THE ELASTIC PHOTOPRODUCTION OF NEUTRAL MESONS AT HELIUM

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CONTENTS

I. INTRODUCTION

I § 1. The photoproduction of mesons at nuclei.

The problem of investigating the angular distributions and cross-sections for the production of mesons by high energy electromagnetic radiation at various nuclei has been receiving a great deal of attention from both theoretical and experimental physicists in recent years. Much of the earlier work in this field was concerned with the photoproduction of mesons at hydrogen since analysis of the results of experiments on this reaction gives important information about the properties of mesons and the nature of the meson-nucleon interaction, which is partially obscured by other factors in the results obtained from work on the photoproduction of mesons at complex nuclei. The theoretical analysis of the dependence of the cross-section for the photo-production of mesons at nuclei on these various factors is, however, of considerable interest and forms the principal subject of the work to be outlined in the following chapters. Although much of the discussion is confined to the particular case of the elastic production of neutral mesons at helium, many of the results and methods are readily applicable to this and other possible photoproduction processes at nuclei containing any number of nucleons.

There are three main factors which must be taken into account in the theoretical analysis of the cross-section for the photoproduction of mesons at complex nuclei. The first factor is the form taken by the initial and final nuclear states. The initial state of the system is, of course, given, but the final state is determined by the charge of the meson which is produced and the manner in which the nucleus is disintegrated (if at all) with the result that there are usually a number of competing processes which can occur. The cross-sections for these processes may differ quite considerably owing to the variation in the magnitude of the form factor, which includes the effect of the initial and final nuclear momentum distributions, the operation of the Pauli Exclusion Principle, the difference in the binding energies of the final states and the Coulomb interaction of the outgoing meson with the residual particles. In helium, for instance, the following reactions are possible.

$$\gamma + He^4 \rightarrow He^4 + \pi^0 \quad \text{or} \quad \gamma + He^4 \rightarrow H^3 + n + \pi^+$$

$$\rightarrow He^3 + n + \pi^0 \qquad \rightarrow D^2 + 2n + \pi^+$$

$$\rightarrow H^3 + p + \pi^0 \qquad \rightarrow p + 3n + \pi^+$$

$$\rightarrow D^2 + D^2 + \pi^0 \qquad \rightarrow He^3 + p + \pi^-$$

$$\rightarrow D^2 + p + n + \pi^0 \qquad \rightarrow D^2 + 2p + \pi^-$$

$$\rightarrow 2p + 2n + \pi^0 \qquad \rightarrow n + 3p + \pi^-$$

From a theoretical point of view the elastic photo-production of neutral mesons from helium ($\gamma + He^4 \rightarrow He^4 + \pi^0$) is the most interesting of the above processes, since the initial and final nuclear states are relatively simple both in their spin and space dependence which considerably reduces the mathematical labour involved in the handling of the wave-functions and thus allows a more detailed investigation of the other factors involved to be undertaken. From the experimental point of view there is, as will be seen in the next section, considerable difficulty in selecting out the elastic production process at high energies, although it is the dominant process at energies just above threshold.

The second factor which is of importance in the analysis of the many-body photoproduction problem is the method by which the wave-equation for the system is solved. It is, of course, impossible to solve this wave-equation exactly and the most suitable form of approximation must be found. The most convenient method of approach is to describe the transition operator for the many-body problem in terms of transition operators involving only two particles and to use phenomenological forms for the matrix elements of the latter. Such a description is the Impulse Approximation of Chew, which is described in some detail in Chapter II and which is applied to the problem of the elastic production of neutral mesons at helium in Chapter III. Chapters IV - VIII are devoted to the development and application of a method of reduction of the transition operator which includes some of the effects neglected in the impulse approximation.

The third main factor which occurs in the analysis of the problem is the form to be adopted for the matrix elements of the two particle transition operators. In the application of the impulse approximation it is necessary to know the matrix element of the transition operator for the photoproduction of a meson at a single nucleon and the choice of this matrix element is extensively discussed in Chapter III. When the corrections to the impulse approximation are considered, the matrix element of the transition operator for the scattering of a meson by a nucleon is also needed, and the most suitable choice for this matrix element is examined in Chapter VI.

I § 2. Experimental results on the photoproduction of mesons at hydrogen and helium.

The best form for the matrix element of the single nucleon photoproduction operator is determined in Chapter III by comparison with the experimental results on the photoproduction of mesons at hydrogen. The most recent results in this field have been submitted from Walker et al (36) and Tollestrup et al (34), who examined the production of charged mesons at hydrogen, and from Oakley and Walker (30) and Walker et al (35), who examined the production of neutral mesons at hydrogen.

The two groups engaged in the investigation of the production of charged mesons used different techniques for finding the angular distribution of the mesons produced. The first group (Walker et al (36)) determined the distribution of mesons at a given angle and energy by passing them through a magnetic spectrometer which selected mesons of the required energy, and these were then passed into a chamber containing two liquid scintillation counters. The energy range of the photon covered was 200 - 470 MeV. The second group (Tollestrup et al (34)) detected and identified the mesons by measuring their range and ionisation in a scintillation counter telescope over an incident photon energy range 230 - 450 MeV and at several laboratory angles.

Both groups analysed their results in the form

$$\frac{d\sigma}{d\Omega_+} = A_+ + B_+ \cos\theta + C_+ \cos^2\theta$$

where θ is the meson angle in the centre of momentum system, and the coefficients A_+, B_+ and C_+ deduced from the two experiments were in fairly good agreement.

Four experimental methods can be used to measure the cross-section for the neutral meson photoproduction reaction

$$\gamma + p \longrightarrow p + \pi^0 \quad ; \pi^0 \longrightarrow 2\gamma.$$

(i) Both of the decay γ -rays may be detected in coincidence.

(ii) The proton and one γ -ray may be detected in coincidence.

(iii) One γ -ray alone may be detected.

(iv) Only the recoil proton may be detected.

Walker et al (35) used method (ii) in their experiment on neutral meson production at hydrogen. Method (iv) was employed by Oakley and Walker (30) in their experiments and appears to be the most satisfactory of the four possible methods, provided that the photoproduction process is the sole contributing factor to the proton flux which is measured. Oakley and Walker examined the reaction over an energy range 260 - 450 MeV and expressed their results in the form

$$\frac{d\sigma}{d\Omega_0} = A_0 + B_0 \cos\theta + C_0 \cos^2\theta$$

where θ is the meson angle in the c.m. system.

The values of A_+ and A_0 deduced by these workers are employed in Chapter III paragraph 4, where the most suitable form of the single nucleon photoproduction matrix element is examined.

Not many experimental results on the elastic photoproduction of neutral mesons at helium are at present available. The principal results up to the time of writing have come from Osborne and de Saussure at the Massachusetts Institute of Technology (31, 32) and Goldwasser et al at Illinois (21). The latter detected the meson by measuring both the decay γ-rays in coincidence and examined the reaction from threshold up to a meson energy of about 50 MeV. Their results indicate that in this range and for equal neutral meson energies in the c.m. system the differential cross-section for production in helium is about four times that for production in hydrogen at an angle of 80° to the incident beam in the laboratory system.

Osborne and de Saussure at M.I.T. used the beam from an electron-synchrotron with a peak energy of about 350 MeV. The electrons were made to strike a thin tungsten wire and the resulting beam of photons after being defined by a lead collimator was passed into a tank containing medically pure helium at a pressure of one atmosphere. Photographic plates were arranged inside the tank with one edge parallel to the direction of the beam in such a way that particles would be incident upon them at grazing angles. The energy and direction of the particles were deduced from observation of the tracks. The photographic plates were developed so that only those tracks caused by doubly charged particles could be discerned and thus the only reactions which had to be considered were:

(i) $\gamma + He^4 \longrightarrow He^4 + \gamma$

(ii) $\gamma + He^4 \longrightarrow He^3 + n$

(iii) $\gamma + He^4 \longrightarrow \pi + He^3 + \text{nucleon}$

(iv) $\gamma + He^4 \longrightarrow \pi^0 + He^4$

The cross-section for the Compton scattering process (i) is small while the energy of the recoil nucleus He^3 in the process (iii) is low with the result that most of these nuclei are stopped before they reach the photographic plates. The recoil nuclei from the reactions (ii) and (iv) cannot, however, be distinguished. A low energy run with a maximum energy of 160 MeV was carried out and it was deduced that the distribution of the recoil nuclei in the reaction (ii) was proportional to $\sin^2\theta$ in the laboratory system. Now because of the kinematics of the elastic photoproduction process the recoil He^4 nuclei do not occur at angles of greater than 60° in the laboratory system and hence if the background of He^3 tracks can be assumed to have a distribution of the form $A \sin^2\theta$ then A may be deduced by counting the tracks between 60° and 180° and the background can then be subtracted in the range θ = 0 to 60°. This subtraction procedure is the weak point in the experiment and their results may well be suspect on this account. The results obtained by de Saussure and Osborne (32) for the differential cross-section for the elastic photoproduction of neutral mesons at helium at 90° in the laboratory system are compared with the corresponding theoretical results in Chapters III and VIII.

II. THE IMPULSE APPROXIMATION

As a fairly large proportion of this thesis is concerned with the application of and corrections to the impulse approximation, it was thought worth while to give a brief preliminary survey of the development of the approximation and of some of its previous applications.

Essentially, the impulse approximation consists in replacing the transition operator for an interaction between a particle and a complex nucleus by the sum of the transition operators for the interactions between the incident particle and the individual nucleons within the nucleus. This approximation was first introduced by Chew (9) in his analysis of the elastic scattering of mesons at deuterium and was later employed by Chew and Lewis (13) in their investigation of the photoproduction of mesons at deuterium. However, the method was not put on a firm theoretical foundation until Chew and Goldberger (12) clearly brought out the relationship between the total transition operator for the photoproduction reaction and the individual nucleon operators in such a way that the terms neglected in assuming the impulse approximation could be interpreted and analysed. This was a generalisation of some previous work along these lines by Chew and Wick (14) and Ashkin and Wick (1).

Chew and Goldberger (12) discuss in some detail the errors caused by omitting the effect of the potential binding the nucleons during the interaction and also of the multiple scattering of the incident particle within the target nucleus. They define the total transition operator for the scattering of the particle at the nucleus to be T^+ where

$$T^+ = V + V\frac{1}{E_a + i\varepsilon - H_o - V}V \qquad 2.1$$

Here $H_o = K + U$, K is the total kinetic energy operator for the nucleons.

U is the nuclear binding potential.

$V = \sum_{k=1}^{N} V_k$ is the interaction potential between the incoming particle and the nucleus. It is assumed that V may be expressed as a sum over all the nucleons ($k = 1 \ldots N$) of V_k the interaction potential between the incident particle and the k^{th} nucleon.

$H_o \Phi_a = E_a \Phi_a$, where Φ_a is the initial state of the system and ε is a small positive constant (29).

The transition operator for scattering at the single nucleon (k) is defined to be

$$t_k^+ = V_k + V_k\frac{1}{E_\ell + i\varepsilon - K - V_k}V_k \qquad 2.2$$

where E_ℓ is an eigenvalue of the operator K.

After some manipulation of the operators they show that (their eq. (25))

$$T^+ = \sum_{k=1}^{N} \left\{ t_k^+ + V \frac{1}{E_\alpha + i\varepsilon - H_0 - V} [U, \omega_k^+] + \left[1 + V \frac{1}{E_\alpha + i\varepsilon - H_0 - V} \right] (V - V_k)(\omega_k^+ - 1) \right\} \qquad 2.3$$

where $t_k^+ = V_k \omega_k^+$

Omission of the second and third terms in equation (2.3) leads to the impulse approximation

$$T^+ = \sum_{k=1}^{N} t_k^+ \qquad 2.4$$

By repeated use of an identity relationship between the operators $\frac{1}{E_\alpha + i\varepsilon - H_0 - V}$ and $\frac{1}{E_\ell + i\varepsilon - K - V_k}$ the second term in equation (2.3) can be expanded as a power series in U and may thus be associated with the correction due to the binding energy of the nucleons. In a similar manner the third term may also be developed into a series, the first term of which may be interpreted as representing two consecutive scatterings of the incident particle within the target nucleus, the second as representing three such scatterings and so on. This last term then describes the multiple scattering of the incident particle within the target nucleus. The formal methods developed by Chew and Goldberger, although they separate out the various corrections to the impulse approximation, suffer from the defect that a pertubation like expansion of the terms must be carried out before evaluation of the matrix elements of the terms is possible and such an expansion may not be valid.

A considerable number of calculations on the photoproduction of mesons at deuterium have been performed using the impulse approximation (13), (18), (19), (23), (28). Chew and Lewis (13) examined, in particular, the elastic production of π° mesons at deuterium and derived the differential cross-section for this reaction,

$$\frac{d\sigma}{d\Omega} = 4\,\frac{q q_0}{(2\pi)^2}\left(\frac{2}{3}K^2 + L^2\right)F(p) \qquad 2.5$$

where $F(p) = \int_{-\infty}^{\infty} u^2(\rho)\, e^{\frac{1}{2} i p \rho}\, d\rho$ is the form factor,

$u(\rho)$ is the ground state wave function of the deuteron,

p is the momentum transferred to the deuteron during the reaction,

and q_0, q are the energy and momentum of the meson.

The scattering matrix element for the photoproduction of neutral mesons at a single nucleon (j) is taken to be of the form $\langle q|T_j|\nu\rangle = \sigma(j)\underset{\sim}{K}+L$, where $\sigma(j)$ is the spin operator of the j^{th} nucleon and $\underset{\sim}{K}$ and L are functions of the photon energy, momentum and polarization, and of the meson momentum and energy. It is assumed here that the scattering matrix elements for production of a neutral meson at a proton and neutron are equal both in magnitude and phase. The differential cross-section (2.5) calculated on the basis of the impulse approximation gives results which tend to be rather higher than the experimentally observed points (8).

Some calculations on the elastic photoproduction of π° mesons at helium have been carried out by Osborne and de Saussure (32) in connection with the interpretation of their experimental work on this problem. They employ the impulse approximation and an independent particle model for the nuclear wave-functions but do not obtain satisfactory agreement between their theoretical and experimental results. The theoretical cross-section for the elastic photoproduction of neutral mesons at helium has also been studied by Yamaguchi (38). He has calculated the differential cross-section for this process on the basis of the impulse approximation and using an Irving-type wave function (22) for the helium nucleus, at photon energies of 166 and 200 MeV. His results at 200 MeV are consistent with the corresponding results presented at the end of Chapter III. The theoretical results computed by Osborne and de Saussure and Yamaguchi appeared while the calculations outlined in Chapter III were still in progress and are included in the more comprehensive analysis outlined in that chapter.

III. THE APPLICATION OF THE IMPULSE APPROXIMATION TO THE CALCULATION OF THE DIFFERENTIAL CROSS-SECTION FOR THE ELASTIC PHOTOPRODUCTION OF NEUTRAL MESONS AT HELIUM.

It is not expected that the impulse approximation will give very satisfactory results when applied to problems involving the α -particle since this nucleus is a tightly bound structure. The multiple scattering of the meson within the helium nucleus will probably alter the differential cross-section for the reaction considerably, particularly in the region of energy where the scattering phase-shifts are large, while the binding energy correction may be quite important at energies near the threshold for meson production. However, it happens that the differential cross-section which includes the effect of the multiple scattering of the meson is most easily expressed in terms of the differential cross-section derived on the basis of the impulse approximation so that a knowledge of the latter is essential in deducing the former. The elastic process is selected for reasons which have already been outlined in Chapter I, namely that the calculations for this process are the least complicated by the nuclear wave-functions used in the description of the initial and final nuclear states so that a more detailed investigation of some of the other aspects of the problem can be made.

III § 1. Reduction of the total transition operator

The initial state of the system consists of a photon of momentum $\underset{\sim}{\nu}$ and energy ν (in units $\hbar = c = 1$), and an α -particle at rest in the laboratory system. The final state contains a neutral meson of momentum $\underset{\sim}{q}$ and energy q_0 and an α-particle recoiling with momentum $\underset{\sim}{D}$. The rest mass m_π of the meson is taken to be 135 MeV.

Let T be the transition operator for the photoproduction of mesons at helium and let T_i be the transition operator for the photoproduction of mesons at a single nucleon (i)

Then according to the impulse approximation

$$T = T_1 + T_2 + T_3 + T_4 \qquad 3.1$$

The protons are denoted by the subscripts 1 and 2, the neutrons by the subscripts 3 and 4.

The matrix elements of the operator T must be taken and averaged over the initial and final momentum distributions of the nucleons, for although the nucleons are treated as 'free' during the interaction, they still retain the momentum distribution imposed upon them by the nuclear binding potential.

Let the nucleons initially have the configuration space wave function $\psi_i(12;34)$ and the momentum space wave function $\phi_i(\underset{\sim}{k}_1,\underset{\sim}{k}_2;\underset{\sim}{k}_3,\underset{\sim}{k}_4)$ where $\underset{\sim}{k}_1$, $\underset{\sim}{k}_2$, $\underset{\sim}{k}_3$, $\underset{\sim}{k}_4$ are possible initial momenta for the nucleons, and let the nucleons finally have the configuration space wave-function $\psi_f(12;34)$ and the momentum space wave-function $\phi_f(\underset{\sim}{k}_1',\underset{\sim}{k}_2';\underset{\sim}{k}_3',\underset{\sim}{k}_4')$ where $\underset{\sim}{k}_1'$, $\underset{\sim}{k}_2'$, $\underset{\sim}{k}_3'$, and $\underset{\sim}{k}_4'$ are possible final momenta for the nucleons.

All the wave-functions are normalised and depend upon the spin as well as the space coordinates of the nucleons. They are anti-symmetric under the interchange of 1 and 2 and of 3 and 4 in compliance with the conditions of the Pauli Exclusion Principle.

Taking the appropriate matrix element of the operator T and averaging over the initial and final nuclear momentum distributions of the nucleus, the following expression is obtained for the transitien probability amplitude for the photoproduction of a meson of momentum $\underset{\sim}{q}$ by a photon of momentum $\underset{\sim}{\nu}$.

$$\langle q|T|\nu\rangle = \int d\underset{\sim}{k}_1' d\underset{\sim}{k}_2' d\underset{\sim}{k}_3' d\underset{\sim}{k}_4' d\underset{\sim}{k}_1 d\underset{\sim}{k}_2 d\underset{\sim}{k}_3 d\underset{\sim}{k}_4 \; \phi_f^*(\underset{\sim}{k}_1', \underset{\sim}{k}_2', \underset{\sim}{k}_3', \underset{\sim}{k}_4') \times$$

$$\Big[\delta(\underset{\sim}{k}_1' + \underset{\sim}{q} - \underset{\sim}{k}_1 - \underset{\sim}{\nu})\, \delta(\underset{\sim}{k}_2' - \underset{\sim}{k}_2)\, \delta(\underset{\sim}{k}_3' - \underset{\sim}{k}_3)\, \delta(\underset{\sim}{k}_4' - \underset{\sim}{k}_4) \langle \underset{\sim}{q}, \underset{\sim}{k}_1' | T_1 | \underset{\sim}{\nu}, \underset{\sim}{k}_1 \rangle$$

$$+ \delta(\underset{\sim}{k}_1' - \underset{\sim}{k}_1)\, \delta(\underset{\sim}{k}_2' + \underset{\sim}{q} - \underset{\sim}{k}_2 - \underset{\sim}{\nu})\, \delta(\underset{\sim}{k}_3' - \underset{\sim}{k}_3)\, \delta(\underset{\sim}{k}_4' - \underset{\sim}{k}_4) \langle \underset{\sim}{q}, \underset{\sim}{k}_2' | T_2 | \underset{\sim}{\nu}, \underset{\sim}{k}_2 \rangle$$

$$+ \delta(\underset{\sim}{k}_1' - \underset{\sim}{k}_1)\, \delta(\underset{\sim}{k}_2' - \underset{\sim}{k}_2)\, \delta(\underset{\sim}{k}_3' + \underset{\sim}{q} - \underset{\sim}{k}_3 - \underset{\sim}{\nu})\, \delta(\underset{\sim}{k}_4' - \underset{\sim}{k}_4) \langle \underset{\sim}{q}, \underset{\sim}{k}_3' | T_3 | \underset{\sim}{\nu}, \underset{\sim}{k}_3 \rangle$$

$$+ \delta(\underset{\sim}{k}_1' - \underset{\sim}{k}_1)\, \delta(\underset{\sim}{k}_2' - \underset{\sim}{k}_2)\, \delta(\underset{\sim}{k}_3' - \underset{\sim}{k}_3)\, \delta(\underset{\sim}{k}_4' + \underset{\sim}{q} - \underset{\sim}{k}_4 - \underset{\sim}{\nu}) \langle \underset{\sim}{q}, \underset{\sim}{k}_4' | T_4 | \underset{\sim}{\nu}, \underset{\sim}{k}_4 \rangle \Big]$$

$$\times \; \phi_i(\underset{\sim}{k}_1, \underset{\sim}{k}_2, \underset{\sim}{k}_3, \underset{\sim}{k}_4) \qquad 3.2$$

Since it is assumed that momentum is transferred only to the nucleon at which the photon is incident, the remaining three nucleons being left undisturbed

$$\therefore \langle q|T|\nu\rangle = \int d\underset{\sim}{k}_1 d\underset{\sim}{k}_2 d\underset{\sim}{k}_3 d\underset{\sim}{k}_4 \Big[\phi_f^*(\underset{\sim}{k}_1 + \underset{\sim}{\nu} - \underset{\sim}{q}, \underset{\sim}{k}_2; \underset{\sim}{k}_3, \underset{\sim}{k}_4) \langle \underset{\sim}{q}, \underset{\sim}{k}_1 + \underset{\sim}{\nu} - \underset{\sim}{q} | T_1 | \underset{\sim}{\nu}, \underset{\sim}{k}_1 \rangle$$

$$+ \phi_f^*(\underset{\sim}{k}_1, \underset{\sim}{k}_2 + \underset{\sim}{\nu} - \underset{\sim}{q}; \underset{\sim}{k}_3, \underset{\sim}{k}_4) \langle \underset{\sim}{q}, \underset{\sim}{k}_2 + \underset{\sim}{\nu} - \underset{\sim}{q} | T_2 | \underset{\sim}{\nu}, \underset{\sim}{k}_2 \rangle$$

$$+ \phi_f^*(\underset{\sim}{k}_1, \underset{\sim}{k}_2; \underset{\sim}{k}_3 + \underset{\sim}{\nu} - \underset{\sim}{q}, \underset{\sim}{k}_4) \langle \underset{\sim}{q}, \underset{\sim}{k}_3 + \underset{\sim}{\nu} - \underset{\sim}{q} | T_3 | \underset{\sim}{\nu}, \underset{\sim}{k}_3 \rangle$$

$$+ \phi_f^*(\underset{\sim}{k}_1, \underset{\sim}{k}_2; \underset{\sim}{k}_3, \underset{\sim}{k}_4 + \underset{\sim}{\nu} - \underset{\sim}{q}) \langle \underset{\sim}{q}, \underset{\sim}{k}_4 + \underset{\sim}{\nu} - \underset{\sim}{q} | T_4 | \underset{\sim}{\nu}, \underset{\sim}{k}_4 \rangle \Big]$$

$$\times \; \phi_i(\underset{\sim}{k}_1, \underset{\sim}{k}_2; \underset{\sim}{k}_3, \underset{\sim}{k}_4) \qquad 3.3$$

The next step in the simplification of the above expression is to neglect the dependence of the matrix element $\langle q, \underset{\sim}{k}_f | T | \underset{\sim}{\nu}, \underset{\sim}{k}_i \rangle$ on the nuclear momenta $\underset{\sim}{k}_f$ and $\underset{\sim}{k}_i$, which allows the matrix element to be removed from underneath the integral sign. This is equivalent to assuming that the matrix element may be evaluated as if the nucleon were at rest before the reaction and the recoil of the nucleon could be neglected. The second assumption is the usual adiabatic approximation in which the nucleon is considered to have effectively infinite mass. The first assumption neglects the motion of the nucleons within the nucleus. The average value of the nucleon momentum $\underset{\sim}{k}_j$ is zero so that if the matrix element of T_j is expanded as a power series in $\underset{\sim}{k}_j$ the first term to contribute to the integral over $\underset{\sim}{k}_j$ will be the term containing k_j^2 and this should give a much smaller contribution than the integral over the term in the expansion which is independent of $\underset{\sim}{k}_j$.

With these assumptions the matrix element $\langle q, \underset{\sim}{k}_j + \underset{\sim}{\nu} - q | T_j | \underset{\sim}{\nu}, \underset{\sim}{k}_j \rangle$ reduces to $\langle q, 0 | T_j | \underset{\sim}{\nu}, 0 \rangle$ which is abbreviated to $\langle q | T_j | \nu \rangle$. The matrix element may now be written

$$\langle q | T | \nu \rangle = \sum_{1,2,3,4} \int d\underset{\sim}{k}_1 d\underset{\sim}{k}_2 d\underset{\sim}{k}_3 d\underset{\sim}{k}_4 \, \phi_f^*(\underset{\sim}{k}_1 + \underset{\sim}{\nu} - q, \underset{\sim}{k}_2 ; \underset{\sim}{k}_3, \underset{\sim}{k}_4) \qquad 3.4$$

$$\times \langle q | T_1 | \nu \rangle \, \phi_i(\underset{\sim}{k}_1, \underset{\sim}{k}_2 ; \underset{\sim}{k}_3, \underset{\sim}{k}_4)$$

where the matrix elements $\langle q|T_j|\nu\rangle$ are retained under the integral sign because they contain the spin operators $\sigma(j)$ which do not commute with ϕ_f or ϕ_i .

When ϕ_f and ϕ_i are replaced by their Fourier transforms the expression for $\langle q|T|\nu\rangle$ becomes

$$\langle q|T|\nu\rangle = \sum_{j=1}^{4}\int d\underline{r}_1 d\underline{r}_2 d\underline{r}_3 d\underline{r}_4\, \psi^*(12;34)\, e^{-i\underline{D}.\underline{R}} \langle q|T_j|\nu\rangle\, e^{i(\underline{\nu}-\underline{q})\underline{r}_j}\, \psi(12;34) \qquad 3.5$$

where $\underline{R} = \frac{1}{4}(\underline{r}_1+\underline{r}_2+\underline{r}_3+\underline{r}_4)$ is the coordinate of the centre of mass of the α -particle.

In the above formalism distinction has been made between the matrix elements $\langle q|T_j|\nu\rangle,\ j=1 ---- 4$ for the different nucleons. Present experimental evidence seems to indicate that the matrix elements for production of mesons at protons and neutrons are the same both in magnitude and phase (26, 32). Assuming this equality

$$\langle q|T|\nu\rangle = \sum^{4}\int d\underline{r}_1 d\underline{r}_2 d\underline{r}_3 d\underline{r}_4\, \psi^*(12;34)\, e^{-i\underline{D}.\underline{R}} (\sigma(j).\underline{K}_0 + L_0)\, e^{i(\underline{\nu}-\underline{q})\underline{r}_j}\, \psi(12;34)$$

$$\langle q|T|\nu\rangle = \sum_{j=1}^{4}\int d\underline{r}_1 d\underline{r}_2 d\underline{r}_3 d\underline{r}_4\, \psi^*(12;34)\, e^{-i\underline{D}.\underline{R}} (\sigma(j).\underline{K}_0 + L_0)\, e^{i(\underline{\nu}-\underline{q})\underline{r}_j}\, \psi(12;34)$$

Bef

where $\langle q|T_j|\nu\rangle = \sigma(j)\underline{K}_0 + L_0$ 3.6

choice must be made for the α -particle wave-function.

III § 2. Nuclear-wave functions

The α -particle wave-function which has been denoted by ψ (12;34) contains both a configuration space and spin wave-function. The spin wave-function is particularly simple as it is the combination of four angular momentum wave-functions of the type $Y^m_{\frac{1}{2}}(j)\left(m=\pm\frac{1}{2}\right)$ which has total angular momentum zero and is in addition anti-symmetric under the interchange of particles 1 and 2 and of 3 and 4. The required wave-function is $X_4(12)$ $X_4(34)$ where

$$X_4(i,j)=\frac{1}{2^{\frac{1}{2}}}\left(Y^{\frac{1}{2}}_{\frac{1}{2}}(i)Y^{-\frac{1}{2}}_{\frac{1}{2}}(j)-Y^{\frac{1}{2}}_{\frac{1}{2}}(j)Y^{-\frac{1}{2}}_{\frac{1}{2}}(i)\right) \qquad 3.7$$

and $Y^{\frac{1}{2}}_{\frac{1}{2}}(i)$ is the spin wave function of a single nucleon (i) corresponding to spin up and $Y^{-\frac{1}{2}}_{\frac{1}{2}}(i)$ is the spin wave function corresponding to spin down. Then

$$\psi\,(12;34) = \psi(\underline{r}_1,\underline{r}_2,\underline{r}_3,\underline{r}_4)X_4(12)\;X_4(34) \qquad 3.8$$

Here $\psi(\underline{r}_1,\underline{r}_2,\underline{r}_3,\underline{r}_4)$ is a symmetric configuration space wave-function ($\underline{r}_j$ is the position coordinate of the j^{th} nucleon). Two types of functions have been considered in looking for the best representation of $\psi(\underline{r}_1,\underline{r}_2,\underline{r}_3,\underline{r}_4)$; firstly Gaussian wave-functions (22) of the form

$$\psi(\underline{r}_1,\underline{r}_2,\underline{r}_3,\underline{r}_4) = N_G^{\frac{1}{2}} e^{-\mu_{\alpha G}^2(r_{12}^2+r_{13}^2+r_{14}^2+r_{23}^2+r_{24}^2+r_{34}^2)} \quad 3.9$$

$N_G^{\frac{1}{2}}$ is a normalizing factor, $\underline{r}_{ij} = \underline{r}_j - \underline{r}_i$ is the relative coordinate of nucleons i and j, and $\mu_{\alpha G}$ is a parameter which is adjusted to give the best fit to the binding energy or radius of the α-particle. The use of these functions lead to integrals which may be fairly easily performed. The Fourier transform of (3.9) is for instance

$$\phi(\underline{k}_1,\underline{k}_2,\underline{k}_3,\underline{k}_4) = (2\pi)^3\delta(\underline{k}_1+\underline{k}_2+\underline{k}_3+\underline{k}_4)\frac{\pi^{\frac{9}{2}}}{\mu_{\alpha G}^9 2^6} e^{-\frac{1}{16\mu_{\alpha G}^2}(k_1^2+k_2^2+k_3^2+k_4^2)} \quad 3.10$$

Their principal defect lies in their bad asympotic behaviour as they fall off too rapidly with increasing separation of the nucleons. Experiments on the scattering of protons at carbon and oxygen (15) have illustrated, however, that the Gaussian wave-functions represent the behaviour of these heavier nuclei quite well, but it is doubtful whether they give an equally true picture of the lighter nuclei.

