FLAT PLATES SUBJECTED TO TRANSVERSE LOADING - Analytical and Experimental Studies.

By

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Prefatory Note.

This original investigation was commenced in the year 1930 and was almost completed in the summer of 1938, but, with the ominous shadow of war in the background, the Author had to abandon it in favour of a more urgent investigation of the structural requirements for Air-raid Shelters at this University. Under the present emergency conditions, it has not been found possible to publish it, but it is hoped that this may be accomplished at an early date.

At the start of the work, few solutions were available for line and concentrated loads, but since the year 1939, several important contributions on the same subject have been made, mainly by American authors, which afford interesting comparisons with some of the arithmetical solutions. The comparisons are contained in an Appendix.

The work was carried out in the James Watt Engineering Laboratories of this University, and acknowledgments are hereby made to Professor Gilbert Cook, Regius Professor of Civil Engineering and Mechanics, and to Dr Alexander Thom for their advice and encouragement.

The test plate, levers, and the material for the manufacture of the heavier connecting blocks in the apparatus, were gifted by Messrs Sir William Arrol & Co., Ltd., and the framed structure was welded by my colleague Dr James Orr. I thank them for their kindly co-operation.

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Notation.

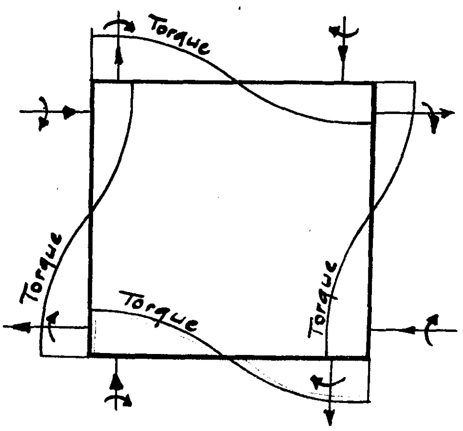
x, y, z	rectangular coordinates.
w	vertical deflection of plate, positive in the direction of increasing z.
p	uniformly distributed load per unit of area, or, uniformly distributed load per unit of length in the case of line loading.
$\overline{W}$	concemtrated load.
I	Moment of Inertia per unit of length.
E	Modulus of Elasticity.
σ	Poisson's Ratio.
h	thickness of plate
K	Flexural Rigidity of Plate, $= I\,E/(1 - \sigma^2)$
∇^2	Laplace operator $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$
2ζ	$\nabla^2 w$
C	Constant, p/K
M_x , M_y	Flexural couples per unit of length normal to the direction indicated by the subscript.
S_x , S_y	Shearing Force per unit of length normal to the direction indicated by the subscript
R_x , R_y	Reaction per unit of length normal to the direction indicated by the subscript.
T	Torsional couple per unit of length
P_c	Corner Load
n, N	Linear dimensions related to the size of the network. $N = 4\,n$, and $N/2$ is the dimension of each small square.

Other symbols are defined in the text.

References to other publications are given at the end of this volume.

The Fields are contained in Volume 2.

Boundary torques are shown in the following manner, the usual right-hand screw convention being used.



The torques shown cause uplift at all corners of the plate.

Plan of Plate.

Introduction.

In the first part of this Thesis, arithmetical methods of analyses are applied to problems of thin flat plates or slabs subjected to transverse loading. Three types of loading are considered, viz.,uniform pressure, uniformly distributed line loads, and concentrated loads. Boundary correction values, which replace the usual complementary functions of the more orthodox analyses,are also established, and their uses illustrated for clamped and simply-supported edge conditions.

The solutions to the various problems are given in a series of Fields which are contained separately in Vol.2. On each Field, two sets of values are marked, one representing half the sum of the curvatures and the other the deflections at the points in question.

A testing machine, whereby concentrated loads are applied to plates or slabs, is described in Part 2. The experimental tests which were made for the purpose of checking the solutions for line and concentrated loading are also recorded in this section.

The theory underlying the problem is dealt with adequately in the standard treatises on the mathematical theory of elasticity, and the fundamental differential equation and the expressions for flexural and torsional couples, shearing forces, etc.,are assumed without further proof, use being made of them as required.

This investigation is an endeavour to obtain general solutions for some of the standard loading conditions, which can be adapted to suit degrees of fixity intermediate between the fully clamped and simply--supported edge conditions, and also to provide a means of estimating deflections, bending moments, etc., in plates of polygonal shape. Skew slabs may also be included in this category.

The arithmetical method of solution of equations of the type $\nabla^4 w$ = Constant has been used in preference to alternative orthodox methods involving trigonometric series, and the method has been extended to cover line and concentrated loading. It is not claimed that the arithmetical method is quicker than the orthodox analytical methods, but it is best suited to this particular investigation since values throughout the entire area of the plate are required. The Author is not aware of any other method which could be applied to the unsymmetrical boundary correction fields.

When the correction fields were completed, it was noted that the values were reciprocated on the various fields and that the usual Theorem of Reciprocal Deflections was valid also for bending moments and shearing forces. If this had been known beforehand, the labour would have been reduced considerably. The correction fields are reproduced as they were when the above theorem was discovered. The discrepancies which appear in some of the values are small and o-f no practical significance.

Uniformly distributed loading is considered in the first instance. This is followed by the series of $\nabla^4 w = 0$ fields which were refesred to previously as boundary correction fields since that is their real purpose. Line loading, point or concentrated loading, and some applications based on the principle of superposition, are then considered in turn. In respect of the latter, it is fairly obvious that values from different fields may be added to, or subtracted from, one another, provided the same shape of field and size of network are used in all cases.

The square plate or field is used throughout since this forms the most convenient basis for all other simply-supported plates. The fine network is also chosen for this reason and not because of its increased accuracy.

Part 1.

A rectangular element of a thin plate subjected to a downward pressure p is shown in Fig.1. M_x and M_y are the flexural couples per unit length for planes normal to the axes of x and y. S_x and S_y are the shearing forces per unit length, and T is the torsional couple per unit length for the same two planes.

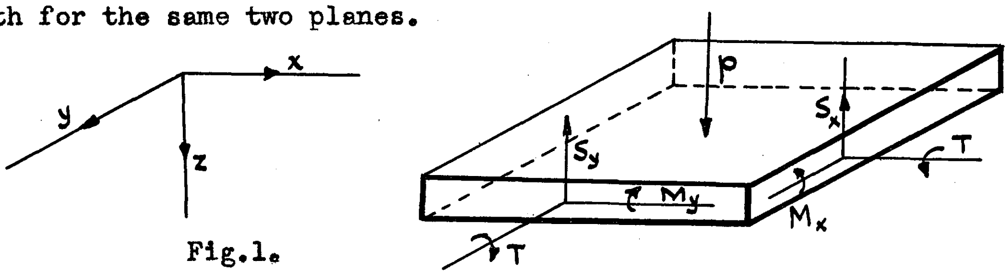


Fig.1.

If the vertical deflections w are small in comparison with the plate thickness and the effects of direct stress and shearing force are neglected in computing the strain energy in the bent plate, the fundamental differential equation which then underlies the problem of a thin plate subjected to transverse loading is,

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p(1-\sigma^2)}{IE}.$$

In the above, σ is Poisson's Ratio and E and I are respectively the Modulus of Elasticity of the plate material and the Moment of Inertia of the plate per unit length.

It can also be shown that,[(1)]

$$M_x = -K\left[\frac{\partial^2 w}{\partial x^2} + \sigma\frac{\partial^2 w}{\partial y^2}\right],$$
$$M_y = -K\left[\frac{\partial^2 w}{\partial y^2} + \sigma\frac{\partial^2 w}{\partial x^2}\right],$$
$$T = -K(1-\sigma)\frac{\partial^2 w}{\partial x \partial y},$$
$$S_x = K\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = K\frac{\partial}{\partial x}(\nabla^2 w),$$
$$S_y = K\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = K\frac{\partial}{\partial y}(\nabla^2 w),$$

where K, the flexural rigidity of the plate, is $I\,E/(1-\sigma^2)$.

If p, E and I are constant throughout, the fundamental differential equation reduces to $\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = p/K$.

i.e., $\nabla^4 w$ = Constant.

Many solutions to problems associated with this fundamental equation have been obtained by making use of trigonometric series. These are referred to later as orthodox methods of analysis. An arithmetical method of solution developed by Thom[(2)] has also been applied successfully in various fields of hydro-dynamical and aeronautical research. In this investigation it is used throughout and, whilst a brief description of the method as applied to the problem of the flat plate is given later, for fuller details reference should be made to Thom's original publications.

The Arithmetical Method of solution of $\nabla^4 w$ = constant.

Denoting the constant by C, (i.e. $C = p/K$ in the particular case of the flat plate), then the equation becomes $\nabla^4 w = C$. This may be re-written $\nabla^2(\nabla^2 w) = C$, and, if $2\zeta = \nabla^2 w$, then $\nabla^2\zeta = C/2$.

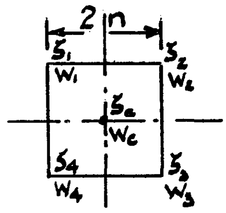


Fig.2

Considering a square of side 2 n, Fig.2, the central value $\zeta_c = \zeta_m - n^2C/4$, ...(1a), where ζ_m = the mean of the corner values.

Corresponding formulae for the w values are

$$w_c = w_m - \tfrac{1}{2}(n^2\ 2\zeta_c)$$

$$= w_m - (n^2\zeta_c), \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1b),$$

where w_m = the mean of the corner w values.

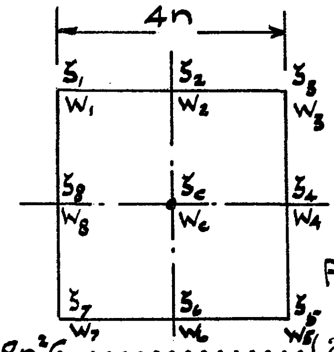


Fig.3.

Also, for a square of side 4n, Fig. 3,

$$12\zeta_c = (\zeta_1+\zeta_3+\zeta_5+\zeta_7) + 2(\zeta_2+\zeta_4+\zeta_6+\zeta_8) - 8n^2C; \ldots\ldots\ldots (2a)$$

and,

$$12W_c = (W_1+W_3+W_5+W_7) + 2(W_2+W_4+W_6+W_8) - 32n^2\zeta_c - 4n^4C \quad \ldots (2b).$$

Or, if ζ_{odd} and ζ_{even} are represented by a and b respectively, and w_{odd} and w_{even} by A and B, the above formulae may be re-written, in short,

$$12\zeta_c = \sum a + 2\sum b - 8n^2C, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2c)$$

and,

$$12\ w_c = \sum A + 2\sum B - 32\ n^2\zeta_c - 4n^4\ C. \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2d).$$

And, finally, for a square of side 8 n, Fig.4,

$$476\zeta_c = 9\sum a + 32\sum b + 46\sum c - 1152\ n^2C, \ldots\ldots\ldots\ldots\ldots\ldots (3a),$$

and,

$$476\ w_c = 9\sum A + 32\sum B + 46\sum C - 16\ n^2\ (18v + 28e + 104\zeta_c + 36\ n^2C), \ldots (3b)$$

where a,b,e,v and e denote ζ values, and A,B,C are w values at the points shown in Fig. 4.

(ζ values are written above the lines and to the right of the point in question, w values being immediately below the ζ values).

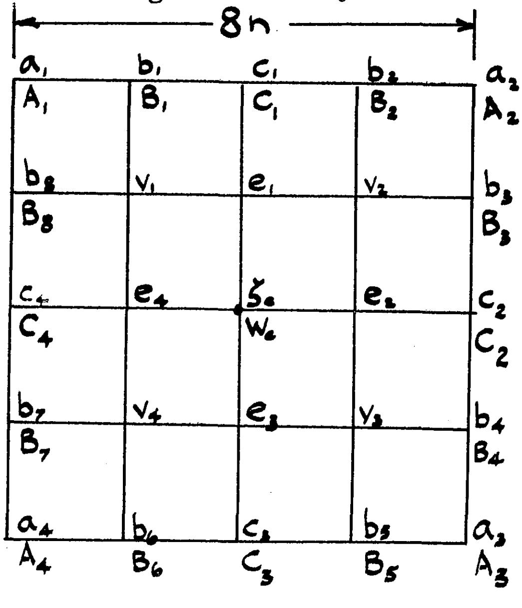


Fig. 4.

A solution is obtained by choosing, when necessary, plausible values for ζ and w in the first instance and obtaining new values for these, square by square, using the above formulae until, finally, the changes in the values are small enough to be neglected. The ζ values should be settled before proceeding to the w values. An "exact" solution is , however, not obtained, but, as in the analogous "Joint-Relaxation Method of Structural Mechanics," (3) the solution can be carried to a degree of accuracy which is ~~a~~ consistent with the assumptions which are made in the basic approach to the problem.

A good choice of initial or plausible values, as they have been described above, will reduce greatly the somewhat tedious arithmetical work, and, although no hard and fast rules can be given, values which have been settled say to the first place of decimals for the coarser networks will aid in the selection of plausible values for the finer ones. The process is described more fully in the following example.

<u>Square plate of uniform thickness, simply-supported along all four edges and loaded with a uniform pressure.</u>

For the plate A B C D, of side 8 N, Fig.5, the boundary conditions are as follows,

Deflections and Bending Moments are to be zero at all points on the boundaries, AB,BC,CD,DA. i.e. the ζ and w values are zero on these boundaries, and plausible values therefore need not be selected.

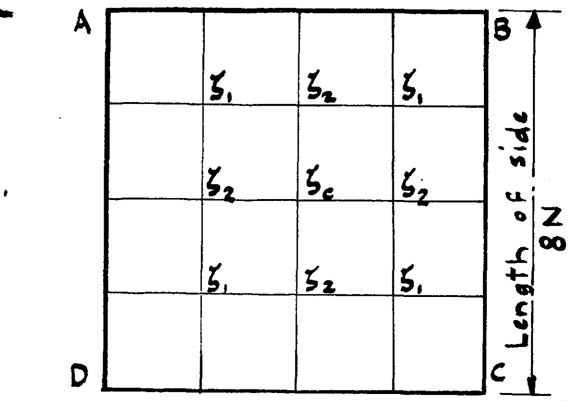


Fig. 5.

For the ζ field, using Formula (3a),

$$476\zeta_c = 0 - 1152\, N^2 C,$$

$$\therefore \quad \zeta_c = -2.42\, N^2 C.$$

(Note. This will be a settled ζ_c value for this particular network).

A plausible value is now selected for ζ_2 and the value of ζ_1 found by applying Formula (2a) to the corner square. ζ_2 is then recalculated and the process repeated until the values of ζ_1 and ζ_2 have settled. The squares are then subdivided, plausible values being inserted where necessary, and the values recalculated, row by row, until a second, and more accurate, ζ field is obtained. etc.etc.

The w values are obtained in a similar manner. In this case Formulae (2b) and (3b) are used, the part involving tho ζ values being first evaluated and noted against each point for reiterative use.

Field No. 1 shows the values obtained when the plate has been divided into 256 squares. It may be noted that ζ_c has settled to $-2.3618\ N^2\ C$, whereas the first value obtained above was $-2.42\ N^2C$.

When subdividing the fields into smaller squares, care must be taken to use the correct value of n in the equations. Thus, with reference to point A on Field No. 1, the ζ and w values are as shown in Figure 6. (Throughout this work 8 N is used to denote the side of the plate).

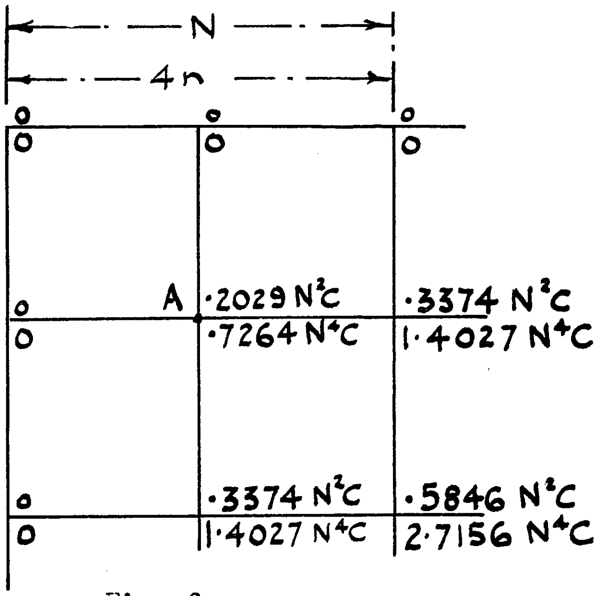


Fig. 6.

Using Formula (2a),

$$12\zeta_c = -(0.5846 + 4 \times 0.3374)\ N^2\ C - 8\ n^2\ C$$

$$= -1.9342\ N^2\ C - 8\ n^2\ C,$$

and, for N = 4n,

$$12\zeta_c = -1.9342\ N^2\ C - 0.5\ N^2\ C.$$

$$\therefore \zeta_c = -0.2029\ N^2\ C.$$

Using Formula (2b),

$$12\ w_c = (2.7156 + 4 \times 1.4027)N^4\ C - 32\ n^2(-0.2029\ N^2\ C) - 4\ n^4\ C$$

$$= 8.3264\ N^4\ C + 0.4058\ N^4\ C - \tfrac{1}{64}N^4\ C.$$

$$= 8.7166\ N^4\ C.$$

$$\therefore w_c = 0.7264\ N^4\ C.$$

Similarly, for N = 4 n, Formulae (3a) and (3b) become,

$$476\zeta_c = 9\sum a + 32\sum b + 46\sum c - 72\ N^2\ C. \qquad \text{(3c)}$$

$$476\ w_c = 9\sum A + 32\sum B + 46\sum C - N^2\left[18\sum v + 28\sum e + 104\zeta_c + 2.25\ N^2\ C\right] \qquad \text{(3d)}$$

(The part inside the bracket in (3d) is evaluated from the settled ζ field).

Comparison of Field No. 1 with Analytical Solutions.

The following expression is given by Prescott [1] for the deflection of a square plate of side A:-

$$\frac{\pi^6 w}{16 A^4 C} = \frac{1}{2^2} \sin\frac{\pi x}{A} \sin\frac{\pi y}{A} + \frac{1}{2^3 3^6} \sin\frac{3\pi x}{A} \sin\frac{3\pi y}{A} + \cdots$$

$$+ \frac{1}{3(3^2+1^2)^2}\left[\sin\frac{3\pi x}{A}\sin\frac{\pi y}{A} + \sin\frac{\pi x}{A}\sin\frac{3\pi y}{A}\right]$$

$$+ \frac{1}{5(5^2+1^2)^2}\left[\sin\frac{5\pi x}{A}\sin\frac{\pi y}{A} + \sin\frac{\pi x}{A}\sin\frac{5\pi y}{A}\right] + \cdots\cdots$$

$$+ \frac{1}{3.5.(5^2+3^2)^2}\left[\sin\frac{3\pi x}{A}\sin\frac{5\pi y}{A} + \sin\frac{5\pi x}{A}\sin\frac{3\pi y}{A}\right]$$

$$+ \text{etc.}$$

This gives the central deflection w, for the centre point of the plate distant $x = A/2$, $y = A/2$ from the origin, $= 16.65\ N^4\ C$, whereas the corresponding value from Field No. 1 is $16.72\ N^4\ C$.

Also, at the centre of Field No.1, $\zeta = -\ 2.3618\ N^2\ C$.

$$\therefore \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = 2\zeta = -\ 2 \times 2.3618\ N^2\ C.$$

From symmetry, $\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial y^2}$ at the centre of the plate, and hence

$$M_x = M_y = -K\left(\frac{\partial^2 w}{\partial x^2} + \sigma\frac{\partial^2 w}{\partial y^2}\right)$$

$$= (1+\sigma)\ 2.3618\ p\ N^2$$

$$= (1+\sigma)\ 0.0369\ p\ A^2, \text{ where A is equal to 8 N.}$$

The above analytical solution gives a corresponding value of $(1+\sigma)\ 0.0368\ p\ A^2$.

Another solution, for a plate 2a by 2a by thickness t, is given by Professor Inglis [4] as,

$$w = \frac{Ca^4}{8}\left[\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{y^2}{a^2}\right) + A_1\left\{f_1\left(\frac{x}{a}\right)\cos\frac{\pi y}{2a} + f_1\left(\frac{y}{a}\right)\cos\frac{\pi x}{2a}\right.\right.$$

$$\left.\left. + A_3\left\{f_3\left(\frac{x}{a}\right)\cos\frac{3\pi y}{2a} + f_3\left(\frac{y}{a}\right)\cos\frac{3\pi x}{2a}\right.\right.\right],$$

where $A_1 = 0.052178$ and $A_3 = -\ 1.308959 \times 10^{-6}$

For $\sigma = 0.3$, this gives a flexural couple at the centre of amount $0.1915\ p\ a^2$ and a central deflection of $0.065\ a^4\ C$, whereas the corresponding values from Field No.1. are $0.192\ p\ a^2$ and $0.0653\ a^4\ C$, respectively.

It may be seen, therefore, that the arithmetical method compares favourably with analytical methods, and Field No. 1. is therefore a very complete solution to the particular case of the simply supported square plate carrying a uniformly distributed load.

Bending Moments, Shearing Forces, Torque etc., are obtained from the ζ and w values thus :-

Bending Moments

Since $M_x = -K\left[\frac{\partial^2 w}{\partial x^2} + \sigma \frac{\partial^2 w}{\partial y^2}\right]$,

and,
$$M_y = -K\left[\frac{\partial^2 w}{\partial y^2} + \sigma \frac{\partial^2 w}{\partial x^2}\right],$$

it is therefore necessary to separate the components $\frac{\partial^2 w}{\partial x^2}$ and $\frac{\partial^2 w}{\partial y^2}$ of the ζ values.

They can be estimated by using a suitable formula for mechanical differentiation of the w field. e.g.,

the Stirling formula, where the differences are symmetrical as regards the direction of increasing and decreasing arguments,

$$f''(a) = \frac{1}{h^2}\left[\Delta^2 f(a) - \frac{1}{12}\Delta^4 f(a) + \frac{1}{90}\Delta^6 f(a) - \cdots\right] \text{-----}(D1)$$

or, the Gregory-Newton formula,

$$f''(a) = \frac{1}{h^2}\left[\Delta^2 f(a) - \Delta^3 f(a) + \frac{11}{12}\Delta^4 f(a) - \frac{5}{6}\Delta^5 f(a) + \cdots\right] \cdots\cdots(D2).$$

Alternatively, graphical differentiation of the w curve may usefully be employed.

Whichever method is adopted, the sum of the estimated quantities will, in general, differ from the 2ζ value at the point in question because of the limitations of the methods. But since the ζ values are obtained by squaring, it is recommended that full weight be given to them. The estimated quantities should therefore be adjusted to give the required sum. Thus, if estimations a and b are obtained for $\frac{\partial^2 w}{\partial x^2}$ and $\frac{\partial^2 w}{\partial y^2}$ at a point where the ζ value is c , the adjusted values would be a(2c/a+b) and b(2c/a+b).

Shearing Force.

The shearing force per unit of length is,

$$S_x = K\frac{\partial}{\partial x}(\nabla^2 w),$$
$$= 2K\frac{\partial \zeta}{\partial x}, \text{ since } 2\zeta = \nabla^2 w.$$

Similarly, $S_y = 2K\frac{\partial \zeta}{\partial y}$.

In this case, the values of $\frac{\partial \zeta}{\partial x}$ and $\frac{\partial \zeta}{\partial y}$ are obtained by mechanical differentiation of the ζ field using the Gregory - Newton formula,

$$f'(a) = \frac{1}{h}\left[\Delta^1 f(a) - \frac{1}{2}\Delta^2 f(a) + \frac{1}{3}\Delta^3 f(a) - \cdots\cdots\right] \cdots\cdots\cdots (D3)$$

Torque

The torsional couple per unit of length is,

$$T = -K(1-\sigma)\frac{\partial^2 w}{\partial x \partial y} .$$

The gradients $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ are obtained by means of (D 3) above, and by plotting them to scale the values of $\frac{\partial^2 w}{\partial x \partial y}$ can be estimated graphically.

Note :- In the difference formulae, D_1, D_2, and D_3, h represents equal increments of x or y.

Shearing Force along the Boundary.

The boundary values of $\frac{\partial S}{\partial y}$ are noted on Field No.1. The shearing force values at corresponding points are therefore as in Fig.7.

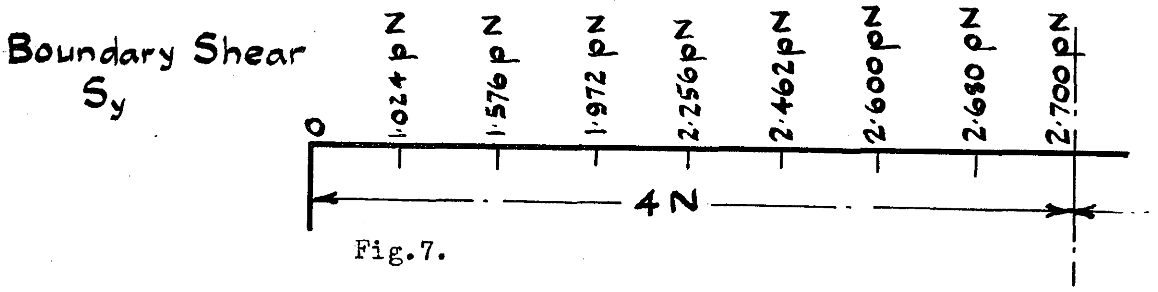


Fig.7.

The analytical solutions, previously referred to, give a corresponding value of 2.704 p N as the maximum shear at the centre of the support.

A useful check on the values shown in Fig.7, is obtained by integrating along the boundary. Using Simpson's Rule for areas, and the above values, 64.1 p N^2 is obtained, whereas the total load applied to the plate is 64.0 p N^2.

Torque along the Boundary.

The estimated values of T are shown in Fig.8. This is a maximum at the corners of the plate where a value $3(1-\sigma)$ p N^2 is reached.

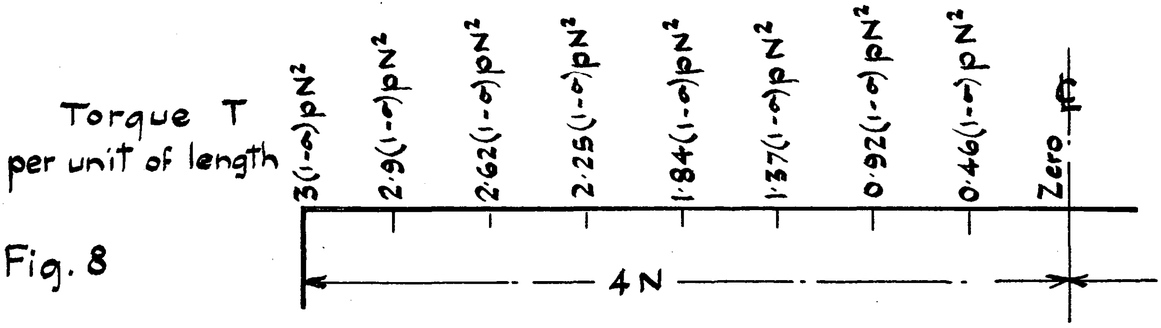


Fig. 8

Reaction along the supports and Corner Load.

At first sight, it would appear that the shearing force along the boundary must also be the reaction on the support, since, as noted above, the total applied load, 64 p N^2, is obtained by integrating the shearing force along the boundary.

It is well known, however, that a plate, loaded and supported as specified in this problem, will not remain in contact with the supports throughout their entire length unless the corners are held down.

The reason for the above behaviour is explained by considering the torque along the boundary. A thin section of the plate in contact with the support is shown in Fig. 9.

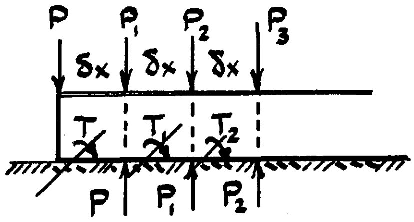


Fig. 9.

The torque on an element of length δx is T δx, where T is the torque per unit of length.

Equilibrium of the element is maintained by applying equal and opposite point loads P at the diagonal corners of the element so that

$P\delta x = T\delta x$, and therefore $P = T$.

Similarly, for the adjoining elements, $P_1 = T_1$, $P_2 = T_2$, etc. etc.

The induced reaction is therefore $(P - P_1)/\delta x = \delta P/\delta x$

$= \delta T/\delta x$, per unit of length.

An amount equal to $\frac{dT}{dx}$ should therefore be added to the shearing force to give the reaction on the supports.

The expressions for Reaction thus become,

$$R_x = S_x + \frac{\partial T}{\partial y},$$

$$= K\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + K(1-\sigma)\frac{\partial^3 w}{\partial x\,\partial y^2}, \quad \text{and,}$$

$$R_y = K\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + K(1-\sigma)\frac{\partial^3 w}{\partial y\,\partial x^2}.$$

(R_x and $M_x = 0$, define the boundary conditions of a free edge).

It is therefore necessary to find the gradients of the Torque curve, and the graphical method is sufficiently accurate for this purpose.

The induced reaction values, for the square plate under discussion, are shown in Fig,10.

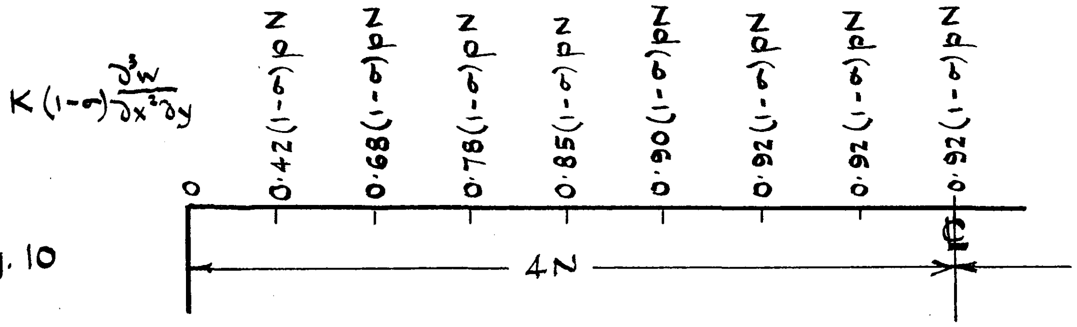


Fig. 10

The total reaction on the supports is therefore the sum of the values at corresponding points on Figs. 7 & 10.

Corner Load

For the corner load, P_c, which must be supplied at the corners of the plate, it is necessary to consider both boundaries at right angles to one another, and hence,

$P_c = 2\,T_c$, where T_c = Torque at corner, per unit of length. This gives, for the simply supported square plate, $P_c = 6\,(1-\sigma)\,p\,N^2$,

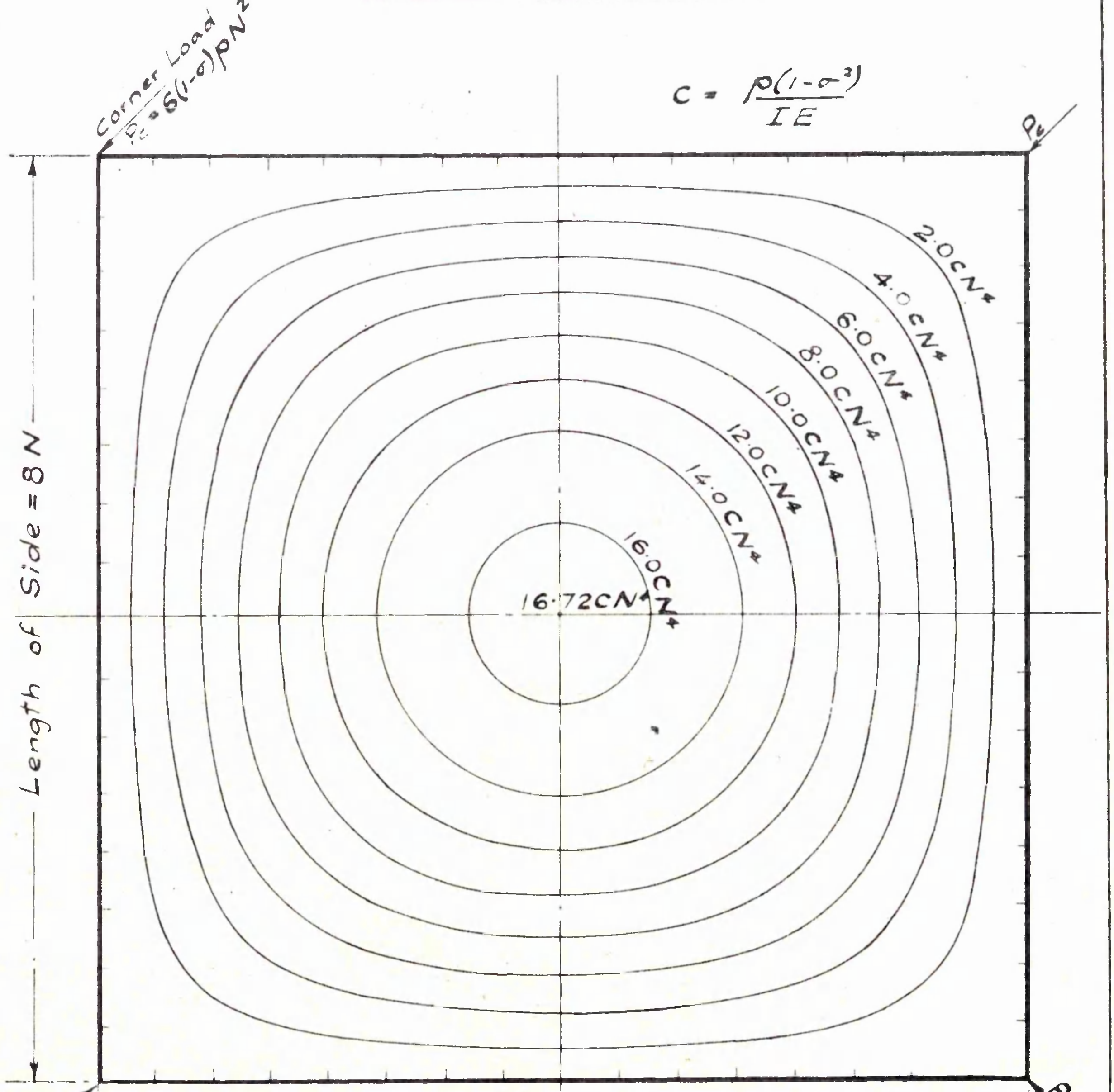
$= .094\,(1-\sigma)\,\overline{W}$, where $\overline{W}$ is the total load on the plate.

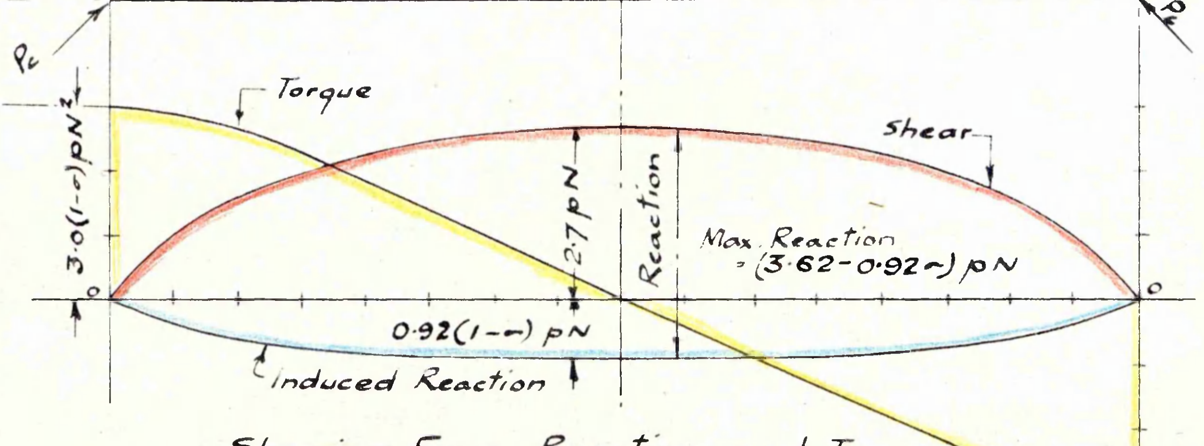
The various quantities discussed in the above are plotted to scale on Diagram No,1. Deflection Contours are also plotted to scale and these, together with the ζ and w values on Field No. 1, complete the useful design data for the square plate as specified.

— Diagram No. 1 —

Square Plate, of uniform thickness, simply supported at the edges and loaded with a uniformly distributed load.

— Deflection Contours —





Shearing Force, Reaction and Torque Distribution along Support

Solution of $\nabla^4 w = 0$

The method of procedure in general use, when analytical solutions are required to problems of thin flat plates bent by transverse forces, may be briefly described thus:-

An algebraic expression, known as the particular integral, which satisfies the fundamental differential equation $\nabla^4 w = p(1 - \alpha^2)/IE$, is found in the first place. To this is added another algebraic expression, the complementary function, for $\nabla^4 w = 0$, so that the complete solution to the problem is obtained. The solution is said to be complete if the loading and boundary conditions are satisfied. In general, the loading conditions are looked after by the particular integral and those of the boundaries by the complementary function.

With the arithmetical method of solution, the same procedure can be followed, as may be seen by an inspection of the various squares formulae hitherto used, and, if settled fields of ζ and w values are available which satisfy $\nabla^4 w = 0$, these can be made to do the same duty as the complementary functions in an orthodox analysis.

Fields Nos. 2 to 43 give ζ and w values for $\nabla^4 w = 0$, when ζ values are applied at certain points on the boundaries of a square plate of side 8 N. The squares formulae applicable to them are.-

$$12\zeta_c = \Sigma a + 2\Sigma b \quad \ldots\ldots\ldots\ldots\ldots\ldots(2e),$$

$$12\, w_c = \Sigma A + 2\Sigma B - 2N^2\zeta_c \quad \ldots\ldots\ldots\ldots\ldots\ldots(2f),$$

$$476\zeta_c = 9\Sigma a + 32\Sigma b + 46\Sigma c \quad \ldots\ldots\ldots\ldots\ldots(3e), \text{ and,}$$

$$476\, w_c = 9\Sigma A + 32\Sigma B + 46\Sigma C - N^2(18\Sigma v + 28\Sigma e + 104\zeta_c) \ldots\ldots\ldots(3f).$$

In work of this kind, a calculating machine is a necessity, but the labour can also be reduced greatly by adopting a routine system from the start.

Thus, with reference to Fig.11, row 1 is completed using Formulae (2e) & (2f) and this is followed in turn by rows 2, 3, etc., Formulae (3e) & (3f) being applied to all points except the end ones in each row. In all cases the work is carried out from left to right.

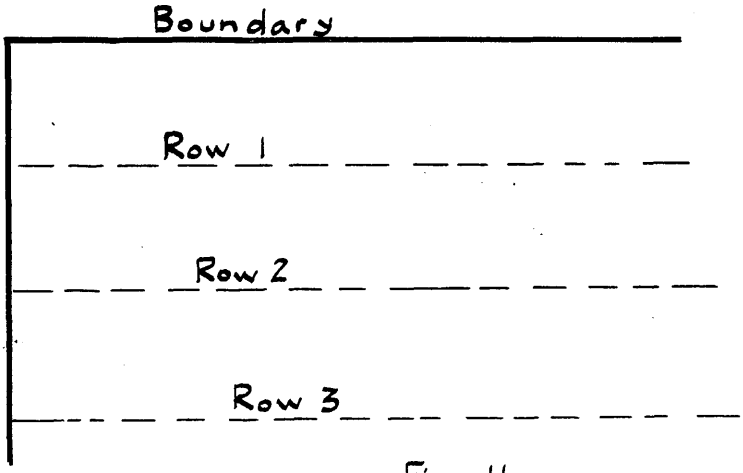


Fig. 11

After one complete traverse of the field is made, the differences between the original and the new values are abstracted to a separate sheet to give a field of differences. The work is then carried out on the difference field, a table of corrections for differences being prepared beforehand. It is advisable, however, to transfer the difference corrections from the difference field to the actual field at frequent intervals, and then re-calculate to obtain another new set of differences.

When Fields Nos. 2 to 10 are settled to a reasonable degree, each in turn can be grouped to give the symmetrical arrangements shown in Fields Nos. 11 to 19. The latter, being symmetrical about the diagonals, can be settled more or less completely with a minimum of labour as compared with the former.

A check on the accuracy of the work is afforded by the summation of ζ values from corresponding points on each of the Fields, 11 to 19. This is shown on Field No. 20. The ζ values should have summed to 10 units at every point on the Field. It may be noted that this is realised as far as practical purposes are concerned, the range being 9.994 to 10.001.

A check on the w values is also obtained by comparing the summation of corresponding w values on Fields 11 to 19 with the values obtained by multiplying the ζ values on Field No. 1 by 40. The reason for this is explained by reference to Formula (1a), viz.,

$$\zeta_c = \zeta_m - n^2 C/4 ,$$

The w values on Field No. 20 can therefore be obtained from the basic squares Formula, $w_c = w_m - n^2\ 10$.

The values marked in red on Field No. 20 are 40 times the ζ values of Field No. 1, C being taken as unity. The differences between both sets of values are negligible.

Fields No. 2 to 10 are further corrected by distributing the small differences obtained by subtracting the original and settled values of Fields Nos. 11 to 19. If these corrections are small, the Fields may be taken as being settled.

Fields Nos. (21-28), (29-35), (36-43), for different grouping of the ζ values on the boundary are obtained from Fields Nos. 2 to 10.

Theorem of Reciprocity.

A most useful extension of the ordinary theorem of reciprocal deflections is revealed on Fields Nos. 2 to 9.

With reference to Fig. 12, it may be noted, by comparing the above Fields, that when a ζ value of -10 units is applied at the point A_1, the ζ and w values along the row $B_1 B_2$ are the same as those at corsesponding points along the row $A_1 A_2$ when the ζ value of -10 units is applied at the point B_1.

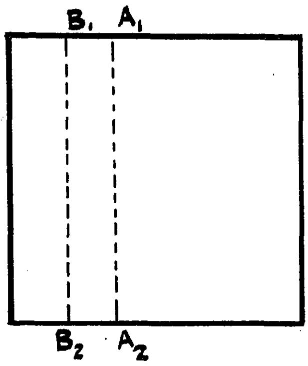


Fig.12.

The following alternative method, whereby Fields 2 to 9 are built from Fields No.2 and part of No.3, reduces considerably the arithmetical work, and, if the full significance of the above Theorem had been appreciated at the start of this investigation, it would have been used instead of that already described.

Alternative Method of obtaining Fields Nos,2 to 9.

Field No. 2 and the left- hand half of Field No.3 are obtained in the previous manner. The remaining Fields are then obtained from these by using superposition and other legitimate devices.

(Two diagrams on tracing cloth are provided in the pocket of the back cover. These are useful when superimposing one field, or part thereof, on another to get a new field.)

When a Field is completed, the values which are reciprocated are transferred from that Field to the other respective Fields in the first instance.

To complete the Fields several methods are available but the following has been proved to be satisfactory in practice.

(L and R are used below to denote the left and right- hand halves of the Fields.)

Field No.3 L is superimposed on Field No.2 so that the ζ values on the boundaries coincide. Field No. 9 R is then obtained by subtracting the values on the Fields.

Field No.2 is superimposed on Field No.3 L as shown in Fig.(a). The values necessary to complete Field No. 9 are then obtained by subtraction.

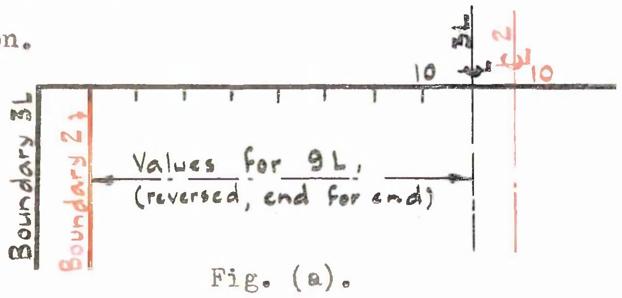


Fig. (a).

Field No.9 L is superimposed on Field No.2 so that the ζ values on the boundaries coincide. Field No.3 is then completed by subtracting the values on the Fields.

Field No.8 L is obtained by folding Field No. 3 about the first line to the right of the centre line, Fig.(b), and then subtracting corresponding values.

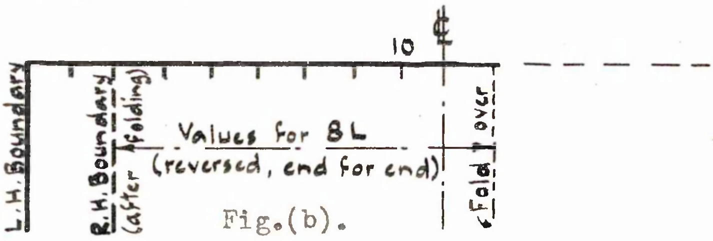


Fig.(b).

Field No.8 L is superimposed on Field No.2 so that the ζ values on the boundaries coincide. Field No. 4 R is then obtained by subtracting the values on the Fields.

Field No.4 is completed by superimposing Field No.8 L on Field No.2 as in Fig.(c) and then adding the values over the length d.

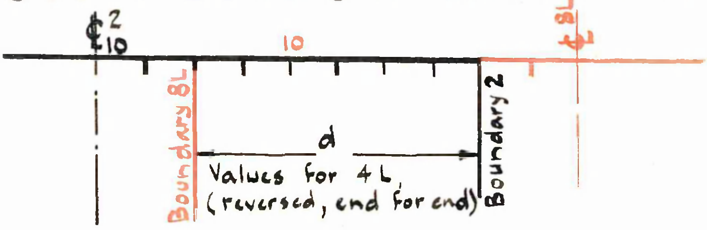


Fig.(c)

Field No.4 L is superimposed on Field No.2 so that the ζ values on the boundaries coincide. The values necessary to complete Field No.8 are then obtained by subtraction.

The values on the end rows of Field No.3 R are re-written, end for end, and placed on Field No.4 as in Fig.(d). The values necessary to complete Field No. 7 L are then obtained by subtraction.

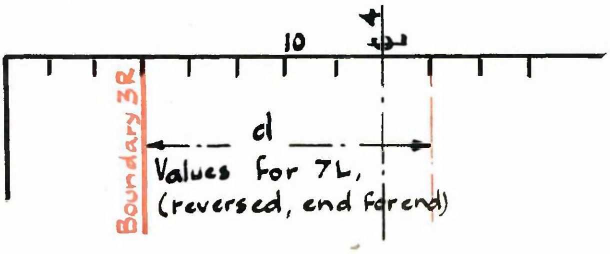


Fig. (d).

Field No. 7 L is superimposed on Field No.2 so that the ζ values on the boundaries coincide. Field No.5 R is then obtained by subtraction.

Field No.7 L is superimposed on Field No. 2 as in Fig. (e). Field No.5 is then completed by adding the values over the length d.

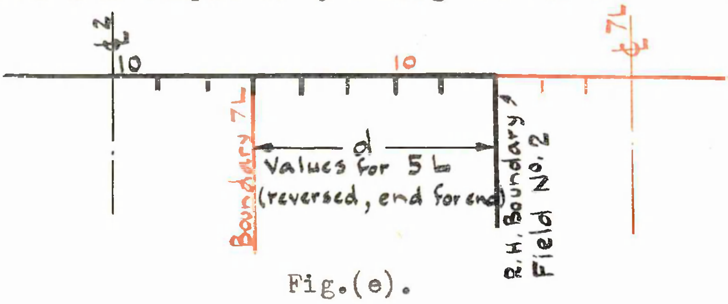


Fig.(e).

Field No.5 L is superimposed on Field No.2 so that the ζ values on the boundaries coincide. Field No.7 is then completed by subtraction.

Field No. 6 L is obtained by folding Field No.4 about the sedond row to theright of the centre line, Fig.(f), and then subtracting the values over the length d.

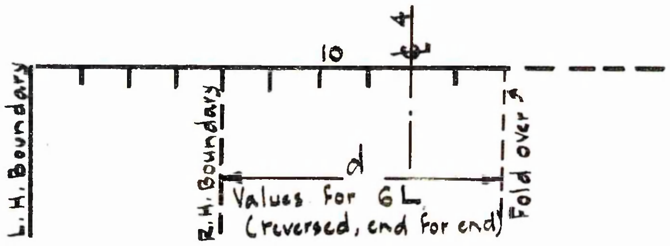


Fig.(f).

Field No. 6 L is superimposed on Field No.2 so that the ζ values on the boundary of each field coincide. Field No.6 R is then obtained by subtracting the values on the fields.

It may be noted that symmetrical arrangements, as on Fields Nos. 36 to 42, are readily obtained by superimposing portions of Field No.2 on one another. It appears, however, that Field No.3 L is necessary in addition to Field No.2 when dealing with the unsymmetrical cases discussed above.

Rectangular Fields.

Rectangular Fields, 8 N by 4 N, etc., etc., are readily obtained by folding Fields Nos. 2 to 10 about their centre lines and then subtracting corresponding values. These Fields are not reproduced in this Thesis, but they were used in the solution of the problem which follows later in connexion with the point load at the quarter-point of an axis of symmetry (page 49).

Triangular Plates.

By folding Fields Nos 2 to 10 about their diagonals and then subtracting corresponding values, Fields similar to the above are obtained for a right-angled isosceles triangle.

Uses of the $\nabla^4 w = 0$ Fields.

The uses of the various $\nabla^4 w = 0$ fields are demonstrated in the following practical problems.

Square plate of uniform thickness, clamped along the edges and loaded with a uniform pressure.

The conditions to be fulfilled in this particular problem are as follows: -

$\nabla^4 w$ = constant,

$$\left.\begin{aligned} w &= 0 \\ \frac{\partial w}{\partial x} &= 0 \\ \frac{\partial w}{\partial y} &= 0 \end{aligned}\right\}, \text{ at all points on the boundaries.}$$

Field No. 1 satisfies all the conditions except the last two, but the gradients $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ can be altered by applying ζ values to the boundaries of this Field. The problem therefore resolves itself into finding the ζ values which will make the gradients of Field No. 1 zero on the boundaries.

The gradients on the boundaries of Fields Nos. 11 to 18, and Field No. 1, as obtained by mechanical differentiation are marked on the respective Fields.

If a, b, c, ...h, are the required values of ζ at the points shown in Fig. 13g, the following eight equations are obtained.

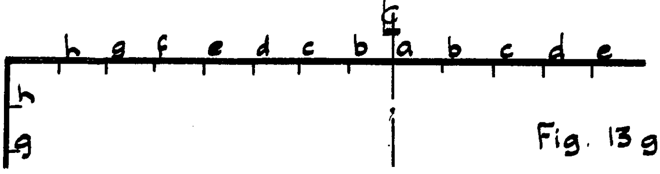


Fig. 13g

0.382a+0.648b+0.481c+0.388d+0.312e+0.239f+0.163g+0.083h = 3.481 1

0.324a+0.622b+0.520c+0.398d+0.316e+0.241f+0.164g+0.084h = 3.421 2

0.240a+0.519b+0.542c+0.451d+0.332e+0.248f+0.169g+0.087h = 3.243 3

0.194a+0.398b+0.452c+0.478d+0.389e+0.268f+0.180g+0.092h = 2.954 4

0.157a+0.316b+0.334c+0.389d+0.418e+0.327f+0.201g+0.103h = 2.547 5

0.120a+0.241b+0.250c+0.269d+0.328e+0.358f+0.264g+0.125h = 2.035 6

0.081a+0.164b+0.169c+0.181d+0.203e+0.262f+0.297g+0.193h = 1.429 7

0.041a+0.083b+0.086c+0.092d+0.110e+0.116f+0.181g+0.242h = 0.741 8

(The actual gradients on Fields Nos. 11 to 18 and Field No. 1, have been divided by 2 when forming the above equations).

Since the gradients are obtained by mechanical differentiation, any attempt to solve these equations using recognised methods such as that described by Morris[(5)] is likely to result in absurd values being obtained for the quantities a to h, and the following trial and error method is to be preferred.

Initial Values of b, d, f, and h, in terms of the other unknowns are taken as follows:-

$b = 0.75a + 0.25c$

$d = 0.75c + 0.375e - 0.125a$

$f = 0.75e + 0.375g - 0.125c$

$h = 0.75g - 0.125e.$

These values are substituted in equations 1, 3, 5, and 7, to give 4 equations for 4 unknowns, which are solved by direct elimination on the calculating machine. The values of a, c, e, and g, so found are then substituted in equations 2, 4, 6, and 8, from which new values are obtained for b, d, f, and h. These now replace the approximations used initially, and the above process is repeated until the desired degree of accuracy is obtained. A plot of the approximate values of a, b, c, d, etc., should be made from time to time to ensure that they lie on a regular curve. This precaution prevents wrong values from being obtained.

The values finally selected are,

$a = 1.668$

$b = 1.632$

$c = 1.502$

$d = 1.298$

$e = 1.028$

$f = 0.718$

$g = 0.403$

$h = 0.089,$

and a comparison of the actual and substituted values of the previous equations is given in the table below.

Equation	Required Value	Value obtained by substituting the above.
1	3. 486	3.481
2	3. 425	3.421
3	3. 242	3.243
4	2. 945	2. 954
5	2. 539	2.547
6	2. 029	2.035
7	1. 427	1.429
8	0. 743	0.741

Settled Fields, of ζ and w values, for the ζ values, a to h, on the boundary, are then obtained from Fields Nos. 11 to 19.

These are then added algebraically to Field No. 1.

The solution thus obtained for the clamped square plate loaded with a uniformly distributed load is shown on Field No.44.

Bending Moments.

The maximum value of the fixing moment occurs at the centre points of the supports and is equal to $-3.336\ p\ N^2$, per unit of length.

At the centre of the plate, $M_x = M_y = (1 + \sigma)1.125\ p\ N^2$, per unit of length.

Deflection.

The maximum deflection at the centre of the plate $= 5.173\ N^4\ C$. This is less than one-third of the deflection of the simply supported plate.

Shearing Force along the Boundary.

The values at various points on the boundary are shown in Fig. 13. These are obtained as previously described for the simply supported plate.

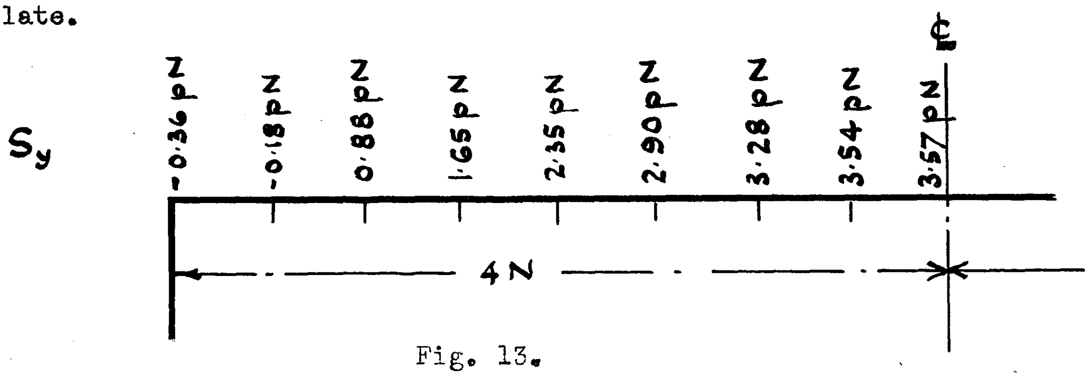


Fig. 13.

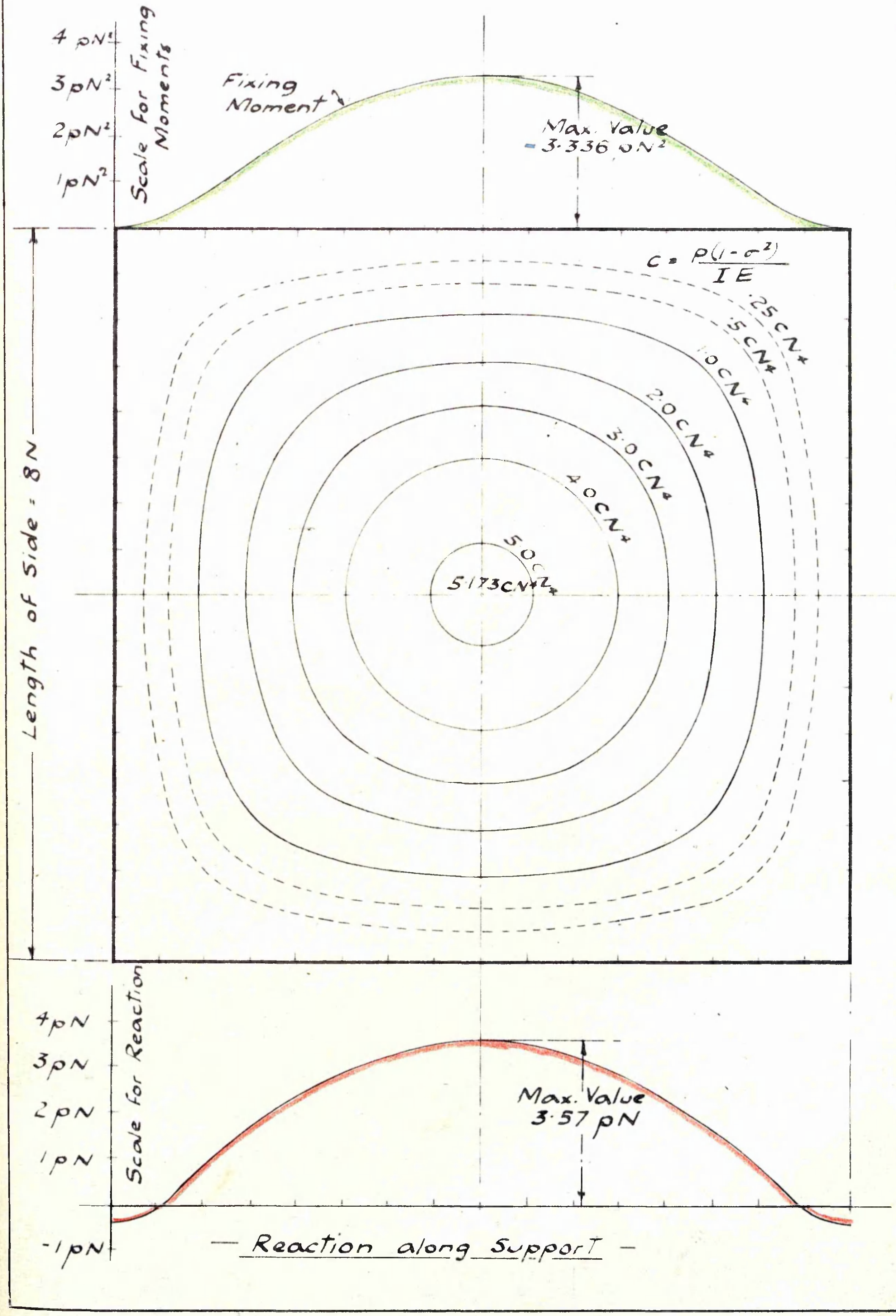
No special significance is attached to the small negative shear at the corners of the plate since it may be due to imperfections in the solution to the problem. The total applied load of $64\ p\ N^2$ is however obtained by integrating, as formerly, along the boundaries.

Contours of deflection, together with the distribution of shearing force and bending moment along the boundaries, are shown in Diagram No. 2. (page 22)

Diagram No 2

Square Plate, of uniform thickness, clamped along the edges and loaded with a uniformly distributed load.

Deflection Contours, Etc.



A solution to the problem of the clamped plate is given by Professor Inglis,(4) and, in the discussion of his paper, an interesting resumé of the historical development of this special problem is contributed by Professor A. N. Krilloff. An extract is as follows:-

"The problem, of which Professor Inglis has just exposed his beautiful solution, is of long date. The fundamental differential equation was given by Mademoiselle Sophie Germain in 1812, if I remember rightly. Some ten years afterwards this problem was taken over by Navier, the founder of the mathematical theory of elasticity, who worked out the solution for the supported rectangular plate. Then it was treated by Poisson, and by Kirchhoff, who in 1850 gave the general boundary conditions.

Some twenty-five years ago the problem of the clamped plate was proposed by the Académie des Sciences, of Paris, as a subject for their highest mathematical distinction, the "Grand Prix de mathématique." This fact illustrates the difficulty of the problem and its importance. Several of the most celebrated mathematicians took part in the competition, among them, Monsieur Jacques Hadamard, Signor Tullio Levi Civita, Herr Dr. Korn, and the late Walter von Ritz. The prize was awarded to Monsieur Hadamard.

All these authors considered the problem from a purely abstract mathematical point of view, their investigations could hardly be used by practical engineers and constructors, because the solutions were not adapted for numerical computation. Only the method of von Ritz yielded practical applications, but the numerical work was very laborious.

About the same time Professor B. M. Koialovitch, of St. Petersburg, took this problem as the subject of his dissertation for the degree of Doctor of Mathematics. Being a professor at the Technological Institute, he considered the practical applications also, and developed a method of successive approximations for the numerical calculations, which he illustrated by the clamped 1 : 2 plate as an example............

.....About the year 1908 Professor S.P. Timoshenko and his pupils at the Polytechnic Institute of St. Petersburg applied Ritz's method, and modified it in such a manner as to greatly simplify the calculations. Their investigations are published in the Transactions of the Polytechnic Institute, in Russian, of course, which means for Western Europe almost the same as Chinese!"

A comparison between the results obtained from Professor Inglis' solution and those given by the arithmetical method is tabulated below.

	Inglis	Arithmetical
Max. central deflection	5.186 p N^4/K	5.173 p N^4/K
Max. B.M. at centre of support	-3.3715 p N^2	-3.336 p N^2
Max. B.M. at centre of plate. (σ = 0.3)	+1.466 p N^2	+1.463 p N^2
Max. Shear at centre of edge	3.6728 p N	3.57 p N
Negative Shear at corner of plate	-0.6244 p N	-0.36 p N

It may be noted that, with the exception of the negative shear at the corners of the plate, the agreement between the respective values is good. The deflections throughout the plate surface also compare very favourably with those given by Professor Inglis.

<u>Square plate, simply supported along the edges and loaded with a uniformly distributed load, applied along a centre line.</u>

A square plate, A,B,C,D, of side 8 N, simply supported along the edges AB,BC,CD,DA, and loaded with a load of amount $\overline{W}$, uniformly distributed along the centre line a-a, is shown in Fig. 14.

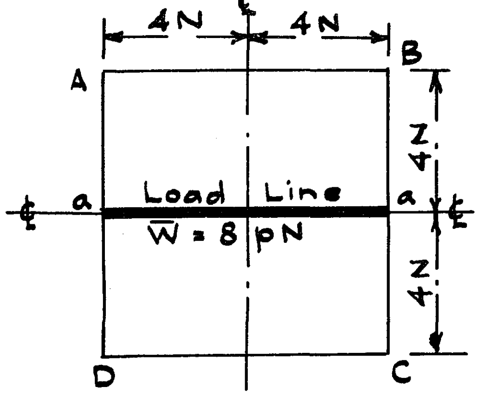


Fig. 14.

The boundary conditions to be fulfilled in this problem are, w = 0 and ζ = 0 at all points on the boundaries.

