TWO PROBLEMS ON DIVISOR FUNCTIONS

A Thesis presented on application for
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by

Eira Joan Scourfield

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PREFACE.

The thesis is divided into five chapters. The first chapter serves as an introduction to the two problems to be solved. The second and third chapters contain proofs of the two results needed in order to solve the first problem, and a solution of the first problem and some corollaries to it are deduced in the fourth chapter. The fifth chapter contains a solution to the second problem. The five Theorems to be proved are stated in the first chapter; all other results to be derived or quoted are called Lemmas. The Theorems are referred to throughout by the numbers assigned to them in the first chapter, but the numbering of lemmas and equations begins afresh in each chapter.

The work in this thesis is claimed as original except in the places where reference to another author's work is made. In the text of the thesis a reference is denoted by a number in square brackets, and full details of the paper or book referred to in this way are given at the end of the thesis.

The problems investigated in this thesis arose from suggestions made to me by Professor R.A. Rankin, and I wish to express my thanks to him for his guidance and all his valuable advice.

CONTENTS.

CHAPTER 1.

1. Introduction.

During the development of the theory of numbers, which is basically the study of properties of integers, much attention has been given to the divisibility properties of integers.
If n is a positive integer, the divisor functions $d(n)$ and $\sigma_\nu(n)$ are defined by

$$d(n) = \sum_{d|n} 1, \qquad \sigma_\nu(n) = \sum_{d|n} d^\nu,$$

where the sums are over all positive (integral) divisors of n; thus $d(n)$ is the number of positive divisors of n and $\sigma_\nu(n)$ is the sum of the νth powers of the positive divisors of n.
One way of examining the divisibility properties of n is to investigate the divisor functions $d(n)$ and $\sigma_\nu(n)$. This approach has stimulated interest in the properties of the divisor functions themselves. Many results about these functions have been obtained, so that some of their properties are well known, but there remain problems to which an answer has not yet been found. The aim of the following chapters is to provide solutions to two of the problems which may be raised.

2. A problem on the divisibility of $\sigma_\nu(n)$.

The first problem that we shall consider is concerned with the function $\sigma_\nu(n)$ when ν itself is a positive integer.

In this case $\sigma_\nu(n)$ is a positive integer for all positive integers n, and hence an investigation of the divisibility properties of $\sigma_\nu(n)$ is feasible. In particular one may ask whether or not $\sigma_\nu(n)$ is, or is not, divisible by some given positive integer k for almost all n. We shall now describe two related problems which arise in this way and which lead us on to Theorem 1, one of the two main results of this thesis.

Let $N(\nu,k;x)$ denote the number of positive integers $n \leq x$ for which $\sigma_\nu(n)$ is not divisible by k. Then one of the problems arising from the above discussion is that of estimating $N(\nu,k;x)$ when ν and k are fixed positive integers independent of x. Although we shall not be concerned primarily with this problem, it provides the background to Theorem 1 and from this Theorem we shall be able to deduce estimates for $N(\nu,k;x)$ in some cases. Hence we begin by describing the known results for $N(\nu,k;x)$.

In 1935 G.N. Watson published a paper [1] in which he showed that, when ν is odd,

$$N(\nu,k;x) = O\left(x(\log x)^{-1/\phi(k)}\right) \tag{1}$$

as $x \to \infty$, where $\phi(k)$ is Euler's function. It follows from this result that, when ν is odd, $\sigma_\nu(n)$ is almost always divisible by k. Two further questions are immediately suggested by this result: What is the corresponding result when ν is even? Is it possible to improve on (1), and, in particular, can one obtain an asymptotic equation which is satisfied by $N(\nu,k;x)$?

These questions were considered by R.A. Rankin in a paper [2], published in 1961, in which he proved that (1) can be replaced by an asymptotic equation when k is a prime, and consequently he deduced that the function on the right of (1) can be replaced by a smaller function of x in other cases too. Let q be a prime, and write

$$h = \frac{q-1}{(\nu,q-1)},$$

where $(\nu,q-1)$ denotes the highest common factor of ν and q-1. Then, more precisely, Rankin proved that, as $x\to\infty$,

$$N(\nu,q;x)\sim\begin{cases} A_1^{(0)}x & \text{if } q \text{ and } h \text{ are odd} \\ A_2^{(0)}x(\log x)^{-1/h} & \text{if } q \text{ is odd and } h \text{ is even} \\ A_3^{(0)}x^{\frac{1}{2}} & \text{if } q=2, \end{cases} \tag{2}$$

where $A_1^{(0)}$, $A_2^{(0)}$ are positive constants depending on ν and q, and where $A_3^{(0)}=3/2$. When q is odd, the proof of this result in the case when h is odd is more straightforward than in the case when h is even. If $q\neq 2$, the case when ν is odd is always included in the second part of Rankin's result; for this case Watson's result, given by (1), provides an estimate, less precise than (2), for $N(\nu,q;x)$. However when ν is even and $q\neq 2$, either the first or the second part of Rankin's result may apply.

As a consequence of Theorem 1, given below, we shall be able to continue this line of investigation a stage further by obtaining in chapter 4 an asymptotic equation, to replace (1), in the case when $k=q^m$, where q is a prime and m is a positive integer, and in some other cases too.

We turn now to the result to be proved in the next three chapters. Let q be a prime and m a positive integer, and assume that both are fixed and independent of x. Define γ by $q^{\gamma} \| \nu$ (where the notation $\|$ means that $q^{\gamma} | \nu$ but $q^{\gamma+1} \nmid \nu$, so that q^{γ} is the highest power of q dividing ν), and write

$$m' = [m/(\gamma+1)]$$

(where the square brackets indicate that the integer part is taken). Denote by $D_m(\nu,q;x)$ the number of positive integers $n \leq x$ for which $q^m \| \sigma_\nu(n)$. Then we have

<u>Theorem 1.</u>

(i) <u>If q and h are both odd, then, as $x \to \infty$,</u>

$$D_m(\nu,q;x) \sim A_1^{(m)} x.$$

(ii) <u>If q is odd and h is even, then, as $x \to \infty$</u>,

$$D_m(\nu,q;x) \sim A_2^{(m)} x(\log\log x)^{m'}(\log x)^{-1/h}.$$

(iii) <u>As $x \to \infty$</u>,

$$D_m(\nu,2;x) \sim A_3^{(m)} x(\log\log x)^{m-1}(\log x)^{-1}.$$

$A_1^{(m)}$, $A_2^{(m)}$, $A_3^{(m)}$ are positive constants depending only on ν, q and m.

Before briefly outlining the main stages in the proof of this theorem, we shall show how $N(\nu,q^m;x)$ may be expressed in terms of $D_r(\nu,q;x)$ (where $r<m$) so that the connection between Theorem 1 and the earlier discussion will become apparent. First of all we observe that $N(\nu,q;x)$ may be regarded as the number of positive integers $n \leq x$ for which $q^0 \| \sigma_\nu(n)$, so that we can write

$$N(\nu,q;x) = D_0(\nu,q;x).$$

If $m \geqslant 2$, it follows from the definition that $N(\nu, q^m; x)$ is the number of positive integers $n \leqslant x$ for which one of $q^r \| \sigma_\nu(n)$, $r = 0, 1, 2, \ldots, m-1$, holds, and hence

$$N(\nu, q^m; x) = \sum_{r=0}^{m-1} D_r(\nu, q; x). \tag{3}$$

We now discuss briefly the proof of Theorem 1.

Define

$$a_m(n) = \begin{cases} 1 & \text{if } q^m \| \sigma_\nu(n) \\ 0 & \text{otherwise.} \end{cases}$$

Then clearly, if we take x to be an integer (which we can do without loss of generality),

$$D_m(\nu, q; x) = \sum_{n=1}^{x} a_m(n), \tag{4}$$

and hence in order to prove Theorem 1 we must obtain an estimate for the sum on the right of (4). We do this in two stages. First of all we express the generating function

$$f_m(s) = \sum_{n=1}^{\infty} a_m(n) n^{-s} \quad ,$$

$s = \sigma + it$ being a complex variable, in terms of the Riemann zeta-function and Dirichlet L-functions, and then we obtain the required result from this. (This technique is a standard one for this type of problem in the theory of numbers; however the difficulties to be overcome at each stage will vary according to the problem being discussed.) This is the approach used by Rankin to prove (2), and it is also the method used by Watson to prove (1) except that Watson replaced the sum representing $N(\nu, k; x)$, which is a sum of the type appearing on the right of (4), by one which had fewer zero terms in it.

The generating function $f_m(s)$ is given by the following result, which is proved in chapter 2.

Theorem 2.

(i) If q and h are both odd,

$$f_m(s) = \zeta(s)g(s),$$

where $\zeta(s)$ is the Riemann zeta-function and $g(s)$ is holomorphic for $\sigma>\frac{1}{2}$ and bounded for $\sigma\geq\frac{1}{2}+\delta$ for any $\delta>0$.

(ii) If q is odd and h is even,

$$f_m(s) = \{\zeta(s)\}^{1-\frac{1}{h}}\sum_{u=0}^{m'}\{\log\zeta(s)\}^{u} H_u(s),$$

where each $H_u(s)$ $(0\leq u\leq m')$ is a sum of products of Dirichlet L-functions associated with non-principal characters, the logarithms of such functions, and a function satisfying the conditions on $g(s)$ in (i).

(iii) If $q=2$,

$$f_m(s) = \sum_{u=0}^{m}\{\log\zeta(s)\}^{u} H_u(s),$$

where each $H_u(s)$ $(0\leq u\leq m)$ satisfies the conditions given in (ii).

In order to prove Theorem 1 (i) we need only use Theorem 2 (i) and the Wiener-Ikehara Theorem (which is stated in Lemma 9 of chapter 2). However in order to prove the rest of Theorem 1, we shall need to prove another result. Let

$$h(s) = \{\zeta(s)\}^{1-\beta}\{\log\zeta(s)\}^{u} H(s),$$

where $0<\beta<1$, u is a non-negative integer and $H(s)$ is a product of powers of Dirichlet L-functions associated with non-principal characters, non-negative powers of logarithms of such functions, and a function holomorphic for $\sigma>\frac{1}{2}$ and bounded for $\sigma\geq\frac{1}{2}+\delta$ for any $\delta>0$. Furthermore suppose that $h(s)$ can be expressed in the form

$$h(s) = \sum_{n=1}^{\infty} b(n)n^{-s},$$

where $b(n)\geq 0$. Then, in chapter 3, we shall prove

Theorem 3. (i) If $0<\beta<1$ and $u\geq 1$, then

$$\sum_{n=1}^{x} b(n) = \frac{H(1)}{\Gamma(1-\beta)} \frac{x(\log\log x)^{u}}{(\log x)^{\beta}} + O\left(\frac{x(\log\log x)^{u-\frac{1}{2}}}{(\log x)^{\beta}}\right),$$

where $\Gamma(1-\beta)$ is the Gamma-function.

(ii) If $0<\beta<1$ and $u=0$, then

$$\sum_{n=1}^{x} b(n) = \frac{H(1)}{\Gamma(1-\beta)} \frac{x}{(\log x)^{\beta}} + O\left(\frac{x}{(\log x)^{\frac{1}{2}(1+\beta)}}\right).$$

(iii) If $\beta=1$ and $u\geq 2$, then

$$\sum_{n=1}^{x} b(n) = uH(1)\frac{x(\log\log x)^{u-1}}{\log x} + O\left(\frac{x(\log\log x)^{u-\frac{3}{2}}}{\log x}\right).$$

(iv) If $\beta=1$ and $u=1$, then

$$\sum_{n=1}^{x} b(n) = H(1)\cdot\frac{x}{\log x} + O\left(\frac{x(\log\log x)^{\frac{1}{2}}}{(\log x)^{3/2}}\right).$$

(v) If $\beta=1$ and $u=0$, then

$$\sum_{n=1}^{x} b(n) = O\left(\frac{x}{(\log x)^{3/2}}\right).$$

In chapter 4 we shall obtain the result of Theorem 1 (ii) and (iii) from Theorems 2 and 3, and we shall deduce from Theorem 1 some results for $N(\nu,k;x)$.

3. A problem on a generalisation of $d(n)$.

We turn now to the second problem to be considered in this thesis, and it will be discussed fully in chapter 5. (The contents of chapter 5 form the substance of a paper [3] already published.) We shall be concerned with a generalisation of the divisor function $d(m)$; we shall replace m by a polynomial $f(n)$ (or, in other words, we shall restrict m to those positive integers which can be expressed in the form $m=f(n)$, where n is also a positive integer), and we shall count only the divisors belonging to a certain congruence class.

More precisely, let $f(n)=an^2+bn+c$ be an irreducible quadratic polynomial with integer coefficients, and let D denote the discriminant b^2-4ac of $f(n)$. We shall assume that $(D,k)=1$, and that, for all positive integers n, $f(n)$ is positive and coprime with k, where k is a fixed integer greater than 1. We denote by $d(m;h)$ the number of positive divisors d of a positive integer m which satisfy $d\equiv h(\mathrm{mod}\ k)$. Then we shall prove

<u>Theorem 4.</u> <u>If $(h,k)=1$, then</u>

$$\sum_{n=1}^{x} d(f(n);h) = A_1 x\log x + O(x \log\log x),$$

<u>where x is a large positive integer, and A_1 is a positive constant depending on k, h and the coefficients of f.</u>

The method used to prove this theorem may also be used to show that

$$\sum_{n=1}^{x} d(f(n)) = A_2\, x \log x + O(x \log\log x), \tag{5}$$

where A_2 is a positive constant depending on the coefficients of f.

This result is mentioned in a paper [4] by Erdös as an unpublished result of Bellman and Shapiro.

The problem of proving a result analogous to (5), or Theorem 4, for irreducible polynomials $g(n)$ of degree greater than 2 appears to be very difficult. An important step in this direction is the paper [4] due to Erdös in which he proves that

$$B_1 x \log x < \sum_{n=1}^{x} d(g(n)) < B_2 x \log x,$$

where B_1 and B_2 are positive constants depending only on the coefficients and degree of g, and $x \geq 2$.

For certain polynomials $f(n)$ we may deduce from Theorem 4 a result, analogous to (5), for the function $r(f(n))$, where $r(m)$ denotes the number of representations of a positive integer m as the sum of two integer squares. We shall assume for this result that $f(n)$ is odd for all positive integers n and that $f(n) \equiv 1 \pmod 4$ for at least some positive integers n; furthermore we shall assume that $D = -\mu^2$, where μ is a positive integer.

Then we have

Theorem 5.

$$\sum_{n=1}^{x} r(f(n)) = A_3 x \log x + O(x \log\log x),$$

where A_3 is a positive constant depending on the coefficients of f.

If $f(n) \equiv 3 \pmod 4$, then it is well known that $r(f(n))=0$; hence if $f(n) \equiv 3 \pmod 4$ for all positive integers n, then

$$\sum_{n=1}^{x} r(f(n)) = 0.$$

In § 8 of chapter 5 we shall illustrate, by means of the example $f(n)=n^2+1$, how the case when $f(n)$ is sometimes even may be dealt with.

CHAPTER 2.

1. Introduction.

The object of this chapter is to prove Theorem 2, which was stated in chapter 1, and hence we shall be considering the function

$$f_m(s) = \sum_{n=1}^{\infty} a_m(n)n^{-s}.$$

Our first step will be to find the order of p^{ν}, where p is a prime, modulo powers of q, and using this we shall be able to find the positive integers α for which $a_m(p^{\alpha})=1$ (see Lemma 4).
Since, as we shall see, $a_m(n)$ is not multiplicative, the next stage is to obtain an expression, given in Lemma 7, for $a_m(n)$ in terms of $a_r(p^{\alpha})$ $(0\leq r\leq m)$. We shall then be in a position to deduce an expression for $f_m(s)$, given in Lemma 8, and finally we show, in §5 to §8, that this is equivalent to an expression of the required type.

We have already mentioned that Theorem 1(i) follows immediately from Theorem 2(i) and the Wiener-Ikehara Theorem, and we shall give the details of this deduction in §5.

2. Preliminary results.

If q is odd, let g be a primitive root (mod q^r) for all positive integers r. (This is possible, for if g_1 is a primitive root (mod q) then either g_1 or g_1+q is a primitive root (mod q^r) for all $r\geq 1$.)

Then every prime p, $p \neq q$, satisfies a congruence relation of the form

$$p \equiv g^{c_p^{(r)}} \pmod{q^r} \quad \text{where } 1 \leq c_p^{(r)} \leq \phi(q^r) = q^{r-1}(q-1), \tag{1}$$

$\phi(q^r)$ being Euler's function. Similarly if $q=2$ and $r \geq 3$, every odd prime p satisfies a congruence relation of the form

$$p \equiv \pm 5^{c_p^{(r)}} \pmod{2^r} \quad \text{where } 1 \leq c_p^{(r)} \leq 2^{r-2}, \tag{2}$$

the + or − sign being taken according as $p \equiv 1$ or $3 \pmod 4$.

Throughout this section we assume that $p \neq q$. Write

$$\kappa_r = \begin{cases} r-1 & \text{if } q \neq 2 \\ r-2 & \text{if } q=2; \end{cases} \tag{3}$$

then in either case $c_p^{(r)}$ satisfies $1 \leq c_p^{(r)} \leq q^{\kappa_r}(q-1)$. Unless otherwise stated, we shall assume that when $q=2$ $\kappa_r \geq 1$, so that $r \geq 3$. We define $\varepsilon_p^{(r)}$ to be the highest power of q dividing $c_p^{(r)}$, so that

$$q^{\varepsilon_p^{(r)}} \parallel c_p^{(r)} \quad ;$$

clearly $0 \leq \varepsilon_p^{(r)} \leq \kappa_r$.

<u>Lemma 1.</u> <u>If $r_2 \geq r_1$</u>,

$$c_p^{(r_2)} \equiv c_p^{(r_1)} \pmod{q^{\kappa_{r_1}}(q-1)} \tag{4}$$

<u>and</u>

$$\varepsilon_p^{(r_1)} = \min(\varepsilon_p^{(r_2)}, \kappa_{r_1}). \tag{5}$$

<u>Also</u>

$$\varepsilon_p^{(r+1)} = \varepsilon_p^{(r)} \text{ or } \varepsilon_p^{(r)}+1. \tag{6}$$

<u>If $q \neq 2$, $r \geq 2$ and $q \mid c_p^{(r)}$</u>, <u>then</u>

$$c_p^{(2)} = qc_p \quad , \tag{7}$$

<u>where $c_p = c_p^{(1)}$</u>. <u>If $r>2$ and $q^{r-1} \parallel c_p^{(r)}$</u>, <u>then</u>

$$c_p^{(r)} = q^{r-1}c_p \quad . \tag{8}$$

<u>Proof</u>. All these results are consequences of the above definitions. If q is odd, (4) follows since

$$p \equiv g^{c_\rho^{(r_2)}} \equiv g^{c_\rho^{(r_1)}} \pmod{q^{r_1}}.$$

and g is a primitive root; the case $q=2$ is similar. The second result follows from the first since $\varepsilon_\rho^{(r_i)} < \kappa_{r_i}$, $i=1,2$, and similarly the third result follows from the second.

It follows from (5) on putting $r_1=2$ and $r_2=r$ that, if $q\neq 2$ and $q\,|\,c_\rho^{(r)}$, then $q\,|\,c_\rho^{(2)}$. On putting $r_1=1$ and $r_2=2$ in (4), we obtain

$$c_\rho^{(2)} = c_\rho + u(q-1),$$

where u is an integer satisfying $0\leq u<q$. Hence if $q\nmid c_\rho^{(2)}$, then $q\,|\,(c_\rho - u+uq)$, so that $q\,|\,(c_\rho - u)$. Since $|c_\rho - u|<q$, it follows that $u=c_\rho$. This proves (7).

If $\varepsilon_\rho^{(r)}=r-1$, it follows from (5) that $\varepsilon_\rho^{(r-1)}=r-2$, $\varepsilon_\rho^{(r-2)}=r-3,\ldots,$ $\varepsilon_\rho^{(2)}=1$. If $3\leq i\leq r$, we have from (4) that

$$c_\rho^{(i)} = c_\rho^{(i-1)} + u_i\, q^{i-2}(q-1) \quad \text{where} \quad 0\leq u_i \leq q-1.$$

If $c_\rho^{(i-1)}=q^{i-2}c_\rho$ and $q^{i-1}\,\|\,c_\rho^{(i)}$, it follows that $u_i=c_\rho$ and $c_\rho^{(i)}=q^{i-1}c_\rho$. On putting $i=3,4,\ldots,r$ and using (7) we obtain (8).

We recall that $q^\gamma\,\|\,\nu$, and that $h=(q-1)/(\nu,q-1)$; clearly $h=1$ if $q=2$. We define t by

$$q^t\,\|\,(p^\nu-1).$$

Furthermore if $q=2$ and $p\equiv 3 \pmod 4$, then we define t' by

$$2^{t'}\,\|\,((-p)^\nu-1);$$

thus $t'=t$ when ν is even, but $t'\geq 2$ and $t=1$ when ν is odd. Clearly, when $q=2$, $t\geq 1$. We assume now that $r\geq t$, and that $r\geq 3$ when $q=2$; then the next lemma gives us an expression for the order of $p^\nu \pmod{q^r}$.

We adopt the convention that the order of $p^\nu \pmod{q^t}$ is 1; if $r<t$, then the order of $p^\nu \pmod{q^t}$ is not defined. If $r>t$, then clearly the order of $p^\nu \pmod{q^r}$ must exceed 1.

<u>Lemma 2.</u> <u>The order of $p^\nu \pmod{q^r}$ is</u>

$$\lambda_\rho^{(r)} h/(h,c_\rho),$$

where

$$\lambda_\rho^{(r)} = \begin{cases} q^{\kappa_r-\gamma-\varepsilon_\rho^{(r)}} & \text{if} \quad \kappa_r-\gamma-\varepsilon_\rho^{(r)} \geqslant 0 \\ 1 & \text{if} \quad \kappa_r-\gamma-\varepsilon_\rho^{(r)} \leqslant 0 \end{cases}$$

<u>except when $q=2$, ν is odd, $p\equiv 3 \pmod 4$ and $p^\nu \equiv -1 \pmod{2^r}$, in which case</u>

$$\lambda_\rho^{(r)} = 2.$$

<u>Proof.</u> Suppose first that $q\neq 2$. We shall obtain the result from the representation of p in the form (1). The order of $g \pmod{q^r}$ is $\phi(q^r)$ by the definition of a primitive root. Hence the order of $g^\nu \pmod{q^r}$, $h^{(r)}$ say, is given by

$$h^{(r)} = \frac{\phi(q^r)}{(\nu,\phi(q^r))} = \frac{q^{r-1}(q-1)}{(\nu,q^{r-1}(q-1))} = \frac{q^{r-1}}{(\nu,q^{r-1})}\,\frac{q-1}{(\nu,q-1)}$$

$$= \begin{cases} q^{r-1-\gamma}h & \text{if} \quad r-1-\gamma \geqslant 0 \\ h & \text{if} \quad r-1-\gamma \leqslant 0. \end{cases}$$

It follows that if $r-1-\gamma \geqslant 0$ the order of $g^{\nu c_\rho^{(r)}} \pmod{q^r}$, that is the order of $p^\nu \pmod{q^r}$, is equal to

$$\frac{h^{(r)}}{(h^{(r)},c_\rho^{(r)})} = \frac{q^{r-1-\gamma}h}{(q^{r-1-\gamma}h,c_\rho^{(r)})} = \frac{q^{r-1-\gamma}}{(q^{r-1-\gamma},c_\rho^{(r)})}\,\frac{h}{(h,c_\rho^{(r)})} = q^{r-1-\gamma-\varepsilon_\rho^{(r)}}\,\frac{h}{(h,c_\rho)}$$

provided that $r-1-\gamma-\varepsilon_\rho^{(r)} \geq 0$; we have used the fact that $(h,c_\rho^{(r)})=(h,c_\rho)$, which follows since $c_\rho^{(r)} \equiv c_\rho \pmod{q-1}$ by (4) and $h|(q-1)$. If $r-1-\gamma \leq 0$, 1 replaces $q^{r-1-\gamma}$ everywhere, and if $r-1-\gamma-\varepsilon_\rho^{(r)} \leq 0$, 1 replaces $q^{r-1-\gamma-\varepsilon_\rho^{(r)}}$; thus in either of these cases the order of $p^\nu \pmod{q^r}$ is

$$h/(h,c_\rho).$$

Suppose next that $q=2$ and $p\equiv 1 \pmod 4$. Then, as above, the order of $5^\nu \pmod{2^r}$ is

$$2^{r-2}/(\nu,2^{r-2}) = 2^{r-2-\gamma},$$

and hence the order of $5^{\nu c_\rho^{(r)}} \pmod{2^r}$, that is the order of $p^\nu \pmod{2^r}$, is

$$2^{r-2-\gamma}/(c_\rho^{(r)},2^{r-2-\gamma}) = 2^{r-2-\gamma-\varepsilon_\rho^{(r)}} = \lambda_\rho^{(r)} \geq 1.$$

Since $r \geq t$ and $h=1$, so that $h/(h,c_\rho)=1$, the result follows.