In order to better the asymptotic behaviour of these functions Irving (22) examined wave-functions of the type

$$\psi(\underline{r}_1,\underline{r}_2,\underline{r}_3,\underline{r}_4) = N_I^{\frac{1}{2}} \frac{e^{-\mu_{\alpha I}(r_{12}^2+r_{13}^2+r_{14}^2+r_{23}^2+r_{24}^2+r_{34}^2)^{\frac{1}{2}}}}{(r_{12}^2+r_{13}^2+r_{14}^2+r_{23}^2+r_{24}^2+r_{34}^2)^{n}} \qquad 3.11$$

n is integral or half integral and $\mu_{\alpha I}$ is chosen to give the best fit to the binding energy or radius of the α- particle. The value $n = \frac{1}{2}$ leads to the simplest mathematical analysis and is adopted from now on.

The Fourier transform of (3.11) (with $n = \frac{1}{2}$) is

$$\phi(\underset{\sim}{k}_1,\underset{\sim}{k}_2,\underset{\sim}{k}_3,\underset{\sim}{k}_4) = (2\pi)^3\,\delta(\underset{\sim}{k}_1+\underset{\sim}{k}_2+\underset{\sim}{k}_3+\underset{\sim}{k}_4)\, N_I^{1\frac{1}{2}} \frac{1}{(\mu_{\alpha I}^2+\alpha^2)^4} \qquad 3.12$$

where $N_I^{1\frac{1}{2}}$ is a normalizing constant and $\alpha^2 = \frac{1}{4}(k_1^2+k_2^2+k_3^2+k_4^2)$

The parameters $\mu_{\alpha I}$ and $\mu_{\alpha G}$ may be fixed either by computing the binding energy or the radius of the α- particle using the appropriate wave-function. Adopting the former course Irving (25) has found the following set of values for

$\frac{1}{\mu_{\alpha I}} = 1.7 \times 10^{-13}$cm, giving a binding energy of 28 MeV for He^4.

$\frac{1}{\mu_{\alpha I}} = 2.0 \times 10^{-13}$cm, giving a binding energy of 20.5 MeV for He^4.

and $\frac{1}{\mu_{\alpha I}} = 2.5 \times 10^{-13}$cm, giving a binding energy of 12.5 MeV for He^4.

The value $\frac{1}{\mu_{\alpha G}}$ = 3·65 x 10^{-13} cm leads to a value of 23 MeV for the binding energy of the α- particle.

$\mu_{\alpha I}$ and $\mu_{\alpha G}$ have also been evaluated by the second method. The root mean square of the radius (R) of the α- particle is defined to be proportional to the radius of the circumscribed sphere of the tetrahedron whose sides are equal to the root mean square separation of the nucleons (s), the constant of proportionality (ρ_0) being greater than or equal to unity. From the geometry of the figure

$$R = \rho_0 \frac{3^{\frac{1}{2}}}{2^{\frac{3}{2}}} s , \quad \rho_0 \geqslant 1 \qquad 3.13$$

An elementary calculation shows that $s_G = \frac{3^{\frac{1}{2}}}{2^{\frac{3}{2}} \mu_{\alpha G}}$, $s_I = \frac{7^{\frac{1}{2}}}{3^{\frac{1}{2}} \mu_{\alpha I}}$

Hofstadter et al (24) have estimated the value R = 1·4 x 10^{-13} cm for the r.m.s. of the radius of the α- particle. Then

$$\frac{1}{\mu_{\alpha G}} = \frac{3.7}{\rho_0} \times 10^{-13} \text{ cm} \leqslant 3.7 \times 10^{-13} \text{ cm} \qquad 3.14a$$

$$\frac{1}{\mu_{\alpha I}} = \frac{1.50}{\rho_0} \times 10^{-13} \text{ cm} \leqslant 1.50 \times 10^{-13} \text{ cm} \qquad 3.14b$$

If the mean values of the separation of the nucleons and the mean value of the radius of the α- particle are used in the evaluation of $\mu_{\alpha G}$ and $\mu_{\alpha I}$ instead of the r.m.s.

values of these quantities, then

$$\frac{1}{\mu_{\alpha G}} = \frac{3.33}{\rho_0} \times 10^{-13}\text{cm} \leq 3.33 \times 10^{-13} \text{ cm} \qquad 3.15a$$

and $$\frac{1}{\mu_{\alpha I}} = \frac{1.39}{\rho_0} \times 10^{-13}\text{cm} \leq 1.39 \times 10^{-13} \text{ cm} \qquad 3.15b$$

The values of $\frac{1}{\mu_{\alpha G}}$ calculated from the binding energy and radius of the α-particle are in fairly good agreement ($\rho_0 \simeq 1$) but there is a slight discrepancy between the two different values for $\frac{1}{\mu_{\alpha I}}$. The impulse approximation calculations have been performed using the values $\frac{1}{\mu_{\alpha G}} = 3.3 \times 10^{-13}$ cm and $\frac{1}{\mu_{\alpha I}} = 1.7 \times 10^{-13}$ cm, for the nuclear parameters, although it may be that the value $\frac{1}{\mu_{\alpha I}} = 1.4 \times 10^{-13}$ cm corresponds more closely with $\frac{1}{\mu_{\alpha G}} = 3.3 \times 10^{-13}$ cm.

III § 3. <u>The matrix element for the photoproduction of a meson at a single nucleon</u>

The most convenient form of the matrix element for the photoproduction of a meson at a single nucleon for the purpose of employment in the impulse approximation calculations is one which is both relatively simple in form and gives results which agree with the most recent experimental results on the photoproduction of mesons at hydrogen. Chew (11) has developed a matrix element which satisfies these criteria quite well.

Chew uses the non-relativistic approximation to pseudo-vector coupling between the meson field and the nucleon, which is considered to be fixed at the origin ('static' approximation). The resultant interaction Hamiltonian is

$$h = (4\pi)^{\frac{1}{2}} \frac{f}{m_\pi} \int d\tau \, \rho(r) \sum_{\lambda=1}^{3} \tau_\lambda \underset{\sim}{\sigma} . \underset{\sim}{\nabla} \phi_\lambda(r) \qquad 3.16$$

Here $\rho(r)$ is the source function of the nucleon, τ_λ is a component of the isotopic spin operator of the nucleon (equation 6.5), ϕ_λ is a component of the meson wave function, $\underset{\sim}{\sigma}$ is the spin operator of the nucleon and f is the coupling constant.

In his treatment of the scattering of a meson by a nucleon (the matrix element for the photoproduction process is closely related to the matrix element for this process), Chew (10) neglects the recoil energy of the nucleon with the result that the integrals involved in the equations for the matrix elements diverge at high energies. If the nucleon had instead been treated relativistically the energy denominator would have ensured the convergence of the integrals at high momenta.

To eliminate this divergence Chew introduces a cut-off momentum above which $V(k)$ the Fourier transform of the source function $\rho(r)$, is zero and below which $V(k)$ is unity. If the cut-off momentum is denoted by Kmax, the corresponding cut-off energy is $\omega_{max} = (K_{max}^2 + m_\pi^2)^{\frac{1}{2}}$ and the theory contains two constants f^2 and ω_{max}, which must be determined by comparison with experiment.

By applying to the static approximation renormalisation procedures originally developed in connection with relativistic quantum electrodynamics by Dyson (17), Chew reduces the effective value of the coupling constant f with the result that many graphs which previously gave large contributions can be neglected or treated by perturbation methods. However, certain series of graphs remain in which resonance is possible in the intermediate states and these must be summed to give integral equations. These graphs are characterised by the fact that a meson appears alone with the nucleon in the intermediate states of the transition. The integral equations may be resolved into isotopic spin and angular momentum states by the use of projection operators and from the solution of the separated equations the scattering phase shifts may be calculated.

Of these the phase shift δ_{33} for the total angular momentum $\frac{3}{2}$, total isotopic spin $\frac{3}{2}$, state, is the largest and by equating the values for δ_{33} calculated from his theory and the corresponding experimental values at certain energies Chew arrives at the values $f^2 = 0.058$ and $\omega_{max} = 5.6\, m_\pi$ for the coupling constant squared and the cut-off energy.

The relationship between the matrix elements for photoproduction and scattering of a meson at a single nucleon may readily be found.

Let $t = h + h \frac{1}{a-h} h$ be the transition operator for the scattering of a meson at a nucleon where h is as defined in (3.16).

Here $a = E - H_0 + i\varepsilon$, H_0 is the sum of the free field Hamiltonians and ε is a small positive constant (in accordance with the formalism of Lippmann and Schwinger (29)).

If T is the transition operator for the photoproduction of a meson at a single nucleon and H is the interaction Hamiltonian between the photon field and the nucleon and meson fields then

$$T = (h+H) + (h+H)\frac{1}{a-h-H}(h+H) \qquad 3.17$$

$$= a\frac{1}{a-h} H \frac{1}{a-h} a$$

since H may be treated as a small pertubation. The factor $\frac{1}{a-h}a$ describes the interaction between the nucleon and its surrounding cloud of virtual mesons before the electromagnetic interaction and is taken into account by using physical quantities corresponding to 'real' nucleons rather than 'bare' nucleons. Hence

$$\begin{aligned} T &= a\frac{1}{a-h}H \\ &= H + t\frac{1}{a}H \end{aligned} \qquad 3.18$$

on eliminating h. Hence taking matrix elements

$$\langle q|T|\nu\rangle = \langle q|H|\nu\rangle + \int \frac{dq'}{(2\pi)^3}\langle q|t|q'\rangle\frac{1}{a(q')}\langle q'|H|\nu\rangle \qquad 3.19$$

Here $|\nu\rangle$ is a state containing a nucleon and a photon of momentum $\underset{\sim}{\nu}$ and $|q\rangle$ is a state containing a nucleon and a meson of momentum $\underset{\sim}{q}$ (in the c.m. system). The matrix element of T is the sum of two terms, the first of which is the Born approximation and the second of which takes into account the effect of the scattering of the meson at the nucleon after its production. This latter term is most important in the resonance region (200 MeV $\leq q_0 \leq$ 400 MeV), where the scattering of the meson through the $J = \frac{3}{2}$, $I = \frac{3}{2}$ (J = total angular momentum, I = total isotopic spin) state is dominant.

The various partial waves may be selected out by using a multipole expansion of the vector potential $\underset{\sim}{A}$ occurring in the interaction H .

Chew (11) considers corrections to the Born approximation arising from scattering of the meson through the $\left(\frac{3}{2}, \frac{3}{2}\right)$ state. The Born approximation for charged meson production is

$$\langle q|T^B_{\pm}|\nu\rangle = 2\pi i e \frac{2^{\frac{1}{2}} f}{m_\pi (q_0 \nu)^{\frac{1}{2}}} \left(\underset{\sim}{\sigma} . \underset{\sim}{\varepsilon} - \frac{2 \underset{\sim}{\sigma} . (\underset{\sim}{q} - \underset{\sim}{\nu})(\underset{\sim}{q} . \underset{\sim}{\varepsilon})}{(m_\pi^2 + (\underset{\sim}{q} - \underset{\sim}{\nu})^2)^{\frac{1}{2}}} \right) \qquad 3.20$$

$\underset{\sim}{\sigma}$ is the spin of the nucleon, $\underset{\sim}{\varepsilon}$ the polarization vector of the photon.

ν , $\underset{\sim}{\nu}$ are the energy and momentum of the photon q_0 and $\underset{\sim}{q}$ of the meson.

$\langle q|T^B_{\pm}|\nu\rangle$ is taken to be approximately

$$\langle q|T^B_{\pm}|\nu\rangle = 2\pi i e \frac{2^{\frac{1}{2}} f}{m_\pi (q_0 \nu)^{\frac{1}{2}}} \underset{\sim}{\sigma} . \underset{\sim}{\varepsilon}$$

while the Born approximation for the production of neutral mesons $\langle q|T^B_0|\nu\rangle$ is zero. Angular momentum conservation shows that the meson scattered in the $J = \frac{3}{2}$ state may arise from a magnetic dipole or electric quadrupole electromagnetic transition. General angular momentum arguments (5, 11, 20) lead to the following expressions for the matrix elements for charged and neutral meson production

$$\langle q|T^{+}|\nu\rangle = ie\frac{2^{\frac{1}{2}}f}{m_\pi}\frac{2\pi}{(q_0\nu)^{\frac{1}{2}}}\Big(\underline{\sigma}.\underline{\varepsilon} + 2M_1 i\frac{\underline{q}.\underline{\nu}\times\underline{\varepsilon}}{q\nu} + (M_1+E_2)\frac{(\underline{\sigma}.\underline{\nu})(\underline{\varepsilon}.\underline{q})}{q\nu} - (M_1-E_2)\frac{(\underline{\sigma}.\underline{\varepsilon})(\underline{q}.\underline{\nu})}{q\nu}\Big) \quad 3.22$$

and

$$\langle q|T_0|\nu\rangle = ie\frac{2f}{m_\pi}\frac{2\pi}{(q_0\nu)^{\frac{1}{2}}}\Big(2M_1 i\frac{\underline{q}.\underline{\nu}\times\underline{\varepsilon}}{q\nu} + (M_1+E_2)\frac{(\underline{\sigma}.\underline{\nu})(\underline{\varepsilon}.\underline{q})}{q\nu} - (M_1-E_2)\frac{(\underline{\sigma}.\underline{\varepsilon})(\underline{q}.\underline{\nu})}{q\nu}\Big) \quad 3.23$$

M_1 and E_2 are the effective matrix elements for the transition through the $\left(\frac{3}{2},\frac{3}{2}\right)$ state due to the magnetic dipole and electric quadrupole interactions respectively.

Salzmann (11) assumes the forms:

$$M_1 = m_1\frac{m_\pi\nu}{q^2}e^{i\delta_{33}}\sin\delta_{33} \quad 3.24a$$

$$E_2 = e_2\frac{m_\pi\nu}{q^2}e^{i\delta_{33}}\sin\delta_{33} \quad 3.24b$$

for M_1 and E_2 where m_1 and e_2 are real energy independent constants.

If it is assumed that the photon is incident along the negative z axis and that the nucleon is lying at the origin then the following expressions are obtained by averaging the matrix elements (3.22) and (3.23) over the photon polarization ($\varepsilon_z = 0$ since $\underline{\nu}.\underline{\varepsilon} = 0$ and $\overline{\varepsilon_x^2} = \overline{\varepsilon_y^2} = \frac{1}{2}$, $\overline{\varepsilon_x\varepsilon_y} = 0$).

$$|\underset{\sim}{K}_+|^2_{Av.} = 2\pi^2 a^2 \Big(\frac{2}{m_\pi^2 q_0 \nu} - 4 m_1 \Big(1 - \frac{e_2}{m_1}\Big) \frac{1}{m_\pi^2 q q_0} \cos\delta_{33} \sin\delta_{33} \cos\theta$$
$$+ \frac{\nu}{q_0 q^4} \sin^2\delta_{33} m_1^2 \Big(2\Big(1 - \frac{e_2}{m_1}\Big)^2 \cos^2\theta + \Big(1 + \frac{e_2}{m_1}\Big)^2 \sin^2\theta \Big) \qquad 3.25a$$

$$|L_+|^2_{Av.} = 8\pi^2 a^2 m_1^2 \frac{\nu}{q_0 q^4} \sin^2\delta_{33} \sin^2\theta \qquad 3.25b$$

$$|\underset{\sim}{K}_0|^2_{Av.} = 4\pi^2 a^2 m_1^2 \frac{\nu}{q_0 q^4} \sin^2\delta_{33} \Big(2\Big(1 - \frac{e_2}{m_1}\Big)^2 \cos^2\theta + \Big(1 + \frac{e_2}{m_1}\Big)^2 \sin^2\theta \Big) \qquad 3.25c$$

$$|L_0|^2_{Av.} = 16\pi^2 a^2 m_1^2 \frac{\nu}{q_0 q^4} \sin^2\delta_{33} \sin^2\theta \qquad 3.25d$$

where $\langle q|T^+|\nu\rangle = \underset{\sim}{\sigma}.\underset{\sim}{K}_+ + L_+$, $\langle q|T_0|\nu\rangle = \underset{\sim}{\sigma}.\underset{\sim}{K}_0 + L_0$, $a^2 = 2e^2f^2$
and $\cos\theta = \frac{\underset{\sim}{q}.\underset{\sim}{\nu}}{q\nu}$

The above relationships (3.25) have been derivedon the assumption that the nucleon recoil energy can be neglected, i.e. that the nucleon has infinite mass. To allow for the finite mass of the nucleon equations (3.25) are taken to be correct in the centre of momentum system of the photon and the nucleon. Henceforth quantities measured in the c.m. (centre of momentum) system will be distinguished by the subscript c. The relationships between quantities in the c.m. and laboratory systems are derived in Appendix A.

As the principal object of the problem is not to examine the validity of the single particle matrix element developed from any specific theory but rather to investigate the best method of reducing the transition operator for the interaction into terms of transition operators involving only two particles, and to examine the effect of using different momentum distributions for the nucleons in the α -particle, the constants f , m_1 and e_2 are treated as variable parameters and their optimum values are determined by comparison with recent experimental results.

If $\frac{d\sigma_+}{d\Omega_{q_c}}$ is the differential cross-section for the photoproduction of charged pions in the c.m. system and $\frac{d\sigma_0}{d\Omega_{q_c}}$ is the corresponding quantity for neutral pion production then

$$\frac{d\sigma_+}{d\Omega_{q_c}} = A_+ + B_+\cos\theta_c + C_+\cos^2\theta_c \tag{3.26}$$

where $A_+ = a^2\left(x_1^+ + x_2^+ m_1^2\left(2 + \frac{1}{2}\left(1 + \frac{e_2}{m_1}\right)^2\right)\right)$

$$B_+ = -2a^2 x_3^+ m_1\left(1 - \frac{e_2}{m_1}\right)$$

$$C_+ = a^2 m_1^2 x_2^+\left(\left(1 - \frac{e_2}{m_1}\right)^2 - \frac{1}{2}\left(1 + \frac{e_2}{m_1}\right)^2 - 2\right)$$

$$x_1^+ = \frac{q_c}{m_{\pi^+}^2 v_c}$$

$$x_2^+ = \frac{v_c}{q_c^3}\sin^2\delta_{33}$$

$$x_3^+ = \frac{1}{m_{\pi^+} q_c}\cos\delta_{33}\sin\delta_{33} \tag{3.27}$$

and $\frac{d\sigma_0}{d\Omega_{q_c}} = A_0 + B_0\cos\theta_c + C_0\cos^2\theta_c$

where $A_0 = a^2 m_1^2 x_2^0 \left(4 + \left(1 + \frac{e_2}{m_1}\right)^2\right)$

$B_0 = 0$

$C_0 = a^2 m_1^2 x_2^0 \left(2\left(1 - \frac{e_2}{m_1}\right)^2 - 4 - \left(1 + \frac{e_2}{m_1}\right)^2\right) \quad ; x_2^0 = \frac{V_c}{q_c^3} \sin^2 \delta_{33}$

x_2^+ and x_2^0 differ slightly on account of the difference between the masses of the charged and neutral mesons.

The energy relationship $q_{oc} \simeq V_c$ has been used in obtaining these cross-sections. This is a fairly good approximation in the c.m. system where the initial and final energies of the nucleon are practically equal.

The most accurately determined quantities experimentally are A_+ and A_0, and from these the constants a^2 and $m_1^2\left(1 + \frac{1}{4}\left(1 + \frac{e_2}{m_1}\right)^2\right)$ can be determined.

The experimental results which give the most reliable values for the quantities A_+ and A_0 over the required energy range have already been discussed in Chapter I § 2. The theoretical results for A_+ and A_0 can be calculated using Chew's coupling constant ($f^2 = 0{\cdot}058$) and the functions $(M_1, E_2) = (m_1, e_2) \frac{m_\pi V_c}{q_c^2} e^{i\delta_{33}} \sin \delta_{33}$. The differential cross-section for neutral pion production at 90° in the c.m. system calculated in this way is much lower than the experimental values at all energies. Moreover, the ratio A_o / χ_2^o , where A_o is the Oakley and Walker experimental result, is by no means constant, contrary to the prediction of the Chew theory, particularly at high energies ($\nu > 210$ MeV) where the ratio systematically decreases with increasing energy. Various trial functions for M_1 and E_2 of the form $\frac{\nu_c}{q_c^n} \cdot e^{i\delta_{33}} \sin\delta_{33}$ have been tested and the value of n which gives the best agreement with the experimental results is $n = \frac{5}{2}$

then $$M_1' = e^{i\delta_{33}} \sin\delta_{33} m_1' \frac{m_\pi \nu_c}{q_c^{5/2}} \qquad 3.29a$$

$$E_2' = e^{i\delta_{33}} \sin\delta_{33} e_2' \frac{m_\pi \nu_c}{q_c^{5/2}} \qquad 3.29b$$

where $m_1'^2$ and $e_2'^2$ are constants with the dimensions of an energy. Incorporating these new values

$$A_+ = a'^2 y_1^+ + 2a'^2 m_1'^2 \left(1 + \frac{1}{4}\left(1 + \frac{e_2'}{m_1'}\right)^2\right) y_2^+ \qquad 3.30a$$

and $$A_o = 4a'^2 m_1'^2 \left(1 + \frac{1}{4}\left(1 + \frac{e_2'}{m_1'}\right)^2\right) y_2^o \qquad 3.30b$$

where $y_1^+ = x_1^+ = \frac{q_c}{m_{\pi_+}^2 \nu_c}$

$$y_2^+ = \frac{\nu_c}{q_c^4} \sin^2\delta_{33} \quad , \quad y_2^o = \frac{\nu_c}{q_c^4} \sin^2\delta_{33}$$

and $a'^2 = 2e^2 f'^2$ where f' is chosen to give the best fit to the experimental results.

The ratio A_0 / y_2^0 turns out to be very nearly constant at energies above ν = 200 MeV with a slight tendency to decrease at higher energies, although not nearly so pronounced as the corresponding tendency in A_0 / x_2^0 at these energies. On the other hand below 200 MeV A_0 / y_2^0 drops quickly as might be expected since $\sin \delta_{33}$ behaves like q_c^3 near threshold. In this region of low meson kinetic energy the experimental results are much better fitted by a $\frac{\nu_c}{q_c^3}$ dependence for A_0, although with somewhat larger values of m_1 and e_2 than those derived by Chew. However, the energy range ν = 200 - 350 MeV is the more important from the point of view of the present investigation, and consequently the energy dependence y_2^0 leading to the expression (3.30b) for A_0 is adopted. The average value of the ratio $A_0 / y_2^0 = 4a'^2 m_1'^2 \left(1 + \frac{1}{4}\left(1 + \frac{e_2'}{m_1'}\right)^2\right)$ over the range ν = 180 to 260 MeV turns out to be 2.9×10^{-22} $(\text{MeV})^3$ $(\text{cm})^2$.

The theoretically calculated form for the differential cross-section for the production of positive mesons based on equation (3.26) (with Chew's values for a, m_1 and e_2), agrees fairly well with the experimental data of Walker et al (36), although the theoretical results are rather too low at high photon energies. Moreover the ratio

$$\left\{ A_+ - 2a^2 m_1^2 \left(1 + \tfrac{1}{4}\left(1 + \tfrac{e_2}{m_1}\right)^2\right) x_2^+ \right\} \Big/ x_1^+$$

shows a systematic increase at the higher end of the energy range which suggests that the contribution of the 'scattering' part of the matrix element is not large enough in comparison with the contribution of the Born approximation. The ratio

$$\left\{ A_+ - 2a'^2 m_1'^2 \left(1 + \tfrac{1}{4}\left(1 + \tfrac{e_2'}{m_1'}\right)^2\right) y_2^+ \right\} \Big/ x_1^+$$

is more nearly constant and leads to an average value of $a'^2 = 2{\cdot}6 \times 10^{-25}\,(\mathrm{Mev})^2(\mathrm{cm.})^2$ (Chew's corresponding value is $a^2 = 3{\cdot}26 \times 10^{-25}\,(\mathrm{Mev})^2(\mathrm{cm})^2$). Then

$$m_1' = 14{\cdot}9\,(\mathrm{Mev})^{\frac{1}{2}} \text{ when } \frac{e_2'}{m_1'} = 0 \qquad 3.31a$$

and

$$m_1' = 11{\cdot}8\,(\mathrm{Mev})^{\frac{1}{2}} \text{ when } \frac{e_2'}{m_1'} = 1 \qquad 3.31b$$

III § 4. The differential cross-section for the elastic photoproduction of mesons at helium

It has already been shown that (equation (3.6))

$$\langle q|T|\nu\rangle = \sum_{j=1}^{4} \int d\underset{\sim}{r_1} d\underset{\sim}{r_2} d\underset{\sim}{r_3} d\underset{\sim}{r_4} \, \psi_f^*(12;34)(\underset{\sim}{\sigma}(j).\underset{\sim}{K_o}+L_o) \times e^{i(\underset{\sim}{\nu}-\underset{\sim}{q})\underset{\sim}{r_j}} \psi_i(12;34) \qquad 3.32$$

where $\langle q|T_j|\nu\rangle = \underset{\sim}{\sigma}(j).\underset{\sim}{K_o}+L_o$. Using (3.8) the above expression reduces to

$$\langle q|T|\nu\rangle = \sum_{j=1}^{4} \chi_4^+(12)\chi_4^+(34)(\underset{\sim}{\sigma}(j).\underset{\sim}{K_o}+L_o)\chi_4(12)\chi_4(34) \times \int d\underset{\sim}{r_1} d\underset{\sim}{r_2} d\underset{\sim}{r_3} d\underset{\sim}{r_4} \, \psi^*(\underset{\sim}{r_1},\underset{\sim}{r_2},\underset{\sim}{r_3},\underset{\sim}{r_4}) e^{-i\underset{\sim}{D}.\underset{\sim}{R}+i(\underset{\sim}{\nu}-\underset{\sim}{q})\underset{\sim}{r_j}} \psi(\underset{\sim}{r_1},\underset{\sim}{r_2},\underset{\sim}{r_3},\underset{\sim}{r_4})$$

$$= 4L_o \int d\underset{\sim}{r_1} d\underset{\sim}{r_2} d\underset{\sim}{r_3} d\underset{\sim}{r_4} \, \psi^*(\underset{\sim}{r_1},\underset{\sim}{r_2},\underset{\sim}{r_3},\underset{\sim}{r_4}) e^{-i\underset{\sim}{D}.\underset{\sim}{R}+i(\underset{\sim}{\nu}-\underset{\sim}{q})\underset{\sim}{r_1}} \psi(\underset{\sim}{r_1},\underset{\sim}{r_2},\underset{\sim}{r_3},\underset{\sim}{r_4})$$

The matrix element $\underset{\sim}{K_o}$ for the photoproduction of neutral mesons accompanied by spin flip of the nucleon does not contribute to the elastic photoproduction process.

Now $\psi(\underset{\sim}{r_1},\underset{\sim}{r_2},\underset{\sim}{r_3},\underset{\sim}{r_4})$ is a function only of the relative coordinates $\underset{\sim}{s_j} = \underset{\sim}{r_j} - \underset{\sim}{r_i}$ of the nucleons and not of the centre of mass coordinate $\underset{\sim}{R} = \frac{1}{4}(\underset{\sim}{r_1}+\underset{\sim}{r_2}+\underset{\sim}{r_3}+\underset{\sim}{r_4})$ Hence introducing the change of coordinates

$$\underset{\sim}{R} = \frac{1}{4}(\underset{\sim}{r}_1+\underset{\sim}{r}_2+\underset{\sim}{r}_3+\underset{\sim}{r}_4) \; , \; \underset{\sim}{s}_2 = \underset{\sim}{r}_2-\underset{\sim}{r}_1 \; , \; \underset{\sim}{s}_3 = \underset{\sim}{r}_3-\underset{\sim}{r}_1 \; , \; \underset{\sim}{s}_4 = \underset{\sim}{r}_4-\underset{\sim}{r}_1$$

the expression for $\langle q|T|v\rangle$ becomes

$$\langle q|T|v\rangle = 4L_o(2\pi)^3\delta(\underset{\sim}{v}-\underset{\sim}{q}-\underset{\sim}{D})F \qquad 3.33$$

where $$F = \int d\underset{\sim}{s}_2 d\underset{\sim}{s}_3 d\underset{\sim}{s}_4 \psi^2(\underset{\sim}{s}_2,\underset{\sim}{s}_3,\underset{\sim}{s}_4)\, e^{-\frac{1}{4}i\underset{\sim}{p}(\underset{\sim}{s}_2+\underset{\sim}{s}_3+\underset{\sim}{s}_4)} \qquad 3.34$$

is the form factor and $\underset{\sim}{p} = \underset{\sim}{v}-\underset{\sim}{q}$

F is normalised such that $F = 1$ when $p = 0$.

Then
$$\frac{d\sigma}{d\underset{\sim}{q}\,d\underset{\sim}{D}} = \frac{1}{(2\pi)^2}\,\delta(E_f-E_i)\,\delta(\underset{\sim}{p}-\underset{\sim}{D})\,16|L_o|^2|F|^2$$

Here $\frac{d\sigma}{d\underset{\sim}{q}\,d\underset{\sim}{D}}$ is the differential cross-section for the production of a meson of momentum, $\underset{\sim}{q}$ by a photon of momentum $\underset{\sim}{v}$, such that the α-particle recoils with momentum $\underset{\sim}{D}$.

E_f is the final energy of the system and E_i the initial energy.

Then $E_i = v + 4M - \varepsilon_\alpha$

and $E_f = q_o + 4M - \varepsilon_\alpha + \frac{D^2}{8M}$

ε_α is the binding energy of the α-particle.

$$\therefore \frac{d\sigma}{d\underset{\sim}{q}} = \frac{1}{(2\pi)^2}\,\delta\left(v-q_o-\frac{p^2}{8M}\right)16|L_o|^2|F|^2$$

and $$\frac{d\sigma}{d\Omega_q} = \frac{1}{(2\pi)^2} \frac{q q_0}{1 + \frac{q_0}{4Mq}(q - \nu\cos\theta)} 16 |L_0|^2 |F|^2 \qquad 3.35$$

with $$\nu = q_0 + \frac{|\underline{k} - \underline{q}|^2}{8M} \qquad 3.36$$

It will be observed that the above energy relationship (3.36) is inconsistent with the corresponding energy relationship for production of a meson of momentum $\underline{q}$ at a single nucleon, namely $\nu = q_0 + \frac{|\underline{k}-\underline{q}|^2}{2M}$ and hence it must be assumed that Chew's formulae for M_1 and E_2 or the slightly adapted formulae (3.29) are still valid off the energy shell. The deviation from the energy shell is not large as the recoil factors $\frac{|\underline{k}-\underline{q}|^2}{2M}$ and $\frac{|\underline{k}-\underline{q}|^2}{8M}$ are in general both fairly small in the energy region considered. By writing the functions M_1 and E_2 in the form (3.24) and M_1' and E_2' in the form (3.29) it has already been assumed that only 'on the energy shell' scattering contributes to the integral in (3.19) and it is, therefore, the Born approximation matrix element $\langle q|H|\nu\rangle$ which must be evaluated off the energy shell.

The form factor F will now be computed.

