Thesis Submitted for the Degree of D. Sc. in the University of Glasgow

MacRoberts' E-Functions.

— by —

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Preface

Befor writing the contents of this thesis I want to express my gratitute to Professor T. M. MacRobert of Glasgow University for his kindness in advising me during the course of this work. My thanks are due to him for the interest he has taken and the encouragement he has given to me. I cannot miss this apportunity of expressing my thanks to his E-function which brought all the results included in this thesis

The thesis is a sequel to my thesis for the Ph.D. from the same university of Glasgow. Both theseses have the same name "MacRoberts' E-functions". For the definit-ions and properties of these functions, the reader is referred to MacRobert, "Functions of a Complex Variable" third edition. This work will be denoted by the letters C.V.

It is impossible to insert in this preface all the results in the thesis or even to refer to them. All the results are new and this can be certified by Professor MacRobert.

The thesis is divided into four chapters. The first chapter is devoted to integrals involving E-functions. In chapter II linear relations between E-functions are established. In this chapter known theorems such as Kummer's theorem, Saalchütze's theorem, and Euler transformation, ... etc, are deduced as particular cases from E-function expansions. By means of the two formulae [(34) CHAP. I. and (6) CHAP. II. in this thesis] namely

$$\cos(n\pi)\, E\left(\tfrac{1}{2}+n, \tfrac{1}{2}-n :: 2x\right) = \sqrt{(2\pi x)}\, e^{x} K_n(x), \quad \ldots (1)$$

$$E(:: n+1 : x) = x^{\frac{1}{2}n} J_n(2/\sqrt{x}); \quad \ldots (2)$$

many expansions of $K_n(x)$ and $J_n(x)$ are deduced.

In chapter III., formulae expressing the product of two E-functions expressed as the sum of two E-functions are established. By means of (1) formulae giving the value of the products $K_m(xe^{i\pi})K_n(x)$ and $K_m(x)K_n(x)$ are deduced. They are formulae (11) and (15) respectively.

The last chapter is devoted to integrals involving Bessel Functions. They are deduced from the theorems given in CHAP. I. and CHAP. III by means of (1) and (2). In this chapter known integrals such as the Weber and Schafheitlin integral, Nicholsons integral, an integral due to Hanumanta Rao and many integrals given in Watson "Theory of Bessel Functions"; are deduced.

Also the integral

$$\int_0^\infty e^{-\lambda}\lambda^{k-1}K_m(\lambda)K_n(b/\lambda)\,d\lambda,$$

is evaluated in §8 of this chapter; while in §9 a large number of integrals involving the products of three Bessel Functions are evaluated. In §10 which is the last section in this chapter integrals involving the products of four Modified Bessel Functions of the Second Kind are evaluated.

A matter of space and time prevented me from adding more deductions from the theorems of chapters I, II and III, and also prevented me from pointing out that certain work on Whittaker's function and known transformations can easily be obtained from Professor MacRoberts' theorems and my theorems.

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CHAPTER I

INTEGRALS INVOLVING E-FUNCTIONS

§1. An integral involving an E-function. The formula to be proved is

$$\int_0^\infty e^{-\lambda}\lambda^{k-1}E(p;\alpha_r:q;\rho_s:\lambda z)\,d\lambda =$$

$$\frac{\pi}{\sin k\pi}\left\{\begin{array}{l} E(p;\alpha_r:1-k,\rho_1,\ldots,\rho_q:e^{\pm i\pi}z) \\ -\,z^{-k}E(p;\alpha_r+k:1+k,\rho_1+k,\ldots,\rho_q+k:e^{\pm i\pi}z)\end{array}\right\}, \quad \ldots(1)$$

where $p \geqq q+1$, $R(\alpha_r+k)>0$, $r=1,2,3,\ldots,p$ and $|\mathrm{amp}\, z|<\pi$.

If $p \leqq q$, the result holds provided that the integral is convergent.

The following formulae will be required in the proof:

If $R(\alpha_{p+1})>0$

$$\int_0^\infty e^{-\mu}\mu^{\alpha_{p+1}-1}E(p;\alpha_r:q;\rho_s:z/\mu)\,d\mu = E(p+1;\alpha_r:q;\rho_s:z), \quad \ldots(2)$$

$$\frac{1}{2\pi i}\int e^{\zeta}\zeta^{-\rho_{q+1}}E(p;\alpha_r:q;\rho_s:z\zeta)\,d\zeta = E(p;\alpha_r:q+1;\rho_s:z), \quad \ldots(3)$$

where the contour starts at $-\infty$ on the ξ-axis, passes positively round the origin and returns to $-\infty$ on the ξ-axis, the initial value of amp ξ being $-\pi$.

If $R(\beta)>0$

$$\Gamma(\alpha)\int_0^\infty e^{-\lambda}\lambda^{\beta-1}(1+\lambda/z)^{-\alpha}\,d\lambda=\sum_{\alpha,\beta}\Gamma(\beta-\alpha)\Gamma(\alpha)z^{\alpha}F(\alpha;\alpha-\beta+1;z), \quad \ldots (4)$$

<u>Proof of the formula</u>: In (4) replace $\underline{\alpha}$ by $\underline{\alpha_1}$, $\underline{\beta}$ by $(k+\alpha_1)$, $\underline{z}$ by $(1/z)$, and it can be written

$$\int_0^\infty e^{-\lambda}\lambda^{k-1}E(\alpha_1::\lambda z)\,d\lambda=\Gamma(k)\Gamma(\alpha_1)F(\alpha_1;1-k;1/z)$$
$$+\Gamma(-k)\Gamma(\alpha_1+k)z^{-k}F(\alpha_1+k;1+k;1/z)$$

where $R(\alpha_1+k)>0$.

Assuming that $|\text{amp}\, z|<\pi$, this can be written

$$\int_0^\infty e^{-\lambda}\lambda^{k-1}E(\alpha_1::\lambda z)\,d\lambda=\frac{\pi}{\sin(k\pi)}\left\{E(\alpha_1:1-k:e^{\pm i\pi}z)-z^{-k}E(\alpha_1+k:1+k:e^{\pm i\pi}z)\right\}$$

where $R(\alpha_1+k)>0$.

This is a particular case of formula (1). On replacing $\underline{z}$ by z/μ and applying formula (2) repeatedly; and then replacing $\underline{z}$ by $(z\xi)$

and applying formula (3) repeatedly; the general case is obtained.

Another Proof of (1): In this proof the following formulae are also required

$$E(p;\alpha_r:q;\rho_s:z)=\sum_{r=1}^{p}\prod_{s=1}^{p}{}'\Gamma(\alpha_s-\alpha_r)\Gamma(\alpha_r)\left\{\prod_{t=1}^{q}\Gamma(\rho_t-\alpha_r)\right\}^{-1}z^{\alpha_r}$$
$$\times F\left\{\begin{matrix}\alpha_r,\alpha_r-\rho_1+1,\dots,\alpha_r-\rho_q+1;(-1)^{p-q}z\\ \alpha_r-\alpha_1+1,\dots*\dots,\alpha_r-\alpha_p+1\end{matrix}\right\}, \quad \dots(5),$$

where $p\geqslant q+1$; when $p=q+1$, $|z|<1$,

$$e^{\pm i\pi\alpha_r}\sin(\alpha_r+k)\pi-e^{\pm i\pi(\alpha_r+k)}\sin(\pi\alpha_r)=\sin(\pi k), \quad \dots(6)$$

Proof of the formula: On applying (5) on the L.H.S. of (1), it becomes

$$\sum_{r=1}^{p} I_r, \quad \text{where}$$

$$I_r=\prod_{s=1}^{p}{}'\Gamma(\alpha_s-\alpha_r)\left\{\prod_{t=1}^{q}\Gamma(\rho_t-\alpha_r)\right\}^{-1}\Gamma(\alpha_r)\Gamma(\alpha_r+k)z^{\alpha_r}F\left\{\begin{matrix}\alpha_r,\alpha_r+k,\alpha_r-\rho_1+1,\dots,\alpha_r-\rho_q+1;(-1)^{p-q}z\\ \alpha_r-\alpha_1+1,\dots*\dots,\alpha_r-\alpha_p+1\end{matrix}\right\}$$

where $r=1,2,3,\dots,p$.

Now if $p\geqslant q+2$, the R.H.S. of (1) is equal to

$$\frac{1}{\sin k\pi}\sum_{r=1}^{p}\left[e^{\pm i\pi\alpha_r}\sin(\alpha_r+k)\pi-e^{\pm i\pi(\alpha_r+k)}\sin(\alpha_r\pi)\right]I_r=\sum_{r=1}^{p}I_r \text{ by (6).}$$

Thus (1) is proved for the case $p \geqslant q+2$. The restriction $p \geqslant q+2$ can be changed to $p \geqslant q+1$ by analytical continuation.

Applications of formula (1). In (1) take $p=q=0$, then it becomes

$$\int_0^\infty \exp\{-x-1/zx\}\, x^{k-1}\, dx = \frac{\pi}{\sin k\pi}\left[E(:1-k:e^{\pm i\pi}z) - z^{-k}E(:1+k:e^{\pm i\pi}z)\right], \quad (7)$$

where $R(z)>0$. [Gray, Mathieu and MacRobert "Bessel Functions, p. 51, (31)],

From (7) one gets the known formula

$$\int_0^\infty e^{-\lambda-k^2/\lambda}\frac{d\lambda}{\sqrt{\lambda}} = \sqrt{\pi}\, e^{-2k}, \quad \ldots \quad (8)$$

where $R(k^2)>0$.

§2. An integral involving the product of two E-functions: The formula to be proved is

$$\int_0^\infty \lambda^{k-1} E(p;\alpha_r:q;\rho_s:\lambda)\, E(l;\beta_t:m;\sigma_u:z/\lambda)\, d\lambda =$$

$$\frac{\pi}{\sin k\pi}\left[z^k E\left\{\begin{matrix}\alpha_1,\ldots,\alpha_p,\beta_1-k,\ldots,\beta_l-k:e^{\pm i\pi}z\\ 1-k,\rho_1,\ldots,\rho_q,\sigma_1-k,\ldots,\sigma_m-k\end{matrix}\right\} - E\left\{\begin{matrix}\alpha_1+k,\ldots,\alpha_p+k,\beta_1,\ldots,\beta_l:e^{\pm i\pi}z\\ 1+k,\rho_1+k,\ldots,\rho_q+k,\sigma_1,\ldots,\sigma_m\end{matrix}\right\}\right], \ldots (9)$$

where $p \geqslant q+1$, $l \geqslant m+1$, $R(\alpha_r+k)>0$ $r=1,2,3,\ldots,p$, $R(\beta_t-k)>0$, $t=1,2,3,\ldots,l$.

and $|\operatorname{amp} z| < \pi$.

For other values of p, q, l, m the result holds if the integral converges.

Proof of (9): In (1) with $p \geqslant q+1$ replace λ by (λ/z), where for the moment z is taken real and positive, and the formula can be written

$$\int_0^\infty \lambda^{k-1} E(p; \alpha_r : q; \rho_s : \lambda) E(:: z/\lambda)\, d\lambda =$$

$$\frac{\pi}{\sin k\pi}\left[z^k E(p; \alpha_r : 1-k, \rho_1, \ldots, \rho_q : e^{\pm i\pi} z) - E(p; \alpha_r + k : 1+k, \rho_1 + k, \ldots, \rho_q + k : e^{\pm i\pi} z)\right]$$

where $R(\alpha_r + k) > 0$, $r = 1, 2, 3, \ldots, p$ and we can take $|\operatorname{amp} z| < \pi$. This is a special case of formula (9), and the general case can be deduced in the same manner as was that of formula (1).

§3. A second integral involving the product of two E-functions: The formula to be proved is

$$\int_0^\infty \lambda^{-\alpha_{p+1}-1} E(p; \alpha_r : q; \rho_s : \lambda) E(l; \beta_t : m; \sigma_u : z\lambda)\, d\lambda =$$

$$\pi^{p-q} z^{\alpha_{p+1}} \sum_{r=1}^{p+1} \prod_{t=1}^{q} \sin(\rho_t - \alpha_r)\pi \left\{\prod_{s=1}^{p+1}{}' \sin(\alpha_s - \alpha_r)\pi\right\}^{-1} z^{-\alpha_r}$$

$$\times E\left\{\begin{matrix}\alpha_r, \alpha_r + \beta_1 - \alpha_{p+1}, \ldots, \alpha_r + \beta_l - \alpha_{p+1}, \alpha_r - \rho_1 + 1, \ldots, \alpha_r - \rho_q + 1 : e^{\pm i\pi(p-q)} z \\ \alpha_r - \alpha_1 + 1, \ldots * \ldots, \alpha_r - \alpha_{p+1} + 1, \alpha_r + \sigma_1 - \alpha_{p+1}, \ldots, \alpha_r + \sigma_m - \alpha_{p+1}\end{matrix}\right\}, \ldots (10)$$

where $p \geqslant q+1$, $l \geqslant m+1$, $R(\alpha_{p+1})>0$, $R(\alpha_r+\beta_t-\alpha_{p+1})>0$, $r=1,2,3,\dots,p$ and $t=1,2,3,\dots,l$, and $|\operatorname{amp} z|<\pi$. For other values of p, q, l, m, the formula is valid if the integral is convergent.

The following formulae will be required in the proof:—

$$E(p;\alpha_r:q;\rho_s:z)=\frac{\Gamma(\alpha_1)\cdots\Gamma(\alpha_p)}{\Gamma(\rho_1)\cdots\Gamma(\rho_q)}F\left(p;\alpha_r:q;\rho_s:-\frac{1}{z}\right), \quad \cdots(11)$$

where $p\leqslant q$, $z\neq 0$. If $p=q+1$, $|z|>1$.

If $p\geqslant q+1$, it follows from (5) and (11) that

$$E(p;\alpha_r:q;\rho_s:z)=\pi^{p-q-1}\sum_{r=1}^{p}\prod_{t=1}^{q}\sin(\rho_t-\alpha_r)\pi\left\{\prod_{s=1}^{p}{}'\sin(\alpha_s-\alpha_r)\pi\right\}^{-1}z^{\alpha_r}$$

$$\times E\left\{\begin{matrix}\alpha_r,\alpha_r-\rho_1+1,\dots,\alpha_r-\rho_q+1:e^{\pm i\pi(p-q-1)}\dfrac{1}{z}\\ \alpha_r-\alpha_1+1,\dots*\dots,\alpha_r-\alpha_p+1\end{matrix}\right\}, \quad \cdots(12)$$

<u>Proof of the formula</u>: In (2) put $u=\lambda z$ and then replace λ by $(1/\lambda)$ and z by $(1/z)$, so obtaining

$$z^{-\alpha_{p+1}}\int_0^\infty e^{-\alpha_{p+1}-1}E(p;\alpha_r:q;\rho_s:\lambda)E(::\lambda z)\,d\lambda=E(p+1;\alpha_r:q;\rho_s:1/z)$$

where $R(\alpha_{p+1})>0$. Hence on applying formula (12), formula (10), with $l = m = 0$ is obtained.

The general case is deduced in the same manner as the general case of formula (1)

<u>A Second proof of (10)</u>: This proof is by induction and in it the following formulae will be required

$$\sum_{h=1}^{p+1} \frac{\left(e^{\pm i\pi p}\right)^{\alpha_h}}{\prod_{j=1}^{p+2}{}' \sin(\alpha_j-\alpha_h)\pi} + \frac{\left(e^{\pm i\pi p}\right)^{\alpha_{p+2}}}{\prod_{j=1}^{p+1} \sin(\alpha_j-\alpha_{p+2})\pi} = 0 \quad , \ldots \ldots , (13),$$

$$\int_C e^{\xi}\xi^{\alpha_{p+1}-1} E(p;\alpha_r:q;\rho_s:z/\xi)\,d\xi =$$
$$e^{i\alpha_{p+1}\pi}E(p+1;\alpha_r:q;\rho_s:z e^{-i\pi}) - e^{-i\alpha_{p+1}\pi}E(p+1;\alpha_r:q;\rho_s:z e^{-i\pi}), \ldots (14),$$

[C.V., p. 380, ex 104] and the C is the same contour of (3).

<u>Proof of formula (13)</u>. To prove (13), consider the integral

$$\frac{1}{2\pi i}\int_\Gamma \frac{e^{\pm i\pi p z}}{\prod_{j=1}^{p+2} \sin(z-\alpha_j)\pi}\,dz,$$

where the contour of integration being a circle with centre at the origin, the radius R of the circle being chosen so that the circle does not pass through any of the poles of the integrand.

Write $Re^{i\phi}$ for $\underline{z}$ with $0 \leqslant \phi \leqslant \pi$, then since [C.V. p 36 $\frac{1}{2}$, ex 31]

$$|\operatorname{cosec} z| \leqslant M e^{-y} \quad y \geqslant 0,$$

$$|\operatorname{cosec} z| \leqslant M e^{y} \quad y \leqslant 0, \quad \text{where} \quad z = x + iy,$$

and $\underline{M}$ is a positive number independent of $\underline{z}$; the integrand is $O[\exp\{\pm p\pi R \sin\phi - (p+2) R\pi \sin\phi\}]$. Hence the integral round the semicircle above the real axis can be made to tend to zero as $R \to \infty$.

Similarly, when $\underline{z}$ describes the semicircle below the real axis, it can be shown that the integral tends to zero when $R \to \infty$. Therefore from Cauchy's theorem, the sum of the residues at the poles of the integrand is equal to zero. Hence (13) is proved.

Proof of the formula (10). When $p = q = 0$, (10) becomes (2). Now in proving (10) consider first of all the case in which $q = 0$; then it is required to prove that

$$\int_0^\infty \lambda^{-\alpha_{p+1}-1} E(p;\alpha_r::z\lambda)\, E(l;\beta_t:m;\sigma_u:\lambda)\, d\lambda =$$

$$\pi^p \sum_{r=1}^{p+1} \left\{ \prod_{s=1}^{p+1}{}' \sin(\alpha_s-\alpha_r)\pi \right\}^{-1} z^{\alpha_r} E\left\{ \begin{matrix} \alpha_r, \alpha_r+\beta_1-\alpha_{p+1}, \ldots, \alpha_r+\beta_l-\alpha_{p+1} : e^{\pm i\pi p}\frac{1}{z} \\ \alpha_r-\alpha_1+1, \ldots * \ldots, \alpha_r-\alpha_{p+1}+1, \alpha_r+\sigma_1-\alpha_{p+1}, \ldots, \alpha_r+\sigma_m-\alpha_{p+1} \end{matrix} \right\}, \ldots (15)$$

where $l \geqslant m+1$, $R(\alpha_{p+1})>0$, $R(\alpha_r+\beta_t-\alpha_{p+1})>0$, $r=1,2,3,\ldots,p$, $t=1,2,3,\ldots,l$ and $|\text{amp}\, z|<\pi$.

Let it be assumed that (15) is valid for a particular value of p. Then replacing z in (15) by (z/u), multiplying both sides by $e^{-u}u^{\alpha_{p+2}-1}$; then the L.H.S. of (15), on applying (2), becomes the L.H.S. of (15) with $(p+1)$ in place of p. While the R.H.S., on applying (1), becomes if $l \geqslant m+1$, $R(\alpha_{p+1})>0$, $R(\alpha_r+\beta_t-\alpha_{p+1})>0$, $r=1,2,3,\ldots,p$, $t=1,2,3,\ldots,l$, $|\text{amp}\, z|<\pi$

$$\pi^{p+1} \sum_{r=1}^{p+1} \left\{ \prod_{s=1}^{p+2}{}' \sin(\alpha_s-\alpha_r)\pi \right\}^{-1} z^{\alpha_r}$$

$$\times \left[E\left\{ \begin{matrix} \alpha_r, \alpha_r+\beta_1-\alpha_{p+1}, \ldots, \alpha_r+\beta_l-\alpha_{p+1} : e^{\pm i\pi(p+1)}\frac{1}{z} \\ \alpha_r-\alpha_1+1, \ldots * \ldots, \alpha_r-\alpha_{p+2}+1, \alpha_r+\sigma_1-\alpha_{p+1}, \ldots, \alpha_r+\sigma_m-\alpha_{p+1} \end{matrix} \right\} \right.$$
$$\left. -\left(\frac{e^{\pm i\pi p}}{z}\right)^{\alpha_r-\alpha_{p+2}} E\left\{ \begin{matrix} \alpha_{p+2}, \alpha_{p+2}+\beta_1-\alpha_{p+1}, \ldots, \alpha_{p+2}+\beta_l-\alpha_{p+1} : e^{\pm i\pi(p+1)}\frac{1}{z} \\ 1+\alpha_{p+2}-\alpha_1, \ldots, 1+\alpha_{p+2}-\alpha_{p+1}, \alpha_{p+2}+\sigma_1-\alpha_{p+1}, \ldots, \alpha_{p+2}+\sigma_m-\alpha_{p+1} \end{matrix} \right\} \right]$$

$$= \pi^{p+1} \sum_{r=1}^{p+2} \left\{ \prod_{s=1}^{p+2}{}' \sin(\alpha_s - \alpha_r)\pi \right\}^{-1} z^{\alpha_r} E\left\{ \begin{matrix} \alpha_r, \alpha_r + \beta_1 - \alpha_{p+1}, \ldots, \alpha_r + \beta_\ell - \alpha_{p+1} : e^{\pm i\pi(p+1)} \frac{1}{z} \\ \alpha_r - \alpha_1 + 1, \ldots * \ldots, \alpha_r - \alpha_{p+2} + 1, \alpha_r + \alpha_1 - \alpha_{p+1}, \ldots, \alpha_r + \alpha_p - \alpha_{p+1} \end{matrix} \right\}$$

by (13). But this is the R.H.S. of (15) with $(p+1)$ in place of p. But (15) holds when $p=0$, hence it holds for all positive integral values of p.

Again in (15) replace z by (ξz), multiply by $e^{\xi} \xi^{-\rho_1}$, then applying (3) to the L.H.S. and (14) to the R.H.S. repeatedly; the general case is obtained. Hence (10) is proved.

§ 4. An integral of an E-function expressed as a sum of E-functions.

The formula to be proved is

$$\int_0^\infty e^{-\lambda} \lambda^{k-1} E(p; \alpha_r : q; \rho_s : \lambda^m z) = \pi \operatorname{cosec} k\pi \, (2\pi)^{\frac{1}{2}m - \frac{1}{2}} m^{k-\frac{1}{2}}$$

$$\times E\left(p; \alpha_r : 1 - \frac{k}{m}, 1 - \frac{k+1}{m}, \ldots, 1 - \frac{k+m-1}{m}, \rho_1, \ldots, \rho_q : e^{\pm i m\pi} m^m z\right)$$

$$+ 2^{\frac{1}{2} - \frac{1}{2}m} \pi^{\frac{1}{2} + \frac{1}{2}m} \sum_{\nu=0}^{m-1} \frac{(-1)^{\nu+1} m^{-\frac{1}{2}-\nu}}{\sin\left(\frac{k+\nu}{m}\right)\pi \prod_{s=1}^{\nu} \sin \frac{s\pi}{m} \prod_{t=1}^{m-\nu-1} \sin \frac{t\pi}{m}} z^{-(k+\nu)/m}$$

$$\times E\left\{ \begin{matrix} p; \alpha_r + \frac{k+\nu}{m} : e^{\pm i m\pi} m^m z \\ 1 + \frac{k+\nu}{m}, 1 + \frac{1}{m}, \ldots, 1 + \frac{\nu}{m}, 1 - \frac{1}{m}, \ldots, 1 - \frac{m-\nu-1}{m}, \rho_1 + \frac{k+\nu}{m}, \ldots, \rho_q + \frac{k+\nu}{m} \end{matrix} \right\}, \ldots (16)$$

where $\underline{m}$ is a positive integer, $p \geq q+1$, $R(m\alpha_r + k) > 0$, $r = 1, 2, 3, \dots, p$ and $|\operatorname{amp} z| < \pi$. For other values of $\underline{p}$ and $\underline{q}$, the result holds if the integral is convergent.

The following formulae are required in the proof:

If $\underline{m}$ is a positive integer and if $R(k) > 0$

$$\int_0^\infty e^{-\lambda}\lambda^{k-1} E(p;\alpha_r:q;\rho_s:z/\lambda^m)\,d\lambda = m^{k-\frac{1}{2}}(2\pi)^{\frac{1}{2}-\frac{1}{2}m} E(p+m;\alpha_r:q;\rho_s:z/m^m),\ \dots (17)$$

where $\alpha_{p+\nu} = (k+\nu-1)/m$, $\nu = 1, 2, 3, \dots, m$ [Ragab F. M., Proc. Glasg. Math. Assoc. Vol. I. p. 131 formula (1); also thesis for Ph.D. p. (52) equ. (11)];

If $\underline{m}$ is a positive integer

$$\sin\frac{k\pi}{m}\sin\frac{(k+1)\pi}{m}\cdots\sin\frac{(k+m-1)\pi}{m} = 2^{1-m}\sin(k\pi),\ \dots (18).$$

Proof of the formula: Consider the special case of (16) when $p=1$, $q=0$, then the L.H.S. of (16) becomes

$$\int_0^\infty e^{-\lambda}\lambda^{k-1} E(\alpha_1::\lambda^m z)\,d\lambda = z^{\alpha_1}\int_0^\infty e^{-\lambda}\lambda^{k+m\alpha_1-1} E\{\alpha_1::1/(\lambda^m z)\}\,d\lambda$$

$$= m^{k+m\alpha_1-\frac{1}{2}}(2\pi)^{\frac{1}{2}-\frac{1}{2}m} z^{\alpha_1} E\{m+1; \alpha_r :: 1/(zm^m)\},$$

where $\alpha_{\nu+1} = (k+\nu-1)/m + \alpha_1$, $\nu = 1, 2, \ldots, m$; by (4).

On applying (12) and (18) this becomes (16) with $p=1, q=0$. The general case can be deduced in the same manner as was that of formula (1). From (16) many particular cases can be deduced. For instance if, in (16); $p=q=0$, one gets the following integral

$$\int_0^\infty \exp\{-\lambda - 1/\lambda^m z\} \lambda^{k-1} d\lambda = \pi \operatorname{cosec}(k\pi)(2\pi)^{\frac{1}{2}m-\frac{1}{2}} m^{k-\frac{1}{2}}$$

$$\times E\left(: 1-\frac{k}{m}, 1-\frac{k+1}{m}, \ldots, 1-\frac{k+m-1}{m} : e^{\pm im\pi} m^m z\right)$$

$$+ 2^{\frac{1}{2}-\frac{1}{2}m} \pi^{\frac{1}{2}+\frac{1}{2}m} \sum_{\nu=0}^{m-1} \frac{(-1)^{\nu+1} m^{-\frac{1}{2}-\nu}}{\sin\left(\frac{k+\nu}{m}\right)\pi \prod_{s=1}^{\nu} \sin\left(\frac{s\pi}{m}\right) \prod_{t=1}^{m-\nu-1} \sin\frac{t\pi}{m}} z^{-(k+\nu)/m}$$

$$\times E\left(: 1+\frac{k+\nu}{m}, 1+\frac{1}{m}, 1+\frac{2}{m}, \ldots, 1+\frac{\nu}{m}, 1-\frac{1}{m}, 1-\frac{2}{m}, \ldots, 1-\frac{m-\nu-1}{m} : e^{\pm im\pi} m^m z\right), \quad \ldots (19)$$

where $R(z) > 0$ and $\underline{m}$ is any positive integer. Thus the value of the integral is found to be the sum of $(m+1)$ generalized hypergeometric functions of the type ${}_0F_m[; \ldots ; (-1)/(e^{\pm mi} m^m z)]$.

§5. A second integral of an E-function expressed as the sum of E-functions.

The formula to be proved is

$$\int_0^1 \lambda^{\alpha-1}(1-\lambda)^{\rho-\alpha-1} E(p;\alpha_r:q;\rho_s: z\lambda^m)\,d\lambda =$$

$$\frac{\sin\rho\pi}{\sin\alpha\pi}\Gamma(\rho-\alpha)\,m^{\alpha-\rho} E\left\{\begin{matrix}\alpha_1,\dots,\alpha_p, 1-\rho/m, 1-(\rho+1)/m,\dots, 1-(\rho+m-1)/m : z\\ 1-\alpha/m, 1-(\alpha+1)/m,\dots, 1-(\alpha+m-1)/m, \rho_1,\dots,\rho_q\end{matrix}\right\}$$

$$-2^{1-m}\Gamma(\rho-\alpha)\,m^{\alpha-\rho}\sum_{\nu=0}^{m-1}\frac{\sin(\rho-\alpha)\pi}{\sin\left(\frac{\alpha+\nu}{m}\right)\pi\prod_{s=1}^{\nu}\sin\frac{s\pi}{m}\prod_{t=1}^{m-\nu-1}\sin\frac{t\pi}{m}}\,z^{-(\alpha+\nu)/m}$$

$$\times E\left\{\begin{matrix}\alpha_1+\frac{\alpha+\nu}{m},\dots,\alpha_p+\frac{\alpha+\nu}{m}, 1+\frac{\alpha-\rho+\nu}{m}, 1+\frac{\alpha-\rho+\nu-1}{m},\dots, 1+\frac{\alpha-\rho+\nu-m+1}{m} : z\\ 1+\frac{\alpha+\nu}{m}, 1+\frac{1}{m}, 1+\frac{2}{m},\dots, 1+\frac{\nu}{m}, 1-\frac{1}{m}, 1-\frac{2}{m},\dots, 1-\frac{m-\nu-1}{m}, \rho_1+\frac{\alpha+\nu}{m},\dots,\rho_q+\frac{\alpha+\nu}{m}\end{matrix}\right\}$$

where m is a positive integer, $p \geqq q+1$, $R(\alpha+m\alpha_r)>0$, $r=1,2,3,\dots,p$; ... (20)
$R(\rho-\alpha)>0$, and $|\operatorname{amp} z|<\pi$. For other values of p and q the formula is valid if the integral is convergent.

The following formula is required in the proof:

If m is a positive integer and if $R(\rho)>R(\alpha)>0$,

$$\int_0^1 \lambda^{\alpha-1}(1-\lambda)^{\rho-\alpha-1} E(p;\alpha_r:q;\rho_s:z/\lambda^m)\,d\lambda$$

$$=\Gamma(\rho-\alpha)\,m^{\alpha-\rho} E(p+m;\alpha_r:q+m;\rho_s:z), \ldots\ldots \qquad (21)$$

where $\alpha_{p+\nu}=(\alpha+\nu-1)/m$, $\rho_{q+\nu}=(\rho+\nu-1)/m$, $\nu=1,2,3,\ldots,m$.
[MacRobert, T. M., Proc. Glasg. Math. Assoc. Vol II., p. (34) equ. 7];

<u>Proof of the formula</u>: When $p=1, q=0$, the integral on the left of (20) can be written

$$z^{\alpha_1}\int_0^1 \lambda^{\alpha+m\alpha_1-1}(1-\lambda)^{\rho-\alpha-1} E\{\alpha_1::1/(z\lambda^m)\}\,d\lambda$$

$$=z^{\alpha_1}\Gamma(\rho-\alpha)\,m^{\alpha-\rho} E\left\{\begin{matrix}\alpha_1, \alpha_1+\frac{\alpha}{m}, \alpha_1+\frac{\alpha+1}{m}, \ldots, \alpha_1+\frac{\alpha+m-1}{m} \\ \alpha_1+\frac{\rho}{m}, \alpha_1+\frac{\rho+1}{m}, \ldots, \alpha_1+\frac{\rho+m-1}{m}\end{matrix} : \frac{1}{z}\right\} \text{ by } (21).$$

On applying (12) and (18), this gives (20) with $p=1$, $q=0$. The general case can be deduced in the same manner as was that of formula (1).

Also many particular cases can be deduced from (20). For instance, if in (20), $p=q=0$, the value of the integral

$$\int_1^\infty e^{-\lambda^m/z}\lambda^{-\rho}(\lambda-1)^{\rho-\alpha-1}\,d\lambda$$

where $R(z)>0$, $R(\rho-\alpha)>0$, is found to be the R.H.S. of (20) with all the linear expressions involving $\alpha_1, \alpha_2, \ldots, \alpha_p, \rho_1, \rho_2, \ldots, \rho_q$ omitted.

§6. Some double integrals: The first of these is

$$\int_0^\infty\int_0^\infty \lambda^{m-1}\mu^{n-1}(1+\lambda+\mu)^{-k} E(p; \alpha_r : q; \rho_s : \lambda z/\mu)\, d\lambda\, d\mu =$$

$$\frac{\pi\,\Gamma(k-m-n)}{\sin m\pi\,\Gamma(k)}\left\{ E\begin{bmatrix} n, \alpha_1, \ldots, \alpha_p : e^{\pm i\pi} z \\ 1-m, \rho_1, \ldots, \rho_q \end{bmatrix} - z^{-m} E\begin{bmatrix} m+n, \alpha_1+m, \ldots, \alpha_p+m : e^{\pm i\pi} z \\ 1+m, \rho_1+m, \ldots, \rho_q+m \end{bmatrix}\right\}$$

where $p \geqslant q+1$, $R(z)>0$, $R(n)>0$, $R(k-n)>0$, $R(m+\alpha_r)>0$, $r=1,2,3,\ldots,p$, $R(k-n+\alpha_r)>0$, $r=1,2,3,\ldots,p$ and $|\operatorname{amp} z|<\pi$. $\quad \ldots (22)$

Formula (22) will be derived from the following formula

$$\frac{1}{\Gamma(\rho_{q+1}-\alpha_{p+1})}\int_0^1 t^{-\rho_{q+1}}(1-t)^{\rho_{q+1}-\alpha_{p+1}-1} E(p; \alpha_r : q; \rho_s : zt)\, dt =$$

$$\frac{\sin(\alpha_{p+1}\pi)}{\sin(\rho_{q+1}\pi)} E(p+1; \alpha_r : q+1; \rho_s : x)$$

$$+ \frac{\sin(\alpha_{p+1}-\rho_{q+1})\pi}{\sin(\rho_{q+1}\pi)} z^{\rho_{q+1}-1} E(p+1; \alpha_r-\rho_{q+1}+1 : 2-\rho_{q+1}, \rho_1-\rho_{q+1}+1, \ldots, \rho_q-\rho_{q+1}+1 : z)$$

$\ldots (23)$

where $p \geqslant q+1$, $R(\rho_{q+1}-\alpha_{p+1})>0$, $R(\alpha_r-\rho_{q+1})>-1$, $r=1,2,3,\ldots,p$.

To prove (22) consider the special case in which $p=q=0$, then the L.H.S. of (22) is equal to

$$\int_0^\infty\int_0^\infty e^{-\mu/(\lambda z)}\lambda^{m-1}\mu^{n-1}(1+\lambda+\mu)^{-k}d\lambda\, d\mu = \frac{1}{\Gamma(k)}\int_0^\infty \lambda^{m-1}d\lambda\int_0^\infty e^{-\mu/(\lambda z)}\mu^{n-k-1}E(k::\tfrac{\mu}{1+\lambda})\,d\mu.$$

Here replace μ by $(\lambda z\mu)$ and get

$$\frac{z^{n-k}}{\Gamma(k)}\int_0^\infty e^{-\mu}\mu^{n-k-1}d\mu\int_0^\infty \lambda^{m+n-k-1}E(k::\tfrac{\lambda\mu z}{1+\lambda})\,d\lambda.$$

The second integral can be written

$$\int_0^1 t^{m+n-k-1}(1-t)^{k-m-n-1}E(k::t\mu z)\,dt,$$

where $\lambda = t/(1-t)$, $R(k)>R(m+n)>0$; and from (23), this is equal to

$$\Gamma(k-m-n)(\mu z)^{k-m-n}E(m+n::\mu z).$$

On applying (16), (22) with $p=q=0$ is obtained. The general case is derived in the usual way.

Next consider the integral

$$\int_0^\infty\int_0^\infty e^{-\lambda\mu-\lambda/\mu}\lambda^{l+m+k-1}\mu^{l+m-k-1}E(p;\alpha_r:q;\rho_s:\lambda^n z/\mu^n)\,d\lambda\, d\mu \quad ,\ldots(24)$$

where n is a positive integral, $p\geqslant q+1$, $R(l+m+k+n\alpha_r)>0$,

$r=1,2,3,\ldots,p$, $|\operatorname{amp} z|<\pi$.

On replacing μ by $(\lambda\mu)$ this can be written

$$\int_0^\infty \lambda^{2l+2m-1}\,d\lambda \int_0^\infty e^{-\lambda^2\mu-1/\mu}\mu^{l+m-k-1}E(p;\alpha_r:q;\rho_s:z/\mu^n)\,d\mu,$$

and on changing the order of integration and replacing λ by $(\lambda/\sqrt{\mu})$ in the inner integral it becomes

$$\tfrac{1}{2}\Gamma(l+m)\int_0^\infty e^{-1/\mu}\mu^{-k-1}E(p;\alpha_r:q;\rho_s:\tfrac{z}{\mu^n})\,d\mu=\tfrac{1}{2}\Gamma(l+m)\int_0^\infty e^{-\mu}\mu^{k-1}E(p;\alpha_r:q;\rho_s:z\mu^n)\,d\mu.$$

This last integral is the integral in (16)° with n in place of m.

Now consider the integral

$$\int_0^\infty\int_0^\infty e^{-\lambda\mu-\lambda/\mu}\lambda^{l+m+k-1}\mu^{l+m-k-1}E(p;\alpha_r:q;\rho_s:\lambda^n\mu^n z)\,d\lambda\,d\mu, \quad \ldots(25)$$

where n is a positive integer, $p\geqq q+1$, $R(l+m+k+n\alpha_r)>0$, $r=1,2,3,\ldots,p$ and $|\operatorname{amp} z|<\pi$.

To prove (25), replace μ by (μ/λ) and get

$$\text{L.H.S.}=\int_0^\infty \lambda^{2k-1}\,d\lambda\int_0^\infty e^{-\mu-\lambda^2/\mu}\mu^{l+m-k-1}E(p;\alpha_r:q;\rho_s:\mu^n z)\,d\mu.$$

Here change the order of integration, replace λ by $(\lambda\sqrt{\mu})$ and so get

$$\frac{1}{2}\Gamma(k)\int_0^\infty e^{-u}\mu^{l+m-1}E(p;\alpha_r:q;\rho_s:\mu^n z)\,d\mu.$$

The last integral is the integral in (16) with $\underline{n}$ in place of $\underline{m}$ and $(l+m)$ in place of $\underline{k}$.

Finally consider the integral

$$\int_0^\infty\int_0^\infty e^{-\lambda\mu-\lambda/\mu}\lambda^{l+m+k-1}\mu^{l+m-k-1}E(p;\alpha_r:q;\rho_s:\lambda^{-n}\mu^{-n}z)\,d\lambda\,d\mu \quad \cdots (26).$$

where $\underline{n}$ is a positive integer, $p \geqslant q+1$, $R(l+m+k)>0$, and $|\operatorname{amp} z|<\pi$.

Here replace μ by (μ/λ) and get

$$\int_0^\infty \lambda^{2k-1}d\lambda\int_0^\infty e^{-\mu-\lambda^2/\mu}\mu^{l+m-k-1}E(p;\alpha_r:q;\rho_s:\mu^{-n}z)\,d\mu.$$

On changing the order of integration and replacing $\underline{\lambda}$ by $(\lambda\sqrt{\mu})$ this becomes

$$\frac{1}{2}\Gamma(k)\int_0^\infty e^{-\mu}\mu^{l+m-1}E(p;\alpha_r:q;\rho_s:\mu^{-n}z)\,d\mu$$

$$=\frac{1}{2}\Gamma(k)\,n^{l+m-\frac{1}{2}}(2\pi)^{\frac{1}{2}-\frac{1}{2}n}E(p+n;\alpha_r:q;\rho_s:z/n^n) \text{ by } (17),$$

where $\alpha_{p+\nu}=(l+m+\nu-1)/n$, $\nu=1,2,3,\cdots,n$.

§7. <u>An integral involving an E-function and a Modified Bessel Function of the Second Kind:</u>

The formula to be proved is

$$\int_0^\infty \lambda^{m-1} K_n(\lambda) E(p; \alpha_r : q; \rho_s : z/\lambda)\, d\lambda =$$

$$2^{\alpha_1+\alpha_2+\cdots+\alpha_p-\rho_1-\rho_2-\cdots-\rho_q+q-p+m-2} \cdot \pi^{\frac{1}{2}(q-p+1)}$$

$$\times \left[E\left(\begin{matrix} \frac{m+n}{2}, \frac{m-n}{2}, \frac{\alpha_1}{2}, \ldots, \frac{\alpha_p}{2}, \frac{\alpha_1+1}{2}, \ldots, \frac{\alpha_p+1}{2} \\ \frac{1}{2}, \frac{\rho_1}{2}, \ldots, \frac{\rho_q}{2}, \frac{\rho_1+1}{2}, \ldots, \frac{\rho_q+1}{2} \end{matrix} ; e^{\pm i\pi} 4^{q-p} z^2 \right) - \frac{2^{p-q}}{z} E\left(\begin{matrix} \frac{m+n+1}{2}, \frac{m-n+1}{2}, \frac{\alpha_1+1}{2}, \ldots, \frac{\alpha_p+1}{2}, \frac{\alpha_1+2}{2}, \ldots, \frac{\alpha_p+2}{2} \\ \frac{3}{2}, \frac{\rho_1+1}{2}, \ldots, \frac{\rho_q+1}{2}, \frac{\rho_1+2}{2}, \ldots, \frac{\rho_q+2}{2} \end{matrix} ; e^{\pm i\pi} 4^{q-p} z^2 \right) \right], \quad \cdots (27)$$

where $R(m \pm n) > 0$, $|\mathrm{amp}\, z| < \pi$.

The following formula is required in the proof [C.V., p. 372, eq (62) i] namely if $R(m \pm n) > 0$

$$\int_0^\infty \lambda^{m-1} K_n(\lambda)\, d\lambda = 2^{m-2}\, \Gamma\left(\frac{m+n}{2}\right) \Gamma\left(\frac{m-n}{2}\right), \quad \cdots\cdots \quad (28)$$

<u>Proof of the formula</u>: Consider the integral

$$\int_0^\infty \lambda^{m-1} K_n(\lambda) E(\quad : q; \rho_s : z/\lambda)\, d\lambda$$

where $R(m \pm n) > 0$. It can be written

$$\int_0^\infty \lambda^{m-1} K_n(\lambda) \frac{1}{\Gamma(\rho_1)\Gamma(\rho_2)\cdots\Gamma(\rho_q)} F(; \rho_1, \rho_2, \ldots, \rho_q; -\lambda/z)\, d\lambda$$

$$= \int_0^\infty \lambda^{m-1} K_n(\lambda) \left\{ \begin{aligned} &\frac{1}{\Gamma(\rho_1)\cdots\Gamma(\rho_q)} F\left(; \frac{1}{2}, \frac{\rho_1}{2}, \frac{\rho_1+1}{2}, \dots, \frac{\rho_q}{2}, \frac{\rho_q+1}{2}; \lambda^2 z^{-2} 4^{-q-1}\right) \\ &- \frac{1}{\Gamma(\rho_1+1)\cdots\Gamma(\rho_q+1)} \left(\frac{\lambda}{z}\right) F\left(; \frac{3}{2}, \frac{\rho_1+1}{2}, \frac{\rho_1+2}{2}, \dots, \frac{\rho_q+1}{2}, \frac{\rho_q+2}{2}; \lambda^2 z^{-2} 4^{-q-1}\right) \end{aligned} \right\} d\lambda$$

On expanding, integrating term by term and applying (28), the value of the integral is found to be

$$2^{m-2}\Gamma\left(\frac{m+n}{2}\right)\Gamma\left(\frac{m-n}{2}\right) \frac{\pi^{\frac{1}{2}}}{\Gamma(\frac{1}{2})\Gamma(\rho_1)\cdots\Gamma(\rho_q)} F\left(\begin{matrix}\frac{1}{2}m+\frac{1}{2}n, \frac{1}{2}m-\frac{1}{2}n; 1/z^2 4^q \\ \frac{1}{2}, \frac{\rho_1}{2}, \frac{\rho_1+1}{2}, \dots, \frac{\rho_q}{2}, \frac{\rho_q+1}{2}\end{matrix}\right)$$

$$-2^{m-1}\Gamma\left(\frac{m+n+1}{2}\right)\Gamma\left(\frac{m-n+1}{2}\right) \frac{\pi^{1/2}}{\Gamma(\frac{3}{2})\Gamma(\rho_1+1)\cdots\Gamma(\rho_q+1)}\left(\frac{1}{2z}\right) F\left(\begin{matrix}\frac{1}{2}+\frac{1}{2}m+\frac{1}{2}n, \frac{1}{2}+\frac{1}{2}m-\frac{1}{2}n; 1/z^2 4^q \\ \frac{3}{2}, \frac{\rho_1+1}{2}, \frac{\rho_1+2}{2}, \dots, \frac{\rho_q+1}{2}, \frac{\rho_q+2}{2}\end{matrix}\right)$$

$$= 2^{m-2-\rho_1-\rho_2-\cdots-\rho_q+q}\,\pi^{\frac{1}{2}(q+1)} \left\{ \begin{aligned} &E\left(\frac{m+n}{2}, \frac{m-n}{2} : \frac{1}{2}, \frac{\rho_1}{2}, \frac{\rho_1+1}{2}, \dots, \frac{\rho_q}{2}, \frac{\rho_q+1}{2} : e^{\pm i\pi} 4^q z^2\right) \\ &- \frac{2^{-q}}{z} E\left(\frac{m+n+1}{2}, \frac{m-n+1}{2} : \frac{\rho_1+1}{2}, \frac{\rho_1+2}{2}, \dots, \frac{\rho_q+1}{2}, \frac{\rho_q+2}{2}, \frac{3}{2} : e^{\pm i\pi} 4^q z^2\right) \end{aligned} \right\}.$$

On making repeated applications of (14), formula (24) is obtained.

<u>Note</u>. The method can be employed to express the integral as the sum of any number of E-functions.

<u>A second proof of (24)</u>. In this proof the following formulae are required

$$\frac{1}{2\pi i}\int_C e^{\xi}\xi^{-\rho_{q+1}} E(p;\alpha_r:q;\rho_s:z\,\xi^n)\,d\xi = (2\pi)^{\frac{1}{2}n-\frac{1}{2}} n^{\frac{1}{2}-\rho_{q+1}} E(p;\alpha_r:q+n;\rho_s:n^n z)$$

where n is any positive integer, and $\rho_{q+\nu+1} = \dfrac{\rho_{q+1}+\nu}{n}$, $\nu = 0, 1, 2, \cdots n-1$, $\quad\cdots(29)$
[MacRobert, T.M., Proc. Glasg. Math. Assoc. Vol. II. p. (35) form. (8)],
and the contour C is the same contour of (3);

$$F(\alpha,\beta;\gamma;z)(=)\ \frac{\Gamma(\gamma-\alpha-\beta)\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} F(\alpha,\beta;\alpha+\beta-\gamma+1;1-z)$$
$$+\ \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)(1-z)^{\gamma-\alpha-\beta}}{\Gamma(\alpha)\Gamma(\beta)} F(\gamma-\alpha,\gamma-\beta;\gamma-\alpha-\beta+1;1-z), \cdots(30)$$

[C.V., p. 249],

$$F\left(\tfrac{1}{2}+n, \tfrac{1}{2}-n; \tfrac{1}{2}+m; \frac{x-1}{2x}\right) = \left(\frac{x+1}{2x}\right)^{m-\frac{1}{2}} x^{m+n} F\left(\frac{m+n+1}{2}, \frac{m+n}{2}; \tfrac{1}{2}+m; 1-x^2\right), \cdots(31)$$

[C.V., p. 305, where $(1/x)$ is written for x, $(m-\frac{1}{2})$ for m and $(-n-\frac{1}{2})$ for n],

$$F(\alpha,\beta;\gamma;z) = (1-z)^{-\alpha} F\left(\alpha,\gamma-\beta;\gamma;\frac{z}{z-1}\right), \cdots\cdots(32)$$

$$F(\alpha,\beta;\gamma;z) = (1-z)^{\gamma-\alpha-\beta} F(\gamma-\alpha,\gamma-\beta;\gamma;z), \quad \ldots\ldots (33),$$

$$\cos(n\pi) E\left(\tfrac{1}{2}+n, \tfrac{1}{2}-n :: 2z\right) = \sqrt{(2\pi z)}\, e^{z} K_n(z), \quad \ldots\ldots (34),$$

[B.V., p. 351, equ. (4)].

<u>Proof of the formula</u>: (1) and (34) give if $R(z)>0$, $R(m\pm n)>0$.

$$\int_0^\infty e^{-\lambda/z} \lambda^{m-1} K_n(\lambda)\, d\lambda = \frac{\cos n\pi}{\cos m\pi}\sqrt{\left(\frac{\pi}{2}\right)}$$

$$\times\left[2^{\frac{1}{2}-m} E\left(\begin{matrix} m+n, m-n : \\ m+\frac{1}{2}\end{matrix} \frac{e^{\pm i\pi}2z}{(z+1)}\right) - \left(\frac{z}{z+1}\right)^{m-\frac{1}{2}} E\left(\begin{matrix}\frac{1}{2}+n, \frac{1}{2}-n : \\ \frac{3}{2}-m\end{matrix} \frac{e^{\pm i\pi}2z}{z+1}\right)\right], \quad \ldots (35)$$

Now apply (30) then (35) becomes

$$\frac{\cos m\pi}{\cos n\pi}\sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty e^{-\lambda/z}\lambda^{m-1}K_n(\lambda)\,d\lambda = 2^{\frac{1}{2}-m}\frac{\Gamma(m+n)\Gamma(m-n)\Gamma(\frac{1}{2}-m)}{\Gamma(\frac{1}{2}+n)\Gamma(\frac{1}{2}-n)} F\left(\begin{matrix}m+n, m-n; \\ \frac{1}{2}+m\end{matrix} \frac{z-1}{2z}\right)$$

$$+ 2^{\frac{1}{2}-m}\Gamma(m-\tfrac{1}{2})\left(\frac{z-1}{2z}\right)^{\frac{1}{2}-m} F\left(\tfrac{1}{2}+n, \tfrac{1}{2}-n; \tfrac{3}{2}-m; \frac{z-1}{2z}\right)$$

$$- \frac{\Gamma(\frac{1}{2}+n)\Gamma(\frac{1}{2}-n)\Gamma(\frac{1}{2}-m)}{\Gamma(1-m-n)\Gamma(1-m+n)}\left(\frac{z}{1+z}\right)^{m-\frac{1}{2}} F\left(\tfrac{1}{2}+n, \tfrac{1}{2}-n; \tfrac{1}{2}+m; \frac{z-1}{2z}\right)$$

$$- \Gamma(m-\tfrac{1}{2})\left(\frac{z}{1+z}\right)^{m-\frac{1}{2}}\left(\frac{z-1}{2z}\right)^{\frac{1}{2}-m} F\left(1-m-n, 1-m+n; \tfrac{3}{2}-m; \frac{z-1}{2z}\right).$$

Now transform the fourth hypergeometric function in the last expression by means of (33), then it will cancel with the second hypergeometric function, apply the same transformation (33) to the first hypergeometric function, add the result to the third one and get

$$\frac{\cos m\pi}{\cos n\pi}\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-\lambda/z}\lambda^{m-1}K_n(\lambda)\,d\lambda = \left[\frac{\Gamma(m+n)\Gamma(m-n)}{\Gamma(\frac{1}{2}-n)\Gamma(\frac{1}{2}+n)} - \frac{\Gamma(\frac{1}{2}+n)\Gamma(\frac{1}{2}-n)}{\Gamma(1-m-n)\Gamma(1-m+n)}\right]\times J$$

where $J \equiv \left(\frac{z}{z+1}\right)^{m-1/2}\Gamma(\tfrac{1}{2}-m)\,F\left(\tfrac{1}{2}+n, \tfrac{1}{2}-n; \tfrac{1}{2}+m; \frac{z-1}{2z}\right)$.

Simplifying the quantity between brackets [ ], the last expression becomes if $R(z)>0$, $R(m\pm n)>0$

$$\int_0^\infty e^{-\lambda/z}\lambda^{m-1}K_n(\lambda)\,d\lambda = \sqrt{\left(\frac{\pi}{2}\right)}\frac{\Gamma(m+n)\Gamma(m-n)}{\Gamma(\frac{1}{2}+m)}\left(\frac{z}{1+z}\right)^{m-\frac{1}{2}}F\left(\begin{matrix}\frac{1}{2}+n, \frac{1}{2}-n; \frac{z-1}{2z}\\ \frac{1}{2}+m\end{matrix}\right)$$

$$= \frac{\sqrt{\pi}}{\sqrt{2}}\cdot\frac{\Gamma(m+n)\Gamma(m-n)}{\Gamma(\frac{1}{2}+m)}\frac{z^{m+n}}{2^{m-1/2}}F\left(\tfrac{1}{2}+\tfrac{m}{2}+\tfrac{n}{2}, \tfrac{n}{2}+\tfrac{m}{2}; \tfrac{1}{2}+m; 1-z^2\right) \text{ by (31)},$$

$$= \sqrt{\left(\frac{\pi}{2}\right)}\frac{\Gamma(m+n)\Gamma(m-n)}{\Gamma(\frac{1}{2}+m)}\frac{2^{\frac{1}{2}-m}}{z}F\left(\tfrac{1}{2}+\tfrac{m}{2}+\tfrac{n}{2}, \tfrac{1}{2}+\tfrac{m}{2}-\tfrac{n}{2}; \tfrac{1}{2}+m; 1-\tfrac{1}{z^2}\right) \text{ by (32)}.$$

But

$$\frac{\Gamma(m+n)\Gamma(m-n)}{\Gamma(\frac{1}{2}+m)} = 2^{2m-2}\Gamma(\tfrac{1}{2}-m)\left[\frac{\Gamma(\frac{m+n}{2})\Gamma(\frac{m-n}{2})}{\Gamma(\frac{1}{2}-\frac{m+n}{2})\Gamma(\frac{1}{2}-\frac{m-n}{2})} - \frac{\Gamma(\frac{1}{2}+\frac{m+n}{2})\Gamma(\frac{1}{2}+\frac{m-n}{2})}{\Gamma(1-\frac{m+n}{2})\Gamma(1-\frac{m-n}{2})}\right].$$

Substituting in the last expression then it becomes if $R(x)>0, R(m\pm n)>0$

$$\int_0^\infty e^{-\lambda/z}\lambda^{m-1}K_n(\lambda)\,d\lambda = \sqrt{\pi}\,2^{m-2}\Gamma(\tfrac{1}{2}-m)$$

$$\times\left[\frac{\Gamma(\frac{m+n}{2})\Gamma(\frac{m-n}{2})}{\Gamma(\frac{1}{2}-\frac{m+n}{2})\Gamma(\frac{1}{2}-\frac{m-n}{2})}\left(\frac{1}{z}\right)F\left(\frac{1}{2}+\frac{m}{2}+\frac{n}{2},\frac{1}{2}+\frac{m}{2}-\frac{n}{2};\frac{1}{2}+m;1-\frac{1}{z^2}\right)\right.$$
$$\left.-\frac{\Gamma(\frac{1}{2}+\frac{m+n}{2})\Gamma(\frac{1}{2}+\frac{m-n}{2})}{\Gamma(1-\frac{m+n}{2})\Gamma(1-\frac{m-n}{2})}\left(\frac{1}{z}\right)F\left(\frac{1}{2}+\frac{m}{2}+\frac{n}{2},\frac{1}{2}+\frac{m}{2}-\frac{n}{2};\frac{1}{2}+m;1-\frac{1}{z^2}\right)\right]$$

$$=\sqrt{\pi}\,2^{m-2}\left[\frac{\Gamma(\frac{m+n}{2})\Gamma(\frac{m-n}{2})\Gamma(\frac{1}{2}-m)}{\Gamma(\frac{1}{2}-\frac{m+n}{2})\Gamma(\frac{1}{2}-\frac{m-n}{2})}F\left(\frac{m}{2}+\frac{n}{2},\frac{m}{2}-\frac{n}{2};\frac{1}{2}+m;1-\frac{1}{z^2}\right)\right.$$
$$\left.-\frac{\Gamma(\frac{1}{2}+\frac{m+n}{2})\Gamma(\frac{1}{2}+\frac{m-n}{2})\Gamma(\frac{1}{2}-m)}{\Gamma(1-\frac{m+n}{2})\Gamma(1-\frac{m-n}{2})}\left(\frac{1}{z}\right)F\left(\frac{1+m+n}{2},\frac{1+m-n}{2};\frac{1}{2}+m;1-\frac{1}{z^2}\right)\right]\quad\text{by (33),}$$

$$=\sqrt{\pi}\,2^{m-2}\left[\frac{\Gamma(\frac{m+n}{2})\Gamma(\frac{m-n}{2})}{\Gamma(\frac{1}{2})}F\left(\frac{m}{2}+\frac{n}{2},\frac{m}{2}-\frac{n}{2};\frac{1}{2};\frac{1}{z^2}\right)\right.$$
$$-\Gamma(m-\tfrac{1}{2})\left(1-\frac{1}{z^2}\right)^{\frac{1}{2}-m}F\left(\frac{1}{2}-\frac{m}{2}-\frac{n}{2},\frac{1}{2}-\frac{m}{2}+\frac{n}{2};\frac{3}{2}-m;1-\frac{1}{z^2}\right)$$
$$-\frac{\Gamma(\frac{1}{2}+\frac{m+n}{2})\Gamma(\frac{1}{2}+\frac{m-n}{2})}{\Gamma(3/2)}\left(\frac{1}{z}\right)F\left(\frac{1}{2}+\frac{m}{2}+\frac{n}{2},\frac{1}{2}+\frac{m}{2}-\frac{n}{2};\frac{3}{2};\frac{1}{z^2}\right)$$
$$\left.+\Gamma(m-\tfrac{1}{2})\left(1-\frac{1}{z^2}\right)^{\frac{1}{2}-m}\left(\frac{1}{z}\right)F\left(1-\frac{m}{2}-\frac{n}{2},1-\frac{m}{2}+\frac{n}{2};\frac{3}{2}-m;1-\frac{1}{z^2}\right)\right]\quad\text{by (30).}$$

Again transform the second hypergeometric function in the last bracket [ ] by means of (33), then the result will cancel with the fourth one, so getting

$$\int_0^\infty e^{-\lambda/z}\lambda^{m-1}K_n(\lambda)\,d\lambda = \sqrt{(\pi)}\,2^{m-2}\left[E\left(\frac{m+n}{2},\frac{m-n}{2};\frac{1}{2}:e^{\pm i\pi}z^2\right)-\left(\frac{1}{z}\right)E\left(\frac{m+n+1}{2},\frac{1+m-n}{2};\frac{3}{2}:e^{\pm i\pi}z^2\right)\right]$$

where $R(z)>0$, $R(m\pm n)>0$. , ---(36)

But (36) is exactly (27) with $p=q=0$. If in (36) z is replaced by (z/μ), and applying (2) to the L.H.S. and (14) to the R.H.S. repeatedly; then replacing z by (ξz) and applying (3) to the L.H.S. and (29) to the R.H.S. repeatedly; then (27) is obtained.

Corollaries: On replacing λ by (λ/i) and z by $ze^{-i\pi/2}$ in (27), and making use of the formula

$$K_n(t) = i^n G_n(it), \qquad \text{.......(37)}$$

it is found that if $R(m\pm n)>0$, $R(z)>0$

$$i^{n-m}\int_0^\infty \lambda^{m-1}G_n(\lambda)\,E(p;\alpha_r:q;\rho_s:z/\lambda)\,d\lambda =$$

$$2^{\alpha_1+\alpha_2+\cdots+\alpha_p-\rho_1-\rho_2-\cdots-\rho_q+q-p+m-2}\,.\,\pi^{\frac{1}{2}(q-p+1)}$$

$$\times\left[E\left(\frac{m+n}{2},\frac{m-n}{2},\frac{\alpha_1}{2},\ldots,\frac{\alpha_p}{2},\frac{\alpha_1+1}{2},\ldots,\frac{\alpha_p+1}{2};\frac{1}{2},\frac{\rho_1}{2},\ldots,\frac{\rho_q}{2},\frac{\rho_1+1}{2},\ldots,\frac{\rho_q+1}{2}:e^{\pm i\pi}4^{q-p}x^2e^{-i\pi}\right)\right.$$
$$\left.-\frac{2^{p-q}}{ze^{-i\pi/2}}E\left(\frac{m+n+1}{2},\frac{m-n+1}{2},\frac{\alpha_1+1}{2},\ldots,\frac{\alpha_p+1}{2},\frac{\alpha_1+2}{2},\ldots,\frac{\alpha_p+2}{2};\frac{3}{2},\frac{\rho_1+1}{2},\ldots,\frac{\rho_q+1}{2},\frac{\rho_1+2}{2},\ldots,\frac{\rho_q+2}{2}:e^{\pm i\pi}4^{q-p}x^2e^{-i\pi}\right)\right] \qquad ,\text{--(38)}.$$

Similarly, on replacing λ by (λi) and z by $(ze^{i\pi/2})$, it is found that

$$i^{m+n}\int_0^\infty \lambda^{m-1} G_n(\lambda e^{i\pi}) E(p;\alpha_p:q;\rho_s:z/\lambda)\,d\lambda =$$

$$2^{(\alpha_1+\alpha_2+\cdots+\alpha_p-\rho_1-\rho_2-\cdots-\rho_q+q-p+m-2)}\cdot \pi^{\frac{1}{2}(q-p+1)}$$

$$\times\left[E\left(\frac{m+n}{2},\frac{m-n}{2},\frac{\alpha_1}{2},\ldots,\frac{\alpha_p}{2},\frac{\alpha_1+1}{2},\ldots,\frac{\alpha_p+1}{2}:\frac{1}{2},\frac{\rho_1}{2},\ldots,\frac{\rho_q}{2},\frac{\rho_1+1}{2},\ldots,\frac{\rho_q+1}{2}:e^{\pm i\pi}4^{q-p}z^2e^{i\pi}\right)\right.$$
$$\left.-\frac{2^{p-q}}{ze^{i\pi/2}}E\left(\frac{1+m+n}{2},\frac{1+m-n}{2},\frac{\alpha_1+1}{2},\ldots,\frac{\alpha_p+1}{2},\frac{\alpha_1+2}{2},\ldots,\frac{\alpha_p+1}{2}:\frac{3}{2},\frac{\rho_1+1}{2},\ldots,\frac{\rho_q+1}{2},\frac{\rho_1+2}{2},\ldots,\frac{\rho_q+2}{2}:e^{\pm i\pi}4^{q-p}z^2e^{i\pi}\right)\right]$$

where $R(z)>0$, $R(m\pm n)>0$. ,....(39)

Hence on applying the formula

$$\pi i J_n(t) = G_n(t) - i^{2n} G_n(te^{i\pi}), \quad \ldots\ldots\ldots (40)$$

it is found that

$$i\pi\int_0^\infty \lambda^{m-1} J_n(\lambda) E(p;\alpha_p:q;\rho_s:z/\lambda)\,d\lambda =$$

$$2^{(\alpha_1+\alpha_2+\cdots+\alpha_p-\rho_1-\rho_2-\cdots-\rho_q+q-p+m-2)}\,\pi^{\frac{1}{2}(q-p+1)}\,i^{m-n}$$

$$\times\left[E\left(\frac{m+n}{2},\frac{m-n}{2},\frac{\alpha_1}{2},\ldots,\frac{\alpha_p}{2},\frac{\alpha_1+1}{2},\ldots,\frac{\alpha_p+1}{2}:\frac{1}{2},\frac{\rho_1}{2},\ldots,\frac{\rho_q}{2},\frac{\rho_1+1}{2},\ldots,\frac{\rho_q+1}{2}:e^{\pm i\pi}4^{q-p}z^2e^{-i\pi}\right)\right.$$
$$\left.-\frac{2^{p-q}}{ze^{-i\pi/2}}E\left(\frac{m+n+1}{2},\frac{m-n+1}{2},\frac{\alpha_1+1}{2},\ldots,\frac{\alpha_p+1}{2},\frac{\alpha_1+2}{2},\ldots,\frac{\alpha_p+2}{2}:\frac{3}{2},\frac{\rho_1+1}{2},\ldots,\frac{\rho_q+1}{2},\frac{\rho_1+2}{2},\ldots,\frac{\rho_q+2}{2}:e^{\pm i\pi}4^{q-p}z^2e^{-i\pi}\right)\right]$$

$$-2^{(\alpha_1+\alpha_2+\cdots+\alpha_p-\rho_1-\rho_2-\cdots-\rho_q+q-p+m-2)}\cdot\pi^{\frac{1}{2}(q-p+1)}\,i^{n-m}$$

$$\times\left[E\left(\frac{m+n}{2},\frac{m-n}{2},\frac{\alpha_1}{2},\ldots,\frac{\alpha_p}{2},\frac{\alpha_1+1}{2},\ldots,\frac{\alpha_p+1}{2}:\frac{1}{2},\frac{\rho_1}{2},\ldots,\frac{\rho_q}{2},\frac{\rho_1+1}{2},\ldots,\frac{\rho_q+1}{2}:e^{\pm i\pi}4^{q-p}z^2e^{i\pi}\right)\right.$$
$$\left.-\frac{2^{p-q}}{ze^{i\pi/2}}E\left(\frac{m+n+1}{2},\frac{m-n+1}{2},\frac{\alpha_1+1}{2},\ldots,\frac{\alpha_p+1}{2},\frac{\alpha_1+2}{2},\ldots,\frac{\alpha_p+2}{2}:\frac{3}{2},\frac{\rho_1+1}{2},\ldots,\frac{\rho_q+1}{2},\frac{\rho_1+2}{2},\ldots,\frac{\rho_q+2}{2}:e^{\pm i\pi}4^{q-p}z^2e^{+i\pi}\right)\right]$$

,....(41);

where $R(m+n)>0$, $R(z)>0$, $R(\frac{3}{2}-m+\alpha_r)>0$, $r=1,2,3,\ldots,p$.

§ 8. A second integral involving the product of an E-function and a modified Bessel Function of the Second Kind. The formula to be proved is

$$\int_0^\infty \lambda^{k-1} K_n(\lambda) E(p;\alpha_r:q;\rho_s:\lambda^2 z)\,d\lambda =$$

$$2^{k-2}\frac{\pi^2}{\sin(\frac{k+n}{2})\pi\sin(\frac{k-n}{2})\pi}E(p;\alpha_r:1-\frac{k+n}{2},1-\frac{k-n}{2},\rho_1,\ldots,\rho_q:4z)$$

$$+\sum_{n,-n}\frac{2^{-n-2}\pi^2}{\sin(\frac{k+n}{2})\pi\sin n\pi}z^{-(k+n)/2}E(p;\alpha_r+\frac{k+n}{2}:1+\frac{k+n}{2},1+n,\rho_1+\frac{k+n}{2},\ldots,\rho_q+\frac{k+n}{2}:4z), \quad \ldots (42)$$

where $p\geq q+1$, $R(k+n+2\alpha_r)>0$, $r=1,2,3,\ldots,p$ and $|\mathrm{amp}\,z|<\pi$. For other values of p and q the formula holds if the integral is convergent.

The following formula is required in the proof:

$$\int_0^\infty \lambda^{k-1}K_n(\lambda)E(p;\alpha_r:q;\rho_s:z/\lambda^{2m})\,d\lambda =$$

$$(2\pi)^{1-m}2^{k-2}m^{k-1}E[p+m;\alpha_r:q;\rho_s:z/(2m)^{2m}]$$

where m is any positive integer, $R(k\pm n)>0$, $\quad \ldots (43)$

$\alpha_{p+2\nu+1} = (k+n+2\nu)/(2m)$, $\alpha_{p+2\nu+2} = (k-n+2\nu)/(2m)$, $\nu = 0, 1, 2, \ldots m-1$.

[Ragab F. M., Proc. Glasg. Math. Assoc. Vol-I. p. 119, form. (1)].

<u>Proof of the formula</u>: When $p=1$, $q=0$, the integral in (42) becomes

$$\int_0^\infty \lambda^{k-1} K_n(\lambda) E(\alpha_1 :: \lambda^2 z)\, d\lambda = z^{\alpha_1} \int_0^\infty \lambda^{k+2\alpha_1-1} K_n(\lambda) E\{\alpha_1 :: 1/(z\lambda^2)\}\, d\lambda$$

$$= 2^{k+2\alpha_1-2} z^{\alpha_1} E\left(\alpha_1, \alpha_1 + \frac{k+n}{2}, \alpha_1 + \frac{k-n}{2} :: 1/4z\right) \text{ by (43) with } m=1,$$

$$= 2^{k-2} \frac{\pi^2}{\sin\left(\frac{k+n}{2}\right)\pi \sin\left(\frac{k-n}{2}\right)\pi} E\left(\alpha_1 : 1-\frac{k+n}{2}, 1-\frac{k-n}{2} : 4z\right)$$

$$+ \sum_{n,-n} \frac{2^{-n-2}\pi^2}{\sin\left(\frac{k+n}{2}\right)\pi \sin n\pi} z^{-(k+n)/2} E\left(\alpha_1 + \frac{k+n}{2} : 1 + \frac{k+n}{2}, 1+n : 4z\right)$$

by (12).

From this (42) can be derived in the usual way.

<u>Corollaries</u>: In (42) write (λ/i) for λ and $(ze^{i\pi})$ for z, and apply (37), so getting if $R(z) > 0$, $p \geqq q+1$, $R(k \pm n + 2\alpha_r) > 0$, $r = 1, 2, 3, \ldots, p$,

$$i^{n-k}\int_0^\infty \lambda^{k-1} G_n(\lambda) E(p;\alpha_r : q; \rho_s : z\lambda^2)\,d\lambda =$$

$$2^{k-2}\frac{\pi^2}{\sin\left(\frac{k+n}{2}\right)\pi \sin\left(\frac{k-n}{2}\right)\pi} E\left(p;\alpha_r : 1-\frac{k+n}{2}, 1-\frac{k-n}{2}, \rho_1, \ldots, \rho_q : 4e^{i\pi}z\right)$$

$$+\sum_{n,-n}\frac{2^{-n-2}\,\pi^2}{\sin\left(\frac{k+n}{2}\right)\pi \sin(n\pi)}\left(e^{i\pi}z\right)^{-\frac{k+n}{2}} E\left\{\begin{matrix}\alpha_1+\frac{k+n}{2}, \ldots, \alpha_p+\frac{k+n}{2} : 4e^{i\pi}z \\ 1+\frac{k+n}{2}, 1+n, \rho_1+\frac{k+n}{2}, \ldots, \rho_q+\frac{k+n}{2}\end{matrix}\right\}; \quad \cdots (44)$$

Similarly writing (λi) for λ and $(ze^{-i\pi})$ for z, then (42) becomes

$$i^{n+k}\int_0^\infty \lambda^{k-1} G_n(\lambda e^{i\pi}) E(p;\alpha_r : q; \rho_s : z\lambda^2)\,d\lambda =$$

$$2^{k-2}\frac{\pi^2}{\sin\left(\frac{k+n}{2}\right)\pi \sin\left(\frac{k-n}{2}\right)\pi} E\left(p;\alpha_r : 1-\frac{k+n}{2}, 1-\frac{k-n}{2}, \rho_1, \ldots, \rho_q ; 4e^{-i\pi}z\right)$$

$$+\sum_{n,-n}\frac{2^{-n-2}\,\pi^2}{\sin\left(\frac{k+n}{2}\right)\pi \sin n\pi}\left(e^{-i\pi}z\right)^{-\frac{k+n}{2}} E\left\{\begin{matrix}\alpha_1+\frac{k+n}{2}, \ldots, \alpha_p+\frac{k+n}{2} : 4e^{-i\pi}z \\ 1+\frac{k+n}{2}, 1+n, \rho_1+\frac{k+n}{2}, \ldots, \rho_q+\frac{k+n}{2}\end{matrix}\right\}$$

where $R(k\pm n+2\alpha_r)>0$, $r=1,2,3,\ldots,p$, $p\geqslant q+1$, $|\text{amp}\, z|<\pi$. $\quad \cdots (45)$

Hence, on applying (40), it is found that if $p\geqslant q+3$, $R(k+n+2\alpha_r)>0$ $r=1,2,3,\ldots,p$, $R\left(\frac{3}{2}-k\right)>0$, $|\text{amp}\, z|<\pi$

$$i\pi\int_0^\infty \lambda^{k-1} J_n(\lambda)\, E(p;\alpha_r:q;\rho_s:z\lambda^2)\,d\lambda =$$

$$i^{k-n}\left[\frac{2^{k-2}\,\pi^2}{\sin\left(\frac{k+n}{2}\right)\pi\,\sin\left(\frac{k-n}{2}\right)\pi}\, E\left(p;\alpha_r:1-\frac{k+n}{2},1-\frac{k-n}{2},\rho_1,\dots,\rho_q:4e^{i\pi}z\right) + \sum_{n,-n}\frac{2^{-n-2}\,\pi^2}{\sin\left(\frac{k+n}{2}\right)\pi\,\sin(n\pi)}\,(e^{i\pi}z)^{-\frac{k+n}{2}}\, E\left\{\begin{matrix}\alpha_1+\frac{k+n}{2},\dots,\alpha_p+\frac{k+n}{2}:4e^{i\pi}z\\ 1+\frac{k+n}{2},1+n,\rho_1+\frac{k+n}{2},\dots,\rho_q+\frac{k+n}{2}\end{matrix}\right\}\right]$$

$$-\,i^{n-k}\left[\frac{2^{k-2}\,\pi^2}{\sin\left(\frac{k+n}{2}\right)\pi\,\sin\left(\frac{k-n}{2}\right)\pi}\, E\left(p;\alpha_r:1-\frac{k+n}{2},1-\frac{k-n}{2},\rho_1,\dots,\rho_q:4e^{-i\pi}z\right) + \sum_{n,-n}\frac{2^{-n-2}\,\pi^2}{\sin\left(\frac{k+n}{2}\right)\pi\,\sin(n\pi)}\,(e^{-i\pi}z)^{-\frac{k+n}{2}}\, E\left\{\begin{matrix}\alpha_1+\frac{k+n}{2},\dots,\alpha_p+\frac{k+n}{2}:4e^{-i\pi}z\\ 1+\frac{k+n}{2},1+n,\rho_1+\frac{k+n}{2},\dots,\rho_q+\frac{k+n}{2}\end{matrix}\right\}\right], \quad \cdots (46).$$

Now when $p \leqq q+2$ i.e. the E-functions on the R.H.S. of (46) are generalized hypergeometric functions, then (46) can be simplified in the form

$$\int_0^\infty \lambda^{k-1} J_n(\lambda)\, E(p;\alpha_r:q;\rho_s:z\lambda^2)\,d\lambda =$$

$$2^{k-1}\Gamma\left(\frac{k+n}{2}\right)\Gamma\left(1-\frac{k+n}{2}\right) E\left(p;\alpha_r:1-\frac{k+n}{2},1-\frac{k-n}{2},\rho_1,\dots,\rho_q:-4z\right)$$

$$+\,2^{-n-1}\,\Gamma\left(-\frac{k+n}{2}\right)\Gamma\left(1+\frac{k+n}{2}\right) z^{-\frac{k+n}{2}}\, E\left\{\begin{matrix}\alpha_1+\frac{k+n}{2},\dots,\alpha_p+\frac{k+n}{2}:-4z\\ 1+\frac{k+n}{2},1+n,\rho_1+\frac{k+n}{2},\dots,\rho_q+\frac{k+n}{2}\end{matrix}\right\}, \quad \cdots (47),$$

where $|\operatorname{amp} z| < \pi$, $R(\frac{3}{2}-k) > 0$, $R(k+n+2\alpha_r) > 0$ $r = 1, 2, 3, \ldots, p$. If $p = q+1$ or $p = q+2$. If $p \leqq q$ (44) holds provided that the integral is convergent.

