Some Problems on the Osculating Conic at a Point on a Plane Curve.

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Notation.

ρ = radius of curvature.

s = arc of a given curve measured up to a given point (x, y).

$\rho' = \frac{d\rho}{ds}$, $\rho'' = \frac{d^2\rho}{ds^2}$.

ψ = inclination of tangent to x-axis.

α = inclination of axis of aberrancy to tangent.

P = normal chord of osculating conic.

R = axis of aberrancy.

R.H. = rectangular hyperbola.

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Section I.

Introduction.

(1) **Osculating Conic.**

The general equation of a conic contains five disposable constants, and therefore a conic can be found to pass through five given points; and the solution is unique unless four or five of the given points are collinear. It follows that at an ordinary point on a given curve S it is possible to construct a conic which has five consecutive points in common with the curve at the point.

This conic is the Conic of Closest Contact or the Osculating Conic at the point.

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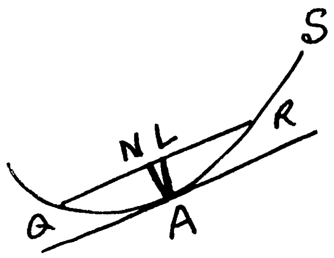
(2) **Aberrancy of Curvature.**

The semi-diameter of the osculating conic at a given point $A(x, y)$ on S does not in general coincide with the normal at A. The angle between the normal and the semi-diameter is taken to measure the deviation from the circular form at A.

This deviation is called by Salmon the Aberrancy of the Curvature at A, and the semidiameter of the osculating conic is called the Axis of Aberrancy (Salmon, Higher Plane Curves, para. 407).

Thus if QR is a chord close to A and parallel to the tangent at A, its mid-point L will lie on the axis of aberrancy, but in general the normal at A will cut QR at another point N. The angle NAL is then the angle of aberrancy.



It is shown by Salmon that if δ be the measure of this angle, then

$$\tan\delta = p - \frac{(1+p^2)r}{3q^2},$$ where p, q and r

are the first, second and third derivatives of y with respect to x (H.P.C., para. 404).

It is shown in Section III (10) that if ρ is the radius of curvature at A, and α the angle between the tangent at A and the axis of aberrancy, then $\rho' = 3\cot\alpha$ $(\rho' \equiv \frac{d\rho}{ds})$.
Hence

$$\tan\delta = \cot\alpha = \frac{1}{3}\frac{d\rho}{ds},$$ and since

$$\rho = \frac{(1+p^2)^{3/2}}{q}, \quad \frac{ds}{dx} = (1+p^2)^{\frac{1}{2}},$$ the result

stated follows.

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(3) Equation of Osculating Conic.

Methods for finding the equation of the osculating conic at a point on a cubic and at a point of a curve of any degree are given by Salmon (H.P.C., paras. 236-238, and para. 409).

The method of finding the equation of the osculating conic at a point of a curve whose coordinates are given in terms of a parameter "t" is given by Hilton (Plane Algebraic Curves, X, 2, example 15).

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(4) Sextactic Contact.

The curvature at A may be such that the osculating conic has contact of the fifth order at A. The contact is then said to be sextactic.

There are thirty-six points on the general cubic where the contact is of this order (Salmon, para. 155, and Hilton, chap. XVI). The condition to be satisfied by the coordinates of a point on a curve in order that the osculating conic may have sextactic contact has been investigated by Cayley (Phil. Trans., 1865, page 545). This condition is mentioned by Salmon (H.P.C., para. 410), who also states the number of sextactic points of a given curve (para. 420).

The method of finding the parameters of the sextactic points of a curve whose coordinates are given in terms of a parameter "t" is given in Hilton (Plane Algebraic Curves, X, 2, example 14).

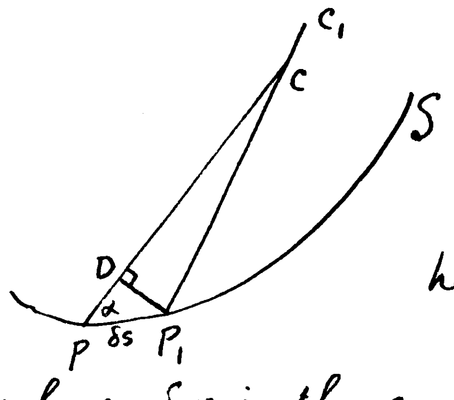
(5) Edwards in his "Elementary Treatise on the Differential Calculus," Chap. X, paras. 354 and 356 gives the expressions for $\frac{d\rho}{ds}$ in terms of the angle of aberrancy, and for the axis of aberrancy in terms of ρ and the ~~axis~~ angle of aberrancy. The latter expression is equivalent to the expression for the axis of aberrancy given in Section III (11c), viz. $R = \dfrac{\rho \sin\alpha}{1 + \frac{d\alpha}{d\psi}}$, where α is the complement of the angle of aberrancy.

(6) Also in Chapter XVII, para. 507, ex. 4, Edwards gives the differential equation for the general conic in the form $9y_2^2 y_5 - 45 y_2 y_3 y_4 + 40 y_3^3 = 0$, an equation due to Monge, and which is given in Section III (18), as the condition for sextactic contact of curves of the type $y = f(x)$.

Edwards draws attention to an interpretation of this result by Mr. A. Mukhopadhyay (Journal of the Asiatic Society of Bengal, vol. LVIII, Part 1), viz. that the expression for the radius of curvature of the locus of centres of aberrancy contains the left-hand member of Monge's equation. A point of a curve whose coordinates satisfy Monge's equation is a sextactic point and therefore corresponds to a point of zero curvature on the locus of centres of aberrancy. This condition is given in Section III, para 16, in the form $\dfrac{dR}{ds} + \cos\alpha = 0$.

That the expression for the radius of curvature of the locus of centres of aberrancy contains $\dfrac{dR}{ds} + \cos\alpha$ as a factor is easily seen.

Using the notation of the figure of Section III (16), let $PC = R$, $P_1C_1 = R + \delta R$, arc $PP_1 = \delta s$, $\alpha = \angle CPP_1$, $CD \perp CP$.



Then $C_1P_1 = C_1C + CP_1$,
$= C_1C + CD$.
But $CD = R - PD = R - \delta s.\cos\alpha$
$\therefore C_1C = \delta R + \delta s.\cos\alpha$,
hence $\rho_{LOCUS} = \mathcal{L}\dfrac{C_1C}{\delta\varphi} = \left(\dfrac{dR}{ds} + \cos\alpha\right)\dfrac{ds}{d\varphi}$,
where $\delta\varphi$ is the angle PCP_1, and since $\delta\varphi = \dfrac{\delta s.\sin\alpha}{R}$; this gives $\rho_{LOCUS} = R\operatorname{cosec}\alpha\left(\dfrac{dR}{ds} + \cos\alpha\right)$.

Other References.

Morley, Frank: The Contact Conics of the Plane Quintic Curve.

White, H.S.: Plane Curves of the Third Order.

Section II.
Contact conics at a point on a plane curve.

(1) *Circle of curvature.*

If A, B, C, are three points on a plane curve S, it is possible to construct a circle to pass through A, B, C. When B and C move to coincidence with A, the construction is still possible, and the circle so obtained is called the Circle of Curvature at A on S. Its radius (ρ) is called the Radius of Curvature, and its centre the Centre of Curvature at A. This circle is said to have three-point contact at A with the given curve.

If the coordinates of A referred to a set of rectangular axes (ξ, η) be (x, y), and the direction-cosines of the tangent at A be (l, m), then the centre of curvature is $(x - m\rho, y + l\rho)$, and the equation of the circle of curvature is $(\xi - x + m\rho)^2 + (\eta - y - l\rho)^2 = \rho^2$. The centre of curvature lies on the normal at A, $\dfrac{\xi - x}{m} + \dfrac{\eta - y}{l} = 0$, at a distance ρ from (x, y).

The locus of the centres of curvature is a second curve called the Evolute of S.

If we find the envelope of the normal we obtain the locus of the centres of curvature. Hence the evolute may be regarded also as the envelope of the normals.

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(2) *Conics of Four-point contact.*

If A, B, C, D, are four points of S, it is not in general possible to construct a circle to pass through A, B, C, D. A certain condition must be fulfilled, viz., the coordinates of D must be such that they satisfy the equation of the circle through A, B, C.

(3) Similarly, if B, C, D, move to coincidence with A, it is not in general possible to have a circle which has four-point contact with S at A. To find the condition that the circle of curvature may have four-point contact at (x, y) on S, let us take s to measure the arc of S from some fixed point up to A, and let us assume that x, y, l, m, ρ are functions of s. We will make use of the formulae

$$x' = \frac{dx}{ds} = l,\ y' = m,\ l^2+m^2=1,\ l' = -\frac{m}{\rho},\ m' = \frac{l}{\rho}.$$

The equation of any circle through (x, y) may be written

$$F(\xi, \eta) \equiv (\xi - x)^2 + (\eta - y)^2 + 2g(\xi - x) + 2f(\eta - y) = 0,$$

where x, y, f, g are functions of s.

The conditions for four-point contact of this circle with **S** at (x, y) are

$$F(x,y) = 0,\ F'(x,y) = 0,\ F''(x,y) = 0,\ F'''(x,y) = 0.$$

$F(x,y) = 0$, identically.

$$F'(\xi,\eta) \equiv 2(\xi - x)\xi' + 2(\eta - y)\eta' + 2g\xi' + 2f\eta'.$$

$$F''(\xi,\eta) \equiv 2(\xi - x)\xi'' + 2(\eta - y)\eta'' + 2g\xi'' + 2f\eta'' + 2\xi'^2 + 2\eta'^2.$$

$$F'''(\xi,\eta) \equiv 2(\xi - x + g)\xi''' + 2(\eta - y + f)\eta''' + 6(\xi'\xi'' + \eta'\eta'').$$

Substituting $\xi' = l,\ \xi'' = -\frac{m}{\rho},\ \xi''' = \frac{m\rho'}{\rho^2} - \frac{l}{\rho^2},$

$$\eta' = m,\ \eta'' = \frac{l}{\rho},\ \eta''' = -\frac{l\rho'}{\rho^2} - \frac{m}{\rho^2},$$

and applying the above conditions, we find

$$gl + fm = 0$$

$$-mg + fl + \rho = 0$$

$$g(m\rho' - l) + f(-l\rho' - m) = 0$$

The first two conditions give $f = -l\rho$, $g = m\rho$, and the third then gives $\rho' = 0$. Hence, at a point of S where $\rho' = 0$, the circle of curvature has four-point contact with S. Under certain conditions the contact may be of a higher order.

4) But without the necessity for any condition as to the curvature at A, it is still possible to find a conic to have four-point contact with S at A. The general equation of a conic has five disposable constants, and therefore a conic can be found to satisfy five conditions, each involving one relation between the constants. Any conic of four-point contact at A fulfils four conditions, and therefore if one more is given, the conic is completely determined. The remaining condition may be that for a parabola, or that for a rectangular hyperbola or for a conic of given eccentricity, or for a conic whose centre is on a given line, &c.

5) The equation of a conic of four-point contact at (x, y) on S may be found in the same way as that for the circle of four-point contact.

The equation is of the form

$$F(\xi, \eta) \equiv a(\xi - x)^2 + 2h(\xi - x)(\eta - y) + b(\eta - y)^2 + 2g(\xi - x) + 2f(\eta - y) = 0,$$

and the required conditions are

$$F(x, y) = 0, \quad F'(x, y) = 0, \quad F''(x, y) = 0, \quad F'''(x, y) = 0.$$

$F(x, y) = 0$, identically.

$$F'(x, y) = 2a(\xi - x)l + 2h(\xi - x)m + 2h(\eta - y)l + 2b(\eta - y)m + 2gl + 2fm, \text{ with } \xi = x,\ \eta = y.$$

$$F''(x, y) = 2a(\xi - x)l' + 2al^2 + 2h(\xi - x)m' + 4hlm + 2h(\eta - y)l' + 2b(\eta - y)m' + 2bm^2 + 2gl' + 2fm', \text{ with } \xi = x,\ \eta = y.$$

$$F'''(x, y) = 2a(\xi - x)l'' + 6all' + 2h(\xi - x)m'' + 6hlm' + 6hl'm + 2h(\eta - y)l'' + 2b(\eta - y)m'' + 6bmm' + 2gl'' + 2fm'', \text{ with } \xi = x,\ \eta = y.$$

Substituting $\xi = x$, $\eta = y$, $l' = -\dfrac{m}{\rho}$, $m' = \dfrac{l}{\rho}$,

$$l'' = \frac{m\rho' - l}{\rho^2}, \quad m'' = -\frac{l\rho' + m}{\rho^2},$$

we obtain the conditions

$$gl + fm = 0,$$
$$fl - gm + \rho(al^2 + 2hlm + bm^2) = 0,$$
$$3\rho\{h(l^2-m^2) + (b-a)lm\} + \rho'(mg - fl) - (gl+fm) = 0,$$
$$\text{i.e. } 3\{h(l^2-m^2) + (b-a)lm\} + \rho'(al^2 + 2hlm + bm^2) = 0.$$

Since we require only the ratios $a : h : b : f : g$, we may take $f = -l\rho$, $g = m\rho$. We may then solve for a and b the equations

$$al^2 + bm^2 + (2hlm - 1) = 0,$$
$$a(\rho' l^2 - 3lm) + b(\rho' m^2 + 3lm) + h(2\rho' lm + 3l^2 - 3m^2) = 0.$$

These give

$$a = \frac{\rho' m + 3l - 3hm}{3l}, \quad b = \frac{-\rho' l + 3m - 3hl}{3m}.$$

These, with $f = -l\rho$, $g = m\rho$, give the equation of the general conic of four-point contact at (x, y). This equation contains one disposable constant h.

(6) The centre $(\bar{\xi}, \bar{\eta})$ of this conic is given by

$$\bar{\xi} - x = \frac{hf - bg}{ab - h^2} = \frac{3\rho lm(\rho' l - 3m)}{(\rho' m + 3l)(-\rho' l + 3m) - 9h},$$

$$\bar{\eta} - y = \frac{gh - af}{ab - h^2} = \frac{3\rho lm(\rho' m + 3l)}{(\rho' m + 3l)(-\rho' l + 3m) - 9h}.$$

If we eliminate h between these equations we obtain the locus of the centres of all conics having four-point contact with S at (x, y). The result is $\dfrac{\xi - x}{\rho' l - 3m} = \dfrac{\eta - y}{\rho' m + 3l}$, a straight line through (x, y).

(7) <u>Parabola of Four-point contact.</u>

The condition that the conic of four-point contact may be a parabola is $ab - h^2 = 0$, i.e. $\qquad 9h = (\rho' m + 3l)(-\rho' l + 3m)$.

In that case $a = \frac{1}{9}(3l + \rho' m)^2$; $b = \frac{1}{9}(3m - \rho' l)^2$;

and the equation of the parabola of four-point contact is

$$(3l+\rho'm)^2(\xi-x)^2+2(3l+\rho'm)(3m-\rho'l)(\xi-x)(\eta-y)$$
$$+(3m-\rho'l)^2(\eta-y)^2+18\rho\{m(\xi-x)-l(\eta-y)\}=0.$$

(8) Rectangular Hyperbola (R.H.) of Four-point contact.

The condition that the conic of four-point contact at (x,y) may be a R.H. is $a+b=0$, i.e. $3h=6lm-\rho'(l^2-m^2)$. In that case

$$a=\tfrac{1}{3}(3l^2-3m^2+2\rho'lm),\ b=-\tfrac{1}{3}(3l^2-3m^2+2\rho'lm),$$

and the equation of the R.H. of four-point contact at (x,y) is

$$\{3(l^2-m^2)+2\rho'lm\}\{(\xi-x)^2-(\eta-y)^2\}$$
$$+2\{6lm-\rho'(l^2-m^2)\}(\xi-x)(\eta-y)+6\rho\{m(\xi-x)-l(\eta-y)\}=$$

(9) Conic of Four-point contact of Given Eccentricity.

If we wish to find the conic of eccentricity e which has four-point contact with S at (x,y) we use the formula

$$\frac{(a-b)^2+4h^2}{ab-h^2}+\frac{e^4}{e^2-1}=0,\ \text{and}$$

substituting for a and b we find h in terms of e

(10) If we wish to find the conic of four-point contact whose centre lies on the straight line $p\xi+q\eta+r=0$, we substitute for ξ and η the coordinates of the centre $\bar{\xi}$ and $\bar{\eta}$, and find

$$p\left\{x+\frac{3\rho lm(\rho'l-3m)}{(\rho'm+3l)(-\rho'l+3m)-9h}\right\}$$
$$+q\left\{y+\frac{3\rho lm(\rho'm+3l)}{(\rho'm+3l)(-\rho'l+3m)-9h}\right\}+r=0,$$

from which h can be found in terms of p,q,r, and of functions of s. The result is

$$9h = \frac{3\rho lm\{p(\rho' l - 3m) + q(\rho' m + 3l)\}}{px + qy + r} + (\rho' m + 3l)(-\rho' l + 3m)$$

From this we can find a and b also in terms of the same quantities, and substituting for a, h, b, f, g, in the equation of the general conic of four-point contact we find the equation of the required conic.

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(11) <u>Loci connected with the Parabola of four-point contact at (x, y).</u>

The general equation of a parabola through (x, y) may be written

$$\alpha^2(\xi - x)^2 + 2\alpha\beta(\xi - x)(\eta - y) + \beta^2(\eta - y)^2 + 2g(\xi - x) + 2f(\eta - y) = 0.$$

The equation of the axis is

$$\alpha(\xi - x) + \beta(\eta - y) = -\frac{\alpha g + \beta f}{\alpha^2 + \beta^2}.$$

That for the tangent at the vertex is

$$\beta(\xi - x) - \alpha(\eta - y) = -\frac{(\alpha g + \beta f)^2}{2(\alpha^2 + \beta^2)(\alpha f - \beta g)}.$$

That for the directrix is

$$\beta(\xi - x) - \alpha(\eta - y) = -\frac{(f^2 + g^2)}{2(\alpha f - \beta g)}.$$

The coordinates of the focus are

$$\xi = x - \frac{\beta(f^2 - g^2) + 2\alpha fg}{2(\alpha^2 + \beta^2)(\alpha f - \beta g)},$$

$$\eta = y - \frac{\alpha(f^2 - g^2) - 2\beta fg}{2(\alpha^2 + \beta^2)(\alpha f - \beta g)}$$

The coordinates of the vertex are

$$\xi = x + \frac{\alpha^2 g(\beta g - 2\alpha f) - \beta f^2(\beta^2 + 2\alpha^2)}{2(\alpha^2 + \beta^2)^2(\alpha f - \beta g)},$$

$$\eta = y + \frac{\alpha g^2(\alpha^2 + 2\beta^2) + \beta^2 f(2\beta g - \alpha f)}{2(\alpha^2 + \beta^2)^2(\alpha f - \beta g)}.$$

(12) Substituting $\alpha = \rho' m + 3l$, $\beta = -\rho' l + 3m$, $g = 9\rho m$, $f = -9\rho l$, we obtain

the corresponding results for the parabola of four-point contact at (x, y).

(13) The equation of the axis is

$$(3l + \rho' m)(\xi - x) + (3m - \rho' l)(\eta - y) + \frac{9\rho\rho'}{9 + \rho'^2} = 0,$$

which is of course parallel to the line obtained as the locus of centres of all conics having four-point contact with S at (x, y).

To find the envelope of the axes of all parabolas of four-point contact of the curve S, we must regard the constants in this equation as functions of the arc s. We differentiate the equation with regard to s, and solving the two equations for $\xi - x$ and $\eta - y$, we obtain the coordinates of a point on the envelope in terms of functions of s.

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(14) The equation of the tangent at the vertex is

$$(3m - \rho' l)(\xi - x) - (3l + \rho' m)(\eta - y) = \frac{3\rho\rho'^2}{2(9 + \rho'^2)},$$

and the envelope of this line can be found in the same way.

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(15) The equation of the directrix is

$$(3m - \rho' l)(\xi - x) - (3l + \rho' m)(\eta - y) = \frac{3}{2}\rho,$$

from which also can be found the envelope of the directrix. The equation obtained on differentiating with regard to s is $(\xi - x)(3l - l\rho\rho'' + m\rho') + (\eta - y)(3m - m\rho\rho'' - l\rho') = \frac{1}{2}\rho\rho'$. On solving these equations we find

$\xi = x + \frac{1}{2} m\rho$, $\eta = y - \frac{1}{2} l\rho$, a point on the normal whose distance from (x, y) is equal to half the corresponding radius of curvature, and on the side remote from the centre of curvature.

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(16) The coordinates of the focus are

$$\xi = x - \frac{3\rho(l\rho' + 3m)}{2(9+\rho'^2)}, \quad \eta = y - \frac{3\rho(m\rho' - 3l)}{2(9+\rho'^2)}.$$

The elimination of s between these equations will give the equation of the locus of the foci of all parabolas of four-point contact of S.

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(17) The coordinates of the vertex are

$$\xi = x - \frac{3\rho\rho'(18l + 3m\rho' + l\rho'^2)}{2(9+\rho'^2)^2},$$

$$\eta = y - \frac{3\rho\rho'(18m - 3l\rho' + m\rho'^2)}{2(9+\rho'^2)^2}.$$

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(18) <u>Results for the Rectangular Hyperbola of four-point contact at (x, y).</u>

The centre is $\bar{\xi} = x + \dfrac{hf - bg}{ab - h^2} = x - \dfrac{3\rho(l\rho' - 3m)}{9+\rho'^2}$,

$$\bar{\eta} = y + \frac{gh - af}{ab - h^2} = y - \frac{3\rho(m\rho' + 3l)}{9+\rho'^2}.$$

The length of the semi-diameter through (x, y) is $\dfrac{3\rho}{\sqrt{9+\rho'^2}}$, and the equation of the semidiameter is $\dfrac{\xi - x}{l\rho' - 3m} = \dfrac{\eta - y}{m\rho' + 3l}$.

(19) The envelope of the semi-diameter is found by solving the equations

$$(m\rho' + 3l)(\xi - x) - (l\rho' - 3m)(\eta - y) = 0,$$

and $(l\rho' + m\rho\rho'' - 3m)(\xi - x) + (m\rho' - l\rho\rho'' + 3l)(\eta - y) = 3\rho$.

These give $\dfrac{\xi - x}{3\rho(l\rho' - 3m)} = \dfrac{\eta - y}{3\rho(m\rho' + 3l)} = \dfrac{1}{9 + \rho'^2 - 3\rho\rho''}$

i.e. $\xi = x + \dfrac{3\rho(l\rho' - 3m)}{9 + \rho'^2 - 3\rho\rho''}$,

$$\eta = y + \frac{3\rho(m\rho' + 3l)}{9 + \rho'^2 - 3\rho\rho''}.$$

(20) The normal at (x, y) being $\dfrac{\xi - x}{m} + \dfrac{\eta - y}{l} = 0$,

we see that the centre of the R.H. of four-point contact at (x, y) lies on the normal provided $\dfrac{l\rho' - 3m}{m} + \dfrac{m\rho' + 3l}{l} = 0$, i.e. $\rho' = 0$.

(21) In general, the centre of the conic of four-point contact does not lie on the normal. If δ be the acute angle between the normal and the semi-diameter through (x, y), then

$$\cos\delta = -\frac{m(l\rho' - 3m) - l(m\rho' + 3l)}{\sqrt{9 + \rho'^2}}$$

$$= \frac{3}{\sqrt{9 + \rho'^2}}, \text{ or } \tan\delta = \frac{\rho'}{3}.$$

This deviation of the semi-diameter through (x, y) from the normal to S at (x, y) is called Aberrancy, and the angle between these two lines is called the angle of aberrancy.

(22) Axis of Aberrancy.

We see that the locus of the centres of the R.H.s of four-point contact of S is not the same as the envelope of their semi-diameters through their points of contact with S. In the case of the circles of curvature of S we know that the locus of the centres of curvature coincides with the envelope of the radii of curvature, this locus being the Evolute of S. Corresponding then to the evolute we have two loci associated with the R.H.s of four-point contact.

We have seen that the system of conics which have four-point contact with S at (x, y) have a common diameter passing through (x, y). The envelope of this diameter must then be the same for all, and the point on this envelope whose coordinates have already been found must be the centre of a particular conic of the system, since these coordinates satisfy the equation of the common diameter. This conic will be shown later to be the conic of five-point contact with S at (x, y).

Thus at every point of S it is possible to construct a conic the locus of whose centre coincides with the envelope of its semi-diameter through (x, y).

Thus of the two loci associated with the R.H.s of four-point contact it is more natural to regard the envelope of the semi-diameter through (x, y) as corresponding to the evolute of the curve S.

This common semi-diameter is called the Axis of Aberrancy at (x, y), and the centre of the conic of five-point contact just mentioned is called the Centre of Aberrancy. The distance of (x, y) from the centre of aberrancy is taken to measure the axis of aberrancy.

(23) The conic of five-point contact at (x, y) is called the Conic of Closest Contact or the Osculating Conic at (x, y).

Conics of Five-point Contact.

(24) If A, B, C, D, E are five points on S, it is not in general possible to construct through these a circle,

or a parabola, or a rectangular hyperbola, or a conic of given eccentricity. In the case of a circle, two conditions are now required, and in the case of any of the others one condition is required.

But without the necessity for imposing any conditions, it is still in general possible to construct a conic through the five points.

If we assume that B, C, D, E, move to coincidence with A then a conic can be constructed so as to have five-point contact with S at A. Its nature will be determined by the curvature at A. Two conditions as to the curvature at A will ensure that this conic is a circle, one condition that it will be a parabola, or a rectangular hyperbola, or a conic of given eccentricity, &c. If we find the general form of the equation of the conic of five-point contact at (x, y), these conditions can easily be determined. This will be done in the following section.

Applications.

(25) We will apply the results obtained for the parabola and rectangular hyperbola of four-point contact to the curve $ay^2 = x^3$, the coordinates of any point of which may be written in terms of a parameter "t"; $x = at^2$; $y = at^3$.

In this case $l = 2(4+9t^2)^{-\frac{1}{2}}$, $m = 3t(4+9t^2)^{-\frac{1}{2}}$;

$\rho = \frac{at}{6}(4+9t^2)^{3/2}$, $\rho' = \frac{2}{3t}(1+9t^2)$, $\rho'' = \frac{2(9t^2-1)}{3at^3}(4+9t^2)^{-\frac{1}{2}}$;

$$9 + \rho'^2 - 3\rho\rho'' = \frac{1}{9t^2}(4+9t^2)^2,$$

$$l\rho' - 3m = \frac{1}{3t}(4+9t^2)^{\frac{1}{2}}; \quad m\rho' + 3l = 2(4+9t^2)^{\frac{1}{2}},$$

$$\frac{3\rho}{9+\rho'^2 - 3\rho\rho''} = \frac{9}{2}at^3(4+9t^2)^{-\frac{1}{2}}.$$

For the parabola of four-point contact at "t".

The equation of the axis is

$$2(\xi - x) - \frac{1}{3t}(\eta - y) + \frac{9at^2(1+9t^2)}{1+36t^2} = 0,$$

i.e. $$2\xi - \frac{1}{3t}\eta + \frac{at^2(22+63t^2)}{3\ (1+36t^2)} = 0.$$

The tangent at the vertex is

$$-\frac{1}{3t}(\xi - x) - 2(\eta - y) = \frac{at(1+9t^2)^2}{1+36t^2},$$

i.e. $$\frac{\xi}{3t} + 2\eta + \frac{at\ (2+12t^2+27t^4)}{3\ (1+36t^2)} = 0.$$

The directrix is

$$-\frac{1}{3t}(\xi - x) - 2(\eta - y) = \frac{at}{4}(4+9t^2),$$

i.e. $$\frac{\xi}{3t} + 2\eta + \frac{1}{12}at(8+3t^2) = 0.$$

The envelope of the directrix is found by solving this and its derivative with respect to t, i.e. from $\xi + 6t\eta + \frac{1}{4}at^2(8+3t^2) = 0$,

and $6\eta + 4at + 3at^3 = 0$.

The result is $\xi = \frac{1}{4}at^2(8+9t^2)$,

$\eta = -\frac{at}{6}(4+3t^2)$, and the equation is

$$4a\xi(3\xi - 4a)^2 = 9\eta^2(729\eta^2 - 216a\xi + 32a^2).$$

The coordinates of the focus are

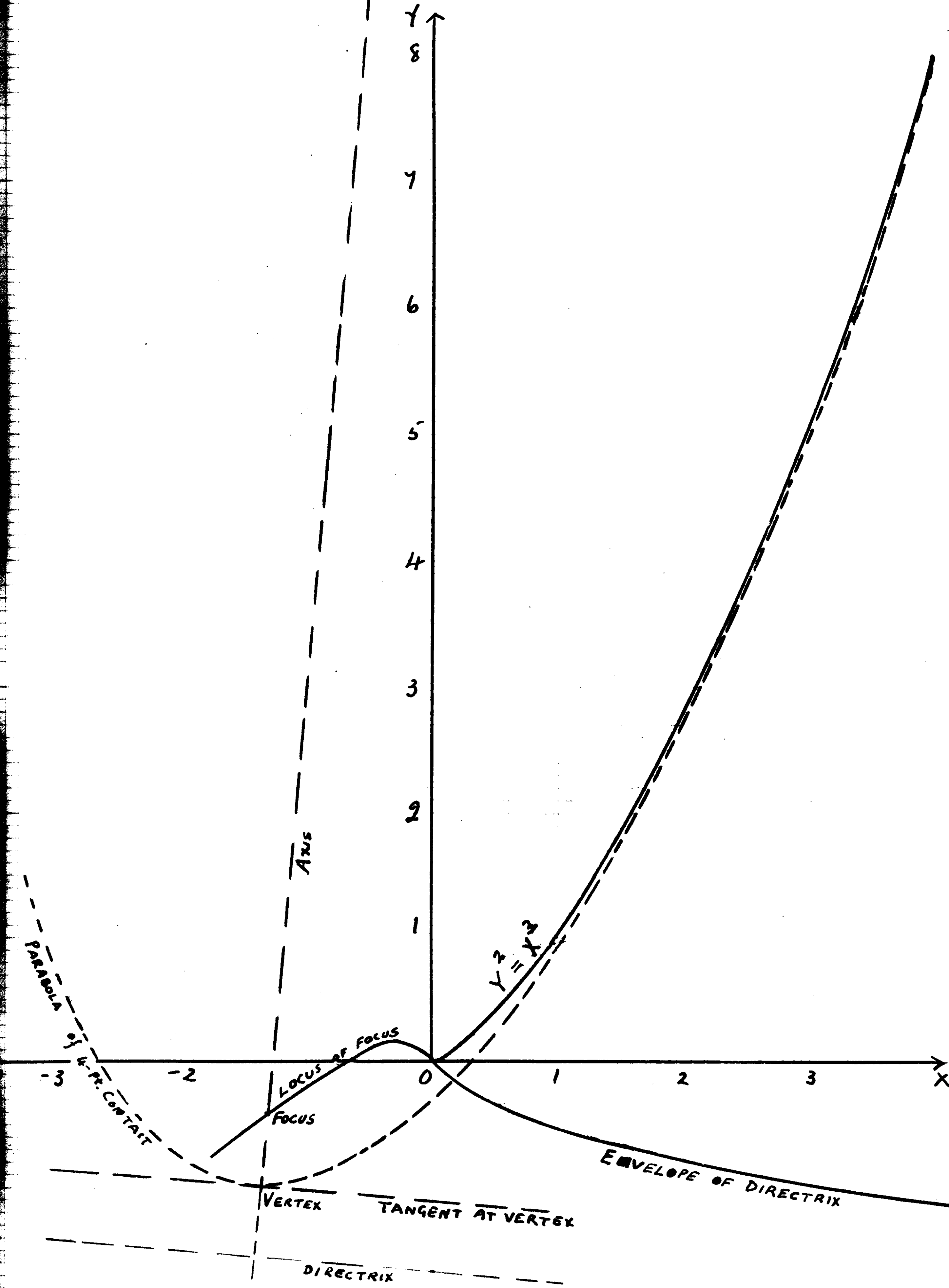
$$\xi = -\frac{at^2(8+45t^2)}{4(1+36t^2)},$$

$$\eta = \frac{at^3(20-9t^2)}{2(1+36t^2)}.$$

The equation of the parabola of closest contact at "t" is

$$4\xi^2 - \frac{4}{3t}\xi\eta + \frac{\eta^2}{9t^2} + \frac{7}{3}at^2\xi - \frac{44}{9}at\eta - \frac{2}{9}a^2t^4 = 0,$$ an

equation of the sixth degree in t, of which the term in t^5 is missing. Hence through any point


Y
8
7
6
5
4
3
2
1
-3
-2
0
1
2
3
X
AXIS
$Y^2 = X^3$
LOCUS OF FOCUS
FOCUS
PARABOLA of 4-PT. CONTACT
VERTEX
TANGENT AT VERTEX
DIRECTRIX
ENVELOPE OF DIRECTRIX

there pass six of the parabolas of four-point contact. Their parameters are such that $t_1+t_2+t_3+t_4+t_5+t_6=0$, and the "$t$"s are found by solving the equation of the parabola for t, ξ and η being given coordinates.

The coordinates of the vertex are

$$\xi = -\frac{at^2(2+144t^2+405t^4)}{(1+36t^2)^2},$$

$$\eta = -\frac{at^3(9t^2-2)(18t^2+5)}{(1+36t^2)^2}.$$

(26) For the rectangular hyperbola of four-point contact at "t" we have the following results.

Coordinates of the centre

$$\bar{\xi}-x = -\frac{3\rho(l\rho'-3m)}{9+\rho'^2} = -\frac{3}{2}\,at^2\frac{(4+9t^2)}{1+36t^2},$$

$$\bar{\eta}-y = -\frac{3\rho(m\rho'+3l)}{9+\rho'^2} = -\frac{9at^3(4+9t^2)}{1+36t^2};$$

$$\text{i.e. } \bar{\xi} = \frac{5}{2}\,at^2\frac{(9t^2-2)}{36t^2+1}, \quad \bar{\eta} = -\frac{5at^3(9t^2+4)}{36t^2+1}.$$

The length of the semi-diameter through (x, y) is $\dfrac{3}{2}\dfrac{at^2(4+9t^2)}{\sqrt{1+36t^2}}$, and the equation of the semi-diameter is $\dfrac{\xi-x}{\frac{1}{3t}} = \dfrac{\eta-y}{2}$,

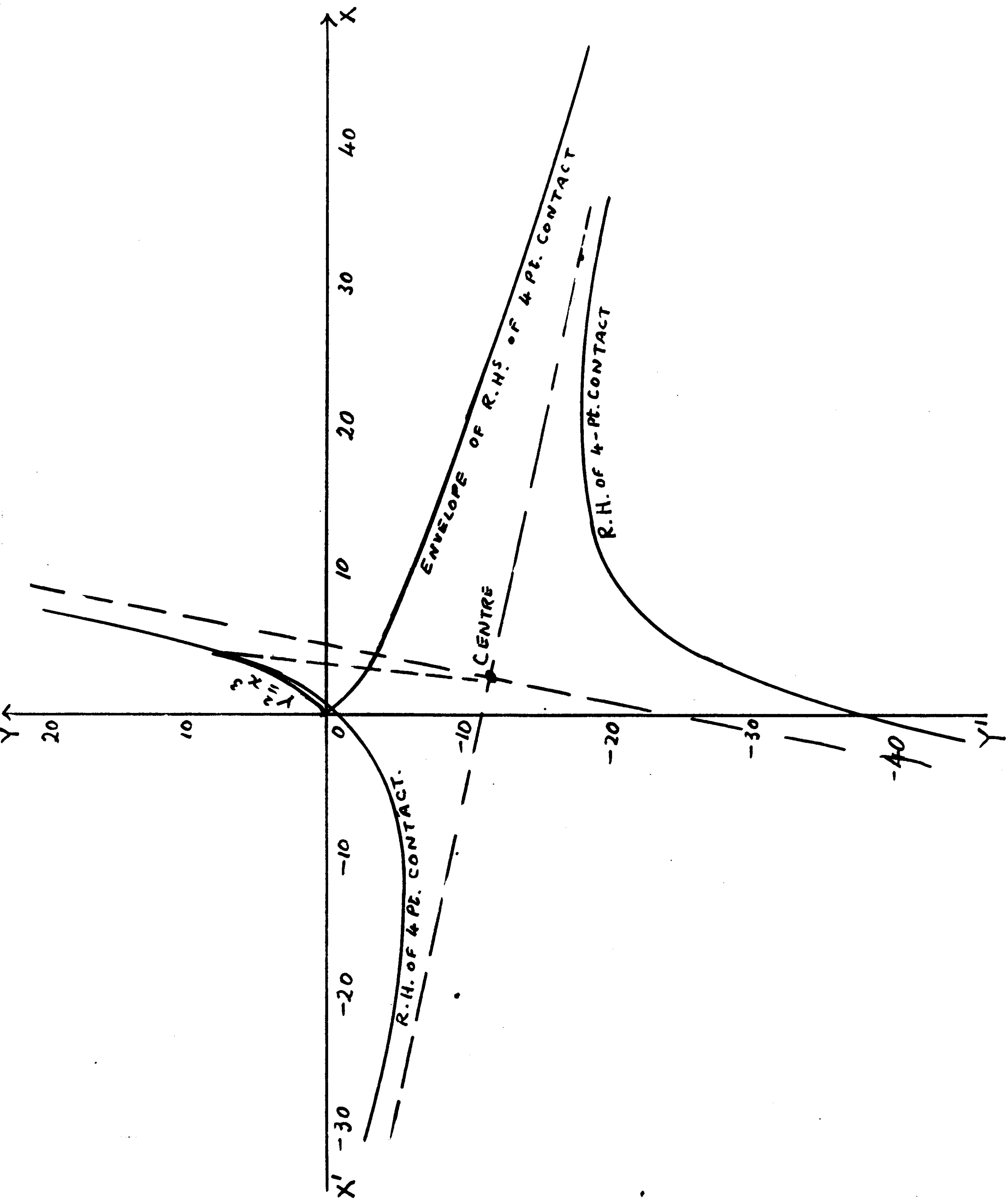
$$\text{i.e. } 6t\xi-\eta = 5at^3.$$

From this equation and from

$$6\xi = 15at^2,$$

we can find the envelope of the semi-diameter through "t". The result is $\xi = \frac{5}{2}at^2$, $\eta = 10at^3$, and the equation of the envelope is

$$32\xi^3 = 5a\eta^2,$$ another semi-cubical parabola.


X
X'
Y
Y'
ENVELOPE OF R.HS. OF 4 Pt. CONTACT
R.H. OF 4-Pt. CONTACT
R.H. OF 4 Pt. CONTACT.
CENTRE
$Y = x^3$
40
30
20
10
0
-10
-20
-30
-40

The equation of the R. H. of four-point contact at "t" is

$$5(\xi-x)^2-5(\eta-y)^2+\left(12t-\frac{4}{3t}\right)(\xi-x)(\eta-y)$$
$$+(\xi-x)(12at^2+27at^4)-(\eta-y)(8at+18at^3)=0,$$

i.e.
$$5(\xi^2-\eta^2)+\left(12t-\frac{4}{3t}\right)\xi\eta+at^2\left(\frac{10}{3}+15t^2\right)\xi$$
$$-20at\left(\frac{1}{3}+t^2\right)\eta-a^2t^4\left(\frac{1}{3}+2t^2\right)=0.$$

This is an equation of the seventh degree in t, in which the coefficient of t^6 is zero. Hence through any point there pass seven of the R. H.^{s} of four-point contact. Their parameters are such that $t_1+t_2+t_3+t_4+t_5+t_6+t_7=0$, and the "$t$"^{s} are found from the equation in t, ξ and η being given.

---

(27) <u>Envelopes of Parabolas and Rectangular Hyperbolas of four-point contact.</u>

Two conics intersect in four points, and since the conics we are considering have four-point contact with S, two adjacent conics of four-point contact at (x, y) will have three coincident points in common at (x, y), and therefore will intersect at a fourth real point. The envelopes we are discussing will be the locus of this point, in addition to the curve S.

To find this locus we use the equation of the conic, differentiate with respect to s, and solve the two equations simultaneously. Three of the roots will be equal to (x, y). The fourth root will the one sought.

---

(28) The equation of the parabola of four-point contact being
$$(3l+\rho'm)^2(\xi-x)^2+(3m-\rho'l)^2(\eta-y)^2$$
$$+2(3l+\rho'm)(3m-\rho'l)(\xi-x)(\eta-y)+18\rho\{m(\xi-x)-l(\eta-y)\}=0,$$
the equation obtained on differentiating with respect to s is

$$2(3l+\rho'm)(-3m+\rho\rho''m+\rho'l)(\xi-x)^2+2(3m-\rho'l)(3l-\rho\rho''l+\rho'm)(\eta-y)^2$$
$$+2\{(3l+\rho'm)(3l-\rho\rho''l+\rho'm)+(3m-\rho'l)(-3m+\rho'l+\rho\rho''m)\}(\xi-x)(\eta-y)$$
$$+12\rho\rho'\{m(\xi-x)-l(\eta-y)\}=0.$$

Taking this result from $\frac{2}{3}\rho'$ times the first we obtain the equation

$$(9+\rho'^2-3\rho\rho'')\{m(3l+\rho'm)(\xi-x)^2-l(3m-\rho'l)(\eta-y)^2$$
$$+(3m^2-3l^2-2lm\rho')(\xi-x)(\eta-y)\}=0,$$

which splits up into $m(\xi-x)-l(\eta-y)=0$, the equation of the tangent at (x,y),

and $(3l+\rho'm)(\xi-x)+(3m-\rho'l)(\eta-y)=0$, the equation of the diameter through (x,y). The fourth point of intersection of the two adjacent parabolas is therefore the intersection of this line with the line at infinity. This might have been inferred, since the line at infinity is an envelope of all parabolas.

---

(29) The equation of the R.H. of four-point contact being

$$\{3(l^2-m^2)+2\rho'lm\}\{(\xi-x)^2-(\eta-y)^2\}$$
$$+2\{6lm-\rho'(l^2-m^2)\}(\xi-x)(\eta-y)+6\rho\{m(\xi-x)-l(\eta-y)\}=0,$$

the equation obtained on differentiating with respect to s is

$$(-6lm+\rho\rho''lm+\rho'l^2-\rho'm^2)\{(\xi-x)^2-(\eta-y)^2\}$$
$$+(6l^2-6m^2-\rho\rho''l^2+\rho\rho''m^2+4\rho'lm)(\xi-x)(\eta-y)+2\rho\rho'\{m(\xi-x)-l(\eta-y)\}=0$$

Multiplying the first equation by ρ' and subtracting three times the second we obtain the equation

$$(18+2\rho'^2-3\rho\rho'')\left[lm\{(\xi-x)^2-(\eta-y)^2\}-(l^2-m^2)(\xi-x)(\eta-y)\right]=0,$$

which gives $m(\xi-x)-l(\eta-y)=0$, the tangent at (x,y),

and $l(\xi-x)+m(\eta-y)=0$, the normal at (x,y).

The fourth point of intersection of the two adjacent R.H^s. is therefore the point where the normal at (x,y) meets the R.H. of four-point contact at (x,y). The coordinates of this point are

$$\xi=x+2\rho m, \quad \eta=y-2\rho l.$$

Hence the normal at (x,y) meets the R.H. of four-point contact and the circle of curvature at equal distances in opposite directions.

(30) Differential Equations for a Parabola and for a R. H.
In the process of finding the envelope of the parabolas of four-point contact of S we obtained the factor $9+\rho'^2-3\rho\rho''$ as the multiplier of one of the equations. If this quantity is zero the equation vanishes identically, i.e. the two adjacent parabolas coincide. This means that the parabola of four-point contact at (x, y) has at least five coincident points in common with S. Moreover, if $9+\rho'^2-3\rho\rho''$ is zero at all points of S, all the parabolas of four-point contact will be coincident. In fact they must coincide with S itself, since the parabolas of four-point contact of a given parabola must be the parabola itself.

Hence the equation $9+\rho'^2-3\rho\rho''=0$ must be the differential equation of a parabola.

In the same way $18+2\rho'^2-3\rho\rho''=0$ must be the differential equation of a R. H.

(31) Envelope of the R. H.^{s} of four-point contact of the curve $a\eta^2=\xi^3$.

Using the quantities previously given, we find the coordinates of a point on the envelope, viz.,

$$\xi = x+2\rho m = at^2+at^2(4+9t^2) = at^2(5+9t^2),$$

$$\eta = y-2\rho l = at^3-\tfrac{2}{3}at(4+9t^2) = -\frac{at}{3}(8+15t^2).$$

On eliminating t we find the equation of the envelope

$$a\xi(75\xi+8a)^2 = 9\eta^2(729\eta^2-945a\xi+5a^2).$$

Section III.

(1) To find the equation of the conic of closest contact at a point (x, y) on a plane curve (S).

Let the coordinates (x, y) of the point be considered as functions of the length (s) of the arc of S measured from some fixed point on it.

Let (l, m) be the direction-cosines of the tangent to S at (x, y).

Then $x' = \frac{dx}{ds} = l$; $y' = \frac{dy}{ds} = m$; $l^2 + m^2 = 1$.

Also $x'' = l' = \frac{dl}{ds} = -\frac{m}{\rho}$; $y'' = m' = \frac{dm}{ds} = \frac{l}{\rho}$, where ρ is the radius of curvature of S at (x, y).

If the equation of the required conic is

$$F(\xi, \eta) \equiv a(\xi - x)^2 + b(\eta - y)^2 + 2h(\xi - x)(\eta - y) + 2g(\xi - x) + 2f(\eta - y) = 0,$$

and if this conic is to have contact of the fourth order with S at (x, y), we must have $F(x, y) = 0$, $F'(x, y) = 0$, $F''(x, y) = 0$, $F'''(x, y) = 0$, $F^{(iv)}(x, y) = 0$, where differentiations are with respect to s.

We have $F'(\xi, \eta) \equiv 2a(\xi - x)\xi' + 2b(\eta - y)\eta' + \cdots$ &c.

$\equiv 2a(\xi - x)l + 2b(\eta - y)m + \cdots$ &c.

$$\therefore (\xi - x)(al + hm) + (\eta - y)(bm + hl) + (gl + fm) = 0 \quad \cdots\cdots (1)$$

Differentiating equation (1), we have

$$(\xi - x)\left(-\frac{am}{\rho} + \frac{hl}{\rho}\right) + (\eta - y)\left(\frac{bl}{\rho} - \frac{hm}{\rho}\right) + (al^2 + 2hlm + bm^2) + \left(-g\frac{m}{\rho} + f\frac{l}{\rho}\right) = 0,$$

or

$$(\xi - x)(hl - am) + (\eta - y)(bl - hm) + (fl - gm) + \rho(al^2 + 2hlm + bm^2) = 0 \quad \cdots (2)$$

Differentiating equation (2) we have

$$(\xi-x)\left(-\frac{hm}{\rho}-\frac{al}{\rho}\right)+(\eta-y)\left(-\frac{bm}{\rho}-\frac{hl}{\rho}\right)+\left(-\frac{fm}{\rho}-\frac{gl}{\rho}\right)$$

$$+\{h(l^2-m^2)+blm-alm\}+\rho'(al^2+2hlm+bm^2)$$
$$+2(-alm+blm+hl^2-hm^2)=0,$$

or, by (1),

$$3\{(b-a)lm+h(l^2-m^2)\}+\rho'(al^2+2hlm+bm^2)=0, \quad \cdots (3)$$

Differentiating equation (3) we have

$$3\{(b-a)(l^2-m^2)-4hlm\}+\rho\rho''(al^2+2hlm+bm^2)$$
$$+2\rho'(-alm+blm+hl^2-hm^2)=0,$$

or, by (3),

$$9\{(b-a)(l^2-m^2)-4hlm\}-(2\rho'^2-3\rho\rho'')(al^2+2hlm+bm^2)=0 \quad \text{-----} (4)$$

Putting $\xi=x$, $\eta=y$ in these equations, we find

$$gl+fm=0, \quad \text{----} (5)$$

$$\rho(al^2+2hlm+bm^2)-gm+fl=0, \quad \text{----} (6)$$

$$a(\rho'l^2-3lm)+h\{2\rho'lm+3(l^2-m^2)\}+b(m^2\rho'+3lm)=0, \quad \cdots (7)$$

$$a\{l^2(2\rho'^2-3\rho\rho'')+9(l^2-m^2)\}+2hlm(2\rho'^2-3\rho\rho''+18)$$
$$+b\{m^2(2\rho'^2-3\rho\rho'')-9(l^2-m^2)\}=0, \quad \text{----} (8)$$

From equations (7) and (8) we find the ratios $a:h:b$. We may therefore take

$$a=9+6\rho'lm+m^2(2\rho'^2-3\rho\rho''),$$
$$b=9-6\rho'lm+l^2(2\rho'^2-3\rho\rho''),$$
$$h=-3\rho'(l^2-m^2)-lm(2\rho'^2-3\rho\rho'').$$

Substituting in (6), we find

$$gm-fl=9\rho.$$

Hence, by (5),

$$g=9\rho m,$$
$$f=-9\rho l.$$

The equation of the conic of closest contact is therefore

$$F(\xi,\eta)\equiv(\xi-x)^2\{9+6\rho'lm+m^2(2\rho'^2-3\rho\rho'')\}$$
$$+(\eta-y)^2\{9-6\rho'lm+l^2(2\rho'^2-3\rho\rho'')\}$$
$$-2(\xi-x)(\eta-y)\{3\rho'(l^2-m^2)+lm(2\rho'^2-3\rho\rho'')\}$$
$$+18\rho\{m(\xi-x)-l(\eta-y)\}=0.$$

(2) If the conic of closest contact at (x, y) is an ellipse, we must have (corresponding to the condition $ab - h^2 > 0$) the condition
$$9 - 3\rho\rho'' + \rho'^2 > 0.$$
For the conic to be a hyperbola we must have
$$9 - 3\rho\rho'' + \rho'^2 < 0.$$
For the conic to be a parabola we must have
$$9 - 3\rho\rho'' + \rho'^2 = 0,$$
and the equation of the parabola of closest contact at (x, y) is
$$(3l + m\rho')^2(\xi - x)^2 + (3m - l\rho')^2(\eta - y)^2 + 2(3m - l\rho')(3l + m\rho')(\xi - x)(\eta - y) + 18\rho\{m(\xi - x) - l(\eta - y)\} = 0,$$
which is also the equation of the parabola of four-point contact at (x, y).

For the conic to be a circle, we must have $\rho' = \rho'' = 0$, and the equation of the circle of curvature is
$$(\xi - x)^2 + (\eta - y)^2 + 2\rho\{m(\xi - x) - l(\eta - y)\} = 0.$$
For the conic to be a rectangular hyperbola, we must have
$$18 - 3\rho\rho'' + 2\rho'^2 = 0,$$
and the equation of the rectangular hyperbola of closest contact at (x, y) is
$$\{3(l^2 - m^2) + 2\rho' lm\}\{(\xi - x)^2 - (\eta - y)^2\} - 2\{\rho'(l^2 - m^2) - 6lm\}(\xi - x)(\eta - y) + 6\rho\{m(\xi - x) - l(\eta - y)\} = 0,$$
which is also the equation of the rectangular hyperbola of four-point contact at (x, y).

If S reduces to a point circle, $\rho = \rho' = \rho'' = 0$, and the conic of closest contact is also the point circle, whose equation is
$$(\xi - x)^2 + (\eta - y)^2 = 0.$$

If S reduces to a straight line, $\rho = \infty$, $\rho' = \rho'' = 0$, and the conic of closest contact becomes the straight line through (x, y) and the line at infinity, whose equations are

$$m(\xi - x) - l(\eta - y) = 0 \text{ or } \infty.$$

If (x, y) is a point of inflexion of S, ρ, ρ', ρ'' are all ~~parallel~~ infinite, and the conic of closest contact becomes the pair of coincident tangents at (x, y), whose equation is

$$m^2(\xi - x)^2 - 2lm(\xi - x)(\eta - y) + l^2(\eta - y)^2 = 0.$$

---

(3) The coordinates of the centre of the conic of closest contact at (x, y) are

$$\bar{\xi} = x + \frac{3\rho(l\rho' - 3m)}{9 + \rho'^2 - 3\rho\rho''}, \quad \bar{\eta} = y + \frac{3\rho(m\rho' + 3l)}{9 + \rho'^2 - 3\rho\rho''}.$$

---

(4) The distance of (x, y) from $(\bar{\xi}, \bar{\eta})$, or the length of the semi-diameter through (x, y) of the conic of closest contact is

$$\frac{3\rho\sqrt{9 + \rho'^2}}{9 + \rho'^2 - 3\rho\rho''}.$$

---

(5) To find the envelope of the line joining the point (x, y) to the centre of the conic of closest contact.

The equation of the straight line joining (x, y) and $(\bar{\xi}, \bar{\eta})$ is

$$\frac{\xi - x}{\bar{\xi} - x} = \frac{\eta - y}{\bar{\eta} - y} \quad \text{or} \quad \frac{\xi - x}{l\rho' - 3m} = \frac{\eta - y}{m\rho' + 3l},$$

i.e. $(\xi - x)(m\rho' + 3l) - (\eta - y)(l\rho' - 3m) = 0$, --- (1)

Differentiating with respect to the parameter s, we have

$$(\xi - x)\left(m\rho'' + \frac{l\rho'}{\rho} - \frac{3m}{\rho}\right) - (\eta - y)\left(l\rho'' - \frac{m\rho'}{\rho} - \frac{3l}{\rho}\right)$$
$$- l(m\rho' + 3l) + m(l\rho' - 3m) = 0,$$

i.e. $(\xi - x)(m\rho\rho'' + l\rho' - 3m) + (\eta - y)(-l\rho\rho'' + m\rho' + 3l) - 3\rho = 0$, --- (2)

Solving (1) and (2) we find

$$\xi = x + \frac{3\rho(l\rho' - 3m)}{9+\rho'^2 - 3\rho\rho''}, \quad \eta = y + \frac{3\rho(m\rho' + 3l)}{9+\rho'^2 - 3\rho\rho''},$$

i.e. $\xi = \bar{\xi}$ and $\eta = \bar{\eta}$.
Hence the envelope of the semidiameter through (x, y) of the conic of closest contact is also the locus of the centre of this conic

---

(6) To find the length of the normal chord through (x, y) of the conic of closest contact at (x, y).

The equation of the chord is

$$\frac{\xi - x}{-m} = \frac{\eta - y}{l},$$ and, if P is the length of the normal chord, we may write

$\xi - x = -mP$, $\eta - y = lP$, and substitute these in the equation of the conic. This substitution gives

$$P = \frac{18\rho}{9 + 2\rho'^2 - 3\rho\rho''}.$$

---

(7) The condition that the centre of the conic of closest contact should lie on the normal is $\rho' = 0$.

The conditions that it will coincide with the centre of curvature are $\rho' = \rho'' = 0$, i.e. at a point where the conic of closest contact is a circle.

---

(8) To find a curve such that the diameter through (x, y) of the conic of closest contact is parallel to a given line.

In the equation $\dfrac{\xi - x}{l\rho' - 3m} = \dfrac{\eta - y}{m\rho' + 3l}$, we put the gradient equal to k, or to $\cot\alpha$, where α is constant.

i.e. $\dfrac{m\rho'+3l}{l\rho'-3m} = \cot\alpha$. This gives $\dfrac{d\rho}{ds} = 3\tan(\psi+\alpha)$, where $\tan\psi = \dfrac{dy}{dx}$. Using the relation $ds = \rho\, d\psi$, this gives on integration $\rho = a\sec^3(\psi+\alpha)$, the equation of a parabola, which, therefore, is the only solution. Of course, the conics of closest contact of a parabola are the parabola itself, and its axes of aberrancy are its diameters, which are parallel to a given line. The result, therefore, of the investigation is of little interest.

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(9) <u>To find a curve whose axes of aberrancy pass through a fixed point, e.g., the origin.</u>

We know that the axes of aberrancy of a conic are its diameters, and therefore the given condition is satisfied by the central conics. To find whether any other types of curves satisfy the given condition, we put $\xi = \eta = 0$ in the equation $\dfrac{\xi - x}{l\rho'-3m} = \dfrac{\eta - y}{m\rho'+3l}$, and find the solutions of the resulting equation

$$y(\rho' dx - 3\,dy) = x(\rho' dy + 3\,dx),$$

$$\text{i.e. } \rho' y^2 \frac{y\,dx - x\,dy}{y^2} = 3(y\,dy + x\,dx),$$

i.e. $-\rho' r\, d\theta = 3\,dr$, where $x = r\cos\theta$, $y = r\sin\theta$,

i.e. $\dfrac{d\rho}{ds} = -3\cot\varphi$, where φ is the angle between the radius vector and the tangent.

This equation may be transformed by the substitutions $\cot\varphi = \dfrac{\sqrt{r^2-p^2}}{p}$, $\dfrac{dr}{ds} = \dfrac{\sqrt{r^2-p^2}}{r}$, $\rho = r\dfrac{dr}{dp}$, where p is the perpendicular from the origin on the tangent. The equation becomes $\dfrac{d\rho}{\rho} = -3\dfrac{dp}{p}$, i.e. $\rho = cp^{-3}$, i.e. $r\dfrac{dr}{dp} = cp^{-3}$. Integrating, this gives

$$p^2 = \frac{c}{k - r^2},$$ the equation of a central conic. This, therefore, is the only solution.

---

(10) To find a curve for which the direction of the diameter at (x, y) of its conic of closest contact makes a given angle with the tangent.

Let α be the angle between the lines

$$\frac{\xi - x}{l} = \frac{\eta - y}{m} \text{ and } \frac{\xi - x}{l\rho' - 3m} = \frac{\eta - y}{m\rho' + 3l},$$

then $$\frac{l(l\rho' - 3m) + m(m\rho' + 3l)}{\sqrt{9 + \rho'^2}} = \cos\alpha,$$

or $\rho' = 3\cot\alpha = \text{const.}$, the equation of an equiangular spiral, which, therefore is the only solution. This curve, and its locus of centres of conics of closest contact are shown in Section V.

When $\alpha = 0$, the equiangular spiral becomes a straight line, and when $\alpha = \frac{\pi}{2}$ it becomes a circle.

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(11) If the intrinsic equation of the curve is given, the calculation of the quantities $\rho, \rho', \rho'', 9 - 3\rho\rho'' + \rho'^2$ is simple, and the nature of the conic of closest contact can readily be found.

Thus if $s = f(\psi)$ be the intrinsic equation of a given curve, $\rho = \frac{ds}{d\psi} = f'(\psi)$, $\rho' = \frac{f''(\psi)}{f'(\psi)}$,

$$\rho\rho'' = \frac{f'(\psi)f'''(\psi) - (f''(\psi))^2}{(f'(\psi))^2},$$

$$9 - 3\rho\rho'' + \rho'^2 = \frac{9(f'(\psi))^2 - 3f'(\psi)f'''(\psi) + 4(f''(\psi))^2}{(f'(\psi))^2}.$$

E.g., for the catenary $s = c\tan\psi$,

$$\begin{aligned} 9 + \rho'^2 - 3\rho\rho'' &= 3 - 2\tan^2\psi \\ &> 0, \text{ when } \tan^2\psi < \tfrac{3}{2}, \\ &= 0, \text{ when } \tan^2\psi = \tfrac{3}{2}, \\ &< 0, \text{ when } \tan^2\psi > \tfrac{3}{2}. \end{aligned}$$

For the equiangular spiral $s = ae^{n\psi}$,

$$9 + \rho'^2 - 3\rho\rho'' = 9 + n^2 > 0.$$

(11a) The expression for the axis of aberrancy may be written in various other forms.

For example if we write $\rho' = \frac{1}{\rho}\frac{d\rho}{d\psi}$, $\rho\rho'' = \frac{1}{\rho}\frac{d^2\rho}{d\psi^2} - \frac{1}{\rho^2}\left(\frac{d\rho}{d\psi}\right)^2$ we find

$$R = \frac{3\rho^2\sqrt{9\rho^2+\rho_1^2}}{4\rho_1^2+9\rho^2-3\rho\rho_2},$$

where $\rho_1 \equiv \frac{d\rho}{d\psi}$, and $\rho_2 \equiv \frac{d^2\rho}{d\psi^2}$.

(11b) If the equation of a given curve is of the form $y = f(x)$, we may substitute

$$\rho = \frac{(1+p^2)^{3/2}}{q},$$

$$\rho' = 3p - \frac{(1+p^2)r}{q^2},$$

$$\rho\rho'' = \frac{(1+p^2)(3q^4 - tq - tp^2q - 2pq^2r + 2r^2 + 2r^2p^2)}{q^4},$$

where $p \equiv \frac{dy}{dx}$, $q \equiv \frac{d^2y}{dx^2}$, $r \equiv \frac{d^3y}{dx^3}$, $t \equiv \frac{d^4y}{dx^4}$,

and obtain

$$R = \frac{3q\sqrt{r^2+(3q^2-rp)^2}}{3tq-5r^2}.$$

(11c) The equation $\rho' = 3\cot\alpha$ gives $\rho'' = -3\operatorname{cosec}^2\alpha\,\frac{d\alpha}{ds}$, and R may therefore be written

$$R = \frac{\rho\operatorname{cosec}\alpha}{\operatorname{cosec}^2\alpha\left(1+\rho\frac{d\alpha}{ds}\right)} = \frac{\rho\sin\alpha}{1+\frac{d\alpha}{d\psi}},$$

where $\frac{d\rho}{d\psi} = 3\rho\cot\alpha$,

$$\text{i.e. } R = \frac{\sin\alpha}{\frac{1}{\rho} + 3\cot\alpha\,\frac{d\alpha}{d\rho}}$$

Conics of Six-Point Contact.

(12) If we seek the condition that six adjacent points on the curve S should lie on a conic, we differentiate equation (4) with respect to s. This is equivalent to the condition $F^{(v)}(x,y)=0$, that the conic of five-point contact should have contact of the fifth order.

The result is

$$9\{(b-a)(2ll'-2mm')-4h(lm'+l'm)\}$$
$$-(al^2+2hlm+bm^2)(\rho'\rho''-3\rho\rho''')$$
$$-(2\rho'^2-3\rho\rho'')(2all'+2hm'l+2hl'm+2bmm')=0$$

Substituting $l'=-\frac{m}{\rho}$, $m'=\frac{l}{\rho}$ we find, on rearranging,

$$\{36+2(2\rho'^2-3\rho\rho'')\}\{lm(a-b)-(l^2-m^2)h\}$$
$$-\rho(\rho'\rho''-3\rho\rho''')(al^2+bm^2+2hlm)=0,$$

or, by equation (3),

$$(36+4\rho'^2-6\rho\rho'')\rho'-3\rho(\rho'\rho''-3\rho\rho''')=0,$$

i.e. $$36\rho'+4\rho'^3-9\rho\rho'\rho''+9\rho^2\rho'''=0,$$

or $$\frac{2}{3}\frac{d}{ds}(\log\rho)=\frac{d}{ds}\log(18+2\rho'^2-3\rho\rho'').$$

This is the general condition for six-point contact.

(13) The conditions that the conic of closest contact may be a circle of six-point contact are $\rho'=\rho''=\rho'''=0$.

The conditions for a parabola of six-point contact are $9+\rho'^2-3\rho\rho''=0$, and $\rho'\rho''+3\rho\rho'''=0$.

The conditions for a R.H. of six-point contact are $18+2\rho'^2-3\rho\rho''=0$, and $\rho'\rho''-3\rho\rho'''=0$.

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(14) If every point on the curve S is such that the conic of closest contact has six-point contact with S, then the above condition must be satisfied at every point of S. The equation of such a curve may therefore be found by integrating the equation.

A first integral is $2\rho'^2-3\rho\rho''=\lambda\rho^{2/3}-18$.

i.e. $\dfrac{du^2}{d\rho} - \dfrac{4}{3\rho}u^2 = \dfrac{2}{3\rho}(18 - \lambda\rho^{2/3})$, where $u = \dfrac{d\rho}{ds}$.

A second integral is

$$u^2 = -9 + \lambda\rho^{2/3} + \lambda_1\rho^{4/3},$$

or $$\frac{ds}{d\rho} = \pm\frac{1}{\sqrt{\lambda_1\rho^{4/3} + \lambda\rho^{2/3} - 9}}$$

$$= \pm\sqrt{\frac{-4k}{c^2\rho^{4/3} + 2c\rho^{2/3} + 36k}},$$

where $\lambda_1 = -\dfrac{c^2}{4k}$, $\lambda = -\dfrac{c}{2k}$.

This latter form of the equation will be found completely solved in Section IV. The solutions are the conic sections, so that there are no curves apart from these such that the osculating conic at every point has six-point contact with the curve. In fact, the equation $36\rho' + 4\rho'^3 - 9\rho\rho'\rho'' + 9\rho^2\rho''' = 0$ is the differential equation of the general conic.

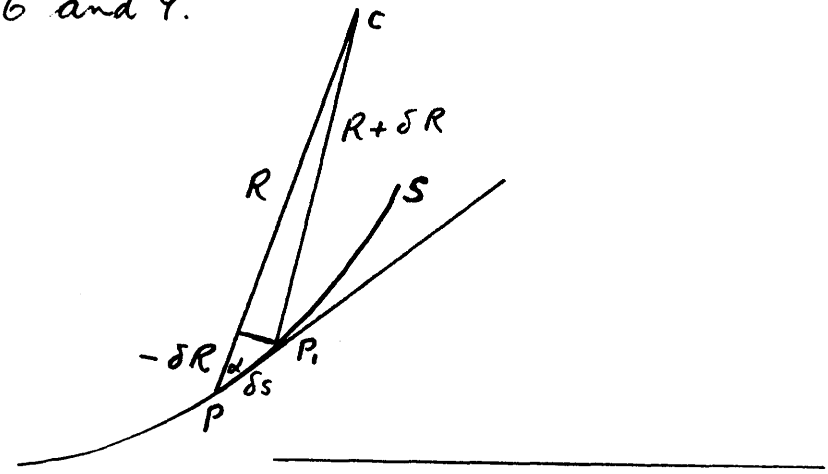
(15) That S must be a conic section under the conditions we are considering may also be seen geometrically. For if A, B, C, D, E, F, G, are seven adjacent points of S, then, if A, B, C, D, E, F, lie on a conic, it must be the conic determined by B, C, D, E, F. Similarly, if B, C, D, E, F, G lie on a conic, it also must be the conic determined by B, C, D, E, F. This argument may then be extended to further adjacent points H, K, L, &c. Thus, if every point of S is such that the conic of closest contact has six-point contact with S, these conics must be coincident with one another and therefore also with S.

(16) The condition for six-point contact may also be derived as follows:- If the conic of closest contact at a point P has contact with S of the fifth order at P, then it must coincide with the conic of closest contact at the point adjacent to P. That is, two adjacent centres of aberrancy must coincide; so that, if R is the length of the axis of aberrancy, and α the angle it makes with the tangent at P, it is evident from the figure that we must have

$$-\frac{dR}{ds} = \cos\alpha = \frac{\rho'}{\sqrt{9+\rho'^2}}.$$

On substituting for R and differentiating, we arrive at the required condition. This condition corresponds to a cusp on the locus of centres of aberrancy.

Many cases will be found among the graphs of the loci of centres of aberrancy, e.g. in the cases of the catenary and the catenary of equal strength, Section V, nos. 6 and 7.



(17) This condition can also be expressed by the equation

$$36\rho^2\rho_1 + 9\rho^2\rho_3 + 40\rho_1^3 - 45\rho\rho_1\rho_2 = 0,$$ where $\rho_1 = \frac{d\rho}{d\psi}$, $\rho_2 = \frac{d^2\rho}{d\psi^2}$, $\rho_3 = \frac{d^3\rho}{d\psi^3}$, a form suitable for application when the ρ, ψ equation of a given curve has been found.

(18) The condition may also be expressed by the equation

$$9y_2^2 y_5 + 40y_3^3 - 45y_2 y_3 y_4 = 0,$$ where

$y_2 = \frac{d^2y}{dx^2}$, $y_3 = \frac{d^3y}{dx^3}$, &c., a form suitable for application when the equation of a given curve can be expressed in the form $y = f(x)$.

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(19) The condition for sextactic contact, viz.,

$36\rho' + 4\rho'^3 - 9\rho\rho'\rho'' + 9\rho^2\rho''' = 0$, has been found on the supposition that ρ and its derivatives are continuous. It is not necessarily true at points for which any of the quantities $\rho, \rho', \rho'', \rho'''$ are discontinuous. The condition, therefore, can only be considered as true for points at which the conics of sextactic contact are non-degenerate.

At a point of inflexion, the conic of closest contact degenerates into the pair of coincident tangents at the point, each of which meets the curve in three coincident points. At a point of inflexion, then, the contact of a curve with its conic of closest contact may be regarded as sextactic.

The same conclusion may be arrived at for a cusp.

At a node, the conic of closest contact has six points in common with the curve. But the contact here cannot be regarded as sextactic, since the six points of contact are not adjacent points, five of them being adjacent points on one branch of the curve, and the other being a distinct point on the second branch.

(20) Conversely, the condition $36\rho' + 4\rho'^3 - 9\rho\rho'\rho'' + 9\rho^2\rho''' = 0$ is not sufficient to ensure sextactic contact. For the left-hand side of the equation may contain a factor which is a factor also of each of the coefficients in the equation of the conic of closest contact. For example, if we calculate the coefficients for the conic of closest contact at "t" on the semi-cubical parabola $x = at^2$, $y = at^3$, by substitution in the general equation of the conic, we find that the quantity $9t^2+4$ occurs as a factor of each coefficient, and also as a repeated factor of the left-hand side of the equation of condition for sextactic contact.

Suppose we find, on substitution of $\rho, \rho', \rho'', x, y, l, m$ in the general equation of the conic of closest contact, that the coefficients have a common factor Δ. Then, if the result of the substitution is

$$F(\xi, \eta) \equiv a(\xi - x)^2 + 2h(\xi - x)(\eta - y) + \&c. = 0,$$

we have on division by Δ,

$$\frac{F}{\Delta} \equiv \frac{a}{\Delta}(\xi - x)^2 + \&c. = 0.$$

If now, we differentiate this equation with respect to the parameter s, we obtain

$$\frac{\partial}{\partial s}\left(\frac{F}{\Delta}\right) \equiv \frac{1}{\Delta}\frac{\partial F}{\partial s} - \frac{F}{\Delta^2}\Delta' = 0.$$

$$\text{i.e. } \frac{\partial}{\partial s}\left(\frac{F}{\Delta}\right) \equiv \left(\frac{a_1}{\Delta} - \frac{a\Delta'}{\Delta^2}\right)(\xi - x)^2 + \&c. = 0,$$

where $\frac{\partial F}{\partial s} \equiv a_1(\xi - x)^2 + 2h_1(\xi - x)(\eta - y) + \&c. = 0.$

It may also happen that the coefficients in the equation $\frac{\partial}{\partial s}\left(\frac{F}{\Delta}\right) = 0$ have a common factor θ, necessarily introduced in the process of differentiation with respect to s of functions not explicitly expressed as functions of s.

On division by θ we find

$$\frac{1}{\theta}\frac{\partial}{\partial s}\left(\frac{F}{\Delta}\right) \equiv \left(\frac{a_1}{\Delta\theta} - \frac{a\Delta'}{\Delta^2\theta}\right)(\xi - x)^2 + \&c. = 0.$$

At a sextactic point the ratios of the coefficients in the equations $\frac{F}{\Delta} = 0$ and $\frac{1}{\theta}\frac{\partial}{\partial s}\left(\frac{F}{\Delta}\right) = 0$ must be equal, i.e. we must have

$$\frac{\frac{a}{\Delta}}{\frac{a_1}{\Delta\theta} - \frac{a\Delta'}{\Delta^2\theta}} = \&c. = \frac{g/\Delta}{\frac{g_1}{\Delta\theta} - \frac{g\Delta'}{\Delta^2\theta}} = \frac{f/\Delta}{\frac{f_1}{\Delta\theta} - \frac{f\Delta'}{\Delta^2\theta}}$$

Substituting for a and g and f the general forms $a = 9 + 6\rho' lm + m^2(2\rho'^2 - 3\rho\rho'')$, $g = 9\rho m$, $f = -9\rho l$, we find

$$a_1 = \frac{6\rho'(l^2 - m^2) + 4\rho'^2 lm + \rho m^2(\rho'\rho'' - 3\rho\rho''')}{\rho}$$

$g_1 = 6\rho' m$, $f_1 = -6\rho' l$, and the condition for sextactic contact becomes

$$\frac{\frac{\{9 + 6\rho' lm + m^2(2\rho'^2 - 3\rho\rho'')\}}{\Delta}}{\frac{\{6\rho'(l^2 - m^2) + 4\rho'^2 lm + \rho m^2(\rho'\rho'' - 3\rho\rho''')\}}{\Delta\rho\theta}} = \frac{\frac{9}{\Delta}}{\frac{6\rho'}{\Delta\theta\rho}}$$

i.e. $\frac{1}{\Delta^2\theta\rho}(36\rho' + 4\rho'^3 - 9\rho\rho'\rho'' + 9\rho^2\rho''') = 0$. So that, when the left-hand side has been reduced as far as possible by cancelling, the equation may be used to find the parameters of the sextactic points.

Envelope of Osculating Conics.

(21) If we wish to find the envelope of the osculating conics of S, we must regard the constants in the equation $F(\xi,\eta)=0$ as functions of the parameter s. We differentiate the equation with respect to this parameter, and solve the two equations to find the coordinates of a point on the envelope in terms of s.

$F(\xi,\eta)\equiv 0$ gives

$$(\xi-x)^2\{9+6\rho' lm+m^2(2\rho'^2-3\rho\rho'')\}$$
$$+(\eta-y)^2\{9-6\rho' lm+l^2(2\rho'^2-3\rho\rho'')\}$$
$$-2(\xi-x)(\eta-y)\{3\rho'(l^2-m^2)+lm(2\rho'^2-3\rho\rho'')\}$$
$$+18\rho\{m(\xi-x)-l(\eta-y)\}=0.$$

$\frac{d}{ds}F(\xi,\eta)=0$ gives

$$(\xi-x)^2\{6\rho'(l^2-m^2)+4\rho'^2 lm+\rho m^2(\rho'\rho''-3\rho\rho''')\}$$
$$+(\eta-y)^2\{6\rho'(l^2-m^2)-4\rho'^2 lm+\rho l^2(\rho'\rho''-3\rho\rho''')\}$$
$$-2(\xi-x)(\eta-y)\{-12\rho' lm+2\rho'^2(l^2-m^2)+\rho\, lm(\rho'\rho''-3\rho\rho''')\}$$
$$+12\rho\rho'\{m(\xi-x)-l(\eta-y)\}=0.$$

Multiplying the first equation by $\frac{2}{3}\rho'$ and subtracting the second from it, we find

$$(36\rho'+4\rho'^3-9\rho\rho'\rho''+9\rho^2\rho''')\{m^2(\xi-x)^2-2lm(\xi-x)(\eta-y)+l^2(\eta-y)^2\}=0.$$

This gives $m(\xi-x)-l(\eta-y)=0$, twice, whence it is evident that the four points of intersection coincide in the point (x,y).

(22) The condition for six-point contact of the curve and its osculating conic also appears as a solution. This implies that if S is a conic the osculating conic at any point coincides with the conic itself. The argument here is the same as in the cases of the parabola and rectangular hyperbola of four-point contact.

(23) It is evident geometrically also that the envelope of the osculating conics of a curve

must be the curve itself. For if A, B, C, D, E, F, be six adjacent points on the given curve, then the conic which passes through A, B, C, D, E, will have four adjacent points in common with the conic which passes through B, C, D, E, F. These adjacent conics, therefore, can have no other point of intersection.

Section IV.

To find a curve whose osculating conics are similar conics.

(1) The conics $ax^2+2hxy+by^2+2gx+2fy+c=0$,

$$a'x^2+2h'xy+b'y^2+2g'x+2f'y+c'=0,$$

are similar if $\dfrac{ab-h^2}{(a+b)^2}=\dfrac{a'b'-h'^2}{(a'+b')^2}$.

(2) For the osculating conic at a point (x, y) we have

$$ab-h^2 \equiv 9(9+\rho'^2-3\rho\rho''),$$

and

$$(a+b)^2 \equiv (18+2\rho'^2-3\rho\rho'')^2.$$

The differential equation may therefore be written

$$\frac{9+\rho'^2-3\rho\rho''}{(18+2\rho'^2-3\rho\rho'')^2}=k,$$

where k may be positive or negative.

The substitution $u=9+\rho'^2$ gives $\dfrac{u-\frac{3}{2}\rho\frac{du}{d\rho}}{\left(2u-\frac{3}{2}\rho\frac{du}{d\rho}\right)^2}=k$,

or $3\rho\dfrac{du}{d\rho}=\left(-\dfrac{1}{k}+4u\right)\pm\sqrt{\dfrac{1}{k}\left(\dfrac{1}{k}-4u\right)}$.

This may be solved by substituting kv^2 for $\frac{1}{k}-4u$, and the solution is $c\rho^{2/3}=\sqrt{1-4ku}\pm 1$.

Substituting for u we get

$$\frac{ds}{d\rho}=\pm\sqrt{\frac{-4k}{c^2\rho^{4/3}+2c\rho^{2/3}+36k}}.$$

This may be changed to the (ψ, ρ) form by writing $ds=\rho\, d\psi$. This gives

$$d\psi=\pm\frac{d\rho}{\rho}\sqrt{\frac{-4k}{c^2\rho^{4/3}+2c\rho^{2/3}+36k}}.$$

Writing $\rho^{2/3}=v^{-1}$, the equation becomes

$$\pm d\psi=\frac{3}{2}\,dv\sqrt{\frac{-4k}{c^2+2cv+36kv^2}}.$$

(3) When k is positive this gives $\pm 2\psi=\sin^{-1}\dfrac{v+\frac{c}{36k}}{\sqrt{\frac{c^2}{36k}\left(\frac{1}{36k}-1\right)}}$

or $$\rho^{-2/3}=-\frac{c}{36k}\pm\frac{c}{36k}\sqrt{1-36k}\,\sin 2\psi$$

i.e. $\rho^{-2/3} = a^2(1 + b\sin 2\psi)$, where $|b| < 1$, since when k is positive, c must be negative for real solutions.

The equation $\rho^{-2/3} = a^2(1 + d^2 \sin 2\psi)$, $d^2 < 1$, corresponds to that of the ellipse

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1,\ \alpha^2 > \beta^2;\ \text{when}\ a^2 = \frac{\alpha^2+\beta^2}{2(\alpha\beta)^{4/3}},$$

$$d^2 = \frac{\alpha^2-\beta^2}{\alpha^2+\beta^2}.$$

The equation $\rho^{-2/3} = a^2(1 - d^2\sin 2\psi)$, $d^2 < 1$, corresponds to that of the ellipse

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1,\ \beta^2 > \alpha^2,\ \text{when}\ a^2 = \frac{\alpha^2+\beta^2}{2(\alpha\beta)^{4/3}},\ d^2 = \frac{\beta^2-\alpha^2}{\beta^2+\alpha^2}.$$

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(4) When k is negative we have $\pm 2\psi = \sin^{-1}\dfrac{v + \frac{c}{36k}}{\frac{-c}{36k}\sqrt{1-36k}}$,

which can be written $v = -\dfrac{c}{36k}\left(1 \pm \sqrt{1-36k}\,\sin 2\psi\right)$,

or $\rho^{-2/3} = a(1 + b\sin 2\psi)$, where $|b| > 1$, and <u>a</u> may be positive or negative. This leads to

$\rho^{-2/3} = a'^2(1 + b\sin 2\psi)$, where b must be chosen so that $1 + b\sin 2\psi$ is positive, or to

$\rho^{-2/3} = a'^2(b\sin 2\psi - 1)$, where b must be chosen so that $b\sin 2\psi - 1$ is positive.

The equations $\rho^{-2/3} = \pm a^2(1 + b\sin 2\psi)$, $b > 1$, correspond to the hyperbolas $\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} = \pm 1$, $\alpha^2 > \beta^2$; and the equations $\rho^{-2/3} = \pm a^2(1 - b\sin 2\psi)$, $b > 1$, correspond to the hyperbolas $\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} = \pm 1$, $\beta^2 > \alpha^2$. The angle between the tangent and x-axis is $\frac{\pi}{4} + \psi$.

(5) If $36K = 1$, all these solutions reduce to ρ = const., and when $\rho' = \rho'' = 0$, $36K = 1$ satisfies the original differential equation.

If in the equation $1 - 4k(9 + \rho'^2) = (c\rho^{2/3} + 1)^2$ we put $36k = 1$, we must have also $\rho' = 0$, $\rho^{-2/3} = -c$, which verifies the conclusion just arrived at.

(6) If in the equation $1-4k(9+\rho'^2)=(c\rho^{2/3}+1)^2$ we put $c=0$, we must also have $k=0$, and if $k=0$, we must have $\rho^{2/3}=0$ or $-\frac{2}{c}$, unless $c=0$.

When c and k are taken to vanish together, we may write $c=\lambda k$, and the equation may be written $-4k(9+\rho'^2)=\lambda^2 k^2 \rho^{4/3}+2\lambda k\rho^{2/3}$,

$$\text{i.e.} \quad -4(9+\rho'^2)=\lambda^2 k\rho^{4/3}+2\lambda\rho^{2/3},$$

and when k tends to zero, this leaves

$-36-4\rho'^2=2\lambda\rho^{2/3}$, the solution of which is of the form $\rho=a\sec^3\psi$, the equation of a parabola.

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(7) When c is infinite, we must also have k negative infinite. If c and k tend to infinity in such a way that $\frac{c}{k}=$ a finite non-zero quantity λ, we may write $c=k\lambda$ when k is very large. This yields the solution $\rho=0$.

If, however, when k is infinite $\frac{c}{\sqrt{-k}}\to\lambda$, we may write $c=\lambda\sqrt{-k}$, when k is very large. This gives the equation $4(9+\rho'^2)=\lambda^2\rho^{4/3}$; the solution of which is of the form $\rho^{-2/3}=a\sin 2\psi$, the equation of a rectangular hyperbola.

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(8) The discriminant of the equation $\dfrac{u-\frac{3}{2}\rho\frac{du}{d\rho}}{\left(2u-\frac{3}{2}\rho\frac{du}{d\rho}\right)^2}=k$ is $\left(\frac{1}{k}-4u\right)^2-4u\left(4u-\frac{1}{k}\right)$. Equating this to zero we find the solution $u=\frac{1}{4k}$, which satisfies the differential equation. This gives $\rho'=$ const., the equation of an equiangular spiral. This is the only solution of the problem we are considering, apart from the conic sections.

Section V.

Examples of curves and of the loci of the centres of their conics of closest contact.

(1) The coordinates of any point on the semi-cubical parabola $ay^2 = x^3$ may be written

$$x = at^2, \quad y = at^3.$$

It follows that

$$l = 2(4+9t^2)^{-\frac{1}{2}}, \quad m = 3t(4+9t^2)^{-\frac{1}{2}}; \quad l' = -\frac{18}{a}(4+9t^2)^{-2};$$

$$\rho = \frac{at}{6}(4+9t^2)^{3/2}, \quad \rho' = \frac{2}{3t}(1+9t^2), \quad \rho'' = \frac{2(9t^2-1)}{3at^3}(4+9t^2)^{-\frac{1}{2}};$$

$$9 - 3\rho\rho'' + \rho'^2 = \frac{(4+9t^2)^2}{9t^2} > 0;$$

$$\frac{3\rho}{9+\rho'^2-3\rho\rho''} = \frac{9}{2}at^3(4+9t^2)^{-\frac{1}{2}}.$$ Thus the conics of closest contact are ellipses.

$$l\rho' - 3m = \frac{1}{3t}(4+9t^2)^{\frac{1}{2}}, \quad m\rho' + 3l = 2(4+9t^2)^{\frac{1}{2}};$$

$$\bar{\xi} = at^2 + \tfrac{3}{2}at^2 = \tfrac{5}{2}at^2, \quad \bar{\eta} = at^3 + 9at^3 = 10at^3;$$

and the locus of the centre of the conic of closest contact is a second semi-cubical parabola whose equation is $32x^3 = 5ay^2$.

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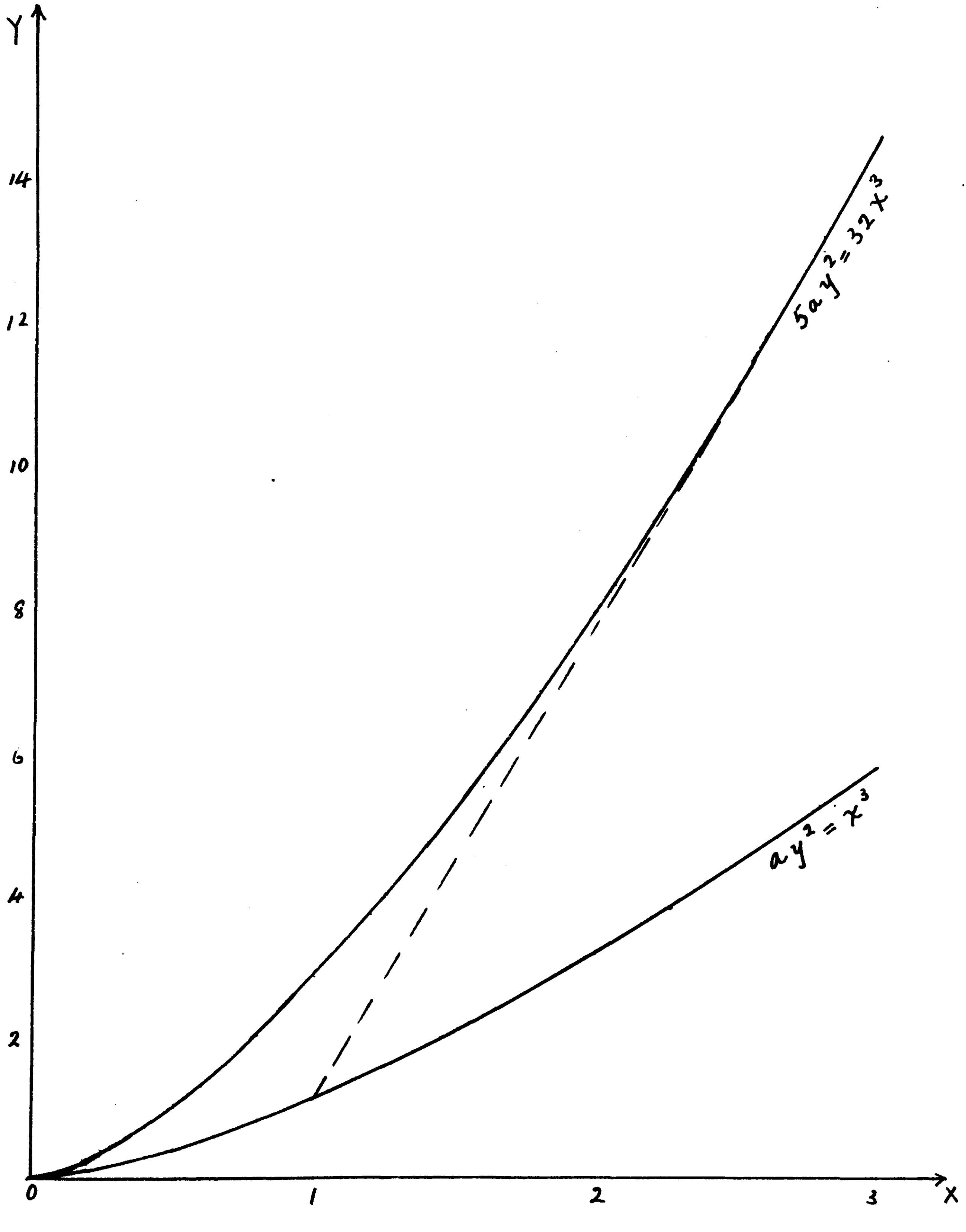
Data for the graphs of $ay^2 = x^3$ and of $5ay^2 = 32x^3$ when $a = \frac{4}{5}$; i.e. for the graphs of $(10y)^2 = (5x)^3$ and of $y^2 = (2x)^3$.

x	0	·2	·4	·6	·8	1·0	1·2	1·4
$y = \frac{1}{10}(5x)^{\frac{3}{2}}$	0	·10	·28	·52	·80	1·12	1·48	1·86
$y = (2x)^{\frac{3}{2}}$	0	·25	·72	1·31	2·02	2·83	3·72	4·68
x	1·6	1·8	2·0	2·2	2·4	2·6	2·8	3·0
$y = \frac{1}{10}(5x)^{\frac{3}{2}}$	2·24	2·70	3·16	3·66	4·16	4·70	5·24	5·80
$y = (2x)^{3/2}$	5·72	6·83	8·00	9·24	10·5	11·9	13·3	14·7



Graph of $ay^2 = x^3$ and of the locus of the centres of its conics of closest contact.

If we substitute for $x, y, l, m, \rho, \rho', \rho''$ in the equation of the conic of closest contact at (x, y), $F(\xi, \eta) = 0$, we obtain the equation of the conic of closest contact at the point "t" of the semi-cubical parabola.

The result is

$$45t^2\xi^2 + 5\eta^2 - 24t\xi\eta + 15at^4\xi - 40at^3\eta - a^2t^6 = 0.$$

As a verification it can easily be seen that it passes through $\xi = at^2$, $\eta = at^3$, and that the coordinates of its centre are $\bar{\xi} = \frac{5}{2}at^2$, $\bar{\eta} = 10at^3$.

The equation shows that through any given point (ξ, η) there pass six of the conics of closest contact. If these correspond to the parameters t_r (1 to 6) we must have $\sum_1^6 t_r = 0$, since the coefficient of t^5 in the equation is zero.

Let the conic of closest contact at "t" meet the cubic at the points "T". Then the values of T are given by

$$5T^6 - 24tT^5 + 45t^2T^4 - 40t^3T^3 + 15t^4T^2 - t^6 = 0.$$

It follows that $\sum \frac{1}{T} = 0$, and since five of the roots of this equation are all equal to t, we must have for the remaining root $\frac{1}{T} + \frac{5}{t} = 0$, or $T = -\frac{t}{5}$. Therefore the conic of closest contact of the cubic (at^2, at^3) at "t" meets the curve again at "$-\frac{t}{5}$", a point on the second branch of the curve. We notice also that when $t = 0$, all the six values of T are zero, or the conic of closest contact at the origin has six coincident points in common with the semi-cubical parabola. In that case the equation of the conic of closest contact reduces to $\eta^2 = 0$, the pair of coincident tangents at the origin.

Similarly, the conic composed of the two coincident lines at infinity meets the curve at six coincident points.

If we calculate the quantity $36\rho' + 4\rho'^3 - 9\rho\rho'\rho'' + 9\rho^2\rho'''$ for the semi-cubical parabola (at^2, at^3), we obtain the expression $\dfrac{(4+9t^2)^3}{t^3}$, which implies that there are no real values of t giving parameters of sextactic points, whereas there are imaginary values of t.

If we use the method of Section III, para. 20, to find the parameters of the sextactic points, we must substitute for ρ, ρ', ρ'', &c. in terms of t in the general equation of the conic of closest contact. On doing this we find that the terms contain a common factor which may be taken as $\dfrac{4+9t^2}{t^2}$, for small values of t, i.e. $\Delta = \dfrac{4+9t^2}{t^2}$.

On division by Δ, and differentiation with respect to s, we find that the terms of the resulting equation have a common factor $\theta = \dfrac{1}{at(4+9t^2)^{\frac{1}{2}}}$.

The condition for sextactic contact is then

$$\frac{at(4+9t^2)^{\frac{1}{2}}}{\frac{(4+9t^2)^2}{t^4} \times 6at(4+9t^2)^{3/2}} \times \frac{(4+9t^3)^3}{t^3} = 0, \text{ i.e. } t=0.$$

But when t is large, we may take Δ as $t^4(4+9t^2)$, and the condition then is

$$\frac{at(4+9t^2)^{\frac{1}{2}}}{t^8(4+9t^2)^2\, 6at(4+9t^2)^{3/2}} \times \frac{(4+9t^2)^3}{t^3} = 0,$$

$$\text{i.e. } \frac{1}{t^{11}} = 0, \text{ or } t = \infty.$$

Thus the only sextactic points are the cusp and the point of inflexion at infinity.

If we find the pole (x_1, y_1) of the straight line $p\xi + q\eta + r = 0$ with respect to the conic of closest contact at "t", we obtain the coordinates

$$x_1 = \frac{at^2(5r - 10pat^2 - 4qat^3)}{2r + 5pat^2 + 20qat^3},$$

$$y_1 = \frac{at^3(40r - 8pat^2 - 5qat^3)}{2(2r + 5pat^2 + 20qat^3)}.$$

The locus of (x_1, y_1) is thus in general a curve of the sixth degree, but for certain values of p, q, and r it degenerates into a cuspidal cubic with the same cusp and same inflexional tangent as the original curve $a\eta^2 = \xi^3$.

When $p = q = 0$, we find $x_1 = \frac{5}{2}at^2$; $y_1 = 10at^3$; and the corresponding locus is $32\xi^3 = 5a\eta^2$.

When $r = q = 0$, we find $x_1 = -2at^2$; $y_1 = -\frac{4}{5}at^3$; and the corresponding locus is $2\xi^3 + 25a\eta^2 = 0$.

When $p = r = 0$, we find $x_1 = -\frac{at^2}{5}$; $y_1 = -\frac{1}{8}at^3$; and the corresponding locus is $125\xi^3 + 64a\eta^2 = 0$.

If now the figure is projected so that the straight line $px + qy + r = 0$ becomes the new line at infinity, the cubic projects into the general cuspidal cubic, and the locus of the centres of the conics of closest contact is now a curve of the sixth degree.

But if the tangent at the cusp becomes the new line at infinity the corresponding locus will be a cuspidal cubic with the same inflexional tangent and cusp at infinity as the projection of the cubic. The equation of the cubic in that case will be of the form $a^2\eta = \xi^3$.

In the same way if the general cuspidal cubic is projected in such a way that the line joining the cusp and the point of inflexion becomes the new line at infinity the locus of the centres of the conics of closest contact of the projection will be a cuspidal cubic whose cusp and inflexion are at infinity. The equation of the cubic in that case will be of the form $\xi^2\eta = a^3$.

For the curve $a^2\eta = \xi^3$, the equation of the locus of the centres of the conics of closest contact is $64a^2\eta + 125\xi^3 = 0$.

For the curve $\xi^2\eta = a^3$, the corresponding equation is $25\xi^2\eta + 2a^3 = 0$.

1a) As an example of a nodal cubic we may take the Folium of Descartes whose equation is $x^3+y^3 = xy$. Any point on the folium may be written in terms of a parameter "t", viz., $x = \dfrac{t}{1+t^3}$, $y = \dfrac{t^2}{1+t^3}$, from which we can find

the equation of the conic of closest contact, viz.

$$(5t^9+5t^6+10t^3+1)x^2 - 2(5t^8+t^5+5t^2)xy + (t^{10}+10t^7+5t^4+5t)y^2 - (t^{10}+5t^4)x - (5t^6+1)y + t^5 = 0.$$

The centre of this conic is given by

$$\bar{\xi} = \frac{hf-bg}{ab-h^2} = \frac{t(t^3+5)}{10(t^3+1)}, \quad \bar{\eta} = \frac{gh-af}{ab-h^2} = \frac{5t^3+1}{10t(t^3+1)}.$$

The locus of the centres of the conics of closest contact is thus a curve of the fifth degree whose equation is

$$12500xy(x^3+y^3-xy) - 625(x^3+y^3) + 1125xy - 27 = 0.$$

The pole of the straight line $px+qy+r=0$ with respect to the conic of closest contact at "t" is

$$x_1 = \frac{\frac{r}{2}t^2(t^3+1)(t^3+5) + p/4\,(4t^3-1) + \frac{5}{4}qt^4}{5rt(t^3+1)^2 + \frac{p}{2}t^2(t^3+1)(t^3+5) + q/2\,(t^3+1)(5t^3+1)},$$

$$y_1 = \frac{r/2\,(t^3+1)(5t^3+1) + \frac{5}{4}pt^4 - q/4\,t^5(t^3-4)}{5rt(t^3+1)^2 + p/2\,t^2(t^3+1)(t^3+5) + q/2\,(t^3+1)(5t^3+1)},$$

from which it follows that the locus of (x_1, y_1) is in general a curve of the eighth degree.

When $p = q = 0$ it reduces to a curve of the fifth degree, as is also the case when $p = q = 1, r = 3$, i.e., when the straight line $px + qy + r = 0$ becomes the asymptote.

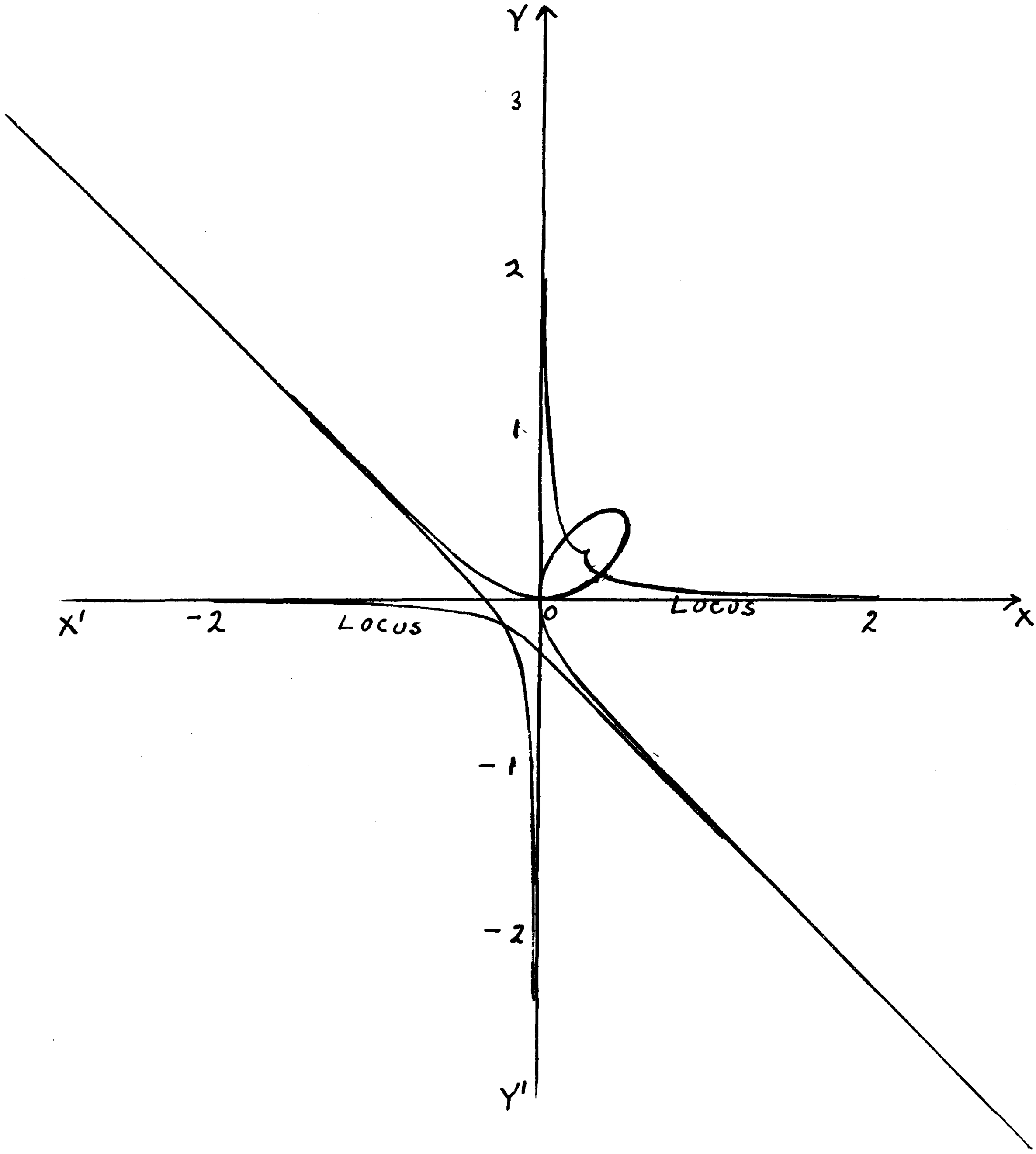
It follows that when the folium is projected so that the straight line $px + qy + r = 0$ becomes the line at infinity, the locus of the centres of the conics of closest contact of the projection which is the general nodal cubic will be in general a curve of the eighth degree. But when the line at infinity is the inflexional tangent or the line through the three inflexions this curve degenerates into one of the fifth degree.

The equation of the conic of closest contact at "t" on the folium is of the 10th degree in t. It follows that through any point in the plane there pass ten of the conics of closest contact. To find where the conic of closest contact at "t" meets the folium, let T be the parameter of the point (other than "t" itself), and substitute $\frac{T}{1+T^3}$ for x and $\frac{T^2}{1+T^3}$ for y in the equation of the conic. This gives $t^5T^6 - (5t^6+1)T^5 + \&c. = 0$.

It follows that $\Sigma T = 5t + \frac{1}{t^5}$.

But $\Sigma T = 5t + T$, hence $T = \frac{1}{t^5}$. Hence the conic meets the folium again at the point "$\frac{1}{t^5}$".

If the contact of the conic at "t" is of the fifth order, we must have $t^6 = 1$. $t^3 = -1$ gives the three inflexions, and $t^3 = 1$ gives the three other points on the folium at which the conic of closest contact has sextactic contact.



Graph of $x^3 + y^3 = xy$,
and the locus of the centres
of its conics of closest contact.

(1b) The general non-singular cubic can be projected so that its equation may be written $y^2 = 4x^3 - g_2 x - g_3$. The coordinates of any point on this cubic can be expressed in terms of a parameter by the use of Weierstrassian Elliptic Functions, i.e. we may write $x = \wp u \equiv \wp$, $y = \wp' u \equiv \wp'$, where $\wp' u = \frac{d}{du}\wp u$, and $(\wp' u)^2 = 4(\wp u)^3 - g_2 \wp u - g_3$.

The equation of any conic can be written $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$. Where this meets the given cubic we have

$$a\wp^2 + 2h\wp\wp' + b\wp'^2 + 2g\wp + 2f\wp' + c = 0.$$

To find the conic of closest contact at the point "u", we differentiate this equation four times with respect to u, and solve the five equations for the ratios $a : h : b : f : g : c$.

By successive differentiations and eliminations we find for the ratio $h : f$ the equation

$$h\left\{16\wp'^4(\wp'^2 - \wp\wp'') + (\wp''^2 - 12\wp\wp'^2)(-5\wp\wp''^2 + 4\wp'^2\wp'' + 12\wp^2\wp'^2)\right\}$$

$$= f\left\{16\wp'^4\wp'' + (\wp''^2 - 12\wp\wp'^2)(5\wp''^2 - 12\wp\wp'^2)\right\}, \text{---} \quad (\mathrm{I})$$

and by substitution in the remaining equations we obtain for the coefficients expressions of the following degrees in $\wp u$:

a, 13^{th}; b, 12^{th}; c, 15^{th}; f, $13\frac{1}{2}^{\text{th}}$; g, 14^{th}; h, $12\frac{1}{2}^{\text{th}}$.

From these we obtain

$$ab - h^2 = -A^4\left\{\tfrac{3}{4}\wp A^2 + \tfrac{3}{2}\wp'^2\wp'' A + 5\wp'^6\right\},$$

$$fh - bg = \wp(ab - h^2) + A^5\left\{\frac{\wp'' A}{16} + \frac{\wp'^4}{4}\right\},$$

$$gh - af = \wp'(ab - h^2) + A^5\left(\tfrac{3}{4}\wp A + \tfrac{1}{4}\wp'^2\wp''\right)\wp';$$

where A is written for $\wp''^2 - 12\wp\wp'^2$.

Hence the coordinates of the centre of the conic of closest contact at "u" are

$$\bar{\xi} = \frac{hf - bg}{ab - h^2} \equiv \frac{\text{expression of degree 10}}{\text{expression of degree 9}},$$

and $\bar{\eta} = \dfrac{gh - af}{ab - h^2} \equiv \dfrac{\text{expression of degree } 10\tfrac{1}{2}}{\text{expression of degree } 9}$.

The locus of the centre is thus a curve of degree 21 in (ξ, η).

The equation $ab - h^2 = 0$, (neglecting the case $A^4 = 0$), gives nine values of $\wp u$ to each of which there correspond two values of $\wp' u$; i.e. there are 18 points on the curve at which the conic of closest contact is a parabola.

At a point of inflexion $A = 0$, and we find $\bar{\xi} = \wp u$, $\bar{\eta} = \wp' u$. The locus therefore passes through the nine points of inflexion (including that at infinity).

To find the sextactic points we must differentiate equation (I) and eliminate $h : f$ between the resulting equation and (I).

This results in an equation of degree $14\tfrac{1}{2}$ in $\wp u$. It is satisfied by $A = 0$, i.e. by the eight finite points of inflexion; and by $\wp' = 0$, i.e. by the three points where the cubic cuts the x-axis. These are also the points of contact of the tangents from the point of inflexion at infinity.

On division by $A\wp'$ there remains an equation of degree 12, which leads to 24 other points on the cubic which are sextactic.

