Thesis submitted for the degree of Doctor of Philosophy in the University of Glasgow,

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PART I.

Pair Creation by a Photon in the Field of an Electron.

PART II.

The Reactions $\pi^+ + t \rightleftarrows p + d$.

by

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PART I.

Pair Creation by a Photon in the Field of an Electron.

PREFACE to PART I.

The author was asked , by Dr. F. Cole of Cornell University, to verify the differential cross-section, quoted by Borsellino (9), for pair production by a photon in the field of a free electron. This differential cross-section was found to be misprinted (see § 6 of this thesis), but the correct form was later found in another paper by Borsellino. (Consiglio Nazionale delle Ricerche, Roma. N212 (1948).) It was noted, however, that Borsellino had omitted the possibility of the initially-present electron absorbing the real photon. This thesis contains the calculation of the cross-section for the process taking this possibility into consideration and, like Borsellino, neglecting exchange terms.

Borsellino based his calculation on the method described in Heitler's "Quantum Theory of Radiation". This thesis contains the derivation of the differential cross-section by the Feynman method (26); this derivation is the core of the author's original contribution to the problem investigated in Part I of this thesis.

In the integration of the differential cross-section, Borsellino's result is accepted for the integration of the terms which he considered. The integration of the other terms, over the four variables involved, is performed in the order used by Borsellino, but is otherwise also original. The expressions arising in the integration are unwieldy and the calculation is extremely long, so that only the results of the integrations over each variable in turn have been quoted.

In the first chapter of the thesis, § 1 is an introduction to the problem of pair creation in the field of an electron and § 2 gives a survey of previous experimental and theoretical treatments of this problem. It is pointed out in § 2 that Votruba (10) has studied the problem using the complete matrix element, but his analysis was so complicated that he could not obtain a complete excitation curve. In this thesis such a curve is obtained for the part of the matrix element considered. § 3 lists the method that has been used to obtain the correct numerical factors in the differential cross-section as Feynman has not made these clear in his papers.

Chapter II contains, in § 4, the statement of the problem considered in the thesis and an explanation of the relation between this and previous calculations of the cross-section for pair production in the field of an electron. In § 5, the square of the matrix element is calculated in the relativistically-invariant notation of Feynman. In § 6 this is translated to the usual three-dimensional notation. Equation (6.11) is the differential cross-section for the process with exchange terms neglected.

The integration of the differential cross-section is carried out in Chapter III. It is found that, except at low γ-ray energies, the contribution to the total cross-section from these additional terms in the matrix element is negligible compared to the contribution from those terms integrated by Borsellino. The effect of including the exchange terms is discussed in § 9. It is shown that, at high energies, Borsellino's formula is approximately correct.

The appendix contains a justification of the use of the Feynman projection operator and this part of the thesis is also due to the author.

In conclusion, the author would like to thank Prof. J.C. Gunn for helpful discussion on the calculation, the Department of Scientific and Industrial Research for a maintenance allowance during the first and third years of his research and Glasgow and Cornell Universities for an exchange scholarship which allowed him to study at Cornell University during his second year of research.

Contents of PART I.

CHAPTER I.

§1. Introduction.

A beam of photons passing through matter may lose intensity through individual quanta producing pairs, the usual production being in the Coulomb field of a nucleus of charge Ze. The differential cross-section for this method of production has been formulated by Bethe and Heitler (1) and by Racah (2). Analytic integration is simple only in the limiting cases of small and large photon energy. For small photon energy, Racah found the total cross-section

$$\sigma(k) = Z^2\left(\frac{e^2}{\hbar c}\right)\left(\frac{e^2}{mc^2}\right)^2 \frac{\pi}{12}\left(\frac{k-2mc^2}{mc^2}\right)^3 \tag{1.1}$$

and Bethe and Heitler found, for $k, \omega, \omega_0 \gg mc^2$,

$$\sigma(k) = Z^2\left(\frac{e^2}{\hbar c}\right)\left(\frac{e^2}{mc^2}\right)^2\left(\frac{28}{9}\log\frac{2k}{mc^2} - \frac{218}{27}\right) \tag{1.2}$$

where k, ω, ω_0 are the energies of the photon, electron and positron respectively. m is the electron rest mass and $\hbar, e, c$ have their usual meaning. Born's first approximation has been used and this is valid if the velocities of the pair constituents are greater than $Z\left(\frac{e^2}{\hbar c}\right)c$, this condition being satisfied if the velocities of the particles are near that of light and $Z \lesssim 60$.

In actual fact, the quantum does not feel the full effect of the nuclear Coulomb field, since the nucleus is screened by the atomic electrons. This has been taken into account by Bethe and Heitler by replacing Z^2 by $\{Z - F(q)\}^2$ where $F(q)$ is the atomic form factor given by $F(q) = \int \rho(r)\, e^{\frac{i}{\hbar c}\vec{q}\cdot\vec{r}}\, d\vec{r}$, $\rho(r)$ is the electron density at a distance r from the nucleus and $\vec{q}$ is the momentum transferred to the atom. We express all our momenta in energy units, e.g. we speak of the momentum $\vec{q}$ instead of the strictly correct $\frac{1}{c}\vec{q}$. Bethe and Heitler found, assuming a Fermi distribution for $\rho(r)$, that when $k, \omega, \omega_0 \gg mc^2$ screening was effective only if $\frac{\omega\omega_0}{k} > \frac{1}{2}\frac{134}{Z^{1/3}}\, mc^2$, and the differential cross-section became

$$d\sigma(k) = Z^2\left(\frac{e^2}{\hbar c}\right)\left(\frac{e^2}{mc^2}\right)^2 d\omega \left[(\omega^2+\omega_0^2)\left\{\varphi_1(\gamma) - \frac{4}{3}\log Z\right\} + \frac{2}{3}\omega\omega_0\left\{\varphi_2(\gamma) - \frac{4}{3}\log Z\right\}\right] \quad (1.3)$$

where $\gamma = \frac{100\, mc^2\, k\, Z^{-\frac{1}{3}}}{\omega\omega_0}$ and φ_1, φ_2 , which decrease with increasing γ , were given graphically. γ determines the effect of screening and if $\gamma = 0$, screening is complete, while if $\gamma \gg 1$ (i.e. for energies $\ll 137\, mc^2 Z^{-\frac{1}{3}}$) screening has no effect. Thus above formula (1.2) is correct only if $k \gg mc^2$ but $\ll 134\, mc^2 Z^{-\frac{1}{3}}$. For $k \gg 134\, mc^2 Z^{-\frac{1}{3}}$ (complete screening),

$$\sigma(k) = Z^2\left(\frac{e^2}{\hbar c}\right)\left(\frac{e^2}{mc^2}\right)^2\left\{\frac{28}{9}\log\left(183\,Z^{-\frac{1}{3}}\right) - \frac{2}{27}\right\}. \quad (1.4)$$

For other values of k, integration was carried out numerically and the final Bethe-Heitler result is shown in fig.1.

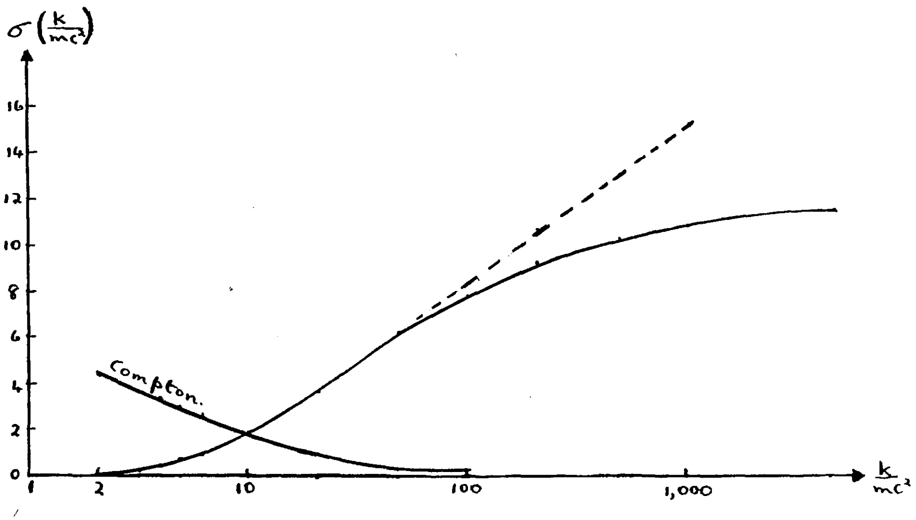

Fig.1. Bethe-Heitler integrated cross-section (in units of $Z^2\frac{e^2}{\hbar c}\left(\frac{e^2}{mc^2}\right)^2$) for lead with screening (——) and without (------). For comparison, the Compton cross-section has been inserted on the same scale.

Using these screening factors is equivalent to studying pair production in the field of an atom and Bethe showed that the most probable processes were those in which the atom took up a small recoil ($< mc^2$).

Although nuclear pair production is the more probable process, it is also possible that a pair may be created in the field of one of the atomic electrons. The possibility of this process was first pointed out by

Perrin (3) who, however, made no attempt to determine its probability. The differences between the two processes are; (1) the electron is not able to take away a large momentum for negligible kinetic energy as the nucleus can, so that the thresholds for the two processes will be different; (2) as a result of the large velocity of recoil of the electron, retardation effects will become important i.e. the electron can emit or absorb transverse quanta; (3) in electronic pair production there are two identical particles (electrons) finally, so that exchange effects come into play. (By exchange effects, we mean the effects due to the operation of the Pauli Exclusion Principle.) If one electron has small energy and the photon and pair constituents large energy, then the effects of exchange and retardation will be small and we would expect the cross-section for production in the field of an electron to be roughly equal to that for production in the field of a bare proton i.e. to expression (1.2) with $Z = 1$ i.e.

$$\sigma_{el}(k) = \left(\frac{e^2}{\hbar c}\right)\left(\frac{e^2}{mc^2}\right)^2 \left\{ \frac{28}{9} \log \frac{2k}{mc^2} - \frac{218}{27} \right\}. \tag{1.5}$$

We study now, following Perrin, the kinematics of pair production in the field of a free electron. The threshold for the reaction is the minimum energy, k_0, of the photon incident on a stationary free electron, which allows the reaction to take place. To find k_o , we transform this laboratory system to that in which the total momentum is zero. If k is the energy of the photon in the laboratory system and if $\beta c \left(= \frac{k}{k + mc^2} \right)$ is the velocity of transformation from one system to the other, then in the new system $k' - \frac{m\beta c^2}{\sqrt{1-\beta^2}} = 0$ where k' , the energy (or momentum) of the photon in this zero momentum system, is given by $k' = k\sqrt{\frac{1-\beta}{1+\beta}}$. The three particles present finally can now all have zero momentum and k_o is given by $k_o' + \left(\frac{mc^2}{\sqrt{1-\beta_o^2}} - mc^2 \right) = 2mc^2$ where $k_o' = k_o\sqrt{\frac{1-\beta_o}{1+\beta_o}}$, $\beta_o = \frac{k_o}{k_o + mc^2}$ and $2mc^2$ is the energy required to create a pair whose constituents have zero momentum. This gives $k_o = 4mc^2$ and $\beta_o = \frac{4}{5}$. Thus the threshold for the reaction is $4mc^2$ and when $k = 4mc^2$ the three final particles move, in the laboratory system, in the direction of the incident photon with equal velocities $\frac{4}{5}c$ i.e. with equal kinetic energies $\frac{2}{3}mc^2$.

If $\vec{q}$, $\vec{p}$ and $\vec{p_o}$ are the momenta of the final electrons and positron respectively in laboratory system, and $W = \sqrt{q^2 + m^2c^4}$, $\omega = \sqrt{p^2 + m^2c^4}$ and $\omega_o = \sqrt{p_o^2 + m^2c^4}$ the corresponding energies, then

$$\vec{q} = \vec{k} - \vec{p} - \vec{p_o} \tag{1.6}$$

$$\sqrt{q^2+m^2c^4} = k + mc^2 - \sqrt{p^2+m^2c^4} - \sqrt{p_0^2+m^2c^4} \tag{1.7}$$

and q takes its maximum and minimum values when all momenta are parallel and $p_0 = p$. This means that the turning values of q satisfy the equation

$$k+mc^2 = \sqrt{q^2+m^2c^4} + \sqrt{k^2-2kq+q^2+4m^2c^4}, \tag{1.8}$$

giving

$$q_{\frac{1}{2}} = \frac{k(k-mc^2) \mp (k+mc^2)\sqrt{k(k-4mc^2)}}{2k+mc^2} \tag{1.9}$$

and the corresponding energies

$$W_{\frac{1}{2}} = \frac{k^2-m^2c^4 \mp k\sqrt{k(k-4mc^2)}}{2k+mc^2}. \tag{1.10}$$

Thus, if $k > 4mc^2$, the energy of each final particle must be greater than, or equal to, W_1 and less than, or equal to, W_2, since the equations (1.5) and (1.6) are symmetrical in the three energies and momenta.

To find the maximum angle, $\theta_{max.}$ - measuring angles from the direction of the incident photon - at which any particle can come off in laboratory system we first find the maximum angle θ at which a particle with energy W can come off. This maximum angle, θ, occurs when $\vec{p} = \vec{p}_0$ and we then have (see fig.2.)

$$4p^2 = k^2 - 2kq\cos\theta + q^2$$

and

$$2\sqrt{p^2+m^2c^4} = k + mc^2 - W.$$

$$\therefore \cos\theta = \frac{mc^2(mc^2+W) + k(W-mc^2)}{kq} \tag{1.11}$$

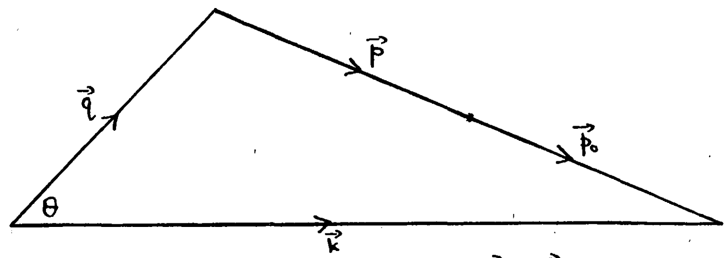

Fig.2. Vector diagram when $\vec{p} = \vec{p}_0$.

If we now allow W to vary within its specified limits, we find

$$\cos\theta_{max} = 2\sqrt{\frac{mc^2}{k}} \; . \tag{1.12}$$

Therefore final particles must come off at an angle between 0 and $\cos^{-1} 2\sqrt{\frac{mc^2}{k}}$.

This process would therefore appear in a cloud chamber as two electron tracks and one positron track, all starting from one apex, all making angles less than $\cos^{-1} 2\sqrt{\frac{mc^2}{k}}$ with the incident photon direction and all having energies lying between above values of W_1 and W_2. These forks are known as triplets and are not to be confused with the triplet forks appearing at the end of an electron track, these latter corresponding to pair creation by an electron in the field of a nucleus. If the cloud chamber were filled with a gas of atomic number Z, and if we neglected screening, the cross-sections for nuclear and electronic pair production would be respectively $C_1 Z^2$ and $C_2 Z$, where C_1 and C_2 are independent of the gas. Nuclear pair production would appear as a pair of tracks, so the ratio of triplets to pairs should be $\frac{C_2}{C_1 Z}$. Assuming $C_1 = C_2$ i.e. that cross-sections for pair production in the fields of electron and proton are equal, we would therefore expect one triplet for every 5 to 10 pairs, if gas were air, so that electronic pair production would be quite noticeable. We are interested in actually finding the value of C_2.

§2. Survey of Previous Work.

(a) Theoretical.

The first attempt to ascribe a probability to the process of pair production in the field of a free electron was by Wheeler and Lamb (4) who obtained the cross-section by an inverted type of Weizsäcker-Williams(5) calculation. They used the known cross-section (Racah (6)) for pair production by an electron in the field of a heavy nucleus, worked in the rest system of the electron, i.e. considered pair creation by a nucleus in the field of an electron, and replaced the nucleus by its equivalent electromagnetic field. This gave, as the cross-section for all momentum transfers,

$$\sigma(k) = \left(\frac{e^2}{\hbar c}\right)\left(\frac{e^2}{mc^2}\right)^2 \left\{ \frac{28}{9} \log \frac{2k}{mc^2} - \frac{28}{9} \log a - \frac{148}{24} \right\} \quad (2.1)$$

where $\alpha(\approx 1)$ is the indeterminate constant introduced by the Weizsäcker-Williams method. This agrees in substance with (1.5), a result we had guessed using only small momentum transfers. This shows that when $k \gg 4mc^2$, small momentum transfers are the most important, large momentum transfers merely altering the constant in (1.5).

For considerations of pair creation in the field of a bound electron, we have to distinguish between three cases. (a) If the recoil momentum of the electron is so small (< momentum of the electron in the orbit) that the atom cannot be excited or ionised, then this should properly be classed as a recoil of the atom as a whole and the process treated as pair creation in the field of a screened nucleus. (Bethe and Heitler)
(b) If the momenta of both electrons finally are much larger than mc^2, the atom is left ionised and the final particles can be treated as being free. This process would appear as a triplet in a cloud chamber. This is the problem with which we shall deal, except that we shall also treat the initial electron as being free.
(c) If the recoil momentum is much smaller than mc^2 but is sufficient to excite the atom, the process would appear in a cloud chamber as a pair, but would actually be pair creation in the field of an electron. This is

the problem Wheeler and Lamb discussed fully and they found

$$d\sigma = \left(\frac{e^2}{\hbar c}\right)\left(\frac{e^2}{mc^2}\right)^2 Z \frac{mc^2}{k^3}\left[(\omega_0^2+\omega)^2\left\{\psi_1(\gamma) - \frac{8}{3}\log Z\right\}\right.$$
$$\left. + \frac{2\omega_0\omega}{3}\left\{\psi_2(\gamma) - \frac{8}{3}\log Z\right\}\right] \tag{2.2}$$

where γ is the same as in the Bethe-Heitler theory. The screening factors $\psi_1(\gamma)$ and $\psi_2(\gamma)$ were calculated on the basis of the Fermi-Thomas model of the atom and also by using atomic wave functions for hydrogen (for all energies) and for nitrogen (for high energies). To compare their integrated cross-section with that of Bethe and Heitler, they recalculated the Bethe-Heitler screening factors using the atomic wave functions and found that for high energy quanta producing pairs in air (nitrogen screening factors used),

$$\sigma_{total} = \sigma_{B-H} + \sigma_{W-L} = (41 \cdot 9 + 6 \cdot 94)\, 10^{-26}\ \text{cm}^2.$$

This means that the cross-section for pair creation in the field of the electron of a hydrogen atom is slightly greater than that for creation in the field of the proton. Wheeler and Lamb are the only people who have studied the problem for bound electrons; the following work will assume that all the particles are free.

Watson (7) has considered pair production in the electrostatic field of a free electron, using Dirac one-electron perturbation theory. On this theory the initial state consists of a photon and two electrons, one of positive and one of negative energy and the final state consists of two positive energy electrons. Let us denote the momenta of the initial photon by $\vec{k}$ and of the initial electrons by $\vec{P}_0$ and $\vec{p}_0$, the momenta of the final electrons by $\vec{q}$, $\vec{p}$ and intermediate electron momenta by dashes. If V denotes the Coulomb interaction of two electrons and U the absorption of the photon by an electron, then the scheme of intermediate states is -

$$\vec{k}, \vec{P}_0\ \vec{p}_0 \xrightarrow{V} \vec{k}, \vec{p}', \vec{q} \xrightarrow{U} \vec{p}, \vec{q} \qquad (2.3a)$$

$$\vec{k}, \vec{P}_0, \vec{p}_0 \xrightarrow{V} \vec{k}, \vec{p}, \vec{q}' \xrightarrow{U} \vec{p}, \vec{q} \qquad (2.3b)$$

$$\vec{k}, \vec{P}_0, \vec{p}_0 \xrightarrow{U} \vec{P}_0', \vec{p}_0 \xrightarrow{V} \vec{p}, \vec{q} \qquad (2.3c)$$

$$\vec{k}, \vec{P}_0, \vec{p}_0 \xrightarrow{U} \vec{P}_0, \vec{p}_0' \xrightarrow{V} \vec{p}, \vec{q}\,. \qquad (2.3d)$$

The virtual electron can have either positive or negative energy. Watson allowed for exchange effects but did so twice, in that he used a wave function anti-symmetric in the two electrons and, having obtained his differential cross-section, he added its (equal) values when q was small (with p , p_0 large) and when p was small (with q, p_0 large). His results should therefore be divided by 2. He showed that for $k \gg 4mc^2$ the most important contribution to the cross-section came from cases in which one final electron had small momentum and the positron and other electron had large momenta. As the γ -ray energy increased the energy of the slow electron tended to zero, so that the process would appear in a cloud chamber as a pair. He evaluated the total cross-section approximately for $k \gg 16mc^2$ then extrapolated his curve back to $k = 4mc^2$. The total cross-section for $k \gg 4mc^2$ he found to be

$$\sigma(k) = \frac{29}{9}\left(\frac{e^2}{\hbar c}\right)\left(\frac{e^2}{mc^2}\right)^2 \log\left(\frac{2k}{mc^2}\right) . \tag{2.4}$$

Nemirovsky (8) used (2.3a) and (2.3d) but included the retarded interaction of the two electrons along with the Coulomb interaction. This means that he considered only the direct transitions of the initially-present positive energy electron to its final state. Like Watson he allowed for exchange. For incident γ -ray energies close to the threshold (4 to $6\,mc^2$) he integrated his differential cross-section to obtain

$$\sigma(k) = 0.084\left(\frac{e^2}{\hbar c}\right)\left(\frac{e^2}{mc^2}\right)^2\left(k+mc^2\right)\left[\frac{1}{\sqrt{mc^2}} - \frac{3}{\sqrt{2k+mc^2}}\right] . \tag{2.5}$$

For large photon energies the dependence of the cross-section on γ -ray energy was similar to that for pair production in the field of an unscreened nucleus and the most probable processes were those of small momentum transfer.

Borsellino (9) carried out a calculation similar to that of Nemirovsky but did not allow for exchange. Analytic integration gave*

$$\sigma(k) = \left(\frac{e^2}{\hbar c}\right)\left(\frac{e^2}{mc^2}\right)^2 \frac{\pi\sqrt{3}}{216}\left(\frac{k-4mc^2}{mc^2}\right)^2 \qquad \text{for } k \sim 4mc^2 \quad (2.6)$$

$$\sigma(k) \doteq \left(\frac{e^2}{\hbar c}\right)\left(\frac{e^2}{mc^2}\right)^2 \left[\frac{28}{9}\log\frac{2k}{mc^2} - \frac{218}{27} - \frac{mc^2}{k}\left\{\frac{4}{3}\left(\log\frac{2k}{mc^2}\right)^3 - 3\left(\log\frac{2k}{mc^2}\right)^2 + 6.84\log\frac{2k}{mc^2} + 21.51\right\}\right] \qquad \text{for intermediate energies} \quad (2.7)$$

$$\sigma(k) = \left(\frac{e^2}{\hbar c}\right)\left(\frac{e^2}{mc^2}\right)^2 \left\{\frac{28}{9}\log\frac{2k}{mc^2} - \frac{218}{27}\right\} \qquad \text{for } k \gg 4mc^2. \quad (2.8)$$

Thus for $k \gg 4mc^2$ his cross-section is the same as (1.5) i.e. cross-sections for pair creation in the fields of a free electron and a bare proton are equal i.e. $C_1 = C_2$. For lower energies, $C_2 < C_1$. His final excitation curve, calculated numerically, is shown in fig.3. To compare the results of these last two authors, we shall evaluate

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* Index of power of $\left(\frac{k-4mc^2}{mc^2}\right)$ is misprinted as 3 instead of 2 in equation (56) of Nuovo Cimento.

their cross-sections for the production in the field of a single free electron at a γ -ray energy of $5.2\,mc^2$. These are 0.38×10^{-29} cm^2 (Nemirovsky) and 2.17×10^{-29} cm^2 (Borsellino). Thus, at this low energy, exchange

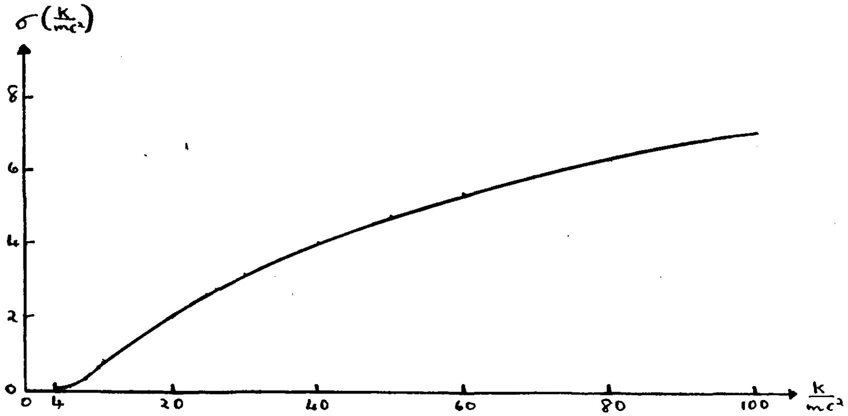

Fig.3. Borsellino's integrated cross-section in units of $(\frac{e^2}{\hbar c})(\frac{e^2}{mc^2})^2$.

effects reduce the total cross-section by a factor between 5 and 6. At high energies we have seen that small momentum transfers are the most important so that exchange effects will be small, and the two authors agree. Watson's results are not at all reliable for low energies as he got these by extrapolation. He also agrees with Borsellino and Nemirovsky for high γ -ray energies, since small momentum transfers mean that the velocity of recoil is small and retardation effects are small. Thus all three authors give, in essence, (1.5) for the total cross-section for $k \gg 4mc^2$.

Votruba (10) used, for the matrix element, (2,3a,b, c,d) taking into account exchange of transverse quanta (retarded interaction) as well as Coulomb interaction, and also including exchange effects. He was unable to obtain a complete excitation curve but made various approximations. For the case of $k = 4mc^2 + \nu mc^2$, $\nu \ll 1$, he found

$$\sigma(k) = \left(\frac{e^2}{\hbar c}\right)\left(\frac{e^2}{mc^2}\right)^2 \frac{\pi\sqrt{3}}{972} \left(\frac{k - 4mc^2}{mc^2}\right)^2 \tag{2.9}$$

in agreement, except for the multiplicative constant, with Borsellino (2.6). For $k \gg 4mc^2$, $q \ll mc^2$ and $p, p_0 \gg mc^2$, he found

$$\sigma(k) = \left(\frac{e^2}{\hbar c}\right)\left(\frac{e^2}{mc^2}\right)^2 \left[\frac{28}{9} \log \frac{2k}{mc^2} - \frac{102}{9}\right] \tag{2.10}$$

but the constant $\frac{102}{9}$ was not reliable. This again, is in agreement, except for the additive constant, with Borsellino (2.8). For $k \gg 4mc^2$, $p_0 \ll mc^2$ and $q, p \gg mc^2$ he obtained

$$\sigma(k) = \left(\frac{e^2}{\hbar c}\right)\left(\frac{e^2}{mc^2}\right)^2 \frac{2mc^2}{k} \log \frac{k}{mc^2} \tag{2.11}$$

where the number 2 was not reliable. This contribution tends to zero as k tends to infinity. When $k \gg 4mc^2$ and $p_0, p, q \gg mc^2$, he obtained

$$\sigma(k) \sim \left(\frac{e^2}{\hbar c}\right)\left(\frac{e^2}{mc^2}\right)^2 \frac{1}{64} S \tag{2.12}$$

where $\zeta(\sim 1)$ is independent of k , so that when he summed all the contributions for $k \gg 4mc^2$, this term merely altered the constant in (2.10) and not the dependence on k . Thus Votruba, with Wheeler and Lamb, has shown that for high incident γ -ray energy, small momentum recoils are the most important, large momentum transfers merely altering the additive constant. The case of $k \gg 4mc^2$ and $q, p, p_0 \ll mc^2$ is prohibited by the kinematics of the reaction.

(b) Experimental.

The experiments dealing with pair creation in the field of an electron fall into two classes: (i) those studying the process directly; (ii) those measuring total absorption or total pair production cross-sections for γ -rays.

(i) The experimentalists observing the process directly have all used cloud chambers as the detecting tool. In their cloud chamber photographs they looked for triplets and early attempts at finding them (11) failed, probably due to the fact that the recoil momentum was small and this electron track did not show up. Da Silva (12) found one triplet starting in a lead foil, but this evidence was not satisfactory since the apex of the triplet was not clearly visible. One triplet was also found by Shinohara and Hatoyama (13) who used a gas-filled chamber so that the apex was clear, but they made no attempt to check the conservation of energy. They did measure the energies of the three final particles and found that one electron had much smaller energy than the other two particles. Ogle and Kruger (14) reported the finding of two triplets and verified that energy and momentum were conserved.

Phillips and Kruger (15) used γ -ray energies between 6 and 7 Mev (their results are to be taken for 6.5 Mev) and studied the relative numbers of pairs and triplets in a cloud chamber filled with methane, air or argon. They found a mean value for $\frac{C_2}{C_1}$ of 0.255 ± 0.018 and also found that one electron had always much smaller energy than the positron or the other electron. An interesting result given by them is the energy distribution of the recoil electron (one with small energy), ~~is~~ shown in fig.4. This shows that the main contribution to the cross-section comes from small momentum transfers. Votruba gives no values for the cross-section at this energy and Borsellino's value of $\frac{C_2}{C_1}$ is about 0.4. Neither of them shows that small momentum transfers are the important ones for this small energy.