Finally we suppose that $q=2$ and $p\equiv 3 \pmod 4$; then $-p\equiv 1 \pmod 4$. If $r>t'$ we have from above that the order of $(-p)^\nu \pmod{2^r}$ is $2^{r-2-\gamma-\varepsilon_\rho^{(r)}} = \lambda_\rho^{(r)} > 1$. Thus, since $(-1)^{\nu 2^{r-2-\gamma-\varepsilon_\rho^{(r)}}} = +1$, the order of $p^\nu \pmod{2^r}$ is $\lambda_\rho^{(r)}$ in this case.
If $r \leq t'$, $(-p)^\nu \equiv 1 \pmod{2^r}$, and hence $p^\nu \equiv (-1)^\nu \pmod{2^r}$. When ν is even, this means that $r=t=t'$ (since we assume that $r \geq t$), and the order of $p^\nu \pmod{2^r}$ is $\lambda_\rho^{(r)}=1$. However when ν is odd, $p^\nu \equiv -1 \pmod{2^r}$ and the order of $p^\nu \pmod{2^r}$ is 2. This completes the proof of the lemma.

We observe that in all cases $\lambda_\rho^{(r+1)} = q\lambda_\rho^{(r)}$ or $\lambda_\rho^{(r)}$. This follows in the special case mentioned for $q=2$ since $\lambda_\rho^{(r)}=2$ for $r \leq t'$ and $\lambda_\rho^{(t'+1)}=2$.

Otherwise the truth of the remark follows from (6); in fact

$$\lambda_p^{(r+1)} = q\,\lambda_p^{(r)} \quad \text{or} \quad \lambda_p^{(r)} \quad \text{according as} \quad \varepsilon_p^{(r+1)} = \varepsilon_p^{(r)} \quad \text{or} \quad \varepsilon_p^{(r)}+1.$$

Corollary 1. If $r>t$, and $t\geq 3$ if $q=2$ or $t\geq 1$ otherwise, then the order of $p^\nu \pmod{q^r}$ is q^{r-t}.

Proof. If $q\neq 2$, then since $q^t \parallel (p^\nu -1)$ and $t\geq 1$, the expression giving the order of $p^\nu \pmod{q^t}$ must equal 1, so that $h/(h,c_p)=1$, whence $h|c_p$, and $\lambda_p^{(t)}=1$. If $q=2$, then since $t\geq 3$, $p\equiv 3 \pmod 4$ and ν odd cannot both hold; thus the order of $p^\nu \pmod{q^t}$ is 1 and so $\lambda_p^{(t)}=1$. For all q and for $r>t$ we have from the lemma that the order of $p^\nu \pmod{q^r}$ is $\lambda_p^{(r)}=q^{\kappa_r-\gamma-\varepsilon_p^{(r)}}>1$. Hence since $\lambda_p^{(t+1)}>1$ but $\lambda_p^{(t)}=1$, we have from the remark preceding the corollary that $\lambda_p^{(t+1)}=q$. Thus by the lemma

$$\kappa_t-\gamma-\varepsilon_p^{(t)}=0 \quad\text{and}\quad \kappa_{t+1}-\gamma-\varepsilon_p^{(t+1)}=\kappa_t+1-\gamma-\varepsilon_p^{(t+1)}=1, \tag{9}$$

giving $\varepsilon_p^{(t+1)}=\varepsilon_p^{(t)}$. On putting $r_1=t+1$ and $r_2=r$ in (5) and noting that $\varepsilon_p^{(t+1)}=\varepsilon_p^{(t)}\leq\kappa_t<\kappa_{t+1}$, we obtain $\varepsilon_p^{(r)}=\varepsilon_p^{(t)}$. Hence, by (9),

$$\gamma+\varepsilon_p^{(r)}=\gamma+\varepsilon_p^{(t)}=\kappa_t,$$

and

$$\lambda_p^{(r)}=q^{\kappa_r-\gamma-\varepsilon_p^{(r)}}=q^{\kappa_r-\kappa_t}=q^{r-t}.$$

Corollary 2. If $t=2$ or $t=1$ and $p\equiv 3 \pmod 8$, then the order of $p^\nu \pmod{2^r}$ is 2^{r-2}. If $t=1$ and $p\equiv 7 \pmod 8$, then the order of $p^\nu \pmod{2^r}$, that is $\lambda_p^{(r)}$, is given by

$$\lambda_p^{(r)}=2 \ \text{ if } 3\leq r\leq t' \quad\text{and}\quad \lambda_p^{(r)}=2^{r-t'} \ \text{ if } r>t'.$$

Proof. If $t=1$ or 2, then ν must be odd, so that $\gamma=0$; for if ν is even, $p^\nu\equiv 1 \pmod 8$ and $t\geq 3$. If $t=2$, so that $4\parallel(p^\nu-1)$, then $p\equiv 5 \pmod 8$, and if $t=1$ and $p\equiv 3 \pmod 8$, then $p\equiv -5 \pmod 8$;

in either case $c_p^{(3)} = 1$, $\varepsilon_p^{(3)} = 0$ and $\lambda_p^{(3)} = 2$. By (4)

$$\varepsilon_p^{(r)} \equiv c_p^{(3)} \pmod 2$$

for $r \geq 3$, and hence $c_p^{(r)}$ is odd, so that $\varepsilon_p^{(r)} = 0$. Thus by the lemma

$$\lambda_p^{(r)} = 2^{\kappa_r} = 2^{r-2}.$$

If $t=1$, then $p \equiv 3 \pmod 4$ and hence the only possibility remaining is $t=1$ and $p \equiv 7 \pmod 8$. Since ν is odd and $2^{t'} \| (p^\nu + 1)$, $2^{t'} \| (p+1)$, and hence $p \equiv -5^{2^{t'-2}} \pmod{2^{t'}}$, giving $c_p^{(t')} = 2^{t'-2}$ and $\varepsilon_p^{(t')} = t'-2$. By the lemma, $\lambda_p^{(r)} = 2$ for $3 \leq r \leq t'$. As in Corollary 1, $\lambda_p^{(t'+1)} = 2$, and $\varepsilon_p^{(t'+1)} = \varepsilon_p^{(t')} = t'-2$, and also $\varepsilon_p^{(r)} = \varepsilon_p^{(t')} = t'-2$ for $r > t'$. Hence by the lemma

$$\lambda_p^{(r)} = 2^{r-t'} \quad \text{for} \quad r > t'.$$

We define $\mu_p^{(r)}$ to be the order of $p^\nu \pmod{q^{r+t}}$; thus, by the previous lemma and its corollaries,

$$\mu_p^{(r)} = \begin{cases} \lambda_p^{(r)} \dfrac{h}{(h, c_p)} = \lambda_p^{(r)} \mu_p & \text{if } p^\nu \not\equiv 1 \pmod q \text{ and } q \neq 2 \\ q^r & \text{if } p^\nu \equiv 1 \pmod q \text{ and } q \neq 2 \\ 2^r & \text{if } q=2 \text{ and } r+t > t \geq 2 \\ 2^{r-1} & \text{if } q=2,\ r \geq 2,\ t=1 \text{ and } p \equiv 3 \pmod 8 \\ \lambda_p^{(r+1)} & \text{if } q=2,\ r \geq 2,\ t=1 \text{ and } p \equiv 7 \pmod 8, \end{cases} \tag{10}$$

where in the first case $\lambda_p^{(r)}$ is given by the lemma and in the last case $\lambda_p^{(r+1)}$ is given by Corollary 2. Note that if $q=2$, $\mu_p = \mu_p^{(1)} = 2$ provided $t \geq 2$, and if $q > 2$, $\mu_p \geq 2$ always. If $q=2$ and $t=1$, $\mu_p^{(2)} = 2$; for completeness we define $\mu_p = 2$ in this case also.

Lemma 3. (i) If $q\neq 2$, and h is even then $\mu_p^{(r)}=2$ and $\mu_p^{(r+1)}=q\mu_p^{(r)}$ if and only if $r\geq\gamma+1$ and p is congruent to one of $\phi(q^{\gamma+1})(\nu,q-1)$ elements of a reduced residue system (mod q^{r+1}).

(ii) Let $q=2$. If $t\geq 2$, then $\mu_p=2$ and for all $r\geq 1$ $\mu_p^{(r+1)}=2\mu_p^{(r)}$. If $t=1$ and $p\equiv 3 \pmod 8$, then $\mu_p^{(2)}=2$ and for all $r\geq 2$ $\mu_p^{(r+1)}=2\mu_p^{(r)}$. If $t=1$ and $p\equiv 7 \pmod 8$, then for $2\leq r\leq t'$ $\mu_p^{(r)}=2$ and for all $r\geq t'$ $\mu_p^{(r+1)}=2\mu_p^{(r)}$.

Proof. (i) We can assume that $t=0$; for if $t\geq 1$, $\mu_p=q>2$, so that $\mu_p^{(r)}>2$ for all $r\geq 1$. Clearly if $\mu_p^{(r)}=2$, then $\mu_p=2$ and $\lambda_p^{(r)}=1$. Now $\mu_p=h/(h,c_p)$, and we see that $\mu_p=2$ if and only if c_p is an odd multiple of $\frac{1}{2}h$; this occurs when

$$c_p=\tfrac{1}{2}h(2u-1) \quad \text{where} \quad 1\leq u\leq(\nu,q-1),$$

the bounds for u following since $1\leq c_p\leq q-1$, so that $\frac{1}{2}(\frac{2}{h}+1)\leq u\leq\frac{1}{2}(2(q-1)h^{-1}+1)=(\nu,q-1)+\frac{1}{2}$, and u is an integer. Thus there are exactly $(\nu,q-1)$ values of c_p which are such that $\mu_p=2$, and hence $\mu_p=2$ if and only if p is congruent to one of $(\nu,q-1)$ elements of a reduced residue system (mod q).

We now find the number of values of $c_p^{(r+1)}$, corresponding to a given value of c_p, for which $\mu_p^{(r+1)}=q\mu_p^{(r)}=q\mu_p$. Clearly $\lambda_p^{(r+1)}=q$ but $\lambda_p^{(r)}=\lambda_p^{(r-1)}=\ldots=\lambda_p^{(1)}=1$; thus

$$(r+1)-1-\gamma-\varepsilon_p^{(r+1)}=1 \quad\text{and}\quad r-1-\gamma-\varepsilon_p^{(r)}=0,$$

giving $\varepsilon_p^{(r+1)}=\varepsilon_p^{(r)}=r-1-\gamma$ provided $r\geq\gamma+1$. By (5)

$$\varepsilon_p^{(r-\gamma)}=\min(\varepsilon_p^{(r)},r-1-\gamma)=r-1-\gamma,$$

and hence $\varepsilon_p^{(r+1)}=\varepsilon_p^{(r)}=\varepsilon_p^{(r-1)}=\ldots=\varepsilon_p^{(r-\gamma)}=r-1-\gamma$.

Therefore $q^{r-1-\gamma} \| c_\rho^{(r-\gamma)}$, and by (8) $c_\rho^{(r-\gamma)} = q^{r-1-\gamma} c_\rho$; thus to each c_ρ there corresponds exactly one $c_\rho^{(r-\gamma)}$. Now by (4)

$$c_\rho^{(r+1)} = c_\rho^{(r-\gamma)} + uq^{r-1-\gamma}(q-1) \quad \text{where} \quad 0 \leqslant u < q^{\gamma+1},$$

so that

$$c_\rho^{(r+1)} = q^{r-1-\gamma}(c_\rho + u(q-1)).$$

Hence $q^{r-1-\gamma} \| c_\rho^{(r+1)}$ implies that $q \nmid (c_\rho - u)$. This means that u can take any value between 0 and $q^{\gamma+1}-1$ except

$$c_\rho,\ c_\rho + q,\ \dots,\ c_\rho + (q^\gamma - 1)q,$$

and thus u and hence $c_\rho^{(r+1)}$ can take $q^{\gamma+1} - q^\gamma = \phi(q^{\gamma+1})$ values for each given value of c_ρ.

Hence $\mu_p^{(r+1)} = q\mu_p^{(r)} = 2q$ if and only if p is congruent to one of $\phi(q^{\gamma+1})(\nu, q-1)$ elements of a reduced residue system (mod q^{r+1}) provided $r \geqslant \gamma+1$. If $r < \gamma+1$, we observe that $\mu_p^{(r+1)} = \mu_p^{(r)} = \mu_p$ for all p, so that no p satisfies the required conditions.

(ii) This result follows immediately from the definition of $\mu_p^{(r)}$. We observe that in the last case $p \equiv 2^{t'} - 1 \pmod{2^{t'+1}}$ and $t' \geqslant 3$.

3. The evaluation of $\sum_{\alpha=1}^{\infty} a_r(p^\alpha)$.

We have already defined (in chapter 1).

$$a_r(n) = \begin{cases} 0 & \text{if } q^r \nparallel \sigma_\nu(n) \\ 1 & \text{if } q^r \| \sigma_\nu(n) \end{cases}$$

for $r \geqslant 1$; we also define $a_0(n)$ by $a_0(n) = 0$ or 1 according as q divides or does not divide $\sigma_\nu(n)$.

Clearly the definition implies that $a_0(1) = 1$, $a_r(1) = 0$ for $r \geqslant 1$.

In this section and the next we shall write $a(n)$ for $a_0(n)$ in order to simplify the notation when this is convenient. The results of this section and the next which involve only $a(n)$, and not $a_r(n)$ for $r\geq 1$, are all proved by Rankin [2]; Lemmas 4 and 5, parts (i) and (ii), and Lemma 6 are proved in the first part of §2 of this paper for $q\neq 2$, and the corresponding result for $q=2$ is mentioned in the last but one paragraph of the paper. We give the proofs here for the sake of completeness.

The next lemma enables us to determine the form of α when $a_r(p^\alpha)=1$, $r\geq 0$.

<u>Lemma 4.</u> (i) <u>If $p\neq q$, $a(p^\alpha)=1$ if and only if $\alpha\neq u\mu_\rho -1$ for any positive integer u.</u>

(ii) <u>$a(q^\alpha)=1$ for all α.</u>

(iii) <u>If $p\neq q$, $r\geq 1$, $r\neq 1$ when $q=2$ and $2 \parallel (p^\nu-1)$, and $\mu_\rho^{(r+1)} = q\mu_\rho^{(r)}$, then $a(p^\alpha)=1$ if and only if $\alpha=u\mu_\rho^{(r)}-1$ where $(u,q)=1$.</u>

(iv) <u>If $r\geq 1$ and either $p=q$ or $r=1$ when $q=2$ and $2 \parallel (p^\nu-1)$ or $\mu_\rho^{(r+1)} = \mu_\rho^{(r)}$, then $a(p^\alpha)=0$ for all α.</u>

<u>Proof.</u> We have

$$\sigma_\nu(p^\alpha) = 1+p^\nu+p^{2\nu}+\dots+p^{\alpha\nu} = (p^{\nu(\alpha+1)}-1)/(p^\nu-1),$$

and $q^t \parallel (p^\nu-1)$ where $t\geq 0$. For any $r\geq 0$, $q^r \parallel \sigma_\nu(p^\alpha)$ implies that $q^{r+t} \parallel (p^{\nu(\alpha+1)}-1)$, and this occurs if and only if the order of $p^\nu \pmod{q^{r+t}}$, that is $\mu_\rho^{(r)}$ by definition, divides $(\alpha+1)$ but the order of $p^\nu \pmod{q^{r+t+1}}$, that is $\mu_\rho^{(r+1)}$, does not.

[We recall that the order of $p^\nu \pmod{q^t}$ is 1, and we use this convention also when t=0.]

(i) If q=2 and $2 \parallel (p^\nu - 1)$, then $a(p^\alpha)=1$ if and only if $4 \nmid (p^{\nu(\alpha+1)}-1)$, and this is so if $\alpha+1$ is odd, so that $\mu_p = 2 \nmid (\alpha+1)$. Otherwise if $p \neq q$, $a(p^\alpha)=1$ if and only if $\mu_p \nmid (\alpha+1)$ and the result follows.

(ii) $\sigma_\nu(q^\alpha) \equiv 1 \pmod q$, and this gives the result.

(iii) If the given conditions are satisfied, then from above $a_r(p^\alpha)=1$ if and only if

$$\mu_p^{(r)} \mid (\alpha+1) \quad \text{but} \quad \mu_p^{(r+1)} \nmid (\alpha+1).$$

Since $\mu_p^{(r+1)} = q\mu_p^{(r)}$, the result follows.

(iv) This part is an immediate consequence of the proof of (ii) if p=q and of (iii) if $\mu_p^{(r+1)} = \mu_p^{(r)}$. If q=2, r=1 and t=1, so that ν is odd and $p \equiv 3 \pmod 4$, the result follows if $4 \nmid (p^{\nu(\alpha+1)}-1)$ for any value of $\alpha+1$; but $2 \parallel (p^{\nu(\alpha+1)}-1)$ if $\alpha+1$ is odd and $8 \mid (p^{\nu(\alpha+1)}-1)$ if $\alpha+1$ is even.

<u>Lemma 5.</u> (i) <u>If $p \neq q$,</u>

$$\sum_{\alpha=0}^{\infty} a(p^\alpha)p^{-\alpha s} = (1-p^{-(\mu_p-1)s})/(1-p^{-s})(1-p^{-\mu_p s}).$$

(ii) $$\sum_{\alpha=0}^{\infty} a(q^\alpha)q^{-\alpha s} = (1-q^{-s})^{-1}.$$

(iii) <u>If $p \neq q$, $r \geq 1$, $r \neq 1$ when $q=2$ and $2 \parallel (p^\nu - 1)$, and $\mu_p^{(r+1)} = q\mu_p^{(r)}$, then</u>

$$\sum_{\alpha=1}^{\infty} a_r(p^\alpha)p^{-\alpha s} = (1-p^{-(q-1)\mu_p^{(r)} s})p^{-(\mu_p^{(r)}-1)s}/(1-p^{-\mu_p^{(r)} s})(1-p^{-q\mu_p^{(r)} s}).$$

(iv) <u>If $r \geq 1$ and either $p=q$ or $r=1$ when $q=2$ and $2 \parallel (p^\nu - 1)$ or $\mu_p^{(r+1)} = \mu_p^{(r)}$, then</u>

$$\sum_{\alpha=1}^{\infty} a_r(p^\alpha)p^{-\alpha s} = 0.$$

Proof. This lemma follows from the previous one.

(i) If $p \neq q$,

$$\sum_{\alpha=0}^{\infty} a(p^{\alpha})p^{-\alpha s} = \sum_{\alpha=0}^{\infty} p^{-\alpha s} - \sum_{u=1}^{\infty} p^{-(u\mu_\rho - 1)s} = \frac{1}{(1-p^{-s})} - \frac{p^{-(\mu_\rho - 1)s}}{(1-p^{-\mu_\rho s})}$$

$$= (1-p^{-(\mu_\rho - 1)s})/(1-p^{-s})(1-p^{-\mu_\rho s}).$$

(ii) $$\sum_{\alpha=0}^{\infty} a(q^{\alpha})q^{-\alpha s} = \sum_{\alpha=0}^{\infty} q^{-\alpha s} = (1-q^{-s})^{-1}.$$

(iii) If all the given conditions hold,

$$\sum_{\alpha=1}^{\infty} a_r(p^{\alpha})p^{-\alpha s} = \sum_{\substack{u=1 \\ (u,q)=1}}^{\infty} p^{-(u\mu_\rho^{(r)} - 1)s} = p^{s}\left\{\sum_{u=1}^{\infty} p^{-u\mu_\rho^{(r)} s} - \sum_{u=1}^{\infty} p^{-uq\mu_\rho^{(r)} s}\right\}$$

$$= p^{s}\left\{\frac{p^{-\mu_\rho^{(r)} s}}{1-p^{-\mu_\rho^{(r)} s}} - \frac{p^{-q\mu_\rho^{(r)} s}}{1-p^{-q\mu_\rho^{(r)} s}}\right\} = \frac{(1-p^{-(q-1)\mu_\rho^{(r)} s})p^{-(\mu_\rho^{(r)} - 1)s}}{(1-p^{-\mu_\rho^{(r)} s})(1-p^{-q\mu_\rho^{(r)} s})}.$$

(iv) In this case

$$\sum_{\alpha=1}^{\infty} a_r(p^{\alpha})p^{-\alpha s} = 0.$$

4. The generating functions.

It is well known that $\sigma_\nu(n)$ is multiplicative, so that

$$\sigma_\nu(n) = \prod_{p^\alpha \| n} \sigma_\nu(p^\alpha),$$

the product being over all distinct primes dividing n. From this it follows that a(n) is multiplicative; for $q \nmid \sigma_\nu(n)$ if and only if $q \nmid \sigma_\nu(p^\alpha)$ for every $p^\alpha \| n$.

Hence

$$a(n) = \prod_{p^{\alpha} \| n} a(p^{\alpha}).$$

Let

$$f(s) = \sum_{n=1}^{\infty} a(n)n^{-s};$$

then we have

Lemma 6.

$$f(s) = \zeta(s) \prod_{p \neq q} (1-p^{-(\mu_p -1)s})/(1-p^{-\mu_p s}),$$

where $\zeta(s)$ is the Riemann zeta-function. In particular if $q=2$

$$f(s) = (1+2^{-s})\zeta(2s).$$

Proof. Since $a(n)$ is multiplicative, we have by Lemma 5 (i) and (ii)

$$f(s) = \sum_{n=1}^{\infty} a(n)n^{-s} = \prod_{p} \left\{ \sum_{\alpha=0}^{\infty} a(p^{\alpha})p^{-\alpha s} \right\}$$

$$= (1-q^{-s})^{-1} \prod_{p \neq q} \frac{1-p^{-(\mu_p -1)s}}{(1-p^{-s})(1-p^{-\mu_p s})}$$

$$= \zeta(s) \prod_{p \neq q} \frac{1-p^{-(\mu_p -1)s}}{1-p^{-\mu_p s}}.$$

If $q=2$, $\mu_p =2$ by definition and

$$f(s) = (1-2^{-s})^{-1} \prod_{p \neq 2} (1-p^{-2s})^{-1} = (1+2^{-s})\zeta(2s).$$

However, although $a(n)$ is multiplicative, $a_m(n)$, $m\geq 1$, is not; for $q^m \| \sigma_\nu(n)$ certainly does not hold if $q^m \| \sigma_\nu(p^\alpha)$ for all $p^\alpha \| n$ (unless $n=p^\alpha$). Nevertheless we can obtain an expression for $a_m(n)$ in terms of $a(n_1)$ and $a_r(p^\alpha)$, where $n_1 | n$, $p^\alpha \| n$ and $r\leq m$. In the following lemma we assume that $p_i^{\alpha_i} \| n$ for all i (with or without a suffix), and that two primes p with different suffixes are distinct. Furthermore when we consider a set of primes $p_{i_1}, p_{i_2}, \ldots, p_{i_k}$, the order in which they are written is significant.

