}(x) & Q^{-m}_{n+m-1}(x) \end{vmatrix} = \frac{(n-1)!}{\Gamma(n+2m+1)}, \tag{18}$$

where <u>n</u> is a positive integer, was proved by Gegenbauer by using the theory of continued fractions. It can be proved by using the recurrence relations for the Legendre functions, for it is known that

$$(2n+1)z\,Q^m_n(z) = (n-m+1)\,Q^m_{n+1}(z) + (n+m)\,Q^m_{n-1}(z),$$
$$(2n+1)\,\xi\,P^{-m}_n(\xi) = (n+m+1)\,P^{-m}_{n+1}(\xi) + (n-m)\,P^{-m}_{n-1}(\xi).$$

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* Denkschriften der Kaiserlichen Akademie der Wissenschaften, XLVIII., p. 304, equ. (85), 1884.

Thus multiplying the first by $P_n^{-m}(\xi)$ and the second by $Q_n^m(z)$, substracting and writing $(n+m)$ for (n), we have

$$(2m+2n-1)(z-\xi)P_{m+n}^{-m}(\xi)\,Q_{m+n}^{m}(z) =$$

$$\left\{(n+1)P_{m+n}^{-m}(\xi)\,Q_{m+n+1}^{m}(z) - (2m+n+1)P_{m+n+1}^{-m}(\xi)\,Q_{m+n}^{m}(z)\right\}$$

$$-\left\{n\,P_{m+n-1}^{-m}(\xi)\,Q_{m+n}^{m}(z) - (2m+n)\,P_{m+n}^{-m}(\xi)\,Q_{m+n-1}^{m}(z)\right\}.$$

If $z=\xi=x$, the L.H.S. of this equation is zero. Thus giving $\underline{n}$ the values $(n-1)$, $(n-2)$, $1, \ldots, 0$, in succession, it is seen that any quantity between $\{\ \}$ with $z=\xi=x$ is equal to

$$-2m\,P_m^{-m}(x)\,Q_{m-1}^{m}(x) = -1, \qquad (B)$$

since

$$P_m^{-m}(x) = (x^2-1)^{\frac{1}{2}m}\, 2^{-m}/\Gamma(m+1) \quad ,$$

$$Q_{m-1}^{m}(x) = (x^2-1)^{-\frac{1}{2}m}\, 2^{m-1}\,\Gamma(m) .$$

Expanding the determinant and changing $Q_{n+m}^{-m}(x)$, $Q_{n+m-1}^{-m}(x)$ to $Q_{n+m}^{m}(x)$ and $Q_{n+m-1}^{m}(x)$ by the relation

$$\frac{Q_n^m(x)}{\Gamma(n+m+1)} = \frac{Q_n^{-m}(x)}{\Gamma(n-m+1)} \text{, and using (B),}$$

(18) is obtained.

APPENDIX

SOME PARTICULAR CASES OF E-FUNCTION FORMULAE

§1. Introductory. In this appendix, it will be shown that various known formulae are particular cases of E-function formulae. Simple proofs, by the E-function formulae, of some ~~known~~ known formulae are also given.

§2. Formulae and proofs. In the first place, Lommel's function $S_{\mu,\nu}(z)$, defined when either of the numbers $\mu \mp \nu$ is an odd positive integer [see Watson Bessel Functions, p. 347] is a particular case of an E-function.

In fact the following equations holds

$$S_{\mu,\nu}(z) = \frac{z^{\mu-1}}{\Gamma(\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu)\Gamma(\frac{1}{2}-\frac{1}{2}\mu+\frac{1}{2}\nu)} E\left(\tfrac{1}{2}-\tfrac{1}{2}\mu-\tfrac{1}{2}\nu, \tfrac{1}{2}-\tfrac{1}{2}\mu+\tfrac{1}{2}\nu, \overset{\text{one}}{1};:\frac{z^2}{4}\right), \quad (1)$$

where $\mu-\nu$ or $\mu+\nu$ is an odd positive integer.
Its recurrence formulae stated in Watson's p. 348 are particular cases of recurrence formulae for the E-function. Also many formulae for this function obtained by Meijer in the Proc. of Akademie te Amsterdam (Vol. $\overline{XLI}$., 1938; and Vol. $\overline{XXXVIII}$., 1935, etc...) are particular cases of, or can be easily proved, by E-function formulae. For example, the formula

$$S_{\mu,\nu}(z) = \frac{2^{\mu}}{z}\int_0^{\infty} e^{-v+\frac{z^2}{8v}}\, W_{\frac{1}{2}\mu,\frac{1}{2}\nu}\left(\frac{z^2}{4v}\right) v^{\frac{1}{2}\mu}\, dv. \quad (2)$$

[Meijer, Proc. Akad. te Amsterdam $\overline{XXXVIII}$., p. 630, 1935.]

is simply a particular case of (13) CH. II.

Also the properties and asymptotic expansions of functions which are expressed in terms of $S_{\mu,\nu}(z)$ such as Neuman's polynomials $O_n(z)$, Schläfli's polynomial $S_n(z)$ and Gegenbauer's polynomial $A_{n,m}(z)$, [see Watson p. 350, 351], can easily be obtained from E-functions' theorems and formulae.

Again in (5) CH. III. take $p=1$, $q=0$, and then using the formula

$$E(\alpha::z) = \Gamma(\alpha)(1+1/z)^{-\alpha}, \tag{3}$$

[C.-V., p. 353], it can be seen that

$$\int_0^\infty \frac{x^{\rho-1}J_\nu(ax)}{(x^2+k^2)^{\mu+1}}\,dx = \left(\frac{2}{a}\right)^{\rho}\frac{1}{\Gamma(\mu+1)\,k^{2\mu+2}}\int_0^\infty x^{\rho-1}J_\nu(2x)E\left(\mu+1::\frac{a^2k^2}{4x^2}\right)dx$$

$$= \frac{a^\nu k^{\rho+\nu-2\mu-2}\,\Gamma(\tfrac{1}{2}\rho+\tfrac{1}{2}\nu)\,\Gamma(\mu+1-\tfrac{1}{2}\rho-\tfrac{1}{2}\nu)}{2^{\nu+1}\,\Gamma(\mu+1)\,\Gamma(\nu+1)}\,{}_1F_2\left(\frac{\rho+\nu}{2};\frac{\rho+\nu}{2}-\mu,\nu+1;\frac{a^2k^2}{4}\right)$$

