# SUBJECTS OF THESIS

## I.

SOME OF THE TEXT-BOOKS USED IN THE TEACHING OF ELEMENTARY MATHEMATICS IN SCOTLAND PRIOR TO THE YEAR 1800.

## II.

DEVELOPMENT OF THE MATHEMATICAL CURRICULUM IN THE SCOTTISH SCHOOLS DURING THE SAME PERIOD.

by

DUNCAN KIPPEN WILSON, M.A., B.Sc.

# PREFACE.

In dealing with the first part of my subject I have confined myself in the main to text-books written by Scotsmen and published in Scotland. In a few cases where there is clear evidence that a text-book, written by an English or a foreign author, was actually used in Scotland, I have departed from this rule. In my search for these books I have visited most of the bigger libraries in Glasgow, Edinburgh, Aberdeen St. Andrews and Perth. The reviews of the books are, of course, the result of actual reading. In the case of the earlier text-books I have indulged in a more detailed description than in those of the eighteenth century, where the differences from the modern books are less marked. In Appendix C. I have noted some books on elementary mathematics, copies of which I have been unable to obtain.

A question of some interest is that of whether text-books were actually handled by the pupils themselves. On this point, there is sufficient evidence of their use by the students of the Scottish Universities and Commercial Schools, but, as regards the pupils in parochial and grammar schools and Academies, I am inclined to the view that, until the beginning of the nineteenth century, mathematical text-books were not as a rule put into the hands of the scholars. My reasons for this conclusion are two-fold/

two-fold. In the first place, lists of the text-books used in the famous Grammar Schools of Scotland prior to 1800 contain no mathematical works. In the second place, an examination of some mathematical note-books compiled by pupils of Perth Academy towards the end of the eighteenth century, shows these to be so exhaustive, and so laden with matter that would naturally have appeared in a text-book, that one is forced to the conclusion that these note-books were meant to serve as text-books. The cheaper text-books on arithmetic, such as the Cocker, Melrose, and Gray, were doubtless exceptions to this general rule.

# CHAPTER I.

## MATHEMATICAL TEACHING IN SCOTLAND PRIOR TO THE FOUNDING OF THE UNIVERSITIES IN THE FIFTEENTH CENTURY.

In treating of the development of the mathematical curriculum in Scottish Schools, I have found it convenient to distinguish five different historical periods. The first covers the years from the beginning of Scottish education, with the settlement of Columba in Iona, to the founding of St. Andrews University in 1411. The second includes 150 years from the founding of the universities to the Reformation. The third period extends from the Reformation in 1560 to about 1620. The fourth includes the remainder of the seventeenth century. The fifth covers the eighteenth century, but historically the Revolution of 1688 marks the division between the fourth and fifth periods.

In the early years with which this chapter deals, the schools, whether Cathedral, Abbey or Collegiate institutions, were directly under the control of the Church. And we must look at the records of the ancient monasteries, convents and abbeys if we hope to get information as to the subjects of study in the Scottish schools of that age.

A/

A good many of the registers and chartularies have been published by the Bannatyne, Maitland and Spalding clubs, but in most of them we look in vain for any reference to the books and manuscripts contained in the libraries. Perhaps the most famous of these libraries was that at Iona, but, according to Dr. Jamieson, "practically no authentic record of its contents can be given".X [1] The monastery is commonly reputed to have been burned by the Danes in 797 and again in 801. All that can be definitely asserted is that "in the ninth century the only book they had (referring to the works of the fathers) was one of the writings of Chrysostom".X [1] This is interesting in view of the fact recorded by the anonymous biographer of Chrysostom that he was acquainted with "the observation and exact calculation of Easter".X [2]

The oldest library of which we have definite records is that of the Culdees in the Priory of Lochleven. This consisted of sixteen books, of which eleven are Latin texts of books of the Old and New Testaments, or of ordinances and regulations for worship. Of the remainder, one contains the works of Origen, which are philosophical and theological in character. Another is the Liber Sententiarum of St. Bernard, which is also a theological work. There is a "Bibliotheca"/

X[1] An Historical Account of the Ancient Culdees of Iona - John Jamieson, D.D., p. 316.

X[2] Anony. Vit. Chrysost. V. Toland's Nazarenus, p. 56. (q. Jamieson)

"Bibliotheca", which is "probably the Vulgate of St. Jerome". X[1] Finally, there are two books "Interpretationes Dictionum" and "Collectio Sententiarum". None of these sixteen books, so far as they are known, contains any of the theoretical or practical arithmetic or geometry which was current in Europe at the date (1152 A.D.) on which the charter "granting the isle of Lochleven to the Priory of St. Andrews, with all its pertinents including these books", was written.

A much more pretentious library is that of the Cathedral Church of Glasgow, catalogued in 1432. Particulars of the 165 books, contained therein, are given in Archaeologia Scotica, Vol. II in a paper "communicated by Mr. Dillon" 1822. Out of the whole list, only one book definitely contains a part devoted to Mathematics, viz. the Origins or Etymologies of Isidore, Bishop of Seville, (unless one unnamed book of St. Augustine included his references to the subject). The only other writers of books on Mathematics who appear in the list are Bede and Boethius; Bede contributes a book of forty homilies on the Gospels, and the "Historia Ecclesiastica". Boethius is represented by the "De Consolatione". Finally, it should be added that one of the volumes is a Latin translation of the "De celo et mundo" of Aristotle.

The/

X[1] Scotland in the Middle Ages - Cosmo Innes, p. 333.

The same volume of the Archaeologia Scotica contains a list of books and manuscripts which belonged to the Franciscan Convent in Aberdeen at the time of the Reformation, but the only notable books in the collection are the Hist. Naturalis Isiodori, and the Geographia Pomponii Melae.

The Registrum Episcopatus Aberdonensis contains two catalogues of books compiled in the years 1436 and 1464 respectively.[x1] Out of 190 books mentioned there is not one devoted to Mathematics.

Now the libraries at Glasgow and Aberdeen were extensive, considering the expense and difficulty in obtaining manuscripts at the period, so that we are forced to the conclusion that the man of learning in the fourteenth century in Scotland was satisfied in his pursuit of mathematical knowledge with the encyclopaedias of Isidore and possibly St. Augustine. Isidore (570-636), Bishop of Seville was the most learned man of his age, but he believed that the "monks were better without any knowledge outside the Scriptures and the writings of the fathers". [x2] He compiled an encyclopaedia in twenty books entitled S. Isidori Hispalensis Episcopi Originum sive Etymologiarum. It was one of the first books on arithmetic to be printed, the first edition appearing in 1472 [x3] at Augsburg. The edition I have read is dated 1617. Books I and II deal with the Trivium and Book III with the Quadrivium of Arts. The whole of the contents/

x1 Volume II, pp. 127 and 154 - published by Maitland Club.

x2 History of Western Education - Wm. Boyd, p. 110.

x3 Rara Arithmetica - D. E. Smith, p. 8.

contents of Book III occupy some ten very large pages, the greater portion of which is concerned with music and astronomy. The arithmetic covers only two pages and geometry is treated in half a page. The arithmetic, the author tells us, is based on the writings of Pythagoras, Nicomachus, Apuleius and Boethius. The book is more of the nature of a dictionary of words than a text-book; it contains some ridiculous versions of the etymology of the Latin nomenclature.

The universality of number in human affairs and the sacredness of certain numbers, such as six and forty, are seriously discussed. The remainder of the treatise consists of an account of four different systems by which numbers can be investigated. The first is the separation into odd and even; here there is a further extension into (a) evenly even (pariter par) i.e. powers of two, like 32 and 64; (b) evenly odd i.e. numbers which are divisible by two but not by four; (c) oddly even i.e. numbers which are divisible continuously by two at least twice, but the final quotient is not unity; (d) oddly odd i.e. powers of odd numbers such as 25, 49, etc. Odd numbers, again, may be classified as simplex (prime) or compositus: two composite numbers which are prime to one another are called mediocres. Again, a number is either superfluus (abundant), diminutus (deficient), or perfectus according as the sum of all the factors, prime and composite, is greater than, less than or equal to the number itself. The three perfect numbers 6, 28 and 496 are given in the text. Thereafter the question of/

of the relationships between two unequal numbers, provides a second line of investigation. Two numbers may be equal or unequal. If unequal, the one stands to the other in the relationship of "major" or "minor". If the relationship is the <u>major</u> one, then it is either multiplex superparticularis, superpartiens, multiplex superparticularis or multiplex superpartiens. If the relationship is <u>minor</u>, then it is submultiplex, subsuperparticularis, subsuperpartiens, submultiplex superparticularis or submultiplex superpartiens. One or two examples of each of these types of ratio are given.

The third method of considering number is the geometrical one, i.e. we may classify numbers as linear, superficial or solid. It is worthy of note here that Isidore calls a complete square a "circularis numerus" and a complete cube a "sphaericus numerus".

The last chapter of the arithmetic is a short one devoted to the solution of the problem of finding the arithmetical, geometric and musical (harmonic) mean of two numbers.

The part of the treatise devoted to geometry is little more than a list of names of the better known, plane figures and solids; these are not even fully defined. The point, line, straight line, surface and circle are given definitions on Euclidean lines, and that completes the geometrical part of the quadrivium. Considerably more space is allowed for the treatment of music and astronomy as these had definitely an utilitarian purpose, but the discussion is non-mathematical.

In/

In Book V of the "Originorum" an account of the calendar is given. This consists of definitions of such words as day and night, week, months of the year (with etymology of the names), season, year - olympiad, age and cycle. A table is given of the six ages of the world from Adam to the date of writing!

The whole of the arithmetic and geometry of Isidore could be acquired in perhaps one or two hours; the book represented a very threadbare version of the "ἀριθμητικη" of the Greeks, as transmitted by Boethius in his "Arithmetica Boetii", and it had little cultural or educative value. It is unlikely, however, that the arithmetic taught in the schools of Scotland in the fourteenth or earlier centuries was confined to the theoretical side. The teaching of arithmetic is largely dictated by its utilitarian side, and in the life of the clergy of those times that side would assert itself in connection with two aspects of life. They would require to be able to calculate the date of Easter in each year, and secondly some at least would have to acquire a knowledge of "writing and keeping accompts".

As regards the former, a number of books have been written on the subject and they are known generally as "Computi". One of the best known of these is included in the works of the Venerable Bede, most of which were in the library of Glasgow Cathedral. Among the scientific works of Bede which have not been rejected by modern authorities as spurious are the "De Natura Rerum" and the "De/

"De Temporibus", but these are merely abridgements of a larger work entitled "De Temporum Ratione".

The first chapter of this work is headed "De Computo vel Loquela Digitorum" and it gives one of the very rare accounts of finger-reckoning which are extant. This system must have had considerable vogue, for, as late as the fifteenth century, in an English manuscript, there is a description of finger reckoning substantially the same as that of Bede. Indeed, it is said that, in the monastic and cathedral schools of Europe down to the fourteenth century the teaching of arithmetic was confined to finger-reckoning.X[1] David Eugene Smith, however, considers the "De Loquela" ~~as~~ probably spurious.X[2]

Here is the code as indicated in the book:-

No. 1 is indicated by holding up the thumb and three fingers of left hand, keeping fourth finger shut.

" 2. Thumb and first and second fingers open, third and fourth shut.

" 3. Thumb and first finger open, others shut.

" 4. Thumb, first and fourth open, second and third shut.

" 5. Second finger closed, others open.

" 6/

? Kunst

X[1] Villicus Geschichte de Reckenhurst, p. 81. Wien, 1897.

X[2] Rara Arithmetica, p. 131.

No. 6. Third finger closed, others open.

" 7. Like 1, but in closing the fourth finger, only the middle joint is crooked, the other remaining straight.

" 8. Like 2, and No. 9 like 3, with the same difference as in the case of 7 and 1.

For 10 and its multiples up to 90, a new system is started, still with the left hand:-

10 : All the fingers are stretched out except the first which is bent round to touch the middle joint of the thumb.

20 : All the fingers straight, thumb pointing to partition of first and second fingers.

30 : Like 10, but forefinger touches end of thumb.

40 : Cross thumb and forefinger, keep others straight.

50 : Like 20, but thumb comes to partition of second and third fingers.

60 : Like 40, but thumb is crooked over first finger.

70 : Bend the first finger, and set the thumb between first and second fingers.

80 : Like 40, but finger is over thumb instead of thumb over finger.

90 : Bend the first finger and set the end of it in the innermost joint of the thumb.

100 to 900 are similar to 10 to 90, but the right hand is substituted for the left.

1000 to 9000 are similar to 1 to 9, with the right hand for the left.

There is here a curious inversion of what one would have considered the natural arrangement, and in the fifteenth century manuscript/

manuscript referred to, the more natural arrangement appears, i.e. 100 to 900 correspond to 1 to 9, and 1000 to 9000 correspond to 10 to 90.

Bede carries the system on from 10,000 to 90,000 by placing the fingers and thumb of left hand in various positions on the breast, and from 100,000 to 900,000 by placing right hand on breast, and he is able to indicate numbers up to 10 millions by using both hands on the breast. Details would be tedious.

A few chapters of the "De Temporum Ratione" are devoted to "tables". As regards divisions of time, Bede points out that Nature decrees that a solar year shall consist of $365\frac{1}{4}$ days, while a lunar year consists of 354 days. Custom again makes a month consist of 30 days which does not fit in with either solar or lunar year.

The table of money values applies pari passu to weights. The pound ("libra"), whether of money or weight is divided into twelve ounces, and separate names are given for 11, 10, 9, 8 down to 1 ounce. The biblical units of length, area, etc. are also explained.

The remainder of this treatise is devoted to an account of the Jewish and Roman calendars, and to the calculation of the date of Easter. It contains a full account of the "Decenno-novalis" or nineteen-years lunar cycle, and the twenty-eight years solar cycle, from which he deduces a period of 532 years after which the relative positions of sun and moon are again the same. He gives the date of each/

each of the new moons in the first year of the Decennonovalis and the variations or epacts for each successive year of the cycle, and explains how the date of Easter is determined. From the calculations involved we can acquire some idea as to the stage of advance in arithmetic at the beginning of the eighth century. Numbers are indicated sometimes by the Latin words, at other times by Roman letters. (In one chapter, through an oversight, the editor commits the anachronism of introducing the Arabic notation). The arithmetical work is restricted to addition, subtraction, multiplication and division of integers. These operations would be performed [redacted], multiplication by continued addition, and division by continued subtraction. The most difficult division is by 59, and it is usually the remainder that is required. Thus in "Caput XXII" instructions are given for finding the age of the moon on any particular day of the year: count the number of days from the 1st of January to the date in question, add the age of the moon on the 1st of Jamuary, divide by 59, and the remainder, if less than 30 is the age of the moon; if greater than 30, subtract 30 from the remainder to get the age of the moon. To aid the neophyte in the summation of the number of days from the 1st of January, Bede gives a table showing how many days elapse between the 1st of January and the Calends, Nones and Ides of each month. "Si ergo scire vis, verbi gratia, anno quo per Calendas Januarias nona est luna, quota sit luna in Calenda Maias, dicito, Maias in Calendas CXXI tolle Calendas remanent CXX/

CXX, adde IX fiunt CXXIX partire per LIX quinquagies novies bini centumdecusoctus: tolle CXVIII remanent XI undecima est luna in Calendas Maias." It is evident from the book that even this amount of arithmetic was too great for most of the readers of the manuscript. We find Bede explaining (for the benefit of those who are unable to count) how to read the result from a book of tables constructed for the purpose.

In "Caput LII" we get an insight into his method of division: he wishes to calculate the epact for the year in which he is writing. DCCXXV : "hos partire per XIX. decies novies triceni DLXX, decies novies octoni cenquinquiais dipondius, remanent III. hos item multiplica per XI fuint XXXIII tolle XXX remanent III. Tres sunt epactae." It is clear from that that he took 30 times 19 from the abacus, leaving a remainder of 155, then 8 times 19 from that again leaving 3.

An instance of the important part played by the fingers in the Arithmetic of those days is provided in Chapter 55, where there is put forward a method of keeping track of the 19 lunar years and 28 years solar cycle. In one hand there are 19 joints counting the nails; these in order starting with the lower joint of the thumb can be taken to represent the 19 separate years of the lunar cycle. Again in the two hands there are 28 joints (nails not being counted this time); these can represent each of the 28 solar years.

At the end of the treatise Bede gives a brief history of the world/

world and states that, at the time he wrote, 4,680 years had passed since the Creation.

From this brief account of the Computus as revealed by Beda, it seems that the calculations were considered beyond the capacity of any but expert mathematicians, and that, in practice, the date of Easter was found from books of tables. The absence of any indication of a text-book on the Computus in the libraries mentioned above leads one to the conclusion that probably very few, if any, schools in Scotland were sufficiently advanced in mathematical study for the calculations necessary.

Some knowledge of practical arithmetic was essential for another purpose, namely the keeping of accounts. The monasteries and other religious houses must have had clerks trained for this purpose. Their calculations were not written down, but the results were. While apparently these "sums" were done in an early stage by finger-reckoning, it is certain that, before the end of the fourteenth century, the abacus had replaced the laborious finger counting. In the Grammar Schools, too, by the end of the period under review, the elements of counting must have been taught. Cosmo Innes in his "Sketches of Early Scottish History" says "We find merchants writing and keeping accounts and corresponding with foreigners in their own language, who must have received their education early in that (15th) century".[X] The Abacus/

X Sketches of Early Scottish History, p. 272.

Abacus was constructed first by means of sand sprinkled on the floor or table, later by a table cloth marked in squares or a frame containing a number of counters strung on wire. According to Dr. Poole,[X] the method of calculating by the Abacus was probably introduced into England by Adelard of Bath at the beginning of the twelfth century, when the Exchequer was established as a method of auditing accounts. There are several text-books on the use of the Abacus, but it is not surprising to find no reference to them in Scottish records, since the processes of addition and subtraction by means of that instrument are so easy that they would not appear in books. It is probable that at the date we are considering, these operations alone were known, multiplication and division being done by continued applications of the simpler processes. For the sake of completeness, I propose to give a short account of one English book dealing with the Abacus. It is entitled "Accomptynge by Counters" and the author is the ill-starred Robert Recorde. Though written in 1543, it refers to a method of counting older than any of the pen methods, so that I prefer to include it here.

Like all the books of this brilliant teacher, it takes the form of a dialogue between master and pupil: the former has already imparted instruction in counting with the pen; this method is to be adopted when no pen is available, and is understandable by those who can neither read nor

write/

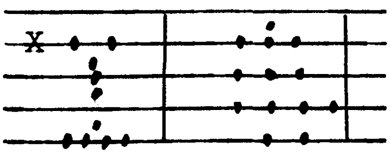
X The Exchequer in the Twelfth Century - R. L. Poole, Ch. III.

write.

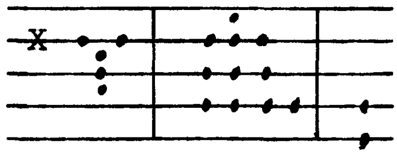
Pupil is given a tablet containing nine horizontal parallel lines in groups of threes. He has also as many counters as he requires. The bottom line represents units, the next tens and so on; the fourth and seventh lines are marked with a cross to indicate thousands and millions. To save counters it is agreed that one counter between the lines represents 5, Thus 1683 would appear thus:

Here is an addition sum:-

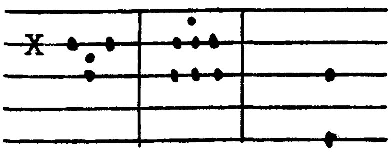
2659 + 8342



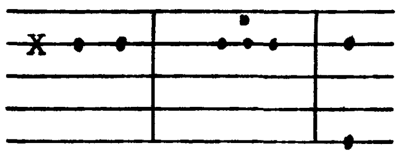
1st stage



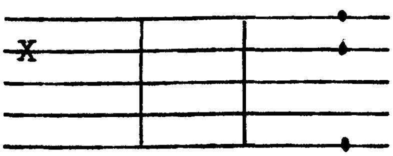
2nd stage



3rd stage



Last stage



Ans. 11001.

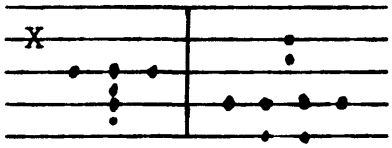
Subtraction/

Subtraction is done similarly, but this time they start from the top, though Recorde admits that some "begin at the nethermoste" as in addition. Care must be taken to check subtraction by addition and vice-versa.

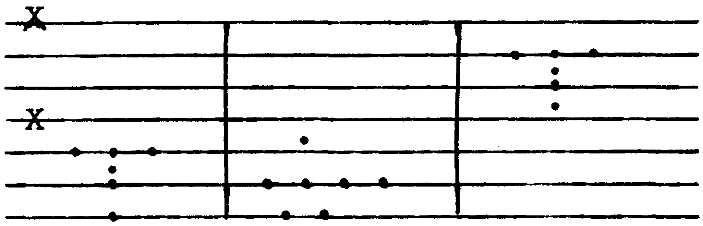
Here is a multiplication

1542 X 365

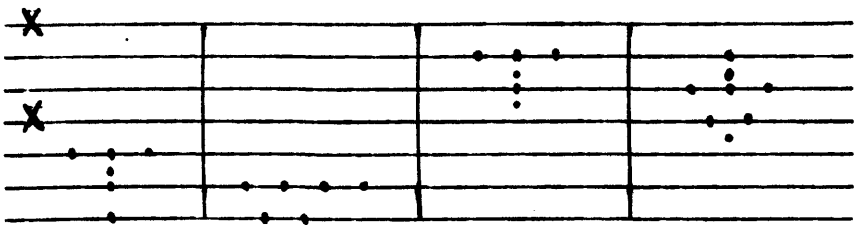
Set these numbers down, the multiplier before the multiplicand:



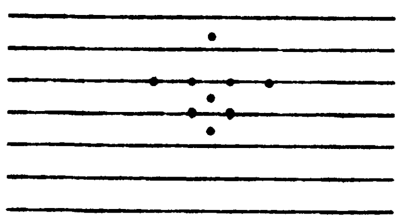

Now multiply the thousands first by 365 and set it down thus:




In the next space there is a counter representing 500. Take $\frac{1}{2}$ of the multiplier and call it thousands ($182\frac{1}{2}$), put a counter in the space below for the odd $\frac{1}{2}$.

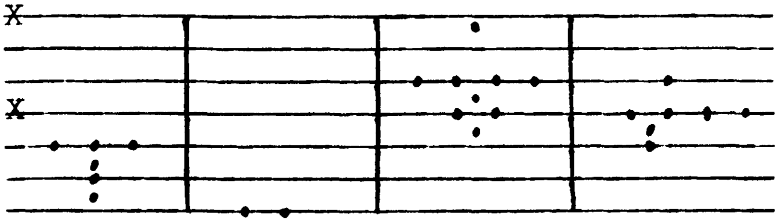



Put this in one sum and it becomes

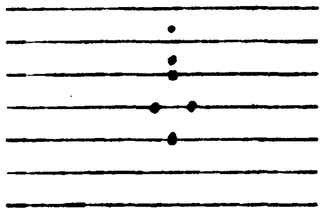


Now/

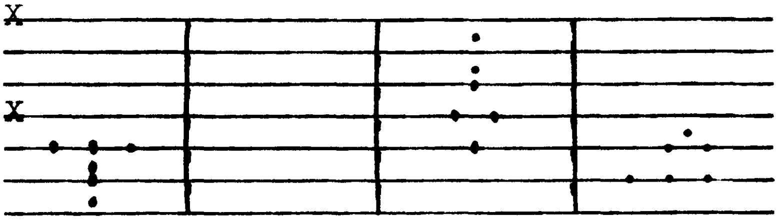
Now take the next figure



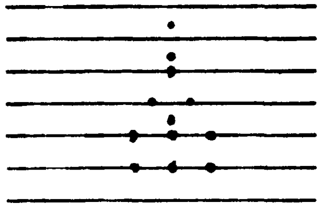
In one line



Now multiply by 2



Answer

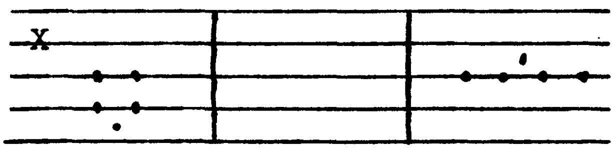


562830

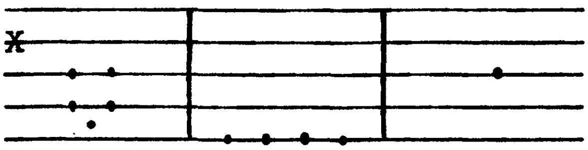
In division he proceeds as follows:-

"If 225 shepe cost 45 l'i, what dyd every sheep cost?"

He sets down the divisor first, the dividend afterwards, leaving a space between for the quotient



He begins at the highest line of the dividend and finds how often the divisor goes into 900s. It goes 4 times; 4 X 2 = 8. He takes from 9 and there remains 1

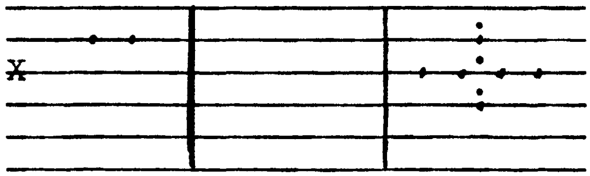


Now/

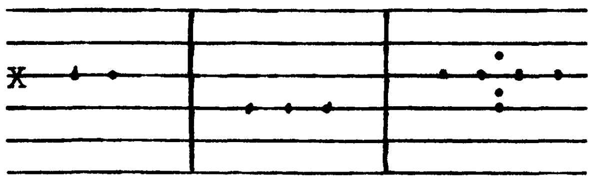
Now 4 X 2 = 8.  Take 8 from 10, there remains 2


4 X 5 - 20 cuts out the two counters left.  Answer "Every sheep costs 4s."

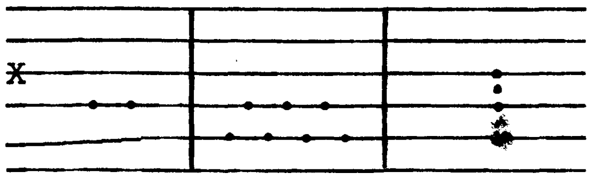
If necessary, he alters the place of the divisor to get it opposite the highest place in the dividend:  e.g. 69600 is to be divided by 200.  The first part of the answer is set down thus:



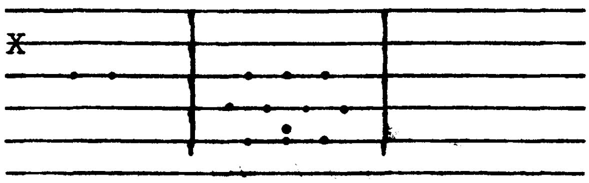

2 goes into 6 three times, but the 3 is not put in the units place but is raised two places;  next time the divisor is set one place lower.




Next stage




Last stage

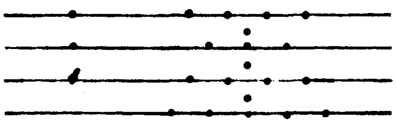



In/

In every case division is verified by multiplication and vice-versa.

The remainder of this book is taken up with a description of merchant's casting. In setting out a sum of money with the counters the lowest line stands for pence, a counter in the middle represents sixpence; the next line represents shillings, a counter in the space above counting ten. Counters are put at the left hand side to represent five. Next line represents pounds and a counter in the space above counts ten. Next line above stands for scores of pounds. Merchants set the following counters for

198 l'i 19s 11d'



The farthings, if any, are put down in a corner at the right hand side, one counter for each farthing; when writing down a sum of money,

q = one counter means one farthing
ob = two counters means one halfpenny
obq = three counters means three farthings

"Accomptynge by Counters", though it was written after the invention of printing, and after the adoption of the Arabic notation gives us an insight into the methods of teaching arithmetic in the earliest schools of Scotland at a time before the Arabic notation was known or widely used. The method of "Accomptynge by Counters" survived in England till the seventeenth century (X).

It/

(X) "The Earliest Arithmetics in English" - Robert Steele.

A review of the teaching of arithmetic in the early middle ages would not be complete without some reference to the new science, known then as "Algorismus" which was struggling to oust the Abacus. The Hindu-Arabic numerals, introduced into Europe in the Twelfth Century, were in universal use among scientific men by the end of the thirteenth century: by the year 1350 at latest "these numerals were familiar both to mathematicians and to merchants".[X] Though doubtless the system of numeration was known previously, text-books on Algorism do not appear in Scotland until the fifteenth century, with the establishment of the Universities.

To sum up, the content of mathematical teaching in Scotland in the pre-University era seems to have included in Arithmetic a few of the most elementary properties of number and ratio, the meaning of arithmetical, geometric and harmonic proportion (as applied to three numbers) and a knowledge of the calendar and of the use of the Abacus. In Geometry, it involved instruction in the shape of the commoner plane figures and solids, together with some of the simpler definitions from Euclid.

---

X Short Account of the History of Mathematics - W. W. Ball, p. 161.

# CHAPTER II.

## FROM THE FOUNDING OF ST. ANDREWS UNIVERSITY TO THE REFORMATION.

Two outstanding events mark this period in European history: the Renaissance and the invention of printing. This was an epoch of great advances in Scottish education, not only in the Universities but in the schools also. Apart from the Cathedral Schools, there were three distinct types of scholastic institutions in Scotland at this time.

The burgh school was by far the most important of these; by the end of the fifteenth century burgh schools had been "planted in every considerable town in Scotland".[X1] The subjects of instruction in these schools are enumerated in the famous Act of James IV (1496) for the compulsory education of the eldest sons and heirs of burgesses and freeholders "fra thai be auct or nyne zeires of age quhill they be competentlie foundit and have perfite Latyne"; thereafter they had to remain at the schools of art and "jure".[X2] Though/

X1 Burgh Schools of Scotland - Grant, p. 25.

X2 Acts of Parliament of Scotland, 1496 c 3 11238, q. Grant.

Though Latin was the chief subject of instruction, both at this period and later, in the Grammar Schools, yet there is one reference to the teaching of arithmetic. In one of the statutes of the Grammar School of Aberdeen under the date 1553, it is stated that boys had to acquire a moderate knowledge of arithmetic, "numerandi artem modice praelibent".X[1] We have no information as to what constituted a moderate course of arithmetic; it would certainly be practical arithmetic, including the Abacus and possibly the elements of algorism. That arithmetic was reckoned of little value educationally, is clear from the fact that the Rectors of the Grammar Schools, though claiming a monopoly in the teaching of Latin in their own towns, yet permitted other schools to teach reading, writing and arithmetic. The schools in which these subjects were taught, were called English or Lecture schools, but little is known about them prior to the Reformation. Of one such school in Stirling it was laid down that masters were not allowed "to lair ony barnis abon sax yeiris without licence of the said master (William Gullein, Master of the Grammar School of Stirling) except tham that leris to reid and wryt and lay compt".X[2]

As to the third type of lay school, the Sang school, no other subject was taught there than Music "meaners and vertew".

Before/

X[1] Miscellany of the Spalding Club V. 400, q. Grant.

X[2] Hutchison "High School of Stirling", p. 12.

Before I discuss the development of mathematical teaching in the Monastery schools, I propose to give an account of the position of that subject in the Universities during the fifteenth century. That mathematics occupied a very subordinate position in the Arts curriculum, is abundantly clear. The main subjects of study were Logic and Philosophy. If you wish to know the subjects taught in our Universities during their infancy "you have simply to know what are the writings of Aristotle".x[1] "The Arts curriculum was not on the ordinary mediaeval lines. The principal subjects belonged to Aristotle's theoretical and not to his practical class of studies, and were not included in the sacred seven liberal arts. The trivium, indeed, was taught, but its subjects occupied a subordinate position, and scarcely any provision was made for the quadrivium".x[2]

The position of mathematics in the University curriculum is made clear by a statute of the Faculty of Arts, Glasgow University, (dated 1482 or 1483) with reference to books. After setting forth the books that are compulsory, e.g. "liber Universalium Porphurii, liber Predicamentorum Aristotelis, duo libri Peri Hermeneias ejusdem. In nova logica, duo libri priorum analyticorum, duo posteriorum..... In Philosophia, octo libri Physicorum, tres decelo et mundo, duo de generacione/

x[1] Rectorial Address to Aberdeen University, 1882. A. Bain, p.12.

x[2] Some Notes on the History of University Education in Scotland - Glasgow Arch. Soc., p. 2.

generacione et corruptione.....", the statute gives a list of optional ones. "Item audiantur libri extraordinarii in toto vel in parte, ubi facultas mature dispensabit, si fiat defectus; scilicet in Logica, textus Petri Hispani cum syncategorimatibus, .......In Philosophia, tres libri metheorologicorum; tractatus de spera, sine dispensacione, sex libri Ethicorum si legantur, perspectiva, algorismus et principia geometrie si legantur." X[1] These were the days of Regents, who taught every subject in the curriculum, and the probability is that the optional subjects were taught when the Regent happened to know something about them. This statute furnishes us with some information on text-books. The "Tractatas de Spera" is a well-known work by Sacrobosco. "Algorismus" is a generic name for the Hindu-Arabic arithmetic, but one of the most famous books on the subject was written by Sacrobosco, and it is probable that his treatise is the book referred to. X[3] As regards the "Principia Geometrie" it is hardly likely that the geometry of Euclid is intended, but rather the version of it transmitted by Boethius. A short account of these text-books will be given later. The association of perspective with algorism and geometry suggests that these subjects may have been contained in a "Compendium" such as that of Faber Stapulensis, which was "very popular in the University of Paris at the opening of the sixteenth century/ X[2]

X[1] Munimenta Alme Universitatis Glasguensis - Liber Statuorum Facultatis Arcum Studij.

X[2] Rara Arithmetica - D. E. Smith, p. 82.

X[3] A list of books presented to Glasgow University in 1483 associates "De Spera et Algorismo" - Life of Melville - McCrie p 431.

century". The book contained an introduction to the arithmetics of Boethius and Jordanus, and to geometry and perspective. I fancy, however, that, if one book was intended by the regulation quoted, the singular *legatur* would have been used, instead of the plural *legantur*; moreover, the compendium by Faber Stapulensis was not printed until 1488, and the arithmetical part of the book is concerned with the theory of numbers and ratios, and not with algorism.

There is one reference to the teaching of mathematics in King's College, Aberdeen, (1549). It occurs in connection with a visitation of the Chancellor, William, Bishop of Aberdeen, when regulations were made for the appointment of a Regent, "qui etiam sit in artibus eruditus et doctus ac qui possit cursum regentie sue perficere ante fluxum sexennii: et quod de cetero regens sic electus in artibus compleat logicalia primo anno; 2° anno phisicam et naturalem philosophiam cum tractatu de sphaera; tertio anno arithmeticam, geometriam et cosmographiam cum moral philosophia....." X[1]. This shows that, by the middle of the sixteenth century, mathematics had a definite place in the University curriculum, but, curiously enough, the course in Natural Philosophy preceded that in Mathematics.

Further ~~light~~ information on the text-books of the middle of the sixteenth century has come down to us from the Records/

X[1] Fasti Aberdonensis, p. 265.

records of the monastery of Kinloss. From 1531 to 1537 and again from 1540 to 1543, the monks of Kinloss had the benefit of instruction from an Italian scholar, Ferrerius. "John Ferrerius was a native of Piedmont, who, being in Paris, in pursuit of his studies at the University there, became the friend of Robert Reid, then on his return from Rome to Scotland, carrying with him the Papal bulls which conferred on him the Abbacy of Kinloss; Ferrerius was induced by the Abbot to accompany him to Scotland in 1528." (X) From 1531 to 1536 Ferrerius resided in Kinloss studying and teaching. "In 1537 ..... he resolved, before returning to Italy, to preserve to posterity an account of his lectures and the authors on whose works he had prelected." These include a number of the works of Cicero and Virgil, Aristotle's Ethics and Logic, Rhetoric of Melancthon, etc., and in addition they contain the following:-

Item. Sphaeram a Sacrobosco.
Item. Arithmeticam nostram.
Item. libros quinque Physicorum Aristotelis.

Again, Ferrerius in an account of his lectures to the monks of Kinloss says of the subjects taught:-

Item. Praxin arithmetices quam nostra marte scripsimus.
Item. Sphaeram Joannis Sacrobosco.
Item. Paraphrasin Fabri in octo libros Physicorum Aristotelis.

He also mentions "commentaries written by me in Kinloss."

Item/

(X) Records of the monastery of Kinloss edited by John Stuart, LL.D. Preface page XIII et seq.

Item. De vera cometae significatione ad Jacobum quintum Scotorum Regem. Lib I.

Item. De numerorum praxi Lib I. and a number of other books including

Item. Annotationum in Physicen Aristotelis Lib I.

Item. Commentariorum in quatuor libros Apodixium Euclides per Boethium versorum Lib 4.

From these references it is clear that Ferrerius gave his pupils an education of the "then" University standard. Unfortunately though several of his books have been preserved, the "Arithmeticam Nostram" and the book of Euclid seem to have been lost. I can find no trace of them, among the works of Ferrerius which have survived, in any of the catalogues of the most famous libraries in Europe. As he used the "Sphaeram a Sacrobosco", I imagine he would also be acquainted with the Tractatus de arte numerandi by the same author, and that his arithmetic would be based on it, though probably he would carry his pupils a little further than is done in that elementary work. It is greatly to be regretted that the "Arithmeticam Nostram" did not survive the excesses of the Reformation, as it would have provided us with information as to the state of mathematical teaching at the time.

The "Euclides per Boethium" appears among the works of Boethius which were published by Teubner in 1871. One of the books is entitled "Ars Geometrica"; according to Heath (X) it is not the work of Boethius, but was put together from his works in the eleventh century. It/

(X) Heath's Euclid, page 92.

It is in the main a statement of the definitions, axioms, postulates and propositions of the first four books of Euclid, without proofs. The writer confuses postulates with axioms, and axioms with definitions. After completing the enunciations of these theorems, he actually states the first three constructions of Book I (to show that he could do it presumably), and then proceeds, as becomes the practical Roman, to apply Euclid's theorems to mensuration. For this he has first to diverge into Arithmetic, explaining what are digits, articles and composite numbers, how the Abacus of Pythagoras can be used to express any number up to $10^{11}$ with the help of the "Apices" or "caracteres" which he invented himself. These apices, representing the digits 1 to 9, bear a certain resemblance to the modern digits.

Thereafter tables of length, area and solidity are given, and the remainder of the book is taken up with calculations of the areas of triangles, quadrilaterals and regular polygons (up to the decagon), the lengths of whose sides are simple integral numbers. In general these areal measurements are estimated by the help of Pythagoras's Theorem (and the extended Pythagoras), taking the square root of any number to an exact integer, but no explanations are given of the processes, and mistakes and misprints are numerous. In the case of the triangles, his estimates of the areas are either approximately or actually correct, but the areas of the regular polygons are nowhere near the truth. The Ars Geometrica of Boethius is a ~~[illegible]~~ substitute for Euclidean/

Euclidean Geometry.

The Arithmetic of Boethius, published along with his Geometry, was a standard text-book in Europe throughout the Middle Ages. There is little doubt that the "Arithmetica et Geometria" of the third year course in King's College, refers to the works of Boethius, in one or other of the many editions which appeared after the discovery of printing. The edition that I have read bears no date, but it corresponds to one described by David Eugene Smith, in Rara Arithmetica, p. 65, published in Paris in 1514. Bound together are the Arithmetic of Jordanus Nemorarius in ten books, a treatise on Music by Faber Stapulensis, the Epitome of the Arithmetica Boetij by Faber Stapulensis, a short account of the Rithmimachie, an arithmetical game, and the Geometry of Boethius. The Arithmetic of Boethius, dealing in great detail with the Greek theories of number and ratio as expounded by Nicomachus, is so well known that a detailed account of it would be superfluous. Nearly thirty editions of this famous book were printed between 1488 and 1570. In these Arabic numerals are used in place of the Roman symbols of the original text, so that a knowledge of algorism was a pre-requisite for the sixteenth century student of Boethius. One of the best known of the algoristic arithmetics came from the pen of a writer whose works were widely read in Scotland, the famous Sacrobosco/

Sacrobosco. John Holywood or Halifax was born probably [redacted] at Halifax in Yorkshire. He is said to have studied at Oxford and to have settled in Paris in 1230. He died there probably in 1256. The "Tractatus De Spera" was first published at Ferrara in 1472, and twentyfour more editions of it appeared before 1500. At least fortyfive further editions saw the light between 1500 and 1647, the date at which the last was published at Leyden. It was also translated into Italian 1537-1550, German 1516, 1519 and 1533 and Spanish. Sacrobosco also wrote "De Arte numerandi". These two works were standard authorities for four hundred years. (X)

Two further works of Sacrobosco were "De Computo Ecclesiastico" and a treatise on the Astrolabe. The fact that the "De Spera" and "De Arte numerandi" were used in Scottish Universities and in the Monastery Schools of Scotland is clear from what has been said above and from the contents of the Library of the Monastery of Kinloss

[redacted]

The copy of Sacrobosco's <u>Spera Mundi</u> in Glasgow University shows no publisher's name or date of publication; it is bound in leather on strong wooden boards and was gifted to Glasgow University in 1703 by one, Robert Woddrow of Eastwood; it bears the names Jo. Stirling, Principal and Johannes Gibsonus, also the words "Henricus Gibsonne.. ........meus est 1553." The blank word, I presume, is liber. There are four treatises bound together: all are in mediaeval Latin and the "Spera Mundi" is liberally interwritten with notes, leaving us in/

(X) History of Mathematics, F. Cajori. p 134 in no doubt that it was used as a text-book by a student.

The first treatise is a Cosmographia by Pomponius Mella and gives a geographical description of Europe, Asia and Africa.

The second treatise is a translation of a geographical work by Dionysius.

The third is entitled "Johannis de Sacrobosco anglici viri clarissimi Spera mundi feliciter incipit."

The fourth is named "Gerardi cremonensis viri clarissimi Theorica planetaru feliciter incipit." The latter has not been so carefully studied as the "Spera Mundi".

The Latin of all these books is difficult to read as the inflections are often dropped.

The fourth treatise bears the date MCCCLXXVIII "Impressa Venetiis per Francisca renner de Hailbrun."

I do not propose to enter into details of the Spera Mundi, as it is not really a mathematical treatise, but merely a digest of the Ptolemaic theory of the motions of the heavenly bodies round the earth as centre. The first chapter discusses Euclid's and Theodosius' definition of a sphere, explains the meaning of the axis and the poles, equator, horizon, etc., and gives an account of all the different spheres on which the heavenly bodies were supposed to rotate round the fixed earth.

The second chapter gives an account of the imaginary circles on the terrestrial and celestial sphere.

The/

The third chapter explains the rising and setting of the planets and the different lengths of night and day in different places.

Chapter IV deals with eclipses of the sun and moon.

Though this book is not a mathematical one, I have mentioned it here, as it gives us a clue to the amount of mathematics that would be required of a student who would hope to understand the Ptolemaic theory as here expounded. It demands a knowledge of the elementary properties of the sphere which follow from the definition, of axis and pole, of great and small circles, of the measurement of the angle between two planes, etc., but does not require any knowledge of spherical trigonometry.

There is a section at the end of this very interesting volume written in very poor Latin; no author's name is given, and it deals with the great mediaeval problem of fixing the date of Easter. It is called the "Computus Cyrometralis", and is written in the form of Latin verse, giving mnemonics by which the complicated rules for finding Easter may be remembered. ~~He~~ Author states rules for finding the Dominical letter of any year and for determining the day in which to celebrate such festivals as Septuagesima, Quadragesima (i.e. Lent), Pascha (Easter) and Rogationes (Pentecost), and shows when the Calends, Nones and Ides fall in each month. From the frequency of the use of the year 1330 in examples, that would appear to be the date of the work.

As regards the purely mathematical work in the first century of

he/

the Scottish Universities, it has been made clear already that the algorismus formed one of the optional subjects of study. The most popular work on algorism is that by Sacrobosco, which, under the title of "De Arte Numerandi Tractatus" was well known to Scottish students; it was actually translated into English. In "The Earliest Arithmetics in English" edited by Robert Steele, there is included an English translation of the De Arte Numerandi, and it is stated there that Sacrobosco's book was printed in 1488, 1490, 1501, 1503, 1510, 1517, 1521, 1522, 1523, 1582, and in "Rara Mathematica" in 1838 and 1841. The edition I have read is that of Halliwell's "Rara Mathematica", and is dated 1838. It comprises some 26 pages in Latin. The author gives the mediaeval division of integral numbers into digit, article, and composite. An article is a whole number divisible by ten. A composite is any number lying between two consecutive articles. There are nine rules in arithmetic:- "Numeratio, additio, subtractio, mediatio, duplatio, multiplicatio, divisio, progressio, et radicum extractio."

Under "numeratio" we get an account of the Arabic system. In "additio" he does not seem to contemplate the addition of more than two numbers; the top number is deleted and the answer written above, as it is found. This idea of deleting figures is carried on into subtraction, where, if the lower figure is greater than the upper, one is deducted from the next figure following (which has ten times the value/

value), that figure is deleted, and a figure one less is written in its place.

The most characteristic parts of the book are, perhaps, the sections devoted to Mediation and Duplation. In Mediation you start at the <u>right</u> hand of the number: if that digit is even, divide it by 2 and set down the answer; if odd, divide the next lower digit by 2 and set down the answer, adding at the right-hand side the "figura dimidii", which is written thus (di), or resolve that unit into sixty minutes and take half of 60. (Here we see the influence of the sexagesimal arithmetic of the Greeks). In proceeding with the rest of the mediation, if you are later compelled to divide an odd number, instead of putting down (di), you add 5 to the preceding figure, i.e. the one to the right. You must not forget the "figura dimidii", as that may be required in the next process.

Duplation, which is, of course, the converse of mediation, and can be used to verify the results of the latter, curiously enough starts at the left-hand side, as do multiplication and division. In the much hackneyed words of old

> "Subtrahis aut addis dextris vel mediabis
> A leva dupla, divide, multiplicaque
> Extrahe radicem semper sub parte sinistra."

Throughout the book Sacrobosco merely states the rules; he seldom works out any examples. The chapter devoted to multiplication is incomplete, some portions having apparently been lost. The author shows first how to multiply digit by digit, and for this purpose the reader/

reader requires to be acquainted with the multiplication table up to 5 times 5. If one digit is greater than 5 his rule is, subtract the lesser digit from the article of its own denomination as often as the greater digit is less than 10. Here he gives an actual example: 4 times 8. 8 from 10 leaves 2, subtract 4 twice from 40. Answer is 32.

After explaining how to multiply digit by article or composite, article by article, etc., he goes on to state the rules for multiplication in general, again giving no examples. To indicate the method I shall set down an example worked as stated in the rules:-

347 - multiplicand. 258 - multiplier.

$$\begin{array}{ccccc} & 1 & 3 & & \\ & 2 & 4 & & \\ & 8 & 3 & 5 & \\ 1 & 2 & 0 & 5 & \\ 6 & 5 & 4 & 2 & 6 \\ \cancel{3} & \cancel{4} & \cancel{7} & & \\ \cancel{2} & \cancel{5} & \cancel{8} & & \\ & \cancel{2} & \cancel{5} & \cancel{8} & \\ & & \cancel{2} & \cancel{5} & \cancel{8} \end{array}$$

Product 8 9 5 2 6

Each figure of the multiplier is multiplied separately by the "last" figure 3 of the multiplicand, answer being set above the figure of the multiplier concerned. Then you must anteriorate the multiplier (anteriorandae per unam differentiam) one place to the right. Then multiply the separate digits by the 4 of the multiplicand setting the digit part of each individual product above the figure of the multiplier concerned and the article part one place to the left. Though 258 is called the multiplier, it seems more like the multiplicand/

multiplicand.