In this example, the solution is obtained by making use of the particular integral and complementary function methods. For the first part, the supports AD and BC are considered to be removed and the plate is regarded as a simple span loaded with a central load $\overline{W}$. The curvatures and deflections throughout the released boundaries are then eliminated by applying equal and opposite values from the $\nabla^4 w = 0$ fields.

For a beam, of unit width and span L, loaded with a concentrated load of amount p, Fig. 15, the deflection w of a point distant y from the centre is given by the equation,

$$w = -\frac{p}{2IE}\left(\frac{L}{4}y^2 - y^3/6\right) + pL^3/48\,IE$$

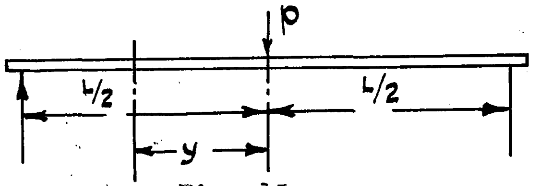


Fig. 15.

Using 8 N instead of L and replacing EI by K, the particular integral for the plate is,

$$w = -\frac{p}{2K}\left[2Ny^2 - y^3/6\right] + 32\,pN^3/3K \quad , \quad \text{where} \quad p = \frac{\overline{W}}{8N}.$$

$$\therefore \frac{\partial w}{\partial y} = -\frac{p}{2K}\left[4Ny - y^2/2\right], \text{ and,}$$

$$\frac{\partial^2 w}{\partial y^2} = -\frac{p}{2K}\left[4N - y\right].$$

$$\therefore \zeta = \frac{1}{2}\frac{\partial^2 w}{\partial y^2} = -\frac{p}{4K}\left[4N - y\right].$$

ζ and w values, as tabulated below, are obtained from the above.

y	ζ values. - pN/K	w values pN^3/K
0	1.000	10.666
0.5N	0.875	10.427
1.0N	0.750	9.750
1.5N	0.625	8.698
2.0N	0.500	7.333
2.5N	0.375	5.719
3.0N	0.250	3.917
3.5N	0.125	1.990
4.0N	0.000	0.000

To make the ζ and w values everywhere zero on the boundary, a settled field for $\nabla^4 w = 0$ is required with the boundary values shown in Fig.16.

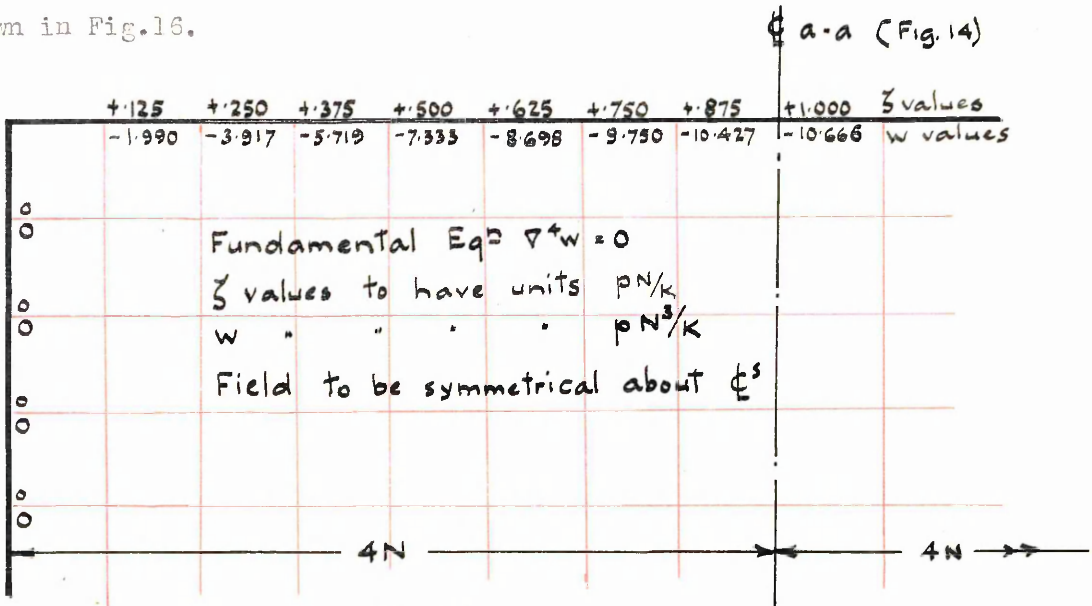


Fig. 16.

This Field may be obtained, either, by gathering values from Fields Nos. 21 to 28, or, by squaring in a similar manner to that already described. If the first method is adopted, the ζ values are obtained first. The w values are made up of two parts,

(a) those produced by the ζ values on the boundary, and

(b) those due to the w values on the boundary. With regard to (b), the 10 on the boundary of Fields Nos. 21 to 28 is regarded as a w value and the ζ Field is used.

Thus, to find the ζ and w values at the centre point of the Field outlined in Fig. 16.

ζ value.

1 Field No.	2 1/10 Central Value ζ, Fields 21 to 28	3 Boundary Value Fig. 16.	4 Col. 2 × Col. 3.
21	.0518	1.000	0.0518
22	.1007	0.875	0.0881
23	.0935	0.750	0.0701
24	.0817	0.625	0.0511
25	.0671	0.500	0.0336
26	.0512	0.375	0.0192
27	.0344	0.250	0.0086
28	.0175	0.125	0.0022
		Total	0.3247

The required ζ value is therefore 0.3247 p N/K.

w values

(a)

1 Field No.	2 1/10 Central Value w, Fields 21 to 28	3 Boundary Value Fig. 16.	4 Col. 2 × Col. 3.
21	.4721	1.000	0.4721
22	.9217	0.875	0.8065
23	.8658	0.750	0.6494
24	.7704	0.625	0.4815
25	.6472	0.500	0.3236
26	.5036	0.375	0.1889
27	.3447	0.250	0.0862
28	.1763	0.125	0.0220
		Total	3.0302 (a)

(b)

1 Field No.	2 1/10 Central Value ζ, Fields 21 to 28	3 Boundary Value Fig.16.	4 Col. 2 × Col. 3.
21	.0518	10.666	0.5530
22	.1007	10.427	1.0500
23	.0935	9.750	0.9116
24	.0817	8.698	0.7106
25	.0671	7.333	0.4920
26	.0512	5.719	0.2928
27	.0344	3.917	0.1347
28	.0175	1.990	0.0348
		Total	4.1795 (b)

Adding (a) & (b), the required w value is - 7.2097 p N^3/K.

The remaining values at points throughout the field are obtained in the same manner, and when the field is completed it is added algebraically to the field of the particular integral.

The final Field is shown in Field No.45. It may be noted that the ζ and w values are zero at all points on the boundaries and the fundamental equation $\nabla^4 w = 0$ is satisfied at all points except those on the load line.

The ζ and w values for points on the load line can be calculated, if desired, from the following formulae,

$$12\zeta_c = \sum a + 2\sum b - p\,N/K \qquad \text{(2g)}$$

$$12w_c = \sum A + 2\sum B - 2\,N^2\zeta_c - p\,N^3/16\,K \qquad \text{(2h)}$$

The above are obtained thus:-

For a square of side 4 n, Fig. 17, loaded with a line load of amount p per unit of length across the centre line 1-1, the central ζ value is obtained by considering squares 1 to 4 in turn.

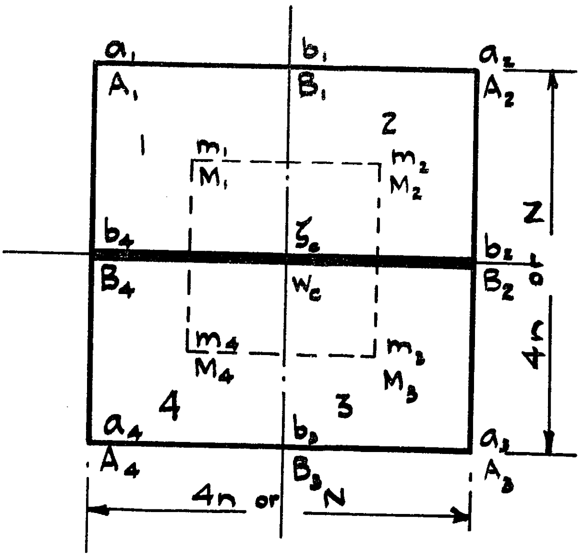


Fig. 17.

$$4\,m_1 = a_1 + b_1 + \zeta_c + b_4$$

$$4\,m_2 = a_2 + b_2 + \zeta_c + b_1$$

$$4\,m_3 = a_3 + b_3 + \zeta_c + b_2$$

$$4\,m_4 = a_4 + b_4 + \zeta_c + b_3$$

$\therefore\quad 4\sum m = \sum a + 2\sum b + 4\,\zeta_c$, and for the central square, shown in a dotted line in Fig. 17,

$4\zeta_c = \sum m - p\, n/K$, by analogy with a beam with fixed ends.

If $N = 4\,n$,

$$4\zeta_c = \sum m - p\, N/4K, \text{ and therefore,}$$

$$16\,\zeta_c = 4\,m - p\, N/K,$$

$$= \sum a + 2\sum b + 4\zeta_c - p\, N/K.$$

$$\therefore\quad 12\,\zeta_c = \sum a + 2\sum b - p\, N/K. \qquad \text{(2g)}$$

Similarly, for the w values, taking squares 1 to 4 as above,

$4\sum M = \sum A + 2\sum B + 4\,w_c - 4\,n^2\sum m$, and, for the central square,

$4\,w_c = \sum M - 4\,n^2\zeta_c$, and therefore,

$$16\,w_c = 4\sum M - 16\,n^2\zeta_c$$

$$= \sum A + 2\sum B + 4\,w_c - 4\,n^2\sum m - 16\,n^2\zeta_c$$

$$\therefore\ 12\,w_c = \sum A + 2\sum B - 2\,N^2\zeta_c - p\,N^3/16\,K. \qquad \text{(2h)}$$

An example, illustrating the use of formulae (2g) and (2h), is as follows.-

The central square of Field No. 45 is reproduced in Fig.18. It will be remembered that the values are negative and the w values positive.

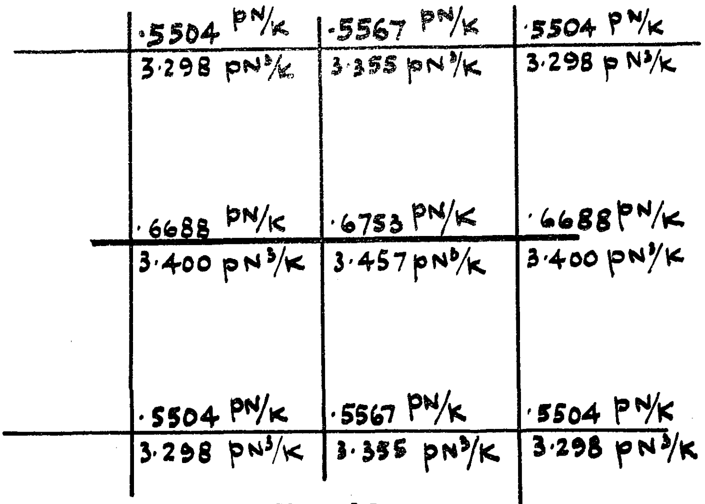


Fig. 18.

$12\,\zeta_c = \Sigma a + 2\Sigma b - p\,N/K$

$\therefore 12\,\zeta_c = -\,8.1036\ p\,N/12\,K$

$\therefore \zeta_c = -\,0.6753\ p\,N/K.$

$\Sigma a =$	$-\ 2.2016$	pN/K
$2\Sigma b =$	$-\ 4.9020$	"
$-pN/K =$	$-\ 1.0000$	"
Total	$-\ 8.1036$	"

$12\,w_c = \Sigma A + 2\Sigma B - 2\,N^2\zeta_c - p\,N^3/16\,K$

$\Sigma A =$	13.1920	pN^3/K
$2\Sigma B =$	27.0200	"
$-\,2\,N^2 =$	1.3506	"
	41.5626	"
$-\,p\,N^3/16K$	$.0625$	"
	41.5001	"

$\therefore w_c = 41.5001\ p\,N^3/12K$

$= 3.458\ p\,N^3/K.$

Deflections, shearing forces etc., are obtained from the settled Field No. 45, in a similar manner to that described for the simply supported plate.

The maximum deflection, at the centre point of the plate, is $3.457\ p\,N^3/K = 0.432\ \overline{W}N^2/K$, where $\overline{W}$ is the total applied load.

Bending Moments at the centre of the plate.

At the centre point, $\zeta_c = -\,0.6753\ p\,N/K$, and therefore,

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 2\zeta_c = -\,1.3506\ p\,N/K\,.$$

In this example, $\frac{\partial^2 w}{\partial y^2}$ and $\frac{\partial^2 w}{\partial x^2}$ are unequal, but by using the expression D 1, on page 10, they are estimated as 0.875 p N/K and 0.466 p N/K respect--ively, $\frac{\partial^2 w}{\partial y^2}$ being at right angles to the load line.

The bending moments, M_y and M_x are then,

$M_y = (\ 0.875 + \sigma\ 0.476\ )\ p\,N$, per unit of length,

$M_x = (\ 0.476 + \sigma\ 0.875\ )\ p\,N$, per unit of length.

Shearing Force along the boundaries.

The shearing forces S_x and S_y are shown in Fig. 19. The maximum intensity occurs at the centre point of the boundaries at right angles to the load line. This maximum value is 1.37 p, per unit of length, where $p = \overline{W}/8N$.

The applied load, 8 p N, is obtained by integrating the values along the boundaries.

Shearing Force along the boundaries

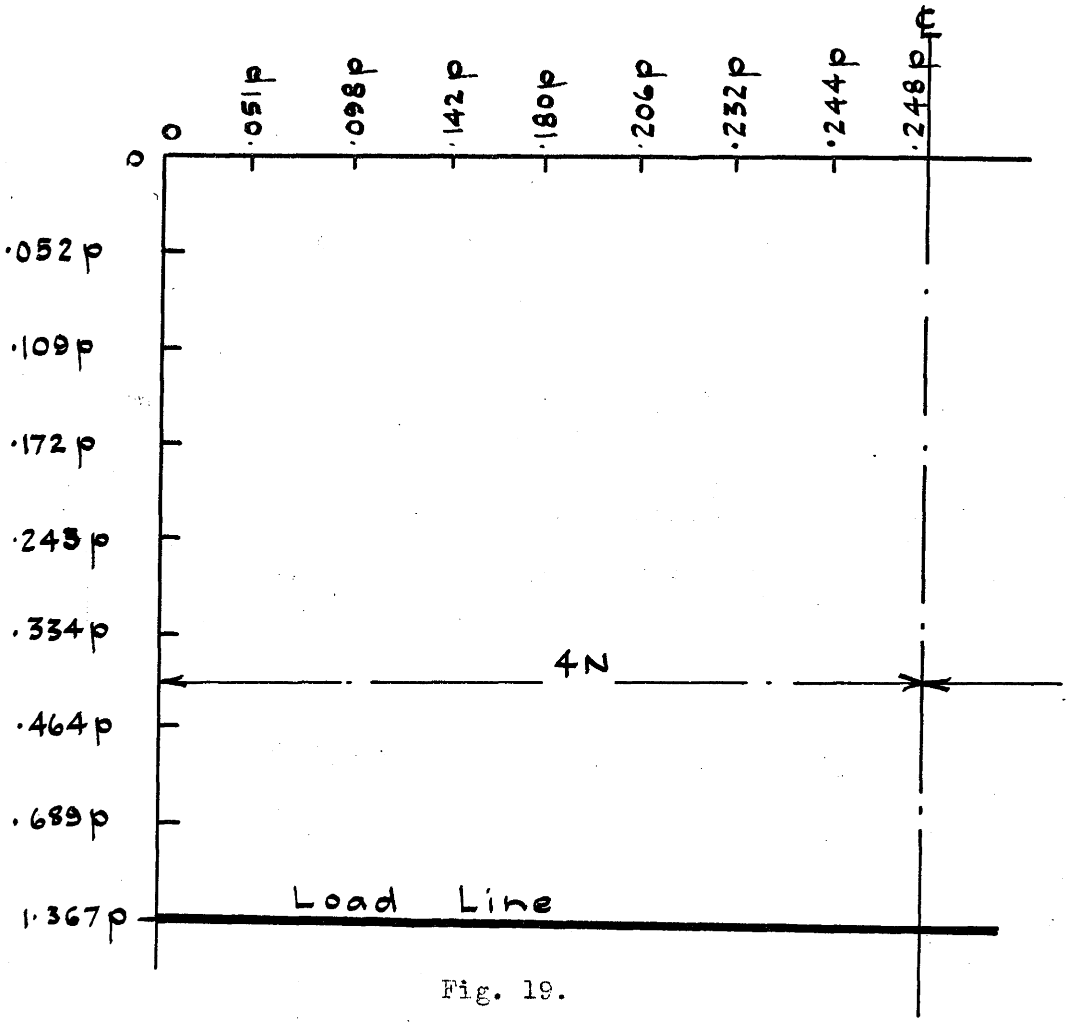


Fig. 19.

Torque along the boundaries

The torques per unit of length, along the boundaries of the plate, are shown in Fig.20.

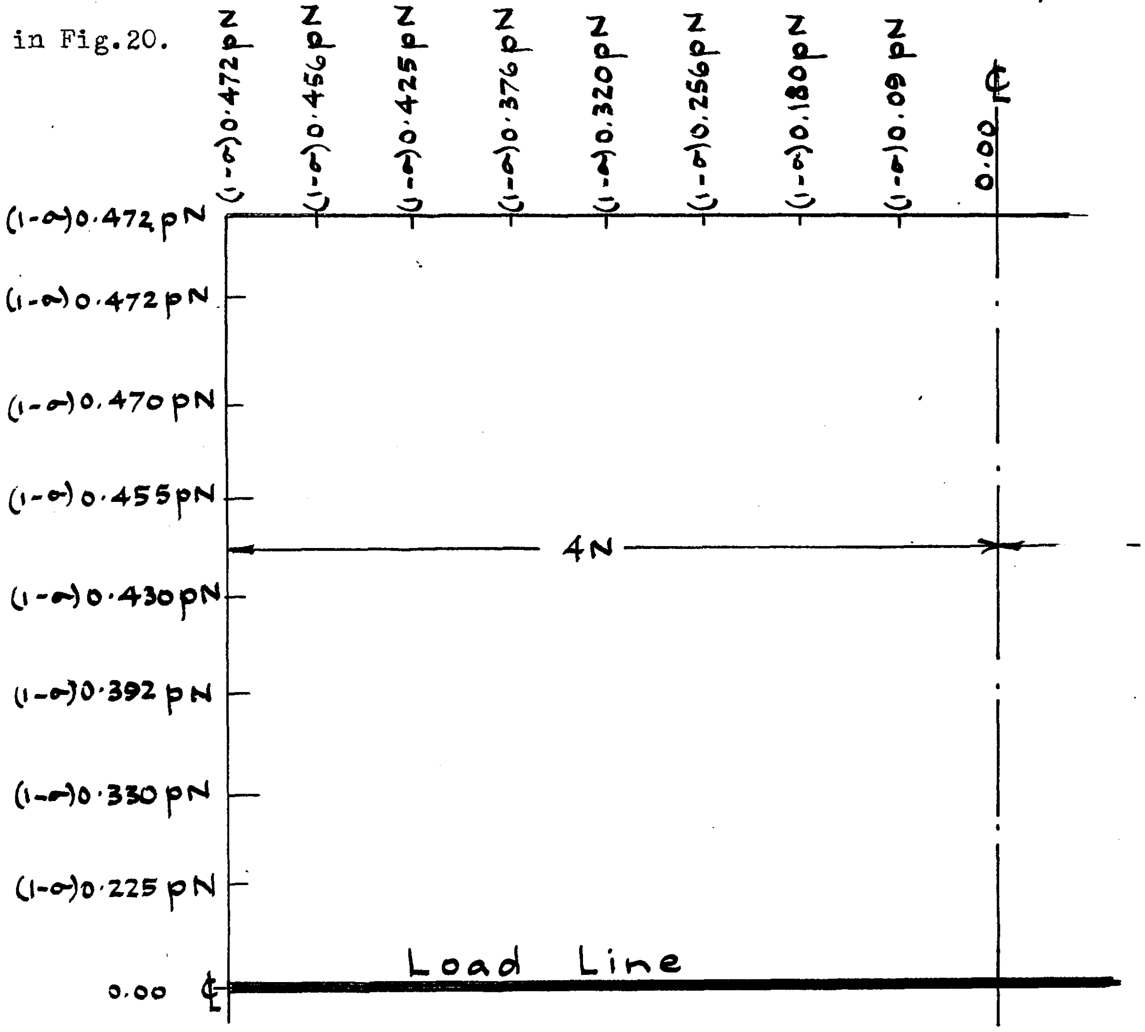


Fig.20.

The corner load $= 2\,T_c = (1 - \sigma)\ 0.944\ p\ N,$

$= (1 - \sigma)\ 0.118\ \overline{W}$, where $\overline{W}$ = total load on the plate.

Reaction along the boundaries.

The reactions, $R_x = S_x + \frac{\partial T}{\partial y}$, and $R_y = S_y + \frac{\partial T}{\partial x}$, are obtained in the same manner as previously described for the uniformly loaded plate. The maximum values are,

$R_x = (2.02 - \sigma 0.65)p$, at the centre point of the boundary at right- angles to the load line,

and,

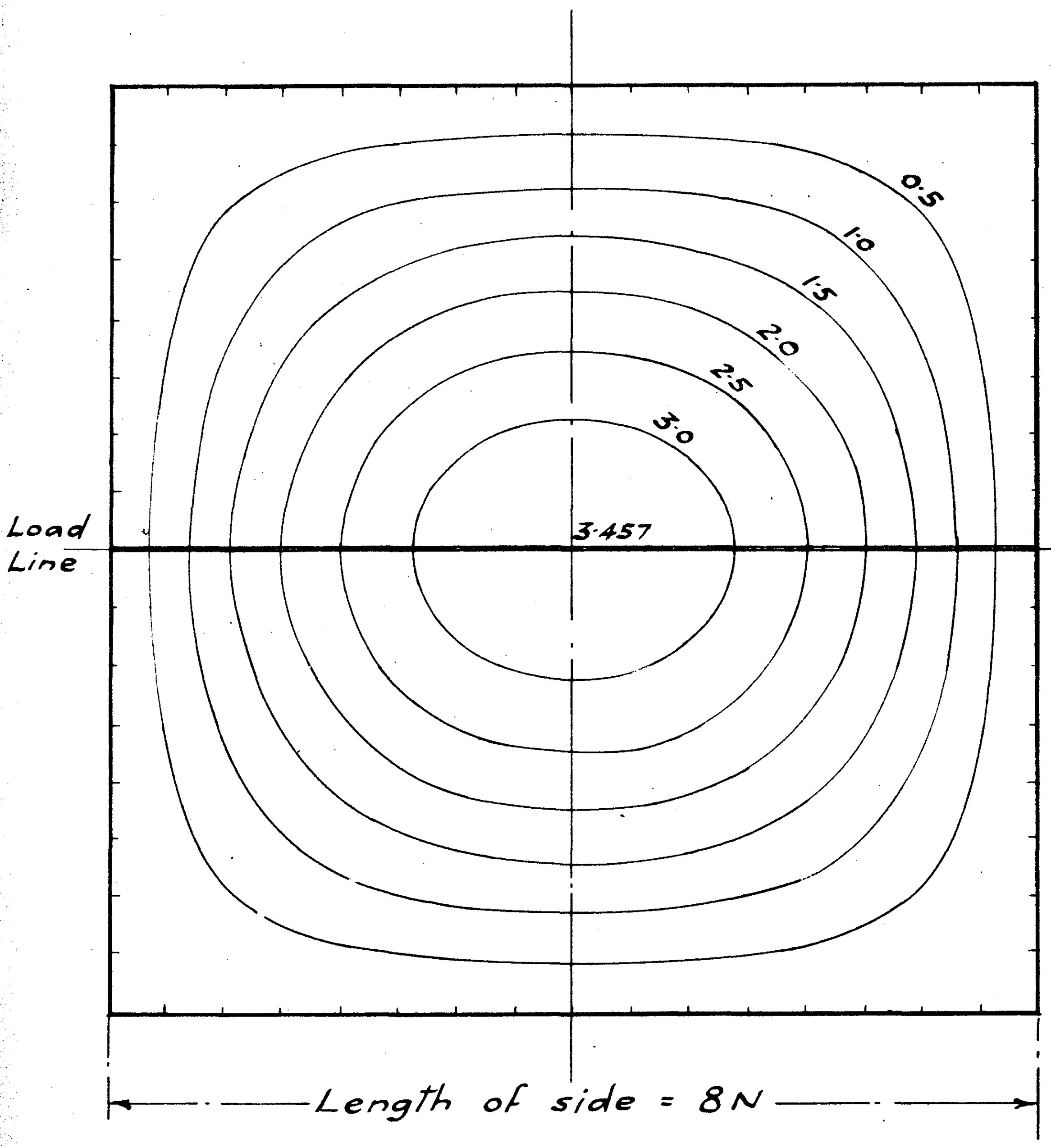
$R_y = (0.43 - \sigma 0.18)p$, at the centre point of the boundary parallel to the load line.

Contours of deflection are plotted on Diagram No. 3 , and the distrib- -ution of Shearing Force, Torque, and Reaction, along the respective boundaries, are shown to scale on Diagram No. 4 .

Diagram No. 3

Square Plate, of uniform thickness, simply supported at the edges and loaded with a uniformly distributed line load across a centre line.

— Deflection Contours —



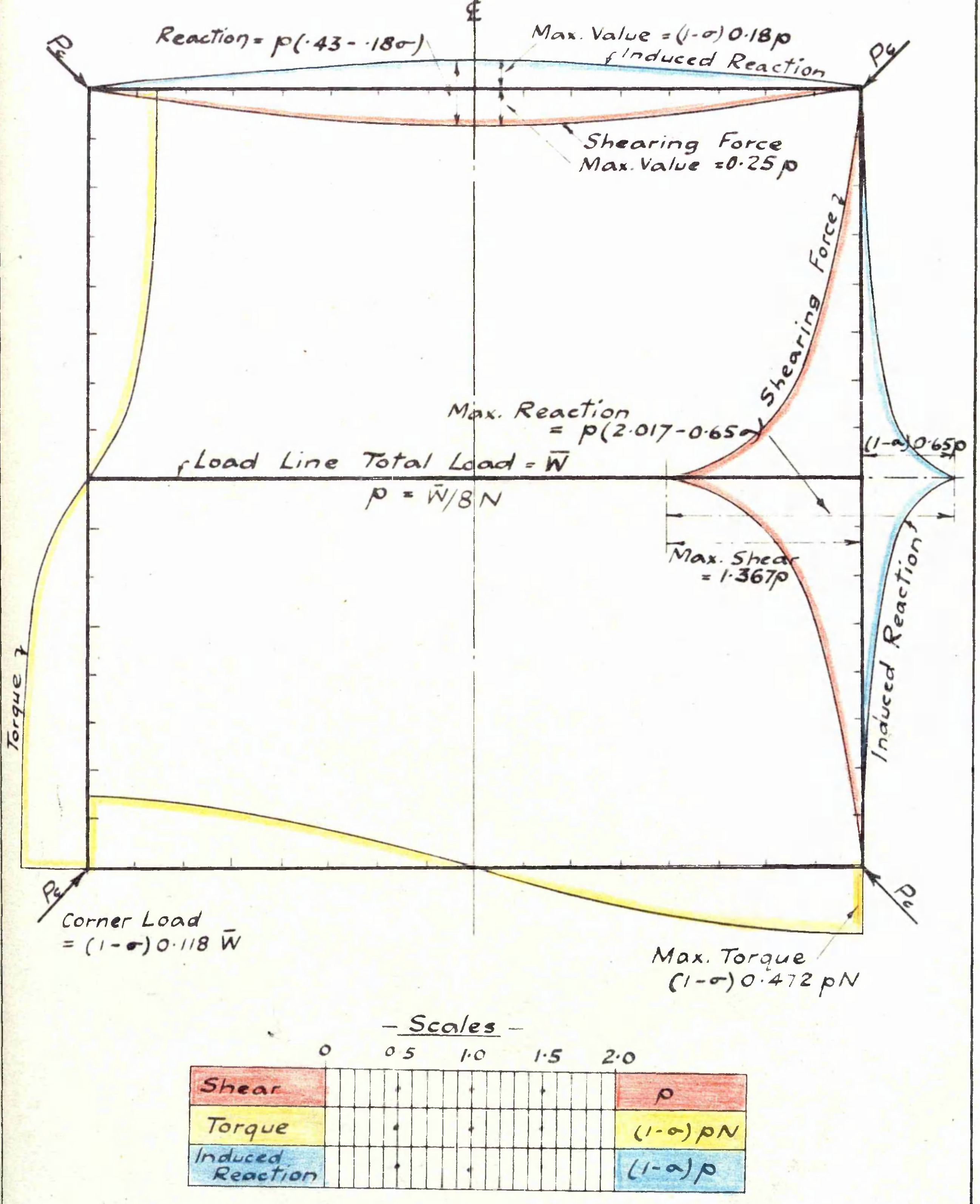
Maximum Deflection = 3·457 pN^3/K

Total Load = 8pN

Diagram No. 4

Square Plate, of uniform thickness, simply supported along the edges, and loaded with a uniformly distributed line load across a centre line.

Distribution of shearing force, reaction and torque along the edges



Square Plate of side 8 N, simply supported along the four edges, and loaded with a uniformly distributed load along a line parallel to an edge and distant 2 N therefrom.

A B C D, Fig. 21, represents a square plate of side 8 N, simply supported along the edges and loaded with a uniformly distributed load of amount $\overline{W}$ along the line b-b.

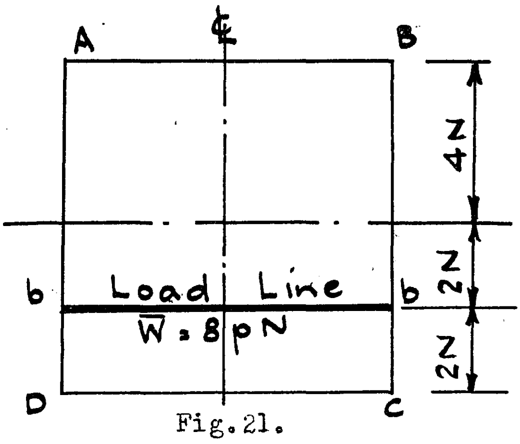


Fig. 21.

The boundary conditions in this problem are the same as in the preceding one and the method of solution is also similar.

With the supports AD and BC removed, then, for a beam of span L and unit width, loaded with a concentrated load p at the quarter point of the span, the following expressions for deflection are available. Fig. 22.

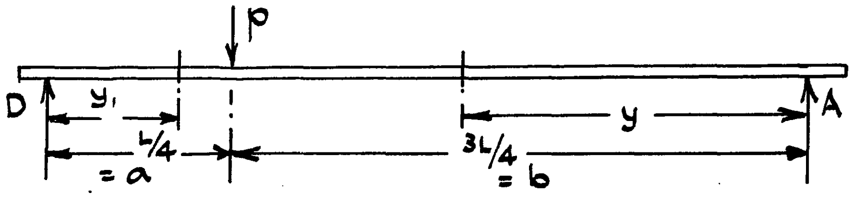


Fig. 22.

For sections on the long segment distant y from A,

$$w = \frac{pay}{IEL}\left(\frac{b^2 + 2ab - y^2}{6}\right),$$

and, for sections on the short segment distant y_1 from D,

$$w_1 = \frac{pby_1}{IEL}\left(\frac{a^2 + 2ab - y_1^2}{6}\right).$$

Substituting 2N for a, and 6N for b, and using the flexural rigidity K for the plate, the above become,

$$w = \frac{py}{24K}(60N^2 - y^2), \text{ and,}$$

$$w_1 = \frac{py_1}{8K}(28N^2 - y_1^2).$$

$$\therefore \frac{\partial^2 w}{\partial y^2} = \frac{-py}{4K} = 2\zeta, \text{ and,}$$

$$\frac{\partial^2 w_1}{\partial y_1^2} = \frac{-3py_1}{4K} = 2\zeta.$$

The ζ and w values, at regular intervals of N/2, are obtained from the above and are as tabulated on the next page.

Section		ζ Values $-pN/K$	w values pN^3/K
x	y		
0		0	0
0.5N		0.1875	1.7344
1.0N		0.3750	3.3750
1.5N		0.5625	4.8281
2.0N	6.0N	0.7500	6.0000
	5.5N	0.6875	6.8177
	5.0N	0.6250	7.2917
	4.5N	0.5625	7.4531
	4.0N	0.5000	7.3333
	3.5N	0.4375	6.9635
	3.0N	0.3750	6.3750
	2.5N	0.3125	5.5990
	2.0N	0.2500	4.6667
	1.5N	0.1875	3.6094
	1.0N	0.1250	2.4583
	0.5N	0.0625	1.2448
	0	0	0

The corrections for the above boundary values are obtained in the same manner as in the previous example. By adding these to the particular integral fields the solution to the problem is obtained. This is shown on Field No.46. (Because of the lack of symmetry in this case, the labour necessary in grouping correction values from the $\nabla^4 w = 0$ fields is very great. In actual practice, however, only a few values along the load and centre lines would be required for design purposes).

It may be noted that the ζ and w values along the line 1-1 on Field No. 45 are reciprocated along the line 2-2 on Field No. 46. Field No.45 can therefore be built up from Field No,46 thus:-

A section through the centre line of Field No.46 is shown in Fig. 23a. By extending this to AC and BD as shown in red, a field is obtained for a rectangular plate, 24 N by 8 N, loaded with two upward loads at J and G and a downward load at F, A and B being points of contra-flexure. If the ζ and w values on the part EF are added algebraically to those in the part HG, Field No.45 is obtained.

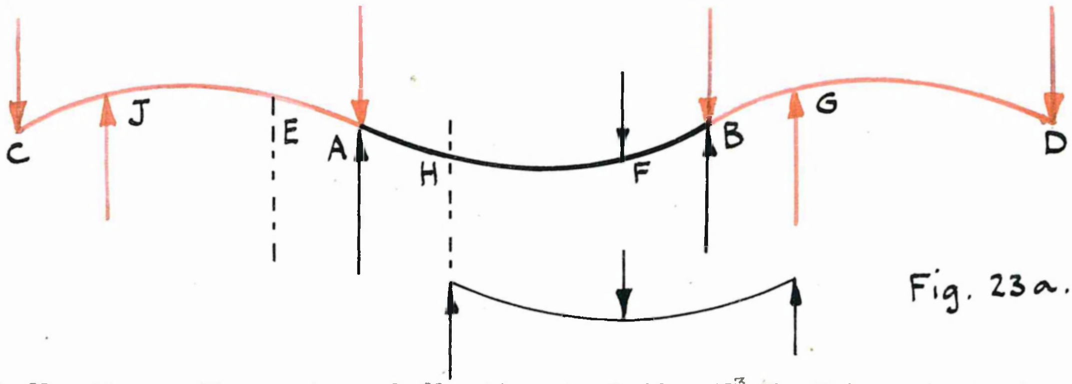


Fig. 23a.

<u>Deflection.</u> The maximum deflection is 2.49 p N^3/K. This occurs at a point on the centre line of the plate approximately mid-way between the load and the centre point. Deflection Contours are plotted to scale on Diagram No.5.

<u>Shearing Force.</u> The shearing force values at points on the boundaries are shown in Fig.23. A maximum value of 1.323 p, per unit of length, is reached

at the point where the load line meets the boundary.

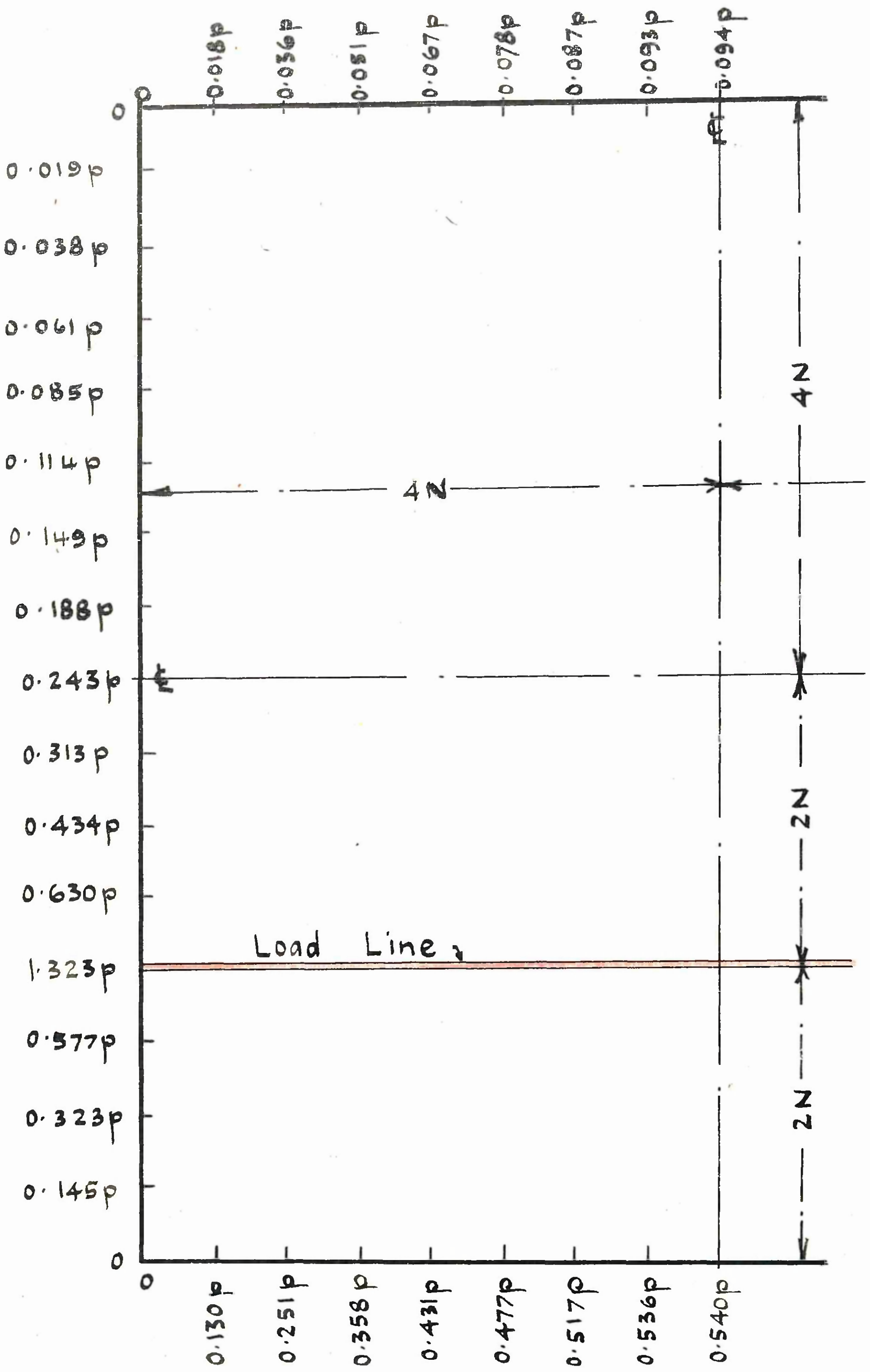


Fig. 23

Torque and Reaction along the Boundaries. The maximum torsional couples per unit of length occur at the corners of the plate nearest to the load line. At these corners the torques are $(1-\alpha)0.6$ pN, at the others $(1-\alpha)$ 0.24 pN. The corner loads are therefore, $(1-\alpha)$ 1.20 pN and $(1-\alpha)$ 0.48 pN. The maximum reaction, per unit of length, is equal to $(1.383 - 0.56\alpha)$p.

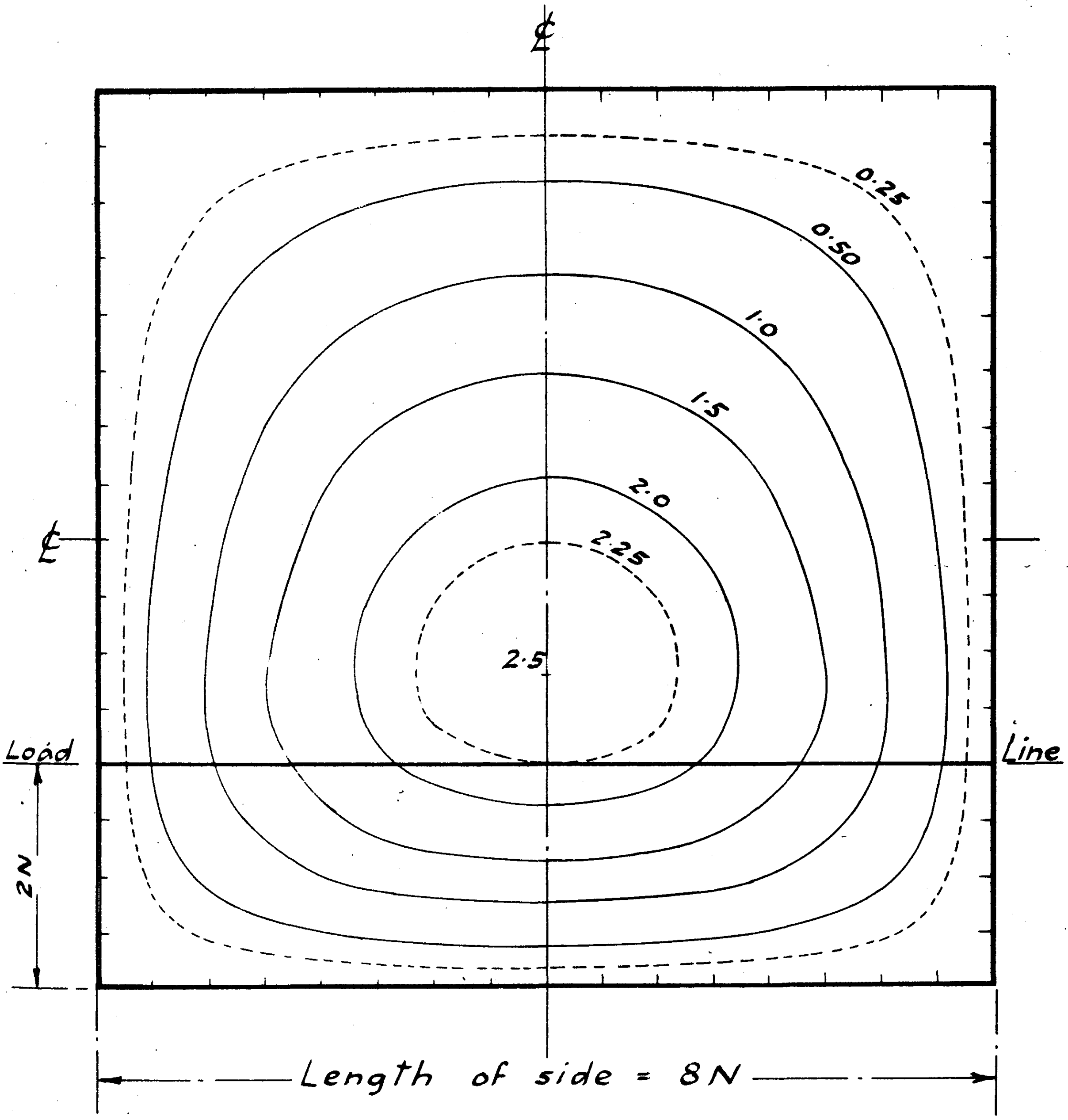
The distribution of shearing force, reaction, torque etc., are plotted on Diagram No.6.

Other useful solutions, with two line loads in various positions on the plate, are readily obtained by using superposition methods similar to that previously described and illustrated on page 35.

Diagram No. 5

Square Plate, of side 8 N and uniform thickness, simply supported at the edges and loaded with a uniformly distributed line load across a line parallel to an edge and distant 2 N therefrom.

— Deflection Contours —



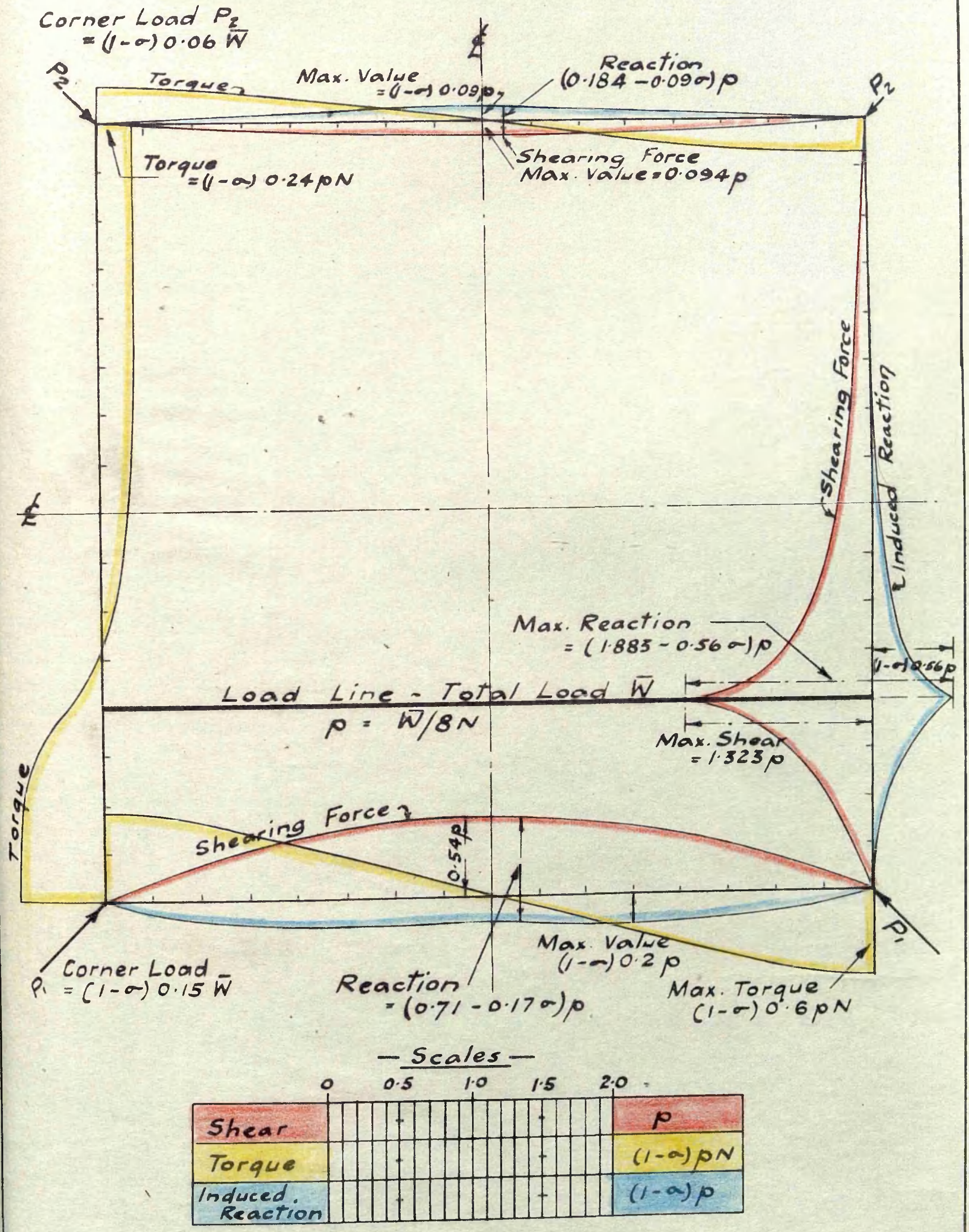
Maximum Deflection = $2{\cdot}5\,pN^3/k$

Total Load = $8\,pN$

Diagram No. 6

Square Plate, of side 8N and uniform thickness, simply supported at the edges and loaded with a uniformly distributed line load across a line parallel to an edge and distant 2N therefrom.

Distribution of shearing force, reaction and torque along the edges.



<u>Square plate of uniform thickness, simply supported along the edges and loaded with a central concentrated load of amount $\overline{W}$.</u>

The fundamental equation for the deformation w in this problem is $\nabla^4 w = 0$. This applies to all points except under the load.

The ζ and w values are zero on the boundaries.

With a point load, the stresses at the point of contact are infinite, but a point load exists only in a mathematical sense. In every practical case the load is distributed over an area of contact. The problem is then solved by assuming that the load is uniformly distributed and that $\nabla^4 w$ = a constant is the fundamental differential equation throughout the area of contact. Or, the load may be concentrated as a line load over a small length of the plate, in which case the formulae developed previously are applicable.

The results depend to some extent on the size and shape of the area of concentration, but the method which is described later is readily extended to meet most cases provided the solutions are known for the particular area or length assumed. If the loaded area is a square, the solution for the fully loaded simply supported square plate is necessary; if a circular or elliptical area is assumed, the solution for the fully loaded simply-supported circular or elliptical plate is necessary, etc., etc.

To get a solution to the specified problem a value of ζ equal to - 10 units is applied in the first instance at the centre point of a square field of side 8N and a settled field obtained for $\nabla^2\zeta = 0$. In this case the formulae applicable to every square but the central square are (2e) and (3c). The central value, $\zeta = -10$, is kept constant and no squaring is made on the central square. From the settled ζ field, the w values satisfying $\nabla^4 w = 0$ are obtained using formulae (2f) and 3(d), but, in this case squaring is allowed to take place across the central square. The settled fields thus obtained are shown on Field No. 47.

The ζ field represents to some scale the required ζ field for the concentrated load $\overline{W}$, but the w field requires further modification since no account was taken of any load term when squaring across the central square.

The boundary values of $\frac{\partial\zeta}{\partial x}$, Field No. 47, also represent to some scale the shearing force along the supports, and the central load, which must be applied to give $\zeta = -10$ at the centre, is therefore obtained by integrating them along the boundaries.

Using Simpson's Rule for areas, the value of $\sum\left(\frac{\partial \zeta}{\partial x}\,dy\right) = 1.333 \times 11.356$. The central ζ value, for a load $\bar{W}$ at the centre, is therefore,

$$\frac{10}{K}\left(\frac{1/2\ \bar{W}}{1.333 \times 11.356}\right), \text{ since } \zeta = 1/2(\nabla^2 w).$$

$\therefore\ \zeta = -0.331\ \bar{W}/K$ is the required value at the centre of the plate loaded with the concentration $\bar{W}$.

The remaining ζ values throughout the field are obtained pro rata from Field No. 47, and are shown in Field No. 48.

In view of later remarks, it may be noted at this stage, that if the load W were assumed to be uniformly distributed as a line load over the length N/2 of the plate, the ζ value as obtained from Formula (2g) would have been $-0.328\ \bar{W}/K$.

The w values are then established as follows:-

A preliminary field is prepared using 0.0331 x (the w values on Field No.47). This field is then corrected for the load term in the squares formula.

Thus, in the square of side 4 n, Fig. 24, the concentrated load is applied at the centre point D.

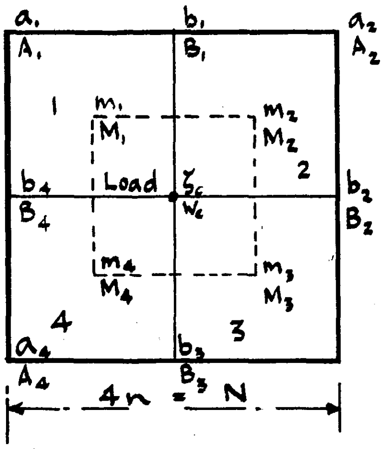


Fig. 24.

By taking the squares 1 to 4 in turn, as formerly,

$4\sum m = \sum a + 2\sum b + 4\zeta_c$.

In the central square $\nabla^4 w \neq 0$, and therefore,

$4\zeta_c = (m_1 + m_2 + m_3 + m_4) - Z$, where Z is a load term.

$\therefore\ 4\zeta_c = \sum m - Z,$

$\therefore\ 16\zeta_c = \sum a + 2\sum b - 4Z + 4\zeta_c$

$\therefore\ 12\zeta_c = \sum a + 2\sum b - 4Z.$ ....(a)

Similarly, for the w values,

$12\,w_c = \sum A + 2\sum B - 32\,n^2\zeta_c - 4\,n^2 Z$.................(b)

For N = 4 n, (a) and (b) become,

$12\zeta_c = \sum a + 2\sum b - 4\,Z$.................................(2j) and,

$12\ w_e = \Sigma A + 2\Sigma B - 2\ N^2\zeta_e - N^2 Z/4$ ....(2k).

To correct for the last term in the above, $N^2\ Z/48$ is subtracted from the central value in the preliminary field and the field adjusted to suit this deduction by the method of differences.

The Z value is found by applying (2j) to the central square. (i.e. the ζ values on Field No. 48.) Therefore,

$$\left[-\ 12 \times 0.331\right] = \left[-\ (4 \times 0.1476) - (8 \times 0.1703)\right] - 4\ Z/(K/\bar{W})$$

$\therefore\ Z = +\ 0.5049\ \bar{W}/K$, and,

$N^2 Z/48 = 0.0105\ \bar{W}\ N^2/K$.

The final fields are shown on Field No.48.

Bending Moments under the Load.

As mentioned earlier, infinite stresses are produced by a mathematical point load, but finite values are obtained by assuming that the load is distributed over an equivalent small area. In the following analysis it is assumed that:-

(1) the equivalent area is a square of side C,

(2) the thin plate theory is valid throughout the equivalent area,

(3) the ζ and w values of Field No. 48 are sufficiently accurate at all points except the load point.

The central square, A B C D, of side N, loaded with a uniform concentration over the square of side C is shown in Figure 25.

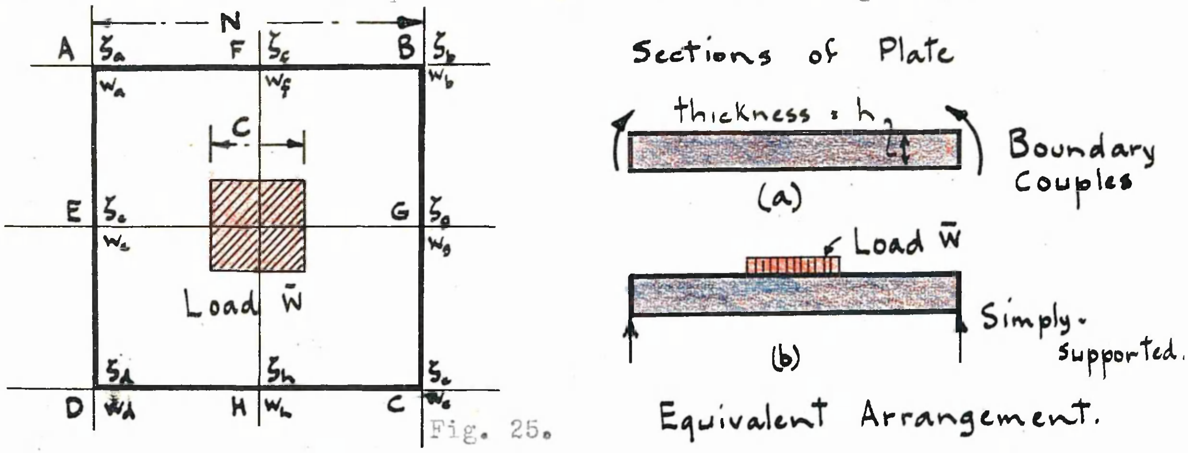


Fig. 25.

The isolated part is regarded as being equivalent to the sum of,

(a) a square plate bent by couples applied along the boundaries, $\nabla^4 w = 0$, being the fundamental equation, and,

(b) a square plate simply supported along the boundaries and loaded with the loading in square C, $\nabla^4 w$ = a constant, being the fundamental equation.

In (a), the ζ boundary values are taken from Field No. 48, and the central ζ value is calculated from the available squares formulae. For (b) the solution is obtained by applying the load over the whole area, A B C D, in the first instance.

Thus, the central square with the ζ values of Field No. 48 is shown in Fig. 26.

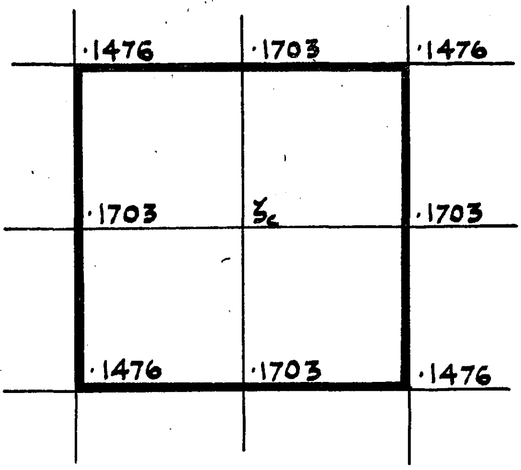


Fig. 26.

(a) ζ_c calculated from Formula (2e) = 0.1627 $\overline{W}/K$

(b) From Field No. 1, the ζ value at the centre of a uniformly loaded square plate of side N = 0.0369 $\overline{W}/K$

Adding (a) and (b) the ζ value of 0.1996 $\overline{W}/K$ is obtained at the centre of a square plate of side 8 N when a load is concentrated over a central square of side N. By symmetry, $M_x = M_y$, and therefore, if A is used instead of 8 N to denote the full plate dimension, then, for a ratio C/A = 1/8, $M_x = M_y = (1 + \sigma)\ 0.1996\ \overline{W}$, per unit of length.

This is repeated with a larger square from Field No. 48, say, for example, of side 2 N, Fig. 27.

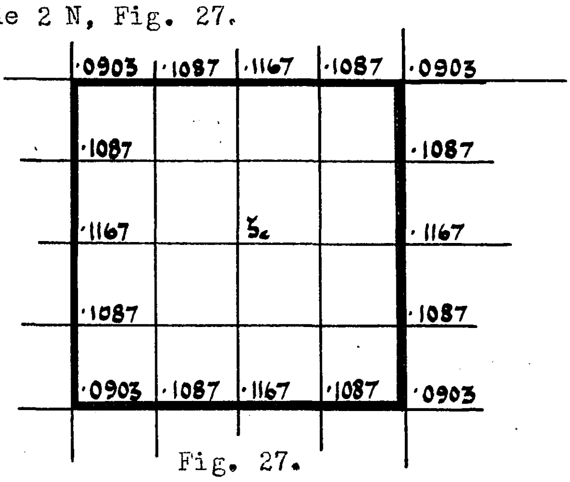


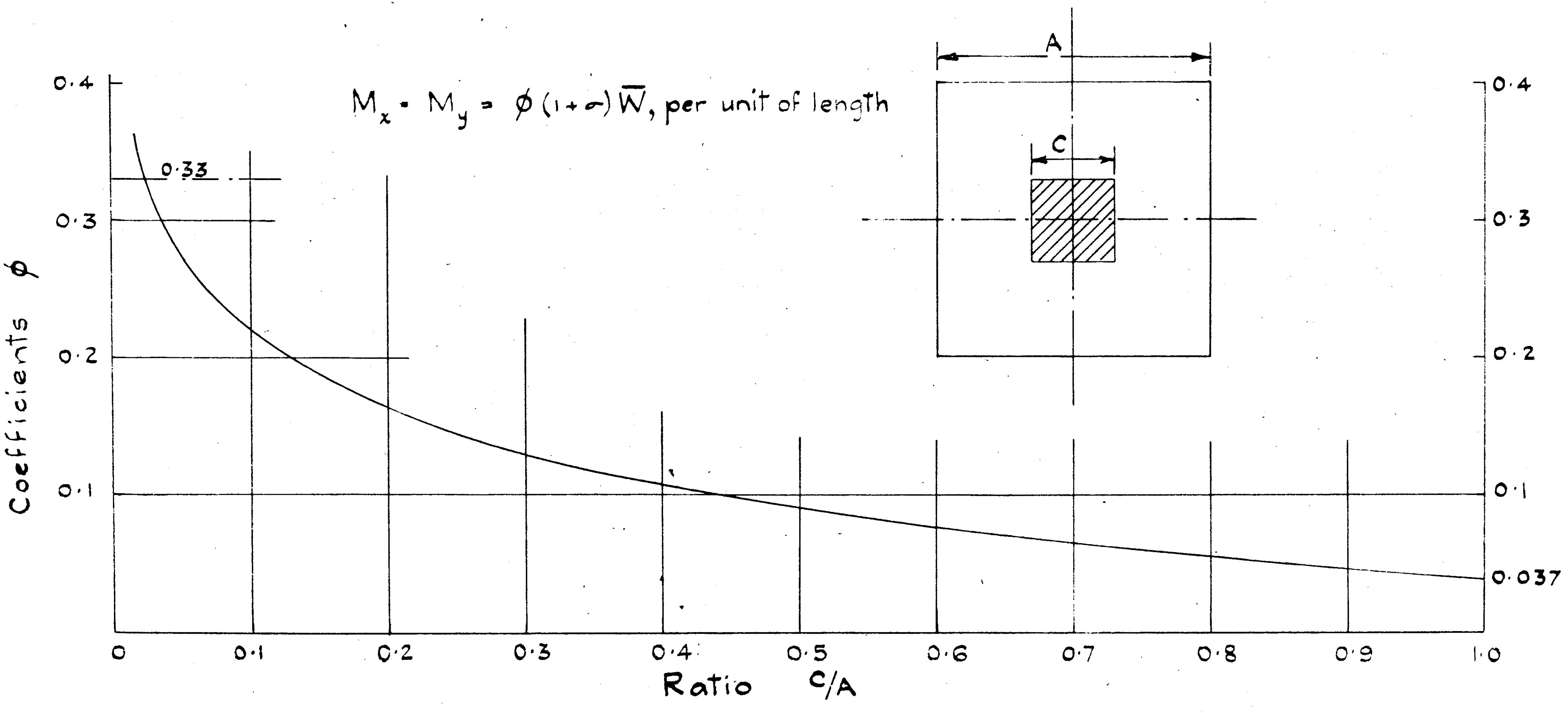
Fig. 27.

Again, the ζ_c value is found from Formula (3c). This gives $\zeta_c = 0.1104\ \overline{W}/K$. To this is added 0.0369 $\overline{W}/K$, and therefore for a ratio C/A = 1/4, $M_x = M_y = (1 + \sigma)\ 0.1473\ \overline{W}$, per unit of length. Incidentally, (0.1473 + 0.1104) gives the coefficient 0.258 for the ratio C/A = 1/16, and, (0.1473 + 0.1627) gives the coefficient 0.31 for C/A = 1/32.

In the above manner, bending moment coefficients for ratios C/A ranging from a very small finite value to unity are obtained. These are plotted on Diagram No. 7 .

Diagram No. 7

Bending Moment at the centre of a simply-supported Square Plate.
Load $\overline{W}$ uniformly distributed over a central square of side C.



If the load is concentrated over a circular area of diameter D, a cylindrical section of the plate is isolated in the first instance. Fig. 28.