Gaerttner and Yeater (16) also measured the ratio of the numbers of pairs to triplets, using an air-filled cloud chamber and the continuous γ-ray spectrum

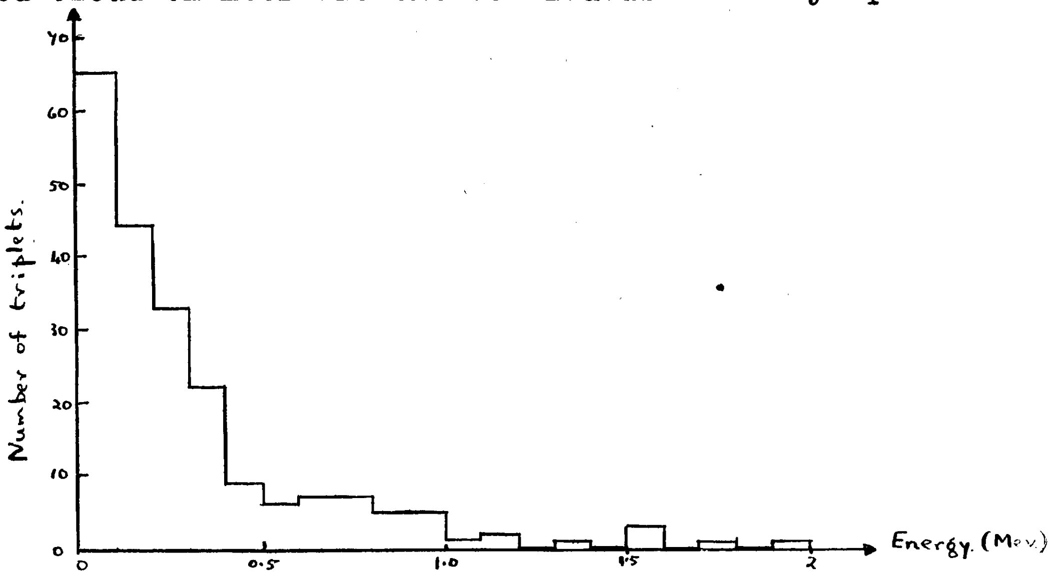

Fig.4. Distribution of the measured values of the kinetic energy of the low energy electron, as measured by Phillips and Kruger.

from a 100 Mev betatron. For the energy range 5 to 20 Mev (mean energy 11 Mev) they found $\frac{C_2}{C_1} = 0.75 \pm 0.19$ (compared to Borsellino's 0.61) and for range 20 to 100 Mev, $\frac{C_2}{C_1} = 0.8 \pm 0.16$ (compared to Borsellino's value between 0.80 and 0.86 and Wheeler and Lamb value between 1 and 1.1). They, also, found a preponderance of recoils with momentum transfer $< mc^2$, a fact which theory has not predicted for the energy range 5 to 20 Mev.

Koch and Carter (17) measured the energy distribution of bremsstrahlung from 19.5 Mev electrons by allowing the bremsstrahlung to create pairs and triplets in the gas (air) contained in a cloud chamber. From the 10,300 pictures they analysed, they found 1,300 cases of nuclear pair production and 33 of electronic. Their results for air are shown in fig.5. Fig.(5a) shows the

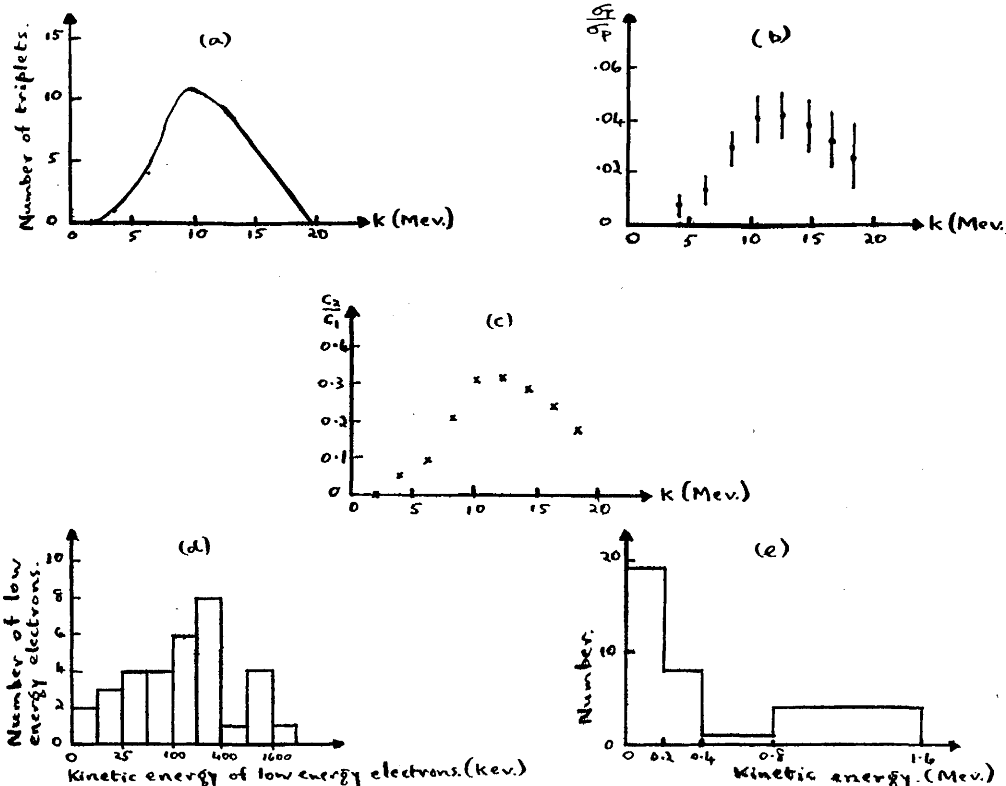

Fig.5. Results of Koch and Carter.

energy distribution of the triplets and is the best smooth curve representing the data. Fig.(5b) shows their values of ratio of number of triplets to number of pairs. They do not quote a value of $\frac{(Z)^2_{effective}}{(Z)_{effective}}$ but assuming it is the same as that of Phillips and Kruger i.e. 7.35, a plot of $\frac{C_2}{C_1}$ can be drawn and this is shown in fig.(5c). They found that one electron had smaller energy than the other particles and fig.(5d) shows the energy distribution of this low energy electron. Fig.(5e) contains the results of fig.(5d) put into a form which can be compared to the corresponding graph of Phillips and Kruger. The two are seen to be similar.

As yet, no one has used electronic apparatus for observing the process directly. It would be useful to repeat the above experiments for the mono-energetic 17.6 Mev γ -rays from the $Li^7(p,\gamma)Be^8$ reaction, observing the creation in the gas of the chamber. As we have seen, creation in plates inserted in the chamber hides the apex of the triplet. A gas of low Z would make the number of triplets more comparable to the number of pairs (though reducing both) and would allow better comparison with theory, since the Born approximation would be valid.

(ii) We shall deal briefly with the measurements of total absorption cross-sections, since these experiments are more in the nature of tests of the validity of the Born approximation for nuclear pair production. The cross-section for electronic pair creation is, for high Z values, so much smaller than the nuclear pair creation cross-section, that wide variations in it make little difference to the total absorption cross-section. For low energies the Compton scattering cross-section also masks variations in the electronic pair production cross-section.

Total absorption cross-sections have been measured by Adams (18) using 11.04, 13.73 and 19.10 Mev γ -rays, by Lawson (19) using 88 Mev γ -rays, by Walker (20) using 17.6Mev γ -rays, ~~and~~ by de Wire, Ashkin and Beach (21) using 280 Mev γ -rays and by Rosenblum, Shrader and Warner (22) using 5.3, 10.3 and 17.6 Mev γ -rays. These workers used the results of either Borsellino or Wheeler and Lamb for the electronic pair production cross-section and added to this the Klein-Nishina (23) value for the Compton scattering cross-section and the Bethe-Heitler (1) value for the nuclear pair production cross-section. Some also took account of the atomic photo-electric and the nuclear photo-disintegration cross-sections. This gave the theoretical total absorption cross-section and discrepancies between it and the experimental one were attributed to the break-down in validity of the Born approximation in the calculation of Bethe and Heitler. To obtain a rough idea of the orders of magnitude involved we quote the results of Walker in Table 1.

There have also been experiments designed to measure the total pair production cross-section alone. Walker (24) has done this with 17.6 Mev γ -rays and found a value for $\frac{C_2}{C_1}$ of 0.8 ± 0.3 compared to

Element.	Compton Cross-section.	Pairs (unscreened nucleus).	Pairs (screened nucleus).	Pairs (electron).	Total σ (theory).	Total σ (experimental).
C	0·2004	0·1100	0·1085	0·0124	0·3213	0·323 ± 1·4%
Al	0·434	0·516	0·505	0·024	0·966	0·942 ± 1·1%
Cu	0·968	2·569	2·489	0·060	3·514	3·62 ± 0·6%
Sn	1·64	7·64	7·33	0·10	9·10	8·96 ± 1·0%
Pb	2·44	20·54	19·51	0·14	22·58	20·56 ± 0·6%

Table 1. Cross-sections (in units 10^{-24} cm^2) obtained by Walker for 17.6 Mev γ -rays. Total σ for Pb includes a contribution of 0.16 for atomic photo-electric effect.

Borsellino's 0.7 and the Wheeler and Lamb value of about unity. To arrive at this, he used $\sigma \propto (Z^2 + \frac{C_2}{C_1} Z)\{1 - S(Z)\}$ where $S(Z)$ measures the effect of screening. He then plotted $\frac{\sigma_{observed}}{Z\{1-S(Z)\}}$ against Z , obtaining a straight line for small Z , and the intercept on the Z -axis gave value of $\frac{C_2}{C_1}$.

Emigh (25) used a cloud chamber to determine the relative nuclear pair production cross-sections in various elements, which were inserted as plates in the cloud chamber. He was using γ -rays from a 300 Mev betatron and assumed that in the region 50 to 300 Mev, any triplets formed would be counted as pairs since the low energy electron would not be observed. Thus his nuclear pair production cross-section had to be corrected for triplets and to do this he used the theory of Wheeler and Lamb. He then studied the deviations of these cross-sections from the values predicted by Bethe and Heitler. This experiment gives, therefore, no information about electronic pair production.

Thus the experiments which give a measured value of $\frac{c_2}{c_1}$ are those which observe the process directly and those which study the dependence on Z of the total pair production cross-section. The experiments show that small momentum transfers are the most important for all energies of the incident γ -ray. They also show that $\frac{c_2}{c_1} < 1$.

§3. Rules for Calculating by the Feynman Method.

The study of the problem, presented in this thesis, will be based on Feynman (26) quantum electrodynamics which is now widely used. The correct numerical factors are not, however, made clear in Feynman's papers and we shall list in this section, following Peshkin (27), the rules whereby these are obtained. We shall use units in which $\hbar = c = 1$; $\underset{\sim}{k}$ will denote a four-vector and $\vec{k}$ a three-vector; $\not{k}$ will denote $k_4\gamma_4 - k_1\gamma_1 - k_2\gamma_2 - k_3\gamma_3$ where the γ 's are the usual Feynman matrices.

(i) Draw all the possible graphs for the process and obtain the matrix element M for each by writing γ_σ for the emission or absorption of a real photon polarised in the σ -direction, $\frac{e^2}{\pi} \frac{\gamma_\mu - - - - - \gamma_\mu}{k^2}$ for a virtual photon with momentum-energy $\underset{\sim}{k}$ and $\frac{1}{\not{p} - m}$ for a virtual electron with momentum-energy $\underset{\sim}{p}$. Summing over μ from 1 to 4 $(\gamma_\mu\gamma_\mu = \gamma_4\gamma_4 - \gamma_1\gamma_1 - \gamma_2\gamma_2 - \gamma_3\gamma_3 = 4)$ means summing over the directions of polarisation of the virtual quantum.

(ii) The probability amplitude for a transition from the state 1 to the state 2 of an electron, involving a real photon polarised in the σ -direction will be $\propto (\bar{u}_2 M_\sigma u_1)$ where u_1 , u_2 are the Dirac spinors describing electrons with momenta-energies $\underset{\sim}{p_1}$ and $\underset{\sim}{p_2}$ respectively and $\bar{u} = u^+\gamma_4$.

Find $|M|^2 = (\bar{u}_1 \bar{M}_\sigma u_2)(\bar{u}_2 M_\sigma u_1)$ where $\bar{M}_\sigma$ is M_σ with the order of the γ -matrices reversed and with explicit appearance of i changed to $-i$. Summing this over σ from 1 to 4 means summing over the directions of polarisation of the real photon. If u_1 and u_2 are positive energy states then we have, after summation over the spins of electrons 1 and 2,

$$|M|^2 = \frac{1}{(2m)^2} \, \mathrm{Spur}\left[(p_1+m)\,\bar{M}_\sigma\,(p_2+m)\,M_\sigma\right]$$ (See Appendix)

(iii) Multiply $|M|^2$ by $\left(\frac{e}{2\pi}\right)^2 \frac{d\vec{k}}{k}$ for each real photon in the final state, by $\frac{2\pi e^2}{k}$ for each real photon in the initial state, by $\frac{m}{E}\,\frac{d\vec{p}}{2\pi}$ for each real electron in the final state, and by $\frac{m}{E}(2\pi)^2$ for each real electron in the initial state. Here E, k are electron and photon energies respectively, and the appropriate values of these and of $\vec{p}$, $\vec{k}$ have to be used. The factors $\frac{m}{E}$ arise from the fact that we have normalised to $\bar{u}u = u^+\gamma_4 u = 1$ instead of the usual $u^+u = 1$. On our scale, $u^+u = \frac{E}{m}$.

(iv) Multiply by $\frac{1}{2}\,\delta(p_f - p_i)$ when the final momentum-energy p_f equals the initial momentum-energy p_i.

(v) To obtain the differential cross-section in natural units, multiply by $\frac{2}{v}$, where v is the collision velocity for the initial state.

(vi) To obtain the differential cross-section in cm^2, remove a factor $\left(\frac{e^2}{m}\right)^2$ which becomes $\left(\frac{e^2}{mc^2}\right)^2$. In the remaining dimensionless factor make the usual transformation from natural units to ordinary units.

CHAPTER II.

§4. Outline of Problem.

We consider the problem in which the initial state consists of a stationary electron and a photon of momentum $\vec{k}$ and energy k, and the final state consists of a positron of momentum $\vec{p}_0$ and energy ω_0,

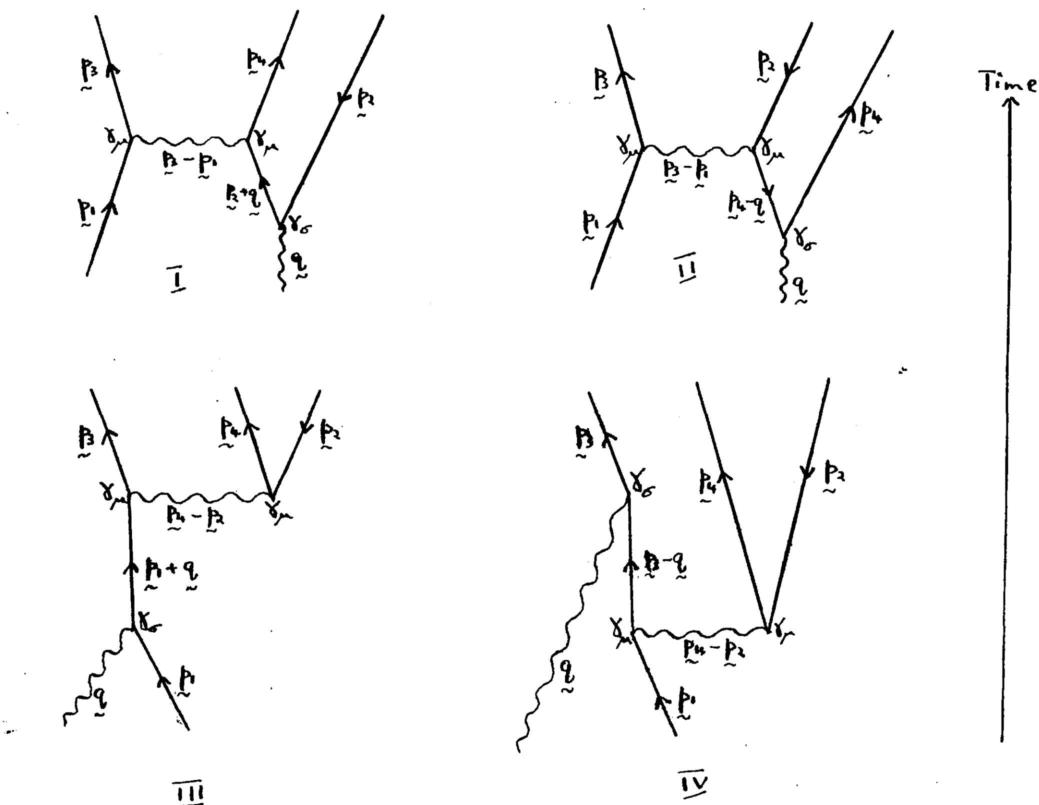

Fig.6. Feynman diagrams for the process.

and two electrons of momenta $\vec{q}$ and $\vec{p}$ and energies W and ω. The graphs for this process are shown in fig.6, where we have labelled the electron and photon lines in a

4-dimensional notation. The electron with momentum-energy $\underset{\sim}{p}_2$ is in a negative energy state and is drawn moving backwards in time. This is equivalent to a positron with momentum-energy $-\underset{\sim}{p}_2$ moving forwards in time. From these four diagrams we get another four, Ie, IIe, IIIe and IVe, by exchanging the states 3 and 4. These exchange matrix elements have to be subtracted, in accordance with the Pauli Exclusion Principle, from the first four. Thus the total matrix element is given by

$$M = \sum_{i=\mathrm{I}}^{\mathrm{IV}} M_i - \sum_{i=\mathrm{I}e}^{\mathrm{IV}e} M_i \qquad (4.1)$$

where M_i is the matrix element associated with the i^{th} diagram.

Watson (7) used $M = \sum_{i=\mathrm{I}}^{\mathrm{IV}} M_i' - \sum_{i=\mathrm{I}e}^{\mathrm{IV}e} M_i'$, where M_i' contained only the Coulomb part of the interaction between the two electrons. Nemirovsky (8) used $M = M_{\mathrm{I}} + M_{\mathrm{II}} - M_{\mathrm{I}e} - M_{\mathrm{II}e}$, while Borsellino used only $M = M_{\mathrm{I}} + M_{\mathrm{II}}$. Votruba used $M = \sum_{i=\mathrm{I}}^{\mathrm{IV}} M_i - \sum_{i=\mathrm{I}e}^{\mathrm{IV}e} M_i$, but was unable to find a complete excitation curve. We shall use $M = \sum_{i=\mathrm{I}}^{\mathrm{IV}} M_i$ and, following the order of integration of Borsellino, arrive at a complete excitation curve.

To assess how important $M_{\mathrm{III}} + M_{\mathrm{IV}}$ is compared to $M_{\mathrm{I}} + M_{\mathrm{II}}$, we shall assume that the positron and three electrons have roughly the same energy and that the matrix elements of all the diagrams, except for the energy denominators, are also roughly equal. The energy denominators arising from the virtual electron lines will then all be roughly equal, and the relevant energy denominators will be those arising from the virtual photon lines. These are $\frac{1}{(p_3-p_1)^2}$ for $M_{I} + M_{II}$ and $\frac{1}{(p_4-p_2)^2}$ for $M_{III} + M_{IV}$. Now p_1, p_3 and p_4 all correspond to positive energy electrons, whereas p_2 corresponds to a negative energy electron. Thus $(p_4-p_2)^2$ will be greater than $(p_3-p_1)^2$, so that $M_{III}+M_{IV}$ will be smaller than $M_{I}+M_{II}$. Actually it will be found that

$$(p_3-p_1)^2 = 2\{m^2-(p_1 \cdot p_3)\} = 2\{m^2-mW\}$$

and

$$(p_4-p_2)^2 = 2\{m^2-(p_2 \cdot p_4)\} = 2\{m^2-mW+k\omega_0-(\vec{k}\cdot\vec{p}_0)+k\omega-(\vec{k}\cdot\vec{p})\}.$$

$k\omega_0-(\vec{k}\cdot\vec{p}_0)+k\omega-(\vec{k}\cdot\vec{p})$ is positive and increases as k increases. Therefore as k increases, $M_{III}+M_{IV}$ will become less important compared to $M_{I}+M_{II}$.

Keeping p_2 as a positive energy particle and changing the sign of k would give the matrix element for electron-electron bremsstrahlung.

§5. Square of Matrix Element.

Let u_1, u_2, u_3 and u_4 be the Dirac spinors describing the states of electrons with momenta-energies $\underset{\sim}{p}_1$, $\underset{\sim}{p}_2$, $\underset{\sim}{p}_3$ and $\underset{\sim}{p}_4$ respectively. Then $\not{p}_i u_i = m u_i$ and $\underset{\sim}{p}_i^2 = m^2$ for $i = 1,2,3,4$. Let $\underset{\sim}{k}$ be the momentum-energy of the real photon, so that $\underset{\sim}{k}^2 = 0$. Let the incident real photon be polarised in the σ -direction. Then from the graphs of § 4 and the rules of § 3 we can immediately write down the matrix elements.

$$M_{\mathrm{I}} = \frac{e^2}{\pi} \{\bar{u}_3 \gamma_\mu u_1\} \{\bar{u}_4 \gamma_\mu (\not{p}_2 + \not{k} - m)^{-1} \gamma_\sigma u_2\} \frac{1}{(\not{p}_3 - \not{p}_1)^2}$$

$$= \frac{e^2}{\pi} \{\bar{u}_3 \gamma_\mu u_1\} \{\bar{u}_4 \gamma_\mu \frac{\not{p}_2 + \not{k} + m}{(\not{p}_2 + \not{k})^2 - m^2} \gamma_\sigma u_2\} \frac{1}{(\not{p}_3 - \not{p}_1)^2}$$

$$= \frac{e^2}{\pi} \{\bar{u}_3 \gamma_\mu u_1\} \{\bar{u}_4 \gamma_\mu \frac{(-\gamma_\sigma \not{p}_2 + 2 p_{2\sigma} + \not{k}\gamma_\sigma + m\gamma_\sigma)}{\not{p}_2^2 + 2(\underset{\sim}{p}_2 \cdot \underset{\sim}{k}) + \underset{\sim}{k}^2 - m^2} u_2\} \frac{1}{(\not{p}_3 - \not{p}_1)^2}$$

where we have used $\not{p}_2 \gamma_\sigma + \gamma_\sigma \not{p}_2 = 2 p_{2\sigma}$. Since $\not{p}_2 u_2 = m u_2$, $\underset{\sim}{k}^2 = 0$ and $\underset{\sim}{p}_2^2 = m^2$, we find

$$M_{\mathrm{I}} = \frac{e^2}{\pi} \{\bar{u}_3 \gamma_\mu u_1\} \{\bar{u}_4 \gamma_\mu (2 p_{2\sigma} + \not{k}\gamma_\sigma) u_2\} \frac{1}{2(\underset{\sim}{p}_2 \cdot \underset{\sim}{k})(\not{p}_3 - \not{p}_1)^2} .$$

Similarly

$$M_{\mathrm{II}} = -\frac{e^2}{\pi} \{\bar{u}_3 \gamma_\mu u_1\} \{\bar{u}_4 (2 p_{4\sigma} - \gamma_\sigma \not{k}) \gamma_\mu u_2\} \frac{1}{2(\underset{\sim}{p}_4 \cdot \underset{\sim}{k})(\not{p}_3 - \not{p}_1)^2} .$$

Therefore

$$M_{\mathrm{I}} + M_{\mathrm{II}} = \frac{e^2}{\pi} \{\bar{u}_3 \gamma_\mu u_1\} \{\bar{u}_4 [2(\underset{\sim}{p}_4 \cdot \underset{\sim}{k}) p_{2\sigma} \gamma_\mu + (\underset{\sim}{p}_4 \cdot \underset{\sim}{k}) \gamma_\mu \not{k} \gamma_\sigma - 2(\underset{\sim}{p}_2 \cdot \underset{\sim}{k}) p_{4\sigma} \gamma_\mu$$

$$+ (\underset{\sim}{p}_2 \cdot \underset{\sim}{k}) \gamma_\sigma \not{k} \gamma_\mu] u_2\} \frac{1}{2(\underset{\sim}{p}_2 \cdot \underset{\sim}{k})(\underset{\sim}{p}_4 \cdot \underset{\sim}{k})(\not{p}_3 - \not{p}_1)^2}$$

$M_{III} + M_{IV}$ is obtained from $M_I + M_{II}$ by making the replacements $\not p_1 \rightleftarrows \not p_2$ and $\not p_3 \rightleftarrows \not p_4$. Therefore

$$M_{III} + M_{IV} = \frac{e^2}{\pi}\{\bar{u}_4 \gamma_\mu u_2\}\{\bar{u}_3 [2(p_3 \cdot k) p_{1\sigma}\gamma_\mu + (p_3 \cdot k)\gamma_\mu \not k \gamma_\sigma - 2(p_1 \cdot k) p_{3\sigma}\gamma_\mu + (p_1 \cdot k)\gamma_\sigma \not k \gamma_\mu] u_1\} \frac{1}{2(p_1 \cdot k)(p_3 \cdot k)(\not p_4 - \not p_2)^2}.$$

We now square M, but defer discussion of the cross-product term $2(\overline{M_I + M_{II}})(M_{III} + M_{IV})$ until § 8.

$$|M_I + M_{II}|^2 = \frac{e^4}{\pi^2} \frac{1}{4(p_2 \cdot k)^2 (p_4 \cdot k)^2 (\not p_3 - \not p_1)^4} \{\bar{u}_1 \gamma_\nu u_3\}\{\bar{u}_3 \gamma_\mu u_1\}$$

$$\times \{\bar{u}_2 [2(p_4 \cdot k) p_{2\sigma}\gamma_\nu + (p_4 \cdot k)\gamma_\sigma \not k \gamma_\nu - 2(p_2 \cdot k) p_{4\sigma}\gamma_\nu + (p_2 \cdot k)\gamma_\nu \not k \gamma_\sigma] u_4\}$$

$$\times \{\bar{u}_4 [2(p_4 \cdot k) p_{2\sigma}\gamma_\mu + (p_4 \cdot k)\gamma_\mu \not k \gamma_\sigma - 2(p_2 \cdot k) p_{4\sigma}\gamma_\mu + (p_2 \cdot k)\gamma_\sigma \not k \gamma_\mu] u_2\}.$$

Summing over the spins of the electrons with momenta-energies p_3, p_4 and p_2, averaging over the spin of the other electron and averaging over the directions of polarisation of the incident quantum, we obtain

$$|M_{I} + M_{II}|^2 = \frac{e^4}{\pi^2}\,\frac{1}{4(p_2\cdot k)^2(p_4\cdot k)^2(\not{p}_3-\not{p}_1)^2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{(2m)^4}$$
$$\times\,\mathrm{Spur}\left[(\not{p}_1+m)\,\gamma_\nu\,(\not{p}_3+m)\,\gamma_\mu\right]$$
$$\times\,\mathrm{Spur}\Big[(\not{p}_2+m)\{2(p_4\cdot k)\,p_{2\sigma}\gamma_\nu + (p_4\cdot k)\,\gamma_\sigma\not{k}\gamma_\nu - 2(p_2\cdot k)\,p_{4\sigma}\gamma_\nu + (p_2\cdot k)\,\gamma_\nu\not{k}\gamma_\sigma\}$$
$$(\not{p}_4+m)\{2(p_4\cdot k)\,p_{2\sigma}\gamma_\mu + (p_4\cdot k)\,\gamma_\mu\not{k}\gamma_\sigma - 2(p_2\cdot k)\,p_{4\sigma}\gamma_\mu + (p_2\cdot k)\,\gamma_\sigma\not{k}\gamma_\mu\}\Big]$$

where summation from 1 to 4 of the indices μ , ν and σ is to be understood. In evaluating the spurs we remember that the spur of a product of an odd number of γ -matrices vanishes and that the spur of a product of γ -matrices is not altered by cyclic permutation of the γ -matrices. We use the following results of Feynman:

$$\gamma_\sigma\gamma_\sigma = 4\ ; \qquad \gamma_\sigma\not{A}\gamma_\sigma = -2\not{A}\ ; \qquad \gamma_\sigma\not{A}\not{B}\gamma_\sigma = 4(A\cdot B);$$

$$\gamma_\sigma\not{A}\not{B}\not{C}\gamma_\sigma = -2\not{C}\not{B}\not{A}\ ;\quad \not{A}\not{B}+\not{B}\not{A} = 2(A\cdot B);\quad \not{p}\gamma_\mu+\gamma_\mu\not{p} = 2p_\mu\ ;$$

$$k^2 = 0\ ; \qquad p_2^2 = p_4^2 = m^2;$$

$$\mathrm{Spur}\,[\not{A}\not{B}] = 4(A\cdot B);$$

$$\mathrm{Spur}\,[\not{A}\not{B}\not{C}\not{D}] = 4(A\cdot B)(C\cdot D) - 4(A\cdot C)(B\cdot D) + 4(A\cdot D)(B\cdot C);$$

$$\mathrm{Spur}\,[\gamma_\nu\gamma_\mu] = 4\,\delta_{\mu\nu};$$

$$\mathrm{Spur}\,[\not{q}\gamma_\mu] = 4q_\mu\,.$$

Here $\delta_{44}=1,\ \delta_{11}=\delta_{22}=\delta_{33}=-1$ and all other $\delta_{\mu\nu}=0$.