$$+\frac{a^{2\mu+2-\rho}\,\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}\rho-\mu-1)}{2^{2\mu+3-\rho}\,\Gamma(\mu+2+\tfrac{1}{2}\nu-\tfrac{1}{2}\rho)}\,{}_1F_2\left(\mu+1;\mu+2+\frac{\nu-\rho}{2},\mu+2-\frac{\nu+\rho}{2};\frac{a^2k^2}{4}\right), \quad (4)$$

where $\underline{a}$ is real and positive and $-R(\nu) < R(\rho) < 2R(\mu)+\frac{7}{2}$.
This is Hankel's integral and is proved in Watson p. 434 where many special cases are deduced.
Another application of (4) is to evaluate the integral

$$\int_0^\infty \left[\sin\left(\frac{\rho+\nu}{2}-\mu\right)\pi\, J_\nu(ax) - \sin\left(\frac{\rho-\nu}{2}-\mu\right)\pi\, J_{-\nu}(ax)\right]\frac{x^{\rho-1}}{(x^2+k^2)^{\mu+1}}$$

$= \sin\left(\frac{\rho+\nu}{2}-\mu\right)\pi \displaystyle\int_0^\infty \frac{x^{\rho-1}J_\nu(ax)}{(x^2+k^2)^{\mu+1}}\,dx$ − same expression with $\underline{-\nu}$ instead of $\underline{\nu}$.

Substituting from (4), it is found that the second and the fourth series cancel, and so, introducing the function $Y_\nu(z)$ defined by the equation

$$Y_\nu(z) = \frac{J_\nu(z)\cos\nu\pi - J_{-\nu}(z)}{\sin\nu\pi}, \tag{5}$$

it is found that

$$\int_0^\infty \left[\cos\left(\frac{\rho-\nu}{2}-\mu\right)\pi\, J_\nu(ax) + \sin\left(\frac{\rho-\nu}{2}-\mu\right)\pi\, Y_\nu(ax)\right]\frac{x^{\rho-1}\,dx}{(x^2+k^2)^{\mu+1}}$$

$$= \frac{\pi\, k^{\rho-2\mu-2}}{2\sin\nu\pi\,\Gamma(\mu+1)}$$

$$\times\left[\frac{(\frac{1}{2}ak)^{\nu}\,\Gamma(\frac{\rho+\nu}{2})}{\Gamma(\nu+1)\Gamma(\frac{\rho+\nu}{2}-\mu)}\,{}_1F_2\left(\frac{\rho+\nu}{2};\ \frac{\rho+\nu}{2}-\mu,\ \nu+1;\ \frac{a^2k^2}{4}\right)\right.$$

$$\left.-\frac{(\frac{1}{2}ak)^{-\nu}\,\Gamma(\frac{\rho-\nu}{2})}{\Gamma(1-\nu)\Gamma(\frac{\rho-\nu}{2}-\mu)}\,{}_1F_2\left(\frac{\rho-\nu}{2};\ \frac{\rho-\nu}{2}-\mu,\ 1-\nu;\ \frac{a^2k^2}{4}\right)\right], \tag{6}$$

which is in Watson p. 428, (1).

Also in (10) CH. III. take $p=1$, $q=0$, $\alpha_1=\mu+1$ and $r=1$; then it can be seen after long reduction

$$\int_0^\infty \frac{x^{\rho-1} J_\nu(ax)}{(x^4+4k^4)^{\mu+1}}\,dx = \frac{\frac{1}{2}\pi\,(k\sqrt{2})^{\rho-4\mu-4}}{\sin\left(\frac{\rho+\nu}{2}-2\mu\right)\pi\,\Gamma(\mu+1)}$$

$$\times\left[\sum_{m=0}^{\infty}\frac{(ak/\sqrt{2})^{\nu+2m}\,\Gamma\left(\frac{\rho+\nu}{4}+\frac{1}{2}m\right)}{m!\,\Gamma(\nu+m+1)\,\Gamma\left(\frac{\rho+\nu}{4}-\mu+\frac{1}{2}m\right)}\cos\left(\frac{\rho+\nu}{4}-\mu+\frac{1}{2}m\right)\pi+\right.$$

$$\left.\sum_{m=0}^{\infty}\frac{(-1)^m\,(ak/\sqrt{2})^{4\mu+4-\rho+4m}\,\Gamma(\mu+m+1)}{m!\,\Gamma\left(2\mu-\frac{1}{2}\rho+\frac{1}{2}\nu+2m+3\right)\Gamma\left(2\mu-\frac{1}{2}\rho-\frac{1}{2}\nu+2m+3\right)}\right], \quad (4)$$

where $4R(\mu)+\frac{11}{2}>R(\rho)>-R(\nu)$ which is in Watson p. 435.

A special case of interest of (5) CH. III. is the following formula

$$\int_0^\infty y^{\nu} J_{\nu-1}(2y) E\left(\alpha,\beta;\nu: \frac{x^2}{y^2}\right) dy = x^{\alpha+\beta} K_{\beta-\alpha}(2x), \tag{8}$$

where $R(\nu)>0$, $R(\alpha) \geqslant R(\beta) > \frac{1}{2}\nu - \frac{1}{4}$.

This can easily be obtained by substituting in (5) CH. III. and noting that the last series vanishes because of

$$\frac{1}{\Gamma(\rho_t - \alpha_r)} = \frac{1}{\Gamma(\nu-\nu)} = \frac{1}{\Gamma(0)},$$

since $\alpha_1 = \alpha$, $\alpha_2 = \beta$, $\alpha_3 = \nu$, $\alpha_4 = 1$, $\rho_1 = \nu$.

Formula (8) was stated in another form with $F\left(\alpha,\beta;\nu;\frac{-y^2}{x^2}\right)$ instead of $\frac{\Gamma(\nu)}{\Gamma(\alpha)\Gamma(\beta)} E\left(\alpha,\beta;\nu:\frac{x^2}{y^2}\right)$ by N. Bose, Mathem-atiche Zeitschrift, $\overline{\underline{LIV}}$., p. 162, (6), 1951.

