THE
MATHEMATICAL THEORY
of
HEAT CONDUCTION

BY THE METHOD OF WAVE TRAINS

A THESIS
SUBMITTED TO THE UNIVERSITY OF GLASGOW
FOR THE DEGREE OF D.Sc.

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PREFACE.

The wave train theory of heat conduction in solids in the sense in which the term is used in the present work, was propounded by Dr. G. Green of Glasgow in a paper in the Philosophical Magazine in April 1927.* This paper was followed at regular intervals by seven others, four by Dr. Green* and three by the writer.†

In the original paper which may be regarded as the most fundamental, the solutions for certain standard conduction problems were obtained on the hypothesis of this mode of heat transference as a form of wave motion. The medium was conceived of as being traversed by systems of temperature wave trains, and such matters as the reflection of these trains at fixed boundaries, and their transmission across surfaces of discontinuity of medium were fully investigated.

The facility with which wave train solutions of the

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* see footnote p. 3. Chap. I.

† Phil. Mag. xv. (May 1933) ; xviii (July 1934) ; xviii Suppl., (Nov. 1934) :
In the sequel references to these papers are indicated by R. I, II, III

heat-conduction equation - and summations of such solutions - could be obtained, easily adjustable to take into account quite complicated boundary conditions, prompted the general survey of the whole field of investigation usually covered in the standard works on the subject. This survey is represented by the papers of the series numbered G. II, III, IV, V, and R. I, II, III.

In papers G. II, III, and IV Dr. Green had dealt fully with problems of heat flow in rods, spheres and cylinders. Many particular cases were fully worked out and in addition some results of a general character were obtained, the special aspects of which were left over for future consideration.* At the suggestion of Dr Green these aspects were taken up by the writer and their discussion forms the matter of the papers R. I, II, and III. of the series.

These three papers in their published form hardly do justice to the subjects with which they deal. At the time of their compilation for the press the results obtained appeared to represent applications only of the fundamental theorems and processes explained in the previous papers, and while many considerations of physical and mathematical interest arose in the course of their investigation, these could not be dealt with satisfactorily in the restricted space allowed by the

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*see e.g G.I p 800 ; G III pp 255; 258, 260 ; Special cases of the general problem fully solved in G II pp 716 et seq, are treated in Chapter IV of this work by the application of first principles. <u>See also footnote p v</u>

publishers. Moreover, with the experience gained in the latter part of the period over which the work has been spread, it became apparent that certain improvements in the treatment of some of the earlier problems were required. Thus for example, the solutions of all the problems involving rods, cylinders or spheres reduce to the evaluation of a type of contour integral. The treatment of this integral in the published papers was not altogether satisfactory. It is hoped that this matter has now been represented in a more acceptable manner.

The thesis now submitted, incorporating all the writer's contributions to the literature, represents a more or less continuous work. The separate introductory statements necessary in the case of published writings appearing at wide intervals, are no longer required and have been omitted. On the other hand, a fullness of treatment formerly possible only in the writer's notebook has been introduced and allows the inclusion of further illustrative examples, a fuller description of the processes and a greater degree of mathematical detail. These improvements were regarded as necessary and their inclusion should help to make the work more readable.

The first chapter is of introductory character and presents such a selection of fundamental processes and results as are required in the later chapters. Here, as elsewhere, the indebtedness of the author to the writings of Dr. Green is apparent and the necessary acknowledgment is indicated.

Chapter II is devoted to the study of one-dimensional flow. The brief treatment of the semi-infinite solid in which several well-known results are derived does not appear in any of the published papers. The discussion of the finite rod when one end is kept at a uniform temperature, is substantially that given in the corresponding published paper but the processes are more fully explained and the integration treatment more satisfactorily disposed of.

Included also in this chapter is the theory of the cooling of a rod from a given initial state under prescribed end conditions. This part of the work forms a section of Dr. Green's third paper but its inclusion in a chapter having pretensions to completeness seemed warranted. The inclusion seems further warranted by the fact of the agreement of the results with those obtained by Green using analysis differing at least in detail from that used by the writer.

The investigation of the flow of heat across a surface of separation of two media of different conductivities seemed to be more satisfactory when an initial heat distribution was prescribed than when - as in the original paper - a definite temperature was maintained at a boundary surface. It is shown however, that the results in the latter case can be readily deduced from those in the former. The whole of this section of the chapter, dealing with two-medium problems has been rewritten. Certain useful results that lead to simplifications of what would otherwise be complicated summation processes, have been exhibited as definite theorems and may be found useful in dealing with other problems of this class.

There is also included in this chapter, as in the corresponding published paper, an original investigation of the theory of a well known experimental method of measuring conductivity.[†]

Chapter III deals with radial flow in a sphere. This subject has been so fully dealt with in one of the other papers[*] that no more work of a fundamental character was included than that required to give the necessary theorems to be used in the main - though restricted - problem of the chapter. Thus the treatment does not differ considerably from that given in the author's second paper. The revised treatment of the contour integrals required to give the complete solution of the problem, leads to the same results exactly as those previously obtained.

The fourth chapter deals with radial flow in infinitely long cylinders. Considerable space is given to the study of linear and cylindrical surface sources; from the solutions representing sources of the latter type many well known results,

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[*] G III p. 245 et seq.

[†]. N.B. When the writer first approached Dr Green for advice as to a definite line of Study in Mathematical Physics, this problem (see p. 37) was the one originally suggested. The investigation led in due course to the paper I.

usually obtained by entirely different methods, have been built up. The Bessel Function analysis is of special interest and is shown in a degree of detail quite impossible in the corresponding published part of the work. Of special note are (1), the discussion of flow in a cylindrical tube, the inner surface of which is kept at a constant temperature, - as by steam circulation - while at the outer surface heat losses take place by radiation; and (2), the treatment of the cylindrical core surrounded by a coaxial sheath of different conductivity. These problems, clearly of some practical importance, have not previously been solved.

The section of this chapter on spherico-cylindrical analogues serves the purpose of linking together as a mathematical unity the various parts of the work, while the brief reference to permanent sources gives some indication of how the methods we have employed in connection with heat-conduction might be applied to obtain results in other branches of Mathematical Physics.

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CHAPTER I.

INTRODUCTORY.

One of the commonest problems in Mathematical Physics is that of determining the "state" at any point in a medium when the general law indicating how that state varies from point to point is given, and when definite initial and boundary conditions are specified. The "state" may be literally the mere displacement of a particle of the medium from an equilibrium configuration or it may be the divergence from some normal condition at points in the medium due to the existence of special circumstances at other points in the medium or at its boundaries. Frequently it is defined by means of a differential equation, the solution of which, modified to take into account the initial and boundary conditions, may be regarded as the objective of the investigation.

In many cases it happens that the state can be explained on the hypothesis of the medium as the vehicle of systems of wave trains. The problem, e.g., might be that of finding the motion of a bounded mass of liquid arising from a single impulse applied at a point on the boundary. In such a case we might regard the complicated after effect at any point in the medium as due to the propagation through it of the wave of disturbance initiated at the impulse centre. If the liquid were of infinite extent there would be little difficulty in accounting for the effect at any point. The single analytical expression for this initial wave would give the representation required. In practical affairs however, we have to take into account the existence of the boundaries of the medium. When a system of wave trains of any type is incident on a boundary there arises the complication of the reflected trains. Rarely indeed will the prescribed condition at a boundary be accounted for by the incidence on it of the primary disturbance train. We must therefore postulate a system of reflected trains of such a type that the resultant effect of the incident and reflected systems at the boundary shall be the maintenance of the condition specified there. Depending on the geometrical form of the wave fronts in the incident train and the form of the boundary, the determination of the reflected system may be a matter of great difficulty. If we suppose that this difficulty has been overcome it becomes possible, using the correct combination of incident and reflected systems to represent fully the effect at any point in the medium due to the original disturbance.

The process implied in these remarks is that usually adopted in connection with the solution of problems in Hydrodynamics, Acoustics, or Electromagnetic Theory. In these branches of Mathematical Physics a wave genesis of transmitted effects seems a most natural one when we consider how many of the phenomena are governed by the fundamental wave-equation

$$\frac{\partial^2\phi}{\partial t^2} = k^2\nabla^2\phi$$

The conception of flow of heat in a conducting medium as a form of wave motion has recently received much attention at the hands of G. Green.[*] In cases where effects are due to applied surface temperatures or heat distributions of periodic type the conception is a perfectly natural one. General physical considerations would suggest that the propagated effects are likewise periodic and that the mode of transmission is the train of waves emanating fromthe temperature or heat source. Even when the physical property in question is not vibratory it can frequently be represented by a summation of periodic terms each of which can be identified with a wave train passing through the medium. The problem of the mathematical theory is the determination ofthe particular summatiom of wave trains that will give the effects

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[*] Phil. Mag. iii. Suppl. (April 1927), I ; v. (April 1928), II ; ix (Feb. 1930), III ; xii Suppl. (Aug. 1931), IV ; xviii (Oct 1934), V. In the sequel references to these papers will be denoted by G.I, G.II. etc.

correctly at all points and at all instants of time and will account satisfactorily for the boundary and initial conditions.

To illustrate what is meant by a temperature wave-train in a conducting medium we take an example from one-dimensional flow. The matter forms more correctly a part of the next chapter but is introduced for the purpose of drawing attention to some general considerations of great importance in all later parts of the work.

In one-dimensional flow the differential equation to be solved is

$$\frac{\partial v}{\partial t} = \kappa \frac{\partial^2 v}{\partial x^2} \qquad --- \ (1)$$

the notation being that usually adopted in heat-conduction theory.

If we suppose that v varies with the time according to the factor e^{ikt} where k represents the frequency of a periodic vibration, and search for solutions of the type $e^{ikt} V$, V being a function of x alone, it is found that V must satisfy the equation

$$\frac{\partial^2 V}{\partial x^2} - \frac{ik}{\kappa} V = 0 \ , \qquad --- \ (2)$$

the solutions of which are given by $V = e^{\pm\sqrt{\frac{ik}{\kappa}}\,x}$.

Thus we have, θ_0 being some constant,

$$v_1 = \theta_0 e^{ikt - x\sqrt{\frac{ik}{\kappa}}} \ ; \qquad v_2 = \theta_0 e^{ikt + x\sqrt{\frac{ik}{\kappa}}} \qquad -- \ (3)$$

The real and imaginary parts of v_1 and v_2 are likewise solutions of (1) and are exhibited as

$$R(v_1),\ I(v_1) = \theta_0 e^{-x\sqrt{\frac{k}{2\kappa}}} \left\{ \cos\left(kt - x\sqrt{\frac{k}{2\kappa}}\right),\quad \sin\left(kt - x\sqrt{\frac{k}{2\kappa}}\right) \right\} \quad \ldots (4)$$

$$R(v_2),\ I(v_2) = \theta_0 e^{x\sqrt{\frac{k}{2\kappa}}} \left\{ \cos\left(kt + x\sqrt{\frac{k}{2\kappa}}\right),\quad \sin\left(kt + x\sqrt{\frac{k}{2\kappa}}\right) \right\} \quad \ldots (5)$$

Each of the two solutions expressed in (4) represents a continuous train of waves of determinate length and frequency advancing in the positive sense of x. The amplitude decays exponentially as the depth advanced. An examination of the form taken by the first of (4) e.g. shows that the solution $R(v_1)$ is that for the case of flow in the semi-infinite medium when the face $x=0$ is kept at the temperature $\theta_0 \cos kt$.

Similarly the two solutions contained in (5) represent trains of the same character advancing in the negative sense of x. There are thus four fundamental trains altogether, two representing effects propagated in the positive direction, the remaining two effects propagated in the negative direction. It will be shown later that in all the other types of regions of space considered there are the four fundamental wave-trains having wave-fronts of the appropriate geometrical form and corresponding in all other respects to the plane waves obtained here.

The solutions (4) and (5) are appropriate in problems involving finite or semi-infinite rods when a temperature is prescribed at one end. The question of the correct manner of combining such solutions to represent flow in specific cases is that of placing the necessary restriction on the parameter k. In problems involving finite rods e.g. it will be found that when the boundary conditions are taken into account only certain values of k with corresponding solutions of the types (4) and (5) are admissible. When these values of k are determined there still remains the question of the amount in which the various periodic terms corresponding to the admissible values of k appear in the solution. Numerous illustrations of how these questions are investigated are given in the later chapters.

THE INSTANTANEOUS PLANE SOURCE.

As a suitable introductory case we might show that the ordinary solutions for plane instantaneous temperature and heat sources can be expressed in terms of the fundamental wave trains. When it is recalled that any initial state e.g can be represented by the appropriate distribution of instantaneous ($t=0$) sources the possibility of expressing all solutions of whatever type in terms of wave-trains is at once realised.

We begin by considering the solution v of (1) given by

$$v = \frac{\theta_0}{\pi}\int_0^\infty e^{ikt - x\sqrt{\frac{ik}{\kappa}}}\,dk = \frac{\theta_0}{\pi}\int_0^\infty e^{-x\sqrt{\frac{k}{2\kappa}}}\cos\left(kt - x\sqrt{\frac{k}{2\kappa}}\right)dk$$
$$+\frac{i\theta_0}{\pi}\int_0^\infty e^{-x\sqrt{\frac{k}{2\kappa}}}\sin\left(kt - x\sqrt{\frac{k}{2\kappa}}\right)dk \quad \cdot(6)$$

The integrals appearing here have been evaluated by Green* without resorting to contour integration. The following demonstration is given for the purpose of introducing a process of which frequent use is made.

Let the function $f(\lambda) \equiv \frac{\theta_0}{\pi} e^{-\kappa\lambda^2 t - ix\lambda} 2i\kappa\lambda$ be integrated along the contour in the λ plane $(\lambda = \xi + i\eta = \rho e^{i\theta})$ consisting of (fig.1.)

(1) the line $\theta = -\frac{\pi}{4}$ from O to B where $OB = R$,

(2) the arc BA of the circle $|\lambda| = R$

(3) the real ξ axis from A to O.

Since $f(\lambda)$ has no singularity within the contour, we have

$$\int_{OBAO} f(\lambda)\,d\lambda = 0.$$

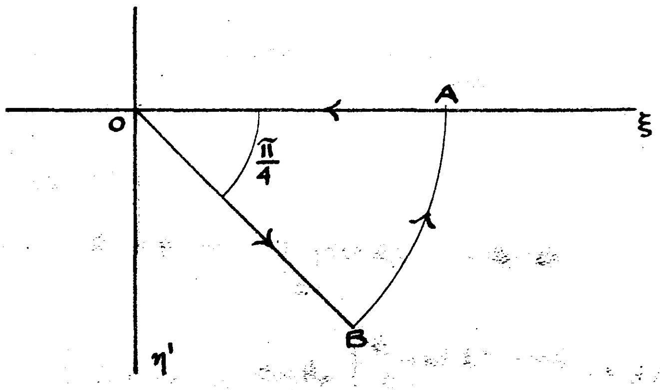

fig. 1.

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* G.I. p. 786.

On (i) $\lambda = \xi e^{-\frac{i\pi}{4}} = \xi \frac{1-i}{\sqrt{2}}$; $\lambda^2 = -i\xi^2$

$$\int_{OB} f(\lambda)d\lambda = \frac{2\kappa\theta_0}{\pi}\int_0^R e^{i\kappa t\xi^2 - ix\xi\frac{1-i}{\sqrt{2}}}\xi d\xi$$

$$= \frac{2\kappa\theta_0}{\pi}\int_0^R e^{-\frac{x\xi}{\sqrt{2}}}\left[\cos\left(\kappa t\xi^2 - \frac{x\xi}{\sqrt{2}}\right) + i\sin\left(\kappa t\xi^2 - \frac{x\xi}{\sqrt{2}}\right)\right]\xi d\xi$$

$$= \frac{\theta_0}{\pi}\int_0^\infty e^{-x\sqrt{\frac{k}{2\kappa}}}\left[\cos\left(kt - x\sqrt{\tfrac{k}{2\kappa}}\right) + i\sin\left(kt - x\sqrt{\tfrac{k}{2\kappa}}\right)\right]dk$$

when $R \longrightarrow \infty$.