Including the point at infinity, there are 36 sextactic points, of which nine are the points of inflexion. The others lie on nine straight lines, each of these containing three of the points which are also points of contact of the tangents drawn from one of the inflexions.

These well-known results are discussed in Hilton, Plane Algebraic Curves, Chap. XVI, and in Salmon, Higher Plane Curves, Chap. V.

The representation of the coordinates on the general cubic by means of Elliptic Functions is also discussed by Hilton, Chap XVI.

The subject of the sextactic points on the general cubic may also be discussed with reference to the general condition of sextactic contact for curves whose equations are of the form $y = f(x)$, viz. $9y_2^2 y_5 + 40y_3^3 - 45y_2 y_3 y_4 = 0$, where $y_2 \equiv \frac{d^2y}{dx^2}$, &c.

Writing the equation of the cubic in the form $y^2 = 4x^3 - g_2 x - g_3$, we find

$$2yy_1 = 12x^2 - g_2;$$

$$y_2 = \frac{12xy^2 - \frac{1}{4}(12x^2-g_2)^2}{y^3};$$

$$\equiv A/y^3;$$

$$y_3 = \frac{y^2\frac{dA}{dx} - \frac{3}{2}A(12x^2-g_2)}{y^5} \equiv \frac{B}{y^5};$$

$$y_4 = \frac{y^2\frac{dB}{dx} - \frac{5}{2}B(12x^2-g_2)}{y^7} \equiv \frac{C}{y^7};$$

$$y_5 = \frac{y^2\frac{dC}{dx} - \frac{7}{2}C(12x^2-g_2)}{y^9} \equiv \frac{D}{y^9}.$$

Substitution of y_2, y_3, y_4, y_5 in the general condition gives

$9A^2D + 40B^3 - 45ABC = 0$; an equation of the 18^{th} degree in x. The equation is satisfied by $y^4 = 0$, giving three sextactic points. There remain twelve other values of x, each giving two sextactic points. This equation does not give the points of inflexion.

(2) The coordinates of any point on the curve $s = c\tan^2\psi$ may be written

$$x = 2c\sec\psi,\quad y = c\left\{\tan\psi\sec\psi - \log\tan\left(\frac{\pi}{4}+\frac{\psi}{2}\right)\right\}.$$

It follows that $l = \cos\psi$, $m = \sin\psi$;

$$l' = \frac{-\cos^3\psi}{2c},\quad m' = \frac{\cos^4\psi}{2c\sin\psi};$$

$$\rho = 2c\tan\psi\sec^2\psi,\quad \rho' = 3\tan\psi + \cot\psi,$$

$$\rho\rho'' = 3\sec^2\psi - \operatorname{cosec}^2\psi;\quad 9+\rho'^2-3\rho\rho'' = 9+4\cot^2\psi;$$

$$\frac{3\rho(l\rho'-3m)}{9+\rho'^2-3\rho\rho''} = \frac{6c\sec\psi}{9+4\cot^2\psi},\quad \frac{3\rho(m\rho'+3l)}{9+\rho'^2-3\rho\rho''} = \frac{6c\sin\psi(3+\cos^2\psi)}{\cos^4\psi(9+4\cot^2\psi)};$$

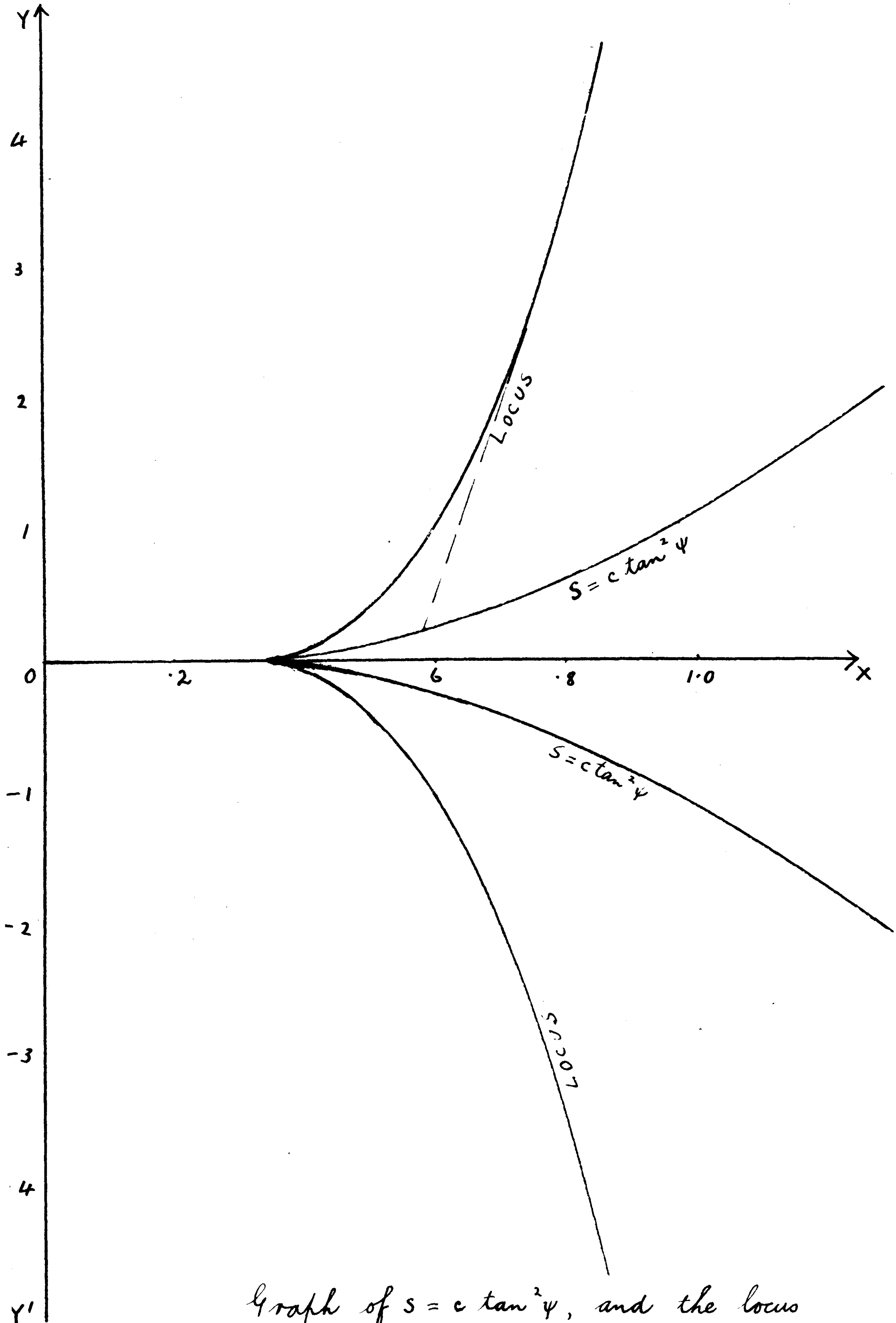
$$\bar{\xi} = 2c\sec\psi + \frac{6c\sec\psi}{9+4\cot^2\psi},$$

$$\bar{\eta} = c\left\{\tan\psi\sec\psi - \log\tan\left(\frac{\pi}{4}+\frac{\psi}{2}\right)\right\} + \frac{6c\sin\psi\,(3+\cos^2\psi)}{\cos^4\psi\,(9+4\cot^2\psi)}.$$

Data for the graph of $s = c\tan^2\psi$ $(c = \frac{1}{6})$

x	·335	·339	·345	·355	·368	·385	·407
y	·000	·001	·002	·005	·011	·020	·034
$\bar{\xi}$	·337	·346	·361	·382	·408	·440	·478
$\bar{\eta}$	·001	·006	·020	·049	·098	·178	·306

x	·435	·471	·519	·581	·667	·789	·975	1·29	1·92
y	·055	·089	·141	·223	·358	·595	1·05	2·07	5·04
$\bar{\xi}$	·524	·580	·650	·740	·860	1·03	1·28	1·70	2·55
$\bar{\eta}$	·511	·850	1·44	2·52	4·72	9·75	23·5	73·1	—



Graph of $s = c \tan^2\psi$, and the locus of its centres of conics of closest contact.

(3) For the exponential curve $y = e^x$, we have

$l = (1+e^{2x})^{-\frac{1}{2}}$, $m = e^x(1+e^{2x})^{-\frac{1}{2}}$;

$l' = -e^{2x}(1+e^{2x})^{-2}$, $m' = e^x(1+e^{2x})^{-2}$;

$\rho = e^{-x}(1+e^{2x})^{3/2}$, $\rho' = 2e^x - e^{-x}$, $\rho'' = (2e^x+e^{-x})(1+e^{2x})^{-\frac{1}{2}}$;

$9 - 3\rho\rho'' + \rho'^2 = -2(e^x+e^{-x})^2 < 0$. Thus the conics of closest contact are hyperbolas.

$$\frac{3\rho}{9+\rho'^2-3\rho\rho''} = -\frac{3}{2}e^{x/2}(e^x+e^{-x})^{-\frac{1}{2}};$$

$l\rho' - 3m = -(e^x+e^{-x})(1+e^{2x})^{-\frac{1}{2}}$;

$m\rho' + 3l = 2(e^{2x}+1)^{\frac{1}{2}}$;

$\bar{\xi} = x + \frac{3}{2}$, $\bar{\eta} = -2e^x$, and the locus of the centre of the conic of closest contact is a second exponential curve whose equation is

$y = -2e^{x-3/2}$.

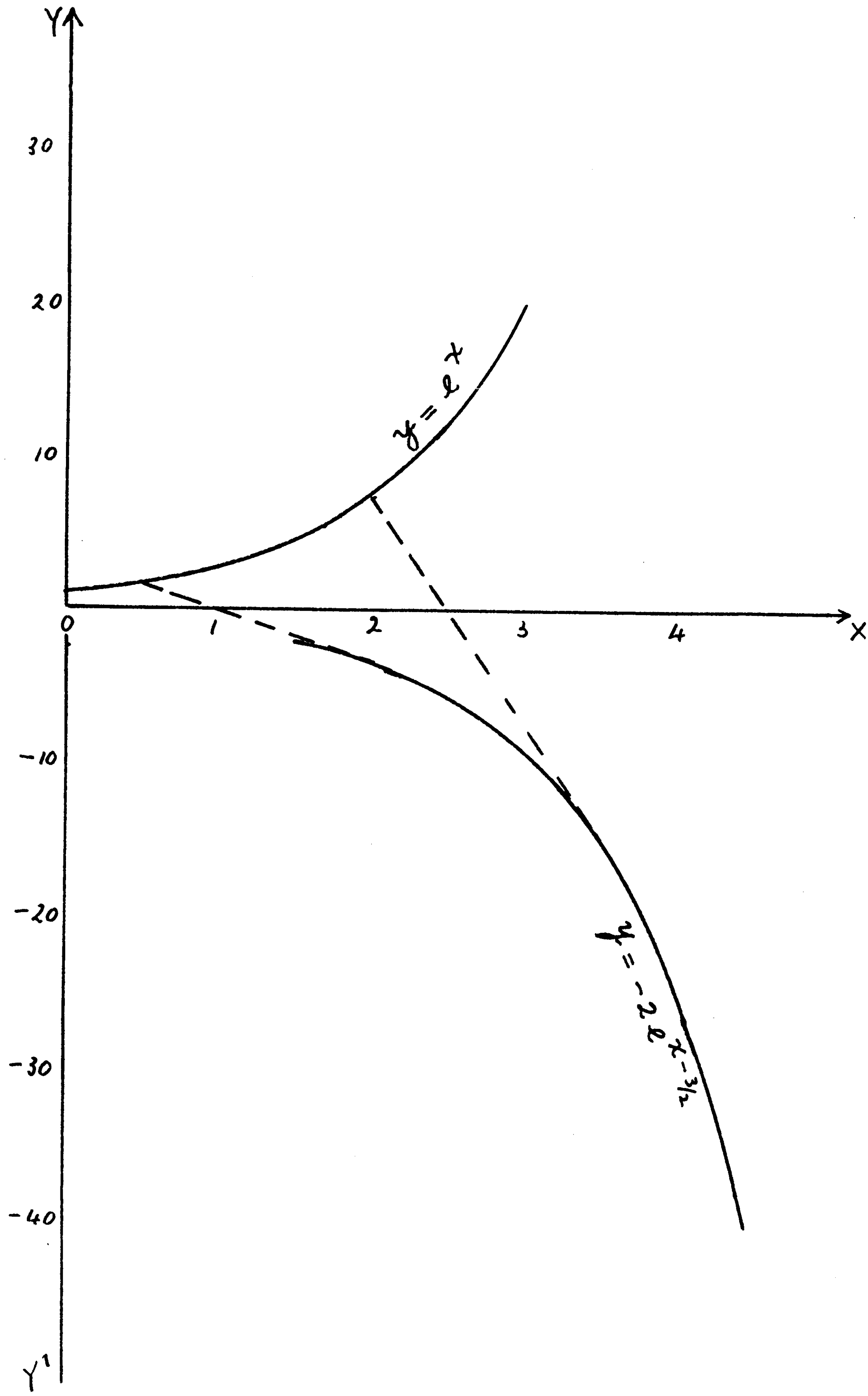
Data for the graphs of $y = e^x$ and of $y = -2e^{x-3/2}$.

x	0	·2	·4	·6	·8	1·0	1·2	1·4	1·6	1·8
y	1·00	1·22	1·49	1·82	2·23	2·72	3·32	4·06	4·95	6·05
$\bar{\xi}$	1·5	1·7	1·9	2·1	2·3	2·5	2·7	2·9	3·1	3·3
$\bar{\eta}$	−2·00	−2·44	−2·98	−3·64	−4·46	−5·44	−6·64	−8·12	−9·90	−12·10
x	2·0	2·2	2·4	2·6	2·8	3·0				
y	7·39	9·02	11·02	13·47	16·48	20·09				
$\bar{\xi}$	3·5	3·7	3·9	4·1	4·3	4·5				
$\bar{\eta}$	−14·78	−18·04	−22·04	−26·74	−32·96	−40·18				

The equation of the conic of closest contact at (x, y) is

$$2e^{2x}\xi^2 - \eta^2 + 8e^x\xi\eta + 2e^{2x}(5-2x)\xi - 8e^x(2+x)\eta + e^{2x}(2x^2-10x+17) = 0,$$

from which it can be found that the conic of closest contact has six-point contact with the curve when $x = -\infty$, at which point the conic degenerates into the pair of coincident straight lines $\eta^2 = 0$.



Graph of $y = e^x$ and of the locus of the centres of conics of closest contact.

(4) The equation of the involute of a circle may be written $s = \frac{1}{2} a \psi^2$.

From this the Cartesian coordinates of any point on the curve may be found from the formulae

$$x = \int \cos\psi \, ds \; ; \quad y = \int \sin\psi \, ds.$$

These give $x = a(\psi \sin\psi + \cos\psi)$, $y = a(-\psi\cos\psi + \sin\psi)$.

It follows that $l = \cos\psi$, $m = \sin\psi$;

$$l' = \frac{-\sin\psi}{a\psi}, \quad m' = \frac{\cos\psi}{a\psi};$$

$$\rho = a\psi, \quad \rho' = \frac{1}{\psi}, \quad \rho\rho'' = -\frac{1}{\psi^2}, \quad 9 + \rho'^2 - 3\rho\rho'' = 9 + \frac{4}{\psi^2};$$

$$\frac{3\rho(l\rho' - 3m)}{9 + \rho'^2 - 3\rho\rho''} = \frac{3a\psi^2(\cos\psi - 3\psi\sin\psi)}{4 + 9\psi^2},$$

$$\frac{3\rho(m\rho' + 3l)}{9 + \rho'^2 - 3\rho\rho''} = \frac{3a\psi^2(\sin\psi + 3\psi\cos\psi)}{4 + 9\psi^2};$$

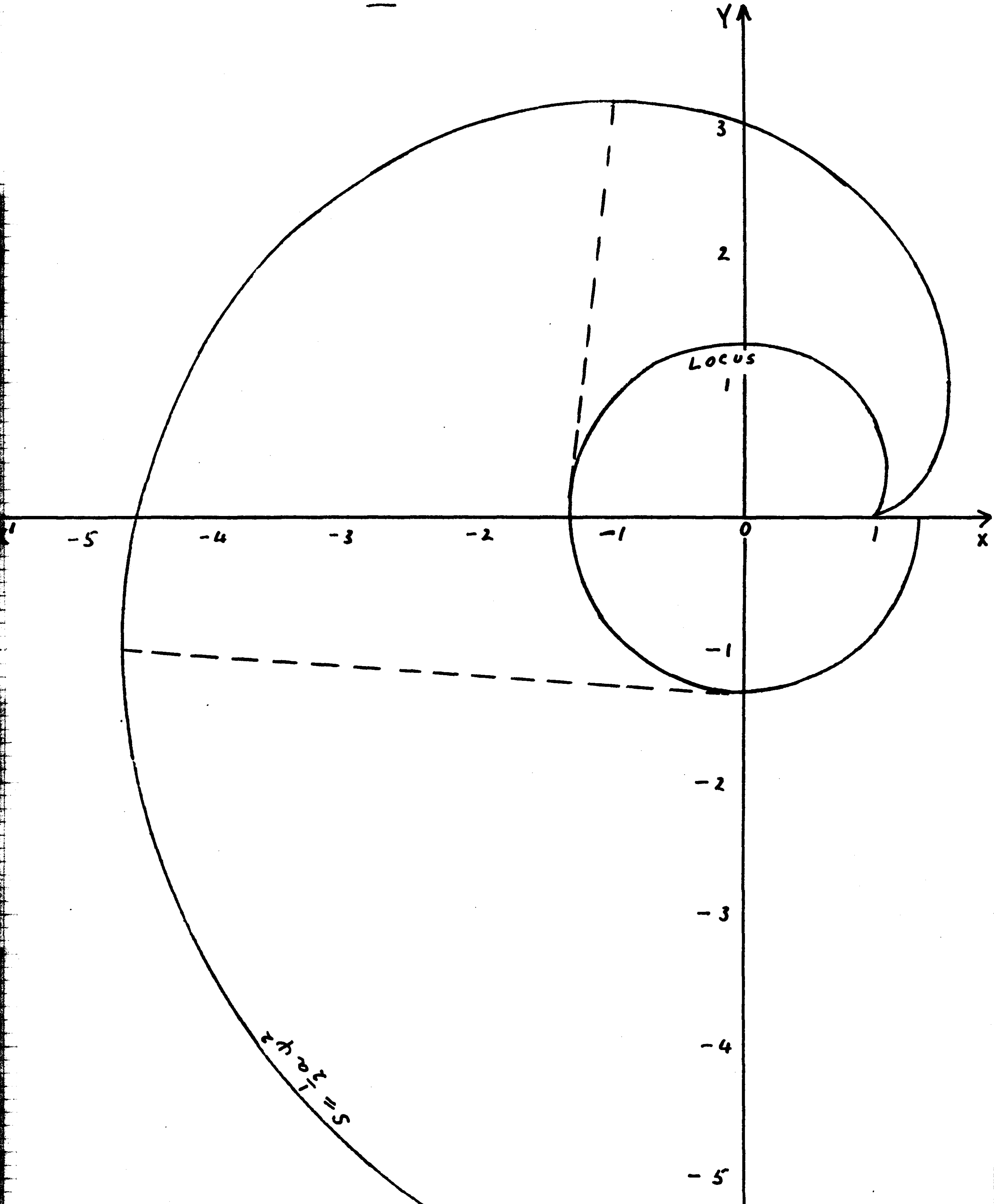
$$\bar{\xi} = \frac{4a(\cos\psi + \psi\sin\psi + 3\psi^2\cos\psi)}{4 + 9\psi^2},$$

$$\bar{\eta} = \frac{4a(\sin\psi - \psi\cos\psi + 3\psi^2\sin\psi)}{4 + 9\psi^2}.$$

The locus of the centres of the conics of closest contact approximates very rapidly to a circle of radius $\frac{4a}{3}$ with its centre at the centre of the generating circle of the involute.

Data for the involute $s = \frac{1}{2} a \psi^2$ $(a = 1)$.

x	1·41	1·57	1·31	·44	−1·00	−2·13	−3·23	−4·13
y	·34	1·00	1·91	2·77	3·14	2·94	2·30	1·23
$\bar{\xi}$	·88	·24	−·49	−1·05	−1·32	−1·28	−1·09	−·67
$\bar{\eta}$	·92	1·28	1·22	·80	·14	−·34	−·76	−1·10
x	−4·64	−4·71	−4·64	−4·03	−2·82	−1·09		
y	−·20	−1·00	−1·83	−3·48	−4·92	−5·92		
$\bar{\xi}$	−·32	−·09	·14	·59	·97	1·22		
$\bar{\eta}$	−1·29	−1·32	−1·32	−1·19	−·92	−·52		



Graph of $s = \frac{1}{2} a \psi^2$, and the locus of the centres of its conics of closest contact.

(5) The coordinates of any point on the spiral $r = a\theta$ may be written $x = a\theta\cos\theta$, $y = a\theta\sin\theta$; we find

$$l = (\cos\theta - \theta\sin\theta)(1+\theta^2)^{-\frac{1}{2}}; \quad m = (\sin\theta + \theta\cos\theta)(1+\theta^2)^{-\frac{1}{2}};$$

$$l' = \frac{-(2+\theta^2)(\sin\theta + \theta\cos\theta)}{a(1+\theta^2)^2}, \quad \rho = \frac{a(1+\theta^2)^{3/2}}{2+\theta^2};$$

$$\rho' = \frac{\theta(4+\theta^2)}{(2+\theta^2)^2}, \quad \rho'' = \frac{8-6\theta^2-\theta^4}{a(2+\theta^2)^3(1+\theta^2)^{\frac{1}{2}}},$$

$$9 + \rho'^2 - 3\rho\rho'' = \frac{(1+\theta^2)^2(120+58\theta^2+9\theta^4)}{(2+\theta^2)^4} > 0$$

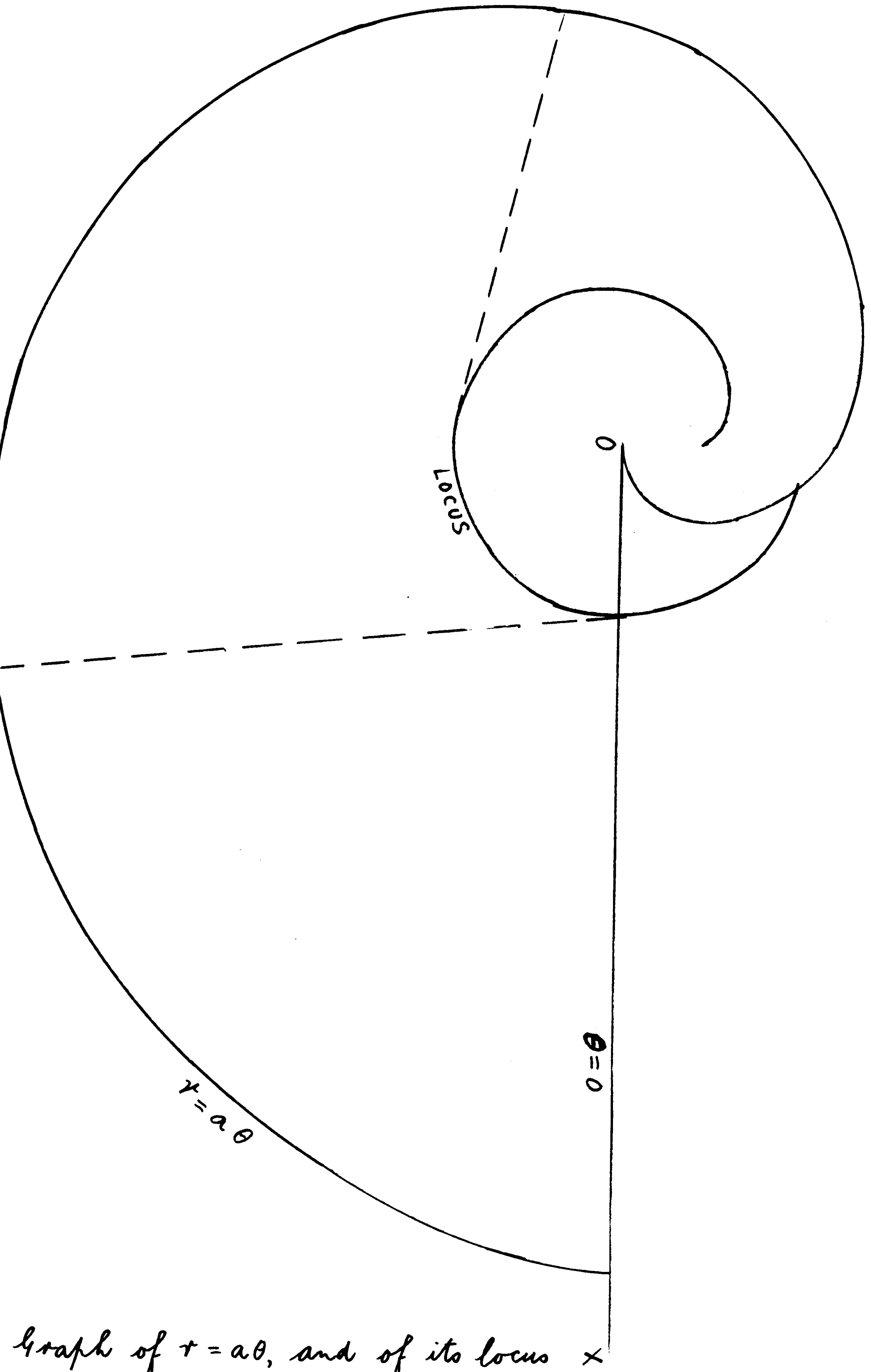
$$\bar{\xi} = \frac{8a\theta\cos\theta(9+2\theta^2) - 12a\sin\theta(2+\theta^2)(3+\theta^2)}{120+58\theta^2+9\theta^4}$$

$$\bar{\eta} = \frac{8a\theta\sin\theta(9+2\theta^2) + 12a\cos\theta(2+\theta^2)(3+\theta^2)}{120+58\theta^2+9\theta^4}$$

The conics of closest contact are ellipses, and the locus of the centres of the conics approximates very rapidly to a circle of radius $\frac{4a}{3}$ whose centre is the origin.

<u>Data for the graphs of $r = a\theta$ and of its locus of centres of conics of closest contact ($a=1$).</u>

θ° (approx.)	0	17	$34\frac{1}{2}$	$51\frac{1}{2}$	$68\frac{1}{2}$	86	103	$120\frac{1}{2}$
r	0	·3	·6	·9	1·2	1·5	1·8	2·1
θ° (locus)	90	91	$98\frac{1}{2}$	$110\frac{1}{2}$	$126\frac{1}{2}$	145	$163\frac{1}{2}$	183
r (locus)	·60	·64	·75	·86	·95	1·05	1·11	1·17
θ° (approx.)	$134\frac{1}{2}$	$154\frac{1}{2}$	172	189	$206\frac{1}{2}$	$223\frac{1}{2}$	$240\frac{1}{2}$	
r	2·4	2·7	3·0	3·3	3·6	3·9	4·2	
θ° (locus)	$201\frac{1}{2}$	$220\frac{1}{2}$	$239\frac{1}{2}$	258	277	$295\frac{1}{2}$	$313\frac{1}{2}$	
r (locus)	1·20	1·22	1·25	1·26	1·27	1·28	1·29	
θ° (approx.)	258	275	291	$308\frac{1}{2}$	$325\frac{1}{2}$	343	360	
r	4·5	4·8	5·1	5·4	5·7	6·0	6·3	
θ° (locus)	332	350	$366\frac{1}{2}$	385	$402\frac{1}{2}$	$420\frac{1}{2}$	$438\frac{1}{2}$	
r (locus)	1·30	1·30	1·30	1·31	1·31	1·31	1·32	



Graph of $r = a\theta$, and of its locus of centres of conics of closest contact

(6) For the catenary $y = c\cosh\frac{x}{c}$, we have

$l = \operatorname{sech}\frac{x}{c}$, $m = \tanh\frac{x}{c}$;

$l' = -\frac{1}{c}\operatorname{sech}^2\frac{x}{c}\tanh\frac{x}{c}$, $m' = \frac{1}{c}\operatorname{sech}^3\frac{x}{c}$;

$\rho = c\cosh^2\frac{x}{c}$, $\rho' = 2\sinh\frac{x}{c}$, $\rho'' = \frac{2}{c}$;

$9 + \rho'^2 - 3\rho\rho'' = 5 - 2\cosh^2\frac{x}{c}$

$l\rho' - 3m = -\tanh\frac{x}{c}$, $m\rho' + 3l = \operatorname{sech}\frac{x}{c} + 2\cosh\frac{x}{c}$;

$$\bar{\xi} = x - \frac{3c}{2}\cdot\frac{\sinh\frac{2x}{c}}{4 - \cosh\frac{2x}{c}}, \quad \bar{\eta} = c\cdot\frac{11\cosh\frac{x}{c} + \cosh\frac{3x}{c}}{4 - \cosh\frac{2x}{c}}.$$

The conics of closest contact are ellipses for values of x which satisfy $\cosh\frac{2x}{c} < 4$, hyperbolas for values of x satisfying $\cosh\frac{2x}{c} > 4$. At the points where x satisfies $\cosh\frac{2x}{c} = 4$, the conics of closest contact are parabolas.

The locus of the centres of the conics of closest contact of the catenary has a pair of symmetrically-placed asymptotes whose equations are

$$\frac{\eta - c\sqrt{2.5}}{\xi - \frac{c}{2}\cosh^{-1}(4)} = -2\sqrt{6}, \text{ and } \frac{\eta - c\sqrt{2.5}}{\xi + \frac{c}{2}\cosh^{-1}(4)} = 2\sqrt{6}.$$

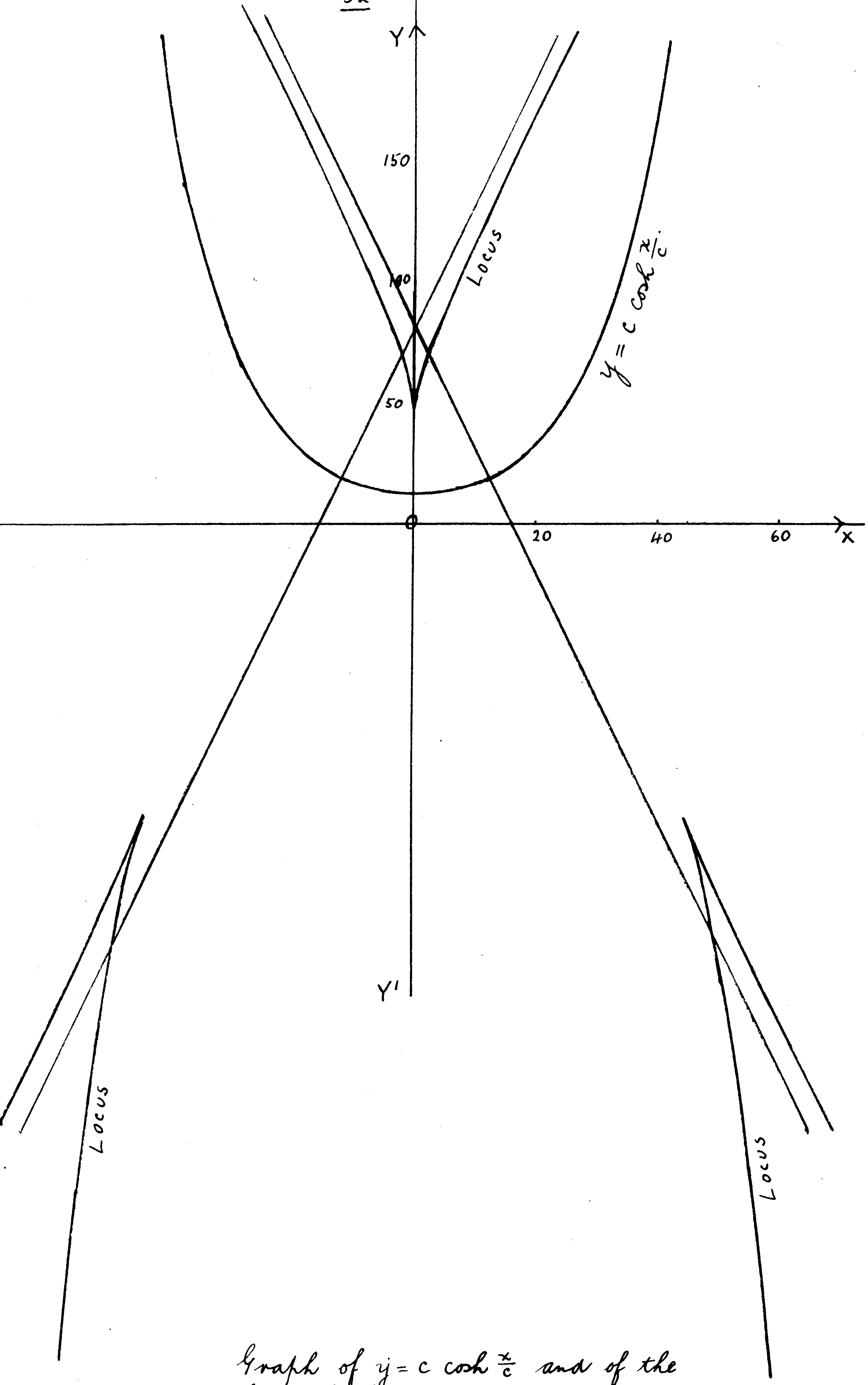
These are shown in pencil on the graph.

Data for the catenary $y = c\cosh\frac{x}{c}$ $(c = 12)$.

x	0	2	4	6	8	10	12	14	16	18
y	12	12.17	12.67	13.53	14.77	16.41	18.52	21.14	24.34	28.23
$\bar{\xi}$	0	−.08	−.66	−2.62	−8.10	−26.6	−262	90.4	55.9	47.7
$\bar{\eta}$	48.0	50.1	57.0	72.1	105.4	202	1365	−358	−184	−140.4

x	20	22	24	26	28	30	32	34
y	32.9	38.5	45.1	53.1	62.4	73.6	86.8	102.4
$\bar{\xi}$	45.0	44.6	45.1	46.2	47.5	49.0	50.7	52.5
$\bar{\eta}$	−124.9	−121.5	−125.0	−134.0	−147.5	−166.0	−189.2	−218

x	36	38	40	42	44
y	120.8	142.6	168.8	198.9	254.9
$\bar{\xi}$	54.4	56.3	58.2	60.2	62.1
$\bar{\eta}$	−252	−295	−345	−414	−782



Graph of $y = c \cosh \frac{x}{c}$ and of the locus of the centres of its conics of closest contact.

(4) For the catenary of equal strength $y = a \log \sec \frac{x}{a}$ we have:–

$$l = \cos \frac{x}{a}, \quad m = \sin \frac{x}{a},$$

$$l' = -\frac{1}{a} \sin \frac{x}{a} \cos \frac{x}{a}, \quad m' = \frac{1}{a} \cos^2 \frac{x}{a};$$
$$\rho = a \sec \frac{x}{a}, \quad \rho' = \tan \frac{x}{a}, \quad \rho'' = \frac{1}{a} \sec \frac{x}{a};$$
$$9 + \rho'^2 - 3\rho\rho'' = 6 - 2\tan^2 \frac{x}{a}.$$

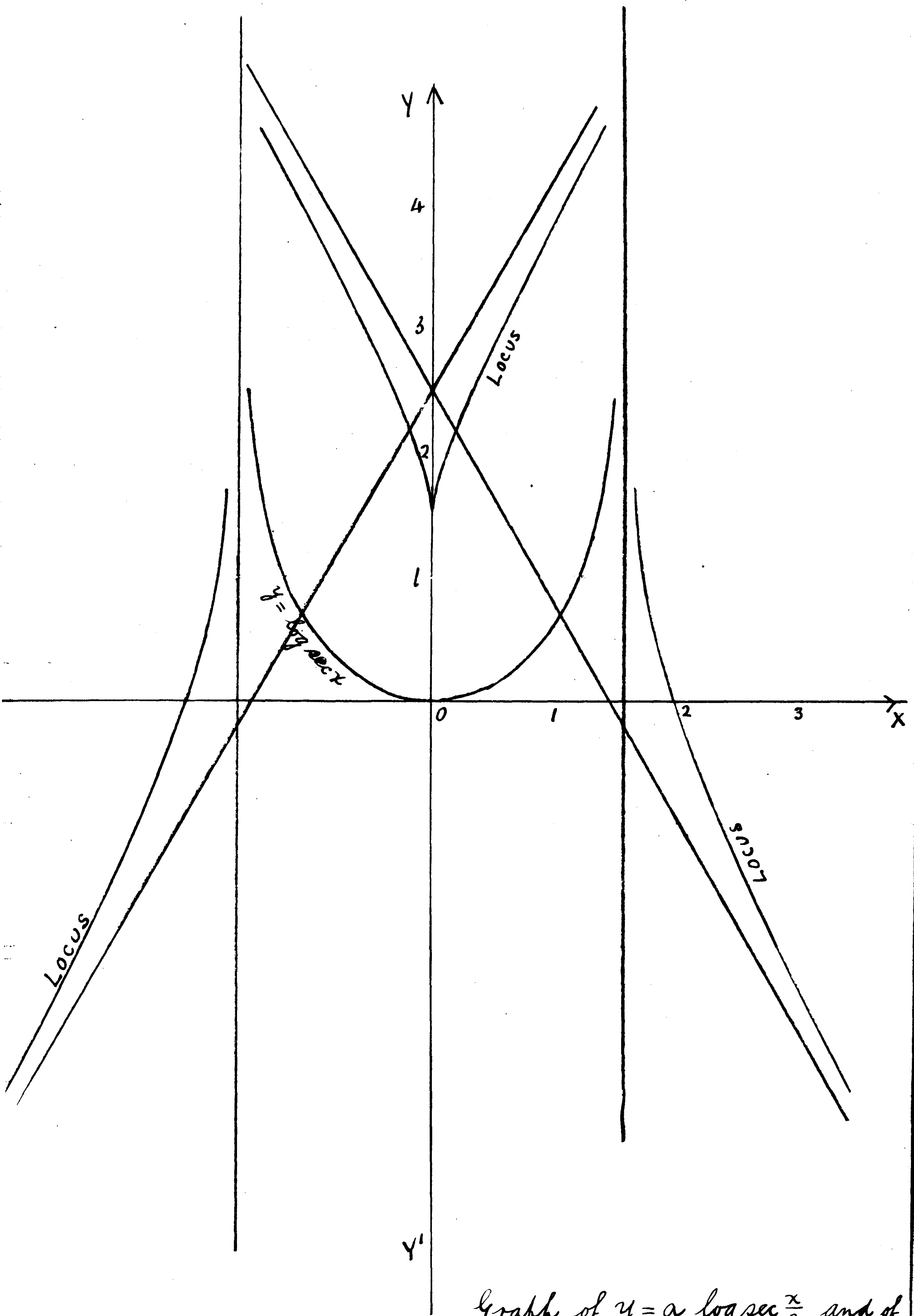
$$\bar{\xi} = x - \frac{3a \sin \frac{2x}{a}}{2(1 + 2\cos \frac{2x}{a})}, \quad \bar{\eta} = y + \frac{3a(2 + \cos \frac{2x}{a})}{2(1 + 2\cos \frac{2x}{a})}.$$

The conics of closest contact are ellipses for values of x which satisfy $\tan^2 \frac{x}{a} < 3$, hyperbolas for values of x satisfying $\tan^2 \frac{x}{a} > 3$. At the points where x satisfies $\tan \frac{x}{a} = \pm\sqrt{3}$, the conics of closest contact are parabolas, and the axes of these parabolas are asymptotes to the locus of the centres of the conics of closest contact. The equations of these asymptotes are

$$\sqrt{3}(\eta - a \log 2) = \pi a \pm 3\xi.$$

Data for the curve $y = a \log \sec \frac{x}{a}$ $(a = 1)$.

x	·087	·175	·262	·349	·436	·524	·611	·698
y	·004	·015	·035	·062	·098	·144	·200	·266
$\bar{\xi}$	−·001	−·003	−·013	−·032	−·051	−·126	−·224	−·396
$\bar{\eta}$	1·51	1·55	1·61	1·70	1·83	2·01	2·29	2·68
x	·785	·873	·960	1·05	1·13	1·22	1·31	1·40
y	·347	·439	·556	·693	·860	1·07	1·35	1·45
$\bar{\xi}$	−·715	−1·397	−3·49	∞	+5·15	+3·04	+2·35	1·98
$\bar{\eta}$	3·35	4·64	8·44	∞	−6·25	−2·43	−·97	−·09
x	1·48	1·57						
y	2·44	∞						
$\bar{\xi}$	1·75	1·57						
$\bar{\eta}$	+·84	∞						



Graph of $y = a \log \sec \frac{x}{a}$ and of the locus of the centres of its conics of closest contact.

(8) The equation of the cardioid may be written $s = 4a\left(1-\cos\frac{\psi}{3}\right)$, from which ~~we find~~ the Cartesian coordinates may be found. These are

$$x = a\left(\cos\frac{2\psi}{3} - \frac{1}{2}\cos\frac{4\psi}{3}\right),\quad y = a\left(\sin\frac{2\psi}{3} - \frac{1}{2}\sin\frac{4\psi}{3}\right).$$

It follows that $l = \cos\psi$, $m = \sin\psi$;

$$l' = -\frac{3}{4a}\frac{\sin\psi}{\sin\frac{\psi}{3}},\quad m' = \frac{3}{4a}\frac{\cos\psi}{\cos\frac{\psi}{3}};$$

$$\rho = \frac{4a}{3}\sin\frac{\psi}{3},\quad \rho' = \frac{1}{3}\cot\frac{\psi}{3},\quad \rho\rho'' = -\frac{1}{9}\operatorname{cosec}^2\frac{\psi}{3};$$

$$9 + \rho'^2 - 3\rho\rho'' = \frac{4}{9}\left(21 + \cot^2\frac{\psi}{3}\right);$$

$$\frac{3\rho(l\rho' - 3m)}{9+\rho'^2-3\rho\rho''} = \frac{3a\left(5\cos\frac{4\psi}{3} - 4\cos\frac{2\psi}{3}\right)}{21+\cot^2\frac{\psi}{3}},$$

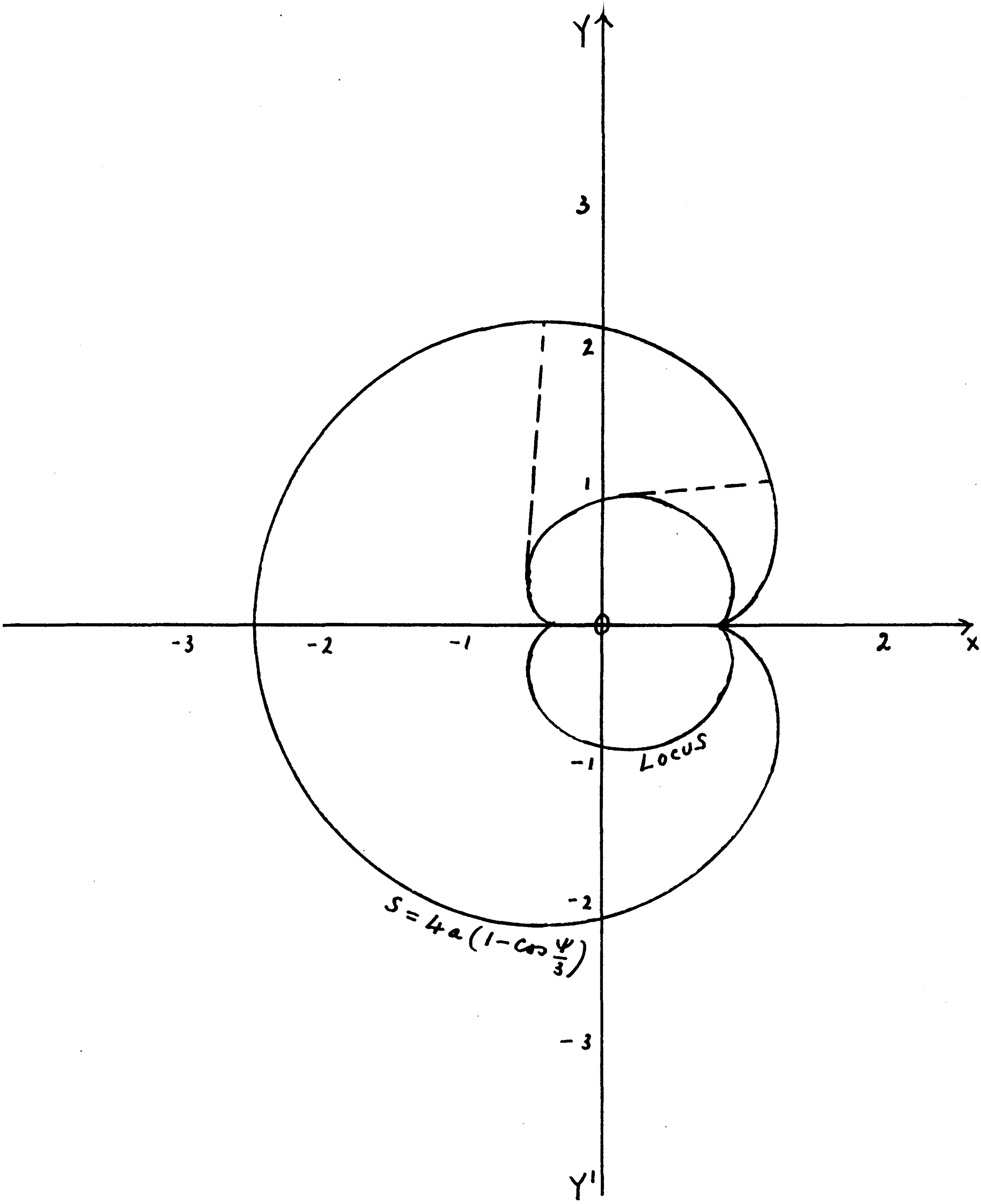
$$\frac{3\rho(m\rho' + 3l)}{9+\rho'^2-3\rho\rho''} = \frac{3a\left(5\sin\frac{4\psi}{3} - 4\sin\frac{2\psi}{3}\right)}{21+\cot^2\frac{\psi}{3}};$$

$$\bar{\xi} = a\left(\cos\frac{2\psi}{3} - \frac{1}{2}\cos\frac{4\psi}{3}\right) + \frac{3a\left(5\cos\frac{4\psi}{3} - 4\cos\frac{2\psi}{3}\right)}{21+\cot^2\frac{\psi}{3}},$$

$$\bar{\eta} = a\left(\sin\frac{2\psi}{3} - \frac{1}{2}\sin\frac{4\psi}{3}\right) + \frac{3a\left(5\sin\frac{4\psi}{3} - 4\sin\frac{2\psi}{3}\right)}{21+\cot^2\frac{\psi}{3}}.$$

Data for the graph of $s = 4a\left(1-\cos\frac{\psi}{3}\right)$ and for the locus of the centres of its conics of closest contact $\left(a=\frac{5}{3}\right)$

x	·83	·86	·93	1·03	1·13	1·22	1·25	1·21	1·04	
y	0	·00	·03	·11	·25	·45	·72	1·03	1·36	
$\bar{\xi}$	·83	·88	·93	·89	·75	·55	·36	·10	−·10	
$\bar{\eta}$	0	·01	·21	·45	·66	·82	·90	·91	·86	
x	·83	·49	·07	−·42	−·93	−1·42	−1·86	−2·20	−2·42	−2·50
y	1·67	1·93	2·10	2·17	2·10	1·89	1·56	1·11	·57	0
$\bar{\xi}$	−·30	−·43	−·50	−·53	−·53	−·49	−·45	−·40	−·37	−·36
$\bar{\eta}$	·46	·63	·48	·34	·22	·15	·06	·03	·02	0



Graph of $s = 4a(1-\cos\frac{\psi}{3})$ and the locus of the centres of its conics of closest contact.

(8a) As an example of a tricuspidal quartic with real cusps we may take the curve whose equation is given in terms of a parameter "t", viz., $x = \frac{1}{(t-1)^2}$, $y = \frac{1}{(t+1)^2}$. This quartic has a real cusp at the origin ($t = \infty$), and two real cusps at infinity ($t = \pm 1$).

The equation of the conic of closest contact at "t" is

$$\frac{1}{20}(t-1)^6(5t^2-18t-3)x^2 + \frac{1}{20}(t+1)^6(5t^2+18t-3)y^2$$
$$-\frac{1}{2}(t^2-1)^3(t^2-33)xy$$
$$-2(t-1)^3(4t^2+9t+3)x + 2(t+1)^3(4t^2-9t+3)y - \frac{4}{5}(t^2-6) = 0$$

The coordinates of the centre of this conic are

$$\bar{\xi} = \frac{5(t^2+6t-3)}{2(t-1)^2(t^2-21)}, \quad \bar{\eta} = \frac{5(t^2-6t-3)}{2(t+1)^2(t^2-21)}.$$

When $t = \infty$, the conic degenerates into the pair of coincident tangents at the cusp at the origin, viz. $(y-x)^2 = 0$.

When $t = 1$, the conic degenerates into the pair of coincident tangents at one of the cusps at infinity, viz $(4y-1)^2 = 0$.

Similarly, when $t = -1$, the conic degenerates into the pair of coincident tangents at the second cusp at infinity, viz., $(4x-1)^2 = 0$.

When $t = \pm\sqrt{21}$, the values of $\bar{\xi}$ and $\bar{\eta}$ are infinite, and the corresponding conics of closest contact are parabolas.

The locus of the centres of the conics of closest contact is a curve of the sixth degree with cusps at $t = \pm 1, 0, \pm 3, \pm\infty$, and asymptotes parallel to $100y + (586 \pm 126\sqrt{21})x = 0$, through the points $x = \frac{11 \pm \sqrt{21}}{200}$, $y = \frac{11 \mp \sqrt{21}}{200}$.

If we find the condition that the conic of closest contact at "t" should have six-point contact with the quartic we obtain the equation $t(t^2-9)(t^2-1) = 0$, the coefficient of the term in t^6 being zero. The roots $t = \infty, 1, -1$, correspond to the cusps, and the roots, $t = 0, \pm 3$, correspond to the three sextactic points.

When $t = 0$, the equation of the conic of closest contact is $x^2 + y^2 + 110xy - 40x - 40y - 32 = 0$.

When $t = 3$, the equation of the conic of closest contact is $-16x^2 + 8192y^2 + 2560xy - 440x + 640y - 1 = 0$, and when $t = -3$, it is $8192x^2 - 16y^2 + 2560xy + 640x - 440y - 1 = 0$.

(8b) The cardioid $s = 4a(1 - \cos\frac{\psi}{3})$ is also a tri-cuspidal quartic with one finite real cusp, and a pair of imaginary cusps at infinity (at the circular points). Its equation may be written

$$(x^2+y^2)^2 + 2ax(x^2+y^2) = a^2y^2, \text{ where } x + \tfrac{a}{2}$$

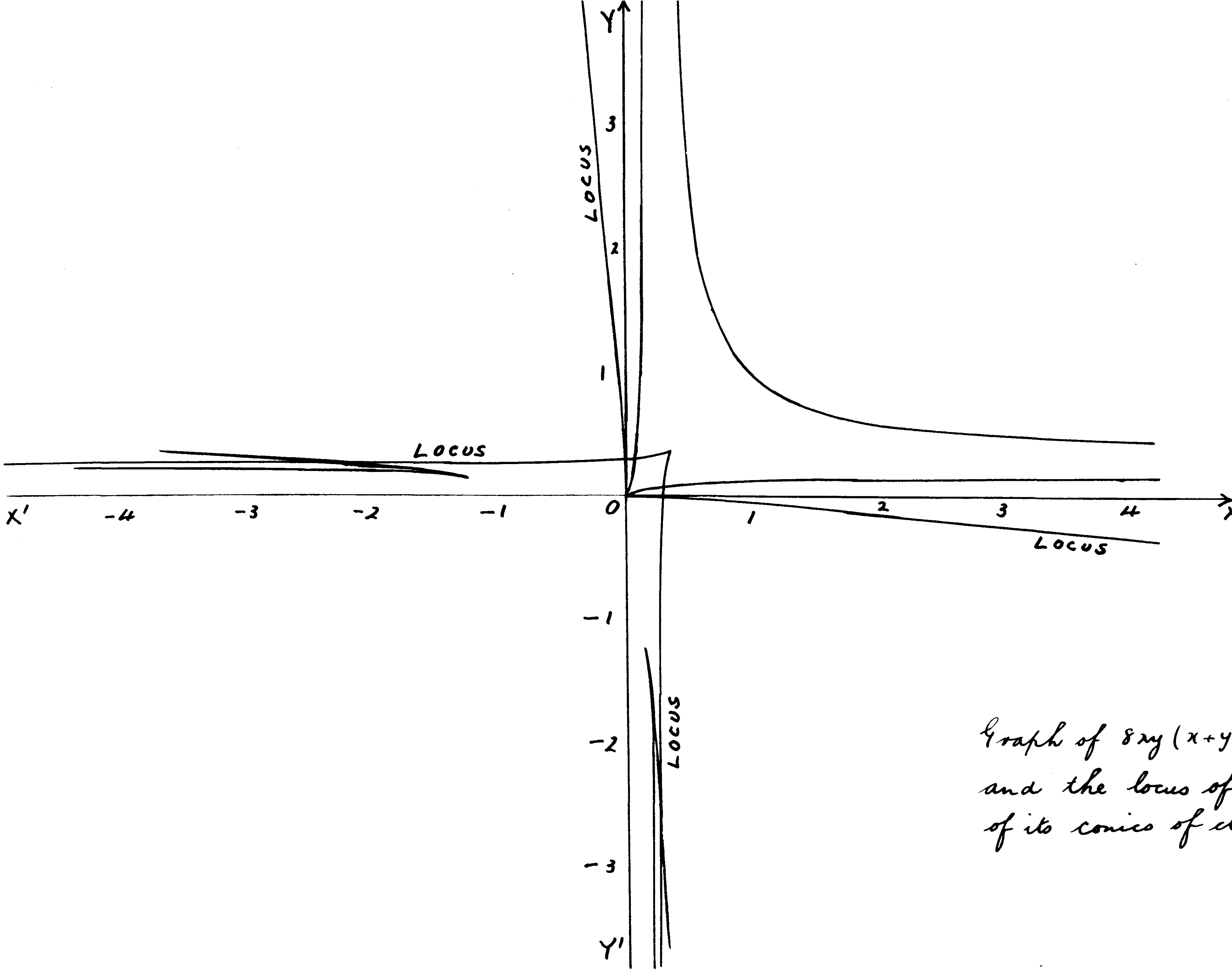
is written for $a(\cos\frac{2\psi}{3} - \frac{1}{2}\cos\frac{4\psi}{3})$.

The coordinates $(\bar{\xi}, \bar{\eta})$ of the centre of the conic of closest contact at "ψ" are

$$\bar{\xi} = -\frac{5a(t-1)(2t^2+2t-3)}{22-20t}, \quad \bar{\eta}^2 = \frac{25a^2(1-t^2)^3}{(11-10t)^2},$$

where the parameter "t" is written for $\cos\frac{2\psi}{3}$. The elimination of "t" leads to the equation of a curve of the sixth degree.

(8c) The three-cusped hypocycloid $\rho = \sin 3\psi$ is a curve of this class. It has three finite real cusps arranged symmetrically at the vertices of an equiangular triangle. The locus of the centres of its conics of closest contact is a curve with six real cusps arranged symmetrically at the vertices of a regular hexagon, alternate cusps coinciding with the cusps of the hypocycloid. This curve is of the tenth degree. It is shown in Section XXIII, no. 11.



Graph of $8xy(x+y-2xy) = (x-y)^2$, and the locus of the centres of its conics of closest contact.

If we find the pole of the straight line $px+qy+r=0$ with respect to the conic of closest contact at the point "t" on the tricuspidal quartic $x=\frac{1}{(t-1)^2}$, $y=\frac{1}{(t+1)^2}$, we obtain the coordinates

$$x_1=\frac{-2(t+1)^4(5t^2-12t+3)p-10(t^2-3)(t^2-1)^2q+5(t^2-1)^2(t+1)^2(t^2+6t-3)r}{5(t^2-1)^2(t+1)^2(t^2+6t-3)p+5(t^2-1)^2(t-1)^2(t^2-6t-3)q+2(t^2-1)^4(t^2-21)r}$$

$$y_1=\frac{-10(t^2-3)(t^2-1)^2p-2(t-1)^4(5t^2+12t+3)q+5(t^2-1)^2(t-1)^2(t^2-6t-3)r}{5(t^2-1)^2(t+1)^2(t^2+6t-3)p+5(t^2-1)^2(t-1)^2(t^2-6t-3)q+2(t^2-1)^4(t^2-21)r}.$$

This gives a locus which is in general of the 10^{th} degree. When $p=0$, or $q=0$, or $r=0$, it reduces to a locus of the eighth degree, and when $p=r=0$, or $p=q=0$, or $q=r=0$, it reduces to a locus of the sixth degree.

It follows that for the general tricuspidal quartic the locus of the centres of the conics of closest contact is a curve of the 10^{th} degree in general. When, however, one of the cusps is at infinity, this locus reduces to a curve of the eighth degree, and when two of the cusps are at infinity, it reduces to a curve of the sixth degree.

(8d) As an example of a quartic with three biflecnodes we may take the curve whose equation is $x^2+y^2=x^2y^2$. Any point on the curve may be written in terms of a parameter θ, $x=\sec\theta$, $y=\operatorname{cosec}\theta$. The equation of the conic of closest contact at "θ" is

$$c^6(5s^2-4c^4)x^2+8s^3c^3(s^2c^2-10)xy+s^6(5c^2-4s^4)y^2$$
$$+8c^3(c^2+10s^4)x+8s^3(s^2+10c^4)y-(4+5s^2c^2)=0,$$

where $c\equiv\cos\theta$, $s\equiv\sin\theta$.

The coordinates of the centre are

$$\bar{\xi}=\frac{+4(2s^4+5c^2)}{5c(4-s^2c^2)},\quad \bar{\eta}=\frac{4(2c^4+5s^2)}{5s(4-s^2c^2)}.$$

The elimination of θ between these equations leads to an equation of the 12^{th} degree in $(\bar{\xi},\bar{\eta})$.

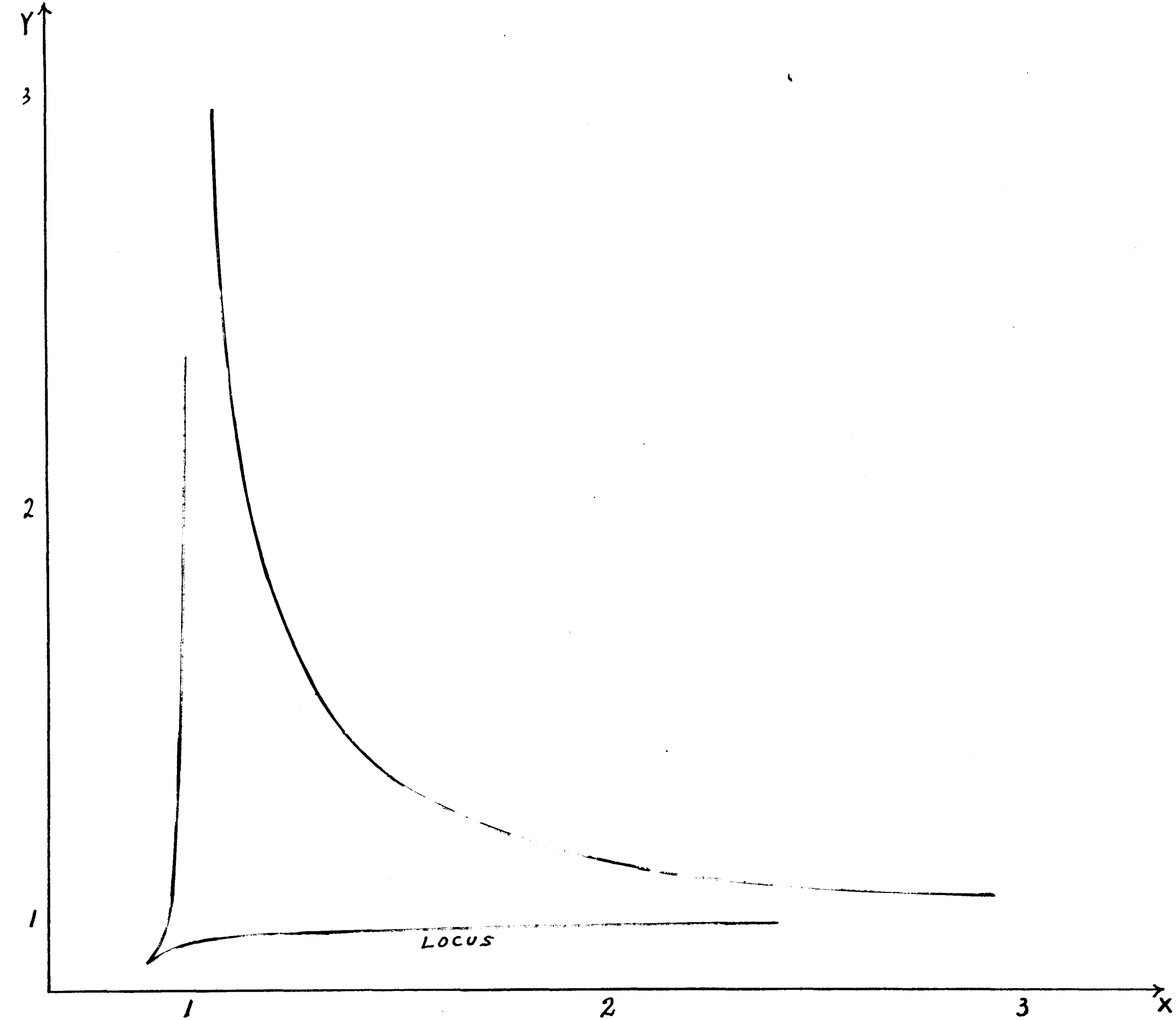
The pole of the straight line $px+qy+r=0$ with regard to the conic of closest contact at "θ" is

$$x_1=\frac{-5s^2(s^2+4c^4)p+4sc(5s^2c^2-2)q-4s^2c(2s^4+5c^2)r}{-4cs^2(2s^4+5c^2)p-4sc^2(2c^4+5s^2)q+5s^2c^2(s^2c^2-4)r},$$

$$y_1=\frac{4sc(5s^2c^2-2)p-5c^2(c^2+4s^4)q-4sc^2(2c^4+5s^2)r}{-4cs^2(2s^4+5c^2)p-4sc^2(2c^4+5s^3)q+5s^2c^2(s^2c^2-4)r}.$$

The elimination of θ between these equations leads to an equation of the 16^{th} degree in (x_1, y_1), from which we infer that the locus of the centres of the conics of closest contact of the general quartic with three biflecnodes is a curve of the 16^{th} degree, which degenerates into a curve of the 12^{th} degree, when the line at infinity contains two of the biflecnodes.

To find the positions of the sextactic points we equate the ratio of any pair of coefficients in the equation of the conic of closest contact to the ratio of the derivatives of these coefficients



Graph of $x^2+y^2=x^2y^2$, and the locus of the centres of its conics of closest contact.

with respect to θ. This gives the equation

$$s^2c^2(4s^2c^2-1)(s^2c^2+2)=0.$$

The real sextactic points are therefore given by $\sin 4\theta=0$, or $\theta=0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$.
The equation $s^2c^2+2=0$ gives eight other imaginary points at which the contact is sextactic.

(8e) The coordinates of any point on the quartic $x=(y-x^2)^2$ may be written in terms of a parameter "t", $x=t^2$, $y=t+t^4$. The equation of the conic of closest contact at "t" is

$$(1024t^{10}-800t^7+20t^4-10t)x^2+(224t^5+20t^2)xy$$
$$+(1-40t^3)y^2+(1120t^9-380t^6+20t^3-1)x$$
$$+(-1024t^{10}+240t^7+10t^4)y+(280t^{11}-55t^8-2t^5)=0,$$

and the coordinates of the centre are

$$\bar{\xi}=\frac{14t^3-\frac{1}{2}}{10t}, \quad \bar{\eta}=\frac{t^2(-128t^6+66t^3+5)}{10t}.$$

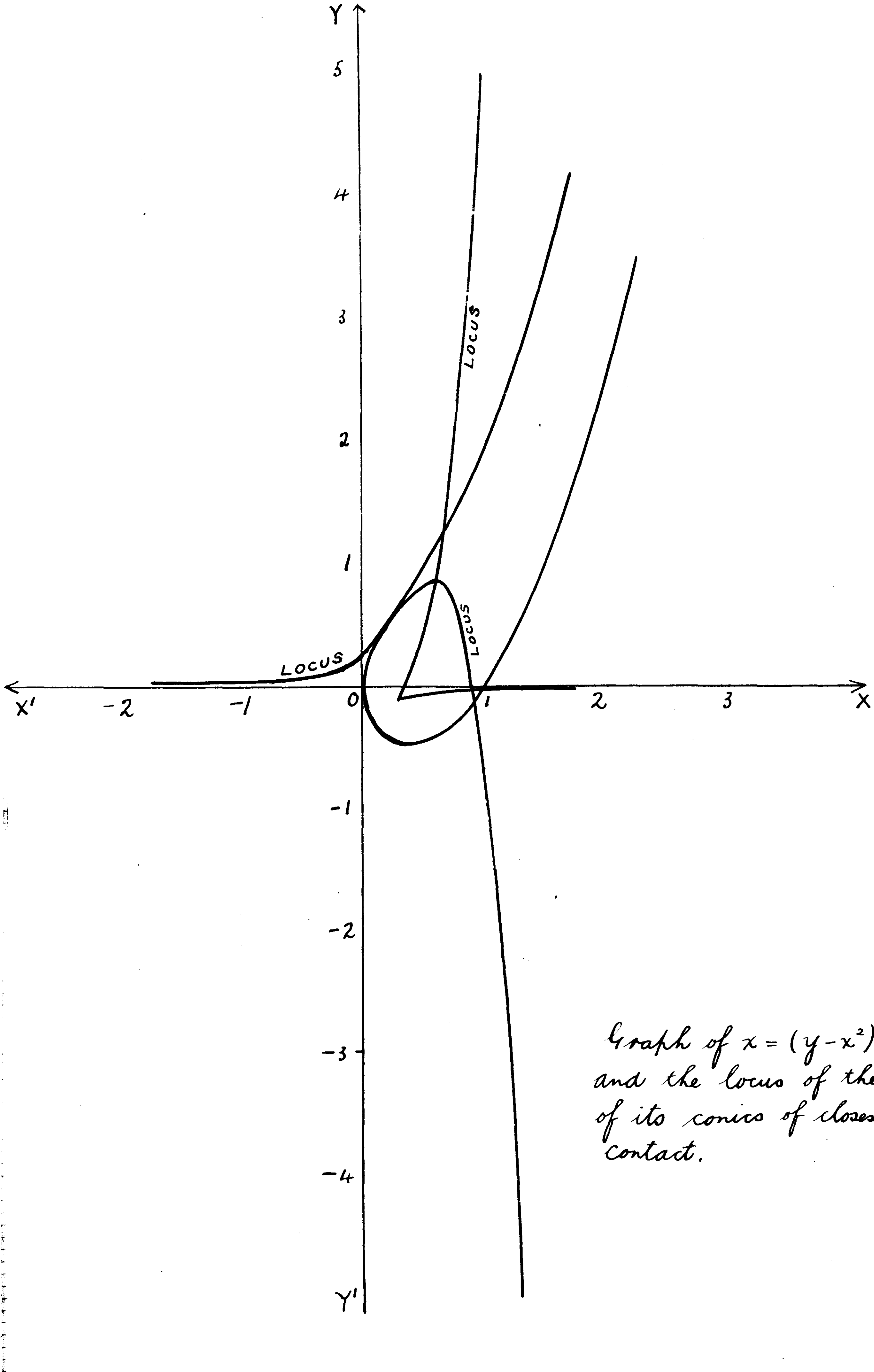
The locus of the centre is thus a curve of the eighth degree.

The pole of the straight line $px+qy+r=0$ with respect to the conic of closest contact at "t" is

$$x_1=\frac{p(-64t^8-2t^5)+q(-70t^7-\frac{5}{2}t^4)+r(-14t^3+\frac{1}{2})}{p(-14t^3+\frac{1}{2})+q(128t^8-66t^5-5t^2)+r(-10t)},$$

$$y_1=\frac{p(-70t^7-\frac{5}{2}t^4)+q(70t^9-110t^6+2t^3-\frac{1}{4})+r(128t^8-66t^5-5t^2)}{p(-14t^3+\frac{1}{2})+q(128t^8-66t^5-5t^2)+r(-10t)},$$

from which it follows that the locus of (x_1, y_1) is in general a curve of the ninth degree. If then the given curve is projected so that the straight line $px+qy+r=0$ becomes the line at infinity, the locus of the centres of the conics of closest contact of the projection will be



Graph of $x = (y - x^2)^2$, and the locus of the centres of its conics of closest contact.

in general a curve of the ninth degree. This will degenerate into a curve of the eighth degree when the line at infinity contains the cusp of the quartic.

To find the sextactic points of the quartic $x=(y-x^2)^2$ we make the ratio of a pair of coefficients in the equation of the conic of closest contact at "t" equal to the ratio of the derivatives of these coefficients with respect to "t". This gives the equation $t(56t^3+1)(8t^3-1)=0$. $t=0$ gives the conic of closest contact at the origin, viz., the parabola $y^2-x=0$. $8t^3-1=0$ gives the real and the two imaginary points of inflexion. $56t^3+1=0$ gives the real and the two imaginary other points at which the contact is sextactic. The double point at infinity is also a sextactic point.

(8f) The coordinates of any point on the quartic $y=(1-x^2)^2$ may be written $x=t$, $y=(1-t^2)^2$. The equation of the conic of closest contact at "t" is

$$(-8+80t^2-160t^4+240t^6-280t^8)x^2+(80t^3-112t^5)xy$$
$$+(1+5t^2)y^2+(-80t^3+16t^5+80t^7+240t^9)x$$
$$+(-6+30t^2-140t^4+210t^6)y+(5-35t^2+170t^4-190t^6-15t^8-63t^{10})=0.$$

Through any given point there pass ten of the conics of closest contact, and the sum of the reciprocals of their parameters is zero.

The coordinates of the centre of the conic of closest contact at "t" are

$(1+4t^2)\bar{\xi}=10t^3$, $(1+4t^2)\bar{\eta}=3-9t^2+25t^4-35t^6$, and the locus of the centre is a curve of the sixth degree.

The coordinates of the pole of the straight line $px+qy+r=0$ with respect to the conic of closest contact at "t" are

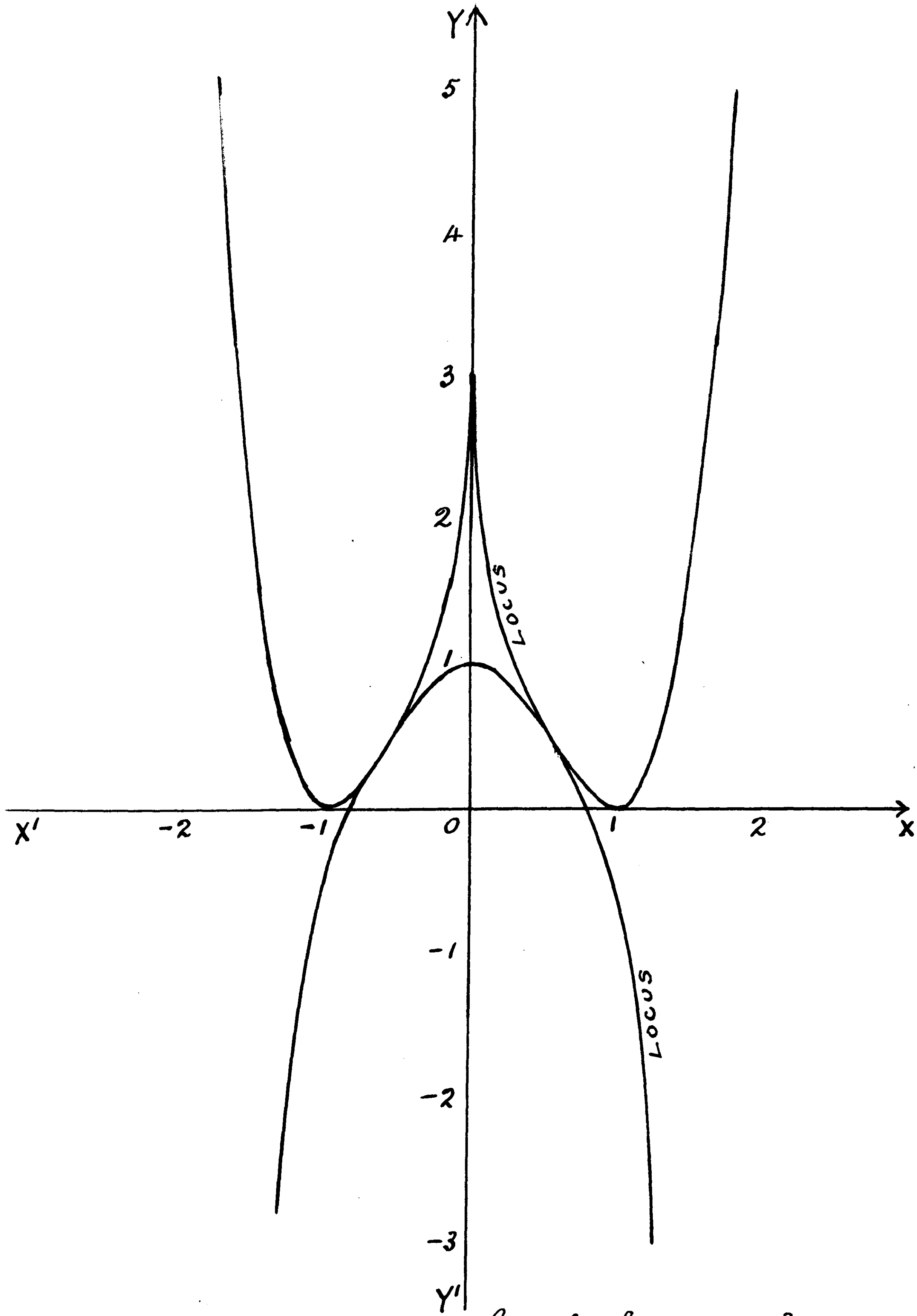
$$x_1 = \frac{p(\frac{1}{2}-4t^2+\frac{35}{2}t^4)+q(10-12t^2-14t^4)+r(10t^3)}{p(10t^3)+q(3-9t^2+25t^4-35t^6)+r(1+7t^2)},$$

$$y_1 = \frac{p(10-12t^2-14t^4)+5q(1-t^2)^5+r(3-9t^2+25t^4-35t^6)}{p(10t^3)+q(3-9t^2+25t^4-35t^6)+r(1+7t^2)};$$

and the locus of (x_1, y_1) is in general a curve of the tenth degree, but for the line $x=0$ it is a curve of the fourth degree, and for the line $y=0$ it is a curve of the fifth degree.

If the given curve is projected in any way we may infer that the locus of the centres of the conics of closest contact of the projection will be in general a curve of the tenth degree. This will degenerate into a curve of the sixth degree if the line at infinity contains the double point, and into a curve of the fifth degree when the line at infinity is a bitangent, and into a curve of the fourth degree when the line at infinity is the tangent at the cusp.

The sextactic points may be found as in the previous example, and the equation giving these points is $t^2(7t^2+3)(3t^2-1)=0$. The solution $3t^2-1=0$ gives the two real points of inflexion; $t^2=0$ gives the sextactic point at $x=0$, and the equation of the corresponding conic of closest contact is the hyperbola $-8x^2+y^2-6y+5=0$; $7t^2+3=0$ gives the two other imaginary sextactic points. The double point at infinity is also a sextactic point.



Graph of $y = (1-x^2)^2$, and the locus of the centres of its conics of closest contact.

(19) The coordinates of a point on the tractory $s = c \log \operatorname{cosec}\psi$ may be written

$$x = -c\cos\psi - c\log\tan\frac{\psi}{2}, \quad y = -c\sin\psi.$$

It follows that $l = \cos\psi$, $m = \sin\psi$;

$$l' = \frac{\sin^2\psi}{c\cos\psi}, \quad m' = -\frac{\sin\psi}{c};$$

$\rho = -c\cot\psi$, $\rho' = \sec\psi\operatorname{cosec}\psi$, $\rho\rho'' = \operatorname{cosec}^2\psi - \sec^2\psi$;

$$9 + \rho'^2 - 3\rho\rho'' = \frac{13 - 12\cos 2\psi - 9\cos^2 2\psi}{\sin^2 2\psi}$$

$$= -\frac{(3\cos 2\psi + 2 - \sqrt{17})(3\cos 2\psi + 2 + \sqrt{17})}{\sin^2 2\psi}.$$

$9 + \rho'^2 - 3\rho\rho''$ is negative when $\cos 2\psi > \frac{\sqrt{17}-2}{3}$, i.e. $|\psi| < 22°28\frac{1}{2}'$,
and positive when $\cos 2\psi < \frac{\sqrt{17}-2}{3}$, i.e. $|\psi| > 22°28\frac{1}{2}'$,
and zero when $\psi = \pm 22°28\frac{1}{2}'$.

$$\bar{\xi} = x + \frac{3c\cot\psi\operatorname{cosec}\psi + 9c\cos\psi}{\dfrac{13 - 12\cos 2\psi - 9\cos^2 2\psi}{\sin^2 2\psi}},$$

$$\bar{\eta} = y - \frac{3c(2\operatorname{cosec}\psi - 3\sin\psi)}{\dfrac{13 - 12\cos 2\psi - 9\cos^2 2\psi}{\sin^2 2\psi}}.$$

The locus of the centres of the conics of closest contact has a pair of symmetrically-placed asymptotes corresponding to $9 + \rho'^2 - 3\rho\rho'' = 0$. When $c = \frac{1}{3}$, their equations are

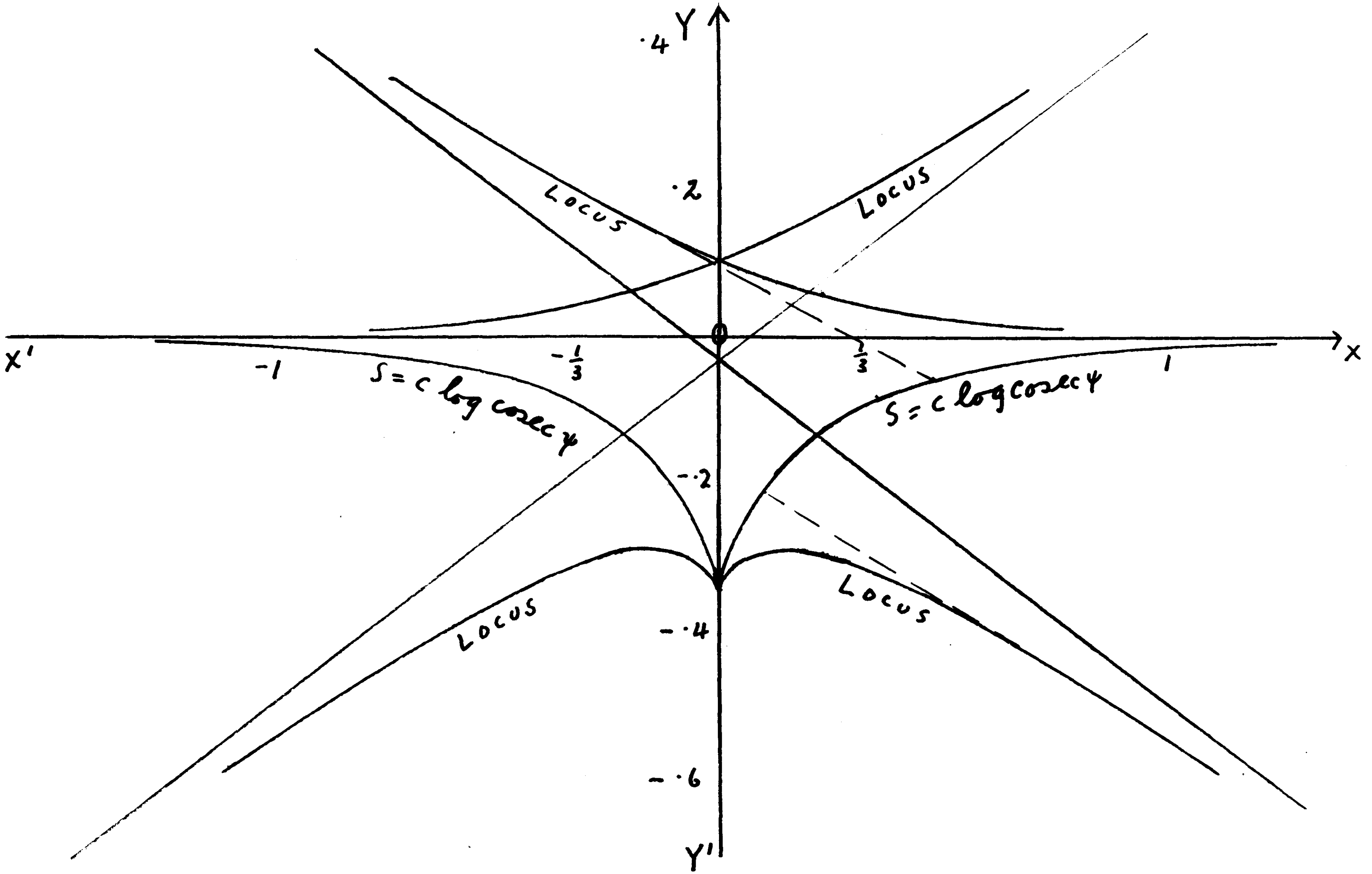
$$\eta + \cdot 023 = \pm \cdot 453\,\xi.$$

The x-axis is also an asymptote.

Data for the tractory $s = c\log\operatorname{cosec}\psi$ $(c = \frac{1}{3})$.

x	1·25	1·02	·88	·79	·71	·48	·35	·27	·20
y	−·006	−·012	−·017	−·023	−·029	−·058	−·086	−·114	−·141
$\bar{\xi}$	·77	·51	·37	·26	·17	−·19	−·69	−4·05	2·28
$\bar{\eta}$	·0124	·0246	·036	·049	·063	·156	·332	1·80	−1·37

x	·15	·11	·08	·058	·040	·026	·016	0
y	−·170	−·203	−·226	−·255	−·273	−·289	−·302	−·333
$\bar{\xi}$	1·11	·67	·46	·32	·22	·17	·11	0
$\bar{\eta}$	−·586	−·413	−·343	−·310	−·292	−·288	−·289	−·333



Graph of $s = c \log \operatorname{cosec} \psi$, and the locus of its centres of conics of closest contact.

(10) The coordinates of a point on the equiangular spiral $r = ae^{n\theta}$ may be written

$$x = ae^{n\theta}\cos\theta, \quad y = ae^{n\theta}\sin\theta.$$

It follows that $l = \dfrac{n\cos\theta - \sin\theta}{(1+n^2)^{1/2}}$, $m = \dfrac{n\sin\theta + \cos\theta}{(1+n^2)^{1/2}}$;

$$l' = \frac{-n\sin\theta - \cos\theta}{ae^{n\theta}(1+n^2)}, \quad m' = \frac{n\cos\theta + \sin\theta}{(1+n^2)\,ae^{n\theta}};$$

$$\rho = ae^{n\theta}(1+n^2)^{\frac{1}{2}}, \quad \rho' = n, \quad \rho'' = 0;$$

$$9 - 3\rho\rho'' + \rho'^2 = 9 + n^2 > 0; \quad \frac{3\rho}{9 - 3\rho\rho'' + \rho'^2} = \frac{3ae^{n\theta}(1+n^2)^{\frac{1}{2}}}{9+n^2};$$

$$l\rho' - 3m = \frac{n^2\cos\theta - 4n\sin\theta - 3\cos\theta}{(1+n^2)^{1/2}},$$

$$m\rho' + 3l = \frac{n^2\sin\theta + 4n\cos\theta - 3\sin\theta}{(1+n^2)^{1/2}};$$

$$\bar{\xi} = \frac{4nae^{n\theta}}{9+n^2}(n\cos\theta - 3\sin\theta), \quad \bar{\eta} = \frac{4nae^{n\theta}}{9+n^2}(n\sin\theta + 3\cos\theta);$$

or $\bar{\xi} = 4\cos\alpha\, e^{-3\alpha\cot\alpha}\, ae^{3\varphi\cot\alpha}\cos\varphi$, $\bar{\eta} = 4\cos\alpha\, e^{-3\alpha\cot\alpha}\, ae^{3\varphi\cot\alpha}\sin\varphi$,

where $n = 3\cot\alpha$, and $\varphi = \theta + \alpha$.

Thus the locus of the centres of the conics of closest contact is a second equiangular spiral of the same angle, and its graph is formed from that of $r = ae^{n\theta}$ by multiplying each radius vector by a constant factor $4\cos\alpha\, e^{-3\alpha\cot\alpha}$.

If O is the pole, and P is the point whose polar coordinates are (r, θ), then Q, the centre of the conic of closest contact at P, will have coordinates $(\bar{r}, \varphi)$, where $\varphi = \theta + \alpha$, and $\bar{r} = 4a\cos\alpha\, e^{-3\alpha\cot\alpha}\, e^{n\varphi}$. It follows that the angle POQ equals α.

From Section I (10) we see that the direction of the tangent PT makes a constant angle with the direction of PQ. This constant angle also equals α, since $\rho' = n = 3\cot\alpha$.

Hence, as in the diagram, $\angle POQ = \angle TPQ = \alpha$. It follows that the circle POQ will touch the spiral $r = ae^{n\theta}$ at P. On calculating PQ and finding the radius of this circle, we get the result that the radius of curvature at P is the diameter of the circle POQ.

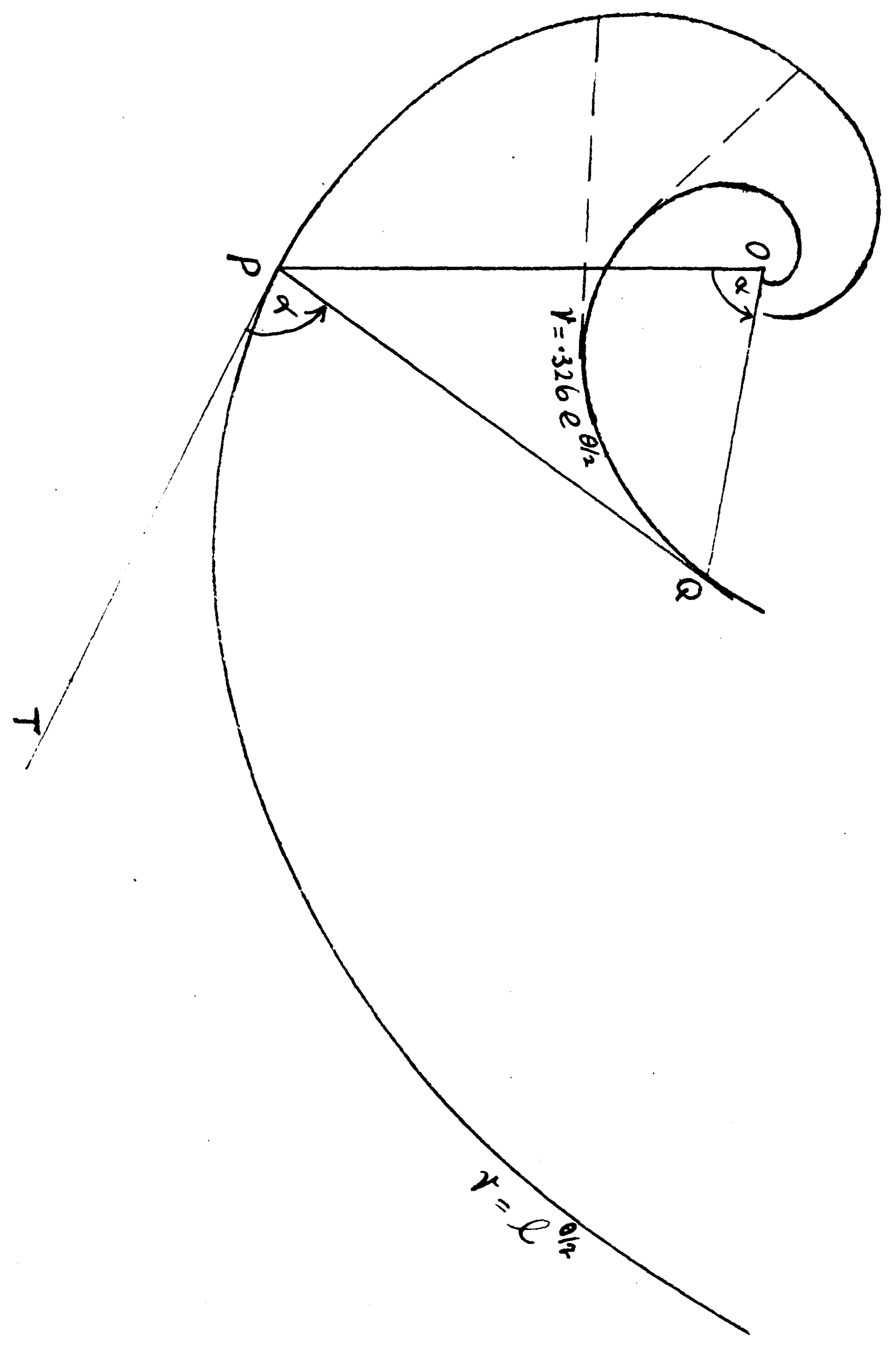
Thus if C is the centre of curvature of a point P on the spiral, a circle on PC as diameter will pass through the pole O, and if we draw a chord through O making with OP an angle α this chord will meet the circle at Q, the centre of the conic of closest contact at P.

Data for the equiangular spiral $r = ae^{n\theta}$ $(a = 1, n = \frac{1}{2})$
$\varphi^\circ = \theta^\circ + 80\frac{1}{2}^\circ$.

θ°	0	10	20	30	40	50	60	70	80	90
r	1·00	1·09	1·19	1·30	1·42	1·55	1·69	1·84	2·01	2·19
r (locus)	·33	·36	·39	·42	·46	·50	·55	·60	·65	·71
θ°	100	110	120	130	140	150	160	170	180	
r	2·39	2·61	2·85	3·11	3·39	3·70	4·04	4·41	4·81	
r (locus)	·78	·85	·93	1·01	1·11	1·21	1·32	1·44	1·57	
θ°	190	200	210	220	230	240	250	260	270	
r	5·25	5·80	6·25	6·82	7·44	8·12	8·86	9·67	10·45	
r (locus)	1·71	1·89	2·04	2·22	2·42	2·65	2·89	3·15	3·40	
θ°	280	290	300	310	320	330	340	350	360	
r	11·51	12·56	13·71	14·96	16·32	17·81	19·44	21·21	23·14	
r (locus)	3·75	4·09	4·47	4·91	5·32	5·77	6·33	6·91	7·54	

The equation of the osculating conic at (x, y) is

$$(\xi - x)^2 \{\sin^2\theta(9 + 3n^2 + 2n^4) + 2\sin\theta\cos\theta\, n(5n^2 - 3) + \cos^2\theta(9 + 17n^2)\}$$
$$+ (\eta - y)^2 \{\sin^2\theta(9 + 17n^2) - 2\sin\theta\cos\theta\, n(5n^2 - 3) + \cos^2\theta(9 + 3n^2 + 2n^4)\}$$
$$- 2(\xi - x)(\eta - y)\{\cos^2\theta\, n(5n^2 - 3) + 2\sin\theta\cos\theta\, n^2(n^2 - 7) - \sin^2\theta\, n(5n^2 - 3)\}$$
$$+ 18\, ae^{n\theta}(1 + n^2)\{(n\sin\theta + \cos\theta)(\xi - x) - (n\cos\theta - \sin\theta)(\eta - y)\}$$
$$= 0.$$



Graph of $r = e^{\theta/2}$, and the locus of the centres of its conics of closest contact.

Section VI.

Problems on the osculating conics.

(1) The expressions for the length (R) of the axis of aberrancy at a point (x, y) on a plane curve, viz.,

$$R = \frac{3\rho\sqrt{9+\rho'^2}}{9+\rho'^2-3\rho\rho''} \quad \text{and } R = \frac{\rho\sin\alpha}{1+\frac{d\alpha}{d\psi}}.$$

and for the length (P) of the normal chord at (x, y) of the osculating conic, viz.,

$$P = \frac{18\rho}{9+2\rho'^2-3\rho\rho''},$$

lead to a number of very interesting problems represented by easily-obtained differential equations. Some of these equations can be solved by elementary methods, although many require the aid of Elliptic Integrals for their solution. A great variety of interesting curves can be obtained by representing the solutions of these equations graphically.

It is the main purpose of this book to examine these equations and their solutions, and to represent graphically a number of typical solutions obtained by giving definite values to the constants of integration.

In some cases the coordinates (x, y) of the current point of the graph can be represented by parametric equations, but in the majority of cases no simple expressions are to be found for these coordinates. In these cases sufficiently accurate determinations of the coordinates can be found by methods of graphical integration.

If ψ is the inclination of the tangent at (x, y) to the x-axis, then, following the usual notation, we have

$$\frac{dx}{ds} = \cos\psi, \quad \frac{dy}{ds} = \sin\psi,$$

i.e. $x = \int \cos\psi \, ds$, $y = \int \sin\psi \, ds$. Hence, if graphs can be drawn representing $\cos\psi$ and $\sin\psi$ as functions of s, the areas between these curves and the s-axis between certain limits will measure the coordinates x and y corresponding to the variable limit of s.

Again, if ρ be the radius of curvature at (x, y) we have $\frac{ds}{d\psi} = \rho$, or $ds = \rho \, d\psi$. Hence we obtain the expressions

$\int \rho \cos\psi \, d\psi$ and $\int \rho \sin\psi \, d\psi$ for x and y.

That is we can measure the coordinates x and y if we can represent $\rho \cos\psi$ and $\rho \sin\psi$ graphically as functions of ψ. This can be done in all cases where a solution of an equation can be represented by an equation of the form $\rho = f(\psi)$.

(2) In many cases we find equations of circles and parabolas occurring as solutions of the differential equations. Remembering that the conic of closest contact of a given conic is the conic itself, we notice that the axes of aberrancy of a circle are its radii, which are of constant length, and the axes of aberrancy of a parabola are its diameters which are parallel and of infinite length.

(3) In many cases also we find the equation of the equiangular spiral occurring as a solution of the differential equations. This remarkable curve has so many properties associated with osculating conics and axes of

aberrancy that it has been found convenient to collect these in one section. This list of properties will be found in Section XXIV.

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(4) Solutions of the Equation $R = k$,

$$\text{or}\quad \frac{3\rho\sqrt{9+\rho'^2}}{9+\rho'^2-3\rho\rho''} = k.$$

Obvious solutions are the equation of a circle or of a point circle ($k=0$), and the equation of a parabola ($k=\infty$), if we regard the diameters of a parabola as being constant. The straight line regarded as the limiting case of a circle may also be taken as a solution in the latter case ($k=\infty$).

Another interesting elementary solution is the equation $ay = \log\cosh\frac{ax}{\sqrt{3}}$.

The discussion of this problem will be found in Section VII, and specimens of the curves satisfying the equation are shown in Section VIII.

---

(5) Solutions of the Equation $R = k\rho$,

$$\text{or}\quad \frac{3\rho\sqrt{9+\rho'^2}}{9+\rho'^2-3\rho\rho''} = k\rho.$$

Obvious solutions are the equation of a circle ($k=1$), and of a parabola ($k=\infty$), and of an equiangular spiral ($\rho' = \text{const.}$)

An interesting solution is $\rho^3 = \sec\varphi - 1$, $\psi = -\cot\frac{\varphi}{2}$, represented by a curve for which R is equal to, but not coincident with, ρ.

Other solutions are discussed in Section IX, and illustrated in Section X.

---

(6) Solutions of $9+\rho'^2-3\rho\rho''=$ const., i.e. of $R \sin\alpha = k\rho$, where α is the inclination of the axis of aberrancy to the tangent.

This is satisfied by the equation of a circle ($k=1$), for obviously the projection of the radius of a circle on the normal is equal to the radius. The equation of an equiangular spiral is also a solution.

Another obvious solution is the equation of a parabola ($k=\infty$), for the projection of the diameter of a parabola on the normal is infinite.

An interesting solution is the equation $\rho^2 = as^3$, or $s = \dfrac{4}{a\psi^2}$.

Other solutions are discussed in Section XI and illustrated in Section XII.

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(7) Solutions of the Equation $9+\rho'^2-3\rho\rho''=\lambda\rho$, i.e. of the equation $R \sin\alpha = k$.

Obvious solutions are the equations of a circle ($\rho=k$), point circle ($k=0$), straight line ($k=\infty$), and parabola ($k=\infty$).

Another simple solution is the equation $\rho = -\frac{1}{2}\sec^2\frac{3\psi}{2}$, which also belongs to the class of curves $\alpha = a\psi+\beta$.

Other solutions are discussed in Section XIII and illustrated in Section XIV.

---

(8) Solutions of $9+\rho'^2-3\rho\rho''=k\rho'$, or of $R\cos\alpha=\lambda\rho$.

The equation is obviously satisfied by the equations of a circle and of a point circle ($\lambda=0$). The equation of the equiangular spiral is also a solution, as is the equation of a parabola ($\lambda=\infty$).

Another interesting solution is the equation

$$c\rho = \frac{e^{3\psi}}{\psi^3}.$$

Other solutions are discussed in Section XV and illustrated in Section XVI.

---

(9) Solutions of $9+\rho'^2-3\rho\rho'' = k\rho\rho'$, or of $R\cos\alpha = \lambda$.

This is obviously satisfied by the equations of a circle ($\lambda = 0$) and of a parabola ($\lambda = \infty$). The only other solution appears to be a particular case of Riccati's equation, which is fully discussed in Section XVII.

---

(10) Solutions of the Equation $P = k$,

$$\text{or } \frac{18\rho}{9+2\rho'^2-3\rho\rho''} = k.$$

An obvious solution is the equation of a circle, whose normal chords are of constant length. The straight line as the limiting case of a circle may be regarded as a solution ($k = \infty$).

Other simple solutions are the equations $s = a\tan\frac{3\psi}{2\sqrt{2}}$, and $\rho^{2/3} = a\sec\sqrt{2}\,\psi$. In the latter case the osculating conic has an asymptote parallel to the normal chord at (x, y).

Other solutions are discussed in Section XVIII and illustrated in Section XIX.

---

(11) Solutions of the Equation $P = k\rho$,

$$\text{or } \frac{18\rho}{9+2\rho'^2-3\rho\rho''} = k\rho.$$

Obvious solutions are the equation of a circle, or point circle ($k=0$), or straight line ($k=\infty$), and the equation of an equiangular spiral.

Another interesting solution is $\rho = as^3$ or $\frac{1}{s^2} = -2a\psi$, for $k = 2$. In this case the normal chord coincides with the diameter of the circle of curvature.

The equation $\rho^{2/3} = a \sec \sqrt{2}\,\psi$ of the previous paragraph is also a solution of the present problem.

Other solutions are discussed in Section XX and illustrated in Section XXI.

(12) Solutions of $\alpha = a\psi$.

The general solution is $\rho = C \sin^{3/a} a\psi$, which includes some well-known curves.

When $a = -1$, we find that the angle between the x-axis and the tangent is equal to the angle between the axis of aberrancy and the tangent. This evidently defines a parabola for which the axes of aberrancy are parallel to the axis of the parabola.

When $a = -2$, we find that the angle between the x-axis and the tangent is half the angle between the axis of aberrancy and the tangent. This obviously defines the rectangular hyperbola $xy = -c^2$, and corresponds to the fact that the part of the tangent intercepted between the asymptotes is bisected at the point of contact.

When $a = -3$, we find R = const. The solution in this case is the curve $y = k \log \cosh \frac{x}{k\sqrt{3}}$, referred to different axes. This equation is mentioned in paragraph 4.

The solutions of $\rho = C \sin^{3/a} a\psi$ lead to a great variety of interesting curves, many of which are discussed and illustrated in Section XXIII.

Section VII.

To find a curve for which the length of the axis of aberrancy is constant.

(1) Writing the constant in the form $\frac{3}{2k}$, the equation to be solved is

$$\frac{3\rho\sqrt{9+\rho'^2}}{9+\rho'^2-3\rho\rho''} = \frac{3}{2k},$$

and a first solution is found by putting $9+\rho'^2 = u^2$, or $u\frac{du}{d\rho} = \rho'\frac{d\rho'}{d\rho} = \rho''$.
The equation becomes $\frac{du}{d\rho} - \frac{1}{3\rho}u + \frac{2k}{3} = 0,$

the solution of which is $u = c\rho^{\frac{1}{3}} - k\rho,$

or
$$s = \int\frac{d\rho}{\sqrt{k^2\rho^2 - 2kc\rho^{4/3} + c^2\rho^{2/3} - 9}}.$$

---

(2) If $c=0$, a complete solution is readily obtained. We find
$$s - s_0 = \frac{1}{k}\cosh^{-1}\frac{k\rho}{3} \quad\text{or}\quad \rho = \frac{3}{k}\cosh k(s-s_0).$$

If we put $\rho = \frac{ds}{d\psi}$, we obtain the further solution
$$ks = \log\frac{\tan\frac{3}{2}(\psi+\alpha)}{\tan {}^3\!/\!_2\,\alpha}.$$

The graph of this equation for the case $k=\frac{1}{2}$, $\alpha = \pi/6$, is shown in the next section. The cartesian equation of the curve is $ky = \log\cosh\left(kx\tan\frac{\pi}{6}\right)$.

---

(3) The particular case $k=0$ gives the equation $u = c\rho^{1/3}$
or
$$\frac{d\rho}{ds} = \sqrt{c^2\rho^{2/3}-9}.$$

On substituting $\rho = \frac{ds}{d\psi}$ this equation becomes
$$\frac{d\psi}{d\rho} = \frac{1}{\rho\sqrt{c^2\rho^{2/3}-9}},$$
the solution of which is $\rho = \left(\frac{3}{c}\right)^3\sec^3(\psi-\alpha)$, the equation of a parabola.

---

(4) If we substitute $\rho = v^{3/2}$ in the integral for s, we obtain
$$\frac{2ks}{3} = \int \frac{v\,dv}{\sqrt{v\left(v^3 - \frac{2c}{k}v^2 + \frac{c^2}{k^2}v - \frac{9}{k^2}\right)}}.$$

(5) The condition for three equal roots of the cubic equation
$$v^3 - \frac{2c}{k}v^2 + \frac{c^2}{k^2}v - \frac{9}{k^2} = 0$$
cannot be satisfied, but the condition for two equal roots can be found in terms of c and k. It is
$$4c^3 = 243k.$$
The repeated root is $\frac{c}{3k}$ $(\equiv a)$, and the non-repeated root is $\frac{81}{c^2}$ $(\equiv b)$, and is positive.

When c has this particular value, the solution of the equation can be completed without the aid of elliptic integrals.

In this case
$$\frac{2ks}{3} = \int \frac{(v-a+a)\,dv}{\sqrt{(v-a)^2(v^2-bv)}}$$
$$= \int \frac{dv}{\sqrt{\left(v-\frac{b}{2}\right)^2 - \frac{b^2}{4}}} + a\int \frac{dv}{(v-a)\sqrt{v(v-b)}}$$
The second integral may be evaluated by the substitution $\frac{v-b}{v} = y^2$; and there are two forms of the solution according as $ab - a^2 >$ or < 0.

If $ab - a^2 > 0$,
$$\frac{2}{3}ks = \log\left(v - \frac{b}{2} + \sqrt{v^2 - bv}\right) + \frac{2a}{\sqrt{ab-a^2}}\tan^{-1}\sqrt{\frac{a}{b-a}\,\frac{v-b}{v}}.$$

If $ab - a^2 < 0$
$$\frac{2}{3}ks = \log\left(v - \frac{b}{2} + \sqrt{v^2 - bv}\right) - \frac{2a}{\sqrt{a^2-ab}}\tanh^{-1}\sqrt{\frac{a}{a-b}\cdot\frac{v-b}{v}}.$$

But since $b \equiv \frac{81}{c^2}$ and $4c^3 = 243k$, it follows that $b = \frac{4c}{3k} = 4a$, and that therefore $ab - a^2 > 0$.

Making the substitution $b = 4a$, the solution is
$$\frac{2}{3}ks = \log\left(v - 2a + \sqrt{v^2 - 4av}\right) + \frac{2}{\sqrt{3}}\tan^{-1}\sqrt{\frac{v-4a}{3v}}.$$
The graph of this equation for the case $k = \frac{3}{2}$, $a = 1$ is shown in the next section.

The equation then is
$$s = \log\left(v - 2 + \sqrt{v^2 - 4v}\right) + \frac{2}{\sqrt{3}}\tan^{-1}\sqrt{\frac{v-4}{3v}}.$$

(6) In the most general case, the equation
$f(v) \equiv v^3 - \frac{2c}{k}v^2 + \frac{c^2}{k^2}v - \frac{9}{k^2} = 0$ has three distinct roots. If we differentiate $f(v)$, we find
$f'(v) = 3v^2 - \frac{4c}{k}v + \frac{c^2}{k^2}$, which vanishes when
$v = \frac{c}{3k}$ or $\frac{c}{k}$.
These values are either both negative or both positive.
When $v = \frac{c}{3k}$, $f(v) = \frac{4}{27}\frac{c^3}{k^3} - \frac{9}{k^2}$.
When $v = \frac{c}{k}$, $f(v) = -\frac{9}{k^2}$.
If c and k have opposite signs both these values are negative, and only one root of $f(v)=0$ is real. This real root is positive, since $f(0)$ is negative and $f(\infty)$ is positive.