Another formula obtained by Bose [Loc. Cit., p. 161, (3)] is the following formula

$$\int_0^1 (1+t)^{-\beta} E(\alpha,\beta :: k(t+1))\,dt = E(\alpha,\beta-1 :: t), \tag{9}$$

where $R(\beta) > 1$. This is clearly a particular case of (15) CH. III. Again, the formula

$$\int_0^\infty \frac{x^\nu K_\nu(ax)}{x^2+k^2}\,dx = \frac{\pi^2 k^{\nu-1}}{4\cos\nu\pi}\left\{H_{-\nu}(ak) - \Upsilon_{-\nu}(ak)\right\}, \tag{10}$$

where $R(\nu) > -\frac{1}{2}$, [Watson, p. 426 (9)], and $H_{-\nu}(x)$ is Struve's function [Watson p. 328] and $\Upsilon_{-\nu}(x)$ is given by (5), is a particular case of (1) CH. III. In fact the integral is equal to $(2/a)^{\nu+1}\frac{1}{k^2}E(1,\frac{1}{2},\nu+\frac{1}{2} :: a^2k^2/4)$ which gives the three hypergeometric series on the R.H.S. of (10).

If the E-functions in (52) CH. III. are changed to Whittaker's function $W_{k,m}(x)$ and Weber's function $D_{-m}(x)$ by the formulae

$$E\left(\tfrac{1}{2}-k+m,\ \tfrac{1}{2}-k-m :: x\right) = \Gamma\left(\tfrac{1}{2}-k+m\right)\Gamma\left(\tfrac{1}{2}-k-m\right) x^{-k} e^{\frac{1}{2}x} W_{k,m}(x), \quad (11)$$

[E.V., p. 351, (15)] and

$$D_{-m}(x) = 2^{\frac{1}{4}-\frac{1}{2}m} x^{-\frac{1}{2}} W_{\frac{1}{4}-\frac{1}{2}m,-\frac{1}{4}}\left(\frac{x^2}{2}\right), \quad (12)$$

the following formulae can be obtained:

$$\int_0^\infty e^{\frac{1}{2}s} J_\nu\left(2\sqrt{(xs)}\right) s^{\frac{1}{2}\nu-m-\frac{1}{2}} W_{k,m}(s)\, ds =$$

$$\frac{\Gamma(\nu-2m+1)}{\Gamma\left(\frac{1}{2}-k+m\right)} x^{\frac{1}{2}\left(m-k-\frac{3}{2}\right)} e^{\frac{1}{2}x} W_{\frac{1}{2}\left(k+3m-\nu-\frac{1}{2}\right),\ \frac{1}{2}\left(\nu+k-m+\frac{1}{2}\right)}(x), \quad (13)$$

where $R(\nu)>-1$, $R(\nu-2m)>-1$, $R(2m-2k-\nu)>-\frac{1}{2}$,
which was proved by the method of Hankel transformation by Erdelyi [Proc. of the Camb. Philo. Soc., XXXIV, part (3) p. 28, 1938], and

$$\int_0^\infty \xi^{n+\frac{1}{2}} e^{\frac{\xi^2}{2a}} D_{-2m-1}\left(\sqrt{\tfrac{2}{a}}\,\xi\right) J_{n+\frac{1}{2}}(2\xi)\,d\xi$$

$$= \frac{\Gamma(n+1)}{\Gamma(m+1)} e^{\frac{a}{2}} a^{\frac{1}{2}(m+n+\frac{1}{2})} 2^{-(m+\frac{3}{2})} W_{-\frac{1}{2}(n+m+\frac{1}{2}),\,\frac{1}{2}(n-m+\frac{1}{2})}(a), \quad (14)$$

where $R(m) > -\frac{1}{2}$, $R(n) > -1$, which was obtained by Shastri, [Mathematische Zeitschrift, XLIV., p. 490, 1939]. In the same way, the formula

$$\int_0^\infty x^{n-\frac{1}{2}} e^{\frac{x^2}{4}} J_{n-\frac{1}{2}}(ax) D_{-m}(x)\,dx = \frac{(2a)^{n-\frac{1}{2}}\Gamma(n)}{\sqrt{\pi}\,\Gamma(m)} \int_0^\infty e^{-\frac{1}{2}x^2} x^{m-1}(x^2+a^2)^{-n}\,dx, \quad (15)$$

can be deduced. This was obtained by Varma [<u>Proc. of the Camb. Phil. Soc.</u> XXXIII., p. 210, 1937], who did not put $\int_0^\infty e^{-\frac{1}{2}x^2} x^{m-1}(x^2+a^2)^{-n}\,dx$ in the form of E-function; it was evaluated by Jeffreys in terms of two Kummer function ${}_1F_1$.

§3. Some more formulae: In this section, many formulae will be proved by the help of the formula

$$\int_0^\infty e^{-t} t^{\gamma-1} E(\alpha, \beta :: tz)\,dt = \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\gamma)\Gamma(\beta+\gamma)}{\Gamma(\alpha+\beta+\gamma)} F\left(\begin{matrix}\alpha, \beta \\ \alpha+\beta+\gamma\end{matrix}; \frac{z-1}{z}\right), \quad (16)$$

where $R(\alpha+\gamma)>0$, $R(\beta+\gamma)>0$, $R(z)>\frac{1}{2}$, [C.V., eq 113, p. 381].

put $z = 2$, $\alpha = \frac{1}{2}+n$, $\beta = \frac{1}{2}-n$; then summing the hypergeometric on the right by the formula

$$F\left(\begin{matrix}\alpha, 1-\alpha \\ c\end{matrix}; \frac{1}{2}\right) = \frac{\Gamma(\frac{1}{2}c)\Gamma(\frac{1}{2}c+\frac{1}{2})}{\Gamma(\frac{1}{2}c+\frac{1}{2}\alpha)\Gamma(\frac{1}{2}+\frac{1}{2}c-\frac{1}{2}\alpha)}, \quad (17)$$

[Bailey, Camb. Tracts on Math., p. 11], and changing the E-function to the modified Bessel function $K_n(t)$ by (36) CH. III., (16) becomes

$$\int_0^{\infty} K_n(t)\, t^{\mu-1}\, dt = 2^{\mu-2}\,\Gamma\left(\frac{\mu-n}{2}\right)\Gamma\left(\frac{\mu+n}{2}\right), \tag{18}$$

where $R(\mu \pm n) > 0$, which is a known formula, (4) CH. II.