(a) <u>Employing Gaussian Wave-Functions</u>

The Gaussian wave-function as defined in equation (3.9) is

$$\psi(\underline{r}_1,\underline{r}_2,\underline{r}_3,\underline{r}_4) = N_G^{\frac{1}{2}}\, e^{-\mu_{\alpha G}^2(r_{12}^2+r_{13}^2+r_{14}^2+r_{23}^2+r_{24}^2+r_{34}^2)}$$

Using the coordinate transformation

$$\underline{R} = \tfrac{1}{4}(\underline{r}_1+\underline{r}_2+\underline{r}_3+\underline{r}_4) \qquad , \underline{\rho}_1 = -\underline{r}_1 + \tfrac{1}{3}(\underline{r}_2+\underline{r}_3+\underline{r}_4)$$

$$\underline{\rho}_2 = -\underline{r}_2 + \tfrac{1}{2}(\underline{r}_3+\underline{r}_4) \qquad , \underline{\rho}_3 = -\underline{r}_3 + \underline{r}_4$$

then $$F_G = \frac{\int d\underline{\rho}_1 d\underline{\rho}_2 d\underline{\rho}_3\, e^{-2\mu_{\alpha G}^2(3\rho_1^2+\frac{8}{3}\rho_2^2+2\rho_3^2)}\, e^{-\frac{3}{4}i\underline{p}\cdot\underline{\rho}_1}}{\int d\underline{\rho}_1 d\underline{\rho}_2 d\underline{\rho}_3\, e^{-2\mu_{\alpha G}^2(3\rho_1^2+\frac{8}{3}\rho_2^2+2\rho_3^2)}}$$

$$= e^{-\frac{3p^2}{128\mu_{\alpha G}^2}} \qquad 3.37$$

(b) <u>Employing Irving-type Wave-Functions</u>

The Irving-type wave-function as defined in equation (3.11) is

$$\psi(\underline{r}_1,\underline{r}_2,\underline{r}_3,\underline{r}_4) = N_I^{\frac{1}{2}}\, \frac{e^{-\mu_{\alpha I}(r_{12}^2+r_{13}^2+r_{14}^2+r_{23}^2+r_{24}^2+r_{34}^2)^{\frac{1}{2}}}}{(r_{12}^2+r_{13}^2+r_{14}^2+r_{23}^2+r_{24}^2+r_{34}^2)^{\frac{1}{2}}}$$

and the corresponding form factor F_I is given by (using the same coordinate transformation as in (a))

$$F_I = \frac{\int d\underline{\rho}_1 d\underline{\rho}_2 d\underline{\rho}_3\, \frac{e^{-2\mu_{\alpha I}(3\rho_1^2+\frac{8}{3}\rho_2^2+2\rho_3^2)^{\frac{1}{2}}}}{(3\rho_1^2+\frac{8}{3}\rho_2^2+2\rho_3^2)}\, e^{-\frac{3}{4}i\underline{p}\cdot\underline{\rho}_1}}{\int d\underline{\rho}_1 d\underline{\rho}_2 d\underline{\rho}_3\, \frac{e^{-2\mu_{\alpha I}(3\rho_1^2+\frac{8}{3}\rho_2^2+2\rho_3^2)^{\frac{1}{2}}}}{(3\rho_1^2+\frac{8}{3}\rho_2^2+2\rho_3^2)}}$$

which reduces to

$$F_I = \frac{7}{48}\left(\frac{15\tan^{-1}x}{x^7} - \frac{1}{x^6(1+x^2)^3}(15+40x^2+33x^4)\right) \qquad 3.38$$

where $x = \frac{3^{\frac{1}{2}}p}{8\mu_{\alpha I}}$

It is easily verified that $\lim_{x\to 0} F_I = 1$, which is, of course, required by the normalisation condition on F .

Equation (3.35) may now be written:

$$\frac{d\sigma}{d\Omega_q} = \frac{16}{(2\pi)^2} A(q,v)|L_o|^2|F|^2 \qquad 3.39$$

where

$$A(q,v) = \frac{qq_o}{1+\frac{q_o}{4Mq}(q - v\cos\theta)} \qquad 3.40$$

and

$$|L_o|^2 = \begin{cases} 16\pi^2 a^2 m_1^2 \frac{v_c}{q_{oc}q_c^4}\sin^2\delta_{33}\sin^2\theta_c & 3.41a \\ 16\pi^2 a'^2 m_1'^2 \frac{v_c}{q_{oc}q_c^5}\sin^2\delta_{33}\sin^2\theta_c & 3.41b \end{cases}$$

The value of m_1' will depend on the value assumed for $\frac{e_2'}{m_1'}$. Two cases are considered:

(a) $m_1' = m_{11}' = 14.9\ (\text{Mev})^{\frac{1}{2}}$ when $\frac{e_2'}{m_1'} = 0$

and (b) $m_1' = m_{12}' = 11.8\ (\text{Mev})^{\frac{1}{2}}$ when $\frac{e_2'}{m_1'} = 1$

Then
$$|L_o|^2 = \begin{cases} 16\pi^2 a^2 m_1^2 B(q_c,q,v_c)\sin^2\delta_{33}\sin^2\theta & 3.42a \\ 16\pi^2 a'^2 m_{11}'^2 B'(q_c,q,v_c)\sin^2\delta_{33}\sin^2\theta & 3.42b \\ 16\pi^2 a'^2 m_{12}'^2 B'(q_c,q,v_c)\sin^2\delta_{33}\sin^2\theta & 3.42c \end{cases}$$

where $B(q_c, q, \nu_c) = \dfrac{q^2 \nu_c}{q_{oc} q_c^6}$ 3.43a

and $B'(q_c, q, \nu_c) = \dfrac{q^2 \nu_c}{q_{oc} q_c^7}$ 3.43b

and F may assume the forms F_G and F_I given by equations (3.37) and (3.38).

The meson energy corresponding to a given photon energy is calculated from the overall energy conservation condition (3.36).

$$\nu = q_o + \frac{\nu^2 + q^2 - 2q\nu\cos\theta}{8M}$$

The recoil factor is relatively small and the equation may be conveniently solved for q_o by successive approximations to give

$$q_o = \nu - \delta_1 + \delta_2 \qquad 3.44$$

where

$$\delta_1 = \frac{1}{4M}\left(\frac{m_\pi^2}{2} + (\nu^2 - m_\pi^2) f\right); \; \delta_2 = \frac{\nu}{4M} f \delta_1 \; ; \; f = 1 - \frac{\nu\cos\theta}{(\nu^2 - m_\pi^2)^{\frac{1}{2}}}$$

Combination of the three possible values of $|L_o|^2$ with the two possible values of F gives rise to six different expressions for the differential cross-section which are detailed below.

(i) $\frac{d\sigma}{d\Omega_q}(1) = 64\, a^2 m_1^2 AB\, |F_G|^2 \sin^2\delta_{33} \sin^2\theta$ 3.45a

(ii) $\frac{d\sigma}{d\Omega_q}(2) = 64\, a'^2 m_{12}'^2 AB' |F_G|^2 \sin^2\delta_{33} \sin^2\theta$ 3.45b

(iii) $\frac{d\sigma}{d\Omega_q}(3) = 64\, a'^2 m_{11}'^2 AB' |F_G|^2 \sin^2\delta_{33} \sin^2\theta$ 3.45c

(iv) $\frac{d\sigma}{d\Omega_q}(4) = 64\, a^2 m_1^2 AB\, |F_I|^2 \sin^2\delta_{33} \sin^2\theta$ 3.45d

(v) $\frac{d\sigma}{d\Omega_q}(5) = 64\, a'^2 m_{12}'^2 AB' |F_I|^2 \sin^2\delta_{33} \sin^2\theta$ 3.45e

(vi) $\frac{d\sigma}{d\Omega_q}(6) = 64\, a'^2 m_{11}'^2 AB' |F_I|^2 \sin^2\delta_{33} \sin^2\theta$ 3.45f

The value of $64 a^2 m_1^2$, using $f^2 = 0.058$, $e^2 = 1/137$ and $m_1 = \cdot 58$ is $7.1 \times 10^{-24}\ (\text{Mev})^2 (\text{cm})^2$.

The value of $64 a'^2 m_{11}'^2$, using $a'^2 = 2.6 \times 10^{-25}\ (\text{Mev.})^2 (\text{cm})^2$ and $m_{11}' = 14.9\ (\text{Mev.})^{\frac{1}{2}}$ is $3.7 \times 10^{-21}\ (\text{Mev})^3 (\text{cm.})^2$

The value of $64 a'^2 m_{12}'^2$, using $a'^2 = 2.6 \times 10^{-25}\ (\text{Mev.})^2 (\text{cm.})^2$ and $m_{12}' = 11.8\ (\text{Mev})^{\frac{1}{2}}$ is $2.3 \times 10^{-21}\ (\text{Mev})^3 (\text{cm.})^2$

The values $\frac{1}{\mu_{\alpha G}} = 3.3 \times 10^{-13}$ cm. and $\frac{1}{\mu_{\alpha I}} = 1.7 \times 10^{-13}$ cm. are used in the evaluation of the form factors, although there is some indication that the latter may be slightly too high.

The cross-sections (i) - (vi) are calculated over the range 180 - 320 MeV of the energy of the incident photon in the laboratory system at intervals of 20 MeV and at meson laboratory angles of 45°, 60°,

90° and 135°. From the results at these angles, together with the fact that the cross-section is zero at 0° and 180° an estimate of the angular distribution can be made except near the upper end of the photon energy range, where the maximum of the angular distribution occurs below 45° so that the angular distribution is not well determined at these high energies by the set of angles examined.

The behaviour of the scattering phase-shift δ_{33} above meson kinetic energies of about 180 MeV is not accurately known. The phase-shift is a linear function of the meson energy in this region and an estimate of δ_{33} for energies greater than 180 MeV can be obtained by linear extrapolation. This approximation affects the differential cross-section at 90° for photon energies greater than 280 MeV; at 45° and 60° for photon energies greater than 300 MeV; and at 135° for photon energies greater than 250 MeV.

The form factors F_G and F_I are most conveniently calculated as functions of $p\,(=|\nu-q|)$ the range of p being chosen to include the maximum and minimum values of $|\nu-q|$ in the energy range and at the angles examined. Within this range the form factor F_G turns out to be greater for a given value of p than the form factor F_I .

The results of the calculations on the differential cross-sections (i), (ii) and (iii) for the elastic photoproduction of neutral mesons at helium are presented graphically in figures (3.1) to (3.4), in which the differential cross-sections $\frac{d\sigma}{d\Omega_q}(1)$ $\frac{d\sigma}{d\Omega_q}(2)$ and $\frac{d\sigma}{d\Omega_q}(3)$ are plotted as functions of the incident photon energy at laboratory angles of 45°, 60°, 90° and 135°. The experimental points determined by Osborne and de Saussure at 90° are included for comparison in figure (3.3). The angular distributions for various photon energies and for each form of the cross-section may readily be deduced from this set of curves. The corresponding set of curves representing the cross-sections $\frac{d\sigma}{d\Omega_q}(4)$, $\frac{d\sigma}{d\Omega_q}(5)$ and $\frac{d\sigma}{d\Omega_q}(6)$, which include the factor $|F_I|^2$ instead of $|F_G|^2$ are very similar to the curves in figures (3.1) to (3.4), the main difference being that the former are smaller in magnitude than the latter by about thirty percent at each angle and energy. The shapes of the curves representing corresponding cross-sections calculated on the basis of Irving and Gaussian wave-functions are practically identical and, in particular, the maxima occur at nearly the same energies. The difference in magnitude would be decreased by using the value (3.14b) for $\mu_{\alpha I}$ which may correspond more closely to the value of $\mu_{\alpha G}$ used in the calculation of F_G .

A glance at the graphs reveals that the differential cross-sections differ considerably in magnitude. Of the three, $\frac{d\sigma}{d\Omega_q}$ (3), which has been calculated on the assumption that the electric quadrupole contribution to the single particle matrix element is negligible, is the largest at each angle and energy, while the cross-section $\frac{d\sigma}{d\Omega_q}$ (1) is a factor of two or three smaller, as would be expected, since the single nucleon photoproduction cross-section derived using Chew's form for M_1 and E_2 and his constants is considerably lower than the corresponding cross-section derived from the adapted forms of M_1 and E_2 and constants which are evaluated by comparison with experimental results. The maximum of the curve $\frac{d\sigma}{d\Omega_q}$ (1) tends to occur at slightly greater energies than that of the other two, since the function X_2^0 (equation (3.27)) does not fall off so rapidly at high energies as the function Y_2^0 (equation 3.30)).

Comparison with the experimental results of

Osborne and de Saussure (32) at 90^{o} (fig.(3.3)) seems to indicate that the cross-section $\frac{d\sigma}{d\Omega_q}$(3) gives the most satisfactory agreement between theory and experiment. The theoretical results are somewhat lower than the experimental and fall away more rapidly on either side of the maximum, which occurs at approximately the same energy (250 MeV) in both cases. However, as pointed out in the introductory paragraph to this Chapter, it is not expected that the impulse approximation should give very satisfactory results for this reaction and the inclusion of some of the corrections to the impulse approximation should render the comparison between theory and experiment more profitable.

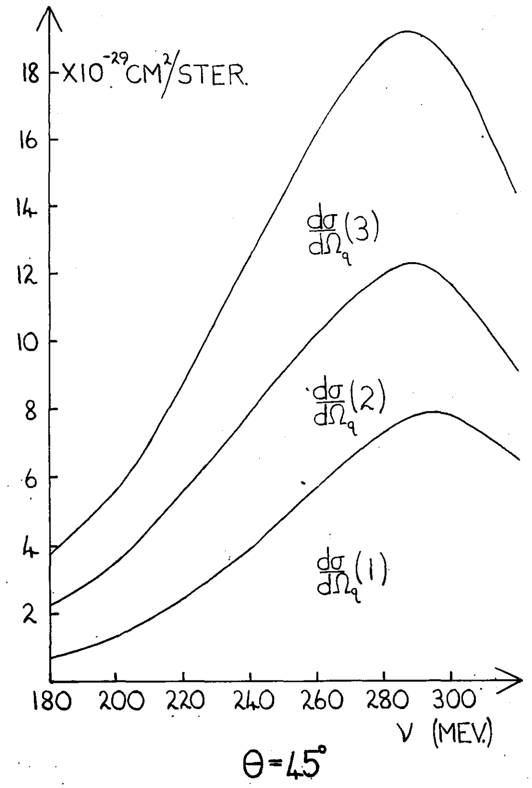

18
16
14
12
10
8
6
4
2
XIO^{-29} CM^2/STER.
$\frac{d\sigma}{d\Omega_q}(3)$
$\frac{d\sigma}{d\Omega_q}(2)$
$\frac{d\sigma}{d\Omega_q}(1)$
180 200 220 240 260 280 300
ν (MEV.)
θ=45°

Fig. 3.1.

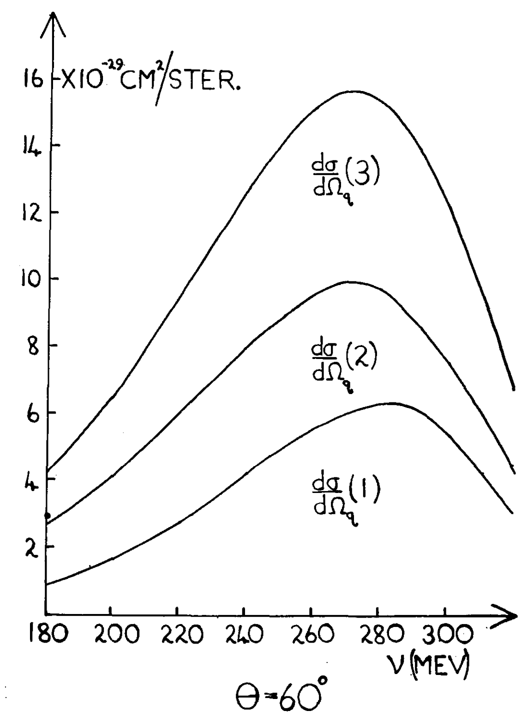
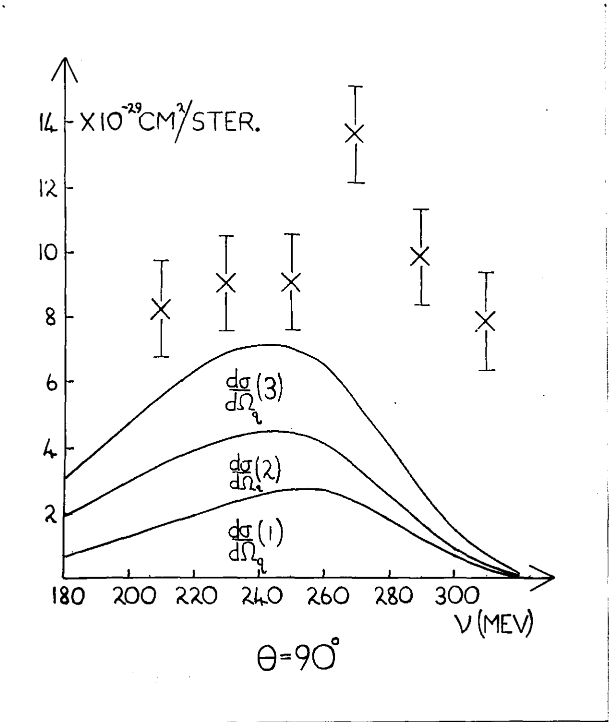
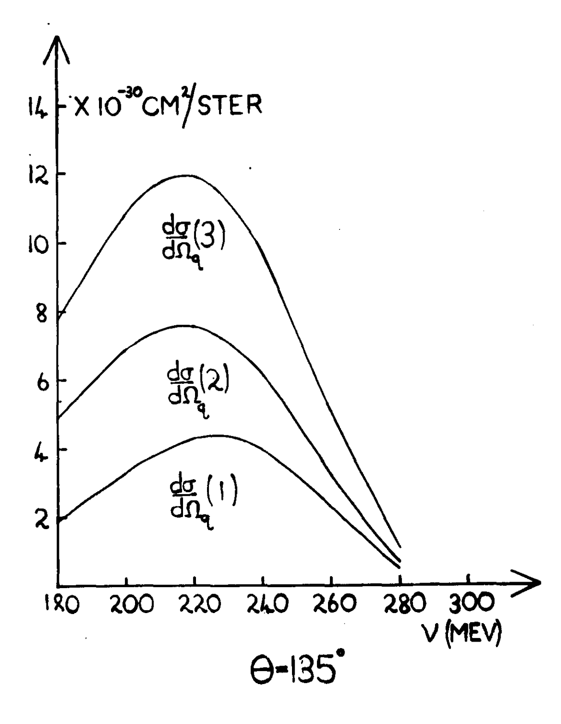

16
14
12
10
8
6
4
2
XIO^{-29} CM^2/STER.
$\frac{d\sigma}{d\Omega_q}(3)$
$\frac{d\sigma}{d\Omega_q}(2)$
$\frac{d\sigma}{d\Omega_q}(1)$
180 200 220 240 260 280 300
ν(MEV)
θ=60°

Fig. 3.2.



14
12
10
8
6
4
2
X 10^-29 CM²/STER.
dσ/dΩq (3)
dσ/dΩq (2)
dσ/dΩq (1)
180 200 220 240 260 280 300
ν (MEV)
θ=90°

Fig. 3.3.



14
12
10
8
6
4
2
X 10^-30 CM²/STER
dσ/dΩq (3)
dσ/dΩq (2)
dσ/dΩq (1)
180 200 220 240 260 280 300
ν (MEV)
θ-135°

Fig. 3.4.

IV. AN INTRODUCTORY SURVEY OF PREVIOUS WORK ON THE MULTIPLE SCATTERING OF MESONS AT NUCLEI.

In the discussion of the impulse approximation (Chapter II) brief reference was made to the various errors which are implicit in this method. Several authors have examined the nature of these errors and estimated the magnitude of the correction to the impulse approximation solution which arises from them. Chew and Goldberger (12) have systematically reduced the transition operator for the scattering of a particle at a complex nucleus and have interpreted the terms which are neglected in assuming the impulse approximation. Watson (37) has examined the general problem of the scattering of a meson at a complex nucleus, and Brueckner and Watson (7), Chappelear (8), and Brueckner (4), have examined the particular case of the scattering of a meson at two nucleons in some detail.

Chew and Goldberger (12) show that one of the principal errors in assuming the impulse approximation arises from the neglect of the multiple scattering of the particle within the target nucleus and manage to separate this effect in the term

$\left[1+V\frac{1}{E_a+i\varepsilon-H_0-V}\right](V-V_k)(\omega_k^+-1)$ (equation (2.3)).
However, as it stands, the matrix element of this term cannot be evaluated since the operator $E_a+i\varepsilon-H_0-V$ still appears in the denominator. The authors show that, by use of an operator identity, this term may be split up into a term which describes double scattering of the incident particle and a series of terms which contain higher order effects. Presumably by repeated application of the operator identity, terms describing three consecutive scatterings of the incident particle and so on may be isolated, but this method of development of the multiple scattering term is equivalent to a perturbation treatment of the problem which is not valid at small separations of the nucleons and a different approach which describes the multiple-scattering effect in terms of integral equations is advisable.

Watson (37) has adopted such an approach in his discussion of the multiple-scattering of mesons at complex nuclei, and the application of the multiple scattering correction to the impulse approximation solution for the photoproduction of mesons in complex nuclei. Much of his discussion is concerned with the solution of the multiple-scattering equations for large nuclei and the relation of this solution to the corresponding solution derived on the basis of various optical models for the nucleus, but his method of obtaining and the approximations made in his derivation of the equations are of considerable interest from the point of view of the methods developed in Chapter V.

Watson assumes that the reaction is induced by particles whose energies are large compared with the binding energies of the nucleons in the nucleus and on this basis can neglect the nuclear binding potential. The total Hamiltonian H which he employs is written in the form

$$H = H_0 + R + V + H' \qquad 4.1$$

where H_0 is the sum of the free field Hamiltonians,

R is the operator which permits re-absorption of the meson $R = \sum_\mu R_\mu$ where R_μ is the operator for absorption of a meson by a pair of nucleons and the summation over μ is taken over all possible nucleon pairs,

$V = \sum_\alpha V_\alpha$ where V_α is the scattering potential between the meson field and the α^{th} nucleon,

and H' is the interaction Hamiltonian between the photon field and nucleon and meson fields.

Watson assumes that once the meson has been absorbed by an operator R_μ, the interaction is finished, since he neglects the possibility of spontaneous re-emission of the meson by the operator R. The operator R also gives rise to a scattering term $\Delta = R\frac{1}{a}R$, which is a many body scattering operator.

The Hamiltonian (4.1) leads to a transition operator T, which, treating H' as a small perturbation, is given by

$$T = (H'+R+V)(1+\Omega\frac{1}{a}H') \qquad 4.2$$

where the operator Ω satisfies the equation

$$\Omega = 1 + \frac{1}{a}(R+V)\Omega \qquad 4.3$$

The above equation is most readily solved in terms of the auxiliary equation

$$\Omega_s = 1 + \frac{1}{b}V\Omega_s \qquad 4.4$$

where $b = a - \Delta$ and $a = E - H_0 + i\varepsilon$ (Lippmann and Schwinger (29))

and $$\Omega = (1+\frac{1}{a}R)\Omega_s(1+\frac{1}{b}\Delta) \qquad 4.5$$

The formal solution of (4.4) is

$$\Omega_s = 1 + \frac{1}{b}\sum_{\alpha_1} t_{\alpha_1}\Omega_s(\alpha_1) \qquad 4.6a$$

$$\Omega_s(\alpha_1) = 1 + \frac{1}{b}\sum_{\alpha_2\neq\alpha_1} t_{\alpha_2}\Omega_s(\alpha_2) \qquad 4.6b$$

(4.6) represents a set of coupled integral equations. Here

$$t_{\alpha_1} = V_{\alpha_1} + V_{\alpha_1}\frac{1}{b - V_{\alpha_1}}V_{\alpha_1}$$

which Watson shows is approximately equivalent in the range of meson energy considered to the operator for the scattering of a meson at a free nucleon

$$t^{\circ}_{\alpha_1} = V_{\alpha_1} + V_{\alpha_1}\frac{1}{a - V_{\alpha_1}}V_{\alpha_1}$$

Suitable phenomenonological forms are chosen for the matrix elements of the operators R, Δ and $t^{\circ}_{\alpha_1}$. Watson does not attempt to obtain the general solution of the multiple scattering equations (4.6), but examines approximate forms of the solution for large numbers of nucleons. If re-absorption of the meson is not permitted (R = 0) the equations (4.2) - (4.6) reduce to forms which are similar to those considered in Chapter V.

Brueckner and Watson (7) have discussed the multiple scattering of a meson between two nucleons in their investigation of the potential which gives rise to nuclear forces, while Brueckner (4) has investigated the magnitude of the correction due to multiple scattering effects to the impulse approximation solution for the scattering of a meson by two nucleons. Chappelear

(8) has extended Brueckner's treatment to include the multiple scattering effects in the elastic photo-production of neutral mesons at deuterium. Although the work of these authors differs quite considerably in detail, the equations involved, the approximations made and the form of the solutions obtained are fundamentally the same in each case. As the treatment of the multiple-scattering problem described in Chapters V and VI was originally based on Chappelear's approach to the two-nucleon problem, a summary of his work is now given. The various approximations which he makes during the development of his solution are presented without comment which will be reserved until the corresponding approximations are made in dealing with the general many-body problem.

The following definitions are needed in the analysis which follows.

Let $h_{1,2}$ be the interaction terms in the Hamiltonian between the meson and the nucleon fields (the nucleons are numbered 1 and 2),

$H_{1,2}$ be the interaction terms in the Hamiltonian between the meson and nucleon fields and the photon field.

Let $h = h_1 + h_2$, $H = H_1 + H_2$, $H^1 = h + H$ and $H_T = H_0 + H^1$ where H_0 is the sum of the free field Hamiltonians.

Then T the transition operator for the problem is

$$T = H' + H' \frac{1}{a - H'} H' \qquad \text{where} \quad a = E - H_0 + i\varepsilon \tag{4.7}$$

(Lippmann and Schwinger (29)).

The corresponding transition operator for the single nucleon photoproduction problem is

$$T_i = (h_i + H_i) + (h_i + H_i) \frac{1}{a - h_i - H_i} (h_i + H_i) \tag{4.8}$$

and
$$t_i = h_i + h_i \frac{1}{a - h_i} h_i \tag{4.9}$$

is the transition operator for the scattering of a meson at a single nucleon. Then treating H, H_1 and H_2 as small perturbations and retaining only those terms which are linear in them T is equivalent to

$$T = \left\{1 + h \frac{1}{a-h}\right\} \left[\left\{1 + h_1 \frac{1}{a - h_1}\right\}^{-1} T_1 \left\{1 + \frac{1}{a - h_1} h_1\right\}^{-1} + \left\{1 + h_2 \frac{1}{a - h_2}\right\}^{-1} T_2 \left\{1 + \frac{1}{a - h_2} h_2\right\}^{-1} \right] \left\{1 + \frac{1}{a-h} h\right\} \tag{4.10}$$

Chappelear makes the approximation of putting the factors $\left\{1 + \frac{1}{a - h_i} h_i\right\}^{-1} \left\{1 + \frac{1}{a - h} h\right\}$ $(i = 1, 2)$ in the above expression equal to unity on the grounds that they are largely taken into account by use of the correct wave-function for the deuteron.

Then

$$T = a\frac{1}{a-h}\left[a\frac{1}{a-h_1}\right]^{-1}T_1 + a\frac{1}{a-h}\left[a\frac{1}{a-h_2}\right]^{-1}T_2 \qquad 4.11$$

The operator $\frac{1}{a-h}$ must be expressed in terms of the single nucleon transition operators t_1 and t_2 and the matrix element of the resultant expression evaluated. Owing to the relative simplicity of the two nucleon problem, the reduction process may be carried out in a fairly straightforward manner which is not applicable to the general problem. Eliminating h_i by the relationship $h_i = \frac{1}{1+t_i\frac{1}{a}}\,t_i$ equation (4.17) becomes

$$T = T_{11} + T_{12} + (1 \rightleftarrows 2) \qquad 4.12$$

where

$$T_{11} = y_1 T_1 \qquad 4.13a$$

$$T_{12} = y_2 t_2 \frac{1}{a} T_1 \qquad 4.13b$$

$$y_1 = \left(1 - t_1\frac{1}{a}t_2\frac{1}{a}\right)^{-1} \qquad 4.14a$$

$$y_2 = \left(1 - t_2\frac{1}{a}t_1\frac{1}{a}\right)^{-1} \qquad 4.14b$$

The form adopted for the matrix element of t_i is

$$\langle q_1|t_i|q_2\rangle = b_i(q_2)\,q_1 q_2\, e^{i(q_2-q_1)r_i} \qquad 4.15$$

q_2 and q_1 are the initial and final momenta of the scattered meson and $b_i(q_2)$ is dependent on the meson energy and also contains the isotopic dependence of the scattering operator t_i .

Three assumptions have been made about the nature of the scattering transition in order to derive the form (4.15) for the scattering matrix element; firstly that only 'on the energy shell' scattering need be taken into account; secondly that there is a resonance in the ($\frac{3}{2}$, $\frac{3}{2}$) state of angular momentum and isotopic spin (Brueckner and Watson (5)) and lastly that there is no spin flip of the nucleon.

The form chosen by Chappelear for the matrix element of the photoproduction operator T_i is

$$\langle q|T_i|\nu\rangle = (\alpha_i + \gamma_i \cdot \underline{q})\, e^{i(\underline{\nu}-\underline{q})\cdot \underline{r}_i} \qquad 4.16$$

α_i and γ_i are energy dependent and are also matrices in charge space. The contribution of the term α_i to the total photoproduction matrix element turns out to be small and hence this term may be neglected.

Using (4.15) the integral equations (4.14) may be solved for y_1 and y_2 and the matrix elements of T_{11} and T_{12} deduced. The results are

$$\langle q|T_{11}|\nu\rangle = e^{i(\underline{\nu}-\underline{q})\cdot \underline{r}_1}\, \underline{q}\left[\frac{1}{1-b_1b_2f^2}\left(\gamma + b_1b_2\frac{g(f+h)}{1-b_1b_2h^2}\,\underline{\rho}(\underline{\rho}\cdot\gamma)\right)\right] \qquad 4.17$$

and $$\langle q|T_{12}|\nu\rangle = -e^{i(\underset{\sim}{\nu}\cdot\underset{\sim}{r_1}-\underset{\sim}{q}\cdot\underset{\sim}{r_2})}\left[\frac{1}{1-b_1b_2f^2}\,b_2\left(f\chi+g(1+b_1b_2fh)\frac{1}{1-b_1b_2h^2}f(\rho\chi)\right)\right] \quad 4.18$$

where $\rho = \underset{\sim}{r_2}-\underset{\sim}{r_1}$ is the relative separation of the nucleons

$b_i = b_i(q)$ evaluated on the energy shell

$$h = f + g\rho^2$$

$$f = -\frac{q_0}{2\pi}\frac{1}{\rho}\frac{d}{d\rho}\frac{e^{iq\rho}}{\rho} \quad 4.19$$

$$g = -\frac{q_0}{2\pi}\frac{1}{\rho}\frac{d}{d\rho}\frac{1}{\rho}\frac{d}{d\rho}\frac{e^{iq\rho}}{\rho} \quad 4.20$$

and q_0 and q are the energy and momentum of the meson on the energy shell.

In order to obtain a numerical result Chappelear introduces phenomenological forms for χ_i and b_i in the equations (4.17) and (4.18) and a matrix representation for the isotopic spin operators. He then averages the matrix element (4.12) over the ground state wave-function of the deuteron which is chosen to have the configuration space dependence

$$\Phi_D(\rho) = \left(\frac{28\alpha}{9}\right)^{\frac{1}{2}}\left(\frac{e^{-\alpha\rho}-e^{-7\alpha\rho}}{\rho}\right)$$

where α = 45.5 MeV.