In this case we may write $f(v) \equiv (v-d^2)(v^2+gv+h)$ and the equation $\frac{2ks}{3} = \int \frac{v\,dv}{\sqrt{(v^2-vd^2)(v^2+gv+h)}}$

can only be solved with the aid of elliptic integrals. For example, putting $k = \frac{\sqrt{6}}{8}$, $c = \frac{\sqrt{6}}{4}$, we find

$$\frac{\sqrt{6}\,s}{12} = \int \frac{v\,dv}{\sqrt{v(v-6)(v^2+2v+16)}},$$

an example in which the cubic $f(v)$ has one real factor and a pair of imaginary factors. The substitution $v = \frac{-6+2\zeta}{1+\zeta}$ gives the equation in

the form
$$\frac{\sqrt{6}}{12}s = \frac{2}{3\sqrt{5}}\int \frac{d\zeta}{\sqrt{(1-\zeta^2/9)(1+\frac{3}{5}\zeta^2)}}$$
$$+ \frac{8}{3\sqrt{5}}\int \frac{d\zeta}{(\zeta^2-1)\sqrt{(1-\zeta^2/9)(1+\frac{3}{5}\zeta^2)}}$$
$$- \frac{4}{3\sqrt{5}}\int \frac{d(\zeta^2-1)}{(\zeta^2-1)\sqrt{(1-\zeta^2/9)(1+\frac{3}{5}\zeta^2)}}$$

If we substitute $1 - \frac{\zeta^2}{9} = \sin^2\theta$ in the first and second of these integrals and $\zeta^2 - 1 = \frac{1}{\omega}$ in the third the solution is

$$\frac{\sqrt{6}}{12}s = -\frac{\sqrt{2}}{8}\int \frac{d\theta}{(1-\frac{9}{8}\sin^2\theta)\sqrt{(1-\frac{27}{32}\sin^2\theta)}} - \frac{\sqrt{2}}{4}\int \frac{d\theta}{\sqrt{(1-\frac{27}{32}\sin^2\theta)}}$$
$$+ \frac{1}{2}\int \frac{d\omega}{\sqrt{(\omega+\frac{1}{8})^2 - (\frac{1}{4})^2}},$$

or $\frac{\sqrt{6}}{12} s = -\frac{\sqrt{2}}{8}\,\Pi\left(\theta, \frac{3\sqrt{6}}{8}, -\frac{9}{8}\right) - \frac{\sqrt{2}}{4}\,F\left(\theta, \frac{3\sqrt{6}}{8}\right)$
$+ \frac{1}{2}\cosh^{-1}\left(4w+\frac{1}{2}\right),$
where $\sin^2\theta = 1 - \frac{\zeta^2}{9}$, $w = \frac{1}{\zeta^2 - 1}$, $\zeta = \frac{6+v}{2-v}$, $v = \rho^{2/3}$.

---

(7) If c and k have the same signs, the turning values of $f(v)$ will be both negative if $\frac{4}{27}\frac{c^3}{k^3} - \frac{9}{k^2} < 0$, i.e. if $\frac{c^3}{k} < \frac{243}{4}$. There is still one real root, a positive one, so that the solution in this case is the same as in the previous case.

---

(8) But if $\frac{4}{27}\frac{c^3}{k^3} - \frac{9}{k^2} > 0$, the turning values of $f(v)$ have opposite signs, and the equation $f(v) = 0$ has three different positive real roots p^2, q^2, r^2; such that $0 < p^2 < \frac{c}{3k} < q^2 < \frac{c}{k} < r^2$; and the equation to be integrated may be written

$$\frac{2ks}{3} = \int \frac{v\,dv}{\sqrt{v(v-p^2)(v-q^2)(v-r^2)}}$$

For example, putting $k = 3$, $c = 6$, we find

$$2s = \int \frac{v\,dv}{\sqrt{v(v-1)\left(v - \frac{3-\sqrt{5}}{2}\right)\left(v - \frac{3+\sqrt{5}}{2}\right)}},$$

which can be solved with the aid of elliptic integrals.

---

(9) To complete the problem of finding the equation of a curve whose axes of aberrancy are of constant length we must examine the differential equations for singular solutions.

The ρ'-discriminant of the equation

$\rho'^2 = k^2\rho^2 - 2kc\rho^{4/3} + c^2\rho^{2/3} - 9$ is the right-hand side equated to zero. This gives solutions of the form $\rho = \text{const.}$, which satisfy this differential equation. Also a value of c can be found such that one of these solutions is $\rho = \frac{3}{2k}$, an equation which satisfies the original differential equation. One solution, therefore, of the problem is a circle of radius $\frac{3}{2k}$, as is of course evident.

Section VIII.

Examples of curves whose axes of aberrancy are of constant length.

(1) The coordinates of a point on the curve

$$y = a \log \cosh \frac{x}{a\sqrt{3}} \quad \text{may be written}$$

$$x = \frac{a\sqrt{3}}{2} \log \frac{1+\sqrt{3}\tan\psi}{1-\sqrt{3}\tan\psi}, \quad y = -\frac{a}{2}\log(1-3\tan^2\psi).$$

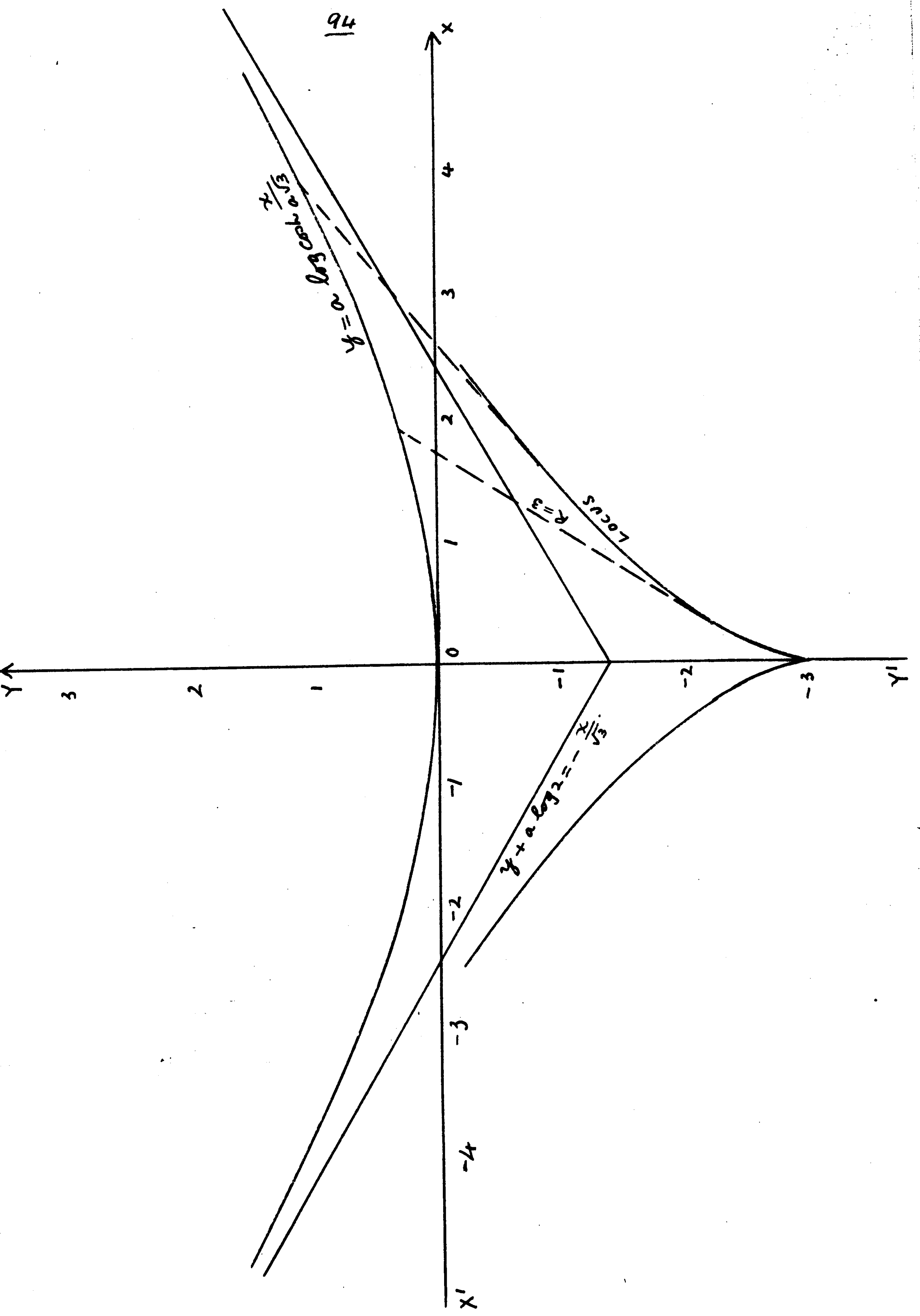
It follows that $s = a \log \tan\left(\frac{3\psi}{2}+\frac{\pi}{4}\right)$;
$l = \cos\psi$, $m = \sin\psi$; $\rho = 3a \sec 3\psi$, $\rho' = 3\tan 3\psi$,
$\rho'' = \frac{3}{a}\sec 3\psi$, $9 + \rho'^2 - 3\rho\rho'' = -18\sec^2 3\psi$;
$l\rho' - 3m = 3\sec 3\psi \sin 2\psi$, $m\rho' + 3l = 3\sec 3\psi \cos 2\psi$;
$\bar{\xi} = x - \frac{3}{2}a \sin 2\psi$, $\bar{\eta} = y - \frac{3}{2}a\cos 2\psi$.
The osculating conics are hyperbolas, and the equations for $\bar{\xi}$ and $\bar{\eta}$ show that the axis of aberrancy is of constant length $\frac{3}{2}a$.

The two curves have a pair of common asymptotes whose equations are

$$y + a \log 2 = \pm \frac{x}{\sqrt{3}}.$$

Data for the graph of $y = a \log \cosh \frac{x}{a\sqrt{3}}$ and for the locus of its centres of aberrancy (For $a = 2$).

ψ°	0	3	6	9	12	15	18	21	24	27	30
x	0	·32	·64	·98	1·34	1·74	2·20	2·78	3·54	4·80	∞
y	0	·01	·03	·08	·15	·24	·38	·58	·90	1·51	∞
$\bar{\xi}$	0	·00	·01	·05	·12	·24	·44	·77	1·31	2·37	∞
$\bar{\eta}$	−3	−2·97	−2·9	−2·77	−2·59	−2·36	−2·05	−1·65	−1·10	−·26	∞



Graph of $y = a \log \cosh \frac{x}{a\sqrt{3}}$, and the locus of its centres of aberrancy.

(2) Graph of the equation

$$\frac{2ks}{3} = \int \frac{v\,dv}{\sqrt{v\left(v^3 - \frac{2c}{k}v^2 + \frac{c^2}{k^2}v - \frac{9}{k^2}\right)}}, \quad \text{where } v = \rho^{\frac{2}{3}},$$

for the case where $4c^3 = 243k$, and $\frac{c}{3k} = a$.
The solution is

$$\frac{2K}{3}s = \log \frac{v - 2a + \sqrt{v^2 - 4av}}{2} + \frac{2}{\sqrt{3}} \tan^{-1}\sqrt{\frac{v-4a}{3v}}.$$

When $K = \frac{3}{2}$, $a = 1$, and the solution is

$$s = \log \frac{v - 2 + \sqrt{v^2 - 4v}}{2} + \frac{2}{\sqrt{3}} \tan^{-1}\sqrt{\frac{v-4}{3v}}.$$

To construct the curve, values of $s, \rho, \frac{1}{\rho}$ can be calculated for a series of values of v. If the values of $\frac{1}{\rho}$ are plotted against the values of s, the values of ψ can be found by graphical integration, using the formula $\psi = \int \frac{1}{\rho}\,ds$.
Values of $\cos\psi$, $\sin\psi$ can next be found from tables, and by using approximate methods of integration, the coordinates x and y can be found using the formulae

$$x = \int \cos\psi\,ds, \quad y = \int \sin\psi\,ds.$$

The locus of the centres of aberrancy can be constructed graphically from the curve, since the length of the axis of aberrancy is unity.

The (ψ, ρ) equation can also be found by writing $\rho = \frac{ds}{d\psi}$, or $ds = \rho\,d\psi$ in the integral equation $\int \frac{2K}{3}\,ds = \int \frac{v\,dv}{(v-a)\sqrt{v(v-4a)}}$. The solution is then

$$\psi = \frac{2}{\sqrt{3}} \tan^{-1}\sqrt{\frac{v-4a}{3a}} - \tan^{-1}\sqrt{\frac{v-4a}{4a}}.$$

When $v = \infty$, $\psi = \cdot 2428$ radian, and the graph has an asymptote in this direction. This is also an asymptote of the locus of centres of aberrancy.

Since $\sqrt{9+\rho'^2} = u = c\rho^{1/3} - k\rho$
$$= -kv^{\frac{1}{2}}\left(v - \frac{c}{k}\right),$$
and since v in this problem must be equal to or greater than $4a = \frac{4c}{3k}$, we must take the negative sign with the radical in $\sqrt{9+\rho'^2}$.
If we write $\rho' = 3\cot\alpha$, we have $\sqrt{9+\rho'^2} = -3\operatorname{cosec}\alpha$, and the coordinates of the centre of aberrancy are:– $\bar{\xi} = x - R\cos(\psi+\alpha)$, $\bar{\eta} = y - R\sin(\psi+\alpha)$.
Since, in the problem considered, $R = 1$, the coordinates $\bar{\xi}, \bar{\eta}$ can be calculated from these formulae.

The osculating conics are hyperbolas.
For we have $9+\rho'^2 = k^2\rho^2 - 2ck\rho^{4/3} + c^2\rho^{2/3}$
$$= \frac{9}{4}\rho^2 - \frac{27}{2}\rho^{4/3} + \frac{81}{4}\rho^{2/3},$$
$$\rho'' = \rho'\frac{d\rho'}{d\rho} = \frac{9}{4}\rho - 9\rho^{1/3} + \frac{27}{4}\rho^{-\frac{1}{3}}$$
$\therefore\ 9+\rho'^2 - 3\rho\rho'' = -\frac{9}{2}\rho^{4/3}(\rho^{2/3}-3)$. But for real values of s, we must have $\rho^{2/3} \geq 4$. Hence
$9+\rho'^2-3\rho\rho''$ is negative.

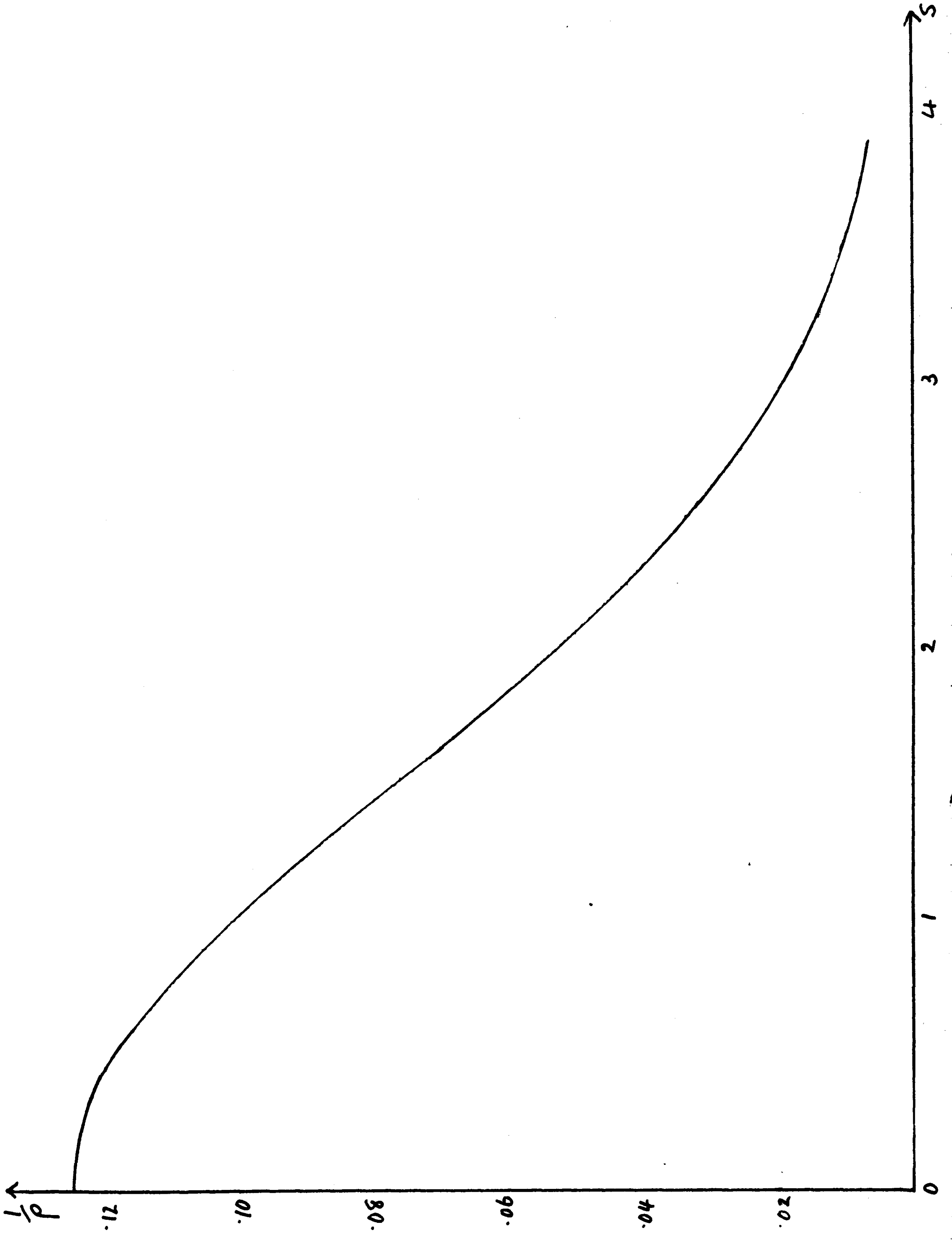
Data for the graph of $s = \log\dfrac{v-2+\sqrt{v^2-4v}}{2} + \dfrac{2}{\sqrt{3}}\tan^{-1}\sqrt{\dfrac{v-4}{3v}}$.

For the $(s, \frac{1}{\rho})$ diagram :–

v	4	4·1	4·2	4·3	4·4	4·5	4·6	4·7	4·8	4·9
s	0	·419	·585	·717	·822	·913	·994	1·067	1·134	1·196
$\frac{1}{\rho}$	·125	·121	·116	·112	·109	·105	·102	·098	·095	·092
v	5·0	5·25	5·50	5·75	6·00	7	8	9	10	12
s	1·254	1·385	1·495	1·598	1·688	1·986	2·211	2·398	2·549	2·800
$\frac{1}{\rho}$	·089	·083	·078	·073	·068	·054	·044	·037	·032	·024
v	14	16	18	22	25	28				
s	3·003	3·170	3·314	3·550	3·697	3·824				
$\frac{1}{\rho}$	·019	·016	·013	·0097	·0080	·0070				

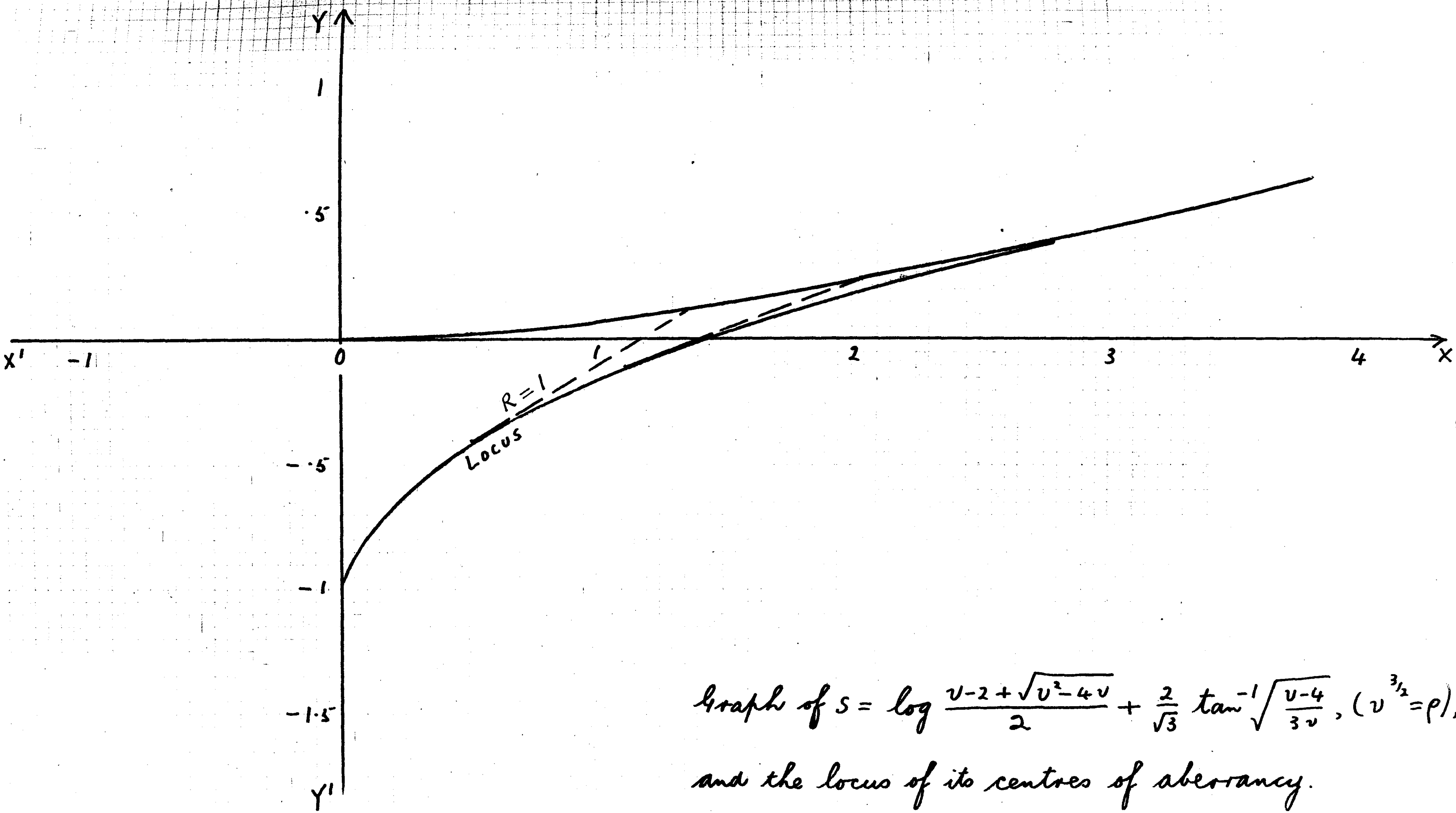
For the graph :–

s	0	·5	1·0	1·5	2·0	2·5	3·0	3·5	3·875
x	0	·50	1·00	1·49	1·98	2·47	2·96	3·45	3·81
y	0	·015	·06	·13	·22	·32	·43	·54	·63



For the graph of $s = \log \frac{v-2+\sqrt{v^2-4v}}{2} + \frac{2}{\sqrt{3}} \tan^{-1}\sqrt{\frac{v-4}{3v}}$, $(v^{\frac{3}{2}}=\rho)$

$(s, \frac{1}{\rho})$ diagram.



Graph of $s = \log \dfrac{v-2+\sqrt{v^2-4v}}{2} + \dfrac{2}{\sqrt{3}} \tan^{-1}\sqrt{\dfrac{v-4}{3v}}$, $(v^{3/2}=\rho)$,

and the locus of its centres of aberrancy.

(3) Graph of the equation $\frac{2k}{3}s = \int \frac{\sqrt{v}\,dv}{\sqrt{v^3 - \frac{2c}{k}v^2 + \frac{c^2}{k^2}v - \frac{9}{k^2}}}$,

where $v = \rho^{2/3}$ for the case when $k = 3$, $c = 6$.

The equation can be written

$$s = \int \frac{\sqrt{\frac{v}{4}}\,dv}{\sqrt{(v-1)(v - \frac{3+\sqrt{5}}{2})(v - \frac{3-\sqrt{5}}{2})}}, \text{ and}$$

for real values of s we must have

$$\frac{3-\sqrt{5}}{2} \leqq v \leqq 1, \text{ or } v \geqq \frac{3+\sqrt{5}}{2}.$$

If we write λ for $\frac{\sqrt{\frac{v}{4}}}{\sqrt{(v-1)(v^2-3v+1)}}$, then λ is infinite when v has one of the values $\frac{3\pm\sqrt{5}}{2}$, 1, and λ is zero when v is infinite.

The graph of λ therefore has three vertical asymptotes $v = \frac{3-\sqrt{5}}{2}$, $v = 1$, $v = \frac{3+\sqrt{5}}{2}$, and one horizontal asymptote $\lambda = 0$.

There is a minimum on the graph at $v = \cdot 594$.

The value of s near $v = \frac{3-\sqrt{5}}{2}$ can be found by writing $v = \frac{3-\sqrt{5}}{2} + \delta$, and expanding λ as far as the term in δ^2. In this result we substitute $\delta = \cdot 068$ and find $\int_{\frac{3-\sqrt{5}}{2}}^{\cdot 450} \lambda\,dv = \cdot 1446$.

Similarly near $v = 1$, we may write $v = 1 - \delta$. In this way we find $\int_{\cdot 9}^{1} \lambda\,dv = \cdot 3165$.

Near $v = \frac{3+\sqrt{5}}{2}$, the substitution $v = \frac{3+\sqrt{5}}{2} + \delta$ leads to the value of the integral $\int_{\frac{3+\sqrt{5}}{2}}^{2\cdot 7} \lambda\,dv$. The value is $\cdot 2413$.

The values of $\int_{\cdot 45}^{\cdot 90} \lambda\,dv$ and $\int_{2\cdot 7}^{12} \lambda\,dv$ can be found by graphical integration by calculating the areas under the graph of λ between these pairs of limits.

Corresponding values of s, v, $\frac{1}{\rho}$ can then be calculated, and using the formula $\psi = \int \frac{ds}{\rho}$, approximate values of ψ can be found by graphical integration from the $(s, \frac{1}{\rho})$ diagram. This enables a table of corresponding values of s, $\cos\psi$, $\sin\psi$ to be calculated. From the graphs $(s, \cos\psi)$ and $(s, \sin\psi)$, the coordinates x and y can be found approximately by graphical integration using the formulae

$$x = \int \cos\psi \, ds, \quad y = \int \sin\psi \, ds.$$

In this example $9+\rho'^2 = 9\rho^2 - 36\rho^{4/3} + 36\rho^{2/3}$,

$$-3\rho\rho'' = -27\rho^2 + 72\rho^{4/3} - 36\rho^{2/3},$$

$$9+\rho'^2 - 3\rho\rho'' = 18\rho^{4/3}(2-\rho^{2/3}).$$

The osculating conics are ellipses for points for which $\rho < 2.828$, and hyperbolas for points for which $\rho > 2.828$. Therefore for that part of the curve corresponding to $\frac{3-\sqrt{5}}{2} \leq v \leq 1$, the osculating conics are ellipses, and for that part of the curve corresponding to $v \geq \frac{3+\sqrt{5}}{2}$, they are hyperbolas.

The axis of aberrancy is $\dfrac{3\rho\sqrt{9+\rho'^2}}{9+\rho'^2-3\rho\rho''} = \dfrac{1}{2}$,

which is the value of $\frac{3}{2K}$ when $K = 3$. The locus of the centres of aberrancy is shown on the graph.

Since $\dfrac{2K}{3} s = \displaystyle\int \frac{\sqrt{v}\, dv}{\sqrt{v(v-\frac{c}{K})^2 - \frac{9}{K^2}}}$, we see that, when v is large, $\frac{2K}{3}s$ is very closely represented by $\displaystyle\int \frac{dv}{v-\frac{c}{K}}$. That is, we can write, in the present example,

$s = \frac{1}{2}\log\frac{v-2}{6}$, for finding increments of s from $v = 8$ upwards.

The same consideration shows that ψ remains finite when $v \to \infty$. This is seen by considering the integral $\int \frac{ds}{\rho}$ when ρ is large. Substituting $ds = \rho\, d\psi$ in the approximate solution $\frac{2K}{3}ds = \dfrac{dv}{v-\frac{c}{K}}$, we find

$$\int \frac{2K}{3}\, d\psi = \int \frac{dv}{v^{3/2}\left(v - \frac{c}{K}\right)} = \frac{-2w^2\, dw}{1 - \frac{c}{K} w^2}, \text{ where } v = w^{-2}.$$

The solution is $\dfrac{c\psi}{3} = w - \dfrac{\sqrt{\frac{K}{c}}}{2} \log \dfrac{1 + w\sqrt{\frac{c}{K}}}{1 - w\sqrt{c/K}} + \text{const.}$

In the present example $c = 6$ and $\frac{c}{K} = 2$, and when $v = 8$, $w = \frac{1}{2\sqrt{2}}$, $\psi = \cdot 223$, and we find

$$\int_{v=8}^{v=\infty} d\psi = \frac{1}{4}\left(\frac{\log 3}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) = \cdot 017.$$

The value of ψ at infinity is therefore $\cdot 240$. The graph therefore has a pair of symmetrically-placed asymptotes in the directions $\psi = \pm \cdot 240$.

<u>Data for the graph of</u> $2s = \displaystyle\int \frac{\sqrt{v}\, dv}{\sqrt{(v-1)(v^2 - 3v + 1)}}$.

v	·382	·45	·5	·6	·7	·8	·9	1	2·618	2·7
λ	∞	1·178	1	·923	·978	1·148	1·590	∞	∞	1·445
s	0	·145	·200	·295	·390	·495	·629	·946	·946	1·187
$\frac{1}{\rho}$	4·24	3·31	2·83	2·15	1·70	1·40	1·18	1·00	·237	·225

v	2·8	2·9	3·0	3·1	3·2	3·4	3·7	4·0	5	6	8	10	12
λ	·940	·733	·612	·531	·471	·387	·313	·258	·169	·126	·083	·063	·05
s	1·32	1·39	1·46	1·51	1·56	1·65		1·84	2·05	2·20	2·40	2·54	2·65
$\frac{1}{\rho}$	·214	·203	·193	·184	·175	·160		·125	·090	·068	·044	·032	·024

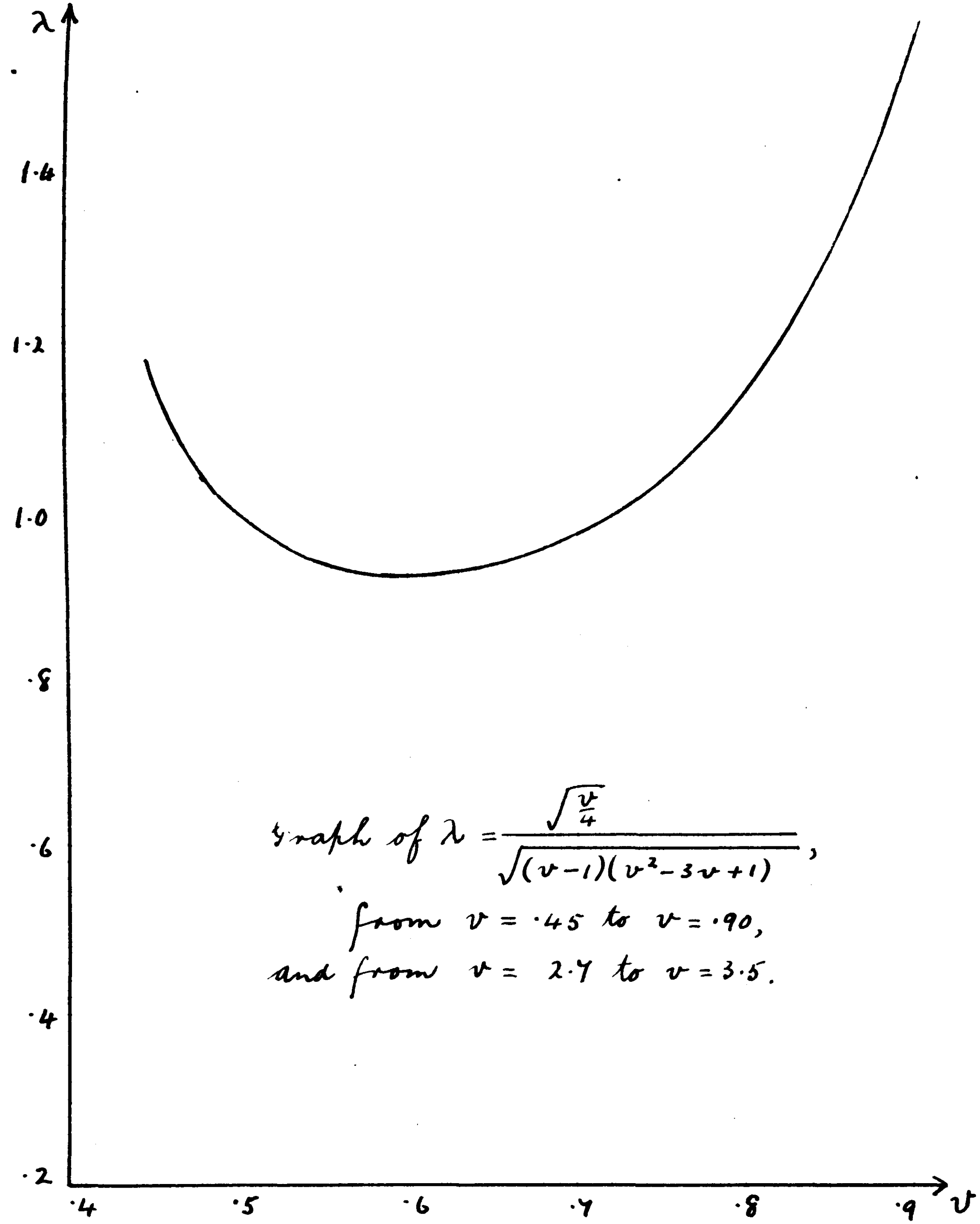
<u>Values of s, ψ, $\cos\psi$, $\sin\psi$.</u>

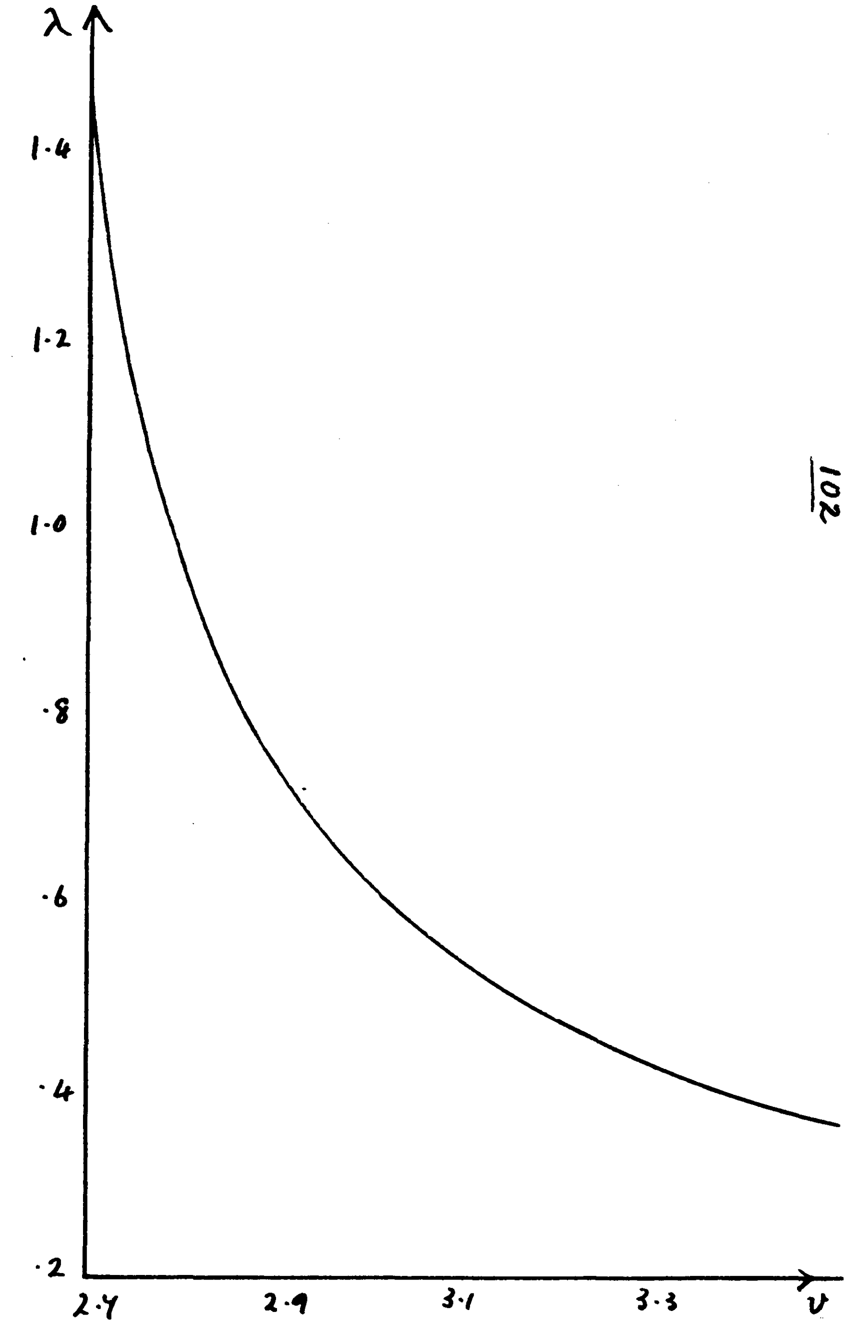
s	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	·946	1
ψ	−1·84	−1·21	−·80	−·52	−·31	−·124	0	·010
$50 \sin\psi$	−48·2	−46·8	−35·7	−24·9	−15·3	−6·2	0	·5
$50 \cos\psi$	−13·2	17·5	35	43·3	47·6	49·6	50	50

s	$1\frac{1}{3}$	$1\frac{2}{3}$	2	$2\frac{1}{3}$	$2\frac{2}{3}$
ψ	·084	·150	·193	·220	·233
$50 \sin\psi$	4·4	7·5	9·6	10·9	11·5
$50 \cos\psi$	49·8	49·4	49·1	48·8	48·7

<u>Values of s, x, y.</u>

s	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	·946	$1\frac{1}{3}$	$1\frac{2}{3}$	2	$2\frac{1}{3}$	$2\frac{2}{3}$
x	−·737	−·721	−·634	−·503	−·328	−·166	0	·333	·665	·993	1·320	1·645
y	·507	·349	·211	·110	·043	·007	0	·017	·057	·114	·183	·258





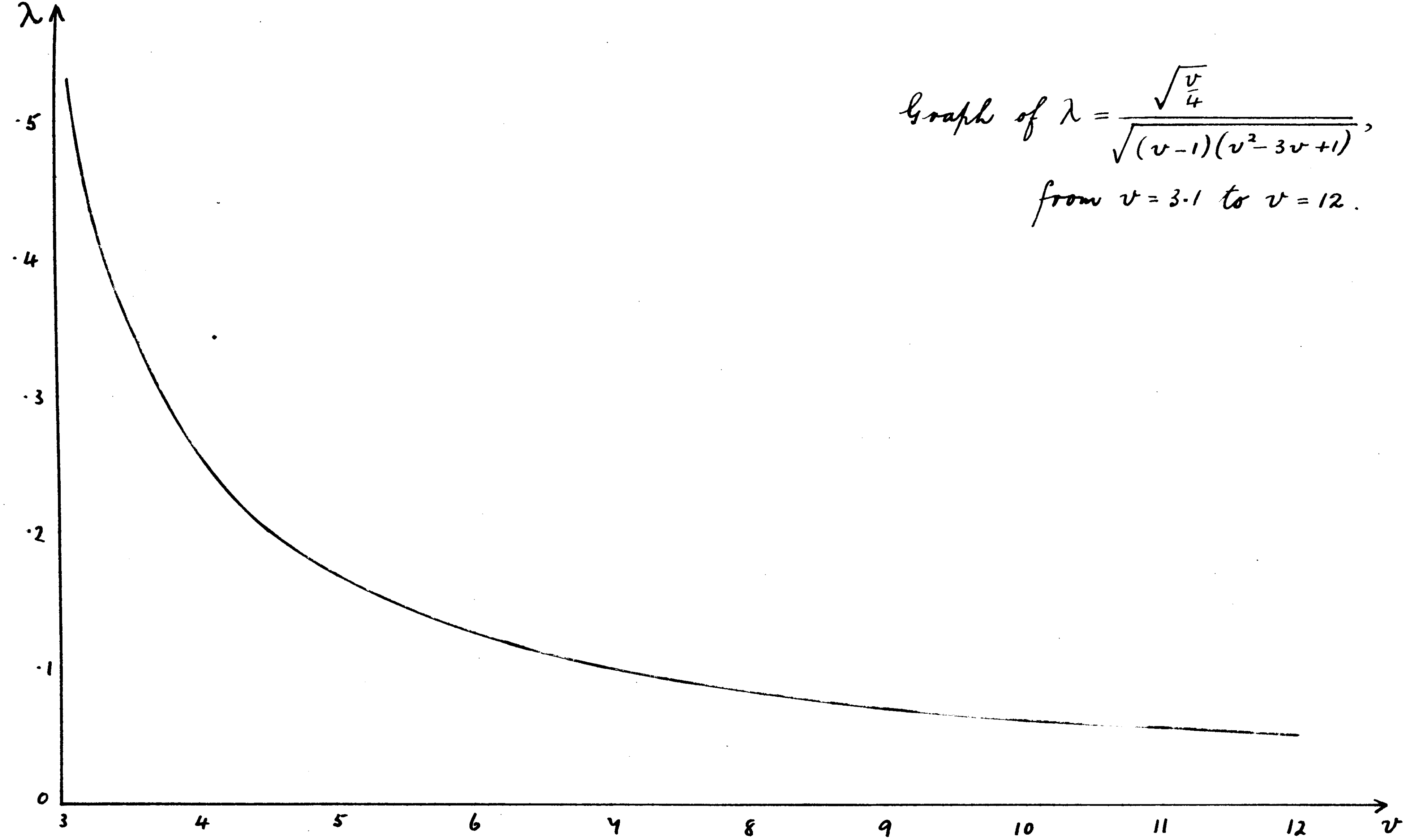
Graph of $\lambda = \dfrac{\sqrt{\frac{v}{4}}}{\sqrt{(v-1)(v^2-3v+1)}}$,

from $v = \cdot 45$ to $v = \cdot 90$,

and from $v = 2\cdot 7$ to $v = 3\cdot 5$.

Graph of $\lambda = \dfrac{\sqrt{\frac{v}{4}}}{\sqrt{(v-1)(v^2-3v+1)}}$,

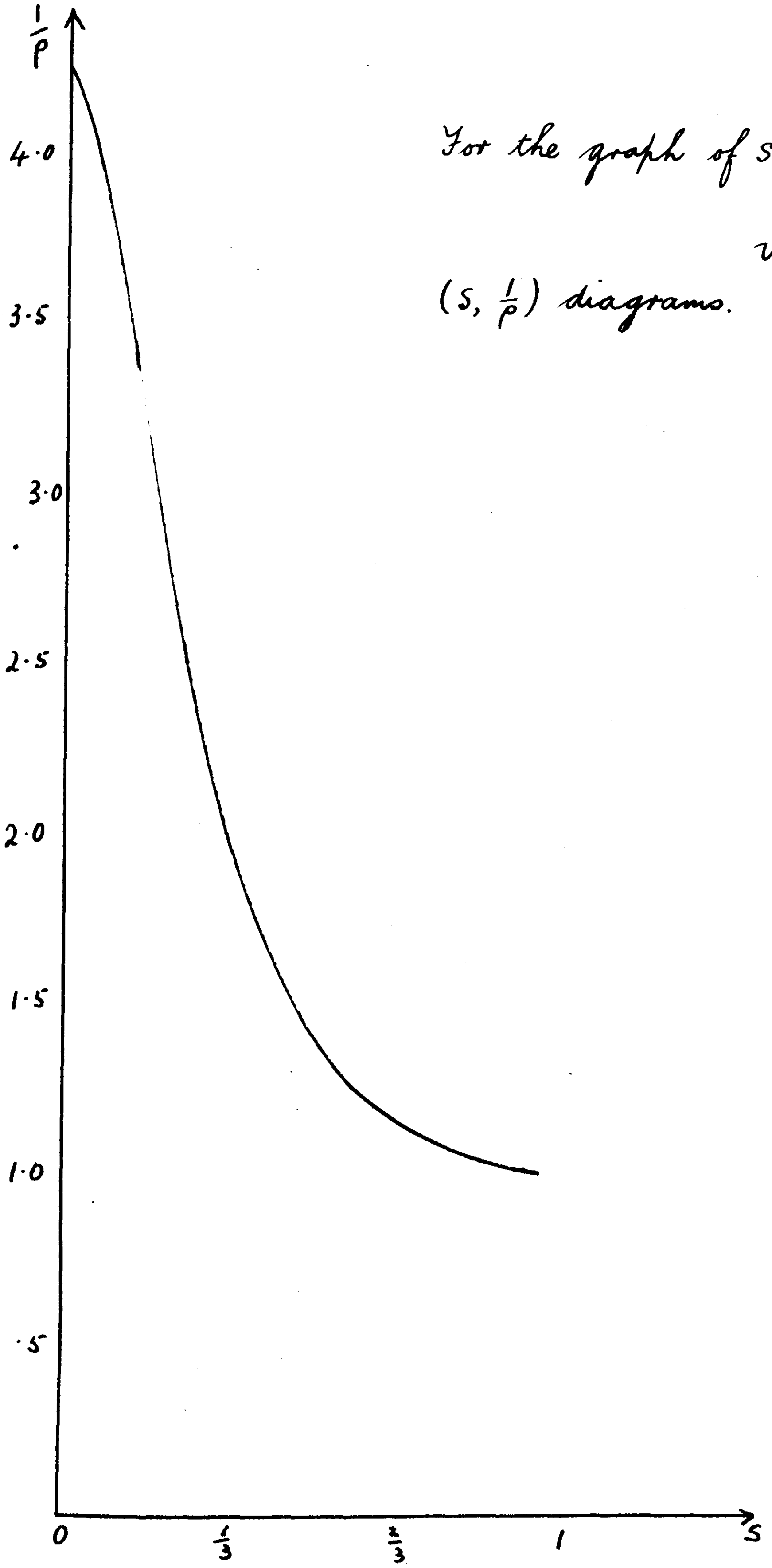
from $v = 3{\cdot}1$ to $v = 12$.

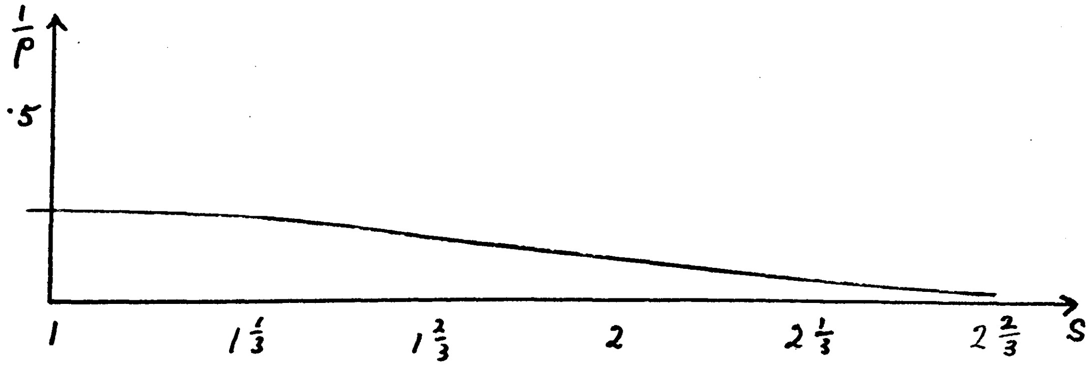


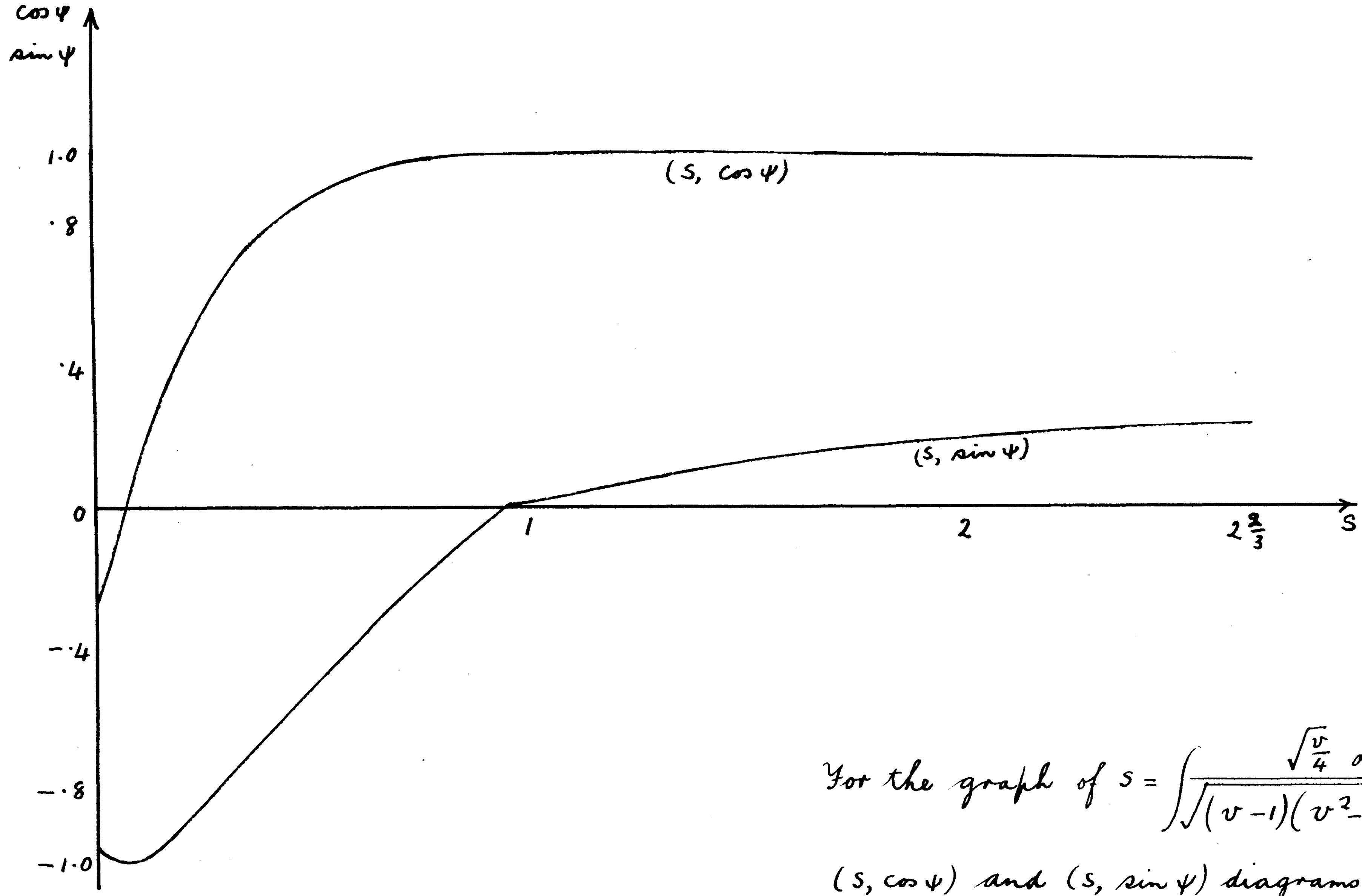
For the graph of $s = \int \frac{\sqrt{\frac{v}{4}}\;dv}{\sqrt{(v-1)(v^2-3v+1)}}$,

$v = \rho^{2/3}$.

$(s, \frac{1}{\rho})$ diagrams.

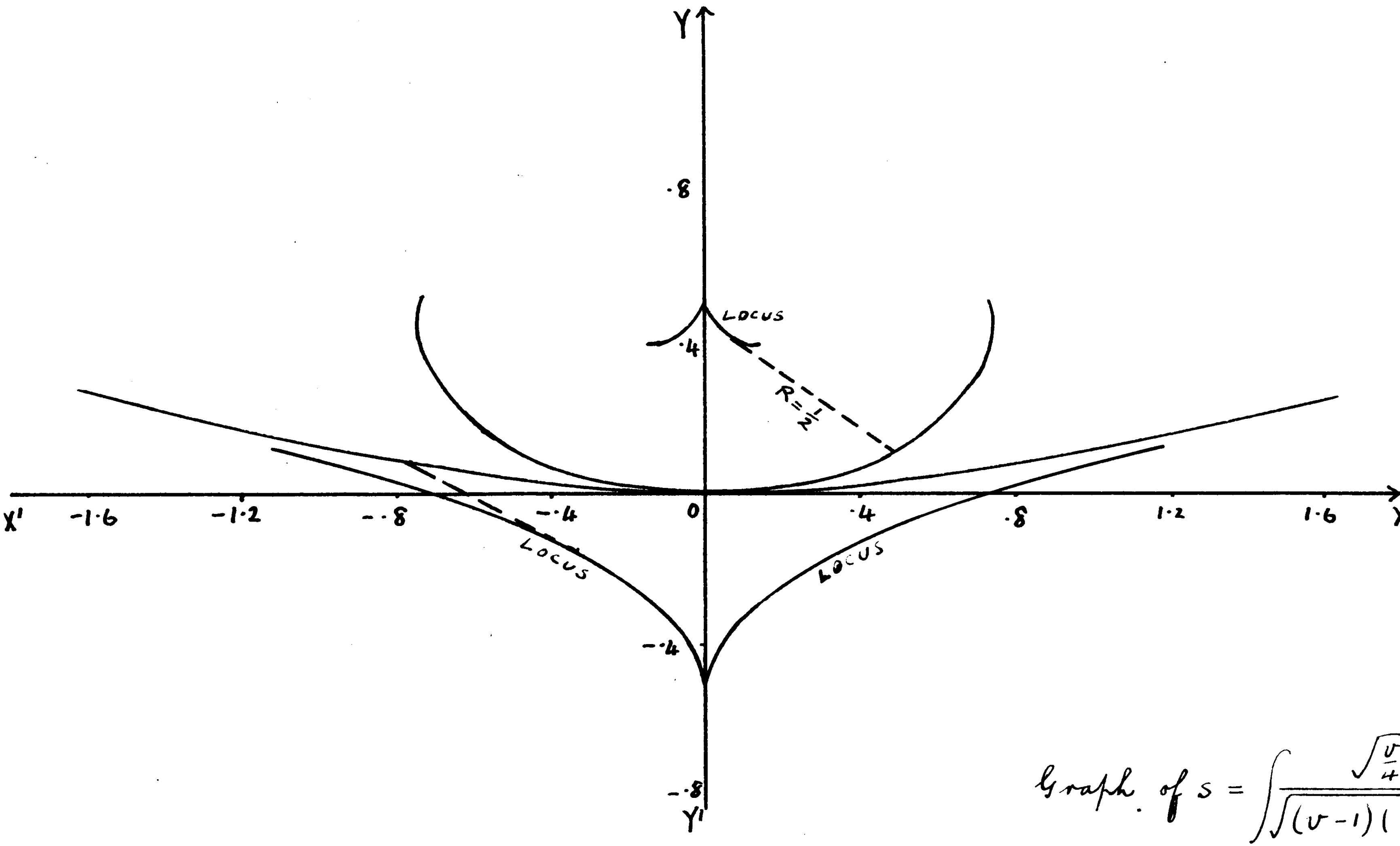






For the graph of $s = \int \frac{\sqrt{\frac{v}{4}}\, dv}{\sqrt{(v-1)(v^2-3v+1)}}$, $v^{\frac{3}{2}} = p$.

(s, cos ψ) and (s, sin ψ) diagrams.



Graph of $s = \int \frac{\sqrt{\frac{v}{4}}\, dv}{\sqrt{(v-1)(v^2-3v+1)}}$, $(v = \rho^{2/3})$ and the locus of the centres of its osculating conics.

(4) Graph of the equation $\frac{2K}{3} s = \int \frac{\sqrt{v}\, dv}{\sqrt{v^3 - \frac{2c}{K} v^2 + \frac{c^2}{K^2} v - \frac{9}{K^2}}}$

where $v = \rho^{2/3}$, for the case when $K = -c = \frac{3}{2}$.

The equation can be written

$$s = \int \frac{\sqrt{v}\, dv}{\sqrt{(v-1)(v^2+3v+4)}},$$ and for real values of ρ and s we must have $v \geqq 1$.

If we write λ for $\frac{\sqrt{v}}{\sqrt{(v-1)(v^2+3v+4)}}$, then λ is infinite when $v = 1$, and zero when v is infinite. The graph of λ therefore has a vertical asymptote $v = 1$, and a horizontal asymptote $\lambda = 0$.

Near $v = 1$, we can write $v = 1 + \delta$, and find the value of $\int_{1.0}^{1.1} \lambda\, dv$, by expanding λ in powers of δ up to δ^2. The result is $\cdot 2248$. The graph of λ can then be used to extend this integral to $v = 9$.

When $v \geqq 9$, the integral $s = \int_9^v \lambda\, dv$ is very closely represented by that of

$$\int_9^v \frac{dv}{1+v} = \log \frac{1+v}{10}.$$ This result can be used to extend s for values of v from 9 to 99.

In this example $\rho'^2 = \frac{9}{4}\rho^2 + \frac{9}{2}\rho^{4/3} + \frac{9}{4}\rho^{2/3} - 9,$

$$\rho\rho'' = \frac{9}{4}\rho^2 + 3\rho^{4/3} + \frac{3}{4}\rho^{2/3},$$

$$9 + \rho'^2 - 3\rho\rho'' = -\frac{9}{2}\rho^{4/3}(1 + \rho^{2/3}).$$

Thus the osculating conics are hyperbolas. The locus of the centres of these is shown on the graph.

The axis of aberrancy is $\frac{3\rho\sqrt{9+\rho'^2}}{+9+\rho'^2-3\rho\rho''} = 1$, which is the value of $\frac{3}{2K}$ when $K = \frac{3}{2}$.

The curve has a pair of symmetrically-placed asymptotes inclined at $\psi = \pm 40^\circ 36'$ to the x-axis. That ψ remains finite can be seen by considering the value of the integral $\int \frac{ds}{\rho}$ when ρ is large. We have seen that when v is large, s is very closely represented by $\int \frac{dv}{1+v}$. Therefore ψ is similarly represented by

$$\int \frac{dv}{v^{3/2}(1+v)} = \int \frac{-2w^2\,dw}{1+w^2}, \text{ where } v = w^{-2}.$$

Corresponding to the limits 9 and ∞ for v, we have the limits $\frac{1}{3}$ and 0 for w. Hence

$$\psi = \int_9^\infty \frac{dv}{v^{3/2}(1+v)} = 2\int_0^{\frac{1}{3}} \frac{w^2\,dw}{1+w^2} = 2(w - \tan^{-1} w)$$
$$= \cdot 0232.$$

Adding this to the value of ψ at $v = 9$, we find $\psi\Big|_{v=1}^{v=\infty} = \cdot 7093.$

Data for the graph of $s = \int \frac{\sqrt{v}\,dv}{\sqrt{(v-1)(v^2+3v+4)}}$.

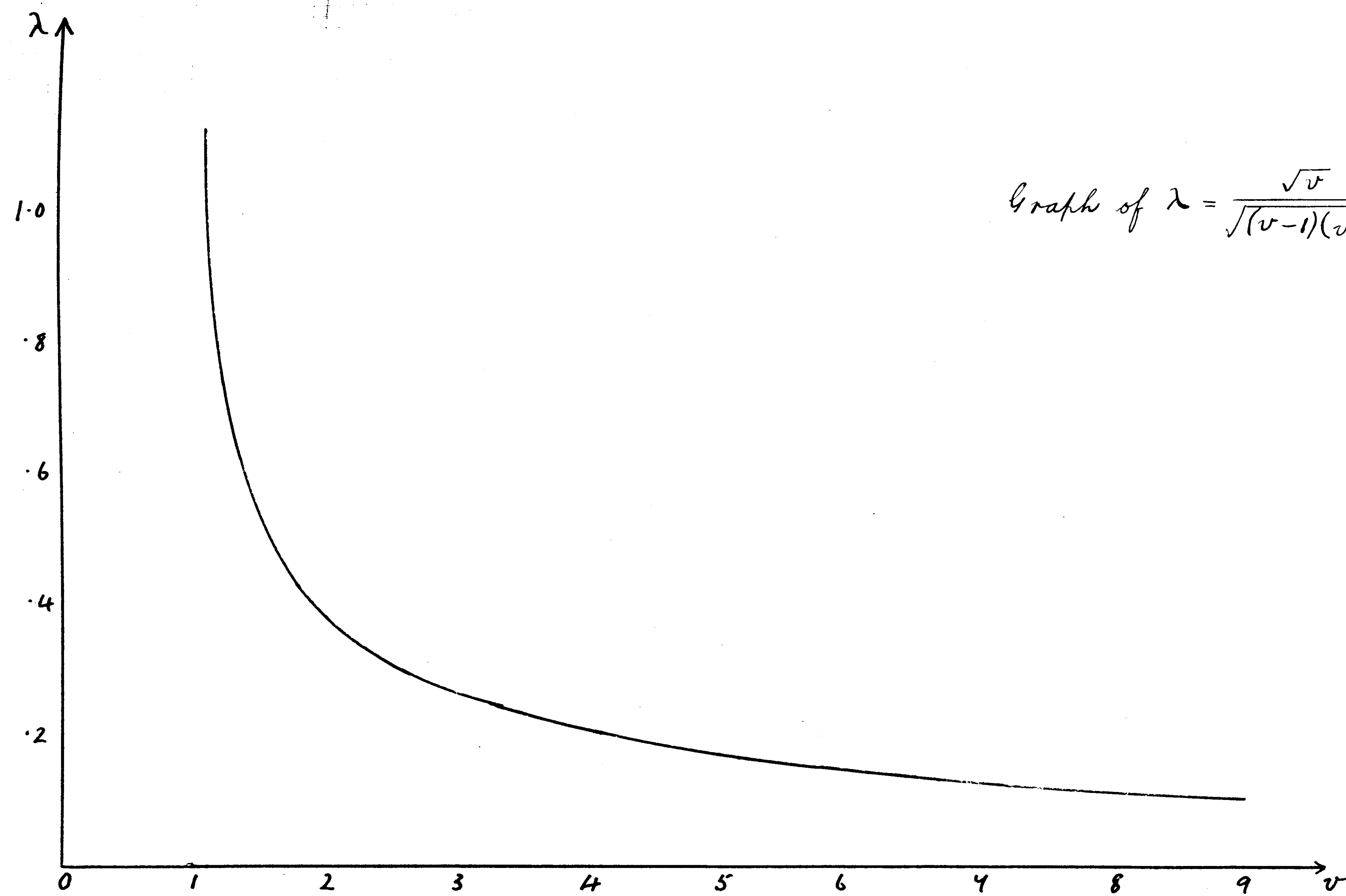
v	1	1·1	1·2	1·3	1·4	1·5	1·6	1·7	1·8	1·9	2	3
λ	∞	1·137	·773	·672		·528			·422		·378	·261
s	0	·225	·321	·393	·457	·513	·564	·610	·654	·695	·734	1·05
$\frac{1}{\rho}$	1	·840	·760	·678	·601	·547	·495	·450	·415	·382	·353	·193

v	4	5	6	7	8	9	19	29	49	69	99
λ	·204	·169		·126		·100					
s	1·29	1·47	1·63	1·76	1·88	1·99	2·68	3·09	3·60	3·94	4·29
$\frac{1}{\rho}$	·125	·089	·068	·054	·044	·037	·012	·0064	·0029	·0017	·0010

s	0	·2	·4	·6	·8	1·0	1·2	1·4	1·6
ψ	0	·193	·350	·462	·538	·588	·624	·658	·665
$50\cos\psi$	50	49·2	47·0	44·7	43·0	41·6	40·6	39·9	39·3
$50\sin\psi$	0	9·5	17·2	22·3	25·6	27·7	29·2	30·2	30·9

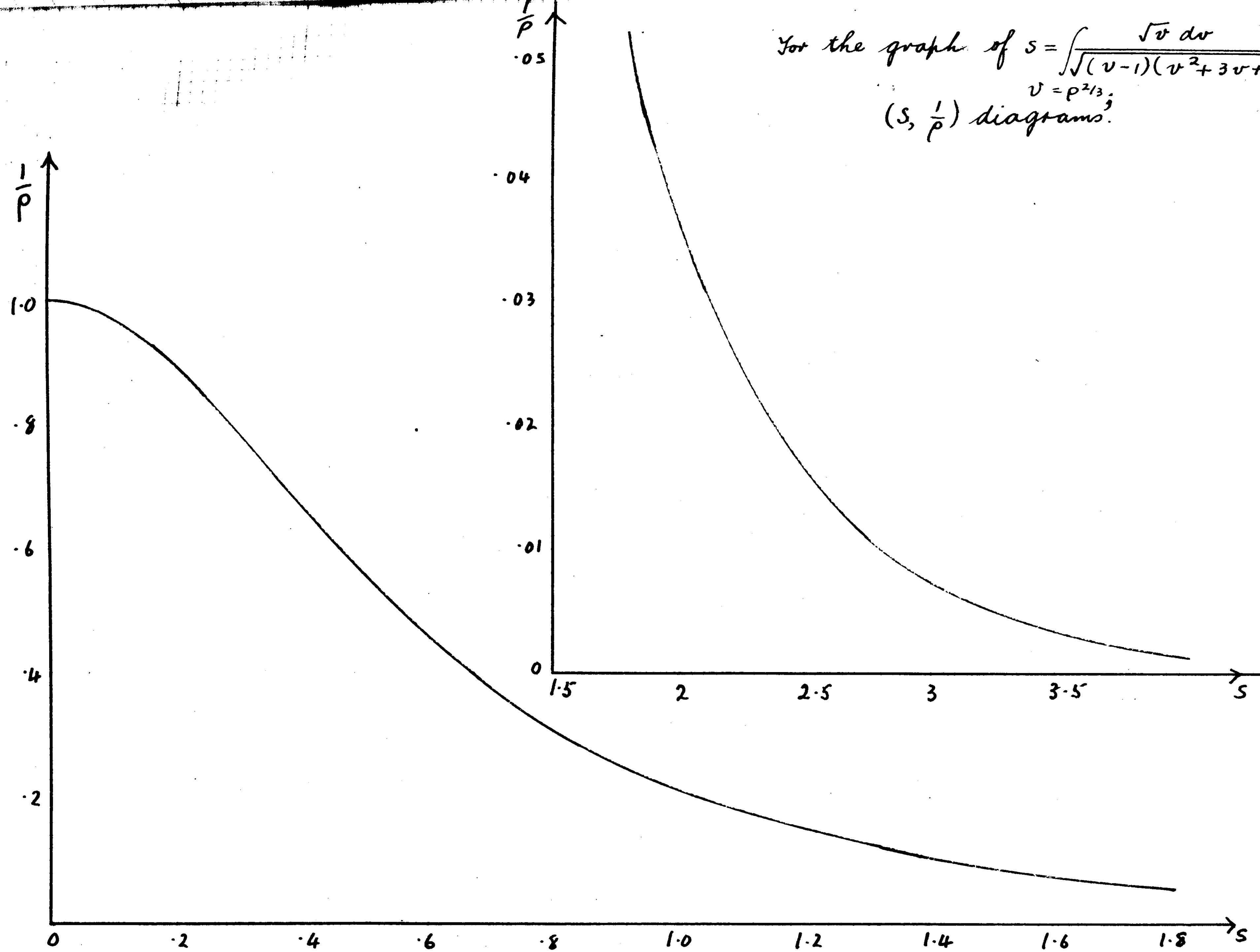
s	1·8	2·05	2·30	2·55	2·80	3·05	3·30	3·55	3·80	4·05
ψ	·678	·689	·695	·700	·703	·705	·707	·708	·708	·709
$50\cos\psi$	39·0	38·6	38·4	38·2	38·1	38·1	38	38	38	38
$50\sin\psi$	31·3	31·8	32·0	32·2	32·3	32·4	32·5	32·5	32·5	32·5

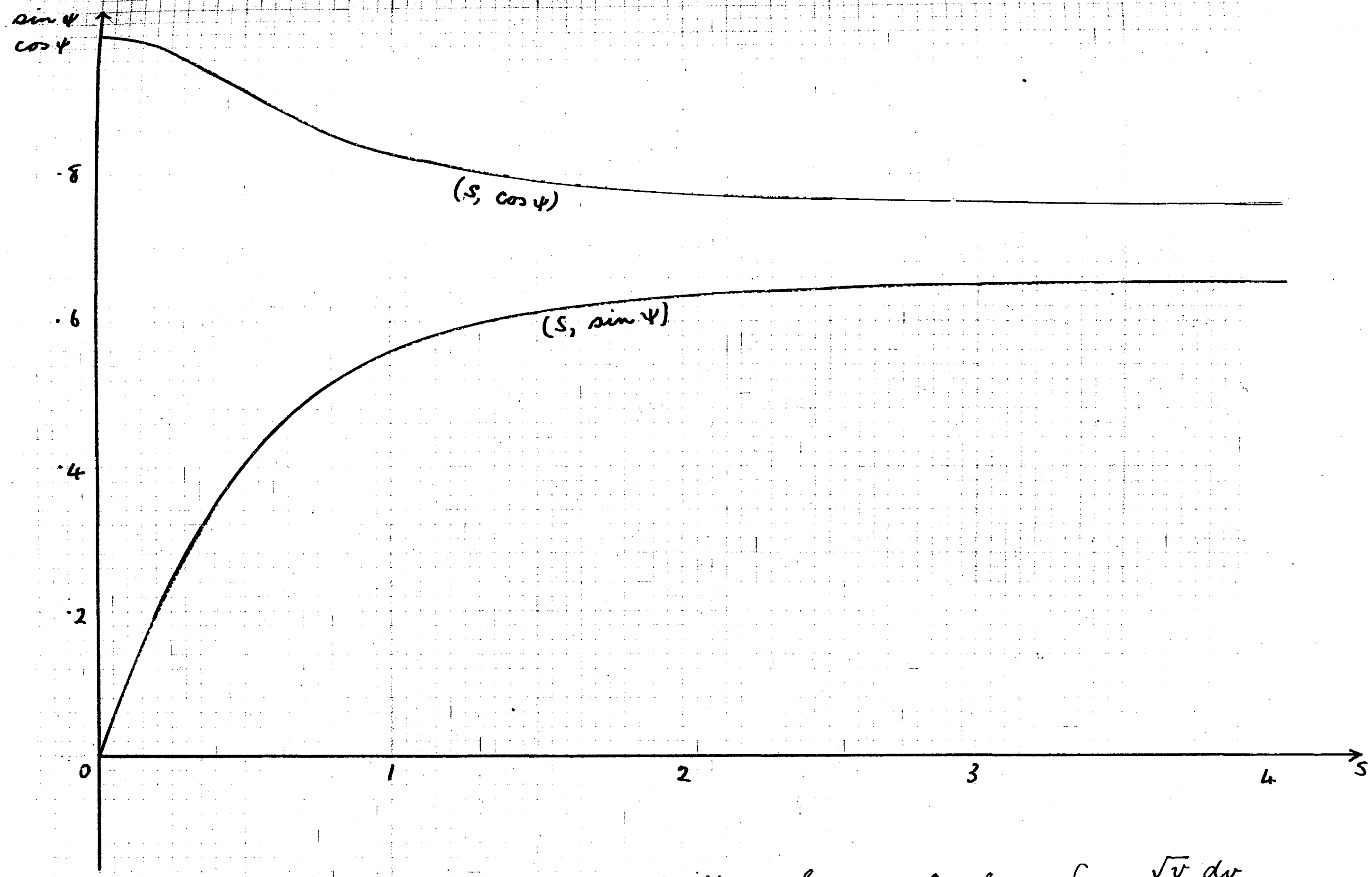
s	0	·5	1·0	1·5	2·0	2·5	3·0	3·5	4·0
x	0	·48	·92	1·32	1·71	2·10	2·48	2·86	3·24
y	0	·11	·36	·65	·97	1·29	1·61	1·94	2·26



Graph of $\lambda = \dfrac{\sqrt{v}}{\sqrt{(v-1)(v^2+3v+4)}}$.

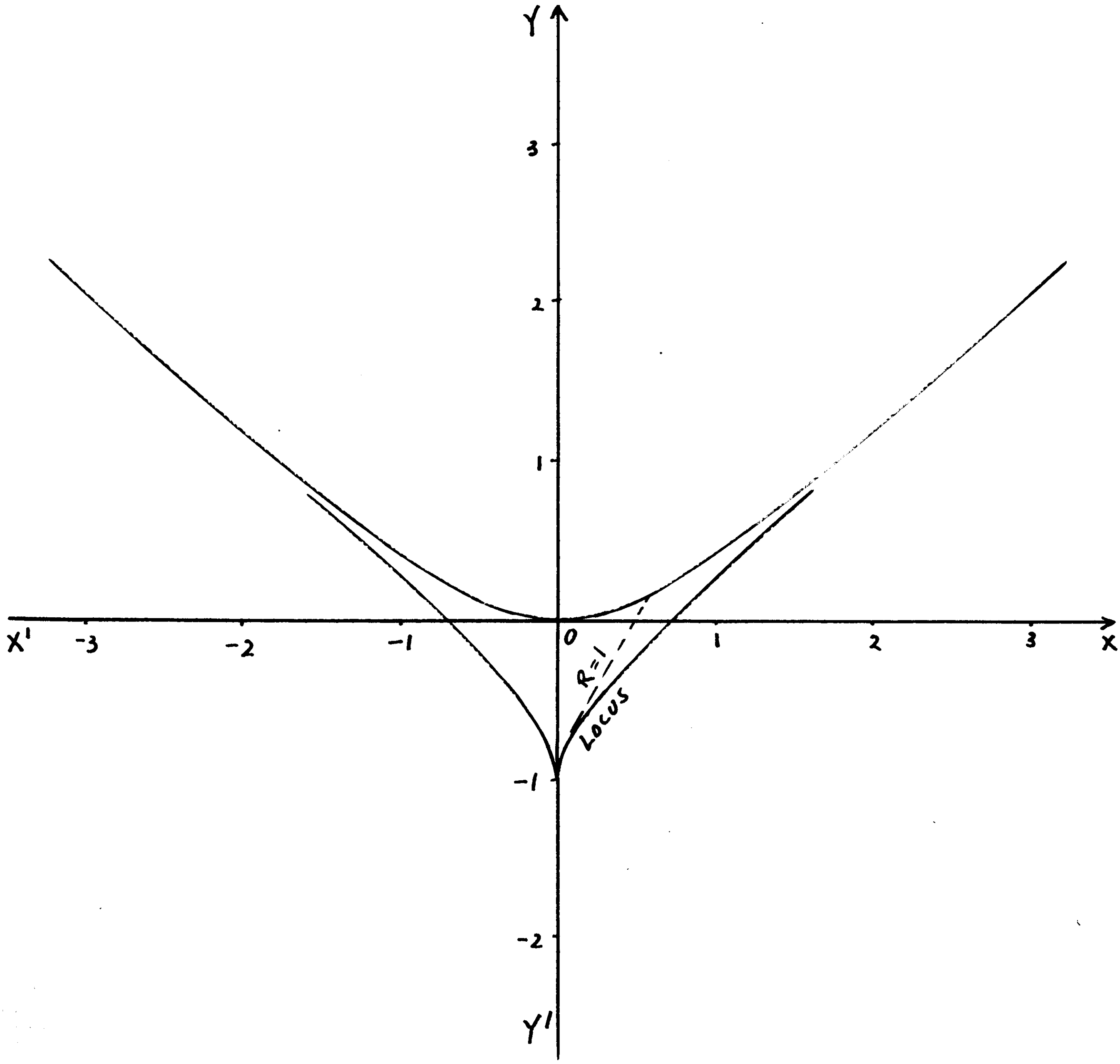
For the graph of $s = \int \frac{\sqrt{v}\,dv}{\sqrt{(v-1)(v^2+3v+4)}}$
$v = \rho^{2/3}$;
$(s, \frac{1}{\rho})$ diagrams.





For the graph of $s = \int \frac{\sqrt{v}\, dv}{\sqrt{(v-1)(v^2+3v+4)}}$, $v = \rho^{2/3}$;

$(s, \cos\psi)$ and $(s, \sin\psi)$ diagrams.



Graph of $s = \int \frac{\sqrt{v}\, dv}{\sqrt{(v-1)(v^2+3v+4)}}$, $v = \rho^{2/3}$, and the locus of the centres of its osculating conics.

Section IX.

To find a curve at any point of which the length of the axis of aberrancy is proportional to the radius of curvature at that point.

(1) Writing the constant of proportionality in the form $\frac{3}{\kappa}$, the differential equation is

$$\frac{3\rho\sqrt{9+\rho'^2}}{9+\rho'^2-3\rho\rho''} = \frac{3\rho}{\kappa},$$ and a first solution

is found by writing $9+\rho'^2 = u^2$, or $u\frac{du}{d\rho} = \rho''$.
The equation becomes $\frac{d\rho}{\rho} = \frac{3\,du}{u-\kappa}$, the solution

of which is $u-\kappa = c\rho^{1/3}$,

$$\text{or } \rho'^2 = \kappa^2 + 2\kappa c\rho^{1/3} + c^2\rho^{2/3} - 9.$$

---

(2) The particular case $\kappa=0$ gives $u = c\rho^{1/3}$, an equation which was discussed in Section VII. The curve is the parabola.

---

(3) The particular case $c=0$ gives $\rho' =$ const., and the curve is the equiangular spiral, which has been discussed in Section V.

---

(4) The condition for equal roots of $\kappa^2+2\kappa c\rho^{\frac{1}{3}}+c^2\rho^{2/3}-9=0$ gives $c=0$ or $\kappa = \infty$.

---

(5) The discriminant of the equation $\rho'^2 = c^2\rho^{\frac{2}{3}}+2\kappa c\rho^{\frac{1}{3}}+\kappa^2-9$ is the right-hand side equated to zero. This gives $\rho^{1/3} = \frac{-\kappa\pm 3}{c}$. This satisfies the differential equation, and when $\kappa=3$, it satisfies the original equation also. Obviously, the condition of the problem is satisfied by a circle, when the constant of proportionality becomes equal to unity.

---

(6) In the equation $ds = \dfrac{d\rho}{\sqrt{c^2\rho^{2/3} + 2Kc\rho^{1/3} + K^2 - 9}}$

substitute $\rho^{1/3} = \dfrac{3\sec\varphi - K}{c}$, and we find

$$\frac{c^3}{3}s = \int (9\sec^3\varphi - 6K\sec^2\varphi + K^2\sec\varphi)\,d\varphi,$$

the solution of which is

$$\frac{c^3}{3}s = \frac{9}{2}\tan\varphi\sec\varphi - 6K\tan\varphi + (K^2 + \tfrac{9}{2})\log\tan(\tfrac{\pi}{4} + \ldots$$

---

(7) Substituting $\rho\, d\psi$ for ds, we find

$$d\psi = \frac{d\rho}{\rho\sqrt{c^2\rho^{2/3} + 2Kc\rho^{1/3} + K^2 - 9}},$$

and using the same substitution we find ψ in terms of φ. The results are

$$\psi = \frac{6}{\sqrt{9-K^2}}\tan^{-1}\sqrt{\frac{3+K}{3-K}}\tan\frac{\varphi}{2}, \quad (K^2 < 9);$$

$$\psi = -\frac{3}{\sqrt{K^2-9}}\log\frac{\sqrt{K-3} + \sqrt{K+3}\tan\frac{\varphi}{2}}{\sqrt{K-3} - \sqrt{K+3}\tan\frac{\varphi}{2}}, \quad (K^2 > 9).$$

$$\psi = -\cot\frac{\varphi}{2}, \qquad (K = 3).$$

---

(8) The curve $s = \displaystyle\int \frac{d\rho}{\sqrt{c^2\rho^{2/3} + 2Kc\rho^{1/3} + K^2 - 9}}$ has a pair of symmetrically-placed asymptotes, for the equation for s in terms of φ shows that s tends to infinity when φ tends to $\frac{\pi}{2}$, and the equation for ψ in terms of φ shows that ψ tends to a finite value when $\varphi \to \frac{\pi}{2}$.

---

(9). For values of φ between $90°$ and $180°$ we may substitute $\theta = 180° - \varphi$ in the equations for s and ψ, and take values of θ from $90°$ to $0°$. This gives for ρ, s, ψ the equations:-

$$c\rho^{1/3} = -3\sec\theta - K,$$

$$\frac{c^3}{3}s = \frac{9}{2}\tan\theta\sec\theta + 6K\tan\theta - (K^2 + \tfrac{9}{2})\log\cot(\tfrac{\pi}{4} + \tfrac{\theta}{2}\ldots$$

$$\psi = \frac{6}{\sqrt{9-K^2}}\tan^{-1}\sqrt{\frac{3+K}{3-K}}\cot\frac{\theta}{2}, \quad (K^2 < 9),$$

$$= -\frac{3}{\sqrt{K^2-9}}\log\frac{\sqrt{K-3} + \sqrt{K+3}\cot\frac{\theta}{2}}{\sqrt{K-3} - \sqrt{K+3}\cot\theta/2}, \quad (K^2 > 9).$$

$$= -\tan\frac{\theta}{2}, \qquad (K^2 = 9).$$

(10) If we write $c\rho^{1/3} = 3\sec\varphi - \kappa$, then since $u = c\rho^{1/3} + \kappa$, we may take $\sqrt{9+\rho'^2} = u = 3\sec\varphi$, and $\rho' = 3\tan\varphi$. This gives $\dfrac{l\rho' - 3m}{\sqrt{9+\rho'^2}} = \sin(\varphi - \psi)$, and $\dfrac{m\rho' + 3l}{\sqrt{9+\rho'^2}} = \cos(\varphi - \psi)$.

These may be written $\cos(\psi + \frac{\pi}{2} - \varphi)$ and $\sin(\psi + \frac{\pi}{2} - \varphi)$. Moreover, the axis of aberrancy has length $\frac{3}{\kappa}\rho$. Hence, the coordinates of the centre of aberrancy may be written

$$\bar{\xi} = x + \frac{3}{\kappa}\rho\cos(\psi + \tfrac{\pi}{2} - \varphi),$$
$$\bar{\eta} = y + \frac{3}{\kappa}\rho\sin(\psi + \tfrac{\pi}{2} - \varphi).$$

Therefore, $\frac{\pi}{2} - \varphi$ is the angle which the direction of the axis of aberrancy makes with the direction of the tangent to the curve, or, the inclination of the axis of aberrancy to the normal is φ.

The subsidiary angle φ, therefore, used in the solution of this problem, is the angle of aberrancy.

The values of $\bar{\xi}$ and $\bar{\eta}$ may be calculated from the above formulae.

(11) When $\kappa = 3$, we have the result that the axis of aberrancy is equal to the radius of curvature. The curve satisfying this condition is shown in example (3) of the following section.

(12) The elimination of s and φ gives the solutions in the forms:-

(i) $\rho^{-\frac{1}{3}} = \pm\dfrac{3c}{\kappa^2 - 9}\cosh\dfrac{\sqrt{\kappa^2-9}}{3}\psi - \dfrac{\kappa c}{\kappa^2 - 9}$, $(\kappa^2 > 9)$,

(ii) $\sqrt{\dfrac{2c}{3}}\,\psi = 2\sqrt{\rho^{-\frac{1}{3}} + \dfrac{c}{6}}$, or $\rho^{-\frac{1}{3}} = \dfrac{c}{6}(\psi^2 - 1)$, $(\kappa^2 = 9)$,

(iii) $\rho^{-\frac{1}{3}} = \dfrac{\kappa c}{9 - \kappa^2} + \dfrac{3c}{9-\kappa^2}\cos\dfrac{\sqrt{9-\kappa^2}}{3}\psi$, $(\kappa^2 < 9)$.

Section X.

Examples of curves such that, at any point (x, y), the axis of aberrancy is proportional to the radius of curvature at the point.

(1) In the equation $ds = \dfrac{d\rho}{\sqrt{c^2\rho^{2/3} + 2Kc\rho^{1/3} + K^2 - 9}}$, put $c = 1$, $K = 2$,

and the equation for s is

$$s = \frac{27}{2}\tan\varphi\sec\varphi - 36\tan\varphi + \frac{51}{2}\log\tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)$$

while $\rho^{1/3} = 3\sec\varphi - 2$,

and $\psi = \dfrac{6}{\sqrt{5}}\tan^{-1}\left(\sqrt{5}\tan\dfrac{\varphi}{2}\right)$.

The graph of the function satisfying these equations was constructed by calculating a number of corresponding values of s and ψ for values of φ ranging from $0°$ to $90°$. From the ψ values corresponding values of $\sin\psi$ and $\cos\psi$ were found from tables. The $(s, \cos\psi)$ and $(s, \sin\psi)$ curves were drawn, and from these, by graphical integration, corresponding approximate values of the coordinates x and y were calculated.

Since the axis of aberrancy is one and a half times the length of the radius of curvature, it was possible by calculating a few values of ρ to draw a part of the locus of centres of aberrancy.

The curve has a pair of symmetrically-placed asymptotes, for when $\varphi = \frac{\pi}{2}$, $s = \infty$, and $\psi = 3{\cdot}08$ radians.

For values of φ between $90°$ and $180°$ we may subtitute $\theta = 180° - \varphi$, and take values of θ from $90°$ to $0°$. This gives the new forms for s, ρ, ψ:-

$$s = \frac{27}{2}\tan\theta\sec\theta + 36\tan\theta - \frac{51}{2}\log\cot\left(\frac{\pi}{4} + \frac{\theta}{2}\right),$$

$$\rho^{1/3} = -3\sec\theta - 2,$$

$$\psi = \frac{6}{\sqrt{5}}\tan^{-1}\left(\sqrt{5}\cot\frac{\theta}{2}\right).$$

This gives a second part of the graph, starting from the origin, and having as asymptotes a pair of straight lines parallel to the asymptotes in the first part of the graph. Both branches meet their asymptotes very slowly.

The curvature of the second part of the graph is so small that on the scale used it is impossible to show any part of the locus of the centres of the osculating conics.

Data for the graph of $s = \int \dfrac{d\rho}{\sqrt{\rho^{2/3} + 4\rho^{1/3} - 5}}$.

$\varphi = 0°$ to $90°$.

s	0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1
$50\cos\psi$	50	49·8	49·0	47·9	46·3	44·2	41·8	39·3	36·2	32·6	28·7
$50\sin\psi$	0	4·8	9·6	14·3	18·9	22·9	26·8	30·3	33·7	37·5	40·9
x	0	·10	·20	·30	·39	·48	·57	·65	·72	·79	·85
y	0	·00	·02	·04	·08	·12	·17	·22	·29	·36	·44

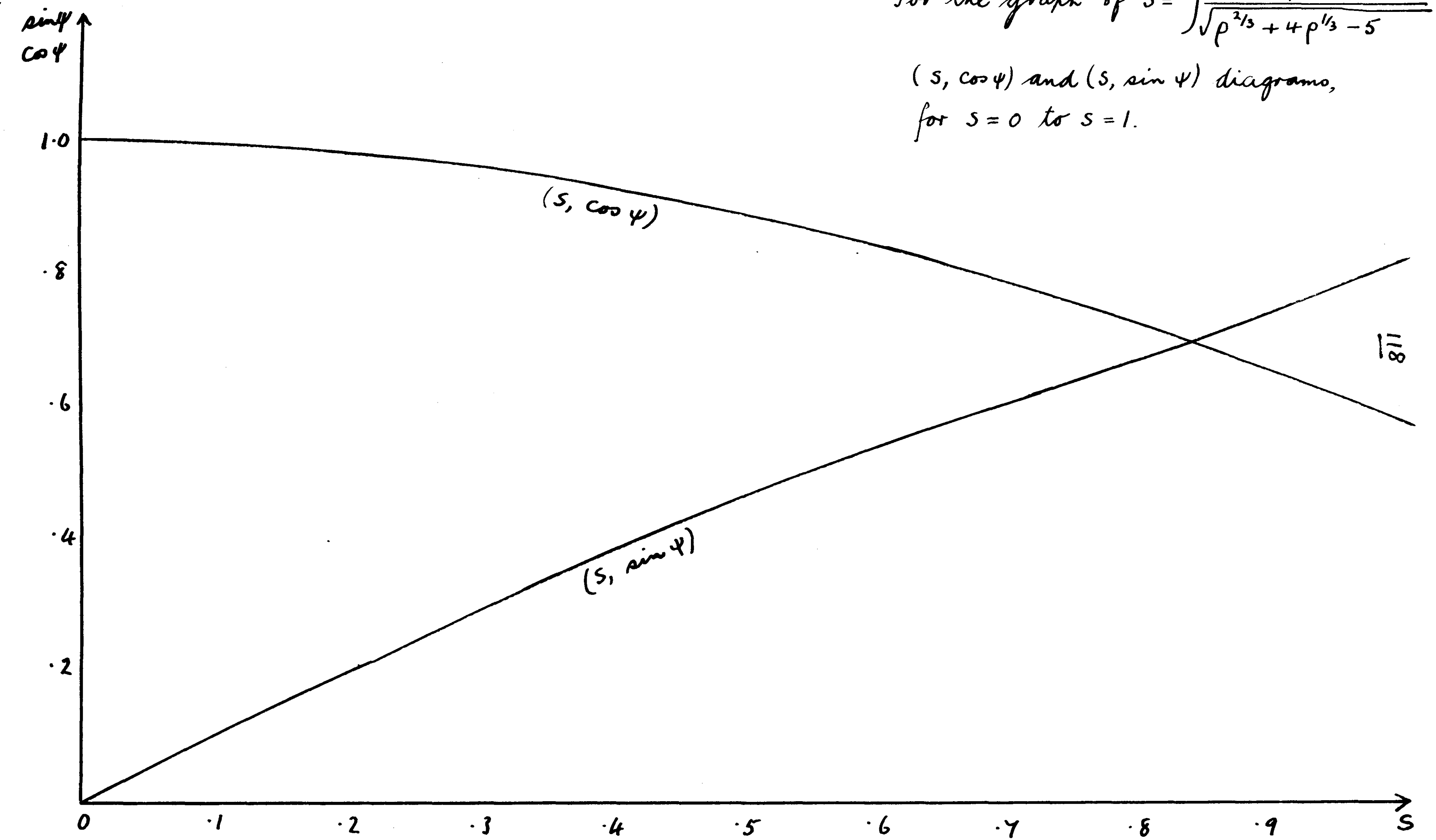
s	1	2	3	4	5	6	7	8	9	10	11	12	13
$40\cos\psi$	22·7	7·9	−3·3	−10·2	−15	−18·2	−20·7	−22·7	−24·3	−25·4	−26·4	−27·2	−27·8
$40\sin\psi$	33·2	39·4	39·9	38·7	37·1	35·8	34·4	33·1	31·8	30·9	30·1	29·1	28·5
x	·85	1·23	1·29	1·12	·81	·39	−·09	−·64	−1·23	−1·85	−2·49	−3·16	−3·85
y	·44	1·34	2·33	3·31	4·26	5·17	6·05	6·89	7·70	8·48	9·25	9·99	10·71

$\varphi = 90°$ to $180°$.

s	0	2	4	6	8	10	12	14	16	18	20
$50\cos\psi$	23·9	24·6	25·1	25·8	26·5	27·2	27·9	28·5	29·0	29·7	30·5
$50\sin\psi$	43·9	43·5	43·0	42·7	42·2	41·8	41·5	41·0	40·5	40·0	39·7
$-x$	0	·97	1·96	2·98	4·03	5·10	6·20	7·33	8·48	9·66	10·86
$-y$	0	1·45	3·48	5·19	6·89	8·57	10·24	11·89	13·52	15·14	16·72

For the graph of $s = \int \frac{d\rho}{\sqrt{\rho^{2/3} + 4\rho^{1/3} - 5}}$

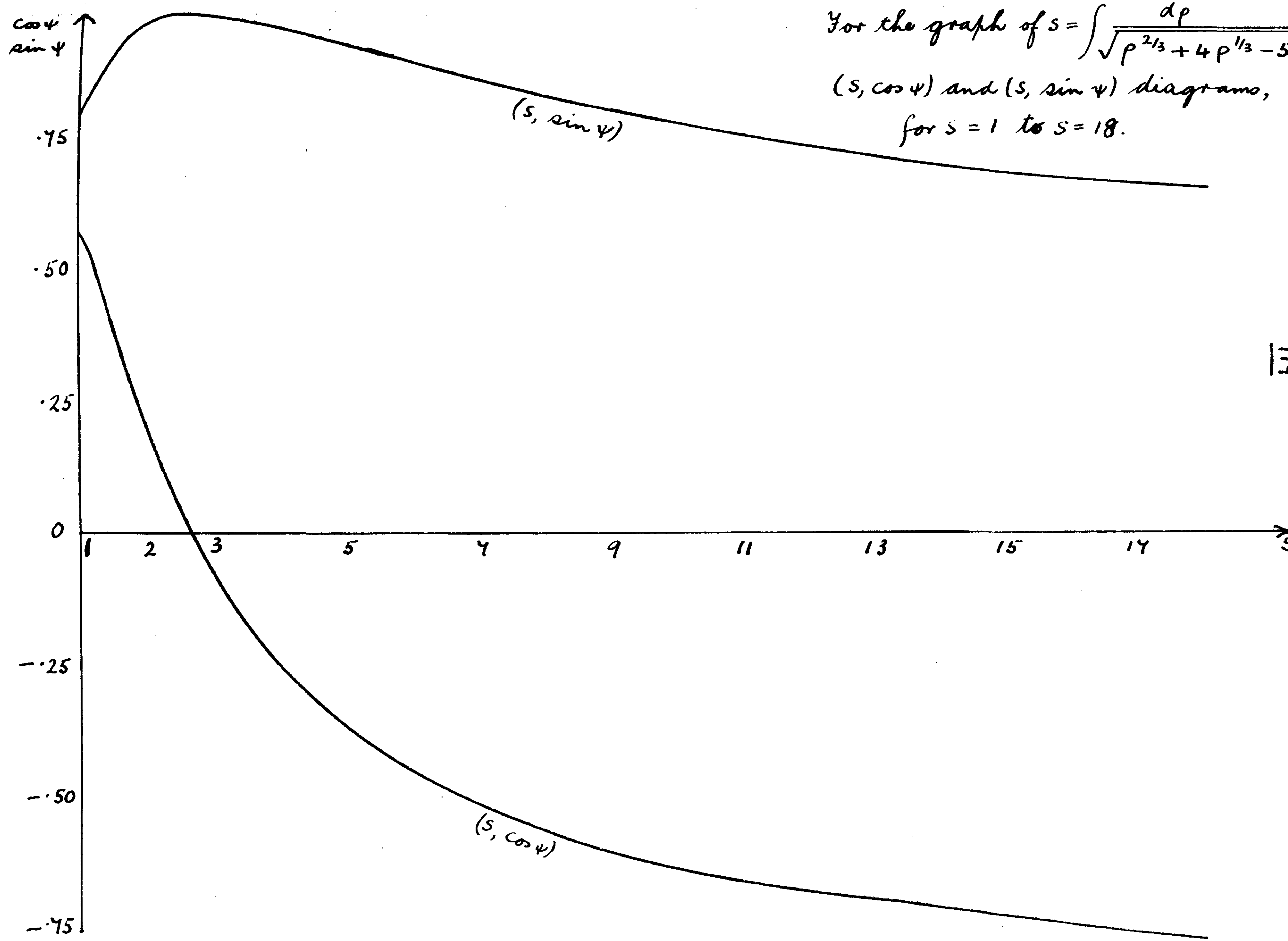
$(s, \cos\psi)$ and $(s, \sin\psi)$ diagrams,
for $s = 0$ to $s = 1$.

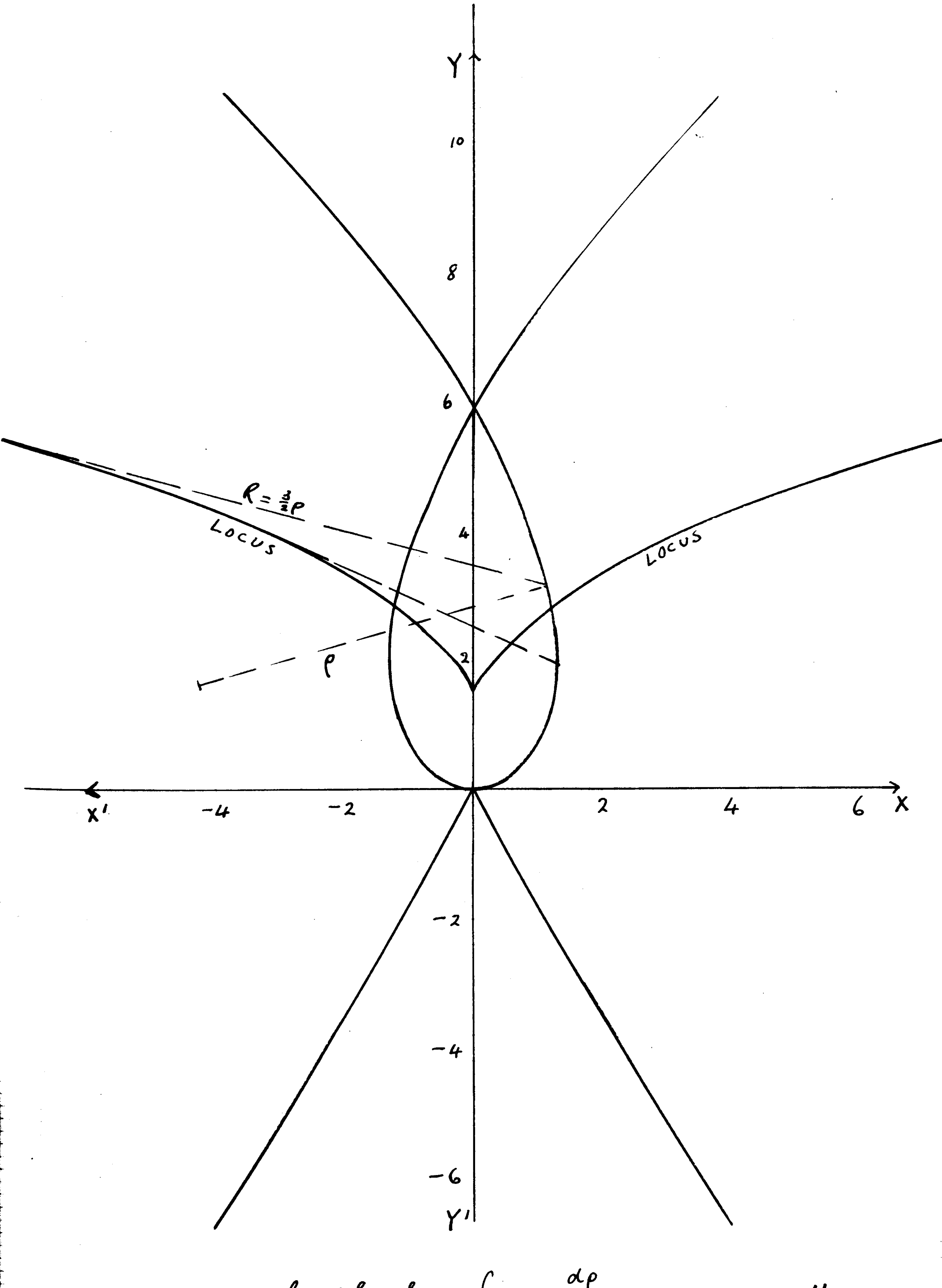


For the graph of $s = \int \frac{d\rho}{\sqrt{\rho^{2/3} + 4\rho^{1/3} - 5}}$

$(s, \cos\psi)$ and $(s, \sin\psi)$ diagrams,

for $s = 1$ to $s = 18$.





Graph of $s = \int \frac{d\rho}{\sqrt{\rho^{2/3} + 4\rho^{1/3} - 5}}$, and the locus of its centres of aberrancy.

(2) Graph of $s = \int \frac{d\rho}{\sqrt{c^2\rho^{2/3} + 2kc\rho^{1/3} + k^2 - 9}}$, when $k = 4\frac{1}{2}$, $c = 3$,

i.e. of the equation $s = \int \frac{\frac{2}{3}\,d\rho}{\sqrt{4\rho^{2/3} + 12\rho^{1/3} + 5}}$.

The substitution $\rho^{1/3} = \sec\varphi - 1{\cdot}5$ gives the solution
$s = \frac{1}{2}\tan\varphi\sec\varphi - 3\tan\varphi + \frac{11}{4}\log\tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)$,
$\psi = -\frac{2}{\sqrt{5}}\log\frac{1+\sqrt{5}\tan\varphi/2}{1-\sqrt{5}\tan\varphi/2}$; the positive sign may be taken.

Notes (1) When $\varphi = 2\tan^{-1}\left(\frac{1}{\sqrt{5}}\right)$, ψ is infinite, but s is finite. Therefore, the curve makes an infinite number of revolutions round this point, before finally reaching it. When φ just exceeds this value, the current point again makes an infinite number of revolutions round the point and finally goes off to infinity.

(2) When $\varphi = 90°$, s is infinite, but ψ is finite, and has the value $\frac{2}{\sqrt{5}}\log\frac{\sqrt{5}+1}{\sqrt{5}-1} = {\cdot}86$ radian. The curve therefore has a pair of symmetrically-placed asymptotes in the directions $\pm{\cdot}86$ radian.

(3) When $\varphi = 0°$, $\psi = 0°$, $s = 0$.

(4) When $\varphi = 180°$, $\psi = 0°$, $s = 0$.

(5) When φ just exceeds $90°$, the current point moves from infinity, from the direction $+{\cdot}86$ radian. The curve therefore has two pairs of parallel asymptotes.

(6) When φ is negative, we get results symmetrical with those in (1), (2) and (5). The y-axis is a line of symmetry for the whole figure.

Details

(1) When $\varphi = 2\tan^{-1}\frac{1}{\sqrt{5}} = 48°12'$, $s = {\cdot}131$, $\psi = \infty$.

(2) When $\varphi = 90°$, $s = \infty$, $\psi = 49°20'$.

The points where ψ becomes infinite are marked A on the large scale graph of the equation.

Since the axis of aberrancy is $\frac{2}{3}$ the length of the corresponding radius of curvature, it is possible to trace approximately by graphical means the locus of the centres of aberrancy. Part of this locus is shown also on the large scale graph.

Data for the graph of $s = \frac{2}{3}\int \frac{dp}{\sqrt{4p^{2/3}+12p^{1/3}+5}}$.

Values of φ, s, ψ, $\cos\psi$, $\sin\psi$ for $\varphi = 0°$ to $\varphi = 85°$.

$\varphi°$	0	5	10	15	20	25	30	35	40
s	0	·023	·044	·063	·082	·098	·111	·121	·126
ψ (rad.)	0	·175	·354	·543	·745	·973	1·24	1·57	2·04
$\cos\psi$	1·00	·98	·95	·86	·73	·56	·33	0	−·45
$\sin\psi$	·00	·17	·35	·52	·68	·83	·94	1	·89
$\varphi°$	45	50	55	60	65	70	75	80	85
s	·130	·132	·135	·157	·246	·55	2·1	6·0	40
ψ (rad.)	2·92	3·46	2·31	2·02	1·51	1·35	1·19	1·06	·95
$\cos\psi$	−·97	−·95	−·67	−·44	·06	·22	·37	·49	·59
$\sin\psi$	·22	−·31	·74	·90	1·00	·98	·93	·87	·81

Values of s, x, y, for $\varphi = 0°$ to $\varphi = 85°$.

$1000x$	0	9·98	19·9	29·7	39·3	48·6	57·6	66·1	74·2	81·6	87·6
$1000y$	0	·3	1·5	3·4	6·1	10·8	15·3	20·5	26·7	33·7	41·8
$1000x$	91·8	93·7	90·2	90·2	81·8	75·8	71·5	68·3	66	64·5	63·5
$1000y$	50·8	60·4	69·1	69·1	71·8	81·9	90	99·6	109	119	129
$1000x$	62·9	62·7	62·8	63·2	63·8	64·6	65·5	66·5	67·7	69	
$1000y$	139	149	159	169	179	189	199	209	219	229	

s	¼	½	¾	1	1¼	1½	1¾	2	2¼	2½	2¾	3
x	·06	·10	·16	·22	·30	·38	·46	·55	·65	·74	·84	·95
y	·16	·41	·65	·90	1·16	1·41	1·65	1·89	2·13	2·37	2·60	2·84
s	3¼	3½	3¾	4	4¼	4½	4¾	5				
x	1·05	1·16	1·27	1·38	1·50	1·61	1·73	1·85				
y	3·07	3·24	3·45	3·68	3·90	4·12	4·35	4·56				
s	5	10	15	20	25	30	35	40				
x	1·85	4·33	7·00	9·75	12·55	15·45	18·30	21·20				
y	4·6	9·2	13·5	17·7	22·0	26·2	29·9	34·0				

For values of φ between $90°$ and $180°$, we may put $\varphi = 180 - \theta$, and solve the differential equations for θ. This leads to the following expressions for ρ, s, and ψ :—

$$\rho = -\sec\theta - 1.5,$$

$$s = \tfrac{1}{2}\tan\theta\sec\theta + 3\tan\theta + \tfrac{11}{4}\log\tan\left(\tfrac{\pi}{4} + \tfrac{\theta}{2}\right),$$

$$\psi = -\frac{2}{\sqrt{5}}\log\frac{1+\sqrt{5}\cot\theta/2}{1-\sqrt{5}\cot\theta/2}$$; the positive sign may be taken

This gives a second part of the graph with a tangent $y = 0$ at the origin, and with arms stretching to infinity in the directions $\psi = \cdot 86$ and $\psi = -\cdot 86$ rad.