In certain cases if $z = \frac{1}{2}$, the hypergeometric function on the right of (16) can be summed by the formula

$$F\left[\begin{matrix}\alpha, \beta\ ; -1 \\ 1+\alpha-\beta\end{matrix}\right] = \frac{\Gamma(1+\alpha-\beta)\,\Gamma(1+\frac{1}{2}\alpha)}{\Gamma(1+\alpha)\,\Gamma(1+\frac{1}{2}\alpha-\beta)}, \tag{19}$$

[Bailey, Camb. Tracts on math., p. 9].

Thus if, in (16), $\gamma = 1-2\beta$, $z = \frac{1}{2}$, it becomes

$$\int_0^{\infty} e^{-t}\, t^{-2\beta}\, E(\alpha, \beta :: \tfrac{1}{2}t)\, dt = 2^{\alpha-2\beta-1}\sqrt{\pi}\,\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha}{2}-\beta+\frac{1}{2}\right)\operatorname{cosec}\pi\beta, \tag{20}$$

where $R(1-\beta) > 0$, $R(\alpha-2\beta+1) > 0$.

If, in (20), t^2 is written for t and $\alpha = -\frac{1}{2}m - \frac{1}{2}$, $\beta = -\frac{1}{2}m$ and the E-function is changed to $D_{m+1}(t)$ by (11) and (12), then it becomes

$$\int_0^\infty e^{-\frac{3}{4}t^2} t^m D_{m+1}(t)\,dt = (\sqrt{2})^{-1-m}\,\Gamma(m+1)\sin\left(\tfrac{1}{4}-\tfrac{1}{4}m\right)\pi, \tag{21}$$

where $R(m) > -1$.

This was proved by Watson [Proc. Lond. Math. Soc., Ser. 2, VIII., pp 393 – 427].

Again the formula

$$\int_0^\infty W_{\mu-2m,\mu}(t)\,t^{\mu-1}e^{-\frac{1}{2}t}\cos\{2\sqrt{xt}\}\,dt = \frac{\sqrt{\pi}\,\Gamma(2\mu+\frac{1}{2})}{\Gamma(2m+1)}\,M_{2\mu-m,m}(x)\,x^{-m-\frac{1}{2}}\,e^{-\frac{x}{2}}, \tag{22}$$

where for convergence $R(2\mu+\frac{1}{2})>0$, was obtained by A. Erdelyi [Compositio Mathematica, IV., p. 419, (5.6)] by using the Jacobian transformation of the Theta function. It can be proved by using (16) as follows

Expand $\cos\{2\sqrt{xt}\}$ in a series and change $W_{\mu-2m,\mu}(t)$ to an E-function by (11) and integrate term by term after changing the order of integration and summation, then the L.H.S. of (22) becomes

$$\frac{1}{\Gamma(\frac{1}{2}+2m)\Gamma(\frac{1}{2}+2m-2\mu)}\int_0^\infty e^{-t}t^{2\mu-2m-1}E(\tfrac{1}{2}+2m,\tfrac{1}{2}+2m-2\mu::t)\cos\{2\sqrt{xt}\}\,dt$$

$$=\frac{1}{\Gamma(\frac{1}{2}+2m)\Gamma(\frac{1}{2}+2m-2\mu)}\sum_{r=0}^{\infty}\frac{(-1)^r(2\sqrt{x})^r}{(2r)!}\int_0^\infty e^{-t}t^{2\mu-2m+r-1}E(\tfrac{1}{2}+2m,\tfrac{1}{2}+2m-2\mu::t)\,dt$$

Using (16) with $z=1$, $\alpha=\frac{1}{2}+2m$, $\beta=\frac{1}{2}+2m-2\mu$, and $\gamma=2\mu-2m+r$, it is found that the L.H.S. of (22) is equal to

$$\frac{\sqrt{\pi}\,\Gamma(2\mu+\frac{1}{2})}{\Gamma(2m+1)}F(2\mu+\tfrac{1}{2};1+2m;-x).$$

Here use (28) CH. III. and the formula

$$M_{k,m}(z)=z^{m+\frac{1}{2}}e^{-\frac{1}{2}z}F(\tfrac{1}{2}-k+m;1+2m;z) \qquad (23)$$

[C.V., p. 351, 16], and (22) is obtained.

Similarly the formula

$$\int_0^\infty e^{-\frac{1}{2}y}\, y^{\mu+\frac{1}{2}\nu-\frac{1}{2}}\, W_{m+\nu+\frac{1}{2},m}(y)\, J_{2m+2\mu+\nu}\{2\sqrt{xy}\}\,dy$$

$$= \frac{\Gamma(2\mu+\nu+1)}{\Gamma(2\mu+1)}\, x^{m+\frac{1}{2}\nu-\frac{1}{2}}\, e^{-\frac{1}{2}x}\, M_{\mu+\nu+\frac{1}{2},\mu}(x), \qquad (24)$$

where $R(2m+2\mu+\nu) > -1$, $R(3\mu+\nu) > -1$,
[Erdelyi, Journ. of the Lond. Math. Soc. XIII., pp. 146–154, (207), 1938], can be proved as was (22) by expanding the Bessel function instead of the trignometric function and using (23) and (28) CH. III.

In the same way; i.e. by expanding the Bessel function and integrating term by term using (16) with $z=1$ after changing the order of integration and summation;

it is easy to obtain the formula

$$\int_0^\infty t^{l-1} e^{-\frac{1}{2}t} I_\nu(2a\sqrt{t})\, W_{k,m}(t)\, dt =$$

$$\frac{\Gamma(\frac{1}{2}\nu+l+m+\frac{1}{2})\,\Gamma(\frac{1}{2}\nu+l-m+\frac{1}{2})}{\Gamma(\nu+1)\,\Gamma(\frac{1}{2}\nu+l-k+1)}\, a^\nu {}_2F_2\left(\begin{matrix}\frac{1}{2}\nu+l+m+\frac{1}{2}, \frac{1}{2}\nu+l-m+\frac{1}{2}; a^2\\ \nu+1, \frac{1}{2}\nu+l-k+1\end{matrix}\right), \quad (25)$$

where $R(l+\frac{1}{2}\nu-m+\frac{1}{2})>0$.

This was proved by Goldstein [Proc. Lond. Math. Soc., Ser. 2., XXXIV., p. 121, (91), 1934].