On (ii) $\lambda = Re^{i\theta}$ and

$$\int_{BA} f(\lambda)d\lambda = \frac{2\kappa i\theta_0}{\pi}\int_{-\frac{\pi}{4}}^{0} e^{-\kappa tR^2(\cos 2\theta + i\sin 2\theta) - ixR(\cos\theta + i\sin\theta)}\, iR^2 e^{2i\theta}d\theta.$$

The modulus of this last integral is

$$\int_{-\frac{\pi}{4}}^{0} e^{-\kappa tR^2\cos 2\theta + xR\sin\theta}R^2 d\theta = \int_0^{\frac{\pi}{4}} e^{-\kappa tR^2\cos 2\theta - xR\sin\theta}R^2 d\theta$$

$$\leqq \int_0^{\frac{\pi}{4}} e^{-\kappa tR^2\cos 2\theta - xR\,2\theta/\pi}R^2 d\theta, \text{ i.e } \leqq \frac{1}{2}\int_0^{\frac{\pi}{2}} e^{-\kappa tR^2\sin\phi - xR(\frac{1}{2} - \frac{\phi}{\pi})}R^2 d\phi$$

$$\leqq \frac{1}{2}R^2 e^{-\frac{xR}{2}}\int_0^{\frac{\pi}{2}} e^{-\kappa tR^2 2\phi/\pi + xR\phi/\pi}d\phi$$

$$= \frac{\pi}{2(2\kappa tR^2 - xR)}\left(e^{-\frac{xR}{2}} - e^{-\kappa tR^2}\right)$$

Thus when $R \longrightarrow \infty$ $\int_{BA} f(\lambda)d\lambda \longrightarrow 0.$

Finally $$\int_{AO} f(\lambda)d\lambda = \frac{2\kappa\theta_0}{\pi}\int_\infty^0 e^{-\kappa t\xi^2 - ix\xi}\, i\xi d\xi$$

$$= -\frac{2\kappa\theta_0}{\pi}\int_0^\infty e^{-\kappa t\xi^2}\sin x\xi.\xi d\xi - i\frac{2\kappa\theta_0}{\pi}\int_0^\infty e^{-\kappa t\xi^2}\cos x\xi.\xi d\xi.$$

i.e. $$\int_{A_0} f(\lambda)\,d\lambda = -\frac{\theta_0 x}{2\sqrt{\pi\kappa t^3}}\,e^{-\frac{x^2}{4\kappa t}} - i\,\frac{\sqrt{\kappa}.\theta_0}{\pi\sqrt{t}}\left\{\frac{1}{\sqrt{\kappa t}} - \frac{x}{\kappa t}\,e^{-\frac{x^2}{4\kappa t}}\int_0^{\frac{x}{2\sqrt{\kappa t}}} e^{\alpha^2}\,d\alpha\right\}^{*}$$

Thus, noting that the integral round the contour is zero we have, on equating real and imaginary parts

$$R(v) = \frac{\theta_0}{\pi}\int_0^\infty e^{-x\sqrt{\frac{k}{2\kappa}}}\cos\left(kt - x\sqrt{\frac{k}{2\kappa}}\right)dk = \frac{\theta_0 x}{2\sqrt{\pi\kappa t^3}}\,e^{-\frac{x^2}{4\kappa t}} \quad \ldots\ (7)$$

$$I(v) = \frac{\theta_0}{\pi}\int_0^\infty e^{-x\sqrt{\frac{k}{2\kappa}}}\sin\left(kt - x\sqrt{\frac{k}{2\kappa}}\right)dk = \frac{\sqrt{\kappa}\,\theta_0}{\pi\sqrt{t}}\left\{\frac{1}{\sqrt{\kappa t}} - \frac{x}{\kappa t}\,e^{-\frac{x^2}{4\kappa t}}\int_0^{\frac{x}{2\sqrt{\kappa t}}} e^{\alpha^2}\,d\alpha\right\} \ldots (8)$$

The interpretation of the results (7) and (8) is postponed until we investigate a solution of (1) of a slightly different type from those given in (4) or (5). The solution referred to is

$$\left.\begin{aligned} x > 0; \quad & v_o = \frac{q}{2K\sqrt{\frac{ik}{\kappa}}}\,e^{ikt - x\sqrt{\frac{ik}{\kappa}}} \\ x < 0; \quad & v_i = \frac{q}{2K\sqrt{\frac{ik}{\kappa}}}\,e^{ikt + x\sqrt{\frac{ik}{\kappa}}} \end{aligned}\right\} \quad \ldots\ (9).$$

These results give

$$\left(-K\frac{\partial v_o}{\partial x}\right)_{x=0} = \frac{q}{2}\,e^{ikt}$$

$$\left(K\frac{\partial v_i}{\partial x}\right)_{x=0} = \frac{q}{2}\,e^{ikt}$$

The solutions v_o and v_i accordingly represent flow in an infinite medium due to the existence at the plane $x = 0$ of a heat

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* These results are obtained by differentiating with regard to x the two well-known integrals

$$\frac{2\kappa\theta_0}{\pi}\int_0^\infty e^{-\kappa t\xi^2}\cos x\xi\,d\xi = \frac{\theta_0\sqrt{\kappa}}{\sqrt{\pi t}}\,e^{-\frac{x^2}{4\kappa t}}$$

$$\frac{2\kappa\theta_0}{\pi}\int_0^\infty e^{-\kappa t\xi^2}\sin x\xi\,d\xi = \frac{2\sqrt{\kappa}\,\theta_0}{\pi\sqrt{t}}\,e^{-\frac{x^2}{4\kappa t}}\int_0^{\frac{x}{2\sqrt{\kappa t}}} e^{\alpha^2}\,d\alpha.$$

See e.g. MacRobert, Functions of a Complex Variable (1917). p. 73.

source emitting at the rate of qe^{ikt} units per unit area per second, half of the emission taking place in the positive direction the other half in the negative direction.

Now consider the solution given by

$$v = \frac{q}{2\pi K}\int_0^\infty e^{ikt - x\sqrt{\frac{ik}{\kappa}}}\,\frac{dk}{\sqrt{\frac{ik}{\kappa}}} \qquad \cdots (10).$$

$$\text{i.e}\quad v = \frac{q\sqrt{\kappa}}{2\pi K}\int_0^\infty \frac{e^{-x\sqrt{\frac{k}{2\kappa}}}}{k^{\frac{1}{2}}}\cos\left\{kt - x\sqrt{\frac{k}{2\kappa}} - \frac{\pi}{4}\right\}dk$$

$$+\frac{iq\sqrt{\kappa}}{2\pi K}\int_0^\infty \frac{e^{-x\sqrt{\frac{k}{2\kappa}}}}{k^{\frac{1}{2}}}\sin\left\{kt - x\sqrt{\frac{k}{2\kappa}} - \frac{\pi}{4}\right\}dk \qquad ..(11)$$

The evaluation of the integrals appearing here has been given by Green* but it is again instructive to obtain the results by the contour integral process explained in connection with (6) above.

We consider the integration round the contour of fig. 1 of the function given by

$$f(\lambda) = \frac{q\kappa}{\pi K}\,e^{-\kappa\lambda^2 t - ix\lambda}$$

Thus we find

$$\int_{OB} f(\lambda)\,d\lambda = \frac{q\kappa}{\pi K}\int_0^\infty e^{-\frac{x\xi}{\sqrt{2}} + i\left(\kappa t\xi^2 - \frac{x\xi}{\sqrt{2}} - \frac{\pi}{4}\right)}\,d\xi$$

$$= \frac{q\sqrt{\kappa}}{2\pi K}\int_0^\infty e^{-x\sqrt{\frac{k}{2\kappa}} + i\left(kt - x\sqrt{\frac{k}{2\kappa}} - \frac{\pi}{4}\right)}\,\frac{dk}{k^{1/2}}$$

= the expression on the right of (11).

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* ibid.

Also it can be shown that when $R \to \infty$ the integral along the arc $BA \to 0$.

Along OA

$$\int f(\lambda)d\lambda = \frac{q\kappa}{\pi K}\int_0^\infty e^{-\kappa t\xi^2 - ix\xi}d\xi$$

$$= \frac{q\sqrt{\kappa}}{2K\sqrt{\pi t}}e^{-\frac{x^2}{4\kappa t}} - i\,\frac{q\sqrt{\kappa}}{\pi K\sqrt{t}}e^{-\frac{x^2}{4\kappa t}}\int_0^{\frac{x}{2\sqrt{\kappa t}}} e^{\alpha^2}d\alpha.$$

where we have made use of the results given in the footnote to p. 9.

Thus, when we make use of the fact that the integral round the contour is zero, we have

$$R(v) = \frac{q\sqrt{\kappa}}{2\pi K}\int_0^\infty \frac{e^{-x\sqrt{\frac{k}{2\kappa}}}}{k^{1/2}}\cos\left\{kt - x\sqrt{\frac{k}{2\kappa}} - \frac{\pi}{4}\right\}dk = \frac{q\sqrt{\kappa}}{2K\sqrt{\pi t}}e^{-\frac{x^2}{4\kappa t}} \quad (12).$$

$$I(v) = \frac{q\sqrt{\kappa}}{2\pi K}\int_0^\infty \frac{e^{-x\sqrt{\frac{k}{2\kappa}}}}{k^{1/2}}\sin\left\{kt - x\sqrt{\frac{k}{2\kappa}} - \frac{\pi}{4}\right\}dk = \frac{-q\sqrt{\kappa}}{K\pi\sqrt{t}}e^{-\frac{x^2}{4\kappa t}}\int_0^{\frac{x}{2\sqrt{\kappa t}}} e^{\alpha^2}d\alpha \ldots (13).$$

The result on the right of (12) is recognisable at once as the well known solution for the instantaneous plane source of strength q at $x=0$. The equivalent solution - that expressed by the integral - indicates how effects due to such a source are ultimately explainable in terms of the fundamental wave-trains given in (9).

Commenting further on these results we observe (i). The solution on the right of (12) gives $v=0$ when $t=0, x\neq 0.$ This can be readily shown using the evaluated form, or from the equivalent integral form, viz:-

$$\int_0^\infty \frac{e^{-x\sqrt{\frac{k}{2\kappa}}}}{k^{1/2}}\cos\left(x\sqrt{\frac{k}{2\kappa}}+\frac{\pi}{4}\right)dk$$

$$=2\sqrt{\kappa}\int_0^\infty e^{-x\xi}(\cos x\xi-\sin x\xi)\,d\xi$$

$$=0\,,\ \text{as required} \qquad x>0.$$

(ii). The solution on the right of (13) gives a certain temperature distribution when $t=0$. When $t\to 0$, the upper limit of integration $\frac{x}{2\sqrt{\kappa t}}$ becomes very great and the value of the integral is sufficiently given by the first few terms of its asymptotic expansion. Thus the solution becomes*

$$v=\frac{-q\sqrt{\kappa}}{K\pi\sqrt{t}}\left\{\frac{\sqrt{\kappa t}}{x}+\frac{2(\kappa t)^{3/2}}{x^3}+\frac{12(\kappa t)^{5/2}}{x^5}+\cdots\right\}$$

$$\longrightarrow\frac{-q\kappa}{K\pi}\frac{1}{x}\quad \text{when}\quad t\to 0,\quad x\neq 0.$$

The same result is obtained from the integral form of the solution, viz:-

$$-\frac{q\sqrt{\kappa}}{2\pi K}\int_0^\infty e^{-x\sqrt{\frac{k}{2\kappa}}}\sin\left(x\sqrt{\frac{k}{2\kappa}}+\frac{\pi}{4}\right)\frac{dk}{k^{1/2}}$$

$$=-\frac{q\kappa}{\pi K}\int_0^\infty e^{-x\xi}(\sin x\xi+\cos x\xi)\,d\xi=-\frac{q\kappa}{\pi K}\frac{1}{x}\,,\qquad x\neq 0.$$

(iii). When $x=0$ the solution on the right of (12) becomes

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* see e.g. Bromwich, Infinite Series (1908) p. 352.

$$v = \frac{q\sqrt{\kappa}}{2K\sqrt{\pi t}}, \qquad t \neq 0.$$

while the corresponding solution of (13) is

$$x = 0, \quad v = 0, \quad t \neq 0.$$

It appears, in fact, that we have the purely mathematical results

$$\int_0^\infty \frac{\cos(kt - \frac{\pi}{4})}{\sqrt{k}}\,dk = \sqrt{\frac{\pi}{t}},$$

$$\int_0^\infty \frac{\sin(kt - \frac{\pi}{4})}{\sqrt{k}}\,dk = 0,$$

$$\text{Whence} \int_0^\infty \frac{\sin kt}{\sqrt{k}}\,dk = \int_0^\infty \frac{\cos kt}{\sqrt{k}}\,dk = \sqrt{\frac{\pi}{2t}}.$$

- results known on other grounds to be correct.

INSTANTANEOUS AND CONTINUOUS DOUBLETS.

The interpretation of the result (7) obtained at an earlier stage is now considered. If we have an instantaneous plane source of strength q per unit area at $x=0$ and a sink of corresponding strength at $x=-\Delta x$ the effect at any point is given by

$$v = \frac{q\,\Delta x . x}{4K\sqrt{\pi\kappa t^3}}\,e^{-\frac{x^2}{4\kappa t}} = \frac{Mx}{4K\sqrt{\pi\kappa t^3}}\,e^{-\frac{x^2}{4\kappa t}},$$

where we suppose that as $\Delta x \to 0$, q increases in such a way that $q\Delta x \to$ a definite limit M the strength of the doublet.

It now becomes apparent that the solution given by (7) is that corresponding to an instantaneous doublet of strength

$2K\theta_0$ "located" at $x=0$. The definite integral form of that result shows how such a doublet is explainable in terms of the fundamental wave-trains.

If the doublet is continuous, i.e. if heat is supplied at the uniform rate of q_0 units per unit area at the face $x=0$ and at an equal negative rate at $x=-\Delta x$ the result is given by

$$v = \frac{\theta_0 x}{2\sqrt{\pi K}} \int_0^t e^{-\frac{x^2}{4K(t-t')}} \frac{dt'}{(t-t')^{3/2}}$$

$$= \theta_0 \left\{ 1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{Kt}}} e^{-\alpha^2} d\alpha \right\}$$

It is apparent from this form that so long as $x > 0$ the effect of the continuous doublet is the same as if the face $x=0$ were kept at the uniform temperature θ_0 throughout.

The result (8) appears so far to be only of theoretical importance. It stands in the same relationship to (7) as (13) does to (12). All the problems discussed in the following pages bear directly on the results (7) and (12). The corresponding investigations based on the solutions (8) and (13) have not yet been carried out.

SUMMARY of CHAPTER I.

Expressions for the instantaneous initial plane source, and the instantaneous or continuous plane doublet are obtained in terms of the fundamental wave-trains. Since any initial temperature state prescribed throughout a medium can be represented by the appropriate distribution of instantaneous sources and any initial or continuous surface temperature by the corresponding distribution of initial or continuous doublets, the possibility of obtaining the solution of any problem in heat conduction in terms of wave trains is thereby indicated.