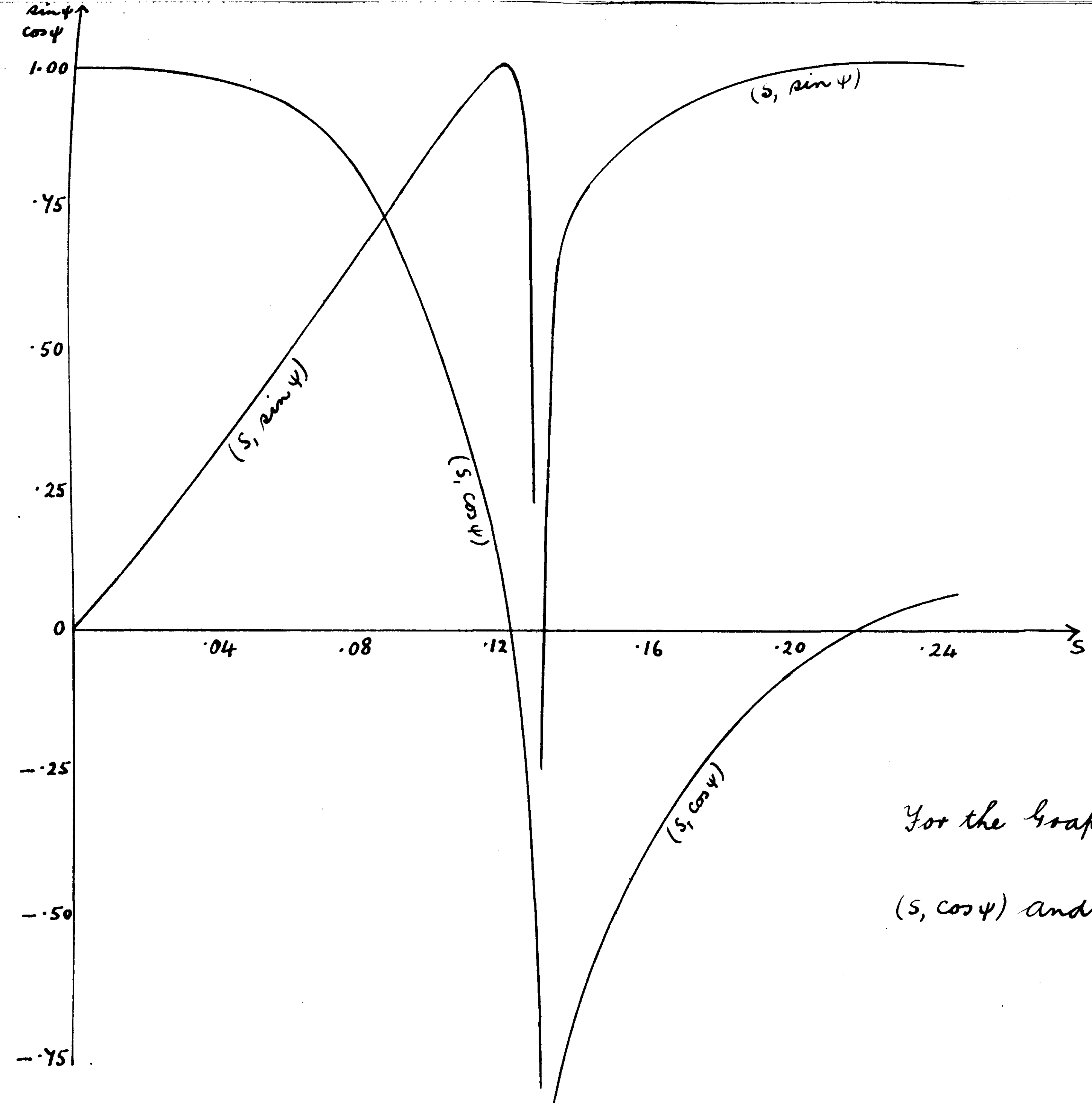
Values of s, ψ, $\cos\psi$, $\sin\psi$ for $\varphi = 100°$ to $\varphi = 180°$.

$\varphi°$	180	175	170	165	160	155	150	145
s	0	·55	1·10	1·67	2·27	2·90	3·58	4·32
ψ (rad.)	0	·035	·071	·106	·141	·178	·216	·254
$50\cos\psi$	50	50	49·9	49·7	49·5	49·2	48·8	48·4
$50\sin\psi$	0	1·8	3·5	5·3	7·0	8·9	10·7	12·6

$\varphi°$	140	135	130	125	120	115	110	105	100
s	5·16	6·13	7·28	8·70	10·55	13·11	17·00	24·5	40
ψ (rad.)	·294	·335	·378	·426	·472	·524	·579	·640	·705
$50\cos\psi$	47·8	47·2	46·5	45·6	44·6	43·3	41·8	40·1	38·1
$50\sin\psi$	14·5	16·4	18·5	20·6	22·7	25·0	27·4	29·8	32·4

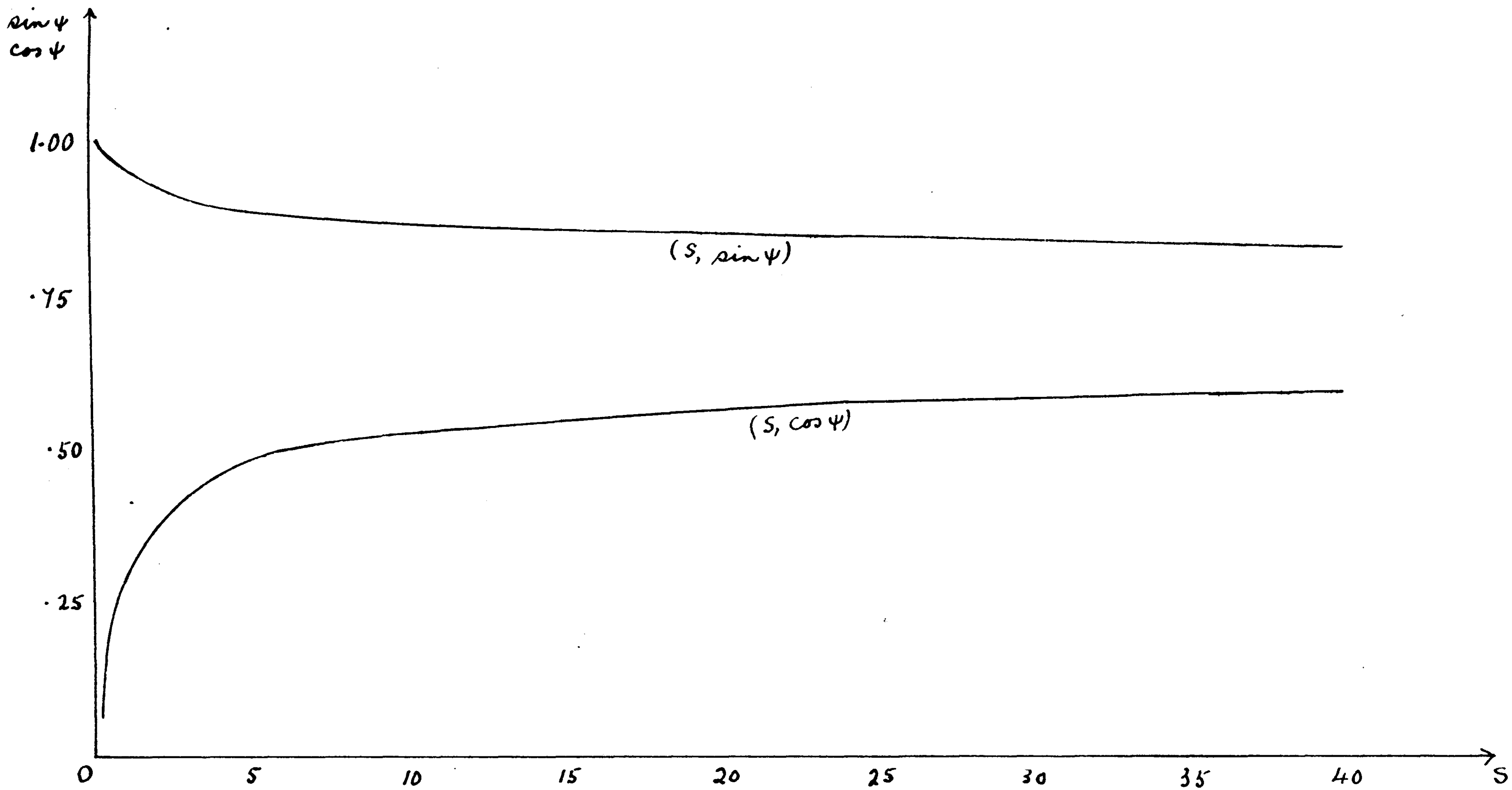
Values of s, x, y for $\varphi = 100°$ to $\varphi = 180°$.

s	0	4	8	12	16	20	24	28	32	36	40
x	0	3·94	7·73	11·33	14·78	18·10	21·36	24·55	27·70	30·80	33·86
y	0	·47	1·72	3·54	5·50	7·72	10·06	12·48	14·96	17·48	20·06



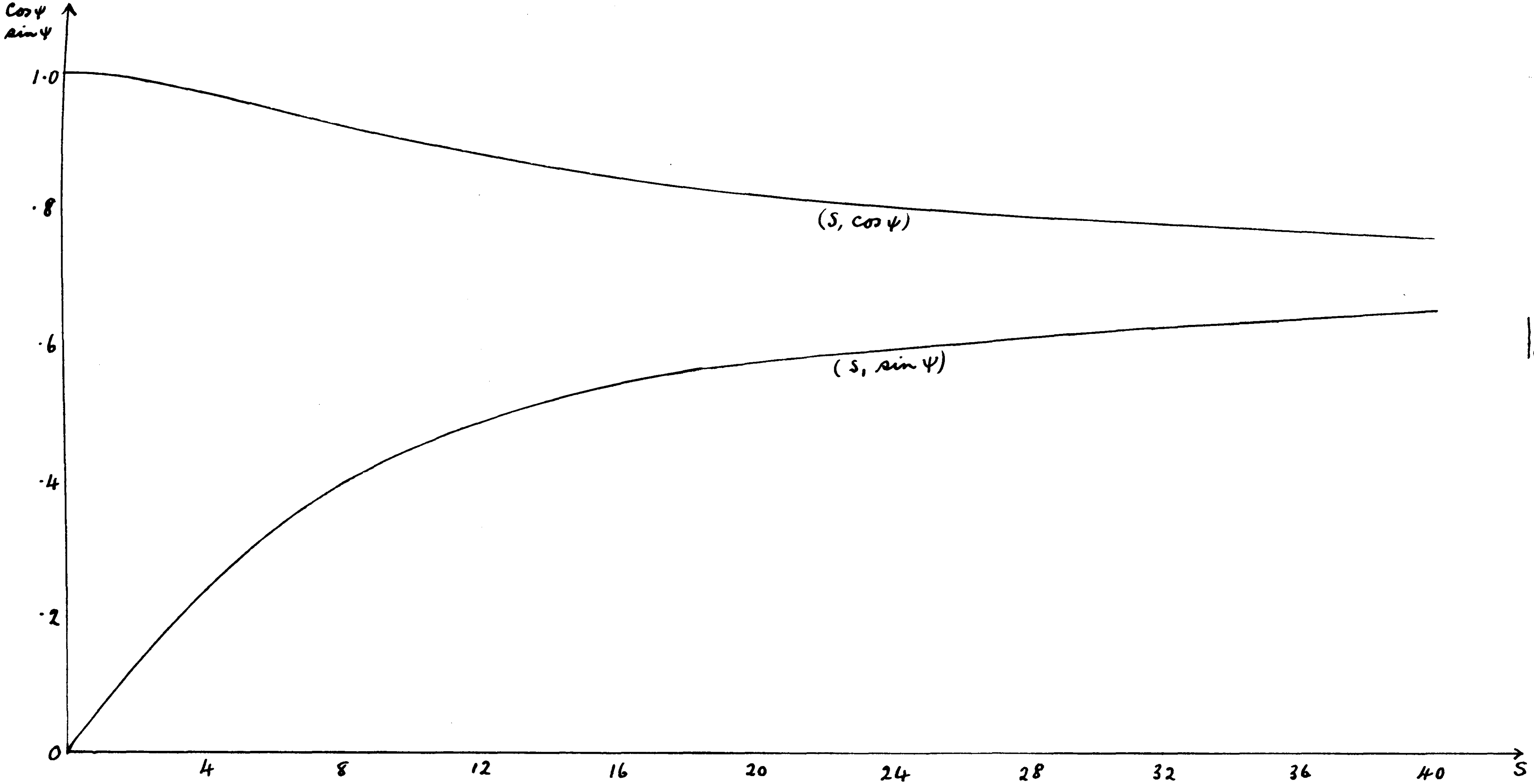
For the Graph of $S = \frac{2}{3}\int \frac{d\rho}{\sqrt{4\rho^{2/3} + 12\rho^{1/3} + 5}}$;

$(S, \cos\psi)$ and $(S, \sin\psi)$ diagrams for $S = 0$ to $\cdot 25$, $(\varphi < 90^\circ)$.



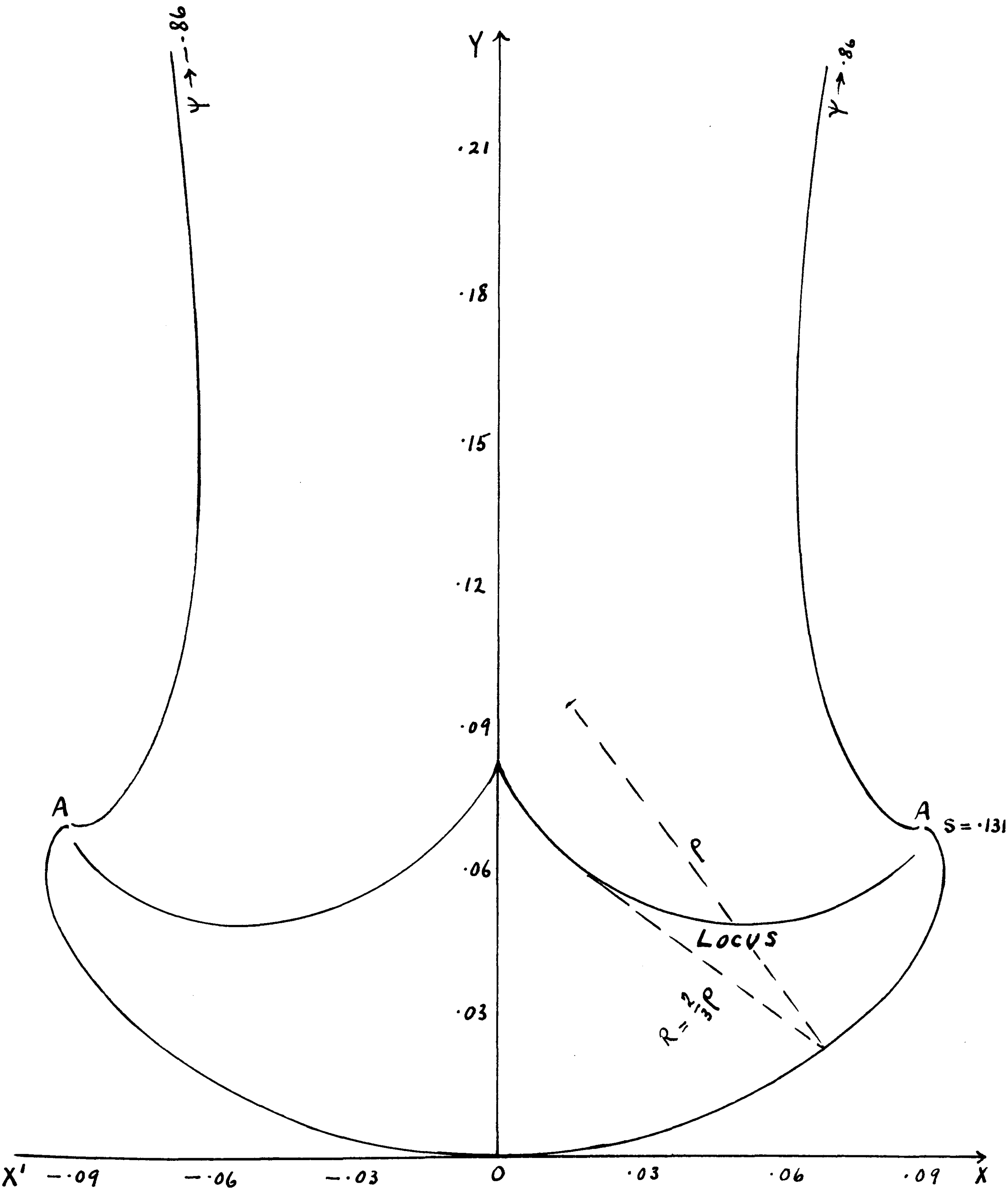
For the Graph of $S = \frac{2}{3}\int \frac{d\rho}{\sqrt{4\rho^{2/3} + 12\rho^{1/3} + 5}}$;

$(S, \cos\psi)$ and $(S, \sin\psi)$ diagrams, for $S = \cdot 25$ to $S = 40$ ($\varphi < 90°$)

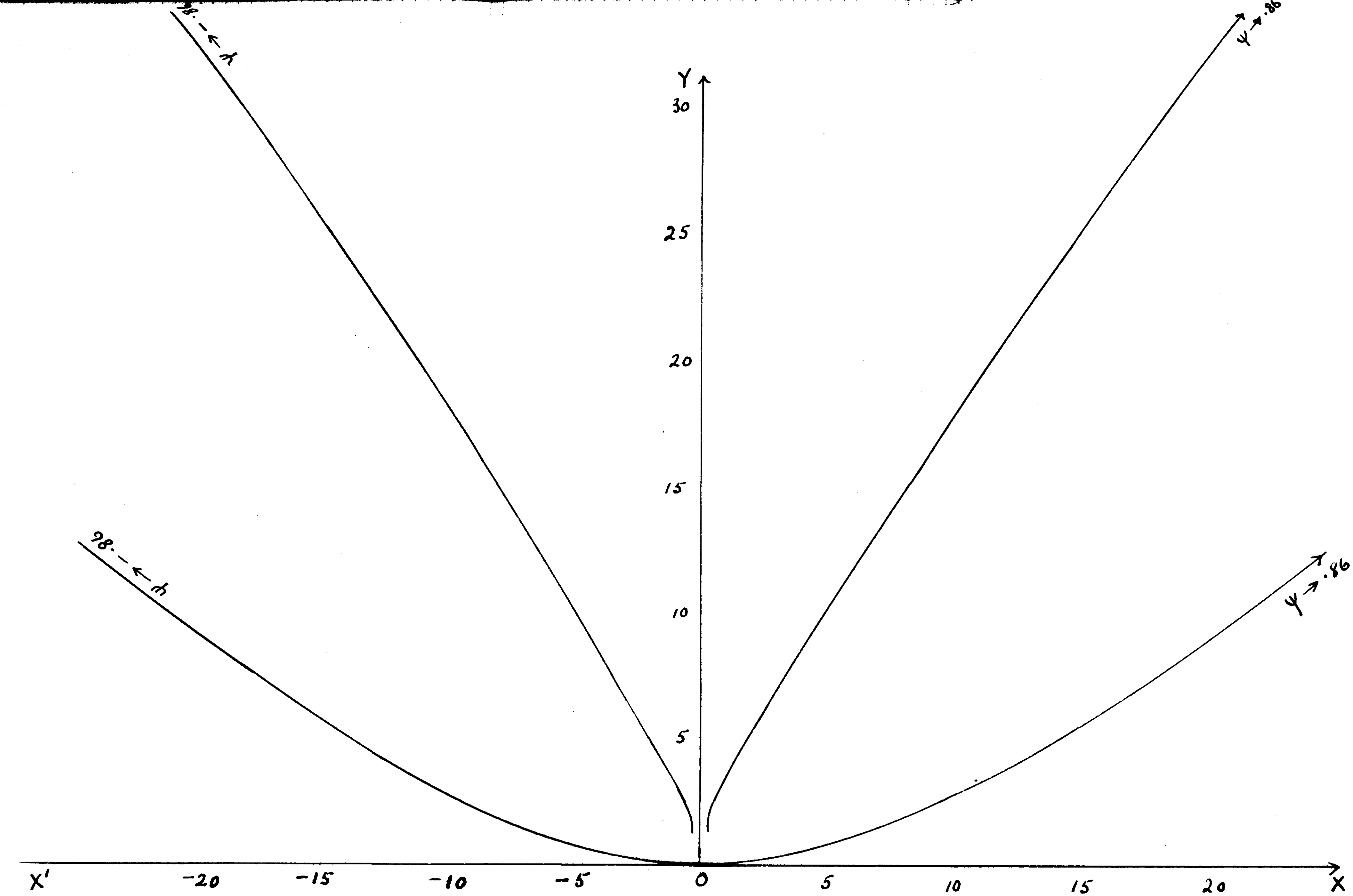


For the Graph of $S = \frac{2}{3}\int \frac{d\rho}{\sqrt{4\rho^{2/3} + 12\rho^{1/3} + 5}}$;

$(S, \cos\psi)$ and $(S, \sin\psi)$ diagrams, for $S = 0$ to 40, $(90^\circ \leqq \varphi < 180^\circ)$.



Graph of $s = \frac{2}{3}\int \frac{d\rho}{\sqrt{4\rho^{2/3} + 12\rho^{1/3} + 5}}$; for $s = 0$ to $s = \cdot 25$; $(\varphi < 90^\circ)$
with part of the locus of the centres of aberrancy.



Graph of $s = \frac{2}{3}\int \frac{d\rho}{\sqrt{4\rho^{2/3} + 12\rho^{1/3} + 5}}$; for $s = \cdot 25$ to 40, and for $\rho \geqq \frac{125}{8}$.

(3) Graph of the equation $s = \int \frac{d\rho}{\sqrt{c^2\rho^{2/3} + 2Kc\rho^{\frac{1}{3}} + K^2 - 9}}$, when $c = k = 3$, i.e. of the equation $3s = \int \frac{d\rho}{\sqrt{\rho^{2/3} + 2\rho^{1/3}}}$.

The solution is
$$s = \tfrac{1}{2}\tan\varphi \sec\varphi + \tfrac{3}{2}\log\tan\left(\tfrac{\pi}{4} + \tfrac{\varphi}{2}\right) - 2\tan\varphi,$$
$$\psi = -\cot\tfrac{\varphi}{2},$$
$$\rho^{\frac{1}{3}} = \sec\varphi - 1.$$

Notes. (1) When $\varphi = 0$, ψ is negative infinite, and s is zero. The curve therefore makes an infinite number of revolutions round the point $\varphi = 0$, and finally goes off to infinity.

(2) When $\varphi = 90°$, s is infinite, but ψ is finite, equal to -1 radian. The curve therefore has an asymptote in this direction.

(3) When $\varphi = 180°$, $s = 0$, $\psi = 0°$.

(4) When φ increases from $90°$, the current point moves from infinity from the direction $\psi = -1$. The curve therefore has a pair of parallel asymptotes.

(5) When φ is negative, we get results symmetrical with those in (1), (2) and (4). The curve therefore has a second pair of parallel asymptotes, and the x-axis is a line of symmetry for the whole figure.

Data for the graph of $3s = \int \frac{d\rho}{\sqrt{\rho^{2/3} + 2\rho^{1/3}}}$.

$\varphi°$	65	70	75	80	81	82	83	84	85
s	·51	1·13	2·79	8·64	11·37	15·32	21·33	30·92	47·44
$-\psi$ (rad.)	1·57	1·43	1·30	1·19	1·17	1·15	1·13	1·11	1·09
$\cos\psi$	·00	·14	·26	·37	·39	·41	·43	·44	·46
$-\sin\psi$	1	·99	·96	·93	·92	·91	·90	·90	·89

x, y values, for $\varphi = 0^\circ$ to $\varphi = 85^\circ$.

x	0	.94	2.71	4.68	6.75	8.89	11.08	13.34	15.61	17.9[illegible]
$-y$	0	4.72	9.54	14.13	18.68	23.2	27.7	32.2	36.6	41.[illegible]

For values of φ between 90° and 180° we may substitute $\varphi = 180^\circ - \theta$, and solve the differential equations in terms of θ. This gives for ρ, s, and ψ the following forms:-

$$\rho^{\frac{1}{3}} = -\sec\theta - 1,$$

$$s = \tfrac{1}{2}\tan\theta\sec\theta + 2\tan\theta + \tfrac{3}{2}\log\tan\left(\tfrac{\pi}{4} + \tfrac{\theta}{2}\right).$$

$$\psi = -\tan\tfrac{\theta}{2}.$$

This gives a second part of the graph, with two branches meeting at the origin, and having the x-axis as a common tangent.

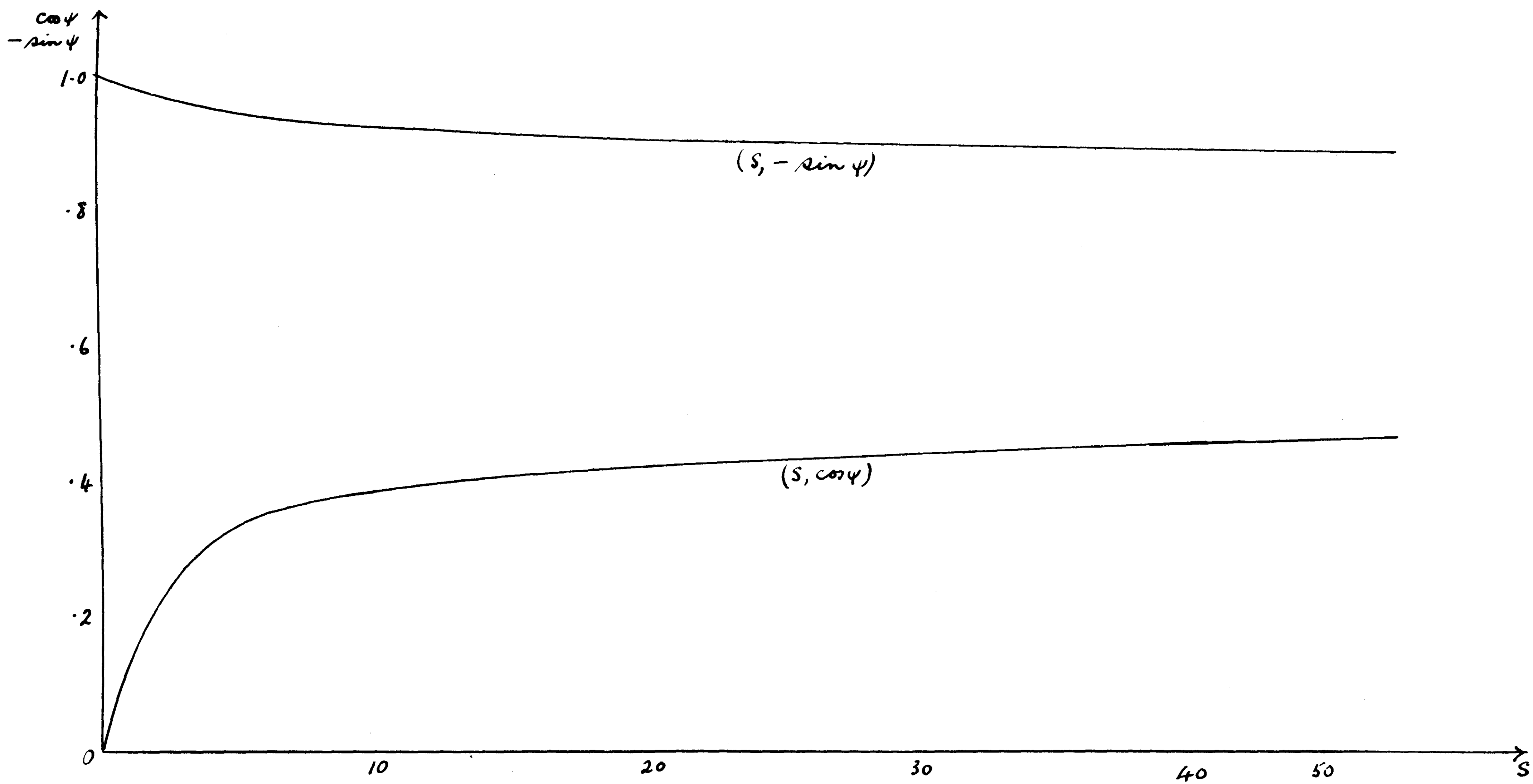
Values of $\varphi, s, \psi, \cos\psi, \sin\psi$ for $\varphi = 98^\circ$ to 180°.

φ°	180	175	170	165	160	155	150	145	140	135	13[illegible]
s	0	.35	.71	1.07	1.46	1.87	2.31	2.81	3.37	4.03	4.8[illegible]
$-\psi$ (rad.)	0	.043	.088	.132	.176	.222	.268	.315	.364	.414	.46[illegible]
$\cos\psi$	1	1.00	1.00	.99	.98	.98	.96	.95	.93	.92	.89
$-\sin\psi$	0	.044	.088	.131	.175	.22	.26	.31	.36	.40	.45

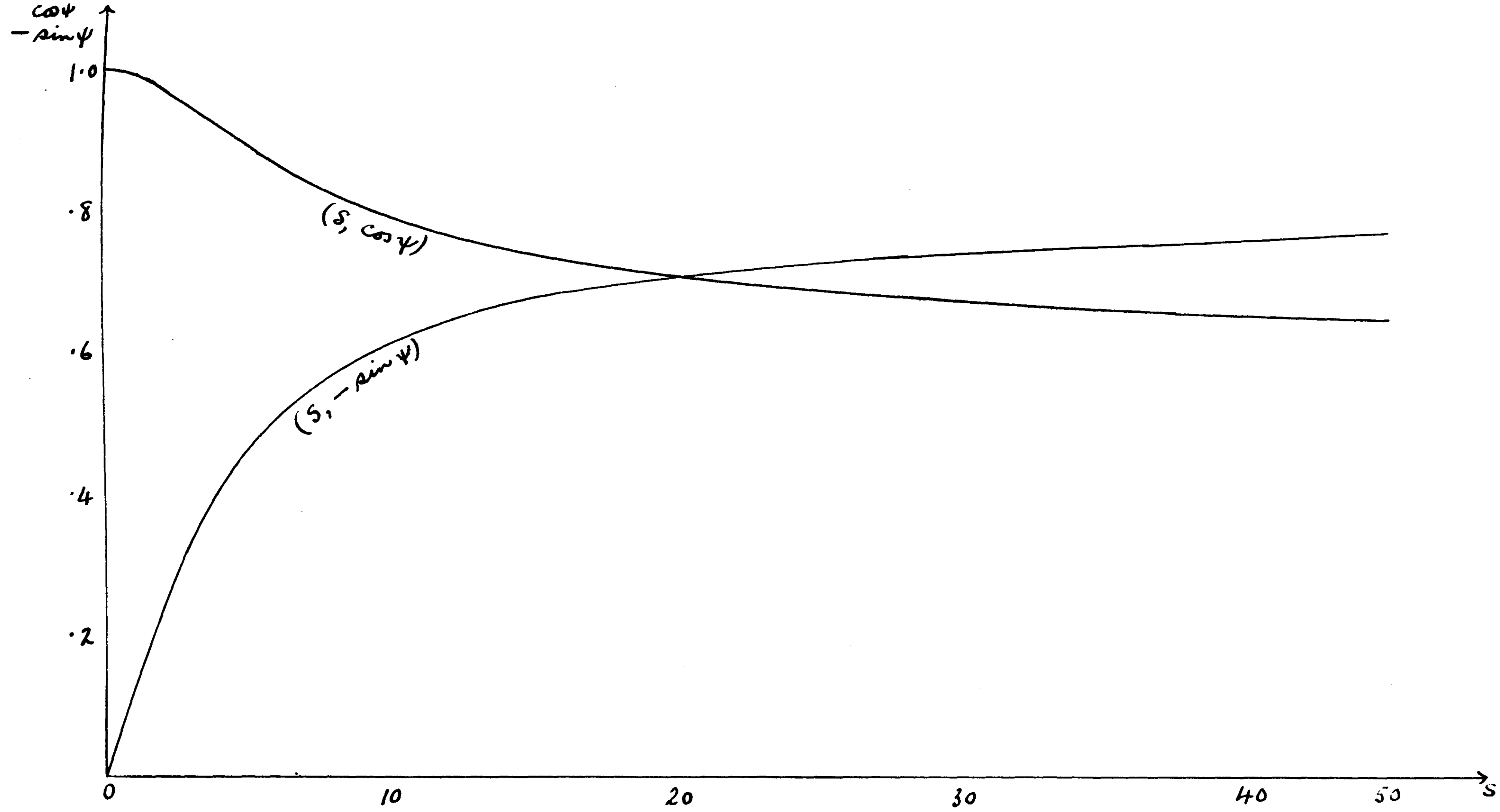
φ°	125	120	115	110	105	100	99	98
s	5.83	7.17	9.09	12.12	17.7	31.3	36.6	43.8
$-\psi$ (rad.)	.521	.577	.637	.700	.767	.839	.854	.869
$\cos\psi$	.87	.84	.80	.76	.72	.67	.66	.65
$-\sin\psi$	.50	.55	.59	.64	.69	.74	.75	.76

Values of x, y for $\varphi = 98^\circ$ to 180°.

x	0	4.73	8.93	12.75	16.36	19.85	23.26	26.60	29.89	33.13
$-y$	0	1.25	4.04	7.22	10.67	14.25	18.10	21.65	25.43	29.25

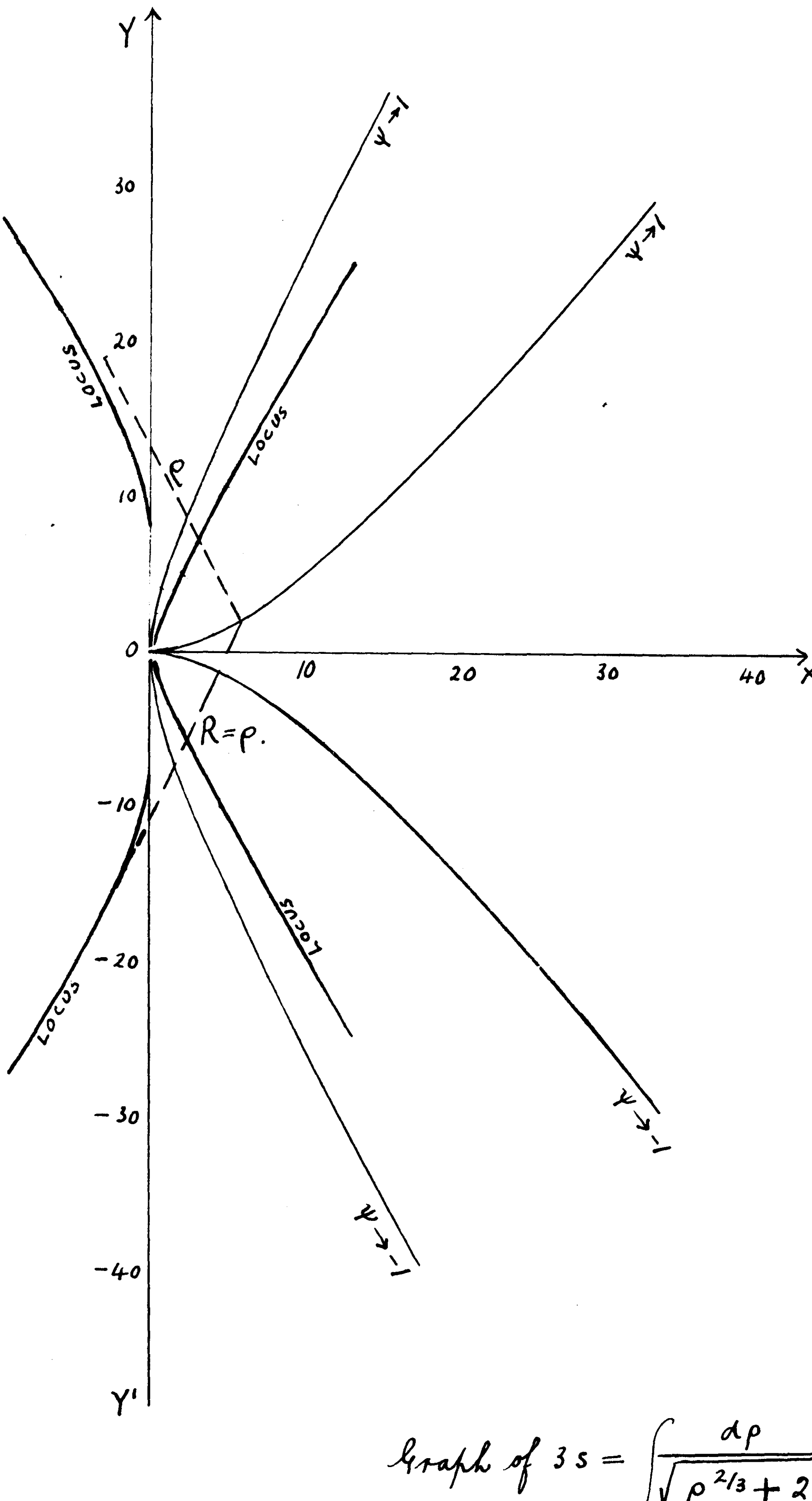


For the Graph of $3\sigma = \int \frac{d\rho}{\sqrt{\rho^{2/3} + 2\rho^{1/3}}}$;
$(s, \cos\psi)$ and $(s, -\sin\psi)$ diagrams, for $\varphi = 0^\circ$ to 85°.



For the Graph of $3s = \int \frac{d\rho}{\sqrt{\rho^{2/3} + 2\rho^{1/3}}}$;

$(s, \cos\psi)$ and $(s, -\sin\psi)$ diagrams, for $\varphi = 98°$ to $180°$.



Graph of $3s = \int \frac{d\rho}{\sqrt{\rho^{2/3} + 2\rho^{1/3}}}$, and the locus of its centres of aberrancy.

Section XI.

To find the curves for which $9+\rho'^2-3\rho\rho''=$ a constant.

(1) Writing the constant in the form $9+K$, the equation to be solved is $\rho'^2-K=3\rho\rho''$. Putting $\rho'^2-K=u$, we find $2\rho''=\frac{du}{d\rho}$, and the solution is $\rho'^2=c\rho^{2/3}+K$.

(2) $c=0$ gives $\rho'^2=K$, an equilateral spiral.

(3) $K=0$ gives $\rho'=\sqrt{c}\,\rho^{1/3}$, which leads to $\rho^{2/3}=\frac{2\sqrt{c}}{3}s$, and again to $s=\frac{4}{\left(\frac{2\sqrt{c}}{3}\right)^3}\cdot\frac{1}{\psi^2}\equiv\frac{a}{\psi^2}$, or $\rho=-\frac{2a}{\psi^3}$.

(4) $K=0$, $c=0$, gives $\rho'=0$, a circle.

(5) K positive, c positive gives $s=\int\frac{d\rho}{\sqrt{K+c\rho^{2/3}}}$ or $\psi=\int\frac{d\rho}{\rho\sqrt{K+c\rho^{2/3}}}$ the solution of which is $\rho^{1/3}=-\sqrt{\frac{K}{c}}\operatorname{cosech}\frac{\sqrt{K}}{3}\psi$.

(6) K positive, c negative gives $s=\int\frac{d\rho}{\sqrt{-c}\sqrt{\frac{K}{-c}-\rho^{2/3}}}$,

or $\psi=\int\frac{d\rho}{\sqrt{-c}.\rho\sqrt{\frac{K}{-c}-\rho^{2/3}}}$, the solution of which is

$\rho^{1/3}=\sqrt{\frac{K}{-c}}\operatorname{sech}\frac{\sqrt{K}}{3}\psi$.

(7) K negative gives $s=\int\frac{d\rho}{\sqrt{-K}\sqrt{\frac{c}{-K}\rho^{2/3}-1}}$ or $\psi=\int\frac{d\rho}{\sqrt{-K}.\rho\sqrt{\frac{c}{-K}\rho^{2/3}-1}}$, the solution of which is $\rho^{1/3}=\sqrt{\frac{-K}{c}}\sec\frac{\sqrt{-K}}{3}\psi$.

(8) The particular value $K=-9$ gives $\rho^{1/3}=\frac{3}{\sqrt{c}}\sec\psi$, or $\rho=a\sec^3\psi$, the equation of a parabola.

(9) Geometrical Interpretation of the Equation $9+\rho'^2-3\rho\rho''=$ const.

The axis of aberrancy R is of length

$$\frac{3\rho\sqrt{9+\rho'^2}}{9+\rho'^2-3\rho\rho''} = \frac{9\rho\operatorname{cosec}\alpha}{9+\rho'^2-3\rho\rho''},$$ where α is the angle which the axis of aberrancy makes with the corresponding tangent.

Hence if $9+\rho'^2-3\rho\rho''$ is const., we have

$$R\sin\alpha = \lambda\rho, \text{ where } \lambda \text{ is a constant.}$$

But $R\sin\alpha$ is the projection of R on the normal. Hence $9+\rho'^2-3\rho\rho''=$ const. is the condition that the projection of the axis of aberrancy on the normal should be proportional to the radius of curvature.

Section XII.

Examples of curves for which $9+\rho'^2-3\rho\rho''=$ constant.

(1) **Graph of the equation $s = \frac{a}{\psi^2}$ when $a=1$.**

When $\psi=0$, $s=\infty$, and when $\psi=\infty$, $s=0$. The current point therefore comes from infinity in the direction $\psi=0$, and makes an infinite number of revolutions round the point $s=0$.

Data for the graph of $s=\frac{1}{\psi^2}$.

s	11·11	10·40	9·78	9·19	8·16	7·30	6·25	4·93	4·00
ψ	·30	·31	·32	·33	·35	·37	·40	·45	·50
$25\cos\psi$	23·9	23·8	23·7	23·7	23·5	23·4	23·0	22·5	22·0
$25\sin\psi$	7·4	7·6	7·9	8·1	8·6	9·0	9·7	10·9	12·0

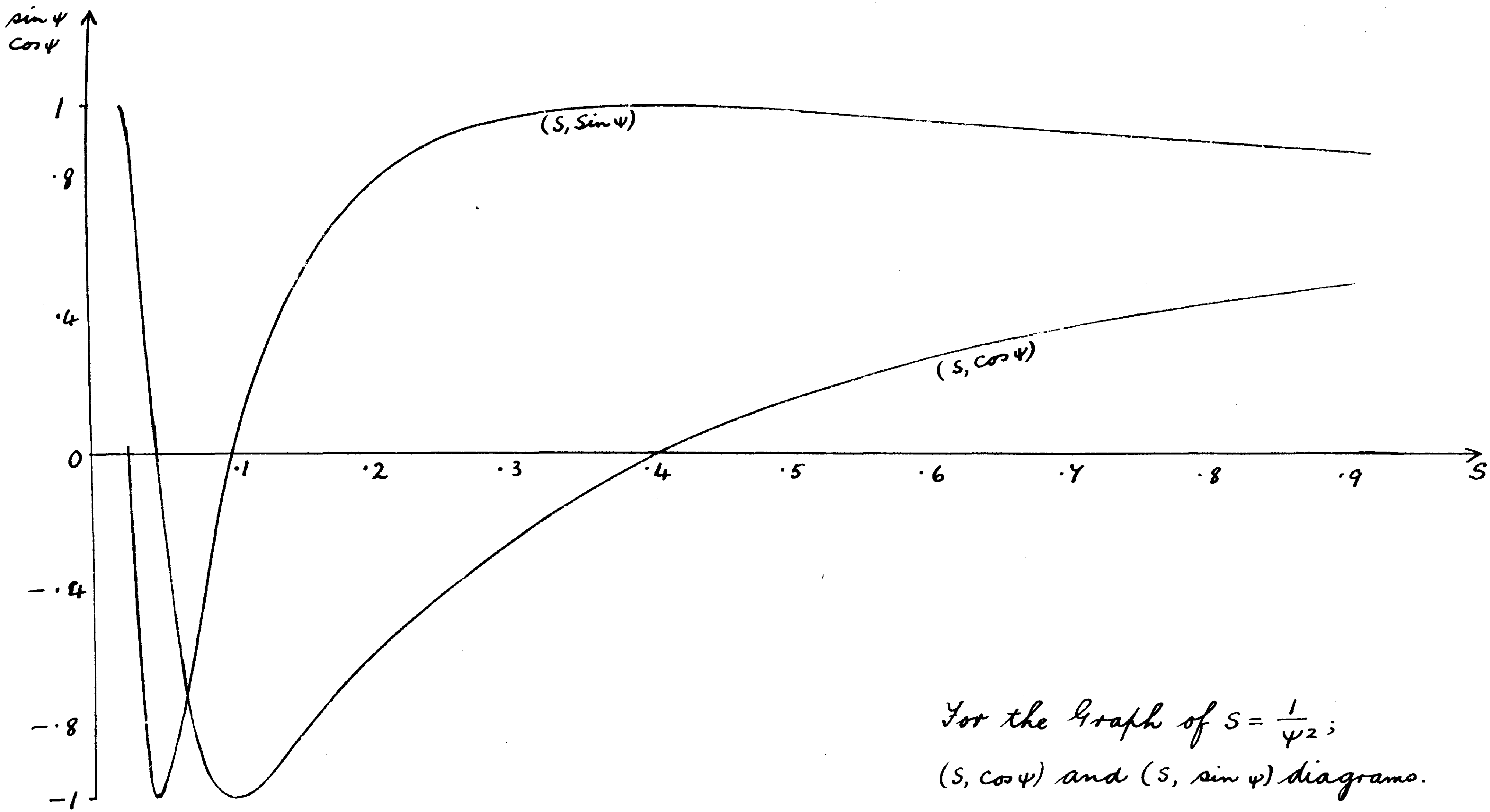
s	2·78	2·04	1·56	1·24	1·00	·69	·51	·39	·31	·25	·16
ψ	·6	·7	·8	·9	1·0	1·2	1·4	1·6	1·8	2·0	2·5
$25\cos\psi$	20·6	19·1	17·4	15·5	13·5	9·0	4·3	−·7	−5·7	−10·4	−20·0
$25\sin\psi$	14·1	16·1	17·9	19·6	21·0	23·3	24·6	25	24·3	22·7	14·9

s	·111	·083	·0625	·0495	·0400	·0331	·0278	·0253
ψ	3·0	3·5	4·0	4·5	5·0	5·5	6·0	6·28
$25\cos\psi$	−24·8	−23·4	−16·3	−5·2	7·1	17·8	24·0	25
$25\sin\psi$	3·5	−8·8	−19·0	−24·4	−24·0	−17·6	−6·9	0

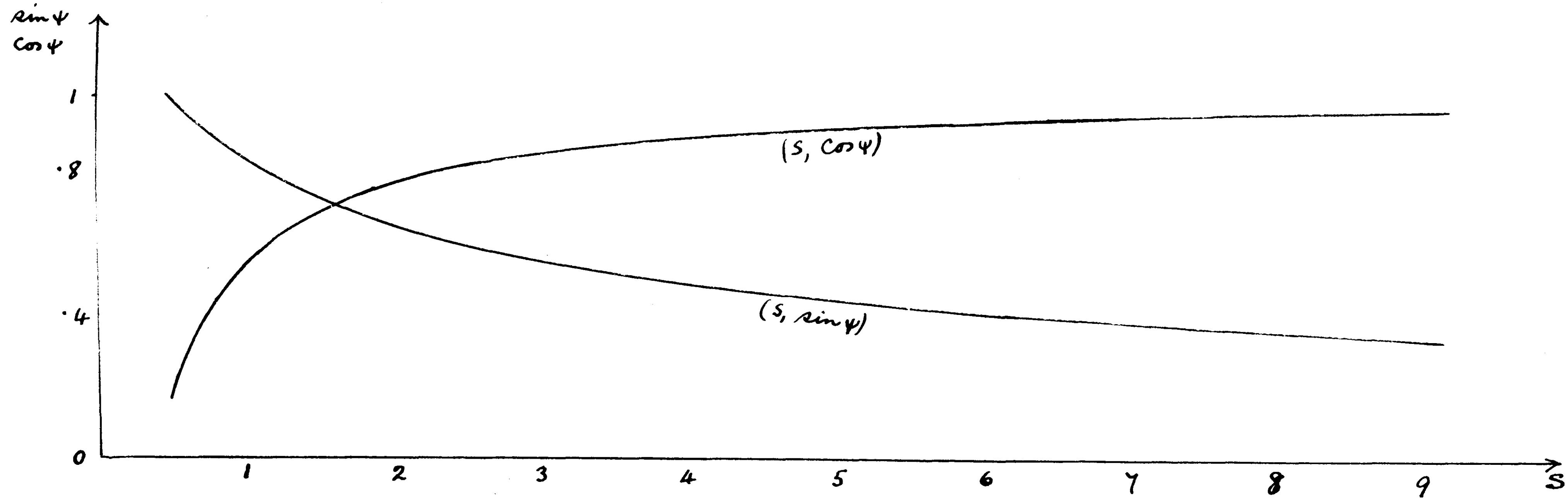
s	·025	·03	·04	·05	·06	·07	·08	·09	·10	·11	·12	·13
x	0	·0044	·0103	·0107	·0067	−·0001	−·0087	−·0181	−·0280	−·0379	−·0476	−·0519
y	0	−·0009	−·0078	−·0179	−·0267	−·0338	−·0390	−·0420	−·0431	−·0426	−·0406	−·0379

s	·14	·15	·20	·25	·30	·35	·40	·50	·60	·70	·80
x	−·0656	−·0738	−·1086	−·1336	−·1504	−·1598	−·1628	−·1548	−·1328	−·1008	−·0604
y	−·0340	−·0293	·0037	·0463	·0933	·1425	·1923	·2915	·3887	·4835	·5751

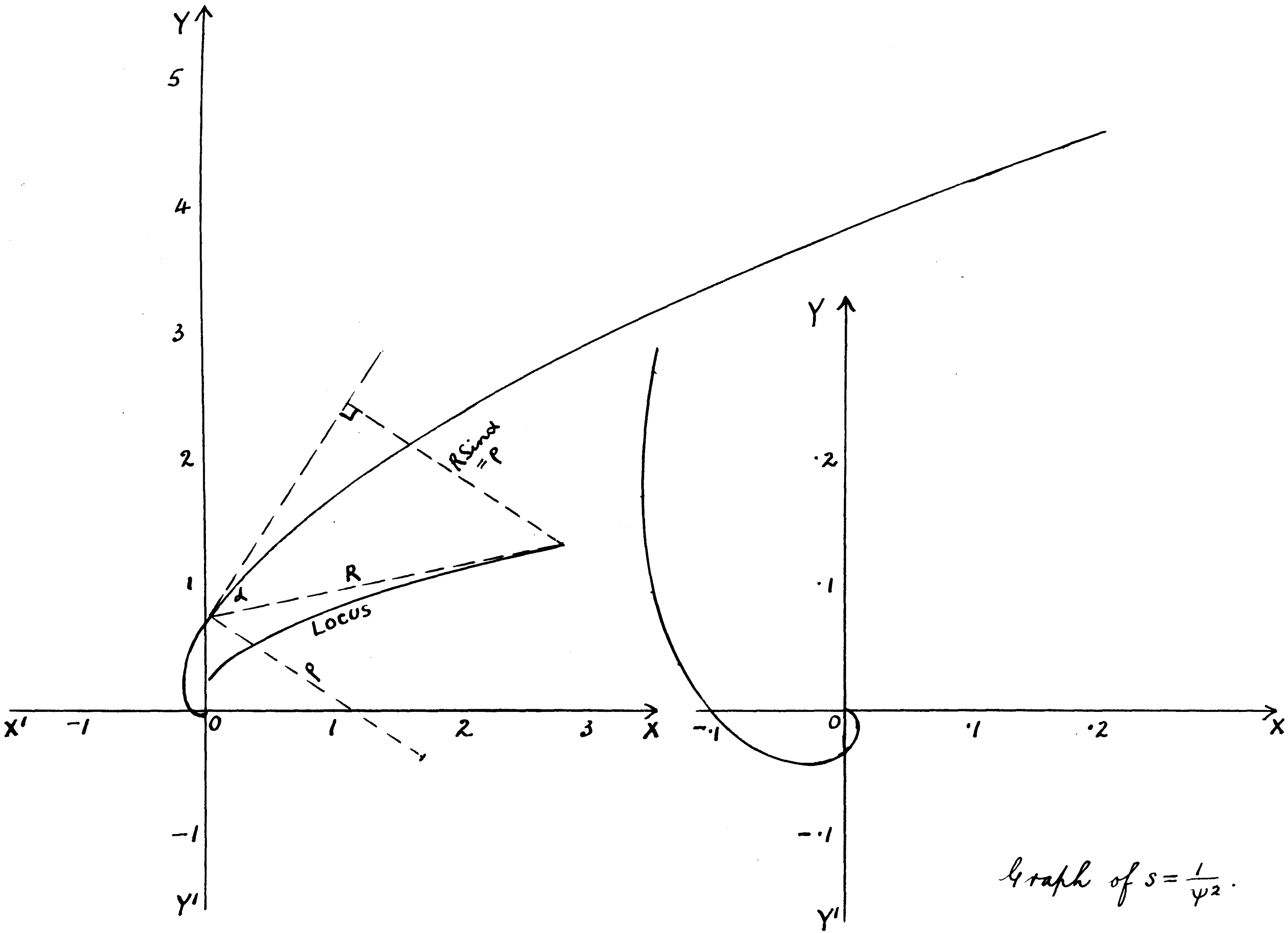
s	·90	1	2	3	4	5	6	7	8	9
x	−·0140	·0372	·717	1·52	2·38	3·27	4·18	5·10	6·04	6·98
y	·6631	·7483	1·484	2·08	2·59	3·05	3·46	3·85	4·20	4·54



For the Graph of $S = \frac{1}{\psi^2}$;
$(S, \cos\psi)$ and $(S, \sin\psi)$ diagrams.



For the graph of $s = \frac{1}{\psi^2}$;
$(s, \cos\psi)$ and $(s, \sin\psi)$ curves.



Graph of $s = \frac{1}{\psi^2}$.

(2) Graph of the equation $\rho^{-\frac{1}{3}} = \sqrt{\frac{c}{k}} \sinh \frac{\sqrt{k}}{3}\psi$, when $\sqrt{\frac{c}{k}} = 2$, and $\frac{\sqrt{k}}{3} = \frac{1}{4}$.

The equation is $\rho^{-\frac{1}{3}} = 2 \sinh \frac{\psi}{4}$.

When $\psi = 0$, $\rho = \infty$, and when $\psi = \infty$, $\rho = 0$. The curve therefore has an asymptote in the direction $\psi = 0$. The current point moves from infinity in the direction $\psi = 0$, and makes an infinite number of revolutions round the point $\rho = 0$, finally arriving at that point.

Data for the graph of $\rho^{-\frac{1}{3}} = 2 \sinh \frac{\psi}{4}$.

ψ, ρ, values.

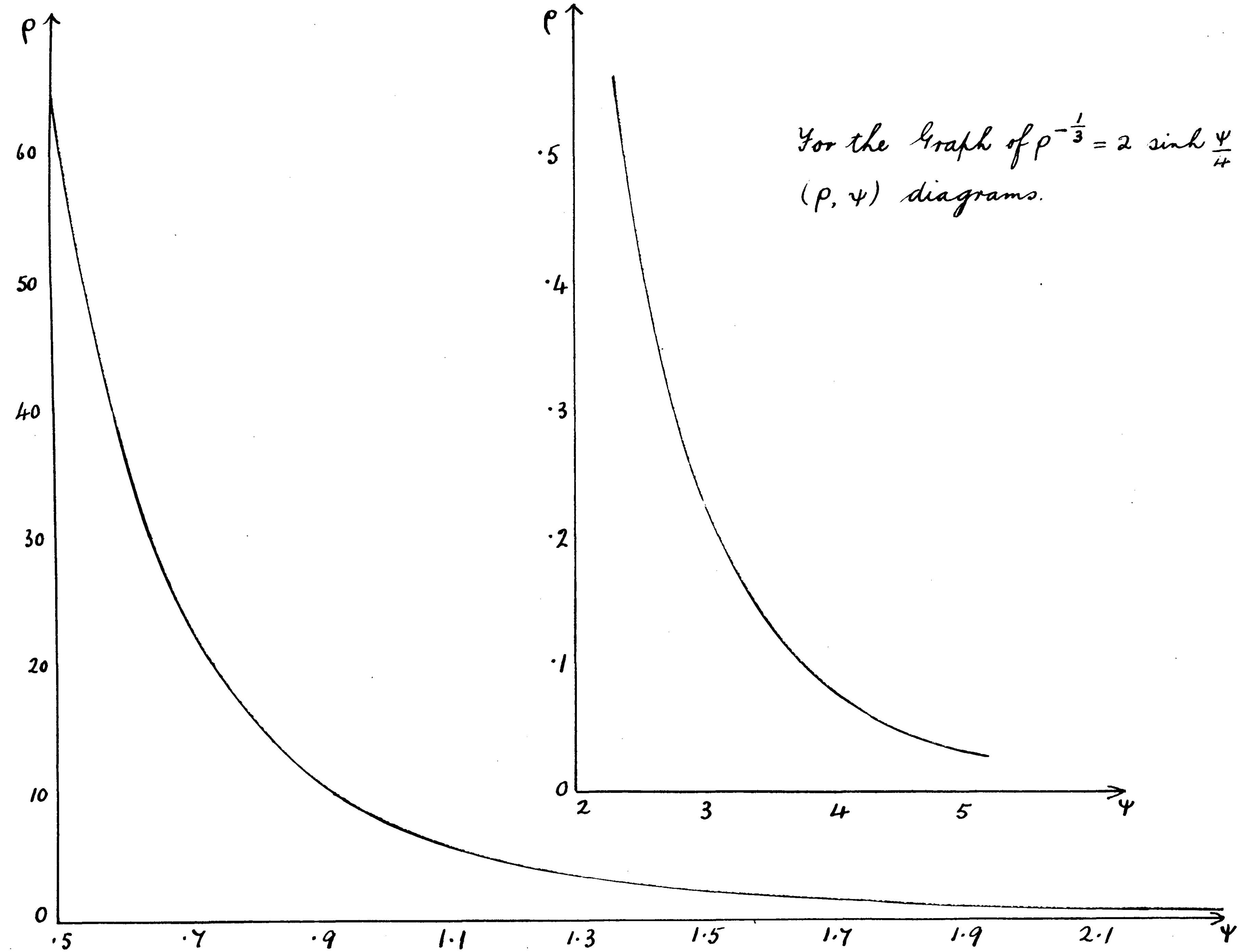
ψ	·4	·5	·52	·56	·60	·64	·68	·72	·76	·80	·90	1·0	1·1		
ρ	125	64	55·2	45·5	36·7	30·4	25·2	21·1	18·2	15·7	10·8	7·79	5·80		
ψ	1·2	1·4	1·6	1·8	2·0	2·3	2·4	2·6	2·8	3·2	3·6	4·0	4·4	4·8	5·2
ρ	4·65	2·75	1·80	1·25	·882	·560	·485	·370	·287	·178	·116	·077	·052	·036	·026

$s, \psi, \cos\psi, \sin\psi$ values.

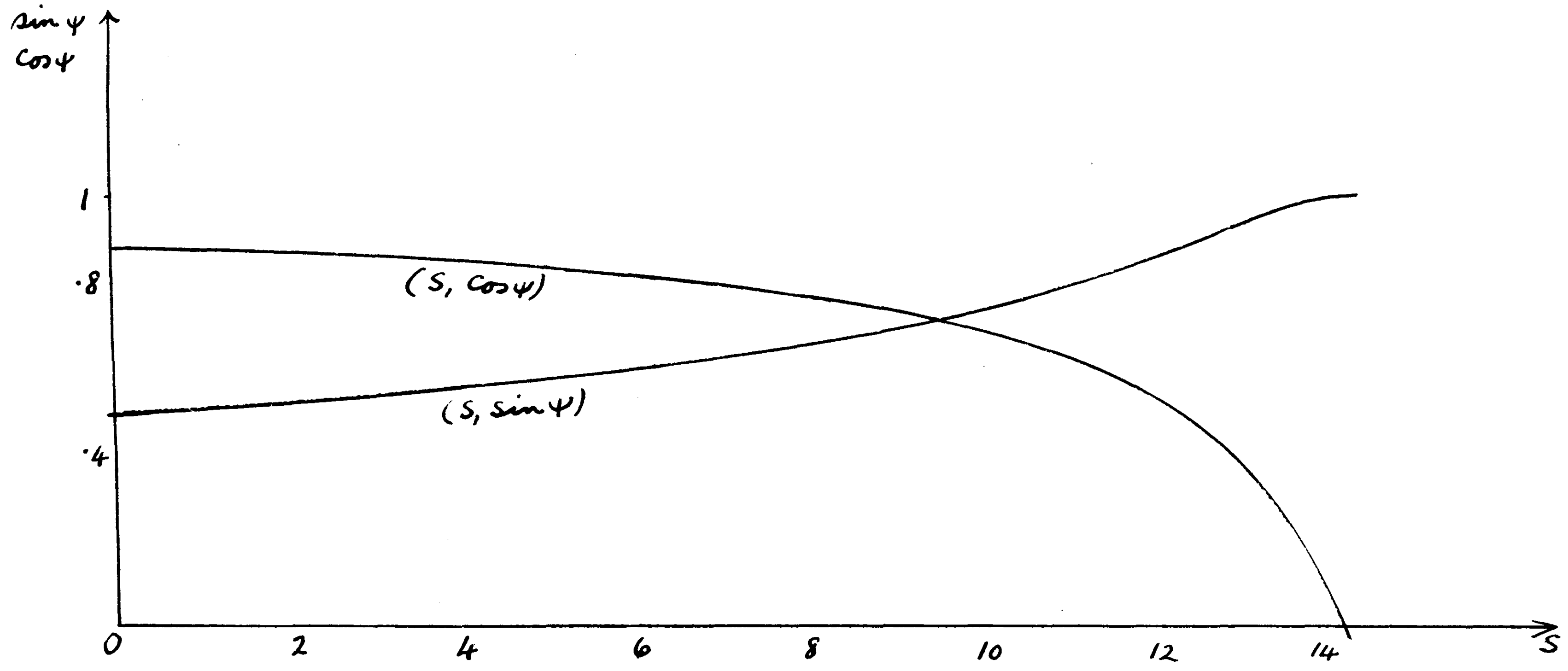
s	0	1·19	2·24	3·18	4·05	4·82	5·53	6·16	6·73	7·26	9·63	10·92
ψ	·50	·52	·54	·56	·58	·60	·62	·64	·66	·68	·80	·90
$25\cos\psi$	22·0	21·7	21·5	21·2	20·9	20·6	20·4	20·0	19·8	19·4	17·4	15·5
$25\sin\psi$	12·0	12·4	12·8	13·3	13·7	14·1	14·5	14·9	15·3	15·7	17·9	19·6
s	12·5	13·7	14·2	14·6	14·9	15·0	15·1	15·2	15·3	15·3	15·3	15·4
ψ	1·10	1·40	1·6	1·9	2·2	2·5	2·8	3·0	3·5	4·0	4·5	5·0
$25\cos\psi$	11·4	4·3	−·7	−8·1	−14·7	−20·0	−23·6	−24·8	−23·4	−16·3	−5·2	7·1
$25\sin\psi$	22·3	24·6	25·0	23·7	20·2	14·9	8·4	3·5	−8·8	−19·0	−24·4	−24·0

s, x, y values.

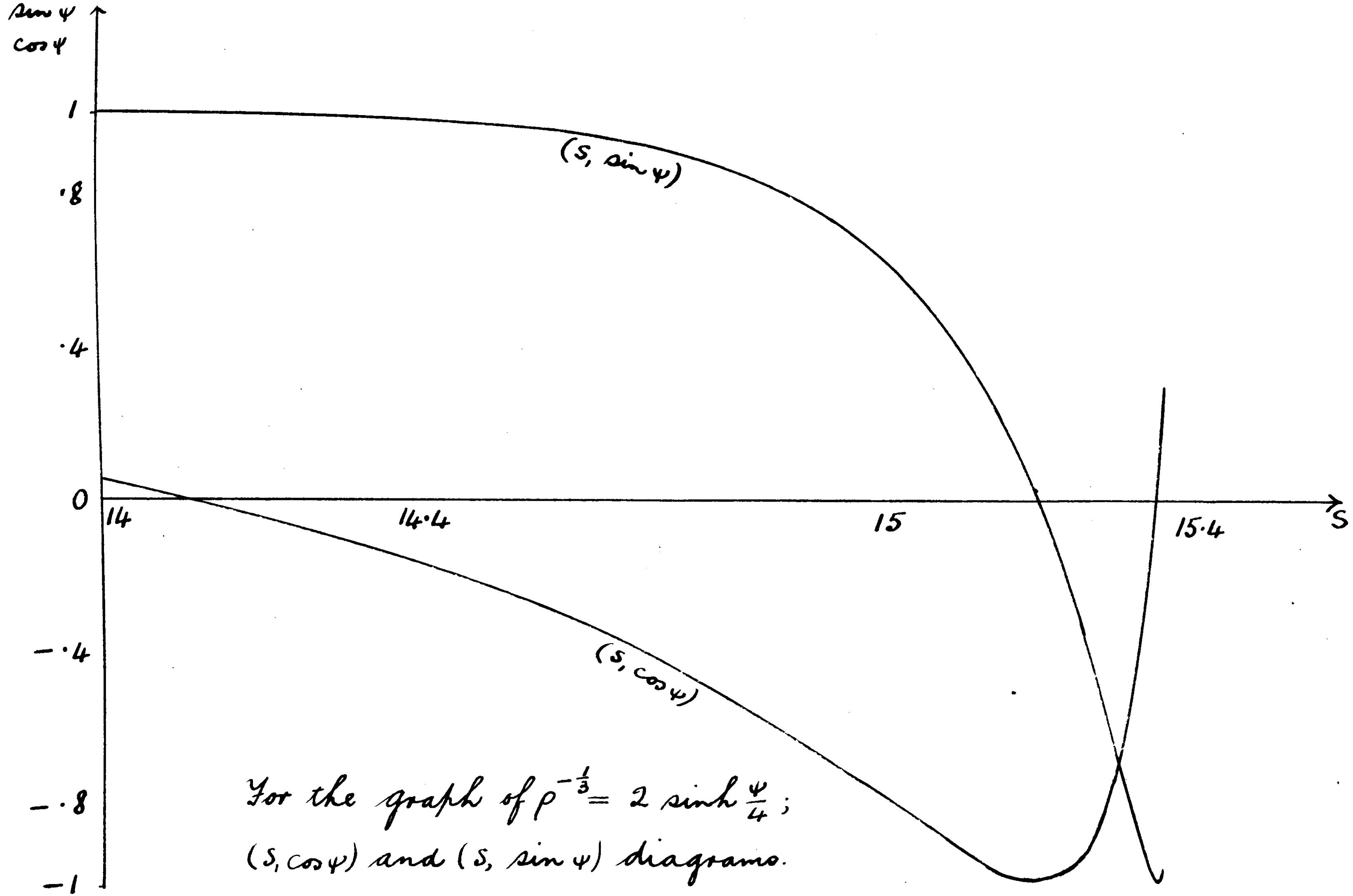
s	0	2	4	6	8	10	12	14	14·2	14·4	14·6
x	0	1·74	3·42	5·06	6·63	8·09	9·30	10·0	10·0	9·98	9·93
y	0	·99	2·06	3·19	4·43	5·81	7·39	9·27	9·47	9·67	9·86
s	14·8	15·0	15·2	15·3							
x	9·85	9·72	9·54	9·45							
y	10·04	10·20	10·27	10·24							



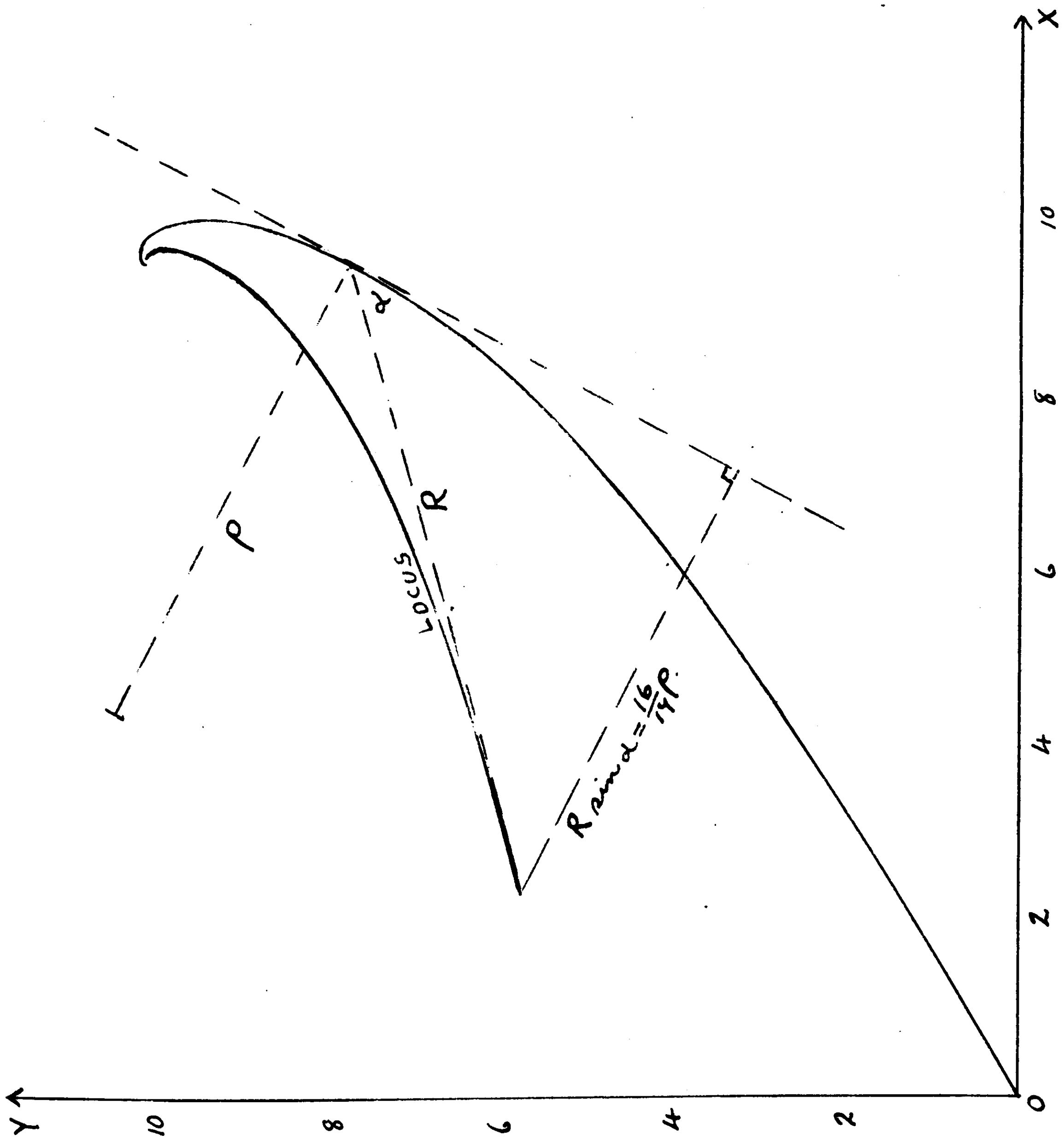
For the Graph of $\rho^{-\frac{1}{3}} = 2 \sinh \frac{\psi}{4}$;
(ρ, ψ) diagrams.



For the graph of $\rho^{-\frac{1}{3}} = 2 \sinh \frac{\psi}{4}$;
$(S, \cos\psi)$ and $(S, \sin\psi)$ diagrams.



For the graph of $\rho^{-\frac{1}{3}} = 2 \sinh \frac{\psi}{4}$;
$(S, \cos\psi)$ and $(S, \sin\psi)$ diagrams.



Graph of $\rho^{-\frac{1}{3}} = 2\sinh\frac{\psi}{4}$.

(3) Graph of the equation $\rho^{-\frac{1}{3}} = \sqrt{\frac{-c}{\kappa}} \cosh \frac{\sqrt{\kappa}}{3}\psi$, when $\sqrt{\frac{-c}{\kappa}} = 2$, $\frac{\sqrt{\kappa}}{3} = \frac{1}{4}$; i.e. of the equation $\rho^{-\frac{1}{3}} = 2\cosh\frac{\psi}{4}$.

When $\psi = 0$, $\rho = \frac{1}{8}$, and when $\psi = \infty$, $\rho = 0$. ρ diminishes steadily from $\rho = \frac{1}{8}$ to $\rho = 0$ while ψ increases from 0 to ∞. The current point makes an infinite number of revolutions round the point $\rho = 0$ before finally arriving at that point.

Data for the graph of $\rho^{-\frac{1}{3}} = 2\cosh\frac{\psi}{4}$.

ψ, ρ values.

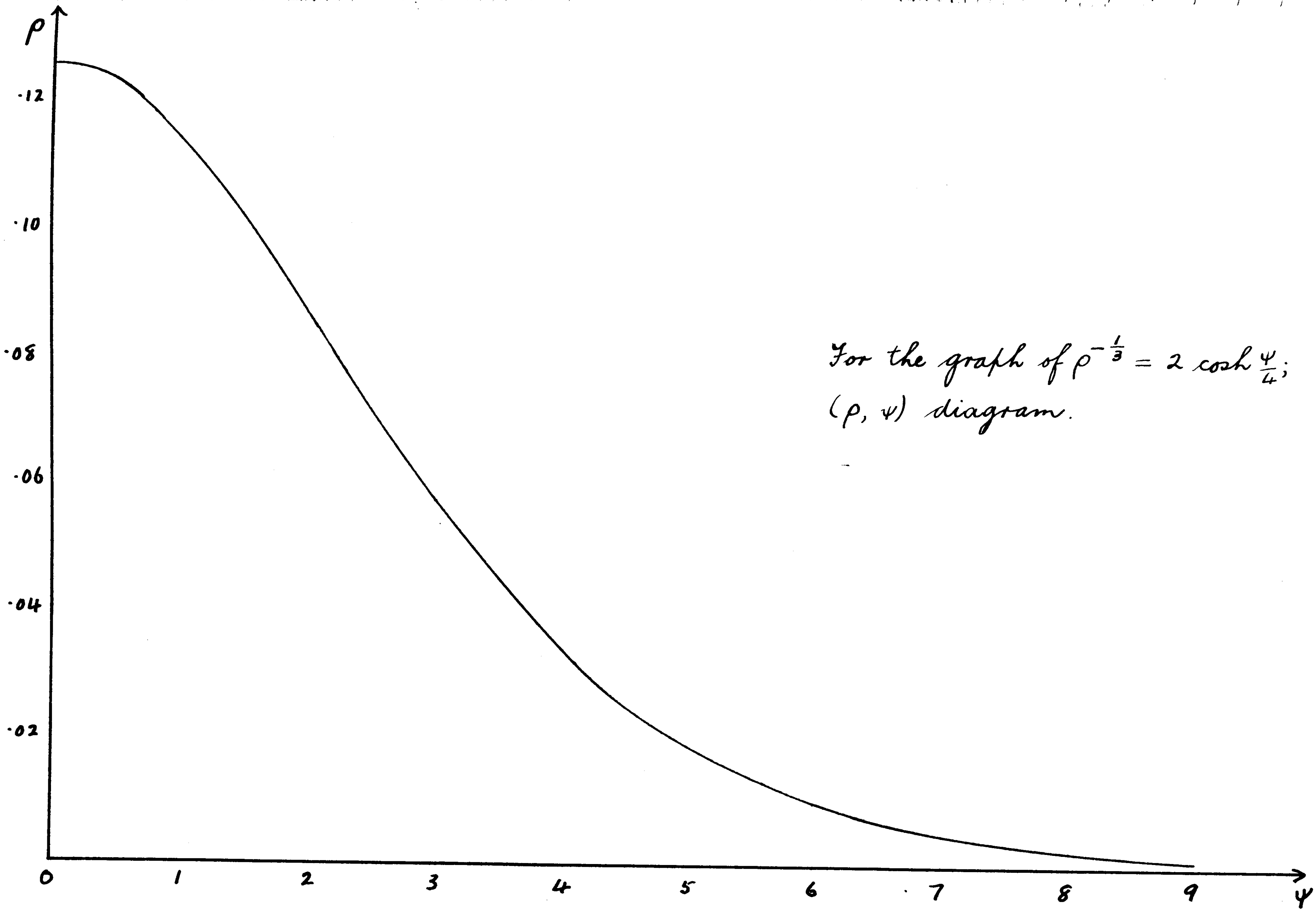
ψ	·4	·8	1·2	1·6	2·0	2·4	2·8	3·2	3·6
ρ	·124	·118	·110	·099	·087	·075	·063	·052	·043
ψ	4·0	4·4	4·8	5·2	6·0	6·8	7·6	8·4	9·0
ρ	·034	·027	·021	·0164	·0096	·0055	·00314	·00176	·00114

s, ψ, $\cos\psi$, $\sin\psi$

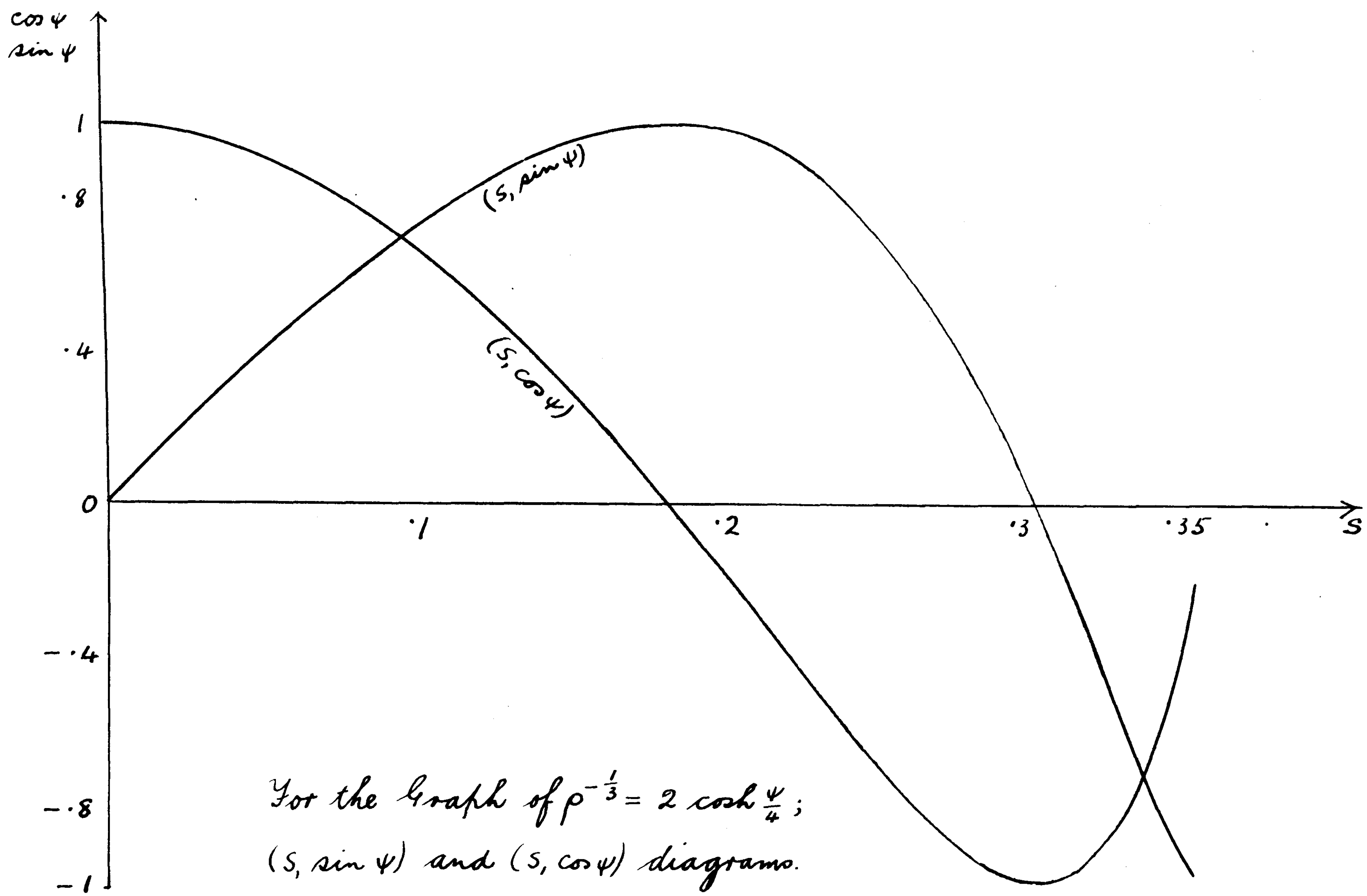
$20s$	0	1·49	2·88	4·10	5·09	5·88	6·48
ψ	0	·6	1·2	1·8	2·4	3·0	3·6
$25\cos\psi$	25	20·6	9·0	−5·7	−18·4	−24·8	−22·4
$25\sin\psi$	0	14·1	23·3	24·4	16·9	3·5	−11·0
$20s$	6·91	7·33	7·51	7·63	7·71	7·76	7·79	7·81
ψ	4·2	5·1	5·7	6·3	6·9	7·5	8·1	8·7
$25\cos\psi$	−12·2	9·4	20·9	25·0	20·4	8·6	−6·1	−18·7
$25\sin\psi$	−21·8	−23·1	−13·8	·4	14·5	23·5	24·2	16·6

s, x, y, $\bar{\xi}$, $\bar{\eta}$.

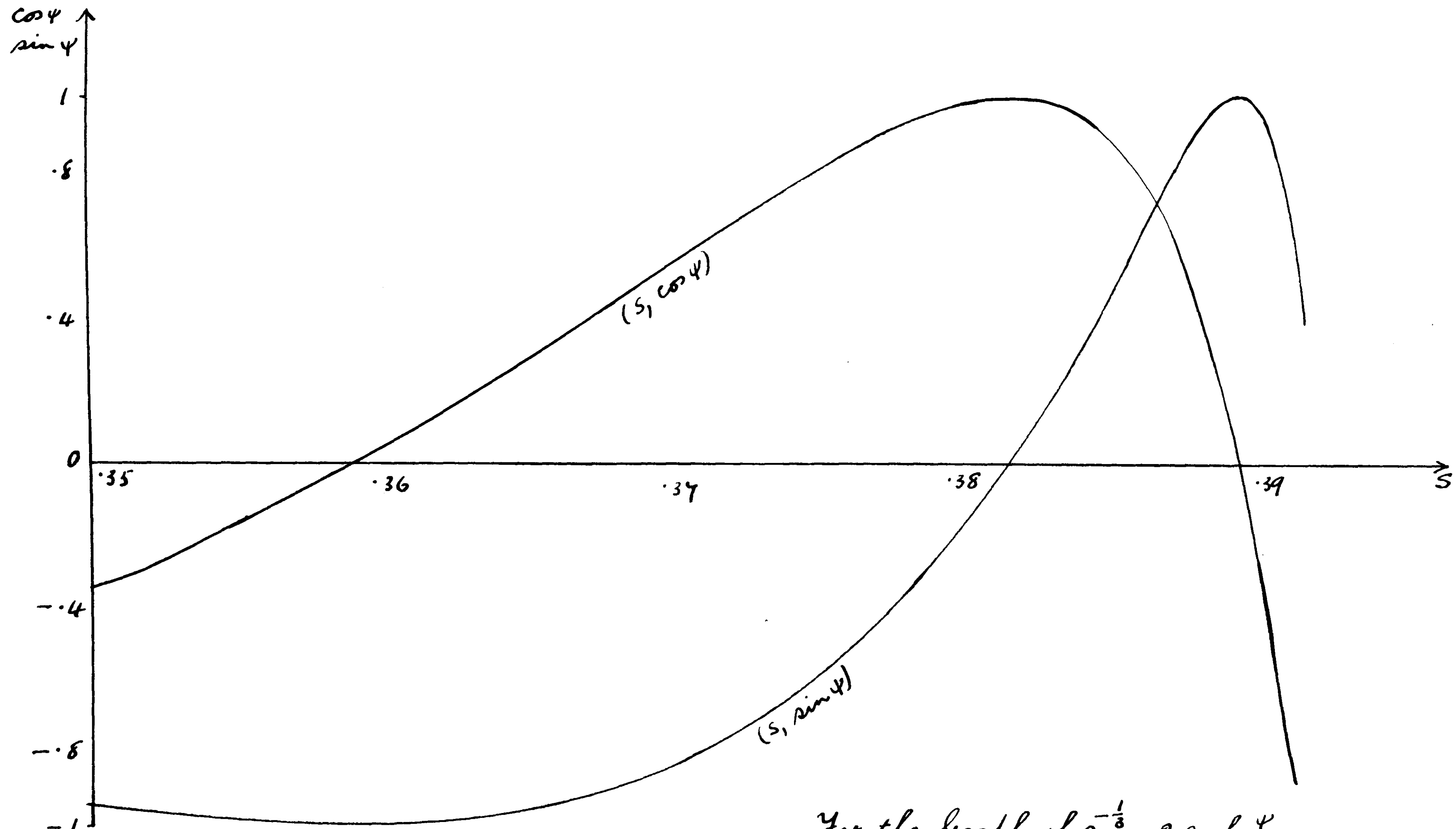
s	0	·05	·10	·15	·20	·25	·30	·35
$25x$	0	1·22	2·24	2·86	2·95	2·40	1·31	·35
$25y$	0	·26	·97	2·04	3·27	4·37	4·89	4·31
$25\bar{\xi}$	0	·00	·10	·32	·78	1·27	1·48	·99
$25\bar{\eta}$	2·94	2·92	2·79	2·62	2·65	2·97	3·61	4·21
s	·355	·36	·365	·37	·375	·38	·385	·39
$25x$	·32	·31	·33	·38	·47	·59	·71	·75
$25y$	4·19	4·07	3·94	3·83	3·74	3·70	3·72	3·83
$25\bar{\xi}$	·91	·81	·72	·69	·58	·57	·60	·72
$25\bar{\eta}$	4·21	4·19	4·16	4·12	4·01	3·91	3·83	3·80



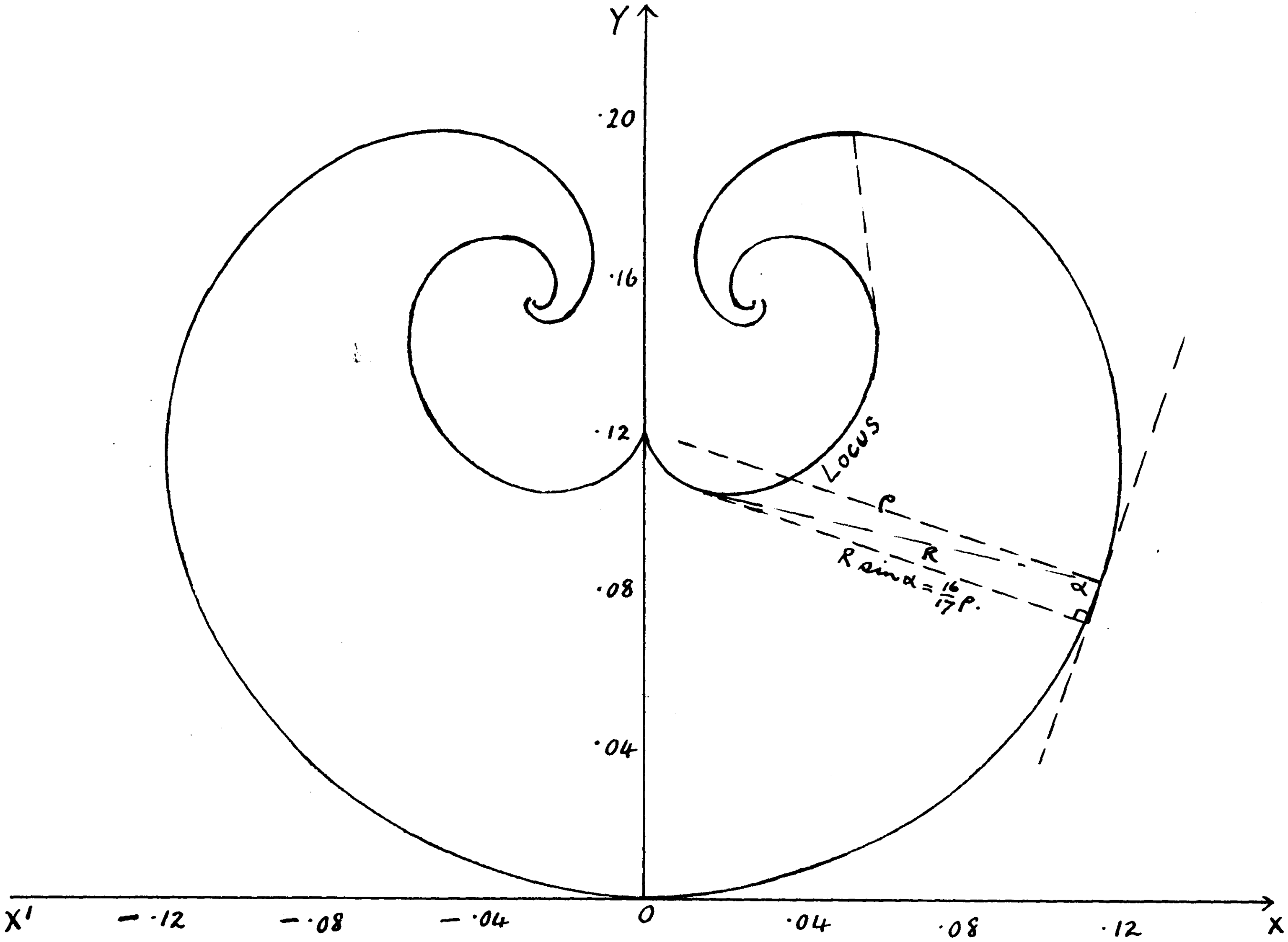
For the graph of $\rho^{-\frac{1}{3}} = 2\cosh\frac{\psi}{4}$; (ρ, ψ) diagram.



For the Graph of $\rho^{-\frac{1}{3}} = 2\cosh\frac{\psi}{4}$;
$(s, \sin\psi)$ and $(s, \cos\psi)$ diagrams.



For the Graph of $\rho^{-\frac{1}{3}} = 2\cosh\frac{\psi}{4}$;
$(S, \cos\psi)$ and $(S, \sin\psi)$ diagrams.



Graph of $\rho^{-\frac{1}{3}} = 2\cosh\frac{\psi}{4}$;
and the locus of its centres of aberrancy.

(4) Graph of the equation $\rho^{\frac{1}{3}} = \sqrt{\frac{-K}{c}} \sec \frac{\sqrt{-K}}{3}\psi$, when $\sqrt{\frac{-K}{c}} = 1$, and $\frac{\sqrt{-K}}{3} = \frac{1}{5}$, i.e. of the equation $\rho^{\frac{1}{3}} = \sec\frac{\psi}{5}$.

When $\psi = 0$, $\rho = 1$, and when $\psi = \frac{5\pi}{2}$, $\rho = \infty$.
The curve goes off to infinity in the directions $\psi = \pm\frac{5\pi}{2}$. ρ steadily increases from 1 to ∞ while ψ increases from 0 to $\frac{5\pi}{2}$, and diminishes steadily from ∞ to 1 while ψ increases from $-\frac{5\pi}{2}$ to 0.

The coordinates of the centre of aberrancy can be calculated from the equations :-

$$\bar{\xi} = x + \frac{\rho\cos\psi . \tan\frac{\psi}{5} - 5\rho\sin\psi}{4.8},$$

$$\bar{\eta} = y + \frac{\rho\sin\psi\tan\frac{\psi}{5} + 5\rho\cos\psi}{4.8},$$ using the values

of $\rho\cos\psi$, $\rho\sin\psi$ from the graphs corresponding to the x and y values in the following table.

Data for the Graph of $\rho^{1/3} = \sec\frac{\psi}{5}$.

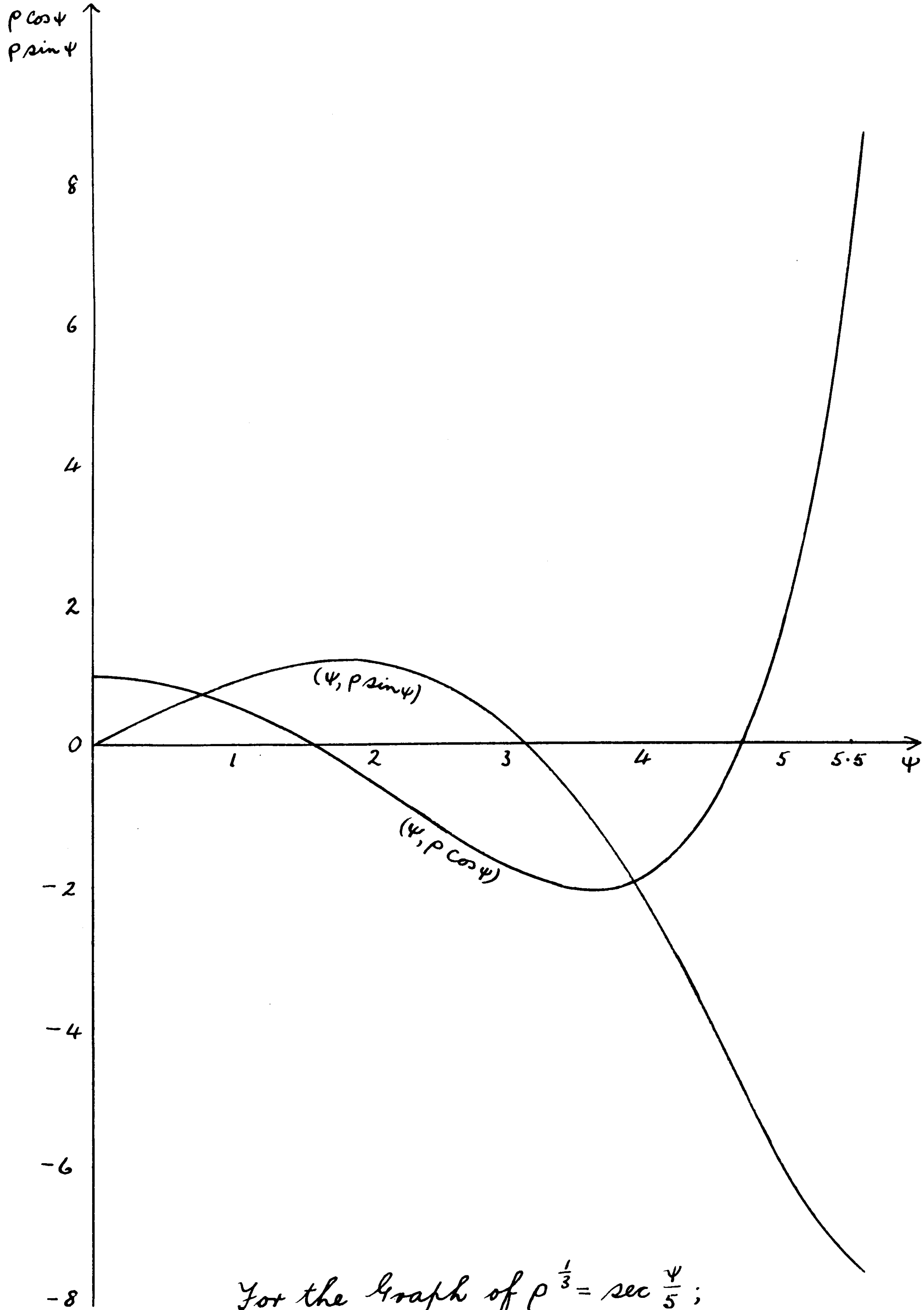
ψ, $\rho\cos\psi$, $\rho\sin\psi$ values.

ψ	0	·349	·698	1·047	1·397	1·745	2·094	2·444
$\rho\cos\psi$	1	·95	·79	·53	·20	−·21	−·66	−1·11
$\rho\sin\psi$	0	·34	·66	·93	1·11	1·19	1·14	·93

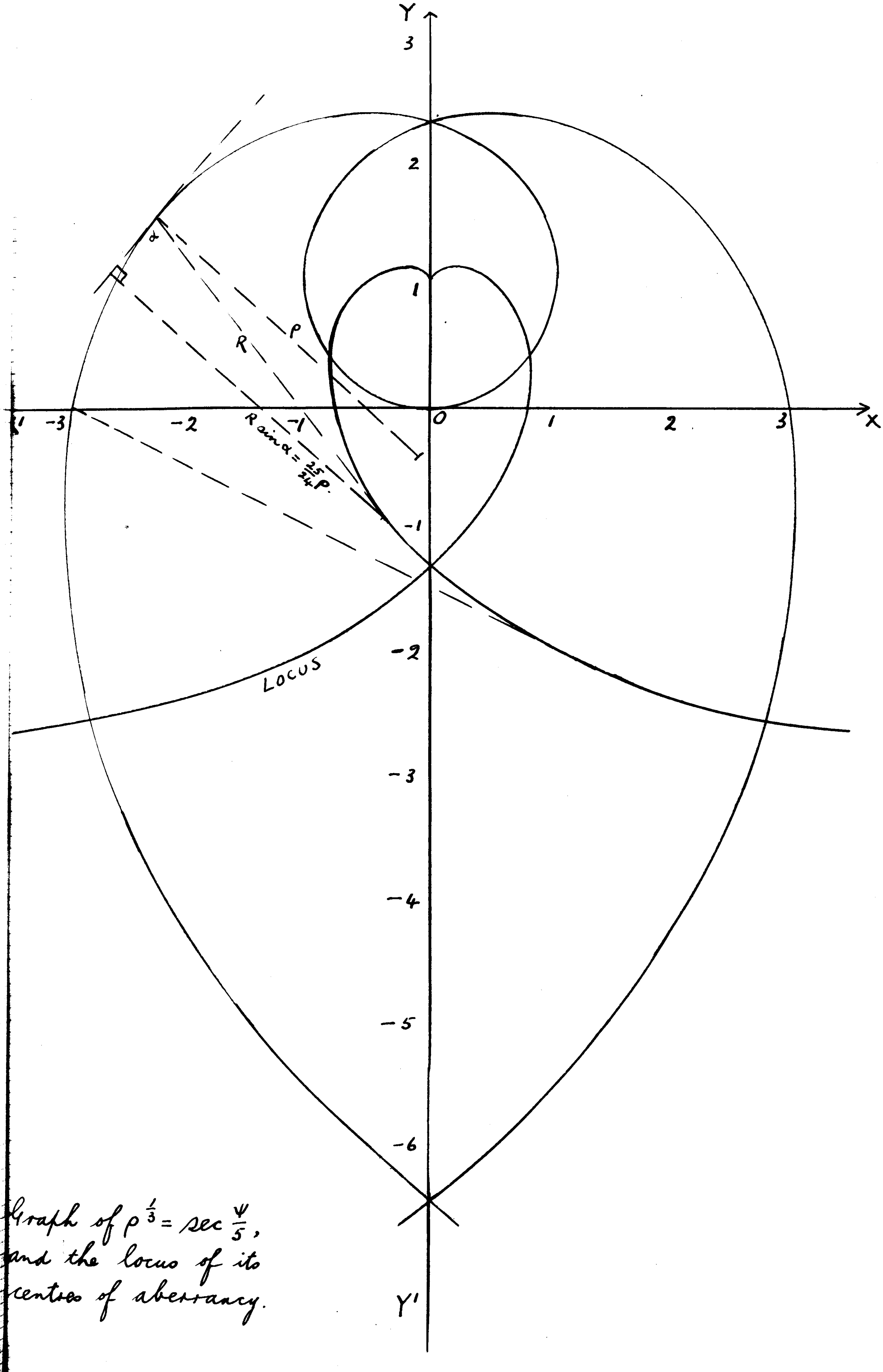
ψ	2·793	3·142	3·491	3·840	4·189	4·538	4·887	5·236	5·585	6·283
$\rho\cos\psi$	−1·54	−1·89	−2·09	−2·06	−1·67	−·74	·99	4·00	9·10	33·9
$\rho\sin\psi$	·56	0	−·76	−1·73	−2·89	−4·22	−5·63	−6·93	−7·63	0

x, y, $\bar{\xi}$, $\bar{\eta}$ values.

x	0	·48	·85	1·03	·93	·49	−·26	−1·24	−2·26	−2·96	−2·81	−·61	·23
y	0	·12	·47	·99	1·58	2·11	2·40	2·28	1·57	·02	−2·52	−5·94	−6·7
$\bar{\xi}$	0	·00	−·07	−·18	−·32	−·59	−·78	−·78	−·38	1·02	4·04		
$\bar{\eta}$	1·04	1·05	1·13	1·17	1·12	·96	·65	−·02	−·92	−1·96	−2·72		



For the Graph of $\rho^{\frac{1}{3}} = \sec\frac{\psi}{5}$;
$(\psi, \rho\cos\psi)$ and $(\psi, \rho\sin\psi)$ diagrams.



Graph of $\rho^{\frac{1}{3}} = \sec \frac{\psi}{5}$, and the locus of its centres of aberrancy.

Section XIII.

To find a curve such that the projection of the axis of aberrancy at any point on the normal at that point is constant.

(1) The length of the axis of aberrancy is given by $R = \dfrac{\rho \sin\alpha}{1+\frac{d\alpha}{d\psi}}$, where $\frac{d\rho}{ds} = 3\cot\alpha$.

The required differential equation is $R\sin\alpha = k$.
Since $3\cot\alpha = \frac{1}{\rho}\frac{d\rho}{d\psi}$, we have

$$-3\,\mathrm{cosec}^2\alpha\,\frac{d\alpha}{d\psi} = -\frac{1}{\rho^2}\left(\frac{d\rho}{d\psi}\right)^2 + \frac{1}{\rho}\frac{d^2\rho}{d\psi^2}.$$

Substituting for R, $\sin^2\alpha$, $\frac{d\alpha}{d\psi}$ in terms of ρ, $\frac{d\rho}{d\psi}$ and $\frac{d^2\rho}{d\psi^2}$, the equation becomes

$$\frac{3\rho}{k} = 3 + \frac{4}{3\rho^2}\left(\frac{d\rho}{d\psi}\right)^2 - \frac{1}{\rho}\frac{d^2\rho}{d\psi^2}, \quad \text{-----} \quad (1)$$

---

(2) The substitution $\frac{d\rho}{d\psi} = u$ gives the form

$$-3 + \frac{3\rho}{k} = \frac{4}{3\rho^2}u^2 - \frac{u}{\rho}\frac{du}{d\rho},$$

$$\text{i.e.}\quad 2\rho^{-5/3}\left(3 - \frac{3\rho}{k}\right) = \frac{d}{d\rho}\left(u^2\rho^{-8/3}\right),$$

whence, integrating,

$$u^2 = -9\rho^2 - \frac{18}{k}\rho^3 + 9c\rho^{8/3}.$$

---

(3) $K = 0$ gives the solution $\rho = 0$.

---

(4) $K = \infty$ gives $\frac{1}{\rho}\frac{d\rho}{d\psi} = \sqrt{-9 + 9c\rho^{2/3}} = \frac{d\rho}{ds}$. It follows that $9 + \rho'^2 - 3\rho\rho'' = 0$. The solution, therefore, in this case is ~~the~~ a parabola.

---

(5) $c = \infty$ gives the solution $\rho = 0$.

---

(6) $c = 0$ gives $\frac{1}{\rho}\frac{d\rho}{d\psi} = \sqrt{-9 - \frac{18}{k}\rho} = \frac{d\rho}{ds}$, the solution of which, for $k = 1$, is $\rho = -\frac{1}{2}\sec^2\frac{3\psi}{2}$, or

$$\frac{d\rho}{ds} = 3\cot\left(\frac{\pi}{2} - 3\frac{\psi}{2}\right).\ \text{It follows}$$

that $\alpha = \frac{\pi}{2} - \frac{3\psi}{2}$, a linear function of ψ. The graph of this equation, therefore, belongs to the class of curves discussed in Section XXII, and illustrated in Section XXIII. Its graph is the same as that for the case $\alpha = -\frac{3}{2}\psi$, but referred to different axes.

---

(7) The discriminant of the equation $u^2 = -9\rho^2 - \frac{18}{k}\rho^3 + 9c\rho^{8/3}$ equated to zero gives solutions of the form $\rho =$ const. The particular solution $\rho = k$ satisfies this for a certain value of c, and also satisfies the differential equation ①.