<u>Lemma 7.</u> <u>If $m\geq 1$,</u>

$$a_m(n) = \sum_{r_1+\cdots+r_k=m} \{\rho^*(r_1,\ldots,r_k)\}^{-1} \sum_{p_{i_1},\ldots,p_{i_k}} a_{r_1}(p_{i_1}^{\alpha_{i_1}})a_{r_2}(p_{i_2}^{\alpha_{i_2}})\ldots$$
$$\ldots a_{r_k}(p_{i_k}^{\alpha_{i_k}})a(np_{i_1}^{-\alpha_{i_1}} p_{i_2}^{-\alpha_{i_2}}\ldots p_{i_k}^{-\alpha_{i_k}}),$$

<u>where (i) the set $r_1, r_2, \ldots, r_k$ runs through all the unrestricted partitions of m which are such that $1\leq r_1\leq r_2\leq\ldots\leq r_k$ and k, the number of parts, does not exceed the number of primes dividing n,</u>

(ii) <u>the set $p_{i_1}, p_{i_2}, \ldots, p_{i_k}$ runs through all sets consisting of k of the distinct primes dividing n,</u>

<u>and</u> (iii) $\rho^*(r_1,\ldots,r_k)$ <u>is defined below.</u>

<u>Proof.</u> Since $\sigma_\nu(n)$ is multiplicative, $q^m \| \sigma_\nu(n)$, so that $a_m(n)=1$, if and only if $q^{r_j} \| \sigma_\nu(p_{i_j}^{\alpha_{i_j}})$ for $j=1, 2,\ldots,k$, where $r_1+r_2+\ldots+r_k=m$, and $q\nmid\sigma_\nu(np_{i_1}^{-\alpha_{i_1}} p_{i_2}^{-\alpha_{i_2}}\ldots p_{i_k}^{-\alpha_{i_k}})$. If this occurs, $a_{r_j}(p_{i_j}^{\alpha_{i_j}})=1$ for $j=1, 2,\ldots,k$ and $a(np_{i_1}^{-\alpha_{i_1}} p_{i_2}^{-\alpha_{i_2}}\ldots p_{i_k}^{-\alpha_{i_k}})=1$, and we have

$$1 = a_m(n) = a_{r_1}(p_{i_1}^{\alpha_{i_1}})a_{r_2}(p_{i_2}^{\alpha_{i_2}})\ldots a_{r_k}(p_{i_k}^{\alpha_{i_k}})a(np_{i_1}^{-\alpha_{i_1}} p_{i_2}^{-\alpha_{i_2}}\ldots p_{i_k}^{-\alpha_{i_k}}). \qquad (11)$$

Clearly (11) will hold for only one set $r_1,\dots,r_k$ satisfying $r_1+\dots+r_k=m$; denote this set by $\hat{r}_1,\dots,\hat{r}_k$.

Define $M(r_1,p_{i_1};r_2,p_{i_2};\dots;r_k,p_{i_k}) = a_{r_1}(p_{i_1}^{\alpha_{i_1}})a_{r_2}(p_{i_2}^{\alpha_{i_2}})\dots a_{r_k}(p_{i_k}^{\alpha_{i_k}})a(np_{i_1}^{-\alpha_{i_1}}p_{i_2}^{-\alpha_{i_2}}\dots p_{i_k}^{-\alpha_{i_k}})$ (so that the expression on the right of (11) is $M(\hat{r}_1,p_{i_1};\hat{r}_2,p_{i_2};\dots;\hat{r}_k,p_{i_k})$). If we use a different partition of m or a different set of primes $p_{i_1},p_{i_2},\dots,p_{i_k}$ dividing n or both, then we obtain an expression of the form

$$M(r_1',p_{i_1}';r_2',p_{i_2}';\dots;r_{k'}',p_{i_{k'}}')$$

which in general is essentially different from, and not just a rearrangement of the terms in, $M(r_1,p_{i_1};r_2,p_{i_2};\dots;r_k,p_{i_k})$; with the exception of the case discussed in the next paragraph, if

$$M(r_1,p_{i_1};r_2,p_{i_2};\dots;r_k,p_{i_k})=1,\ M(r_1',p_{i_1}';r_2',p_{i_2}';\dots;r_{k'}',p_{i_{k'}}')=0. \qquad (12)$$

However if the r_j are not all distinct, and if we rearrange the set of primes $p_{i_1},p_{i_2},\dots,p_{i_k}$ in such a way that

$$M(r_1,p_{i_1}';r_2,p_{i_2}';\dots;r_k,p_{i_k}') = M(r_1,p_{i_1};r_2,p_{i_2};\dots;r_k,p_{i_k}),$$

the terms on the left being just a rearrangement of the terms on the right, then we can regard these two expressions as being equivalent, and clearly if

$$M(r_1,p_{i_1};r_2,p_{i_2};\dots;r_k,p_{i_k})=1,\quad M(r_1,p_{i_1}';r_2,p_{i_2}';\dots;r_k,p_{i_k}')=1.$$

Denote by $\rho^*(r_1,\dots,r_k)$ the number of different ordered sets of primes $p_{i_1}',p_{i_2}',\dots p_{i_k}'$ which give rise to an expression which is equivalent to

$$M(r_1,p_{i_1};r_2,p_{i_2};\dots;r_k,p_{i_k});$$

we shall now calculate $\rho^*(r_1,\dots,r_k)$.

Suppose that the set $r_1,\dots r_k$ contains τ distinct elements $r^*_1, r^*_2, \dots, r^*_\tau$ (so that $r^*_1<r^*_2<\dots<r^*_\tau$ and $1\leq\tau\leq k$) occurring $\ell_1, \ell_2, \dots, \ell_\tau$ times respectively in the set; clearly $\ell_1+\ell_2+\dots+\ell_\tau=k$. We have $M(r_1,p_{i_1};r_2,p_{i_2};\dots;r_k,p_{i_k}) = M(r^*_1,p_{i_1};\dots;r^*_1,p_{i_{\ell_1}};r^*_2,p_{i_{\ell_1+1}};\dots;r^*_\tau,p_{i_{\ell_1+\dots+\ell_\tau}})$. The set of primes $p_{i_1},p_{i_2},\dots,p_{i_{\ell_1}}$ can be arranged among themselves in $\ell_1!$ ways without essentially altering $M(r_1,p_{i_1};\dots;r_k,p_{i_k})$, and similarly the remaining $\tau-1$ sets of primes can each be arranged among themselves. However interchanging two primes which are paired off above with different values of r^*, for example interchanging p_{i_1} and $p_{i_{\ell_1+1}}$, will essentially alter $M(r_1,p_{i_1};\dots;r_k,p_{i_k})$. Thus the whole set of primes $p_{i_1},p_{i_2},\dots,p_{i_k}$ can be arranged in exactly $\ell_1!\ell_2!\dots\ell_\tau!$ ways without essentially altering $M(r_1,p_{i_1};\dots;r_k,p_{i_k})$. Hence

$$\rho^*(r_1,r_2,\dots,r_k) = \ell_1!\ell_2!\dots\ell_\tau!.$$

It follows that, if $a_m(n)=1$,

$$\sum_{r_1+\dots+r_k=m}\ \sum_{p_{i_1},\dots,p_{i_k}} M(r_1,p_{i_1};\dots;r_k,p_{i_k})$$

$$= \sum_{r_1+\dots+r_k=m}\ \sum_{p_{i_1},\dots,p_{i_k}} a_{r_1}(p_{i_1}^{\alpha_{i_1}})a_{r_2}(p_{i_2}^{\alpha_{i_2}})\dots a_{r_k}(p_{i_k}^{\alpha_{i_k}})a(np_{i_1}^{-\alpha_{i_1}}p_{i_2}^{-\alpha_{i_2}}\dots p_{i_k}^{-\alpha_{i_k}})$$

$$= \rho^*(\hat{r}_1,\hat{r}_2,\dots,\hat{r}_k) = \rho^*(\hat{r}_1,\hat{r}_2,\dots,\hat{r}_k)a_m(n),$$

where the set $\hat{r}_1,\hat{r}_2,\dots,\hat{r}_k$ is the particular set occurring on the right of (11), and where the conditions on the summations are as given in the statement of the Lemma. Since the inner sum is zero unless $r_j=\hat{r}_j$, $j=1,2,\dots,k$, (which follows from (12)), the result of the lemma follows in the case when $a_m(n)=1$.

If $a_m(n)=0$, all expressions of the type

$$M(r_1, p_{i_1}; r_2, p_{i_2}; \ldots; r_k, p_{i_k}),$$

with $r_1+r_2+\ldots+r_k=m$, will be zero since at least one of its terms must be zero, and the result of the lemma follows in this case also.

We are now in a position to find the generating function

$$f_m(s) = \sum_{n=1}^{\infty} a_m(n)n^{-s}$$

for $m\geq 1$.

Lemma 8. If ν is odd and $q\neq 2$ or if ν is even,

$$f_m(s) = f(s) \sum_{r_1+\cdots+r_k=m} \{\rho^*(r_1,\ldots,r_k)\}^{-1} \sum_{p_1} P(p_1,\mu_{p_1}^{(r_1)};s) \sum_{p_2} P(p_2,\mu_{p_2}^{(r_2)};s)\ldots \sum_{p_k} P(p_k,\mu_{p_k}^{(r_k)};s),$$

where

$$P(p,\mu_p^{(r)};s) = \frac{(1-p^{-s})(1-p^{-\mu_p s})(1-p^{-(q-1)\mu_p^{(r)} s})p^{-(\mu_p^{(r)}-1)s}}{(1-p^{-(\mu_p-1)s})(1-p^{-\mu_p^{(r)} s})(1-p^{-q\mu_p^{(r)} s})},$$

where the set $r_1, r_2, \ldots, r_k$ runs through all the unrestricted partitions of m, and where the sum over p_i $(i=1,2,\ldots,k)$ is over all primes except $q, p_1, p_2, \ldots, p_{i-1}$ and those for which $\mu_{p_i}^{(r_i+1)} = \mu_{p_i}^{(r_i)}$.

If ν is odd and $q=2$, the same result holds provided that each sum over p satisfies the conditions above and the additional condition that $p\equiv 1 \pmod 4$ if $r=1$.

Proof. By Lemma 7,

$$f_m(s) = \sum_{n=1}^{\infty}\Big\{ \sum_{r_1+\cdots+r_k=m} \{\rho^*(r_1,\ldots,r_k)\}^{-1} \sum_{p_{i_1},\ldots,p_{i_k}|n} a_{r_1}(p_{i_1}^{\alpha_{i_1}})a_{r_2}(p_{i_2}^{\alpha_{i_2}})\ldots$$

$$a_{r_k}(p_{i_k}^{\alpha_{i_k}})a(np_{i_1}^{-\alpha_{i_1}}p_{i_2}^{-\alpha_{i_2}}\ldots p_{i_k}^{-\alpha_{i_k}})\Big\}n^{-s}$$

$$= \sum_{n=1}^{\infty} \sum_{r_1+\cdots+r_k=m} \{\rho^*(r_1,\ldots,r_k)\}^{-1} \sum_{p_{i_1}|n} a_{r_1}(p_{i_1}^{\alpha_{i_1}})p_{i_1}^{-\alpha_{i_1}s} \sum_{p_{i_2}|n} a_{r_2}(p_{i_2}^{\alpha_{i_2}})p_{i_2}^{-\alpha_{i_2}s}\ldots$$

$$\sum_{p_{i_k}|n} a_{r_k}(p_{i_k}^{\alpha_{i_k}})p_{i_k}^{-\alpha_{i_k}s}\, a(np_{i_1}^{-\alpha_{i_1}}p_{i_2}^{-\alpha_{i_2}}\ldots p_{i_k}^{-\alpha_{i_k}})(np_{i_1}^{-\alpha_{i_1}}p_{i_2}^{-\alpha_{i_2}}\ldots p_{i_k}^{-\alpha_{i_k}})^{-s}$$

$$= \sum_{r_1+\cdots+r_k=m} \{\rho^*(r_1,\ldots,r_k)\}^{-1} \sum_{p_1} \sum_{\alpha_1=1}^{\infty} a_{r_1}(p_1^{\alpha_1})p_1^{-\alpha_1 s} \sum_{\substack{p_2\\ p_2\neq p_1}} \sum_{\alpha_2=1}^{\infty} a_{r_2}(p_2^{\alpha_2})p_2^{-\alpha_2 s}$$

$$\ldots \sum_{\substack{p_k\\ p_k\neq p_1,\ldots,p_{k-1}}} \sum_{\alpha_k=1}^{\infty} a_{r_k}(p_k^{\alpha_k})p_k^{-\alpha_k s} \sum_{\substack{n=1\\ (n,p_1\cdots p_k)=1}}^{\infty} a(n)n^{-s}, \qquad (13)$$

where each sum over p is over all primes except the ones indicated.

By Lemmas 5(i) and 6, we have if $p_i \neq q$, $i=1,2,\ldots,k$,

$$\sum_{\substack{n=1\\ (n,p_1\cdots p_k)=1}}^{\infty} a(n)n^{-s} = \prod_{\substack{p\\ p\neq p_1,\ldots,p_k}} \Big\{\sum_{\alpha=0}^{\infty} a(p^{\alpha})p^{-\alpha s}\Big\} = f(s)\Big\{\prod_{i=1}^{k}\Big\{\sum_{\alpha_i=0}^{\infty} a(p_i^{\alpha_i})p_i^{-\alpha_i s}\Big\}\Big\}^{-1}$$

$$= f(s)\prod_{i=1}^{k}(1-p_i^{-s})(1-p_i^{-\mu_{p_i}s})(1-p_i^{-(\mu_{p_i}-1)s})^{-1}. \qquad (14)$$

We do not need to consider the above sum with any p_i equal to q; for if $p_j=q$, $a_{r_j}(p_j^{\alpha_j})=a_{r_j}(q^{\alpha_j})=0$ for all α_j by Lemma 4(iv) and the corresponding term on the right of (13) is zero.

If we now use (14) and apply Lemma 5 (iii) and (iv) to (13), we shall obtain the result of this lemma. We note that if ν is odd, q=2, r=1 and $p\equiv 3 \pmod 4$, then $2\,\|\,(p^{\nu}-1)$, and so by Lemma 5(iv) $\sum_{\alpha=1}^{\infty} a_1(p^{\alpha})p^{-\alpha s}=0$ in this case; hence if ν is odd and q=2 we have to include in each sum over p the additional condition that $p\equiv 1 \pmod 4$ if r=1.

5. Proofs of Theorems 1(i) and 2(i).

In this section we shall assume that q and h are both odd. It follows from (10) that μ_ρ cannot be even, so that $\mu_\rho \geq 3$ and hence $\mu_\rho^{(r)} \geq 3$ for all $r \geq 1$. From the definition of $P(p, \mu_\rho^{(r)}; s)$, we have

$$|P(p,\mu_\rho^{(r)};s)| \leq \frac{(1+p^{-\sigma})(1+p^{-\mu_\rho\sigma})(1+p^{-(q-1)\mu_\rho^{(r)}\sigma})p^{-(\mu_\rho^{(r)}-1)\sigma}}{(1-p^{-(\mu_\rho-1)\sigma})(1-p^{-\mu_\rho^{(r)}\sigma})(1-p^{-q\mu_\rho^{(r)}\sigma})} \leq \Omega(\sigma)p^{-(\mu_\rho^{(r)}-1)\sigma},$$

where $\Omega(\sigma)$, a function of $\sigma = \mathrm{Re}\, s$ only, is obtained by using the inequalities $p \geq 2$, $\mu_\rho \geq 3$, $\mu_\rho^{(r)} \geq 3$. Hence

$$\left| \sum_{\substack{p_i \neq q, p_1, \ldots, p_{i-1} \\ \mu_{p_i}^{(r_i+1)} = q\mu_{p_i}^{(r_i)}}} P(p_i, \mu_{p_i}^{(r_i)}; s) \right| \leq \sum_p |P(p, \mu_\rho^{(r_i)}; s)| \leq \Omega(\sigma) \sum_p p^{-(\mu_\rho^{(r_i)}-1)\sigma} \leq \Omega(\sigma) \sum_p p^{-2\sigma} \tag{15}$$

which is convergent for $\sigma > \frac{1}{2}$; thus the sum on the left must be absolutely convergent for $\sigma > \frac{1}{2}$.

Since $\mu_\rho \geq 3$, the infinite product in the expression for $f(s)$, given in Lemma 6, is also absolutely convergent for $\sigma > \frac{1}{2}$. Hence it follows from Lemmas 6 and 8 and above that

$$f_m(s) = \zeta(s)g(s),$$

where $g(s)$ is holomorphic for $\sigma > \frac{1}{2}$ and bounded for $\sigma \geq \frac{1}{2} + \delta$ $(\delta > 0)$. This completes the proof of Theorem 2(i).

We shall now show that Theorem 1(i) follows from Theorem 2(i) and the Wiener-Ikehara Theorem which we state in

Lemma 9. *If $\Phi(\tau)$ is a non-negative, non-decreasing function in $0 \leq \tau < \infty$ such that the integral*

$$F(s) = \int_0^\infty e^{-s\tau} \Phi(\tau) d\tau$$

converges for $\sigma>1$, and if for some constant B and some function $G(t)$, where $t=\text{Im } s$,

$$\lim_{\sigma\to 1+}\left\{F(s) - \frac{B}{s-1}\right\} = G(t)$$

uniformly in every finite interval $-a\leq t\leq a$, then

$$\lim_{\tau\to\infty} \Phi(\tau)e^{-\tau} = B.$$

This is given in §17 of Chapter V of Widder [5].

To deduce the result from this, let

$$sF(s) = f_m(s) = \sum_{n=1}^{\infty} a_m(n)n^{-s},$$

$$\Phi(\tau) = S(e^{\tau}) = \sum_{n=1}^{[e^{\tau}]} a_m(n),$$

$$B = g(1);$$

then in order to prove Theorem 1(i) we need to estimate $S(x)$, for

$$S(x) = \sum_{n=1}^{x} a_m(n) = D_m(\nu,q;x).$$

Clearly $f_m(s)$ is holomorphic for $\sigma>1$, so that

$$f_m(s) = \sum_{n=1}^{\infty} a_m(n)n^{-s} = \sum_{n=1}^{\infty}\{S(n)-S(n-1)\}n^{-s} = \int_1^{\infty} y^{-s}dS(y)$$

$$= \int_0^{\infty} e^{-\tau s}dS(e^{\tau}) = s\int_0^{\infty} e^{-\tau s}S(e^{\tau})d\tau = s\int_0^{\infty} e^{-\tau s}\Phi(\tau)d\tau$$

converges for $\sigma>1$. Since $\zeta(s)-(s-1)^{-1}$ is holomorphic for $\sigma>0$ (see Lemma 1(i) of chapter 3) and $g(s)$ is holomorphic for $\sigma>\frac{1}{2}$, it follows that $f_m(s)s^{-1}-g(1)(s-1)^{-1}$ is holomorphic for $\sigma>\frac{1}{2}$, so that

$$\lim_{\sigma\to 1+}\{f_m(s)s^{-1} - g(1)(s-1)^{-1}\} = G(t)$$

uniformly in every finite interval $-a\leq t\leq a$. It is obvious that $\Phi(\tau)=S(e^{\tau})$ is non-negative and non-decreasing in $0\leq\tau<\infty$.

Hence the conditions of Lemma 9 are satisfied and an application of it yields

$$\lim_{\tau \to \infty} \Phi(\tau) e^{-\tau} = g(1),$$

whence

$$\lim_{x \to \infty} S(x) x^{-1} = g(1).$$

Thus as $x \to \infty$

$$S(x) = \sum_{n=1}^{x} a_m(n) = D_m(\nu, q; x) \sim g(1)x,$$

which is Theorem 1(i).

6. Proof of Theorem 2(ii) for $m' \geq 1$.

We assume in this section that q is odd, h is even and $m' = [m/(\delta+1)] \geq 1$. Then it follows from the proof of Lemma 3 (i) that for any positive integer r there exist primes p for which $\mu_p^{(r)} = 2$. For such a prime p we have by the definition of $P(p, \mu_p^{(r)}; s)$ that

$$P(p, \mu_p^{(r)}; s) = P(p, 2; s) = \frac{(1-p^{-s})(1-p^{-2s})(1-p^{-2(q-1)s})p^{-s}}{(1-p^{-s})(1-p^{-2s})(1-p^{-2qs})}$$

$$= \left\{ 1 - \frac{p^{-2(q-1)s} - p^{-2qs}}{1-p^{-2qs}} \right\} p^{-s}. \tag{16}$$

Hence

$$\sum_{\substack{p_i \neq q, p_1, \cdots, p_{i-1} \\ \mu_{p_i}^{(r_i+1)} = q\mu_{p_i}^{(r_i)}}} P(p_i, \mu_{p_i}^{(r_i)}; s) = \sum_{\substack{p \neq q \\ \mu_p^{(r_i+1)} = q\mu_p^{(r_i)} \\ \mu_p^{(r_i)} = 2}} p^{-s} + \psi_i(s), \tag{17}$$

where the sums on the left and the right are non-empty if and only if $r_i \geq \delta+1$ by Lemma 3(i) and, by the arguments used at the beginning of §5, $\psi_i(s)$ is holomorphic for $\sigma > \frac{1}{2}$ and bounded for $\sigma \geq \frac{1}{2} + \delta$ for any $\delta > 0$.

Lemma 10. Assume that $r \geq \gamma+1$. Let b_j, $j=1,2,\ldots,$ $\phi(q^{\gamma+1})(\nu, q-1)$, be the distinct elements of a reduced residue system (mod q^{r+1}), which occur in the proof of Lemma 3(i), and let χ be a character and χ_0 the principal character (mod q^{r+1}). If $L(s,\chi)$ is the Dirichlet L-series associated with the character χ, and $G(s,\chi)$ is a function which is holomorphic for $\sigma > \frac{1}{2}$ and bounded for $\sigma \geq \frac{1}{2}+\delta$, for any $\delta > 0$, then

$$\sum_{\substack{p \neq q \\ \mu_p^{(r+1)} = q\mu_p^{(r)} \\ \mu_p^{(r)} = \lambda}} p^{-s} = (\nu, q-1) q^{\gamma - r} \{ \log \zeta(s) + \log (1-q^{-s}) \}$$

$$+ q^{-r}(q-1)^{-1} \sum_{j=1}^{\phi(q^{\gamma+1})(\nu, q-1)} \left\{ \sum_{\chi \neq \chi_0} \frac{\log L(s,\chi)}{\chi(b_j)} + \sum_{\chi} \frac{G(s,\chi)}{\chi(b_j)} \right\}$$

where the sum over χ is over all characters χ (mod q^{r+1}) except, when indicated, χ_0.

Proof. By Lemma 3(i)

$$\sum_{\substack{p \neq q \\ \mu_p^{(r+1)} = q\mu_p^{(r)} \\ \mu_p^{(r)} = \lambda}} p^{-s} = \sum_{j=1}^{\phi(q^{\gamma+1})(\nu, q-1)} \sum_{p \equiv b_j (\mathrm{mod}\, q^{r+1})} p^{-s}, \qquad (18)$$

and the b_j can be determined from the proof of Lemma 3(i).

Now

$$\log L(s,\chi) = \sum_{p \neq q} \sum_{u=1}^{\infty} \frac{1}{u} \frac{\chi(p^u)}{p^{us}}$$

(see for example equation (1) of section 2, § 14 of Hasse [6])

and hence

$$\log L(s,\chi) = \sum_p \frac{\chi(p)}{p^s} + \sum_p \sum_{u=2}^{\infty} \frac{1}{u} \frac{\chi(p^u)}{p^{us}} = \sum_p \chi(p) p^{-s} - G(s,\chi)$$

where $G(s,\chi)$ satisfies the conditions given in the statement of the lemma.

Thus by a well known property of characters (see for example equation 2' of section 2, § 13 of Hasse [6])

$$\phi(q^{r+1}) \sum_{p \equiv b_j \pmod{q^{r+1}}} p^{-s} = \sum_{\chi} \sum_{p} \frac{\chi(p)\, p^{-s}}{\chi(b_j)} = \sum_{\chi} \frac{\log L(s,\chi)}{\chi(b_j)} + \sum_{\chi} \frac{G(s,\chi)}{\chi(b_j)}$$

where the sum over χ is over all characters $\chi \pmod{q^{r+1}}$. Since

$$L(s,\chi_0) = (1-q^{-s})\,\zeta(s) \tag{19}$$

(see for example section 1, § 14 of Hasse [6]), and $\chi_0(b_j)=1$,

$$\sum_{p \equiv b_j \pmod{q^{r+1}}} p^{-s} = \left\{ \phi(q^{r+1}) \right\}^{-1} \left\{ \log\left\{(1-q^{-s})\,\zeta(s)\right\} + \sum_{\chi \neq \chi_0} \frac{\log L(s,\chi)}{\chi(b_j)} + \sum_{\chi} \frac{G(s,\chi)}{\chi(b_j)} \right\}. \tag{20}$$

This together with (18) gives the result of the lemma.

The next lemma is proved by Rankin [2] (in the paragraphs containing equations (12) to (14)).