The instantaneous source and doublet solutions are particular cases of the general solution

$$v = \frac{1}{\pi}\int_0^\infty e^{ikt - x\sqrt{\frac{ik}{\kappa}}} f(k)\,dk \;;^{*}$$

the heat source solution being that obtained when

$$f(k) = \frac{q}{2K\sqrt{\frac{ik}{\kappa}}} \;,$$

and the doublet solution that obtained when

$$f(k) = \text{const} = \theta_0.$$

The question of the reflection or transmission of wave trains is considered in the succeeding chapters. By suitable choice of $f(k)$, the train

$$e^{ikt - x\sqrt{\frac{ik}{\kappa}}} f(k)$$

may be made to represent a train reflected or transmitted at a boundary however simple or complicated the boundary condition may be.

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[*] For a general discussion of solutions of this type see H. Bateman "Partial Differential Equations of Mathematical Physics" (1932) p. 214 et seq.

CHAPTER II.

PROBLEMS INVOLVING ONE-DIMENSIONALFLOW.

The typical case considered is that of a rod of finite or semi-infinite length and of uniform cross section. The direction of heat flow is that of the length of the rod. The flow may be due to one end of the rod being maintained at some prescribed temperature or again it may be due to the cooling off of the rod from a given initial temperature on account of heat losses of a known character from the boundaries.

Many practical methods of determining conductivity depend on temperature observations on a heated rod. The full theory of some of these methods is a suitable subject of enquiry in the present connection and one well-known method is examined at some length.

In the earlier problems discussed the rod is of one material throughout its length. Opportunity is taken at a later stage of investigating the case of a rod consisting of two sections of different conductivity. It is found that the wave-train analysis of heat flow is specially suitable in such cases as this where we have a surface of separation of two different media.

(α). The Semi-infinite Rod.

Some aspects of the semi-infinite rod have already been considered. To introduce the subject of reflection of temperature wave trains we consider the case represented by

$$v = 0, \quad t = 0, \quad x \neq x_1$$

$$v = 0, \quad x = 0\,;$$

at the plane $x = x_1$ at $t = 0$ an instantaneous source of strength q per unit area.

Let us suppose to begin withthat the source is periodic and of strength qe^{ikt} per unit area. Then first effects are accounted for by the wave train solution given by

$$\left.\begin{aligned} x < x_1, \quad v_{1i} &= \frac{q}{2K\sqrt{\frac{ik}{\kappa}}}\, e^{ikt-(x_1-x)\sqrt{\frac{ik}{\kappa}}} \\ x > x_1, \quad v_{1o} &= \frac{q}{2K\sqrt{\frac{ik}{\kappa}}}\, e^{ikt-(x-x_1)\sqrt{\frac{ik}{\kappa}}} \end{aligned}\right\} \quad \cdots (14)$$

It will be found that these expressions satisfy the required conditions at the surface $x = x_1$ viz.

$$v_i = v_o \qquad \text{and} \qquad -K\left(\frac{\partial v_o}{\partial x} - \frac{\partial v_i}{\partial x}\right) = qe^{ikt} \qquad ---- \quad (15)$$

If any addition be made to either of the expressions in (14) to represent secondary effects, it is clear that an equal addition must be made to the other if these conditions are to be maintained. Thus e.g. it is apparent that the train v_i on arrival at the plane $x=0$ violates the zero temperature condition there. We must therefore suppose that a corresponding reflected train is set up, the joint effect of the incident-reflected pair at this surface being the maintenance of the required condition. A suitable form of train is clearly

$$-\frac{q}{2K\sqrt{\frac{ik}{\kappa}}}\,e^{ikt-(x_1+x)\sqrt{\frac{ik}{\kappa}}}\,.$$

Thus we find that all the conditions are satisfied by the solution

$$\left.\begin{aligned} x<x_1\,, \quad v_i &= \frac{q}{2K\sqrt{\frac{ik}{\kappa}}}\,e^{ikt}\left\{e^{-(x_1-x)\sqrt{\frac{ik}{\kappa}}} - e^{-(x_1+x)\sqrt{\frac{ik}{\kappa}}}\right\} \\ x>x_1\,, \quad v_o &= \frac{q}{2K\sqrt{\frac{ik}{\kappa}}}\,e^{ikt}\left\{e^{-(x-x_1)\sqrt{\frac{ik}{\kappa}}} - e^{-(x+x_1)\sqrt{\frac{ik}{\kappa}}}\right\} \end{aligned}\right\} \quad \cdots \quad (16)$$

If now we take the solution given by

$$R\,\frac{1}{\pi}\int_0^\infty (v_i\,, v_o)\,dk$$

it is clear that we will obtain that for the joint effect of the instantaneous source of strength q at $x=x_1$ and the equal "image" sink at $x=-x_1$, these sources now being supposed placed in the doubly infinite medium with no boundary at $x=0$. The effect is physically the same as that due to the original source in presence of the reflecting boundary.

If we write $\sqrt{\frac{ik}{\kappa}} = i\lambda$ and apply the integration process to v_0 in (16) we find that the required solution is given by

$$x > x_1 , \qquad v_0 = R\frac{q}{2\pi K}\int_{OB} e^{-\kappa\lambda^2 t - xi\lambda} 2i \sin x_1\lambda\kappa d\lambda$$

$$= R\frac{2q\kappa}{\pi K}\int_{OB} e^{-\kappa\lambda^2 t - xi\lambda} i \sin x_1\lambda \, d\lambda \quad - - - - (17).$$

where the path of integration is the line OB of fig.1. By adopting the method explained in connection with the corresponding evaluation in Chapter I. we find that the required solution is

$$x > x_1 , \qquad v_0 = R\frac{2q\kappa}{\pi K}\int_0^\infty e^{-\kappa t\xi^2 - ix\xi} i \sin x_1\xi d\xi$$

$$= \frac{2q\kappa}{\pi K}\int_0^\infty e^{-\kappa t\xi^2} \sin x\xi \sin x_1\xi d\xi$$

$$= \frac{q\sqrt{\kappa}}{2K\sqrt{\pi t}}\left\{e^{-\frac{(x-x_1)^2}{4\kappa t}} - e^{-\frac{(x+x_1)^2}{4\kappa t}}\right\} \quad - - - - (18).$$

with, clearly, exactly the same result for the region $x < x_1$.

If e.g.an initial state is prescribed throughout the rod given by

$$t = 0, \qquad v = f(x) , \qquad 0 \leqq x \leqq \infty,$$

we replace q by $\frac{K}{\kappa}f(x_1)dx_1$ and integrate with regard to x_1 from 0 to ∞. In this way we obtain

$$v = \frac{1}{2\sqrt{\pi\kappa t}}\int_0^\infty f(x_1)\left\{e^{-\frac{(x-x_1)^2}{4\kappa t}} - e^{-\frac{(x+x_1)^2}{4\kappa t}}\right\}dx_1$$

$$= \frac{1}{\sqrt{\pi}}\left\{\int_{-\frac{x}{2\sqrt{\kappa t}}}^{\infty} f(x+2\sqrt{\kappa t}\,\alpha)e^{-\alpha^2}d\alpha - \int_{\frac{x}{2\sqrt{\kappa t}}}^{\infty} f(-x+2\sqrt{\kappa t}\,\alpha)e^{-\alpha^2}d\alpha\right\}.^{*} \quad \ldots (19)$$

It is of interest to notice the form taken by this result when the integration with regard to x_1 is effected prior to that with regard to ξ. From (18) we see that the required form is

$$v = \frac{2}{\pi}\int_0^\infty {}_\wedge d\xi\int_0^\infty f(x_1)\sin x\xi.\sin x_1\xi\,dx_1. \qquad / \; e^{-\kappa t\xi^2}$$

With $t=0$ in this equation, we have at once Fourier's well known integral theorem

If now we take the particular case $f(x) = \text{constant} = v_0$ the result (19) yields

$$v = \frac{2v_0}{\sqrt{\pi}}\int_0^{\frac{x}{2\sqrt{\kappa t}}} e^{-\alpha^2}d\alpha. \qquad - - - - - (20).$$

It should be noted also that if we take the solution given by

$$v = v_0\left\{1 - \frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2\sqrt{\kappa t}}} e^{-\alpha^2}d\alpha\right\}, \qquad - - - - (21).$$

we have the solution corresponding to flow in the semi-infinite solid originally at the uniform temperature zero, when the face $x=0$ is kept at the temperature v_0 throughout. In writing down this solution we have been guided by the general principle that if v is a solution of

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* The result in this form is given by Byerly, "Fourier's Series etc" (1893) p. 84.

$$\frac{\partial v}{\partial t} = \kappa \frac{\partial^2 v}{\partial x^2}$$

$$v = 0, \quad x = 0, \quad ; \qquad v = f(x), \quad t = 0$$

then $\quad u = f(x) - v \quad$ is a solution of

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}$$

$$u = 0, \quad t = 0, \qquad u = f(0), \quad x = 0 \quad ; \quad f''(x) = 0.$$

The rod of finite length a.

We examine in the first place some solutions obtained by supposing that the end $x = 0$ of the rod is kept at the periodic temperature $\theta_0 e^{ikt}$. The condition obtaining at the end $x = a$ may conveniently be stated later.

As we have seen in the previous chapter, a temperature wave train is propagated along from the heated end of the rod giving a first effect at distance x indicated by

$$v_1 = \theta_0 e^{ikt - x\sqrt{\frac{ik}{\kappa}}} \qquad \text{- - - - (22)}$$

This train arrives at the boundary $x = a$ with the value $\theta_0 e^{ikt - a\sqrt{\frac{ik}{\kappa}}}$ It is exceedingly improbable that this is the temperature condition prescribed there. We must therefore suppose that the train (22) is reflected at the boundary, thereby setting up a negatively travelling train whose effect v_2 is given by

$$v_2 = A\theta_0 e^{ikt - (2a - x)\sqrt{\frac{ik}{\kappa}}}, \qquad \text{- - - - - - (23).}$$

where A is a constant depending on the condition at the surface $x = a$. The total effect at any point (including $x = a$) is now given by $(v_1 + v_2)$. It is apparent however that the train (23), converging towards $x = 0$ would on its arrival, violate the condition at that surface (viz. $v = \theta_0 e^{ikt}$).

We must therefore again suppose that there is a corresponding reflected train. If we call this reflected train v_3 it is clear that a necessary condition is that at $x=0$, v_2 and v_3 should just neutralise one another. Clearly then v_3 is given by

$$v_3 = -A\theta_0 e^{ikt-(2a+x)\sqrt{\frac{ik}{\kappa}}} \quad . \quad . \quad . \quad . \quad (24)$$

In writing down the expressions for v_2 and v_3 we have been guided by the physical consideration of phase equality at the boundary in any incident-reflected pair.

Continuing the process, we see that when the train (~~3~~24) arrives at $x=a$ it again sets up a negatively travelling train. Denoting it by v_4 we have

$$v_4 = -A^2\theta_0 e^{ikt-(4a-x)\sqrt{\frac{ik}{\kappa}}}, \quad . \quad . \quad . \quad (25).$$

So far the effect at any point is $(v_1+v_2+v_3+v_4)$, but it is apparent that the process we have indicated goes on indefinitely. The complete system of trains within the medium is conveniently visualised by means of the following scheme. ($\rho = \theta_0 e^{ikt}$).

Positive Trains.	Negative Trains.
$\rho e^{-x\sqrt{\frac{ik}{\kappa}}}$	$A\rho e^{-(2a-x)\sqrt{\frac{ik}{\kappa}}}$
$-A\rho e^{-(2a+x)\sqrt{\frac{ik}{\kappa}}}$	$-A^2\rho e^{-(4a-x)\sqrt{\frac{ik}{\kappa}}}$
$+A^2\rho e^{-(4a+x)\sqrt{\frac{ik}{\kappa}}}$	$+A^3\rho e^{-(6a-x)\sqrt{\frac{ik}{\kappa}}}$
. . .	

Thus for the total effect of the periodic temperature $\theta_0 e^{ikt}$ at $x=0$ we have

$$v = \theta_0 \sum_0^\infty (-1)^n A^n e^{ikt-(2na+x)\sqrt{\frac{ik}{\kappa}}} - \theta_0 \sum_1^\infty (-1)^n A^n e^{ikt-(2na-x)\sqrt{\frac{ik}{\kappa}}} \quad \ldots (26).$$

The result (26) is perfectly general. For special cases, depending on the nature of the condition at the end $x=a$ we modify it by giving to A the appropriate form.

(i). The end $x=a$ kept at zero temperature.

If v_n, v_{n+1} represent respectively any incident train and the corresponding reflected train we have

$$x=a, \qquad v_n + v_{n+1} = 0.$$

$$\text{whence} \quad A^n(1+A) = 0 \quad \text{or} \quad A = -1.$$

In this case the result (26) becomes

$$v = \theta_0 \sum_0^\infty e^{ikt-(2na+x)\sqrt{\frac{ik}{\kappa}}} - \theta_0 \sum_1^\infty e^{ikt-(2na-x)\sqrt{\frac{ik}{\kappa}}}. \quad \ldots (27)$$

This is the form of result required when the periodic temperature $\theta_0 e^{ikt}$ is maintained at $x=0$. The real or the imaginary part of the right hand side of (27) is taken according as this temperature is $\theta_0 \cos kt$ or $\theta_0 \sin kt$.

Important results are obtained from (27) when we take the solution v' indicated by

$$v' = \frac{\theta_0}{\pi} \int_0^\infty v \, dk.$$

The solution required is

$$v' = \frac{\theta_0}{\pi}\sum_{0}^{\infty}\int_0^{\infty} e^{ikt-(2na+x)\sqrt{\frac{ik}{\kappa}}}\,dk \;-\; \frac{\theta_0}{\pi}\sum_{1}^{\infty}\int_0^{\infty} e^{ikt-(2na-x)\sqrt{\frac{ik}{\kappa}}}\,dk \quad \ldots \quad (28)$$

Using the result (7), Chap.I., taking the real part of each side, we have

$$R(v') = \frac{\theta_0}{2\sqrt{\pi\kappa t^3}}\left\{\sum_{0}^{\infty}(2na+x)e^{-\frac{(2na+x)^2}{4\kappa t}} - \sum_{1}^{\infty}(2na-x)e^{-\frac{(2na-x)^2}{4\kappa t}}\right\} \quad \ldots (29)$$

If we denote the terms of the first summation by s_0, s_1, s_2 &c, and those in the second summation by t_1, t_2, t_3 &c, we see that s_0 represents an instantaneous doublet of strength $2K\theta_0$ at $x=0$, t_1 an equal and opposite doublet at the image of s_0 in the plane $x=a$, s_1 an equal positive doublet at the image of t_1 in the plane $x=0$ and so on. The series in fact give the distribution of doublets in an infinite medium equivalent to an original doublet of strength $2K\theta_0$ at $x=0$ in presence of the boundary $x=a$ kept at zero temperature. See fig. 2 .

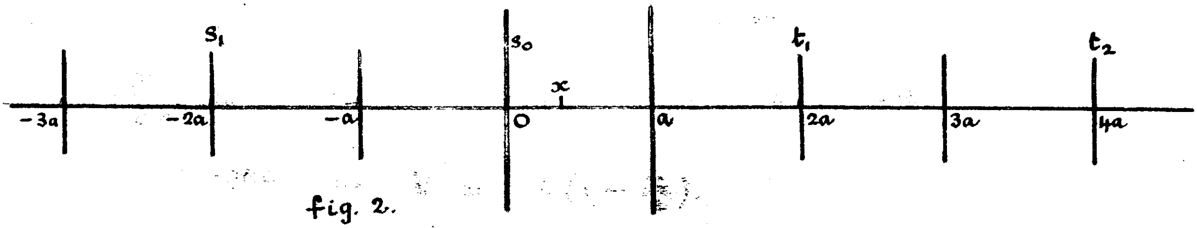

fig. 2.