---

(8) If we substitute $\rho^{-\frac{1}{3}} = v$, the equation for u

$$u^2 = 9\rho^2\left(-1 - \frac{2\rho}{k} + c\rho^{2/3}\right) \text{ becomes}$$

$$\pm d\psi = \frac{\sqrt{v}\,dv}{\sqrt{-v^3 + cv - 2/k}} = \frac{v\,dv}{\sqrt{v(-v^3 + cv - 2/k)}}.$$

The equation ① shows that $\frac{1}{k}$ is merely a multiplier of ρ, and that change of sign of k is equivalent merely to a change of sign of ρ. We may therefore take $k = +1$, when k is finite and non-zero. The equation for ψ then is $\pm d\psi = \frac{v\,dv}{\sqrt{v(-v^3 + cv - 2)}}$, which can be solved with the aid of elliptic integrals.

Let $f(v) = -v^3 + cv - 2$,
then $f'(v) = -3v^2 + c$,
and $f''(v) = -6v$.

The equation $f(v) = 0$ has a pair of equal roots when $c = 3$. The condition for three equal roots cannot be satisfied.

---

(9) When $c = 3$, $f(v) = -v^3 + 3v - 2 = (v-1)(v-1)(-v-2)$, and we have $\pm\psi = \int \frac{v\,dv}{(v-1)\sqrt{-v(v+2)}}$. If we put

$1 + v = \sin\theta$, we find $\pm\psi = \int \left(1 - \frac{1}{2 - \sin\theta}\right) d\theta$

$$= \theta - \frac{2}{\sqrt{3}} \tan^{-1} \frac{2\tan\frac{\theta}{2} - 1}{\sqrt{3}} + \text{const.}$$

---

(10) <u>c positive.</u>

Since $f'(v) = -3v^2 + c$, $f(v)$ has two real turning values given by $v^2 = \frac{c}{3}$. Also $f(\infty)$ is negative, $f(-\infty)$ is positive, and $f(0)$ is negative.

We have $f\left(\sqrt{\frac{c}{3}}\right) = \frac{2}{3} \frac{c\sqrt{c}}{\sqrt{3}} - 2$, so that when $c > 3$, $f(v) = 0$ has two positive real roots, and one negative real root, and $f(v)$ may be written
$f(v) \equiv -(v + h^2)(v - k^2)(v - m^2)$, where
$-h^2 < -\sqrt{\frac{c}{3}} < 0 < k^2 < \sqrt{\frac{c}{3}} < m^2$. Moreover, the expression $\sqrt{v(-v^3 + cv - 2)}$ requires that, when v is negative, $f(v)$ must be negative also, and that when v is positive, $f(v)$ must be positive also. Hence for real solutions in this case, we must have

$$-h^2 \leqq v \leqq 0, \text{ or } k^2 \leqq v \leqq m^2.$$

For example when $c = 5$, we have
$f(v) = -(v + \sqrt{2} + 1)(v - \sqrt{2} + 1)(v - 2)$, and for real solutions $-(\sqrt{2} + 1) \leqq v \leqq 0$, or $\sqrt{2} - 1 \leqq v \leqq 2$.

---

(11) But when $c = 3$, the two positive real roots become coincident, so that we can only have $-h^2 \leqq v \leqq 0$, where $h^2 = 2$.

---

(12) When $c < 3$, $f(v) = 0$ has a pair of imaginary roots, and one real negative root. In this case real solutions are given only by
$-h^2 \leqq v \leqq 0$, k^2 and m^2 being imaginary.

For example, when $c = \frac{11}{12}$, we have
$$f(v) = -\left(v+\tfrac{3}{2}\right)\left(v^2 - \tfrac{3v}{2} + \tfrac{4}{3}\right),$$ and for real solutions $-\frac{3}{2} \leqq v \leqq 0$.

---

(13) <u>c negative.</u>

In this case $f'(v)$ is always negative, hence there are no real turning values, and, as in the previous paragraph, we must have for real solutions $-h^2 \leqq v \leqq 0$.

For example, when $c = -1$, we have
$$f(v) = -(v+1)(v^2 - v + 2),$$ and for real solutions in this case we must have
$$-1 \leqq v \leqq 0.$$

---

(14) Since the equation $u^2 = -9\rho^2 - \frac{18}{k}\rho^3 + 9c\rho^{8/3}$ can be written $\rho'^2 = -9 - \frac{18}{k}\rho + 9c\rho^{2/3}$, it follows that $9 + \rho'^2 - 3\rho\rho'' = \frac{9\rho}{k}$. Thus the solution of the problem of this section is also the solution of the problem of finding a curve for which the quantity $9 + \rho'^2 - 3\rho\rho''$ is proportional to ρ.

Section XIV. Curves for which $R \sin\alpha = k$.

Graph of the equation $R \sin\alpha = -2$.

(1) The solution is $\rho = \operatorname{cosec}^2 \frac{3\psi}{2}$.

It follows that $\rho' = 3 \cot(-\frac{3\psi}{2})$,

$$\rho\rho'' = \frac{9}{2} \operatorname{cosec}^2 \frac{3\psi}{2},$$

$$9 + \rho'^2 - 3\rho\rho'' = -\frac{9}{2} \operatorname{cosec}^2 \frac{3\psi}{2},$$

$$R = 2 \operatorname{cosec} \frac{3\psi}{2};$$

$$\bar{\xi} = x + 2\operatorname{cosec}\frac{3\psi}{2} \cos\frac{\psi}{2}, \quad \bar{\eta} = y - 2\operatorname{cosec}\frac{3\psi}{2} \sin\frac{\psi}{2};$$

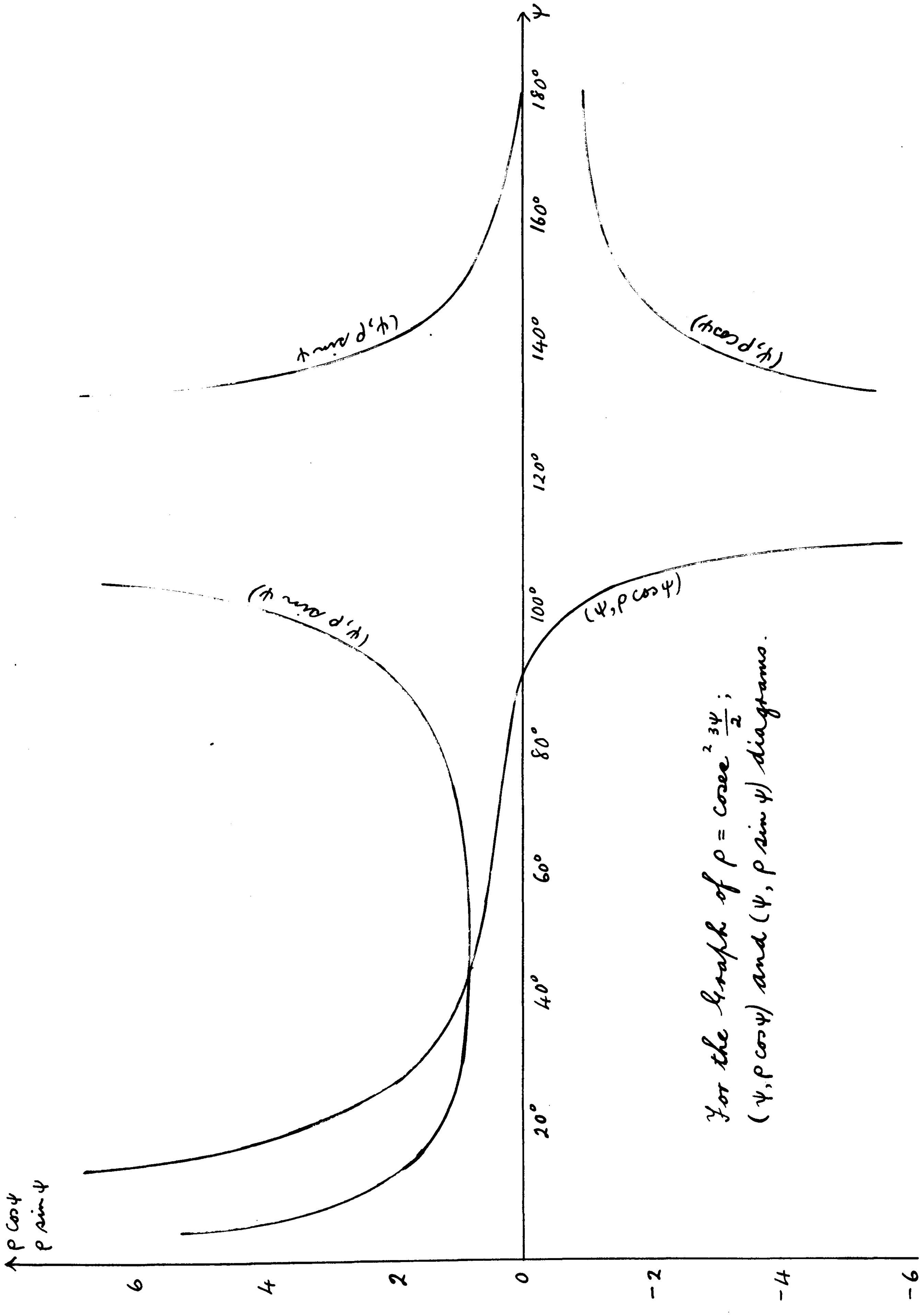
$$x = \int \operatorname{cosec}^2 \frac{3\psi}{2} \cos\psi \, d\psi = -\frac{2}{9}\cot\frac{\psi}{2} + \frac{2\sqrt{3}}{9} \log \frac{\sqrt{3} + \tan\frac{\psi}{2}}{\sqrt{3} - \tan\frac{\psi}{2}} - \frac{4}{9}\frac{\sin\psi}{1 + 2\cos\psi},$$

$$y = \int \operatorname{cosec}^2 \frac{3\psi}{2} \sin\psi \, d\psi = \frac{2}{9} \log \frac{1 - \cos\psi}{1 + 2\cos\psi} + \frac{2}{3}\frac{1}{1 + 2\cos\psi}.$$

The angle between the x-axis and the tangent is two-thirds of the angle between the negative direction of the tangent and the positive direction of the axis of aberrancy.

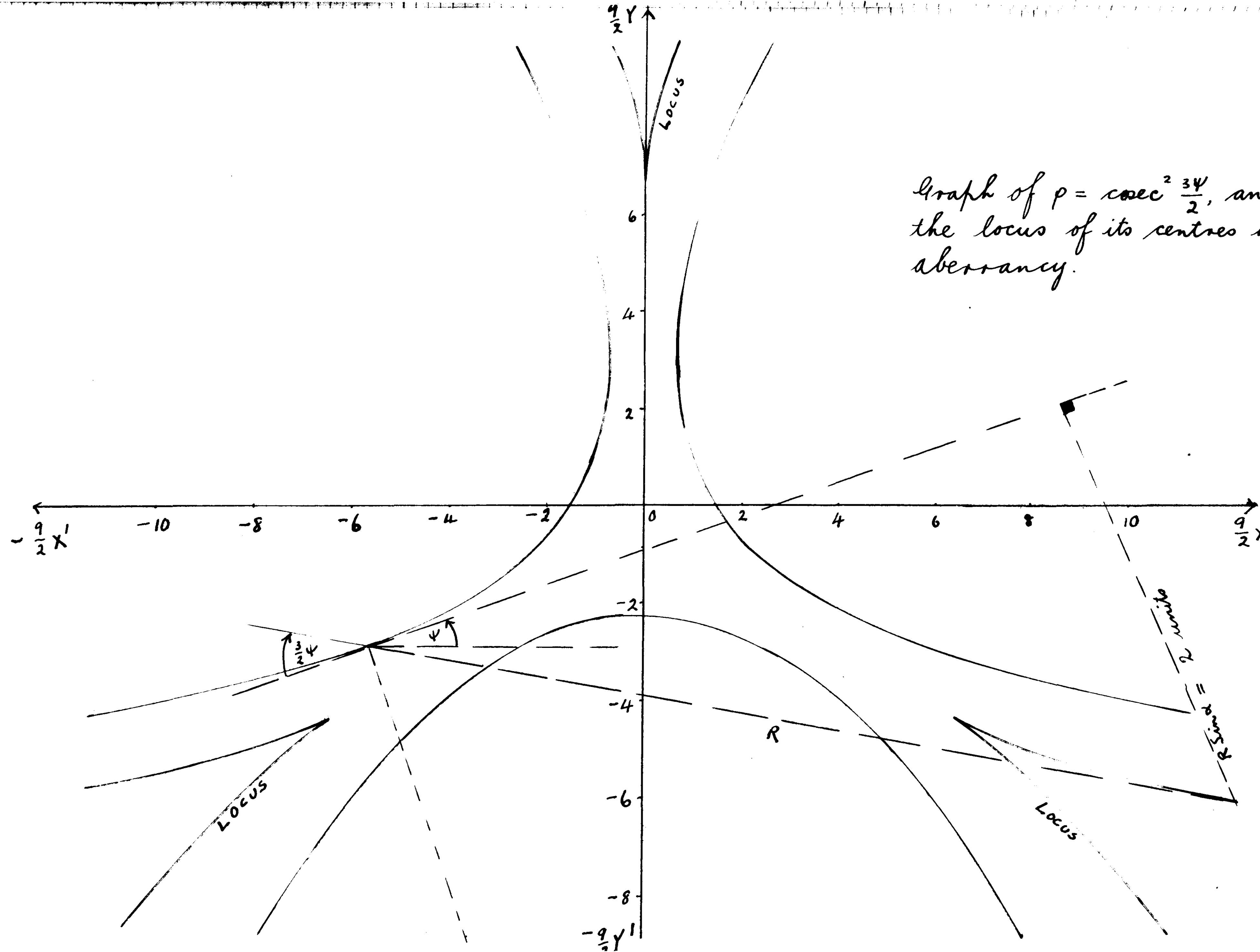
ψ°	0	5	10	15	20	25	30	35	40	45
$\rho\cos\psi$	∞	58.5	14.7	6.60	3.76	2.45	1.73	1.30	1.02	.83
$\rho\sin\psi$	∞	5.12	2.59	1.77	1.37	1.14	1	.91	.86	.83
ψ°	50	55	60	65	70	75	80	85	90	95	100
$\rho\cos\psi$	.69	.58	.50	.43	.37	.30	.23	.14	0	−.24	−.69
$\rho\sin\psi$	.82	.83	.87	.92	1.01	1.13	1.31	1.58	2	2.69	3.94
ψ°	105	110	115	120	125	130	135	140	145	150
$\rho\cos\psi$	−1.77	−5.11	−24.8	$-\infty$	−33.67	−9.60	−4.83	−3.06	−2.21	−1.73
$\rho\sin\psi$	6.60	14.03	53.20	∞	48.08	11.44	4.83	2.57	1.55	1
ψ°	155	160	165	170	175	180				
$\rho\cos\psi$	−1.44	−1.25	−1.13	−1.06	−1.01	−1				
$\rho\sin\psi$	.67	.46	.30	.19	.09	0				

$\frac{9}{2}x$	$-\infty$	−11.37	−5.56	−3.56	−2.52	−1.86	−1.40	−1.06	−.82	−.72
$\frac{9}{2}y$	$-\infty$	−4.27	−2.83	−1.92	−1.20	−.55	.11	.84	1.74	3.00
$\frac{9}{2}\bar{\xi}$	∞	23.26	12.16	8.73	7.25	6.58	6.39	6.58	7.14	8.25
$\frac{9}{2}\bar{\eta}$	$-\infty$	−7.30	−5.95	−5.21	−4.75	−4.48	−4.39	−4.50	−4.88	−6.01
$\frac{9}{2}x$	−.94	−2.59	$-\infty$	8.78	4.62	2.84	1.69	.80	0	
$\frac{9}{2}y$	5.14	9.94	∞	−9.75	−4.44	−3.16	−2.62	−2.38	−2.31	
$\frac{9}{2}\bar{\xi}$	10.63	17.36	∞	−5.91	−1.54	−.46	−.11	−.01	0	
$\frac{9}{2}\bar{\eta}$	−8.65	−18.54	$-\infty$	21.77	12.47	9.13	7.61	6.90	6.69	



For the Graph of $\rho = \operatorname{cosec}^2 \frac{3\psi}{2}$; $(\psi, \rho\cos\psi)$ and $(\psi, \rho\sin\psi)$ diagrams.

Graph of $\rho = \operatorname{cosec}^2 \frac{3\psi}{2}$, and the locus of its centres of aberrancy.



(2) Graph of the equation $u^2 = 9\rho^2\left(-1 - \frac{2\rho}{k} + c\rho^{2/3}\right)$, when $k = 1$, $c = 3$; i.e. of the equation

$$\pm\psi = \int\left(1 - \frac{1}{2-\sin\theta}\right)d\theta$$

$$= \theta - \frac{2}{\sqrt{3}}\tan^{-1}\frac{2\tan\frac{\theta}{2} - 1}{\sqrt{3}} + \frac{\pi}{2} - \frac{2\pi}{3\sqrt{3}},$$

where $\rho^{\frac{1}{3}} = \dfrac{1}{\sin\theta - 1}$.

From the (ψ, θ) equation, or by graphical integration from the graph of $1 - \dfrac{1}{2-\sin\theta}$, corresponding values of $\theta, \psi, \rho, \rho\cos\psi, \rho\sin\psi$ can be found, and from the $(\psi, \rho\cos\psi)$ and $(\psi, \rho\sin\psi)$ diagrams corresponding values of the coordinates x and y can be calculated.

Since $\rho'^2 = \dfrac{u^2}{\rho^2} = -9 - 18\rho + 27\rho^{2/3}$, it follows that $9 + \rho'^2 - 3\rho\rho'' = 9\rho$, and since $\rho^{-\frac{1}{3}} = \sin\theta - 1$, and therefore negative, it follows that at every point on the curve the osculating conic is a hyperbola.

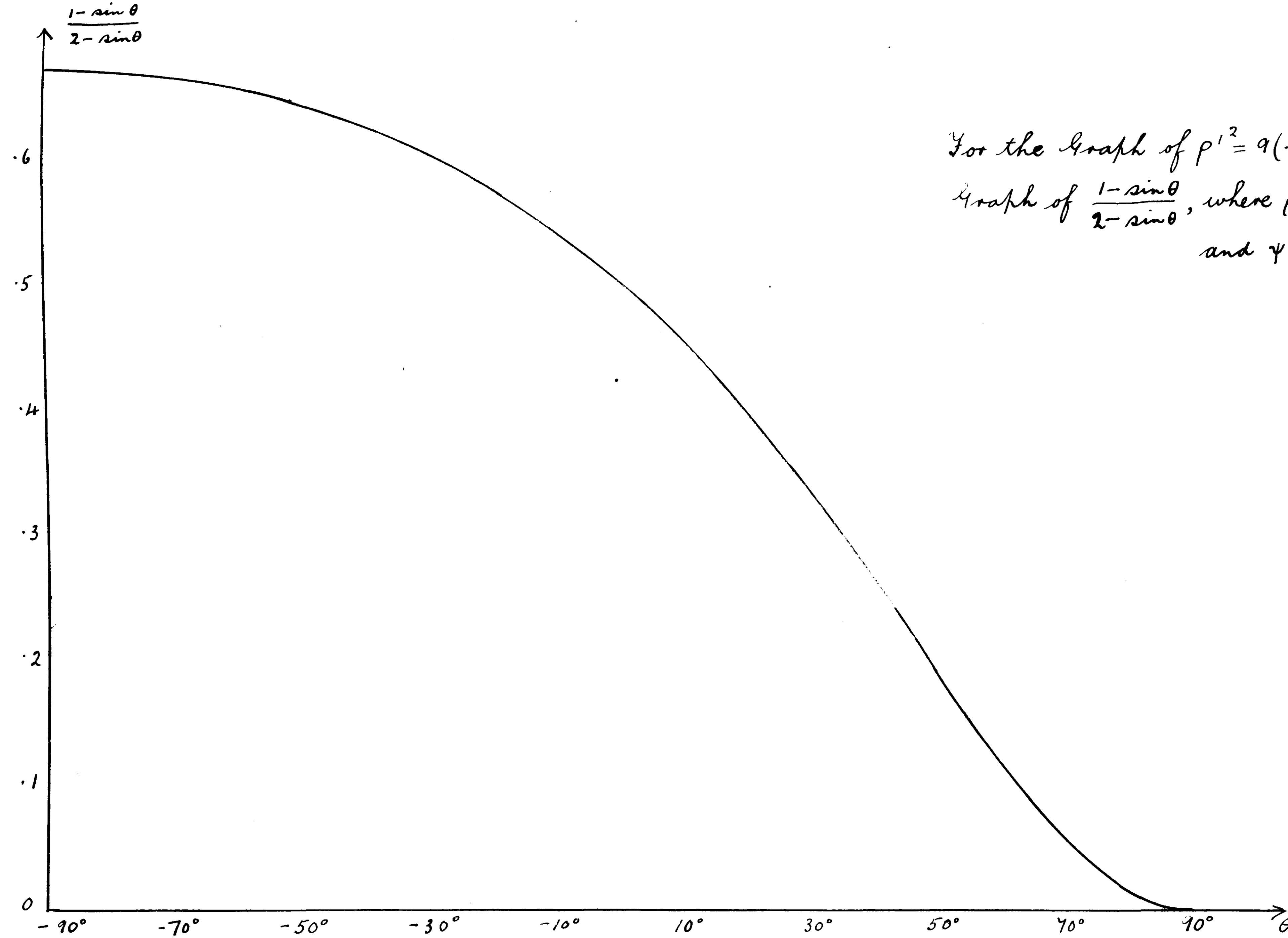
Moreover, since the axis of aberrancy is of length $R = \operatorname{cosec}\alpha$, and $\operatorname{cosec}^2\alpha = \frac{1}{9}(\rho'^2 + 9)$ $= -2\rho + 3\rho^{2/3}$, we can calculate α and R for the values of ρ already calculated, and so find the positions of the centres of aberrancy.

Data for the Graph of $\rho'^2 = -9 - 18\rho + 27\rho^{2/3}$.

θ°	-90	-80	-70	-60	-50	-40	-30
ψ°	0	6°40′	13°17′	19°50′	26°14′	32°35′	38°42′
$-\rho^{1/3}$	·5000	·5039	·5155	·5359	·5662	·6088	·6667
$-\rho\cos\psi$	·1250	·1272	·1332	·1447	·1628	·1901	·2314
$-\rho\sin\psi$	·0000	·0149	·0315	·0522	·0804	·1215	·1854

θ°	-20	-10	0	10	20	30	40	50	60
ψ°	$44^\circ 34'$	$50^\circ 7'$	$55^\circ 19'$	$60^\circ 5'$	$64^\circ 20'$	$67^\circ 59'$	$70^\circ 58'$	$73^\circ 14'$	$74^\circ 47'$
$-\rho^{\frac{1}{3}}$	·7452	·8519	1·	1·210	1·520	2·	2·800	4·274	7·463
$-\rho\cos\psi$	·2944	·3965	·5690	·8837	1·521	2·998	7·161	22·51	109·1
$-\rho\sin\psi$	·2912	·4744	·8222	1·536	3·164	7·416	20·49	74·74	401·1

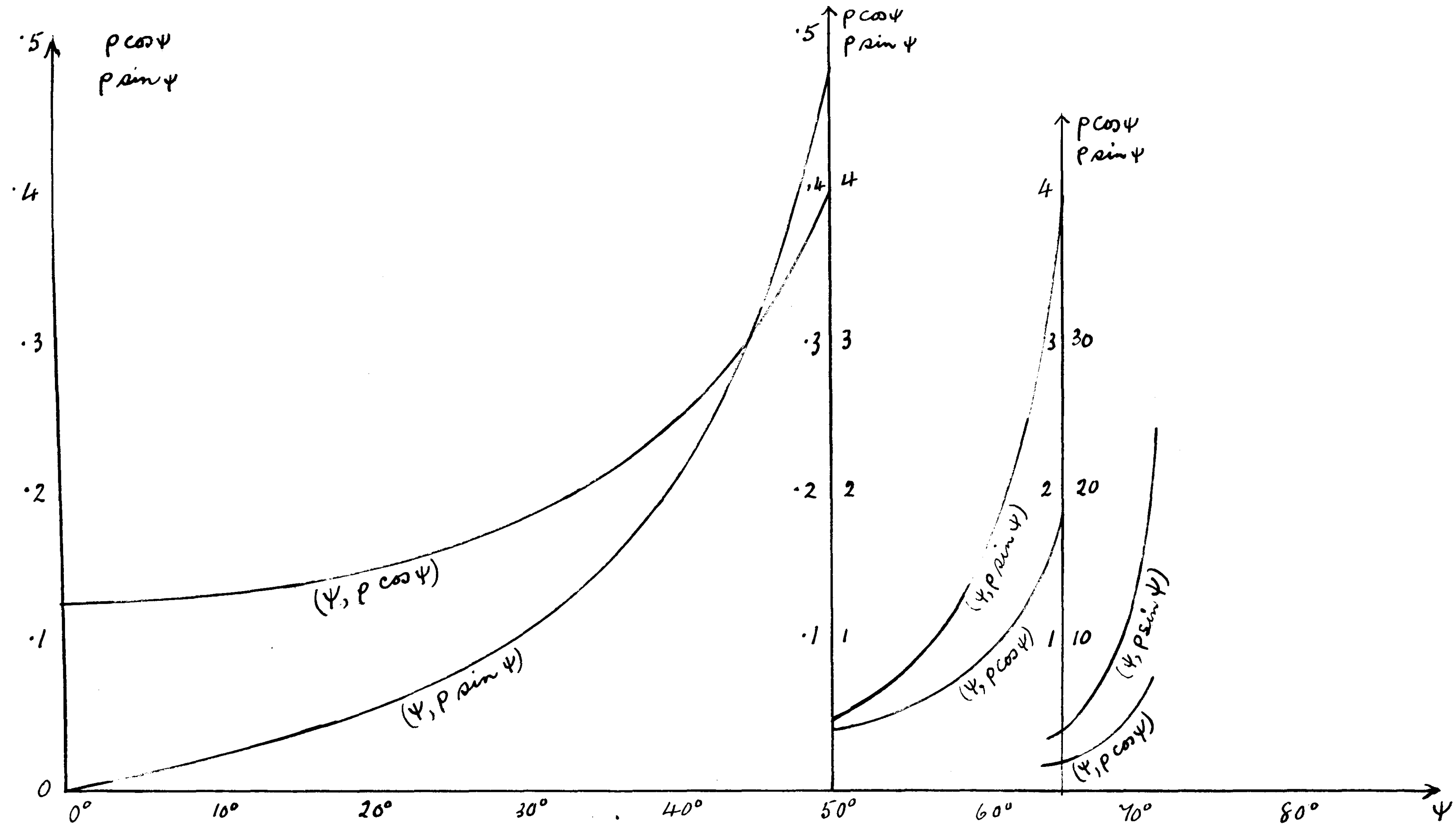
x	0	·022	·046	·074	·111	·164	·268	·375	·654	1·84
y	0	·002	·008	·022	·048	·103	·258	·470	1·09	4·77



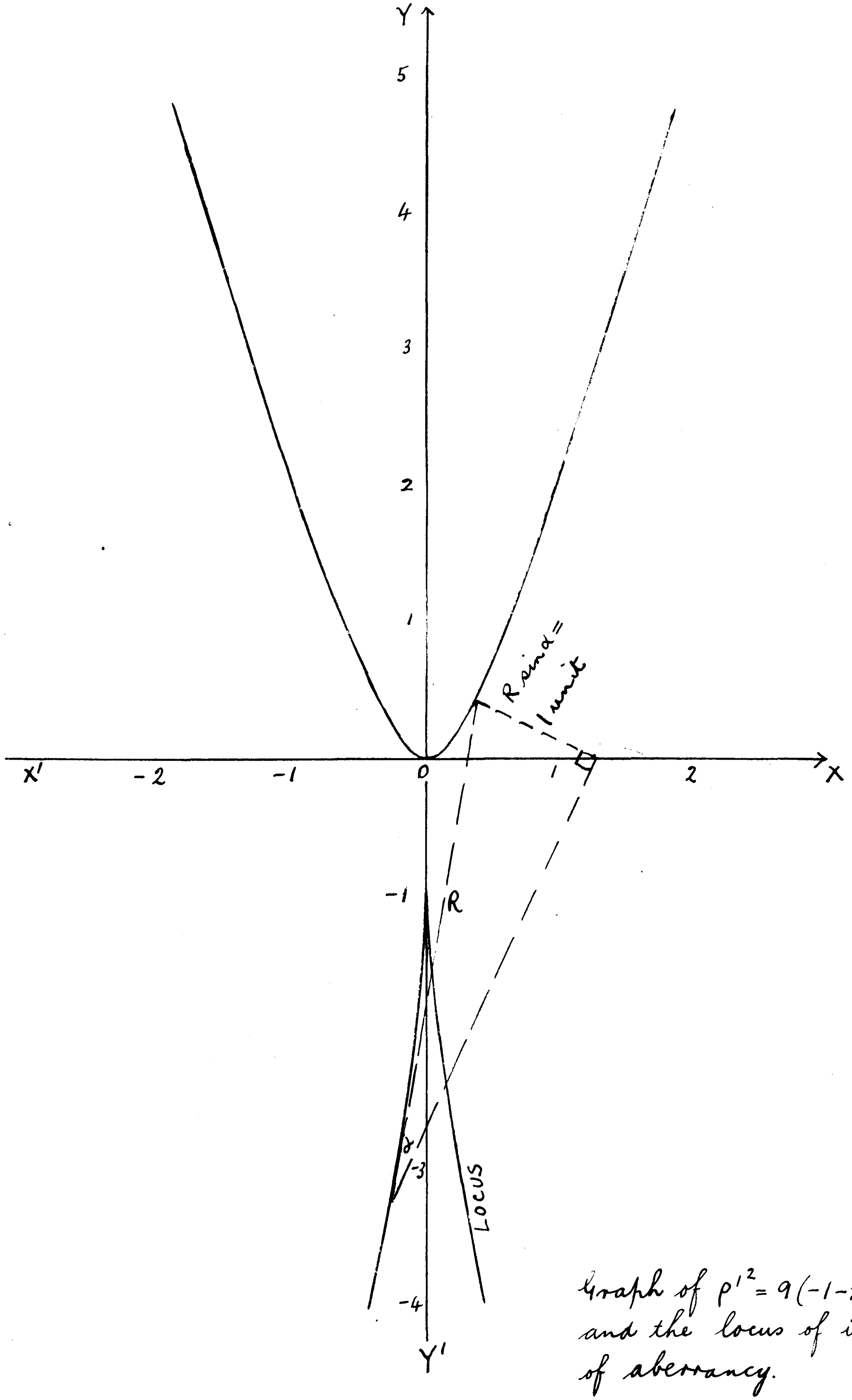
For the Graph of $\rho'^2 = 9(-1-2\rho+3\rho^{2/3})$;

Graph of $\frac{1-\sin\theta}{2-\sin\theta}$, where $\rho^{-\frac{1}{3}} = \sin\theta - 1$,

and $\psi = \int_{-\frac{\pi}{2}}^{\theta} \frac{1-\sin\theta}{2-\sin\theta}\, d\theta$.



For the Graph of $\rho'^2 = 9(-1-2\rho+3\rho^{2/3})$;
$(\psi, \rho\cos\psi)$ and $(\psi, \rho\sin\psi)$ diagrams.



Graph of $p'^2 = 9(-1-2p+3p^{2/3})$, and the locus of its centres of aberrancy.

(3) Graph of the Equation $9 + p'^2 = -18p - 9p^{2/3}$.

The solution may be written $\psi = \int \dfrac{\sqrt{w}\,dw}{\sqrt{(1-w)(w^2+w+2)}}$, $(w = -p^{-1/3})$.

If we put λ for $\dfrac{\sqrt{w}}{\sqrt{(1-w)(w^2+w+2)}}$, then $\psi = \int \lambda\,dw$.

λ is zero when $w = 0$, and infinite when $w = 1$, and is real only for values of w such that $0 \leqq w \leqq 1$. Hence p must be negative, and since $9 + p'^2 - 3pp'' = 9p$, the osculating conics must be hyperbolas.

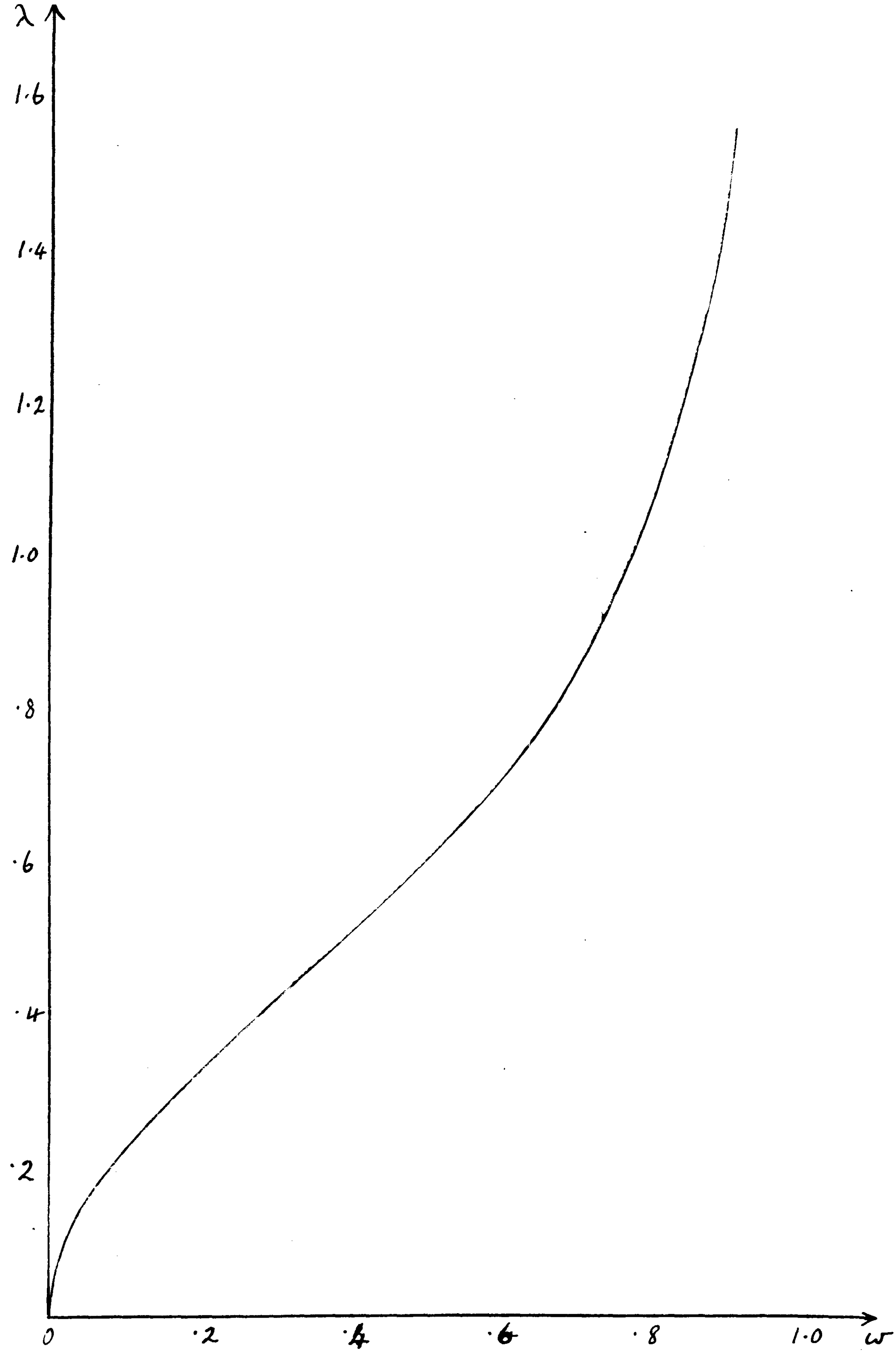
To calculate the increment of ψ from $w = \cdot 9$ to $w = 1$, we substitute $\delta = 1 - w$, and expand λ in terms of δ as far as the term in δ^2. The result is $I\Big]_{\cdot 9}^{1} = \delta^{1/2}\left(1 - \frac{\delta}{24} - \frac{29\delta^2}{640}\right)\Big]_0^{\cdot 1} = \cdot 3148$. The graph of λ may then be used to extend this integral from $w = 0$ to $w = \cdot 9$.

Corresponding values of $p, \psi, p\cos\psi, p\sin\psi$ may then be calculated. From the $(\psi, p\cos\psi)$, $(\psi, p\sin\psi)$ diagrams, corresponding x and y values may be calculated by graphical integration.

Since $\operatorname{cosec}^2\alpha = p^{2/3}(-1 - 2p^{1/3}) = \frac{1}{w^2}\left(-1 + \frac{2}{w}\right)$, values of α corresponding to the w values can be calculated. Finally since the axis of aberrancy has length $\operatorname{cosec}\alpha$, the positions of the centres of aberrancy can be found on the graph.

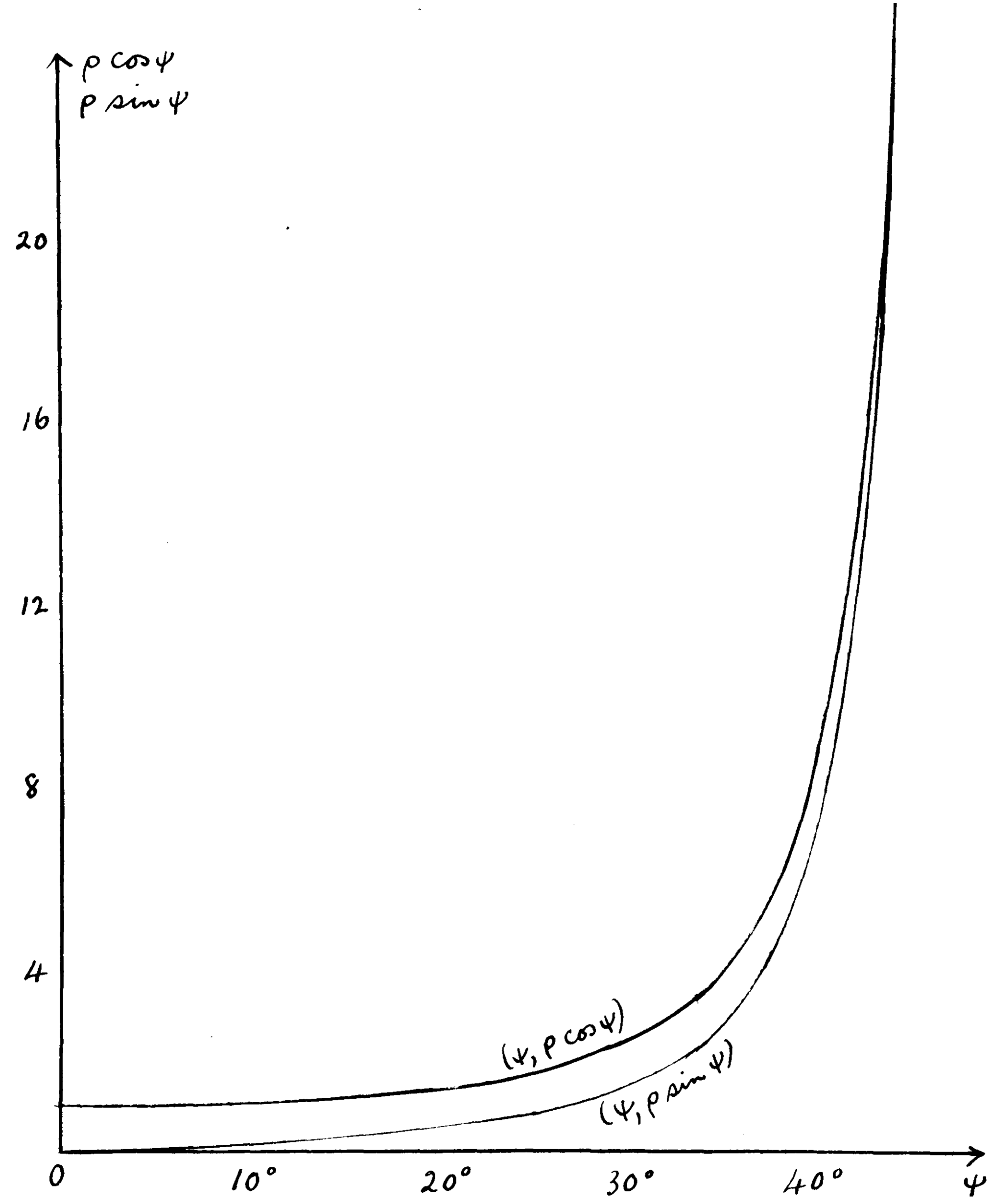
λ	0	·3341	·4235	·5102	·6032	·7119	·8553	1·079	1·558	∞
w	0	·2	·3	·4	·5	·6	·7	·8	·9	1
$-p^{1/3}$	∞	5	$3\frac{1}{3}$	$2\frac{1}{2}$	2	$1\frac{2}{3}$	$1\frac{3}{7}$	$1\frac{1}{4}$	$1\frac{1}{9}$	1
ψ	49°41′	47°13′	45°2′	42°20′	39°8′	35°23′	30°53′	25°20′	18°2′	0°
$p\cos\psi$	∞		26·17	11·54	6·204	3·773	2·502	1·765	1·304	1
$p\sin\psi$	∞		26·20	10·52	5·048	2·680	1·496	·8356	·4245	0
$\alpha°$		3°48′	7°15′	11°32′	16°45′	23°6′	30°54′	40°48′	54°27′	90°

$\psi°$	0	10	20	30	40	41	42	43	44	45
x	0	·19	·41	·73	1·46	1·60	1·77	1·99	2·28	2·66
y	0	·024	·094	·24	·75	·87	1·02	1·21	1·47	1·82

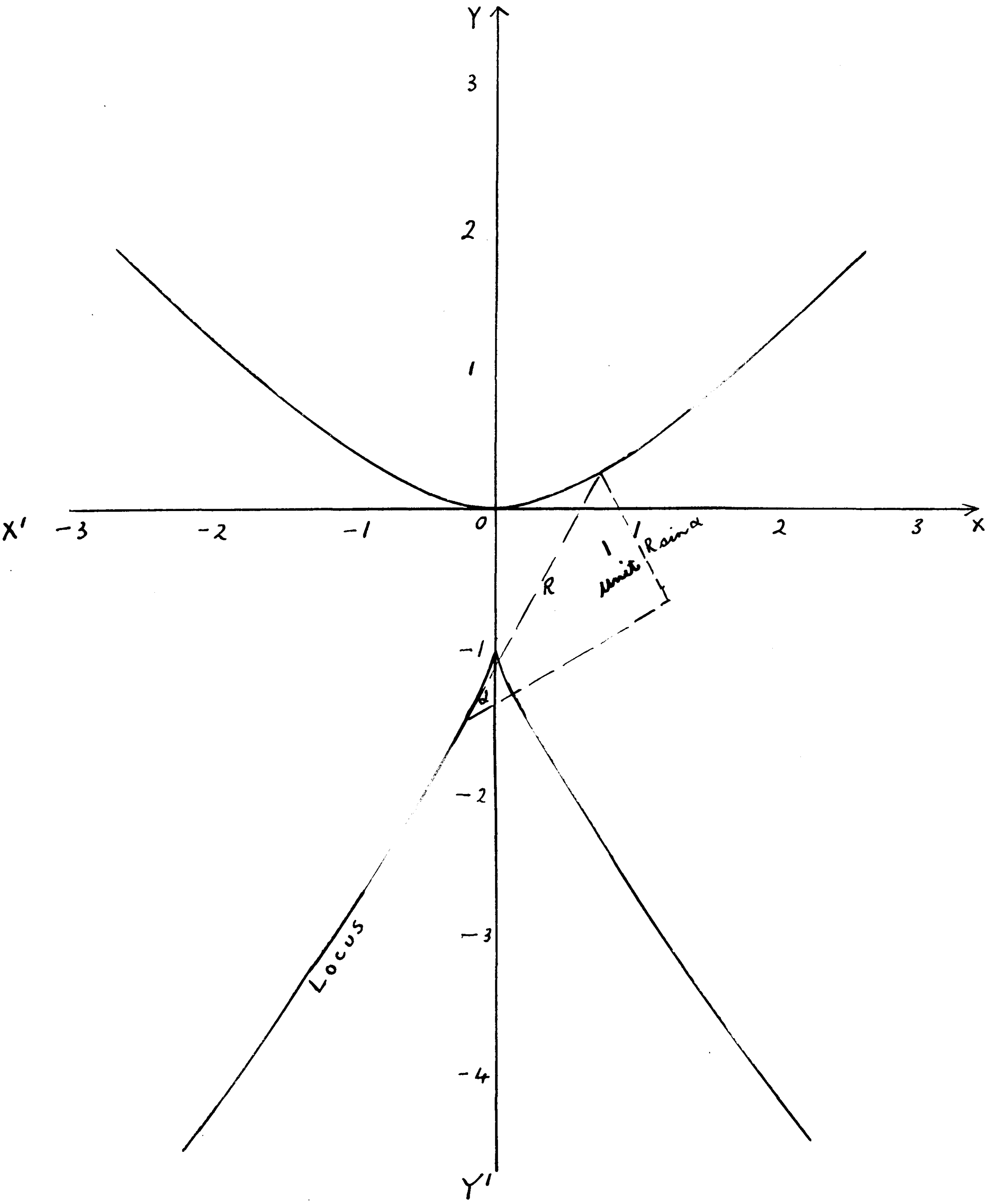


For the graph of $\rho'^2 = -9 - 18\rho - 9\rho^{2/3}$;

Graph of $\lambda = \dfrac{\sqrt{\omega}}{\sqrt{(1-\omega)(\omega^2+\omega+2)}}$, $(\omega = -\rho^{-\frac{1}{3}})$.



For the Graph of $\rho'^2 = -(9 + 18\rho + 9\rho^{2/3})$
$(\psi, \rho\cos\psi)$ and $(\psi, \rho\sin\psi)$ diagrams.



Graph of $\rho'^2 = -(9 + 18\rho + 9\rho^{2/3})$, and the locus of its centres of aberrancy.

(4) Graph of the equation $p'^2 = -9 - 18p + 9cp^{2/3}$, when $c = 5$.

The equation may be written $\psi = \pm \int \frac{v\,dv}{\sqrt{-v(v-2)(v^2+2v-1)}}$, where $v = p^{-1/3}$.

For real solutions we must have either $\sqrt{2}-1 \leqq v \leqq 2$, or $0 \geqq v \geqq -\sqrt{2}-1$.

Let $\lambda = \frac{v}{\sqrt{-v(v-2)(v^2+2v-1)}}$, then λ is infinite when

$v = -\sqrt{2}-1$, $\sqrt{2}-1$, 2, and zero when $v = 0$.

To find $\int \lambda\,dv$ near $v = 2$, we put $v = 2 - \delta$, and find $\int_{1.9}^{2.0} \lambda\,dv$ by expanding λ in powers of δ as

far as the term in δ^2. This gives

$$\int_{1.9}^{2.0} \lambda\,dv = \int_0^{.1} \lambda\,d\delta = \sqrt{\frac{2}{7}} \int_0^{.1} \left(\delta^{-\frac{1}{2}} + \frac{5}{28}\delta^{\frac{1}{2}} + \frac{103}{49.32}\delta^{3/2}\right) d\delta.$$

The result is $\cdot 334$ radian.

Similarly, near $\sqrt{2}-1$, we may substitute $v = \sqrt{2}-1+\delta$, and calculate $\int_{\sqrt{2}-1}^{.5} \lambda\,dv$. The result is $\cdot 185$ radian.

The graph of λ may be used to calculate $\int_{.5}^{1.9} \lambda\,dv$. From the corresponding values of ψ and v, tables of values of $p\cos\psi$, $p\sin\psi$ can be calculated, and from the graphs of these, corresponding pairs of coordinates (x, y) can be found.

The part of the figure which corresponds to v negative can be obtained in a similar way. We find $\int_{-\sqrt{2}-1}^{-2\cdot 3} \lambda\,dv = -\cdot 297$ radian, and this

integral can be extended by the methods of graphical integration from $v = -2\cdot 3$ to $v = 0$.

Since $9 + p'^2 - 3pp'' = 9p$, the osculating conics for that part of the curve corresponding to

v positive are ellipses, and for that part of the curve corresponding to v negative, they are hyperbolas.

Since $\operatorname{cosec}^2\alpha = \rho^{2/3}(5-2\rho^{1/3})$, and since the axes of aberrancy is of length $\operatorname{cosec}\alpha$, it is possible to plot the centres of aberrancy.

Data for the graph of λ.

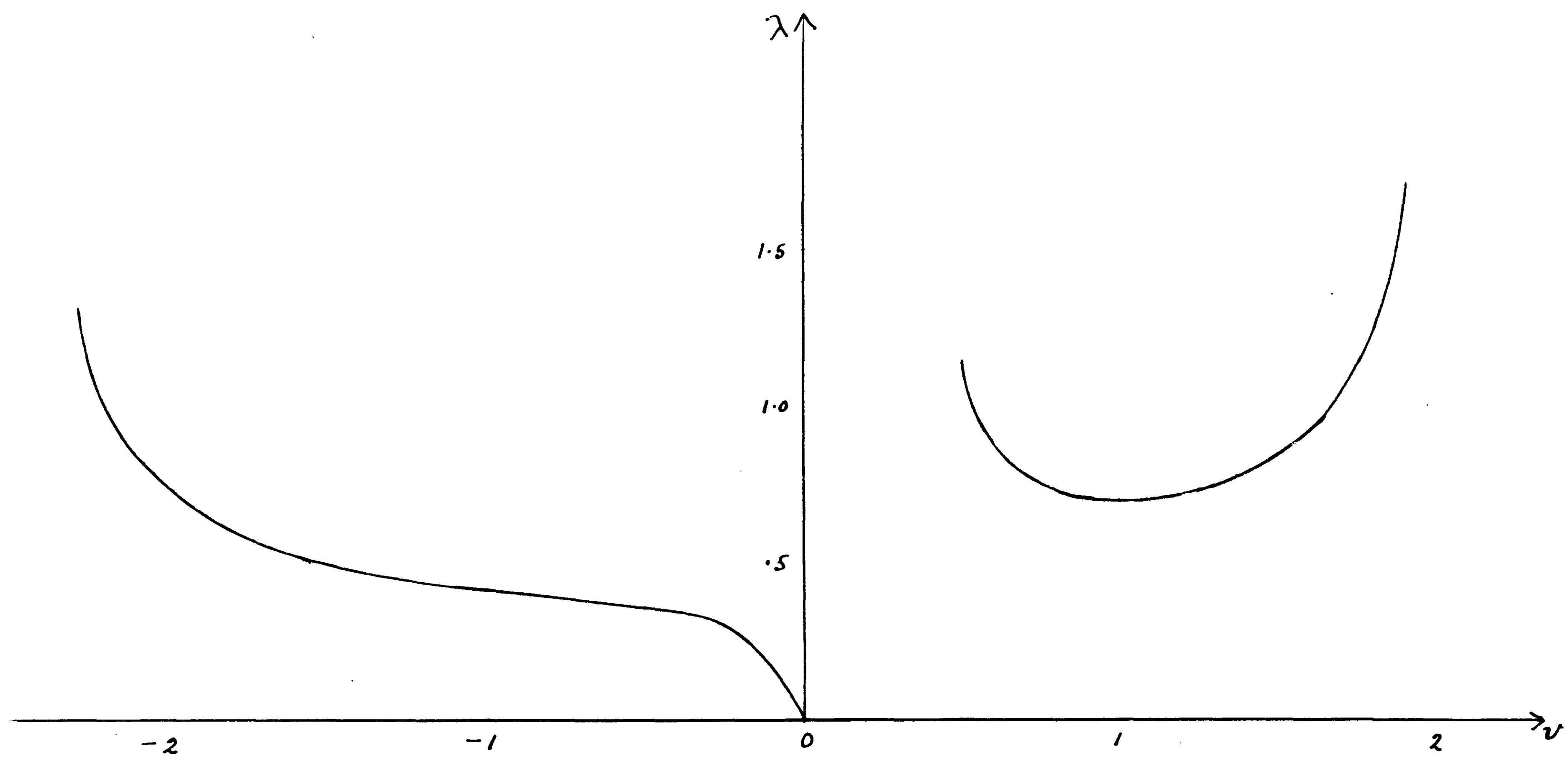
v	·5	·7	·9	1·1	1·3	1·5	1·7	1·9	
λ	1·154	·7778	·7128	·7122	·7514	·8403	1·034	1·721	
v	−2·3	−2·2	−2·0	−1·6	−1·2	−·8	−·4	−·2	0
λ	−1·313	−·9671	−·7071	−·5206	−·4373	−·3819	−·3187	−·2587	0

Data for the case v positive.

v	·4142	·5	·7	·9	1·1	1·3	1·5	1·7	1·9	2·0
$\psi°$	0	10°36′	21°6′	29°36′	37°42′	46°12′	55°12′	66°18′	80°12′	101°18′
$\rho^{1/3}$	2·4142	2	1·4286	1·1111	·9091	·7692	·6667	·5882	·5263	·5
$\rho\cos\psi$	14·06	7·861	2·720	1·192	·5944	·3149	·1692	·0818	·0248	−·0245
$\rho\sin\psi$	0	1·471	1·050	·6774	·4594	·3284	·2434	·1863	·1436	·1227
α	90°	30°	28°34′	35°	41°18′	46°48′	51°36′	60°24′	73°	90°
R	1	2	2·091	1·744	1·516	1·372	1·277	1·150	1·046	1
x	0	2·20	3·07	3·35	3·46	3·51	3·53	3·55	3·56	3·55
y	0	·22	·47	·62	·70	·76	·80	·83	·86	·88

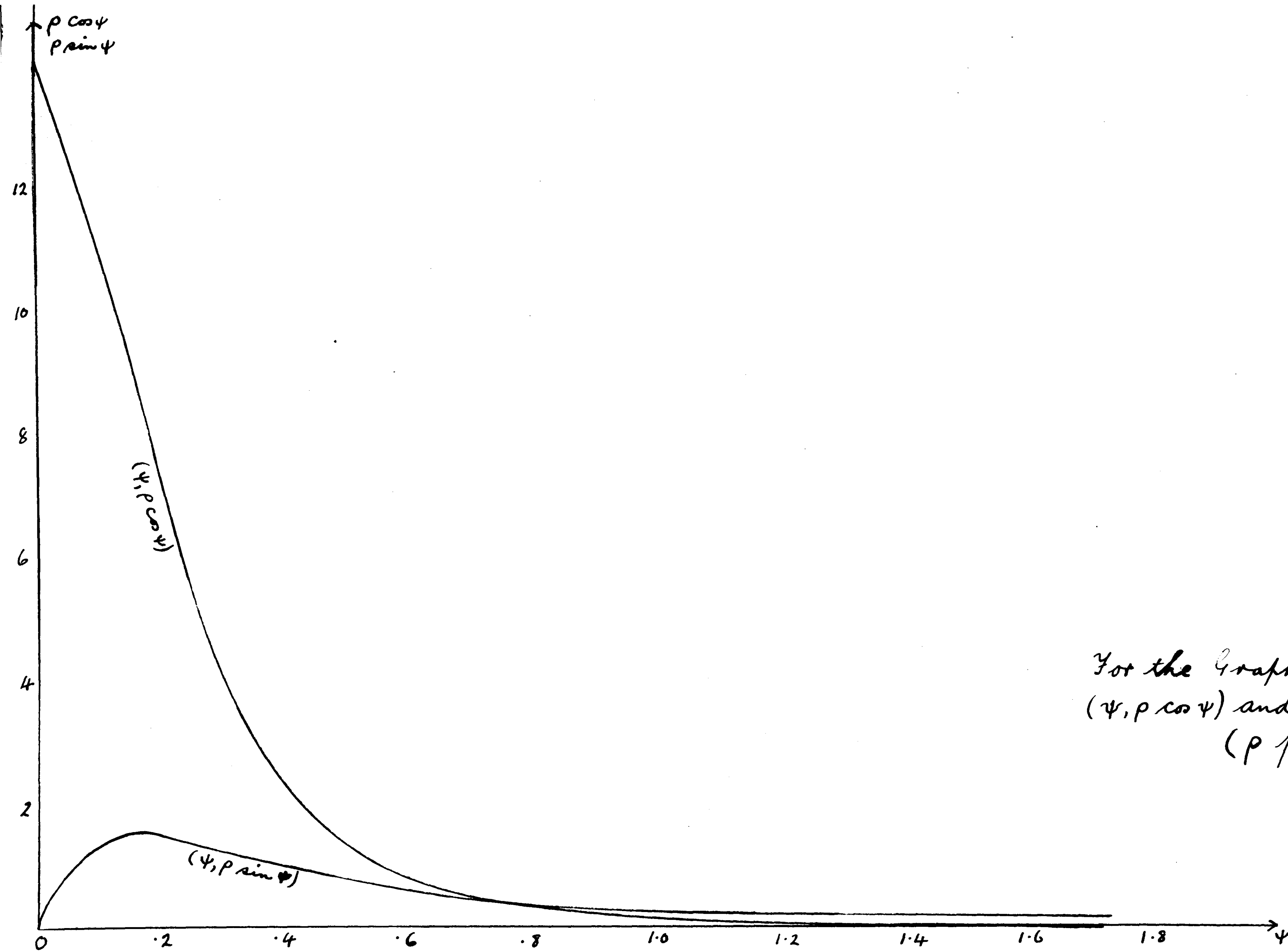
Data for the case v negative.

$-v$	2·4142	2·300	2·25	2	1·75	1·5	1·25	1	·75	·5	·25
$\psi°$	0°	17°	20°24′	32°48′	41°48′	49°18′	56°	62°6′	67°42′	72°54′	77°30′
$-\rho^{1/3}$	·4142	·4348	·4444	·5	·5714	·6667	·8	1	1·3333	2	4
$-\rho\cos\psi$	·0711	·0786	·0823	·1051	·1390	·1972	·2864	·4679	·9	2·352	13·86
$-\rho\sin\psi$	0	·0240	·0306	·0677	·1243	·2248	·4245	·8837	2·194	7·645	62·5
α	90°	71°42′	68°	54°42′	44°54′	36°36′	29°6′	22°12′	15°42′	9°36′	3°48′
R	1	1·0533	1·0785	1·2253	1·4167	1·6772	2·0562	2·6466	3·6955	6	15
x	0	·013	·026	·042	·062	·091	·111	·138	·183	·264	·337
$-y$	0	·001	·005	·013	·027	·056	·082	·127	·210	·402	·637

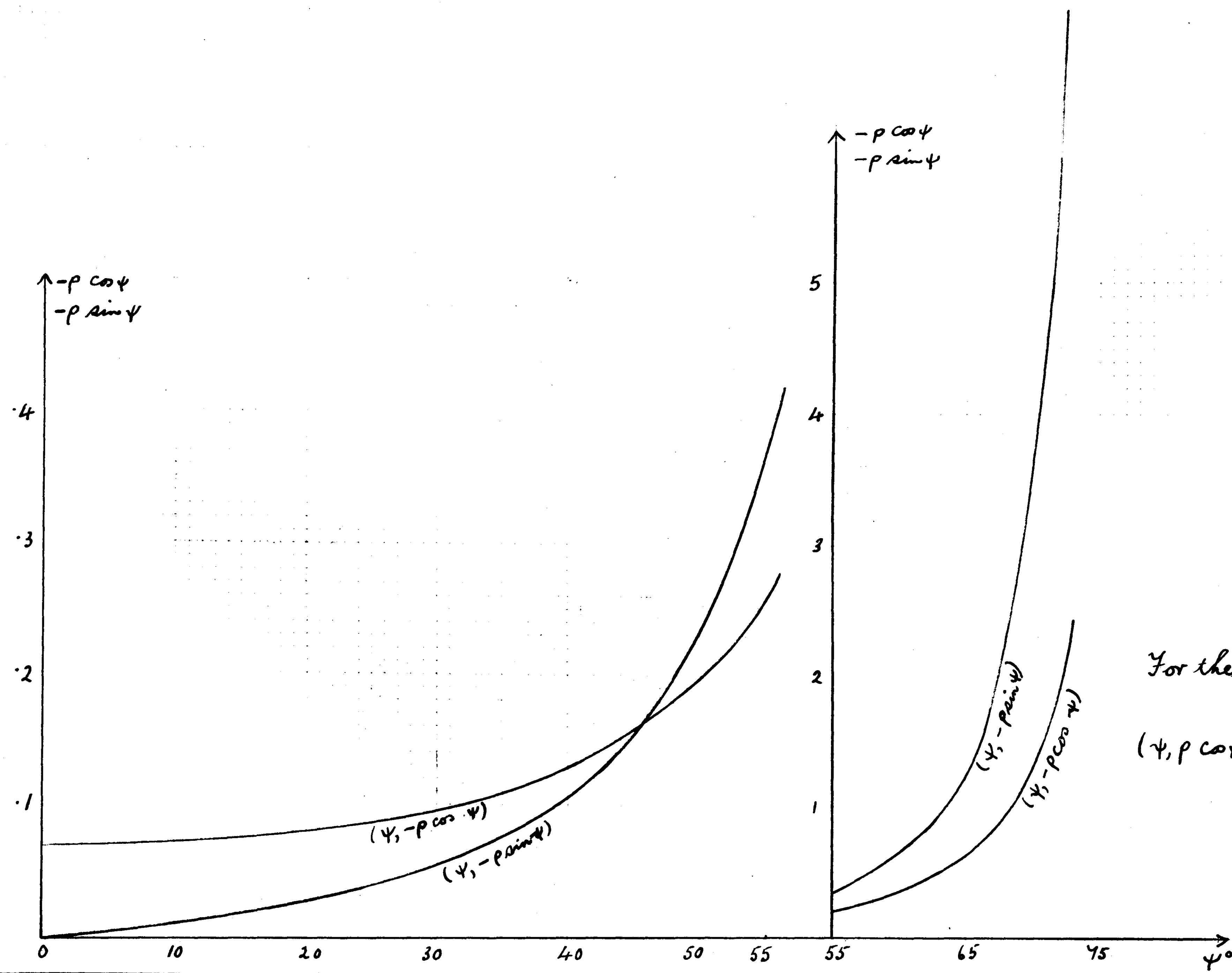


For the Graph of $9+\rho'^2=-18\rho+45\rho^{2/3}$;

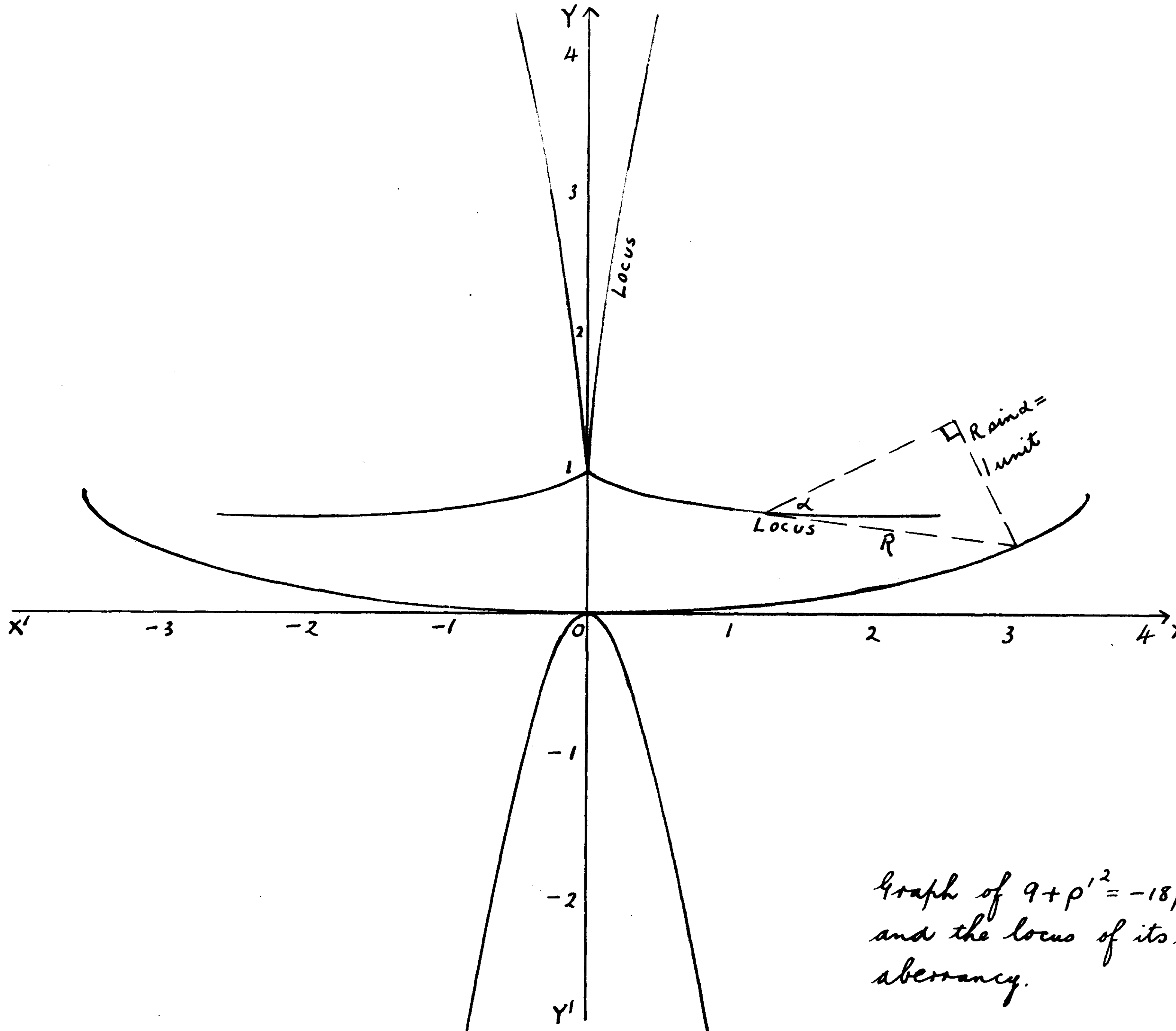
Graph of $\lambda=\dfrac{v}{\sqrt{-v(v-2)(v^2+2v-1)}}$, $(v=\rho^{-\frac{1}{3}})$.



For the Graph of $9+\rho'^2 = -18\rho + 45\rho^{2/3}$;
$(\psi, \rho\cos\psi)$ and $(\psi, \rho\sin\psi)$ diagrams,
(ρ positive).



For the Graph of
$$9+\rho'^2=-18\rho+45\rho^{2/3};$$
$(\psi, \rho\cos\psi)$ and $(\psi, \rho\sin\psi)$ diagrams;
(ρ negative).



Graph of $9+\rho'^2 = -18\rho + 45\rho^{2/3}$, and the locus of its centres of aberrancy.

Section XV.

To find a curve for which the projection of the axis of aberrancy at any point on the corresponding tangent is proportional to the radius of curvature at that point.

(1) The equation is $R\cos\alpha = \dfrac{\rho}{k}$,

$$\text{i.e. } \frac{3\rho\sqrt{9+\rho'^2}}{9+\rho'^2-3\rho\rho''} \times \frac{\rho'}{\sqrt{9+\rho'^2}} = \frac{\rho}{k},$$

$$\text{i.e. } 9+\rho'^2-3\rho\rho'' = 3k\rho'.$$

(2) A first integral solution is obtained by writing $\rho' = u$, $\rho'' = u\dfrac{du}{d\rho}$. The equation becomes

$$9+u^2-3ku = 3\rho u\frac{du}{d\rho},$$

$$\text{i.e. } \frac{2}{3}\frac{d\rho}{\rho} = \frac{du\,(2u-3k+3k)}{u^2-3ku+9},$$

$$\text{i.e. } \tfrac{2}{3}\log c\rho = \log(u^2-3ku+9) + 3k\int\frac{du}{u^2-3ku+9}$$

$$\text{i.e. } \tfrac{2}{3}\log c\rho = \log(u^2-3ku+9) + \frac{k}{\sqrt{1-k^2/4}}\tan^{-1}\frac{u-\frac{3k}{2}}{3\sqrt{1-k^2/4}}, \quad (k^2<4);$$

$$\text{or } = \log(u^2-6u+9) - \frac{6}{u-3}, \qquad (k=2);$$

$$\text{or } = \log(u^2-3ku+9) + \frac{k}{2\sqrt{k^2/4-1}}\log\frac{u-\frac{3k}{2}-3\sqrt{\frac{k^2}{4}-1}}{u-\frac{3k}{2}+3\sqrt{\frac{k^2}{4}-1}},$$

$$\left[(k^2>4),\ \left(u-\tfrac{3k}{2} > 3\sqrt{\tfrac{k^2}{4}-1}\right)\right];$$

$$\text{or } = \log(-u^2+3ku-9) + \frac{k}{2\sqrt{k^2/4-1}}\log\frac{3\sqrt{k^2/4-1}-u+\frac{3k}{2}}{3\sqrt{\frac{k^2}{4}-1}+u-\frac{3k}{2}},$$

$$\left[(k^2>4),\ \left(u-\tfrac{3k}{2} < 3\sqrt{\tfrac{k^2}{4}-1}\right)\right].$$

(3) A second integral solution can be found by writing $d\rho = u\,ds = u\rho\,d\psi$, or $d\psi = \dfrac{d\rho}{u\rho}$. This gives

$$\frac{1}{3}\,d\psi = \frac{du}{9-3ku+u^2},$$

$$\text{or } k\psi = \int\frac{3k\,du}{\left(u-\frac{3k}{2}\right)^2+9\left(1-k^2/4\right)}.$$

We have, therefore

$$3\sqrt{1-k^2/4}\,\tan\sqrt{1-k^2/4}\,\psi = u - \tfrac{3k}{2}, \ (k^2<4);$$

or

$$-\frac{3}{\psi} = u-3, \quad (k=2);$$

or

$$\pm e^{2\sqrt{k^2/4-1}\,\psi} = \frac{u-\frac{3k}{2}-3\sqrt{\frac{k^2}{4}-1}}{u-\frac{3k}{2}+3\sqrt{k^2/4-1}}, \ (k^2>4);$$

(4) These lead to the various forms of the complete solutions, viz.,

$$c\rho = e^{\frac{3k}{2}\psi}\sec^3(\sqrt{1-k^2/4}\,\psi), \quad (k^2<4);$$

$$\text{or} = \frac{e^{3\psi}}{\psi^3}, \quad (k=2);$$

$$\left.\begin{aligned}\text{or} &= e^{\frac{3k}{2}\psi}\operatorname{cosech}^3(\sqrt{k^2/4-1}\,\psi),\\ \text{or} &= e^{\frac{3k}{2}\psi}\operatorname{sech}^3(\sqrt{k^2/4-1}\,\psi),\end{aligned}\right\}\ (k^2>4).$$

---

(5) Since c is obviously a multiplier of ρ, we may take $c=1$ in these equations, except when c is zero or infinite.

When c is infinite, we find the solution $\rho=0$.

When c is zero, we find $u^2-3ku+9=0$,

i.e. $u-\frac{3}{2}k = \pm 3\sqrt{k^2/4-1}$,

the equations of a pair of equiangular spirals.

The particular value zero of k gives the solution $\rho = \sec^3\psi$, the equation of a parabola.

The particular value infinity of k gives the solution $\rho'=0$, the general equation of a circle.

The particular value $k=1$ gives the solution

$$\rho = e^{\frac{3}{2}\psi}\sec^3\tfrac{\sqrt{3}}{2}\psi.$$

The particular value $k=3$, gives the two solutions $\rho = e^{\frac{9}{2}\psi}\operatorname{cosech}^3\frac{\sqrt{5}}{2}\psi$,

and $\rho = e^{\frac{9}{2}\psi}\operatorname{sech}^3\frac{\sqrt{5}}{2}\psi$.

Section XVI.

Examples of curves for which the projection of the axis of aberrancy at any point on the corresponding tangent is proportional to the radius of curvature at that point.

(1) Graph of the equation $c\rho = e^{\frac{3k}{2}\psi} \sec^3 \sqrt{1-k^2/4}\,\psi$, when $c = k = 1$; i.e. of the equation $\rho = e^{\frac{3}{2}\psi} \sec^3 \frac{\sqrt{3}}{2}\psi$.

This gives $u = \rho' = \frac{d\rho}{ds} = \frac{1}{\rho}\frac{d\rho}{d\psi} = \frac{3}{2}\left(1 + \sqrt{3}\tan\frac{\sqrt{3}}{2}\psi\right)$,

$$\rho\rho'' = \frac{du}{d\psi} \qquad = \frac{9}{4}\sec^2\frac{\sqrt{3}}{2}\psi,$$

$$9 + \rho'^2 - 3\rho\rho'' \qquad = \frac{9}{2}\left(1+\sqrt{3}\tan\frac{\sqrt{3}}{2}\psi\right)$$
$$= 3\rho'.$$

We may take values of ψ such that $-\frac{\pi}{2} \leq \frac{\sqrt{3}}{2}\psi \leq \frac{\pi}{2}$. For the equation $\rho = e^{\sqrt{3}\left(\frac{\sqrt{3}}{2}\psi \pm n\pi\right)} \sec^3\left(\frac{\sqrt{3}}{2}\psi \pm n\pi\right)$ gives $\rho = \pm e^{\pm n\pi\sqrt{3}} e^{\frac{3\psi}{2}} \sec^3 \frac{\sqrt{3}}{2}\psi$,

$= \text{const.} \times e^{\frac{3\psi}{2}} \sec^3 \frac{\sqrt{3}}{2}\psi$. The complete figure, therefore, consists of an infinite number of similar branches extending from infinity in the direction $-\frac{\pi}{2} = \frac{\sqrt{3}}{2}\psi$ to infinity in the direction $\frac{\sqrt{3}}{2}\psi = \frac{\pi}{2}$.

If for ψ we write $\psi_1 - \frac{\pi}{3\sqrt{3}}$, we find
$u = \frac{3}{2}\left\{1 + \sqrt{3}\tan\left(\frac{\sqrt{3}}{2}\psi_1 - \frac{\pi}{6}\right)\right\}$, and u vanishes when $\psi_1 = 0$; ρ also has a minimum when $\psi_1 = 0$.

The osculating conics are ellipses for points on the curve which satisfy $2\frac{\pi}{3} \geqq \frac{\sqrt{3}}{2}\psi_1 > 0$, and hyperbolas for points which satisfy $0 > \frac{\sqrt{3}}{2}\psi_1 \geqq \frac{2\pi}{3}$. When $\psi_1 = 0$, the osculating conic is a parabola.

The projection of the axis of aberrancy on the tangent is equal to the radius of curvature.

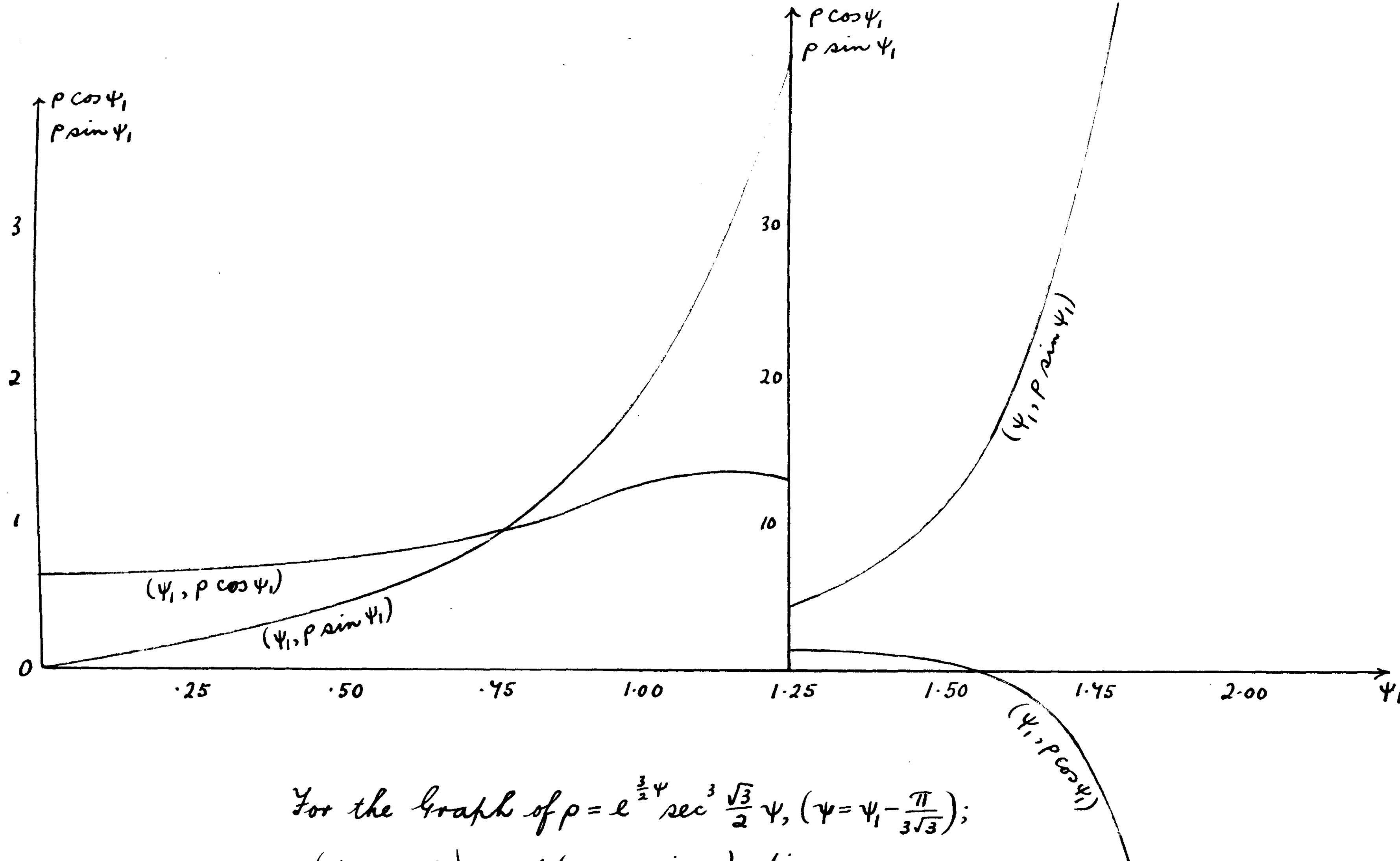
Data for the graph of $\rho = e^{\frac{3}{2}\psi} \sec^3 \frac{\sqrt{3}}{2}\psi$, $\psi = \psi_1 - \frac{\pi}{3\sqrt{3}}$.

ψ_1	0	.2017	.4031	.6045	.8059	1.0073
ρ	.6221	.6588	.7738	1.0000	1.416	2.206
$\rho\cos\psi_1$	.6221	.6454	.7117	.8228	.9806	1.179
$\rho\sin\psi_1$	.0000	.1319	.3036	.5683	1.021	1.865
ψ_1	1.2087	1.4101	1.6115	1.8315	2.0155	
ρ	3.812	7.445	17.38	48.99	207.5	
$\rho\cos\psi_1$	1.353	1.191	−.7074	−11.75	−89.30	
$\rho\sin\psi_1$	3.563	7.350	17.36	47.55	187.4	

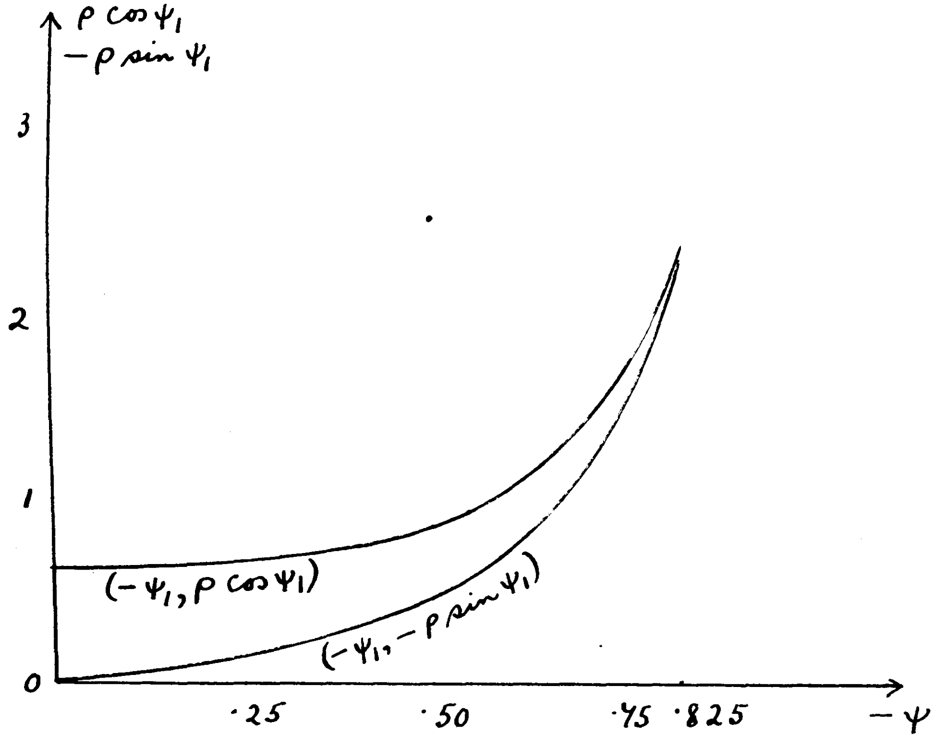
x	0	.16	.34	.55	.82	1.15	1.31	1.45	1.45	1.11	.57
y	0	.02	.09	.25	.58	1.27	1.98	3.03	4.79	8.12	10.68
ψ_1	0°	14°18′	28°36′	43′	57°18′	71°36′	78°48′				
α	90°	77°18′	68°48′	58°37′	51°9′	43°47′	40°14′				
R	∞	3.41	2.39	2.44	3.45	6.00	8.58				

ψ_1	0	−.2011	−.4025	−.6045	−.8065	−1.0073	−1.2087
ρ	.6221	.6642	.8315	1.304	3.012	17.06	∞
$\rho\cos\psi_1$	.6221	.6507	.7649	1.073	2.086	9.109	
$\rho\sin\psi_1$	0	−.1330	−.3263	−.7409	−2.173	−14.42	

x	0	−.16	−.35	−.66	−.81
y	0	.02	.10	.33	.47
ψ_1	0	−14°18′	−28°30′	−43°	−47°16′
α	−90°	−73°44′	−54°	−33°34′	−26°32′
R	∞	2.5	1.70	2.64	4.48

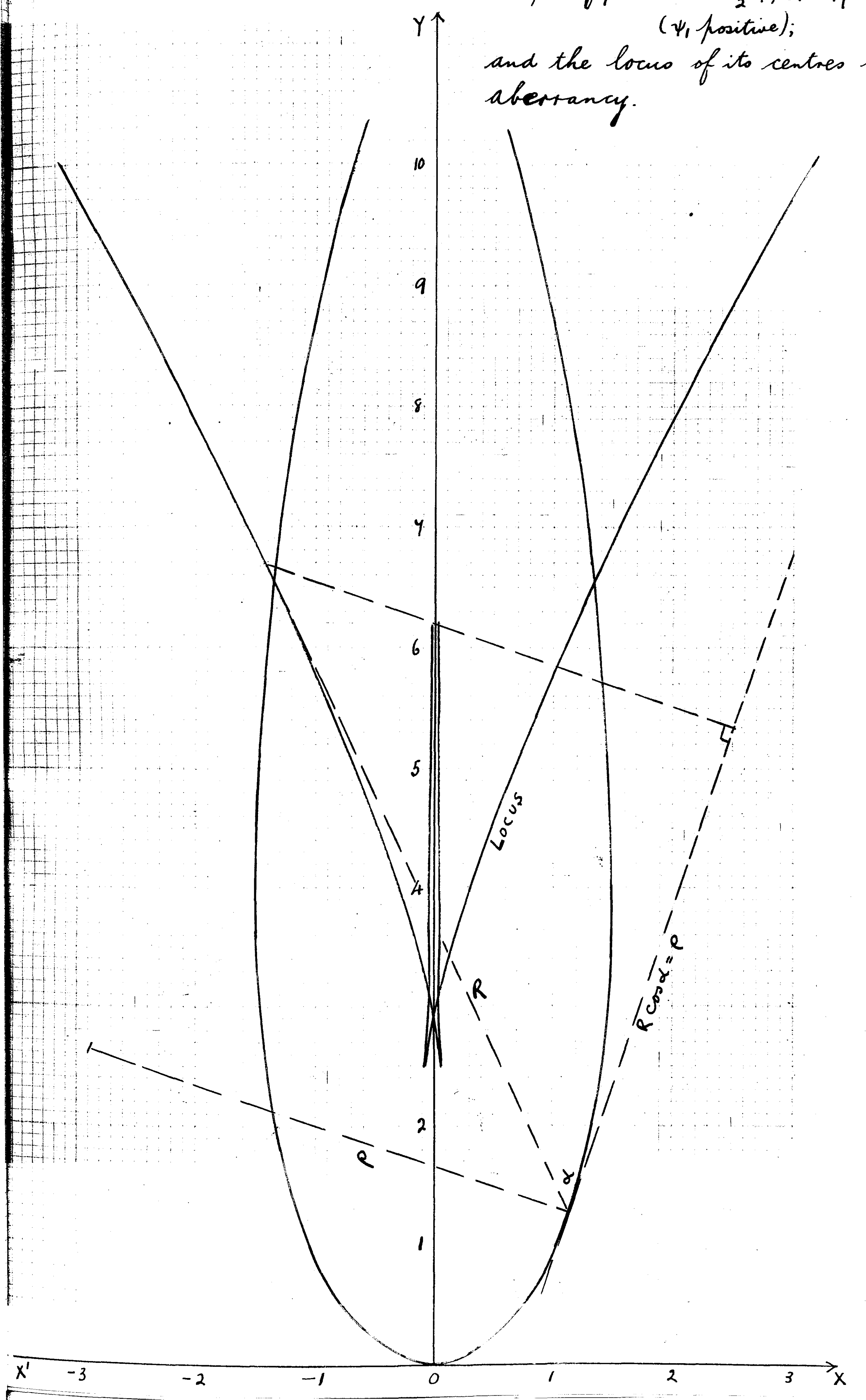


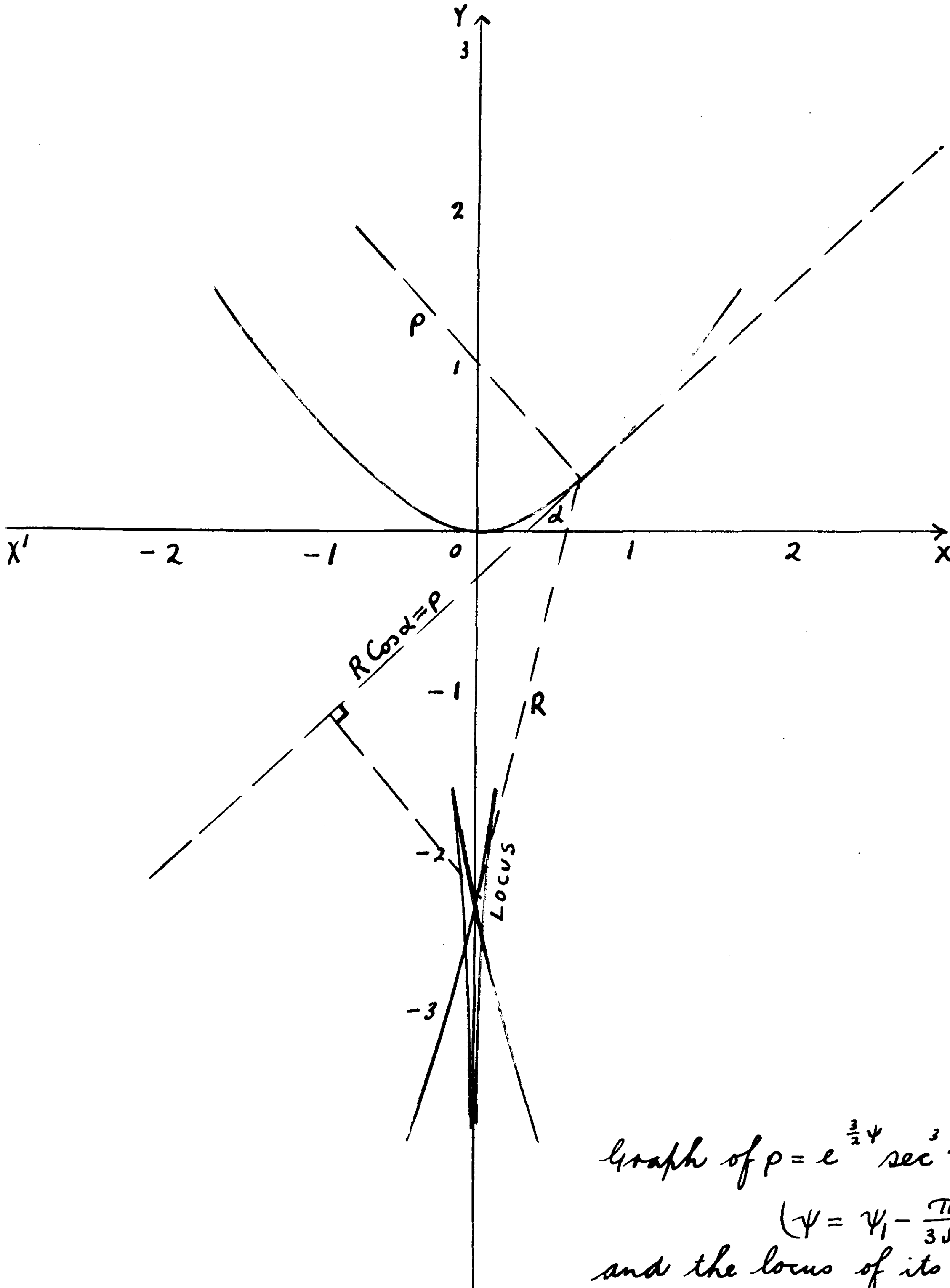
For the Graph of $\rho = e^{\frac{3}{2}\psi} \sec^3 \frac{\sqrt{3}}{2}\psi$, $\left(\psi = \psi_1 - \frac{\pi}{3\sqrt{3}}\right)$;
$(\psi_1, \rho\cos\psi_1)$ and $(\psi_1, \rho\sin\psi_1)$ diagrams;
(ψ_1 positive).



For the Graph of $\rho = e^{\frac{3}{2}\psi} \sec^3 \frac{\sqrt{3}}{2} \psi$, $\left(\psi = \psi_1 - \frac{\pi}{3\sqrt{3}}\right)$;
$(\psi_1, \rho \cos \psi_1)$ and $(\psi_1, \rho \sin \psi_1)$ diagrams;
(ψ_1 negative).

Graph of $\rho = e^{\frac{3}{2}\psi} \sec^3 \frac{\sqrt{3}}{2}\psi$, $(\psi = \psi_1 - \frac{\pi}{3\sqrt{3}})$;
(ψ_1 positive);
and the locus of its centres of aberrancy.





Graph of $\rho = e^{\frac{3}{2}\psi} \sec^3 \frac{\sqrt{3}}{2}\psi$,
$\left(\psi = \psi_1 - \frac{\pi}{3\sqrt{3}}\right)$; ($\psi_1$ neg.),
and the locus of its centres of aberrancy.

(2) Graph of the equation $\rho = \dfrac{e^{3\psi}}{\psi^3}$.

This gives
$$\rho' = u = \frac{1}{\rho}\frac{d\rho}{d\psi} = 3\left(1-\frac{1}{\psi}\right);$$
$$\rho\rho'' = \frac{3}{\psi^2};$$
$$9+\rho'^2-3\rho\rho'' = 18\left(1-\frac{1}{\psi}\right) = 6\rho'.$$

ρ' is positive for all values of ψ except for such that $0 \leqq \psi \leqq 1$. The osculating conics are hyperbolas for points such that $0 \leqq \psi < 1$, and ellipses for points such that $\psi \leqq 0$ or >1. When $\psi = 1$, the osculating conic is a parabola.

When $\psi = 0$, $\rho = \infty$; when $\psi = 1$, $\rho = e^3$; when $\psi = \infty$, $\rho = \infty$; when $\psi = -\infty$, $\rho = 0$. The curve therefore consists of two branches.

Starting with the value $\psi = -\infty$, the current point makes an infinite number of revolutions round the point $\rho = 0$, and finally goes off to infinity in the direction $\psi = 0$.

Starting with the value $\psi = 0$, the current point moves from $\psi = 0$ from infinity with $\rho = \infty$, to the point $\psi = 1$ where ρ has a minimum e^3. It then makes an infinite number of revolutions in a positive direction, with ρ continuously increasing from e^3 to ∞.

The projection of the axis of aberrancy on the tangent is equal to half the length of the corresponding radius of curvature.

Data for the graph of $\rho = \dfrac{e^{3\psi}}{\psi^3}$.

ψ (rad.)	0	·1	·2	·3	·4	·5	·6	·7	·8	·9
$\rho\cos\psi$	∞	1343	223·2	87·02	47·77	31·5	23·12	18·22	14·99	12·69
$\rho\sin\psi$	∞	134·9	45·22	26·92	20·2	17·2	15·82	15·34	15·44	15·99
ψ (rad.)	1·0	1·1	1·2	1·3	1·4	1·5	1·6	1·7	1·8	1·9
$\rho\cos\psi$	10·85	9·241	7·676	6·017	4·130	1·866	−·8624	−4·3	−8·626	−14·08
$\rho\sin\psi$	17·3	18·16	19·73	21·67	23·95	26·61	29·65	33·11	36·98	41·21

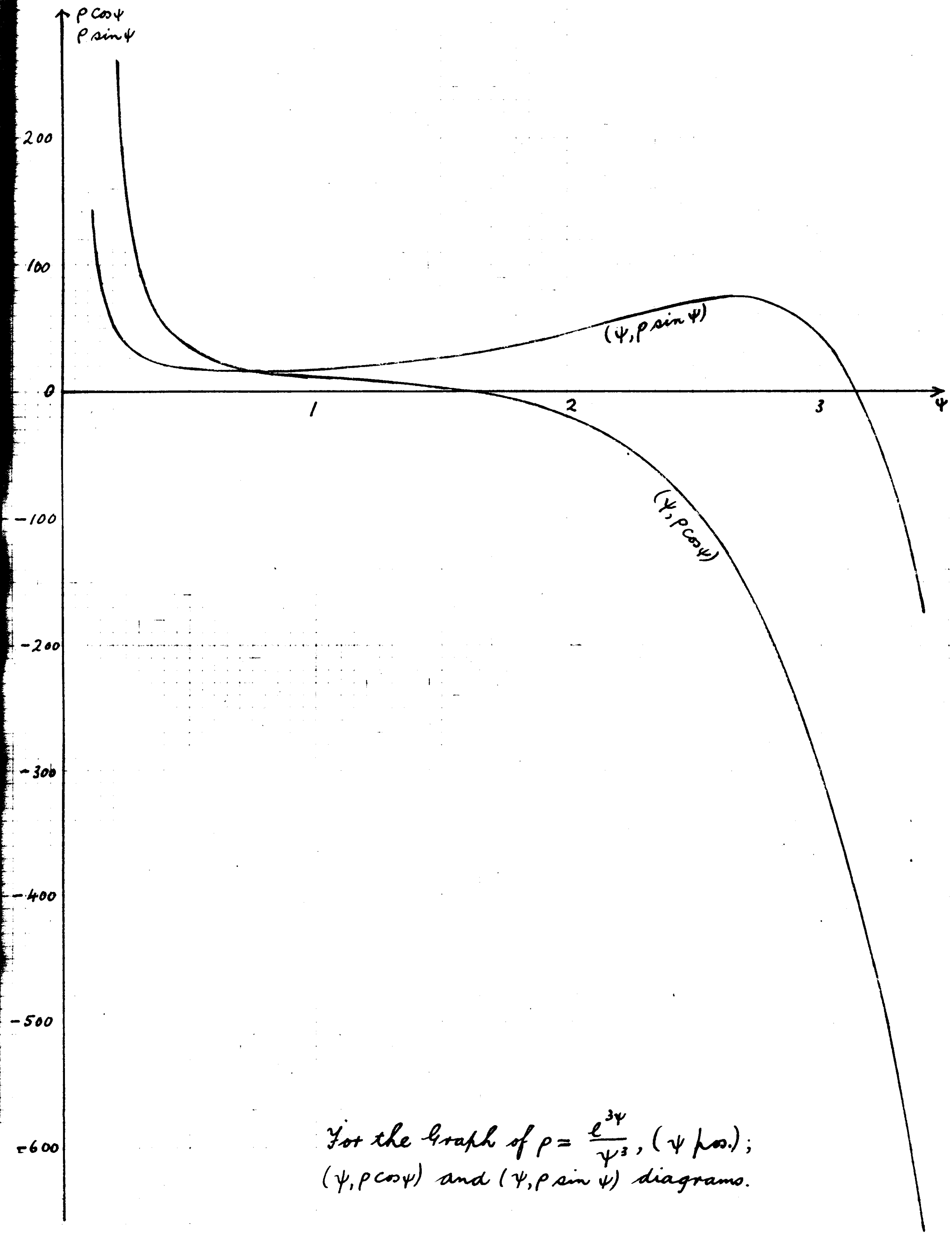
ψ (rad.)	2.0	2.1	2.2	2.3	2.4	2.5	2.6	2.7
$\rho \cos \psi$	−20.99	−29.7	−40.61	−54.34	−71.41	−92.72	−119	−151.2
$\rho \sin \psi$	45.85	50.76	55.85	60.82	65.45	69.28	71.53	71.50
ψ (rad.)	2.8	2.9	3.14	3.2	3.4	3.6		
$\rho \cos \psi$	−190.8	−297.2	−397.9	−449.9	−661.6	−942.1		
$\rho \sin \psi$	67.81	42.30	0	−26.19	−174.8	−464.9		

$\frac{x}{50}$	0	.17	.48	.64	.71	.64	.47	.13	−.51
$\frac{y}{50}$	0	.04	.16	.34	.55	.92	1.19	1.52	1.88
ψ°	11°27′	14°19′	28°39′	57°18′	85°57′	114°36′	128°55′	143°14′	157°34′
α°	−14°2′	−18°26′	−45°	±90°	71°34′	63°26′	60°58′	59°2′	57°32′
$\frac{R}{50}$	2.35		.51	∞	.84	1.13		2.25	
$\frac{x}{50}$	−1.64	−2.31	−3.12	−4.13	−5.34				
$y/50$	2.18	2.24	2.23	2.11	1.85				
ψ°	171°54′	177°38′	183°22′	189°6′	194°50′				
α°	56°18′	55°53′	55°30′	55°5′	54°47′				
$R/50$	5.41		7.96		11.87				

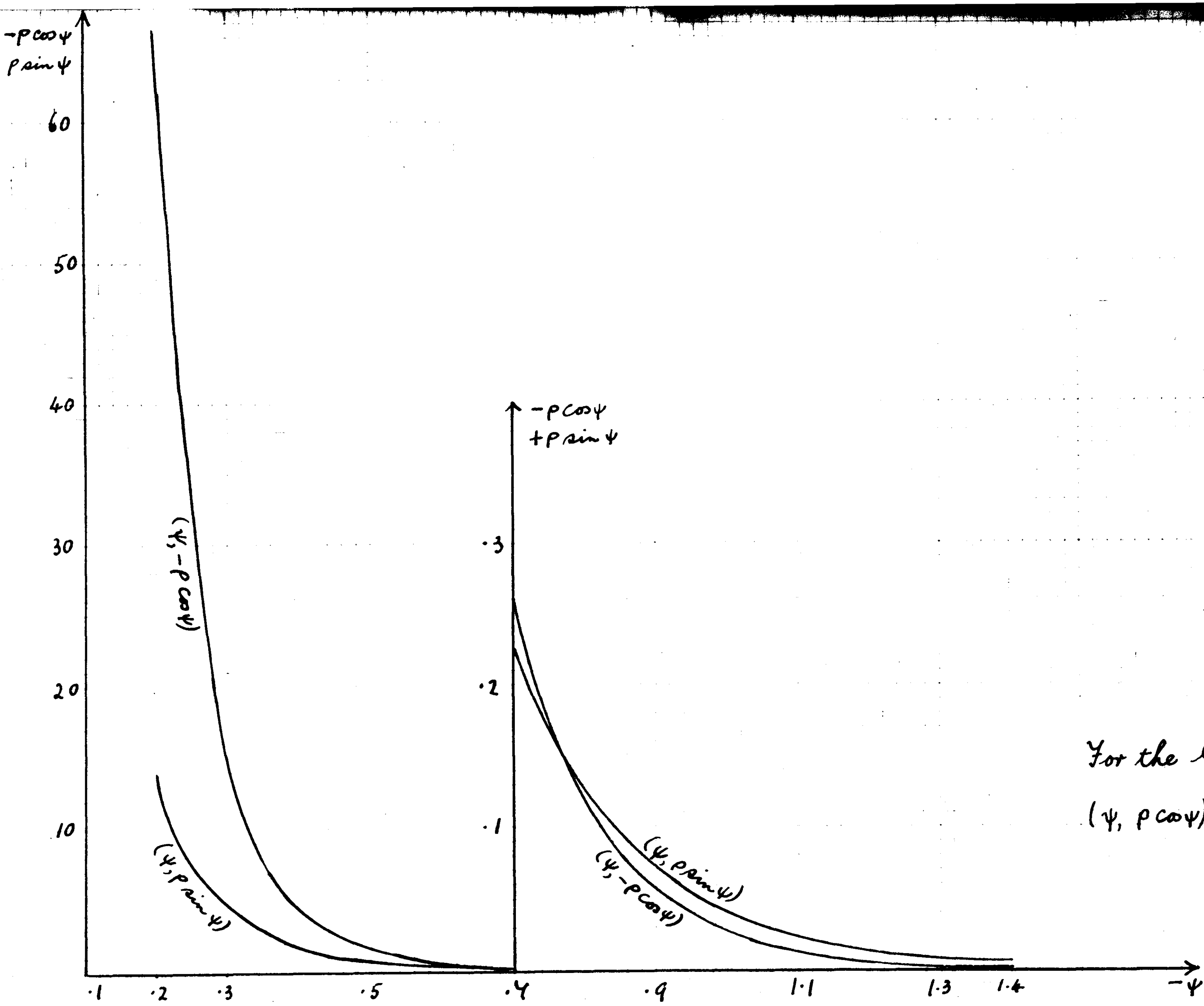
When ψ is negative we may take $\rho = -\dfrac{e^{3\psi}}{\psi^3}$.