Let

$$\mathrm{Spur}\,X = \mathrm{Spur}\left[(\not{p}_1+m)\,\gamma_\nu\,(\not{p}_3+m)\,\gamma_\mu\right]$$

$$= 4\left[(p_{1\nu}p_{3\mu}+p_{1\mu}p_{3\nu}) + \{m^2-(\underset{\sim}{p_1}\cdot\underset{\sim}{p_3})\}\,\delta_{\mu\nu}\right].$$

Let

$$\begin{aligned}\mathrm{Spur}\,Y = \mathrm{Spur}\Big[(\not{p}_2+m)\big\{&2(\underset{\sim}{p_4}\cdot\underset{\sim}{k})\,p_{2\sigma}\gamma_\nu + (\underset{\sim}{p_4}\cdot\underset{\sim}{k})\,\gamma_\sigma K\gamma_\nu \\ &-2(\underset{\sim}{p_2}\cdot\underset{\sim}{k})\,p_{4\sigma}\gamma_\nu + (\underset{\sim}{p_2}\cdot\underset{\sim}{k})\,\gamma_\nu \not{K}\gamma_\sigma\big\} \\ +(\not{p}_4+m)\big\{&2(\underset{\sim}{p_4}\cdot\underset{\sim}{k})\,p_{2\sigma}\gamma_\mu + (\underset{\sim}{p_4}\cdot\underset{\sim}{k})\,\gamma_\mu K\gamma_\sigma \\ &-2(\underset{\sim}{p_2}\cdot\underset{\sim}{k})\,p_{4\sigma}\gamma_\mu + (\underset{\sim}{p_2}\cdot\underset{\sim}{k})\,\gamma_\sigma\not{K}\gamma_\mu\big\}\Big]\end{aligned}$$

then

$$\begin{aligned}\frac{\mathrm{Spur}\,Y}{4(\underset{\sim}{p_2}\cdot\underset{\sim}{k})^2(\underset{\sim}{p_4}\cdot\underset{\sim}{k})^2} = 4\Big[&A(p_{2\nu}p_{4\mu}+p_{2\mu}p_{4\nu}) - A(\underset{\sim}{p_2}\cdot\underset{\sim}{p_4})\,\delta_{\mu\nu} \\ &+B(p_{2\nu}k_\mu+p_{2\mu}k_\nu) - B(\underset{\sim}{p_2}\cdot\underset{\sim}{k})\,\delta_{\mu\nu} \\ &+C(p_{4\nu}k_\mu+p_{4\mu}k_\nu) - C(\underset{\sim}{p_4}\cdot\underset{\sim}{k})\,\delta_{\mu\nu} \\ &+D\,\delta_{\mu\nu} + \frac{2m^2k_\nu k_\mu}{(\underset{\sim}{p_2}\cdot\underset{\sim}{k})(\underset{\sim}{p_4}\cdot\underset{\sim}{k})} \\ &-\frac{2p_{2\nu}p_{2\mu}}{(\underset{\sim}{p_2}\cdot\underset{\sim}{k})} + \frac{2p_{4\nu}p_{4\mu}}{(\underset{\sim}{p_4}\cdot\underset{\sim}{k})}\Big]\end{aligned}$$

where

$$A = \frac{m^2}{(\underset{\sim}{p_2}\cdot\underset{\sim}{k})^2} + \frac{m^2}{(\underset{\sim}{p_4}\cdot\underset{\sim}{k})^2} - \frac{2(\underset{\sim}{p_2}\cdot\underset{\sim}{p_4})}{(\underset{\sim}{p_2}\cdot\underset{\sim}{k})(\underset{\sim}{p_4}\cdot\underset{\sim}{k})} - \frac{1}{(\underset{\sim}{p_2}\cdot\underset{\sim}{k})} + \frac{1}{(\underset{\sim}{p_4}\cdot\underset{\sim}{k})}$$

$$B = -\frac{m^2}{(\underset{\sim}{p_4}\cdot\underset{\sim}{k})^2} - \frac{1}{(\underset{\sim}{p_4}\cdot\underset{\sim}{k})} + \frac{(\underset{\sim}{p_2}\cdot\underset{\sim}{p_4})}{(\underset{\sim}{p_2}\cdot\underset{\sim}{k})(\underset{\sim}{p_4}\cdot\underset{\sim}{k})}$$

$$C = \frac{m^2}{(p_2\cdot k)^2} - \frac{1}{(p_2\cdot k)} - \frac{(p_2\cdot p_4)}{(p_2\cdot k)(p_4\cdot k)}$$

$$D = -\frac{2m^2(p_2\cdot p_4)}{(p_2\cdot k)(p_4\cdot k)} + \frac{m^4}{(p_4\cdot k)^2} + \frac{m^4}{(p_2\cdot k)^2} + \frac{m^2}{(p_2\cdot k)} - \frac{m^2}{(p_4\cdot k)} .$$

To evaluate $\mathrm{Spur}\,X\ \mathrm{Spur}\,Y$ we use:

$$\delta_{\mu\nu}\,\delta_{\mu\nu} = 4 \; ; \qquad \delta_{\mu\nu}(A_\nu B_\mu + A_\mu B_\nu) = 2(A\cdot B) \; ;$$

$$(A_\nu B_\mu + A_\mu B_\nu)(C_\nu D_\mu + C_\mu D_\nu) = 2\left\{(A\cdot C)(B\cdot D) + (A\cdot D)(B\cdot C)\right\} .$$

We write $(\not{p}_3-\not{p}_1)^2 = 2\{m^2-(p_1\cdot p_3)\}$ $\because$ $p_1^2 = p_3^2 = m^2$.

The final result is:

$$|M_{\mathrm{I}} + M_{\mathrm{II}}|^2 = \frac{e^4}{\pi^2}\,\frac{1}{8m^4}\,\frac{1}{\{m^2-(p_1\cdot p_3)\}^2}\times$$

$$\times\Bigg[\!\!\Bigg[\frac{1}{(p_2\cdot k)}\Big\{m^4 + m^2(p_2\cdot p_4) - 2(p_1\cdot p_2)(p_2\cdot p_3) - (p_1\cdot p_2)(p_3\cdot p_4) - (p_1\cdot p_4)(p_2\cdot p_3)$$
$$- (p_1\cdot p_4)(p_3\cdot k) - (p_1\cdot k)(p_3\cdot p_4) + m^2(p_4\cdot k)\Big\}$$

$$+ \frac{1}{(p_4\cdot k)}\Big\{-m^4 - m^2(p_2\cdot p_4) + (p_1\cdot p_2)(p_3\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3) + 2(p_1\cdot p_4)(p_3\cdot p_4)$$
$$- (p_1\cdot p_3)(p_3\cdot k) - (p_1\cdot k)(p_2\cdot p_3) + m^2(p_2\cdot k)\Big\}$$

$$+ \frac{m^2}{(p_2\cdot k)^2}\Big\{-m^2(p_2\cdot p_4) + (p_1\cdot p_3)(p_3\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3) - m^2(p_4\cdot k) + (p_1\cdot p_4)(p_3\cdot k)$$
$$+ (p_1\cdot k)(p_3\cdot p_4) + 2m^4 - m^2(p_1\cdot p_3)\Big\}$$

$$+ \frac{m^2}{(p_4\cdot k)^2}\Big\{-m^2(p_2\cdot p_4) + (p_1\cdot p_2)(p_3\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3) + m^2(p_2\cdot k) - (p_1\cdot p_2)(p_3\cdot k)$$
$$- (p_1\cdot k)(p_2\cdot p_3) + 2m^4 - m^2(p_1\cdot p_3)\Big\}$$

$$+ \frac{1}{(p_2\cdot k)(p_4\cdot k)}\Big\{(p_2\cdot p_4)\big[m^2(p_2\cdot p_4) - (p_1\cdot p_3)(p_3\cdot p_4) - (p_1\cdot p_4)(p_2\cdot p_3) - m^2(p_2\cdot k)$$
$$+ (p_1\cdot p_2)(p_3\cdot k) + (p_1\cdot k)(p_2\cdot p_3) - 2m^4 + m^2(p_1\cdot p_3)\big]$$
$$+ (p_2\cdot p_4)\big[m^2(p_2\cdot p_4) - (p_1\cdot p_2)(p_3\cdot p_4) - (p_1\cdot p_4)(p_2\cdot p_3) + m^2(p_4\cdot k)$$
$$- (p_1\cdot p_4)(p_3\cdot k) - (p_1\cdot k)(p_3\cdot p_4) - 2m^4 + m^2(p_1\cdot p_3)\big]$$
$$+ 2m^2(p_1\cdot k)(p_3\cdot k)\Big\}\Bigg]\!\!\Bigg]$$

$|M_{III}+M_{IV}|^2$ is obtained from this by making the replacements $p_1 \rightleftarrows p_2$ and $p_3 \rightleftarrows p_4$.

§6. The Differential Cross-Section.

To pass over into the notation used in the first sentence of § 4, we put

$$p_1 = (m, \vec{0}) \quad , \quad p_2 = (-w_0, -\vec{p}_0) \quad , \quad k = (k, \vec{k}),$$

$$p_3 = (W, \vec{q}) \quad , \quad p_4 = (\omega, \vec{p}).$$

Conservation of energy and momentum give

$$-\vec{p}_0 + \vec{k} = \vec{p} + \vec{q} \tag{6.1}$$

and

$$m - w_0 + k = W + \omega \tag{6.2}$$

$$\text{i.e.} \quad m - \sqrt{p_0^2+m^2} + k = \sqrt{q^2+m^2} + \sqrt{p^2+m^2}.$$

Let $\quad k w_0 - (\vec{k}\cdot\vec{p}_0) = (k \cdot n_0)$ and $k\omega - (\vec{k}\cdot\vec{p}) = (k \cdot n)$. (6.3)

As an illustration of the change from four-dimensional to three-dimensional notation, we evaluate $(p_2 \cdot p_3)$.

$$(p_2 \cdot p_3) = -w_0 W + (\vec{p}_0 \cdot \vec{q}).$$

Now $\quad \vec{p}_0 + \vec{q} = \vec{k} - \vec{p}$

$$\therefore \quad 2(\vec{p}_0 \cdot \vec{q}) = k^2 + p^2 - 2(\vec{k}\cdot\vec{p}) - p_0^2 - q^2$$

$$= k^2 + \omega^2 - m^2 + 2(\underset{\sim}{k}\cdot\underset{\sim}{\Pi}) - 2k\omega - \omega_0^2 + m^2 - W^2 + m^2$$

$$= (k-\omega)^2 - (W - m + \omega_0)^2 + 2(\underset{\sim}{k}\cdot\underset{\sim}{\Pi}) + 2m^2 - 2mW + 2\omega_0 W - 2m\omega_0 .$$

$$\therefore \quad (\underset{\sim}{p_2}\cdot\underset{\sim}{p_3}) = m^2 - mW - m\omega_0 + (\underset{\sim}{k}\cdot\underset{\sim}{\Pi})$$

$$= (\underset{\sim}{k}\cdot\underset{\sim}{\Pi}) - m(k-\omega) .$$

The other four-dimensional scalar products can be evaluated similarly and we find:

$(\underset{\sim}{p_1}\cdot\underset{\sim}{p_2}) = -m\omega_0$ $\qquad (\underset{\sim}{p_1}\cdot\underset{\sim}{p_3}) = mW$

$(\underset{\sim}{p_1}\cdot\underset{\sim}{p_4}) = m\omega$ $\qquad (\underset{\sim}{p_1}\cdot\underset{\sim}{k}) = mk$

$(\underset{\sim}{p_2}\cdot\underset{\sim}{p_3}) = (\underset{\sim}{k}\cdot\underset{\sim}{\Pi}) - m(k-\omega)$ $\qquad (\underset{\sim}{p_2}\cdot\underset{\sim}{p_4}) = mW - (\underset{\sim}{k}\cdot\underset{\sim}{\Pi_0}) - (\underset{\sim}{k}\cdot\underset{\sim}{\Pi})$

$(\underset{\sim}{p_2}\cdot\underset{\sim}{k}) = -(\underset{\sim}{k}\cdot\underset{\sim}{\Pi_0})$ $\qquad (\underset{\sim}{p_3}\cdot\underset{\sim}{p_4}) = -(\underset{\sim}{k}\cdot\underset{\sim}{\Pi_0}) + m(k-\omega_0)$

$(\underset{\sim}{p_3}\cdot\underset{\sim}{k}) = mk - (\underset{\sim}{k}\cdot\underset{\sim}{\Pi_0}) - (\underset{\sim}{k}\cdot\underset{\sim}{\Pi})$ $\qquad (\underset{\sim}{p_4}\cdot\underset{\sim}{k}) = (\underset{\sim}{k}\cdot\underset{\sim}{\Pi})$.

After this change we have:

$$|M_{I} + M_{II}|^2 = \frac{e^4}{\pi^2}\,\frac{1}{8m^4}\,\frac{1}{(m^2-mW)^2}\times$$

$$\times\Big[\!\!\Big[\,\frac{2m^2(m-W)(-3m+W+\omega_0)}{(\underset{\sim}{k}\cdot\underset{\sim}{\Pi}_0)} + \frac{2m^2(m-W)(-3m+W+\omega)}{(\underset{\sim}{k}\cdot\underset{\sim}{\Pi})}$$

$$+ m(W-2m)\frac{(\underset{\sim}{k}\cdot\underset{\sim}{\Pi})}{(\underset{\sim}{k}\cdot\underset{\sim}{\Pi}_0)} + m(W-2m)\frac{(\underset{\sim}{k}\cdot\underset{\sim}{\Pi}_0)}{(\underset{\sim}{k}\cdot\underset{\sim}{\Pi})} + \frac{m^4\{(3m-W)(m-W)+2\omega(W-m+\omega)\}}{(\underset{\sim}{k}\cdot\underset{\sim}{\Pi}_0)^2}$$

$$+ \frac{m^4\{(3m-W)(m-W)+2\omega_0(W-m+\omega_0)\}}{(\underset{\sim}{k}\cdot\underset{\sim}{\Pi})^2} - \frac{2m^3}{(\underset{\sim}{k}\cdot\underset{\sim}{\Pi}_0)(\underset{\sim}{k}\cdot\underset{\sim}{\Pi})}\Big[(m-W)(\omega_0^2+\omega^2)$$

$$+ 2m\omega_0\omega + (m-W)\{-m^2-m(k+m)-(k-m)(m-W)\}\Big]\,\Big]\!\!\Big] \qquad (6.4)$$

and $$|M_{III}+M_{IV}|^2 = \frac{e^4}{\pi^2}\,\frac{1}{8m^4}\,\frac{1}{\{m^2-mW+(\underset{\sim}{k}\cdot\underset{\sim}{\Pi}_0)+(\underset{\sim}{k}\cdot\underset{\sim}{\Pi})\}^2}\times$$

$$\times\Big[\!\!\Big[\,\frac{1}{k}\Big[m\{W(3m-W)+kW+4\omega\omega_0\} + (\underset{\sim}{k}\cdot\underset{\sim}{\Pi}_0)(k-m-2\omega_0)$$
$$+(\underset{\sim}{k}\cdot\underset{\sim}{\Pi})(k-m-2\omega) - \frac{1}{m}(\underset{\sim}{k}\cdot\underset{\sim}{\Pi}_0)^2 - \frac{1}{m}(\underset{\sim}{k}\cdot\underset{\sim}{\Pi})^2\Big]$$

$$+\frac{m}{\{kW-(\vec{k}\cdot\vec{q})\}}\Big[-m\{W(3m-W)-3k(m-W)-mk+4\omega\omega_0\}$$
$$+(\underset{\sim}{k}\cdot\underset{\sim}{\Pi}_0)(k+W-m+2\omega_0)+(\underset{\sim}{k}\cdot\underset{\sim}{\Pi})(k+W-m+2\omega)-\frac{2}{m}(\underset{\sim}{k}\cdot\underset{\sim}{\Pi}_0)(\underset{\sim}{k}\cdot\underset{\sim}{\Pi})\Big]$$

$$+\frac{m}{k^2}\Big[m\{(3m-W)(m-W)-kW-2\omega\omega_0\}+(\underset{\sim}{k}\cdot\underset{\sim}{\Pi}_0)(2m-k+2\omega_0)$$
$$+(\underset{\sim}{k}\cdot\underset{\sim}{\Pi})(2m-k+2\omega)+\frac{1}{m}(\underset{\sim}{k}\cdot\underset{\sim}{\Pi}_0)^2+\frac{1}{m}(\underset{\sim}{k}\cdot\underset{\sim}{\Pi})^2\Big]$$

$$+\frac{m^3}{\{kW-(\vec{k}\cdot\vec{q})\}^2}\Big[m\{(3m-W)(m-W)+k(m-W)+mk-2\omega\omega_0\}$$
$$+(\underset{\sim}{k}\cdot\underset{\sim}{\Pi}_0)(k-W+2m)+(\underset{\sim}{k}\cdot\underset{\sim}{\Pi})(k-W+2m)\Big]$$

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$$+\frac{m}{k\{kW-(\vec{k}\cdot\vec{q})\}}\Big[-2mW\{(3m-W)(m-W)+k(m-W)-2ww_0\}$$
$$-W(\underset{\sim}{k}\cdot\underset{\sim}{\Pi_0})(4m-W+2w_0)-W(\underset{\sim}{k}\cdot\underset{\sim}{\Pi})(4m-W+2w)$$
$$-\frac{W}{m}(\underset{\sim}{k}\cdot\underset{\sim}{\Pi_0})^2-\frac{W}{m}(\underset{\sim}{k}\cdot\underset{\sim}{\Pi})^2-2(\underset{\sim}{k}\cdot\underset{\sim}{\Pi_0})(\underset{\sim}{k}\cdot\underset{\sim}{\Pi})\Big]\Big]. \quad (6.5)$$

From the rules of § 3, we find

$$d\sigma = |M|^2\frac{2\pi e^2}{k}\cdot\frac{m}{m}(2\pi)^2\cdot\frac{m}{W}\frac{d\vec{q}}{2\pi}\cdot\frac{m}{w}\frac{d\vec{p}}{2\pi}\cdot\frac{m}{w_0}\frac{d\vec{p}_0}{2\pi}\cdot\frac{1}{2}\delta(E_f-E_i)\,\delta(\vec{p}_f-\vec{p}_i)\cdot 2$$

$$= |M|^2\frac{e^2m^3}{kww_0W}\,d\vec{q}\,d\vec{p}\,d\vec{p}_0\,\delta(\vec{p}_f-\vec{p}_i)\,\delta(E_f-E_i)$$

$$= |M|^2\frac{e^2m^3}{kww_0W}\,d\vec{q}\,d\vec{p}\,\delta(E_f-E_i), \quad (6.6)$$

where conservation of momentum has been used in $|M|^2$.

$$\delta(E_f-E_i)\,d\vec{p} = p^2\frac{dp}{dE_f}\,d\Omega_p$$

$$E_f = W+w_0+w.$$

Let
$$\vec{\eta} = \vec{p}_0+\vec{p}\,. \quad (6.7)$$

Therefore
$$p_0^2 = \eta^2+p^2-2\eta p\cos\theta_{\eta p}\,.$$

In finding $\frac{dp}{dE_f}$ we hold $\vec{q}$ (and therefore $\vec{\eta}$) and $\theta_{\eta p}$ constant.

$$\frac{dE_f}{dp} = \frac{p}{\omega} + \frac{p-\eta\cos\theta_{\eta p}}{\omega_0} = \frac{p\omega_0 + p\omega - \eta\omega\cos\theta_{\eta p}}{\omega\omega_0}$$

$$\therefore \quad p^2\frac{dp}{dE_f} = \frac{p^2\omega\omega_0}{p\omega_0+p\omega-\eta\omega\cos\theta_{\eta p}} .$$

To measure $d\Omega_p$ we take $\vec{\eta}$ as our polar axis.

$$\therefore \quad d\Omega_p = -d(\cos\theta_{\eta p})\, d\varphi$$

where φ is the angle between the $(\vec{p}, \vec{\eta}, \vec{p}_0)$ plane and the $(\vec{k}, \vec{\eta}, \vec{q})$ plane.

$$\therefore \quad p^2\frac{dp}{dE_f}d\Omega_p = -\frac{p^2\omega\omega_0}{p\omega_0+p\omega-\eta\omega\cos\theta_{\eta p}}\, d(\cos\theta_{\eta p})\, d\varphi$$

$$= -\frac{p^2\omega\omega_0}{p\omega_0+p\omega-\eta\omega\cos\theta_{\eta p}}\, \frac{d(\cos\theta_{\eta p})}{d\omega}\, d\omega\, d\varphi .$$

Writing $\omega_0+\omega = \varepsilon$, (6.8)

we also have ~~(6.9)~~

$$2\eta p\cos\theta_{\eta p} = \eta^2 + 2\varepsilon\omega - \varepsilon^2 \qquad (6.9)$$

$$\therefore \quad \frac{d\omega}{d(\cos\theta_{\eta p})} = \frac{\eta p^2}{p\varepsilon - \omega\eta\cos\theta_{\eta p}}$$

$$\therefore \quad p^2\frac{dp}{dE_f}\, d\Omega_p = -\frac{p^2\omega\omega_0}{p\varepsilon-\omega\eta\cos\theta_{\eta p}} \cdot \frac{p\varepsilon-\omega\eta\cos\theta_{\eta p}}{\eta p^2}\, d\omega\, d\varphi$$

$$= -\frac{\omega\omega_0}{\eta}\, d\omega\, d\varphi . \qquad (6.10)$$

Therefore, $d\sigma = -|M|^2\frac{e^2m^3}{k\omega\omega_0 W}\, d\vec{q}\, \frac{\omega\omega_0}{\eta}\, d\omega\, d\varphi$

$$\text{i.e.} \quad d\sigma = -\left\{ |M_{I} + M_{II}|^2 + 2\,\overline{(M_{I} + M_{II})}\,(M_{III} + M_{IV}) + |M_{III} + M_{IV}|^2 \right\}$$

$$\times \frac{e^2 m^3}{k W} \frac{q^2}{\eta}\, dq\, d\Omega_q\, dw\, d\varphi. \qquad (6.11)$$

Borsellino gives*

$$d\sigma = -|M_{I} + M_{II}|^2 \frac{e^2 m^3}{k W} \frac{q^2}{\eta}\, dq\, d\Omega_q\, dw\, d\varphi. \qquad (6.12)$$

We shall accept his integration of this and go on to integrate the other two terms.

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* Equation (29) of Nuovo Cimento contains misprints. Denominator in first line should be $(\underset{\sim}{k}\cdot\underset{\sim}{\Pi}_0)^2$ instead of $(\underset{\sim}{k}\cdot\underset{\sim}{\Pi}_0)$ and denominator in second line should be $(\underset{\sim}{k}\cdot\underset{\sim}{\Pi})^2$ instead of $(\underset{\sim}{k}\cdot\underset{\sim}{\Pi}_0)$. Apart from these, our formula for $|M_{I} + M_{II}|^2$ agrees with Borsellino's.

CHAPTER III.

§7. The Total Cross-Section.

In this section we perform the integration of

$$d\sigma = -\left|M_{\overline{III}} + M_{\overline{IV}}\right|^2 \frac{e^2 m^3}{kW} \frac{q^2}{\eta}\, dq\, d\Omega_q\, dw\, d\varphi .$$

(a) Integration over φ.

The range of integration is 0 to 2π. The factors which depend on φ are $(\underset{\sim}{k}\cdot\underset{\sim}{\Pi_0})$ and $(\underset{\sim}{k}\cdot\underset{\sim}{\Pi})$.
However $(\underset{\sim}{k}\cdot\underset{\sim}{\Pi_0}) + (\underset{\sim}{k}\cdot\underset{\sim}{\Pi}) = k\epsilon - (\vec{k}\cdot\vec{\eta})$ and is therefore independent of φ.

$$(\underset{\sim}{k}\cdot\underset{\sim}{\Pi_0}) = A_0 - B_0\cos\varphi \quad \text{and} \quad (\underset{\sim}{k}\cdot\underset{\sim}{\Pi}) = A - B\cos\varphi ,$$

where $A_0 = k\omega_0 - k p_0 \cos\theta_{\eta k}\cos\theta_{\eta p_0}$

$$B_0 = k p_0 \sin\theta_{\eta p_0}\sin\theta_{\eta k}$$

$$A = k\omega - k p\cos\theta_{\eta k}\cos\theta_{\eta p}$$

$$B = k p \sin\theta_{\eta p}\sin\theta_{\eta k} .$$

We use

$$\int_0^{2\pi} (\underset{\sim}{k}\cdot\underset{\sim}{\Pi_0})\,d\varphi = 2\pi A_0 \quad ; \quad \int_0^{2\pi} (\underset{\sim}{k}\cdot\underset{\sim}{\Pi})\,d\varphi = 2\pi A \quad ;$$

$$\int_0^{2\pi} (\underset{\sim}{k}\cdot\underset{\sim}{\Pi_0})^2 d\varphi = \pi(2A_0^2 + B_0^2) \quad ; \quad \int_0^{2\pi} (\underset{\sim}{k}\cdot\underset{\sim}{\Pi})^2 d\varphi = \pi(2A^2 + B^2) \, ;$$

$$\int_0^{2\pi} (\underset{\sim}{k}\cdot\underset{\sim}{\Pi_0})(\underset{\sim}{k}\cdot\underset{\sim}{\Pi})\,d\varphi = \frac{1}{2}\int_0^{2\pi}\left[\left\{(\underset{\sim}{k}\cdot\underset{\sim}{\Pi_0}) + (\underset{\sim}{k}\cdot\underset{\sim}{\Pi})\right\}^2 - (\underset{\sim}{k}\cdot\underset{\sim}{\Pi_0})^2 - (\underset{\sim}{k}\cdot\underset{\sim}{\Pi})^2\right] d\varphi$$

$$= \frac{\pi}{2}\left[2\left\{k\epsilon - (\vec{k}\cdot\vec{\eta})\right\}^2 - (2A_0^2 + B_0^2) - (2A^2 + B^2)\right].$$

Taking $\int_0^{2\pi} (\underset{\sim}{k}\cdot\underset{\sim}{\Pi_0})(\underset{\sim}{k}\cdot\underset{\sim}{\Pi})\,d\varphi = \pi(2AA_0 + BB_0)$

leads to complications in the integration over ω.

We have

$$\int_0^{2\pi} |M_{\overline{III}} + M_{\overline{IV}}|^2 \, d\varphi$$

$$= \frac{e^4}{\pi} \frac{1}{(8m^4)} \frac{1}{\{(m-W)(m+k)+(\vec{k}.\vec{q})\}^2} \times$$

$$\times \Big[\!\!\Big[\; \frac{1}{k}\Big[2m\{W(3m-W)+kW+4ww_0\} + 2A_0(k-m-2w_0) + 2A(k-m-2w) - \frac{1}{m}(2A_0^2+B_0^2) - \frac{1}{m}(2A^2+B^2)\Big]$$

$$+ \frac{m}{\{kW-(\vec{k}.\vec{q})\}}\Big[-2m\{W(3m-W)-3k(m-W)-mk+4ww_0\} + 2A_0(k+W-m+2w_0) + 2A(k+W-m+2w) - \frac{2}{m}\{k\epsilon-\vec{k}.\vec{q}\}^2 + \frac{1}{m}(2A_0^2+B_0^2) + \frac{1}{m}(2A^2+B^2)\Big]$$

$$+ \frac{m}{k^2}\Big[2m\{(3m-W)(m-W)-kW-2ww_0\} + 2A_0(2m-k+2w_0) + 2A(2m-k+2w) + \frac{1}{m}(2A_0^2+B_0^2) + \frac{1}{m}(2A^2+B^2)\Big]$$

$$+ \frac{m^3}{\{kW-(\vec{k}.\vec{q})\}^2}\Big[2m\{(3m-W)(m-W)+k(m-W)+mk-2ww_0\} + 2A_0(k-W+2m) + 2A(k-W+2m)\Big]$$

$$+ \frac{m}{k\{kW-(\vec{k}.\vec{q})\}}\Big[-4mW\{(3m-W)(m-W)+k(m-W)-2ww_0\} - 2WA_0(4m-W+2w_0) - 2WA(4m-W+2w) - \frac{W}{m}(2A_0^2+B_0^2) - \frac{W}{m}(2A^2+B^2) - 2\{k\epsilon-(\vec{k}.\vec{q})\}^2 + (2A_0^2+B_0^2) + (2A^2+B^2)\Big] \; \Big]\!\!\Big].$$

(b) Integration over ω.

In integration over ω , $\vec{q}$ is held constant, so that

$$\vec{p}+\vec{p}_0 = \vec{K}-\vec{q} = \vec{\eta} = \text{constant},$$

$$\omega+\omega_0 = k+m-W = \varepsilon = \text{constant}.$$

Turning values of p will be when $\vec{p}$ and $\vec{p}_0$ lie in the same direction, so that these turning values satisfy the equation

$$\varepsilon = \sqrt{p^2+m^2} + \sqrt{(\eta-p)^2+m^2}$$

with solutions

$$p = \frac{\eta}{2} \pm \frac{\varepsilon}{2}\sqrt{\frac{\varepsilon^2-\eta^2-4m^2}{\varepsilon^2-\eta^2}}.$$

The corresponding energies

$$\begin{matrix}\omega_2\\ \omega_1\end{matrix} = \frac{\varepsilon}{2} \pm \frac{\eta}{2}\sqrt{\frac{\varepsilon^2-\eta^2-4m^2}{\varepsilon^2-\eta^2}}$$

are the limits for integration over ω.

A_0 , A , B_0 and B have to be expressed in terms of ω and ω_0.

$$A \doteq k\omega - \frac{(\vec{k}\cdot\vec{\eta})(\vec{p}\cdot\vec{\eta})}{\eta^2}$$

$$= k\omega - \frac{(\vec{k}\cdot\vec{\eta})(\eta^2-\varepsilon^2+2\omega\varepsilon)}{2\eta^2} \qquad \text{from} \qquad (6.9)$$

$$= a'' + b''\omega$$

where

$$a'' = \frac{\varepsilon^2-\eta^2}{2\eta^2}(\vec{k}\cdot\vec{\eta})$$

and

$$b'' = \frac{k\eta^2-\varepsilon(\vec{k}\cdot\vec{\eta})}{\eta^2}.$$

Similarly

$$B^2 = a''' + b'''\omega + c'''\omega^2$$

where

$$a''' = -\left\{k^2\eta^2 - (\vec{k}.\vec{\eta})^2\right\}\left\{\frac{m^2}{\eta^2} + \frac{(\epsilon^2-\eta^2)^2}{4\eta^4}\right\},$$

$$b''' = \frac{\epsilon(\epsilon^2-\eta^2)}{\eta^4}\left\{k^2\eta^2 - (\vec{k}.\vec{\eta})^2\right\},$$

$$c''' = -\frac{(\epsilon^2-\eta^2)}{\eta^4}\left\{k^2\eta^2 - (\vec{k}.\vec{\eta})^2\right\}.$$

Since

$$\omega + \omega_0 = \epsilon$$

and

$$\omega_1 + \omega_2 = \epsilon$$

we have

$$\int_{\omega_1}^{\omega_2} f(\omega_0)\,d\omega = \int_{\omega_1}^{\omega_2} f(\omega)\,d\omega.$$

It is not, however, advantageous to use this and instead we proceed as follows.

Using $\omega + \omega_0 = \epsilon$, we express $\int_0^{2\pi} |M_{\overline{III}} + M_{\overline{IV}}|^2 d\varphi$ in the form $Z_1 + Z_2 \omega\omega_0$ where Z_1 and Z_2 are constants in the integration.