The next chapter is on Division, but the manuscript is very incomplete. The method suggested in the book is not the "Scratch" method (which will be described later), but is just a kind of continued subtraction. Sacrobosco says that the dividend and divisor are set down with the last figure (i.e. the left-hand one) of the divisor below the last figure of the dividend (unless it be greater than the latter). The initial figure of the quotient is got by subtracting the last figure of the divisor from the last one or last two of dividend as often as possible, and is entered straight above the first figure of the divisor. Then all the figures of the lower line are subtracted that number of times from those above them. Here there is a hiatus in the manuscript, because he goes on to talk about anteriorating the divisor without explaining what that means in this connection, and about the occasional need to insert a zero in the quotient. Then he says that you should go on taking divisor from dividend until you come to the place where first figure of divisor is under first figure of dividend. If there is a remainder, set it to one side and write it on the slate (tabula). There is a translation of this work among "Steele's Earliest Arithmetics in English", and the writer there sets down three examples (the Latin manuscript shows none). These examples represent the finished article; the most complicated/

complicated one appears thus:-

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| The Residue | | | | | | 1 | 2 |
| The quocient | | | | 2 | 0 | 0 | 4 |
| To be dyvydede | 8 | 8 | 6 | 3 | 7 | 0 | 4 |
| The dyvysor | 4 | 4 | 2 | 3 | | | |

It will be observed that the "residue" is not set to one side. From the evidence of the text-book it is doubtful whether its author could have performed a really difficult long division.

The next chapter is on "Progressions". A Progression is defined as a collection of quantities which have a common difference of 1 or 2. If the common difference is 1, the progression is "natural or continuous". If the common difference is 2, the progression is "broken or discontinuous". He shows how to sum each of the following three progressions to an even or an odd number of terms:-

(a) 1, 2, 3, 4, 5, 6, 7..........
(b) 2, 4, 6, 8, 10..........
(c) 1, 3, 5, 7, 9,..........

In the next chapter he elaborates the ancient Greek theory of number, and explains the meaning of linear, artificial and solid numbers.

It is interesting to note that this is the ~~last~~ best section of the book, though it deals with a subject which is more abstruse to us, but less abstruse to his contemporaries than division.

In the last two chapters Sacrobosco lays down the rules for extracting square root and cube root, but without any examples to demonstrate/

demonstrate the process, so that it must have been exceedingly difficult to follow his teaching. I shall quote an example of each from the English translation to show how he meant the answer to be written down:-

(1) a square root

| | | | | | |
|---|---|---|---|---|---|
| The residue | | | | | |
| To be quadrede | 4 | 1 | 2 | 0 | 9 |
| The double | | 4 | 0 | | |
| The under double | 2 | | 0 | | 3 |

(2) a cube root using Latin terms

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Residuum | | | | | | | 5 |
| Cubicandus | 8 | 3 | 6 | 5 | 4 | 3 | 2 |
| Triplum | | | 6 | 0 | | | |
| Subtriplum | 2 | | | 0 | | | 3 |

It will be observed that, as in division, the examples chosen are particularly easy ones. In either rule, if a remainder is left, this is just set aside, the process of extracting the square root being defined as that of finding the root of a square number, or of the greatest square number, which is less than the given number. In some of the later books we shall see how a closer approximation to the root was reached.

This, then, is the "De arte numerandi Tractatus"; it is written in good Latin, but, owing to the omission of examples, it must have been difficult to follow. From it one may deduce the state of advancement of mathematics in the first century of the Scottish Universities.

---

# DEVELOPMENT OF MATHEMATICAL TEACHING IN THE SCHOOLS BETWEEN 1400 AND 1560.

One of the features of this period was the founding of English or Lecture Schools, devoted to the teaching of reading, writing and arithmetic. These institutions were largely outwith the control of the burgh authorities, and were intended for the education of those who were unable to benefit by the classical course of the Grammar School. The work in arithmetic was purely utilitarian in scope and very elementary in character. Arithmetic was taught, mainly, by the Abacus, but by the end of the period algorism was beginning to replace Abacism. A similar course was introduced into some at least of the Grammar Schools at this time. There is no evidence of any instruction in Geometry or the other branches of Mathematics in either of these types of schools.

As regards the Cathedral schools considerable progress in mathematical teaching took place in the first half of the sixteenth century. The theoretical arithmetic of the Greeks, as expounded in the Latin text of Boethius, formed part of the curriculum, but algorism, as expounded by Sacrobosco, was the foundation of the theoretical course. The algoristic arithmetic covered the four elementary rules as applied to integers, square root and cube root, with one or two very simple arithmetical progressions.

The course in geometry included the enunciations of the more important/

important propositions in the first four books of Euclid, with applications to the measurement of the areas of triangles and regular polygons. Further, the Ptolemaic theory of the planets, as expounded by Sacrobosco, was carefully studied.

Probably such a course was the exception rather than the rule, as few of the Cathedral schools would be fortunate enough to obtain a teacher of the outstanding scholarship of John Ferrerius.

# CHAPTER III.

## THE ARITHMETIC AND GEOMETRY OF RAMUS.

We get an insight into the state of mathematical teaching in Scotland in the ~~first~~ second half of the sixteenth century from the life of Andrew Melville and the diary of his nephew, James. The latter received his early education at a little village school in Logie, near Montrose, and, after five years there, attended the Grammar School at Montrose. The diary gave full details of his studies. These were confined largely to Latin Grammar, with some French and Greek, and there is no mention of arithmetic or mathematics. It is perfectly clear that mathematics was not then a school subject, and that such arithmetic as was taught was of a very elementary character. Naturally as all the text books on the subject were written in Latin, it was essential that the pupil should acquire a thorough mastery of and familiarity with that language before proceeding to the University, where his studies of Logic, Ethics, Natural Philosophy and Mathematics followed.

Andrew Melville, the older and more renowned of the two members of the family, was born in 1545, and was educated at the Grammar/

Grammar School, Montrose, and at St. Andrews University. Later he studied under Petrus Ramus in the University of Paris, and was much influenced by the doctrines of that teacher, who was one of the first to rebel against the Aristotelian tradition of infallibility which had for centuries laid a deadening hand on scientific progress. Melville later studied and taught at Geneva, and, when he returned to Scotland in 1574, was recognised to be one of the foremost teachers of his time. He was at once appointed Principal of Glasgow University, and he made a thorough reorganisation there. Although Principal, he took a big part in the teaching work: among other things "he taught the elements of Euclid with the Arithmetic and Geometry of Ramus, the Geography of Dionysius and the Cosmographia of Honter: in Natural Philosophy he made use of Fernelius." (1)

Melville continued the study of Aristotle, but he included the works of Ramus and Talaeus who led the revolt against the scholastic tradition. "The modern spirit appears in his lectures on geography and history, in the Arithmetic and Geometry of Ramus: in the Natural Philosophy of Fernelius." (2)

In Melville's four years curriculum the elements of Arithmetic and Geometry were introduced in the second year prior/

(1) "The Life of Andrew Melville" by Thomas McCrie (1819) p.72
(2) Story of the University of Edinburgh by Sir Alex. Grant. Vol 1 p.82

prior to the study of Natural Philosophy. Melville was transferred to St. Andrews University later, and introduced a similar course there. Methematics formed part of the course at St. Mary's College, St. Andrews, immediately after its establishment in 1554, but before Melville's advent, "according to the old plan of teaching in universities, mathematics formed preposterously the last part of the course." (1) Knox, in his first Book of Discipline agreed with Melville and appointed mathematics to be taught before physics.

Petrus Ramus is mentioned as one of the writers whose books were studied in the Grammar School of Stirling (2), but, I fancy that does not refer to his arithmetic and geometry, but to some of his versions of Latin writers.

Petrus Ramus or Pierre de la Ramée was born in the village of Vermandois at the beginning of the 16th century, some say in 1515, others put it as early as 1502. His grandfather was a refugee who came from the neighbourhood of Liége and settled in Picardy, and was engaged in the coalmines. His father was too poor to educate him, and at first he acted as a herder of cattle. Ultimately at the age of 28, he ran off to Paris, as he was keen on learning; there his uncle provided money to send him to the Collège/

(1) McCrie's Life of Andrew Melville. p 239.
(2) History of the High School of Stirling by A.F. Hutchison. p 40

Collège de Navarre. After completing the language and rhetoric course, he entered on the study of philosophy, but he soon realised the futility of the Aristotelian science, so called. In his oration prior to receiving the degree of "maître ès arts," he undertook to prove that Aristotle was not infallible. In 1543 he propounded a new theory of logic to replace that of Aristotle, and taught in the college of Ave Maria. His teaching brought him into collision with the college authorities, and he was forbidden to lecture any more. In 1544 he produced an edition of Euclid. Next year, under the good offices of the Cardinal of Lorraine, he became principal of the college at Presles, and was allowed to teach mathematics and science. In 1551 he was appointed Professor of Philosophy and Rhetoric in the Collège de France, and in 1562 King Charles IX gave him power to reform the Universities. In 1568 he visited England and thereafter he lectured for some time in the University of Heidelberg, where, influenced by Calvin, he became a Protestant. He returned to France in 1571, and was murdered in the massacre of St. Bartholomew in 1574. His Arithmeticae was written in 1555, and was later revised by Tobias Steger, Lazarus Schoner and Villebrond Snellius.

The copy of this famous text-book in Glasgow University Library/

Library is entitled

Petri Rami

Arithmeticae Libri Duo;

Geometriae Septem et Viginti

A Lazara Schonero recogniti et aucti

Francofurti

Apud Andreae Wecheli heredes

Claudium Marnium et Joannem Aubrium

MDXCIX.

The first preface to the book is written by the editor, Lazarus Schoner; he claims that Ramus is the first writer on elementary mathematics since Euclid and Theon, and maintains the superiority of Ramus over Euclid in his insistence on the importance of arithmetic. Ramus's arithmetical proofs and his use of inductive methods are to be preferred to Euclid's "reductio ad absurdum" type of proof. Ramus treats arithmetic first before proceeding to geometry, while Euclid roams back and forward between the two subjects.

The book commences with two sections on arithmetic. Liber I is devoted to the four rules, reduction and vulgar fractions. He gives a "proof" of Euclid's axiom "Si aequalibus addantur aequales/

aequales, toti erunt aequales et contra" by arithmetic. His method of carrying out addition is the modern one, but in subtraction he works from the left-hand side.

The compilation of the multiplication table up to nine times nine was a serious business in these days, and Ramus tackles the more difficult part of it by his favourite method of cross-multiplication; e.g., if he wishes to find nine times eight, he treats nine as seven plus two and eight as five plus three and we have

| | | | |
|---|---|---|---|
| 9 X 8 | 7 | 2 | 21 6 |
| | 5 | 3 | 35 10 |
| | | | 72 |

He teaches the modern method in long multiplications, but in a little note at the end he refers to a method of multiplication which he calls "prosthaphaeresis." I will give one example of the process. He wishes to multiply 8 by 6, and for that purpose divides 6 into two parts 3 and 3. One of these parts he adds to the 8, and the other he subtracts; then he sets down the sum thus:

| Totus 8 | Prosthaph. | | Partes | Facti |
|---|---|---|---|---|
| 6 | 11 | - | 3 | 33 |
| | 5 | - | 3 | 15 |
| 48 | | | | 48 |

Though/

Though Ramus only gives simple examples, one would suppose that the process was devised for cases such as 99 X 51.

In division Ramus has compiled a table like the converse of the multiplication table. His method of division is the "Scratch" method that was universal in these days, and of which I shall set down one example:

Divide 76945 by 234

1st step

7
~~7~~ 6 ~~9~~ 4 5 | 3
~~2~~ ~~3~~ ~~4~~
~~7~~ ~~0~~ ~~2~~

2nd step

0
2 ~~7~~ 6
~~7~~ ~~6~~ ~~9~~ ~~4~~ 5 | 32
~~2~~ ~~3~~ ~~4~~ ~~4~~
~~7~~ ~~0~~ ~~2~~ ~~8~~
~~2~~ ~~3~~
~~4~~ ~~6~~

3rd step

1
~~0~~ 9
~~2~~ ~~7~~ ~~6~~ 3
~~7~~ ~~6~~ ~~9~~ ~~4~~ ~~5~~ | 328
~~2~~ ~~3~~ ~~4~~ ~~4~~ ~~4~~
~~7~~ ~~0~~ ~~2~~ ~~8~~ 2
~~2~~ ~~3~~ ~~3~~
~~4~~ ~~6~~ ~~7~~
~~2~~
~~1~~ ~~8~~

Quotient is 328 and remainder 193

In his treatment of fractions he manages on every possible occasion/

occasion to arrange a cross-multiplication. Thus in finding the L.C.M. of two quantities, he proceeds thus:

12 and 8 are the numbers

$$\begin{array}{ccc} & \underline{\quad 24 \quad} & \\ & 12 \quad\quad 8 & \\ 4 & 3 \quad\quad 2 & \end{array}$$

Reduction of fractions to a common denominator

$$\frac{3}{4} \qquad \frac{5}{6} \qquad \frac{9}{12} \qquad \frac{10}{12}$$

$$2) \qquad 2 \underset{12}{\underline{\qquad\quad}} 3$$

To his joy he gets in two crosses this time.

So also in a division of fractions

$$\frac{3}{4} \qquad \frac{2}{3} \qquad \frac{\overset{1}{\cancel{9}}}{8} \quad \left( 1 \frac{1}{\cancel{8}} \right.$$

He is much handicapped by the fact that he has no symbols for the fundamental operations. To indicate addition he has to use the Latin "et". Subtract 3 from 7, is write 3 de 7 manet 4. Multiplication is written 2 multiplicati per $\frac{1}{3}$, Division dividantur 5 per 2 . If he wishes to express a fraction of a fraction, he uses a comma: $\frac{1}{2}$ of $\frac{1}{2}$ is written $\frac{1}{2}$, $\frac{1}{2}$.

The second book of arithmetic shows clearly the influence of the geometrical tradition still prevailing in arithmetic, handicapped/

handicapped as that science was, owing to the lack of signs. The book deals with ratio and proportion; comparison of two quantities may be made either by difference or by division. The former is called Arithmetical Proportion, the latter Geometrical Proportion. The same distinction in extended form applies to arithmetical and geometric progression, which are also treated in the book. Comparison of quantities by difference leads him to the solution of various problems on mixtures and to the type of problem known as "alligation", e.g. Mix two things, one costing 17 per lb. and the other 24 per lb. so that the mixture is worth 22 per lb.:

| | | |
|---|---|---|
| | 17 | 2 |
| 22 | | |
| | 24 | 5 |

Difference of 24 and 22 is 2, difference of 17 and 22 is 5, therefore take 2 lbs. of the cheaper one and 5 lbs. of the dearer.

The subject of ratio is treated very fully in this book, and with his system of adding, subtracting, multiplying and dividing ratios, the author is able to solve a great many types of problem which are now treated by fractions. The importance attached to the subject is manifest from the imposing system of nomenclature attached to different individual ratios. These names have not altogether died out, c.f. the use of the word sesquioxide/

sesquioxide in chemistry.

He divides ratios into Prime and Conjunct.

A prime ratio has one term unity, a conjunct has not.

A prime ratio may be (a) multiplex, (b) submultiplex, (c) fractional.

In multiplex prime ratio, antecedent is an integer, consequent is 1; these ratios are further subdivided into double, triple, quadruple, &c.

In submultiple prime ratio, antecedent is 1, consequent is an integer, and such ratios may be subdouble, subtriple, &c.

Fractional prime ratios are either

(a) superparticular, where antecedent is 1 and a fraction whose numerator is 1, e.g., $1\frac{1}{2}$ to 1 is called sesquialtera, $1\frac{1}{3}$ to 1 is called sesquitertia (consequent = 1), or

(b) subsuperparticular, the inverse of (a), or

(c) superpartient where consequent is 1 and antecedent is 1 with a fraction whose numerator is not 1, e.g., $1\frac{2}{3}$ to 1 is called superbitertia, or

(d) subsuperpartient, the inverse of (c).

Conjunct ratios are either ($\alpha$) multiplex superparticular, e.g. 5 to 2, 7 to 2, 7 to 3, 9 to 2, where the result of dividing the antecedent by the consequent gives a whole number greater than 1 together with a fraction whose numerator is 1, e.g. $2\frac{1}{2}$, $3\frac{1}{2}$, $2\frac{1}{3}$, &c., or

($\beta$)/

($\beta$) multiplex subsuperparticular, the inverse of ($\alpha$), or
($\gamma$) multiplex superpartient, where division of antecedent by consequent gives a whole number greater than 1 together with a fraction whose numerator is not 1, e.g., 8 to 3, 12 to 5, 24 to 5, or
($\delta$) multiplex subsuperpartient, the inverse of ($\gamma$).

The definition of prime ratio given above is not quite general enough: it should read "Where the quotient of the antecedent by the consequent, or vice-versa, (whichever is the greater) is either a whole number or an improper fraction lying between 1 and 2 in value..." This is not brought out clearly in the book, the names are simply attached to these various ratios, so that they may be referred to later. A knowledge of these names is required for the proper understanding of the chapters that follow.

Though Ramus has all these beautiful names, he has no symbol for ratio.

He solves various practical problems by addition, subtraction, multiplication and division of ratios, e.g. the problems of exchange and of compound proportion are treated by multiplication or division of ratios. I shall set out his solution of the cistern problem, which is certainly as old as 1000 A.D. (1)

"One/

(1) Cajori - History of Mathematics.

"One pipe fills a cistern in 4 hours, another empties it in 11 hours. How long does it take to fill when both are going?"

| | Impletio | Vacuatio | | | |
|---|---|---|---|---|---|
| Lacus | 1 | 1 | 11 | 4 | 7 |
| Horae | 4 | 11 | 44 | 44 | 44 |

$\frac{44}{7}$ horae.

After a full treatment of ratio, Ramus solves some problems on what he calls arithmetic proportion and continued arithmetic proportion, i.e. Arithmetical Progression. He shows how to find any term of an A.P. and the sum of the same. Here is his setting down of the problem of finding the 10th term of the "progressio quaternaria", i.e., 1, 5, 9, &c.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| i | ii | iii | iiii | v | vi | vii | viii | ix | x | 9 |
| | | | | | | | | | 37 | 4 |
| 4 | | | | | | | | | | 36 |

Rule: Take 1 from the number of the term, multiply by the common difference, and add the first term. The setting down is similar to that used by Napier in his "De Arte Logistica" ~~(see p.~~ ). Napier would probably be familiar with this book.

Next he treats Geometrical Proportion and states the "aurea regula". His treatment of direct proportion is in line with that followed in most of the later text-books, i.e., his infallible rule is that you make the known thing in the questional clause the third term, the quantity with the same name in the conditional clause/

clause the first term and then apply the golden rule that product of extremes = product of means. In his treatment of indirect proportion, or, as he calls it, reciprocal proportion, he differs from later writers. In order to apply his golden rule in the same form to this, he makes his unknown the third term of the proportion. In reciprocal proportion, antecedent of first ratio is to antecedent of the other ratio as the consequent of the latter is to the consequent of the former.

Example: If 15 oxen plough a field in 8 days, how long will 20 oxen take?

| Oxen | Oxen | D | D |
|---|---|---|---|
| 15 | 20 | ? | 8 |

Applying golden rule, answer is 6. Ramus has no sign for ratio and just sets down the quantities as shown.

Compound proportion is treated as a multiplication of ratios, or alternatively as two or more simple proportions.

Problems of mixtures, including the celebrated problem of the gold crown of Hiero solved by Archimedes, are treated as the addition of ratios, as is also the Regula Societatis, i.e. problem of proportional parts and partnerships.

A great many of the common problems of arithmetic are treated by ratio, which is the keynote of Ramus's arithmetic; even practice is included here.

Example/

Example: If 8 ells cost 19 shillings, what will 23 ells cost?

| Ells | Shillings | Ells | A. |
|---|---|---|---|
| 8 | 19 | 23 | |
| | | 8 | 19 |
| | | 8 | 19 |
| | | 4 | $9\frac{1}{2}$ |
| | | 2 | $4\frac{3}{4}$ |
| | | 1 | $2\frac{3}{8}$ |
| | | Total | $54\frac{5}{8}$ |

In one chapter which looks like an interpolation, "De Aequatione" he deals with what was later called the Rule of False, the only method available for many problems at a time when algebra was in its infancy. This will appear from an example:

"Find a number such that if you add one-third of itself to it and then sub_tract one sixth of this new number from the answer your remainder is 100." (The Latin in which this is expressed is beautifully concise in comparison with the English translation.)

He tries 144 and gets a remainder 160, i.e. 60 too much
He tries 108 and gets a remainder 120, i.e. 20 too much

$$\begin{array}{ccc} 144 & \searrow\!\!\!\nearrow & 60 \\ 108 & \nearrow\!\!\!\searrow & 20 \end{array}$$

He now takes 60 times 108 and subtracts 144 times 20, getting 3600; he divides this by 40, the difference of the two errors, and gets 90, the desired number.

No proof is given of this rule, but it was a favourite with writers on arithmetic, though of little practical value. The advent/

advent of algebra killed it by the nineteenth century.

He reveals some considerable mathematical intuition in his treatment of geometric progression, achieving some good results doubtless by inductive methods. He shows how to write down any term of the series 2, 4, 8 ...... The 12th term is the square of the 6th. The 29th is the square of the 12th multiplied by the 5th.

He sums a series of which he is given the 1st, 2nd and last terms, e.g., 16, 24 ........81

Subtract 1st from 2nd, and also from the last. Then as the difference of the 1st and 2nd is to the 1st, so is the difference of the 1st and last to the sum of all the terms but the last.

8 is to 16 as 65 is to 130

$\therefore$ sum of whole series = 130 + 81

This completes the arithmetic of Ramus.

The editor, Lazarus Schoner here interpolates a work of his own on Figurate Numbers. This is a most interesting part of the work, as it contains a development of the Greek theory of number in the Arabic notation. The Greek ἀριθμητικη was properly a study of the theory of number. The study of calculation for practical purposes they designated λογιστικη

With the Greeks the theory of numbers had a geometrical basis; here we see the modern spirit modifying the geometrical tendency and/

and feeling out for the algebraic theory of indices which was soon to follow. The result is the production of a kind of hybrid theory, which looks rather strange to us. Thus it seems odd to read that two figurate numbers are "Similar" when they have their "sides proportional." Thus 6 and 24.

$$\underbrace{2 \quad 3}_{6} \qquad\qquad \underbrace{4 \quad 6}_{24}$$

2 to 3 as 4 to 6, 6 and 24 are similar figurate numbers. Numbers are either linear (e.g. 7) or superficial (e.g. 6) or solid (e.g. 30). This classification is rather illogical, as the same number might belong to more than one class. Thus 30 considered as 6 times 5 is a superficial number, but considered as 2 times 3 times 5 is a solid number. The same trouble arises over his classification of superficial numbers: these are either equilateral (i.e. squares) or inequilateral (rectangles). But he classifies equilateral into explicable and inexplicable. The explicable equilaterals are those whose radix is rational: inexplicable equilaterals have a radix which is irrational. These inexplicable equilaterals may be considered in another connection as inequilateral. The word Radix, he says, was introduced by the Arabs, who also called a square of a number "Zensus". The side of an inexplicable square number is called the "Latus surdum" (from which our word <u>surd</u> is derived) because a broken trumpet is said/

said to give deaf sounds or missed beats (surdos ictus).

A good part of this book is taken up with Involution and Evolution, the former being developed from Euclid's proposition that the square on the whole line = the sum of the squares on the segments + twice the product of the segments (which theorem is "proved" by an arithmetical example in the true spirit of Ramus, the reformer). The theory of extraction of the square root he calls "Analysis quadrati" and it is taken up with reference to the "gnomon." When a number is the difference of two square numbers it is called a "Gnomon". Thus all the odd numbers are gnomons

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 3 | | 5 | | 7 | | 9 | | 11 | |
| 1 | | 4 | | 9 | | 16 | | 25 | | 36 |

In extracting the square root, he uses the "abacus quadrati" to find the greater segment of the radix or latus. The abacus quadrati is just a table of the squares of the numbers from 1 to 9. Suppose the number is 144. From the abacus quadrati we find the greater segment to be 10. To find the smaller segment, there remains the gnomon 44. A gnomon is made up of two equal rectangles and a square: one of the sides of the two equal rectangles is 10. Twice 10 gives 20: the other side is the smaller segment, and the gnomon contains also the square of the smaller segment. The lesser segment is obviously 2, which is put down twice as follows/

follows:-

$$\cancel{1}\ \cancel{4}\ \cancel{4}\ \ \underline{\lfloor 1\ 2}$$
$$\cancel{2}\ \cancel{2}$$
$$\cancel{4}\ \cancel{4}$$

It is interesting also to see how he proceeds in the case of an irrational at a time when no decimal system existed. If the number is irrational the root is written with an L in front: thus L2 or L3 for what we call $\sqrt{2}$ or $\sqrt{3}$. He approximates to the value of the latus as follows.

Find the latus of 148. Proceeding as before, there is a remainder of 4. The answer, he says, will be nearly $12\frac{4}{25}$ this number being a little too small. His justification for the 25 is two-fold. In the first place, 12 is the greater segment now: twice 12 should be taken on account of the two rectangles of the gnomon: the fraction would then be $\frac{4}{24}$ were it not for the square part of the gnomon: allowing for that he adds 1 to the denominator and makes $\frac{4}{25}$

His second reason is the square of latus 12 is 144
the square of latus 13 is 169
difference is 25
148 is 4 above 144
$\therefore$ its latus must be nearly $12\frac{4}{25}$

Napier writing at about the same time gives the correct root as lying between $12\frac{4}{24}$ and $12\frac{4}{25}$.

The extraction of cube root is treated on similar lines, being based on a theorem that if a number is cut into two segments the/

the cube of the whole number is equal to the sum of the cubes of the segments together with six solids (three of which are made up of the square of the greater segment with the lesser segment and the other three of the square of the lesser with the greater). The part in brackets appears very concisely in Latin by comparison with the cumbrous translation: "quorum terni segmentis et alternis eorumdem quadratis continentur." This proposition, he says, does not appear in Euclid, but is quoted by Cardan and "proved" arithmetically by Ramus.

The extraction of cube root is based on the solid gnomon, which is the difference of two cubes, and presents no new features. Irrational cube roots are written thus: 1c5 = $\sqrt[3]{5}$. Approximate cube root is found thus. Suppose number is 252. From the abacus it appears that 6 is the nearest whole number leaving remainder 36. Take three times the square of 6, add three times 6 and 1 for the small cube, and you get the gnomon 127. Therefore approximate root, which is too small is $6\frac{36}{127}$.

Following this section, Schoner gives an account of figurate numbers of higher order than the third, and in the absence of indices has to introduce names:

4th power is Biquadraticus bq.
5th " " Solidus ∫ (Napier calls it sursolid)
6th " " Quadraticubus qc.
7th " " Bisolidus b∫.
8th " " Triquadratus tq.
9th " " Cubicubus cc.

In/

In a note on figurate progressions, he gives as an example the geometric series where the ratio is subdupla; putting in 1 as the 0th term:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | i | ii | iii | iiii | v | vi | vii | viii | ix |
| 1 | 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 |
| | $\ell$ | q | c | bq | $\int$ | qc | b$\int$ | tq | cc |

He speaks of the numbers in the top line as "indices"; it seems strange that when he was so obviously feeling out for a convenient system of notation he should have failed to strike the modern system. Further on he almost reaches the decimal system, when he combines the above integral and fractional series in one, thus:-

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| vi | v | iiii | iii | ii | i | 0 | i | ii | iii | iiii | v | vi |
| 64 | 32 | 16 | 8 | 4 | 2 | 1 | $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{1}{8}$ | $\frac{1}{16}$ | $\frac{1}{32}$ | $\frac{1}{64}$ |

The left-hand part he calls "integra", the right-hand "minuta". Below he quotes the sexagesimal series of the astronomer:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| iiii | iii | ii | i | 0 | i | ii | iii | iiii |
| 12960000 | 216000 | 3600 | 60 | 1 | $\frac{1}{60}$ | $\frac{1}{3600}$ | $\frac{1}{216000}$ | $\frac{1}{129600000}$ |

These series are fundamental in relation to his further elaboration of algebraic quantities, particularly when he comes to treat of quantities which have what we would call negative indices. He calls the 0th term the "species" and points out that the introduction of the 0th term is an improvement on Euclid who called the index of that term 1.

1/

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| i | ii | iii | iiii | v | vi | vii | viii | ix | x | Euclid |
| 1 | 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 | |
| | i | ii | iii | iiii | v | vi | vii | viii | ix | Figurates. |

---

In the next section he explains the notation he adopts for surd numbers and shows how to operate on them.

$\sqrt{2}$ is written L2. $\sqrt[3]{6}$ is Lc 6 and so on. As a result it is impossible to have a coefficient in front of the surd sign. $2\sqrt{2}$ has to be L8.

For purposes of addition and subtraction it is necessary to discriminate between two "symmetrical" and two "unsymmetrical" surds. Napier introduces the same classification, but uses the name "commensurable" and "incommensurable". Two surds are symmetrical or commensurable, when the two numbers contain a common factor and are such that, when each is divided by this factor, the quotients are numbers whose roots are rational.

e.g. L 12 and L 27 L c 24 and L c 81

Such numbers are of course capable of addition and subtraction which are performed as follows:-

| | | | |
|---|---|---|---|
| | L 27 | L 12 | |
| L 3) | 9 | 4 | |
| | 3 | 2 | |
| Additio | 5 | 1 | subductio |
| | 25 | 1 | |
| | L 75 | L3 | |

If/

If the surds are incommensurable they are just written down with the sign of addition or subtraction between them. Here for the first time in the book appear the signs ——+—— which is plus and ———— which is minus.

Thus L 6 ——+—— L 8

The next chapter of this part of the book deals with algebra. His notation for powers and coefficients is so much similar to that for surds as to be confusing.

2 L stands for what we would call 2 X

3 Q " " " " " " $3 X^2$

5 C " " " " " " $5 X^3$

He shows how to add and subtract such quantities, reserving the use of his signs for mixtures of powers.

Thus 5 Q 12 Q 15 Q additi faciunt 32 Q

but 3 Q adde 9 L totus erit 9 L ——+—— 3 Q

In multiplication and division he refers to the "indices" quoted in the series, page , and quotes Diophantus, as his authority for the rule of adding or subtracting the indices.

12 Q ...... ii

<u>3 C</u> ...... iii

36 ∫ ...... v.

Our X he calls the "numerus" and to prove the above he assumes the numerus to be 2.

12 Q/

| | | |
|---|---|---|
| 12 Q | ----- 48 | 36 |
| 3 C | ----- 24 | 32 |
| | 192 | 72 |
| | 96 | 108 |
| | 1152 | 1152 |

Here is the division of 12 $X^4$ by 6

12 BQ ---- iiii ( iiii over 2 BQ

6 0 ---- 0

Rules for extraction of square and cube root follow.

Cube root of 27 QC is 3 Q,

but square root of 16 C is L 16 C.

The double meaning of the symbols must have caused endless confusion, and great care would be necessary in printing. When his answer is a negative power, further confusion arises. Thus 32 C divided by 4 BQ gives $\frac{8}{1}$ L. I am not clear whether this is a misprint and the L should be below the line, or whether the 1 below the line indicates the fact that the answer is what he calls a "minutum primum", i.e. the first power on the minuta side of the series. It is probably a misprint for $\frac{8}{1\ L}$.

The next section of the book is labelled Algebra, and is by Ramus himself, amended by Schoner. He discusses the derivation of the word and says that Regiomontanus called it the "ars rei et census", i.e. the study of the unknown and its square ( X and $X^2$ in modern language ). The Italians call it "ars de la cosa." It is the part of arithmetic, which shows the rules by/

by which the figurate numbers are operated on. The two main sections of algebra are Numeration and Equation. Under numeration he treats of the application of the four rules to figurate numbers whether these are surds or powers. In the main this part is an extension of the work treated in the previous section, including, as it does, expressions with more than one term. I will set down one example of division:

$$\begin{array}{llll} & \quad - 40\ L & & \\ 30\ Q - 58\ L & + \ 24 & & \underline{\lfloor 6\ L - 8} \\ \cancel{5}\ \cancel{L} - \cancel{3} & & & \\ & \cancel{5}\ \cancel{L} - \cancel{3} & & \end{array}$$

(i.e. $30\ X^2 - 58\ X + 24 \div 5\ X - 3 = 6\ X - 8$)

$$\begin{array}{ll} \cancel{48} + L\ \cancel{43}2 + L\ \cancel{3}84 + L\ \cancel{72} & \underline{\lfloor 6 + L\ 6} \\ \cancel{8} + L\ \cancel{12} \qquad\qquad \cancel{8} + L\ \cancel{12} & \end{array}$$

[i.e. $(48 + \sqrt{432} + \sqrt{384} + \sqrt{72}) \div (8 + \sqrt{12}) = 6 + \sqrt{6}$]

It is somewhat surprising to find that he is able to extract the root of an expression of two terms, one rational and one irrational. He gives no authority for the rule which he states. I shall show his working first and state the rule afterwards/

afterwards:-

latus of 23 —+— L 448

dimid. $11\frac{1}{2}$ L 112 $11\frac{1}{2}$ ~~23~~ 7 ( L 7 segment. minus

quad. $132\frac{1}{4}$ 112 $4\frac{1}{2}$ ~~16~~

differentia $20\frac{1}{4}$ 16 ( 4 segment. majus

quaesitum latus 4 —+— L 7.

Rule: Take $\frac{1}{2}$ of both less and greater segments; square each of these halves and subtract them (answer = $20\frac{1}{4}$); take square root of the answer (= $4\frac{1}{2}$) together with half of the lesser segment (- $11\frac{1}{2}$); add and subtract these and extract the root of each answer.

It would be interesting to know where Ramus got his authority for this rule which requires the ability to solve these general equations:

$$x + y = a$$
$$4xy = b$$

--------------

The second book of algebra deals with equations, which provided in those days the real justification for the existence of algebra. There is no solution of equations for their own sake: each equation is arrived at as the solution of a problem. One example will illustrate the nomenclature and show the handi-cap/

handicap the author suffered in his ignorance of signs.

"Quis est numerus ex cujus $\frac{2}{3}$ quadruplicata fiunt 11?
Esto pro quaesito 1 L $\frac{2}{3}$ per 4 multiplicata faciet $\frac{4}{3}$ L
unde concludito $\frac{4}{3}$ L aequat 11 ergo 1 L aequat $\frac{33}{4}$ i.e.
$8\frac{1}{4}$."

This, the first type of equation consists of one term on each side, the left hand term may be of any degree.

The other type of equation solved, the "Aequatio Secunda", contains two terms on one side and one term on the other. He shows how to solve such quadratic equations by completing the square. A chapter is devoted to each of the three types.

1st type 9 Q + 5 L aequat 294

2nd type 6 L + 9 aequat 1 Q

3rd type 1 Q + 21 aequat 10 L

For each type he states a separate rule without attempting to bring them all under one heading. Thus the rule for the last one

1 Q + 21 aequat 10 L

Divide 10 by 2; square the quotient; subtract 21. This leaves 4. Take square root; gives 2. Add 2 to half of 10 and subtract it and you get 7 and 3 for the solutions of the problem from which the above equation was derived.

At this time and later, none but positive solutions were accepted, but Ramus admits an irrational solution, e.g. in the problem to divide a line 6 units long in extreme and mean ratio/

ratio. He is precluded by his notation from introducing more than one independent variable in the solution of any problem. One problem he solves involves a biquadratic

$$9 \text{ bq} + 5 \text{ q} \quad \text{aequat } 294.$$

This constitutes the algebra of that early epoch, a science hampered by the lack of a satisfactory notation, but one devised for the solution of problems, and therefore of considerable value in practice.

The next section of the book is again written by Schoner, and is devoted to the subject of "Sexagesimal logistic." As this part of arithmetic was mainly of use to astronomers, it was sometimes called astronomical arithmetic; being the forerunner of the decimal system, it is worthy of some reference here. Sexagesimal arithmetic was invented by Ptolemy, and the equivalent decimal system is of European origin; Purbach and Regiomontanus are due some credit for it. The notation is interesting. Here is an example:-

| | | | | | |
|---|---|---|---|---|---|
| ii œ | i œ | 0 | i | ii | iii |
| 3 | 15 | 7 | 50 | 34 | 23 |

In words this would be "three sexagesimal seconds (or 3 q or 3 integral seconds), 15 sexagesimal firsts ( or 15 L), 7 parts or units, 50 minutes (or first scruples or $\frac{50}{1L}$), 34 seconds (or $\frac{34}{1q}$), 23 thirds (or $\frac{23}{1C}$). The diphthong œ is simply a mark used to distinguish the integral parts from the fractional/

fractional parts. The notation is remarkable and shows how the ground was prepared for the introduction of positive and negative indices and decimals. Addition and subtraction of such numbers is obvious; the answer was written above the power symbols. For multiplication it was necessary to construct another abacus or multiplication table, showing the products of all numbers up to 59 times 59, such products being expressed in the sexagesimal system. Thus the 13 times table would appear thus:

| | | | |
|---|---|---|---|
| 13 times 1 | 0 | 13 | |
| 13 times 2 | 0 | 26 | |
| 13 times 3 | 0 | 39 | |
| 13 times 4 | 0 | 52 | |
| 13 times 5 | 1 | 5 | |
| 13 times 10 | 2 | 10 | |
| 13 times 14 | 3 | 2 | and so on. |

The table is in the form of a rectangle built up of two unequal triangles. The lower and larger triangle contains all the products of the numbers 1 to 59 by each of the numbers 1 to 30. The upper triangle is inverted to fit over this one and contains all the products of the numbers from 59 down to 31 by the numbers from 31 to 59. The result is that a space is left between these triangles and on each side of this diagonal division there are seen all the squares of the numbers 1 to 59, which are required for the extraction of the square root.

Here I append one example of multiplication, one of division and one of root extraction:-

| | | | | | | |
|---|---|---|---|---|---|---|
| Multiply | 0 | i | ii | iii | iiii | |
| | | 59 | 8 | 11 | 22 | by 24 |

0/

| | | | | |
|---|---|---|---|---|
| 0 | i | ii | iii | iiii |
| | 59 | 8 | 11 | 22 |
| | | | | 0 |
| | | | | 24 |
| | 3 | 4 | 8 | 48 |
| 23 | 36 | 12 | 24 | |
| 23 | 39 | 16 | 32 | 48 |
| 0 | i | ii | iii | iiii |

Multiplier is here 24 units and therefore no trouble arises with regard to the shifting of the sexagesimal place. 24 X 22 is got from the abacus, and similarly with the other products. Doubtless with experience in looking up the table proficiency would be acquired in the operation.

Division:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Divide | ii œ | i œ | 0 | i | ii | by | 1 œ | 0 |
| | 12 | 12 | 47 | 10 | 21 | | 12 | 13 |

He looks up the powers of 12 in the abacus and finds that the most likely one is 59. Taking 59 as a probable quotient he multiplies it by 1œ / 12 0 / 13 and finds to his relief that the product is 12 0 47; thus 59 is the first part of the quotient

| | | | | | |
|---|---|---|---|---|---|
| | 12 | 0 | | | o |
| 11œ | 1œ | 0 | 1 | 11 | 59 |
| ~~12~~ | ~~12~~ | ~~47~~ | 10 | 21 | |
| | 1œ | 0 | | | |
| | ~~12~~ | ~~13~~ | | | |
| ~~12~~ | ~~0~~ | ~~47~~ | | | |

| | | | | |
|---|---|---|---|---|
| This leaves | iœ | 0 | i | ii |
| | 12 | 0 | 10 | 21 |

By trial he finds 12 X 59 is too great and tries 12 X 58 which is the correct quotient. Next stages would be

iiœ/

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | ~~11~~ | | | | | | |
| | ~~12~~ | ~~0~~ | ~~36~~ | | | | | |
| iiœ | 1œ | 0 | i | ii | | 0 | i | ii |
| ~~12~~ | ~~12~~ | 47 | ~~10~~ | ~~21~~ | | 59 | 58 | 57 |
| | | 1œ | 0 | | | | | |
| | | ~~12~~ | ~~13~~ | | | | | |
| ~~12~~ | ~~0~~ | ~~47~~ | | | | | | |
| | | 1œ | 0 | | | | | |
| | | ~~12~~ | ~~13~~ | | | | | |
| | ~~11~~ | ~~48~~ | ~~34~~ | | | | | |
| | | 1œ | 0 | | | | | |
| | | ~~12~~ | ~~13~~ | | | | | |
| | | ~~11~~ | ~~36~~ | ~~21~~ | | | | |

If you consider the possibility of a divisor of four orders, with the probability of making at least one unsuccessful trial quotient at each step of the division, if you consider the possibilities of error that may arise in looking up an abacus with 59 columns in it, it is possible to conceive of the waste of time and energy perpetrated by the astronomers of the period and to understand the joy with which they hailed the invention of logarithms.

Here is an example of evolution, a simple one containing an even number of orders:-

| i | ii | iii | iiii |
|---|---|---|---|
| 17 | 12 | 33 | 4 |

Owing to the construction of the abacus as explained above, one looks on either side of the gap for a number whose square is near to

| i | ii |
|---|---|
| 17 | 12 |

32 squared is seen to be

| i | ii |
|---|---|
| 17 | 4 |

| | | | | | | |
|---|---|---|---|---|---|---|
| | ~~8~~ | | | | | |
| i | ii | iii | iiii | | i | ii |
| ~~17~~ | ~~12~~ | ~~33~~ | ~~4~~ | | 32 | 8 |
| ~~17~~ | ~~4~~ | | | | | |
| | ~~1~~ | ~~4~~ | | | | |
| 0 | ~~8~~ | | | | | |
| | 0 | ~~32~~ | | | | |
| | | ~~1~~ | ~~4~~ | | | |

We/

We double 32 and get $\overset{o}{1}\ \ \overset{1}{4}$. We guess that the next term will be $\underset{8}{\text{ii}}$ This is multiplied by the $\overset{o}{1}\ \ \overset{i}{4}$ and the square of $\underset{8}{\text{ii}}$ viz. $\underset{1}{\text{iii}}\ \ \underset{4}{\text{iiii}}$ is appended in accordance with the usual rule for evolution.

The author tells us that Ptolemy and Theon did not carry their logistics beyond the minutes. Schoner also explains reduction from lower to higher denominations and reduction of vulgar fractions to sexagesimals. He expresses surds approximately in the same system, e.g. finding latus of 10800, he gets 103 and a remainder of 191: this remainder is multiplied by 60 and the root found to nearest minute

$$\begin{array}{r} 191 \\ 60 \\ \hline 11460 \end{array} \qquad\qquad \begin{array}{r} 13 \\ 11460 \\ 2060 \\ 1030 \\ 103 \end{array} \ \ \underline{|55}$$

leaving 130 of a remainder which may be carried to seconds.

Ramus was the leader of a revolt against the theory of the infallibility of Aristotle, and in his Geometry which includes twentyseven books he carries on a warfare against the teaching of Euclid. As some modern writers have considered that pure Euclidean Geometry is too difficult for schoolboys, so Ramus held that it was beyond the capacity of the university students of his day. That he was correct in that assumption I have little doubt, for geometry in Scottish Universities at least was not a compulsory subject of study. To Ramus. geometry is literally "earth measurement." The propositions of Euclid are undoubtedly of value in a practical way, and Ramus has made here a very/

very complete collection of these theorems and constructions. The rigid proofs are to him of no value, and in general he avoids them. His agility in that direction is aided by an inordinate number of axioms, but when these fail he refers the reader to Euclid for the full proof. On very rare occasions he gives the Euclidean proof; some of his substitutes for a real proof are very thin.

As a kind of glossary of all the information that is required by a surveyor, or gauger, the book is excellent. Not being tied down to a logical sequence, he is able to collect all the theorems about triangles in general in one book, those about equilateral figures in another, those about circles in a third and so on. The theory of similar triangles appears alongside that about congruent triangles and so on, making this geometry a complete and excellent book of reference. The first four books contain a copious number of definitions, many of which are really axioms. Some of his definitions are remarkably general. Parallelism he treats as the equality of perpendiculars, and applies his definition to curved lines as well as straight, and to surfaces, and even solids. Angles may be made by the intersection of two lines, whether straight or curved, or by three or more surfaces. The word centre applies to all figures, and means really the centre of gravity of the figure considered as a lamina. Radius has the same general meaning, but, of course, it is only in a circle that the radii are/

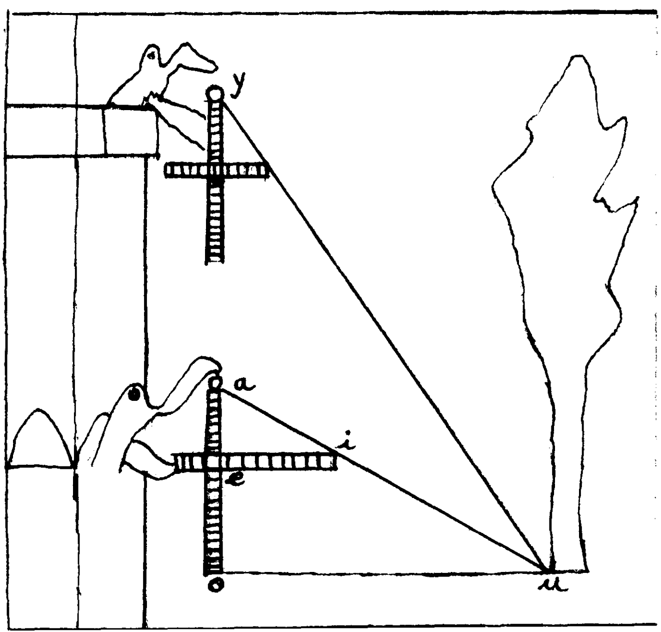
are all equal. Diameter has the same general significance and in a solid is an axis. Another general idea is that of Prime figures; the triangle is a prime figure, because other rectilineal figures can be split up into triangles; the pyramid is prime because other rectilineal solids can be split up into pyramids. Another very general definition (or rather axiom) states that similar figures are in the ratio of their homologous sides raised to a power equal to that of the number of dimensions of the figure. He gives a "proof" of this by arithmetic.

The first two congruency theorems follow easily from the axiom that if the two arms of one angle are respectively equal to the two arms of another angle, the <u>angles</u> are congruent (and therefore the bases are equal) and conversely.