Applications: Take $p = q = 0$ in (42) and (44), then they give the following two integrals

$$\int_0^\infty \exp\{-1/(z\lambda^2)\}\lambda^{k-1}K_n(\lambda)\,d\lambda =$$

$$2^{k-2}\Gamma\left(\frac{k+n}{2}\right)\Gamma\left(\frac{k-n}{2}\right){}_0F_2\left(;1-\frac{k+n}{2}, 1-\frac{k-n}{2}; -1/4z\right)$$

$$+\sum_{n,-n} 2^{-n-2}\Gamma\left(-\frac{k+n}{2}\right)\Gamma(-n)\, z^{-\frac{k+n}{2}}\, {}_0F_2\left(;1+\frac{k+n}{2}, 1+n; -1/4z\right), \quad \ldots \ldots (48)$$

where $R(z) > 0$.

Also $$\int_0^\infty \exp\{-1/(z\lambda^2)\}\lambda^{k-1}J_n(\lambda)\,d\lambda =$$

$$2^{k-1}\frac{\Gamma\left(\frac{k+n}{2}\right)}{\Gamma\left(1-\frac{k-n}{2}\right)}{}_0F_2\left(;1-\frac{k+n}{2}, 1-\frac{k-n}{2}; \frac{1}{4z}\right) + 2^{-n-1}\frac{\Gamma\left(-\frac{k+n}{2}\right)}{\Gamma(1+n)} z^{-\frac{k+n}{2}}{}_0F_2\left(;1+\frac{k+n}{2}, 1+n; \frac{1}{4z}\right)$$

$$, \ldots \ldots (49)$$

where $R(\frac{3}{2}-k) > 0$, $R(z) > 0$.

§9. A third integral involving an E-function and a Bessel Function:

The formula to be proved is

$$\int_0^\infty \lambda^{m-1} K_n(\lambda) E(p;\alpha_r:q;\rho_s:z\lambda)\,d\lambda =$$

$$2^{\alpha_1+\alpha_2+\cdots+\alpha_p-\rho_1-\rho_2-\cdots-\rho_q+q-p+m-2}\,\pi^{\frac{1}{2}(q-p-1)}$$

$$\times\left[A E\left\{ \begin{matrix} \frac{\alpha_1}{2},\ldots,\frac{\alpha_p}{2},\frac{\alpha_1+1}{2},\ldots,\frac{\alpha_p+1}{2} : e^{\pm i\pi}4^{2+q-p}z^2 \\ \frac{1}{2}, 1-\frac{m+n}{2}, 1-\frac{m-n}{2}, \frac{\rho_1}{2},\ldots,\frac{\rho_q}{2}, \frac{\rho_1+1}{2},\ldots,\frac{\rho_q+1}{2} \end{matrix} \right\} \right.$$

$$+B\left[\frac{2^{p-q-2}}{z}\right] E\left\{ \begin{matrix} \frac{\alpha_1+1}{2},\ldots,\frac{\alpha_p+1}{2}, \frac{\alpha_1+2}{2},\ldots,\frac{\alpha_p+2}{2} ; e^{\pm i\pi}4^{2+q-p}z^2 \\ \frac{3}{2}, \frac{3}{2}-\frac{m+n}{2}, \frac{3}{2}-\frac{m-n}{2}, \frac{\rho_1+1}{2},\ldots,\frac{\rho_q+1}{2}, \frac{\rho_1+2}{2},\ldots,\frac{\rho_q+2}{2} \end{matrix} \right\}$$

$$\left. +\sum_{n,-n} C_n \left[\frac{2^{p-q-2}}{z}\right]^{m+n} E\left\{ \begin{matrix} \frac{\alpha_1+m+n}{2},\ldots,\frac{\alpha_p+m+n}{2}, \frac{\alpha_1+m+n+1}{2},\ldots,\frac{\alpha_p+m+n+1}{2} : e^{\pm i\pi}4^{2+q-p}z^2 \\ 1+\frac{m+n}{2}, \frac{1}{2}+\frac{m+n}{2}, 1+n, \frac{\rho_1+m+n}{2},\ldots,\frac{\rho_q+m+n}{2}, \frac{\rho_1+m+n+1}{2},\ldots,\frac{\rho_q+m+n+1}{2} \end{matrix} \right\} \right]$$

$$, \quad \cdots (50)$$

where

$$A \equiv \{\Gamma(\tfrac{1}{2})\}^2 \Gamma(\tfrac{m+n}{2})\Gamma(1-\tfrac{m+n}{2})\Gamma(\tfrac{m-n}{2})\Gamma(1-\tfrac{m-n}{2}),$$

$$B \equiv \Gamma(-\tfrac{1}{2})\Gamma(\tfrac{3}{2})\Gamma(\tfrac{m+n}{2}-\tfrac{1}{2})\Gamma(\tfrac{3}{2}-\tfrac{m+n}{2})\Gamma(\tfrac{m-n}{2}-\tfrac{1}{2})\Gamma(\tfrac{3}{2}-\tfrac{m-n}{2}),$$

$$C_n \equiv \Gamma(-\tfrac{m+n}{2})\Gamma(1+\tfrac{m+n}{2})\Gamma(\tfrac{1}{2}-\tfrac{m+n}{2})\Gamma(\tfrac{1}{2}+\tfrac{m+n}{2})\Gamma(-n)\Gamma(1+n),$$ and also

$R(m\pm n+\alpha_r)>0$, $r=1,2,3,\ldots,p$, $p\geqslant q+1$, $|\mathrm{amp}\, z|<\pi$. For other values of $\underline{p}$ and $\underline{q}$, (50) is valid if the integral is convergent.

<u>Proof of the formula</u>: Consider first the special case of (50) with

$p=1, q=0$ and $\alpha_1=\alpha$, then the L.H.S. of (50) becomes if $\alpha_1=\alpha$

$$\int_0^\infty \lambda^{m-1} K_n(\lambda) E(\alpha::z\lambda)\,d\lambda = z^\alpha \int_0^\infty \lambda^{m+\alpha-1} K_n(\lambda) E(\alpha::1/z\lambda)\,d\lambda$$

$$= 2^{\alpha+m-3}(2z)^\alpha \left[\begin{array}{l} E\left(\frac{\alpha+m+n}{2}, \frac{\alpha+m-n}{2}, \frac{\alpha}{2}, \frac{\alpha+1}{2} ; \frac{1}{2} : e^{\pm i\pi} \times 1/4z^2\right) \\ -(2z) E\left(\frac{\alpha+m+n+1}{2}, \frac{\alpha+m-n+1}{2}, \frac{\alpha+1}{2}, \frac{\alpha+2}{2} ; \frac{3}{2} : e^{\pm i\pi}\frac{1}{4z^2}\right) \end{array} \right] \text{by (27).}$$

where $R(z)>0$, $R(m\pm n+\alpha)>0$.

From (12), so that the expression becomes if $R(m\pm n+\alpha)>0$, $|\operatorname{amp} z|<\pi$

$$\int_0^\infty \lambda^{m-1} K_n(\lambda) E(\alpha::z\lambda)\,d\lambda =$$

$$\frac{2^{\alpha+m-3}}{\pi} A\left[e^{\pm i\pi\left(\frac{\alpha}{2}\right)} \sin\left(\frac{\alpha+1}{2}\right)\pi - e^{\pm i\pi\left(\frac{\alpha+1}{2}\right)} \sin\frac{\pi\alpha}{2}\right] E\left\{\begin{array}{l} \frac{\alpha}{2}, \frac{\alpha+1}{2} : e^{\pm i\pi} 4z^2 \\ 1-\frac{m+n}{2}, 1-\frac{m-n}{2}, \frac{1}{2} \end{array}\right\}$$

$$+\frac{2^{\alpha+m-3}}{\pi} B\left(\frac{1}{2z}\right)\left[e^{\pm i\pi\left(\frac{\alpha}{2}\right)} \sin\left(\frac{\alpha+1}{2}\right)\pi - e^{\pm i\pi\left(\frac{\alpha+1}{2}\right)} \sin\frac{\pi\alpha}{2}\right] E\left\{\begin{array}{l} \frac{\alpha+1}{2}, \frac{\alpha+2}{2} : e^{\pm i\pi} 4z^2 \\ \frac{3}{2}-\frac{m+n}{2}, \frac{3}{2}-\frac{m-n}{2}, \frac{3}{2} \end{array}\right\}$$

$$+\frac{2^{\alpha+m-3}}{\pi} \sum_{n,-n} C_n \left(\frac{1}{2z}\right)^{m+n} \left[e^{\pm i\pi\left(\frac{m+n+\alpha}{2}\right)} \sin\left(\frac{\alpha+m+n+1}{2}\right)\pi - e^{\pm i\pi\left(\frac{1+m+n+\alpha}{2}\right)} \sin\left(\frac{\alpha+m+n}{2}\right)\pi \right]$$

$$\times E\left(\frac{\alpha+m+n}{2}, \frac{\alpha+m+n+1}{2} ; 1+\frac{m+n}{2}, \frac{1}{2}+\frac{m+n}{2}, 1+n : e^{\pm i\pi} 4z^2\right).$$

But since $e^{\pm i\pi\frac{x}{2}} \sin\left(\frac{x+1}{2}\right)\pi - e^{\pm i\pi\left(\frac{x+1}{2}\right)} \sin\left(\frac{\pi x}{2}\right) = 1$,

then the last expression becomes if $R(z)>0$, $R(m\pm n+\alpha)>0$,

$$\int_0^\infty \lambda^{m-1} K_n(\lambda) E(\alpha::\lambda z)\,d\lambda = 2^{\alpha+m-3}\pi^{-1} A E\left(\frac{\alpha}{2}, \frac{\alpha+1}{2} : 1-\frac{m+n}{2}, 1-\frac{m-n}{2}, \frac{1}{2} : e^{\pm i\pi} 4z^2\right)$$

$$+ 2^{\alpha+m-3}\pi^{-1} B\left(\frac{1}{2z}\right) E\left(\frac{\alpha+1}{2}, \frac{\alpha+2}{2} : \frac{3}{2}-\frac{m+n}{2}, \frac{3}{2}-\frac{m-n}{2}, \frac{3}{2} : e^{\pm i\pi} 4z^2\right)$$

$$+ 2^{\alpha+m-3}\pi^{-1} \sum_{n,-n} C_n \left(\frac{1}{2z}\right)^{m+n} E\left(\frac{\alpha+m+n}{2}, \frac{\alpha+m+n+1}{2} : 1+\frac{m+n}{2}, \frac{1}{2}+\frac{m+n}{2}, 1+n : e^{\pm i\pi} 4z^2\right).$$

From this (50) can be derived by replacing z in the last expression by (z/μ) and applying (17) repeatedly, and then replacing z by ξz and applying (29) repeatedly in the same manner as was of formula (27) second proof.

Corollaries: On replacing λ by (λ/i) in (50) and z by $z\,e^{i\pi/2}$ and applying (34), it is found that

$$i^{n-m}\int_0^\infty \lambda^{m-1} G_n(\lambda) E(p;\alpha_r:q;\rho_s: z\lambda)\,d\lambda =$$

$$2^{\alpha_1+\alpha_2+\cdots+\alpha_p-\rho_1-\rho_2-\cdots-\rho_q+q-p+m-2}\,\pi^{\frac{1}{2}(q-p-1)}$$

$$\left[A\, E\left\{ \begin{matrix} \frac{\alpha_1}{2}, \ldots, \frac{\alpha_p}{2}, \frac{\alpha_1+1}{2}, \ldots, \frac{\alpha_p+1}{2} \\ \frac{1}{2}, 1-\frac{m+n}{2}, 1-\frac{m-n}{2}, \frac{\rho_1}{2}, \ldots, \frac{\rho_q}{2}, \frac{\rho_1+1}{2}, \ldots, \frac{\rho_q+1}{2} \end{matrix} : e^{\pm i\pi} 4^{2+q-p} z^2 e^{i\pi} \right\} \right.$$

$$+ B\left[\frac{2^{p-q-2}}{z\,e^{i\pi/2}}\right] E\left\{ \begin{matrix} \frac{\alpha_1+1}{2}, \ldots, \frac{\alpha_p+1}{2}, \frac{\alpha_1+2}{2}, \ldots, \frac{\alpha_p+2}{2} \\ \frac{3}{2}, \frac{3}{2}-\frac{m+n}{2}, \frac{3}{2}-\frac{m-n}{2}, \frac{\rho_1+1}{2}, \ldots, \frac{\rho_q+1}{2}, \frac{\rho_1+2}{2}, \ldots, \frac{\rho_q+2}{2} \end{matrix} : e^{\pm i\pi} 4^{2+q-p} z^2 e^{i\pi} \right\}$$

$$\left. + \sum_{n,-n} C_n \left[\frac{2^{p-q-2}}{z e^{i\pi/2}}\right]^{m+n} E\left\{ \begin{matrix} \frac{\alpha_1+m+n}{2}, \ldots, \frac{\alpha_p+m+n}{2}, \frac{\alpha_1+m+n+1}{2}, \ldots, \frac{\alpha_p+m+n+1}{2} \\ 1+\frac{m+n}{2}, \frac{1}{2}+\frac{m+n}{2}, 1+n, \frac{\rho_1+m+n}{2}, \ldots, \frac{\rho_q+m+n}{2}, \frac{\rho_1+m+n+1}{2}, \ldots, \frac{\rho_q+m+n+1}{2} \end{matrix} : e^{\pm i\pi} 4^{2+q-p} z^2 e^{i\pi} \right\} \right], \quad \cdots (51)$$

where $R(m \pm n + \alpha_r) > 0$, $r = 1, 2, 3, \ldots, p$, $|\mathrm{amp}\, z| < \pi$, $p \geqq q+1$

Similarly on replacing λ by (λi) and z by $ze^{-i\pi/2}$, it is found that

$$i^{m+n}\int_0^\infty \lambda^{m-1} G_m(\lambda e^{i\pi}) E(p;\alpha_r:q;\rho_s:z\lambda)\,d\lambda =$$

$$2^{\alpha_1+\alpha_2+\cdots+\alpha_p-\rho_1-\rho_2-\cdots-\rho_q+q-p+m-2}\,\pi^{\frac{1}{2}(q-p-1)}$$

$$\times\left[\begin{array}{l} A E\left\{\begin{matrix}\frac{\alpha_1}{2},\ldots,\frac{\alpha_p}{2},\frac{\alpha_1+1}{2},\ldots,\frac{\alpha_p+1}{2} : e^{\pm i\pi} 4^{2+q-p} z^2 e^{-i\pi} \\ \frac{1}{2}, 1-\frac{m+n}{2}, 1-\frac{m-n}{2}, \frac{\rho_1}{2},\ldots,\frac{\rho_q}{2},\frac{\rho_1+1}{2},\ldots,\frac{\rho_q+1}{2}\end{matrix}\right\} \\ +B\left[\frac{2^{p-q-2}}{ze^{-i\pi/2}}\right] E\left\{\begin{matrix}\frac{\alpha_1+1}{2},\ldots,\frac{\alpha_p+1}{2},\frac{\alpha_1+2}{2},\ldots,\frac{\alpha_p+2}{2} : e^{\pm i\pi} 4^{2+q-p} z^2 e^{-i\pi} \\ \frac{3}{2},\frac{3}{2}-\frac{m+n}{2},\frac{3}{2}-\frac{m-n}{2},\frac{\rho_1+1}{2},\ldots,\frac{\rho_q+1}{2},\frac{\rho_1+2}{2},\ldots,\frac{\rho_q+2}{2}\end{matrix}\right\} \\ +\sum_{n,-n} C_n\left[\frac{2^{p-q-2}}{ze^{-i\pi/2}}\right]^{m+n} E\left\{\begin{matrix}\frac{\alpha_1+m+n}{2},\ldots,\frac{\alpha_p+m+n}{2},\frac{\alpha_1+m+n+1}{2},\ldots,\frac{\alpha_p+m+n+1}{2} : e^{\pm i\pi} 4^{2+q-p} z^2 e^{-i\pi} \\ 1+\frac{m+n}{2},\frac{1}{2}+\frac{m+n}{2},1+n,\frac{\rho_1+m+n}{2},\ldots,\frac{\rho_q+m+n}{2},\frac{\rho_1+m+n+1}{2},\ldots,\frac{\rho_q+m+n+1}{2}\end{matrix}\right\}\end{array}\right], \quad \cdots (52)$$

where $R(m\pm n+\alpha_r)>0$, $r=1,2,3,\ldots,p$, $|\text{amp}\, z|<\pi$, $p \geqq q+1$.

Hence on applying (40) it is found that if $p \geqq q+2$, $|\text{amp}\, z|<\pi$, $R(m+n+\alpha_r)>0$, $r=1,2,3,\ldots,p$, $R\left(\frac{3}{2}-m\right)$

$$i\pi\int_0^\infty \lambda^{m-1} J_n(\lambda)\, E(p; \alpha_r : q; \rho_s : z\lambda)\, d\lambda =$$

$$2^{\alpha_1+\alpha_2+\cdots+\alpha_p-\rho_1-\rho_2-\cdots-\rho_q+q-p+m-2}\cdot \pi^{\frac{1}{2}(q-p-1)}\cdot i^{m-n}$$

$$\times\Bigg[A E\left\{ \begin{matrix} \frac{\alpha_1}{2}, \ldots, \frac{\alpha_p}{2}, \frac{\alpha_1+1}{2}, \ldots, \frac{\alpha_p+1}{2} : e^{\pm i\pi} 4^{2+q-p} z^2 e^{i\pi} \\ \frac{1}{2}, 1-\frac{m+n}{2}, 1-\frac{m-n}{2}, \frac{\rho_1}{2}, \ldots, \frac{\rho_q}{2}, \frac{\rho_1+1}{2}, \ldots, \frac{\rho_q+1}{2} \end{matrix} \right\}$$

$$+ B\left[\frac{2^{p-q-2}}{z e^{i\pi/2}}\right] E\left\{ \begin{matrix} \frac{\alpha_1+1}{2}, \ldots, \frac{\alpha_p+1}{2}, \frac{\alpha_1+2}{2}, \ldots, \frac{\alpha_p+2}{2} : e^{\pm i\pi} 4^{2+q-p} z^2 e^{i\pi} \\ \frac{3}{2}, \frac{3}{2}-\frac{m+n}{2}, \frac{3}{2}-\frac{m-n}{2}, \frac{\rho_1+1}{2}, \ldots, \frac{\rho_q+1}{2}, \frac{\rho_1+2}{2}, \ldots, \frac{\rho_q+2}{2} \end{matrix} \right\}$$

$$+ \sum_{n,-n} C_n \left[\frac{2^{p-q-2}}{z e^{i\pi/2}}\right]^{m+n} E\left\{ \begin{matrix} \frac{\alpha_1+m+n}{2}, \ldots, \frac{\alpha_p+m+n}{2}, \frac{\alpha_1+m+n+1}{2}, \ldots, \frac{\alpha_p+m+n+1}{2} : e^{\pm i\pi} 4^{2+q-p} z^2 e^{i\pi} \\ 1+\frac{m+n}{2}, \frac{1}{2}+\frac{m+n}{2}, 1+n, \frac{\rho_1+m+n}{2}, \ldots, \frac{\rho_q+m+n}{2}, \frac{\rho_1+m+n+1}{2}, \ldots, \frac{\rho_q+m+n+1}{2} \end{matrix} \right\} \Bigg]$$

$$- 2^{\alpha_1+\alpha_2+\cdots+\alpha_p-\rho_1-\rho_2-\cdots-\rho_q+q-p+m-2}\cdot \pi^{\frac{1}{2}(q-p-1)}\cdot i^{n-m}$$

$$\times\Bigg[A E\left\{ \begin{matrix} \frac{\alpha_1}{2}, \ldots, \frac{\alpha_p}{2}, \frac{\alpha_1+1}{2}, \ldots, \frac{\alpha_p+1}{2} : e^{\pm i\pi} 4^{2+q-p} z^2 e^{-i\pi} \\ \frac{1}{2}, 1-\frac{m+n}{2}, 1-\frac{m-n}{2}, \frac{\rho_1}{2}, \ldots, \frac{\rho_q}{2}, \frac{\rho_1+1}{2}, \ldots, \frac{\rho_q+1}{2} \end{matrix} \right\}$$

$$+ B\left[\frac{2^{p-q-2}}{z e^{-i\pi/2}}\right] E\left\{ \begin{matrix} \frac{\alpha_1+1}{2}, \ldots, \frac{\alpha_p+1}{2}, \frac{\alpha_1+2}{2}, \ldots, \frac{\alpha_p+2}{2} : e^{\pm i\pi} 4^{2+q-p} z^2 e^{-i\pi} \\ \frac{3}{2}, \frac{3}{2}-\frac{m+n}{2}, \frac{3}{2}-\frac{m-n}{2}, \frac{\rho_1+1}{2}, \ldots, \frac{\rho_q+1}{2}, \frac{\rho_1+2}{2}, \ldots, \frac{\rho_q+2}{2} \end{matrix} \right\}$$

$$+ \sum_{n,-n} C_n \left[\frac{2^{p-q-2}}{z e^{-i\pi/2}}\right]^{m+n} E\left\{ \begin{matrix} \frac{\alpha_1+m+n}{2}, \ldots, \frac{\alpha_p+m+n}{2}, \frac{\alpha_1+m+n+1}{2}, \ldots, \frac{\alpha_p+m+n+1}{2} : e^{\pm i\pi} 4^{2+q-p} z^2 e^{-i\pi} \\ 1+\frac{m+n}{2}, \frac{1}{2}+\frac{m+n}{2}, 1+n, \frac{\rho_1+m+n}{2}, \ldots, \frac{\rho_q+m+n}{2}, \frac{\rho_1+m+n+1}{2}, \ldots, \frac{\rho_q+m+n+1}{2} \end{matrix} \right\} \Bigg], \ldots \quad (53)$$

When $2p \leqq 2q+3$ i.e when the E-functions on the R.H.S. of (53) are generalized hypergeometric function, then (53) can be simplified in the form

$$\int_0^\infty \lambda^{m-1} J_n(\lambda) E(p; \alpha_r : q; \rho_s : z\lambda)\, d\lambda =$$

$$2^{\alpha_1+\alpha_2+\cdots+\alpha_p-\rho_1-\rho_2-\cdots-\rho_q+q-p+m-1} \cdot \pi^{\frac{1}{2}(q-p-1)}$$

$$\left[\begin{aligned} &\pi\,\Gamma\left(\frac{m+n}{2}\right)\Gamma\left(1-\frac{m+n}{2}\right) E\left\{ \begin{matrix} \frac{\alpha_1}{2}, \ldots, \frac{\alpha_p}{2}, \frac{\alpha_1+1}{2}, \ldots, \frac{\alpha_p+1}{2} : 4^{2+q-p} z^2 \\ \frac{1}{2}, 1-\frac{m+n}{2}, 1-\frac{m-n}{2}, \frac{\rho_1}{2}, \ldots, \frac{\rho_q}{2}, \frac{\rho_1+1}{2}, \ldots, \frac{\rho_q+1}{2} \end{matrix} \right\} \\ &-\pi\,\Gamma\left(\frac{m+n-1}{2}\right)\Gamma\left(\frac{3}{2}-\frac{m+n}{2}\right)\left[\frac{2^{p-q-2}}{z}\right] E\left\{ \begin{matrix} \frac{\alpha_1+1}{2}, \ldots, \frac{\alpha_p+1}{2}, \frac{\alpha_1+2}{2}, \ldots, \frac{\alpha_p+2}{2} : 4^{2+q-p} z^2 \\ \frac{3}{2}, \frac{3}{2}-\frac{m+n}{2}, \frac{3}{2}-\frac{m-n}{2}, \frac{\rho_1+1}{2}, \ldots, \frac{\rho_q+1}{2}, \frac{\rho_1+2}{2}, \ldots, \frac{\rho_q+2}{2} \end{matrix} \right\} \\ &+\Gamma\left(-\frac{m+n}{2}\right)\Gamma\left(1+\frac{m+n}{2}\right)\Gamma\left(\frac{1}{2}-\frac{m+n}{2}\right)\Gamma\left(\frac{1}{2}+\frac{m+n}{2}\right)\left[\frac{2^{p-q-2}}{z}\right]^{m+n} \\ &\times E\left\{ \begin{matrix} \frac{\alpha_1+m+n}{2}, \ldots, \frac{\alpha_p+m+n}{2}, \frac{\alpha_1+m+n+1}{2}, \ldots, \frac{\alpha_p+m+n+1}{2} : 4^{2+q-p} z^2 \\ 1+\frac{m+n}{2}, \frac{1}{2}+\frac{m+n}{2}, 1+n, \frac{\rho_1+m+n}{2}, \ldots, \frac{\rho_q+m+n}{2}, \frac{\rho_1+m+n+1}{2}, \ldots, \frac{\rho_q+m+n+1}{2} \end{matrix} \right\} \end{aligned} \right], \quad \ldots\ldots (54)$$

where $R(m+n+\alpha_r)>0$ $r=1,2,3,\ldots,p$, $R\left(\frac{3}{2}-m\right)$ when $p=q+1$. When $p \leqq q$, (54) is valid if the integral converges.

§ 10. An E-function integral: The formula to be proved is

$$\int_0^\infty e^{-\lambda} \lambda^{k-1} E(p; \alpha_r : q; \rho_s : z/\sqrt{\lambda})\, d\lambda = 2^{\alpha_1+\alpha_2+\cdots+\alpha_p-\rho_1-\rho_2-\cdots-\rho_q+q-p} \cdot \pi^{\frac{1}{2}(q-p+1)}$$

$$\left[\begin{aligned} &E\left(k, \frac{\alpha_1}{2}, \ldots, \frac{\alpha_p}{2}, \frac{\alpha_1+1}{2}, \ldots, \frac{\alpha_p+1}{2} : \frac{1}{2}, \frac{\rho_1}{2}, \ldots, \frac{\rho_q}{2}, \frac{\rho_1+1}{2}, \ldots, \frac{\rho_q+1}{2} : e^{\pm i\pi} 4^{1+q-p} z^2\right) \\ &-\left[\frac{2^{p-q-1}}{z}\right] E\left(k+\frac{1}{2}, \frac{\alpha_1+1}{2}, \ldots, \frac{\alpha_p+1}{2}, \frac{\alpha_1+2}{2}, \ldots, \frac{\alpha_p+2}{2} : \frac{3}{2}, \frac{\rho_1+1}{2}, \ldots, \frac{\rho_q+1}{2}, \frac{\rho_1+2}{2}, \ldots, \frac{\rho_q+2}{2} : e^{\pm i\pi} 4^{1+q-p} z^2\right) \end{aligned} \right], \ldots (55),$$

where $R(k)>0$.

The following formula will be required in the proof:—

$$\int_0^\infty \exp\left\{-\mu/(\sqrt{2}\cdot z) - \tfrac{1}{2}\mu^2\right\}\mu^{2k-1}\,d\mu = \frac{2^{3k-1}\,z^{2k}}{\sqrt{\pi}}\,E\left\{k, k+\tfrac{1}{2} :: 1/4z^2\right\}, \quad\ldots (56)$$

where $R(k) > 0$.

[(56) is formula (17) with $p = q = 0$ and $n = 2$. Also obtained by E. T. Whittaker, Proc. Lond. Math. Soc. (1), (35), 1903, pp. 417–427].

<u>Proof of the formula</u>: The L.H.S. of (55) with $p = q = 0$ is equal to

$$\int_0^\infty e^{-\sqrt{\lambda}/z - \lambda}\lambda^{k-1}\,d\lambda = 2^{1-k}\int_0^\infty \exp\left\{-\mu/(\sqrt{2}\cdot z) - \tfrac{1}{2}\mu\right\}\mu^{k-1}\,d\mu$$

$$= \frac{(2z)^{2k}}{\sqrt{\pi}}E\left\{k, k+\tfrac{1}{2} :: 1/4z^2\right\} \text{ by } (56),$$

$$= \sqrt{\pi}\left[E(k : \tfrac{1}{2} : e^{\pm i\pi} 4z^2) - 1/(2z)\,E(k+\tfrac{1}{2} : \tfrac{3}{2} : e^{\pm i\pi} 4z^2)\right] \text{ by } (12).$$

From this (55) can be derived in the same manner as formula (50).

<u>§11. A second E-function integral</u>: The formula to be proved is

$$\int_0^\infty e^{-\lambda}\lambda^{k-1}E(p;\alpha_r:q;\rho_s:z\sqrt{\lambda})\,d\lambda = 2^{\alpha_1+\alpha_2+\cdots+\alpha_p-\rho_1-\rho_2-\cdots-\rho_q+q-p}\cdot\pi^{\frac{1}{2}(q-p-1)}$$

$$\times\left[\begin{array}{l} \pi\Gamma(k)\Gamma(1-k)E\left(\frac{\alpha_1}{2},\ldots,\frac{\alpha_p}{2},\frac{\alpha_1+1}{2},\ldots,\frac{\alpha_p+1}{2} : \frac{1}{2}, 1-k, \frac{\rho_1}{2},\ldots,\frac{\rho_q}{2},\frac{\rho_1+1}{2},\ldots,\frac{\rho_q+1}{2} : 4^{1+q-p}z^2\right) \\ -\pi\Gamma\left(\frac{k}{2}-\frac{1}{2}\right)\Gamma\left(\frac{3}{2}-\frac{k}{2}\right)\left[\frac{2^{p-q-1}}{z}\right]E\left(\frac{\alpha_1+1}{2},\ldots,\frac{\alpha_p+1}{2},\frac{\alpha_1+2}{2},\ldots,\frac{\alpha_p+2}{2} : \frac{3}{2},\frac{3}{2}-k,\frac{\rho_1+1}{2},\ldots,\frac{\rho_q+1}{2},\frac{\rho_1+2}{2},\ldots,\frac{\rho_q+2}{2} : 4^{1+q-p}z^2\right) \\ +\Gamma(-k)\Gamma(1+k)\Gamma\left(\frac{1}{2}+k\right)\Gamma\left(\frac{1}{2}-k\right)\left[\frac{4^{p-q-1}}{z^2}\right]^k \\ \times E\left\{\begin{array}{l} k+\frac{\alpha_1}{2},\ldots,k+\frac{\alpha_p}{2},k+\frac{\alpha_1+1}{2},\ldots,k+\frac{\alpha_p+1}{2} : 4^{1+q-p}z^2 \\ 1+k, \frac{1}{2}+k, k+\frac{\rho_1}{2},\ldots,k+\frac{\rho_q}{2},k+\frac{\rho_1+1}{2},\ldots,k+\frac{\rho_q+1}{2} \end{array}\right\} \end{array}\right]$$

$$, \quad\ldots (57)$$

where $R(k+\frac{1}{2}\alpha_r)>0$, $r=1,2,3,\dots,p$, $|\mathrm{amp}\, z|<\pi$, $p \geqq q+1$. For other values of p and q, the formula holds provided that the integral is convergent.

To prove (54), consider the special case $q=0$, $p=1$ with $\alpha_1=\alpha$, then the L.H.S. of (54) becomes if $R(k+\frac{\alpha}{2})>0$, $|\mathrm{amp}\, z|<\pi$.

$$\int_0^\infty e^{-\lambda}\lambda^{k-1}E(\alpha::z\sqrt{\lambda})\,d\lambda = 2^{\alpha-1}\pi^{-1}\Gamma(-k)\Gamma(1+k)\Gamma(\tfrac{1}{2}-k)\Gamma(\tfrac{1}{2}+k)(1/z^2)^k$$
$$\times\left[e^{\pm i\pi(k+\frac{\alpha}{2})}\sin(k+\tfrac{\alpha+1}{2})\pi - e^{\pm i\pi(k+\frac{\alpha+1}{2})}\sin(k+\tfrac{\alpha}{2})\pi\right]E\left\{\begin{matrix}k+\frac{\alpha}{2}, k+\frac{\alpha+1}{2} : z^2\\ 1+k, \frac{1}{2}+k\end{matrix}\right\}$$
$$+2^{\alpha-1}\pi^{-1}\{\Gamma(\tfrac{1}{2})\}^2\Gamma(k)\Gamma(1-k)\left[e^{\pm i\pi\frac{\alpha}{2}}\sin(\tfrac{\alpha+1}{2})\pi - e^{\pm i\pi(\frac{\alpha+1}{2})}\sin\tfrac{\pi\alpha}{2}\right]E\left\{\begin{matrix}\frac{\alpha}{2}, \frac{\alpha+1}{2} : z^2\\ 1-k, \frac{1}{2}\end{matrix}\right\}$$
$$-2^{\alpha-1}\left[\tfrac{1}{z}\right]\Gamma(k-\tfrac{1}{2})\Gamma(\tfrac{3}{2}-k)\left[e^{\pm i\pi\frac{\alpha}{2}}\sin(\tfrac{\alpha+1}{2})\pi - e^{\pm i\pi(\frac{\alpha+1}{2})}\sin\tfrac{\pi\alpha}{2}\right]E\left\{\begin{matrix}\frac{\alpha+1}{2}, \frac{\alpha+2}{2} : z^2\\ \frac{3}{2}, \frac{3}{2}-k\end{matrix}\right\}.$$

But since $e^{\pm i\pi\frac{x}{2}}\sin(\frac{x+1}{2})\pi - e^{\pm i\pi(\frac{x+1}{2})}\sin\frac{\pi x}{2} = 1$,

then the last expression becomes

$$\int_0^\infty e^{-\lambda}\lambda^{k-1}E\{\alpha::z\sqrt{\lambda}\}\,d\lambda = 2^{\alpha-1}\Gamma(k)\Gamma(1-k)E(\tfrac{\alpha}{2},\tfrac{\alpha+1}{2}:\tfrac{1}{2},1-k:z^2)$$
$$-2^{\alpha-1}\Gamma(k-\tfrac{1}{2})\Gamma(\tfrac{3}{2}-k)(1/z)E(\tfrac{\alpha+1}{2},\tfrac{\alpha+2}{2}:\tfrac{3}{2},\tfrac{3}{2}-k:z^2)$$
$$+2^{\alpha-1}\pi^{-1}\Gamma(-k)\Gamma(1+k)\Gamma(\tfrac{1}{2}-k)\Gamma(\tfrac{1}{2}+k)(1/z^2)^k E(k+\tfrac{\alpha}{2},k+\tfrac{\alpha+1}{2}:1+k,\tfrac{1}{2}+k:z^2),$$

where $R(k+\frac{1}{2}\alpha)>0$, $|\mathrm{amp}\, z|<\pi$.

This is a special case of formula (54) and the general case can be deduced in the same manner as was that of formula (50).

§12. An integral involving the product of two E-functions expressed as a sum of three E-functions. The formula to be proved is

$$\int_0^\infty e^{-\lambda}\lambda^{k-1}E(\gamma,\delta::\lambda)E(p;\alpha_r:q;\rho_s:z\lambda)\,d\lambda =$$

$$\frac{\pi\,\Gamma(\gamma)\Gamma(\delta)\sin(\gamma+\delta+k)\pi}{\sin(\delta+k)\pi\,\sin(\gamma+k)\pi}E\left\{\begin{matrix}\alpha_1,\dots,\alpha_p,1-\gamma-\delta-k : e^{\pm i\pi}z\\ 1-\gamma-k,1-\delta-k,\rho_1,\dots,\rho_q\end{matrix}\right\}$$

$$-\sum_{\gamma,\delta}\frac{\pi\,\Gamma(\gamma)\Gamma(\delta)\sin(\delta\pi)}{\sin(\delta-\gamma)\pi\,\sin(\gamma+k)\pi}z^{-\gamma-k}E\left\{\begin{matrix}\alpha_1+\gamma+k,\dots,\alpha_p+\gamma+k,1-\delta : e^{\pm i\pi}z\\ 1+\gamma+k,1+\gamma-\delta,\rho_1+\gamma+k,\dots,\rho_q+\gamma+k\end{matrix}\right\}, \quad\dots(58)$$

where $R(\gamma+k+\alpha_r)>0$, $R(\delta+k+\alpha_r)>0$, $r=1,2,3,\dots,p$, $p\geqslant q+1$, $|\mathrm{amp}\,z|<\pi$. For other values of p and q, the formula holds provided that the integral is convergent.

The following formula is required in the proof

$$\int_0^\infty e^{-nt}\,t^{nk-1}E(n\gamma,n\delta::nt)E(p;\alpha_r:q;\rho_s:z/t^n)\,dt =$$

$$\frac{\Gamma(n\gamma)\Gamma(n\delta)}{(2\pi)^{\frac{1}{2}n-\frac{1}{2}}\sqrt{n}}E(p+2n;\alpha_r:q+n;\rho_s:z), \quad\dots\dots\dots(59)$$

where n is any positive integer and $\alpha_{p+2\nu+1}=\gamma+k+\nu/n$, $\alpha_{p+2\nu+2}=\delta+k+\nu/n$, $\rho_{q+\nu+1}=\gamma+\delta+k+\nu/n$, $\nu=0,1,2,\dots,n-1$, $R(k+\gamma)>0$, $R(\delta+k)>0$,

[Ragab, F. M. Proc. of the Glasg. Math. Assoc. Vol. I., p. 134 equ. (1)]

To prove (58), consider the special case with $p=1$ and $q=0$, then we have

$$\int_0^\infty e^{-\lambda}\lambda^{k-1}E(\gamma,\delta::\lambda)E(\alpha::z\lambda)\,d\lambda = z^\alpha\int_0^\infty e^{-\lambda}\lambda^{k+\alpha-1}E(\gamma,\delta::\lambda)E(\alpha::1/z\lambda)\,d\lambda$$

$$=\Gamma(\gamma)\Gamma(\delta)z^\alpha E(\alpha,\alpha+\gamma+k,\alpha+\delta+k;\alpha+\gamma+\delta+k:1/z)$$ by (59) with $n=1$, where $R(\gamma+k+\alpha)>0$, $R(\delta+k+\alpha)>0$, $|\operatorname{amp} z|<\pi$.

On applying (12), this becomes (58) with $p=1$, $q=0$. The general case can be obtained in the usual way.

Corollaries: In (58) take $p=q=0$, and get if $R(z)>0$

$$\int_0^\infty \exp\{-\lambda-1/z\lambda\}\lambda^{k-1}E(\gamma,\delta::\lambda)\,d\lambda =$$

$$\frac{\Gamma(\gamma)\Gamma(\delta)\Gamma(\gamma+k)\Gamma(\delta+k)}{\Gamma(\gamma+\delta+k)}\,{}_1F_2(1-\gamma-\delta-k;1-\gamma-k,1-\delta-k;1/z)$$

$$+\sum_{\gamma,\delta}\Gamma(\gamma)\Gamma(1-\gamma-k)\Gamma(\delta-\gamma)z^{-\gamma-k}\,{}_1F_2(1-\delta;1+\gamma+k,1+\gamma-\delta;1/z), \quad \ldots\ldots (60)$$

The particular case of (60) when $k=-\gamma-\delta$ gives if $R(z)>0$

$$\int_0^\infty e^{-\lambda-1/z\lambda}\lambda^{-\gamma-\delta-1}E(\gamma,\delta::\lambda)\,d\lambda = 2\Gamma(\gamma)\Gamma(\delta)z^{\frac{1}{2}\gamma+\frac{1}{2}\delta}K_{\gamma-\delta}(2\sqrt{z}), \quad \ldots (61)$$

(61) was proved by another method by me [Proc. Glasg. Math. Assoc. Vol. I, p. 135 formula (1).

A generalization of formula (58). If in the proof of (58), the general formula (59) is used instead of being used with $n=1$, then on applying (12) and (18); the following generalization of (58) is obtained.

$$\int_0^\infty e^{-\lambda}\lambda^{k-1}E(\gamma,\delta::\lambda)\,E(p;\alpha_r:q;\rho_s:z\lambda^m)\,d\lambda =$$

$$\frac{\pi^{\frac{1}{2}m+\frac{1}{2}}\, m^{k-\frac{1}{2}}\,\Gamma(\gamma)\Gamma(\delta)\sin(\gamma+\delta+k)\pi}{2^{\frac{1}{2}-\frac{1}{2}m}\sin(\gamma+k)\pi\,\sin(\delta+k)\pi}$$

$$\times E\left\{\begin{matrix}\alpha_1,\ldots,\alpha_p,1-\frac{\gamma+\delta+k}{m},1-\frac{\gamma+\delta+k+1}{m},\ldots,1-\frac{\gamma+\delta+k+m-1}{m} : e^{\pm i m\pi} m^m z\\ 1-\frac{\gamma+k}{m},1-\frac{\gamma+k+1}{m},\ldots,1-\frac{\gamma+k+m-1}{m},1-\frac{\delta+k}{m},1-\frac{\delta+k+1}{m},\ldots,1-\frac{\delta+k+m-1}{m},\rho_1,\ldots,\rho_q\end{matrix}\right\}$$

$$+\, 2^{\frac{1}{2}-\frac{1}{2}m}\,\pi^{\frac{1}{2}+\frac{1}{2}m}\,\Gamma(\gamma)\Gamma(\delta)$$

$$\times\sum_{\gamma,\delta}\left[\sum_{\nu=0}^{m-1}\frac{(-1)^{\nu+1}\prod_{\mu=0}^{m-1}\sin\left(\frac{\delta+\mu-\nu}{m}\right)\pi\; m^{-\frac{1}{2}-\gamma-\nu}\, z^{-\frac{1}{m}(k+\gamma+\nu)}}{\sin\left(\frac{\gamma+k+\nu}{m}\right)\pi\prod_{s=1}^{\nu}\sin\frac{s\pi}{m}\prod_{t=1}^{m-\nu-1}\sin\frac{t\pi}{m}\prod_{\mu=0}^{m-1}\sin\left(\frac{\delta+\mu-\gamma-\nu}{m}\right)\pi}\right.$$

$$\times E\left\{\begin{matrix}\alpha_1+\frac{\gamma+k+\nu}{m},\ldots,\alpha_p+\frac{\gamma+k+\nu}{m},1+\frac{\nu-\delta}{m},1+\frac{\nu-\delta-1}{m},\ldots\ldots\\ 1+\frac{\gamma+k+\nu}{m},1+\frac{1}{m},1+\frac{2}{m},\ldots,1+\frac{\nu}{m},1-\frac{1}{m},1-\frac{2}{m},\ldots,1-\frac{m-\nu-1}{m},\ldots\ldots\end{matrix}\right.$$

$$\left.\left.\begin{matrix}\ldots\ldots,1+\frac{\nu-\delta-m+1}{m} : e^{\pm i m\pi} m^m z\\ \ldots\ldots,1+\frac{\gamma+\nu-\delta}{m},1+\frac{\gamma+\nu-\delta-1}{m},\ldots,1+\frac{\gamma+\nu-\delta-m+1}{m},\rho_1+\frac{\gamma+k+\nu}{m},\ldots,\rho_q+\frac{\gamma+k+\nu}{m}\end{matrix}\right\}\right],\quad\text{---- (62)}$$

where $R(\gamma+k+m\alpha_r)>0$, $R(\delta+k+m\alpha_r)>0$, $r=1,2,3,\dots,p$. $|\mathrm{amp}\, z|<\pi$, $p \geqq q+1$, and m is any positive integer. For other values of p, and q the formula holds provided that the integral is convergent.

Applications: Many particular cases can be deduced from (62). For instance if in (62), $p=q=0$, the value of the integral

$$\int_0^\infty \exp\{-\lambda-1/(z\lambda^m)\}\lambda^{k-1}E(\gamma,\delta::\lambda)\,d\lambda,$$

where $R(z)>0$, is found to be the R.H.S. of (62) with all the linear expressions involving $\alpha_1,\alpha_2,\dots,\alpha_p, \rho_1,\rho_2,\dots,\rho_q$ omitted.

§ 13. Integrals involving a simple type of E-function: The first formula to be proved is

$$\int_0^\infty e^{-\lambda^2}\lambda^{k-1}E(\gamma,\delta::\lambda^2)E(p;\alpha_r:q;\rho_s:z/\lambda)\,d\lambda =$$

$$2^{\alpha_1+\alpha_2+\dots+\alpha_p-\rho_1-\rho_2-\dots-\rho_q+q-p-1}\cdot\pi^{\frac{1}{2}(q-p+1)}\,\Gamma(\gamma)\Gamma(\delta)$$

$$\times\left[E\left\{\begin{matrix}\frac{\alpha_1}{2},\dots,\frac{\alpha_p}{2},\frac{\alpha_1+1}{2},\dots,\frac{\alpha_p+1}{2},\gamma+\frac{1}{2}k,\delta+\frac{1}{2}k : e^{\pm i\pi}4^{1+q-p}z^2\\ \frac{1}{2},\frac{\rho_1}{2},\dots,\frac{\rho_q}{2},\frac{\rho_1+1}{2},\dots,\frac{\rho_q+1}{2},\gamma+\delta+\frac{1}{2}k\end{matrix}\right\}\right.$$

$$\left.-\left[\frac{2^{p-q-1}}{z}\right]E\left\{\begin{matrix}\frac{\alpha_1+1}{2},\dots,\frac{\alpha_p+1}{2},\frac{\alpha_1+2}{2},\dots,\frac{\alpha_p+2}{2},\frac{1}{2}+\gamma+\frac{1}{2}k,\frac{1}{2}+\delta+\frac{1}{2}k : e^{\pm i\pi}4^{1+q-p}z^2\\ \frac{3}{2},\frac{\rho_1+1}{2},\dots,\frac{\rho_q+1}{2},\frac{\rho_1+2}{2},\dots,\frac{\rho_q+2}{2},\frac{1}{2}+\gamma+\delta+\frac{1}{2}k\end{matrix}\right\}\right]$$

where $R(k+2\gamma)>0$, $R(k+2\delta)>0$, $|\mathrm{amp}\, z|<\pi$, $\quad\dots(63)$

The following formula is required in the proof:-

$$E(\gamma,\delta::\lambda^2)=\Gamma(\gamma)\int_0^\infty e^{-\mu}\mu^{\delta-1}\left(1+\frac{\mu}{\lambda^2}\right)^{-\gamma}d\mu, \quad \ldots\ldots \quad (64)$$

[G.V., p. 348] where $R(\delta)>0$.

To prove (63) consider the special case when $p=q=0$, then the L.H.S. of (63) becomes if $R(\gamma+2k)>0$, $R(\delta+2k)>0$, $R(z)>0$

$$\Gamma(\gamma)\int_0^\infty e^{-\lambda^2-\lambda/z}\lambda^{k-1}d\lambda\int_0^\infty e^{-\mu}\mu^{\delta-1}\left(1+\frac{\mu}{\lambda^2}\right)^{-\gamma}d\mu \quad \text{by (64).}$$

Here put $\mu=\lambda^2\xi$ and change the order of integration, so getting

$$\Gamma(\gamma)\int_0^\infty \xi^{\delta-1}(1+\xi)^{-\gamma}d\xi\int_0^\infty e^{-\lambda^2(1+\xi)-\lambda/z}\lambda^{k+2\delta-1}d\lambda$$

$$=\Gamma(\gamma)\int_0^\infty \xi^{\delta-1}(1+\xi)^{-\gamma-\delta-\frac{1}{2}k}d\xi\int_0^\infty e^{-\lambda^2-\frac{\lambda}{z\sqrt{(1+\xi)}}}\lambda^{k+2\delta-1}d\lambda$$

$$=\frac{\Gamma(\gamma)(2z)^{k+2\delta}}{2\sqrt{\pi}}\int_0^\infty \xi^{\delta-1}(1+\xi)^{-\gamma}E\left\{\delta+\frac{k}{2},\delta+\frac{k+1}{2}::\frac{1}{4z^2(1+\xi)}\right\}d\xi \quad \text{by (56).}$$

Now write $(1-\xi)/\xi$ for ξ and evaluate this last integral by means of (23), so getting if $R(k+2\gamma)>0$, $R(k+2\delta)>0$, $R(z)>0$

$$\int_0^\infty e^{-\lambda^2-\lambda/z}\lambda^{k-1}E(\gamma,\delta::\lambda^2)\,d\lambda =$$

$$\frac{\Gamma(\gamma)\Gamma(\delta)\sin(\pi\gamma)}{2\sqrt{\pi}\sin(\gamma-\delta)\pi}(2z)^{k+2\delta}E\left\{\begin{matrix}\delta+\frac{k}{2},\delta+\frac{k+1}{2},1-\gamma;\frac{1}{4z^2}\\ 1+\delta-\gamma\end{matrix}\right\}$$

$$-\frac{\Gamma(\gamma)\Gamma(\delta)\sin(\pi\delta)}{2\sqrt{\pi}\sin(\gamma-\delta)\pi}(2z)^{k+2\gamma}E\left\{\begin{matrix}\gamma+\frac{k}{2},\gamma+\frac{k+1}{2},1-\delta;\frac{1}{4z^2}\\ 1+\gamma-\delta\end{matrix}\right\}.$$

Now expand each E-function on the R.H.S. of the last expression by means of (12), add, then two series will cancel, and so getting

$$\int_0^\infty e^{-\lambda^2-\lambda/z}\lambda^{k-1}E(\gamma,\delta::\lambda^2)\,d\lambda =$$

$$\frac{\sqrt{\pi}}{2}\Gamma(\gamma)\Gamma(\delta)\left[\frac{\sin(\gamma+\frac{1}{2}k)\pi\sin(\gamma\pi)-\sin(\delta+\frac{1}{2}k)\pi\sin(\delta\pi)}{\sin(\gamma+\delta+\frac{1}{2}k)\pi\sin(\gamma-\delta)\pi}\right]$$

$$\times E\left(\gamma+\tfrac{1}{2}k,\delta+\tfrac{1}{2}k:\tfrac{1}{2},\gamma+\delta+\tfrac{1}{2}k:e^{\pm i\pi}4z^2\right)$$

$$-\frac{\sqrt{\pi}}{2}\Gamma(\gamma)\Gamma(\delta)\left[\frac{\sin(\gamma+\frac{1}{2}k+\frac{1}{2})\pi\sin(\gamma\pi)-\sin(\delta+\frac{1}{2}k+\frac{1}{2})\pi\sin(\delta\pi)}{\sin(\frac{1}{2}+\gamma+\delta+\frac{1}{2}k)\pi\sin(\gamma-\delta)\pi}\right].$$

$$\times E\left(\gamma+\frac{k+1}{2},\delta+\frac{k+1}{2}:\frac{3}{2},\frac{1}{2}+\gamma+\delta+\frac{1}{2}k:e^{\pm i\pi}4z^2\right),$$

where $R(k+2\gamma)>0$, $R(k+2\delta)>0$, $R(z)>0$.

But since the each quantity between brackets in the last expression {[ ] brackets}; then formula (63) with $p=q=0$

is obtained. The general case can be deduced in the usual way as formula (50) by making repeated applications of (17) and (29).

<u>A second formula</u>: The formula to be proved is

$$2^{\rho_1+\rho_2+\cdots+\rho_q-\alpha_1-\alpha_2-\cdots-\alpha_p+p-q+1}\cdot\pi^{\frac{1}{2}(p-q-3)}$$

$$\times\int_0^\infty e^{-\lambda^2}\lambda^{k-1}E(\gamma,\delta::\lambda^2)E(p;\alpha_r:q;\rho_s:z\lambda)\,d\lambda =$$

$$\frac{\Gamma(\gamma)\Gamma(\delta)\sin(\gamma+\delta+\frac{k}{2})\pi}{\sin(\gamma+\frac{1}{2}k)\pi\sin(\delta+\frac{1}{2}k)\pi}E\left\{\begin{matrix}\frac{\alpha_1}{2},\dots,\frac{\alpha_p}{2},\frac{\alpha_1+1}{2},\dots,\frac{\alpha_p+1}{2},1-\gamma-\delta-\frac{1}{2}k:4^{1+q-p}z^2\\ \frac{1}{2},1-\gamma-\frac{1}{2}k,1-\delta-\frac{1}{2}k,\frac{\rho_1}{2},\dots,\frac{\rho_q}{2},\frac{\rho_1+1}{2},\dots,\frac{\rho_q+1}{2}\end{matrix}\right\}$$

$$+\frac{\Gamma(\gamma)\Gamma(\delta)\sin(\frac{1}{2}-\gamma-\delta-\frac{k}{2})\pi}{\cos(\gamma+\frac{1}{2}k)\pi\cos(\delta+\frac{1}{2}k)\pi}\left[\frac{2^{p-q-1}}{z}\right]E\left\{\begin{matrix}\frac{\alpha_1+1}{2},\dots,\frac{\alpha_p+1}{2},\frac{\alpha_1+2}{2},\dots,\frac{\alpha_p+2}{2},\frac{3}{2}-\gamma-\delta-\frac{k}{2}:4^{1+q-p}z^2\\ \frac{3}{2},\frac{3}{2}-\gamma-\frac{1}{2}k,\frac{3}{2}-\delta-\frac{k}{2},\frac{\rho_1+1}{2},\dots,\frac{\rho_q+1}{2},\frac{\rho_1+2}{2},\dots,\frac{\rho_q+2}{2}\end{matrix}\right\}$$

$$-\sum_{\gamma,\delta}\left[\frac{2\Gamma(\gamma)\Gamma(\delta)\sin\delta\pi}{\sin(k+2\gamma)\pi\sin(\delta-\gamma)\pi}\left[\frac{4^{p-q-1}}{z^2}\right]^{\gamma+\frac{1}{2}k}\right.$$

$$\left.\times E\left\{\begin{matrix}\gamma+\frac{k}{2}+\frac{\alpha_1}{2},\dots,\gamma+\frac{k}{2}+\frac{\alpha_p}{2},\gamma+\frac{k}{2}+\frac{\alpha_1+1}{2},\dots,\gamma+\frac{k}{2}+\frac{\alpha_p+1}{2},1-\delta:4^{1+q-p}z^2\\ 1+\gamma+\frac{k}{2},\frac{1}{2}+\gamma+\frac{k}{2},1+\gamma-\delta,\gamma+\frac{k}{2}+\frac{\rho_1}{2},\dots,\gamma+\frac{k}{2}+\frac{\rho_q}{2},\gamma+\frac{k}{2}+\frac{\rho_1+1}{2},\dots,\gamma+\frac{k}{2}+\frac{\rho_q+1}{2}\end{matrix}\right\}\right]$$

where $R(k+2\gamma+\alpha_r)>0$, $R(k+2\delta+\alpha_r)>0$, $r=1,2,3,\dots,p$, $\quad\dots(65)$

$p \geq q+1$. For other values of p and q, the formula holds provided that the integral is convergent.

To prove (65), consider the special case with $p=1$, $q=0$; then the L.H.S. of (65) becomes if $\alpha_1=\alpha$ and $R(k+2\gamma+\alpha)>0$, $R(k+2\delta+\alpha)>0$, $|\mathrm{amp}\, z|<\pi$

$$2^{2-\alpha}\pi^{-1}\int_0^\infty e^{-\lambda^2}\lambda^{k-1}E(\gamma,\delta::\lambda^2)E(\alpha::z\lambda)\,d\lambda=$$

$$2^{2-\alpha}\pi^{-1}z^{\alpha}\int_0^\infty e^{-\lambda^2}\lambda^{k+\alpha-1}E(\gamma,\delta::\lambda^2)E(\alpha::1/z\lambda)\,d\lambda=$$

$$\pi^{-1}\Gamma(\gamma)\Gamma(\delta)z^{\alpha}E\left(\frac{\alpha}{2},\frac{\alpha+1}{2},\gamma+\frac{k}{2}+\frac{\alpha}{2},\delta+\frac{k}{2}+\frac{\alpha}{2}:\frac{1}{2},\gamma+\delta+\frac{k}{2}+\frac{\alpha}{2}:\frac{e^{\pm i\pi}}{z^2}\right)$$

$$-\pi^{-1}\Gamma(\gamma)\Gamma(\delta)z^{(\alpha+1)}E\left(\frac{\alpha+1}{2},\frac{\alpha+2}{2},\gamma+\frac{k}{2}+\frac{\alpha+1}{2},\delta+\frac{k}{2}+\frac{\alpha+1}{2}:\frac{3}{2},\gamma+\delta+\frac{k}{2}+\frac{\alpha+1}{2}:\frac{e^{\pm i\pi}}{z^2}\right)$$

by (63).

Now expand each E-function on the R.H.S. of the last expression by means of (12), add and get

$$2^{2-\alpha}\pi^{-1}\int_0^\infty e^{-\lambda^2}\lambda^{k-1}E(\gamma,\delta::\lambda^2)E(\alpha::z\lambda)\,d\lambda=$$

$$\frac{\Gamma(\gamma)\Gamma(\delta)\sin(\gamma+\delta+\tfrac{1}{2}k)\pi}{\sin(\gamma+\tfrac{1}{2}k)\pi\sin(\delta+\tfrac{1}{2}k)\pi}\left[e^{\pm\frac{i\pi\alpha}{2}}\sin\left(\frac{\alpha+1}{2}\right)\pi - e^{\pm i\pi\left(\frac{\alpha+1}{2}\right)}\sin\frac{\pi\alpha}{2}\right]$$

$$\times E\left(\frac{\alpha}{2},\frac{\alpha+1}{2},1-\gamma-\delta-\tfrac{1}{2}k:\tfrac{1}{2},1-\gamma-\tfrac{1}{2}k,1-\delta-\tfrac{1}{2}k:z^2\right)$$

$$+\frac{\Gamma(\gamma)\Gamma(\delta)\sin\left(\frac{1}{2}-\gamma-\delta-\frac{k}{2}\right)\pi}{\cos(\gamma+\tfrac{1}{2}k)\pi\cos(\delta+\tfrac{1}{2}k)\pi}\left(\frac{1}{z}\right)\left[e^{\pm i\pi\frac{\alpha}{2}}\sin\left(\frac{\alpha+1}{2}\right)\pi - e^{\pm i\pi\left(\frac{\alpha+1}{2}\right)}\sin\left(\frac{\pi\alpha}{2}\right)\right]$$

$$\times E\left(\frac{\alpha+1}{2},\frac{\alpha+2}{2},\frac{3}{2}-\gamma-\delta-\tfrac{1}{2}k:\frac{3}{2},\frac{3}{2}-\gamma-\tfrac{1}{2}k,\frac{3}{2}-\delta-\tfrac{1}{2}k;z^2\right)$$

$$-\sum_{\gamma,\delta}\left[\frac{2\Gamma(\gamma)\Gamma(\delta)\sin(\delta\pi)}{\sin(k+2\gamma)\pi\sin(\delta-\gamma)\pi}\left(\frac{1}{z^2}\right)^{\gamma+\frac{k}{2}}\right.$$

$$\times\left[e^{\pm i\pi\left(\gamma+\frac{k+\alpha}{2}\right)}\sin\left(\gamma+\frac{k+\alpha+1}{2}\right)\pi - e^{\pm i\pi\left(\gamma+\frac{k+\alpha+1}{2}\right)}\sin\left(\gamma+\frac{k+\alpha}{2}\right)\pi\right]$$

$$\left.\times E\left\{\begin{matrix}\frac{\alpha}{2}+\gamma+\frac{k}{2},\ \gamma+\frac{k}{2}+\frac{\alpha+1}{2},\ 1-\delta\ :\ z^2\\ 1+\gamma+\frac{k}{2},\ \frac{1}{2}+\gamma+\frac{k}{2},\ 1+\gamma-\delta\end{matrix}\right\}\right]$$

where $R(k+\alpha+2\gamma)>0$, $R(k+\alpha+2\delta)>0$, $|\text{amp}\, z|<\pi$.

But since $e^{\pm i\pi\frac{x}{2}}\sin\left(\frac{x+1}{2}\right)\pi - e^{\pm i\pi\left(\frac{x+1}{2}\right)}\sin\frac{\pi x}{2} = 1$

then each quantity in the last expression between brackets [ ] is equal to unity. Thus (65) is obtained with $p=1$, $q=0$. The general case can be deduced in the usual way as (63)

§14. Integrals involving the product of an E-function and a Modified Bessel Functions of the First Kind: The first formula to be proved is

$$\sqrt{\pi}\cos(m\pi)\int_0^\infty e^{-\mu} I_n(\mu)\,\mu^{m-1} E(p;\alpha_r:q;\rho_s:z/\mu)\,d\mu =$$

$$2^{-m}\sin(m-n)\pi\, E\left(\alpha_1,\ldots,\alpha_p, m+n, m-n : \tfrac{1}{2}+m, \rho_1,\ldots,\rho_q : 2z\right)$$

$$-\,2^{-1/2}\cos(n\pi)\,z^{m-\frac{1}{2}} E\left\{\begin{matrix}\alpha_1+\frac{1}{2}-m,\ldots,\alpha_p+\frac{1}{2}-m, \frac{1}{2}+n, \frac{1}{2}-n : 2z\\ \frac{3}{2}-m, \rho_1+\frac{1}{2}-m,\ldots,\rho_q+\frac{1}{2}-m\end{matrix}\right\}, \quad \ldots (66)$$

where $R(n+m)>0$, $R(\alpha_r-m+\frac{1}{2})>0$, $r=1,2,3,\ldots,p$, $|\mathrm{amp}\, z|<\pi$.

The following formula is required in the proof:

$$\sqrt{(2\pi)}\cos(m\pi)\Gamma(\alpha)\int_0^\infty e^{-\mu} I_{n+\frac{1}{2}}(\mu)\,\mu^{m-\frac{1}{2}}(z+\mu)^{-\alpha}\,d\mu =$$

$$\sin(m-n)\pi\cdot 2^{-m} z^{-\alpha} E(m+n+1, m-n, \alpha : m+1 : 2z)$$

$$-\sin(n\pi)\,z^{m-\alpha} E(-n, n+1, \alpha-m : 1-m : 2z), \quad \ldots (67)$$

where $R(z)>0$, $R(m+n)>-1$, $R(\alpha-m)>0$ [B.V., p. 379, ex. 103].

Proof of the formula: If in (64), $(m-\frac{1}{2})$, $(n-\frac{1}{2})$ are written for m and n respectively, then (64) becomes (66) with $p=1$, $q=0$. The general case of (66) can then be deduced in the usual way.

The second formula to be proved is

$$\frac{\sin(n+m)\pi}{\sqrt{\pi}} 2^m \int_0^\infty e^{-\mu} I_n(\mu)\, \mu^{m-1} E(p;\alpha_r : q; \rho_s : z\mu)\, d\mu =$$

$$E(\alpha_1,\dots,\alpha_p, \tfrac{1}{2}-m : 1-n-m, 1+n-m, \rho_1,\dots,\rho_q : e^{\pm i\pi}\tfrac{z}{2})$$

$$-(2/z)^{m+n} E\left\{\begin{matrix}\alpha_1+m+n,\dots,\alpha_p+m+n, \frac{1}{2}+n : e^{\pm i\pi}(\frac{z}{2})\\ 1+n+m, 1+2n, \rho_1+n+m,\dots,\rho_q+n+m\end{matrix}\right\}, \quad \dots (68),$$

where $R(n+m+\alpha_r)>0$, $r=1,2,3,\dots,p$, $R(\frac{1}{2}-m)>0$, $|\operatorname{amp} z|<\pi$ and $p \geqq q+1$. For other values of p and q the formula holds if the integral is convergent.

To prove (68) consider the special case with $p=1$, $q=0$; then the L.H.S. becomes if $\alpha_1=\alpha$

$$2^m \frac{\sin(n+m)\pi \cdot z^\alpha}{\sqrt{\pi}} \int_0^\infty e^{-\mu} I_n(\mu)\, \mu^{m+\alpha-1} E(\alpha :: 1/z\mu)\, d\mu =$$

$$\frac{\sin(n+m)\pi \sin(m+\alpha-n)\pi}{\pi\cos(m+\alpha)\pi}(z/2)^\alpha E(\alpha, m+n+\alpha, m-n+\alpha : \tfrac{1}{2}+m+\alpha : 2/z)$$

$$-\frac{\cos(n\pi)\sin(n+m)\pi}{\pi\cos(m+\alpha)\pi}(z/2)^{\frac{1}{2}-m} E(\tfrac{1}{2}-m, \tfrac{1}{2}+n, \tfrac{1}{2}-n : \tfrac{3}{2}-m-\alpha : 2/z) \text{ by (66).}$$

where $R(m+n+\alpha)>0$, $R(\frac{1}{2}-m)>0$, $|\operatorname{amp} z|<\pi$.

Now expand each E-function on the R.H.S. of the last expression by means of (12), add, then two series will cancel so getting if $R(m+n+\alpha)>0$, $R(\frac{1}{2}-m)>0$, $|\operatorname{amp} z|<\pi$

$$2^m \frac{\sin(m+n)\pi}{\sqrt{\pi}} \int_0^\infty e^{-\mu} I_n(\mu)\, \mu^{m-1} E(\alpha :: z\mu)\, d\mu =$$

$$\left[\frac{\sin(m+\alpha-n)\pi \cos m\pi - \sin(\alpha\pi)\cos(n\pi)}{\sin(m-n)\pi \cdot \cos(m+\alpha)\pi}\right] E\left(\tfrac{1}{2}-m, \alpha : 1+n-m, 1-n-m : e^{\pm i\pi}\cdot\left(\tfrac{z}{2}\right)\right)$$

$$-\left[\frac{\cos n\pi \sin(m+\alpha-n)\pi - \cos n\pi \sin(m+\alpha+n)\pi}{-\cos(m+\alpha)\pi \sin(2n\pi)}\right]\left(\frac{2}{z}\right)^{m+n} E\left\{\begin{matrix}\tfrac{1}{2}+n, m+\alpha+n : e^{\pm i\pi}\cdot\left(\tfrac{z}{2}\right) \\ 1+m+n, 1+2n\end{matrix}\right\}.$$

Since each quantity in the last expression between [ ] is equal to unity: then (68) is obtained with $p=1$, $q=0$. The general case can be obtained in the usual way.

The third formula to be proved is

$$\int_0^\infty e^{-\mu} I_n(\mu)\, \mu^{m-1} E(p; \alpha_r : q; \rho_s : z/\mu^2)\, d\mu =$$

$$\frac{1}{2\sqrt{2}\cdot\pi} \frac{\sin(m-n)\pi}{\cos(m\pi)} E\left\{ \begin{matrix} \alpha_1, \ldots, \alpha_p, \frac{m+n}{2}, \frac{m+n+1}{2}, \frac{m-n}{2}, \frac{m-n+1}{2} ; z \\ \frac{3}{4}+\frac{m}{2}, \frac{1}{4}+\frac{m}{2}, \rho_1, \ldots, \rho_q \end{matrix} \right\}$$

$$+ \frac{1}{4\sqrt{2}\cdot\pi} \cdot \frac{\cos(n\pi)}{\sin(\frac{m}{2}-\frac{1}{4})\pi} z^{\frac{m}{2}-\frac{1}{4}} E\left\{ \begin{matrix} \alpha_1+\frac{1}{4}-\frac{m}{2}, \ldots, \alpha_p+\frac{1}{4}-\frac{m}{2}, \frac{1}{4}+\frac{n}{2}, \frac{3}{4}+\frac{n}{2}, \frac{1}{4}-\frac{n}{2}, \frac{3}{4}-\frac{n}{2} : z \\ \frac{5}{4}-\frac{m}{2}, \frac{1}{2}, \rho_1+\frac{1}{4}-\frac{m}{2}, \ldots, \rho_q+\frac{1}{4}-\frac{m}{2} \end{matrix} \right\}$$

$$- \frac{1}{4\sqrt{2}\cdot\pi} \cdot \frac{\cos(n\pi)}{\sin(\frac{3}{4}-\frac{m}{2})\pi} z^{\frac{m}{2}-\frac{3}{4}} E\left\{ \begin{matrix} \alpha_1+\frac{3}{4}-\frac{m}{2}, \ldots, \alpha_p+\frac{3}{4}-\frac{m}{2}, \frac{3}{4}+\frac{n}{2}, \frac{5}{4}+\frac{n}{2}, \frac{3}{4}-\frac{n}{2}, \frac{5}{4}-\frac{n}{2} : z \\ \frac{7}{4}-\frac{m}{2}, \frac{3}{2}, \rho_1+\frac{3}{4}-\frac{m}{2}, \ldots, \rho_q+\frac{3}{4}-\frac{m}{2} \end{matrix} \right\}$$

where $R(n+m)>0$, $R(2\alpha_r - m + \frac{1}{2})>0$, $r=1,2,3,\ldots,p$, $|\operatorname{amp} z|<\pi$. ... (69)

The following formula is required in the proof [E.V., p. 268] namely

$$I_n(\mu) = \frac{1}{\Gamma(\frac{1}{2})\Gamma(n+\frac{1}{2})} \left(\frac{\mu}{2}\right)^n \int_{-1}^{+1} e^{-\mu\lambda}(1-\lambda^2)^{n-\frac{1}{2}}\, d\lambda, \qquad \ldots\ldots (70)$$

where $R(n+\frac{1}{2})>0$.

To prove (69) consider the special case with $p=q=0$. Substitute for $I_n(\mu)$ from (70) and change the order of integration. Thus the L.H.S. of (69) becomes if $R(m+n)>0$, $R(z)>0$

$$\int_0^\infty e^{-\mu-\mu^2/z}\mu^{m-1} I_n(\mu)\,d\mu = \frac{2^{-n}}{\Gamma(\tfrac12)\Gamma(n+\tfrac12)}\int_{-1}^{+1}(1-\lambda^2)^{n-\frac12}d\lambda\int_0^\infty e^{-\mu(1+\lambda)-\mu^2/z}\mu^{m+n-1}d\mu$$

$$= \frac{2^{m-1}}{\sqrt{\pi}\,\Gamma(\tfrac12)\Gamma(n+\tfrac12)}\int_{-1}^{+1}(1-\lambda)^{n-\frac12}(1+\lambda)^{-m-\frac12}E\left\{\frac{m+n}{2}, \frac{m+n+1}{2} :: \frac{z(1+\lambda)^2}{4}\right\} \text{ by } (56).$$

In this last integral write $(2\lambda-1)$ for $\underline{\lambda}$, and then evaluate it by means of (20) and so obtain (69) with $p=q=0$. The general case can then be deduced in the usual way.

The fourth formula to be proved is

$$\int_0^\infty e^{-\mu}\mu^{m-1} I_n(\mu) E(p;\alpha_r:q;\rho_s:z\mu^2)\,d\mu =$$

$$\frac{\pi}{\sqrt{2}\cdot\sin(m+n)\pi}E\left\{\begin{matrix}\alpha_1,\dots,\alpha_p, \frac14-\frac{m}{2}, \frac34-\frac{m}{2} : z\\ 1-\frac{m+n}{2}, \frac12-\frac{m+n}{2}, \rho_1,\dots,\rho_q, 1+\frac{n-m}{2}, \frac12+\frac{n-m}{2}\end{matrix}\right\}$$

$$-\frac{\pi}{2\sqrt{2}\sin\left(\frac{m+n}{2}\right)\pi}z^{-\frac{m+n}{2}}E\left\{\begin{matrix}\alpha_1+\frac{m+n}{2},\dots,\alpha_p+\frac{m+n}{2}, \frac14+\frac{n}{2}, \frac34+\frac{n}{2} : z\\ 1+\frac{m+n}{2}, \frac12, \rho_1+\frac{m+n}{2},\dots,\rho_q+\frac{m+n}{2}, 1+n, \frac12+n\end{matrix}\right\}$$

$$+\frac{\pi}{2\sqrt{2}\cos\left(\frac{m+n}{2}\right)\pi}z^{-\frac{m+n+1}{2}}E\left\{\begin{matrix}\alpha_1+\frac{m+n+1}{2},\dots,\alpha_p+\frac{m+n+1}{2}, \frac34+\frac{n}{2}, \frac54+\frac{n}{2} : z\\ 1+\frac{m+n+1}{2}, \frac32, \rho_1+\frac{m+n+1}{2},\dots,\rho_q+\frac{m+n+1}{2}, \frac32+n, 1+n\end{matrix}\right\}$$

$$, \dots\dots (71)$$

where $R(n+m+2\alpha_r)>0$, $r=1,2,3,\dots,p$., $R(\frac{1}{2}-m)>0$, $|\operatorname{amp} z|<\pi$, $p \geqslant q+1$. For other values of p and q, the formula holds provided that the integral is convergent.