Also in (16), if $\alpha+\beta+\gamma=0$, the R.H.S. vanishes because $\frac{1}{\Gamma(0)}$, and the formula becomes after changing the E-function to Whittaker's function

$$\int_0^\infty e^{-\frac{1}{2}t} t^{k-2}\, W_{k,m}(t)\, dt = 0, \quad (26)$$

[Goldstein, Loc. Cit., p. 113, (50)].

Again, if in (16), $\gamma = 2k$, $\alpha = \frac{1}{2} - k + m$, $\beta = \frac{1}{2} - k - m$, it becomes

$$\int_0^\infty x^{k-1} e^{-\frac{1}{2}x} W_{k,m}(x)\, dx = \Gamma(\tfrac{1}{2}+k+m)\,\Gamma(\tfrac{1}{2}+k-m), \qquad (27)$$

where $R(k \pm m + \frac{1}{2}) > 0$, [Goldstein, Loc. Cit., p. 113, (52)].

If, in (16) $t(1+z)$ is written for t, $\frac{2}{1+z}$ for z, $(\frac{1}{2}+n)$ for α, $(\frac{1}{2}-n)$ for β and $(m-\frac{1}{2})$ for γ, then on applying (36) CH. III. and the formulae

$$P_n^m(z) = \frac{1}{\Gamma(1-m)} \left(\frac{z+1}{z-1}\right)^{\frac{1}{2}m} F\left(-n, n+1; 1-m; \frac{1-z}{2}\right), \qquad (28)$$

[S.H., p. 314, (3)],

$$T_n^m(z) = \frac{1}{\Gamma(1-m)} \left(\frac{z+1}{1-z}\right)^{\frac{1}{2}m} F\left(-n, n+1; 1-m; \frac{1-z}{2}\right), \qquad (29)$$

[S.H., p. 314, (4)], (16) becomes

$$\int_0^\infty e^{-tz} K_n(t)\, t^{m-1}\,dt = \sqrt{\frac{\pi}{2}}\,\Gamma(m+n)\,\Gamma(m-n) \begin{cases} (z^2-1)^{\frac{1}{4}-\frac{1}{2}m}\, P_{n-\frac{1}{2}}^{\frac{1}{2}-m}(z), & |z|>1 \\ (1-z^2)^{\frac{1}{4}-\frac{1}{2}m}\, T_{n-\frac{1}{2}}^{\frac{1}{2}-m}(z), & |z|<1 \end{cases}, \quad (30)$$

where $R(m\pm n)>0$, $R(z)>0$.

This was proved by MacRobert [C.V. ex 62 p. 372].

Finally some formulae for Bateman's k-function and $W_{k,m}(x)$ obtained by Shabade [Bulletin, Calcutta, Math., Soc., $\overline{\text{XXIII}}$, p. 133, 1931], are easily obtained by the help of (16).

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AN INTEGRAL INVOLVING THE PRODUCT OF A BESSEL FUNCTION AND AN E-FUNCTION

by FOUAD M. RAGAB

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The formula *

$$4\int_0^\infty \lambda^{m-1}K_n(2\lambda)E(p\,;\alpha_r:q\,;\rho_s:x\lambda^{-2})\,d\lambda$$
$$=E(p+2\,;\alpha_r:q\,;\rho_s:x),\ \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where $\alpha_{p+1}=\frac{1}{2}(m+n)$, $\alpha_{p+2}=\frac{1}{2}(m-n)$, $R(m\pm n)>0$ and x is real and positive, was given by MacRobert (*Phil. Mag.*, Ser. 7, XXXI, p. 258). From it the formula (6) below will be deduced.

In (1) let it be assumed that $R(m\pm n)>0$, $R(\frac{3}{2}-m+2\alpha_r)>0$, $r=1, 2, \ldots, p$, and let amp λ decrease by $\frac{1}{2}\pi$, amp x decreasing simultaneously by π, finally writing λ/i in place of λ and $xe^{-i\pi}$ in place of x: then, noting that

$$K_n(t)=i^nG_n(it),\ \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

we have

$$4i^{n-m}\int_0^\infty \lambda^{m-1}G_n(2\lambda)E(p\,;\alpha_r:q\,;\rho_s:x\lambda^{-2})\,d\lambda$$
$$=E(p+2\,;\alpha_r:q\,;\rho_s:xe^{-i\pi}).\ \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Similarly, on increasing amp λ by $\frac{1}{2}\pi$ and amp x by π, we have

$$4i^{n+m}\int_0^\infty \lambda^{m-1}G_n(2\lambda e^{i\pi})E(p\,;\alpha_r:q\,;\rho_s:x\lambda^{-2})\,d\lambda$$
$$=E(p+2\,;\alpha_r:q\,;\rho_s:xe^{i\pi})\ \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Hence, on applying the formula

$$\pi iJ_n(t)=G_n(t)-i^{2n}G_n(te^{i\pi}),\ \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

it is found that

$$4i\pi\int_0^\infty \lambda^{m-1}J_n(2\lambda)E(p\,;\alpha_r:q:\rho_s:x\lambda^{-2})\,d\lambda$$
$$=i^{m-n}E(p+2\,;\alpha_r:q\,;\rho_s:xe^{-i\pi})$$
$$-i^{-m+n}E(p+2\,;\alpha_r:q\,;\rho_s:xe^{i\pi}),\ \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

where $R(m+n)>0$, $R(\frac{3}{2}-m+2\alpha_r)>0$, $r=1, 2, \ldots, p$, and x is real and positive.

In particular, if $p\geqq q-1$, formula (6) can be written

$$2\pi\int_0^\infty \lambda^{m-1}J_n(2\lambda)E(p\,;\alpha_r:q\,;\rho_s:x\lambda^{-2})\,d\lambda$$
$$=\sum_{r=1}^{p+1}\frac{\prod_{s=1}^{p+2}{}' \Gamma(\alpha_s-\alpha_r)}{\prod_{t=1}^{q}\Gamma(\rho_t-\alpha_r)}\Gamma(\alpha_r)\sin\left(\tfrac{1}{2}m-\tfrac{1}{2}n-\alpha_r\right)\pi\, x^{\alpha_r}$$
$$\times F\left\{\begin{matrix}\alpha_r,\ \alpha_r-\rho_1+1,\ \ldots,\ \alpha_r-\rho_q+1\ ;\ (-1)^{p-q+1}x\\ \alpha_r-\alpha_1+1,\ \ldots * \ldots,\ \alpha_r-\alpha_{p+2}+1\end{matrix}\right\},\ \ldots\ldots\ldots(7)$$

where $R(m+n)>0$, $R(\frac{3}{2}-m+2\alpha_r)>0$, $r=1, 2, \ldots, p$, and x is real and positive. It should be noted that the $(p+2)$th term on the right of (7) does not appear because $\alpha_{p+2}=\frac{1}{2}(m-n)$.