<u>Lemma 11.</u> <u>Let g be a primitive root (mod q) and let $\chi(n)$ be the character defined by</u>

$$\chi(n) = e(\beta/h) \quad \text{for} \quad n \equiv g^{\beta} \pmod{q},$$

<u>where $e(z)=\exp(2\pi i z)$. Then</u>

$$f(s) = \zeta(s)\left\{F(s)\right\}^{1/h}\psi(s),$$

<u>where $\psi(s)$ is holomorphic for $\sigma>\frac{1}{2}$ and bounded for $\sigma\geq\frac{1}{2}+\delta$ for any</u> $\delta>0$, <u>and where</u>

$$F(s) = \prod_{r=1}^{h}\left\{L(s,\chi^r)/L(s,\chi^{2r})\right\} = \prod_{p|h}(1-p^{-s})^{-1}\left\{\zeta(s)\right\}^{-1}\prod_{r=1}^{h-1}L(s,\chi^r)\left\{\prod_{r=1}^{\frac{1}{2}h-1}L(s,\chi^{2r})\right\}^{-2}.$$

Proof. Suppose that $p\neq q$ and $h \nmid c_\rho$; then $\mu_\rho = h/(h,c_\rho)$. It follows that

$$(c_\rho, \tfrac{1}{2}h) = \begin{cases} \tfrac{1}{2}(c_\rho, h) & \text{if } \mu_\rho \text{ is odd} \\ (c_\rho, h) & \text{if } \mu_\rho \text{ is even.} \end{cases}$$

Hence, if $c_\rho' = c_\rho/(c_\rho, h)$, so that $(c_\rho', \mu_\rho)=1$,

$$\prod_{r=1}^{h}\left\{\frac{(1-\chi^r(p)p^{-s})}{(1-\chi^{2r}(p)p^{-s})}\right\} = \prod_{r=1}^{h}\left\{\frac{(1-p^{-s}e(c_\rho r/h))}{(1-p^{-s}e(2c_\rho r/h))}\right\} = \prod_{r=1}^{\mu_\rho}\left\{\frac{(1-p^{-s}e(c_\rho' r/\mu_\rho))}{(1-p^{-s}e(2c_\rho' r/\mu_\rho))}\right\}^{h/\mu_\rho}.$$

If μ_ρ is odd, this last expression equals 1. However if μ_ρ is even, this expression equals

$$\left\{\frac{\prod_{r=1}^{\mu_\rho}(1-p^{-s}e(c_\rho' r/\mu_\rho))}{\prod_{r=1}^{\frac{1}{2}\mu_\rho}(1-p^{-s}e(c_\rho' r/\tfrac{1}{2}\mu_\rho))^2}\right\}^{h/\mu_\rho} = \left\{\frac{(1-p^{-\mu_\rho s})}{(1-p^{-\frac{1}{2}\mu_\rho s})^2}\right\}^{h/\mu_\rho}.$$

It follows that

$$F(s) = \prod_{p\neq q}\left\{\prod_{r=1}^{h}\left\{\frac{(1-\chi^r(p)p^{-s})}{(1-\chi^{2r}(p)p^{-s})}\right\}\right\}^{-1} = \prod_{\substack{p\neq q\\ \mu_\rho \text{ even}}}\left\{\frac{(1-p^{-\frac{1}{2}\mu_\rho s})^2}{(1-p^{-\mu_\rho s})}\right\}^{h/\mu_\rho}.$$

By Lemma 6,

$$f(s) = \zeta(s)\{F(s)\}^{1/h}\,\frac{\prod_{p\neq q,\,\mu_\rho\neq 2}(1-p^{-(\mu_\rho-1)s})\prod_{p\neq q,\,\mu_\rho\text{ even}}(1-p^{-\mu_\rho s})^{1/\mu_\rho}}{\prod_{p\neq q}(1-p^{-\mu_\rho s})\prod_{\substack{p\neq q,\,\mu_\rho\neq 2\\ \mu_\rho\text{ even}}}(1-p^{-\frac{1}{2}\mu_\rho s})^{2/\mu_\rho}}$$

$$= \zeta(s)\{F(s)\}^{1/h}\psi(s),$$

say. Since $\psi(s)$ is an infinite product of factors of the type $(1-p^{-us})$ with $u\geq 2$, $\psi(s)$ is holomorphic for $\sigma>\frac{1}{2}$ and bounded for $\sigma\geq\frac{1}{2}+\delta$.

To complete the proof of the Lemma, we observe that, since χ is a character (mod h),

$$F(s) = \prod_{r=1}^{h-1} L(s,\chi^r) \Big\{ \prod_{\substack{r=1 \\ r \neq \frac{1}{2}h}}^{h-1} L(s,\chi^{2r}) \Big\}^{-1} \{L(s,\chi^{\lambda h})\}^{-1}$$

$$= \prod_{p|h} (1-p^{-s})^{-1} \{\zeta(s)\}^{-1} \prod_{r=1}^{h-1} L(s,\chi^r) \Big\{ \prod_{r=1}^{\frac{1}{2}h-1} L(s,\chi^{2r}) \Big\}^{-2}$$

by the result giving (19).

We are now able to complete the proof of Theorem 2(ii) when $m' \geq 1$.

From (17) and Lemmas 8 and 10 we obtain

$$f_m(s) = f(s) \sum_{\substack{r_1+\cdots+r_k=m \\ r_i \geq \gamma+1 \ (1\leq i\leq k)}} \{\rho^*(r_1,\ldots,r_k)\}^{-1} \prod_{i=1}^{k} \Big\{ (\nu, q-1) q^{\gamma-r_i} \{\log \zeta(s) + \log(1-q^{-s})\}$$

$$+ q^{-r_i} (q-1)^{-1} \sum_{j=1}^{\phi(q^{\gamma+1})(\nu,q-1)} \Big\{ \sum_{\chi^{(r_i)} \neq \chi_0^{(r_i)}} \frac{\log L(s,\chi^{(r_i)})}{\chi^{(r_i)}(b_j)} + \sum_{\chi^{(r_i)}} \frac{G(s,\chi^{(r_i)})}{\chi^{(r_i)}(b_j)} \Big\} + \psi_i(s) \Big\} \qquad (21)$$

where $\chi^{(r_i)}$ is a character (mod q^{r_i+1}), and otherwise the notation is the same as before. Clearly the term on the right containing the highest power of $\log \zeta(s)$ will occur when the product contains its maximum number of terms, so that k takes its maximum value. Now k will be greatest when the r_i are as small as possible and this will occur when the r_i are as near to the value $\gamma+1$ as possible; however the r_i cannot all equal $\gamma+1$ unless $(\gamma+1)|m$. Thus the maximum value of k is

$$[m/(\gamma+1)] = m',$$

and in this case

$$r_i = \gamma+1+r_i', \quad i=1,2,\ldots,m',$$

where

$$0 \le r_i' \le m-m'(\gamma+1) < \gamma+1 \quad \text{and} \quad \sum_{i=1}^{m'} r_i' = m-m'(\gamma+1).$$

The r_i' can be chosen in $\hat{\rho}(m,\gamma)$ ways, where $\hat{\rho}(m,\gamma)$ is the number of unrestricted partitions of $m-m'(\gamma+1)$ into at most m' parts. Hence the number of ways of choosing the r_i when k takes its maximum value is $\hat{\rho}(m,\gamma)$. This means that the term on the right of (21) which contains the highest power of $\log \zeta(s)$ is

$$f(s)\rho(m,\gamma)(\nu,q-1)^{m'} q^{m'\gamma-m} \{\log \zeta(s)\}^{m'}, \tag{22}$$

where

$$\rho(m,\gamma) = \sum_{r_1'+r_2'+\cdots+r_{m'}'=m-m'(\gamma+1)} \{\rho^*(\gamma+1+r_1', \gamma+1+r_2', \ldots, \gamma+1+r_{m'}')\}^{-1},$$

the sum having $\hat{\rho}(m,\gamma)$ terms. The remaining terms will be of the form

$$f(s)\{\log \zeta(s)\}^u \prod_{i=1}^{v} \{\log L(s,\chi^{(r_i)})\ \eta_i(s)\} \tag{23}$$

where $0 \le u < m'$, $0 \le v \le m'-u$, and $1 \le r_i \le m-u(\gamma+1)$, where the r_i are not necessarily all distinct, and $\chi^{(r_i)}$ is a non-principal character (mod q^{r_i+1}), and where $\eta_i(s)$, a function of s and the characters occurring in (21), is holomorphic for $\sigma > \frac{1}{2}$ and bounded for $\sigma \ge \frac{1}{2}+\delta$ for any $\delta > 0$. Hence from Lemma 11 and equations (21) to (23) we obtain

$$f_m(s) = \{\zeta(s)\}^{1-1/h} \sum_{u=0}^{m'} \{\log \zeta(s)\}^u H_u(s),$$

where $H_u(s)$, $0 \le u \le m'$, satisfies the conditions of Theorem 2(ii), and $H_{m'}(s)$ can be obtained from (22) and Lemma 11.

7. Proof of Theorem 2(ii) for m'=0.

We assume now that q is odd, h is even and $m'=0$. The last condition means that $m \leq \gamma$, so that $q^m | \nu$. If $r \leq m \leq \gamma$, then by (10) and Lemma 3(i),

$$\mu_p^{(r)} = 2 \qquad \text{and} \qquad \mu_p^{(r+1)} = q\mu_p^{(r)}$$

cannot both hold. For if $p^\nu \not\equiv 1 \pmod{q}$, $\mu_p^{(r)} = \lambda_p^{(r)} \mu_p = \mu_p$ for $r \leq \gamma + 1$. On the other hand, if $p^\nu \equiv 1 \pmod{q}$, $\mu_p^{(r)} = q^r \geq q \geq 3$. Hence, as in the case when both q and h are odd,

$$\sum_{\substack{p_i \neq q, p_1, \dots, p_{i-1} \\ \mu_{p_i}^{(r_i+1)} = q\mu_{p_i}^{(r_i)}}} P(p_i, \mu_{p_i}^{(r_i)}; s)$$

is absolutely convergent for $\sigma > \frac{1}{2}$ when $r_i \leq m$ by the arguments which led to (15).

Thus by Lemma 8

$$f_m(s) = f(s)\,\eta(s),$$

where $\eta(s)$ is holomorphic for $\sigma > \frac{1}{2}$ and bounded for $\sigma \geq \frac{1}{2} + \delta$. Since h is even there exist primes p for which $\mu_p = 2$, and hence by Lemma 11

$$f_m(s) = \{\zeta(s)\}^{1-1/h} H_0(s),$$

where $H_0(s)$ satisfies the conditions of Theorem 2(ii).

8. Proof of Theorem 2(iii).

We assume now that $q=2$. From Lemma 3(ii) we observe that

$$\mu_p^{(r)} = 2 \quad \text{and} \quad \mu_p^{(r+1)} = 2\mu_p^{(r)}$$

do not both hold unless either $r=1$ and either ν is even or ν is odd and $p\equiv 1 \pmod 4$, or $r=2$, ν is odd and $p\equiv 3 \pmod 8$, or $r\geq 3$, ν is odd and $p\equiv 2^r-1 \pmod{2^{r+1}}$; thus for every odd prime p there is exactly one value of r for which $\mu_p^{(r)}=2$ and $\mu_p^{(r+1)}=2\mu_p^{(r)}$.
When $q=2$ and $\mu_p^{(r)}=2$, (16) still holds, and hence by the arguments used in §5 and §6 it follows that

$$\sum_{\substack{p_i\neq 2, p_1, \ldots, p_{i-1} \\ \mu_{p_i}^{(r_i+1)} = 2\mu_{p_i}^{(r_i)}}} P(p_i, \mu_{p_i}^{(r_i)}; s) = \sum_{\substack{p\neq 2 \\ \mu_p^{(r_i+1)}=2\mu_p^{(r_i)} \\ \mu_p^{(r_i)}=2}} p^{-s} + \psi_i(s) \tag{24}$$

where $\psi_i(s)$ is holomorphic for $\sigma>\frac{1}{2}$ and bounded for $\sigma\geq\frac{1}{2}+\delta$ for any $\delta>0$. We observe from Lemma 3(ii) that the sum on the left of (24) is never empty; this is true even when the sum on the right of (24) is empty. From the remarks at the beginning of this section it follows that

$$\sum_{\substack{p\neq 2 \\ \mu_p^{(r_i+1)}=2\mu_p^{(r_i)} \\ \mu_p^{(r_i)}=2}} p^{-s} = \begin{cases} \sum\limits_{p\neq 2} p^{-s} & \text{if } \nu \text{ is even and } r_i=1 \\ 0 & \text{if } \nu \text{ is even and } r_i>1 \\ \sum\limits_{p\equiv 2^{r_i}-1 \pmod{2^{r_i+1}}} p^{-s} & \text{if } \nu \text{ is odd and } r_i\geq 1. \end{cases} \tag{25}$$

Suppose first that ν is even. Now

$$\sum_{p\neq 2} p^{-s} = \log\zeta(s) - \sum_p \sum_{u\geq 2}^{\infty} \frac{1}{up^{us}} - 2^{-s} = \log\zeta(s) + G(s),$$

where $G(s)$ is holomorphic for $\sigma>\frac{1}{2}$ and bounded for $\sigma\geq\frac{1}{2}+\delta$ for any $\delta>0$ (see section 4, §12 of Hasse [6]). Hence we obtain from (24), (25) and Lemmas 6 and 8 that

$$f_m(s) = (1+2^{-s})\zeta(2s) \sum_{\substack{r_1+\cdots+r_k=m \\ r_1=\cdots=r_j=1 \\ r_i>1\ (j+1\le i\le k)}} \Big\{\{\rho^*(r_1,\ldots,r_k)\}^{-1} \prod_{i=1}^{j}(\log\zeta(s) + G(s) + \psi_i(s)) \prod_{i=j+1}^{k}\psi_i(s)\Big\}$$

$$= \sum_{u=0}^{m}\{\log\zeta(s)\}^u H_u(s),$$

where $H_m(s)= \frac{1}{m!}(1+2^{-s})\zeta(2s)$ and where $H_u(s)$, $0\le u\le m$, satisfies the conditions of Theorem 2(iii); in fact here each $H_u(s)$ is holomorphic for $\sigma>\frac{1}{2}$ and bounded for $\sigma\ge\frac{1}{2}+\delta$ and does not involve any L-functions.

Suppose next that ν is odd. Then by (20)

$$\sum_{p\equiv 2^r-1 \pmod{2^{r+1}}} p^{-s} = 2^{-r}\Big\{\log\zeta(s) + \log(1-2^{-s}) + \sum_{\chi\ne\chi_0}\frac{\log L(s,\chi)}{\chi(2^r-1)} + \sum_{\chi}\frac{G(s,\chi)}{\chi(2^r-1)}\Big\} \tag{26}$$

where χ runs through all characters (mod 2^{r+1}) except, when indicated, χ_0. Hence from (24), (25), (26) and Lemmas 6 and 8

$$f_m(s) = (1+2^{-s})\zeta(2s) \sum_{r_1+\cdots+r_k=m}\{\rho^*(r_1,\ldots,r_k)\}^{-1}\prod_{i=1}^{k}\Big[2^{-r_i}\Big\{\log\zeta(s) + \log(1-2^{-s}) + \sum_{\chi^{(r_i)}\ne\chi_0^{(r_i)}}\frac{\log L(s,\chi^{(r_i)})}{\chi^{(r_i)}(2^{r_i}-1)} + \sum_{\chi^{(r_i)}}\frac{G(s,\chi^{(r_i)})}{\chi^{(r_i)}(2^{r_i}-1)}\Big\} + \psi_i(s)\Big\}$$

where $\chi^{(r_i)}$ is a character (mod 2^{r_i+1}). The argument to be used now is similar to that used in §6. The maximum value of k is m which occurs when $r_1=r_2=\ldots=r_k=1$, and thus the highest power of $\log\zeta(s)$ appearing on the right is $\{\log\zeta(s)\}^m$. Hence

$$f_m(s) = \sum_{u=0}^{m}\{\log\zeta(s)\}^u H_u(s),$$

where $H_m(s)=\frac{1}{m!}2^{-m}(1+2^{-s})\zeta(2s)$ and $H_u(s)$ satisfies the conditions of Theorem 2(iii).

CHAPTER 3.

1. Introduction.

In chapter 1 we defined $h(s)$ to be a function which can be expressed both as an infinite sum of the form

$$h(s) = \sum_{n=1}^{\infty} b(n)n^{-s},$$

where $b(n)\geqslant 0$, and as a product of the form

$$h(s) = \{\zeta(s)\}^{1-\beta}\{\log \zeta(s)\}^{u} H(s),$$

where $0<\beta\leqslant 1$, u is a non-negative integer, and $H(s)$ is a product of powers of Dirichlet L-functions associated with non-principal characters, non-negative powers of the logarithms of such functions, and a function holomorphic for $\sigma>\frac{1}{2}$ and bounded for $\sigma\geqslant\frac{1}{2}+\delta$ for any $\delta>0$. More precisely, we can write $H(s)$ in the form

$$H(s) = \prod_{i=1}^{\lambda_1}\{\log L(s, \chi_1^{(i)})\}^{v_i}\prod_{i=1}^{\lambda_3}\{L(s, \chi_2^{(i)})\}^{w_i}\psi(s),$$

where the v_i, $i=1,2,\ldots,\lambda_1$, are non-negative integers, the w_i, $i=1,2,\ldots,\lambda_2$, are positive integers, the w_i, $i=\lambda_2+1,\ \lambda_2+2,\ \ldots,\ \lambda_3$, are negative integers, where the $\chi_j^{(i)}$, for $j=1,2$ and all i, are non-principal characters (mod $k_j^{(i)}$), and where $\psi(s)$ is holomorphic for $\sigma>\frac{1}{2}$ and bounded for $\sigma\geqslant\frac{1}{2}+\delta$ for all $\delta>0$.

The object of this chapter is to obtain an estimate for

$$\sum_{n=1}^{x} b(n),$$

where x is a large positive integer, in terms of x and constants. The result to be obtained has been stated in Theorem 3 of chapter 1.

The method used to prove this theorem follows in principle one of the methods used to prove the Prime Number Theorem (given, for example, in Landau [7]). Briefly, we find a certain contour Γ inside and on which $h(s)$ is holomorphic, and we integrate the function $x^s h(s) s^{-2}$ round this contour (see §3). In this way we are able to obtain an estimate for

$$\sum_{n=1}^{x} b(n)\log(x/n),$$

given in Lemma 8, and then to deduce the required estimate for $\sum_{n=1}^{x} b(n)$ (see §4).

2. Preliminary lemmas.

In the following two lemmas $c_1, c_2, \ldots$ denote positive constants; in Lemma 2 these constants depend on the non-principal character χ occurring in the statement of the lemma. In these lemmas, we shall state some properties of $\zeta(s)$ and of $L(s, \chi)$ which we shall need to use in order to determine the behaviour of $h(s)$. These properties are all proved in Landau [7].

Lemma 1.

(i) $\zeta(s) - (s-1)^{-1}$ is holomorphic for $\sigma > 0$.

(ii) There exists c_1 such that $\zeta(s) \neq 0$ for $|t| \geq 3$, $\sigma \geq 1 - c_1\{\log |t|\}^{-9}$, and for $|t| \leq 3$, $\sigma \geq 1 - c_1\{\log 3\}^{-9}$.

(iii) There exists c_2 such that

$$|\zeta(s)| < c_2 \log |t|$$

for $|t| \geq 3$, $\sigma \geq 1 - \{\log |t|\}^{-1}$; and c_3 such that

$$|\log \zeta(s)| < c_3 \{\log |t|\}^9$$

for $|t| \geq 3$, $\sigma \geq 1 - c_1 \{\log |t|\}^{-9}$.

(iv) <u>There exist c_4, c_5 and c_6 such that</u>

$$|\zeta(s)| < c_4 \quad \text{and} \quad |\log \zeta(s)| < c_5$$

<u>for</u> $|t| \leq 3, \quad 1-c_1\{\log 3\}^{-9} \leq \sigma \leq 1-c_6 < 1.$

The properties given in parts (i), (ii) and (iii) above are contained in §42 and §43, §46 to §48, and §64 of Landau [7]; part (iv) follows immediately from the rest of the lemma.

<u>Lemma 2.</u> <u>Let χ be a non-principal character (mod k); then:</u>

(i) $L(s, \chi)$ <u>is holomorphic for $\sigma > 0$.</u>

(ii) <u>There exist c_7, c_8, c_9, and c_{10} such that</u>

$$|L(s, \chi)| < c_7 \log |t|$$

<u>for</u> $|t| \geq 3, \quad \sigma \geq 1 - \{\log |t|\}^{-1}$, <u>and</u>

$$|L(s, \chi)| > c_8\{\log |t|\}^{-5} \quad \text{and} \quad |\log L(s, \chi)| < c_9\{\log |t|\}^{7}$$

<u>for</u> $|t| \geq 3, \quad \sigma \geq 1-c_{10}\{\log |t|\}^{-7}$.

(iii) <u>There exist c_{11}, c_{12} and c_{13} such that</u>

$$0 < c_{11} < |L(s, \chi)| < c_{12} \quad \text{and} \quad |\log L(s, \chi)| < c_{13}$$

<u>for</u> $|t| \leq 3, \quad 1-c_{10}\{\log 3\}^{-7} \leq \sigma \leq 1.$

With the exception of the bound for $|\log L(s, \chi)|$, the properties given in parts (i) and (ii) above are contained in §114 and §116 to §117 of Landau [7]. The bound for $|\log L(s, \chi)|$ can be deduced from that of $|L'(s, \chi)/L(s, \chi)|$ (which is $c_{14}\{\log |t|\}^{7}$, as is given in §117 of Landau [7]) in the same way as the bound for $|\log \zeta(s)|$ is deduced from that of $|\zeta'(s)/\zeta(s)|$ (see §64 of Landau [7]). Thus if $|t| \geq 3, \; 1-c_{10}\{\log |t|\}^{-7} \leq \sigma \leq 2,$

$$|\log L(s,\chi) - \log L(2+it,\chi)| = \left|\int_{2+it}^{s} \frac{L'(z,\chi)}{L(z,\chi)}\, dz\right| < 2c_{14}\{\log|t|\}^7,$$

so that

$$|\log L(s,\chi)| < \log \zeta(2) + 2c_{14}\{\log|t|\}^7 ;$$

if $|t|\geq 3$, $\sigma \geq 2$,

$$|\log L(s,\chi)| = \left|\sum_{m,p} \frac{\chi(p^m)}{mp^{ms}}\right| < \sum_{m,p} (mp^{2m})^{-1} = \log \zeta(2).$$

Hence for $|t|\geq 3$, $\sigma \geq 1-c_{10}\{\log|t|\}^{-7}$,

$$|\log L(s,\chi)| < c_9\{\log|t|\}^7 .$$

Part (iii) of thelemma follows immediately from the rest of the lemma. We observe that the lower bound for $|L(s,\chi)|$ implies that $L(s,\chi)\neq 0$.

The next lemma is an immediate consequence of Lemmas 1 and 2 and the definition of h(s). We observe that, if $\sigma\geq\frac{1}{2}+\delta$, for any $\delta>0$, $|\psi(s)|<c_{15}$ (since $\psi(s)$ is bounded). For suitable positive constants d_1, d_2, d_3, (the actual values depending on the constants of the previous two lemmas and the definition of h(s)), we have

Lemma 3.

(i) The function h(s) is holomorphic for $|t|\geq 3$, $\sigma\geq 1-d_1\{\log|t|\}^{-9}$, and for $|t|\leq 3$, $\sigma\geq 1-d_1\{\log 3\}^{-9}$ except for a singularity at s=1.

(ii) $|h(s)| < d_2\{\log|t|\}^{\kappa}$

for $|t|\geq 3$, $\sigma\geq 1-d_1\{\log|t|\}^{-9}$, where $\kappa>0$, and

$$|h(s)| < d_3$$

for $|t|\leq 3$, $\sigma=1-d_1\{\log 3\}^{-9}$.

It follows from Lemmas 1 and 2 and the definition of h(s) that we may take

$$K = (1-\beta) + 9u + \sum_{i=1}^{\lambda_1} 7 v_i + \sum_{i=1}^{\lambda_2} w_i + \sum_{i=\lambda_2+1}^{\lambda_3} 5|w_i| ,$$

and that the constants d_2 and d_3 are products of the constants c. The constants d_1 must be chosen so that all parts of Lemma 1 and, for all characters χ appearing in the definition of h(s), all parts of Lemma 2 are applicable in the corresponding regions of Lemma 3.