Now $\quad \int_{\omega_1}^{\omega_2} (\epsilon - 2\omega)\, d\omega = 0$

and

$$\therefore \int_{\omega_1}^{\omega_2} \omega\omega_0 \, d\omega = \int_{\omega_1}^{\omega_2} \omega(\epsilon - \omega)\, d\omega$$

$$= \int_{\omega_1}^{\omega_2} \left(\tfrac{1}{2}\epsilon^2 - \omega^2\right) d\omega$$

$$= \left[\tfrac{1}{2}\epsilon^2 - \tfrac{1}{3}\left\{\frac{3\epsilon^2}{4} + \frac{\eta^2}{4} - \frac{m^2\eta^2}{\epsilon^2 - \eta^2}\right\}\right] \int_{\omega_1}^{\omega_2} d\omega$$

$$= \left[\frac{\epsilon^2}{4} - \frac{\eta^2}{12} + \frac{m^2\eta^2}{3(\epsilon^2 - \eta^2)}\right] \int_{\omega_1}^{\omega_2} d\omega .$$

We therefore replace $Z_1 + Z_2\omega\omega_0$ by $Z_1 + Z_2\left\{\frac{\epsilon^2}{4} - \frac{\eta^2}{12} + \frac{m^2\eta^2}{3(\epsilon^2 - \eta^2)}\right\}$ and to integrate over ω we merely multiply this by

$$\omega_2 - \omega_1 = \eta \sqrt{\frac{\epsilon^2 - \eta^2 - 4m^2}{\epsilon^2 - \eta^2}} .$$

Expressing the integrand in the form $Z_1 + Z_2 \omega\omega_0$, we have

$$\int_0^{2\pi} |M_{\overline{III}} + M_{\overline{IV}}|^2 d\varphi = \frac{e^4}{\pi} \frac{1}{8m^4} \frac{1}{\{(m - W)(m + k) + (\vec{k}.\vec{q})\}^2} \times$$

$$\times \Big[\!\Big[\frac{1}{k^2}\Big[2m\{kW(3m - W) + k^2W + m(3m - W)(m - W) - mkW\} + (2a'' + b''\epsilon)\{2m^2 - (m - k)^2\}$$

$$- 4a''k\epsilon - 4b''k\epsilon^2 + 4a''m\epsilon + 4b''m\epsilon^2 + \frac{(m - k)}{m}\{4a''^2 + 4a''b''\epsilon + 2b''^2\epsilon^2 + 2a''' + b'''\epsilon + c'''\epsilon^2\}$$

$$+ \omega\omega_0\left\{8mk - 4m^2 + 8b''k - 8mb'' + 2\left(\frac{k - m}{m}\right)(2b''^2 + c''')\right\}\Big]$$

$$+ \frac{m}{k\{kW - (\vec{k}.\vec{q})\}}\Big[-2m\{kW(3m - W) - 3k^2(m - W) - mk^2 + 2W(3m - W)(m - W) + 2kW(m - W)\}$$

$$+ 2k(k + W - m)(2a'' + b''\epsilon) - 2W(4m - W)(2a'' + b''\epsilon) + 4k\epsilon a'' + 4W\epsilon a'' + 4kb''\epsilon^2$$

$$-4\overline{W}b''\varepsilon^2 - 2\left(\frac{m+k}{m}\right)\left\{k\varepsilon-(\vec{k}.\vec{\eta})\right\}^2 + \frac{\varepsilon}{m}\left\{4a''^2+4a''b''\varepsilon+2b''^2\varepsilon^2+2a'''+b'''\varepsilon+c'''\varepsilon^2\right\}$$

$$+\omega\omega_0\left\{-8mk+8m\overline{W}-8kb''+8\overline{W}b''-2\frac{\varepsilon}{m}(2b''^2+c''')\right\}\Big]$$

$$+\frac{m^3}{\{k\overline{W}-(\vec{k}.\vec{q})\}^2}\left[2m\left\{(3m-\overline{W})(m-\overline{W})+k(m-\overline{W})+mk\right\}+2(k-\overline{W}+2m)(2a''+b''\varepsilon)-4m\omega\omega_0\right]\Big]\Big] .$$

We have .

$$2a''+b''\varepsilon = k\varepsilon-(\vec{k}.\vec{\eta})$$

and

$$4a'''+2b'''\varepsilon+c'''\varepsilon^2 = \left\{k^2\eta^2-(\vec{k}.\vec{\eta})^2\right\}\left\{\frac{\varepsilon^2-\eta^2-4m^2}{\eta^2}\right\}.$$

After these replacements and integration over ω,

$$\frac{1}{\eta}\int_{\omega_1}^{\omega_2}d\omega\int_0^{2\pi}d\varphi\,\left|M_{i\vec{\eta}}+M_{i\vec{\nu}}\right|^2 = \alpha+\beta$$

where

$$\alpha = \frac{e^4}{\pi}\,\frac{1}{8m^4}\,\frac{1}{\{(m-\overline{W})(m+k)+(\vec{k}.\vec{q})\}^2}\sqrt{\frac{\varepsilon^2-\eta^2-4m^2}{\varepsilon^2-\eta^2}}\times$$

$$\times\Big[\Big[\frac{1}{k^2}\Big[m\left\{2k\overline{W}(3m-\overline{W})+2k^2\overline{W}+2m(3m-\overline{W})(m-\overline{W})-2mk\overline{W}+2k\varepsilon^2-m\varepsilon^2\right\}$$

$$+2\left\{k\varepsilon-(\vec{k}.\vec{\eta})\right\}\left\{k(k-m)+m(2m-k)-k\varepsilon+2m\varepsilon\right\}+\left\{k\varepsilon-(\vec{k}.\vec{\eta})\right\}^2\left(\frac{m-k}{m}\right)\Big]$$

$$+\frac{m}{k\{k\overline{W}-(\vec{k}.\vec{q})\}}\Big[-2m\left\{k\overline{W}(3m-\overline{W})-3k^2(m-\overline{W})-mk^2+2\overline{W}(3m-\overline{W})(m-\overline{W})\right.$$

$$\left.+2k\overline{W}(m-\overline{W})+k\varepsilon^2-\overline{W}\varepsilon^2\right\}+2\left\{k\varepsilon-(\vec{k}.\vec{\eta})\right\}\left\{k(k+\overline{W}-m)\right.$$

$$\left.-\overline{W}(4m-\overline{W})+k\varepsilon-\overline{W}\varepsilon\right\}+\left\{k\varepsilon-(\vec{k}.\vec{\eta})\right\}^2\left\{-\frac{k+\overline{W}+m}{m}\right\}\Big]$$

$$+\frac{m^3}{\{k\overline{W}-(\vec{k}.\vec{q})\}^2}\Big[m\left\{2(3m-\overline{W})(m-\overline{W})+2k(m-\overline{W})+2mk-\varepsilon^2\right\}$$

$$+2\left\{k\varepsilon-(\vec{k}.\vec{\eta})\right\}(k-\overline{W}+2m)\Big]\quad\Big]\Big]$$

and

$$\beta = -\frac{1}{3}\frac{e^4}{\pi}\frac{1}{8m^4}\frac{1}{\{(m-W)(m+k)+(\vec{k}\cdot\vec{q})\}^2}\cdot\frac{(\epsilon^2-\eta^2-4m^2)^{3/2}}{(\epsilon^2-\eta^2)^{1/2}}\times$$

$$\times\Big[\!\Big[\ \frac{1}{k^2}\Big[m^2-mk-k^2-\frac{(m-k)}{4m}(\epsilon^2-\eta^2)\Big]$$

$$+\frac{m}{k\{kW-(\vec{k}\cdot\vec{q})\}}\Big[\frac{1}{2}\{-4m^2+6m(m-W)+3mk-(m-W)^2\}-\frac{\epsilon}{4m}(\epsilon^2-\eta^2)\Big]$$

$$+\frac{m^3}{\{kW-(\vec{k}\cdot\vec{q})\}^2}\Big[\frac{2m-\epsilon}{2}\Big]\ \Big]\!\Big].$$

(c) Integration over Ω_q.

Let us change the variable to

$$x = k\epsilon-(\vec{k}\cdot\vec{\eta})$$
$$= kq\cos\theta_{kq}-k(W-m)$$

Then

$$d\Omega_q = 2\pi\sin\theta_{kq}\,d\theta_{kq}$$
$$= -\frac{2\pi}{kq}\,dx.$$

In § 1, we proved that the maximum angle at which a particle of energy W could come off, was given by

$$\cos\theta_{kq} = \frac{m(m+W)+k(W-m)}{kq}$$

and the minimum angle was $\theta_{kq}=0$. Therefore the limits of integration for x are $x_1 = m(W+m)$

$$x_2 = k(q-W+m).$$

The lower limit x_1 could also have been obtained from the requirement that $\sqrt{\frac{\epsilon^2-\eta^2-4m^2}{\epsilon^2-\eta^2}}$ has to be real.

The factors which depend on x are η^2, $(\vec{k}\cdot\vec{\eta})$ and $(\vec{k}\cdot\vec{q})$.

$$(\vec{k}\cdot\vec{\eta}) = k\epsilon-x$$
$$(\vec{k}\cdot\vec{q}) = x+k(W-m)$$
$$\eta^2 = |\vec{k}-\vec{q}|^2 = \epsilon^2+2m(W-m)-2x$$

$$\varepsilon^2-\eta^2 = 2\left\{x-m(W-m)\right\}$$

$$\sqrt{\frac{\varepsilon^2-\eta^2-4m^2}{\varepsilon^2-\eta^2}} = \sqrt{\frac{x-m(W+m)}{x-m(W-m)}}$$

$$(m-W)(m+k)+(\vec{k}\cdot\vec{q}) = x-m(W-m)$$

$$kW-(\vec{k}\cdot\vec{q}) = mk-x.$$

After this change of variable, we split up into partial fractions and find that

$$\alpha = \frac{e^4}{\pi}\frac{1}{8m^4}\left\{\frac{x-m(W+m)}{x-m(W-m)}\right\}^{\frac{1}{2}}\left[\alpha_0+\frac{\alpha_1}{\{x-m(W-m)\}}+\frac{\alpha_2}{\{x-m(W-m)\}^2}+\frac{\alpha_3}{mk-x}+\frac{\alpha_4}{(mk-x)^2}\right]$$

and

$$\beta = -\frac{1}{3}\frac{e^4}{\pi}\frac{1}{8m^4}\left\{\frac{x-m(W+m)}{x-m(W-m)}\right\}^{3/2}\left[\beta_0+\frac{\beta_1}{\{x-m(W-m)\}}+\frac{\beta_2}{mk-x}+\frac{\beta_3}{(mk-x)^2}\right],$$

where

$$\alpha_0 = -\frac{k-m}{mk^2}$$

$$\alpha_1 = \frac{1}{k^2\varepsilon^2}\left\{4m^2(W-m)^2-4mk\varepsilon(W-m)+2mk(W-m)^2+k\varepsilon^3+k^3\varepsilon\right\}$$

$$\alpha_2 = \frac{1}{k\varepsilon}\left\{2m^2(k^2+\varepsilon^2)\right\}$$

$$\alpha_3 = \frac{1}{k\varepsilon^2}\left\{-4m^2\varepsilon+4m^2(W-m)-4m\varepsilon(W-m)+2m(W-m)^2+\varepsilon^3+\varepsilon(W-m)^2\right\}$$

$$\alpha_4 = \frac{1}{\varepsilon}\left\{4m^3+m^2\varepsilon\right\}$$

$$\beta_0 = \frac{k-m}{mk^2}$$

$$\beta_1 = \frac{1}{k^2\varepsilon^2}\left\{2m^2(W-m)^2-4mk\varepsilon(W-m)-k\varepsilon^3-k^3\varepsilon\right\}$$

$$\beta_2 = \frac{1}{k\varepsilon^2}\left\{-2m^2\varepsilon + 2m^2(W-m) + 2m\varepsilon^2 - 4m\varepsilon(W-m) - \varepsilon(W-m)^2 - \varepsilon^3\right\}$$

$$\beta_3 = \frac{1}{\varepsilon}\left\{2m^3 - m^2\varepsilon\right\}.$$

To carry out this integration we let $y = \sqrt{\frac{x-m(W+m)}{x-m(W-m)}}$.
Then $dx = 4m^2 \frac{y}{(1-y^2)^2}dy$ and limits are $y_1 = 0$ and

$$y_2 = \sqrt{\frac{k(q-W+m)-m(W+m)}{k(q-W+m)-m(W-m)}}.$$

Then

$$\int_{x_1}^{x_2}dx \int_{\omega_1}^{\omega_2}\frac{d\omega}{\eta}\int_0^{2\pi}d\varphi\,|M_{\text{III}}+M_{\text{IV}}|^2$$

$$= \frac{e^4}{\pi}\frac{1}{8m^4}\int_0^{y_2}dy\left[\frac{4m^2 d_0 y^2}{(1-y^2)^2} + \frac{2d_1 y^2}{(1-y^2)} + d_2\frac{y^2}{m^2} + \frac{4md_3 y^2}{(1-y^2)\{(\varepsilon-2m)-\varepsilon y^2\}}\right.$$

$$+\frac{4d_4 y^2}{\{(\varepsilon-2m)-\varepsilon y^2\}^2} - \frac{1}{3}\frac{4m^2\beta_0 y^4}{(1-y^2)^2} - \frac{1}{3}\frac{2\beta_1 y^4}{(1-y^2)} - \frac{1}{3}\frac{4m\beta_2 y^4}{(1-y^2)\{(\varepsilon-2m)-\varepsilon y^2\}}$$

$$\left. - \frac{1}{3}\frac{4\beta_3 y^4}{\{(\varepsilon-2m)-\varepsilon y^2\}^2}\right].$$

These integrals are elementary and give

$$\mathcal{J} = \int_{x_1}^{x_2}dx\int_{\omega_1}^{\omega_2}\frac{d\omega}{\eta}\int_0^{2\pi}d\varphi\,|M_{\text{III}}+M_{\text{IV}}|^2$$

$$= \frac{e^4}{\pi}\frac{1}{6m^4}\left[C_0\left\{\frac{2y_2}{(1-y_2^2)} - \log\left|\frac{1+y_2}{1-y_2}\right|\right\} + C_1\left\{-2y_2 + \log\left|\frac{1+y_2}{1-y_2}\right|\right\} + C_2\left\{\frac{y_2^3}{3}\right\}\right.$$

$$+ C_3\left\{\left(\frac{\varepsilon-2m}{\varepsilon}\right)^{\frac{1}{2}}\log\left|\frac{\sqrt{\varepsilon-2m}+\sqrt{\varepsilon}\,y_2}{\sqrt{\varepsilon-2m}-\sqrt{\varepsilon}\,y_2}\right| - \log\left|\frac{1+y_2}{1-y_2}\right|\right\}$$

$$\left. + C_4\left\{\frac{2y_2}{(\varepsilon-2m)-\varepsilon y_2^2} - \frac{1}{\sqrt{\varepsilon(\varepsilon-2m)}}\log\left|\frac{\sqrt{\varepsilon-2m}+\sqrt{\varepsilon}\,y_2}{\sqrt{\varepsilon-2m}-\sqrt{\varepsilon}\,y_2}\right|\right\}\right]$$

where $C_0 = -\frac{m(k-m)}{k^2}$

$$C_1 = \frac{1}{k^2\varepsilon^3}\Big[2m^3k(W-m) - 2m^2\varepsilon^2(W-m) - 2mk\varepsilon^2(W-m) + mk\varepsilon(W-m)^2 + k\varepsilon^4 + k^3\varepsilon^2\Big]$$

$$C_2 = \frac{1}{k^2\varepsilon^2}\Big[m^2(W-m)^2 - 2mk\varepsilon(W-m) + k^3\varepsilon + k\varepsilon^3\Big]$$

$$C_3 = \frac{1}{k\varepsilon^3}\Big[2m^3(W-m) - 2m^2\varepsilon^2 + 2m\varepsilon(W-m)^2 - mk^2\varepsilon + \varepsilon^4 + \varepsilon^2(W-m)^2\Big]$$

$$C_4 = \frac{1}{\varepsilon^3}\left[m^2(\varepsilon+m)^2\right].$$

(d) __Integration over q.__

$$\int d\Omega_q \frac{d\omega}{\eta}\, d\varphi\, |M_{I\bar{J}} + M_{\bar{J}}|^2 = -\frac{2\pi}{kq}\, g$$

$$\therefore d\sigma = \frac{e^2m^3}{kW}\, q^2 dq\, \frac{2\pi}{kq}\, g$$

$$= \frac{e^6}{m}\frac{1}{3k^2}\frac{I}{W}\, q\, dq \qquad \text{where } I = \frac{6m^4\pi}{e^4}\, g$$

Now $W^2 = q^2 + m^2 \qquad \therefore W\,dW = q\,dq$

$$\therefore d\sigma = \frac{e^6}{m}\frac{1}{3k^2}\, I\, dW.$$

Integration over q is now replaced by integration over W, which can be done numerically. In §1, we proved that the limits of integration for W are

$$\begin{matrix} W_1 \\ W_2 \end{matrix} = \frac{k^2 - m^2 \mp k\sqrt{k(k-4m)}}{2k+m}.$$

These could also have been obtained from the requirement that y_2 has to be real.

In cm^2, $\sigma = \left(\frac{e^2}{mc^2}\right)^2 \left(\frac{e^2}{\hbar c}\right) \Gamma(\gamma)$

where

$$\Gamma(\gamma) = \frac{1}{3\gamma^2} \int_{W_1'}^{W_2'} I' \, dW' .$$

γ and W' are the energies of the photon and electron respectively in units of mc^2,

$$I' = C_0' \left\{ \frac{2y_2'}{(1-y_2'^2)} - \log\left|\frac{1+y_2'}{1-y_2'}\right| \right\} + C_1' \left\{ -2y_2' + \log\left|\frac{1+y_2'}{1-y_2'}\right| \right\} + C_2' \left\{ \frac{y_2'^3}{3} \right\}$$

$$+ C_3' \left\{ \left(\frac{\epsilon'-2}{\epsilon'}\right)^{\frac{1}{2}} \log\left|\frac{\sqrt{\epsilon'-2} + \sqrt{\epsilon'}\,y_2'}{\sqrt{\epsilon'-2} - \sqrt{\epsilon'}\,y_2'}\right| - \log\left|\frac{1+y_2'}{1-y_2'}\right| \right\}$$

$$+ C_4' \left\{ \frac{2y_2'}{(\epsilon'-2) - \epsilon' y_2'^2} - \frac{1}{\sqrt{\epsilon'(\epsilon'-2)}} \log\left|\frac{\sqrt{\epsilon'-2} + \sqrt{\epsilon'}\,y_2'}{\sqrt{\epsilon'-2} - \sqrt{\epsilon'}\,y_2'}\right| \right\},$$

$$C_0' = -\frac{\gamma-1}{\gamma^2}$$

$$C_1' = \frac{1}{\gamma^2 \epsilon'^3} \left\{ 2\gamma(W'-1) - 2\epsilon'^2(W'-1) - 2\gamma\epsilon'^2(W'-1) + \gamma\epsilon'(W'-1)^2 + \gamma\epsilon'^4 + \gamma^3\epsilon'^2 \right\}$$

$$C_2' = \frac{1}{\gamma^2 \epsilon'^2} \left\{ (W'-1)^2 - 2\gamma\epsilon'(W'-1) + \gamma^3\epsilon' + \gamma\epsilon'^3 \right\}$$

$$C_3' = \frac{1}{\gamma \epsilon'^3} \left\{ 2(W'-1) - 2\epsilon'^2 + 2\epsilon'(W'-1)^2 + \gamma^2\epsilon' + \epsilon'^4 + \epsilon'^2(W'-1)^2 \right\}$$

$$C_4' = \frac{1}{\epsilon'^3} \left\{ (\epsilon'+1)^2 \right\}$$

$$\epsilon' = \gamma - W' + 1$$

$$y_2' = \sqrt{\frac{\gamma\sqrt{W'^2-1} - \gamma(W'-1) - (W'+1)}{\gamma\sqrt{W'^2-1} - \gamma(W'-1) - (W'-1)}} .$$

Values of I' were plotted against the corresponding values of W' for various γ values, and three typical curves are shown in fig.7. As γ increased, the curves shifted towards the high energy end. The total cross-section, for a given value of γ, was obtained, in units of $\left(\frac{e^2}{\hbar c}\right)\left(\frac{e^2}{mc^2}\right)^2$, by measuring the area under the appropriate curve and dividing by $3\gamma^2$. From these graphs, a final complete excitation curve was drawn and this is shown in fig.8.

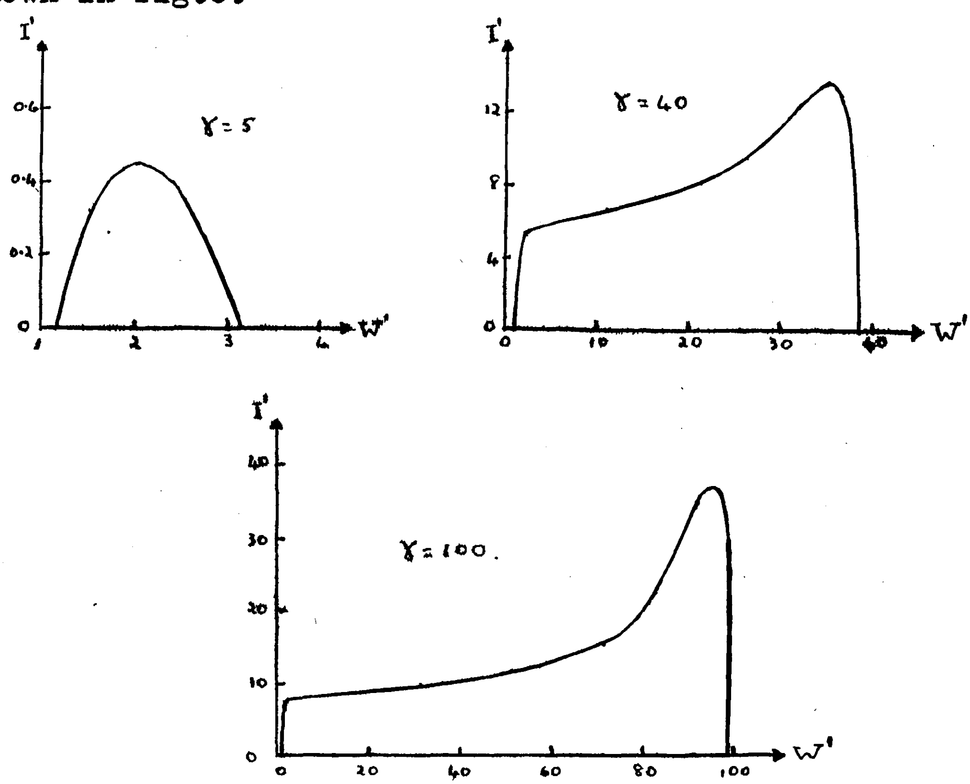

Fig.7. Plots of I' against W' for three of the chosen values of γ.

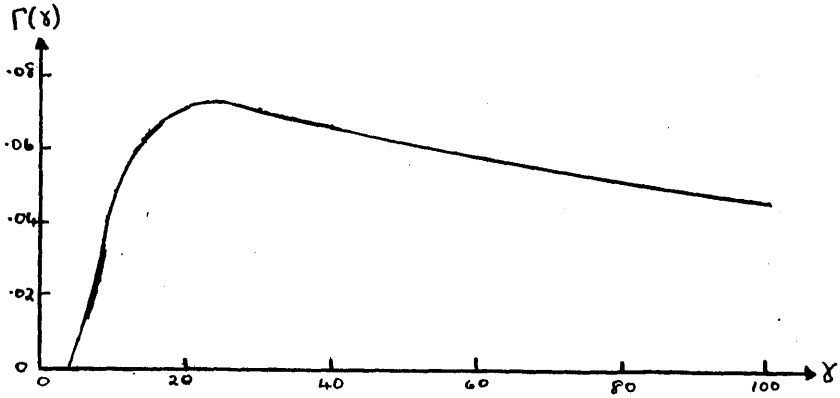

Fig.8. Total cross-section, in units of $\left(\frac{e^2}{\hbar c}\right)\left(\frac{e^2}{mc^2}\right)^2$, as a function of γ.

§8. The Cross-Product Term.

The cross-product term between $M_{\mathrm{I}} + M_{\mathrm{II}}$ and $M_{\overline{\mathrm{III}}} + M_{\overline{\mathrm{IV}}}$ is

$$2\,\overline{(M_{\overline{\mathrm{III}}} + M_{\overline{\mathrm{IV}}})}\,(M_{\mathrm{I}} + M_{\mathrm{II}})$$

$$= 2\,\frac{e^4}{\pi^2}\,\frac{1}{4(p_1\cdot k)(p_2\cdot k)(p_3\cdot k)(p_4\cdot k)}\;\frac{1}{(\not{p}_3-\not{p}_1)^2(\not{p}_4-\not{p}_2)^2}\,\times$$

$$\times\{\bar{u}_3\gamma_\mu u_1\}\{\bar{u}_1\left[2(p_3\cdot k)p_{1\sigma}\gamma_\nu + (p_3\cdot k)\gamma_\sigma\not{k}\gamma_\nu - 2(p_1\cdot k)p_{3\sigma}\gamma_\nu + (p_1\cdot k)\gamma_\nu\not{k}\gamma_\sigma\right]u_3\}$$

$$\times\{\bar{u}_4\left[2(p_4\cdot k)p_{2\sigma}\gamma_\mu + (p_4\cdot k)\gamma_\mu\not{k}\gamma_\sigma - 2(p_2\cdot k)p_{4\sigma}\gamma_\mu + (p_2\cdot k)\gamma_\sigma\not{k}\gamma_\mu\right]u_2\}\{\bar{u}_2\gamma_\nu u_4\}.$$

We study the symmetry properties of this under the transformation $-p_2 \rightleftarrows p_4$.

$$\{\bar{u}_4\left[2(p_4\cdot k)p_{2\sigma}\gamma_\mu + (p_4\cdot k)\gamma_\mu\not{k}\gamma_\sigma - 2(p_2\cdot k)p_{4\sigma}\gamma_\mu + (p_2\cdot k)\gamma_\sigma\not{k}\gamma_\mu\right]u_2\}\{\bar{u}_2\gamma_\nu u_4\}$$

$$\to \{\bar{u}_2\left[2(p_2\cdot k)p_{4\sigma}\gamma_\mu - (p_2\cdot k)\gamma_\mu\not{k}\gamma_\sigma - 2(p_4\cdot k)p_{2\sigma}\gamma_\mu - (p_4\cdot k)\gamma_\sigma\not{k}\gamma_\mu\right]u_4\}\{\bar{u}_4\gamma_\nu u_2\}$$

$$= -\{\bar{u}_2\left[2(p_4\cdot k)p_{2\sigma}\gamma_\mu + (p_4\cdot k)\gamma_\sigma\not{k}\gamma_\mu - 2(p_2\cdot k)p_{4\sigma}\gamma_\mu + (p_2\cdot k)\gamma_\mu\not{k}\gamma_\sigma\right]u_4\}\{\bar{u}_4\gamma_\nu u_2\}.$$

The other factors remain unchanged.

$\overline{(M_{\bar{i}\bar{j}}+M_{\bar{i}j})}(M_{\bar{j}}+M_{\bar{j}})$, after the summation over spins, involves the factor

$$\mathrm{Spur}\left[(\not{p}_4+m)\left\{2(\underset{\sim}{p_4}\cdot\underset{\sim}{k})p_{2\sigma}\gamma_\mu+(\underset{\sim}{p_4}\cdot\underset{\sim}{k})\gamma_\mu\not{k}\gamma_\sigma-2(\underset{\sim}{p_3}\cdot\underset{\sim}{k})p_{4\sigma}\gamma_\mu+(\underset{\sim}{p_3}\cdot\underset{\sim}{k})\gamma_\sigma\not{k}\gamma_\mu\right\}(\not{p}_2+m)\gamma_\nu\right]$$

$$=\mathrm{Spur}\left[(\not{p}_2+m)\left\{2(\underset{\sim}{p_4}\cdot\underset{\sim}{k})p_{2\sigma}\gamma_\mu+(\underset{\sim}{p_4}\cdot\underset{\sim}{k})\gamma_\sigma\not{k}\gamma_\mu-2(\underset{\sim}{p_3}\cdot\underset{\sim}{k})p_{4\sigma}\gamma_\mu+(\underset{\sim}{p_3}\cdot\underset{\sim}{k})\gamma_\mu\not{k}\gamma_\sigma\right\}(\not{p}_4+m)\gamma_\nu\right].$$

Under the above transformation, this factor becomes

$$-\mathrm{Spur}\left[(\not{p}_2+m)\left\{2(\underset{\sim}{p_4}\cdot\underset{\sim}{k})p_{2\sigma}\gamma_\mu+(\underset{\sim}{p_4}\cdot\underset{\sim}{k})\gamma_\sigma\not{k}\gamma_\mu-2(\underset{\sim}{p_3}\cdot\underset{\sim}{k})p_{4\sigma}\gamma_\mu+(\underset{\sim}{p_3}\cdot\underset{\sim}{k})\gamma_\mu\not{k}\gamma_\sigma\right\}(\not{p}_4+m)\gamma_\nu\right].$$

Thus the contribution from the cross-product term changes sign under the transformation $-\underset{\sim}{p_2}\rightleftarrows\underset{\sim}{p_4}$. This transformation is equivalent to $\omega_0,\vec{p}_0\rightleftarrows\omega,\vec{p}$, so that after the integration over φ we can write the cross-product term as $f(\omega_0,\omega)=-f(\omega,\omega_0)$. The limits of integration over ω (ω_1 and ω_2) are such that $\omega_1+\omega_2=\varepsilon$. We also have $\omega+\omega_0=\varepsilon$, so that

$$\int_{\omega_1}^{\omega_2} f(\omega_0,\omega)\,d\omega=-\int_{\varepsilon-\omega_1}^{\varepsilon-\omega_2} f(\omega_0,\varepsilon-\omega_0)\,d\omega_0$$

$$=\int_{\omega_1}^{\omega_2} f(\omega_0,\varepsilon-\omega_0)\,d\omega_0$$

$$=\int_{\omega_1}^{\omega_2} f(\omega,\varepsilon-\omega)\,d\omega$$

$$=\int_{\omega_1}^{\omega_2} f(\omega,\omega_0)\,d\omega$$

$$=-\int_{\omega_1}^{\omega_2} f(\omega_0,\omega)\,d\omega$$

$$\int_{\omega_1}^{\omega_2} f(\omega_0,\omega)\,d\omega=0.$$

Thus the cross-product term gives a zero contribution to the total cross-section. The contribution to the differential cross-section is zero when the constituents of the pair have equal momenta, and when this is not the case, the contribution from $\omega = \frac{\varepsilon}{2} - \chi$, $\omega_0 = \frac{\varepsilon}{2} + \chi$ is cancelled by the contribution from $\omega = \frac{\varepsilon}{2} + \chi$, $\omega_0 = \frac{\varepsilon}{2} - \chi$.

§9. Conclusion.

We have calculated the correction to the cross-section obtained by Borsellino (9) (fig.3) when one takes into account the possibility of the absorption of the real γ -ray by the initially present electron. As was suggested in § 4, the contribution from $M_{\overline{III}} + M_{\overline{IV}}$ is always less than that from $M_{\overline{I}} + M_{\overline{II}}$, and decreases in importance as the γ -ray energy increases. At γ -ray energies of 2.5 and 5 Mev, we have to add corrections of 30% and 7.8% respectively to Borsellino's results, but by 20 Mev the correction has dropped to less than 2%. The cross-section obtained from these new terms alone has a maximum at a γ -ray energy of about 12.5 Mev.