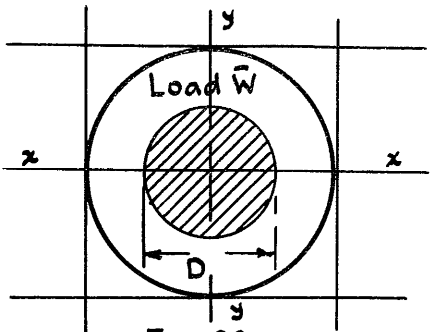


Fig. 28

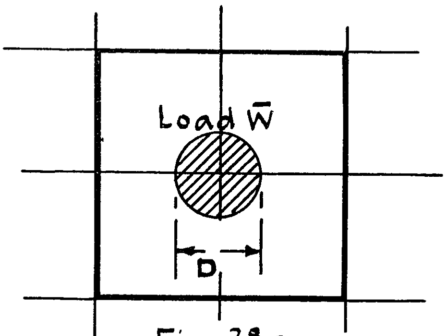


Fig. 28a.

If the circle is not too large in diameter, it is reasonable to assume that the value of ζ at the centre, produced by the boundary couples, part (a), is the same as the ζ value at the point where the circle cuts the x or y axis.

At the centre of a simply-supported circular plate, the radial and tangential flexural couples are both equal to $(3+\sigma)\bar{W}/16\pi$, where $\bar{W}$ is the total uniformly distributed load.

The bending moments under the load are found thus:-

For D = N and $\sigma = 0.3$,

M_a, part (a), $\nabla^4 w = 0$, $= 0.17\,(1+\sigma)\,\bar{W} \qquad = 0.221\ \bar{W}$

M_b, part (b), $\nabla^4 w = \text{constant}$, $= (3+\sigma)\,\bar{W}/16\pi \qquad = \underline{0.0657\ \bar{W}}$

$\therefore$ for a ratio D/A = 1/8, $M_x = M_y \qquad = 0.2867\ \bar{W}$

This is repeated with D = 2 N and D = 4 N for ratios D/A = 1/4 and 1/2, and these values are then used, as formerly, in conjunction with an isolated square section, Fig. 28 a, to obtain the solutions for smaller values of the above ratios. The results thus obtained, with different values of Poisson's ratio, are plotted on Diagram No. 8 .

For a high concentration of load, Westergaard[6] gives,

$D_e = 2\sqrt{0.4\,D^2 + h^2} - 1.35\,h$, where h is the plate thickness,

as the equivalent diameter D_e of the loaded disc to be used with the thin plate theory. For a point load, the equivalent diameter is therefore 0.65 h.

If the load is concentrated over a short length of a centre line, the same procedure may be followed using the previous line load solution. Thus, for a loaded length L and a ratio L/A = 1/8,

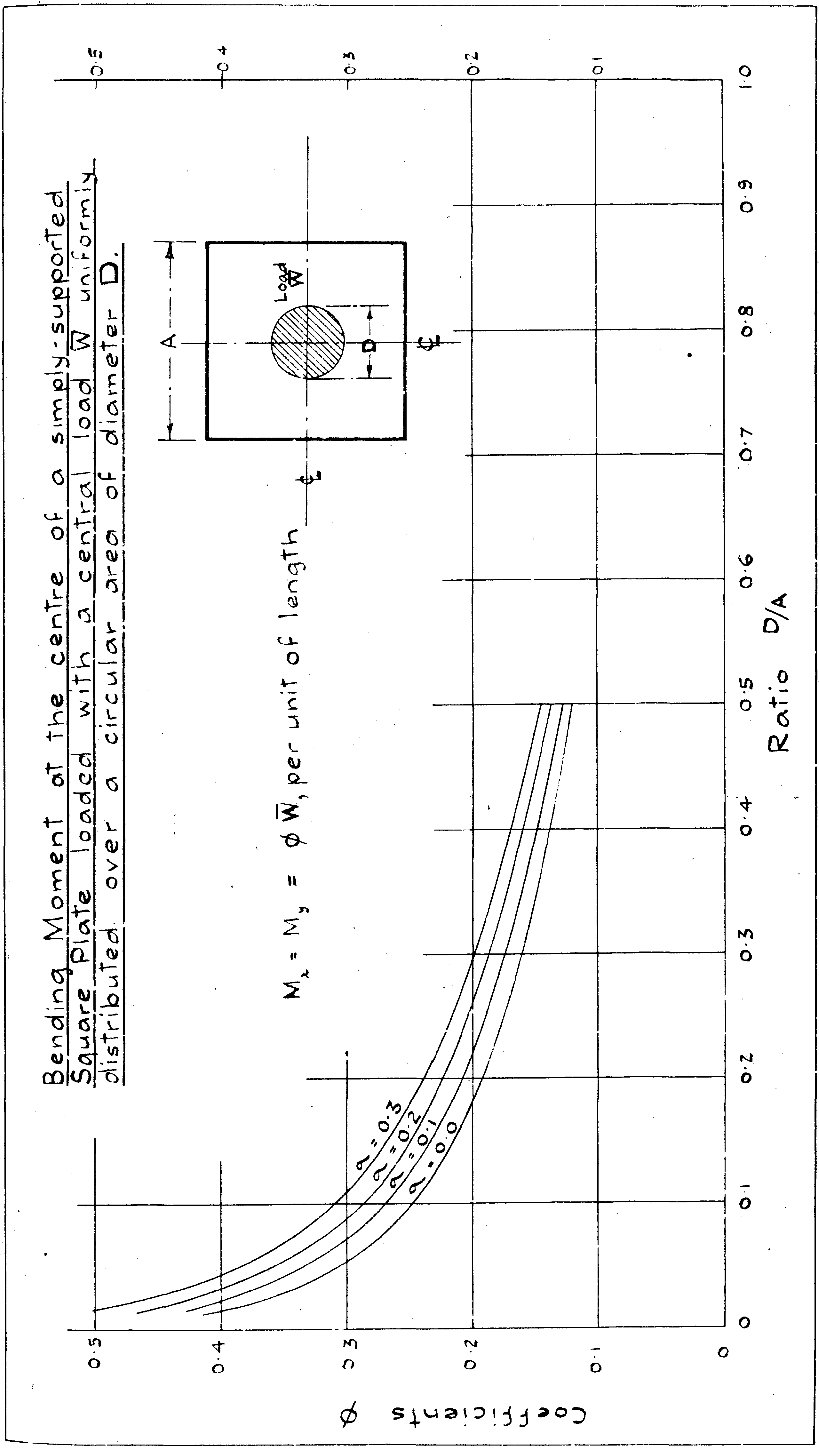
$$M_y = (1+\sigma)\,0{\cdot}1627\,\bar{W} + (0{\cdot}875 + \sigma\,0{\cdot}476)\,\bar{W}/8 = (0{\cdot}272 + 0{\cdot}222\,\sigma)\,\bar{W},$$

and, for L/A = 1/4,

$$M_y = (1+\sigma)\,0{\cdot}1104\,\bar{W} + (0{\cdot}875 + \sigma\,0{\cdot}476)\,\bar{W}/8 = (0{\cdot}22 + 0{\cdot}17\,\sigma)\,\bar{W}.$$

The above are slightly inaccurate because of the assumptions of symmetry when computing the flexural couples produced by the ζ values on the boundary of the isolated square. With a line load, the fields are

Diagram No. 8



Bending Moment at the centre of a simply-supported Square Plate loaded with a central load $\overline{W}$ uniformly distributed over a circular area of diameter D.
$M_x = M_y = \phi \overline{W}$, per unit of length
A
D
Load $\overline{W}$
℄
℄
$\alpha = 0.3$
$\alpha = 0.2$
$\alpha = 0.1$
$\alpha = 0.0$
0.5
0.4
0.3
0.2
0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
Ratio D/A
Coefficients ϕ

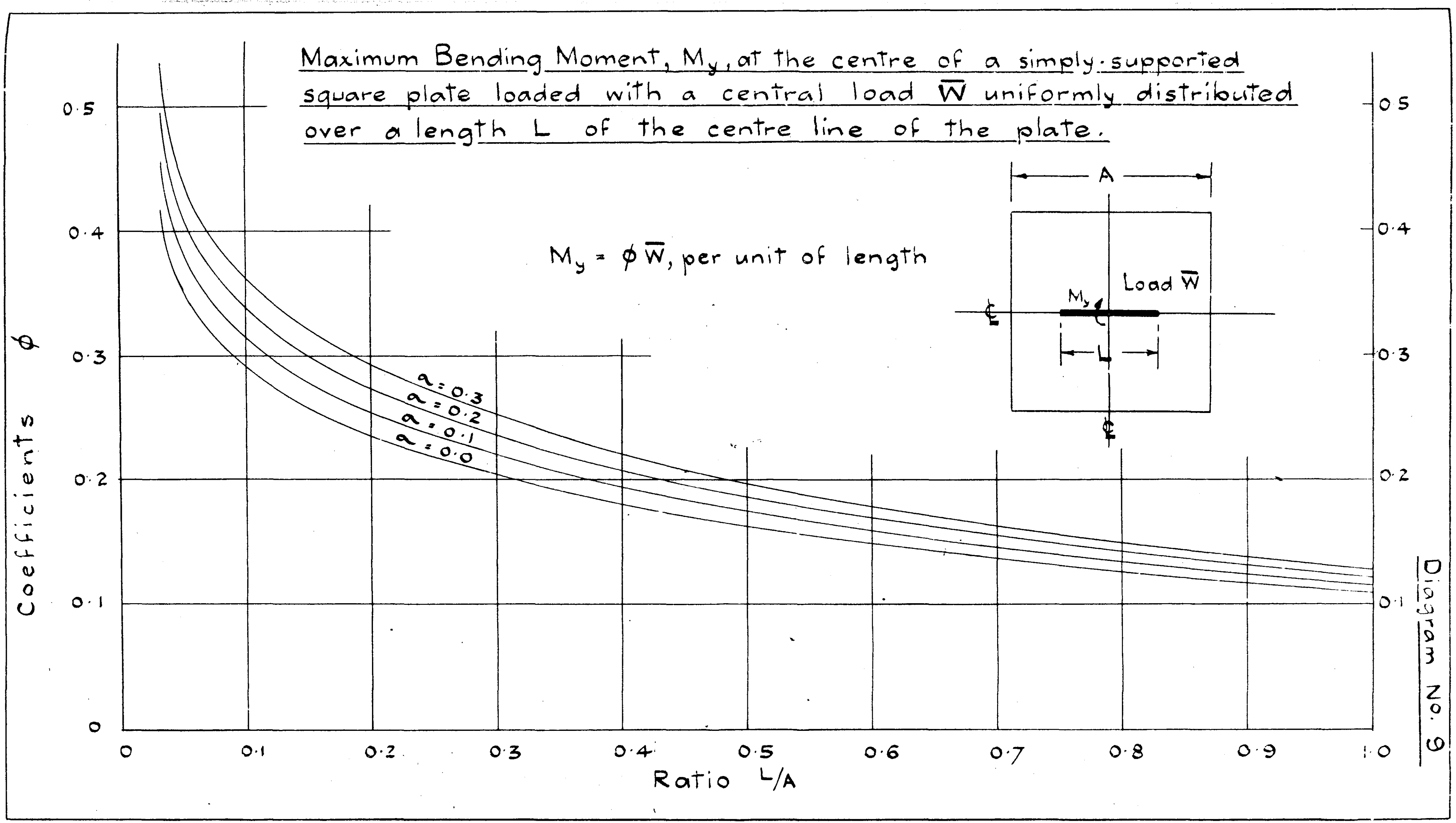


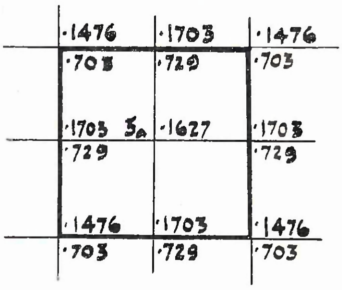
Diagram No. 9
Maximum Bending Moment, M_y, at the centre of a simply-supported square plate loaded with a central load W̄ uniformly distributed over a length L of the centre line of the plate.
M_y = φ W̄, per unit of length
A
Load W̄
M_y
L
℄
α = 0.3
α = 0.2
α = 0.1
α = 0.0
Coefficients φ
Ratio L/A
0
0·1
0·2
0·3
0·4
0·5
0·6
0·7
0·8
0·9
1·0

symmetrical about the respective centre lines and not , in addition, about the diagonals as in Field No.48. It is thought, however, that the inaccuracy is negligible and Diagram No.9 has been prepared accordingly to meet the case of the line load concentration.

Deflections.

It is of interest to note the effect of distributions of load on the deflection values of Field No.48.

Considering, in the first instance, a uniform distribution throughout the central square of side N. As formerly, this portion of the plate is isolated from Field No.48 and is as shown in Fig.29.



ζ values have units $-\overline{W}/K$

w values do. do. $\overline{W}\ N^2/K$

Fig.29.

Applying Formula (2f), $w_a = 0.7475\ \overline{W}\ N^2/K$, where w_a is the deflection as in case (a), page 41 .

Also, for a square of side N, the central deflection produced by the uniformly distributed load $= w_b = 16.7206\ \overline{W}N^2/(64)^2K = 0.0041\ \overline{W}\ N^2/K$.

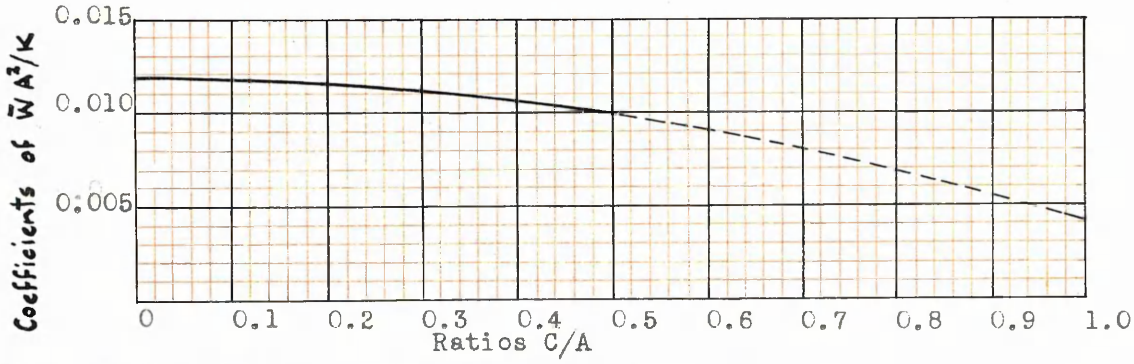
$\therefore w_a + w_b = 0.7516\ \overline{W}\ N^2/K$ is the required deflection at the centre of the plate loaded as above.

Or, if A and C are used to denote, as formerly, the plate and loaded square dimensions, then, for $C/A = 1/8$, $w = 0.0118\ \overline{W}\ A^2/K$.

It follows from the above that $(0.7475 + 0.0118)\overline{W}\ N^2/K$ is the deflection at the centre of the plate when the load is distributed over a square of side N/8.

i.e. for $C/A = 1/64$, $w = 0.01185\ \overline{W}\ A^2/K$.

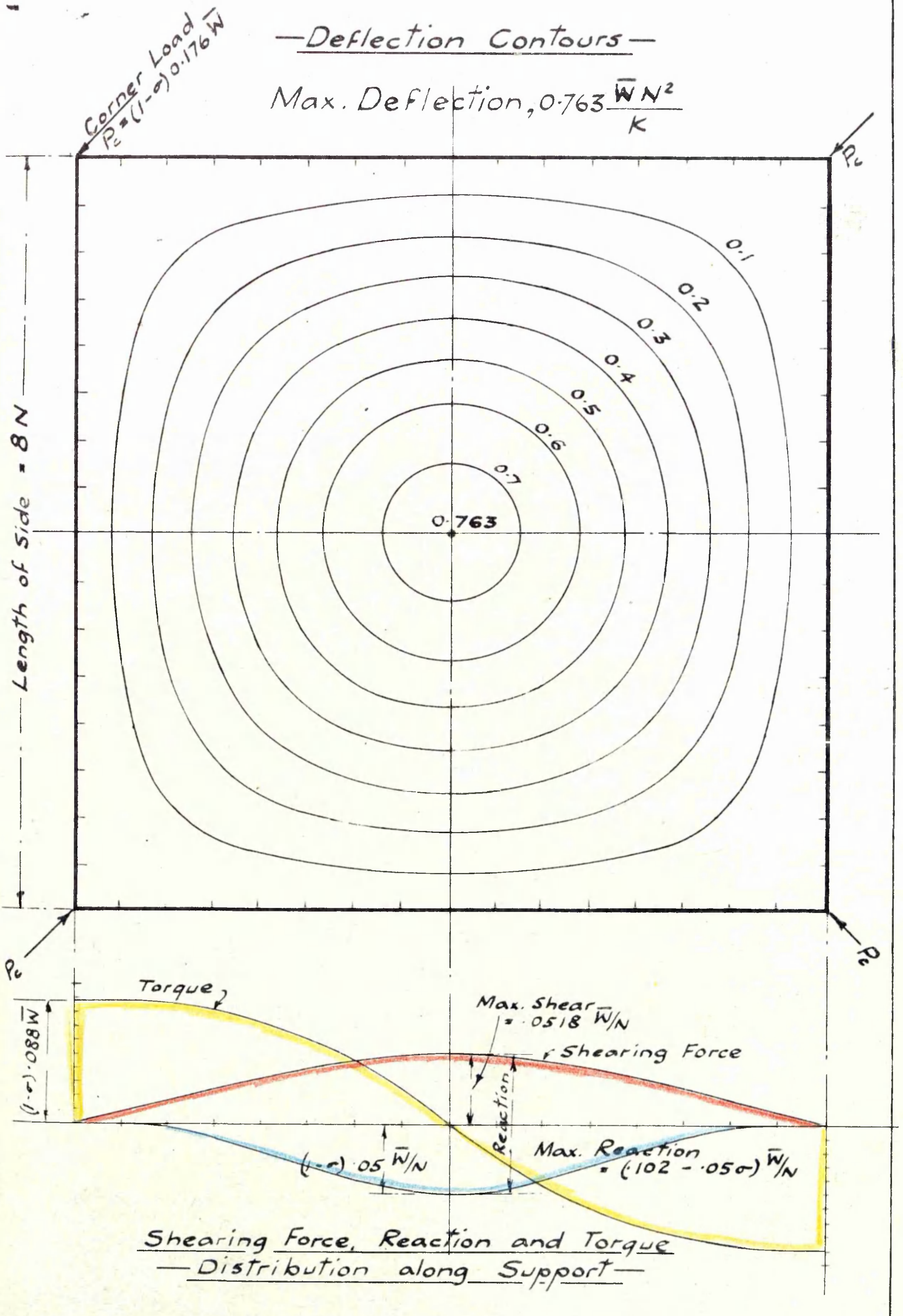
Similar calculations give central deflection values of $0.0114\ \overline{W}\ A^2/K$ and $0.0099\ \overline{W}\ A^2/K$ for $C/A = 1/4$ and $1/2$ respectively. The above values are plotted below and it may be noted that deflections are not particularly sensitive to changes in the area of concentration of the load.



Contours of Deflection and the distribution of Shearing Force, Torque, etc., along the boundary are shown in Diagram No.10.

Diagram № 10

Square Plate of uniform thickness, simply supported at the edges and loaded with a central concentrated load $\overline{W}$.



Square plate, of uniform thickness, simply supported along the edges and loaded with a concentrated load at the quarter point on an axis of symmetry.

The solution to this problem must satisfy the same requirements as in the preceding case of the central concentrated load. It may be obtained in the same manner, but advantage may also be taken of the available $\nabla^4 w = 0$ fields and the previous solution on Field No. 48. The latter method is described below.

E F C D, Fig. 30, is a part of Field No. 48, forming a rectangle 8 N by 6 N, in which the ζ and w values are zero along EC, FD, EF. To complete the square of side 8 N, the portion A B F E, shown in red, is added.

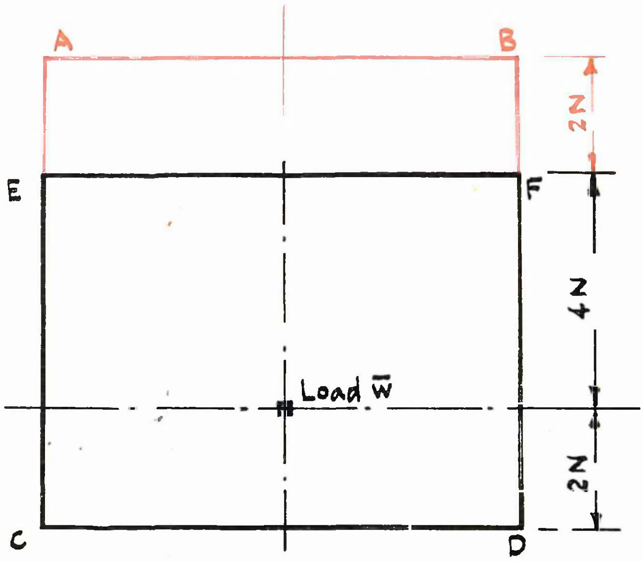


Fig. 30.

The ζ and w values in the added portion must satisfy the fundamental differential equation and must not produce any discontinuity along the line EF. These requirements are met by producing the field C D F E to AB with values of ζ and w equal numerically but of opposite sign to those at corresponding points in the part ED.

The preliminary field thus obtained is shown in Fig. 31. It satisfies the loading but not the boundary conditions. The boundary values are corrected by applying equal and opposite values and a correction field is built up from Fields Nos. 2, 36, 37, 38, ...43. In this case, however, the equal and opposite nature of the boundary values along AB and CD, makes it possible to use the rectangular fields, 8 N by 4 N, referred to previously, on page 18.

By adding the preliminary and correction fields algebraically, the solution to this particular problem is obtained. This is shown in Field No. 49.

Deflection contours, obtained from the above field, are plotted on Diagram No. 11, and the distribution of shearing force, etc., along the boundaries, on Diagram No. 12.

0	.0088	.0173	.0263	.0356	.0448	.0531	.0591	.0613
0	-.083	-.164	-.241	-.312	-.374	-.424	-.456	-.467
0	.0065	.0130	.0197	.0263	.0326	.0379	.0416	.0429
0	-.064	-.127	-.186	-.241	-.288	-.324	-.348	-.356
0	.0043	.0087	.0130	.0173	.0212	.0244	.0265	.0273
	-.044	-.086	-.127	-.164	-.195	-.219	-.235	-.240
0	.0022	.0043	.0065	.0085	.0104	.0119	.0129	.0133
0	-.022	-.044	-.064	-.083	-.098	-.111	-.118	-.121
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	-.0022	-.0043	-.0065	-.0085	-.0104	-.0119	-.0129	-.0133
0	.022	.044	.064	.083	.098	.111	.118	.121
	other							
	values							
		as						
			on					
			Field No. 48.				Load $\bar{W}$	-.3310 .763
0	-.0085	-.0173	-.0263	-.0356	-.0448	-.0531	-.0591	-.0613
0	.083	.164	.241	.312	.374	.424	.456	.467

ζ values have units $\bar{W}/K$

w values have units $\bar{W}\,N^2/K$

$\nabla^4 w = 0$, at all points except the load point.

Fig. 31.

Bending Moments under the Load.

The bending moments under the load are obtained in a similar manner to that previously described. In this case, however, $\partial^2 w/\partial y^2 \neq \dfrac{\partial^2 w}{\partial x^2}$, and it is necessary to separate these components from the ζ values in order to compute the respective bending moments M_x and, M_y.

Thus, for the square of side N with the load concentrated over a central square of side N/4, the ζ and w values at the boundary, abstracted from Field No. 49, are shown in Fig. 32.

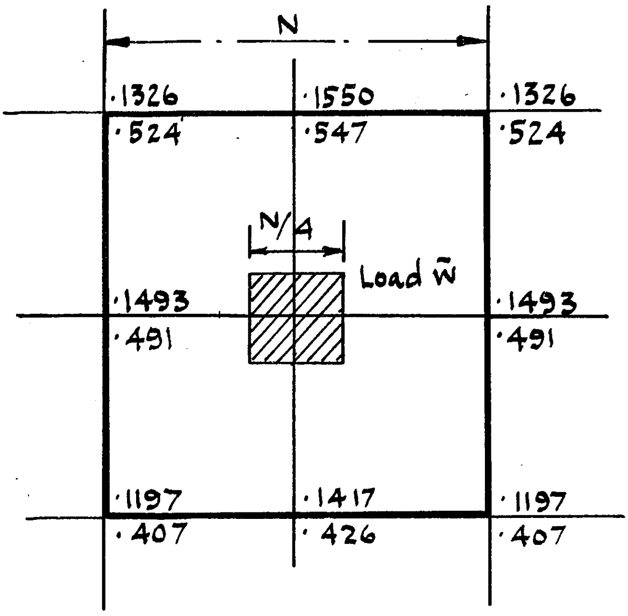


Fig. 32.

The ζ value at the centre of the square, Formula (2e), $= -0.1434\ \bar{W}/K$
and the w value at the centre, Formula (2f), $= \cdot 505\ \bar{W}N^2/K$.
The data necessary to evaluate $\dfrac{\partial^2 w}{\partial y^2}$ and $\dfrac{\partial^2 w}{\partial x^2}$ are thus available, and by arithmetical differentiation of the w values,

$$\frac{\partial^2 w}{\partial y^2} = -0{\cdot}16\,\bar{W}/K \quad \text{and}$$
$$\frac{\partial^2 w}{\partial x^2} = -0{\cdot}13\,\bar{W}/K.$$

From Diagram No. 7, for C/A = 1/4, $M_y = (1 + \sigma)\ 0.15\ \bar{W}$.

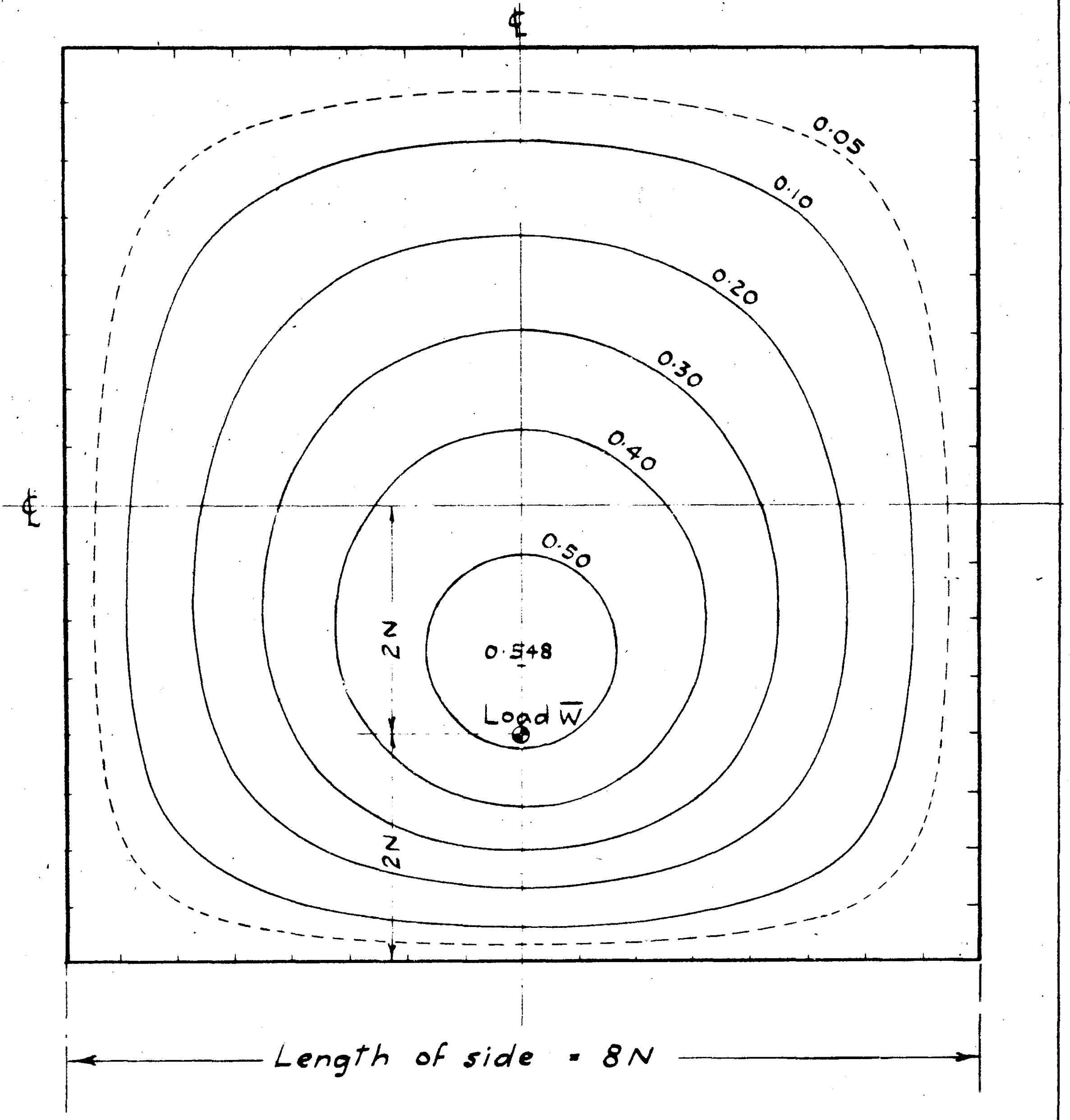
The maximum bending moment under the load is therefore,

$$M_y = (0.16 + 0.13\sigma) + 0.15\ (1 + \sigma)\ \bar{W} \text{ per unit of length,}$$
$$= (0.31 + 0.28\sigma)\ \bar{W}, \text{ per unit of length.}$$

Diagram No. 11

Square Plate, of uniform thickness, simply supported at the edges and loaded at the quarter point with a concentrated load $\overline{W}$.

— Deflection Contours —

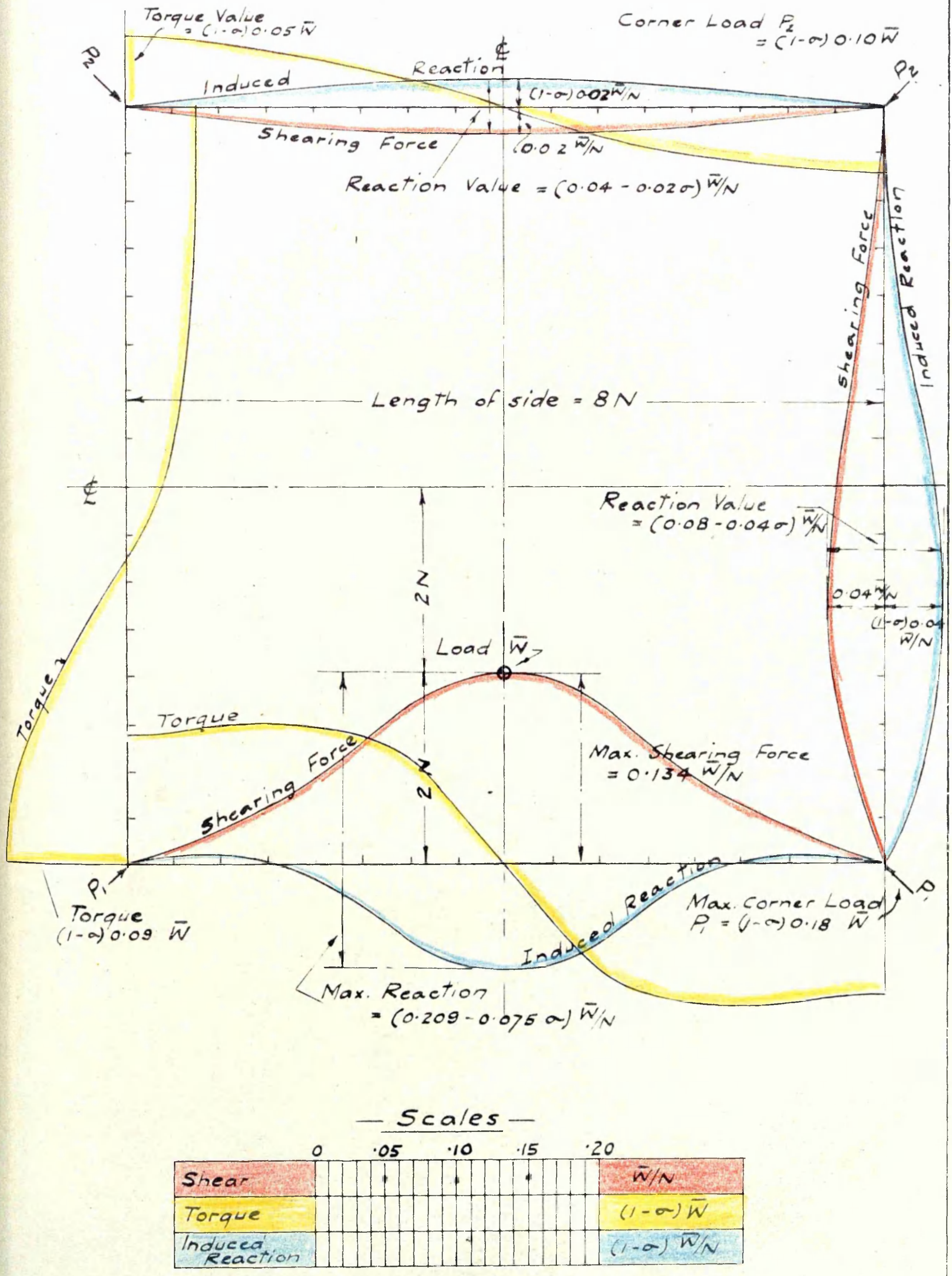


Max. Deflection = $0{\cdot}548\ \overline{W} N^2/K$

Diagram No. 12

Square Plate, of uniform thickness, simply supported at the edges and loaded with a concentrated load $\bar{W}$ at the quarter point.

Distribution of Shearing Force, Reaction and Torque along the edges.



Square Plate of side 8 N simply supported along the boundaries and symmetrically loaded with four concentrated loads each of amount $\bar{W}$ at the quarter points.

The loaded plate is shown in Fig. 33.

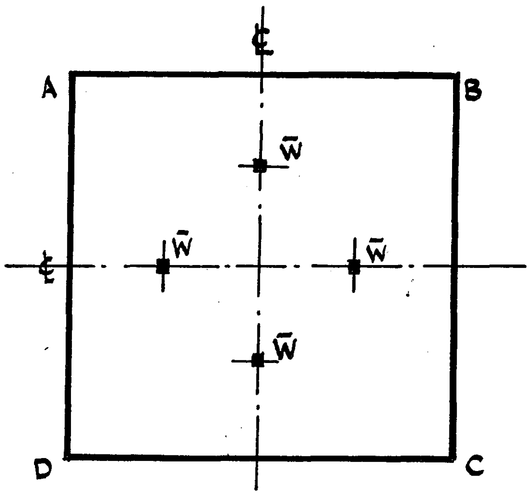


Fig. 33.

The ζ and w values are obtained directly by adding the values from four fields similar to Field No. 49. In this case the field is symmetrical also about the diagonals, and a closer approximation is readily obtained by further squaring. The settled field is shown on Field No. 50.

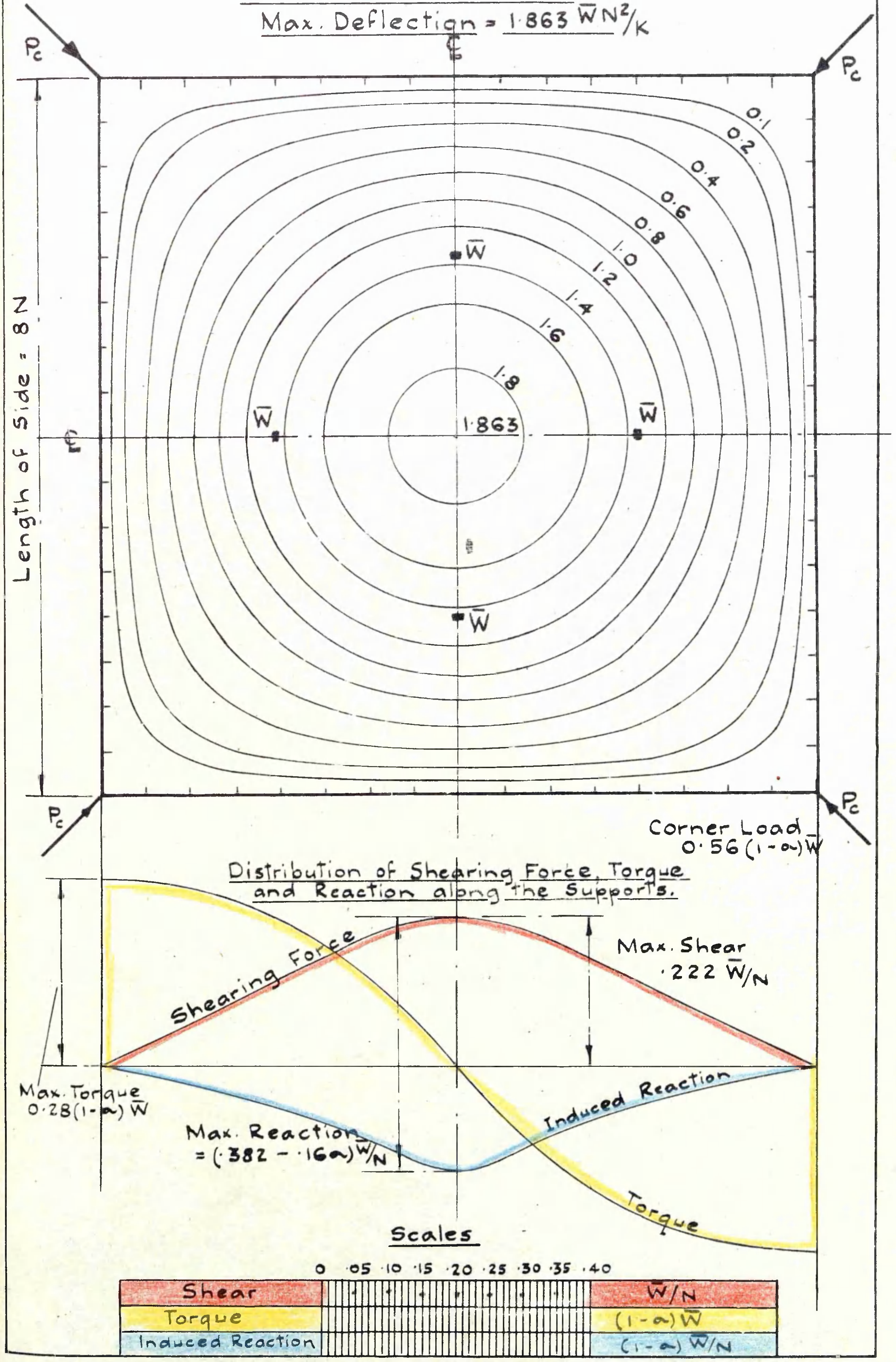
Deflection contours etc., are plotted on Diagram No. 13.

Diagram No. 13

Square Plate of uniform thickness, simply-supported at the edges and loaded with 4 concentrated loads, each of amount $\bar{W}$, at the quarter-points of the ℄s

— Deflection Contours —

Max. Deflection = $1{\cdot}863\ \bar{W}N^2/K$



Square Plate of side 8 N, loaded completely with a uniformly distributed load, simply supported on the boundaries and propped at the centre and quarter points so that deflections are zero at all supports.

The plate is shown in Fig. 34, the intermediate props being at the points E, F, G, H and J.

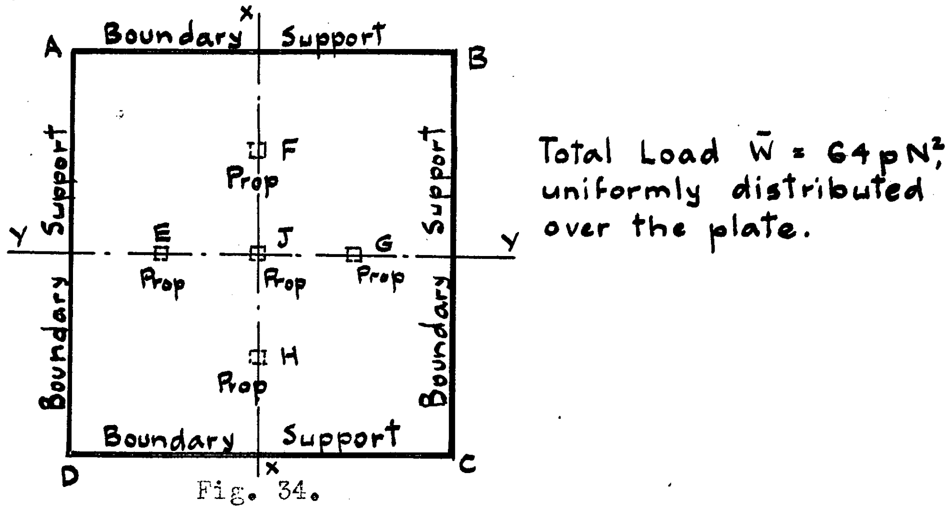


Fig. 34.

The solution to this problem is obtained by combining the previous solutions of Fields Nos. 1, 48 and 50.

With the props removed, the deflections at the centre and quarter points, from Field No. 1, are 16.7206 p N^4/K and 12.0950 p N^4/K respectively. If the loads on the central and quarter points are A and B respectively, then,

$$0.763\ A + 1.8625\ B = 16.7206\ p\ N^2, \text{ and,}$$

$$0.467\ A + 1.3854\ B = 12.0950\ p\ N^2.$$

$\therefore A = 3.406\ p\ N^2$, and,

$B = 7.582\ p\ N^2$.

Of the total load $\bar{W}$ on the plate, $\bar{W} = 64\ p\ N^2$, 0.0534 $\bar{W}$ is taken by the central prop, 0.1183 $\bar{W}$ by each of the quarter points props and the remainder, 0.4734 $\bar{W}$ is carried on the boundary supports.

The ζ and w values are obtained by multiplying the values on Field No. 48 by 3.406 p $N^2/\bar{W}$ and those on Field No. 50 by 7.582 p $N^2/\bar{W}$ and subtracting the sum of the fields thus formed from Field No. 1. The resulting field is given on Field No. 51.

Deflection contours etc., are plotted on Diagram No. 14.

Bending Moments.

The maximum negative bending moment occurs over the quarter point supports, and the value of this bending moment is ascertained, as formerly, by considering the area of the support in contact with the plate. In this case, because of the lack of symmetry and the wide variations in the ζ values it is necessary to subdivide the loaded square in order to obtain a more accurate estimation of the quantities $\frac{\partial^2 w}{\partial x^2}$ and $\frac{\partial^2 w}{\partial y^2}$ than that which would be given by applying Formulae (2e) and (2f) to the isolated square. It is simpler, therefore, to determine the values of the Bending Moments from the separate fields, viz., Fields Nos. 1, 48 & 50, and to add these algebraically. If the prop reaction at F is uniformly distributed over a square area of side N/4, the following are obtained:-

$$M_x = -(1.52 + \sigma 1.04)\, p N^2,$$

and,

$$M_y = -(1.04 + \sigma 1.52)\, p N^2,$$ per unit of length, where the subscripts x and y denote the flexural couples for planes perpendicular to the axes of X and Y shown in Fig. 34.

Shearing Force, Torque etc.

The boundary values of these quantities are given on Diag. No. 14. They were obtained by mechanical differentiation of Field No. 51, but, in this case, the squares were sub-divided for a distance N from the boundary.

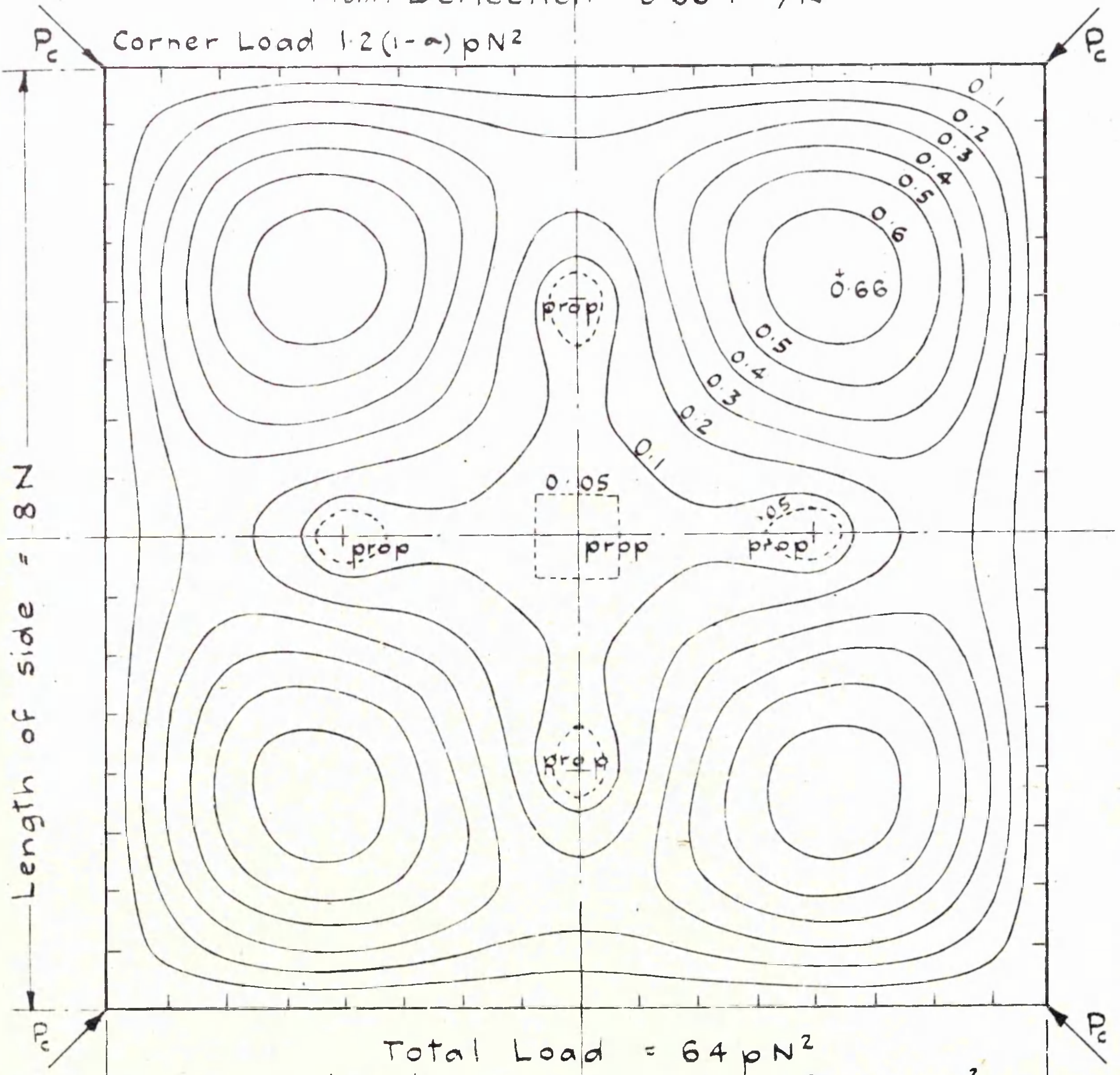
(This problem is interesting also from the constructional side of reinforced concrete where floor slab shuttering may be removed a few days after the concrete has been poured provided the floor is propped to avoid excessive deflection. Over-propping may result in a reversal of the bending moments with dangerous consequences unless double reinforcement is provided).

Diagram No. 14

Square Plate, of uniform thickness, simply-supported on the edges and propped at the centre and quarter points of the $\mathcal{C}_s$, and loaded with a uniformly distributed load.

Deflection Contours

Max. Deflection = $0 \cdot 66 \; pN^4/K$

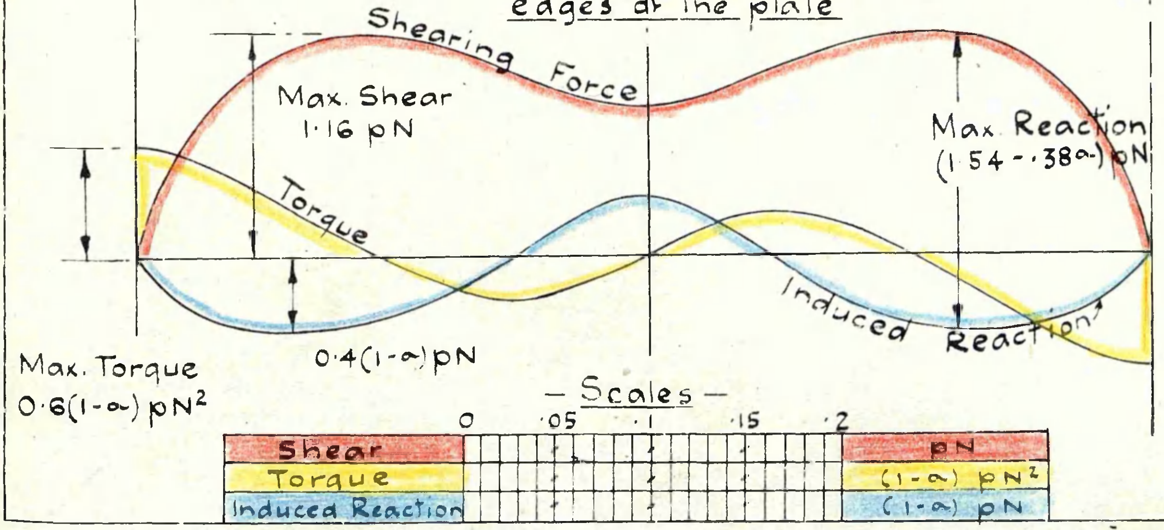


Total Load = $64\,pN^2$

Load on centre prop = $3 \cdot 406\,pN^2$

Load on quarter-point prop = $7 \cdot 582\,pN^2$

Distribution of Shear, Torque and Reaction on the edges of the plate



Rectangular plate, 8 N by 4 N, simply-supported on the boundaries and loaded with a line load along the long axis of the plate. Fig. 35 .

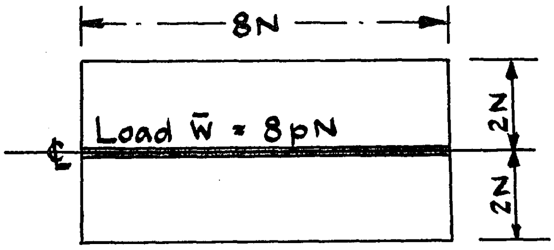


Fig. 35

The solution to this problem is shown on Field No. 52. It is obtained by folding Field No. 46 about its centre line and subtracting corresponding values.

Deflection contours, etc., are plotted on Diagram No. 15 .

The maximum bending moments are at the centre point of the plate, the values being,

$$M_y = (0.812 + 0.119\sigma)\ p\ N, \text{ and,}$$
$$M_x = (0.119 + 0.812\sigma)\ p\ N, \text{ per unit of length.}$$

Rectangular plate, 8 N by 4 N, simply-supported on the boundaries and loaded with a concentrated load at the centre point.

The solution to this problem is shown on Field No. 53. It is obtained by folding Field No. 49 about its centre line and subtracting corresponding values.

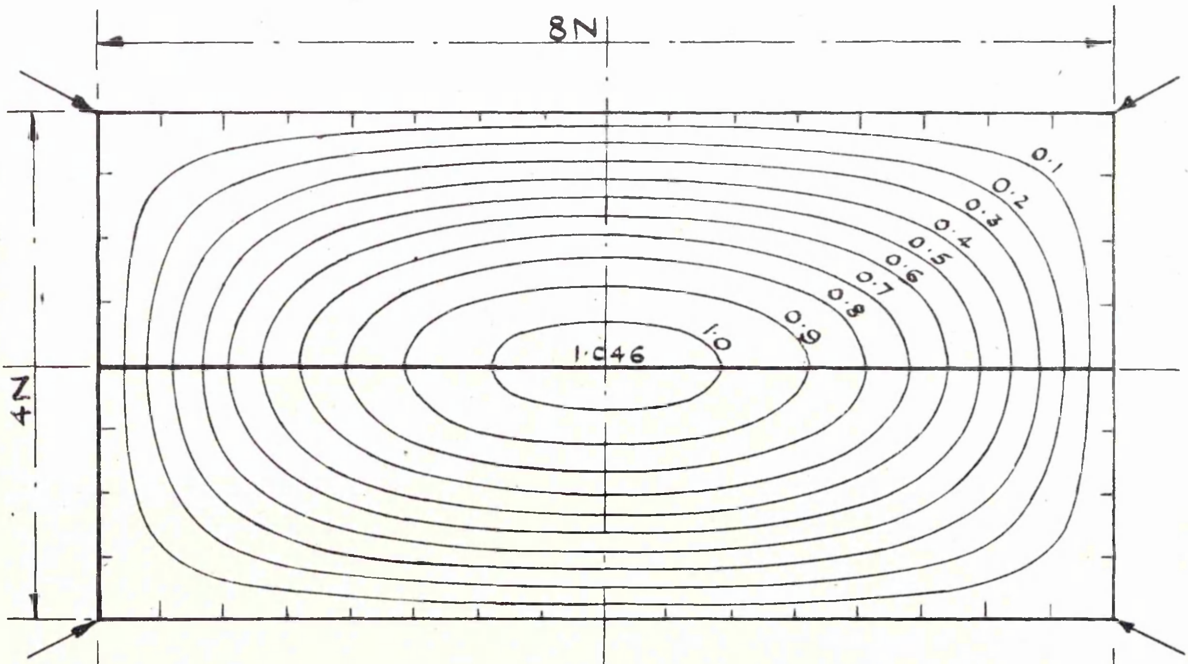
Deflection contours etc., are plotted on Diagram No. 16 .

The bending moments under the load are calculated as in the case of the concentrated load at the quarter-point of the square plate.

Diagram No. 15

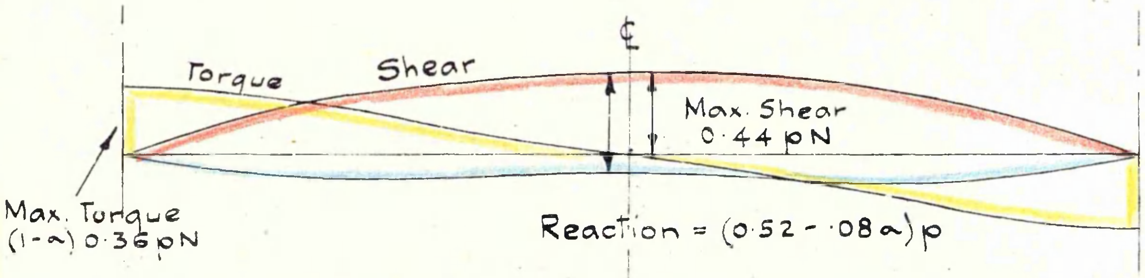
Rectangular Plate, of uniform thickness, simply-supported at the edges and loaded with a uniformly distributed load across the longer centre line

Deflection Contours

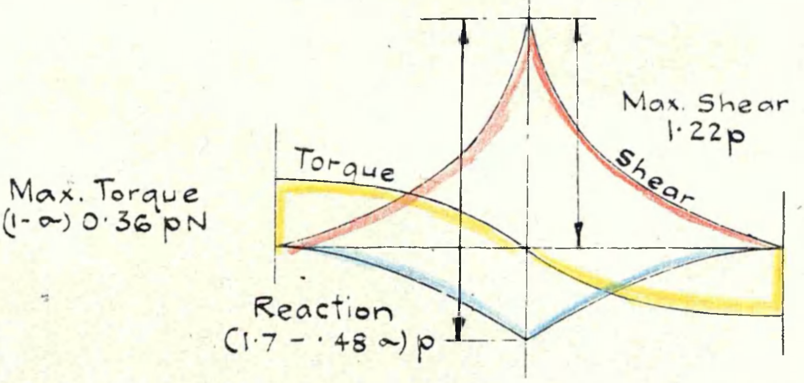


Corner Load
$(1-\alpha)\,0.72\,pN$

Total Load = $8\,pN$
Max. Deflection = $1.046\ pN^3/K$



Distribution of Shear, Torque and Reaction ~ Long Edge



Distribution of Shear, Torque and Reaction - Short Edge

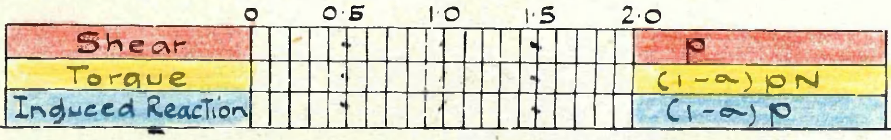
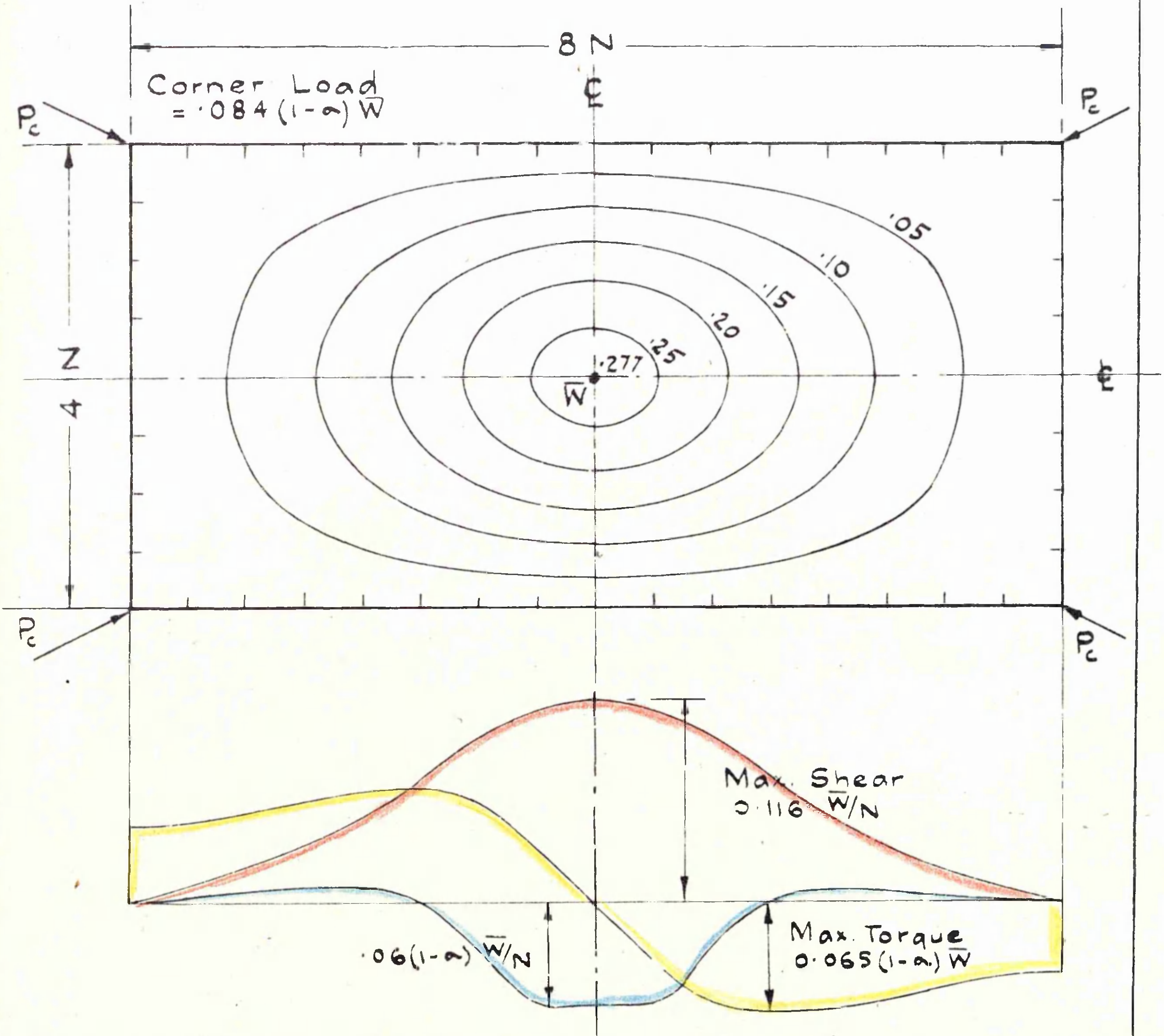


Diagram No. 16

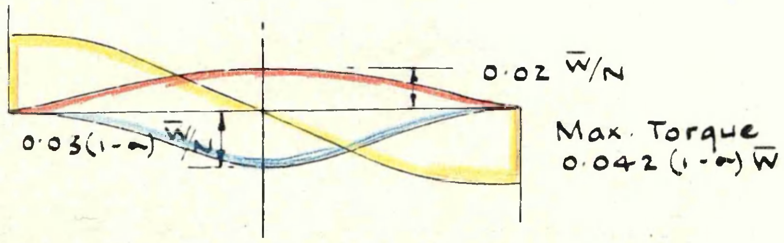
Rectangular Plate, of uniform thickness, simply-supported at the edges and loaded at the centre with a concentrated load $\overline{W}$

Deflection Contours

Max. Deflection = $0.277\ \overline{W}N^2/K$

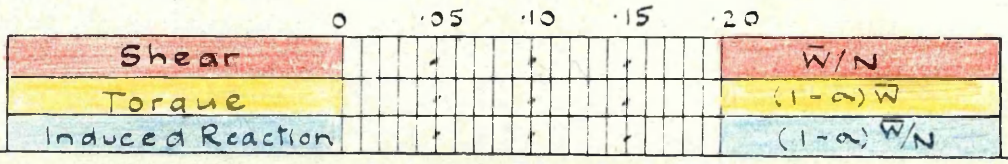


Distribution of Shear, Torque and Reaction ~ Long Edge



Distribution of Shear, Torque and Reaction ~ Short Edge

— Scales —



<u>Effect of the size of the network on the accuracy of the results.</u>

It is noted previously, that a network of 16 squares gives a ζ value of $-2.42\ C\ N^2$ at the centre of the simply-supported square plate loaded with a uniformly distributed load. This is about 2.5 % greater than the true value, and a similar network gives the central deflection also about 5 % in excess of that obtained by using 256 squares. As these are limits which are in accordance with the design requirements of many practical problems, it is of interest to investigate further the effects of the size of the network on the accuracy of the results.

For the $\nabla^4 w = 0$ fields, Nos.3 to 10, a coarser network is readily obtained by folding them about their respective centre lines and subtracting corresponding values. Fields Nos. 54 to 58 and 59 to 61 are the resulting fields for networks of 64 and 16 squares respectively, and Field No.62 gives the solution to the particular problem of the concentrated load at the centre of the simply-supported square plate when a network of 64 squares is also used. The latter was obtained by extracting the values from a central square of side 4 N from Field No.48 and liquidating the boundary values by means of Fields Nos. 54 to 58.

A comparison of the results obtained from Fields Nos.48 and 62 is as follows:-

Square plate of side A, Central Concentrated Load $\overline{W}$			
Description	Network 256 squares	Network 64 squares	% difference.
Max. Deflection.	$0.01192\ \overline{W}A^2/K$	$0.01222\ \overline{W}A^2/K$	2.5
Bending Moment $M_x = M_y$ ($C/A = 1/16$).	$0.258\ (1+\sigma)\overline{W}$	$0.255\ (1+\sigma)\overline{W}$	1.2
Max. Boundary Shear	$0.4144\ \overline{W}/A$	$0.384\ \overline{W}/A$	7.3

The agreement between the respective values is very good with the exception of the shearing forces or other quantities which are computed from the gradients of the ζ and w fields. It may be noted, however, by comparing the actual ζ and w values of Fields Nos.48 and 62, that a very close agreement between the respective values of shearing force, torque, etc., may be obtained by subdividing the boundary squares of Field No.62 when determining the boundary gradients $\frac{\partial \zeta}{\partial x}$ and $\frac{\partial w}{\partial x}$ by arithmetical differentiation. In the above comparison, no subdivision of the boundary squares was made.