If $m=\beta+1$, $n=\beta-1$ and $\rho_q=\beta$, then

$$\alpha_{p+1}=\beta,\ \alpha_{p+2}=1$$

and

$$\alpha_r-\alpha_{p+1}+1=\alpha_r-\beta+1,$$

which cancels $\alpha_r-\rho_q+1$ on the right of (7).

Also

$$\alpha_r-\alpha_{p+2}+1=\alpha_r,$$

which cancels α_r on the right of (7).

* For the properties of the E-functions see MacRobert, *Functions of a Complex Variable*, third edition.

Again, $\rho_q - \alpha_{p+1} = 0$, so that

$$\frac{1}{\Gamma(\rho_q - \alpha_{p+1})} = 0,$$

and therefore the last term on the right of (7) disappears.

Finally, noting that

$$\Gamma(\alpha_{p+2} - \alpha_r)\,\Gamma(\alpha_r) = \frac{\pi}{\sin(\alpha_r \pi)},$$

that

$$\sin(\tfrac{1}{2}m - \tfrac{1}{2}n - \alpha_r)\pi = \sin(\alpha_r \pi),$$

and that

$$\frac{\Gamma(\alpha_{p+1} - \alpha_r)}{\Gamma(\rho_q - \alpha_r)} = 1,$$

we have, if $p \geqq q-1$,

$$2\int_0^\infty \lambda^\beta J_{\beta-1}(2\lambda)\,E(p\,;\,\alpha_r : q\,;\,\rho_s : x\lambda^{-2})\,d\lambda$$

$$= \sum_{r=1}^{p} \frac{\prod\limits_{s=1}^{p}{}' \,\Gamma(\alpha_s - \alpha_r)}{\prod\limits_{t=1}^{q-1} \Gamma(\rho_t - \alpha_r)}\, x^{\alpha_r} F\left(\begin{matrix}\alpha_r - \rho_1 + 1, \,\dots\,, \alpha_r - \rho_{q-1} + 1\,;\; (-1)^{p-q+1} x \\ \alpha_r - \alpha_1 + 1, \,\dots * \dots\,, \alpha_r - \alpha_p + 1\end{matrix}\right), \quad \text{............(8)}$$

where $\rho_q = \beta$, $R(\beta) > 0$, $R(\tfrac{1}{2} - \beta + 2\alpha_r) > 0$, $r = 1, 2, \dots, p$.

It should be noted that β does not appear on the right of (8).

This result was given, for the case $p = q+1$, by Meijer (*Proc. Akad. te Amsterdam*, XXXIX, 1936, p. 397).

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Proceedings of the Glasgow Mathematical Association—Vol. I, Part 2

GENERALISATIONS OF SOME INTEGRALS INVOLVING BESSEL FUNCTIONS AND E-FUNCTIONS

by FOUAD M. RAGAB

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§ 1. *Introductory.* In § 3 a generalisation of the formula [MacRobert, *Phil. Mag.*, Ser. 7, XXXI, p. 258]

$$4\int_0^\infty \lambda^{m-1} K_n(2\lambda) E(p\,;\, \alpha_s : q\,;\, \rho_t : x\lambda^{-2})\, d\lambda$$
$$= E(p+2\,;\, \alpha_s : q\,;\, \rho_t : x), \qquad (1)$$

where $\alpha_{p+1}=\frac{1}{2}m+\frac{1}{2}n$, $\alpha_{p+2}=\frac{1}{2}m-\frac{1}{2}n$, $R(m\pm n)>0$, and x is real and positive, will be established. In the course of the proof Hardy's formula [*Mess. of Maths.*, LVI, (1927), p. 190],

$$\int_0^\infty K_n(x) K_n\left(\frac{b}{x}\right) dx = \pi K_{2n}(2\sqrt{b}), \qquad (2)$$

where $R(b)>0$, will be required. This was originally proved by an application of Mellin's Inversion Formula. An alternative proof is given in § 2, and some related formulae are deduced.

§ 2. *Proof of Hardy's Formula.* Denote the integral on the left of (2) by $F(b)$; then

$$F'(b) = \int_0^\infty K_n(x) K_n'\left(\frac{b}{x}\right)\frac{1}{x}\, dx,$$

and

$$F''(b) = \int_0^\infty K_n(x) K_n''\left(\frac{b}{x}\right)\frac{1}{x^2}\, dx;$$

so that

$$b^2 F''(b) = \int_0^\infty K_n(x) \left\{\left(\frac{b^2}{x^2}+n^2\right) K_n\left(\frac{b}{x}\right) - \frac{b}{x} K_n'\left(\frac{b}{x}\right)\right\} dx$$
$$= n^2 F(b) - bF'(b) + b^2\int_0^\infty K_n(x) K_n\left(\frac{b}{x}\right)\frac{dx}{x^2}.$$

But, on replacing x by b/x in $F(b)$, it is seen that

$$F(b) = b\int_0^\infty K_n(x) K_n\left(\frac{b}{x}\right)\frac{1}{x^2}\, dx.$$

Hence

$$b^2 F''(b) + bF'(b) - (b+n^2) F(b) = 0. \qquad (3)$$

Now in the equation

$$x^2 y'' + xy' - (x^2+4n^2) y = 0, \qquad (4)$$

with solutions $K_{2n}(x)$ and $I_{2n}(x)$, put $x=2\sqrt{b}$, and it reduces to (3). Therefore

$$\int_0^\infty K_n(x) K_n\left(\frac{b}{x}\right) dx = AK_{2n}(2\sqrt{b}) + BI_{2n}(2\sqrt{b}).$$

Here let $b\to\infty$, and it is seen that B must be zero. Thus

$$\int_0^\infty K_n(x) \left\{I_{-n}\left(\frac{b}{x}\right) - I_n\left(\frac{b}{x}\right)\right\} dx$$
$$= \frac{A}{2\cos n\pi}\left\{I_{-2n}(2\sqrt{b}) - I_{2n}(2\sqrt{b})\right\}.$$

Now assume that $R(n)>0$, multiply by b^n and let $b\to 0$; then

$$\frac{2^n}{\Gamma(1-n)}\int_0^\infty K_n(x) x^n\, dx = \frac{A}{2\cos n\pi\,\Gamma(1-2n)}.$$

But, if $R(m\pm n)>0$,

$$\int_0^\infty K_n(x) x^{m-1}\, dx = 2^{m-2}\,\Gamma\left(\frac{m+n}{2}\right)\Gamma\left(\frac{m-n}{2}\right). \qquad (5)$$

Therefore

$$\frac{2^{2n-1}}{\Gamma(1-n)}\Gamma(n+\tfrac{1}{2})\Gamma(\tfrac{1}{2})=\frac{A}{2\cos n\pi\Gamma(1-2n)},$$

and from this it follows that $A=\pi$.