The result (29) may be rewritten in the form

$$R(v') = \frac{\theta_0}{2\sqrt{\pi\kappa t^3}}\sum_{-\infty}^{\infty}(x+2na)e^{-\frac{(x+2na)^2}{4\kappa t}}, \quad \ldots \quad (30)$$

which, by adaptation of a well known transformation,* can be exhibited

$$R(v') = \frac{2\kappa\theta_0}{a}\sum_1^\infty \frac{n\pi}{a} e^{-\frac{n^2\pi^2}{a^2}\kappa t} \sin\frac{n\pi x}{a} . \qquad . \; . \; . \; (31)$$

We can build up from this solution that corresponding to a continuous doublet of strength $2\kappa\theta_0 f(t)$ at the end $x=0$ of the rod, operative from the instant $t=0$ onwards. All we have to do is to replace θ_0 by $f(t')dt'$, t by $(t-t')$ and integrate with regard to t' from 0 to t. In particular, if $f(t) =$ constant $= v_0$, say, the result would become

$$v' = \frac{2\kappa v_0}{a}\sum_1^\infty \frac{n\pi}{a}\sin\frac{n\pi x}{a}\int_0^t e^{-\frac{n^2\pi^2}{a^2}\kappa(t-t')}dt'$$

$$= \frac{2v_0}{\pi}\sum_1^\infty \frac{1}{n}\sin\frac{n\pi x}{a}\left(1-e^{-\frac{n^2\pi^2}{a^2}\kappa t}\right) \qquad . \; . \; . \; (32)$$

When t is indefinitely great i.e. when the steady state has become established this result takes the form

$$V' = \frac{2v_0}{\pi}\sum_1^\infty \frac{1}{n}\sin\frac{n\pi x}{a}$$

This state however is that given by the system

$$\frac{\partial^2 v}{\partial x^2} = 0 \quad ; \quad v = v_0,\ x = 0 \quad ; \quad v = 0,\ x = a.$$

whence $$V' = v_0\left(1-\frac{x}{a}\right).$$

Thus the result (32) may be shown

$$v' = v_0\left(1-\frac{x}{a}\right) - \frac{2v_0}{\pi}\sum_1^\infty e^{-\kappa\frac{n^2\pi^2}{a^2}t}\sin\frac{n\pi x}{a} \qquad . \; . \; . \; (33).$$

The identity of the two forms for V' given by

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* see e.g. Carslaw p.159.

$$V' = v_0\left(1-\frac{x}{a}\right) = \frac{2v_0}{\pi}\sum_1^\infty \frac{1}{n}\sin\frac{n\pi x}{a}$$

is simply a restricted case of Fourier's Theorem.

The method we have adopted above of building up solutions from the fundamental equation (26) is only suitable in cases where the reflection constant **A** is of very simple character. In the cases that we go on to consider, this constant - or, as we should more properly designate it - this operator, may be a more or less complicated f~~o~~unction of **k**. As a consequence the integrations corresponding to those we have in (28) would become almost intractable. We accordingly consider a useful alternative procedure for the obtaining of the results shown above, principally with a view to its adoption in the further problems to be investigated.

The procedure consists in summing the infinite series in (26) before considering the solutions obtained by means of the **k** integrations. Thus, writing $i\lambda$ for $\sqrt{\frac{ik}{\kappa}}$ in (26) we find that this result takes the form

$$v = \theta_0 e^{ikt-ix\lambda}\sum_0^\infty(-1)^n A^n e^{-2nai\lambda} - \theta_0 e^{ikt+ix\lambda}\sum_1^\infty(-1)^n A^n e^{-2nai\lambda}$$

$$= \frac{\theta_0 e^{ikt-ix\lambda}}{1+Ae^{-2ai\lambda}} + \frac{A\theta_0 e^{ikt-(2a-x)i\lambda}}{1+Ae^{-2ai\lambda}}$$

$$= \theta_0 e^{ikt}\,\frac{e^{-ix\lambda} + Ae^{-i(2a-x)\lambda}}{1+Ae^{-2ai\lambda}} \qquad \ldots \quad (34)$$

This again is a fundamental result. If e.g. as in the case discussed above $A = -1$, (34) becomes

$$v = \theta_o e^{ikt} \frac{e^{-ix\lambda} - e^{-(2a-x)i\lambda}}{1 - e^{-2ai\lambda}}$$

$$= \theta_o e^{ikt} \frac{\sin(a-x)\lambda}{\sin a\lambda} \qquad \ldots \quad (35)$$

Again the result v' defined in (28) takes the form

$$v' = \frac{2ik\theta_o}{\pi} \int_{OB} e^{-\kappa\lambda^2 t} \frac{\sin(a-x)\lambda}{\sin a\lambda} \lambda d\lambda , \qquad \ldots \quad (36)$$

where the path of integration is the infinite radius $\theta = -\frac{\pi}{4}$ in the λ plane. ($\lambda = Re^{i\theta}$).

Now integrate the function

$$f(\lambda) \equiv e^{-\kappa\lambda^2 t} \frac{\sin(a-x)\lambda}{\sin a\lambda} \lambda$$

round the contour consisting of the path OB, the arc BA, and the real axis AO indented at the points given by $a\lambda = n\pi$; see fig. 3.

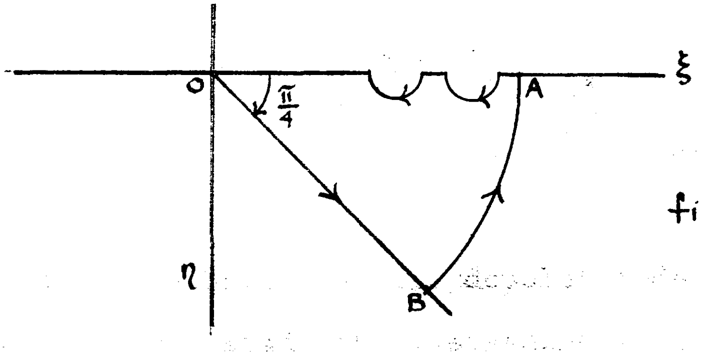

fig. 3.

We demonstrate in the first place that when $R \to \infty$ the value of the integral along the arc $BA \to 0$.

When R is very great it can be readily shown that

$$\left|\frac{\sin(a-x)\lambda}{\sin a\lambda}\right| \doteqdot \frac{\cosh\{(a-x)R\sin\theta\}}{\cosh\{aR\sin\theta\}}$$

$$\doteqdot \frac{e^{-(a-x)R\sin\theta}}{e^{-aR\sin\theta}} \quad ; \; -\frac{\pi}{2} \leqq \theta \leqq 0.$$

$$= e^{xR\sin\theta}$$

accordingly we have $\left|\int_{BA} f(\lambda)d\lambda\right| \doteqdot \int_{-\frac{\pi}{4}}^{0} e^{-\kappa tR^2\cos 2\theta + xR\sin\theta} R^2 d\theta$

the convergence of which when $R \to \infty$ has already been demonstrated. Thus when $R \to \infty$ and when the radii of the indents $\longrightarrow 0$,

$$\int_{OB} f(\lambda)d\lambda = \int_0^\infty f(\xi)d\xi + \sum \int_{indent} f(\lambda)\,d\lambda.$$

The value of the integral round the indent at $a\lambda = n\pi$ is

$$-\pi i \,\frac{n\pi}{a}\, e^{-\frac{n^2\pi^2}{a^2}\kappa t}\, \frac{1}{a} \sin\frac{n\pi x}{a}$$

Thus from (36) we have

$$v' = \frac{2i\kappa\theta_0}{\pi}\int_0^\infty e^{-\kappa t\xi^2} \frac{\sin(a-x)\xi}{\sin a\xi}\,\xi d\xi + \frac{2\kappa\theta_0}{a}\sum_1^\infty \frac{n\pi}{a} e^{-\frac{n^2\pi^2}{a^2}\kappa t} \sin\frac{n\pi x}{a}, \cdots (37)$$

or, taking the real part of this result, that which gives the solution for the case of the instantaneous doublet of strength $2\kappa\theta_0$ at $x=0$,

$$v' = \frac{2\kappa\theta_0}{a}\sum_1^\infty \frac{n\pi}{a} e^{-\frac{n^2\pi^2}{a^2}\kappa t} \sin\frac{n\pi x}{a}.$$

in agreement with the result obtained by the former method. From this point onwards the procedure is exactly as previously indicated. While we have used the doublet mode of expression, it need hardly be repeated that everything is just as if the end $x=0$ of the rod were kept at the temperature θ_0 throughout.

(ii). The end $x=a$ of the rod radiating to a medium at zero temperature.

Analytically this condition is expressed by

$$x = a, \quad -K\frac{\partial v}{\partial x} = hv \qquad \ldots (38)$$

where h is the coefficient of surface emissivity.

If we take as typical train emanating from the end $x=0$ kept at the periodic temperature $\theta_0 e^{ikt}$, and as continuation reflected from the surface $x=a$ the following expressions respectively

$$\rho e^{-xi\lambda}, \quad A\rho e^{-(2a-x)i\lambda}, \qquad \left(i\lambda = \sqrt{\frac{ik}{\kappa}}\right)$$

we find that A, to satisfy the above condition is given by

$$A = \frac{Ki\lambda - h}{Ki\lambda + h}. \qquad \ldots (39)$$

We have now to modify the fundamental result (34) by giving to A the above form. Properly reduced we find that the required result is

$$v = \theta_0 e^{ikt}\frac{K\lambda\cos(a-x)\lambda + h\sin(a-x)\lambda}{K\lambda\cos a\lambda + h\sin a\lambda} \qquad \ldots (40).$$

Proceeding at once to the solution $v' = \frac{\theta_0}{\pi}\int_0^\infty v\,dk$, whose real part represents the effect due to the instantaneous doublet of strength $2K\theta_0$ at $x=0$, we have

$$v' = \frac{2\kappa i\theta_0}{\pi}\int_{OB} e^{-\kappa\lambda^2 t}\frac{K\lambda\cos(a-x)\lambda + h(\sin(a-x)\lambda}{K\lambda\cos a\lambda + h\sin a\lambda}\lambda\,d\lambda \quad \ldots (41).$$

where the path of integration is again the line OB of fig. 1.

The integration is effected by integrating the function

$$f(\lambda) = \frac{K\lambda\cos(a-x)\lambda + h\sin(a-x)\lambda}{K\lambda\cos a\lambda + h\sin a\lambda}\lambda e^{-\kappa\lambda^2 t}$$

round a contour like that in fig. 3 , the only difference being that the semi-circular indents are round the points given by the positive roots of the equation

$$K\lambda \cos a\lambda + h \sin a\lambda = 0. \qquad \ldots (42)$$

Using the same theory as in corresponding previous evaluations we have

$$\int_{OB} f(\lambda)d\lambda = \int_0^\infty f(\xi)d\xi + \pi i \sum_1^\infty e^{-\kappa\lambda^2 t} \frac{K\lambda\cos(a-x)\lambda + h\sin(a-x)\lambda}{\frac{d}{d\lambda}\{K\lambda\cos a\lambda + h\sin a\lambda\}}\lambda$$

$$= \int_0^\infty f(\xi)d\xi - \pi i \sum_1^\infty e^{-\kappa\lambda^2 t} \frac{(K^2\lambda^2+h^2)\lambda}{h(K+ah)+aK^2\lambda^2} \sin x\lambda \qquad \ldots (43)$$

where the summation is with regard to all the positive roots of the equation (42).

Thus, confining our attention to the real part of the result (43), we obtain

$$R(v') = 2\kappa\theta_0 \sum_1^\infty e^{-\kappa\lambda^2 t} \frac{(K^2\lambda^2+h^2)\lambda}{h(K+ah)+aK^2\lambda^2} \sin x\lambda \qquad \ldots (44)$$

The effect of the continuous doublet of strength $2K\theta_0$ at $x=0$ is given by

$$v_1 = 2\theta_0 \sum_1^\infty \frac{K^2\lambda^2+h^2}{h(K+ah)+aK^2\lambda^2} \frac{\sin x\lambda}{\lambda}\left(1 - e^{-\kappa\lambda^2 t}\right) \qquad \ldots (45)$$

The corresponding "steady state" solution is obtained by making $t \to \infty$. This state however, obtained from first principles is

$$\phi = \theta_0\left(1 - \frac{hx}{K+ah}\right)$$

Thus v_1 can be exhibited in the form

$$v_1 = \theta_0\left\{1 - \frac{hx}{K+ah} - 2\sum_1^\infty e^{-\kappa\lambda^2 t}\frac{(K^2\lambda^2+h^2)}{h(K+ah)+aK^2\lambda^2}\,\frac{\sin x\lambda}{\lambda}\right\} \quad \dots \quad (46)$$

The equivalence indicated by

$$1 - \frac{hx}{K+ah} = 2\sum_1^\infty \frac{(K^2\lambda^2+h^2)}{h(K+ah)+aK^2\lambda^2}\,\frac{\sin x\lambda}{\lambda},$$

the summation being with regard to all the positive roots of the equation (42) is, in fact, a well known result.*

In the foregoing problems relating to the finite rod the flow has been due to the maintenance of a prescribed temperature at one end, the temperature at all points of the rod except at this end being initially zero. A return to this class of problem is made later in the discussion of a practical method of determining conductivity.

It is equally important that we consider the case where an initial temperature state is prescribed throughout the rod. The results for certain simple initial temperature distributions are readily obtained from those shown above by using the principle indicated in connection with the corresponding investigation involving the semi-infinite rod. (see p.20). For general purposes however it is necessary that we indicate in outline how such problems are treated by the application of first principles.

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* see e.g. Carslaw pp 74 et seq.

Cooling of a finite rod from a given initial temperature state.

Let the initial atate of the rod be given by

$$t = o, \quad v = f(x) \;; \quad 0 \leqq x \leqq a. \qquad \dots (47)$$

The condition obtaining at the ends $x=o$ and $x=a$ of the rod may conveniently be specified later.

The procedure is indicated by the following steps which are considered in the order named.

(i). The effect due to a periodic source of strength qe^{ikt} per unit area at the plane $x=x_1$ is fouhd.

(ii). From this effect, that due to an instantaneous plane source of strength q at $x=x_1$ is obtained.

(iii). If q be replaced by $\frac{K}{\kappa}f(x_1)dx_1$ in the result giving the effect (ii)$_{\wedge}^{*}$, we obtain that due to the initial temperature state prescribed above.

As previously indicated (P 17), the first effects of the periodic source at $x=x_1$ are given by

$$\left.\begin{aligned} x < x_1, \quad & v_{1_i} = \frac{q}{2Ki\lambda} e^{ikt-(x_1-x)i\lambda} = \rho e^{-(x_1-x)i\lambda} \\ x > x_1, \quad & v_{1_o} = \frac{q}{2Ki\lambda} e^{ikt-(x-x_1)i\lambda} = \rho e^{-(x-x_1)i\lambda} \end{aligned}\right] \quad \dots (48)$$

where $i\lambda = \sqrt{\frac{ik}{\kappa}}$; $\rho = \frac{q}{2Ki\lambda} e^{ikt}$.

Similarly, as previously remarked, if any addition be made to either of these trains to represent secondary reflection effects,

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* insert " and this result integrated with regard to x_1 from o to a".

an equal addition must be made to the other so that the dual condition at the source surface may be maintained. Thus we begin with the original negative train and write down the contributions made by it and its continuations by reflections at the surfaces $x=0$ and $x=a$ to v_i and v_o the resultant effect in both parts $x \lessgtr x_1$ of the field.

$x < x_1$		$x > x_1$	
	$pe^{-(x_1-x)i\lambda}$		
$Fpe^{-(x_1+x)i\lambda}$		$Fpe^{-(x_1+x)i\lambda}$	
	$AFpe^{-(2a+x_1-x)i\lambda}$		$AFpe^{-(2a+x_1-x)i\lambda}$
$AF^2pe^{-(2a+x_1+x)i\lambda}$		$AF^2pe^{-(2a+x_1+x)i\lambda}$	
	$A^2F^2pe^{-(4a+x_1-x)i\lambda}$		$A^2F^2pe^{-(4a+x_1-x)i\lambda}$
$A^2F^3pe^{-(4a+x_1+x)i\lambda}$		. . .	

where **A** and **F** are the reflection coefficients applicable at the boundaries $x=a$, $x=0$ respectively.