Chappelear has calculated the angular distribution of the mesons at two photon energies which correspond to meson-nucleon scattering phase shifts δ_{33} of about 45° and 90°. He finds that the multiple scattering depresses the cross-section by roughly the same factor at all angles. The experimental results obtained by Silverman and Stearns(33) tend to agree rather better with the cross-section which includes the multiple scattering effect than with the impulse approximation cross-section. Unfortunately, no experimental results are available in the region of forward angles ($< 90^{o}$) where the difference between the theoretical cross-sections is more marked, so that it is not possible to differentiate with any assurance between the validity of the two methods.

V. DERIVATION OF THE MULTIPLE SCATTERING EQUATIONS.

§ 1. The Integral Equations

In Chapter IV it was seen that various authors have managed to estimate the magnitude of the multiple scattering effect in two special cases; firstly when only two nucleons are involved, in which case the reduction of the total transition operator is very much simplified, and secondly, when the nucleus is large, in which case it has been shown that the solution of the multiple scattering equations can be identified with the solution of the multiple scattering problem derived on the basis of optical models for the nucleus. In this Chapter a method will be examined whereby the multiple scattering equations for any number of particles may be reduced to a series of linear simultaneous equations, with the aid of approximations similar in character to those employed by Chappelear (8).

The multiple scattering of a meson between A nucleons, numbered 1 to A, is considered. The following definitions are required:

$h_{1...A}$ = interaction terms in the Hamiltonian between the meson field and the nucleon field.

$H_{1,2 \cdots A}$ = interaction terms in the Hamiltonian between the meson and nucleon fields and the radiation field.

$$h = h_1 + h_2 + \cdots\cdots + h_A$$

$$H = H_1 + H_2 + \cdots\cdots + H_A$$

$H_0 = K + H_\pi + H_v$ where K is the kinetic energy operator for the nucleons

H_π is the free field Hamiltonian for the meson field, and

H_v is the free field Hamiltonian for the radiation field

$$a = E - H_0 + i\varepsilon$$

where ε is a small positive constant and E is the eigenvalue of H_0 corresponding to the initial state of the system

$$H' = h + H$$

$T = H' + H' \dfrac{1}{a - H'} H'$ is the transition operator for the problem 5.1

$T_i = a \dfrac{1}{a - h_i - H_i} (h_i + H_i)$ is the transition operator for the photo-production of a meson at nucleon i, 5.2

and $t_i = h_i + h_i \dfrac{1}{a - h_i} h_i$ is the transition operator for the interaction of the meson field with nucleon i. 5.3

It will be observed from the above that there has been no potential explicitly included for the binding of the nucleons.

The transition operator T may be written as

$$T = a\frac{1}{a-h-H}(h+H) \qquad 5.4$$

Since the coupling between the radiation and nucleon and meson fields is weak only those terms which are linear in H need be retained as the contribution of terms quadratic or of higher powers in H is negligible. The term independent of H does not, of course, contribute to the photoproduction process at all and is, therefore, omitted.

To this approximation

$$\begin{aligned} T &= a\frac{1}{a-h}H\frac{1}{a-h}a \\ &= \sum_{i=1}^{A} a\frac{1}{a-h}H_i\frac{1}{a-h}a \end{aligned} \qquad 5.5$$

and in a similar manner T_i reduces to

$$T_i = a\frac{1}{a-h_i}H_i\frac{1}{a-h_i}a \qquad 5.6$$

Equation (5.6) is used to eliminate H_i from (5.5). Then

$$T = \sum_{i=1}^{A} a\frac{1}{a-h}(a-h_i)\frac{1}{a}T_i\frac{1}{a}(a-h_i)\frac{1}{a-h}a \qquad 5.7$$

Now the operator $\frac{1}{a}(a-h_i)\frac{1}{a-h}a$ in equation (5.7) above describes the exchange of mesons between the nucleons before the interaction T_i and these processes may be taken into account by using a reasonable wave-function for the nucleons initially. It is also possible that a virtual meson which is produced at one of the nucleons by this operator might not be re-absorbed at another until after the interaction T_i. If the binding of the nucleons is neglected during the photoproduction reaction this type of process need not be taken into account, and a fairly good approximation to T may be obtained by putting $\frac{1}{a}(a-h_i)\frac{1}{a-h}a$ equal to unity and using the correct wave-function for the initial nucleus.

Then

$$T = \sum_{i=1}^{A} a\frac{1}{a-h}(a-h_i)\frac{1}{a}T_i \qquad 5.8$$

The meson-nucleon scattering operator is

$$\begin{aligned} t_i &= h_i + h_i\frac{1}{a-h_i}h_i \\ &= a\frac{1}{a-h_i}h_i \\ &= t_{si} + t_{ai} \end{aligned} \qquad 5.9$$

Where $t_{si} = a \frac{1}{a - h_i \frac{1}{a} h_i} h_i \frac{1}{a} h_i$ 5.10a

and $t_{ai} = a \frac{1}{a - h_i \frac{1}{a} h_i} h_i$ 5.10b

t_{si} can describe the scattering of a meson or the absorption or production of an even number of mesons, while t_{ai} can describe the absorption or production of an odd number of mesons.

The total transition operator T can be expressed as

$$T = \sum_{i=1}^{A} y_i T_i \qquad 5.11$$

where $y_i = a \frac{1}{a-h} (a - h_i) \frac{1}{a}$

$$= \frac{1}{1 - h\frac{1}{a}h\frac{1}{a}} \left(1 - h\frac{1}{a}h_i\frac{1}{a}\right) + \frac{1}{1 - h\frac{1}{a}h\frac{1}{a}} \left(h\frac{1}{a} - h_i\frac{1}{a}\right) \qquad 5.12$$

The interaction T_i produces a single meson and the second term in the above expression for y_i can only represent the production or absorption of an odd number of mesons and hence must give rise to a photodisintegration process or a process involving the multiple production of mesons and neither of these reactions is being considered. Thus for the photoproduction of a single meson

$$y_i = \frac{1}{1-h\frac{1}{a}h\frac{1}{a}}\left(1-h\frac{1}{a}h_i\frac{1}{a}\right) \qquad 5.13$$

For reasonably high photon energies the binding energy of the nucleons is small in comparison with the energy of the photoproduced meson, and hence processes which lead to the exchange of virtual mesons between the nucleons while the photoproduced meson is being scattered between the individual nucleons may be neglected. To this approximation not more than one meson is being exchanged between the nucleons at a given time, and the problem of expressing the operator y_i in terms of the operators t_i is considerably simplified.

A typical term from the expansion of y_i in powers of h_j can be written as

$$T_{bz} = U_{bc}\frac{1}{a}\cdots\cdots\frac{1}{a}U_{pq}\frac{1}{a}U_{rs}\cdots\cdots\cdots\frac{1}{a}U_{yz}\frac{1}{a}$$

where $U_{pq} = h_p\frac{1}{a}h_q$. U_{pq} can represent the following reactions:

(i) The absorption of a meson at nucleon q followed by the absorption of a meson at nucleon p.

(ii) The production of a meson at nucleon q followed by the production of a meson at nucleon p.

(iii) The absorption of a meson at nucleon q followed by the production of a meson at nucleon p.

(iv) The production of a meson at nucleon q followed by the absorption of a meson at nucleon p.

It is assumed that the potential h_j has been renormalised, so that graphs contributing to the self energy of the nucleons need not be considered. This, together with the assumption that the binding energy can be neglected during the multiple scattering process, implies that only those graphs which describe the exchange of the photoproduced meson between the nucleons or the scattering of the photoproduced meson at one of the nucleons need be retained for consideration.

The term T_{bz} may be written

$$T_{bz} = U_{bc}\frac{1}{a} \cdots U_{pq}V.$$

The case $q \neq p$ is considered. The operator V contains an even number (2M) of annihilation (A) and creation (C) operators (A + C = 2M). Since one meson is present initially, V must give rise to an odd number of mesons (C - A + 1 = 2(C - M) + 1). If this number is greater than one the graphs which can result from the action of the operator U_{pq} contribute either to self-energy effects, which have been taken into account in the renormalisation of h_j, or to processes in which two or more mesons are simultaneously being exchanged between the nucleons and these have been excluded by the assumption that the binding energy of the nucleons can be neglected (these conclusions are not necessarily correct in the case $q = p$) and hence V must give rise to a single meson. For similar reasons it can be shown that h_q cannot produce a second meson but must absorb the meson produced by the operator V. This immediately excludes the possibilities (ii) and (iv) in which a meson is produced at the nucleon q. The possible reaction (i) assumes that at least three mesons must be present as a result of the action of the operator V and for the reasons outlined above, must also be excluded.

The process described in (iii) (again $q \neq p$) is only important for close separations of the nucleons, where the weighting factor arising from the integration over the initial and final nuclear wave-functions is small, and can, therefore, be neglected. Then

$$U_{pq} = \delta_{pq} h_p \frac{1}{a} h_p$$

$$= \delta_{pq} u_p \qquad 5.14$$

and

$$u = h \frac{1}{a} h$$

$$= \sum_{j=1}^{A} u_j \qquad 5.15$$

Therefore, from equation (5.13)

$$y_i = \frac{1}{1-\sum_{j=1}^{A} u_j \frac{1}{a}} \left(1 - u_i \frac{1}{a}\right) \qquad 5.16$$

But from equation (5.10a)

$$u_i = \frac{1}{1+t_{si}\frac{1}{a}} t_{si}$$

$$= z_i t_{si} \qquad 5.17$$

$$\text{where} \quad z_i = \frac{1}{1+t_{si}\frac{1}{a}} \qquad 5.18$$

and hence substituting (5.17) for $u_j (j=1 \ldots A)$ in (5.16)

$$y_i = \frac{1}{-A+1+\sum_{j=1}^{A} z_j} z_i \qquad 5.19$$

and the problem reduces to the solution of the integral equations(5.18) and (5.19).

As the binding energy has been neglected, matrix elements of t_{si} which correspond to the production or absorption of two or more mesons can be omitted, and only the matrix element which corresponds to the scattering of a single meson need be retained.

V § 2. Single nucleon transition matrix elements.

Before the integral equations (5.18) and (5.19) can be solved suitable forms must be chosen for the matrix elements of the operators t_{si} and T_i . At the present stage these matrix elements are defined only to the extent necessary for the solution of the integral equations and a more detailed examination of their energy dependence and isotopic spin dependence will follow at a later stage (Chapter VI).

The choice of the single nucleon scattering matrix element is determined by the following series of approximations and limitations:

(1) Only scattering 'on the energy shell' is taken into account. 'Off the energy shell scattering' will be of greatest importance at small nucleon separations where the weighting factor arising from the integrations over the initial and final nuclear wave-functions is small.

(2) A phenomenological form for the matrix element derived from a phase shift analysis of the meson-nucleon scattering differential cross-section which involves only s- and p-wave mesons (see, for example, reference (2)) is assumed.

(3) In the above form all the phase shifts except δ_{33} are put equal to zero (Brueckner and Watson (5)).

(4) The possibility of spin-flip of the nucleon is neglected.

The validity of these approximations will be examined more closely in Chapter VI.

Incorporating these approximations, the matrix element of the operator t_{si} in momentum space is represented by:-

$$\langle q_1 | t_{si} | q_2 \rangle_{m.s.} = b_i(q_2)\, q_1 \cdot q_2 \qquad 5.20$$

where q_2 and q_1 are the initial and final momenta of the meson and $b_i(q_2)$ is a function of the energy of the meson and is also a matrix in isotopic spin space. The postscript m.s. serves to differentiate the momentum space matrix elements from the corresponding configuration space matrix elements which will be introduced shortly.

The form of the matrix element of the single nucleon photoproduction operator T_i has already been extensively discussed (Chapter III). Two forms of $\langle q | T_i | \nu \rangle$ with slightly different energy dependences and values for the coupling constant and multipole strengths were described, but both of these have the general dependence

$$\langle q | T_i | \nu \rangle_{m.s} = \alpha_i + \gamma_i \cdot q \qquad 5.21$$

where $\underset{\sim}{\nu}$ is the momentum of the incident photon, q is the momentum of the meson, and α_i and γ_i are functions of the photon energy and are also matrices in charge space. γ_i depends in addition on the photon polarisation vector and the nucleon spin $\underset{\sim}{\sigma}(i)$ (equations (3.22) and (3.23)).

To ascertain the forms of the matrix elements (5.20) and (5.21) in configuration space, the discussion is temporarily limited to the particular case A = 4 and analysis along the lines of Chapter III equation (3.2) to equation (3.5) is carried out. The argument may easily be extended to include any value of A, but a considerable degree of simplification in the notation is gained by restricting the discussion to the case A = 4. The matrix element of a typical operator $t_{s3}\frac{1}{a}t_{s2}\frac{1}{a}\times T_1$ which might arise from the expansion of the operator $y_1 T_1$ is averaged over the initial and final momentum distributions of the four nucleons. The matrix element may be expanded as

$$\langle q_3|t_{s3}\frac{1}{a}t_{s2}\frac{1}{a}T_1|\nu\rangle_{ms} = \int \frac{dq_1}{(2\pi)^3}\frac{dq_2}{(2\pi)^3}\langle q_3|t_{s3}|q_2\rangle_{ms}\frac{1}{a(q_2)}\times$$

$$\langle q_2|t_{s2}|q_1\rangle_{m.s.}\frac{1}{a(q_1)}\langle q_1|T_1|\nu\rangle_{m.s.}$$

and the required average written as

$$\int \frac{d\underline{q}_1}{(2\pi)^3}\frac{d\underline{q}_2}{(2\pi)^3}\int d\underline{k}_1 d\underline{k}_2 d\underline{k}_3 d\underline{k}_4\, \phi_f^*(\underline{k}_1+\underline{v}-\underline{q}_1, \underline{k}_2+\underline{q}_1-\underline{q}_2, \underline{k}_3+\underline{q}_2-\underline{q}_3, \underline{k}_4)$$

$$\times \langle q_3|t_{s3}|q_2\rangle_{m.s.}\frac{1}{a(q_2)}\langle q_2|t_{s2}|q_1\rangle_{m.s}$$

$$\times \frac{1}{a(q_1)}\langle q_1|T_1|v\rangle_{m.s.}\,\phi_i(\underline{k}_1,\underline{k}_2,\underline{k}_3,\underline{k}_4)$$

where the notation is similar to that of equation (3.2).

Replacing the momentum space wave-functions by their Fourier transforms the expression becomes

$$\int \frac{d\underline{q}_1}{(2\pi)^3}\frac{d\underline{q}_2}{(2\pi)^3}\int d\underline{r}_1 d\underline{r}_2 d\underline{r}_3 d\underline{r}_4\, \psi_f^*(\underline{r}_1,\underline{r}_2,\underline{r}_3,\underline{r}_4)\langle q_3|t_{s3}|q_2\rangle_{m.s.}e^{i(\underline{q}_2-\underline{q}_3)\underline{r}_3}\frac{1}{a(q_2)}\langle q_2|t_{s2}|q_1\rangle_{m.s.}$$

$$\times e^{i(\underline{q}_1-\underline{q}_2)\underline{r}_2}\frac{1}{a(q_1)}\langle q_1|T_1|v\rangle_{m.s.}e^{i(\underline{v}-\underline{q}_1)\underline{r}_1}\psi_i(\underline{r}_1,\underline{r}_2,\underline{r}_3,\underline{r}_4)$$

$$= \int \frac{d\underline{q}_1}{(2\pi)^3}\frac{d\underline{q}_2}{(2\pi)^3}\int d\underline{r}_1 d\underline{r}_2 d\underline{r}_3 d\underline{r}_4\, \psi_f^*(\underline{r}_1,\underline{r}_2,\underline{r}_3,\underline{r}_4)\langle q_3|t_{s3}|q_2\rangle\frac{1}{a(q_2)}\langle q_2|t_{s2}|q_1\rangle$$

$$\times \frac{1}{a(q_1)}\langle q_1|T_1|v\rangle\,\psi_i(\underline{r}_1,\underline{r}_2,\underline{r}_3,\underline{r}_4)$$

$$= \int d\underline{r}_1 d\underline{r}_2 d\underline{r}_3 d\underline{r}_4\, \psi_f^*(\underline{r}_1,\underline{r}_2,\underline{r}_3,\underline{r}_4)\langle q_3|t_{s3}\frac{1}{a}t_{s2}\frac{1}{a}T_1|v\rangle\psi_i(\underline{r}_1,\underline{r}_2,\underline{r}_3,\underline{r}_4)$$

where the configuration space matrix elements $\langle q_1|t_{si}|q_2\rangle$ and $\langle q_1|T_i|v\rangle$ are defined by

$$\langle q_1|t_{si}|q_2\rangle = \langle q_1|t_{si}|q_2\rangle_{m.s.}e^{i(\underline{q}_2-\underline{q}_1)\underline{r}_i} = b_i(q_2)\,\underline{q}_1.\underline{q}_2\,e^{i(\underline{q}_2-\underline{q}_1)\underline{r}_i} \qquad 5.22$$

and $$\langle q|T_i|v\rangle = \langle q|T_i|v\rangle_{m.s.}e^{i(\underline{v}-\underline{q})\underline{r}_i} = (\alpha_i+\underline{\gamma}_i.\underline{q})\,e^{i(\underline{v}-\underline{q})\underline{r}_i} \qquad 5.23$$

In the above discussion the matrix element of the operator $\frac{1}{a}$ is taken to be diagonal in momentum space and of the form

$$\langle q_1 | \tfrac{1}{a} | q_2 \rangle = (2\pi)^3 \delta(q_1 - q_2) \frac{1}{a(q_2)} \,, \text{ where } a(q_2) = E - \omega(q_2) + i\epsilon \qquad 5.24$$

on the assumption that the recoil energy of the nucleon may be neglected (adiabatic approximation). $\omega(q_2) = (q_2^2 + m_\pi^2)^{\frac{1}{2}}$ ($\equiv q_{20}$) is the energy of the meson.

V § 3. Reduction of the Integral Equations

The auxiliary integral equation (5.18) for the operator z_j can now be solved with the aid of the scattering matrix element (5.22). Equation (5.18) can be re-arranged to give

$$z_i = 1 - z_i t_{si} \frac{1}{a} \qquad 5.25$$

$$\therefore \langle q_1 | z_i | q_2 \rangle = (2\pi)^3 \delta(q_1 - q_2) - \int \frac{dq'}{(2\pi)^3} \langle q_1 | z_i | q' \rangle \langle q' | t_{si} | q_2 \rangle \frac{1}{a(q_2)} \qquad 5.26$$

Substituting (5.22) for $\langle q' | t_{si} | q_2 \rangle$ and introducing the auxiliary function

$$\underset{\sim}{S}_i(q_1) = \int \frac{dq_2}{(2\pi)^3} \langle q_1 | z_i | q_2 \rangle \, q_2 \, e^{-i q_2 . \underset{\sim}{r}_i} \qquad 5.27$$

(5.26) becomes

$$\langle q_1 | z_i | q_2 \rangle = (2\pi)^3 \delta(q_1 - q_2) - \frac{e^{i q_2 . \underset{\sim}{r}_i}}{a(q_2)} \, q_2 . \underset{\sim}{S}_i(q_1) \, b_i(q_2) \qquad 5.28$$

This equation may be solved for $S_i(q_1)$ by multiplying each side by $\frac{1}{(2\pi)^3} q_2 e^{-iq_2 \cdot r_i}$ and integrating over q_2. This leads to the result

$$S_i(q_1) = G_i q_1 e^{-iq_1 \cdot r_i} \qquad 5.29$$

where $$G_i = \frac{1}{1 + J_i} \qquad 5.30$$

and $$J_i = \frac{1}{6\pi^2} \int_0^\infty q^4 \frac{b_i(q)}{a(q)} dq \qquad 5.31$$

and hence

$$\langle q_1 | z_i | q_2 \rangle = (2\pi)^3 \delta(q_1 - q_2) - b_i(q_2) G_i e^{i(q_2 - q_1) r_i} q_1 q_2 \frac{1}{(a(q_2))} \qquad 5.32$$

The operator z_i can now be expressed in the form

$$z_i = 1 - t^r_{si} \frac{1}{a} \qquad 5.33$$

where

$$\langle q_1 | t^r_{si} | q_2 \rangle = b_i(q_2) G_i e^{i(q_2 - q_1) r_i} q_1 q_2 \qquad 5.34$$

$$= G_i \langle q_1 | t_{si} | q_2 \rangle$$

since b_i and G_i commute. The factor G_i acts as a type of renormalizing constant. In terms of the 'normalized' scattering transition operators t^r_{si} the operator y_i is

$$y_i = \frac{1}{1 - \sum_{j=1}^{A} t^r_{sj} \frac{1}{a}} \left(1 - t^r_{si} \frac{1}{a}\right) \qquad 5.35a$$

$$= a \frac{1}{a - t^r_s} (a - t^r_{si}) \frac{1}{a} \qquad 5.35b$$

where $t^r_s = \sum_{j=1}^{A} t^r_{sj}$

Comparison with the expression (5.15) for y_i reveals that (5.35b) could be obtained directly from (5.15) by the substitution of the renormalized operators t^r_{si} for the scattering potential u_i ($i = 1 \ldots A$).

The integral equation (5.35a) is most conveniently written as

$$y_i = z_i + \sum_{j=1}^{A} t^r_{sj} \frac{1}{a} y_i \qquad 5.36$$

which has matrix elements

$$\langle q_1 | y_i | q_2 \rangle = \langle q_1 | z_i | q_2 \rangle + \sum_{j=1}^{A} e^{-i q_1 . r_j} G_j \, q_1 . R_{ij}(q_2) \qquad 5.37$$

with $$R_{ij}(q_2) = \int \frac{dq_1}{(2\pi)^3} \frac{b_j(q_1)}{a(q_1)} e^{i q_1 . r_j} q_1 \langle q_1 | y_i | q_2 \rangle \qquad 5.38$$

The set of equations which the R's satisfy may be deduced by multiplying each side of equation (5.37) on the left by $\frac{1}{(2\pi)^3} \frac{b_j(q_1)}{a(q_1)} q_1 e^{i q_1 . r_j}$ and integrating over q_1. The reduction of these equations to their simplest terms involves the evaluation of an integral of the type

$$\begin{aligned} J_{jk} &= \int \frac{dq}{(2\pi)^3} \frac{b_k(q)}{a(q)} e^{i q . r_{jk}} (\alpha . q)(\beta . q) \qquad r_{jk} = r_k - r_j \qquad 5.39 \\ &= -(\alpha . \nabla_{jk})(\beta . \nabla_{jk}) \int \frac{dq}{(2\pi)^3} \frac{b_k(q)}{a(q)} e^{i q . r_{jk}} \\ &= (\alpha . \nabla_{jk})(\beta . \nabla_{jk}) \frac{b_k q_0}{(2\pi)} \frac{e^{i q_0 r_{jk}}}{r_{jk}} \quad , \quad r_{jk} = |r_{jk}| \end{aligned}$$

where all the energy dependent quantities are evaluated on the energy shell.

$$\therefore J_{jk} = b_k\left\{(\underset{\sim}{\alpha}.\underset{\sim}{\beta})\frac{1}{r_{jk}}\frac{d}{dr_{jk}}\frac{q_0}{2\pi}\frac{e^{iqr_{jk}}}{r_{jk}} + (\underset{\sim}{\alpha}.\underset{\sim}{r}_{jk})(\underset{\sim}{\beta}.\underset{\sim}{r}_{jk})\frac{1}{r_{jk}}\frac{d}{dr_{jk}}\frac{1}{r_{jk}}\frac{d}{dr_{jk}}\frac{q_0}{2\pi}\frac{e^{iqr_{jk}}}{r_{jk}}\right\}$$

$$= -b_k\left\{(\underset{\sim}{\alpha}.\underset{\sim}{\beta})f_{jk} + (\underset{\sim}{\alpha}.\underset{\sim}{r}_{jk})(\underset{\sim}{\beta}.\underset{\sim}{r}_{jk})\, g_{jk}\right\} \qquad 5.40$$

where $$f_{jk} = -\frac{q_0}{2\pi}\frac{1}{r_{jk}}\frac{d}{dr_{jk}}\frac{e^{iqr_{jk}}}{r_{jk}} \qquad 5.41$$

and $$g_{jk} = \frac{1}{r_{jk}}\frac{d}{dr_{jk}}f_{jk} \qquad 5.42$$

Using (5.40) the 'R' equations become

$$G_k \underset{\sim}{R}_{ik}(q_2) = \underset{\sim}{I}_{ik}(q_2) - \sum_{j\neq k} b_k G_j\left(f_{jk}\underset{\sim}{R}_{ij}(q_2) + g_{jk}\underset{\sim}{r}_{jk}(\underset{\sim}{r}_{jk}.\underset{\sim}{R}_{ij}(q_2)\right) \qquad 5.43$$

since $$J_{kk} = \int\frac{d\underset{\sim}{q}}{(2\pi)^3}\frac{b_k(q)}{a(q)}(\underset{\sim}{\alpha}.\underset{\sim}{q})(\underset{\sim}{\beta}.\underset{\sim}{q})$$

$$= (\underset{\sim}{\alpha}.\underset{\sim}{\beta})J_k \text{ by } (5.31)$$

Here $$\underset{\sim}{I}_{ik}(q_2) = \int\frac{d\underset{\sim}{q}_1}{(2\pi)^3}\frac{b_k(q_1)}{a(q_1)}e^{i\underset{\sim}{q}_1.\underset{\sim}{r}_k}\underset{\sim}{q}_1\langle q_1|Z_i|q_2\rangle \qquad 5.44$$

The matrix element of the transition operator T can be expressed most conveniently in terms of functions which are closely related to

(i) $$\underset{\sim}{S}_{ij} = \int\frac{d\underset{\sim}{q}_2}{(2\pi)^3}\underset{\sim}{R}_{ij}(q_2)e^{-i\underset{\sim}{q}_2.\underset{\sim}{r}_i}(\underset{\sim}{q}_2.\underset{\sim}{\gamma}_i) \qquad 5.45$$

and (ii) $$\underset{\sim}{L}_{ij} = \int\frac{d\underset{\sim}{q}_2}{(2\pi)^3}\underset{\sim}{R}_{ij}(q_2)e^{-i\underset{\sim}{q}_2.\underset{\sim}{r}_i}\alpha_i \qquad 5.46$$

and the simultaneous equations satisfied by these functions may readily be deduced from equation (5.43).

(i) Equations for the $\underset{\sim}{S}_{ij}$

Multiplying each side of (5.43) by $\frac{1}{(2\pi)^3} e^{-i q_2 . r_i}(q_2 . \gamma_i)$ on the right and integrating with respect to q_2 leads to

$$G_k \underset{\sim}{S}_{ik} + \sum_{j=1}^{A} (1-\delta_{jk}) b_k G_j \left(f_{jk} \underset{\sim}{S}_{ij} + g_{jk} r_{jk} (r_{jk} . \underset{\sim}{S}_{ij}) \right)$$
$$= \int \frac{dq_2}{(2\pi)^3} \underset{\sim}{I}_{ik}(q_2) e^{-i q_2 . r_i} (q_2 . \gamma_i)$$
$$= \delta_{ik} J_i G_i \gamma_i - (1-\delta_{ik}) \left[b_k G_i \left(f_{ik} \gamma_i + g_{ik} r_{ik} (r_{ik} . \gamma_i) \right) \right] \quad 5.47$$

by (5.44), (5.32), (5.40) and (5.31).

If the change of variables $\underset{\sim}{P}_{ij} = G_j (\underset{\sim}{S}_{ij} + \delta_{ij} \gamma_i)$ is introduced the set of equations (5.47) assume the simple form

$$\underset{\sim}{P}_{ik} + \sum_{j=1}^{A} (1-\delta_{jk}) b_k \left(f_{jk} \underset{\sim}{P}_{ij} + g_{jk} r_{jk} (r_{jk} . \underset{\sim}{P}_{ij}) \right) = \gamma_i \delta_{ik} \quad 5.48$$

for $i = 1 \ldots A$, $k = 1 \ldots A$.

(ii) Equations for the $\underset{\sim}{L}_{ij}$

In a similar manner to the above a set of equations may be obtained for the functions $\underset{\sim}{L}_{ij}$ by a suitable transformation of (5.43), giving

$$G_k \underset{\sim}{L}_{ik} + \sum_{j=1}^{A} (1-\delta_{jk}) b_k G_j \left(f_{kj} \underset{\sim}{L}_{ij} + g_{kj} r_{jk} (r_{jk} . \underset{\sim}{L}_{ij}) \right) \quad 5.49$$
$$= -i (1-\delta_{ik}) b_k r_{ki} f_{ki} \alpha_i$$

which may be further simplified by the change of variables $\underset{\sim}{M}_{ij} = G_j \underset{\sim}{L}_{ij}$

$$\underset{\sim}{M}_{ik} + \sum_{j=1}^{A} (1-\delta_{jk}) b_k \left(f_{jk} \underset{\sim}{M}_{ij} + g_{jk} \underset{\sim}{r}_{jk} (\underset{\sim}{r}_{jk} \cdot \underset{\sim}{M}_{ij}) \right) = -i(1-\delta_{ik}) b_k \underset{\sim}{r}_{ki} f_{ki} \alpha_i \quad 5.50$$

for $i = 1 \cdots A,\ k = 1 \cdots A.$

The total transition operator T is related to y_i by

$$T = \sum_{i=1}^{A} y_i T_i$$

$\therefore \langle q_1 | T | \nu \rangle = \sum_{i=1}^{A} \langle q_1 | y_i T_i | \nu \rangle$ where $\underset{\sim}{\nu}$ is the momentum of the incident photon and q_1 is the momentum of the observed meson.

$$\therefore \langle q_1 | T | \nu \rangle = \sum_{i=1}^{A} \int \frac{dq_2}{(2\pi)^3} \langle q_1 | y_i | q_2 \rangle \langle q_2 | T_i | \nu \rangle \quad 5.51$$

$$= \sum_{i=1}^{A} \int \frac{dq_2}{(2\pi)^3} \left(\langle q_1 | z_i | q_2 \rangle + \sum_{j=1}^{A} e^{-iq_1 \cdot r_j} G_j q_1 \cdot \underset{\sim}{R}_{ij}(q_2) \right) \times (\alpha_i + \gamma_i \cdot q_2) e^{i(\nu - q_2) r_i} \quad 5.52$$

by (5.37) and (5.23)

$$\therefore \langle q_1 | T | \nu \rangle = \sum_{i=1}^{A} e^{i\nu \cdot r_i} \Big[\Big(\alpha_i e^{-iq_1 \cdot r_i} + \sum_{j=1}^{A} e^{-iq_1 \cdot r_j} G_j q_1 \cdot \underset{\sim}{L}_{ij} \Big) + \Big(G_i (q_1 \cdot \gamma_i) e^{-iq_1 \cdot r_i} + \sum_{j=1}^{A} e^{-iq_1 \cdot r_j} G_j q_1 \cdot \underset{\sim}{S}_{ij} \Big) \Big] \quad 5.53$$

$$= \sum_{i=1}^{A} \Big[\alpha_i e^{-iq_1 \cdot r_i} + \sum_{j=1}^{A} e^{-iq_1 \cdot r_j} q_1 \cdot (\underset{\sim}{M}_{ij} + \underset{\sim}{P}_{ij}) \Big] \quad 5.54$$

$$\therefore \langle q_1|T|\nu\rangle = \sum_{i=1}^{A}\sum_{j=1}^{A} e^{i(\underline{\nu}.\underline{r}_i - q_i.\underline{r}_j)}\left[\alpha_i\delta_{ij} + q_i(\underline{M}_{ij} + \underline{P}_{ij})\right] \quad 5.55$$

The renormalization constants G_j do not appear either in the above expression for the probability amplitude for the photoproduction of a meson at the A nucleons or in the equations (5.48) and (5.50) for the functions $\underline{P}_{ij}$ and $\underline{M}_{ij}$ respectively. This lends considerable weight to the interpretation of G_j as a type of renormalization constant.