His parallel theorems are developed from the definition mentioned above, coupled with a postulate that if two lines never meet they are parallel and with Euclid's fifth postulate. After collecting in Books 5 to 8 most of the theorems about triangles, including those on similar triangles, he proceeds in Book 9 to apply these to the measurement of heights and distances. The instrument with which these measurements were made is, he says, "perantiquium" and is called the "Radius" or "Baculus Jacob". It was used by Archimedes and Hipparchus, and was mentioned in the writings of Pliny and Vergil. The Arabians used it and in later years Regiomontanus amongst others. The radius consists of two arms of unequal length called the Index and the Transversarium. The index/

index is $2\frac{1}{10}$ times the other, and they are divided into 2100 equal parts and 1000 equal parts respectively, or sometimes 4200 and 2000, or 210 and 100. The index is of stronger material than the transverse, as it has to support the latter. Both are usually made of bronze or very good wood. By an arrangement of tubes it is possible to slide the transverse up or down through the whole length of the index and to move it backwards and forwards at right angles to the index. There is also a little pin called a cochlea (snail) from its shape, to keep the transverse in position. A cursor slides along the outside of the transverse and is fitted with a sight. In using it, close one eye, rest the cheekbone on the radius, and keep the hand steady.

With the help of 15 diagrams Ramus shows how to measure all manner of heights and distances both from above and below. The diagrams suggest warlike operations by land and sea. Two examples will suffice

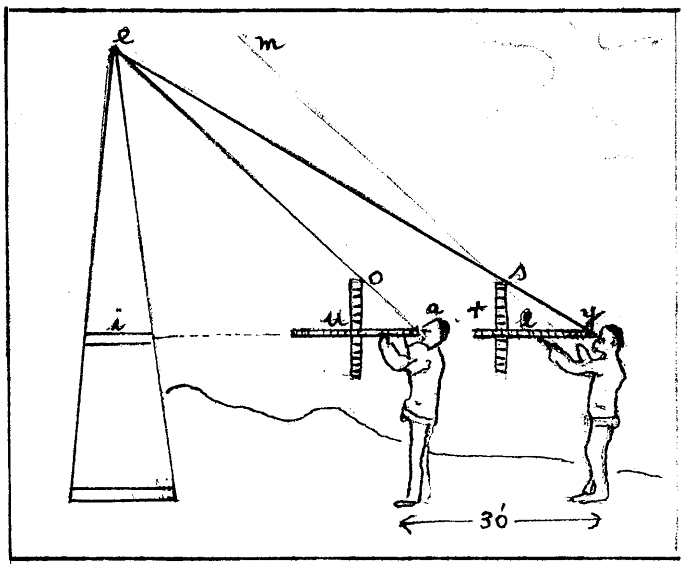


Here the object is to measure how far the tree is from the foot of the tower. Suppose the top of the index is 4' above the ground. Read ae the number of parts of the index and ei the number of parts of the transverse. Then

" ae ad ei sic ao ad ou."

Hence we can find ou.

A second estimate can be got from the upper window whose height is known.




Here we have to find the distance observer is from the middle of the tower ai and the height of the spire ei. Through s draw lsm parallel to aoe. Then triangles oua and srl are similar. But ou equals sr, because the segment of the transverse is the same in each of the two positions. Therefore triangles oua and/

and <u>srl</u> are congruent.

Now y r ad y i sic s r a d e i

i.e. sic o u ad e i

But o u ad e i sic a u or l r ad e i

∴ y r - l r ad y i - a i sic o u ad e i

i.e. y l ad y a sic o u ad e i

i.e. sic y r ad y i

Now y l is just the difference of the two segments of the indices.

y a is the distance between the two observing stations.

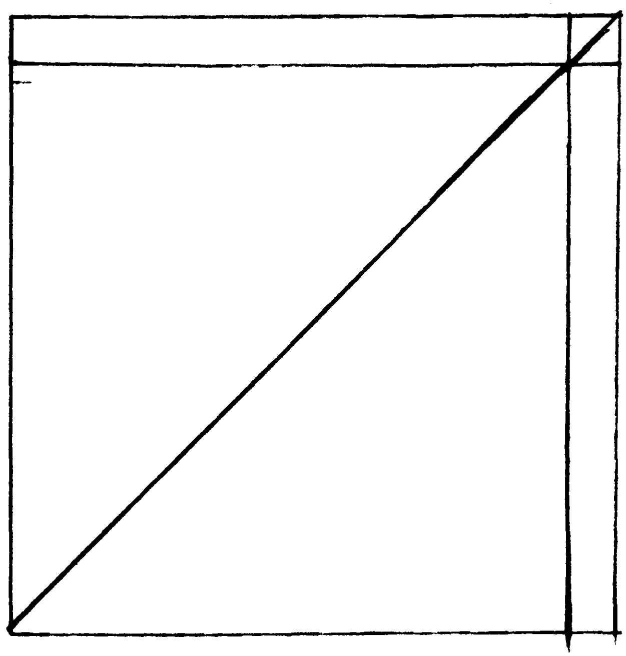
y r is measured and y i can be calculated from the proportion, and thereafter e i can be obtained.

These two examples will show that the author had a very good grasp of the application of the principle of similar triangles to practical mathematics. The instrument used, though capable of many adaptations would scarcely give accurate results as it would take a strong man of iron nerve to keep it steady in position when pressed into the cheekbone.

---

After this excursion into surveying, Ramus returns to a discussion of four-sided figures, which are divided into parallelograms and trapezia, a trapezium being any four-sided figure. which is not a parallelogram. The generic name for four-sided figures is "Quadrangle". The line that we call a diagonal he calls a diameter/

diameter, a name which applies to any line through the centre of the parallelogram. The name "Diagonal" is reserved by Ramus for the figure we call the parallelogram about the diagonal of a parallelogram. He makes very little distinction between a parallelogram and a rectangle, using the former word often when the latter would be necessary. In discussing the square he quotes Pythagoras's Theorem and its converse in very concise Latin: "Si basis trianguli subtedit rectu, aeque potest cruribus et contra", 47 and 48, p. 1, but instead of spending time proving it he gives rules for finding three numbers which would form the lengths of sides of a right angled triangle, and lays emphasis on the Euclidean theorem on which our method of extracting the square root of a number is based. Here he provides a justification of his method for approximating to the square root of 148 as described earlier. The argument is more or less as follows:-



We/

We have found 12 to be the integral part of the root: the next higher number is 13: the gnomon in the figure consists of two complements, each 12 by 1 and a "diagonal" 1 by 1, i.e. 25 in all. Since the given number is 4 above the square of 12, its latus or square root should be nearly 12 $\frac{4}{25}$. Ramus shows by arithmetical considerations that this is too small, and here he really has recourse to decimal fractions without the notation for them.

He says $148 = \frac{1480000}{10000}$

The square root of this numerator is 1216, leaving a remainder. The square root of the denominator is 100.

$\therefore$ "latus de 148 aequatur $\frac{1216}{100}$ aequatur 12 $\frac{16}{100}$" or 12 $\frac{4}{25}$
Since there is a remainder this answer must be too low.

Later Ramus treats of the circle and his nomenclature is different from ours. A "segment" is a part of a circle bounded by a periphery and a straight line or lines. There are two classes of segment (1) a sector (2) a sectio, the latter being what we call a segment.

As an example of the kind of "proof" he will accept, the theorem that angles at the centre or circumference of equal circles are in the same ratio as the arcs they subtend is "proved" thus. He proves that if the angles are equal the arcs are equal, and says that $\therefore$ if the angles are unequal, the arcs are unequal, hence proposition follows.

A/

A good deal of space is devoted to what he calls "Ascription of triangulati", i.e. a discussion of polygons inscribed in and circumscribed about a circle, as this leads up to the calculation of the area of a circle.

Euclid's theorems on the geometry of planes are quoted mostly without proof in Book 20, and the remaining seven books are taken up with a discussion of solids, pyramids, prisms and polyhedra mista "(i.e. solids made up of two pyramids back to back)" the sphere, cone and cylinder. In every case he quotes only the theorems which are required in measuring the areas of their surfaces, and their "solidity", i.e. volume. In connection with the cube he gives a justification for his rule for approximating to the cube root of a number which is not a complete cube:

e.g. the cube root of 17616 is 26, leaving a remainder of 40. This is the numerator of a fraction, of which the denominator is $3 \times 26^2 + 3 \times 26 + 1$ i.e. 2107

Approximate cube root is $26\frac{40}{2107}$ but this is a little too big, for if we express 17616 as $\frac{17616000000}{1000000}$ and extract the cube root of this we get $\frac{2601}{100}$ with a considerable remainder, 19712199 $26\frac{1}{100} +$

The answer $26\frac{40}{2107}$ is therefore a little too big, he says, but this is due to the fact that $26\frac{1}{100}$ is a bad approximation.

Ramus's geometry then is of the nature of a textbook on practical/

practical mathematics and is intended to be the ground-work of the surveyor and the gauger. He does not go into the details of land surveying, ~~and~~ measurement of casks, etc., as is done in some of the later books to which I shall have occasion to refer, but on every page he lays stress on the practical nature of his mathematics as opposed to the rigid logical abstractions of Euclid and the Greeks. Though this was one of the books laid down by the Senate of Glasgow University during the reign of James VI. of Scotland for reading by the Arts students, it is very questionable whether the standard of the book was reached by many of the students. The advent of Andrew Melville to the Principalship brought about a big advance in the University which had formerly been in rather a moribund condition, but in the years which followed the abortive attempt to found a sound system of education as outlined by Knox in the Book of Discipline, the standard attained receded considerably. If one remembers that mathematics was not taught in the schools (beyond a very elementary course in arithmetic) and in the Universities formed only a part of one year's work, viz. the third year of the course (1), it is impossible to believe that the students could/

(1) Munimenta Alme Universitatis Glasguensis Statutes promulgated in the reign of James VI, with regard to curriculum.

Annus Tertius: Mathemata initio, Arithmeticam inquam et Geometriam; deinde Aristotelis Logica, Ethica, Politica Ciceronis Officia, Selectos Platonis. Dialogos cum iudicio praelegito.

could have absorbed all the work of the book I have just detailed. Nevertheless I feel sure that this book had some considerable influence in directing the energies of students towards the practical side of mathematics, and when, in the eighteenth century, mathematics began to find a place in the curriculum of a few of the schools, the tendency towards the practical side of the subject was most marked. It is for that reason that I have given a more complete account of this text-book than of most of those which follow.

I have seen another earlier edition, dated 1592, also edited by Schoner; it is identical with the one I have described, except that the part dealing with geometry is not included. There is also in the Simson collection an edition dated 1599, edited by Schoner, and containing another work as well. The latter was written by Ramus in 1566, and is entitled "Petri Rami scholorum mathematicarum libri unus et triginta." It consists of an introduction in the form of an exhortation addressed to Catherine de Medici to encourage the study of mathematics; there is a short section on the arithmetic of integers and fractions, and the remainder consists of a critical, historical survey of the theorems of the fourteen books of Euclid. From this it is clear that Ramus considered the great geometer to be needlessly abstruse in his proofs. The enunciations of most of Euclid's theorems are quoted with the numbers attached.

In Edinburgh University library there is a copy of the three books of the Arithmetic of Ramus. It is published in Paris, dated 1557/

1557, and is labelled "Editio Secunda". There are parallel passages in Greek and Latin. In the main it is similar to the first book mentioned, but contains no geometry, and the parts by Lazarus Schoner are, of course, absent.

There is also an edition of the Arithmetic and Geometry dated 1627, but unnumbered.

---

# CHAPTER IV.

## JOHN NAPIER.

The works of Ramus have a greater significance for us in view of the fact that the great Lord Napier was one of his pupils. (1) Napier's books were almost the first mathematical works printed in Scotland. The only one that preceded them was "A Newe Treatise of the right reckoning of yeares and ages of the world" by Robert Pont, dated 1599; it deals with the calendar. (2)

The chief mathematical treatises published by Napier in his lifetime were the "Canon Mirificus" and the "Rabdologia". The former is outwith the scope of my paper. The Rabdologiae seu Numerationis per Virgulas was published in Edinburgh in 1617. It consists of two separate sections and a considerable appendix. The first section deals with the invention of "Neper's Rods" for the simplification of multiplication, division and root extraction. This device is so well known that it would be superfluous to describe it here, but the remainder of the book contains much that is of interest.

Part II contains some curious sets of tables which do not seem of much practical use. The first of these enables one by a simple rule of three calculation to solve such a problem as this:

"Find/

(1) Life of John Napier, by Mark Napier, 1834. p 106
(2) Chapters in the History of Bookkeeping - David Murray. p 311

> "Find the length of the side of a square whose area will be equal to the area of an equilateral triangle or regular polygon whose side is given," and conversely.

The polygons range from the pentagon to the decagon. Another table concerns the diameter of the circle circumscribing a regular polygon. No indication is given as to the method by which the tables were constructed, but doubtless use was made of the tables of sines and tangents of the half angle of the polygon. A third table shows how to find the length of the edge of a cube whose volume is equal to that of another regular solid of given length of edge. Again there is a table showing the densities of various metals in "drachmae per cochlearem". In every case these tables are set out in an equal number of rows and columns and are built round the figure 1000 which serves as a kind of unit for each of the entities included. In the first table, for instance, the figure 1000 is taken in turn for the side of a triangle, square, pentagon, etc. up to decagon, and the table shows the length of the side of <u>anyone</u> of these figures which would have the same area as that of <u>any other</u> whose side was 1000 units; thus the information supplied is comprehensive enough.

The appendix, which is nearly as long as the rest of the book, contains an account of two other inventive ideas. The first is really an extension of the use of the rods, by means of which a multiplication of two numbers, each of ten digits, can be performed; it involves the construction of some two hundred rods of specified dimensions/

dimensions. Half of these are thick and the other half thin. The thicker ones have the multiplication table inscribed in a special manner on them, each ten times over. The thin ones are used in the actual process to blot out the parts of that table that are not required. The whole multiplication can be done by the rods alone and the final answer read off by diagonal addition.

The same apparatus can be used to perform a complex division sum by taking the inverse of the divisor and carrying through a multiplication as before. The inverse is got by looking up the divisor in a table of sines, finding the angle corresponding and turning up the complement of that angle in the table of secants. As an alternative method you may look up the divisor in a table of tangents and get its inverse from the complement of the angle in that table itself. In either case your final answer after multiplication must be divided by the radius which in most tables was taken as ten millions.

The second part of the appendix to the Rabdologia contains an entire new system of arithmetic, which Napier calls "Arithmetica Localis". The basis of this scheme is that all numbers can be written as the sums of powers of 2 (including 1 as a power). These powers are represented by letters of the alphabet:

a stands for 1
b " " 2
c " " 4
d " " 8 and so on.

When the English alphabet is exhausted, the Greek one is brought into use, so that it is possible by means of this notation to express/

express any number up to $2^{46}$, and doubtless by borrowing other alphabets even higher. The translation of any number into the new notation (translatio vulgarium numerorum in locales) can be done by continued subtraction of known powers of 2.

e.g. $19 - 16 = 3 \qquad 3 - 2 = 1 \qquad 1 - 1 = 0$

∴ 19 is a b e.

Another method is to divide the number continuously by 2, subtracting 1 where necessary for exact division. Then wherever in the division 1 had to be subtracted that letter comes in the final result.

Thus 1611 - 1 = 1610. Enter a in the local number.

| | | |
|---|---|---|
| 1610 ÷ 2 = 805 | 805 - 1 = 804 | Enter b in local number. |
| 804 ÷ 2 = 402 | | |
| 402 ÷ 2 = 201 | 201 - 1 = 200 | " d " " " |
| 200 ÷ 2 = 100 | | |
| 100 ÷ 2 = 50 | | |
| 50 ÷ 2 = 25 | 25 - 1 = 24 | " g " " " |
| 24 ÷ 2 = 12 | | |
| 12 ÷ 2 = 6 | | |
| 6 ÷ 2 = 3 | 3 - 1 = 2 | " k " " " |
| 2 ÷ 1 = 1 | 1 - 1 = 0 | " l " " " |

Hence 1611 is a b d g k l.

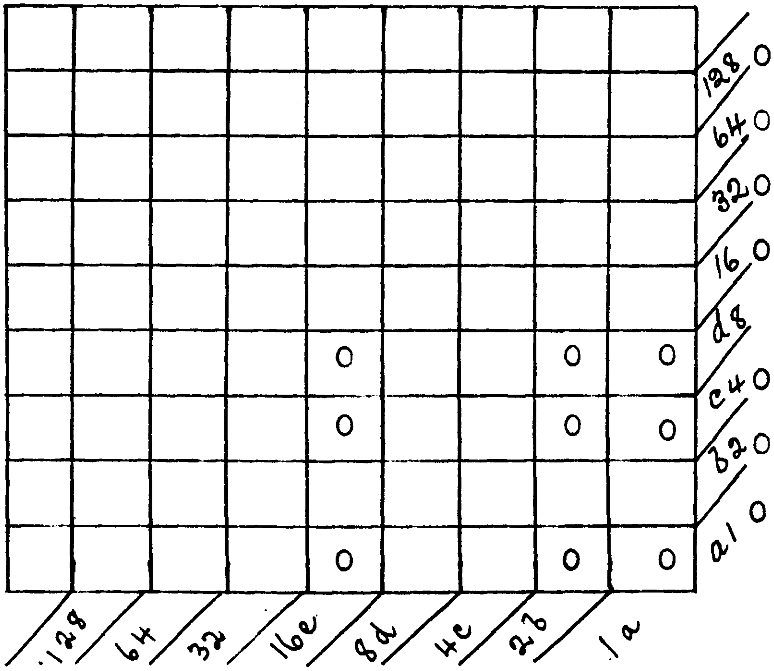
Addition is done as shown in the following example:-

59 + 47 = abdef + abcdf<br>
= aabbcddeff<br>
= bcceeg<br>
= bdfg<br>
= 2 + 8 + 32 + 64<br>
= 106

To/

To make the process worth while a table is given by which the number can be read off directly from the letters and vice-versa. The table contains the letters for 1 to 9, 10 to 90, 100 to 900, etc.

The multiplication of such expressions is done by means of still another table which permits of its application to factors, each with fourteen digits. I shall explain the method with much smaller ones. Take the product of 19 X 13 and construct a square table thus:




19 = a b e 13 = a c d

When a is multiplied by a we put O in the square where row a and column a meet; when d is multiplied by e we put O in the square corresponding to row d and column e. This explains all the Os. in the centre of the diagram. To add these together we imagine them/

them to move diagonally upwards at an angle of $45^{\circ}$ to the horizontal. When they reach the side we put 0 at the number reached. Thus the 0 corresponding to d X e represents 128, the next below it 64. The lowest one in that column encounters another 0 in its progress diagonally and therefore represents 32, not 16. Similarly the next two represent two 8s and so on. The answer is got by adding up the 0s at the right hand side or by translating a b c e f g h from the reduction table already referred to.

By an inversion of the process Napier is able to do division, and he extends the use of the "areal" table even to the extraction of the square root.

The whole of the book illustrates the extraordinary fertility of the brain which invented logarithms and indicates how deepseated was his desire to rescue mathematics from the thralldom of interminable calculations.

------------

In addition to the Canon and the Rabdologia Napier left behind him an unfinished work called De Arte Logistica. This was not published until the nineteenth century, but I include it here because of the light it throws on the state of advance of Algebra and Arithmetic in Napier's time. The book is luxuriously bound.. Mark Napier, a descendant of the Laird of Merchiston, is the editor. He believes that it must have been written about the year 1594, some years/

years before the discovery of logarithms, as it contains no reference to them, where such would have been expected in the case of a book written after the great discovery. Probably Napier left this work unfinished in order to concentrate on the greater task.

"De Arte Logistica" is dedicated to "Mr. Henrie Briggs, Professor of Geometry at Oxforde" and is written in beautiful classical Latin. It covers a good deal of the same ground as the arithmetical and algebraic parts of Ramus's book. Napier employs a different notation for roots and powers. Here are his root signs:-

| | | |
|---|---|---|
| Square root | Radix Quadrata | $\surd\mathcal{E}$ |
| Cube root | Radix Cubica | $\surd\mathcal{C}$ |
| Fourth root | Radix Quadrati Quadrata | $\surd\mathcal{E}\mathcal{E}$ |
| Fifth Root | Radix Supersolida | $\surd\mathcal{B}$ |
| Sixth Root | Radix Quadrati Cubica | $\surd\mathcal{E}\mathcal{C}$ |

etc.

He also suggests another notation which is interesting. The first nine numbers are arranged thus:-

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 4 | 5 | 6 |
| 7 | 8 | 9 |

Then ┘ stands for the Radix.

⊔ " " " Radix quadrata. $\sqrt{\ }$

└ " " " " cubica. $\sqrt[3]{\ }$

⊐ " " " " quadrati quadrata. $\sqrt[4]{\ }$

□ " " " " supersolida. $\sqrt[5]{\ }$

⊏ " " " " quadrati cubica. $\sqrt[6]{\ }$

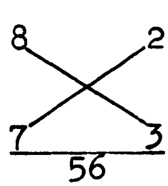
┐ stands for the Radix secunda supersolida $\sqrt[7]{\ }$

⊓ " " " " quadrati quadrati quadrata $\sqrt[8]{\ }$

┌ " " " " cubicicubica $\sqrt[9]{\ }$

This notation was capable of indefinite extension by the addition of 0 at the top right hand corner, so that ┘$^0$ stood for $\sqrt[11]{\ }$, but he does not use the notation himself. In his treatment of algebraic powers he calls the unknown that is introduced for the purpose of solving problems the "positio" and indicates it by 1 ℞, R being the first letter of the Latin Res. The square of this is 1 ℰ, the cube is 1 𝒞 and so on. Napier's notation advances beyond that of Ramus in that he is able to express powers of more than one variable or even of constant quantities. Thus 4 $a^2$ $b^3$ is written 4 a ℰ b 𝒞. $\sqrt[3]{x^6} = x^2$ is written thus $\sqrt{}$𝒞 . 0 ℰ 𝒞 = 0 ℰ the 0 standing apparently for a coefficient 1.

Napier used the signs + − X ÷ freely. His methods of adding, subtracting, multiplying and dividing are similar to those of Ramus. He shows in mediaeval fashion how the multiplication table up to nine times nine was constructed by the method of "defectus a denario". Thus 8 times 7 is got thus. The defect of 8 from 10 is 2, of 7 from 10 is 3




Take/

Take either 2 from 7 or 3 from 8 to get the tens digit and multiply 2 by 3 to get the units digit.

In connection with evolution he constructs a table by which the process shown for square and cube root could be generalised up to the 11th root or further. The table (of binomial coefficients) he calls the "tabula supplementorum". I reproduce it as far as the eighth power:-

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | | | | | | | |
| 2 | 1 | | | | | | |
| 3 | 3 | 1 | | | | | |
| 4 | 6 | 4 | 1 | | | | |
| 5 | 10 | 10 | 5 | 1 | | | |
| 6 | 15 | 20 | 15 | 6 | 1 | | |
| 7 | 21 | 35 | 35 | 21 | 7 | 1 | |
| 8 | 28 | 56 | 70 | 56 | 28 | 8 | 1 |

Napier divides algebra into two sections. Positive algebra deals with equations and problems. Nominative algebra deals with the study of rationals (i.e. arithmetic) and the study of irrationals (i.e. geometry).

Napier's treatment of what he calls commensurable and incommensurable surds has been already referred to; it is more general than that of Ramus. He shows how to divide by an irrational denominator containing two or more irrationals by rationalising it, and he gives a notation for what he calls <u>Universal</u> surd quantities, i.e. those where the root sign covers a function containing roots. He uses no brackets, but a dot has to suffice; thus $\sqrt{10 + \sqrt{2}}$ is written $\sqrt{\mathcal{E}} \cdot 10 + \sqrt{\mathcal{E}}\, 2$

I/

I have given a short account of this book by Napier, though it was not a text-book used in Scotland, because it exhibits one of the many efforts which were being made at that period to evolve a simple form of algebraic symbols.

---

# APPENDIX TO CHAPTER IV.

## DEVELOPMENT OF THE MATHEMATICAL CURRICULUM IN SCOTTISH SCHOOLS BETWEEN 1560 AND 1620.

The period of the Scottish Reformation was one exceedingly fruitful of ideas for the advance of education; magnificent reforms were conceived, but, being too advanced for that age, they failed to reach fulfilment. Andrew Melville reformed the three existing Universities from the foundations, abolishing regents and establishing specialist teachers, with a consequent widening and deepening of the curriculum. Yet his new system had a short life. In every University except Glasgow regenting was soon restored. In Glasgow, "the system (of Melville) went on, in appearance at least, for fifty years." X Had the reforms proved permanent, the raising of the standard of mathematical teaching in the Universities would have reacted on the schools.

In the second decade of the seventeenth century the name and fame of "matchlesse Merchiston", the first Scottish mathematician to publish his writings in Scotland, must have attracted many students to the study of mathematics.

The/

X Rectorial Address to Aberdeen University, 1882 - A. Bain, p. 16.

The advance in the study of theoretical arithmetic and mathematics in the Universities had little immediate effect on the schools With the destruction of the Cathedral schools, these branches of study passed out of the curriculum. The Scottish reformers under Knox, however, propounded a scheme of universal education based on the system of Calvin, which "made ample provision for the study of the vernacular and of practical arithmetic." [x1] Though the provisions of the "First Book of Discipline" did not reach fruition at that time, they led to the founding of some parochial schools, where reading, writing and arithmetic were taught. The English or Lecture schools, moreover, increased in number and influence.

As regards the curriculum in arithmetic in these schools, we have little information, since no mathematical text-books were published in Scotland before the time of Napier. We know, however, from the works of Recorde, published in the sixteenth century, that the abacus and the Arabian arithmetic were both taught. Recorde laid stress on the latter, but he taught the former also. "Now that you have learned the common kyndes of Arithmetyke with the penne you shall se the same arte in couters." [x2]

The extent of the arithmetical teaching would be conditioned by the requirements of the age; commercial arithmetic, as we now understand it, came into being in Scotland late in the seventeenth century/

[x1] History of Western Education - W. Boyd, p. 209.

[x2] The Ground of Artes - Recorde, p. 116.

century; and the "profession of public accountant did not emerge till the eighteenth century. The auditor, however, had long been known", X and from the earliest times, trade had been encouraged. All that was required of an auditor was that he should keep a note of receipts and disbursements, and should be able to state the sum required on either side to balance the accounts. Hence one would infer that practical arithmetic, as taught in the schools at this time, would include little beyond numeration, with addition and subtraction of integers and compound quantities. This would involve some acquaintance with tables of money, weights and measures, together with an ability to multiply and divide by simple numbers like 12, 20, 28, etc. Multiplication and division were greatly assisted by adventitious aids like "Napier's Bones". In the absence of such devices, the former operation would be performed by continued addition, the latter by continued subtraction. Problems of exchange would be beyond the work of the school.

The above is an outline of the probable "content" of arithmetical teaching in Scottish schools during the period under review. In the absence of "native" text-books, it is impossible to say whether or not a wider course was covered in any particular school.

---

X Chapters in the History of Bookkeeping - D. Murray, p. 52.

# CHAPTER V.

## TEACHING OF MATHEMATICS IN SCOTTISH SCHOOLS AND UNIVERSITIES IN THE SEVENTEENTH CENTURY.

Though the beginning of the seventeenth century witnessed many remarkable discoveries in mathematics and science, it has to be confessed that the science of number, the oldest branch of mathematics, made little progress in Scottish schools until the end of the ~~seventeenth~~ century. It has been mentioned that a little counting was taught to the choristers in the Sang schools in mediaeval times, but after the Reformation these schools sank into obscurity. On the other hand a type of school arose in which the vernacular was taught: these were usually known as "English" schools, but sometimes as "Scots" schools; their work was elementary, and they were meant to act as feeders to the Grammar Schools. Thus in the Paisley Council Records on 10th November, 1684, we find that one, David Tavendale was appointed "Doctor to the Arithmetic and Scots School." Originally this work was carried on in the Grammar School, but from the above date a separate building was used. In many Grammar Schools in Scotland, from early in the seventeenth century, it was recognised that the Rector's assistant or second master (who was called Doctor) should "superintend/

"superintend the initiatory English, writing and arithmetic classes". (1)

We find a reference to the teaching of arithmetic in the Town Records of Stirling. "30th January, 1697, it was resolved that a qualified person who could teach writing and arithmetic should be settled in the Grammar School as formerlie". (2)

Arithmetic is known to have been taught in the Grammar School of Irvine as early as 1665, of Wigtown in 1686 and Dunbar, 1690,(3) and, as mentioned previously, in Aberdeen Grammar School in the sixteenth century. The first reference to the teaching of arithmetic in other Scottish schools occurs in the eighteenth century, but quite possibly it may have been taught long before the Town Councils condescended to recognise it as part of the curriculum of the schools. There is no doubt whatever that Arithmetic, as a school subject, was considered beneath contempt, not alone in Scotland, but also in England in those days. Thus Brinsley in his Ludus Literarius (1612) makes Spondeus say "For I am much troubled about this that my readers and others above them are much to seeke in all matters of numbers, whether in figures or in letters. Insomuch as when they heare the chapters named in the church, many of them cannot turn to them, much less to the verse". Philoponus says later "you shall have scholars almost ready to go to the University, who can yet/

(1) History of the Paisley Grammar School by Robt.Brown, p.49.
(2) History of the High School of Stirling by A. F. Hutchison. p.83
(3) Grant: Burgh Schools of Scotland. p.398

yet hardly tell you the number of pages, chapters and other divisions in their books to find what they should".

Again, we find Charles Hoole, an English educationist of classical leanings in his "A new Discovery of the old Art of Teaching Schoole",(1660) advocating the establishment of what he calls "The Petty School" which was to serve as a preparatory school for those who are to go to a Grammar School, but also for those to whom the Latin tongue is thought to be unnecessary. Instead of dismissing such as "incapable of learning", the Petty School is to teach them writing and arithmetic. The Statutes of Charterhouse School (founded in 1612) say "It shall be the (Master's) care to teach the scholars to cipher and cast an accompt, especially those that are less capable of learning and fittest to be put to trade".

Such was the position of arithmetic in the early seventeenth century. Towards the end of the century, however, after the close of the religious and civil wars, there was an undoubted development of arithmetical teaching, and many text-books were published and eagerly purchased. The reaction against the classical tradition began with Bacon, Milton, Dury and others in England, Comenius and St. Jean Baptiste de la Salle in France and Francke in Germany. Scotland felt the influence of this movement by the end of the seventeenth century, and the following century witnessed a considerable modernisation of the curriculum.

Mathematics as a branch of study in Scottish schools is of comparatively/

comparatively modern growth. The oldest notice of pure mathematics in the burgh records occurs in 1660.

"In 1660, James Corss, mathematician, presents a petition to the council of Glasgow, asking as a native of Glasgow permission to teach mathematics in his old burgh in the native tongue. The Council grant him permission to open a school for teaching mathematics and promise to give him their best encouragement". (X)

Whether this effort bore fruit we do not know, and it is perhaps significant that no other reference to the teaching of mathematics occurs until 1718.

I have found no text-books that were written for use in schools or commercial colleges until near the end of the century, and can come to no conclusion as to the degree of advancement in the teaching of arithmetic prior to 1680 ~~and mathematics~~. If we judge by the fact that the most elementary parts of arithmetic were included in the University course, we would infer that very little was taught in the schools. In the Elementary Schools established by Francke in 1688 and later, the course in arithmetic embraced the Four Rules, the Rule of Three, and the meaning of a fraction, so that, in all probability, arithmetic, so far as it was taught in Scottish schools throughout the greater part of the seventeenth century, would not include any more than the Four Rules in integers.

---

As/

(X) Burgh Records of Glasgow quoted in Grant's Burgh Schools, p. 398.

As regards the work done by the Universities of Scotland in Mathematics, it is not the purpose of this paper to deal with that, except in so far as it throws light on the work of the schools.

Speaking generally, the seventeenth century was a period when the universities languished owing to civil and religious commotion, but it is abundantly clear that the teaching of arithmetic and mathematics was recognised to be the function of the university and not the school. Knox's Book of Discipline visualised a university devoted to the extent of two-thirds of its time to mathematical and physical science, the literary and classical training having been acquired in the Grammar School. The course, outlined previously, prevailed in the Scottish Universities until the middle of the seventeenth century. In 1645 Commissioners from the four universities met and decided to draw up a course of study to be obligatory on all, each one to share in the drawing up of the scheme. In 1648 another Commission met and carried out the scheme. The course agreed on included:

First Year: October until November - Latin; November to June - Greek. "the remanent time of that year after the month of June to be spent in learning the elements of the Hebrew tongue that at last they may be able to read the elements of Arithmetic, the four species at least". (X)

Second Year: Logic and Rhetoric.

Third Year: The elements of Geometry and the first and second books of the arithmetic.

(X) History of Edinburgh University, by Andrew Dalzel, 1862. Vol II p151

FOURTH YEAR: the other four books of the Arithmetic, the DeCoelo, elements of Astronomy and Geography, de Ortu et Interitu, the Meteors and de Anima.

This course was adopted with slight modifications in the different universities.

Throughout the greater part of the century the regent system of teaching prevailed, and it was really the beginning of the next century before the system of electing professors for individual subjects became common. In Edinburgh University a professor of mathematics, Andrew Young, was appointed in 1620, but survived only three years. Again in 1640 we find one appointed, who lived till 1662: he was Thomas Crawford. James Gregory held the chair of mathematics for a year, 1674-75 and his nephew, David, 1683, followed in 1692 by the brother of the last-named, James Gregory. The first man definitely appointed Professor of Mathematics in Glasgow University was George Sinclair, and his appointment dates from 1691. He had been a regent some 25 years before, but was deposed in 1666 for refusing to take the "oathe of alleadgeance" and submit to "the churche government as it is now established by law." X Previously no doubt, certain regents had specialised in mathematics and given extra lectures outside the course to those able and willing/

X Munimenta Universitatis Glasguensis. Vol 2 p 336 willing to benefit by them.

In 1687 a new scheme was drawn up by St. Andrews University. Here the first year contains practical Arithmetic, which is to be taught "rather by frequent practices and examples than often repeated rules." The elements of Geometry may be begun if the pupils are fit for it.

The Second or Logic year is to contain some lessons in Geometry, as being a fine form of logic.

The Third year "the elements of Geometry should be completed together with some practices of the Geometry....... for these geometrical practices they must necessarily understand the plain Trigonometry which can easily be taught them in a few days." X Physics is also tackled.

Fourth year: the rest of the Physics, Cosmography, Optics, Spherical Trigonometry and Mechanics.

It is clear that while progress in the teaching of mathematics was slow throughout the 17th century, there was a considerable advance in the status of the subject towards the end of the century, as was only natural at a time when epoch-making discoveries were being made in mathematics and science.

X See Dalzel's History of Edinburgh University. p. 219

# CHAPTER VI.

## THE TRISSOTETRAS.

After Napier's time, the first book on elementary mathematics of Scottish origin is one bearing the above curious title. It is the work of "Sir Thomas Urquhart, of Cromartie, Knight," a graduate of Aberdeen University.

The title of the book is a sufficient guide to the nature of the contents:

> "The Trissotetras or A most exquisite table for resolving all manner of Triangles whether Plaine or Sphericall, Rectangular or Obliquangular" and so on.

He claims its "importance in the study of Fortification, Dyaling, Navigation, Surveying, Architecture, the Art of Shadowing, taking of Heights and Distances, the use of both the Globes, Perspective, the skill of making of maps, the Theory of the Planets, &c."

"commented on with all possible brevity and perspicuity in the hiddest and most researched mysteries...." Superlatives are overworked by the author, both here and in his dedication to his mother, as well as in a foreword eulogising Lord Napier. The book was published in London in 1645.

In assessing the worth of this book (one might almost call it an effusion) one has to put oneself in the place of the author/

author. Trigonometry at this date was apparently an exceedingly abstruse subject owing to the lack of symbols. The Sine, originally the whole chord, was now the half chord of the double arc; the Tangent was a straight line drawn from one extremity of the arc perpendicular to the diameter intersecting the secant at the other extremity of the arc. The secant was the prolonged Radius, terminated on the tangent. The name Cosecant was comparatively recent, being attributed by Ball [x] to Rheticus, 1596. Cosine and Cotangent were names coined by Gunther, 1620. The Cosine, Cosecant and Co-tangent are defined in this book, but these names are not used. The present-day contractions for these terms were not introduced until a century later (Euler, 1748). The modern system of naming the sides of a triangle a, b, c, and the angles A, B, C, was unknown. Every rule had to be stated fully in words. In right-angled triangles the names are easy: hypotenuse or subtendent, side, base and perpendicular. In an oblique-angled triangle one side is the base and the others are simply described as "the side opposite the given angle" or "the side opposite the required angle," and so on. The introduction of logarithms enormously facilitated the calculation, but rather aggravated the complexity of the statements of the different rules. In the solution of triangles any one of the sides can be taken as the Radius (sometimes called the Whole Line i.e. Sine/

x Ball: a short history of mathematics. p. 215

Sine of $90^{\circ}$). The rule known to us as the Tan of the half-angle was thus expressed and doubtless thus had to be committed to memory: "If from the summe of the logarithm of the difference of the sides and Tangent of halfe the summe of the opposite Angles be subduced the aggregat or summe of the Logarithms of the two proposed sides, the remainder thereof will prove the Logarithm of the Tangent of halfe the difference of the opposite Angles: the which joyned to the one and abstracted from the other affords us the measure of the angle we require." If this is the kind of statement required of the formulae for the plane triangle, it may be conceived that those for the spherical triangle were not very concise.

The Trissotetras is an attempt to solve the problem of memorising these rules by a device of the nature of the famous mnemonic "Barbara Celarent, Darii, Ferio" used for the fifteen Syllogistic Moods in Logic. By his device the author succeeds in compressing all the rules for solution of plane triangles, four for right-angled triangles and four for obliquangled triangles into one page, and those for spherical triangles into three. These four pages had to be memorised by the student (no inconsiderable feat even with mnemonics) before he could become skilled in the use of trigonometry. History does not relate whether the device attained any popularity, but it is perhaps interesting to quote the prefatory remarks of a person who/

who signs himself "J.A." and who thought that the student could master this system in a month "instead of three quarters of a yeere usually by Professors allowed to their Schollers for the right conceiving of this Science."

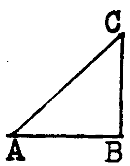
"A secret (in my opinion) so precious that (as the author spoke of Marchiston) I may with the like pertinence avouch of him that his Countrey and Kindred would not have been more honoured by him had he purchased millions of gold.......than for this subtile and divine invention which will outlast the continuance of any inheritance," etc.

What is this immortal device? The Trissotetras gets its name from the fact that the subject naturally divides itself into sections which have either three (tris) or four (tetras) subsections within them. The author in addition to coining the above word has also perpetrated the manufacture of many words of Greek or of mixed origin. Most of these are, I fancy, mere affectation on the part of Sir Thomas Urquhart, Knight. It is not my intention to consider the device, known as the Trissotetras in detail, but I shall give one or two examples to explain the scheme. In the first instance one has to relearn one's alphabet, for each letter of the alphabet (and some combinations of letters) has a meaning of its own.

A is an angle, B the base, E a side, I an angle required, M a tangent complement (i.e. a cotangent) I a tangent/

tangent and so on. One of the ingenious devices consists in making the vowels represent things which often occur, so that the combination of letters will make something pronounceable. To apply this alphabet: here is one of the "Axioms" of Trigonometry. "The Subtendent (i.e. hypotenuse) in plain Triangles may be either Radius or Secant and the Ambients (i.e. the base and perpendicular) either Radius, Sine or Tangent." In other words, you can choose any side of a right-angled triangle as the radius, if you care. This axiom is expressed by one word <u>RULERST</u>. Looking up the alphabet shorthand dictionary, we find R = radius, u = subtendent side, l = secant (the first clause of the axiom); e = side, r = radius, s = sine, t = tangent, (the second clause of the axiom).

One branch of this axiom is labelled <u>VRADESSO</u>, which word means "when the hypotenuse is radius the sides are sines of their opposite angles."




With AC as radius,

BC is the sine of angle A.

The "tan of the half-angle" formula quoted above reduces to one word <u>GREDIFTAL</u>. GRE = sum or aggregate of sides, DIF = the difference, T = tangent, AL = halfe. <u>GREDIFTAL</u> means "as the summe of the two sides is to their difference so is the tangent of the halfe sum of the opposite Angles to the Tangent/

Tangent of half their difference." It is an open question whether the Charybdis of the shorthand notation is preferable to the Scylla of the verbose statement.

Again the well-known formula $\frac{\sin A}{a} = \frac{\sin B}{b}$ when used to find the side b in the author's shorthand is

Sapeg - Eg - Sapyr Yr

Sapeg means the sine of the angle opposite the side given

- is to :

Eg side given

- so is

Sapyr sine of angle opposite side required (Yr)

- to

Yr side required.

For those acquainted with Napier's Canon this means: "If to the summe of the Logarithm of the side given and of the Sine of the Angle opposite to the side required, we add the logarithm of difference of the Secant complement from the Radius, we get the logarithm of the side required."

These examples will be sufficient to give an idea of the misplaced ingenuity of the author of this remarkable book; it is scarcely necessary to give further details. No examples to illustrate the various methods of solving triangles are given. The book is written in the high-flown language of the period and is rather prolix. The technical words used are mostly original/

original and a "Lexicidion" which is given at the end of the book is a very necessary addition. Judging by the definitions there, the author had no great knowledge of mathematics. Thus Logarithms are defined as "those artificiall numbers by which, with addition and subtraction onely, we work the same effects, as by other numbers, with multiplication and division."

An Oblong is "a parallelogram or square more long than large."

A Parallelogram is "an oblong, long square, rectangle or figure made of parallel lines."

One would infer that his knowledge of Euclid went no further than the enunciations. Pythagoras's Theorem he calls "Bucarnon" from the famous mythical story of the sacrifice of an ox by its discoverer.

Here ends this curious work, which I fancy would probably have failed to reach publication if the author had not had a "handle" to his name.

# CHAPTER VII.

## THE WORKS OF GEORGE SINCLAIR.

George Sinclair was one of the four regents in Glasgow University at the time of the restoration of King Charles II. He held this office from 1654 to 1666. As a regent, his duty was to teach all the subjects of the curriculum, but he must have had a special fondness for the mathematical and physical science side, as he has left several works on these subjects.

On 14th February, 1665, the Lord Archbishop of Glasgow produced an order under which all professors and regents had to take the "oathe of alleadgeance," under pain of deprivation of office. On the 17th of March, 1666, the Lord Archbishop and a commission of visitation "required Mr. George Sinclair to give obedience to the actis of the said commissione relating to the taking of the oathe of alledgeance and subscryveing his submitting to and owneing the churche government as it is now established by law."[x] George Sinclair, after obtaining a few days to consider his position, resigned his office of Regent on the 21st of March, 1666. For some years after these events he seems to have resided in Edinburgh, probably doing private teaching, but after the Revolution of 1688, he returned to Glasgow/

[x] Munimenta Universitatis Glasguensis, p. 336.

Glasgow University as regent in 1689, and held that office till 23rd March, 1691, when he was presented to the "profession of Mathematicks and Experimentall Philosophy" in the University at a salary of "six hundred merks yearly".[x] The professorship "hath a far lesse salary and profite attending it" than the regentship, and in consideration of that and of his stand against "prelacy" the Senatus promised "that so soon as Mr. George Sinclair shall be removed by death or otherwayes......the said Mr. George Sinclair or his assigneyes shall have payed to them nine hundred merks Scots money". He held the office of professor till 1699.

Sinclair wrote several books: "Tyrocinia Mathematica", dated 1661; "Ars Nova et Magna" written a few years later. The latter deals with Hydrostatics and was ridiculed by James Gregory, the greatest of that wonderful family, who was then Professor of Mathematics in St. Andrews. Sinclair also published "Principles of Astronomy and Navigation, &c." in 1688. He had a considerable reputation as a mathematician and "demonologist", but he showed no great originality, and, according to Gregory, was ignorant of the geometry of Euclid.

The Tyrocinia Mathematica, written in Latin and published during his first spell as regent, was intended for the use of students of Glasgow University, and in all likelihood it contains most of the work in mathematics and physics required of/

[x] Munimenta Universitatis Glasguensis.

of the students who studied under Sinclair. It is divided into four perfectly distinct sections. The first section deals with arithmetic. Though the title suggests that we are to have new light on the mathematics, there is little of originality in the treatment of arithmetic. The section deals only with integers, and is confined to numeration and the four fundamental rules. The author has an original but rather useless division of numbers into "perfect" and "imperfect". Perfect numbers are those which have three (or presumably a multiple of three) digits, e.g. 432. Imperfect numbers have not the periods so complete, e.g. 34. This seems a very weak idea of perfection in number compared with the ancient Greek "perfect" number, i.e. one such that the sum of all its factors was exactly equal to the number itself, e.g. $6(= 1 + 2 + 3)$.

In the performance of the four fundamental rules in those days, great stress is laid on the necessity for checking the answer at the end. The method of "casting out the nines" was recommended in this book and was a very popular one for nearly two centuries afterwards, until its weaknesses were pointed out. It seems to us like a waste of time. In addition, for example, you take each number separately, add the digits and set down the remainders of the division of each such sum by nine. Then you add all these remainders and cast out the nines from this sum and write down the remainder of this process. Lastly, you sum the/

the digits of the answer and cast out the nines from that. Your remainder now should be equal to your former remainder. The weaknesses of this type of check on the answer seem pretty obvious, but the process had a long life. According to Charles Hutton[x], the method was first given by Dr. Wallis in his Arithmetic, published in 1657. It is applied here to the checking of addition, subtraction, multiplication and division.

A good deal of space in the book we are considering is devoted to a description of Napier's Bones, and an explanation of their use in multiplication and division. These were explained by Napier in his Rhabdologia, and, as is well known, consist of the multiplication table written on blocks of wood, which can be so chosen that we can set down the product of any considerable multiplicand by each of the digits 1, 2, 3, up to 9 in turn. E.g. if the multiplicand is, say, 56743, he finds by Napier's Bones the products 2 X 56743, 3 X 56743, etc. up to 9 X 56743, and uses these in multiplying by the ordinary process. Multiplication and division were still considered to be operations of exceeding difficulty, and many textbooks of later date than this recommend in cases of complex multiplicands or divisors the setting down of the products of such operators by each of the digits in turn.

Apart from the use of the rods, Sinclair differs from modern/

[x] Course of Mathematics for the Royal Military Academy - Charles Hutton, 1811.

modern writers on multiplication in one or two ways. He uses the term "Dirigens" or "directing figure" to indicate each individual digit of the multiplicand and he multiplies first by the highest dirigens, i.e. left-hand digit. He never sets down the multiplicand in his sum since this is already set up in the rods. If there are zeros in the multiplier he writes down in the sum as many zeros as there are figures in the preceding or succeeding row. Here is the setting down of the product of 6596 by 2050.

```
   2050
  ------
  13192
   00000
    32980
     00000
  --------
  13521800
```

The holy rite of checking the result by casting out the nines must not be omitted, and it is set down in the form of a cross, thus

```
   2
 8 X 7
   2
```

8 is the remainder when nines are cast out of 6596,

7 " " " " " " " " " 2050

8 X 7 = 56. Cast out nines and 2 is the remainder. Cast out nines from the answer and 2 is again the remainder, therefore answer is correct.

The process of division is set down in a manner similar to the/

the old-fashioned scratch method already referred to, but, as he makes use of the rods to find the partial ~~quotients~~ products, the old setting down is slightly modified; the divisor itself is not set down at all. Here we have to divide 4568 by 22:

```
 0
01   |14
4568 (27 14/22
4454
 1
```

There is no scratching out of figures; this represents a transition from the old to the modern Italian method of division.