Thus, resulting from the original negative train alone we have the partial effects

$$x < x_1, \qquad pe^{-x_1 i\lambda}\frac{\{Fe^{-xi\lambda}+e^{xi\lambda}\}}{1-AFe^{-2ai\lambda}}$$

$$x > x_1, \qquad Fpe^{-x_1 i\lambda}\frac{\{e^{-ix\lambda}+Ae^{-(2a-x)i\lambda}}{1-AFe^{-2ai\lambda}}\}$$

Similarly if we tabulate the original positive train and its various continuations by reflection at $x=a$ and $x=0$ we obtain

$x < x_1$		$x > x_1$	
		$pe^{-(x-x_1)i\lambda}$	
	$Ape^{-(2a-x-x_1)i\lambda}$		$Ape^{-(2a-x-x_1)i\lambda}$
$AFpe^{-(2a+x-x_1)i\lambda}$		. . .	

Hence the partial effect arising from the original positive train is

$$x < x_1 , \qquad Ape^{-(2a-x_1)i\lambda} \frac{\{Fe^{-xi\lambda} + e^{xi\lambda}\}}{1 - AFe^{-2ai\lambda}} ,$$

$$x > x_1 , \qquad pe^{x_1 i\lambda} \frac{e^{-xi\lambda} + Ae^{-(2a-x)i\lambda}}{1 - AFe^{-2ai\lambda}} .$$

Thus the total effect due to the <u>periodic</u> surface source is given by

$$x < x_1 , \qquad v_i = \frac{Fe^{-xi\lambda} + e^{xi\lambda}}{1 - AFe^{-2ai\lambda}} \left\{ pe^{-x_1 i\lambda} + Ape^{-(2a-x_1)i\lambda} \right\}$$

$$x > x_1 , \qquad v_o = \frac{e^{-xi\lambda} + Ae^{-(2a-x)i\lambda}}{1 - AFe^{-2ai\lambda}} \left\{ pe^{x_1 i\lambda} + Fpe^{-x_1 i\lambda} \right\}$$

$$\left.\begin{aligned} \text{i.e} \quad x < x_1 , \quad v_i &= p \frac{\{Fe^{-xi\lambda} + e^{xi\lambda}\}\{e^{-x_1 i\lambda} + Ae^{-(2a-x_1)i\lambda}\}}{1 - AFe^{-2ai\lambda}} \quad . \quad . \\ &= p\phi(x, x_1) \quad \text{say}, \\ x > x_1 , \quad v_o &= p\phi(x_1, x). \end{aligned}\right\} \quad (49)$$

These results are perfectly general. For special cases we give to A and F forms appropriate to the conditions prescribed at $x = a$ and $x = 0$. As previously we take two cases :-

(1). Both ends at zero temperature.

(2). Both ends radiating to a medium at zero temperature.

In the first case we readily find $A = F = -1$

so that the first of (49) becomes

$$x < x_1 , \quad v_i = p \cdot \frac{2i \sin x\lambda \{e^{-x_1 i\lambda} - e^{-(2a-x_1)i\lambda}\}}{1 - e^{-2ai\lambda}} = p \frac{2i \sin x\lambda . \sin(a - x_1)\lambda}{\sin a\lambda} \quad . . (50)$$

with a corresponding form for v_o.

In the second case we find $A = F = \frac{Ki\lambda - h}{Ki\lambda + h}$.

and the first of (49) becomes after necessary reduction

$$x < x_1, \qquad v_i = \rho \frac{2i(K\lambda \cos x\lambda + h \sin x\lambda)\{K\lambda \cos(a-x_1)\lambda + h \sin(a-x_1)\lambda\}}{2Kh\lambda \cos a\lambda + (h^2 - K^2\lambda^2)\sin a\lambda} \quad \ldots (51).$$

$$= \rho F(\lambda), \text{ say.}$$

Proceeding now to the results for the case of the instantaneous source at $x = x_1$ we find that these are

(1). <u>Ends at zero temperature.</u>

$$v_i = \frac{q\kappa}{K\pi} \int_{OB} e^{-\kappa\lambda^2 t} \frac{2i \sin x\lambda \sin(a-x_1)\lambda}{\sin a\lambda} d\lambda \quad \ldots (52)$$

where the path of integration is the line OB of fig. 1.

The evaluation of the integral is again effected by an integrattion round the complete contour of fig. 3.

It is found also that the real part of the result is that which arises from the integrals round the indents at the points given by $a\lambda = n\pi$. Thus the result required is

$$x < x_1, \qquad v = \frac{2q\kappa}{Ka} \sum_1^{\infty} e^{-\kappa\lambda^2 t} \sin x\lambda \sin x_1\lambda \ , \quad a\lambda = n\pi \quad \ldots (53)$$

with, clearly, exactly the same result for the region $x > x_1$.

(2). <u>Ends radiating to medium at zero temperature.</u>

In this case we have

$$v_i = \frac{q\kappa}{K\pi} \int_{OB} e^{-\kappa\lambda^2 t} F(\lambda) d\lambda \quad \ldots (54)$$

For the evaluation of the integral here, we take as closed contour the path of fig. 3, on the understanding that the indents are round the points given by the positive roots of the equation

$$\phi(\lambda) \equiv 2K\lambda h \cos a\lambda - (K^2\lambda^2 - h^2)\sin a\lambda = 0. \quad \ldots \quad (55)$$

At a root of this equation the factor $K\lambda\cos(a-x_1)\lambda + h\sin(a-x_1)\lambda$ of the numerator of $F(\lambda)$ becomes $K\lambda\cos x_1\lambda + h\sin x_1\lambda$.

Thus when we take the real part of the result (54), giving the solution for the case of the instantaneous source at $x = x_1$ we obtain, noting that at a root of (55), $\phi'(\lambda) = -\{a(K^2\lambda^2+h^2)+2Kh\}$

$$v_i = \frac{2q\kappa}{K}\sum_1^\infty e^{-\kappa\lambda^2 t}\frac{(K\lambda\cos x\lambda + h\sin x\lambda)(K\lambda\cos x_1\lambda + h\sin x_1\lambda)}{a(K^2\lambda^2+h^2)+2Kh} \quad \ldots \quad (56)$$

where the summation is with regard to all the positive roots of the equation (55).

Finally, corresponding to the initial distribution given by

$$v = f(x), \quad t = 0, \quad 0 \leqq x \leqq a.$$

we have (1). Ends at zero temperature.

from (53).

$$v = \frac{2}{a}\sum_1^\infty e^{-\frac{n^2\pi^2}{a^2}\kappa t}\sin\frac{n\pi x}{a}\int_0^a f(x_1)\sin\frac{n\pi x_1}{a}dx_1. \quad \ldots \quad (57)$$

Clearly with $t=0$, this result reduces to the familiar half-range Fourier series for $f(x)$.

(2). Radiation at both ends.

from (56).

$$v = 2\sum_1^\infty e^{-\kappa\lambda^2 t}\frac{(K\lambda\cos x\lambda + h\sin x\lambda)}{a(K^2\lambda^2+h^2)+2Kh}\int_0^a f(x_1)(K\lambda\cos x_1\lambda + h\sin x_1\lambda)dx_1 \ldots (58)$$

The form taken by this result when $t=0$ is a well known theorem.*

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* Carslaw. ibid.

Investigation of the theory of an experimental method of determining conductivity.*

The investigation may be regarded as an application of the theory given on pp21-31 above, where we have considered the flow of heat in a rod, one end of which is maintained at a constant temperature while at the other end heat losses take place according to some simple law.

One end of a rod of relatively low conductivity is kept at a constant temperature from the commencement of the experiment. To the other end is soldered a copper ball. It is assumed that the whole ball instantaneously takes the temperature of the end of the rod to which it is fixed. Part of the heat passing from the rod to the ball is used in raising the temperature of the ball, while the remaining part is radiated from the surface of the ball to a medium which we may suppose kept at zero temperature. Thus we have, if ϖ is the cross sectional area of the rod, M the mass of the ball, s the specific heat of the ball, S the surface area of the ball, h the coefficient of surface emissivity,

$$-K\varpi\frac{\partial v}{\partial x} = Ms\frac{\partial v}{\partial t} + hSv.$$

$$\text{or} \quad -K\frac{\partial v}{\partial x} = q\frac{\partial v}{\partial t} + pv. \qquad q = \frac{Ms}{\varpi}, \quad p = \frac{hS}{\varpi} \quad \ldots (59)$$

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* see J.H. Gray, Proc. Roy. Soc. 1894. Methods based on that of Gray have been in use in the Physical Laboratory at Glasgow University for some time. (See footnote † introd. p V above.)

The problem to be solved is that in which the end $x=0$ of the rod is kept at a constant temperature θ_0 while the condition at the end $x=a$ is that specified in the equation(59). The procedure to be adopted is that indicated in several previous cases. Thus the solution for heat flow in the rod due to the instantaneous creation of the temperature θ_0 at $x=0$ at the instant $t=0$ is given by

$$v=\frac{2i\kappa\theta_0}{\pi}\int_{OB} e^{-\kappa\lambda^2 t}\,\frac{e^{-ix\lambda}+Ae^{-i(2a-x)\lambda}}{1+Ae^{-2ai\lambda}}\,\lambda\,d\lambda \quad \ldots (60)$$

where A is the coefficient in any wave train reflected from the surface $x=a$ under the condition expressed by (59). By taking as incident and reflected trains respectively the forms

$$\theta_0 e^{ikt-ix\lambda}\ , \qquad A\theta_0 e^{ikt-i(2a-x)\lambda}$$

we find quite readily that A is given by

$$A=\frac{Ki\lambda-(p-q\kappa\lambda^2)}{Ki\lambda+(p-q\kappa\lambda^2)} \quad \ldots\ldots (61)$$

When this value of A is inserted in the integral in (60), the result we then have is

$$v=\frac{2i\kappa\theta_0}{\pi}\int_{OB} e^{-\kappa\lambda^2 t}\,\frac{K\lambda\cos(a-x)\lambda+(p-q\kappa\lambda^2)\sin(a-x)\lambda}{K\lambda\cos a\lambda+(p-q\kappa\lambda^2)\sin a\lambda}\,\lambda\,d\lambda \ldots (62)$$

By using the contour of fig. 3 where the understanding is that the semicircular indents are round the points given by the positive roots of the equation

$$\tan a\lambda=-\frac{K\lambda}{p-q\kappa\lambda^2}, \quad \ldots\ldots (63)$$

we find that the above result yields after reduction

$$v = 2\kappa\theta_0 \sum A_m e^{-\kappa\lambda^2 t} \lambda \sin x\lambda \quad \ldots \quad (64)$$

where
$$A_m = \frac{K^2\lambda^2 + (p - q\kappa\lambda^2)^2}{a\left\{K^2\lambda^2 + (p - q\kappa\lambda^2)^2\right\} + K(p + q\kappa\lambda^2)}$$

and where the summation is with regard to all the positive roots of the equation (63).

When the temperature θ_0 is maintained at the end $x = 0$ from the instant $t=0$ onwards the last result above is replaced by

$$v = 2\theta_0 \sum A_m (1 - e^{-\kappa\lambda^2 t}) \frac{\sin x\lambda}{\lambda}$$

or
$$v = \theta_0\left(1 - \frac{px}{K + pa}\right) - 2\theta_0 \sum A_m e^{-\kappa\lambda^2 t} \frac{\sin x\lambda}{\lambda} \quad \ldots \quad (65)$$

where we have again availed ourselves of the device of filling in the first term from the direct consideration of the steady state.

To estimate the practical importance of the result (65), we must consider how many terms of the series need be retained having regard to the numerical values of the various constants involved. For the sake of definiteness we make a brief arithmetical examination, taking values suggested by the apparatus used by Gray in the original investigation. Thus e.g. for an iron rod- selecting a relatively poor conductor-

$K = 0.167$, $\kappa = 0.2016$, $a = 6.28$; radius of rod 0.2 ; radius of copper ball 2.75 ; $h = 0.0003$. With these values we find that the equation (63) becomes

$$\tan 6.28\lambda = -\frac{0.001358}{0.001845 - \lambda^2} \quad \ldots \quad (66)$$

The first relevant root of this equation is given by $\lambda = 0.04534$; and the succeeding roots approximately by $\lambda_n = \frac{n}{2}$ $n = 1, 2, 3, \text{\&c}$ to a degree of accuracy that increases as n increases. Thus if it is the intention to observe the temperature at $x = a$, i.e. the temperature of the ball, we see that for this value of x all the terms on the right of (65) after the second are practically zero. In fact we may exhibit this result in the form

$$V_a = \theta_0 \left\{ \frac{K}{K + \mu a} - 2A_1 \frac{\sin a\lambda_1}{\lambda_1} e^{-\kappa\lambda_1^2 t} - \ldots \right\} \quad \ldots \quad (67)$$

or numerically

$$V_a = \theta_0 \left\{ 0.1049 - 0.1069\, e^{-0.0004145t} + \cdots \right\}$$

It would seem quite a justifiable conclusion from the practical point of view to take as a working formula

$$V_a = \theta_0 \left\{ \frac{K}{K + \mu a} - B_1 e^{-\kappa\lambda_1^2 t} \right\} \quad \ldots \quad (68)$$

The comparison of this result with that obtained by assuming - as in ordinary laboratory practice - a uniform temperature gradient along the bar, is of considerable interest. If in (59) we put $\frac{\theta_0 - v_a}{a}$ for $-\frac{\partial v}{\partial x}$ and integrate this equation we get

$$V_a = \frac{K\theta_0}{K+ha}\left\{1 - e^{-\frac{K+ha}{aq}t}\right\}, \qquad (69)$$

or numerically, with the same data as before,

$$V_a = \theta_0\left\{0\cdot1049 - 0\cdot1049\, e^{-0\cdot0004155t}\right\}.$$

The conclusion reached in (68) would seem to indicate some such process as follows, for the determination of the constants K and h.

We notice that $K\theta_0/(K+ha)$, the steady temperature, can be observed directly or inferred from a temperature-time curve. Let the value be θ_0/f. Then $h = K(f-1)/a$. Next we have

$$\frac{\partial v}{\partial t} = \kappa\lambda_1^2 B_1 \theta_0 e^{-\kappa\lambda_1^2 t},$$

so that if we take the ratio g_1/g_2 of the values of $\frac{\partial v}{\partial t}$ at two instants t_1, t_2 we get

$$\kappa\lambda_1^2 = \frac{1}{t_2 - t_1}\log\frac{g_1}{g_2};$$

g_1 and g_2 may be determined from the temperature-time curve, and in this way $\kappa\lambda_1^2$ calculated. Finally, λ_1 is the smallest root of the equation

$$\tan a\lambda = \frac{-K\lambda}{h - q\kappa\lambda^2} = \frac{-c\rho\lambda}{\frac{c\rho(f-1)}{a} - q\lambda^2}$$

in which all the quantities are now known. The root may be obtained graphically and, since we know $\kappa\lambda_1^2$, κ can now be calculated. The value of h is obtained by putting the calculated value of κ in the formula $p = K(f-1)/a$.

In view of possible applications of the theory given, it should be remembered that the conclusions reached apply only to apparatus of the particular design specified. For rods of different length and section and for a ball of greater or smaller surface area we would have to consider again the special form taken by the general result (64). The assumption also that the entire ball at once takes the temperature of the end of the rod would require further examination. In that respect the theory is still incomplete.