<u>Proof of the formula.</u> To prove (71) consider the special case when $p=1$, $q=0$, then the L.H.S. becomes if $\alpha_1=\alpha$, $R(m+n+2\alpha)>0$, $R(\frac{1}{2}-m)>0$, $|\operatorname{amp} z|<\pi$

$$\int_0^\infty e^{-\mu} I_n(\mu)\,\mu^{m-1} E(\alpha::z\mu^2)\,d\mu = z^\alpha \int_0^\infty e^{-\mu} I_n(\mu)\,\mu^{m+2\alpha-1} E(\alpha::1/z\mu^2)\,d\mu$$

$$= \frac{\sin(m+2\alpha-n)\pi}{2\sqrt{2}\cdot\pi\cos(m+2\alpha)\pi}\, z^\alpha E\left\{\begin{matrix}\alpha+\frac{m+n}{2},\ \alpha+\frac{m+n+1}{2},\ \alpha+\frac{m-n}{2},\ \alpha+\frac{m-n+1}{2},\ \alpha : \frac{1}{z}\\ \frac{3}{4}+\frac{m}{2}+\alpha,\ \frac{1}{4}+\frac{m}{2}+\alpha\end{matrix}\right\}$$

$$+ \frac{\cos n\pi}{4\sqrt{2}\cdot\pi\sin(\frac{m}{2}+\alpha-\frac{1}{4})\pi}\, z^{-(\frac{m}{2}-\frac{1}{4})} E\left\{\begin{matrix}\frac{1}{4}-\frac{m}{2},\ \frac{1}{4}+\frac{n}{2},\ \frac{3}{4}+\frac{n}{2},\ \frac{1}{4}-\frac{n}{2},\ \frac{3}{4}-\frac{n}{2} : \frac{1}{z}\\ \frac{5}{4}-\frac{m}{2}-\alpha,\ \frac{1}{2}\end{matrix}\right\}$$

$$- \frac{\cos n\pi}{4\sqrt{2}\cdot\pi\sin(\frac{3}{4}-\frac{m}{2}-\alpha)\pi}\, z^{-(\frac{m}{2}-\frac{3}{4})} E\left\{\begin{matrix}\frac{3}{4}-\frac{m}{2},\ \frac{3}{4}+\frac{n}{2},\ \frac{5}{4}+\frac{n}{2},\ \frac{3}{4}-\frac{n}{2},\ \frac{5}{4}-\frac{n}{2} : \frac{1}{z}\\ \frac{7}{4}-\frac{m}{2}-\alpha,\ \frac{3}{2}\end{matrix}\right\}$$

by (69).

Now expand each E-function on the R.H.S. of the last expression by means of (12), then after some simplification the last two terms will vanish and (71) is obtained with $p=1$, $q=0$. The general case can then be deduced in the usual way.

§ 15. Integrals involving the product of two Bessel Functions and an E-function: The first formula to be proved is

$$\int_0^\infty x^{\rho-1} K_\nu(x) K_\mu(x) E(p; \alpha_r : q; \rho_s : z/x^2)\, dx =$$

$$\frac{\sqrt{\pi}}{4} E\left\{ \begin{matrix} \alpha_1, \ldots, \alpha_p, \frac{\rho+\nu+\mu}{2}, \frac{\rho+\nu-\mu}{2}, \frac{\rho+\mu-\nu}{2}, \frac{\rho-\nu-\mu}{2} \\ \rho_1, \ldots, \rho_q, \frac{\rho}{2}, \frac{\rho+1}{2} \end{matrix} : z \right\}, \quad \ldots\ldots (72)$$

where $R(\rho \pm \nu \pm \mu) > 0$, $|\mathrm{amp}\, z| < \pi$, $p \geqslant q+1$. For other values of p and q the formula holds if the integral is convergent.

The following formula is required in the proof:

$$K_\nu(x) K_\mu(x) = \frac{1}{4\sqrt{\pi}\cdot x} \sum_{i, -i} \frac{1}{i} E\left(\frac{1+\nu+\mu}{2}, \frac{1+\nu-\mu}{2}, \frac{1+\mu-\nu}{2}, \frac{1-\mu-\nu}{2} : \frac{1}{2} : e^{\pm i\pi} x^2 \right)$$

[This thesis CHAP. III. formula (15)]. $\quad \ldots (73)$

To prove (72) substitute from (73) and write (z/λ) for x^2, so getting if $R(\rho \pm \nu \pm \mu) > 0$, $p \geqslant q+1$, $|\mathrm{amp}\, z| < \pi$

$$\int_0^\infty x^{\rho-1} K_\nu(x) K_\mu(x) E(p; \alpha_r : q; \rho_s : z/x^2)\, dx =$$

$$\frac{z^{\frac{\rho}{2}-\frac{1}{2}}}{8\sqrt{\pi}}\int_0^\infty \lambda^{-\frac{\rho}{2}-\frac{1}{2}} E(p;\alpha_r:q;\rho_s:\lambda)\left[\sum_{i,-i}\frac{1}{i}E\left(\frac{1+\nu+\mu}{2},\frac{1+\nu-\mu}{2},\frac{1+\mu-\nu}{2},\frac{1-\mu-\nu}{2}:\frac{1}{2}:\frac{e^{\pm i\pi}z}{\lambda}\right)\right]d\lambda$$

Now evaluate this last integral by means of (9) taking the case with $e^{-i\pi}$ in formula (9) with the term corresponding to $(+i)$ in $\sum\limits_{i,-i}$ and the case with $e^{+i\pi}$ with the term corresponding to $(-i)$ and thus (72) is obtained. The restrictions necessary for (9) can be removed by analytical continuation.

As the direct special case of (72), take $p=q=0$, and get

$$\int_0^\infty e^{-\lambda^2/z}\lambda^{\rho-1}K_\nu(\lambda)K_\mu(\lambda)\,d\lambda = \frac{\sqrt{\pi}}{4}E\left(\frac{\rho+\mu+\nu}{2},\frac{\rho+\mu-\nu}{2},\frac{\rho+\nu-\mu}{2},\frac{\rho-\nu-\mu}{2}:\frac{\rho}{2},\frac{\rho+1}{2}:z\right), \quad \cdots (74)$$

where $R(z)>0$, $R(\rho\pm\nu\pm\mu)>0$.

The second integral to be evaluated is

$$\int_0^\infty \lambda^{\rho-1}J_\nu(\lambda)J_\mu(\lambda)E(p;\alpha_r:q;\rho_s:z/\lambda^2)\,d\lambda$$

where $R(\rho+\nu+\mu)>0$, $R(2+2\alpha_r-\rho)>0$, $r=1,2,3,\ldots,p$, $|\operatorname{amp} z|<\pi$.

This integral can be evaluated if $p\geqq q+1$ by substituting for $J_\nu(x)J_\mu(x)$ from the formula [Watson "Bessel Functions"] p. 78 namely

$$J_\nu(\lambda)J_\mu(\lambda) = \lambda^{\mu+\nu}\cdot\frac{1}{\sqrt{\pi}}E\left(\frac{1+\mu+\nu}{2}, \frac{2+\mu+\nu}{2}; 1+\nu, 1+\mu, 1+\mu+\nu : 1/\lambda^2\right), \cdots (75),$$

and then applying (10).

In the same way, by applying (9) instead of (10), the integral

$$\int_0^\infty \lambda^{\rho-1} J_\nu(\lambda)J_\mu(\lambda)E(p; \alpha_r : q; \rho_s : z\lambda^2)\,d\lambda,$$

where $p \geqq q+1$, $R(\rho+\mu+\nu+2\alpha_r) > 0$, $r = 1, 2, 3, \ldots, p$, $R(2-\rho) > 0$, $|\operatorname{amp} z| < \pi$; can be evaluated.

Finally, from (9), (10), (73) and (75), integrals of the form

$$\int_0^\infty \lambda^{\rho-1} J_\nu(\lambda)J_\mu(\lambda)K_n(z\lambda)K_m(z\lambda)\,d\lambda,$$

where $R(\rho+\nu+\mu\pm m\pm n) > 0$, $R(z) > 0$, and

$$\int_0^\infty \lambda^{\rho-1} J_\nu(\lambda)J_\mu(\lambda)K_n\left(\frac{z}{\lambda}\right)K_m\left(\frac{z}{\lambda}\right)d\lambda,$$

where $R(2-\rho\pm m\pm n) > 0$, $R(z)$;

can be evaluated.

CHAPTER II.

LINEAR RELATIONS BETWEEN E-FUNCTIONS

§1. First formula. The formula to be proved is

$$\sum_{r=0}^{2n} {}^{2n}C_r \frac{(b;r)}{(\frac{1}{2}b+\frac{1}{2}-n;r)} (2x)^{-r} E\left(\begin{matrix}\frac{1}{2}+\frac{1}{2}r, 1+\frac{1}{2}r, \frac{1}{2}b+\frac{1}{2}-n+r, \alpha_1+r, \ldots, \alpha_p+r : x \\ \frac{1}{2}+\frac{1}{2}b+r, \frac{1}{2}-n+r, 1+r, \rho_1+r, \ldots, \rho_q+r\end{matrix}\right)$$

$$= \frac{(2n)!\; 2^{-2n}}{n!(\frac{1}{2}-\frac{1}{2}b;n)} E(p;\alpha_r:q;\rho_s:x), \quad \ldots \ldots (1)$$

The formulae required in the proof are the Barnes' integral

$$E(p;\alpha_r:q;\rho_t:x) = \frac{1}{2\pi i}\int \frac{\Gamma(\xi)\prod_{r=1}^{p}\Gamma(\alpha_r-\xi)}{\prod_{t=1}^{q}\Gamma(\rho_t-\xi)} x^{\xi}\, d\xi, \quad \ldots (2)$$

and the Whipple's formula [Whipple, F. J. W., Proc. Lond. Math. Soc. (2) (23), 1923, p. 113]

$$F\left(\begin{matrix}\alpha, \beta, \gamma; 1 \\ \frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2}, 2\gamma\end{matrix}\right) = \frac{\Gamma(\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2})\Gamma(\frac{1}{2})\Gamma(\gamma+\frac{1}{2})\Gamma(\gamma-\frac{1}{2}\alpha-\frac{1}{2}\beta+\frac{1}{2})}{\Gamma(\frac{1}{2}\alpha+\frac{1}{2})\Gamma(\frac{1}{2}\beta+\frac{1}{2})\Gamma(\gamma-\frac{1}{2}\alpha+\frac{1}{2})\Gamma(\gamma-\frac{1}{2}\beta+\frac{1}{2})}, \quad \ldots (3)$$

Proof of the formula. From (2) the E-function on the L.H.S. of (1) is equal to

$$\frac{1}{2\pi i}\int \frac{\Gamma(\xi)\Gamma(\frac{1}{2}+\frac{1}{2}r-\xi)\Gamma(1+\frac{1}{2}r-\xi)\Gamma(\frac{1}{2}b+\frac{1}{2}-n+r-\xi)\ \prod\Gamma(\alpha_t+r-\xi)}{\Gamma(\frac{1}{2}+\frac{1}{2}b+r-\xi)\Gamma(\frac{1}{2}-n+r-\xi)\Gamma(1+r-\xi)\ \prod\Gamma(\rho_s+r-\xi)}\, x^{\xi}\, d\xi.$$

Here replace ξ by $(\xi+r)$, not that

$$\Gamma(\tfrac{1}{2}-\tfrac{1}{2}r-\xi)\Gamma(1-\tfrac{1}{2}r-\xi)=\Gamma(\tfrac{1}{2})\Gamma(1-r-2\xi)\,2^{r+2\xi}$$

$$\frac{\Gamma(\frac{1}{2})\Gamma(1-2\xi)\,2^{r+2\xi}}{(-1)^r(2\xi;r)}=\frac{2^r\,\Gamma(\frac{1}{2}-\xi)\Gamma(1-\xi)}{(-1)^r\,(2\xi;r)},$$

and it can be seen that the L.H.S. of (1) is equal to

$$\frac{1}{2\pi i}\int \frac{\Gamma(\xi)\Gamma(\frac{1}{2}-\xi)\Gamma(\frac{1}{2}b+\frac{1}{2}-n-\xi)\prod\Gamma(\alpha_t-\xi)}{\Gamma(\frac{1}{2}+\frac{1}{2}b-\xi)\Gamma(\frac{1}{2}-n-\xi)\ \prod\Gamma(\rho_s-\xi)}\, F\left(\begin{matrix}-2n,\ b,\ \xi\ ;\ 1\\ \frac{1}{2}b+\frac{1}{2}-n,\ 2\xi\end{matrix}\right)x^{\xi}\,d\xi.$$

Now by (3), the generalized hypergeometric function is equal to

$$\frac{\Gamma(\frac{1}{2}b+\frac{1}{2}-n)\Gamma(\frac{1}{2})\Gamma(\xi+\frac{1}{2})\Gamma(\xi+\frac{1}{2}-\frac{1}{2}b+n)}{\Gamma(\frac{1}{2}-n)\Gamma(\frac{1}{2}b+\frac{1}{2})\Gamma(\xi+n+\frac{1}{2})\Gamma(\xi-\frac{1}{2}b+\frac{1}{2})},$$

and noting that

$$\Gamma(\tfrac{1}{2}+\xi)\,\Gamma(\tfrac{1}{2}-\xi) = \pi/\cos\xi\pi$$

$$\Gamma(\tfrac{1}{2}-n-\xi)\,\Gamma(\tfrac{1}{2}+n+\xi) = \pi/\cos(n+\xi)\pi$$

$$\Gamma(\tfrac{1}{2}-n-\xi+\tfrac{1}{2}b)\,\Gamma(\tfrac{1}{2}+n+\xi-\tfrac{1}{2}b) = \pi/\cos(n+\xi-\tfrac{1}{2}b)\pi,$$

$$\Gamma(\tfrac{1}{2}-\xi+\tfrac{1}{2}b)\,\Gamma(\tfrac{1}{2}+\xi-\tfrac{1}{2}b) = \pi/\cos(\xi-\tfrac{1}{2}b)\pi,$$

it is found that the expression reduces to

$$\frac{\Gamma(\tfrac{1}{2}b+\tfrac{1}{2}-n)\,\Gamma(\tfrac{1}{2})}{\Gamma(\tfrac{1}{2}-n)\,\Gamma(\tfrac{1}{2}b+\tfrac{1}{2})} \times R.H.S. \text{ of } (2);$$

and from this the result follows

An alternative proof: When $p \leqq q$ the E-functions in (1) can be expressed as generalized hypergeometric functions. On picking out the terms in x^{-m} on the left and summing by means of formula (3) the term in x^{-m} on the right is obtained. The restriction on p can be removed by applying formula (2) of CHAP. I. repeatedly if necessary.

Applications: Many particular cases can be deduced from (1). The most important of these are those cases when $x=-1$ and the E-function on the R.H.S. of (1) is a generalized hypergeometric function which can be summed by the know theorems which sum the generalized function. For instance when $p = q = 0$ (1) gives

$$e^{x} = \frac{n!\,(\tfrac{1}{2}-\tfrac{1}{2}b;n)}{(2n)!\,2^{-2n}} \sum_{r=0}^{2n} \frac{{}^{2n}C_r\,(b;r)\,(-2/x)^{-r}}{(\tfrac{1}{2}b+\tfrac{1}{2}-n;r)} E\left(\begin{matrix}\tfrac{1}{2}+\tfrac{1}{2}r,\ 1+\tfrac{1}{2}r,\ \tfrac{1}{2}b+\tfrac{1}{2}-n+r : -1/x\\ \tfrac{1}{2}+\tfrac{1}{2}b+r,\ \tfrac{1}{2}-n+r,\ 1+r\end{matrix}\right)$$

, . . (·4)

Again in (1) take $p=2$, $q=1$ with $\rho_1=\gamma$, and apply Gauss's theorem, and get

$$\sum_{r=0}^{2n} \frac{{}^{2n}C_r\,(b;r)\,(-2)^{-r}\,\Gamma(\tfrac{1}{2}+\tfrac{1}{2}r)\,\Gamma(1+\tfrac{1}{2}r)\,\Gamma(\tfrac{b}{2}+\tfrac{1}{2}-n+r)\,\Gamma(\alpha+r)\,\Gamma(\beta+r)}{(\tfrac{1}{2}b+\tfrac{1}{2}-n;r)\,\Gamma(\tfrac{1}{2}+\tfrac{1}{2}b+r)\,\Gamma(\tfrac{1}{2}-n+r)\,\Gamma(1+r)\,\Gamma(\gamma+r)}$$

$$\times\ {}_5F_4\left(\begin{matrix}\tfrac{1}{2}+\tfrac{1}{2}r,\ 1+\tfrac{1}{2}r,\ \tfrac{b}{2}+\tfrac{1}{2}-n+r,\ \alpha+r,\ \beta+r;1\\ \tfrac{1}{2}+\tfrac{1}{2}b+r,\ \tfrac{1}{2}-n+r,\ 1+r,\ \gamma+r\end{matrix}\right)$$

$$= \frac{(2n)!\,2^{-2n}\,\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma-\alpha-\beta)}{n!\,(\tfrac{1}{2}-\tfrac{b}{2};n)\,\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}, \quad \ldots\ldots \quad (5)$$

where $R(\gamma)>0$, $R(\gamma-\alpha-\beta)>0$.

In (1) take $p=0$, $q=1$ with $\rho_1=\mu+1$, and write $(4/x^2)$ for x, apply the formula

$$E(::\mu+1:z) = z^{\frac{1}{2}\mu}\,J_\mu(2/\sqrt{z}), \quad \ldots\ldots \quad (6)$$

so getting

$$J_\mu(x) = \frac{n!\,(\frac{1}{2}-\frac{b}{2};n)}{(2n)!\;2^{-2n}} \sum_{r=0}^{2n} \frac{{}^{2n}C_r\,(b;r)\,\Gamma(\frac{1}{2}+\frac{1}{2}r)\,\Gamma(1+\frac{1}{2}r)\,\Gamma(\frac{1}{2}b+\frac{1}{2}-n+r)}{2^r\,(\frac{1}{2}b+\frac{1}{2}-n;r)\,\Gamma(\frac{1}{2}+\frac{b}{2}+r)\,\Gamma(\frac{1}{2}-n+r)\,\Gamma(1+r)\,\Gamma(1+\mu+r)}$$

$$\times\left(\frac{x^2}{4}\right)^{\frac{1}{2}\mu+r} {}_3F_4\left(\begin{matrix}\frac{1}{2}+\frac{1}{2}r,\,1+\frac{1}{2}r,\,\frac{1}{2}b+\frac{1}{2}-n+r;\;-x^2/4\\ \frac{1}{2}+\frac{1}{2}b+r,\,\frac{1}{2}-n+r,\,1+r,\,1+\mu+r\end{matrix}\right), \quad\ldots(7)$$

A third proof of formula (1). This proof is by induction. The following formulae are required in this proof

$$\alpha_1 E(p;\alpha_r:q;\rho_s:x) = E(\alpha_1+1,\alpha_2,\ldots,\alpha_p:q;\rho_s:x) + \frac{1}{x}E(p;\alpha_r+1:q;\rho_s+1:x), \quad\ldots(8)$$

$$(\rho_1-1)E(p;\alpha_r:q;\rho_s:x) = E(p;\alpha_r:\rho_1-1,\rho_2,\ldots,\rho_q:x) + \frac{1}{x}E(p;\alpha_r+1:q;\rho_s+1:x), \quad\ldots(9)$$

and

$$(\alpha_1-1)E(\alpha_1-1,\alpha_2+1,\alpha_3,\ldots,\alpha_p:q;\rho_s:x) + (\alpha_1-\alpha_2-1)x^{-1}E(\alpha_1,\alpha_2+1,\ldots,\alpha_p+1:q;\rho_s+1:x)$$
$$= \alpha_2 E(p;\alpha_r:q;\rho_s:x) \quad\ldots\ldots(10)$$

[Formulae (9), (10) and (8) are to be found in B. V., p. 356].

When $n=0$, the formula is obvious. To prove (1) when $n=1$, consider the L.H.S. of (1) with $n=1$. It can be written

$$E\left(\begin{matrix}\frac{1}{2},1,\frac{1}{2}b-\frac{1}{2},\alpha_1,\ldots,\alpha_p:x\\ \frac{1}{2}b+\frac{1}{2},-\frac{1}{2},1,\rho_1,\ldots,\rho_q\end{matrix}\right) + \frac{2b}{(b-1)x}E\left(\begin{matrix}1,\frac{3}{2},\frac{1}{2}b+\frac{1}{2},\alpha_1+1,\ldots,\alpha_p+1:x\\ \frac{1}{2}b+\frac{3}{2},\frac{1}{2},2,\rho_1+1,\ldots,\rho_q+1\end{matrix}\right)$$

$$+\frac{b}{(b-1)x^2}E\left(\frac{3}{2},2,\frac{b}{2}+\frac{3}{2},\alpha_1+2,\ldots,\alpha_p+2:\frac{b}{2}+\frac{5}{2},\frac{3}{2},3,\rho_1+2,\ldots,\rho_q+2:x\right).$$

Now apply (9) to the first E-function in the last expression with $(1/2)$ in the role of ρ_1, so getting

$$\text{L.H.S. of (1) with } n=1 = -\tfrac{1}{2}E\left(\tfrac{1}{2}, 1, \tfrac{1}{2}b-\tfrac{1}{2}, \alpha_1, \ldots, \alpha_p : \tfrac{1}{2}b+\tfrac{1}{2}, \tfrac{1}{2}, 1, \rho_1, \ldots, \rho_q : x\right)$$

$$-\frac{1}{x}E\left(\tfrac{3}{2}, 2, \tfrac{1}{2}b+\tfrac{1}{2}, \alpha_1+1, \ldots, \alpha_p+1 : \tfrac{1}{2}b+\tfrac{3}{2}, \tfrac{3}{2}, 2, \rho_1+1, \ldots, \rho_q+1 : x\right)$$

$$+\frac{2b}{(b-1)x}\left[E\left(\begin{matrix}1, \tfrac{3}{2}, \tfrac{1}{2}b+\tfrac{1}{2}, \alpha_1+1, \ldots, \alpha_p+1 : x \\ \tfrac{1}{2}b+\tfrac{3}{2}, \tfrac{1}{2}, 2, \rho_1+1, \ldots, \rho_q+1\end{matrix}\right) + \frac{1}{2x}E\left(\begin{matrix}\tfrac{3}{2}, 2, \tfrac{b}{2}+\tfrac{3}{2}, \alpha_1+2, \ldots, \alpha_p+2 : x \\ \tfrac{b}{2}+\tfrac{5}{2}, \tfrac{3}{2}, 3, \rho_1+2, \ldots, \rho_q+2\end{matrix}\right)\right].$$

Sum the quantity between the brackets [ ] by means of (10) with $(\tfrac{3}{2})$ and 1 in the roles of $\underline{\alpha_1}$ and $\underline{\alpha_2}$ respectively, so getting

$$\text{L.H.S. of (1) with } n=1 = -\tfrac{1}{2}E\left(\tfrac{1}{2}, 1, \tfrac{b}{2}-\tfrac{1}{2}, \alpha_1, \ldots, \alpha_p : \tfrac{b}{2}+\tfrac{1}{2}, \tfrac{1}{2}, 1, \rho_1, \ldots, \rho_q : x\right)$$

$$+\frac{1}{(b-1)x}E\left(\tfrac{b}{2}+\tfrac{1}{2}, \alpha_1+1, \ldots, \alpha_p+1 : \tfrac{b}{2}+\tfrac{3}{2}, \rho_1+1, \ldots, \rho_q+1 : x\right)$$

$$=\frac{1}{(1-b)}E(p; \alpha_r : q; \rho_s : x) \text{ by (8).}$$

Thus (1) is proved for the special case $n=1$

If (8) is multiplied by (ρ_1-1) and (9) by α_1, then replacing $\underline{\rho_1}$ by (ρ_1+1) after substraction, the following formula is obtained.

$$E\left(\begin{matrix}\alpha_1+1, \alpha_2, \dots, \alpha_p : x \\ \rho_1+1, \rho_2, \dots, \rho_q\end{matrix}\right) = \frac{\alpha_1}{\rho_1} E(p; \alpha_r : q; \rho_s : x) + \frac{\alpha_1 - \rho_1}{\rho_1 x} E(p; \alpha_r + 1 : \rho_1 + 2, \rho_2 + 1, \dots, \rho_q + 1 : x) \quad , \dots (11).$$

(11) will be applied to prove the general case.

Now assume that (1) holds for a particular value of $\underline{n}$, multiply by $(1+2n)/(1-b+2n)$: then the R.H.S. becomes the R.H.S. of (1) with $(n+1)$ in place of $\underline{n}$; while the L.H.S., on applying (11), with $(\frac{1}{2}-n+\frac{b}{2}+r-1)$ and $(\frac{1}{2}-n+r-1)$ in the roles of $\underline{\alpha_1}$ and $\underline{\rho_1}$, becomes

$$\frac{(1+2n)}{(1-b+2n)}\left\{\sum_{r=0}^{2n} \frac{{}^{2n}C_r\,(b;r)\,(\frac{1}{2}-n+\frac{b}{2}+r-1)}{(1-n+\frac{b}{2}+\frac{1}{2};r)(\frac{1}{2}-n+r-1)}(2x)^{-r} I_r + \sum_{r=0}^{2n} \frac{{}^{2n}C_r\,(b/2)\,(2x)^{-r}}{(1-n+\frac{b}{2}+\frac{1}{2};r)\,(\frac{1}{2}-n+r-1)\,x} J_r\right\}$$

where

$$I_r \equiv E\left(\begin{matrix}\frac{1}{2}+\frac{r}{2}, 1+\frac{r}{2}, \frac{1}{2}b+\frac{1}{2}-n+r-1, \alpha_1+r, \dots, \alpha_p+r : x \\ \frac{1}{2}+\frac{b}{2}+r, \frac{1}{2}-n+r-1, 1+r, \rho_1+r, \dots, \rho_q+r\end{matrix}\right) \text{ and}$$

$$J_r \equiv E\left(\begin{matrix}\frac{3}{2}+\frac{r}{2}, 2+\frac{r}{2}, \frac{1}{2}b+\frac{1}{2}-n+r, \alpha_1+r+1, \dots, \alpha_p+r+1 : x \\ \frac{1}{2}+\frac{b}{2}+r+1, \frac{1}{2}-n+r+1, 2+r, \rho_1+r+1, \dots, \rho_q+r+1\end{matrix}\right)$$

In the second of these two series keep the last term corresponding to $(r=2n)$, replace $\underline{r}$ by $(r-1)$, then this series becomes

$$\frac{(b;2n)\,(2x)^{-2n}}{(1-n+\frac{b}{2}+\frac{1}{2};2n)\,(n-\frac{1}{2})}\left(\frac{b}{2x}\right) J_{2n}$$

$$+\sum_{r=1}^{2n} \frac{{}^{2n}C_{r-1}\,(b;r-1)\,(2x)^{-r+1}}{(-n+\frac{b}{2}+\frac{1}{2};r-1)(\frac{1}{2}-n+r-2)}\left(\frac{b}{2x}\right) E\left(\begin{matrix}1+\frac{r}{2}, \frac{3}{2}+\frac{r}{2}, \frac{b}{2}+\frac{1}{2}-n+r-1, \alpha_1+r, \dots, \alpha_p+r : x \\ \frac{1}{2}+\frac{b}{2}+r, \frac{1}{2}-n+r, 1+r, \rho_1+r, \dots, \rho_q+r\end{matrix}\right).$$

On applying (11) with $(\frac{1}{2}+\frac{r}{2})$ and $(\frac{1}{2}-n+r-1)$ in the roles of $\underline{\alpha_1}$ and $\underline{\rho_1}$, then the series in the last expression is found to be equal to

$$\sum_{r=1}^{2n} \frac{{}^{2n}C_{r-1}\,(b;r-1)\,(\frac{1}{2}+\frac{1}{2}r)\,(b/2)}{(-n+\frac{b}{2}+\frac{1}{2};r-1)\,(\frac{1}{2}-n+r-2)(\frac{1}{2}-n+r-1)}\,(2x)^{-r+1}\,I_r$$

$$+\sum_{r=1}^{2n} \frac{{}^{2n}C_{r-1}\,(b;r-1)\,(2x)^{-r+1}}{(-n+\frac{b}{2}+\frac{1}{2};r-1)}\cdot\frac{(b/2x)}{(\frac{1}{2}-n+r-2)}\cdot\frac{(n-\frac{1}{2}r+1)}{(\frac{1}{2}-n+r-1)x}\,J_r$$

Similarly, on keeping the last term corresponding to $(r=2n)$ in the second of the last two series, replacing $\underline{r}$ by $(r-1)$; then applying (11) with $(\frac{1}{2}+\frac{r}{2})$ and $(\frac{1}{2}-n+r-1)$ in the roles of $\underline{\alpha_1}$ and $\underline{\rho_1}$ respectively. So that the second series in the last expression will be found to be equal to

$$\frac{{}^{2n}C_{2n-1}\,(b;2n-1)}{(-n+\frac{b}{2}+\frac{1}{2};2n-1)}\,\frac{(2x)^{-2n+1}}{(n-\frac{3}{2})(2x)}\cdot\frac{b}{(n-\frac{1}{2})x}\,J_{2n}$$

$$+\sum_{r=2}^{2n} \frac{{}^{2n}C_{r-2}\,(b;r-2)\,(2x)^{-r+2}}{(-n+\frac{b}{2}+\frac{1}{2};r-2)} \cdot \frac{(b/2x)}{(\frac{1}{2}-n+r-3)} \cdot \frac{(n-\frac{r}{2}+\frac{3}{2})}{(\frac{1}{2}-n+r-2)x} \cdot \frac{\frac{1}{2}+\frac{r}{2}}{(\frac{1}{2}-n+r-1)}\, I_r$$

$$+\sum_{r=2}^{2n} \frac{{}^{2n}C_{r-2}\,(b;r-2)\,(2x)^{-r+2}}{(-n+\frac{b}{2}+\frac{1}{2};r-2)} \cdot \frac{(b/2x)}{(\frac{1}{2}-n+r-3)} \cdot \frac{(n-\frac{r}{2}+\frac{3}{2})}{(\frac{1}{2}-n+r-2)x} \cdot \frac{(n-\frac{r}{2}+1)}{(\frac{1}{2}-n+r-1)x}\, J_r.$$

Proceeding thus we find that the coefficient of I_r where $r=1,2,3,\ldots,2n$ is equal to

$$\{(1+2n)/(1-b+2n)\}\cdot(2x)^{-r}$$

$$\times\left\{\frac{{}^{2n}C_r\,(b;r)\,(\frac{1}{2}-n+\frac{b}{2}+r-1)}{(-n+\frac{b}{2}+\frac{1}{2};r)\,(\frac{1}{2}-n+r-1)} + \frac{b(1+r)}{2}\sum_{m=1}^{r} \frac{{}^{2n}C_{r-m}\,(b;r-m)\,(2n-r+3;m-1)}{(-n+\frac{b}{2}+\frac{1}{2};r-m)\,(\frac{1}{2}-n+r-m-1;m+1)}\right\}.$$

Now the last sum is equal to

$$\frac{(2n)!\,\Gamma(\frac{1}{2}-n-1)}{(2n-r+2)!\,\Gamma(\frac{1}{2}-n+r)} \sum_{m=1}^{r} \frac{(b;r-m)\,(-n-\frac{1}{2};r-m)}{(r-m)!\,(-n+\frac{b}{2}+\frac{1}{2};r-m)}\,(2n-r+m+1),$$

and since $-\frac{b}{2}(2n-r+m+1)=(b+r-m)(-n-\frac{1}{2}+r-m)-(r-m)(-n+\frac{b}{2}-\frac{1}{2}+r-m)$,

so that the last sum is equal to $-\dfrac{2}{b}\,\dfrac{(2n)!\,\Gamma(-n-\frac{1}{2})}{(2n-r+2)!\,\Gamma(\frac{1}{2}-n+r)}$

$$\times\left\{\sum_{m=1}^{r} \frac{(b;r-m+1)\,(-n-\frac{1}{2};r-m+1)}{(r-m)!\,(-n+\frac{b}{2}+\frac{1}{2};r-m)} - \sum_{m=1}^{r-1} \frac{(b;r-m)\,(-n-\frac{1}{2};r-m)}{(r-m-1)!\,(-n+\frac{b}{2}+\frac{1}{2};r-m-1)}\right\},$$

$$= -\frac{2}{b}\,\frac{(2n)!\,\Gamma(-n-\frac{1}{2})}{(2n-r+2)!\,\Gamma(\frac{1}{2}-n+r)}\cdot\frac{(b;r)(-n-\frac{1}{2};r)}{(r-1)!(-n+\frac{b}{2}+\frac{1}{2};r-1)}.$$

Thus the coefficient of I_r, $r=1,2,3,\dots,2n$ is equal to

$$\frac{(1+2n)(2x)^{-r}}{(1-b+2n)}\cdot\frac{(2n)!\,(b;r)}{r!\,(2n-r+2)!\,(-n+\frac{b}{2}+\frac{1}{2};r-1)(-n-\frac{1}{2}+r)}$$

$$\times\left\{(2n-r+2)(2n-r+1)-r(r+1)\right\} = \frac{{}^{2n+2}C_r\,(b;r)\,(2x)^{-r}}{(-n+\frac{b}{2}-\frac{1}{2};r)}.$$

Similarly the coefficient of J_{2n} will be found to be equal to

$$\frac{(1+2n)(b+2n)(b;2n)(2x)^{-2n-1}}{(-n+\frac{b}{2}-\frac{1}{2};2n+1)}.$$

Adding the last results the L.H.S. of (1) will be equal to

$$\sum_{r=0}^{2n}\frac{{}^{2n+2}C_r\,(b;r)\,(2x)^{-r}}{(-n+\frac{b}{2}-\frac{1}{2};r)}I_r+\frac{(1+2n)(b+2n)(b;2n)}{(-n+\frac{b}{2}-\frac{1}{2};2n+1)}(2x)^{-2n-1}J_{2n}$$

Again apply (11) with $(1+n)$ and $(\frac{1}{2}+n)$ in the roles of α_1 and ρ_1 to the last E-function by J_{2n}; then after a little reduction the L.H.S. becomes the L.H.S. of (1) with $(n+1)$ in place of n. But (1) is true for $n=0$ and $n=1$. Hence (1) is true for all positive integral values of $\underline{n}$.

§ 2. Second formula: The formula to be proved is

$$\sum_{r=0}^{n} \frac{(-1)^r\, {}^nC_r\, (\alpha;r)\,(1+\tfrac{1}{2}\alpha;r)}{(\tfrac{1}{2}\alpha;r)\,(1+\alpha+n;r)}\, x^{-2r}\, E\left\{\begin{matrix}\tfrac{1}{2}+\alpha+2r, 1+2\alpha+2n+2r, \alpha_1+2r, \ldots, \alpha_p+2r : x\\ \tfrac{1}{2}+\alpha+n+2r, 1+2\alpha+4r, \rho_1+2r, \ldots, \rho_q+2r\end{matrix}\right\}$$

$$= 2^{2n}(1+\alpha;n)\,E(p;\alpha_r:q;\rho_s:x), \qquad \ldots\ldots (12)$$

where $\underline{n}$ is any positive integer. To prove (12) consider the special case when $p=q=0$. Then the coefficient of $\left(\frac{-1}{x}\right)^s$ on the L.H.S. is equal to

$$\frac{\Gamma(\frac{1}{2}+\alpha+s)\,\Gamma(1+2\alpha+2n+s)}{\Gamma(\frac{1}{2}+\alpha+n+s)\,\Gamma(1+2\alpha+s)} \times \frac{1}{\lfloor s}$$

$$+ \frac{(-n)(\alpha;1)(1+\frac{1}{2}\alpha;1)}{\lfloor 1\,(\frac{1}{2}\alpha;1)(1+\alpha+n;1)} \cdot \frac{\Gamma(\frac{1}{2}+\alpha+2+s-2)\,\Gamma(1+2\alpha+2n+2+s-2)}{\Gamma(\frac{1}{2}+\alpha+n+2+s-2)\,\Gamma(1+2\alpha+4+s-2)\lfloor s-2}$$

$$+ \cdots\cdots$$

$$= \frac{\Gamma(\frac{1}{2}+\alpha+s)\,\Gamma(1+2\alpha+2n+s)}{\Gamma(\frac{1}{2}+\alpha+n+s)\,\Gamma(1+2\alpha+s)\lfloor s} \cdot F\left(\begin{matrix}\alpha, 1+\frac{1}{2}\alpha, -\frac{1}{2}s, -\frac{1}{2}s+\frac{1}{2}, -n; 1\\ \frac{1}{2}\alpha, 1+\alpha+\frac{1}{2}s, \frac{1}{2}+\alpha+\frac{1}{2}s, 1+\alpha+n\end{matrix}\right).$$

Now sum the last hypergeometric function by means of Dougall's second theorem [Proc. Edin. Math. Soc., $\overline{\underline{XXV}}$, 1906, 10] namely

$$F\left(\begin{matrix}\alpha, 1+\frac{1}{2}\alpha, \beta, \gamma, \delta; 1\\ \frac{1}{2}\alpha, 1+\alpha-\beta, 1+\alpha-\gamma, 1+\alpha-\delta\end{matrix}\right) = \frac{\Gamma(1+\alpha-\beta)\Gamma(1+\alpha-\gamma)\Gamma(1+\alpha-\delta)\Gamma(1+\alpha-\beta-\gamma-\delta)}{\Gamma(1+\alpha)\Gamma(1+\alpha-\beta-\gamma)\Gamma(1+\alpha-\beta-\delta)\Gamma(1+\alpha-\gamma-\delta)} \quad \ldots (13)$$

where $R(\alpha-\beta-\gamma-\delta)>-1$ and one of the parameters β, γ and δ is a negative integer.

The last coefficient $= \dfrac{\Gamma(\frac{1}{2}+\alpha+s)\Gamma(1+2\alpha+2n+s)}{\Gamma(\frac{1}{2}+\alpha+n+s)\Gamma(1+2\alpha+s)\lfloor s}$

$$\times \frac{\Gamma(\alpha+\frac{1}{2}s+1)\Gamma(\alpha+\frac{1}{2}s+\frac{1}{2})\Gamma(\alpha+n+1)\Gamma(\alpha+s+n+\frac{1}{2})}{\Gamma(1+\alpha)\Gamma(\alpha+s+\frac{1}{2})\Gamma(\alpha+\frac{1}{2}s+n+\frac{1}{2})\Gamma(\alpha+\frac{1}{2}s+n+1)}$$

$$= \frac{\Gamma(\frac{1}{2}+s+s)\Gamma(1+2\alpha+2n+s)}{\Gamma(\frac{1}{2}+\alpha+n+s)\Gamma(1+2\alpha+s)\lfloor s} \times \frac{\Gamma(2\alpha+s+1)\Gamma(\alpha+n+1)\Gamma(\alpha+s+n+\frac{1}{2})}{\Gamma(1+\alpha)\Gamma(\alpha+s+\frac{1}{2})\Gamma(1+2\alpha+2n+s)} \cdot 2^{2n}$$

$$= (1+\alpha;n)\frac{1}{\lfloor s}\cdot 2^{2n}$$

$=$ coefficient of $(-\frac{1}{x})^s$ on the R.H.S. of (12) with $p=q=0$. Thus (12) is proved for the case $p=q=0$. The general case can then be deduced in the same manner as formula (1) CHAP. I. Also the restrictions $R(\alpha+n+s-\frac{1}{2})>-1$ necessary for (13) can be removed by analytical continuation.

A second proof of (12). When $n=0$ (12) is obvious. When $n=1$, the L·H·S. of (12) is equal to

$$E\left(\tfrac{1}{2}+\alpha, 3+2\alpha, \alpha_1, \ldots, \alpha_p : \tfrac{3}{2}+\alpha, 1+2\alpha, \rho_1, \ldots, \rho_q : x\right)$$
$$-1/x^2 E\left(\tfrac{5}{2}+\alpha, \alpha_1+2, \ldots, \alpha_p+2 : \tfrac{7}{2}+\alpha, \rho_1+2, \ldots, \rho_q+2 : x\right)$$

$$= (2+2\alpha)E\left(\tfrac{1}{2}+\alpha, 2+2\alpha, \alpha_1, \ldots, \alpha_p : \tfrac{3}{2}+\alpha, 1+2\alpha, \rho_1, \ldots, \rho_q : x\right)$$
$$-1/x\, E\left(\tfrac{3}{2}+\alpha, 3+2\alpha, \alpha_1+1, \ldots, \alpha_p+1 : \tfrac{5}{2}+\alpha, 2+2\alpha, \rho_1+1, \ldots, \rho_q+1 : x\right)$$
$$-1/x^2 E\left(\tfrac{5}{2}+\alpha, \alpha_1+2, \ldots, \alpha_p+2 : \tfrac{7}{2}+\alpha, \rho_1+2, \ldots, \rho_q+2 : x\right) \text{ by } (8).$$

Now apply (10) with the quantities $(\tfrac{3}{2}+\alpha)$, $(1+2\alpha)$ in the roles of α_1 and α_2 respectively to the first E-function in the last expression and get

$$\text{L·H·S. of (12) with } n=1 = 4(1+\alpha)E(p; \alpha_r : q; \rho_s : x)$$
$$+\frac{2(1+\alpha)}{x} E\left(\tfrac{3}{2}+\alpha, \alpha_1+1, \ldots, \alpha_p+1 : \tfrac{5}{2}+\alpha, \rho_1+1, \ldots, \rho_q+1 : x\right)$$
$$-1/x\left[E\left\{\begin{matrix}\tfrac{3}{2}+\alpha, 3+2\alpha, \alpha_1+1, \ldots, \alpha_p+1 : x\\ \tfrac{5}{2}+\alpha, 2+2\alpha, \rho_1+1, \ldots, \rho_q+1\end{matrix}\right\} + \frac{1}{x}E\left\{\begin{matrix}\tfrac{5}{2}+\alpha, 3+2\alpha, \alpha_1+2, \ldots, \alpha_p+2 : x\\ \tfrac{7}{2}+\alpha, 3+2\alpha, \rho_1+2, \ldots, \rho_q+2\end{matrix}\right\}\right]$$

$$= 4(1+\alpha)E(p; \alpha_r; q; \rho_s : x) \text{ by a second application of } (8)$$

to the quantity brackets [ ]. Therefore (12) is proved for the case $n=1$ and it becomes

$$4(1+\alpha)E(p;\alpha_r:q;\rho_s:x) = E\left\{\begin{matrix}\frac{1}{2}+\alpha, 3+2\alpha, \alpha_1, \ldots, \alpha_p : x\\ \frac{3}{2}+\alpha, 1+2\alpha, \rho_1, \ldots, \rho_q\end{matrix}\right\}$$

$$-1/x^2 E\left\{\begin{matrix}\frac{5}{2}+\alpha, \alpha_1+2, \ldots, \alpha_p+2 : x\\ \frac{7}{2}+\alpha, \rho_1+2, \ldots, \rho_q+2\end{matrix}\right\}$$

which can be written in the form

$$4(1+\alpha)E\left\{\begin{matrix}\frac{3}{2}+\alpha, \alpha_1, \ldots, \alpha_p : x\\ 3+2\alpha, \rho_1, \ldots, \rho_q\end{matrix}\right\}$$

$$= E\left\{\begin{matrix}\frac{1}{2}+\alpha, \alpha_1, \ldots, \alpha_p : x\\ 1+2\alpha, \rho_1, \ldots, \rho_q\end{matrix}\right\} - \frac{1}{x^2}E\left\{\begin{matrix}\frac{5}{2}+\alpha, \alpha_1+2, \ldots, \alpha_p+2 : x\\ 5+2\alpha, \rho_1+2, \ldots, \rho_q+2\end{matrix}\right\} \quad \ldots (14)$$

To prove the general formula assume that it is true for a particular value of $\underline{n}$, then it will be proved that it is true when $\underline{n}$ is replaced by $(n+1)$.

Thus the following formula which is (12) with $(\alpha+1)$ instead of α is assumed to be true

$$\sum_{r=0}^{n} \frac{(-1)^r\, {}^nC_r (\alpha+1;r)(\frac{3}{2}+\frac{\alpha}{2};r)}{(\frac{1}{2}\alpha+\frac{1}{2};r)(2+\alpha+n;r)\,x^{2r}} E\left\{\begin{matrix}\frac{3}{2}+\alpha+2r, 3+2\alpha+2n+2r, \alpha_1+2r, \dots, \alpha_p+2r : x \\ \frac{3}{2}+\alpha+n+2r, 3+2\alpha+4r, \rho_1+2r, \dots, \rho_q+2r\end{matrix}\right\}$$

$$= 2^{2n}(2+\alpha;n)\,E(p;\alpha_r:q;\rho_s:x), \quad \dots\dots (15)$$

Multiply both sides of (15) with $4(1+\alpha)$; then the R.H.S. becomes the R.H.S. of (12) with $(n+1)$ in place of $\underline{n}$; while the L.H.S., on applying (14) with $(1+\alpha+2r)$ in the role of $(1+\alpha)$ becomes

$$\sum_{r=0}^{n} \frac{(-1)^r\, {}^nC_r (\alpha+1;r)(\frac{3}{2}+\frac{1}{2}\alpha;r)}{(\frac{1}{2}\alpha+\frac{1}{2};r)(2+\alpha+n;r)} \frac{(1+\alpha)}{(1+\alpha+2r)} x^{-2r}$$

$$\times E\left\{\begin{matrix}\frac{1}{2}+\alpha+2r, 3+2\alpha+2n+2r, \alpha_1+2r, \dots, \alpha_p+2r : x \\ 1+2\alpha+4r, \frac{3}{2}+\alpha+n+2r, \rho_1+2r, \dots, \rho_q+2r\end{matrix}\right\}$$

$$+\sum_{r=0}^{n} \frac{(-1)^{r+1}\, {}^nC_r (\alpha+1;r)(\frac{3}{2}+\frac{\alpha}{2};r)}{(\frac{1}{2}\alpha+\frac{1}{2};r)(2+\alpha+n;r)} \frac{(1+\alpha)}{(1+\alpha+2r)} x^{-2(r+1)}$$

$$\times E\left\{\begin{matrix}\frac{5}{2}+\alpha+2r, 3+2\alpha+2n+2r+2, \alpha_1+2r+2, \dots, \alpha_p+2r+2 : x \\ 5+2\alpha+4r, \frac{7}{2}+\alpha+n+2r, \rho_1+2r+2, \dots, \rho_q+2r+2\end{matrix}\right\}.$$

In the second of these last two series write $(r-1)$ for r, keep the first term of the first series as well as the last term of the second series, add similar terms, noting that

$$\frac{(1+\frac{1}{2}\alpha;r)}{(\frac{1}{2}\alpha;r)} = \frac{\alpha+2r}{\alpha},$$

then the last expression becomes

$$E(\tfrac{1}{2}+\alpha, 3+2\alpha+2n, \alpha_1, \ldots, \alpha_p : 1+2\alpha, \tfrac{3}{2}+\alpha+n, \rho_1, \ldots, \rho_q : x)$$

$$+\frac{(-1)^{n+1}(\alpha+1;n)(\frac{3}{2}+\frac{1}{2}\alpha;n)(1+\alpha)}{(\frac{1}{2}\alpha+\frac{1}{2};n)(2+\alpha+n;n)(1+\alpha+2n)} x^{-2n-2} E\left\{\begin{matrix}\frac{5}{2}+\alpha+2n, 5+2\alpha+4n, \alpha_1+2n+2, \ldots, \alpha_p+2n+2 : x\\ 5+2\alpha+4n, \frac{7}{2}+\alpha+3n, \rho_1+2n+2, \ldots, \rho_q+2n+2\end{matrix}\right\}$$

$$+\sum_{r=1}^{n} \frac{(-1)^r(1+\alpha;r-1)\,x^{-2r}}{(2+\alpha+n;r)}$$

$$E\left\{\begin{matrix}\frac{1}{2}+\alpha+2r, 3+2\alpha+2n+2r, \alpha_1+2r, \ldots, \alpha_p+2r\\ 1+2\alpha+4r, \frac{3}{2}+\alpha+n+2r, \rho_1+2r, \ldots, \rho_q+2r\end{matrix}\right\}\left[(\alpha+r)\cdot {}^{n}C_{r} + (1+\alpha+n+r)\cdot {}^{n}C_{r-1}\right].$$

But the quantity between brackets [ ] is equal to ${}^{n+1}C_r\,(\alpha+2r)$ so that the last expression becomes

$$\sum_{r=0}^{n+1} \frac{(-1)^r\, {}^{n+1}C_r\, (\alpha;r)\,(1+\tfrac{1}{2}\alpha;r)}{(\tfrac{1}{2}\alpha;r)\,(2+\alpha+n;r)}\, x^{-2r}\, E\left\{\begin{matrix}\tfrac{1}{2}+\alpha+2r, 3+2\alpha+2n+2r, \alpha_1+2r, \ldots, \alpha_p+2r : x \\ \tfrac{3}{2}+\alpha+n+2r, 1+2\alpha+4r, \rho_1+2r, \ldots, \rho_q+2r\end{matrix}\right\}.$$

But this is the L.H.S. of (12) with $(n+1)$ in place of n. But (12) is true for $n=0$ and $n=1$. Hence (12) is true for all positive integral values of n.

Particular cases of (12): Take $x=-1$, $p=2$, $q=1$ in (12) and get

$$\sum_{r=0}^{n} \frac{(-1)^r\, {}^{n}C_r\, (\alpha;r)\,(1+\tfrac{1}{2}\alpha;r)\,\Gamma(\tfrac{1}{2}+\alpha+2r)\,\Gamma(1+2\alpha+2n+2r)\,\Gamma(\delta+2r)\,\Gamma(\beta+2r)}{(\tfrac{1}{2}\alpha;r)\,(1+\alpha+n;r)\,\Gamma(\tfrac{1}{2}+\alpha+n+2r)\,\Gamma(1+2\alpha+4r)\,\Gamma(\gamma+2r)}$$

$$\cdot\, {}_4F_3\left(\begin{matrix}\tfrac{1}{2}+\alpha+2r, 1+2\alpha+2n+2r, \delta+2r, \beta+2r ; 1 \\ \tfrac{1}{2}+\alpha+n+2r, 1+2\alpha+4r, \gamma+2r\end{matrix}\right)$$

$$= 2^{2n}(1+\alpha;n)\,\frac{\Gamma(\delta)\,\Gamma(\beta)\,\Gamma(\gamma-\delta-\beta)}{\Gamma(\gamma-\delta)\,\Gamma(\gamma-\beta)}, \quad \ldots\ldots \quad (16)$$

where $R(\gamma)>0$, $R(\gamma-\delta-\beta)>0$.

In (12) take $p=2$, $q=1$, $x=-2$ and apply the formula due to Gauss namely

$$F\left(2\beta, 2\gamma;\ \beta+\gamma+\tfrac{1}{2};\ \tfrac{1}{2}\right) = \frac{\Gamma(\beta+\gamma+\frac{1}{2})\,\Gamma(1/2)}{\Gamma(\beta+\frac{1}{2})\,\Gamma(\gamma+\frac{1}{2})}, \quad \ldots\ldots (17)$$

so getting

$$\sum_{r=0}^{n} (-1)^r \frac{{}^nC_r\,(\alpha;r)(1+\frac{1}{2}\alpha;r)\,\Gamma(\frac{1}{2}+\alpha+2r)\,\Gamma(1+2\alpha+2n+2r)\,\Gamma(2\beta+2r)\,\Gamma(2\gamma+2r)}{(\frac{1}{2}\alpha;r)\,(1+\alpha+n;r)\,\Gamma(\frac{1}{2}+\alpha+n+2r)\,\Gamma(1+2\alpha+4r)\,\Gamma(\beta+\gamma+\frac{1}{2}+2r)}(-2)^{-2r}$$

$$\times\ {}_4F_3\left(\begin{matrix}\frac{1}{2}+\alpha+2r,\ 1+2\alpha+2n+2r,\ 2\beta+2r,\ 2\gamma+2r;\ \frac{1}{2}\\ \frac{1}{2}+\alpha+n+2r,\ 1+2\alpha+4r,\ \beta+\gamma+\frac{1}{2}+2r\end{matrix}\right)$$

$$= 2^{2n+\beta+\gamma-2}\,\pi^{-1/2}\,(1+\alpha;n)\,\Gamma(\beta)\,\Gamma(\gamma), \quad \ldots\ldots (18)$$

Again in (12) write $(1/x^2)$ for x and take $p=1$, $q=2$ and apply the formula [Watson, "Theory of Bessel Functions" p. 147 form (1) § 5.41]

$$\left\{J_\nu(x)\right\}^2 = \frac{1}{\{\Gamma(\nu+1)\}^2}\left(\tfrac{1}{2}x\right)^{2\nu}{}_1F_2\left(\nu+\tfrac{1}{2};\ \nu+1,\ 2\nu+1;\ -x^2\right), \quad \ldots (19)$$

and get

$$\left\{J_\nu(x)\right\}^2 = \frac{2^{-2n}x^{2\nu}}{\sqrt{\pi}\,(1+\alpha;n)}\sum_{r=0}^{n}\frac{(-1)^r\,{}^nC_r\,(\alpha;r)(1+\frac{\alpha}{2};r)\,\Gamma(\frac{1}{2}+\alpha+2r)\,\Gamma(1+2\alpha+2n+2r)\,\Gamma(\nu+\frac{1}{2}+2r)}{(\frac{1}{2}\alpha;r)(1+\alpha+n;r)\,\Gamma(\frac{1}{2}+\alpha+n+2r)\,\Gamma(1+2\alpha+4r)\,\Gamma(\nu+1+2r)\,\Gamma(2\nu+1+2r)}$$

$$\times\ x^{4r}\,{}_3F_4\left(\begin{matrix}\frac{1}{2}+\alpha+2r,\ 1+2\alpha+2n+2r,\ \nu+\frac{1}{2}+2r;\ -x^2\\ \frac{1}{2}+\alpha+n+2r,\ 1+2\alpha+4r,\ \nu+1+2r,\ 2\nu+1+2r\end{matrix}\right), \quad \ldots (20)$$

Also in (12) take $p=2$, $q=0$ with $\alpha_1=\frac{1}{2}+\nu$, $\alpha_2=\frac{1}{2}-\nu$ and write $(2x)$ for x then apply (34) CHAP. I., so getting

$$K_\nu(x)=2^{-2n}\frac{\cos\nu\pi\cdot e^{-x}}{\sqrt{(2\pi x)}\,(1+\alpha;n)}\sum_{r=0}^{n}\frac{(-1)^r\,{}^nC_r\,(\alpha;r)(1+\frac{1}{2}\alpha;r)}{(\frac{1}{2}\alpha;r)\,(1+\alpha+n;r)}(2x)^{-2r}$$

$$\times E\left\{\begin{matrix}\frac{1}{2}+\alpha+2r,\,1+2\alpha+2n+2r,\,\frac{1}{2}+\nu+2r,\,\frac{1}{2}-\nu+2r:2x\\ \frac{1}{2}+\alpha+n+2r,\,1+2\alpha+4r\end{matrix}\right\},\quad\text{----(21)}$$

§3. Third formula: The formula to be proved is

$$\sum_{r=0}^{n}\frac{(-1)^r\,{}^nC_r\,(\alpha;r)(1+\frac{1}{2}\alpha;r)}{(\frac{1}{2}\alpha;r)}E\left\{\begin{matrix}\rho-\alpha,\,\rho+n,\,\alpha_1,\ldots,\alpha_p:x\\ \rho+r,\,\rho-\alpha+n-1,\,\rho_1,\ldots,\rho_q\end{matrix}\right\}$$

$$=(\rho-\alpha-n-1)E(p;\alpha_r:q;\rho_s:x)-1/x\,E(p;\alpha_r+1:q;\rho_s+1;x),\quad\cdots(22)$$

To prove (22) consider the special case with $p=q=0$. The the coefficient of $\left(-\frac{1}{x}\right)^s$ on the L.H.S. is

$$\sum_{r=0}^{n}\frac{(-1)^r\,{}^nC_r\,(\alpha;r)(1+\frac{1}{2}\alpha;r)}{(\frac{1}{2}\alpha;r)\,\lfloor s}\times\frac{\Gamma(\rho-\alpha+s)\,\Gamma(\rho+n+s)}{\Gamma(\rho+r+s)\,\Gamma(\rho-\alpha+n-1+s)}$$

$$=\frac{\Gamma(\rho-\alpha+s)\,\Gamma(\rho+n+s)}{\Gamma(\rho+s)\,\Gamma(\rho-\alpha+n-1+s)}\cdot\frac{1}{\lfloor s}\cdot F\left(\begin{matrix}-n,\,\alpha,\,1+\frac{1}{2}\alpha;\,1\\ \frac{1}{2}\alpha,\,\rho+s\end{matrix}\right)$$

But since
$$\frac{(1+\frac{1}{2}\alpha;r)}{(\frac{1}{2}\alpha;r)}=\frac{\alpha+2r}{\alpha}=1+\frac{2r}{\alpha},$$

$$\therefore F\left(\begin{matrix}-n,\alpha,1+\frac{1}{2}\alpha;1\\ \frac{1}{2}\alpha,\rho+s\end{matrix}\right)=F\left(\begin{matrix}-n,\alpha;1\\ \rho+s\end{matrix}\right)+\frac{(-n)(\alpha)}{(\frac{1}{2}\alpha)(\rho+s)}F\left(\begin{matrix}-n+1,\alpha+1;1\\ \rho+s+1\end{matrix}\right)$$

$$=\frac{\Gamma(\rho+s)\Gamma(\rho+s+n-\alpha)}{\Gamma(\rho+s+n)\Gamma(\rho+s-\alpha)}+\frac{(-2n)}{\rho+s}\cdot\frac{\Gamma(\rho+s+1)\Gamma(\rho-\alpha+s+n-1)}{\Gamma(\rho+s+n)\Gamma(\rho+s-\alpha)}.$$

So that the last coefficient is equal to

$$\frac{1}{\lfloor s}\left\{(\rho+s+n-\alpha-1)-2n\right\}=\frac{1}{\lfloor s}(\rho+s-n-\alpha-1).$$

Also the coefficient of $(-\frac{1}{x})^s$ on the R.H.S. of (22) with $p=q=0$ is equal to

$$\frac{1}{\lfloor s}(\rho-\alpha-n-1)+\frac{1}{\lfloor s-1}=\frac{1}{\lfloor s}(\rho-\alpha-n-1+s).$$

Hence (22) is proved for the special case $p=q=0$. The general formula can be deduced in the usual way.

A second proof of (22): This proof is by induction. The following are required in this proof

$$E(p;\alpha_p:\rho_1-1,\rho_2,\dots,\rho_q;x) = E(\alpha_1+1,\alpha_2,\dots,\alpha_p;q;\rho_s;x)$$
$$-(\alpha_1-\rho_1+1)E(p;\alpha_p:q;\rho_s;x), \quad \dots\dots (23),$$

$$\sum_{r=0}^{n-1}(-1)^r\,{}^{n-1}C_r\,(\alpha+1;r)\,E\left\{\begin{matrix}\rho-\alpha+1,\rho+n+1,\alpha_1,\dots,\alpha_p:x\\ \rho+r+2,\rho-\alpha+n,\rho_1,\dots,\rho_q\end{matrix}\right\} = E(p;\alpha_p:q;\rho_s:x), \quad (24)$$

(23) is obtained by the substraction of (8) and (9); while (24) can be proved by substituting for the E-function on the left from (2), changing the order of integration and summation, and then applying Gauss's theorem

When $n=0$, or $n=1$, (22) can be deduced from (8) and (9). When $n=2$ the L.H.S. of (22) is equal to

$$E(\rho-\alpha,\rho+2,\alpha_1,\dots,\alpha_p:\rho,\rho-\alpha+1,\rho_1,\dots,\rho_q:x)$$
$$-2(2+\alpha)E(\rho-\alpha,\rho+2,\alpha_1,\dots,\alpha_p:\rho+1,\rho-\alpha+1,\rho_1,\dots,\rho_q:x)$$

$$+(\alpha+1)(\alpha+4)E(\rho-\alpha, \alpha_1, \cdots, \alpha_p : \rho-\alpha+1, \rho_1, \cdots, \rho_q : x).$$

Apply (23) with $(\rho-\alpha)$, $(\rho+1)$ in the roles of $\underline{\alpha_1}$ and $\underline{\rho_1}$ to the first E-function in the last expression, then it becomes

$$E(\rho+2, \alpha_1, \cdots, \alpha_p : \rho+1, \rho_1, \cdots, \rho_q : x)$$
$$-(4+\alpha)\, E(\rho-\alpha, \rho+2, \alpha_1, \cdots, \alpha_p : \rho-\alpha+1, \rho+1, \rho_1, \cdots, \rho_q : x)$$
$$+(\alpha+1)(\alpha+4)E(\rho-\alpha, \alpha_1, \cdots, \alpha_p : \rho-\alpha+1, \rho_1, \cdots, \rho_q : x).$$

Again apply (23) with $(\rho-\alpha)$, $(\rho+2)$ in the roles of $\underline{\alpha_1}$ and $\underline{\rho_1}$ to the second E-function in the last expression, then it becomes

$$E(\rho+2, \alpha_1, \cdots, \alpha_p : \rho+1, \rho_1, \cdots, \rho_q : x)$$
$$-(4+\alpha)E(p; \alpha_r : q; \rho_s : x)$$
$$=(\rho-\alpha-3)E(p; \alpha_r : q; \rho_s : x) - 1/x\, E(p; \alpha_r+1 : q; \rho_s+1 : x) \text{ by } (8).$$

For the proof of the general case of (22), suppose that it is true for a particular value of $\underline{n}$ $(n \geq 2)$. Then it will be proved that it is true when $(n+1)$ is written for $\underline{n}$.

Thus the following formula which is (22) with $(\rho+1)$ in place of $\underline{\rho}$ is assumed to be true

$$\sum_{r=0}^{n} \frac{(-1)^r \, {}^nC_r \, (\alpha; r)(1+\tfrac{1}{2}\alpha; r)}{(\tfrac{1}{2}\alpha; r)} E\left\{\begin{matrix} \rho+1-\alpha, \rho+n+1, \alpha_1, \dots, \alpha_p : x \\ \rho+r+1, \rho-\alpha+n, \rho_1, \dots, \rho_q \end{matrix}\right\}$$

$$= (\rho-\alpha-n) E(p; \alpha_r : q; \rho_s : x) - 1/x \, E(p; \alpha_r+1 : q; \rho_s+1 : x), \quad \dots (25);$$

The L.H.S. of (22) with $(n+1)$ in place of $\underline{n}$ is equal to

$$\sum_{r=0}^{n} \frac{(-1)^r \, {}^{n+1}C_r \, (\alpha; r)(1+\tfrac{1}{2}\alpha; r)}{(\tfrac{1}{2}\alpha; r)} E\left\{\begin{matrix} \rho-\alpha, \rho+n+1, \alpha_1, \dots, \alpha_p : x \\ \rho+r, \rho-\alpha+n, \rho_1, \dots, \rho_q \end{matrix}\right\}$$

$$+ \frac{(-1)^{n+1} (\alpha; n+1)(1+\tfrac{1}{2}\alpha; n+1)}{(\tfrac{1}{2}\alpha; n+1)} E\left\{\begin{matrix} \rho-\alpha, \alpha_1, \dots, \alpha_p : x \\ \rho-\alpha+n, \rho_1, \dots, \rho_q \end{matrix}\right\}.$$

Now keep the last term, and apply (23) with $(\rho-\alpha)$, $(\rho+r+1)$ in the roles of $\underline{\alpha_1}$ and $\underline{\rho_1}$ respectively to each term of the last series. Thus the L.H.S. of (22) with $(n+1)$ in place of $\underline{n}$ becomes

$$\sum_{r=0}^{n} \frac{(-1)^r\, {}^{n+1}C_r\, (\alpha;r)(1+\frac{1}{2}\alpha;r)}{(\frac{1}{2}\alpha;r)} E\left\{\begin{matrix} \rho-\alpha+1, \rho+n+1, \alpha_1, \ldots, \alpha_p : x \\ \rho+r+1, \rho-\alpha+n, \rho_1, \ldots, \rho_q \end{matrix}\right\}$$

$$+\sum_{r=0}^{n} \frac{(-1)^{r+1}\, {}^{n+1}C_r\, (\alpha;r)(1+\frac{1}{2}\alpha;r)}{(\frac{1}{2}\alpha;r)} (-\alpha-r) E\left\{\begin{matrix} \rho-\alpha, \rho+n+1, \alpha_1, \ldots, \alpha_p : x \\ \rho+r+1, \rho-\alpha+n, \rho_1, \ldots, \rho_q \end{matrix}\right\}$$

$$+ \frac{(-1)^{n+1}(\alpha;n+1)(1+\frac{1}{2}\alpha;n+1)}{(\frac{1}{2}\alpha;n+1)} E\left\{\begin{matrix} \rho-\alpha, \alpha_1, \ldots, \alpha_p : x \\ \rho-\alpha+n, \rho_1, \ldots, \rho_q \end{matrix}\right\}.$$

If in the second of the last two series $\underline{r}$ is replaced by $(r-1)$, then it becomes

$$\sum_{r=1}^{n} \frac{(-1)^r\, {}^{n+1}C_{r-1}(\alpha;r-1)(1+\frac{\alpha}{2};r-1)}{(\frac{1}{2}\alpha;r-1)} (-\alpha-r+1) E\left\{\begin{matrix} \rho-\alpha, \rho+n+1, \alpha_1, \ldots, \alpha_p : x \\ \rho+r, \rho-\alpha+n, \rho_1, \ldots, \rho_q \end{matrix}\right\}$$

$$+ \frac{(-1)^{n+1}\, {}^{n+1}C_n\, (\alpha;n)(1+\frac{1}{2}\alpha;n)}{(\frac{1}{2}\alpha;n)} (-\alpha-n) E\left\{\begin{matrix} \rho-\alpha, \alpha_1, \ldots, \alpha_p : x \\ \rho-\alpha+n, \rho_1, \ldots, \rho_q \end{matrix}\right\}.$$

On applying (23) again as the last application, the last expression is found to be equal to

$$\sum_{r=1}^{n} \frac{(-1)^{r}\, {}^{n+1}C_{r-1}\,(\alpha;r-1)\,(1+\tfrac{1}{2}\alpha;r-1)}{(\tfrac{1}{2}\alpha;r-1)}(-\alpha-r+1)\,I_r + \frac{(-1)^{n+1}\, {}^{n+1}C_n\,(\alpha;n)\,(1+\tfrac{1}{2}\alpha;n)\,(-\alpha-n)}{(\tfrac{1}{2}\alpha;n)}\,J_n$$

$$+\sum_{r=1}^{n} \frac{(-1)^{r+1}\, {}^{n+1}C_{r-1}\,(\alpha;r-1)(1+\tfrac{1}{2}\alpha;r-1)}{(\tfrac{1}{2}\alpha;r-1)}(-\alpha-r+1)(-\alpha-r)\,E\left\{\begin{matrix}\rho-\alpha,\ \rho+n+1,\ \alpha_1,\dots,\alpha_p : x\\ \rho+r+1,\ \rho-\alpha+n,\ \rho_1,\dots,\rho_q\end{matrix}\right\},$$

where $I_r \equiv E(\rho-\alpha+1,\ \rho+n+1,\ \alpha_1,\dots,\alpha_p : \rho+r+1,\ \rho-\alpha+n,\ \rho_1,\dots,\rho_q : x)$,

$J_n \equiv E(\rho-\alpha,\ \alpha_1,\dots,\alpha_p : \rho-\alpha+n,\ \rho_1,\dots,\rho_q : x)$.

Similarly, on keeping the last term of the second series in the last expression, then replacing r by $(r-1)$ and applying (23) as before, then this series is found to be equal to

$$\sum_{r=2}^{n} \frac{(-1)^{r}\, {}^{n+1}C_{r-2}\,(\alpha;r-2)\,(1+\tfrac{1}{2}\alpha;r-2)}{(\tfrac{1}{2}\alpha;r-2)}(-\alpha-r+2)(-\alpha-r+1)\,I_r$$

$$+\sum_{r=2}^{n} \frac{(-1)^{r+1}\, {}^{n+1}C_{r-2}\,(\alpha;r-2)(1+\tfrac{1}{2}\alpha;r-2)}{(\tfrac{1}{2}\alpha;r-2)}(-\alpha-r+2)(-\alpha-r+1)(-\alpha-r)$$

$$\times E(\rho-\alpha,\ \rho+n+1,\ \alpha_1,\dots,\alpha_p : \rho+r+1,\ \rho-\alpha+n,\ \rho_1,\dots,\rho_q : x)$$

$$+\ \frac{(-1)^{n+1}\, {}^{n+1}C_{n-1}\,(\alpha;n-1)(1+\tfrac{1}{2}\alpha;n-1)}{(\tfrac{1}{2}\alpha;n-1)}(-\alpha-n+1)(-\alpha-n)\,J_n .$$

Proceeding thus, we find that the coefficient of I_r where $r=1,2,3,\dots,n$ is

$$\frac{(-1)^r\,{}^{n+1}C_r\,(\alpha;r)(1+\tfrac{1}{2}\alpha;r)}{(\tfrac{1}{2}\alpha;r)}+\frac{(\alpha;r)}{\alpha}\sum_{m=1}^{r}\frac{(-n-1;r-m)}{(r-m)!}(\alpha+2r-2m),$$

$$\text{since}\quad\frac{(1+\tfrac{1}{2}\alpha;r)}{(\tfrac{1}{2}\alpha;r)}=\frac{\alpha+2r}{\alpha}.$$

Also the coefficient of J_n is found to be equal to

$$-\frac{(-1)^n(\alpha;n+1)(1+\tfrac{1}{2}\alpha;n+1)}{(\tfrac{1}{2}\alpha;n+1)}+\frac{(-1)^n\,{}^{n+1}C_n\,(\alpha;n)(1+\tfrac{1}{2}\alpha;n)}{(\tfrac{1}{2}\alpha;n)}(\alpha+n)$$

$$+(\alpha;n+1)\sum_{m=1}^{n}\frac{(-n-1;n-m)}{(n-m)!}\cdot\frac{(\alpha+2n-2m}{\alpha}.$$

Now the coefficient of I_r can be written

$$\frac{(-1)^r\,{}^{n+1}C_r\,(\alpha;r)(1+\tfrac{\alpha}{2};r)}{(\tfrac{1}{2}\alpha;r)}+(\alpha;r)\sum_{m=1}^{r}\frac{(-n-1;r-m)}{(r-m)!}+\frac{2(\alpha;r)}{\alpha}\sum_{m=1}^{r-1}\frac{(-n-1;r-m)}{(r-m-1)!}$$

$$=\frac{(-1)^r\,{}^{n+1}C_r\,(\alpha;r)(1+\tfrac{\alpha}{2};r)}{(\tfrac{1}{2}\alpha;r)}+(\alpha;r)\left\{\sum_{m=1}^{r}\frac{(-n;r-m)}{(r-m)!}-\sum_{m=1}^{r-1}\frac{(-n;r-m-1)}{(r-m-1)!}\right\}$$

$$-\frac{2(\alpha;r)}{\alpha\cdot n}\left\{\sum_{m=1}^{r-1}\frac{(-n-1;r+1-m)}{(r-m-1)!}-\sum_{m=1}^{r-2}\frac{(-n-1;r-m)}{(r-m-2)!}\right\},$$

$$=\frac{(-1)^r\,{}^{n+1}C_r(\alpha;r)(1+\frac{1}{2}\alpha;r)}{(\frac{1}{2}\alpha;r)}+(\alpha;r)\frac{(-n;r-1)}{(r-1)!}-\frac{2(\alpha;r)}{\alpha\cdot n}\cdot\frac{(-n-1;r)}{(r-2)!}.$$

From this, it is easily seen that coefficient of J_n is equal to zero.

But since

$$^{n+1}C_r = {}^nC_r + {}^nC_{r-1};$$

then the coefficient of I_r, $r=1,2,3,\ldots,n$ is equal to

$$(-)^r\frac{{}^nC_r(\alpha;r)(1+\frac{1}{2}\alpha;r)}{(\frac{1}{2}\alpha;r)}+(-1)^r\frac{{}^nC_{r-1}(\alpha;r)(1+\frac{1}{2}\alpha;r)}{(\frac{1}{2}\alpha;r)}+(-1)^{r-1}\,{}^nC_{r-1}(\alpha;r)$$

$$-2(-1)^r\,{}^{n+1}C_r\cdot\frac{(\alpha;r)(r)!}{\alpha\cdot n(r-2)!},$$

$$=(-1)^r\frac{{}^nC_r(\alpha;r)(1+\frac{1}{2}\alpha;r)}{(\frac{1}{2}\alpha;r)}+(-1)^r(\alpha+1;r-1)\,{}^nC_{r-1}\left[\alpha+2r-\alpha\right]$$

$$+2(-1)^{r+1}\,{}^{n+1}C_r\frac{(\alpha;r)(r)!}{\alpha\cdot n(r-2)!}$$

$$= (-1)^r \frac{{}^nC_r(\alpha;r)(1+\frac{1}{2}\alpha;r)}{(\frac{1}{2}\alpha;r)} + 2(-1)^r(\alpha+1;r-1)r\left[{}^nC_{r-1} - (r-1)\frac{{}^{n+1}C_r}{n}\right]$$

$$= (-1)^r \frac{{}^nC_r(\alpha;r)(1+\frac{1}{2}\alpha;r)}{(\frac{1}{2}\alpha;r)} + 2(-1)^r(\alpha+1;r-1)\,{}^{n-1}C_{r-1}.$$

Therefore the L.H.S. of (22) with $(n+1)$ in place of $\underline{n}$ is equal to

$$\sum_{r=1}^{n} \frac{(-1)^r\,{}^nC_r(\alpha;r)(1+\frac{\alpha}{2};r)}{(\frac{1}{2}\alpha;r)} E\left\{\begin{matrix}\rho-\alpha+1, \rho+n+1, \alpha_1, \ldots, \alpha_p : x\\ \rho+r+1, \rho-\alpha+n, \rho_1, \ldots, \rho_q\end{matrix}\right\}$$

$$-2\sum_{r=0}^{n-1}(-1)^r\,{}^{n-1}C_r(\alpha+1;r)E\left\{\begin{matrix}\rho-\alpha+1, \rho+n+1, \alpha_1, \ldots, \alpha_p : x\\ \rho+r+2, \rho-\alpha+n, \rho_1, \ldots, \rho_q\end{matrix}\right\}$$

$$= (\rho-\alpha-n-2)E(p;\alpha_r:q;\rho_s:x) - 1/x\,E(p;\alpha_r+1:q;\rho_s+1:x) \text{ by}$$

(24) and (25),

= R.H.S. of (22) with $(n+1)$ in place of $\underline{n}$. But (22) is true for $n=0$, $n=1$ and $n=2$. Hence it is true for all positive integral values of $\underline{n}$.