For these new terms, and neglecting exchange as we have done, high momentum transfers to the recoil electron become more important as the γ -ray energy increases, and, even at low energies, high momentum transfers are as important as low. However, at high energies, our correction to Borsellino's result is so small that his finding of the low momentum transfers being the more important will be unaltered.

It is expected that exchange effects will be important only at low energies and will reduce the value of the cross-section by a factor of about 5 or 6. For $\gamma = 8$ (4 Mev γ -ray) Koch and Carter (17) find an experimental value of about .05 for $\frac{c_2}{c_1}$. Borsellino's value is .225 and ours is .018, so that, neglecting exchange, $\frac{c_2}{c_1}$ is 0.243. Thus, for this energy, exchange effects would have to reduce the cross-section for triplet production by a factor of about 5. For $\gamma = 13$(6.5 Mev γ -ray), the experimental value of $\frac{c_2}{c_1}$ is 0.19(the mean of the values obtained by Phillips and Kruger (15) and Koch and Carter (17)). Borsellino's value is 0.408 and , adding our contribution of 0.023, we have for the theoretical value, neglecting exchange, 0.431, so that, at this energy exchange effects would have to reduce the cross-section by a factor of 2 or 3. We see that, as the γ -ray energy increases, the effect of exchange decreases, as does our contribution, so that at very high energies the cross-section calculated by Borsellino will be approximately correct.

APPENDIX.

The Projection Operator in Feynman Formalism.

It is not at all clear from Feynman's papers that $\frac{\not p+m}{2m}$ is the correct operator projecting negative energy states into positive energy states. Dirac's equation is

$$\not p\, u = + m\, u \tag{A.1}$$

whether u describes a positive or a negative energy particle. If the energy of the particle is positive $p = (E, \vec{p})$, while if it is negative, $p = (-E, \vec{p})$. In the usual formalism of Dirac, this is

$$\begin{aligned} \{E - (\vec{\alpha}\cdot\vec{p})\} u_+ &= \beta m\, u_+ \\ \{-E - (\vec{\alpha}\cdot\vec{p})\} u_- &= \beta m\, u_- \end{aligned} \tag{A.2}$$

where we have used the definition of Feynman's γ's $(\gamma_\mu = (\beta, \beta\vec{\alpha}))$ and where u_+ and u_- denote positive and negative energy states respectively. In this formalism, the mass is always positive.

However, as Feynman states, $\not p$ does have the eigenvalues $\pm m$, but the corresponding eigenfunctions are those of positive and negative mass, not of positive and negative energy. We shall describe positive mass states by the spinor u_+, negative mass states by u_-. In this formalism, the energy is always positive.

We have for the solution of Dirac's equation,

$\psi_s = \mu_s e^{i\vec{p}\cdot\vec{r} - iEt}$, $s = 1, 2, 3, 4$, where μ is a spinor whose components satisfy the equations

$$
\begin{aligned}
&(E-m)\mu_1 \qquad\qquad - p_z\mu_3 - (p_x - i p_y)\mu_4 = 0\\
&\qquad (E-m)\mu_2 - (p_x + i p_y)\mu_3 + p_z\mu_4 = 0\\
&-p_z\mu_1 - (p_x - i p_y)\mu_2 + (E+m)\mu_3 = 0 \qquad\qquad (A.3)\\
&-(p_x + i p_y)\mu_1 + p_z\mu_2 \qquad + (E+m)\mu_4 = 0 .
\end{aligned}
$$

These equations have solutions only if $E^2 - m^2 - p^2 = 0$. If, to obtain a solution, we choose a sign for $E_{\pm} = \pm\sqrt{p^2 + m^2}$, we get

$$
\begin{array}{lccccl}
 & s=1 & s=2 & s=3 & s=4 & \\
\mu = U_+ = \Big(& 1 & 0 & \dfrac{p_z}{E_+ + m} & \dfrac{p_x + i p_y}{E_+ + m} & \Big) \text{ for } s_z = \tfrac{1}{2}\\
\mu = U_+ = \Big(& 0 & 1 & \dfrac{p_x - i p_y}{E_+ + m} & -\dfrac{p_z}{E_+ + m} & \Big) \text{ for } s_z = -\tfrac{1}{2}\\
\mu = U_- = \Big(& \dfrac{p_z}{E_- - m} & \dfrac{p_x + i p_y}{E_- - m} & 1 & 0 & \Big) \text{ for } s_z = \tfrac{1}{2}\\
\mu = U_- = \Big(& \dfrac{p_x - i p_y}{E_- - m} & -\dfrac{p_z}{E_- - m} & 0 & 1 & \Big) \text{ for } s_z = -\tfrac{1}{2} .
\end{array}
\qquad (A.4)
$$

If, however, we choose a sign for $m_{\pm} = \pm\sqrt{E^2 - p^2}$, we get

$$
\begin{array}{lccccl}
\mu = u_+ = \Big(& 1 & 0 & \dfrac{p_z}{E + m_+} & \dfrac{p_x + i p_y}{E + m_+} & \Big) \text{ for } s_z = \tfrac{1}{2}\\
\mu = u_+ = \Big(& 0 & 1 & \dfrac{p_x - i p_y}{E + m_+} & -\dfrac{p_z}{E + m_+} & \Big) \text{ for } s_z = -\tfrac{1}{2}\\
\mu = u_- = \Big(& \dfrac{p_z}{E - m_-} & \dfrac{p_x + i p_y}{E - m_-} & 1 & 0 & \Big) \text{ for } s_z = \tfrac{1}{2}\\
\mu = u_- = \Big(& \dfrac{p_x - i p_y}{E - m_-} & -\dfrac{p_z}{E - m_-} & 0 & 1 & \Big) \text{ for } s_z = -\tfrac{1}{2} .
\end{array}
\qquad (A.5)
$$

In (A.4), m is always positive, while in (A.5), E is always positive. Thus $u_+ = U_+$, which is to be expected since m and E are both positive in the two cases. If in U_-, we replace E_- by E and m by m_-, we obtain u_-. This means that a spinor, representing a particle with negative energy and positive mass, can be interpreted as representing a particle with positive energy and negative mass.

Hence the operator $\frac{\not{p}+m}{2m}$ which projects the negative mass states into the positive mass states, can be interpreted as projecting the negative energy states into the positive energy states. Thus Feynman's summation over positive and negative mass states can be replaced by summation over positive and negative energy states, using the same projection operator.

That this projection operator is correct can also be seen as follows.

If the probability amplitude for a transition from the state 1 to the state 2 is $(u_2^+ \beta M u_1)$, then the transition probability is $(u_1^+ M^+ \beta u_2)(u_2^+ \beta M u_1)$ where M^+ is M with the order of the α's and β's reversed and explicit appearance of i changed to $-i$. Let us denote summation over the spins of the positive energy states of the electron j by $\sum_j{}'$, and summation over the spins of both positive and negative energy states by $\sum_j$. Carrying out the summation over the spins of particles 1 and2, which we assume both to be in positive energy states, we obtain

$$X = \sum_1{}' \sum_2{}' (u_1^+ M^+ \beta u_2)(u_2^+ \beta M u_1),$$ where we assume the normalisation $u_1^+ u_1 = u_2^+ u_2 = 1$.

$$= \sum_1{}' \sum_2 \left\{ u_1^+ M^+ \beta \left(\frac{E_2 + (\vec{\alpha}.\vec{p}_2) + \beta m}{2E_2} \right) u_2 \right\} \left\{ u_2^+ \beta M u_1 \right\},$$ using the normal energy projection operator.

$$= \sum_1{}' \left\{ u_1^+ M^+ \beta \left(\frac{E_2 + (\vec{\alpha}.\vec{p}_2) + \beta m}{2E_2} \right) \beta M u_1 \right\}$$

$$= \sum_1 \left\{ u_1^+ \left(\frac{E_1 + (\vec{\alpha}.\vec{p}_1) + \beta m}{2E_1} \right) M^+ \beta \left(\frac{E_2 + (\vec{\alpha}.\vec{p}_2) + \beta m}{2E_2} \right) \beta M u_1 \right\}$$

$$= \mathrm{Spur} \left[\left(\frac{E_1 + (\vec{\alpha}.\vec{p}_1) + \beta m}{2E_1} \right) M^+ \beta \left(\frac{E_2 + (\vec{\alpha}.\vec{p}_2) + \beta m}{2E_2} \right) \beta M \right]$$

$$= \mathrm{Spur} \left[\left(\frac{\beta E_1 - \beta(\vec{\alpha}.\vec{p}_1) + m}{2E_1} \right) \beta M^+ \beta \left(\frac{\beta E_2 - \beta(\vec{\alpha}.\vec{p}_2) + m}{2E_2} \right) M \right] . \tag{A.6}$$

Let us now denote summations over the spins of mass states by S's , analogous to the Σ's . The matrix element $(u_2^+ \beta M u_1)$ is, in the Feynman formalism, $(\bar{u}_2 M u_1)$. The transition probability is

$$(\bar{u}_1 \bar{M} u_2)(\bar{u}_2 M u_1) = (u_1^+ \beta \bar{M} u_2)(u_2^+ \beta M u_1)$$

where we assume the normalisation $\bar{u}_1 u_1 = \bar{u}_2 u_2 = 1$.

For the moment, we shall neglect the difference in normalisation, and equate $(u_1^+ M^+ \beta u_2)(u_2^+ \beta M u_1)$ and $(\bar{u}_1 \bar{M} u_2)(\bar{u}_2 M u_1)$, so that

$$\beta \bar{M} = M^+ \beta$$

$$\text{or } \bar{M} = \beta M^+ \beta .$$

M^+ is M with the order of the α's and β's reversed and explicit appearance of i changed to $-i$, so that $\bar{M}$ is M with the order of the γ's reversed and explicit appearance of i changed to $-i$. We now have u_1 and u_2 representing positive mass particles.

Then
$$\begin{aligned} Y &= \sum_1{}' \sum_2{}' (\bar{u}_1 \bar{M} u_2)(\bar{u}_2 M u_1) \\ &= \sum_1{}' S_2 \left\{ \bar{u}_1 \bar{M} \left(\frac{p_2 + m}{2m}\right) u_2 \right\} \left\{ \bar{u}_2 M u_1 \right\} \\ &= \sum_1{}' \left\{ \bar{u}_1 \bar{M} \left(\frac{p_2+m}{2m}\right) M u_1 \right\} \\ &= \sum_1 \left\{ \bar{u}_1 \left(\frac{p_1+m}{2m}\right) \bar{M} \left(\frac{p_2+m}{2m}\right) M \right\} \\ &= \mathrm{Spur} \left[\left(\frac{p_1+m}{2m}\right) \bar{M} \left(\frac{p_2+m}{2m}\right) M \right] \end{aligned}$$

To normalise this to $u^+ u = 1$, we have to multiply by $\frac{m^2}{E_1 E_2}$, so that

$$\begin{aligned} Y &= \mathrm{Spur} \left[\left(\frac{p_1+m}{2E_1}\right) \bar{M} \left(\frac{p_2+m}{2E_2}\right) M \right] \\ &= \mathrm{Spur} \left[\left(\frac{\beta E_1 - \beta(\vec{\alpha} \cdot \vec{p}_1) + m}{2E_1} \right) \beta M^+ \beta \left(\frac{\beta E_2 - \beta(\vec{\alpha} \cdot \vec{p}_2) + m}{2E_2} \right) M \right] \\ &= X \end{aligned}$$

Thus the two methods give the same result.

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PART II.

The Reactions $\pi^+ + t \rightleftarrows p + d$.

PREFACE to PART II.

The problem of π^+ meson production in proton-proton (p - p) collisions has been considerably investigated, and it has been found that the probability of the two final nucleons forming a deuteron (d) is very high. The reaction $p+p \rightarrow \pi^+ + d$ has been studied by a phenomenological weak-coupling treatment. Prof. J.C. Gunn suggested to the author that it would be possible to carry out a similar treatment of the problem of π^+ meson production in proton-deuteron collisions, with the formation of a triton (t). The number of nucleons involved in this reaction, $p+d \rightarrow \pi^+ + t$, is not sufficient to make such a calculation prohibitive. Part II of this thesis contains calculations of the differential and total cross-sections for this reaction.

The problem of π^+ meson production in proton-deuteron collisions is of interest as it is the simplest process by which an estimate can be made of the probability of π^+ meson production in proton-neutron collisions. This information, however, (see § 2), is obtained only from the reactions in which the final nucleons are not all bound. It is found, in this thesis, that the 'two-body' reaction is more profitable as a method of studying the properties of the triton than as a method of studying the mechanism of meson production.

The first chapter consists of a survey of previous work required for an understanding of the problem. §§ 1 and 2 are reviews of previous studies of π^+ meson production in nucleon-nucleon collisions: § 1 deals with the production in proton-proton collisions and § 2 with the production in proton-neutron collisions. It is noted in § 2 that the proton-neutron production cross-section is not small enough to allow this method of π^+ meson production to be neglected in the proton-deuteron production process. Previous theoretical studies of the reaction $p+d \rightarrow \pi^+ + t$, based on the impulse approximation, have neglected this method of production. These studies, and the experimental information available on this reaction, are summarised in § 3.

The second chapter (§§ 4 - 8), all of which is the author's own work, contains the phenomenological weak-coupling treatment of the reactions $\pi^+ + t \rightleftarrows p+d$. This follows the lines of a calculation by Cheston (13) of the cross-sections for the reactions $\pi^+ + d \rightleftarrows p+p$. The cross-section for the reaction $\pi^+ + t \rightarrow p+d$ is calculated, and that for the inverse reaction is obtained by detailed balancing.

§ 4 describes the kinematics of the reactions, which have been dealt with separately, since they rest on a sure foundation. § 5 contains the general formalism which would have to be applied to any treatment of the problem based on weak-coupling perturbation theory. For clarity, this part of the calculation has not been made completely general; the triton was described by the Irving wave function (31) and the deuteron by a Yukawa-type wave function. Any repetition of the calculation could, however, start from the end of §5: the only alteration required would be in the wave functions inserted for the description of the deuteron and the triton.

The calculation is carried out for scalar mesons with scalar coupling in §6 and for pseudoscalar mesons with pseudovector coupling in §7. Only these theories have been considered, since the experimental ratio of the cross-sections for the reactions $p+p \rightleftarrows \pi^{+}+d$ shows that the π^{+} meson has spin zero. Analyses of other meson processes indicate that the meson has odd parity, but it was found ((11) and (13)) that a phenomenological weak-coupling treatment of the processes $p+p \rightleftarrows \pi^{+}+d$ gave better agreement with experiment for scalar mesons than for pseudoscalar. It was therefore thought advisable to consider both parities of the meson.

For scalar mesons, the calculation was carried out using an Irving and Gaussian wave function to describe the triton. With the former, the angular distribution is not unreasonable but the total cross-section is much too large. With the Gaussian wave function, the total cross-section is roughly in agreement with experiment but the angular distribution is unreasonable. These results are discussed in §8 and show that, in the high momentum region, the Irving wave function is too large, the Gaussian too small. After these findings, it was not considered advisable to carry out the pseudoscalar calculation fully, and in this theory, only protons of energy 340 Mev were considered. The results are discussed in §8. No experimental excitation curve is available, and the calculated one for scalar mesons is not therefore discussed.

In the appendix, the integrals occurring in the text are evaluated. This is also the author's own work.

In conclusion, the author would like to express his thanks to Prof. J.C. Gunn for suggesting this problem and for fruitful discussions during the solving of it. He would also like to thank the Nuffield Foundation for a Studentship, during the tenure of which this work was performed.

Contents of PART II.

CHAPTER I.

§1. Introduction.

A beam of photons passing through matter may lose intensity through individual quanta producing pairs, the usual production being in the Coulomb field of a nucleus of charge Ze. The differential cross-section for this method of production has been formulated by Bethe and Heitler (1) and by Racah (2). Analytic integration is simple only in the limiting cases of small and large photon energy. For small photon energy, Racah found the total cross-section

$$\sigma(k) = Z^2\left(\frac{e^2}{\hbar c}\right)\left(\frac{e^2}{mc^2}\right)^2 \frac{\pi}{12}\left(\frac{k-2mc^2}{mc^2}\right)^3 \tag{1.1}$$

and Bethe and Heitler found, for $k, \omega, \omega_0 \gg mc^2$,

$$\sigma(k) = Z^2\left(\frac{e^2}{\hbar c}\right)\left(\frac{e^2}{mc^2}\right)^2\left(\frac{28}{9}\log\frac{2k}{mc^2} - \frac{218}{27}\right) \tag{1.2}$$

where k, ω, ω_0 are the energies of the photon, electron and positron respectively. m is the electron rest mass and $\hbar, e, c$ have their usual meaning. Born's first approximation has been used and this is valid if the velocities of the pair constituents are greater than $Z\left(\frac{e^2}{\hbar c}\right)c$, this condition being satisfied if the velocities of the particles are near that of light and $Z \lesssim 60$.

In actual fact, the quantum does not feel the full effect of the nuclear Coulomb field, since the nucleus energy gives the total $\frac{d\sigma}{d\Omega}$ at the appropriate angle; such integrations give the results shown in Table 1. Assuming that two-thirds of the mesons are due to the reaction $p+p \rightarrow \pi^+ + d$, the angular distribution in the

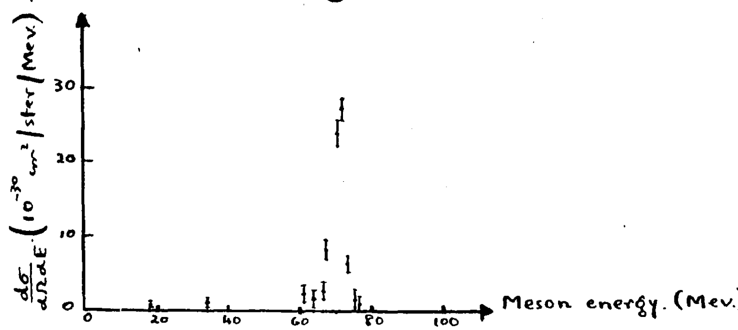

Fig.1. π^+ spectrum at 0° for 340 Mev protons. (Laboratory system)

Laboratory system angle.	$\left(\frac{d\sigma}{d\Omega}\right)_{Lab.}$ (10^{-28} cm²/ster).
0°	2.00 ± 0.40
18°	$1.6^{+0.3}_{-0.2}$
30°	$0.58^{+0.12}_{-0.16}$
60°	$0.07^{+0.04}_{-0.02}$

Table 1. Differential cross-section for 340 Mev incident protons.

centre of mass system for this reaction is of the form

$$\frac{d\sigma}{d\Omega} = (A + B\cos^2\theta)\,10^{-29}\ \text{cm}^2/\text{ster.}$$

with $A \sim .25$ and $B \sim 3$.

Durbin, Loar and Steinberger (4) and Clark, Roberts and Wilson (5), studying the inverse reaction $\pi^+ + d \rightarrow p + p$ have approximately verified this $\cos^2\theta$ distribution for different meson energies. Their results are shown in table 2. (iii) Integration of $\frac{d\sigma}{d\Omega}$ over angle for a particular proton energy gives the total cross-section at this energy. This has been done for proton energies, in the laboratory system, of 345, 365 and 380 Mev by

π^+ meson energy. (Mev.)	$\frac{d\sigma}{d\Omega}$ for reaction $\pi^+ + d \rightarrow p + p$ ($10^{-28}\ cm^2/ster.$)
25	$9(\cos^2\theta + 0.22)$
40	$18(\cos^2\theta + 0.2)$
53	$21.5(\cos^2\theta + 0.18)$

Table 2. Angular distributions for reaction $\pi^+ + d \rightarrow p + p$.

Passman, Block and Havens (6) who measured the meson yield at 90° in the laboratory system for these energies and assumed the above form of the angular distribution. The excitation curve they obtained is shown in fig.2 (results (a)) and shows a variation approximately proportional to T^2, where T is the meson kinetic energy in the centre of mass system. Schulz, Hamlin, Jakobson and Merritt (7) found a variation proportional to $T^{3/2}$. However, they measured only mesons in the peak, so that their excitation curve is for the reaction $p + p \rightarrow \pi^+ + d$ alone. The excitation curve for this reaction alone can be obtained by detailed balancing with spin zero mesons from the results, quoted above, on the inverse reaction and this is also shown in fig.2. (results (b))

(b) Theoretical.

The theoretical methods of studying meson production in nucleon-nucleon collisions fall into four classes,

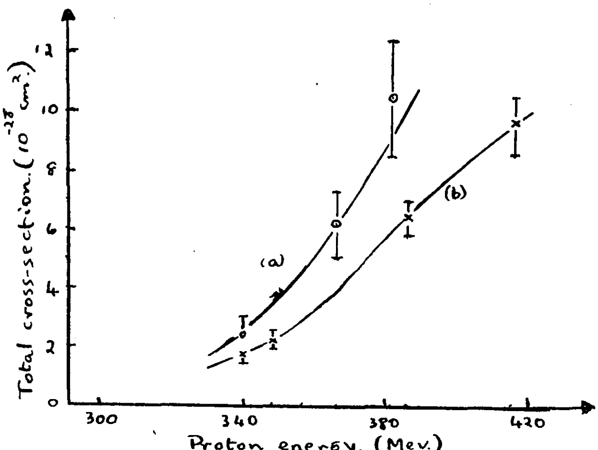

Fig.2. Excitation curves, (a) including $p+p \rightarrow \pi^+ + d$ and $p+p \rightarrow \pi^+ + p + n$ and (b) for $p+p \rightarrow \pi^+ + d$ only.

(i) the pure field-theoretic method as used by Morette (8) and Brueckner (9), (ii) the phenomenological weak-coupling method as used by Foldy and Marshak (10), Gunn, Power and Touschek (11), Fujimoto and Yamaguchi (12) and Cheston (13), (iii) the purely phenomenological method as used by Watson and Brueckner (14) and (iv) the isobaric state method as used by Matsuyama and Miyazawa (15) and Brueckner and Watson (16).

The first of these assumes that the exchange of momentum between the nucleons is, to lowest order, due to the exchange of one meson, so that the process is of the third order in the mesonic coupling constant. In this method it is difficult to take into account processes in which the final two nucleons are bound, and it has been restricted in use to dealing with cases in which plane waves can be taken for the nucleons initially and finally. This means that the results of this method will be uncertain for low relative energies of the two nucleons in the final state, where the experimental results are available. In the second method, the force between the nucleons is represented by a phenomenological potential, but the usual operators of field theory are used for describing the emission of the meson. This means that the final result is only of first order in the mesonic coupling constant; part of the uncertainty in the calculation has been transferred from field theory to nuclear physics, although the error caused by a weak-coupling treatment still remains, but to a lesser extent. Power (17) has partially justified the use of a phenomenological potential, a method which is not obviously correct since the π -mesons themselves are partly responsible for the potential. In the third method, the purely phenomenological, a proper relativistically invariant form is assumed for the transition matrix element and an analysis is made of the conditions of charge symmetry and conservation of angular momentum and parity. On this basis, Watson and Brueckner (14) have studied all the possible nucleon-nucleon production processes for the cases in which the mesons are emitted in s or p waves. The fourth method assumes that one of the nucleons is excited to an isobaric state and then decays emitting a meson. Of these four methods, we shall be chiefly interested in the second and we now illustrate its use by giving a brief survey of Cheston's (13) work, as we shall be using his approach in dealing with the problem of π^+ meson production in proton-deuteron collisions where the final nucleons form a triton. We shall restrict ourselves to scalar mesons with scalar coupling (henceforth denoted by $S(S)$) and pseudoscalar mesons with pseudovector coupling (denoted by $PS(PV)$).

Cheston considered the reaction $\pi^+ + d \rightarrow p + p$ and obtained the cross-section for the inverse process by detailed balancing. He restricted himself to mesons of energies greater than 5 Mev (so that the Coulomb barrier of the deuteron could be neglected) and less than 100 Mev (so that the nucleons would have energies less than 500 Mev and could be treated non-relativistically). The interaction operators were therefore used in their non-relativistic forms, these being obtained by the method of Foldy and Wouthuysen (18). For the wave function describing the two protons he took appropriately anti-symmetrised plane waves, i.e. he neglected their interaction. He stated that allowance for this would change the cross-section by only a few per cent. (Gunn, Power and Touschek (11) have taken this interaction into account and we shall discuss their results below). Cheston used the Hulthén wave function for the deuteron and allowed, in ps(pv) theory, for a small admixture of the $^3D_1$ state with the $^3S_1$ state. This allowance made only a small difference to his results and we shall henceforth neglect it. He calculated the angular distribution of the protons in the centre of mass system for meson energies of 5, 22.7, 50 and 100 Mev. His results for 22.7 Mev mesons (which correspond to 340 Mev protons in the laboratory system) are shown in fig.3. In the centre of mass system the angular distribution of the protons from the reaction $\pi^+ + d \rightarrow p + p$ is the same as that of the mesons from the reaction $p + p \rightarrow \pi^+ + d$. Fig.3 therefore ~~therefore~~ represents also the angular distribution of the mesons, in the centre of mass system, produced by protons whose laboratory energy is 340 Mev.

Cheston found that at the lowest energy the angular distribution for $PS(PV)$ theory was much more isotropic than for $S(S)$ theory, and that as the energy of the

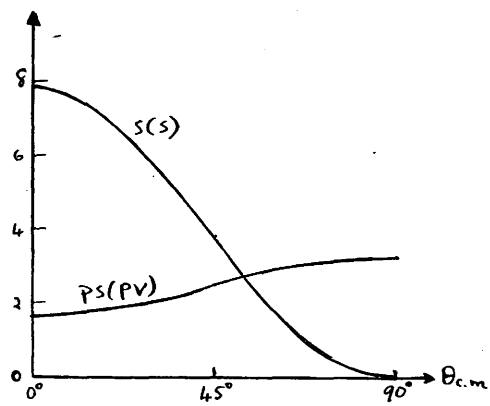

Fig.3. Angular Distributions for 340 Mev protons, as calculated by Cheston.

meson (and hence the proton) increased both distributions became less isotropic. This can be explained in the following way.

Consider the reaction $p+p \rightarrow \pi^{+}+d$. The final two-nucleon state is a $^{3}S_{1}$ state, and the initial two-nucleon state is limited, by the Pauli Exclusion Principle, to being one of $^{1}S_{0}$, $^{3}P_{2,1,0}$, $^{1}D_{2}$, etc.. Now, at low energies, the mesons will be emitted predominantly in s -states and for this there are no possible transitions for $S(S)$ theory, but the transition $^{3}P_{1} \rightarrow {}^{3}S_{1}$ is allowed in $PS(PV)$ theory. Thus for scalar mesons, the meson has to be emitted in a p -state leading to anisotropy, whereas for pseudoscalar mesons the meson can be emitted in an s-state leading to isotropy. The fact that the $ps(pv)$ angular distribution is not quite isotropic, even at low energies, shows that p-state mesons are also competing, this being due to the gradient operator in the interaction. The increase

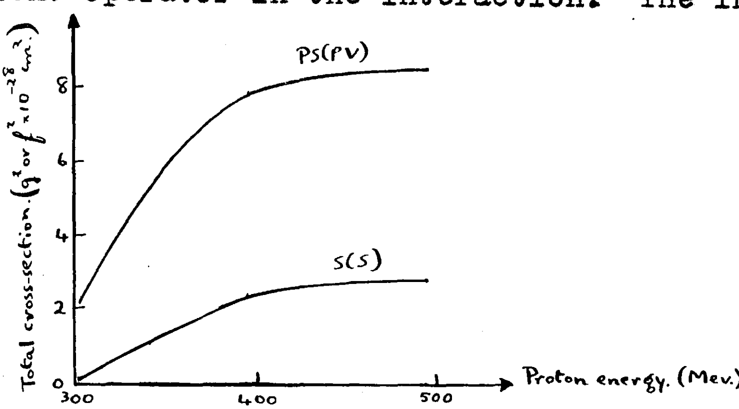

Fig.4. Excitation curves calculated by Cheston.

in anisotropy with increasing energy is caused by the increasing importance of the meson being emitted with higher angular momentum. Fig.4 shows the excitation curve obtained by Cheston for the reaction $p+p \rightarrow \pi^+ + d$.

Gunn, Power and Touschek (11), (G.P.T.), considered the reactions $p+p \rightarrow \pi^+ + d$ and $p+p \rightarrow \pi^+ + p + n$ and took into account the interaction of the two protons in the initial state. This interaction they took to be zero for odd parity states; for even parity states they considered the interaction to be only in the s-state. To describe the internucleon interaction, they used a Hulthén potential $U(r) = J e^{-\varkappa r}\left[1 - e^{-\varkappa r}\right]^{-1}$ with $\frac{1}{\varkappa} = 1.14 \times 10^{-13}$ cm., ${}^{(3)}J = 46.6$ Mev and ${}^{(1)}J = 24.2$ Mev where ${}^{(3)}J$ and ${}^{(1)}J$ refer to the spin triplet and singlet states respectively. For the angular distribution of the π^+ mesons in the centre of mass system they found a $\cos^2\theta$ distribution for $S(S)$ theory; for $PS(PV)$ theory they found that interference between the s - and d -waves

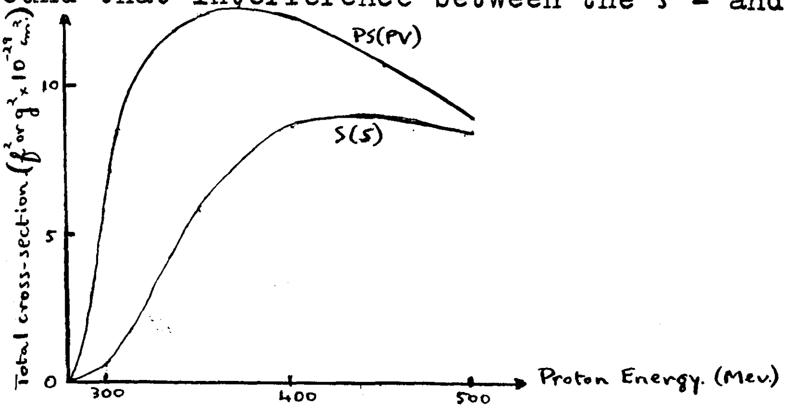

Fig.5. Excitation curves calculated by Gunn, Power and Touschek for reaction $p+p \rightarrow \pi^+ + d$.

led to a cancellation in the forward direction. Their calculated excitation curves are shown in fig.5. Their total cross-sections were sensitive to the internucleon potential chosen and they found that for a square well potential ${}^{(3)}J$ = 41 Mev and a = 1.85 x 10^{-13} cm., the cross-sections at 350 Mev were reduced by a factor 25.