The multiple scattering problem has now been reduced to that of solving a series of linear simultaneous equations in the components of the vectors $\underline{P}_{ij}$ and $\underline{M}_{ij}$. These equations contain the matrices b_k in charge space which may be of considerable dimensions even when A is small (see for example Chapter VI, where the matrices b_k are examined for the particular case A = 4). The principal results of this paragraph are summarised in equations (5.55), (5.48) and (5.50), namely

$$\langle q_1|T|\nu\rangle = \sum_{i=1}^{A}\sum_{j=1}^{A} e^{i(\underline{\nu}.\underline{r}_i - q_i.\underline{r}_{ij})}\left\{\alpha_i\delta_{ij} + q_i(\underline{M}_{ij} + \underline{P}_{ij})\right\} \quad 5.56$$

where $$\underline{P}_{ik} + \sum_{j=1}^{A}(1-\delta_{jk})b_k\left(f_{jk}\underline{P}_{ij} + g_{jk}r_{jk}(r_{jk}.\underline{P}_{ij})\right) = \gamma_i\delta_{ik} \quad 5.57$$

$$\underline{M}_{ik} + \sum_{j=1}^{A}(1-\delta_{jk})b_k\left(f_{jk}\underline{M}_{ij} + g_{jk}r_{jk}(r_{jk}\underline{M}_{ij})\right) \quad 5.58$$

$$= -ib_k\alpha_i\underline{r}_{ki}f_{ki}(1-\delta_{ik})$$

Henceforward, the equations (5.57) will be referred to as the 'P' equations and the equations (5.50) as the 'M' equations.

V § 4. The multiple scattering equations for the scattering of a meson at A nucleons.

The multiple scattering equations for the scattering of a meson at A nucleons are very similar in character to the multiple scattering equations for the photoproduction of a meson at A nucleons, because of the basic similarity of the total transition operators for these two interactions. The total transition operator for the scattering reaction is

$$S = h + h\frac{1}{a-h}h \qquad 5.59$$

where h and a are defined at the beginning of this Chapter.

With the aid of the approximations which led to the expression (5.16) it can be shown that

$$S = \frac{1}{1-u\frac{1}{a}}u \qquad 5.60$$

where only those terms which can represent the scattering of a single meson at the individual nucleons have been retained.

Now from (5.33) $z_i = 1 - t^r_{si}\frac{1}{a}$

but from (5.17) $z_i = 1 - u_i\frac{1}{a}$

$$\therefore \qquad u_i = t^r_{si} \qquad 5.61$$

and hence $$S = \frac{1}{1 - t^r_s\frac{1}{a}}\, t^r_s$$

$$= \sum_{i=1}^{A} S_i \qquad 5.62$$

where $$S_i = \frac{1}{1 - t^r_s\frac{1}{a}}\, t^r_{si} \qquad 5.63$$

Solving the integral equation (5.63) for the matrix elements of the operator S_i by methods similar to those employed in the previous paragraph it can be shown that

$$\langle q_1|S_i|q_2\rangle = \sum_{i=1}^{A}\sum_{j=1}^{A} e^{i(q_2.r_i - q_1.r_j)}\, q_1.I_{ij}(q_2) \qquad 5.64$$

where

$$I_{ik}(q_2) + \sum_{j=1}^{A} b_k(1-\delta_{jk})\left(f_{jk}\, I_{ij}(q_2) + g_{jk}\, r_{jk}(r_{jk}.I_{ij}(q_2))\right) = \delta_{ik} b_i q_2 \qquad 5.65$$

for $i = 1 \cdots\cdots A$

and $k = 1 \cdots\cdots A$.

VI. THE ISOTOPIC SPIN AND ENERGY DEPENDENCE OF THE MESON-NUCLEON SCATTERING AND SINGLE NUCLEON PHOTOPRODUCTION MATRIX ELEMENTS

So far the matrix elements $\langle q_1 | t_{si} | q_2 \rangle$ and $\langle q | T_i | \nu \rangle$ have been defined only to the extent necessary for the derivation of the multiple scattering equations. To facilitate the further examination of the equations and their solutions the isotopic spin and energy dependence of these matrix elements will now be determined.

VI § 1. Isotopic spin formalism

The concept of isotopic spin arose from the observation that nuclear forces are charge independent. For this reason the proton and neutron may be considered as two states of the same particle, the nucleon and similarly the positive negative and neutral mesons may be considered as three states of the one particle, these states being differentiated by the 'Z component of the isotopic spin' which is closely related to the charge of the particles. The role of isotopic spin in the theory of nuclear structure is analogous to that of spin in the theory of atomic states, insofar as it is related to the number of permissible quantum states available to the particles. Pursuing this analogy further a two-dimensional isotopic spin space may be associated with the nucleons in which the protons have the representation

$$p \equiv Y_{\frac{1}{2}}^{\frac{1}{2}} \text{ or } \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad 6.1$$

and the neutrons have the representation

$$n \equiv Y_{\frac{1}{2}}^{-\frac{1}{2}} \text{ or } \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad 6.2$$

the quantum number m in the spherical harmonic $Y_{\frac{1}{2}}^{m}$ being chosen such that the charge q of the nucleon is given by $q = m + \frac{1}{2}$. (6.1) and (6.2) define the isotopic spin wave-functions of the nucleon.

In a similar manner a three dimensional isotopic spin space may be associated with the three possible charge states of the meson in which the mesons have the 'wave-functions'.

$$\pi^{+} \equiv Y_{1}^{1} \text{ or } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \ \pi^{0} \equiv Y_{1}^{0} \text{ or } \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; \ \pi^{-} \equiv Y_{1}^{-1} \text{ or } \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad 6.3$$

the value of m in this case being equal to the charge of the meson.

Operators in isotopic spin space are defined in analogy with the shift operators in angular momentum theory (see, for example, reference (3) p.782). An arbitrary phase factor occurs in the derivation of the result of acting with these operators on the angular momentum wave-functions and is chosen to be zero here in order to allow the combination of isotopic spin wave-functions by means of Clebsch-Gordan coefficients (16). The isotopic spin operators for the nucleons are defined as follows

$$\tau^+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} ; \tau^- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} ; \tau^0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad 6.4$$

If $\tau_1 = \tau^+ + \tau^- \; ; \tau_2 = i(\tau^- - \tau^+) \; ; \tau_3 = \tau_0$ 6.5

$$\text{then } [\tau_i, \tau_j] = 2i\varepsilon_{ijk}\tau_k \qquad 6.6$$

where the commutator bracket $[\; , \;]$ and the symbol ε_{ijk} have their usual meanings. The corresponding operators for the mesons are

$$\ell^+ = \frac{1}{2^{\frac{1}{2}}} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} ; \ell^- = \frac{1}{2^{\frac{1}{2}}} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} ; \ell^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad 6.7$$

If $\ell_1 = \ell^+ + \ell^- \; ; \ell_2 = i(\ell^- - \ell^+) \; ; \ell_3 = \ell_0$ 6.8

$$\text{then } [\ell_p, \ell_q] = i\varepsilon_{pqr}\ell_r \qquad 6.9$$

In terms of meson annihilation and creation operators (a_k, b_k and c_k annihilate positive, negative and neutral mesons of momentum $\underset{\sim}{k}$,respectively) ℓ^+, ℓ^- and ℓ_0 may be written

$$\ell^+ = \frac{1}{2^{\frac{1}{2}}}(a_{\underset{\sim}{k}}^* c_{\underset{\sim}{k}} + c_{\underset{\sim}{k}}^* b_{\underset{\sim}{k}}); \ell^- = \frac{1}{2^{\frac{1}{2}}}(c_{\underset{\sim}{k}}^* a_{\underset{\sim}{k}} + b_{\underset{\sim}{k}}^* c_{\underset{\sim}{k}}); \ell_0 = (a_{\underset{\sim}{k}}^* a_{\underset{\sim}{k}} - b_{\underset{\sim}{k}}^* b_{\underset{\sim}{k}}) \qquad 6.10$$

If the vector $\underset{\sim}{U}$ is defined to have components

$$U_1 = \frac{1}{2^{\frac{1}{2}}}(b_{\underset{\sim}{k}} - a_{\underset{\sim}{k}}); U_2 = -\frac{i}{2^{\frac{1}{2}}}(a_{\underset{\sim}{k}} + b_{\underset{\sim}{k}}); U_3 = c_{\underset{\sim}{k}} \qquad 6.11$$

then $\underset{\sim}{\ell}$ can be conveniently expressed as

$$\underset{\sim}{\ell} = i\underset{\sim}{U} \times \underset{\sim}{U}^* \qquad 6.12$$

The components of $\underset{\sim}{U}$ satisfy the commutation relationship $[U_i, U_j^*] = \delta_{ij}$ all other commutators being zero.

The values of the commutators $[\ell_i, \phi_j]$ are required for the further development of the properties of isotopic spin. ϕ_j $(j=1,2,3)$ are components of the meson field operators defined by

$$\phi_1 = \frac{1}{2^{\frac{1}{2}}}(\phi + \phi^*); \phi_2 = \frac{i}{2^{\frac{1}{2}}}(\phi - \phi^*); \phi_3 = \phi_0 \qquad 6.13$$

ϕ , ϕ^* and ϕ_0 are the usual operators associated with charged and scalar meson fields. If the evaluation of these commutators is attempted using the expansion $\phi = \sum_{\underset{\sim}{k}} \frac{1}{(2V\omega_{\underset{\sim}{k}})^{\frac{1}{2}}}(a_{\underset{\sim}{k}} e^{i\underset{\sim}{k}.\underset{\sim}{x}} + b_{\underset{\sim}{k}}^* e^{-i\underset{\sim}{k}.\underset{\sim}{x}})$ it is found that the values obtained are such that the components of the vector $\frac{1}{2}\underset{\sim}{\tau} + \underset{\sim}{\ell}$ do not commute with $\underset{\sim}{\tau}.\underset{\sim}{\phi}$. Now it is essential for the development of the properties of isotopic spin that the commutators $\left[\frac{1}{2}\tau_j + \ell_j, \tau . \phi\right]$ should be zero. There are two possible courses open in order to obtain the required values for the commutators $[\ell_i, \phi_j]$.

(1) The isotopic spin operator for the meson may be re-defined as $\underset{\sim}{\ell} = \int \phi \times \underset{\sim}{\pi}\, d^3x$ where $\underset{\sim}{\pi}$ is the momentum conjugate to ϕ. With this definition, however, meson isotopic spin wave-functions cannot be correctly combined with nucleon isotopic spin wave-functions by means of Clebsch-Gordan coefficients. Instead the required combination rules may be obtained by operating on the state $|\psi(N)\rangle$, representing the isotopic spin state of N nucleons, with the operator $(\underset{\sim}{\tau}(1) + \cdots\cdots + \underset{\sim}{\tau}(N)).\phi$ which gives rise to a combination of isotopic spin states representing N nucleons and one meson, with the same isotopic spin as the initial state $|\psi(N)\rangle$. Other states with different total isotopic spin may be constructed by using the orthogonality relationships between states with the same z-component of, but different total, isotopic spin. The combination coefficients obtained in this way have the same magnitude as the corresponding Clebsch-Gordan

coefficients but differ in sign in certain cases.

(2) The field operator ϕ may be expanded as

$$\phi = \sum_{\underset{\sim}{k}} \frac{1}{(2V\omega_{\underset{\sim}{k}})^{\frac{1}{2}}} \left(-a_{\underset{\sim}{k}} e^{i\underset{\sim}{k}.\underset{\sim}{x}} + b^*_{\underset{\sim}{k}} e^{-i\underset{\sim}{k}.\underset{\sim}{x}} \right) \qquad 6.14$$

The transformation $a_{\underset{\sim}{k}} \to -a_{\underset{\sim}{k}}$ implied by this redefinition leaves the results which lead to the physical interpretation of $a_{\underset{\sim}{k}}$ unchanged (e.g. $[a_{\underset{\sim}{k}}, a^*_{\underset{\sim}{k}'}] = \delta_{\underset{\sim}{k}\underset{\sim}{k}'}$ $N_{\underset{\sim}{k}} = a^*_{\underset{\sim}{k}} a_{\underset{\sim}{k}}$). This form of ϕ leads to commutation relationships with $\underset{\sim}{\ell}$ (as defined in equation (6.7)), which ensure the conservation of the vector quantity $\underset{\sim}{I} = \frac{1}{2}\underset{\sim}{\tau} + \underset{\sim}{\ell}$. These commutation relationships are

$$[\ell_p, \phi_q] = i\varepsilon_{pqr}\phi_r \qquad 6.15$$

The interaction Hamiltonian between the nucleon and meson fields is on the PS-PS theory (Kemmer (27)) (that is pseudoscalar mesons with pseudoscalar coupling of the meson and nucleon fields)

$$H_i = g\bar{\psi}\gamma_5 \underset{\sim}{\tau}.\underset{\sim}{\phi}\psi \qquad 6.16$$

Here $\psi = \begin{bmatrix} \psi_P \\ \psi_N \end{bmatrix}$; $\tau^+\psi = \begin{bmatrix} \psi_P \\ 0 \end{bmatrix}$ etc.

If the system contains N nucleons and P mesons the interaction terms in the Hamiltonian are

$$H_i(N,P) = g\bar{\psi}\gamma_5(\underset{\sim}{\tau}(1)+\cdots+\underset{\sim}{\tau}(N))(\underset{\sim}{\phi}(1)+\cdots+\underset{\sim}{\phi}(P))\psi \qquad 6.17$$

The isotopic spin of this system is defined to be

$$\underset{\sim}{I} = \frac{1}{2}\sum_{n=1}^{N}\underset{\sim}{\tau}(n) + \sum_{p=1}^{P}\underset{\sim}{\ell}(p) \qquad 6.18$$

It follows immediately from the commutation relationships

$$[\tau_i(a), \tau_j(b)] = 2i\,\delta_{ab}\,\varepsilon_{ijk}\,\tau_k(a) \qquad 6.19a$$

$$[\ell_i(a), \phi_j(b)] = i\,\delta_{ab}\,\varepsilon_{ijk}\,\phi_k(a) \qquad 6.19b$$

$$[\tau_i(a), \ell_j(b)] = 0 \qquad 6.19c$$

that each component of $\underset{\sim}{I}$ is conserved during the interaction $H_i(N,P)$. Now I_3, $\underset{\sim}{I}^2$ and H_i commute with each other and, hence, two new quantum numbers m_Q (or I_3) and I may be introduced which are conserved during the interaction $H_i(N,P)$, where

$$I_3\psi(I,m_Q) = m_Q\psi(I,m_Q);\; \underset{\sim}{I}^2\psi(I,m_Q) = I(I+1)\psi(I,m_Q) \qquad 6.20$$

and $\psi(I,m_Q)$ is the isotopic spin wave-function of the meson-nucleon system. If A = number of nucleons, Z = number of protons, M_+ = number of positive mesons, and M_- = number of negative mesons, then $m_Q = Z - \frac{A}{2} + M_+ - M_-$.

With the definitions and properties of isotopic spin which have been established it is now possible to construct the charge matrix b_k . The gradient coupling Hamiltonian defined in equation (3.16) has the same isotopic spin properties as the Hamiltonian defined in this Chapter (equation (6.16)) and hence isotopic spin is conserved by the scattering transition operator t_{si}. If $b_k(q)$ is written as $b(q)c(k)$ where $c(k)$ is an energy independent matrix in charge space and $b(q)$ is a scalar in charge space, then each term in $c(k)$ must contain one annihilation and one creation operator for the meson which is scattered and the most general form of $c(k)$ satisfying this requirement is

$$c(k) = \sum_{i,j=1}^{3} \alpha_{ij} U_i U_j^* \qquad 6.21$$

where $\alpha_{ij} = \alpha_{ij}(1)\tau_1(k) + \alpha_{ij}(2)\tau_2(k) + \alpha_{ij}(3)\tau_3(k) + \alpha_{ij}(4)\tau_4(k)$

and the $\alpha_{ij}(a)$, $a = 1 \cdots 4$ are scalars in charge space. Imposition of the condition that isotopic spin is to be conserved during the transition $\left([c(k), \underset{\sim}{I}] = 0\right)$ determines that

$$c(k) = a\, \underset{\sim}{U} . \underset{\sim}{U}^* + \underset{\sim}{\tau}(k) . \underset{\sim}{\ell}$$

, where a is a constant. 6.22

The operator $\underset{\sim}{U}.\underset{\sim}{U}^*$ is equal to $a_k a_{\underset{\sim}{k}}^* + b_k b_{\underset{\sim}{k}}^* + c_k c_{\underset{\sim}{k}}^*$ and is effectively unity.

One of the assumptions made when the general form of the matrix element $\langle q_1 | t_{si} | q_2 \rangle$ was derived was that the scattering takes place through the $J = \frac{3}{2}, I = \frac{3}{2}$ state. Hence, the constant a is to be chosen so that $c(k)$ projects out the $I = \frac{1}{2}$ state and, therefore, a satisfies the equation

$$(\psi_{\frac{1}{2}}^{\pm\frac{1}{2}} | a + \underset{\sim}{\tau}(k).\underset{\sim}{\ell} | \psi_{\frac{1}{2}}^{\pm\frac{1}{2}}) = 0 \qquad 6.23$$

where ψ_I^m is the isotopic spin state with z-component of isotopic spin m and total isotopic spin I. Then

$$\psi_{\frac{1}{2}}^{\frac{1}{2}} = \left(\frac{2}{3}\right)^{\frac{1}{2}}(n,\pi^+) - \left(\frac{1}{3}\right)^{\frac{1}{2}}(p,\pi^0) \qquad 6.24a$$

$$\psi_{\frac{1}{2}}^{-\frac{1}{2}} = \left(\frac{1}{3}\right)^{\frac{1}{2}}(n,\pi^0) - \left(\frac{2}{3}\right)^{\frac{1}{2}}(p,\pi^-) \qquad 6.24b$$

where (n,π^+), for instance, is the isotopic spin state of a neutron and positive meson. (6.23) gives a = 2. Therefore

$$b_i(q) = b(q)(2 + \tau(\underset{\sim}{q}).\underset{\sim}{\ell}) \qquad 6.25$$

where $b(q)$ contains the energy dependence of the transition

VI§2. Matrix representations for the scattering operators in charge space.

As an illustration of the methods of finding the most suitable matrix representation of the operators

$C(i)$, the particular example of the elastic photoproduction of neutral mesons at helium, which is to be studied in greater detail in subsequent Chapters, will be examined. The particular example of a meson which is initially produced at nucleon p and after several scatterings between the nucleons is eventually scattered out from nucleon q is considered. Let $|i\rangle$ be the initial isotopic spin state of the four nucleons and $|f\rangle$ be the final spin state of the nucleon-meson system. Then if the isotopic spin operator associated with the photoproduction operator T_p is $d(p)$ the isotopic spin dependence of the matrix element for this particular transition is

$$\langle f|c(q)\ldots c(r)\ldots d(p)|i\rangle$$

$$= \sum_{n_1 \ldots n_j = 1}^{N} \langle f|c(q)|n_1\rangle\langle n_1|\ldots\langle n_f|c(r)|n_g\rangle\ldots|n_j\rangle\langle n_j|d(p)|i\rangle \qquad 6.26$$

Here $|n\rangle\,(n=1\ldots N)$ is a complete set of basis states for the description of the isotopic spin states of four nucleons and a meson. There are forty eight such basis states altogether, but consideration of charge conservation $(I_3 = 0)$ immediately reduces the number of these for which C has non-zero matrix elements to fourteen.

Let $c(q)_{jk} = \langle j | c(q) | k \rangle$, $d(p)_{ji} = \langle j | d(p) | i \rangle$

Then (6.26) is closely related to the matrix product

$$M = \begin{bmatrix} m_{11} \\ \vdots \\ m_{14,1} \end{bmatrix} = \begin{bmatrix} c(q)_{11} & \cdots & c(q)_{1,14} \\ \vdots & & \vdots \\ c(q)_{14,1} & \cdots & c(q)_{14,14} \end{bmatrix} \times \cdots \begin{bmatrix} c(r)_{11} & \cdots & c(r)_{1,14} \\ \vdots & & \vdots \\ c(r)_{14,1} & \cdots & c(r)_{14,14} \end{bmatrix} \times \cdots \begin{bmatrix} d(p)_{1i} \\ \vdots \\ d(p)_{14,i} \end{bmatrix} \qquad 6.27$$

If $|f\rangle = \sum_{i=1}^{M_1} A_i |n_i\rangle$ where the A_i are constants and $M_1 \leq 14$ then the matrix element (6.26) reduces to

$$\langle f | c(q) \ldots d(p) | i \rangle = \sum_{i=1}^{M} m_{i,1} A_i \qquad 6.28$$

The matrix elements of the operators $c(i)$ and $d(i)$ are evaluated in two representations.

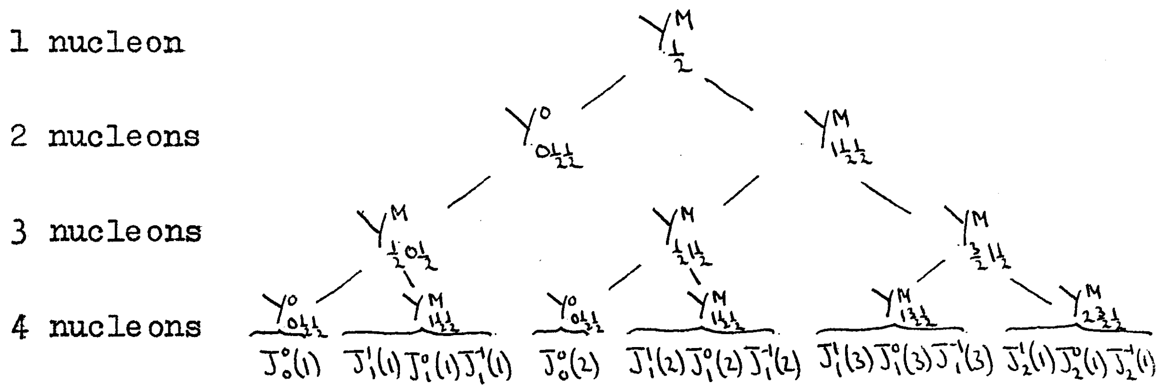
(i) <u>The First Matrix Representation of the Operator</u>

The basis states in this representation which give non-zero matrix elements are

$$J_0^0(1)\pi^0, J_0^0(2)\pi^0 ; J_1^1(1)\pi^-, J_1^1(2)\pi^-, J_1^1(3)\pi^- ; J_1^0(1)\pi^0, J_1^0(2)\pi^0, J_1^0(3)\pi^0 ;$$

$$J_1^{-1}(1)\pi^+, J_1^{-1}(2)\pi^+, J_1^{-1}(3)\pi^+ ; J_2^1(1)\pi^-, J_2^0(1)\pi^0, J_2^{-1}(1)\pi^+ \qquad 6.29$$

$J_J^M(k)$ is an isotopic spin eigenstate of four nucleons with charge quantum number M and total isotopic spin J. These states are built up according to the following scheme.



Four 14 x 14 matrices for C(1), C(2), C(3) and C(4) are obtained in this representation.

(ii) <u>The Second Matrix Representation of the Operator $C(i)$</u>

The dimensions of the matrices in the first representation may be considerably reduced by taking into account the conservation of the total isotopic spin I during the scattering process. In order to obtain this reduction a second representation of the meson-nucleon states is employed in which the basis states $Y^{0}_{I,J\ell}$ are eigenfunctions of $\underset{\sim}{I}^2 = (\underset{\sim}{J} + \underset{\sim}{\ell})^2$ and $I_3 = J_3 + \ell_3$ with eigenvalues I and O respectively. The relationship between the two sets of basis states is given by

$$Y^{0}_{1,01}(1) = J^{0}_{0}(1)\pi^{0} \qquad 6.30a$$

$$Y^{0}_{1,01}(2) = J^{0}_{0}(2)\pi^{0} \qquad 6.30b$$

$$Y^{0}_{0,11}(i) = \frac{1}{3^{\frac{1}{2}}}\left(J^{1}_{1}(i)\pi^{-} - J^{0}_{1}(i)\pi^{0} + J^{-1}_{1}(i)\pi^{+}\right) \quad i=1,2,3 \qquad 6.30c$$

$$Y^{0}_{1,11}(i) = \frac{1}{2^{\frac{1}{2}}}\left(J^{1}_{1}(i)\pi^{-} - J^{-1}_{1}(i)\pi^{+}\right) \quad i=1,2,3 \qquad 6.30d$$

$$Y^{0}_{2,11}(i) = \frac{1}{6^{\frac{1}{2}}}\left(J^{1}_{1}(i)\pi^{-} + 2J^{0}_{1}(i)\pi^{0} + J^{-1}_{1}(i)\pi^{+}\right) \quad i=1,2,3 \qquad 6.30e$$

$$Y^{0}_{1,21}(1) = \frac{1}{10^{\frac{1}{2}}}\left(3^{\frac{1}{2}}J^{1}_{2}(1)\pi^{-} - 2J^{0}_{2}(1)\pi^{0} + 3^{\frac{1}{2}}J^{-1}_{2}(1)\pi^{+}\right) \qquad 6.30f$$

$$Y^{0}_{2,21}(1) = \frac{1}{2^{\frac{1}{2}}}\left(J^{1}_{2}(1)\pi^{-} - J^{-1}_{2}(1)\pi^{+}\right) \qquad 6.30g$$

$$Y^{0}_{3,21}(1) = \frac{1}{5^{\frac{1}{2}}}\left(J^{1}_{2}(1)\pi^{-} + 3^{\frac{1}{2}}J^{0}_{2}(1)\pi^{0} + J^{-1}_{2}(1)\pi^{+}\right) \qquad 6.30h$$

From the properties of the operator $C(i)$ it follows that the only non-zero matrix elements of $C(i)$ in the second representation occur between states with $I = 1$, namely (6.30a), (6.30b), (6.30d) and (6.30f), which reduces the dimensions of the matrix $C(i)$ to 6 x 6. It is, however, necessary to evaluate all the matrix elements in the first representation as they are needed in the formation of the second matrix.