The four elementary rules applied to integers complete the arithmetic introduced in this book, and apparently represent all that was taught in the University of Glasgow at that time. It seems rather strange that students should be taken through a somewhat comprehensive scheme in astronomy and should be expected to use logarithms in the solution of ordinary and spherical triangles with such a meagre grounding in arithmetic. At this period arithmetic was fighting for a place in the University curriculum and had hardly attained a place at all in the schools. Thus we find that a Royal Commission on 7th September, 1667 recommended "some good abrigement of all the pairts of the mathematicks at least arithmeticke, geometrie, geographie and astronomie /

astronomie be taucht by each of the maisters in their cours"[x]
(at the universities).

This book of Sinclair contains, I fancy, most of the mathematical and physical science course that he taught at that period.

The second section is nominally a treatise on the geometry of the sphere, but is actually an account of the Ptolemaic System of Astronomy. It is based on Sacrobosco's work, but is more detailed, and includes the most important imaginary circles in the heavens and the earth and the natural and artificial divisions of time. The next section, "Praxes Astronomical" is curiously enough more mathematical than the "Geometry of the Sphere", but the mathematics is that of the N.C.O. instructor in the Artillery. Sinclair presupposes that spherical trigonometry and the compilation of tables of logs, sines, tangents, etc. is much too difficult for his pupils, and to replace that deficiency he puts in their hands a mathematical instrument calculated to solve all the necessary problems without understanding them. What the slide-rule is to the Artillery instructor, the Sector was to Sinclair. He explains the geometrical meanings of the sine, tangent and secant (he never uses the cosine, cotangent or cosecant). These quantities are, of course, lines not ratios, and the student is shown how to measure the sine of any angle less than $90^{\circ}$ and the tangent of any/

[x] Munimenta Universitatis Glasguensis, p. 483.

any angle less than 45°. He is then presented with a sector of brass or wood, each limb of which is graduated in lines (i.e. units of length), sines up to the "whole sine" or Radius, and tangents up to tan 45°, i.e. again the radius. With this sector and a pair of compasses he is able to tackle any problem in the rule of three involving sines or tangents of angles. The problems involved deal with measurement and calculation of the sun's altitude, declination, amplitude, azimuth, etc. The measurements are done with the quadrant and the calculations with the sector. In most cases they involve the use of one or other of the formulae for solution of right-angled spherical triangles. These formulae are quoted in the form of proportion sums, no explanation is vouchsafed, and the student is merely shown how to use them in solving the problem on the sector. One example of the twenty different rules will show the method. I shall give a free translation.

EIGHTH LESSON.

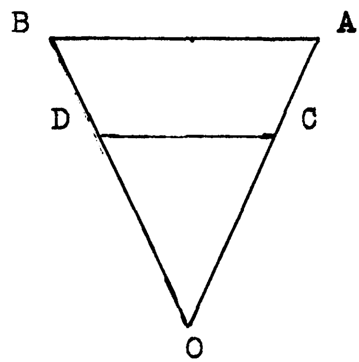
To find the declination of the sun by trigonometry from table of sines given the distance from the nearest point of the equinox and the obliquity of the ecliptic (the sixth lesson showed how to measure the latter).

As the whole sine : sine of greatest obliquity of ecliptic
:: sine of distance of sun from nearest point of equinox
: sine of required quantity.

(in modern symbols $\sin A = \dfrac{\sin a}{\sin c}$ $C = 90^\circ$)

e.g./

e.g. I wish to know the sun's declination on 10th - 11th May. The time since the vernal equinox represents 60°. Greatest obliquity of ecliptic = 23° 30'.




The figure represents the sector of which OA or OB is the radius or "whole sine". With the compasses I measure sine 23° 30' and then extend the two arms of the sector till AB = sine 23°30'. I then look at sine 60°, which is at the point C, and measure with the compasses from C to D, where D is at sine 60° also, but on the other leg of the sector. Lastly I stretch the compasses from C to D, and then place one end of the compasses at O, and the other along the leg of the sector: the point where the compass cuts the sector gives me on the line of sines the angle I require ($\because$ OA : AB :: OC : CD). Angle required is 20° 15'.

It will be fairly clear from that example that the mathematical development expected of the student is remarkably small in comparison with the complexity of the astronomical work he had to undertake, but, of course, since the earliest times, astronomy was the one branch of physical science which had been developed/

developed on practical lines, and I suspect that the average student of those days could have taught a good deal of astronomy to his fellow of these days.

The third book deals with geography nominally, but in reality it is more geometrical than geographical. In some respects this is the most interesting part of the work, as the practical problems solved are necessarily related to the science of Navigation, a subject which gained great prominence in the mathematical curriculum of the Scottish schools in the eighteenth and early nineteenth centuries. The treatment of the subject is rather archaic even for 1661, for he makes no reference to Mercator's Projection, which was in pretty general use in scientific circles thirty years before this date. Moreover, the earth is considered to hang suspended in the middle of the world, surrounded by air and shining with every kind of brightness, and Ovid is quoted as his "authority":

> "Stat vi terra sua.........
> Ponderibus librata suis immobilis haeret."

He gives an account of all the imaginary circles on the earthly sphere, and the situation as regards latitude and longitude of the chief capital towns of the world is quoted from the Lansbergian tables. A good deal of space is devoted to a rather useless account of the "climates". These are zones of the earth bounded to North and South by parallels of latitude such that the longest/

longest day in summer in one zone is half an hour shorter than in the next zone North of it. Likewise space is devoted to an explanation of antipodeans, perioeci and antoeci.

He shows how the size of the earth was computed taking $1^{\circ}$ of elevation of the pole as equal to 60 Scottish miles.

The practical problems in geography that follow are of a very elementary nature and principally concern the measurement of the distance between two places on the earth's surface. Thus he calculates the distance between two places of same longitude but different latitude, and vice-versa, the amplitude of the torrid and temperate zones, etc. One or two of the sixteen problems involve the use of trigonometry. Thus Problem 6 is to find the distance between two places, one on the equator, the other towards either pole, the difference of longitude being less than $90^{\circ}$.

> RULE: As whole radius is to complement of difference, so is complement of given latitude to complement of required distance.

This way of setting down the rule is rather misleading to us, for the words "sine of" must be understood before the word complement in each case, so that he is really using the formula $\underline{\cos c = \cos a \cos b}$ for a spherical triangle right-angled at C. Using the line of sines on the sector as before he obtains as the fourth term in the proportion the complement of the required angular distance which is then turned into miles at 60 miles to every/

every degree.

It will be seen that none of the problems solved is of any use in navigation, but the problem of finding the longitude of a place had not been solved at this date. While the author works out one example of each of the problems tackled, he never sets down any either here or in the arithmetic for the pupils to solve.

The last section of this book is entitled "Tractatus Echometricas" and is a semi-scientific account of echoes.

The book as a whole is disappointing in its lack of originality, and if we may accept the course it covers as a reasonable approximation to the mathematics required for graduation at this period, it leaves a poor impression on the mind. There seems little doubt that, owing to civil and religious turmoil, this was a lean period in the history of the Scottish Universities as compared with the epoch of Andrew Melville and his successors.

Just before he was restored to his post of Regent in Glasgow University, Sinclair wrote another book with a high-sounding title, "The Principles of Astronomy and Navigation", to which is added a "Discovery of the Secrets of Nature which are found in the Mercurial-Weather Glass, etc." as also a "New Proposal for buoying up a Ship of any Burden from the Bottom of the Sea." This work was published in Edinburgh and was dedicated to the Lord/

Lord Provost and Magistrates thereof, so that it is likely that Sinclair held a teaching appointment of some nature in the Capital.

The first part of this book is mainly an English version of the astronomical and geographical parts of the Tyrocinia, but without the mathematical parts of that work. Its claim to lay down the principles of Navigation is not justified, and it is, in the main, a somewhat extended version of the Sphaera of Sacrobosco.

## CHAPTER VIII.

Contemporary with Sinclair was a mathematician of a different stamp. James Gregory, the inventor of the Reflecting Telescope, was professor of Mathematics in St. Andrews from 1670 to 1674, and afterwards for a year in Edinburgh, where he died in 1675. His principal works were the "Optica Promota" and "Geometriae Pars Universalis", but these lie outwith the scope of this paper, which deals with books on elementary mathematics. Gregory had a tremendous reputation as a mathematician, and I have no doubt that the great development in the teaching of mathematics in the period we are just about to consider owed something to the world-wide reputation of James Gregory and his nephew, David.

Two books which were published in the eighties of the seventeenth century indicate the foundation of private commercial schools at that time in Edinburgh: reference has already been made to the establishment of such a school in Glasgow. In the Advocate's Library I found a book called Idea Rationaria, published in 1683 and written by Robert Colinson. This book deals entirely with book-keeping, but there is also a copy of a book on Arithmetic in the same library, which must be a very rare specimen, as very few copies of it were/

were printed in the first instance. It is entitled "The Scots Arithmetician or Arithmetick in all its parts", by James Paterson, Arithmetician, Edinburgh, 1685. The author was ambitious, for he set out to cover "Arithmetick in all its parts", and he details these parts: Vulgar or Decimal, Algebraical or Analitical, Sexagenary or Circular, Logarithmical or Artificial, Instrumental or Mechanical, each of these sections to be complete as regards both integers and fractions." He was, however, an old man when he produced this book, and, though he tells us to expect all the sections, he seems to have covered only the first, viz. Vulgar and Decimal Arithmetick for integers and fractions. He was a teacher in the Parish of Balrishean, Barony of Dunluce, County Antrim, Ireland before settling down in Edinburgh, where he set up a shop for the sale of books and Mathematical Instruments, such as "Cross Staffs, Quadrants, Scales, Spiral Lines, Dials, Uncels for weighing money, portable Ink, Sea Compasses, etc. at the head of the Cowgate, in the sign of the Cross-Staff and Quadrant." The book, like the Tyrocinia of Sinclair, is in very small print, which, along with the"fewness of the number printed", apparently enhanced its price. The author shows a rather regrettable fondness for doggerel rhyme, and in the dedication prints an Acrostic in heroic couplet, each line in turn beginning with a letter of James Paterson, Arithmetician, supposed to be written in praise of the author by one "J.H., a lover/

lover of the Mathematicks," but possibly written by J.P., a "Lover of Advertisement." The work covers most of the arithmetic required in a commercial career, viz. the four rules, proportion, exchange, partnership, mixtures and practice. Interest, simple or compound, and the problems rising out of it are not included. Undoubtedly the characteristic feature of the book is that the author introduces each rule by an atrocious piece of doggerel. Evidently in his teaching he believed greatly in committing rules to memory, and as an aid to that process he inserts verses such as this:

"To learn a-right to multiplie
The Table get in memorie
Then first set down multiplicand
And nixt, let multiplier stand
Multiplicand then multiplie
First place of multiplier by
Units set down: tens keep in minde
To add when ye oecasion finde"

and so on. Sometimes the rhyme seems more difficult to understand than the rule itself. Thus the Compound Rule or Double Rule of Three seems like a conundrum:

"For Compound either back or fore
Thus multiplie for Divisore
If fore, first two with back disjunct
If back, last two with fore conjunct
For divident the other three
In Quotient, the sixth will be."

This puzzle is worth unravelling, as it will serve to indicate how proportion was tackled in these days: the aim in every case was to make the process fool-proof. In simple proportion you had/

had to decide whether the rule was direct or inverse, or, as it was put, does "more give more" or "more give less". There were only three terms set down in a proportion and the second or middle term had the same name as the answer: the first term was then the other quantity in the conditional clause, and the third term the corresponding quantity in the principal clause. All this was automatic and independent of any reasoning as to direct or inverse.

Ex. If 15 men earn £36, what will 20 men earn?

| men | pounds | men |
|---|---|---|
| 15 | 36 | 20 |

The terms were set down thus side by side without symbols. Now you have to decide whether "more gives more" or "more gives less". If it be direct proportion, then product of 2nd and 3rd terms is taken as dividend, 1st term as divisor, and the quotient is the answer. Direct proportion is called "fore" in the rhyme above. If the proportion is indirect or "back", take the product of 1st and 2nd terms as dividend and the third term as divisor, then the quotient is the answer.

In the Double Rule of Three, five terms are set down. The third term is of the same name as the answer, the first and the fourth are of like name, also the 2nd and 5th. The 1st and 5th terms are called "disjunct" i.e. separated, the 2nd and 4th "conjunct", The whole rhyme thus means: If both proportions are/

are direct, product of 1st and 2nd terms is the divisor, and product of 3rd, 4th and 5th the dividend. If both are inverse, product of 4th and 5th terms is the divisor and product of 1st, 2nd and 3rd the dividend. If the first proportion is direct and the second inverse, the divisor is the product of the two which are disjunct, i.e. the 1st and 5th. If first proportion is inverse and second direct, the divisor is the product of the two conjunct terms, i.e. the 2nd and 4th.

Each of the rules treated in the book is introduced by a jingle of its own, and apparently the teacher of these days expected his pupils to commit all these rules to memory, so that in the end they must have had some terrible lumber in their brains. One can scarcely conceive that the meaning of the above rule would remain as long as the sound of the jingle itself.

"The Scots Arithmetician" has an interest for us in tracing the evolution of the modern symbols for fractions, vulgar and decimal. Paterson mentions two notations for vulgar fractions, viz.

$$\begin{array}{ccc} N & & D \\ 1 & : & 2 \end{array} \quad \text{and} \quad \frac{N\ 356}{D\ 587}$$

He says that the latter is the usual notation, but that personally he prefers the former. While he says so, however, he uses more frequently another notation altogether, viz. a comma placed between numerator and denominator, thus 2'5 $(= \frac{2}{5})$.

Decimal fractions, he says, are shown thus:

$$:6 = \frac{6}{10} \qquad :25 = \frac{25}{100} \qquad :005 = \frac{005}{1000}$$

This/

This leads to a confusion with the notation for vulgar fractions which is all the more perplexing in that he uses the comma notation to represent decimals as well. $\overset{c}{\phantom{0}}07 = \frac{7}{100}$ Perhaps the direction of the comma indicated a decimal as distinct from a vulgar fraction, but if so the book is full of misprints. Decimals are also indicated thus 32|8 (= 32.8), the method which became popular just after this period, but the author is not very certain in their use, so that most of his examples are of concrete quantities, with which he evidently feels on safer ground. His unit is the Roman Ducat, which contains 10 Julios and 100 Bajoches. With this notation he would write 542.65 as follows:

| D | | J.B. |
|---|---|---|
| 542 | : | 65 |

With these he is able to perform the fundamental operations. Thus he multiplies as follows 37876738.53 by .3

| Ducats | | J.B. |
|---|---|---|
| 37876738 | : | 53 |
| | : | 3 |
| 11363021 | : | 559 |

In this as in other examples he shows a misconception of the meaning of multiplication for in the above he is not really multiplying by $\frac{3}{10}$, but by 3 Julios. Later he gaily multiplies lbs. ozs. by lbs. ozs. without considering the meaninglessness of the process.

The/

The modern method of division is introduced here, so that this book represents an advance on that of Sinclair. The process is set out in detail:

```
     Divisor Dividend          31338
      37867 ) 9876758 ( 260   -------
   1. Product 75734            37867

   Resolvend  230335
   2. Product 227202               8
                                5  X  5
   Remainder   31338               4

       Proof  987658
```

The proof is by multiplication, but to make assurance doubly sure, verification by casting out the nines is carried through. Evidently the use of Napier's Bones ~~is~~ was still popular in 1685.

As in many of the books I have read, the section on the Rule of Three is the happiest of all. The simplicity and generality of the Golden Rule seem to have impressed themselves on the brains of the early arithmeticians. In the rules based on proportion, such as Fellowship and the Medial Rule, i.e. Averages, he is not so happy and the examples chosen constitute in many cases rather useless problems. It sounds more academic than practical to calculate how an apothecary would require to mix certain ingredients, some hot, some cold to various degrees, some dry, some wet. Yet this is a favourite type of problem in the old arithmetics. Two other rules, to which I shall often have to make reference, are now obsolete, but were a source of much/

much difficulty in those days, when algebraic processes were little understood. They are explained without proof in this work. The first is "alligation" (spelt by Paterson with one <u>l</u>), which is best explained by an example:

> Suppose I have wheat at 32/- the bushel, oats at 28/- barley at 20/- and pease at 16/-. How may a mixture be made to sell at 27/- the bushel?

The prices are set down with the required price opposite, thus:

```
        ( 32     )  11
        (        )
        ( 28 )   )   7
  27/-  (    )   )
        ( 20 )   )   1
        (        )
        ( 16     )   5
```

The 32/- is 5/- above and the 16/- is 11/- below the price required. If I grouped these two together, I should have to take, therefore 11 bushels at 32/- and 5 at 16/- to get the required mean price. Similarly I take 7 of oats to 1 of barley. Clearly this problem admits of a manifold infinity of solutions but these are reduced to a few in the case of compound alligation, where a definite quantity of the mixture has to be made up. The author of this book makes no reference to the multiplicity of possible solutions, but accepts the rule as it was given to him with the easiest and most obvious one.

The other rule to which I wish to refer is called here "Supposition", but more usually "Position" or the "Rule of False" (?)

In "Single Position" we assume a certain number to be the solution/

solution of some problem, then work out the quantities and verify the answer from the data. If the answer is wrong, we alter our supposition proportionally. In the Compound Rule of Supposition or the Rule of Double Position we make first one assumption, then a second one, and find the error in each case. According as these errors are in the same or in a different direction, we proceed to the finding of the correct solution.

> e.g. "Suppose a mason agrees to work 20 days at 12 sh. the day and that he is to rebat 1 lb. 10sh. for each day that he is idle, and having received in the end 7 lb. 16 sh., I demand how many days he was idle".

The two assumptions made were (1) 10 days working, (2) 12 days working:

```
          10 X 12  =  120              12 X 12  =  144
          10 X 30  =  300               8 X 30  =  240
                     ----                          ----
                     -180                           -96
                      156                           156
                     ----                          ----
      (1) Error      -336          (2)Error        -252
                       12                            10
                     ----                          ----
              336)   4032                          2520
              252)   2520
                 )           d.    sh.
               84)   1512 ( 18 X 12  =  216
                       84    2 X 30  =   60
                     ----               ---
                      672               156
                      672
```

With the first assumption the error was 336 shillings,
" " second " " " " 252 "

Both errors were in the same direction. I then multiply the

first/

first error by the second assumption (336 X 12) and the second error by the first assumption (252 X 10) and subtract the products. I also subtract the two errors, and, on dividing these two, I get 18 days as the correct answer, the verification of which is shown. On the other hand, if the errors had been unlike, after cross multiplication I add instead of subtracting.

This ingenious rule was handed down from generation to generation without proof. The reasoning would have followed these lines. An increase of 2 in the assumption has caused a reduction of 84 in the error; what further increase would wipe out the 252 error still remaining? The rule has, of course, the same effect and the teachers of those days dearly loved a cast-iron rule.

An interesting survival from sixteenth century arithmetic finds a place in this book, but is excluded by all the later writers. This is the rule called Ceres and Virginum, sometimes known as Regula Virginum or Regula Cecis or simply the Rule of Drinks.[X] It arose from the prevailing custom of charging men, women and maidens varying prices for drinks. To facilitate the calculation a definite rule was laid down. The questions which Paterson solves by this rule are more general than the usual "rule of drinks".

The type of problem dealt with is of this nature; you buy, say, 12 pecks of oats and barley (called "bear" in this book) for 4 lbs./

X Educational Significance of Sixteenth Century Arithmetic, L. L. Jackson, p. 153.

4 lbs. money. Oats cost 6 sh. and barley 8 sh. the peck. To find how many of each, calculate the price of the whole at the cheaper rate and subtract from 80 sh. Then subtract the two rates, and divide your previous difference by this one. This is the sort of rule that a schoolboy of average intelligence would discover for himself, though he might not be able to say how he arrived at his answer.

This text-book concludes with a slight introduction to practice and with a table which was most probably the most useful part of the book to the business man for whom it was written, viz. "A Table of Annual Rent at 6 L per cent per annum" for principals from 1 sh. up to 10000 L.

On the whole it is a disappointing book: obviously intended to be purely utilitarian (there can be no other excuse for the awful doggerel inflicted on the reader). It, at the same time, deals with problems which are largely of theoretical interest rather than of practical importance. The treatment of fractions is, as stated, unsatisfactory, partly because of the want of a proper notation, and partly because the author is by no means clear as to what a decimal fraction is. He has also a complete misconception of the meaning of multiplication. On the other hand, the work covers a good deal of ground, and has the estimable advantage, rare at that period, of being written in English.

---

# APPENDIX TO CHAPTER VIII.

## DEVELOPMENT OF THE MATHEMATICAL CURRICULUM IN SCOTTISH SCHOOLS BETWEEN 1620 AND 1700.

So far as the Universities were concerned, progress in the first half of this period was disappointingly slow. One would have expected that the fame of John Napier would have attracted Scots students towards mathematical study, but the greater part of the century was a time of civil and religious disturbances, which, with the consequent purging of the universities by one side or the other, disastrously retarded advance in all directions. The latter part of the period was one made ever memorable by the discoveries of Isaac Newton. Under the inspiration of the two Gregories, mathematical teaching in the Scottish Universities threw off the shackles of the egregious regenting system, and advanced to a higher plane. The progress in the Universities naturally inspired advances in the schools.

Before the end of the seventeenth century, a fairly considerable course of practical arithmetic had been introduced in the parochial schools and in some of the grammar schools. The abacus/

abacus as a practical instrument was relegated to the infant school: arithmetic now included numeration, the four rules in integers and vulgar fractions, together with the Rule of Three Simple and Rule of Three Compound. Towards the end of the century, finite decimal fractions were in general use. In the early part of this period, multiplication and division were performed with the help of Napier's Bones, but, before the end of the century, the modern methods of performing these operations were in general use.

A review of the development of mathematical teaching in Scotland in the seventeenth century would not be complete without a reference to the establishment of commercial schools or academies. These institutions were modelled on the "Reckoning Schools" established in Italy, Germany, France and the Netherlands during the previous century. The first of which we have definite record was established in Edinburgh about 1680 by Robert Colinson, author of "Idea Rationaria", a treatise on Bookkeeping: Colinson was trained in Holland. About the same time James Paterson, author of the "Scots Arithmetician", started a similar school in Edinburgh. In 1695, "a teacher of the art of navigation, bookkeeping, arithmetic and writing" was appointed by the Town Council of Glasgow.[X]

The first Scottish bank, the Bank of Scotland, was founded in 1695/

X Town Council Minutes, 7th December, 1695, XV. p. 84, quoted by D. Murray.

1695, and with the development of home and foreign trade at this time there was a real demand for teaching in practical arithmetic and mathematics. Early in the eighteenth century, many such commercial schools were established in Glasgow, Edinburgh and elsewhere, and the majority of the text-books reviewed later were written by the teachers therein.

The mathematical curriculum of the seventeenth century commercial academies followed these lines:-

> In arithmetic, numeration is followed by an account of the tables of weights and measures, length, area, volume and time. Addition and subtraction of integers, vulgar and decimal fractions and compound quantities, follow. Multiplication and division of integers are taught in modern fashion, as are the various rules of vulgar fractions. There follow reduction from one name to another; transformation of vulgar fractions into finite decimals; simple and compound Rule of Three, with applications to problems in Partnership and mixtures of commodities. Some algebraic problems are solved by the Rule of False. Practice, Simple Interest and easy Exchange problems complete the course in arithmetic.

The commercial academies, even at this early stage, provided some training in algebra, trigonometry and logarithms, surveying, gauging and navigation, but I have no information as to the extent of that teaching in the seventeenth century.

These academies were intended for the training of boys who had completed a grammar school education, and who intended to follow the career of a business man, banker, accountant or factor. At this early stage the course was strictly utilitarian. In the eighteenth/

eighteenth century, with the advent of some really distinguished teachers, it took on a more cultural aspect, and the commercial academies filled a room in the educational edifice left vacant through the concentration of the Grammar Schools on the Classics.

---

# CHAPTER IX.

## THE MATHEMATICAL WORKS OF THE REV. GEORGE BROWN, KILMAURS.

A remarkably interesting figure is the author of three works on Mathematics which appeared at the end of the 17th and beginning of the 18th centuries. George Brown graduated M.A. at King's College, Aberdeen on 13th July, 1675. He became a schoolmaster at Fordyce, and about 1686 he was appointed minister of Kilmaurs. About this time he seems to have got into trouble for exercising ministerial functions before he was properly qualified by law to do so, and was haled before the Privy Council who ordered him to find "Baile before he can have the Benefite of his licence." X This he was unable to do and was accordingly banished from Edinburgh. In 1698 he appeared again before the Privy Council, this time as a supplicant, asking them to recall their sentence of banishment and to grant him a patent for an instrument called the Rotula Arithmetica. He seems to have been successful in both quests, though the Lords/

X Acta of the Privy Council of Scotland - 1698, [redacted]
[redacted] "School life in Scotland" by Dr. Insh.

Lords declined his offer to show them how to use the Rotula! He published a book in 1700 giving an account of the Rotula, and another in 1701 on "Decimal Arithmetick." In 1717-18 he wrote his most valuable work "Arithmetica Infinita, or the Accurate Accomptant's best Companion."

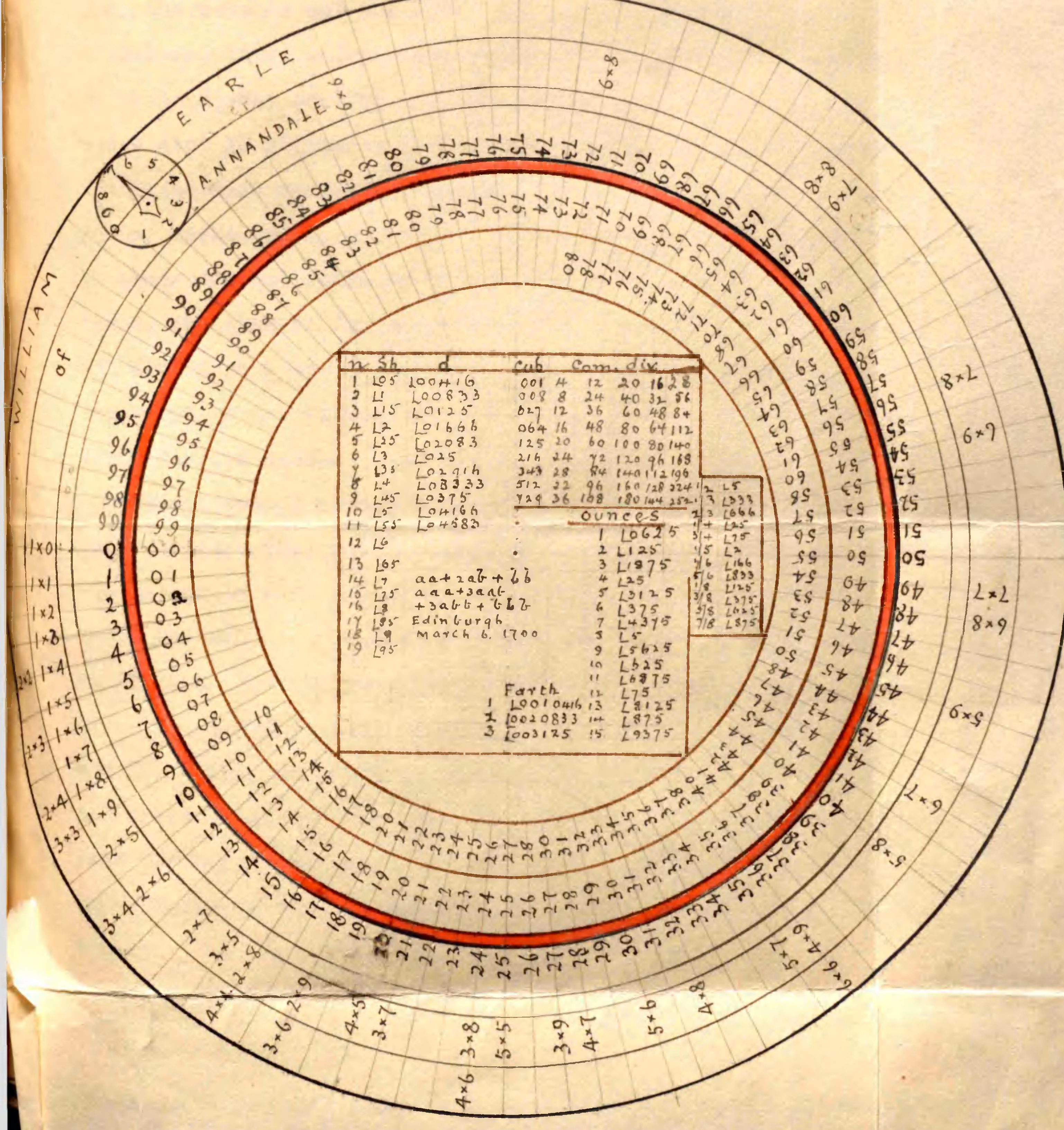
---

## THE ROTULA ARITHMETICA.

The first reference I found to this instrument was in an "Account of the life and writings of John Napier of Merchiston" by David Stewart, Earl of Buchan and Walter Minto, LL.D. This book was published in Perth in 1787, and in giving an account of Napier's Rods they refer to Brown's Rotula and say that a specimen of this calculating machine was then to be seen in the Advocate's Library. I found on inquiry that no such instrument had been in the possession of the Library for many years. Ultimately I discovered a specimen of the Rotula conforming in every detail to the description in the inventor's book, in the Archaeological Museum in Edinburgh, and I was informed by Mr. Callender, the Curator, who kindly allowed me to examine the instrument that there is another similar specimen in Kirkwall. The one in Edinburgh bears the name "William Earle of Annandale" and the heading "Edinburgh, March 6, 1700" a little over a year after the patent was granted.

It/

# Sketch of Rotula



It consists of two concentric circular discs, the inner of which is movable and the outer fixed. The discs are of brass and the frame is of hard wood, the whole being solidly built. Both the movable and the fixed circles are graduated into 100 divisions, these being respectively on the outer and inner edges of their discs, so that they lie against one another. The outer disc is 12 inches in diameter, the inner about 8 inches. The fixed disc contains a pointer with a dial divided into ten parts, marked 0, 1, 2.....9. As the inner disc is rotated it comes automatically to a stop when 99 is opposite 0 on the fixed, and then if forced round farther it turns the pointer on the dial from 0 to 1, indicating thereby the number of hundreds traversed by the movable disc. It must be understood that the divisions are natural numbers, not logarithmic ones, so that the instrument can be used for additions and subtractions. It seems strange that one should have recourse to an instrument to enable one to add, say, 8 and 5, but the author claims that "when the rotulas are become common, the mother may teach her children at home as much Arithmetick as may serve them all their lives." Addition and subtraction are easy. In Addition, you put the first number to be added on the movable opposite zero on the fixed disc, and read your answer on the movable opposite the other number to be added, on the fixed. The reverse process of subtraction is equally obvious, but if the numbers exceed 100, it would/

would have to be done a figure at a time, and possibly the number in the units place in the Discharge (i.e. the subtrahend) may be greater than that in the Charge (or minuend). E.g., we have to subtract, say, an 8 from a 5. We set 5 on the movable against 8 on the fixed, and instead of reading opposite zero on the fixed we read at 10 on the fixed and get 7 as the difference. The rest of the process is obvious; he calls 10 in the above rule "the Denominator of the Species", i.e. of the decimal and Arabic systems, but if you are carrying through a subtraction, say, of pence from pence, then 12 becomes the denominator of the species; in shillings 20 would be the denominator, and so on.

Let us see how multiplication is tackled. The multiplication table is too difficult to be memorised! So it is set down on the rotula, i.e. on the fixed disc; outside the number on two outer concentric circles you find the multiplication table. Opposite 6 on the fixed scale you will find 2 X 3, opposite 24 you will find both 3 X 8 and 4 X 6, and so on, up to 81 = 9 X 9. With the rotula you require to learn multiplication by a single digit (as in Napier's Rods), the rest of the process being carried out on paper in the usual way. Suppose then we had to multiply 542 by 8, we go round the circle on the fixed disc until we find 8 times 2. We set down 6 on our paper and in order to carry the 1 to the next place we turn the movable circle so that/

that the division 01 on it is opposite the zero on the fixed. Now we go along looking for 8 times 4 on the fixed, and opposite this on the movable we find 33, since the movable is 1 place in front of the other. We set 3 on our paper and turn 03 on the movable to zero on the fixed and look for 8 X 5, opposite which we find 43. Hence our answer is 4336. It will be clear from that description that the rotula acts as a multiplication table up to 9 times 9 and little more.

Three methods are suggested for division. The first concerns cases where the divisor is fairly small. Suppose we wish to divide 8865 by 9. We first set everything to zero, i.e. "Rectify" the instrument. We then take the first "Dividual", i.e. the partial dividend 88. Turn round the movable slowly until 88 on it is opposite some number on the fixed which has factors containing a 9 in them. Set down the other factor, in this case also 9, as the first figure in the answer. Then opposite zero on the fixed scale you will find 7 on the movable. This 7 is the remainder, and is to be carried on to the next figure which becomes 76. Now rectify again and turn round until 76 comes opposite a number containing a 9 factor. The other factor in the "cell" 8 goes in the answer, and so on.

When the divisor is greater than 9 a different method is employed. Suppose we have to divide 865 by 13. We put in a peg on the fixed disc opposite 13 (there are little perforations for/

for the purpose. We note the point of the movable opposite this peg, viz. 13, and turn it round to zero, and count one. Again we note the point opposite 13 on the fixed and turn it round to zero, counting two. We keep on doing this until the number 86 on the movable has passed the 13 on the fixed and find we have made six movements. 6 is the first figure in the answer. Now we look along the fixed scale from 0 until we spot our 86 on the movable and read the number on the fixed opposite 86; in this case it is 8, .. 8 isour remainder, and we proceed to divide 85 by 13 for the next figure. This seems rather a lengthy process but doubtless speed would be acquired by practice.

In long division the rotula is merely an aid to the ordinary paper method. It is used to find all the products of the divisor by the nine digits as in Napier's Rods. If the divisor contains two digits only, these can easily be read off; e.g. suppose divisor is 19. Put 19 on the movable to the stop (i.e. zero on the fixed)opposite 19 on the fixed, read off 38 on the movable, opposite 38 on the fixed read 57 on the movable, and so on. Then use these in dividing by the ordinary method.

```
19 ) 796857 ( 41939 16/19
38     36
57     178
76       75
95       187
114        16
133
152
171
```

One/

One could imagine this might be useful in an office where a good deal of division by the same divisor had to be done. When the inventor however begins to apply his instrument to direct division by a divisor containing 4 digits, he is stretching it beyond the limits of its usefulness. Space forbids the elaboration of the method, but its leisureliness can be inferred from this indication. He wishes to divide 15036948 by 1986, and to do so he has to try 9, 8 and 7 in turn as possible first figures for his quotient before alighting on the correct one. This adaptation of his instrument would scarcely be popular with "mothers teaching their children Arithmetick."

The use of the rotula is not confined to integers, and its author applied it to compound quantities by the simple process of turning these into decimals of the higher denomination. His insistence on the use of decimals Ishall refer to again in discussing his later books. Here it appears in the form of tables which fill up the centre part of the movable disc. On that are engraved tables showing what decimal of a £ is represented by 1.....11 pence and 1.....19 shillings; it also shows 1....15 ounces as decimals of 1 lb, 1, 2 and 3 farthings as decimals of £1, and the commonest vulgar fractions up to $\frac{7}{8}$ as decimals.

Yet another table in the same space gives the tables of divisors in common use such as 4, 12, 20, 16 and 28, up to 9 times in each; another gives the cubes of the first nine numbers/

numbers, and there is even given the square and cube of a + b, thus: aa + 2ab + bb and aaa + 3aab + 3abb + bbb.

The end of the book shows the solution of problems in rule of three, interest, fellowship, etc., and he concludes with the naive remark, "If any Difficulty occur in these lessons, it may be easily overcome and at a very Reasonable Rate by a little converse with the Author."

The Rotula is a wonderfully ingenious device, but could scarcely justify its claim to be an improvement ~~of~~ on the logarithmic instruments. Its importance historically lies in the evidence it supplies of the fact that mathematically Scotland had not advanced very far by the end of the seventeenth century; the great advance was yet to come.

## A COMPENDIOUS BUT A COMPLEAT SYSTEM OF DECIMAL ARITHMETICK.

This book, published in 1701, is a companion volume to the one on the Rotula, and covers a good deal of the same ground. In many respects it is a remarkable book, as its author must have been a striking personality. The period under review was a transition one, and witnessed a considerable development in the popularising of mathematical study, doubtless owing to the tremendous reputation acquired by Sir Isaac Newton. This work actually/

actually personifies that transition, as it commences with finger reckoning which, we have seen, was the first arithmetic of our ancestors, and ends with the use of logarithms in the solution of problems in Compound Interest and Annuities. His finger reckoning is confined to the numbers 6, 7, 8 and 9, the first five digits being considered fit for mental work. For numbers greater than 5, a shut hand represents 5 itself and one, two, three or four open fingers indicate 6, 7, 8 and 9 respectively. To add, express one digit on the left hand, the other on the right, then the sum of the open fingers shows how much the sum of the numbers exceeds 10.

The only case of subtraction to which this form of counting need be applied is that in which a digit greater than 5 has to be subtracted from another smaller digit. Put the digit to be subtracted on the left hand and count the shut fingers, then add that number to the digit from which you were subtracting. Thus, e.g. 8 from 1. 8 shows two shut fingers. $2 + 1 = 3$.

In multiplication the sum of the open fingers on both hands gives the tens figure, and the product of the shut fingers gives the units figure, e.g. 9 X 7. 4 + 2 open fingers, 1 X 3 shut fingers.

In division the divisor is put on the left hand, then on the right we open up as many fingers as, with those open on the left will make up the tens digit of the dividend. The right hand tells/

tells us the quotient, provided the product of the shut fingers does not exceed the units figure of the dividend. e.g. 87 divided by 9 leaves four fingers open on the right hand, representing the figure 9, which is the quotient since the product of the shut fingers is 1 X 1.

But in the case of 61 ÷ 9, quotient should be 7 by the rule, but the product of the shut fingers is now 3 X 1, which is greater than the units figure: hence the true quotient is one less than 7.

With this concession to a traditional method of counting which had been an unconscionable time in dying, Brown proceeds to elaborate his arithmetic proper, the outstanding feature of which is his enthusiasm for the introduction of decimal fractions. He was undoubtedly a pioneer of the decimal, and had a clear idea of the theory of its use and of its value in calculation. Throughout the book he never uses vulgar fractions; his notation is the modern one, advocated by Napier, but not then in universal use, but Brown, evidently unwilling to discard the old sexagesimal units, retains sometimes the letters a, b, c, d, etc. above the consecutive decimal digits. Thus:

$\overset{a}{\cdot 1}\overset{b}{1}\overset{c}{1}\overset{d}{1}$ i.e. one prime, one second, one third, one fourth.

Brown introduces infinite decimals, but only those with one repeated figure. This is due to the fact that his chief use for decimals/

decimals is to express shillings and pence as decimals of £1. His application of the four rules to such finite and infinite decimals is on the usual lines, but he gives a rather neat, and I think, original method for multiplying a single repeater by a whole number.

Multiply in the usual way, and when the sum of the two digits in your answer is less than 9, set it down as your figure and carry the tens digit, e.g. .0041$\dot{6}$ X 7. 7 times 6 gives 42. Add the 4 and 2. Set down 6 and carry 4.

On the other hand if the sum of the two digits exceeds 9, set down the excess over 9, and carry one more than the tens digit. Thus .0041$\dot{6}$ X 8. First figure is 3 and carriage 4. 5 This is, of course, just dividing by 9, but in this simple form would doubtless appeal to the arithmeticians of Brown's day.

Brown shows quite accurately how to multiply by a decimal whether finite or infinite, but, like his predecessor, he is evidently not clear as to the meaning of the operation, as we find him quoting as a"Famous Example" the multiplication of

3L 19sh. 6d. by 3L 19sh. 6d.

Division is still a laborious process, but is done in the modern way, ultra-modern as regards the setting down, for he does not put down the separate products of the divisor with the digits in the quotient, but merely the result of the subsequent subtraction, i.e. he uses what was then known as the "Short Italian/

Italian" method. It is far from short, however, as he often tries each of the digits in turn from 9 downwards before he gets the correct one. Thus if he were dividing 908 by 19, he would find that the 1 goes into 9 nine times; this is a trial quotient but will not do, because 9 X 9 is greater than the next figure 0; so he works down through 8, 7, 6, 5 to 4 before he hits on the right figure.

In division of decimals, the method used here and right through the eighteenth century is to ignore the decimal point until the very end of the process and then to place the decimal in such a position that the sum of the number of places in the divisor and the quotient is equal to the number of places in the dividend. He shows how to divide by a repeating decimal without reducing to a vulgar fraction.

From the nature of the work covered in the remainder of the book it is clear that "Decimal Arithmetick" was written for the use of the business man, as it consists of such subjects as Exchange, Rates and Valuation, Simple and Compound Interest, Present Worth and Discount, Annuities. The usual method of solving such problems was to compile a table on the assumption of a sum of £1 for various rates and times, and to calculate what is required to produce the given sum by simple division. Brown, however, does not provide these tables, but shows how they may be worked out by the use of logarithms.

"Decimal Arithmetick" represents a considerable advance on the/

the work of Sinclair and Paterson. It is by no means exhaustive, and the author, in completely ignoring the rules of vulgar fractions, erred through over-enthusiasm for the newer decimals. He lived before his time, and even now we have not come round to his way of thinking, but he had surely some part in inaugurating a movement towards the decimal. Unfortunately his followers were led into that morass known as the repeating, circulating and later recurring decimal, so that the beautiful simplicity of the decimal system was obscured, and men must have turned in disgust from its needless intricacies.

---

ARITHMETICA INFINITA

OR

THE ACCURATE ACCOMPTANT'S BEST COMPANION, 1717-18.

---

Of all the books on elementary mathematics that I have read, this one is unique in that it is not out-of-date even in the twentieth century. The life-work of the author was the popularising of decimal fractions. In this book we have a costing table showing the price of from 1 to 9 articles at every rate from $\frac{1}{4}$d. to 19 sh. 11$\frac{3}{4}$d.; the prices are expressed in pounds correct to the seventh decimal place, so that it is possible to calculate/

calculate the cost of any number of articles up to 100 millions. The decimal is indicated by the sign $\lfloor\quad$ or by sticking a pin in the appropriate place, the latter method being the more useful when we wish to price a large number of articles, as for example 9648, when we have to read the 9 row shifting the decimal three places for 9000, two places for 600, and so on.

The costing table is continued beyond £1, but, with the economy of the true mathematician (which might not appeal to the business clerk using the tables) Brown gives the rates up to £99, omitting those from £1 to £9, and every multiple of £5. If you wish to price so many articles at £1, £2, £3......£9 each, you double that figure, take the shillings table shifting the decimal point one place to the right (since £8 = 10 X 16/-.) For the same reason he omits multiples of £5, taking, say £15 as 100 times 3/-. Again ½ an article will be got from the table for 5 articles and so on. To complete the scheme he gives the usual rule for converting a decimal of a pound into shillings and pence, and conversely, in the latter case, to seven decimal places. Here, if the second and third decimal places found in the usual way are 00, 25, 50 or 75, they are said to be "perfect" and need no addition; if they are not perfect, the "excess" is written down by subtracting 25, 50 or 75 from the decimal

e.g. 1/8 = $\lfloor\underline{083}$ excess is 8

11/1½ = $\lfloor\underline{556}$ " " 6 and so on

Now/

Now to continue this to the fifths, i.e. 5th decimal place, we take this "excess" and multiply it by 4, adding one if it be 24 or more, 2 if 48 or more, 3 if 72 or more.   Then

$$1/8 \;=\; \lfloor\underline{08333}$$

$$11/1\tfrac{1}{2} \;=\; \lfloor\underline{55625}$$

The second one is now perfect, but the first one would be continued to the seventh place by further application of the same rule:    Excess = 8    4 X 8 = 32 + 1 = 33

The number 1 which is added here he calls the "perfective" written with a symbol $\mathcal{P}$.   The excess is denoted by the letter "x".   He gives the reason for multiplying the excess by 4; where the number is not perfect there is an error of $\frac{1}{24}$ of a unit in the third decimal place, for every one of the "excess" over the "perfect" number and $\frac{1}{24}$ = approximately $\lfloor\underline{04}$ .   This again leaves an error of $\frac{1}{24}$ of a unit in the fifth place where that is not"perfect" and so on.   In this way the author has compiled a very complete and extraordinarily useful costing table:  the only criticism is that from the modern point of view, while his figures are correct if taken to the seventh place, they are not correct to the third place, that is to say, his rule of adding 1 for every 24 farthings does not, in 50% of the cases give the answer correct to the third place;   it should be, of course, 1 for 13 to 37 inclusive, 2 for 37 to 63 inclusive, and so on.   Doubtless, in consequence, the user of the table/

table in the eighteenth century would discover an error of $\frac{1}{4}$d. often enough, for all the arithmeticians of this epoch in approximating simply dropped the extra figures whether the next one was more or less than 5. Thus $\underline{|8979166}$ with Brown is $\underline{|897}$ correct to the third place. Apart from that detail, which would not prevent anyone at the present day from making accurate use of the tables, this book is a remarkably good instrument for use in commerce. It also contains an interest table per annum and per diem for every $\frac{1}{4}$% from 1% to $9\frac{3}{4}$%, a table of guineas (from which it is interesting to learn that that coin varied in value from 20/6d. to 22/6d.), an exchange table, and the cost of a year's living at so much per day. The last table of all shows what decimal of £1 one can spend per day if one's income per year is any figure whatever. He gives the decimal of £1 up to 9 places.

The Arithmetica Infinita seems to have been the most popular of Brown's books, but, so far as I know, it reached only one edition. One suspects that its author was a little before his time, and that his book has received less attention than it deserves.

# CHAPTER X.

## COCKER'S ARITHMETIC.

This book, though written by an Englishman, finds a place in my paper because of its extraordinary popularity in Scotland. Cocker was to the Scotland of the eighteenth century what Gray was to the Scotland of the nineteenth century, and a copy of his Arithmetic was to be found in most households in the latter half of the century. Thus in an article on Scottish education amongst the poorer classes in Burns's time, it is said that "the penmanship of Butterworth and the Arithmetic of Cocker may be studied by men in the humblest walks of life".[X] Robert Burns was taught arithmetic by his father by candle light (1768) and it is very probable that a copy of this book was in possession of the family.

It is somewhat puzzling to account for the popularity of Cocker and for its proverbial reputation for accuracy when one peruses some of the copies available. Far better text-books were written by Malcolm, Mair and others, yet they failed to displace the other, and we find Mair and Fisher editing editions of Cocker as well as writing arithmetics of their own. According to De Morgan, Cocker's extraordinary popularity was due to his introduction of the modern method of/

X Life and Writings of Burns, edited by Alexander Peterkin, New York, 1824. p. 111

of division in preference to the old "scratch" method described previously. His aim was to teach by a simple, easy method "suitable to the meanest capacity", and by setting down all his rules concisely, and omitting any proof of them, he may be said to have attained that object. Another feature was the introduction of examples to be worked by the pupils. There is little doubt that the phenomenal sale of the book was due to these features, and when it was replaced by Melrose and Gray, the reason probably was that these books went a step further in the conciseness of their statements of rules and in their multiplicity of examples.