Both Cheston and G.P.T., using the phenomenological weak-coupling method, found that scalar mesons gave a better fit to the experimental $\cos^2\theta$ distribution than did pseudoscalar mesons. However, as G.P.T. pointed out, better agreement may be found using pseudoscalar theory by choosing a more suitable internucleon potential. G.P.T. also found, for low energies, a better fit to the experimental excitation curve with scalar mesons, though the rise in the cross-section with energy was not so steep as experiment requires.

§2. π^+ Meson Production in Proton-Neutron Collisions.

No direct experimental information on the reaction $p+n \rightarrow \pi^+ + n + n$ has been obtained, since a neutron source of sufficient intensity and energy resolution is not yet available. Information about this process has to be deduced from meson production in nucleon-nucleus collisions, or from other nucleon-nucleon production processes on the assumption of charge independence.

Analyses of meson production in nucleon-nucleus collisions have, so far, been based on the impulse approximation. This approximation was first introduced by Chew (19) in his treatment of the inelastic scattering of high energy neutrons by deuterons. In a later paper, Chew and Wick (20) clarified the actual approximations which are made. These are (i) that the incident nucleon never interacts strongly with more than one nucleon of the nucleus at the same time, (ii) that the amplitude of the incident wave arriving at a nucleon is nearly the same as if that nucleon were alone and (iii) that during the interaction the binding forces between the nucleons of the nucleus are negligible. The third approximation is based on the assumption that the 'collision time' is so short that forces, other than those of interaction between the two participating nucleons, can be neglected. It is this suddenness of interaction which gives the approximation its name. The waves from the different scattering centres (nucleons) are added, but each individually is the same as that produced by a single nucleon. Thus the many-body problem is reduced to a superposition of two-body problems. This method can be extended to meson production in nucleon-nucleus collisions (see Noyes (21)) which becomes a superposition of nucleon-nucleon production processes. If, for example, we knew the π^+ meson yield from protons on nuclei and from protons on protons, we could estimate the π^+ meson yield from protons on neutrons. The only effects of the non-participating nucleons are to give a momentum distribution in the initial state to the participating nucleon of the nucleus and to limit the possible final states of the two participating nucleons through the Exclusion Principle.

'Charge independence' denotes the equality of proton-proton, neutron-neutron and proton-neutron interaction forces for states of the same spin and parity and is reflected in the invariance of the interaction energy under rotations in isotopic spin space. This principle can be extended to include phenomena involving pions, the 'quanta' of the nuclear field. In this, the coupling constants for π^+ and π^- mesons are equal and the coupling constant for π° mesons is $\frac{1}{\sqrt{2}}$ that of the charged mesons but is opposite in sign for neutrons and protons. An isotopic spin $\vec{L}$ is ascribed to the mesons and the states $\omega(1^+), \omega(1^0), \omega(1^-)$ which contain one π^+, one π°, one π^- respectively, correspond to $L = 1$ and $L_3\,\omega(1^+) = -\omega(1^+)$, $L_3\,\omega(1^0) = 0$, $L_3\,\omega(1^-) = \omega(1^-)$. In charge independent theory, the total isotopic spin $\vec{J}$ is a constant of the motion. ($\vec{J} = \vec{T} + \vec{L}$, where $\vec{T}$ is the isotopic spin operator of the nucleons). On the basis of this theory, relations between the various nucleon-nucleon production cross-sections can be established. In the following we denote the cross-section for π^+ meson production in proton-neutron collisions by σ_{pn}^+ and use similar notation for the other nucleon-nucleon production cross-sections.

As an illustration of the method of estimating σ_{pn}^+ by using charge independence and data on nucleon-nucleus production, we review the analysis given by Passman, Block and Havens (22) of charged meson production in proton-deuteron collisions. They found experimentally, using a 380 Mev beam of protons, a yield of π^+ mesons twenty-five times larger than the yield of π^- mesons at an angle of 90° in the laboratory system. The possible nucleon-nucleon processes producing these mesons are (a) $p+p \to \pi^+ + d$ or $p+p \to \pi^+ + p + n$, (b) $p+n \to \pi^+ + n + n$ and (c) $p+n \to \pi^- + p + p$. Charge independence gives, apart from small Coulomb effects, $\sigma_{pn}^+ = \sigma_{pn}^-$. (Actually, charge symmetry alone gives $\sigma_{pn}^+ = \sigma_{pn}^-$, where charge symmetry denotes the equality of only proton-proton and neutron-neutron interaction forces.) The small yield of π^- mesons indicates that σ_{pn}^- is small compared to σ_{pp}^+ and so therefore is σ_{pn}^+. The operation of the Exclusion Principle will however, in proton-deuteron production, inhibit process (c) more than processes (a) and (b), since in it there are three like particles (protons) finally. Noyes (21) showed that the $\frac{\pi^+}{\pi^-}$ ratio at 0° to a 345 Mev proton beam could be as large as 8.2 because of this effect. The cross-section σ_{pn}^+ is not therefore as small as the above analysis suggests and is probably about one-third or one-quarter of σ_{pp}^+. The above analysis assumes that the angular distributions of the π^+ mesons from proton-proton collisions and of the π^- mesons from proton-neutron collisions are the same. That these angular distributions are probably not very dissimilar can be seen from the approximate equality of the $\frac{\pi^+}{\pi^-}$ ratio at 0° (Dudziack (23)) and 90° (Passman, Block and Havens (24)) obtained from experiments in which carbon was bombarded with protons.

An estimate of σ_{np}^+ can also be obtained from nucleon-nucleus production without the use of the charge-independent theory. (It should be noted that charge-independent theory has not as yet been tested rigourously by experiment.) As an illustration of how this may be done, we review the analysis of Passman, Block and Havens (25) who studied the π^+ mesons produced at 90° to a 381 Mev proton beam incident on a carbon target. They allowed for the absorption of the mesons in the carbon nucleus before escape and for the enhancement of the π^+ meson production caused by the nucleonic momentum distribution in carbon. (σ_{pp}^+ has a steep excitation curve so that the increased cross-section for protons moving towards the incident beam more than compensates for the decreased cross-section for protons moving in the opposite direction.). With these considerations they found a ratio of 9 for the carbon to hydrogen cross-sections at this angle and energy. There are 6 protons in carbon, so that the 6 neutrons must give one-third of the

π^+ mesons observed, and hence $\sigma_{np}^+ \sim \frac{1}{2}\sigma_{pp}^+$. Passman, Block and Havens have calculated the expected shape of the π^+ meson spectrum at 90°, using the impulse approximation. They assumed that the π^+ mesons were produced only in proton-proton collisions and to describe this they used the experimental data described in §1. They found the best agreement with experiment with a Gaussian momentum distribution for the nucleons in carbon, $N(p) = e^{-p^2/\alpha^2}$ with $\frac{\alpha^2}{2M}$ = 14 - 19 Mev, M = nucleon mass. (Compare the work of Cladis, Hess and Moyer (26).)

Similar analyses can be carried out for the yields of $\pi^\pm$ and π^0 mesons produced by protons or neutrons incident on nuclei using the relations between the various nucleon-nucleon production cross-sections of $\pi^\pm$ and π^0 mesons for charge independent theory. Analysis of π^+ meson production in neutron-nuclei collisions would not require the use of the charge independent theory. These analyses confirm the above conclusion that σ_{np}^+ is less than σ_{pp}^+ but not by an order of magnitude and we shall not enter into them here.

§3. $\underline{p + d \rightarrow \pi^+ + t}$.

(a) Experimental.

Passman, Block and Havens (22) have studied the yield at 90° (laboratory system) of π^+ mesons produced by

381 Mev protons incident on deuterium. The π^+ meson energy spectrum had no peak near 90 Mev; this indicated that the reaction $p+d \to \pi^+ + t$ was not very strong.

Frank, Bandtel, Madley and Moyer (27) have studied the reaction $p+d \to \pi^+ + t$ using 341 Mev protons. They measured the differential cross-sections for meson production at centre of mass angles of 30°, 50°, 70°, 90°, 130°, and 150° and found that $\frac{d\sigma}{d\Omega}$ was constant from 180° to 90° and rose from 90° to 0°, the value of $\left(\frac{d\sigma}{d\Omega}\right)_{0^\circ} \Big/ \left(\frac{d\sigma}{d\Omega}\right)_{90^\circ}$ being between 5 and 10. They estimated the total cross-section to be about 5×10^{-30} cm^2.

(b) Theoretical.

Ruderman (28) has studied the reaction $p+d \to \pi^+ + t$ and shown that, on the assumption of charge independence, $\frac{d\sigma(p+d \to He^3 + \pi^0)}{d\sigma(p+d \to t + \pi^+)} = \frac{1}{2}$. In his calculation he used the impulse approximation and assumed that π^+ mesons were produced only in proton-proton collisions. We have seen in §2 that this assumption is not satisfactory as σ^+_{pn} is not negligible compared to σ^+_{pp}. For his triton wave function he assumed that the two participating nucleons were very close, i.e. he wrote

$$\psi_t\left(x_1 - x_2,\ x_3 - \frac{x_1+x_2}{2}\right) \sim \frac{\psi_d(x_1-x_2)}{\psi_d(0)}\ \psi_t(0, x_3 - x_1)$$

where x_1 and x_2 are the co-ordinates of the participating nucleons and x_3 and x_1 are the co-ordinates of the neutrons in the triton. For the spatial parts of the wave functions he took

$$\psi_d(r) = \frac{e^{-\beta r} - e^{-\gamma r}}{r}\left(\frac{\beta\gamma(\beta+\gamma)}{2\pi(\beta-\gamma)^2}\right)^{\frac{1}{2}}$$

and $$\psi_t(\rho, r) = \frac{\alpha^3}{8\pi}\, e^{-\alpha r}$$

with $\gamma = 6\beta, \quad \beta = \frac{0.32}{\lambda}, \quad \alpha = \frac{1.6}{\lambda}, \quad \lambda = 1.4 \times 10^{-13}$ cm.

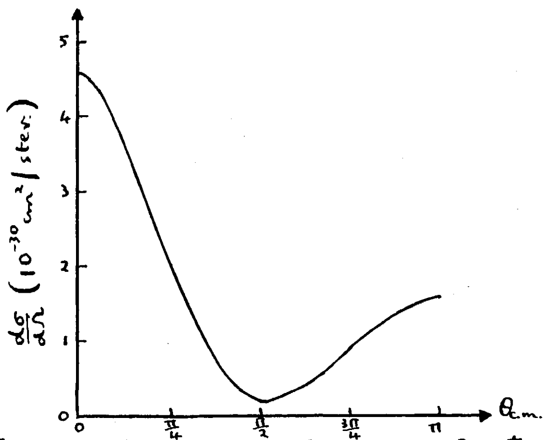

Fig.6. Angular distribution of π^+ mesons from $p+d \rightarrow \pi^+ + t$ as calculated by Ruderman for 345 Mev protons.

Fig.6. shows his results for the angular distribution of the π^+ mesons in the centre of mass system, calculated for incident protons of energy 345 Mev in the laboratory system. The corresponding meson energy in the centre of mass system is about 80 Mev. (Ruderman used 90 Mev, a figure obtained by treating the nucleons non-relativistically). He used the extrapolated value of $\sigma(p+p \rightarrow \pi^+ + d)$ for this

meson energy and estimated the total cross-section to be about 1.3 x 10^{-29} cm^2. We see that, compared with experiment, his value of $\left(\frac{d\sigma}{d\Omega}\right)_{0^\circ} \Big/ \left(\frac{d\sigma}{d\Omega}\right)_{90^\circ}$ is too large: it is about 30. He found a rise in $\frac{d\sigma}{d\Omega}$ from 90° to 180° which is not found by experiment and his total cross-section is too large by a factor 2.6.

Bludman (29) has repeated Ruderman's calculation using a hard core to suppress the high momentum components of the deuteron wave function. He used an improved triton wave function, not quoted in his abstract, on the assumption that two of the nucleons are very close. The result of this is to reduce the total cross-section and to flatten the angular distribution in the backward direction. Bludman found, for a proton energy of 341 Mev, a total cross-section of 9.1 x 10^{-30} cm^2 without the core and 5.2 x 10^{-30} cm^2 with the core.

CHAPTER II.

§4. Kinematics of Reaction.

Let us consider the general problem in which a particle of mass M_1 and velocity β_1 is incident on a stationary particle of mass M_2 and particles of mass μ and M_i are produced. We treat these particles relativistically and use natural units, $\hbar = c = 1$. The velocity of transformation from this system to the centre of mass system is $v = \dfrac{\gamma_1 \beta_1}{\frac{M_2}{M_1} + \gamma_1}$ where $\gamma_1 = \dfrac{1}{\sqrt{1-\beta_1^2}}$.
The velocity of M_1 in the centre of mass system is $\dfrac{\beta_1 \gamma_1}{\frac{M_1}{M_2} + \gamma_1}$ and that of M_2 is $\dfrac{\gamma_1 \beta_1}{\frac{M_2}{M_1} + \gamma_1}$. Denoting centre of mass quantities by dashes, we have the total energy

$$E' = \sqrt{(M_1+M_2)^2 + 2T_1 M_2}$$

where T_1 is the kinetic energy of M_1 in the laboratory system. The total kinetic energy in the centre of mass system is

$$T' = \sqrt{(M_1+M_2)^2 + 2T_1 M_2} - (M_1 + M_2) . \tag{4.1}$$

At threshold, $E_t' = \mu + M_i$, therefore the threshold energy is

$$T_{1_t} = \frac{(\mu + M_i)^2 - (M_1 + M_2)^2}{2M_2} . \tag{4.2}$$

The kinetic energy of M_2 in the centre of mass system is

$$T_1' = \frac{M_2 T_1 - M_1 T'}{T' + (M_1 + M_2)} , \tag{4.3}$$

so that the momentum of either particle is given by

$$p'^2 = \frac{M_2^2 T_1 (T_1 + 2M_1)}{(T' + M_1 + M_2)^2} \quad (4.5)$$

For the particular case of $p + d \rightarrow \pi^+ + t$ we assume that $M_1 = M$, $M_2 = 2M$, $M_i = 3M$, where M is the mass of a proton. We treat the triton non-relativistically and the other particles relativistically. From conservation of energy, we have

$$T' = \omega' + \frac{k'^2}{6M} - B, \quad (4.6)$$

where ω' is the meson energy, $\vec{k}'$ the meson momentum (which is equal and opposite to that of the triton) and B is the binding energy of a proton in a triton. B = 6.2 Mev. We have

$$T_1 = \frac{T'^2}{4M} + \frac{3}{2} T' \quad (4.7)$$

$$T_1' = \frac{M(2T_1 - T')}{T' + 3M} \quad (4.8)$$

and

$$p'^2 = \frac{4M^2 T_1 (T_1 + 2M)}{(T' + 3M)^2} \quad (4.9)$$

The differential cross-section for the reaction $p + d \rightarrow \pi^+ + t$, $\frac{d\sigma_{prod.}}{d\Omega'}$, is obtained from the differential cross-section for the reaction $\pi^+ + t \rightarrow p + d$, $\frac{d\sigma_{abs.}}{d\Omega'}$, by detailed balancing. $\frac{d\sigma_{prod.}}{d\Omega'}$ is the differential cross-section for the production of a meson of momentum k' into the solid angle $d\Omega'$ by a proton of momentum p'. $\frac{d\sigma_{abs.}}{d\Omega'}$ is the differential cross-section for the production of a proton of momentum p' into the solid angle $d\Omega'$ by a meson of momentum k'. Then $\frac{d\sigma_{prod.}}{d\Omega'} \Big/ \frac{d\sigma_{abs.}}{d\Omega'} = \frac{1}{3}\frac{k'^2}{p'^2}$ if we assume that the meson has zero spin.

Some numerical values obtained from these kinematical considerations are given in table 3: we have used $M = 938$ Mev and $\mu = 141$ Mev.

Kinetic energy of meson. (c.m. system.) $(\omega'-\mu)$ (Mev.)	Proton energy. (c.m. system.) T_1' (Mev.)	Proton energy. (lab. system.) T_1 (Mev.)	Total kinetic energy. (c.m. system.) T' (Mev.)	$\frac{d\sigma_{prod.}}{d\Omega'} \Big/ \frac{d\sigma_{abs.}}{d\Omega'}$.
10	95.4	223.6	145.3	5.16×10^{-3}
30	109.5	254.2	166.5	1.44×10^{-2}
50	123.2	291.0	184.4	2.25×10^{-2}
78.3	142.9	340.0	218.2	3.26×10^{-2}
100	154.9	348.0	241.6	3.94×10^{-2}

Table 3. Numerical values obtained from kinematics.

The threshold for the reaction $p + d \to \pi^+ + t$, obtained from (4.2) using the exact values of M_1, M_2 and M_i, is 209 Mev.

The other kinematical problem involved is the transformation of the meson angular distribution to the laboratory system. This is given by (see Gunn (30)),

$$\frac{d\sigma_{prod.}}{d\Omega} = \frac{k^2}{k'}\frac{1}{\gamma(k - v\omega\cos\theta)}\left(\frac{d\sigma_{prod.}}{d\Omega'}\right) \tag{4.10}$$

where $\gamma = \frac{1}{\sqrt{1-v^2}}$. The relationships between centre of mass and laboratory quantities are

$$\theta = \tan^{-1}\left\{\frac{k'\sin\theta'}{\gamma(k'\cos\theta' + v\omega')}\right\}$$

$$\omega = \gamma(\omega' + vk'\cos\theta')$$

and $$k^2 = \omega^2 - \mu^2.$$

θ'	θ	ω (Mev.)	k (Mev.)	$\frac{k^2}{k'}\frac{1}{\gamma(k-v\omega\cos\theta)}$
0°	0°	266.3	226.0	1.81
30°	22° 30'	260.9	219.5	1.71
45°	34° 10'	254.4	211.8	1.60
90°	72° 30'	225.7	176.2	1.12
135°	120° 30'	197.0	137.5	0.68
180°	180°	185.0	119.8	0.51

Table 4. Values of laboratory system quantities for 340 Mev protons.

Table 4 shows the numerical values for these, for 340 Mev protons in the laboratory system, for which $k' = 168.0$ Mev, $\omega' = 219.3$ Mev, $v = 0.235$.

§ 5. General Formalism.

We consider the reaction in which a triton (t) absorbs a meson (π^+) of mass μ , momentum $\vec{k}$ and energy ω , and disintegrates into a proton (p) and a deuteron (d). To avoid difficulties associated with the Coulomb barrier of the triton we restrict ourselves to mesons of energy greater than 10 Mev in the centre of mass system. In order to be able to treat the nucleons non-relativistically we consider the mesons to have energy less than 100Mev. We treat the mesons relativistically.

To lowest order in weak-coupling perturbation theory, the matrix element of the interaction Hamiltonian causing a transition from an initial state with three nucleons (ψ_o) and a π^+ meson (φ) to a final state with three nucleons (ψ_f) is

$$H'_{fo} = \int d\vec{r}_1\, d\vec{r}_2\, d\vec{r}_3\ \psi_f^+(\vec{r}_i, s_i, \tau_i)\left\{\sum_{i=1}^{3} O^{(i)}\varphi(\vec{r}_i)\right\}\psi_o(\vec{r}_i, s_i, \tau_i)\,. \qquad (5.1)$$

After the Fourier expansion of the meson field $\varphi(\vec{r}_i)$

$$O^{(i)}\varphi(\vec{r}_i) = g\sqrt{\frac{2\pi}{\omega}}\, Q^{(i)}\beta^{(i)} e^{i\vec{k}\cdot\vec{r}_i} \qquad \text{for } S(S) \quad (5.2)$$

and $$O^{(i)}\varphi(\vec{r}_i) = \frac{if}{\mu}\sqrt{\frac{2\pi}{\omega}}\, Q^{(i)}\left\{(\vec{\sigma}^{(i)}\cdot\vec{k}) - \rho_1^{(i)}\omega\right\} \qquad \text{for } PS(PV) \quad (5.3)$$

Here $\vec{\sigma}^{(i)}$, $\beta^{(i)}$, $\rho_1^{(i)}$ are the usual Dirac matrices acting on the i^{th} nucleon; $Q^{(i)}$ is the operator which acts on the final state nuclear wave function, ψ_f, changes the i^{th} nucleon, if it is a proton, to a neutron, and gives zero if the i^{th} nucleon is a neutron; τ_i and s_i are the isotopic and spin variables respectively of the i^{th} nucleon; we work in units in which $\hbar = c = 1$.

For the energies we are considering the nucleons can be treated non-relativistically, so we use the non-relativistic approximation of Foldy and Wouthuysen (18). The even operators $\beta^{(i)}$ and $\vec{\sigma}^{(i)}$ are replaced by 1 and the 2 x 2 Pauli spin matrices, $\vec{\sigma}^{(i)}$, respectively. The odd operator $\rho_1^{(i)}$ is replaced by $\frac{i}{2M}\vec{\sigma}^{(i)}\cdot(\overleftarrow{\nabla}^{(i)} - \overrightarrow{\nabla}^{(i)})$ where $\overleftarrow{\nabla}^{(i)}$ is the gradient operator with respect to $\vec{r}_i$ acting on the final nuclear wave function. $\overrightarrow{\nabla}^{(i)}$ is the similar operator acting on the initial nuclear wave function. These wave functions are now the appropriate ordinary Schrodinger wave functions.

Carrying out an integration by parts for the $\overrightarrow{\nabla}^{(i)}$ part of $\rho_1^{(i)}$ we obtain

$$-\frac{i}{2M}\overrightarrow{\nabla}^{(i)} = \frac{i}{2M}\overleftarrow{\nabla}^{(i)} - \frac{\vec{k}}{2M}.$$

Therefore $\rho_1^{(i)}$ is replaced by $\vec{\sigma}^{(i)}\cdot\left(\frac{i}{M}\overleftarrow{\nabla}^{(i)} - \frac{\vec{k}}{2M}\right)$.

Thus, in the non-relativistic form,

$$O^{(i)} = g\sqrt{\frac{2\pi}{\omega}}\, Q^{(i)} \qquad \text{for } S(S) \qquad (5.4)$$

and

$$O^{(i)} = \frac{if}{\mu}\sqrt{\frac{2\pi}{\omega}}\, Q^{(i)}\left\{\vec{\sigma}^{(i)}\cdot\vec{k} - \frac{i\omega}{M}\overleftarrow{\nabla} + \frac{\omega}{2M}\vec{k}\right\} \qquad \text{for } PS(PV). \qquad (5.5)$$

For the triton wave function, we take the properly anti-symmetrised ^2S space-symmetric wave function

$$\psi_0(\vec{r}_i, s_i, \tau_i) = \varphi_t(\vec{r}_i)\,\frac{1}{\sqrt{6}}\left\{(\alpha_2-\alpha_1)\beta_3 + (\alpha_3-\alpha_2)\beta_1 + (\alpha_1-\alpha_3)\beta_2\right\} \qquad (5.6)$$

for the z -component of spin = $+\frac{1}{2}$, and

$$\psi_0(\vec{r}_i, s_i, \tau_i) = \varphi_t(\vec{r}_i)\frac{1}{\sqrt{6}}\left\{(\alpha_2'-\alpha_1')\beta_3 + (\alpha_3'-\alpha_2')\beta_1 + (\alpha_1'-\alpha_3')\beta_2\right\} \qquad (5.6)$$

for the z -component of spin = $-\frac{1}{2}$. $\left.\begin{matrix}\alpha_i\\ \alpha_i'\end{matrix}\right\}$ is the product of three spin wave functions $v(1)\,v(2)\,v(3)$ such that the i^{th} nucleon has spin component $\left.\begin{matrix}-\frac{1}{2}\\ +\frac{1}{2}\end{matrix}\right\}$ and the other two have spin components $\left.\begin{matrix}+\frac{1}{2}\\ -\frac{1}{2}\end{matrix}\right\}$. Similarly $\left.\begin{matrix}\beta_i\\ \beta_i'\end{matrix}\right\}$ is the product of three isotopic spin wave functions $u(1)\,u(2)\,u(3)$ such that the i^{th} nucleon represents a $\left.\begin{matrix}\text{proton}\\ \text{neutron}\end{matrix}\right\}$ and the other two represent $\left.\begin{matrix}\text{neutrons}\\ \text{protons}\end{matrix}\right\}$.

Let $\vec{R} = \frac{1}{3}(\vec{r}_1 + \vec{r}_2 + \vec{r}_3)$, the co-ordinates of the centre of mass of the triton, and

$\vec{r}_{ij} = \vec{r}_j - \vec{r}_i$, the relative co-ordinates of the three nucleons. We separate the triton wave function into a part which describes the centre of mass motion and and a part which depends on the relative co-ordinates of the nucleons. As a first choice for the latter part, let us take the wave function studied by Irving (31). This gives for the triton wave function

$$\varphi_t(\vec{r}_i) = \sqrt{N_t}\;\frac{e^{-\alpha_t(r_{21}^2 + r_{31}^2 + r_{32}^2)^{\frac{1}{2}}}}{(r_{21}^2 + r_{31}^2 + r_{32}^2)^{\frac{1}{2}}}\; e^{i\vec{P}_0\cdot\vec{R}}\,, \qquad (5.7)$$

where $\vec{P}_0$ is the momentum associated with the centre of mass of the triton. N_t is a normalisation factor equal to $\frac{8\sqrt{3}}{\pi^3}\,\alpha_t^4$.

For the final state, we assume that particles 2 and 3 form the deuteron and that particle 1 is the emitted proton. The wave function for this system is

$$\Psi_f(\vec{r}_i, s_i, \tau_i) = \varphi_{p+d}(\overline{\vec{r}_2, \vec{r}_3}, \vec{r}_1)\ {}^{2s+1}\sigma_{m_s}(\overline{23}, 1)\ {}^1(\tau)_0(2,3)\, u_-(1) \qquad (5.8)$$

where a bar above two variables denotes symmetry in them. The other possibilities (particles 2 and 3 being the emitted proton) are taken into account by multiplying the matrix element by 3.

${}^{2s+1}\sigma_{m_s}(\overline{23},1)$ can be ${}^2\sigma_{\pm\frac{1}{2}}$ or ${}^4\sigma_{\pm\frac{3}{2},\pm\frac{1}{2}}$ where

$$
\begin{aligned}
{}^2\sigma_{\frac{1}{2}}(\overline{23},1) &= \tfrac{1}{\sqrt{3}}\left[\tfrac{1}{\sqrt{2}}\alpha_2 + \tfrac{1}{\sqrt{2}}\alpha_3 - \sqrt{2}\,\alpha_1\right] \\
{}^2\sigma_{-\frac{1}{2}}(\overline{23},1) &= \tfrac{1}{\sqrt{3}}\left[\tfrac{1}{\sqrt{2}}\alpha_2' + \tfrac{1}{\sqrt{2}}\alpha_3' - \sqrt{2}\,\alpha_1'\right] \\
{}^4\sigma_{\frac{3}{2}}(\overline{23},1) &= \gamma \quad \{= v_+(1)\, v_+(2)\, v_+(3)\} \\
{}^4\sigma_{\frac{1}{2}}(\overline{23},1) &= \tfrac{1}{\sqrt{3}}\left[\alpha_1 + \alpha_2 + \alpha_3\right] \\
{}^4\sigma_{-\frac{1}{2}}(\overline{23},1) &= \tfrac{1}{\sqrt{3}}\left[\alpha_1' + \alpha_2' + \alpha_3'\right] \\
{}^4\sigma_{-\frac{3}{2}}(\overline{23},1) &= \gamma' \quad \{= v_-(1)\, v_-(2)\, v_-(3)\}.
\end{aligned}
$$

We also have ${}^1(\tau)_0(2,3)\, u_-(1) = \frac{1}{\sqrt{2}}(\beta_2' - \beta_3')$.

The wave function for the final state is anti-symmetric in the indices 2 and 3, as is the wave function for the initial state, so that

$$\int d\vec{r}_2\, d\vec{r}_3\, \Psi_f^+ \left[f^{(3)}\right] \Psi_0 = \int d\vec{r}_2\, d\vec{r}_3\, \Psi_f^+ \left[f^{(2)}\right] \Psi_0 .$$

Therefore $O^{(3)}\varphi(\vec{r}_3)$ can be replaced by $O^{(2)}\varphi(\vec{r}_2)$.
Let $\vec{R}' = \frac{1}{2}(\vec{r}_2 + \vec{r}_3)$, the co-ordinates of the centre of mass of the deuteron.

In the final state wave function, we again extract the motion of the centre of mass of the deuteron. The emitted proton is described by a plane wave. This means that we assume that there is no interaction between the proton and the deuteron. Such an interaction could be included by inserting a phase shift in the proton wave function. To describe the deuteron we use a Yukawa-type wave function $\sqrt{N_\alpha}\,\frac{e^{-\alpha_\alpha r_{23}}}{r_{23}}$. With these assumptions we write

$$\varphi_{p+d}(\vec{r}_2, \vec{r}_3, \vec{r}_1) = \sqrt{N_\alpha}\,\frac{e^{-\alpha_\alpha r_{23}}}{r_{23}}\, e^{i\vec{P}'\cdot\vec{R}'}\, e^{i\vec{p}\cdot\vec{r}_1} \qquad (5.9)$$

where $\vec{P}'$ is the momentum of the centre of mass of the deuteron and $\vec{p}$ is the momentum of the emitted proton. N_α is the normalisation factor, and is equal to $\frac{1}{2\pi}\alpha_\alpha$.

We have now assigned all the necessary momenta and conservation of these gives

$$\vec{k} + \vec{P}_0 = \vec{p} + \vec{P}'. \qquad (5.10)$$

We work in the centre of mass system, for which

$$\vec{k} + \vec{P}_0 = 0. \qquad (5.11)$$

We now have

$$H'_{fo} = 3\int d\vec{r}_1\, d\vec{r}_2\, d\vec{r}_3\, \sqrt{N_d}\, \frac{e^{-\alpha_d r_{23}}}{r_{23}}\, e^{-i\vec{p}\cdot\vec{r}_1}\, e^{-i\vec{P}'\cdot\frac{1}{2}(\vec{r}_2+\vec{r}_3)}$$

$$\times\, {}^{2s+1}\sigma_{m_s}(\overline{23},1)\, \frac{1}{\sqrt{2}}(\beta_2'-\beta_3')\left\{ O^{(1)} e^{i\vec{k}\cdot\vec{r}_1} + 2\,O^{(2)} e^{i\vec{k}\cdot\vec{r}_2} \right\}$$

$$\times\, \sqrt{N_t}\, \frac{e^{-\alpha_t (r_{21}^2+r_{31}^2+r_{32}^2)^{\frac{1}{2}}}}{(r_{21}^2+r_{31}^2+r_{32}^2)^{\frac{1}{2}}}\, e^{i\vec{P}_0\cdot\frac{1}{3}(\vec{r}_1+\vec{r}_2+\vec{r}_3)}$$

$$\times\, \frac{1}{\sqrt{6}}\left\{ (\alpha_2-\alpha_1)\beta_3 + (\alpha_3-\alpha_2)\beta_1 + (\alpha_1-\alpha_3)\beta_2 \right\} \qquad (5.12)$$

when the triton has z -component of spin equal to $+\frac{1}{2}$. Let us change the variables from $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_3$ to $\vec{R}'$, $\vec{\rho}_1$ and $\vec{r}_{23}$ where

$\vec{R}' = \frac{1}{2}(\vec{r}_2+\vec{r}_3)$, the position co-ordinates of the centre of mass of the deuteron,

$\vec{\rho}_1 = \vec{r}_1 - \frac{1}{2}(\vec{r}_2+\vec{r}_3)$, the position co-ordinates of the emitted proton relative to the centre of mass of the deuteron,

and $\vec{r}_{23} = \vec{r}_3 - \vec{r}_2$, the relative co-ordinates of the deuteron.