−ψ	.1	.2	.3	.4	.5	.6	.7	.8
$\rho \cos \psi$	928.7	67.26	14.37	4.336	1.567	.6308	.2733	.1234
$-\rho \sin \psi$	93.26	13.62	4.446	1.833	.8555	.4316	.2301	.1272
−ψ	.9	1.0	1.1	1.2	1.3	1.4		
$\rho \cos \psi$	.0579	.0269	.0122	.00573	.0025	.0009		
$-\rho \sin \psi$	.0730	.0419	.0240	.0147	.0089	.0054		

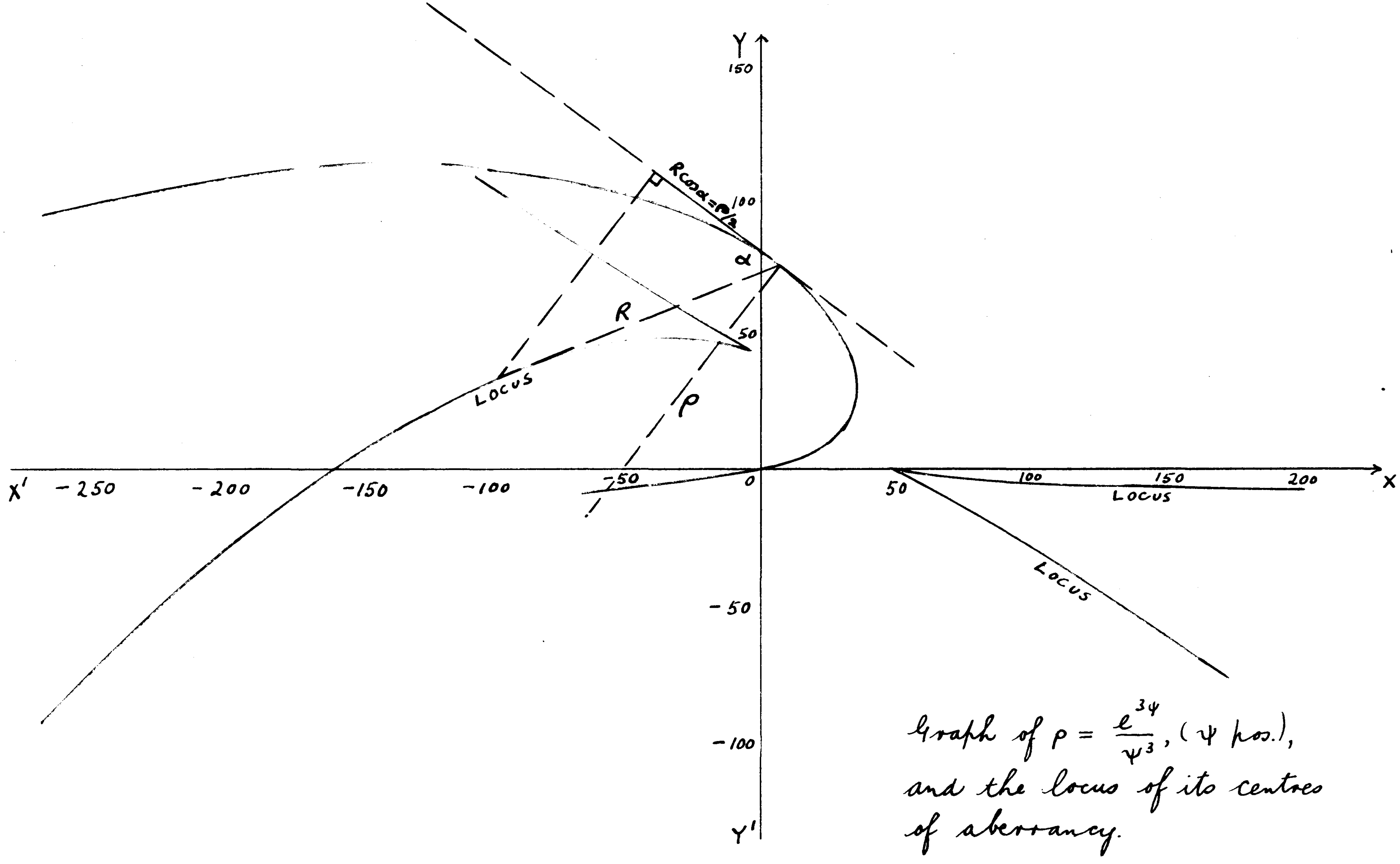
−x	0	3.5	4.31	4.59	4.70	4.74	4.77	4.77	4.78	4.78
y	0	.76	1.05	1.17	1.22	1.25	1.28	1.28	1.29	1.29
−ψ°	11°27′	17°11′	22°55′	28°39′	34°23′	40°6′	51°34′	63°1′	74°29′	80°13′
α°	9°28′	13°	15°54′	18°26′	20°33′	22°23′	25°21′	27°39′	29°29′	30°15′
R	34.8	7.72	2.45	.94	.40	.19	.05			



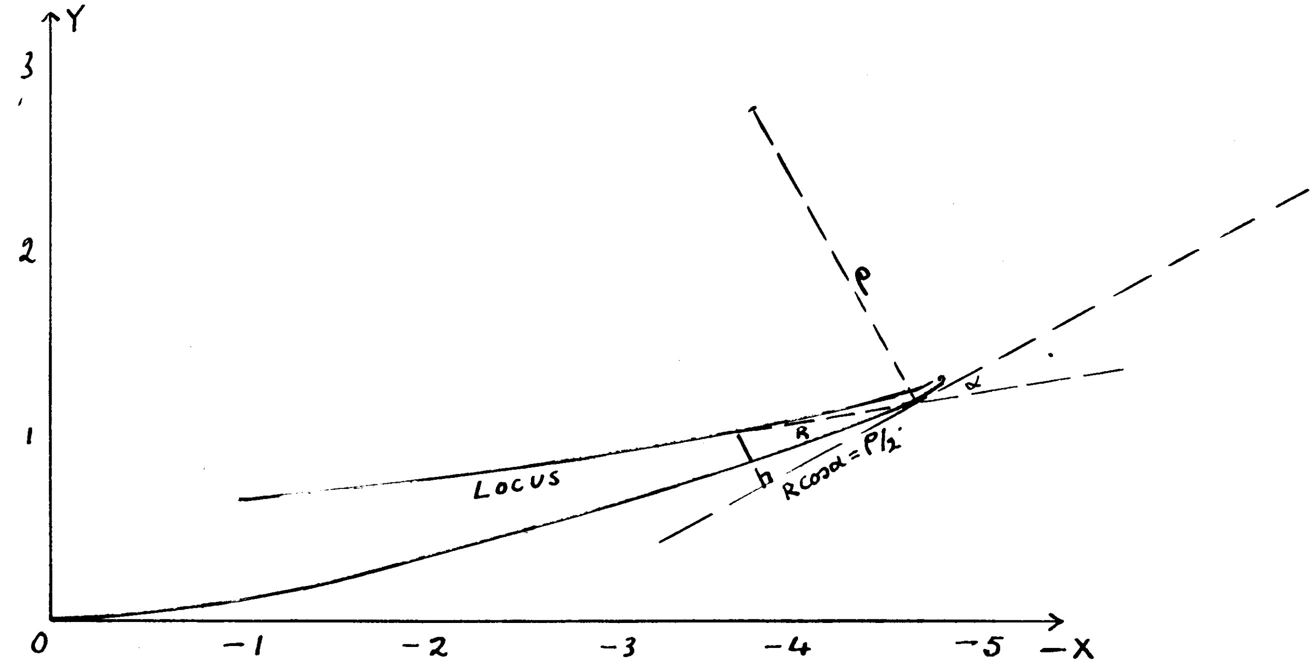
For the Graph of $\rho = \frac{e^{3\psi}}{\psi^3}$, ($\psi$ pos.); ($\psi, \rho\cos\psi$) and ($\psi, \rho\sin\psi$) diagrams.



For the Graph of $\rho = \dfrac{e^{3\psi}}{\psi^3}$, ($\psi$ neg.); (ψ, $\rho\cos\psi$) and (ψ, $\rho\sin\psi$) diagrams.



Graph of $\rho = \dfrac{e^{3\psi}}{\psi^3}$, ($\psi$ pos.), and the locus of its centres of aberrancy.



Graph of $\rho = -\frac{e^{3\psi}}{\psi^3}$, ($\psi$ neg.), and the locus of its centres of aberrancy.

(3) Graph of the equation $\rho = e^{\frac{9}{2}\psi} \operatorname{cosech}^3 \frac{\sqrt{5}}{2}\psi$.

This gives $u = \rho' = \frac{1}{\rho}\frac{d\rho}{d\psi} = \frac{9}{2} - \frac{3\sqrt{5}}{2} \coth \frac{\sqrt{5}}{2}\psi$,

$$\begin{aligned} \rho\rho'' &= \tfrac{15}{4} \operatorname{cosech}^2 \tfrac{\sqrt{5}}{2}\psi, \\ 9 + \rho'^2 - 3\rho\rho'' &= 9\left(\tfrac{9}{2} - \tfrac{3\sqrt{5}}{2} \coth \tfrac{\sqrt{5}}{2}\psi\right), \\ &= 9\rho'. \end{aligned}$$

ρ' is positive for all negative values of ψ, and for all positive values of ψ such that $e^{\sqrt{5}\psi} > \frac{3+\sqrt{5}}{3-\sqrt{5}}$, i.e. $\psi > \cdot 8606$.

When $e^{\sqrt{5}\psi} = \frac{3+\sqrt{5}}{3-\sqrt{5}}$, $u = 0$

When $0 \leqq \sqrt{5}\psi < \log_e \frac{3+\sqrt{5}}{3-\sqrt{5}}$, u is negative.

The osculating conic at $\psi = \frac{1}{\sqrt{5}} \log_e \frac{3+\sqrt{5}}{3-\sqrt{5}}$ is a parabola; at any point such that $0 \leqq \psi < \cdot 8606$ it is a hyperbola; and at any other point it is an ellipse.

The equation for ρ may be written

$\rho^{-\frac{1}{3}} = e^{-\frac{3}{2}\psi}\left(e^{\frac{\sqrt{5}}{2}\psi} - e^{-\frac{\sqrt{5}}{2}\psi}\right)$, neglecting the factor 2,

i.e. $\rho^{-\frac{1}{3}} = e^{-\cdot 382\psi} - e^{-2\cdot 618\psi}$, from which values of $\rho \cos\psi$ and $\rho \sin\psi$ can be calculated.

Data for the graph of $\rho = \frac{1}{8} e^{\frac{9}{2}\psi} \operatorname{cosech}^3 \frac{\sqrt{5}}{2}\psi$.

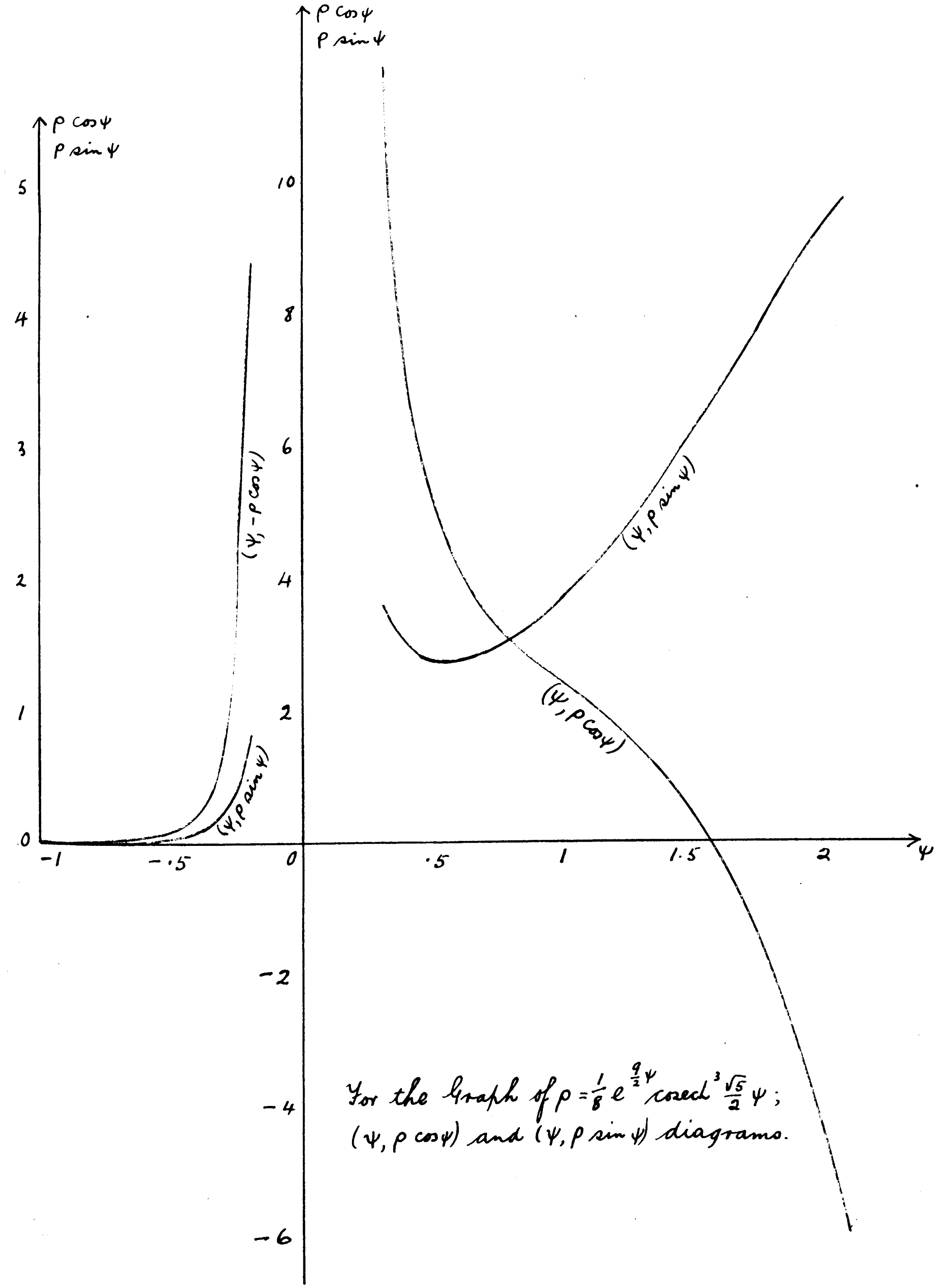
ψ	0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1·0
$\rho\cos\psi$	∞	137·8	26·27	11·52	7·076	5·110	4·091	3·448	3·011	2·681	2·406
$\rho\sin\psi$	∞	13·84	5·322	3·562	2·992	2·789	2·793	2·904	3·100	3·378	3·758
ψ	1·1	1·2	1·3	1·4	1·5	1·6	1·7	1·8	1·9	2·0	2·1
$\rho\cos\psi$	2·090	1·775	1·404	·9661	·4383	−·199	−·968	−1·904	−2·978	−4·285	−5·753
$\rho\sin\psi$	4·108	4·564	5·058	5·603	6·183	6·825	7·454	8·160	8·714	9·358	9·831

$-\psi$	0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1·0
$-\rho\cos\psi$	∞	56·6	4·34	·775	·193	·0567	·0169	·0063	·00225	·00081	·00029
$\rho\sin\psi$	∞	5·68	·878	·240	·0815	·0309	·0116	·00546	·00232	·00103	·00048

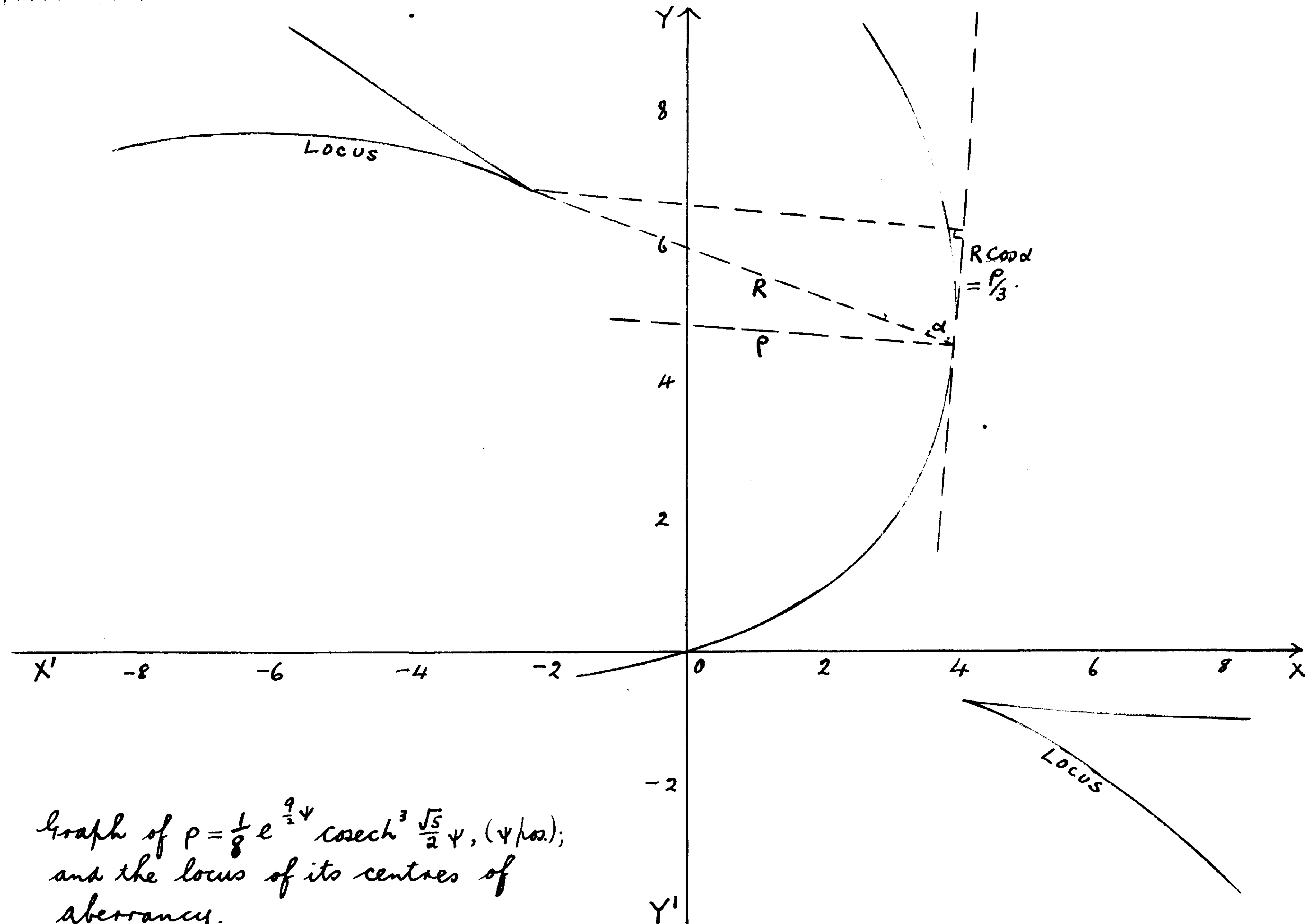
x	0	.88	1.48	2.31	2.91	3.39	3.75	3.93
y	0	.33	.61	1.18	1.81	2.56	3.47	4.59
ψ	17°11′	22°55′	28°39′	40°6′	51°34′	63°1′	74°29′	85°57′
α	−27°3′	−40°34′	−54°51′	−78°2′	88°	80°4′	75°55′	73°12′
R	4.51	3.37	3.37	7.25	41.1	8.91	6.71	6.56

x	3.95	3.89	3.75	3.51	3.15	2.64
y	5.24	5.96	6.74	7.59	8.49	9.45
ψ	91°40′	97°24′	103°8′	108°52′		
α	72°18′	71°44′	71°7′	70°40′		
R	6.80	7.26	7.74	8.43		

Since $\rho' = u = 3\cot\alpha$, it follows that $\cot\alpha = \frac{3}{2} - \frac{\sqrt{5}}{2}\coth\frac{\sqrt{5}}{2}\psi$, from which α can be calculated. Then since $R = \frac{\rho}{3}\sec\alpha$, values of R can be calculated. The centres of aberrancy can then be found corresponding to any of the above points on the curve from a knowledge of R and $\psi + \alpha$.



For the Graph of $\rho = \frac{1}{8} e^{\frac{9}{2}\psi} \operatorname{cosech}^3 \frac{\sqrt{5}}{2}\psi$; $(\psi, \rho\cos\psi)$ and $(\psi, \rho\sin\psi)$ diagrams.



Graph of $\rho = \frac{1}{8} e^{\frac{9}{2}\psi} \operatorname{cosech}^3 \frac{\sqrt{5}}{2}\psi$, ($\psi$ pos.); and the locus of its centres of aberrancy.

(4) Graph of the equation $\rho = \frac{1}{8} e^{\frac{9}{2}\psi} \operatorname{sech}^3 \frac{\sqrt{5}}{2}\psi$.

This gives
$$u = \rho' = \frac{1}{\rho}\frac{d\rho}{d\psi} = \frac{9}{2} - \frac{3\sqrt{5}}{2} \tanh \frac{\sqrt{5}}{2}\psi,$$
$$\rho\rho'' = -\frac{15}{4} \operatorname{sech}^2 \frac{\sqrt{5}}{2}\psi,$$
$$9 + \rho'^2 - 3\rho\rho'' = 9\left(\frac{9}{2} - \frac{3}{2}\sqrt{5} \tanh \frac{\sqrt{5}}{2}\psi\right),$$
$$= 9\rho'$$

ρ' is positive for all values of ψ, positive and negative. at all points of the curve, the osculating conics are ellipses.

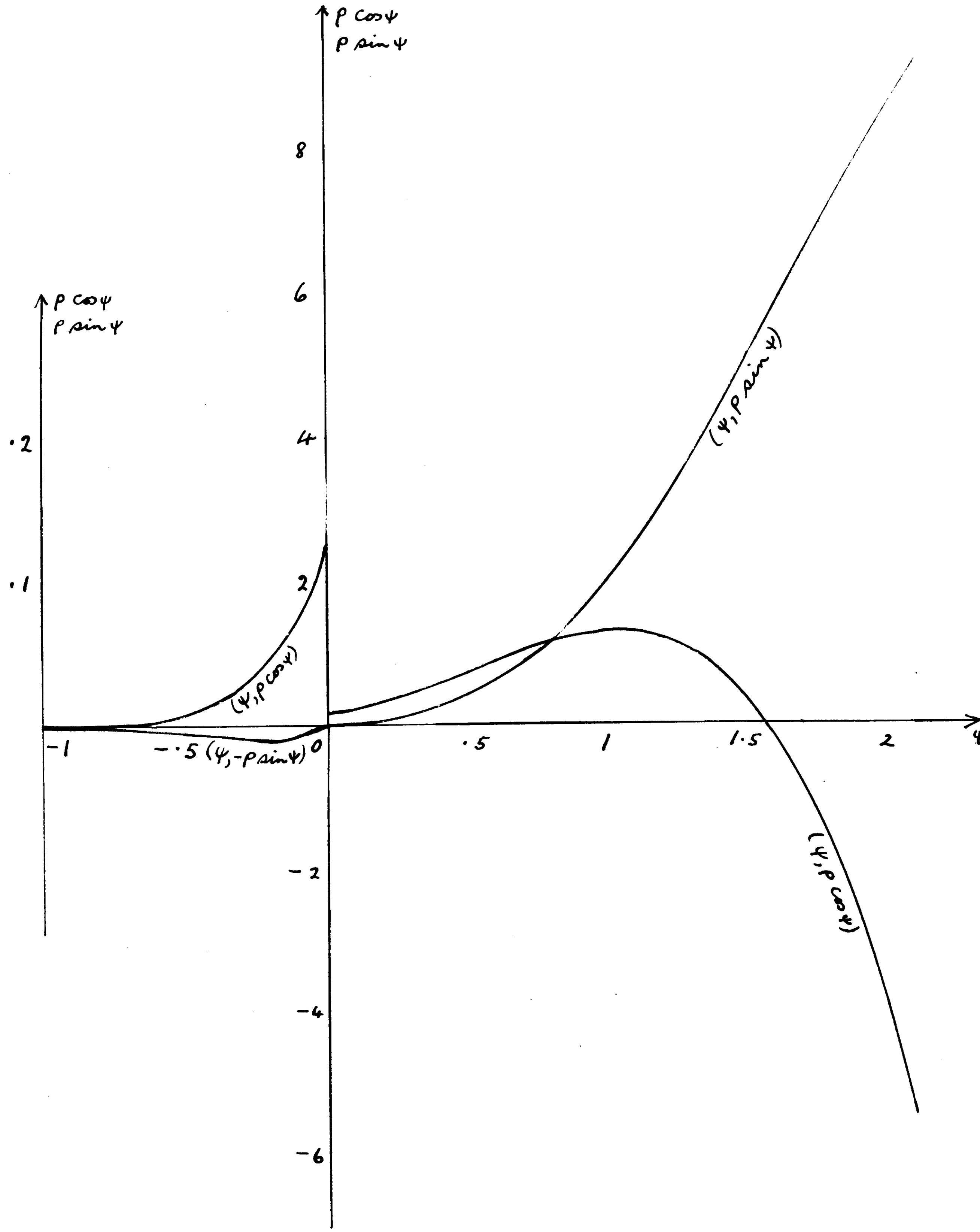
The equation for ρ may be written
$$\rho^{-\frac{1}{3}} = e^{-3/2\psi}\left(e^{\frac{\sqrt{5}}{2}\psi} + e^{-\frac{\sqrt{5}}{2}\psi}\right) = e^{-.382\psi} + e^{-2.618\psi}.$$

Data for the graph of $\rho = \frac{1}{8} e^{\frac{9}{2}\psi} \operatorname{sech}^3 \frac{\sqrt{5}}{2}\psi$.

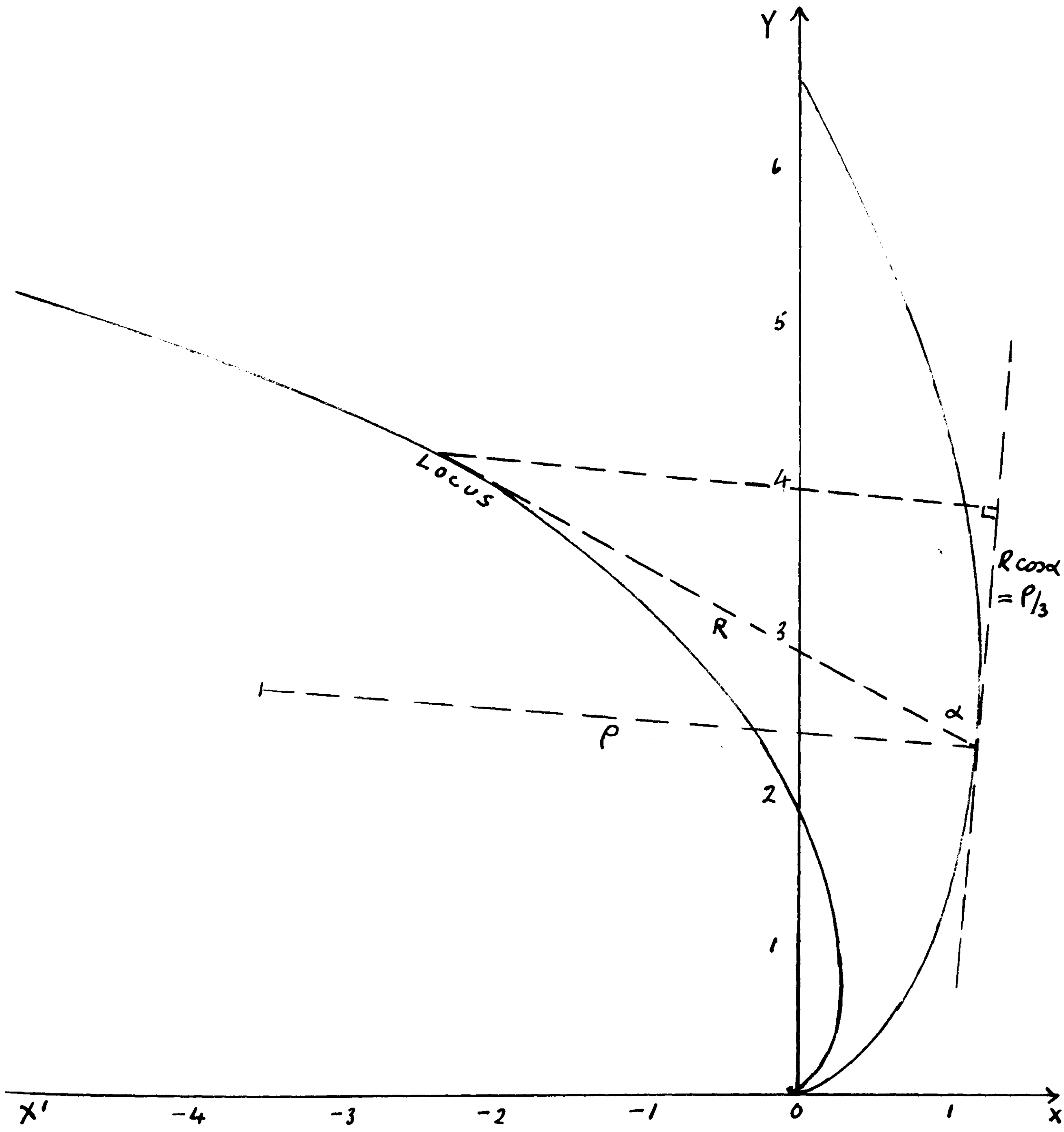
ψ	0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1·0
$\rho\cos\psi$	·125	·192	·280	·390	·522	·665	·819	·966	1·096	1·196	1·254
$\rho\sin\psi$	0	·0192	·0567	·121	·221	·363	·560	·816	1·128	1·508	1·954
ψ	1·1	1·2	1·3	1·4	1·5	1·6	1·7	1·8	1·9	2·0	2·1
$\rho\cos\psi$	1·249	1·175	1·011	·744	·355	−·167	−·846	−1·695	−2·733	−3·966	−5·457
$\rho\sin\psi$	2·456	3·022	3·642	4·315	5·012	5·754	6·515	7·266	7·998	8·662	9·315

$-\psi$	0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1·0
$\rho\cos\psi$	·125	·0779	·0463	·0262	·0142	·0074	·0037	·0018	·0008	·0004	·00015
$-\rho\sin\psi$	0	·0078	·0094	·0081	·0060	·0040	·0025	·0015	·0008	·0005	·00024

x	−·025	−·025	−·02	0	·07	·25	·56	·93
y	·004	·0035	·002	0	·01	·11	·41	1·07
ψ°	−57°18′		−17°12′	0	17°12′	34°24′	51°34′	68°45′
α°	22°36′		28°14′	33°42′	41°18′	49°48′	57°	62°26′
R	·0001		·0104	·0501	·181	·397	1·178	2·335
x	1·18	1·18	1·16	1·04	·82	·48	·02	
y	2·26	2·80	3·42	4·11	4·87	5·71	6·61	
ψ°	85°54′			103°12′				
α°	65°32′			67°				
R	4·04			6·36				



For the Graph of $\rho = \frac{1}{8} e^{\frac{9}{2}\psi} \operatorname{sech}^3 \frac{\sqrt{5}}{2}\psi$;
$(\psi, \rho\cos\psi)$ and $(\psi, \rho\sin\psi)$ diagrams.



Graph of $\rho = \frac{1}{8} e^{\frac{9}{2}\psi} \operatorname{sech}^3 \frac{\sqrt{5}}{2}\psi$,
and the locus of its centres of aberrancy.

Section XVII.

To find the equation of a curve such that the projection of the axis of aberrancy at any point of it on the corresponding tangent is constant.

(1) If R is the length of the axis of aberrancy;
α is the angle which the axis of aberrancy makes with the corresponding tangent;
k is the given projection;
then the required condition is $R\cos\alpha = k$,
where $R = \dfrac{3\rho\sqrt{9+\rho'^2}}{9+\rho'^2-3\rho\rho''}$, $\rho' = 3\cot\alpha$.

The equation may be written

$$\frac{3\rho\sqrt{9+\rho'^2}}{9+\rho'^2-3\rho\rho''} = \frac{k\sqrt{9+\rho'^2}}{\rho'}$$

or $\quad 9+\rho'^2-3\rho\rho'' = \frac{3}{k}\rho\rho'.$

(2) If we put $u = \dfrac{d\rho}{d\psi}$, we find $\rho' = \dfrac{u}{\rho}$, $\rho\rho'' = \dfrac{u}{\rho}\dfrac{du}{d\rho} - \dfrac{u^2}{\rho^2}$,
and the equation becomes

$$\frac{u}{k} = 3 + \frac{4}{3}\frac{u^2}{\rho^2} - \frac{u}{\rho}\frac{du}{d\rho},$$

$$\text{or } \frac{1}{k} - \frac{3}{u} = \frac{4}{3}\frac{u}{\rho^2} - \frac{1}{\rho}\frac{du}{d\rho} = -\rho^{\frac{1}{3}}\frac{d}{d\rho}\left(u\rho^{-4/3}\right).$$

(3) If we substitute $Y = \frac{k}{3}u\rho^{-4/3} + \frac{1}{2}\rho^{2/3}$,
and $X = \rho^{-2/3} + \dfrac{Y^2}{k^2}$,

the equation becomes $\dfrac{dY}{dX} + Y^2 = k^2X$.

---

(4) $k=0$ gives $u=0$, or $\rho = \text{const.}$, the equation of a circle.
$k=\infty$ gives $9+\rho'^2-3\rho\rho''=0$, the equation of a parabola.
It is evident that $\frac{1}{k}$ is merely a multiplier of ρ, and therefore merely determines the scale of the figure. Thus, when k is finite and non-zero, it may be taken equal to unity.
Change of sign of k is equivalent to

change of sign of ρ, and corresponds to a change of direction of the x-axis through an angle of 180°. Therefore k may be considered positive.

When $k = +1$, the equations become

$$u = \frac{d\rho}{d\psi},$$
$$Y = \tfrac{1}{3} u \rho^{-4/3} + \tfrac{1}{2}\rho^{2/3}$$
$$X = \rho^{-2/3} + Y^2$$
$$\frac{dY}{dX} + Y^2 = X.$$

This is Riccati's equation $\frac{dY}{dX} + bY^2 = cX^m$ for the case $b = c = m = 1$.

(5) The substitution $Y = \frac{1}{W}\frac{dW}{dX}$ gives the equation in the linear form

$$\frac{d^2W}{dX^2} - XW = 0.$$

This has a solution of the form $W = AW_1 + BW_2$, where

$$W_1 = \left(1 + \frac{X^3}{\lfloor 3} + \frac{1.4}{\lfloor 6}X^6 + \frac{1.4.7}{\lfloor 9}X^9 + \ldots\right),$$

$$\text{and } W_2 = \left(X + \frac{2}{\lfloor 4}X^4 + \frac{2.5}{\lfloor 7}X^7 + \frac{2.5.8}{\lfloor 10}X^{10} + \ldots\right).$$

This gives $$Y = \frac{1}{W}\frac{dW}{dX} = \frac{\frac{d}{dX}(AW_1 + BW_2)}{AW_1 + BW_2},$$
$$= \frac{\frac{d}{dX}(W_1 + CW_2)}{W_1 + CW_2}.$$

But, since $X = 0$ requires $Y = 0$, $\rho^{-2/3} = 0$, we must have $C = 0$, i.e.

$$Y = \frac{1}{W_1}\frac{dW_1}{dX} = \frac{\frac{X^2}{\lfloor 2} + \frac{1.4}{\lfloor 5}X^5 + \frac{1.4.7}{\lfloor 8}X^8 + \frac{1.4.7.10}{\lfloor 11}X^{11} + \ldots}{1 + \frac{X^3}{\lfloor 3} + \frac{1.4}{\lfloor 6}X^6 + \frac{1.4.7}{\lfloor 9}X^9 + \frac{1.4.7.10}{\lfloor 12}X^{12} + \ldots}$$

The solution just given for the equation $\frac{dY}{dX} + Y^2 = X$ is given in Forsyth's Treatise on Differential Equations, page 146.

(6) From a table of corresponding values of X and Y a table of values of $\frac{1}{u}$ and ρ can be calculated.

Then by integration from the $(\rho, \frac{1}{u})$ graph, values of ψ can be calculated corresponding to known values of ρ, using the formula $\psi = \int \frac{d\rho}{u}$. This enables coordinates for the graphs $(\psi, \rho\cos\psi)$ and $(\psi, \rho\sin\psi)$ to be calculated. From these graphs the coordinates (x, y) of the required curve can be found.

---

(7) When $u = 0$, ρ is finite and $Y = \frac{1}{2}\rho^{2/3}$, $X - Y^2 = \bar{\rho}^{2/3}$; i.e. $2XY - 2Y^3 - 1 = 0$, which is the condition that $\frac{d^2Y}{dX^2} = 0$. Hence, when $u = 0$, $\frac{dY}{dX}$ has a maximum, i.e. ρ has a minimum. The value of ρ when $u = 0$ is approximately 1.347.

(8) From the equation $u = 3 + \frac{4}{3}\frac{u^2}{\rho^2} - \frac{u}{\rho}\frac{du}{d\rho}$, we see that when u is small, we have approximately $\psi = \int \frac{d\rho}{u} = \int \frac{1}{\rho}\frac{du}{3-u} = -\frac{1}{\bar{\rho}}\log\frac{3-u}{3}$, since the variation of ρ is small.

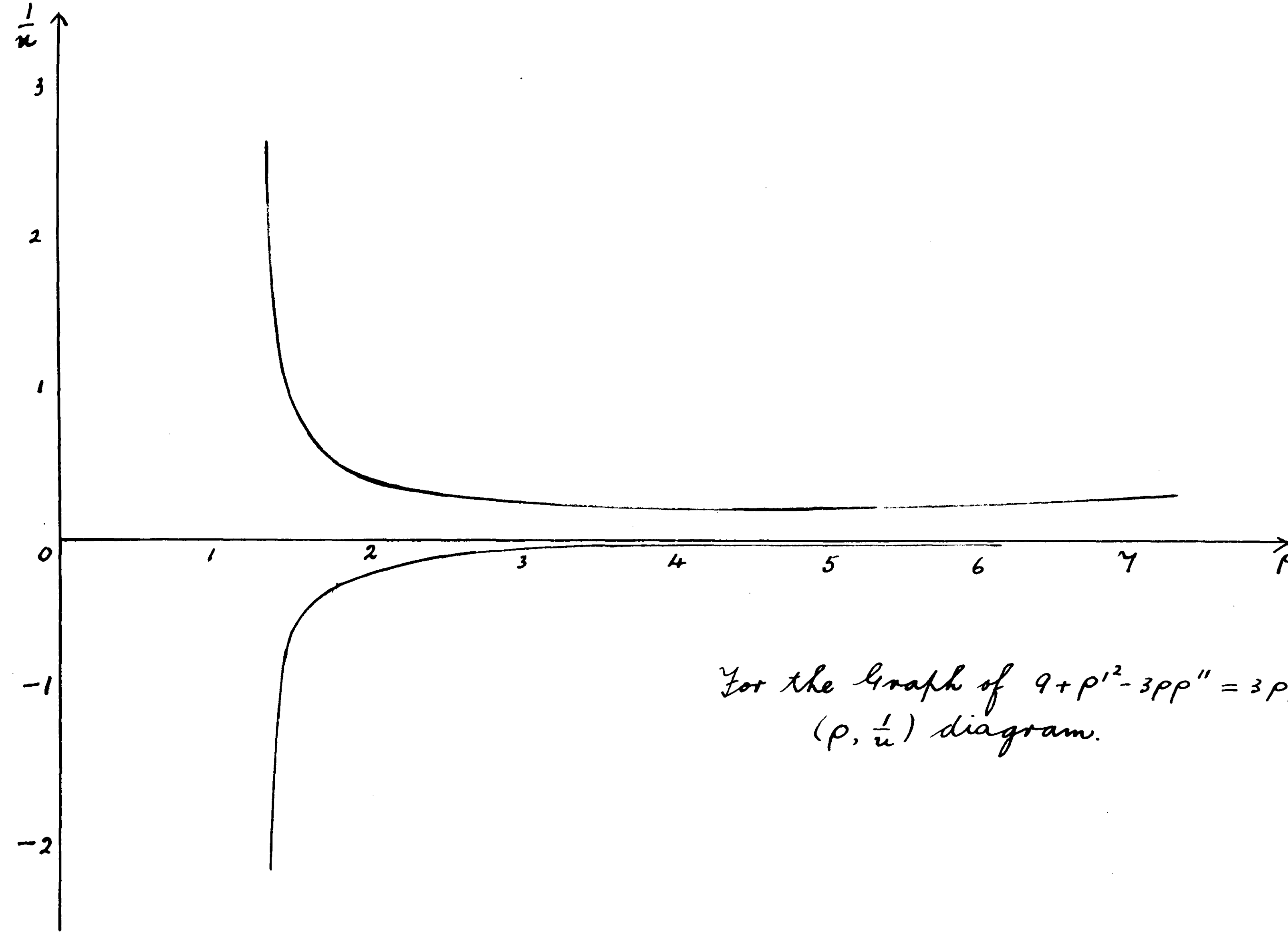
This shows that ψ is finite when $u = 0$.

Data for the Graph of $9 + \rho'^2 - 3\rho\rho'' = 3\rho\rho'$.

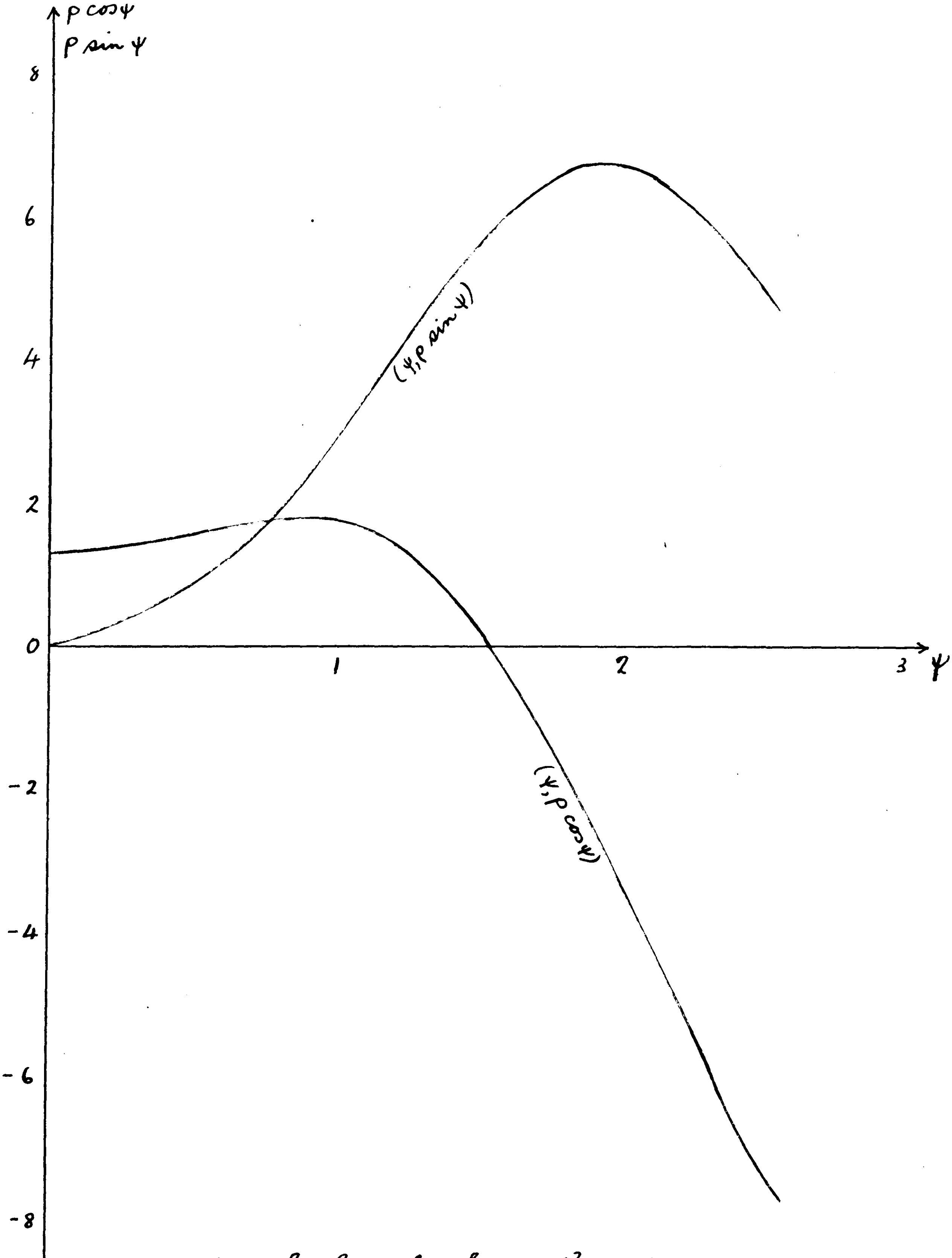
ρ	∞	31·62	11·20	6·152	4·051	2·960	2·331	1·934
u	$-\infty$	−1498	−187·2	−55·17	−23·05	−11·60	−6·515	−3·907
$\frac{1}{u}$	0	−·00067	−·00534	−·01813	−·04339	−·08621	−·1535	−·2568
ρ	1·682	1·517	1·423	1·362	1·347	1·349	1·368	1·416
u	−2·412	−1·479	−·8392	−·4551	0	·02946	·3782	·7089
$\frac{1}{u}$	−·4146	−·6760	−1·192	−2·198	∞	33·94	2·645	1·410
ρ	1·602	1·888	2·290	2·966	3·632	4·304	4·969	6·064
u	1·393	2·079	2·944	3·789	4·346	4·556	4·384	4·238
$\frac{1}{u}$	·7178	·4810	·3396	·2640	·2301	·2195	·2279	·2360

x	−1.41	−1.02	−.59	−.28	0	.35	.72	1.14	1.59
y	.67	.37	.13	.03	0	.05	.20	.50	1.03
ψ	−.7	−.6	−.4	−.2	0	.25	.50	.75	1.0
ρ		3.6	1.86	1.44	1.35	1.45	1.79	2.38	3.30
u		−17.2	−3.5	−1.0	0	.89	1.90	3.05	4.14
α°		−32°8′	−57°54′	−76°54′	±90°	78°24′	70°30′	66°52′	67°24′
R	1.03	1.18	1.88	4.41	∞	4.97	3.00	2.55	2.60

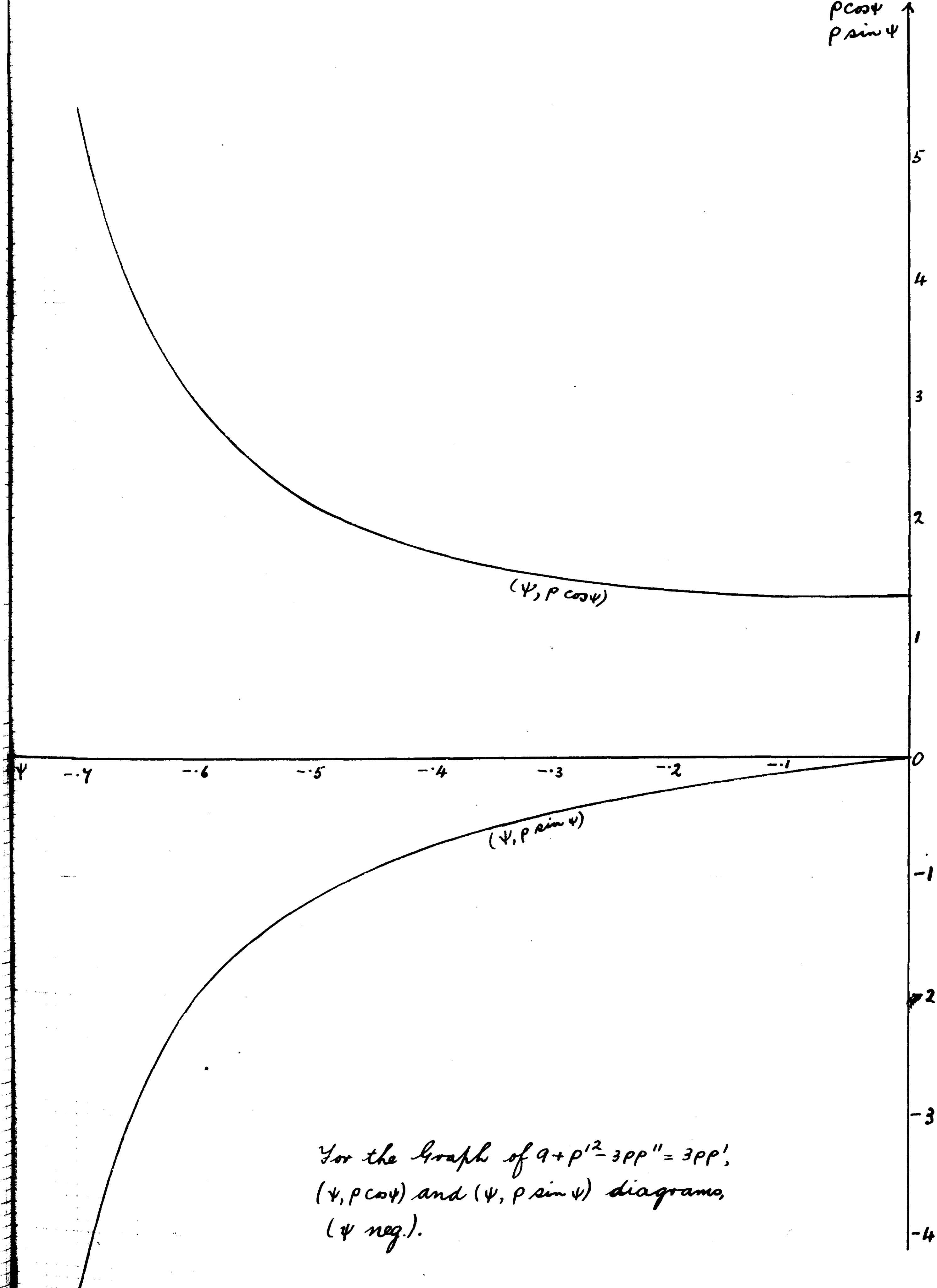
x	1.92	2.17	2.24	2.08	1.62	.84	−.25	−1.66
y	1.69	2.59	3.69	4.96	6.30	7.62	8.84	9.88
ψ	1.2	1.4						
ρ	4.20	5.14						
u	4.58	4.33						
α°	70°	74°18′						
R	2.92	3.70						



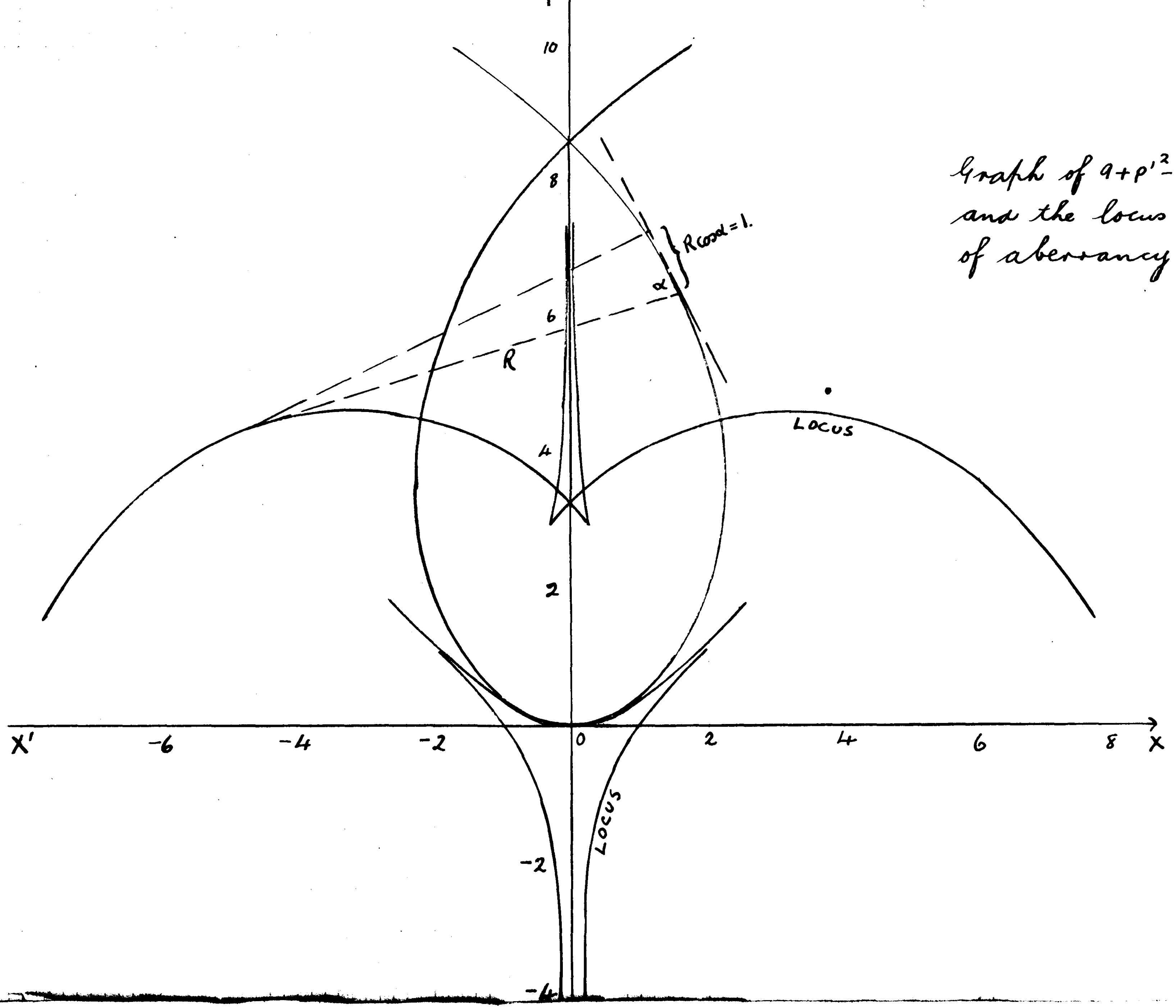
For the Graph of $9+\rho'^2-3\rho\rho''=3\rho\rho'$,
$(\rho, \frac{1}{u})$ diagram.



For the Graph of $9 + \rho'^2 - 3\rho\rho'' = 3\rho\rho'$;
$(\psi, \rho \cos\psi)$ and $(\psi, \rho \sin\psi)$ diagrams; (ψ pos.)



For the Graph of $9+\rho'^2-3\rho\rho''=3\rho\rho'$,
$(\psi, \rho\cos\psi)$ and $(\psi, \rho\sin\psi)$ diagrams,
(ψ neg.).



Graph of $9+p'^2-3pp''=3pp'$, and the locus of its centres of aberrancy.

Section XVIII.

To find the equation of a curve at any point of which the normal chord of the osculating conic is of constant length.

(1) The length of the chord is $\dfrac{18\rho}{9+2\rho'^2-3\rho\rho''}$, and the constant may be written $\frac{18}{K}$. The differential equation of the required curve therefore is $K\rho = 9+2\rho'^2-3\rho\rho''$, and a first solution is obtained by writing $\rho'^2 = u$, i.e. $\frac{du}{d\rho} = 2\rho''$.

This gives $\dfrac{du}{d\rho} - \dfrac{4}{3\rho}u = \dfrac{18-2K\rho}{3\rho}$,

the solution of which is $u = c\rho^{4/3} + 2K\rho - \frac{9}{2}$,

$$\text{i.e., } s = \int \frac{d\rho}{\sqrt{c\rho^{4/3}+2K\rho-9/2}}.$$

(2) The discriminant of the equation $\rho'^2 = c\rho^{4/3}+2K\rho-\frac{9}{2}$ equated to zero gives solutions of the form $\rho = \text{const.}$, and c may be chosen so that $\rho = \frac{9}{K}$ satisfies this differential equation and the original one also.

(3) The particular case $c = \infty$ gives $\rho'^2 = \lambda\rho^{4/3}$ where $\lambda \to \infty$. The solution of this is $\rho = \dfrac{\lambda^{3/2}s^3}{27}$, and $\rho \to \infty$ for finite non-zero values of s.

In this case the limiting form of the required curve is a straight line.

(4) If $K \to \infty$, the equation becomes $\rho'^2 = \lambda\rho$ where $\lambda \to \infty$. The solution in this case is $\rho = \dfrac{\lambda s^2}{4}$, and the conclusion is the same as in the previous case.

(5) The particular case $c = 0$ gives $s = \displaystyle\int \frac{d\rho}{\sqrt{2K\rho - 9/2}}$, the solution of which is $2Ks = 2\sqrt{2K\rho - 9/2}$, or $\rho = \dfrac{9}{4K} + \dfrac{K}{2}s^2$.

Using the relation $\frac{d\psi}{ds} = \frac{1}{\rho}$ we have also
$$\psi = \int \frac{ds}{\frac{K}{2}\left(s^2 + \frac{9}{2K^2}\right)},$$
the solution of which is
$$s = \frac{3}{K\sqrt{2}} \tan \frac{3\psi}{2\sqrt{2}}.$$
The constants of integration are chosen so that $s = 0$ and $\psi = 0$ when $\rho = \frac{9}{4K}$.

---

(6) The particular case $K = 0$ gives $s = \int \frac{d\rho}{\sqrt{c\rho^{4/3} - 9/2}}$.

Putting $d\psi = \frac{ds}{\rho}$, we obtain $\psi = \int \frac{d\rho}{\rho\sqrt{c\rho^{4/3} - 9/2}}$. This may be solved by putting $\rho^{2/3} = \frac{1}{w}$, and the solution obtained is $\rho^{2/3} = \frac{3}{\sqrt{2c}} \sec\sqrt{2}\psi$, and $\rho = \left(\frac{3}{\sqrt{2c}}\right)^{3/2}$ when $\psi = 0$.

In this case the length of the normal chord is infinite. This means that the osculating conic at any point of the curve is a hyperbola with an asymptote parallel to the normal at the point.

---

(7) In the equation $ds = \frac{d\rho}{\sqrt{c\rho^{4/3} + 2K\rho - 9/2}}$ write $\rho = v^3$ and we obtain $ds = \frac{3v^2\,dv}{\sqrt{cv^4 + 2Kv^3 - 9/2}}$. The condition for equal roots of the equation $cv^4 + 2Kv^3 - \frac{9}{2} = 0$ gives $3K^4 + 8c^3 = 0$, and the repeated root is $-\frac{3K}{2c}$. The quartic expression can then be written
$$\left(v + \frac{3K}{2c}\right)^2\left(cv^2 - Kv + \frac{3K^2}{4c}\right) = -c\left(v + \frac{3K}{2c}\right)^2\left(-v^2 + \frac{K}{c}v - \frac{3K^2}{4c^2}\right)$$
where the factor $-c$ is positive. The last factor is negative so that the square root of the quartic is imaginary. There is no real solution in this case.

The conditions for three or four repeated roots of $cv^4 + 2Kv^3 - \frac{9}{2} = 0$ cannot be satisfied by finite values of K and c.

---

(8) If $f(v) \equiv cv^4 + 2Kv^3 - \frac{9}{2}$,

$$f'(v) \equiv v^2(4cv + 6K),$$
$$f''(v) \equiv v(12cv + 12K),$$
$$f'''(v) \equiv 24cv + 12K.$$

The function $f(v)$ has a turning value at $v = -\frac{3K}{2c}$, and points of inflexion at $v = 0$ and $v = -\frac{K}{c}$.

If c is positive, $f(v)$ is negative at each of these points, and $f(v) = 0$ has only two real roots, one positive and one negative.

---

(9) <u>c positive, K positive.</u>

In this case the equation for s may be written

$$s = \int \frac{3v^2\,dv}{\sqrt{c(v+a^2)(v-b^2)(v^2+gv+h)}},$$ where a and b are real, and $a^2 > \frac{3K}{2c}$. Since the factors of $v^2 + gv + h$ are imaginary, this expression is positive. Therefore $(v+a^2)(v-b^2)$ must be positive also for real values of s. That is we must have $-a^2 \geqq v$, or $b^2 \leqq v$, and the values of p range from negative infinity to $-a^6$, and from b^6 to positive infinity.

If $K = \frac{45}{32}$, and $c = \frac{27}{16}$, we get the equation

$$s = \int \frac{4v^2\,dv}{\sqrt{(v-1)(v+2)(3v^2+2v+4)}},$$ which may be solved with the aid of elliptic integrals.

---

(10) <u>c positive, K negative.</u>

This is equivalent to changing the sign of v under the radical. The equation may be written

$$s = \int \frac{3v^2\,dv}{\sqrt{c(v-a^2)(v+b^2)(v^2-gv+h)}},$$ where $a^2 > -\frac{3K}{2c}$.

Here we must have $v \geqq a^2$ or $v \leqq -b^2$.

Putting $K = -\frac{45}{32}$ and $c = \frac{27}{16}$, we find

$$s = \int \frac{4v^2\,dv}{\sqrt{(v+1)(v-2)(3v^2-2v+4)}},$$ an equation of

similar form to that in the previous paragraph.

(11) c negative, κ positive.
In this case the turning value of $f(v)$ will be a maximum, and will be positive if $3\kappa^4 > -8c^3$. If $3\kappa^4 < -8c^3$, the expression under the radical is negative for all values of v, and no real solution exists. The case $3\kappa^4 = -8c^3$ has been discussed in paragraph 7. If $3\kappa^4 > -8c^3$, $f(v)$ has a maximum at $v = -\frac{3\kappa}{2c}$, a positive quantity.

Since $f(0)$ is negative, it follows that $f(v) = 0$ has a pair of positive real roots, and a pair of imaginary roots. Thus the equation for s is of the form
$$s = \int \frac{3v^2\,dv}{\sqrt{-c(a^2-v)(v-b^2)(v^2+gv+h)}}.$$

For real values of s in this case we must have $a^2 \leqq v \leqq b^2$, assuming $a^2 < b^2$.

If $c = -\frac{63}{16}$, $\kappa = \frac{135}{32}$ we get the equation $s = \int \frac{4v^2\,dv}{\sqrt{(2-v)(v-1)(4v^2+6v+4)}}$.

(12) c negative, κ negative.
If $3\kappa^4 > -8c^3$, $f(v) = 0$ will have a pair of real negative roots, and a pair of imaginary roots. The equation for s is of the form $s = \int \frac{3v^2\,dv}{\sqrt{c(v+a^2)(v+b^2)(v^2-gv+h)}}$, and for real values of s we must have $-a^2 \leqq v \leqq -b^2$, if $a^2 > b^2$.

If $c = -\frac{63}{16}$, $\kappa = -\frac{135}{32}$, we get the equation
$$s = \int \frac{4v^2\,dv}{\sqrt{(2+v)(-v-1)(4v^2-6v+4)}},$$
an equation of similar form to that in the previous paragraph.

The solutions of chief interest are those contained in paragraphs (5), (6), (9), and (11).

Section XIX.

Examples of curves such that at any point the length of the normal chord of the osculating conic is constant.

(1) The graph of $s = \frac{3}{K\sqrt{2}} \tan \frac{3\psi}{2\sqrt{2}}$ for $K = \frac{3}{\sqrt{2}}$.
The (ρ, s) equation in this case is $\rho = \frac{3\sqrt{2}}{4}(1+s^2)$.
The quantity $9 + \rho'^2 - 3\rho\rho'' = \frac{9}{4}(1-s^2)$. The conics of closest contact are ellipses for points for which $s^2 < 1$, hyperbolas where $s^2 > 1$, and parabolas where $s = \pm 1$.

s is infinite when $\frac{3\psi}{2\sqrt{2}} = \pm\frac{\pi}{2}$ or $\psi = \pm\frac{\pi\sqrt{2}}{3}$. The curve therefore has a pair of symmetrically-placed asymptotes inclined to the x-axis at angles $\pm 1{\cdot}4809$ radians, $= \pm 84°51'$.

At the points where $s = \pm 1$, the corresponding ψ values are $\psi = \pm {\cdot}7405 = \pm 42°26'$.

The normal at (x, y) on this curve meets the corresponding osculating conic at a point whose distance from (x, y) is constant and equal to $\frac{18}{K} = 6\sqrt{2}$.
In this case $\rho = \frac{3\sqrt{2}}{4}(1+s^2)$, $\rho' = \frac{3\sqrt{2}}{2}s$, $\rho'' = \frac{3\sqrt{2}}{2}$.
At the origin $\rho = \frac{3\sqrt{2}}{4}$, $\rho' = 0$, $\rho'' = \frac{3\sqrt{2}}{2}$,

$$x = 0, \quad y = 0, \quad l = 1, \quad m = 0,$$

and the osculating conic is the ellipse $\xi^2 + \frac{1}{4}\eta^2 = \frac{3\sqrt{2}}{2}\eta$.
This ellipse is dotted in in the figure, as also is the osculating hyperbola at the point where $\psi = 80°$, $s = 11{\cdot}1$.

The graph is constructed by first calculating series of values of $s\left(= \tan \frac{3\psi}{2\sqrt{2}}\right)$, $\cos\psi$, $\sin\psi$ corresponding to a series of values of ψ. The values of $\cos\psi$ and $\sin\psi$ are then plotted against the values of s. From these graphs the corresponding pairs of coordinates x and y are found by

Graphical integration.

Data for the graph of $s = \tan \frac{3\psi}{2\sqrt{2}}$.

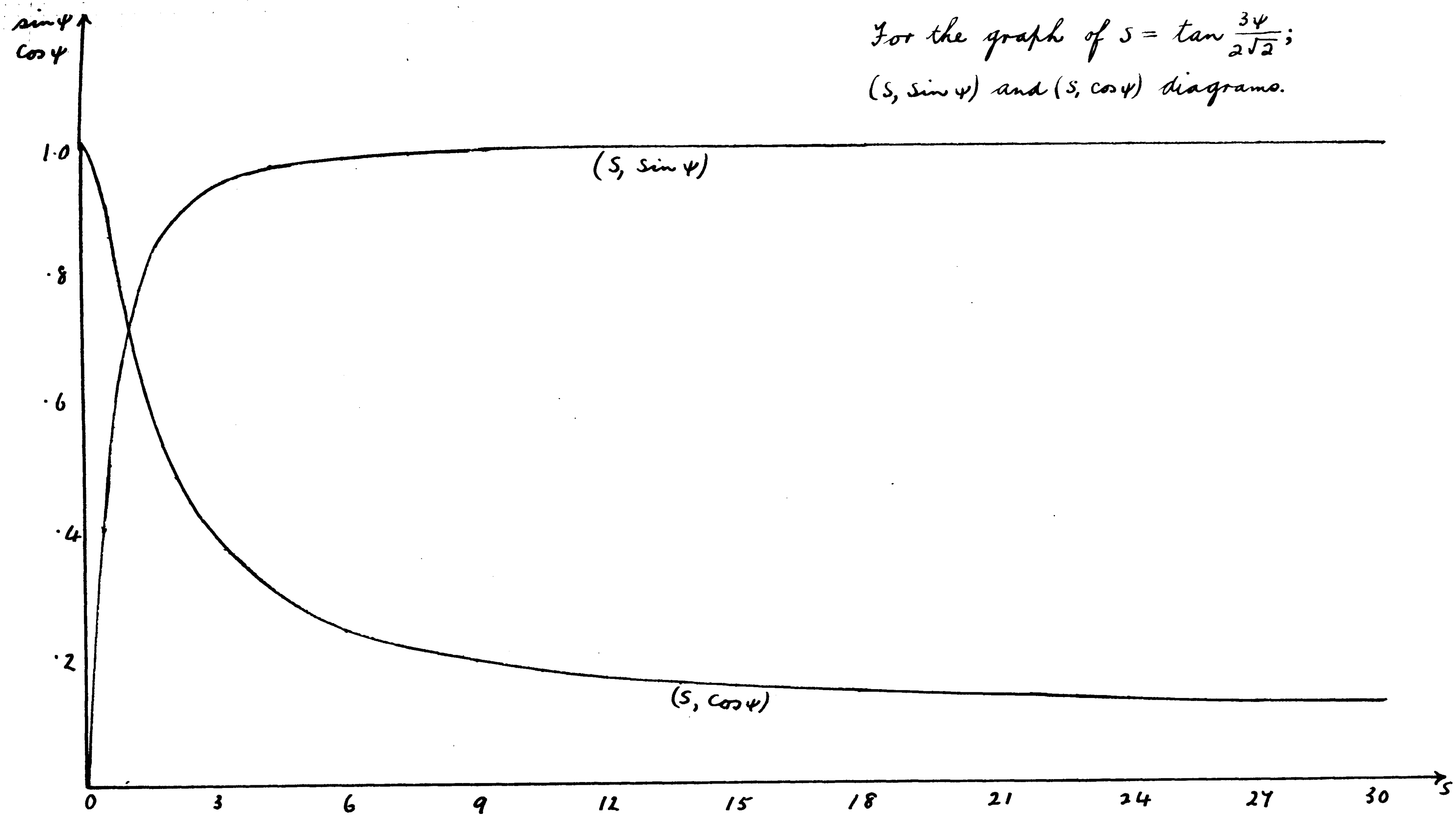
Values of ψ, s, $\cos\psi$, $\sin\psi$.

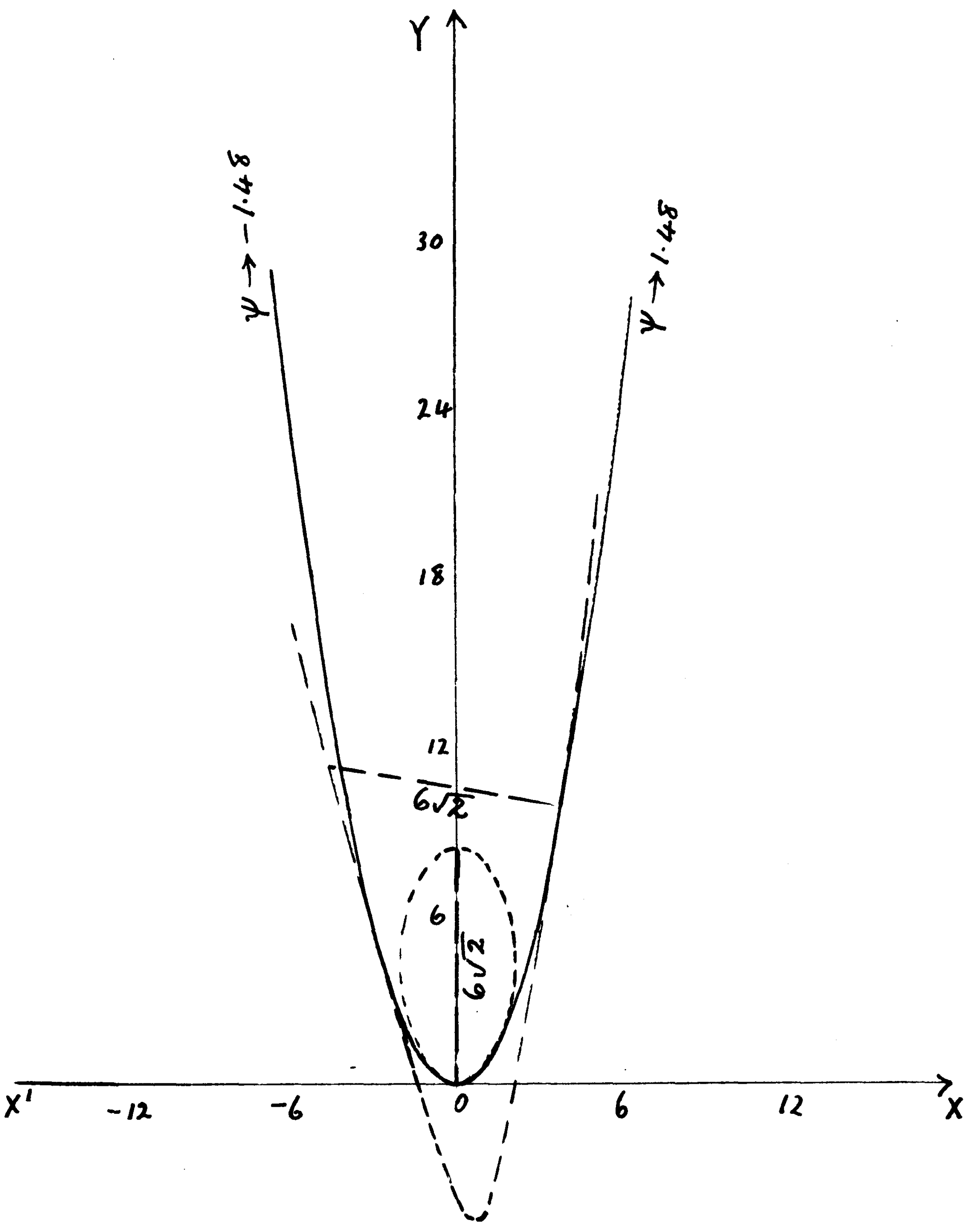
$\psi°$	0	10	20	30	40	50	60	65
s	0	·187	·388	·620	·914	1·33	2·02	2·60
$50\cos\psi$	50	49·2	47·0	43·3	38·3	32·1	25·0	21·1
$50\sin\psi$	0	8·7	17·1	25·0	32·1	38·3	43·3	45·3
$\psi°$	70	75	80	81	82	83	84	
s	3·55	5·41	11·1	14·0	18·9	28·9	63·7	
$50\cos\psi$	17·1	12·9	8·7	7·8	7·0	6·1	5·2	
$50\sin\psi$	47·0	48·3	49·2	49·4	49·5	49·6	49·7	

Values of s, x, y.

s	0	·6	1·2	1·8	2·4	3·0	3·6	4·2	4·8	5·4	6·0
x	0	·58	1·04	1·30	1·69	1·94	2·16	2·35	2·52	2·69	2·84
y	0	·16	·53	1·01	1·53	2·07	2·64	3·21	3·78	4·36	4·94
s	0	3	6	9	12	15	18	21	24	27	30
x	0	1·94	2·84	3·47	4·01	4·48	4·92	5·34	5·74	6·12	6·49
y	0	2·07	4·94	7·87	10·8	13·8	16·7	19·7	22·7	25·7	28·6

For the graph of $s = \tan \frac{3\psi}{2\sqrt{2}}$; $(s, \sin\psi)$ and $(s, \cos\psi)$ diagrams.





Graph of the equation $s = \tan \frac{3\psi}{2\sqrt{2}}$.

(2) Graph of the equation $\rho^{2/3} = \frac{3}{\sqrt{2c}} \sec\sqrt{2}\,\psi$, where $\frac{3}{\sqrt{2c}} = 1$.

ρ and s are infinite when $\sqrt{2}\,\psi = \frac{\pi}{2}$ or $\psi = 1{\cdot}111 = 63°39'$. The curve therefore has a pair of symmetrically-placed asymptotes in the directions $\psi = \pm 1{\cdot}111$.

In this case $\rho = (\sec\sqrt{2}\,\psi)^{3/2}$, $\rho' = \frac{3}{\sqrt{2}} \tan\sqrt{2}\,\psi$, $\rho\rho'' = 3\sec^2\sqrt{2}\,\psi$, $9 + \rho'^2 - 3\rho\rho'' = -\frac{9}{2}\tan^2\sqrt{2}\,\psi$.
Thus, at every point except the origin, the osculating conic is a hyperbola. At the origin the osculating conic is a parabola whose equation is $\xi^2 = 2\eta$.

In the figure this parabola is dotted in. It evidently has its axis along the normal at the origin. It represents the curve very closely for values of x from 0 to ± 1. The osculating hyperbola at $\psi = 45°$ is also dotted in. This hyperbola has an asymptote parallel to the normal at $\psi = 45°$.

The equation enables corresponding values of ψ, ρ, $\cos\psi$, $\sin\psi$ to be tabulated. Values of s corresponding to these are found from the (ψ, ρ) diagram. Values of x and y are found from the $(s, \cos\psi)$ and $(s, \sin\psi)$ diagrams.

Data for the graph of $\rho^{2/3} = \sec\sqrt{2}\,\psi$.

ψ, ρ values.

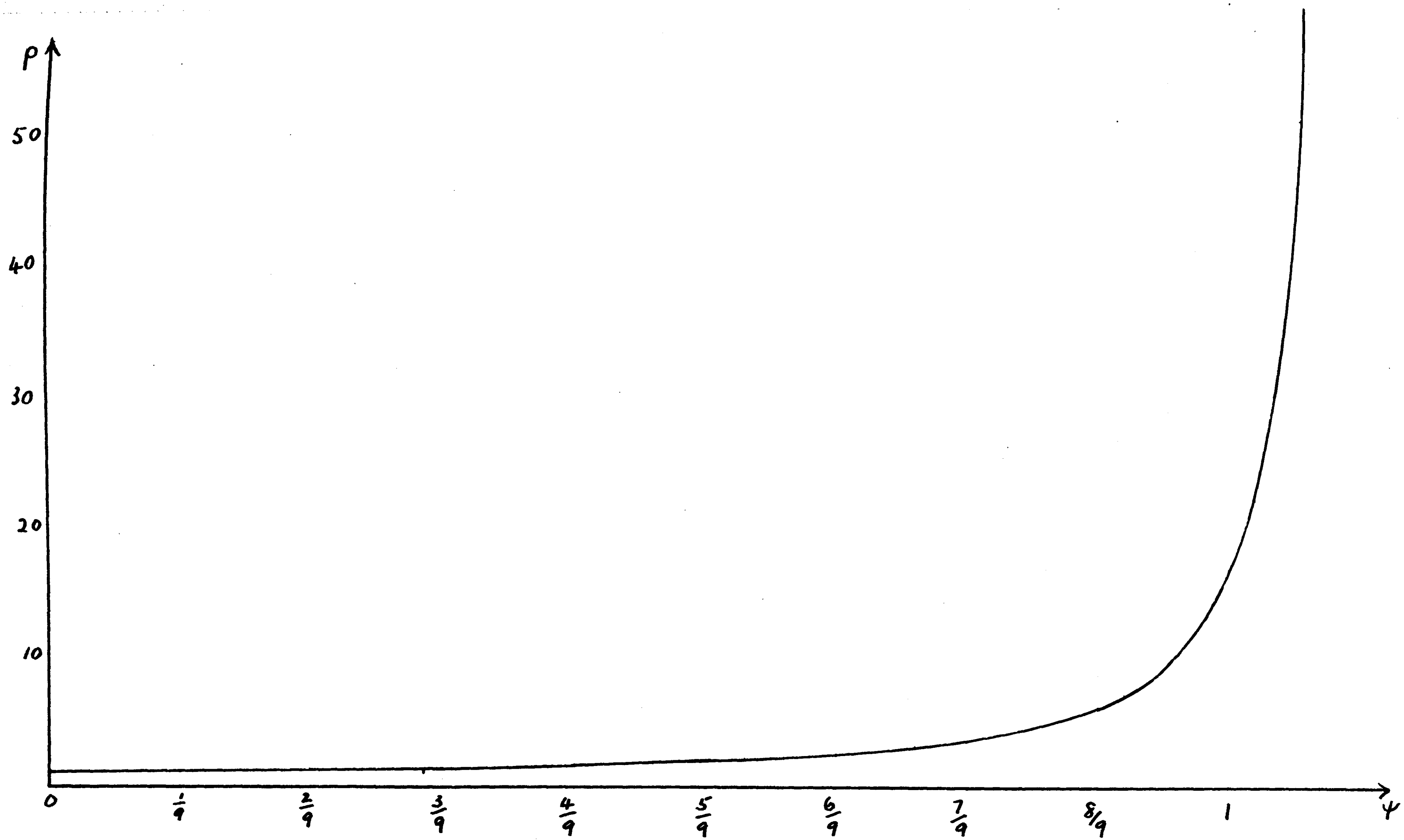
$\psi°$	0	10	20	30	40	50	55	56	57	58	59	60	61	62	63
ρ	1	1·05	1·21	1·58	2·45	5·27	10·3	12·3	15·2	19·3	26	37·2	59·8	120	469

ψ, s, $\cos\psi$, $\sin\psi$ values.

ψ	0	·175	·349	·524	·698	·785	·873	·960	·995	1·03	1·08
s	0	·11	·23	·36	·50	·67	·89	1·19	2·45	3·14	4·24
$50\sin\psi$	0	5·5	11·0	16·4	21·5	26·4	30·7	35·1	42·1	42·7	43·5
$50\cos\psi$	50	49·7	48·8	47·2	45·1	42·5	39·3	35·6	27·0	26·1	24·6

s, x, y values.

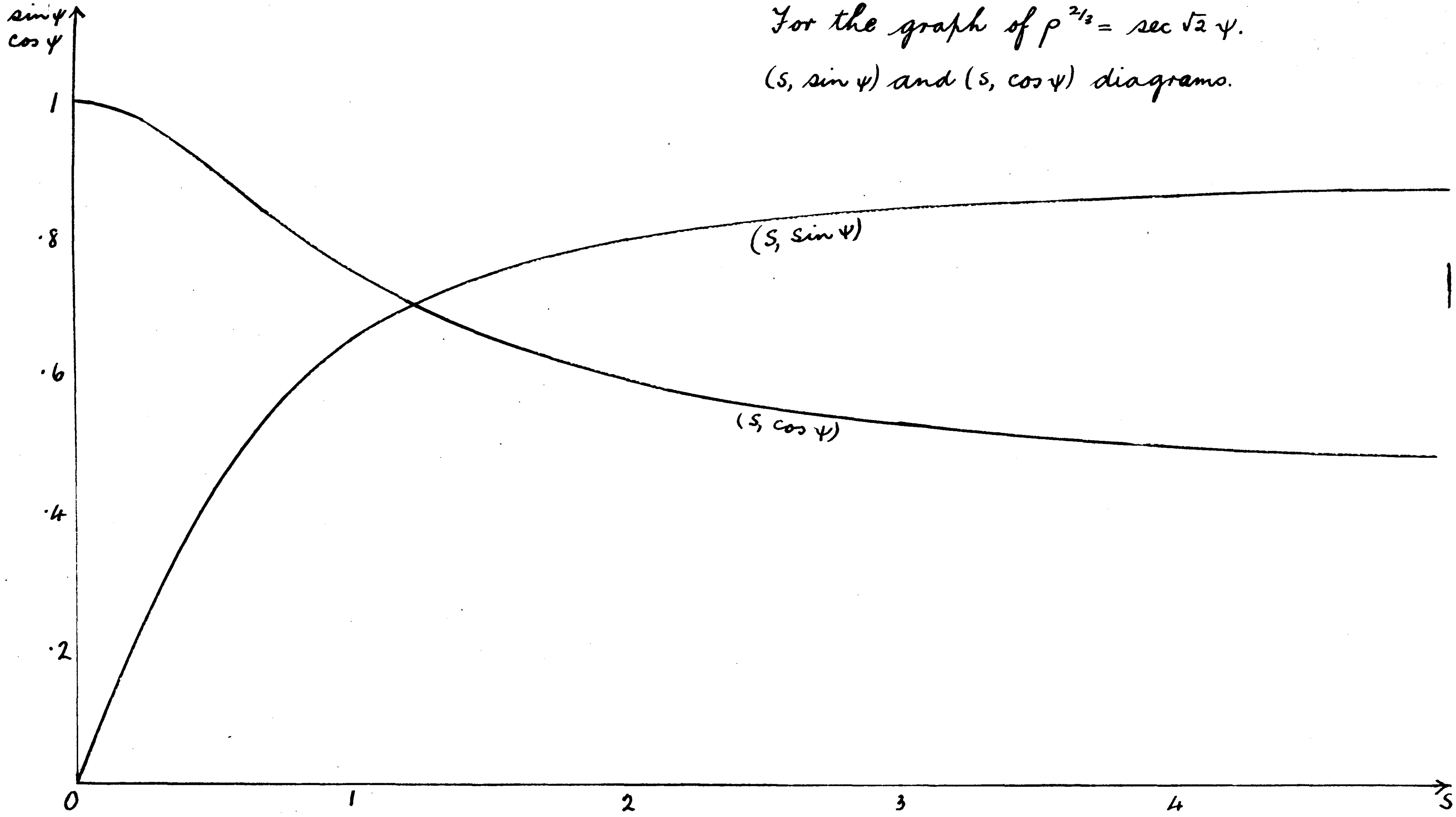
s	0	·5	1	1·5	2	2·5	3	3·5	4	4·5	5
x	0	·48	·90	1·25	1·57	1·86	2·13	2·39	2·64	2·89	3·13
y	0	·11	·39	·75	1·14	1·54	1·96	2·39	2·82	3·26	3·69

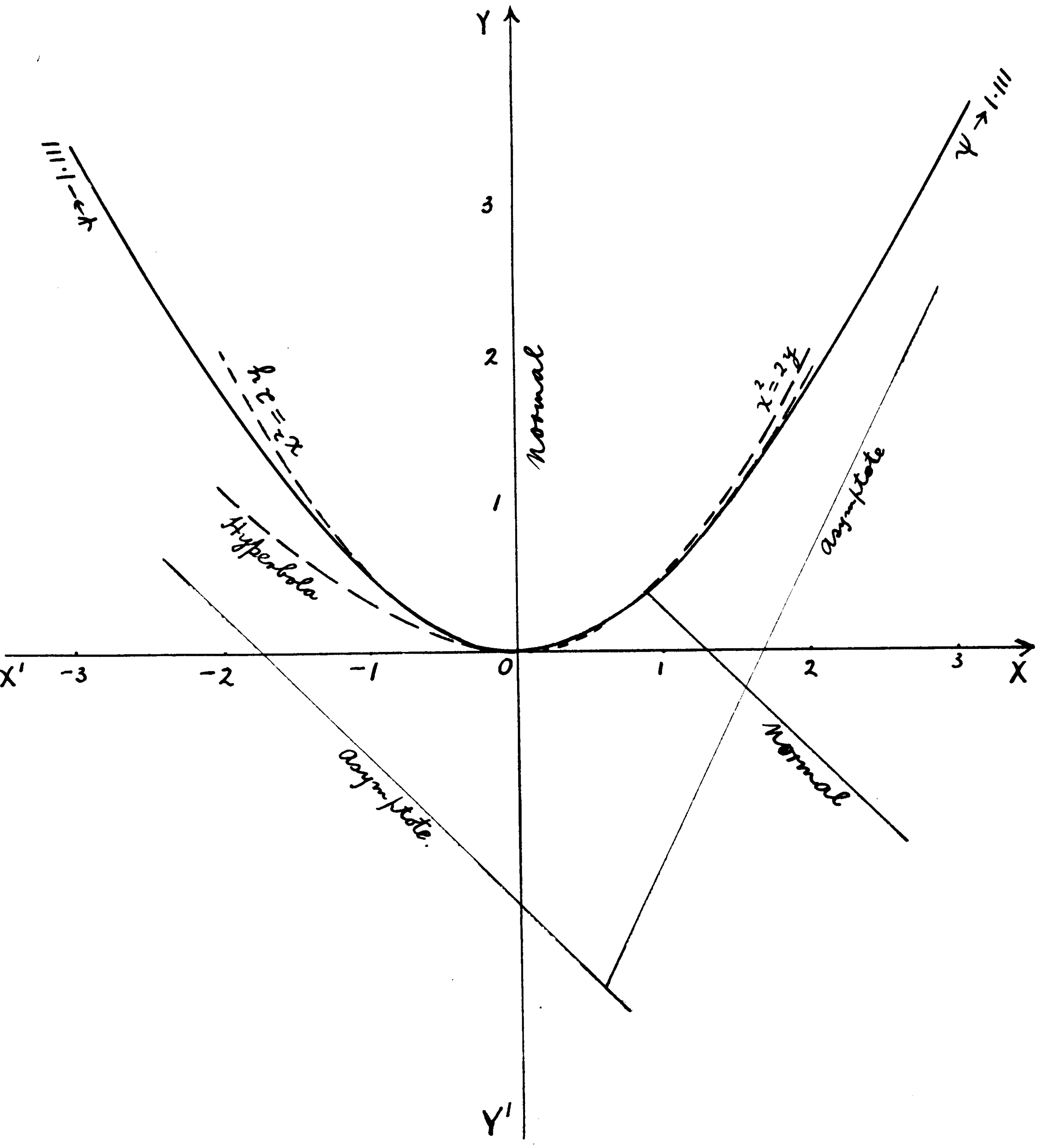


For the Graph of $\rho^{2/3} = \sec\sqrt{2}\,\psi$. (ρ, ψ) diagram.

For the graph of $\rho^{2/3} = \sec \sqrt{2}\,\psi$.

$(s, \sin\psi)$ and $(s, \cos\psi)$ diagrams.





Graph of $\rho^{2/3} = \sec\sqrt{2}\,\psi$.

(3) Graph of the equation $s = \int \frac{4v^2\,dv}{\sqrt{(v-1)(v+2)(3v^2+2v+4)}}$, $(v = \rho^{\frac{1}{3}})$.

If we put λ for $\frac{4v^2}{\sqrt{(v-1)(v+2)(3v^2+2v+4)}}$, then λ becomes

infinite when v tends to 1 or to -2. At all other points λ is finite. When v tends to $\pm$ infinity, λ tends to $\frac{4}{\sqrt{3}}$. The graph of λ therefore has three asymptotes, $v = 1$, $v = -2$, $\lambda = 2.31$. No part of the graph lies between $v = 1$ and $v = -2$. There is only one turning point, where $v^3 = 6.4$ or $v = 1.854$. There is a point of inflexion to the right of this, at $v = 3.4$.

To evaluate $\int \lambda\,dv$ near $v = 1$, we may substitute $v = 1 + \delta$. Then λ is approximately $\frac{4}{3\sqrt{3}}\left(\delta^{-\frac{1}{2}} + \frac{25}{18}\delta^{\frac{1}{2}} + \frac{5}{216}\delta^{\frac{3}{2}}\right)$, and the integral

$$\int \lambda\,dv = \frac{4\sqrt{3}}{9}\left(2\delta^{\frac{1}{2}} + \frac{25}{27}\delta^{3/2} + \frac{1}{108}\delta^{5/2}\right).$$

From this we find $\int_1^{1.1} \lambda\,dv = \cdot 489$. The integral from $v = 1.1$ to $v = 5$ may be obtained approximately from the graph of λ.

To evaluate $\int \lambda\,dv$ near $v = -2$, we may substitute $v = -2 - \delta$. Then λ is approximately $\frac{2}{3}\left(4\delta^{-\frac{1}{2}} - \frac{125}{72}\delta^{3/2}\right)$, and the integral

$$\int \lambda\,dv = -\frac{2}{3}\left(8\delta^{\frac{1}{2}} - \frac{25}{36}\delta^{5/2}\right).$$

From this we find $\int_{-2}^{-2.1} \lambda\,dv = -1.684$, and the graph of λ may be used to extend this integral from $v = -2.1$ to $v = -5$.

The area under the graph of λ gives the value of s corresponding to a given value of v. We assume that s is zero when $v = 1$ and when $v = -2$. From a table of corresponding values of s and $\frac{1}{\rho}$ $(= v^{-3})$, the $(s, \frac{1}{\rho})$ diagram can be drawn, from which, by graphical integration, corresponding ψ values can be found. Then, as in previous problems, from the $(s, \cos\psi)$ and $(s, \sin\psi)$ diagrams, corresponding pairs of values of x and y can be found.

In this example we have $\rho' = \frac{3}{4}\sqrt{3\rho^{4/3}+5\rho-8}$, $\rho'' = \frac{9}{32}(4\rho^{\frac{1}{3}}+5)$
At the origin on the graph we have $\rho = 1$ or $\rho = -8$, $\rho' = 0$, $\rho'' = \frac{81}{32}$ or $-\frac{27}{32}$, $2\rho'^2 - 3\rho\rho'' = -\frac{243}{32}$ or $-\frac{81}{4}$, $l = 1$, $m = 0$, $x = 0$, $y = 0$.

Corresponding to $\rho = 1$, we have the osculating conic $9\xi^2 + \frac{45}{32}\eta^2 - 18\eta = 0$, and corresponding to $\rho = -8$, we have the osculating conic $9\xi^2 - \frac{45}{4}\eta^2 + 144\eta = 0$.
The normals at the origin meet these conics at distances $\frac{64}{5}$ and $\frac{64}{5}$, which is the value of $\frac{18}{k}$. These conics are dotted in in the figure.

Values of v, s, $\frac{1}{\rho}$, for $v = 1$ to $v = 5$.

v	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2	3	4	5
s	0	.49	.73	.94	1.13	1.32	1.50	1.68	1.86	2.03	2.21	4.04	5.95	7.92
$\frac{50}{\rho}$	50	37.6	28.9	22.7	18.3	14.8	12.2	10.2	8.8	7.5	6.3	1.9	.8	.4

Values of s, ψ, $\sin\psi$, $\cos\psi$, x, y, for $v = 1$ to $v = 5$.

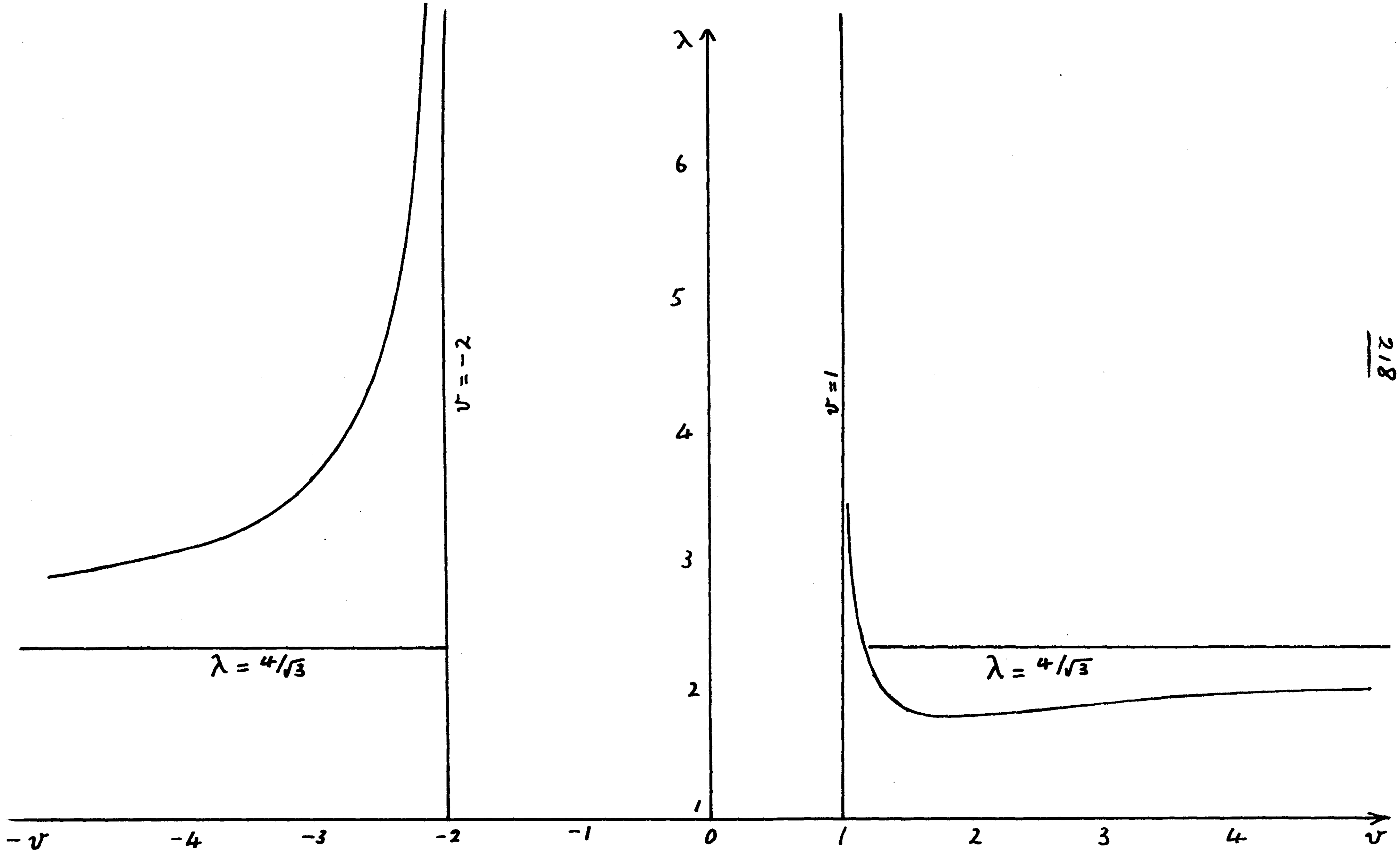
s	0	.5	1	1.5	2	3	4	5	6	7	8
ψ	0	.434	.724	.891	.991	1.098	1.152	1.183	1.203	1.217	1.227
$50\cos\psi$	50	45.4	37.5	31.5	27.8	22.8	20.3	18.9	18.0	17.3	16.9
$50\sin\psi$	0	21.0	33.1	38.9	41.8	44.5	45.7	46.3	46.7	46.9	47.1
x	0		.90		1.54	2.04	2.47	2.87	3.23	3.59	3.93
y	0		.37		1.14	2.00	2.90	3.82	4.75	5.69	6.63

Values of v, s, $\frac{1}{\rho}$, for $v = -2$ to $v = -5$.

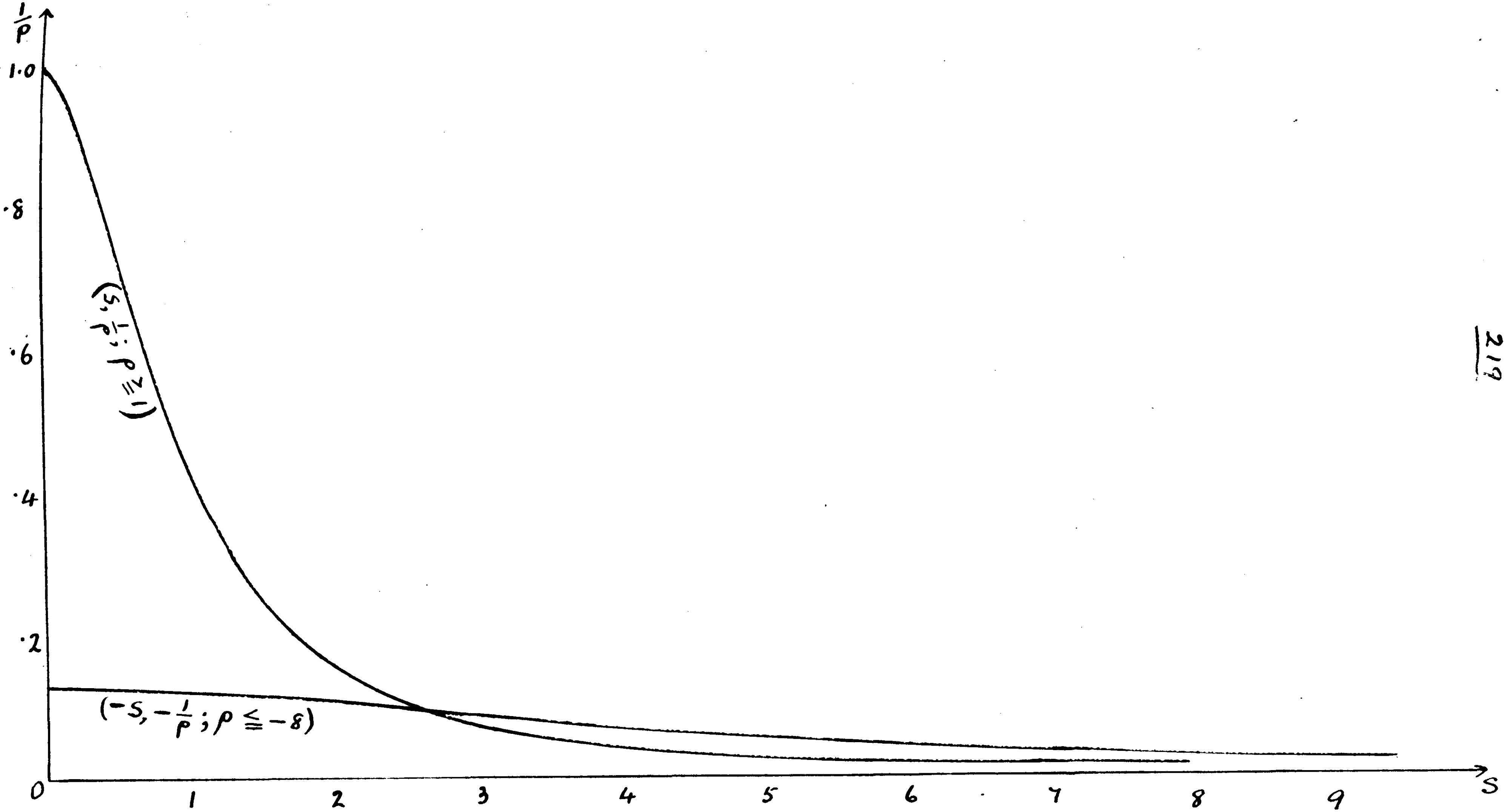
$-v$	2	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	3	4	5
$-s$	0	1.69	2.45	3.03	3.54	4.01	4.44	4.86	5.25	5.63	6	9.37	12.3
$-\frac{50}{\rho}$	6.3	5.4	4.7	4.1	3.6	3.2	2.8	2.5	2.3	2.1	1.9	.8	.4

Values of s, ψ, $\cos\psi$, $\sin\psi$, x, y for $v = -2$ to $v = -5$.

$-s$	0	1	2	3	4	5	6	7	8	9
ψ	0	.121	.231	.325	.399	.458	.504	.537	.564	.585
$50\cos\psi$	50	49.6	48.7	47.4	46.1	44.9	43.8	43.0	42.3	41.7
$50\sin\psi$	0	6.0	11.5	16.0	19.5	22.1	24.1	25.6	26.7	27.6
$-x$	0	1.00	1.98	2.94	3.88	4.79	5.67	6.54	7.39	8.23
$-y$	0	.06	.24	.51	.87	1.28	1.74	2.24	2.76	3.31

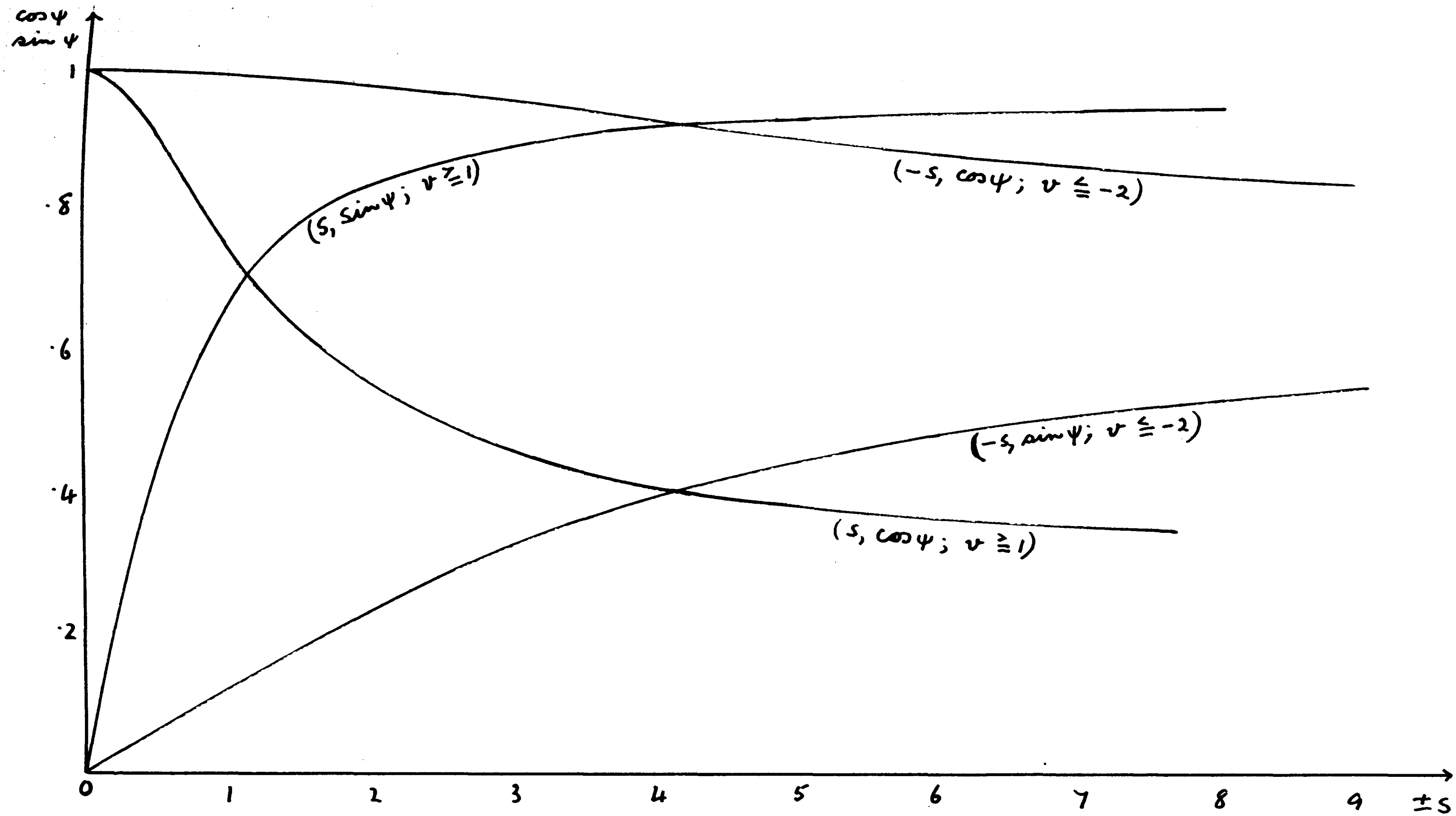


Graph of $\lambda = \dfrac{4v^2}{\sqrt{(v-1)(v+2)(3v^2+2v+4)}}$.



For the graph of $s = \int \frac{4v^2\,dv}{\sqrt{(v-1)(v+2)(3v^2+2v+4)}}$, $(v^3 = \rho)$;

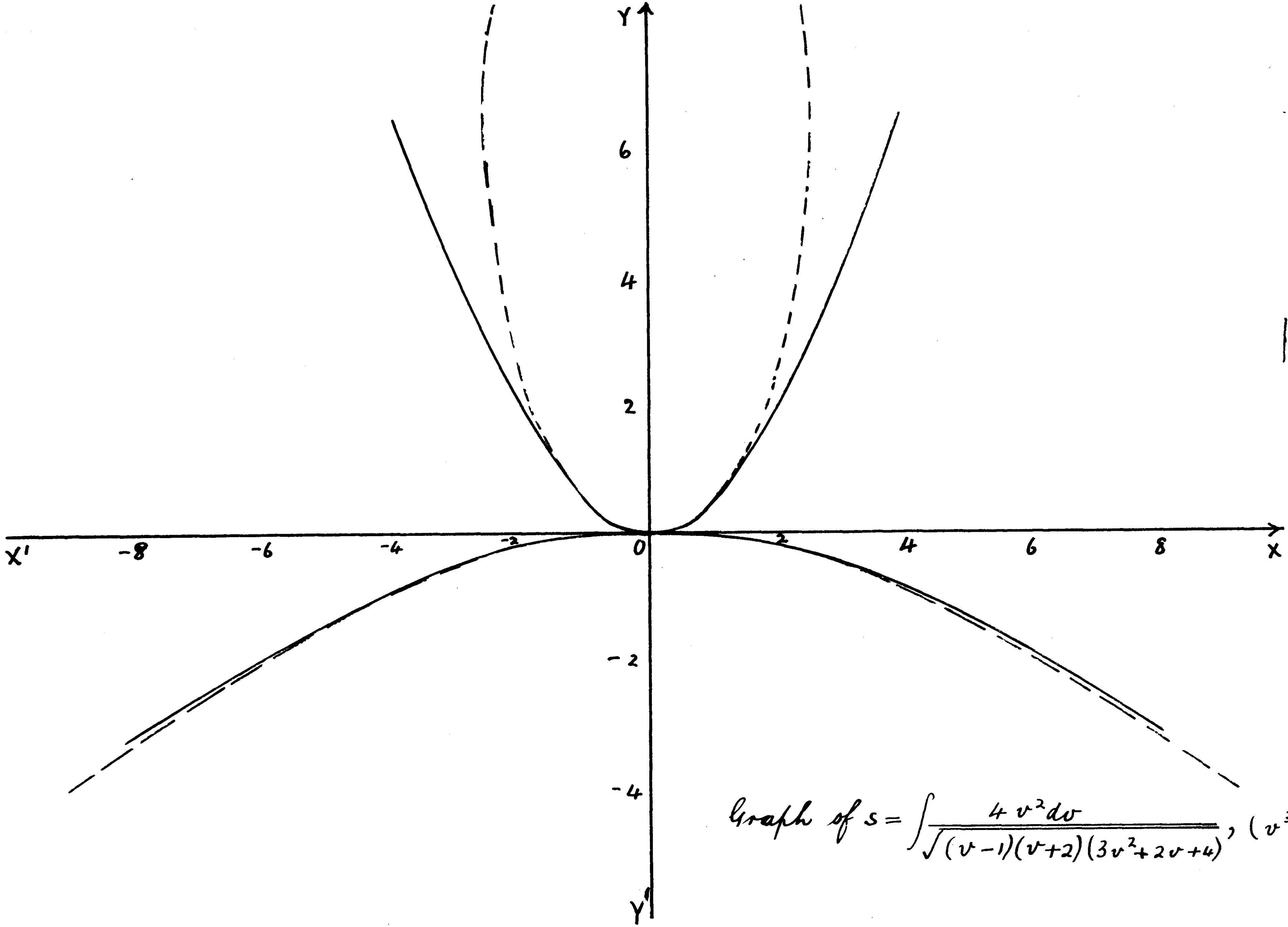
$(s, \frac{1}{\rho})$ diagram for $\rho \geqq 1$;
$(-s, -\frac{1}{\rho})$ diagram for $\rho \leqq -8$.