It is concluded, therefore, that for many practical problems, especially those relating to simply-supported rectangular plates, the network hitherto used is unnecessarily close and the increase in accuracy is not justified by the amount of additional labour which must be expended.

The main reason for using the fine network in the previous examples was not to afford the above comparison, but to provide a selection of settled fields having different loading conditions from which portions, rectangular or polygonal, may be cut, together with a comprehensive series of $\nabla^4 w = 0$ fields for liquidating the boundary values on the primary field. It is possible to get a direct solution by solving several sets of simultaneous equations in the first instance, but when the number of unknowns becomes large it is probably simpler to revert to the usual method of squaring. In this event, a judicious selection from one or more of the available $\nabla^4 w = 0$ fields will leave small residuals only on the boundaries which are more easily eliminated than the full boundary values.

Fields Nos 54 to 58 are also usefully employed in squaring the correction fields. A rectangular plate, length/breadth ratio 1.5/1, simply-supported at the edges and loaded with a central concentrated load, $\bar{W}$, is shown in Fig. 36. This part is taken from Field No.48 and has the overall dimensions of 4N by 6N, each small square being of side N/2. To eliminate the boundary values, or the residuals if the available correction fields are used, plausible values are selected in the first instance for the points on the line GH and these, together with the values on the boundaries AG, HD, and DA, enable ζ and w values to be computed for points on the line EF by using the values on Fields Nos.54 to 58. Square EFCB is then taken and new values obtained for the line GH and this process is repeated until the changes in the values on GH and EF are negligible. Intermediate values are then filled in using the above Fields and the correction field is added to the primary one.

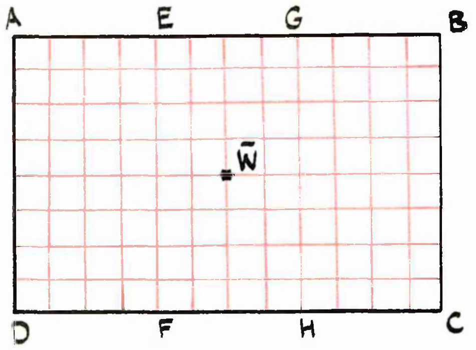


Fig. 36.

Fields Nos.59 to 61 are also useful for filling in values in isolated squares.

<u>Slabs continuous over several supports.</u>

It may be seen by reference to Fig. 31, page 50 that if the boundary values on that field are liquidated by adding a similar field with the values reversed end for end, a settled field is obtained which has a load at each of the quarter points of the Y axis. Fig. 37.

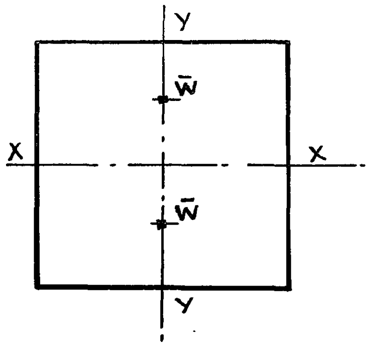


Fig.37

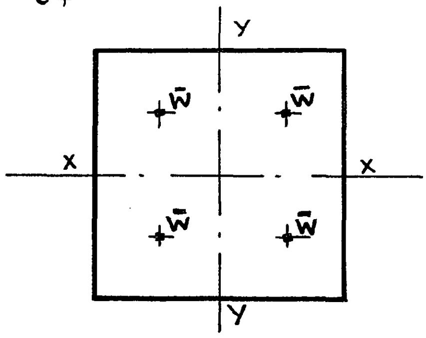


Fig.38.

Similarly, by extending the original field for various distances, other fields having the double load symmetrically placed on one of the centre lines are readily obtained. It follows, also, that if these fields are in turn extended in the other direction, settled field are made available which have 4 concentrated loads symmetrically placed with respect to the X and Y axes. Fig.38.

It is easy, therefore, to get solutions for slabs which are fully loaded and supported at the edges and at intermediate points, but although this loading gives the worst conditions at the intermediate supports it does not meet the case of partial loading which produces the most unfavourable effects at the centres of the intermediate panels. In the latter case, it is advantageous to combine the results from different sizes of plates, each of which has been divided into the same size of network. This is demonstrated in the following example.

A flat slab floor, A B C D, is shown in Fig.39. This floor is supported at the edges and at intermediate columns, E to M, which prevent deflection of the floor at these points and do not resist bending moments but merely take direct load.

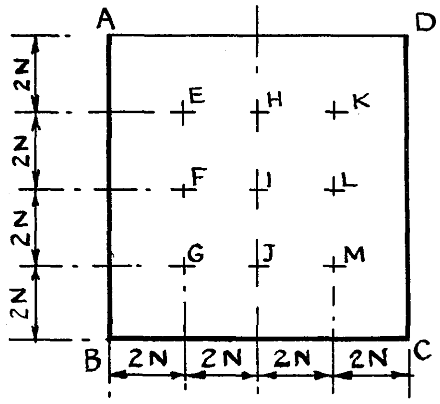


Fig. 39.

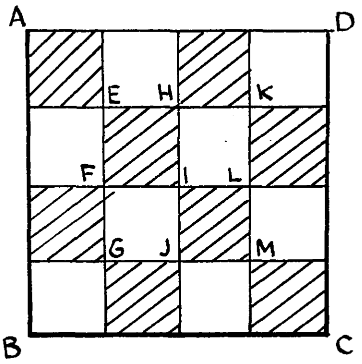


Fig.39a.

For a uniformly distributed load throughout the entire floor area, it is necessary to solve 3 equations to find the loads on the columns E, H, and I, and to combine three fields with the primary one as in the previous propped plate problem.

For superimposed loading on any part of the floor, the most unfavourable conditions at the centres of the panels occur when the areas shown hatched in Fig.39a are loaded only. To obtain a primary field in this case, i.e. a field having the proper loading conditions but with the intermediate supports removed, it is necessary to use a square field of side 2N which has been settled with a network of 16 squares. By extending this field as in the previous examples, i.e. using a series of spans with upward and downward loads alternately, the field shown in Fig. 39b is obtained, which, when combined with Field No.1 gives the field shown in Fig.39c. It is only necessary to halve the ζ and w values on this field in order to get the required primary field.

A further difficulty arises in the case of the column loads. The loads on each of the columns at F,H,L and J are equal and, similarly, for E and M, and G and K. It is therefore necessary to obtain the solution for a plate or slab loaded with two loads symmetrically placed on a diagonal. Fig.39f. This is found by using the previous method of alternate upward and downward loads as illustrated in Figs.39d, 39e and 39f.

The necessary data for finding the loads on columns E,H,K and I are now available and the method of procedure is similar to that previously described.

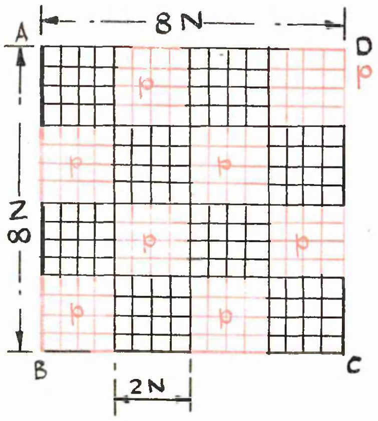


Fig.39b.

Areas loaded with downward loads are shown in black.

Areas loaded with upward loads are shown in red.

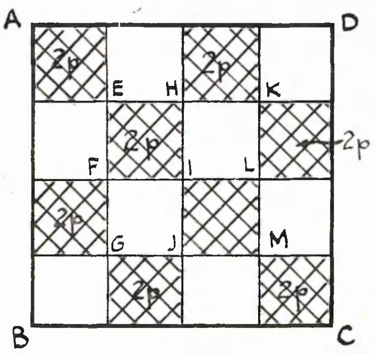


Fig.39c.

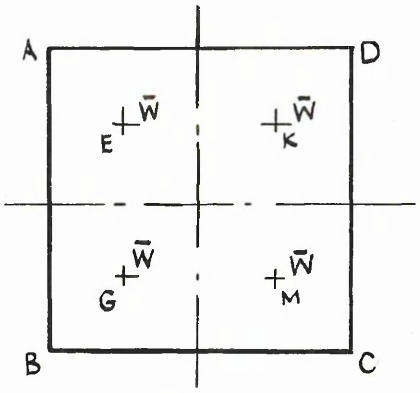


Fig.39d.

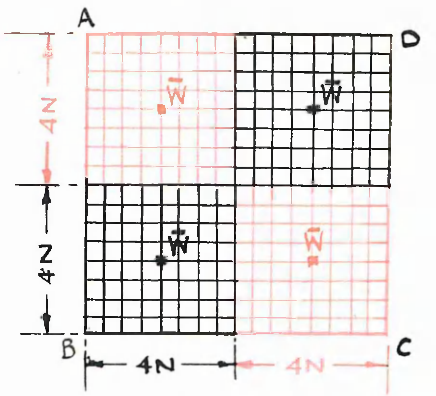


Fig.39e.

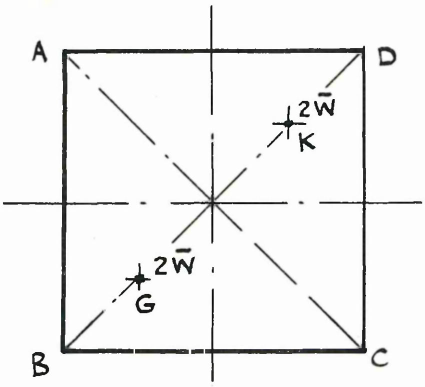


Fig.39f.

Beams as intermediate supports.

A very common type of bridge decking or warehouse floor consists of ribs and slabs, and it is possible to extend the previous solutions to meet this important practical case.

It is interesting to compare, in the first instance, the results of the line load solutions with those for a series of equal concentrated loads $\bar{W}$ applied thus :-

(a) along a centre line of the square plate, Fig.40.

(b) along a line parallel to the centre line, Fig.40a.

(c) along the longer centre line of the rectangular plate, Fig.40b.

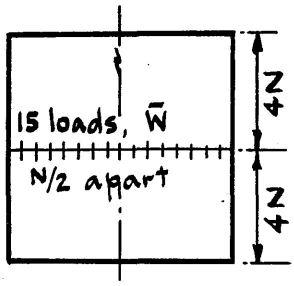


Fig. 40

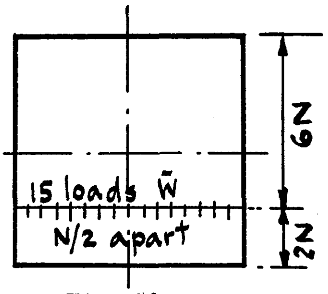


Fig.40a.

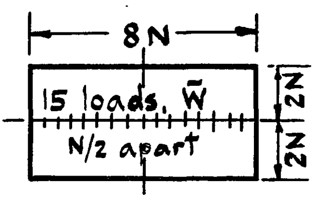


Fig.40b.

A line of 'values' from Field No.48 is shown in Fig.40c.

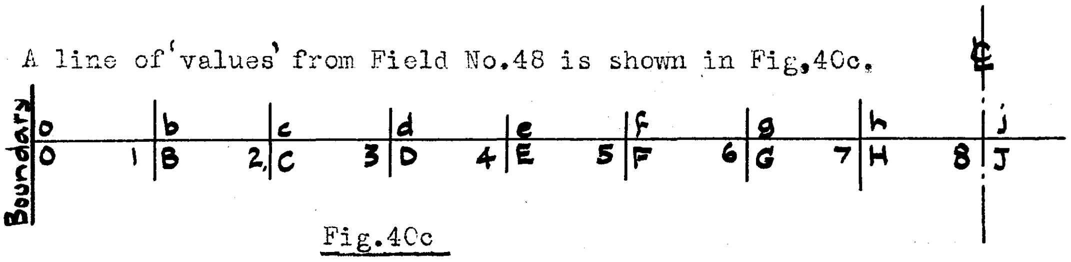


Fig.40c

'Values' for the points 1 to 8 are readily obtained for the specified loading conditions thus.

1. (j+H)

2. (j+h) + (h+g) = (j+2h+g)

3. (j+2h +g) + (g+f) = (j+2h+2g+f)

etc.,etc.,

8. 2(b+c+d+e+f+g+h) + j.

Similarly for the deflection values, and also for the rectangular plate, Field No.53.

The fields thus formed are not reproduced in detail but the loaded line ζ and w values are tabulated below together with quarter line values for (a) and (c) and centre line values for (b). A comparison is also made with the line load solutions on the assumption that the concentrated load $\bar{W}$ is distributed over a length N/2.

Case (a), 15 loads each of amount W and uniformly spaced at intervals N/2 along the centre line.

1. Centre line values.

								℄	
0	.5013	.7883	.9895	1.1353	1.2395	1.3097	1.3503	1.3636	$\overline{W}/K$
0	1.492	2.381	4.111	5.148	5.971	6.567	6.928	7.049	$\overline{W}N^2/K$

Assuming $\overline{W}$ is distributed over a length N/2.

								℄	
0	.2507	.3942	.4948	.5676	.6197	.6549	.6752	.6818	pN/K
0	.746	1.441	2.056	2.574	2.986	3.284	3.464	3.524	pN^3/K

Centre line values, Field No.45.

								℄	
0	.2482	.3906	.4901	.5625	.6141	.6488	.6688	.6753	pN/K
0	.735	1.417	2.021	2.530	2.930	3.221	3.400	3.457	pN^3/K

Max. Difference - ζ values = 1 %
w values = 2 % .

2. Values along a line mid-way between the centre line and the edge and parallel to the load line.

Assuming $\overline{W}$ is distributed over a length N/2.

								℄	
0	.0602	.1163	.1652	.2055	.2364	.2582	.2711	.2754	pN/K
0	.462	.902	1.300	1.644	1.920	2.123	2.246	2.278	pN^3/K

Corresponding values, Field No.45.

								℄	
0	.0598	.1154	.1638	.2037	.2342	.2557	.2683	.2725	pN/K
0	.455	.888	1.279	1.619	1.887	2.085	2.206	2.244	pN^3/K

Max.Difference - ζ values = 1 %
w values = 1.5% .

Case (b), Similar loading as above but load applied along the quarter line.

1. Load line values, assuming $\overline{W}$ is distributed over a length N/2.

								℄	
0.	.2295	.3527	.4349	.4919	.5312	.5569	.5715	.5762	pN/K
0	.506	.970	1.373	1.709	1.968	2.155	2.268	2.305	pN^3/K

Corresponding values, Field No.46

								℄	
0	.2275	.3496	.4308	.4873	.5260	.5512	.5655	.5702	pN/K
0	.495	.950	1.347	1.669	1.923	2.105	2.213	2.249	pN^3/K

Max. Difference ζ values = 1 %
w values = 2.5 % .

Centre line values as in 2 above.

Case (c) Similar loading as above applied along the longer centre line of a rectangular plate 8N x 4N.

1. Load line values, assuming $\overline{W}$ is distributed over a length N/2.

								℄	
0	.2083	.3113	.3751	.4162	.4426	.4589	.4678	.4706	pN/K
0	.265	.498	.691	.840	.951	1.026	1.071	1.086	pN^3/K

Corresponding values, Field No.53

								℄	
0	.2064	.3084	.3713	.4120	.4380	.4537	.4625	.4653	pN/K
0	.256	.483	.672	.815	.917	.989	1.032	1.046	pN^3/K

2. Values along a line parallel to the load line and mid-way between the load line and an edge.

								℄	
0	.0663	.1217	.1624	.1902	.2085	.2199	.2261	.2281	pN/K
0	.173	.329	.462	.567	.645	.698	.729	.740	pN^3/K

Corresponding Values, Field No.53.

								℄	
0	.0658	.1206	.1607	.1882	.2061	.2173	.2235	.2254	pN/K
0	·167	·319	·447	·548	·622	·673	·703	·713	pN^3/K

Max. Difference, 1. and 2. ~ ζ values = 1% ; w values = 3·8%

The agreement between the above values, particularly the ζ values, is sufficiently close to justify the method being applied to other line loading conditions and especially to slabs which are continuous over intermediate beams. The total loads are not quite the same in both of the above cases, there being a small difference due to small lengths N/4 at each end of the line not being loaded in the $\overline{W}$ system. It is assumed, however, that any load on these end portions is transferred to the boundary supports without affecting the slab, and although further refinements will eliminate the discrepancy they are hardly justified from a practical point of view. The beam loading is therefore considered as being equivalent to a series of concentrated loads spaced N/2 apart and the amount of each concentration is obtained by equating slab and beam deflections at each assumed point of concentration.

For the primary field, three loading conditions are necessary, viz.,

(a), all spans loaded,

(b), alternate spans loaded,

(c), adjacent spans, and thereafter each alternate span, loaded,

and it is only in the simpler symmetrical cases of two or three intermediate spans that solutions are readily available.

The problems are also considerably more difficult when the boundary supports deflect under load. The simply supported plate is, in fact, analagous to the statically determinate beam in the sense that the ζ and w values being zero on the boundaries are known beforehand. For a free edge the boundary conditions are,

$$M_x = 0\ ;\quad \therefore \frac{\partial^2 w}{\partial x^2} + \sigma \frac{\partial^2 w}{\partial y^2} = 0,$$

$$R_x = 0\ ;\quad \therefore \frac{\partial}{\partial x}\left[\frac{\partial^2 w}{\partial x^2} + (2-\sigma)\frac{\partial^2 w}{\partial y^2}\right] = 0,$$

and for a slab supported on a flexible beam which does not resist torsion,

$$M_x = 0\ ;\quad \therefore \frac{\partial^2 w}{\partial x^2} + \sigma \frac{\partial^2 w}{\partial y^2} = 0,$$

and,

$$K\frac{\partial}{\partial x}\left[\frac{\partial^2 w}{\partial x^2} + (2-\sigma)\frac{\partial^2 w}{\partial y^2}\right] = IE\frac{\partial^4 w}{\partial y^4},$$

where IE is the flexural rigidity of the supporting beam.

Trial and error methods ultimately yield the correct ζ and w values on the free or deflected boundary, but from certain preliminary investigations with test fields, the Author is inclined to regard the arithmetical method as not being particularly suited to this special case. Since the technique of the method has not been fully developed, it is not possible to give a definite opinion in the meantime.

Part 2. - Experimental Work.

Description of Apparatus.

A testing machine, specially designed for applying loads to plates or slabs is shown on Drawing No. 1, page 70.

The supporting frame is 6 ft. long, 3 ft. wide and about 4 ft. high, and is made from 6 in. by 3 in. by 12 lb/ft. rolled steel joists welded together to form a rigid structure.

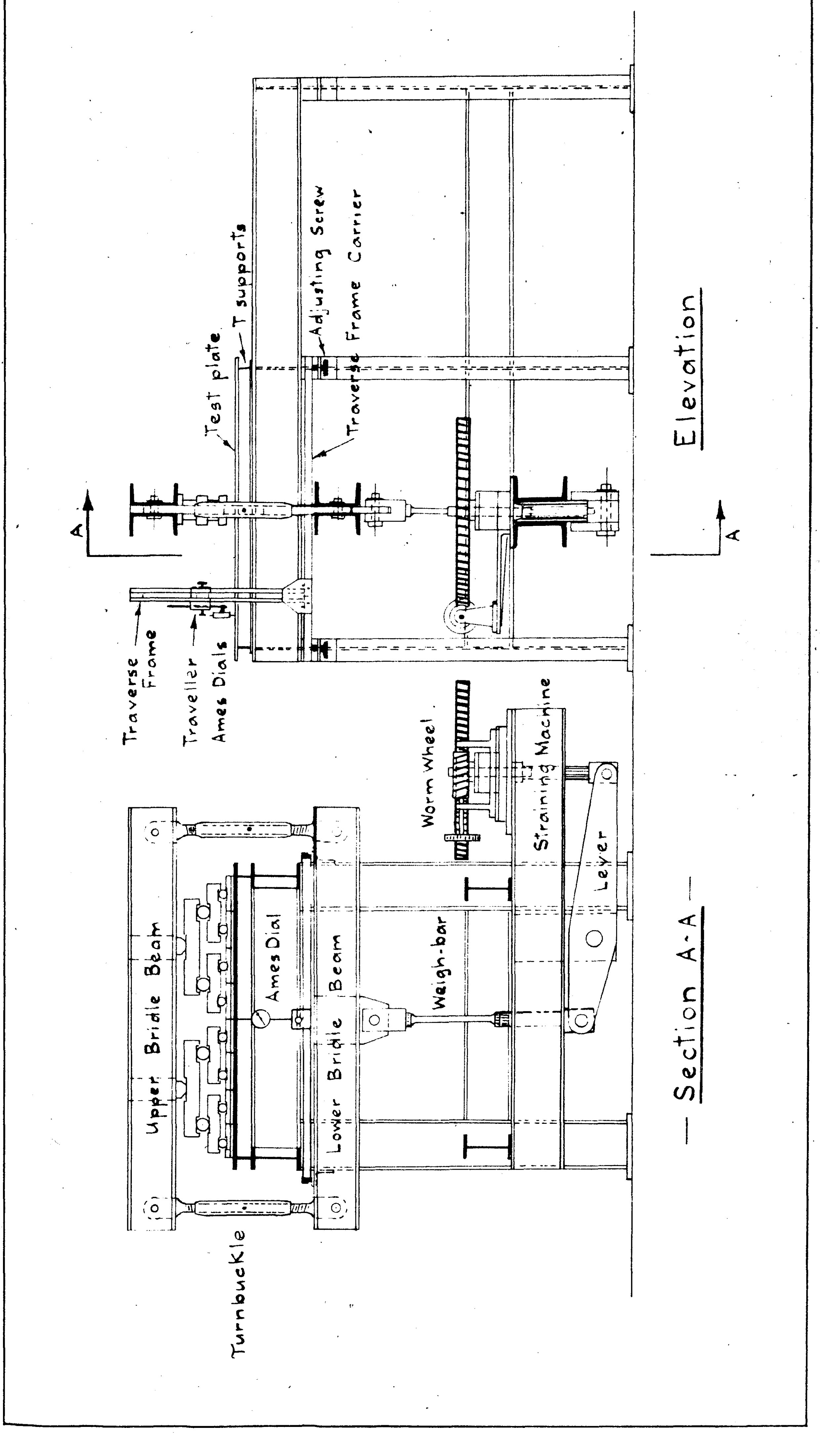
The supports for the plate to be tested are T sections, 2 in. by 2 in. by 1/4 in. thick, resting on top of the frame and held in position by small set screws. To provide a complete bearing throughout the required boundary supports it is necessary to file the vertical legs of the T's to suit the irregularities in the test plate. It is important to make sure that the proper support conditions are realised as far as it is practicable, and considerable care is necessary with this part of an experiment.

Rectangular plates, 6 ft. by 3 ft. maximum size, with or without intermediate supports, are readily accommodated in the testing machine.

The loading device consists of a screw straining machine carried on two 7 in. by 3 in. by 14.22 lb/ft. rolled steel channels underslung on the steel frame. The load is applied to the test plate by means of bridle beams made from rolled steel channels of the same section as the above. These span across the plate and frame and are connected at their ends by turnbuckles which permit adjustments for height to be made. The load is transferred from the upper bridle beam to the test plate, either by means of an arrangement of beams and rollers in the case of a line load, or by packing and a central strut in the case of a concentrated load. The lower bridle beam is connected at its centre to a weigh-bar which is attached to the short arm of a lever, ratio 2 to 1, with its fulcrum on a block fixed to the supporting channels of the straining machine. The longer arm of the lever is connected to the straining machine. The latter is described in a publication by Professor Gilbert Cook (7) who kindly granted the use of this part of the apparatus for the plate tests. Small increments of load can be applied by means of a worm wheel on the straining machine.

The weigh-bar is made from mild steel. It was calibrated in a 10 ton Buckton testing machine, the extension on a 4 in. gauge length measured by means of an Ewing Extensometer giving the load on the weigh-bar. The calibration values for the weigh-bar used throughout the experiments are given in the following table and the calibration curve

Drawing No. 1


Elevation
Test plate
T supports
Adjusting Screw
Traverse Frame Carrier
Traverse Frame
Traveller
Ames Dials
A
A
Section A-A
Upper Bridle Beam
Lower Bridle Beam
Ames Dial
Weigh-bar
Worm Wheel
Straining Machine
Lever
Turnbuckle

is plotted on Drawing No. 2, page 72.

Diameter of Weigh-Bar, 3/4 in. (nominal).

Extensometer Reading Divisions	Load lb.
33.6	1000
48.5	5000
59.6	8000

The above calibration was checked from time to time to ensure that no variation had developed during the repeated loading and unloading of the test plate.

The deflection of the loaded plate is measured by means of Ames Dials which are graduated in thousandths of an inch. All dials used in the experiments were compared with a standard dial which had been checked previously in a dividing machine. The dials showed remarkable agreement and no corrections were necessary.

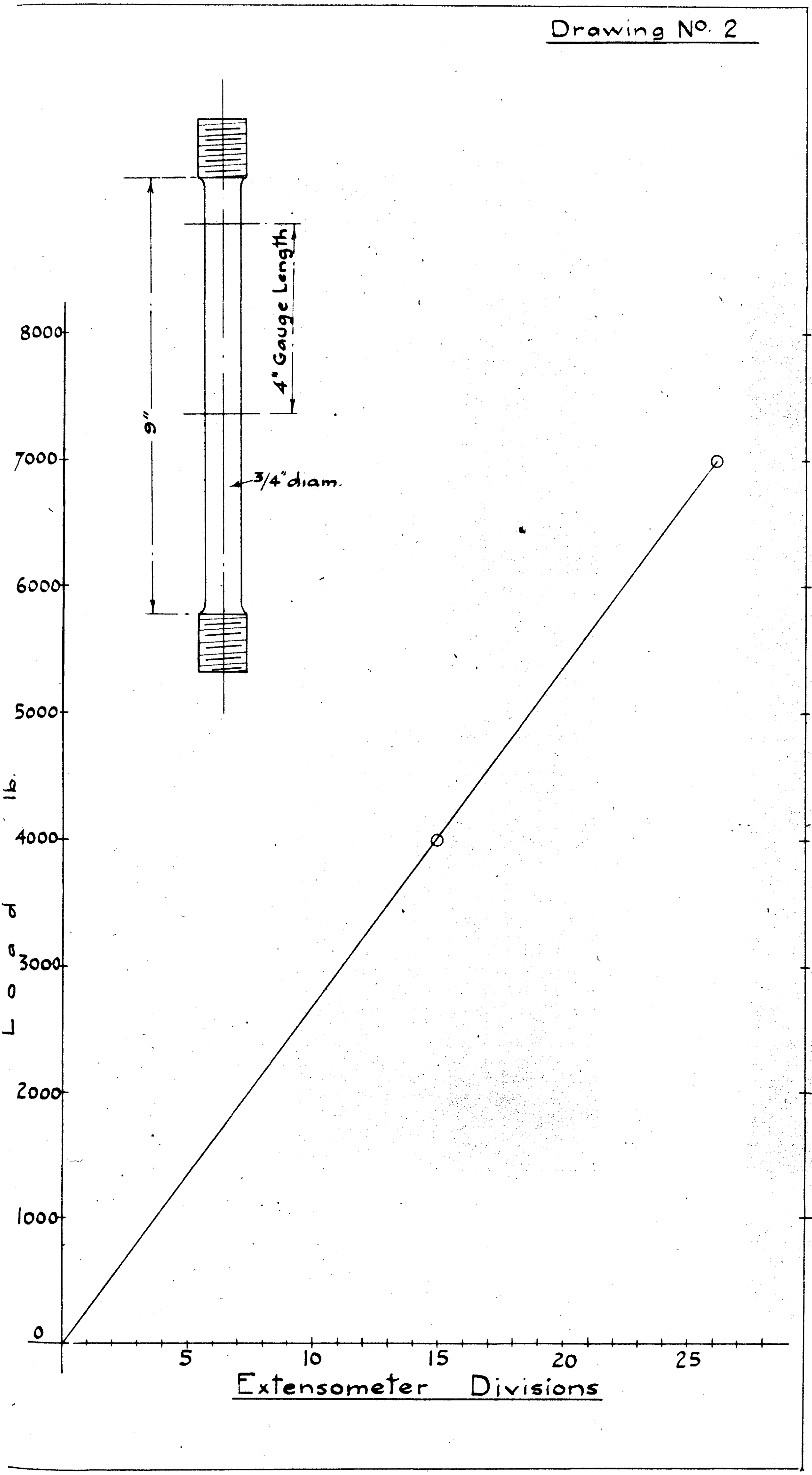
The dials are attached to traveller blocks on a movable traverse frame which is supported and locked in position on a carrier frame. Supports for the carrier frame are provided by angle cleats fixed to the main frame of the structure. Adjusting screws through the cleats are also provided to give a four point bearing. Any point on the plate surface, except the load points, is thus reached with comparative ease. The general arrangement of the above is shown on Drawing No. 3, page 73.

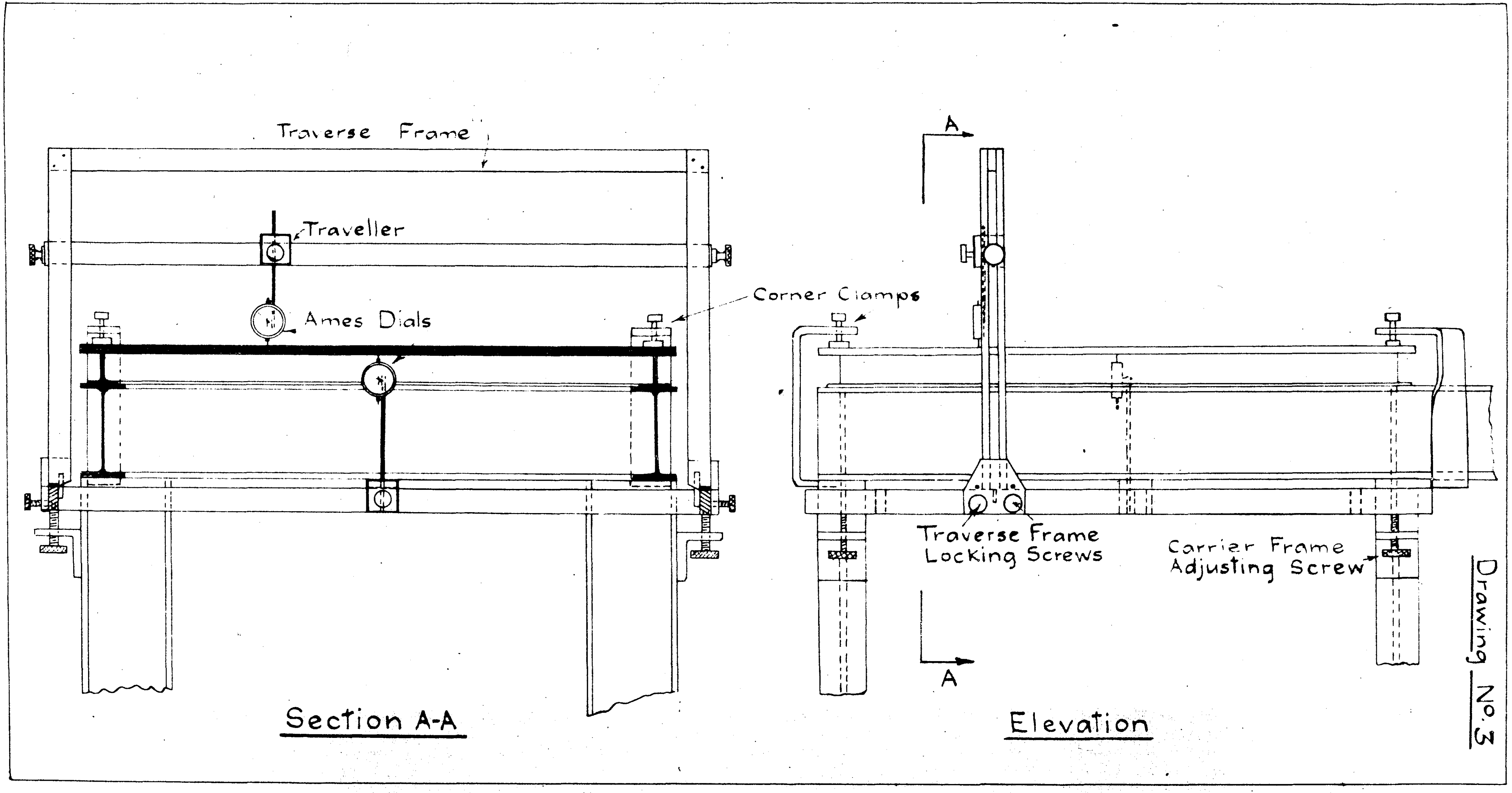
Method of Testing.

The surface of the test plate was divided into squares of 4.5 in. side by means of fine pencil lines, and the deflections at the corners of the squares thus formed were measured in the following manner.

A small amount of load was applied initially to eliminate any irregularities in the seating of the plate and this load was maintained on the plate throughout the complete test. The deflection dials were then set to zero and a test load was applied to the plate. The difference in the dial readings, before and after the application of the test load, gave the deflection of the point on the plate surface. The loading was then returned to the original value and the dials were read again to ensure that no creep or other errors had taken place during the test. The dials were then moved to other points and the above process was repeated until the complete plate area had been explored.

Throughout the tests on steel plates no troubles of any kind were experienced with creep or other errors. The extensometer and dial zero


Drawing No. 2
4" Gauge Length
9"
3/4" diam.
Load lb.
8000
7000
6000
5000
4000
3000
2000
1000
0
5
10
15
20
25
Extensometer Divisions


Traverse Frame
Traveller
Ames Dials
Corner Clamps
A
A
Traverse Frame
Locking Screws
Carrier Frame
Adjusting Screw
Section A-A
Elevation
Drawing No. 3

readings agreed perfectly, and, eventually, the precautionary measurements mentioned above were dispensed with and the dials were moved to other points while the test load was on the plate. In this case the dials recorded the rise in the plate when the load was removed.

The repeated loading and unloading of the plate was unavoidable and, as a further precaution against errors in load measurement, a dial was kept in the same position throughout the test. The most convenient place for this dial was at the centre point of the load line, the dial being underneath the plate and easily read in conjunction with the extensometer. A movement of the traverse frame altered the reading on this dial but the readings before and after the movement were made the same by adjusting the zero on the dial. In this way the introduction of errors because of the repeated loadings was reduced to a minimum.

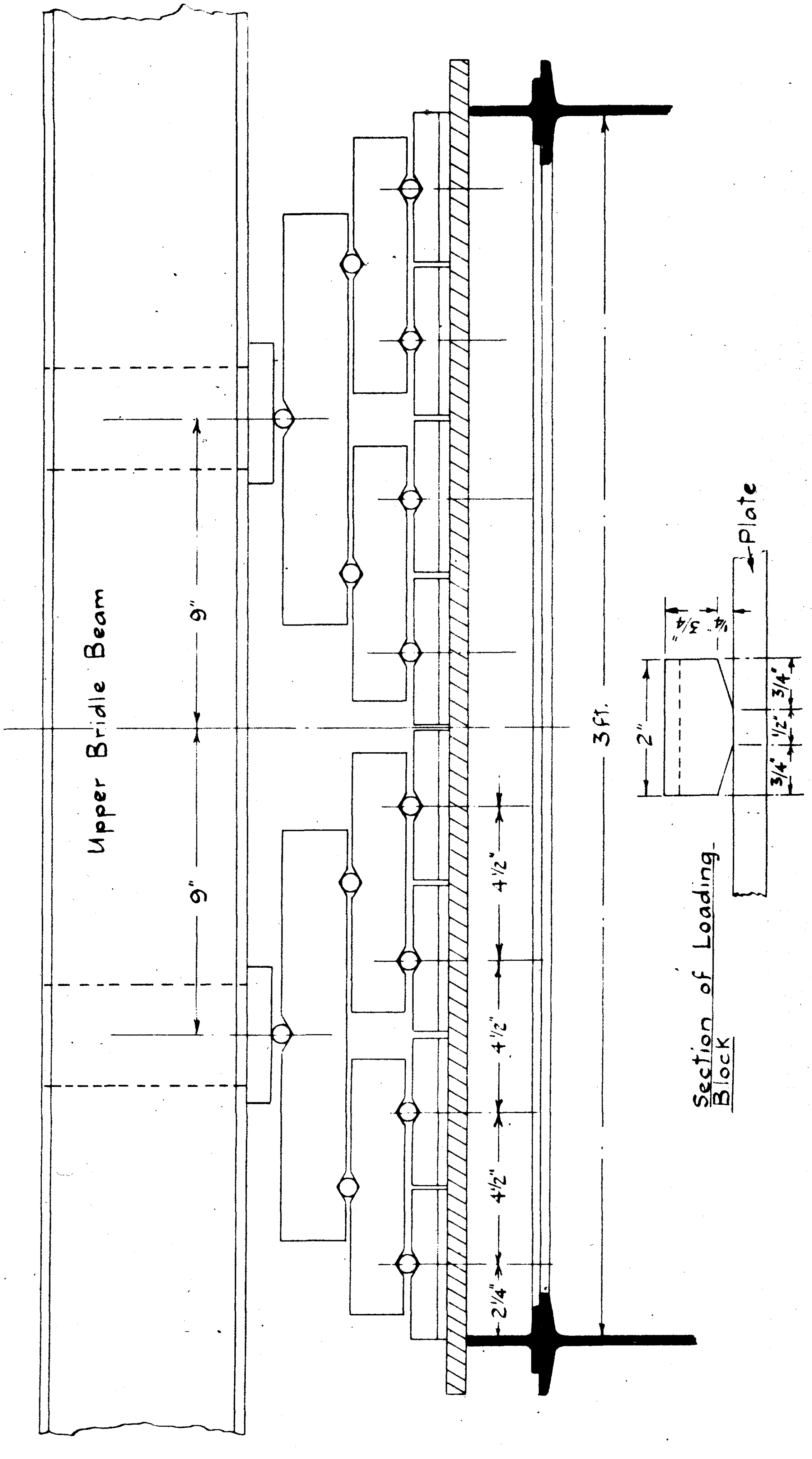
The following experiments were made with a steel plate 3 ft. square and 1/2 in. thick.

Square plate, simply supported along the edges and loaded with a uniformly distributed load applied across a centre line.

An attempt to get the specified loading conditions was made with the arrangement shown in Drawing No. 4/ page 75. This arrangement had the advantage of being easily erected but preliminary test with it gave measured deflections considerably less than the theoretical values. The theoretical solutions are based on the assumption that only the bending and torsional stresses induce the strain energy stored in the plate, the effects of shearing and direct stresses being neglected, but they are accepted as being good approximations provided the deflections are small in comparison with the plate thickness.

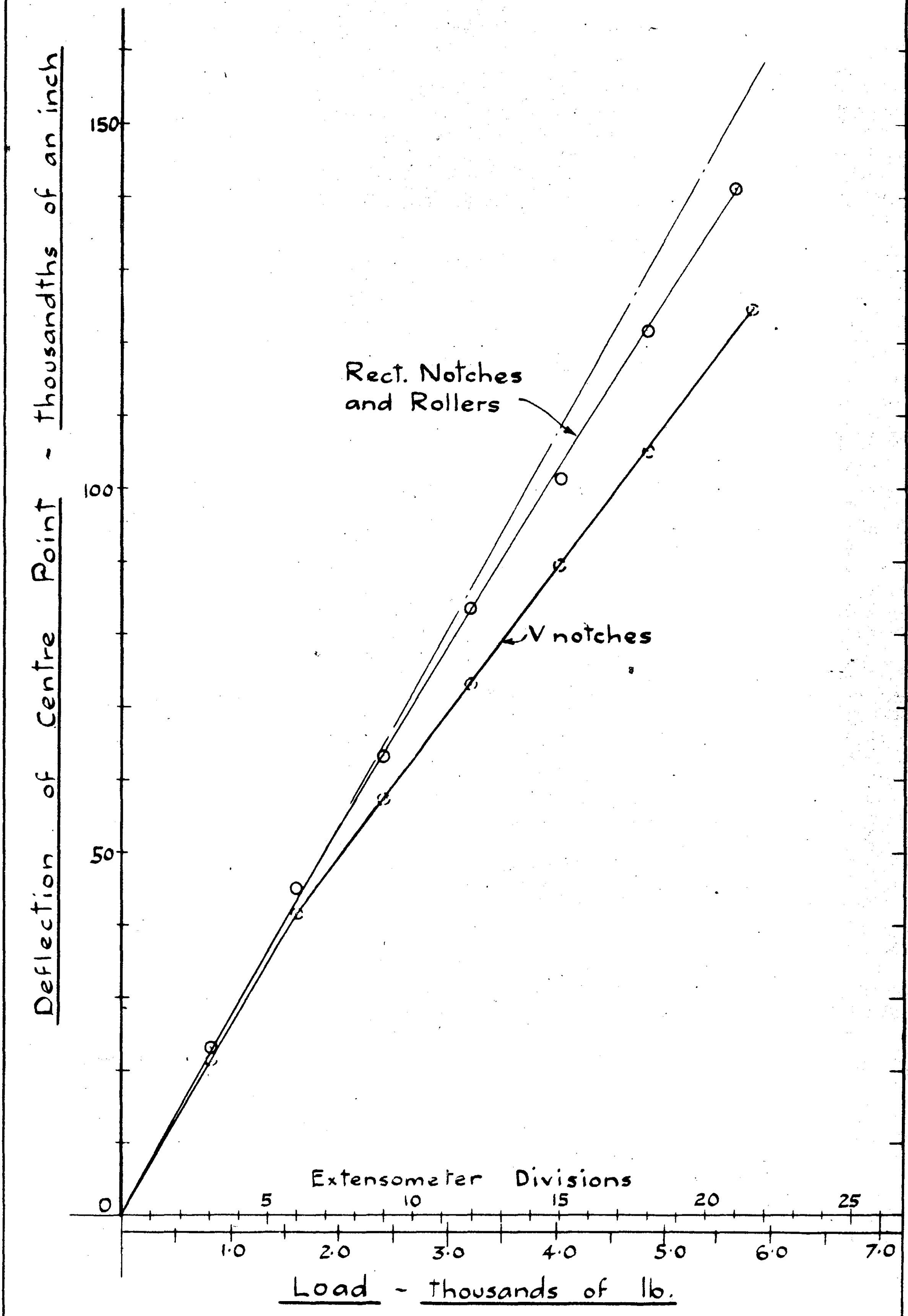
To ascertain the cause of the above discrepancy, the deflection of the centre point of the plate, for various increments of load, was measured and the results are plotted on Diagram No. 5/ page 76. No corrections have been made for the sinking of the supports under load in this case. The break in the deflection-load curve suggested that some shift had taken place during the test. At one stage in the loading of the plate a sharp, metallic noise was heard, but this was attributed to the settling of the straining machine beams and no special notice was taken of it. On further investigation it was found that it was produced by a movement of the load blocks resting on the plate, and it is therefore probable, that after a certain stage in the loading was reached, part of the load increment passed directly to the supports by arch action and did not affect the

Drawing No. 4



Upper Bridle Beam
9"
9"
3 ft.
4 1/2"
4 1/2"
4 1/2"
2 1/4"
Plate
1/4"
3/4"
2"
3/4"
1/2"
3/4"
Section of Loading Block

Drawing No. 5



Deflection of Centre Point - thousandths of an inch
150
100
50
0
Rect. Notches and Rollers
V notches
Extensometer Divisions
5
10
15
20
25
1.0
2.0
3.0
4.0
5.0
6.0
7.0
Load - thousands of lb.

plate under test.

Accordingly, to eliminate these frictional troubles the V notches were machined out and larger diameter rollers were provided as shown in Drawing No. 6, page 78.

The load-deflection curve for the centre point with this loading arrangement is also plotted on Drawing No. 5, page 76, and it will be seen that the deflection is more nearly directly proportional to load.

The distribution of load was not checked, but, as far as all practicable requirements are concerned, no great errors are introduced by making the assumption that the loads on each lower roller are equal.

During the preliminary testing, it was noticed, also, that the corners of the plate rose off the supports and only the central portions for a length of about 10.5 in. on either side of the centre lines remained in contact. Corner clamps, as illustrated in Drawing No. 3, page 73, were provided and no further troubles were experienced in this respect.

Measurements of the deflections of all points on a square net work of 4.5 in. side were taken for an applied load of 5650 lb. The deflections of corresponding points agreed extremely well and the mean values are shown in Fig. No. 41 .

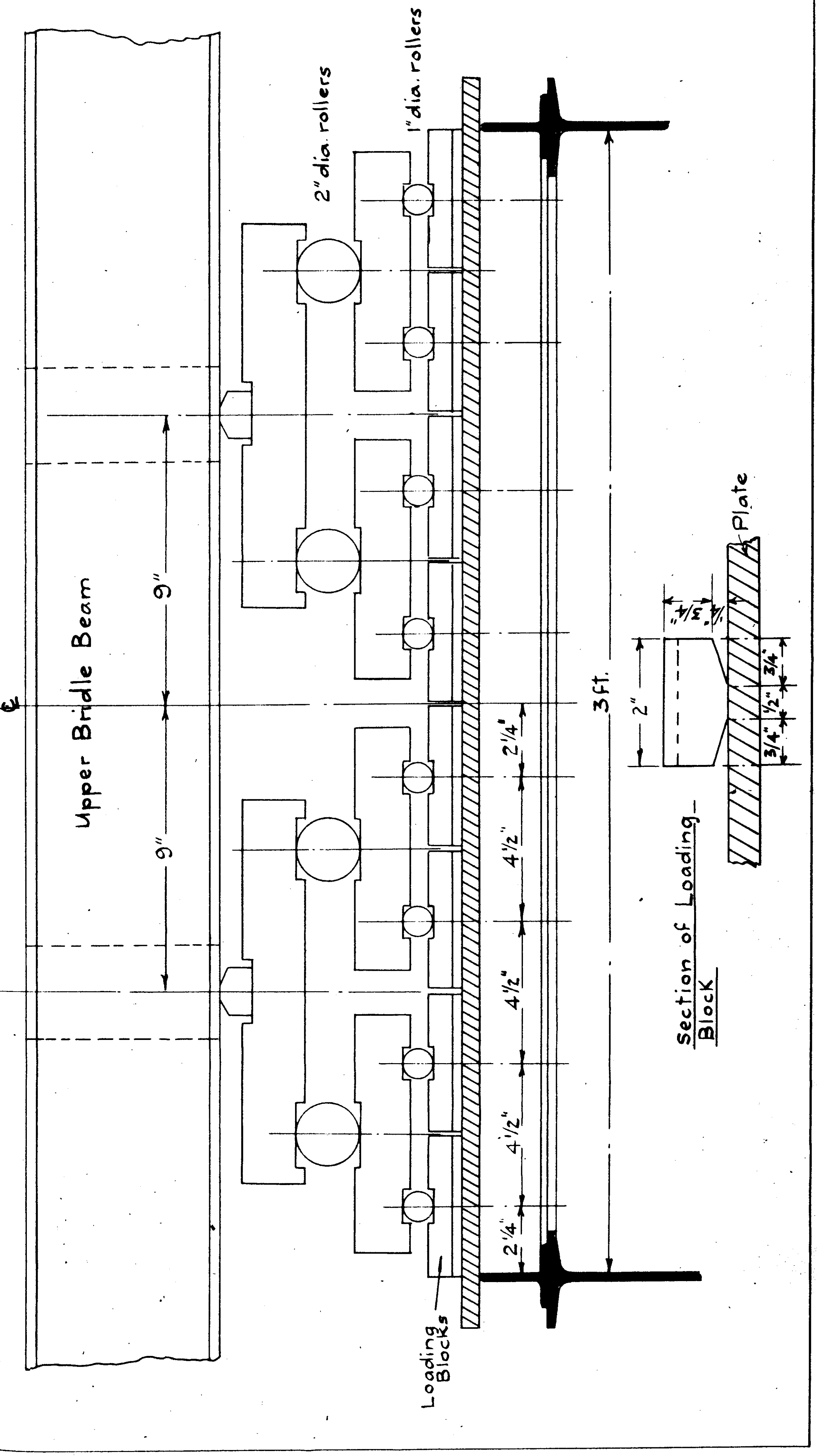
1.05	1.55	2.3	3.0	3.0
1.6	20.0	35.4	45.1	48.8
2.6	38.0	66.7	85.6	92.2
3.1	52.1	92.3	107.7	126.8
3.5	60.0	104.5	132.3	141.0

L o a d L i n e

Deflections in thousandths of an inch. Load = 5650 lb.

Fig. 41.

Drawing No. 6


Upper Bridle Beam
℄
9"
9"
2" dia. rollers
1" dia. rollers
Loading Blocks
3 ft.
2 1/4"
4 1/2"
4 1/2"
4 1/2"
2 1/4"
Plate
Section of Loading Block
2"
1/4"
3/4"
3/4"
1/2"
3/4"

Comparison of measured and calculated deflections.

Because of the deflections of the supports under the test load, a direct comparison of the measured and theoretical deflections is not available. A comparison may be obtained, either, by correcting the measured deflections for support yield, or, by correcting the boundary of the theoretical solution to agree with the test conditions. The former method has been chosen and the corrections were obtained from Fields Nos. 21 to 28. The effects of curvature, on the boundaries were neglected, but it is thought that this is not a serious omission since the boundary deflections are comparatively small.

The corrected field is shown in Fig. 42, the corrected measured deflections being in all cases written in black above the line.

The values in red, below the line, are those obtained from the theoretical solution to this problem, Field No. 45, and they are based on the assumption of equal central, measured and calculated, deflections, the amount of the load and the flexural rigidity of the plate being ignored in the meantime. (i.e. the deflection values on Field No. 45, are multiplied by $\frac{138 \cdot 3}{3 \cdot 457} \doteqdot 40$).

A useful check on the solution of the fundamental differential equation is afforded by comparing the black and red values. It may be noted that the agreement between them is practically complete.

To calculate the theoretical values, the modulus of elasticity and Poisson's ratio for the plate material must be known. Assuming these as 30×10^6 lb/in^2 and 0.28 respectively, then for a square plate of 3 ft. side and 1/2 in. thick, loaded with a total load of 5650 lb. the value of $pN^3(1-\sigma^2)/IE = 0.0421''$. The theoretical deflections are therefore obtained by multiplying the coefficients of w on Field No. 45 by the above value. The values thus obtained are marked in green on Fig. 42.

The theoretical deflections are in excess of the measured values at all points on the plate surface. At the centre point this difference amounts to 6.7 thousandths of an inch. (i.e. about 4.6 per cent. of the theoretical deflection.)

Comparison of Measured and Theoretical Deflections.

Steel plate, 3 ft. square and 1/2 inch thick, simply supported on the four edges and loaded with a central line load of 5650 lb.

	18.0 18.3 19.3	32.7 33.4 35.1	42.1 43.3 45.3	45.8 46.5 49.0	
	35.8 35.5 37.4	64.2 64.6 68.0	82.8 83.5 87.8	89.4 89.8 94.4	
	49.6 49.8 52.5	89.7 90.0 94.5	115.0 115.0 120.8	124.1 124.0 130.0	
℄	57.2 56.8 59.6	101.8 101.8 106.5	129.6 129.0 135.5	138.3 138.3 145.0	load line

Measured deflections, corrected for boundary deflection, are shown in black. Red values are based on equal calculated and measured deflections of the centre point.

Theoretical deflections are shown in green.

Deflections in thousandths of an inch.

Fig. 42.

Steel plate, 3 ft. square and 1/2 inch thick, simply supported along the edges and loaded with a uniformly distributed load along a line parallel to one edge and distant 9 inches therefrom.

The steel plate was kept on the same supports as in the previous experiment and the straining machine and the beam and roller arrangement on Drawing No. 6, page 78, were moved to give the specified load position. Corner clamps were also provided to prevent corner uplift.

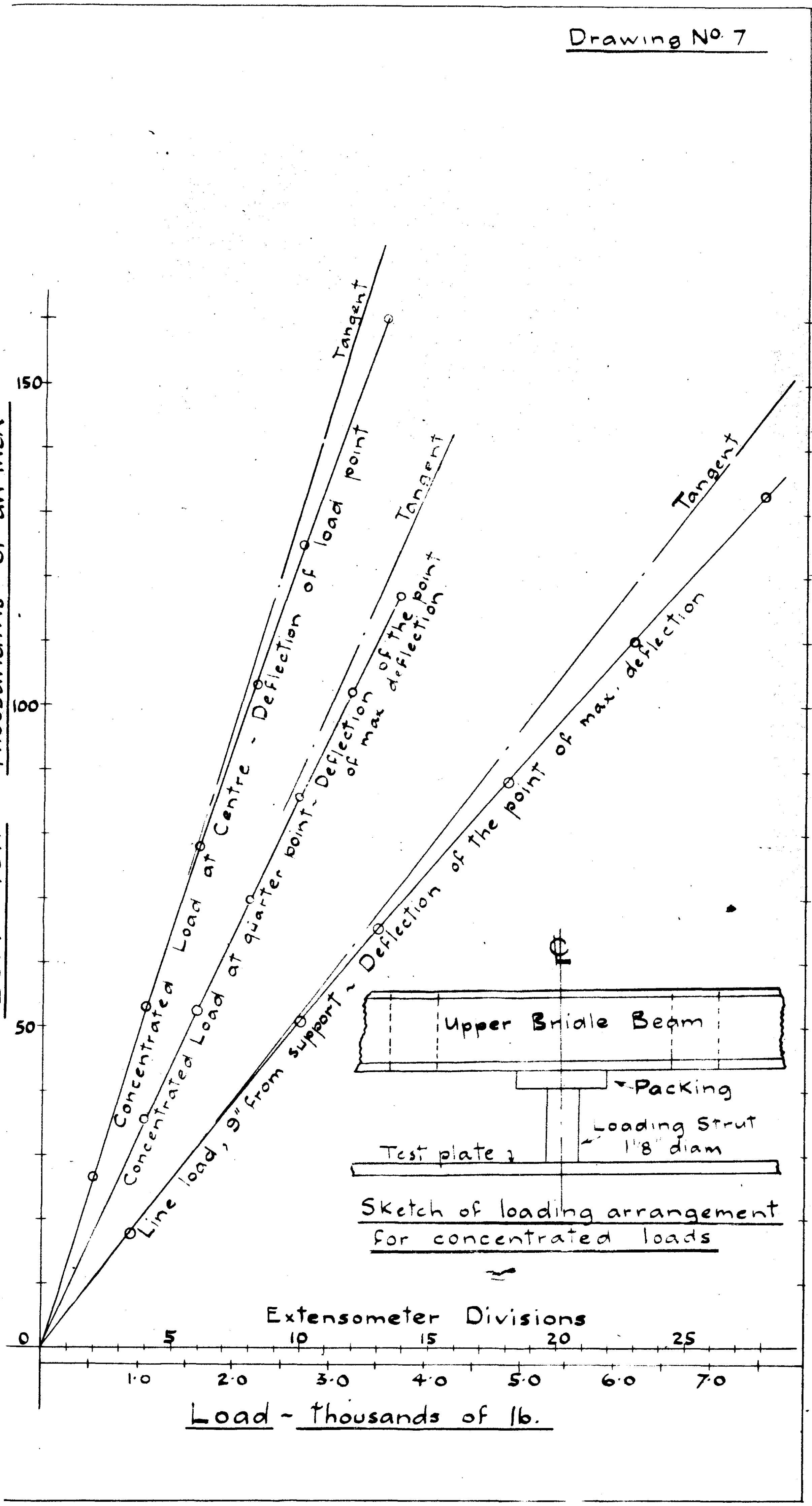
The load-deflection curve, for the point of maximum deflection on the plate, is shown on Drawing No. 7, page 82. This has not been corrected for sinking of the supports under load.

The deflections of the corner points on a 4.5 inch square net work were measured in the same manner as in the preceding experiment and the average values for corresponding points are given in Fig. 43. The total applied load in this case was 7540 lb.

0.4	1.1	1.5	2.8	3.5	
1.4	13.5	23.9	31.3	34.2	
2.0	26.2	46.8	60.7	66.0	
2.5	38.6	68.6	88.5	95.0	
2.5	49.0	87.0	111.5	119.5	
2.5	55.5	97.8	123.3	132.0	
2.5	52.5	91.2	111.8	120.0	Load Line
2.5	32.4	55.5	68.5	72.5	
1.5	2.7	4.5	4.5	4.5	

Deflections in thousandths of an inch. Total load = 7540 lb.

Fig. 43.



Drawing No. 7
Deflection ~ thousandths of an inch
0
50
100
150
Tangent
Concentrated Load at Centre - Deflection of load point
Tangent
Concentrated Load at quarter point - Deflection of the point of max deflection
Tangent
Line load, 9" from support ~ Deflection of the point of max. deflection
Upper Bridle Beam
Packing
Loading Strut 1'8" diam
Test plate
Sketch of loading arrangement for concentrated loads
Extensometer Divisions
5
10
15
20
25
1·0
2·0
3·0
4·0
5·0
6·0
7·0
Load ~ thousands of lb.

Comparison of measured and theoretical delfections.

The correction values for the boundary deflections, again neglecting the effects of curvatures, may be obtained by summing values from Fields Nos. 2, 29 to 35, and 36 to 43, but the labour involved is very great and they are obtained more readily by direct squaring using the methods and formulae already described for the solutions of $\nabla^4 w = 0$. (Fields 54 to 58 have since been made available and may be used instead of the above)

The corrections thus obtained are shown in Fig. 44.

				℄	
0.4	1.1	1.5	2.8	3.5	
1.4	1.6	2.0	2.4	2.7	
2.0	2.0	2.2	2.4	2.5	
2.5	2.3	2.4	2.5	2.5	
℄ 2.5	2.5	2.6	2.6	2.7	
2.5	2.6	2.8	2.9	2.9	
2.5	2.8	3.0	3.2	3.3	Load Line
2.5	2.9	3.5	3.8	3.8	
1.5	2.7	4.5	4.5	4.5	

Corrections for boundary deflections in thousandths of an inch.

Fig. 44.

Applying the above corrections to the measured values on Fig. 43, the values, marked in black above the lines in Fig. 45, are obtained. The values marked in red on the same Figure are again based on the assumption of equal theoretical and measured deflections of the point of maximum deflection. (i.e. the red values are obtained by multiplying the coefficients of w on Field No. 46 by $\frac{129.1}{2.487} \doteqdot 52$.) The agreement between both values is very good.

With E and σ 30×10^6 lb/in.^{2} and 0.28 respectively, then for a square plate of 3 ft. side and 1/2 inch thick, loaded with a line load of 7540 lb. the value of $\frac{pN^3(1-\sigma^2)}{IE} = 0.0562''$. Multiplying the coefficients of w on Field No. 46 by the above value, the theoretical deflections are obtained and are marked in green on Fig. 45 . It may be noted that they are greater than the measured values at all points on the plate. At the point of maximum deflection the difference is 10.5 thousandths of an inch which is equivalent to 7.5 per cent of the measured value.

11.9	21.9	28.9	31.5
12.1	22.3	29.0	31.4
13·1	24·1	31·4	33·9
24.2	44.6	58.3	63.5
24.2	44.3	57.9	62.5
26·2	48·0	62·6	67·5
36.3	66.2	86.0	92.5
36.0	65.8	85.3	92.1
38·9	71·1	92·2	99·5
46.5	84.4	108.9	116.8
46.1	84.0	108.2	116.5
49·8	90·8	117·0	126·0
52.9	95.0	120.4	129.1
52.6	94.3	120.1	129.1
56·8	101·8	130·0	139·6
49.7	88.2	108.6	116.7
49.3	86.7	109.0	116.7
53·3	93·9	118·0	126·1
29.5	52.0	64.7	68.7
28.7	50.6	63.9	68.3
31·0	54·8	69·2	73·9

Load Line

Note.- To accommodate the rows of figures the distances at right-angles to the load line are exaggerated on this figure.

Measured deflections, corrected for boundary deflection, are shown in black.

Red values are based on equal calculated and measured deflections of the centre point.

Theoretical deflections are shown in green.

Deflections in thousandths of an inch. Total load = 7540 lb.

Fig. 45.

Steel plate, 3 ft. square and 1/2 inch thick, simply supported along the edges and loaded with a concentrated load at the centre of the plate.

The loading arrangement is shown on Drawing No. 7, page 82, the load being distributed over an area of 1 square inch of the plate area. Corner clamps were provided to prevent corner uplift.

The deflections were measured over the same net work and in the same manner as previously described. The load deflection graph for the centre point of the plate is shown in Diagram No. 7 . This, also, has not been corrected for sinking of the supports.

The mean values, for an applied load of one ton, are shown on Fig. 46.

				℄
0.3	0.5	1.2	1.5	1.6
0.5	12.2	23.1	30.8	32.8
1.2	23.1	43.0	58.0	64.1
1.5	30.8	58.0	79.8	89.5
℄ 1.6	32.8	64.1	89.5	103.0

Deflections in thousandths of an inch   Central Load = 2240 lb.

Fig. 46.

The correction values for the boundary deflections are readily obtained from Fields Nos. 11 to 19.

The corrected measured values are marked in black on Fig. 47 .

The values, marked in red are based on the previous assumption of equal theoretical and measured deflections of the centre point of the plate. It may be noted that the agreement between both sets of values is very good.

The theoretical solution, Field No. 48, gives the central deflection, for a high concentration of load, = $0.763\ \bar{W}\ N^2/K$.

With E and σ equal to $30 \times 10^6 lb/in^2$ and 0.28 respectively, then for a plate 3 ft. square and 1/2 inch thick, loaded with a load $\bar{W}$ = 2240 lb., the value of $\bar{W} N^2/K$ is equal to 0.134 inches. Multiplying the coefficients of the w values on Field No. 48 by 0.134, the theoretical deflections, shown in green, are obtained. The maximum theoretical deflection is 0.1025 inches, which is about 1 per cent greater than the corrected measured value.

The agreement between the various values is very good in this particular case. The load deflection graph on Drg. No 7, page 82 shows that the variation of deflection with load is very nearly linear for a range of central deflections from zero to 0.160 inches.

				$\mathbb{C}$
	11.3 11.4 11.5	21.9 21.9 22.1	29.4 29.0 29.2	31.3 31.9 32.1
	21.9 21.9 22.1	41.8 41.5 41.8	56.7 56.2 56.6	62.8 62.0 62.5
	29.4 29.0 29.2	56.7 56.2 56.6	78.5 78.0 78.5	88.2 88.0 88.6
$\mathbb{C}$	31.3 31.9 32.1	62.8 62.0 62.5	88.2 88.0 88.6	101.7 101.7 102.3

Measured deflections, corrected for boundary deflection, are shown in black.

Red values are based on equal calculated and measured deflections of the centre point of the plate.

Theoretical deflections are shown in green.

Deflections in thousandths of an inch.

Fig. 47.

<u>Steel plate, 3 ft. square and 1/2 inch thick, simply supported along the edges and loaded with a concentrated load at the 1/4 point on an axis of symmetry.</u>

The loading arrangement of the preceding experiment was used also in this case, and, since both experiments are similar the test results only are given. Corner uplift was again prevented.

The load-deflection curve, for the point of maximum deflection of the plate, is shown on Drawing No. 7, page 82. This has not been corrected for the boundary deflections.