Particular cases: In (22) take $x=-1$, $p=2$, $q=1$, so getting

$$\sum_{r=0}^{n} \frac{(-1)^r \, {}^nC_r (\alpha;r)(1+\frac{1}{2}\alpha;r)\Gamma(\rho-\alpha)\Gamma(\rho+n)\Gamma(\beta)\Gamma(\delta)}{(\frac{1}{2}\alpha;r)\,\Gamma(\rho+r)\Gamma(\rho-\alpha+n-1)\Gamma(\gamma)} \, {}_4F_3\left(\begin{matrix}\rho-\alpha, \rho+n, \beta, \delta; 1\\ \rho+r, \rho-\alpha+n-1, \gamma\end{matrix}\right)$$

$$=(\rho-\alpha-n-1)\,\frac{\Gamma(\beta)\Gamma(\delta)\Gamma(\gamma-\beta-\delta)}{\Gamma(\gamma-\beta)\,\Gamma(\gamma-\delta)} + \frac{\Gamma(\beta+1)\Gamma(\delta+1)\Gamma(\gamma-\beta-\delta-1)}{\Gamma(\gamma-\beta)\Gamma(\gamma-\delta)}, \quad \cdots\cdots (26)$$

where $R(\gamma)>0$, $R(\gamma-\beta-\delta)>0$, $R(\gamma-\beta-\delta-1)>0$.

Also in (22) take $p=0$, $q=1$ and apply (6) after writting $(4/x^2)$ for and get

$$(4/x^2)^{\frac{1}{2}\mu}\left[(\rho-\alpha-n-1)\,J_\mu(x) - \frac{x}{2}\,J_{\mu+1}(x)\right] =$$

$$\sum_{r=0}^{n} \frac{(-1)^r \, {}^nC_r(\alpha;r)(1+\frac{\alpha}{2};r)\Gamma(\rho-\alpha)\Gamma(\rho+n)}{(\frac{1}{2}\alpha;r)\,\Gamma(\rho+r)\Gamma(\rho-\alpha+n-1)\Gamma(\mu+1)} \, {}_2F_3\left(\begin{matrix}\rho-\alpha, \rho+n; -x^2/4\\ \rho+r, \rho-\alpha+n-1, \mu+1\end{matrix}\right), \quad \cdots\cdots (27).$$

In (22) write $(1/x^2)$ for $\underline{x}$ and apply (19) which can be written in the form

$$E(\nu+\tfrac{1}{2} : \nu+1, 2\nu+1 : 1/x^2) = \sqrt{\pi}.\, x^{-2\nu}\left\{J_\nu(x)\right\}^2, \quad \cdots\cdots (28)$$

so getting

$$\sum_{r=0}^{n} \frac{(-1)^r {}^nC_r (\alpha;r)(1+\tfrac{1}{2}\alpha;r)\Gamma(\rho-\alpha)\Gamma(\rho+n)\Gamma(\nu+\tfrac{1}{2})}{(\tfrac{1}{2}\alpha;r)\Gamma(\nu+1)\Gamma(2\nu+1)\Gamma(\rho+r)\Gamma(\rho-\alpha+n-1)}\, {}_3F_4\left(\begin{matrix}\rho-\alpha, \rho+n, \nu+\tfrac{1}{2}; -x^2\\ \rho+r, \rho-\alpha+n-1, \nu+1, 2\nu+1\end{matrix}\right)$$

$$+ \frac{\Gamma(\nu+\tfrac{3}{2})}{\Gamma(\nu+2)\Gamma(2\nu+2)} x^2 {}_1F_2(\nu+\tfrac{3}{2}; \nu+2, 2\nu+2; -x^2)$$

$$= \sqrt{\pi}\, x^{-2\nu}(\rho-\alpha-n-1)\left\{J_\nu(x)\right\}^2, \qquad (29).$$

§4. Fourth formula: The formula to be proved is

$$\frac{1}{\Gamma(h)} \sum_{n=0}^{\infty} \frac{1}{\lfloor n} {}_1F_1(-n;h;1)\, x^{-n} E(h+n, \alpha_1+n, \dots, \alpha_p+n : q; \rho_t+n : x) = E(p; \alpha_r : q; \rho_s : x) \qquad (30)$$

To prove (30) consider the special case with $p=q=0$; then the L.H.S. of (30) becomes

$$\sum_{n=0}^{\infty} \frac{(h;n)}{\lfloor n} F(-n;h;1)\, x^{-n}(1+1/x)^{-h-n} = \left(\frac{x}{1+x}\right)^h \sum_{n=0}^{\infty} \frac{(h;n)}{\lfloor n}(1+x)^{-n} F(-n;h;1)$$

$$= \left(\frac{x}{1+x}\right)^h \sum_{n=0}^{\infty} \frac{(h;n)}{\lfloor n}(1+x)^{-n} \sum_{r=0}^{n} \frac{(-1)^r\, {}^nC_r}{(h;r)} = \left(\frac{x}{1+x}\right)^h \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r\,(h;r)} \sum_{n=r}^{\infty} \frac{(h;n)}{\lfloor n-r}(1+x)^{-n}$$

$$= \left(\frac{x}{1+x}\right)^h \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r\,(h;r)} \sum_{n=0}^{\infty} \frac{(h;n+r)}{\lfloor n}(1+x)^{-n-r}$$

$$= \left(\frac{x}{1+x}\right)^h \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r} (1+x)^{-r} \sum_{n=0}^{\infty} \frac{(h+r;n)}{\lfloor n} (1+x)^{-n}$$

$$= \left(\frac{x}{1+x}\right)^h \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r} (1+x)^{-r} \left[1-\frac{1}{1+x}\right]^{-h-r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r} x^{-r} = e^{-\frac{1}{x}} = E(\;::x)$$

= R.H.S. of (30) with $p=q=0$.

Thus (30) is proved for the special case $p=q=0$. The general case can be deduced in the usual way.

As a particular case of (30) write $(2x)$ for $\underline{x}$ and apply (34) CHAP-I., and get

$$K_\mu(x) = \frac{\cos\mu\pi \cdot e^{-x}}{\sqrt{(2\pi x)}\,\Gamma(h)} \sum_{n=0}^{\infty} \frac{1}{\lfloor n} {}_1F_1(-n;h;1)(2x)^{-n} E(h+n, \tfrac{1}{2}+\mu+n, \tfrac{1}{2}-\mu+n::2x)$$

$$\text{---}(31).$$

Also from (6), (30) gives

$$J_\mu(x) = \frac{1}{\Gamma(h)} \left(\frac{x}{2}\right)^\mu \sum_{n=0}^{\infty} \frac{1}{\lfloor n} {}_1F_1(-n;h;1) \left(\frac{x}{2}\right)^{2n} \frac{\Gamma(h+n)}{\Gamma(\mu+n+1)} {}_1F_1(h+n;\mu+n+1;-x^2/4)$$

$$\text{---}(32).$$

If $x=-1$, $p=2$, $q=1$; then (30) gives

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\lfloor n} {}_1F_1(-n;h;1)\, E(h+n, \alpha+n, \beta+n : \gamma+n ; -1) = \frac{\Gamma(h)\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} \quad , \ldots (33).$$

§5. Fifth formula: The formula to be established is

$$\sum_{s=0}^{\infty} \frac{1}{\Gamma(\frac{1}{2}\alpha_r + s + \frac{1}{2}) \lfloor s} (2x)^{-2s} E(p; \alpha_r + 2s ; q ; \rho_t + 2s ; x).$$

$$= \frac{2^{\alpha_r - 1}}{\sqrt{\pi}} E\left(\alpha_1, \alpha_2, \ldots, \frac{\alpha_r}{2}, \ldots, \alpha_p ; q ; \rho_t : \frac{x}{2}\right)$$

where $r = 1, 2, 3, \ldots, p$, and $p \geqslant 1$. $\quad , \ldots (34)$

To prove (34) consider the special case with $p=1$, $q=0$ and $\alpha_1 = \alpha$, then the L.H.S. of (34) is equal to

$$\sum_{s=0}^{\infty} \frac{\Gamma(\alpha+2s)}{\Gamma(\frac{\alpha}{2}+\frac{1}{2}+s)} \cdot \frac{1}{\lfloor s} (2x)^{-2s} \left(1+\frac{1}{x}\right)^{-\alpha-2s} = \frac{2^{\alpha-1}\Gamma(\frac{\alpha}{2})}{\sqrt{\pi}} \sum_{s=0}^{\infty} \frac{x^{-2s}}{\lfloor s} (\tfrac{1}{2}\alpha; s) \left(\frac{x}{1+x}\right)^{\alpha+2s}$$

$$= \frac{2^{\alpha-1}\Gamma(\frac{1}{2}\alpha)}{\sqrt{\pi}} \left(\frac{x}{1+x}\right)^{\alpha} \left[1 - \frac{1}{(1+x)^2}\right]^{-\frac{1}{2}\alpha} = \frac{2^{\alpha-1}}{\sqrt{\pi}} E\left(\frac{\alpha}{2} :: \frac{1}{2}x\right)$$

$= $ R.H.S. of (34) with $p=1$, $q=0$.

Thus (34) is proved for this special case. The general case can then be deduced in the usual way.

Applications: (34) gives at once if $\rho_1 = \alpha_1$ and $p = q = 1$, the known expansion

$$ {}_1F_1(a; 2a : 2x) = e^x {}_0F_1(; a+\tfrac{1}{2} ; x^2/4), \quad \cdots \cdots (35)$$

from which the formula [B.V., p., 344, form. (14)] namely

$$I_n(z) = \frac{1}{\Gamma(n+1)} \left(\frac{z}{2}\right)^n e^{-z} {}_1F_1(n+\tfrac{1}{2} ; 2n+1 ; 2z), \quad \cdots \cdots (36);$$

is deduced.

If in (34) $x = -2$, $r=1$, $p=2$, $q=1$ with $\alpha_1 = \alpha$, $\alpha_2 = \beta$, $\rho_1 = \gamma$, then it gives

$$\sum_{s=0}^{\infty} \frac{\Gamma(\alpha+2s)\Gamma(\beta+2s)(-4)^{-2s}}{\lfloor s \, \Gamma(\frac{1}{2}\alpha+\frac{1}{2}+s)\Gamma(\gamma+2s)} F\left(\begin{matrix}\alpha+2s, \beta+2s; \frac{1}{2} \\ \gamma+2s\end{matrix}\right) = \frac{2^{\alpha-1}\Gamma(\frac{\alpha}{2})\Gamma(\beta)\Gamma(\gamma-\frac{\alpha}{2}-\beta)}{\sqrt{\pi}\,\Gamma(\gamma-\frac{\alpha}{2})\Gamma(\gamma-\beta)}$$

$$, \cdots (37),$$

where $R(\gamma - \frac{\alpha}{2} - \beta) > 0$.

If in (34), $(4x)$ is written for x; then applying (34) CHAP. I. one gets

$$K_\mu(x) = \frac{\cos \mu\pi \, e^{-x}}{2^{2\mu+1/2}\sqrt{x}} \sum_{s=0}^{\infty} \frac{1}{\Gamma(1+\mu+s)} \cdot \frac{1}{\lfloor s} (8x)^{-2s} E(1+2\mu+2s, \tfrac{1}{2}-\mu+2s : 4x), \cdot\cdot (38).$$

§6. Sixth formula: The formula to be proved is

$$\sum_{s=0}^{\infty} \frac{1}{s!}(1-1/\lambda)^s w^{-s} E(p ; \alpha_r+s ; q ; \rho_t+s : w) = E(p ; \alpha_r : q, \rho_t : w\lambda), \cdot\cdot (39)$$

where $R(\lambda) > \frac{1}{2}$ and $|\text{amp}\, z| < \pi$.

To prove (39) consider the special case with $p = q = 0$. Then the L.H.S.

of (39) is equal to exp $[-1/\lambda w]$ which is $E[::w\lambda]$ and the general case can then be deduced in the usual way.

Another simple proof of (39) is by substituting for the E-function in the L.H.S. of (39) from (2), writing $(\xi+s)$ for $\underline{\xi}$ and then changing the order of integration and summation.

<u>Applications of (39):</u>— If in (39) $4/(z+h)$ is written for $\underline{\lambda w}$ and $(4/z)$ for $\underline{w}$, and (6) applied; then (39) with $p=0$, $q=1$ gives the first form of Lommel's expansion for $(z+h)^{\pm\frac{1}{2}\nu}J_\nu\{\sqrt{(z+h)}\}$, [Watson, "Theory of Bessel Functions" p. 140, equations (1) and (2)] namely

$$(z+h)^{-\frac{1}{2}\nu}J_\nu\{\sqrt{(z+h)}\}=\sum_{m=0}^{\infty}\frac{(-\frac{1}{2}h)^m}{m!}z^{-\frac{1}{2}(\nu+m)}J_{\nu+m}(\sqrt{z}), \quad \ldots\ldots (40).$$

It may be noted that the other expansion

$$(z+h)^{\frac{1}{2}\nu}J_\nu\{\sqrt{(z+h)}\}=\sum_{m=0}^{\infty}\frac{(\frac{1}{2}h)^m}{m!}z^{\frac{1}{2}(\nu-m)}J_{\nu-m}(\sqrt{z}), \quad \ldots\ldots (41)$$

can be deduced from the formula [Ragab. F.M., thesis for Ph.D., p 20 eqn (20)] namely

$$\lambda^{1-\rho_t}E(p;\alpha_r:q;\rho_t:\lambda z)=\sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{1}{\lambda}-1\right)^m E(p;\alpha_r:\rho_1,\rho_2,\ldots,\rho_t-m,\ldots,\rho_q:z), \ldots (42)$$

where $R(\lambda)>\frac{1}{2}$ and $t=1,2,3,\ldots,q$.

Other deductions from (39) are the following expansions

$$\sum_{s=0}^{\infty}(-1/x^2)^s\frac{1}{s!}E\left(p;\alpha_r+s:q;\rho_t+s:\frac{x}{2}\right)=E\left\{p;\alpha_r:q;\rho_t:\frac{x^2}{1+2x}\right\}, \quad\ldots\ldots (43)$$

and

$$\sum_{s=0}^{\infty}(1/x^2)^s\frac{1}{s!}E\left(p;\alpha_r+s:q;\rho_t+s:\frac{x^2}{1+2x}\right)=E\left(p;\alpha_r:q;\rho_t:\frac{x}{2}\right), \quad\ldots\ldots (44).$$

Combining (34) CHAP. I. and (6) this chapter with (43) and (44), we obtain more expansions of $K_\nu(x)$ and $J_\nu(x)$.

§ 7. <u>Seventh formula</u>: The formula to be proved is

$$\sum_{r=0}^{\infty}\frac{(\alpha;r)(1+\frac{1}{2}\alpha;r)}{r!\,(\frac{1}{2}\alpha;r)}x^{-3r}E\left\{\begin{matrix}\alpha_1+3r,\ldots,\alpha_p+3r:x\\1+3\alpha+6r,\rho_1+3r,\ldots,\rho_q+3r\end{matrix}\right\}$$

$$=\frac{2^{1-3\alpha}}{3}\cdot\frac{\sqrt{\pi}}{\Gamma(1+\alpha)}E\left\{\begin{matrix}\alpha,\alpha_1,\ldots,\alpha_p:\frac{4}{3}x\\\frac{3}{2}\alpha,\frac{3}{2}\alpha+\frac{1}{2},\rho_1,\ldots,\rho_q\end{matrix}\right\}, \quad\ldots\ldots (45).$$

To prove (45), substitute for the E-function in the L.H.S. of it from (2) so getting

$$\text{L.H.S. of }(45)=\sum_{r=0}^{\infty}\frac{(\alpha;r)(1+\frac{1}{2}\alpha;r)}{r!\,(\frac{1}{2}\alpha;r)}x^{-3r}\frac{1}{2\pi i}\int\frac{\Gamma(\xi)\prod\Gamma(\alpha_t+3r-\xi)}{\Gamma(1+3\alpha+6r-\xi)\prod\Gamma(\rho_s+3r-\xi)}x^{\xi}d\xi.$$

Here replace ξ by $(\xi+3r)$, note that

$$\frac{\Gamma(\xi+3r)}{\Gamma(1+3\alpha-\xi+3r)} = \frac{\Gamma(\xi)}{\Gamma(1+3\alpha-\xi)} \cdot \frac{\left(\frac{\xi}{3};r\right)\left(\frac{\xi+1}{3};r\right)\left(\frac{\xi+2}{3};r\right)}{\left(\alpha+\frac{1}{3}-\frac{\xi}{3};r\right)\left(\alpha+\frac{2}{3}-\frac{\xi}{3};r\right)\left(\alpha+1-\frac{\xi}{3};r\right)},$$

and it can be seen that the L.H.S. of (45) is equal to

$$\frac{1}{2\pi i}\int \frac{\Gamma(\xi)}{\Gamma(1+3\alpha-\xi)} F\left(\begin{matrix}\alpha, 1+\frac{1}{2}\alpha, \frac{\xi}{3}, \frac{\xi+1}{3}, \frac{\xi+2}{3}\\ \frac{1}{2}\alpha, \alpha+\frac{1}{3}-\frac{\xi}{3}, \alpha+\frac{2}{3}-\frac{\xi}{3}, \alpha+1-\frac{\xi}{3}\end{matrix}; 1\right) x^{\xi} d\xi.$$

Now by (13), the generalized hypergeometric function is equal to

$$\frac{\Gamma(1+\alpha-\frac{\xi}{3})\Gamma(\frac{2}{3}+\alpha-\frac{\xi}{3})\Gamma(\frac{1}{3}+\alpha-\frac{\xi}{3})}{\Gamma(1+\alpha)\Gamma(\frac{2}{3}+\alpha-\frac{2}{3}\xi)\Gamma(\frac{1}{3}+\alpha-\frac{2}{3}\xi)\Gamma(\alpha-\frac{2}{3}\xi)}$$

$$= \frac{2^{1-3\alpha}}{3} \frac{\sqrt{\pi}\;\Gamma(\xi)\Gamma(\alpha-\xi)}{\Gamma(1+\alpha)\Gamma(\frac{3}{2}\alpha-\xi)\Gamma(\frac{3}{2}\alpha+\frac{1}{2}-\xi)} \cdot \left(\frac{4}{3}\right)^{\xi},$$

by using the formula [C.V., p., 154, equa. (5)], namely

$$\Gamma(z)\Gamma(z+\tfrac{1}{m})\cdots\Gamma(z+\tfrac{m-1}{m}) = (2\pi)^{\frac{1}{2}(m-1)} m^{\frac{1}{2}-mz}\,\Gamma(mz), \quad \ldots (46)$$

Thus the L.H.S. of (45) is equal to

$$\frac{2^{1-3\alpha}}{3} \frac{\sqrt{\pi}}{\Gamma(1+\alpha)} \cdot \frac{1}{2\pi i}\int \frac{\Gamma(\xi)\Gamma(\alpha-\xi)\Gamma(\alpha_1-\xi)\cdots\Gamma(\alpha_p-\xi)}{\Gamma(\frac{3}{2}\alpha-\xi)\Gamma(\frac{3}{2}\alpha+\frac{1}{2}-\xi)\Gamma(\rho_1-\xi)\cdots\Gamma(\rho_q-\xi)} \left(\frac{4x}{3}\right)^{\xi} d\xi,$$

and from this (45) follows. The restrictions necessary for (13) can be removed by analytical continuation.

<u>An alternative proof of (45)</u>: First suppose that $p \leqslant q+1$; then the E-functions in the L.H.S. are generalized hypergeometric functions and the L.H.S. of (45) is equal to

$$\sum_{r=0}^{\infty} \frac{(\alpha;r)(1+\frac{1}{2}\alpha;r)}{r!(\frac{1}{2}\alpha;r)\,x^{3r}} \sum_{m=0}^{\infty} \frac{\Gamma(\alpha_1+3r+m)\cdots\Gamma(\alpha_p+3r+m)}{m!\,\Gamma(1+3\alpha+6r+m)\Gamma(\rho_1+3r+m)\cdots\Gamma(\rho_q+3r+m)} \left(-\frac{1}{x}\right)^m.$$

Here write $\underline{n}$ for $(3r+m)$ and change the order of summation, and get

$$\text{L.H.S. of } (45) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1+n)\cdots\Gamma(\alpha_p+n)}{n!\,\Gamma(1+3\alpha+n)\Gamma(\rho_1+n)\cdots\Gamma(\rho_q+n)} \left(-\frac{1}{x}\right)^n \times F\left(\begin{matrix}\alpha, 1+\frac{1}{2}\alpha, -\frac{n}{3}, -\frac{n}{3}+\frac{1}{3}, -\frac{n}{3}+\frac{2}{3}; 1\\ \frac{1}{2}\alpha, 1+\alpha+\frac{n}{3}, \frac{2}{3}+\alpha+\frac{n}{3}, \frac{1}{3}+\alpha+\frac{n}{3}\end{matrix}\right),$$

and on applying (13) again; (45) is obtained for the case $p \leqslant q+1$. The restriction $p \leqslant q+1$ can then be removed in the same manner as formula (1) this chapter. Also the restrictions necessary for (13) can be removed by analytical continuation.

<u>Particular cases</u>: In (45) take $p=q=0$, and get

$$\frac{2^{1-3\alpha}\sqrt{\pi}\,\Gamma(\alpha)}{3\,\Gamma(1+\alpha)\Gamma(\frac{3}{2}\alpha)\Gamma(\frac{3}{2}\alpha+\frac{1}{2})}\,{}_1F_2\left(\begin{matrix}\alpha; -\frac{3}{4}x\\ \frac{3}{2}\alpha, \frac{3}{2}\alpha+\frac{1}{2}\end{matrix}\right) = \sum_{r=0}^{\infty} \frac{(\alpha;r)(1+\frac{1}{2}\alpha;r)\,x^{-3r}}{r!(\frac{1}{2}\alpha;r)\Gamma(1+3\alpha+6r)}\,{}_0F_1\left(;1+3\alpha+6r; -\frac{1}{x}\right), \quad \cdots (47)$$

Again in (45) take $p=2$, $q=0$ and $x=-1$, apply Gauss's theorem and get

$$\sum_{r=0}^{\infty} \frac{(-1)^{3r}(\gamma;r)(1+\frac{1}{2}\gamma;r)\Gamma(\alpha+3r)\Gamma(\beta+3r)\Gamma(1+3\gamma-\alpha-\beta)}{r!(\frac{1}{2}\gamma;r)\Gamma(1+3\gamma-\alpha+3r)\Gamma(1+3\gamma-\beta+3r)} =$$

$$\frac{2^{1-3\gamma}}{3}\cdot\frac{\sqrt{\pi}\,\Gamma(\gamma)\Gamma(\beta)\Gamma(\alpha)}{\Gamma(1+\gamma)\Gamma(\frac{3}{2}\gamma)\Gamma(\frac{3}{2}\gamma+\frac{1}{2})}\,{}_3F_2\left(\begin{matrix}\gamma, \alpha, \beta; \frac{3}{4}\\ \frac{3}{2}\gamma, \frac{3}{2}\gamma+\frac{1}{2}\end{matrix}\right), \ldots (48)$$

where $R(1+3\gamma)>0$, $R(1+3\gamma-\alpha-\beta)>0$.

§ 8. Generalizations of known recurrence relations. The first formula is

$$\Gamma(\rho_1)\sum_{r=0}^{n}(-1)^r\,{}^nC_r\,\frac{1}{\Gamma(\rho_1-n+r)}\,x^{-r}E(\alpha_1+r,\ldots,\alpha_p+r:\rho_1+r,\ldots,\rho_q+r:x)$$

$$=E(p;\alpha_r:\rho_1-n,\rho_2,\ldots,\rho_q:x), \ldots\ldots (49).$$

This formula generalizes formula (9). Since the E-function is symmetrical in the ρ's, formula (49) is equivalent to q similar relations.

(49) can easily be proved by substituting from (2) for the E-function in the L.H.S. of (49), writing $(\xi+r)$ for ξ, changing the order of integration and summation and then applying Gauss's theorem.

(49) can also be proved by induction for suppose that it is true for a particular value of n, then it will be proved that it holds when n is replaced by $(n+1)$.

Now (9) can be written

$$E(p;\alpha_r:\rho_1-1,\rho_2,\ldots,\rho_q:x)=(\rho_1-1)E(p;\alpha_r:q;\rho_s:x)-1/x\,E(p;\alpha_r+1:q;\rho_s+1:x) \quad \ldots (50).$$

Also the following formula which is (49) with (ρ_1-1) instead of $\underline{\rho}$ is assumed to be true, namely

$$\Gamma(\rho_1-1)\sum_{r=0}^{n}(-1)^r\,{}^nC_r\,\frac{1}{\Gamma(\rho_1-n+r-1)}\,x^{-r}E(\alpha_1+r,\ldots,\alpha_p+r:\rho_1+r-1,\rho_2+r,\ldots,\rho_q+r:x)$$

$$=E(p;\alpha_r:\rho_1-n-1,\rho_2,\ldots,\rho_q:x), \ldots \ldots \ldots$$

Now apply (50) to each term on the left of the last formula, so getting

$$E(p;\alpha_r:\rho_1-n-1,\rho_2,\ldots,\rho_q:x)=$$

$$\Gamma(\rho_1-1)\sum_{r=0}^{n}(-1)^r\,{}^nC_r\,\frac{(\rho_1+r-1)}{\Gamma(\rho_1-n+r-1)}\,x^{-r}E(\alpha_1+r,\ldots,\alpha_p+r:\rho_1+r,\rho_2+r,\ldots,\rho_q+r:x)$$

$$+\Gamma(\rho_1-1)\sum_{r=0}^{n}(-1)^{r+1}\,{}^nC_r\,\frac{1}{\Gamma(\rho_1-n+r-1)}\,x^{-r-1}E(\alpha_1+r+1,\ldots,\alpha_p+r+1:\rho_1+r+1,\ldots,\rho_q+r+1:x).$$

In the second of the last two series, write $(r-1)$ for $\underline{r}$, add the two series applying the formula

$${}^nC_r(\rho_1+r-1)+{}^nC_{r-1}(\rho_1-n+r-2)={}^{n+1}C_r(\rho_1-1),$$

then (49) is proved with $(n+1)$ in place of $\underline{n}$. But (49) is true for $n=1$, hence by induction, it is true for all positive integral values of $\underline{n}$.

<u>Particular cases:</u> In (49) take $\rho_1=\alpha_1$, then it becomes

$$ {}_1F_1(\rho;\rho-n;x)=e^x\,{}_1F_1(-n;\rho-n;-x), \qquad \ldots \ldots \quad (51)$$

which is a particular case of the known transformation

$$ {}_1F_1(\alpha;\beta;x)=e^x\,{}_1F_1(\beta-\alpha;\beta;-x), \qquad \ldots \ldots \quad (52).$$

Again in (49) take $p=2$, $q=1$, $x=-1$ and get

$$ {}_3F_2\left(\begin{matrix}\alpha_1, \alpha_2, -n \\ \rho_1-n, 1+\alpha_1+\alpha_2-\rho_1\end{matrix}\right)=\frac{\Gamma(1+\alpha_1+\alpha_2-\rho_1)\Gamma(1+\alpha_1-\rho_1+n)\Gamma(1+\alpha_2-\rho_1+n)\Gamma(1-\rho_1)}{\Gamma(1+n-\rho_1)\Gamma(1+\alpha_2-\rho_1)\Gamma(1+\alpha_1-\rho_1)\Gamma(1+\alpha_1+\alpha_2+n-\rho_1)}, \ldots (53),$$

which is Saalchutz's theorem.

Also $p=3$, $q=2$ in (49) with $x=-1$, give if $\rho_1=\alpha_1$

$$ {}_3F_2\left(\begin{matrix}\alpha_1, \alpha_2, \alpha_3;1 \\ \alpha_1-n, \rho_2\end{matrix}\right)=\frac{\Gamma(\rho_2)\Gamma(\rho_2-\alpha_1-\alpha_2)}{\Gamma(\rho_2-\alpha_2)\Gamma(\rho_2-\alpha_3)}\,{}_3F_2\left(\begin{matrix}-n, \alpha_2, \alpha_3;1 \\ 1+\alpha_2+\alpha_3-\rho_2, \alpha_1-n\end{matrix}\right), \ldots (54)$$

where $R(\rho_2-\alpha_2-\alpha_3-n)>0$.

(54) was given by G. H. Hardy "A chapter from Ramanujan's note Book" [Proc. Camb. Phil. Soc. (2), 1923, 498, (5.2).].

The second formula to be proved is

$$\sum_{r=0}^{n}(-1)^r\,{}^nC_r\,\frac{(\alpha_1;n)}{(\alpha_1;r)}\,x^{-r}E(\alpha_1+r,\ldots,\alpha_p+r:\rho_1;\rho_s+r:x)=E(\alpha_1+n,\alpha_2,\ldots,\alpha_p:\rho;\rho_s:x), \quad\ldots(55).$$

(55) with $n=1$ gives (8) which can be written in the form

$$\alpha_1E(\alpha_1,\alpha_2,\ldots,\alpha_p:\rho;\rho_s:x)-1/x\,E(\alpha_1+1,\ldots,\alpha_p+1:\rho;\rho_s+1:x)$$

$$=E(\alpha_1+1,\alpha_2,\ldots,\alpha_p:\rho;\rho_s:x), \quad\ldots(56).$$

Also (55) is equivalent to p recurrence relations.

(55) can easily be proved by substituting from (2) for the E-function in the left, writing $(\xi+r)$ for ξ, changing the order of integration and summation and then applying Gauss's theorem.

To prove (55) by induction, assume it is true for a particular value of n, then it will be proved that it is true when n is replaced by $(n+1)$. So that the following formula which is (55) with (α_1+1) instead of α_1 is assumed to be true

$$\sum_{r=0}^{n}(-1)^r\,{}^nC_r\,\frac{(\alpha_1+1;n)}{(\alpha_1+1;r)}\,x^{-r}E(\alpha_1+r+1,\alpha_2+r,\ldots,\alpha_p+r:\rho;\rho_s+r:x)=E(\alpha_1+n+1,\alpha_2,\ldots,\alpha_p:\rho;\rho_s:x).$$

Here apply (56) to each term on the left with (α_1+r) in the role of α_1 so getting

$$E(\alpha_1+n+1, \alpha_2, \ldots, \alpha_p ; q ; \rho_s ; x) =$$

$$\sum_{r=0}^{n} (-1)^r \, {}^nC_r \frac{(\alpha_1 ; n+1)}{(\alpha_1 ; r)} x^{-r} E(\alpha_1+r, \alpha_2+r, \ldots, \alpha_p+r ; q ; \rho_s+r ; x)$$

$$+ \sum_{r=0}^{n} (-1)^{r+1} \, {}^nC_r \frac{(\alpha_1 ; n+1)}{(\alpha_1 ; r+1)} x^{-r-1} E(\alpha_1+r+1, \alpha_2+r+1, \ldots, \alpha_p+r+1 : q ; \rho_s+r+1 ; x).$$

In the second of these two series, write $(r-1)$ for $\underline{r}$, then add the two series applying the identity

$$ {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r ,$$

then the last expression becomes (55) with $(n+1)$ in place of $\underline{n}$. But (55) is true for $n=1$; therefore it is true for all positive integral values of $\underline{n}$.

As a particular case of (55) write $(-1/x)$ for $\underline{x}$ and take $p=2$, $q=1$, with $\alpha_1 = \rho_1 = \rho$, and get if $\beta = -n$

$$F(\beta, \alpha ; \rho ; x) = (1-x)^{-\alpha} F\left(\alpha, \rho-\beta ; \rho ; \frac{x}{x-1}\right), \quad \ldots\ldots \quad (57)$$

which is the known Euler-Transformation.

Also if in (55) $x=-1$, $p=2$, $q=1$; then it becomes Gauss's theorem again.

The third formula to be proved in this section is

$$\sum_{r=0}^{n} {}^{n}C_{r}\, x^{-r} E(\alpha_1+n, \alpha_2+r, \ldots, \alpha_p+r : q ; \rho_s+r : x) =$$

$$(\alpha_1 ; n) E(p ; \alpha_r : q ; \rho_s : x), \quad \ldots\ldots (58)$$

which also generalizes (56).

It can be proved as before by substituting from (2), writing $(\xi+r)$ for ξ, changing the order of integration and summation, and then applying Gauss's theorem.

To prove it by induction, assume that it is true for a particular value of $\underline{n}$. Multiply both sides by (α_1+n) and apply (56) to each term on the left so getting

$$(\alpha_1 ; n+1) E(p ; \alpha_r : q ; \rho_s ; x) =$$

$$\sum_{r=0}^{n} {}^{n}C_{r}\, x^{-r} E(\alpha_1+n+1, \alpha_2+r, \ldots, \alpha_p+r : q ; \rho_s+r : x)$$

$$+ \sum_{r=0}^{n} {}^{n}C_{r}\, x^{-r-1} E(\alpha_1+n+1, \alpha_2+r+1, \ldots, \alpha_p+r+1 : q ; \rho_s+r+1 : x).$$

In the second of the last two series write $(r-1)$ for r, add the two series using the identity

$$^{n}C_{r} + {}^{n}C_{r-1} = {}^{n+1}C_{r},$$

then (58) is proved to be true when $(n+1)$ is written for n. But since it is true when $n=1$, then it is true for all positive integral values of n.

§ 9. Recurrence relations: The two formulae to be proved are

$$E(\alpha_1-1, 2-\alpha_1, \alpha_2, \ldots, \alpha_p : q ; \rho_s : x) - E(\alpha_1+1, -\alpha_1, \alpha_2, \ldots, \alpha_p : q ; \rho_s : x)$$
$$= 4(\alpha_1-\tfrac{1}{2})x^{-1}E(\alpha_1, 1-\alpha_1, \alpha_2+1, \ldots, \alpha_p+1 : q ; \rho_s+1 : x), \quad \ldots (59)$$

and

$$E(\alpha_1-1, 2-\alpha_1, \alpha_2, \ldots, \alpha_p : q ; \rho_s : x) + E(\alpha_1+1, -\alpha_1, \alpha_2, \ldots, \alpha_p : q ; \rho_s : x)$$
$$= -2E(\alpha_1, 1-\alpha_1, \alpha_2, \ldots, \alpha_p : q ; \rho_s : x)$$
$$-2/x\, E(\alpha_1, 1-\alpha_1, \alpha_2+1, \ldots, \alpha_p+1 : q ; \rho_s+1 : x)$$
$$+4/x^2 E(1+\alpha_1, 2-\alpha_1, \alpha_2+2, \ldots, \alpha_p+2 : q ; \rho_s+2 : x), \quad \ldots (60).$$

To prove (59), apply (10) with α_1 and $1-\alpha_1$ in the roles of α_1 and α_2 respectively to the second E-function on the L.H.S. of (59), then it becomes

$$E(\alpha_1-1, 2-\alpha_1, \alpha_2, \ldots, \alpha_p : q, ; \rho_s : x) =$$

$$-E(\alpha_1, 1-\alpha_1, \alpha_2, \ldots, \alpha_p : q, ; \rho_s : x) - 2/x\, E(\alpha_1, 2-\alpha_1, \alpha_2+1, \ldots, \alpha_p+1 : q, ; \rho_s+1 : x), \quad \ldots (61)$$

Similarly apply (10) with $(1-\alpha_1)$, α_1 in the roles of α_1 and α_2 respectively to the second E-function in the left of (59), then it becomes

$$E(\alpha_1+1, -\alpha_1, \alpha_2, \ldots, \alpha_p : q, ; \rho_s : x) =$$

$$-E(1-\alpha_1, \alpha_1, \alpha_2, \ldots, \alpha_p : q, ; \rho_s : x) - 2/x\, E(1-\alpha_1, 1+\alpha_1, \alpha_2+1, \ldots, \alpha_p+1 : q, ; \rho_s+1 : x), \quad \ldots (62).$$

Substract (61) and (62) then (59) is obtained by applying the formula [C.V., p 356 ex 2., iv] namely

$$(\alpha_2-\alpha_1) E(p ; \alpha_r : q, ; \rho_s : x) = E(\alpha_2+1, \alpha_1, \ldots, \alpha_p : q, ; \rho_s : x)$$

$$- E(\alpha_1+1, \alpha_2, \ldots, \alpha_p : q, ; \rho_s : x), \quad \ldots\ldots (63)$$

To prove (60) add (61) and (62); so getting

$$\text{L.H.S. of (60)} = -2E(\alpha_1, 1-\alpha_1, \alpha_2, \ldots, \alpha_p : q ; \rho_s : x)$$
$$-2/x\, E(\alpha_1, 2-\alpha_1, \alpha_2+1, \ldots, \alpha_p+1 : q ; \rho_s+1 : x)$$
$$-2/x\, E(1+\alpha_1, 1-\alpha_1, \alpha_2+1, \ldots, \alpha_p+1 : q ; \rho_s+1 : x).$$

Add the last two E-functions, applying the formula [C.V., e42, p. 356] namely

$$(\alpha_1+\alpha_2)E(p ; \alpha_r : q ; \rho_s : x) = E(\alpha_1+1, \alpha_2, \ldots, \alpha_p : q ; \rho_s : x)$$
$$+E(\alpha_1, \alpha_2+1, \alpha_3, \ldots, \alpha_p : q ; \rho_s : x) + \frac{2}{x}E(p ; \alpha_r+1 : q ; \rho_s+1 : x), \quad \text{(64)},$$

with $(1-\alpha_1)$ and $\underline{\alpha_1}$ in the roles of $\underline{\alpha_1}$ and $\underline{\alpha_2}$ respectively.

<u>Note</u>: (59) and (60) can also be proved by substituting from (2) as in (55).

As a good application of these two formulae; take $p=2$, $q=0$ and apply (34) CHAP. I. and also the formula [C.V., e42. p. 356]

namely
$$\frac{d}{dz}E(p;\alpha_r:q;\rho_s:z) = \frac{1}{z^2}E(p;\alpha_r+1:q;\rho_s+1:z),$$
so getting
$$K_{\nu-1}(x) - K_{\nu+1}(x) = -2\nu/x\, K_\nu(x),$$
$$K_{\nu-1}(x) + K_{\nu+1}(x) = -2K_\nu'(x),$$
$$xK_\nu'(x) + \nu K_\nu(x) = -xK_{\nu-1}(x),$$
$$xK_\nu'(x) - \nu K_\nu(x) = -xK_{\nu+1}(x).$$

§10. More linear relations :– The first formula to be proved is
$$\sum_{r=0}^{n} \frac{{}^nC_r(\alpha;r)(1+\tfrac{1}{2}\alpha;r)}{(\tfrac{1}{2}\alpha;r)}(2x)^{-r}$$
$$\times E\left\{\begin{matrix}\frac{\alpha}{2}+r, \frac{\alpha+1}{2}+r, \frac{1}{2}+\frac{n}{2}+\frac{r}{2}, \frac{n}{2}+\frac{r}{2}, 1+\alpha+n+r, \alpha_1+r, \ldots, \alpha_p+r : x\\ \frac{\alpha}{2}+\frac{n}{2}+r, \frac{\alpha+1}{2}+\frac{n}{2}+r, n+r, \frac{1}{2}+r, 1+\alpha+2r, \rho_1+r, \ldots, \rho_q+r\end{matrix}\right\}$$
$$= E(p;\alpha_r:q;\rho_s:x), \quad \ldots\ldots \quad (65).$$

The following formula is required in the proof:

$$ {}_4F_3\left[\begin{matrix} \alpha, 1+\frac{1}{2}\alpha, \beta, -n ; 1 \\ \frac{1}{2}\alpha, 1+\alpha-\beta, 1+2\beta-n \end{matrix}\right] = \frac{(\alpha-2\beta)_n(-\beta)_n}{(1+\alpha-\beta)_n(-2\beta)_n}, \quad \ldots \quad (66) $$

[Bailey, W. N., Cambri. Tracts in Math. (32), p. 30, equ. (1.3)].

To prove (65), consider the special case with $p=q=0$, then the coefficient of $(-\frac{1}{x})^s$ on the L.H.S. is equal to

$$ \frac{\Gamma(\frac{\alpha}{2}+s)\Gamma(\frac{\alpha+1}{2}+s)\Gamma(\frac{1}{2}+\frac{n}{2}+s)\Gamma(\frac{n}{2}+s)\Gamma(1+\alpha+n+s)}{\Gamma(\frac{\alpha}{2}+\frac{n}{2}+s)\Gamma(\frac{\alpha+1}{2}+\frac{n}{2}+s)\Gamma(n+s)\Gamma(\frac{1}{2}+s)\Gamma(1+\alpha+s)} \cdot \frac{1}{\underline{\lfloor s}} $$

$$ + \frac{(-1)^1(-n;1)(\alpha;1)(1+\frac{1}{2}\alpha;1)}{(\frac{1}{2}\alpha;1)\underline{\lfloor 1}\cdot(-1)^1} \cdot 2^{-1} $$

$$ \times \frac{\Gamma(\frac{\alpha}{2}+1+s-1)\Gamma(\frac{\alpha+1}{2}+1+s-1)\Gamma(\frac{1}{2}+\frac{n}{2}+\frac{1}{2}+s-1)\Gamma(\frac{n}{2}+\frac{1}{2}+s-1)\Gamma(1+\alpha+n+1+s-1)}{\Gamma(\frac{\alpha}{2}+\frac{n}{2}+1+s-1)\Gamma(\frac{\alpha+1}{2}+\frac{n}{2}+1+s-1)\Gamma(n+1+s-1)\Gamma(\frac{1}{2}+1+s-1)\Gamma(1+\alpha+2+s-1)\underline{\lfloor s-1}} $$

$$ + \cdots\cdots $$

$$ + \cdots $$

$$\frac{\Gamma(\frac{\alpha}{2}+s)\Gamma(\frac{\alpha+1}{2}+s)\Gamma(1+\alpha+n+s)\Gamma(\frac{1}{2}+\frac{n}{2}+s)\Gamma(\frac{n}{2}+s)}{\Gamma(\frac{\alpha}{2}+\frac{n}{2}+s)\Gamma(\frac{\alpha+1}{2}+\frac{n}{2}+s)\Gamma(n+s)\Gamma(\frac{1}{2}+s)\Gamma(1+\alpha+s)\,\underline{|s}}$$

$$\times\frac{2^{-n-2s}\Gamma(1-\frac{n}{2}-s)\Gamma(\frac{1}{2}-\frac{n}{2}-s)}{\sqrt{\pi}\,\Gamma(1-n-2s)}\;{}_4F_3\left(\begin{matrix}\alpha,\ 1+\frac{1}{2}\alpha,\ -s,\ -n\ ;1\\ \frac{1}{2}\alpha,\ 1+\alpha+s,\ 1-2s-n\end{matrix}\right),$$

by using the relation

$$(\alpha;-r)=(-1)^r/(1-\alpha;r) \quad \text{where } \underline{r} \text{ is any positive integer}$$

or zero.

Now substitute for the generalized hypergeometric function from (66) with $\beta=-s$ and get

Coefficient of $(-\frac{1}{x})^s$ on the L.H.S. of (65) $=\frac{1}{\underline{|s}}=$ coefficient of $(-\frac{1}{x})^s$ on the R.H.S. Thus (65) is proved for the special case $p=q=0$. The general case can then be deduced in the usual way.

<u>Particular cases of (65)</u>. (65) in combination of (34) CHAP. I. gives

$$K_\mu(x)=\frac{\cos\mu\pi.\,e^{-x}}{\sqrt{(2\pi x)}}\sum_{r=0}^{n}\frac{{}^nC_r(\alpha;r)(1+\frac{1}{2}\alpha;r)}{(\frac{1}{2}\alpha;r)}(4x)^{-r},$$

$$\times E\left\{\begin{matrix}\frac{\alpha}{2}+r,\ \frac{\alpha+1}{2}+r,\ \frac{1}{2}+\frac{n+r}{2},\ \frac{n+r}{2},\ 1+\alpha+n+r,\ \frac{1}{2}+\mu+r,\ \frac{1}{2}-\mu+r:2x\\ \frac{\alpha+n}{2}+r,\ \frac{\alpha+n+1}{2}+r,\ n+r,\ \frac{1}{2}+r,\ 1+\alpha+2r\end{matrix}\right\} \quad (67)$$

In (65) write $(4/x^2)$ for x, take $p=0$, $q=1$, and apply (6) so getting

$$J_\mu(x) = \left(\frac{x}{2}\right)^\mu \sum_{r=0}^{n} \frac{{}^nC_r (\alpha;r)(1+\frac{1}{2}\alpha;r)\,(8/x^2)^{-r}}{(\frac{1}{2}\alpha;r)}$$

$$\times \frac{\Gamma(\frac{\alpha}{2}+r)\Gamma(\frac{\alpha+1}{2}+r)\Gamma(\frac{1}{2}+\frac{n}{2}+\frac{r}{2})\Gamma(\frac{n}{2}+\frac{r}{2})\Gamma(1+\alpha+n+r)}{\Gamma(\frac{\alpha+n}{2}+r)\Gamma(\frac{\alpha+n+1}{2}+r)\Gamma(n+r)\Gamma(\frac{1}{2}+r)\Gamma(1+\alpha+2r)\Gamma(1+\mu+r)}$$

$$\times {}_5F_6\left(\begin{matrix}\frac{\alpha}{2}+r, \frac{\alpha+1}{2}+r, \frac{1+n+r}{2}, \frac{n+r}{2}, 1+\alpha+n+r; -x^2/4 \\ \frac{\alpha+n}{2}+r, \frac{\alpha+n+1}{2}+r, n+r, \frac{1}{2}+r, 1+\alpha+2r, 1+\mu+r\end{matrix}\right), \quad \cdots (68).$$

Also $x=-1$ in (65) gives if $p=2$, $q=1$

$$\sum_{r=0}^{n} \frac{(-2)^{-r}\, {}^nC_r(\alpha;r)(1+\frac{1}{2}\alpha;r)\Gamma(\frac{\alpha}{2}+r)\Gamma(\frac{\alpha+1}{2}+r)\Gamma(\frac{1+n+r}{2})\Gamma(\frac{n+r}{2})\Gamma(1+\alpha+n+r)\Gamma(\beta+r)\Gamma(\delta+r)}{(\frac{1}{2}\alpha;r)\Gamma(\frac{\alpha+n}{2}+r)\Gamma(\frac{\alpha+n+1}{2}+r)\Gamma(n+r)\Gamma(\frac{1}{2}+r)\Gamma(1+\alpha+2r)\Gamma(\gamma+r)}$$

$$\times {}_7F_6\left(\begin{matrix}\frac{\alpha}{2}+r, \frac{\alpha+1}{2}+r, \frac{1+n+r}{2}, \frac{n+r}{2}, 1+\alpha+n+r, \beta+r, \delta+r; 1 \\ \frac{\alpha+n}{2}+r, \frac{\alpha+n+1}{2}+r, n+r, \frac{1}{2}+r, 1+\alpha+2r, \gamma+r\end{matrix}\right)$$

$$= \frac{\Gamma(\beta)\Gamma(\delta)\Gamma(\gamma-\beta-\delta)}{\Gamma(\gamma-\beta)\Gamma(\gamma-\delta)}, \quad \cdots (69)$$

where $R(\gamma)>0$, $R(\gamma-\beta-\delta)>0$.

Again write $(1/x^2)$ for $\underline{x}$ in (65) and apply (28) so getting

$$\{J_\nu(x)\}^2 = \frac{x^{2\nu}}{\sqrt{\pi}} \sum_{r=0}^{n} \frac{{}^nC_r(\alpha;r)(1+\tfrac{1}{2}\alpha;r)\Gamma(\frac{\alpha}{2}+r)\Gamma(\frac{\alpha+1}{2}+r)\Gamma(\frac{1+n+r}{2})\Gamma(\frac{n+r}{2})\Gamma(1+\alpha+n+r)\Gamma(\nu+\frac{1}{2}+r)(x^2/2)^r}{(\frac{1}{2}\alpha;r)\Gamma(\frac{\alpha+n}{2}+r)\Gamma(\frac{\alpha+n+1}{2}+r)\Gamma(n+r)\Gamma(\frac{1}{2}+r)\Gamma(1+\alpha+2r)\Gamma(1+\nu+r)\Gamma(2\nu+1+r)}$$

$$\times {}_6F_7\left(\begin{matrix}\frac{\alpha}{2}+r, \frac{\alpha+1}{2}+r, \frac{1+n+r}{2}, \frac{n+r}{2}, 1+\alpha+n+r, \frac{1}{2}+\nu+r; -x^2 \\ \frac{\alpha+n}{2}+r, \frac{\alpha+n+1}{2}+r, n+r, \frac{1}{2}+r, 1+\alpha+2r, 1+\nu+r, 1+2\nu+r\end{matrix}\right), \quad \ldots (70)$$

The second formula to be proved is

$$\sum_{r=0}^{n} {}^nC_r(\alpha;r)(2x)^{-r}$$

$$\times E\left\{\begin{matrix}\frac{\alpha}{2}+r, \frac{\alpha+1}{2}+r, \frac{1}{2}\alpha+n+r, \frac{1+n+r}{2}, \frac{n+r}{2}, 1+\alpha+n+r, 1+\frac{\alpha}{2}+r, \alpha_1+r, \ldots, \alpha_p+r : x \\ \frac{\alpha+n}{2}+r, \frac{\alpha+n+1}{2}+r, n+r, \frac{1}{2}+r, 1+\alpha+2r, 1+\frac{\alpha}{2}+n+r, \frac{1}{2}\alpha+r, \rho_1+r, \ldots, \rho_q+r\end{matrix}\right\}$$

$$= E(p;\alpha_r : q;\rho_s : x), \quad \ldots (71).$$

To prove (71) consider the special case with $p=q=0$. Then (71) can be proved in the same manner as formula (65) except instead of applying (66), the following formula [Bailey, W.N., Cambr. Tracts in Math. (32), p. 30 equ. (1.2)] namely

$$ {}_3F_2\left(\begin{matrix}\alpha,\ \beta,\ -n\ ;1\\ 1+\alpha-\beta,\ 1+2\beta-n\end{matrix}\right) = \frac{(\alpha-2\beta)_n\,(1+\tfrac{1}{2}\alpha-\beta)_n\,(-\beta)_n}{(1+\alpha-\beta)_n\,(\tfrac{1}{2}\alpha-\beta)_n\,(-2\beta)_n}, \quad\ldots (72), $$

is applied with $\beta=-s$.
Also (71) gives particular cases as those given by (65).
The third formula to be proved is

$$ \sum_{r=0}^{\infty} \frac{(\alpha;r)\,(1+\beta;r)}{r!\,(\beta;r)}\, x^{-r} E\left\{\begin{matrix}1+2\beta-\alpha+r,\ \beta+r,\ \beta+\tfrac{1}{2}+r,\ \alpha_1+r,\ \ldots,\ \alpha_p+r : x\\ \beta-\tfrac{\alpha}{2}+r,\ \beta-\tfrac{\alpha}{2}+\tfrac{1}{2}+r,\ 1+2\beta+2r,\ \rho_1+r,\ \ldots,\ \rho_q+r\end{matrix}\right\} $$

$$ = 2^{-\alpha}\left(1-\frac{\alpha}{2\beta}\right) E(p;\alpha_r:q;\rho_s:x), \quad\ldots\ldots (73). $$

The following formula is required in the proof:

$$ {}_3F_2\left[\begin{matrix}\alpha,\ 1+\beta,\ \gamma\ ;1\\ \beta,\ 1+2\beta-\gamma\end{matrix}\right] = \frac{\Gamma(\beta)\,\Gamma(1+\beta-\frac{\alpha}{2})\,\Gamma(1+2\beta-\gamma)\,\Gamma(2\beta-\alpha-2\gamma)}{\Gamma(1+\beta)\,\Gamma(\beta-\frac{\alpha}{2})\,\Gamma(1+2\beta-\alpha-\gamma)\,\Gamma(2\beta-2\gamma)}, \quad (74). $$

[Bailey, W. N. Cambr. Tracts in Math. (32) p. (45) form. (2).].
To prove (73) consider the special case with $p=q=0$. Thus the coefficient of $(-\frac{1}{x})^s$ on the L.H.S. is equal to

$$\frac{\Gamma(1+2\beta-\alpha+s)\,\Gamma(\beta+s)\,\Gamma(\beta+\frac{1}{2}+s)}{\Gamma(\beta-\frac{\alpha}{2}+s)\,\Gamma(\beta-\frac{\alpha}{2}+\frac{1}{2}+s)\,\Gamma(1+2\beta+s)}\cdot\frac{1}{\lfloor s}$$

$$+\frac{(\alpha;1)(1+\beta;1)}{\lfloor 1\,(\beta;1)(-1)^1}\cdot\frac{\Gamma(1+2\beta-\alpha+1+s-1)\,\Gamma(\beta+1+s-1)\,\Gamma(\beta+\frac{1}{2}+1+s-1)}{\Gamma(\beta-\frac{\alpha}{2}+1+s-1)\,\Gamma(\beta-\frac{\alpha}{2}+\frac{1}{2}+1+s-1)\,\Gamma(1+2\beta+2+s-1)\,\lfloor s-1}$$

$$+\cdots\cdots$$

$$+\cdots\cdots$$

$$=\frac{\Gamma(1+2\beta-\alpha+s)\,\Gamma(\beta+s)\,\Gamma(\beta+\frac{1}{2}+s)}{\Gamma(\beta-\frac{\alpha}{2}+s)\,\Gamma(\beta-\frac{\alpha}{2}+\frac{1}{2}+s)\,\Gamma(1+2\beta+s)}\cdot\frac{1}{\lfloor s}\;{}_3F_2\left(\begin{matrix}\alpha, 1+\beta, -s; 1\\ \beta, 1+2\beta+s\end{matrix}\right).$$

Now substitute for the generalized hypergeometric function from (7.4) and get

Coefficient of $(-\frac{1}{x})^s$ on the L.H.S. of (7.1) with $p=q=0$ is equal to

$$\frac{\Gamma(1+2\beta-\alpha+s)\,\Gamma(\beta+s)\,\Gamma(\beta+\frac{1}{2}+s)}{\lfloor s\,\Gamma(\beta-\frac{\alpha}{2}+s)\,\Gamma(\beta-\frac{\alpha}{2}+\frac{1}{2}+s)\,\Gamma(1+2\beta+s)}$$

$$\times\frac{\Gamma(\beta)\,\Gamma(1+\beta-\frac{\alpha}{2})\,\Gamma(1+2\beta+s)\,\Gamma(2\beta-\alpha+2s)}{\Gamma(1+\beta)\,\Gamma(\beta-\frac{\alpha}{2})\,\Gamma(1+2\beta-\alpha+s)\,\Gamma(2\beta+2s)}=2^{-\alpha}\left(1-\frac{\alpha}{2\beta}\right)\frac{1}{\lfloor s}.$$

But this is the coefficient of $(-\frac{1}{x})^s$ on the R.H.S. of (7.3).

Thus formula (73) with $p=q=0$ is proved. The general formula can then be deduced in the usual way.

Particular cases: In (73) take $p=2, q=0$, and apply (34) CHAP. I. so getting

$$K_\mu(x) = \frac{2^\alpha (2\beta)\cos\mu\pi . e^{-x}}{\sqrt{(2\pi x)}(2\beta-\alpha)} \sum_{r=0}^{\infty} \frac{(\alpha;r)(1+\beta;r)}{\lfloor r\,(\beta;r)} (2x)^{-r}$$

$$E\left\{ \begin{matrix} 1+2\beta-\alpha+r, \beta+r, \beta+\frac{1}{2}+r, \frac{1}{2}+\mu+r, \frac{1}{2}-\mu+r : 2x \\ \beta-\frac{\alpha}{2}+r, \beta-\frac{\alpha}{2}+\frac{1}{2}+r, 1+2\beta+2r \end{matrix} \right\}, \quad \ldots (75)$$

Also (73) in combination with (6) gives

$$J_\mu(x) = \left(\frac{x}{2}\right)^\mu \frac{2^{\alpha+1}\beta}{(2\beta-\alpha)} \sum_{r=0}^{\infty} \frac{(\alpha;r)(1+\beta;r)}{\lfloor r\,(\beta;r)} \left(\frac{x^2}{4}\right)^r$$

$$\times \frac{\Gamma(1+2\beta-\alpha+r)\Gamma(\beta+r)\Gamma(\beta+\frac{1}{2}+r)}{\Gamma(\beta-\frac{\alpha}{2}+r)\Gamma(\beta-\frac{\alpha}{2}+\frac{1}{2}+r)\Gamma(1+2\beta+2r)\Gamma(1+\mu+r)}$$

$$\times {}_3F_4\left(\begin{matrix} 1+2\beta-\alpha+r, \beta+r, \beta+\frac{1}{2}+r ; -x^2/4 \\ \beta-\frac{\alpha}{2}+r, \beta-\frac{\alpha}{2}+\frac{1}{2}+r, 1+2\beta+2r, 1+\mu+r \end{matrix} \right), \quad \ldots (76).$$

Also (73) in combination with (28) gives

$$\{J_\nu(x)\}^2 = \frac{2^{\alpha+\beta}\beta . x^{2\nu}}{(2\beta-\alpha)\sqrt{\pi}} \sum_{r=0}^{\infty} \frac{(\alpha;r)(1+\beta;r)\Gamma(1+2\beta-\alpha+r)\Gamma(\beta+r)\Gamma(\frac{1}{2}+\beta+r)\Gamma(\nu+\frac{1}{2}+r) . x^{2r}}{\lfloor r\,(\beta;r)\Gamma(\beta-\frac{\alpha}{2}+r)\Gamma(\beta-\frac{\alpha}{2}+\frac{1}{2}+r)\Gamma(1+2\beta+2r)\Gamma(1+\nu+r)\Gamma(1+2\nu+r)}$$

$$\times {}_4F_5\left(\begin{matrix}1+2\beta-\alpha+r, \beta+r, \beta+\frac{1}{2}+r, \nu+\frac{1}{2}+r; -x^2 \\ \beta-\frac{\alpha}{2}+r, \beta-\frac{\alpha}{2}+\frac{1}{2}+r, 1+2\beta+2r, 1+\nu+r, 1+2\nu+r\end{matrix}\right), \dots (74)$$

Again $x=-1$ with $p=2$, $q=1$ give (73) in the form

$$\sum_{r=0}^{\infty} \frac{(-1)^r (\alpha;r)(1+\beta;r)}{\lfloor r\,(\beta;r)} \cdot \frac{\Gamma(1+2\beta-\alpha+r)\Gamma(\beta+r)\Gamma(\beta+\frac{1}{2}+r)\Gamma(b+r)\Gamma(c+r)}{\Gamma(\beta-\frac{\alpha}{2}+r)\Gamma(\beta-\frac{\alpha}{2}+\frac{1}{2}+r)\Gamma(1+2\beta+2r)\Gamma(d+r)}$$

$$\times {}_5F_4\left(\begin{matrix}1+2\beta-\alpha+r, \beta+r, \beta+\frac{1}{2}+r, b+r, c+r; 1 \\ \beta-\frac{\alpha}{2}+r, \beta-\frac{\alpha}{2}+\frac{1}{2}+r, 1+2\beta+2r, d+r\end{matrix}\right)$$

$$= 2^{-\alpha}\left(1-\frac{\alpha}{2\beta}\right) \frac{\Gamma(b)\Gamma(c)\Gamma(d-b-c)}{\Gamma(d-b)\Gamma(d-c)}, \dots\dots (78)$$

where $R(d)>0$, $R(d-b-c)>0$.

The last formula to be proved in this section is

$$\sum_{r=0}^{n} \frac{(\alpha;r)(1+\tfrac{1}{2}\alpha;r)(\beta;r)(\gamma;r)}{r!\,(\tfrac{1}{2}\alpha;r)(1+\alpha-\beta;r)(1+\alpha-\gamma;r)}\, x^{-r}\, E\left\{\begin{matrix}1+\alpha-\gamma+r,\ 1+\alpha-\beta+r,\ \alpha_1+r,\ \ldots,\ \alpha_p+r : x\\ 1+\alpha-\beta-\gamma+r,\ 1+\alpha+2r,\ \rho_1+r,\ \ldots,\ \rho_q+r\end{matrix}\right\}$$

$$= \frac{\Gamma(1+\alpha-\gamma)\Gamma(1+\alpha-\beta)}{\Gamma(1+\alpha)\Gamma(1+\alpha-\beta-\gamma)}\, E(p;\alpha_r:q;\rho_s:x), \quad \ldots (79).$$

When $\underline{\gamma}$ is a negative integer $(-n)$, (79) becomes formula (1) CHAP. I. in the thesis for Ph.D. The proof of (79) is exactly the same as the second proof of formula (1) in the thesis for Ph.D. except instead of using formula (7) there or (13) this chapter; the following formula [Bailey, W.N. Cambr. Tract. in Math. (32) p. (27), § 4.4 eqn. (1)] namely

$${}_5F_4\left(\begin{matrix}\alpha,\ 1+\tfrac{1}{2}\alpha,\ \beta,\ \gamma,\ \delta;\ 1\\ \tfrac{\alpha}{2},\ 1+\alpha-\beta,\ 1+\alpha-\gamma,\ 1+\alpha-\delta\end{matrix}\right) = \frac{\Gamma(1+\alpha-\beta)\Gamma(1+\alpha-\gamma)\Gamma(1+\alpha-\delta)\Gamma(1+\alpha-\beta-\gamma-\delta)}{\Gamma(1+\alpha)\Gamma(1+\alpha-\gamma-\delta)\Gamma(1+\alpha-\beta-\delta)\Gamma(1+\alpha-\beta-\gamma)}, \quad \ldots (80),$$

where neither of the parameters $\underline{\beta}$, $\underline{\gamma}$, or $\underline{\delta}$ needs to be a negative integer, is used.

<u>Particular cases of (79)</u>: In (79) take $p=q=0$, and get

$$e^x = \frac{\Gamma(1+\alpha)\Gamma(1+\alpha-\beta-\gamma)}{\Gamma(1+\alpha-\gamma)\Gamma(1+\alpha-\beta)} \times \sum_{r=0}^{\infty} \frac{(\alpha;r)(1+\tfrac{1}{2}\alpha;r)(\beta;r)(\gamma;r)}{r!\,(\tfrac{1}{2}\alpha;r)(1+\alpha-\beta;r)(1+\alpha-\gamma;r)} \left(-\frac{1}{x}\right)^{-r}$$

$$\times \frac{\Gamma(1+\alpha-\gamma+r)\,\Gamma(1+\alpha-\beta+r)}{\Gamma(1+\alpha-\beta-\gamma+r)\,\Gamma(1+\alpha+2r)}\ {}_2F_2\left(\begin{matrix}1+\alpha-\gamma+r, 1+\alpha-\beta+r; x\\ 1+\alpha-\beta-\gamma+r, 1+\alpha+2r\end{matrix}\right), \cdots (81).$$

Also (79) in combination with (34) CHAP. I. gives

$$K_\mu(x) = \frac{\cos\mu\pi\cdot\Gamma(1+\alpha)\,\Gamma(1+\alpha-\beta-\gamma)}{\sqrt{(2\pi x)}\,\Gamma(1+\alpha-\gamma)\,\Gamma(1+\alpha-\beta)}\cdot e^{-x}$$

$$\times \sum_{r=0}^{\infty} \frac{(\alpha;r)(1+\tfrac{1}{2}\alpha;r)(\beta;r)(\gamma;r)}{r!\,(\tfrac{1}{2}\alpha;r)(1+\alpha-\beta;r)(1+\alpha-\gamma;r)}\,(2x)^{-r}\,E\left\{\begin{matrix}1+\alpha-\gamma+r, 1+\alpha-\beta+r, \tfrac{1}{2}+\mu+r, \tfrac{1}{2}-\mu+r : 2x\\ 1+\alpha-\beta-\gamma+r, 1+\alpha+2r\end{matrix}\right\}$$

$$, \cdots (82)$$

Also (79) in combination with (6) gives

$$J_\mu(x) = \frac{\Gamma(1+\alpha)\,\Gamma(1+\alpha-\beta-\gamma)}{\Gamma(1+\alpha-\gamma)\,\Gamma(1+\alpha-\beta)}\left(\frac{x}{2}\right)^{\mu}\cdot\sum_{r=0}^{\infty}\frac{(\alpha;r)(1+\tfrac{1}{2}\alpha;r)(\beta;r)(\gamma;r)}{r!\,(\tfrac{1}{2}\alpha;r)(1+\alpha-\beta;r)(1+\alpha-\gamma;r)}\left(\frac{x}{2}\right)^{2r}$$

$$\times \frac{\Gamma(1+\alpha-\gamma+r)\,\Gamma(1+\alpha-\beta+r)}{\Gamma(1+\alpha-\beta-\gamma+r)\,\Gamma(1+\alpha+2r)\,\Gamma(1+\mu+r)}\ {}_2F_3\left(\begin{matrix}1+\alpha-\gamma+r, 1+\alpha-\beta+r; -x^2/4\\ 1+\alpha-\beta-\gamma+r, 1+\alpha+2r, 1+\mu+r\end{matrix}\right), \cdots (83).$$

Again (79) in combination with (28) gives

$$\{J_\nu(x)\}^2 = \frac{\Gamma(1+\alpha)\,\Gamma(1+\alpha-\beta-\gamma)}{\sqrt{\pi}\;\Gamma(1+\alpha-\gamma)\Gamma(1+\alpha-\beta)}\, x^{2\nu} \sum_{r=0}^{\infty} \frac{(\alpha;r)(1+\frac{1}{2}\alpha;r)(\beta;r)(\gamma;r)}{r!(\frac{1}{2}\alpha;r)(1+\alpha-\beta;r)(1+\alpha-\gamma;r)}\, x^{2r}$$

$$\times \frac{\Gamma(1+\alpha-\gamma+r)\,\Gamma(1+\alpha-\beta+r)\,\Gamma(\frac{1}{2}+\nu+r)}{\Gamma(1+\alpha-\beta-\gamma+r)\Gamma(1+\alpha+2r)\Gamma(1+\nu+r)\Gamma(1+2\nu+r)}\, {}_3F_4\left(\begin{matrix}1+\alpha-\gamma+r,\, 1+\alpha-\beta+r,\, \frac{1}{2}+\nu+r;\, -x^2\\ 1+\alpha-\beta-\gamma+r,\, 1+\alpha+2r,\, 1+\nu+r,\, 1+2\nu+r\end{matrix}\right), \quad \ldots (84)$$

§ 11. Further recurrence relations: The following are just combinations of the recurrence formulae given in § 8 this chapter and the known formulae given in p. 356, C.V.,

Thus by the substraction of (49) and (55), one gets

$$\sum_{r=0}^{n} (-1)^r\, {}^nC_r \left[\frac{(\alpha_1;n)}{(\alpha_1;r)} - \frac{\Gamma(\rho_1)}{\Gamma(\rho_1-n+r)}\right] E(\alpha_1+r, \ldots, \alpha_p+r : \rho_1+r, \ldots, \rho_q+r : x)$$

$$= E(\alpha_1+n, \alpha_2, \ldots, \alpha_p : q; \rho_s : x) - E(p; \alpha_r : \rho_1-n, \rho_2, \ldots, \rho_q : x), \quad \ldots (85),$$

which gives when $n=1$ the formula

$$(\alpha_1-\rho_1)E(p;\alpha_r : \rho_1+1, \rho_2, \ldots, \rho_q : x) = E(\alpha_1+1, \alpha_2, \ldots, \alpha_p : \rho_1+1, \rho_2, \ldots, \rho_q : x)$$
$$- E(p;\alpha_r : q; \rho_s : x), \quad \ldots \ldots (86).$$

(86) with $\underline{\alpha_2}$ instead of $\underline{\alpha_1}$ becomes

$$(\alpha_2-\rho_1)E(p;\alpha_r:\rho_1+1,\rho_2,\dots,\rho_q:x)=E(\alpha_1,\alpha_2+1,\dots,\alpha_p:\rho_1+1,\rho_2,\dots,\rho_q:x)$$
$$-E(p;\alpha_r:q;\rho_s:x), \quad \dots\dots (87).$$

Multiply (86) by $(\alpha_2-\rho_1)$, (87) by $(\alpha_1-\rho_1)$, subtract and get

$$(\alpha_1-\alpha_2)E(p;\alpha_r:q;\rho_s:x)+(\rho_1-\alpha_1)E(\alpha_1,\alpha_2+1,\alpha_3,\dots,\alpha_p:\rho_1+1,\rho_2,\dots,\rho_q:x)$$
$$-(\rho_1-\alpha_2)E(\alpha_1+1,\alpha_2,\alpha_3,\dots,\alpha_p:\rho_1+1,\rho_2,\dots,\rho_q:x)=0, \quad \dots\dots (88)$$

Also from (86) and (8), it is easily seen that

$$\alpha_2 E(p;\alpha_r:q;\rho_s:x)-(\rho_1-\alpha_1)E(\alpha_1,\alpha_2+1,\alpha_3,\dots,\alpha_p:\rho_1+1,\rho_2,\dots,\rho_q:x)$$
$$-E(\alpha_1+1,\alpha_2+1,\alpha_3,\dots,\alpha_p:\rho_1+1,\rho_2,\dots,\rho_q:x)-\frac{1}{x}E(p;\alpha_r+1:q;\rho_s+1:x)=0, \quad \dots (89).$$

Again from (86), (8) and (9), one gets

$$\alpha_1 E(p;\alpha_r:q;\rho_s:x)-\rho_1 E(\alpha_1+1,\alpha_2,\dots,\alpha_p:\rho_1+1,\rho_2,\dots,\rho_q:x)$$
$$-(\rho_1-\alpha_1)x^{-1}E(p;\alpha_r+1:\rho_1+2,\rho_2+1,\dots,\rho_q+1:x), \quad \dots\dots (90)$$

The next formula to be proved is

$$\alpha_1 E(p;\alpha_r:q;\rho_s:x)+(\rho_1-\alpha_2)x^{-1}E(\alpha_1,\alpha_2,\alpha_3+1,\dots,\alpha_p+1:\rho_1,\rho_2+1,\dots,\rho_q+1:x)$$
$$-E(\alpha_1+1,\alpha_2,\alpha_3,\dots,\alpha_p:q;\rho_s:x)-x^{-1}E(\alpha_1+1,\alpha_2,\alpha_3+1,\dots,\alpha_p+1:\rho_1,\rho_2+1,\dots,\rho_q+1:x)$$
$$-(\rho_1-\alpha_1)(\rho_1-\alpha_2)x^{-1}E(\alpha_1,\alpha_2,\alpha_3+1,\dots,\alpha_p+1:q;\rho_s+1:x)=0, \quad \dots (91).$$

To prove (91) add the first and third E-functions by means of (8), add the result to the fourth E-function applying (86), so getting

$$
\begin{aligned}
\text{L.H.S. of (91)} &= (\alpha_2-\rho_1)x^{-1}E(\alpha_1+1, \alpha_2, \alpha_3+1, \dots, \alpha_p+1 : \rho_1+1, \rho_2+1, \dots, \rho_q+1 : x) \\
&\quad -(\alpha_2-\rho_1)x^{-1}E(\alpha_1, \alpha_2, \alpha_3+1, \dots, \alpha_p+1 : \rho_1, \rho_2+1, \dots, \rho_q+1 : x) \\
&\quad +(\alpha_2-\rho_1)(\rho_1-\alpha_1)x^{-1}E(\alpha_1, \alpha_2, \alpha_3+1, \dots, \alpha_p+1 ; q ; \rho_s+1 : x),
\end{aligned}
$$

which is equal to zero by applying (86) once more. Thus (91) is proved.