For the graph of $s = \int \frac{4v^2\,dv}{\sqrt{(v-1)(v+2)(3v^2+2v+4)}}$, $(\rho = v^3)$;

$(s, \cos\psi)$ and $(s, \sin\psi)$ diagrams for $\rho \geqq 1$;

$(-s, \cos\psi)$ and $(-s, \sin\psi)$ diagrams for $\rho \leqq -8$.



Graph of $s = \int \frac{4v^2\,dv}{\sqrt{(v-1)(v+2)(3v^2+2v+4)}}$, $(v^3=\rho)$

(4) Graph of the equation $s = \int \frac{4v^2\,dv}{\sqrt{(2-v)(v-1)(7v^2+6v+4)}}$, $v^3 = \rho$.

If we write the equation $s = \int \lambda\,dv$, then λ tends to infinity when v tends to 1 or to 2. At all other points λ is finite. The graph of λ therefore has two asymptotes, $v = 1$ and $v = 2$. There is a minimum at $v^3 = \frac{32}{15}$ or $v = 1{\cdot}29$.

To evaluate $\int \lambda\,dv$ near $v = 1$, we put $v = 1 + \delta$. Then λ is approximately $\frac{4}{\sqrt{17}}\,\delta^{-\frac{1}{2}}\left(1 + \frac{65}{34}\delta + \frac{2815}{2312}\delta^2\right)$.
From this we find $\int_1^{1\cdot1} \lambda\,dv = {\cdot}654$.

To evaluate $\int \lambda\,dv$ near $v = 2$, we put $v = 2 - \delta$. Then λ is approximately $\frac{2}{\sqrt{11}}\left(4\delta^{-\frac{1}{2}} - \frac{5}{11}\delta^{\frac{1}{2}} + \frac{295}{968}\delta^{3/2}\right)$.
From this we find $\int_{1\cdot9}^{2} \lambda\,dv = 1{\cdot}519$, and the graph of λ may be used to find the value of the integral from $v = 1{\cdot}1$ to $v = 1{\cdot}9$.

In this example $\rho' = \frac{3}{4}\sqrt{-7\rho^{4/3} + 15\rho - 8}$,

$$\rho'' = \frac{9}{32}\left(-\frac{28}{3}\rho^{1/3} + 15\right).$$

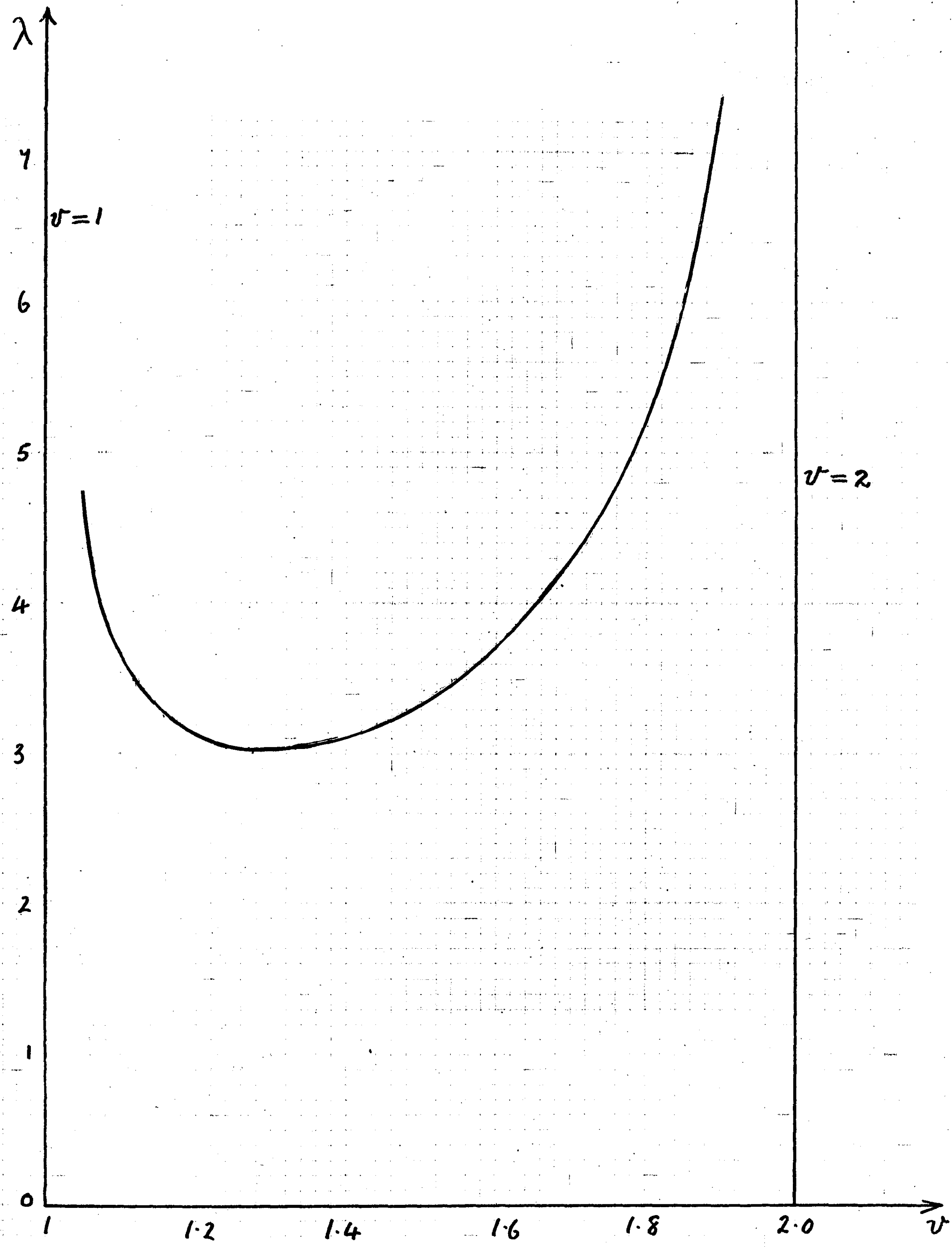
At the origin on the graph we have $\rho = 1$, $\rho' = 0$, $\rho'' = \frac{51}{32}$, $2\rho'^2 - 3\rho\rho'' = -\frac{153}{12}$, $l = 1$, $m = 0$, $x = 0$, $y = 0$. The osculating conic at the origin is $32\xi^2 + 15\eta^2 = 64\eta$. The normal at the origin meets this conic at a distance of $\frac{64}{15}$, which is the value of $\frac{18}{k}$. This conic is dotted in in the figure.

Values of v, s, $\frac{1}{\rho}$ for $v = 1$ to $v = 2$.

v	1·0	1·1	1·2	1·3	1·4	1·5	1·6	1·7	1·8	1·9	2·0
s	0	·65	·99	1·29	1·60	1·92	2·27	2·67	3·15	3·77	5·29
$\frac{50}{\rho}$	50	37·6	28·9	22·8	18·3	14·8	12·2	10·2	8·6	7·6	6·3

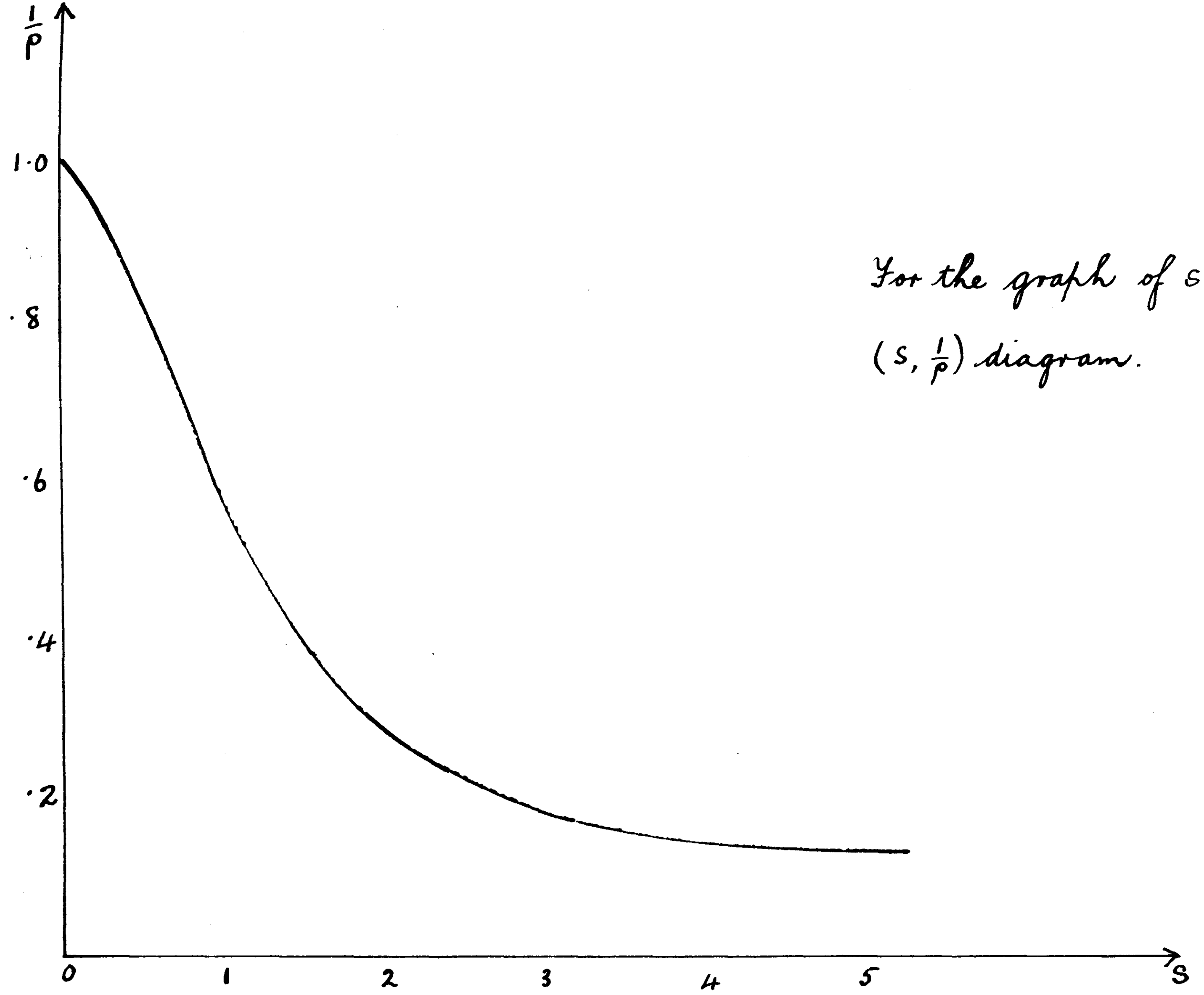
s, ψ, $\cos\psi$, $\sin\psi$, x, y.

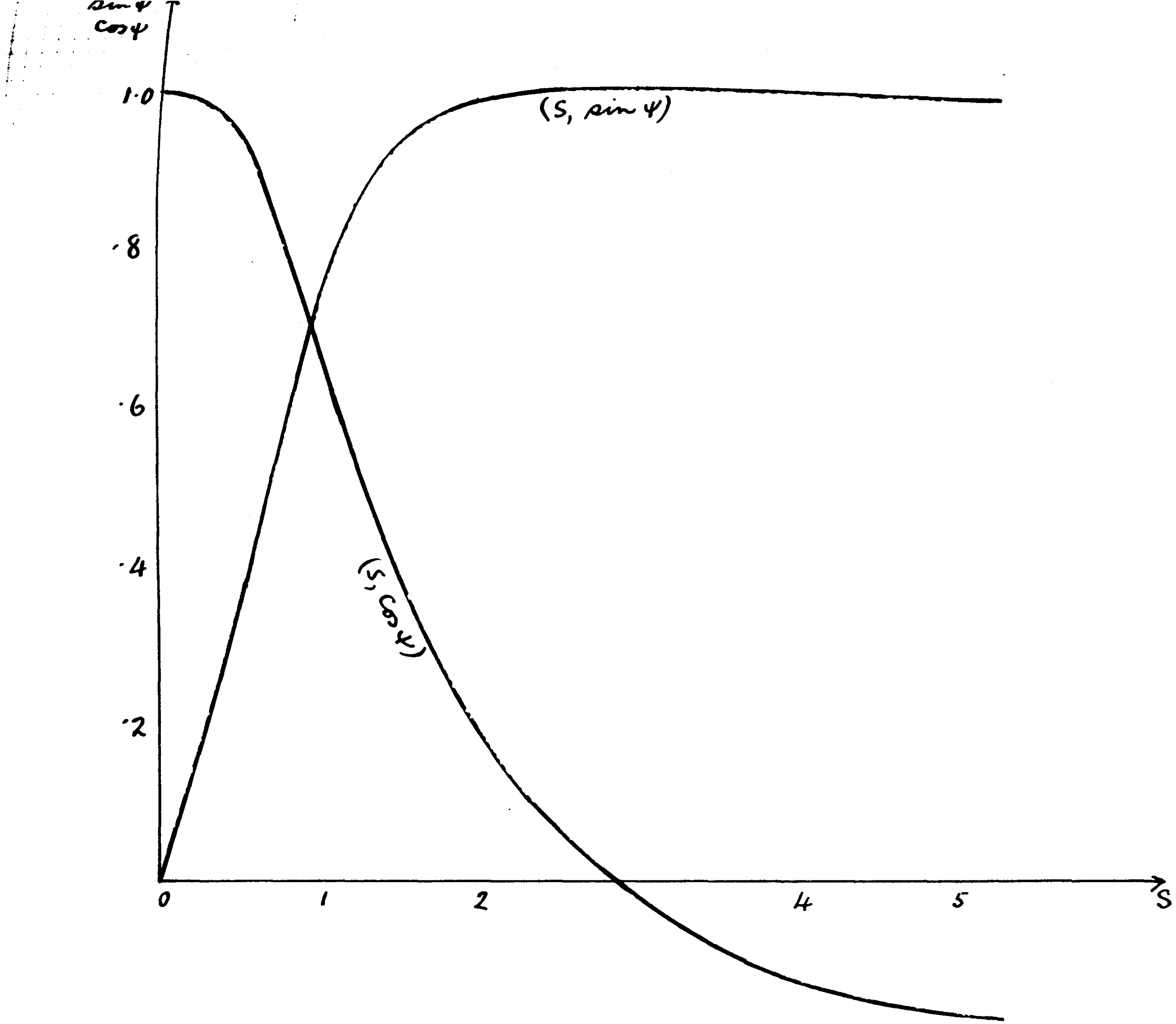
s	0	·5	1	1·5	2	2·5	3	3·5	4	4·5	5
ψ	0	·46	·81	1·06	1·23	1·35	1·45	1·54	1·61	1·68	1·75
$50\cos\psi$	50	44·8	34·4	24·6	17·0	10·9	6·0	1·7	−2·0	−5·5	−8·8
$50\sin\psi$	0	22·2	36·3	43·5	47·0	48·8	49·6	50	50	49·7	49·2
x	0	·49	·90	1·17	1·31	1·38	1·39	1·36	1·31	1·24	1·17
y	0	·085	·35	·77	1·25	1·74	2·24	2·74	3·24	3·74	4·23



Graph of $\lambda = \dfrac{4v^2}{\sqrt{(2-v)(v-1)(4v^2+6v+4)}}$, $(v^3=\rho)$.

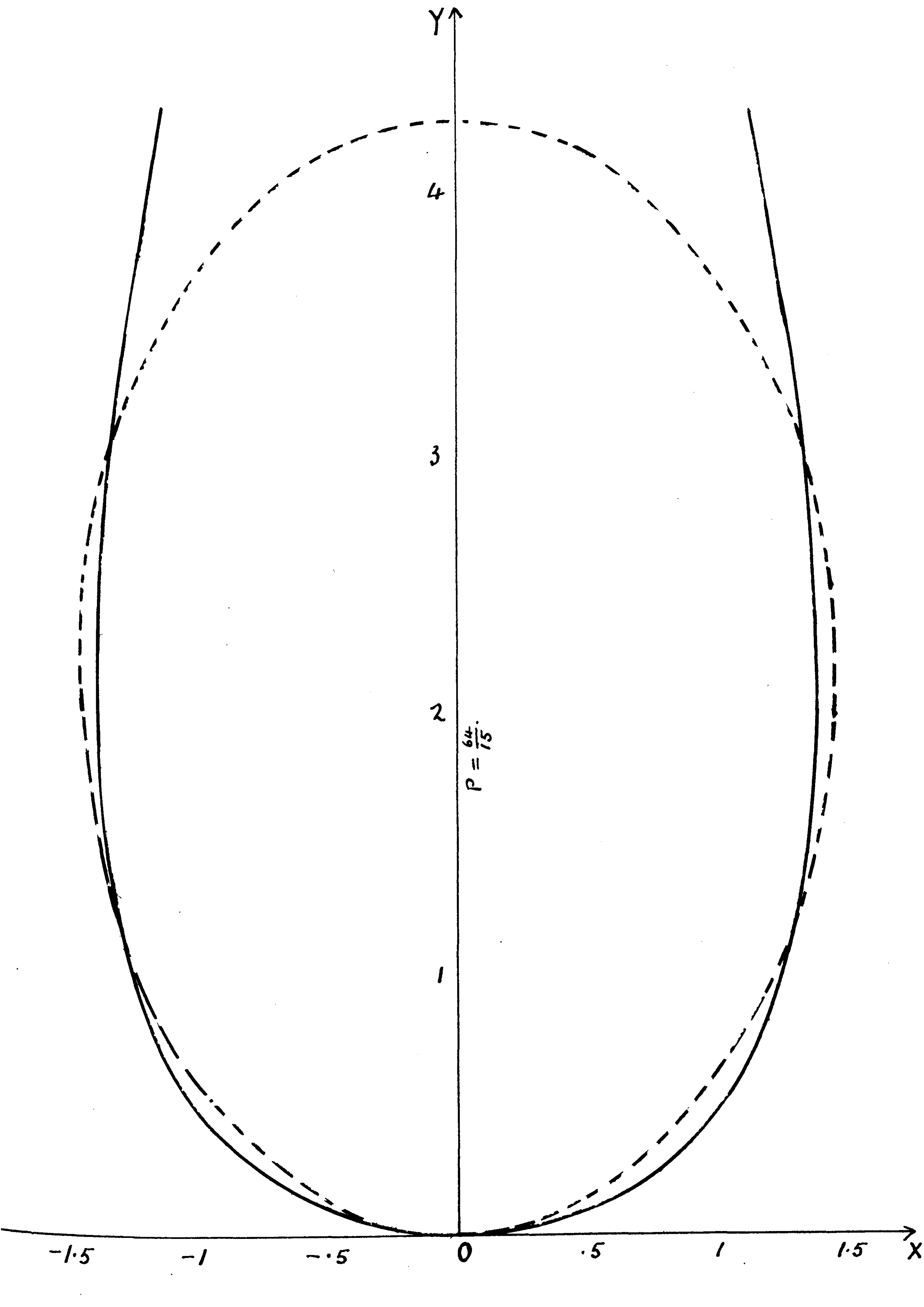
For the graph of $s = \int \frac{4v^2\,dv}{\sqrt{(2-v)(v-1)(7v^2+6v+4)}}$, $v^3 = \rho$;
$(s, \frac{1}{\rho})$ diagram.





For the graph of $s = \int \frac{4 v^2 \, dv}{\sqrt{(2-v)(v-1)(7v^2+6v+4)}}$
$(v^3 = \rho)$.

$(s, \cos\psi)$ and $(s, \sin\psi)$ diagrams.



Graph of $s = \int_1^2 \frac{4v^2\,dv}{\sqrt{(2-v)(v-1)(4v^2+6v+4)}}$,

$(v^3 = \rho)$.

Section XX.

To find the equation of a curve such that, at any point of it, the length of the normal chord of its osculating conic is proportional to the radius of curvature at the point.

(1) Writing the constant in the form $\frac{18}{9+2K}$, the equation of the required curve is $\frac{18\rho}{9+2\rho'^2-3\rho\rho''} = \frac{18\rho}{9+2K}$, or

$\rho'^2 - K = \frac{3}{2}\rho\rho''$. Putting $u = 2\rho'^2 - 2K$, we find $\frac{du}{d\rho} = 4\rho''$. This gives $u = \frac{3}{4}\rho\frac{du}{d\rho}$, the solution of which is $u = 2c\rho^{4/3}$, i.e. $\rho'^2 = c\rho^{4/3} + K$.

(2) $c = \infty$ gives $\rho = 0$ or $\rho = \infty$.

(3) $c = 0$ gives $\rho = \infty$ or $\rho'^2 = K$. This is the equation of an equilateral spiral $\rho = \sqrt{K}s$, which therefore is a curve satisfying the given condition. Its graph is shown in section V.

(4) $K = \infty$ gives $\rho'^2 = \infty$, or $\rho = \infty$.

(5) The discriminant of $\rho'^2 = c\rho^{4/3} + K$ equated to zero gives solutions $\rho =$ constant, which satisfy the original differential equation when $K = 0$.

(6) $K = 0$ gives $\rho'^2 = c\rho^{4/3}$, the solution of which is of the form $\rho = as^3$, when ρ and s are taken to vanish together. This may be written $\frac{1}{s^2} = -2a\psi$, where ψ is infinite when s is zero.

In this case the extremity of the normal chord of the osculating conic coincides with the extremity of the diameter of the circle of curvature.

(7) K positive, c positive.

The equation is $s = \int \frac{d\rho}{\sqrt{c\rho^{4/3}+K}}$. Writing $d\psi = \frac{ds}{\rho}$, we find $\psi = \int \frac{\frac{1}{\rho}d\rho}{\sqrt{c\rho^{4/3}+K}}$.

This may be solved by the substitution $\rho^{2/3} = v^{-1}$.
This gives the solution $-\frac{2\sqrt{K}}{3}\psi = \sinh^{-1} v\sqrt{\frac{K}{c}}$,
which can also be written $\rho^{2/3} = -\sqrt{\frac{K}{c}} \operatorname{cosech} \frac{2\sqrt{K}}{3}\psi$,
where ρ is infinite when ψ is zero.

---

(8) K positive, c negative.
In this case $d\psi = \dfrac{d\rho}{\rho\sqrt{c\rho^{4/3}+K}}$. The substitution $\rho^{2/3}=v^{-1}$

gives the solution $-\frac{2\sqrt{K}}{3}\psi = \cosh^{-1} v\sqrt{\frac{K}{-c}}$, which can be written $\rho^{2/3} = \sqrt{\frac{K}{-c}} \operatorname{sech} \frac{2\sqrt{K}}{3}\psi$, where $\rho^{2/3} = \sqrt{\frac{K}{-c}}$ when $\psi = 0$.

---

(9) K negative.
In this case we find $-\frac{2\sqrt{-K}}{3}\psi + K_1 = \displaystyle\int \frac{dv}{\sqrt{-c/K - v^2}} = \sin^{-1} v\sqrt{\frac{-K}{c}}$.

This may be written in the form $\rho^{2/3} = \sqrt{-\frac{K}{c}} \sec \frac{2\sqrt{-K}}{3}\psi$,
where $\rho^{2/3} = \sqrt{\frac{-K}{c}}$ when $\psi = 0$.

---

(10) The particular case $K = -9/2$ gives $\frac{18}{9+2K} = \infty$, and the solution in this case is

$$\rho^{2/3} = \frac{3}{\sqrt{2c}} \sec \sqrt{2}\,\psi,$$ an equation

which has been discussed in Sections XVIII and XIX.

Section XXI.

Examples of curves at any point of which the length of the normal chord of the osculating conic is proportional to the length of the corresponding radius of curvature.

(1) Graph of $\rho = \frac{1}{2} s^3$ or $s^2 = -\frac{1}{\psi}$.

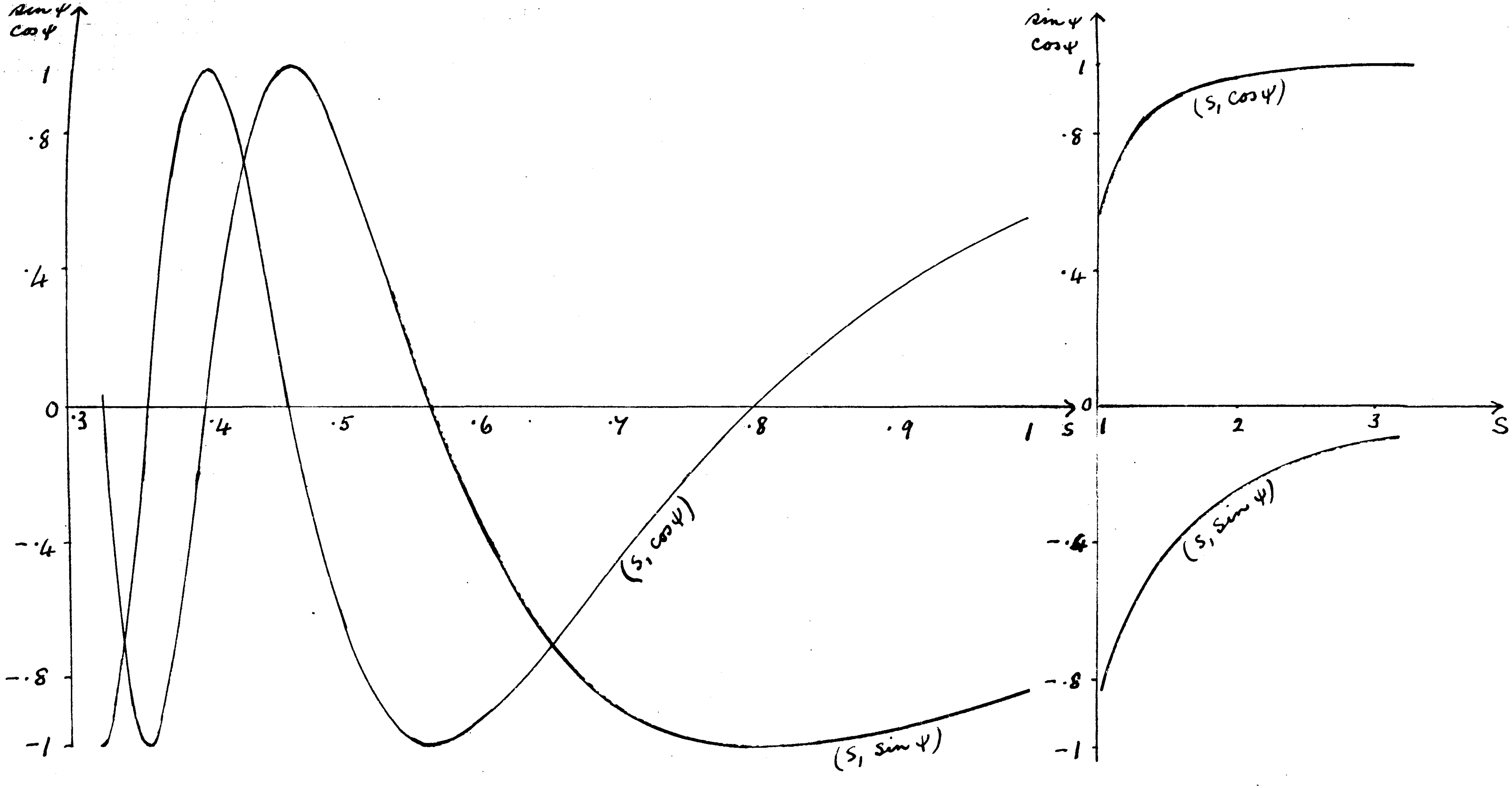
When s is zero, $\psi = -\infty$ and $\rho = 0$. The curve makes an infinite number of revolutions round the point $\rho = 0$, and finally goes off to infinity in the direction $\psi = 0$. The curve has an asymptote in this direction.

In this case $\rho = \frac{1}{2} s^3$, $\rho' = \frac{3}{2} s^2$, $\rho'' = 3s$, $9 + \rho'^2 - 3\rho\rho'' = 9(1 - \frac{1}{4} s^4)$. At the point $s = \sqrt{2}$, marked A on the graph, the osculating conic is a parabola.

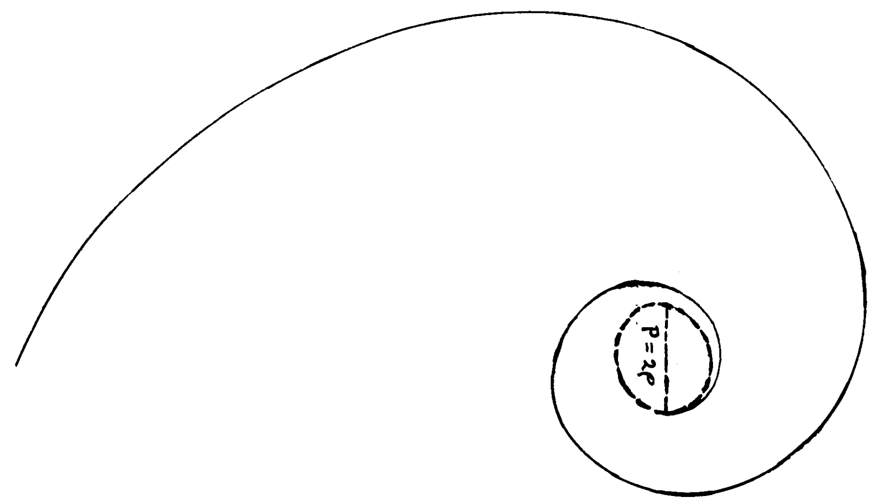
At the point $\psi = -4\pi$, we have $\rho = \cdot 01122$, $2\rho'^2 - 3\rho\rho'' = 0$, $l = +1$, $m = 0$, $3\rho' = \cdot 3582$, $s = \cdot 2821$. The equation of the osculating conic referred to that point as origin is $\xi^2 + \eta^2 - \cdot 08\,\xi\eta - \cdot 02245\,\eta = 0$. This cuts the radius of curvature at a distance $\cdot 02245$, i.e. at a distance 2ρ from $\psi = -4\pi$. This conic is shown on the figure.

Data for the graph of $s^2 = -1/\psi$.

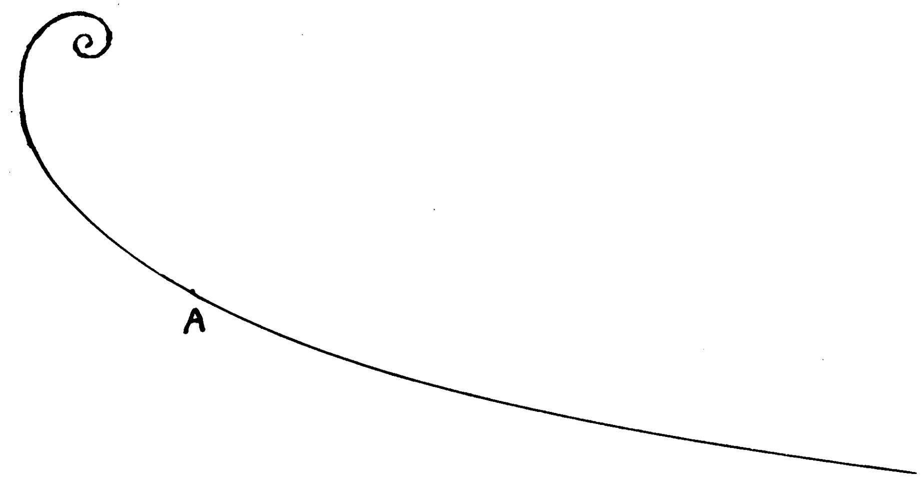
s	·28	·30	·32	·34	·36	·38	·40	·42	·44	·46
$2{\cdot}5x$	·017	·052	·022	−·021	−·035	−·010	·037	·083	·116	·128
$2{\cdot}5y$	−·058	−·030	·009	−·002	−·048	−·089	−·104	−·087	−·049	0
s	·48	·50	·54	·56	·58	·60	·62	·64	·66	·68
$2{\cdot}5x$	·118	·092	·009	−·040	−·089	−·138	−·183	−·229	−·260	−·291
$2{\cdot}5y$	·048	·091	·145	·153	·151	·137	·116	·087	·052	·013
s	·70	·90	1·2	1·6	2·0	2·4	2·8			
$2{\cdot}5x$	−·316	−·326	·117	·982	1·93	2·90	3·88			
$2{\cdot}5y$	−·031	−·518	−1·11	−1·61	−1·92	−2·12	−2·27			



For the graph of $\rho = \frac{1}{2} s^3$;
$(s, \cos\psi)$ and $(s, \sin\psi)$ diagrams.



Scale 25" = 1 unit.



Scale $2\frac{1}{2}$" = 1 unit.

Graph of $\rho = \frac{1}{2} s^3$.

(2) Graph of the equation $\rho^{-2/3} = \sqrt{\frac{c}{K}} \sinh \frac{2\sqrt{K}}{3}\psi$, when $K = \frac{9}{2}$, and $\frac{c}{K} = 4$, i.e. of the equation $\rho^{-2/3} = 2 \sinh \sqrt{2}\,\psi$.

When ψ is zero, ρ is infinite, and when ψ is infinite, ρ is zero. The curve therefore has an asymptote in the direction $\psi = 0$. The current point moves from infinity in the direction $\psi = 0$, and after making an infinite number of revolutions round $\rho = 0$ arrives at that point. The convergence to $\rho = 0$ is very rapid, the length of arc from $\rho = \cdot 008$ to $\rho = 0$ being only $\cdot 00347$ unit, while ψ increases from $2{\cdot}275$ to infinity.

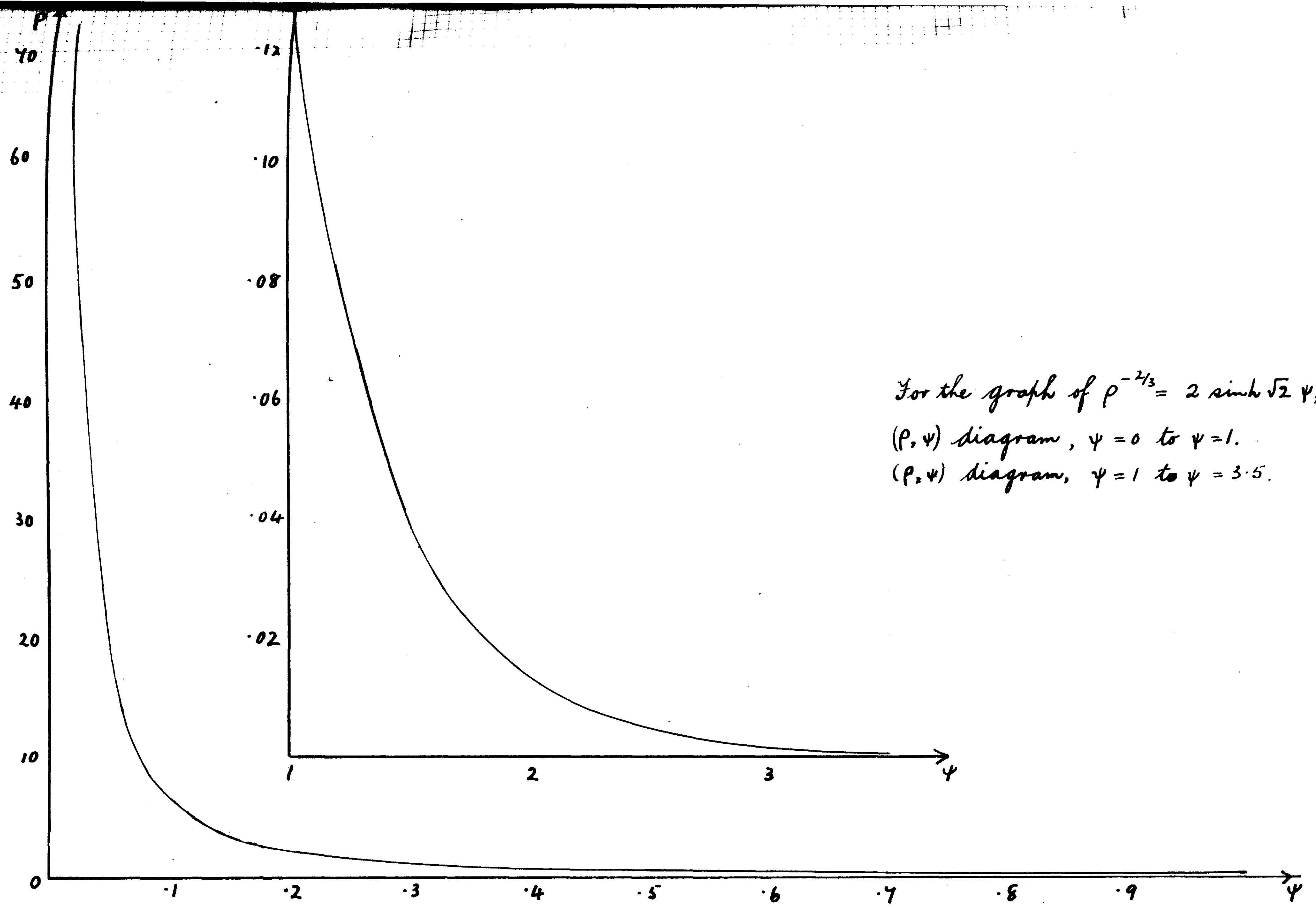
In this case $\rho'^2 = 18\rho^{4/3} + \frac{9}{2}$, $\rho'' = 12\rho^{\frac{1}{3}}$, $9 + \rho'^2 - 3\rho\rho'' = \frac{27}{2} - 18\rho^{4/3}$. At the point $\rho^{4/3} = \cdot 75$, i.e. $\rho = \cdot 8060$, the osculating conic is a parabola. The point is marked A on the graph.

When $\psi = \frac{\pi}{2}$, we have $l = 0$, $m = 1$, $\rho^{-2/3} = 2 \sinh \frac{\pi}{\sqrt{2}}$, $\rho = \cdot 035$, $\rho' = -2{\cdot}17$, $2\rho'^2 - 3\rho\rho'' = 9$, and the equation of the osculating conic is $2\xi^2 + \eta^2 - 1{\cdot}45\,\xi\eta + \cdot 07\xi = 0$, referred to the point $\psi = \frac{\pi}{2}$ as origin. The normal at $\xi = \frac{\pi}{2}$ meets this conic at a distance $\cdot 035$, which is the value of ρ. Since $\frac{18}{9+2K} = 1$, the osculating conic at any point of this curve passes through the corresponding centre of curvature.

Part of the conic $2\xi^2 + \eta^2 - 1{\cdot}45\,\xi\eta + \cdot 07\xi = 0$ is shown in the figure, and a part also of the circle of curvature is drawn in.

Values of s, x, y.

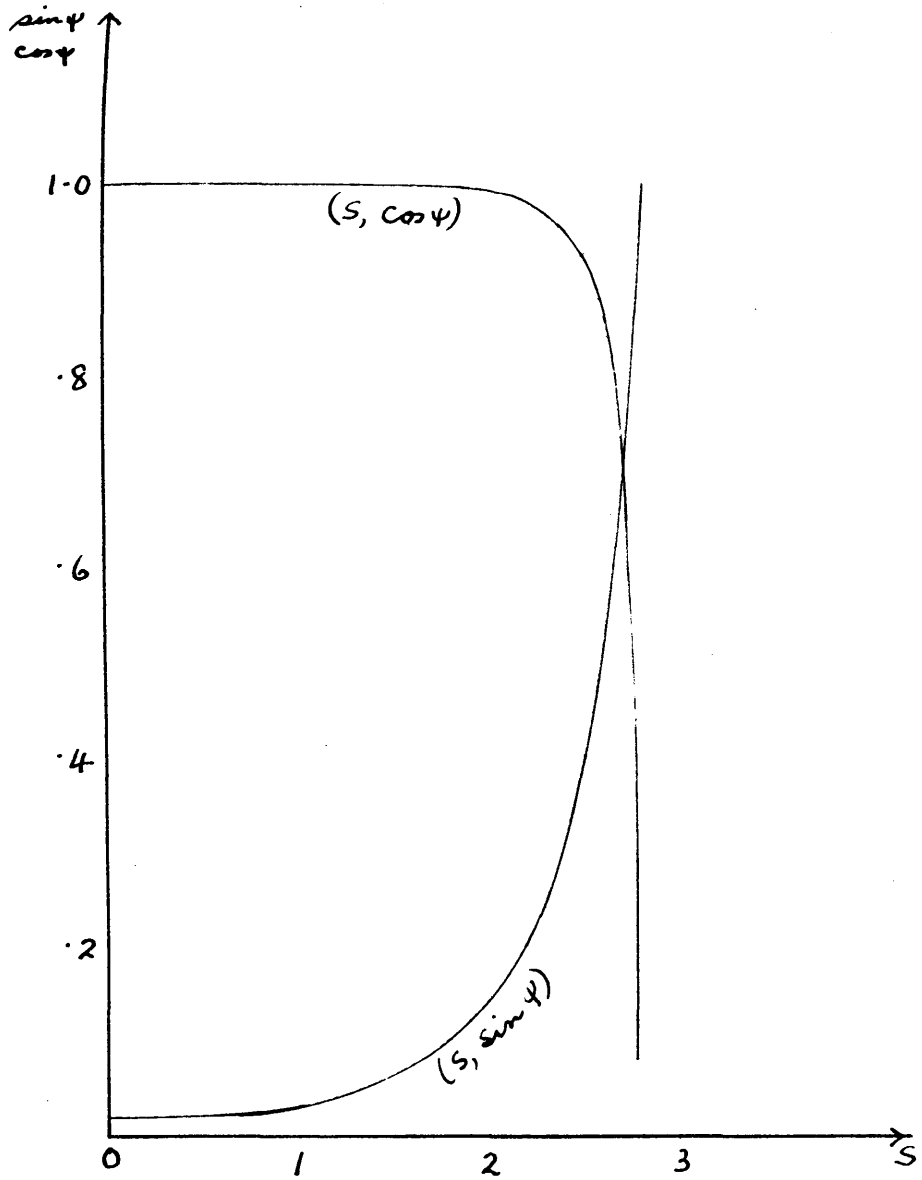
s	0	·2	·4	·6	·8	1·0	1·2	1·4	1·6	1·8
x	0	·2	·4	·6	·8	1·0	1·2	1·4	1·6	1·79
y	0	·00	·0[illegible]	·01	·01	·02	·02	·03	·04	·06
s	2·0	2·2	2·4	2·6	2·8						
x	1·99	2·19	2·38	2·57	2·68						
y	·08	·12	·17	·25	·40						
$100s$	1	1·5	2·0	2·5	3·0	3·5	4·0	4·5	5·0	5·5	5·75
$100x$	0	·224	·424	·600	·740	·844	·900	·888	·776	·516	·302
$100y$	0	·448	·908	1·38	1·86	2·34	2·84	3·34	3·89	4·25	4·37



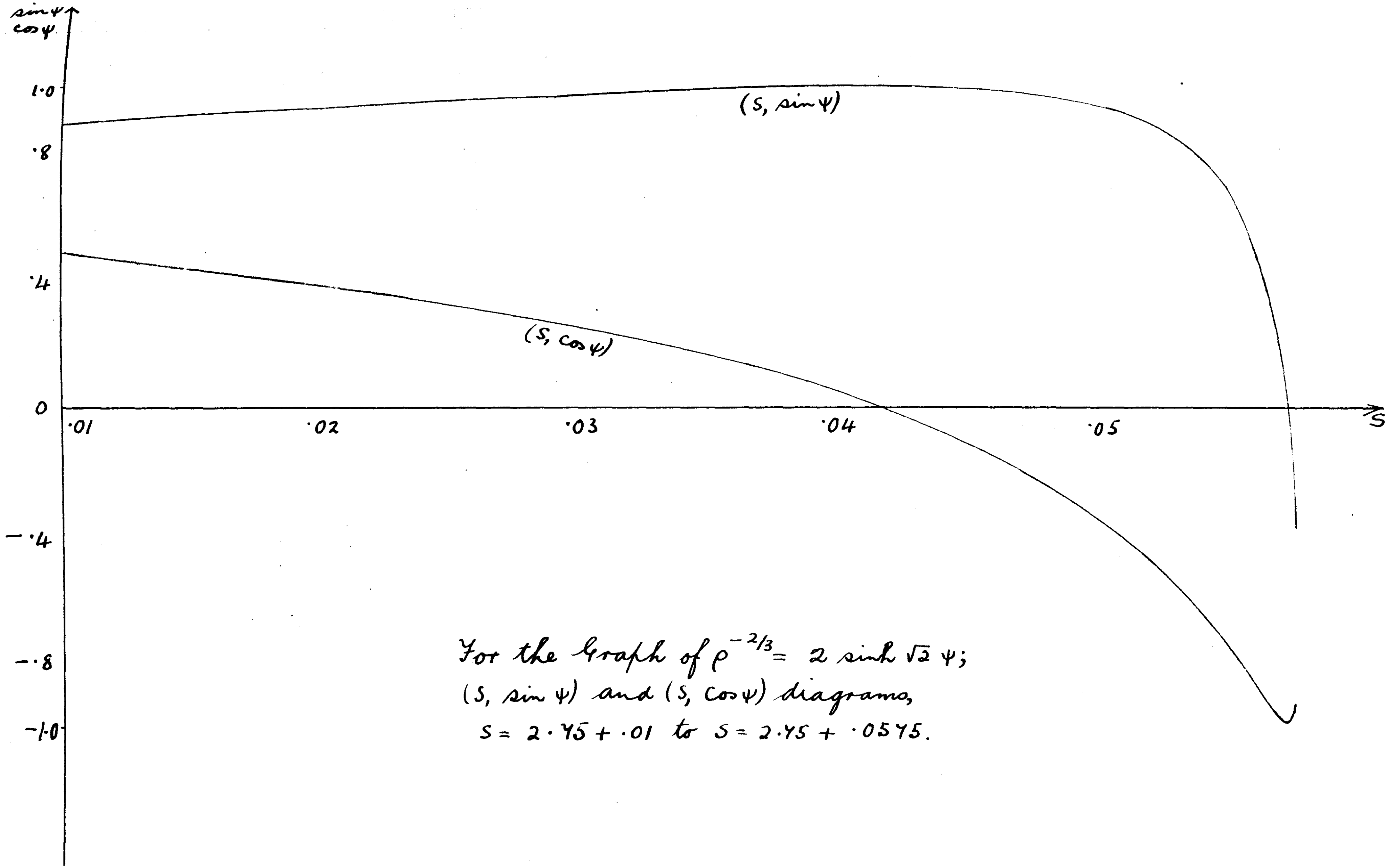
For the graph of $\rho^{-2/3} = 2 \sinh \sqrt{2}\, \psi$;

(ρ, ψ) diagram, $\psi = 0$ to $\psi = 1$.

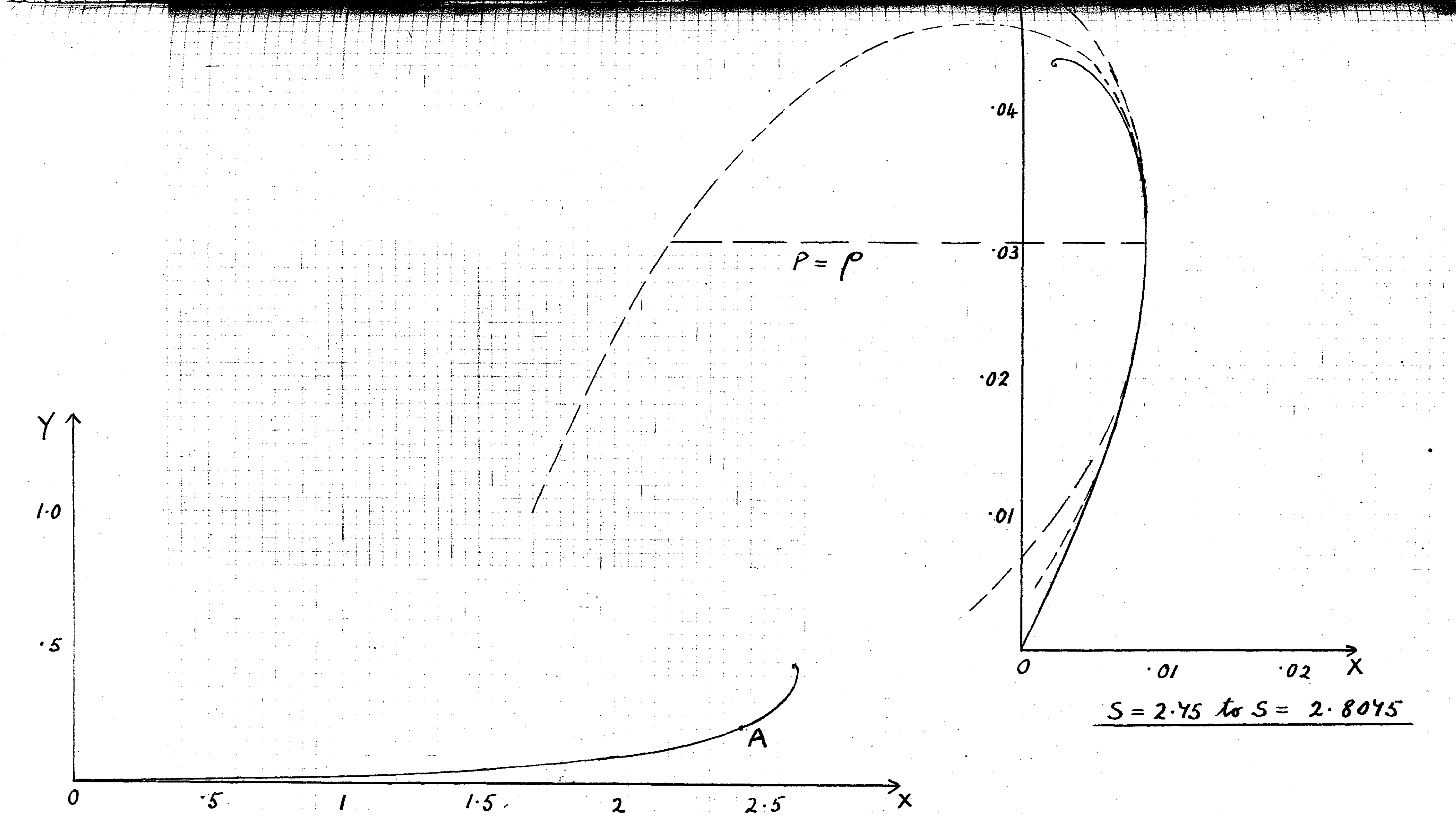
(ρ, ψ) diagram, $\psi = 1$ to $\psi = 3{\cdot}5$.



For the Graph of $\rho^{-2/3} = 2 \sinh \sqrt{2}\,\psi$;
$(S, \sin\psi)$ and $(S, \cos\psi)$ diagrams,
from $S = 0$ to $S = 2{\cdot}75$.



For the Graph of $\rho^{-2/3} = 2 \sinh \sqrt{2}\,\psi$;
(S, sin ψ) and (S, cos ψ) diagrams,
$S = 2\cdot 75 + \cdot 01$ to $S = 2\cdot 75 + \cdot 0575$.



S = 0 to S = 2·8

Graph of $\rho^{-2/3} = 2 \sinh \sqrt{2}\, \psi$.

(3) Graph of the equation $s = \int \frac{d\rho}{\sqrt{c\rho^{4/3} + K}}$, where $K = 9/2$, $c = -18$. The (ρ, ψ) equation is $\rho^{-2/3} = 2 \cosh \sqrt{2}\,\psi$.

ρ is finite, its maximum value being $\frac{1}{2\sqrt{2}}$, and occurring when $\psi = 0$.

When $\psi = \pm\infty$, $\rho = 0$. The curve therefore makes an infinite number of revolutions round these points, before finally arriving at them.

For large values of ψ the curve approximates very closely to that of $\rho^{-2/3} = 2 \sinh \sqrt{2}\,\psi$ or to that of $\rho^{-2/3} = e^{\sqrt{2}\,\psi}$.

The whole length of the curve from $\psi = -\infty$ to $\psi = +\infty$ is finite, and equal to about $\cdot 61$ unit.

The y-axis is a line of symmetry of the figure when the tangent at $\psi = 0$ is taken as x-axis.

In this case we have $\rho'^2 = -18\rho^{4/3} + \frac{9}{2}$; $\rho'' = -12\rho^{1/3}$; $9 + \rho'^2 - 3\rho\rho'' = \frac{27}{2} + 18\rho^{4/3}$.

The osculating conics are ellipses.

At $\psi = 0$, we have $l = 1$, $m = 0$;

$\rho^{-2/3} = 2$, $\rho = \cdot 3536$, $\rho' = 0$, $2\rho'^2 - 3\rho\rho'' = 9$, and the equation of the osculating conic at $\psi = 0$ is $\xi^2 + 2\eta^2 = \cdot 707\eta$. The normal at $\psi = 0$ meets this conic at a distance $\cdot 3535$, which is the value of ρ at that point.

The osculating conic at any point passes through the corresponding centre of curvature.

Part of the ellipse $\xi^2 + 2\eta^2 - \cdot 707\eta = 0$ is shown in the figure, along with part of the circle of curvature at that point.

ψ, ρ values for the graph of $\rho^{-2/3} = 2\cosh\sqrt{2}\,\psi$.

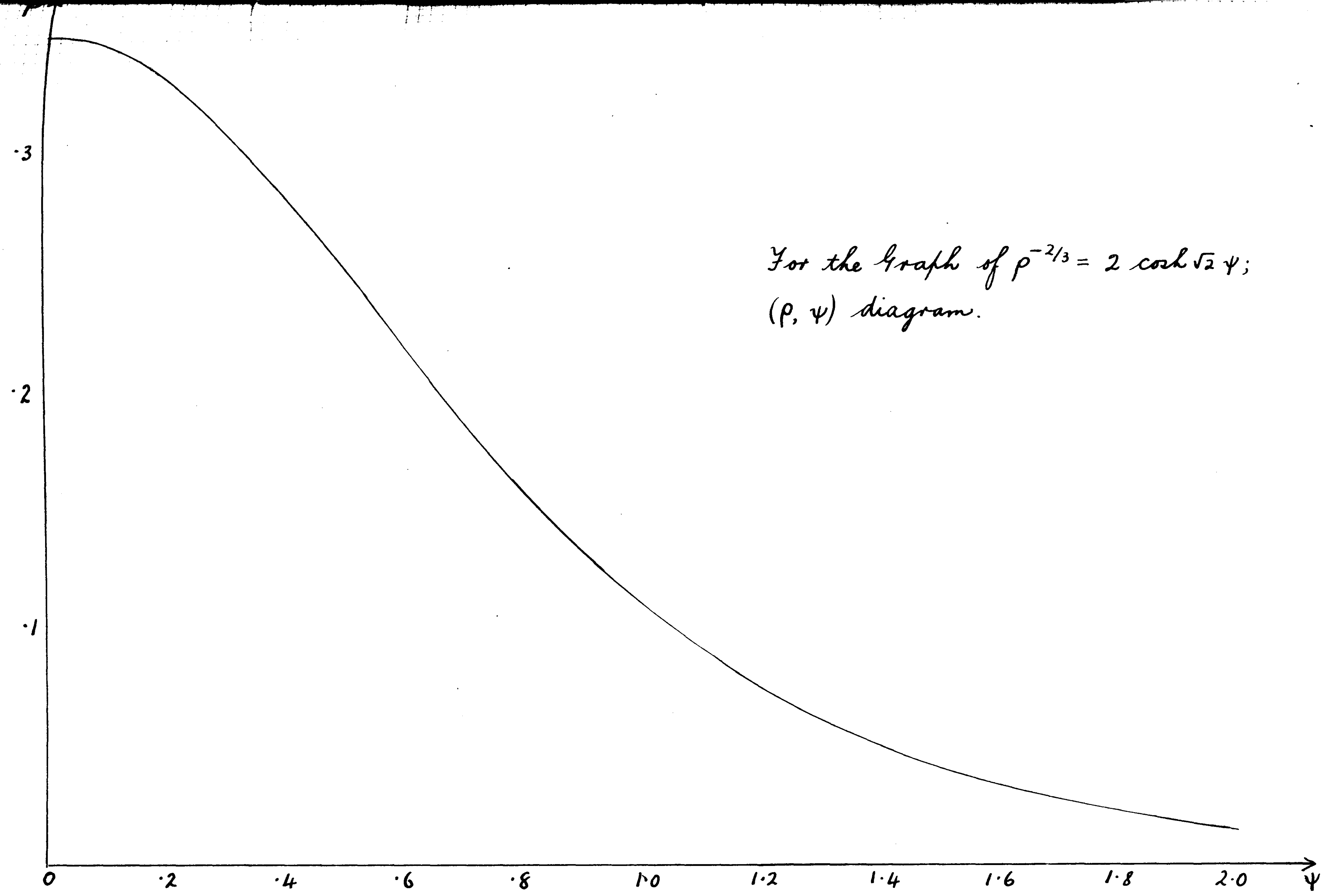
ψ	0	·088	·177	·354	·500	·707	·900	1·061	1·200	1·414	1·600
ρ	·354	·350	·337	·295	·253	·185	·132	·098	·075	·049	·033
ψ	1·768	2·000	2·121								
ρ	·023	·014	·011								

Values of s, ψ, $\cos\psi$, $\sin\psi$.

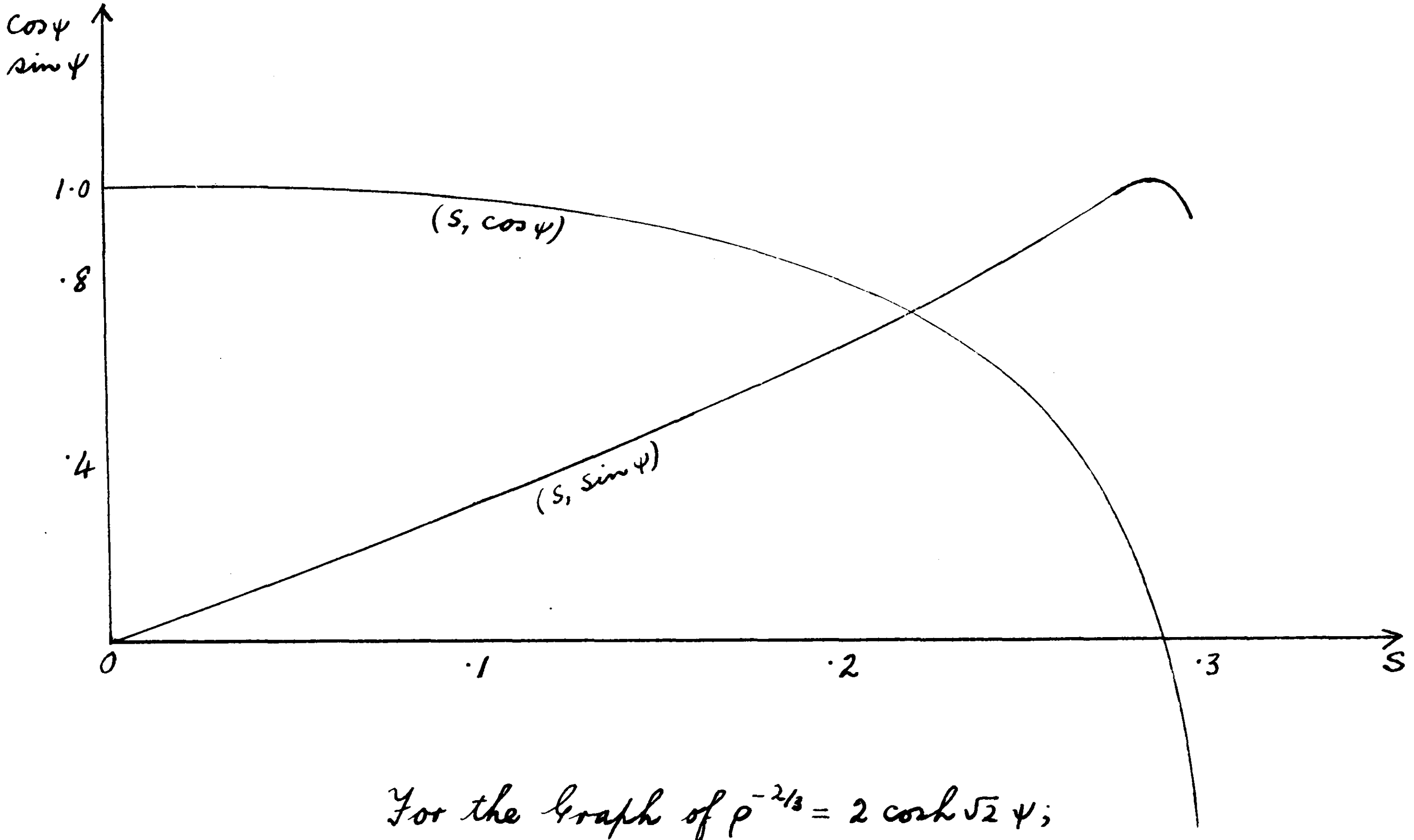
S	0	·069	·131	·181	·224	·251	·269	·281	·289	·295	·298
ψ	0	·2	·4	·6	·8	1·0	1·2	1·4	1·6	1·8	2·0
$50\cos\psi$	50	49·0	46·1	41·3	34·9	27·0	18·1	8·5	−1·5	−11·4	−20·8
$50\sin\psi$	0	10	19·5	28·3	35·8	42·1	46·6	49·3	50	48·7	45·5

Values of s, x, y.

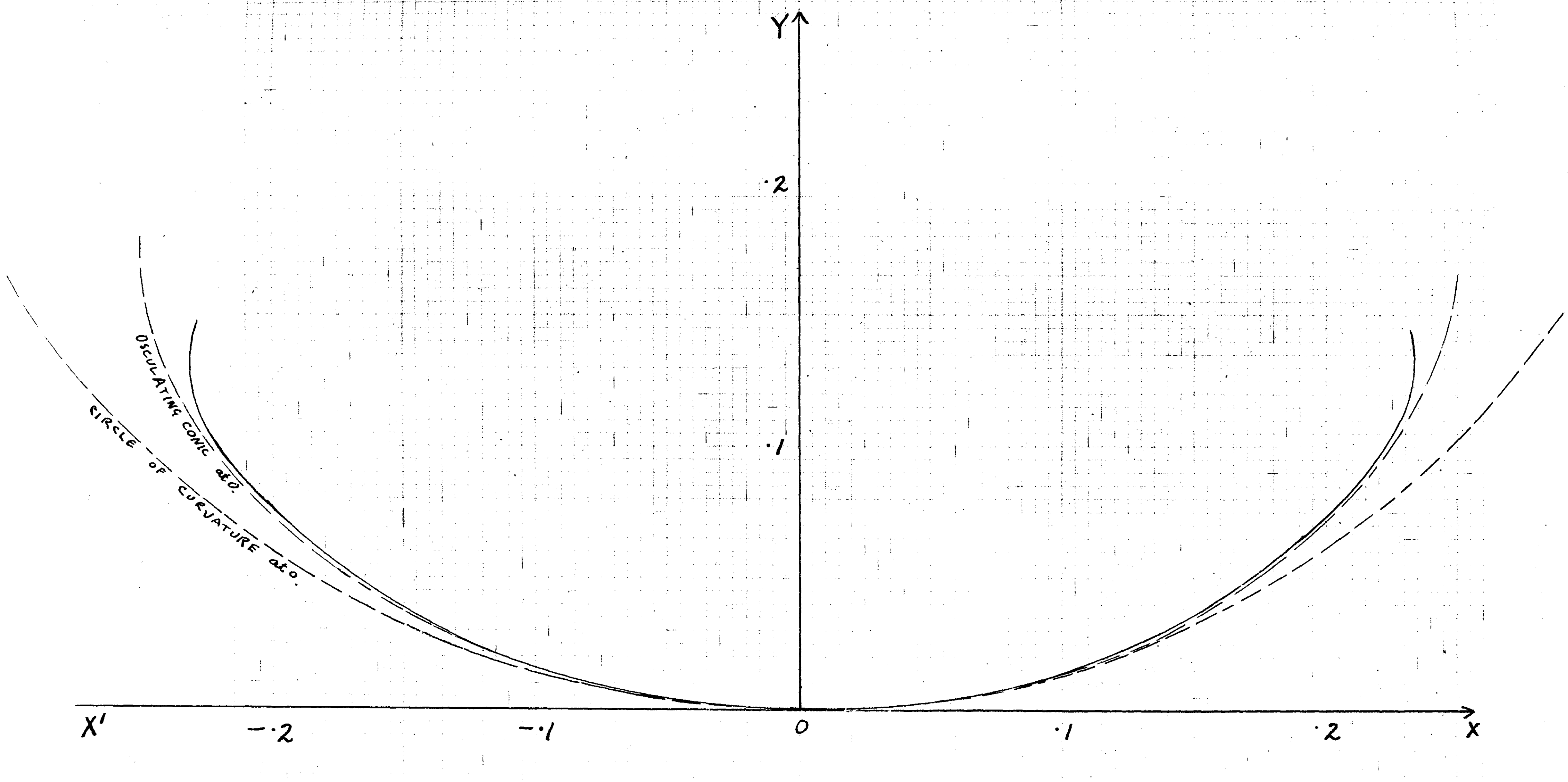
s	0	·05	·10	·15	·20	·250	·255	·260
x	0	·050	·098	·145	·187	·221	·224	·226
y	0	·003	·014	·033	·061	·097	·101	·105
s	·265	·270	·275	·280	·285	·290	·295	·300
x	·230	·231	·233	·234	·234	·234	·234	·232
y	·110	·115	·119	·124	·129	·134	·139	·143



For the Graph of $\rho^{-2/3} = 2 \cosh \sqrt{2}\, \psi$;
(ρ, ψ) diagram.



For the Graph of $\rho^{-2/3} = 2\cosh\sqrt{2}\,\psi$;
$(s, \cos\psi)$ and $(s, \sin\psi)$ diagrams.



Graph of $\rho^{-2/3} = 2 \cosh \sqrt{2}\, \psi$.

(4) Graph of the equation $s = \int \frac{d\rho}{\sqrt{c\rho^{4/3}+k}}$, when

$c = -k = \frac{81}{484}$, i.e. of the equation $\rho^{2/3} = \sec\frac{3\psi}{11}$.

ρ and s are infinite when $\psi = \pm\frac{11\pi}{6}$. The curve therefore has a pair of symmetrically-placed asymptotes in the directions $\pm\frac{11\pi}{6}$. The y-axis is a line of symmetry when the tangent at $\psi = 0$ is the x-axis.

In this case we have $\rho'^2 = \frac{81}{484}(\rho^{4/3}-1)$, $\rho'' = \frac{27}{242}\rho^{1/3}$, $9 + \rho'^2 - 3\rho\rho'' = \frac{4275}{484} - \frac{81\rho^{4/3}}{484}$. At the point $\rho = 19.58$, $\psi = 299°20'$, the osculating conic is a parabola. This point is marked A on the graph.

At $\psi = 0$, we have $l = 1$, $m = 0$, $\rho = 1$, $\rho' = 0$, $3\rho\rho'' = \frac{81}{242}$, $2\rho'^2 - 3\rho\rho'' = -\frac{81}{242}$, and the equation of the osculating conic at the origin is $9\xi^2 + \frac{2097}{242}\eta^2 - 18\eta = 0$. The normal at O meets this at a distance $\frac{484}{233}$, which is the value of $\frac{18\rho}{9+2k}$ at the origin.

The normal at any point of the curve meets the osculating conic at a distance whose ratio to the corresponding radius of curvature is $\frac{484}{233}$.

Data for the graph of $\rho^{2/3} = \sec\frac{3\psi}{11}$.

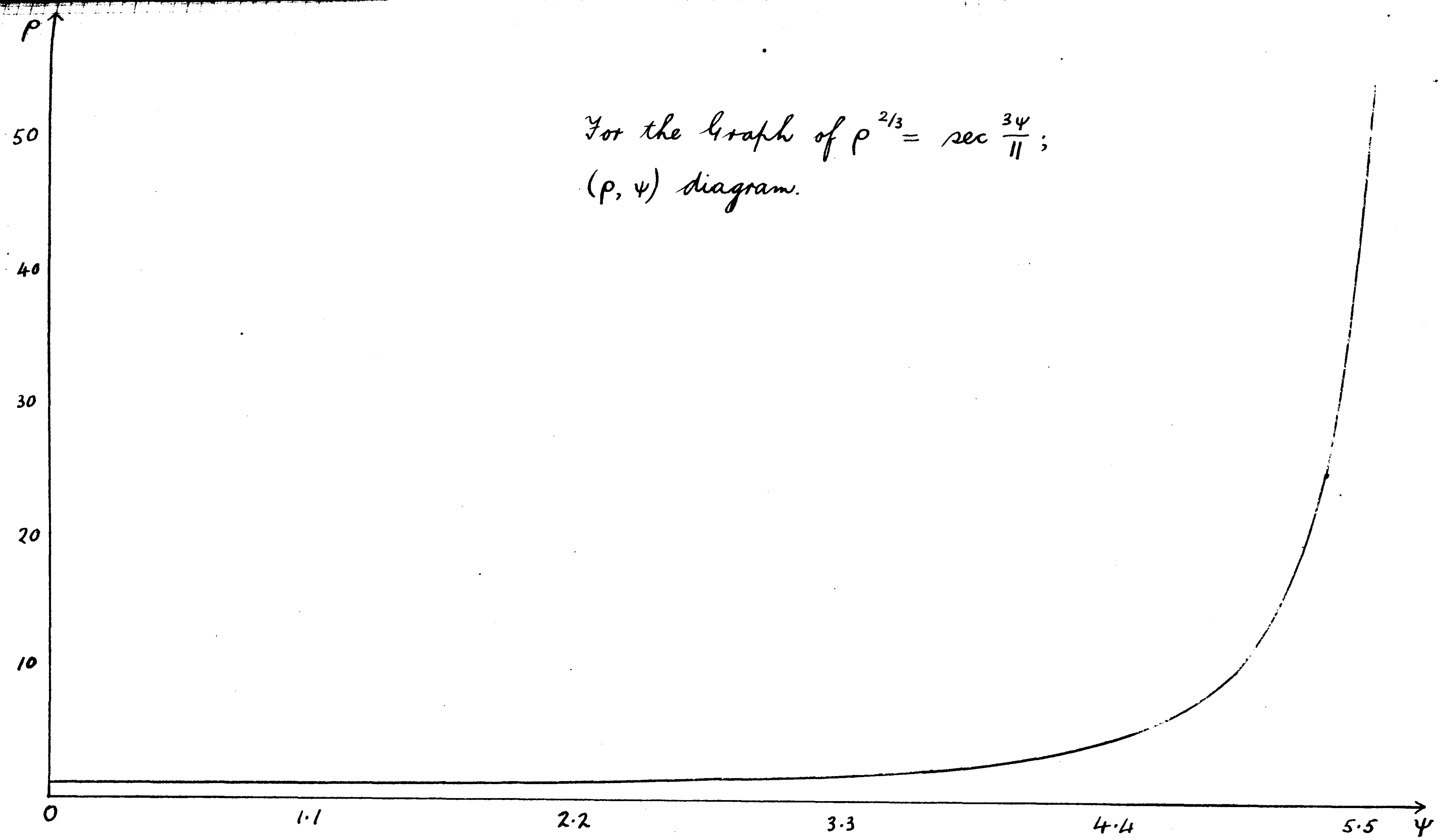
$\frac{3\psi°}{11}$	0	6	12	18	24	30	36	42	48	54	60	66	72	78
ρ	1	1·008	1·034	1·078	1·146	1·241	1·374	1·56	1·83	2·22	2·83	3·86	5·82	10·5

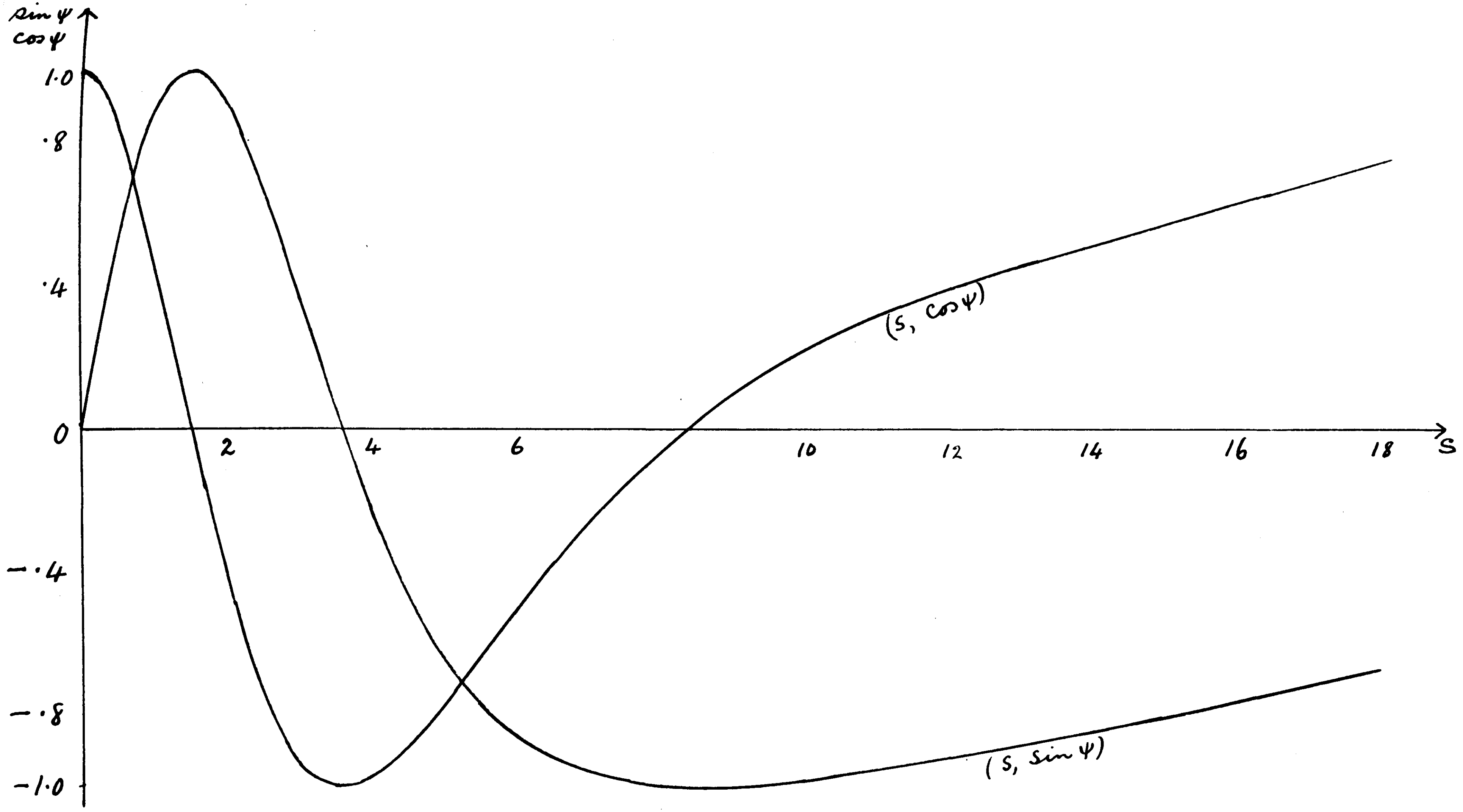
s	0	·51	1·03	1·60	2·23	2·98	3·88	5·05	6·85	10·24	17·76
$\psi°$	0	31·5	63	94·5	126·7	157·9	189·1	220·7	252·3	283·8	315
$50\cos\psi$	50	42·6	22·7	−3·9	−29·8	−46·3	−49·4	−37·9	−15·2	11·9	35·5
$50\sin\psi$	0	26·1	44·6	49·8	40·1	18·8	−7·9	−32·6	−47·6	−48·6	−35·2

s	0	·40	·8	1·2	1·6	2·0	2·4	2·8	3·2	3·6	4·0
x	0	·38	·70	·89	·93	·83	·61	·29	−·08	−·47	−·87
y	0	·04	·27	·62	1·02	1·40	1·73	1·98	2·13	2·18	2·14

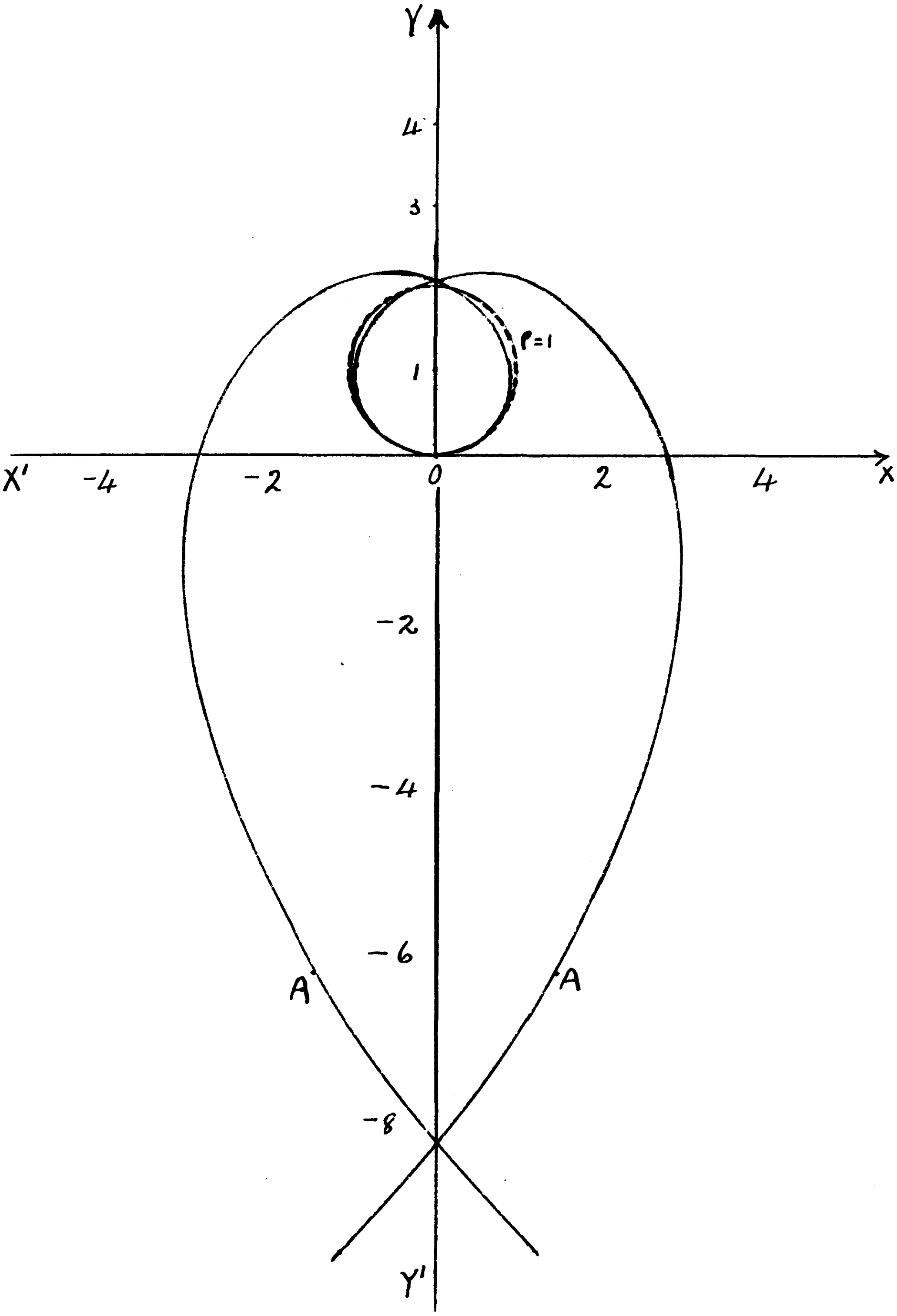
s	4·4	4·8	5·2	5·6	6·0	6·4	6·8	7·2	7·6	8·0	9	10
x	−1·24	−1·59	−1·89	−2·16	−2·39	−2·57	−2·71	−2·83	−2·90	−2·95	−2·94	−2·79
y	2·01	1·82	1·57	1·27	·94	·59	·21	−·17	−·56	−·96	−1·96	−2·94

s	11	12	13	14	15	16	17	18
x	−2·53	−2·20	−1·79	−1·33	−·81	−·23	·4	1·09
y	−3·71	−4·66	−5·57	−6·45	−7·29	−8·09	−8·85	−9·56


For the Graph of $\rho^{2/3} = \sec \frac{3\psi}{11}$;
(ρ, ψ) diagram.
ρ
50
40
30
20
10
0
1·1
2·2
3·3
4·4
5·5
ψ



For the Graph of $\rho^{2/3} = \sec \frac{3\psi}{11}$;
$(s, \cos\psi)$ and $(s, \sin\psi)$ diagrams.



Graph of the equation
$\rho^{2/3} = \sec \frac{3\psi}{11}$.

Section XXII.

To find a curve such that the angle between the tangent and the axis of aberrancy is a linear function of the angle between the x-axis and the axis of aberrancy.

(1) The coordinates of the centre of aberrancy are
$$\bar{\xi} = x + \frac{3\rho(l\rho' - 3m)}{9+\rho'^2 - 3\rho\rho''}, \quad \bar{\eta} = y + \frac{3\rho(m\rho' + 3l)}{9+\rho'^2 - 3\rho\rho''},$$
and the axis of aberrancy is $R = \dfrac{3\rho\sqrt{9+\rho'^2}}{9+\rho'^2 - 3\rho\rho''}$.

The expression for $\bar{\xi} - x$ may be written $\dfrac{3\rho\sqrt{9+\rho'^2}}{9+\rho'^2-3\rho\rho''} \times \dfrac{l\rho'-3m}{\sqrt{9+\rho'^2}}$,

i.e. $R\dfrac{l\rho'-3m}{\sqrt{9+\rho'^2}}$.

If we write $\rho' = 3\cot\alpha$, $\sqrt{9+\rho'^2} = +3\operatorname{cosec}\alpha$, we may put $\dfrac{l\rho'-3m}{\sqrt{9+\rho'^2}} = \cos(\psi+\alpha)$, where $l = \cos\psi$, $m = \sin\psi$,

and similarly we may put $\dfrac{m\rho'+3l}{\sqrt{9+\rho'^2}} = \sin(\psi+\alpha)$.

Thus we get $\bar{\xi} = x + R\cos(\psi+\alpha)$, $\bar{\eta} = y + R\sin(\psi+\alpha)$.
Thus the axis of aberrancy is inclined at an angle α to the tangent to the curve.

---

(2) Let α be a linear function of ψ, i.e. $\alpha = a\psi + \beta$.
Then $\rho' = 3\cot(a\psi+\beta)$; $\rho = c\sin^{\frac{3}{a}}(a\psi+\beta)$;
$$\rho\rho'' = -3a\operatorname{cosec}^2(a\psi+\beta);$$
$$9+\rho'^2 - 3\rho\rho'' = 9(1+a)\operatorname{cosec}^2(a\psi+\beta); \quad R = \frac{c}{1+a}\{\sin(a\psi+\beta)\}^{1+\frac{3}{a}};$$
$$\bar{\xi} = x + R\cos(\overline{a+1}\,\psi+\beta), \quad \bar{\eta} = y + R\sin(\overline{a+1}\,\psi+\beta).$$
Also $\dfrac{dx}{ds} = l = \cos\psi$, and $ds = \rho\, d\psi$,
$$\therefore\ x = \int\rho\cos\psi\, d\psi = c\int\cos\psi\sin^{3/a}(a\psi+\beta)\,d\psi,$$
$$\text{and } y = \int\rho\sin\psi\, d\psi = c\int\sin\psi\sin^{3/a}(a\psi+\beta)\,d\psi.$$

---

(3) If we rotate the axes in a negative direction through an angle $\frac{\beta}{a}$, and write

$x_1 = x\cos\frac{\beta}{a} - y\sin\frac{\beta}{a}$, $y_1 = x\sin\frac{\beta}{a} + y\cos\frac{\beta}{a}$,

$\bar{\xi}_1 = \bar{\xi}\cos\frac{\beta}{a} - \bar{\eta}\sin\frac{\beta}{a}$, $\bar{\eta}_1 = \bar{\xi}\sin\frac{\beta}{a} + \bar{\eta}\cos\frac{\beta}{a}$, then (x_1, y_1) and $(\bar{\xi}_1, \bar{\eta}_1)$ are the coordinates of (x, y) and $(\bar{\xi}, \bar{\eta})$ respectively referred to the new axes of coordinates. The new value of ψ will be ψ_1 where $\psi_1 = \psi + \frac{\beta}{a}$, or $\psi = \psi_1 - \frac{\beta}{a}$. This substitution leads to the forms:-

$$\alpha = a\psi_1;\ \rho' = 3\cot a\psi_1;\ \rho = c\sin^{3/a} a\psi_1;$$
$$\rho\rho'' = -3a\operatorname{cosec}^2 a\psi_1;\ 9 + \rho'^2 - 3\rho\rho'' = 9(1+a)\operatorname{cosec}^2 a\psi_1;$$
$$R = \frac{c}{1+a}(\sin a\psi_1)^{1+3/a};$$
$$\bar{\xi}_1 = x_1 + R\cos(a+1)\psi_1,\ \bar{\eta}_1 = y_1 + R\sin(a+1)\psi_1;$$
$$x_1 = \int\rho\cos\psi_1\, d\psi_1 = c\int\cos\psi_1 \sin^{3/a} a\psi_1\, d\psi_1,$$
$$y_1 = \int\rho\sin\psi_1\, d\psi_1 = c\int\sin\psi_1\sin^{3/a} a\psi_1\, d\psi_1.$$

The effect of this substitution is to make the angle α proportional to the angle which the tangent makes with a particular straight line which is chosen as the new x-axis. Therefore, unless $a = 0$, it is sufficient to examine the solutions of $\rho' = 3\cot a\psi$, the line $\psi = 0$ being taken as x-axis.

(4) If $a = 0$, the more general form gives $\rho' = 3\cot\beta$, an equiangular spiral, or a circle, or a straight line.

---

(5) The value $a = -1$ in $\rho' = 3\cot a\psi$ gives a parabola.

(6) The equation $\rho' = 3\cot(-2\psi)$ satisfies $18 + 2\rho'^2 - 3\rho\rho'' = 0$, and represents the rectangular hyperbola $xy = -\frac{1}{2}c^2$.

(7) The equation $\rho' = 3\cot(-3\psi)$ gives $R = \frac{c}{2}$. The curve therefore belongs to the class of curves for which R is constant. It is discussed in Sections VII and VIII. In this case the angle between the x-axis and the tangent is one-third of the angle between the negative direction of the tangent and the direction of R.

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(8) If we desire to find the points on the curve $\rho = c \sin^{3/a} a\psi$ at which the conic of closest contact has six-point contact, we calculate ρ', ρ'', ρ''' and substitute in the equation

$$36\rho' + 4\rho'^3 - 9\rho\rho'\rho'' + 9\rho^2\rho''' = 0.$$

We find

$$\rho = c \sin^{3/a} a\psi;$$
$$\rho' = 3 \cot a\psi;$$
$$\rho'' = -\frac{3a}{c} \operatorname{cosec}^{(2+\frac{3}{a})} a\psi;$$
$$\rho\rho''' = \frac{3a^2}{c}\left(2 + \frac{3}{a}\right) \cot a\psi \operatorname{cosec}^{(2+3/a)} a\psi;$$

and the required condition is

$$\cot a\psi \operatorname{cosec}^2 a\psi \,(2 + 3a + a^2) = 0.$$

This equation is satisfied when $a = -1$ or $a = -2$. This implies that the curves $\rho = c \sin^{-3}\psi$ and $\rho = c\sin^{-3/2} 2\psi$ have contact of at least the fifth order at every point with their conics of closest contact, and therefore must be conic sections. As we have already seen they are equations of a parabola and of a rectangular hyperbola respectively.

The equation is also satisfied when $a\psi = \frac{\pi}{2}(2n-1)$, at which points $\rho' = 0$. Therefore the points on the curve $\rho = c \sin^{3/a} a\psi$ at which ρ is a maximum or a minimum are such that the corresponding conics of closest contact have six-point contact with the curve.

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(9) If we apply the result obtained in Section III, para. 20, we find in this case $\Delta = \operatorname{cosec}^2 a\psi$, $\Theta = \frac{1}{\rho}$, and the equation of condition for sextactic points is

$$\frac{\cot a\psi \operatorname{cosec}^2 a\psi}{\operatorname{cosec}^4 a\psi} = 0,$$

i.e. $\sin 2a\psi = 0$, which includes the points $\rho = 0$, which are cuspidal, as well as the points $\cot a\psi = 0$ found above.

Section XXIII.

Examples of curves such that the angle between the tangent and the axis of aberrancy is proportional to the angle between the x-axis and the tangent.

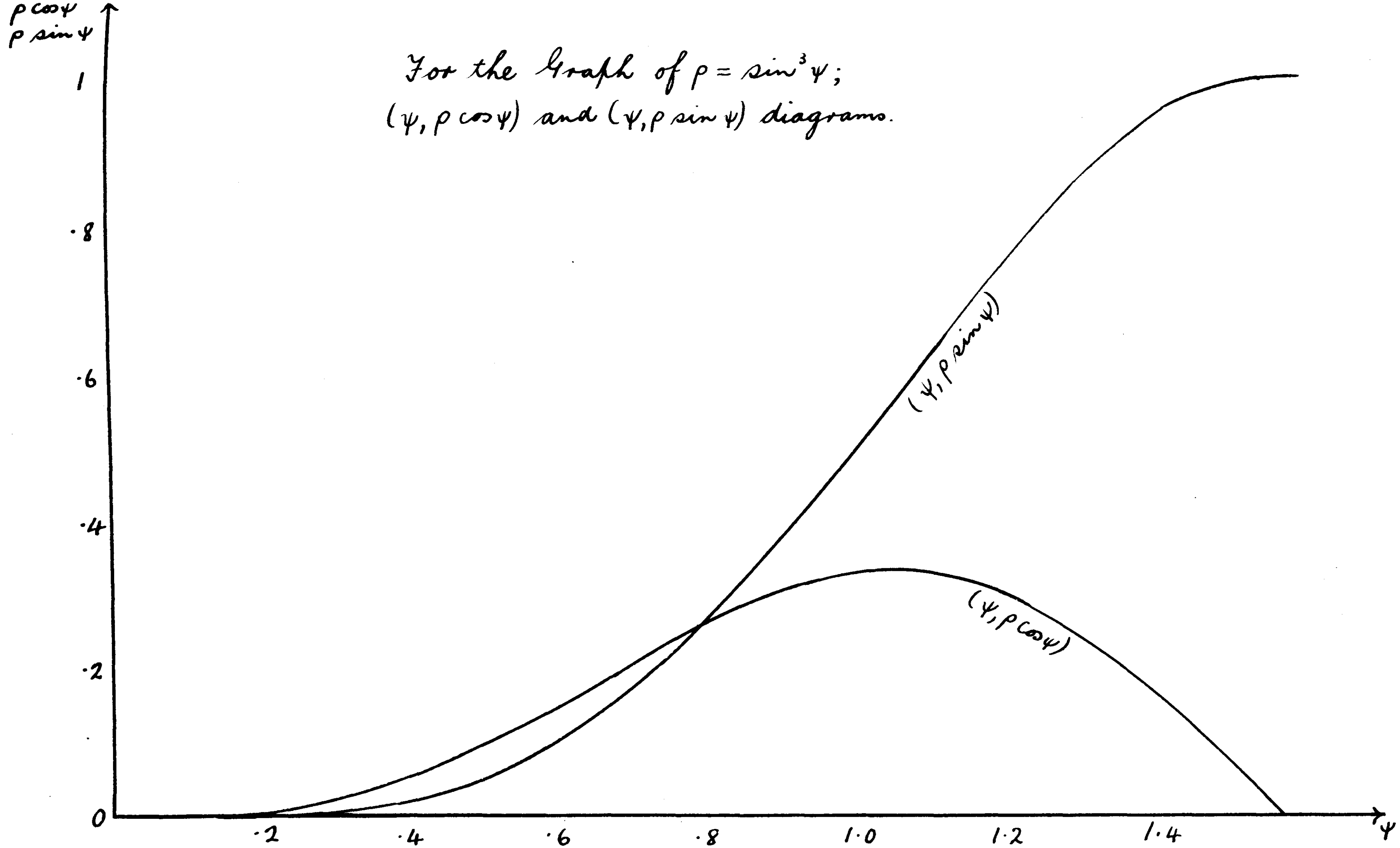
(1) Graph of the equation $\rho' = 3 \cot a\psi$, when $a = 1$.
We have $\frac{d\rho}{\rho d\psi} = 3 \cot \psi$, and therefore $\rho = c \sin^3 \psi$.
It follows that $\rho\rho'' = -3 \operatorname{cosec}^2 \psi$; $9 + \rho'^2 - 3\rho\rho'' = 18 \operatorname{cosec}^2 \psi$;
$R = \frac{c}{2} \sin^4 \psi$;
$\bar{\xi} = x + \frac{c}{2} \sin^4 \psi \cos 2\psi$, $\bar{\eta} = y + \frac{c}{2} \sin^4 \psi \sin 2\psi$;
$x = \int \rho \cos \psi \, d\psi = \frac{c}{4} \sin^4 \psi$, $y = \frac{c}{32}(12\psi - 8 \sin 2\psi + \sin 4\psi)$.
The angle between the x-axis and the tangent is equal to the angle between the tangent and the axis of aberrancy.

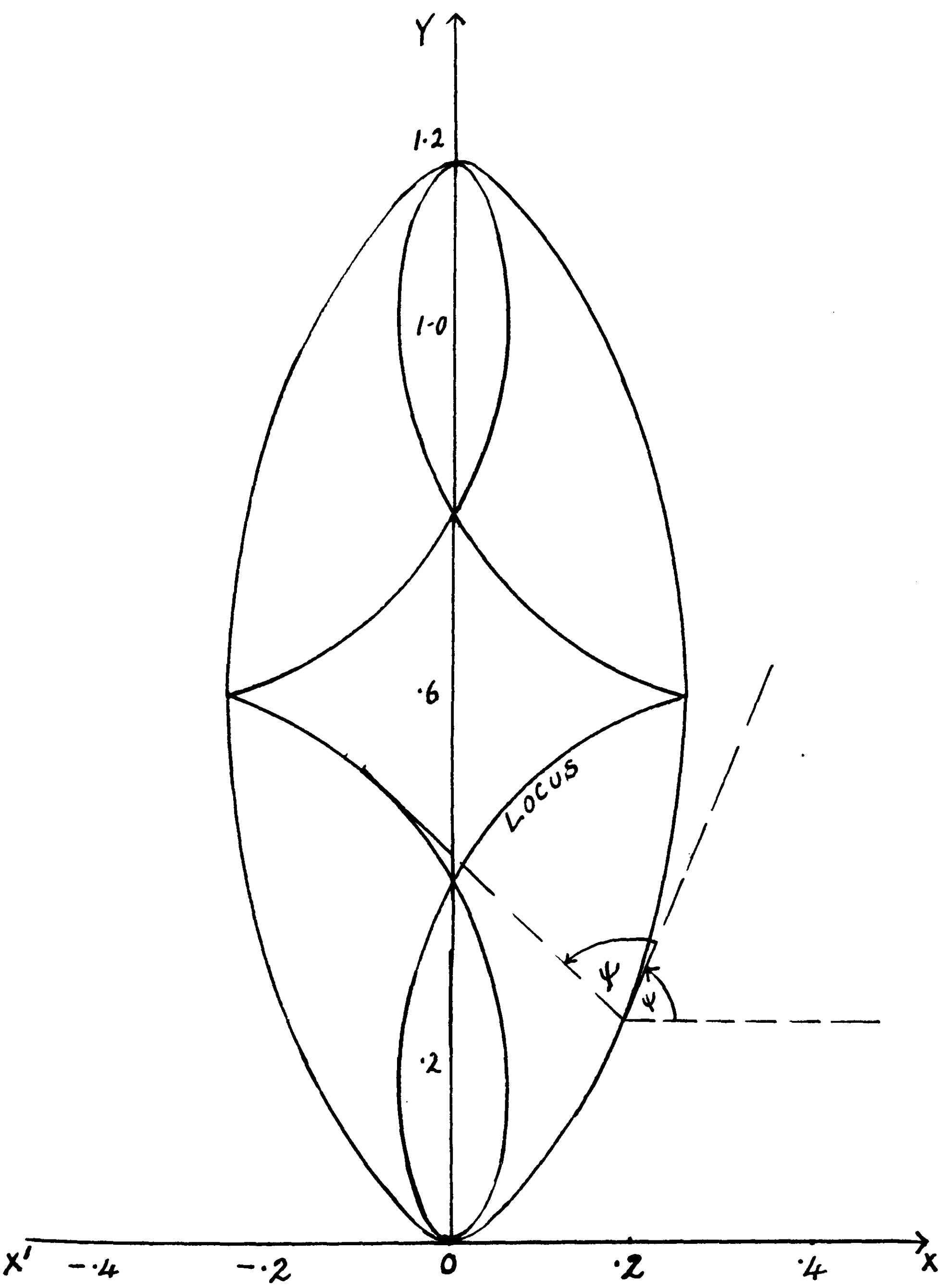
The part of the graph which lies to the left of the y-axis corresponds to $\alpha = \psi - \pi$, and is such that the angle between the tangent and the axis of aberrancy is equal to the angle between the negative x-axis and the tangent.

ψ	0	·088	·174	·262	·350	·436	·524	·610	·698
$\rho \cos\psi$	0	·001	·005	·010	·038	·068	·108	·154	·204
$\rho \sin\psi$	0	·000	·002	·004	·014	·026	·062	·108	·170
ψ	·786	·872	·960	1·048	1·134	1·222	1·309	1·396	1·484	1·571
$\rho \cos\psi$	·250	·290	·316	·324	·314	·284	·234	·166	·086	0
$\rho \sin\psi$	·250	·344	·450	·562	·674	·780	·870	·946	·984	1

$10x$	0	·00	·01	·03	·07	·15	·27	·42	·62
$10y$	0	·00	·00	·01	·03	·07	·14	·26	·47
$10\bar{\xi}$	0	·01	·03	·08	·15	·31	·46	·57	·62
$10\bar{\eta}$	0	·00	·01	·06	·13	·34	·65	1·10	1·72
$10x$	·81	1·11	1·39	1·67	1·92	2·15	2·32	2·44	2·50
$10y$	·70	1·04	1·49	2·02	2·65	3·36	4·15	5·02	5·89
$10\bar{\xi}$	·51	·34	−·02	−·50	−1·07	−1·62	−2·12	−2·40	−2·50
$10\bar{\eta}$	2·40	3·16	3·92	4·61	5·15	5·54	5·77	5·87	5·89

For the Graph of $\rho = \sin^3\psi$;
$(\psi, \rho\cos\psi)$ and $(\psi, \rho\sin\psi)$ diagrams.





Graph of $\rho = \sin^3\psi$, and the locus of its centres of aberrancy.

Graph of the equation $\rho' = 3 \cot a\psi$, when $a = \frac{1}{2}$.

(2) We have $\frac{d\rho}{\rho\, d\psi} = 3 \cot \frac{1}{2}\psi$, and therefore $\rho = c \sin^6 \frac{\psi}{2}$.

It follows that $\rho\rho'' = -\frac{3}{2}\operatorname{cosec}^2\frac{\psi}{2}$; $9+\rho'^2-3\rho\rho'' = \frac{27}{2}\operatorname{cosec}^2\frac{\psi}{2}$;

$R = \frac{2c}{3}\sin^7\frac{\psi}{2}$;

$\bar{\xi} = x + \frac{2c}{3}\sin^7\frac{\psi}{2}\cos\frac{3\psi}{2}$, $\bar{\eta} = y + \frac{2c}{3}\sin^7\frac{\psi}{2}\sin\frac{3\psi}{2}$;

$$x = c\int \sin^6\frac{\psi}{2}\cos\psi\, d\psi = \frac{c}{8}\left(-\frac{15}{8}\psi + \frac{13}{4}\sin\psi - \sin 2\psi + \frac{1}{4}\sin 3\psi - \frac{1}{32}\sin 4\psi\right)$$

$$y = c\int \sin^6\frac{\psi}{2}\sin\psi\, d\psi = \frac{c}{2}\sin^8\frac{\psi}{2}.$$

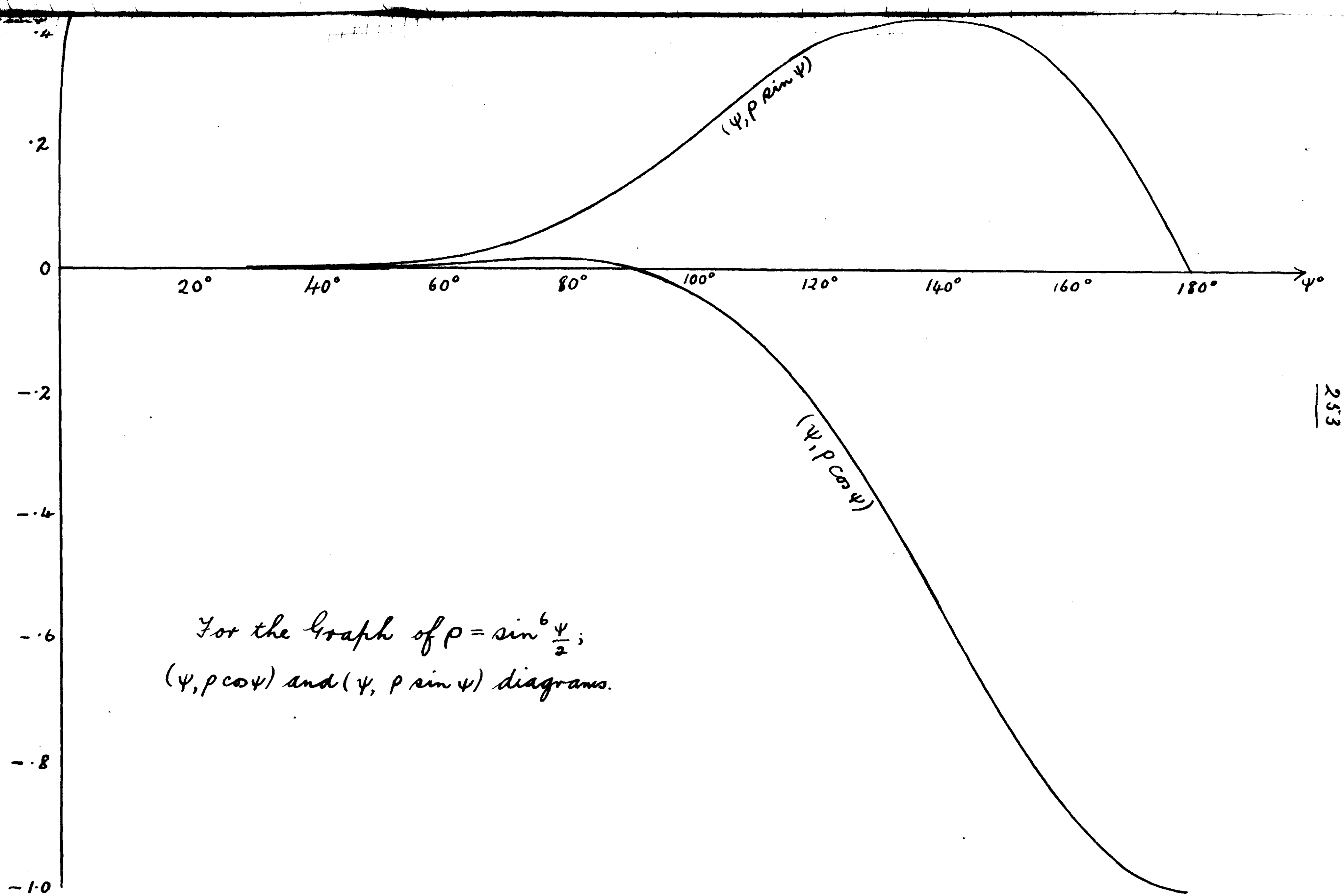
The angle between the x-axis and the tangent is twice the angle between the tangent and the direction of R.