The averages of the measured deflections at corresponding points throughout the plate surface, for a total load of 3690 lb., are given in Fig. 48 .

				℄
0	0.5	0.8	2.0	3.5
0.9	10.5	19.5	26.5	29.1
2.1	21.0	38.1	50.4	55.0
2.3	30.3	55.7	73.9	80.8
℄ 2.3	36.7	70.0	93.7	103.0
2.3	39.5	74.6	104.7	117.5
				119.0 (Max.)
2.3	34.0	65.8	95.2	113.0 Load Point
1.6	20.5	40.0	58.1	67.2
0.8	0.8	2.8	4.4	4.5

<u>Deflections in thousandths of an inch.</u> <u>Total Load = 3690 lb.</u>

Fig. 48.

The corrections for the above boundary deflections are shown in Fig. 49 . These are obtained in a similar manner to those for the line load across the quarter of the plate.

0	0.5	[illegible]	2.0	3.5
0.9	1.3	1.6	2.1	2.4
2.1	2.0	2.1	2.3	2.4
2.3	2.3	2.3	2.4	2.4
2.3	2.4	2.5	2.5	2.6
2.3	2.3	2.6	2.8	2.9
2.3	2.3	2.6	3.1	3.2
1.6	1.9	2.6	3.4	3.7
0.8	0.8	2.8	4.4	4.5

Corrections for boundary deflections in thousandths of an inch.

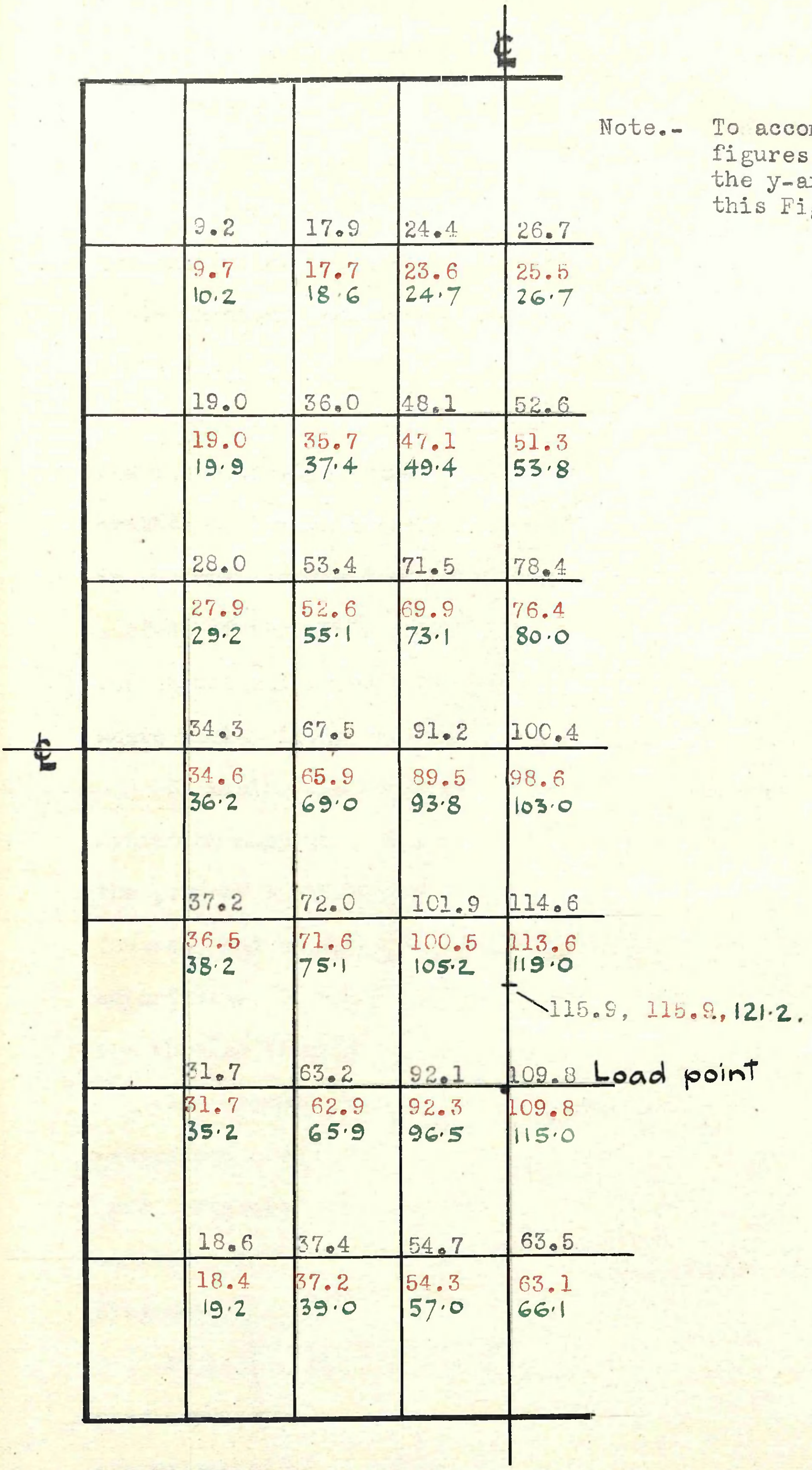
Fig. 49 .

Comparison of measured and theoretical deflections.

The corrected measured values are shown in black on Fig. 50. The values in red, on the same figure, are based, as formerly, on equal theoretical and measured deflections of the point of maximum deflection. (i.e. the red values are obtained by multiplying the w values of Field No. 49 by 115.9/.549). In the central portion of the field, the red values are less than the corrected measured values by 2 thousandths of an inch, but near the load point they are in good agreement.

Using the previous values of E and α , $\bar{W} N^2/K$ = 0.221 inches, and, multiplying the w coefficients of Field No. 49 by this value, the theoretical deflections, in inches, are obtained. These are marked in green on Fig. 50 . The maximum deflection is about 4.6 per cent greater than the measured value.

Note.- To accommodate the rows of figures the distances along the y-axis are exaggerated on this Figure.



Measured deflections, corrected for boundary deflection, are shown in black.

Red values are based on equal calculated and measured deflections of the point of maximum deflection.

Theoretical deflections are shown in green.

Deflections in thousandths of an inch. Total Load = 3690 lb.

Fig. 50 .

Conclusion.

The main purpose of the foregoing experiments is to afford a comparison between the deflection values of the arithmetical analyses and those from actual tests with line and concentrated loading. It was expected that the theoretical values, as in other recorded experiments(8), would be greater than the test results and that the difference would increase with load, but the Author was unable to find any record of experimental work which would furnish a comparison throughout the entire surface of the loaded plate.

The agreement between the two sets of values, marked in black and red respectively, leaves little doubt about the efficiency of the arithmetical method of analysis. When a fine network is used, the method is somewhat long and tedious, but it gives a solution for the entire surface of the plate which is, in effect, analogous with influence lines for structural frameworks, etc. This is an extremely valuable property where moving loads are concerned and should not be overlooked.

By making use of the Theorems of Reciprocity and methods of super--position many important problems are readily solved once the values on the primary field are known. The fields themselves prove the Reciprocity Theorem, and an analytical proof is therefore considered as being superfluous. It may be noted from Fields Nos. 45 & 46 and 48 & 49 that the theorem is valid for deflections,bending moments,and shearing force.

The arithmetical method may also be used with advantage where simply--supported plates of polygonal shape or floor slabs with re-entrant angles are concerned, and many interesting solutions for the isoceles right--angled triangle are obtained by folding the square fields about their diagonals and subtracting corresponding values.

It is not claimed that the method is suitable for all classes of practical problems. With coarser networks, however, many closer approximations than those in current use may be made without undue labour, and,in this respect, it may be mentioned that all degrees of fixity, from the fully clamped to the simply supported edge condition, are readily handled.

It is hoped that the Fields may also prove useful in other branches of research where solutions of fundamental differential equations of a similar type are necessary.

--o--o--0--o--o--

Appendix

As mentioned in the Prefatory Note, several publications are now available which make it possible to compare the various results.

Line Load across the centre of the plate.

(a) Square plate of side a and thickness h, $\sigma = 0.3$.

$w_{max.} = 0.0736\ p\ a^3/E\ h^3$ ....(Timoshenko)(9)

Corresponding value, arithmetical method, $0.0736\ p\ a^3/E\ h^3$.

(b) Rectangular plate, 2a x a x thickness h, line load applied along the centre line of length 2a, $\sigma = 0.3$.

$w_{max.} = 0.1779\ p\ a^3/E\ h^3$ ....(Timoshenko)(9)

Corresponding value, arithmetical method, $0.179\ p\ a^3/E\ h^3$.

$$\left.\begin{aligned} M_y &= 0.107\ \overline{W} \\ M_x &= 0.044\ \overline{W} \end{aligned}\right\} \text{..................................} \quad \text{(do.)}$$

Corresponding values, arithmetical method, $0.106\ \overline{W}$ and $0.045\ \overline{W}$.

Concentrated Load at the centre of the plate.

Square plate of side a, thickness h, $\sigma = 0.3$

$w_{max.} = 0.1265\ \overline{W}a^2/Eh^3$ ....(do.)

Corresponding value, arithmetical method, $0.13\ \overline{W}a^2/Eh^3$.

Rectangular plate, 2a x a x thickness h, $\sigma = 0.3$.

$w_{max.} = 0.1803\ \overline{W}a^2/Eh^3$, ...(do.)

Corresponding value, arithmetical method, $0.189\ \overline{W}a^2/Eh^3$.

Partially Loaded Plate.

For the load $\overline{W}$ concentrated over a square of side C, the values of the ϕ coefficients on Diagram No. 7, page 43, are in complete agreement with Timoshenko's for the range C/A = 0.1 to C/A = 1.0 .

For the line load, Diagram No.8, page 46, the ϕ coefficients are less than Timoshenko's for the smaller ratios of L/A but they agree at L/A = 1. It was mentioned on page 44 that assumptions of symmetry had been made, and that slight errors had been introduced as a result. The maximum difference is at L/A = 0.1, Diagram No.8 gives $\phi = 0.36$ whereas Timoshenko's value is 0.378.

A complete solution for the simply supported square plate with a concentrated load at its centre is given by Holl (10). The Marcus method, based on the membrane analogy, was used in this case, and a network of 64 squares was taken. A comparison of the several results is interesting in view of the closer mesh of 256 squares on Field No. 48.

In this example the dimensions of the plate are 2a by 2a by thickness h and the thickness/span ratio, h/2a, is 0.10, $\alpha = 0.20$.

In another publication, Holl[11] gives the following values for the equivalent diameter D_e:-

D_e = 0.055 A for point loads, **where A is the plate length or span.**

D_e = 0.060 A, when the actual load is concentrated over a circle of diameter D = 0.02 A = 0.2 h.

D_e = 0.074 A, as above, D = 0.060 A = 0.6 H.

D_e = D, for values of D greater than 0.08 A.

It is assumed that these values have been used instead of Westergaard's.

Quantity	Holl	Arithmetical
Max. deflection	$0.0522\ \bar{W}a^2/K$	$0.0478\ \bar{W}a^2/K$
$M_x = M_y$	$0.3499\ \bar{W}$	$0.345\ \bar{W}$
Max. Shear, boundary.	$0.2205\ \bar{W}/a$	$0.2072\ \bar{W}/a$
Max. Reaction.	$0.368\ \bar{W}/a$	$0.368\ \bar{W}/a$
Max. Torque, boundary.	$0.0688\ \bar{W}/a$	$0.0704\ \bar{W}/a$
Corner Load.	$0.1376\ \bar{W}$	$0.1408\ \bar{W}$.

With the exception of the maximum deflection, the quantities are in good agreement. It may be noted that Timoshenko's value for the maximum deflection of the above plate is $0.0445\ \bar{W}a^2/K$.

The Marcus method is apparently very similar to the arithmetical one. The membrane analogy on which it is based is,

(a) the deflections of a membrane loaded with loads proportional to those on a given plate may be considered as the sum of the principal moments of the actual plate,

and,

(b), a second membrane may be loaded with elastic weights proportional to these moment sums, and, subject to appropriate boundary conditions, the deflection of the latter membrane will be proportional to the deflections of the actual plate under the given load system.

It is easy to prove that this gives the same formulae (1a) and (1b) for the single square of side 2n but thereafter the methods differ—the arithmetical method groups the squares whereas the Marcus method uses finite differences to get two sets of simultaneous equations. In the example given by Holl[10] it was necessary to solve 2 sets of equations for 10 unknowns in each case, and in view of this, it does not appear to the Author as being simpler or quicker than the arithmetical method used in this Thesis.

Tests of Plaster-Model slabs subjected to concentrated loads have been made by Newmark and Lepper.[12] These give an interesting comparison with the theoretical analysis for the concentrated load on a simply supported square plate.

It is claimed by the above authors, "that when specimens of pottery plaster are made under the proper conditions the stress-strain relation is practically linear up to the point of rupture; the material is relatively weak in tension; and failure seems to occur at a limiting tensile stress, but for practical purposes may be considered to be nearly independent of the magnitude of the other principal stresses. When plaster is used for tests the necessity for measuring strains is eliminated, since the intensity of the maximum tensile stress occurring in the test specimen is equal to the strength of the plaster, and this fairly definite stress corresponds to the ultimate load on the specimen just before rupture. One may determine relative stresses for given loads on different specimens by a comparison of ultimate loads."

The values of Poisson's ratio, determined as the ratio of lateral to longitudinal curvature of certain test beams, varied from 0.15 to 0.26 for seven tests with an average of 0.20.

As a result of these experiments, a coefficient, which may be interpreted as the maximum moment due to a unit load of 1 lb. at the various load positions, was determined. In the case of a plaster slab, 12 inches square and 1 inch thick, loaded with a concentrated load at the centre over a circular area of 1 inch diameter, the coefficient is given as 0.331, and for similar loading at the quarter point of a centre line the coefficient is 0.267.

The corresponding values from the arithmetical analyses, using Westergaard's formula for the equivalent diameter, are 0.305 and 0.29 respectively.

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BASIC CURVE METHODS IN ROAD-CURVE DESIGN

By W. MacGregor*, B.Sc.

Discussed in writing, April, 1942

Transition Curves

To those who are familiar with the advantages to be gained by using basic curve methods in the design and setting-out of road bends, it is strange indeed that despite the numerous papers which have been published in recent years on highway alignment and design, none has dealt with the subject from the aspect of the basic curve. In new road schemes, the running chainage should be maintained throughout the works, and, although much has also been written about this for the case where circular curves alone form the bend, little attention has so far been given to it when transitions are introduced. It will, in general, be agreed that setting-out methods that will allow any point or station on the bend to be readily located are to be preferred to others. In this respect, basic curve methods will meet all requirements, and their use will also give a quick solution in location problems, such as making the bend pass through a pre-selected point. It is not the intention to describe any new type of transition in the present paper, or to dispute the accepted methods of design. The spiral and the lemniscate will be described from the aspect of the basic curve and formulæ correlating speed and curvature will be given for both.

The basic curve method was introduced by Thom,[1] who pointed out that in transition work all spirals have the same

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[1]See bibliography, p. 330.

form and differ only in their size or scale and in the length of curve which is used. The basic curve is defined by its equation. Its linear quantities are non-dimensional, but it may best be visualized as being a model of the full-size curve to a scale of 1/K. Lengths on the full-size curve are therefore obtained by multiplying the corresponding basic curve values by K. This applies equally to the lemniscate and to the spiral, but to avoid confusion in the respective formulæ, the use of K will be restricted to the spiral and the multiplier for the lemniscate will be denoted by J. The basic curves are shown together in Fig. 1, the numbers marked thereon being the so-called lengths

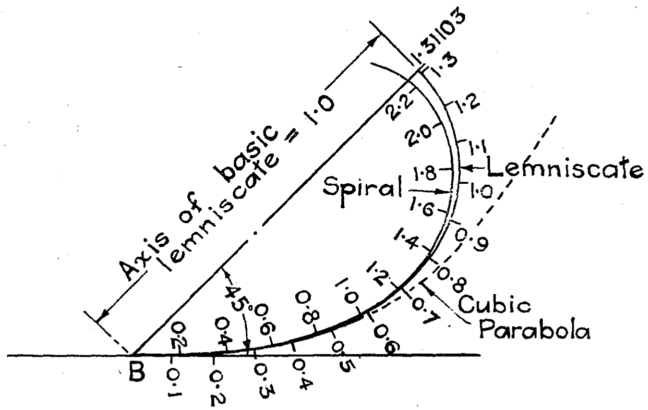


Fig. 1.

from the tangent point B to the point in question. These, when multiplied by K or J respectively, give the distances on the full-size curve. It will be proved later that, for the same initial rate of change of acceleration, $J=\sqrt{3}K$, and to give a comparison between the lemniscate and spiral, scales in the ratio $\sqrt{3}:1$ have been used in plotting the respective curves.

In improvement schemes especially, the multipliers, K and J depend mainly on the extent of the ground available, whereas in new road design, the optimum values can be found by considering the requirements of speed and curvature in addition to those of the site. The speed and curvature requirement herein adopted is the rate of change of acceleration. This quantity will be denoted by C ft. per sec. per sec. in 1 sec., and it will be assumed that the speed of the vehicle is constant throughout the bend in question, v and V being used for speeds in ft. per sec. and miles per hour respectively.

The Centrifugal Ratio, F. On a vehicle travelling in a curve of radius R at a speed v, the radial acceleration is v^2/R. If the centre of gravity of the vehicle is at height h above road level and b is the width of wheel base, then on level road surfaces, overturning of the vehicle will occur if $(v^2/gR)h > b/2$, and side-slip will take place when $(v^2/gR) > \mu$, the coefficient of friction between the tyres and the road. If $\mu < b/2h$ side-slip will take place first; this is the general tendency with road vehicles. On roads superelevated at an angle γ to the horizontal, v^2/gR must not exceed $(\mu + \tan\gamma)/(1 - \mu\tan\gamma)$ or slipping will result.

The important ratio, v^2/gR, has come to be known as the Centrifugal Ratio, and throughout this paper it will be denoted by F. Substituting 32·2ft. per sec. per sec. for g and keeping R in ft. and V in m.p.h.,

$$F = V^2/15R. \tag{1}$$

Extensive research is being constantly carried out by the Road (Materials and Construction) Board of the Department of Scientific and Industrial Research on the development and maintenance of non-skid surfaces, and much valuable data has been published from the results of tests on various types of surfacing materials under wet and dry conditions.[2] An excellent resumé of this work has been given by Pidgeon.[3] In general, it appears that a value of F=0·25 is about the maximum which should be used in the design of modern roads; with this the radial acceleration is 8 ft. per sec. per sec.

The Rate of Change of Acceleration, C. As a result of experiments on railway curves, Shortt[4] suggested that a value of 1 ft. per sec. per sec. in 1 sec. was about the maximum rate at which the acceleration could be acquired without the passengers in a railway carriage experiencing a sensation of discomfort. This value is the standard in present-day railway practice, and many engineers have adopted it also for the design of road curves. In road work, however, it is not a universal standard. Many feel that it is too low; that it gives transitions which are unduly long; and that values of about 2 ft. per sec. per sec. in 1 sec. are not only permissible but indeed desirable. In 1939, a comprehensive review of the position was made by Orchard[5], and since the publication of that paper the validity of C as a factor in the design has been questioned.[6] It should not be overlooked that, apart from its validity from the aspect of comfort, the value of C serves as a useful and convenient means of measuring the sharpness of a bend. More experimental work, with all types of transport vehicles, must be made before the position is finally cleared and a maximum permissible value of C established. In the meantime, therefore, the choice of value must be left to the discretion of the engineer.

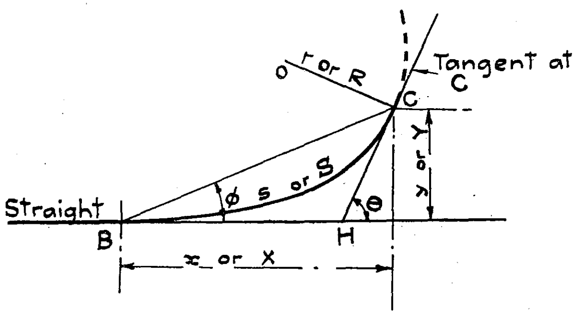


Fig. 2.

THE SPIRAL TRANSITION CURVE

Referring to Fig. 2, the basic spiral is defined by the equation $d\theta/ds=s$, where s is the distance measured along the basic curve. (2)

$\therefore$ $rs=1$, r being the radius of curvature at the point distant s from the origin (3)

For the full-size curve, $S=Ks$, $R=Kr$, and the above become

$$d\theta/dS=s/K=S/K^2, \tag{4}$$

and

$$RS=K^2. \tag{5}$$

Since K is constant numerically, the radius of the full-size spiral is therefore inversely proportional to the distance along the curve; consequently the centrifugal ratio F will vary at a uniform rate when the vehicle is on the transition. Integrating (2) and (3), respectively,

$$\theta = s^2/2, \tag{6}$$

$$= S^2/2K^2. \tag{7}$$

It may be noted that the multiplier K for the spiral is, in effect, the length of the full-size curve which gives a deviation angle $\theta = \frac{1}{2}$ radian.

For the rate of change of acceleration, C, we have,

$$C = \frac{d}{dt}(v^2/R) = v^3/K^2, \tag{8}$$

and therefore, provided the speed of the vehicle is constant, C will also be constant throughout the length of the transition. Re-writing (8), and changing the speed units to m.p.h.,

$$K = 1{\cdot}775\sqrt{(V^3/C)} \text{ ft.} \tag{8a}$$

If Q denotes the rate of turning of the steering wheel in degrees per sec., and G and B are, respectively, the gear ratio of the steering wheel and the length of the wheelbase of the vehicle, then,

$$Q = 26{\cdot}7 \text{ GBC}/V^2. \tag{9}$$

From this it will be noted that if the steering wheel is turned at a steady rate, the vehicle will describe a spiral curve. Since G and B are not uniform for all transport vehicles, Q is best used, not as a prime factor in the design of the bend, but rather as a check on the selected value of C.

With selected values of V and C, the multiplier K is obtained from (8) or (8a), but to complete the data for the design of the bend the length of the transition or the limiting radius must be ascertained. These depend on the value of F, and on the magnitude of the actual deviation angle between the straights. Each bend must therefore receive individual consideration before the optimum values are fixed.

It may be possible to form the bend entirely with two transitions, each being the mirror image of the other about the line bisecting the intersection angle between the straights. On these wholly transitional bends, however, the driver of the vehicle has no respite from turning the steering wheel. On entering the curve he must turn the wheel throughout the entire length of the first transition and thereafter unwind until the next straight is reached. The majority of motorists do not like this arrangement, and it should not be used indiscriminately. Even a short length of circular arc between the transitions is in certain respects an easement curve, and, in the Author's opinion, it adds to the appearance.

On the other hand, it should not be overlooked that the layout which develops the maximum permissible values of C and F, will approach the intersection point of the straights more closely than any other arrangement; incidentally, this will also be the shortest length of curved road but, paradoxically as it may seem, the longest route. This important site requirement may well justify the wholly transitional layout, but in the past the Author has noted a tendency to use this arrangement simply because the calculations are to some extent simplified.

Nevertheless, it is most helpful in the preliminary work to know if a particular bend can be made wholly transitional with the selected values of F and C. Since, in this event, each transition must contribute equally to the deviation angle, the limiting value of the latter is,

$$2\theta = \alpha = (gF)^2/vC \text{ radians.} \tag{10}$$

An arrangement of the above which is more suited for practical use is,

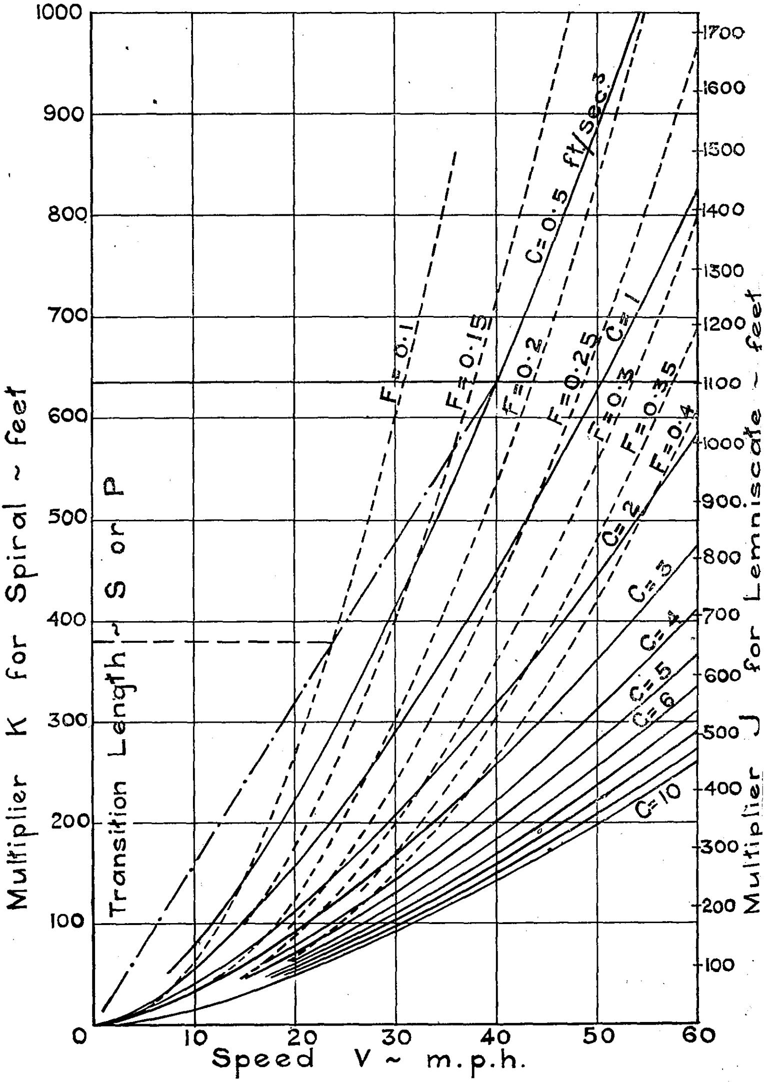
$$\alpha = 40508\ F^2/VC, \text{ where } \alpha \text{ is the limiting value of the deviation angle in degrees,} \tag{11}$$

and a further modification, which is useful in certain problems, is

$$K = V^2\sqrt{\alpha}/113{\cdot}5\ F. \tag{11a}$$

If the deviation angle of the bend under consideration is less than the limiting value found from (11), the bend can be wholly transitional and the value of F used in the substitution may not even be developed. If, however, it is greater, an intermediate length of circular arc will be necessary. At their junction, transition and arc must have the same radius and a common tangent. The locus of the common centre of curvature will be the line bisecting the intersection angle between the straights (Fig. 4).

The length of the spiral throughout which the value of F will not be exceeded, and which is, in fact, the length of the spiral which can be used as a transition, is readily found by substi-



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1200
1100
1000
900.
800
700
600
500
400
300
200
100
Multiplier K for Spiral ~ feet
Transition Length ~ S or P
Multiplier J for Lemniscate ~ feet
Speed V ~ m.p.h.
C=0.5 ft/sec.3
C=1
C=2
C=3
C=4
C=5
C=6
C=10
F=0.1
F=0.15
F=0.2
F=0.25
F=0.3
F=0.35
F=0.4

Fig. 3.

tuting K^2/S for the radius R in (1). This gives,

$$S=15F(K/V)^2. \qquad (12)$$

Care must be taken when using (12), for whether this length will be used in whole or in part depends, as already noted, on the magnitude of the actual deviation angle.

Fig. 3 shows in graphical form the relationship between the design factors V, C and F, the multiplier K and the limiting length of the transition S. It has been constructed by using (8a) and (12) and instructions in its use are given in the accompanying footnote. The higher values of C and F have been included, not because they are likely to be used in road practice, but because they show very clearly the effects of an increase in speed, and may therefore be useful to those who may wish to carry out further investigations of the maximum value of C. It is thought that the diagram, apart from its use in the preliminary work, will give a clearer perspective of the factors which form the basis of the design of transition curves.

In setting-out circular curves, the advantages of the degree system are too well known to require further amplification. A curve of D° is one in which an arc of 100 ft. subtends an angle of D° at the centre. It will be noted that a slight departure from usual custom has been made by using the arc and not the chord length of 100 ft. The relationship between R and D is therefore, R=5729·58/D ft., and in the preliminary work the approximation,

$$D=86000\ F/V^2 \qquad (1a)$$

may be used as an alternative to (1).

A certain amount of latitude can be allowed in fixing the values of K, S or R in order that calculations and setting-out

---

The intersection of the vertical line corresponding to the speed standard with the curve of the selected value of C gives the multipliers K and J. Thus, for V=40 m.p.h. and C=0·5 ft. per sec. per sec. in 1 sec., K=635 ft. and J=1100 ft.

The intersection of the curve of F and the straight line drawn through the origin to the point as found above gives the lengths S and P throughout which the value of F will not be exceeded. Thus, for V=40 m.p.h., C=0·5 ft. per sec. per sec. in 1 sec. and F=0·1, S and P are both equal to 378 ft. If Prof. Royal-Dawson's unit chord system[7] is used, the length of the unit chord is very nearly K/6.

may be facilitated. Whatever adjustments are made to these quantities, the fundamental relationship, $RS=K^2$, must be strictly observed.

Tangent Distances. When the values of K, S and R have been finally settled, and the deviation angle α between the straights has been measured by theodolite, accurate calculations must be made to determine tangent distances, etc. Referring to Fig. 4, which shows the centre line of the road for the general case of a bend that requires an intermediate length of circular arc, sufficient data is required to establish the points B, E, H, J, C, G and F. The importance of these points demands that

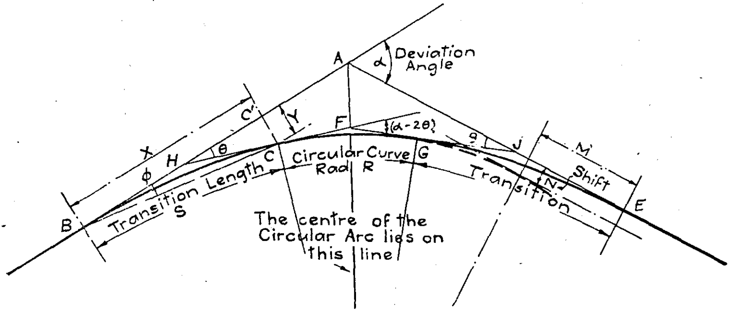


Fig. 4.

pegs, when once fixed, should be referenced carefully so that they can be replaced quickly in the event of their being moved or destroyed during the execution of the works.

The value of θ is obtained from (6) or (7), it being a matter of choice whether the calculations are made for the basic or the full-size curve in the first instance. The deviation angle for the circular arc $(\alpha-2\theta)$ readily follows. (In the wholly transitional layout, the points C and G would coincide and the above procedure would be reversed, s or S being obtained from (6) or (7) respectively, by substituting $\theta=\alpha/2$.)

The sub-tangent lengths, CF and $FG=R\tan(\alpha/2-\theta)$.

For the spiral (Fig. 2),

$$X=Kx=K\sqrt{2\theta}\ (1-\theta^2/10+\theta^4/216-\theta^6/9360+\ldots\ldots\ldots)$$

and $$Y=Ky=K\sqrt{2\theta}\ (\theta/3-\theta^3/42+\theta^5/1320-\ldots\ldots\ldots)$$

Substituting $\theta = s^2/2$, these become

$$X = Ks(1 - s^4/40 + s^8/3456 - s^{12}/599040 + \ldots\ldots\ldots)$$

and
$$Y = Ks(s^2/6 - s^6/336 + s^{10}/42240 - \ldots\ldots\ldots)$$

$$\therefore \tan\varphi = Y/X = \theta/3 + \theta^3/105 + \theta^5/5997 + \theta^7/198700 + \ldots\ldots\ldots$$

The rather formidable aspect of the above expressions has doubtless restricted the general use of the spiral in the past. Adequate tables are now available, those by Thom[1] being specially prepared to suit the basic curve method of design. They give values of x, y and φ, for values of s ranging from zero to 2·4, at intervals which are sufficiently close to give interpolated values of φ to single seconds of arc.

When X and Y have been found,

$$HC = Y/\sin\theta, \qquad HA = HF\cos(\alpha/2 - \theta)/\cos\alpha/2,$$
$$HC' = Y/\tan\theta, \qquad AF = HF\sin\theta/\cos\alpha/2,$$
$$HF = HC + CF, \qquad BA = BC' - C'H + HA.$$

The points can now be located and pegged and the running chainages of B, C, G and E obtained.

The Shift Method of Obtaining Tangent Distances. If no transitions had been inserted, the circular arc would have been tangential to the straights, but with transitions, Fig. 4, the minimum distance between the extended curve and the straights is known as the shift N, and the distance from tangent point to the shift point is M which can, for convenience, be called the shift point distance.

$$N = Y - R(1 - \cos\theta), \tag{13}$$

and

$$M = X - R\sin\theta. \tag{14}$$

The tangent distances are therefore,

$$AB = AE = (R + N)\tan\alpha/2 + M. \tag{15}$$

This method is undoubtedly quicker than the previous one but no intermediate checks on the field work will be available until the complete bend has been set out. It is, however, very useful when the shorter transitions lengths are used.

Setting-out the Curves. The transitions BC and EG will usually be set out with the theodolite stationed at the tangent points B and E respectively, Fig. 4. Intermediate points on the transitions are located by turning the deflection angles from the straights and chaining the distances along the curve. To avoid confusion with the total length of the transition, s_1 and S_1 will be used to denote the distances of the intermediate points on the basic and full-size curves respectively, and the deflection angles to these points will be distinguished by a similar subscript. The values of φ_1 may be obtained by substituting the values of s_1 or S_1 in the previous series. This is laborious and is not suited for work in the field. Thom[1] gives the approximation,

$$\varphi_1, \text{ in minutes of arc} = 573 s_1^2 - 1{\cdot}213 s_1^6 - \ldots\ldots\ldots ;$$

this is sufficiently accurate for values of s_1 not greater than 1.

If the running chainage is to be maintained, the S_1 distances will be known in the first place. These depend on the chainage of the tangent point and on the station interval and are readily reduced to basic curve values by dividing by K. If the running

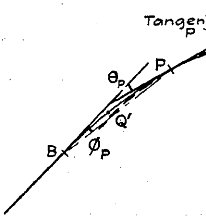


Fig. 5.

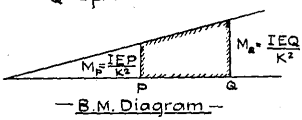


Fig. 5 (a).

chainage is not to be maintained, the basic curve distances may be selected to facilitate the setting out. With the tables[1] the calculations can be made in the field and the curves set out with little delay.

The intermediate circular curve would be set out in the usual way. Except to point out that, if the shift method of obtaining tangent distances has been used, the tangent at C would be located by stationing the instrument at this point, sighting back on B, transiting and turning through an angle $(\theta - \varphi)$, no further explanation need be given.

The Law of the Osculating Circle. An osculating circle at a point P on the spiral is the circle which is tangential to the spiral at P and which has the same radius of curvature as the spiral at this point (Fig. 5). The law states that the rate of divergence of the spiral from the osculating circle is approximately the same as the rate of divergence of the spiral from the tangent at the origin. The following demonstration may be of interest to students versed in the Mechanics of Structures.

An important theorem, used to determine beam deflections, states that the deflection of Q from the tangent at P, where P and Q are points on a beam subjected to bending, is equal to the moment of the area of the portion of the bending moment diagram between P and Q about the point Q divided by IE, where I and E have their usual significations.

Considering the spiral as a bent beam, and using the well-known relationship $M/I=E/R$, the bending moment at the point distant S from the start of the curve is $M=IES/K^2$. The bending moment diagram is therefore a straight line, Fig. 5(a). Let P and Q denote also the distances of the points along the spiral. The deflection of the point Q from the tangent at P is aQ and, from the above, $aQ=(2P+Q)(Q-P)^2/6K^2$.

For the osculating circle at P, of radius K^2/P, the deflection of a point $(Q-P)$ distant from P is $ab=P(Q-P)^2/2K^2$. $\therefore\ bQ=aQ-ab=(Q-P)^3/6K^2$, which, by comparison with the previous expression for Y, is approximately the same as the deflection of a point, distant $(Q-P)$ from the origin, from the tangent at the origin.

This law is of extreme value when obstacles interfere with the line of sight from the instrument stations at B or E in Fig. 4, since it will allow points on the spiral to be located from another point on the curve. Thus if P is to be the new instrument station, and Q the point to be located, the deflection angle from the tangent at P to the point Q, is equal to the deflection angle for a length $(Q-P)$ of the spiral from the straight at the beginning, *plus* the deflection angle from the tangent for a length $(Q-P)$ on the osculating circle. The latter has a radius of K^2/P and consequently the osculating circle deflection angles are, in minutes of arc, $1719\ P(Q-P)/K^2$. If, however, the point to be located is Q', lying between B and P, the deflection angle from the tangent is equal to the osculating circle deflection angle *minus* the spiral deflection angle. The tangent at P would be located by sighting the point B and turning through the angle $(\theta_p-\varphi_p)$.

The rules are not strictly accurate and corrections, in seconds of arc, have been given by Thom.

The Lemniscate Transition Curve

In recent years the lemniscate of Bernouilli has been used by road engineers not only as a transition curve but also in the complicated clover-leaf flyover junctions. Referring to Fig. 6,

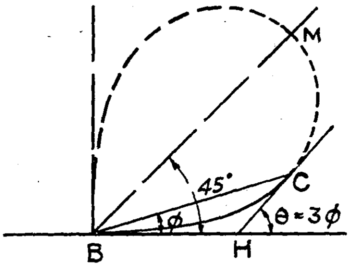


Fig. 6.

BC=P, the polar ray to the point C;
R=Radius of curvature at C;
BM is the axis of the lemniscate, and on the full-size curve this is equal in length to the multiplier J;
CH is the tangent at the point C;
φ is the polar deflection angle for the ray BC;
θ is the angle turned through by the curve in length BC.
The equations for the full-size lemniscate are,

$$P = 3R \sin 2\varphi, \tag{16}$$

and

$$P = J\sqrt{\sin 2\varphi}. \tag{17}$$

Another important property of the lemniscate is that the angle θ is exactly 3φ. For the basic lemniscate BM is unity and the above become,

$$p = 3r \sin 2\varphi, \tag{18}$$

and

$$p = \sqrt{\sin 2\varphi}. \tag{19}$$

From these it follows that,

$$pr = \tfrac{1}{3} \text{ (compare } rs = 1 \text{ in the basic spiral)} \tag{20}$$

and, $$PR = J^2/3 \text{ (compare } RS = K^2 \text{ in the full-size spiral)} \tag{21}$$

To Determine the Multiplier J *for the Lemniscate.* For the lemniscate, by successive differentiation, the rate of change of acceleration, C, is $(3v^3/J^2) \cos 2\varphi$. C is therefore not constant in value, but is a maximum at the beginning of the curve where φ is zero, and thereafter decreases until zero value would be obtained if $\varphi=45°$ were reached. A certain amount of easement in the rate of turning of the steering wheel (9), is therefore obtained with the lemniscate.

Using the maximum value of C as the criterion,

$$J=\sqrt{3\,(v^3/C)}\text{ ft.,} \tag{22}$$

or, with m.p.h. units,

$$J=3{\cdot}075\sqrt{(V^3/C)}\text{ ft.} \tag{22a}$$

A comparison of these with (8) and (8a) shows that, for equal values of C, $J=\sqrt{3}K$, and, accordingly, the scale to suit the lemniscate is shown on the right-hand side of Fig. 3.

The maximum length of the lemniscate which may be used as a transition is decided, as in the case of the spiral, by the centrifugal ratio F. Substituting $J^2/3P$ for the radius R in (1),

$$P=5F\ (J/V)^2. \tag{23}$$

A comparison of (12) and (23) shows that, for $J=\sqrt{3}K$, the lengths S and P, in the spiral and lemniscate respectively, are the same. Fig. 3 can therefore be used for both curves. It may be noted, however, that S is a curved length whereas P is a chord length.

For wholly transitional bends, the formulæ equivalent to (11) and (11a) of the spiral are readily obtained by substituting $\varphi=\alpha/6$ in the lemniscate equations (16) and (17). This gives

$$\sin\alpha/3=236\ F^2/VC, \tag{24}$$

and

$$J=V^2\sqrt{(\sin\alpha/3)}/5F. \tag{24a}$$

It may be noted that $\sin\alpha/3$ replaces $\alpha/3$ in the corresponding spiral formulæ. The remarks concerning the selection of the values of K, S and R apply equally to the lemniscate values J, P and R, the relationship $PR=J^2/3$ being strictly observed.

Tangent Distances, etc. Once the essentials J, P and R have been finally fixed, the various tangent distances, etc., can be determined as follows:

With the notation of Fig. 7, φ is found from (17) or (19), then

$$BH=P\sin 2\varphi/\sin 3\varphi,\text{ and }HC=P\sin\varphi/\sin 3\varphi.$$

The sub-tangent length for the circular curve $CF=R\tan(\alpha/2-3\varphi)$. Triangle HAF can now be solved for the lengths HA and

AF, and the tangent distances AB and AE obtained by adding the lengths BH and HA.

(In the special case of a wholly transitional bend the procedure is reversed, $\varphi=\alpha/6$ being substituted in (17) or (19) to get the lengths P or p respectively. The tangent distance for the basic curve is then $p \cos 2\varphi/\cos 3\varphi$; this, when multiplied by J, gives the full-size tangent distance.)

Shift Methods. In view of the ease with which the tangent distances, etc., can be obtained from the properties of the lemniscate, it is only where the check points C and G are not

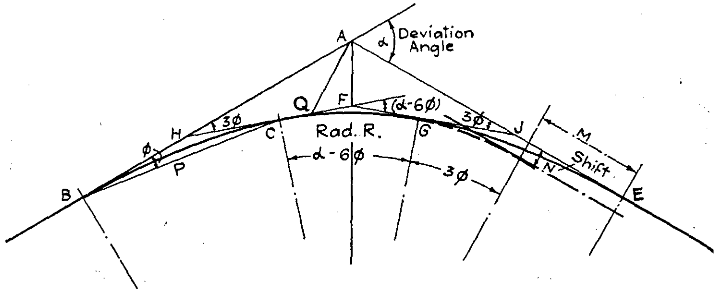


Fig. 7

to be located beforehand that any advantage is to be gained by using the shift methods. For the full-size curve, Fig. 7, the shift is,

$$N=P \sin \varphi - R(1-\cos 3\varphi)$$
$$=R(3 \cos \varphi - \cos 3\varphi - 2)/2, \qquad (25)$$

and the shift point distance is

$$M=R(3 \sin \varphi + \sin 3\varphi)/2 \qquad (26)$$
$$=(P \cos \varphi)/2 + R \sin^3 \varphi. \qquad (26a)$$

The tangent distances are

$$AB=AE=(R+N) \tan \alpha/2 + M. \qquad (27)$$

The Length of the Lemniscate. To distinguish the lemniscate from the spiral the length of the curve will be denoted by l or L instead of s and S. For the basic lemniscate

$$dl=r\,d\theta=3r\,d\varphi.$$

$$\therefore\ l=\int_0^{\phi} 3r\,d\varphi=\int_0^{\phi} d\varphi/\sqrt{\sin 2\varphi} \qquad (28)$$

This integral can be obtained in the form of an infinite series which, unfortunately, does not converge rapidly for the higher values of φ, and the arithmetical work involved precludes its use as a really practical formula. An approximate expression for the length has been given by Prof. Royal-Dawson.[7]

It may be noted, however, that by making the substitution, $\cos^2 A=\sin 2\varphi$, in (28),

$$l=\frac{1}{\sqrt{2}}\left[\int_0^{\pi/2}\frac{dA}{\sqrt{1-\frac{1}{2}\sin^2 A}}-\int_0^A\frac{dA}{\sqrt{1-\frac{1}{2}\sin^2 A}}\right].$$

These are elliptic integrals of the first kind, the modulus being 45° in each case. The first one is known as the complete integral and has the value 1·85407468....... Tables of values of the second to twelve decimal places are published[8] for values of A, at 1/2° intervals, between the limits 0 and 90°. Values of l and φ which have been obtained by the Author on the above basis are given in Table I. The corresponding values of p and r have not been included since they can be obtained from tables of the natural trigonometrical functions by using the relationships, $p=\sqrt{\sin 2\varphi}=\cos A$, and $r=(\sec A)/3$.

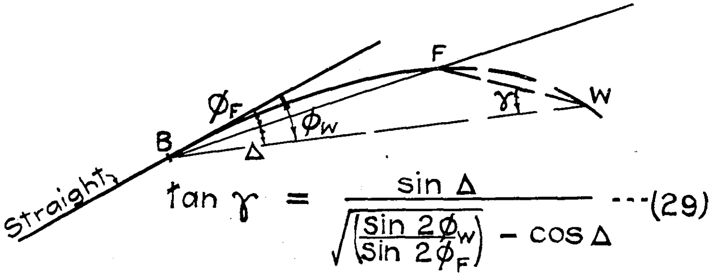
Setting-out the Lemniscate. To maintain the running chainage the lengths L along the curve will be known when the chainage of the tangent point B is found. These become basic curve lengths on dividing by J, and the deflection angles may then be found by interpolation in Table I, linear interpolation being sufficiently accurate for all practical purposes.

If the running chainage is not to be maintained, equal curve lengths corresponding to something of the order of 0·05 to 0·10 of the basic curve would be satisfactory, the values of φ being again found from the Table. It may be noted, however, that at the beginning of the curve the values of φ are small and the difference between the curve length and polar ray is of no practical significance. Accordingly the approximations $2\varphi=\sin 2\varphi$ and $l=p$ may be made in (19), which then becomes

$$\varphi=1719\ l^2 \text{ minutes of arc.} \tag{19a}$$

The approximations compensate one another to a certain extent and (19a) is sufficiently accurate for values of φ not exceeding about 4°. A comparison of the true and approximate values is interesting. For a length of 0·4 of the basic curve (19a) would give a deflection angle of 4° 35′ 2″, whereas the true value as found by interpolation in Table I is 4° 34′ 47″. The error in the approximation is therefore some 15″ of arc, and for a multiplier of 1430 ft. this would produce an error in alignment of less than 1 in.

Allowance for Curvature. By keeping the peg interval small, the difference between chord and curve length is negligible as far as practical setting out is concerned. It is only in the higher values of φ that the differences become appreciable. It may be of interest to record that, if for an estimate of the length of the basic curve from 0 to 45° deflection angle, a chord length of 0·05 had been used instead of a curve length the error in the estimation would have been of the order 1/2400, which, in surveying work, represents an accuracy more than sufficient



$$\tan \gamma = \frac{\sin \Delta}{\sqrt{\left(\frac{\sin 2\phi_W}{\sin 2\phi_F}\right)} - \cos \Delta} \quad \cdots (29)$$

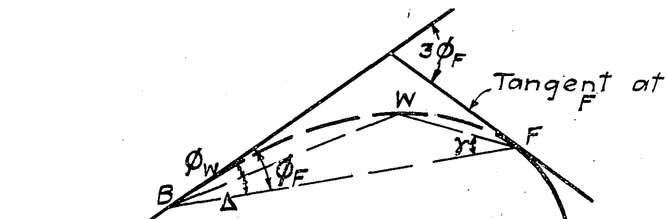
Fig. 8.

for ordinary chaining purposes. And, in addition, at station 1·00 on the basic curve, deflection angle 27° 44′ 16″, the difference between this angle and that of the point which would have been obtained if twenty chord lengths each of 0·05 had been used instead of curve lengths, is less than 1′ of arc.

Setting out the Lemniscate from a Point on the Curve. One great disadvantage of the lemniscate, as compared with the spiral, is that the osculating circle law is not sufficiently accurate unless the points P and Q, Fig. 5, are close to each other. Referring to Fig. 8, let the curve be set out from the tangent point B on the straight, as far as the station F. To complete the setting-out of the transition with the theodolite stationed at F the deflection angles, from the polar ray BF produced, to the remaining stations must be calculated. Thus for station W, the deflection angle is $(\Delta + \gamma)$, where Δ is the difference between the original deflection angles from the straight at the beginning of the curve. Expression (29), Fig. 8, will give the values of γ directly, but it is not specially suited for accurate logarithmic calculation, and the following method may be preferable. By solving triangle BWF, and taking logs, we have,

$$\log \sin (\Delta+\gamma)-\log \sin \gamma=(\log \sin 2\varphi_W-\log \sin 2\varphi_F)/2. \qquad (29a)$$

If an approximate value is known for γ to begin with, this can be solved very quickly using trial and error methods. $\gamma=(2\varphi_F+Z)$, where $Z=1719\,p\,\delta l$, minutes of arc, p and δl being the basic curve polar ray BF and the basic curve length FW respectively, is a good approximation provided the lengths δl are not too large. (It underestimates the value of γ as δl increases.)



$$\tan \gamma = \frac{\sin \Delta}{\sqrt{\left(\frac{\sin 2\phi_F}{\sin 2\phi_W}\right)} - \cos \Delta} \quad \cdots (30)$$

Fig. 9.

On the other hand, when W is between F and B, expression (30), Fig. 9, should be used instead of (29), and the deflection angle from the *tangent* at F is $(2\varphi_F-\gamma)$. The alternative, corresponding to (29a), is

$$\log \sin (\Delta+\gamma)-\log \sin \gamma=(\log \sin 2\varphi_F-\log \sin 2\varphi_W)/2 \qquad (30a)$$

and in this case the approximation for γ is $\gamma=(2\varphi_F-Q)$, where $Q=1719\,\delta l\,(3p-\delta l)$, minutes of arc, and p and δl are as already defined.

It may be noted in connection with the above, that Q is the angle as found from the osculating circle law, whereas Z is the deflection angle for a length δl on a circle having a radius three times that of the osculating circle at the point F. An example which illustrates the above is included in the Appendix.

The Cubic Parabola Transition Curve

This curve is the first approximation to the spiral, and, with the same notation, its equation is $Y=X^3/6RS=X^3/6K^2$. For low values of φ, the following approximations hold with sufficient accuracy for practical purposes: $X=S$; the shift $=S^2/24R$; the shift point distance is $S/2$. From these it will be seen that the transition bisects the shift. Tangent distances are found as formerly, and the curve may be set out by offsets from the straight, or by deflection angles φ, as before, the values of φ and θ being obtained from the expressions $\tan \varphi = Y/X = X^2/6K^2$, and $\tan \theta = X^2/2K^2$, respectively. The cubic parabola is not suitable for the higher values of φ, in fact the radius of curvature increases after the point corresponding to $\varphi=8^\circ 3'$ is reached.

Model Curves

An excellent idea of the position of the bend can be obtained graphically by using a scale model of the basic curve. This will ensure that the proposed values of the multiplier and radius or length of transition will meet the site requirements in addition to those of speed and curvature, and will allow amendments to be made to these quantities before the final calculations are begun. In this way, not only may unnecessary work be avoided, but, in addition, an excellent check on the accuracy of the arithmetical work itself will be available. The basic curve, together with the locus of the centres of curvature, can be drawn to a suitable scale on tracing paper or cloth, but the durability of a celluloid model should not be overlooked.

For purposes of illustration, the lemniscate will be taken as the transition in question, but the remarks apply equally to the spiral. The Author has found a basic model whose axis is 10 inches in length, to be sufficient to cover all combinations of transition and arc likely to be met in practice and to give, with careful work, accuracies to a foot or so. The model curve is shown in Fig. 10 and the data for its accurate construction is given in Table II.

For *direct* use the proposed values of P and R are converted into basic curve values by dividing by J. These are marked carefully on the basic model and the latter is then positioned on an accurate drawing of the straights so that the respective straights

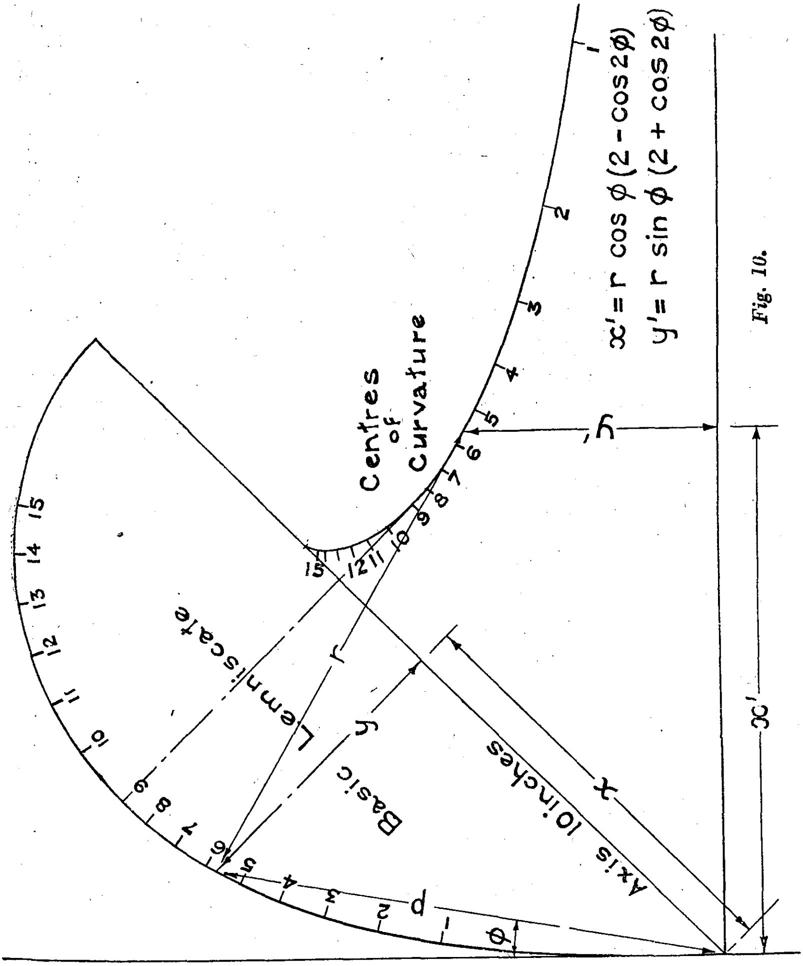


Fig. 10.

coincide and the centre of curvature point corresponding to r is on the line bisecting the intersection angle between the straights. With this point as centre, the circular arc is pencilledin lightly with compasses. In the special case of a wholly transitional bend the centres of curvature line will be tangential to the line bisecting the intersection angle. Tangent and other distances from the intersection point may now be measured with the basic curve scale, and these when multiplied by J will give the corresponding lengths on the full-size bend.

In improvement schemes especially it may happen that the bend must pass through a definite point. Here the multiplier is found primarily from the site condition and not by considering F and C, although, when the multiplier has been fixed, the latter considerations fix the speed standard for the bend. In this event the use of the model will allow the optimum value of J to be selected. Let the point through which the curve must pass be Q, Fig. 7. Q may be fixed in position by measuring the angle BAQ and the distance AQ. Many combinations of transitions and circular arc will satisfy the site condition, but, with respect to speed and curvature, the optimum arrangement may be found by placing the basic model in the manner previously described on an accurate drawing of the straights to which has been added the line corresponding to the direction AQ. The intersection of the model bend with this line fixes the basic length corresponding to AQ. This basic length and also the radius of curvature r at the end of the transition are scaled from the drawing with the basic curve scale. The multiplier J is then the actual distance AQ divided by the corresponding basic length, and the radius of the full-size circular arc is $R = Jr$. The speed standard for the limiting values of F and C can now be found. Various positions are thus tried until the optimum arrangement is found. The calculations and setting out are then made in the usual manner. If, however, it is found that, due to inaccuracy in the graphical work, the curve does not pass through Q, but through Q′ also on the line AQ, and the error QQ′ is too great to be ignored, the multiplier should be altered in the ratio AQ/AQ′. All lengths, tangent distances radius, etc., are proportional to the multiplier and the necessary alterations are readily made.

Analytical solutions are possible, but these are very laborious, and, except when Q happens to be at the centre of the bend, when the algebraic work is simplified considerably, they cannot be recommended.

Widening at Bends

The calculations for the layout of the bend should in general be made for the line corresponding to the centre line of the carriageway under consideration. To get the kerb lines the centre line may be moved parallel to itself as shown in Fig. 11.

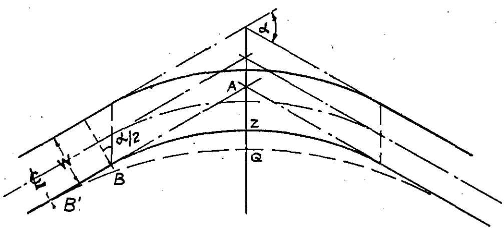


Fig. 11.

In this way the widening of the carriageway is automatic, and the same setting-out table can be used for the kerbs. The distance between the kerbs, measured in a direction parallel to the line bisecting the intersection angle between the straights, is constant and equal to W sec $\alpha/2$, this being the normal width at the centre of the bend. If additional widening is required the simplest method is to increase the multiplier on the inside curve. To illustrate this point, let ZQ be the additional width required, i.e. $ZQ=(W_C—W \sec \alpha/2)$, where W_C is the required width of carriageway at the centre of the bend. The new multiplier is then J(AQ/AZ), the distances being calculated from the original layout, or they can be often obtained with sufficient accuracy from the basic model. The increase in tangent distance, BB′, is T(ZQ/AZ) where T is the original tangent distance AB. The increase in chord length is found in a similar way. For large deviation angles the additional widening produced by the parallel shift method may be excessive. Thus, for a carriageway of normal width 30 ft., and a deviation angle α of 60°, the increase in width by the parallel shift method would be 4·64 ft. In this case, the scale of the inside curve will need to be reduced ; this should not escape attention when the values are being originally selected.

Reverse Bends

These are mainly confined to improvement schemes and in consequence the limits of the land available have to be considered in their design. A typical example is shown in Fig. 12, α_2 being greater than α_1, and, although the lemniscate notation has been used for purposes of illustration, the spiral can be applied with equal facility.

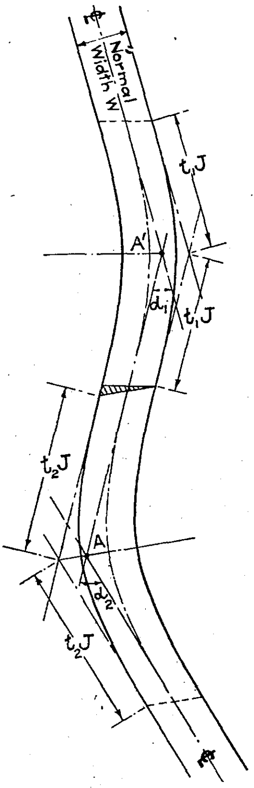


Fig. 12.

It should not be assumed that two wholly transitional bends will give the best solution without in the first instance assessing the relative merits of several alternative arrangements. If the deviation angles are unequal, two wholly transitional bends of the same scale or multiplier will result in a greater value of F being reached on the bend with the larger deviation angle. It is true that the calculations are simplified if wholly transitional bends are used, and it is possible to obtain this arrangement and equal values of F at the same time by the use of different multipliers. (24a) shows that J is proportional to $\sqrt{\sin \alpha/3}$ and simple proportion will give the desired result. This arrangement, however, departs from uniformity and cannot be recommended.

A more logical method would be to provide the same values of C and F throughout, and these conditions can only be fulfilled by having the same length of transition on each bend. The best combination of transition and arc can be obtained with the model curve in the manner previously described, but the basic tangent lengths must then be known before the multiplier can be found. In the preliminary work these may be scaled from the drawing. Denoting these by t_1 and t_2 respectively, it is readily seen from Fig. 12 that the multiplier is $D/(t_1+t_2)$, where D is the actual distance AA′ between the intersection points, minus a length which will allow the kerbs to be accommodated. If the parallel shift method is adopted, the deduction is $W(\tan\alpha_2/2 - \tan\alpha_1/2)/2$. It will be understood that the final values of t_1 and t_2 must be calculated; the model curve has only been used to facilitate the selection of the basic quantities p and r. When J has been found the speed standard can be obtained from Fig. 3. The drainage of the hatched portion of the roadway in Fig. 12 may require special consideration when the gradient of the road is flat.

Superelevation

Throughout the bend the adverse camber on the outside must be replaced by superelevating or banking the roadway. The most usual method of effecting this is to keep the crown of the road at its normal profile level and to raise and lower the outer and inner channels and kerbs respectively. For small amounts of superelevation the inside camber may be preserved over the length where its slope is greater than the required cross-fall. The change-over from the normal cambered section to the superelevated one must be worked in carefully at the beginning of the transition to give an appearance pleasing to the eye, and for this purpose parabolic vertical curves on the kerbs can be used. The lengths of these depend on the scale of the transition and the amount of superelevation to be provided. Excellent examples are given by Collins and Hart[9], and further details need not be discussed here.

On flat gradients a minimum cross-fall of $\frac{1}{4}$ in. per ft. width should be provided to give adequate drainage to the road. The maximum amount, however, depends on the classes of vehicle the road has to serve. The extremely dangerous conditions which would be brought about by the combination of excessive cross-fall, slippery road surface and wind pressure, should be considered, and in no case should the safety of slow moving traffic, such as a horse-drawn cart of hay, be endangered. For this reason cross falls greater than 1 in. per ft. width are rarely exceeded. Assuming that the coefficient of friction between the tyres and the road were zero, a cross-fall of 12 F in. per ft. would be required to prevent side-slip. Even if it were possible to provide this amount, it is illogical to base any argument on a condition which could only arise on ice-bound roads. In any event, for the higher values of F, a third of this quantity, say 4 F in. per ft., is about the maximum quantity that can be applied.

In fixing the amount of superelevation several other factors should be taken into account. Of these the chief is the effect of gradient on the relative positions of the wheels of the vehicle. The outer wheels of a car travelling on an up-grade will be higher up the grade than the inner ones and an effect equivalent to an increase in superelevation is obtained. The reverse is true when the car is travelling on the down grade.

Vertical Curves

For the purpose of providing transitions at all changes of gradient the parabolic curve is most commonly used. Gradients are usually expressed in the tangential percentage system, and for purposes of definition a gradient of $\pm a$ per cent. will be taken to mean one which rises or falls through a vertical height of a ft. in a horizontal length of 100 ft., the latter distance being measured in the direction of the running chainage. The grades are, in general, fixed by considering drainage and control levels through which the finished road must pass, but at bridge crossings, especially those of large span and skew, it is often possible to fix the tangent grades after the vertical curve itself has been determined. By making the centre of the bridge coincide with the highest point of the curve, the bridge design is simplified and unnecessary dead load in the form of concrete or other filling is eliminated. Simultaneous changes in horizontal and vertical alignment, especially at summits, should be avoided wherever possible.

To connect the gradients of $+a$ per cent. and $-b$ per cent., Fig. 13, by means of a parabolic curve, the length L must be known. This is the projected length on the horizontal plane, not the curve length, and it is fundamental if a parabola with a vertical axis is to be used that, no matter what values

a and b have, the tangent points must be equally spaced a horizontal distance of L/2 on each side of the intersection point. Disregard of this leads to hybrid curves.

In summit curves the length L should be fixed by considering the visibility, E, for a height of eye of h ft., as shown in Fig. 14(a). The Ministry of Transport requirements in this respect, based on a value of 3 ft. 9 in. for h, are as follows[10] :

Trunk roads, E=600 ft. ; Class I roads, E=500 ft. ; Class II roads, E=450 ft. ; unclassified roads, E=300 ft.