[illegible] $= f(x)$ [illegible]

[illegible]

already been found [illegible] the result (64) [illegible]

Hence the result is reproduced.

$$[illegible]$$

[illegible] the same expression, [illegible] interchanged}

PROBLEMS INVOLVING CHANGE OF MEDIUM.

The case contemplated is that in which we have a rod of length $(a+b)$, one section of which, from $x=0$ to $x=a$ is of one material, the other section from $x=a$ to $x=b$ being of another material of different conductivity. The first section is referred to as medium I and has conductivity and diffusivity K_1 and κ_1 respectively. The second section is denoted by medium II and to this the corresponding constants K_2, κ_2 apply.

The problem ultimately to be solved is indicated by the following system.

$$0 \leqq x \leqq a ; \quad \frac{\partial v_1}{\partial t} = \kappa_1 \frac{\partial^2 v_1}{\partial x^2} \quad : \quad a \leqq x \leqq b ; \quad \frac{\partial v_2}{\partial t} = \kappa_2 \frac{\partial^2 v_2}{\partial x^2} \quad \cdots (70)$$

$$t=0, \quad v_1 = f_1(x), \quad 0 \leqq x \leqq a \quad : \quad v_2 = f_2(x), \quad a \leqq x \leqq b.$$

The temperature condition applying at the exposed surfaces $x=0$ and $x=b$ may conveniently be specified later.

We begin by considering the effect of a periodic surface source of strength qe^{ikt} per unit area at the surface $x=x_1$ within Medium_I. This effect, so far as the first medium is concerned has already been found. (See the result (49) above.) For convenience the result is reproduced.

$$\left.\begin{array}{ll} x < x_1, & v_i = p\dfrac{\{Fe^{-xi\lambda} + e^{xi\lambda}\}\{e^{-x_1 i\lambda} + Ae^{-(2a-x_1)i\lambda}\}}{1 - AFe^{-2ai\lambda}} \\ x > x_1, & v_o = p\{\text{the same expression, } x, x_1 \text{ interchanged}\} \end{array}\right\} \cdots (71)$$

In these expressions the various symbols and operators have the meanings previously assigned to them.

So far as the second medium is concerned we have now to take into account that each positively travelling train comprised in v_0 is on arrival at the boundary $x=a$ partially transmitted to medium II. The transmitted train by repeated reflections at the boundaries $x=b$ and $x=a$ builds up the complete first-effect system in medium II.

Theorem. (1). Any positively travelling train $\alpha e^{-ix\lambda}$ in medium I builds up by partial transmission at $x=a$ the effect v_2 in medium II given by

$$v_2 = A'\alpha\, e^{-\ell}\,\frac{\psi}{S_2}$$

where A' is the coefficient for transmission at $x=a$, B and C' are the coefficients for reflection at the surfaces $x=b$ and $x=a$ respectively within the second medium.

$$\ell = \{a+\mu(b-a)\}i\lambda \quad ; \quad \mu = \sqrt{\frac{K_1}{K_2}}$$

$$\psi = e^{\mu(b-x)i\lambda} + B e^{-\mu(b-x)i\lambda},$$

$$S_2 = 1 - BC' e^{-\mu(2b-2a)i\lambda}.$$

The theorem is readily established. The original transmitted train is

$$A'\alpha e^{-ai\lambda-\mu(x-a)i\lambda}.$$

The first reflected train from $x=b$ is $A'B\alpha e^{-ai\lambda-\mu(2b-x-a)i\lambda}$.

The next train, that reflected from $x=a$ is $A'BC'\alpha e^{-ai\lambda-\mu(2b+x-3a)i\lambda}$.

The corresponding expressions for the further trains set up by reflections at the boundaries are easily written down. If the positive and the negative trains be summed separately, the result as stated above is at once obtained.

Theorem. (2). Any negatively travelling train $Be^{-\mu(b-x)i\lambda}$ in medium II, when partially retransmitted across the surface $x=a$ gives an effect v_1 in medium I where

$$v_1 = BCe^{-l}\frac{\phi}{S_1} ,$$

C being the coefficient for retransmission,

and $$\phi = Fe^{-ix\lambda} + e^{ix\lambda} ,$$

$$S_1 = 1 - AFe^{-2ai\lambda} .$$

The proof follows at once by writing down the transmitted continuation of the train $Be^{-\mu(b-x)i\lambda}$ as $CBe^{-\mu(b-a)i\lambda-(a-x)i\lambda}$.

Reverting now to the problem proper, we now observe that all the positively travelling elements in the effect v_0 indicated in (7') above, may be regarded as one comprehensive train

$$\rho\frac{\phi_1}{S_1}e^{-ix\lambda} \qquad \left(\phi_1 = Fe^{-ix_1\lambda} + e^{ix_1\lambda}\right).$$

By the theorem I, we see that the partial transmission of this train across the surface $x=a$ gives the effect v_{21} in medium II where

$$v_{21} = A'\rho e^{-l}\frac{\phi_1\psi}{S_1 S_2}.$$

Again all the negatively travelling trains comprised in this sum comstitute the single train

$$A'\rho e^{-\ell}\frac{\phi_1 B e^{-\mu(b-x)i\lambda}}{S_1 S_2}$$

and by theorem II, the retransmission of this train across the surface $x=a$ gives the first secondary effect v_{12} in medium I where

$$v_{12}=\rho A'BC e^{-2\ell}\frac{\phi_1\phi}{S_1^2 S_2}.$$

Continuing the process and noticing that the positively travelling element in ϕ is $Fe^{-ix\lambda}$, we find that the next-order effect in medium II is

$$v_{22}=\rho A'^2 BCF e^{-3\ell}\frac{\phi_1\psi}{S_1^2 S_2^2}.$$

The process continues indefinitely. The following table shows the successive order effects in the two media.

Effect No.	Medium I	Medium II.
1	$\rho\frac{\phi_1}{S_1}\left\{e^{-xi\lambda}+Ae^{-(2a-x)i\lambda}\right\}$	$A'\rho e^{-\ell}\frac{\phi_1\psi}{S_1S_2}$
2	$\rho A'BC e^{-2\ell}\frac{\phi_1\phi}{S_1^2S_2}$	$\rho A'^2BCF e^{-3\ell}\frac{\phi_1\psi}{S_1^2S_2^2}$
3	$\rho A'^2B^2C^2F e^{-4\ell}\frac{\phi_1\phi}{S_1^3S_2^2}$	$\rho A'^3B^2C^2F^2 e^{-5\ell}\frac{\phi_1\psi}{S_1^3S_2^3}$

It is at once seen that the successive order effects in medium II are given by the terms of an infinite geometrical progression. If the anomalous first effect in medium I be excluded a like remark applies to this medium also. This effect is partly reduced to conformity by rewriting it in the form

$$\rho\frac{\phi_1}{F}\cdot\frac{Fe^{-xi\lambda}+e^{ix\lambda}-e^{ix\lambda}+AFe^{-(2a-x)i\lambda}}{1-AFe^{-2ai\lambda}}$$

$$=\rho\frac{\phi_1\phi}{FS_1}-\rho\frac{\phi_1}{F}e^{ix\lambda}$$

Thus summing the series we have

$$v_1=-\rho\frac{\phi_1}{F}e^{ix\lambda}+\rho\frac{\phi\phi_1}{F}\frac{S_2}{S_1S_2-A'BCFe^{-2l}}, \quad \cdot\ \cdot\ \cdot \quad (72)$$

$$v_2=A'\rho e^{-\ell}\phi_1\psi\frac{1}{S_1S_2-A'BCFe^{-2l}}. \quad \cdot\ \cdot\ \cdot \quad (73)$$

These results apply when the periodic source is located in medium I. For general purposes it is equally important that we obtain the corresponding results when the source is in medium II. It is unnecessary to reproduce the details of the analysis for this case; we remark that the anticipated reciprocity between the sets of results is fully borne out and is at once apparent in the final form ~~they~~ these results take. Thus we have

$$v_1=C\rho'e^{-\ell}\psi_1\phi\frac{1}{S_1S_2-A'BCFe^{-2l}} \quad * \quad - - - \quad (74).$$

$$v_2=-\rho'\frac{\psi_1}{B}e^{\mu(b-x)i\lambda}+\rho'\frac{\psi\psi_1}{B}\frac{S_1}{S_1S_2-A'BCFe^{-2l}}. \quad - - - \quad (75).$$

$$\left(\rho'=\frac{q}{2K_2\mu i\lambda}e^{ikt}\right)$$

The results we have obtained are perfectly general. Particular problems are solved by giving to the reflection constants F and B forms appropriate to the conditions prevailing at the boundaries $x=0$ and $x=b$. The coefficients A, A', C, C' applicable at $x=a$

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* $\psi_1 = e^{\mu(b-x_1)i\lambda}+Be^{-\mu(b-x_1)i\lambda}$

are independent of the end conditions and always have the same form. Thus if we take forms for v_1 and v_2 indicated by

$$\begin{array}{ll} \text{Source in Medium I.} & \begin{cases} v_1 = e^{-ix\lambda} + Ae^{-(2a-x)i\lambda} \\ v_2 = A'e^{-ai\lambda - \mu(x-a)i\lambda} \end{cases} \end{array}$$

$$\begin{array}{ll} \text{Source in Medium II.} & \begin{cases} v_2 = e^{-\mu(b-x)i\lambda} + C'e^{-\mu(b+x-2a)i\lambda} \\ v_1 = Ce^{-\mu(b-a)i\lambda - (a-x)i\lambda} \end{cases} \end{array}$$

We have to choose the various constants so that at $x = a$ we may have

$$v_1 = v_2$$

$$K_1 \frac{\partial v_1}{\partial x} = K_2 \frac{\partial v_2}{\partial x}$$

In this way we find

$$\left.\begin{array}{lll} A = \dfrac{K_1 - K_2\mu}{K_1 + K_2\mu} = \dfrac{\sigma-1}{\sigma+1} \;; & C = \dfrac{2K_2\mu}{K_1+K_2\mu} = \dfrac{2}{\sigma+1} & \\ & & \sigma = \dfrac{K_1}{K_2\mu} \\ A' = \dfrac{2K_1}{K_1 + K_2\mu} = \dfrac{2\sigma}{\sigma+1} \;; & C' = -A = -\dfrac{\sigma-1}{\sigma+1} & \end{array}\right\} \cdots (76)$$

with, clearly, such simple relationships as $AC' - CA' = -1$.

Next let us consider the case

Both ends of the rod kept at zero temperature.

In this case we readily find that the coefficients F and B are given by $F = B = -1$.

also

$$\psi = 2i \sin \mu(b-x)\lambda$$

$$\phi = 2i \sin x\lambda$$

with corresponding forms for ψ_1 and ϕ_1.

Similarly $S_1 = 1 + Ae^{-2ai\lambda}$; $S_2 = 1 + c'e^{-2\mu(b-a)i\lambda}$.

When these simplified forms are inserted in (72),(73),(74),(75) we find, after considerable reduction that these results may be shown as

Source in Medium I.

$$\left\{\begin{aligned} v_1 &= \frac{qe^{ikt}}{2K_1 i\lambda} 2i\sin x_1\lambda \,\frac{\sin(a-x)\lambda\cos\mu(b-a)\lambda + \sigma\cos(a-x)\lambda\sin\mu(b-a)\lambda}{f_1(\lambda)}, \quad \text{---(77)} \\ v_2 &= \frac{\sigma q e^{ikt}}{2K_1 i\lambda} 2i\sin x_1\lambda \,\frac{\sin\mu(b-x)\lambda}{f_1(\lambda)}. \quad \text{---(78)} \end{aligned}\right.$$

where

$$f_1(\lambda) = \sin a\lambda\cos\mu(b-a)\lambda + \sigma\cos a\lambda\sin\mu(b-a)\lambda. \quad \text{---(81)}$$

also : Source in Medium II.

$$\left\{\begin{aligned} v_1 &= \frac{qe^{ikt}}{2K_2\mu i\lambda} 2i\sin\mu(b-x_1)\lambda \,\frac{\sin x\lambda}{f_1(\lambda)}, \quad \text{-----(79)} \\ v_2 &= \frac{qe^{ikt}}{2K_2\mu i\lambda} 2i\sin\mu(b-x_1)\lambda \,\frac{\sin a\lambda\cos\mu(x-a)\lambda + \sigma\cos a\lambda\sin\mu(x-a)\lambda}{f_1(\lambda)}. \quad \text{---(80)} \end{aligned}\right.$$

Likewise when we proceed to solutions representing instantaneous initial sources, obtained by taking the real part of the solution of the type

$$v = \frac{1}{\pi}\int_0^\infty v_1 dk$$

we obtain from the first of the above set

$$v_1 = \frac{q\kappa_1}{\pi K_1}\int_{OB} e^{-\kappa_1\lambda^2 t}\, 2i\sin x_1\lambda \frac{\left\{\begin{aligned}&\sin(a-x)\lambda\cos\mu(b-a)\lambda \\ &+\sigma\cos(a-x)\lambda\sin\mu(b-a)\lambda\end{aligned}\right\}}{f_1(\lambda)} d\lambda. \quad (82)$$

where the path of integration is the line OB of fig. 1 . To evaluate the integral we take as a closed contour that of fig. 3, on the understanding that the indents are round the points given by the roots of the equation

$$f_1(\lambda) \equiv \sin a\lambda \cos\mu(b-a)\lambda + \sigma\cos a\lambda \sin\mu(b-a)\lambda = 0 \qquad \ldots (83)$$

It is found that the real part of the required integral is that which arises from the sum of the integrals round the semi-circular indents.

At a root of (83) we find that the expression

$$\sin(a-x)\lambda\cos\mu(b-a)\lambda + \sigma\cos(a-x)\lambda\sin\mu(b-a)\lambda$$

becomes

$$\frac{\sin\mu(b-a)\lambda}{\sin a\lambda}\sin x\lambda ,$$

while the corresponding expression in (80) becomes

$$\frac{\sin a\lambda}{\sin\mu(b-a)\lambda}\sin\mu(b-x)\lambda$$

Thus the complete set of results required is

$$\text{Inst. Source in Medium I} \left\{ \begin{array}{ll} v_1 = -\dfrac{2q\kappa_1}{K_2\mu}\sum_1^\infty e^{-\kappa_1\lambda^2 t}\dfrac{\sin x_1\lambda\sin x\lambda\sin\mu(b-a)\lambda}{\sin a\lambda\, f_1'(\lambda)} & (84). \\ v_2 = -\dfrac{2q\kappa_1}{K_2\mu}\sum_1^\infty e^{-\kappa_1\lambda^2 t}\dfrac{\sin x_1\lambda\sin\mu(b-x)\lambda}{f_1'(\lambda)} & (85). \end{array} \right.$$

$$\text{Inst. Source in Medium II} \left\{ \begin{array}{ll} v_1 = -\dfrac{2q\kappa_1}{K_2\mu}\sum_1^\infty e^{-\kappa_1\lambda^2 t}\dfrac{\sin\mu(b-x_1)\lambda\sin x\lambda}{f_1'(\lambda)} & (86). \\ v_2 = -\dfrac{2q\kappa_1}{K_2\mu}\sum_1^\infty e^{-\kappa_1\lambda^2 t}\dfrac{\sin\mu(b-x_1)\lambda\sin\mu(b-x)\lambda\sin a\lambda}{\sin\mu(b-a)\lambda\cdot f_1'(\lambda)} & (87). \end{array} \right.$$

where the summations are with regard to all the positive roots of the equation (83).