The matrices C(1), C(2), C(3) and C(4) computed in the second representation are

	$Y^0_{1,01}(1)$	$Y^0_{1,01}(2)$	$Y^0_{1,11}(1)$	$Y^0_{1,11}(2)$	$Y^0_{1,11}(3)$	$Y^0_{1,21}(1)$
$Y^0_{1,01}(1)$	2	0	0	$\left(\frac{2}{3}\right)^{\frac{1}{2}}$	$-\frac{2}{3^{\frac{1}{2}}}$	0
$Y^0_{1,01}(2)$	0	2	$\left(\frac{2}{3}\right)^{\frac{1}{2}}$	$-\frac{2^{\frac{3}{2}}}{3}$	$-\frac{2}{3}$	0
$Y^0_{1,11}(1)$	0	$\left(\frac{2}{3}\right)^{\frac{1}{2}}$	2	$\left(\frac{1}{3}\right)^{\frac{1}{2}}$	$\left(\frac{1}{6}\right)^{\frac{1}{2}}$	$-\left(\frac{5}{6}\right)^{\frac{1}{2}}$
$Y^0_{1,11}(2)$	$\left(\frac{2}{3}\right)^{\frac{1}{2}}$	$-\frac{2^{\frac{3}{2}}}{3}$	$\left(\frac{1}{3}\right)^{\frac{1}{2}}$	$\frac{4}{3}$	$\frac{1}{3.2^{\frac{1}{2}}}$	$-\frac{1}{3}\left(\frac{5}{2}\right)^{\frac{1}{2}}$
$Y^0_{1,11}(3)$	$-\frac{2}{3^{\frac{1}{2}}}$	$-\frac{2}{3}$	$\left(\frac{1}{6}\right)^{\frac{1}{2}}$	$\frac{1}{3.2^{\frac{1}{2}}}$	$\frac{7}{6}$	$-\frac{5^{\frac{1}{2}}}{6}$
$Y^0_{1,21}(1)$	0	0	$-\left(\frac{5}{6}\right)^{\frac{1}{2}}$	$-\frac{1}{3}\left(\frac{5}{2}\right)^{\frac{1}{2}}$	$-\frac{5^{\frac{1}{2}}}{6}$	$\frac{1}{2}$

C(1)

$Y^0_{1,01}(1)$	2	0	0	$-\left(\frac{2}{3}\right)^{\frac{1}{2}}$	$\frac{2}{3^{\frac{1}{2}}}$	0
$Y^0_{1,01}(2)$	0	2	$-\left(\frac{2}{3}\right)^{\frac{1}{2}}$	$-\frac{2^{\frac{3}{2}}}{3}$	$-\frac{2}{3}$	0
$Y^0_{1,11}(1)$	0	$-\left(\frac{2}{3}\right)^{\frac{1}{2}}$	2	$-\left(\frac{1}{3}\right)^{\frac{1}{2}}$	$-\left(\frac{1}{6}\right)^{\frac{1}{2}}$	$\left(\frac{5}{6}\right)^{\frac{1}{2}}$
$Y^0_{1,11}(2)$	$-\left(\frac{2}{3}\right)^{\frac{1}{2}}$	$-\frac{2^{\frac{3}{2}}}{3}$	$-\left(\frac{1}{3}\right)^{\frac{1}{2}}$	$\frac{4}{3}$	$\frac{1}{3.2^{\frac{1}{2}}}$	$-\frac{1}{3}\left(\frac{5}{2}\right)^{\frac{1}{2}}$
$Y^0_{1,11}(3)$	$\frac{2}{3^{\frac{1}{2}}}$	$-\frac{2}{3}$	$-\left(\frac{1}{6}\right)^{\frac{1}{2}}$	$\frac{1}{3.2^{\frac{1}{2}}}$	$\frac{7}{6}$	$-\frac{5^{\frac{1}{2}}}{6}$
$Y^0_{1,21}(1)$	0	0	$\left(\frac{5}{6}\right)^{\frac{1}{2}}$	$-\frac{1}{3}\left(\frac{5}{2}\right)^{\frac{1}{2}}$	$-\frac{5^{\frac{1}{2}}}{6}$	$\frac{1}{2}$

C(2)

	$Y^0_{1,01}(1)$	$Y^0_{1,01}(2)$	$Y^0_{1,11}(1)$	$Y^0_{1,11}(2)$	$Y^0_{1,11}(3)$	$Y^0_{1,21}(1)$
$Y^0_{1,01}(1)$	2	0	$-2^{\frac{1}{2}}$	0	0	0
$Y^0_{1,01}(2)$	0	2	0	$\frac{2^{\frac{1}{2}}}{3}$	$\frac{4}{3}$	0
$Y^0_{1,11}(1)$	$-2^{\frac{1}{2}}$	0	1	0	0	0
$Y^0_{1,11}(2)$	0	$\frac{2^{\frac{1}{2}}}{3}$	0	$\frac{7}{3}$	$-\frac{2^{\frac{1}{2}}}{3}$	$\frac{10^{\frac{1}{2}}}{3}$
$Y^0_{1,11}(3)$	0	$\frac{4}{3}$	0	$-\frac{2^{\frac{1}{2}}}{3}$	$\frac{7}{6}$	$-\frac{5^{\frac{1}{2}}}{6}$
$Y^0_{1,21}(1)$	0	0	0	$\frac{10^{\frac{1}{2}}}{3}$	$-\frac{5^{\frac{1}{2}}}{6}$	$\frac{1}{2}$

C(3)

$Y^0_{1,01}(1)$	2	0	$2^{\frac{1}{2}}$	0	0	0
$Y^0_{1,01}(2)$	0	2	0	$2^{\frac{1}{2}}$	0	0
$Y^0_{1,11}(1)$	$2^{\frac{1}{2}}$	0	1	0	0	0
$Y^0_{1,11}(2)$	0	$2^{\frac{1}{2}}$	0	1	0	0
$Y^0_{1,11}(3)$	0	0	0	0	$\frac{5}{2}$	$\frac{5^{\frac{1}{2}}}{2}$
$Y^0_{1,21}(1)$	0	0	0	0	$\frac{5^{\frac{1}{2}}}{2}$	$\frac{1}{2}$

C(4)

VI § 3. The isotopic spin and energy dependence of the matrix element $\langle q|T_1|\nu\rangle$

The form adopted for the single particle photoproduction matrix element, $\langle q|T_1|\nu\rangle_{m.s.}$, for example, is that proposed by Chew (11) or the slightly adapted form extensively discussed in Chapter III. In either case the matrix element may be written

$$\langle q|T_1|\nu\rangle_{m.s.} = \left(\tau^-(1)\,a^*_{\underline{k}} + \tau^+(1)\,b^*_{\underline{k}}\right)\alpha + \left(\tfrac{1}{2^{\frac{1}{2}}}\left(\tau^-(1)\,a^*_{\underline{k}} + \tau^+(1)\,b^*_{\underline{k}}\right) + c^*_{\underline{k}}\right)\underline{\gamma}\cdot\underline{q} \qquad 6.32$$

where α as before is either

$$\frac{2\pi i a}{m_\pi (q_0\nu)^{\frac{1}{2}}}\,\underline{\sigma}\cdot\underline{\varepsilon} \qquad 6.33a$$

or

$$\frac{2\pi i a'}{m_\pi (q_0\nu)^{\frac{1}{2}}}\,\underline{\sigma}\cdot\underline{\varepsilon} \qquad 6.33b$$

and $\underline{\gamma}$ is either

$$\frac{2^{\frac{3}{2}}\pi i a}{m_\pi (q_0\nu)^{\frac{1}{2}}}\left(2iM_1\frac{\underline{\nu}\times\underline{\varepsilon}}{q\nu} + (M_1+E_2)\frac{(\underline{\sigma}\cdot\underline{\nu})\,\underline{\varepsilon}}{q\nu} - (M_1-E_2)\frac{(\underline{\sigma}\cdot\underline{\varepsilon})\,\underline{\nu}}{q\nu}\right) \qquad 6.34a$$

or

$$\frac{2^{\frac{3}{2}}\pi i a'}{m_\pi (q_0\nu)^{\frac{1}{2}}}\left(2iM_1'\frac{\underline{\nu}\times\underline{\varepsilon}}{q\nu} + (M_1'+E_2')\frac{(\underline{\sigma}\cdot\underline{\nu})\,\underline{\varepsilon}}{q\nu} - (M_1'-E_2')\frac{(\underline{\sigma}\cdot\underline{\varepsilon})\,\underline{\nu}}{q\nu}\right) \qquad 6.34b$$

The various quantities appearing in the above equations have already been defined in Chapter III, equations (3.20), (3.24), (3.25) and (3.29). $(\underline{\gamma}\cdot\underline{q})$ is the enhancement term due to scattering through the $I=\frac{3}{2}$, $J=\frac{3}{2}$ state and the operator $O_{\frac{3}{2}}(1) = \frac{1}{2^{\frac{1}{2}}}\left(\tau^-(1)\,a^*_{\underline{k}} + \tau^+(1)\,b^*_{\underline{k}}\right) + c^*_{\underline{k}}$ as might be expected, projects out the $I=\frac{1}{2}$ component of isotopic spin, that is $\langle \psi_{\frac{1}{2}}^{\pm\frac{1}{2}} | O_{\frac{3}{2}} | p \rangle = 0$ and $\langle \psi_{\frac{1}{2}}^{\pm\frac{1}{2}} | O_{\frac{3}{2}} | n \rangle = 0$

The matrix elements of the operators

$$O_B(j) = \left(\tau^-(j) a_{\underline{k}}^* + \tau^+(j) b_{\underline{k}}^*\right) \qquad 6.35a$$

and

$$O_{\frac{3}{2}}(j) = \frac{1}{2^{\frac{1}{2}}}\left(\tau^-(j) a_{\underline{k}}^* + \tau^+(j) b_{\underline{k}}^*\right) + c_{\underline{k}}^* \quad j = 1 \cdots 4 \qquad 6.35b$$

must be computed between the six basis states which define the matrices $C(j)$ and the ground state of the α -particle. The spin-isotopic spin wave-function of the α -particle is

$$\psi_\alpha = \frac{1}{6^{\frac{1}{2}}} \begin{vmatrix} \alpha(1)p(1) & \alpha(2)p(2) & \alpha(3)p(3) & \alpha(4)p(4) \\ \alpha(1)n(1) & \alpha(2)n(2) & \alpha(3)n(3) & \alpha(4)n(4) \\ \beta(1)p(1) & \beta(2)p(2) & \beta(3)p(3) & \beta(4)p(4) \\ \beta(1)n(1) & \beta(2)n(2) & \beta(3)n(3) & \beta(4)n(4) \end{vmatrix} \qquad 6.36$$

where α , β are the space spin wave-functions for spin up and spin down respectively, and p , n are the isotopic spin wave-functions for the proton and neutron respectively.

Then

$$\psi_\alpha = \frac{1}{6^{\frac{1}{2}}}\Big(S(12;34) I_4(12) I_4(34) - S(13;24) I_4(13) I_4(24) + S(14;23) I_4(14) I_4(23)\Big) \qquad 6.37$$

where

$$S(12;34) = \alpha(1)\alpha(2)\beta(3)\beta(4) + \beta(1)\beta(2)\alpha(3)\alpha(4) \qquad 6.38$$

and $I_4(12) = \frac{1}{2^{\frac{1}{2}}}\left(p(1)n(2) - n(1)p(2)\right)$ is the singlet isotopic spin wave-function for two nucleons 6.39

or
$$\psi_\alpha = \frac{1}{6^{\frac{1}{2}}}\left\{ S(12;34)J_0^0(1) - \frac{1}{2}S(13;24)\left(3^{\frac{1}{2}}J_0^0(2) + J_0^0(1)\right) + \frac{1}{2}S(14;23)\left(3^{\frac{1}{2}}J_0^0(2) - J_0^0(1)\right)\right\} \qquad 6.40$$

The required matrix elements are readily evaluated. The matrix for $O_{\frac{3}{2}}(1)$ in the second representation is for instance

$$\begin{bmatrix} \langle Y^0_{1,01}(1)|O_{\frac{3}{2}}(1)|\psi_\alpha\rangle \\ \langle Y^0_{1,01}(2)|O_{\frac{3}{2}}(1)|\psi_\alpha\rangle \\ \langle Y^0_{1,11}(1)|O_{\frac{3}{2}}(1)|\psi_\alpha\rangle \\ \langle Y^0_{1,11}(2)|O_{\frac{3}{2}}(1)|\psi_\alpha\rangle \\ \langle Y^0_{1,11}(3)|O_{\frac{3}{2}}(1)|\psi_\alpha\rangle \\ \langle Y^0_{1,21}(1)|O_{\frac{3}{2}}(1)|\psi_\alpha\rangle \end{bmatrix} = \frac{1}{6^{\frac{1}{2}}}S(12;34)\begin{bmatrix} 1 \\ 0 \\ 0 \\ 6^{-\frac{1}{2}} \\ -3^{-\frac{1}{2}} \\ 0 \end{bmatrix} - \frac{1}{2.6^{\frac{1}{2}}}S(13;24)\begin{bmatrix} 1 \\ 3^{\frac{1}{2}} \\ 2^{-\frac{1}{2}} \\ -6^{-\frac{1}{2}} \\ -2\,3^{-\frac{1}{2}} \\ 0 \end{bmatrix} + \frac{1}{2.6^{\frac{1}{2}}}S(14;23)\begin{bmatrix} -1 \\ 3^{\frac{1}{2}} \\ 2^{-\frac{1}{2}} \\ -3^{\frac{1}{2}}2^{-\frac{1}{2}} \\ 0 \\ 0 \end{bmatrix} \qquad 6.41$$

and similar expressions can be obtained for the remaining seven matrices.

VI § 4. The energy dependence of the scattering matrix element $\langle q_1|t_{si}|q_2\rangle$

In paragraph 2 of this Chapter the matrix element $\langle q_1|t_{si}|q_2\rangle$ was reduced to

$$\langle q_1|t_{si}|q_2\rangle = b(q_2)(2+\underset{\sim}{\tau}^{(i)}.\underset{\sim}{\ell})(\underset{\sim}{q}_1.\underset{\sim}{q}_2)e^{i(\underset{\sim}{q}_2-\underset{\sim}{q}_1)\underset{\sim}{r}_i} \qquad 6.42$$

The dependence of the function b on the meson energy will be deduced by comparing the differential cross-section for meson-nucleon scattering derived from (6.42) with the corresponding cross-section derived from a phase shift analysis. The particular reaction $p+\pi^+\rightarrow p+\pi^+$ (or (π^+,π^+)) is chosen for this comparison. Then

$$\langle p\pi^+|2+\underset{\sim}{\tau}.\underset{\sim}{\ell}|p\pi^+\rangle = \langle p\pi^+|2+2(\tau^+\ell^-+\tau^-\ell^+)+\tau_0\ell_0|p\pi^+\rangle \qquad 6.43$$
$$= 3$$

and
$$\langle q_1|t_{si}|q_2\rangle_{(\pi^+,\pi^+)} = 3b(q)\,\underset{\sim}{q}_1.\underset{\sim}{q}_2\,e^{i(\underset{\sim}{q}_2-\underset{\sim}{q}_1)\underset{\sim}{r}_i} \qquad 6.44$$

which leads to the differential cross-section

$$\frac{d\sigma}{dq_1 d\underset{\sim}{D}} = \frac{1}{(2\pi)^2}\,\delta(q_{10}-q_{20})\,\delta(\underset{\sim}{q}_2-\underset{\sim}{q}_1-\underset{\sim}{D})\,9|b(q)|^2 q_1^2 q_2^2\cos^2\theta\,\frac{q_{20}}{q_2} \qquad 6.45$$

in the laboratory system for the reaction $p+\pi^+\rightarrow p+\pi^+$. $\underset{\sim}{D}$ is the recoil momentum of the nucleon. The recoil energy of the nucleon has been neglected. Integration over $\underset{\sim}{D}$ and q_{10} leads to (putting $q_{10}=q_{20}=q_0$)

$$\frac{d\sigma}{d\Omega_q}(\pi^+,\pi^+) = 9\left|\frac{bq_0}{2\pi}\right|^2 q^4\cos^2\theta \qquad 6.46$$

The corresponding cross-section constructed from a phase shift analysis (reference (2) p.67) is, in the centre of mass system of the incoming meson and the nucleon,

$$\frac{d\sigma}{d\Omega_{q_c}}(\pi^+,\pi^+) = \frac{1}{4q_c^2}\left\{|a+b\cos\theta_c|^2 + |c|^2\sin^2\theta_c\right\} \qquad 6.47$$

where $$a = e^{2i\delta_3}-1 \qquad 6.48a$$

$$b = 2e^{2i\delta_{33}}+e^{2i\delta_{31}}-3 \qquad 6.48b$$

$$c = e^{2i\delta_{33}}-e^{2i\delta_{31}} \qquad 6.48c$$

a is the contribution of the s-waves mesons to the scattering amplitude, b arises from scattering of p-wave mesons without spin-flip of the nucleon, and c arises from the scattering of p-wave mesons with spin-flip of the nucleon. Now only scattering through the $(\frac{3}{2},\frac{3}{2})$ state was considered in the derivation of $\langle q_1|t_{si}|q_2\rangle$ and this is equivalent to the approximations $\delta_3 \simeq \delta_{31} \simeq 0$ in the energy range considered (meson kinetic energies from zero up to about 180 Mev). In this range $|\delta_3| < 20°$ and $|\delta_{31}| < 15°$ and hence the approximation is not unreasonable. Equating δ_3 and δ_{31} to zero

$$a = 0\ ,\ b = 4e^{i\delta_{33}}\sin\delta_{33}\ ,\ c = 2e^{i\delta_{33}}\sin\delta_{33} \qquad 6.49$$

$$\frac{d\sigma}{d\Omega_{q_c}}(\pi^+,\pi^+) = \frac{|e^{i\delta_{33}}\sin\delta_{33}|^2}{q_c^2}(4\cos^2\theta_c+\sin^2\theta_c) \quad 6.50$$

If spin-flip is neglected the expression becomes

$$\frac{d\sigma}{d\Omega_{q_c}}(\pi^+,\pi^+) = \frac{|e^{i\delta_{33}}\sin\delta_{33}|^2}{q_c^2}\, 4\cos^2\theta_c \quad 6.51$$

This last approximation is poor near $\theta_c = 90°$ and $\theta_c = 270°$ although the differential cross-section is much smaller at these angles than it is near $\theta_c = 0°$ or $180°$. The transformations between the c.m. quantities and the corresponding quantities in the laboratory system are (to a close approximation)

$$q_c = \frac{q}{1+\frac{q_0}{M}} \quad ; \quad \cos\theta_c = \left(1+\frac{q_0}{M}\right)\cos\theta \quad 6.52$$

Hence (6.51) gives

$$\frac{d\sigma}{d\Omega_q}(\pi^+,\pi^+) = \frac{|e^{i\delta_{33}}\sin\delta_{33}|^2}{q^2}\, 4\cos^2\theta\left(1+\frac{q_0}{M}\right)^5 \quad 6.53$$

Comparison of equations (6.46) and (6.53) show that

$$\frac{b q_0}{(2\pi)} = \frac{2}{3}\,\frac{e^{i\delta_{33}}\sin\delta_{33}}{q^3}\left(1+\frac{q_0}{M}\right)^{\frac{5}{2}} \quad 6.54$$

and therefore

$$\langle q_1|t_{si}|q_2\rangle = \frac{2\pi}{q_0}\,\frac{2(2+\tau^{(i)}\cdot\ell)}{3}\,\frac{e^{i\delta_{33}}\sin\delta_{33}}{q^3}\left(1+\frac{q_0}{M}\right)^{\frac{5}{2}} \quad 6.55$$
$$\times\, q_1 q_2\, e^{i(q_2-q_1)r_i}$$

VII. SOLUTION OF THE MULTIPLE SCATTERING EQUATIONS

§1. Solution for two nucleons

It is of some interest to obtain the solution of the equations for the multiple scattering of a meson at two nucleons, firstly because comparison with the results obtained by Chappelear (equations (4.17) and (4.18)) affords a check on the derivation of the equations, and secondly because the form of the solution for the two nucleon problem acts as a guide to the form expected for the solution of the equations for larger values of A. The case A = 2 is the only one in which the exact solution may be obtained easily.

(i) Solution of the 'P' equations for A = 2

The 'P' equations for the case A = 2 are (with i = 1)

$$\underset{\sim}{P_{11}} + b_1\left(f_{12}\underset{\sim}{P_{12}} + g_{12}\underset{\sim}{r_{12}}(\underset{\sim}{r_{12}}\cdot\underset{\sim}{P_{12}})\right) = \chi_1 \qquad 7.1a$$

$$\underset{\sim}{P_{12}} + b_2\left(f_{21}\underset{\sim}{P_{11}} + g_{21}\underset{\sim}{r_{21}}(\underset{\sim}{r_{21}}\cdot\underset{\sim}{P_{11}})\right) = 0 \qquad 7.1b$$

together with a similar set of equations obtained by interchanging the subscripts 1 and 2 in the above.

If $\underset{\sim}{r}_{12} = \underset{\sim}{\rho}$, $f_{12} = f_{21} = f$, $g_{12} = g_{21} = g$

$$\underset{\sim}{P}_{11} + b_1(f\underset{\sim}{P}_{12} + g\underset{\sim}{\rho}(\underset{\sim}{\rho}.\underset{\sim}{P}_{12})) = \underset{\sim}{\chi}_1 \qquad 7.2a$$

$$\underset{\sim}{P}_{12} + b_2(f\underset{\sim}{P}_{11} + g\underset{\sim}{\rho}(\underset{\sim}{\rho}.\underset{\sim}{P}_{11})) = 0 \qquad 7.2b$$

This last set of equations has the solution

$$\underset{\sim}{P}_{11} = \frac{1}{1-b_1b_2f^2}\left(\underset{\sim}{\chi}_1 + b_1b_2\frac{g(f+h)}{1-b_1b_2h^2}\underset{\sim}{\rho}(\underset{\sim}{\rho}.\underset{\sim}{\chi}_1)\right) \qquad 7.3a$$

and

$$\underset{\sim}{P}_{12} = -\frac{1}{1-b_2b_1f^2}b_2\left(f\underset{\sim}{\chi}_1 + g(1+fhb_1b_2)\frac{1}{1-b_1b_2h^2}\underset{\sim}{\rho}(\underset{\sim}{\rho}.\underset{\sim}{\chi}_1)\right) \qquad 7.3b$$

where $h = f + \rho^2 g$

and these results agree with the results obtained by Chappelear's method in equations (4.17) and (4.18). The impulse approximation solution is simply $\underset{\sim}{P}_{11} = \underset{\sim}{\chi}_1$ and $\underset{\sim}{P}_{12} = 0$.

(ii) <u>Solution of the 'M' equations for A = 2</u>

In the notation of the previous paragraph the 'M' equations become

$$\underset{\sim}{M}_{11} + b_1(f\underset{\sim}{M}_{12} + g\underset{\sim}{\rho}(\underset{\sim}{\rho}.\underset{\sim}{M}_{12})) = 0 \qquad 7.4a$$

$$\underset{\sim}{M}_{12} + b_2(f\underset{\sim}{M}_{11} + g\underset{\sim}{\rho}(\underset{\sim}{\rho}.\underset{\sim}{M}_{11})) = -ib_2\alpha_1 f\underset{\sim}{\rho} \qquad 7.4b$$

Hence $$\underset{\sim}{M}_{11} = i\rho b_1 \frac{fh}{1-b_2 b_1 h^2} b_2 \alpha_1 \qquad 7.5a$$

and $$\underset{\sim}{M}_{12} = -i\rho \frac{f}{1-b_2 b_1 h^2} b_2 \alpha_1 \qquad 7.5b$$

The impulse approximation solution is $\underset{\sim}{M}_{11} = \underset{\sim}{M}_{12} = 0$.

The transition probability amplitude for the photoproduction of a meson at two nucleons is obtained by substituting the functions (7.3) and (7.5) together with the corresponding solutions for $\underset{\sim}{M}_{21}$, $\underset{\sim}{M}_{22}$, $\underset{\sim}{P}_{21}$ and $\underset{\sim}{P}_{22}$ in the expression (5.55) for $\langle q|T|\nu\rangle$.

VII § 2. Solution of the multiple-scattering equations for A = 4.

The four nucleon 'P' equations derived from equation (5.48) with A = 4 and i = 1 may be written in the following simplified notation

$$\underset{\sim}{x}_1 + b_1 H_1 \underset{\sim}{x}_2 + b_1 H_2 \underset{\sim}{x}_3 + b_1 H_3 \underset{\sim}{x}_4 = \underset{\sim}{\gamma} \qquad 7.6a$$

$$b_2 H_1 \underset{\sim}{x}_1 + \underset{\sim}{x}_2 + b_2 H_4 \underset{\sim}{x}_3 + b_2 H_5 \underset{\sim}{x}_4 = 0 \qquad 7.6b$$

$$b_3 H_2 \underset{\sim}{x}_1 + b_3 H_4 \underset{\sim}{x}_2 + \underset{\sim}{x}_3 + b_3 H_6 \underset{\sim}{x}_4 = 0 \qquad 7.6c$$

$$b_4 H_3 \underset{\sim}{x}_1 + b_4 H_5 \underset{\sim}{x}_2 + b_4 H_6 \underset{\sim}{x}_3 + \underset{\sim}{x}_4 = 0 \qquad 7.6d$$

where $\underset{\sim}{x}_1$, $\underset{\sim}{x}_2$, $\underset{\sim}{x}_3$ and $\underset{\sim}{x}_4 = \underset{\sim}{P}_{11}, \underset{\sim}{P}_{12}, \underset{\sim}{P}_{13}$ and $\underset{\sim}{P}_{14}$ 7.7

respectively.

$$H_1, H_2, H_3, H_4, H_5, H_6 \;=\; H_{12}, H_{13}, H_{14}, H_{23}, H_{24}, H_{34} \qquad 7.8$$

respectively

and

$$H_{ij}\,\underline{x} = f_{ij}\,\underline{x} + g_{ij}\,\underline{r}_{ij}(\underline{r}_{ij}\cdot\underline{x}) \qquad 7.9$$

b_1, b_2, b_3 and b_4 are (equation (6.31)) matrices in charge space with dimensions 6 x 6 if the photoproduction of neutral mesons is considered. The labour involved in the manipulation of these matrices during the process of solving the set of equations (7.6) is prohibitive if adequate computing facilities are not available, and hence the charge exchange processes are neglected at this stage in order to obtain a numerical result which will at least give a qualitative indication of the magnitude of the multiple scattering correction. It is difficult to estimate the error caused by this approximation, but it is probably no worse than that incurred by neglecting the spin-flip term in the scattering matrix element, which is a comparable type of approximation. Hence neglecting the charge exchange terms the expression for $b_i(q)$ becomes

$$b_i(q) = \frac{2\pi}{q_0}\,\frac{\lambda(\lambda+\tau_0(i)\,\ell_0)}{3}\,\frac{e^{i\delta_{33}}\sin\delta_{33}}{q^3}\left(1+\frac{q_0}{M}\right)^{\frac{5}{2}} \qquad 7.10$$

But the matrix element of the operator $\tau_0(i)\ell_0$ between states which contain a neutral meson is always zero and, therefore,

$$b_i(q) = \frac{2\pi}{q_0}\,\frac{A}{3q^3}\,e^{i\delta_{33}} \qquad i = 1 \cdots 4 \tag{7.11}$$

where A is a function of the meson energy defined by

$$A = 4\sin\delta_{33}\left(1+\frac{q_0}{M}\right)^{\frac{5}{2}} \tag{7.12}$$

The photoproduction matrix element $\langle q|T_i|\nu\rangle$ is also considerably simplified by this approximation, since there is now no production of s-wave mesons. The simplified form of $\langle q|T_i|\nu\rangle$ is

$$\langle q|T_i|\nu\rangle = \gamma\, \underset{\sim}{q}\, e^{i(\underset{\sim}{\nu}-\underset{\sim}{q})\underset{\sim}{r_i}} \tag{7.13}$$

where γ has been defined in equation (6.34). The fact that α does not appear in this expression for the photoproduction matrix element means that the 'M' equations for the simplified problem need not be solved.

The set of equations (7.6) has now been reduced to a set of twelve linear simultaneous equations in the components of the vectors $\underset{\sim}{x}_1 \cdots \underset{\sim}{x}_4$. The simple appearance of the set of equations (7.6) is deceptive as the algebra of the operators $H_i\,(i=1\cdots 6)$ is quite complex.

Equations (7.6c) and (7.6d) can be solved to give $\underset{\sim}{\chi}_3$ and $\underset{\sim}{\chi}_4$ in terms of $\underset{\sim}{\chi}_1$ and $\underset{\sim}{\chi}_2$ since these equations regarded as equations in $\underset{\sim}{\chi}_3$ and $\underset{\sim}{\chi}_4$ are similar in type to the deuteron 'P' equations discussed in § 1. However, the equations for $\underset{\sim}{\chi}_1$ and $\underset{\sim}{\chi}_2$ which result from substituting for $\underset{\sim}{\chi}_3$ and $\underset{\sim}{\chi}_4$ in (7.6a) and (7.6b) the solutions of (7.6c) and (7.6d) are quite unmanageable and it seems advisable to look for some approximation whereby the equations (7.6) can be simplified.

In order to investigate the relative importance of the terms in these equations it is necessary to know the magnitude of the functions f_{jk} and g_{jk} for various separations of the nucleons. f_{jk} has been defined (equation (5.41)) to be

$$f_{jk} = -\frac{q_0}{2\pi}\frac{1}{r_{jk}}\frac{d}{dr_{jk}}\frac{e^{iqr_{jk}}}{r_{jk}}$$

$$= \frac{q_0}{2\pi}\frac{e^{iqr_{jk}}}{r_{jk}^3}\left(1-i(qr_{jk})\right) \qquad 7.14$$

and multiplying by $b_j(q)$

$$b_j f_{jk} = \frac{A}{3y_{jk}^3} e^{i(\delta_{33}+y_{jk})}(1-iy_{jk}) \qquad 7.15$$

$$= \frac{A}{3y_{jk}^3} e^{i(\delta_{33}+y_{jk}-\tan^{-1}y_{jk})}(1+y_{jk}^2)^{\frac{1}{2}}$$

where

$$y_{jk} = qr_{jk} \qquad 7.16$$

Similarly g_{jk} has been defined (equation (5.42)) to be

$$g_{jk} = \frac{1}{r_{jk}} \frac{d}{dr_{jk}} f_{jk} \qquad 7.17$$

$$= \frac{q_0}{2\pi} \frac{1}{r_{jk}^5} e^{iqr_{jk}} \left(-3 + 3iqr_{jk} + (qr_{jk})^2\right)$$

and therefore

$$b_j g_{jk} r_{jk}^2 = \frac{A}{y_{jk}^3} e^{i(\delta_{33} + y_{jk})} \left(-1 + \frac{1}{3} y_{jk}^2 + i y_{jk}\right) \qquad 7.18$$

$$= \frac{A}{y_{jk}^3} e^{i\left(\delta_{33} + y_{jk} + \pi - \tan^{-1} \frac{y_{jk}}{1 - \frac{1}{3} y_{jk}^2}\right)} \left(1 + \frac{1}{3} y_{jk}^2 + \frac{1}{9} y_{jk}^4\right)^{\frac{1}{2}} \qquad 7.19$$

To obtain a rough estimate of the magnitude of the coefficients of $\underset{\sim}{x}_j$ in the equations (7.6), the mean value of $b_j g_{jk} r_{jk} (r_{jk} \quad)$ is taken to be $\frac{1}{3} b_j g_{jk} r_{jk}^2$

The impulse approximation solution to the problem is $\underset{\sim}{x}_1 = \psi$ and $\underset{\sim}{x}_2 = \underset{\sim}{x}_3 = \underset{\sim}{x}_4 = 0$. This solution will be valid when $|b_j f_{jk}|$ and $\frac{1}{3} |b_j g_{jk} r_{jk}^2|$ are much smaller than one. In the region $0 \leq r_{jk} \leq 4 \times 10^{-13}$ cm., $220 \leq \nu \leq 260$ Mev., the moduli of these functions are roughly comparable in magnitude and take on the values; ∞ at $r_{jk} = 0$; 1 at $r_{jk} = 10^{-13}$ cm. ; $\cdot 3$ at $r_{jk} = 1\cdot 8 \times 10^{-13}$ cm. and $\cdot 1$ at $r_{jk} = 3\cdot 1 \times 10^{-13}$ cm. In the region $r_{jk} > 1\cdot 5 \times 10^{-13}$ cm. ($\nu \leq 260$ Mev.), the solution is tending towards the impulse approximation solution, while for small values of r_{jk} ($r_{jk} < 10^{-13}$ cm.) the impulse approximation is certainly inaccurate. As the root mean square radius of the α- particle is only 1.4 x 10^{-13} cm. it appears that the impulse approximation solution for the elastic photoproduction of neutral mesons at helium will have little more than qualitative significance.

For small r_{jk} ($< 10^{-13}$ cm.) the functions $b_j f_{jk}$ and $\frac{1}{3} b_j g_{jjk} r_{jk}^2$ assume the approximate form $\frac{A}{3y_{jk}^3}$ and are proportional to r_{jk}^{-3}. Then $|x_j|$ ($j = 1 \cdots 4$) will be proportional to $\frac{y_{jk}^3}{A}$ and will be negligibly small for the greater part of this region. It is necessary then in deriving an approximate form of the equations (7.6) to look for a set of equations whose solution tends to the impulse approximation solution for $r_{jk} \geqslant 1.5 \times 10^{-13}$ cm., and is negligibly small for $r_{jk} < 10^{-13}$ cm.. Such a set of equations is

$$\underset{\sim}{x_1} + b(H_1 \underset{\sim}{x_2} + H_2 \underset{\sim}{x_3} + H_3 \underset{\sim}{x_4}) = \underset{\sim}{f} \qquad 7.20a$$

$$\underset{\sim}{x_2} = -bH_1 \underset{\sim}{x_1} \qquad 7.20b$$

$$\underset{\sim}{x_3} = -bH_2 \underset{\sim}{x_1} \qquad 7.20c$$

$$\underset{\sim}{x_4} = -bH_3 \underset{\sim}{x_1} \qquad 7.20d$$

It is obvious that this set of equations (7.20) tends towards the impulse approximation $\chi_1 = \psi$, $\chi_2 = \chi_3 = \chi_4 = 0$ for large nucleon separations. It may also be shown that for $r_{jk} < 10^{-13}$ cm., $|\chi_1|$ is proportional to $\frac{y_{jk}^6}{A^2}$ and $|\chi_2|$, $|\chi_3|$ and $|\chi_4|$ are proportional to $\frac{y_{jk}^3}{A}$. It would seem then that the solution of the approximate set (7.20) has the same general trends in behaviour as the solution of the complete set (7.6), although, of course, it is difficult to compare their detailed behaviour particularly in the region $10^{-13} \leq r_{jk} \leq 2 \times 10^{-13}$ cm. The indications are that the magnitude of χ_1 as derived from (7.20) will be underestimated in this region of nucleon separation.