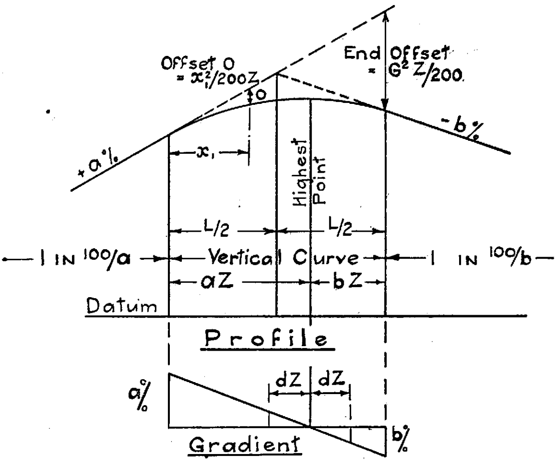


Fig. 13.

Denoting the algebraic difference of the percentage grades by G, then L=ZG, where Z is a multiplier in feet units. Strict attention must be paid to algebraic conventions when determining the numerical value of G. Thus, for the case shown in Fig. 13, $G=(a)-(-b)=a+b$. Referring to Fig. 14(a), with the origin at the highest point, the general equation of the parabola is $y=-cx^2$, where c is constant numerically. Differentiating this expression,

$$dy/dx=-2\,cx \text{ and } d^2y/dx^2=-2c.$$

The gradient at any point on the curve may be scaled or calculated from the diagram shown in Fig. 13. The highest point on the curve, or the lowest in the case of a sag, is aZ and

bZ ft. distant from the respective ends of the curve; these can therefore be located at once.

At $x=-aZ$, $dy/dx=a/100$; and at $x=+bZ$, $dy/dx=-b/100$.

$\therefore\ a/100=2caZ$ and $-b/100=-2cbZ$.

By subtracting the above, $c=1/(200Z)$, and the general equation of the curve is therefore,

$$y=\pm x^2/(200\ Z), \tag{31}$$

the + sign being used for sags, and — sign for summits.

The radius of curvature is approximately $R=100\ Z$; this is sufficiently accurate for all curves likely to be used in practice. (When drawing vertical curves on the profile, the radius of the curve in inches is approximately $100\ Zn/m^2$, where 1 in.$=m$ ft. and 1 in.$=n$ ft. are the scales for chainage and height respectively.)

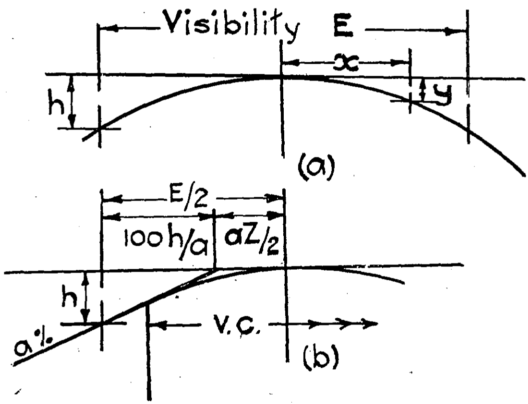


Fig. 14.

To determine the multiplier Z. Substituting the values $E/2$ and h, for x and y respectively, in (31), $E=\sqrt{(800hZ)}$

For $h=3{\cdot}75$ ft. this becomes

$$Z=E^2/3000\text{ ft.} \tag{32}$$

For the various road classes, (32) gives values of Z of 120, 83, 67·5 and 30 ft. Strictly speaking, (32) is valid only when L is greater than E, and on flat gradients it will underestimate the visibility. In this event, Fig. 14b, either $E/2=aZ/2+100h/a$ or $E/2=bZ/2+100h/b$ may be used instead. To avoid unnecessary complication, it is suggested that the value of the steeper gradient be taken. From the aspect of visibility a vertical curve will not be necessary when the steeper gradient is less than $200h/E$ per cent. Substituting the values of E for the appropriate road classes, the limiting gradients are: Trunk roads, 1·25 per cent.; Class I, 1·50 per cent.; Class II, 1·67 per cent.; unclassified, 2·5 per cent.

On sags and summits, where visibility is no longer a factor, Z is found by limiting the amount of the radial acceleration, v^2/R, to a value of 2 to 3 ft. per sec. per sec. Replacing R by 100 Z, then, for speeds in m.p.h. and an acceleration of 2·6 ft. per sec. per sec.,

$$Z = V^2/120 \tag{33}$$

$\therefore$ for $V = 60$ m.p.h., Z should not be less than 30 ft.

In the Author's opinion the minimum length of curve, irrespective of visibility or speed requirements, should not be less than 200 ft. If an arbitrary length is chosen Z can be obtained from $Z = L/G$.

Finished Road Levels. Finished road levels may be calculated from (31), it being remembered that the highest point is aZ ft. distant from the start of the curve. This method, however, is not readily applied to the case where the curve connects two gradients of the same sign. The finished levels in this case are more readily determined by the method of vertical offsets from the grade line produced. With the notation of Fig. 13, the offsets are proportional to the squares of the distances from the tangent points, and the expression,

$$0 = x_1^2/200Z \tag{34}$$

will give the required values.

Channel Grading. On gradients flatter than 1 in 200, channel grading may be used to give proper drainage to the road. In general, if the limiting gradient is d per cent., channel grading over a length of dZ feet on each side of the turning point of the curve will be necessary.

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TABLE I—BASIC LEMNISCATE

$$l=\int_0^\phi \frac{d\varphi}{\sqrt{\sin 2\varphi}};\quad \cos^2 A = \sin 2\varphi;\ p = \cos A;\ r = (\sec A)/3.$$

A	l	φ Deg.	φ Min.	φ Sec.
90°	0	0	0	0
89·5	0·008727	0	0	08
89	0·017452	0	0	32
88·5	0·026177	0	01	11
88	0·034900	0	02	06
87·5	0·043619	0	03	16
87	0·052336	0	04	42
86·5	0·061049	0	06	24
86	0·069757	0	08	22
85·5	0·078459	0	10	35
85	0·087156	0	13	03
84·5	0·095847	0	15	47
84	0·104530	0	18	47
83·5	0·113205	0	22	02
83	0·121872	0	25	32
82·5	0·130530	0	29	17
82	0·139178	0	33	18
81·5	0·147816	0	37	33
81	0·156444	0	42	04
80·5	0·165060	0	46	50
80	0·173664	0	51	50
79·5	0·182256	0	57	06
79	0·190834	1	02	36
78·5	0·199399	1	08	20
78	0·207951	1	14	20
77·5	0·216487	1	20	33
77	0·225009	1	27	01
76·5	0·233515	1	33	43
76	0·242005	1	40	39
75·5	0·250479	1	47	50
75	0·258935	1	55	14
74·5	0·267375	2	02	52
74	0·275797	2	10	43
73·5	0·284201	2	18	48
73	0·292586	2	27	07
72·5	0·300953	2	35	38
72	0·309300	2	44	23
71·5	0·317628	2	53	21
71	0·325936	3	02	32
70·5	0·334223	3	11	56
70	0·342491	3	21	32
69·5	0·350738	3	31	21
69	0·358963	3	41	22
68·5	0·367168	3	51	35
68	0·375350	4	02	00
67·5	0·383512	4	12	38
67	0·391651	4	23	27
66·5	0·399768	4	34	28
66	0·407863	4	45	40
65·5	0·415935	4	57	04
65	0·423985	5	08	39
64·5	0·432012	5	20	26
64	0·440015	5	32	23
63·5	0·447996	5	44	31
63	0·455954	5	56	50
62·5	0·463889	6	09	19
62	0·471800	6	21	59
61·5	0·479687	6	34	49
61	0·487552	6	47	49
60·5	0·495392	7	00	59
60	0·503209	7	14	20

TABLE I—*cont'd*

A	l	φ Deg.	Min.	Sec.	A	l	φ Deg.	Min.	Sec.
59·5	0·511003	7	27	49	44·5	0·734061	15	17	22
59	0·518773	7	41	29	44	0·741155	15	34	50
58·5	0·526519	7	55	18	43·5	0·748229	15	52	24
58	0·534242	8	09	16	43	0·755283	16	10	04
57·5	0·541941	8	23	23	42·5	0·762317	16	27	50
57	0·549617	8	37	40	42	0·769331	16	45	40
56·5	0·557268	8	52	05	41·5	0·776326	17	03	37
56	0·564897	9	06	39	41	0·783301	17	21	39
55·5	0·572502	9	21	22	40·5	0·790257	17	39	45
55	0·580083	9	36	13	40	0·797194	17	57	58
54·5	0·587642	9	51	13	39·5	0·804112	18	16	15
54	0·595176	10	06	21	39	0·811012	18	34	37
53·5	0·602688	10	21	37	38·5	0·817893	18	53	04
53	0·610177	10	37	02	38	0·824757	19	11	35
52·5	0·617642	10	52	34	37·5	0·831602	19	30	12
52	0·625085	11	08	14	37	0·838430	19	48	53
51·5	0·632505	11	24	01	36·5	0·845240	20	07	38
51	0·639902	11	39	56	36	0·852033	20	26	28
50·5	0·647276	11	55	58	35·5	0·858809	20	45	23
50	0·654628	12	12	08	35	0·865568	21	04	21
49·5	0·661958	12	28	25	34·5	0·872310	21	23	24
49	0·669266	12	44	49	34	0·879037	21	42	31
48·5	0·676551	13	01	20	33·5	0·885747	22	01	41
48	0·683815	13	17	57	33	0·892441	22	20	56
47·5	0·691057	13	34	42	32·5	0·899120	22	40	15
47	0·698277	13	51	33	32	0·905783	22	59	37
46·5	0·705476	14	08	30	31·5	0·912431	23	19	03
46	0·712654	14	25	34	31	0·919064	23	38	33
45·5	0·719810	14	42	44	30·5	0·925682	23	58	06
45	0·726946	15	00	00	30	0·932286	24	17	43

TABLE I—*cont'd*

A	l	φ Deg.	Min.	Sec.	A	l	φ Deg.	Min.	Sec.
29·5	0·938876	24	37	23	14·5	1·131123	34	48	08
29	0·945452	24	57	06	14	1·137389	35	09	01
28·5	0·952014	25	16	53	13·5	1·143649	35	29	55
28	0·958562	25	36	43	13	1·149902	35	50	51
27·5	0·965097	25	56	35	12·5	1·156149	36	11	47
27	0·971619	26	16	31	12	1·162391	36	32	46
26·5	0·978128	26	36	30	11·5	1·168627	36	53	45
26	0·984625	26	56	32	11	1·174857	37	14	45
25·5	0·991109	27	16	37	10·5	1·181082	37	35	47
25	0·997581	27	36	44	10	1·187302	37	56	49
24·5	1·004041	27	56	54	9·5	1·193517	38	17	53
24	1·010490	28	17	07	9	1·199728	38	38	57
23·5	1·016927	28	37	22	8·5	1·205935	39	00	03
23	1·023353	28	57	40	8	1·212138	39	21	09
22·5	1·029768	29	18	00	7·5	1·218337	39	42	16
22	1·036173	29	38	23	7	1·224532	40	03	23
21·5	1·042567	29	58	48	6·5	1·230724	40	24	32
21	1·048951	30	19	15	6	1·236913	40	45	40
20·5	1·055325	30	39	45	5·5	1·243099	41	06	50
20	1·061689	31	00	16	5	1·249283	41	28	00
19·5	1·068044	31	20	50	4·5	1·255464	41	49	11
19	1·074389	31	41	26	4	1·261643	42	10	22
18·5	1·080726	32	02	03	3·5	1·267820	42	31	33
18	1·087054	32	22	43	3	1·273996	42	52	45
17·5	1·093373	32	43	24	2·5	1·280171	43	13	58
17	1·099684	33	04	08	2	1·286344	43	35	10
16·5	1·105987	33	24	52	1·5	1·292516	43	56	22
16	1·112282	33	45	39	1	1·298687	44	17	35
15·5	1·118569	34	06	27	0·5	1·304858	44	38	48
15	1·124850	34	27	17	0	1·311029	45	00	00

TABLE II—MODEL CURVE VALUES

φ Degrees	p	x	y	r	x'	y'	Ref.
0	0	0	0	∞	∞	0	
1·5	0·229	0·166	0·157	1·457	1·459	0·114	
3	0·323	0·240	0·216	1·031	1·035	0·162	1
4·5	0·396	0·301	0·257	0·843	0·851	0·198	2
6	0·456	0·354	0·287	0·731	0·743	0·228	3
7·5	0·509	0·404	0·310	0·655	0·672	0·254	4
9	0·556	0·450	0·327	0·600	0·621	0·277	5
10·5	0·599	0·493	0·339	0·557	0·584	0·298	6
12	0·638	0·535	0·347	0·523	0·555	0·317	7
13·5	0·674	0·575	0·352	0·495	0·533	0·334	8
15	0·707	0·612	0·354	0·471	0·516	0·350	9
18	0·767	0·683	0·348	0·435	0·492	0·377	10
21	0·818	0·747	0·333	0·408	0·478	0·401	11
24	0·862	0·805	0·309	0·387	0·470	0·420	12
27	0·899	0·855	0·278	0·371	0·466	0·435	13
30	0·931	0·899	0·241	0·358	0·465	0·448	14
33	0·956	0·935	0·199	0·349	0·466	0·457	15
36	0·975	0·963	0·153	0·342	0·468	0·464	
39	0·989	0·984	0·103	0·337	0·469	0·468	
42	0·997	0·996	0·052	0·334	0·471	0·471	
45	1·000	1·000	0	0·333	0·471	0·471	

If the axis is made 10 inches, these values should be multiplied by 10 in. Fig. 10.

APPENDIX

Example 1. Referring to Fig. 7, $\alpha=38° 30' 0''$. Determine a layout for the centre line of a road to suit the following requirements :

Speed standard, 60 m.p.h. ; C and F, 1 ft. per sec. per sec. in 1 sec. and 0·20 respectively ; lemniscate transitions are to be used, and the running chainage is to be maintained. Station interval 100 ft.

Preliminary Work. From Fig. 3, for V=60 and C=1, the multiplier J is 1430 ft. and for F=0·20, the length P=567 ft. $\therefore \sin 2\varphi=(567/1430)^2=0{\cdot}157$ (slide-rule value), and $2\varphi=9° 2'$. The maximum value of α for a wholly transitional bend is therefore $27° 6'$ and the bend cannot be wholly transitional for the specified values of F and C. (This is an alternative method to (24).) An intermediate length of circular arc will therefore be used. For $V=60$, and $F=0{\cdot}2$, the degree of the curve (1a), is $4{\cdot}76°$.

Since the running chainage is to be maintained, J will be made 1428·57 ft.; this has as its reciprocal 0·0007, and the conversion to basic lengths is thereby facilitated. The degree of the curve will be altered to $4{\cdot}5°$, the radius of this is 1273·2 ft. P is therefore, from (21), 534·28 ft.

At this stage the model curve should be used in the manner described. From this it is found that the basic tangent distance is 0·502 and the distance from the intersection point to the centre of the bend is 0·060. Multiplying these by J, distances of 717 and 85·6 ft. respectively are obtained. Assuming these are satisfactory from the point of view of site requirements the final accurate calculations may now be made.

Final Calculations. These were made with 7-figure logarithms but economy of paper precludes them from being given in detail, and only the final result in each step will be given.

Substituting the selected values of P and J in (17),

$$\varphi=4°\ 1'\ 13'',$$

and therefore,

$$2\varphi=8°\ 2'\ 26''\ ;\quad 3\varphi=12°\ 3'\ 39''\ ;\quad 6\varphi=24°\ 7'\ 18''\ \text{and}$$
$$(\alpha-6\varphi)=14°\ 22'\ 42''.$$

Solving triangle BHC,

$$BH=P\sin 2\varphi/\sin 3\varphi=357{\cdot}66\text{ ft.},$$

and $$HC=P\sin\varphi/\sin 3\varphi=179{\cdot}27\text{ ft.}$$

The sub-tangent distance

$$CF=R\tan(\alpha/2-3\varphi)=160{\cdot}60\text{ ft.}$$
$$\therefore\ HF=HC+CF=339{\cdot}87\text{ ft.}$$

From triangle HAF,

$$HA=HF\sin HFA/\sin HAF=357{\cdot}17\text{ ft.},$$

and $$AF=HF\sin 3\varphi/\sin HAF=75{\cdot}22\text{ ft.}$$

The distance from A to the centre of the bend is

$$AF+R\left[\sec\left(\frac{\alpha}{2}-3\varphi\right)-1\right]=85{\cdot}23\text{ ft.}$$

The tangent distance AB=(HA+HB)=714·83 ft. It may be noted that the basic model gave 85·6 and 717 ft. for these quantities.

The various points can now be located from the straights and the running chainage of the tangent point B found on the ground. Assuming that the chainage of B is $4+^{33\cdot5}$, i.e. B is 433·5 ft. from the beginning of the work, the curve lengths and chainages of the intermediate tangent points are as follows: For $\varphi=4°\ 1'\ 13''$, interpolation in Table I gives for the basic curve length, $l=0{\cdot}37473$.

$$\therefore\ L=Jl=535{\cdot}33 \text{ ft.},$$

and for the circular curve, the total angle to be turned through is 14° 22′ 42″, which means a length of

$$100\times14{\cdot}37833/4{\cdot}5=319{\cdot}52 \text{ ft.}$$

The chainages of the various tangent points are therefore:

Chainage of B=433·5.

Chainage of C=433·5+535·33=$9+^{68\cdot83}$.

Chainage of G=968·83+319·52=$12+^{88\cdot35}$.

Chainage of E=1288·35+535·33=$18+^{23\cdot68}$.

The deflection angles for the curves may now be found. These are as tabulated.

For the first transition curve,

Chainage	Length L ft.	Basic length l=L/J	φ
$4+^{33\cdot5}$	0	0	0
$5+^{00}$	66·5	0·04655	0° 3′ 44″
$6+^{00}$	166·5	0·11655	0° 23′ 21″
$7+^{00}$	266·5	0·18655	0° 59′ 49″
$8+^{00}$	366·5	0·25655	1° 53′ 7″
$9+^{00}$	466·5	0·32655	3° 3′ 14″
$9+^{68\cdot83}$	535·33	0·37473	4° 1′ 13″

For the circular curve,

Chainage	Arc length ft.	Deflection Angle	Vernier Setting
$9+^{68\cdot83}$	0	0	359° 17′ 55″
$10+^{00}$	31·17	0° 42′ 5″	0° 0′ 0″
$11+^{00}$	131·17	2° 57′ 5″	2° 15′ 0″
$12+^{00}$	231·17	5° 12′ 5″	4° 30′ 0″
$12+^{88\cdot35}$	319·52	7° 11′ 21″	6° 29′ 16″

For the second transition,

Chainage	Length L	Basic length l=L/J	φ
$18^{+23\cdot68}$	0	0	0
18^{+00}	23·68	0·016576	0° 0′ 28″
17^{+00}	123·68	0·086576	0° 12′ 53″
16^{+00}	223·68	0·156576	0° 42′ 7″
15^{+00}	323·68	0·226576	1° 28′ 14″
14^{+00}	423·68	0·296576	2° 31′ 10″
13^{+00}	523·68	0·366576	3° 50′ 51″
$12^{+88\cdot35}$	535·33	0·37473	4° 1′ 13″

The first transition curve will be set out from the point B, the deflection angles being referred to the tangent at B. The circular curve will be set out from the station C, the deflection angles being referred to the tangent at C. In this case the special vernier settings shown in the table eliminate the need to set to seconds at the intermediate stations. The line of sight would be directed along the tangent at C, that is, the line CF in Fig. 7, with the vernier reading 359° 17′ 55″. The second transition would be set out from E, the deflection angles being referred to the straight at E. Being a left-hand curve from the instrument man's point of view these will need to be deducted from 360°.

Example 2. Given that the deflection angles φ to stations corresponding to 0·8, 0·9 and 1·30 respectively, on the basic curve are 18° 5′ 22″, 22° 42′ 50″ and 44° 22′ 6″ Find the values of γ to locate stations 0·9 and 1·3 with the instrument set up at 0·8

By direct substitution in (29) the required values are 38° 22′ 58″ and 47° 34′ 2″ respectively.

To illustrate the alternative method, however, γ= 36° 10′ 44″+1719 $p\delta l$ minutes of arc. At 0·8, p=0·768 and for 0·9, δl=0·10. ∴ γ=36° 10′ 44″+2° 12′=38° 22′ 44″, a value which is only 14″ in error. For 1·30, however, δl=0·50 and the first approximation for γ is 36° 10′ 44″+11°=47° 10′ 44″. Keeping in mind that the rule underestimates the values a first approximation γ=47° 12′ 0″ will be tried. Then from (29a),

$$\log \sin 88^\circ\ 44'\ 12'' = 9{\cdot}9998944$$
$$\log \sin 36^\circ\ 10'\ 44'' = 9{\cdot}7710789$$
$$\overline{0{\cdot}2288155}$$

$$\therefore\ (a-b)/2 = 0{\cdot}1144078$$

$$\log \sin (\gamma+\Delta) - \log \sin \gamma = (c)$$

$$\text{Try } \gamma = 47^\circ\ 12'$$
$$\Delta = 26^\circ\ 16'\ 44''$$
$$\therefore\ (\gamma+\Delta) = 73^\circ\ 28'\ 44''$$

log sin 73° 28′ 44″	=9·9816895	Difference for 1′		= 37
log sin 47° 12′ 0″	=9·8655362	Do.	do.	=116
	0·1161533			79
(c), above	0·1144078	∴ Correction=17455/794=2		
Error	=0·0017455			

Repeating the above with $\gamma = 47^\circ\ 12' + 22' = 47^\circ\ 34'$, it will b found that the error in this value of γ is about $2''$ of arc.

Example 3. Given that the levels at A and B, chainag $161+^{16}$ and $171+^{16}$, the two controlling points at a brid crossing, must be 410·63 ft. above datum level, determine th chainages and levels at each end of a parabolic vertical curv to give approach grades of $+3{\cdot}16$ and $-2{\cdot}40$ per cent., th visibility to comply with the requirements of the Ministry o Transport for Trunk road schemes.

From (32), $Z = 120$ ft., and therefore the length of the curv will be $ZG = 667{\cdot}20$ ft. This is greater than the visibility an (32) is therefore valid. The distances of the highest point wi be aZ and bZ from the tangent points, that is, 379·20 an 288·00 ft. respectively.

From the site requirement the chainage of the highest poi will be midway between A and B, i.e. at chainage $161+^{66}$ f The chainages of the beginning and end of the curve are ther fore $157+^{86\cdot8}$ and $164+^{54}$.

The general equation of the curve (31) is $y = x^2/24000$, an the level of the highest point $= 410{\cdot}63 + (50)^2/24000 = 410{\cdot}734$ f above datum level. The level at the beginning of the curve

$$=410{\cdot}734-(379{\cdot}20)^2/24000=404{\cdot}74,$$

and the level at the end of the curve

$$=410{\cdot}734-(288)^2/24000=407{\cdot}28.$$

The intersection point of the grades will be at chainage $161+^{20\cdot40}$ ft., the level of this point being 415·28 ft. Finished levels on the curve can be found from (31).

Discussion

Mr. E. H. Cornelius: The Author has stated in his opening paragraph that it is not his intention "to dispute accepted methods of design." This uncritical attitude is responsible, perhaps, for some of the conclusions reached and inferences drawn which need the following constructive criticism.

The Multipliers are simply the parameters of the curves discussed. As such they possess many interesting properties other than serving as mere multipliers. However, the Author's handling of curves by means of their parameters is the nearest approach yet made to what has become standard practice in the case of the circle, which is recognised by either of its two parameters, the radius or diameter. Specifications will be simplified very materially when it is possible to use the parameter as the full and complete description of the curve designed for any deviation point whatever be the deviation angle. This is quite possible in the case of the lemniscate which needs only the parameter, J, to describe the curve as fully as the radius or diameter would describe a circle.

The Centrifugal Ratio. For the safety of vehicles, the highway engineer's true criterion for maintenance of his road is the amount of reliance he can place on frictional resistances. It is unwise to cloak this by the use of the centrifugal ratio in the manner described. Writing μ as tan φ, the Author's equation becomes $\tan(\gamma+\varphi)=(v^2/g)(1/R)$. It is written in this form to extend the use of the parameter or multiplier which is the constant, (v^2/g), of a curve of curvatures of the original curve. A very simple geometrical construction will separate γ from φ and, then, tan φ becomes valuable as a figure by which a judgment can be formed as to the safety of the road. If, moreove tan φ is used, not as a coefficient but merely as a reliance o friction, it becomes a useful criterion for the purpose intende

The Rate of Change of Acceleration, C. The equations stat to be, for the spiral $C=v^3/K^2$ and for the lemniscate $C=(3v^3/J$ $\cos 2\varphi$, are obtained by the *scalar differentiation* of the radi accelerations, A, from the equation $A=v^2/R$. The fundament property of A is that it is a *vector quantity* and therefore th rate of change cannot be obtained by scalar differentiatio In the case of the lemniscate this has led to the erroneous con clusion that C "is a maximum where φ is zero and thereaft decreases until zero value would be obtained if $\varphi=45$ we reached." Just the opposite is the case as may be seen b writing $A=v^2/R$ in the form $A=v^2(1/R)$ where $(1/R)$ is th curvature and v is the constant velocity. At the beginning o a lemniscate the curvature is very small whereas when $\varphi=4$ is reached its change becomes very rapid comparatively. Con sequently C=vectorial dA/dt over a period of one second very small at the beginning of the curve becoming a maximu in the vicinity of $\varphi=45°$. The Author's Fig. 3 will requi alteration materially.

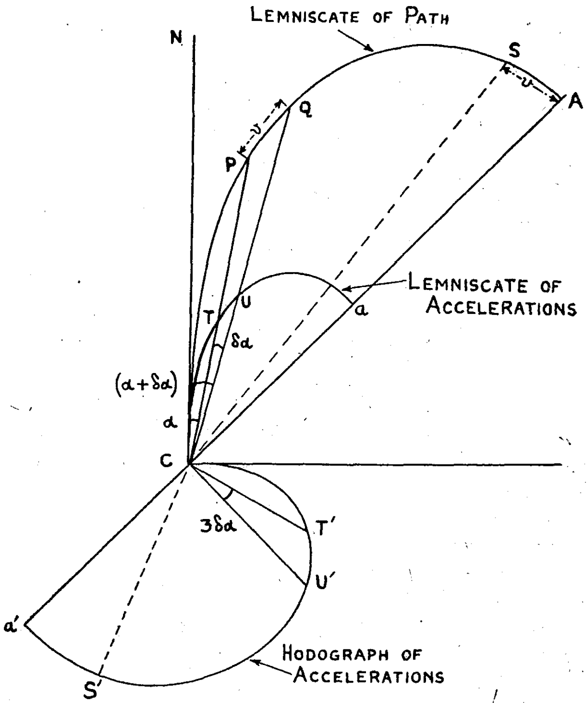
The correct method of evaluating C=vectorial dA/dt is b drawing the Hodograph of Accelerations. For the lemnisca this is shown in Fig. 15 and follows from a simple geometric construction which depends on the axiom that equal vecto have equal magnitudes and either the same or parallel directio and on the properties of the lemniscate that

(i) The vectorial, tangential and intrinsic angles are in th ratio of 1 : 2 : 3.

(ii) The magnitudes of radial accelerations when plott along the vectorial angles, α, describe the lemniscate of acceler tions, $A=(3v^2/J)\sqrt{\sin 2\alpha}$ whose parameter is $(3v^2/J)$. He v is the constant velocity and J is the parameter of the lemni cate of the path.

(iii) Parallel directions through the centre C of the lemnisca of the path to the radii of curvature (perpendicular to tangent will be found along rays through C drawn 90° out of phase wi the intrinsic angles swept by tangents with the axis of referen

The hodograph of accelerations is obtained by plotting th values $A=(3v^2/J)\sqrt{\sin 2\alpha}$ of the lemniscate of acceleratio along rays through the centre or origin of the lemniscate of the path at angles sweeping $(90°+3\alpha)$ with the axis of reference.



CA = parameter, J. $Ca = Ca' = 3v^2/J$.
$PQ = SA = v$ ft. $CP = J\sqrt{\sin 2a}$
$CQ = J\sqrt{\sin 2(\alpha+\delta\alpha)}$. $CT = (3v^2/J)\sqrt{\sin 2\alpha} = CT'$.
$CU = (3v^2/J)\sqrt{\sin 2(\alpha+\delta\alpha)} = CU'$.

Arcs $a'S'$ and $T'U'$ are vectorial $\dfrac{dA}{dt}$ over one second.

CN is the axis of reference.

$\angle NCT' = (90+3\alpha)$. $\angle T'CU' = 3\delta\alpha$.

Fig. 15.

The arc of the hodograph between two radius vectors one second apart subtends at the origin an angle which is $3\delta\alpha$ when the constant velocity, v, during that second, describes the arc (the lemniscate of the path subtended by the angle $\delta\alpha$. It is th length of the arc of the hodograph which in one second giv the value C=vectorial dA/dt.

The Law of the Osculating Circle. The inference that th bending moment diagram of a beam bent in the shape of spiral must be a straight line is not tenable. In fact, it is pr bably another spiral whose parameter is the product of th parameter of the beam spiral and the flexural rigidity, EI. I the relation $M/I=E/R$, $M=EI(1/R)$ and $(1/R)$ is the curvatu of the beam spiral.

The Lemniscate Transition Curve. The Author sugge evaluating the parameter or multiplier of the lemniscate of th path from the rate of change of radial acceleration, C. This a very tedious process and quite the wrong way to use C. I correct function is as a test of results obtained independentl

Denoting the vectorial angle by α to avoid confusion wi the angle of friction, φ, the Author's equations (16) and (1 become

$$P=3R\sin 2\alpha \quad (16) \quad \text{and} \quad P=J\sqrt{\sin 2\alpha} \quad (1$$

Combining these two, $J=3R\sqrt{\sin 2\alpha}$. This gives the val of the parameter J when R is known and R is found fro $\tan(\gamma+\varphi)=v^2/gR$.

Using the lemniscate as the curve of the path, the procedu for design would be somewhat as follows:

(i) Decide on values to be given to $\tan\gamma$ and $\tan\varphi$.

(ii) Select the point on the curve where R is likely to be minimum and evaluate R from $\tan(\gamma+\varphi)=v^2/gR$.

(iii) Evaluate J from $J=3R\sqrt{\sin 2\alpha}$ and adjust it to suit th conditions obtaining at the site.

(iv) Plot the curve and draw the lemniscate and hodogra of accelerations and evaluate C(max). This is the test.

Since drivers like to just feel the curve, C may safely be allow to go as high as 4 ft. per sec. per sec. in 1 sec. Passengers wh are not preoccupied with driving may experience some slig discomfort but the criterion of safety is that the driver must just able to feel the curve and must be permitted to do th without discomfort. (Shortt's standard of unity is from th point of view of the passenger.)

(v) When a final decision regarding J has been reached, then multipliers or parameters of the lemniscate of curvature, $(1/R)=(3/J)\sqrt{\sin 2\alpha}$ and of the lemniscate of superelevation, $\tan(\gamma+\varphi)=(v^2/g)(3/J)\sqrt{\sin 2\alpha}$ may be found quite readily as the equations indicate the multipliers of the lemniscate of unit parameter whose radius vectors are $p=\sqrt{\sin 2\alpha}$.

(vi) The design is completed by separating $\tan\gamma$ from $\tan\varphi$ for selected points along the curve located by the vectorial angle α, and so finding the gradient of superelevation of the inner and outer kerbs.

Model Curves. The scale model of the basic curve is very useful in ensuring that the curve suits the conditions obtaining at the site. There its utility appears to cease. If, however, the scale model be used as a unit curve, a simple alteration of the scale results in the evaluation of such other multipliers as may be necessary. The vast difference in magnitudes must receive careful consideration. For example, if the parameter of the lemniscate of the path be 1,000 ft. then at 60 m.p.h. the lemniscate of the superelevation will have a parameter of only 0·726. Again the Cartesian co-ordinates of the lemniscate of the path will be very small in the vicinity of the origin compared with the parameter.

Superelevation. The correct amount of superelevation is dependant principally on the amount of the reliance on friction. The effect of the relative positions of the wheels is negligible because the length of the wheelbase is small compared with the radius of curvature. On the other hand, the reliance on friction (itself a superelevation) is, when used logically, a valuable corrective to any tendency to use excessive superelevations and extravagant radii of curvature. Frictional resistance to rolling is smaller than to slipping. Consequently it is illogical to cater for high speeds needing high reliances on rolling friction while denying frictional resistances to side slipping at those or indeed any speeds. The superelevation gradient is calculated very easily in the case of the lemniscate once a decision has been reached as to the maximum superelevation desirable.

Vertical Curves. The parabolic vertical curve has the vital defect of possessing at its junction with a straight a finite curvature. The change of curvature from the zero of the straight to the finite of the curve is instantaneous and consequently there must be a shock or blow disruptive to the road. It is illogical to prescribe conservative values for C, the rate of change in radial acceleration, for horizontal curves and to allow it to rise to unpredictable heights in vertical curves.

The justification for a parabolic curve seems to be that the parabola is the path of a particle projected with some initial velocity when under the influence of gravity alone. It is not known generally that the lemniscate can be justified on the ground that a particle under the influence of gravity alone will travel along a lemniscate in the same time as it takes to travel along a radius vector. In the case of a lemniscate time and therefore power are saved on gradients and, since the curvature is zero at the centre or origin, a junction with a straight can be made without the development of disruptive shock. The evaluation of the rate of change in radial acceleration is as direct as in the case of horizontal curves. Incidentally an interesting corollary is that lemniscate cambers will drain roads as quickly as straight crossfalls while limiting the steepness of the incline in the most used part of the cross section.

General. The Author's approach to the problem of designing road curves may be regarded as a decided advance on the methods employed up to date. The multiplier or parameter should be capable of defining a type of curve as completely as the radius or diameter defines a circle. This is true in the case of the lemniscate.

It is important to realise the value of C, the rate of change in radial acceleration, as a test of the design. The test can be applied to both horizontal and vertical curves and it is logical to apply it in this way. When the multiplier or parameter has been fixed then such matters as visibility on vertical curves and the length of the curve are settled automatically.

In all cases the important dimension is the minimum radius of curvature. Little can be done until this has been evaluated and it is best found from considerations governing the superelevation desirable and the reliance to be placed on frictional resistances. In this connection it is important to realise that the circle cum transition curve is unnecessarily wasteful and on the site does not look any better than the wholly transitional curve.

In these comments the lemniscate has been used freely because it lends itself to a greater extent than any of the other curves to handling by the method of multipliers. Indeed, the unit curve, $p=\sqrt{\sin 2\alpha}$, is capable of solving graphically most of the problems in highway curve design.

Prof. F. G. Royal-Dawson : This paper affords an excellent study of transition principles from a purely mathematical standpoint. It is perhaps open to question whether it is not too exclusively mathematical in outlook for easy assimilation by the average engineer engaged in location work : what the latter needs is a method giving a series of simple polar deflections for equal chords of easily determinable length, and involving minimum ready-reckoning for each individual curve. This question hinges primarily on the choice of a unit of measurement. Taking the lemniscate, the Author's unit is the major axis, from which he deduces, by recourse to elliptic integrals, a most valuable series of curve lengths with their corresponding polar deflections, embodied in Table I. This is an essential prelude to the study of the lemniscate, but Table I as it stands is obviously not adapted for direct field use. It was from similar considerations that the writer evolved his " unit-chord " system, taking as the unit of measurement a polar ray of 16 minutes deflection, thus obtaining for a quarter-chord sequence the simple square-law series 1′, 4′, 9′, 16′, 25′, etc., for *all* transitions, without modification up to 4°, and needing only slight corrections thereafter for the majority of road curves. By this method the major axis becomes 10·3648 units, by which the Author's figures should be multiplied to produce like results.

The Author's choice of unit for the spiral is more subtle. It involves two factors, $rs=1$ (where s is used instead of l), that is $RS=rKsK=K^2$, a constant. The Author rightly points out that the writer's unit is very nearly K/6. In short, the unit length ($s=1$) is equivalent to the writer's 5·9841 units, so that the latter's constant $RL=(5{\cdot}9841)^2=35{\cdot}81$. Incidentally, it will be found that the ratio $10{\cdot}3648/5{\cdot}9841=\sqrt{3}$, which confirms the Author's comparison of basic unit lengths.

From a practical point of view, considering the material similarity of the two curves for a considerable portion of their length, there is no reason why the same unit should not be used for both. The spiral is so divided in Table XV of "Road Curves," from which it will be seen that the polar ray at 45° is 10·0199 units, as compared with 10·3648 for the lemniscate. Again, if a 10-inch model (1 unit=1 inch) be taken, it will be found that at 7·534 inches (the 15° mark on the lemniscate) the lateral difference is only 0·006 inch (less than the point of a sharp pencil), while for a 36-inch model (1 unit=3·6 inches) the corresponding lateral difference is only 0·022 (about 1/50th) inch. So that for drawing office purposes the contours of the curves are practically identical within a range of 15° (equivalent to a 90° bend).

The Author's diagram, Fig. 3, is ingenious, but its utility or otherwise to the engineer would depend upon the method to which he was accustomed. On the other hand, the basic lemniscate, Fig. 10, with its attendant locus of centres of curvature, has a definite element of usefulness in helping to determine the main features of a proposed curve on a scale plan, especially in showing whether a central circular arc would be necessary or not, in a given case. In principle it goes a step further than the process described in pp. 39 and 40 of "Road Curves."

Regarding the value of "C" to be adopted for design purposes, the writer's views have been fully expressed,[11] so no more need be said on the subject, except to point out that the question is not one of "experiment," in the backyard sense of the term, but of intelligent observation of actual traffic movements on the open road.

Turning to other points the Author suggests that on a wholly transitional curve the driver has no respite from turning the steering-wheel, and that the majority of motorists do not like this arrangement. As regards the first suggestion, it will be found that in actual practice the driver takes a respite whenever he wants it, whatever the curve may be, and that in fact the whole operation of steering consists of intermittent hand movements interspersed with respites or rests while following the general course of the curve. The second objection seems to be largely imaginary, existing in the minds, not of motorists as such, but mainly of engineers accustomed to the "spiral degree" method of setting out, with its inevitable centra circular arc. The ordinary motorist who has no personal knowledge of the technology of any curve on which he happens to find himself, cannot tell one from another, except as between long and short, flat and sharp, symmetrical and irregular, according to environment.

On the question of maintaining a through chainage when staking out a transition curve, in theory this is quite feasible, whatever unit may be used, but it cannot be pretended that it facilitates the construction of the transition, as the Author's own worked-out example shows, nor that it even serves any useful purpose. Chain pegs, like milestones, are merely records of through distances which are liable to be changed at any time by realignment schemes in other portions of the route. The incorporation of through chainage as an integral part of the setting out of transition curves is therefore in general a waste of time and labour, except perhaps in the case of curves which are mainly circular and of large radius, when the transition is comparatively short, entailing little calculation.

With regard to setting out transitions from intermediate points, if these points are restricted to half or quarter chords, as in the writer's unit-chord system, the required deflection in both directions can be read off direct from tables up to 9 unit chord points in "Road Curves"[12] and for further distances in "Motorways."[13]

On the question of vertical curves the Ministry of Transport requirements quoted by the Author are probably under revision. So far as summit curves are concerned, the writer's independent investigations give the following desirable values of Z according to speeds, for 60 m.p.h, 165; for 45 m.p.h., 93; and for 30 m.p.h., 41. In regard to sags or valley curves, for which the use of transitions is advocated, the writer has evolved definite figures for the impact factor, which tend to show that the minimum radius for 60 m.p.h. should be not much less than 3,000 ft., and preferably much more. If transitions are not used, the circular radius should be round about 9,000 ft. This means $Z=90$ against the Author's proposed 30, which seems a somewhat low figure for a non-transitional curve at that speed.

Mr. H. W. S. HUSBANDS, M.C.: If it is desired to use the spiral or lemniscate the basic curve method is no doubt useful, but the writer sees no necessity for their use. As a railway engineer he has always used a circular arc and cubic parabola transition, which is sufficiently accurate over the short length necessary for the transition curve. The use of spirals and lemniscates is surely an unnecessary refinement, and a curve transitional throughout is a contradiction in terms; if there is not even a small length of circular arc, it must result in a kink at the centre. The absurdity of lengthening the route in order to continue to a much sharper minimum radius with a so-called through transition is apparent, and it is satisfactory to note that the Author prefers a main circular arc. The writer considers a circular arc with constant acceleration to be safer than a transition with changing acceleration, but agrees with the Author that a rate of change of 2 ft. per sec. per sec. in 1 sec. is permissible.

The parabolic vertical curve can be set out very simply by bisecting the perpendicular from the interesection point to the chord joining the tangent points and quartering the offset at the centre of successive chords. It might be mentioned that the parabola is practically indistinguishable from a circle for angles of deflection less than about 15°.

Mr. H. A. Warren, M.Sc.(Eng.): The paper forms a useful summary of principles and formulae relating to transition curves in present orthodox practice. It is, however, not easy to see in what way the Author's "basic" methods are in any way more basic, or simple, or useful than those outlined in previous and similar publications.[14] The chief points for criticism are the continued use of "maximum rate of gain of centrifugal acceleration" as a design factor, and the insufficient attention paid to its value and the effects of its value on design. It is easy to say "the choice of value must be left to the discretion of the engineer" and then proceed with the mathematical presentation of the spiral and the lemniscate, but this is really very little help to the practising engineer.

The simplest experiments[15] with actual road vehicles will show that the value of C can reach 50 ft. per sec. per sec. in 1 sec. or higher, practically to infinity, without any danger or even discomfort whatever, and that the idea that C cannot much exceed unity is completely illusory. The reason why the value of C is a prime consideration is that when the higher values are used the length of the transition curve necessary becomes so small that other considerations such as the smooth application of superelevation without undue vertical acceleration, provide the limiting factor in design and influence both the shape and the length. Moreover the shapes of the lemniscate and the spiral for short length transitions are so indistinguishable from the cubic parabola that the bulk of the elaborate mathematics for setting out the former curves is rendered useless, especially when it is remembered that there is in any case no compulsion on the motorist to follow the kerb lay-out at all. One cannot avoid the feeling that the practical highway engineer looking to papers such as this for guidance will find an abundance of mathematics but little that will help him in assessing the prime factors affecting design. In the matter of transition curves in general there is need for less theorizing and more experiment.

The numerical example No. 1 given in the Appendix will be used to illustrate these remarks on how far from reality is the standard of $C=1$. The radius of the curve entered is 1,273 ft., and assuming a wheel-base of 7·8 ft. and a steering gear ratio of 6 to 1, the angle turned by the steering column will be 2·1°. The length of the transition curve according to the paper is 535·3 ft. and the speed is 88 ft. per sec., so that the time occupied in turning the steering column is 6·1 secs. The motorist is thus asked to take 6·1 secs. over the negligible task of turning the steering wheel through 2·1°. If he does not occupy this time he will not keep constant distance from the kerb which has been laid out with such precision on $C=1$ principles. In actual fact the wheel turns through the 2·1° by a mere " twitch " practically instantaneously, and since a transition is traced only whilst the wheel is being turned, the length is reduced to a very small value. The necessity or even desirability of transitions at high speeds is much overrated, but at low speeds, where the wheel can be turned through very large angles and therefore requires considerable time, the provision of transition curves becomes a practical desirability, as for example at 90° street intersections. Curiously enough this aspect has received the least attention.

Author's Reply

Mr. MacGregor: Mr. Cornelius has made an extremely valuable contribution to the paper and it will well repay anyone to make a close study of his remarks on superelevation and the extended use of multipliers. The reason for the statement to which he has taken exception was that the Author did not wish to add to the controversy already in existence between two well-known writers on curve design until experiment and experience had proved conclusively that the new basis of design was better than the old. Petrol rationing has, however, made it impossible for the experimental work to be carried out. The chief point of difference is in the meaning attached to the rate of change of acceleration C. Mr. Cornelius has stated that the equations, $C=v^3/K^2$ and $C=(3v^3/J^2)\cos 2\varphi$, are wrong because they have been obtained by *scalar* differentiation, and that the conclusions reached therefrom are also wrong. At the beginning of the lemniscate the curvature is small, whereas at $\varphi=45°$ the curvature is relatively large, but—and this is the important point—the rate of change of curvature is a maximum at the beginning. A glance at the basic model, Fig. 10, will show this to be the case. Mr. Cornelius has seemingly confused curvature with rate of change of curvature. The *vectorial* rate of change of acceleration is quite irrelevant to the problem. This may be seen by considering the change in acceleration and its effect on the comfort of the passenger when the vehicle is travelling with speed v in a circle of radius R. The acceleration is then constant in magnitude and is equal to v^2/R; it is directed towards the centre of the circle and therefore the hodograph of acceleration is another circle also of radius v^2/R. The *scalar* rate of change of acceleration is zero, since the acceleration is constant in magnitude, but the *vectorial* rate of change is v^3/R^2. Once the circular part of the road bend is reached, the passenger has adjusted himself relative to the car and feels a constant pressure on certain parts of his anatomy. By giving him time on the transition to brace himself to meet this force he experiences no discomfort unless the force is great, and by keeping the value of the centrifugal ratio, F, below 0·25, the force exerted by the car on the passenger will not exceed one-quarter of his weight. On the transition he has had to adjust himself to meet the increase in force only, and the direction of the line of action does not worry him, for it appears to be constant. For the spiral, v^3/K^2 is Shortt's formula, since $RS=K^2$ and the acceleration is acquired at a uniform rate. In the lemniscate the rate of change of acceleration is not constant. This may be seen by referring to the lemniscate of accelerations in Fig. 15. Here CT represents the radial acceleration at P, CU that at Q and Ca that at A. It is obvious that if the distances PQ and SA on the lemniscate of the path represent the distances travelled by the car in 1 second, then the rate of change of acceleration (*scalar*) in the vicinity of PQ is considerably greater than in the vicinity of SA. It may also be noted by reference to Prof. Royal-Dawson's remarks that Fig. 3 requires no alteration on this account. The process of determining the multiplier is therefore not a tedious one, for it consists simply of reading the value from Fig. 3. Mr. Cornelius will no doubt revise his design methods in the light of the above and the laborious task of drawing hodographs of acceleration need be done no longer. Whether the designer chooses P and R in preference to J and R makes little difference to the problem, the fundamental property $PR=J^2/3$ must be observed in both cases. In the demonstration of the law of the osculating circle by analogy with beam deflections, the bending moment diagram plotted on a base S is a straight line. The law can, of course, be proved from the usual theorems of curvature.

The mathematical reader will have no difficulty in following Mr. Cornelius's description of the extended use of the multiplier and basic curve to superelevation problems. His treatment of superelevation is perfectly logical and the Author agrees that it is illogical to cater for high speeds, needing high reliances on rolling friction, while denying frictional resistances to side slipping at those or indeed any speeds. His method would not apply over the portion of the transition where μ is in itself greater than v^2/gR. This might be taken to indicate that the procedure of introducing the superelevation so that it is sweet to the eye is legitimate. The engineer must use his own discretion when selecting μ and it is probable that once the maximum value of γ has been found he will grade in the superelevation without further reference to μ. The Author considered the question of fitting transition curves to vertical curves but came to the conclusion that in view of the large radii generally used they would be an unnecessary refinement. Settlement of the bank material would also affect them and consequently he did not pursue the matter any further. The interesting property of the lemniscate to which Mr. Cornelius has referred would appear to be true only for the case where the axis of the lemniscate is inclined at 45° to the vertical.

The Author appreciates the criticism of his paper by Prof. Royal-Dawson. It will in general be agreed that Prof. Royal-Dawson has been mainly responsible for the introduction of scientific principles into road curve design and the Author has found his publications to be of immense value. Regarding the relative merits of basic curve and unit-chord methods, it all depends on the system to which one is accustomed and on the weight placed on the maintenance of the running chainage. In new works, alterations in the alignment of the route will seldom occur and the Author has found that there is less likelihood of mistakes being made if the running chainage is maintained. Profiles and cross-sections have to be taken and grade levels calculated before the contract drawings are completed and the levelling party will have less trouble if pegs are put in at even chainages. In improvement schemes there is not the same need to retain the running chainage, but, as in the first case, Table I, together with a slide rule, will give the engineer all the information he requires for setting out the lemniscate. He can take any length of chord he pleases provided he respects the difference between curve and chord length. Thus for chord lengths of 50 ft. on a lemniscate with $J = 1{,}000$ ft., he would find by interpolation the values of φ corresponding to $l = 0{\cdot}05$, $0{\cdot}10$, $0{\cdot}15$, etc.

Since the engineer who uses the spiral is unlikely to change over to the lemniscate and vice versa, the Author does not think there is likely to be any confusion of the multipliers K and J. He cannot agree that the same unit should have been chosen for each curve; the selected units appear to him to be the natural ones for the respective curves. The similarity of the curves over a considerable portion of their lengths has been commented upon, and, as Prof. Royal-Dawson points out, the difference between the curves can hardly be detected on a 10-in. model. In this connection, it should be noted that Prof.

Royal-Dawson is referring to his own 10-in. model which has an axis length of 10·3648 in. The Author's basic model has an axis length of 10 in. exactly and the 15° mark will therefore be at 7·269 in. along the curve. The spiral enthusiast can, of course, obtain his basic model by using the tables[1] referred to. The Author should also have stated that there is no need to go on decreasing the radius so that a bend can be made wholly transitional. It is axiomatic that the smaller the centrifugal ratio the safer the bend, and this can be met in most cases by introducing a length of circular arc.

Prof. Royal-Dawson's remarks on vertical curves should be noted for future reference. The determination of finished road levels on a sag curve will be somewhat tedious in comparison with the non-transitional case and the Author therefore recommends the use of the higher multiplier.

The Author agrees with Mr. Husbands that a cubic parabola is sufficiently accurate for the shorter length transitions in railway work, but in roads the minimum radius of the bend may be considerably smaller than that in use on railways and the use of the spiral and lemniscate is then justifiable. The method of quartering referred to by Mr. Husbands will be familiar to most engineers. In general, however, grade levels have to be computed for odd chainages and the Author has found direct substitution in equations (31) or (34) to be the quicker method.

Mr. Warren's chief point of criticism is that the Author has continued to use C as a design factor when he, from the simplest experiments with actual road vehicles, has deduced that the value of C can reach 50 ft. per sec. per sec in 1 sec. or higher, practically to infinity, without any danger or even discomfort whatever. The experiments are described in Mr. Warren's paper[15]; in brief, they consisted of measuring by *tacheometric* means the track of a car when turns ranging from "natural" to "forced" were made in a large car park. The track was recorded by means of a sharp, thin trail of water which issued, about one inch above the ground, from flexible tubing connected to a drum of water mounted in the passenger's seat. The vehicles passed through a straight lane formed by two parallel steel bands, 100 ft. long, and spaced about 2 ft. wider than the wheel track of the vehicle being tested. At the end of this lane ranging rods stood on either side and immediately the posts were passed the driver was supposed to turn in a comfortable path. The instrument was stationed at the point of intersection of the water trail and the line joining the ranging rods, and tracks which showed that the vehicle had turned slightly before or after the posts were not recorded.

From a plot of the observed values to a scale of 1 in. $=10$ ft. Mr. Warren found that elusive quantity, the third derivative of a curve, or, in engineer's language, C. His values, although given in some instances to two decimal places, are inconsistent, and range from 0·79 to 58·4 ft. per sec. per sec. in 1 sec. As a result he concluded that "unless the above results can be explained away, present designs based on $C=1$ are just so much scrap." The results require little explaining away, for anyone with experience in graphical differentiation would have told him that the values of C obtained in this manner might be inaccurate to the extent of several hundred per cent. It would thus be easy to dismiss the subject, but since the practical highway engineer may accept Mr. Warren's invitation to plot the results and see for himself, it may be better to point out the idiosyncrasies of the experimental evidence.

By plotting the deflection angles on a base of radius vector then, with the notation used in the present paper, if a lemniscate of axis length J ft. has been described, $P=J\sqrt{\sin 2\varphi}$, and a curve similar to (a), Fig. 16, would be obtained. If no transition is introduced and the vehicle describes a circular arc, the chord becomes proportional to the sine of the deflection angle and curve (a) will be almost a straight line. If, however, the radius increases from a finite value, then a curve similar to (b) will be obtained. It should be noted that curve (a) is concave upwards whereas curve (b) is convex upwards. The curves 1, 2, 4, 6, 8 and 9 in Fig. 16 are those obtained from Mr. Warren's experimental data. It will be seen that curves 1, 4, and 6 show signs that some form of transition may have been followed and it is significant that these were described as "natural" turns. For 4 Mr. Warren deduced that $C=58·4$ ft. per sec. per sec. in 1 sec., whereas for the "easy and comfortable" turn 2 his value was (5·16). The brackets are his, and they signify that the value was obtained by guessing the shift, though perhaps the description of the turn may have misled him into imagining that C should have been low on that account. The Author would have concluded that, if the data were free from experimental errors, C must have been infinite in run 2. Runs 8 and 9, for which the published values of C are 18 and (4·34) were described as "forced" and "very comfortable" respectively. The Author would also have concluded that 9 was even more forced than 8; that C could quite well have been infinite in both instances; and that Mr. Warren should have sufficient justification to revise the opinion expressed by him that "once it

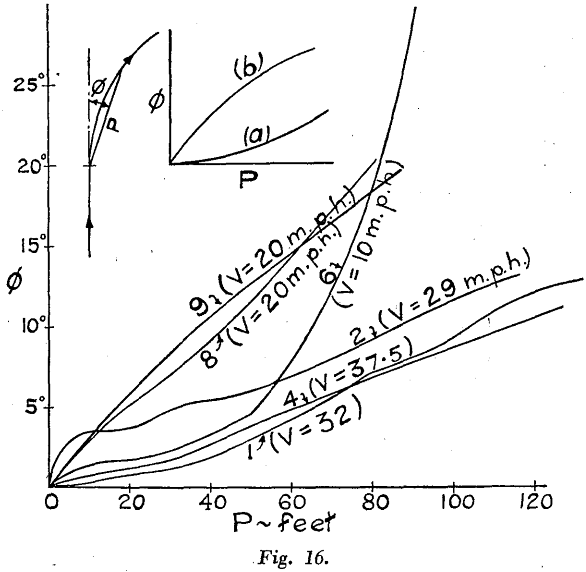


Fig. 16.

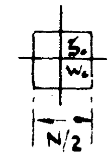
is realized that it is *impossible* for the road vehicle to turn other than by a transition curve, the most humorous absurdity of laying out the kerb otherwise will be apparent to everyone."

In the Author's opinion the data are not free from experimental errors, but there are so many potential sources of error that he is unable to say which has made the major contribution. Skidding may explain the type (b) characteristics, but from the general outline of the curves in Fig. 16, especially 2, it appears however that the instrument was stationed not at the tangent point but at some point along the curve. Mr.

Warren's insistence on the instrument station being on the line joining the ranging rods was most unfortunate. Other factors which merit some attention and correction are the centrifugal and aerodynamic effects on the fine jet of water. It might well be that only when the vehicle was travelling along the straight or on a circular curve, when conditions would become stabilized, that anything like the path of the car was described by the jet. The Author is therefore unwilling to advise anyone to discard the present basis of design until further experiments with all types of transport vehicles and drivers have pointed the way to a better basis of design, and he agrees with Prof. Royal-Dawson that the experiments should consist of intelligent observation of actual traffic movements on the open road.

Mr. Warren's remarks on how far from practical reality is the standard C=1, appear to the Author to be a logical case for the continued use of this standard when site conditions permit. The driver of the vehicle is given 6·1 secs to meet the changing curvature of the road, in fact it is not beyond the realm of possibility that with proper superelevation the car will round the bend itself. By using low values of C the average highway engineer will find that he has sufficient length in the transition to work in the superelevation without having to resort to a transition of the form $y=Ax^4-Bx^5$. At any rate no matter which value is used for the multiplier, or how it has been obtained, the basic curve method of design will facilitate the setting-out of the spiral, lemniscate and cubic parabola; this is but one of its many advantages.

— Squares Formulae —



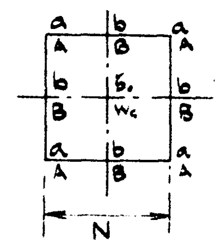
Single Square of side N/2.

$\nabla^4 w = C$

$$\zeta_c = \zeta_m - N^2 C/64 \qquad \text{(1c)}$$
$$w_c = w_m - N^2 \zeta_c/16 \qquad \text{(1d)}$$

$\nabla^4 w = 0$

$$\zeta_c = \zeta_m \qquad \text{(1e)}$$
$$w_c = w_m - N^2 \zeta_c/16 \qquad \text{(1f)}$$



4 single Squares grouped to form Square of side N

$\nabla^4 w = C$

$$12\zeta_c = \Sigma a + 2\Sigma b - N^2 C/2 \qquad \text{(2c)}$$
$$12 w_c = \Sigma A + 2\Sigma B - 2N^2\zeta_c - N^4 C/64 \qquad \text{(2d)}$$

$\nabla^4 w = 0$

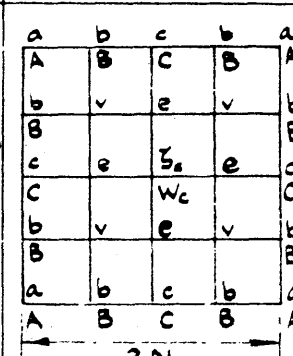
$$12\zeta_c = \Sigma a + 2\Sigma b \qquad \text{(2e)}$$
$$12 w_c = \Sigma A + 2\Sigma B - 2N^2\zeta_c \qquad \text{(2f)}$$

Line Load uniformly distributed over ℄ of square

$$12\zeta_c = \Sigma a + 2\Sigma b - pN/K \qquad \text{(2g)}$$
$$12 w_c = \Sigma A + 2\Sigma B - 2N^2\zeta_c - pN^3/16K \qquad \text{(2h)}$$

Point Load at centre of square

$$12\zeta_c = \Sigma a + 2\Sigma b - 4Z \qquad \text{(2j)}$$
$$12 w_c = \Sigma A + 2\Sigma B - 2N^2\zeta_c - N^2 Z/4 \qquad \text{(2K)}$$



16 single squares grouped to form Square of side 2N

$\nabla^4 w = C$

$$476\zeta_c = 9\Sigma a + 32\Sigma b + 46\Sigma c - 72N^2 C \qquad \text{(3c)}$$
$$476 w_c = 9\Sigma A + 32\Sigma B + 46\Sigma C - N^2\left[18\Sigma v + 28\Sigma e + 104\zeta_c + 2{\cdot}25 C N^2\right] \qquad \text{(3d)}$$

$\nabla^4 w = 0$

$$476\zeta_c = 9\Sigma a + 32\Sigma b + 46\Sigma c \qquad \text{(3e)}$$
$$476 w_c = 9\Sigma A + 32\Sigma B + 46\Sigma C - N^2\left[18\Sigma v + 28\Sigma e + 104\zeta_c\right] \qquad \text{(3f)}$$

Field No. 1

$\nabla^4 w = C = p/K$

ζ values
w values

Gradient $\partial w/\partial y$		1·482 CN³	2·858 CN³	4·070 CN³	5·094 CN³	5·908 CN³	6·486 CN³	6·842 CN³	6·962 CN³ ℄
0·512 pN/K	ζ	·2029	·3374						
A	w	·7264	1·4027						
0·788 pN/K	ζ	·3374	·5846	·7680					
	w	1·4027	2·7156	3·8828					
0·986 pN/K	ζ	·4341	·7680	1·0230	1·2149				
	w	2·0021	3·8828	5·5608	6·9833				
1·128 pN/K	ζ	·5048	·9039	1·2149	1·4521	1·6264			
	w	2·5079	4·8699	6·9833	8·7792	10·2122			
1·231 pN/K	ζ	·5555	1·0021	1·3547	1·6264	1·8273	1·9655		
	w	2·9096	5·6548	8·1164	10·2122	11·8869	13·1045		
1·300 pN/K	ζ	·5897	1·0686	1·4499	1·7456	1·9655	2·1171	2·2059	
	w	3·2004	6·2236	8·9384	11·2530	13·1045	14·4517	15·269.	
1·34 pN/K	ζ	·6095	1·1071	1·5053	1·8152	2·0462	2·2059	2·2996	2·3304
	w	3·3764	6·5679	9·4363	11·8838	13·8429	15·2691	16·1349	16·4250
1·350 pN/K	ζ	·6160	1·1197	1·5234	1·8380	2·0728	2·2352	2·3304	2·3618 ζ
℄	w	3·4353	6·6831	9·6030	12·0950	14·0903	15·5431	16·4250	16·7206 w

Values of $\partial\zeta/\partial x$

4 N

Small squares have length of side N/2

ζ values are negative and have units CN²

w values are positive and have units CN⁴

Field No 2

$\nabla^4 w = 0$

℄ 10

.023	.052	.093	.159	.291	.616	1.781	2.711	ζ values
.148	.300	.467	.656	.882	1.168	1.559	1.786	w values
.042	.093	.162	.268	.448	.781	1.234	1.469	
.277	.564	.870	1.203	1.575	1.987	2.363	2.527	
.055	.119	.199	.310	.469	.678	.881	.971	
.379	.766	1.167	1.585	2.016	2.429	2.748	2.871	
.060	.128	.207	.305	.424	.553	.659	.702	
.446	.895	1.348	1.801	2.236	2.618	2.886	2.984	
.060	.124	.196	.277	.364	.448	.509	.532	
.480	.956	1.427	1.880	2.294	2.639	2.866	2.948	
.056	.116	.176	.240	.304	.360	.400	.414	
.484	.962	1.423	1.856	2.240	2.543	2.742	2.811	
.050	.101	.151	.203	.250	.291	.317	.327	
.467	.924	1.360	1.761	2.106	2.376	2.548	2.608	
.042	.086	.127	.168	.204	.233	.253	.259	
.433	.856	1.256	1.617	1.924	2.160	2.310	2.361	℄
.035	.071	.105	.137	.166	.187	.201	.206	
.390	.768	1.124	1.443	1.711	1.915	2.042	2.086	
.028	.057	.085	.110	.131	.148	.158	.162	
.339	.669	.976	1.250	1.481	1.651	1.759	1.796	
.022	.045	.067	.086	.103	.115	.122	.126	
.286	.561	.819	1.047	1.237	1.382	1.467	1.498	
.017	.035	.051	.066	.078	.088	.092	.095	
.229	.451	.657	.839	.990	1.104	1.174	1.197	
.012	.025	.037	.047	.057	.063	.067	.068	
.172	.339	.494	.630	.743	.827	.879	.897	
.008	.016	.024	.031	.036	.041	.044	.044	
.115	.226	.328	.420	.495	.551	.585	.597	
.004	.008	.012	.015	.018	.020	.022	.022	
.058	.113	.164	.210	.248	.275	.293	.298	

℄

8N

If the ζ values are -ve, the w values are +ve, and vice versa
Multiply the ζ units by N^2 to get w units

Field No. 3

$\nabla^4 w = 0$

10 ℄

·036	·080	·151	·285	·611	1·778	2·709	1·781	·617	·294	·163	·098	·060	·035	·017
·180	·369	·578	·821	1·124	1·531	1·773	1·559	1·183	·910	·699	·526	·378	·245	·121
·063	·140	·252	·438	·773	1·228	1·466	1·233	·783	·453	·275	·174	·109	·064	·030
·335	·683	1·053	1·460	1·902	2·307	2·499	2·361	2·013	1·630	1·287	·985	·713	·465	·229
·079	·171	·289	·454	·668	·876	·968	·883	·682	·476	·321	·214	·140	·085	·040
·449	·906	1·375	1·853	2·310	2·669	2·831	2·747	2·466	2·093	1·703	1·328	·975	·639	·316
·082	·174	·280	·406	·540	·652	·698	·660	·558	·432	·318	·225	·153	·094	·045
·516	1·034	1·546	2·035	2·469	2·789	2·935	2·885	2·663	2·331	1·946	1·546	1·149	·759	·377
·078	·161	·250	·344	·432	·500	·528	·507	·449	·372	·291	·215	·152	·096	·046
·542	1·078	1·594	2·069	2·468	2·755	2·891	2·864	2·687	2·401	2·043	1·649	1·239	·827	·412
·070	·141	·214	·283	·345	·389	·409	·398	·363	·311	·252	·196	·142	·091	·045
·537	1·060	1·554	1·999	2·365	2·623	2·750	2·738	2·597	2·352	2·028	1·658	1·258	·844	·423
·060	·119	·177	·231	·276	·307	·322	·316	·294	·258	·215	·171	·127	·083	·041
·508	1·001	1·459	1·866	2·194	2·425	2·544	2·544	2·430	2·220	1·935	1·596	1·220	·824	·414
·049	·098	·144	·186	·219	·243	·255	·252	·237	·211	·180	·146	·110	·072	·036
·465	·913	1·327	1·690	1·983	2·189	2·298	2·304	2·213	2·034	1·786	1·484	1·142	·775	·391
·040	·079	·116	·149	·174	·192	·201	·200	·190	·171	·148	·121	·093	·062	·031
·413	·810	1·174	1·493	1·749	1·929	2·026	2·036	1·963	1·814	1·602	1·336	1·033	·704	·356
·032	·063	·092	·116	·136	·151	·158	·158	·150	·137	·119	·099	·076	·051	·026
·356	·697	1·011	1·283	1·502	1·656	1·741	1·754	1·695	1·572	1·392	1·167	·904	·617	·313
·025	·049	·072	·090	·106	·116	·122	·122	·117	·107	·095	·078	·061	·041	·021
·297	·582	·842	1·067	1·249	1·379	1·451	1·463	1·417	1·318	1·170	·983	·764	·523	·265
·019	·038	·055	·068	·079	·088	·092	·092	·089	·081	·072	·060	·047	·031	·016
·237	·464	·672	·852	·996	1·100	1·157	1·168	1·134	1·056	·940	·791	·616	·421	·213
·013	·027	·038	·048	·057	·063	·067	·066	·065	·059	·052	·044	·034	·024	·012
·177	·347	·502	·636	·745	·822	·866	·874	·851	·793	·706	·595	·465	·318	·161
·008	·017	·025	·032	·036	·040	·042	·042	·041	·038	·034	·029	·022	·015	·008
·118	·231	·334	·423	·495	·546	·576	·583	·566	·528	·472	·397	·310	·213	·109
·005	·008	·012	·015	·018	·020	·021	·021	·021	·019	·017	·014	·011	·007	·004
·060	·116	·167	·211	·247	·273	·288	·291	·283	·264	·236	·198	·155	·107	·054

℄

8 N ℄

If the ζ values are −ve, the w values are +ve, and vice versa
Multiply the ζ units by N^2 to get w units.