These results are of very general character in the sense that we can obtain at once from them the solutions correspomding to any initial distribution prescribed throughout the whole rod. If e.g. the initial state is that given by

$$t=0,\quad v=\phi_1(x),\quad 0\leqq x\leqq a\ ;\qquad v=\phi_2(x),\ a\leqq x\leqq b$$

we replace q in (84) and (86) by $\frac{K_1}{\kappa_1}\phi_1(x_1)dx_1$ and integrate with regard to x_1 from o to a ; replace q in (85) and (87) by $\frac{K_2}{\kappa_2}\phi_2(x_1)dx_1$ and integrate with regard to x_1 from a to b . The total effect in the first medium is then obtained by adding the first and third of these results, that in the second medium by adding the second and fourth.

We show the form the results take for two selected cases.

(1).

$$\phi_1(x) = \phi_2(x) = \text{const} = v_0 \text{ say.}$$

It is found that at a root of (83)

$$f_1'(\lambda) = -\frac{\sigma a \sin^2\mu(b-a)\lambda \;+\; \mu(b-a)\sin^2 a\lambda}{\sin a\lambda \,\sin\mu(b-a)\lambda}$$

and finally the total effect in the first medium due to the initial temperature v_0 throughout the whole rod is given by

$$v_1 = 2v_0\sum_1^\infty e^{-\kappa_1\lambda^2 t}\frac{\sin\mu(b-a)\lambda\,\sin x\lambda}{\sigma a\sin^2\mu(b-a)\lambda+\mu(b-a)\sin^2 a\lambda}\,\frac{1}{\lambda}\left\{\sigma\sin\mu(b-a)\lambda+\sin a\lambda\right\}\ \cdots(88).$$

and v_2 the total effect in the second medium is given by

$$v_2 = 2v_0\sum_1^\infty e^{-\kappa_1\lambda^2 t}\frac{\sin a\lambda\,\sin\mu(b-x)\lambda}{\sigma a\sin^2\mu(b-a)\lambda+\mu(b-a)\sin^2 a\lambda}\,\frac{1}{\lambda}\left\{\sigma\sin\mu(b-a)\lambda+\sin a\lambda\right\}\ \cdots(89).$$

(2).

$$\phi_1(x) = \theta_0\left\{1-\frac{K_2x}{K_1(b-a)+K_2a}\right\}$$

$$\phi_2(x) = \theta_0\left\{\frac{K_1(b-x)}{K_1(b-a)+K_2a}\right\}$$

When the integration process is worked out for this case we find that the required results are

$$v_1 = -2\theta_0 \sum_1^{\infty} e^{-\kappa_1 \lambda^2 t} \frac{\sigma \operatorname{cosec}^2 a\lambda}{\frac{d}{d\lambda}\left(\cot\mu(b-a)\lambda + \sigma \cot a\lambda\right)} \frac{\sin x\lambda}{\lambda} \qquad \ldots (90)$$

$$v_2 = -2\theta_0 \sum_1^{\infty} e^{-\kappa_1 \lambda^2 t} \frac{\sigma \operatorname{cosec} a\lambda \operatorname{cosec} \mu(b-a)\lambda}{\frac{d}{d\lambda}\left(\cot\mu(b-a)\lambda + \sigma \cot a\lambda\right)} \frac{\sin \mu(b-x)\lambda}{\lambda} \qquad \ldots (91)$$

If now we take the solution given by

$$V_1 = \phi_1(x) - v_1$$

$$V_2 = \phi_2(x) - v_2$$

we find that V_1, V_2 satisfy the conditions

$$\left.\begin{array}{l} V_1 = \theta_0, \ x = 0. \\ V_1 = 0, \ t = 0. \end{array}\right\} \qquad \left.\begin{array}{l} V_2 = 0, \ t = 0 \\ V_2 = 0, \ x = b \end{array}\right\} \quad V_1 = V_2, \ x = a$$

Thus V_1, V_2 are the solutions for the case where the end $x=0$ is kept at the constant temperature θ_0 throughout, the whole rod being originally at the temperature zero. The result, in the form shown, is in entire agreement with that obtained directly for this problem in the author's Phil. Mag. paper.*

In the above illustrations it is assumed that both ends of the rod are kept at zero temperature. We now indicate briefly the corresponding results for the case

Both ends radiating to a medium at zero temperature.

We readily find

$$F = \frac{i\lambda - H_1}{i\lambda + H_1} = -e^{-2i\theta_1}$$

$$B = \frac{i\lambda - H_2}{i\lambda + H_2} = -e^{-2i\theta_2}$$

---

* R.I. p.954.

where $\tan\theta_1 = \frac{\lambda}{H_1}$, $\tan\theta_2 = \frac{\lambda}{H_2}$; $H_1 = \frac{h_1}{K_1}$, $H_2 = \frac{h_2}{\mu K_2}$

h_1 and h_2 being the emissivities at the ends $x = 0$ and $x = b$ respectively. The subsidiary angles θ_1, θ_2 are introduced merely to simplify the remaining parts of the analysis.

Thus after reduction we find

$$S_1S_2 - A'BCFe^{-2l} = \qquad (c = b-a)$$

$$(1-A)\,2i\,e^{-i\{\theta_1+\theta_2+\lambda[a+\mu(b-a)]\}}\left\{\sin(\theta_1+a\lambda)\cos(\theta_2+\mu c\lambda)+\sigma\cos(\theta_1+a\lambda)\sin(\theta_2+\mu c\lambda)\right\}$$

Also $S_2\phi$ becomes

$$e^{-i\{\theta_1+\theta_2+\mu c\lambda\}}\,2i\sin(\theta_1+x\lambda)\left\{(1-A)\cos(\theta_2+\mu c\lambda)+i(1+A)\sin(\theta_2+\mu c\lambda)\right\}$$

whence we have

$$\frac{S_2\phi}{S_1S_2-A'BCFe^{-2l}} = \frac{\sin(\theta_1+x\lambda)\left\{\cos(\theta_2+\mu c\lambda)+i\sigma\sin(\theta_2+\mu c\lambda)\right\}}{e^{-ai\lambda}\left\{\sin(\theta_1+a\lambda)\cos(\theta_2+\mu c\lambda)+\sigma\cos(\theta_1+a\lambda)\sin(\theta_2+\mu c\lambda)\right\}}$$

Thus ultimately the result (72), giving the effect in the first medium may be shown as

$$v_1 = \rho\,2i\sin(\theta_1+x_1\lambda)\ \frac{\sin(a-x)\lambda\cos(\theta_2+\mu c\lambda)+\sigma\cos(a-x)\lambda\sin(\theta_2+\mu c\lambda)}{F_1(\lambda)} \quad \ldots (92)$$

where $F_1(\lambda) = \sin(\theta_1+a\lambda)\cos(\theta_2+\mu c\lambda)+\sigma\cos(\theta_1+a\lambda)\sin(\theta_2+\mu c\lambda).$ ... (93)

Similarly the result (73), giving the effect in the second medium due to the periodic source in the first, becomes

$$v_2 = \sigma\rho\,2i\sin(\theta_1+x_1\lambda)\ \frac{\sin\{\theta_2+\mu(b-x)\lambda\}}{F_1(\lambda)} \quad \ldots (94).$$

In the same way when the periodic source is in the <u>second</u> medium we find from (74) and (75) that the effects are given by/

$$v_1 = \sigma \rho' 2i \sin(\theta_1 + x\lambda) \frac{\sin\{\theta_2 + \mu(b-x_1)\lambda\}}{F_1(\lambda)} \quad \cdots (95)$$

$$v_2 = \rho' 2i \sin\{\theta_2 + \mu(b-x_1)\lambda\} \frac{\sin\{\theta_1 + a\lambda\}\cos\mu(x-a)\lambda + \sigma\cos\{\theta_1 + a\lambda\}\sin\mu(x-a)\lambda}{F_1(\lambda)} \quad \cdots (96)$$

Finally the solutions for the corresponding <u>instantaneous</u> sources are given by

$$\text{Inst Source in Medium I} \begin{cases} v_1 = -\frac{2q\kappa_1}{\mu K_2} \sum_1^\infty e^{-\kappa_1\lambda^2 t} \frac{\sin(\theta_1 + x_1\lambda)\sin(\theta_1 + x\lambda)\sin\{\theta_2 + \mu c\lambda\}}{\sin(\theta_1 + a\lambda) F_1'(\lambda)} & \ldots (97) \\ v_2 = -\frac{2q\kappa_1}{\mu K_2} \sum_1^\infty e^{-\kappa_1\lambda^2 t} \frac{\sin(\theta_1 + x_1\lambda)\sin\{\theta_2 + \mu(b-x)\lambda\}}{F_1'(\lambda)} & \ldots (98) \end{cases}$$

$$\text{Inst Source in Medium II} \begin{cases} v_1 = -\frac{2q\kappa_1}{\mu K_2} \sum_1^\infty e^{-\kappa_1\lambda^2 t} \frac{\sin\{\theta_2 + \mu(b-x_1)\lambda\}\sin(\theta_1 + x\lambda)}{F_1'(\lambda)} & \ldots (99) \\ v_2 = -\frac{2q\kappa_1}{\mu K_2} \sum_1^\infty e^{-\kappa_1\lambda^2 t} \frac{\sin\{\theta_2 + \mu(b-x_1)\lambda\}\sin\{\theta_2 + \mu(b-x)\lambda\}\sin(\theta_1 + a\lambda)}{\sin\{\theta_2 + \mu c\lambda\} F_1'(\lambda)} & \cdot (100) \end{cases}$$

where the summations are with regard to all the positive roots of the equation $F_1(\lambda) = 0$

i.e the equation $\frac{\lambda + H_1 \tan a\lambda}{\lambda + H_2 \tan \mu c\lambda} + \sigma \frac{H_1 - \lambda \tan a\lambda}{H_2 - \lambda \tan \mu c\lambda} = 0.$

From the solutions indicated above we could obtain as previously those corresponding to any arbitrary initial heat distribution. The details for the working out in any particular case become very laborious. The principal interest, clearly, is the striking form analogy between the set~~s~~ of results (97)-(100) and the set (84)-(87) . Clearly the latter could be deduced from the former by writing $\theta_1 = \theta_2 = 0$ throughout.

This section of the chapter is concluded with some observations on the types of problems whose complete solutions can conveniently be obtained from the various sets of results arrived at above.

The result (87) e.g., giving the effect in medium II of a given initial heat distribution in that medium could be used to calculate the insulation afforded by placing the end of that medium in contact with a layer of low conducting lagging. The lagging corresponds to medium I in the analysis given above. The temperature gradient at $x=a$ within medium II at any instant can be calculated and its value in the circumstances compared with what it would be if the end were kept at zero temperature, or radiated according to any other law. The calculation of the protection afforded for various thicknesses of the lagging and various values of the relative conductivities would give results of practical value.

Similarly the result (87) could be used to estimate the inefficiency of a heater over the end of which has formed a layer of low conducting scale. The temperature gradient at $x=a$ - a measure of the heat passing to the heater proper - could be calculated and this result compared with the direct heat current at $x=a$ if there were no low conducting layer.

Again the result (86) could be used to calculate the epoch or distance of any initial temperature "disturbance" at a remote point in medium II say. If e.g. a layer at distance x_1 in medium II was brought suddenly at some past instant to a given high temperature θ_0 say, all the other parts of the two media being at zero temperature, it is clear that observations of the gradient at $x=0$ in medium I now would enable us to infer the epoch of the original disturbance. The relation of this problem to the classical "Age of the Earth" problem investigated by Kelvin is at once apparent. In the original calculations a uniform conductivity throughout the Earth was assumed. Later calculations by Perry and Heaviside were based on the assumption of an inner nucleus of one material surrounded by a concentric layer of different conductivity. The plane flow results of the present chapter are not, however, at once applicable to the classical problem; the necessary spherical analysis is given in a later chapter.

The present chapter is concluded by an examination of the case where medium II in the general problem discussed above extends to infinity.

The second medium extending to infinity.

It is possible to deduce the results for this case from those already obtained by simply making **b** infinitely great but it is more instructive to investigate the problem from first principles. The case is simpler in the respect that wave trains on transmission from medium I to medium II across the surface of separation at $x=a$ proceed without interruption to infinity. There is thus no complication corresponding to retransmission effects from medium II to medium I.

If, therefore we have a periodic source at $x=x_1$ within medium I, the total effect in this medium is given by what was called the first-effect in the previous investigation.

Thus;- $$x<x_1, \quad v_{1_i}=\rho\frac{\{Fe^{-ix\lambda}+e^{ix\lambda}\}\{e^{-ix_1\lambda}+Ae^{-(2a-x_1)i\lambda}\}}{1-AFe^{-2ai\lambda}} \quad \ldots (101)$$

$$= \rho f(x, x_1)$$

$$x>x_1, \quad v_{1_o}= \rho f(x_1, x) \quad \ldots (102)$$

Clearly also we have

$$v_2=A'\rho\frac{\{Fe^{-ix_1\lambda}+e^{ix_1\lambda}\}e^{-ai\lambda-\mu(x-a)i\lambda}}{1-AFe^{-2ai\lambda}} \quad (103)$$

When the periodic source is located in medium II we have the corresponding results

$$v_1= C\rho'\,\frac{e^{-(x_1-a)\mu i\lambda-ai\lambda}(Fe^{-ix\lambda}+e^{ix\lambda})}{1-AFe^{-2ai\lambda}} \quad \ldots (104).$$

v_2/

$$a \leqq x \leqq x_1 , \qquad v_{2_i} = \rho' e^{-(x_1-x)\mu i\lambda} + C'\rho' e^{-(x_1+x-2a)\mu i\lambda} + \frac{FA'C\rho' e^{-2ai\lambda-(x_1+x-2a)\mu i\lambda}}{1-AFe^{-2ai\lambda}} \quad \ldots (105)$$

$x_1 \leqq x$, $v_{2_o} =$ same expression, r, r_1 interchanged.

The first term in v_{2_i} is recognised as the original negative train emanating from the source, the second as the positive train set up by the partial reflection of the first at the surface $x = a$, the third as the positive train set up by the partial retransmission from medium I of the trains reflected at the surface $x = 0$ on incidence of the transmitted part of the original negative train.

If we confine our attention to the case where the end of the rod is kept at zero temperature we have $F = -1$, while the various other reflection and transmission coefficients have the forms already found. In these circumstances the above results become after some reduction

Source in Medium I

$$v_{1_i} = \frac{qe^{ikt}}{2K_1 i\lambda} 2i\sin x\lambda \, \frac{e^{-ix_1\lambda} + Ae^{-(2a-x_1)i\lambda}}{1 + Ae^{-2ai\lambda}} \quad \ldots$$

$$= \frac{qe^{ikt} 2i\sin x\lambda}{2K_1 i\lambda} \, \frac{\sigma\cos(a-x_1)\lambda + i\sin(a-x_1)\lambda}{\sigma\cos a\lambda + i\sin a\lambda} \quad \ldots (106)$$

$v_{1_o} =$ same expression, r, r_1 interchanged.