The part of the graph which lies below the x-axis corresponds to $\alpha = \psi + \pi$, and is such that the angle between the tangent and the axis of aberrancy is half that between the negative x-axis and the tangent.

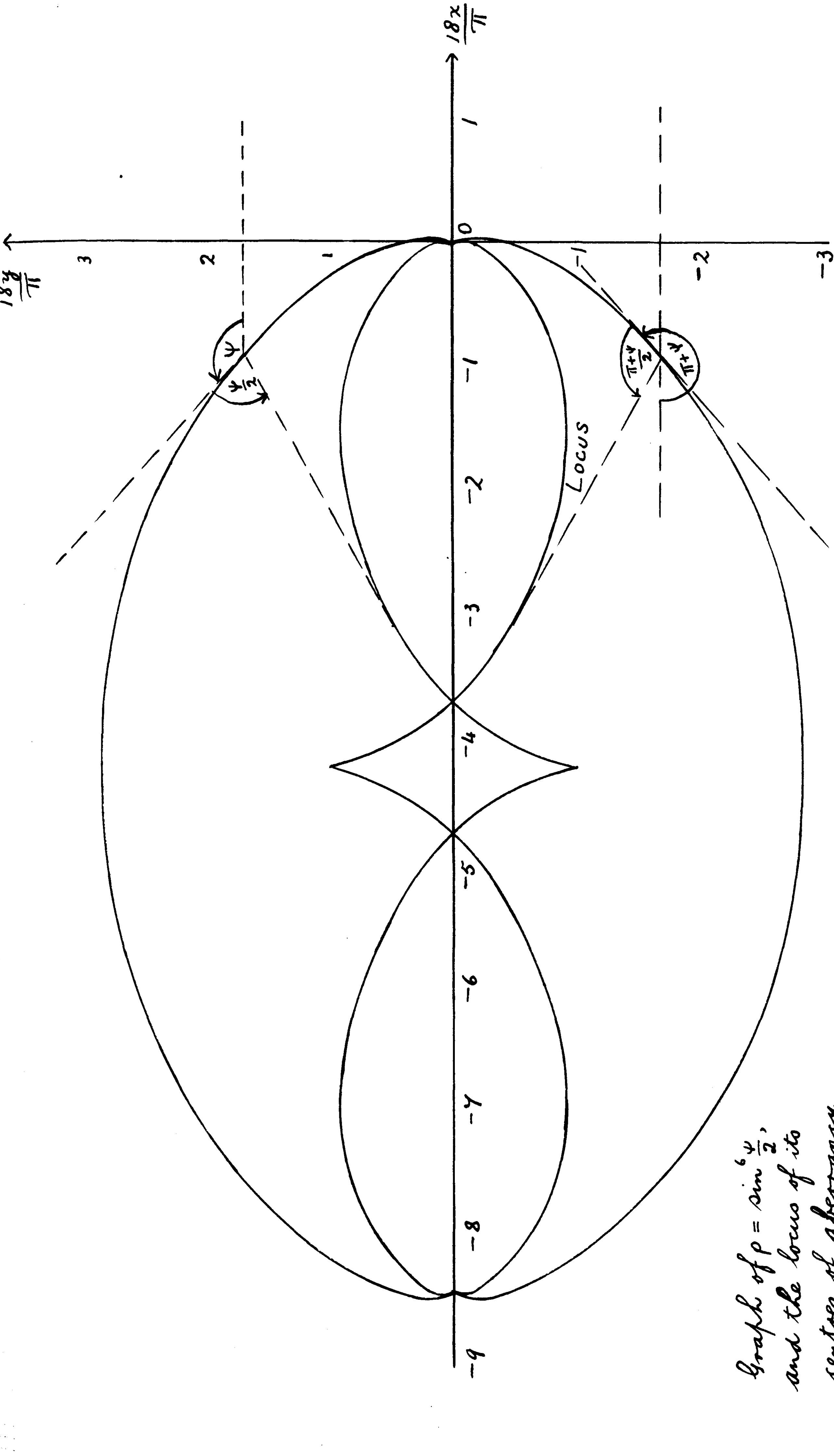
Data for the graph of $\rho' = 3\cot\frac{\psi}{2}$, $\rho = \sin^6\frac{\psi}{2}$.

ψ (deg.)	30	40	50	60	70	80	90	100
$\rho\cos\psi$	·000	·001	·004	·008	·012	·012	0	−·035
$\rho\sin\psi$	·000	·001	·004	·014	·033	·070	·125	·199
ψ (deg.)	110	120	130	140	150	160	170	180
$\rho\cos\psi$	−·103	−·211	−·356	−·528	−·703	−·858	−·962	−1
$\rho\sin\psi$	·284	·365	·425	·443	·406	·312	·170	0

$\frac{18}{\pi}x$	·01	·02	·03	·04	·02	−·05	−·21	−·49
$\frac{18}{\pi}y$	·01	·04	·09	·18	·35	·59	·91	1·31
$\frac{18}{\pi}\bar{\xi}$	·00	·00	−·06	−·20	−·49	−·96	−1·61	−2·39
$\frac{18}{\pi}\bar{\eta}$	·04	·12	·24	·42	·65	·84	·91	·81
$\frac{18}{\pi}x$	−·93	−1·55	−2·33	−3·24	−4·22			
$\frac{18}{\pi}y$	1·74	2·17	2·53	2·77	2·85			
$\frac{18}{\pi}\bar{\xi}$	−3·07	−3·72	−4·05	−4·21	−4·22			
$\frac{18}{\pi}\bar{\eta}$	·50	·00	−·45	−·84	−·98			



For the Graph of $\rho = \sin^6 \frac{\psi}{2}$;
$(\psi, \rho\cos\psi)$ and $(\psi, \rho\sin\psi)$ diagrams.



Graph of $\rho = \sin^6 \frac{\psi}{2}$, and the locus of its centres of aberrancy.

Graph of the equation $\rho' = 3 \cot a\psi$, when $a = \frac{1}{3}$.

(3) The equation is $\frac{d\rho}{\rho} = \frac{3}{9} \cot\frac{\psi}{3}\, d\psi$, the solution of which is $\rho = c \sin^9 \frac{\psi}{3}$.

It follows that $\rho\rho'' = -\operatorname{cosec}^2\frac{\psi}{3}$; $9 + \rho'^2 - 3\rho\rho'' = 12 \operatorname{cosec}^2\frac{\psi}{3}$;

$R = \frac{3c}{4} \sin^{10}\frac{\psi}{3}$;

$$x = c\int \sin^9\frac{\psi}{3} \cos\psi\, d\psi,$$

$$y = c\int \sin^9\frac{\psi}{3} \sin\psi\, d\psi;$$

$$\bar{\xi} = x + \frac{3c}{4} \sin^{10}\frac{\psi}{3} \cos\frac{4\psi}{3},$$

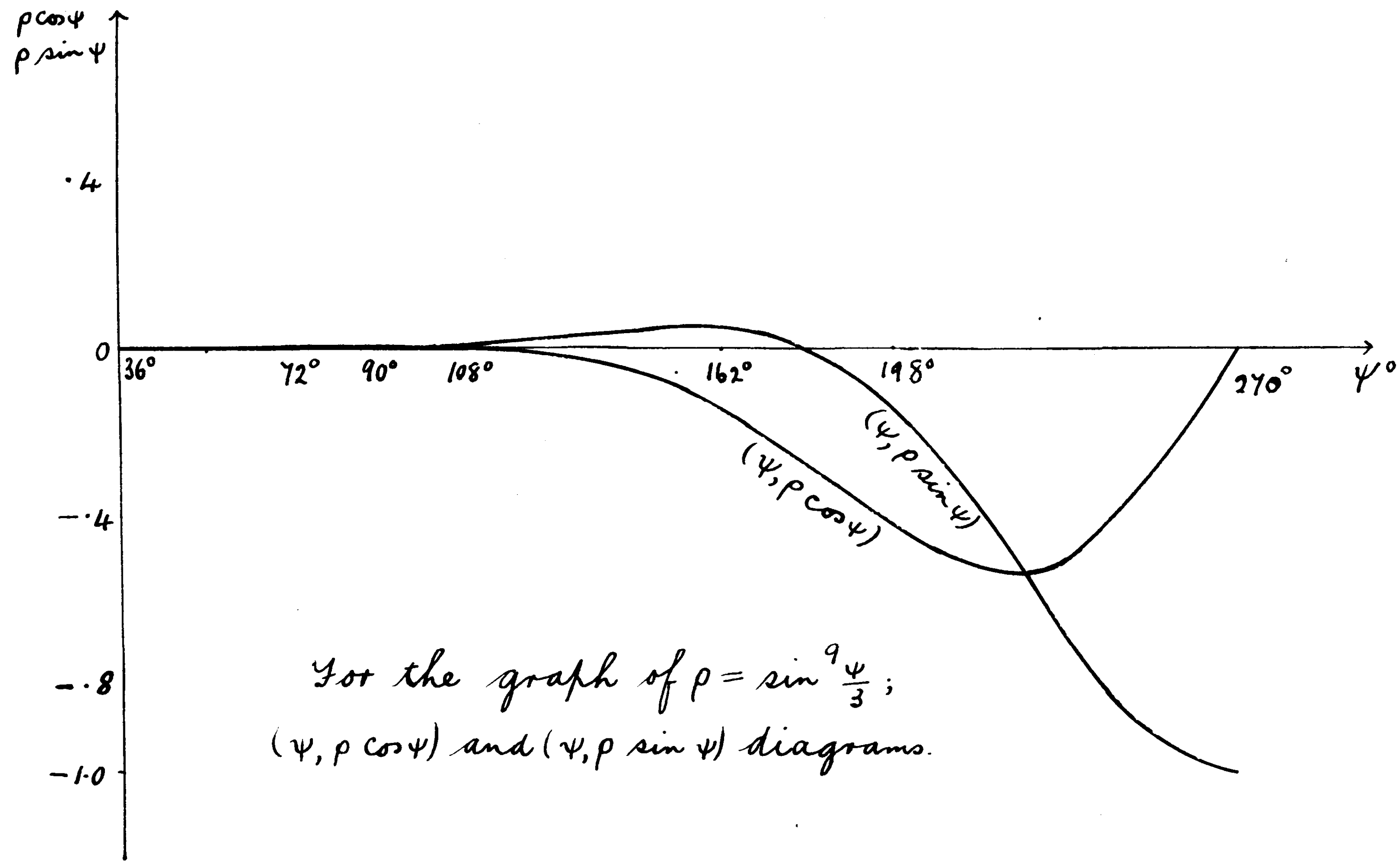
$$\bar{\eta} = y + \frac{3c}{4} \sin^{10}\frac{\psi}{3} \sin\frac{4\psi}{3}.$$

The angle between the x-axis and the tangent is three times the angle between the tangent and the direction of R.

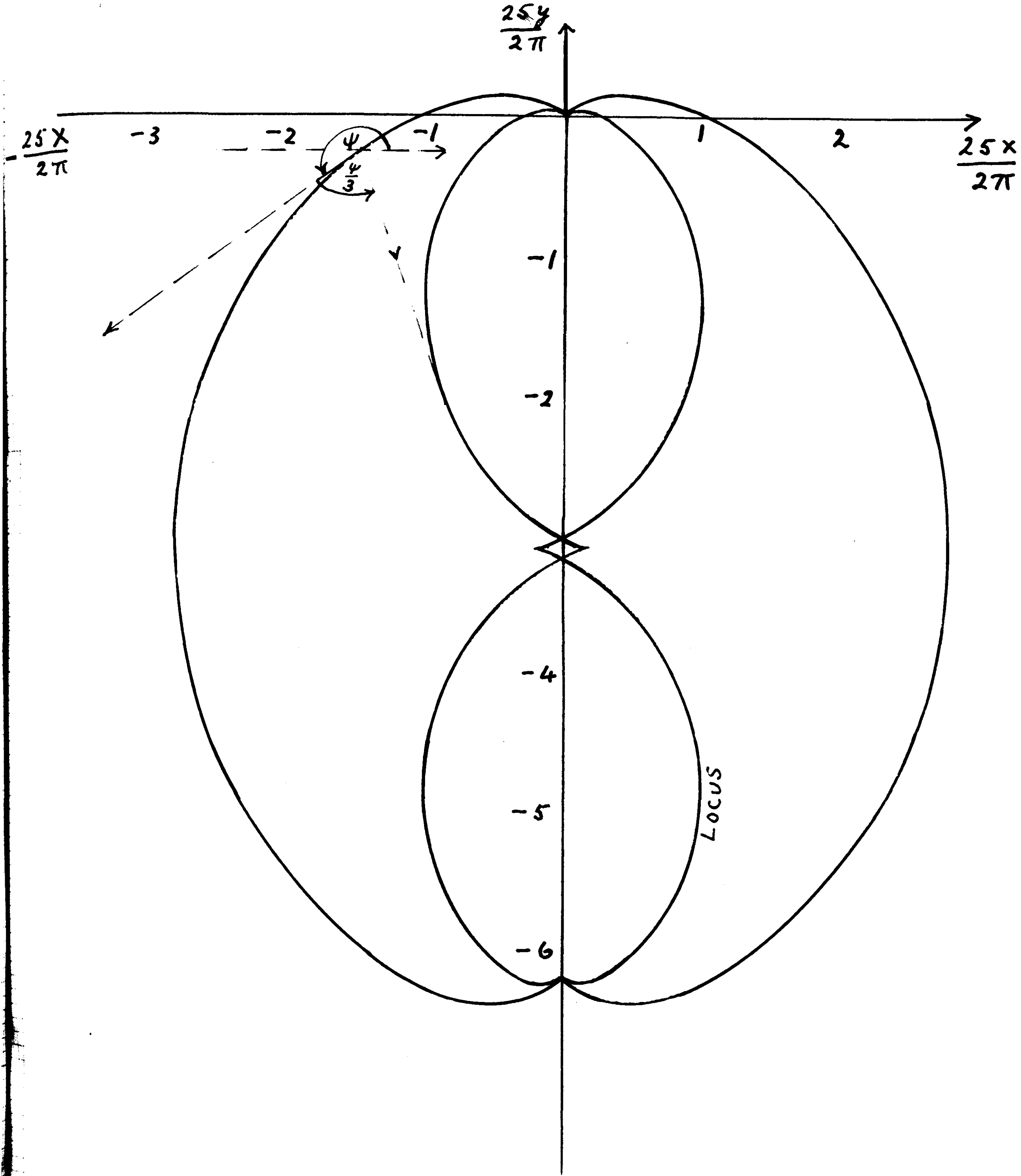
Data for the graph of $\rho' = 3 \cot\frac{\psi}{3}$, $\rho = \sin^9\frac{\psi}{3}$.

ψ (deg.)	72	90	108	126	144	162	180	198	216	234	252	270
$\rho\cos\psi$	·000	0	−·003	−·016	−·056	−·141	−·274	−·421	−·515	−·482	−·294	0
$\rho\sin\psi$	·000	·002	·008	·022	·041	·046	0	−·137	−·344	−·663	−·905	−1

$\frac{25x}{2\pi}$	·00	·00	−·01	−·06	−·18	−·43	−·86	−1·45	−2·10	−2·61	−2·81
$\frac{25y}{2\pi}$	·00	·00	·02	·06	·11	·15	·08	−·23	−·90	−1·91	−3·11
$\frac{25\bar{\xi}}{2\pi}$	·00	−·01	−·06	−·21	−·47	−·78	−·99	−·89	−·50	−·03	·17
$\frac{25\bar{\eta}}{2\pi}$	·00	·01	·03	·03	−·10	−·46	−1·12	−1·95	−2·68	−3·06	−3·11



For the graph of $\rho = \sin^9 \frac{\psi}{3}$;
$(\psi, \rho\cos\psi)$ and $(\psi, \rho\sin\psi)$ diagrams.



Graph of $\rho = \sin^9 \frac{\psi}{3}$, and the locus of its centres of aberrancy.

Graph of the equation $\rho' = 3\cot\frac{3}{2}\psi$.

(4) The solution is $\rho = c\sin^2\frac{3\psi}{2}$. It follows that
$\rho\rho'' = -\frac{9}{2}\operatorname{cosec}^2\frac{3\psi}{2}$; $9+\rho'^2-3\rho\rho'' = \frac{45}{2}\operatorname{cosec}^2\frac{3\psi}{2}$;
$R = \frac{2c}{5}\sin^3\frac{3\psi}{2}$;

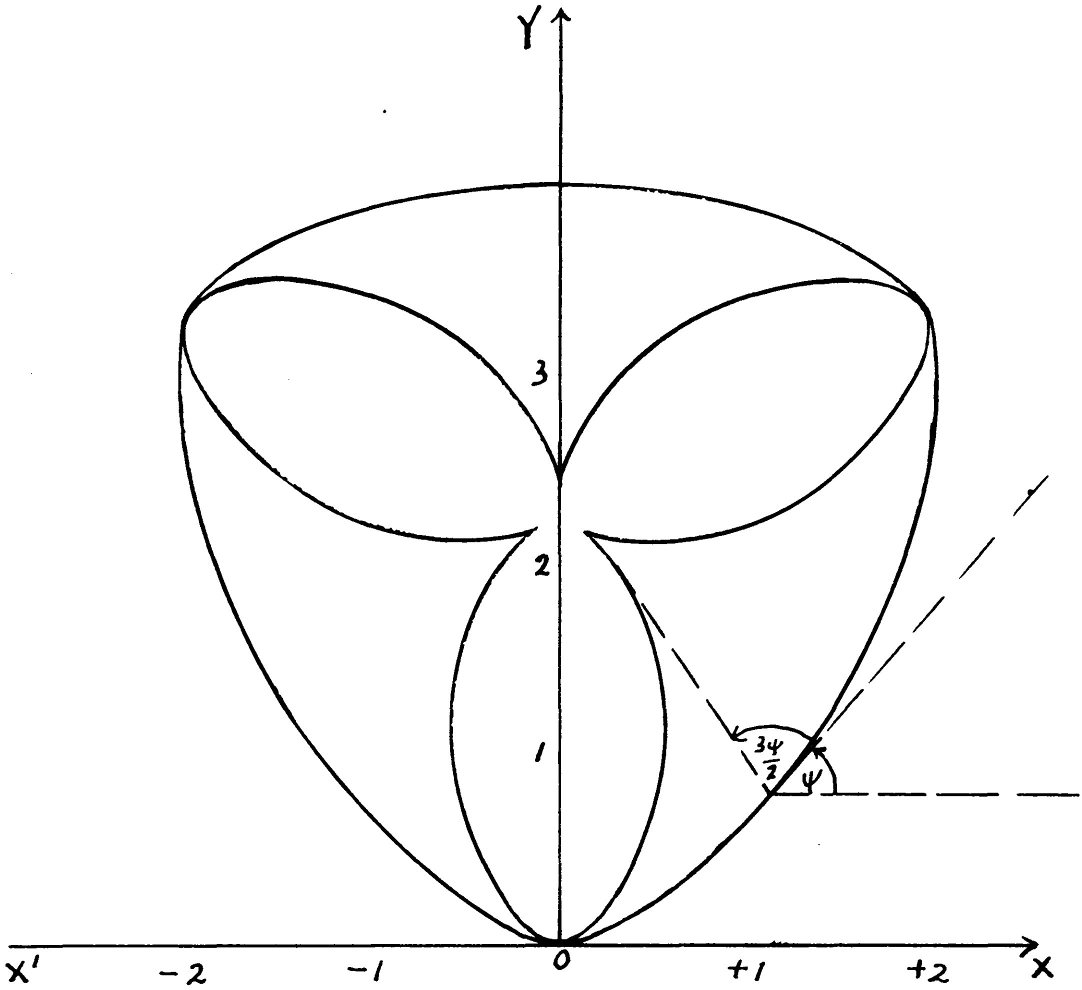
$$x = c\int\sin^2\frac{3\psi}{2}\cos\psi\,d\psi = \frac{c}{2}\left(\sin\psi - \frac{1}{8}\sin 4\psi - \frac{1}{4}\sin 2\psi\right),$$
$$y = c\int\sin^2\frac{3\psi}{2}\sin\psi\,d\psi = \frac{c}{2}\left(\frac{9}{8} - \cos\psi + \frac{1}{8}\cos 4\psi - \frac{1}{4}\cos 2\psi\right);$$

$$\bar{\xi} = x + \frac{2c}{5}\sin^3\frac{3\psi}{2}\cos\frac{5\psi}{2} = \frac{7c}{80}\left(4\sin\psi - 2\sin 2\psi + \sin 4\psi - \frac{4}{7}\sin 7\psi\right),$$
$$\bar{\eta} = y + \frac{2c}{5}\sin^3\frac{3\psi}{2}\sin\frac{5\psi}{2} = -\frac{7c}{80}\left(4\cos\psi + 2\cos 2\psi + \cos 4\psi - \frac{4}{7}\cos 7\psi - \frac{45}{7}\right).$$

The angle between the x-axis and the tangent is two-thirds of the angle between the tangent and the axis of aberrancy.

Data for the graph of $\rho' = 3\cot\frac{3\psi}{2}$, $\rho = 4\sin^2\frac{3\psi}{2}$.

x	0	·016	·116	·35	·41	1·13	1·52	1·80	1·96	2·00
y	0	·002	·03	·14	·40	·82	1·38	1·99	2·56	3·00
$\bar{\xi}$	0	·04	·25	·48	·53	·30	·11	·37	·98	1·60
$\bar{\eta}$	0	·014	·19	·69	1·42	2·00	2·18	2·12	2·21	2·60
x	1·98	1·95	1·95	1·94	1·86	1·65	1·25	·68	0	
y	3·26	3·36	3·38	3·39	3·46	3·61	3·79	3·94	4·00	
$\bar{\xi}$	1·91	1·96	1·96	1·92	1·67	1·12	·46	·07	0	
$\bar{\eta}$	3·07	3·33	3·38	3·40	3·49	3·46	3·12	2·63	2·40	



Graph of $\rho = 4 \sin^2 \frac{3\psi}{2}$, and the locus of its centres of aberrancy.

Graph of the equation $\rho' = 3 \cot a\psi$, when $a = 2$.

(5) The equation is $\frac{d\rho}{\rho} = 3 \cot 2\psi \, d\psi$, the solution of which is $\rho = c \sin^{3/2} 2\psi$.
This gives $\rho\rho'' = -6 \operatorname{cosec}^2 2\psi$; $9 + \rho'^2 - 3\rho\rho'' = 27 \operatorname{cosec}^2 2\psi$;
$R = \frac{c}{3} \sin^{5/2} 2\psi$;

$$x = c \int \sin^{3/2} 2\psi \cos\psi \, d\psi, \quad y = c \int \sin^{3/2} 2\psi \sin\psi \, d\psi;$$

$$\bar{\xi} = x + \frac{c}{3} \sin^{5/2} 2\psi \cos 3\psi,$$

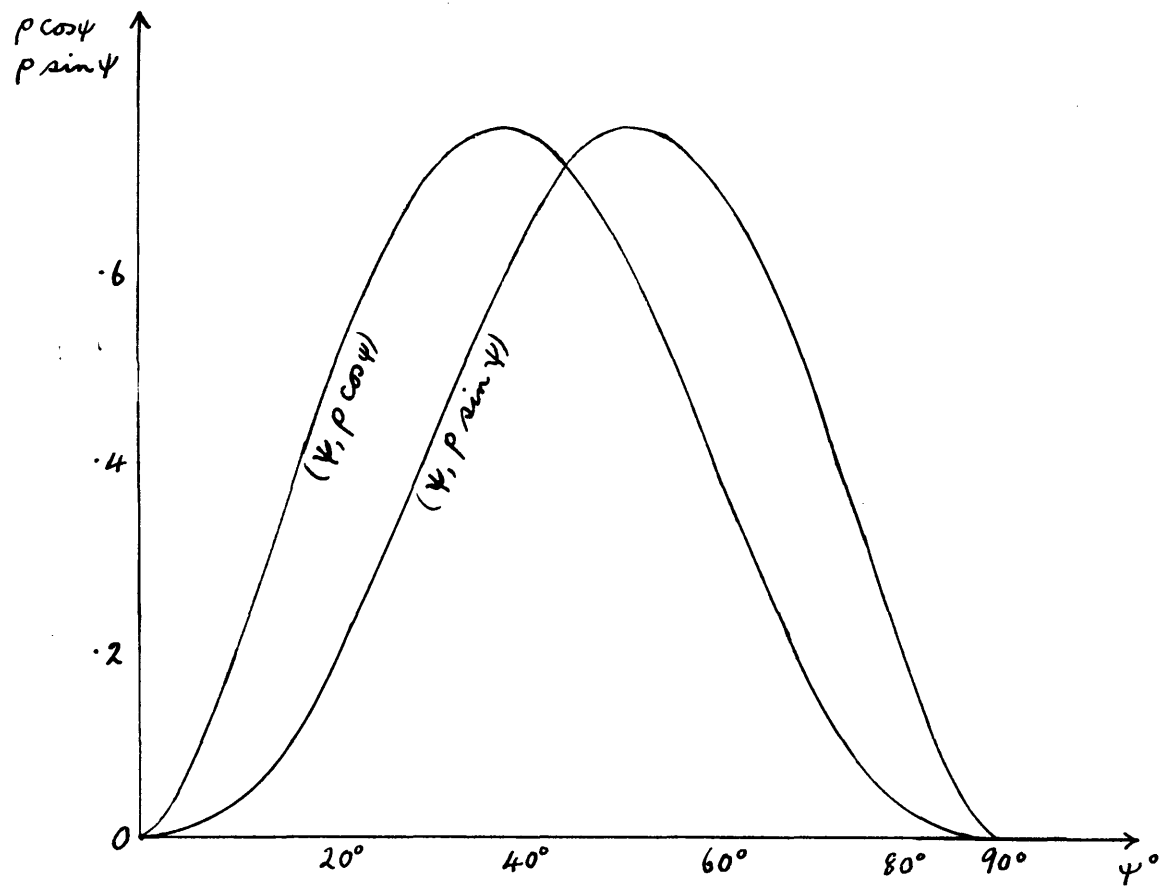
$$\bar{\eta} = y + \frac{c}{3} \sin^{5/2} 2\psi \sin 3\psi.$$

The angle between the x-axis and the tangent is half that between the tangent and the axis of aberrancy.

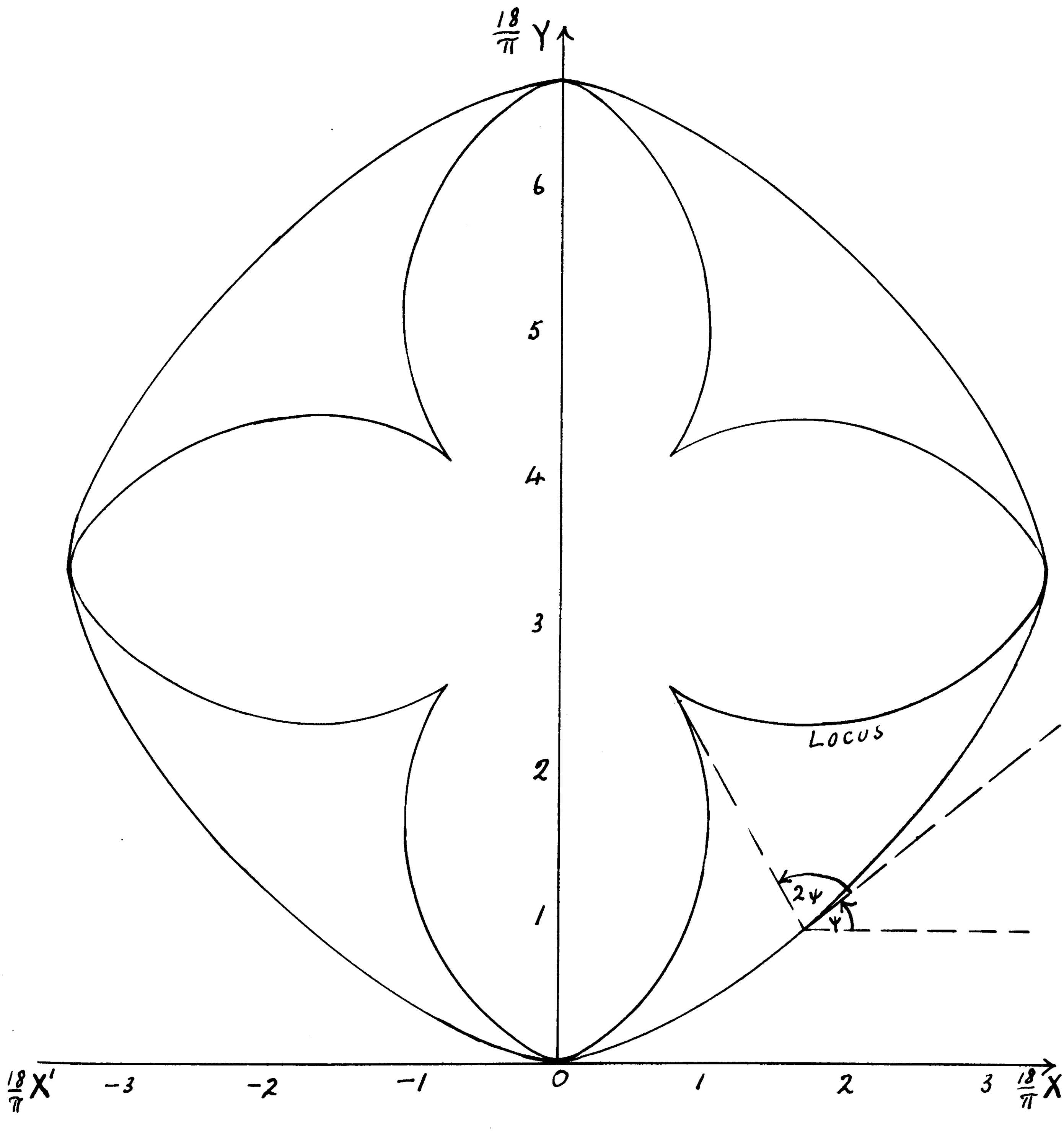
Data for the graph of $\rho' = 3 \cot 2\psi$, $\rho = \sin^{3/2} 2\psi$.

ψ (deg.)	0	10	20	30	40	50	60	70	80	90
$\rho \cos\psi$	0	·197	·484	·698	·749	·628	·403	·176	·035	0
$\rho \sin\psi$	0	·035	·176	·403	·628	·749	·698	·484	·197	0

$\frac{18}{\pi} x$	0	·10	·44	1·03	1·75	2·44	2·96	3·25	3·35	3·37
$\frac{18}{\pi} y$	0	·02	·12	·41	·93	1·62	2·34	2·93	3·27	3·37
$\frac{18}{\pi} \bar{\xi}$	0	·21	·76	1·03	·83	·85	1·62	2·70	3·29	3·37
$\frac{18}{\pi} \bar{\eta}$	0	·08	·67	1·75	2·52	2·54	2·34	2·61	3·16	3·37



For the Graph of $\rho = \sin^{3/2} 2\psi$;
$(\psi, \rho\cos\psi)$ and $(\psi, \rho\sin\psi)$ diagrams.



Graph of $\rho = \sin^{3/2} 2\psi$, and of the locus of its centres of aberrancy.

Graph of the equation $\rho' = 3 \cot a\psi$, when $a = \frac{2}{3}$.

(6) The equation is $\frac{d\rho}{\rho} = 3 \cot \frac{2}{3}\psi$, the solution of which is $\rho = c \sin^{9/2} \frac{2\psi}{3}$. It follows that

$$\rho\rho'' = -2 \operatorname{cosec}^2 \tfrac{2}{3}\psi; \quad 9 + \rho'^2 - 3\rho\rho'' = 15 \operatorname{cosec}^2 \tfrac{2\psi}{3};$$

$$R = \frac{3c}{5} \sin^{11/2} \frac{2\psi}{3};$$

$$x = c \int \sin^{9/2} \frac{2\psi}{3} \cos\psi \, d\psi,$$

$$y = c \int \sin^{9/2} \frac{2\psi}{3} \sin\psi \, d\psi;$$

$$\bar{\xi} = x + \frac{3c}{5} \sin^{11/2} \frac{2\psi}{3} \cos \frac{5\psi}{3},$$

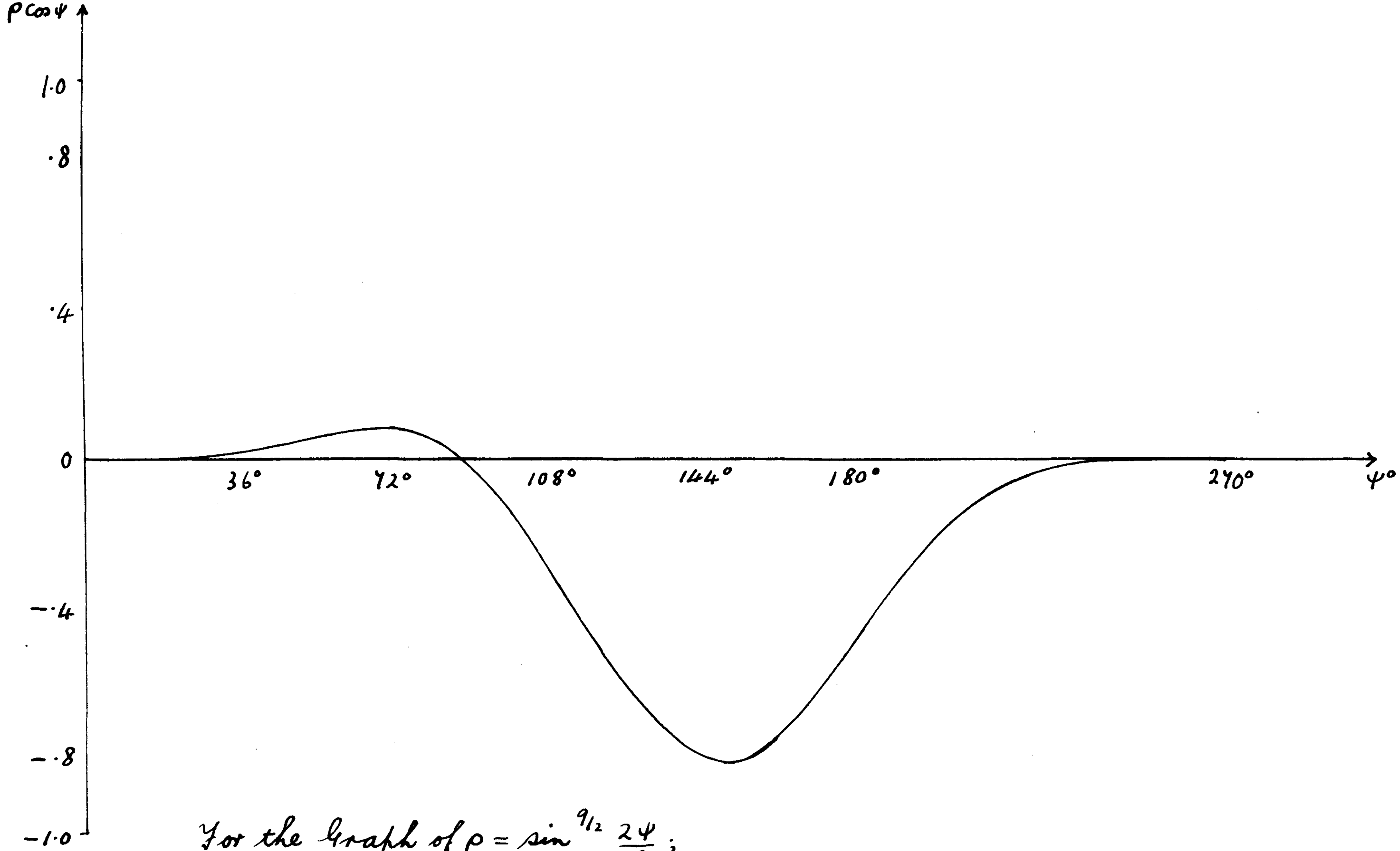
$$\bar{\eta} = y + \frac{3c}{5} \sin^{11/2} \frac{2\psi}{3} \sin \frac{5\psi}{3}.$$

The angle between the tangent and the positive direction of the axis of aberrancy is two-thirds of the angle between the x-axis and the tangent.

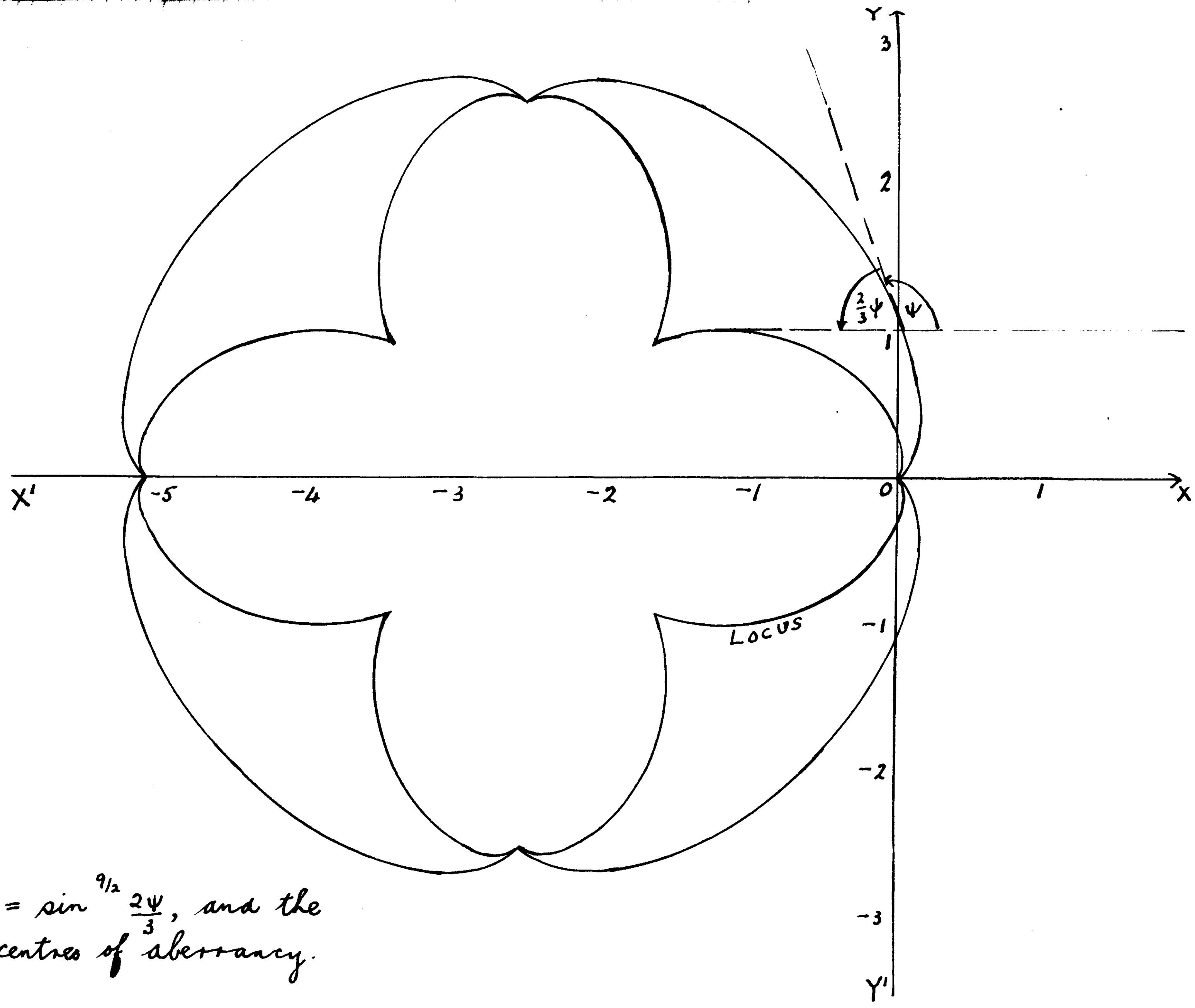
Data for the graph of $\rho = \sin^{9/2} \frac{2\psi}{3}$.

ψ°	0	18	36	54	72	90	108	126
$\rho\cos\psi$	0	·001	·014	·054	·081	0	−·247	−·573
$\rho\sin\psi$	0	·000	·010	·074	·250	·523	·759	·789
ψ°	144	162	180	198	216	234	252	270
$\rho\cos\psi$	−·789	−·759	−·523	−·250	−·074	−·010	−·000	0
$\rho\sin\psi$	·573	·247	0	−·081	−·054	−·014	−·001	0

$\frac{25}{3\pi}x$	0	0	·01	·04	·10	·14	·05	−·31
$\frac{25}{3\pi}y$	0	0	0	·03	·16	·47	1·01	1·67
$\frac{25}{3\pi}\bar{\xi}$	0	0	·02	·04	−·06	−·48	−1·16	−1·65
$\frac{25}{3\pi}\bar{\eta}$	0	0	·01	·12	·43	·83	1·01	·92
$\frac{25}{3\pi}x$	−·88	−1·54	−2·08	−2·39	−2·52	−2·55	−2·55	
$\frac{25}{3\pi}y$	2·24	2·60	2·69	2·65	2·59	2·56	2·55	
$\frac{25}{3\pi}\bar{\xi}$	−1·65	−1·54	−1·72	−2·12	−2·43	−2·44	−2·55	
$\frac{25}{3\pi}\bar{\eta}$	·90	1·39	2·07	2·49	2·59	2·57	2·55	



For the Graph of $\rho = \sin^{9/2} \frac{2\psi}{3}$;
$(\psi, \rho \cos\psi)$ diagram.



Graph of $\rho = \sin^{9/2} \frac{2\psi}{3}$, and the locus of its centres of aberrancy.

Graph of the equation $\rho' = 3\cot a\psi$, when $a = \frac{5}{2}$.

(7) The equation is $\frac{d\rho}{\rho} = 3\cot\frac{5\psi}{2}\,d\psi$, the solution of which is $\rho = c\sin^{6/5}\frac{5\psi}{2}$.

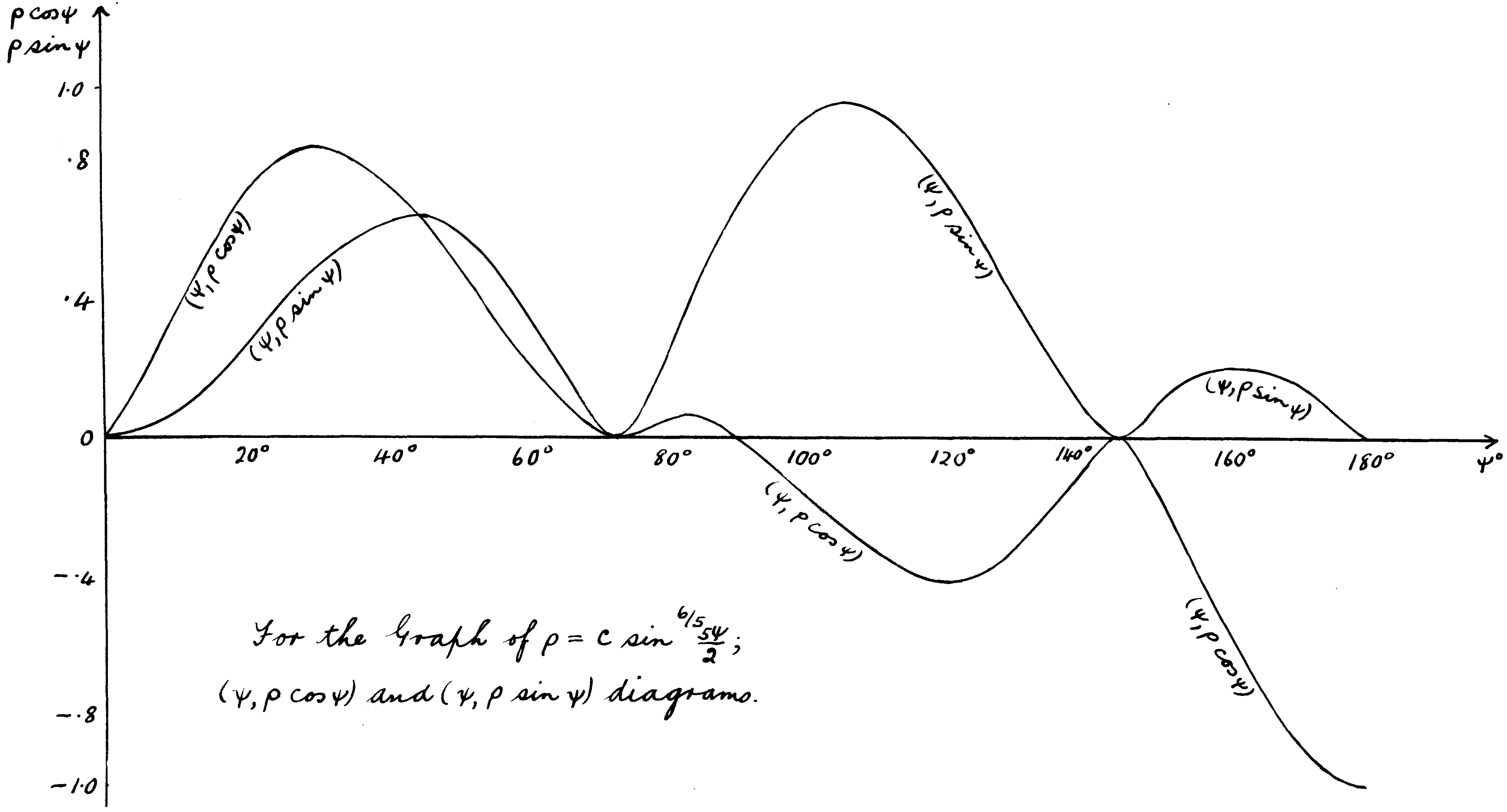
It follows that
$$-3\rho\rho'' = \frac{45}{2}\operatorname{cosec}^2\frac{5}{2}\psi;$$
$$9+\rho'^2-3\rho\rho'' = \frac{63}{2}\operatorname{cosec}^2\frac{5\psi}{2};$$
$$R = \frac{2c}{7}\sin^{11/5}\frac{5\psi}{2};$$
$$x = c\int\sin^{6/5}\frac{5\psi}{2}\cos\psi\,d\psi,\quad y = c\int\sin^{6/5}\frac{5\psi}{2}\sin\psi\,d\psi;$$
$$\bar{\xi} = x + \frac{2c}{7}\sin^{11/5}\frac{5\psi}{2}\cos\frac{7\psi}{2},$$
$$\bar{\eta} = y + \frac{2c}{7}\sin^{11/5}\frac{5\psi}{2}\sin\frac{7\psi}{2}.$$

The angle between the x-axis and the tangent is two-fifths of the angle between the tangent and the axis of aberrancy.

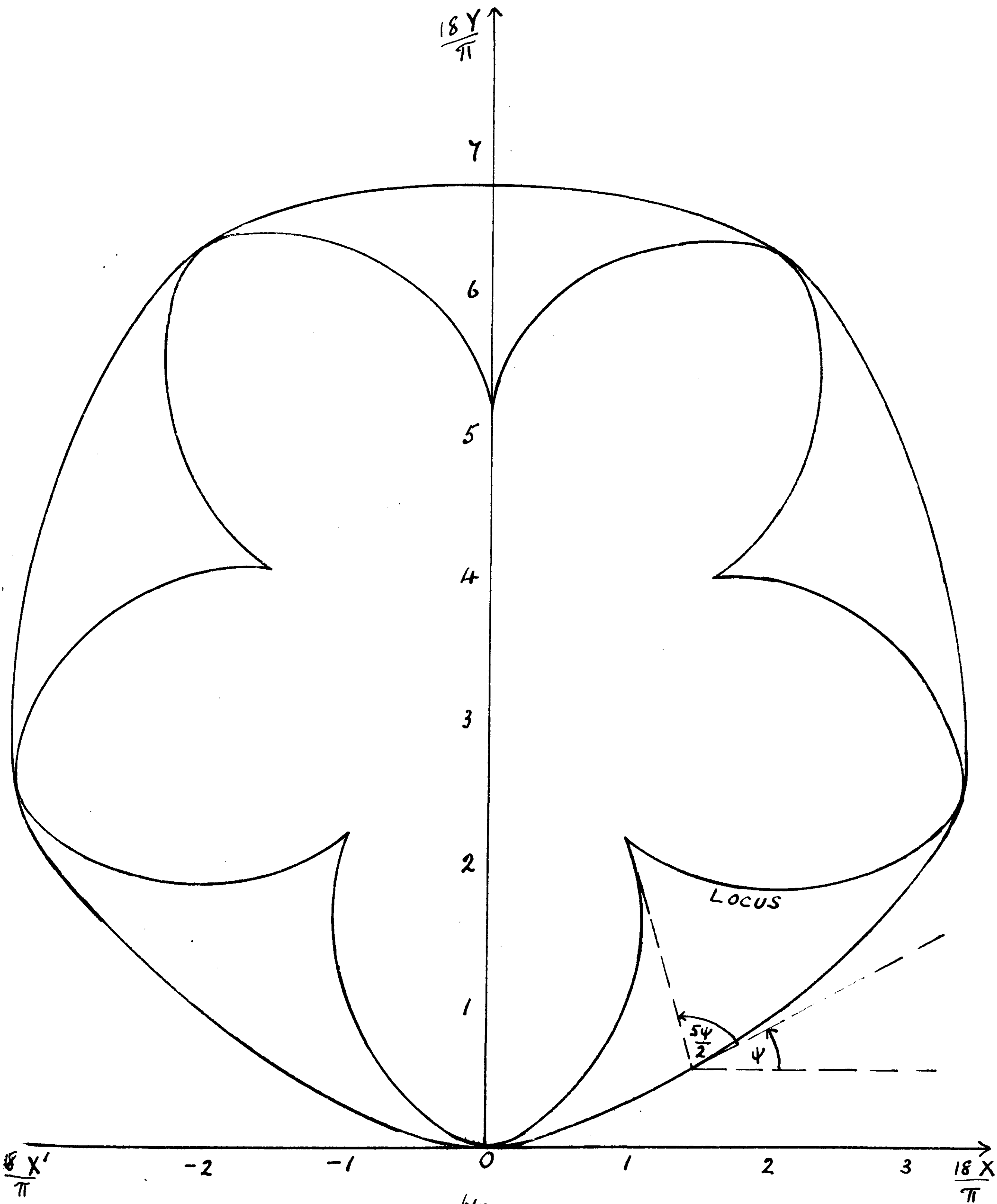
Data for the graph of $\rho = \sin^{6/5}\frac{5\psi}{2}$.

ψ (deg.)	0	10	20	30	40	50	60	70	80	90
$\rho\cos\psi$	0	·350	·683	·831	·735	·506	·218	·018	·479	0
$\rho\sin\psi$	0	·062	·249	·480	·617	·603	·377	·050	·216	·660
ψ (deg.)	100	110	120	130	140	150	160	170	180	
$\rho\cos\psi$	−·160	−·341	−·421	−·330	−·094	−·171	−·553	−·875	−1	
$\rho\sin\psi$	·908	·935	·729	·393	·079	·099	·213	·154	0	

$\frac{18x}{\pi}$	0	·164	·692	1·476	2·264	2·892	3·244	3·340	3·356	
$\frac{18y}{\pi}$	0	·020	·168	·548	1·108	1·740	2·232	2·424	2·496	
$\frac{18\bar{\xi}}{\pi}$	0	·366	1·004	1·083	1·079	1·840	2·935	3·337	3·329	
$\frac{18\bar{\eta}}{\pi}$	0	·161	1·024	2·013	2·103	1·832	2·054	2·417	2·648	
$\frac{18x}{\pi}$	3·400	3·312	3·052	2·660	2·272	2·064	2·004	1·652	·928	0
$\frac{18y}{\pi}$	2·964	3·772	4·716	5·568	6·124	6·332	6·360	6·520	6·704	6·784
$\frac{18\bar{\xi}}{\pi}$	2·860	1·916	1·581	2·063	2·314	2·086	1·923	1·070	·141	0
$\frac{18\bar{\eta}}{\pi}$	3·504	4·018	4·030	4·535	5·644	6·305	6·338	6·308	5·624	5·147



For the Graph of $\rho = c \sin^{6/5} \frac{5\psi}{2}$;
$(\psi, \rho \cos\psi)$ and $(\psi, \rho \sin\psi)$ diagrams.



Graph of $\rho = \sin^{6/5}\frac{5\psi}{2}$, and the locus of its centres of aberrancy.

Graph of the equation $\rho' = 3 \cot a\psi$ when $a = 4$.

(8) The equation is $\frac{d\rho}{\rho} = 3 \cot 4\psi \, d\psi$, the solution of which is $\rho = c \sin^{3/4} 4\psi$. It follows that

$-3\rho\rho'' = 36 \operatorname{cosec}^2 4\psi$; $9 + \rho'^2 - 3\rho\rho'' = 45 \operatorname{cosec}^2 4\psi$;

$R = \frac{c}{5} \sin^{7/4} 4\psi$;

$x = c \int \sin^{3/4} 4\psi \cos\psi \, d\psi$, $y = \int \sin^{3/4} 4\psi \sin\psi \, d\psi$;

$\bar{\xi} = x + \frac{c}{5} \sin^{7/4} 4\psi \cos 5\psi$,

$\bar{\eta} = y + \frac{c}{5} \sin^{7/4} 4\psi \sin 5\psi$.

The angle between the x-axis and the tangent is one-quarter of the angle between the tangent and the axis of aberrancy.

If $\sin 4\psi$ is negative, the value of ρ may be taken as $c \sin^{3/4}(-4\psi)$, and the corresponding value of R is $\frac{\rho}{5} \sin 4\psi$, which will be negative when $\sin 4\psi$ is negative.

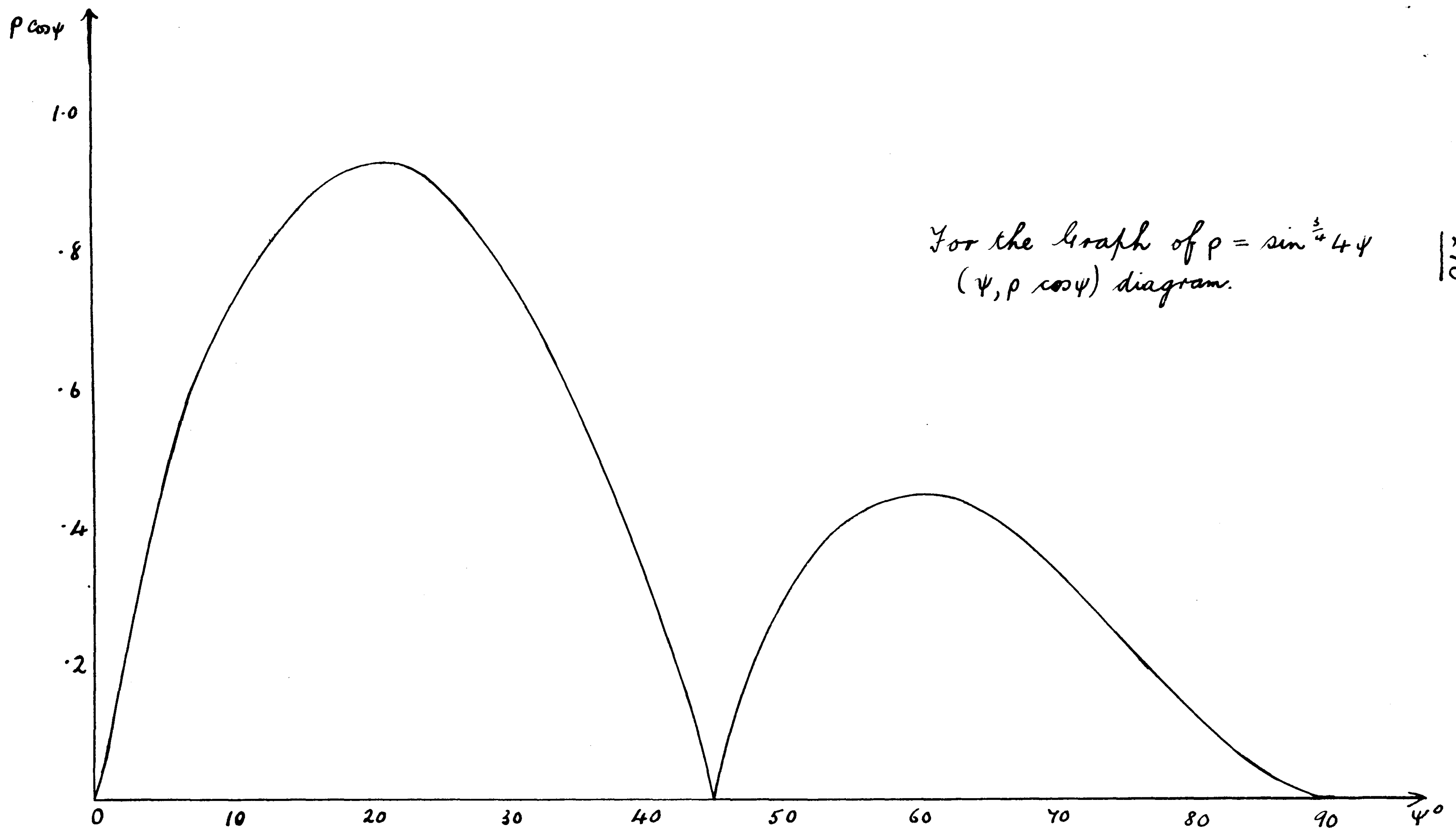
Data for the Graph of $\rho = \sin^{3/4} 4\psi$.

ψ (deg.)	0	5	10	15	20	25	30	35	40	45	50
$\rho \cos\psi$	0	·446	·707	·867	·929	·896	·777	·588	·343	0	·288
$\rho \sin\psi$	0	·039	·125	·232	·338	·418	·449	·412	·288	0	·343

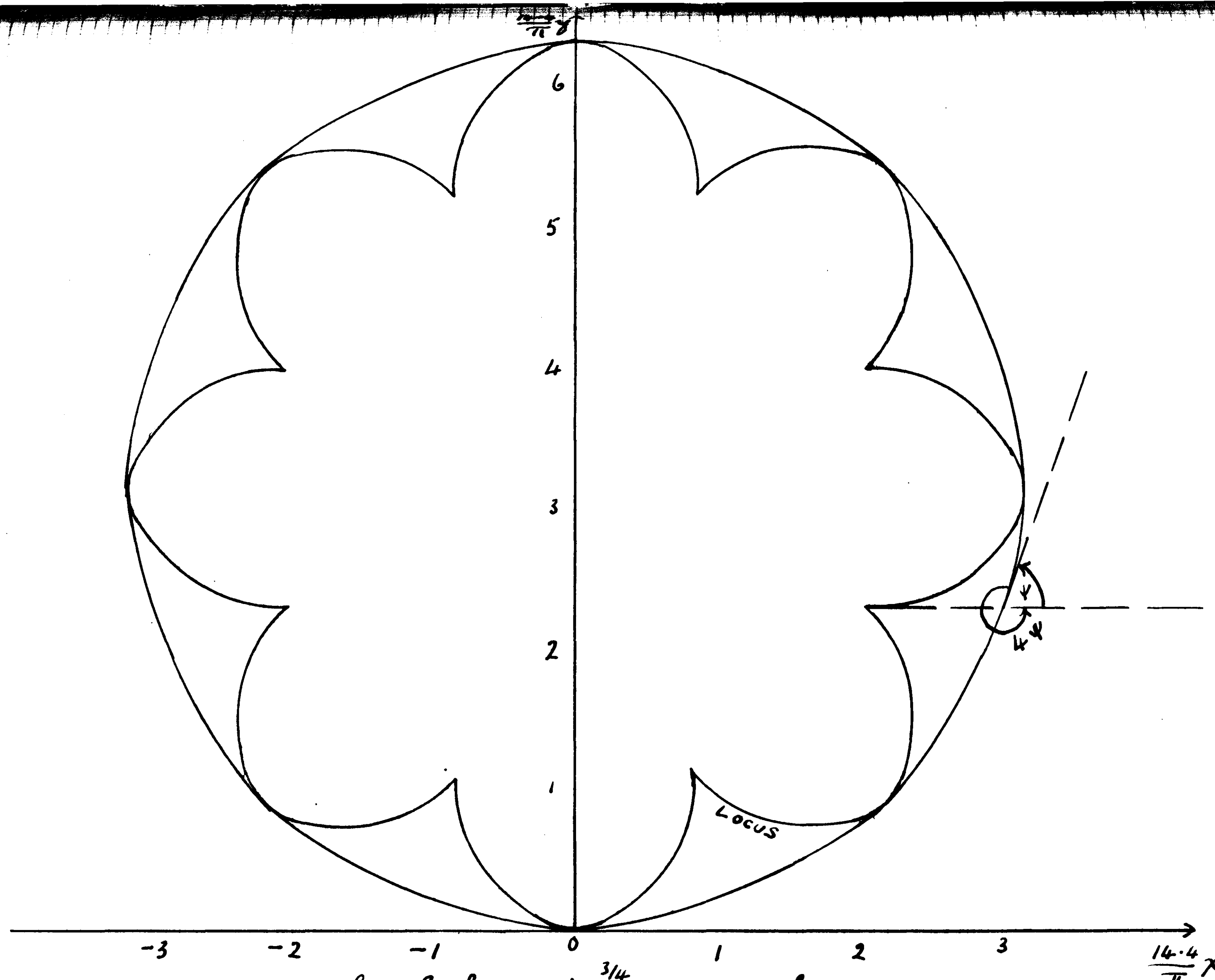
ψ (deg.)	55	60	65	70	75	80	85	90
$\rho \cos\psi$	·412	·449	·418	·338	·232	·125	·039	0
$\rho \sin\psi$	·588	·777	·896	·929	·867	·707	·446	0

$\frac{14.4}{\pi} x$	0	·08	·32	·64	1·00	1·37	1·70	1·97	2·16	2·23
$\frac{14.4}{\pi} y$	0	·01	·04	·11	·22	·37	·54	·72	·86	·92
$\frac{14.4}{\pi} \bar{\xi}$	0	·21	·59	·82	·85	·86	1·09	1·55	2·03	2·23
$\frac{14.4}{\pi} \bar{\eta}$	0	·07	·36	·79	1·10	1·10	·90	·75	·81	·92

$\frac{14.4}{\pi} x$	2·30	2·44	2·61	2·79	2·94	3·05	3·12	3·15	3·16
$\frac{14.4}{\pi} y$	1·00	1·18	1·45	1·79	2·16	2·52	2·83	3·07	3·16
$\frac{14.4}{\pi} \bar{\xi}$	2·34	2·40	2·26	2·06	2·06	2·36	2·80	3·09	3·16
$\frac{14.4}{\pi} \bar{\eta}$	1·13	1·60	2·07	2·30	2·31	2·33	2·56	2·95	3·16



For the Graph of $\rho = \sin^{\frac{3}{4}} 4\psi$
$(\psi, \rho \cos\psi)$ diagram.



Graph of $\rho = \sin^{3/4} 4\psi$, and the locus of its centres of aberrancy.

Graph of the equation $\rho' = 3 \cot a\psi$, when $a = -4$.

(9) The equation is $\frac{d\rho}{\rho} = -3 \cot 4\psi \, d\psi$, the solution of which is $\rho = c \operatorname{cosec}^{3/4} 4\psi$. It follows that

$\rho\rho'' = 12 \operatorname{cosec}^2 4\psi$; $9 + \rho'^2 - 3\rho\rho'' = -27 \operatorname{cosec}^2 4\psi$;

$R = \frac{c}{3} \sin^{\frac{1}{4}} 4\psi$;

$x = c \int \operatorname{cosec}^{3/4} 4\psi \cos\psi \, d\psi$, $y = c \int \operatorname{cosec}^{3/4} 4\psi \sin\psi \, d\psi$;

$\bar{\xi} = x + \frac{c}{3} \sin^{\frac{1}{4}} 4\psi \cos 3\psi$; $\bar{\eta} = y - \frac{c}{3} \sin^{\frac{1}{4}} 4\psi \sin 3\psi$.

The angle between the x-axis and the tangent is one quarter of the angle between the negative direction of the tangent and the positive direction of the axis of aberrancy.

The values of $\int \operatorname{cosec}^{3/4} 4\psi \cos\psi \, d\psi$ and $\int \operatorname{cosec}^{\frac{3}{4}} 4\psi \sin\psi \, d\psi$ near $\psi = 0$ can be found by writing $1 - \frac{1}{2}\psi^2$ for $\cos\psi$ and $\psi - \frac{1}{6}\psi^3$ for $\sin\psi$. These approximations give

$$\int_{0^\circ}^{5^\circ} \operatorname{cosec}^{3/4} 4\psi \cos\psi \, d\psi = .7696.$$

Similarly, near 45°, approximations can be found by writing $\psi = 45 \pm \delta$, and using the first two terms of the expansions of $\cos\delta$ and $\sin\delta$. These approximations give:-

$$\int_{40^\circ}^{45^\circ} \operatorname{cosec}^{3/4} 4\psi \cos\psi \, d\psi = .5527, \text{ and}$$

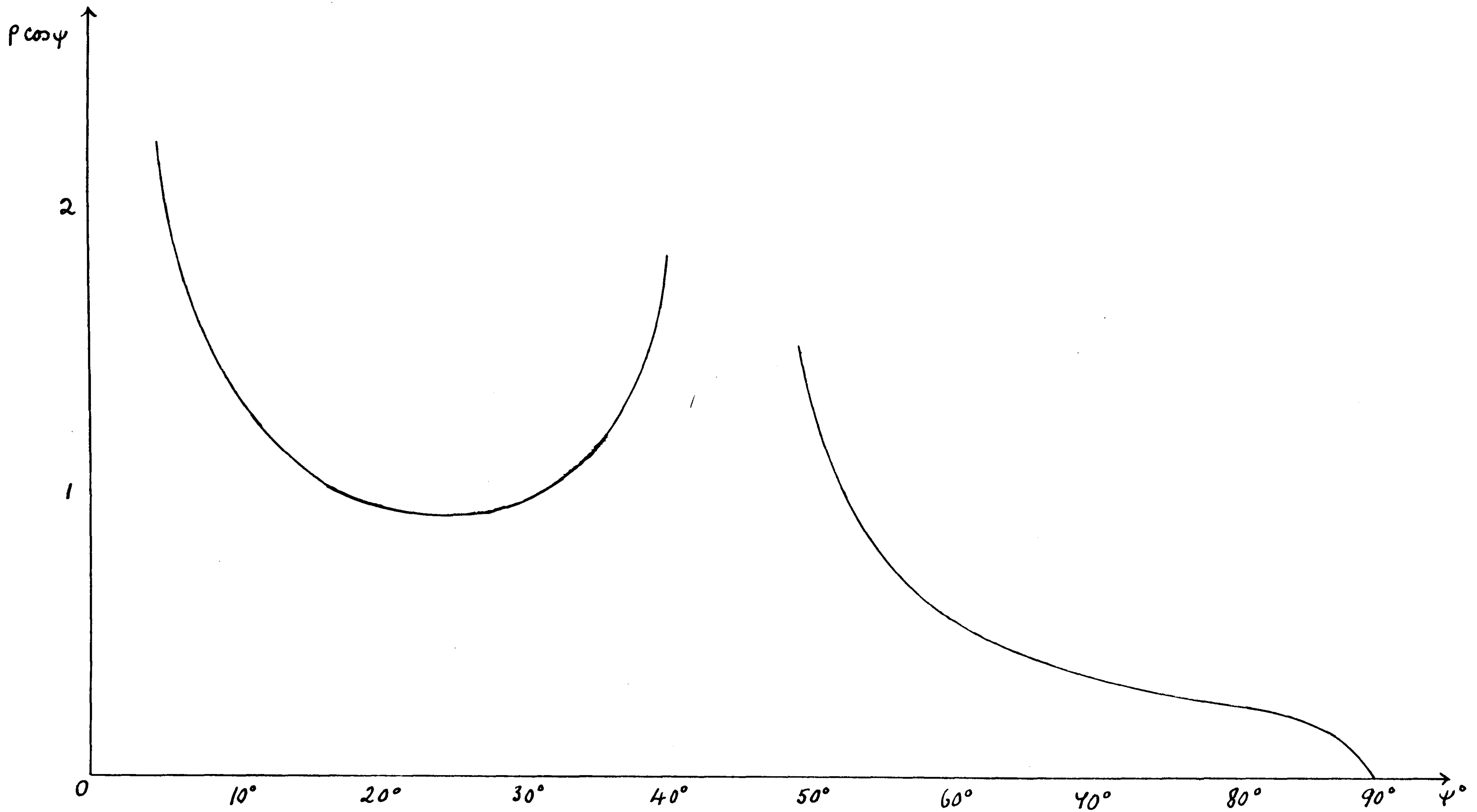
$$\int_{45^\circ}^{50^\circ} \operatorname{cosec}^{3/4} 4\psi \cos\psi \, d\psi = .5339.$$

Data for the graph of $\rho = \operatorname{cosec}^{3/4} 4\psi$.

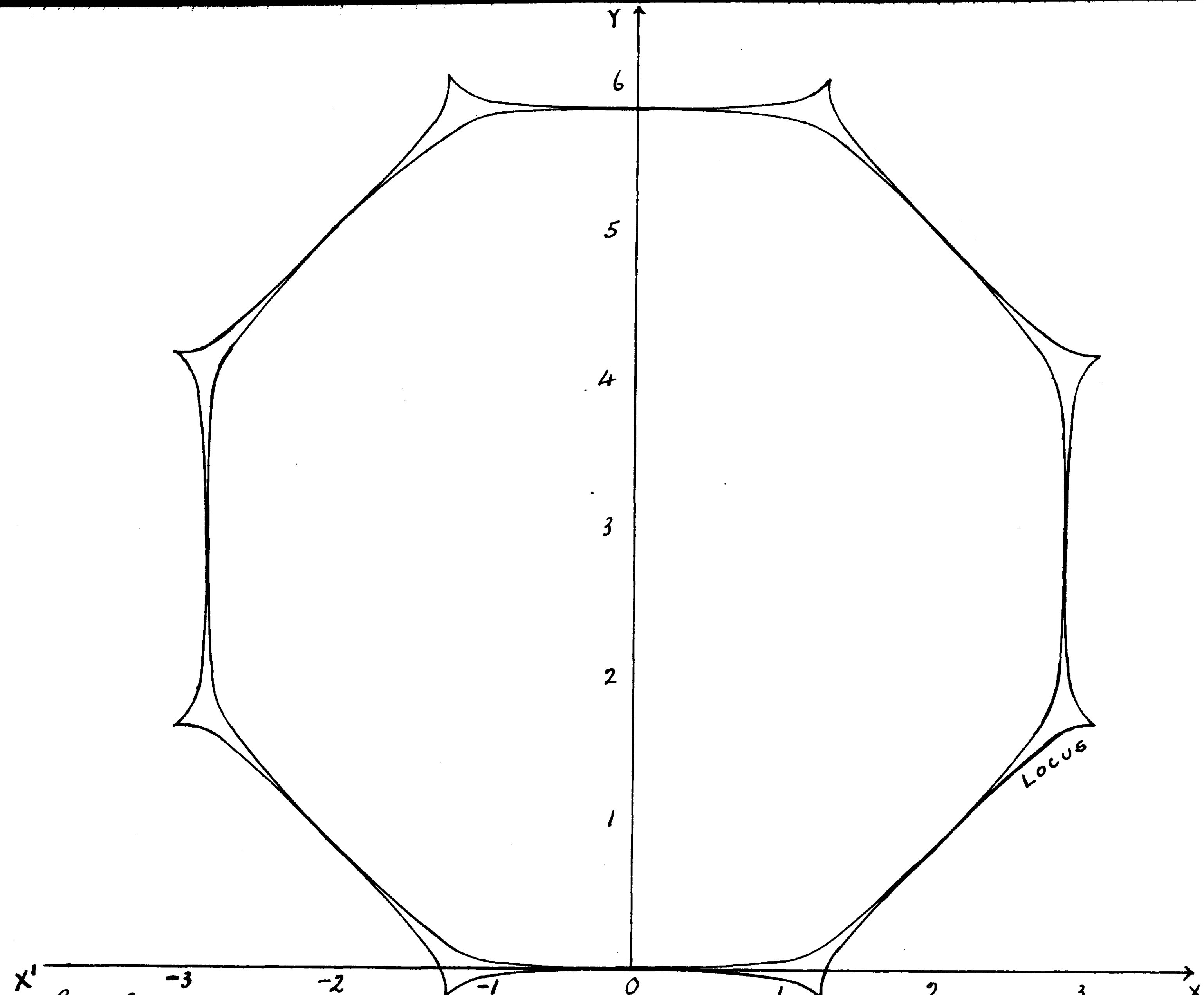
ψ°	0	5	10	15	20	25	30	35	40	45	50
$\rho\cos\psi$	∞	2.23	1.37	1.08	.95	.92	.96	1.14	1.41	∞	1.44
ψ°	55	60	65	70	75	80	85	90			
$\rho\cos\psi$	.80	.56	.43	.35	.29	.24	.19	0			

x	0	.77	.92	1.02	1.11	1.19	1.27	1.36	1.48	2.04
y	0	.01	.03	.05	.08	.12	.16	.22	.31	.85
$\bar{\xi}$	0	1.02	1.18	1.25	1.28	1.28	1.27	1.28	1.35	2.04
$\bar{\eta}$	0	−.06	−.12	−.18	−.21	−.20	−.16	−.09	.09	.85

x	2.57	2.66	2.72	2.77	2.80	2.83	2.85	2.87	2.88
y	1.40	1.52	1.61	1.69	1.77	1.86	1.97	2.11	2.88



For the graph of $\rho = \operatorname{cosec}^{3/4} 4\psi$;
$(\psi, \rho\cos\psi)$ diagram.



Graph of $p = \operatorname{cosec}^{3/4} 4\psi$, and the locus of its centres of aberrancy.

Graph of the equation $\rho = 3 \cot a\psi$, when $a = \frac{3}{4}$.

(10) The equation is $\frac{d\rho}{\rho} = 3 \cot \frac{3}{4}\psi \, d\psi$, the solution of which is $\rho = c \sin^4 \frac{3}{4}\psi$. It follows that

$$-3\rho\rho'' = \frac{27}{4} \operatorname{cosec}^2 \frac{3}{4}\psi \; ; \; 9 + \rho'^2 - 3\rho\rho'' = \frac{63}{4} \operatorname{cosec}^2 \frac{3}{4}\psi \; ;$$

$$R = \frac{4c}{7} \sin^5 \frac{3\psi}{4} \; ;$$

$$x = c \int \sin^4 \frac{3\psi}{4} \cos\psi \, d\psi = c\left[+\frac{3}{8}\sin\psi - \frac{1}{10}\sin\frac{5\psi}{2} - \frac{1}{2}\sin\frac{\psi}{2} + \frac{1}{64}\sin 4\psi + \frac{1}{32}\sin 2\psi\right],$$

$$y = c \int \sin^4 \frac{3\psi}{4} \sin\psi \, d\psi = c\left[-\frac{3}{8}\cos\psi + \frac{1}{10}\cos\frac{5\psi}{2} - \frac{1}{2}\cos\frac{\psi}{2} - \frac{1}{64}\cos 4\psi + \frac{1}{32}\cos 2\psi + \frac{243}{320}\right];$$

$$\bar{\xi} = x + \frac{4c}{7}\sin^5\frac{3\psi}{4}\cos\frac{7\psi}{4},$$

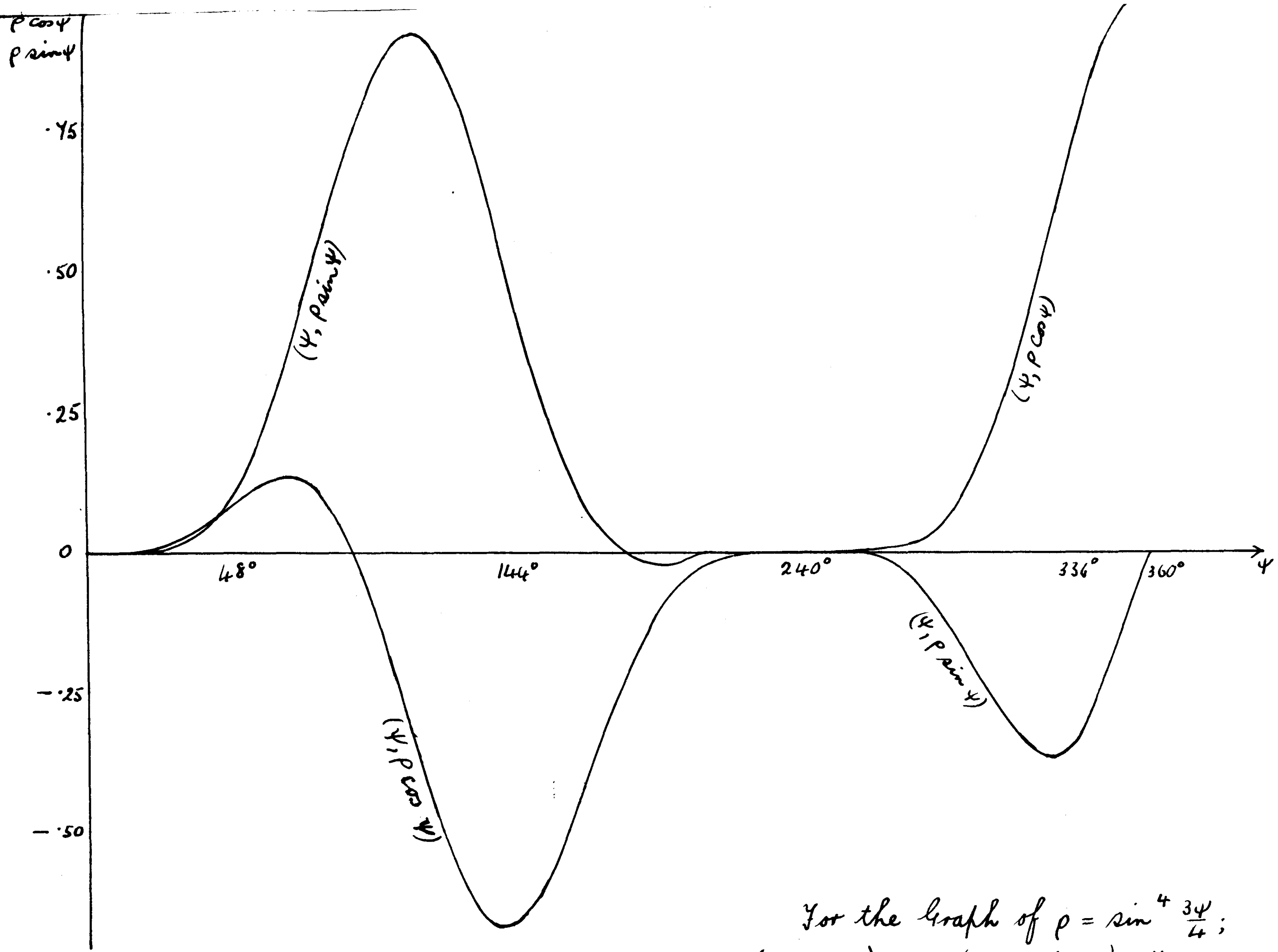
$$\bar{\eta} = y + \frac{4c}{7}\sin^5\frac{3\psi}{4}\sin\frac{7\psi}{4}.$$

The angle between the tangent and the axis of aberrancy is three-quarters of the angle between the x-axis and the tangent.

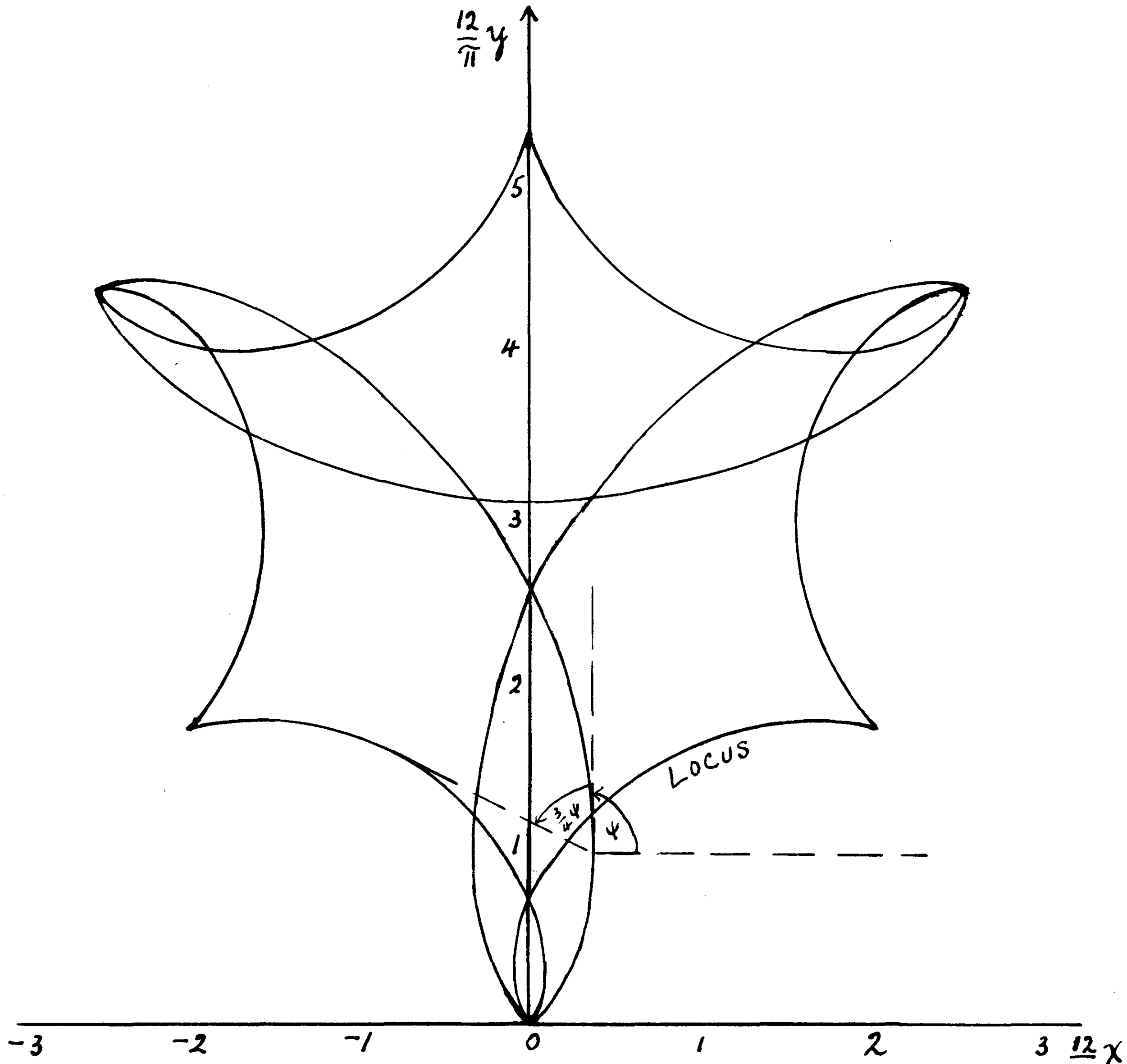
Data for the Graph of $\rho = \sin^4 \frac{3\psi}{4}$.

ψ (deg.)	0	12	24	36	48	60	72	84	96	108	
$\rho\cos\psi$	0	·001	·008	·034	·080	·125	·132	·066	−·086	−·294	
$\rho\sin\psi$	0	·000	·004	·025	·089	·217	·408	·627	·814	·905	
ψ (deg.)	120	132	144	156	168	180	192	204	216	228	
$\rho\cos\psi$	−·500	−·637	−·662	−·576	−·419	−·250	−·117	−·039	−·007	−·000	
$\rho\sin\psi$	·866	·707	·485	·256	·089	0	−·025	−·017	−·005	−·000	
ψ (deg.)	240	252	264	276	288	300	312	324	336	348	360
$\rho\cos\psi$	0	−·000	−·001	·004	·037	·125	·287	·510	·747	·931	1
$\rho\sin\psi$	0	−·001	−·009	−·042	−·114	−·217	−·319	−·371	−·333	−·198	0

$\frac{12x}{\pi}$	0	0	·06	·26	·34	−·14	−1·12	−2·03
$\frac{12}{\pi}y$	0	0	·06	·40	1·40	2·80	3·93	4·35
$\frac{12}{\pi}\bar{\xi}$	0	·01	·08	−·17	−1·33	−2·03	−1·63	−1·72
$\frac{12}{\pi}\bar{\eta}$	0	·01	·21	1·01	1·75	1·71	2·31	3·66
$\frac{12}{\pi}x$	−2·44	−2·52	−2·52	−2·52	−2·51	−2·29	−1·47	·00
$\frac{12}{\pi}y$	4·37	4·34	4·34	4·34	4·26	3·91	3·36	3·06



For the Graph of $\rho = \sin^4 \frac{3\psi}{4}$;
$(\psi, \rho \cos\psi)$ and $(\psi, \rho \sin\psi)$ diagrams.



Graph of $\rho = \sin^4 \frac{3\psi}{4}$, and the locus of its centres of aberrancy.

Graph of the equation $\rho' = 3 \cot a\psi$, when $a = 3$.

(11) We have $\dfrac{d\rho}{\rho\, d\psi} = 3 \cot 3\psi$, and therefore $\rho = c \sin 3\psi$.
It follows that $\rho\rho'' = -9 \operatorname{cosec}^2 3\psi$; $9 + \rho'^2 - 3\rho\rho'' = 36 \operatorname{cosec}^2 3\psi$;
$R = \frac{c}{4} \sin^2 3\psi$;

$$\bar{\xi} = x + \frac{c}{4}\sin^2 3\psi \cos 4\psi, \quad \bar{\eta} = y + \frac{c}{4}\sin^2 3\psi \sin 4\psi;$$

$$x = \int c \sin 3\psi \cos\psi \, d\psi = \frac{c}{8}(3 - \cos 4\psi - 2\cos 2\psi),$$

$$y = \int c \sin 3\psi \sin\psi \, d\psi = \frac{c}{8}(2\sin 2\psi - \sin 4\psi).$$

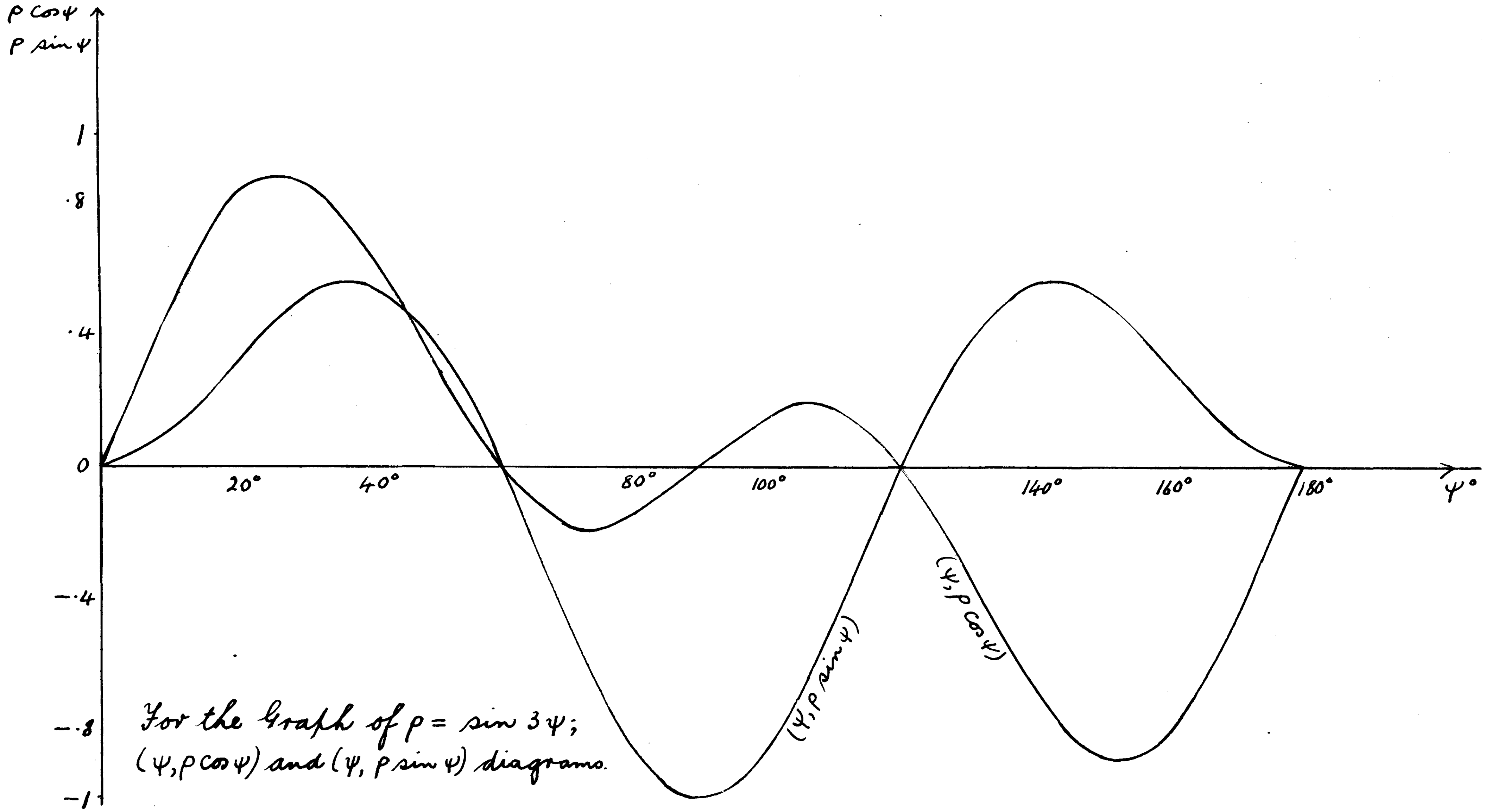
The angle between the x-axis and the tangent is one-third of the angle between the tangent and the axis of aberrancy.

The part of the graph which lies to the left of the y-axis corresponds to $\alpha = \psi - \pi$, and is such that the angle between the tangent and the axis of aberrancy is three times the angle between the negative x-axis and the tangent.

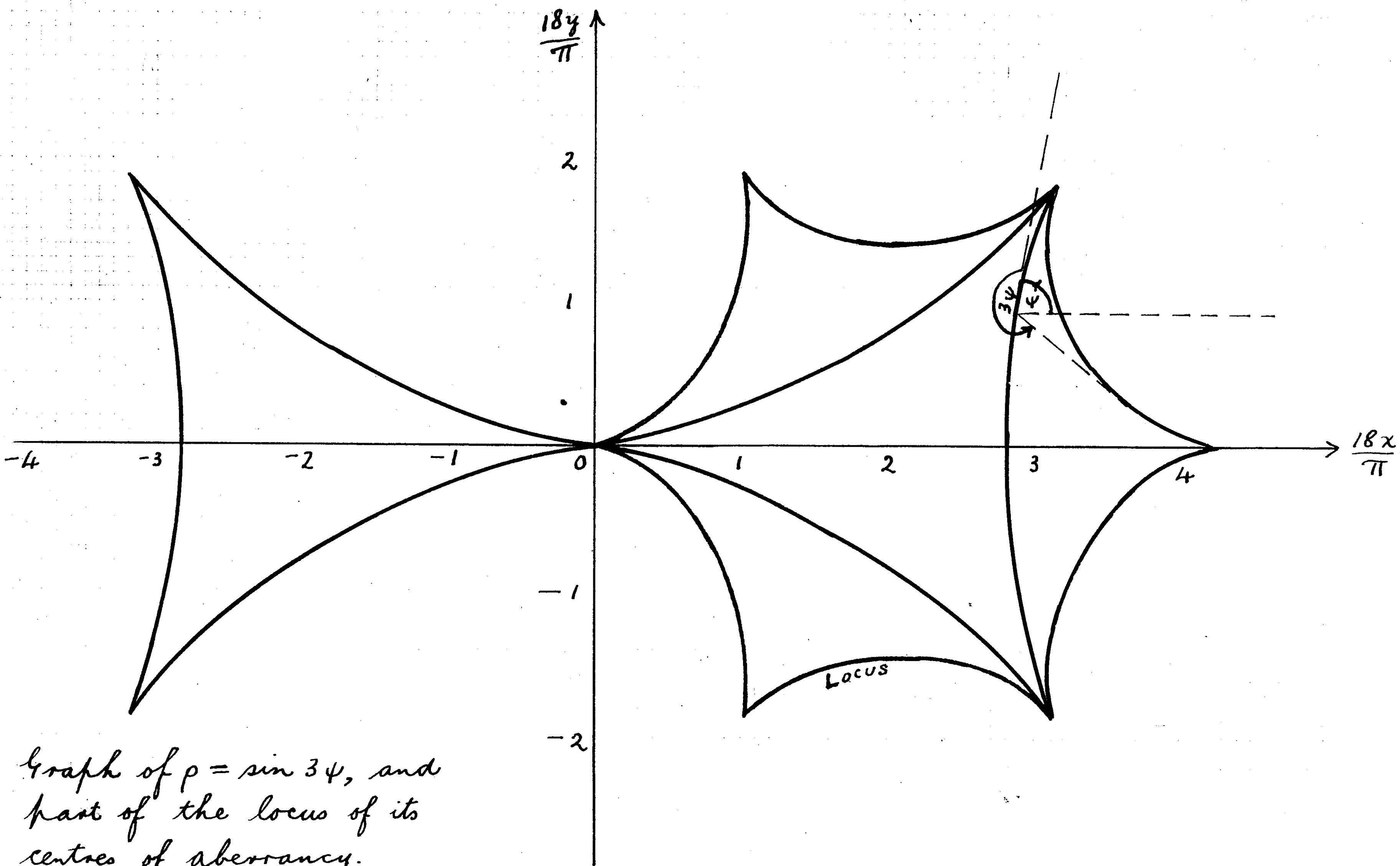
Data for the graph of $\rho' = 3 \cot 3\psi$, $\rho = \sin 3\psi$.

ψ (deg.)	0	10	20	30	40	50	60	70	80	90
$\rho \cos\psi$	0	·492	·814	·866	·663	·321	0	−·171	−·150	0
$\rho \sin\psi$	0	·087	·296	·500	·557	·383	0	−·470	−·853	−1
ψ (deg.)	100	110	120	130	140	150	160	170	180	
$\rho \cos\psi$	·150	·171	0	−·321	−·663	−·866	−·814	−·492	0	
$\rho \sin\psi$	−·853	−·470	0	·383	·557	·500	·296	·087	0	

$\frac{18x}{\pi}$	0	·25	·90	1·74	2·50	3·00	3·16	3·07	2·91	
$\frac{18y}{\pi}$	0	·04	·24	·63	1·16	1·63	1·82	1·59	·93	
$\frac{18\bar{\xi}}{\pi}$	0	·52	1·09	1·02	1·49	2·66	3·16	3·13	3·73	
$\frac{18\bar{\eta}}{\pi}$	0	·27	1·30	1·87	1·53	1·51	1·82	1·24	·24	
$\frac{18x}{\pi}$	2·84	2·91	3·07	3·16	3·00	2·50	1·74	·90	·25	0
$\frac{18y}{\pi}$	0	−·93	−1·59	−1·82	−1·63	−1·16	−·63	−·24	−·04	0
$\frac{18\bar{\xi}}{\pi}$	4·27	3·73	3·13	3·16	2·66	1·49	1·02	1·09	·52	0
$\frac{18\bar{\eta}}{\pi}$	0	−·24	−1·24	−1·82	−1·51	−1·53	−1·87	−1·30	−·27	0



For the Graph of $\rho = \sin 3\psi$;
$(\psi, \rho\cos\psi)$ and $(\psi, \rho\sin\psi)$ diagrams.



Graph of $\rho = \sin 3\psi$, and part of the locus of its centres of aberrancy.

Graph of the equation $\rho' = 3\cot a\psi$, when $a = -\frac{1}{2}$.

(12) We have $\dfrac{d\rho}{\rho\, d\psi} = -3\cot\dfrac{\psi}{2}$, and therefore $\rho = c\,\mathrm{cosec}^6\dfrac{\psi}{2}$.

It follows that $\rho\rho'' = \frac{3}{2}\,\mathrm{cosec}^2\frac{\psi}{2}$; $9+\rho'^2-3\rho\rho'' = \frac{9}{2}\,\mathrm{cosec}^2\frac{\psi}{2}$,
$R = -2c\,\mathrm{cosec}^5\dfrac{\psi}{2}$;

$$\bar{\xi} = x - 2c\,\mathrm{cosec}^5\frac{\psi}{2}\cos\frac{\psi}{2} = x - 2c\,\mathrm{cosec}^4\frac{\psi}{2}\cot\frac{\psi}{2},$$

$$\bar{\eta} = y - 2c\,\mathrm{cosec}^5\frac{\psi}{2}\sin\frac{\psi}{2} = y - 2c\,\mathrm{cosec}^4\frac{\psi}{2};$$

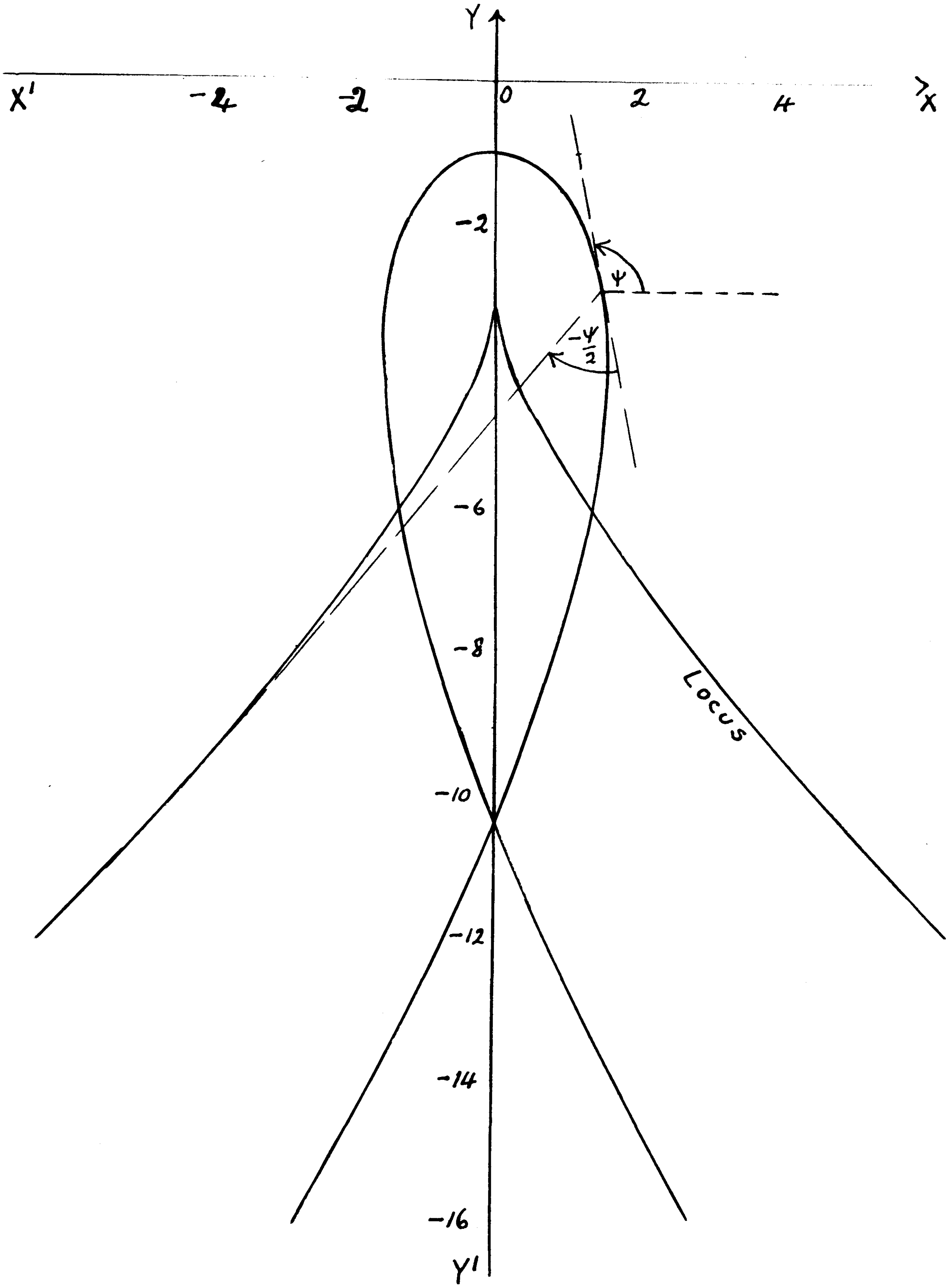
$$x = \int c\,\mathrm{cosec}^6\frac{\psi}{2}\cos\psi\, d\psi = 2c\cot\frac{\psi}{2}\left(1 - \frac{1}{5}\cot^4\frac{\psi}{2}\right),$$

$$y = \int c\,\mathrm{cosec}^6\frac{\psi}{2}\sin\psi\, d\psi = -c\,\mathrm{cosec}^4\frac{\psi}{2}.$$

It follows that $\bar{\eta} = 3y$.
In this case the angle between the x-axis and the tangent is twice the angle between the negative direction of the tangent and the axis of aberrancy.

Data for the graph of $\rho' = 3\cot(-\frac{1}{2}\psi)$, $\rho = \mathrm{cosec}^6\frac{\psi}{2}$.

x	-2.78	$.48$	1.42	1.60	1.51	1.33	1.13
y	-16	-9.24	-5.86	-4.00	-2.90	-2.20	-1.78
$\bar{\xi}$			-12.44	-6.40	-3.36	-1.78	$-.92$
$\bar{\eta}$			-17.58	-12.00	-8.70	-6.60	-5.34
x	$.92$	$.73$	$.54$	$.35$	$.18$	$.00$	
y	-1.48	-1.28	-1.15	-1.06	-1.02	-1.00	
$\bar{\xi}$	$-.38$	$-.20$	$-.08$	$-.02$	$.00$	$.00$	
$\bar{\eta}$	-4.44	-3.84	-3.45	-3.18	-3.06	-3.00	



Graph of $\rho = \operatorname{cosec}^6 \frac{\psi}{2}$, and the locus of its centres of aberrancy.

Section XXIV.

Properties of the equiangular spiral $r = ae^{\theta\cot\lambda}$.

(1) $\rho = r\operatorname{cosec}\lambda = a\operatorname{cosec}\lambda\, e^{-\lambda\cot\lambda}\, e^{\psi\cot\lambda}$, where $\psi = \lambda + \theta$; $\rho = s\cot\lambda$.

(2) The length of the axis of aberrancy is $\dfrac{3\operatorname{cosec}\lambda}{\sqrt{9+\cot^2\lambda}}\cdot r$.

(3) The angle between the tangent and the axis of aberrancy is $\cot^{-1}\dfrac{\rho'}{3} = \cot^{-1}\dfrac{\cot\lambda}{3}$. This is also the angle subtended at the pole by the line joining corresponding points on the spiral and the locus of its centres of aberrancy.

(4) The equation of the locus of centres of aberrancy is $r = 4\cos\alpha\, e^{-3\alpha\cot\alpha}\, ae^{3\theta\cot\alpha}$ where $3\cot\alpha = \cot\lambda$, i.e.

$$r = \frac{4\cot\lambda}{\sqrt{9+\cot^2\lambda}}\, e^{-\cot\lambda\cdot\cot^{-1}\frac{\cot\lambda}{3}}\, ae^{\theta\cot\lambda},$$

and corresponding to the point $(ae^{\theta\cot\lambda}, \theta)$ on the spiral, we have the point whose vectorial angle is $\theta + \cot^{-1}\dfrac{\cot\lambda}{3}$ on the locus of the centres of aberrancy. This locus is therefore a second equiangular spiral of the same angle as the first, and is formed by multiplying each radius vector of the first by the constant factor $4\cos\alpha\, e^{-3\alpha\cot\alpha}$, ($3\cot\alpha = \cot\lambda$).

(5) The axis of aberrancy is proportional to the radius vector (r), to the radius of curvature (ρ), and to the arc (s) measured from $\rho = 0$. It is also proportional to the normal chord of the osculating conic.

(6) The osculating conics are similar conics, the eccentricity being given by $e^4 + \frac{4}{9}\cot^2\lambda\,(e^2-1) = 0$, or $e = \sqrt{2-\sec^2\frac{\alpha}{2}}$ where $3\cot\alpha = \cot\lambda$.

(7) It belongs to the class of curves for which the quantity $9 + \rho'^2 - 3\rho\rho''$ is constant, i.e.

the class of curves for which the projection of the axis of aberrancy on the normal is proportional to the corresponding radius of curvature.

(8) It belongs to the class of curves for which the projection of the axis of aberrancy on the tangent is proportional to the corresponding radius of curvature, i.e., to the class of curves for which $9+\rho'^2-3\rho\rho''$ is proportional to ρ'.

(9). The line joining the centre of aberrancy at a point A on the spiral and the centre of the rectangular hyperbola of closest contact at A is bisected at A.