The equations (7.20) have the solution

$$\chi_1 = \frac{1}{f}\psi + \frac{1}{f}\sum_{i,j=1}^{3}\frac{A_{ij}}{A}\rho_i(\rho_j\cdot\psi) \qquad 7.21a$$

$$\chi_2 = -b\left\{\frac{f_1}{f}\psi + \sum_{i,j=1}^{3}\rho_i\frac{A_{ij}}{A}(\rho_j\cdot\psi)\left(\frac{f_1}{f} + \frac{g_1}{\alpha_1}\delta_{i1}\right)\right\} \qquad 7.21b$$

$$\chi_3 = -b\left\{\frac{f_2}{f}\psi + \sum_{i,j=1}^{3}\rho_i\frac{A_{ij}}{A}(\rho_j\cdot\psi)\left(\frac{f_2}{f} + \frac{g_2}{\alpha_2}\delta_{i2}\right)\right\} \qquad 7.21c$$

$$\chi_4 = -b\left\{\frac{f_3}{f}\psi + \sum_{i,j=1}^{3}\rho_i\frac{A_{ij}}{A}(\rho_j\cdot\psi)\left(\frac{f_3}{f} + \frac{g_3}{\alpha_3}\delta_{i3}\right)\right\} \qquad 7.21d$$

where $f = 1 - b^2(f_1^2 + f_2^2 + f_3^2)$, $f_1 = f_{12}$, $f_2 = f_{13}$, $f_3 = f_{14}$ 7.22

$$\rho_i = \alpha_i^{\frac{1}{2}} s_i, \; i = 1,2,3. \quad s_1 = r_{21}, \; s_2 = r_{31}, \; s_3 = r_{41} \qquad 7.23$$

$$\alpha_i = b^2 q_i (f_i + h_i) \qquad i = 1,2,3 \tag{7.24}$$

$$h_i = f_i + q_i s_i^2 \qquad i = 1,2,3 \tag{7.25}$$

$$\rho_1 \cdot \rho_2 = \rho_1 \rho_2 \cos\theta_3 \ ; \ \rho_1 \cdot \rho_3 = \rho_1 \rho_3 \cos\theta_2 \ ; \ \rho_2 \cdot \rho_3 = \rho_2 \rho_3 \cos\theta_1 \tag{7.26}$$

$$\begin{aligned} A &= f^3 - f^2(\rho_1^2 + \rho_2^2 + \rho_3^2) + f(\rho_1^2\rho_2^2 \sin^2\theta_3 + \rho_1^2\rho_3^2 \sin^2\theta_2 + \rho_2^2\rho_3^2 \sin^2\theta_1) \\ &\quad - \rho_1^2\rho_2^2\rho_3^2(1 + 2\cos\theta_1\cos\theta_2\cos\theta_3 - \cos^2\theta_1 - \cos^2\theta_2 - \cos^2\theta_3) \end{aligned} \tag{7.27}$$

$$A_{11} = f^2 - (\rho_2^2 + \rho_3^2) f + \rho_2^2 \rho_3^2 \sin^2\theta_1 \tag{7.28}$$

$$A_{22} = f^2 - (\rho_1^2 + \rho_3^2) f + \rho_1^2 \rho_3^2 \sin^2\theta_2 \tag{7.29}$$

$$A_{33} = f^2 - (\rho_1^2 + \rho_2^2) f + \rho_1^2 \rho_2^2 \sin^2\theta_3 \tag{7.30}$$

$$A_{12} = \rho_1 \rho_2 (f \cos\theta_3 + \rho_3^2(\cos\theta_1 \cos\theta_2 - \cos\theta_3)) \tag{7.31}$$

$$A_{13} = \rho_1 \rho_3 (f \cos\theta_2 + \rho_2^2(\cos\theta_1 \cos\theta_3 - \cos\theta_2)) \tag{7.32}$$

$$A_{23} = \rho_2 \rho_3 (f \cos\theta_1 + \rho_1^2(\cos\theta_2 \cos\theta_3 - \cos\theta_1)) \tag{7.33}$$

VIII. THE MULTIPLE SCATTERING CORRECTION TO THE DIFFERENTIAL CROSS-SECTION FOR THE ELASTIC PHOTOPRODUCTION OF MESONS AT HELIUM

§ 1. The photoproduction matrix element

By equation (5.55) the matrix element of the total transition operator T is

$$\langle q|T|v\rangle = \sum_{i=1}^{4}\sum_{j=1}^{4} e^{i(\underline{v}\cdot\underline{r}_i - \underline{q}\cdot\underline{r}_j)}\, \underline{q}\cdot\underline{P}_{ij} \qquad 8.1$$

and this must be averaged over the initial and final wave-functions of the α- particle.

Let $\Psi(12;34) = \chi_4(12)\chi_4(34)\psi(\underline{r}_1,\underline{r}_2,\underline{r}_3,\underline{r}_4)$ be the α- particle wave-function where $\psi(\underline{r}_1,\underline{r}_2,\underline{r}_3,\underline{r}_4)$ is a symmetric, normalised, configuration space wave-function.

Then the transition probability amplitude M is

$$M = \int d\underline{r}_1 d\underline{r}_2 d\underline{r}_3 d\underline{r}_4\, \psi^2(\underline{r}_1,\underline{r}_2,\underline{r}_3,\underline{r}_4)\, e^{-i\underline{D}\cdot\underline{R}}\, \chi_4^+(12)\chi_4^+(34) \times \sum_{ij=1}^{4} e^{i(\underline{v}\cdot\underline{r}_i - \underline{q}\cdot\underline{r}_j)}\, \underline{q}\cdot\underline{P}_{ij}\, \chi_4(12)\chi_4(34) \qquad 8.2$$

where $\underline{D}$ is the recoil momentum, and $\underline{R}$ the centre of mass coordinate of the α- particle. Each term in (equation (7.21)) depends linearly on γ , and γ can be written as $\chi = \chi_0 + \chi(\underset{\sim}{\sigma}(i))$ where $\chi(\underset{\sim}{\sigma}(i))$ is dependent linearly on the spin operator $\underset{\sim}{\sigma}(i)$ of the i^{th} nucleon and χ_0 is independent of $\underset{\sim}{\sigma}(i)$. Then

$$X_4^+(12)X_4^+(34)\,\chi(\sigma)\,X_4(12)X_4(34) = 0 \qquad 8.3a$$

and

$$X_4^+(12)X_4^+(34)\,\chi_0\,X_4(12)X_4(34) = \chi_0 \qquad 8.3b$$

$(\underset{\sim}{q}.\chi_0)$ is identical with L_0 of equation (3.23) Therefore

$$\chi_0 = -\frac{2^{\frac{5}{2}}\pi a}{m_\pi (q_0 v)^{\frac{1}{2}}} M_1 \frac{\underset{\sim}{v} \times \underset{\sim}{\varepsilon}}{qv} \qquad 8.4a$$

or

$$\chi_0 = -\frac{2^{\frac{5}{2}}\pi a'}{m_\pi (q_0 v)^{\frac{1}{2}}} M_1' \frac{\underset{\sim}{v} \times \underset{\sim}{\varepsilon}}{qv} \qquad 8.4b$$

Then

$$M = \int d\underset{\sim}{r_1} d\underset{\sim}{r_2} d\underset{\sim}{r_3} d\underset{\sim}{r_4}\, \psi^2(\underset{\sim}{r_1},\underset{\sim}{r_2},\underset{\sim}{r_3},\underset{\sim}{r_4})\, e^{-i\underset{\sim}{Q}.\underset{\sim}{R}} \sum_{ij=1}^{4} e^{i(\underset{\sim}{v}.\underset{\sim}{r_i} - \underset{\sim}{q}.\underset{\sim}{r_j})}\, \underset{\sim}{q}.\underset{\sim}{P}_{ij}(\chi_0) \qquad 8.5$$

From the symmetry of this expression it can be seen first of all that each nucleon contributes equally to M and hence M reduces to

$$M = 4\int d\underset{\sim}{r_1} d\underset{\sim}{r_2} d\underset{\sim}{r_3} d\underset{\sim}{r_4}\, \psi^2(\underset{\sim}{r_1},\underset{\sim}{r_2},\underset{\sim}{r_3},\underset{\sim}{r_4})\, e^{-i\underset{\sim}{Q}.\underset{\sim}{R} + i\underset{\sim}{v}.\underset{\sim}{r_1}} \sum_{j=1}^{4} e^{-i\underset{\sim}{q}.\underset{\sim}{r_j}}\, \underset{\sim}{q}.\underset{\sim}{x}_j(\chi_0) \qquad 8.6$$

and secondly that the contributions of nucleons 2, 3 and 4 to the integral in (8.6) are equal. Therefore,

$$M = 4\int d\underset{\sim}{r_1} d\underset{\sim}{r_2} d\underset{\sim}{r_3} d\underset{\sim}{r_4}\, \psi^2(\underset{\sim}{r_1},\underset{\sim}{r_2},\underset{\sim}{r_3},\underset{\sim}{r_4})\, e^{-i\underset{\sim}{Q}.\underset{\sim}{R} + i\underset{\sim}{v}.\underset{\sim}{r_1}} \qquad 8.7$$

$$\times \left(e^{-i\underset{\sim}{q}.\underset{\sim}{r_1}}\, \underset{\sim}{q}.\underset{\sim}{x}_1(\chi_0) + 3 e^{-i\underset{\sim}{q}.\underset{\sim}{r_2}}\, \underset{\sim}{q}.\underset{\sim}{x}_2(\chi_0) \right)$$

A Gaussian wave-function is chosen to represent the α- particle (equation (3.9)) and the coordinate transformation

$$\underset{\sim}{R} = \frac{1}{4}(\underset{\sim}{r_1}+\underset{\sim}{r_2}+\underset{\sim}{r_3}+\underset{\sim}{r_4})\ ;\ \underset{\sim}{s_1} = \underset{\sim}{r_2}-\underset{\sim}{r_1}\ ;\ \underset{\sim}{s_2} = \underset{\sim}{r_3}-\underset{\sim}{r_1}\ ;\ \underset{\sim}{s_3} = \underset{\sim}{r_4}-\underset{\sim}{r_1} \qquad 8.8$$

is used to give

$$M = 4(2\pi)^3\delta(\underset{\sim}{k}-\underset{\sim}{q}-\underset{\sim}{D})N\int d\underset{\sim}{s_1}d\underset{\sim}{s_2}d\underset{\sim}{s_3}\, e^{-2\mu_{\alpha G}^2\left(3(s_1^2+s_2^2+s_3^2)-2(\underset{\sim}{s_1}.\underset{\sim}{s_2}+\underset{\sim}{s_1}.\underset{\sim}{s_3}+\underset{\sim}{s_2}.\underset{\sim}{s_3})\right)}$$
$$\times\left\{e^{-\frac{1}{4}i(\underset{\sim}{k}-\underset{\sim}{q})(\underset{\sim}{s_1}+\underset{\sim}{s_2}+\underset{\sim}{s_3})}(\underset{\sim}{q}.\underset{\sim}{\chi_1})+3e^{-\frac{1}{4}i(\underset{\sim}{k}-\underset{\sim}{q})(\underset{\sim}{s_1}+\underset{\sim}{s_2}+\underset{\sim}{s_3})-i\underset{\sim}{q}.\underset{\sim}{s_1}}(\underset{\sim}{q}.\underset{\sim}{\chi_2})\right\} \qquad 8.9$$

where it is understood that $\underset{\sim}{\chi_1}$ and $\underset{\sim}{\chi_2}$ depend on ψ_0 (and not on $\psi = \psi_0 + \psi(\underset{\sim}{r})$).

VIII § 2. The tetrahedron model for the α-particle

The integrals in the expression (8.9)for the transition probability amplitude M are difficult to perform primarily because of the dependence of $\underset{\sim}{\chi_1}$ and $\underset{\sim}{\chi_2}$ on the angles between the relative coordinates $\underset{\sim}{s_1}$, $\underset{\sim}{s_2}$ and $\underset{\sim}{s_3}$ of the nucleons. In order to evaluate the expression (8.9) for M integrations over nine variables would have to be performed by numerical methods. If a change of variables were introduced such that the angles θ_1 , θ_2 and θ_3 of equation (7.26) were three of the new variables, it is possible that the numerical work might be reduced to the evaluation of a six-dimensional integral, but the amount of calculation involved would still be prohibitive. It is obvious that the qualitative nature of the matrix element $\langle q | T | v \rangle$ as developed in Chapter VII does not warrant such an extensive programme of numerical calculation, and hence it is advisable to investigate the possibility of finding a model for the α- particle which would reduce this programme considerably.

The model developed with this end in view is the tetrahedron model for the α-particle. In this model it is assumed that the nucleons are placed at the vertices of a regular tetrahedron which is only allowed to expand or contract in such a way that the sides remain equal. These restrictions remove five of the nine original degrees of freedom of the α-particle (neglecting the motion of the centre of mass of the nucleus). The angles θ_1 , θ_2 and θ_3 are now each 60° with the result that, since $\underset{\sim}{\chi}_1$ and $\underset{\sim}{\chi}_2$ are dependent only on these angles, only one numerical integration, namely that over the relative separations of the nucleons, remains to be done. The tetrahedron assumptions outlined above in fact increase the similarity of the α -particle problem to the deuteron problem both in the nature of the nuclear wave-function employed (which is dependent on only one relative separation in each case) and in the nature of the solutions to the multiple scattering equations.

The symmetry of the model may be more fully exploited by changing the system of axes. Let the original set of axes be Ox, Oy, Oz with the nucleon 1 lying at the origin O, the axes being fixed by the directions of the incident photon and the observed meson in a way which is not specified at the moment. The new z axis Oz', with direction cosines ($\sin\theta\cos\phi$, $\sin\theta\ \sin\phi$, $\cos\theta$) with respect to the original set of axes, is chosen to be the axis of the tetrahedron passing through O (i.e. Oz' makes equal angles with $\underset{\sim}{s}_1$, $\underset{\sim}{s}_2$ and $\underset{\sim}{s}_3$); the new x axis Ox' is chosen to lie in the plane zOz' and to be perpendicular to Oz' ($z'\hat{O}x' = 90°$) and the new y axis Oy' is chosen to complete the right-handed orthogenal set Ox', Oy', Oz'.

The new set of axes have the following direction cosines with respect to the original set

$$Ox' ; (l_1, m_1, n_1) ; (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta) \quad 8.10a$$

$$Oy' ; (l_2, m_2, n_2) ; (-\sin\phi \ , \ \cos\phi \ , 0 \ ) \quad 8.10b$$

$$Oz' ; (l_3, m_3, n_3) ; (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \quad 8.10c$$

while with respect to the set Ox', Oy', Oz' , $\underset{\sim}{s}_1$, $\underset{\sim}{s}_2$ and $\underset{\sim}{s}_3$ have direction cosines

$$\underset{\sim}{s}_1 \quad \left(\left(\tfrac{1}{3}\right)^{\frac{1}{2}}\cos\psi \quad , \left(\tfrac{1}{3}\right)^{\frac{1}{2}}\sin\psi \quad , \left(\tfrac{2}{3}\right)^{\frac{1}{2}} \right) \quad 8.11a$$

$$\underset{\sim}{s}_2 \quad \left(-\left(\tfrac{1}{3}\right)^{\frac{1}{2}}\cos(60-\psi), \left(\tfrac{1}{3}\right)^{\frac{1}{2}}\sin(60-\psi), \left(\tfrac{2}{3}\right)^{\frac{1}{2}} \right) \quad 8.11b$$

$$\underset{\sim}{s}_3 \quad \left(-\left(\tfrac{1}{3}\right)^{\frac{1}{2}}\cos(60+\psi), -\left(\tfrac{1}{3}\right)^{\frac{1}{2}}\sin(60+\psi), \left(\tfrac{2}{3}\right)^{\frac{1}{2}} \right) \quad 8.11c$$

that is with respect to the new axes $\underset{\sim}{s}_1$, $\underset{\sim}{s}_2$ and $\underset{\sim}{s}_3$ have spherical polar coordinates $(s, \cos^{-1}(\frac{2}{3})^{\frac{1}{2}}, \psi)$, $(s, \cos^{-1}(\frac{2}{3})^{\frac{1}{2}}, 120+\psi)$ and $(s, \cos^{-1}(\frac{2}{3})^{\frac{1}{2}}, 240+\psi)$ respectively. With respect to the original axes $\underset{\sim}{s}_1$, $\underset{\sim}{s}_2$ and $\underset{\sim}{s}_3$ have spherical polar coordinates (s_1, θ_1, ϕ_1), (s_2, θ_2, ϕ_2) and (s_3, θ_3, ϕ_3).

Hence in the evaluation of the transition probability M, $\int_{-\infty}^{\infty} d\underset{\sim}{s}_1 d\underset{\sim}{s}_2 d\underset{\sim}{s}_3$ is replaced by

$$\int_0^{\infty} s^2 ds \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi \int_0^{2\pi} d\psi \qquad 8.12$$

The relation between the two coordinate systems is given by

$$\sin\theta_1\cos\phi_1 = \frac{1}{3^{\frac{1}{2}}}\left(\cos\theta\cos\phi\cos\psi - \sin\phi\sin\psi + 2^{\frac{1}{2}}\sin\theta\cos\phi\right) \quad 8.13a$$

$$\sin\theta_1\sin\phi_1 = \frac{1}{3^{\frac{1}{2}}}\left(\cos\theta\sin\phi\cos\psi + \cos\phi\sin\psi + 2^{\frac{1}{2}}\sin\theta\sin\phi\right) \quad 8.13b$$

$$\cos\theta_1 = \frac{1}{3^{\frac{1}{2}}}\left(-\sin\theta\cos\psi + 2^{\frac{1}{2}}\cos\theta\right) \quad 8.13c$$

with the corresponding relationships for θ_2, ϕ_2 and θ_3, ϕ_3 obtained by replacing ψ in the above relationships by $120^\circ+\psi$ and $240^\circ+\psi$ respectively.

With the simplifying assumptions of the tetrahedron model the Gaussian form of the α- particle wave-function reduces to

$$\psi_\alpha = N_T^{\frac{1}{2}} e^{-12\mu_{\alpha T}^2 s^2} \quad 8.14$$

Here $\mu_{\alpha T}$ is an adjustable parameter which is chosen so that a reasonable fit is obtained to the radius of the α- particle when it is described by the above wave-function. In Chapter III it was shown that the relationship between the root mean square radius $R_{r.m.s.}$ and the root mean square separation of the nucleons

$S_{r.m.s.}$ is

$$R_{r.m.s.} = \rho_0 \frac{3^{\frac{1}{2}}}{2^{\frac{3}{2}}} S_{r.m.s}\ ,\ \rho_0 \gg 1. \qquad 8.15$$

$S_{r.m.s.}$ calculated on the basis of the tetrahedron model is $\left(\frac{1}{8\mu_{\alpha T}^2}\right)^{\frac{1}{2}}$ while $S_{r.m.s.}$ obtained with the use of Gaussian wave-functions is $\left(\frac{3}{8\mu_{\alpha G}^2}\right)^{\frac{1}{2}}$. Then by (8.15)

$$\mu_{\alpha T} = \frac{1}{3^{\frac{1}{2}}}\mu_{\alpha G} \qquad 8.16$$

and the values of $\mu_{\alpha G}$ which give the best fit to the radius or binding energy of the α- particle have already been discussed (equations (3.13a), (3.14a)). Using the wave-function (8.14) to represent the α- particle the transition probability amplitude $M_{I.A}$ for the elastic photoproduction of neutral mesons at helium, evaluated on the basis of the impulse approximation and the tetrahedron model is

$$M_{I.A.} = 4(2\pi)^3\delta(\underset{\sim}{k}-\underset{\sim}{q}-\underset{\sim}{D})N_T\int_0^\infty s^2 ds\, e^{-12\mu_{\alpha T}^2 s^2}\int_0^\pi \sin\theta\, d\theta \qquad 8.17$$

$$\times\int_0^{2\pi} d\phi\, e^{-i\left(\frac{3}{8}\right)^{\frac{1}{2}}(\underset{\sim}{k}-\underset{\sim}{q})\underset{\sim}{z}' s}\int_0^{2\pi} d\psi\,(q.\chi_0)$$

where

$$\frac{1}{N_T} = \int_0^\infty s^2 ds\, e^{-12\mu_{\alpha T}^2 s^2}\int_0^\pi \sin\theta\, d\theta\int_0^{2\pi} d\phi\int_0^{2\pi} d\psi \qquad 8.18$$

$$= \frac{\pi^{\frac{5}{2}}}{4.3^{\frac{3}{2}}\mu_{\alpha T}^3}$$

and $\underset{\sim}{z}'$ is a unit vector along the Oz' direction.

Then by selecting the Oz axis to be parallel to $\underset{\sim}{v}-\underset{\sim}{q}$

$$M_{1A} = 4(2\pi)^3\delta(\underset{\sim}{v}-\underset{\sim}{q}-\underset{\sim}{D})(\underset{\sim}{q}\cdot\underset{\sim}{\gamma}_0)N_T 8\pi^2 \times \int_0^\infty s^2 ds\, e^{-12\mu_{\alpha T}^2 s^2} j_0\left(\left(\tfrac{3}{8}\right)^{\frac{1}{2}} ps\right) \tag{8.19}$$

$$= 4(2\pi)^3\delta(\underset{\sim}{v}-\underset{\sim}{q}-\underset{\sim}{D})(\underset{\sim}{q}\cdot\underset{\sim}{\gamma}_0)N_T \frac{\pi^2}{3^{\frac{3}{2}}\mu_{\alpha T}^3} \times \int_0^\infty x^2 dx\, e^{-x^2} j_0\left(\frac{3^{\frac{1}{2}}px}{2^{\frac{5}{2}}\mu_{\alpha G}}\right) \tag{8.20}$$

by (8.16)

$$\therefore M_{1A} = 4(2\pi)^3\delta(\underset{\sim}{v}-\underset{\sim}{q}-\underset{\sim}{D})(\underset{\sim}{q}\cdot\underset{\sim}{\gamma}_0)e^{-\frac{3p^2}{128\mu_{\alpha G}^2}}$$

$$= 4(2\pi)^3\delta(\underset{\sim}{v}-\underset{\sim}{q}-\underset{\sim}{D})L_0 e^{-\frac{3p^2}{128\mu_{\alpha G}^2}} \tag{8.21}$$

M_{1A} is identical with the impulse approximation solution obtained when the α- particle is represented by a Gaussian wave-function, as may be seen on comparison of (8.21) with (3.33) and (3.37). It does not seem unreasonable to hope, therefore, that the tetrahedron model will also give a solution for the multiple scattering correction to the impulse approximation, which is not far removed from that obtained by integrating over the full configuration space $\int_{-\infty}^{\infty} d\underset{\sim}{s}_1 d\underset{\sim}{s}_2 d\underset{\sim}{s}_3$

VIII § 3. <u>The multiple-scattering correction to the impulse approximation for the elastic photo-production of π^o mesons at helium on the tetrahedron model.</u>

On the tetrahedron model the definitions (7.22) to (7.26) are altered to

$$\theta_1=\theta_2=\theta_3=60^\circ\ ;\ s_1=s_2=s_3=s\ ;\ f_1=f_2=f_3=e$$
$$g_1=g_2=g_3=g\ ;\ \alpha_1=\alpha_2=\alpha_3=\alpha\ ;\ \rho_1=\rho_2=\rho_3=\rho \qquad 8.22$$

$$A_{12}=A_{13}=A_{23}=\frac{1}{4}\rho^2(2f-\rho^2) \qquad 8.23$$

$$A_{11}=A_{22}=A_{33}=\frac{1}{4}(2f-3\rho^2)(2f-\rho^2) \qquad 8.24$$

$$A \qquad\qquad =\frac{1}{4}(2f-\rho^2)^2(f-2\rho^2) \qquad 8.25$$

$$\frac{A_{12}}{A}=\frac{A_{13}}{A}=\frac{A_{23}}{A}=\frac{\rho^2}{(2f-\rho^2)(f-2\rho^2)}=C \qquad 8.26$$

$$\frac{A_{11}}{A}=\frac{A_{22}}{A}=\frac{A_{33}}{A}=\frac{2f-3\rho^2}{(2f-\rho^2)(f-2\rho^2)}=B \qquad 8.27$$

Hence the solutions to the equations ($\underset{\sim}{\chi}_1$ and $\underset{\sim}{\chi}_2$) which are required in the evaluation of the matrix element are

$$(q\,\chi_1)=\frac{1}{f}(q\cdot\chi_0)+\frac{1}{f}(B-C)\big((q\cdot\rho_1)(\rho_1\cdot\chi)+(q\cdot\rho_2)(\rho_2\cdot\chi)+(q\cdot\rho_3)(\rho_3\cdot\chi)\big)$$
$$+\frac{6C}{\rho^2}(q\cdot z')(\chi_0\cdot z') \qquad 8.28$$

where $\underset{\sim}{z}'$ as before is a unit vector in the direction Oz'

and $$(q.\underline{x_2}) = (q.\underline{t_1}) + (q.\underline{t_2}) \qquad 8.29$$

where $$(q.\underline{t_1}) = -be(q.\underline{x_1}) \qquad 8.30$$

and $$(q.\underline{t_2}) = -\frac{bg}{\alpha}(q.\rho_1)\left\{(B-C)(\rho_1.\underline{\rho_0}) + C(\rho_1+\rho_2+\rho_3).\underline{\rho_0}\right\} \qquad 8.31$$

The matrix element M (8.9) may be conveniently separated into

$$M = M_1 + M_2(t_1) + M_2(t_2) \qquad 8.32$$

where $$M_1 = 4(2\pi)^3\delta(\underline{\nu}-\underline{q}-\underline{D})N_T\int_0^\infty s^2 ds\, e^{-12\mu_{\ell T}^2 s^2}\int_0^\pi \sin\theta\, d\theta$$
$$\times\int_0^{2\pi} d\phi\int_0^{2\pi} d\psi\, e^{-\frac{1}{4}i(\underline{\nu}-\underline{q})(\underline{s_1}+\underline{s_2}+\underline{s_3})}(q.\underline{x_1}) \qquad 8.33$$

and $$M_2(t_i) = 12(2\pi)^3\delta(\underline{\nu}-\underline{q}-\underline{D})N_T\int_0^\infty s^2 ds\, e^{-12\mu_{\ell T}^2 s^2}\int_0^\pi \sin\theta\, d\theta$$
$$\times\int_0^{2\pi} d\phi\int_0^{2\pi} d\psi\, e^{-\frac{1}{4}i(\underline{\nu}-\underline{q})(\underline{s_1}+\underline{s_2}+\underline{s_3})-i\underline{q}.\underline{s_1}}(q.\underline{t_i}) \qquad 8.34$$
$$i = 1, 2$$

to facilitate the evaluation of the integrations over the angles θ , ϕ and ψ .

The calculations are rather simpler for the meson angle of 90° in the laboratory system and the evaluation of the matrix element M is restricted to this particular case.

(i) Evaluation of M_1 for mesons produced at 90° in the laboratory system

From (8.33) $$M_1 = 4(2\pi)^3\,\delta(\underset{\sim}{\nu}-\underset{\sim}{q}-\underset{\sim}{D})\,N_T\,I_1 \qquad 8.35a$$

where $$I_1 = \int_0^\infty s^2 ds\, e^{-12\mu_{T}^2 s^2} J_1 \qquad 8.35b$$

$$J_1 = \int_0^\pi \sin\theta\, d\theta\, K_1 \qquad 8.35c$$

$$K_1 = \int_0^{2\pi} d\phi\, e^{-i\left(\frac{3}{8}\right)^{\frac{1}{2}}(\underset{\sim}{\nu}-\underset{\sim}{q})\underset{\sim}{z}'s} L_1 \qquad 8.35d$$

and $$L_1 = \int_0^{2\pi} d\psi\,(\underset{\sim}{q}.\underset{\sim}{\chi_1}) \qquad 8.35e$$

The Oz axis is chosen to be parallel to $\underset{\sim}{\delta_1} = \left(\frac{3}{8}\right)^{\frac{1}{2}}\underset{\sim}{p}$, $(\underset{\sim}{p}=\underset{\sim}{\nu}-\underset{\sim}{q})$. Then by substituting (8.28) for $(\underset{\sim}{q}.\underset{\sim}{\chi_1})$ it can readily be shown that

$$K_1 = e^{-i\delta_1 s\cos\theta}\left(2\pi^2(\underset{\sim}{q}.\underset{\sim}{\chi_0})D + 3\pi^2(\underset{\sim}{q}.\underset{\sim}{\chi_0})E\sin^2\theta + 3\pi^2 q_z \chi_{0z} E\,(2\cos^2\theta - \sin^2\theta)\right) \qquad 8.36$$

with $$D = \frac{1}{f}\left(2+(B-C)\rho^2\right) \qquad 8.37$$

and $$E = \frac{1}{f}(B+3C)\rho^2 \qquad 8.38$$

Now $$q_z\chi_{0z} = \frac{(\underset{\sim}{q}.\underset{\sim}{p})(\underset{\sim}{\chi_0}.\underset{\sim}{p})}{p^2} = -\frac{(\underset{\sim}{q}.\underset{\sim}{p})(\underset{\sim}{q}.\underset{\sim}{\chi_0})}{p^2}$$ since $\underset{\sim}{\chi_0}.\underset{\sim}{\nu} = 0$ $(p = |\underset{\sim}{p}|)$

$$\therefore\; K_1 = e^{-i\delta_1 s\cos\theta}\pi^2(\underset{\sim}{q}.\underset{\sim}{\chi_0})\left(2D + 3E\sin^2\theta - 3E\frac{(\underset{\sim}{q}.\underset{\sim}{p})}{p^2}(2\cos^2\theta - \sin^2\theta)\right) \qquad 8.39$$

and $J_1 = 4\pi^2 (q.\chi_0) \left[j_0(\delta_1 s)(D+E) + E\left(1+3\frac{q.p}{p^2}\right) j_2(\delta_1 s) \right], \delta_1 = |\delta_1|$ 8.40

For the production of mesons at 90° to the incident photon beam in the laboratory system and over the range of photon energy considered (200-260 MeV) the factor $\left(1+3\frac{q.p}{p^2}\right)$ is approximately zero, and therefore

$$I_1 = \frac{4\pi^2 (q.\chi_0)}{(12\mu_{dG}^2)^{\frac{3}{2}}} \left[\{D, 0^{y_1}, 0\} + \{E, 0^{y_1}, 0\} \right] \quad 8.41$$

where $\{G, \ell^y, n\} = \int_0^\infty x^2 dx\, e^{-\frac{1}{3}x^2} G\, j_\ell(y) \alpha^n$ 8.42

$$\alpha = \frac{qx}{6\mu_{dG}} = \frac{1}{3^{\frac{1}{2}} z^{-1}} \quad 8.43$$

and $y_1 = \frac{\delta_1 x}{2.3^{\frac{1}{2}} \mu_{dG}}$ 8.44

$j_\ell(x)$ is the spherical Bessel function of order ℓ .