<u>Lemma 4</u>. <u>If</u> $|s-1| \leq d_1 \{\log 3\}^{-9}$,

$$h(s)s^{-2} - H(1)(s-1)^{\beta-1} \{-\log (s-1)\}^{u}$$

$$= \sum_{j=0}^{u} \binom{u}{j} \{-\log (s-1)\}^{u-j} (s-1)^{\beta} \sum_{k=1}^{\infty} \omega_{jk} (s-1)^{k-1} ,$$

<u>where the ω_{jk} are constants, and</u> $\sum_{k=1}^{\infty} \omega_{jk} (s-1)^{k-1}$ <u>is convergent.</u>

<u>Proof</u>. By Lemma 1(i), $(s-1)\zeta(s)$ is holomorphic for $\sigma > 0$, and

$$\lim_{s \to 1} (s-1)\zeta(s) = 1; \tag{1}$$

also, by Lemma 1(ii), it is certainly true that if $|s-1| \leq d_1\{\log 3\}^{-9}$

$$\zeta(s) \neq 0.$$

Hence $K(s) = \log \zeta(s) + \log (s-1)$ is holomorphic when $|s-1| \leq d_1 \{\log 3\}^{-9}$, and

$$\lim_{s \to 1} K(s) = 0. \tag{2}$$

Now

$$h(s)s^{-2} = \{\zeta(s)\}^{1-\beta} \{\log \zeta(s)\}^{u} H(s)s^{-2},$$

where $H(s)s^{-2}$, being the product of a function holomorphic for $\sigma > \frac{1}{2}$ and bounded for $\sigma \geq \frac{1}{2} + \delta$, powers of L-functions associated with non-principal characters and positive powers of the logarithms of such L-functions, is holomorphic when $|s-1| < d_1 \{\log 3\}^{-9}$ by Lemma 2. We may write

$$h(s)s^{-2} = (s-1)^{\beta-1}\{-\log (s-1) + K(s)\}^u \{(s-1)\,\zeta(s)\}^{1-\beta} H(s)\, s^{-2}$$

$$= (s-1)^{\beta-1} \sum_{j=0}^{u} \binom{u}{j} \{-\log (s-1)\}^{u-j} \{K(s)\}^j \{(s-1)\,\zeta(s)\}^{1-\beta} H(s)s^{-2}.$$

For all j, $\{K(s)\}^j \{(s-1)\,\zeta(s)\}^{1-\beta} H(s)s^{-2}$ is holomorphic when $|s-1| < d_1 \{\log 3\}^{-9}$, and hence it can be expanded as a convergent power series of the form

$$\sum_{k=0}^{\infty} \omega_{jk}\,(s-1)^k.$$

From (1) and (2) we have that

$$\omega_{j0} = \lim_{s\to 1} \{\{K(s)\}^j \{(s-1)\,\zeta(s)\}^{1-\beta} H(s)s^{-2}\} = 0$$

for all $j \geq 1$, and that

$$\omega_{00} = \lim_{s\to 1} \{(s-1)\,\zeta(s)\}^{1-\beta} H(s)s^{-2} = H(1).$$

Thus we have shown that

$$h(s)s^{-2} = H(1)(s-1)^{\beta-1} \{-\log (s-1)\}^u$$

$$+ (s-1)^{\beta-1} \sum_{j=0}^{u} \binom{u}{j} \{-\log (s-1)\}^{u-j} \sum_{k=1}^{\infty} \omega_{jk}(s-1)^k,$$

and the result of the Lemma follows immediately.

3. An estimate for $\sum_{n=1}^{x} b(n)\log(x/n)$.

Lemma 5.

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} y^s s^{-2} ds = \begin{cases} 0 & \text{if } 0<y\le 1 \\ \log y & \text{if } y\ge 1. \end{cases}$$

This is proved in §49 of Landau [7].

Lemma 6.

$$\sum_{n=1}^{x} b(n)\log(x/n) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} x^s h(s) s^{-2} ds.$$

Proof.

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} x^s h(s) s^{-2} ds = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} s^{-2} \sum_{n=1}^{\infty} b(n)(x/n)^s ds$$

$$= \frac{1}{2\pi i} \sum_{n=1}^{\infty} b(n)\int_{2-i\infty}^{2+i\infty} (x/n)^s s^{-2} ds = \sum_{n=1}^{x} b(n)\log(x/n)$$

by Lemma 5.

Our next aim is to estimate the integral appearing on the right in Lemma 6. To do this we cut the complex plane along the real axis from the point s=1 to the left. Let Γ be the contour $\bar{A}ABCDE\bar{E}\bar{D}\bar{C}\bar{B}\bar{A}$, where the vertices above the real axis are defined by

$$A = 2 + ix^2, \quad B = 1 - d_1\{\log x^2\}^{-9} + ix^2,$$

$$C = 1 - d_1\{\log 3\}^{-9} + 3i, \quad D = 1 - d_1\{\log 3\}^{-9},$$

$$E = 1 - \delta$$

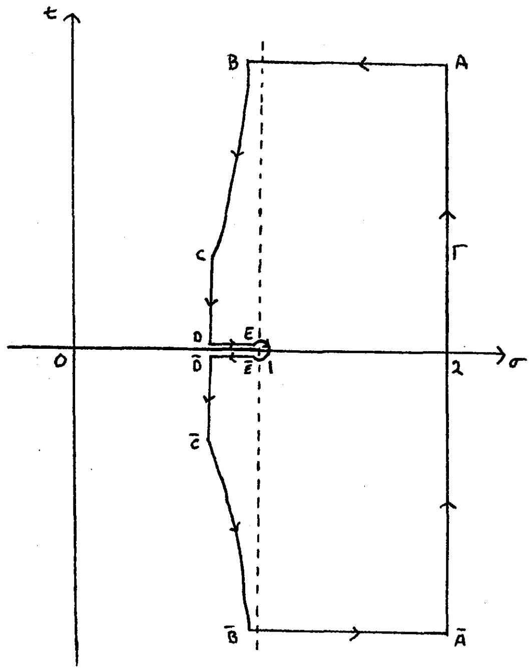
for a small, positive δ (which will tend to zero later), and where $\bar{A},\bar{B},\bar{C},\bar{D},\bar{E}$ are the complex conjugates of A,B,C,D,E; neighbouring vertices are joined by straight lines except for

B,C which are joined by the curve

$$\sigma = 1 - d_1\{\log t\}^{-9} \quad (3 \le t \le x^2),$$

$\bar{B},\bar{C}$ which are joined by the curve which is the image in the real axis of the curve BC, and E,$\bar{E}$ which are joined by the circle

$$|s-1| = \delta .$$



The constant d_1 has been chosen so that h(s) is holomorphic in the region bounded by Γ (see Lemma 3(i)); hence by Cauchy's Theorem

$$\int_\Gamma x^s h(s) s^{-2} ds = 0,$$

so that

$$\int_{\bar{A}A} x^s h(s) s^{-2} ds = -\int_{ABCDE\bar{E}\bar{D}\bar{C}\bar{B}\bar{A}} x^s h(s) s^{-2} ds. \tag{3}$$

<u>Lemma 7.</u>

$$\sum_{n=1}^{x} b(n)\log(x/n) = -\frac{1}{2\pi i}\left\{\int_{DE}+\int_{\bar{E}\bar{D}}\right\} x^s h(s) s^{-2} ds + O\left(xe^{-(\log x)^{\frac{1}{11}}}\right).$$

Proof. From Lemma 6 and (3) we obtain

$$\sum_{n=1}^{x} b(n)\log(x/n) = \frac{1}{2\pi i}\left(\int_{2-i\infty}^{2-ix^2} + \int_{\bar{A}A} + \int_{2+ix^2}^{2+i\infty}\right) x^{s}h(s)s^{-2}ds$$

$$= \frac{1}{2\pi i}\left(\int_{2-i\infty}^{2-ix^2} - \int_{ABCDE\bar{E}\bar{D}\bar{C}\bar{B}\bar{A}} + \int_{2+ix^2}^{2+i\infty}\right) x^{s}h(s)s^{-2}ds. \tag{4}$$

We now show that all the integrals on the right except those over DE and $\overline{ED}$ are sufficiently small in absolute value to be included in the error term of the lemma.

(i) By Lemma 3(ii),

$$\left|\int_{2+ix^2}^{2+i\infty} x^{s}h(s)s^{-2}ds\right| \leqslant \int_{x^2}^{\infty} x^2|h(s)||s^{-2}|dt < x^2\int_{x^2}^{\infty} d_2(\log t)^{K} t^{-2}dt$$

$$< x^2\int_{x^2}^{\infty} d_2 t^{-2+\varepsilon}dt = O(x^2x^{-2+2\varepsilon}) = O(x^{2\varepsilon})$$

for any $\varepsilon > 0$.

(ii) Since $|AB|<2$, we have by Lemma 3(ii)

$$\left|\int_{AB} x^{s}h(s)s^{-2}ds\right| < x^2 d_2 (\log x^2)^{K} x^{-4} 2 = O(x^{-2+\varepsilon})$$

for any $\varepsilon > 0$.

(iii) On BC, $\sigma = 1-d_1(\log t)^{-9}$ and $3 \leqslant t \leqslant x^2$; hence

$$\left|\int_{BC} x^{s}h(s)s^{-2}ds\right|$$

$$\leqslant \int_{3}^{x^2} \frac{x^{1-d_1(\log t)^{-9}}\,|h(1-d_1(\log t)^{-9}+it)|}{\{1-d_1(\log t)^{-9}\}^2+t^2}\,|9d_1(\log t)^{-10}t^{-1}+i|\,dt$$

$$= O\left(x\int_{3}^{x^2} x^{-d_1(\log t)^{-9}}(\log t)^{K} t^{-2}dt\right) = O\left(x(\log x)^{K}\int_{3}^{x^2} x^{-d_1(\log t)^{-9}} t^{-2}dt\right)$$

by Lemma 3(ii). Let $y = \exp\{(\log x)^{1/10}\}$; then

$$\int_3^{x^2} x^{-d_1} (\log t)^{-9} t^{-2} dt = \left(\int_3^y + \int_y^{x^2}\right) x^{-d_1} (\log t)^{-9} t^{-2} dt$$

$$= O\left(x^{-d_1} (\log y)^{-9}\right) + O\left(\int_y^{x^2} t^{-2} dt\right)$$

$$= O\left(\exp\{-d_1 (\log y)^{-9} \log x\}\right) + O\left(y^{-1}\right)$$

$$= O\left(e^{-d_1 (\log x)^{\frac{1}{10}}}\right) + O\left(e^{-(\log x)^{\frac{1}{10}}}\right).$$

Therefore

$$\left|\int_{BC} x^s h(s) s^{-2} ds\right| = O\left(x (\log x)^{\kappa} e^{-d_1 (\log x)^{\frac{1}{10}}}\right) + O\left(x (\log x)^{\kappa} e^{-(\log x)^{\frac{1}{10}}}\right)$$

$$= O\left(x e^{-(\log x)^{\frac{1}{11}}}\right)$$

since

$$\lim_{x\to\infty}\left\{\frac{(\log x)^{\kappa} e^{-d_1 (\log x)^{\frac{1}{10}}}}{e^{-(\log x)^{1/11}}}\right\} = \lim_{\xi\to\infty}\left\{\frac{\xi^{\kappa} e^{-d_1 \xi^{\frac{1}{10}}}}{e^{-\xi^{1/11}}}\right\} = \lim_{\eta\to\infty}\left\{\frac{\eta^{110\kappa} e^{\eta^{10}}}{e^{d_1 \eta^{11}}}\right\} = 0,$$

and similarly

$$\lim_{x\to\infty}\left\{\frac{(\log x)^{\kappa} e^{-(\log x)^{\frac{1}{10}}}}{e^{-(\log x)^{1/11}}}\right\} = 0.$$

(iv) By Lemma 3(ii)

$$\left|\int_{CO} x^s h(s) s^{-2} ds\right| = O\left(x^{1-d_1} (\log 3)^{-9}\right).$$

(v) By Lemma 4

$$\left|\int_{\varepsilon\bar{\varepsilon}} x^s h(s) s^{-2} ds\right| = O\left(x^{1+\delta} |(\log \delta)^u| \delta^{\beta-1} 2\pi\delta\right) = O\left(x^{1+\delta} |(\log \delta)^u| \delta^{\beta}\right);$$

since $\beta > 0$,

$$\lim_{\delta\to 0}\left\{x^{1+\delta} |(\log \delta)^u| \delta^{\beta}\right\} = 0,$$

and thus

$$\lim_{\delta\to 0}\left\{\int_{\varepsilon\bar{\varepsilon}} x^s h(s) s^{-2} ds\right\} = 0.$$

by Lemma 3(ii). Let $y = \exp\{(\log x)^{1/10}\}$; then

$$\int_3^{x^2} x^{-d_1} (\log t)^{-9} t^{-2} dt = \left(\int_3^{y} + \int_y^{x^2}\right) x^{-d_1} (\log t)^{-9} t^{-2} dt$$

$$= O\left(x^{-d_1} (\log y)^{-9}\right) + O\left(\int_y^{x^2} t^{-2} dt\right)$$

$$= O\left(\exp\{-d_1 (\log y)^{-9} \log x\}\right) + O\left(y^{-1}\right)$$

$$= O\left(e^{-d_1 (\log x)^{\frac{1}{10}}}\right) + O\left(e^{-(\log x)^{\frac{1}{10}}}\right).$$

Therefore

$$\left|\int_{BC} x^s h(s) s^{-2} ds\right| = O\left(x(\log x)^{\kappa} e^{-d_1 (\log x)^{\frac{1}{10}}}\right) + O\left(x(\log x)^{\kappa} e^{-(\log x)^{\frac{1}{10}}}\right)$$

$$= O\left(x e^{-(\log x)^{\frac{1}{11}}}\right)$$

since

$$\lim_{x\to\infty} \left\{\frac{(\log x)^{\kappa} e^{-d_1 (\log x)^{\frac{1}{10}}}}{e^{-(\log x)^{1/11}}}\right\} = \lim_{\xi\to\infty}\left\{\frac{\xi^{\kappa} e^{-d_1 \xi^{\frac{1}{10}}}}{e^{-\xi^{1/11}}}\right\} = \lim_{\eta\to\infty}\left\{\frac{\eta^{110\kappa} e^{\eta^{10}}}{e^{d_1 \eta^{11}}}\right\} = 0,$$

and similarly

$$\lim_{x\to\infty}\left\{\frac{(\log x)^{\kappa} e^{-(\log x)^{\frac{1}{10}}}}{e^{-(\log x)^{1/11}}}\right\} = 0.$$

(iv) By Lemma 3(ii)

$$\left|\int_{CO} x^s h(s) s^{-2} ds\right| = O\left(x^{1-d_1} (\log 3)^{-9}\right).$$

(v) By Lemma 4

$$\left|\int_{\varepsilon\bar{\varepsilon}} x^s h(s) s^{-2} ds\right| = O\left(x^{1+\delta} |(\log \delta)^u| \delta^{\beta-1} 2\pi\delta\right) = O\left(x^{1+\delta} |(\log \delta)^u| \delta^{\beta}\right);$$

since $\beta > 0$,

$$\lim_{\delta\to 0}\left\{x^{1+\delta} |(\log \delta)^u| \delta^{\beta}\right\} = 0,$$

and thus

$$\lim_{\delta\to 0}\left\{\int_{\varepsilon\bar{\varepsilon}} x^s h(s) s^{-2} ds\right\} = 0.$$

By symmetry the bounds for the integrals along curves in the lower half plane are the same as the bounds for the corresponding integral (i), (ii), (iii) or (iv) in the upper half plane. Hence by (4) and (i) to (v) we have

$$\sum_{n=1}^{x} b(n)\log(x/n) = -\frac{1}{2\pi i}\left\{\int_{DE} + \int_{\bar{E}\bar{D}}\right\} x^s h(s) s^{-2} ds + O\left(xe^{-(\log x)^{\frac{1}{11}}}\right)$$

where in the integrals along DE and $\bar{E}\bar{D}$ we assume that δ has tended to zero.

<u>Lemma 8.</u> <u>Let</u> $\sum_1(x) = \sum_{n=1}^{x} b(n)\log(x/n)$.

(i) <u>If $0<\beta<1$ and $u\geq 1$, then</u>

$$\sum_1(x) = \frac{H(1)}{\Gamma(1-\beta)} \frac{x(\log\log x)^u}{(\log x)^\beta} + O\left(\frac{x(\log\log x)^{u-1}}{(\log x)^\beta}\right).$$

(ii) <u>If $0<\beta<1$ and $u=0$, then</u>

$$\sum_1(x) = \frac{H(1)}{\Gamma(1-\beta)} \frac{x}{(\log x)^\beta} + O\left(\frac{x}{(\log x)}\right).$$

(iii) <u>If $\beta=1$ and $u\geq 2$, then</u>

$$\sum_1(x) = uH(1)\frac{x(\log\log x)^{u-1}}{\log x} + O\left(\frac{x(\log\log x)^{u-2}}{\log x}\right).$$

(iv) <u>If $\beta=1$ and $u=1$, then</u>

$$\sum_1(x) = H(1)\frac{x}{\log x} + O\left(\frac{x\log\log x}{(\log x)^2}\right).$$

(v) <u>If $\beta=1$ and $u=0$, then</u>

$$\sum_1(x) = O\left(\frac{x}{(\log x)^2}\right).$$

<u>Proof</u>. Let $\theta=1-d_1(\log 3)^{-9}$. Suppose first that β and u satisfy the conditions of (i), (ii) or (iii). If $\theta\leq s\leq 1$, then by Lemma 4

$$
\begin{aligned}
&|x^s h(s)s^{-2} - H(1)x^s\{-\log(s-1)\}^u(s-1)^{\beta-1}| \\
&= |x^s \sum_{j=0}^{u}\binom{u}{j}\{-\log(s-1)\}^{u-j}(s-1)^{\beta}\sum_{k=1}^{\infty}\omega_{jk}(s-1)^{k-1}| \\
&= O\left(x^s \sum_{j=0}^{u} |\{\log(s-1)\}^{u-j}(s-1)^{\beta}|\,|\sum_{k=1}^{\infty}\omega_{jk}(s-1)^{k-1}|\right) \\
&= O\left(x^s \sum_{j=0}^{u} |\{\log(s-1)\}^{u-j}(s-1)^{\beta}|\right)
\end{aligned}
$$

since $\sum_{k=1}^{\infty}\omega_{jk}(s-1)^{k-1}$ is convergent. When $\theta\leq s\leq 1$, $|\{\log(s-1)\}^{u-j}(s-1)^{\beta}|=O(1)$ since $\beta>0$; hence

$$\int_{\theta}^{1} x^s \sum_{j=0}^{u} |\{\log(s-1)\}^{u-j}(s-1)^{\beta}|ds = O\left(\int_{\theta}^{1} x^s ds\right) = O(x/\log x).$$

Hence

$$
\begin{aligned}
\left\{\int_{D\varepsilon}+\int_{\bar{\varepsilon}\bar{D}}\right\} x^s h(s)s^{-2}ds = &\int_{\theta}^{1} H(1)x^{s^+}\{-\log(s^+-1)\}^u(s^+-1)^{\beta-1}ds^+ \\
&- \int_{\theta}^{1} H(1)x^{s^-}\{-\log(s^- -1)\}^u(s^- -1)^{\beta-1}ds^- + O(x/\log x),
\end{aligned}
$$

where s^+ indicates the upper edge and s^- indicates the lower edge of the cut. Now $(s^+-1) = (1-s^+)e^{\pi i}$ and $(s^- -1) = (s^+-1)e^{-2\pi i}$ $= (1-s^+)e^{-\pi i}$; it follows that

$$
\begin{aligned}
\left\{\int_{D\varepsilon}+\int_{\bar{\varepsilon}\bar{D}}\right\} x^s h(s)s^{-2}ds = H(1)\int_{\theta}^{1} x^{s^+}(1-s^+)^{\beta-1}\Big\{&\{-\log(1-s^+)-\pi i\}^u e^{\pi i(\beta-1)} \\
&-\{-\log(1-s^+)+\pi i\}^u e^{-\pi i(\beta-1)}\Big\}ds^+ + O\left(\frac{x}{\log x}\right)
\end{aligned}
$$

$$
\begin{aligned}
= H(1)\int_{\theta}^{1} x^s(1-s)^{\beta-1}\left\{\sum_{m=0}^{u}\binom{u}{m}\{-\log(1-s)\}^{u-m}\{(-\pi i)^m e^{\pi i(\beta-1)} - (\pi i)^m e^{-\pi i(\beta-1)}\}\right\}ds \\
+ O\left(\frac{x}{\log x}\right)
\end{aligned}
$$

$$= H(1)\sum_{m=0}^{u}\binom{u}{m}\left\{(-\pi i)^m e^{\pi i(\beta-1)} - (\pi i)^m e^{-\pi i(\beta-1)}\right\}\int_\theta^1 x^s(1-s)^{\beta-1}\{-\log(1-s)\}^{u-m}ds$$
$$+ O\left(\frac{x}{\log x}\right) \qquad (5)$$

on writing s for s^+.

Assume now that the conditions of case (i) are satisfied, so that $0<\beta<1$ and $u\geq 1$, and consider the integral

$$I = \int_\theta^1 x^s(1-s)^{\beta-1}\{-\log(1-s)\}^y ds$$

where $0\leq y\leq u$ and y is an integer. On using the substitution

$s=1-\dfrac{\eta}{\log x}$, we obtain

$$I = x(\log x)^{-\beta}\int_0^{(1-\theta)\log x} x^{-\eta/\log x}\eta^{\beta-1}\{\log\log x - \log\eta\}^y d\eta$$

$$= x(\log x)^{-\beta}\sum_{r=0}^{y}(-1)^r\binom{y}{r}(\log\log x)^{y-r}\int_0^{(1-\theta)\log x} e^{-\eta}\eta^{\beta-1}(\log\eta)^r d\eta.$$

Now

$$\int_0^{(1-\theta)\log x} e^{-\eta}\eta^{\beta-1}(\log\eta)^r d\eta = \int_0^\infty e^{-\eta}\eta^{\beta-1}(\log\eta)^r d\eta + O\left((\log x)^{\beta-1+\varepsilon}x^{-1+\theta}\right)$$

for any $\varepsilon>0$ satisfying $\varepsilon<1-\beta$. If $r=0$, the integral on the right is $\Gamma(\beta)$; for all r the integral on the right is absolutely convergent - when $r=0$ this is well known and it can be proved in a similar manner when $r>0$. It follows that

$$I = \Gamma(\beta)x(\log\log x)^y(\log x)^{-\beta} + O\left(x(\log\log x)^{y-1}(\log x)^{-\beta}\right) \qquad (6)$$

unless $y=0$, in which case the error term is $O\left(\dfrac{x^\theta}{\log x}\right)$. Hence by (5) we have

$$\left\{\int_{D\bar{E}}+\int_{\bar{E}\bar{D}}\right\} x^s h(s)s^{-2}ds = H(1)\Gamma(\beta)2i\ \sin\pi(\beta-1)\ \frac{x(\log\log x)^u}{(\log x)^\beta}+O\left(\frac{x(\log\log x)^{u-1}}{(\log x)^\beta}\right)$$

$$= \frac{-2\pi iH(1)}{\Gamma(1-\beta)}\ \frac{x(\log\log x)^u}{(\log x)^\beta} + O\left(\frac{x(\log\log x)^{u-1}}{(\log x)^\beta}\right)$$

since $\Gamma(\beta)\Gamma(1-\beta)=\pi/\sin\pi\beta$ (see, for example, 3·124 of Titchmarsh [8]). Part (i) of the lemma now follows from Lemma 7.

Similarly in case (ii), when $0<\beta<1$ and $u=0$, the integral I is given by

$$I = \int_\theta^1 x^s(1-s)^{\beta-1}ds = \Gamma(\beta)x(\log x)^{-\beta} + O(x^\theta/\log x).$$

As above it follows that

$$\left\{\int_{D\bar{E}}+\int_{\bar{E}\bar{D}}\right\} x^s h(s)s^{-2}ds = \frac{-2\pi iH(1)}{\Gamma(1-\beta)}\ \frac{x}{(\log x)^\beta} + O\left(\frac{x}{\log x}\right),$$

and on using Lemma 7 we obtain the result of Lemma 8(ii).