Cocker, a Londoner, was born in 1636 and died in 1676. He wrote many books. "The tutor to Writing and Arithmetic" was published in 1664, but his famous Arithmetic appeared posthumously in 1678, with a second edition the next year, a fourth in 1682, and thereafter it was constantly reprinted not only in London, but in Dublin, Glasgow and Edinburgh. I have seen the 42nd edition, dated 1725 & the 52nd edition, dated 1748. The latter edition is supposed to be corrected and amended by George Fisher, who himself wrote a popular book on Arithmetic. I have also read two editions revised and corrected by John Mair of Perth Academy; these are dated 1765 and 1771, but the foreword by Mair is dated 1751. The number of the edition is not stated in either case. The one revised by Fisher is so full of misprints as to be almost unintelligible and to make Cocker a byeword for inaccuracy, but Mair's edition is a great improvement/

improvement on the other.

The style of the book is very pretentious, the definitions are abstruse and too diffuse. Thus we find him saying "Hence it is that Unit is Number; for the Part is of the same Matter that is its Whole, the Unit is Part of the Multitude of Units; therefore the Unit is of the same Matter that is the Multitude of Units, but the Matter of the Multitude of Units is Number. Therefore the Matter of Units is Number" and so on. The beginning of the book, containing these definitions and an account of the ancient theory of numbers, Lineal, Superficial, Solid, Perfect and Imperfect, etc. is the weakest part. The sections on Rule of Three, Fellowship, Alligation and Practice are well written. Vulgar fractions are comprehensively treated, but decimals are not included. In dealing with Profit and Loss, which he calls "Loss and Gain", he reckons the percentage sometimes on the Buying Price and sometimes on the Selling Price; he does not use the term "per cent", but speaks of "gaining 15L in laying out 100L" or "loseth after the rate of 12L in receiving 100L". The rule for treating a problem in exchange involving a number of currencies is delightfully simple; you just arrange the terms in two columns, see that the same medium of exchange does not appear twice in one column; then one column will have one less term than the other. Take the product of the items in the column which has most terms in it, as dividend, the product of the items in the other column as divisor, and the quotient is the answer/

answer. This is a good example of the type of rule favoured by Cocker. Occasionally the author worries about not providing a proof of the rules he states, but is content to verify his answer as a proof of the rule. In the case of the rule for Double Position, he refers the reader, for a proof, to Wingate's Arithmetic or Oughtred's "Clavis Mathematica".

John Mair's edition of Cocker is greatly superior to the earlier one; he says in his preface that as the modern editions have been "so carelessly printed..........that the book tends to mislead rather than instruct" he has gone back to the edition of 1697. This edition of Mair's is superior too in the matter of printing.

With all its imperfections, Cocker's book must find an honoured place in the history of the development of arithmetic; it helped to destroy the feeling that the processes of that science were too difficult for the man of average intelligence, or even the boy of school age, so that in the eighteenth century arithmetic found its place with the other two Rs in the curriculum of the Parochial Schools.

---

Another book on Arithmetic, published in 1714, was definitely intended for the use of pupils in Scottish Schools. It is entitled "Arithmetick Compendiz'd, or a Short Treatise of Arithmetick; composed for the use of the Charity Schools erected by the Society in Scotland for propagating Christian Knowledge".

In the first edition, of which there is a copy in the British Museum/

Museum, the author is described simply as a "Member of the Society". The second edition was published in 1721; a copy is to be found in the Scottish National Library, the preface being signed in ink by "David Spence, Treasurer to the Old Bank of Scotland".

The first edition contains 87 pages and measures $6\frac{1}{4}$ inches by $3\frac{3}{4}$ inches; it is a mere introduction to Arithmetic, designed for cheapness, and aiming at the training of the "Merchant, Factor, Chamberlain, or Tradesman". It deals with the four rules in integers and vulgar and decimal fractions, and includes a section on the Rule of Three in integers. In discussing numeration the author uses the term Digit, Article, and Mixt Number as in Sacrobosco's Algorism. Infinite decimals are not used; when they occur, they are restricted to five or six decimal places. The book throughout is strictly utilitarian, but few examples for working out are provided.

The second edition is identical in contents with the first except that it contains "Tables of Interest" for the use of bankers. These tables are drawn up in a curious way; it seems that the official rate of interest varied at different times between 5, $5\frac{1}{2}$, and 6 per cent. The principal sum taken is £100; interest is given quarterly at each term from Martinmas, 1669 to Martinmas, 1733, and is cumulative, so that for any intermediate dates it can be found by subtraction. It is interesting to find that the Whitsunday term, up to 1690, occurred at various dates in May and June, but thereafter it was fixed by law to fall on

15th/

15th May.

A second interest table is included in the book; it shows the return on £100 for any number of days from 1 to 30 at 6, $5\frac{1}{2}$, 5, $4\frac{1}{2}$ and 4 per cent.

"Arithmetick Compendiz'd" is a simple little book of no great merit, but it is interesting, historically, in that it provides us with clear information as to the course in Arithmetic followed in the parochial and charity schools early in the eighteenth century.

# CHAPTER XI.

## PLACE OF MATHEMATICS IN THE CURRICULUM OF THE SCHOOLS 1700-1750.

During this period the grammar schools in the University towns maintained their old curriculum which was confined to the study of Latin. Thus in 1696 we find the Master of Glasgow Grammar School, asked to give an account of his method of teaching, saying that the work was carried on "accordingto a standard formula observed over three hundred years in that school".[X] In 1718 the curriculum of the Edinburgh Grammar School was changed, but it was still restricted to Latin. In George Heriot's Hospital only a little arithmetic was taught. Boys entering the grammar schools at the age of eight or nine were expected to have had a little training in reading, writing and very elementary arithmetic.

On the other hand, grammar schools in districts outside the sphere of the Universities, included the study of the three Rs in their curriculum. I have already mentioned some schools where arithmetic is definitely known to have been taught in the seventeenth century, The Burgh Records show that that subject was in the curriculum of the grammar schools in the following towns at the dates mentioned: Dysart, 1708; Dundee, 1712; Forfar, 1713; St. Andrews, 1714/

X Munimenta Univers. Glas. vol 2 p. 536

1714; Ayr and Selkirk, 1721; Perth, 1729; Dingwall, 1730; Haddington, 1731; Kilmarnock, 1745; Kinghorn, 1746; Dumbarton, 1747; and Kirkcudbright, 1748. The dates mentioned are those when definite reference to the teaching of arithmetic appears in the Burgh Records, but doubtless, in many cases, that subject was taught long before any reference was made to the fact.

Mathematics was taught in Perth Grammar School before 1718, at which date we find in the Burgh Records that "Patrick Stobbie was Professor of Mathematics at the Grammar School of Perth." [X] In 1721 the Doctor of the Grammar School at Ayr had to equip himself in Mathematics.[X]

Mathematics was taught in Dunbar Grammar School in 1734 and in Dundee in 1735.[X] I have no record of the extent of the study of mathematics at this period, but from my knowledge of the text-books then available, it appears to have been confined to the elements of Euclid and Trigonometry, sufficient for simple land surveying, and, possibly, gauging, the use of equations in the solution of problems, and last, but not least, navigation. The last mentioned subject was zealously taught in the towns on the Scottish seaboard from the beginning of the eighteenth century. We have references in the Burgh Records to that subject in Dunbar, 1721; Ayr, 1727 and Dundee, 1735.[X] It is clear from Sinclair's "Statistical Account of Scotland", to which reference will be made later, that navigation was a popular subject in many schools throughout the eighteenth century. By/

X Grant: Burgh Schools of Scotland. p 399

By the middle of the century, mathematics and science were on the eve of challenging the supremacy of classics in the grammar schools, and arithmetic had found a place in the curriculum of the majority of the parish schools which had been established throughout the land by that time.

Turning to individual schools of which we have records, we find that arithmetic was taught in George Heriot's Hospital early in the eighteenth century: there is a reference in the records of the Trust for the year 1731, which shows that a resolution was hastily carried, but soon abandoned, that "the teaching of bookkeeping be set aside, as writing and arithmetic are the finishing accomplishments of the Hospital".(1) Again we find that in March, 1739, a Committee was appointed to inquire "how far it might be convenient to employ a certain person to teach the boys practical geometry and drawing".(2) It does not appear that such a committee ever met, and for seventy years afterwards the teacher of writing was also the arithmetic master. In 1751, Thomas Heriot, Dean of Guild, left money to endow a prize to a boy "who shall upon examination be found to excel the rest of the boys in the common rules of Arithmetic".(3) The first recognised teacher of mathematics was appointed in the year 1810.

In the Edinburgh Grammar School, there is no reference to the teaching of Arithmetic until the end of the eighteenth century, but the writing master evidently included that in his department.

In/

(1) Steven's History of Heriot's Hospital, p. 100.
(2) " " " " " p. 101.
(3) " " " " " p. 108.

In the Grammar School of Glasgow, thenineteenth century had arrived before a class for Writing and Arithmetic was established (1816). Previously the pupils who wished to do so visited a Writing Academy outside. There is, however, a note in the Minutes of the Council of 1st October, 1739, showing the award of a salary of £8. 6s. 8d. to a Mr. James Stirling for teaching "Arithmetic in its various branches, Bookkeeping, Navigation and the parts of Practical Mathematics useful and necessary to be taught."[1]. This teacher had, of course, no connection with the Grammar School.

In Aberdeen there were schools for writing and arithmetic outside the Grammar School at least as early as 1607. Again in 1628, Wedderburn, then Rector of Aberdeen Grammar School "agreit with Andro Hewat.....to instruct and learne his scollares to wreit and to schaw thame some principles of arithmetic everie day betwixt ten and twelff in the foirnoone."[2] On 23rd October, 1710, there ~~is~~ was a recommendation to parents to see that their boys "be taught to read English perfectly and to write weell and somewhat of arithmetick and musick."[2] before they enter the Grammar School at the age of nine. Undoubtedly there were public schools for writing, drawing and mathematics in the town under the control of the Council, as the Grammar School itself was; pupils of the Grammar School could attend these for parallel instruction. This was the state of affairs at the end of the eighteenth century.

Robert/

(1) Historical account of the Grammar School of Glasgow by James Clelland, LL.D. p.8
(2) Bon Record - H. F. Morland Simpson. p 59

Robert Gordon's Hospital, founded in 1729 on the model of Heriot's Hospital aimed at providing instruction in arithmetic and bookkeeping, and, in the deed of foundation, it was set forth that instruction would be provided in these subjects by the schoolmasters if they were able, and if not by visiting masters. In 1751 the Governors recommended that "as the Boyes may come to knowledge and have a genius for the Mathematicks or Navigatione, both the Masters be obliged to teach the Boyes in these Sciences as they may be capable," (1) and new teachers appointed after that date were required to be accomplished in these subjects.

In the Grammar School of Stirling, there was a person qualified to teach writing and arithmetic certainly prior to 1697. In 1719 a separate school was established for young boys and girls in which they could learn the rudiments of the three Rs. It was soon incorporated into the Grammar School again. In 1727 there were two doctors for these subjects.(2) In 1747 the school for writing, arithmetic and bookkeeping started a separate existence in a different part of the same building as the Grammar School, and continued so for fifty years. The versatility of the teachers of the day is shown by the fact that the teacher of these subjects was also the precentor.

To sum up, from 1700 to 1750, in the University towns, a knowledge of elementary arithmetic was deemed essential for a pupil entering/

(1) History of Robert Gordon's Hospital - Robert Anderson. p.38
(2) History of the High School of Stirling - A. F. Hutchison. p.90

entering the Grammar School. When he entered these portals, he could only continue his studies in that subject outside the school. In these towns also there were many Academies in which the youth of the nation were being trained for the "Counting-House". In many of the Grammar Schools of the smaller towns, Arithmetic was taught by the Rector's assistant, along with writing and bookkeeping. Mathematics, apart from Arithmetic, was only beginning to find a place in the Grammar Schools of these towns.

# CHAPTER XII.

## THE WORKS OF JOHN WILSON.

John Wilson, A.L.M. describes himself as "Teacher of Mathematicks and consequently Navigation; also Arithmetick separately at particular hours" in Edinburgh, from which one may presume that he conducted an Academy for the training of youths who had passed through a Grammar School, and required to gain a knowledge of arithmetic and mathematics, particularly on the practical side.

Wilson wrote on the subject of Architecture as well as on purely mathematical work, and all of his books were "well received by the public".(X)

In 1714 he published a book on "Trigonometry with an introduction to the use of both Globes and Projection of the Sphere", etc. The copy I have seen is, curiously enough, bound along with a book on the doctrine of Fractions by Samuel Cunn, the latter being printed by a London printer. Both books appeared in the same year, and the arithmetical book is interesting in that its author claims to be the first "to make a systematic study of Circulations", i.e. recurring decimals which have more than one repeating figure. He admits the priority of the Rev. Mr. Brown in his "System of Decimal Arithmetick", in/

(X) Preface to Wilson's Arithmetic written by John Mair.

in the use of interminate decimals, but points out, what I have already mentioned, that Brown uses only such factors as will produce a single "Repetend". I have not found any reference to Cunn's work in the books of subsequent writers on repeating decimals in Scotland. On the other hand, Wilson's work on Trigonometry was a standard work on the subject throughout the eighteenth century and is quoted with respect even at the end of that epoch. I have also seen a copy of the book, dated 1714, in which it is not bound along with Cunn's Arithmetic, and I am disinclined to consider the possibility of that Arithmetic having been in use in Scotland.

Wilson, in his preface, tells us that he is a teacher, and that the method he lays down is that which he has adopted in his teaching. If so, one is inclined to believe that he would be a very successful teacher; he is clear and concise, and his language is simple, but expressive; his curt, crisp style of writing is characteristic. As a step towards clearness and simplicity, he sets forth the symbols, that he proposes to use, on the front page. He uses all the trigonometrical ratios, and contracts some of them more than we do. Thus $\underline{S}$ or $\underline{s}$ stands for sine, $\underline{T}$ or $\underline{t}$ for tangent, $\varsigma$ or $\sigma$ for cosine, 7 or cot for cotangent. Other symbols, which are now in disuse are:-

: : : for Arithmetical Proportion

$\div\!\div$ for continued Geometrical Proportion, e.g.

a : b :: b : c :: c : d

$\dfrac{\cdot\ \cdot\ \cdot}{\cdot\ \cdot\ \cdot}$ /

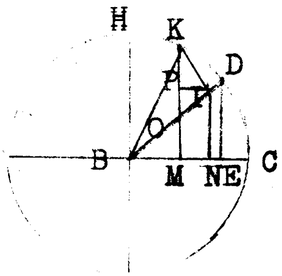
$\frac{\cdot \quad \cdot \quad \cdot}{\cdot \quad \cdot \quad \cdot}$ for continued Arithmetical Proportion

i.e. a - b = b - c = c - d etc.

Most of the common symbols of our time are in use at the period now being considered, but the Cartesian notation of powers is not yet established, so that $DC^2$ (the square on the line DC) appears as $DC_q$; the letters used are printed rather small in size, so that the book is somewhat difficult to read. Doubtless the small print was in the interests of economy, as was another feature of this and other mathematical books of this date, viz. that, instead of placing the diagram of a geometrical theorem on the page to which it refers, a collection of fifteen to twenty diagrams is made on a page inserted some distance ahead. The book follows the lines of Euclid by setting out theorems and problems numbered in order. The aim of the first part is to show how the table of Sines and Tangents is constructed. These are, of course, lines, not ratios, in a circle of radius equal to ten million units. To construct a table of sines, he requires first to establish certain theorems for the sine of half the arc in terms of the sine of the whole arc, and for the sine of the sum and difference of two arcs (spelt "arches").

I shall quote one of these proofs to show the notation used:;

"Given: Sines of two arches to find the sine of their sum




Given/

Given KI and DE the sines of two arches KD and DC.

Sought KM the Sine of their sum KC.

Complete the Parallelogram PINM.

The Triangles BDE, BIN, BOM, OIK, OIP, and KIP are all similar.

Therefore BD.DE :: BI.IN(= PM)(4 e 6)

and BD.BE :: KI.K~~O~~P (4 e 6)

and KP + PM = KM Q.E.F.

This is all the proof that is given, the figures in brackets referring to Euclid. DE represents the sine of the "first arch", BE its cosine, KI the sine of the "second arch" and BI its cosine.

A single dot is the symbol for ratio throughout this book.

In constructing his table of sines he first shows that the sine of $30^\circ$ is one half of the Radius, since the Radius is the chord of $60^\circ$, and the sine is the half chord of double the angle. Then by using the relation between sine of an angle and sine of half the angle, he finds sine $15^\circ$, sine $7\frac{1}{2}^\circ$ and so on down to the 2048th part of $30^\circ$ which is $52''\ 44'''\ 3''''\ 45'''''$ i.e. 52 seconds, 44 thirds, 3 fourths, 45 fifths, in sexagesimals. Now he assumes that for such small angles, the sine is proportional to the angle and since $30^\circ = 1800'$

then $1800 \cdot 2048 :: 52''\ 44'''\ 3''''\ 45''''' : 1'$

$:: \text{sine } 52''\ 44'''\ 3''''\ 45''''' : \text{sine } 1'$

Hence he builds up the table from sine of one minute, two minutes, etc./

etc. up to $30^\circ$; from $30^\circ$ to $60^\circ$ he fills up the table by use of the following theorem:-

> "If two arches be equally distant from $30^\circ$, the square of the Sine of their common Distance is one-third of the square of the Difference of the Sines of these two Arches."

From $60^\circ$ to $90^\circ$ he builds up the table by making use of the proposition:-

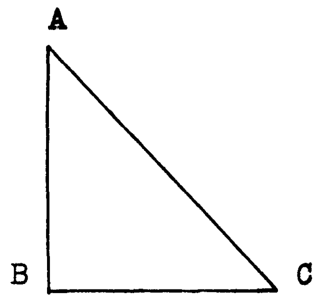
> "The Difference of the Sines of two Arches equally distant from $60^\circ$ is equal to the Sine of their common Distance."

Geometrical proofs of these theorems are given. The table of sines, now completed, provides cosines when required, and the table of tangents is then constructed from the relation:-

Radius . Tangent :: Cosine . Sine

Wilson devotes a section to the explanation of the construction and use of tables of Logarithms or "Artificiall Numbers" and "Logarithmick or Artificial Sines" before expounding the solution of "Plain Triangles".

Here it was the practice to devote a section first to solving right angled triangles, and the crux of the method consisted in the choosing of the proper side of the triangle to represent the radius. Suppose, for instance, the two sides containing the right angle are given and we wish to find one of the two acute angles.




Given/

Given AB and BC, to find angle ACB.

We assume CB as the Radius, C representing the centre, then BA will correspond to the tangent of the angle ACB, and we have

$$BC \;.\; R^a \;::\; AB \;.\; {}^{t}C \qquad R^a = 10,000,000 .$$

From this ratio with the help of a table of tangents the angle C may be found.

If we had been given AC and AB, we should have chosen AC as radius and said

$$AC \;.\; R^a \;::\; AB \;:\; {}^{s}C$$

The method of solution of ordinary triangles (called "oblique-angled") differs little from that of the present-day except with regard to the case where the three sides are given, and the angles have to be found. Here he draws a perpendicular from the vertex to the base, and works out the length of one of the segments of the base so formed by means of the theorem that the base is to the sum of the sides as the difference of the sides is to the difference of the segments of the base. These segments found, the problem reduces to the solution of two right-angled triangles. The supplement of an angle is called its "Complement to 180°."

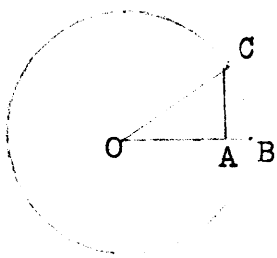
The solution of "Spherique" triangles is based on 43 geometrical propositions, for which he provides only 9 diagrams! Wilson makes no use of Napier's "Rule of Circular Parts", so that he has as many as sixteen separate cases for the solution of right-angled spherical triangles alone. The angle of a spherical triangle is indicated/

indicated by a single capital letter, the side by two letters. The solution of the case where the hypotenuse and one angle adjacent are given appears thus:-

| | Given | Sought | Proportions |
|---|---|---|---|
| Case 16 | BC Hyp. $\angle^{le}$ C | B | R : $\sigma\!\!^{-}$ BC :: $t_C$ . $\urcorner B$ |

(i.e. Radius : cos BC :: tan C : cot B)

In the solution of oblique angled spherical triangles he makes considerable use of the versed sine of an angle, i.e. the part of the diameter lying between the sine and the circumference, or, in other words, the radius - the cosine.




AB is the versed sine of the arc CB.

The case where the three sides or three angles of the spherical triangle are given is solved by the use of the versed sine of an angle, which can easily be obtained from the table of sines.

In one respect this text-book shows that the influence of the mediaeval theories of astronomy had not yet evaporated. Wilson gives a full account of the use of the two Globes, one Terrestrial and the other Celestial. Each of these is hung in a Brazen Meridian, so that any point of either sphere can be brought to this circle/

circle which will then represent the meridian of that point. This meridian is divided into quadrants. The artificial globe is also equipped with a "Wooden Horizon" containing a circle divided like the Ecliptic into Signs and Degrees. The space on this horizon (in the economical manner that we observed in Brown's Rotula) contains two calendars, viz. the Julian and Gregorian, each divided into months and days, "answering to the Divisions on the former Circle". The Julian calendar commences on March 11th, the Gregorian on March 22nd. The Wooden Horizon also shows the thirtytwo points of the compass.

With these globes Wilson is able to solve a great many problems. With the Terrestrial he finds the latitude and longitude of any place, and conversely the part of the earth which has a given latitude and longitude. He measures the distance between any two points on the earth's surface (a) in degrees, (b) in miles, counting 60 miles to one degree.

With the Celestial Globe, he measures the sun's Declination, Right Ascension, etc., the "Hourand Minute" of the sun's rising and setting; also similar problems for the stars. For example, to find the sun's place on the Ecliptic at any time, consult the Wooden Horizon for that date and you find in what sign and degree the sun then is; spot that sign and degree on the ecliptic of the Celestial Globe and you have placed the sun.

A considerable space in the book is devoted to such mediaeval problems/

problems as the calculation of the Golden Number, Epact and Dominical Letter for any year. The Golden Number tells in what year of the nineteen year cycle of the moon we have arrived; to get it add 1 to the year, divide by 19, and note the remainder. The Epact is the age of the moon on the 1st of January; to find the Epact of the year, multiply the Golden Number by 11 (the difference in days between the solar and lunar year) and take the remainder when this product is divided by 30.

For the year quoted, 1713, Golden Number is 4. Therefore Epact is remainder when 44 is divided by 30, i.e. 14.

The calculation of the Dominical Letter is done according to the "Old Style" calendar. Take the year, add $\frac{1}{4}$ of it (for leap years), then add 5, divide by 7, note remainder, and subtract it from 8.

Thus

$$\begin{array}{rl}
4\,\underline{|1713} & \\
428 & - 1 \\
\underline{1713} & \\
\underline{2141} & \\
\underline{5} & \\
\underline{|2146} & \\
306 & - 4
\end{array}$$

$$8 - 4 = 4$$

This means that the Dominical Letter for 1713 was D. This, of course, applied to the old style calendar.

The rule for the New Style calendar which came into vogue in 1752 is given in the second edition of Wilson's Arithmetic.

In/

In Leap Year, of course, there were two Dominical Letters, one for the period up to February 24th (when apparently the extra day was wont to be added) and the other for the remainder of the year.

The next subject to engage attention is the construction of dials and the laying out of hour lines thereon. These dials are horizontal or vertical, but there are many varieties of vertical dial. One perpendicular to the horizon and facing North or South, he calls "Direct Erect". Another looking North or South, but with its face falling back from the Zenith is known as "Direct Reclining". An erect vertical dial, whose face looks East or West is called a "Meridian Dial".

In the next section Wilson tackles by means of trigonometry the type of problem for which the Celestial Globe was used. Here is an example:-

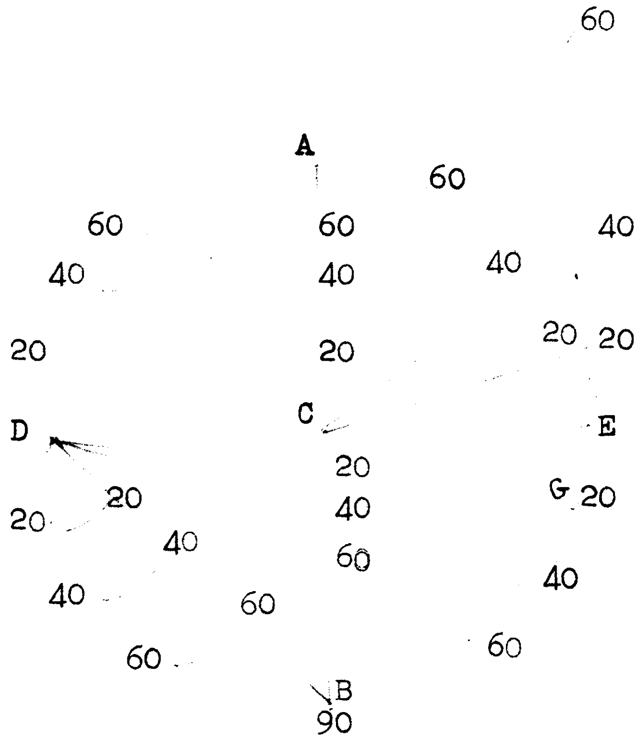
Problem is this: "given the Sun's Altitude, Azimuth and Declination, to find the Hour of the Day in any given Latitude."

In the Triangle PZ⊙, $^{s}$P⊙ . $^{s}$PZ⊙ :: $^{s}$Z⊙ . $^{s}$ZP⊙, the Measure of the Hour from 12".

This is all the solution given in the text, but it is understood that HR stands for the horizon, EQ for the equator, ZN for Zenith and Nadir, Pp for the Ecliptic. Thus P⊙ is the complement of the Declination, PZ⊙ is the Azimuth, Z⊙ is the complement of the altitude. Hence the angle ZPO can be found in degrees and so on.

It will be seen that the statement of the solution is mathematically concise, and no examples are worked out.

An excellent account is given of the stereographic projection of the sphere, and for the applications of projection he constructs a scale on which are shown simultaneously a line of Sines, Tangents, Secants, Semi-tangents and Chords. For this purpose a circle is drawn and the circumference is divided into four quadrants and each of these into $90^{\circ}$. The scale shows Sines up to $90^{\circ}$, Tangents and Secants up to $70^{\circ}$, Semi-tangents up to $90^{\circ}$ i.e. tan 45, and Chords up to $90^{\circ}$. The Semi-tangent seems unnecessary, but is used in the practical applications of projection. I shall show the scale for three angles, viz. $20^{\circ}$, $40^{\circ}$ and $60^{\circ}$:-




The sines are got from the quadrant AD by drawing perpendiculars from each $1^\circ$ on that quadrant to AC. If we measure from C on this line towards A, we get the value of all the sines from $0^\circ$ to $90^\circ$; they are set off ready for use.

The tangents and secants are obtained from the quadrant AE, a tangent being drawn at E to the circle and the tangents measured from E and the secants from C to the line. Beyond $70^\circ$ the rule does not extend.

The Semi-tangents are obtained from the quadrant EB, by joining each $1^\circ$ on this quadrant to D. The angle EDG is $\frac{1}{2}$ of $20^\circ$ and the semi-tangents are therefore measured on CB.

The Chords are got from quadrant DB by drawing arcs with centre/

centre D and radius equal to the chord of the required angle to cut DB; DB is the line of chords. The construction of such a scale of a size which would be of use must have been a serious task.

With the aid of this scale, Wilson shows how to project the celestial sphere on various planes such as the "Plan of the Horizon of Edinburgh Lat. $56^{o}$".

A short reference to Orthographic Projection where the point of projection is at infinity concludes the book proper, but a short appendix is devoted to the practical applications of Trigonometry to the measurement of heights and distances and to navigation. The apparatus that he uses for measurement of heights and distances is simple, (1) a Line of Leagues, i.e. a ruler divided off into units and tenths and by diagonals into hundredths, (2) a quadrant with "Plum Line", (3) a Graphometer.

Only four very simple examples are given of the measurement of heights and distances.

As regards navigation, he confines himself to giving an account of the "Plain Chart" (where the degrees of Longitude are equal at all Latitudes), and of the Mercator Projection with the "Tables of Meridional Parts" drawn up by "Mr. Wright". Navigation is treated very shortly and hurriedly.

Wilson's Trigonometry was recognised as a standard work on the subject throughout the greater part of the eighteenth century; judging/

judging by the quality of the writing, the author was a man of outstanding mathematical ability, and the book must have been very useful as a reference book for teachers. On the other hand, owing to the absence of examples explanatory of the rules, it would scarcely be a popular text-book, and I have not found any copies of a subsequent edition. The printing is poor, the diagrams are few, small and crowded together.

Wilson is the author also of an "Introduction to Arithmetic", dated 1741. Along with the works of Malcolm and later of Mair, it marks a definite advance in the study of decimals. The author admits his indebtedness to the "Reverend George Brown, Minister of the Gospel" for pointing out the usefulness of decimal fractions. Brown, however, "advances no theory of the Doctrine; that, with the few cases in which he is deficient, I have supplied here." The deficiency, as already mentioned, relates to the matter of "Infinite Circulates", i.e. decimals in which the repeating part contains more than one digit. "This Arithmetic of Infinite Circulates I have laid down as methodically and distinctly as I could". Method, clearness and conciseness are the characteristics of Wilson's work, and the author reveals himself as one possessed of an abundant supply of common sense. It is clear from the preface that Brown's advocacy of decimals had met with a mixed reception, some business men evidently preferring to stick to the old methods of calculation by Practice on the ground that the decimal prices of the "Arithmetica Infinita"/

Infinita" are not exact enough and that they are losing money by adopting them. Wilson points out that they can continue the decimals as far as is necessary to ensure accuracy. I may remark, in passing, that the modern method of approximating to decimals, by considering whether the decimal place beyond the one required contains a digit greater or less than 5, does not appear to have been used in any of the eighteenth century text-books.

A good deal of space is devoted to the explanation of short methods of multiplying and dividing by numbers containing a series of digits all nines, as these processes are often required in the treatment of repeating decimals: e.g. to divide 372 by 99, we say 3 is the quotient and add 3 to the 72 for the remainder:-

$$\begin{array}{rl} & 75 \text{ Remainder} \\ 99\ )\ & \cancel{3}72 \\ & 3 \text{ Quotient} \end{array}$$

Wilson, unlike many of the writers on arithmetic, shows the reason for this process.

An interesting feature of his chapter on Division is the quotation of a Latin "Monastich" on the process:-

1 2 3 4
"Dic quot. multiplica, subduc, transferque sequentem"

which is translated thus:-

First ask how oft the Divisor's got?
The answer gives the figure in the quot.
Subtract the product of these two and then
Bring down the next, and ask "How oft again"?

In his treatment of "Rule of Three" Wilson makes an interesting suggestion/

suggestion, which would have led up to the modern version of that rule, if it had been followed out. He lays down the rule as it was universally practised in his time, where the second term is of the same name as the thing sought, and the first and third terms are set down automatically, with a different operation afterwards for direct and inverse proportion. After treating that method fully, he suggests as an alternative putting the number of the same species with the number sought as the third term: he then gives four rules for the placing of the first and second terms as "more gives more, less gives less, more gives less or less gives more". Allowing for the archaic phraseology, that is practically the Rule of Three as it was taught at the end of the nineteenth century; by a curious perversion of idea and process, Wilson considers this method as if it made all problems in proportion <u>Direct</u>, whereas it simply reduces the <u>operation</u> in every case to that performed in Direct Proportion.

Wilson works out several examples in this way, but evidently it did not catch on at the time; he does not extend it to the "Rule of Five, Seven, Nine or Eleven", though he treats each of these Compound Proportions in the way mentioned in the Chapter on Paterson's "Scots Arithmetician".

Wilson must have been an inspiring teacher, as witness the following topical illustration of the necessity for reducing fractions to a common denominator preparatory to addition:

"7/

"7 Lowlandmen and 8 Highlandmen do not make either
15 Lowlandmen or 15 Highlandmen, but they do make
15 Scotsmen".

The decimal point has now established itself, but the methods of indicating repeaters are not the same as we use. A "Single Repetend" is shown thus:-

$$\frac{5}{12} = .41\not{6}$$

This is called an Infinite Repeating Decimal;

$$\frac{11}{35} = .3,142857,$$

This is called an Infinite Circulating Decimal. He shows by a consideration of possible remainders that all vulgar fractions reduce either to finite decimals or to one of these types. When these repeaters run to 16, 18 or more figures, he suggests stopping at the seventh place and treating them as finite. It is regrettable that that suggestion came to naught and that mathematics pursued the Will o' the Wisp of repeating decimals. The failure of the proposal was probably due to the absence of the modern method of approximating by the next figure.

Wilson suggests some interesting properties of the sum of the digits in the circulating part which he had discovered inductively, but space forbids my entering on this fascinating subject. He shows all the rules for multiplication and division of "pure and mixt circulating decimals" (i.e. those without and with a non-circulating part), sticking to decimals throughout so long as the multiplier or divisor is not itself a circulating decimal. In the latter/

latter case he has recourse to vulgar fractions, which necessitates the use of the short methods of multiplying and dividing by a number with a series of nines in it. As may be supposed, some of the calculations undertaken are monumental.

In an appendix to the work we have an account of the duodecimal method of calculating areas, as required by masons, carpenters, etc. From that it appears that 1 inch is divided into 12 parts called lines. The nomenclature is somewhat confusing when we come to areas; 1 square foot is called a "superficial foot", but 1 "superficial" inch is not 1 square inch, but represents a rectangle one foot long and one inch wide. "1 superficial line" is one-twelfth of a superficial inch, so that one superficial line is what we call one square inch. To add to the confusion, the term one square inch was used also, though not by tradesmen. Here is the multiplication by duodecimals of 22 feet, 4 inches by 19 feet, 7 inches, each superficial unit being twelve times the one to the right of it:-

| Feet | Inches | Lines |
|---|---|---|
| 22 | 4 | 0 |
| 19 | 7 | 0 |
| 418 | 0 | 0 |
| 6 | 4 | 0 |
| 12 | 10 | 0 |
| | 2 | 4 |
| 437 | 4 | 4 |

Thus 4 X 7 = 28 superficial lines = 2 inches 4 lines.

Wilson/

Wilson proposes to abolish this method and reduce length and breadth to a decimal of one foot before multiplying, so that he would get his answer in square feet and then reduce the decimal part to "square inches". His example, in which he reduces 25 feet 7 inches and 17 feet 5 inches to decimals and obtains an answer with ten significant figures in it, would not appeal to the tradesman, who, with training, could doubtless carry out the above process quite easily.

To sum up the account of this book, I may say that it is very well printed, is clearly and concisely written, never loses sight of the practical side of arithmetic and illustrates all the work with plenty of examples. It marks a step in the development of decimals. It is regrettable that the desire for a complete system of decimals should have led to the development to extremes of the ugly repeating decimal, and that the author did not further elaborate his idea of approximating. He does not give ~~and~~ the method of contracted multiplication or division. A second edition of Wilson's Arithmetic appeared after his death in 1752. It was edited by John Mair, who expresses great respect for the ability of the author. This edition is fuller than the first one.

---

# CHAPTER XIII.

## MALCOLM'S ARITHMETIC .

Alexander Malcolm was a teacher of arithmetic and bookkeeping in Edinburgh in 1718, but seems to have resided later in Aberdeen. He published two books, viz. "A New Treatise of Arithmetick and Bookkeeping" in 1718 and "A New System of Arithmetick, Theorical and Practical," in 1730. As regards the arithmetical part of the former, it suggests itself as the groundwork on which the more ambitious work was built. The most interesting fact about the former book is that infinite decimals are not mentioned. In reducing from a vulgar fraction to a decimal Malcolm stops at the sixth place when the result is not finite. It is clear then that the theory of recurring decimals must have been promulgated in Scotland between 1718 and 1730, and Alexander Malcolm, ~~if~~ though not the originator, was one of the pioneers in the development of that theory. As the greater includes the less, I shall say no more about the earlier work than that it displays in embryo the fine qualities of keen analytical research and philosophic thought which later appeared fully developed in the "New System of Arithmetick, Theorical and Practical".

As/

As a text-book, the latter work is on a different plane from any of the others with which I have to deal. It presents the greatest possible contrast to Cocker's Arithmetic in that Malcolm demonstrates by algebra the truth of every one of his rules. The book is complete on the theoretical side. For instance, he explains the summation of infinite series before he introduces the repeating or circulating decimal. Yet while Cocker ran to over a hundred editions this work reached only two, dated 1730 and 1731 respectively. There was no popular demand for a real mathematical textbook on arithmetic, and doubtless the criticism was made that the book was more a treatise on algebra than on the less generalised science. Malcolm is almost apologetic in bringing forth a new arithmetic, as it was firmly believed that the last word had been said on the subject by two English writers, Hill and Hatton. He gives a brief but interesting account of the history of the science, from which we learn that the theory of circulating decimals originated with Dr. Wallis.

With a treatise of the size of this one I cannot do more than indicate some of the salient features. His attitude towards the prevailing method of checking the answer by casting out the nines is that a "true sum will always appear true by this ~~truth~~ method and to make a false sum true there must be at least two errors and these opposite to one another". He demonstrates the theory of the method which suffered a progressive decline in popularity from this date.

A/

A fetish is made of simple methods of division and multiplication in special cases. It appears that "Neper's Rods" were still in general use at this time. Malcolm has one original contribution to the simplification of big multiplications. In line with Napier's artificial aids, it was customary in these cases to make a table containing the products of the multiplicand by each of the digits 1 to 9. To simplify this still further, Malcolm suggests that it is only necessary to multiply by 1, 2 and 5, the other products being got from these by addition.

It is interesting to learn from the tables of money that the coins current in Scotland in 1730 were:-

(1) Copper - the farthing and halfpenny.

(2) Silver - penny, twopence, fourpence, sixpence, shilling, half-crown and crown.

(3) Gold - half-a-guinea and guinea.

One of the many excellent demonstrations of arithmetical rules in the book is that in which Malcolm justifies the process of finding the G.C.M. by continued division. He calls this the "highest number which will divide both numerator and denominator," and he proves (1) that the method does give a number which will divide the two exactly, (2) that it is the greatest possible integer that will do so, and (3) that when numerator and denominator are divided by the number so found the fraction is reduced to its lowest terms. This process was in much greater use then than now, since/

since cancelling in the course of a calculation was rarely resorted to.

While dealing with involution and evolution in arithmetic, he gives a proof of the Binomial Theorem for an integral index, and lays down nineteen theorems on positive integral and fractional "indexes". He shows how to extract the square root, cube root, fourth root and even the fifth root, by use of the binomial expansion. Surds he always writes down in the form of fractional indices.

Thereafter he launches into a full account of proportion and progression; the latter is just "continuous proportion." True to Napierian tradition, he treats arithmetical and geometric series along parallel lines, and also gives an account of arithmetical proportion, the symbolic expression of which is $a \,.\, b : c \,.\, d$ That means that $a - b = c - d$.

The symbols used in arithmetrical and geometric progression, a, d, s, and n and a, r, s and n are the same as ours, except that a is always the least term, and it may be the last and not the first. The greatest term is represented in each kind of progression by $l$.

Malcolm excludes negative numbers saying "It is true that there is no such thing in nature as a Number less than 0, and so this will presently be looked on as chimerical." This means that a may be 0 or any positive quantity, l may be any number so long as it is not less/

less than a or d and not greater than s, the sum of the series. d may be any number not exceeding l or s. s may be any number not less than a or d or l. In geometric series a must be less than s and not greater than l.

An unusual feature of this book is the exposition of Harmonic proportion and Harmonic series with an account of ~~its~~ their application in music.

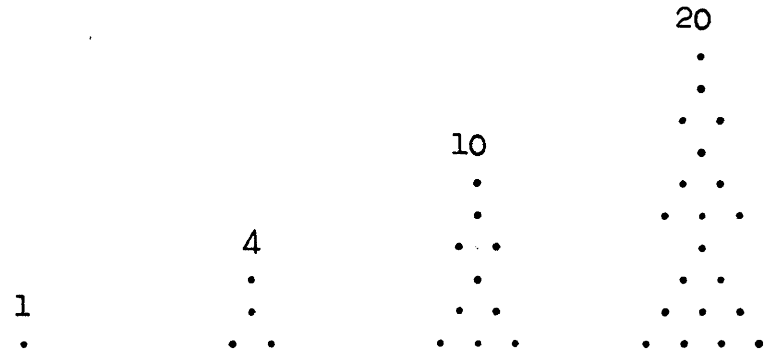
In the next section we find that Malcolm has not cast off the arithmetical garments of the Middle Ages, for we get an account of the "ancient Pythagorean Speculations about numbers and geometrical figures whence they pretended to discover many Mysteries and Secrets of Nature". This rather futile but fascinating subject is fully treated. Thus every power of two is shown to be a "deficient" number (i.e. one such that the sum of all the factors of it is less than the number itself) because the sum of 1 + 2 + 4 + 8 +.......n terms is $2^n - 1$, i.e. the sum of all the factors of $2^n$ is one less than $2^n$. Again we are introduced to Triangular numbers, which are formed by summing the series 1, 2, 3, 4, 5......to 1, 2, 3, 4, 5 etc. terms, giving the new series 1, 3, 6, 10, 15, 21.......These are called triangular because they form triangles when set down thus:-

```
 3        6          10            15
. .     . . .      . . . .      . . . . .
 .       . .        . . .        . . . .
          .          . .          . . .
                      .            . .
                                    .
```

If the process be repeated with this series, we get a series of Triangular Pyramidal numbers 1, 4, 10, 20, 35 etc.

These/

These can be set down to represent triangular pyramids, thus:-




This and other arithmetical progressions lead to many fascinating theorems of which there are thirtysix in this exhaustive chapter. I shall mention one.

Take the series of triangular numbers 1, 3, 6, 10.......
and square each term and you get the series

1, 9, 36, 100.........

Subtract each term of this series ~~of~~ from the one after it and you get cubes of the "Natural Progression:"

1, 8, 27, 64..........

He proves this true for <u>n</u> terms by writing down the $(n - 1)^{th}$ and the $n^{th}$ triangular numbers $\frac{n^2 - n}{2}$ and $\frac{n^2 + n}{2}$, squaring these and subtracting. Thus he gets $n^3$ for the $n^{th}$ term of the last series.

Later he treats of infinite series and of convergence which he calls "encreasing or decreasing limitedly", and thereafter he introduces circulating decimals, which, unlike many of his contemporaries, he marks in the modern way with dots. Sometimes the circulate starts before the decimal point, as thus $\dot{4}\cdot 2857\dot{1}$. When the/

the circulates have to be operated on by multiplication or division, Malcolm recommends their reduction to vulgar fractions.

An excellent account is given of the origin and use of logarithms, and an unusual feature of a book on arithmetic is the treatment of Combinations and Permutations. These are classified into three varieties:-

(1) Changes or Permutations.
(2) Choices or Elections.
(3) Compositions or Limited Elections, where there are restrictions on the choice.

No symbols are used in naming these different varieties, but the general expressions for Changes and Elections are given, and a table is constructed for the latter in every case up to 9 articles out of 12. Tables of all kinds were very popular in the eighteenth century, and doubtless this one would be used by people to whom the general formula was unintelligible.

Having now completed the theory of arithmetic, Malcolm devotes the rest of the book to the "Applications of the Doctrine of Proportion in the Common Affairs of Life and Commerce". Here for the first time he provides the reader with unworked examples.

The feature of his treatment of Proportion is that he makes no distinction between Direct and Inverse Proportion: his method is to write down the first, second and third terms in the automatic fashion of the time, with the second term having the same name as the answer, then consider whether the answer is to be greater or less than the second term, and accordingly multiply the second term/

term by the greater or less of the other two and divide by the less or greater. One of the applications of proportion is to simple interest, and this is done by a formula simpler even than ours:-

$$n = p\ t\ r$$

where $\underline{n}$ represents interest, $\underline{r}$ is the interest on £1 in one year.

It is very probable that the failure of this book to catch the popular fancy is to be traced to the fact that Malcolm lived before his time; thus in his treatment of Interest, Annuities, etc., he makes considerable use of algebraic formulae, and the taste of the period inclined towards the statement of cast-iron rules in words. Algebra had not yet come into its kingdom.

Malcolm enters into a lengthy discussion as to the rule for finding the present worth of an annuity which is to run for so many years. He quotes the two rules laid down by two English writers, Kersey and Moreland respectively, which give entirely different results for the purchase price of the annuity. Like the Aberdonian of fiction, Malcolm objects to both of these rules on the ground that the purchaser is only allowed to reckon simple interest on the money paid down. Though he objects to the principle, he obligingly shows how questions can be worked out according to the English rules, and thereafter propounds his own solution, allowing for compound interest. His method became the standard one, i.e. - find the principal for which the annuity is one year's interest: find the present worth of this principal as a sum due at the end of the time; subtract/

subtract this present worth from its principal, and the remainder is the present worth of the annuity. In commercial practice, of course, all such problems were worked out from books of tables, showing the price of a £1 annuity for various rates and times.

The last section of Malcolm's work is typical of the writer: he discusses what he calls the "Equation of Payments", i.e. the problem of the man who has two or more debts due at different times; what time could he pay the whole so that neither creditor nor debtor would lose on the transaction. Various English writers had solved this problem previously; the popular method was the easy one adopted by Cocker and will be seen from this example:-

> "If 40L are due after six months, and 70L after four months, when should the whole debt be paid?"
>
> 40 X 6 = 240 70 X 4 = 280 Add 240 and 280 and divide by the whole debt. Answer is 4·72 months.

Other solutions have been given by Kersey and by Sir Thomas Moreland, but Malcolm puts forward his own solution. He objects to Cocker's solution on the ground that on the debts paid before they were due, debtor is only entitled to interest on the present worth of the debts, i.e. to discount, whereas on those due before they were paid, creditor is entitled to interest. Cocker's solution assumes interest in both cases. Malcolm lays down his rule which in its wording is so complex as to be almost unintelligible, leading as it does to a quadratic equation. After a few examples he proves the truth of his rule by algebra, thus:-

"Let/

"Let the Debt first payable be called $\underline{d}$
the distance of its Term of Payment $\underline{t}$
the last payable debt $\underline{D}$
the Distance of its Term $\underline{T}$
the Distance of the equated time $\underline{x}$
one year's interest of 1L $\underline{r}$

Then the Interest of $\underline{d}$ for time $\overline{x - t} = dr \times \overline{x - t}$
$= drx - drt$

The <u>Discount</u> of D for Time $T - x = \dfrac{D\,T\,r - D\,r\,x}{1 + T\,r - r\,x}$

~~This~~ Equating of these expressions leads to a quadratic, giving two solutions, of which you have to pick out the valid one. He concludes "Some will be ready to think that I have taken too much Pains about a question of no great Moment in Business and that in common Affairs any of the rules may do without any considerable error. I do freely own the last part; yet I believe what is done may prove a very useful exercise for a Student, and in all cases, I think, <u>Truth is worth the knowing.</u>" The last sentence constituted an admirable epilogue for this book. Throughout its 623 closely printed pages the author kept to his vow that nothing was to be written down which could not be demonstrated logically, and he produced a text-book on Arithmetic which is almost, if not quite, unique.