From (2) other formulae of the same type can be derived as follows.

In (2) let amp b decrease by $\frac{1}{2}\pi$, finally writing b/i in place of b; then, since

$$K_n(t)=i^n G_n(it), \qquad \text{.......(6)}$$

the formula becomes

$$\int_0^\infty K_n(x)\,G_n\left(\frac{b}{x}\right)dx=\pi i^{-n}\,K_{2n}(2\sqrt{b}\,.\,e^{-i\pi/4}), \qquad \text{.......(7)}$$

provided that $-\frac{5}{2}<R(n)<\frac{5}{2}$.

Similarly, on replacing b by ib, it is seen that

$$\int_0^\infty K_n(x)\,G_n\left(\frac{b}{x}e^{i\pi}\right)dx=\pi i^{-n}\,K_{2n}(2\sqrt{b}\,.\,e^{i\pi/4}), \qquad \text{.......(8)}$$

where $-\frac{5}{2}<R(n)<\frac{5}{2}$.

Hence, using the formula

$$\pi i J_n(t)=G_n(t)-i^{2n}\,G_n(te^{i\pi}), \qquad \text{.......(9)}$$

it follows that

$$i\int_0^\infty K_n(x)\,J_n\left(\frac{b}{x}\right)dx=i^{-n}\,K_{2n}(2\sqrt{b}\,.\,e^{-i\pi/4}) - i^n\,K_{2n}(2\sqrt{b}\,.\,e^{i\pi/4}), \qquad \text{.......(10)}$$

where $-5<R(n)<5$. This formula also is given in Hardy's paper.

Again, let amp x and amp b increase simultaneously by $\frac{1}{2}\pi$, so that x becomes ix and b becomes ib; then

$$\int_0^\infty G_n(xe^{i\pi})\,J_n\left(\frac{b}{x}\right)dx=K_{2n}(2\sqrt{b}\,.\,e^{i\pi/2}) - i^{-2n}K_{2n}(2\sqrt{b}), \qquad \text{.......(11)}$$

where $-\frac{1}{2}<R(n)<\frac{5}{2}$.

Similarly, if amp x and amp b decrease simultaneously by $\frac{1}{2}\pi$, (10) becomes

$$\int_0^\infty G_n(x)\,J_n\left(\frac{b}{x}\right)dx=i^{-2n}\,K_{2n}(2\sqrt{b}\,.\,e^{i\pi/2}) - K_{2n}(2\sqrt{b}), \qquad \text{.......(12)}$$

where $-\frac{1}{2}<R(n)<\frac{5}{2}$.

Finally, from (9), (11) and (12) it follows that

$$\begin{aligned}\pi i\int_0^\infty J_n(x)\,J_n\left(\frac{b}{x}\right)dx &= i^{-2n}\,K_{2n}(2\sqrt{b}\,.\,e^{-i\pi/2}) - i^{2n}\,K_{2n}(2\sqrt{b}\,.\,e^{i\pi/2})\\ &= G_{2n}(2\sqrt{b}) - i^{4n}\,G_{2n}(2\sqrt{b}\,.\,e^{i\pi})\\ &= \pi i\,J_{2n}(2\sqrt{b}),\end{aligned}$$

so that

$$\int_0^\infty J_n(x)\,J_n\left(\frac{b}{x}\right)dx=J_{2n}(2\sqrt{b}), \qquad \text{.......(13)}$$

where $R(n)>-\frac{1}{2}$. This formula was given by Bateman (*Proc. Camb. Phil. Soc.*, XXI, (1908), p. 186).

§ 3. *Generalisation of the Integral.* The formula to be proved is

$$\begin{aligned}2^{2^r+r+1}\pi^{2^r-1}\int_0^\infty \lambda^{2^r m-1}\,K_{2^r n}(2^{r+1}\lambda)\,E(p\,;\;\alpha_s:\;q\,;\;\rho_t:\;x\lambda^{-2^{r+1}})\,d\lambda&\\ =E(p+2^{r+1}\,;\;\alpha_s:\;q\,;\;\rho_t:\;x),&\qquad \text{.......(14)}\end{aligned}$$

where $r=0, 1, 2, \ldots$, and

$$\left.\begin{aligned}\alpha_{p+2k+1}&=\tfrac{1}{2}m+\tfrac{1}{2}n+\frac{k}{2^r}\\ \alpha_{p+2k+2}&=\tfrac{1}{2}m-\tfrac{1}{2}n+\frac{k}{2^r}\end{aligned}\right\}k=0, 1, 2, \ldots, 2^r-1, \qquad \text{.......(14')}$$

$R(m\pm n)>0$ and x is real and positive.