We turn now to case (iii), so that we assume that $\beta=1$ and $u\geq 2$. Then (5) becomes

$$\left\{\int_{D\bar{E}}+\int_{\bar{E}\bar{D}}\right\} x^s h(s)s^{-2}ds = H(1)\sum_{m=0}^{u}\binom{u}{m}\left\{(-\pi i)^m-(\pi i)^m\right\}\int_\theta^1 x^s\{-\log(1-s)\}^{u-m}ds$$

$$+ O(x/\log x);$$

We note that the term corresponding to $m=0$ is zero. Now $\Gamma(1)=1$, and hence by (6)

$$I = \int_\theta^1 x^s\{-\log(1-s)\}^y ds = \frac{x(\log\log x)^y}{\log x} + O\left(\frac{x(\log\log x)^{y-1}}{\log x}\right)$$

unless $y=0$, in which case

$$I = \frac{x-x^\theta}{\log x}.$$

Hence

$$\left\{\int_{0\bar{\epsilon}} + \int_{\bar{\epsilon}\bar{0}}\right\} x^s h(s) s^{-2} ds = -2\pi i u H(1) \frac{x(\log\log x)^{u-1}}{\log x} + O\left(\frac{x(\log\log x)^{u-2}}{\log x}\right),$$

and the result of part (iii) follows from Lemma 7.

If $\beta=1$ and $u=1$, so that we are considering case (iv), then by Lemma 4 we have that when $\theta \leq s \leq 1$

$$|x^s h(s) s^{-2} - H(1) x^s \{-\log(s-1)\}|$$

$$= |x^s (s-1) \{ \{-\log(s-1)\} \sum_{k=1}^{\infty} \omega_{0k} (s-1)^{k-1} + \sum_{k=1}^{\infty} \omega_{1k} (s-1)^{k-1} \}|$$

$$= O\left(x^s |s-1| \{ |-\log(s-1)| + 1\}\right)$$

since the two infinite sums are convergent. On putting $s=1-\eta/\log x$, we have

$$\int_{\theta}^{1} x^s |s-1| ds = \frac{x}{(\log x)^2} \int_{0}^{(1-\theta)\log x} e^{-\eta} \eta d\eta = O\left(\frac{x}{(\log x)^2}\right),$$

and

$$\int_{\theta}^{1} x^s |(s-1)\{-\log(s-1)\}| ds = \frac{x}{(\log x)^2} \int_{0}^{(1-\theta)\log x} e^{-\eta} \eta |\log\log x - \log(-\eta)| d\eta$$

$$= O\left(\frac{x \log\log x}{(\log x)^2}\right)$$

from above and since

$$\left| \int_{0}^{(1-\theta)\log x} e^{-\eta} \eta \log \eta \, d\eta \right| = O(1).$$

Hence

$$\left\{\int_{0\bar{\epsilon}} + \int_{\bar{\epsilon}\bar{0}}\right\} x^s h(s) s^{-2} ds = \int_{\theta}^{1} H(1) x^{s^+} \{-\log(s^+ - 1)\} ds^+$$

$$- \int_{\theta}^{1} H(1)x^{s^-}\{-\log(s^- - 1)\}\,ds^- + O\left(\frac{x\log\log x}{(\log x)^2}\right)$$

$$= H(1)\int_{\theta}^{1} x^{s^+}\{(-\log(1-s^+) - \pi i) - (-\log(1-s^+) + \pi i)\}\,ds^+$$

$$+ O\left(\frac{x\log\log x}{(\log x)^2}\right)$$

$$= -2\pi i H(1)\int_{\theta}^{1} x^s ds + O\left(\frac{x\log\log x}{(\log x)^2}\right)$$

$$= -2\pi i H(1)\frac{x}{\log x} + O\left(\frac{x\log\log x}{(\log x)^2}\right).$$

Part (iv) of the lemma now follows from Lemma 7.

If $\beta=1$ and $u=0$, so that we are considering part (v) of the lemma, $h(s)=H(s)$. Hence $h(s)$ is holomorphic inside the contour $\bar{A}ABCD\bar{D}\bar{C}\bar{B}\bar{A}$, where the complex plane is no longer cut so that $\bar{D}=D$, and where the rest of the contour is the same shape as the corresponding part of Γ. Integrating round this contour using the results of Lemma 7, we obtain

$$\sum_{n=1}^{x} b(n)\log(x/n) = O\left(xe^{-(\log x)^{\frac{1}{11}}}\right) = O\left(x/(\log x)^2\right),$$

which is part (v) of the lemma. This completes the proof of Lemma 8.

4. Proof of Theorem 3.

Lemma 9. Suppose that

$$\sum_{n=1}^{x} b(n)\log(x/n) = \frac{Bx^{\alpha}(\log\log x)^{\gamma}}{(\log x)^{\beta}} + O\left(\frac{x^{\alpha}(\log\log x)^{\gamma_1}}{(\log x)^{\beta_1}}\right),$$

where $b(n)\geq 0$, $B>0$, $\alpha>0$, $B,\alpha,\beta,\gamma,\beta_1,\gamma_1$, are non-negative constants, and where $\gamma_1<\gamma$ and $\beta\leq\beta_1<\beta+2$ or $\gamma_1\geq\gamma$ and $\beta<\beta_1\leq\beta+2$. Then

$$\sum_{n=1}^{x} b(n) = \frac{B\alpha x^{\alpha}(\log\log x)^{\gamma}}{(\log x)^{\beta}} + O\left(\frac{x^{\alpha}(\log\log x)^{\frac{1}{2}(\gamma+\gamma_1)}}{(\log x)^{\frac{1}{2}(\beta+\beta_1)}}\right).$$

Proof. We have already defined $\sum_1(x)=\sum_{n=1}^{x} b(n)\log(x/n)$; let $\sum_2(x)=\sum_{n=1}^{x} b(n)$. By hypothesis

$$\textstyle\sum_1(x) = \frac{Bx^{\alpha}(\log\log x)^{\gamma}}{(\log x)^{\beta}} + O\left(\frac{x^{\alpha}(\log\log x)^{\gamma_1}}{(\log x)^{\beta_1}}\right). \tag{7}$$

Let $\delta=\delta(x)=o(1)$ be a positive function of x to be chosen later.

Then

$$\textstyle\sum_1(x(1+\delta)) = \frac{Bx^{\alpha}(1+\delta)^{\alpha}(\log\log x(1+\delta))^{\gamma}}{(\log x(1+\delta))^{\beta}} + O\left(\frac{x^{\alpha}(1+\delta)^{\alpha}(\log\log x(1+\delta))^{\gamma_1}}{(\log x(1+\delta))^{\beta_1}}\right).$$

Now $\log x(1+\delta) = \log x+O(\delta)$, and

$$\log\log x(1+\delta) = \log\log x + \log\left\{1+\frac{\log(1+\delta)}{\log x}\right\} = \log\log x + O\left(\frac{\delta}{\log x}\right);$$

Hence

$$\textstyle\sum_1(x(1+\delta)) = B(1+\delta)^{\alpha}\frac{x^{\alpha}(\log\log x)^{\gamma}}{(\log x)^{\beta}}\left\{1 + O\left(\frac{\delta}{\log x\log\log x}\right) + O\left(\frac{\delta}{\log x}\right)\right\}$$

$$+ O\left(\frac{x^{\alpha}(\log\log x)^{\gamma_1}}{(\log x)^{\beta_1}}\right)$$

$$= \frac{Bx^{\alpha}(\log\log x)^{\gamma}}{(\log x)^{\beta}}\left\{1+\alpha\delta + O(\delta^2) + O\left(\frac{\delta}{\log x}\right)\right\}$$

$$+ O\left(\frac{x^{\alpha}(\log\log x)^{\gamma_1}}{(\log x)^{\beta_1}}\right). \tag{8}$$

where $b(n) \geq 0$, $B>0$, $\alpha>0$, $B, \alpha, \beta, \gamma, \beta_1, \gamma_1$, are non-negative constants, and where $\gamma_1 < \gamma$ and $\beta \leq \beta_1 < \beta+2$ or $\gamma_1 \geq \gamma$ and $\beta < \beta_1 \leq \beta+2$. Then

$$\sum_{n=1}^{x} b(n) = \frac{B\alpha x^{\alpha}(\log\log x)^{\gamma}}{(\log x)^{\beta}} + O\left(\frac{x^{\alpha}(\log\log x)^{\frac{1}{2}(\gamma+\gamma_1)}}{(\log x)^{\frac{1}{2}(\beta+\beta_1)}}\right).$$

Proof. We have already defined $\Sigma_1(x) = \sum_{n=1}^{x} b(n)\log(x/n)$; let $\Sigma_2(x) = \sum_{n=1}^{x} b(n)$. By hypothesis

$$\Sigma_1(x) = \frac{Bx^{\alpha}(\log\log x)^{\gamma}}{(\log x)^{\beta}} + O\left(\frac{x^{\alpha}(\log\log x)^{\gamma_1}}{(\log x)^{\beta_1}}\right). \quad (7)$$

Let $\delta = \delta(x) = o(1)$ be a positive function of x to be chosen later.

Then

$$\Sigma_1(x(1+\delta)) = \frac{Bx^{\alpha}(1+\delta)^{\alpha}(\log\log x(1+\delta))^{\gamma}}{(\log x(1+\delta))^{\beta}} + O\left(\frac{x^{\alpha}(1+\delta)^{\alpha}(\log\log x(1+\delta))^{\gamma_1}}{(\log x(1+\delta))^{\beta_1}}\right).$$

Now $\log x(1+\delta) = \log x + O(\delta)$, and

$$\log\log x(1+\delta) = \log\log x + \log\left\{1 + \frac{\log(1+\delta)}{\log x}\right\} = \log\log x + O\left(\frac{\delta}{\log x}\right);$$

Hence

$$\Sigma_1(x(1+\delta)) = B(1+\delta)^{\alpha}\frac{x^{\alpha}(\log\log x)^{\gamma}}{(\log x)^{\beta}}\left\{1 + O\left(\frac{\delta}{\log x\log\log x}\right) + O\left(\frac{\delta}{\log x}\right)\right\}$$

$$+ O\left(\frac{x^{\alpha}(\log\log x)^{\gamma_1}}{(\log x)^{\beta_1}}\right)$$

$$= \frac{Bx^{\alpha}(\log\log x)^{\gamma}}{(\log x)^{\beta}}\left\{1 + \alpha\delta + O(\delta^2) + O\left(\frac{\delta}{\log x}\right)\right\}$$

$$+ O\left(\frac{x^{\alpha}(\log\log x)^{\gamma_1}}{(\log x)^{\beta_1}}\right). \quad (8)$$

By definition,

$$\Sigma_1(x(1+\delta)) - \Sigma_1(x) = \sum_{n=1}^{x(1+\delta)} b(n)\log \frac{x(1+\delta)}{n} - \sum_{n=1}^{x} b(n)\log \frac{x}{n}$$

$$= \log (1+\delta) \sum_{n=1}^{x} b(n) + \sum_{n=x+1}^{x(1+\delta)} b(n)\log \frac{x(1+\delta)}{n}$$

$$\geq \log (1+\delta) \Sigma_2(x) \tag{9}$$

since the second sum is not negative. Similarly

$$\Sigma_1(x(1+\delta)) - \Sigma_1(x) = \log(1+\delta) \sum_{n=1}^{x(1+\delta)} b(n) + \sum_{n=x+1}^{x(1+\delta)} b(n)\log \frac{x}{n}$$

$$\leq \log(1+\delta) \Sigma_2(x(1+\delta)) \tag{10}$$

since the second sum is not positive.

By (7), (8) and (9),

$$\Sigma_2(x) \leq \frac{\Sigma_1(x(1+\delta)) - \Sigma_1(x)}{\log (1+\delta)}$$

$$= \frac{Bx^{\alpha}(\log\log x)^{\gamma}}{(\log x)^{\beta}} \left\{ \alpha\delta + O(\delta^2) + O\left(\frac{\delta}{\log x}\right) + O\left(\frac{(\log\log x)^{\gamma_1-\gamma}}{(\log x)^{\beta_1-\beta}}\right)\right\}(\log(1+\delta))^{-1}$$

$$= \frac{Bx^{\alpha}(\log\log x)^{\gamma}}{(\log x)^{\beta}} \left\{ \alpha + O(\delta) + O\left((\log x)^{-1}\right) + O\left(\frac{(\log\log x)^{\gamma_1-\gamma}}{\delta(\log x)^{\beta_1-\beta}}\right)\right\}. \tag{11}$$

By (7), (8) and (10),

$$\Sigma_2(x(1+\delta)) \geq \frac{\Sigma_1(x(1+\delta)) - \Sigma_1(x)}{\log(1+\delta)}$$

$$= \frac{Bx^{\alpha}(\log\log x)^{\gamma}}{(\log x)^{\beta}} \left\{ \alpha + O(\delta) + O\left((\log x)^{-1}\right) + O\left(\frac{(\log\log x)^{\gamma_1-\gamma}}{\delta(\log x)^{\beta_1-\beta}}\right)\right\}. \tag{12}$$

If we replace x by $x/(1+\delta)$ in (12), we obtain

$$\sum_2(x) \geq \frac{Bx^{\alpha}(\log\log x)^{\gamma}}{(\log x)^{\beta}} \left\{\frac{1+O\left(\frac{\delta}{\log x}\right)}{(1+\delta)^{\alpha}}\right\} \left\{\alpha + O(\delta) + O\left((\log x)^{-1}\right) + O\left(\frac{(\log\log x)^{\gamma_1-\gamma}}{\delta(\log x)^{\beta_1-\beta}}\right)\right\}$$

$$= \frac{Bx^{\alpha}(\log\log x)^{\gamma}}{(\log x)^{\beta}}\left\{\alpha + O(\delta) + O\left((\log x)^{-1}\right) + O\left(\frac{(\log\log x)^{\gamma_1-\gamma}}{\delta(\log x)^{\beta_1-\beta}}\right)\right\}. \tag{13}$$

We now choose δ so that all the error terms of (11) and (13) are of a smaller order of magnitude than the first term; since $\beta < \beta_1 \leq \beta+2$, we can take $\delta = x^{-1}[x\delta']$, where

$$\delta' = \left\{(\log\log x)^{\frac{1}{2}(\gamma-\gamma_1)}(\log x)^{\frac{1}{2}(\beta_1-\beta)}\right\}^{-1},$$

and then the error terms of (11) and (13) are

$$O\left(\frac{x^{\alpha}(\log\log x)^{\gamma}}{(\log x)^{\beta}} \frac{(\log\log x)^{\frac{1}{2}(\gamma_1-\gamma)}}{(\log x)^{\frac{1}{2}(\beta_1-\beta)}}\right) = O\left(\frac{x^{\alpha}(\log\log x)^{\frac{1}{2}(\gamma+\gamma_1)}}{(\log x)^{\frac{1}{2}(\beta+\beta_1)}}\right).$$

Hence by (11) and (13)

$$\frac{B\alpha x^{\alpha}(\log\log x)^{\gamma}}{(\log x)^{\beta}} + O\left(\frac{x^{\alpha}(\log\log x)^{\frac{1}{2}(\gamma+\gamma_1)}}{(\log x)^{\frac{1}{2}(\beta+\beta_1)}}\right) \leq \sum_2(x)$$

$$\leq \frac{B\alpha x^{\alpha}(\log\log x)^{\gamma}}{(\log x)^{\beta}} + O\left(\frac{x^{\alpha}(\log\log x)^{\frac{1}{2}(\gamma+\gamma_1)}}{(\log x)^{\frac{1}{2}(\beta+\beta_1)}}\right),$$

so that

$$\sum_2(x) = \frac{B\alpha x^{\alpha}(\log\log x)^{\gamma}}{(\log x)^{\beta}} + O\left(\frac{x^{\alpha}(\log\log x)^{\frac{1}{2}(\gamma+\gamma_1)}}{(\log x)^{\frac{1}{2}(\beta+\beta_1)}}\right),$$

which is the result of the lemma.

We observe that if $\beta_1 > \beta+2$, the result of the lemma holds provided that we replace the error term by

$$O\left(x^{\alpha}(\log\log x)^{\gamma}/(\log x)^{\beta+1}\right).$$

<u>Corollary</u>. <u>If</u> $\sum_{n=1}^{x} b(n)\log(x/n) = O\left(x/(\log x)^2\right)$, <u>then</u>

$$\sum_{n=1}^{x} b(n) = O\left(x/(\log x)^{\frac{3}{2}}\right) .$$

<u>Proof</u>. From (9) and (10) we have

$$\Sigma_2(x) \leqslant \frac{\Sigma_1(x(1+\delta)) - \Sigma_1(x)}{\log(1+\delta)} \leqslant \Sigma_2(x(1+\delta)),$$

and by hypothesis $\dfrac{\Sigma_1(x(1+\delta)) - \Sigma_1(x)}{\log(1+\delta)} = O\left(x(\log x)^{-2}\delta^{-1}\right)$.

Hence as above

$$O\left(x(\log x)^{-2}\delta^{-1}\right) \leqslant \Sigma_2(x) \leqslant O\left(x(\log x)^{-2}\delta^{-1}\right) ,$$

giving

$$\Sigma_2(x) = O\left(x(\log x)^{-2}\delta^{-1}\right) = O\left(x(\log x)^{-\frac{3}{2}}\right)$$

if we choose $\delta = x^{-1}[x\delta']$ and $\delta' = (\log x)^{-\frac{1}{2}}$.

We can now deduce the result of Theorem 3 from Lemmas 8 and 9. If we take

(i) $\alpha=1$, $\gamma=u\geqslant 1$, $\gamma_1=u-1$, $\beta_1=\beta<1$

(ii) $\alpha=1$, $\gamma=\gamma_1=u=0$, $\beta<\beta_1=1$

(iii) $\alpha=1$, $\gamma=u-1\geqslant 1$, $\gamma_1=u-2$, $\beta=\beta_1=1$

(iv) $\alpha=1$, $\gamma=u-1=0$, $\gamma_1=1$, $\beta=1$, $\beta_1=2$

in Lemma 9 and use the corresponding part of Lemma 8 for the estimate for $\Sigma_1(x)$, then we obtain, in turn, the first four parts of Theorem 3; thus we have:

(i) $$\sum_{n=1}^{x} b(n) = \frac{H(1)}{\Gamma(1-\beta)} \frac{x(\log\log x)^u}{(\log x)^\beta} + O\left(\frac{x(\log\log x)^{u-\frac{1}{2}}}{(\log x)^\beta}\right)$$

when $0<\beta<1$ and $u\geqslant 1$.

(ii) $$\sum_{n=1}^{x} b(n) = \frac{H(1)}{\Gamma(1-\beta)} \frac{x}{(\log x)^{\beta}} + O\left(\frac{x}{(\log x)^{\frac{1}{2}(1+\beta)}}\right)$$

when $0<\beta<1$ and $u=0$.

(iii) $$\sum_{n=1}^{x} b(n) = uH(1) \frac{x(\log\log x)^{u-1}}{\log x} + O\left(\frac{x(\log\log x)^{u-\frac{3}{2}}}{\log x}\right)$$

when $\beta=1$ and $u\geq 2$.

(iv) $$\sum_{n=1}^{x} b(n) = H(1) \frac{x}{\log x} + O\left(\frac{x(\log\log x)^{\frac{1}{2}}}{(\log x)^{3/2}}\right)$$

when $\beta=1$ and $u=1$.

Finally from Lemma 8(v) and the corollary to Lemma 9, we obtain

(v) $$\sum_{n=1}^{x} b(n) = O\left(\frac{x}{(\log x)^{3/2}}\right)$$

when $\beta=1$ and $u=0$. This completes the proof of Theorem 3.

CHAPTER 4.

1. Proof of Theorem 1 (ii) and (iii).

If q is odd and h is even, then by Theorem 2(ii),

$$f_m(s) = \{\zeta(s)\}^{1-1/h} \sum_{u=0}^{m'} \{\log \zeta(s)\}^u H_u(s),$$

where each $H_u(s)$ $(0\leq u\leq m')$ is a sum of functions satisfying the conditions imposed on $H(s)$ in Theorem 3. Hence, if $m'\geq 1$, we have from Theorem 3(i) and (ii) (with $\beta=1/h<1$) that

$$D_m(\nu,q;x) = \sum_{n=1}^{x} a_m(n) = \frac{H_{m'}(1)}{\Gamma(1-\frac{1}{h})} \frac{x(\log\log x)^{m'}}{(\log x)^{1/h}} + O\left(\frac{x(\log\log x)^{m'-\frac{1}{2}}}{(\log x)^{1/h}}\right),$$

where the constant $H_{m'}(1)$ is given by (22) and Lemma 11 of §6, chapter 2. Similarly, if $m'=0$, we have from Theorem 3(ii) that

$$D_m(\nu,q;x) = \frac{H_0(1)}{\Gamma(1-\frac{1}{h})} \frac{x}{(\log x)^{1/h}} + O\left(\frac{x}{(\log x)^{\frac{1}{2}(1+\frac{1}{h})}}\right),$$

where $H_0(1)$ may be obtained from §7 and Lemma 11 of §6, chapter 2. The result of Theorem 1(ii) now follows.

If $q=2$, then, by Theorem 2(iii),

$$f_m(s) = \sum_{u=0}^{m} \{\log \zeta(s)\}^u H_u(s),$$

where each $H_u(s)$ $(0\leq u\leq m)$ is a sum of functions satisfying the conditions imposed on $H(s)$ in Theorem 3. Hence, if $m\geq 2$, we have from Theorem 3(iii), (iv) and (v) that

$$D_m(\nu,2;x) = mH_m(1)\,\frac{x(\log\log x)^{m-1}}{\log x} + O\left(\frac{x(\log\log x)^{m-\frac{3}{2}}}{\log x}\right),$$

and, if $m=1$, we have from Theorem 3 (iv) and (v) that

$$D_m(\nu,2:x) = H_1(1)\,\frac{x}{\log x} + O\left(\frac{x(\log\log x)^{\frac{1}{2}}}{(\log x)^{3/2}}\right);$$

in either case

$$H_m(1) = \begin{cases} \frac{1}{m!}(1+2^{-1})\zeta(2) = \frac{\pi^2}{4}\frac{1}{m!} & \text{if } \nu \text{ is even} \\ \\ \frac{1}{2^m m!}(1+2^{-1})\zeta(2) = \frac{\pi^2}{4}\frac{1}{2^m m!} & \text{if } \nu \text{ is odd.} \end{cases}$$

The result of Theorem 1(iii) now follows.

2. An asymptotic expression for $N(\nu,q^m;x)$.

We have already seen ((3) of chapter 1) that

$$N(\nu,q^m;x) = \sum_{r=0}^{m-1} D_r(\nu,q;x), \tag{1}$$

where $D_0(\nu,q;x)=N(\nu,q;x)$ is given by (2) of chapter 1. Our first corollary follows from this and Theorem 1. Assume that $m\geq 2$; then we have

Corollary 1. As $x\to\infty$,

$$N(\nu,q^m;x)\sim \begin{cases} B_1^{(m)}x & \text{if } q \text{ and } h \text{ are both odd} \\ B_2^{(m)}x(\log\log x)^{\left[\frac{m-1}{\gamma+1}\right]}(\log x)^{-\frac{1}{h}} & \text{if } q \text{ is odd and } h \text{ is even} \\ B_3^{(m)}x(\log\log x)^{m-2}(\log x)^{-1} & \text{if } q=2, \end{cases}$$

where

$$B_1^{(m)} = \sum_{r=0}^{m-1} A_1^{(r)}, \quad B_2^{(m)} = \sum_{r=\left[\frac{m-1}{\gamma+1}\right](\gamma+1)}^{m-1} A_2^{(r)}, \quad B_3^{(m)} = A_3^{(m-1)} .$$

Proof. If $q=2$, or if q is odd, h is even and $q\nmid\nu$ (so that $\gamma=0$), then, from (1) and Theorem 1(ii) and (iii), we obtain

$$N(\nu,q^m;x)\sim D_{m-1}(\nu,q;x),$$

and the result follows in these cases.

If q and h are both odd, the result follows immediately from (1) and Theorem 1(i).

Finally suppose that q is odd, h is even and $q \mid \nu$ (so that $\gamma \geq 1$). Then, by (1) and Theorem 1(ii),

$$N(\nu,q^m;x) = \sum_{r=0}^{m-1} A_\lambda^{(r)} \frac{x(\log\log x)^{\left[\frac{r}{\gamma+1}\right]}}{(\log x)^{1/h}} + O\left(\frac{x}{(\log x)^{1/h}} \sum_{r=\gamma+1}^{m-1} (\log\log x)^{\left[\frac{r}{\gamma+1}\right]-\frac{1}{\lambda}}\right)$$

$$+ O\left(\frac{x}{(\log x)^{\frac{1}{2}(1+\frac{1}{h})}}\right) \tag{2}$$

on using the estimates for the error terms given in the previous section; we observe that the sum in the first error term is non-empty only if $m \geq \gamma+2$, and that, if this is so, $\left[\frac{r}{\gamma+1}\right] \geq 1$ for at least one value of r satisfying $0 \leq r \leq m-1$. The highest power of log log x appearing on the right of (2) is $(\log\log x)^{\left[\frac{m-1}{\gamma+1}\right]}$.