Alexander Malcolm A.M. also wrote "A Treatise of Musick, speculative, practical and historical", which was published in Edinburgh, 1721 and "A Treatise of Bookkeeping" which, like the Arithmetic, was published in London, 1731.

## CHAPTER XIV.

## TWO TEXT-BOOKS INSPIRED BY DAVID GREGORY.

David Gregory was born in the Upper Kirkgate, Aberdeen in 1661X and educated at the Grammar School there and also for a short time at either Marischal College or King's College. It was at Edinburgh University, however, that he took his degree A.M. in 1683. Actually before graduating he was appointed to the Chair of Mathematics in Edinburgh, an office which had not been filled since the death of his uncle, James Gregory, in 1675. David Gregory's reputation as a mathematician was testified to by Sir Isaac Newton, and he was soon lost to Edinburgh University, being appointed Savilian Professor of Astronomy in the University of Oxford in 1691. He died in 1708. During the period in which he taught mathematics in Edinburgh, verbatim copies of his lectures, delivered in Latin, were taken by some of his students; these were preserved and apparently handed down to later generations of students. Ultimately they were published, under the auspices of Professor McLaurin, ~~probably~~ with additions. The arithmetical and algebraic part was published in the year 1736 in the original Latin under the title "Arithmeticae et Algebrae Compendium". The name of the author is not stated, but the book is marked "In usum juventutis Academicae Edinburgi". This book, unlike/

X The Academic Gregories - A.G. Stewart, p.52

unlike Gregory's other one on Practical Geometry, does not appear to have proved popular, as I can trace only one edition.

The Compendium is chiefly interesting in that it gives us an insight into the course of teaching in Edinburgh University at the time it was written. In the first instance we learn that the teaching of Arithmetic started from the very elements. The four rules are taught with painstaking thoroughness; this will be evident when I mention that he suggests checking an addition of four numbers by subtracting the first, second and third in turn from the sum, when the fourth should be left. The casting out of the nines is not used by Gregory for the purpose of verification: no reference is made to that method. The modern notation is used for vulgar fractions, but the dot for a decimal fraction is replaced by a comma. Nothing is said about interminate decimals beyond the advice to go so far in writing them down that the remaining places can safely be ignored. None of the practical applications of arithmetic are given.

Throughout the algebra indices are used. Negative numbers are defined simply as being less than zero, and the rule of signs is justified on the reasoning that 3, 2, 1, 0, -1, -2, -3, etc. form a diminishing series:

hence, since $+3 \times +3 = 9$, $+3 \times +2 = 6$, $+3 \times +1 = 3$, $+3 \times 0 = 0$

$\therefore$ $+3 \times -1$ must be equal to the next lower number in the series, i.e. $-3$.

Similarly/

Similarly,

since $-3 \times +3 = -9$, $-3 \times +2 = -6$, $-3 \times +1 = -3$, $-3 \times 0 = 0$ here we have an ascending series, and consequently $-3 \times -1$ must give the next higher number in the series, i.e. $+3$.

In his treatment of ratio, Gregory distinguishes between "Arithmetical Ratio", i.e. the difference of two numbers, and "Geometrical Ratio".

In his treatment of surds, it is interesting to note the advance that has been made in notation since Napier. Two notations are suggested, one the modern one, the other as follows:- $\overline{2|}^{\,2}$ = square root of 2; $\overline{3|}^{\,3}$ = cube root of 3. Throughout, the modern notation is used, and as a numerical coefficient can be introduced before the root sign, Napier's rules for addition and subtraction of surds are unnecessary.

These preliminary rules for operating with algebraic quantities have paved the way for the main purpose of the science, viz. the solution of problems by means of equations. In the latter Gregory uses $\underline{x}$ and $\underline{y}$ for the unknowns. In this section, unlike most contemporary writers, he illustrates his rules by literal equations instead of actual numerical problems.

A curious relic from the earlier algebra is the solution of a set of problems inspired by the Second Book of Euclid. In these he imagines two quantities $\underline{z}$ and $\underline{x}$ of which $\underline{z}$ is the greater, and he writes down $\underline{s}$ for their sum, $\underline{d}$ for their difference, R (rectangulum)

(rectangulum) stands for their product, $\underline{s}^q$ for the sum of their squares ($z^2 + x^2$) and $\underline{d}^q$ for the difference of their squares. In the various problems you are given two of these seven quantities, and are required to find the remaining five. These problems are then applied in the general solution of the quadratic equation. Here, as was the usual practice, Gregory distinguishes four different cases:-

(1) when the roots are both positive: here you get an equation of the nature

$$x^2 - sx + bc = 0$$

where $\underline{s}$ stands for the sum of the roots b and c. In solving such an equation you are given the sum of two numbers and their product, to find the numbers. This has been solved previously and gives as solution

$$x = \tfrac{1}{2}s \pm \sqrt{\tfrac{1}{4}s^2 - bc}$$

(2) when one root is positive and one negative, the positive being greater than the negative. The equation is

$$x^2 - dx - bc = 0$$

where d is the difference of the roots. The solution is

$$x = +\tfrac{1}{2}d \pm \sqrt{bc + \tfrac{1}{4}d^2}$$

(3) when one root is positive and one negative, the negative root being the greater numerically. The equation is

$$x^2 + dx - bc = 0$$

Solution $$x = -\tfrac{1}{2}d \pm \sqrt{-bc + \tfrac{1}{4}d^2}$$

(4) both roots negative:

Equation $$x^2 + sx + bc = 0$$

Solution $$x = -\tfrac{1}{2}s \pm \sqrt{-bc + \tfrac{1}{4}s^2}$$

Gregory devotes some space to the utilisation of algebra in the/

the solution of some of the problems and theorems of Euclid which are suited to such treatment, e.g. the second book deductions from Pythagoras and some trigonometrical problems. In the latter it is interesting to find that the tangent is referred to as a "ratio"; this is the only book of the period where I have found such a conception explicit instead of implicit.

Another unusual feature is the solution of indeterminate problems, among others the famous one of finding sets of three integers which would form sides of a right-angled triangle; for this he obtains the equation

$$\frac{v^2}{4y} = x$$

where v is the number of units in the greater of the two sides containing the right angle, (x + y) is the hypotenuse, (x - y) the other side.

Thereafter an account is given of negative and fractional indices, with one or two examples of rationalisation of the denominator of a fraction containing surds. Here the methods and notation are entirely modern, with the exception that the only form of brackets used is the vinculum. In the extraction of the square root of a binomial surd he makes use of the notation referred to above, $s^q$, $d^q$, and R. The problem is to find z and x, being given $s^q$ and R, and a general solution of this problem has been already/

already given.

e.g. to find $\sqrt{27 + \sqrt{704}}$

we are given $s^q = 27$ $\quad 2R = \sqrt{704}$

$\therefore$ $s^{q^2} = 729$

$4R^2 = 704$

$\therefore$ $s^{q^2} - 4R^2 = 25$

$\sqrt{s\sqrt{s^q - 4R^2}} = 5 = d^q$

But $z^2 = \dfrac{s^q + d^q}{2} = 16$

$x^2 = \dfrac{s^q - d^q}{2} = 11$

$\therefore$ root $= 4 + \sqrt{11}$

The book concludes with a short account of the theory of the Cubic and Biquadratic equations and the various methods for their solution including that by approximation.

The "Arithmeticae et Algebrae Compendium" was not meant to be a popular text-book; it is academic in tone and contents and divorced from practical applications. It is interesting as showing the big advance that was made in the teaching of mathematics in Scottish universities towards the end of the seventeenth century.

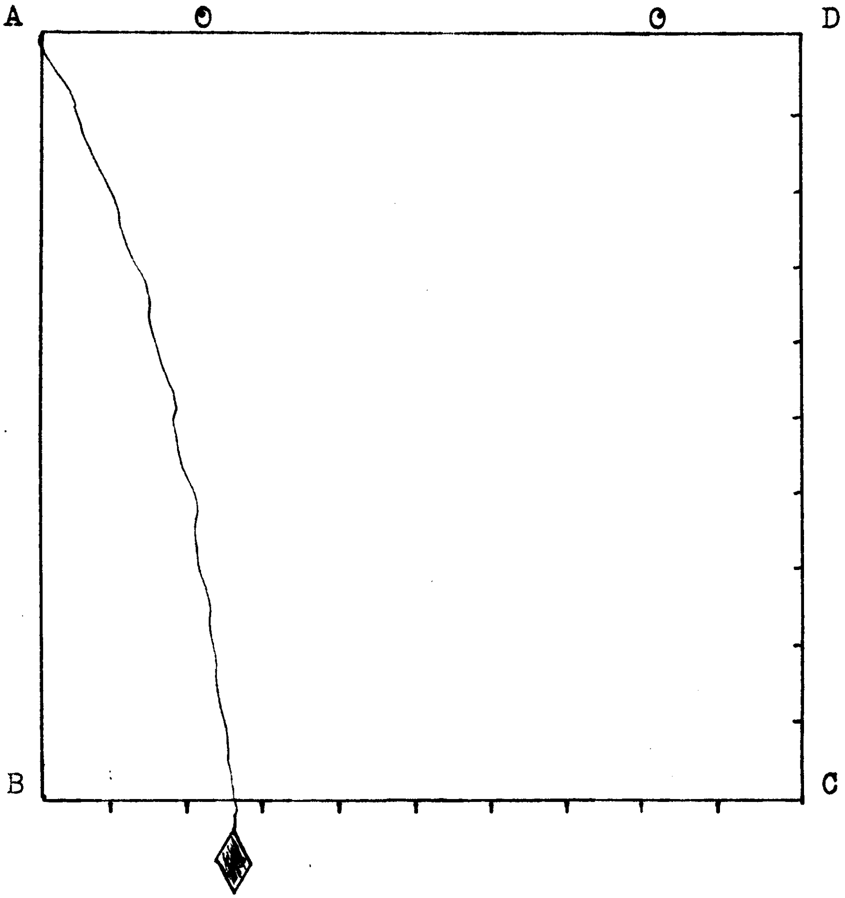
David Gregory, in 1703, published in Oxford a text of Euclid containing fifteen books, the Data, the Harmonica, Canons, Phaenomena, Optica, Catoptrica, De Divisionibus and a fragment "De Levi et Ponderoso". This contained the Greek text with a Latin/

Latin translation alongside. I have not found any evidence that this text of Euclid was used in the universities of Scotland. Gregory did teach the first six books and the eleventh and twelfth of Euclid, together with some plane trigonometry. This is made clear from McLaurin's preface to Gregory's "Treatise of Practical Geometry". This book was written in Latin about the year 1685, and in this form was handed down by successive students, being ultimately translated into English. A bookseller persuaded Colin McLaurin to publish a translation with some additions of his own, and this was done in 1745. It is an exceedingly readable little book, and was deservedly popular throughout the remainder of the eighteenth century. McLaurin's additions, which bring the book up to date, are throughout printed in inverted commas. The book falls naturally into three parts, dealing respectively with the measurement of length, superficies and solidity.

For the measurement of heights and distances the instruments used were the Geometrical Square, the Plain Mirror, Two Staffs, Four Staffs, the Quadrant, the Graphometer, and the Protractor with Line of Equal Parts.

A short account of the Geometrical Square and of the Graphometer may be necessary here. The geometrical square was made of brass or wood of four rulers jointed together at right angles. Two of the sides of the square were graduated each into 100 equal parts, or, if it was large, into 1000 parts. A third side contained two sights/

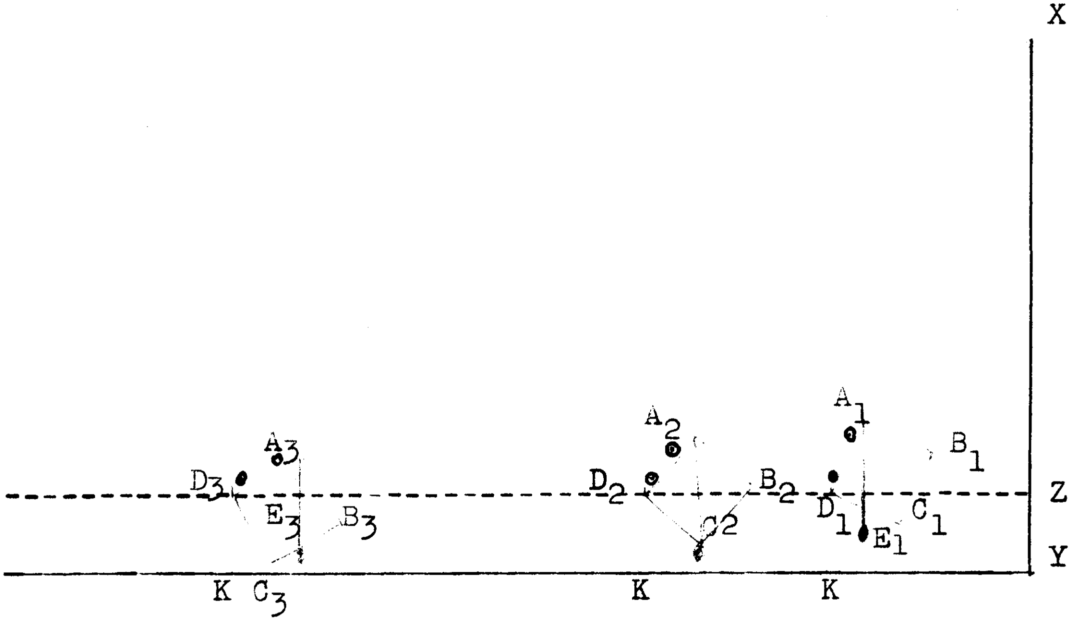
sights, and from one vertex there hung a plumb-line.




One example will explain the use of this instrument. The problem is to measure the height of a tower X Y.

Measure/

Measure a distance from Y to K.




We hold the instrument at the height of the eye, with D next the eye and the sights aligned on X. Three results may be obtained:-

(1) the plumb-line may fall on DC say at $E_1$.
Then triangle $A_1D_1E_1$ is similar to triangle $XD_1Z$

$\therefore XZ : D_1Z :: A_1D_1 : D_1E_1$

The ratio $A_1D_1$ to $D_1E_1$ is read off from the scale, so that XZ can be found.

(2) the plumb line may pass through C.
Here since $D_2C_2 = A_2D_2$
$\therefore XZ = D_2Z$

(3) the plumb line may fall on BC, say at $E_3$.
Then $\triangle A_3B_3E_3$ is similar to $\triangle XD_3Z$
As in (1) XZ can be calculated.

The instrument could be used without the plumb-line as it was provided with what was called an "index": this was a wooden bracket pivoted at A and capable of being moved round and of remaining/

remaining in any position. On this index there were also two sights. With the help of the index, the geometrical square could be used to measure the distance of an inaccessible point or that between two inaccessible points.


Suppose F is the inaccessible point, and we wish to measure AF. We set the instrument horizontally, with the fixed sights pointing to F, and turn the index so that it is at right angles to the line with the fixed sights AD. This gives us some convenient place B in the line of the index. We then proceed to B with our instrument, set the fixed sights back on A, and turn the index until it points to F. We measure RS on the scale.

Then R S : R B :: A F : B A

Hence/

Hence we can find AF if we measure AB.

---

The graphometer is used for measuring angles. It consists of a kind of protractor of brass or wood, which can rotate round a fulcrum at its centre on a ball and socket. It has two sights along a diameter, and, by means of a movable ruler pivoted at the centre, also containing two sights, angles can be measured whose vertex is at the instrument. There is a compass needle in a box at the centre. At the time this book was published these cruder instruments had been replaced by the Theodolite, of which McLaurin adds an account. Surveying by means of the Plane Table is explained.

---

The second part of the book shows how to measure the areas of rectilineal figures, circles, ellipses and the total surfaces of rectilinear solids and of the cylinder, cone and sphere.

The third part deals with the measurement of the "solidity" of these same solids. For application to the gauging of casks, Gregory gives the rules for measuring the solid content of the frustum of a cone cut by a plane parallel to the base, and of a spheroid and the segments of the spheroid by planes perpendicular to its axis. In each of these cases, whether in measurement of area or volume, the rule is clearly stated with a reference to the Euclidean theorem on which it is based. The last proposition explains how to measure the/

the volume of an irregular body by immersing it in a prismatic vessel of water, and measuring the rise in height of the water.

Probably the most interesting part of this book is the account given by McLaurin of the units used in the measurement of lines, areas and volumes. It includes a table of a great many foreign units of length expressed in English inches to three decimal places. The Scots Chain, McLaurin says, should be 24 ells in length, so that 10 square chains may represent one Scots acre, and we may get the benefit of the simplicity attained by the English Chain, called Gunter's Chain. The Scots Fall is 36 square ells, and with 40 falls to one rood and 4 roods to one acre, we get a chain which, when squared, is one-tenth of one acre.

At the end of the book, McLaurin gives various empirical rules for computing the contents of vessels of certain shapes and sizes in units of capacity. Thus the content of a cylindrical vessel in gallons is found "by multiplying the square of the diameter of the vessel by its height (each in inches) and their product by the decimal fraction ·0027851.

Gregory's "Treatise on Practical Geometry" is an excellent book, clearly and concisely written, well arranged and beautifully printed. It is much more of a popular text-book than the "Arithmeticae et algebrae compendium". It was a recognised text-book for generations of students, and exercised a considerable influence on the work of the schools.

I/

I have read five editions of the book. The first edition appeared in 1745 and the second in 1751. The seventh edition is dated 1769, the tenth 1787 and the eleventh 1798. The contents of all the editions that I have seen are identical, practically word for word. That alone, spread as the editions were over more than half a century, is a tribute to the excellence of the writing.

A short reference may be made here to McLaurin's "Treatise of Algebra" in three parts, which was published posthumously in 1748 ~~[illegible]~~. This is, of course, a famous and well-known work, representing the author's course of lectures in Edinburgh University. The second and third parts are beyond the scope of this paper, but the first part contains the "fundamental Rules and Operations". There are a few points of interest. McLaurin's symbols are mostly the modern ones, but he uses $\sqsubset$ for "is greater than" and $\sqsupset$ for "is less than".

At the beginning of the book he is sparing in his use of indices, preferring to write powers out in full, but later he uses indices entirely. He develops the theory of negative and positive quantities from the directions right and left on a line. In his treatment of arithmetical and geometric progressions he represents a transition between the old and the new; instead of having a least and greatest term as in the old method, he uses the symbol <u>a</u> always for the first term and <u>x</u> for the last term in an A.P. In/

In a G.P. a is the first term and y the last term. Apart from the use of these extra symbols x and y, his treatment is the modern one.

In solving linear, simultaneous equations in two (or three) variables, he uses the old method of elimination by expressing one of the variables twice in terms of the other. He gives a general solution of such equations in two and in three variables, and solves all subsequent equations of that nature by means of the general solution.

For the quadratic equation in one variable he gives two general solutions (1) of the equation $y^2 + ay = b$

(2) " " " $y^2 - ay = b$

but he does not use this general solution in solving other quadratics: he prefers to solve each from first principles by completing the square.

---

Notices of the first and second editions of Gregory's "Practical Geometry" appear in the Scots Magazine in May, 1745 and January, 1751, respectively. From the same paper it appears that McLaurin's "Treatise of Algebra" was first published in 1748.

---

# CHAPTER XV.

## FISHER'S "ARITHMETICK".

This popular work is entitled "Arithmetick in the plainest and most concise Methods hitherto extant" X by George Fisher, Accomptant, Glasgow. The edition I have seen is the tenth, and is dated 1756. The first edition must have appeared early in the eighteenth century, as decimal and vulgar fractions are relegated to a minor place at the end of the book, and repeating or circulating decimals are not mentioned. It seems to have attained remarkable popularity in England, as most of the later editions between the first and tenth were published in London. These must have been produced rather hurriedly, as they seem to have been full of errata, a fault very noticeable in the author's edition of Cocker to which I have referred. The tenth edition, like the first, is guaranteed by Fisher to be free from errors, a claim which is not justified. There was at least one later edition, dated 1780.

The author reveals himself throughout as a very careful and painstaking teacher; he sets down all the steps in every arithmetical operation with meticulous care and considerable elaboration, dotting/

X An edition (No. not stated) was advertised in the Scots Magazine, August, 1761 and another in the Glasgow Mercury, October 15th, and November 12th, 1778. The price, two shillings, accounts for its popularity.

dotting all the "i"s and stroking all the "t"s. He betrays a Dickensian sense of humour at times, as when making out a bill for a Brewer's Clerk from:

"Laurence Lick-Spiggot, Frank Froth, Sam Swigg, Ben Bumper, Henry Here's t'ye and Stephen Stout".

From his treatment of division of integers, we learn that the "Scratch" method had not disappeared from the schools by 1756, but was gradually being ousted by the modern method, which he calls the "First Italian way of Division". He recommends the use of the "Short Italian Way" in which the products of divisor and part of quotient are not written down, but only the results of the subtraction.

The great aim of the author of this book is to make arithmetic as simple as possible; his rules are stated in the plainest words. Thus the watchwords for division are: (1) Seek; (2) Multiply; (3) Subtract. He is greatly enamoured of the art of cancelling, particularly in the Rule of Three. He claims to have discovered an original method for multiplying by 112, whereby mistakes are eliminated. It is no great discovery, but seems foolproof. Suppose number to be multiplied is 246. It is set down thus:

$$\begin{array}{r} 246\phantom{00} \\ 246\phantom{00} \\ 246\phantom{0} \\ \underline{246\phantom{}} \\ \underline{27552} \end{array}$$

Again in the calculation of interest he suggests various simple but/

but inaccurate methods for the case where the time is in days, taking 30 days to the month and working at 6%, other rates being found by practice. At every point he is looking for possible simplifications and for rules which are of practical use. Thus he omits such subjects as "Alligation" and the "Rule of False" as being of no practical use: on the other hand we find him multiplying 2s.4d. by 3s.6d., obtaining 8s.2d. He is so fond of this particular multiplication that he shows four ways of performing it, one by vulgar fractions, one by decimals and two by a kind of duodecimals, thus:-

| s. | d. |
|---|---|
| 2 | 4 |
| 3 | 6 |
| 6 | |
| 1 | 0 |
| 1 | 0 |
| | 2 |
| 8 | 2 |

or thus

| s. | d. | |
|---|---|---|
| 2 | 4 | |
| | 3 | 6 |
| 7 | 0 | |
| 1 | 2 | |
| 8 | 2 | |

The general impression conveyed by this book is that, while the author was not a mathematician, he was an excellent and painstaking teacher; one is not surprised that the book proved exceedingly popular. Dr. Murray[X] says that there is another book bearing the same author's name in the form of a ready reckoner, containing instruction in spelling, reading, writing, arithmetic and easy bookkeeping. This was first published in 1743 and ran to over thirty editions in Scotland, England and the United States, the/

X Chapters in the History of Bookkeeping, by David Murray. p.308

the last one being published in 1853. The sixteenth edition appeared in August, 1762. A full account of the contents is given in an advertisement in the Edinburgh Courant, September 27th, 1762.

"The Instructor or Young Man's Best Companion. Containing spelling, reading, writing and arithmetic, merchant accompts, and a short and easy method of bookkeeping. The method of measuring all sorts of tradesmen's work. The practical Gauger, the art of dialling, ....... A compendium of geography and astronomy and interest tables, etc."

The price was 2s.6d., and the book is in the direct line of succession to Isidore's Etymologies and the Compendium of Faber Stapulensis.

The idea of publishing books of tables was very prevalent in the eighteenth century. I have read several books of the ready reckoner type, containing tables of the cost of any number of articles up to 10,000 at any price from ¼d. to £1, and such other tables as are required for calculating interest, brokerage, etc. Usually they have a table for the thrifty housewife, showing the total cost of living per week, month or year, at so much per day.

In Baillie's Institution, Glasgow, there is a copy (picked up from the street) of a book with the resounding title "Panarithmologia", by William Leybourn, Glasgow, the thirteenth edition, dated 1761./

1761. That such a book should have reached thirteen editions or more is a revelation of the general ignorance of arithmetic prevailing among the business men of the period.

I found another similar book in the Sandeman Library, Perth, entitled "The Trader's Assistant", dated 1784, and written, printed, and sold by Robert Morrison, Bookseller, Perth. The author is so confident of the accuracy of his work that he "engages to pay to the discoverer of every error in these Money Tables FIVE SHILLINGS sterling". The number of the edition is not stated, but there must have been more than one, as the order of the table of contents at the front is not the order followed in the book.

In Edinburgh University Library I found another entitled "New and Correct Tables" by John Thomson, Writer in Edinburgh. It deals with the cost of grain in quantities from one lippy to a thousand bolls at prices varying from five shillings to thirty shillings per boll. It also contains a table for reducing English measures to the Edinburgh and Linlithgow standards, and a comparison of the weights and measures of England and Scotland. This ready reckoner was published in Edinburgh in 1761.

In 1768 the same writer produced tables on interest at 4, 4½ and 5 per cent for various periods from one day to ten years, together with tables of exchange and commission. The latter achieved another edition in 1775, while the former was still being republished in 1777. Ultimately the two sets of tables were/

were merged in a more ambitious book "The Universal Calculator", published in 1784, 1788, and 1792.

Other books of this nature are

X (1) "Two Tables for finding the Annual Rent of any Sum"
by D.S., June, 1745.

(2) "Tables of Simple and Compound Interest"
by John Lawrie, 1776 (this is fully reviewed in the Edinburgh Magazine and Review, August, 1776, and in the Scots Magazine, July, 1776.)

X (3) "Tables of Interest"
by John Morrison, published in 1789 and 1791.

X (4) "Mathematical Tables containing Logarithms and a Traverse Table"
by John Brown, 1789.

X (5) "Mathematical Tables"
by Robert Hamilton, 1791.

From the periodical literature of the eighteenth century, it is evident that these books were widely used.

---

Note: Robert Burns used a "Ready Reckoner or Tradesman's sure Guide".

X Notice in Scots Magazine.

# CHAPTER XVI.

## THE ARITHMETICS OF MAIR AND HAMILTON.

It is very appropriate that from Perth Academy, which took a great part in the modernising of the school curriculum in the eighteenth century, there came two of the best books on Arithmetic produced at that time. The first of these was the work of John Mair, A.M., who, after acting as Rector of Ayr Grammar School for ten years, was appointed Rector of Perth Academy in 1761. He was the first Master of Mathematics definitely appointed in that school. He died in March 1769, and was succeeded by Robert Hamilton. These two men were responsible for the extraordinary development in mathematical teaching that took place in Perth Academy at that period.

John Mair wrote "Bookkeeping methodiz'd according to the Italian form", 1741, "Bookkeeping modernised", "Arithmetic Rational and Practical", "An Introduction to Latin Syntax", 1755, and "The tyro's dictionary, Latin and English". Each of these books attained considerable popularity and they were republished till well on in the nineteenth century.

He also wrote a "Radical Vocabulary" and published some Latin/

Latin texts of Caesar and Sallust and an edition of Corderey's Colloquies, and lastly "A Brief Survey of the Terraqueous Globe".

The Arithmetic was first published in 1766; there is a copy of the first edition in Edinburgh University Library, and in Perth Academy there is a third edition copy, dated 1777. I have also read the fifth edition, dated 1794.

It contains 615 pages and is a remarkably comprehensive work which earned for its author a tremendous reputation as a mathematician, so much so that he had almost become proverbial like Cocker, whose arithmetic he edited. Thus we find Halbert, in his Arithmetic published in 1779, propounding a complicated problem in verse, and concluding that

> "Perhaps, if alive at this date,
> It ev'n might perplex Mr. Mair."

Mair's Arithmetic shows very clearly the influence of the Rev. George Brown and of Alexander Malcolm. To the former's "Arithmetica Infinita", Mair admits his indebtedness, and he has imbibed a love for the use of decimals from that source. The finite decimal had now found a place, but the circulating decimal was proving a fractious infant, unpopular with ordinary students.

Throughout the work Mair borrows freely from Malcolm, but he does not introduce algebra to prove arithmetical rules. He bases his whole system on twelve fundamental principles or axioms and four corollaries. The axioms are the first axioms of Euclid applied to the Arabic system of numeration, while the corollaries derived/

derived from them relate to the peculiar properties of the number 9 when acting as a divisor. On these he develops the four fundamental rules, which, as he says, contain the whole of arithmetic; thereafter we get the applications of arithmetic in other sciences. Among these applications he treats of progression without the use of algebraic symbols, and as a consequence is compelled to introduce in the rules a most cumbrous language, which could have been avoided by the introduction of a little algebra. If this part of the subject were to be introduced at all, it should have been treated algebraically. That is a criticism from a modern point of view, but unquestionably to the eighteenth century reader Mair was an improvement on Malcolm.

A book on Arithmetic in those days was almost encyclopaedic; in the matter of tables of weights and measures, etc., a mass of information had to be provided. Mair is fond of the use of casting out the nines, though recognising the weaknesses of the method. He even checks the addition of two sums of money in pounds, shillings, pence and farthings by it.

| £ | S | D | F |
|---|---|---|---|
| 48 | 17 | 10 | 1 |
| 55 | 18 | 7 | 2 |
| 104 | 16 | 5 | 3 |

To check this, add 4 and 8, cast out 9 and leave 3.
$3 + 5 + 5 = 13$ " " 9 " " 4
Reduce to shillings $= 80$ " " 9 " " 8
$8 + 1 = 9$ " " 9 " " 0
$7 + 1 + 8 = 16 = 1 + 6 = 7$, reduce to pence, and so on.

The/

The casting out of the nines must have attained great favour, when it was seriously recommended as a method of checking a sum such as that.

In its general lines the book followed closely Malcolm's Arithmetic, but Mair set down a good many examples for working by the students, and thereby elaborated an idea which developed rapidly in later books, until ultimately the popular type of arithmetic consisted of little else than exercises.

One problem which Mair sets out very clearly is that of calculating areas and "solidity" by Duodecimal arithmetic. The difficulty arises when the dimensions are given in feet, inches and lines, where a line is one-twelfth of an inch. It was the custom of the time to multiply each of the separate units together, so that we are confronted with the problem of multiplying feet by inches, or feet by lines, or inches by lines. Thus, for instance, when feet are multiplied by lines, the answer is divided by 144 to give square feet, and any remainder, being one foot long and one line broad, represents square inches. Again, in calculating volume or "solidity", it will be necessary to multiply square feet by lines: the product has again to be divided by 144 to give "solid" feet, but this time the remainder, being one square foot in area and one line in thickness, is not "solid" inches, but has to be multiplied by 12 to give solid inches, since each unit of the remainder represents 12 solid inches. Again, if feet are multiplied/

multiplied by square lines, the result divided by 12 gives solid inches, and any remainder is multiplied by 144 to give solid lines. These examples will indicate the complexity of the system; Mair recommends the adoption of decimals in place of the recognised divisions.

In connection with duodecimal division, on page 344, there is an interesting interpolation in pen by some reader, showing the use of two signs Λ standing for ten, and V standing for eleven. The division of two quantities in the duodecimal scale is carried out:-

| | | 1 | 11 | 111 |
|---|---|---|---|---|
| Dividend is | 65 | 9 | 11 | 0 |

i.e. 65 units, 9 firsts, 11 seconds, 0 thirds in scale of 12.

| | | 1 | 11 | 111 |
|---|---|---|---|---|
| Divisor is | 8 | 1 | 10 | 11 |

```
8·1ΛV ) 55·9V ( 8·0992  2V2
                        ----
                        81ΛV
        55334
        -----
         67800
         61523
         -----
          62990
          61523
          -----
           14690
           1439Λ
           -----
             2V2
```

Sexagesimal arithmetic has almost disappeared, and it receives a bare mention at the end of the section on decimals. These are treated very fully, the methods of operating with circulating decimals being explained in great detail. In a reference to the Arithmetica Infinita, on page 336, he says that Brown's/

Brown's tables are still extant, and are "recommended by Dr. John Keill, professor of astronomy in the university of Oxford."

Mair distinguishes repeating decimals by drawing a stroke through the one repeating figure: $\frac{1}{3} = \cancel{3}$. Circulating decimals are marked off by commas, thus

$$\frac{3}{14} = \cdot 2{,}142857,$$

A section is devoted to approximate decimals, and here it is interesting to find mention of the modern method of approximating "by five or over", which Mair summarily rejects as inaccurate, saying that "rather than adopt a method so irregular and at the same time so inaccurate, it is better to allow the product one decimal place more than properly you have occasion for."

As an authority on bookkeeping, Mair naturally devoted a good deal of space to Commercial Arithmetic, the problems of exchange in particular receiving detailed explanation. Holland was, at that time, the business centre of the world; next in importance came London, Hamburg and Venice.

Though Mair avoids incursions into the realm of algebra as far as possible, yet he has to explain evolution and progression before treating of Interest and Annuities. He actually shows how to extract roots up to the ninth root of integers; for these rules he uses algebraic binomial formulae, but eschews the Cartesian notation, merely repeating the letter, to represent a power, even/

even up to seven times.

A fairly complete account is given of arithmetical and geometric progression, but here he follows the old-fashioned method of stating all the rules in words, and is consequently long-winded, and sometimes even ambiguous. Thus one of his rules in connection with arithmetical progression, in which he seeks to find the common difference of such a series, says "The difference of the extremes divided by the number of terms minus unity quotes the common difference". This, to us, sounds ambiguous, though perhaps to his readers it was not so, as "quotes" means "gives as the result of division". The weakness of the arithmetical method of treating progression is glaring in some cases. Thus, for instance, he tackles the problem in which he is given the first term and the common ratio of a G.P., and he wishes to find some remote term, say the 30th. He writes down a few terms of the series, then takes the greatest of these, say the fourth term; he squares the fourth term and divides the answer by the first term, and thus obtains the seventh term. Now he repeats the process, and gets the thirteenth term, and with another similar operation gets the twentyfifth term. From there by use of the common ratio, he creeps up to the thirtieth term.

Logarithms are not used in this book, and the rules for simple interest, annuities, etc., are all stated in words; this, the method to which men were accustomed, was the popular one.

On the theoretical side Mair follows closely on Malcolm. The neglect/

neglect of algebraic notation and of logarithms in the treatment of interest leads to some clumsy methods. Thus the problem of finding how many years a certain principal will take to amount to a certain sum at compound interest is tackled as follows:

> Divide the amount by the principal, divide this again by the amount of £1 for 1 year at the given rate continually until the quotient comes to unity. The number of these continued divisions tells you the number of years.

What happens when the quotient never comes to unity is not explained, but, in any case, the statement of these rules was to many of his pupils of academic interest, as tables are supplied from which all the problems can be automatically solved.

An interesting account is given of the tables of vital statistics compiled by Hr. Halley and De Moivre, for use in problems on insurance and deferred annuities.

A feature of the book is the detailed account given of the measurement of the areas of all the well-known figures, including even the parabola and the cycloid. The measurement of the surface and the solidity of most of the regular solids, even to the parabolic conoid and spindle is also given, together with an account of surveying and gauging. The last-mentioned subjects formed an important part of the curriculum of the schools, as I shall show later.

Taken as a whole, this work well deserved the popularity which it achieved for its author. It is a model as regards arrangement, and gives a complete account of all the arithmetical processes and their/

their practical applications. He does not reach the high philosophical level of Malcolm, but he was a great educationist, and played no small part in raising mathematics from its position as the Cinderella of school subjects to its present honoured place.

## HAMILTON'S "A SHORT SYSTEM OF ARITHMETIC AND BOOKKEEPING.

Robert Hamilton was the son of Gavin Hamilton, a Baillie in Edinburgh. He succeeded John Mair as Rector of Perth Academy in 1769, and taught there with great success for ten years. He resigned in July 1779 on being appointed Professor of Natural Philosophy in Marischal College, Aberdeen. X

Hamilton's "Introduction to Merchandize" was published in Edinburgh in 1777, and was dedicated to the Lord Provost, Baillies and Council of Perth: it is in two volumes, the second of which deals with bookkeeping. Volume I on Arithmetic and Algebra is an excellent book. It is, of course, strictly utilitarian in conception, but the trend of arithmetical teaching was essentially so at that period. Hamilton was a clear thinker, and he had little patience with the branches of arithmetic which are of no practical use. He is rather contemptuous of repeating and circulating decimals, preferring to use approximations; he gives an account of the infinite/

X See Appendix A.

infinite decimal, but advises the student to omit that section. He does not bow down in silent worship of the "Golden Rule": in fact, he suggests that a great many problems in proportion can be treated as examples of vulgar fractions. This is just the modern Unitary method, so that Hamilton was more than a century before his time.

Here is an example:-

"If 3 yards cost 15s.9d., what will 7 yards cost at the same rate?"

| | s. | d. | |
|---|---|---|---|
| 3 ) | 15 | 9 | Price of 3 yards |
| | 5 | 3 | " " 1 yard |
| | | 7 | |
| £1 | 16 | 9 | " " 7 yards" |

"Many other instances might be adduced, where the operation and the reason of it are equally obvious. These are generally, though unnecessarily referred to the rule of proportion."

In his actual treatment of the Rule of Three, he uses the modern way in which the third term is of the same kind as the answer. This was tentatively suggested by Mair, though the latter did not follow out his suggestion. The use of x for the unknown fourth term is not yet in evidence, but otherwise Hamilton's is the nineteenth century method.

The "Introduction to Merchandize" is largely a textbook of a mercantile education, and was written for the use of the students under his care, to supply ample material for his teaching: hence throughout/

throughout there are plenty of unworked examples. One point in which Hamilton improves on other writers is in his relegation of those parts of Arithmetic which are best treated by Algebra to a later stage of the book.

He rejects useless instruments like the "Rule of False" and "Medial Position" in favour of algebraic solutions by equations. Simple and Compound Interest are treated by formulae.

A curiosity of the book is that the author ~~of the book~~ has corrected typographical errors by pen to save the trouble of printing a table of "errata".

He heartily dislikes the method of "casting out the nines" as a check on the ordinary operations. In connection with his account of decimal approximation, he refers to the method of adding "one to the last decimal place when the following figure would have been 5 or upwards." Though he considers that this "gives an exacter answer", still he does not recommend it or use it. Perhaps he was frightened of being too unorthodox. Heterodoxy is evident again in his treatment of the calculation of areas and volumes; he clearly perceives the advantages of using square feet and square inches, cubic feet and cubic inches, but, owing to its universal use by practical men, he is impelled to teach the duodecimal method. In doing so he states very clearly the units involved both in areas and volumes. In the former case as the units proceed by powers of 12, there are five different units involved/

involved, if we do not go above square feet:-

| | | | |
|---|---|---|---|
| I | Superficial | foot, | i.e. square foot or a space 12' by 1" |
| II | " | inch, | i.e. space 1 foot long, 1 inch broad. |
| III | " | part, | i.e. square inch, or a space 1 foot long and 1 part broad. |
| IV | " | second | i.e. space 1 inch long and 1 part broad. |
| V | " | third | i.e. square part. |

A "part" is, of course, what Mair calls a "line", i.e. the twelfth part of an inch.

During his treatment of this subject, Hamilton metaphorically puts up a prayer for the introduction of a decimal system of weights and measures, a prayer soon to be answered in France, but not yet in his own country.

In the part of the book which deals shortly with algebra it is noteworthy that the Cartesian notation of power by indices has now come into greater favour, but the old clumsy method is not yet extinct. Algebra includes merely the four rules, involution, evolution and equations, the last-mentioned being used entirely for the solution of problems. The equations are simple or quadratic in one variable with some literals introduced. Progressions are treated as by Malcolm, using symbols, but Hamilton, despite his utilitarian aim, is led into the solution of the summing of certain other series by inductive methods. For instance, he takes the series 1, 3, 6, 10, 15, 21....... Each term is the sum of the progression 1, 2, 3, 4, 5.....to 1, 2, 3.....terms. This he calls an irregular progression, the nth term of which is known to be $\frac{n(n-1)}{2}$. He then observes/

observes, as it were by accident, that

$1 \times 2 \times 3 = 6 \quad 2 \times 3 \times 4 = 24 \quad 3 \times 4 \times 5 = 60 \quad 4 \times 5 \times 6 = 120$

If these numbers be divided by 6, the quotients are the sums of the series to 1, 2, 3, and 4 terms. Hence the sum to n terms should be

$$\frac{n(n+1)(n+2)}{6}$$

and by inductive methods he shows the truth of this answer. Such series must have been popular curiosities at the time, for in Halbert's "Practical Figurer", which appeared two years later, the same series, with all its terms multiplied by 2, is summed by similar methods, and Halbert claims his solution as absolutely original!

The third part of Hamilton's book is devoted to an account of British weights and measures. In it he makes an interesting suggestion that the preservation of the standards of length should be done by recording the distance between two mountains, as that would not vary in centuries. He gives an historical account of the English and Scots standards of weight, length and capacity.

To sum up, the "Introduction to Merchandize" is one of the best books on arithmetic in my period: its author reveals himself as a most enlightened and original teacher, with ideas far in advance of his time, anxious to cut out the dead-weight of ancient rules and unnecessarily elaborate calculations. The book does not appear to have attained a popularity worthy of its deserts, but in 1788, Hamilton, who was then Professor of Philosophy in the Marischal College/

College, Aberdeen, published an abridgement of it under the title, A Short System of Arithmetic and Bookkeeping". This later book is marred by many misprints, and is inferior in clearness and precision to the more ambitious work. It follows more orthodox lines than the "Introduction to Merchandize".

## REFERENCES TO THE BOOKS OF MAIR AND HAMILTON IN CONTEMPORARY PERIODICALS.

The Scots Magazine contains many references to the publication of the popular works of John Mair. The "Arithmetic Rational and Practical" was in the first place published in three parts. Part I appeared in January, 1764, and contained the doctrine of vulgar fractions: Part II appeared in December, 1765, and dealt with decimals: Part III, treating of practical applications, was published in August, 1766. The prices of the three separate volumes were 3s., 2s.6d., and 3s., respectively, and the complete book cost 7s.6d. The second edition came out in 1772; a notice of the third edition appears in the Glasgow Mercury, January 8th, 1778. Mair's edition of Cocker's Arithmetic was sold for one shilling; it appeared in July, 1751 and again in November, 1761. A notice anent Hamilton's "Introduction to Merchandise" appears in the Scots Magazine of January, 1778. Volume I was sold at 4s.6d., but/

but evidently was unattractive at that price. When the much shorter and inferior "Short System of Arithmetic and Bookkeeping" was published, the price was 2s.6d. Hamilton published also a book of tables of logarithms and of various other things useful in Navigation, Practical Geometry, and Commercial Arithmetic. This came out in 1791, and is contained in a list of books "deemed suitable for University students", advertised by William Creech, Publisher in the Edinburgh Evening Courant, October 31st, 1791.

# CHAPTER XVII.

## SOME MORE EIGHTEENTH CENTURY BOOKS ON ARITHMETIC.

Among those books which have survived is "The Universal Accountant and Complete Merchant", by William Gordon of the Academy, Glasgow. I have seen the first and third editions, the former dated 1763, the latter 1770. There are two volumes, but the second deals with bookkeeping. The "Academy", Glasgow was a commercial college for youths who had passed through the Grammar School, and in doing so had acquired only the rudiments of counting. The book is designed for young men who are "destined for the counting-house". It seems to have achieved a considerable sale, both in Scotland and elsewhere. Gordon was evidently the Head of the Academy at which, he says, "youth are boarded and instructed in languages, ancient and modern, Writing, Figuring, Geography, Geometry, Algebra and Merchants' Accounts....to qualify them for the Counting House, Army, Navy or Mechanical Employments."

The most interesting feature of the book is the sketch given by the author of the ideal course of study at such an Academy as his. He lays great emphasis on the necessity for a thorough grounding in the/

the Classics before entrance into this commercial college. Thereafter the pupil will be taught the arithmetic of integers and fractions, together with involution and evolution. He will then switch over to geometry and algebra, returning afterwards to those parts of arithmetic for which the higher mathematics is required. Practice in the application of these will be obtained from practical mensuration, the drawing of plans and maps, navigation and geography. Thereafter he will proceed to the technical part of his training, bookkeeping and commercial arithmetic. With such academies in the hands of teachers who were adequately remunerated, "our teachers would be men of understanding, our young men would be senators, and our 'merchants would be princes'".

As far as the mathematics is concerned, there is little that is worthy of mention in this book. The author does not appear to have been a very sound mathematician, and generally contents himself with a statement of rules for working out certain types of problems. He prefers approximate decimals to recurring ones, but teaches all the usual rules for operating with the latter. Again, in estimating areas by duodecimals, he uses the usual duodecimal units for the "superficies", but he does the multiplication by a kind of practice. Suppose he had to multiply 4 feet, 8 inches by 5 feet, 7 inches, he would proceed thus:-

Feet/

| Feet | Inches | Parts | |
|---|---|---|---|
| 4 | 8 | - | |
| 5 | 7 | - | |
| 23 | 4 | | |
| 2 | 4 | - | 6" = $\frac{1}{2}$ of 1 foot |
| | 4 | 8 | 1" = $\frac{1}{6}$ of 6 inches |
| 26 | 0 | 8 | |

i.e. in our units, 26 sq. feet, 8 sq. inches.

At one part of the work, in attempting to be original, he gives a rule which is quite wrong. The problem is to find the Greatest Common Measure of four or five numbers, and his rule is to add together all the numbers except the least, and take the G.C.M. of the sum and of the least number. As it so happens, the example set down by Gordon can be solved correctly by this wrong method. His numbers are 12, 16, 24, 36 and 52 and the G.C.M. of 12 and 128 is 4, but if the first two numbers had been 16 and 20 instead of 12 and 16, the rule would have failed, as in many cases it is bound to fail.

This is only one of several mistakes in the book.

Gordon is the only writer that I have read who adopts the "5 or over" rule in approximating to a decimal.

It is evident that Algebra (which, he says, is called The New Analysis or Specious Arithmetic or Numerical Geometry) has not yet acquired any degree of popularity. Gordon's treatment of it is brief, and he advises his pupils to "omit this part if in a hurry". He uses aa and $a^2$ for a squared, and root signs are indicated thus/

thus $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$, etc.

Equations are the only useful part of algebra, in Gordon's opinion, and these only for problem solution. None of his equations contain more than one power of the unknown variable. This variable is usually expressed by a vowel <u>a</u> or <u>e</u> or <u>i</u>, while constants are expressed by consonants.

Gordon is original in admitting the possibility of negative roots on the ground that negative numbers have a meaning in bankruptcy.

One has a feeling that to Gordon algebra was an unfamiliar weapon of which he was suspicious, and that he was happier when dealing with numerical quantities. This means that throughout the book we are treated to these long-winded, clumsy statements of interminable rules, which are a feature of seventeenth and eighteenth century arithmetic books.

Some interesting points emerge from his treatment of mensuration and gauging with which the arithmetical part of the book concludes. In the first place, as becomes a practical man, Gordon teaches his pupils to multiply by the sliding rule or Gunter's Scale. Another point is that the words Trapezium, Pentagon, Hexagon, etc. are used to mean respectively a Quadrilateral, Regular Pentagon, Regular Hexagon, etc. This is a survival from the Geometry of Ramus, two centuries earlier, but it was the usual nomenclature

"The Universal Accountant" is perhaps more interesting for the very/

very lengthy account it gives of the commercial activities of the age, and of the laws relating to insurance, the stock exchamge, etc., but that is outwith my sphere. Mathematically the book is not to be rated very highly, but, for practical purposes, it seems to have been suited to the age in which it appeared.