The formula to be proved is

$$
\begin{aligned}
&(\rho_1-\alpha_1)E(p ; \alpha_r : q ; \rho_s : x) + \frac{\alpha_2}{x}E(\alpha_1, \alpha_2, \alpha_3+1, \dots, \alpha_p+1 : \rho_1, \rho_2+1, \dots, \rho_q+1 : x) \\
&-(\rho_1-\alpha_1)(\alpha_1-1)E(\alpha_1-1, \alpha_2, \alpha_3, \dots, \alpha_p ; q ; \rho_s : x) - x^{-1}E(p ; \alpha_r+1 : q ; \rho_s+1 : x) \\
&\quad -x^{-2}E(\alpha_1+1, \alpha_2+1, \alpha_3+2, \dots, \alpha_p+2 : \rho_1+1, \rho_2+2, \dots, \rho_q+2 : x) = 0 \quad \text{, --- (92).}
\end{aligned}
$$

To prove (92) add the first and third E-functions on the L.H.S. of (92) by means of (8). Also add the second and the last

E-functions by means of (8), so getting

$$\text{L.H.S. of } (92) = x^{-1}E(\alpha_1, \alpha_2+1, \alpha_3+1, \dots, \alpha_p+1 : \rho_1, \rho_2+1, \dots, \rho_q+1 : x)$$
$$-(\rho_1-\alpha_1)\,x^{-1}E(\alpha_1, \alpha_2+1, \alpha_3+1, \dots, \alpha_p+1 : q ; \rho_s+1 : x)$$
$$-x^{-1}E(p ; \alpha_r+1 : q ; \rho_s+1 : x),$$

which is equal to zero by applying (86). Thus (92) is proved. In the same way by applying (8) and (86), the following formula is deduced

$$E(\alpha_1+1, \alpha_2, \dots, \alpha_p : q ; \rho_s : x) + x^{-1}E(\alpha_1, \alpha_2+1, \dots, \alpha_p+1 : \rho_1, \rho_2+1, \dots, \rho_q+1 : x)$$
$$-(2\alpha_1-\rho_1)E(p ; \alpha_r : q ; \rho_s : x)$$
$$-(\rho_1-\alpha_1)(\alpha_1-1)E(\alpha_1-1, \alpha_2, \dots, \alpha_p : q ; \rho_s : x) = 0, \quad \dots \dots (93)$$

The next formula to be proved is

$$(2\alpha_2-\rho_1)E(p ; \alpha_r : q ; \rho_s : x) + (\rho_1-\alpha_2)(\alpha_2-1)E(\alpha_1, \alpha_2-1, \alpha_3, \dots, \alpha_p : q ; \rho_s : x)$$
$$-E(\alpha_1, \alpha_2+1, \alpha_3, \dots, \alpha_p : q ; \rho_s : x) - x^{-1}E(\alpha_1, \alpha_2+1, \alpha_3+1, \dots, \alpha_p+1 : \rho_1, \rho_2+1, \dots, \rho_q+1 : x)$$
$$+x^{-1}(\alpha_2-\alpha_1)E(\alpha_1, \alpha_2, \alpha_3+1, \dots, \alpha_p+1 : \rho_1, \rho_2+1, \dots, \rho_q+1 : x) = 0, \quad \dots (94)$$

To prove (94), substitute from (8) for the second E-function in the last expression, add the result to the first E-function. Also substitute from (63) for the last E-function, so getting

$$\text{L.H.S. of } (94) = \alpha_2 E(p;\alpha_r:q;\rho_s:x) + (\rho_1-\alpha_2)x^{-1}E(\alpha_1+1,\alpha_2,\alpha_3+1,\dots,\alpha_p+1:q;\rho_s+1:x)$$
$$-E(\alpha_1,\alpha_2+1,\alpha_3,\dots,\alpha_p:q;\rho_s:x) - x^{-1}E(\alpha_1+1,\alpha_2,\alpha_3+1,\dots,\alpha_p+1:\rho_1,\rho_2+1,\dots,\rho_q+1:x).$$

In the last expression add the first and third E-functions by means of (8), then by applying (86), formula (94) follows.

The last formula to be proved in this section is

$$(\rho_1-1)E(p;\alpha_r:q;\rho_s:x) - E(p;\alpha_r:\rho_1-1,\rho_2,\dots,\rho_q:x)$$
$$-x^{-1}E(\alpha_1,\alpha_2,\alpha_3+1,\dots,\alpha_p+1:\rho_1-1,\rho_2+1,\dots,\rho_q+1:x)$$
$$-(\rho_1-\alpha_1)(\rho_1-\alpha_2)x^{-1}E(\alpha_1,\alpha_2,\alpha_3+1,\dots,\alpha_p+1:q;\rho_s+1:x)$$
$$+(2\rho_1-\alpha_1-\alpha_2-1)x^{-1}E(\alpha_1,\alpha_2,\alpha_3+1,\dots,\alpha_p+1:\rho_1,\rho_2+1,\dots,\rho_q+1:x)=0$$
$$, \dots (95)$$

To prove (95), add the first two E-functions by means of (9). Also substitute from (9) for the third E-function so getting

$$\text{L.H.S. of (95)} = x^{-1}E(p;\alpha_r+1:q;\rho_s+1:x) + x^{-2}E\left(\begin{matrix}\alpha_1+1, \alpha_2+1, \alpha_3+2, \ldots, \alpha_p+2 : x\\ \rho_1+1, \rho_2+2, \ldots, \rho_q+2\end{matrix}\right)$$

$$+(\rho_1-\alpha_2)\left[\begin{matrix}E(\alpha_1, \alpha_2, \alpha_3+1, \ldots, \alpha_p+1 : \rho_1, \rho_2+1, \ldots, \rho_q+1 : x)\\ -(\rho_1-\alpha_1)E(\alpha_1, \alpha_2, \alpha_3+1, \ldots, \alpha_p+1 : q; \rho_s+1 : x)\end{matrix}\right]$$

$$-\alpha_1 x^{-1}E(\alpha_1, \alpha_2, \alpha_3+1, \ldots, \alpha_p+1 : \rho_1, \rho_2+1, \ldots, \rho_q+1 : x).$$

Now substitute from (86) for the quantity between brackets [ ] and from (8) for the last E-function so getting

$$\text{L.H.S. of (95)} = x^{-1}E(p;\alpha_r+1:q;\rho_s+1:x)$$

$$-x^{-1}E(\alpha_1+1, \alpha_2, \alpha_3+1, \ldots, \alpha_p+1 : \rho_1, \rho_2+1, \ldots, \rho_q+1 : x)$$

$$+(\rho_1-\alpha_2)x^{-1}E(\alpha_1+1, \alpha_2, \alpha_3+1, \ldots, \alpha_p+1 : \rho_1+1, \rho_2+1, \ldots, \rho_q+1 : x),$$

which is equal to zero by applying (86) once more. Thus (95) is proved.

CHAPTER III.

PRODUCTS OF TWO E-FUNCTIONS EXPRESSED AS THE SUM OF TWO E-FUNCTIONS

§1. First theorem: The formula to be proved is

$$E(\alpha,\beta::ze^{i\pi})-E(\alpha,\beta::ze^{-i\pi})=\frac{(2\pi i)\,z^{\alpha+\beta-1}e^{-z}}{\Gamma(1-\alpha)\Gamma(1-\beta)}E(1-\alpha,1-\beta::z),\ \dots(1).$$

The following formulae are required in the proof

$$F(\alpha;\rho;z)=e^{z}F(\rho-\alpha;\rho;-z),\ \dots\dots\dots\ (2)$$

$$E(\alpha,\beta::z)=\sum_{\alpha,\beta}\Gamma(\beta-\alpha)\Gamma(\alpha)z^{\alpha}F(\alpha;\alpha-\beta+1;z),\ \dots\dots\dots\ (3)$$

[G.V., p. 348; equ. (11)].

Proof of the formula: Expand each of the two E-functions on the L.H.S. by means of (3), add similar terms, so getting

$$E(\alpha,\beta::ze^{i\pi})-E(\alpha,\beta::ze^{-i\pi})=(2\pi i)\left[\frac{\Gamma(\beta-\alpha)}{\Gamma(1-\alpha)}z^{\alpha}F\left(\begin{matrix}\alpha;-z\\1+\alpha-\beta\end{matrix}\right)+\frac{\Gamma(\alpha-\beta)}{\Gamma(1-\beta)}z^{\beta}F\left(\begin{matrix}\beta;-z\\1+\beta-\alpha\end{matrix}\right)\right]$$

$$=(2\pi i)\frac{z^{\alpha+\beta-1}e^{-z}}{\Gamma(1-\alpha)\Gamma(1-\beta)}\left\{\Gamma(\alpha-\beta)\Gamma(1-\alpha)z^{1-\alpha}F\left({1-\alpha\,;z \atop 1+\beta-\alpha}\right)+z^{1-\beta}\Gamma(\beta-\alpha)\Gamma(1-\beta)F\left({1-\beta\,;z \atop 1+\alpha-\beta}\right)\right\}\text{ by (2).}$$

On applying (3) again, (1) is obtained.

§2. Second theorem: The formula to be proved is

$$E(\alpha,\beta::z)E(1-\alpha,1-\beta::z)=\frac{1}{2\pi i}\,\frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(1-\alpha)\Gamma(1-\beta)}{\sqrt{(\pi)}}\,e^{z}\left(\frac{z}{2}\right)^{1-\alpha-\beta}$$
$$\times\left\{E\left(\alpha,\beta,\frac{\alpha+\beta}{2},\frac{\alpha+\beta+1}{2}:\alpha+\beta:e^{i\pi}\cdot\frac{z^2}{4}\right)-E\left(\alpha,\beta,\frac{\alpha+\beta}{2},\frac{\alpha+\beta+1}{2}:\alpha+\beta:e^{-i\pi}\cdot\frac{z^2}{4}\right)\right\}$$

The following formula is required in the proof , ... (4)

$$\Gamma(\tfrac{1}{2})E(\alpha,\beta::iz)E(\alpha,\beta::-iz)=2^{\alpha+\beta-1}\Gamma(\alpha)\Gamma(\beta)E\left(\alpha,\beta,\frac{\alpha+\beta}{2},\frac{\alpha+\beta+1}{2}:\alpha+\beta:\frac{z^2}{4}\right)$$

[C.V., p. 380, ex (111) i.].

, ... (5)

Proof of the formula: Substitute from (5) for each of the two E-functions on the R.H.S. of (4), so getting

$$E\left(\alpha,\beta,\frac{\alpha+\beta}{2},\frac{\alpha+\beta+1}{2}:\alpha+\beta:e^{i\pi}\cdot\frac{z^2}{4}\right)-E\left(\alpha,\beta,\frac{\alpha+\beta}{2},\frac{\alpha+\beta+1}{2}:\alpha+\beta:e^{-i\pi}\cdot\frac{z^2}{4}\right)$$

$$=\frac{\Gamma(\frac{1}{2})}{2^{\alpha+\beta-1}\Gamma(\alpha)\Gamma(\beta)}\left\{E(\alpha,\beta::e^{i\pi}z)E(\alpha,\beta::z)-E(\alpha,\beta::z)E(\alpha,\beta::e^{-i\pi}z)\right\},$$

$$\frac{\Gamma(\frac{1}{2})}{2^{\alpha+\beta-1}\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{(2\pi i)\, z^{\alpha+\beta-1} e^{-z}}{\Gamma(1-\alpha)\Gamma(1-\beta)} \cdot E(\alpha,\beta::z)E(1-\alpha,1-\beta::z) \text{ by (1).}$$

Substituting in the R.H.S. of (4), then (4) is obtained.

<u>An alternative proof</u>: In this proof, the following formulae are required [C.V., p. 380, eq. 110] namely

$$F(\alpha;\rho;z)F(\alpha;\rho;-z) = F(\alpha,\rho-\alpha;\rho,\tfrac{1}{2}\rho,\tfrac{1}{2}\rho+\tfrac{1}{2};z^2/4), \quad \ldots (6),$$

$$F(\alpha;\rho;z)F(\alpha-\rho+1;2-\rho;-z) = F(\tfrac{1}{2}+\tfrac{1}{2}\rho-\alpha,\tfrac{1}{2}-\tfrac{\rho}{2}+\alpha;\tfrac{1}{2},\tfrac{1}{2}+\tfrac{\rho}{2},\tfrac{3}{2}-\tfrac{\rho}{2};\tfrac{z^2}{4})$$
$$+\frac{(\rho-1)(\rho-2\alpha)}{\rho(2-\rho)}zF(1+\tfrac{\rho}{2}-\alpha,1-\tfrac{\rho}{2}+\alpha;\tfrac{3}{2},1+\tfrac{\rho}{2},2-\tfrac{\rho}{2};\tfrac{z^2}{4}), \quad \ldots (7).$$

<u>Proof of (4)</u>: Expand each of the E-functions on the L.H.S. of (4) by means of (3). Multiply and then apply (2), and get

$$e^{-z}E(\alpha,\beta::z)E(1-\alpha,1-\beta::z) =$$
$$\Gamma(\beta-\alpha)\Gamma(\alpha-\beta)\Gamma(\alpha)\Gamma(1-\alpha)\, z\, F(\alpha;1+\alpha-\beta;z)F(\beta;1+\beta-\alpha;-z)$$
$$+\Gamma^2(\beta-\alpha)\Gamma(\alpha)\Gamma(1-\beta)\, z^{1+\alpha-\beta} F(\alpha;1+\alpha-\beta;z)F(\alpha;1+\alpha-\beta;-z)$$
$$+\Gamma^2(\alpha-\beta)\Gamma(\beta)\Gamma(1-\alpha)\, z^{1+\beta-\alpha} F(\beta;1+\beta-\alpha;z)F(\beta;1+\beta-\alpha;-z)$$

$$+\ \Gamma(\alpha-\beta)\Gamma(\beta-\alpha)\Gamma(\beta)\Gamma(1-\beta)\, z\, F(\beta; 1+\beta-\alpha; z)\, F(\alpha; 1+\alpha-\beta; -z).$$

Now multiply each two hypergemetric functions by means of (6) and (4) and get

$$e^{-z} E(\alpha, \beta :: z) E(1-\alpha, 1-\beta :: z) = \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(1-\alpha)\Gamma(1-\beta)}{(2\pi i)\cdot\sqrt{\pi}}\left(\frac{z}{2}\right)^{1-\alpha-\beta}$$

$$\left[\frac{\sqrt{\pi}\,\Gamma(\frac{1}{2}\beta-\frac{1}{2}\alpha)\Gamma(\frac{1}{2}\alpha-\frac{1}{2}\beta)}{\Gamma(\frac{1}{2}\alpha+\frac{1}{2}\beta)\Gamma(1-\frac{1}{2}\alpha-\frac{1}{2}\beta)}\left(\frac{z}{2}\right)^{\alpha+\beta} F\left(\begin{matrix}1-\frac{\alpha}{2}-\frac{\beta}{2}, \frac{1}{2}\alpha+\frac{1}{2}\beta; z^2/4\\ \frac{1}{2}, 1+\frac{1}{2}\beta-\frac{1}{2}\alpha, 1+\frac{1}{2}\alpha-\frac{1}{2}\beta\end{matrix}\right)\right.$$

$$+\frac{\sqrt{\pi}\,\Gamma^2(\beta-\alpha)}{\Gamma(1-\alpha)\Gamma(\beta)\,2^{\alpha+\beta-1}}\, z^{2\alpha} F\left(\begin{matrix}\alpha, 1-\beta; z^2/4\\ 1+\alpha-\beta, \frac{1}{2}+\frac{1}{2}\alpha-\frac{1}{2}\beta, 1+\frac{1}{2}\alpha-\frac{1}{2}\beta\end{matrix}\right)$$

$$+\frac{\sqrt{\pi}\,\Gamma^2(\alpha-\beta)}{\Gamma(1-\beta)\Gamma(\alpha)\,2^{\alpha+\beta-1}}\, z^{2\beta} F\left(\begin{matrix}\beta, 1-\alpha; z^2/4\\ 1+\beta-\alpha, \frac{1}{2}+\frac{1}{2}\beta-\frac{1}{2}\alpha, 1+\frac{1}{2}\beta-\frac{1}{2}\alpha\end{matrix}\right)$$

$$\left.-\frac{\sqrt{\pi}\,\Gamma(-\frac{1}{2}+\frac{\beta}{2}-\frac{\alpha}{2})\Gamma(-\frac{1}{2}+\frac{\alpha}{2}-\frac{\beta}{2})}{\Gamma(-\frac{1}{2}+\frac{\alpha}{2}+\frac{\beta}{2})\Gamma(\frac{1}{2}-\frac{\alpha}{2}-\frac{\beta}{2})}\,\frac{z^{\alpha+\beta+1}}{2^{\alpha+\beta}} F\left(\begin{matrix}\frac{3}{2}-\frac{\alpha}{2}-\frac{\beta}{2}, \frac{1}{2}+\frac{\alpha}{2}+\frac{\beta}{2}; \frac{z^2}{4}\\ \frac{3}{2}, \frac{3}{2}+\frac{\beta}{2}-\frac{\alpha}{2}, \frac{3}{2}+\frac{\alpha}{2}-\frac{\beta}{2}\end{matrix}\right)\right]$$

$= e^{-z} \times$ R.H.S. of (4) by apply (5) CHAP. I. Thus (4) is proved.

§3. Third theorem: The formula to be proved is

$$E(\alpha, 1-\alpha :: iz)\, E(\beta, 1-\beta :: -iz) =$$

$$\frac{\sqrt{\pi}\sin\left(\frac{\alpha}{2}+\frac{\beta}{2}\right)\pi \cos\left(\frac{\alpha}{2}-\frac{\beta}{2}\right)\pi}{\sin(\alpha\pi)\sin(\beta\pi)} E\left(\frac{1}{2}+\frac{\alpha}{2}-\frac{\beta}{2}, \frac{1}{2}+\frac{\beta}{2}-\frac{\alpha}{2}, \frac{\alpha}{2}+\frac{\beta}{2}, 1-\frac{\alpha}{2}-\frac{\beta}{2} : \frac{1}{2} : \frac{z^2}{4}\right)$$

$$+\frac{2\sqrt{\pi}\, e^{\frac{i\pi}{2}} \cos\left(\frac{\alpha}{2}+\frac{\beta}{2}\right)\pi \sin\left(\frac{\alpha}{2}-\frac{\beta}{2}\right)\pi}{z \cdot \sin(\alpha\pi)\sin(\beta\pi)} E\left(1+\frac{\alpha}{2}-\frac{\beta}{2}, 1+\frac{\beta}{2}-\frac{\alpha}{2}, \frac{1}{2}+\frac{\alpha}{2}+\frac{\beta}{2}, \frac{3}{2}-\frac{\alpha}{2}-\frac{\beta}{2} : \frac{3}{2} : \frac{z^2}{4}\right)$$

$$, \quad \ldots (8)$$

(8) will be derived by means of the following subsidiary formula

$${}_1F_1(\alpha; 2\alpha; x) \times {}_1F_1(\beta; 2\beta; -x) =$$

$${}_2F_3\left[\tfrac{1}{2}(\alpha+\beta), \tfrac{1}{2}(\alpha+\beta+1); \alpha+\tfrac{1}{2}, \beta+\tfrac{1}{2}, \alpha+\beta; x^2/4\right], \quad \ldots (9).$$

Proof of the subsidiary formula: In the product

$${}_1F_1(\alpha; 2\alpha; x) \times {}_1F_1(\beta; 2\beta; -x),$$

the coefficient of x^n is easily seen to be

$$\frac{(\alpha; n)}{n!\,(2\alpha; n)}\, {}_3F_2\left[\begin{matrix} -n, \beta, 1-2\alpha-n; 1 \\ 2\beta, 1-\alpha-n \end{matrix}\right],$$

so that on applying (3) CHAP. II to sum the generalized hypergeometric function; (9) is obtained.

Proof of the third theorem (8). Expand each of the two E-functions on the R.H.S. of (8) by means of (5) CHAP. I, sum any two similar terms and get.

$$R.H.S. \text{ of } (8) = \Gamma(1-2\alpha)\Gamma(1-2\beta)\Gamma(\alpha)\Gamma(\beta)\, e^{\frac{i\pi}{2}(\alpha-\beta)} z^{\alpha+\beta} {}_2F_3\left[\begin{matrix}\frac{1}{2}(\alpha+\beta),\ \frac{1}{2}(\alpha+\beta+1);\ -z^2/4\\ \alpha+\frac{1}{2},\ \beta+\frac{1}{2},\ \alpha+\beta\end{matrix}\right]$$

$$+\Gamma(2\alpha-1)\Gamma(2\beta-1)\Gamma(1-\alpha)\Gamma(1-\beta)\, e^{\frac{i\pi}{2}(\beta-\alpha)} z^{2-\alpha-\beta} {}_2F_3\left[\begin{matrix}\frac{1}{2}(2-\alpha-\beta),\ \frac{1}{2}(3-\alpha-\beta);\ -\frac{z^2}{4}\\ \frac{3}{2}-\alpha,\ \frac{3}{2}-\beta,\ 2-\alpha-\beta\end{matrix}\right]$$

$$+\Gamma(1-2\alpha)\Gamma(\alpha)\Gamma(2\beta-1)\Gamma(1-\beta)\, e^{\frac{i\pi}{2}(\alpha+\beta-1)} z^{1+\alpha-\beta} {}_2F_3\left[\begin{matrix}\frac{1}{2}(1+\alpha-\beta),\ \frac{1}{2}(2+\alpha-\beta);\ \frac{-z^2}{4}\\ 1+\alpha-\beta,\ \frac{3}{2}-\beta,\ \frac{1}{2}+\alpha\end{matrix}\right]$$

$$+\Gamma(2\alpha-1)\Gamma(1-\alpha)\Gamma(1-2\beta)\Gamma(\beta)\, e^{\frac{i\pi}{2}(1-\alpha-\beta)} z^{1+\beta-\alpha} {}_2F_3\left[\begin{matrix}\frac{1}{2}(1-\alpha+\beta),\ \frac{1}{2}(2-\alpha+\beta);\ \frac{-z^2}{4}\\ \frac{3}{2}-\alpha,\ \frac{1}{2}+\beta,\ 1-\alpha+\beta\end{matrix}\right].$$

Now substitute for each ${}_2F_3$ in the last expression from (9), then

(8) is obtained by another application of (5) CHAP. I.

<u>Particular cases</u>: When $\alpha=\beta$, then the second term on the R.H.S. of (8) vanishes and one is left with

$$E(\alpha, 1-\alpha :: iz) \times E(\alpha, 1-\alpha :: -iz)$$

$$= \sqrt{\pi}\,\operatorname{cosec}(\pi\alpha)\, E(\tfrac{1}{2}, \alpha, 1-\alpha :: z^2/4), \quad \cdots\cdots (10)$$

which can also be deduced from (5).

<u>Formula involving the product of two Modified Bessel Functions of the Second Kind</u>: In (8) write $(2x)$ for z, and apply (34) CHAP. I., so getting

$$K_n(x)K_m(xe^{i\pi}) =$$

$$\frac{1}{(2ix)\sqrt{\pi}}\cdot\frac{\cos\left(\frac{m}{2}+\frac{n}{2}\right)\pi\cos\left(\frac{n}{2}-\frac{m}{2}\right)\pi}{1}E\left(\frac{1}{2}+\frac{m}{2}+\frac{n}{2}, \frac{1}{2}+\frac{m}{2}-\frac{n}{2}, \frac{1}{2}+\frac{n}{2}-\frac{m}{2}, \frac{1}{2}-\frac{n}{2}-\frac{m}{2} : \frac{1}{2} : e^{i\pi}x^2\right)$$

$$+ e^{\frac{i\pi}{2}}\cdot\frac{\cos\left(\frac{1}{2}+\frac{m}{2}+\frac{n}{2}\right)\pi\sin\left(\frac{n}{2}-\frac{m}{2}\right)\pi}{(2x^2)\sqrt{\pi}}E\left(1+\frac{m}{2}+\frac{n}{2}, 1+\frac{m}{2}-\frac{n}{2}, 1+\frac{n}{2}-\frac{m}{2}, 1-\frac{n}{2}-\frac{m}{2} : \frac{3}{2} : e^{i\pi}x^2\right), \quad \cdots\cdots (11).$$

Applications of (11) will be given in the next chapter.

§4. Fourth theorem: The formula to be proved is

$$2\pi i\, E(\alpha, 1-\alpha :: z)\, E(\beta, 1-\beta :: z) =$$

$$\frac{\pi^{\frac{3}{2}} e^{z}}{\sin(\alpha\pi)\sin(\pi\beta)}\left[\begin{array}{l} E\left(\tfrac{1}{2}\alpha+\tfrac{1}{2}\beta, \tfrac{1}{2}+\tfrac{1}{2}\alpha-\tfrac{1}{2}\beta, \tfrac{1}{2}+\tfrac{1}{2}\beta-\tfrac{1}{2}\alpha, 1-\tfrac{1}{2}\alpha-\tfrac{1}{2}\beta : \tfrac{1}{2} : e^{i\pi}\cdot\frac{z^2}{4}\right) \\ -E\left(\tfrac{1}{2}\alpha+\tfrac{1}{2}\beta, \tfrac{1}{2}+\tfrac{1}{2}\alpha-\tfrac{1}{2}\beta, \tfrac{1}{2}+\tfrac{1}{2}\beta-\tfrac{1}{2}\alpha, 1-\tfrac{1}{2}\alpha-\tfrac{1}{2}\beta : \tfrac{1}{2} : e^{-i\pi}\cdot\frac{z^2}{4}\right) \end{array}\right] \quad \cdots (12).$$

The following formula is required in the proof

$$E(\alpha, 1-\alpha :: z)\, E(\beta, 1-\beta :: e^{i\pi} z) =$$

$$\frac{\sqrt{\pi}\,\sin\left(\frac{\alpha}{2}+\frac{\beta}{2}\right)\pi\,\cos\left(\frac{\alpha}{2}-\frac{\beta}{2}\right)\pi}{\sin(\alpha\pi)\sin(\beta\pi)} E\left(\frac{1}{2}+\frac{\alpha}{2}-\frac{\beta}{2}, \frac{1}{2}+\frac{\beta}{2}-\frac{\alpha}{2}, \frac{\alpha}{2}+\frac{\beta}{2}, 1-\frac{\alpha}{2}-\frac{\beta}{2} : \frac{1}{2} : e^{i\pi}\cdot\frac{z^2}{4}\right)$$

$$-\frac{2\sqrt{\pi}\,\cos\left(\frac{\alpha}{2}+\frac{\beta}{2}\right)\pi\,\sin\left(\frac{\alpha}{2}-\frac{\beta}{2}\right)\pi}{z\,\sin(\alpha\pi)\sin(\beta\pi)} E\left(1+\frac{\alpha}{2}-\frac{\beta}{2}, 1+\frac{\beta}{2}-\frac{\alpha}{2}, \frac{1}{2}+\frac{\alpha}{2}+\frac{\beta}{2}, \frac{3}{2}-\frac{\alpha}{2}-\frac{\beta}{2} : \frac{3}{2} : e^{i\pi}\cdot\frac{z^2}{4}\right)$$

[(13) is formula (8) with $\left(z e^{\frac{i\pi}{2}}\right)$ written for z and α, β interchanged]; $\cdots$ (13)

$$(2\pi i)e^{-z} E(\beta, 1-\beta :: z) = \Gamma(\beta)\Gamma(1-\beta)\left[E(\beta, 1-\beta :: e^{i\pi} z) - E(\beta, 1-\beta :: e^{-i\pi} z)\right], \cdots (14),$$

[(14) is formula (1) with $\alpha = 1-\beta$]

Proof of the formula: By (14), the L.H.S. of (12) is equal to

$$(2\pi i)E(\alpha, 1-\alpha :: z)\, E(\beta, 1-\beta :: z) =$$

$$e^{z}\Gamma(\beta)\Gamma(1-\beta)E(\alpha, 1-\alpha :: z)E(\beta, 1-\beta :: e^{i\pi}z)$$

$$- e^{z}\Gamma(\beta)\Gamma(1-\beta)E(\alpha, 1-\alpha :: z)E(\beta, 1-\beta :: e^{-i\pi}z).$$

Now substitute for the products on the R.H.S. of the last expression from (13) and from (8) with $(ze^{-\frac{i\pi}{2}})$ written for z, and get

$$2\pi i E(\alpha, 1-\alpha :: z)\, E(\beta, 1-\beta :: z) =$$

$$e^{z}\Gamma(\beta)\Gamma(1-\beta)$$

$$\times\left[\frac{\sqrt{\pi}\sin\left(\frac{\alpha}{2}+\frac{\beta}{2}\right)\pi\cos\left(\frac{\alpha}{2}-\frac{\beta}{2}\right)\pi}{\sin(\alpha\pi)\sin(\beta\pi)}E\left(\frac{1}{2}+\frac{\alpha}{2}-\frac{\beta}{2}, \frac{1}{2}+\frac{\beta}{2}-\frac{\alpha}{2}, \frac{\alpha}{2}+\frac{\beta}{2}, 1-\frac{\alpha}{2}-\frac{\beta}{2} : \frac{1}{2} : e^{i\pi}\frac{z^2}{4}\right)\right.$$

$$\left.-\frac{2\sqrt{\pi}\cos\left(\frac{\alpha}{2}+\frac{\beta}{2}\right)\pi\sin\left(\frac{\alpha}{2}-\frac{\beta}{2}\right)\pi}{z\,\sin(\alpha\pi)\sin(\beta\pi)}E\left(1+\frac{\alpha}{2}-\frac{\beta}{2}, 1+\frac{\beta}{2}-\frac{\alpha}{2}, \frac{1}{2}+\frac{\alpha}{2}+\frac{\beta}{2}, \frac{3}{2}-\frac{\alpha}{2}-\frac{\beta}{2} : \frac{3}{2} : e^{i\pi}\frac{z^2}{4}\right)\right]$$

$$
-e^{z}\Gamma(\beta)\Gamma(1-\beta)
$$

$$
\times\left[\frac{\sqrt{\pi}\sin\left(\frac{\alpha}{2}+\frac{\beta}{2}\right)\pi\cos\left(\frac{\alpha}{2}-\frac{\beta}{2}\right)\pi}{\sin(\alpha\pi)\sin(\beta\pi)}E\left(\frac{1}{2}+\frac{\alpha}{2}-\frac{\beta}{2},\frac{1}{2}+\frac{\beta}{2}-\frac{\alpha}{2},\frac{\alpha}{2}+\frac{\beta}{2},1-\frac{\alpha}{2}-\frac{\beta}{2}:\frac{1}{2}:e^{-i\pi}\cdot\frac{z^2}{4}\right)\right.
$$

$$
\left.-\frac{2\sqrt{\pi}}{z}\frac{\cos\left(\frac{\alpha}{2}+\frac{\beta}{2}\right)\pi\sin\left(\frac{\alpha}{2}-\frac{\beta}{2}\right)\pi}{\sin(\alpha\pi)\sin(\beta\pi)}E\left(1+\frac{\alpha}{2}-\frac{\beta}{2},1+\frac{\beta}{2}-\frac{\alpha}{2},\frac{1}{2}+\frac{\alpha}{2}+\frac{\beta}{2},\frac{3}{2}-\frac{\alpha}{2}-\frac{\beta}{2}:\frac{3}{2}:e^{-i\pi}\frac{z^2}{4}\right)\right].
$$

But it can easily be proved that

$$
\frac{1}{z}\left[\begin{array}{l} E\left(1+\frac{\alpha}{2}-\frac{\beta}{2},1+\frac{\beta}{2}-\frac{\alpha}{2},\frac{1}{2}+\frac{\alpha}{2}+\frac{\beta}{2},\frac{3}{2}-\frac{\alpha}{2}-\frac{\beta}{2}:\frac{3}{2}:e^{i\pi}\cdot\frac{z^2}{4}\right) \\ -E\left(1+\frac{\alpha}{2}-\frac{\beta}{2},1+\frac{\beta}{2}-\frac{\alpha}{2},\frac{1}{2}+\frac{\alpha}{2}+\frac{\beta}{2},\frac{3}{2}-\frac{\alpha}{2}-\frac{\beta}{2}:\frac{3}{2}:e^{-i\pi}\cdot\frac{z^2}{4}\right)\end{array}\right]
$$

$$
=\frac{1}{2}\left[\begin{array}{l} E\left(\frac{1}{2}+\frac{\alpha}{2}-\frac{\beta}{2},\frac{1}{2}+\frac{\beta}{2}-\frac{\alpha}{2},\frac{\alpha}{2}+\frac{\beta}{2},1-\frac{\alpha}{2}-\frac{\beta}{2}:\frac{1}{2}:e^{i\pi}\cdot\frac{z^2}{4}\right) \\ -E\left(\frac{1}{2}+\frac{\alpha}{2}-\frac{\beta}{2},\frac{1}{2}+\frac{\beta}{2}-\frac{\alpha}{2},\frac{\alpha}{2}+\frac{\beta}{2},1-\frac{\alpha}{2}-\frac{\beta}{2}:\frac{1}{2}:e^{-i\pi}\cdot\frac{z^2}{4}\right)\end{array}\right]
$$

So that the last expression becomes

$$(2\pi i)E(\alpha, 1-\alpha::z)E(\beta, 1-\beta::z) =$$

$$\frac{\sqrt{\pi}\,\Gamma(\beta)\Gamma(1-\beta)\,e^{z}}{\sin(\alpha\pi)\sin(\beta\pi)}\left\{\sin\left(\frac{\alpha}{2}+\frac{\beta}{2}\right)\pi\cos\left(\frac{\alpha}{2}-\frac{\beta}{2}\right)\pi-\cos\left(\frac{\alpha}{2}+\frac{\beta}{2}\right)\pi\sin\left(\frac{\alpha}{2}-\frac{\beta}{2}\right)\pi\right\}$$

$$\times\left[\begin{array}{l} E\left(\frac{1}{2}+\frac{\alpha}{2}-\frac{\beta}{2}, \frac{1}{2}+\frac{\beta}{2}-\frac{\alpha}{2}, \frac{\alpha}{2}+\frac{\beta}{2}, 1-\frac{\alpha}{2}-\frac{\beta}{2} : \frac{1}{2} : e^{i\pi}\cdot\frac{z^2}{4}\right) \\ -E\left(\frac{1}{2}+\frac{\alpha}{2}-\frac{\beta}{2}, \frac{1}{2}+\frac{\beta}{2}-\frac{\alpha}{2}, \frac{\alpha}{2}+\frac{\beta}{2}, 1-\frac{\alpha}{2}-\frac{\beta}{2} : \frac{1}{2} : e^{-i\pi}\cdot\frac{z^2}{4}\right) \end{array}\right]$$

from which (12) is obtained.

A second formula involving the product of two Modified Bessel Functions of the Second Kind.

(12) in combination with (34) CHAP. I. gives

$$K_m(x)K_n(x) = \frac{\sqrt{\pi}}{4\pi x}\sum_{i,-i}\frac{1}{i}E\left(\frac{1}{2}+\frac{m}{2}+\frac{n}{2}, \frac{1}{2}+\frac{m}{2}-\frac{n}{2}, \frac{1}{2}+\frac{n}{2}-\frac{m}{2}, \frac{1}{2}-\frac{n}{2}-\frac{m}{2} : \frac{1}{2} : e^{i\pi}x^2\right)$$

, ... (15).

Applications of (15) will be given in the next chapter.

CHAPTER IV.

INTEGRALS INVOLVING BESSEL FUNCTIONS.

§1. Nicholson's integral of $J_\nu^2(x) + Y_\nu^2(x)$: The formula to be proved is

$$J_\nu^2(x) + Y_\nu^2(x) = \frac{8}{\pi^2}\int_0^\infty K_0(2x\sinh t)\cosh(2\nu t)\,dt, \quad \ldots\ldots \quad (1).$$

(1) was proved in Watson "Theory of Bessel Functions" §13.73 by applying Hardy's theorem on generalized integrals. Another proof was given by Dixon and Ferrar [Quart. Journ. of Math., Vol. I., 1930, p. 137]. They proved also the formula

$$J_\nu^2(x) + Y_\nu^2(x) = \frac{8\cos\nu\pi}{\pi^2}\int_0^\infty K_{2\nu}(2x\sinh t)\,dt, \quad \ldots\ldots \quad (2).$$

Bailey [Quart. Journ. of Math., (8), 1937, p. 53] deduced (2) by a method similar to that given by C. S. Meijer [Proc. Lond. Math. Soc., (2). Vol. (40), 1936, pp., 1—22].

In this proof (1) will be deduced from E-functions' formulae. The following formulae will be required in the proof:

$$4\int_0^\infty \lambda^{m-1} K_n(2\lambda) E(p; \alpha_r : q; \rho_s : x\lambda^{-2})\, d\lambda = E(p+2; \alpha_r : q; \rho_s : x), \quad \cdots (3)$$

where $\alpha_{p+1} = \frac{1}{2}(m+n)$, $\alpha_{p+2} = \frac{1}{2}(m-n)$, $R(\frac{1}{2}m \pm \frac{1}{2}n) > 0$.

[C.V., p. 380, ex. 109],

$$E\left(\tfrac{1}{2}-k+m, \tfrac{1}{2}-k-m, \tfrac{1}{2}-k, 1-k : 1-2k : \frac{x^2}{4}\right)$$
$$= 2^{2k}\Gamma(\tfrac{1}{2})\Gamma(\tfrac{1}{2}-k+m)\Gamma(\tfrac{1}{2}-k-m)x^{-2k} W_{k,m}(ix) W_{k,m}(-ix)$$

[C.V., p. 380, ex. (111) ii], $\quad \cdots (4)$

$$F(m, m+\tfrac{1}{2}; \tfrac{1}{2}; x^2) = \tfrac{1}{2}\left[(1+x)^{-2m} + (1-x)^{-2m}\right], \quad \cdots\cdots (5)$$

[Kummer, Journ. Fur. Math. Vol. (15), 1936, p. 48].

The function $Y_\nu(x)$ is defined in Watson "Theory of Bessel Functions" p. 64. Also in p. (73), § 3.6; there is the relation

$$H_\nu^{(1)}(x) H_\nu^{(2)}(x) = J_\nu^2(x) + Y_\nu^2(x), \quad \cdots\cdots (6),$$

where the functions $H_\nu^{(1)}(x)$, $H_\nu^{(2)}(x)$ are related to the Bessel-Function $K_\nu(x)$ by the formulae

$$K_\nu\left(x e^{-\frac{1}{2}\pi i}\right) = \tfrac{1}{2}\pi i\, e^{\frac{1}{2}\nu\pi i} H_\nu^{(1)}(x),$$

$$K_\nu\left(x e^{\frac{1}{2}\pi i}\right) = -\tfrac{1}{2}\pi i\, e^{-\frac{1}{2}\nu\pi i} H_\nu^{(2)}(x).$$

Also it can be proved [Meijer, C.S., Mathem. Annalen, Band 112, 1936, p. 487] that

$$H_\nu^{(1)}(x)\, H_\nu^{(2)}(x) = \frac{2}{\pi x} W_{0,\nu}(2ix)\, W_{0,\nu}(-2ix), \quad \ldots (4)$$

From (3) and (4) it is easily seen that

$$W_{k,m}(ix)\, W_{k,m}(-ix) = \frac{4x^{2k}}{\sqrt{\pi}\cdot 2^{2k}\,\Gamma(\frac{1}{2}-k+m)\,\Gamma(\frac{1}{2}-k-m)}$$

$$\times \int_0^\infty u^{\alpha+\beta-1} K_{\alpha-\beta}(2u)\, E\left(\tfrac{1}{2}+m-k, \tfrac{1}{2}-m-k, \tfrac{1}{2}-k, 1-k : 1-2k, \alpha, \beta : \frac{x^2}{4u^2}\right) du, \quad \ldots (8)$$

where $R(\alpha)>0$, $R(\beta)>0$.

In the formula

$$F\left(\tfrac{1}{2}+m, \tfrac{1}{2}-m; \tfrac{1}{2}; x\right) = (1-x)^{-\frac{1}{2}-m} F\left(\tfrac{1}{2}+m, m; \tfrac{1}{2}; \frac{x}{x-1}\right); \ldots$$

put $x = -\sinh^2 t$ and apply (5), so getting

$$F\left(\tfrac{1}{2}+m, \tfrac{1}{2}-m; \tfrac{1}{2}; -\sinh^2 t\right) = \frac{\cosh 2mt}{\cosh t}, \quad \ldots \ldots \quad (9)$$

<u>Proof of the formula (1).</u> In (8), put $u = \frac{1}{2} x \sinh t$ and then make the following substitutions

$$k = 0, \quad \alpha = \tfrac{1}{2}, \quad \beta = \tfrac{1}{2}.$$

Noting that the E-function on the R.H.S. of (8) is a hypergeometric function, then (1) is obtained by the applications of (6), (7) and (9).

<u>Note</u>: If in (8)

$k = 0$, $\alpha = \frac{1}{2}+m$, and $\beta = \frac{1}{2}-m$, then (2) is obtained.

This was the method of W.N. Bailey "Loc. Cit".

<u>§2: An Integral due to Hanumanta Rao.</u> In (9) CHAP. I, take $p = l = 0$, $q = m = 1$, put $\rho_1 = m+1$, $\sigma_1 = n+1$, replace z by $(16/z^2)$, λ by $(4\lambda^2)$ and k by $\frac{1}{2}(\rho - m + n)$; then from (6) CHAP. II., if $R(m-\rho) > -\frac{3}{2}$, $R(\rho+n) > -\frac{3}{2}$,

$$\int_0^\infty \lambda^{\rho-1} J_m(1/\lambda)\, J_n(z\lambda)\, d\lambda =$$

$$\frac{\Gamma(\frac{1}{2}\rho - \frac{1}{2}m + \frac{1}{2}n)\, z^{m-\rho}}{2^{2m-\rho+1}\,\Gamma(\frac{1}{2}m+\frac{1}{2}n-\frac{1}{2}\rho+1)\,\Gamma(m+1)}\, F\left(; m+1, \frac{m}{2}-\frac{n}{2}-\frac{\rho}{2}+1, \frac{m}{2}+\frac{n}{2}-\frac{\rho}{2}+1; \frac{z^2}{16}\right)$$

$$+ \frac{\Gamma(\frac{1}{2}m - \frac{1}{2}n - \frac{1}{2}\rho)\, z^{n}}{2^{2n+\rho+1}\,\Gamma(\frac{1}{2}m+\frac{1}{2}n+\frac{1}{2}\rho+1)\,\Gamma(n+1)}\, F\left(; n+1, \frac{n}{2}-\frac{m}{2}+\frac{\rho}{2}+1, \frac{m}{2}+\frac{n}{2}+\frac{\rho}{2}+1; \frac{z^2}{16}\right)$$

$, \ldots (10)$

which was given by Hanumanta Rao [Messen. of Math., XLVII, 1918, pp. 134—137].

§ 3. Integrals of the Modified Bessel Function of the Second Kind.

In (9) CHAP. I, take $p=2$, $q=0$, $\alpha_1=\frac{1}{2}+n$, $\alpha_2=\frac{1}{2}-n$ and replace λ by (2λ) and z by $(2z)$ then from (34) CHAP. I, it follows that if $R(k\pm n)>-\frac{1}{2}$, $R(\beta_t-k)>0$, $t=1,2,3,\ldots,l$, $l \geqq m+1$

$$\int_0^\infty \lambda^{k-\frac{1}{2}} e^{\lambda} K_n(\lambda)\, E(l; \beta_t : m; \sigma_u : z/\lambda)\, d\lambda = \sqrt{\left(\frac{\pi}{2}\right)}\, \frac{\cos n\pi}{\sin k\pi}$$

$$\times \left\{ z^k E\left(\begin{matrix}\frac{1}{2}+n, \frac{1}{2}-n, \beta_1-k, \ldots, \beta_l-k : 2e^{\pm i\pi} z \\ 1-k, \sigma_1-k, \ldots, \sigma_m-k\end{matrix}\right) - 2^{-k} E\left(\begin{matrix}\frac{1}{2}+n+k, \frac{1}{2}-n+k, \beta_1, \ldots, \beta_l : 2e^{\pm i\pi} z \\ 1+k, \sigma_1, \ldots, \sigma_m\end{matrix}\right)\right\}$$

$, \ldots (11).$

In particular from (6) CHAP. II., if $l=0$, $m=1$ and σ_1 is replaced by $(m+1)$

$$\int_0^\infty \lambda^{k-\frac{1}{2}m-\frac{1}{2}} e^{\lambda} K_n(\lambda)\, J_m\left(2\sqrt{\tfrac{\lambda}{z}}\right) d\lambda =$$

$$\sqrt{\left(\frac{\pi}{2}\right)}\, \frac{\cos n\pi}{\sin k\pi}\, z^{-\frac{1}{2}m} \begin{Bmatrix} z^{k} E\left(\frac{1}{2}+n, \frac{1}{2}-n : 1-k, m+1-k : 2e^{\pm i\pi} z\right) \\ -2^{-k} E\left(\frac{1}{2}+n+k, \frac{1}{2}-n+k : 1+k, 1+m : 2e^{\pm i\pi} z\right) \end{Bmatrix}, \quad \ldots (12)$$

where $R(\pm n-\frac{1}{2}) < R(k) < R(\frac{1}{2}m+\frac{3}{4})$, $R(z)>0$.

Next in (11) put $l=1$, $m=0$, $\beta_1=\beta$ and get

$$\int_0^\infty \lambda^{k-\frac{1}{2}} (\lambda+z)^{-\beta} e^{\lambda} K_n(\lambda)\, d\lambda =$$

$$\sqrt{\left(\frac{\pi}{2}\right)} \cdot \frac{\cos n\pi}{\sin k\pi}\, \frac{z^{-\beta}}{\Gamma(\beta)} \begin{bmatrix} z^{k} E\left(\frac{1}{2}+n, \frac{1}{2}-n, \beta-k : 1-k : 2e^{\pm i\pi} z\right) \\ -2^{-k} E\left(\frac{1}{2}+n+k, \frac{1}{2}-n+k, \beta : 1+k : 2e^{\pm i\pi} z\right) \end{bmatrix}, \quad \ldots (13)$$

where $R(\mp n-\frac{1}{2}) < R(k) < R(\beta)$ and $|\operatorname{amp} z| < \pi$.

Also in (11) take $l=2$, $m=0$, put $\beta_1 = \frac{1}{2}+m$, $\beta_2 = \frac{1}{2}-m$ and replace z by $(2z)$; then from (34) CHAP-I.

$$\int_0^\infty \lambda^{k-1} e^{\lambda + z/\lambda} K_n(\lambda) K_m(z/\lambda)\, d\lambda =$$

$$\frac{\cos(m\pi)\cos(n\pi)}{2\sqrt{(z)}\cdot \sin k\pi}\left\{\begin{array}{l}(2z)^k E(\frac{1}{2}+n, \frac{1}{2}-n, \frac{1}{2}+m-k, \frac{1}{2}-m-k : 1-k : 4e^{\pm i\pi} z) \\ -2^{-k} E(\frac{1}{2}+n+k, \frac{1}{2}-n+k, \frac{1}{2}+m, \frac{1}{2}-m : 1+k : 4e^{\pm i\pi} z)\end{array}\right\}, \quad \cdots (14)$$

where $R(\pm n - \frac{1}{2}) < R(k) < R(\frac{1}{2} \pm m)$ and $|\operatorname{amp} z| < \pi$.

In (10) CHAP-I., take $p=2$, $q=0$ with $\alpha_1 = \frac{1}{2}+n$, $\alpha_2 = \frac{1}{2}-n$ and replace z by $(z/2)$, $\underline{\lambda}$ by (2λ), (α_{p+1}) by $\underline{k}$, then from (34) CHAP-I.

$$\int_0^\infty \lambda^{-k-\frac{1}{2}} e^{\lambda} K_n(\lambda) E(l; \beta_t : m; \sigma_u : z\lambda)\, d\lambda =$$

$$\frac{2^{k-\frac{1}{2}} \pi^{\frac{3}{2}} \cos(n\pi)}{\cos(n-k)\pi \cos(n+k)\pi} E\left\{\begin{array}{l} k, \beta_1, \ldots, \beta_l : e^{\pm 2i\pi} \frac{z}{2} \\ k+\frac{1}{2}+n, k+\frac{1}{2}-n, \sigma_1, \ldots, \sigma_m \end{array}\right\}$$

$$+ \sum_{n, -n} \frac{2^n \pi^{\frac{3}{2}} \cos(n\pi)}{\sin(k-\frac{1}{2}-n)\pi \sin(-2n\pi)} z^{k-n-\frac{1}{2}} E\left\{\begin{array}{l} \frac{1}{2}+n, \frac{1}{2}+n+\beta_1-k, \ldots, \frac{1}{2}+n+\beta_l-k : e^{\pm 2i\pi} \frac{z}{2} \\ \frac{3}{2}+n-k, 1+2n, \frac{1}{2}+n+\sigma_1-k, \ldots, \frac{1}{2}+n+\sigma_m-k \end{array}\right\}, \quad \cdots (15)$$

where $R(k)>0$, $R(\tfrac{1}{2}\pm n-k+\beta_t)>0$, $t=1,2,3,\ldots,l$, $|\operatorname{amp} z|<\pi$.

In particular if $m=l=0$ and $R(k)>0$, $R(z)>0$

$$\int_0^\infty \lambda^{-k-\frac{1}{2}} \exp\{\lambda - 1/(z\lambda)\} K_n(\lambda)\, d\lambda =$$

$$\frac{2^{k-\frac{1}{2}}\pi^{3/2}\cos(n\pi)}{\cos(n-k)\pi\cos(n+k)\pi} E(k : k+\tfrac{1}{2}-n, k+\tfrac{1}{2}+n : \tfrac{z}{2})$$

$$+\sum_{n,-n} \frac{2^n \pi^{\frac{3}{2}}\cos(n\pi)}{\sin(k-\frac{1}{2}-n)\pi\sin(-2n\pi)} z^{k-n-\frac{1}{2}} E(\tfrac{1}{2}+n : \tfrac{3}{2}+n-k, 1+2n : \tfrac{z}{2}), \quad \ldots (16).$$

In (15) take $l=0$, $m=1$ with $\rho_1=m+1$, then from (6) CHAP. II.

$$\int_0^\infty \lambda^{-k+\frac{1}{2}m-\frac{1}{2}} e^\lambda K_n(\lambda) J_m\{2/\sqrt{(z\lambda)}\}\, d\lambda =$$

$$\frac{2^{k-\frac{1}{2}}\pi^{\frac{3}{2}}\cos(n\pi)}{\cos(n-k)\pi\cos(n+k)\pi} z^{-\frac{1}{2}m} E(k : k+\tfrac{1}{2}-n, k+\tfrac{1}{2}+n, 1+m ; \tfrac{z}{2})$$

$$+\sum_{n,-n} \frac{2^n\pi^{\frac{3}{2}}\cos(n\pi)\cdot z^{k-n-\frac{m}{2}-\frac{1}{2}}}{\sin(k-\frac{1}{2}-n)\pi\sin(-2n\pi)} E(\tfrac{1}{2}+n ; \tfrac{3}{2}+n-k, 1+2n, \tfrac{3}{2}+n+m-k ; \tfrac{z}{2}), \quad \ldots (17)$$

where $R(k)>0$, $R(\tfrac{5}{2}\pm 2n+m-k)>0$, $|\operatorname{amp} z|<\pi$.

Again in (15) take $l=2$, $m=0$, put $\beta_1 = \frac{1}{2}+m$, $\beta_2 = \frac{1}{2}-m$, and replace z by $(2z)$; then from (34) CHAP. I.

$$\int_0^\infty \exp\{\lambda + z\lambda\}\lambda^{-k} K_n(\lambda) K_m(z\lambda)\, d\lambda =$$

$$\frac{2^{k-1}\pi \cos(n\pi)\cos(m\pi)}{\cos(n-k)\pi \cos(n+k)\pi} z^{-\frac{1}{2}} E\left(k, \tfrac{1}{2}+m, \tfrac{1}{2}-m : k+\tfrac{1}{2}-n, k+\tfrac{1}{2}+n : z\right)$$

$$+\sum_{n,-n} \frac{2^{n-\frac{1}{2}}\pi \cos(n\pi)\cos(m\pi)}{\sin(k-\frac{1}{2}-n)\pi \sin(-2n\pi)} z^{-\frac{1}{2}}(2z)^{k-n-\frac{1}{2}} E\left\{\begin{matrix} \frac{1}{2}+n, 1+n+m-k, 1+n-m-k : z \\ \frac{3}{2}+n-k, 1+2n \end{matrix}\right\}, \quad \ldots\ldots (18)$$

where $R(k)>0$, $R(1-k\pm n\pm m)>0$, $R(z)>0$

§4. Integrals involving Bessel Functions of the First Kind. In (1) CHAP. I. take $p=0$, $q=1$ with $\rho_1 = n+1$, and apply (6) CHAP. II. and get

$$z^{\frac{1}{2}n}\int_0^\infty e^{-\lambda}\lambda^{k+\frac{1}{2}n-1} J_n\{2/\sqrt{(\lambda z)}\}\, d\lambda =$$

$$\frac{\Gamma(k)}{\Gamma(1+n)} F\left(; 1-k, n+1; \tfrac{1}{z}\right) + \frac{\Gamma(-k)}{\Gamma(n+k+1)} z^{-k} F\left(; 1+k, n+k+1; \tfrac{1}{z}\right), \quad \ldots (19)$$

where $R(k+\frac{1}{2}n) > -\frac{3}{4}$, $R(z)>0$.

In (a) CHAP. I. take $l=0$, $m=1$, with $\rho_1 = n+1$, then from (6) CHAP. II.

$$\int_0^\infty \lambda^{k-\frac{1}{2}n-1} J_n\left(2\sqrt{\frac{\lambda}{z}}\right) E(p;\alpha_r:q;\rho_s:\lambda)\,d\lambda = \frac{\pi}{\sin k\pi} z^{-\frac{1}{2}n}$$

$$\times\left[z^k E\left\{\begin{matrix}\alpha_1,\ldots,\alpha_p : e^{\pm i\pi}z\\ 1-k,\rho_1,\ldots,\rho_q, 1+n-k\end{matrix}\right\} - E\left\{\begin{matrix}\alpha_1+k,\ldots,\alpha_p+k : e^{\pm i\pi}z\\ 1+k,\rho_1+k,\ldots,\rho_q+k, 1+n\end{matrix}\right\}\right], \quad\ldots(20)$$

where $R(k+\alpha_r)>0$, $r=1,2,3,\ldots,p$, $R(\frac{3}{4}-k+\frac{n}{2})>0$, $|\text{amp}\,z|<\pi$.

In particular when $p=1$, $q=0$; (20) gives

$$\int_0^\infty \frac{x^{\rho-1}J_\nu(ax)}{(x^2+k^2)^{\mu+1}}\,dx = \frac{a^\nu k^{\rho+\nu-2\mu-2}\,\Gamma(\frac{\rho}{2}+\frac{\nu}{2})\Gamma(\mu+1-\frac{\rho}{2}-\frac{\nu}{2})}{2^{\nu+1}\,\Gamma(\mu+1)\,\Gamma(\nu+1)}\,{}_1F_2\left(\begin{matrix}\frac{\rho+\nu}{2}; & \frac{a^2k^2}{4}\\ \frac{\rho+\nu}{2}-\mu, \nu+1\end{matrix}\right)$$

$$+\frac{a^{2\mu+2-\rho}\,\Gamma(\frac{1}{2}\nu+\frac{1}{2}\rho-\mu-1)}{2^{2\mu+2-\rho}\,\Gamma(\mu+2+\frac{\nu}{2}-\frac{\rho}{2})}\,{}_1F_2\left(\begin{matrix}\mu+1; & \frac{a^2k^2}{4}\\ \mu+2+\frac{\nu-\rho}{2}, \mu+2-\frac{\nu+\rho}{2}\end{matrix}\right), \quad\ldots(21)$$

where $-R(\nu) < R(\rho) < 2R(\mu)+\frac{7}{2}$,

which was proved by another method in Watson "Theory of Bessel

Functions" p. 434, formula (1). It is also deduced by me "Thesis for Ph.D." p. 130, equ. (4).

In (10) CHAP. I. take $p=q=0$, $l=0$, $m=1$, and apply (6) CHAP. II., so getting

$$\int_0^\infty e^{-p^2t} J_\nu(at)\, t^{\mu-1}\, dt = \frac{\Gamma(\frac{1}{2}\nu+\frac{1}{2}\mu)\,(\frac{1}{2}a/p)^\nu}{2\,p^\mu\,\Gamma(\nu+1)}\, {}_1F_1\left(\tfrac{1}{2}\nu+\tfrac{1}{2}\mu;\, \nu+1;\, \frac{-a^2}{4p^2}\right), \quad \cdots (22)$$

where $R(\mu+\nu)>0$,

which also was given in Watson "Theory of Bessel Functions" p. 393, formula (2).

Again in (10) CHAP. I. take $l=0$, $m=1$ and apply (6) CHAP. II., so getting

$$2\pi\int_0^\infty \lambda^{m-1} J_n(2\lambda)\, E(p;\alpha_r:q;\rho_s: z\lambda^{-2})\, d\lambda =$$

$$\sum_{r=1}^{p+1} \frac{\prod_{s=1}^{p+2}{}' \,\Gamma(\alpha_s-\alpha_r)}{\prod_{t=1}^{q} \Gamma(\rho_t-\alpha_r)}\, \Gamma(\alpha_r)\, \sin(\tfrac{1}{2}m-\tfrac{1}{2}n-\alpha_r)\pi\; z^{\alpha_r}$$

$$F\left(\begin{matrix}\alpha_r, \alpha_r-\rho_1+1, \ldots, \alpha_r-\rho_q+1 \\ \alpha_r-\alpha_1+1, \ldots * \ldots, \alpha_r-\alpha_{p+2}+1\end{matrix};\, (-1)^{p-q+1}z\right), \quad \cdots (23)$$

where $\alpha_{p+1} = \frac{1}{2}(m+n)$, $\alpha_{p+2} = \frac{1}{2}(m-n)$, $R(m+n)>0$, $R(\frac{3}{2}-m+2\alpha_r)>0$, $r=1,2,3,\dots,p$, $p \geqq q+1$.

(22) was proved by another method by me [Proc. Glasg. Math. Assoc., Vol. I., p. 8 formula (7)].

§ 5. The discontinuous integral of Weber and Schafheitlin :- If

$$I \equiv \int_0^\infty J_m(\lambda x)\, J_n(\lambda)\, \lambda^{-k}\, d\lambda,$$

where x is real and positive, $-1 < R(k) < R(m+n+1)$, then

$$I = \left\{ \begin{array}{ll} \dfrac{\Gamma\left(\frac{m+n-k+1}{2}\right) 2^{-k}}{\Gamma\left(\frac{n-m+k+1}{2}\right)\Gamma(m+1)}\, x^m\, F\left(\begin{matrix}\frac{m+n-k+1}{2}, \frac{m-n-k+1}{2}; x^2 \\ m+1\end{matrix}\right), & 0<x<1 \\[2ex] \dfrac{\Gamma\left(\frac{m+n-k+1}{2}\right) 2^{-k}}{\Gamma\left(\frac{m-n+k+1}{2}\right)\Gamma(n+1)}\, x^{k-n-1}\, F\left(\begin{matrix}\frac{m+n-k+1}{2}, \frac{n-m-k+1}{2}; \frac{1}{x^2} \\ n+1\end{matrix}\right), & x>1 \end{array} \right\}, \quad \dots\dots (24)$$

To prove this take $p=0$, $q=1$, $l=0$, $m=1$ in (10) CHAP. I. and replace (α_{p+1}) by α, ρ_1 by $(n+1)$ and σ_1 by $(m+1)$, then from (6) CHAP. II.

$$\pi x^{\frac{1}{2}m}\int_0^\infty \lambda^{-1-d+\frac{1}{2}m+\frac{1}{2}n} J_m\{2\sqrt{(\lambda x)}\} J_n(2\sqrt{\lambda})\, d\lambda =$$

$$-\sin(n-d)\pi E(d, d-n : m+1 : -x)$$

$$= \frac{\pi\Gamma(d)}{\Gamma(1+n-d)\Gamma(m+1)} F(d, d-n; m+1; \tfrac{1}{x}),$$

provided that $x > 1$.

Now replace λ by $(4/\lambda^2)$, x by $(1/x^2)$ and d by $\frac{1}{2}(m+n-k+1)$ and so obtain the first case of (24).

To obtain the second case interchange m and n, replace λ by (λ/x) and then replace x by $(1/x)$.

§ 6. Integrals involving Bessel Functions: As before (16) CHAP. I. in combination with (6) CHAP. II. gives if $|\operatorname{amp} z| < \pi$, $R(\frac{3}{2} - \frac{2k}{m} + n) > 0$, and m is any positive integer

$$\int_0^\infty e^{-\lambda}\lambda^{k+\frac{nm}{2}-1} J_n\{2/(\lambda^{\frac{m}{2}}\sqrt{z})\}\,d\lambda =$$

$$\frac{\pi(2\pi)^{\frac{m}{2}-\frac{1}{2}} m^{k-\frac{1}{2}}}{\sin(k\pi)z^{n/2}} E\left(:1-\frac{k}{m},1-\frac{k+1}{m},\ldots,1-\frac{k+m-1}{m},n+1:e^{\pm im\pi}m^m z\right)$$

$$+2^{\frac{1}{2}-\frac{1}{2}m}\pi^{\frac{1}{2}+\frac{1}{2}m} z^{-\frac{1}{2}n}\sum_{\nu=0}^{m-1}\frac{(-1)^{\nu+1}m^{-\frac{1}{2}-\nu}}{\sin\left(\frac{k+\nu}{m}\right)\pi\prod_{s=1}^{\nu}\sin\left(\frac{s\pi}{m}\right)\prod_{t=1}^{m-\nu-1}\sin\left(\frac{t\pi}{m}\right)} z^{-(k+\nu)/m}$$

$$\times E\left\{:1+\frac{k+\nu}{m},1+\frac{1}{m},\ldots,1+\frac{\nu}{m},1-\frac{1}{m},1-\frac{2}{m},\ldots,1-\frac{m-\nu-1}{m},1+n+\frac{k+\nu}{m}:e^{\pm im\pi}m^m z\right\} \quad ,\ldots(25)$$

(16) CHAP. I. in combination with (34) CHAP. I. gives

$$\int_0^\infty \exp\{-\lambda+z\lambda^m\}\lambda^{k+\frac{m}{2}-1}K_n\{z\lambda^m\}\,d\lambda =$$

$$\sqrt{\left(\frac{\pi}{2}\right)}\frac{\cos n\pi}{\sin k\pi}(2\pi)^{\frac{1}{2}m-\frac{1}{2}}m^{k-\frac{1}{2}}z^{-\frac{1}{2}}E\left(\frac{1}{2}+n,\frac{1}{2}-n;1-\frac{k}{m},1-\frac{k+1}{m},\ldots,1-\frac{k+m-1}{m}:e^{\pm im\pi}2m^m z\right)$$

$$+2^{\frac{1}{2}-\frac{1}{2}m}\pi^{\frac{1}{2}+\frac{1}{2}m}(2\pi z)^{-\frac{1}{2}}\cos(n\pi)\sum_{\nu=0}^{m-1}\frac{(-1)^{\nu+1}m^{-\frac{1}{2}-\nu}}{\sin\left(\frac{k+\nu}{m}\right)\pi\prod_{s=1}^{\nu}\sin\left(\frac{s\pi}{m}\right)\prod_{t=1}^{m-\nu-1}\sin\frac{t\pi}{m}}(2z)^{-(k+\nu)/m}$$

$$\times E\left\{\begin{matrix}\frac{1}{2}+n+\frac{k+\nu}{m},\frac{1}{2}-n+\frac{k+\nu}{m}:e^{\pm im\pi}(2m^m z)\\ 1+\frac{k+\nu}{m},1+\frac{1}{m},1+\frac{2}{m},\ldots,1+\frac{\nu}{m},1-\frac{1}{m},1-\frac{2}{m},\ldots,1-\frac{m-\nu-1}{m}\end{matrix}\right\} \quad ,\ldots(26),$$

where $R(k+\frac{1}{2}m \pm mn) > 0$, $|\operatorname{amp} z| < \pi$ and m is any positive integer.

In (20) CHAP. I. write $[\lambda/(1+\lambda)]$ for λ and apply (6) CHAP. II., so getting

$$\int_0^\infty \frac{\lambda^{\alpha+\frac{1}{2}nm-1}}{(1+\lambda)^{\rho+\frac{1}{2}nm}} J_n\left\{\frac{2}{\sqrt{z}}\left(\frac{\lambda+1}{\lambda}\right)^{\frac{m}{2}}\right\} d\lambda =$$

$$\frac{\sin(\rho\pi)}{\sin(\alpha\pi)} m^{\alpha-\rho} \Gamma(\rho-\alpha) z^{-\frac{1}{2}n} E\left\{\begin{matrix} 1-\frac{\rho}{m}, 1-\frac{\rho+1}{m}, \dots, 1-\frac{\rho+m-1}{m} : z \\ 1-\frac{\alpha}{m}, 1-\frac{\alpha+1}{m}, \dots, 1-\frac{\alpha+m-1}{m}, 1+n \end{matrix}\right\}$$

$$-2^{1-m}\Gamma(\rho-\alpha) m^{\alpha-\rho} z^{-\frac{1}{2}n} \sum_{\nu=0}^{m-1} \frac{\sin(\rho-\alpha)\pi}{\sin\left(\frac{\alpha+\nu}{m}\right)\pi \prod_{s=1}^{\nu} \sin\frac{s\pi}{m} \prod_{t=1}^{m-\nu-1} \sin\frac{t\pi}{m}} z^{-(\alpha+\nu)/m}$$

$$E\left\{\begin{matrix} 1+\frac{\alpha-\rho+\nu}{m}, 1+\frac{\alpha-\rho+\nu-1}{m}, \dots, 1+\frac{\alpha-\rho+\nu-m+1}{m} : z \\ 1+\frac{\alpha+\nu}{m}, 1+\frac{1}{m}, 1+\frac{2}{m}, \dots, 1+\frac{\nu}{m}, 1-\frac{1}{m}, 1-\frac{2}{m}, \dots, 1-\frac{m-\nu-1}{m}, n+1+\frac{\alpha+\nu}{m} \end{matrix}\right\}, \dots (24)$$

where $R(\frac{3}{2}+\frac{2\alpha}{m}+n) > 0$, $R(\rho-\alpha) > 0$, $|\operatorname{amp} z| < \pi$ and m is any positive integer

Also in (20) CHAP. I., write $[\lambda/(1+\lambda)]$ for λ and apply (34) CHAP. I., so getting if $R(\rho-\alpha) > 0$, $R(\alpha+\frac{m}{2} \pm mn) > 0$, $|\operatorname{amp} z| < \pi$

$$\int_0^\infty \exp\left\{z\left(\frac{\lambda}{1+\lambda}\right)^m\right\} \lambda^{\alpha+\frac{1}{2}m-1}(1+\lambda)^{-\rho-\frac{1}{2}m} K_n\left\{z\left(\frac{\lambda}{1+\lambda}\right)^m\right\} d\lambda =$$

$$\frac{\cos(n\pi)\sin(\rho\pi)}{\sin(\alpha\pi)} m^{\alpha-\rho}(2\pi z)^{-\frac{1}{2}}\Gamma(\rho-\alpha) E\left\{\begin{matrix}\frac{1}{2}+n, \frac{1}{2}-n, 1-\frac{\rho}{m}, 1-\frac{\rho+1}{m}, \dots, 1-\frac{\rho+m-1}{m} : 2z \\ 1-\frac{\alpha}{m}, 1-\frac{\alpha+1}{m}, \dots, 1-\frac{\alpha+m-1}{m}\end{matrix}\right\}$$

$$-2^{1-m} m^{\alpha-\rho}(2\pi z)^{-\frac{1}{2}}\Gamma(\rho-\alpha)\cos n\pi \sum_{\nu=0}^{m-1} \frac{\sin(\rho-\alpha)\pi}{\sin\left(\frac{\alpha+\nu}{m}\right)\pi \prod_{s=1}^{\nu}\sin\frac{s\pi}{m} \prod_{t=1}^{m-\nu-1}\sin\left(\frac{t\pi}{m}\right)}(2z)^{-\frac{\alpha+\nu}{m}}$$

$$\times E\left\{\begin{matrix}1+\frac{\alpha-\rho+\nu}{m}, 1+\frac{\alpha-\rho+\nu-1}{m}, \dots, 1+\frac{\alpha-\rho+\nu-m+1}{m}, \frac{1}{2}+n+\frac{\alpha+\nu}{m}, \frac{1}{2}-n+\frac{\alpha+\nu}{m} : 2z \\ 1+\frac{\alpha+\nu}{m}, 1+\frac{1}{m}, 1+\frac{2}{m}, \dots 1+\frac{\nu}{m}, 1-\frac{1}{m}, 1-\frac{2}{m}, \dots, 1-\frac{m-\nu-1}{m}\end{matrix}\right\}$$

where $\underline{m}$ is any positive integer. , ... (28)

(24) CHAP. I in combination with (6) CHAP. II gives

$$\int_0^\infty \lambda^{k-1} J_m(z\sqrt{\lambda}) K_n(\lambda)\, d\lambda =$$

$$2^{k-\frac{1}{2}m-2}\sqrt{\pi}\left(\frac{z}{2}\right)^m \frac{\Gamma(\frac{1}{2}k+\frac{1}{4}m+\frac{1}{2}n)\Gamma(\frac{1}{2}k+\frac{1}{4}m-\frac{1}{2}n)}{\Gamma(\frac{1}{2}+\frac{1}{2}m)\Gamma(1+\frac{1}{2}m)} {}_2F_3\left(\begin{matrix}\frac{k}{2}+\frac{m}{4}+\frac{n}{2}, \frac{k}{2}+\frac{m}{4}-\frac{n}{2} ; \frac{z^4}{64} \\ \frac{1}{2}, \frac{1}{2}+\frac{m}{2}, 1+\frac{m}{2}\end{matrix}\right)$$

$$-2^{k-\frac{m}{2}-2}\pi\left(\frac{z}{2}\right)^m \frac{\Gamma(\frac{1}{2}+\frac{k}{2}+\frac{m}{4}+\frac{n}{2})\Gamma(\frac{1}{2}+\frac{k}{2}+\frac{m}{4}-\frac{n}{2})}{\Gamma(\frac{3}{2})\Gamma(1+\frac{1}{2}m)\Gamma(\frac{3}{2}+\frac{1}{2}m)}\left(\frac{z^2}{8}\right) {}_2F_3\left(\begin{matrix}\frac{1}{2}+\frac{k}{2}+\frac{m}{4}+\frac{n}{2}, \frac{1}{2}+\frac{k}{2}+\frac{m}{4}-\frac{n}{2} ; \frac{z^4}{64} \\ \frac{3}{2}, 1+\frac{m}{2}, \frac{3}{2}+\frac{m}{2}\end{matrix}\right)$$

, ... (29)

where $R(k+\frac{1}{2}m \pm n) > 0$, $R(z) > 0$.

Also (41) CHAP. I. in combination with (6) CHAP. II. gives

$$\int_0^\infty \lambda^{k-1} J_m(z\sqrt{\lambda})\, J_n(\lambda)\, d\lambda =$$

$$2^{k-\frac{1}{2}m-1}\sqrt{\pi}\left(\frac{z}{2}\right)^m \frac{\Gamma(\frac{1}{2}k+\frac{1}{4}m+\frac{1}{2}n)}{\Gamma(1+\frac{1}{2}n-\frac{1}{2}k-\frac{1}{4}m)\,\Gamma(\frac{1}{2}+\frac{1}{2}m)\,\Gamma(1+\frac{1}{2}m)}$$

$$\times {}_2F_3\left(\frac{1}{2}k+\frac{1}{4}m+\frac{1}{2}n, \frac{1}{2}k+\frac{1}{4}m-\frac{1}{2}n; \frac{1}{2}, \frac{1}{2}+\frac{1}{2}m, 1+\frac{1}{2}m; \frac{-z^4}{64}\right)$$

$$-2^{k-\frac{1}{2}m-2}\pi\left(\frac{z}{2}\right)^{m+2} \frac{\Gamma(\frac{1}{2}k+\frac{1}{4}m+\frac{1}{2}n+\frac{1}{2})}{\Gamma(\frac{3}{2})\,\Gamma(\frac{1}{2}+\frac{1}{2}n-\frac{1}{2}k-\frac{1}{4}m)\,\Gamma(1+\frac{m}{2})\,\Gamma(\frac{3}{2}+\frac{m}{2})}$$

$$\times {}_2F_3\left(\frac{1}{2}+\frac{k}{2}+\frac{m}{4}+\frac{n}{2}, \frac{1}{2}+\frac{k}{2}+\frac{m}{4}-\frac{n}{2}; \frac{3}{2}, \frac{m+2}{2}, \frac{m+3}{2}; \frac{-z^4}{64}\right), \quad \ldots (30)$$

where $R(k+n+\frac{1}{2}m) > 0$, $R(\frac{5}{4}-k) > 0$, $R(z) > 0$.

(24) CHAP. I. in combination with (34) CHAP. I. gives if $R(z) > 0$, $R(k \pm n + \frac{1}{2}) > 0$

$$\int_0^\infty \lambda^{k-1} e^{z/\lambda} K_m(z/\lambda) K_n(\lambda)\, d\lambda = 2^{k-3}\pi^{-1}\cos(m\pi)\, z^{-\frac{1}{2}}$$

$$\times\left[\begin{array}{l} E\left(\frac{1}{4}+\frac{k}{2}+\frac{n}{2}, \frac{1}{4}+\frac{k}{2}-\frac{n}{2}, \frac{1}{4}+\frac{m}{2}, \frac{1}{4}-\frac{m}{2}, \frac{3}{4}+\frac{m}{2}, \frac{3}{4}-\frac{n}{2} : \frac{1}{2} : e^{\pm i\pi}\frac{z^2}{4}\right) \\ -\left(\frac{2}{z}\right)E\left(\frac{3}{4}+\frac{k}{2}+\frac{n}{2}, \frac{3}{4}+\frac{k}{2}-\frac{n}{2}, \frac{3}{4}+\frac{m}{2}, \frac{3}{4}-\frac{m}{2}, \frac{5}{4}+\frac{m}{2}, \frac{5}{4}-\frac{m}{2} ; \frac{3}{2} ; e^{\pm i\pi}\frac{z^2}{4}\right) \end{array}\right]$$

$, \ldots (31)$

(41) CHAP. I. in combination with (34) CHAP. I. gives

$$i\pi\int_0^\infty \lambda^{k-1} e^{z/\lambda} K_m(z/\lambda)\, J_n(\lambda)\, d\lambda =$$

$$2^{k-3}\pi^{-1} i^{k+\frac{1}{2}-n}\cos(m\pi)\, z^{-\frac{1}{2}}$$

$$\times\left[\begin{array}{l} E\left(\frac{1}{4}+\frac{1}{2}k+\frac{1}{2}n, \frac{1}{4}+\frac{1}{2}k-\frac{1}{2}n, \frac{1}{4}+\frac{m}{2}, \frac{1}{4}-\frac{m}{2}, \frac{3}{4}+\frac{m}{2}, \frac{3}{4}-\frac{m}{2} : \frac{1}{2} : e^{\pm i\pi}\frac{z^2}{4}e^{-i\pi}\right) \\ -\left(\frac{2i}{z}\right)E\left(\frac{3}{4}+\frac{k}{2}+\frac{n}{2}, \frac{3}{4}+\frac{k}{2}-\frac{n}{2}, \frac{3}{4}+\frac{m}{2}, \frac{3}{4}-\frac{m}{2}, \frac{5}{4}+\frac{m}{2}, \frac{5}{4}-\frac{m}{2} ; \frac{3}{2} : e^{\pm i\pi}\frac{z^2}{4}e^{-i\pi}\right) \end{array}\right]$$

$$-2^{k-3}\pi^{-1} i^{n-k-\frac{1}{2}}\cos(m\pi)\, z^{-\frac{1}{2}}$$

$$\times\left[\begin{array}{l} E\left(\frac{1}{4}+\frac{k}{2}+\frac{n}{2}, \frac{1}{4}+\frac{k}{2}-\frac{n}{2}, \frac{1}{4}+\frac{m}{2}, \frac{1}{4}-\frac{m}{2}, \frac{3}{4}+\frac{m}{2}, \frac{3}{4}-\frac{m}{2} : \frac{1}{2} : e^{\pm i\pi}\frac{z^2}{4}.e^{i\pi}\right) \\ -\left(\frac{2e^{-\frac{i\pi}{2}}}{z}\right)E\left(\frac{3}{4}+\frac{k}{2}+\frac{n}{2}, \frac{3}{4}+\frac{k}{2}-\frac{n}{2}, \frac{3}{4}+\frac{m}{2}, \frac{3}{4}-\frac{m}{2}, \frac{5}{4}+\frac{m}{2}, \frac{5}{4}-\frac{m}{2} : \frac{3}{2} : e^{\pm i\pi}\frac{z^2}{4}.e^{i\pi}\right) \end{array}\right]$$

$, \ldots (32)$

where $R(z)>0$, $R(k+n+\frac{1}{2})>0$, $R(\frac{3}{2}-k\pm m)>0$.