Field No. 4

$\nabla^4 w = 0$

10 ℄

·058	·136	·274	·604	1·772	2·705	1·778	·615	·293	·165	·101	·064	·041	·025	·012
·222	·458	·725	1·049	1·473	1·730	1·532	1·170	·911	·714	·556	·423	·305	·198	·097
·099	·223	·417	·759	1·217	1·459	1·229	·782	·454	·278	·178	·117	·075	·046	·022
·405	·827	1·276	1·756	2·195	2·418	2·311	1·989	1·633	1·316	1·041	·796	·581	·378	·186
·115	·249	·424	·647	·862	·958	·876	·679	·476	·325	·222	·151	·099	·061	·029
·529	1·062	1·596	2·104	2·512	2·718	2·674	2·433	2·098	1·746	1·411	1·096	·804	·527	·261
·113	·236	·374	·517	·635	·687	·653	·554	·433	·322	·234	·166	·113	·070	·034
·589	1·170	1·725	2·219	2·595	2·796	2·795	2·623	2·337	1·999	1·648	1·299	·961	·633	·315
·101	·203	·309	·407	·481	·515	·502	·448	·374	·296	·225	·166	·116	·073	·036
·601	1·184	1·724	2·191	2·539	2·733	2·763	2·643	2·409	2·103	1·763	1·410	1·053	·699	·348
·085	·169	·249	·319	·371	·396	·392	·362	·313	·259	·204	·155	·112	·071	·036
·579	1·135	1·643	2·072	2·394	2·583	2·632	2·549	2·361	2·094	1·781	1·439	1·085	·724	·362
·070	·136	·199	·250	·289	·309	·308	·290	·259	·222	·180	·139	·102	·067	·033
·537	1·049	1·513	1·902	2·196	2·375	2·435	2·381	2·229	2·002	1·720	1·404	1·067	0·716	·359
·056	·109	·157	·197	·225	·242	·245	·234	·213	·185	·154	·122	·090	·060	·030 ℄
·482	·942	1·355	1·704	1·968	2·134	2·198	2·164	2·043	1·852	1·605	1·320	1·008	·680	·342
·045	·087	·124	·154	·178	·192	·194	·187	·173	·153	·129	·104	·077	·052	·026
·424	·824	1·186	1·490	1·724	1·874	1·938	1·918	1·822	1·663	1·451	1·201	·921	·623	·313
·035	·068	·096	·120	·137	·149	·152	·149	·139	·124	·106	·086	·065	·044	·022
·360	·702	1·012	1·272	1·472	1·604	1·665	1·655	1·581	1·449	1·271	1·055	·814	·552	·278
·027	·052	·075	·092	·106	·115	·118	·116	·109	·098	·084	·069	·053	·036	·018
·299	·581	·836	1·053	1·220	1·332	1·386	1·382	1·324	1·220	1·074	·895	·691	·471	·238
·020	·039	·056	·069	·079	·085	·088	·087	·082	·074	·065	·055	·042	·029	·014
·236	·461	·664	·836	·970	1·060	1·105	1·105	1·062	·981	·866	·723	·561	·381	·193
·014	·028	·040	·050	·057	·061	·064	·063	·059	·054	·048	·039	·030	·021	·010
·176	·344	·496	·623	·723	·791	·827	·828	·796	·737	·652	·546	·424	·289	·146
·009	·018	·026	·032	·036	·039	·041	·040	·039	·035	·030	·027	·020	·013	·007
·116	·228	·329	·414	·480	·525	·550	·551	·531	·491	·436	·365	·284	·194	·098
·004	·009	·012	·015	·018	·020	·021	·021	·020	·018	·016	·013	·010	·007	·004
·058	·113	·164	·206	·240	·262	·274	·276	·266	·246	·219	·183	·142	·097	·049

℄

8 N

If the ζ values are −ve, the w values are +ve, and vice versa
Multiply the ζ units by N^2 to get w units.

Field Nº 5

$\nabla^4 w = 0$

10 ℄

·100	·251	·588	1·760	2·696	1·772	·612	·292	·165	·101	·065	·043	·028	·017	·008	
·278	·577	·928	1·374	1·653	1·473	1·127	·883	·704	·556	·437	·334	·242	·158	·078	
·159	·373	·728	1·195	1·443	1·218	·774	·449	·276	·178	·119	·080	·053	·032	·016	
·491	·998	1·524	2·008	2·267	2·195	1·905	1·577	1·291	1·043	·827	·638	·463	·303	·150	
·171	·370	·608	·833	·939	·861	·669	·470	·322	·222	·155	·107	·072	·044	·021	
·611	1·217	1·785	2·249	2·506	2·510	2·313	2·020	1·709	1·413	1·137	·884	·647	·425	·210	
·152	·311	·472	·601	·662	·634	·542	·425	·318	·234	·169	·121	·082	·051	·024	
·652	1·277	1·839	2·277	2·537	2·593	2·473	2·239	1·951	1·649	1·348	1·060	·782	·516	·255	
·125	·248	·360	·446	·488	·481	·433	·364	·291	·225	·170	·125	·087	·054	·027	
·640	1·242	1·773	2·186	2·445	2·535	2·473	2·297	2·050	1·765	1·466	1·164	·867	·574	·286	
·097	·193	·274	·336	·368	·370	·346	·304	·255	·205	·160	·120	·086	·054	·026	
·596	1·157	1·644	2·027	2·278	2·390	2·372	2·240	2·038	1·785	1·500	1·202	·903	·600	·299	
·076	·148	·208	·256	·283	·289	·277	·251	·217	·181	·145	·112	·080	·052	·026	
·538	1·044	1·482	1·832	2·072	2·193	2·198	2·107	1·942	1·721	1·464	1·185	·895	·599	·299	
·059	·113	·160	·197	·219	·227	·220	·204	·181	·155	·126	·100	·072	·048	·023	℄
·474	·919	1·309	1·621	1·842	1·963	1·989	1·926	1·793	1·605	1·379	1·125	·854	·573	·287	
·045	·087	·123	·151	·169	·176	·175	·166	·149	·129	·107	·086	·063	·041	·021	
·408	·792	1·131	1·405	1·604	1·720	1·753	1·711	1·606	1·449	1·253	1·029	·785	·530	·266	
·035	·066	·093	·116	·131	·138	·137	·131	·120	·105	·090	·072	·053	·035	·018	
·343	·668	·954	1·190	1·362	1·469	1·505	1·479	1·399	1·269	1·103	·910	·698	·471	·237	
·026	·050	·071	·088	·100	·105	·107	·103	·095	·084	·072	·059	·045	·029	·014	
·281	·548	·782	·979	1·124	1·215	1·253	1·237	1·174	1·070	·935	·775	·595	·403	·203	
·019	·037	·053	·066	·074	·080	·081	·078	·072	·065	·057	·046	·034	·023	·012	
·221	·432	·618	·774	·892	·967	·999	·989	·942	·862	·756	·629	·484	·329	·165	
·014	·027	·037	·046	·053	·056	·058	·056	·053	·047	·041	·033	·026	·018	·009	
·163	·320	·458	·574	·662	·720	·746	·741	·707	·650	·571	·475	·367	·250	·125	
·008	·016	·025	·030	·033	·036	·037	·036	·034	·031	·026	·021	·017	·011	·006	
·109	·211	·304	·381	·439	·478	·495	·493	·472	·433	·381	·319	·246	·168	·085	
·005	·008	·012	·015	·016	·017	·018	·018	·017	·015	·013	·010	·008	·005	·003	
·054	·105	·151	·190	·219	·238	·247	·246	·235	·216	·191	·159	·123	·084	·043	

℄

8N

If the ζ values are -ve, the w values are +ve, and vice versa. Multiply the ζ units by N^2 to get w units.

$\nabla^4 w = 0$

10 ℄

194	552	1.737	2.681	1.761	.604	286	160	.099	.065	.044	.030	.020	013	.006
.354	.747	1.227	1.532	1.375	1.049	.823	657	527	423	334	.256	.187	.122	.060
.276	.665	1.154	1.414	1.197	.759	.438	.268	.174	.117	.080	.056	.037	.023	.012
.593	1.190	1.731	2.039	2.009	1.756	1.463	1.206	986	.799	.637	.491	.356	234	116
.255	.529	.777	.898	832	.647	.455	.311	215	151	.106	.075	.051	.032	.015
690	1.337	1.868	2.192	2.251	2.105	1.857	1.588	1.332	1.097	882	685	503	.329	.163
.200	.389	.541	.617	.602	.517	.408	306	.226	.166	.121	.086	.060	.038	.018
.691	1.322	1.834	2.161	2.279	2.219	2.040	1.803	1.550	1.297	1.056	830	.612	.402	.199
.148	.281	.385	442	.446	.407	.344	.277	.216	.165	.125	.091	.063	.040	.020
.644	1.232	1.708	2.033	2.187	2.189	2.072	1.882	1.653	1.409	1.162	920	683	.452	.224
.108	.205	.279	.325	.336	.320	.284	241	.196	.156	.119	.090	.063	041	.020
.579	1.108	1.545	1.857	2.029	2.070	2.003	1.858	1.662	1.439	1.201	.959	.718	.476	.237
.079	.150	.206	.243	.256	.251	.231	.203	.171	.140	110	.085	.060	.039	.019
.508	.976	1.367	1.657	1.834	1.898	1.869	1.763	1.601	1.404	1.184	.954	.716	.478	.239
.059	.111	.154	.182	.197	.197	.187	.168	.146	122	.098	.076	.055	.036	.018
.438	.844	1.189	1.452	1.623	1.701	1.695	1.618	1.488	1.318	1.122	.910	688	.461	.231
.044	083	.115	.139	.151	.154	.149	.137	.121	.103	.084	.067	.049	.032	.016
.371	.717	1.015	1.249	1.405	1.487	1.497	1.444	1.341	1.198	1.028	840	.637	.428	.215
.033	.062	.087	105	.116	.119	117	.110	.099	.085	.071	.057	.041	.028	.014
309	598	.850	1.050	1.190	1.269	1.287	1.251	1.170	1.053	909	.746	.570	383	.193
.024	046	.064	.079	.088	091	.091	086	.078	.068	.058	.046	.034	.023	.012
.250	.487	694	869	979	1.049	1.070	1.048	.986	893	.775	639	.489	.331	.166
.017	.034	.047	.058	.066	.069	.069	.065	060	.053	045	.036	.027	.019	.009
.195	.381	545	677	.774	833	855	839	.794	.721	.628	520	.399	.270	.136
.012	.024	.033	.041	.046	.048	.049	.047	.043	.039	.033	.027	.020	.014	.007
.144	.280	.403	.502	.576	.621	.638	.629	.598	.545	.477	.395	.305	.205	104
.008	.015	.021	.026	.029	.031	.031	.030	028	.025	.021	.018	.014	.009	005
095	185	265	.331	.380	.411	.424	.419	.399	.364	.320	264	.204	.138	.070
.004	.008	.010	.012	.014	015	.015	.015	.014	.013	.011	.008	.006	.005	.003
.047	.092	.132	.164	.189	.205	.211	209	.199	.182	.160	.132	.102	.070	.035

℄

8N ℄

If the ζ values are −ve, the w values are +ve, and vice versa.
Multiply the ζ units by N^2 to get w units.

$\nabla^4 w = 0$

10 ℄

.453	1.681	2.643	1.737	.587	.274	.151	.092	.060	.041	.028	.019	.013	.009	.004
.470	1.005	1.352	1.227	.929	.725	.579	.468	.379	.306	.242	.186	.136	.088	.044
.507	1.056	1.350	1.153	.728	.416	.253	.163	.109	.075	.053	.037	.026	.016	.008
.700	1.325	1.704	1.730	1.527	1.276	1.056	.872	.715	.582	.464	.358	.262	.171	.084
.358	.662	.818	.777	.607	.426	.290	.200	.140	.100	.072	.051	.035	.022	.010
.727	1.342	1.741	1.871	1.788	1.597	1.379	1.170	.979	.805	.647	.504	.370	.243	.120
.236	.427	.534	.541	.471	.374	.282	.208	.155	.114	.084	.060	.041	.026	.013
.670	1.244	1.642	1.833	1.840	1.726	1.550	1.353	1.154	.964	.784	.614	.453	.298	.148
.156	.284	.363	.386	.360	.309	.251	.197	.152	.115	.087	.065	.045	.028	.014
.593	1.108	1.489	1.709	1.777	1.725	1.598	1.431	1.245	1.055	.868	.686	.508	.337	.168
.106	.195	.254	.280	.275	.250	.214	.177	.142	.111	.085	.064	.046	.029	.014
.512	.966	1.318	1.545	1.646	1.643	1.561	1.428	1.266	1.088	.904	.719	.537	.356	.178
.074	.137	.183	.207	.209	.199	.177	.152	.126	.103	.080	.062	.045	.028	.014
.438	.831	1.148	1.367	1.485	1.511	1.465	1.364	1.227	1.069	.897	.720	.541	.360	.180
.053	.098	.133	.154	.161	.157	.145	.128	.109	.091	.073	.056	.042	.027	.014
.370	.707	.986	1.188	1.309	1.354	1.332	1.259	1.148	1.011	.856	.692	.522	.350	.175 ℄
.038	.071	.097	.115	.123	.123	.116	.105	.092	.078	.064	.049	.036	.024	.013
.309	.593	.832	1.013	1.130	1.184	1.178	1.128	1.038	.924	.788	.641	.486	.326	.164
.028	.052	.071	.086	.094	.096	.092	.085	.077	.065	.056	.043	.032	.021	.011
.254	.490	.692	.848	.953	1.007	1.014	.978	.910	.815	.701	.573	.436	.293	.147
.020	.039	.052	.064	.070	.072	.071	.067	.060	.053	.044	.036	.026	.017	.009
.205	.396	.560	.692	.783	.832	.844	.821	.768	.693	.600	.493	.376	.254	.127
.014	.027	.038	.047	.051	.055	.053	.051	.047	.041	.036	.029	.022	.014	.007
.159	.308	.438	.543	.617	.660	.673	.659	.620	.561	.488	.403	.308	.208	.105
.010	.019	.026	.033	.035	.038	.038	.037	.033	.030	.026	.021	.016	.011	.005
.117	.227	.323	.401	.457	.491	.503	.494	.466	.425	.370	.306	.235	.160	.081
.007	.012	.017	.021	.022	.024	.024	.023	.021	.019	.017	.013	.011	.007	.004
.076	.148	.213	.264	.303	.325	.334	.329	.312	.284	.249	.206	.159	.107	.054
.003	.006	.008	.010	.012	.013	.013	.012	.011	.010	.009	.007	.005	.004	.002
.038	.073	.106	.132	.151	.162	.167	.165	.156	.142	.125	.104	.080	.054	.027

℄

8N

If the ζ values are −ve, the w values are +ve, and vice versa.
Multiply the ζ units by N^2 to get w units.

Field Nº 8

$\nabla^4 w = 0$

10

							℄							
1·487	2·546	1·680	·552	·251	·136	·082	·052	·036	·025	·019	·012	008	004	·002
·650	1·073	1·005	·746	·578	·459	370	·302	·246	·199	158	·122	090	058	[illegible]
·780	1·191	1·056	·665	·375	222	·142	094	066	·046	·033	·023	·015	009	005
·733	1·213	1·325	1·191	·998	·827	·686	·566	·467	·379	·303	·234	·171	·112	·056
·406	·648	662	·528	·370	250	172	·119	·085	060	·044	·031	·021	·013	·006
·655	1·129	1·342	1·336	1·217	1·063	·910	·769	·642	·529	·424	·330	241	·159	·079
·228	·381	·428	·388	313	·236	·176	·129	096	·070	·051	·037	·026	016	·007
·555	·991	1·245	1·323	1·280	1·172	1·038	·900	765	636	·516	·404	·297	·196	·097
138	·238	·284	281	·249	·205	162	·125	096	·073	055	·041	·028	·018	008
·465	·851	1·109	1·232	1·246	1·185	1·083	·960	832	·701	·575	454	·336	222	·110
·088	156	·195	·207	·193	170	·142	·116	·092	·071	055	·040	·029	019	·009
·389	·723	·968	1·110	1·159	1·139	1·066	·965	850	·727	603	·479	357	·237	·118
·059	·107	·136	·151	·149	·137	·120	·101	·082	·067	·053	039	·029	·018	·009
325	612	833	·978	1·047	1·051	1·007	·929	831	·718	·602	482	·361	·240	·120
·040	074	·098	·111	·115	·109	·100	·086	·073	060	·047	·036	·027	017	008
·271	·514	·709	·847	922	·943	·920	·862	·781	683	·577	467	350	·234	·117
·029	·055	·071	084	·089	·087	·080	·072	·062	·052	042	·034	·025	·016	008
224	429	598	·721	·797	826	817	·775	·710	·627	534	·433	·328	·220	·110
·022	·040	·053	·063	·068	·067	·065	·059	·052	044	·036	·028	·021	·014	·007
·184	·352	·494	603	·672	·707	·704	·675	·624	·559	·476	·389	·296	·198	·100
017	·029	039	046	·050	·051	·051	·046	·042	036	·031	024	·018	·013	·006
·147	·284	·400	·491	·551	583	·587	·568	·529	·474	·409	·335	·255	·172	·086
·011	·021	029	·034	038	040	·037	·035	·032	·028	·023	·020	014	·010	·005
·114	·220	·311	·384	·435	·463	·469	·457	428	·386	·334	·274	·209	·142	·071
·007	·015	·020	·024	·025	·027	·027	·026	·024	·020	·017	·015	·011	·008	·004
·084	·162	230	284	·322	·345	·351	·343	·323	·292	·253	·209	·160	110	·056
·005	·009	·012	·015	·018	·018	·017	016	·015	·013	·012	009	·008	005	·003
·056	·106	·151	·187	·213	·227	·233	·228	·216	·195	·170	·140	·108	·075	·038
003	·005	·006	·007	·008	009	·009	·008	008	·007	·006	·005	·004	·003	·002
·028	·053	·076	093	·106	114	·116	·114	·108	·097	085	·070	·055	·038	·019

℄ 8N

If the ζ values are −ve, the w values are +ve, and vice versa.
Multiply the ζ units by N^2 to get w units.

Field No 9

$\nabla^4 w = 0$

10

							℄							
2.094	1.487	0.452	0.193	0.100	·058	035	·024	·016	·012	008	006	004	002	001
·604	650	·470	·354	·277	222	·180	·148	·121	·099	·079	061	044	030	016
·685	·180	·306	·275	·158	·097	·063	·043	·031	·021	·016	·012	008	·005	·002
512	·733	697	592	·491	·406	·337	·280	·231	·188	151	·119	·086	·057	·029
·290	·406	·357	255	·171	·115	080	·056	·041	·029	·021	·015	·010	·007	·003
·403	653	·725	·689	611	·529	·451	·381	·319	·263	·212	166	·122	081	·040
·145	228	236	·199	·153	·113	082	·060	·046	·034	027	019	·013	008	·004
·320	555	670	·690	652	·591	519	·449	382	·319	·260	·204	·151	101	050
·082	·136	·155	·147	·126	·100	·079	·061	·047	035	·027	·019	·015	·010	005
·258	·466	591	644	·641	602	·547	483	·418	·352	·290	·229	·1[illegible]0	·113	·056
050	086	·105	108	099	·085	070	·057	·045	·036	027	·020	015	·010	005
·211	·388	513	579	597	581	542	489	·430	369	·305	·243	·182	121	·061
·032	058	073	·079	077	·069	061	051	·042	·034	·026	020	·015	·010	·004
·173	·324	·437	·510	540	539	514	·473	·423	364	306	·246	183	123	061
·022	040	051	058	060	055	051	044	·037	030	024	019	014	010	004
·142	270	371	439	·476	485	471	·441	·398	349	395	238	·180	121	·060
015	·030	037	043	046	044	042	·037	032	026	·021	017	·013	·009	004
·117	·224	·310	·374	412	·426	·420	398	·364	·322	·274	·223	169	·113	·056
·011	·021	·026	·032	034	·035	·033	030	026	·024	019	015	010	·007	004
·096	·183	·257	311	347	·364	·363	·347	·322	287	·245	·200	153	103	052
·008	·015	020	024	·026	·026	·026	·025	·022	·019	015	012	009	006	·003
076	147	·207	254	286	302	304	294	·274	245	·212	·173	132	090	045
·005	012	·014	·018	·020	·021	·020	019	017	·015	012	011	·008	006	·003
·059	116	·161	199	·225	·240	·243	237	·222	·200	·173	·142	·109	·075	037
·004	·008	·010	·012	013	·014	·014	·014	012	·012	·009	008	006	·004	·002
·043	·084	120	148	·167	·180	182	179	·168	·153	·132	·109	085	057	·028
·002	005	007	008	·008	009	009	009	·009	·008	006	006	·004	002	·001
·029	·055	·078	·099	·111	·119	·122	·120	·113	·103	·089	·075	·057	·038	·020
·001	·002	·003	·004	005	·005	·005	004	·004	·004	·003	·003	·002	·001	001
·014	·028	·039	049	·056	·060	·061	·060	·057	·052	·045	·037	·028	·020	·010

℄ (right edge, at row ·022 / ·142)

℄

8N

If the ζ values are -ve, the w values are +ve, and vice versa.
Multiply the ζ units by N^2 to get w units.

Field No. 10

$\nabla^4 w = 0$

·928 / ·223														
·235 / ·156	·217 / ·201													
·089 / ·114	·115 / ·177	·093 / ·188												
·042 / ·086	·063 / ·147	·062 / ·174	·051 / ·175											
·023 / ·068	·037 / ·121	·042 / ·153	·038 / ·163	·032 / ·160										
·014 / ·055	·024 / ·100	·028 / ·131	·028 / ·146	·025 / ·148	·021 / ·141									
·009 / ·044	·016 / ·082	·020 / ·110	·020 / ·125	·019 / ·131	·017 / ·129	·014 / ·121								
·006 / ·036	·011 / ·067	·014 / ·091	·015 / ·107	·014 / ·114	·013 / ·115	·012 / ·111	·010 / ·102							
·004 / ·029	·007 / ·055	·010 / ·075	·011 / ·089	·011 / ·097	·010 / ·099	·009 / ·096	·008 / ·090	·006 / ·081						
·003 / ·023	·005 / ·044	·007 / ·061	·008 / ·073	·008 / ·080	·007 / ·083	·007 / ·081	·006 / ·077	·005 / ·070	·004 / ·061					
·002 / ·018	·004 / ·035	·005 / ·048	·005 / ·058	·005 / ·064	·005 / ·067	·005 / ·066	·004 / ·062	·003 / ·057	·003 / ·051	·002 / ·043				
·001 / ·014	·002 / ·026	·003 / ·037	·003 / ·045	·004 / ·050	·004 / ·052	·003 / ·051	·003 / ·050	·002 / ·046	·002 / ·041	·001 / ·035	·001 / ·027			
·001 / ·010	·001 / ·019	·002 / ·027	·002 / ·033	·002 / ·037	·002 / ·038	·002 / ·038	·001 / ·037	·001 / ·034	·001 / ·031	·001 / ·026	·020	·015		
·006	·012	·001 / ·018	·001 / ·021	·001 / ·024	·001 / ·025	·001 / ·025	·001 / ·024	·001 / ·023	·020	·017	·013	·010	·006	
·003	·006	·008	·010	·012	·012	·012	·012	·011	·010	·008	·007	·005	·003	·001

8 N

If the ζ values are −ve, the w values are +ve, and vice versa.
Multiply the ζ units by N^2 to get w units.

Field No 11

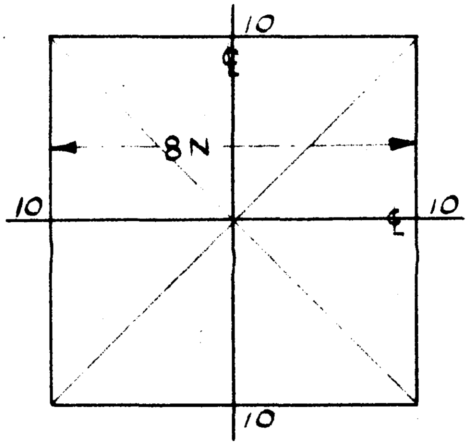
$\frac{\partial w}{\partial y}$ coefficients

$\frac{\partial w}{\partial y}$	0.82	1.63	2.39	3.13	3.68	4.80	6.49	7.63	
	·055	·111	·172	·253	·393	·721	1·887	2·817	ζ values
	·410	·807	1·183	1·541	1·893	2·268	2·709	2·952	w values
		·220	·330	·462	·655	·994	1·448	1·684	
		1·581	2·303	2·969	3·589	4·168	4·642	4·836	
			·473	·617	·789	1·002	1·206	1·295	
			3·321	4·221	5·005	5·656	6·112	6·279	
				·743	·866	·991	1·093	1·134	
				5·281	6·154	6·827	7·263	7·416	
					·933	·997	1·047	1·066	
					7·061	7·732	8·151	8·294	
						1·016	1·035	1·043	
						8·389	8·791	8·927	
							1·036	1·037	
							9·180	9·311	
								1·037	
								9·440	

10 ℄

$-\nabla^4 w = 0-$

Key Diagram



Field No 12

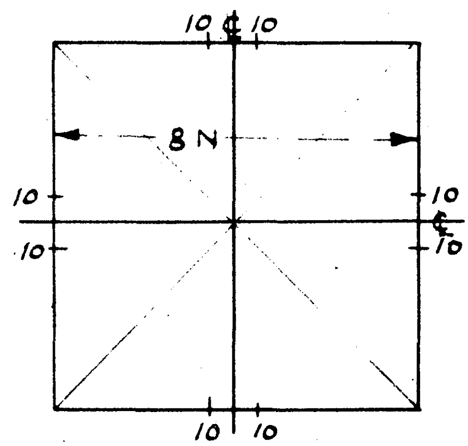
$\frac{\partial w}{\partial y}$ coefficients

$\frac{\partial w}{\partial y}$	1.66	3.28	4.82	6.33	7.96	10.37	12.43	12.96	
	·121	·242	·379	·577	·981	2·283	3·539	3·774	ζ values
	·828	1·627	2·382	3·103	3·823	4·605	5·216	5·411	w values
		·476	·713	1·008	1·466	2·106	2·675	2·892	
		3·181	4·621	5·944	7·163	8·233	8·991	9·265	
			1·004	1·296	1·632	1·995	2·289	2·403	
			6·634	8·397	9·904	11·108	11·904	12·184	
				1·520	1·739	1·948	2·06	2·165	
				10·451	12·121	13·384	14·182	14·457	
					1·847	1·951	2·031	2·060	
					13·864	15·140	15·927	16·194	
						1·980	2·009	2·022	
						16·407	17·176	17·435	
							2·010	2·012	
							17·931	18·184	
								2·011	
								18·435	

10 ℄

$-\nabla^4 w = 0-$

Key Diagram



If the ζ values are -ve, the w values are +ve, and vice versa.
Multiply the ζ units by N^2 to get w units

Field No 13

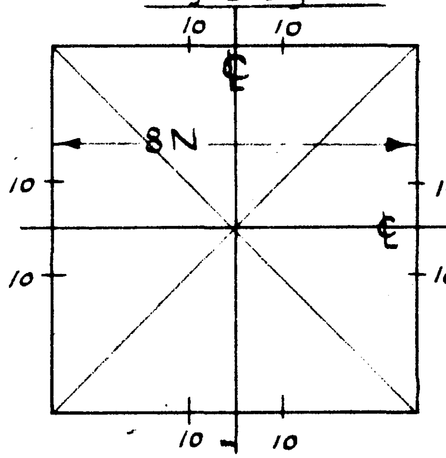
$\partial w/\partial y$ coefficients

							℄	
1.72	3.39	5.00	6.68	9.04	10.84	10.40	9.62	$\partial w/\partial y$ coefficients
.153	310	.506	877	2.087	3.083	2.283	1.442	ζ values
.855	1.674	2.447	3.195	3.971	4.533	4.615	4.540	w values
	598	895	1.305	1.828	2.163	2.103	1.981	
	3.253	4.689	5.981	7.088	7.868	8.238	8.325	
		1.190	1.469	1.739	1.922	1.978	1.979	
		6.637	8.282	9.604	10.547	11.084	11.252	
			1.611	1.748	1.852	1.907	1.923	
			10.167	11.626	12.676	13.299	13.504	
				1.795	1.844	1.877	1.887	
				13.190	14.306	14.972	15.194	
					1.855	1.868	1.873	
					15.460	16.149	16.378	
						1.868	1.869	
						16.850	17.082	
							1.869	
							17.216	

10 ℄ ℄

$-\nabla^4 w = 0-$

Key Diagram



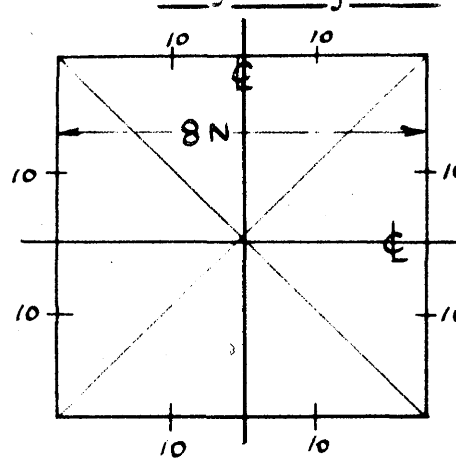
Field No 14

							℄	
1.83	3.62	5.38	7.78	9.57	9.02	7.96	6.76	$\partial w/\partial y$ coefficients
.230	.471	.850	2.038	2.985	2.084	.978	.783	ζ values
.904	1.757	2.556	3.353	3.909	3.961	3.822	3.780	w values
	.867	1.279	1.752	2.005	1.812	1.449	1.292	
	3.358	4.749	5.898	6.685	7.046	7.126	7.127	
		1.483	1.659	1.746	1.694	1.576	1.518	
		6.518	7.907	8.897	9.491	9.771	9.847	
			1.664	1.677	1.657	1.617	1.597	
			9.481	10.639	11.402	11.821	11.951	
				1.659	1.645	1.630	1.623	
				11.942	12.831	13.342	13.508	
					1.639	1.634	1.632	
					13.815	14.390	14.578	
						1.635	1.634	
						15.002	15.202	
							1.634	
							15.408	

10 ℄ ℄

$-\nabla^4 w = 0-$

Key Diagram



If the ζ values are -ve, the w values are +ve, and vice versa.
Multiply the ζ units by N^2 to get w units.

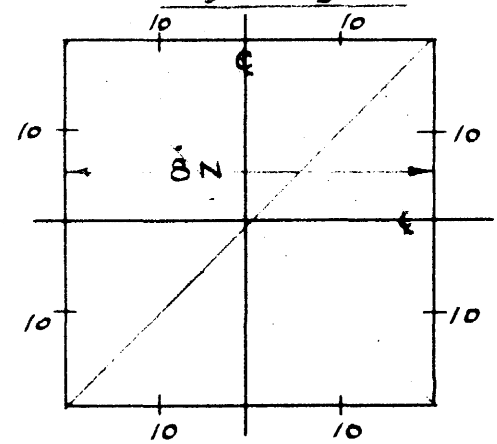
Field No. 15

$\partial w/\partial y$ coefficients

								℄	
$\partial w/\partial y$ coefficients	2·20	4·30	6·55	8·36	7·78	6·65	6·32	6·25	
			10						
ζ	·410	·875	2·064	2·977	2·033	·870	·571	·502	ζ values
w	·994	1·904	2·748	3·309	3·345	3·178	3·095	3·071	w values
ζ		1·424	1·823	1·991	1·720	1·269	·977	·893	
w		3·497	4·712	5·504	5·845	5·899	5·872	5·859	
ζ			1·763	1·716	1·567	1·358	1·194	1·136	
w			6·158	7·161	7·757	8·047	8·161	8·190	
ζ				1·596	1·491	1·381	1·295	1·262	
w				8·374	9·192	9·685	9·936	10·012	
ζ					1·433	1·377	1·334	1·317	
w					10·210	10·870	11·234	11·350	
ζ						1·361	1·345	1·338	
w						11·661	12·110	12·255	
ζ							1·344	1·343	
w							12·612	12·775	
ζ								1·343	℄
w								12·945	

— $\nabla^4 w = 0$ —

Key Diagram

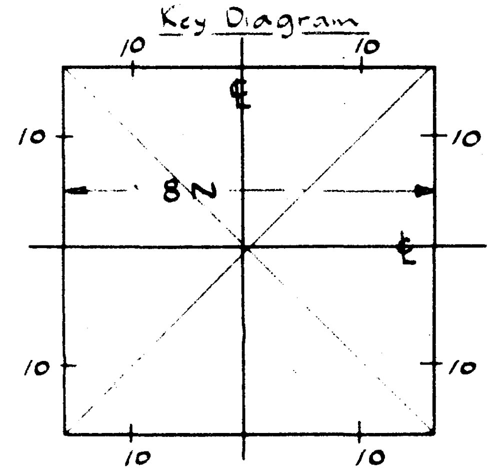


Field No. 16

								℄	
$\partial w/\partial y$ coefficients	2·32	5·24	7·15	6·55	5·36	4·97	4·82	4·77	
			10						
ζ	·922	2·222	3·056	2·044	·835	·495	·372	·340	ζ values
w	1·158	2·137	2·716	2·731	2·538	2·427	2·372	2·355	w values
ζ		2·181	2·116	1·721	1·191	·835	·671	·624	
w		3·505	4·307	4·619	4·636	4·576	4·531	4·516	
ζ			1·793	1·517	1·230	·999	·865	·822	
w			5·337	5·927	6·199	6·303	6·337	6·345	
ζ				1·351	1·192	1·057	·969	·940	
w				6·785	7·309	7·600	7·741	7·783	
ζ					1·126	1·061	1·014	·997	
w					8·053	8·513	8·759	8·836	
ζ						1·044	1·026	1·019	
w						9·106	9·436	9·541	
ζ							1·026	1·024	
w							9·819	9·942	
ζ								1·025	℄
w								10·072	

— $\nabla^4 w = 0$ —

Key Diagram



If the ζ values are −ve, the w values are +ve, and vice versa.
Multiply the ζ units by N^2 to get w units.

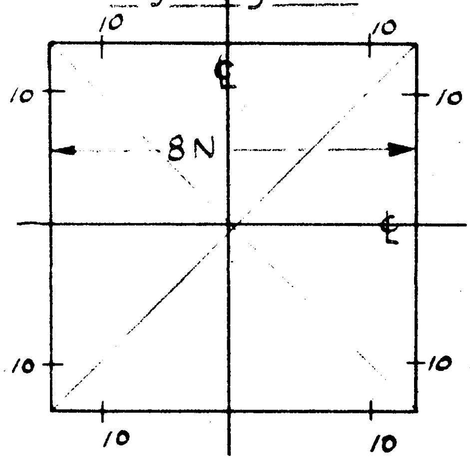
$\partial w/\partial y$ coefficients

	3·93	[illegible]	[illegible]	4·03	3·59	3·38	[illegible]	[illegible] (℄)	
	10							℄	
	2·986	3·350	2·123	·827	·448	·300	·237	·219	ζ values
	1·449	2·102	2·094	1·870	1·736	1·659	1·619	1·607	w values
		2·429	1·774	1·140	·731	·527	·433	·405	
		3·009	3·314	3·302	3·212	3·141	3·100	3·086	
			1·428	1·094	·828	·658	·569	·542	
			3·950	4·225	4·318	4·341	4·344	4·343	
				·959	·819	·711	·647	·626	
				4·773	5·076	5·235	5·311	5·333	
					·769	·717	·681	·669	
					5·563	5·855	6·008	6·056	
						·705	·691	·685	
						6·250	6·467	6·536	
							·690	·689	
							6·726	6·808	
								·690	
								6·895	℄

Field No. 17

— $\nabla^4 w = 0$ —

Key Diagram



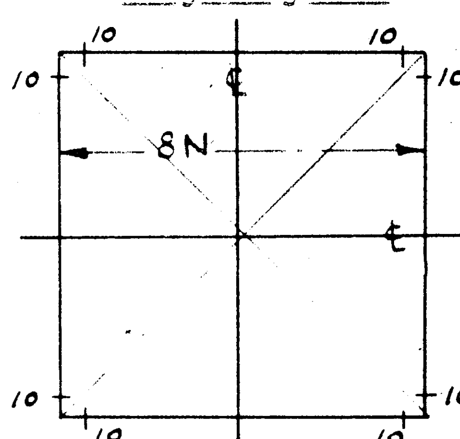
$\partial w/\partial y$ coefficients

	4·83	3·86	2·51	2·05	1·84	1·74	1·68	1·66 (℄)	
	10							℄	
	4·194	2·185	·761	·363	·211	·146	·117	·109	ζ values
	1·285	1·315	1·097	·967	·892	·850	·828	·822	w values
		1·585	·949	·552	·356	·260	·216	·202	
		1·765	1·795	1·721	1·655	1·610	1·586	1·579	
			·768	·561	·416	·329	·286	·272	
			2·101	2·203	2·226	2·227	2·223	2·222	
				·491	·415	·359	·326	·316	
				2·470	2·612	2·685	2·719	2·729	
					·390	·363	·345	·338	
					2·857	2·999	3·075	3·098	
						·357	·350	·347	
						3·200	3·310	3·344	
							·351	·350	
							3·440	3·483	
								·350	
								3·527	℄

Field No. 18

— $\nabla^4 w = 0$ —

Key Diagram



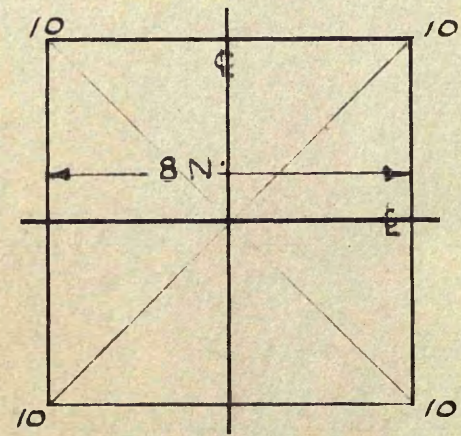
If the ζ values are −ve, the w values are +ve, and vice versa.
Multiply the ζ units by N^2 to get w units.

Field No. 19

							℄	
·930	·235	090	·044	·025	·018	·014	·012	ζ values
·231	·172	·137	118	·106	100	·096	095	w values
	·218	·117	·067	·043	·030	·024	024	
	·231	·223	·208	·196	·189	·185	·184	
		096	·068	·049	·039	·032	·030	
		·256	264	·263	·260	·258	·257	
			·059	·048	·041	·036	·035	
			·293	·306	·312	·315	·315	
				·045	·041	·038	·037	
				·333	·347	·354	·356	
					·039	·038	·038	
					·368	·380	·384	
						·038	·038	
						·394	·399	
							·038	
							·404	

— $\nabla^4 w = 0$ —

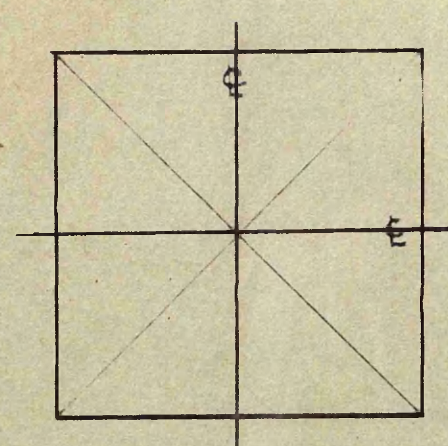
Key Diagram



If the ζ values are −ve the w values are +ve, and vice versa
Multiply the ζ units by N^2 to get w units

Field No. 20

10	10	10	10	10	10	10	℄ 10
10·001	10·001	10·001	10·0	9·999	10·0	9·998	9·998
8·114	13·495	17·360	20·187	22·213	23·581	24·372	24·633
8·116	13·496	17·364	20·192	22·220	23·588	24·380	24·640
	9·998	9·996	9·998	9·995	9·996	9·996	9·997
	23·380	30·713	36·146	40·068	42·730	44·271	44·777
	23·384	30·720	36·156	40·084	42·744	44·284	44·788
		9·998	9·997	9·996	9·996	9·996	9·997
		40·912	48·587	54·173	57·980	60·194	60·919
		40·920	48·596	54·188	57·996	60·212	60·936
			9·994	9·995	9·997	9·996	9·998
			58·065	65·035	69·806	72·587	73·500
			58·084	65·056	69·824	72·608	73·520
				9·997	9·996	9·997	9·994
				73·073	78·593	81·822	82·886
				73·092	78·620	81·848	82·912
					9·996	9·996	9·997
					84·656	88·209	89·378
					84·684	88·236	89·408
						9·998	9·996
						91·954	93·186
						91·984	93·216
							9·997
							94·443
							94·472



Field No. 21

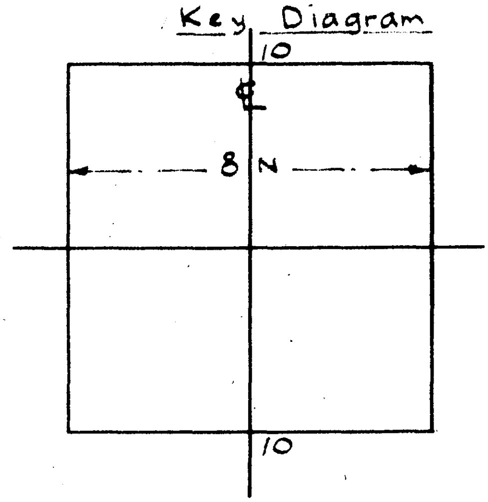
℄ 10

·028	·060	·104	·175	·309	·636	1·802	2·732	ζ values
·205	·414	·631	·865	1·128	1·444	1·852	2·085	w values
·051	·110	·186	·299	·485	·822	1·277	1·513	
·393	·791	1·199	1·623	2·071	2·538	2·949	3·124	
·068	·144	·236	·358	·526	·741	·949	1·040	
·552	1·105	1·660	2·215	2·759	3·257	3·628	3·768	
·079	·163	·259	·372	·503	·641	·753	·797	
·676	1·347	2·006	2·641	3·228	3·722	4·061	4·183	
·084	·171	·264	·363	·466	·562	·632	·658	
·765	1·519	2·246	2·927	3·532	4·017	4·335	4·447	
·085	·172	·260	·350	·435	·508	·558	·576	
·824	1·631	2·400	3·107	3·719	4·196	4·502	4·608	
·085	·171	·257	·340	·415	·477	·518	·532	
·867	1·693	2·485	3·204	3·818	4·291	4·591	4·693	
·085	·171	·255	·336	·408	·467	·505	·518	
·867	1·713	2·512	3·235	3·849	4·321	4·619	4·721	

℄

— $\nabla^4 w = 0$ —

Key Diagram



Field No. 22

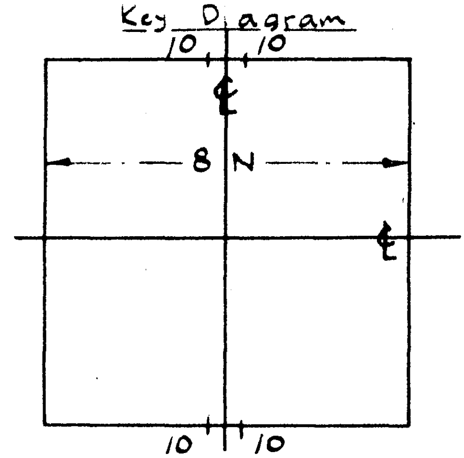
10 ℄

·061	·132	·234	·413	·810	2·110	3·367	3·603	ζ values
·414	·836	1·278	1·757	2·306	2·977	3·526	3·699	w values
·112	·238	·408	·670	1·119	1·760	2·333	2·552	
·790	1·590	2·410	3·265	4·154	5·013	5·653	5·888	
·145	·305	·502	·761	1·097	1·473	1·779	1·897	
1·103	2·210	3·317	4·413	5·464	6·379	7·016	7·244	
·163	·338	·534	·760	1·010	1·254	1·436	1·504	
1·345	2·678	3·984	5·225	6·352	7·276	7·890	8·107	
·171	·348	·534	·729	·923	1·097	1·217	1·261	
1·517	3·008	4·439	5·769	6·932	7·853	8·446	8·655	
·172	·347	·522	·694	·856	·991	1·081	1·113	
1·628	3·220	4·729	6·107	7·287	8·203	8·785	8·986	
·171	·342	·510	·671	·814	·930	1·006	1·033	
1·690	3·338	4·888	6·292	7·480	8·390	8·965	9·161	
·170	·340	·506	·662	·801	·911	·982	1·007	
1·711	3·376	4·940	6·349	7·539	8·448	9·023	9·217	

℄

— $\nabla^4 w = 0$ —

Key Diagram



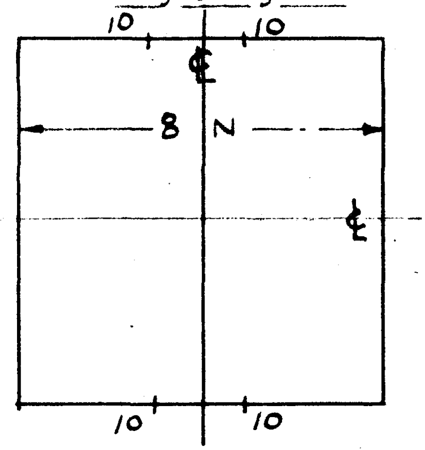
If the ζ values are −ve, the w values are +ve, and vice versa.
Multiply the ζ units by N^2 to get w units

Field No. 23

·076	·175	·338	·696	1·907	2·908	2·111	1·271	ζ values
·427	·867	1·336	1·861	2·486	2·953	2·983	2·892	w values
·135	·299	·537	·933	1·464	1·811	1·763	1·644	
·806	1·626	2·468	3·334	4·152	4·752	5·025	5·080	
·169	·358	·595	·886	1·188	1·399	1·476	1·484	
1·111	2·220	3·318	4·370	5·299	5·993	6·396	6·524	
·181	·372	·584	·805	1·013	1·170	1·258	1·283	
1·333	2·646	3·912	5·078	6·078	6·837	7·300	7·456	
·182	·365	·552	·734	·897	1·025	1·102	1·128	
1·484	2·935	4·305	5·548	6·595	7·388	7·882	8·049	
·178	·352	·523	·682	·8[illegible]9	·928	·994	1·019	
1·580	3·115	4·554	5·839	6·918	7·730	8·237	8·407	
·173	·342	·503	·649	·775	·874	·934	·953	
1·631	3·212	4·687	5·998	7·091	7·912	8·426	8·599	
·171	·338	·494	·639	·758	·853	·915	·935	
1·648	3·244	4·728	6·048	7·145	7·971	8·483	8·658	

$\nabla^4 w = 0$

Key Diagram

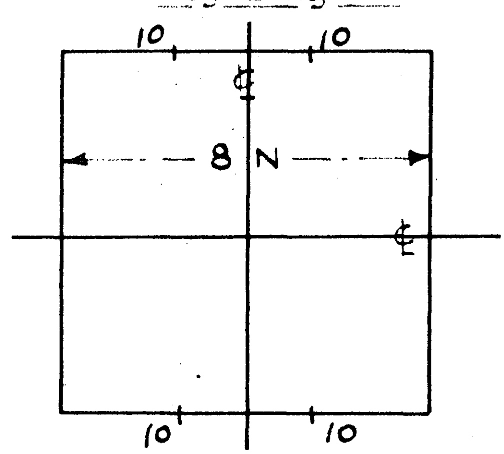


Field No. 24

·115	·282	·636	1·829	2·791	1·906	·811	·619	ζ values
·452	·923	1·445	2·058	2·500	2·485	2·311	2·258	w values
·189	·433	·821	1·328	1·623	1·463	1·121	·970	
·834	1·679	2·537	3·345	3·916	4·149	4·163	4·141	
·215	·457	·741	1·019	1·187	1·187	1·102	1·053	
1·111	2·212	3·259	4·183	4·878	5·292	5·476	5·521	
·209	·424	·641	·832	·960	1·013	1·012	1·004	
1·295	2·553	3·724	4·740	5·534	6·072	6·368	6·457	
·194	·382	·562	·716	·829	·895	·926	·933	
1·410	2·769	4·019	5·105	5·971	6·586	6·949	7·067	
·178	·348	·507	·644	·749	·819	·858	·869	
1·477	2·896	4·198	5·330	6·244	6·907	7·310	7·440	
·167	·328	·474	·604	·704	·775	·817	·833	
1·510	2·962	4·294	5·453	6·392	7·079	7·501	7·637	
·163	·322	·465	·592	·689	·763	·801	·817	
1·524	2·985	4·325	5·493	6·441	7·138	7·565	7·704	

$\nabla^4 w = 0$

Key Diagram



If the ζ values are −ve, the w values are +ve, and vice versa.
Multiply the ζ units by N^2 to get w units

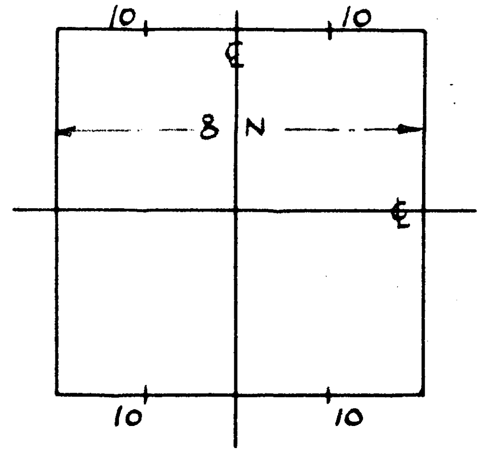
Field No 25

10 ℄

·205	·576	1·774	2·732	1·830	·696	·413	·349	ζ values
·497	1·030	1·647	2·086	2·059	1·859	1·761	1·732	w values
·299	·712	1·225	1·512	1·328	·932	·672	·598	
·873	1·748	2·555	3·127	3·344	3·332	3·272	3·249	
·290	·598	·881	1·041	1·019	·886	·763	·717	
1·100	2·156	3·079	3·772	4·184	4·366	4·425	4·435	
·245	·478	·675	·798	·833	·804	·762	·744	
1·223	2·377	3·388	4·187	4·738	5·071	5·237	5·286	
·204	·391	·548	·657	·716	·733	·731	·727	
1·285	2·500	3·572	4·453	5·105	5·540	5·782	5·860	
·174	·336	·471	·576	·644	·680	·696	·701	
1·318	2·567	3·681	4·613	5·329	5·830	6·122	6·217	
·157	·305	·431	·533	·603	·648	·672	·679	
1·334	2·599	3·735	4·698	5·452	5·987	6·306	6·411	
·153	·296	·419	·518	·591	·637	·665	·671	
1·338	2·609	3·755	4·726	5·490	6·037	6·364	6·472	℄

$\nabla^4 w = 0$

Key Diagram



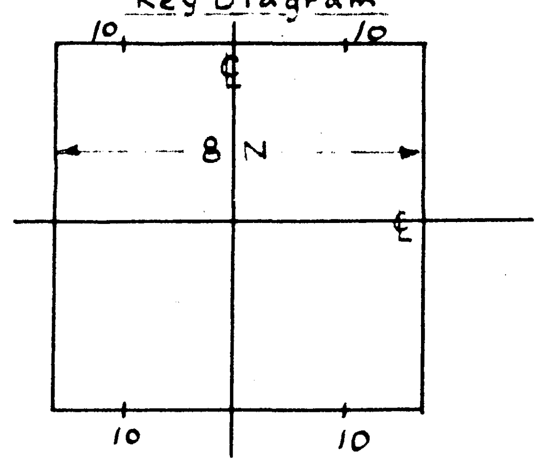
Field No. 26

10 ℄

·461	1·698	2·671	1·774	·636	·337	·234	·207	ζ values
·579	1·221	1·672	1·649	1·445	1·335	1·281	1·265	w values
·524	1·090	1·403	1·225	·821	·536	·408	·373	
·915	1·752	2·336	2·559	2·542	2·469	2·419	2·402	
·384	·713	·896	·882	·742	·595	·502	·473	
1·043	1·970	2·668	3·084	3·264	3·319	3·328	3·328	
·271	·495	·635	·675	·641	·584	·537	·519	
1·082	2·059	2·843	3·392	3·729	3·913	3·998	4·023	
·199	·369	·488	·552	·563	·550	·536	·529	
1·092	2·094	2·935	3·580	4·026	4·307	4·457	4·505	
·158	·299	·404	·473	·511	·522	·524	·523	
1·091	2·107	2·984	3·687	4·206	4·553	4·749	4·811	
·138	·262	·362	·433	·478	·502	·513	·515	
1·090	2·112	3·008	3·742	4·302	4·686	4·909	4·983	
·33	·251	·349	·421	·469	·495	·510	·512	
1·089	2·114	3·016	3·759	4·331	4·729	4·959	5·036	℄

$\nabla^4 w = 0$

Key Diagram



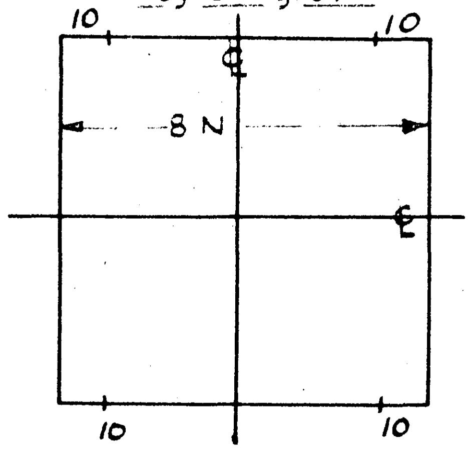
If the ζ values are −ve, the w values are +ve, and vice versa.
Multiply the ζ units by N^2 to get w units.

Field No. 27

10							℄	
1·493	2·558	1·699	·576	·282	·175	·133	·121	ζ values
·724	1·220	1·219	1·032	·927	·868	·839	·831	w values
·792	1·214	1·090	·712	·435	·298	·239	·221	
·881	1·504	1·755	1·752	1·683	1·629	1·600	1·589	
·424	·683	·714	·597	·457	·358	·308	·291	
·875	1·558	1·975	2·161	2·217	2·227	2·225	2·224	
·251	·428	·496	·479	·426	·373	·340	·330	
·838	1·549	2·064	2·386	2·564	2·655	2·698	2·714	
·167	·295	·370	·393	·384	·365	·349	·343	
·809	1·528	2·101	2·511	2·781	2·944	3·029	3·057	
·124	·228	·299	·337	·352	·352	·349	·348	
·790	1·511	2·113	2·579	2·911	3·125	3·244	3·282	
·103	·194	·261	·307	·331	·341	·345	·344	
·779	1·499	2·119	2·612	2·978	3·222	3·363	3·407	
·097	·183	·251	·296	·325	·336	·344	·344	
·775	1·497	2·118	2·628	2·999	3·254	3·401	3·447	℄

— $\nabla^4 w = 0$ —

Key Diagram

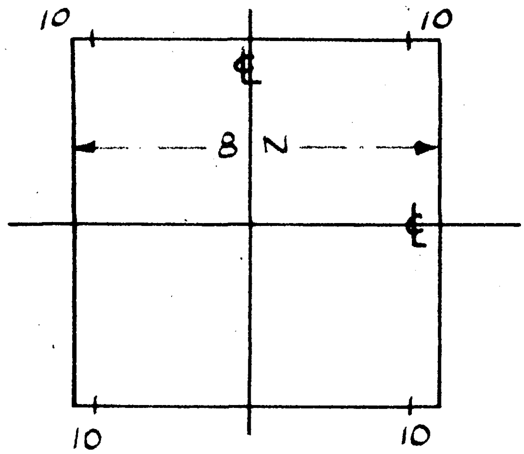


Field No. 28

10							℄	
2·097	1·493	·461	·206	·115	·077	·061	·057	ζ values
·642	·726	·682	·501	·456	·431	·420	·417	w values
·691	·792	·524	·299	·189	·135	·111	·103	
·588	·882	·920	·879	·840	·815	·802	·798	
·299	·424	·384	·290	·215	·171	·147	·141	
·515	·875	1·050	1·112	1·124	1·123	1·122	1·120	
·156	·252	·271	·245	·212	·183	·166	·160	
·466	·841	1·091	1·235	1·311	1·351	1·366	1·373	
·097	·168	·201	·202	·195	·182	·174	·171	
·435	·814	1·102	1·301	1·428	1·503	1·543	1·555	
·068	·124	·157	·175	·181	·178	·175	·175	
·418	·794	1·103	1·334	1·495	1·600	1·658	1·674	
·056	·106	·138	·160	·170	·174	·175	·175	
·408	·783	1·101	1·353	1·532	1·652	1·720	1·743	
·053	·099	·131	·156	·167	·171	·175	·175	
·405	·780	1·102	1·355	1·548	1·670	1·739	1·763	℄

— $\nabla^4 w = 0$ —

Key Diagram

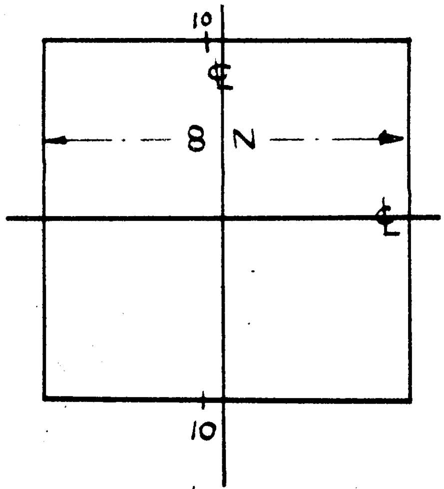


If the ζ values are −ve, the w values are +ve, and vice versa.
Multiply the ζ units by N^2 to get w units.

Field No 29

$\nabla^4 w = 0$

·040	·089	·163	·301	·629	1·798	2·730	1·802	·638	·313	·180	·113	·071	·043	·020
·240	·485	·745	1·033	1·371	1·803	2·060	1·850	1·465	1·173	·934	·725	·533	·352	·175
·072	·158	·277	·469	·809	1·268	1·508	1·275	·826	·493	·309	·202	·131	·080	·039
·453	·913	1·387	1·883	2·396	2·854	3·075	2·944	2·579	2·169	1·758	1·382	1·023	·677	·338
·092	·198	·327	·502	·725	·938	1·033	·948	·745	·534	·373	·258	·174	·109	·052
·626	1·253	1·877	2·489	3·055	3·492	3·698	3·622	3·318	2·887	2·409	1·923	1·440	·957	·477
·102	·211	·335	·474	·620	·740	·789	·751	·646	·513	·391	·285	·200	·126	·062
·754	1·498	2·218	2·888	3·465	3·889	4·092	4·054	3·797	3·387	2·886	2·338	1·765	1·180	·591
·103	·210	·322	·435	·538	·617	·650	·630	·567	·479	·386	·294	·212	·137	·067
·839	1·659	2·436	3·136	3·718	4·134	4·342	4·327	4·104	3·719	3·214	2·632	2·003	1·349	·677
·102	·204	·305	·399	·482	·541	·567	·556	·513	·448	·372	·295	·216	·142	·071
·892	1·758	2·566	3·282	3·867	4·279	4·492	4·493	4·293	3·924	3·420	2·825	2·163	1·462	·735
·100	·198	·291	·379	·450	·499	·523	·515	·484	·429	·363	·291	·218	·145	·072
·920	1·811	2·634	3·359	3·943	4·355	4·571	4·580	4·394	4·035	3·537	2·932	2·254	1·528	·769
·098	·196	·287	·371	·439	·486	·509	·503	·473	·422	·361	·291	·219	·144	·071
·930	1·826	2·655	3·381	3·966	4·379	4·596	4·609	4·426	4·069	3·572	2·969	2·285	1·550	·782



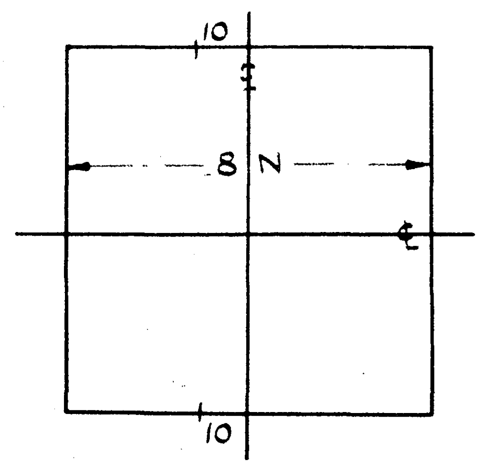
— Key Diagram —

If the ζ values are −ve, the w values are +ve, and vice versa.
Multiply the ζ units by N^2 to get w units.