---

The next book, in chronological order, is "The Practical Figurer" published in Paisley, 1779, and written by William Halbert, schoolmaster at Auchinleck. The edition I have seen is the first, but there must have been several, as the book appears in a list of those published in Paisley under the date 1789.[X]

"The Practical Figurer" is quite a refreshing book, written by a man of a reflective and original turn of mind, who scorns to waste time in stating all the simple rules of elementary arithmetic which can be taught by rote to a child, and prefers to gather together an unusually varied, amusing and instructive collection of problems. These are "useful, subtle and entertaining for gaining the Attention and improving the Genius of Youth". Halbert considers that most of the instruction contained in school books on arithmetic is just "schoolboys' prattle......which might have been useful at the time of the Reformation". He claims that most of his examples are original. A great many are in rhyme. Here is one containing the reference/

X History of the Paisley Grammar School - Robert Brown, 1875. p149 reference to John Mair:-

"A herring, a hen and a ham
  Were set on the table before us
A bottle, a bowl and a dram
  For shillings a head, half a score o's
The fish was a fifth of the fowl
  And both were a fifth of the ham
The bottle a fifth of the whole,
  And so were the bowl and the dram.
We paid for our hunger and thirst
  In equal proportion, to wit
As article last to the first
  In order as formerly wrote:
Exhibit a sensible state
  Of this very nice bill of fare,
Perhaps, if alive at this date,
  It ev'n might perplex Mr. Mair."

Another shorter one reads:-

"I got a million rings of brass
As portion with a bonny lass,
Say what amount this tocher brings
At 18d. each gross of rings.

and another

"What cost 500 horned brutes
At crowns the piece for all their clutes?"

Doubtless rhyming was fashionable in Ayrshire in those days. This work is about the most human of arithmetical text-books, and gives us an insight into the habits and psychology of country people at the end of the eighteenth century, particularly with regard to smuggling. The book covers the usual ground, including the four rules, proportion and its applications, then series, followed by interest and annuities. It deserves a place in this historical survey, not merely for its humour and originality, but also because it/

it set the fashion for producing a text-book in which most of the space was devoted to unworked examples. This tendency was further worked out in the popular arithmetics by Melrose and Gray.

That Halbert possessed remarkable mathematical intuition is shown in his statement of "one Rule, entirely new, called Irregular Progression, never before heard of in Arithmetic". I have already referred to his summing of the progression

2, 6, 12, 20, 30, 42, 56, 72.......

when writing of Hamilton's solution of the same. Here is another series which he reached in solving the problem of a shepherd, with 100 ewe lambs, each producing a ewe lamb when two years old and every year thereafter, who wishes to find how many they would amount to in 20 years. The series is:-

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89........

each term being the sum of the two preceding it. Apparently by guessing, he discovered two laws governing the series:-

(1) Take any term, multiply it by 11, and add the term 5 places in front of it, and you get the term 5 places behind it.

(2) Take any term, multiply it by 123, subtract the term 10 places in front of it, and you get the term 10 places behind it.

By the aid of the second rule he found the 21st term to be 10946 and hence the answer to the problem to be 1094600 for 100 ewe lambs.

This and similar problems, solved by guessing, would help to pass the winter nights in a country schoolhouse.

One/

One of his chapters is devoted to Curiosities, i.e. parlour tricks connected with numbers, such as the formation of magic squares and the well-known trick, where you pretend to write down the answer of a big addition under the following conditions: you ask a person to write down two sets of figures, then you write down a third, he writes a fourth and you a fifth. If you are careful to write down the third line so that each digit of it is the complement to 9 of the digit above it, and the same with the fifth line, then you can state the sum at once. Thus:

| | | |
|---|---|---|
| | 315276 | by putting a 2 in front |
| | 243819 | of the first line and |
| X | 756180 | deducting 2 from the last |
| | 327135 | digit of the same. |
| X | 672864 | |
| | 2315274 | |

Among the Curiosities is a method for finding Perfect Numbers; he takes the G.P.

1, 2, 4, 8........

If you continue this to a <u>prime</u> number of terms, and multiply the last term by the sum of the terms, you get a perfect number. Thus

$2 \times 3 = 6$, $4 \times (1 + 2 + 4) = 28$, $16 \times (1 + 2 + 4 + 8 + 16) = 496$

are all perfect numbers in the Greek sense of the word, i.e. the sum of all the possible factors is equal to the number itself.

Halbert's love for the solution of problems leads to his giving a short account of algebra. This contains the minimum of the rules of algebra necessary for the solution of equations. The problems/

problems treated lead to the solution of simple and simultaneous equations of the first and second degree. He uses vowels for the unknown and ignores negative roots. Finally he gives a method of solving a numerical cubic equation, from which the $a^2$ term has been eliminated, e.g. $a^3 - 5a = 1$. This is done by finding the cube root, but making the correction for the 5a as you go along. It is actually carried out to ten decimal places. Here is the commencement of the process:-

$$
\begin{array}{rrl}
a^3 - 5a = & 1 & (\underline{2{\cdot}3300587396} \\
 & \underline{10} & \quad \text{add } 5a \\
 & 11 & \\
 & \underline{8} & \\
700) & 3{\cdot}000 & \\
387) & & \\
 & \overline{2100} & \\
 & 540 & \\
 & \underline{27} & \\
 & 2667 & \\
108700) & 333000 & \\
4167) & & \\
 & \overline{326100} & \\
 & 6210 & \\
 & \underline{27} & \\
 & 332337 & \\
 & 663000 & \quad \text{etc.}
\end{array}
$$

From the nature of the equation it appears that there will be a real positive root. The integral part of the root is found, by guessing, to be 2; to the 1 on the right hand side, $5 \times 2 = 10$ is added. $2^3 = 8$ is subtracted from this, leaving 3. The next three figures are taken down. Thereafter the 5a part is deducted from the left-hand side instead of being added to the right. By trial and error he finds that $0{\cdot}3$ is the next figure in the answer. Then/

Then $3 \times 20^2 = 12{\cdot}00$

$3 \times 20 \times {\cdot}3 = 1{\cdot}80$

${\cdot}3 \times {\cdot}3 = {\cdot}09$

When this is multiplied by a, i.e. the next figure ·3, it is necessary to subtract 5 times that figure, i.e. 1·500. Instead of doing so he subtracts $\frac{1}{\cdot 3}$ of 1·500 before multiplying by ·3, i.e. he subtracts 5·00

$$\begin{array}{r} 7{\cdot}00 \\ 1{\cdot}80 \\ 0{\cdot}09 \end{array}$$

This multiplied by ·3 gives

$$\begin{array}{r} 2{\cdot}100 \\ {\cdot}540 \\ \underline{{\cdot}027} \\ 2{\cdot}667 \end{array}$$

Subtract 2·667 from 3·000 and proceed to the next place. This seems to be the method used though it does not account for the figure 3·87 at the left-hand side, which one would suppose should have been 1·89. It would have been easier for the writer to add 5 X ·3 to the right-hand side at once, proceeding then in the usual manner for cube root. Thus

| | |
|---|---|
| | $3{\cdot}000$ |
| | $\underline{1{\cdot}500}$ |
| $3 \times 2{\cdot}0^2 = 12{\cdot}00$ | $4{\cdot}500$ |
| $3 \times 2{\cdot}0 \times 3 = 1{\cdot}80$ | |
| ${\cdot}3 \times {\cdot}3 = \underline{{\cdot}09}$ | $\underline{4{\cdot}167}$ |
| $13{\cdot}89$ | ${\cdot}333$ |

I have not seen this method of solving numerical equations in any other/

other book. Though limited to the finding of one root, it could be extended in theory, at least, to equations of higher degree than the third, and though the calculations seem rather appalling to us, the mathematicians of those days did not hesitate to attempt the finding of roots even as high as the ninth by the use of the binomial expansion for that power.

Though the "Practical Figurer" does not appear to have attained a very big circulation, it must have been well received in its own immediate neighbourhood. A list of subscribers is published in the book itself, and many of the names are of schoolmasters in the West country. One name is that of George Douglas, teacher of Mathematics in Ayr, whose book on geometry will be reviewed later. One name, Burness, has caused some speculation, but apparently there is no connection with Robert Burns. The list of names serves to show that Halbert's book was used as a text-book in the schools, though not put in the hands of the pupils. Most of the subscribers took one copy only, while one is recorded as taking four and another six.

The fact that text-books in arithmetic at that time were written for masters and not for pupils is further demonstrated in the preface to the next arithmetic which I have to mention.

"The Institutes of Arithmetic" was written by Alexander Ewing, teacher of mathematics in Edinburgh, 1784, and it professes to be a text-book for the use of boys at school. The crude method by which arithmetic/

arithmetic was taught is laid bare by Ewing when he says in his preface "Such a book(as this) being put into the hands of every learner will render the business of teaching and learning much easier than it is or can be without such assistance; for the teacher, being free of the trouble of dictating rules and examples, will be at leisure to give proper instruction."

One can scarcely say that Ewing has fulfilled his purpose; he even admits himself that the order followed in the book is not the one that a learner should adopt; thus, after giving the rules for addition of integers, he goes on at once to the multiplicity of tables of weights and measures, which it was the unhappy lot of the schoolboy of those days to master.

There is really nothing original about this book; it covers the usual ground, devotes the usual inordinate amount of space to short trick methods of multiplying and dividing, and to infinite decimals One point in which Ewing improves on his predecessors is in his insistence on the desirability of securing the Lowest Common Denominator in the addition of vulgar fractions. The usual practice was to take the product of all the denominators as the new denominator, add, and at the end reduce the answer to its lowest terms. Ewing gives the modern method for finding the L.C.M. and uses it in addition and subtraction of fractions.

The weakest section of the book is that on Proportion, where he uses the term Arithmetical Ratio of two numbers to stand for the difference/

difference of the two on the analogy of Geometrical Ratio. From this is developed the equality of arithmetical ratios, which he calls a Disjunct or Discontinued Proportion, and represents thus:

2 · 4 : 9 · 11

The treatment of series by arithmetic alone, involving as it does a complete statement of two sets of rules, one when there is an odd number and one when an even number of terms, dates back to Sacrobosco, and seems rather out of date at the end of the eighteenth century. Ultimately he is impelled to introduce algebra in summing geometric progressions to an infinite number of terms.

A page is devoted to "Musical Proportion", for further information on which reference should be made to Malcolm's Arithmetic, which must have been looked on as a standard work.

He states the Rule of Three in the old-fashioned way, in which direct and inverse proportion are treated separately, and then refers to Maclaurin's Algebra as containing a better method, by which one rule is sufficient. Maclaurin sets down the second term in the manner usual at the time, but thereafter in settling the place of the first and third terms, he uses the method adopted in modern times for placing the first and second terms; thereafter he multiplies the second and third terms and divides by the first term to get the answer.

When he comes to Compound Proportion, which he calls "Universal", Ewing, like many others, tries to state a cast-iron rule in language/

language which will be understandable, and, like all the others, produces a monstrosity. It is interesting, however, that in a proof of his rule by reducing the problem to one in two or more simple proportions, he introduces the letter x for the fourth term: this is the only case where I have encountered this convention in these early books.

The rest of the book covers the course usual at these times; throughout he gives a fair number of examples, and there are seventy-five miscellaneous problems at the end. On the whole, there is little that is outstanding in the book, and its appeal would be rather limited, apart from the fact that it is small and easily accommodated in the pocket.

Ewing also wrote a book on "Practical Mathematics" which is reviewed later.

---

The next book which comes within my survey is the Concise System of Arithmetic, by A. Melrose: he is described as a teacher in Edinburgh, and the edition I have seen bears no number and is dated 1791. This was the former of two very popular arithmetics produced about this time. Though overshadowed by the more famous Gray's "Introduction to Arithmetic", yet "with Gray's it was in universal use for three generations" X in Scotland.

The book was in two parts, but Part II merely contained answers to/

X Chapters in the history of Bookkeeping by David Murray, p.328.

to the examples in Part I and hints for the solution of some of the more intricate problems. It is easy to see from a perusal of the book why it attained such popularity in Scotland. Though it contains only ninety-six pages, and can conveniently be carried in the pocket, yet it covers all the ground of the usual arithmetic. This is made easy, because Melrose merely states all the rules with extreme brevity, and then sets down plenty of examples suitable for the average pupil. He does not show any of the processes by means of examples himself, yet he covers most of the work required by the ordinary business man of the period.

The first edition is inferior in some respects to the others: it contains a rather jumbled account of English and Scottish weights and measures. In the second and later editions, the two sets of weights and measures are explained separately, and a short account is given of those used in Ireland and the Isle of Man; the second edition also gives an account of the new standard weights and measures proposed for the use of Great Britain and Ireland in a Bill then before Parliament (1816). In the first edition all the usual trick methods for doing particular multiplications and divisions are wisely omitted, but they appear as a supplement in the second edition. In the first edition, the treatment of vulgar and decimal fractions occurs early, and is very short: no account is given of interminate decimals beyond the rules for turning these into vulgar fractions. In the second edition fractions are given a separate section and are treated/

treated in detail. Here, too, some space is devoted to Commission, Insurance, and even the buying and selling of Stock. Both editions devote a considerable space to problems of exchange amongst the various nations of Europe and the United States. The rules for extraction of square and cube root and for the summing of Arithmetical and Geometric Progressions are stated, but no examples are worked out. The same applies to the useless rules for Alligation and Position, and to Duodecimal Multiplication. The second edition is further enlarged to include the usual tables for the solution of problems on Compound Interest and Annuities.

Melrose's Arithmetic was intended, in virtue of its size, cheapness and abundance of unworked examples, to be placed in the hands of the pupils of a large class. It makes no pretence to teach the methods of the science, yet it quite evidently supplied a "felt want", and attained great popularity. The first edition is dated 1791, and the second, though undated, must have appeared in 1816. There were many later editions.

---

The most popular arithmetic of all is Gray's "Introduction to Arithmetic". This, like Melrose's, belongs to the nineteenth century, though one of the editions in the British Museum, possibly the first, is dated 1797. A full account of this book has been given by the late Charles Tweedie in ~~an article~~ a paper to the Edinburgh Mathematical Society, dated 19th October, 1924, so it would be superfluous to/

to say anything of it. I have seen the ninety-fifth edition, which is dated 1877 and published by Oliver and Boyd, Edinburgh. According to Mr. Tweedie, it reached 101 editions.

On the whole, the ninety-fifth edition, shows remarkably little change in arithmetical rules. It contains Addition, Subtraction and Division Tables, as well as the usual Multiplication Table. Casting out the nines is still used as a verification for these processes. Cubic feet are still called Solid feet, but the term Superficial feet has disappeared. Duodecimal arithmetic, by which artificers "cast up the contents of their work" is still retained. Many of the old Scotch measures for land, for wheat and for liquids are still appearing.

Simple and Compound Proportion have been much improved by the device of making the term of the same name as the answer, the third term in the proportion. In Simple Interest the rate is now reckoned per cent and not per unit. The end of the book is very modern in publishing tables of the Metric System.

Melrose and Gray represent the last word in the mechanical teaching of arithmetic, but there is no doubt that they served a very useful purpose, and it ~~might~~ may be that the near future will see a reaction towards the acquiring again of a mechanical proficiency in arithmetic, such as our forefathers possessed.

The next book, though published in London, appears in this paper for two reasons. In the first place, it is the only arithmetic I have read, which is written for the use of ladies, and secondly, the late Professor Gibson[X] says that its author was a graduate of Marischal College, Aberdeen.

The book is entitled "The Young Lady's New Guide to Arithmetic" by John Greig, Teacher of Writing, Geography, etc, and the one I have seen is a second edition, dated 1800. It is published, under one cover, along with

(1) "A Rational System of Arithmetic" by John Skally, Birmingham.
(2) An Appendix to Dowling's Arithmetic.
(3) A Key to Voster's Arithmetic by Daniel McSweeney of Limerick.

Sandwiched amongst these other books, Greig's Guide has had to be curtailed by the omission of worked examples of the simple rules. It is a short and entirely utilitarian book, chiefly interesting for the light it throws on the stage of advancement expected of girls in arithmetic. Greig wrote for young ladies who were meant to be ornamental rather than useful, and it is difficult to take the book seriously. The preface, for instance, is adorned by thirty lines of Heroic Couplet, supposed to be written by a friend of the author in adulation of his abilities, but probably written by himself. This device was adopted by more than one mathematical writer whose works ~~whom~~ I have encountered.

The book is closely and poorly printed, and extends to a mere eighty pages. Alike as regards the course sketched out and the examples/

x History of mathematics in Scotland to the end of the 18th Century - G. A. Gibson in Proceedings of the Edinburgh Mathematical Society Series 2 Vol I 1928.

examples set, this arithmetic shows a strong domestic bias. It includes the four elementary rules for integers and for compound quantities with the tables necessary for shopping, etc., as well as Reduction, Rule of Three, Practice and calculation of Simple Interest. Throughout Greig follows the policy of merely stating the rules and in one or two cases working out an example. Decimal fractions are entirely omitted. The method of writing receipts and Promissory Notes is explained, together with the meaning of Bills of Exchange. After supplying a purely utilitarian course of instruction, Greig feels impelled to provide the young ladies of his time with some interesting problems to occupy their leisure hours. These are rather prehistoric, however, being the questions which occupied the learned Bede, viz. to find when a year is a Leap Year (perhaps that was utilitarian!), to find the Dominical Letter, Golden Number, Epact and the date of Easter, the last by means of a table. Another table shows how many days there are between any two dates in a given year, and still another the day of the first of the month for any year from 1800 to 1848.

That tells us the amount of arithmetic that a lady of fashion would require. In the parish schools of Scotland, however, it seems that the girls got in the main the same teaching as boys. Dr. Murray says that "Arithmetic was carefully and thoroughly taught (to girls) and many distinguished themselves as arithmeticians. In the Academies they often took the same mathematics course/

course as boys". X

Doubtless Greig's "The Young Lady's New Guide to Arithmetic" was written for pupils in private schools.

---

Another book on Arithmetic is "The Accomptant's Guide: an improved system of Practical Arithmetic", by James Morrison, Master of the Mercantile Academy, Glasgow. The first edition of this book was published while the author was actually employed in a "counting-house", so that he must have had practical experience in commerce before becoming a teacher. The edition I have seen is the third: it is stated to be much more comprehensive than the first, and is dated 1806. The book was specially written for the use of schools and business academies. It contains some three to four thousand examples, and is an excellent text-book. In one respect it shows an improvement on most others. The four rules are explained first with reference to integers, and later compound quantities are introduced.

A remark in the Preface throws a flood of light on the teaching methods of the eighteenth century. "The text contains only such matter as is intended for the pupils to copy into their account books. By adopting that plan, the work is not only prevented from swelling to an improper size for the use of schools, but teachers are enabled to reject the explanatory observations without trouble". Evidently the system of teaching involved the dictation of rules which/

X Chapters in the history of Bookkeeping. p 345 which were slavishly copied and committed to memory. Morrison, being mathematically minded, desires to give a reason for each rule, but apologetically prints it in small letters, and suggests that it need not be read.

There are a few points of interest in this book to which brief reference may be made. He tells us that, though the Avoirdupois system of weights was legalised by the Articles of Union for use throughout Great Britain, it was not adopted in Scotland, except for the sale of leather, tallow, soap, flour, bread, candles, groceries, resin, wax, pitch, wrought metals, some Baltic goods and all goods bought from England. The Trone weight, though forbidden by Statute, was still used for many commodities.

Some of the examples given in the book are worthy of a place in Pendlebury's collection; one particularly elaborate exercise involves thirty-four different quantities, and the statement of it occupies half a page.

In the section on Simple Interest there is included a "Perpetual Calendar", showing the Dominical Letter for every year from the beginning of the Christian era to 4000 A.D. There is also a table showing the day of the week on which the first of the month occurs for every month and each of the seven Dominical Letters.

In discussing "Equation of Payments" Morrison refers to the "very laborious" method invented by Malcolm, but says that the simple method is "agreeable to the customs of men in business".

The/

The sections dealing with Commercial Arithmetic, with the exception of that on Insurance, are treated fully and with hosts of excellent examples. "Stock-jobbing" in those days was confined to Government Stock, South Sea Stock, India Stock and Bank Stock.

The book, as a whole, shows the intensely practical nature of the teaching of Arithmetic in the schools and mercantile academies of that period. "The Accomptants' Guide" is an excellent production, judged by the standards and requirements of the period. Morrison is clear, lucid and concise in his statement of rules, and exhibits mathematical precision in giving reasons for all such statements. Some of the examples demand considerable insight and mathematical ability: the book seems to have proved popular.

# REFERENCES IN CONTEMPORARY PERIODICALS

From the Scots Magazine we learn that William Gordon, in addition to the "Universal Accountant", wrote "The General Countinghouse and Man of Business", published in 1766. Furthermore, he was joint author of "The Elements, Analysis and Practice of Arithmetic with Plane Trigonometry" by William Gordon and Robert Dodson, published in Glasgow, 1771. The most interesting feature of the latter work is the section devoted to Trigonometry or rather to the solution of triangles. Of Trigonometry, as we understand it, there is none, but empirical rules are given for solving right angled and oblique angled triangles by arithmetic; these rules, he says, "we cannot pretend to account for here, without entering deeper into the subject than our present plans will admit". The solution of a right angled triangle, when two of the sides are given, is done by Pythagoras' Theorem. When one angle and a side are given, solution is got by means of what is called the "Arithmetical Radius" which "will have the same effect in an arithmetical proportion as the radius in a sinical proportion"; that is to say, the arithmetical radius bears the same ratio to an angle that the hypotenuse bears to the side opposite the angle. Though this method of solution of triangles is given, more reliance is placed on drawing to scale, using the "Scale of Equal Parts" for the sides and the line of chords for the angles. The book/

book includes one or two simple applications of this "Trigonometry" to Surveying and Navigation.

---

Alexander Ewing advertises himself in the Edinburgh Evening Courant of May 4th, 1763, as teaching arithmetic, bookkeeping, practical mathematics, etc., "at the foot of the Horse Wynd, Edinburgh". He enrolled both ladies and gentlemen as pupils. The first edition of his "Institutes of Arithmetic" is dated 1756, another edition 1772. There are notices of the first edition of his "Synopsis of Practical Mathematics" in 1771 and of the second edition in 1779. He also published a "Practical Astronomy" in 1797.

---

# CHAPTER XVIII.

## THREE EIGHTEENTH - CENTURY BOOKS OF EUCLIDEAN GEOMETRY.

The first in point of time and the most famous of these is by Robert Simson, Professor of Mathematics in Glasgow University, from 1711 to 1761. It is unnecessary to give details of his life, but it is interesting to know that, when he was a student at Glasgow University, no lectures in Mathematics were given in the College.[X] He is stated to have mastered Euclid's Elements without instruction. When he was offered the Chair of Mathematics in 1710 at the age of 22, he was allowed to spend a year in London for further study. His "Elements of Euclid" appeared in 1756, both in Latin and in English. I have seen these and the second edition, dated 1762. Others followed in 1767, 1772 and 1775: the tenth is dated 1803, the thirteenth 1806, the twenty-third 1830, the twenty-fourth 1834, and the twenty-sixth 1844. There were thirty successive editions.

Simson claims to have rectified the errors introduced by Theon and others into the text of Euclid, particularly in the Fifth and Eleventh Books. His edition contains the first six books and the Eleventh and Twelfth, these constituting the geometry taught in the University at that period. In the second edition he added the/

X Life and Writings of Robert Simson by Rev. Wm. Trail, 1812. p. 1.

the "Book of Euclid's Data", and in the tenth edition, which I have read, there is a further addition of a section on Plane and Spherical Trigonometry.

Simson was an outstanding Classical scholar as well as a great mathematician. His Euclid is so well known that it would be superfluous to enter into an account of it.

In the theorems on parallels, Simson adopts the Euclidean axiom, and in a note gives a proof of that axiom by means of four separate theorems, based on the axiom that a straight line cannot first come nearer to another straight line and then go farther from it before it cuts it.

Simson retains the word "diameter" as used in the earlier geometries for what we now call the diagonal of a parallelogram. The word "arc" is not used; instead, the word "circumference" served the double purpose of describing the whole and the part.

At the end of Book XII he gives a series of notes indicating where he had departed from the original text of Euclid, and why he had done so.

Two other books on theoretical geometry that were produced in the last decade of the century bear comparison with Simson's great work. The first of these is by Alexander Ingram, Philomath, Edinburgh: its date of publication is 1799, and there appears to have been but one edition. Ingram follows closely on Simson, even to the lettering in his proofs, but he claims to have corrected some of the/

the errors in previous translations overlooked by Simson. He certainly adds some useful propositions not included by Simson; for instance, three propositions are added at the end of the second book which are of considerable use. One of these is the well-known theorem that the sum of the squares on two sides of a triangle is equal to double the sum of the squares on half the base and on the median to the base. Evidently the word "median" had not been invented at that period, as a circumlocution is used in place of that word.

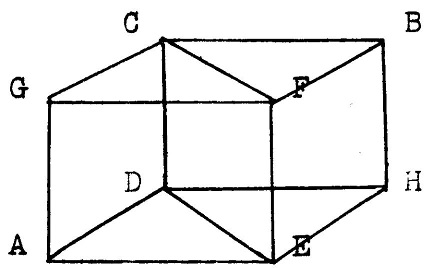
Ingram introduces the definitions of "arch" and "chord" not given by Simson, yet curiously enough he does not use the word chord to simplify the enunciations of the propositions that follow. Ingram uses the words "plane angle" to signify the inclination of two straight lines. The word tangent is not used in the geometry. It is simply "a straight line touching a circle"; this would doubtless be done to prevent confusion with the "tangent" of trigonometry, which, being itself a line and not a ratio, could ~~not~~ be confused with a tangent in geometry.

Again in the third book, Ingram adds some propositions, notably the converses of Euclid III, 22 and 35, and the concurrency of three perpendiculars from the vertices of a triangle to the opposite sides.

Ingram makes considerable alterations in Simson's definitions and axioms in Book V with the intention of rendering these easier/

easier to the student, but the conciseness of the latter is preferable to the circumlocution of the former. These alterations lead to slight variations in the proofs of the theorems. In the Eleventh Book Ingram takes exception to Simson's proof of Proposition 28:

> "If a parallelopiped be cut by a plane passing through the diagonals of two of the opposite planes, it shall be cut in two equal parts."




Simson proves that plane CDEF cuts the solid into two equal parts, by proving triangles CGF and CBF equal to one another; also triangles ADE, DHE: parallelogram CA is equal and similar to BE and GE to CH. Hence, he says, "the prism contained by the two triangles CGF and DAE and the three parallelograms CA, GE, EC ~~are~~ is equal to the prism contained by the two triangles CBF, DHE and the three parallelograms BE, CH, EC, because they are contained by the same number of equal and similar planes alike situated and none of their solid angles are contained by more than three plane angles". Ingram points out that while the triangles GCF, CFB are equal and similar, yet they are not similarly situated in respect of the equal and similar parallelograms GE and CH (except in the case where/

where these are rectangles) for the angle CBH is equal to angle GFE, and angle BCD is equal to FGA, but the angle at B is not equal to the angle CFG. The only case in which the triangular prisms would be similarly situated is that in which angle CBH is equal to AGF, but these angles are supplementary, and therefore the only case in which the proof is sufficient is, as aforesaid, that of a rectangular prism. Ingram substitutes a proof of his own based on a previous theorem that if two triangular prisms have one of the sides of their parallelograms common to both, and the sides opposite to it in the same straight line, the prisms are equal to one another.

In the Twelfth Book, Ingram follows closely on Simson, and indeed his Euclid might quite well have appeared as one of the editions of the latter, with himself as editor. He even puts in Simson's Preface (along with one of his own) at the beginning of the book.

Further, in following the great master he adds notes at the end. In his notes on Book I he makes an effort to get over the difficulty of the parallel theorems by proving Euclid's axiom. This requires seven theorems, and he has to assume an axiom which is at the basis of most of the proofs that have been attempted, i.e. "If two points on one straight line be at unequal distances from another straight line, the two straight lines when produced will meet on the side of the point whose distance is the smaller."

This/

This has to be assumed in the second of his seven theorems.

To the Euclidean geometry, Ingram adds a short account of Plane and Spherical Trigonometry, Logarithms and Practical Geometry. In the Trigonometry there is little change from the beginning of the eighteenth century; the cosine has come into more general use, and the radius is reckoned to be one unit instead of ten millions, so that the sines and cosines are decimal fractions. There is a slight advance towards simplification in notation, but the use of the small letters a, b, c to denote the sides of a triangle has not yet become known. The capital letter P is used to denote half the sum of the three sides, so that the cos $\frac{1}{2}$ A formula appears thus:-

"The rect. BA,AC : rect. contained by P and the difference of P,BC :: $R^2$ : $\cos^2 \frac{1}{2}$ BAC"

The table of sines and tangents is still constructed as in Wilson's Trigonometry so far as finding sin 1' and sin 2' is concerned. Thereafter instead of using the addition formula for sin (A + B), he uses another relationship, which, expressed in modern symbols, is as follows:-

$$\sin A + \sin (A + 2B) = \cos B \times 2 \sin (A + B)$$

Suppose we know sin 1' and sin 2', and we wish to find sin 3';

put $A = 1'$ and $B = 1'$

then $\sin 1' + \sin 3' = \cos 1' \times 2 \sin 2'$

$\therefore \sin 3' = 2 \cos 1' \sin 2' - \sin 1'$

Again/

Again put $A = 2'$ $B = 1'$ and you get sin 4', and so on up to $90^{\circ}$ by 1' at a time. Even with this simplification, the compilation of tables was a monumental job!

In the solution of right-angled spherical triangles, he makes use of the rules with regard to Circular Parts.

Little need be said with regard to the section on Practical Geometry; it covers almost identically the same ground as Gregory's book on the subject, but is much condensed in form. Perhaps the only interesting feature is a note on the "Sliding Rule" used by "Officers of the Revenue". This seems to resemble the modern slide-rule, but gaugers also used a rod which was likewise marked logarithmically, and required a pair of compasses along with it. These instruments were provided with lines by means of which ready calculations could be made of the contents of a barrel in wine or ale gallons from a measurement of its diameter and its "wet inches". There were different lines for each of the common shapes of barrel.

In the "Practical Geometry" section of his book Ingram reverts to a practice which seems to have been abandoned about this time. In previous books on geometry it was the usual practice, in the interests of economy, to print a whole series of diagrams on one page. In his geometry and trigonometry Ingram adopts the modern method of drawing the diagram beside the theorem to which it refers, but in the last section (doubtless to conserve space) he reverts to the old bad way.

Ingram's/

Ingram's Euclid, though not strikingly original, improves in some respects on the master, Simson, by including a number of useful propositions omitted by the latter. It did not compete, however, with that of his contemporary, Playfair, as the latter, by striking out on original lines, supplied the thing that many mathematicians were sighing for, a geometry that was Euclidean in its logical qualities, but was clear of the Euclidean geometrical theory of proportion.

---

John Playfair was Professor in turn of Mathematics and Natural Philosophy in Edinburgh University. The title of his geometry is "Elements of Geometry containing the first six books of Euclid, with two books on the geometry of solids." The first edition appeared in 1795, the fourth is dated 1814, the fifth 1819, the seventh 1826, the eighth 1831, the ninth 1836 and the tenth 1846. Like Simson's, this text-book is so well known that there is no need to give a full account of it; it belongs more to the nineteenth than to the eighteenth century. Playfair's aim was to attune the geometry of Euclid to the state of advancement reached by the mathematical sciences at the end of the eighteenth century, and in particular to state the theory of proportion in terms of algebraic symbols, so as to avoid the "tediousness and circumlocution" associated with Book V. Curiously enough, Playfair/

Playfair lays little stress on his invention of the axiom of parallels that bears his name; he merely says that was put in to simplify matters. In Book II the algebraic notation for squares and rectangles is used. In the last two Books, which are numbered VII and VIII he departs considerably from Euclid, again with simplification in view, omitting a number of propositions included in Euclid Book XI, and putting in several propositions on pyramids which are in Book XII. Again in his eighth Book Playfair omits a good many of the less useful propositions of Book XII, and puts in several helpful theorems. In this last book he collects only propositions dealing with the quadrature of the circle and with solids that depend on the circle. These are in all cases compared with one another and with parallelopipeds. Many of his propositions on the Geometry of Solids are considerably shortened by the assumption (which is not made by Euclid) that "on any given figure as a base a solid may be constituted or conceived to exist, equal to a given solid (because a solid, whatever be its base, as its height may be indefinitely varied, is capable of all degrees of magnitude from nothing upwards)". This assumption simplifies, for instance, the proof of the theorem that if a cone and a cylinder have the same base and altitude, the cone is the third part of the cylinder. Again, with the help of that assumption, Playfair is able to introduce the "Method of Exhaustions" in a simpler form than usual. His aim throughout is to make Euclid easier/

easier without sacrificing his sequence or anything vital in the matter of assumptions.

The section on plane trigonometry is slightly fuller than that of Ingram, but contains nothing new: in the Spherical Trigonometry he does not make use of Napier's Rules of Circular Parts in the first edition, 1795. The letter S is used to represent the sum of two particular sides of a plane triangle, D their difference.

In the fourth edition considerable alterations have been made. Instead of Books VII and VIII he has three supplements, the first dealing with the quadrature of the circles, the second with the intersection of planes as in Euclid, Book XI, the third with the geometry of solids as in the first edition.

A note is added at the end of the Spherical Trigonometry on the use of Napier's Rules of Circular Parts. It is interesting that another stage has been reached in the evolution of signs. In this edition, 1814, Playfair introduces the symbols s and S, the former to stand for the whole sum of the sides, the latter for the whole sum of the angles, and he uses the small letters a, b, c, for the sides, so that $a + b + c = s$ $\quad A + B + C = S$. These symbols are used only for spherical triangles. In plane triangles S still stands for the sum of two particular sides, which are named e.g. BC and CA. D stands for BC - CA.

The seventh edition, published after the author's death, is edited by William Wallace, College of Edinburgh, and dated 1826. It contains/

contains some additions. The letters a, b, and c are used for the sides of any triangle; $a + b + c$ is made equal to $2p$. The use of s and S in spherical triangles has disappeared again.

Playfair, it may be noted, gives no account of Practical Geometry; further, his diagrams are printed alongside the propositions as in modern fashion.

---

# CHAPTER XIX.

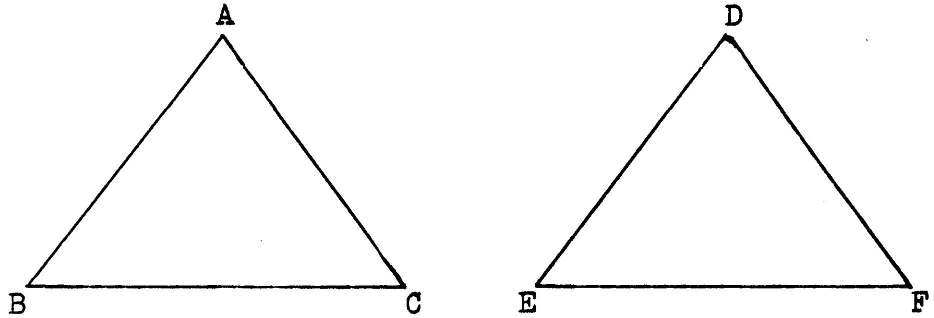
## SOME OTHER EIGHTEENTH- CENTURY BOOKS OF GEOMETRY.

The first of these is entitled "The Elements of Euclid", and is written by "George Douglas, Teacher of Mathematics in the Academy at Ayr". As mentioned before, Douglas was one of the subscribers to Halbert's "Practical Figurer". This Euclid was published in London in 1776, and very few copies of it are extant. One has been preserved in Ayr Academy, one in Glasgow University and one in the University of Edinburgh. The aim of the book, as of many of more recent origin, is to make Euclid more palatable to the pupils of a school, without sacrificing any of the rigour of the proofs, but this, as was to be expected, Douglas has not achieved. His chief objection to Euclid is on the grounds of prolixity, and he has certainly achieved considerable conciseness, but by abandoning in many cases essential steps in the proofs.

In the definitions he is apt to put in too much. He divides four-sided figures into square, oblong, rhombus, rhomboides, and says that all other quadrilateral figures are called Trapezia. Thereafter he proceeds to define a parallelogram. Further a Rhomboides/

Rhomboides is defined as one whose opposite sides <u>and angles</u> are equal. Euclid omits the words underlined. This is typical of the tendency to sacrifice rigour for perspicuity.

Like many modern writers who wish to simplify geometry, Douglas sidesteps Pons Asinorum. He claims that in his proof he has not "used more freedom than is done in the demonstration as it now stands", but this claim will not bear examination. His proof follows these lines




Given Triangle ABC has AB equal to AC: to prove angle ABC equal to angle ACB.

Let a triangle DEF be given, having the sides DE, DF equal to the sides AB and AC, and angle EDF equal to angle BAC. He then proves by Proposition 4 that the triangles are equal in all respects, and that therefore angle ABC is equal to angle DEF and angle ACB is equal to angle DFE.

But the sides AB and DF are equal and the sides AC and DE are also equal: hence by Prop 4, it can be shown that angle ACB is equal to angle DEF and angle ABC is equal to angle DFE.

Hence angle ABC is equal to angle ACB.

Of/

Of course, this proof, like all similar short cuts, involves a hypothetical construction.

Apart from this one, the propositions are proved in general in the Euclidean manner, but the degree of condensation can be inferred from the fact that the 48 propositions of Book I cover only 25 pages, exclusive of three pages of diagrams.

Euclid's parallel axiom is quoted as a corollary to Prop. 17, which proves that any two angles of a triangle are together less than two right angles. But surely the axiom, being the converse of that proposition, can hardly be taken as a corollary on it.

There is little alteration on Euclid's Book II. Douglas tries to simplify some of the enunciations, but is hampered by his ignorance of the expression "to divide a line <u>externally</u>" with the result that his "simplification" is worse than the original. Here is his version of Prop. 6. "If a right line be divided into two equal parts, and <u>another line added to it</u>, the rectangle contained by the whole and the added line as one side of the rectangle and the added line for the other side together with the square of half the line, are equal to the square of the half and added line as one side of the square."

In his proofs here and throughout the book, Douglas sacrifices clearness for conciseness. For instance, he assumes certain figures to be parallelograms or squares, as the case may be, which should have been proved to be such.

His/

His definitions are apt to be vague; for instance, after defining the angle in a segment, he adds

> "But when the right lines containing the angle do receive any part of the circumference, then the angle is said to stand upon that circumference".

The word "circumference" means what we call an arc, but the word <u>receive</u> is surely used in a sense which is now archaic.

Space is conserved on every possible occasion, particularly by putting pairs of converse theorems together. Books IV and V are much shortened. Many references, most of them of a critical nature, are made to Simpson's Euclid. I think this must refer to the famous Robert Simson, rather than to Thomas Simpson, whom Robert Simson quotes. Douglas's chief complaint is that Simson has imagined difficulties to exist where they need not, and he quotes the one point on which Simson was particularly insistent, i.e. his definition of equal and similar solid figures, as those contained by an equal number of similar and equal planes, similarly situated. In his preface Douglas criticises this extra condition on the ground that Simson has not made out the necessity for it, but his criticism seems to be due to the fact that he misunderstood the point that Simson wishes to drive home, that you could have two solid figures, each made up of six different triangles, but with one of them what we might call a reentrant figure; and therefore not similarly situated.

The book is rendered unnecessarily difficult by reason of many misprints/

misprints.

Like the other authors, Douglas completes his book with a section on "Plain and Spherical Trigonometry"; in this he lays down all the usual theorems leading up to the solution of triangles, but strangely enough, he does not show how to solve the triangles themselves: this seems an extraordinary omission. He uses the capital letters S and T to stand for sine and tangent respectively. He gives, however, a note on the construction and use of logarithms and of the table of logarithmic sines and tangents.

The end of this book is taken up with tables:-

(1) of logarithms from 1 to 10,000;

(2) of Artificial Sines, Tangents and Secants; (this is a very complete table, one page for each degree and 60 minutes on each page up to $90^{\circ}$.)

(3) of Natural Sines for every 1' up to $90^{\circ}$;

(4) of Logarithmic Versed Sines;

(5) for the conversion of Sexagesimal fractions into ordinary decimals.

On the whole, we must write this book down as a failure: throughout the author has aimed at economising space to the greatest possible extent with the result that he has sacrificed clearness and logical completeness. This was done to provide room for examples and applications, but, apart from trigonometry, none are given. If he had conserved space by collecting only the more useful of Euclid's propositions, his book would have been more helpful/

helpful perhaps. Douglas is chiefly interesting as the pioneer among the workers for a simplified geometry, but he was not so daring in his flights from Euclid as some of his successors.

---

The next author, William Scott, Edinburgh was much more venturesome in his efforts to render Euclid popular. His book is called "Elements of Geometry in which all the Material Propositions in the first Six, Eleventh and Twelfth books of Euclid are demonstrated with conciseness and perspicuity", 1782, and he certainly succeeds in avoiding many of the difficulties of Euclid by a liberal use of axioms and postulates.

Thus Section I, which deals with the properties of triangles, contains three postulates additional to the usual:

(1) that any given angle may be bisected by a straight line.

(2) that from any given point a straight line may be drawn perpendicular to a given straight line.

(3) that a straight line may be drawn parallel to a given straight line through a point in its plane.

Two very useful axioms are these:-

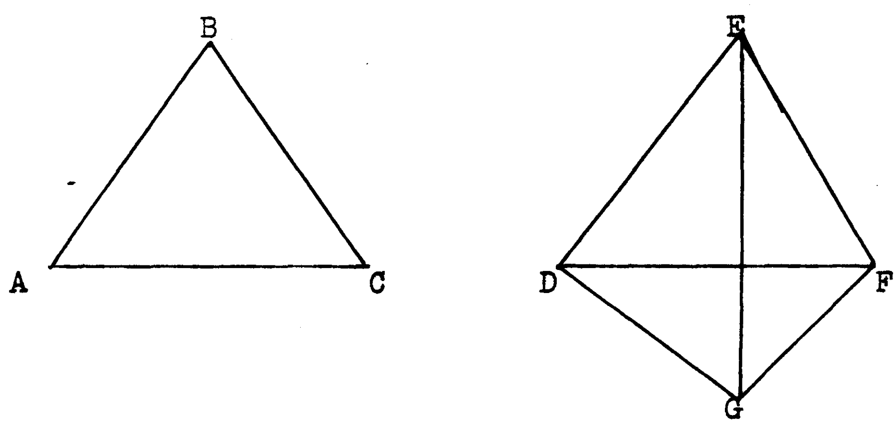
(1) Two different straight lines cannot be drawn from one point to another.

(2) Perpendiculars, let fall upon either of two parallels from the other, are equal.

With the first postulate mentioned, Pons Asinorum is proved easily as in modern text-books. With these two axioms, the parallel theorems/

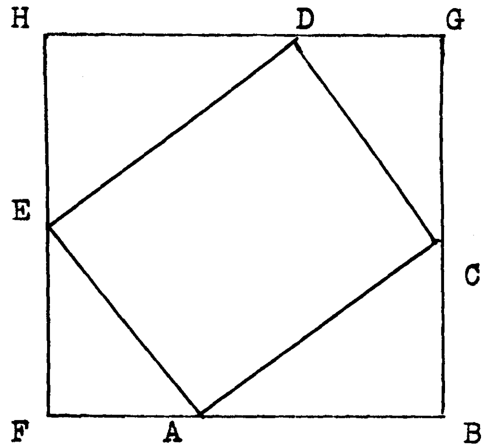
theorems, which come almost at the beginning of Section I, are easily proved. In a list of symbols given at the beginning, it is curious that the symbols for <u>greater than</u> and <u>less than</u> are the exact reverse of those we use.

The order of the theorems in his first section is interesting: he begins with the first congruency theorem, then we get the theorems on adjacent angles, the parallel theorems, the sum of the angles of a triangle and the third congruency theorem. After four inequality theorems, we get the "three-sides" congruency theorem. He gives two proofs for the last-mentioned, but the first one is not valid, as it depends for its proof on a theorem of which he has only proved the converse: his alternative proof is that given in modern books on geometry, to which a diagram will be sufficient reference:-




The second section contains theorems on parallelograms and includes one or two from the second book of Euclid. Two alternative proofs are given of Pythagoras, one the usual proof, the other as/

as follows:-




ABC is the right-angled triangle.

Produce BA to F making AF = CB and similarly

" BC " G " CG = AB

Draw FH perpendicular to FB.

Make FE = AB, EH = BC, HD = AB

BFHG is a square as all four sides are equal.

The four triangles are proved congruent and hence ACDE is a square.

Then FHGB = square on AB + BC

$= AB^2 + BC^2 + 2AB.BC$ (this theorem was proved previously)

But the sum of the four triangles = 2AB.BC

Hence $AB^2 + BC^2$ = FHGB - the four triangles

$= ACDE = AC^2$

In/

In Section IV he gets over most of the difficulties of Euclid, Book V by assuming the theorems as axioms, and the remaining propositions are easily proved from his first axiom that any two quantities of the same kind may be supposed to consist of parts which are all equal to one another. This axiom is made to do overtime in the three following sections in which he has collected the more important theorems of Books VI, XI and XII of Euclid. The common solids, prism, cylinder, pyramid, etc. are defined as formed by the movement of generating surfaces or lines. Here he uses a further axiom. "If any two solids, having the same altitude, have also at all equal altitudes their sections made by planes parallel to the bases equal, the solids themselves are equal." With the help of this and the previous axiom quoted, and by use of the theory of the generating surface, and in case of solids with circular surfaces, the use of infinitesimals, Scott is able to prove most of the theorems about solids that are included in his book with "conciseness and perspicuity". Throughout he confines himself to those theorems which are of obvious application in mensuration.

The remainder of the book is concerned with constructions of lines and figures as in Euclid. These constructions are not interspersed with the theorems, but are collected at the end of the book; the logical difficulties associated with the use of these constructions in the theorems, when they have not been proved, are/

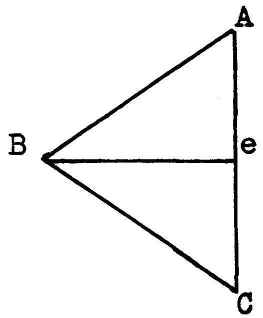
are got over by assuming the possibility of such constructions as an axiom. The whole book contains 146 pages and is of pocket size; the diagrams are small and are placed ~~with~~ twenty to thirty on one page. The purpose of the book is to supply the practical man who has no time for the finer logic of Euclid's Elements with a working knowledge of the most important theorems for use in many practical applications of geometry. Within these limits it doubtless served a useful purpose, though its weaknesses from the point of view of pure reasoning are glaring: as with Douglas's Euclid, condensation is carried too far in the proofs, which are thus rendered more difficult than need be for the beginner. Nevertheless, if geometry were to be made a utilitarian instead of a cultural subject, some such system as that contained in this book would be worthy of consideration. So far as I have seen it only reached one edition.