The Jacobian of this transformation is unity and we now have

$$H'_{fo} = \frac{\sqrt{3}}{2}\sqrt{N_d N_t}\int d\vec{R}'\, e^{i\vec{R}'\cdot(-\vec{p}-\vec{P}'+\vec{k}+\vec{P}_0)} \int d\vec{\rho}_1\, d\vec{r}_{23}\, \frac{e^{-\alpha_d r_{23}}}{r_{23}}$$

$$\times\, e^{i\vec{\rho}_1\cdot(-\vec{p}+\frac{1}{3}\vec{P}_0)}\ {}^{2s+1}\sigma_{m_s}(\overline{23},1)\ (\beta_2'-\beta_3')$$

$$\times \left\{ O^{(1)} e^{i\vec{k}\cdot\vec{\rho}_1} + 2\,O^{(2)} e^{-\frac{1}{2}i\vec{k}\cdot\vec{r}_{23}} \right\} \frac{e^{-\alpha_t(2\rho_1^2+\frac{3}{2}r_{23}^2)^{\frac{1}{2}}}}{(2\rho_1^2+\frac{3}{2}r_{23}^2)^{\frac{1}{2}}}$$

$$\times \left\{ (\alpha_2-\alpha_1)\beta_3 + (\alpha_3-\alpha_2)\beta_1 + (\alpha_1-\alpha_3)\beta_2 \right\} \qquad (5.13)$$

where for $Ps(Pv)$ theory,

$$O^{(1)} = \frac{if}{\mu}\sqrt{\frac{2\pi}{\omega}}\, Q^{(1)}\left\{\vec{\sigma}^{(1)}\cdot\vec{k} - \frac{\omega}{M}\vec{p} + \frac{\omega}{2M}\vec{k}\right\} \tag{5.14}$$
$$= Q^{(1)}\,\Theta^{(1)}, \text{ say}$$

and

$$O^{(2)} = \frac{if}{\mu}\sqrt{\frac{2\pi}{\omega}}\, Q^{(2)}\left\{\vec{\sigma}^{(2)}\cdot\vec{k} - \frac{\omega}{2M}\vec{P}' + \frac{i\omega}{M}\overleftarrow{\nabla}_{\vec{r}_{23}} + \frac{\omega}{2M}\vec{k}\right\} \tag{5.15}$$
$$= Q^{(2)}\,\Theta^{(2)}.$$

$\overleftarrow{\nabla}_{\vec{r}_{23}}$ is the gradient operator with respect to $\vec{r}_{23}$ and acts only on $\frac{e^{-\alpha_d r_{23}}}{r_{23}}$.

From (5.10) and (5.11), (5.13) becomes

$$H'_{fo} = \frac{\sqrt{3}}{2}\sqrt{N_d N_t}\int d\vec{\rho}_1\, d\vec{r}_{23}\, \frac{e^{-\alpha_d r_{23}}}{r_{23}}\, e^{i\vec{\rho}_1\cdot(-\vec{p}-\frac{1}{3}\vec{k})}$$
$$\times\, {}^{2s+1}\sigma_{m_s}(\overline{23},1)(\beta_2'-\beta_3')\left\{O^{(1)}e^{i\vec{k}\cdot\vec{\rho}_1} + 2\,O^{(2)}e^{-\frac{1}{2}i\vec{k}\cdot\vec{r}_{23}}\right\}$$
$$\times\,\frac{e^{-\alpha_t(2\rho_1^2+\frac{3}{2}r_{23}^2)^{\frac{1}{2}}}}{(2\rho_1^2+\frac{3}{2}r_{23}^2)^{\frac{1}{2}}}\left\{(\alpha_2-\alpha_1)\beta_3 + (\alpha_3-\alpha_2)\beta_1 + (\alpha_1-\alpha_3)\beta_2\right\}. \tag{5.16}$$

We have $Q^{(1)}(\beta_2'-\beta_3') = (\beta_3-\beta_2)$

and $\quad Q^{(2)}(\beta_2'-\beta_3') = -\beta_1$.

Inserting these and taking the products of the isotopic spin wave functions, (5.16) becomes

$$H'_{fo} = \frac{\sqrt{3}}{2}\sqrt{N_d N_t}\int d\vec{\rho}_1\, d\vec{r}_{23}\, \frac{e^{-\alpha_d r_{23}}}{r_{23}}\, e^{i\vec{\rho}_1\cdot(-\vec{p}-\frac{1}{3}\vec{k})}$$
$$\times\, {}^{2s+1}\sigma_{m_s}(\overline{23},1)\,\frac{e^{-\alpha_t(2\rho_1^2+\frac{3}{2}r_{23}^2)^{\frac{1}{2}}}}{(2\rho_1^2+\frac{3}{2}r_{23}^2)^{\frac{1}{2}}}$$
$$\times\left\{\Theta^{(1)}e^{i\vec{k}\cdot\vec{\rho}_1}(\alpha_2+\alpha_3-2\alpha_1) + 2\,\Theta^{(2)}e^{-\frac{1}{2}i\vec{k}\cdot\vec{r}_{23}}(\alpha_2-\alpha_3)\right\}. \tag{5.17}$$

We shall defer the squaring of this matrix element and the summations over the spins of the nucleons until §§ 6 and 7, when we shall deal with the specific cases of $S(S)$ and $PS(PV)$ theories. The former of these theories contains no spin operator in $\Theta^{(i)}$, whereas the latter theory does contain the operators $\vec{\sigma}^{(i)}$.

The differential cross-section for the production of a proton is

$$d\sigma = \frac{2\pi}{v_0} \sum |H'_{fo}|^2 \rho(E) \tag{5.18}$$

where v_0 is the velocity of collision in the initial state ($\pi^+ + t$), $\rho(E)$ is the density of final states and $\sum$ denotes an average over the initial nuclear spins and a summation over the final nuclear spins.

To derive an expression for $\rho(E)$ we treat the nucleons non-relativistically.

The final kinetic energy is $E = \frac{3}{4}\frac{p^2}{M}$ and the kinetic energy of the proton emitted is $E_p = \frac{p^2}{2M} = \frac{2}{3}E$. Therefore

$$\rho(E) = \frac{pM}{12\pi^3}\, d\Omega$$

where the proton is emitted into the solid angle $d\Omega$. Angles are measured from the direction of the incident meson.

$$v_0 \doteq \frac{k}{\omega}$$

therefore $$\frac{d\sigma}{d\Omega} = \frac{pM\omega}{6\pi^2 k} \sum |H'_{fo}|^2 \tag{5.19}$$

is the differential cross-section for the production of a proton with momentum $p\ (=\frac{2}{\sqrt{3}}\sqrt{ME})$ into the solid angle $d\Omega$ in the centre of mass system.

§6. Scalar Mesons.

For $S(S)$ theory $\theta^{(1)} = \theta^{(2)} = g\sqrt{\frac{2\pi}{\omega}}$, so that (5.17) becomes

$$H'_{fo} = \frac{\sqrt{3}}{2}\sqrt{N_d N_t}\ g\sqrt{\frac{2\pi}{\omega}} \int d\vec{\rho}_1\, d\vec{r}_{23}\, \frac{e^{-\alpha_d r_{23}}}{r_{23}}\, e^{i\vec{\rho}_1\cdot(-\vec{p}-\frac{1}{3}\vec{k})}$$
$$\times \frac{e^{-\alpha_t(2\rho_1^2+\frac{3}{2}r_{23}^2)^{\frac{1}{2}}}}{(2\rho_1^2+\frac{3}{2}r_{23}^2)^{\frac{1}{2}}}\cdot {}^{2S+1}\sigma_{m_S}(\overline{23},1)$$
$$\times\left\{ e^{i\vec{k}\cdot\vec{\rho}_1}(\alpha_2+\alpha_3-2\alpha_1) + 2e^{-\frac{1}{2}i\vec{k}\cdot\vec{r}_{23}}(\alpha_2-\alpha_3)\right\}. \tag{6.1}$$

${}^{2S+1}\sigma_{m_S}(\overline{23},1)$ is symmetric in 2 and 3 and $(\alpha_2-\alpha_3)$ is anti-symmetric in 2 and 3 , so that the $(\alpha_2-\alpha_3)$ term gives no contribution after squaring and summing over the nuclear spins. No spin changes are allowed in $S(S)$ theory so that the ${}^2S_{\frac{1}{2}}$ state of the triton goes to the ${}^2\sigma_{\frac{1}{2}}(\overline{23},1)$ state of the $(p+d)$ system and the ${}^2S_{-\frac{1}{2}}$ state goes to the ${}^2\sigma_{-\frac{1}{2}}(\overline{23},1)$ state.

Thus

$$\sum|H'_{fo}|^2 = 9g^2\frac{\pi}{\omega}N_d N_t\left|\int d\vec{\rho}_1\, d\vec{r}_{23}\,\frac{e^{-\alpha_d r_{23}}}{r_{23}}\, e^{i\vec{\rho}_1\cdot(-\vec{p}+\frac{2}{3}\vec{k})}\,\frac{e^{-\alpha_t(2\rho_1^2+\frac{3}{2}r_{23}^2)^{\frac{1}{2}}}}{(2\rho_1^2+\frac{3}{2}r_{23}^2)^{\frac{1}{2}}}\right|^2$$

$$= 9g^2\frac{\pi}{\omega}N_d N_t\, I_1^2 \tag{6.2}$$

where the integral I_1 is evaluated in the appendix.

Therefore, from (5.19),

$$\frac{d\sigma}{d\Omega} = \frac{\sqrt{3}}{\pi} g^2 M N_d N_t \frac{\sqrt{ME}}{k} I_1^2 \tag{6.3}$$

$$= \frac{12}{\pi^5} g^2 M \alpha_t^4 \alpha_d \frac{\sqrt{ME}}{k} I_1^2 .$$

$\alpha_d^2 = M W_d$, where W_d is the deuteron binding energy, giving $\alpha_d = 2.32 \times 10^{12}\ cm^{-1}$. α_t is taken to be $4 \times 10^{12}\ cm^{-1}$, a figure obtained by choosing α_t to fit the Coulomb energy of He^3 (see reference (32)). I_1^2 was evaluated numerically for meson energies in the centre of mass system of 10, 30, 50, 78.3 and 100 (Mev) (see § 4) and for angles in the centre of mass system of 0°, 30°, 45°, 90°,

Meson energy. (c.m. system.) (Mev.)	$\left(\frac{d\sigma}{d\Omega}\right)_{0°} \Big/ \left(\frac{d\sigma}{d\Omega}\right)_{90°}$	$\left(\frac{d\sigma}{d\Omega}\right)_{90°} \Big/ \left(\frac{d\sigma}{d\Omega}\right)_{180°}$
10	1.9	1.8
30	3.2	2.4
50	4.3	3.3
78.3	6.5	3.5
100	6.6	5.1

Table 5. Ratios of the differential cross-sections in the centre of mass system.

135° and 180°. Table 5 shows the values of the ratios $\left(\frac{d\sigma}{d\Omega}\right)_{0°} \Big/ \left(\frac{d\sigma}{d\Omega}\right)_{90°}$ and $\left(\frac{d\sigma}{d\Omega}\right)_{90°} \Big/ \left(\frac{d\sigma}{d\Omega}\right)_{180°}$. for the various meson energies.

Fig.7 shows the angular distribution of the protons in the centre of mass system for 78.3 Mev mesons; this is the same angular distribution as that of the mesons

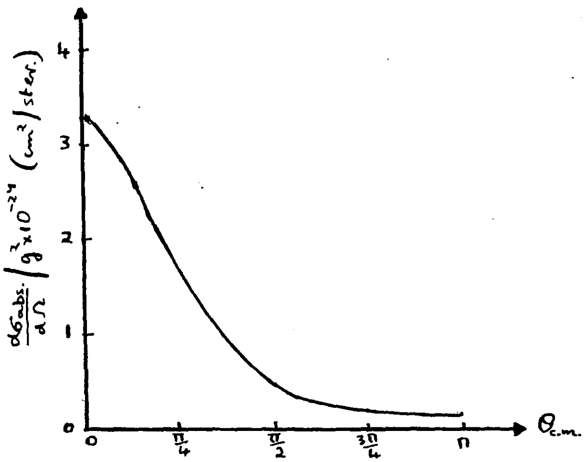

Fig.7. Angular distribution of protons (or mesons) for 78.3 Mev mesons in the centre of mass system.

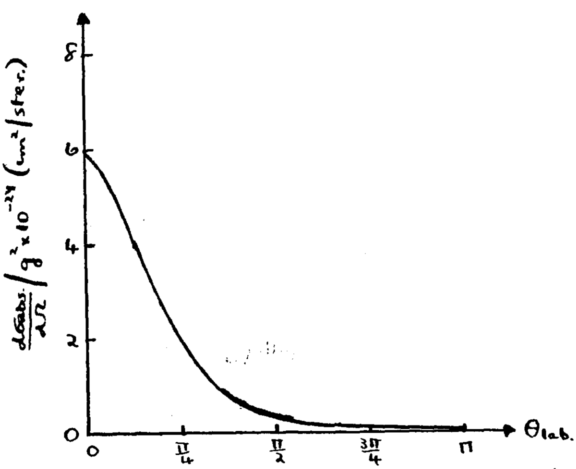

Fig.8. Angular distribution of π^+ mesons from reaction $p+d \rightarrow \pi^+ + t$ in laboratory system for incident protons of energy 340 Mev.

produced in the centre of mass system by protons whose laboratory system energy is 340 Mev. The corresponding angular distribution of the π^+ mesons in the laboratory system was calculated from the data in § 4 and is shown in fig.8.

Meson energy. (c.m. system.) (Mev.)	$\sigma_{abs.}/g^2 \times 10^{-26}$ (cm^2)	$\sigma_{prod.}/g^2 \times 10^{-28}$ (cm^2)
10	8·2	4·2
30	3·3	4·8
50	2·0	4·5
78·3	1·1	3·6
100	0·8	3·1

Table 6. Total cross-sections for reactions $p+d \rightleftarrows \pi^+ + t$.

The total cross-sections for the reaction $\pi^+ + t \to p + d$ are obtained from the above by straightforward numerical integration and are shown in table 6. The total cross-

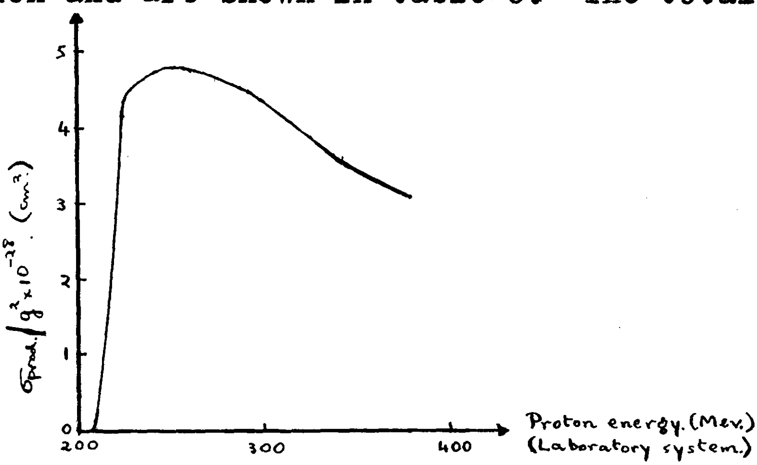

Fig.9. Excitation curve for reaction $p+d \to \pi^+ + t$.

sections for the reaction $p+d \to \pi^+ + t$ are obtained from these by the detailed balancing method explained in § 4 and these are also shown in table 6 and plotted in fig.9. This excitation curve rises steeply and has a maximum of about 5×10^{-28} cm^2 at a laboratory proton energy near 240 Mev. The value of the total cross-section at 340 Mev is 3.6×10^{-28} cm^2 which is to be compared with the experimental value (ref. (27)) of about 5×10^{-30} cm^2. The angular distribution shown in fig.7 has a value for $\left(\frac{d\sigma}{d\Omega}\right)_{0^\circ} \Big/ \left(\frac{d\sigma}{d\Omega}\right)_{90^\circ}$ compatible with the experimental value (ref. (27)). The value of $\left(\frac{d\sigma}{d\Omega}\right)_{90^\circ} \Big/ \left(\frac{d\sigma}{d\Omega}\right)_{180^\circ}$, however, is larger than the experimental value although the angular distribution does tend to show a flattening in the backward direction as found by experiment. (The calculated angular distribution is flat between 120° and 180°).

The total cross-section is too large by a factor of 100. This is probably caused by the fact that both the deuteron and triton wave functions become infinite when the nucleons in these nuclei come close together. This means that we have used too much high momentum component in describing these nuclei. To assess to which nucleus the large cross-section is due, the calculation was repeated for 78.3 Mev mesons, using a Hulthén wave function for the deuteron. This new wave function is $\sqrt{N_d'}\ \frac{e^{-\alpha_d r_{23}} - e^{-\gamma\alpha_d r_{23}}}{r_{23}}$ (where $N_d' = \frac{\gamma\alpha_d}{4\pi}$) and remains finite when $r_{23} = 0$. The angular distribution of the protons in the centre of mass system was practically unaltered in shape and the total cross-section was reduced only by a factor of less than 2. Even the Hulthén wave function, however, might still contain too much high momentum component. Bludman (29) (see §3) found better agreement with experiment by using a 'hard core' for the deuteron i.e. by assuming the wave function was zero for r_{23} less than some critical value $(0{\cdot}38\,\frac{\hbar}{\mu c})$. Bludman, however, found that the total cross-section calculated using the Hulthén wave function with the core was one-half that found using the Hulthén wave function without the core. It is therefore unlikely that even a 'hard core' deuteron wave function would reduce the cross-section found above by the necessary factor of 100. The importance of the hard core in Bludman's calculation was that it flattened the angular distribution in the backward direction. Bludman, in his calculation, used a triton wave function in which the two "participating nucleons" (he used the impulse approximation) were close together and he found a total cross-section equal to the experimental one. It therefore appeared interesting to repeat the above calculation using a different triton wave function. A Gaussian form was chosen for the wave function, namely

$$\sqrt{N_t'}\; e^{-\alpha_t^2\left(r_{31}^2 + r_{21}^2 + r_{23}^2\right)}$$

with $N_t' = \frac{24\sqrt{3}\,\alpha_t'^6}{\pi^3}$ and $\alpha_t' = 2.44 \times 10^{12}\,\mathrm{cm}^{-1}$ (cf. ref. (30)).

This gave

$$\frac{d\sigma}{d\Omega} = \frac{36}{\pi^5} g^2 \alpha_d \alpha_t'^6 M \frac{\sqrt{ME}}{k} \, |I_{14}|^2$$

with

$$I_{14} = \int d\vec{\rho}_1 \, d\vec{r}_{23} \, \frac{e^{-\alpha_d r_{23}}}{r_{23}} \, e^{i\vec{\rho}_1\cdot(-\vec{p}+\frac{2}{3}\vec{k})} \, e^{-\alpha_t'^2(2\rho_1^2+\frac{3}{2}r_{23}^2)}$$

I_{14} is evaluated in the appendix.

The angular distribution derived from this is

$$\frac{d\sigma}{d\Omega} \propto \exp\left\{\frac{2}{3\sqrt{3}\,\alpha_t'^2} \sqrt{ME}\, k \cos\theta\right\}$$

$$\doteq \exp\{10 \cos\theta\},$$

for 78.3 Mev meson in the centre of mass system.

Thus, practically all the mesons produced in the reaction $p+d \to \pi^+ + t$ are produced in a very narrow cone in the forward direction; this is at complete variance with the experimental result. This result will be discussed further in § 8. With this wave function the integration of the differential cross-section can be carried out analytically and gives a total cross-section of $7g^2 10^{-30}\,\mathrm{cm}^2$ for the production of π^+ mesons by 340 Mev protons in the reaction $p+d \to \pi^+ + t$.

§7. Pseudoscalar Mesons.

For PS(PV) theory,

$$\theta^{(1)} = \frac{if}{\mu}\sqrt{\frac{2\pi}{\omega}}\left\{\vec{\sigma}^{(1)}\cdot\vec{k} - \frac{\omega}{M}\vec{P} + \frac{\omega}{2m}\vec{k}\right\}$$

and
$$\theta^{(2)} = \frac{if}{\mu}\sqrt{\frac{2\pi}{\omega}}\left\{\vec{\sigma}^{(2)}\cdot\vec{k} + \frac{\omega}{2M}\vec{P} + \frac{\omega}{2m}\vec{k} + i\frac{\omega}{M}\vec{\nabla}_{\vec{r}_{23}}\right\}$$

where $\vec{\nabla}_{\vec{r}_{23}}$ acts only on $\frac{e^{-d_d r_{23}}}{r_{23}}$.

Equation (5.17) becomes

$$H'_{fo} = \frac{\sqrt{3}}{2}\sqrt{N_d N_t}\,\frac{if}{\mu}\sqrt{\frac{2\pi}{\omega}}\Big[\langle {}^{2s+1}\sigma_{m_s}(\overline{23},1)|\vec{\sigma}^{(1)}|d_2+d_3-2d_1\rangle\cdot(\vec{k}' - \frac{\omega}{M}\vec{P} + \frac{\omega}{2m}\vec{k})\,I_1$$

$$+ 2\langle {}^{2s+1}\sigma_{m_s}(\overline{23},1)|\vec{\sigma}^{(2)}|d_2-d_3\rangle\cdot(\vec{k} + \frac{\omega}{2M}\vec{P} + \frac{\omega}{2m}\vec{k})\,I_2$$

$$+ \frac{2\omega}{M}\langle {}^{2s+1}\sigma_{m_s}(\overline{23},1)|\vec{\sigma}^{(2)}|d_2-d_3\rangle\cdot\vec{I}_3'\Big] \qquad (7.1)$$

where I_1 is as in the S(S) case and

$$I_2 = \int d\vec{\rho}_1\, d\vec{r}_{23}\,\frac{e^{-d_d r_{23}}}{r_{23}}\, e^{i\vec{\rho}_1\cdot(-\vec{P}-\frac{1}{3}\vec{k})}\, e^{-\frac{1}{2}i\vec{k}\cdot\vec{r}_{23}}\,\frac{e^{-d_t(2\rho_1^2+\frac{3}{2}r_{23}^2)^{\frac{1}{2}}}}{(2\rho_1^2+\frac{3}{2}r_{23}^2)^{\frac{1}{2}}}. \qquad (7.2)$$

$$\vec{I}_3' = i\int d\vec{\rho}_1\, d\vec{r}_{23}\left(\vec{\nabla}_{\vec{r}_{23}}\frac{e^{-d_d r_{23}}}{r_{23}}\right) e^{i\vec{\rho}_1\cdot(-\vec{P}-\frac{1}{3}\vec{k})}\, e^{-\frac{1}{2}i\vec{k}\cdot\vec{r}_{23}}\,\frac{e^{-d_t(2\rho_1^2+\frac{3}{2}r_{23}^2)^{\frac{1}{2}}}}{(2\rho_1^2+\frac{3}{2}r_{23}^2)^{\frac{1}{2}}}. \qquad (7.3)$$

These integrals are evaluated in the appendix. $\vec{I}_3'$ is of the form $\frac{\vec{k}}{2} I_3$.

If we let $(\vec{k}' - \frac{\omega}{M}\vec{P} + \frac{\omega}{2m}\vec{k})\,I_1 = \vec{k}_1$

and $2(\vec{k} + \frac{\omega}{2M}\vec{P} + \frac{\omega}{2m}\vec{k})I_2 + 2\cdot\frac{\omega}{2M}\vec{k}\,I_3 = \vec{k}_2$

then
$$H'_{fo} = \frac{\sqrt{3}}{2}\sqrt{N_d N_t}\,\frac{if}{\mu}\sqrt{\frac{2\pi}{\omega}}\Big[\langle {}^{2s+1}\sigma_{m_s}(\overline{23},1)|\vec{\sigma}^{(1)}|d_2+d_3-2d_1\rangle\cdot\vec{k}_1$$

$$+ \langle {}^{2s+1}\sigma_{m_s}(\overline{23},1)|\vec{\sigma}^{(2)}|d_2-d_3\rangle\cdot\vec{k}_2\Big] \qquad (7.4)$$

For the summation over the spins, we use

$$\sigma_x \upsilon_+ = \upsilon_- \qquad \sigma_x \upsilon_- = \upsilon_+$$

$$\sigma_y \upsilon_+ = i\upsilon_- \qquad \sigma_y \upsilon_- = -i\upsilon_+$$

$$\sigma_z \upsilon_+ = \upsilon_+ \qquad \sigma_z \upsilon_- = -\upsilon_-$$

where υ_+, υ_- are the spin wave functions of a nucleon with z-component of spin equal to $+\frac{1}{2}$ and $-\frac{1}{2}$ respectively, and $\vec{\sigma}$ is the spin operator acting on that nucleon.

Summing over the final spins and averaging over the initial spins of the nucleons, we find

$$\Sigma |H'_{f0}|^2 = 3N_\alpha N_t \frac{f^2}{\mu^2}\frac{\pi}{\omega}\left[3k_1^2 + k_2^2 - 2(\vec{k}_1\cdot\vec{k}_2)\right] . \tag{7.5}$$

The differential cross-section is obtained from (5.19) and is

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \frac{4}{\pi^5} f^2 \frac{d_\alpha d_t^4}{\mu^2} M \frac{\sqrt{ME}}{k}\left[3k_1^2 + k_2^2 - 2(\vec{k}_1\cdot\vec{k}_2)\right] \\ &= \frac{4}{\pi^5} f^2 \frac{d_\alpha d_t^4}{\mu^2} M \frac{\sqrt{ME}}{k}\Big[2\left\{\left(I_1+\tfrac{\omega}{2M}I_1\right)\vec{k} - \tfrac{\omega}{M}I_1\vec{p}\right\}^2 \\ &\quad + \left\{\left(I_1 + \tfrac{\omega}{2M}I_1 - 2I_2 - \tfrac{\omega}{M}I_2 - \tfrac{\omega}{M}I_3\right)\vec{k}\right. \\ &\quad \left.-\left(\tfrac{\omega}{M}I_1 + \tfrac{\omega}{M}I_2\right)\vec{p}\right\}^2\Big] .\end{aligned} \tag{7.6}$$

The triton wave function was found to be unsatisfactory in the $S(S)$ case, so it was not considered profitable to evaluate the above differnial cross-section for more than one energy. The meson energy chosen was 78.3 Mev as this is the energy at which the experimental results are available. The shape of the angular distribution obtained is shown in fig.10, which contains also the angular distribution obtained by suppressing the 'odd'

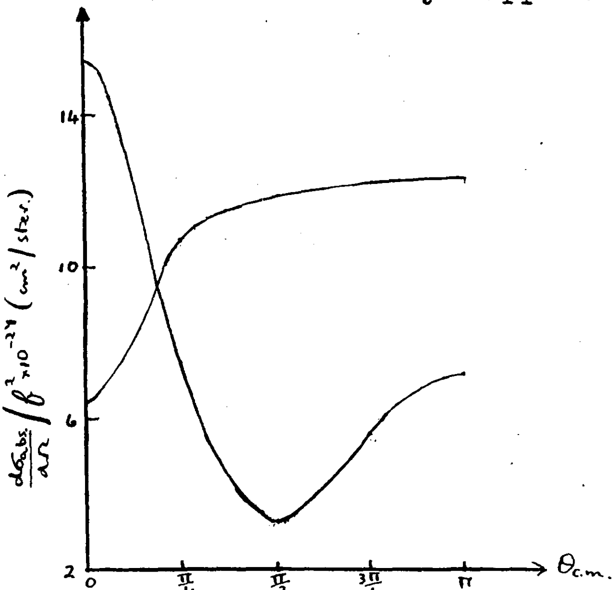

Fig.10. Angular distributions in the centre of mass system calculated on PS(PV) theory for incident 340 Mev protons: (a) includes the whole interaction; (b) includes only the even operator of the interaction.

part of the interaction ($\rho_1 \omega$), which gives rise to all the terms involving $\frac{\omega}{M}$. Neglecting this 'odd' part of the interaction, we have

$$\frac{d\sigma}{d\Omega} = \frac{4}{\pi^5} f^2 \frac{d_a d_r^4}{\mu^2} M \sqrt{ME}\, k \left[3I_1^2 - 4I_1 I_2 + 4I_2^2\right]. \tag{7.7}$$

Integration of (7.6) and (7.7) gave the total cross-section at this energy to be $\sigma_{abs.} = 3.6 f^2 \times 10^{-26}$ cm^2 for the complete interaction and $1.6 f^2 \times 10^{-26}$ cm^2 when the 'odd' part of the interaction was neglected. Detailed balancing (§ 4) gave $\sigma_{prod.} = 12 f^2 \times 10^{-28}$ cm^2 for the former and $5 f^2 \times 10^{-28}$ cm^2 for the latter interaction. Thus, at this energy, the total cross-section calculated in PS(PV) theory is larger than that calculated in S(S) theory. The complete interaction gives an angular distribution greatly different from the experimental one (ref. (27)). The even part of the interaction gives the values 5 and 0.4 for the ratios $\left(\frac{d\sigma}{d\Omega}\right)_{0^\circ} \Big/ \left(\frac{d\sigma}{d\Omega}\right)_{90^\circ}$ and $\left(\frac{d\sigma}{d\Omega}\right)_{90^\circ} \Big/ \left(\frac{d\sigma}{d\Omega}\right)_{180^\circ}$ respectively but the angular distribution shows no tendency to flatten off in the backward direction. The even part of the interaction does however give a better fit to the experimental angular distribution than does the complete interaction, a result which was also found for the reaction $p + p \rightarrow \pi^+ + d$. (see ref. (30)).