In (41) CHAP. I. take $p=q=0$, and get

$$i\pi\int_0^\infty e^{-\lambda/z}\lambda^{m-1}J_n(\lambda)\,d\lambda = 2^{m-1}(i)\sqrt{(\pi)}$$

$$\times\left[\frac{\sqrt{\pi}\,\Gamma(\frac{1}{2}m+\frac{1}{2}n)}{\Gamma(1+\frac{1}{2}n-\frac{1}{2}m)}F(\tfrac{1}{2}m+\tfrac{1}{2}n,\tfrac{1}{2}m-\tfrac{1}{2}n;\tfrac{1}{2};-1/z^2)\right.$$
$$\left.-\frac{\pi\,\Gamma(\frac{1}{2}+\frac{m}{2}+\frac{n}{2})}{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2}+\frac{n}{2}-\frac{m}{2})}\left(\frac{1}{z}\right)F\left(\tfrac{1}{2}+\tfrac{1}{2}m+\tfrac{1}{2}n,\tfrac{1}{2}+\tfrac{1}{2}m-\tfrac{1}{2}n;\tfrac{3}{2};-1/z^2\right)\right], \cdots (33)$$

where $R(z)>0$, $R(m+n)>0$.

Now substitute for each hypergeometric function on the R.H.S. of (33) from the formula [B.V., p. 341, eq (3).] namely

$$F(\alpha,\beta;\gamma;-z)(=)$$

$$\frac{\Gamma(\gamma)\Gamma(\beta-\alpha)\cdot z^{-\alpha}}{\Gamma(\beta)\Gamma(\gamma-\alpha)}F\left(\begin{matrix}\alpha,\alpha-\gamma+1;\\ \alpha-\beta+1\end{matrix}\;\frac{-1}{z}\right)+\frac{\Gamma(\gamma)\Gamma(\alpha-\beta)\cdot z^{-\beta}}{\Gamma(\alpha)\Gamma(\gamma-\beta)}F\left(\begin{matrix}\beta,\beta-\gamma+1;\\ \beta-\alpha+1\end{matrix}\;\frac{-1}{z}\right), \cdots (34),$$

then the resulting second and fourth hypergeometric functions will cancel, then adding the remaining two

(33) becomes

$$\int_0^\infty e^{-\lambda/z}\lambda^{m-1} J_n(\lambda)\,d\lambda = \frac{z^{m+n}\Gamma(m+n)}{2^n\Gamma(n+1)} F\left(\frac{m+n}{2}, \frac{m+n+1}{2}; n+1; -z^2\right), \quad \cdots (35).$$

(35) was given in Watson "Theory of Bessel Functions" p. 385, formula (2)

In (24) CHAP. I. take $p=1$, $q=0$, so getting

$$\int_0^\infty \frac{\lambda^{m-1}}{(z+\lambda)^\alpha} K_n(\lambda)\,d\lambda = 2^{\alpha+m-3}\{1/\Gamma(\alpha)\} z^{-\alpha}$$

$$\times\left\{E\left(\frac{m+n}{2}, \frac{m-n}{2}, \frac{\alpha}{2}, \frac{\alpha+1}{2} : \frac{1}{2} : e^{\pm i\pi}\frac{z^2}{4}\right) - \frac{2}{z}E\left(\frac{m+n+1}{2}, \frac{m-n+1}{2}, \frac{\alpha+1}{2}, \frac{\alpha+2}{2} : \frac{3}{2} : e^{\pm i\pi}\frac{z^2}{4}\right)\right\}$$

where $R(m\pm n)>0$, $|\text{amp}\, z|<\pi$. $\quad , \cdots (36)$

In (41) CHAP. I. take $p=1$, $q=0$, so getting if $|\text{amp}\, z|<\pi$, $R(m+n)>0$, $R(\frac{3}{2}+\alpha-m)>0$

$$i\pi\int_0^\infty \frac{\lambda^{m-1}}{(z+\lambda)^\alpha} J_n(\lambda)\,d\lambda =$$

$$\frac{2^{\alpha+m-3}}{\Gamma(\alpha)z^\alpha} i^{m-n}\left[E\left(\frac{m+n}{2}, \frac{m-n}{2}, \frac{\alpha}{2}, \frac{\alpha+1}{2} : \frac{1}{2} : e^{\pm i\pi} \frac{z^2}{4} e^{-i\pi}\right) - \left(\frac{2e^{i\pi/2}}{z}\right) E\left(\frac{m+n+1}{2}, \frac{m-n+1}{2}, \frac{\alpha+1}{2}, \frac{\alpha+2}{2} ; \frac{3}{2} : e^{\pm i\pi} \frac{z^2}{4} e^{-i\pi}\right)\right]$$

$$-\frac{2^{\alpha+m-3}}{\Gamma(\alpha)z^\alpha} i^{n-m}\left[E\left(\frac{m+n}{2}, \frac{m-n}{2}, \frac{\alpha}{2}, \frac{\alpha+1}{2} ; \frac{1}{2} : e^{\pm i\pi} \frac{z^2}{4} e^{i\pi}\right) - \left(\frac{2e^{-i\pi/2}}{z}\right) E\left(\frac{m+n+1}{2}, \frac{m-n+1}{2}, \frac{\alpha+1}{2}, \frac{\alpha+2}{2} ; \frac{3}{2} : e^{\pm i\pi} \frac{z^2}{4} e^{i\pi}\right)\right], \cdots (37)$$

In (44) CHAP. I. take $p=0$, $q=1$ and apply (6) CHAP. II. and get

$$\int_0^\infty x^{\rho-1} J_\mu(x) J_\nu\left(\frac{b}{x}\right) dx =$$

$$\frac{b^\nu \Gamma(\frac{1}{2}\mu+\frac{1}{2}\rho-\frac{1}{2}\nu)}{2^{2\nu-\rho+1}\Gamma(\nu+1)\Gamma(\frac{1}{2}\mu+\frac{1}{2}\nu-\frac{1}{2}\rho+1)} F\left(;\nu+1, \frac{\nu-\mu-\rho}{2}+1, 1+\frac{\nu+\mu-\rho}{2}; \frac{b^2}{16}\right)$$

$$+\frac{b^{\mu+\rho}\Gamma(\frac{1}{2}\nu-\frac{1}{2}\mu-\frac{1}{2}\rho)}{2^{2\mu+\rho+1}\Gamma(\mu+1)\Gamma(\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2}\rho+1)} F\left(;\mu+1, \frac{\nu-\mu+\rho}{2}+1, \frac{\nu+\mu+\rho}{2}+1; \frac{b^2}{16}\right), \cdots (38)$$

where $-R(\mu+\frac{3}{2}) < R(\rho) < R(\nu+\frac{3}{2})$,

which is again Rao's formula [Messen. of Math. XLVII., 1918], and also was deduced in §2 this chapter

(42) CHAP. I in combination with (34) CHAP. I. gives

$$\int_0^\infty \exp\{z\lambda^2\}\lambda^{k-1}K_m(z\lambda^2)K_n(\lambda)\,d\lambda =$$

$$2^{k-\frac{5}{2}}\sqrt{\left(\frac{\pi}{z}\right)}\,\Gamma\left(\frac{k}{2}+\frac{n}{2}-\frac{1}{2}\right)\Gamma\left(\frac{k}{2}-\frac{n}{2}-\frac{1}{2}\right){}_2F_2\left(\frac{1}{2}+m, \frac{1}{2}-m; \frac{3}{2}-\frac{k+n}{2}, \frac{3}{2}-\frac{k-n}{2}; \frac{-1}{8z}\right)$$

$$+\sum_{n,-n} 2^{-n-2}\pi^{-1/2}\cos(m\pi)\Gamma\left(\frac{1}{2}-\frac{k+n}{2}\right)\Gamma(-n)\Gamma\left(-m+\frac{k+n}{2}\right)\Gamma\left(m+\frac{k+n}{2}\right)(2z)^{-\frac{k+n}{2}}$$

$$\times\,{}_2F_2\left(m+\frac{k+n}{2}, -m+\frac{k+n}{2}; \frac{1}{2}+\frac{k+n}{2}, 1+n; \frac{-1}{8z}\right), \quad \ldots\ldots (39)$$

where $R(k\pm 2m\pm n)>0$, $R(z)>0$.

(47) CHAP. I. in combination with (34) CHAP. I. gives if $R(z)>0$, $R(k\pm 2m+n)>0$, $R(\frac{5}{2}-k)>0$,

$$\int_0^\infty \exp\{z\lambda^2\}\lambda^{k-1}K_m(z\lambda^2)J_n(\lambda)\,d\lambda =$$

$$2^{k-\frac{3}{2}}\sqrt{\left(\frac{\pi}{z}\right)}\frac{\Gamma\left(\frac{k+n}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}-\frac{k-n}{2}\right)}\,{}_2F_2\left(\frac{1}{2}+m, \frac{1}{2}-m; \frac{3}{2}-\frac{k+n}{2}, \frac{3}{2}-\frac{k-n}{2}; \frac{1}{8z}\right)$$

$$+2^{-n-1}\pi^{-1/2}\cos(m\pi)\frac{\Gamma\left(\frac{1}{2}-\frac{k+n}{2}\right)}{\Gamma(1+n)}\Gamma\left(m+\frac{k+n}{2}\right)\Gamma\left(-m+\frac{k+n}{2}\right)(2z)^{-\frac{k+n}{2}}$$

$$\times\,{}_2F_2\left(m+\frac{k+n}{2}, -m+\frac{k+n}{2}; \frac{1}{2}+\frac{k+n}{2}, 1+n; \frac{1}{8z}\right), \quad \ldots (40)$$

In (30) CHAP. I. take $p=q=0$ and get

$$\int_0^\infty e^{-1/z\lambda}\lambda^{m-1}K_n(\lambda)\,d\lambda =$$

$$2^{m-2}\Gamma\left(\frac{m+n}{2}\right)\Gamma\left(\frac{m-n}{2}\right){}_0F_3\left(; \frac{1}{2}, 1-\frac{m+n}{2}, 1-\frac{m-n}{2}; \frac{1}{16z^2}\right)$$

$$+2^{m-2}\pi^{-1/2}\Gamma\left(-\frac{1}{2}\right)\Gamma\left(\frac{m+n}{2}-\frac{1}{2}\right)\Gamma\left(\frac{m-n}{2}-\frac{1}{2}\right)\cdot\left(\frac{1}{4z}\right)$$

$$\times\,{}_0F_3\left(; \frac{3}{2}, \frac{3}{2}-\frac{m+n}{2}, \frac{3}{2}-\frac{m-n}{2}; \frac{1}{16z^2}\right)$$

$$+2^{m-2}\pi^{-1/2}\sum_{n,-n}\Gamma\left(-\frac{m+n}{2}\right)\Gamma\left(\frac{1}{2}-\frac{m+n}{2}\right)\Gamma(-n)\left(\frac{1}{4z}\right)^{m+n}{}_0F_3\left(; 1+\frac{m+n}{2}, \frac{1}{2}+\frac{m+n}{2}, 1+n; \frac{1}{16z^2}\right)$$

$$, \quad \ldots (41)$$

where $R(z) > 0$.

If $m=0$, then (41) gives

$$\int_0^\infty e^{-y} K_n(b/y)\frac{dy}{y} =$$

$$\tfrac{1}{4}\Gamma\left(\tfrac{n}{2}\right)\Gamma\left(-\tfrac{n}{2}\right){}_0F_3\left(;\tfrac{1}{2}, 1-\tfrac{n}{2}, 1+\tfrac{n}{2}; b^2/16\right)$$
$$+\pi^{-1/2}\Gamma\left(-\tfrac{1}{2}\right)\Gamma\left(\tfrac{n}{2}-\tfrac{1}{2}\right)\Gamma\left(-\tfrac{n}{2}-\tfrac{1}{2}\right)\left(\tfrac{b}{16}\right){}_0F_3\left(;\tfrac{3}{2}, \tfrac{3}{2}-\tfrac{n}{2}, \tfrac{3}{2}+\tfrac{n}{2}; \tfrac{b^2}{16}\right)$$
$$+\sum_{n,-n}\frac{1}{4\sqrt{\pi}}\Gamma\left(-\tfrac{n}{2}\right)\Gamma\left(\tfrac{1}{2}-\tfrac{n}{2}\right)\Gamma(-n)\left(\tfrac{b}{4}\right)^n {}_0F_3\left(;1+\tfrac{n}{2}, \tfrac{1}{2}+\tfrac{n}{2}, 1+n; b^2/16\right)$$
$$, \quad \ldots (42)$$

where $R(b)>0$.

But since [Watson "Theory of Bessel Functions" p. 148]

$$I_n(x)\,J_n(x) = \sum_{p=0}^\infty \frac{(-1)^p\left(\tfrac{1}{2}x\right)^{2n+4p}}{p!\,\Gamma(n+p+1)\,\Gamma(n+2p+1)}, \quad \ldots \ldots (42'),$$

then (42) becomes if x is written for $\sqrt{(2bi)}$

$$\int_0^\infty e^{-y} K_n\left(\frac{b}{y}\right)\frac{dy}{y} =$$

$$2\left[\frac{\pi}{2\sin n\pi}\right]^2\left\{e^{-\frac{n\pi i}{2}} I_n(x)\,J_n(x) - e^{\frac{n\pi i}{2}} I_n(x)\,J_{-n}(x) - e^{-\frac{n\pi i}{2}} I_{-n}(x)\,J_n(x) + e^{\frac{n\pi i}{2}} I_{-n}(x)\,J_{-n}(x)\right\}$$
$$= 2K_n\{\sqrt{(2bi)}\}K_n\{\sqrt{(-2bi)}\}, \quad \ldots \ldots (43)$$

where $\underline{b}$ is taken for simplicity to be real and positive. (43) was proved by applying Mellin's inversion formula by Hardy [Messeng. of Math. p. 186, 1924.]. Also by Macdonald, C.V., p. 383, ex. 123. Also Watson "Theory of Bessel Functions" p. 439, formula (1).

In (54) CHAP. I. take $p = q = 0$ and get

$$\int_0^\infty e^{-1/z\lambda} \lambda^{m-1} J_n(\lambda)\, d\lambda =$$

$$2^{m-1} \frac{\Gamma(\frac{1}{2}m + \frac{1}{2}n)}{\Gamma(1 - \frac{1}{2}m + \frac{1}{2}n)}\, {}_0F_3\left(; \tfrac{1}{2}, 1 - \tfrac{m+n}{2}, 1 - \tfrac{m-n}{2}; -1/16z^2\right)$$

$$-2^{m-1}\sqrt{\pi}\, \frac{\Gamma(\frac{m+n}{2} - \frac{1}{2})}{\Gamma(\frac{3}{2})\Gamma(\frac{3}{2} - \frac{m-n}{2})} \left(\frac{1}{4z}\right) {}_0F_3\left(; \tfrac{3}{2}, \tfrac{3}{2} - \tfrac{m+n}{2}, \tfrac{3}{2} - \tfrac{m-n}{2}; -1/16z^2\right)$$

$$+ \frac{2^{m-1}}{\sqrt{\pi}} \cdot \frac{\Gamma(-\frac{m+n}{2})\Gamma(\frac{1}{2} - \frac{m+n}{2})}{\Gamma(1+n)} \left(\frac{1}{4z}\right)^{m+n} {}_0F_3\left(; 1 + \tfrac{m+n}{2}, \tfrac{1}{2} + \tfrac{m+n}{2}, 1+n; -1/16z^2\right)$$

where $R(z) > 0$, $R(\frac{3}{2} - m) > 0$. , ... (44)

In particular if $m = 0$, (44) gives

$$\int_0^\infty e^{-y}\, J_n\left(\frac{b}{y}\right)\frac{dy}{y} =$$

$$\frac{\Gamma(\frac{1}{2}n)}{2\,\Gamma(1+\frac{1}{2}n)}\,{}_0F_3\left(;\tfrac{1}{2}, 1-\tfrac{1}{2}n, 1+\tfrac{1}{2}n; -b^2/16\right)$$

$$-\frac{\sqrt{\pi}\,\Gamma(\frac{n}{2}-\frac{1}{2})}{\Gamma(\frac{3}{2})\Gamma(\frac{3}{2}+\frac{n}{2})}\cdot\left(\frac{b}{8}\right){}_0F_3\left(;\tfrac{3}{2}, \tfrac{3}{2}-\tfrac{n}{2}, \tfrac{3}{2}+\tfrac{n}{2}; -b^2/16\right)$$

$$+\frac{\Gamma(-\frac{n}{2})\Gamma(\frac{1}{2}-\frac{n}{2})}{2\sqrt{\pi}\,\Gamma(n+1)}\left(\frac{b}{4}\right)^n{}_0F_3\left(;1+\tfrac{n}{2}, \tfrac{1}{2}+\tfrac{n}{2}, 1+n; -b^2/16\right), \quad \ldots\ldots (45)$$

where $\underline{b}$ is taken for simplicity real and positive.

(50) CHAP. I. in combination with (34) CHAP. I. gives

$$\int_0^\infty e^{z\lambda}\lambda^{m-1/2}K_n(\lambda)K_\mu(z\lambda)\,d\lambda = 2^{m-\frac{7}{2}}\pi^{-2}\cos(\mu\pi)\cdot z^{-1/2}$$

$$A\,E\left(\tfrac{1}{4}+\tfrac{\mu}{2}, \tfrac{1}{4}-\tfrac{\mu}{2}, \tfrac{3}{4}+\tfrac{\mu}{2}, \tfrac{3}{4}-\tfrac{\mu}{2} : \tfrac{1}{2}, 1-\tfrac{m+n}{2}, 1-\tfrac{m-n}{2} : e^{\pm i\pi}4z^2\right)$$

$$+B\left(\tfrac{1}{2z}\right)E\left(\tfrac{3}{4}+\tfrac{\mu}{2}, \tfrac{3}{4}-\tfrac{\mu}{2}, \tfrac{5}{4}+\tfrac{\mu}{2}, \tfrac{5}{4}-\tfrac{\mu}{2}, \tfrac{3}{2}, \tfrac{3}{2}-\tfrac{m+n}{2}, \tfrac{3}{2}-\tfrac{m-n}{2} : e^{\pm i\pi}4z^2\right)$$

$$+\sum_{n,-n} C_n\left(\tfrac{1}{2z}\right)^{m+n}E\left\{\begin{matrix}\tfrac{1}{4}+\tfrac{m+n+\mu}{2}, \tfrac{1}{4}+\tfrac{m+n-\mu}{2}, \tfrac{3}{4}+\tfrac{m+n+\mu}{2}, \tfrac{3}{4}+\tfrac{m+n-\mu}{2} : e^{\pm i\pi}4z^2\\ 1+\tfrac{m+n}{2}, \tfrac{1}{2}+\tfrac{m+n}{2}, 1+n\end{matrix}\right\}$$

$$, \quad \ldots\ldots (46),$$

where $R(m+\frac{1}{2}\pm n \pm \mu)>0$, $R(z)>0$ and A, B, C_n are given in (50) CHAP. I.

In particular if in (46) $\mu = n$, $z=1$; then it becomes

$$\int_0^\infty e^{\lambda}\lambda^{m-\frac{1}{2}}\{K_n(\lambda)\}^2\,d\lambda =$$

$$2^{m-\frac{7}{2}}\pi^{-2}\cos(n\pi)\,A\,E\left\{\begin{matrix}\frac{1}{4}+\frac{n}{2},\ \frac{1}{4}-\frac{n}{2},\ \frac{3}{4}+\frac{n}{2},\ \frac{3}{4}-\frac{n}{2} : -4\\ \frac{1}{2},\ 1-\frac{m+n}{2},\ 1-\frac{m-n}{2}\end{matrix}\right\}$$

$$+2^{m-\frac{9}{2}}\pi^{-2}\cos(n\pi)\,B\,E\left\{\begin{matrix}\frac{3}{4}+\frac{n}{2},\ \frac{3}{4}-\frac{n}{2},\ \frac{5}{4}+\frac{n}{2},\ \frac{5}{4}-\frac{n}{2} : -4\\ \frac{3}{2},\ \frac{3}{2}-\frac{m+n}{2},\ \frac{3}{2}-\frac{m-n}{2}\end{matrix}\right\}$$

$$+2^{m-\frac{7}{2}}\pi^{-2}\cos(n\pi)\sum_{n,-n} C_n\,2^{-m-n}\,E\left\{\begin{matrix}\frac{1}{4}+\frac{m}{2}+n,\ \frac{1}{4}+\frac{m}{2},\ \frac{3}{4}+\frac{m}{2}+n,\ \frac{3}{4}+\frac{m}{2} : -4\\ 1+\frac{m+n}{2},\ \frac{1}{2}+\frac{m+n}{2},\ 1+n\end{matrix}\right\}$$

where $R(m+\frac{1}{2})>0$, $R(m\pm 2n+\frac{1}{2})>0$. $\qquad\ldots(47)$

(54) CHAP. I. in combination with (34) CHAP. I gives if $R(z)>0$, $R(k\pm m+n)>0$, $R(2-k)>0$,

$$\int_0^\infty e^{z\lambda}\lambda^{k-1}K_m(z\lambda)J_n(\lambda)\,d\lambda = 2^{k-3}\pi^{-2}\cos(m\pi)\,z^{-\frac{1}{2}}$$

$$\times\Bigg[\frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}k-\frac{1}{4}+\frac{n}{2}\right)\Gamma\left(\frac{1}{4}+\frac{m}{2}\right)\Gamma\left(\frac{1}{4}-\frac{m}{2}\right)\Gamma\left(\frac{3}{4}+\frac{m}{2}\right)\Gamma\left(\frac{3}{4}-\frac{m}{2}\right)}{\Gamma\left(\frac{5}{4}-\frac{k}{2}+\frac{n}{2}\right)}$$

$$\times {}_4F_3\left(\frac{1}{4}+\frac{m}{2}, \frac{1}{4}-\frac{m}{2}, \frac{3}{4}+\frac{m}{2}, \frac{3}{4}-\frac{m}{2}; \frac{1}{2}, \frac{5}{4}-\frac{k}{2}-\frac{n}{2}, \frac{5}{4}-\frac{k}{2}+\frac{n}{2}; \frac{-1}{4z^2}\right)$$

$$-\frac{\pi\,\Gamma\left(\frac{k}{2}+\frac{n}{2}-\frac{3}{4}\right)\Gamma\left(\frac{3}{4}+\frac{m}{2}\right)\Gamma\left(\frac{3}{4}-\frac{m}{2}\right)\Gamma\left(\frac{5}{4}+\frac{m}{2}\right)\Gamma\left(\frac{5}{4}-\frac{m}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{7}{4}-\frac{k}{2}+\frac{n}{2}\right)}\left(\frac{1}{2z}\right)$$

$$\times {}_4F_3\left(\frac{3}{4}+\frac{m}{2}, \frac{3}{4}-\frac{m}{2}, \frac{5}{4}+\frac{m}{2}, \frac{5}{4}-\frac{m}{2}; \frac{3}{2}, \frac{7}{4}-\frac{k}{2}-\frac{n}{2}, \frac{7}{4}-\frac{k}{2}+\frac{n}{2}; \frac{-1}{4z^2}\right)$$

$$+\frac{\Gamma\left(\frac{1}{4}-\frac{k+n}{2}\right)\Gamma\left(\frac{3}{4}-\frac{k+n}{2}\right)\Gamma\left(\frac{k+m+n}{2}\right)\Gamma\left(\frac{k-m+n}{2}\right)\Gamma\left(\frac{1+k+m+n}{2}\right)\Gamma\left(\frac{1+k-m+n}{2}\right)}{\Gamma(1+n)}\left(\frac{1}{2z}\right)^{k+n-\frac{1}{2}}$$

$$\times {}_4F_3\left(\frac{k+m+n}{2}, \frac{k-m+n}{2}, \frac{k+m+n+1}{2}, \frac{k-m+n+1}{2}; \frac{3}{4}+\frac{k}{2}+\frac{n}{2}, \frac{1}{4}+\frac{k}{2}+\frac{n}{2}, 1+n; \frac{-1}{4z^2}\right)\Bigg], \quad \cdots (48)$$

In particular if in (48) $n = m$, $z = 1$, then it becomes if $R(k) > 0$, $R(k+2n) > 0$, $R(2-k) > 0$

$$\int_0^\infty e^{\lambda}\lambda^{k-1} K_n(\lambda)\, J_n(\lambda)\, d\lambda =$$

$$2^{k-3}\pi^{-\frac{3}{2}}\cos(n\pi)\frac{\Gamma\left(\frac{1}{2}k-\frac{1}{4}+\frac{n}{2}\right)\Gamma\left(\frac{1}{4}+\frac{n}{2}\right)\Gamma\left(\frac{1}{4}-\frac{n}{2}\right)\Gamma\left(\frac{3}{4}+\frac{n}{2}\right)\Gamma\left(\frac{3}{4}-\frac{n}{2}\right)}{\Gamma\left(\frac{5}{4}-\frac{k}{2}+\frac{n}{2}\right)}$$

$$\times\, {}_4F_3\left(\frac{1}{4}+\frac{n}{2}, \frac{1}{4}-\frac{n}{2}, \frac{3}{4}+\frac{n}{2}, \frac{3}{4}-\frac{n}{2}; \frac{1}{2}, \frac{5}{4}-\frac{k}{2}-\frac{n}{2}, \frac{5}{4}-\frac{k}{2}+\frac{n}{2}; -\frac{1}{4}\right)$$

$$-2^{k-4}\pi^{-1}\cos(n\pi)\frac{\Gamma\left(\frac{k}{2}+\frac{n}{2}-\frac{3}{4}\right)\Gamma\left(\frac{3}{4}+\frac{n}{2}\right)\Gamma\left(\frac{3}{4}-\frac{n}{2}\right)\Gamma\left(\frac{5}{4}+\frac{n}{2}\right)\Gamma\left(\frac{5}{4}-\frac{n}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{7}{4}-\frac{k}{2}+\frac{n}{2}\right)}$$

$$\times\, {}_4F_3\left(\frac{3}{4}+\frac{n}{2}, \frac{3}{4}-\frac{n}{2}, \frac{5}{4}+\frac{n}{2}, \frac{5}{4}-\frac{n}{2}; \frac{3}{2}, \frac{7}{4}-\frac{k}{2}-\frac{n}{2}, \frac{7}{4}-\frac{k}{2}+\frac{n}{2}; -\frac{1}{4}\right)$$

$$+2^{-n-\frac{5}{2}}\pi^{-2}\cos(n\pi)\frac{\Gamma\left(\frac{1}{4}-\frac{k+n}{2}\right)\Gamma\left(\frac{3}{4}-\frac{k+n}{2}\right)\Gamma\left(\frac{k}{2}+n\right)\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{k+1}{2}+n\right)\Gamma\left(\frac{k+1}{2}\right)}{\Gamma(1+n)}$$

$$\times\, {}_4F_3\left(\frac{k}{2}+n, \frac{k}{2}, \frac{k+1}{2}+n, \frac{k+1}{2}; \frac{1}{4}+\frac{k}{2}+\frac{n}{2}, \frac{3}{4}+\frac{k}{2}+\frac{n}{2}, 1+n; -\frac{1}{4}\right)$$

$$, \cdots\cdots (49)$$

§7. Integrals involving products of two Bessel Functions: In (42) CHAP. I. take $p=0$, $q=1$, replace z by $(4/z^2)$, n by m and put $\rho_1 = n+1$; then on applying (6) CHAP. II, it is found that if z is real and positive, $R(k+n\pm m) > -\frac{3}{2}$

$$(2/z)^n \int_0^\infty \lambda^{k+n-1} K_m(\lambda) J_n(z/\lambda)\, d\lambda =$$

$$\frac{2^{k-2}\pi^2}{\sin\left(\frac{k+m}{2}\right)\pi \sin\left(\frac{k-m}{2}\right)\pi} E\left(:1-\frac{k+m}{2}, 1-\frac{k-m}{2}, n+1 : 16/z^2\right)$$

$$+\sum_{m,-m} \frac{2^{-m-2}\pi^2}{\sin\left(\frac{k+m}{2}\right)\pi \sin(m\pi)} \left(\frac{z}{2}\right)^{k+m} E\left(:1+\frac{k+m}{2}, m+1, 1+n+\frac{k+m}{2} : 16/z^2\right).$$

Here replace k by $(k-n)$, then if z is real and positive, $R(k\pm m) > -\frac{3}{2}$

$$\int_0^\infty \lambda^{k-1} K_m(\lambda) J_n(z/\lambda)\, d\lambda =$$

$$\frac{\Gamma\left(\frac{k+m-n}{2}\right)\Gamma\left(\frac{k-m-n}{2}\right)}{2^{2n-k+2}\,\Gamma(n+1)} z^n F\left(;1-\frac{k+m-n}{2}, 1-\frac{k-m-n}{2}, n+1; \frac{-z^2}{16}\right)$$

$$+\sum_{m,-m} \frac{\Gamma\left(\frac{-k-m+n}{2}\right)\Gamma(-m)}{\Gamma\left(1+\frac{k+m+n}{2}\right)} \frac{z^{k+m}}{2^{k+2m+2}} F\left(;1+\frac{k+m-n}{2}, 1+\frac{k+m+n}{2}, m+1; \frac{-z^2}{16}\right), \cdots (50)$$

On applying the formula [C.V., p. 240] namely

$$G_n(z) = \frac{\pi}{2\sin n\pi}\left\{J_{-n}(z) - e^{-in\pi} J_n(z)\right\}, \quad \ldots\ldots \tag{51}$$

it follows that, if $0 \leqslant \operatorname{amp} z \leqslant \pi$, $R(k \pm m) > -\frac{3}{2}$

$$i^n \int_0^\infty \lambda^{k-1} K_m(\lambda)\, G_n(z/\lambda)\, d\lambda =$$

$$\sum_{n,-n} 2^{k+2n-3}\, \Gamma\left(\frac{k+m+n}{2}\right)\Gamma\left(\frac{k-m+n}{2}\right)\Gamma(n)\left(\frac{i}{z}\right)^n F\left(;1-\frac{k+m+n}{2}, 1-\frac{k-m+n}{2}, 1-n; \frac{-z^2}{16}\right)$$

$$+\sum_{m,-m} \frac{\pi}{2\sin n\pi}\, \Gamma\left(\frac{-k-m-n}{2}\right)\Gamma\left(\frac{-k-m+n}{2}\right)\Gamma(-m)\, \frac{z^{k+m}}{2^{k+2m+2}}$$

$$\times F\left(;1+\frac{k+m+n}{2}, 1+\frac{k+m-n}{2}, m+1; \frac{-z^2}{16}\right)\frac{1}{\pi}\left[-\sin\left(\frac{k+m-n}{2}\right)\pi\, e^{\frac{in\pi}{2}} + \sin\left(\frac{k+m+n}{2}\right)\pi\, e^{-\frac{in\pi}{2}}\right].$$

Now the expression in the last bracket is equal to

$$\sin(n\pi)\; i^{-k-m}.$$

Hence on replacing z by (iz) and noting that

$$G_n(iz) = i^{-n} K_n(z),$$

The formula becomes

$$\int_0^\infty \lambda^{k-1} K_m(\lambda) K_n(z/\lambda)\, d\lambda =$$

$$\sum_{m,-m} 2^{k+2m-3}\,\Gamma\left(\frac{k+m+n}{2}\right)\Gamma\left(\frac{k-m+n}{2}\right)\Gamma(n)\, z^{-n} F\left(;1-\frac{k+m+n}{2},1-\frac{k-m+n}{2},1-n;\frac{z^2}{16}\right)$$

$$+\sum_{m,-m} 2^{-k-2m-3}\,\Gamma\left(\frac{-k-m-n}{2}\right)\Gamma\left(\frac{-k-m+n}{2}\right)\Gamma(-m)\, z^{k+m} F\left(;1+\frac{k+m+n}{2},1+\frac{k+m-n}{2},m+1;\frac{z^2}{16}\right)$$

where $R(z)>0$. , ... (52)

(50) CHAP I. in combination with (6) CHAP II. gives

$$2^{2-k}\int_0^\infty \lambda^{k-1} J_m(z/\sqrt{\lambda}) K_n(\lambda)\, d\lambda =$$

$$D z^m {}_0F_5\left(;\frac{1}{2},1-\frac{k}{2}+\frac{m}{4}-\frac{n}{2},1-\frac{k}{2}+\frac{m}{4}+\frac{n}{2},\frac{1}{2}+\frac{m}{2},1+\frac{m}{2};\frac{z^4}{1024}\right)$$

$$+E z^{m+2} {}_0F_5\left(;\frac{3}{2},\frac{3}{2}-\frac{k}{2}+\frac{m}{4}-\frac{n}{2},\frac{3}{2}-\frac{k}{2}+\frac{m}{4}+\frac{n}{2},1+\frac{m}{2},\frac{3}{2}+\frac{m}{2};\frac{z^4}{1024}\right)$$

$$+\sum_{n,-n} H\left(\frac{z^2}{32}\right)^{k+n} {}_0F_5\left(1+\frac{k}{2}-\frac{m}{4}+\frac{n}{2},\frac{1}{2}+\frac{k}{2}-\frac{m}{4}+\frac{n}{2},1+n,\frac{1}{2}+\frac{k}{2}+\frac{n}{2}+\frac{m}{4},1+\frac{k}{2}+\frac{n}{2}+\frac{m}{4};\frac{z^4}{1024}\right)$$

where, $R(z)>0$, $R\left(\frac{3}{2}\pm 2n\pm 2k\right)>0$ and , .. (53)

$$D \equiv 2^{-\frac{3}{2}m}\,\Gamma\left(\tfrac12 k - \tfrac m4 + \tfrac n2\right)\Gamma\left(\tfrac12 k - \tfrac m4 - \tfrac n2\right)\left\{\Gamma(1+m)\right\}^{-1},$$

$$E_1 \equiv 2^{-\frac32 m-4}\,\Gamma\left(-\tfrac12\right)\Gamma\left(\tfrac12 k-\tfrac m4+\tfrac n2-\tfrac12\right)\Gamma\left(\tfrac k2-\tfrac m4-\tfrac n2-\tfrac12\right)\left\{\Gamma\left(\tfrac12\right)\Gamma(m+2)\right\}^{-1},$$

$$H \equiv \Gamma\left(-\tfrac k2+\tfrac m4-\tfrac n2\right)\Gamma\left(\tfrac12-\tfrac k2+\tfrac m4-\tfrac n2\right)\Gamma(-n)\left\{\Gamma\left(\tfrac k2+\tfrac12+\tfrac m4+\tfrac n2\right)\Gamma\left(1+\tfrac k2+\tfrac m4+\tfrac n2\right)\right\}^{-1}$$

Hence on applying (51) this chapter, it follows that if, $0 \leqslant \operatorname{amp} z \leqslant \pi$, $R\left(\frac32 \pm 2n \pm 2k\right) > 0$

$$2^{2-k}\, i^{m}\int_0^\infty \lambda^{k-1}\, G_m(z/\sqrt{\lambda})\, K_n(\lambda)\, d\lambda =$$

$$\sum_{m,-m}\left[\begin{array}{l} 2^{\frac32 m-1}\,\Gamma\left(\tfrac12 k+\tfrac14 m+\tfrac12 n\right)\Gamma\left(\tfrac12 k+\tfrac14 m-\tfrac12 n\right)\Gamma(m)\,(z/i)^{-m} \\ \times\, {}_0F_5\left(;\tfrac12, 1-\tfrac k2-\tfrac m4-\tfrac n2, 1-\tfrac k2-\tfrac m4+\tfrac n2, \tfrac12-\tfrac m2, 1-\tfrac m2; \tfrac{z^4}{1024}\right)\end{array}\right]$$

$$+\sum_{m,-m}\left[\begin{array}{l} 2^{\frac32 m-5}\,\Gamma\left(-\tfrac12\right)\Gamma\left(\tfrac k2+\tfrac m4+\tfrac n2-\tfrac12\right)\Gamma\left(\tfrac k2+\tfrac m4-\tfrac n2-\tfrac12\right)\Gamma(m)\left\{\Gamma\left(\tfrac12\right)(1-m)\right\}^{-1} z^2\left(\tfrac iz\right)^m \\ \times\, {}_0F_5\left(;\tfrac32, \tfrac32-\tfrac k2-\tfrac m4-\tfrac n2, \tfrac32-\tfrac k2-\tfrac m4+\tfrac n2, 1-\tfrac m2, \tfrac32-\tfrac m2; \tfrac{z^4}{1024}\right)\end{array}\right]$$

$$+\sum_{n,-n}\left[\begin{array}{l} \dfrac{\pi}{2\cdot 2i\sin m\pi}\,\Gamma\left(-\tfrac k2+\tfrac m4-\tfrac n2\right)\Gamma\left(-\tfrac k2-\tfrac m4-\tfrac n2\right)\Gamma\left(\tfrac12-\tfrac k2+\tfrac m4-\tfrac n2\right)\Gamma\left(\tfrac12-\tfrac k2-\tfrac m4-\tfrac n2\right)\Gamma(-n) \\ \dfrac{z^{2k+2n}}{2^{5k+2n}}\,{}_0F_5\left(;1+\tfrac k2-\tfrac m4+\tfrac n2, \tfrac12+\tfrac k2-\tfrac m4+\tfrac n2, 1+n, \tfrac12+\tfrac k2+\tfrac m4+\tfrac n2, 1+\tfrac k2+\tfrac m4+\tfrac n2; \tfrac{z^4}{1024}\right) \\ \left\{-\tfrac{1}{\pi^2}\sin\left(\tfrac k2+\tfrac m4+\tfrac n2\right)\pi\,\sin\left(\tfrac k2+\tfrac m4+\tfrac n2-\tfrac12\right)\pi\, e^{-\frac{im\pi}{2}} + \tfrac{1}{\pi^2}\sin\left(\tfrac k2-\tfrac m4+\tfrac n2\right)\pi\,\sin\left(\tfrac k2-\tfrac m4+\tfrac n2-\tfrac12\right)\pi\, e^{\frac{im\pi}{2}}\right\}\end{array}\right]$$

Now the expression in the brackets $\{\quad\}$ is equal to

$$\frac{1}{2}\left[i^{-2k-2n} \sin(m\pi)\right].$$

Hence on replacing z by (iz) and noting that

$G_m(iz) = i^{-m} K_m(z)$, then the formula becomes

$$2^{3-k}\int_0^\infty \lambda^{k-1} K_m(z/\sqrt{\lambda}) K_n(\lambda)\, d\lambda =$$

$$\sum_{m,-m}\left[2^{\frac{3}{2}m}\,\Gamma\left(\tfrac{1}{2}k+\tfrac{1}{4}m+\tfrac{1}{2}n\right)\Gamma\left(\tfrac{1}{2}k+\tfrac{1}{4}m-\tfrac{1}{2}n\right)\cdot z^{-m}\cdot\Gamma(m) \times {}_0F_5\left(;\tfrac{1}{2}, 1-\tfrac{k}{2}-\tfrac{m}{4}-\tfrac{n}{2}, 1-\tfrac{k}{2}-\tfrac{m}{4}+\tfrac{n}{2}, \tfrac{1}{2}-\tfrac{m}{2}, 1-\tfrac{m}{2}; \tfrac{z^4}{1024}\right)\right]$$

$$-\sum_{m,-m}\left[2^{\frac{3}{2}m-4}\,\Gamma\left(-\tfrac{1}{2}\right)\Gamma\left(\tfrac{k}{2}+\tfrac{m}{4}+\tfrac{n}{2}-\tfrac{1}{2}\right)\Gamma\left(\tfrac{k}{2}+\tfrac{m}{4}-\tfrac{n}{2}-\tfrac{1}{2}\right)\Gamma(m)\left\{\Gamma\left(\tfrac{1}{2}\right)(1-m)\right\}^{-1} z^{2-m} \times {}_0F_5\left(;\tfrac{3}{2}, \tfrac{3}{2}-\tfrac{k}{2}-\tfrac{m}{4}-\tfrac{n}{2}, \tfrac{3}{2}-\tfrac{k}{2}-\tfrac{m}{4}+\tfrac{n}{2}, 1-\tfrac{m}{2}, \tfrac{3}{2}-\tfrac{m}{2}, \tfrac{z^4}{1024}\right)\right]$$

$$+\sum_{n,-n}\left[\Gamma(-n)\Gamma\left(-n-k-\tfrac{1}{2}m\right)\Gamma\left(-n-k+\tfrac{1}{2}m\right) 2^{1+2k+2n}\left(z^2/32\right)^{k+n} \times {}_0F_5\left(;1+\tfrac{k}{2}-\tfrac{m}{4}+\tfrac{n}{2}, \tfrac{1}{2}+\tfrac{k}{2}-\tfrac{m}{4}+\tfrac{n}{2}, 1+n, 1+\tfrac{k}{2}+\tfrac{m}{4}+\tfrac{n}{2}, \tfrac{1}{2}+\tfrac{k}{2}+\tfrac{m}{4}+\tfrac{n}{2}; \tfrac{z^4}{1024}\right)\right]$$

where $R(z)>0$, ... (54)

(54) CHAP. I. in combination with (6) CHAP. II. gives

$$\int_0^\infty \lambda^{k-1} J_m(z/\sqrt{\lambda})\, J_n(\lambda)\, d\lambda =$$

$$\frac{2^{k-\frac{3}{2}m-1}\Gamma\left(\frac{k}{2}+\frac{n}{2}-\frac{m}{4}\right)}{\Gamma\left(1-\frac{k}{2}+\frac{n}{2}+\frac{m}{4}\right)\Gamma(1+m)} z^m {}_0F_5\left(;\frac{1}{2}, 1-\frac{k}{2}-\frac{n}{2}+\frac{m}{4}, 1-\frac{k}{2}+\frac{n}{2}+\frac{m}{4}, \frac{1}{2}+\frac{m}{2}, 1+\frac{m}{2}; \frac{-z^4}{1024}\right)$$

$$-\frac{2^{k-\frac{3}{2}m-5}\sqrt{\pi}\,\Gamma\left(\frac{k}{2}+\frac{n}{2}-\frac{m}{4}-\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}+\frac{m}{4}+\frac{n}{2}-\frac{k}{2}\right)\Gamma(m+2)} z^{m+2}$$

$$\times {}_0F_5\left(;\frac{3}{2}, \frac{3}{2}+\frac{m}{4}-\frac{k}{2}-\frac{n}{2}, \frac{3}{2}+\frac{m}{4}-\frac{k}{2}+\frac{n}{2}, 1+\frac{m}{2}, \frac{3}{2}+\frac{m}{2}; \frac{-z^4}{1024}\right)$$

$$+\frac{2^{k-1}\Gamma\left(\frac{m}{4}-\frac{k}{2}-\frac{n}{2}\right)\Gamma\left(\frac{1}{2}+\frac{m}{4}-\frac{k}{2}-\frac{n}{2}\right)}{\Gamma(1+n)\Gamma\left(\frac{1}{2}+\frac{k}{2}+\frac{m}{4}+\frac{n}{2}\right)\Gamma\left(1+\frac{k}{2}+\frac{m}{4}+\frac{n}{2}\right)}\left(\frac{z^2}{32}\right)^{k+n}$$

$$\times {}_0F_5\left(;1+\frac{k}{2}+\frac{n}{2}-\frac{m}{4}, \frac{1}{2}+\frac{k}{2}+\frac{n}{2}-\frac{m}{4}, 1+n, 1+\frac{k}{2}+\frac{n}{2}+\frac{m}{4}, \frac{1}{2}+\frac{k}{2}+\frac{n}{2}+\frac{m}{4}; \frac{-z^4}{1024}\right)$$

where $R\left(\frac{1}{2}m+\frac{3}{2}\right) > R(k) > R\left(-n-\frac{3}{4}\right)$, $R(z)>0$. ,...(55)

§8. Integrals involving the product of an exponential function and Bessel Functions:

(55) CHAP. I. in combination with (6) CHAP. II. gives

$$4\int_0^\infty \exp\{-\lambda^4\}\,\lambda^{k-1} J_n(z\lambda)\,d\lambda =$$

$$\frac{\Gamma(\frac{1}{4}n+\frac{1}{4}k)}{\Gamma(1+n)}\left(\frac{z}{2}\right)^n {}_1F_3\left(\tfrac{1}{4}n+\tfrac{1}{4}k;\tfrac{1}{2},\tfrac{1}{2}+\tfrac{1}{2}n,1+\tfrac{1}{2}n;\frac{z^4}{256}\right)$$

$$-\frac{\sqrt{\pi}\,\Gamma(\frac{1}{4}n+\frac{1}{4}k+\frac{1}{2})}{2^{n+3}\,\Gamma(\frac{3}{2})\,\Gamma(2+n)}z^{n+2}\,{}_1F_3\left(\tfrac{1}{4}n+\tfrac{1}{4}k+\tfrac{1}{2};\tfrac{3}{2},1+\tfrac{1}{2}n,\tfrac{3}{2}+\tfrac{1}{2}n;\frac{z^4}{256}\right), \quad \dots (56)$$

where $R(k+n)>0$, $R(z)>0$.

Also (55) CHAP. I. in combination with (34) CHAP. I. gives

$$\int_0^\infty \exp\{-\lambda^2+z/\lambda\}\,\lambda^{k-1}K_n(z/\lambda)\,d\lambda =$$

$$\frac{\cos n\pi}{2^{5/2}\,\pi\sqrt{z}}E\left(\tfrac{1}{4}+\tfrac{1}{2}k,\tfrac{1}{4}+\tfrac{1}{2}n,\tfrac{1}{4}-\tfrac{1}{2}n,\tfrac{3}{4}+\tfrac{1}{2}n,\tfrac{3}{4}-\tfrac{1}{2}n:\tfrac{1}{2}:e^{\pm i\pi}z^2\right)$$

$$-\frac{\cos n\pi}{2^{5/2}\,\pi}\cdot z^{-3/2}E\left(\tfrac{3}{4}+\tfrac{1}{2}k,\tfrac{3}{4}+\tfrac{1}{2}n,\tfrac{3}{4}-\tfrac{1}{2}n,\tfrac{5}{4}+\tfrac{1}{2}n,\tfrac{5}{4}-\tfrac{1}{2}n:\tfrac{3}{2}:e^{\pm i\pi}z^2\right)$$

$$, \quad \dots (57)$$

where $R(k+\frac{1}{2})>0$, $R(z)>0$.

(57) CHAP. I. in combination with (6) CHAP. II. gives

$$4\int_0^\infty \exp\{-\lambda^4\}\lambda^{k-1} J_n(z/\lambda)\,d\lambda =$$

$$\frac{\Gamma(\frac{1}{4}k-\frac{1}{4}n)}{\Gamma(1+n)}(z/2)^n\,{}_0F_4\left(;\frac{1}{2}, 1-\frac{k-n}{4}, \frac{n+1}{2}, \frac{n+2}{2}; \frac{-z^4}{256}\right)$$

$$-\frac{\sqrt{\pi}\,\Gamma(\frac{1}{4}k-\frac{1}{4}n-\frac{1}{2})}{2^{n+3}\Gamma(\frac{3}{2})\Gamma(n+2)} z^{n+2}\,{}_0F_4\left(;\frac{3}{2}, \frac{3}{2}-\frac{k-n}{4}, \frac{n+2}{2}, \frac{n+3}{2}; \frac{-z^4}{256}\right)$$

$$+\frac{\Gamma(\frac{n}{4}-\frac{k}{4})\Gamma(\frac{1}{2}+\frac{n}{4}-\frac{k}{4})}{\Gamma(\frac{k+n}{4}+\frac{1}{2})\Gamma(\frac{k+n}{4}+1)}\left(\frac{z}{4}\right)^k {}_0F_4\left(;1+\frac{k-n}{4}, \frac{1}{2}+\frac{k-n}{4}, 1+\frac{k+n}{4}, \frac{1}{2}+\frac{k+n}{4}; \frac{-z^4}{256}\right), \quad\ldots(58)$$

where $R(z)>0$, $R(\frac{3}{2}+k)>0$.

(57) CHAP. I. in combination with (34) CHAP. I. gives if $R(k\pm n)>0$, $R(z)>0$

$$\int_0^\infty \exp\{-\lambda^2+z\lambda\}\lambda^{k-1} K_n(z\lambda)\,d\lambda =$$

$$\frac{\cos n\pi}{2^{\frac{5}{2}}\cdot\pi^2\sqrt{z}}\left[\begin{array}{l}\pi\Gamma(\frac{1}{2}k-\frac{1}{4})\Gamma(\frac{3}{4}-\frac{1}{2}k)E(\frac{1}{4}+\frac{n}{2}, \frac{1}{4}-\frac{n}{2}, \frac{3}{4}+\frac{n}{2}, \frac{3}{4}-\frac{n}{2} : \frac{1}{2}, \frac{3}{4}-\frac{1}{2}k : z^2) \\ -\pi\Gamma(\frac{1}{2}k-\frac{3}{4})\Gamma(\frac{5}{4}-\frac{1}{2}k)[\frac{1}{z}]E(\frac{3}{4}+\frac{n}{2}, \frac{3}{4}-\frac{n}{2}, \frac{5}{4}+\frac{n}{2}, \frac{5}{4}-\frac{n}{2} : \frac{3}{2}, \frac{5}{4}-\frac{1}{2}k : z^2) \\ +\Gamma(\frac{1}{4}-\frac{1}{2}k)\Gamma(\frac{3}{4}+\frac{1}{2}k)\Gamma(\frac{3}{4}-\frac{1}{2}k)\Gamma(\frac{1}{4}+\frac{1}{2}k)\,(1/z^2)^{\frac{1}{2}k-\frac{1}{4}} \\ \times E(\frac{1}{2}k+\frac{1}{2}n, \frac{1}{2}k-\frac{1}{2}n, \frac{1}{2}+\frac{1}{2}k+\frac{1}{2}n, \frac{1}{2}+\frac{1}{2}k-\frac{1}{2}n : \frac{3}{4}+\frac{1}{2}k, \frac{1}{4}\mp\frac{1}{2}k : z^2)\end{array}\right], \quad\ldots(59)$$

(60) CHAP. I. in combination with (34) CHAP. I. gives if $k=-\frac{1}{2}$

$$\int_0^\infty \exp\left\{-\frac{y}{2}-\frac{1}{xy}\right\} K_n\left(\frac{1}{xy}\right)\frac{dy}{y} =$$

$$\Gamma(n)\Gamma(-n)\,{}_1F_2\left(\tfrac{1}{2};1-n,1+n;1/x\right)+\sum_{n,-n}\{\Gamma(-n)\}^2 x^{-n}2^{-2n-1}\,{}_1F_2\left(\tfrac{1}{2}+n;1+n,1+2n;\tfrac{1}{x}\right)$$

$$=K_n^{\,2}(1/\sqrt{x})^{*} \qquad ,\ldots(60),$$

where $\underline{x}$ is real and positive; which was given by Macdonald ["Bessel Functions" by Watson p. 439, and l.c., p. 383 eq. (23)].

(58) CHAP. I. in combination with (6) CHAP. II. gives if $R(\frac{3}{2}+l\pm 2n)>0$, $R(z)>0$

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* Footnote: this is easily seen by applying the two formulae [Watson "Theory of Bessel Functions" pp. 147, 148] namely

$$I_\nu^{\,2}(z)=\frac{1}{\Gamma^2(\nu+1)}\left(\frac{z}{2}\right)^{2\nu}{}_1F_2\left(\tfrac{1}{2}+\nu;1+\nu,1+2\nu;z^2\right),$$

$$J_\mu(z)J_\nu(z)=\sum_{m=0}^{\infty}\frac{(-1)^m(\frac{1}{2}z)^{\mu+\nu+2m}(\mu+\nu+m+1)_m}{m!\,\Gamma(\mu+m+1)\Gamma(\nu+m+1)}$$

$$\int_0^\infty e^{-\lambda^2}\lambda^{l-1} J_m(z/\lambda) K_n(\lambda^2)\, d\lambda =$$

$$\frac{\sqrt{\pi}\,\Gamma(\frac{1}{2}l+n-\frac{1}{2}m)\Gamma(\frac{1}{2}l-n-\frac{1}{2}m)}{2^{\frac{1}{2}l+\frac{1}{2}m+1}\Gamma(\frac{1}{2}l+\frac{1}{2}-\frac{1}{2}m)\,\Gamma(1+m)}\,{}_1F_3\left(\frac{1-l+m}{2}; 1-\frac{1}{2}l-n+\frac{1}{2}m, 1-\frac{1}{2}l+n+\frac{1}{2}m, 1+m; \frac{z^2}{2}\right)$$

$$+\sum_{n,-n}\left[\frac{\sqrt{\pi}\,\Gamma(-\frac{1}{2}l-n+\frac{1}{2}m)\Gamma(-2n)}{2^{l+n+1}\Gamma(\frac{1}{2}-n)\Gamma(1+\frac{l}{2}+n+\frac{m}{2})}\, z^{2n+l} \times {}_1F_3\left(\frac{1}{2}+n; 1+\frac{1}{2}l-\frac{1}{2}m+n, 1+2n, 1+\frac{1}{2}l+\frac{1}{2}m+n; \frac{z^2}{2}\right)\right], \cdots (61)$$

(58) CHAP. I. in combination with (34) CHAP. I. gives

$$\int_0^\infty e^{-\lambda(1-z)}\lambda^{l-1} K_m(z\lambda) K_n(\lambda)\, d\lambda =$$

$$\frac{\pi\,\Gamma(l+n-\frac{1}{2})\Gamma(l-n-\frac{1}{2})}{2^l\,\Gamma(l)}\, z^{-\frac{1}{2}}\,{}_3F_2\left(\frac{1}{2}+m, \frac{1}{2}-m, 1-l; \frac{3}{2}-l-n, \frac{3}{2}-l+n; \frac{1}{z}\right)$$

$$+\sum_{n,-n}\frac{\cos(m\pi)\,\Gamma(-2n)\,\Gamma(\frac{1}{2}-n-l)\Gamma(l+m+n)\Gamma(l-m+n)}{2^l\,\Gamma(\frac{1}{2}-n)\,\Gamma(\frac{1}{2}+l+n)\, z^{l+n}}\,{}_3F_2\left\{\begin{matrix} l+m+n, l-m+n, \frac{1}{2}+n; \frac{1}{z} \\ l+n+\frac{1}{2}, 1+2n \end{matrix}\right\}, \cdots (62)$$

where $R(l \pm m \pm n) > 0$, $R(z) > 0$.

In (62) CHAP. I., take $m=2$ and apply (6) CHAP. II., so getting

$$\int_0^\infty e^{-\lambda}\lambda^{k-1}E(\gamma,\delta::\lambda)\,J_n(b/\lambda)\,d\lambda =$$

$$\frac{2^{k-1}\Gamma(\gamma)\Gamma(\delta)\Gamma(\tfrac12\gamma+\tfrac12 k-\tfrac12 n)\Gamma(\tfrac12+\tfrac12\gamma+\tfrac12 k-\tfrac12 n)\Gamma(\tfrac12\delta+\tfrac12 k-\tfrac12 n)\Gamma(\tfrac12+\tfrac12\delta+\tfrac12 k-\tfrac12 n)}{\sqrt{\pi}\,\Gamma(\tfrac12\gamma+\tfrac12\delta+\tfrac12 k-\tfrac12 n)\Gamma(\tfrac12\gamma+\tfrac12\delta+\tfrac12\delta-\tfrac12 n+\tfrac12)\Gamma(1+n)}\left(\frac{b}{4}\right)^n$$

$$\times\,{}_2F_5\left(\begin{matrix}1-\tfrac12\gamma-\tfrac12 k+\tfrac12 n-\tfrac12\delta,\ \tfrac12-\tfrac12\gamma-\tfrac12\delta-\tfrac12 k+\tfrac12 n;\ -b^2/16\\ 1-\tfrac12\gamma-\tfrac12 k+\tfrac12 n,\ \tfrac12-\tfrac12\gamma-\tfrac12 k+\tfrac12 n,\ 1-\tfrac12\delta-\tfrac12 k+\tfrac12 n,\ \tfrac12-\tfrac12\delta-\tfrac12 k+\tfrac12 n,\ 1+n\end{matrix}\right)$$

$$+\sum_{\gamma,\delta}\Bigg[2^{k-1}\frac{\Gamma(\gamma)\Gamma(\delta)\Gamma(\tfrac12 n-\tfrac12\gamma-\tfrac12 k)\Gamma(\tfrac12\delta-\tfrac12\gamma)\Gamma(\tfrac12+\tfrac12\delta-\tfrac12\gamma)}{\Gamma(1+\tfrac12\gamma+\tfrac12 k+\tfrac12 n)\Gamma(\tfrac12\delta)\Gamma(\tfrac12+\tfrac12\delta)}\left(\frac{b}{4}\right)^{\gamma+k}$$

$$\times\,{}_2F_5\left(1-\frac{\delta}{2},\frac12-\frac{\delta}{2};1+\frac{\gamma}{2}+\frac{k}{2}-\frac{n}{2},1+\frac{\gamma}{2}-\frac{\delta}{2},\frac12,\frac12+\frac{\gamma}{2}-\frac{\delta}{2},1+\frac{\gamma}{2}+\frac{k}{2}+\frac{n}{2};\frac{-b^2}{16}\right)$$

$$+\frac{2^{k-1}\Gamma(\gamma)\Gamma(\delta)\Gamma(\tfrac12 n-\tfrac{\gamma+1}{2}-\tfrac{k}{2})\Gamma(-\tfrac12)\Gamma(\tfrac{\delta}{2}-\tfrac{\gamma}{2}-\tfrac12)\Gamma(\tfrac{\delta}{2}-\tfrac{\gamma}{2})}{\sqrt{\pi}\,\Gamma(1+\tfrac{\gamma}{2}+\tfrac{k}{2}+\tfrac{n}{2}+\tfrac12)\Gamma(\tfrac12\delta-\tfrac12)\Gamma(\tfrac12\delta)}\left(\frac{b}{4}\right)^{\gamma+k+1}$$

$$\times\,{}_2F_5\left(\frac32-\frac{\delta}{2},1-\frac{\delta}{2};\frac32+\frac{\gamma}{2}+\frac{k}{2}-\frac{n}{2},\frac32+\frac{\gamma}{2}-\frac{\delta}{2},1+\frac{\gamma}{2}-\frac{\delta}{2},\frac32+\frac{\gamma}{2}+\frac{k}{2}+\frac{n}{2};-\frac{b^2}{16}\right)\Bigg]$$

where $R(\tfrac32+\gamma+k)>0$, $R(\tfrac32+\delta+k)>0$, $|\mathrm{amp}\,b|<\pi$, ... (63).

In (63) take $\gamma = \frac{1}{2}+m$, $\delta = \frac{1}{2}-m$, and apply (34) CHAP. I., so getting

$$\int_0^\infty e^{-\lambda}\lambda^{k-1} K_m(\lambda)\, J_n(b/\lambda)\, d\lambda =$$

$$\frac{2^{-\frac{3}{2}}\Gamma\left(\frac{m}{2}+\frac{k}{2}-\frac{n}{2}\right)\Gamma\left(\frac{1}{2}+\frac{m}{2}+\frac{k}{2}-\frac{n}{2}\right)\Gamma\left(-\frac{m}{2}+\frac{k}{2}-\frac{n}{2}\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}m+\frac{1}{2}k-\frac{1}{2}n\right)}{\Gamma\left(\frac{1}{4}+\frac{1}{2}k-\frac{1}{2}n\right)\Gamma\left(\frac{3}{4}+\frac{1}{2}k-\frac{1}{2}n\right)\Gamma(1+n)}$$

$${}_2F_5\left(\frac{3}{4}-\frac{k}{2}+\frac{n}{2}, \frac{1}{4}-\frac{k}{2}+\frac{n}{2}; 1-\frac{m}{2}-\frac{k}{2}+\frac{n}{2}, \frac{1}{2}-\frac{m}{2}-\frac{k}{2}+\frac{n}{2}, 1+n, 1+\frac{m}{2}-\frac{k}{2}+\frac{n}{2}, \frac{1}{2}+\frac{m}{2}-\frac{k}{2}+\frac{n}{2}; \frac{-b^2}{4}\right)$$

$$+\sum_{m,-m}\Bigg[\frac{2^{-\frac{3}{2}}\sqrt{\pi}\,\Gamma\left(\frac{n}{2}-\frac{m}{2}-\frac{k}{2}\right)\Gamma(-m)\Gamma\left(\frac{1}{2}-m\right)}{\Gamma\left(1+\frac{1}{2}m+\frac{1}{2}n+\frac{1}{2}k\right)\Gamma\left(\frac{1}{4}-\frac{1}{2}m\right)\Gamma\left(\frac{3}{4}-\frac{1}{2}m\right)}\left(\frac{b}{2}\right)^{m+k}$$

$${}_2F_5\left(\frac{3}{4}+\frac{m}{2}, \frac{1}{4}+\frac{m}{2}; 1+\frac{m}{2}+\frac{k}{2}-\frac{n}{2}, 1+m, \frac{1}{2}, \frac{1}{2}+m, 1+\frac{m}{2}+\frac{k}{2}+\frac{n}{2}; \frac{-b^2}{4}\right)$$

$$+\frac{2^{-\frac{3}{2}}\Gamma\left(\frac{n}{2}-\frac{m}{2}-\frac{k}{2}-\frac{1}{2}\right)\Gamma\left(-\frac{1}{2}\right)\Gamma\left(-m-\frac{1}{2}\right)\Gamma(-m)}{\Gamma\left(1+\frac{1}{2}m+\frac{1}{2}n+\frac{1}{2}k+\frac{1}{2}\right)\Gamma\left(-\frac{m}{2}-\frac{1}{4}\right)\Gamma\left(\frac{1}{4}-\frac{m}{2}\right)}\left(\frac{b}{2}\right)^{m+k+1}$$

$${}_2F_5\left(\frac{5}{4}+\frac{m}{2}, \frac{3}{4}+\frac{m}{2}; \frac{3}{2}+\frac{m}{2}+\frac{k}{2}-\frac{n}{2}, \frac{3}{2}, \frac{3}{2}+m, 1+m, \frac{3}{2}+\frac{m}{2}+\frac{k}{2}+\frac{n}{2}; \frac{-b^2}{4}\right)\Bigg]$$

where $R(b)>0$, $R\left(\frac{3}{2}+k\pm m\right)>0$. $\quad\ldots(64)$

On applying (51), it follows that if $0 \leq \operatorname{amp} b \leq \pi$, $R\left(\frac{3}{2}+k \pm m\right)>0$

$$i^{n}\int_0^\infty e^{-\lambda}\lambda^{k-1}K_m(\lambda)\,G_n(b/\lambda)\,d\lambda =$$

$$\sum_{n,-n}\left[\frac{\Gamma\left(\frac{k+m+n}{2}\right)\Gamma\left(\frac{1+k+m+n}{2}\right)\Gamma\left(\frac{-m+k+n}{2}\right)\Gamma\left(\frac{1-m+k+n}{2}\right)\Gamma(n)}{2^{5/2}\Gamma\left(\frac{1}{4}+\frac{1}{2}k+\frac{1}{2}n\right)\Gamma\left(\frac{3}{4}+\frac{1}{2}k+\frac{1}{2}n\right)}\left(\frac{2i}{b}\right)^{n} \times {}_2F_5\left(\begin{matrix}\frac{3}{4}-\frac{1}{2}k-\frac{1}{2}n,\ \frac{1}{4}-\frac{1}{2}k-\frac{1}{2}n;\ -b^2/4 \\ 1-\frac{m}{2}-\frac{k}{2}-\frac{n}{2},\ \frac{1}{2}-\frac{m}{2}-\frac{k}{2}-\frac{n}{2},\ 1-n,\ 1+\frac{m}{2}-\frac{k}{2}-\frac{n}{2},\ \frac{1}{2}+\frac{m}{2}-\frac{k}{2}-\frac{n}{2}\end{matrix}\right)\right]$$

$$+\sum_{m,-m}\left[\frac{\pi\,\Gamma\left(\frac{1}{2}\right)\Gamma\left(-\frac{n}{2}-\frac{k}{2}-\frac{m}{2}\right)\Gamma\left(\frac{n}{2}-\frac{k}{2}-\frac{m}{2}\right)\Gamma(-m)\Gamma\left(\frac{1}{2}-m\right)}{2^{5/2}\sin(n\pi)\,\Gamma\left(\frac{1}{4}-\frac{1}{2}m\right)\Gamma\left(\frac{3}{4}-\frac{1}{2}m\right)}\left(\frac{b}{2}\right)^{m+k} \times {}_2F_5\left(\begin{matrix}\frac{3}{4}+\frac{1}{2}m,\ \frac{1}{4}+\frac{1}{2}m;\ -b^2/4 \\ 1+\frac{m}{2}+\frac{k}{2}-\frac{n}{2},\ \frac{1}{2},\ 1+m,\ \frac{1}{2}+m,\ 1+\frac{m}{2}+\frac{k}{2}+\frac{n}{2}\end{matrix}\right) \times \frac{1}{\pi}\left\{-\sin\left(\frac{k+m-n}{2}\right)\pi\, e^{\frac{in\pi}{2}}+\sin\left(\frac{k+m+n}{2}\right)\pi\, e^{-\frac{in\pi}{2}}\right\}\right]$$

$$
+\sum_{m,-m}\left[\frac{\pi\,\Gamma\left(\frac{-n-m-k-1}{2}\right)\Gamma\left(\frac{n-m-k-1}{2}\right)\Gamma\left(-\frac{1}{2}\right)\Gamma\left(-m-\frac{1}{2}\right)\Gamma(-m)}{2^{5/2}\sin n\pi\,\Gamma\left(-\frac{1}{2}m-\frac{1}{4}\right)\Gamma\left(\frac{1}{4}-\frac{1}{2}m\right)}\left(\frac{b}{2}\right)^{m+k+1}\right.
$$

$$
\times\,{}_2F_5\left(\begin{matrix}\frac{5}{4}+\frac{m}{2},\ \frac{3}{4}+\frac{m}{2};\ -b^2/4\\ \frac{3}{2}+\frac{m}{2}+\frac{k}{2}-\frac{n}{2},\ \frac{3}{2},\ \frac{3}{2}+m,\ 1+m,\ \frac{3}{2}+\frac{m}{2}+\frac{k}{2}+\frac{n}{2}\end{matrix}\right)
$$

$$
\left.\times\frac{1}{\pi}\left\{-\sin\left(\frac{k+m-n+1}{2}\right)\pi e^{\frac{in\pi}{2}}+\sin\left(\frac{k+m+n+1}{2}\right)\pi e^{-\frac{in\pi}{2}}\right\}\right]
$$

Now the two quantities between brackets $\{\ \}$ in the last expression are equal to

$$\sin(n\pi)\cdot i^{-k-m},$$

$$\sin(n\pi)\cdot i^{-k-m-1}, \text{ respectively.}$$

Hence on replacing b by (ib) and noting that

$$G_n(ib) = i^{-n} K_n(b),$$

the formula becomes

$$\int_0^\infty e^{-\lambda}\lambda^{k-1}K_m(\lambda)K_n(b/\lambda)\,d\lambda =$$

$$\begin{aligned}
&\sum_{n,-n}\Bigg[\frac{\Gamma\left(\frac{k+m+n}{2}\right)\Gamma\left(\frac{1+k+m+n}{2}\right)\Gamma\left(\frac{-m+k+n}{2}\right)\Gamma\left(\frac{1-m+k+n}{2}\right)\Gamma(n)}{2^{5/2}\,\Gamma\left(\frac{1}{4}+\frac{1}{2}k+\frac{1}{2}n\right)\Gamma\left(\frac{3}{4}+\frac{1}{2}k+\frac{1}{2}n\right)}\left(\frac{b}{2}\right)^{n} \\
&\qquad \times {}_2F_5\left(\begin{matrix}\frac{3}{4}-\frac{1}{2}k-\frac{1}{2}n,\ \frac{1}{4}-\frac{1}{2}k-\frac{1}{2}n;\ b^2/4 \\ 1+\frac{m}{2}-\frac{k}{2}-\frac{n}{2},\ \frac{1}{2}+\frac{m}{2}-\frac{k}{2}-\frac{n}{2},\ 1-n,\ 1-\frac{m}{2}-\frac{k}{2}-\frac{n}{2},\ \frac{1}{2}-\frac{m}{2}-\frac{k}{2}-\frac{n}{2}\end{matrix}\right)\Bigg] \\
&+\sum_{m,-m}\Bigg[\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(1-\frac{n}{2}-\frac{k}{2}-\frac{m}{2}\right)\Gamma\left(\frac{n}{2}-\frac{k}{2}-\frac{m}{2}\right)\Gamma(-m)\,\Gamma\left(\frac{1}{2}-m\right)}{2^{5/2}\,\Gamma\left(\frac{1}{4}-\frac{m}{2}\right)\Gamma\left(\frac{3}{4}-\frac{m}{2}\right)}\left(\frac{b}{2}\right)^{m+k} \\
&\qquad \times {}_2F_5\left(\begin{matrix}\frac{3}{4}+\frac{1}{2}m,\ \frac{1}{4}+\frac{1}{2}m;\ b^2/4 \\ 1+\frac{m}{2}+\frac{k}{2}-\frac{n}{2},\ \frac{1}{2},\ 1+m,\ \frac{1}{2}+m,\ 1+\frac{m}{2}+\frac{k}{2}+\frac{n}{2}\end{matrix}\right)\Bigg] \\
&+\sum_{m,-m}\Bigg[\frac{\Gamma\left(\frac{-n-m-k-1}{2}\right)\Gamma\left(\frac{n-m-k-1}{2}\right)\Gamma\left(-\frac{1}{2}\right)\Gamma\left(-m-\frac{1}{2}\right)\Gamma(-m)}{2^{5/2}\,\Gamma\left(-\frac{1}{2}m-\frac{1}{4}\right)\Gamma\left(\frac{1}{4}-\frac{1}{2}m\right)}\left(\frac{b}{2}\right)^{m+k+1} \\
&\qquad \times {}_2F_5\left(\begin{matrix}\frac{5}{4}+\frac{m}{2},\ \frac{3}{4}+\frac{m}{2};\ b^2/4 \\ \frac{3}{2}+\frac{m}{2}+\frac{k}{2}-\frac{n}{2},\ \frac{3}{2},\ \frac{3}{2}+m,\ 1+m,\ \frac{3}{2}+\frac{m}{2}+\frac{k}{2}+\frac{n}{2}\end{matrix}\right)\Bigg]
\end{aligned}$$

where $R(b) > 0$. 65

In (63) CHAP. I. take $p=q=0$, $\gamma=\frac{1}{2}+n$, $\delta=\frac{1}{2}-n$, and apply (34) CHAP. I. so getting

$$\int_0^\infty \exp\left\{-\frac{\lambda^2}{2}-\frac{\lambda}{z}\right\}\lambda^{k-1}K_n\left(\frac{\lambda^2}{2}\right)d\lambda =$$

$$\left(\frac{\pi}{2}\right)\frac{\Gamma(\frac{1}{2}k+n)\Gamma(\frac{1}{2}k-n)}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+\frac{1}{2}k)}\,{}_2F_2\left(\tfrac{1}{2}k+n,\tfrac{1}{2}k-n;\tfrac{1}{2},\tfrac{1}{2}+\tfrac{1}{2}k;1/4z^2\right)$$

$$-\left(\frac{\pi}{2}\right)\frac{\Gamma(\frac{1}{2}+\frac{1}{2}k+n)\Gamma(\frac{1}{2}+\frac{1}{2}k-n)}{\Gamma(\frac{3}{2})\Gamma(1+\frac{1}{2}k)}\left(\frac{1}{2z}\right){}_2F_2\left(\frac{1}{2}+\frac{k}{2}+n,\frac{1}{2}+\frac{k}{2}-n;\frac{3}{2},1+\frac{1}{2}k;\frac{1}{4z^2}\right)$$

, ...(66)

where $R(z)>0$, $R(k\pm n)>0$.