— $\nabla^4 w = 0$ —

10

.061	.144	.286	.619	1.790	2.725	1.799	.636	.313	.183	.117	.077	.051	.031	.015
.281	.572	.888	1.255	1.712	1.992	1.806	1.446	1.177	.960	.774	.606	.447	.296	.147
.107	.241	.443	.790	1.254	1.498	1.270	.822	.493	.313	.209	.143	.096	.059	.028
.522	1.055	1.604	2.171	2.675	2.944	2.861	2.540	2.164	1.808	1.477	1.164	.864	.572	.285
.129	.277	.465	.696	.918	1.019	.940	.742	.535	.379	.269	.189	.130	.082	.039
.704	1.405	2.091	2.727	3.235	3.509	3.501	3.262	2.894	2.483	2.063	1.642	1.227	.815	.406
.133	.274	.429	.585	.714	.774	.742	.642	.516	.397	.299	.220	.154	.098	.048
.825	1.632	2.389	3.056	3.564	3.856	3.901	3.728	3.400	2.980	2.513	2.023	1.522	1.015	.508
.128	.256	.383	.499	.588	.630	.619	.564	.482	.394	.310	.235	.168	.110	.054
.899	1.765	2.560	3.243	3.758	4.065	4.149	4.024	3.733	3.323	2.836	2.304	1.744	1.170	.585
.120	.237	.345	.440	.508	.545	.543	.510	.451	.383	.310	.242	.177	.115	.057
.939	1.838	2.655	3.344	3.866	4.187	4.296	4.203	3.941	3.543	3.052	2.494	1.899	1.277	.640
.115	.223	.323	.405	.466	.500	.502	.477	.432	.374	.308	.244	.179	.119	.059
.959	1.873	2.699	3.393	3.920	4.248	4.373	4.299	4.051	3.664	3.171	2.606	1.988	1.339	.672
.112	.218	.314	.395	.450	.484	.490	.467	.426	.370	.308	.245	.180	.120	.060
.964	1.884	2.711	3.408	3.935	4.267	4.396	4.329	4.086	3.703	3.209	2.640	2.017	1.360	.684



— Key Diagram —

If the ζ values are -ve, the w values are +ve, and vice versa
Multiply the ζ units by N^2 to get w units.

— $\nabla^4 w = 0$ —

10 ℄

.104	.259	.600	.775	2.713	1.790	.629	.310	.181	.116	.078	.054	.036	.022	.010
.331	.681	1.079	1.564	1.871	1.711	1.373	1.128	.939	.773	.628	.494	.366	.241	.120
.167	.390	.753	1.226	1.477	1.254	.811	.485	.310	.209	.146	.102	.070	.044	.021
.599	1.208	1.828	2.388	2.707	2.673	2.400	2.071	1.763	1.476	1.209	.956	.709	.471	.234
.185	.397	.644	.879	.991	.917	.727	.526	.375	.269	.195	.140	.097	.062	.030
.775	1.537	2.244	2.823	3.169	3.230	3.060	2.761	2.417	2.063	1.709	1.359	1.015	.675	.336
.172	.349	.525	.666	.735	.714	.623	.502	.387	.299	.225	.166	.116	.075	.037
.874	1.708	2.457	3.051	3.429	3.560	3.473	3.229	2.895	2.511	2.104	1.689	1.266	.844	.421
.152	.299	.430	.533	.588	.586	.540	.467	.386	.309	.242	.183	.131	.084	.042
.921	1.791	2.556	3.165	3.569	3.751	3.726	3.533	3.224	2.836	2.401	1.939	1.463	.978	.489
.133	.259	.367	.452	.499	.509	.483	.435	.375	.311	.250	.192	.139	.089	.045
.940	1.825	2.598	3.217	3.641	3.858	3.875	3.720	3.435	3.050	2.604	2.112	1.601	1.071	.537
.120	.235	.331	.407	.452	.465	.452	.416	.366	.310	.252	.198	.143	.093	.046
.946	1.833	2.614	3.238	3.676	3.911	3.952	3.818	3.549	3.168	2.717	2.215	1.681	1.129	.565
.118	.226	.320	.393	.438	.453	.439	.408	.361	.309	.252	.199	.144	.096	.046
.949	1.839	2.618	3.248	3.683	3.927	3.978	3.852	3.586	3.211	2.757	2.251	1.708	1.147	.575

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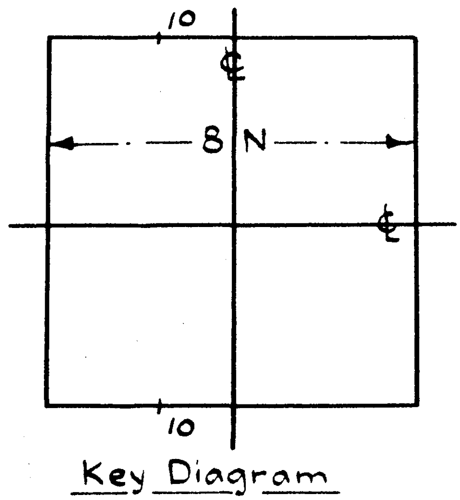

Key Diagram

℄

If the ζ values are -ve, the w values are +ve, and vice versa. Multiply the ζ units by N^2 to get w units.

— $\nabla^4 w = 0$ —

10 ℄

.197	.559	1.747	2.693	1.775	.619	.301	.175	.113	.078	.055	.038	.026	.017	.008
.402	.839	1.359	1.697	1.565	1.264	1.034	866	.726	.606	.495	.390	.289	.192	.095
.283	.680	1.174	1.439	1.227	790	.469	.299	202	.142	.102	073	050	.032	.016
.688	1.376	1.996	2.371	2.388	2.168	1.887	1.625	1.385	1.164	.956	.756	.560	.373	.186
.267	.552	.810	939	879	696	.504	.358	.258	.191	.140	.102	071	.045	.022
.834	1.617	2.271	2.693	2.826	2.725	2.495	2.217	1.930	1.641	1.358	1.079	.808	.534	.267
.217	.422	.588	.675	.668	.586	.476	.372	.285	.219	.166	.122	.087	.056	.027
.887	1.704	2.378	2.837	3.054	3.053	2.894	2.643	2.343	2.019	1.685	1.349	1.010	.673	.336
.172	.328	.450	.521	.534	.499	.436	.363	.295	.234	.183	.137	.098	.064	.032
.895	1.718	2.401	2.894	3.167	3.238	3.142	2.930	2.639	2.302	1.938	1.560	1.171	.782	.391
.141	.267	.367	.430	.453	.439	.401	.350	.295	.241	.191	.147	.105	.069	.034
.888	1.707	2.394	2.908	3.219	3.339	3.290	3.169	2.832	2.492	2.110	1.706	1.287	.860	.430
.123	.233	.321	.381	.408	.405	.380	.339	.292	.243	.195	.151	.109	.071	.035
.879	1.693	2.382	2.905	3.239	3.385	3.365	3.206	2.941	2.602	2.212	1.793	1.353	.906	.454
.118	.223	.308	.365	.394	.393	.373	.336	.291	.243	.196	.153	.110	.073	.036
.876	1.688	2.378	2.905	3.246	3.402	3.389	3.236	2.975	2.636	2.244	1.821	1.376	.922	.462

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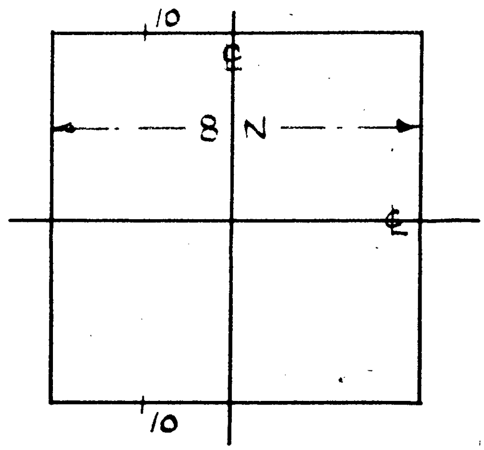


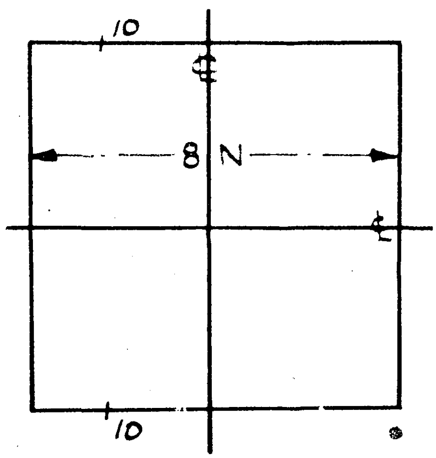
— Key Diagram —

℄

If the ζ values are -ve, the w values are +ve, and vice versa.
Multiply the ζ units by N^2 to get w units

— $\nabla^4 w = 0$ —

		10					℄							
·455	1·686	2·653	1·747	·599	·287	·163	·103	·070	·051	·037	·026	·019	·012	·005
·508	1·079	1·457	1·359	1·079	·887	·746	·633	535	·448	·366	·290	·215	·143	·071
·513	1·068	1·366	1·175	·751	·441	·278	·186	·131	095	·071	·051	·036	·023	·011
·777	1·474	1·916	1·995	1·829	1·602	1·391	1·201	1·028	867	·712	·565	420	·279	·139
·369	·680	·845	·810	643	464	329	·237	·174	·130	·099	·072	·052	·032	·016
·843	1·568	2·063	2·273	2·246	2·088	1·883	1·664	1·446	1·230	1·018	·811	·604	·402	·200
·250	·455	·572	·587	·522	·429	·335	·259	·202	·155	·118	·088	·062	·041	·020
·829	1·553	2·081	2·376	2·457	2·387	2·224	2·012	1·775	1·526	1·272	1·017	·762	·507	·253
·176	·324	·416	·450	·431	·381	·323	·264	·213	·168	·131	·101	·071	·046	·023
·797	1·503	2·050	2·401	2·559	2·558	2·443	2·252	2·014	1·749	1·467	1·179	·885	·590	·295
·134	·248	·325	·366	·369	·346	·305	·262	·218	·176	·141	·107	·078	·051	·025
·766	1·457	2·010	2·394	2·600	2·650	2·574	2·406	2·175	1·903	1·606	1·293	·973	·650	·325
·111	·209	·281	·322	·333	·321	·295	·257	·219	·180	·145	·111	·082	·053	·026
·747	1·425	1·981	2·381	2·616	2·694	2·644	2·491	2·266	1·992	1·686	1·362	1·028	·687	·344
·105	·196	·266	·308	·322	·314	·290	·256	·218	·182	·146	·112	·084	·054	·027
·740	1·414	1·972	2·376	2·618	2·708	2·663	2·518	2·295	2·022	1·712	1·384	1·044	·700	350



Key Diagram

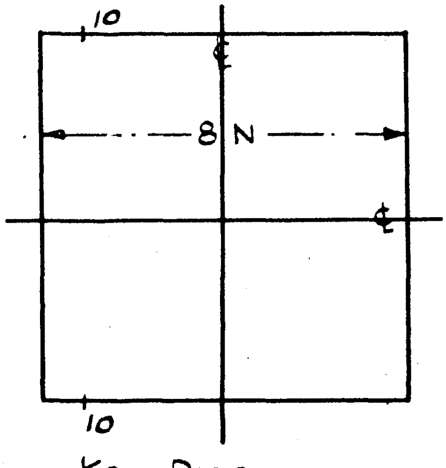
℄

If the ζ values are −ve, the w values are +ve, and vice versa.
Mult by the ζ units by N^2 to get w units.

$\nabla^4 w = 0$

10

1·489	2·550	1·687	·559	·258	·144	·090	·061	·043	·031	·024	·017	·013	·007	·003
·677	1·125	1·080	·839	·683	·573	·486	·416	·353	·296	·243	·192	·145	·095	·047
·784	1·200	1·068	·680	·392	·240	·158	·111	·080	·059	·044	·032	·023	·014	·007
·788	1·319	1·476	1·378	1·211	1·055	·918	·794	·682	·575	·473	·374	·279	·185	·093
·414	·663	·682	·552	·396	·278	·199	·145	·109	·081	·062	·046	·032	·021	·011
·739	1·291	1·576	1·621	1·539	1·407	1·260	1·112	·964	·820	·677	·540	·402	·267	·134
·239	·402	·457	·422	·351	·276	·212	·165	·127	·098	·074	·057	·040	·026	·012
·669	1·211	1·557	1·707	1·715	1·634	1·506	1·357	1·192	1·021	·850	·678	·507	·338	·168
·154	·266	·324	·327	·299	·256	·212	·172	·137	·109	·086	·065	·047	·030	·013
·612	1·135	1·509	1·723	1·797	1·768	1·669	1·528	1·360	1·175	·984	·789	·591	·394	·197
·109	·196	·249	·270	·261	·237	·206	·174	·143	·115	·091	·068	·051	·033	·015
·573	1·076	1·461	1·712	1·831	1·841	1·770	1·641	1·474	1·286	1·079	·867	·652	·436	·218
·087	·160	·207	·234	·237	·223	·200	·172	·144	·118	·094	·072	·054	·033	·016
·549	1·040	1·430	1·698	1·843	1·877	1·823	1·703	1·540	1·345	1·135	·914	·688	·459	·230
·081	·149	·196	·223	·231	·217	·199	·172	·145	·119	·095	·073	·054	·035	·017
·542	1·028	1·418	1·694	1·844	1·887	1·839	1·724	1·561	1·367	1·154	·934	·700	·468	·234



— Key Diagram —

If the ζ values are -ve, the w values are +ve, and vice versa.
Multiply the ζ units by N^2 to get w units.

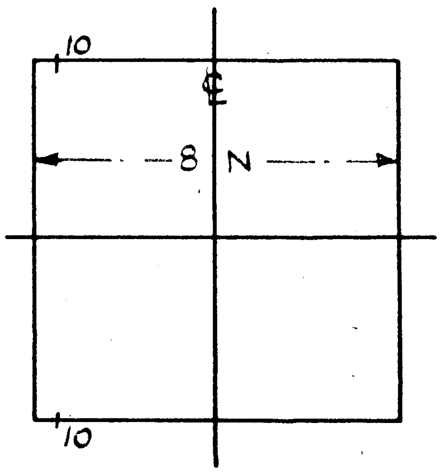
Field No 35

$$\nabla^4 w = 0$$

10 ℄

2.095	1.490	.455	.197	.104	.062	.040	.028	.021	.015	.010	.009	.006	.004	.001
.618	.677	.509	.403	.332	.281	.241	.208	.178	150	.124	.098	.072	.049	.024
.688	.785	.513	.282	.166	.106	.071	.051	.039	.029	.022	.017	.012	007	.004
540	788	777	.688	.601	.525	.459	.399	.344	.291	.239	.191	.143	.095	.048
.294	.414	.367	267	.185	.130	.094	.070	.053	.042	.031	.023	.016	.011	.005
.446	.737	.844	.837	779	.708	.634	.560	.488	.415	.345	.276	.206	.138	.068
.150	.240	250	.216	.173	134	.102	.080	.063	.049	.039	.029	.021	.013	.006
.379	.668	.831	.889	.877	.831	.762	.687	.604	.519	.433	.346	.260	.173	.087
.089	.151	.176	.171	.152	.127	.105	.085	.069	.055	.043	.031	.025	.016	.007
334	.612	799	.898	.927	.905	.851	.777	.692	.599	.502	.402	303	202	.101
.060	.107	.132	.140	.134	.119	.104	.088	.072	.059	.047	.035	.026	.017	.008
.306	.571	.769	.890	945	.944	.905	.837	.752	.655	.551	.443	.334	.224	.112
.048	.087	.110	.122	.123	.114	.102	.087	.073	.061	.047	.037	.028	.018	.009
.290	.548	.748	.884	.952	.965	.933	872	.786	.686	.580	.469	.353	.236	.117
.044	.079	.102	.117	.119	.111	.101	.087	.073	.061	.047	.037	.028	.019	.009
.284	.539	742	.879	.952	.971	943	.882	.797	.699	.590	.477	.360	.241	.120

℄



— Key Diagram —

℄

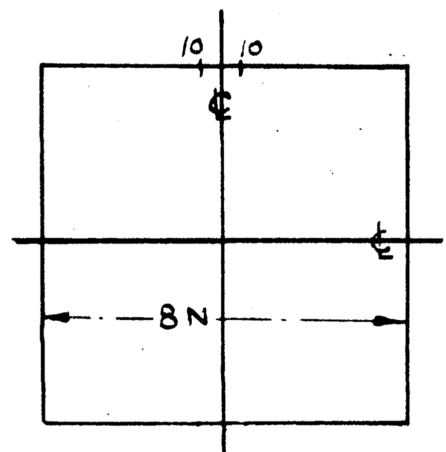
If the ζ values are -ve, the w values are +ve, and vice versa.
Multiply the ζ units by N^2 to get w units

Field No. 36

— $\nabla^4 w = 0$ —

						10	℄	
·052	·116	·211	·384	·774	2·072	3·326	3·561	ζ values
·300	·614	·956	1·348	1·823	2·440	2·955	3·118	w values
·096	·205	·361	·611	1·048	1·681	2·249	2·466	
·564	1·147	1·766	2·445	3·188	3·938	4·512	4·722	
·119	·255	·430	·668	·989	1·351	1·648	1·765	
·765	1·545	2·350	3·181	4·013	4·763	5·298	5·495	
·128	·268	·433	·632	·859	1·084	1·255	1·319	
·894	1·793	2·695	3·582	4·415	5·120	5·598	5·771	
·124	·251	·402	·560	·723	·873	·978	1·015	
·954	1·904	2·833	3·718	4·512	5·156	5·578	5·728	
·115	·232	·355	·479	·598	·701	·772	·797	
·959	1·905	2·813	3·657	4·393	4·975	5·348	5·477	
·101	·202	·302	·401	·491	·565	·616	·632	
·921	1·825	2·680	3·462	4·129	4·646	4·975	5·088	
·084	·170	·253	·331	·400	·455	·492	·503	
·856	1·688	2·470	3·175	3·769	4·224	4·511	4·609	℄
·071	·141	·207	·269	·322	·363	·391	·400	
·768	1·514	2·208	2·829	3·351	3·744	3·990	4·072	
·058	·114	·167	·215	·256	·289	·308	·317	
·668	1·315	1·916	2·450	2·894	3·228	3·437	3·509	
·046	·090	·133	·169	·201	·224	·239	·245	
·562	1·104	1·606	2·050	2·420	2·697	2·868	2·926	
·036	·069	·102	·129	·152	·169	·180	·183	
·451	·885	1·288	1·644	1·936	2·156	2·291	2·337	
·025	·050	·073	·092	·109	·121	·130	·131	
·338	·665	·967	1·231	1·451	1·616	1·718	1·749	
·015	·033	·046	·060	·070	·078	·083	·084	
·227	·443	·644	·820	·966	1·075	1·142	1·166	
·008	·016	·023	·030	·035	·039	·042	·043	
·114	·223	·322	·410	·482	·536	·570	·582	

℄



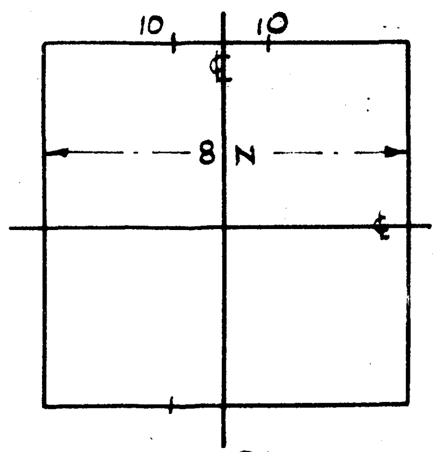
— Key Diagram —

If the ζ values are -ve, the w values are +ve, and vice versa.
Multiply the ζ units by N^2 to get w units.

— $\nabla^4 w = 0$ —

					10		℄	
·069	·160	·315	·668	1·873	2·870	2·071	1·230	ζ values
·320	·657	1·029	1·472	2·028	2·444	2·443	2·340	w values
·120	·269	·492	·875	1·396	1·737	1·683	1·564	
·592	1·205	1·856	2·555	3·236	3·735	3·944	3·978	
·144	·310	·524	·797	1·083	1·283	1·352	1·358	
·789	1·588	2·399	3·200	3·923	4·464	4·772	4·867	
·147	·305	·486	·682	·869	1·010	1·087	1·109	
·904	1·804	2·686	3·519	4·242	4·795	5·133	5·246	
·137	·277	·424	·573	·707	·811	·875	·896	
·948	1·883	2·777	3·600	4·301	4·836	5·172	5·285	
·121	·240	·361	·475	·575	·655	·704	·722	
·941	1·860	2·728	3·511	4·175	4·677	4·992	5·097	
·103	·203	·301	·390	·468	·530	·567	·580	
·896	1·765	2·580	3·307	3·916	4·376	4·664	4·762	
·086	·169	·247	·320	·379	·427	·458	·467	
·824	1·622	2·364	3·024	3·572	3·985	4·241	4·329	℄
·071	·139	·201	·259	·306	·344	·367	·374	
·735	1·447	2·107	2·692	3·175	3·536	3·760	3·836	
·057	·112	·161	·207	·243	·273	·290	·298	
·639	1·255	1·826	2·327	2·743	3·053	3·245	3·309	
·045	·089	·127	·161	·191	·213	·226	·232	
·536	1·052	1·527	1·947	2·293	2·552	2·710	2·763	
·034	·067	·097	·123	·144	·160	·171	·175	
·429	·843	1·225	1·560	1·835	2·041	2·168	2·210	
·024	·049	·071	·088	·104	·115	·123	·126	
·321	·632	·919	1·169	1·375	1·528	1·623	1·657	
·015	·031	·046	·058	·067	·074	·080	·080	
·215	·422	·612	·780	·916	1·017	1·081	1·102	
·007	·015	·022	·028	·034	·038	·041	·042	
·108	·211	·306	·389	·458	·508	·540	·552	

℄



Key Diagram

If the ζ values are -ve, the w values are +ve, and vice versa.
Multiply the ζ units by N^2 to get w units.

— $\nabla^4 w = 0$ —

10 ℄

·107	268	·616	1·804	2·762	1·874	·776	·584	ζ values
·355	·734	1·171	1·709	2·090	2·030	1·831	1·767	w values
·174	·406	·781	1·276	1·563	1·396	1·050	·898	
640	1·300	1·987	2·645	3·095	3·238	3·196	3·155	
·191	·413	·679	·940	1·093	1·084	·991	·940	
822	1·643	2·433	3·133	3·644	3·923	4·023	4·040	
·177	·363	·554	·721	·830	·868	·860	·849	
·908	1·792	2·621	3·337	3·885	4·242	4·426	4·479	
·153	303	·446	·570	658	·706	·724	·728	
·926	1·817	2·641	3·350	3·911	4·301	4·523	4·593	
·124	247	·361	·456	·528	·576	·601	·607	
896	1·757	2·547	3·229	3·779	4·171	4·408	4·481	
·101	·200	·288	·368	428	·470	·494	·501	
·837	1·640	2·378	3·018	3·536	3·912	4·141	4·214	
·082	·161	·232	·296	·345	·381	·400	·409	
762	1·493	2·163	2·747	3·220	3·569	3·782	3·852	℄
·065	·128	·186	·237	·276	·305	·324	·331	
·674	1·322	1·917	2·435	2·867	3·167	3·360	3·422	
·054	·101	·147	·188	·221	·244	·257	·261	
·581	1·139	1·652	2·100	2·466	2·737	2·902	2·959	
·041	·080	·115	·146	·172	·189	·202	·206	
·484	952	1·378	1·754	2·059	2·286	2·427	2·473	
·032	·061	·087	·111	·130	·145	·153	·155	
·387	·760	1·102	1·403	1·648	1·829	1·942	1·979	
·022	·044	·062	·079	·093	·104	·111	·112	
·289	·570	·826	1·049	1·234	1·370	1·454	1·482	
·014	·028	·041	·052	·060	·067	·071	·072	
·193	·378	·550	·699	·821	·911	·967	·987	
·007	·013	·019	·025	·029	·032	·034	·036	
096	·189	·274	·349	·410	·455	·483	·492	

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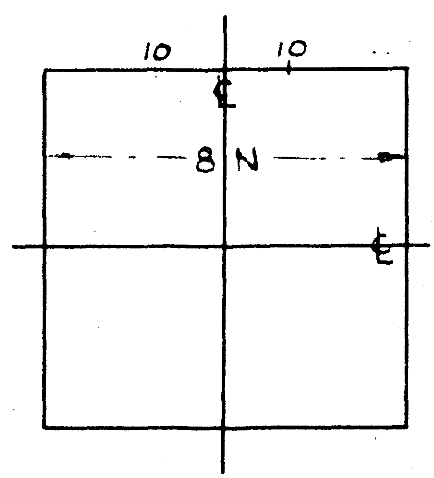

Key Diagram

If the ζ values are -ve, the w values are +ve, and vice versa.
Multiply the ζ units by N^2 to get w units.

Field N° 39

— $\nabla^4 w = 0$ —

10

·199	·564	1·757	2·711	1·805	·669	·385	·320	ζ values
·414	·869	1·413	1·789	1·710	1·472	1·350	·314	w values
·287	·688	1·190	1·469	1·278	·876	·612	·537	
·709	1·425	2·087	2·531	2·645	2·556	2·449	2·412	
·270	·560	828	·973	·939	·799	·670	·622	
·853	1·666	2·371	2·875	3·132	3·201	3·189	3·176	
·218	·426	·601	·703	·723	·683	·633	·613	
891	1·725	2·445	2·990	3·336	3·517	3·589	3·607	
·168	322	449	533	·571	·573	·561	·554	
869	1·683	2·390	2·954	3·350	3·598	3·725	3·764	
·128	·246	·343	·415	·456	476	·480	·481	
816	1·585	2·262	2·817	3·230	3·509	3·665	3·716	
·098	·189	·266	·327	·367	·391	·402	·405	
·747	1·454	2·083	2·610	3·018	3·302	3·469	3·525	
·077	·148	·209	·259	·295	·318	·332	·336	
·669	1·305	1·877	2·363	2·745	3·019	3·182	3·236	℄
·060	·115	164	·205	·236	·257	270	·273	
·587	1·145	1·652	2·088	2·433	2·685	2·837	2·887	
·047	·090	·129	162	·188	·204	·216	·219	
·502	·982	1·419	1·797	2·099	2·322	2·457	2·502	
·03[illegible]	·070	·099	·125	·146	·160	·170	·172	
·417	·817	1·182	1·500	1·755	1·942	2·056	2·096	
·026	·052	·074	·094	·111	·122	·128	·131	
·332	·652	·943	1·196	1·403	1·555	1·648	1·679	
·019	·037	·053	·068	·080	·088	·092	·094	
·248	·484	·708	·896	1·052	1·165	1·236	1·258	
·012	·024	·034	·043	·051	·056	·059	·061	
·165	·324	·469	·596	·699	·776	·823	·838	
·006	·012	·016	·020	·025	·028	·029	·030	
·082	·162	·234	·297	·350	·387	·411	·418	

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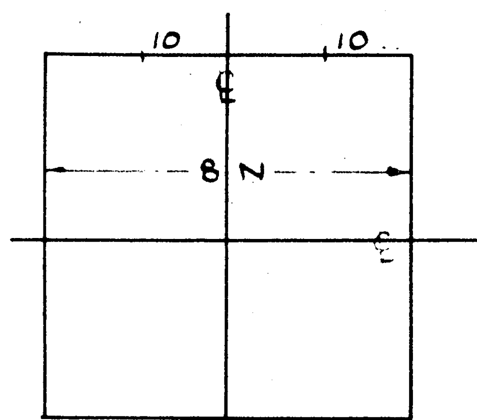


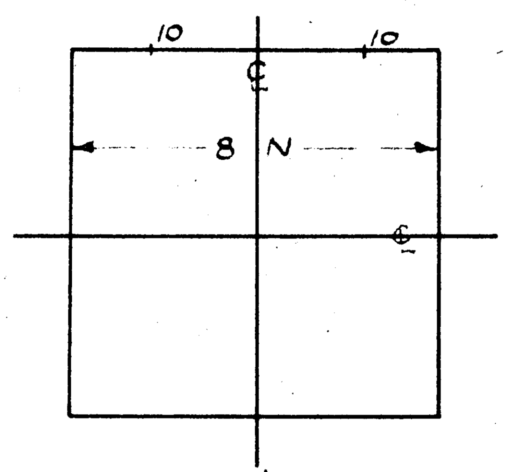
— Key Diagram —

If the ζ values are −ve, the w values are +ve, and vice versa.
Multiply the ζ units by N^2 to get w units.

$\nabla^4 w = 0$

		10					℄	
·456	1·689	2·658	1·756	·615	·315	·210	·183	ζ values
·514	1·094	1·487	1·413	1·170	1·031	·958	·936	w values
·514	1·072	1·375	1·191	·782	·492	·363	·326	
·785	1·497	1·965	2·089	1·990	1·859	1·772	1·744	
·369	·683	·854	·828	·680	·527	·431	·400	
·846	1·584	2·110	2·376	2·436	2·402	2·359	2·340	
·249	·454	·575	·600	·554	·488	·437	·417	
·818	1·543	2·096	2·447	2·624	2·691	2·705	2·706	
·170	·313	·409	·451	·448	·424	·404	·394	
·760	1·444	1·998	2·395	2·644	2·781	2·844	2·862	
·120	·225	·301	·344	·360	·361	·356	·354	
·690	1·323	1·855	2·265	2·551	2·731	2·826	2·856	
·087	·166	·229	·269	·290	·302	·304	·304	
·618	1·192	1·690	2·088	2·383	2·579	2·693	2·727	
·066	·125	·175	·211	·234	·248	·254	·256	
·545	1·057	1·508	1·880	2·165	2·365	2·479	2·518	℄
·050	·096	·134	·164	·188	·202	·209	·210	
·473	·920	1·319	1·655	1·919	2·107	2·217	2·255	
·039	·074	·104	·129	·150	·161	·169	·170	
·401	·784	1·128	1·422	1·655	1·822	1·923	1·956	
·029	·057	·079	·100	·115	·125	·132	·134	
·332	·649	·937	1·185	1·382	1·526	1·613	1·642	
·021	·042	·060	·075	·086	·096	·100	·103	
·264	·517	·747	·946	1·105	1·222	1·294	1·318	
·016	·029	·043	·054	·062	·069	·072	·074	
·197	·386	·557	·708	·828	·916	·970	·988	
·010	·019	·027	·035	·040	·044	·046	·046	
·131	·256	·371	·471	·551	·610	·647	·658	
·004	·009	·014	·017	·020	·022	·023	·023	
·065	·128	·185	·236	·275	·304	·323	·330	

℄



Key Diagram

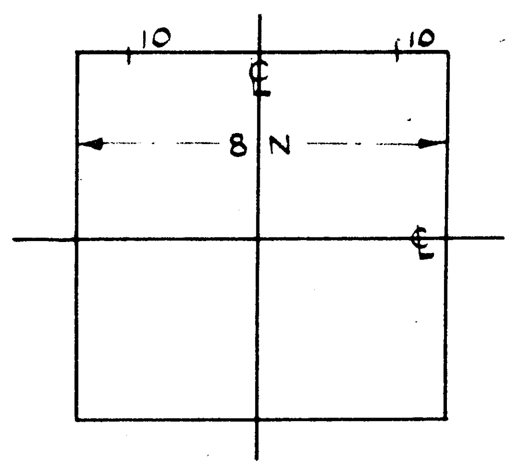
If the ζ values are -ve, the w values are +ve, and vice versa
Multiply the ζ units by N^2 to get w units.

$\nabla^4 w = 0$

10 ℄

1.488	2.550	1.689	.564	.269	.160	.117	.105	ζ values
.678	1.130	1.095	.868	.735	.658	.616	.604	w values
.784	1.200	1.071	.688	.407	.268	.207	.189	
.788	1.325	1.496	1.425	1.301	1.207	1.152	1.132	
.412	.661	.683	.559	.415	.311	.257	.238	
.734	1.288	1.584	1.667	1.641	1.591	1.551	1.538	
.235	.397	.454	.425	.365	.306	.271	.259	
.652	1.187	1.543	1.727	1.796	1.807	1.802	1.800	
.145	.255	.313	.323	.304	.278	.257	.251	
.575	1.073	1.445	1.686	1.821	1.886	1.914	1.920	
.096	.175	.225	.247	.248	.241	.233	.231	
.507	.961	1.324	1.588	1.762	1.866	1.916	1.931	
.067	.123	.165	.189	.201	.203	.202	.201	
.445	.851	1.193	1.459	1.648	1.769	1.837	1.857	
.049	.092	.125	.148	.163	.168	.172	.172	
.388	.748	1.059	1.314	1.499	1.627	1.700	1.724	℄
.036	.070	.096	.117	.130	.138	.142	.143	
.334	.648	.925	1.153	1.330	1.453	1.526	1.549	
.028	.054	.075	.091	.104	.111	.116	.117	
.284	.551	.789	.991	1.148	1.261	1.328	1.351	
.022	.041	.058	.071	.081	.087	.092	.093	
.233	.456	.655	.826	.960	1.057	1.115	1.136	
.016	.031	.043	.054	.062	.068	.069	.071	
.185	.362	.521	.658	.769	.848	.896	.914	
.011	.023	.031	.039	.043	.048	.051	.052	
.140	.270	.391	.494	.575	.636	.673	.686	
.007	.014	.020	.024	.029	.031	.031	.033	
.093	.179	.259	.327	.383	.423	.448	.456	
.004	.007	.010	.012	.014	.015	.016	.017	
.046	.090	.130	.163	.191	.211	.223	.228	

℄



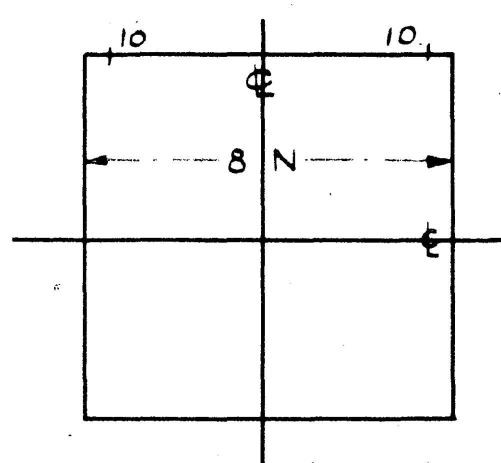
— Key Diagram —

If the ζ values are −ve, the w values are +ve, and vice versa.
Multiply the ζ units by N^2 to get w units

$$\nabla^4 w = 0$$

10

2.094	1.490	.456	.199	.107	.069	.052	.048	ζ values
.618	.679	.514	.415	.356	.320	.301	.296	w values
.688	.785	.514	.286	.174	.118	.093	.085	
.540	.790	.785	.708	.641	.594	.568	.559	
.293	.413	.367	.270	.193	.145	.121	.112	
.443	.734	.846	.855	.824	.791	.771	.762	
.149	.235	.249	.217	.180	.147	.129	.121	
.370	.653	.821	.894	.912	.910	.901	.899	
.086	.146	.171	.166	.153	.136	.126	.121	
.314	.578	.762	.873	.931	.956	.965	.966	
.054	.096	.121	.128	.127	.120	.117	.116	
.271	.509	.694	.822	.903	.949	.972	.979	
.037	.067	.088	.100	.103	.105	.102	.102	
.234	.447	.621	.756	.846	.903	.936	.947	
.026	.049	.065	.078	.083	.086	.087	.087	
.202	.390	.551	.678	.771	.835	.870	.882	
.019	.038	.050	.061	.067	.072	.073	.074	
.173	.337	.480	.597	.686	.748	.783	.797	
.014	.028	.037	.046	.054	.058	.061	.062	
.147	.286	.409	.511	.593	.650	.685	.695	
.010	.021	.030	.036	.042	.046	.048	.049	
.121	.236	.340	.427	.498	.548	.578	.588	
.007	.017	.022	.028	.032	.035	.038	.039	
.096	.188	.270	.341	.398	.440	.465	.475	
.006	.012	.016	.020	.023	.026	.027	.028	
.071	.141	.204	.257	.300	.332	.351	.358	
.004	.007	.011	.013	.014	.017	.017	.017	
.048	.093	.135	.171	.199	.222	.235	.239	
.002	.004	.005	.006	.007	.008	.008	.008	
.024	.047	.067	.086	.100	.111	.118	.120	



Key Diagram

the ζ values are -ve, the w values are +ve, and vice versa
ply the ζ units by N^2 to get w units.

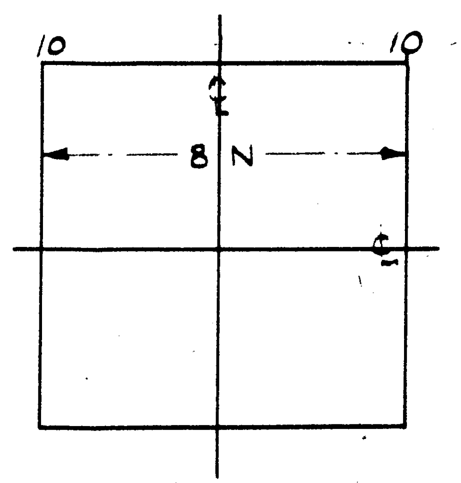
Field No. 43

— $\nabla^4 w = 0$ —

℄

·930	·235	·090	·043	·025	·017	·013	·012	ζ values
·226	·161	·124	·100	·086	·078	·073	·072	w values
235	·217	·116	·065	·041	·029	·023	022	
·162	·213	·196	·173	·156	·144	·137	·134	
·089	·116	·095	·065	·047	·035	·030	·028	
·122	195	·215	211	·201	·192	·185	182	
·042	064	·064	054	043	·036	031	·030	
·096	·168	·207	·219	·221	219	·214	·214	
·023	038	·044	·042	037	·033	·030	·028	
·080	·145	·190	·213	·224	·228	228	·228	
·014	025	030	032	·030	·028	·027	·026	
·067	·125	·169	·198	·215	224	228	·230	
·009	·017	·022	023	·024	024	023	·023	
·056	·107	·148	·176	·197	·210	·217	·221	
·006	·012	·015	·018	·018	·019	·020	·020	
·048	·092	128	157	·177	·192	·200	202	
·004	·008	·011	·013	·014	·015	·015	·016	
·040	·078	109	·135	·154	169	·177	·179	
·003	·005	·008	010	·011	·011	·012	012	
·033	·064	·092	·114	·131	·144	·151	·154	
002	004	·006	·006	·007	·008	·008	·008	
·026	·052	·074	·093	·107	119	·124	·125	
·001	·002	·003	·004	·005	·006	·006	·006	
·020	·039	·057	·072	·085	·093	·097	·100	
·001	·001	·002	·002	·003	·003	·003	·003	
·015	·029	·042	·053	·063	·069	·072	·074	
	·001	·001	·001	·001	·001	·001	·001	
·009	·018	·028	·034	·041	·045	·048	·048	
·004	·009	·014	·017	·020	·022	·023	·024	

℄



— Key Diagram —

If the ζ values are -ve, the w values are +ve, and vice versa.
Multiply the ζ units by N^2 to get w units

Clamped Edges ~ Uniform Load p

$$-\nabla^4 w = C = p/K -$$

	0								℄
ζ	+·089	+·145							
w		·022							
ζ	+·403	+·232	+·147						
w		·078	·260						
ζ	+·718	+·367	+·123	·035	ζ values				
w		·145	·481	·887	w values				
ζ	+1·028	+·517	+·140	·130	·318				
w		·209	·695	1·287	1·874				
ζ	+1·298	+·660	+·176	·183	·442	623			
w		·262	·877	1·632	2·383	3·038			
ζ	+1·502	+·775	+·215	·208	·519	·740	887		
w		·300	1·013	1·892	2·770	3·538	4·127		
ζ	+1·632	+·849	+·243	·219	·561	·806	·970	1·063	
w		·323	1·096	2·052	3·011	3·850	4·494	4·896	
℄ ζ	+1·668	+·874	+·254	·221	·574	·828	997	1·094	1·125
℄ w		·330	1·124	2·107	3·092	3·956	4·619	5·033	5·173

← 4 N →

Small squares have length of side = $N/2$

ζ values are negative except near the boundary

ζ units are CN^2; w units are CN^4.

$$K = IE/(1-\sigma^2)$$

Field No. 45

Line Load along the Centre Line

℄

·0129	0251	·0362	·0456	·0528	·0584	·0618	·0629	ζ values
·116	·228	·332	·419	·491	·544	·579	589	w values
·0266	·0520	·0745	·0934	1083	·1190	·1256	·1277	
·232	·458	662	·834	·977	1·080	1·145	1·165	
·0419	·0815	·1162	·1452	1679	·1843	·1940	·1972	
·347	·681	·980	1·237	1·445	1·602	1·696	1·725	
0598	·1154	·1638	2037	2342	·2557	2683	·2725	Line 1-17
455	·888	1·279	1·619	1·887	2·085	2·206	2·244	
·0819	·1571	·2202	2714	3096	3359	·3517	·3569	
·554	1·084	1·551	1·961	2·284	2·518	2·664	2·713	
1118	·2107	2902	·3514	·3963	·4271	·4450	·4510	
·640	1·248	1·787	2·245	2·612	2·878	3·039	3·094	
·1577	·2846	·3784	·4476	·4972	·5311	·5504	·5567	
·705	1·365	1·951	2·452	2·841	3·128	3·298	3·355	
·2482	·3906	·4901	·5625	·6141	·6488	·6688	·6753	Line Load p/unit of length
·735	1·417	2·020	2·530	2·930	3·221	3·400	3·457	

℄

8 N

ζ values are −ve and have units pN/K

w values are +ve and have units pN^3/K

$$K = \bar{I}E/(1-\sigma^2)$$

Field No. 46

Line Load along the quarter-line.

.0047	.0092	.0133	0170	0199	0221	0236	.0239	ζ values
.059	.115	.168	.214	.252	.279	.298	.304	w values
0096	.0190	.0275	.0347	0407	0451	.0479	0486	
.119	234	.338	.429	.504	.559	.594	.604	
.0150	.0295	.0425	0540	0631	0700	0742	.0756	
.179	352	.508	.644	756	.840	.891	.908	
.0211	.0412	.0595	.0753	.0880	0975	.1030	.1049	
.239	.467	.675	.859	1.006	1.116	1.181	1.203	
.0279	0548	.0787	.0998	.1164	.1285	.1360	1385	
.297	.582	.839	1.068	1.250	1.385	1.469	1.498	
0363	.0710	1019	.1283	.1494	.1644	.1735	.1767	
.353	.694	1.000	1.267	1.484	1.643	1.741	1.774	
.0466	0905	.1296	.1625	.1892	.2063	.2173	.2211	
.407	.795	1.148	1.455	1.698	1.881	1.989	2.027	
.0598	.1154	.1637	.2037	.2341	.2556	2685	.2726	Line 2-2 ℄
.456	889	1.279	1.619	1.887	2.084	2.206	2.246	
0772	.1476	.2068	.2541	.2892	.3138	.3278	3326	
.495	.963	1.385	1.744	2.028	2.239	2.364	2.406	
1021	.1916	.2627	.3164	3555	3817	.3970	.4021	
.520	1.012	1.447	1.814	2.106	2.316	2.444	2.487	
.1426	.2553	3356	.3936	.4341	.4609	4762	4812	
.526	1.016	1.442	1.811	2.085	2.286	2.409	2.450	
.2275	.3496	4308	.4873	5260	5512	5655	.5702	Line Load p/unit of length
.495	950	1.347	1.669	1.923	2.105	2.213	2.249	
.1296	.2299	.2994	3476	.3808	.4025	.4147	.4186	
.408	785	1.111	1.379	1.590	1.742	1.832	1.861	
.0753	.1395	.1881	.2229	.2469	.2624	.2714	2742	
.286	.553	.785	.977	1.126	1.232	1.297	1.317	
.0351	.0662	.0906	1087	.1210	.1294	.1342	.1355	
.146	283	.403	.504	582	.637	.672	.682	

℄

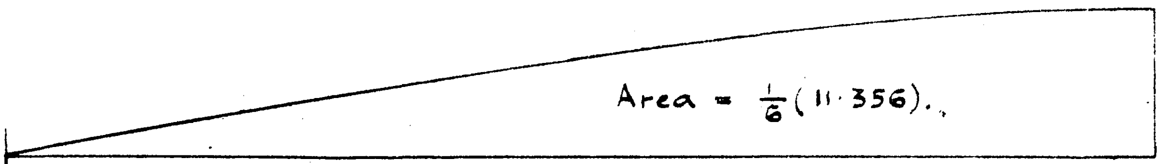
8 N

ζ values are -ve, and have units pN/K

w values are +ve, and have units pN^3/K

$K = IE/(1-\alpha^2)$.

Field No. 47



$\partial\zeta/\partial y$ curve

ζ values
w values

Coeffts $\partial\zeta/\partial y$	0.133	0.262	0.386	0.510	0.622	0.708	0.762	0.784 ℄
	·066 ·670							
	·131 1·33	·262 2·61						
	·195 1·94	·394 3·83	·595 5·63					
	·258 2·50	·522 4·94	·796 7·27	1·075 9·43				
	·315 2·97	·639 5·89	·984 8·69	1·354 11·31	1·747 13·64			
	·361 3·34	·736 6·62	1·146 9·80	1·605 12·80	2·132 15·52	2·729 17·80		
	·391 3·57	·802 7·09	1·257 10·51	1·786 13·77	2·436 16·78	3·285 19·39	4·458 21·36	
℄	·402 3·65	·825 7·25	1·297 10·76	1·852 14·11	2·554 17·22	3·527 19·97	5·145 22·15	10·000 23·55

4 N

Concentrated Load at Centre

$\nabla^4 w = 0$, except at centre.

ζ values
w values

	·0022 ·022							
	·0043 ·044	·0087 ·086						
	·0065 ·064	·0130 ·127	·0197 ·186					
	·0085 ·083	·0173 ·164	·0263 ·241	·0356 ·312				
	·0104 ·098	·0212 ·195	·0326 ·288	·0448 ·374	·0578 ·451			
	·0119 ·111	·0244 ·219	·0379 ·324	·0531 ·424	·0706 ·514	·0903 ·588		
	·0129 ·118	·0265 ·235	·0416 ·348	·0591 ·456	·0806 ·555	·1087 ·641	·1476 ·703	
℄	·0133 ·121	·0273 ·240	·0429 ·356	·0613 ·467	·0845 ·570	·1167 ·660	·1703 ·729	·3310 ·763 Load $\bar{W}$

℄

4 N

Small squares have length of side = N/2
ζ values are -ve, and have units $\bar{W}/K$
w values are +ve, and have units $\bar{W}N^2/K$
$K = IE/(1-\sigma^2)$

Concentrated Load at quarter point.

℄

	·0050	·0047	·0044	·0039	·0033	·0026	·0018	·0009
	·061	·059	·056	·049	·042	·033	·023	·012
ζ values	·0099	·0097	·0090	·0080	·0067	·0053	·0036	·0019
w values	·121	·119	·112	·099	·084	·066	·046	·023
	·0153	·0150	·0140	·0125	·0106	·0081	·0055	·0028
	·182	·178	·167	·151	·127	·100	·068	·035
	·0214	·0210	·0195	·0173	·0145	·0112	·0076	·0038
	·243	·238	·223	·200	·169	·132	·090	·046
	·0286	·0279	·0259	·0229	·0190	·0146	·0098	·0050
	·303	·297	·279	·249	·211	·164	·112	·057
	·0371	·0361	·0334	·0292	·0241	·0183	·0123	·0061
	·362	·354	·331	·295	·249	·194	·132	·067
	·0477	·0463	·0423	·0366	·0296	·0223	·0148	·0074
	·417	·408	·380	·338	·284	·219	·150	·076
	·0613	·0591	·0531	·0448	·0356	·0263	·0173	·0085 ℄
	·467	·456	·424	·374	·312	·241	·164	·083
	·0797	·0759	·0662	·0538	·0415	·0300	·0194	·0095
	·509	·495	·458	·401	·331	·255	·172	·087
	·1069	·0991	·0813	·0626	·0463	·0326	·0208	·0101
	·538	·522	·476	·414	·339	·257	·173	·088
	·1550	·1326	·0947	·0681	·0486	·0335	·0210	·0101
	·547	·524	·473	·404	·328	·248	·167	·083
Load $\bar{W}$	·3096	·1493	·0972	·0672	·0468	·0317	·0197	·0095
	·520	·491	·437	·370	·298	·224	·150	·075
	·1417	·1197	·0828	·0577	·0400	·0270	·0168	·0079
	·426	·407	·363	·306	·246	·184	·123	·061
	·0795	·0725	·0569	·0414	·0291	·0196	·0121	·0057
	·299	·287	·257	·220	·176	·131	·087	·044
	·0368	·0343	·0285	·0213	·0152	·0103	·0064	·0030
	·153	·148	·134	·114	·091	·068	·045	·022

℄

8N

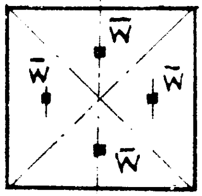
ζ values are -ve, and have units $\bar{W}/K$

w values are +ve, and have units $\bar{W}N^2/K$

$K = IE/(1-\sigma^2)$

Field No. 50

Concentrated Loads at the quarter points.



℄

.0079	ζ values						
.0683	w values.						
.0157	.0314						
.1346	.2651						
.0236	.0470	.0702					
.1970	.3878	.5663					
.0318	.0631	.0934	.1226				
.2538	.4990	.7274	.9314				
.0403	.0803	.1182	.1521	.1820			
.3030	.5955	.8663	1.1047	1.3026			
.0489	.0990	.1477	.1871	.2124	.2294		
.3423	.6729	.9782	1.2428	1.4550	1.6108		
.0559	.1164	.1869	.2413	.2509	.2438	.2444	
.3683	.7251	1.0565	1.3400	1.5565	1.7082	1.8007	
.0587	.1240	.2096	.4022	.2733	.2503	.2456	.2452
.3774	.7438	1.0862	1.3854	1.5937	1.7418	1.8321	1.8625

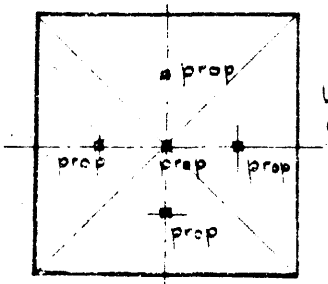
℄

← 4 N →

Small squares have length of side = N/2
ζ values are -ve, and have units $\bar{W}/K$
w values are +ve, and have units $\bar{W}N^2/K$

Field No. 51

— Propped Plate —



Uniform Load on plate = $64\,pN^2$

Quarter point prop Load = $7.58\,pN^2$
Centre point " " = $3.41\,pN^2$

	0·136 ζ values ·134 w values							
	0·204 ·232	0·317 ·413						
	0·233 ·290	0·367 ·510	0·424 ·633					
	0·235 ·301	0·366 ·528	0·417 ·647	0·402 ·654				
	0·214 ·278	0·321 ·475	0·347 ·567	0·320 ·562	0·253 ·474			
	0·178 ·227	0·235 ·376	0·201 ·418	0·146 ·386	0·115 ·322	0·070 ·236		
	0·142 ·182	0·135 ·270	+0·047 ·240	+0·217 ·171	+0·131 ·151	+0·012 ·134	+0·056 ·087	
℄	0·126 ·162	0·087 ·226	+0·212 ·155	+1·421 0·00 prop	+0·287 ·065	+0·060 ·089	+0·112 ·051	+0·625 0·00 prop

4N

ζ values are −ve except where noted, units pN^2/K
w values are +ve and have units pN^4/K

$K = IE/(1-\sigma^2)$

Field No. 52

Rectangular Plate ~ Line Load along ℄

4 N

2 N

·0305	·0571	·0773	·0916	·1011	·1074	·1105	·1115
·087	·168	·236	·290	·330	·358	·374	·378
·0658	·1206	·1607	·1882	·2061	·2173	·2235	·2254
·167	·319	·447	·548	·622	·673	·703	·713
·1147	·2005	·2569	·2937	·3177	·3324	·3404	·3429
·229	·433	·603	·734	834	·902	·941	·953
·2064	·3084	·3713	·4120	·4380	·4537	·4625	·4653
·256	·483	·672	·815	·917	·989	1·032	1·046

Line Load p/unit of length.

ζ values are −ve and have units pN/K

w values are +ve and have units pN^3/K

Field No 53.

Rectangular Plate ~ Concentrated Load at Centre.

4 N

2 N

·0021	·0046	·0077	·0119	·0172	·0239	·0296	·0320
·011	·022	·036	·048	·064	·078	·088	·092
·0039	·0085	·0143	·0223	·0334	·0479	·0629	·0697
·021	·042	·064	·091	·120	·145	·168	·177
·0051	·0112	·0189	·0295	·0452	·0688	·1047	·1264
·026	·055	·084	·118	·155	·195	·228	·244
·0057	·0121	·0205	·0323	·0499	·0777	·1283	·2882
·029	·060	·092	·129	·170	·214	·253	·277

Load $\bar{W}$

ζ values are −ve and have units $\bar{W}/K$

w values are +ve and have units $\bar{W}N^2/K$.

Field No. 54

10

·186	·537	1·713	2·647	ζ values
·283	·603	1·010	1·244	w values
·261	·636	1·109	1·350	
·452	·910	1·312	1·481	
·235	·487	·713	·809	
·485	·932	1·269	1·400	
·173	·336	·461	·509	
·429	·809	1·076	1·174	
·116	·218	·292	·318	
·335	·624	·821	·891	
·070	·131	·172	·187	
·225	·417	·547	·594	
·032	·060	·079	·086	
·113	·208	·273	·294	

4N

Field No. 55

10 ℄

·447	1·670	2·628	1·713	·556	·231	·090
·415	·897	1·189	1·011	·660	·399	·189
·496	1·035	1·318	1·109	·669	·336	·142
·592	1·110	1·385	1·312	1·006	·653	·318
·343	·631	·773	·714	·526	·318	·145
·568	1·030	1·280	1·269	1·050	·724	·363
·216	·387	·477	·461	·372	·246	·121
·467	·843	1·056	1·076	·925	·660	·342
·131	·234	·291	·292	·247	·173	·087
·346	·627	·794	·821	·722	·528	·277
·075	·133	·168	·172	·150	·108	·056
·227	·411	·522	·547	·487	·362	·191
·035	·062	·077	·079	·069	·051	·027
·111	·203	·259	·273	·244	·182	·098

4N

— $\nabla^4 w = 0$ —

If the ζ values are −ve, the w values are +ve, and vice versa.
Multiply ζ units by N^2 to get w units.

Field No. 56

10 ℄

1·484	2·539	1·670	·537	·231	·109	·046
·612	1·000	·895	·603	·399	·244	·116
·773	1·178	1·035	·636	·336	·171	·074
·659	1·070	1·110	·910	·651	·415	·202
·396	·627	·632	·486	·316	·182	·083
·546	·918	1·030	·932	·722	·480	·239
·215	·354	·387	·336	·247	·155	·074
·415	·718	·844	·809	·662	·458	·231
·119	·203	·235	·218	·173	·115	·057
·293	·518	·629	·623	·527	·375	·192
·065	·112	·134	·131	·106	·075	·037
·187	·333	·411	·417	·360	·263	·136
·029	·049	·061	·060	·050	·034	·018
·091	·163	·203	·208	·181	·133	·070

4N

Field No. 57

10 ℄

2·092	1·484	·447	·186	·090	·045	·018
·585	·612	·415	·282	·188	·115	·054
·682	·772	·496	·261	·140	·075	·033
·475	·659	·591	·451	·317	·201	·097
·285	·395	·343	·236	·146	·084	·037
·349	·546	·568	·484	·364	·239	·118
·138	·214	·216	·173	·120	·073	·034
·248	·413	·467	·428	·341	·232	·116
·073	·118	·131	·117	·088	·058	·028
·170	·294	·346	·335	·277	·192	·099
·038	·063	·075	·070	·057	·038	·019
·106	·187	·227	·225	·190	·135	·071
·017	·028	·035	·032	·026	·018	·009
·051	·090	·112	·113	·097	·070	·036

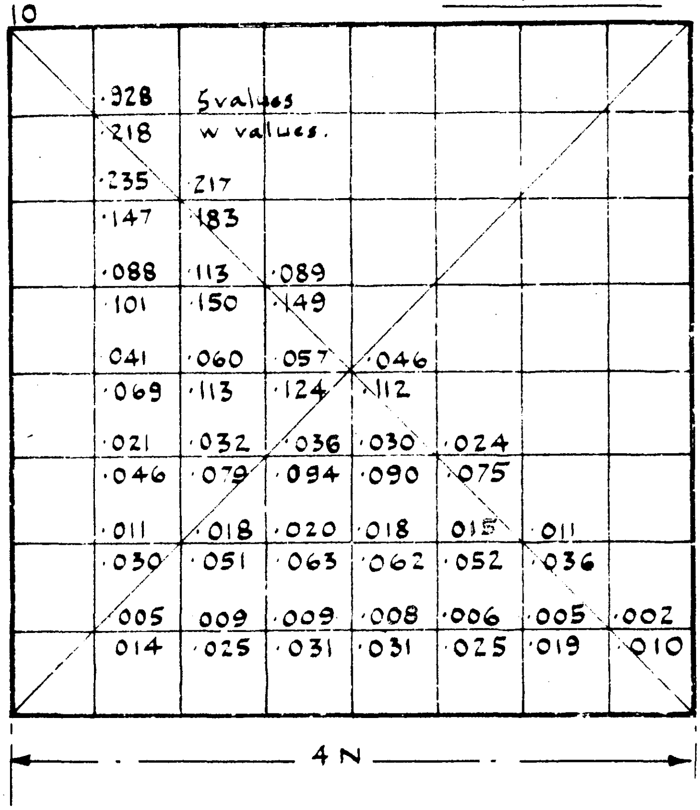
4N

— $\nabla^4 w = 0$ —

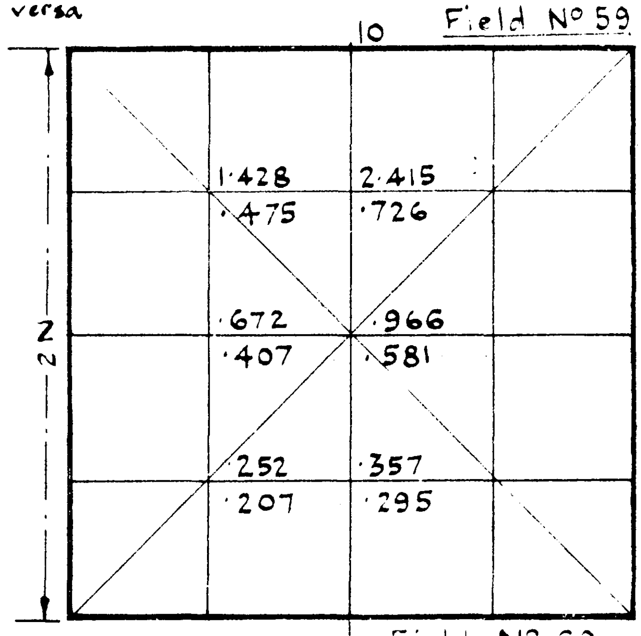
If the ζ values are -ve, the w values are +ve, and vice versa

Multiply the ζ units by N^2 to get w units.

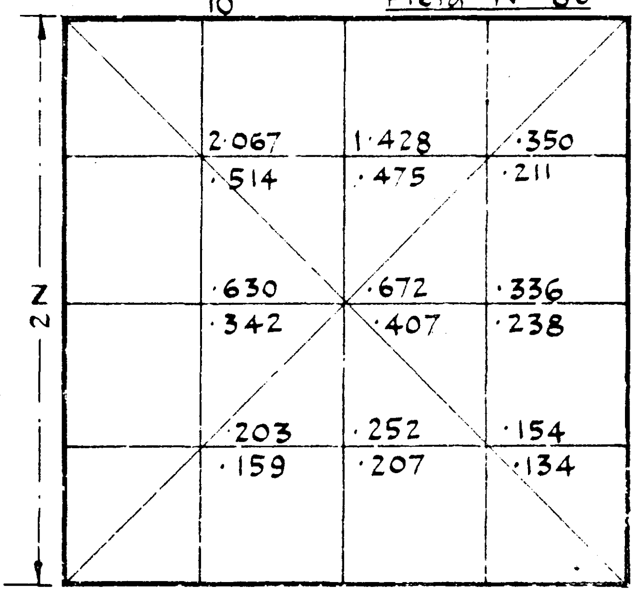
Field No. 58



Field No 59



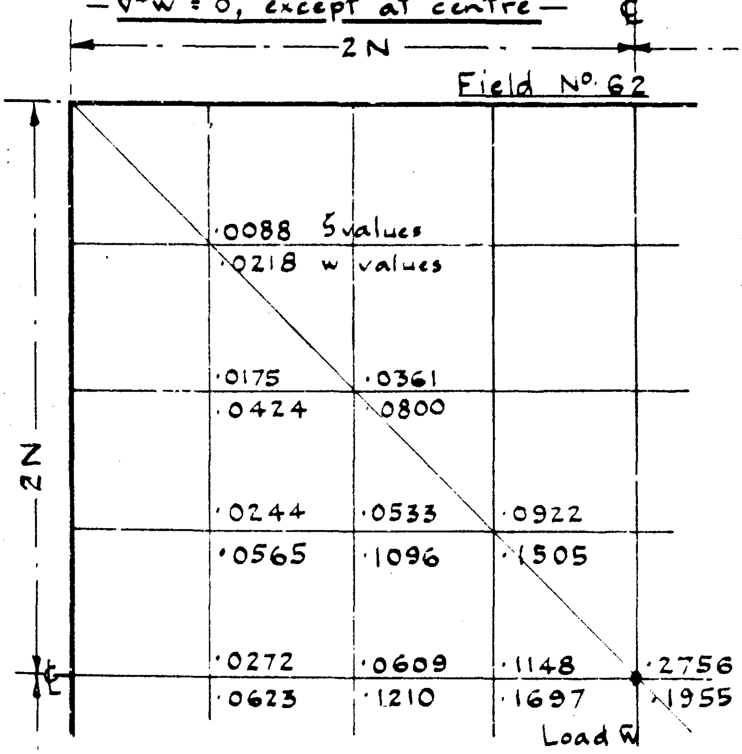
Field No 60



Concentrated Load $\bar{W}$ at centre

— $\nabla^4 w = 0$, except at centre —

Field No. 62



ζ values are -ve, and have units $\bar{W}/K$

w values are +ve and have units $\bar{W}N^2/K$

Field No. 61