Now $\left[\frac{r}{\gamma+1}\right] = \left[\frac{m-1}{\gamma+1}\right]$ when $\left[\frac{m-1}{\gamma+1}\right](\gamma+1) \leq r \leq m-1$. Hence

$$N(\nu,q^m;x) \sim \left\{ \sum_{r=\left[\frac{m-1}{\gamma+1}\right](\gamma+1)}^{m-1} A_\lambda^{(r)} \right\} \frac{x(\log\log x)^{\left[\frac{m-1}{\gamma+1}\right]}}{(\log x)^{1/h}}.$$

This completes the proof of the corollary.

3. Some results for $N(\nu,k;x)$.

In this section we shall deduce some estimates for $N(\nu,k;x)$ when k is divisible by at least two distinct primes. Some results in this direction have already been obtained by Rankin in §4 of his paper [2], and these results are improvements on Watson's estimate for $N(\nu,k;x)$ (see (1) of chapter 1).

Using Theorem 1 and Rankin's methods we can obtain further improvements.

Let

$$k = q_0^{m_0} \prod_{r=1}^{t} q_r^{m_r},$$

where $q_0, q_1, \ldots, q_t$ are primes and $2 = q_0 < q_1 < q_2 < \ldots < q_t$, and where $m_0 \geq 0$ and $m_r \geq 1$ for $1 \leq r \leq t$; we are assuming that k is divisible by at least two distinct primes, so that $t \geq 1$ if $m_0 > 0$ and $t \geq 2$ if $m_0 = 0$. If $q_r^{m_r} | \sigma_\nu(n)$ for some r satisfying $0 \leq r \leq t$, it does not necessarily follow that $k | \sigma_\nu(n)$, but if $k | \sigma_\nu(n)$ then $q_r^{m_r} | \sigma_\nu(n)$ for all r satisfying $0 \leq r \leq t$. It follows that (see (19) and (21) of [2]).

$$\max_{0 \leq r \leq t} N(\nu, q_r^{m_r}; x) \leq N(\nu, k; x) \leq \sum_{r=0}^{t} N(\nu, q_r^{m_r}; x); \tag{3}$$

if the term $N(\nu, 2^0; x)$ occurs, we take its value to be 0. We observe that

$$N(\nu, 2^{m_0}; x) = o(N(\nu, q_r^{m_r}; x))$$

for $r = 1, 2, \ldots, t$ and all possible values of m_0 and m_r.

For $1 \leq r \leq t$, define $h_r = (q_r - 1)/(\nu, q_r - 1)$ and γ_r by $q_r^{\gamma_r} \| \nu$; if all the h_r are even, define

$$\lambda = \max_{1 \leq r \leq t} h_r \quad \text{and} \quad \mu = \max_{\substack{1 \leq r \leq t \\ h_r = \lambda}} \left[\frac{m_r + 1}{\gamma_r + 1}\right].$$

Then we have

<u>Corollary 2.</u> (i) <u>If h_r is odd when $r = i$, for exactly one value of i satisfying $1 \leq i \leq t$, and h_r is even otherwise, then, as $x \to \infty$,</u>

$$N(\nu, k; x) \sim B_1^{(m_i)} x.$$

(ii) *If all the h_r are even, and if the relations $h_r = \lambda$ and $\left[\frac{m_r - 1}{\gamma_{r+1}}\right] = \mu$ hold simultaneously when $r=i$, for exactly one value of i satisfying $1 \le i \le t$, and not otherwise, then, as $x \to \infty$,*

$$N(\nu,k;x) \sim B_2^{(m_i)} \frac{x(\log\log x)^{\mu}}{(\log x)^{1/\lambda}} .$$

Proof. These results follow immediately from (3) and Corollary 1 since, under the conditions stated,

$$\sum_{r=0}^{t} N(\nu, q_r^{m_r}; x) \sim N(\nu, q_i^{m_i}; x) = \max_{0 \le r \le t} N(\nu, q_r^{m_r}; x).$$

The constants $B_1^{(m_i)}$ and $B_2^{(m_i)}$ are given by Corollary 1.

Corollary 3. (i) *If h_r is odd for at least two integers r satisfying $1 \le r \le t$, then*

$$C_1 \le \lim_{x \to \infty} x^{-1} N(\nu,k;x) \le C_2 ,$$

where C_1 and C_2 are positive constants and $C_1 \ne C_2$.

(ii) *If all the h_r are even, and if the relations $h_r = \lambda$ and $\left[\frac{m_r - 1}{\gamma_{r+1}}\right] = \mu$ hold simultaneously for at least two integers r satisfying $1 \le r \le t$, then*

$$C_3 \le \lim_{x \to \infty} \left\{ x^{-1} (\log\log x)^{-\mu} (\log x)^{\frac{1}{\lambda}} N(\nu,k;x) \right\} \le C_4 ,$$

where C_3 and C_4 are positive constants and $C_3 \ne C_4$.

Proof.

(i) Suppose that h_r is odd when $r = r_i$, $i = 1, 2, \ldots, j$, where j satisfies $2 \le j \le t$, and h_r is even otherwise. Then, by Corollary 1, as $x \to \infty$,

$$\sum_{r=0}^{t} N(\nu, q_r^{m_r}; x) \sim \left\{ \sum_{i=1}^{j} B_1^{(m_{r_i})} \right\} x = C_2 x,$$

say, and

$$\max_{0 \le r \le t} N(\nu, q_r^{m_r}; x) \sim \left\{ \max_{1 \le i \le j} B_1^{(m_{r_i})} \right\} x = C_1 x,$$

say; clearly $C_1 \ne C_2$. The result now follows from (3).

(ii) In this case we suppose that $h_r = \lambda$ and $\left[\frac{m_r - 1}{\gamma_r + 1}\right] = \mu$ hold simultaneously only when $r = r_i$, $i = 1, 2, \ldots, j$, where j satisfies $2 \le j \le t$. By Corollary 1, as $x \to \infty$,

$$\sum_{r=0}^{t} N(\nu, q_r^{m_r}; x) \sim \left\{ \sum_{i=1}^{j} B_\lambda^{(m_{r_i})} \right\} \frac{x(\log\log x)^\mu}{(\log x)^{1/\lambda}} = \frac{C_4 x(\log\log x)^\mu}{(\log x)^{1/\lambda}},$$

say, and

$$\max_{0 \le r \le t} N(\nu, q_r^{m_r}; x) \sim \left\{ \max_{1 \le i \le j} B_\lambda^{(m_{r_i})} \right\} \frac{x(\log\log x)^\mu}{(\log x)^{1/\lambda}} = C_3 \frac{x(\log\log x)^\mu}{(\log x)^{1/\lambda}},$$

say; clearly $C_3 \ne C_4$. The result now follows from (3).

CHAPTER 5.

1. Introduction.

The object of this chapter is to prove Theorem 4, and hence to deduce Theorem 5; both these theorems are stated in chapter 1. We are assuming that $f(n)=an^2+bn+c$ is an irreducible polynomial with integer coefficients, and that, for all positive integers n, f(n) is positive and coprime with the fixed positive integer k. The discriminant b^2-4ac of f(n) is denoted by D, and $(D,k)=1$. We denote by $d(m;h)$ the number of positive divisors d of a positive integer m which satisfy $d\equiv h(\text{mod } k)$, and we assume that $(h,k)=1$.

The proof of Theorem 4 will depend ultimately on estimating a multiple sum involving the Jacobi symbol $(a^2k^2D|t)$ (see Lemma 4). It will be convenient to assume that $(a,2k)=1$ and $(D,2)=1$, although these conditions are not at all essential; they merely simplify the notation to be used. The proof of the theorem remains valid even if $2^{2\tau}\,||D$ for some positive integer τ, and a and k are both even, provided that we replace D by $2^{-2\tau}D$ and a by $2^{-\delta}a$, where $2^{\delta}\,||a$, in the working; we shall justify this remark in §3 and we shall assume the truth of it when deducing the corollary given in § 8.

Throughout this chapter $A_1, A_2, A_3 \ldots$ denote positive constants, and they and the constants implied by the O-notation depend at most on k and h and the coefficients of f.

The error terms obtained in Theorems 4 and 5 are certainly not the best possible. In fact, by the method of this chapter, one may show that the error terms are $O(xL_M(x))$, where $L_i(x)$ is defined by

$$L_1(x) = \log x, \quad L_i(x) = \log \{ L_{i-1}(x) \} \text{ for } i \geq 2, \tag{1}$$

and where M is a positive integer independent of x; we shall indicate how this may be done in §2.

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2. The large divisors of f(n).

We shall consider first the large divisors of $f(n)$; by a large divisor (corresponding to a given x) we mean a divisor greater than X, where X is defined to be the least positive integer such that

$$f(n) \leq X^2 \quad \text{for} \quad 1 \leq n \leq x. \tag{2}$$

Clearly there exist positive constants C_1 and C_2 such that

$$C_1 x < X \leq C_2 \; x. \tag{3}$$

If $d_X(m;h)$ denotes the number of positive divisors d of a positive integer m which satisfy $d \equiv h \pmod k$ and $d \leq X$, then we may write

$$d(f(n) \; ; h) = d_X(f(n) \; ;h) + \sum_{\substack{d \mid f(n) \\ d > X \\ d \equiv h \pmod k}} 1. \tag{4}$$

The sum on the right contains what we have called the large divisors of $f(n)$. It may be empty (which is certainly the case if $f(n) \leq X$). If it is not empty, then consider a typical large divisor d of $f(n)$ giving rise to one term of this sum. We have that $f(n) = d\delta$, where by (2)

$$\delta = f(n)/d < X^2 \,/\, X = X \quad \text{and} \quad d\delta \equiv h\delta \equiv f(n) \pmod k. \tag{5}$$

We define h_1 by the congruence $hh_1 \equiv 1 \pmod k$; since $(h,k) = 1$, h_1 is unique modulo k and $(h_1,k) = 1$. The congruence in (5) may now be written in the form $\delta \equiv h_1 f(n) \pmod k$; we observe that, since $(f(n),k) = 1$, $(\delta,k) = 1$.

We now see that to every large divisor d of $f(n)$, with $d \equiv h \pmod k$, there corresponds a unique divisor δ with $\delta < X$ and

$\delta \equiv h_1 f(n) \pmod k$. The correspondence is not one-one, for clearly it is possible for both δ and $f(n)/\delta$ to be less than X. However we may rewrite (4) in the form

$$d(f(n)\ ;h) = d_X(f(n)\ ;h)+d_X(f(n)\ ;h_1 f(n)) - \sum_{\substack{\delta | f(n) \\ f(n)/X \leq \delta \leq X \\ \delta \equiv h_1 f(n) \pmod k}} 1,$$

so that

$$\sum_{n=1}^{x} d(f(n)\ ;h) = \sum_{n=1}^{x} \{ d_X(f(n)\ ;h)+d_X(f(n)\ ;h_1 f(n))\} - \Delta, \qquad (6)$$

where

$$\Delta = \sum_{n=1}^{x} \sum_{\substack{\delta | f(n) \\ f(n)/X \leq \delta \leq X \\ \delta \equiv h_1 f(n) \pmod k}} 1. \qquad (7)$$

We observe that the expression on the right of (6) does not contain any large divisors.

Let $y = [x/\log x]$; then there exists a positive constant C_3 such that

$$f(n) > C_3 y^2 \quad \text{for} \quad y < n \leq x.$$

From (7) it follows that

$$0 \leq \Delta \leq \sum_{n=1}^{y} d_X(f(n)\ ;h_1 f(n)) + \sum_{n=y+1}^{x} \sum_{\substack{\delta | f(n) \\ Y < \delta \leq X \\ \delta \equiv h_1 f(n) \pmod k}} 1$$

$$\leq \sum_{n=1}^{y} d_X(f(n);h_1 f(n)) + \sum_{n=1}^{x} \{ d_X(f(n);h_1 f(n)) - d_Y(f(n);h_1 f(n))\}, \qquad (8)$$

where $Y = [C_3 y^2 X^{-1}]$.

[We may improve the upper bound for Δ by splitting up the sum on the right of (7) into more than two parts in the following way. Write $y_0 = 0$, $y_M = x$ and $y_m = [x/L_m(x)]$ for $1 \le m \le M-1$, where M and $L_m(x)$ are defined at the end of §1; put $Y_0 = 0$, and $Y_m = [C_3^{(m)} y_m^2 \, X_m^{-1}]$ for $1 \le m \le M-1$, where $C_3^{(m)}$ is a positive constant such that $f(n) > C_3^{(m)} \, y_m^2$ for $n > y_m$. Then Δ satisfies

$$0 \le \Delta \le \sum_{m=1}^{M} \sum_{n=y_{m-1}+1}^{y_m} \sum_{\substack{\delta | f(n) \\ Y_{m-1} < \delta \le x \\ \delta \equiv h_1 f(n) \pmod{k}}} 1 \le \sum_{m=1}^{M} \sum_{n=1}^{y_m} \{ d_X(f(n); h_1 f(n)) - d_{Y_{m-1}}(f(n); h_1 f(n)) \} .$$

Using this estimate for Δ instead of (8), we can obtain the improvement of the error terms of Theorems 4 and 5 mentioned in §1.]

We now put $n = mk + \ell$, where $0 \le \ell < k$. Then we may regard $f(n) = f(mk + \ell)$ as a polynomial in m with coefficients depending on ℓ, k and the coefficients of $f(n)$, so that we may write $f(n) = F_\ell(m)$, say. We observe that the discriminant of $F_\ell(m)$ is $k^2 D$, and that $F_\ell(m) \equiv f(\ell) \pmod{k}$. We now have that

$$\sum_{n=1}^{x} d_X(f(n); h_1 f(n)) = \sum_{\ell=0}^{k-1} \sum_{m} d_X(F_\ell(m); h_1 f(\ell)), \tag{9}$$

where in the summation over m, m runs through the integers of the interval $0 \le m \le (x-\ell)/k$ and where $m \ne 0$ if $\ell = 0$; there are corresponding expressions for the other sums of (6) and (8). Hence in order to find an estimate for the right side of (6), our main task must be to consider sums of the type

$$\sum_{m=1}^{z} d_\chi(F_\ell(m)\ ;\ h_2), \tag{10}$$

where $(h_2, k) = 1$.

3. Preliminary results.

This section contains some definitions and lemmas which will be used in estimating the sum (10).

Let $\rho(q)$ denote the number of solutions in m of the congruence

$$F_\ell(m) \equiv 0 \pmod q, \quad 1 \leq m \leq q.$$

Then, if p denotes a prime, σ any positive integer and C_4 a positive constant depending only on k and D, $\rho(q)$ has the following properties:

<u>Lemma 1</u>. (i) $\rho(q_1 q_2) = \rho(q_1)\rho(q_2)$ <u>if</u> $(q_1, q_2) = 1$.

(ii) $\rho(p^\sigma) = \rho(p) \leq 2$ <u>if</u> $p \nmid k^2 D$.

(iii) $\rho(p^\sigma) = \rho(p^{2\varepsilon+1})$ <u>if</u> $p^\varepsilon \,||\, k^2 D$ <u>and</u> $\sigma > 2\varepsilon$.

(iv) $\rho(p^\sigma) \leq C_4$ <u>always</u>.

It is well known that (i) holds, and a proof of this result may be found in §8·2 of [9]; it is easily shown that the inequality $\rho(p) \leq 2$ in (ii) is valid. The rest of (ii), and (iii) are proved by Nagell in [10], pp 346-349, and (iv) follows immediately from (ii) and (iii); Nagell deduces from his result that (iv) holds with $C_4 = 2(k^2 D)^2$.

In several places we shall need to consider separately from other possibilities the case when $4 \mid k$ and $D = -\mu^2$, where μ is a positive integer.

We shall refer to this as condition I : we shall use μ only in this context.

We define $\chi(t)$ by

$$\chi(1) = 1, \quad \chi(t) = \begin{cases} (a^2k^2D \mid t), & t \text{ odd} \\ 0, & t \text{ even} \end{cases}$$

where $(a^2k^2D \mid t)$ is the Jacobi symbol. (If $a=2^{\gamma} a_1$ and $D=4^{\tau} D_1$, where a_1 and D_1 are odd, then, when t is odd,

$$\chi(t) = (a^2k^2D|t) = (a_1^2 k^2 D_1 |t).$$

Hence we can replace a and D by a_1 and D_1 respectively during the rest of the proof of Theorem 4; this justifies the remark made in §1. During the rest of the proof of Theorem 4 we shall assume that a and D are odd.) We have the following result:

Lemma 2. If M is the lowest common multiple of 4 and $|akD|$,

$$\sum_{\substack{t=1 \\ t\equiv h_3 (\mathrm{mod}\, k)}}^{M} \chi(t) = 0$$

except when condition I holds.

Proof. Put

$$g = \begin{cases} K(a^2k^2D) = K(D) & \text{if } D \equiv 1 (\mathrm{mod}\ 4), \\ 4K(a^2k^2D) = 4K(D) & \text{if } D \not\equiv 1 (\mathrm{mod}\ 4), \end{cases}$$

where $K(m)$ denotes the squarefree kernel of m. Then g is the leader of the Jacobi symbol $(a^2k^2D|t)$([6].p129). Hence

$$\chi(t) = \chi_g(t) \quad \chi_0(t)$$

for all t, where χ_g is a character modulo g and χ_0 is the principal character modulo M; χ is therefore a character modulo M, since $g \mid M$.

If ψ runs over all characters modulo k, we have

$$\phi(k) \sum_{\substack{t=1 \\ t\equiv h_3 (\mathrm{mod}\, k)}}^{M} \chi(t) = \sum_{t=1}^{M} \chi(t) \sum_{\psi} \psi(t)\overline{\psi}(h_3) = \sum_{\psi} \overline{\psi}(h_3) \sum_{t=1}^{M} \chi(t)\psi(t),$$

where $\phi(k)$ is Euler's function. Now $\chi\psi$ is a character modulo M and hence the inner sum on the right will equal 0, so that the required result will follow, if we show that, for all characters ψ modulo k, $\chi\psi \neq \chi_0$ except when $D = -\mu^2$ and $4|k$. But $\chi\psi = \chi_0$ implies that $\chi(t) = \chi_0(t)\overline{\psi}(t)$ for all t. It follows that $g|k$; since $(D,k) = 1$, this means that $|K(D)| = 1$ so that $D = -\mu^2$. (Since f is irreducible, $D \neq \mu^2$). Hence $D \not\equiv 1 \pmod 4$, which implies that $4 | g$, so that $4 | k$. Thus $\chi\psi = \chi_0$ only if condition I holds.

<u>Lemma 3.</u>

$$\sum_{\substack{u\leq U \\ (u, 2akD)=1 \\ u\equiv h_4 (\mathrm{mod}\, k)}} 1 = A_4 U + O(1),$$

<u>where</u> $(h_4, k) = 1$.

<u>Proof.</u> We observe that the condition $(u,k) = 1$ is automatically satisfied since $u \equiv h_4 \pmod k$ and $(h_4,k) = 1$. In particular it follows that u is odd if k is even.

Put $\theta = 0$ or 1 according as k is even or odd. Then $\phi(2^\theta a\,|D|)$ of the integers

$$gk + h_4 \quad (0 \leq g < 2^\theta a|D|)$$

are coprime with $2^\theta a|D|$. Thus

$$\sum_{\substack{u < 2^\theta a|D|k \\ (u, 2^\theta aD)=1 \\ u\equiv h_4 (\mathrm{mod}\, k)}} 1 = \phi(2^\theta a\,|D|), \tag{11}$$

whence

$$\sum_{\substack{u \leq U \\ (u, 2akD)=1 \\ u \equiv h_4 \pmod{k}}} 1 = \frac{\phi(2^\theta a \mid D \mid)}{2^\theta a \mid D \mid k} U + O(1).$$

4. Transformation of the sum $\sum_{m=1}^{z} d_\chi(F_\ell(m); h_2)$.

The sum $\sum_{m=1}^{z} d_\chi(F_\ell(m); h_2)$ is the number of solutions in m and q of the congruence

$$F_\ell(m) \equiv 0 \pmod{q}, \quad 1 \leq m \leq z, \quad 1 \leq q \leq X, \quad q \equiv h_2 \pmod{k}.$$

Let $\rho_z(q)$ denote the number of solutions in m of

$$F_\ell(m) \equiv 0 \pmod{q} \qquad (1 \leq m \leq z);$$

then $\rho_q(q) = \rho(q)$ and $\rho_z(q)$ satisfies

$$[z/q]\rho(q) \leq \rho_z(q) \leq ([z/q]+1)\rho(q).$$

It follows that

$$\sum_{m=1}^{z} d_\chi(F_\ell(m); h_2) = \sum_{\substack{q=1 \\ q \equiv h_2 \pmod{k}}}^{X} \rho_z(q)$$

$$= z \sum_{\substack{q=1 \\ q \equiv h_2 \pmod{k}}}^{X} \rho(q)/q + O\left(\sum_{\substack{q=1 \\ q \equiv h_2 \pmod{k}}}^{X} \rho(q) \right). \qquad (12)$$

In order to find an estimate for the right side of (12), we shall need to consider first the sum

$$\sum_{\substack{q=1 \\ q \equiv h_2 \pmod{k}}}^{X} \rho(q).$$

Each integer q may be written as a product rs where $(s, 2akD) = 1$, and where each prime dividing r also divides $2akD$. Then, by Lemma 1(i), we have that

$$\sum_{\substack{q=1 \\ q\equiv h_2 (\mathrm{mod}\ k)}}^{X} \rho(q) = \sum_{r\le X} \rho(r) \sum_{\substack{s\le X/r \\ rs\equiv h_2 (\mathrm{mod}\ k)}} \rho(s). \tag{13}$$

Since $(rs,k) = (h_2,k) = 1$, the condition $rs \equiv h_2 \pmod k$ may be rewritten in the form $s \equiv r_1 h_2 \equiv h_5 \pmod k$, say, where r_1 is an integer, unique modulo k, satisfying $rr_1 \equiv 1 \pmod k$, and where h_5 depends on r and $(h_5,k) = 1$.

Consider now the inner sum on the right of (13). If $s = p_1^{\sigma_1} p_2^{\sigma_2} \ldots p_i^{\sigma_i}$ where $p_1, p_2, \ldots, p_i$ are distinct primes (not dividing 2akD), we have, from Lemmas 1(i) and (ii), that

$$\rho(s) = \rho(p_1^{\sigma_1})\rho(p_2^{\sigma_2})\ldots\rho(p_i^{\sigma_i}) = \rho(p_1)\rho(p_2)\ldots\rho(p_i).$$

Furthermore ([6],p.140)

$$\rho(p) = 1 + (a^2k^2D|p),$$

and hence

$$\rho(s) = \prod_{p|s} \{1 + (a^2k^2D|p)\} = \sum_{\substack{t|s \\ t \text{ squarefree}}} \chi(t) \tag{14}$$

We observe that for the special case $D = -\mu^2$, if $(p,2ak\mu) = 1$, then

$$(a^2k^2D|p) = (-a^2k^2\mu^2 \mid p) = (-1)^{\frac{1}{2}(p-1)}.$$

Thus, if $p \equiv 3 \pmod 4$, $\rho(p) = 0$, and it follows that $\rho(s) = 0$ if $s \equiv 3 \pmod 4$.

The inner sum on the right of (13) is given by

Lemma 4.

$$\Sigma_1 = \sum_{\substack{s\le Z \\ (s,2akD)=1 \\ s\equiv h_5 (\mathrm{mod}\ k)}} \rho(s) = A_5 Z + O(Z^{2/3})$$

unless condition I holds and $h_5 \equiv 3 \pmod 4$, in which case $\Sigma_1 = 0$.

[The exponent $\frac{1}{3}$ may be replaced by any number α satisfying $\frac{1}{2} < \alpha < 1$.]