Also in (65) CHAP. I. take $p=q=0$, $\gamma=\frac{1}{2}+n$, $\delta=\frac{1}{2}-n$, and apply (34) CHAP. I. so getting if $R(z)>0$

$$\int_0^\infty \exp\left\{-\lambda^2-\frac{1}{z\lambda}\right\}\lambda^{k-1}K_n\left(\frac{\lambda^2}{2}\right)d\lambda =$$

$$\left(\frac{\pi}{2}\right)\frac{\Gamma(\frac{1}{2}k+n)\,\Gamma(\frac{1}{2}k-n)}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}k+\frac{1}{2})}\,{}_1F_3\left(\frac{1}{2}-\frac{1}{2}k;\ \frac{1}{2},\ 1-\frac{k}{2}-n,\ 1-\frac{k}{2}+n;\ \frac{-1}{4z^2}\right)$$

$$-\left(\frac{\pi}{4}\right)\frac{\Gamma(\frac{1}{2}k+n-\frac{1}{2})\,\Gamma(\frac{1}{2}k-n-\frac{1}{2})}{\Gamma(\frac{3}{2})\,\Gamma(\frac{1}{2}k)}\left(\frac{1}{z}\right){}_1F_3\left(1-\frac{1}{2}k;\ \frac{3}{2},\ \frac{3}{2}-\frac{k}{2}-n,\ \frac{3}{2}-\frac{k}{2}+n;\ \frac{-1}{4z^2}\right)$$

$$+\sum_{n,-n}\frac{\Gamma(-2n)\,\Gamma(-\frac{1}{2}k-n)\,\Gamma(\frac{1}{2}-\frac{k}{2}-n)}{2\,\Gamma(\frac{1}{2}-n)}\left(\frac{1}{4z^2}\right)^{n+\frac{k}{2}}{}_1F_3\left(\begin{matrix}\frac{1}{2}+n;\ -1/4z^2\\ 1+n+\frac{k}{2},\ \frac{1}{2}+n+\frac{k}{2},\ 1+2n\end{matrix}\right)$$

, . . . (.67)

In (63) CHAP-I. take $p=0$, $q=1$, $\gamma=\frac{1}{2}+n$, $\delta=\frac{1}{2}-n$ and apply both (34) CHAP. I. and (6) CHAP. II. so getting

$$\int_0^\infty e^{-\lambda^2/2}\lambda^{k-1}K_n\left(\frac{\lambda^2}{2}\right)J_m\left(2\sqrt{\frac{\lambda}{z}}\right)d\lambda =$$

$$\frac{\pi}{2^{m+1}z^{\frac{m}{2}}}\cdot\frac{\Gamma(\frac{k}{2}+\frac{m}{4}+n)\,\Gamma(\frac{k}{2}+\frac{m}{4}-n)}{\Gamma(\frac{k}{2}+\frac{m}{4}+\frac{1}{2})\,\Gamma(\frac{m+1}{2})\,\Gamma(\frac{m+2}{2})}\,{}_2F_4\left(\begin{matrix}\frac{k}{2}+\frac{1}{4}m+n,\ \frac{1}{2}k+\frac{1}{4}m-n;\ 1/16z^2\\ \frac{1}{2}+\frac{1}{2}k+\frac{1}{4}m,\ \frac{1}{2}+\frac{1}{2}m,\ 1+\frac{1}{2}m,\ \frac{1}{2}\end{matrix}\right)$$

$$-\frac{\pi^{\frac{3}{2}}}{2^{m+3}}\cdot\frac{\Gamma(\frac{k}{2}+\frac{m}{4}+n+\frac{1}{2})\,\Gamma(\frac{k+1}{2}+\frac{m}{4}-n)}{\Gamma(\frac{3}{2})\,\Gamma(1+\frac{k}{2}+\frac{m}{4})\,\Gamma(\frac{m+2}{2})\,\Gamma(\frac{m+3}{2})}\,z^{-\frac{m}{2}-1}{}_2F_4\left(\begin{matrix}\frac{k+1}{2}+\frac{m}{4}+n,\ \frac{k+1}{2}+\frac{m}{4}-n;\ \frac{1}{16z^2}\\ \frac{3}{2},\ 1+\frac{k}{2}+\frac{m}{4},\ \frac{m+2}{2},\ \frac{m+3}{2}\end{matrix}\right)$$

, - . (68)

where $R(z)>0$, $R(k\pm 2n+\frac{1}{2}m)>0$.

In (65) CHAP. I. take $\gamma=\frac{1}{2}+n$, $\delta=\frac{1}{2}-n$, $p=0$, $q=1$ and apply both (34) CHAP. I. and (6) CHAP. II., so getting

$$\int_0^\infty e^{-\lambda^2/2}\lambda^{k-1}K_n(\lambda^2/2)\,J_m(2/\sqrt{(z\lambda)})\,d\lambda=$$

$$\frac{\pi\Gamma(\frac{k}{2}-\frac{m}{4}+n)\Gamma(\frac{k}{2}-\frac{m}{4}-n)}{2^{1+m}\Gamma(\frac{k}{2}-\frac{m}{4}+\frac{1}{2})\Gamma(\frac{m+1}{2})\Gamma(\frac{m+2}{2})}z^{-\frac{1}{2}m}{}_1F_5\left(\begin{matrix}-\frac{k}{2}+\frac{m}{4}+\frac{1}{2};\ -1/16z^2\\ \frac{1}{2},1+\frac{m}{4}-\frac{k}{2}-n,1+\frac{m}{4}-\frac{k}{2}+n,\frac{1+m}{2},\frac{2+m}{2}\end{matrix}\right)$$

$$-\frac{\pi^{\frac{3}{2}}\Gamma(\frac{k}{2}-\frac{m}{4}-\frac{1}{2}+n)\Gamma(\frac{k}{2}-\frac{m}{4}-\frac{1}{2}-n)}{2^{m+3}\Gamma(\frac{3}{2})\Gamma(\frac{k}{2}-\frac{m}{4})\Gamma(\frac{m+2}{2})\Gamma(\frac{m+3}{2})}z^{-\frac{m}{2}-1}$$

$$\times{}_1F_5\left(\begin{matrix}1-\frac{k}{2}+\frac{m}{4};\ -1/16z^2\\ \frac{3}{2},\frac{3}{2}-\frac{k}{2}+\frac{m}{4}-n,\frac{3}{2}-\frac{k}{2}+\frac{m}{4}+n,\frac{m+2}{2},\frac{m+3}{2}\end{matrix}\right)$$

$$+\sum_{n,-n}\frac{\sqrt{\pi}\Gamma(-n-\frac{k}{2}+\frac{m}{4})\Gamma(\frac{1}{2}-n-\frac{1}{2}k+\frac{m}{4})\Gamma(-2n)}{2\Gamma(\frac{1}{2}-n)\Gamma(\frac{1}{2}+\frac{m}{4}+n+\frac{1}{2}k)\Gamma(1+\frac{m}{4}+n+\frac{1}{2}k)}\left(\frac{1}{16z^2}\right)^{n+\frac{1}{2}k}$$

$$\times{}_1F_5\left(\begin{matrix}\frac{1}{2}+n;\ -1/16z^2\\ 1+n+\frac{k}{2}-\frac{m}{4},\frac{1}{2}+n+\frac{k}{2}-\frac{m}{4},1+2n,\frac{1}{2}+\frac{m}{4}+n+\frac{k}{2},1+\frac{m}{4}+n+\frac{k}{2}\end{matrix}\right)$$

$$,\quad\ldots(69)$$

where $R(z)>0$, $R(\frac{3}{2}\pm 2n+2k)>0$.

In (63) CHAP. I. take $p=2$, $q=0$ with $\gamma=\frac{1}{2}+n$, $\delta=\frac{1}{2}-n$, $\alpha_1=\frac{1}{2}+m$, $\alpha_2=\frac{1}{2}-m$ and apply (34) CHAP. I. and get

$$\int_0^\infty \exp\left\{-\frac{\lambda^2}{2}+\frac{z}{\lambda}\right\}\lambda^{k-1}K_n\left(\frac{\lambda^2}{2}\right)K_m\left(\frac{z}{\lambda}\right)d\lambda =$$

$$\frac{\cos m\pi}{4\sqrt{(2\pi z)}}E\left\{\begin{matrix}\frac{1}{4}+\frac{m}{2}, \frac{1}{4}-\frac{m}{2}, \frac{3}{4}+\frac{m}{2}, \frac{3}{4}-\frac{m}{2}, \frac{1}{4}+\frac{k}{2}+n, \frac{1}{4}+\frac{k}{2}-n : e^{\pm i\pi}z^2 \\ \frac{1}{2}, \frac{3}{4}+\frac{1}{2}k\end{matrix}\right\}$$

$$-\frac{\cos m\pi}{4\sqrt{(2\pi z)}}\left(\frac{1}{z}\right)E\left\{\begin{matrix}\frac{3}{4}+\frac{m}{2}, \frac{3}{4}-\frac{m}{2}, \frac{5}{4}+\frac{m}{2}, \frac{5}{4}-\frac{m}{2}, \frac{3}{4}+\frac{k}{2}+n, \frac{3}{4}+\frac{k}{2}-n : e^{\pm i\pi}z^2 \\ \frac{3}{2}, \frac{5}{4}+\frac{1}{2}k\end{matrix}\right\}$$

where $R(z)>0$, $R(k+\frac{1}{2}\pm 2n)>0$. . . . . (70)

In (65) CHAP. I. take $p=2$, $q=0$ with $\gamma=\frac{1}{2}+n$, $\delta=\frac{1}{2}-n$, $\alpha_1=\frac{1}{2}+m$, $\alpha_2=\frac{1}{2}-m$ and apply (34) CHAP. I., so getting if $R(z)>0$, $R(k\pm 2n\pm m)>0$

$$\int_0^\infty \exp\left\{-\frac{\lambda^2}{2}+z\lambda\right\}\lambda^{k-1}K_n\left(\frac{\lambda^2}{2}\right)K_m(z\lambda)\,d\lambda =$$

$$\frac{1}{4}\sqrt{\left(\frac{\pi}{2z}\right)}\ \frac{\sin\left(\frac{1}{4}+\frac{k}{2}\right)\pi\quad\cos(m\pi)}{\sin\left(n+\frac{k}{2}-\frac{1}{4}\right)\pi\,\sin\left(-n+\frac{k}{2}-\frac{1}{4}\right)\pi}$$

$$\times E\left\{\begin{matrix}\frac{1}{4}+\frac{m}{2},\ \frac{1}{4}-\frac{m}{2},\ \frac{3}{4}+\frac{m}{2},\ \frac{3}{4}-\frac{m}{2},\ \frac{3}{4}-\frac{1}{2}k : z^2\\ \frac{1}{2},\ \frac{5}{4}-n-\frac{k}{2},\ \frac{5}{4}+n-\frac{k}{2}\end{matrix}\right\}$$

$$+\frac{1}{4}\sqrt{\left(\frac{\pi}{2z}\right)}\ \frac{\sin\left(\frac{1}{4}-\frac{k}{2}\right)\pi\quad\cos(m\pi)}{\cos\left(n+\frac{1}{2}k-\frac{1}{4}\right)\pi\,\cos\left(-n+\frac{1}{2}k-\frac{1}{4}\right)\pi}\left(\frac{1}{z}\right)$$

$$\times E\left\{\begin{matrix}\frac{3}{4}+\frac{m}{2},\ \frac{3}{4}-\frac{m}{2},\ \frac{5}{4}+\frac{m}{2},\ \frac{5}{4}-\frac{m}{2},\ \frac{5}{4}-\frac{1}{2}k : z^2\\ \frac{3}{2},\ \frac{7}{4}-n-\frac{1}{2}k,\ \frac{7}{4}+n-\frac{1}{2}k\end{matrix}\right\}$$

$$-\frac{1}{4}\sqrt{\left(\frac{\pi}{2z}\right)}\sum_{n,-n}\left[\frac{2\cos(n\pi)\cos(m\pi)}{\sin(2n+k-\frac{1}{2})\pi\,\sin(-2n\pi)}\left(\frac{1}{z^2}\right)^{n+\frac{1}{2}k-\frac{1}{4}}\right.$$
$$\left.\times E\left\{\begin{matrix}\frac{m}{2}+\frac{k}{2}+n,\ -\frac{m}{2}+\frac{k}{2}+n,\ \frac{1}{2}+\frac{m}{2}+\frac{k}{2}+n,\ \frac{1}{2}-\frac{m}{2}+\frac{k}{2}+n,\ \frac{1}{2}+n : z^2\\ \frac{3}{4}+n+\frac{k}{2},\ \frac{1}{4}+n+\frac{k}{2},\ 1+2n\end{matrix}\right\}\right]$$

, ... (71)

(66) CHAP. I. in combination with (6) CHAP. II. gives

$$\sqrt{\pi}\cos\left(k+\tfrac{1}{2}m\right)\pi\cdot z^{\frac{m}{2}}\int_0^\infty e^{-u}u^{k-1}I_n(u)\,J_m\left(2\sqrt{\tfrac{u}{z}}\right)du=$$

$$2^{-k-\frac{m}{2}}\sin\left(k+\frac{m}{2}-n\right)\pi\,\frac{\Gamma\left(k+\frac{m}{2}+n\right)\Gamma\left(k+\frac{m}{2}-n\right)}{\Gamma\left(\frac{1}{2}+k+\frac{m}{2}\right)\Gamma(m+1)}\,{}_2F_2\left(\begin{matrix}k+\frac{m}{2}+n,\ k+\frac{m}{2}-n;\ \frac{-1}{2z}\\ \frac{1}{2}+k+\frac{m}{2},\ 1+m\end{matrix}\right)$$

$$-2^{-\frac{1}{2}}\pi z^{k+\frac{m}{2}-\frac{1}{2}}\,\frac{1}{\Gamma\left(\frac{3}{2}-k-\frac{m}{2}\right)\Gamma\left(\frac{3}{2}-k+\frac{m}{2}\right)}\,{}_2F_2\left(\begin{matrix}\frac{1}{2}+n,\ \frac{1}{2}-n;\ -1/2z\\ \frac{3}{2}-k-\frac{1}{2}m,\ \frac{3}{2}-k+\frac{1}{2}m\end{matrix}\right), \dots (42)$$

where $R(z)>0$, $R\left(k+n+\frac{m}{2}\right)>0$, $R\left(\frac{5}{2}-2k\right)>0$.

(66) CHAP. I. in combination with (34) CHAP. I. gives

$$\pi\sqrt{2z}\cdot\cos\left(k+\tfrac{1}{2}\right)\pi\int_0^\infty \exp\left\{-u+\frac{z}{u}\right\}u^{k-1}K_m\left(\frac{z}{u}\right)I_n(u)\,du=$$

$$2^{-k-\frac{1}{2}}\cos(k-n)\pi\cos(m\pi)\,E\left(\tfrac{1}{2}+m,\ \tfrac{1}{2}-m,\ k+\tfrac{1}{2}+n,\ k+\tfrac{1}{2}-n : 1+k : 4z\right)$$

$$-2^{-\frac{1}{2}}\cos(n\pi)\cos(m\pi)(2z)^k E\left(\tfrac{1}{2}+m-k,\ \tfrac{1}{2}-m-k,\ \tfrac{1}{2}-n,\ \tfrac{1}{2}+n : 1-k : 4z\right)$$

where $R\left(k+\frac{1}{2}+n\right)>0$, $R(z)>0$, $R\left(\frac{1}{2}\pm m-k\right)>0$. , ... (43),

(68) CHAP. I. in combination with (6) CHAP. II. gives

$$\frac{\sin(n+k-\frac{m}{2})\pi}{\sqrt{\pi}}\, 2^{k-\frac{m}{2}} \int_0^\infty e^{-u} u^{k-1} J_m\{2/\sqrt{(zu)}\} I_n(u)\, du =$$

$$\frac{\Gamma(\frac{1}{2}-k+\frac{m}{2})}{\Gamma(1-n-k+\frac{m}{2})\Gamma(1+n-k+\frac{m}{2})\Gamma(m+1)} z^{-\frac{m}{2}} {}_1F_3\left(\begin{matrix}\frac{1}{2}-k+\frac{m}{2};\ 2/z \\ 1-n-k+\frac{m}{2},\ 1+n-k+\frac{m}{2},\ 1+m\end{matrix}\right)$$

$$-\left(\frac{2}{z}\right)^{k+n-\frac{m}{2}} z^{-\frac{m}{2}} \frac{\Gamma(\frac{1}{2}+n)}{\Gamma(1+n+k-\frac{m}{2})\Gamma(1+n+k+\frac{m}{2})\Gamma(1+2n)}$$

$$\times {}_1F_3\left(\tfrac{1}{2}+n;\ 1+n+k-\tfrac{m}{2},\ 1+n+k+\tfrac{m}{2},\ 1+2n;\ \tfrac{2}{z}\right), \quad \cdots\cdots (74)$$

where $R(\frac{3}{2}+2n+2k)>0$, $R(z)>0$, $R(\frac{1}{2}+\frac{1}{2}m-k)>0$.

(68) CHAP. I. in combination with (34) CHAP. I. gives

$$-\frac{\cos(n+k)\pi}{\sqrt{\pi}\ 2^{1/2-k}\cos(m\pi)} \sqrt{(2\pi z)} \int_0^\infty \exp\{-u+zu\} u^{k-1} K_m(zu) I_n(u)\, du$$

$$=E(\tfrac{1}{2}+m,\ \tfrac{1}{2}-m,\ 1-k : \tfrac{3}{2}-k-n,\ \tfrac{3}{2}-k+n : e^{\pm i\pi} z)$$

$$-(1/z)^{k-\frac{1}{2}+n} E(k+m+n,\ k-m+n,\ \tfrac{1}{2}+n : \tfrac{1}{2}+n+k,\ 1+2n : e^{\pm i\pi} z), \cdots (75)$$

where $R(z)>0$, $R(k\pm m+n)>0$, $R(1-k)>0$.

(69) CHAP. I. in combination with (6) CHAP. II. gives if $R(k+m+n)>0$, $R(z)>0$, $R(2-k)>0$

$$\int_0^\infty e^{-u} u^{k-1} J_m(zu)\, I_n(u)\, du =$$

$$\frac{1}{2\sqrt{2}\cdot\pi}\cdot\frac{\Gamma\left(\frac{1}{4}-\frac{1}{2}k-\frac{1}{2}m\right)\Gamma\left(\frac{3}{4}-\frac{1}{2}k-\frac{1}{2}m\right)\Gamma\left(\frac{1}{2}m+\frac{1}{2}n+\frac{1}{2}k\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}m+\frac{1}{2}n+\frac{1}{2}k\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}n-\frac{1}{2}k-\frac{1}{2}m\right)\Gamma\left(1+\frac{1}{2}n-\frac{1}{2}k-\frac{1}{2}m\right)\Gamma(1+m)}\left(\frac{z}{2}\right)^m$$

$$\times\, {}_4F_3\left(\begin{matrix}\frac{m}{2}+\frac{k}{2}+\frac{n}{2},\ \frac{1}{2}+\frac{m}{2}+\frac{n}{2}+\frac{k}{2},\ \frac{1}{2}+\frac{m}{2}-\frac{n}{2}+\frac{k}{2},\ \frac{m}{2}-\frac{n}{2}+\frac{k}{2};\ \frac{-z^2}{4}\\ \frac{3}{4}+\frac{k}{2}+\frac{m}{2},\ \frac{1}{4}+\frac{k}{2}+\frac{m}{2},\ 1+m\end{matrix}\right)$$

$$+\ \frac{1}{2\sqrt{2}\cdot\pi}\cdot\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{m}{2}+\frac{k}{2}-\frac{1}{4}\right)}{\Gamma\left(\frac{5}{4}-\frac{k}{2}+\frac{m}{2}\right)}\left(\frac{z^2}{4}\right)^{\frac{1}{4}-\frac{k}{2}}$$

$$\times\, {}_4F_3\left(\begin{matrix}\frac{1}{4}+\frac{n}{2},\ \frac{3}{4}+\frac{n}{2},\ \frac{1}{4}-\frac{n}{2},\ \frac{3}{4}-\frac{n}{2};\ \frac{-z^2}{4}\\ \frac{5}{4}-\frac{k}{2}-\frac{m}{2},\ \frac{1}{2},\ \frac{5}{4}-\frac{k}{2}+\frac{m}{2}\end{matrix}\right)$$

$$-\ \frac{1}{2\sqrt{2}\cdot\pi}\cdot\frac{\pi\,\Gamma\left(\frac{m}{2}+\frac{k}{2}-\frac{3}{4}\right)\Gamma\left(\frac{3}{4}+\frac{1}{2}n\right)\Gamma\left(\frac{5}{4}+\frac{1}{2}n\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{n}{2}-\frac{1}{4}\right)\Gamma\left(\frac{n}{2}+\frac{1}{4}\right)\Gamma\left(\frac{7}{4}-\frac{k}{2}+\frac{m}{2}\right)}\left(\frac{z^2}{4}\right)^{\frac{3}{4}-\frac{k}{2}}$$

$$\times\, {}_4F_3\left(\begin{matrix}\frac{3}{4}+\frac{n}{2},\ \frac{5}{4}+\frac{n}{2},\ \frac{3}{4}-\frac{n}{2},\ \frac{5}{4}-\frac{n}{2};\ \frac{-z^2}{4}\\ \frac{7}{4}-\frac{k}{2}-\frac{m}{2},\ \frac{3}{2},\ \frac{7}{4}-\frac{k}{2}+\frac{m}{2}\end{matrix}\right), \quad \ldots\ldots \quad (76)$$

(69) CHAP. I. in combination with (34) CHAP. I. gives

$$\frac{\sqrt{(2\pi z)}}{\cos(m\pi)} \int_0^\infty \exp\left\{-u + \frac{z}{u^2}\right\} u^{k-1} K_m\left(\frac{z}{u^2}\right) I_n(u)\, du =$$

$$\frac{1}{2\sqrt{2}\cdot\pi} \cdot \frac{\sin(k-n)\pi}{\sin k\pi} E\left\{ \begin{matrix} \frac{1}{2}+m, \frac{1}{2}-m, \frac{k+n+1}{2}, \frac{k+n+2}{2}, \frac{k+1-n}{2}, \frac{k+2-n}{2} : 2z \\ \frac{5}{4}+\frac{k}{2}, \frac{3}{4}+\frac{k}{2} \end{matrix} \right\}$$

$$+ \frac{1}{4\sqrt{2}\cdot\pi} \frac{\cos n\pi}{\sin\left(\frac{k}{2}+\frac{1}{4}\right)\pi} (2z)^{\frac{k}{2}+\frac{1}{4}}$$

$$\times E\left\{ \begin{matrix} \frac{1}{4}+m-\frac{k}{2}, \frac{1}{4}-m-\frac{k}{2}, \frac{1}{4}+\frac{n}{2}, \frac{3}{4}+\frac{n}{2}, \frac{1}{4}-\frac{n}{2}, \frac{3}{4}-\frac{n}{2} : 2z \\ \frac{3}{4}-\frac{k}{2}, \frac{1}{2} \end{matrix} \right\}$$

$$- \frac{1}{4\sqrt{2}\cdot\pi} \frac{\cos n\pi}{\sin\left(\frac{1}{4}-\frac{k}{2}\right)\pi} (2z)^{\frac{k}{2}-\frac{1}{4}}$$

$$\times E\left\{ \begin{matrix} \frac{3}{4}+m-\frac{k}{2}, \frac{3}{4}-m-\frac{k}{2}, \frac{3}{4}+\frac{n}{2}, \frac{5}{4}+\frac{n}{2}, \frac{3}{4}-\frac{n}{2}, \frac{5}{4}-\frac{n}{2} : 2z \\ \frac{5}{4}-\frac{k}{2}, \frac{3}{2} \end{matrix} \right\}$$

, . . . (71)

where $R(k+n+1)>0$, $R(z)>0$, $R\left(\frac{1}{2} \pm 2m - k\right)>0$.

(71) CHAP. I. in combination with (6) CHAP. II. gives

$$\int_0^\infty e^{-u} u^{k-1} I_n(u) J_m(z/u)\,du =$$

$$\frac{2^{m-k}}{\sqrt{\pi}} \cdot \frac{\Gamma(k-m+n)\,\Gamma(\tfrac{1}{2}-k+m)}{\Gamma(1+n+m-k)\,\Gamma(1+m)} \left(\frac{z^2}{4}\right)^{\frac{m}{2}}$$

$$\times {}_2F_5\left(\begin{matrix} \frac{1}{4}+\frac{m}{2}-\frac{k}{2},\ \frac{3}{4}+\frac{m}{2}-\frac{k}{2};\ -z^2/4 \\ 1+\frac{m}{2}-\frac{k}{2}-\frac{n}{2},\ \frac{1}{2}+\frac{m}{2}-\frac{k}{2}-\frac{n}{2},\ 1+m,\ 1+\frac{m}{2}+\frac{n}{2}-\frac{k}{2},\ \frac{1}{2}+\frac{m}{2}+\frac{n}{2}-\frac{k}{2} \end{matrix}\right)$$

$$+ \frac{\Gamma(\frac{m}{2}-\frac{k}{2}-\frac{n}{2})\,\Gamma(\frac{1}{4}+\frac{n}{2})\,\Gamma(\frac{3}{4}+\frac{n}{2})}{2\sqrt{2}\,\Gamma(\frac{1}{2})\,\Gamma(1+\frac{k}{2}+\frac{n}{2}+\frac{m}{2})\,\Gamma(1+n)\,\Gamma(\frac{1}{2}+n)} \left(\frac{z^2}{4}\right)^{\frac{k+n}{2}}$$

$$\times {}_2F_5\left(\begin{matrix} \frac{1}{4}+\frac{n}{2},\ \frac{3}{4}+\frac{n}{2};\ -z^2/4 \\ 1+\frac{k}{2}+\frac{n}{2}-\frac{m}{2},\ \frac{1}{2},\ 1+\frac{k}{2}+\frac{n}{2}+\frac{m}{2},\ \frac{1}{2}+n,\ 1+n \end{matrix}\right)$$

$$+ \frac{\Gamma(\frac{m}{2}-\frac{k}{2}-\frac{n}{2}-\frac{1}{2})\,\Gamma(\frac{3}{4}+\frac{n}{2})\,\Gamma(\frac{5}{4}+\frac{n}{2})}{2\sqrt{2}\,\Gamma(\frac{3}{2})\,\Gamma(\frac{3}{2}+\frac{k}{2}+\frac{n}{2}+\frac{m}{2})\,\Gamma(1+n)\,\Gamma(\frac{3}{2}+n)} \left(\frac{z^2}{4}\right)^{\frac{k+n+1}{2}}$$

$$\times {}_2F_5\left(\begin{matrix} \frac{3}{4}+\frac{n}{2},\ \frac{5}{4}+\frac{n}{2};\ -z^2/4 \\ \frac{3}{2}+\frac{k}{2}+\frac{n}{2}-\frac{m}{2},\ \frac{3}{2},\ \frac{3}{2}+\frac{k}{2}+\frac{n}{2}+\frac{m}{2},\ 1+n,\ \frac{3}{2}+n \end{matrix}\right), \quad \ldots (78)$$

where $R(\frac{3}{2}+n+k)>0$, $R(z)>0$, $R(\frac{1}{2}+m-k)>0$.

(41) CHAP. I. in combination with (34) CHAP. I. gives

$$\frac{\sqrt{(2\pi z)}}{\cos(m\pi)}\int_0^\infty \exp\{-u+zu^2\}\, u^{k-1} K_m(zu^2)\, I_n(u)\, du =$$

$$-\frac{\pi}{\sqrt{2}\cdot\sin(k+n)\pi} E\left\{\begin{matrix} \frac{1}{2}+m, \frac{1}{2}-m, \frac{3}{4}-\frac{1}{2}k, \frac{5}{4}-\frac{1}{2}k : 2z \\ \frac{3}{2}-\frac{k+n}{2}, 1-\frac{k+n}{2}, \frac{3}{2}+\frac{n-k}{2}, 1+\frac{n-k}{2} \end{matrix}\right\}$$

$$+\frac{\pi\cdot(2z)^{-\frac{k+n+1}{2}}}{2\sqrt{2}\cos\left(\frac{k+n}{2}\right)\pi} E\left\{\begin{matrix} m+\frac{k+n}{2}, -m+\frac{k+n}{2}, \frac{1}{4}+\frac{n}{2}, \frac{3}{4}+\frac{n}{2} : 2z \\ \frac{1}{2}+\frac{k+n}{2}, \frac{1}{2}, 1+n, \frac{1}{2}+n \end{matrix}\right\}$$

$$+\frac{\pi\cdot(2z)^{-\frac{k+n}{2}}}{2\sqrt{2}\sin\left(\frac{k+n}{2}\right)\pi} E\left\{\begin{matrix} m+\frac{k+n+1}{2}, -m+\frac{k+n+1}{2}, \frac{3}{4}+\frac{n}{2}, \frac{5}{4}+\frac{n}{2} : 2z \\ 1+\frac{k+n}{2}, \frac{3}{2}, \frac{3}{2}+n, 1+n \end{matrix}\right\}$$

where $R(z)>0$, $R(k\pm 2m+n)>0$, $R(\frac{3}{2}-k)>0$. , . . . (79)

In (66) CHAP. I. take $p=q=0$ and get if $R(m+n)>0$, $R(z)>0$

$$\sqrt{\pi}\cos(m\pi)\int_0^\infty \exp\{-u-\tfrac{u}{z}\}\, u^{m-1} I_n(u)\, du =$$

$$2^{-m}\sin(m-n)\pi\, E(m+n, m-n : \tfrac{1}{2}+m : 2z)$$

$$-2^{-1/2}\cos(n\pi)\, z^{m-1/2} E(\tfrac{1}{2}+n, \tfrac{1}{2}-n : \tfrac{3}{2}-m : 2z), \quad . . . (80)$$

In (68) CHAP. I. take $p = q = 0$ and get

$$\frac{\sin(n+m)\pi}{\sqrt{\pi}} 2^m \int_0^\infty \exp\left\{-u - \frac{1}{2u}\right\} u^{m-1} I_n(u)\,du =$$

$$E\left(\tfrac{1}{2}-m : 1-n-m, 1+n-m : \tfrac{-z}{2}\right)$$

$$-(2/z)^{m+n} E\left(\tfrac{1}{2}+n : 1+n+m, 1+2n : \tfrac{-z}{2}\right), \quad \ldots\ldots\ldots (81)$$

where $R(z) > 0$, $R(\tfrac{1}{2}-m) > 0$.

In (69) CHAP. I. take $p = q = 0$ and get if $R(z) > 0$, $R(m+n) > 0$

$$\int_0^\infty \exp\left\{-u - \frac{u^2}{z}\right\} u^{m-1} I_n(u)\,du =$$

$$\frac{1}{2\sqrt{2}\cdot\pi} \cdot \frac{\sin(m-n)\pi}{\cos(m\pi)} E\left\{\begin{matrix} \frac{m+n}{2}, \frac{m+n+1}{2}, \frac{m-n}{2}, \frac{m-n+1}{2} : z \\ \frac{3}{4}+\frac{m}{2}, \frac{1}{4}+\frac{m}{2} \end{matrix}\right\}$$

$$+ \frac{1}{4\sqrt{2}\cdot\pi} \cdot \frac{\cos(n\pi)}{\sin\left(\frac{m}{2}-\frac{1}{4}\right)\pi} z^{\frac{m}{2}-\frac{1}{4}} E\left\{\begin{matrix} \frac{1}{4}+\frac{n}{2}, \frac{3}{4}+\frac{n}{2}, \frac{1}{4}-\frac{n}{2}, \frac{3}{4}-\frac{n}{2} : z \\ \frac{5}{4}-\frac{m}{2}, \frac{1}{2} \end{matrix}\right\}$$

$$- \frac{1}{4\sqrt{2}\cdot\pi} \cdot \frac{\cos n\pi}{\sin\left(\frac{3}{4}-\frac{m}{2}\right)\pi} z^{\frac{m}{2}-\frac{3}{4}} E\left\{\begin{matrix} \frac{3}{4}+\frac{n}{2}, \frac{5}{4}+\frac{n}{2}, \frac{3}{4}-\frac{n}{2}, \frac{5}{4}-\frac{n}{2} : z \\ \frac{7}{4}-\frac{m}{2}, \frac{3}{2} \end{matrix}\right\}$$

$$, \ldots (82)$$

In (41) CHAP. I. take $p = q = 0$ and get if $R(z) > 0$, $R(\frac{1}{2} - m) > 0$

$$\int_0^\infty \exp\left\{-u - \frac{1}{zu^2}\right\} u^{m-1} I_n(u)\, du =$$

$$\frac{\pi}{\sqrt{2}\cdot \sin(m+n)\pi} E\left\{ \begin{matrix} \frac{1}{4} - \frac{m}{2}, \frac{3}{4} - \frac{m}{2} : z \\ 1 - \frac{m+n}{2}, \frac{1}{2} - \frac{m+n}{2}, 1 + \frac{n-m}{2}, \frac{1}{2} + \frac{n-m}{2} \end{matrix} \right\}$$

$$- \frac{\pi}{2\sqrt{2}\cdot \sin\left(\frac{m+n}{2}\right)\pi} z^{-\frac{m+n}{2}} E\left\{ \begin{matrix} \frac{1}{4} + \frac{n}{2}, \frac{3}{4} + \frac{n}{2} : z \\ 1 + \frac{m+n}{2}, \frac{1}{2}, 1+n, \frac{1}{2} + n \end{matrix} \right\}$$

$$+ \frac{\pi}{2\sqrt{2}\cdot \cos\left(\frac{m+n}{2}\right)\pi} z^{-\frac{m+n+1}{2}} E\left\{ \begin{matrix} \frac{3}{4} + \frac{n}{2}, \frac{5}{4} + \frac{n}{2} : z \\ \frac{3}{2} + \frac{m+n}{2}, \frac{3}{2}, \frac{3}{2} + n, 1+n \end{matrix} \right\}, \quad \text{---(83)}$$

From (111) CHAP. III. and (14) CHAP. I., one gets

$$\int_0^\infty e^{-\lambda} \lambda^{k-1} K_m\left(\frac{x}{\lambda} e^{i\pi}\right) K_n\left(\frac{x}{\lambda}\right) d\lambda =$$

$$\frac{2^{k-1}\cos\left(\frac{m}{2} + \frac{n}{2}\right)\pi \cos\left(\frac{n}{2} - \frac{m}{2}\right)\pi}{\pi\,(x e^{i\pi/2})} E\left(\frac{1+m+n}{2}, \frac{1+m-n}{2}, \frac{1+n-m}{2}, \frac{1-n-m}{2}, \frac{k+1}{2}, \frac{k+2}{2} ; \frac{1}{2} : e^{i\pi}\frac{x^2}{4}\right)$$

$$+ \frac{e^{\frac{i\pi}{2}} 2^{k} \cos\left(\frac{1}{2} + \frac{m}{2} + \frac{n}{2}\right)\pi \sin\left(\frac{n}{2} - \frac{m}{2}\right)\pi}{\pi x^2} E\left(\frac{2+n+m}{2}, \frac{2+n-m}{2}, \frac{2+m-n}{2}, \frac{2-n-m}{2}, \frac{k+2}{2}, \frac{k+3}{2} ; \frac{3}{2} : e^{i\pi}\frac{x^2}{4}\right)$$

where $R(k) > -1$, $R(x) > 0$. , ---(84)

In particular when $n=m$, (84) becomes if $R(x)>0$, $R(k+1)>0$

$$\int_0^\infty e^{-\lambda}\lambda^{k-1}K_n\left(\frac{x}{\lambda}e^{i\pi}\right)K_n\left(\frac{x}{\lambda}\right)d\lambda =$$

$$\frac{2^{k-1}\cos n\pi}{\pi(e^{\frac{i\pi}{2}}.x)}\, E\left(\tfrac{1}{2}+n, \tfrac{1}{2}-n, \tfrac{1}{2}, \tfrac{k+1}{2}, \tfrac{k+2}{2} :: \tfrac{x^2}{4}e^{i\pi}\right), \quad \ldots (85)$$

An alternative proof for (85) can be obtained by substituting for $K_n\left(\frac{x}{\lambda}e^{i\pi}\right)K_n\left(\frac{x}{\lambda}\right)$ from the formula [B.V., p. 383, eq. 123] namely if $R(x+y)^2 \geqslant 0$

$$K_n(x)K_n(y) = \frac{1}{2}\int_0^\infty \exp\left\{-\frac{\xi}{2}-\frac{x^2+y^2}{2\xi}\right\}K_n\left(\frac{xy}{\xi}\right)\frac{d\xi}{\xi}, \quad \ldots (86).$$

Thus the L.H.S. of (85) then becomes

$$\int_0^\infty e^{-\lambda}\lambda^{k-1}K_n\left(\frac{x}{\lambda}e^{i\pi}\right)K_n\left(\frac{x}{\lambda}\right)d\lambda =$$

$$\frac{1}{2}\int_0^\infty e^{-\lambda}\lambda^{k-1}d\lambda\int_0^\infty \exp\left\{-\frac{\xi}{2}-\frac{x^2+x^2}{2\xi\lambda^2}\right\}K_n\left(e^{i\pi}\frac{x^2}{\lambda^2\xi}\right)\frac{d\xi}{\xi}.$$

Here apply (34) CHAP. I. and change the order of integration and get

$$\int_0^\infty e^{-\lambda}\lambda^{k-1}K_n\left(\frac{x}{\lambda}e^{i\pi}\right)K_n\left(\frac{x}{\lambda}\right)d\lambda =$$

$$\frac{\cos(n\pi)}{2\sqrt{(2\pi)}\cdot(e^{\frac{i\pi}{2}}x)}\int_0^\infty e^{-\frac{\xi}{2}}\xi^{-\frac{1}{2}}d\xi\int_0^\infty e^{-\lambda}\lambda^{k}E\left(\tfrac{1}{2}+n, \tfrac{1}{2}-n :: \frac{2e^{i\pi}x^2}{\lambda^2\xi}\right)d\lambda$$

$$= \frac{2^{k-1}\cos(n\pi)}{\sqrt{(2\pi)}\, e^{\frac{i\pi}{2}}x\sqrt{\pi}}\int_0^\infty e^{-\frac{\xi}{2}}\xi^{-\frac{1}{2}}E\left(\tfrac{1}{2}+n, \tfrac{1}{2}-n, \tfrac{k+1}{2}, \tfrac{k+2}{2} :: \frac{e^{i\pi}x^2}{2\xi}\right)d\xi$$

$$= \frac{2^{k-1}\cos(n\pi)}{e^{\frac{i\pi}{2}}x\pi}E\left(\tfrac{1}{2}, \tfrac{1}{2}+n, \tfrac{1}{2}-n, \tfrac{k+1}{2}, \tfrac{k+2}{2} :: \frac{e^{i\pi}x^2}{4}\right),$$

by (14) CHAP. I. Thus (86) is proved.

From (15) CHAP. III. and (14) CHAP. I., one gets

$$\int_0^\infty e^{-\lambda}\lambda^{k-1}K_m\left(\frac{x}{\lambda}\right)K_n\left(\frac{x}{\lambda}\right)d\lambda =$$

$$\frac{2^{k-2}}{\pi x}\sum_{i,-i}\frac{1}{i}E\left(\frac{1+m+n}{2}, \frac{1+m-n}{2}, \frac{1+n-m}{2}, \frac{1-n-m}{2}, \frac{k+1}{2}, \frac{k+2}{2} : \frac{1}{2} : e^{i\pi}\frac{x^2}{4}\right), \quad \text{---}(87),$$

where $R(x) > 0$.

From (11) CHAP. III. and (16) CHAP. I. one gets

$$\int_0^\infty e^{-\lambda}\lambda^{k-1}K_m(x\lambda e^{i\pi})K_n(x\lambda)\,d\lambda =$$

$$-\frac{2^{k-2}\pi\cos(\frac{m}{2}+\frac{n}{2})\pi\cos(\frac{n}{2}-\frac{m}{2})\pi}{e^{i\pi/2}\sin(k\pi)x}E\left(\frac{1+m+n}{2},\frac{1+m-n}{2},\frac{1+n-m}{2},\frac{1-n-m}{2}:\frac{3}{2}-\frac{k}{2},1-\frac{k}{2},\frac{1}{2}:-4x^2\right)$$

$$+\frac{\pi\cos(\frac{m}{2}+\frac{n}{2})\pi\cos(\frac{n}{2}-\frac{m}{2})\pi}{4e^{i\pi/2}\cos(\frac{k\pi}{2})x}(e^{i\pi}x^2)^{-\frac{k-1}{2}}E\left(\frac{n+m+k}{2},\frac{m-n+k}{2},\frac{n-m+k}{2},\frac{-n-m+k}{2}:\frac{k}{2}+\frac{1}{2},\frac{k}{2},\frac{1}{2}:-4x^2\right)$$

$$+\frac{\pi\cos(\frac{m}{2}+\frac{n}{2})\pi\cos(\frac{n}{2}-\frac{m}{2})\pi}{8e^{i\pi/2}\sin(\frac{k\pi}{2})x}(e^{i\pi}x^2)^{-\frac{k}{2}}E\left(\frac{n+m+k+1}{2},\frac{n-m+k+1}{2},\frac{m-n+k+1}{2},\frac{-n-m+k+1}{2}:\frac{3}{2},\frac{k+1}{2},\frac{k+2}{2}:-4x^2\right)$$

$$+\frac{e^{\frac{i\pi}{2}}\cos(\frac{1}{2}+\frac{m}{2}+\frac{n}{2})\pi\sin(\frac{n}{2}-\frac{m}{2})\pi}{\sqrt{\pi}(2x^2)}$$

$$\Bigg[\frac{2^{k-2}\pi^{3/2}}{\sin(k\pi)}E\left(1+\frac{m}{2}+\frac{n}{2},1+\frac{m}{2}-\frac{n}{2},1+\frac{n}{2}-\frac{m}{2},1-\frac{n}{2}-\frac{m}{2}:\frac{3}{2},\frac{3}{2}-\frac{k}{2},2-\frac{k}{2}:-4x^2\right)$$

$$+\frac{\pi^{3/2}}{2\sin(k\pi/2)}(x^2e^{i\pi})^{-\frac{k-2}{2}}E\left(\frac{n+m+k}{2},\frac{n-m+k}{2},\frac{m-n+k}{2},\frac{-m-n+k}{2}:\frac{1}{2},\frac{k}{2},\frac{k+1}{2}:-4x^2\right)$$

$$-\frac{\pi^{3/2}}{4\cos(k\pi/2)}(x^2e^{i\pi})^{-\frac{k-1}{2}}E\left(\frac{1+n+m+k}{2},\frac{1+n-m+k}{2},\frac{1+m-n+k}{2},\frac{1-n-m+k}{2}:\frac{3}{2},\frac{k+1}{2},\frac{k+2}{2}:-4x^2\right)\Bigg]$$

where $R(x)>0$, $R(k\pm m\pm n)>0$

, ... (88)

In particular when $n=m$, the last formula becomes

$$\int_0^\infty e^{-\lambda}\lambda^{k-1}K_n(x\lambda e^{i\pi})K_n(x\lambda)\,d\lambda =$$

$$-\frac{2^{k-2}\pi\cos n\pi}{e^{\frac{i\pi}{2}}\sin(k\pi)\cdot x}E\left(\tfrac{1}{2},\tfrac{1}{2}+n,\tfrac{1}{2}-n:1-\tfrac{1}{2}k,\tfrac{3}{2}-\tfrac{1}{2}k:-4x^2\right)$$

$$+\frac{\pi\cos n\pi}{4e^{i\pi/2}x\cos\left(\frac{k\pi}{2}\right)}\left(e^{i\pi}x^2\right)^{-\frac{k-1}{2}}E\left(n+\frac{k}{2},-n+\frac{k}{2},\frac{k}{2}:\frac{1}{2},\frac{k}{2}+\frac{1}{2}:-4x^2\right)$$

$$+\frac{\pi\cos(n\pi)}{8e^{i\pi/2}\sin\left(\frac{k\pi}{2}\right)}\left(e^{i\pi}x^2\right)^{-\frac{k}{2}}E\left(n+\frac{k+1}{2},-n+\frac{k+1}{2},\frac{k+1}{2}:\frac{3}{2},1+\frac{k}{2}:-4x^2\right), \quad \cdots (89)$$

where $R(x)>0$, $R(k)>0$, $R(k\pm 2n)>0$.

From (15) CHAP. III. and (16) CHAP. I., one gets if $R(x)>0$, $R(k\pm m\pm n)>0$

$$\int_0^\infty e^{-\lambda}\lambda^{k-1}K_m(x\lambda)K_n(x\lambda)\,d\lambda =$$

$$\frac{\sqrt{\pi}\,\Gamma\left(\frac{m}{2}+\frac{n}{2}+\frac{k}{2}\right)\Gamma\left(\frac{m}{2}-\frac{n}{2}+\frac{k}{2}\right)\Gamma\left(\frac{n}{2}-\frac{m}{2}+\frac{k}{2}\right)\Gamma\left(-\frac{n}{2}-\frac{m}{2}+\frac{k}{2}\right)}{4\Gamma\left(\frac{1}{2}+\frac{k}{2}\right)\Gamma\left(\frac{k}{2}\right)}x^{-k}$$

$$\times {}_4F_3\left(\frac{m}{2}+\frac{n}{2}+\frac{k}{2},\frac{m}{2}-\frac{n}{2}+\frac{k}{2},\frac{n}{2}-\frac{m}{2}+\frac{k}{2},-\frac{n}{2}-\frac{m}{2}+\frac{k}{2};\frac{1}{2},\frac{k}{2},\frac{k+1}{2};1/4x^2\right)$$

$$-\frac{\pi\,\Gamma\left(\frac{m}{2}+\frac{n}{2}+\frac{k}{2}+\frac{1}{2}\right)\Gamma\left(\frac{m}{2}-\frac{n}{2}+\frac{k}{2}+\frac{1}{2}\right)\Gamma\left(\frac{n}{2}-\frac{m}{2}+\frac{k}{2}+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}-\frac{n}{2}-\frac{m}{2}+\frac{k}{2}\right)}{8\Gamma\left(\frac{3}{2}\right)\Gamma\left(1+\frac{1}{2}k\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}k\right)}\cdot x^{-k-1}$$

$$\times {}_4F_3\left(\frac{1}{2}+\frac{m}{2}+\frac{n}{2}+\frac{k}{2},\frac{1}{2}+\frac{m}{2}-\frac{n}{2}+\frac{k}{2},\frac{1}{2}+\frac{n}{2}-\frac{m}{2}+\frac{k}{2},\frac{1}{2}-\frac{n}{2}-\frac{m}{2}+\frac{k}{2};\frac{3}{2},1+\frac{k}{2},\frac{1}{2}+\frac{k}{2};\frac{1}{4x^2}\right), \quad \cdots (90)$$

In Particular when $n=m$, the last formula becomes

$$\int_0^\infty e^{-\lambda}\lambda^{k-1}\{K_n(x\lambda)\}^2 d\lambda =$$

$$\frac{\sqrt{\pi}\,\Gamma(n+\frac{1}{2}k)\Gamma(-n+\frac{1}{2}k)\Gamma(\frac{1}{2}k)}{4\Gamma(\frac{1}{2}+\frac{1}{2}k)}x^{-k}{}_3F_2\left(n+\tfrac{1}{2}k, -n+\tfrac{1}{2}k, \tfrac{1}{2}k; \tfrac{1}{2}, \tfrac{1}{2}+\tfrac{1}{2}k; \frac{1}{4x^2}\right)$$

$$-\frac{\pi\Gamma(n+\frac{k+1}{2})\Gamma(-n+\frac{k+1}{2})\Gamma(\frac{k+1}{2})}{8\Gamma(\frac{3}{2})\Gamma(1+\frac{1}{2}k)}x^{-k-1}{}_3F_2\left(\begin{matrix}n+\frac{k+1}{2}, -n+\frac{k+1}{2}, \frac{k+1}{2}; \frac{1}{4x^2}\\ \frac{3}{2}, 1+\frac{k}{2}\end{matrix}\right), \quad \cdots(91)$$

where $R(k)>0$, $R(k\pm 2n)>0$, $R(x)>0$.

§9. Integrals involving the product of three Bessel Functions:

From (11) CHAP. III and (43) CHAP. I. with $m=1$, one gets

$$\int_0^\infty \lambda^{k-1}K_l(2\lambda)K_m\left(\frac{x}{\lambda}e^{i\pi}\right)K_n\left(\frac{x}{\lambda}\right)d\lambda =$$

$$\frac{\cos(\frac{m}{2}+\frac{n}{2})\pi\cos(\frac{n}{2}-\frac{m}{2})\pi}{8e^{i\pi/2}\sqrt{\pi}\cdot x}E\left(\frac{1+m+n}{2}, \frac{1+m-n}{2}, \frac{1+n-m}{2}, \frac{1-n-m}{2}, \frac{1+k+l}{2}, \frac{1+k-l}{2} : \frac{1}{2} : x^2e^{i\pi}\right)$$

$$+\frac{e^{\frac{i\pi}{2}}\cos(\frac{1}{2}+\frac{m}{2}+\frac{n}{2})\pi\sin(\frac{n}{2}-\frac{m}{2})\pi}{8\sqrt{\pi}\cdot x^2}E\left(\frac{2+m+n}{2}, \frac{2+m-n}{2}, \frac{2+n-m}{2}, \frac{2-n-m}{2}, \frac{2+k+l}{2}, \frac{2+k-l}{2} : \frac{3}{2} : x^2e^{i\pi}\right)$$

where $R(x)>0$, $R(k\pm l+1)>0$. $\quad \cdots(92)$

When $n=m$ (92) gives if $R(x)>0$, $R(k\pm l+1)>0$

$$\int_0^\infty \lambda^{k-1} K_l(2\lambda) K_n\left(\frac{x}{\lambda}e^{i\pi}\right) K_n\left(\frac{x}{\lambda}\right) d\lambda =$$

$$\frac{\cos n\pi}{8e^{\frac{i\pi}{2}}\sqrt{(\pi)}\cdot x} E\left(\tfrac{1}{2}+n, \tfrac{1}{2}-n, \tfrac{1}{2}, \tfrac{1}{2}+\tfrac{1}{2}k+\tfrac{1}{2}l, \tfrac{1}{2}+\tfrac{1}{2}k-\tfrac{1}{2}l :: x^2 e^{i\pi}\right), \quad \ldots (93)$$

From (15) CHAP. III. and (43) CHAP. I. with $m=1$, one gets

$$\int_0^\infty \lambda^{k-1} K_l(2\lambda) K_m\left(\frac{x}{\lambda}\right) K_n\left(\frac{x}{\lambda}\right) d\lambda =$$

$$\frac{1}{16\sqrt{(\pi)}\cdot x} \sum_{i,-i} \frac{1}{i} E\left(\frac{m+n+1}{2}, \frac{m-n+1}{2}, \frac{n-m+1}{2}, \frac{-n-m+1}{2}, \frac{1+k+l}{2}, \frac{1+k-l}{2} : \frac{1}{2} : e^{i\pi}x^2\right), \quad \ldots (94)$$

where $R(x)>0$ and the necessary restrictions for (43) CHAP. I. can be removed by analytical continuation.

When $n=m$ (94) gives if $R(x)>0$

$$\int_0^\infty \lambda^{k-1} K_l(2\lambda) \left\{K_n\left(\frac{x}{\lambda}\right)\right\}^2 d\lambda =$$

$$\frac{1}{16\sqrt{(\pi)}\cdot x} \sum_{i,-i} \frac{1}{i} E\left(\tfrac{1}{2}+n, \tfrac{1}{2}-n, \tfrac{1}{2}, \tfrac{1}{2}+\frac{k}{2}+\frac{l}{2}, \tfrac{1}{2}+\frac{k}{2}-\frac{l}{2} :: e^{i\pi}x^2\right), \quad \ldots (95)$$

From (11) CHAP. III. and the formula [Ragab F.M., Proc. Glasg. Math. Assoc. Vol. I. p. 8 equ. (6)] namely

$$4i\pi\int_0^\infty \lambda^{m-1} J_n(2\lambda) E(p;\alpha_r:q;\rho_s: x/\lambda^2)\, d\lambda =$$

$$i^{m-n} E(p+2;\alpha_r:q;\rho_s: x e^{-i\pi}) - i^{n-m} E(p+2;\alpha_r:q;\rho_s: x e^{i\pi}), \quad \ldots (96)$$

where $R(m+n)>0$, $R(\frac{3}{2}-m+2\alpha_r)>0$, $r=1,2,3,\ldots,p$, $\alpha_{p+1}=\frac{1}{2}(m+n)$, $\alpha_{p+2}=\frac{1}{2}(m-n)$; one gets

$$\int_0^\infty \lambda^{k-1} J_l(2\lambda) K_m\left(\frac{x}{\lambda}e^{i\pi}\right) K_n\left(\frac{x}{\lambda}\right) d\lambda =$$

$$-\frac{e^{\frac{i\pi}{2}(k+1-l)}\cos\left(\frac{m}{2}+\frac{n}{2}\right)\pi\cos\left(\frac{n}{2}-\frac{m}{2}\right)\pi}{8\,\pi^{3/2}\cdot x}$$

$$\times E\left(\tfrac{1}{2}+\tfrac{m}{2}+\tfrac{n}{2}, \tfrac{1}{2}+\tfrac{m}{2}-\tfrac{n}{2}, \tfrac{1}{2}+\tfrac{n}{2}-\tfrac{m}{2}, \tfrac{1}{2}-\tfrac{n}{2}-\tfrac{m}{2}, \tfrac{1}{2}+\tfrac{k}{2}+\tfrac{l}{2}, \tfrac{1}{2}+\tfrac{k}{2}-\tfrac{l}{2} : \tfrac{1}{2} : x^2\right)$$

$$+\frac{e^{\frac{i\pi}{2}(l-k-1)}\cos\left(\frac{m}{2}+\frac{n}{2}\right)\pi\cos\left(\frac{n}{2}-\frac{m}{2}\right)\pi}{8\,\pi^{3/2}\cdot x}$$

$$\times E\left(\tfrac{1}{2}+\tfrac{m}{2}+\tfrac{n}{2}, \tfrac{1}{2}+\tfrac{m}{2}-\tfrac{n}{2}, \tfrac{1}{2}+\tfrac{n}{2}-\tfrac{m}{2}, \tfrac{1}{2}-\tfrac{n}{2}-\tfrac{m}{2}, \tfrac{1}{2}+\tfrac{k}{2}+\tfrac{l}{2}, \tfrac{1}{2}+\tfrac{k}{2}-\tfrac{l}{2} : \tfrac{1}{2} : e^{2\pi i} x^2\right)$$

$$+\frac{e^{+\frac{i\pi}{2}(k+2-l)}\cos\left(\frac{l}{2}+\frac{m}{2}+\frac{n}{2}\right)\pi\sin\left(\frac{n}{2}-\frac{m}{2}\right)\pi}{8\pi^{3/2}x^2}$$

$$\times E\left(1+\frac{m}{2}+\frac{n}{2},1+\frac{m}{2}-\frac{n}{2},1+\frac{n}{2}-\frac{m}{2},1-\frac{n}{2}-\frac{m}{2},1+\frac{k}{2}+\frac{l}{2},1+\frac{k}{2}-\frac{l}{2}:\frac{3}{2}:x^2\right)$$

$$-\frac{e^{+\frac{i\pi}{2}(l-2-k)}\cos\left(\frac{l}{2}+\frac{m}{2}+\frac{n}{2}\right)\pi\sin\left(\frac{n}{2}-\frac{m}{2}\right)\pi}{8\pi^{3/2}\cdot x^2}$$

$$\times E\left(1+\frac{m}{2}+\frac{n}{2},1+\frac{m}{2}-\frac{n}{2},1+\frac{n}{2}-\frac{m}{2},1-\frac{n}{2}-\frac{m}{2},1+\frac{k}{2}+\frac{l}{2},1+\frac{k}{2}-\frac{l}{2}:\frac{3}{2}:e^{2\pi i}x^2\right)$$

where $R(x)>0$, $R(k\pm l+1)>0$, $R(\frac{3}{2}-k\pm m\pm n)>0$. , ...(97).

In particular when $n=m$, the last formula becomes

$$\int_0^\infty \lambda^{k-1} J_l(2\lambda) K_n\left(\frac{x}{\lambda}e^{i\pi}\right) K_n\left(\frac{x}{\lambda}\right) d\lambda =$$

$$-\frac{e^{\frac{i\pi}{2}(k+1-l)}\cos(n\pi)}{8\pi^{3/2}x} E\left(\frac{1}{2},\frac{1}{2}+n,\frac{1}{2}-n,\frac{1}{2}+\frac{k}{2}+\frac{l}{2},\frac{1}{2}+\frac{k}{2}-\frac{l}{2}::x^2\right)$$

$$+\frac{e^{\frac{i\pi}{2}(l-k-1)}\cos n\pi}{8\pi^{3/2}\cdot x} E\left(\frac{1}{2},\frac{1}{2}+n,\frac{1}{2}-n,\frac{1}{2}+\frac{k}{2}+\frac{l}{2},\frac{1}{2}+\frac{k}{2}-\frac{l}{2}::e^{2\pi i}x^2\right)$$

, . . . (98)

where $R(x)>0$, $R(k+l+1)>0$, $R(\frac{3}{2}-k)>0$, $R(\frac{3}{2}-k\pm 2n)>0$.

From (13) CHAP. III. and (96) one gets

$$\int_0^\infty \lambda^{k-1} J_l(2\lambda) K_m\left(\frac{x}{\lambda}\right) K_n\left(\frac{x}{\lambda}\right) d\lambda =$$

$$\frac{e^{\frac{i\pi}{2}(k-l-1)}}{16x\pi^{3/2}}\left[\begin{array}{l} E\left(\frac{1}{2}+\frac{m}{2}+\frac{n}{2}, \frac{1}{2}+\frac{m}{2}-\frac{n}{2}, \frac{1}{2}+\frac{n}{2}-\frac{m}{2}, \frac{1}{2}-\frac{n}{2}-\frac{m}{2}, \frac{1}{2}+\frac{k}{2}+\frac{l}{2}, \frac{1}{2}+\frac{k}{2}-\frac{l}{2} : \frac{1}{2} : x^2\right) \\ -E\left(\frac{1}{2}+\frac{m}{2}+\frac{n}{2}, \frac{1}{2}+\frac{m}{2}-\frac{n}{2}, \frac{1}{2}+\frac{n}{2}-\frac{m}{2}, \frac{1}{2}-\frac{n}{2}-\frac{m}{2}, \frac{1}{2}+\frac{k}{2}+\frac{l}{2}, \frac{1}{2}+\frac{k}{2}-\frac{l}{2} : \frac{1}{2} : e^{-2\pi i}x^2\right)\end{array}\right]$$

$$-\frac{e^{\frac{i\pi}{2}(l-k-3)}}{16x\pi^{3/2}}\left[\begin{array}{l} E\left(\frac{1}{2}+\frac{m}{2}+\frac{n}{2}, \frac{1}{2}+\frac{m}{2}-\frac{n}{2}, \frac{1}{2}+\frac{n}{2}-\frac{m}{2}, \frac{1}{2}-\frac{n}{2}-\frac{m}{2}, \frac{1}{2}+\frac{k}{2}+\frac{l}{2}, \frac{1}{2}+\frac{k}{2}-\frac{l}{2} : \frac{1}{2} : e^{2i\pi}x^2\right) \\ -E\left(\frac{1}{2}+\frac{m}{2}+\frac{n}{2}, \frac{1}{2}+\frac{m}{2}-\frac{n}{2}, \frac{1}{2}+\frac{n}{2}-\frac{m}{2}, \frac{1}{2}-\frac{n}{2}-\frac{m}{2}, \frac{1}{2}+\frac{k}{2}+\frac{l}{2}, \frac{1}{2}+\frac{k}{2}-\frac{l}{2} : \frac{1}{2} : x^2\right)\end{array}\right]$$

where $R(x)>0$, $R\left(\frac{3}{2}-k\pm m\pm n\right)>0$. ... (99)

When $n=m$, the last formula becomes

$$\int_0^\infty \lambda^{k-1} J_l(2\lambda)\left\{K_n\left(\frac{x}{\lambda}\right)\right\}^2 d\lambda =$$

$$\frac{e^{\frac{i\pi}{2}(l-k-1)}}{16x\pi^{3/2}}\left[\begin{array}{l} E\left(\frac{1}{2}, \frac{1}{2}+n, \frac{1}{2}-n, \frac{1}{2}+\frac{1}{2}k+\frac{1}{2}l, \frac{1}{2}+\frac{1}{2}k-\frac{1}{2}l :: x^2\right) \\ -E\left(\frac{1}{2}, \frac{1}{2}+n, \frac{1}{2}-n, \frac{1}{2}+\frac{1}{2}k+\frac{1}{2}l, \frac{1}{2}+\frac{1}{2}k-\frac{1}{2}l :: e^{-2\pi i}x^2\right)\end{array}\right]$$

$$-\frac{e^{\frac{i\pi}{2}(l-k-3)}}{16x\pi^{3/2}}\left[\begin{array}{l} E\left(\frac{1}{2}, \frac{1}{2}+n, \frac{1}{2}-n, \frac{1}{2}+\frac{1}{2}k+\frac{1}{2}l, \frac{1}{2}+\frac{1}{2}k-\frac{1}{2}l :: e^{2i\pi}x^2\right) \\ -E\left(\frac{1}{2}, \frac{1}{2}+n, \frac{1}{2}-n, \frac{1}{2}+\frac{1}{2}k+\frac{1}{2}l, \frac{1}{2}+\frac{1}{2}k-\frac{1}{2}l :: x^2\right) \end{array}\right], \cdots (100)$$

where $R(x)>0$, $R(\frac{3}{2}-k)>0$, $R(\frac{3}{2}-k\pm 2n)>0$.

From (11) CHAP. III. and (42) CHAP. I., one gets

$$\int_0^\infty \lambda^{k-1} K_l(\lambda) K_m(x\lambda e^{i\pi}) K_n(x\lambda)\, d\lambda =$$

$$\frac{\cos\left(\frac{n}{2}+\frac{m}{2}\right)\pi \ \cos\left(\frac{n}{2}-\frac{m}{2}\right)\pi}{e^{i\pi/2}\cdot(2x)\sqrt{\pi}}$$

$$\times\left[\begin{array}{l} \frac{2^{k-3}\ \pi^2}{\cos\left(\frac{k+l}{2}\right)\pi\ \cos\left(\frac{k-l}{2}\right)\pi} E\left\{\begin{array}{l} \frac{1}{2}+\frac{m}{2}+\frac{n}{2}, \frac{1}{2}+\frac{m}{2}-\frac{n}{2}, \frac{1}{2}+\frac{n}{2}-\frac{m}{2}, \frac{1}{2}-\frac{n}{2}-\frac{m}{2} : -4x^2 \\ \frac{3}{2}-\frac{k+l}{2}, \frac{3}{2}-\frac{k-l}{2}, \frac{1}{2} \end{array}\right\} \\ +\sum_{l,-l} \frac{2^{-l-2}\ \pi^2}{\sin\left(\frac{k+l-1}{2}\right)\pi\ \sin(l\pi)} \left(x^2 e^{i\pi}\right)^{-\left(\frac{k+l-1}{2}\right)} \\ \times E\left\{\begin{array}{l} \frac{l+k+m+n}{2}, \frac{l+k+m-n}{2}, \frac{l+k+n-m}{2}, \frac{l+k-n-m}{2} : -4x^2 \\ \frac{1}{2}+\frac{1}{2}k+\frac{1}{2}l, 1+l, \frac{1}{2}k+\frac{1}{2}l \end{array}\right\} \end{array}\right]$$

$$+\frac{e^{\frac{i\pi}{2}}\cos\left(\frac{1}{2}+\frac{m}{2}+\frac{n}{2}\right)\pi\,\sin\left(\frac{n}{2}-\frac{m}{2}\right)\pi}{(2x^2)\sqrt{\pi}}$$

$$\times\Bigg[\frac{2^{k-4}\,\pi^2}{\sin\left(\frac{k+l}{2}\right)\pi\,\sin\left(\frac{k-l}{2}\right)\pi}E\left\{\begin{matrix}\frac{2+n+m}{2},\frac{2+m-n}{2},\frac{2+n-m}{2},\frac{2-n-m}{2}:-4x^2\\ 2-\frac{k+l}{2},2-\frac{k-l}{2},\frac{3}{2}\end{matrix}\right\}$$

$$+\sum_{l,-l}\frac{2^{-l-2}\,\pi^2}{\sin\left(\frac{k+l-2}{2}\right)\pi\,\sin(l\pi)}\left(e^{i\pi}x^2\right)^{-\left(\frac{k+l-2}{2}\right)}$$

$$\times E\left\{\begin{matrix}\frac{l+k+m+n}{2},\frac{l+k+m-n}{2},\frac{l+k+n-m}{2},\frac{l+k-n-m}{2}:-4x^2\\ \frac{1}{2}k+\frac{1}{2}l,1+l,\frac{1}{2}+\frac{1}{2}k+\frac{1}{2}l\end{matrix}\right\}\Bigg]$$

where $R(x)>0, R(k\pm l\pm m\pm n)>0$. ,...(101)

In particular when $n=m$ the last formula gives if $R(k\pm l)>0, R(x)>0$ $, R(k\pm l\pm 2n)>0$

$$\int_0^\infty \lambda^{k-1}K_l(\lambda)K_n(x\lambda e^{i\pi})K_n(x\lambda)\,d\lambda=$$

$$\frac{\cos(n\pi)\cdot 2^{k-3}\cdot\pi^2}{(2x)\sqrt{\pi}\,e^{i\pi/2}\cos\left(\frac{k+l}{2}\right)\pi\cos\left(\frac{k-l}{2}\right)\pi}E\left(\frac{1}{2},\frac{1}{2}+n,\frac{1}{2}-n:\frac{3}{2}-\frac{k+l}{2},\frac{3}{2}-\frac{k-l}{2}:-4x^2\right)$$

$$+\frac{\cos n\pi}{(2x)\sqrt{\pi}\,e^{i\pi/2}}\sum_{l,-l}\frac{2^{-l-2}\,\pi^2}{\sin\left(\frac{k+l-1}{2}\right)\pi\sin l\pi}\left(x^2e^{i\pi}\right)^{-\left(\frac{k+l-1}{2}\right)}E\left\{\begin{matrix}\frac{k+l}{2}+n,\frac{k+l}{2}-n,\frac{k+l}{2}:-4x^2\\ \frac{1}{2}+\frac{1}{2}l+\frac{1}{2}k,1+l\end{matrix}\right\}$$

,...(102)

From (115) CHAP. III. and (42) CHAP. I., one gets

$$\int_0^\infty \lambda^{k-1} K_l(\lambda) K_m(x\lambda) K_n(x\lambda)\, d\lambda =$$

$$-\sum_{l,-l}\left[\frac{2^{-l-3}\pi^{\frac{3}{2}}\Gamma\left(\frac{k+l+m+n}{2}\right)\Gamma\left(\frac{k+l+m-n}{2}\right)\Gamma\left(\frac{k+l+n-m}{2}\right)\Gamma\left(\frac{k+l-m-n}{2}\right)}{\sin(l\pi)\,\Gamma(\frac{1}{2}+\frac{1}{2}k+\frac{1}{2}l)\,\Gamma(1+l)\,\Gamma(\frac{1}{2}k+\frac{1}{2}l)\, x} x^{-(k+l-1)} \times {}_4F_3\left(\begin{matrix}\frac{k+l+m+n}{2}, \frac{k+l+m-n}{2}, \frac{k+l+n-m}{2}, \frac{k+l-n-m}{2}; \frac{1}{4x^2} \\ \frac{1}{2}+\frac{1}{2}k+\frac{1}{2}l, 1+l, \frac{1}{2}k+\frac{1}{2}l\end{matrix}\right)\right], \cdots (103)$$

where $R(x)>0$, $R(k\pm l\pm m\pm n)>0$.

In particular when $n=m$, the last formula gives

$$\int_0^\infty \lambda^{k-1} K_l(\lambda) \{K_n(x\lambda)\}^2 d\lambda =$$

$$-\sum_{l,-l}\left[\frac{2^{-l-3}\pi^{3/2}\Gamma(\frac{1}{2}k+\frac{1}{2}l+n)\Gamma(\frac{1}{2}k+\frac{1}{2}l-n)\Gamma(\frac{1}{2}k+\frac{1}{2}l)}{\sin(l\pi)\Gamma(\frac{1}{2}+\frac{1}{2}k+\frac{1}{2}l)\Gamma(1+l)\, x} x^{-(k+l-1)} \times {}_3F_2\left(\frac{1}{2}k+\frac{l}{2}+n, \frac{k}{2}+\frac{l}{2}-n, \frac{k}{2}+\frac{l}{2}; \frac{1}{2}+\frac{k}{2}+\frac{l}{2}, 1+l; 1/4x^2\right)\right], \cdots (104)$$

where $R(x)>0$, $R(k+l)>0$, $R(k+l\pm 2n)>0$.

(11) CHAP. III. and (44) CHAP. I. give

$$\int_0^\infty \lambda^{k-1} J_l(\lambda) K_m(x\lambda e^{i\pi}) K_n(x\lambda)\, d\lambda =$$

$$-\frac{2^{k-3}\cos\left(\frac{m}{2}+\frac{n}{2}\right)\pi\cos\left(\frac{n}{2}-\frac{m}{2}\right)\pi\,\Gamma\left(\frac{1+m+n}{2}\right)\Gamma\left(\frac{1+m-n}{2}\right)\Gamma\left(\frac{1+n-m}{2}\right)\Gamma\left(\frac{1-n-m}{2}\right)}{e^{i\pi/2}\cos\left(\frac{k+l}{2}\right)\pi\,\Gamma\left(\frac{3}{2}-\frac{k+l}{2}\right)\Gamma\left(\frac{3}{2}-\frac{k-l}{2}\right)x}$$

$$\times {}_4F_3\left(\frac{1+m+n}{2}, \frac{1+m-n}{2}, \frac{1+n-m}{2}, \frac{1-n-m}{2}; \frac{3}{2}-\frac{k+l}{2}, \frac{3}{2}-\frac{k-l}{2}, \frac{1}{2}; -1/4x^2\right)$$

$$+\frac{\sqrt{\pi}\,2^{-l-2}\cos\left(\frac{m}{2}+\frac{n}{2}\right)\pi\cos\left(\frac{n}{2}-\frac{m}{2}\right)\pi\,\Gamma\left(\frac{k+l+m+n}{2}\right)\Gamma\left(\frac{k+l+m-n}{2}\right)\Gamma\left(\frac{k+l+n-m}{2}\right)\Gamma\left(\frac{k+l-m-n}{2}\right)}{e^{\frac{i\pi}{2}(k+l)}\cos\left(\frac{k+l}{2}\right)\pi\,\Gamma\left(\frac{1}{2}+\frac{1}{2}k+\frac{1}{2}l\right)\Gamma(1+l)\Gamma\left(\frac{1}{2}k+\frac{1}{2}l\right)x^{k+l}}$$

$$\times {}_4F_3\left(\frac{k+l+m+n}{2}, \frac{k+l+m-n}{2}, \frac{k+l+n-m}{2}, \frac{k+l-m-n}{2}; \frac{1+k+l}{2}, \frac{k+l}{2}, 1+l; -1/4x^2\right)$$

$$+\frac{e^{i\pi/2}\,2^{k-4}\sqrt{\pi}\sin\left(\frac{m}{2}+\frac{n}{2}\right)\pi\sin\left(\frac{n}{2}-\frac{m}{2}\right)\pi\,\Gamma\left(\frac{2+m+n}{2}\right)\Gamma\left(\frac{2+m-n}{2}\right)\Gamma\left(\frac{2+n-m}{2}\right)\Gamma\left(\frac{2-n-m}{2}\right)}{\sin\left(\frac{k+l}{2}\right)\pi\,\Gamma\left(\frac{3}{2}\right)\Gamma\left(2-\frac{1}{2}k-\frac{1}{2}l\right)\Gamma\left(2-\frac{1}{2}k+\frac{1}{2}l\right)x^2}$$

$$\times {}_4F_3\left(\frac{2+m+n}{2}, \frac{2+m-n}{2}, \frac{2+n-m}{2}, \frac{2-n-m}{2}; \frac{3}{2}, \frac{4-k-l}{2}, \frac{4-k+l}{2}; -1/4x^2\right)$$

$$-\frac{e^{\frac{i\pi}{2}}\,2^{-l-2}\sqrt{\pi}\sin\left(\frac{m}{2}+\frac{n}{2}\right)\pi\sin\left(\frac{n}{2}-\frac{m}{2}\right)\pi\,\Gamma\left(\frac{k+l+m+n}{2}\right)\Gamma\left(\frac{k+l+m-n}{2}\right)\Gamma\left(\frac{k+l+n-m}{2}\right)\Gamma\left(\frac{k+l-m-n}{2}\right)}{\sin\left(\frac{k+l}{2}\right)\pi\,\Gamma\left(\frac{1}{2}k+\frac{1}{2}l\right)\Gamma(1+l)\Gamma\left(\frac{1}{2}+\frac{1}{2}k+\frac{1}{2}l\right)e^{\frac{i\pi}{2}(k+l-2)}x^{k+l}}$$

$$ {}_4F_3\left(\frac{k+l+m+n}{2}; \frac{k+l+m-n}{2}, \frac{k+l+n-m}{2}, \frac{k+l-m-n}{2}; \frac{k+l+1}{2}, \frac{k+l}{2}, 1+l; -1/4x^2\right), \quad \ldots (105)$$

where $R(k+l\pm m\pm n)>0$, $R(x)>0$, $R\left(\frac{5}{2}-k\right)>0$.

When $n=m$, the last formula becomes

$$\int_0^\infty \lambda^{k-1} J_l(\lambda) K_n(x\lambda e^{i\pi}) K_n(x\lambda)\, d\lambda =$$

$$-\frac{2^{k-3}\cos(n\pi)\,\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+n)\Gamma(\frac{1}{2}-n)}{e^{\frac{i\pi}{2}}\cos(\frac{k+l}{2})\pi\,\Gamma(\frac{3}{2}-\frac{k+l}{2})\Gamma(\frac{3}{2}-\frac{k-l}{2})}\left(\frac{1}{x}\right){}_3F_2\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}+n, \frac{1}{2}-n; -1/4x^2\\ \frac{3}{2}-\frac{k+l}{2}, \frac{3}{2}-\frac{k-l}{2}\end{matrix}\right)$$

$$+\frac{2^{-l-2}\sqrt{\pi}\cos(n\pi)\,\Gamma(\frac{k+l}{2})\Gamma(n+\frac{k+l}{2})\Gamma(-n+\frac{k+l}{2})}{e^{\frac{i\pi}{2}(k+l)}\,\Gamma(1+l)\Gamma(\frac{1}{2}+\frac{1}{2}k+\frac{1}{2}l)\cos(\frac{k+l}{2})\pi\cdot x^{k+l}}\,{}_3F_2\left(\begin{matrix}\frac{k+l}{2}, n+\frac{k+l}{2}, -n+\frac{k+l}{2}; \frac{-1}{4x^2}\\ 1+l, \frac{1+k+l}{2}\end{matrix}\right)$$

where $R(k+l)>0$, $R(k+l\pm 2n)>0$, $R(x)>0$, $R(\frac{5}{2}-k)>0$, ... (106).