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It can be proved by induction; for, assuming that it is valid, it follows that

$$E(p+2^{r+2};\ \alpha_s:\ q;\ \rho_t:\ x)$$
$$=2^{2^r+r+1}\pi^{2^r-1}\int_0^\infty \lambda^{2^r l-1}K_{2^r n}(2^{r+1}\lambda)E(p+2^{r+1};\ \alpha_s:\ q;\ \rho_t:\ x\lambda^{-2^{r+1}})\,d\lambda$$

2^r

where

$$\left.\begin{aligned}\alpha_{p+2^{r+1}+2k+1}&=\tfrac12 l+\tfrac12 n+\frac{k}{2^r}\\ \alpha_{p+2^{r+1}+2k+2}&=\tfrac12 l-\tfrac12 n+\frac{k}{2^r}\end{aligned}\right\}k=0,1,2,\ldots,2^r-1,$$

$$=2^{2^{r+1}+2r+2}\pi^{2^{r+1}-2}\int_0^\infty \lambda^{2^r l-1}K_{2^r n}(2^{r+1}\lambda)\,d\lambda$$
$$\times\int_0^\infty \mu^{2^r m-1}K_{2^r n}(2^{r+1}\mu)E(p;\ \alpha_s:\ q;\ \rho_t:\ x(\lambda\mu)^{-2^{r+1}})\,d\mu.$$

Here replace μ by μ/λ and get

$$2^{2^{r+1}+2r+2}\pi^{2^{r+1}-2}\int_0^\infty \lambda^{2^r(l-m)-1}K_{2^r n}(2^{r+1}\lambda)\,d\lambda$$
$$\times\int_0^\infty \mu^{2^r m-1}K_{2^r n}\left(2^{r+1}\frac{\mu}{\lambda}\right)E(p;\ \alpha_s:\ q;\ \rho_t:\ x\mu^{-2^{r+1}})\,d\mu.$$

Next put $l=m+2^{-r}$ and change the order of integration, so getting

$$2^{2^{r+1}+2r+2}\pi^{2^{r+1}-2}\int_0^\infty \mu^{2^r m-1}E(p;\ \alpha_s:\ \rho_t:\ x\mu^{-2^{r+1}})\,d\mu$$
$$\times\int_0^\infty K_{2^r n}(2^{r+1}\lambda)\,K_{2^r n}\left(2^{r+1}\frac{\mu}{\lambda}\right)d\lambda,$$

q;

where

$$\left.\begin{aligned}\alpha_{p+2^{r+1}+2k+1}&=\tfrac12 m+\tfrac12 n+\frac{2k+1}{2^{r+1}}\\ \alpha_{p+2^{r+1}+2k+2}&=\tfrac12 m-\tfrac12 n+\frac{2k+1}{2^{r+1}}\end{aligned}\right\}\begin{aligned}&k=0,1,2,\ldots,2^r-1\\ &\text{or}\\ &2k+1=1,3,5,\ldots,2^{r+1}-1\end{aligned}$$

?

check in R' copy

But, from (14′),

$$\left.\begin{aligned}\alpha_{p+2k+1}&=\tfrac12 m+\tfrac12 n+\frac{2k}{2^{r+1}}\\ \alpha_{p+2k+2}&=\tfrac12 m-\tfrac12 n+\frac{2k}{2^{r+1}}\end{aligned}\right\}\begin{aligned}&k=0,1,2,\ldots,2^r-1\\ &\text{or}\\ &2k=0,2,4,\ldots,2^{r+1}-2.\end{aligned}$$

?

Therefore

$$\left.\begin{aligned}\alpha_{p+2k+1}&=\tfrac12 m+\tfrac12 n+\frac{k}{2^{r+1}}\\ \alpha_{p+2k+2}&=\tfrac12 m-\tfrac12 n+\frac{k}{2^{r+1}}\end{aligned}\right\}k=0,1,2,\ldots,2^{r+1}-1.$$

Now, from (2), the last integral is equal to

$$\frac{1}{2^{r+1}}\pi K_{2^{r+1}n}(2^{r+2}\sqrt{\mu}).$$

wc $\pi^{2^{r+1}}$

Hence, on replacing μ by λ^2, we have

$$E(p+2^{r+2};\ \alpha_s:\ q;\ \rho_t:\ x)=2^{2^{r+1}+r+2}\pi^{2^{r+1}-1}$$
$$\times\int_0^\infty \lambda^{2^{r+1}m-1}K_{2^{r+1}n}(2^{r+2}\lambda)E(p;\ \alpha_s:\ q;\ \rho_t:\ x\lambda^{-2^{r+2}})\,d\lambda,$$

check in R' copy

which is (14) with $r+1$ in place of r.

But the formula holds when $r=0$: hence it holds for all positive integral values of r.

If in (14) amp λ is decreased by $\tfrac12\pi$ and amp x by $2^r\pi$, it becomes, by (6),

$$2^{2^r+r+1}\pi^{2^r-1}i^{2^r(n-m)}\int_0^\infty \lambda^{2^r m-1}G_{2^r n}(2^{r+1}\lambda)E(p;\ \alpha_s:\ q;\ \rho_t:\ x\lambda^{-2^{r+1}})\,d\lambda$$
$$=E(p+2^{r+1};\ \alpha_s:\ q;\ \rho_t:\ xe^{-i\pi 2^r}),\qquad\qquad(15)$$

where $R(m \pm n)>0$, $R(\frac{3}{2} - 2^r m + 2^{r+1}\alpha_s)>0$, $s=1, 2, \ldots, p$, and x is real and positive.

Similarly, and subject to the same conditions,

$$2^{2^r+r+1}\,\pi^{2^r-1}\,i^{2^r(n+m)} \int_0^\infty \lambda^{2^r m-1} G_{2^r n}(2^{r+1}\lambda e^{i\pi})\, E(p\,;\ \alpha_s : q\,;\ \rho_t : x\lambda^{-2^{r+1}})\, d\lambda$$

$$= E(p+2^{r+1}\,;\ \alpha_s : q\,;\ \rho_t : xe^{i\pi 2^r}), \quad \ldots\ldots (16)$$

and, from (9),

$$2^{2^r+r+1}\,\pi^{2^r}\, i \int_0^\infty \lambda^{2^r m-1} J_{2^r n}(2^{r+1}\lambda)\, E(p\,;\ \alpha_s : q\,;\ \rho_t : x\lambda^{-2^{r+1}})\, d\lambda$$

$$= i^{2^r(m-n)}\, E(p+2^{r+1}\,;\ \alpha_s : q\,;\ \rho_t : xe^{-i\pi 2^r})$$

$$- i^{-2^r(m-n)}\, E(p+2^{r+1}\,;\ \alpha_s : q\,;\ \rho_t : xe^{i\pi 2}). \quad \ldots\ldots (17)$$

2^r / $e^{i\pi 2^r}$

The case in which $r=0$ was given in a previous paper (see page 7).

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where $R(m+n)>0$, $R(\frac{3}{2} - 2^r m + 2^{r+1}\alpha_s)>0$,
$s=1, 2, \ldots, p$, and x is real and positive