Proof. We suppose first that condition I does not hold. By (14), and since the Mobius function $\mu(t)$ satisfies

$$|\mu(t)| = \sum_{v^2 | t} \mu(v),$$

$$\begin{aligned}\Sigma_1 = \sum_{\substack{s \leq Z \\ (s,2akD)=1 \\ s \equiv h_5 (\mathrm{mod}\ k)}} \sum_{\substack{t | s \\ t \text{ squarefree}}} \chi(t) &= \sum_{t \leq Z} \chi(t)\,|\mu(t)| \sum_{\substack{u \leq Z/t \\ (u,2akD)=1 \\ tu \equiv h_5 (\mathrm{mod}\ k)}} 1 \\ &= \sum_{\substack{v \leq \sqrt{Z} \\ (v,2akD)=1}} \mu(v) \sum_{w \leq Z/v^2} \chi(w) \sum_u 1, \end{aligned} \tag{15}$$

where $\sum_u$ stands for the summation over all positive integers u satisfying the conditions $u \leq Z/v^2 w$, $(u,2akD) = 1$ and $uv^2 w \equiv h_5 \pmod k$. We split the sum over w into two parts so that, with the above meaning for $\sum_u$,

$$\sum_{w \leq Z/v^2} \chi(w) \sum_u 1 = \sum_{w \leq (Z/v^2)^{1/3}} \chi(w) \sum_u 1 + \sum_{(Z/v^2)^{1/3} < w \leq Z/v^2} \chi(w) \sum_u 1 = \Sigma_2 + \Sigma_3, \tag{16}$$

say.

In the sum Σ_2, we may suppose that $(w, k) = 1$, since otherwise $\chi(w) = 0$ and the congruence $uv^2 w \equiv h_5 \pmod k$, with $(h_5, k) = 1$, cannot be satisfied. Then the congruence $uv^2 w \equiv h_5 \pmod k$ is equivalent to a congruence of the form $u \equiv h_6 \pmod k$, where $(h_6, k) = 1$ and h_6 is unique modulo k for fixed v and w. From Lemma 3 and since

$$\left| \sum_{w=W_1}^{W_2} \chi(w) \right| = O(1)$$

for any positive integers W_1 and W_2 satisfying $W_1 < W_2$, we obtain

$$\Sigma_2 = A_4 \frac{Z}{v^2} \sum_{w \leq (Z/v^2)^{1/3}} \frac{\chi(w)}{w} + O\Big(\sum_{w \leq (Z/v^2)^{1/3}} |\chi(w)| \Big)$$

$$= A_4 \frac{Z}{v^2} \sum_{w=1}^{\infty} \frac{\chi(w)}{w} + O\left(\frac{Z}{v^2} \left| \sum_{w>(Z/v^2)^{1/3}} \frac{\chi(w)}{w} \right|\right) + O((Z/v^2)^{1/3}).$$

In order to estimate the sum $\sum_{w>(Z/v^2)^{1/3}} \chi(w)\, w^{-1}$, we write

$$T(w) = \sum_{j \leq w} \chi(j),$$

so that $|T(w)| = O(1)$ for all w from above, and then we have

$$\left| \sum_{w>(Z/v^2)^{1/3}} \frac{\chi(w)}{w} \right| = \left| \sum_{w>(Z/v^2)^{1/3}} \frac{T(w)-T(w-1)}{w} \right|$$

$$= \left| \sum_{w>(Z/v^2)^{1/3}} T(w) \left(\frac{1}{w} - \frac{1}{w+1}\right) - \frac{T([(Z/v^2)^{1/3}])}{[(Z/v^2)^{1/3}]+1} \right|$$

$$= O\left(\sum_{w>(Z/v^2)^{1/3}} \frac{1}{w(w+1)}\right) + O\left((Z/v^2)^{-1/3}\right)$$

$$= O((Z/v^2)^{-1/3}).$$

It follows that

$$\Sigma_2 = A_4 L(\chi) Z/v^2 + O((Z/v^2)^{2/3}), \tag{17}$$

where $L(\chi) = \sum_{w=1}^{\infty} \chi(w)/w \neq 0$, the series being convergent ([6],p.222).

In order to estimate Σ_3, we change the order of summation so that

$$\Sigma_3 = \sum_{\substack{u < (Z/v^2)^{2/3} \\ (u,2ak0)=1}} \sum_{\substack{(Z/v^2)^{1/3} < w \leq Z/v^2 u \\ uv^2 w \equiv h_5 \pmod{k}}} \chi(w).$$

The congruence $uv^2 w \equiv h_5 \pmod{k}$ is equivalent to one of the form $w \equiv h_7 \pmod{k}$, where $(h_7, k) = 1$ and h_7 is unique modulo k for fixed u and v. Hence, by Lemma 2,

$$\Sigma_3 = O((Z/v^2)^{2/3}). \tag{18}$$

From equations (15) to (18) we obtain

$$\Sigma_1 = \sum_{\substack{v \leq \sqrt{Z} \\ (v, 2akD)=1}} \mu(v) \{ A_4 L(\chi) Z/v^2 + O((Z/v^2)^{2/3}) \}$$

$$= A_4 L(\chi) Z \sum_{\substack{v=1 \\ (v,2akD)=1}}^{\infty} \frac{\mu(v)}{v^2}$$

$$+ O\left(Z \sum_{v > \sqrt{Z}} \frac{|\mu(v)|}{v^2} + Z^{2/3} \sum_{v=1}^{\infty} \frac{|\mu(v)|}{v^{4/3}} \right). \qquad (19)$$

The error term on the right is $O(Z^{2/3})$; the sum in the main term on the right is given by

$$\sum_{\substack{v=1 \\ (v,2akD)=1}}^{\infty} \frac{\mu(v)}{v^2} = \prod_{p \nmid 2akD} \{ 1 + \mu(p) p^{-2} + \mu(p^2) p^{-4} + \ldots \}$$

$$= \prod_{p \nmid 2akD} (1 - p^{-2}) = \{ \zeta(2) \prod_{p | 2akD} (1 - p^{-2}) \}^{-1}$$

$$= \frac{6}{\pi^2} \prod_{p | 2akD} (1 - p^{-2})^{-1} = A_6 ,$$

say. This, together with (19), gives the result of the Lemma, with $A_5 = A_4 A_6 L(\chi)$, provided condition I does not hold.

In order to complete the proof of the lemma, we have to consider the case which we have so far omitted; thus we now suppose that $D = -\mu^2$ and $k = 2^{\nu} k_1$, where $(k_1, 2) = 1$ and $\nu \geq 2$. We recall that, in the paragraph before Lemma 4, we observed that, when $D = -\mu^2$, $\rho(s) = 0$ if $s \equiv 3 \pmod 4$ and $(s, 2ak\mu) = 1$. Since $4|k$, it follows that, if $h_5 \equiv 3 \pmod 4$,

$$\Sigma_1 = \sum_{\substack{s \leq Z \\ (s, 2ak\mu)=1 \\ s \equiv h_5 \pmod k}} \rho(s) = 0. \qquad (20)$$

If $h_5 \equiv 1 \pmod 4$, then $h_5 + 2k_1 \equiv 3 \pmod 4$, so that (20) holds with h_5 replaced by $h_5 + 2k_1$.

Hence

$$\Sigma_1 = \sum_{\substack{s \le Z \\ (s,2ak\mu)=1 \\ s \equiv h_5 (\mathrm{mod}\, k)}} \rho(s) = \sum_{\substack{s \le Z \\ (s,2ak\mu)=1 \\ s \equiv h_5 (\mathrm{mod}\, k)}} \rho(s) + \sum_{\substack{s \le Z \\ (s,2ak\mu)=1 \\ s \equiv h_5 + 2k_1 (\mathrm{mod}\, k)}} \rho(s) = \Sigma_4 + \Sigma_5,$$

say. The method of the first part of Lemma 4 can now be applied to the sum $\Sigma_4 + \Sigma_5$ provided that we use the following fact instead of Lemma 2. (Lemma 2 cannot be used in this case because condition I holds). If $(h_8, k) = 1$, $h_8 \equiv 1 (\mathrm{mod}\ 4)$ and τ is any positive odd integer, then

$$\sum_{\substack{w < a\mu k \\ w \equiv h_8 (\mathrm{mod}\, k)}} \chi(w) + \sum_{\substack{w < a\mu k \\ w \equiv h_8 + 2k_1\tau (\mathrm{mod}\, k)}} \chi(w) = \sum_{\substack{w < a\mu k \\ (w,a\mu)=1 \\ w \equiv h_8 (\mathrm{mod}\, k)}} 1 - \sum_{\substack{w < a\mu k \\ (w,a\mu)=1 \\ w \equiv h_8 + 2k_1\tau (\mathrm{mod}\, k)}} 1 = \phi(a\mu) - \phi(a\mu) = 0$$

by (11). The constant A_5 obtained for this case equals $2A_4 A_6 L(\chi)$.

5. The sum $\sum_{r \le X} \rho(r) r^{-1}$.

We now complete the evaluation of the right side of (13). If condition I does not hold, then from (13) and Lemma 4 we obtain

$$\sum_{\substack{q=1 \\ q \equiv h_2 (\mathrm{mod}\, k)}}^{X} \rho(q) = A_5 X \sum_{r \le X} \rho(r) r^{-1} + O\left(X^{2/3} \sum_{r \le X} \rho(r) r^{-2/3}\right), \qquad (21)$$

where r runs through the integers divisible only by primes dividing 2aD and satisfying $(r,k) = 1$. Similarly if $4|k$ and $D = -\mu^2$,

$$\sum_{\substack{q=1 \\ q \equiv h_2 (\mathrm{mod}\, k)}}^{X} \rho(q) = A_5 X \sum_{\substack{r \le X \\ r \equiv h_2 (\mathrm{mod}\, 4)}} \rho(r) r^{-1} + O\left(X^{2/3} \sum_{\substack{r \le X \\ r \equiv h_2 (\mathrm{mod}\, 4)}} \rho(r) r^{-2/3}\right); \qquad (22)$$

the extra condition $r \equiv h_2 (\mathrm{mod}\ 4)$ arises from the fact that the inner sum on the right of (13) is non-zero only if $s \equiv 1 (\mathrm{mod}\ 4)$.

Let $2^{\varepsilon_0} a|D| = 2^{\varepsilon_0} p_1^{\varepsilon_1} \ldots p_i^{\varepsilon_i} p_{i+1}^{\varepsilon_{i+1}} \ldots p_j^{\varepsilon_j}$, where the p_ν $(1 \leq \nu \leq j)$ are distinct odd primes, $p_\nu | a$ but $p_\nu \nmid D$ for $1 \leq \nu \leq i$, $p_\nu | D$ for $i < \nu \leq j$, ε_ν $(1 \leq \nu \leq j)$ are positive integers and $\varepsilon_0 = 0$ or 1 according as k is even or odd. Each integer r may be written in the form $r = 2^{\varepsilon_0 \sigma_0} p_1^{\sigma_1} \ldots p_j^{\sigma_j}$, where the σ_ν $(0 \leq \nu \leq j)$ are non-negative integers. Then by Lemma 1 (i) and (iv),

$$\rho(r) = \rho(2^{\varepsilon_0\sigma_0})\rho(p_1^{\sigma_1})\ldots\rho(p_j^{\sigma_j}) \leq (C_4)^{j+1}. \tag{23}$$

We put $p_0 = 2$ and define the integers η_ν $(0 \leq \nu \leq j)$ by

$$p_\nu^{\eta_\nu} \leq X < p_\nu^{\eta_\nu+1}. \tag{24}$$

Then we have that

$$\sum_{\substack{r \leq X \\ (r,k)=1}} \rho(r) r^{-2/3} \leq (C_4)^{j+1} \prod_{\nu=0}^{j} \left\{ \sum_{\sigma_\nu=0}^{\eta_\nu} p_\nu^{-\frac{2}{3}\sigma_\nu} \right\} = O(1),$$

so that the error terms of (21) and (22) are $O(X^{2/3})$.

In order to estimate the main terms of (21) and (22), we have

<u>Lemma 5</u>.

$$\text{(i)} \quad \Sigma_6 = \sum_{\substack{r \leq X \\ (r,k)=1}} \rho(r) r^{-1} = A_7 + O(X^{-1}(\log X)^{j+1}).$$

$$\text{(ii)} \quad \Sigma_7 = \sum_{\substack{r \leq X \\ (r,k)=1}} \rho^*(r) r^{-1} = A_8 + O(X^{-1}(\log X)^{j}),$$

<u>where</u> $\rho^*(r) = \rho(r) \sin \frac{r\pi}{2}$, <u>and k is even.</u>

<u>Proof</u>. (i) Suppose first that k is odd. Then

$$\Sigma_6 = \prod_{\nu=0}^{j} \left\{ \sum_{\sigma_\nu=0}^{\eta_\nu} \rho(p_\nu^{\sigma_\nu}) p_\nu^{-\sigma_\nu} \right\} + O\left(\sum_{X < r \leq X'} \rho(r) r^{-1} \right), \tag{25}$$

where $X' = \prod_{\nu=0}^{j} p_\nu^{\eta_\nu}$. By (23) and (24) the error term on the right is

$$O\left((C_4)^{j+1} X^{-1} \prod_{\nu=0}^{j} \eta_\nu \right) = O(X^{-1}(\log X)^{j+1}).$$

Let

$$S_\nu = \sum_{\sigma_\nu=0}^{n_\nu} \rho(p_\nu^{\sigma_\nu})p_\nu^{-\sigma_\nu},$$

and suppose first that $0 \leqslant \nu \leqslant i$, so that $p_\nu \mid 2a$ but $p_\nu \nmid D$. Then by Lemma 1(ii),

$$S_\nu = 1 + \rho(p_\nu)\sum_{\sigma_\nu=1}^{n_\nu} p_\nu^{-\sigma_\nu} = 1 + \rho(p_\nu)/(p_\nu - 1) + O(X^{-1})$$

$$= E_\nu + O(X^{-1}),$$

say.

Suppose next that $i < \nu \leqslant j$, so that $p_\nu \mid D$. Then by Lemma 1(iii),

$$S_\nu = \sum_{\sigma_\nu=0}^{2t_\nu} \rho(p_\nu^{\sigma_\nu})p_\nu^{-\sigma_\nu} + \rho(p_\nu^{2t_\nu+1}) \sum_{\sigma_\nu=2t_\nu+1}^{n_\nu} p_\nu^{-\sigma_\nu}$$

$$= \sum_{\sigma_\nu=0}^{2t_\nu} \rho(p_\nu^{\sigma_\nu})p_\nu^{-\sigma_\nu} + \rho(p_\nu^{2t_\nu+1})\{p_\nu^{2t_\nu}(p_\nu - 1)\}^{-1}$$

$$+ \quad O(X^{-1})$$

$$= E_\nu + O(X^{-1}),$$

say. The result now follows from (25) with $A_7 = \prod_{\nu=0}^{j} E_\nu$ provided that k is odd.

If k is even, we omit the factor involving p_0 (= 2) in the above, and we obtain the required result with $A_7 = \prod_{\nu=1}^{j} E_\nu$.

(ii) We are assuming that k is even, so that r is always odd. Then

$$\Sigma_7 = \prod_{\nu=1}^{j}\left\{\sum_{\sigma_\nu=0}^{n_\nu} \rho(p_\nu^{\sigma_\nu})\, p_\nu^{-\sigma_\nu} \sin\left(p_\nu^{\sigma_\nu}\frac{\pi}{2}\right)\right\} + O\left(\sum_{X<r\leqslant X'} \rho(r)\, r^{-1}\left|\sin\frac{r\pi}{2}\right|\right).$$

The method used to prove (i) can now be applied to Σ_7 , and this gives the required result with $A_8 = \prod_{\nu=1}^{j} E_\nu'$, where

$$E_\nu' = \begin{cases} E_\nu & \text{if } p_\nu \equiv 1 (\text{mod } 4), \\ 1 - \rho(p_\nu)/(p_\nu + 1) & \text{if } p_\nu \equiv 3 (\text{mod } 4) \text{ and } 1 \leq \nu \leq i, \\ \sum_{\sigma_\nu = 0}^{2\varepsilon_\nu} \rho(p_\nu^{\sigma_\nu})(-p_\nu)^{-\sigma_\nu} - \rho(p_\nu^{2\varepsilon_\nu+1})\{p_\nu^{2\varepsilon_\nu}(p_\nu + 1)\}^{-1} & \text{if} \end{cases}$$

$p_\nu \equiv 3(\text{mod } 4)$ and $i < \nu \leq j$.

If condition I does not hold, we obtain, from (21) and Lemma 5(i),

$$\sum_{\substack{q=1 \\ q \equiv h_2 (\text{mod } k)}}^{X} \rho(q) = A_5 A_7 X + O(X^{2/3}). \tag{26}$$

Suppose now that condition I does hold. Then by Lemma 5(i) and (ii),

$$\begin{aligned} \sum_{\substack{r \leq X \\ (r,k)=1 \\ r \equiv h_2 (\text{mod } 4)}} \rho(r) r^{-1} &= \tfrac{1}{2}\{\Sigma_6 + (-1)^{\frac{1}{2}(h_2-1)} \Sigma_7\} \\ &= \tfrac{1}{2}\{A_7 + (-1)^{\frac{1}{2}(h_2-1)} A_8\} + O(X^{-1}(\log X)^j) \\ &= A_9(h_2) + O(X^{-1}(\log x)^j), \end{aligned}$$

say. Hence, if $4|k$ and $D = -\mu^2$, we have by (22) that

$$\sum_{\substack{q=1 \\ q \equiv h_2 (\text{mod } k)}}^{X} \rho(q) = A_5 A_9 (h_2) X + O(X^{2/3}). \tag{27}$$

6. Proof of Theorem 4.

In this section we shall complete the proof of Theorem 4. We suppose first that condition I does not hold and we deduce from (26) and estimate for the right side of (12). We write

$$T(q) = \sum_{\substack{u=1 \\ u \equiv h_2 (\text{mod } k)}}^{q} \rho(u), \quad T(0) = 0.$$

Then we have

$$\begin{aligned} \sum_{\substack{q=1 \\ q \equiv h_2 (\text{mod } k)}}^{X} \rho(q)/q &= \sum_{q=1}^{X} \{T(q) - T(q-1)\}/q = \sum_{q=1}^{X} T(q)\{q^{-1} - (q+1)^{-1}\} + T(X)/(X+1) \\ &= A_{10} \sum_{q=1}^{X} (q+1)^{-1} + O\left(\sum_{q=1}^{X} q^{-\frac{1}{3}}(q+1)^{-1}\right) + O(1) \end{aligned}$$

$$= A_{10} \log X + O(1), \tag{28}$$

by (26), where $A_{10} = A_5 A_7$.

From (12), (26) and (28) we obtain

$$\sum_{m=1}^{z} d_\chi(F_\ell(m); h_2) = A_{10}\, z \log X + O(z + X), \tag{29}$$

and therefore (9) becomes

$$\sum_{n=1}^{x} d_\chi(f(n); h_1 f(n)) = \sum_{\ell=0}^{k-1} A_{10}[(x-\ell)/k]\log X + O(x)$$

$$= A_{10}\, x \log x + O(x) \tag{30}$$

on using (3). From (8), (9) and (29), the estimate of Δ is

$$\Delta \leqslant A_{10}\{ y \log X + x \log X - x \log Y \} + O(y + Y + x)$$

$$= O(x \log (X/Y)) = O(x \log\log x).$$

Hence from (6) and (30) we obtain

$$\sum_{n=1}^{x} d(f(n); h) = 2A_{10}\, x \log x + O(x \log\log x),$$

which gives the result of the theorem with $A_1 = 2A_{10}$ provided condition I does not hold. More precisely the constant A_1 is given by

$$A_1 = \frac{12}{\pi^2 k} A_7 L(\chi) \prod_{\substack{p \mid 2a\partial \\ (p,k)=1}} \left(\frac{p}{p+1}\right) \prod_{p \mid k} \left(\frac{p^2}{p^2-1}\right),$$

where A_7 is the constant of Lemma 5(i).

If condition I holds so that $4 \mid k$ and $D = -\mu^2$, we use (27) instead of (26), and the required result follows in a similar way with A_1 given by

$$A_1 = A_1(h) = A_5\{ A_9(h) + k^{-1} \sum_{\ell=0}^{k-1} A_9'(h_1 f(\ell))\} ; \tag{31}$$

we recall that h_1 satisfies $hh_1 \equiv 1 \pmod{k}$ and that $A_q(h_2)$ depends on the value of $h_2 \pmod{4}$. This completes the proof of Theorem 4.

7. Proof of Theorem 5.

In order to prove Theorem 5, we use a well known property of $r(m)$, and we apply Theorem 4 with $k = 4$ and $h = 1$ and 3. We assume for this theorem that $D = -\mu^2$.

It is well known ([9], § 16.9) that

$$r(m) = 4\{d(m;1) - d(m;3)\} \quad ;$$

hence by Theorem 4,

$$\sum_{n=1}^{x} r(f(n)) = 4\sum_{n=1}^{x} \{d(f(n);1) - d(f(n);3)\}$$

$$= 4\{A_1(1) - A_1(3)\}\, x \log x + O(x \log\log x),$$

where $A_1(1)$ and $A_1(3)$ are given by (31) with $k = 4$; this is the required result with $A_3 = 4\{A_1(1) - A_1(3)\}$. [If D is not of the form $-\mu^2$, $A_1(1) = A_1(3) = A_1$, and it follows that

$$\sum_{n=1}^{x} r(f(n)) = O(x \log\log x).]$$

We can find the value of A_3 as a product of several terms depending on a and μ. We have

$$A_1(1) - A_1(3) = A_5\{A_q(1) - A_q(3) + \tfrac{1}{4}\sum_{\ell=0}^{3}\{A_q(f(\ell)) - A_q(3f(\ell))\}\} \quad .$$

Since $f(n)$ is always odd and $f(n) \equiv 1 \pmod{4}$ for at least some integers n, there are two cases to consider: (i) $f(\ell) \equiv 1 \pmod{4}$ for $\ell = 0, 1, 2, 3$; (ii) $f(\ell) \equiv 1 \pmod{4}$ for exactly two of

$\ell = 0, 1, 2, 3$ and $f(\ell) \equiv 3 \pmod 4$ for the remaining two of $\ell = 0,$ $1, 2, 3$. In the case (i) we have

$$A_1(1)-A_1(3) = 2A_5\{A_9(1)-A_9(3)\},$$

and in case (ii)

$$A_1(1)-A_1(3) = A_5\{A_9(1)-A_9(3)\}.$$

From the definition of the constants $A_4, \ldots, A_9$ we have that

$$A_5\{A_9(1)-A_9(3)\} = \frac{6}{\pi^2} \quad L(\chi) \prod_{p|2a\mu}\left(\frac{p}{p+1}\right) \sum_r \rho^*(r)r^{-1},$$

where the summation over r runs over all positive integers which are divisible only by odd primes dividing $a\mu$. Hence the constant A_3 of Theorem 5 is given by

$$A_3 = \begin{cases} \dfrac{48}{\pi^2} \quad L(\chi) \displaystyle\prod_{p|2a\mu}\left(\frac{p}{p+1}\right) \sum_r \rho^*(r)r^{-1} & \text{in case (i)}, \\[2ex] \dfrac{24}{\pi^2} \quad L(\chi) \displaystyle\prod_{p|2a\mu}\left(\frac{p}{p+1}\right) \sum_r \rho^*(r)r^{-1} & \text{in case (ii)}. \end{cases} \qquad (32)$$

8. A corollary to Theorem 5.

Our last result is concerned with the polynomial $f(n) = n^2+1$. From Theorem 5 we shall deduce the

Corollary.

$$\sum_{n=1}^{x} r(n^2+1) = \frac{8}{\pi} \; x \log x \; + O\,(x \log \log x).$$

Proof. As it stands, the polynomial n^2+1 does not satisfy all the conditions of Theorem 5, for it is not always odd. However we may write

$$f(2m) = 4m^2 + 1 = f_1(m)$$

and
$$f(2m+1) = 2(2m^2 + 2m + 1) = 2f_2(m);$$

then the discriminants of f_1 and f_2 are -16 and -4 respectively.

For all positive integers m, $f_1(m) \equiv 1 \pmod 4$ and $f_2(m) \equiv 1 \pmod 4$, and hence both f_1 and f_2 satisfy all the conditions of case (i) of Theorem 5. Thus, since for both these polynomials, $L(x) = \frac{1}{4}\pi$ and $\sum_r \rho^*(r)r^{-1} = 1$,

$$\sum_{m=1}^{y} r(f_1(m)) = \frac{8}{\pi} y \log y + O(y \log \log y)$$

and

$$\sum_{m=1}^{y} r(f_2(m)) = \frac{8}{\pi} y \log y + O(y \log \log y).$$

Since $r(2f_2(m)) = r(f_2(m))$, we obtain from above

$$\sum_{n=1}^{x} r(f(n)) = \sum_{m=1}^{[x/2]} r(f_1(m)) + \sum_{m=0}^{[(x-1)/2]} r(f_2(m))$$

$$= \frac{8}{\pi} x \log x + O(x \log \log x),$$

which is the Corollary.

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