(ii) <u>Evaluation of $M_2(t_1)$ for mesons produced at 90° in the laboratory system.</u>

From (8.34) $M_2(t_1) = 12(2\pi)^3 \delta(\underset{\sim}{k} - \underset{\sim}{q} - \underset{\sim}{D}) N_T I_2(t_1)$ 8.45a

where $I_2(t_1) = \int_0^\infty s^2 ds\, e^{-12\mu_{dT}^2 s^2} J_2(t_1)$ 8.45b

$$J_2(t_1) = \int_0^\pi \sin\theta\, d\theta\, K_2(t_1) \quad 8.45c$$

$$K_2(t_1) = \int_0^{2\pi} d\phi \, e^{-i\left(\frac{3}{8}\right)^{\frac{1}{2}}(\underset{\sim}{\nu}-\underset{\sim}{q})\underset{\sim}{z}'s} L_2(t_1) \qquad 8.45d$$

and

$$L_2(t_1) = \int_0^{2\pi} d\psi \, (\underset{\sim}{q}\cdot\underset{\sim}{t}_1) \, e^{-i\underset{\sim}{q}\cdot\underset{\sim}{s}_1} \qquad 8.45e$$

In this case the Oz axis is chosen to be parallel to the vector $\underset{\sim}{q}$ and the Oy axis parallel to $\underset{\sim}{\nu}$ (since $\underset{\sim}{q}\cdot\underset{\sim}{\nu} = 0$) so that $\gamma_y = 0$, since $\underset{\sim}{\gamma}\cdot\underset{\sim}{\nu} = 0$. Then

$$L_2(t_1) = -\pi\,(\underset{\sim}{q}\cdot\underset{\sim}{\gamma}_0)\, e^{-i\left(\frac{2}{3}\right)^{\frac{1}{2}} qs\cos\theta} J_0(\alpha_1 \sin\theta)(D' + 3E'\cos^2\theta) \qquad 8.46$$

where

$$D' = beD \qquad 8.47a$$

$$E' = beE \qquad 8.47b$$

and

$$\alpha_1 = \frac{qs}{3^{\frac{1}{2}}} \qquad 8.48$$

$M_2(t_1)$ may be obtained to a sufficiently close approximation by retaining only the first two terms in the expansion of $J_0(\alpha_1 \sin\theta)$. $J_n(x)$ is the cylindrical Bessel function of order n. Then

$$J_2(t_1) = -\pi(\underset{\sim}{q}\cdot\underset{\sim}{\gamma}_0)\int d\Omega \, e^{-i\underset{\sim}{\delta}\cdot\underset{\sim}{s}} \left[D'\left(1 - \frac{\alpha_1^2}{4}\sin^2\theta\right) + 3E'\cos^2\theta\left(1 - \frac{\alpha_1^2}{4}\sin^2\theta\right)\right] \qquad 8.49$$

where $\delta_x = 0$, $\delta_y = \left(\frac{3}{8}\right)^{\frac{1}{2}}\nu$, $\delta_z = \left(\frac{1}{24}\right)^{\frac{1}{2}} q$, $\delta = \left(\frac{3}{8}\right)^{\frac{1}{2}}\left(\nu^2 + \frac{1}{9}q^2\right)^{\frac{1}{2}}$ 8.50

and $d\Omega = \sin\theta \, d\theta \, d\phi$

The expansion of the plane wave $e^{-i\underset{\sim}{\delta}\cdot\underset{\sim}{s}}$ in terms of spherical harmonics and spherical Bessel functions is employed

$$e^{-i\delta s} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (-i)^l j_l(\delta s) Y_{l,m}(\delta) Y^*_{l,m}(\theta\phi) \qquad 8.51$$

and $e^{-i\delta s}$ can be represented to a close approximation in the energy region under consideration by retaining only these terms up to and including $l=2$. Since the meson is produced at 90° only terms with $m=0$ give a non-zero contribution to $M_2(t_1)$. Then

$$I_2(t_1) = -\frac{4\pi^2(q\chi_0)}{(12\mu^2_{d\alpha})^{\frac{3}{2}}}\Big[\{D', 0, ^y 0\} - \tfrac{1}{6}\{D', 0, ^y 2\} + \{E', 0, ^y 0\} - \tfrac{1}{10}\{E', 0, ^y 2\}$$

$$- P_2(\cos\delta)\Big(\tfrac{1}{6}\{D', 2, ^y 2\} + 2\{E', 2, ^y 0\} - \tfrac{1}{14}\{E', 2, ^y 2\}\Big)\Big] \qquad 8.52$$

where the significance of the bracket notation $\{G, l^y, n\}$ has been explained in equation (8, 42) and $y = \frac{\delta x}{2.3^{\frac{1}{2}}\mu_{d\alpha}}$

(iii) <u>Evaluation of $M_2(t_2)$ for mesons produced at 90° in the laboratory system.</u>

Using a notation analogous to that of the previous section it may readily be deduced that

$$L_2(t_2) = -2\pi b q s^2 (q\chi_0) e^{-i(\frac{2}{3})^{\frac{1}{2}} q s \cos\theta}\Big[\big\{\tfrac{1}{6}(B-C) + \tfrac{1}{2}(B+3C)\cos^2\theta\big\} J_0(\alpha_1 \sin\theta)$$

$$- \frac{2^{\frac{1}{2}}}{3} i (2B+C)\sin\theta\cos\theta\, J_1(\alpha_1\sin\theta) \qquad 8.53$$

$$- \tfrac{1}{6}(B-C)\sin^2\theta\, J_2(\alpha_1\sin\theta)\Big]$$

If the approximations of section (ii) are introduced, namely expanding the cylindrical Bessel functions up to the power α_1^2 and retaining only the $l = 0, 1,$ and 2 terms in the multipole expansion of $e^{-i\delta_{\underline{s}}}$, then

$$I_2(t_2) = -4\pi^2(q.\gamma_0)\Big[\tfrac{1}{3}\{D'',0',0\} - \tfrac{1}{18}\{D'',0',2\} + \tfrac{2}{3}\{E'',0',0\} - \tfrac{1}{15}\{E'',0',2\}$$
$$- \frac{2^{\frac{1}{2}}}{15}\cos\delta\big(2\{D'',1',1\} - \tfrac{1}{7}\{D'',1',3\} + 3\{E'',1',1\} - \tfrac{3}{14}\{E'',1',3\}$$
$$- P_2(\cos\delta)\big(\tfrac{1}{3}\{D'',2',0\} + \tfrac{2}{63}\{D'',2',2\} + \tfrac{4}{3}\{E'',2',0\} - \tfrac{1}{21}\{E'',2',2\}\big)\Big] \quad 8.54$$

where $$D'' = bgs^2 D \quad 8.55$$
$$E'' = bgs^2 E \quad 8.56$$

and $$\cos\delta = \frac{\delta_z}{\delta} \quad 8.57$$

Hence collecting together the results of equations (8.41), (8.52) and (8.54), and neglecting some of the terms included in these equations because their contribution to M is relatively small, the expression for M becomes

$$M = 4(2\pi)^3\delta(\chi - q - \underline{D})N_T\,\frac{4\pi^2(q.\gamma_0)}{(12\mu_G^2)^{\frac{3}{2}}}\times$$
$$\Big[\{D,0',0\} + \{E,0',0\} - 3\{D',0',0\} - 3\{E',0',0\} \quad 8.58$$
$$+ \tfrac{1}{2}\{D',0',2\} - \{D'',0',0\} + \tfrac{1}{6}\{D'',0',2\} - 2\{E'',0',0\}$$
$$+ \frac{2^{\frac{1}{2}}\cos\delta}{15}\big(2\{D'',1',1\} + 3\{E'',1',1\} + P_2(\cos\delta)\big(\{D'',2',0\} + 4\{E'',2',0\}\big)\big)\Big]$$

in which $\alpha = \frac{1}{3^{\frac{1}{2}}} z = \frac{qx}{6\mu_{dG}}$ 8.59

$$y_1 = \frac{\delta_1 x}{2.3^{\frac{1}{2}}\mu_{dG}}, \quad \delta_1 = \left(\frac{3}{8}\right)^{\frac{1}{2}}\left(v^2+q^2\right)^{\frac{1}{2}} \qquad 8.60$$

$$y = \frac{\delta x}{2 3^{\frac{1}{2}}\mu_{dG}}, \quad \delta = \left(\frac{3}{8}\right)^{\frac{1}{2}}\left(v^2+\frac{1}{9}q^2\right)^{\frac{1}{2}} \qquad 8.61$$

$$\cos\delta = \frac{q}{3\left(v^2+\frac{1}{9}q^2\right)^{\frac{1}{2}}} \qquad 8.62$$

and $\left\{G, \ell^{y}, n\right\} = \int_0^\infty x^2 dx\, e^{-\frac{1}{3}x^2} j_\ell(y)\,\alpha^n$ 8.63

From (8.36), (8.37), (8.26) and (8.27)

$$D = \frac{4}{2f-\rho^2} \qquad 8.64$$

$$E = \frac{2\rho^2}{(2f-\rho^2)(f-2\rho^2)} \qquad 8.65$$

where $f = 1-3(be)^2$ 8.66

and $\rho^2 = 2(be)(bgs^2)+(bgs^2)^2$ 8.67

(7.16) and (7.19) give

$$be = \frac{A}{3z^3}\, e^{i(\delta_{33}+z-\tan^{-1}z)}\left(1+z^2\right)^{\frac{1}{2}} \qquad 8.68$$

$$bgs^2 = \frac{A}{z^3}\, e^{i\left(\pi+\delta_{33}+z-\tan^{-1}\frac{z}{1-\frac{1}{3}z^2}\right)}\left(1+\frac{1}{3}z^2+\frac{1}{9}z^4\right)^{\frac{1}{2}} \qquad 8.69$$

The above expression for M becomes very much simpler at energies just above the threshold for production of neutral mesons. In fact it can be shown that M is independent of energy in the range $0 \leq q \leq 50$ MeV of meson momentum which corresponds to the range $137 \leq \nu \leq 147$ MeV of photon energy. It is not, however, expected that the multiple scattering correction as developed here will give accurate results at these low energies as the binding energy of the nucleons will play an important part. The result is included for the sake of completeness and because the behaviour of the functions, D, E, D', E', D'' and E'' near threshold is quite interesting. If M'_{MS} is defined by

$$M = 4(2\pi)^3 \delta(\underset{\sim}{\nu} - \underset{\sim}{q} - \underset{\sim}{D}) N_T \, 4\pi^2 (\underset{\sim}{q}.\underset{\sim}{\gamma}_0) \frac{1}{(12\mu^2_{\alpha G})^{\frac{3}{2}}} M'_{MS} \qquad 8.70$$

Then near threshold

$$M'_{MS} \simeq \{D + E + 3E', O^{y_1}, O\} \qquad 8.71$$

Since in this region $D'' \simeq -3D'$, $E'' \simeq -3E'$ and $y_1 \simeq y$

After some reduction M'_{MS} may be presented in the form

$$M'_{MS} = \int_{1.58}^{2} dx\, x^2 e^{-\frac{1}{3}x^2} j_0(y_1) \frac{2(2 - 2r + \frac{1}{3}r^2)}{(2 - 2r - r^2 + r^3)} + 2\int_2^{\infty} dx\, x^2 e^{-\frac{1}{3}x^2} j_0(y_1) \qquad 8.72$$

$$+ \frac{i\pi}{3}\left((1 - 2^{-\frac{1}{2}}) \left[x^2 e^{-\frac{1}{3}x^2} j_0(y_1) \right]_{x=1.33} + \left[x^2 e^{-\frac{1}{3}x^2} j_0(y_1) \right]_{x=1.5} \right)$$

where $r = \mathrm{Re}\, 3be$

$$= \frac{\alpha'}{x^3} \qquad 8.73$$

and $$\alpha' = \frac{4\sin\delta_{33}}{q_c^3} \times \frac{(12\mu_{\alpha G}^2)^{\frac{3}{2}}}{\left(1+\frac{m_\pi}{M}\right)^{\frac{1}{2}}} \qquad 8.74$$

α' is independent of meson energy in the region under examination.

The third term in equation (8.72) arises from the contribution of 'poles' of D, E and E' at $r = 2^{\frac{1}{2}}$ ($x = 1{\cdot}33$) and the fourth term from 'poles' of E and E' at $r = 1 (x = 1{\cdot}5)$ (equation (8.83)).

The matrix element M of equation (8.32) may be written in general as

$$M_{MS} = 4(2\pi)^3 \delta(\underline{k}-\underline{q}-\underline{D}) N_T \frac{4\pi^2 L_0}{(12\mu_{\alpha G}^2)^{\frac{3}{2}}} M'_{MS} \qquad 8.75$$

and hence the differential cross-section for the elastic photoproduction of neutral mesons at helium (including the multiple-scattering correction) $\frac{d\sigma_{MS}}{d\Omega_q}$ is given by

$$\frac{d\sigma_{MS}}{d\Omega_q} = \frac{1}{(2\pi)^2} \frac{q q_0}{1 + \frac{q_0}{4Mq}(q - v\cos\theta)} 16 \left(\frac{4\pi^2 |L_0|}{(12\mu_{\alpha G}^2)^{\frac{3}{2}}}\right)^2 |M'_{MS}|^2 \qquad 8.76$$

The matrix element $M_{I.A}$ of equation (8.17) may be written as

$$M_{I.A} = 4(2\pi)^3 \delta(\underset{\sim}{\nu} - \underset{\sim}{q} - \underset{\sim}{D}) N_T \frac{4\pi^2 L_0}{(12\mu^2_{dG})^{\frac{3}{2}}} M'_{I.A.} \qquad 8.77$$

where $M'_{I.A} = \int_0^\infty 2x^2 e^{-\frac{1}{3}x^2} dx\, j_0(y_1)$ 8.78

and the differential cross-section $\frac{d\sigma}{d\Omega_q}{}_{I.A.}$ derived on the basis of the impulse approximation is

$$\frac{d\sigma}{d\Omega_q}{}_{I.A.} = \frac{1}{(2\pi)^2} \frac{q q_0}{1 + \frac{q_0}{4Mq}(q - \nu\cos\theta)} 16\left(\frac{4\pi^2 |L_d|}{(12\mu^2_{dG})^{\frac{3}{2}}}\right)^2 |M'_{I.A}|^2 \qquad 8.79$$

The ratio of the two cross-sections of equations (8.76) and (8.79) is

$$\frac{d\sigma}{d\Omega_q}{}_{MS} \Big/ \frac{d\sigma}{d\Omega_q}{}_{IA} = \frac{|M'_{MS}|^2}{|M'_{IA}|^2} \qquad 8.80$$

VIII § 4. Calculation and discussion of results.

The expression (8.58) for M_{MS} is evaluated for incident photon energies of 220, 240 and 260 MeV, which correspond to meson phase shifts (δ_{33}) of about 13°, 20° and 27° respectively, while the expression (8.72), which is valid near the threshold of the reaction, is evaluated at an incident photon energy of 140 MeV, where δ_{33} is very small. D, E, D', E', D'' and E'' are calculated as functions of $z = \frac{qx}{2 \cdot 3^{\frac{1}{2}} \mu_{\alpha G}}$ over the range $0 \leq x \leq 4$ of x at intervals of $\cdot 2$ except in the range $1 \leq x \leq 2$ where the interval is chosen to be $\cdot 1$ because of the rapid variation of the functions in this region, and the required integrals are computed numerically.

The behaviour of the functions D, E, D', E', D'' and E'' is practically independent of the angle at which the meson is emitted since q_0 and hence q and δ_{33} depend on the angle Θ only through the relatively small recoil factor $\frac{p^2}{8M}$ (equation (3.36)). The following general trends in the behaviour of these functions can be deduced from the calculations carried out at $\Theta = 90^{\circ}$. It is found that the contribution to M_{MS} from the three nucleons not involved in the electro-magnetic interaction is small in comparison with the contribution from the nucleon at which the photon is incident, largely because of the cancellation of terms involving D', E', D'', E''. This cancellation process is brought out quite clearly in the expression (8.71) for the matrix element near threshold, and is also important at higher meson energies. It also turns out that the contribution of E to M_{MS} is small in comparison with that of D, partly because the positive and negative contributions from E at different separations s of the nucleons tend to cancel out and partly because $|E|$ is smaller than $|D|$ in any case. The form of the function D, then, is the dominant factor in determining the magnitude of the multiple-scattering cross-section.

It is interesting to examine the change in behaviour of the function D as the energy of the photon is increased from threshold to 260 MeV. At threshold energies

$$D = \frac{4}{2 - r^2} \tag{8.81}$$

except near the point $r = 2^{\frac{1}{2}}$ (which corresponds to $x = 1.33$).

$$r = \frac{A}{q^3} \frac{2^3 3^{\frac{3}{2}} \mu_{\alpha a}^3}{x^3} \tag{8.82}$$

and is practically independent of energy since $\frac{\sin\delta_{33}}{q_c^3}$ is a constant in this region.

When $r = 2^{\frac{1}{2}}(1+\varepsilon_1)$, $\varepsilon_1 \ll 1$

$$\mathrm{Re}\,D = -\frac{\varepsilon_1}{\varepsilon_1^2 + \frac{\delta^2}{2}} \tag{8.83a}$$

and

$$\mathrm{Im}\,D = \frac{\delta}{2^{\frac{1}{2}}(\varepsilon_1^2 + \frac{\delta^2}{2})} \tag{8.83b}$$

where

$$\delta = r\left(\delta_{33} + \frac{1}{3}z^3\right) \ll 1$$

The general form of the real (ReD) and imaginary (ImD) parts of D near threshold as deduced from equations (8.81) and (8.83) is sketched in figure 8.1.

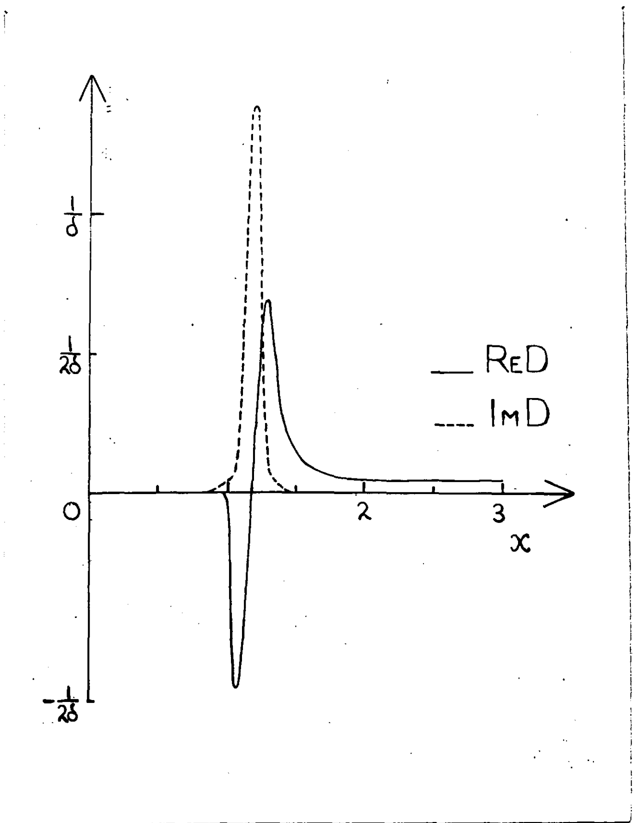


Fig. 8.1.

As the photon energy is increased the maxima and minima of the functions ReD and ImD are progressively reduced in magnitude and at the same time the bases of the peaks spread out over a larger range of x . The abscissae of the maximum and minimum points do not alter much as the energy changes, since in the photon energy region considered the behaviour of the functions $b e$ and $b q s^2$ at these nucleon separations is determined, to a great extent, by the factor $\frac{A}{Z^3}$ which does not vary a great deal in this energy region. These trends in the behaviour of D are illustrated in figure 8.2, in which the real and imaginary parts of D at ν = 220 and 260 MeV are plotted.

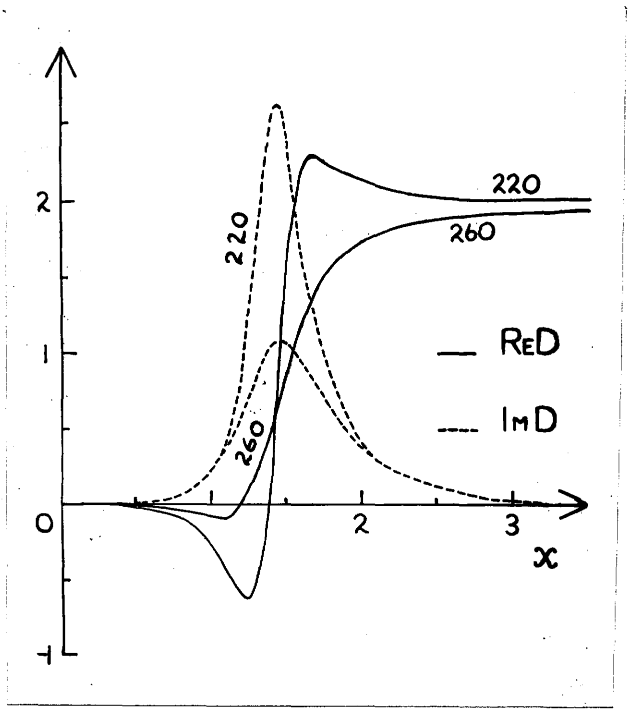


Fig. 8.2.

The real part of $D(\mathrm{Re}\,D)$ is negligible at all the energies examined in the range $0 \leq x \leq 1.3$ (or $0 \leq s \leq 1.3 \times 10^{-13}$ cm.). At a nucleon separation of 2×10^{-13} cm., $\mathrm{Re}\,D$ is very nearly equal to 2, the remaining functions being zero, and this corresponds to the impulse approximation solution to the problem of the scattering of a meson at four nucleons. The multiple scattering of the meson through the $(\frac{3}{2},\frac{3}{2})$ state then appears to suppress the production of the meson up to a separation of the nucleons of 1.3×10^{-13} cm. and for larger separations the photoproduction process is fairly well described by the impulse approximation. The reduction in the cross-section due to this suppression at low energies is partly compensated for by the contribution of the imaginary part of D to the matrix element.

The values for the ratio $\frac{d\sigma_{MS}}{d\Omega_q} / \frac{d\sigma_{I.A}}{d\Omega_q}$ calculated at the energies ν = 140, 220, 240 and 260 MeV are tabulated below.

ν (Mev)	140	220	240	260
$\frac{d\sigma_{MS}}{d\Omega_q} / \frac{d\sigma_{I.A}}{d\Omega_q}$	.63	.53	.38	.16

Fig. 8.3.

The ratio is nearly constant for energies near threshold and then drops rapidly with increasing photon energy.

Using the above values for the ratios of the cross-sections it is possible to compute the multiple-scattering cross-sections $\frac{d\sigma_{MS}}{d\Omega_q}(j)$ corresponding to the impulse approximation cross-sections $\frac{d\sigma_{IA}}{d\Omega_q}(j)$ of equations (3.45). $\frac{d\sigma_{MS}}{d\Omega_q}(3)$, $\frac{d\sigma_{IA}}{d\Omega_q}(3)$ and the experimental results of Osborne and de Saussure are compared in Fig. 8.4.

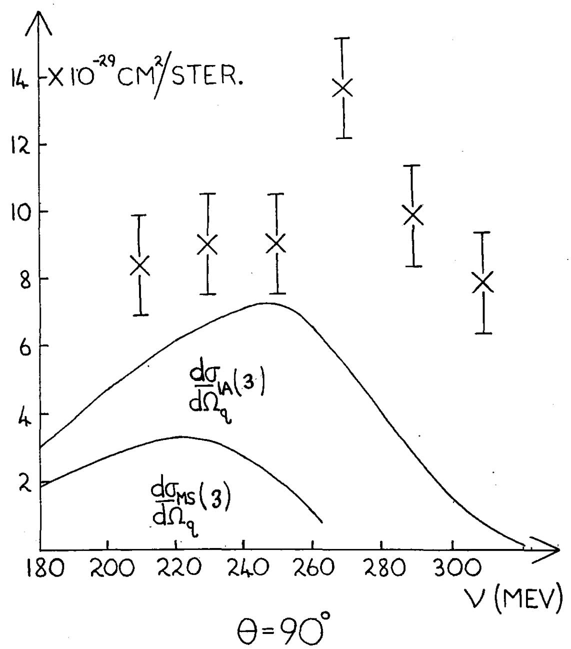


Fig. 8.4.

As can be seen from the graph the theoretically derived cross-sections, particularly the multiple-scattering cross-section, are considerably lower at all photon energies than the experimentally observed points. The experimental results of Osborne and de Saussure seem to be rather on the high side, since even if the dependence of the theoretical cross-section on the form factor and the multiple-scattering correction factor were excluded, the resulting cross-section would not be more than about 10^{-28} cm^2/steradian at 240 MeV and 90° meson angle, which is barely large enough to fit the experimental results.

It is of some interest to compare, for various nucleon separations, the qualitative behaviour of the multiple-scattering correction factor for the photo-production of a meson at four free nucleons with the corresponding correction factor for the multiple-scattering of a meson at two free nucleons which has been discussed in detail by Brueckner (4). If it is assumed that only the function $D(s)$ contributes to the multiple-scattering matrix element for four nucleons and that $D(s)$ is independent of the angle at which the meson is produced, the ratio of the total multiple-scattering cross-section $\sigma_{MS}(s)$ to the total impulse approximation cross-section $\sigma_{IA}(s)$ at the separation s of the nucleons, is given by

$$\frac{\sigma_{MS}(s)}{\sigma_{IA}(s)} = \frac{|D(s)|^2}{4} \tag{8.84}$$

The behaviour of this ratio at ν = 220 MeV $(\delta_{33}=13°)$ is compared in figure (8.5), with the ratio of the total multiple-scattering cross-section for the scattering of a meson at two free nucleons $\sigma_T(s)$ to the corresponding impulse approximation cross-section $\sigma_F(s)$ as computed by Brueckner (4) at δ_{33} = 30°.

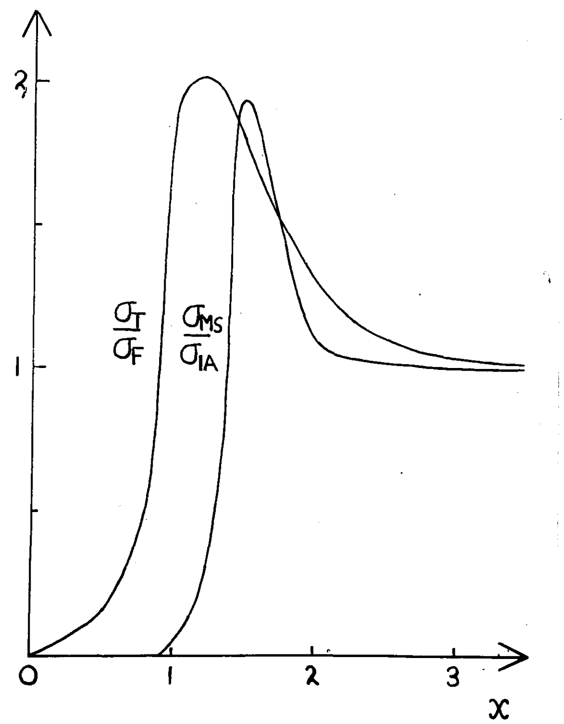


Fig. 8.5.

The similarity in behaviour of the two ratios is quite marked. The ratio $\frac{\sigma_T}{\sigma_F}$ undergoes the same changes in form as the ratio $\frac{\sigma_{MS}}{\sigma_{IA}}$ when the energy of the scattered meson is increased, although the latter changes the more swiftly of the two (for instance the form of the ratios is comparable when $\delta_{33} = 13^0$ in the case of $\frac{\sigma_{MS}}{\sigma_{IA}}$, and $\delta_{33} = 30^0$ in the case of $\frac{\sigma_T}{\sigma_F}$). The similarity between the two ratios has probably been emphasised to some extent by the approximate form of the multiple-scattering equations (7.20) (in which the possibility of the meson being scattered between the nucleons 2 and 3, 2 and 4, and 3 and 4 has been neglected) and by the tetrahedron model, since the effect of these two approximations has been to treat the four nucleons as three deuteron like particles which have the nucleon 1 in common.

It is obvious that the results for the multiple-scattering cross-section which have been deduced in this Chapter can have no more than a qualitative significance. The three approximations of neglecting the charge exchange processes, assuming a modified form of the multiple scattering equations

(7.20) and adopting a very much simplified model for the α- particle have been introduced in order to derive a result which would give a qualitative indication of the correction to the impulse approximation due to multiple-scattering effects without involving an excessive amount of numerical work, and with adequate computing facilities the complete solution of the four nucleon problem on the basis of the sets of equations (5.48) and (5.50) would probably be quite feasible.

The approximations which have been made in the derivation of the multiple-scattering equations have already been discussed in some detail and are comparable to those employed by Brueckner, Watson and Chappelear in their examination of allied problems. The principal merit of the multiple-scattering equations lies in the fact that the problem which was originally described in terms of integral equations is reduced to the solution of a series of linear simultaneous equations which, in theory, may readily be found. It seems quite possible that the method by which the equations were obtained could fairly easily be developed to include s- as well as p-wave meson scattering and $I = \frac{1}{2}$ in addition to $I = \frac{3}{2}$ isotopic spin states. The problem of making some allowance for 'off the energy shell' scattering would be considerably more difficult, although even this might be possible for simple forms of the 'off the energy shell' scattering matrix element. However, these approximations and others which were employed in obtaining the multiple-scattering equations have all been justified to a certain extent and it is not unreasonable, therefore, to hope that these equations include the major part of the correction to the impulse approximation solution for the photoproduction or scattering of mesons at nuclei in the energy range 200 to 400 MeV of the incident particle.

APPENDIX A.

In the examination of the single nucleon photoproduction matrix element which was carried out in Chapter III it was found that the various energies and momenta which appeared in the matrix element derived by Chew (11) had to be transformed from the centre of momentum system of the photon and nucleon to the laboratory system, in which the nucleon is at rest. The relationships involved in this transformation are described below.

Let $\underset{\sim}{\nu}$, ν be the momentum and energy of the photon and $\underset{\sim}{q}$, q_o be the momentum and energy of the meson where $\underset{\sim}{q}\cdot\underset{\sim}{\nu} = q\nu\cos\theta$, all these quantities being measured in the laboratory system. The corresponding quantities in the centre of momentum (c.m.) system are differentiated by the subscript c. M and m_π are the rest masses of the nucleon and meson respectively.

The velocity of the c.m. system relative to the laboratory system is

$$\underset{\sim}{V} = \frac{\underset{\sim}{\nu}}{M+\nu} \qquad \text{A.1}$$

and employing this velocity in the Lorentz transformation between the two systems the following relationships between quantities in the c.m. and laboratory systems may be deduced

$$V_c = \frac{M\nu}{(M^2+2\nu M)^{\frac{1}{2}}} \qquad \text{A.2}$$

$$q_{oc} = \frac{(q_o(M+\nu) - q\nu\cos\theta)}{(M^2+2\nu M)^{\frac{1}{2}}} \qquad \text{A.3}$$

$$q_c = \frac{(q^2(M+\nu)^2+m_K^2\nu^2-2q_oq\nu(M+\nu)\cos\theta+q^2\nu^2\cos^2\theta)^{\frac{1}{2}}}{(M^2+2\nu M)^{\frac{1}{2}}} \qquad \text{A.4}$$

$$\cos\theta_c = \frac{(-q_o\nu + q(M+\nu)\cos\theta)}{(q^2(M+\nu)^2+m_K^2\nu^2-2qq_o\nu(M+\nu)\cos\theta+q^2\nu^2\cos^2\theta)^{\frac{1}{2}}} \qquad \text{A.5}$$

$$\sin\theta_c = \frac{q}{q_c}\sin\theta \qquad \text{A.6}$$

$$\frac{V_c}{q_{oc}q_c^4}\sin^2\theta_c = \frac{q^2V_c}{q_{oc}q_c^6}\sin^2\theta \qquad \text{A.7}$$

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