---

The third book to be considered here is the "Synopsis of Practical Mathematics" by Alexander Ewing, author of the "Institutes of Arithmetic" (to which reference has already been made) and of "Practical Astronomy". Ewing conducted a school in mathematics at Bishop's Land-Close, near Carruber's-Close, Edinburgh, where he had classes for "the soldier, sailor, engineer, surveyor and man of business."

The/

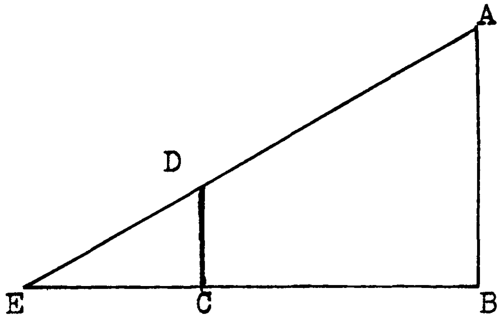
The first edition of the Synopsis of Practical Mathematics is dated 1771; I have also read the fourth edition, which is dated 1799. This is entirely a book of practical applications of geometry, and it follows the general lines of that by Gregory and McLaurin, but contains a good deal more than the latter. Ewing commences with a number of geometrical constructions (without proofs), and shows how to construct lines of chords, sines, tangents, secants and rhumbs (the last mentioned showing the thirty-two points of the compass); a short account of plane trigonometry up to the solution of triangles, and of the Gunter's scale leads up to the mensuration of heights and distances. These measurements are on the usual lines, except that the author claims originality for one method of measuring an angle by use of the chain only. The angles so measured are horizontal.




We wish to measure angle ABC. From B set off a certain distance (say 1 chain) along BA and the same distance along BC. Set up poles at A and C and measure AC. Then if Be is the perpendicular from B to AC, Ae is one half of AC and is therefore known. Then as he says BA : Ae :: Radius : sine ABe. Hence angle ABe is found, and its/

its double is the whole angle required.

A rather crude way of measuring a vertical angle is also suggested:




To measure angle AEB. Lie flat on the ground with the eye at E; get a pole CD and stick it in the ground in a vertical position, so that D and A are in line with the eye; measure EC and CD and get the angle AEB from the relationship

EC : CD :: Radius : tangent of AEB

The instruments used in the measurement are the mirror, two and four poles, quadrant and theodolite.

An interesting account is given of "levelling", i.e. of finding the difference of vertical level between two places. A spirit-level with sights and a telescope is used. Suppose we wish to find the difference of level between two places A and C; we set our level up at B between A and C. At A we erect a pole 10 or 12 feet high, divided into inches and fitted with a movable piece of pasteboard with a black line on it. This pasteboard is moved up and down until the sights on the level at B are aligned on it, and the/

the distance above the ground is then measured. The process is repeated with the pole at C, and the difference between the two distances gives the height of A above or below C. If B is more than 400 yards from either station, a correction has to be made for the difference between the true and apparent level. This is got from a table constructed as follows:-

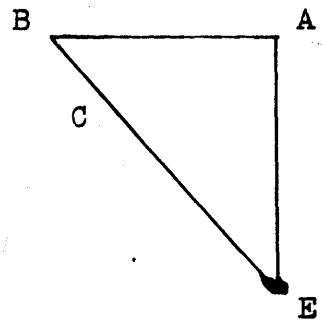



Diagram represents the earth, E its centre, B appears to be on same level at A, but it really differs by BC. He takes AE as 3982 miles and works out BE (by Pythagoras) for a given distance AB. He then gets BC, the difference between the apparent level and the true level for two points AB. This is worked out for various distances, and beyond 400 yards is not negligible.

A fairly complete account is given of the surveying of land but the use of the plane table is not mentioned. The mensuration of solids and all the practical rules for the gauging of casks of various shapes are treated fairly completely. Thereafter some space is devoted to Navigation, but strangely enough no account is given/

given of Spherical Trigonometry. He gives a short but very clear account of four different methods of sailing a ship

(1) by the plain chart where the parallels of latitude are all of the same length as the equator (fairly accurate within the torrid zone).

(2) Parallel sailing: here we find the distance between two meridians in any parallel of latitude from their distance on the equator by multiplying by the cosine of the latitude.

(3) Middle latitude sailing: here the distance between the two meridians in a latitude half-way between that sailed from and that reached is taken as the "departure" (i.e. the distance travelled east or west).

(4) Mercator's sailing.

While Ewing works out examples of each of these different methods, he tells the student that they can all be read from books of tables. He tells shortly how to find the latitude of a ship at sea, and how to measure the variation of the compass.

This completes the navigation course, a much shorter and simpler one than that outlined in John Wilson's Trigonometry.

A section is next devoted to Gunnery. The elements on which this science depends are the "Point of Projection", the Elevation, the Amplitude, Random or Range and the Impetus. The last mentioned is the greatest height to which the ball would travel if fired vertically with maximum charge, and it is assumed that the greatest range with elevation 45° is twice the impetus. Further, it is assumed that the greatest height reached when the elevation is 45° is one-fourth of the greatest amplitude or range. Several rules are/

are quoted (without proof) from the works of writers on projectiles, e.g.

> "The horizontal ranges of equal bodies projected with the same velocity at different elevations are to one another as the sines of twice the angles of elevation".

With these rules he works out various problems on elevation, range, time, etc. There is no reference to wind or air resistance beyond the statement that the latter is nearly proportional to the range.

The "Synopsis of Practical Geometry" is an excellent book of reference, well arranged, with the rules clearly stated and examples worked out in each case. It gives us an insight into the course of study in mathematics that was pursued in a business or professional academy at the end of the eighteenth century by young men who had previously completed their classical education in one of the Grammar Schools. Doubtless this practical course was preceded or accompanied by a course in Euclidean geometry. The book is not highly original (it shows the influence of McLaurin's Practical Geometry to be very marked), but it well merited the popularity it achieved.

The fourth edition (1799) is considerably more complete than the first; it contains a detailed account of the method of constructing tables of sines and tangents, not only by the old methods already mentioned, but from the series for the sine and cosine. In the solution of triangles the capital letters A, B and C are used/

used for the sides and the small letters a, b and c for the angles. P is used for one half of the sum of the sides. Finality had not yet been reached on these conventions, but Ewing uses s and s' for sine and cosine, t and t' for tangent and cotangent.

In this edition the use of the Geometrical Square is explained. (It is interesting to find from a reference that John Wilson's Trigonometry was still in use).

The section on the Mensuration of Solids is greatly extended to include all the regular solids. The section on Navigation is more than doubled; detailed examples are given of the finding of the latitude of a ship at sea and of the variation of the compass. At the end, too, a table is added showing "Difference of Latitude" and "Departure" for every ¼ degree of the compass. In every respect the fourth edition is enlarged and improved, particularly in the working out of examples.

---

In addition to books on pure geometry, there were doubtless a number dealing with applied geometry and trigonometry. There was, for instance, the "Elements of Navigation or the Practical Rules of the Art", written by William Wilson, M.A. and published at Edinburgh in 1773. This is purely a technical book including only the bare minimum of pure geometry and trigonometry; the proofs of the theorems were simplified as much as possible by a liberal use of the reductio ad/

ad absurdum.

Another book of Applied Mathematics was "The Art of Land-measuring" by John Gray, Teacher of Mathematics in Greenock and Land-measurer. This was published at Glasgow in 1757. It is a strictly practical book, assuming a knowledge of elementary mathematics. In it are the practical details of surveying, map-making, laying out land of given area in definite shapes, etc. In an appendix are full descriptions of the instruments used, in which connection it is interesting to note that the author preferred the chain for measuring angles to the theodolite.

---

Another text-book, which falls to be included here, is the "Elements of Trigonometry, plane and spherical, with the Principles of Perspective, and Projection of the Sphere" by John Wright, Edinburgh, 1772. In a note (on this book) which appeared in the Scots Magazine, February, 1772, it is recorded that the author is a teacher of Mathematics in Edinburgh, presumably in a Commercial College. The copy I have read is in the Andersonian Library, Glasgow. It differs from most of the books published at that time in being concerned exclusively with Trigonometry. It is a sound book without any startling originality; the author prefers to follow the usual plan, first defining the trigonometrical quantities, then showing how to construct tables. He describes all the instruments used in drawing figures to scale; considerable space/

space is devoted to the solution of right-angled triangles, while two pages suffice for other triangles. A feature of the book is the section devoted to perspective and the projection of the sphere, with applications to spherical trigonometry. Throughout the book the sides and angles of a triangle are named in full as in geometry.

Though this is an excellent treatise, written by an able mathematician, it seems to have failed to attain popularity, as there is no record of a subsequent edition; probably the price, 5s., for a book devoted to only one branch of Mathematics, was a handicap.

In the preface Wright makes scathing reference to a so-called geometrical construction for an ellipse, given in a "Synopsis of Practical Mathematics recently published". Though no name is mentioned, the author of the unsound theorem was Alexander Ewing, who was, of course, a business rival to John Wright.

---

Another excellent mathematical text-book was written by John West, assistant teacher of Mathematics in the University of St. Andrews, 1784. This is one of the volumes mentioned in an advertisement by William Creech, the Publisher, in 1791, and there recommended as suitable for the use of University students. The title is "Elements of Mathematics, comprehending Geometry, Conic Sections, Mensuration and Spherics, for the use of Schools". The copy I have seen is one, presented by the publisher to "John Anderson, Esq., Professor of Natural Philosophy in the University of Glasgow", and it/

it forms one of the Andersonian Collection.

As will be inferred from its length (417 pages) and from the title, it is comprehensive in scope. The geometry follows generally that of Euclid, but the author allows himself some latitude in the condensation of proofs for the sake of conciseness. This enables him to include four books on Conic Sections. The mensuration is on the usual lines, but the section dealing with the measurement of areas of plane figures is extended to include regular polygons and the conic sections. Mensuration od solids, likewise, includes the parabolic and elliptic conoids.

The geometry of the sphere, omitted from the ordinary Euclidean geometry, is included in the section on Spherics. Considerable symbolism is introduced in the spherical trigonometry, particularly in connection with Napier's Rules, e.g.

$$R \times \cos M = \cot A \times \cot a$$

$$R \times \cos M = S, O \times S, o$$

M is the middle part, A and a are the two parts adjacent to M. O and o are the two remaining parts of the right-angled triangle. S, means sine and T, means tan. The sides of the triangle are indicated by AB, BC, and CA.

The section on spherics includes the stereographic and orthographic projections of the sphere.

Altogether this text-book provides a very complete course in mathematics, and it was, I fancy, the inspiration of the mathematical courses in the Academies of Perth and Dundee at that time. I have no information with regard to any subsequent edition.

# CHAPTER XX.

## TRAILL'S "ELEMENTS OF ALGEBRA".

"The Elements of Algebra" by William Traill is written for the use of students in Scottish Universities. The copy I have perused is a third edition, dated 1789; the first edition appeared in 1778. The author seems to have been Professor of Mathematics in Aberdeen University.[X] In spite of the modesty of the foreword, in which it is stated that this book is not meant "to supersede the perusal of a more complete System of Algebra", this is a fairly useful book.

In the symbols and definitions it is quite modern. Though marred somewhat by many confusing misprints, it is written in a logical and convincing style, and is well printed. No examples are provided for the student, but, as is consistent with a series of notes of lectures, exercises are worked out neatly and concisely, each step being in logical sequence.

As regards its content, the book deals with equations up to and including the Cubic with literal equations which he denotes "Specious", Involution, Evolution, Surds and problems Determinate and Indeterminate.

There are also brief references to the solution of geometrical problems/

X See Appendix B.

problems by algebra, coordinate geometry and graphical work, meaning of Transcendental Curve, etc. These constitute the third part of the book, which is admittedly rather scrappy. Further odds and ends are put into six appendices. The most interesting of these is a discussion of the convention as to the signs of the trigonometrical functions when the angle varies from $0^\circ$ to $360^\circ$. Here it is interesting to record that these functions are treated as ratios, rather than as lines, the radius being taken as unity.

Some points of historical interest follow. The use of indices to express powers is usual, but occasionally these are still written out in full. The vinculum is still the only bracket though Traill says that some writers use parenthesis ( ) as a vinculum. The solution of simultaneous equations is still done by expressing one of the variables in terms of the other, in two different ways and equating the two expressions. The modern method of elimination by multiplication of the equations and subsequent subtraction is, however, given in the appendix to the book.

The general solution of the quadratic is still given in three parts corresponding to the three general equations:-

$$x^2 + ax = b^2 \qquad x^2 - ax = b^2 \qquad x^2 - ax = -b^2$$

It will be observed that in the above classification of quadratics the case $x^2 + ax = -b^2$ is omitted, presumably because the roots here must be negative.

A feature of his system of equations is the systematic way in which/

which he sets down an explanation of each of the steps in the solution.

In the discussion of Arithmetical and Geometric Series, he still uses the old notation for the greatest and least terms, irrespective of whether these are first or last in the series. In some applications of algebra to problems in Physics and Mechanics he has occasion to use the symbol for proportionality. This he does by writing the symbol of equality $=$ between the two quantities. He mentions that some writers, to avoid confusion, use the symbol $\propto$ , but that there is no need for confusion, as the two quantities connected by the sign could not be equal, since they are of different kinds.

Traill's book shows that ~~by~~ before the end of the eighteenth century algebra had attained a considerable place in the elementary mathematics course at ~~the~~ Aberdeen University.X That century saw a gradual revulsion from the extreme geometrical tradition in mathematics with at first a development of the practical applications of geometry, and later a tendency to soften the austerity of Euclidean methods by the introduction of algebra. Even at the end of the century, however, algebra was far from being a familiar tool. As throughout the history of mathematics, the tale of progress in the science is largely the tale of the development of a simplified notation. By the end of my period, the algebraic notation had been greatly simplified, and that of trigonometry was gradually reaching perfection/

X This book is advertised by William Creech as suitable for University students, Edinburgh Evening Courant, Oct. 31,1791.

perfection. Hence my account of the text-books has been, to some extent, taken up with a survey of the different methods used to get over the difficulties of an imperfect form of symbolism, and of the various inventions that have been tried and rejected or amended from time to time.

In the remainder of my paper, I will give an account of the development in the teaching of mathematics in Scottish schools in the second half of the eighteenth century.

# CHAPTER XXI.

## DEVELOPMENT OF MATHEMATICAL TEACHING IN SCOTTISH SCHOOLS 1750-1800.

In the parochial schools the second half of the eighteenth century witnessed a big advance in mathematical teaching. The position at the end of the century can be accurately gauged from the "Statistical Account of Scotland drawn up from the communications of the Ministers of the different Parishes" by Sir John Sinclair, Bart., published in Edinburgh,(1791). Though the account of the schools is by no means complete, it is clear that arithmetic was taught in nearly every school, whether under the control of the heritors, the S.P.C.K. or trustees.

Furthermore, it is remarkable to find how widespread was the teaching of the different branches of mathematics, more particularly on the applied side. There are references to the teaching of navigation in such relatively obscure places as Kirkmaiden in Wigtonshire, Careston in Angus, Lochgoilhead in Argyll, Oldhamstocks in Haddington, Auchterhouse in Angus, Benholme in Kincardine. Geometry, Mensuration, Surveying and Gauging formed part of the school course at Loudoun in Ayrshire, Scoonie in Fifeshire, Yarrow, Kirkwall/

Kirkwall, Dupplin and Aberdalgie in Perthshire. We even find Astronomy in the curriculum at Careston in Angus and at Callander. These subjects were taught both in day schools and in the evening classes.

When the parochial schoolmaster was capable of doing so, he took his pupils through a course sufficient to qualify them for entrance to the University. Thus we find the parish minister of Dalry in Kirkcudbright reporting that the endowed school there was formerly one of the most famous in the South of Scotland, but "learning is now so common that there is scarcely a parish schoolmaster of 10 in Scotland, who is not able to teach Latin and Greek with accompts and some practical parts of the mathematics; in fact everything necessary to prepare the young student for the University, as well as to qualify the man of business for acting his part well in any ordinary occupation". This is, of course, a considerable exaggeration, since in one parish it is reported that the only teacher available was a tradesman: nevertheless, it serves to indicate the extent of the teaching which the capable scholars could obtain.

Navigation appears in the curriculum of most of the schools in the villages on the coast: mensuration and surveying in that of the schools in the inland places. Euclidean geometry is occasionally mentioned among the subjects of study, but usually the practical side is predominant. These branches of mathematics were not taught/

taught to all the pupils, as extra fees were charged for special subjects. We are told, for example, that at Eastwood, Renfrewshire, there were 105 pupils of whom 36 were taught English alone, 23 Writing in addition; 18 more got Arithmetic, 4 Bookkeeping, 2 Mathematics, and 22 Latin. The fee for Mathematics was five shillings per quarter and for Latin four shillings, an indication that Classics no longer occupied the pedestal in the schools.

The development of the teaching of arithmetic and practical mathematics in the parochial schools was one of the features of the eighteenth century, but perhaps the more striking advance in mathematical teaching took place in the Grammar Schools. The middle of the century saw the launching of a new movement to found Academies in Scotland. Academies were established in France early in the seventeenth century, their object being to provide accomplishments suitable for the soldier, courtier, and man of the world. Their curriculum included Physical Science, Mathematics, Geography, Heraldry, and Modern History. The movement attained popularity also in Germany, where the new schools were styled Ritterakademien. Their curricula and social standing caused a transformation in German education and discredited the monopoly of the classical schools. In England the attempt to introduce the French Academy was a failure and such schools as were started were private ventures; Milton, for example, ran one such school. A few academies were established during the religious troubles of the later seventeenth century by professors/

professors driven out from the universities for Nonconformity. These schools had a modern curriculum.

In Scotland the word "Academy" was applied early in the eighteenth century to a number of commercial colleges, which were established for the training of men of business, who, after completing a course at the Grammar School, required some knowledge of commercial arithmetic, bookkeeping, practical geometry, geography, etc. These academies also catered for the needs of prospective officers in the Army and Navy, but they were supplementary to, and in no sense rivals of the grammar schools.

The first academy established in Scotland for the training of boys in modern subjects was that of Turreff in Aberdeenshire. A Mr. William Meston, who had been expelled from the professoriate at Marischal College on political grounds, set up an academy there in 1715 "to teach such sciences as were then taught in the Universities".[X] This institution did not survive for many years, as we are told in 1791 that it had long since been abandoned.[X]

Perth Academy, projected on 24th September, 1760, was definitely established as a rival to the Grammar School of the same city, and it provided an alternative course in which Latin did not figure at all. Full details of the scheme of instruction will be given in the next chapter; in the meantime, it will be sufficient to say that it was surprisingly complete on the side of the physical sciences/

X Statistical Account of Scotland - Vol 17, Sir John Sinclair.

sciences and mathematics. The next Academy, in chronological order, was that founded at Dundee in 1786. According to an advertisement in the Glasgow Mercury, August 26th, 1788, Dundee Academy provided eight courses of instruction:-

(1) French and Italian.
(2) Arithmetic and Bookkeeping.
(3) First class of mathematics, comprehending the Elements of Geometry, Plane Trigonometry, and Mensuration.
(4) Second class of mathematics, including the Elements of Spherics, Conic Sections, Algebra and Fluxions.
(5) Geography and Navigation.
(6) Chemistry and Natural History.
(7) Natural Philosophy.
(8) Perspective, Fortification and Architecture.

The whole of this encyclopaedic course was covered in three years, and "the following is the order of attending the classes reckoned most advantageous for the students":-

1st Year. French, Arithmetic, Bookkeeping and the first class of Mathematics.

2nd Year. Drawing, Perspective, Geography, and the second class of Mathematics.

3rd Year. Architecture, Fortification, Chemistry, Natural History and Natural Philosophy.

It is naively explained that it is "not necessary to take the whole course"

Three years later, as appears from an advertisement in the Edinburgh Evening Courant, September 15th, 1791, the course was substantially the same, except for the omission of Italian and Natural History, but it was "complete in two years". These schemes of instruction are modelled on those of the Commercial Academies/

Academies of Edinburgh and Glasgow. One must presume that a course in English and Latin had been overtaken previously.

At Inverness Academy, which was opened at Whitsunday, 1792, there was a five years' course:-

| | |
|---|---|
| 1st Class. | English Language. |
| 2nd " | Latin, Greek and French. |
| 3rd " | Writing, Arithmetic and Bookkeeping. |
| 4th " | Mathematics, including Elements of Euclid, Algebra, Mensuration and Surveying, Trigonometry, Navigation, Fortification, Gunnery, and Drawing. |
| 5th " | The Rector's Class, wherein are to be taught Natural and Experimental Philosophy and Astronomy. |

Here it is clear that the modern course is the superstructure, Latin the groundwork.

In Perth Academy, as in Dundee, the course in Mathematics and Science was covered in two years.[x1]

Academies were opened at Elgin and Fortrose in 1791. In the latter case the advertised course is less ambitious, but it includes Mathematics, Navigation, Geography, Perspective, etc.[x2]

Banff Academy was opened in 1794 or earlier; there were six masters, each apparently a specialist, one for Classics, Geography and Rhetoric, one for Writing, Arithmetic, Bookkeeping, Mathematics, Navigation and Church Music!, one for English, one for French/

x1 Edinburgh Evening Courant, September 2nd, 1790.

x2 " " " July 2nd, 1795.

French ("Le Chevalier de Villeblanche, a native of France and a man of education"), one for Drawing and one for Instrumental Music and Dancing. [x1] Apparently Banff provided a better balanced education than any of the other Academies of the period.

Ayr Academy was formally opened in 1796 and the prospectus showed the following syllabus of instruction:- Arithmetic, Book-keeping, Euclid, Trigonometry, Geography with the use of both globes, Practical Mathematics, Gunnery, Fortification, Natural Philosophy, Astronomy, Latin, Greek, Writing, Drawing, English, French, Italian, etc.[x2] Long before the Academy was built Ayr had provided a modern education, for, as early as 1746, John Mair taught Arithmetic, Bookkeeping, Algebra, Trigonometry, Euclid, Practical Mathematics, Navigation, Geography, Natural Philosophy and Astronomy.[x3]

Other schools, though not formally opened as Academies, extended their mathematical course in harmony with the new movement. Thus we are told that MacDougall, a teacher of Stirling High School, in 1791 broadened the course in Mathematics to include Surveying, Mensuration, Dialling, Gauging and Navigation.[x4]

The/

x1 Edinburgh Evening Courant, February 9th, 1795.

x2 Grant - Burgh Schools of Scotland, p. 120

x3 ~~Ayr~~ Academy and Burgh Schule p. 57.

x4 History of Stirling High School - A. F. Hutchison. p 161

The Grammar Schools of the University towns remained aloof from the modern movement until the nineteenth century. In Edinburgh and Glasgow instruction in Mathematics was provided by a number of flourishing Commercial Academies. Advertisements of these are frequent in the Edinburgh Evening Courant, Edinburgh Advertiser, Glasgow Mercury, etc. The High School of Edinburgh, doubtless out of deference to the wishes of parents, graciously permitted teachers from these commercial schools to conduct classes in Arithmetic, Bookkeeping and even Mathematics in the High School, but such classes had to be taken at most inconvenient hours and under unpleasant conditions. One teacher, "Mr. G.Paton", who conducted a Commercial Academy at No. 50, South Bridge, in 1795, had been in the habit of taking classes at the High School, but "he had no salary, nor enjoyed any advantage whatever from the appointment", so had decided to conduct the classes in his own academy in future, "as it is but a few paces from the High School, and the apartments are more comfortable, dry and airy than the room which is there allotted for the same purpose".[x1] There is sufficient evidence here of the fight mathematics had to wage, before it achieved a place in the Grammar School of Edinburgh. Some even of the provincial Grammar Schools maintained their classical course until the nineteenth century. Thus, in Kelso Grammar School in 1791, the subjects of instruction are Latin and Greek.[x2] In 1795 a schoolmaster is wanted for/

[x1] Edinburgh Evening Courant, September 19th, 1795.

[x2] " " "

for Cupar, who must be able to teach Latin, Greek and French.[X]

Ultimately the Academies themselves lost their identity and became merged again in the Grammar Schools, but the effect of the movement was to establish all round a more practical course of studies. Early in the nineteenth century the purely classical schools began to appoint special masters for mathematics, and modern subjects gradually attained their present position in the school curriculum. While, during the second half of the eighteenth century, the zeal of the protagonists for the modernisation of education somewhat outran their discretion, the ultimate result was the development of a satisfactory compromise between the classical and the modern side of the Secondary School.

---

X Edinburgh Evening Courant, October 15th, 1795.

# CHAPTER XXII.

## DEVELOPMENT OF THE "CONTENT" OF MATHEMATICAL TEACHING IN SCOTTISH SCHOOLS IN THE EIGHTEENTH CENTURY.

The teaching of arithmetic in the schoolroom in Scotland was really an innovation of the seventeenth century, and, till the end of that century, only a few schools taught that subject at all. What evidence do the text-books afford us as regards the parts of arithmetic that were taught? The conclusion to be drawn from a study of such books is that, by the end of the seventeenth century, the course in arithmetic in the provincial grammar schools included numeration and the four rules applied to integers and compound quantities. Multiplication and division were carried through by the use of Napier's Rods. The capable pupils were taught the rule of three, vulgar fractions, some applications of the rule of three and possibly square root and cube root of integers.

The beginning of the eighteenth century witnessed a considerable advance in the teaching of arithmetic. In attempting to sketch the curriculum from a consideration of the text-books I will put down the maximum course that would have been covered by any school at that time/

time. During the first quarter of the eighteenth century, the modern methods of multiplication and division displaced the use of Napier's Rods, and, in particular, the Italian method of division was popularised by Cocker's text-book. Finite decimals came into general use. The solution of problems involving two or more proportions was evolved under the title "Double Rule of Three" (when the two were treated separately) and "Rule of Five, Seven, Nine, etc." (when the sum was treated as a compound proportion). The rules of Single Fellowship, Double Fellowship, and Alligation (medial or alternate) provided useful exercises for the pupils who were mathematically inclined. A separate section was devoted to the application of vulgar fractions in proportion. Practice became very popular, but perhaps the most curious of all the exercises was that known as <u>single and double position</u>, where questions, easily soluble by algebra, were tackled by a kind of proportionate trial-and-error method. With the development of international trade, Exchange became a matter of considerable importance. Lastly, commercial questions of Loss and Gain, Barter, allowances for Tare and Tret, and the famous "Equation of Payments" may have been included though usually these would be left to the commercial academies.

In the second quarter of the eighteenth century the chief feature was the development of the teaching of Infinite Decimals. Experiments were made with a view to simplifying the rule of three, both/

both simple and compound, and, in particular, to bringing direct and inverse proportion under one rule. Decimals attained great popularity involving their users in some monumental calculations. The estimation of areas and volumes by duo-decimal multiplication was a favourite exercise, though the replacement of the duo-decimal by the decimal was suggested by the more enlightened writers. The application of algebra to the solution of arithmetical problems was still unpopular among teachers.

In the second half of the century the curriculum in mathematics was further extended in the parochial schools as well as in the academies and provincial grammar schools. The course outlined in Halbert's "Practical Figurer" was evidently intended as a model for the country schoolmaster at the end of the eighteenth century. It covers the arithmetic of integers, vulgar and decimal fractions, rule of three, fellowship, practice, alligation, allowances for tare and tret, gain and loss, barter, commission, insurance, exchange, rule of false, extraction of square root and cube root, progressions, interest, present worth and discount and annuities. Furthermore, algebra up to the solution of quadratic and cubic equations was included. Again, the more important propositions of the first six books of Euclid were explained, probably without complete proofs. Thereafter the pupils in country schools undertook a course in mensuration, while those in seaside villages and small coast towns studied the elements of navigation.

The course in mensuration would vary largely in different schools according to the instruments available: it included measurement of length and surveying of fields by dividing them into triangles. The rules were given for the calculation of the areas of triangles, quadrilaterals, regular polygons, the circle with its sectors and segments, the ellipse, parabola and even the cycloid. The areas of the surfaces of cylinder, cone and sphere were also taught. Applications of this part of mensuration were found in the work of the carpenter, bricklayer, plasterer, joiner, painter and glazier. The mensuration of solids included that of the prism, pyramid, cylinder, cone, sphere, spheroid, parabolic conoid, and parabolic spindle. Applications were found here in the work of the mason and timber merchant and in gauging of casks.

The course in navigation included an account of the various imaginary circles on terrestrial and celestial spheres, meaning of longitude, latitude, declination, amplitude, azimuth, right and oblique ascension. The points of the compass were explained and the methods of plain sailing, middle latitude sailing and possibly the use of Mercator's Chart. The construction of dials and the use of the two globes were also taught.

---

Hitherto in outlining the "Content" of the mathematical teaching in the schools of Scotland I have had to rely, for evidence, on the text-books which have been preserved, but, in treating of the mathematical curricula of the academies (to which I now proceed), I am fortunate in having had access to note-books written by pupils of Perth Academy towards the end of the eighteenth century. The first note-book contains actual notes of the lectures delivered by Robert Hamilton, the Rector of Perth Academy, whose text-books have been mentioned previously. The note-book was the work of "Patrick Mitchell, Perth", and it is dated October 18th, 1777. The lectures are on Arithmetic, Gunter's Scale, Practical Geometry, Land Surveying, Geography, Navigation, Trigonometry and Natural Philosophy. The written work is so exhaustive and detailed that one is forced to the conclusion that the notes had to serve as the pupil's text-book.

The arithmetic covers the four rules in integers, fractions and compound quantities, simple and compound proportion, distributive proportion, practice, commission and simple interest. Throughout, the work in arithmetic is closely correlated with bookkeeping and great care is taken to achieve neatness and accuracy. Some curious features are that the symbol  is used to represent the figure <u>4</u> and that the digit <u>1</u> at the beginning of a number is set down thus  The examples are eminently practical in nature.

The use /

The use of Gunter's Scale in multiplication, division, rule of three and trigonometry is next explained.

The section dealing with practical geometry describes the measurement of heights and distances, accessible and inaccessible, by means of the geometrical square, the quadrant, two staffs of different sizes, the graphometer and the theodolite. Actual measurements are carried out in most cases and the calculations are done on paper, using logarithms. One of the distances measured is that between "Perth Hospital and Pithevelis". The calculations involve the solution by Trigonometry of right-angled and oblique-angled triangles; this must have been taught previously.

In Land Surveying we are first given a table showing the number of links to be subtracted from each chain, in the case of an ascending line, to reduce it to the horizontal. The surveying is done by dividing up the land into triangles either from a point in the perimeter or from one or more points inside the field, and instruction is first given in the trigonometrical solution of the area of a triangle. A practical survey of the lands of "Rosemount" and "Burnside" is made.

The course in geography consists of two parts. The first of these is merely a list of all the principal physical features of the different countries in the world. The second is headed General Geography; it represents an introduction to the study of navigation. A detailed account is given of the various imaginary points and circles/

circles on the terrestrial and celestial spheres, and some problems are worked out by the use of the two globes. With this introduction the study of navigation commences. Here he lays down the five quantities that rule the problem:-

I. Difference of Latitude.
II. Difference of Longitude.
III. Easting or Westing, or Departure.
IV. Course.
V. Distance.

The method of measuring latitude is explained and, on the assumption of Plain Sailing, or Middle-Latitude Sailing, the problem reduces to the solution of plane right-angled triangles. The use of Mercator's Chart and the Tables of Meridional Parts is then expounded.

The last section of the mathematical part of the course seems to have been misplaced. It gives an account of plane trigonometry up to the solution of triangles, and must obviously have been overtaken before some of the sections which precede it.

Part II of the notes is concerned with Natural Philosophy:- Newton's Laws and other parts of Mechanics, Astronomy, Precession of the Equinoxes, Tides, Densities of the Planets, a perpetual Almanac, Hydrostatics, Pneumatics, Sound, Light, Magnetism and Electricity, with a little very crude Chemistry. This section consists largely of a statement of facts and theories without reference to experiments.

The note-book does not contain a complete course in Mathematics, for it is certain that the first six books of Euclid in some form/

form or other must have been covered before the course in mensuration and surveying; moreover, the complete absence of algebra is surprising.

---

A more complete set of note-books, which I found in the Glasgow University Mathematical Library, provides a full account of the curriculum in Mathematics at Perth Academy at the very end of the eighteenth century. In this very interesting series of books, written in a beautiful copper-plate, with neat and careful diagrams, there is little external evidence of the time and place of writing. Only one note-book bears a name and date; it is signed at the end under a sketch of a feather-quill "Jacobus Balearis, 1801". Internal evidence leaves us in no doubt that all the books were written about the same date. That they are the work of a pupil of Perth Academy is proved by the repeated reference to Perth, e.g. in Astronomy to the "horizon of Perth". Furthermore, there is a note on page 52 of the Astronomy note-book :-

> "Problem 21. To explain the phenomena of the Peroeci, Antoeci and Antipodes.
> Mr. Gibson gave us thus far in this problem and nothing more".

Now I have ascertained from Mr. Edward Smart, B.A., B.Sc., late Rector of Perth Academy that "Alexander Gibson, who had been a teacher of Mathematics in Dunbar, was appointed Rector of Perth Academy in 1779. His tenure as Rector and teacher of Mathematics and Science lasted for thirty years - he retired in 1809."

Hence/

Hence there can be no doubt that all four note-books, which bear the same handwriting, were the class work of a pupil of Perth Academy at the end of the eighteenth century.

The first of them deals with arithmetic. The subjects treated are as follows:-

Addition (simple and compound including avoirdupois weight, troy weight, apothecaries' weight, sterling money, dry measures, length, area and time); Subtraction; Multiplication (simple and compound); Division; Reduction; Proportion (simple, compound and distributive); Practice; Deduction of Weights; Commission; Simple Interest; Vulgar Fractions; Finite Decimal Fractions; Square and Cube Root; Exchange (with an account of foreign coin-ages).

The second book contains Geometry and Trigonometry; in the former the writer makes a collection of the most important problems and theorems of the first six books of Euclid. In the problems he states the construction; in the theorems he draws the figure and states the particular enunciation; in neither case does he give any proof. All the theorems quoted are those with practical appli-cations. In Trigonometry he sets down the definitions of the func-tions and then proceeds directly to the solution of triangles. Logarithms are used and the result is verified in each case by drawing the triangle to scale.

The third book deals with Astronomy and the first section of it is/

is devoted to the stereographic projection of the sphere. A number of projections which are of importance in geography are carried out. The second section deals with the solution of spherical triangles, using Napier's Analogies. The third section deals with Astronomy proper: measurement and calculation of the declination, ascension, latitude, longitude, amplitude and azimuth of the sun and the stars, refraction and parallax. It also gives a comparison of the Copernican, Ptolemaic and Tychonic systems, and space is devoted to the study of eclipses of the sun and moon. The fourth section deals with Nautical Astronomy and is confined to the determination of the latitude and longitude of a ship at sea and of the variations of the compass.

The fourth note-book contains a very considerable course in Algebra. It begins with the evaluation of algebraic expressions, and the four rules addition, subtraction, multiplication and division, surds being included in the examples. Next we have involution up to the fifth power leading to a statement of the Binomial Theorem. Extraction of the roots of algebraic expressions is carried out as far as the sixth root. There is a short note on the infinite series which result from the evaluation of such an expression as $\frac{a\ x}{a + x}$ or from the binomial expansion of $(a^2 + x^2)^{\frac{1}{2}}$. Equations, simple and quadratic, in one or more variables are treated by the methods that have been indicated when dealing with text-books on algebra. The determination of the nature/

nature of the roots in the case of equations of higher degrees from a consideration of the coefficients of the different terms is dealt with by means of particular examples. The methods of increasing and diminishing the roots of an equation by a given amount, and of eliminating the second term in an equation by a suitable substitution are explained. Equations are solved by the method of approximation and many examples of problems involving equations are given; these include indeterminate problems with a limited number of solutions and literal equations. Next are treated the construction and use of a table of logarithms by means of convergent series. This is followed by arithmetical and geometric progressions with summation of series by the method of differences, combinations and permutations, simple and compound interest, annuities and the "doctrine of chances". Some space is devoted to the solution of geometrical problems by means of algebra, and there is a short reference to coordinate geometry, the equations of the circle and the conic sections being given. The algebra course concludes with a description of the differential and integral calculi under the names of "fluxions" and the "indirect method of fluxions" respectively.

From a consideration of these four note-books it is clear that a very considerable course was overtaken in Perth Academy at the end of the eighteenth century. Moreover, instruction was given in mensuration, surveying, gauging and navigation, as outlined in the/

the earlier book. It seems that the mathematical curriculum in the Academy of that epoch was more extensive than that of the average secondary school of the present day; owing to the fact that the teaching of languages was left to the rival Grammar School time was available for the completion of so considerable a course in mathematics and science.

## CHAPTER XXIII.

## CONCLUSION.

In summing up this paper, I should like to set down some of the general impressions conveyed by an extended reading of the text-books on elementary mathematics in use in Scotland prior to the year 1800.

In the first place, one is conscious of the preponderating influence, in the development of mathematics, exerted over many centuries by the astronomy of the ancients. The mathematics of the Middle Ages was dominated by the Ptolemaic theory of the planets as expounded by Sacrobosco; sexagesimal arithmetic delayed the introduction of the decimal, and made multiplication and division tasks of herculean proportions; even at the present day the sexagesimal system still lives. It is strange to reflect that our forefathers, who had recourse to devices such as Napier's Rods and the Rotula when confronted by a multiplication sum that would not unduly tax a pupil in the upper classes of the elementary school, yet possessed a familiarity with the motions of the heavenly bodies unknown to any but the best of the pupils of our secondary schools.

Another/

Another impression got from my reading has been that of wonder at the amazing fertility of genius exhibited by the great Lord Napier.

Again, one cannot but be struck by the extraordinary importance of the discovery of suitable symbols and contractions in the history of mathematical text-books. The discovery of the Arabic system of numeration, the unfortunate delay in adopting the decimal point, and the notation of indices are trite examples. The difficulties associated with the invention of a suitable notation for the representation of surds, and of recurring decimals are further illustrations of the same point.

Another impression conveyed by the text-books is that of the clear, utilitarian bent of the mathematics taught in Scotland, particularly in the eighteenth century. In the teaching of arithmetic great stress was laid on the cast-iron rules for the solution of problems; every endeavour was made to make the methods foolproof, but there was no demand for an explanation of the theoretical basis of the subject. Any efforts at improvement in textbooks were mainly centred on the simplification of the rules, particularly the Rule of Three. Moreover, the examples usually showed a strong commercial bias, and arithmetic and book-keeping were often twin subjects in the curriculum. That the result was of more value than the process is demonstrated by the popularity of ready reckoners in the commercial circles of those days.

In/

In geometry, too, the practical applications to mensuration, navigation, etc. were deemed of more importance than the cultural value of the pure geometry of Euclid, so far at least as the schools and commercial colleges were concerned. It seems as if the teaching of Petrus Ramus had influenced the schools of Scotland for nearly two centuries, and even the great work of Robert Simson failed to elucidate all the difficulties traditionally associated with Euclidean geometry.

Yet another impression conveyed by the books on arithmetic is of the extraordinary breadth of that subject in the eighteenth century. Emerging from the shackles of the ancient geometry, it still retained some ancient geometrical lumber, and at the other end it contained much that has subsequently been relegated to the algebra text-book.

Lastly, it would not be fair to conclude without paying a tribute to the enlightenment of many of the mathematical teachers of Scotland, whose works have come down to our times; to the far-sightedness of such men as Malcolm, Mair and Hamilton, and to the respect which they inspired in the populace was due the great advance in mathematical teaching in our country during and after their lifetime.

# APPENDIX.

## A.

Robert Hamilton became Professor of Natural Philosophy in Marischal College on the 18th of June, 1779, but in the following year he exchanged duties with Patrick Copland, Professor of Mathematics in the same College. From 1780 to 1817 he acted as Professor of Mathematics, though nominally holding the Chair of Natural Philosophy. In 1817 the formal exchange of Chairs was made and Hamilton acted as Professor of Mathematics till his death on 14th July, 1829.

---

## B.

William Traill, LL.D. was appointed Professor of Mathematics in Marischal College on October 29th, 1766. He was a son of William Traill, Minister of St. Monans; he graduated M.A. at Glasgow in 1763 and LL.D. of Marischal College in 1774; he resigned on 6th April, 1779, obtaining preferment in the Church of Ireland. He died on 3rd February, 1831. In addition to the "Elements of Algebra" which was first published at Aberdeen in 1776, he wrote a "Life of Robert Simson".

---

# APPENDIX.

## C.

## LIST OF OTHER TEXT-BOOKS TO WHICH REFERENCES ARE MADE IN EIGHTEENTH CENTURY PERIODICALS.

(1) "An Epitome of Arithmetic in which the whole art of numbers is explained and demonstrated from a few self-evident Principles or Axioms taken from Numbers only: the Progress of the Operations are generally much shorter than in any other Book of Arithmetic, though at the same time fully as plain if not more so. With an Appendix containing general Rules and Examples for measuring Planes and Solids: as also general rules for Gauging or Dry Gauging Casks and all manner of Utensils, with the most expeditious method of measuring Square and Round Timber, etc. by the Sliding Rule, with several other useful Things not to be found in any other Book of Arithmetic". by John Scot, Excise Officer in Biggar. Advertisement in Edinburgh Evening Courant, June 25 and 28, 1753.

(2) "Hill's Arithmetic". Advertised in Scots Magazine, April, 1760 and July, 1763. (This is possibly the well-known English school-book.)

(3) "The Man of Business and Gentleman's Assistant, containing a treatise of Practical Arithmetic including vulgar and decimal fractions, in which are inserted many concise and valuable rules for the ready casting up of merchandise, never yet published in this Kingdom. Bookkeeping, by Single and Double Entry, together with an essay on English Grammar adapted to the use of gentlemen, merchants, traders and schools". by W. Perry, Master of the Academy at Kelso (Scots Magazine, September, 1774.

(4/

(4) "A complete treatise on practical mathematics, including the nature and use of mathematical instruments. With an appendix on algebra. Principally designed for the use of schools and academies", by John MacGregor, Teacher of Mathematics, Edinburgh (Scots Magazine, December, 1792.

(5) "Complete System of Theoretical and Practical Arithmetic and Bookkeeping" by John Robertson, Edinburgh (Edinburgh Evening Courant, September 10, 1791.)

(6) "Theory of Interest, simple and compound, derived from first principles and applied to Annuities of all descriptions" by the Rev. Mr. David Wilkie, Minister at Cults (Scots Magazine, May, 1794).

# B I B L I O G R A P H Y


| | | |
|---|---|---|
| Abelson, Paul | "The Seven Liberal Arts" | New York, 1906. |
| Adamson, John W. | "Pioneers of Modern Education 1600-1700". | Cambridge, 1905. |
| Anderson, Robert | "History of Robert Gordon's Hospital". | Aberdeen, 1896. |
| Bain, Alexander | "Rectorial Address to the Students of Aberdeen University". | Aberdeen, 1882. |
| Ball, W. W. R. | "A Short History of Mathematics" | London, 1888. |
| Bossut, John | "History of Mathematics to 1750" | London, 1803. |
| Boyd, William | "History of Western Education" | London, 1921. |
| Brown, Robert | "History of the Paisley Grammar School". | Paisley, 1875. |
| Cajori, Florian | "History of Mathematics" | New York, 1919. |
| Chalmers, George | "Life of Ruddiman" | London, 1794. |
| Chapman, George | "Treatise on Education" | London, circa 1790. |
| Cleland, James | "History of the High School of Glasgow" | Glasgow, 1878. |
| Commission Report | "Burgh and Middle Class Schools." | Edinburgh, 1868. |
| Coutts, James | "History of Glasgow University" | Glasgow, 1909. |

Dalzel/

| | | |
|---|---|---|
| Dalzel, Andrew | "History of Edinburgh University". | Edinburgh, 1862. |
| De Morgan, Augustus | "Arithmetical Books" | London, 1847 |
| Graham, Henry G. | "The Social Life of Scotland in the Eighteenth Century" | London, 1899. |
| Grant, Sir Alex. | "History of Edinburgh University" | London, 1884. |
| Grant, James | "History of the Burgh Schools of Scotland". | Edinburgh, 1876. |
| Halliwell, James | "Rara Arithmetica". | London, 1839. |
| Heath, Thomas L. | "The Thirteen Books of Euclid's Elements". | Cambridge, 1908. |
| Hutchison, A. F. | "History of the High School of Stirling". | Stirling, 1904. |
| Insh, George P. | "School Life in Old Scotland". | Edinburgh, 1925. |
| Irving, David | "Life of George Buchanan". | Edinburgh, 1807. |
| Jackson, Lambert L. | "The Educational Significance of Sixteenth Century Arithmetic". | New York, 1906. |
| Jamieson, John | "An Historical Account of the Ancient Culdees of Iona." | Edinburgh, 1811. |
| Kerr, John | "Scottish Education". | Cambridge, 1913. |
| Leslie, John | "The Philosophy of Arithmetic". | Edinburgh, 1820. |
| McCrie, Thomas | "Life of Andrew Melville". | Edinburgh, 1819. |
| — | "Munimenta Alme Universitatis Glasguensis". | Glasgow, 1854. |
| Murray, David | "Chapters in the History of Bookkeeping". | Glasgow, 1930. |


Napier/

| | | |
|---|---|---|
| Napier, Mark | "Memoirs of John Napier". | Edinburgh, 1834. |
| Patrick, David | "Air Academy Burgh Schule" | Ayr, 1895. |
| Peterkin, Alexander | "Life and Writings of Burns" | New York, 1824. |
| Poole, Reginald L. | "The Exchequer in the Twelfth Century". | Oxford, 1912. |
| Potts, Robert | "Elementary Arithmetic (historical)". | London, 1876. |
| Rait, Robert S. | "Some Notes on the History of University Education in Scotland" communicated to Glasgow Archaological Society, December, 1904. | |
| Simpson, H. F. M. | "History of Aberdeen Grammar School". | Aberdeen, 1906. |
| Sinclair, Sir John | "Statistical Account of Scotland". | Edinburgh, 1791. |
| Smith, David E. | "Rara Arithmetica". | Boston, 1908. |
| Steele, Robert | "The Earliest Arithmetics in English". | Oxford, 1922. |
| Steven, Dr. | "History of the High School of Edinburgh". | Edinburgh, 1849. |
| | "History of Heriot's Hospital". | Edinburgh, 1859. |
| Stewart, David and Minto, Walter | "Life, Writings and Inventions of John Napier of Merchistor." | Perth, 1787. |
| Stewart, John | "Records of the Monastery of Kinloss". | Edinburgh, 1872. |
| Strong, John | "History of Secondary Education in Scotland". | Oxford, 1909. |


Trail/

Trail, Rev. Wm. "Life and Writings of Robert Simson". Edinburgh, 1812.

Veitch, J. "Philosophy in the Scottish Universities" Volume II of the First Series of "Mind". Edinburgh, 1877.

Archaeologia Scotica, Volume II, Edinburgh, 1822.

Anonymous "An Account of the Society in Scotland for Propagating Christian Knowledge". Edinburgh, 1714.

# SCOTTISH PERIODICAL LITERATURE

## of the

## EIGHTEENTH CENTURY

The Scots Magazine.

The Edinburgh Magazine and Review.

The Edinburgh Evening Courant.

The Edinburgh Advertiser.

The Caledonian Mercury.

The Glasgow Courier.

The Glasgow Mercury.