From (15) CHAP. III. and (44) CHAP. I., one gets

$$\int_0^\infty \lambda^{k-1} J_l(\lambda) K_m(x\lambda) K_n(x\lambda)\, d\lambda =$$

$$\frac{2^{-l-2}\Gamma(\frac{k+l+m+n}{2})\Gamma(\frac{k+l+m-n}{2})\Gamma(\frac{k+l+n-m}{2})\Gamma(\frac{k+l-n-m}{2})}{\Gamma(\frac{1}{2}+\frac{1}{2}k+\frac{1}{2}l)\Gamma(1+l)\Gamma(\frac{1}{2}k+\frac{1}{2}l)\,x}\sqrt{\pi}\cdot x^{-(k+l-1)}$$

$$\times\, {}_4F_3\left(\frac{k+l+m+n}{2}, \frac{k+l+m-n}{2}, \frac{k+l+n-m}{2}, \frac{k+l-n-m}{2}; \frac{1+k+l}{2}, \frac{k+l}{2}, 1+l; -1/4x^2\right)$$

where $R(k+l\pm m\pm n)>0$, $R(x)>0$. ,-(107)

When $n=m$, the last formula gives

$$\int_0^\infty \lambda^{k-1} J_l(\lambda) \{K_n(x\lambda)\}^2 d\lambda =$$

$$\frac{2^{-l-2}\,\Gamma(n+\frac{k+l}{2})\Gamma(-n+\frac{k+l}{2})\Gamma(\frac{k+l}{2})\sqrt{\pi}}{\Gamma(\frac{1}{2}+\frac{1}{2}k+\frac{1}{2}l)\Gamma(1+l)\,x} x^{-(k+l-1)} {}_3F_2\left(\begin{matrix} n+\frac{k+l}{2}, -n+\frac{k+l}{2}, \frac{k+l}{2}; \frac{-1}{4x^2} \\ \frac{1}{2}+\frac{1}{2}k+\frac{1}{2}l, 1+l \end{matrix}\right)$$

where $R(k+l)>0$, $R(k+l\pm 2n)>0$. $\qquad\qquad$, $\cdots$ (108)

From (11) CHAP. III. and (69) CHAP. I., one gets

$$\int_0^\infty e^{-\lambda}\lambda^{k-1} I_l(\lambda) K_m\left(\frac{x}{\lambda}e^{i\pi}\right) K_n\left(\frac{x}{\lambda}\right) d\lambda =$$

$$\frac{\cos(\frac{m}{2}+\frac{n}{2})\pi \cos(\frac{n}{2}-\frac{m}{2})\pi \sin(k-l)\pi}{2^{5/2}\, e^{i\pi/2}\, \pi^{3/2} \cos(k\pi)\, x}$$

$$\times E\left\{\begin{matrix} \frac{1+m+n}{2}, \frac{1+m-n}{2}, \frac{1+n-m}{2}, \frac{1-n-m}{2}, \frac{k+l+1}{2}, \frac{k+l-1}{2}, \frac{k+l+2}{2}, \frac{k+l-2}{2} : e^{i\pi}x^2 \\ \frac{5}{4}+\frac{k}{2}, \frac{3}{4}+\frac{k}{2}, \frac{1}{2} \end{matrix}\right\}$$

$$+\frac{\cos(\frac{m}{2}+\frac{n}{2})\pi \cos(\frac{n}{2}-\frac{m}{2})\pi \cos(l\pi)}{2^{7/2}\, e^{i\pi/2}\cdot \pi^{3/2}\cdot \sin(\frac{k}{2}+\frac{1}{4})\pi\cdot x} (x^2 e^{i\pi})^{\frac{k}{2}+\frac{1}{4}}$$

$$\times E\left\{\begin{matrix} \frac{1}{4}+\frac{m+n-k}{2}, \frac{1}{4}+\frac{m-n-k}{2}, \frac{1}{4}+\frac{n-m-k}{2}, \frac{1}{4}+\frac{-n-m-k}{2}, \frac{1}{4}+\frac{l}{2}, \frac{3}{4}+\frac{l}{2}, \frac{1}{4}-\frac{l}{2}, \frac{3}{4}-\frac{l}{2} : e^{i\pi}x^2 \\ \frac{3}{4}-\frac{1}{2}k, \frac{1}{4}-\frac{1}{2}k, \frac{1}{2} \end{matrix}\right\}$$

$$- \frac{\cos\left(\frac{m}{2}+\frac{n}{2}\right)\pi \cos\left(\frac{n}{2}-\frac{m}{2}\right)\pi \cos(l\pi)}{2^{7/2}\cdot\pi^{3/2} e^{i\pi/2} \sin\left(\frac{1}{4}-\frac{k}{2}\right)\pi \cdot x} \left(x^2 e^{\pi i}\right)^{\frac{k}{2}-\frac{1}{4}}$$

$$\times E\left\{\begin{matrix} \frac{3}{4}+\frac{m+n-k}{2}, \frac{3}{4}+\frac{m-n-k}{2}, \frac{3}{4}+\frac{n-m-k}{2}, \frac{3}{4}+\frac{-n-m-k}{2}, \frac{3}{4}+\frac{l}{2}, \frac{5}{4}+\frac{l}{2}, \frac{3}{4}-\frac{l}{2}, \frac{5}{4}-\frac{l}{2} : e^{i\pi}x^2 \\ \frac{5}{4}-\frac{1}{2}k, \frac{3}{2}, \frac{3}{4}-\frac{1}{2}k \end{matrix}\right\}$$

$$+ \frac{e^{\frac{i\pi}{2}} \cos\left(\frac{1}{2}+\frac{m}{2}+\frac{n}{2}\right)\pi \sin\left(\frac{n}{2}-\frac{m}{2}\right)\pi \sin(k-l)\pi}{2^{5/2}\cdot\pi^{3/2} \cos(k\pi)\, x^2}$$

$$\times E\left\{\begin{matrix} \frac{2+n+m}{2}, \frac{2+n-m}{2}, \frac{2+m-n}{2}, \frac{2-n-m}{2}, \frac{2+k+l}{2}, \frac{3+k+l}{2}, \frac{2+k-l}{2}, \frac{3+k-l}{2} : e^{i\pi}x^2 \\ \frac{7}{4}+\frac{k}{2}, \frac{5}{4}+\frac{k}{2}, \frac{3}{2} \end{matrix}\right\}$$

$$+ \frac{e^{\frac{i\pi}{2}} \cos\left(\frac{1}{2}+\frac{m}{2}+\frac{n}{2}\right)\pi \sin\left(\frac{n}{2}-\frac{m}{2}\right)\pi \cos(l\pi)}{2^{7/2}\cdot\pi^{3/2}\cdot \sin\left(\frac{k}{2}+\frac{3}{4}\right)\pi \quad x^2} \left(x^2 e^{i\pi}\right)^{\frac{k}{2}+\frac{3}{4}}$$

$$\times E\left\{\begin{matrix} \frac{1}{4}+\frac{m+n-k}{2}, \frac{1}{4}+\frac{m-n-k}{2}, \frac{1}{4}+\frac{n-m-k}{2}, \frac{1}{4}+\frac{-n-m-k}{2}, \frac{1}{4}+\frac{l}{2}, \frac{3}{4}+\frac{l}{2}, \frac{1}{4}-\frac{l}{2}, \frac{3}{4}-\frac{l}{2} : e^{i\pi}x^2 \\ \frac{1}{4}-\frac{k}{2}, \frac{1}{2}, \frac{3}{4}-\frac{k}{2} \end{matrix}\right\}$$

$$+ \frac{e^{\frac{i\pi}{2}} \cos\left(\frac{1}{2}+\frac{m}{2}+\frac{n}{2}\right)\pi \sin\left(\frac{n}{2}-\frac{m}{2}\right)\pi \cos(l\pi)}{2^{7/2}\cdot\pi^{3/2}\cdot \sin\left(\frac{k}{2}+\frac{1}{4}\right)\pi \quad x^2} \left(x^2 e^{i\pi}\right)^{\frac{k}{2}+\frac{1}{4}}$$

$$\times E\left\{\begin{matrix} \frac{3}{4}+\frac{m+n-k}{2}, \frac{3}{4}+\frac{m-n-k}{2}, \frac{3}{4}+\frac{n-m-k}{2}, \frac{3}{4}+\frac{-n-m-k}{2}, \frac{3}{4}+\frac{l}{2}, \frac{3}{4}-\frac{l}{2}, \frac{5}{4}+\frac{l}{2}, \frac{5}{4}-\frac{l}{2} : e^{i\pi}x^2 \\ \frac{3}{4}-\frac{k}{2}, \frac{3}{2}, \frac{5}{4}-\frac{1}{2}k \end{matrix}\right\}$$

where $R(x)>0$, $R(k+l+1)>0$, $R\left(\frac{1}{2}-k\pm m\pm n\right)>0$. $\quad$, - - - (109)

Note: When $n=m$, the last three terms in (109) vanish and one is left with the first three terms with $m=n$. The conditions for convergence then become: $R(x)>0$, $R(k+l+1)>0$, $R(\frac{1}{2}-k)>0$, $R(\frac{1}{2}-k\pm 2n)>0$.

From (15) CHAP-III and (69) CHAP-I, one gets

$$\int_0^\infty e^{-\lambda}\lambda^{k-1} I_l(\lambda) K_m\left(\frac{x}{\lambda}\right) K_n\left(\frac{x}{\lambda}\right) d\lambda = \frac{1}{4x\sqrt{\pi}}$$

$$\times \sum_{i,-i} \frac{1}{i} \Bigg[\frac{\sin(k-l)\pi}{2^{3/2}\cdot\pi\cdot\cos(k\pi)} E\left\{ \begin{matrix} \frac{1+m+n}{2}, \frac{1+m-n}{2}, \frac{1+n-m}{2}, \frac{1-n-m}{2}, \frac{1+k+l}{2}, \frac{2+k+l}{2}, \frac{1+k-l}{2}, \frac{2+k-l}{2} : e^{i\pi}x^2 \\ \frac{5}{4}+\frac{1}{2}k, \frac{3}{4}+\frac{1}{2}k, \frac{1}{2} \end{matrix} \right\}$$

$$+ \frac{\cos(l\pi)}{2^{5/2}\cdot\pi\cdot\sin\left(\frac{k}{2}+\frac{1}{4}\right)\pi} (x^2 e^{i\pi})^{\frac{k}{2}+\frac{1}{4}}$$

$$\times E\left\{ \begin{matrix} \frac{1}{4}+\frac{m+n-k}{2}, \frac{1}{4}+\frac{m-n-k}{2}, \frac{1}{4}+\frac{n-m-k}{2}, \frac{1}{4}+\frac{-n-m-k}{2}, \frac{3}{4}+\frac{l}{2}, \frac{1}{4}+\frac{l}{2}, \frac{1}{4}-\frac{l}{2}, \frac{3}{4}-\frac{l}{2} : e^{i\pi}x^2 \\ \frac{3}{4}-\frac{k}{2}, \frac{1}{2}, \frac{1}{4}-\frac{k}{2} \end{matrix} \right\}$$

$$- \frac{\cos(l\pi)}{2^{5/2}\cdot\pi\cdot\sin\left(\frac{1}{4}-\frac{k}{2}\right)\pi} (x^2 e^{i\pi})^{\frac{k}{2}-\frac{1}{4}}$$

$$\times E\left\{ \begin{matrix} \frac{3}{4}+\frac{m+n-k}{2}, \frac{3}{4}+\frac{m-n-k}{2}, \frac{3}{4}+\frac{n-m-k}{2}, \frac{3}{4}+\frac{-n-m-k}{2}, \frac{3}{4}+\frac{l}{2}, \frac{5}{4}+\frac{l}{2}, \frac{3}{4}-\frac{l}{2}, \frac{5}{4}-\frac{l}{2} : e^{i\pi}x^2 \\ \frac{5}{4}-\frac{1}{2}k, \frac{3}{2}, \frac{3}{4}-\frac{1}{2}k \end{matrix} \right\} \Bigg]$$

where $R(x)>0$, $R(\frac{1}{2}-k\pm m\pm n)>0$. , ---(110)

When $n=m$, the last formula gives

$$\int_0^\infty e^{-\lambda}\lambda^{k-1} I_l(\lambda)\left\{K_n\left(\frac{x}{\lambda}\right)\right\}^2 d\lambda = \frac{1}{4x\sqrt{\pi}} \times \sum_{i,-i} \frac{1}{i} \Bigg[\frac{\sin(k-l)\pi}{2^{3/2}\cdot\pi\cdot\cos(k\pi)} E\left\{\begin{matrix} \frac{1}{2}, \frac{1}{2}+n, \frac{1}{2}-n, \frac{1+k+l}{2}, \frac{2+k+l}{2}, \frac{1+k-l}{2}, \frac{2+k-l}{2} : e^{i\pi}x^2 \\ \frac{5}{4}+\frac{1}{2}k, \frac{3}{4}+\frac{1}{2}k \end{matrix}\right\}$$

$$+ \frac{\cos(l\pi)}{2^{5/2}\cdot\pi\cdot\sin\left(\frac{k}{2}+\frac{1}{4}\right)\pi}\left(x^2e^{i\pi}\right)^{\frac{k}{2}+\frac{1}{4}} \times E\left\{\begin{matrix} \frac{1}{4}-\frac{k}{2}, \frac{1}{4}+n-\frac{k}{2}, \frac{1}{4}-n-\frac{k}{2}, \frac{1}{4}+\frac{l}{2}, \frac{3}{4}+\frac{l}{2}, \frac{1}{4}-\frac{l}{2}, \frac{3}{4}-\frac{l}{2} : e^{i\pi}x^2 \\ \frac{3}{4}-\frac{1}{2}k, \frac{1}{2} \end{matrix}\right\}$$

$$- \frac{\cos(l\pi)}{2^{5/2}\cdot\pi\cdot\sin\left(\frac{1}{4}-\frac{k}{2}\right)\pi}\left(x^2e^{i\pi}\right)^{\frac{k}{2}-\frac{1}{4}} \times E\left\{\begin{matrix} \frac{3}{4}-\frac{k}{2}, \frac{3}{4}+n-\frac{k}{2}, \frac{3}{4}-n-\frac{k}{2}, \frac{3}{4}+\frac{l}{2}, \frac{5}{4}+\frac{l}{2}, \frac{3}{4}-\frac{l}{2}, \frac{5}{4}-\frac{l}{2} : e^{i\pi}x^2 \\ \frac{5}{4}-\frac{1}{2}k, \frac{3}{2} \end{matrix}\right\} \Bigg]$$

where $R(x)>0$, $R(\frac{1}{2}-k)>0$, $R(\frac{1}{2}-k\pm 2n)>0$. $\quad\cdots(111)$

From (11) CHAP. III. and (71) CHAP. I., one gets

$$\int_0^\infty e^{-\lambda}\lambda^{k-1} I_l(\lambda) K_m(x\lambda e^{i\pi}) K_n(x\lambda)\, d\lambda =$$

$$-\frac{\sqrt{\pi}\cos\left(\frac{m}{2}+\frac{n}{2}\right)\pi\cos\left(\frac{n}{2}-\frac{m}{2}\right)\pi}{e^{\frac{i\pi}{2}}2^{3/2}\sin(k+l)\pi\cdot x} E\left\{\begin{matrix}\frac{1+m+n}{2}, \frac{1+m-n}{2}, \frac{1+n-m}{2}, \frac{1-n-m}{2}, \frac{3}{4}-\frac{k}{2}, \frac{5}{4}-\frac{k}{2} : e^{i\pi}x^2\\ \frac{3}{2}-\frac{k+l}{2}, 1-\frac{k+l}{2}, \frac{1}{2}, \frac{3}{2}+\frac{l-k}{2}, 1+\frac{l-k}{2}\end{matrix}\right\}$$

$$+\frac{\sqrt{\pi}\cos\left(\frac{m}{2}+\frac{n}{2}\right)\pi\cos\left(\frac{n}{2}-\frac{m}{2}\right)\pi}{e^{i\pi/2}2^{5/2}\cos\left(\frac{k+l}{2}\right)\pi\cdot x}\left(x^2e^{i\pi}\right)^{-\left(\frac{k+l-1}{2}\right)}$$

$$\times E\left\{\begin{matrix}\frac{n+m+k+l}{2}, \frac{n-m+k+l}{2}, \frac{m-n+k+l}{2}, \frac{-m-n+k+l}{2}, \frac{1}{4}+\frac{l}{2}, \frac{3}{4}+\frac{l}{2} : e^{i\pi}x^2\\ \frac{1}{2}+\frac{1}{2}k+\frac{1}{2}l, \frac{1}{2}, \frac{1}{2}k+\frac{1}{2}l, 1+l, \frac{1}{2}+l\end{matrix}\right\}$$

$$-\frac{\sqrt{\pi}\cos\left(\frac{m}{2}+\frac{n}{2}\right)\pi\cos\left(\frac{n}{2}-\frac{m}{2}\right)\pi}{e^{i\pi/2}2^{5/2}\sin\left(\frac{k+l}{2}\right)\pi\cdot x}\left(x^2e^{i\pi}\right)^{-\left(\frac{k+l}{2}\right)}$$

$$\times E\left\{\begin{matrix}\frac{1+m+n+k+l}{2}, \frac{1+m-n+k+l}{2}, \frac{1+n-m+k+l}{2}, \frac{1-n-m+k+l}{2}, \frac{3}{4}+\frac{l}{2}, \frac{5}{4}+\frac{l}{2} : e^{i\pi}x^2\\ \frac{1}{2}k+\frac{1}{2}l+1, \frac{3}{2}, \frac{1}{2}k+\frac{1}{2}l+\frac{1}{2}, \frac{3}{2}+l, 1+l\end{matrix}\right\}$$

$$-\frac{e^{\frac{i\pi}{2}}\sqrt{\pi}\sin\left(\frac{m}{2}+\frac{n}{2}\right)\pi\sin\left(\frac{n}{2}-\frac{m}{2}\right)\pi}{2^{3/2}\sin(k+l)\pi\cdot x}$$

$$\left.\times E\left\{\begin{matrix}\frac{2+m+n}{2}, \frac{2+m-n}{2}, \frac{2+n-m}{2}, \frac{2-n-m}{2}, \frac{5}{4}-\frac{k}{2}, \frac{7}{4}-\frac{k}{2} : e^{i\pi}x^2\\ 2-\frac{k+l}{2}, \frac{3}{2}-\frac{k+l}{2}, \frac{3}{2}, 2+\frac{l-k}{2}, \frac{3}{2}+\frac{l-k}{2}\end{matrix}\right\}\right\}$$

$$- \frac{e^{\frac{i\pi}{2}} \sqrt{\pi} \sin\left(\frac{m}{2}+\frac{n}{2}\right)\pi \sin\left(\frac{n}{2}-\frac{m}{2}\right)\pi}{2^{5/2} \sin\left(\frac{1}{2}k+\frac{1}{2}l\right)\pi \; x^2} \left(x^2 e^{i\pi}\right)^{-\left(\frac{k+l-2}{2}\right)}$$

$$\times E\left\{ \begin{matrix} \frac{m+n+k+l}{2}, \frac{m-n+k+l}{2}, \frac{n-m+k+l}{2}, \frac{-n-m+k+l}{2}, \frac{1+2l}{4}, \frac{3+2l}{4} : e^{i\pi}x^2 \\ \frac{1}{2}k+\frac{1}{2}l, \frac{1}{2}, \frac{l}{2}+\frac{1}{2}k+\frac{1}{2}l, 1+l, \frac{1}{2}+l \end{matrix} \right\}$$

$$- \frac{e^{\frac{i\pi}{2}} \sqrt{\pi} \sin\left(\frac{m}{2}+\frac{n}{2}\right)\pi \sin\left(\frac{n}{2}-\frac{m}{2}\right)\pi}{2^{5/2} \cos\left(\frac{1}{2}k+\frac{1}{2}l\right)\pi \cdot x^2} \left(x^2 e^{i\pi}\right)^{-\left(\frac{k+l-1}{2}\right)}$$

$$\times E\left\{ \begin{matrix} \frac{1+n+m+k+l}{2}, \frac{1+n-m+k+l}{2}, \frac{1+m-n+k+l}{2}, \frac{1-n-m+k+l}{2}, \frac{3}{4}+\frac{l}{2}, \frac{5}{4}+\frac{l}{2} : e^{i\pi}x^2 \\ \frac{1}{2}+\frac{1}{2}k+\frac{1}{2}l, \frac{3}{2}, 1+\frac{1}{2}k+\frac{1}{2}l, \frac{3}{2}+l, 1+l \end{matrix} \right\}$$

where $R(x)>0$, $R(k+l\pm m\pm n)>0$, $R(\frac{3}{2}-k)>0$ . . . . . . (112).

When $n=m$, the last formula becomes if $R(x)>0$, $R(k+l)>0$, $R(k+l\pm 2n)>0$ and $R(\frac{3}{2}-k)>0$

$$\int_0^\infty e^{-\lambda} \lambda^{k-1} I_l(\lambda) K_n(x\lambda e^{i\pi}) K_n(x\lambda)\, d\lambda =$$

$$- \frac{\sqrt{\pi} \cos n\pi}{e^{i\pi/2} \cdot 2^{3/2} \sin(k+l)\pi} \left(\frac{1}{x}\right) E\left\{ \begin{matrix} \frac{1}{2}, \frac{1}{2}+n, \frac{1}{2}-n, \frac{3}{4}-\frac{1}{2}k, \frac{5}{4}-\frac{1}{2}k : e^{i\pi}x^2 \\ \frac{3}{2}-\frac{k+l}{2}, 1-\frac{k+l}{2}, \frac{3}{2}+\frac{l-k}{2}, 1+\frac{l-k}{2} \end{matrix} \right\}$$

$$+ \frac{\sqrt{\pi} \cos(n\pi)}{e^{i\pi/2}\, 2^{5/2} \cos\left(\frac{k+l}{2}\right)\pi \cdot x} \left(x^2 e^{i\pi}\right)^{-\left(\frac{k+l-1}{2}\right)} E\left\{ \begin{matrix} n+\frac{k+l}{2}, -n+\frac{k+l}{2}, \frac{k+l}{2}, \frac{1}{4}+\frac{l}{2}, \frac{3}{4}+\frac{l}{2} : e^{i\pi}x^2 \\ \frac{1}{2}+\frac{1}{2}k+\frac{1}{2}l, \frac{1}{2}, 1+l, \frac{1}{2}+l \end{matrix} \right\}$$

$$- \frac{\sqrt{\pi} \cos(n\pi)}{e^{i\pi/2}\, 2^{5/2} \sin\left(\frac{k+l}{2}\right)\pi \cdot x} \left(x^2 e^{i\pi}\right)^{-\left(\frac{k+l}{2}\right)} E\left\{ \begin{matrix} n+\frac{k+l+1}{2}, -n+\frac{k+l+1}{2}, \frac{k+l+1}{2}, \frac{3}{4}+\frac{l}{2}, \frac{5}{4}+\frac{l}{2} : e^{i\pi}x^2 \\ 1+\frac{1}{2}k+\frac{1}{2}l, \frac{3}{2}, \frac{3}{2}+l, 1+l \end{matrix} \right\}$$

, ... (113)

From (15) CHAP. III. and (71) CHAP. I. one gets

$$\int_0^\infty e^{-\lambda}\lambda^{k-1} I_l(\lambda) K_m(x\lambda) K_n(x\lambda)\, d\lambda =$$

$$\frac{\sqrt{\pi}\;\Gamma\left(\frac{k+l+m+n}{2}\right)\Gamma\left(\frac{k+l+m-n}{2}\right)\Gamma\left(\frac{k+l+n-m}{2}\right)\Gamma\left(\frac{k+l-n-m}{2}\right)\Gamma\left(\frac14+\frac{l}{2}\right)\Gamma\left(\frac34+\frac{l}{2}\right)}{2^{5/2}\;\Gamma\left(\frac12+\frac12 k+\frac12 l\right)\Gamma\left(\frac12\right)\Gamma\left(\frac12 k+\frac12 l\right)\Gamma(1+l)\Gamma\left(\frac12+l\right)}\, x^{-(k+l-1)}$$

$$\times {}_6F_5\left(\begin{matrix}\frac{k+l+m+n}{2}, \frac{k+l+m-n}{2}, \frac{k+l+n-m}{2}, \frac{k+l-n-m}{2}, \frac14+\frac{l}{2}, \frac34+\frac{l}{2} \\ \frac12+\frac12 k+\frac12 l, \frac12, \frac12 k+\frac12 l, 1+l, \frac12+l\end{matrix}; 1/x^2\right)$$

$$-\frac{\sqrt{\pi}\;\Gamma\left(\frac{k+l+m+n+1}{2}\right)\Gamma\left(\frac{k+l+m-n+1}{2}\right)\Gamma\left(\frac{k+l+n-m+1}{2}\right)\Gamma\left(\frac{k+l-n-m+1}{2}\right)\Gamma\left(\frac34+\frac{l}{2}\right)\Gamma\left(\frac54+\frac{l}{2}\right)}{2^{5/2}\;\Gamma\left(1+\frac12 k+\frac12 l\right)\Gamma\left(\frac32\right)\Gamma\left(\frac12+\frac12 k+\frac12 l\right)\Gamma(1+l)\Gamma\left(\frac32+l\right)}\, x^{-(k+l)}$$

$$\times {}_6F_5\left(\begin{matrix}\frac{1+k+l+m+n}{2}, \frac{1+k+l+m-n}{2}, \frac{1+k+l+n-m}{2}, \frac{1+k+l-n-m}{2}, \frac34+\frac{l}{2}, \frac54+\frac{l}{2} \\ 1+\frac12 k+\frac12 l, \frac32, \frac12+\frac12 k+\frac12 l, 1+l, \frac32+l\end{matrix}; 1/x^2\right)$$

where $R(x)>0$, $R(k+l\pm m\pm n)>0$. . . . . . . (114)

In particular when $n=m$, the last formula becomes if $R(x)>0$, $R(k+l)>0$, $R(k+l\pm 2n)>0$

$$\int_0^\infty e^{-\lambda}\lambda^{k-1} I_l(\lambda)\{K_n(x\lambda)\}^2 d\lambda =$$

$$\frac{\sqrt{\pi}\,\Gamma(n+\frac{k+l}{2})\,\Gamma(-n+\frac{k+l}{2})\,\Gamma(\frac{k+l}{2})\,\Gamma(\frac{1}{4}+\frac{l}{2})\,\Gamma(\frac{3}{4}+\frac{l}{2})\; x^{-(k+l-1)}}{2^{5/2}\,\Gamma(\frac{1}{2}+\frac{1}{2}k+\frac{1}{2}l)\,\Gamma(\frac{1}{2})\,\Gamma(1+l)\,\Gamma(\frac{1}{2}+l)\cdot x}$$

$${}_5F_4\left(\begin{matrix} n+\frac{k+l}{2},\ -n+\frac{k+l}{2},\ \frac{k+l}{2},\ \frac{1}{4}+\frac{l}{2},\ \frac{3}{4}+\frac{l}{2};\ 1/x^2 \\ \frac{1}{2}+\frac{1}{2}k+\frac{1}{2}l,\ \frac{1}{2},\ 1+l,\ \frac{1}{2}+l \end{matrix}\right)$$

$$-\ \frac{\sqrt{\pi}\,\Gamma(n+\frac{k+l+1}{2})\,\Gamma(-n+\frac{k+l+1}{2})\,\Gamma(\frac{k+l+1}{2})\,\Gamma(\frac{3}{4}+\frac{l}{2})\,\Gamma(\frac{5}{4}+\frac{l}{2})}{2^{5/2}\,\Gamma(1+\frac{1}{2}k+\frac{1}{2}l)\,\Gamma(\frac{3}{2})\,\Gamma(1+l)\,\Gamma(\frac{3}{2}+l)\cdot x}\; x^{-(k+l)}$$

$$\times\ {}_5F_4\left(\begin{matrix} n+\frac{k+l+1}{2},\ -n+\frac{k+l+1}{2},\ \frac{k+l+1}{2},\ \frac{3}{4}+\frac{l}{2},\ \frac{5}{4}+\frac{l}{2};\ 1/x^2 \\ 1+\frac{1}{2}k+\frac{1}{2}l,\ \frac{3}{2},\ 1+l,\ \frac{3}{2}+l \end{matrix}\right) \qquad , \cdots\cdots (115)$$

§ 10. Integrals involving the product of four Modified Bessel Functions of the Second Kind.

Apply (11) CHAP-III and the formula

$$K_n(z) = i^n G_n(iz), \cdots\cdots,$$

so getting if $R(k \pm \nu \pm \mu) > -1$, $R(\pm m \pm n - k) > -1$, $R(x) > 0$

$$\int_0^\infty \lambda^{k-1} K_\nu(i\lambda)\, K_\mu\!\left(\frac{\lambda}{i}\right) K_m\!\left(\frac{x}{\lambda}e^{i\pi}\right) K_n\!\left(\frac{x}{\lambda}\right) d\lambda =$$

$$i^{\nu+\mu}\int_0^\infty \lambda^{k-1} G_\nu(e^{i\pi}\lambda)\, G_\mu(\lambda)\, K_m\!\left(\frac{x}{\lambda}e^{i\pi}\right) K_n\!\left(\frac{x}{\lambda}\right) d\lambda =$$

$$\int_0^\infty \lambda^{k-1}\left\{\begin{array}{l}\left[\dfrac{\cos\left(\frac{\nu}{2}+\frac{\mu}{2}\right)\pi\cos\left(\frac{\mu}{2}-\frac{\nu}{2}\right)\pi}{(2\lambda)\sqrt{\pi}} E\left(\frac{1+\nu+\mu}{2}, \frac{1+\nu-\mu}{2}, \frac{1+\mu-\nu}{2}, \frac{1-\mu-\nu}{2} : \frac{1}{2} : \lambda^2\right) \right. \\ \left. - \dfrac{e^{i\pi/2}\cos\left(\frac{1}{2}+\frac{\nu}{2}+\frac{\mu}{2}\right)\pi\sin\left(\frac{\mu}{2}-\frac{\nu}{2}\right)\pi}{(2\lambda^2)\sqrt{\pi}} E\left(\frac{2+\nu+\mu}{2}, \frac{2+\nu-\mu}{2}, \frac{2+\mu-\nu}{2}, \frac{2-\mu-\nu}{2} : \frac{3}{2} : \lambda^2\right)\right] \\ \times\left[\dfrac{\cos\left(\frac{n}{2}+\frac{m}{2}\right)\pi\cos\left(\frac{n}{2}-\frac{m}{2}\right)\pi}{(2ix)\sqrt{\pi}}\lambda E\left(\frac{1+n+m}{2}, \frac{1+n-m}{2}, \frac{1+m-n}{2}, \frac{1-n-m}{2} : \frac{1}{2} : \frac{e^{i\pi}x^2}{\lambda^2}\right) \right. \\ \left. + \dfrac{e^{i\pi/2}\cos\left(\frac{1}{2}+\frac{n}{2}+\frac{m}{2}\right)\pi\sin\left(\frac{n}{2}-\frac{m}{2}\right)\pi}{(2x^2)\sqrt{\pi}}\lambda^2 E\left(\frac{2+m+n}{2}, \frac{2+m-n}{2}, \frac{2+n-m}{2}, \frac{2-n-m}{2} : \frac{3}{2} : \frac{e^{i\pi}x^2}{\lambda^2}\right)\right]\end{array}\right\} d\lambda$$

Now multiply, evaluate each of the resulting four integrals by means of (9) CHAP. I., so getting if $R(x)>0$, $R(k\pm\nu\pm\mu)>-1$, $R(\pm m\pm n-k)>-1$

$$i^{\mu+\nu}\int_0^\infty \lambda^{k-1} G_\nu(e^{i\pi}\lambda)\, G_\mu(\lambda)\, K_m\!\left(\frac{x}{\lambda}e^{i\pi}\right) K_n\!\left(\frac{x}{\lambda}\right) d\lambda =$$

$$\frac{\cos\left(\frac{\nu}{2}+\frac{\mu}{2}\right)\pi \cos\left(\frac{\mu}{2}-\frac{\nu}{2}\right)\pi \cos\left(\frac{m}{2}+\frac{n}{2}\right)\pi \cos\left(\frac{n}{2}-\frac{m}{2}\right)\pi}{e^{i\pi/2}\,(8x)\,\sin(\pi k/2)}$$

$$\times\left[\left(e^{\frac{i\pi}{2}}x\right)^{k} E\left\{\begin{matrix}\frac{1+\nu+\mu}{2}, \frac{1+\nu-\mu}{2}, \frac{1+\mu-\nu}{2}, \frac{1-\mu-\nu}{2}, \frac{1+m+n-k}{2}, \frac{1+m-n-k}{2}, \frac{1+n-m-k}{2}, \frac{1-n-m-k}{2} : x^2 \\ \frac{1}{2}, 1-\frac{1}{2}k, \frac{1}{2}-\frac{1}{2}k\end{matrix}\right\}\right.$$

$$\left.-E\left\{\begin{matrix}\frac{1+\nu+\mu+k}{2}, \frac{1+\nu-\mu+k}{2}, \frac{1+\mu-\nu+k}{2}, \frac{1-\nu-\mu-k}{2}, \frac{1+m+n}{2}, \frac{1+m-n}{2}, \frac{1+n-m}{2}, \frac{1-n-m}{2} : x^2 \\ \frac{1}{2}, 1+\frac{1}{2}k, \frac{1}{2}+\frac{1}{2}k\end{matrix}\right\}\right]$$

$$+\frac{e^{\frac{i\pi}{2}}\cos\left(\frac{\mu}{2}+\frac{\nu}{2}\right)\pi \cos\left(\frac{\mu}{2}-\frac{\nu}{2}\right)\pi \cos\left(\frac{1}{2}+\frac{m}{2}+\frac{n}{2}\right)\pi \sin\left(\frac{n}{2}-\frac{m}{2}\right)\pi}{(8x^2)\,\sin\left(\frac{k+1}{2}\right)\pi}$$

$$\times\left[\left(e^{\frac{i\pi}{2}}x\right)^{k+1} E\left\{\begin{matrix}\frac{1+\nu+\mu}{2}, \frac{1+\nu-\mu}{2}, \frac{1+\mu-\nu}{2}, \frac{1-\nu-\mu}{2}, \frac{1+m+n-k}{2}, \frac{1+n-m-k}{2}, \frac{1+m-n-k}{2}, \frac{1-n-m-k}{2} : x^2 \\ \frac{1}{2}, \frac{1}{2}-\frac{1}{2}k, 1-\frac{1}{2}k\end{matrix}\right\}\right.$$

$$\left.-E\left\{\begin{matrix}\frac{2+\nu+\mu+k}{2}, \frac{2+\nu-\mu+k}{2}, \frac{2+\mu-\nu+k}{2}, \frac{2-\mu-\nu+k}{2}, \frac{2+m+n}{2}, \frac{2+m-n}{2}, \frac{2+n-m}{2}, \frac{2-n-m}{2} : x^2 \\ 1+\frac{1}{2}k, \frac{3}{2}+\frac{1}{2}k, \frac{3}{2}\end{matrix}\right\}\right]$$

$$-\frac{\cos\left(\frac{1}{2}+\frac{\mu}{2}+\frac{\nu}{2}\right)\pi \sin\left(\frac{\mu}{2}-\frac{\nu}{2}\right)\pi \cos\left(\frac{m}{2}+\frac{n}{2}\right)\pi \sin\left(\frac{n}{2}-\frac{m}{2}\right)\pi}{(8x)\,\sin\left(\frac{k-1}{2}\right)\pi}$$

$$\times\left[\left(e^{i\pi/2}x\right)^{k-1} E\left\{\begin{matrix}\frac{2+\nu+\mu}{2}, \frac{2+\nu-\mu}{2}, \frac{2+\mu-\nu}{2}, \frac{2-\mu-\nu}{2}, \frac{2+m+n-k}{2}, \frac{2+m-n-k}{2}, \frac{2+n-m-k}{2}, \frac{2-n-m-k}{2} ; x^2 \\ \frac{3}{2}, \frac{3}{2}-\frac{k}{2}, 1-\frac{k}{2}\end{matrix}\right\}\right.$$

$$\left.-E\left\{\begin{matrix}\frac{1+\nu+\mu+k}{2}, \frac{1+\nu-\mu+k}{2}, \frac{1+\mu-\nu+k}{2}, \frac{1-\nu-\mu+k}{2}, \frac{1+n+m}{2}, \frac{1+n-m}{2}, \frac{1+m-n}{2}, \frac{1-m-n}{2} : x^2 \\ 1+\frac{1}{2}k, \frac{1}{2}+\frac{1}{2}k, \frac{1}{2}\end{matrix}\right\}\right]$$

$$+\frac{\sin\left(\frac{n}{2}+\frac{m}{2}\right)\pi\,\sin\left(\frac{n}{2}-\frac{m}{2}\right)\pi\,\sin\left(\frac{\mu}{2}+\frac{\nu}{2}\right)\pi\,\sin\left(\frac{\mu}{2}-\frac{\nu}{2}\right)\pi}{(8x^2)\,\sin(\pi k/2)}$$

$$\times\left[\left(e^{\frac{i\pi}{2}}x\right)^k E\left\{\begin{matrix}\frac{2+\nu+\mu}{2},\frac{2+\nu-\mu}{2},\frac{2+\mu-\nu}{2},\frac{2-\nu-\mu}{2},\frac{2+m+n-k}{2},\frac{2+m-n-k}{2},\frac{2+n-m-k}{2},\frac{2-n-m-k}{2};x^2\\ \frac{3}{2},1-\frac{1}{2}k,\frac{3}{2}-\frac{1}{2}k\end{matrix}\right\}\right.$$

$$\left.-E\left\{\begin{matrix}\frac{2+\nu+\mu+k}{2},\frac{2+\nu-\mu+k}{2},\frac{2+\mu-\nu+k}{2},\frac{2-\nu-\mu+k}{2},\frac{2+m+n}{2},\frac{2+m-n}{2},\frac{2+n-m}{2},\frac{2-n-m}{2};x^2\\ \frac{3}{2},\frac{3}{2}+\frac{1}{2}k,1+\frac{1}{2}k\end{matrix}\right\}\right] \qquad \ldots (116)$$

The necessary restrictions for (9) CHAP. I. can be removed by analytical continuation.

Again apply (11) CHAP. III. and (15) CHAP. III. and get if $R(x)>0$ $R(\pm m\pm n-k)>-1$

$$i^{\mu+\nu}\int_0^\infty \lambda^{k-1}G_\nu(e^{i\pi}\lambda)\,G_\mu(\lambda)\,K_m\left(\frac{x}{\lambda}\right)K_n\left(\frac{x}{\lambda}\right)d\lambda=$$

$$\int_0^\infty \lambda^{k-1}\left[\frac{\cos\left(\frac{\mu+\nu}{2}\right)\pi\,\cos\left(\frac{\mu-\nu}{2}\right)\pi}{8\pi x}\sum_{i,-i}\frac{1}{i}\left\{\begin{matrix}E\left(\frac{1}{2}+\frac{\nu}{2}+\frac{\mu}{2},\frac{1}{2}+\frac{\nu}{2}-\frac{\mu}{2},\frac{1}{2}+\frac{\mu}{2}-\frac{\nu}{2},\frac{1}{2}-\frac{\mu}{2}-\frac{\nu}{2}:\frac{1}{2}:\lambda^2\right)\\ \times E\left(\frac{1}{2}+\frac{m}{2}+\frac{n}{2},\frac{1}{2}+\frac{m}{2}-\frac{n}{2},\frac{1}{2}+\frac{n}{2}-\frac{m}{2},\frac{1}{2}-\frac{n}{2}-\frac{m}{2}:\frac{1}{2}:e^{i\pi}x^2/\lambda^2\right)\end{matrix}\right\}\right.$$

$$\left.+\frac{e^{\frac{i\pi}{2}}\sin\left(\frac{\mu+\nu}{2}\right)\pi\,\sin\left(\frac{\mu-\nu}{2}\right)\pi}{8\pi x\lambda}\sum_{i,-i}\frac{1}{i}\left\{\begin{matrix}E\left(1+\frac{\nu}{2}+\frac{\mu}{2},1+\frac{\nu}{2}-\frac{\mu}{2},1+\frac{\mu}{2}-\frac{\nu}{2},1-\frac{\mu}{2}-\frac{\nu}{2}:\frac{3}{2}:\lambda^2\right)\\ \times E\left(\frac{1}{2}+\frac{m}{2}+\frac{n}{2},\frac{1}{2}+\frac{m}{2}-\frac{n}{2},\frac{1}{2}+\frac{n}{2}-\frac{m}{2},\frac{1}{2}-\frac{n}{2}-\frac{m}{2}:\frac{1}{2}:\frac{e^{i\pi}x^2}{\lambda^2}\right)\end{matrix}\right\}\right]d\lambda$$

Now multiply, evaluate each integral by means of (9) CHAP. I., and get

$$i^{\mu+\nu}\int_0^\infty \lambda^{k-1} G_\nu(\lambda e^{i\pi}) G_\mu(\lambda) K_m\left(\frac{x}{\lambda}\right) K_n\left(\frac{x}{\lambda}\right) d\lambda =$$

$$\frac{\cos\left(\frac{\mu+\nu}{2}\right)\pi \cos\left(\frac{\mu-\nu}{2}\right)\pi}{8x} x^k E\left\{\begin{matrix}\frac{1+\nu+\mu}{2}, \frac{1+\nu-\mu}{2}, \frac{1+\mu-\nu}{2}, \frac{1-\mu-\nu}{2}, \frac{1+m+n-k}{2}, \frac{1+m-n-k}{2}, \frac{1+n-m-k}{2}, \frac{1-n-m-k}{2} : x^2 \\ \frac{1}{2}, 1-\frac{1}{2}k, \frac{1}{2}-\frac{1}{2}k\end{matrix}\right\}$$

$$+ \frac{e^{i\pi/2}\sin\left(\frac{\mu+\nu}{2}\right)\pi \sin\left(\frac{\mu-\nu}{2}\right)\pi}{8x} x^{k-1} E\left\{\begin{matrix}\frac{2+\nu+\mu}{2}, \frac{2+\nu-\mu}{2}, \frac{2+\mu-\nu}{2}, \frac{2-\nu-\mu}{2}, \frac{2+m+n-k}{2}, \frac{2+m-n-k}{2}, \frac{2+n-m-k}{2}, \frac{2-n-m-k}{2} ; x^2 \\ \frac{3}{2}, \frac{3}{2}-\frac{1}{2}k, 1-\frac{1}{2}k\end{matrix}\right\}$$

, ... (11 Y)

where $R(x)>0$, $R(\pm m \pm n - k) > -1$ and the restrictions necessary for (9) CHAP. I. can be removed by analytical continuation. It may be noted that in obtaining (11 Y), (9) CHAP. I. was applied with $(e^{-i\pi}x^2)$ with $(+i)$ and $(e^{i\pi}x^2)$ with $(-i)$ to obtain $E(\cdots :: +x^2)$ in both cases.

Similarly apply (11) CHAP. III and (15) CHAP. III. and get if $R(x)>0$, $R(k \pm \mu \pm \nu \pm m \pm n) > 0$

$$i^{\mu+\nu}\int_0^\infty \lambda^{k-1} G_\nu(e^{i\pi}\lambda) G_\mu(\lambda) K_m(x\lambda) K_n(x\lambda)\, d\lambda$$

$$\int_0^\infty \lambda^{k-1}\left[\begin{array}{l}\dfrac{\cos\left(\frac{\mu+\nu}{2}\right)\pi\cos\left(\frac{\mu-\nu}{2}\right)\pi}{8\pi x\lambda^2}\displaystyle\sum_{i,-i}\frac{1}{i}\left\{\begin{array}{l}E\left(\frac{1+\nu+\mu}{2},\frac{1+\nu-\mu}{2},\frac{1+\mu-\nu}{2},\frac{1-\mu-\nu}{2}:\frac{1}{2}:\lambda^2\right)\\ \times E\left(\frac{1+m+n}{2},\frac{1+m-n}{2},\frac{1+n-m}{2},\frac{1-n-m}{2}:\frac{1}{2}:e^{i\pi}x^2\lambda^2\right)\end{array}\right\}\\ +\dfrac{e^{i\pi/2}\sin\left(\frac{\mu+\nu}{2}\right)\pi\sin\left(\frac{\mu-\nu}{2}\right)\pi}{8\pi x\lambda^3}\displaystyle\sum_{i,-i}\frac{1}{i}\left\{\begin{array}{l}E\left(1+\frac{\nu}{2}+\frac{\mu}{2},1+\frac{\nu}{2}-\frac{\mu}{2},1+\frac{\mu}{2}-\frac{\nu}{2},1-\frac{\mu}{2}-\frac{\nu}{2}:\frac{3}{2}:\lambda^2\right)\\ \times E\left(\frac{1+m+n}{2},\frac{1+m-n}{2},\frac{1+n-m}{2},\frac{1-n-m}{2}:\frac{1}{2}:e^{i\pi}x^2\lambda^2\right)\end{array}\right\}\end{array}\right]d\lambda.$$

Now multiply, evaluate each integral by means of (10) CHAP-I., and get

$$i^{\mu+\nu}\int_0^\infty \lambda^{k-1}G_\nu(e^{i\pi}\lambda)G_\mu(\lambda)K_m(x\lambda)K_n(x\lambda)\,d\lambda =$$

$$\sum_{\nu,-\nu}\left[\begin{array}{l}\dfrac{\cos\left(\frac{\mu+\nu}{2}\right)\pi\cos\left(\frac{\mu-\nu}{2}\right)\pi\,\Gamma\left(\frac{1+\nu+\mu}{2}\right)\Gamma\left(\frac{k+\mu+\nu+m+n}{2}\right)\Gamma\left(\frac{k+\mu+\nu+m-n}{2}\right)}{8\pi x}\\ \times\dfrac{\Gamma\left(\frac{k+\mu+\nu+n-m}{2}\right)\Gamma\left(\frac{k+\mu+\nu-m-n}{2}\right)\Gamma(-\mu)\Gamma(-\nu)\Gamma(-\mu-\nu)}{\Gamma\left(-\frac{\mu}{2}-\frac{\nu}{2}\right)\Gamma\left(\frac{1+\mu+\nu+k}{2}\right)\Gamma\left(\frac{\mu+\nu+k}{2}\right)}x^{(1-\mu-\nu-k)}\\ \times {}_6F_5\left\{\begin{array}{l}\frac{1+\mu+\nu}{2},\frac{k+\mu+\nu+m+n}{2},\frac{k+\mu+\nu+m-n}{2},\frac{k+\mu+\nu+n-m}{2},\frac{k+\mu+\nu-n-m}{2},\frac{2+\nu+\mu}{2};\frac{-1}{x^2}\\ \frac{1+\mu+\nu+k}{2},1+\mu,1+\nu,1+\mu+\nu,\frac{\mu+\nu+k}{2}\end{array}\right\}\end{array}\right]$$

$$
\begin{aligned}
+\sum_{\nu,-\nu}\Bigg[& \frac{\cos\left(\frac{\mu+\nu}{2}\right)\pi\cos\left(\frac{\mu-\nu}{2}\right)\pi\,\Gamma\left(\frac{1+\nu-\mu}{2}\right)\Gamma\left(\frac{k+\nu-\mu+m+n}{2}\right)\Gamma\left(\frac{k+\nu-\mu+n-m}{2}\right)}{8\pi x} \\
& \times\frac{\Gamma\left(\frac{k+\nu-\mu+m-n}{2}\right)\Gamma\left(\frac{k+\nu-\mu-m-n}{2}\right)\Gamma(\mu)\Gamma(-\nu)\Gamma(\mu-\nu)}{\Gamma\left(\frac{\mu}{2}-\frac{\nu}{2}\right)\Gamma\left(\frac{1-\mu+\nu+k}{2}\right)\Gamma\left(\frac{\nu-\mu+k}{2}\right)}\,x^{(1+\mu-\mu-k)} \\
& \times{}_6F_5\left\{\begin{matrix}\frac{1+\nu-\mu}{2},\frac{k+\nu-\mu+m+n}{2},\frac{k+\nu-\mu+m-n}{2},\frac{k+\nu-\mu+n-m}{2},\frac{k+\nu-\mu-m-n}{2},\frac{2-\mu+\nu}{2};\ \frac{-1}{x^2}\\ \frac{1}{2}+\frac{\nu}{2}-\frac{\mu}{2}+\frac{k}{2},1-\mu,1+\nu,1-\mu+\nu,\frac{\nu}{2}-\frac{\mu}{2}+\frac{k}{2}\end{matrix}\right\}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
-\sum_{\nu,-\nu}\Bigg[& \frac{e^{i\pi/2}\cos\left(\frac{1+\nu+\mu}{2}\right)\pi\sin\left(\frac{\mu-\nu}{2}\right)\pi\,\Gamma\left(1+\frac{\nu+\mu}{2}\right)\Gamma\left(\frac{k+\mu+\nu+m+n}{2}\right)\Gamma\left(\frac{k+\mu+\nu+m-n}{2}\right)}{8\pi x} \\
& \times\frac{\Gamma\left(\frac{k+\mu+\nu+n-m}{2}\right)\Gamma\left(\frac{k+\mu+\nu-m-n}{2}\right)\Gamma(-\mu)\Gamma(-\nu)\Gamma(-\mu-\nu)}{\Gamma\left(\frac{1-\mu-\nu}{2}\right)\Gamma\left(\frac{1+\mu+\nu+k}{2}\right)\Gamma\left(\frac{k+\mu+\nu}{2}\right)}\,x^{(1-\mu-\nu-k)} \\
& \times{}_6F_5\left\{\begin{matrix}\frac{2+\mu+\nu}{2},\frac{k+\mu+\nu+m+n}{2},\frac{k+\mu+\nu+m-n}{2},\frac{k+\mu+\nu+n-m}{2},\frac{k+\mu+\nu-m-n}{2},\frac{1+\mu+\nu}{2};\ \frac{-1}{x^2}\\ \frac{1+\mu+\nu+k}{2},1+\mu,1+\nu,1+\mu+\nu,\frac{\mu+\nu+k}{2}\end{matrix}\right\}\Bigg]
\end{aligned}
$$

$$-\sum_{\nu_1-\nu}\left[\frac{e^{\frac{i\pi}{2}}\cos\left(\frac{1+\mu+\nu}{2}\right)\pi\sin\left(\frac{\mu-\nu}{2}\right)\Gamma\left(1+\frac{\nu-\mu}{2}\right)\Gamma\left(\frac{k+\nu-\mu+m+n}{2}\right)\Gamma\left(\frac{k+\nu-\mu+m-n}{2}\right)}{8\pi x}\right.$$

$$\times\frac{\Gamma\left(\frac{k+\nu-\mu+n-m}{2}\right)\Gamma\left(\frac{k+\nu-\mu-n-m}{2}\right)\Gamma(\mu)\Gamma(-\mu)\Gamma(\mu-\nu)}{\Gamma\left(\frac{1+\mu-\nu}{2}\right)\Gamma\left(\frac{1-\mu+\nu+k}{2}\right)\Gamma\left(\frac{\nu-\mu+k}{2}\right)}\,x^{(1+\mu-\nu-k)}$$

$$\left.\times\,{}_6F_5\left\{\begin{matrix}\frac{2-\mu+\nu}{2},\ \frac{k+\nu-\mu+m+n}{2},\ \frac{k+\nu-\mu+m-n}{2},\ \frac{k+\nu-\mu+n-m}{2},\ \frac{k+\nu-\mu-n-m}{2},\ \frac{1-\mu+\nu}{2};\ \frac{-1}{x^2}\\ \frac{1+\nu-\mu+k}{2},\ 1-\mu,\ 1+\nu,\ 1-\mu+\nu,\ \frac{\nu-\mu+k}{2}\end{matrix}\right\}\right] \quad \text{(118)}$$

where $R(x)>0$, $R(k\pm\nu\pm\mu\pm m\pm n)>0$ and the restrictions necessary for (10) CHAP. I can be changed to the restrictions necessary for (118) by analytical continuation.

Also if $R(x)>0$, $R(k)<2$, $R(k\pm\nu\pm\mu\pm m\pm n)>0$; then (11) CHAP. III. gives

$$i^{\mu+\nu}\int_0^\infty \lambda^{k-1}G_\nu(e^{i\pi}\lambda)G_\mu(\lambda)K_m(x\lambda e^{i\pi})K_n(x\lambda)\,d\lambda=$$

$$\frac{\cos\left(\frac{\mu+\nu}{2}\right)\pi\cos\left(\frac{\mu-\nu}{2}\right)\pi\cos\left(\frac{m+n}{2}\right)\pi\cos\left(\frac{n-m}{2}\right)\pi}{e^{i\pi/2}\,8\pi x}$$

$$\times\int_0^\infty \lambda^{\frac{k}{2}-2}\begin{cases} E\left(\frac{1}{2}+\frac{\nu}{2}+\frac{\mu}{2},\frac{1}{2}+\frac{\nu}{2}-\frac{\mu}{2},\frac{1}{2}+\frac{\mu}{2}-\frac{\nu}{2},\frac{1}{2}-\frac{\mu}{2}-\frac{\nu}{2}:\frac{1}{2}:\lambda\right) \\ \times E\left(\frac{1}{2}+\frac{m}{2}+\frac{n}{2},\frac{1}{2}+\frac{m}{2}-\frac{n}{2},\frac{1}{2}+\frac{n}{2}-\frac{m}{2},\frac{1}{2}-\frac{n}{2}-\frac{m}{2}:\frac{1}{2}:e^{i\pi}x^2\lambda\right)\end{cases}d\lambda$$

$$+e^{\frac{i\pi}{2}}\,\frac{\cos\left(\frac{\mu+\nu}{2}\right)\pi\cos\left(\frac{\mu-\nu}{2}\right)\pi\cos\left(\frac{1+m+n}{2}\right)\sin\left(\frac{n-m}{2}\right)\pi}{8x^2\pi}$$

$$\times\int_0^\infty \lambda^{\frac{k}{2}-\frac{5}{2}}\begin{cases} E\left(\frac{1}{2}+\frac{\mu}{2}+\frac{\nu}{2},\frac{1}{2}+\frac{\mu}{2}-\frac{\nu}{2},\frac{1}{2}+\frac{\nu}{2}-\frac{\mu}{2},\frac{1}{2}-\frac{\nu}{2}-\frac{\mu}{2}:\frac{1}{2}:\lambda\right) \\ \times E\left(1+\frac{m}{2}+\frac{n}{2},1+\frac{m}{2}-\frac{n}{2},1+\frac{n}{2}-\frac{m}{2},1-\frac{n}{2}-\frac{m}{2}:\frac{3}{2}:e^{i\pi}x^2\lambda\right)\end{cases}d\lambda$$

$$-\frac{\cos\left(\frac{1+\mu+\nu}{2}\right)\pi\sin\left(\frac{\mu-\nu}{2}\right)\pi\cos\left(\frac{n-m}{2}\right)\pi\cos\left(\frac{n+m}{2}\right)\pi}{8x\pi}$$

$$\times\int_0^\infty \lambda^{\frac{k}{2}-\frac{5}{2}}\begin{cases} E\left(1+\frac{\nu}{2}+\frac{\mu}{2},1+\frac{\nu}{2}-\frac{\mu}{2},1+\frac{\mu}{2}-\frac{\nu}{2},1-\frac{\mu}{2}-\frac{\nu}{2}:\frac{3}{2}:\lambda\right) \\ \times E\left(\frac{1}{2}+\frac{m}{2}+\frac{n}{2},\frac{1}{2}+\frac{m}{2}-\frac{n}{2},\frac{1}{2}+\frac{n}{2}-\frac{m}{2},\frac{1}{2}-\frac{n}{2}-\frac{m}{2}:\frac{1}{2}:e^{i\pi}x^2\lambda\right)\end{cases}d\lambda$$

$$-\frac{\cos\left(\frac{1+\mu+\nu}{2}\right)\pi\sin\left(\frac{\mu-\nu}{2}\right)\pi\cos\left(\frac{1+n+m}{2}\right)\pi\sin\left(\frac{n-m}{2}\right)\pi}{8\pi x^2}$$

$$\times\int_0^\infty \lambda^{\frac{k}{2}-3}\begin{cases} E\left(1+\frac{\nu}{2}+\frac{\mu}{2},1+\frac{\nu}{2}-\frac{\mu}{2},1+\frac{\mu}{2}-\frac{\nu}{2},1-\frac{\mu}{2}-\frac{\nu}{2}:\frac{3}{2}:\lambda\right) \\ \times E\left(1+\frac{m}{2}+\frac{n}{2},1+\frac{m}{2}-\frac{n}{2},1+\frac{n}{2}-\frac{m}{2},1-\frac{m}{2}-\frac{n}{2}:\frac{3}{2}:e^{i\pi}x^2\lambda\right)\end{cases}d\lambda \quad \ldots (119),$$

Now expand each of the last four integrals by means of (10) CHAP-I and the result will be the sum of twenty generalized hypergeometric functions of the type ${}_6F_5(\cdots;\cdots;-1/x^2)$.

More formulae involving the product of four Bessel Functions:

Formulae (122), (123) below will be deduced from the following formulae

$$\int_0^\infty \lambda^{k-1} E(p;\alpha_r:q;\rho_s:e^{\pm i\pi}\lambda^2)\,E\left(l;\beta_t:m;\sigma_u:\frac{x}{e^{\pm i\pi}\lambda^2}\right)d\lambda =$$

$$i^{\mp k}\frac{\pi x^{k/2}}{2\sin(k\pi/2)}\left\{E\left[\begin{matrix}\alpha_1,\dots,\alpha_p,\beta_1-\frac{k}{2},\dots,\beta_l-\frac{k}{2}:e^{\pm i\pi}x\\ \rho_1,\dots,\rho_q,1-\frac{k}{2},\sigma_1-\frac{k}{2},\dots,\sigma_m-\frac{k}{2}\end{matrix}\right]-x^{-\frac{k}{2}}E\left[\begin{matrix}\alpha_1+\frac{k}{2},\dots,\alpha_p+\frac{k}{2},\beta_1,\dots,\beta_l:e^{\pm i\pi}x\\ \rho_1+\frac{k}{2},\dots,\rho_q+\frac{k}{2},1+\frac{k}{2},\sigma_1,\dots,\sigma_m\end{matrix}\right]\right\},$$

where $R(x)>0$, $R(\frac{1}{2}k+\alpha_r)>0$, $r=1,2,3,\dots,p$ and $R(\beta_t-\frac{k}{2})>0$, $\cdots$ (120)
$t=1,2,3,\dots,l$, $p\geqslant q+1$, $l\geqslant m+1$. For other values of $\underline{p}$ and $\underline{q}$, $\underline{l}$, $\underline{m}$, (120) holds if the integral is convergent;
and if $R(x)>0$, $R(k)>0$, $R(\alpha_r+\beta_t-\frac{1}{2}k)>0$, $r=1,2,3,\dots,p$, $t=1,2,3,\dots,l$, $p\geqslant q+1$, $l\geqslant m+1$ then

$$\int_0^\infty \lambda^{-k-1} E(p;\alpha_r:q;\rho_s:e^{\pm i\pi}\lambda^2)\,E(l;\beta_t:m;\sigma_u:e^{\pm i\pi}x\lambda^2)\,d\lambda =$$

$$\frac{i^{\pm k}\pi^{p-q}\prod_{s=1}^{q}\sin(\rho_s-\frac{k}{2})\pi}{2\prod_{r=1}^{p}\sin(\alpha_r-\frac{k}{2})\pi} E\left\{\begin{matrix}\frac{1}{2}k,\beta_1,\dots,\beta_l,1+\frac{1}{2}k-\rho_1,\dots,1+\frac{1}{2}k-\rho_q : e^{\pm i\pi(p-q)}x \\ 1+\frac{k}{2}-\alpha_1,\dots,1+\frac{k}{2}-\alpha_p,\sigma_1,\dots,\sigma_m\end{matrix}\right\}$$

$$+\, i^{\pm k}\frac{\pi^{p-q}}{2}\sum_{h=1}^{p}\frac{\prod_{j=1}^{q}\sin(\rho_j-\alpha_h)\pi}{\left\{\prod_{r=1}^{p}{}'\sin(\alpha_r-\alpha_h)\pi\right\}\sin(\frac{k}{2}-\alpha_h)\pi}\cdot x^{\frac{k}{2}-\alpha_h}$$

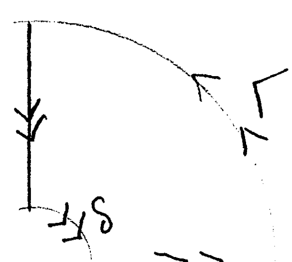
$$\times E\left\{\begin{matrix}\alpha_h,\alpha_h+\beta_1-\frac{1}{2}k,\dots,\alpha_h+\beta_l-\frac{1}{2}k,1+\alpha_h-\rho_1,\dots,1+\alpha_h-\rho_q : e^{\pm i\pi(p-q)}x \\ 1+\alpha_h-\frac{k}{2},1+\alpha_h-\alpha_1,\dots,1+\alpha_h-\alpha_p,\alpha_h+\sigma_1-\frac{k}{2},\dots,\alpha_h+\sigma_m-\frac{k}{2}\end{matrix}\right\}, \quad \dots (121).$$

For other values of p, q, (121) holds if the integral converges.

Proofs of (120) and (121): For (120) consider the function

$$f(z) \equiv z^{k-1}E(p;\alpha_r:q;\rho_s:z^2)\,E(l;\beta_t:m;\sigma_u:x/z^2),$$

integrated round the contour C drawn below;



then if $R(\frac{1}{2}k+\alpha_r)>0$, $r=1,2,3,\ldots,p$ and $R(\beta_t-\frac{1}{2}k)>0$, $t=1,2,3,\ldots,l$, then integrals along the parts δ, $\Gamma \longrightarrow 0$ as $|z|\longrightarrow 0$ and $|z|\longrightarrow\infty$ respectively; Thus

$$\int_0^\infty f(x)\,dx = i\int_0^\infty f(xe^{\frac{i\pi}{2}})\,dx.$$

From this (120) is obtained by applying (9) CHAP. I. with (λ^2) instead of $\underline{\lambda}$. The other form of (120) can be obtained by integr-ating $f(z)$ round the contour C' which is the reflection of C in the real axis.

(121) can be obtained in the same way by using (10) CHAP. I. instead of (9) CHAP. I. and considering

$$f(z) \equiv z^{k-1}E(p;\alpha_r:q;\rho_s:z^2)E(l;\beta_t:m;\sigma_u:xz^2).$$

Now if $R(x)>0$, $R(k\pm\nu\pm\mu\pm m\pm n)>0$; then from (15) CHAP. III.

$$\int_0^\infty \lambda^{k-1} K_\nu(\lambda) K_\mu(\lambda) K_m(x\lambda) K_n(x\lambda)\, d\lambda = \frac{1}{16\pi x}$$

$$\times \int_0^\infty \lambda^{k-3} \Bigg[\Big\{ \sum_{i,-i} \frac{1}{i} E\left(\tfrac{1}{2}+\tfrac{\nu}{2}+\tfrac{\mu}{2}, \tfrac{1}{2}+\tfrac{\nu}{2}-\tfrac{\mu}{2}, \tfrac{1}{2}+\tfrac{\mu}{2}-\tfrac{\nu}{2}, \tfrac{1}{2}-\tfrac{\nu}{2}-\tfrac{\mu}{2} : \tfrac{1}{2} : e^{i\pi}\lambda^2\right) \Big\}$$
$$\times \Big\{ \sum_{i,-i} \frac{1}{i} E\left(\tfrac{1}{2}+\tfrac{m}{2}+\tfrac{n}{2}, \tfrac{1}{2}+\tfrac{m}{2}-\tfrac{n}{2}, \tfrac{1}{2}+\tfrac{n}{2}-\tfrac{m}{2}, \tfrac{1}{2}-\tfrac{n}{2}-\tfrac{m}{2} : \tfrac{1}{2} : e^{i\pi} x^2\lambda^2\right) \Big\} \Bigg] d\lambda.$$

Now multiply, evaluate each of the resulting four integrals by means of (121), so getting if $R(x)>0$, $R(k)<2$, $R(k\pm\nu\pm\mu\pm m\pm n)>0$

$$\int_0^\infty \lambda^{k-1} K_\nu(\lambda) K_\mu(\lambda) K_m(x\lambda) K_n(x\lambda)\, d\lambda =$$

$$\sum_{\nu,-\nu} \Bigg\{ \sum_{\mu,-\mu} \Bigg[\frac{\cos\left(\frac{\pi k}{2}\right)\Gamma\left(\frac{1+\nu+\mu}{2}\right)\Gamma\left(\frac{k+\mu+\nu+m+n}{2}\right)\Gamma\left(\frac{k+\nu+\mu+m-n}{2}\right)\Gamma\left(\frac{k+\mu+\nu+n-m}{2}\right)}{16\pi x}$$
$$\times \frac{\Gamma\left(\frac{k+\mu+\nu-m-n}{2}\right)\Gamma(-\mu)\Gamma(-\nu)\Gamma(-\mu-\nu)\Gamma\left(\frac{1-k-\mu-\nu}{2}\right)}{\Gamma\left(-\frac{\mu}{2}-\frac{\nu}{2}\right)\Gamma\left(\frac{\mu+\nu+k}{2}\right)} x^{(1-k-\mu-\nu)}$$
$$\times {}_6F_5\left\{ \begin{matrix} \frac{1+\mu+\nu}{2}, \frac{k+\mu+\nu+m+n}{2}, \frac{k+\mu+\nu+m-n}{2}, \frac{k+\mu+\nu+n-m}{2}, \frac{k+\mu+\nu-m-n}{2}, \frac{2+\mu+\nu}{2}; \frac{1}{x^2} \\ \frac{1+k+\mu+\nu}{2}, 1+\mu, 1+\nu, 1+\mu+\nu, \frac{\mu+\nu+k}{2} \end{matrix} \right\} \Bigg] \Bigg\}$$

$$+\sum_{\nu,-\nu}\left\{\sum_{\mu,-\mu}\left[\frac{\cos\left(\frac{k}{2}+\mu+\nu\right)\pi\,\Gamma\left(\frac{1+\mu+\nu}{2}\right)\Gamma\left(\frac{k+\mu+\nu+m+n}{2}\right)\Gamma\left(\frac{k+\mu+\nu+m-n}{2}\right)\Gamma\left(\frac{k+\mu+\nu+n-m}{2}\right)}{16\pi x}\right.\right.$$

$$\times\frac{\Gamma\left(\frac{k+\mu+\nu-m-n}{2}\right)\Gamma\left(\frac{1-k-\mu-\nu}{2}\right)\Gamma(-\mu)\Gamma(-\nu)\Gamma(-\mu-\nu)}{\Gamma\left(-\frac{\mu}{2}-\frac{\nu}{2}\right)\Gamma\left(\frac{\mu+\nu+k}{2}\right)}\,x^{(1-k-\mu-\nu)}$$

$$\left.\left.\times\,{}_6F_5\left\{\begin{matrix}\frac{1+\mu+\nu}{2},\ \frac{k+\mu+\nu+m+n}{2},\ \frac{k+\mu+\nu+m-n}{2},\ \frac{k+\mu+\nu+n-m}{2},\ \frac{k+\mu+\nu-m-n}{2},\ \frac{2+\mu+\nu}{2};\ \frac{1}{x^2}\\ \frac{1+k+\mu+\nu}{2},\ 1+\mu,\ 1+\nu,\ 1+\mu+\nu,\ \frac{k+\mu+\nu+2}{2}\end{matrix}\right\}\right]\right\}$$

where $R(x)>0$, $R(k\pm\nu\pm\mu\pm m\pm n)>0$. The restriction $R(k)<2$ can be removed by analytical continuation. ... (122)

Again from (15) CHAP-III, if $R(x)>0$

$$\int_0^\infty \lambda^{k-1}K_\nu(\lambda)K_\mu(\lambda)K_m(x/\lambda)K_n(x/\lambda)\,d\lambda =$$

$$\frac{1}{16\pi x}\int_0^\infty \lambda^{k-1}\left[\left\{\sum_{i,-i}\frac{1}{i}E\left(\frac{1}{2}+\frac{\nu}{2}+\frac{\mu}{2},\ \frac{1}{2}+\frac{\mu}{2}-\frac{\mu}{2},\ \frac{1}{2}+\frac{\mu}{2}-\frac{\nu}{2},\ \frac{1}{2}-\frac{\mu}{2}-\frac{\mu}{2}:\frac{1}{2}:e^{i\pi}\lambda^2\right)\right\}\right.$$

$$\left.\times\left\{\sum_{i,-i}\frac{1}{i}E\left(\frac{1}{2}+\frac{m}{2}+\frac{n}{2},\ \frac{1}{2}+\frac{m}{2}-\frac{n}{2},\ \frac{1}{2}+\frac{n}{2}-\frac{m}{2},\ \frac{1}{2}-\frac{n}{2}-\frac{m}{2}:\frac{1}{2}:\frac{e^{i\pi}x^2}{\lambda^2}\right)\right\}\right]d\lambda.$$

Now multiply, evaluate each integral by means of (120), so getting

$$\int_0^\infty \lambda^{k-1} K_\nu(\lambda) K_\mu(\lambda) K_m(x/\lambda) K_n(x/\lambda)\, d\lambda =$$

$$\frac{\pi \cos\frac{\pi k}{2}}{\sin\frac{\pi k}{2}}\left[x^k E\left\{\begin{matrix}\frac{1+\nu+\mu}{2}, \frac{1+\nu-\mu}{2}, \frac{1+\mu-\nu}{2}, \frac{1-\mu-\nu}{2}, \frac{1+m+n-k}{2}, \frac{1+m-n-k}{2}, \frac{1+n-m-k}{2}, \frac{1-n-m-k}{2} : e^{\pm i\pi}x^2 \\ \frac{1}{2}, 1-\frac{1}{2}k, \frac{1}{2}-\frac{1}{2}k\end{matrix}\right\} \right.$$

$$\left. - E\left\{\begin{matrix}\frac{1+\nu+\mu+k}{2}, \frac{1+\nu-\mu+k}{2}, \frac{1+\mu-\nu+k}{2}, \frac{1-\mu-\nu+k}{2}, \frac{1+m+n}{2}, \frac{1+m-n}{2}, \frac{1+n-m}{2}, \frac{1-n-m}{2} : e^{\pm i\pi}x^2 \\ \frac{1}{2}+\frac{1}{2}k, 1+\frac{1}{2}k, \frac{1}{2}\end{matrix}\right\}\right]$$

$$-\frac{e^{\frac{i\pi k}{2}}\pi}{2\sin\frac{\pi k}{2}}\left[x^k E\left\{\begin{matrix}\frac{1+\nu+\mu}{2}, \frac{1+\nu-\mu}{2}, \frac{1+\mu-\nu}{2}, \frac{1-\mu-\nu}{2}, \frac{1+m+n-k}{2}, \frac{1+m-n-k}{2}, \frac{1+n-m-k}{2}, \frac{1-n-m-k}{2} : e^{\pm i\pi}x^2 e^{2i\pi} \\ \frac{1}{2}, 1-\frac{1}{2}k, \frac{1}{2}-\frac{1}{2}k\end{matrix}\right\} \right.$$

$$\left. - e^{-i\pi k} E\left\{\begin{matrix}\frac{1+\mu+\nu+k}{2}, \frac{1+\mu-\nu+k}{2}, \frac{1+\nu-\mu+k}{2}, \frac{1-\mu-\nu+k}{2}, \frac{1+m+n}{2}, \frac{1+m-n}{2}, \frac{1+n-m}{2}, \frac{1-n-m}{2} : (e^{\pm i\pi}x^2 e^{2i\pi}) \\ \frac{1}{2}, \frac{1}{2}+\frac{1}{2}k, 1+\frac{1}{2}k\end{matrix}\right\}\right]$$

$$-\frac{e^{-\frac{i\pi k}{2}}\pi}{2\sin\left(\frac{\pi k}{2}\right)}\left[x^k E\left\{\begin{matrix}\frac{1+\mu+\nu}{2}, \frac{1+\nu-\mu}{2}, \frac{1+\mu-\nu}{2}, \frac{1-\mu-\nu}{2}, \frac{1+m+n-k}{2}, \frac{1+n-m-k}{2}, \frac{1+m-n-k}{2}, \frac{1-n-m-k}{2} : e^{\pm i\pi}x^2 e^{-2i\pi} \\ \frac{1}{2}, 1-\frac{1}{2}k, \frac{1}{2}-\frac{1}{2}k\end{matrix}\right\} \right.$$

$$\left. - e^{i\pi k} E\left\{\begin{matrix}\frac{1+\nu+\mu+k}{2}, \frac{1+\nu-\mu+k}{2}, \frac{1+\mu-\nu+k}{2}, \frac{1-\mu-\nu+k}{2}, \frac{1+m+n}{2}, \frac{1+m-n}{2}, \frac{1+n-m}{2}, \frac{1-n-m}{2} : e^{\pm i\pi}x^2 e^{-2i\pi} \\ \frac{1}{2}, 1+\frac{1}{2}k, \frac{1}{2}+\frac{1}{2}k\end{matrix}\right\}\right]$$

where $R(x)>0$. The restrictions necessary for (120) can be removed by analytical continuation ..... (123)