§8. Discussion of Results.

(a) Angular Distribution of Mesons from the reaction $p+d\to\pi^{+}+t$.

(i) Irving triton wave function.

The angular distribution of the mesons in the centre of mass system, calculated using the Irving wave function to describe the triton, is shown in fig.7 (for $S(S)$ meson theory) and in fig.10 (for $PS(PV)$ meson theory). Both these curves are for 340 Mev protons in the laboratory system. The experimental angular distribution at this energy is described in § 3. It is seen that $S(S)$ theory gives a better fit to the experimentally-determined angular distribution than does $PS(PV)$ theory. With the 'odd interaction' suppressed, $PS(PV)$ theory gives the peaking in the forward direction, as found by experiment, but does not give the required flattening off in the backward direction. These findings are similar to those for the reaction $p+p\to\pi^{+}+d$, where $S(S)$ theory and $PS(PV)$ with the odd interaction suppressed give better agreement with experiment than does $PS(PV)$ theory.

These results are contrary to those of other meson processes, (photo-production of mesons and π^{-} meson capture processes), which favour pseudoscalar theory.

The calculated angular distributions for the reactions $\pi^{+}+d\rightleftarrows p+p$ can be explained by a partial wave analysis (13) (see § 1). Such an analysis for the reactions $\pi^+ + t \rightleftarrows p + d$ would, however, be unprofitable. This may be seen as follows.

The angular distribution calculated for scalar mesons is anistropic. In scalar theory, the reaction is allowed when both the proton and the meson are in s -states, and if this were dominant, the distribution would be roughly isotropic. In pseudoscalar theory, these values of the angular momenta are forbidden, yet the calculated angular distribution is closer to isotropy. It therefore appears that to explain the calculated angular distributions, higher values of the angular momenta would have to be considered. For a proton of 400 Mev energy, values of ℓ as high as 6 are present. An analysis involving these values of ℓ would not yield specific information.

(ii) Gaussian triton wave function.

When the triton was described by a Gaussian wave function, (§ 6), $S(S)$ theory gave an angular distribution in which the mesons were practically all produced in a very narrow cone in the forward direction. The discussion of this shall be deferred until we have discussed the total cross-section calculated with the Irving triton wave function.

(b) Total Cross-sections of the Reactions $\pi^{+}+t \rightleftarrows p+d$.

(i) Irving triton wave function.

As in the reactions $\pi^{+}+d \rightleftarrows p+p$, it is found that $PS(PV)$ theory gives a larger total cross-section for the reactions $\pi^{+}+t \rightleftarrows p+d$ than does $S(S)$ theory. However, both theories, with the Irving triton wave function, give a total cross-section larger than the experimental one (§ 3) by a factor of about 100. (In this calculation it is assumed that the coupling constants have values of about unity.)

This study of the reactions $\pi^{+}+t \rightleftarrows p+d$ is, however, far from being exact. We shall discuss now the possible sources of error which could lead to the high values obtained for the total cross-section.

Since the coupling constants are not small, the use of a weak-coupling treatment is not rigorous. This procedure, however, as we have seen in § 1, when applied in lowest order to the reactions $\pi^{+}+d \rightleftarrows p+p$, does give total cross-sections not widely different from the experimentally observed cross-sections, if the coupling constants are assumed to be about unity. We would therefore expect similar results for the reactions $\pi^{+}+t \rightleftarrows p+d$ and we shall assume that the errors in these total cross-sections are due to other causes.

The basis of these calculations is that the matrix element for the transition from a state described by the wave function ψ_i to a state described by ψ_f is $M = \langle \psi_f | O | \psi_i \rangle$, where O is the interaction operator causing the transition. We shall first discuss the possible sources of error in this operator O, and then those in ψ_f and ψ_i.

We have used the non-relativistic form of O; however, the velocities of the nucleons are not large enough for this to make more than a 10% difference to the total cross-section. It appears, therefore, that the large errors in the total cross-sections are not due to this operator O.

One important function of the operator O is to determine the form of the angular distribution of the emitted particles and we have discussed this above.

Let us now consider ψ_f, which we take to be the wave function describing the proton-deuteron system. The form of ψ_f used assumed that there was no interaction between these particles. Cheston (13) stated that in the reactions $\pi^+ + d \rightleftarrows p + p$, taking account of the proton-proton interaction altered the total cross-section only slightly, since the protons had high relative energy. The same reasoning applies to the reactions $\pi^+ + t \rightleftarrows p + d$. The proton-deuteron interaction would be taken into consideration by inserting a phase factor in the plane wave used to describe the proton. It is unlikely that such a correction would reduce the total cross-sections by the necessary factor of 100. The only remaining source of error in the calculation lies in the wave function, ψ_i , used to describe the triton. We shall now discuss this.

In the production of mesons in proton-deuteron collisions, large momentum transfers are involved, so that in the reaction $p+d \rightarrow \pi^+ + t$ it is necessary to describe the high momentum components of the triton wave function as accurately as possible. The Irving wave function inadequately describes the high momentum components, as it tends to infinity when ρ_1 and r_{23} tend to zero. This wave function was studied by Irving with the intention of fitting the binding energy of the triton accurately. It is therefore correct only in its asymptotic behaviour, i.e. only in its description of the low momentum components. In momentum space, the Irving wave function is too large in the high momentum region and this could account for the necessary factor of 100.

(ii) Gaussian triton wave function.

When the triton is described by a Gaussian wave function, $S(S)$ theory gives a total cross-section approximately in agreement with the experimental value.

This total cross-section, however, is very sensitive to the triton radius used; the total cross-section decreases with increasing radius of the triton. The Gaussian wave function is actually too small in the high momentum region. This can be seen from the angular distribution it gives, and we now give a discussion of this. (See § 8(a)(ii))

The greater the angle at which the meson is produced, the greater is the momentum transferred. For large-angle scattering, therefore, the high momentum components of the triton wave function are required. At large angles the differential cross-section is found to be too small, so that in momentum space, the triton wave function is too small in the high momentum region. In contradiction to the failure of the Gaussian wave function in this calculation it was noted in § 2 that a Gaussian distribution of momenta of the nucleons in the carbon nucleus gave good agreement with experiment for mesons produced in proton-carbon collisions. (The Fourier transform of the Gaussian wave function is also Gaussian.)

(c) Conclusion.

It would be profitable to repeat the above calculations using a triton wave function which describes the high momentum components accurately. Such a wave function could be found from an analysis of experiments on the scattering of nucleons by tritons.

On the other hand, the above calculations could be used as a means of determining such a wave function. The problem considered in this thesis is more useful for determining the properties of the tritium nucleus than for studying the mechanism of meson production.

APPENDIX.

In this appendix we shall evaluate the integrals occurring in the text and shall denote references to Watson's "Bessel Functions" by W and references to Hobson's "Spherical and Ellipsoidal Harmonics" by H .

$J_n(z)$ is used for the Bessel function of order n and $K_n(z)$ for the Bessel function of purely imaginary argument of order n . The other functions are in standard notation.

(a) Evaluation of I_1 .

Let $\vec{K}_1 = -\vec{p} + \frac{2}{3}\vec{k}$

$$I_1 = \int d\vec{\rho}_1\, d\vec{r}_{23}\, \frac{e^{-d_d r_{23}}}{r_{23}}\, e^{i\vec{\rho}_1\cdot\vec{K}_1}\, \frac{e^{-d_t(2\rho_1^2+\frac{3}{2}r_{23}^2)^{\frac{1}{2}}}}{(2\rho_1^2+\frac{3}{2}r_{23}^2)^{\frac{1}{2}}}$$

$$= \frac{16\pi^2}{K_1}\int_0^\infty d\rho_1 \int_0^\infty dr_{23}\, \rho_1 r_{23}\, e^{-d_d r_{23}} \sin(\rho_1 K_1)\, \frac{e^{-d_t(2\rho_1^2+\frac{3}{2}r_{23}^2)^{\frac{1}{2}}}}{(2\rho_1^2+\frac{3}{2}r_{23}^2)^{\frac{1}{2}}} .$$

Let $r_{23} = \frac{\sqrt{2}}{\sqrt{3}} r$, $\rho_1 = \frac{1}{\sqrt{2}}\rho$, ~~and~~ $\frac{\sqrt{2}}{\sqrt{3}} d_d = d_d'$ and $\frac{1}{\sqrt{2}} K_1 = K_1'$.

Then

$$I_1 = \frac{16\pi^2}{3K_1}\int_0^\infty dr\, r\, e^{-d_d' r} \int_0^\infty d\rho\, \rho \sin(K_1'\rho)\, \frac{e^{-d_t(\rho^2+r^2)^{\frac{1}{2}}}}{(\rho^2+r^2)^{\frac{1}{2}}} .$$

Consider

$$I_1' = \int_0^\infty d\rho\, \rho \sin(K_1'\rho)\, \frac{e^{-d_t(\rho^2+r^2)^{\frac{1}{2}}}}{(\rho^2+r^2)^{\frac{1}{2}}} .$$

Now $J_{\frac{1}{2}}(K_1'\rho) = \sqrt{\dfrac{2}{\pi K_1'\rho}}\, \sin(K_1'\rho)$ (W.p.54) and $$K_{\frac{1}{2}}\left(\alpha_t\sqrt{\rho^2+r^2}\right) = \sqrt{\frac{\pi}{2\alpha_t(\rho^2+r^2)^{\frac{1}{2}}}}\; e^{-\alpha_t(\rho^2+r^2)^{\frac{1}{2}}} \qquad \text{(W.p.80)}$$

Therefore

$$I_1' = \sqrt{\alpha_t k_1'}\int_0^\infty d\rho\; \rho^{3/2}\, J_{\frac{1}{2}}(k_1'\rho)\; K_{\frac{1}{2}}\left(\alpha_t\sqrt{\rho^2+r^2}\right)\frac{1}{(\rho^2+r^2)^{\frac{1}{4}}}$$

$$= \frac{k_1' r}{\sqrt{\alpha_t^2+k_1'^2}}\; K_{-1}\left\{r\sqrt{\alpha_t^2+k_1'^2}\right\} \qquad \text{(W.p.416)}$$

$$= \frac{k_1' r}{\sqrt{\alpha_t^2+k_1'^2}}\; K_1\left\{r\sqrt{\alpha_t^2+k_1'^2}\right\}. \qquad \text{(W.p79)}$$

Therefore

$$I_1 = \frac{16\pi^2}{3\sqrt{2}}\;\frac{1}{(\alpha_t^2+k_1'^2)^{\frac{1}{2}}}\int_0^\infty dr\; r^2 e^{-\alpha_d' r}\; K_1\left\{r\sqrt{\alpha_t^2-k_1'^2}\right\}.$$

Let $r = \frac{x}{\sqrt{\alpha_t^2+k_1'^2}}$, and $\alpha_d'' = \frac{\alpha_d'}{\sqrt{\alpha_t^2+k_1'^2}}$.

Then

$$I_1 = \frac{16\pi^2}{3\sqrt{2}}\;\frac{1}{(\alpha_t^2+k_1'^2)^2}\int_0^\infty dx\; x^2 e^{-\alpha_d'' x}\, K_1(x).$$

Consider $I'' = \int_0^\infty dx\; x^2 e^{-\alpha_d'' x}\, K_1(x)$

and let

$$\alpha_d'' = \cosh\gamma.$$

Then $$I'' = \sqrt{\frac{\pi}{2}}\;\Gamma(2)\,\Gamma(4)\;\frac{P_{\frac{1}{2}}^{-\frac{5}{2}}(\cosh\gamma)}{\sinh^{5/2}\gamma} \qquad \text{(W.p.388)}$$

Now $$Q_2^{-1}(\coth\gamma) = \sqrt{\frac{\pi}{2}}\; e^{-i\pi}\;\Gamma(2)\,(\sinh\gamma)^{\frac{1}{2}}\; P_{\frac{1}{2}}^{-5/2}(\cosh\gamma) \qquad \text{(H.p.427)}$$

and $$\frac{Q_2^{-1}(\coth\gamma)}{\Gamma(2)} = \frac{Q_2^{1}(\coth\gamma)}{\Gamma(4)} \qquad \text{(H.p.196)}$$

so that

$$I_1'' = -\frac{1}{\sinh^3\gamma} Q_2^1(\coth\gamma)$$

$$= \frac{1}{\sinh^4\gamma}\left[3\gamma\coth\gamma - \cosh^2\gamma - 2\right].$$

In the text, $\alpha_\alpha'' < 1$, so γ is replaced by $i\gamma$ and

$$I_1 = \frac{48\pi^2}{\sqrt{2}} \frac{1}{(3\alpha_t^2+3K_1'^2-2\alpha_\alpha^2)^2}\left[\frac{3\sqrt{2}\,\alpha_\alpha}{(3\alpha_t^2+3K_1'^2-2\alpha_\alpha^2)^{\frac{1}{2}}}\cos^{-1}\frac{\sqrt{2}\,\alpha_\alpha}{\sqrt{3}\sqrt{\alpha_t^2+K_1'^2}} - \frac{2\alpha_\alpha^2}{3(\alpha_t^2+K_1'^2)} - 2\right].$$

For the purpose of numerical work we write this as

$$I_1 = \frac{4\pi^2}{24\sqrt{2}\,\alpha_\alpha^4}\left(\frac{18}{3\frac{\alpha_t^2}{\alpha_\alpha^2}+\frac{3K_1'^2}{\alpha_\alpha^2}-2}\right)^2\left[\left(\frac{18}{3\frac{\alpha_t^2}{\alpha_\alpha^2}+\frac{3K_1'^2}{\alpha_\alpha^2}-2}\right)^{\frac{1}{2}}\cos^{-1}\left(\frac{2}{3\frac{\alpha_t^2}{\alpha_\alpha^2}+\frac{3K_1'^2}{\alpha_\alpha^2}}\right)^{\frac{1}{2}} - \frac{2}{3\frac{\alpha_t^2}{\alpha_\alpha^2}+\frac{3K_1'^2}{\alpha_\alpha^2}} - 2\right].$$

$$K_1' = \frac{1}{\sqrt{2}}K_1 = \frac{1}{\sqrt{2}}\left|-\vec{p}+\tfrac{2}{3}\vec{k}\right|.$$

Therefore,

$$K_1'^2 = \frac{2}{3}\left(ME + \frac{1}{3}k^2 - \frac{2}{\sqrt{3}}\sqrt{ME}\,k\cos\theta\right)$$

since $p^2 = \frac{4}{3}ME$.

Here, θ is the angle between the vectors $\vec{p}$ and $\vec{k}$.

(b) <u>Evaluation of I_2</u> .

Let $\vec{K}_2 = -\vec{p} - \frac{1}{3}\vec{k}$ and $\vec{K}_3 = -\frac{1}{2}\vec{k}$.

$$I_2 = \int d\vec{\rho}_1\, d\vec{r}_{23}\, \frac{e^{-\alpha_\alpha r_{23}}}{r_{23}}\, e^{i\vec{\rho}_1\cdot\vec{K}_2}\, e^{i\vec{r}_{23}\cdot\vec{K}_3}\, \frac{e^{-\alpha_t(2\rho_1^2+\frac{3}{2}r_{23}^2)^{\frac{1}{2}}}}{(2\rho_1^2+\frac{3}{2}r_{23}^2)^{\frac{1}{2}}}$$

$$= \frac{16\pi^2}{K_2K_3}\int_0^\infty d\rho_1\int_0^\infty dr_{23}\,\rho_1\, e^{-\alpha_\alpha r_{23}}\sin(K_2\rho_1)\sin(K_3 r_{23})\,\frac{e^{-\alpha_t(2\rho_1^2+\frac{3}{2}r_{23}^2)^{\frac{1}{2}}}}{(2\rho_1^2+\frac{3}{2}r_{23}^2)^{\frac{1}{2}}}.$$

Let $r_{23}=\frac{\sqrt{2}}{\sqrt{3}}r$, $\rho_1=\frac{1}{\sqrt{2}}\rho$, $K_2'=\frac{1}{\sqrt{2}}K_2$, $K_3'=\frac{\sqrt{2}}{\sqrt{3}}K_3$ and $\alpha_\alpha'=\frac{\sqrt{2}}{\sqrt{3}}\alpha_\alpha$.

Then

$$I_2=\frac{16\pi^2}{3\sqrt{2}\,K_2'K_3'}\int_0^\infty dr\, e^{-\alpha_\alpha' r}\sin(K_3'r)\int_0^\infty d\rho\,\rho\sin(K_2'\rho)\,\frac{e^{-\alpha_t(\rho^2+r^2)^{\frac{1}{2}}}}{(\rho^2+r^2)^{\frac{1}{2}}}$$

$$=\frac{16\pi^2}{3\sqrt{2}\,K_3'(\alpha_t^2+K_2'^2)^{\frac{1}{2}}}\int_0^\infty dr\, e^{-\alpha_\alpha' r}\sin(K_3'r)\; r\; K_1\left\{r\sqrt{\alpha_t^2+K_2'^2}\right\}.$$

Let $r=\frac{x}{(\alpha_t^2+K_2'^2)^{\frac{1}{2}}}$, $K_3''=\frac{K_3'}{(\alpha_t^2+K_2'^2)^{\frac{1}{2}}}$ and $\alpha_\alpha''=\frac{\alpha_\alpha'}{(\alpha_t^2+K_2'^2)^{\frac{1}{2}}}$.

Then

$$I_2=\mathcal{I}\left[\frac{16\pi^2}{3\sqrt{2}\,K_3''(\alpha_t^2+K_2'^2)^2}\int_0^\infty dx\, e^{-\alpha_\alpha'' x+iK_3''x}\,x\,K_1(x)\right].$$

Let $\alpha_\alpha''-iK_3''=\cosh\gamma$.

Then

$$I_2'=\int_0^\infty dx\, e^{-(\alpha_\alpha''-iK_3'')x}\,x\,K_1(x)$$

$$=\frac{1}{\sinh^3\gamma}\left(\gamma-\cosh\gamma\sinh\gamma\right)$$

, by an analysis similar to that in (a).

Therefore,

$$I_2=\frac{16\pi^2}{3\sqrt{2}\,K_3''(\alpha_t^2+K_2'^2)^2}\,\frac{1}{(A^2+B^2)^{3/2}}\Bigg[-K_3''(\alpha_\alpha''^2+K_3''^2+1)(A^2+B^2)^{\frac{1}{2}}$$

$$+\left(\frac{\sqrt{A^2+B^2}-A}{2}\right)^{\frac{1}{2}}\left(2A+\sqrt{A^2+B^2}\right)\cosh^{-1}\frac{p+q}{2}$$

$$-\left(\frac{\sqrt{A^2+B^2}+A}{2}\right)^{\frac{1}{2}}\left(2A-\sqrt{A^2+B^2}\right)\cos^{-1}\frac{p-q}{2}\Bigg]$$

where $A = \alpha_\alpha''^2 - K_3''^2 - 1$

$$B = 2\alpha_\alpha'' K_3''$$

$$p = \sqrt{(1+\alpha_\alpha'')^2 + K_3''^2}$$

and $q = \sqrt{(1-\alpha_\alpha'')^2 + K_3''^2}$.

(c) <u>Evaluation of $\vec{I}_3'$</u>.

$$\vec{I}_3' = i\int d\vec{\rho}_1\, d\vec{r}_{23} \left(\vec{\nabla}_{\vec{r}_{23}} \frac{e^{-\alpha_d r_{23}}}{r_{23}}\right) e^{i\vec{\rho}_1\cdot\vec{K}_2}\, e^{i\vec{r}_{23}\cdot\vec{K}_3}\, \frac{e^{-\alpha_t(2\rho_1^2+\frac{3}{2}r_{23}^2)^{\frac{1}{2}}}}{(2\rho_1^2+\frac{3}{2}r_{23}^2)^{\frac{1}{2}}}$$

$$= -i\int d\vec{\rho}_1\, d\vec{r}_{23}\, \frac{e^{-\alpha_d r_{23}}}{r_{23}} (\alpha_d r_{23}+1)\, \vec{r}_{23}\, e^{i\vec{\rho}_1\cdot\vec{K}_2}\, e^{i\vec{r}_{23}\cdot\vec{K}_3}\, \frac{e^{-\alpha_t(2\rho_1^2+\frac{3}{2}r_{23}^2)^{\frac{1}{2}}}}{(2\rho_1^2+\frac{3}{2}r_{23}^2)^{\frac{1}{2}}}.$$

We can write

$$\vec{I}_3' = i\int d\vec{r}_{23}\, \vec{r}_{23}\, e^{i\vec{r}_{23}\cdot\vec{K}_3} f(r_{23})$$

$$= i\int_0^\infty dr_{23} \int_{-1}^1 d\xi \int_0^{2\pi} d\varphi\, e^{ir_{23}K_3\xi} f(r_{23})\, r_{23}^2 \left(r_{23}\sqrt{1-\xi^2}\cos\varphi,\, r_{23}\sqrt{1-\xi^2}\sin\varphi,\, r_{23}\xi\right)$$

where we have taken $\vec{K}_3$ as the z-axis of the $\vec{r}_{23}$ space.

Now $\int_0^{2\pi}\cos\varphi\, d\varphi = \int_0^{2\pi}\sin\varphi\, d\varphi = 0$,

so that the x and y components vanish.

$$I_{3_z}' = i\int_0^\infty dr_{23} \int_{-1}^1 d\xi \int_0^{2\pi} d\varphi\, e^{ir_{23}K_3\xi} f(r_{23})\, r_{23}^3\, \xi$$

$$= \frac{4\pi}{K_3}\int_0^\infty dr_{23}\, f(r_{23})\, r_{23}^2 \left[\cos(r_{23}K_3) - \frac{1}{r_{23}K_3}\sin(r_{23}K_3)\right].$$

We have taken $\vec{K}_3$ as the z-axis, so that $\vec{I}_3{}'$ is a vector in the direction of $\vec{K}_3$ with magnitude I'_{3_z}. We can therefore write

$$\vec{I}_3{}' = -\frac{8\pi^2 \vec{K}_3{}'}{\sqrt{3}\,K_2'\,K_3'^2}\int_0^\infty dr\left[\cos(K_3'r) - \frac{1}{K_3'r}\sin(K_3'r)\right]\frac{e^{-\alpha_d' r}}{r}(\alpha_d' r + 1)$$

$$\times \int_0^\infty d\rho\;\rho\,\sin(K_2'\rho)\,\frac{e^{-\alpha_t(\rho^2+r^2)^{\frac{1}{2}}}}{(\rho^2+r^2)^{\frac{1}{2}}}\;,$$

with the same change of variables as in (b).

Thus,

$$\vec{I}_3{}' = -\frac{8\pi^2\vec{K}_3{}'}{\sqrt{3}\,K_3'^2(\alpha_t^2+K_2'^2)^{\frac{1}{2}}}\int_0^\infty dr\left[\cos(K_3'r) - \frac{1}{K_3'r}\sin(K_3'r)\right]e^{-\alpha_d' r}(\alpha_d' r+1)\,K_1\left\{r\sqrt{\alpha_t^2+K_2'^2}\right\}$$

$$= -\frac{8\pi^2\vec{K}_3{}''}{\sqrt{3}\,K_3''^2(\alpha_t^2+K_2'^2)^{3/2}}\Bigg[\alpha_d''\int_0^\infty dx\;x\,e^{-\alpha_d'' x}\,K_1(x)\cos(K_3''x)$$

$$-\frac{\alpha_d''}{K_3''}\int_0^\infty dx\;e^{-\alpha_d'' x}\,K_1(x)\sin(K_3''x)$$

$$+\int_0^\infty dx\;e^{-\alpha_d'' x}\,K_1(x)\cos(K_3''x)$$

$$-\frac{1}{K_3''}\int_0^\infty dx\;\frac{1}{x}\,e^{-\alpha_d'' x}\,K_1(x)\sin(K_3''x)\Bigg],$$

again with the same change of variables as in (b).

The last of these integrals does not satisfy the conditions necessary for the application of the method used in (a) and (b). We use instead the following method.

We have $\frac{1}{x} K_1(x) = -K_0(x) - K_1'(x)$ (w.p.79)

and $K_1(x) = -K_0'(x)$ (w.p.79)

so that $\frac{1}{x} K_1(x) = -K_0(x) + K_0''(x)$

$$\int_0^\infty dx \frac{1}{x} K_1(x) e^{-d_d''x} \sin(K_3''x)$$

$$= -\int_0^\infty dx K_0(x) e^{-d_d''x} \sin(K_3''x) + \int_0^\infty dx K_0''(x) e^{-d_d''x} \sin(K_3''x)$$

$$= \Big[-K_1(x) e^{-d_d''x} \sin(K_3''x)\Big]_0^\infty - d_d'' \int_0^\infty dx K_1(x) e^{-d_d''x} \sin(K_3''x) + K_3'' \int_0^\infty dx K_1(x) e^{-d_d''x} \cos(K_3''x).$$

$$\therefore \vec{I}_3' = -\frac{8\pi^2 \vec{K}_3''}{\sqrt{3} K_3''^2 (d_r^2 + K_1'^2)^{3/2}} \Big[d_d'' \int_0^\infty dx\, x\, e^{-d_d''x} K_1(x) \cos(K_3''x) + \frac{1}{K_3''} \int_0^\infty dx K_0(x) e^{-d_d''x} \sin(K_3''x) - 1 \Big]$$

$$\int_0^\infty dx\, x\, e^{-d_d''x} K_1(x) \cos(K_3''x) = \Re\, I_2'$$ (see (b)),

and $\int_0^\infty dx K_0(x) e^{-d_d''x} \sin(K_3''x)$ is evaluated in a manner similar to that used for the evaluation of I_2'. The final result is that $\vec{I}_3' = \frac{\vec{k}}{\lambda} I_3$

where
$$I_3 = \frac{16\pi^2}{3\sqrt{2} K_3'' (d_r^2 + K_1'^2)^2} \frac{1}{(A^2+B^2)^{3/2}} \times \Big[-\frac{d_d''^2}{K_3''} (d_d''^2 + K_3''^2 - 1)(A^2+B^2)^{\frac{1}{2}} - \frac{1}{K_3''}(A^2+B^2)^{3/2}$$
$$+ \frac{d_d''}{K_3''}\left(\frac{\sqrt{A^2+B^2}+A}{2}\right)^{\frac{1}{2}} (2A - \sqrt{A^2+B^2}) \cosh^{-1}\frac{p+q}{2} + \frac{1}{K_3''^2}\left(\frac{\sqrt{A^2+B^2}-A}{2}\right)^{\frac{1}{2}} (A^2+B^2) \cosh^{-1}\frac{p+q}{2}$$
$$+ \frac{d_d''}{K_3''}\left(\frac{\sqrt{A^2+B^2}-A}{2}\right)^{\frac{1}{2}} (2A + \sqrt{A^2+B^2}) \cos^{-1}\frac{p-q}{2} - \frac{1}{K_3''^2}\left(\frac{\sqrt{A^2+B^2}+A}{2}\right)^{\frac{1}{2}} (A^2+B^2) \cos^{-1}\frac{p-q}{2} \Big]$$

with A, B, p and q as defined in (b).

For I_2 and I_3 we have

$$k_2^2 = \frac{1}{3}\left(4ME + \frac{1}{3}k^2 + \frac{4}{\sqrt{3}}\sqrt{ME}\, k \cos\theta\right).$$

(d) Evaluation of I_{14}.

$$I_{14} = \int d\vec{\rho}_1\, d\vec{r}_{23}\, \frac{e^{-d_d r_{23}}}{r_{23}}\, e^{i\vec{\rho}_1\cdot\vec{K}_1}\, e^{-d_t^{\prime 2}\left(2\rho_1^2+\frac{3}{2}r_{23}^2\right)}$$

$$= \frac{16\pi^2}{3K_1}\int_0^\infty d\rho\, \rho \sin(K'\rho)\, e^{-d_t^{\prime 2}\rho^2}\int_0^\infty dr\, r\, e^{-d_d' r}\, e^{-d_t^{\prime 2} r^2}.$$

$$\int_0^\infty d\rho\, \rho \sin(K_1'\rho)\, e^{-d_t^{\prime 2}\rho^2}$$

$$= \sqrt{\frac{\pi K_1'}{2}}\int_0^\infty d\rho\, \rho^{3/2} J_{\frac{1}{2}}(K_1'\rho)\, e^{-d_t^{\prime 2}\rho^2}$$

$$= \frac{\sqrt{\pi}}{4}\frac{K_1'}{d_t^{\prime 3}}\, e^{-K_1^{\prime 2}/4d_t^{\prime 2}}. \qquad \text{(W.p.394)}$$

$$\int_0^\infty dr\, r\, e^{-d_d' r}\, e^{-d_t^{\prime 2} r^2}$$

$$= \sqrt{\frac{\pi d_d'}{2}}\,\frac{\pi}{2}\, e^{\frac{1}{4}\pi i}\int_0^\infty dr\, r^{3/2}\left\{J_{-\frac{1}{2}}(i d_d' r) + i J_{\frac{1}{2}}(i d_d' r)\right\} e^{-d_t^{\prime 2} r^2}$$

$$= -\frac{\sqrt{\pi}}{2}\frac{1}{d_t^{\prime 2}}\, e^{d_d^{\prime 2}/4d_t^{\prime 2}}\left[\frac{d_d'}{2d_t'} + \frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{1}{(2n-1)}\left(\frac{d_d^{\prime 2}}{4d_t^{\prime 2}}\right)^n\right]. \qquad \text{(W.p.394 \& p.100)}$$

Therefore

$$I_{14} = -2\,\frac{\sqrt{2}}{3}\pi^3\frac{1}{d_t^{\prime 3}}\exp\left\{-\frac{1}{4d_t^{\prime 2}}\left(K_1^{\prime 2}-d_d^{\prime 2}\right)\right\}$$
$$\times\left[\frac{d_d'}{2d_t'} + \frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{1}{(2n-1)}\left(\frac{d_d^{\prime 2}}{4d_t^{\prime 2}}\right)^n\right].$$

Here

$$k_1'^2 = \frac{2}{3}\left(ME + \frac{1}{3}k^2 - \frac{2}{\sqrt{3}}\sqrt{ME}\ k\cos\theta \right),$$

so that the dependence of Γ_{1G} on θ is given by

$$\Gamma_{1G} \propto \exp.\left\{ \frac{1}{3\sqrt{3}} \frac{1}{d_r'^2} \sqrt{ME}\ k \cos\theta \right\}.$$

$\frac{d_a'^2}{4 d_r'^2} = 0.12$, so that we need take only the first two terms of the series in the square brackets.

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