Source in Medium I

$$\left\{\begin{aligned} v_2 &= (1+A)\frac{qe^{ikt}}{2K_1 i\lambda}\,\frac{2i\sin x_1\lambda\, e^{-ai\lambda-\mu(x-a)i\lambda}}{1+Ae^{-2ai\lambda}} \\ &= \frac{\sigma q_1 e^{ikt}2i\sin x_1\lambda}{2K_1 i\lambda}\,\frac{\cdot\, e^{-\mu(x-a)i\lambda}}{\sigma\cos a\lambda+i\sin a\lambda} \end{aligned}\right. \qquad \ldots \quad (107)$$

Source in Medium II

$$\left\{\begin{aligned} v_1 &= (1-A)\frac{qe^{ikt}}{2K_2\mu i\lambda}\,2i\sin x\lambda\,\frac{e^{-ai\lambda-\mu(x_1-a)i\lambda}}{1+Ae^{-2ai\lambda}} \\ &= \frac{qe^{ikt}2i\sin x\lambda}{2K_2\mu i\lambda}\,\frac{e^{-\mu(x_1-a)i\lambda}}{\sigma\cos a\lambda+i\sin a\lambda} \qquad \ldots \quad (108) \\ v_{2_i} &= \frac{qe^{ikt}}{2K_2\mu i\lambda}\left\{e^{-(x_1-x)\mu i\lambda}+C'e^{-(x_1+x-2a)\mu i\lambda}\right. \\ &\qquad\qquad \left. -\frac{A'Ce^{-2ai\lambda-(x_1+x-2a)\mu i\lambda}}{1+Ae^{-2ai\lambda}}\right\} \\ &= \frac{qe^{ikt}}{2K_2\mu i\lambda}\,2ie^{-\mu(x_1-a)i\lambda}\,\frac{\sin a\lambda\cos\mu(x-a)\lambda+\sigma\cos a\lambda\sin\mu(x-a)\lambda}{\sigma\cos a\lambda+i\sin a\lambda} \\ &\qquad\qquad \ldots \quad (109) \\ v_{2_o} &= \text{same expression } r, r_1 \text{ interchanged.} \end{aligned}\right.$$

Proceeding next to the solutions representing the corresponding instantaneous initial sources, that arising from the first of of the above set is

$$v_1 = \frac{q\kappa_1}{\pi K_1}\int_{OB} e^{-\kappa\lambda^2 t}\,2i\sin x\lambda\,\frac{\sigma\cos(a-x_1)\lambda+i\sin(a-x_1)\lambda}{\sigma\cos a\lambda+i\sin a\lambda}\,d\lambda \qquad \ldots \quad (110)$$

with three others arising respectively from (107) – (109). The contour of integration is again the line $\theta=-\frac{\pi}{4}$ in the λ plane

To evaluate the integrals we must investigate the roots of the equation

$$\sigma \cos a\lambda + i \sin a\lambda = 0. \qquad (111)$$

This equation may be rewritten

$$e^{2(a\lambda + n\pi)i} = \frac{1-\sigma}{1+\sigma}$$

$$\text{whence} \qquad a\lambda = -n\pi - \frac{i}{2}\log\frac{1-\sigma}{1+\sigma}$$

If $\sigma < 1$ the logarithm is a real negative number; if $\sigma > 1$ the logarithm is that of $-\frac{\sigma-1}{\sigma+1}$ and the real part of this log. is again negative. Thus in either case, all the roots of the equation (111) lie in the upper half of the λ plane. If we assume that the integral along the arc of infinite radius is zero, the integral along the line OB may be replaced by that along the real axis. Hence we have

$$v_1 = \frac{q\kappa_1}{\pi K_1}\int_0^\infty e^{-\kappa t\xi^2}\, 2i \sin x\xi\, \frac{\sigma\cos(a-x_1)\xi + i\sin(a-x_1)\xi}{\sigma\cos a\xi + i\sin a\xi}\, d\xi \qquad (112)$$

When we take the real part of this result which is the part required to represent the instantaneous initial source we have

$$v_1 = \frac{2q\kappa_1\sigma}{\pi K_1}\int_0^\infty e^{-\kappa t\xi^2}\, \frac{\sin x_1\xi \sin x\xi}{\sigma^2\cos^2 a\xi + \sin^2 a\xi}\, d\xi. \qquad (113)$$

Similarly

$$v_2 = \frac{2q\kappa_1\sigma}{\pi K_1}\int_0^\infty e^{-\kappa t\xi^2} \sin x_1\xi\, \frac{\sigma\cos a\xi \sin\mu(x-a)\xi + \sin a\xi \cos\mu(x-a)\xi}{\sigma^2\cos^2 a\xi + \sin^2 a\xi}\, d\xi$$

(114)

These results apply when the source is in the first medium. In the same way we obtain, when the source is in the second medium

$$v_1 = \frac{2q\kappa_1}{\pi K_2\mu}\int_0^\infty e^{-\kappa t\xi^2}\sin x\xi\,\frac{\sigma\cos a\xi\,\sin\mu(x_1-a)\xi+\sin a\xi\,\cos\mu(x_1-a)\xi}{\sigma^2\cos^2 a\xi+\sin^2 a\xi}\,d\xi \qquad (115).$$

$$v_2 = \frac{2q\kappa_1}{\pi K_2\mu}\int_0^\infty e^{-\kappa t\xi^2}\,\frac{\{\sigma\cos a\xi\,\sin\mu(x_1-a)\xi+\sin a\xi\,\cos\mu(x_1-a)\xi\}\times\{\sigma\cos a\xi\,\sin\mu(x-a)\xi+\sin a\xi\,\cos\mu(x-a)\xi\}}{\sigma^2\cos^2 a\xi+\sin^2 a\xi}\,d\xi \qquad (116).$$

An immediate verification of these results is obtained if we take the special case $K_1=K_2, \kappa_1=\kappa_2$. It will be found that all four results reduce to

$$v = \frac{2q\kappa}{\pi K}\int_0^\infty e^{-\kappa t\xi^2}\sin x_1\xi\,\sin x\xi\,d\xi, \qquad (117)$$

which is the correct form of solution for the instantaneous source in the semi- infinite medium, when the face at a finite distance is kept at zero temperature.

The full investigation of the solutions (113)–(116) which we leave as definite integrals is not in the meantime attempted. Evaluations by successive approximation based on such assumptions as (1), $\sigma=1+\epsilon$ where ϵ is small, (2), σ very small, (3), σ very large, could be obtained without a great deal of difficulty.

CHAPTER III.

PROBLEMS INVOLVING SPHERICAL FLOW.

When there is complete symmetry about a point, as for example when we have a point source in an infinite medium, or at the centre of a finite sphere whose surface is kept at zero temperature the fundamental equation of heat conduction takes the form

$$\frac{\partial v}{\partial t} = \frac{\kappa}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) \tag{118}$$

By writing $rv = u$ this equation becomes

$$\frac{\partial u}{\partial t} = \kappa\frac{\partial^2 u}{\partial r^2}$$

The wave train solutions of this latter equation have been fully investigated in the previous chapters. If u is any such solution then $\frac{u}{r}$ represents the corresponding spherical wave train solution. Thus the solutions of (118) required are

$$\left.\begin{aligned} v_1 &= \frac{A}{r}e^{ikt - r\sqrt{\frac{ik}{\kappa}}} \\ v_2 &= \frac{A}{r}e^{ikt + r\sqrt{\frac{ik}{\kappa}}} \end{aligned}\right\} \tag{119}$$

The real and imaginary parts of v_1 represent temperature trains travelling outwards from the centre of the sphere, the real and imaginary parts of v_2 trains converging inwards on the centre. If e.g. the flow is due to a periodic point source of strength q_0e^{ikt} at the centre of the sphere we require

$$\mathop{\mathcal{L}}_{r\rightarrow 0}\left(-4\pi r^2 K\frac{\partial v}{\partial r}\right) = q_0e^{ikt}$$

Thus we find

$$A = \frac{q_0}{4\pi K\cancel{r}}$$

Hence, since only the diverging train is required, the solution is

$$v = \frac{q}{4\pi K r}e^{ikt - r\sqrt{\frac{ik}{\kappa}}} \qquad (120)$$

If, on the other hand, the flow is due to a uniform distribution of periodic point sources over the spherical surface $r=r_1$ the rate of emission being qe^{ikt} per unit area, both the converging and the diverging trains are required. Thus we might exhibit the solution as

$$\left.\begin{aligned} r<r_1, \quad v_{1_i} &= \frac{A}{r}e^{ikt-(r_1-r)\sqrt{\frac{ik}{\kappa}}} \\ r>r_1, \quad v_{1_o} &= \frac{A}{r}e^{ikt-(r-r_1)\sqrt{\frac{ik}{\kappa}}} \end{aligned}\right\} \qquad (121)$$

These forms clearly satisfy the condition $v_i=v_o$, $r=r_1$. There is however, the further condition

$$-K\left(\frac{\partial v_o}{\partial r} - \frac{\partial v_i}{\partial r}\right)_{r=r_1} = qe^{ikt} \qquad (122)$$

From this we readily find $A = \dfrac{qr_1}{2K\sqrt{\frac{ik}{\kappa}}}$

It is clear however that the solution v_i where

$$r < r_1 , \qquad v_i = \frac{q r_1}{2Kr\sqrt{\frac{ik}{\kappa}}} e^{ikt-(r_1-r)\sqrt{\frac{ik}{\kappa}}}$$

is an incomplete representation. The expression becomes infinite like $\frac{1}{r}$ at the origin and would accordingly indicate the presence of a source there, contrary to hypothesis. We must therefore suppose that the converging train v_i is reflected at $r=0$, or, what is the same thing - passes through the origin, emanating as the corresponding diverging train v_{2i} , where

$$v_{2i} = \frac{A q r_1}{2Kr\sqrt{\frac{ik}{\kappa}}} e^{ikt-(r_1+r)\sqrt{\frac{ik}{\kappa}}}$$

the constant A being chosen so that the $r=0$ condition is satisfied. This condition may be stated

$$\lim_{r\to 0} r^2 \frac{\partial}{\partial r}\left(v_{1i}+v_{2i}\right) = 0.$$

whence we readily find $\qquad A = -1.$

Accordingly we have

$$r < r_1 , \qquad v_i = \frac{q r_1}{2Kri\lambda} . e^{ikt-(r_1-r)i\lambda} - \frac{q r_1}{2Kri\lambda} e^{ikt-(r_1+r)i\lambda}$$

$$= \frac{q r_1 \, 2i \sin r\lambda}{2Kri\lambda} e^{ikt-r_1 i\lambda} , \quad i\lambda = \sqrt{\frac{ik}{\kappa}} \qquad (123)$$

It will be seen however that the addition of the term v_{2i} to v_{1i} as given in (123) violates the conditions at the source surface $r=r_1$. These however, are at once restored if we add v_{2i} likewise to v_{1o}; thus we obtain

~~It will be noticed that as given by this equation~~ 8

$$r > r_1 , \quad v_0 = \frac{q r_1}{2Kri\lambda} e^{ikt-(r-r_1)i\lambda} - \frac{q r_1}{2Kri\lambda} e^{ikt-(r+r_1)i\lambda}$$

$$= \frac{q r_1\, 2i \sin r_1\lambda}{2Kri\lambda} e^{ikt-ri\lambda} \qquad (123')$$

It will be noticed that v_0 as given by this equation reduces to the correct form for the central point source when $r_1 \to 0$. If we suppose that this limit is reached in such a way that $4\pi r_1^2 q \to q_0$ the result becomes

$$v = \frac{q_0}{4\pi K r} e^{ikt-ri\lambda}$$

in agreement with (120) above.

The result (123) applicable to the case of the spherical surface source can be obtained by integrating the solution (120) for the point source, over the surface of the sphere of radius r_1

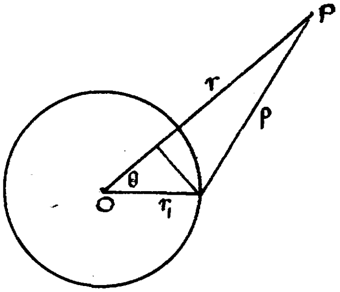

fig. 4.

It is clear that the effect at P due to a surface distribution of density qe^{ikt} is given by

$$v = \frac{q}{4\pi K} \int_0^{\pi} \frac{2\pi r_1 \sin\theta\, r_1\, d\theta}{\rho} e^{ikt-\rho i\lambda}$$

$$\rho^2 = r^2 + r_1^2 - 2rr_1\cos\theta$$

$$= \frac{q r_1 e^{ikt}}{2Kri\lambda} \int_{r-r_1}^{r+r_1} e^{-\rho i\lambda}\, d\rho . = \frac{q r_1\, 2i\sin\lambda r_1}{2Kri\lambda} e^{ikt-i\lambda r}$$

as required.

When the point at which the effect is required is inside the sphere $r = r_1$ the only difference is that the limits in the last written integral are $r_1 - r$ and $r_1 + r$; this has the effect of interchanging r and r_1 in the evaluated result.

Instantaneous point and spherical surface sources.

Consider the solution of (118) given by

$$v = \frac{q}{4\pi^2 K r}\int_0^\infty e^{ikt - ri\lambda}\,dk \qquad \ldots (124)$$

The integral appearing here is evaluated by integrating the function

$$f(\lambda) \equiv \frac{q}{4\pi^2 K r}\, e^{-\kappa\lambda^2 t - ri\lambda}\, 2i\kappa\lambda.$$

round the contour of fig.1.

The integral in (124) is equivalent to

$$\frac{iq\kappa}{2\pi^2 K r}\int_{OB} e^{-\kappa\lambda^2 t - ri\lambda}\,\lambda\, d\lambda$$

It can be shown that the integral along the circular arc $BA \to 0$ as $R \to 0$. Thus, since the integrand has no singularity within the contour we have

$$v = \frac{iq\kappa}{2\pi^2 K r}\int_{OB} e^{-\kappa\lambda^2 t - ri\lambda}\,\lambda\, d\lambda = \frac{iq\kappa}{2\pi^2 K r}\int_0^\infty e^{-\kappa t\xi^2 - ir\xi}\,\xi\, d\xi. \qquad (125)$$

When we take the real part of this result we have

$$v = \frac{q\kappa}{2\pi^2 K r}\int_0^\infty e^{-\kappa t\xi^2} \sin r\xi . \xi\, d\xi = \frac{q e^{-\frac{r^2}{4\kappa t}}}{8K\sqrt{\pi^3 \kappa t^3}} \qquad (126)$$

This is at once recognised as the solution for the case of the instantaneous point source in the infinite medium.

In the same way if we take the solution

$$v=\frac{qr_1}{2\pi Kr}\int_0^\infty e^{ikt-r_1 i\lambda}\,\frac{2i\sin r\lambda}{i\lambda}\,dk\,, \qquad \cdots\cdots \qquad (127)$$

and evaluate the integral by the method just indicated we obtain

$$v=\frac{q\kappa r_1}{\pi Kr}\int_{OB} e^{-\kappa\lambda^2 t-ri\lambda}\,2i\sin r\lambda\,d\lambda=\frac{q\kappa r_1}{\pi Kr}\int_0^\infty e^{-\kappa t\xi^2-ir_1\xi}\,.\,2i\sin r\xi\,d\xi\ldots(128)$$

and on taking the real part of this result

$$v=\frac{q\kappa r_1}{\pi Kr}\int_0^\infty e^{-\kappa t\xi^2}\left\{\cos(r_1-r)\xi-\cos(r_1+r)\xi\right\}d\xi=\frac{qr_1\sqrt{\kappa}}{2Kr\sqrt{\pi t}}\left\{e^{-\frac{(r_1-r)^2}{4\kappa t}}-e^{-\frac{(r_1+r)^2}{4\kappa t}}\right\}..(129)$$

This is the well known solution for the case of the instantaneous surface source within the infinite medium.

It is again possible to deduce this last result by integrating the instantaneous point source solution over the surface of the sphere of radius r_1 .
From a figure it is at once apparent that the result required is

$$v=\frac{2\pi r_1^2}{8K\sqrt{\pi^3\kappa t^3}}\int_0^\pi e^{-\frac{\rho^2}{4\kappa t}}\sin\theta\,d\theta.$$

$$=\frac{qr_1\sqrt{\kappa}}{2Kr\sqrt{\pi t}}\int_{r-r_1}^{r+r_1} e^{-\frac{\rho^2}{4\kappa t}}\,\frac{\rho}{2\kappa t}\,d\rho$$

where $\rho^2=r^2+r_1^2-2rr_1\cos\theta.$

From these observations we at once obtain the result in the form given in (129).

The deduction of the solutions for spherical surface sources from those for point sources of either the periodic or instantaneous type suggests that likewise from the point source solution we might deduce that for the infinite plane source of given surface density.

It is readily seen from a figure that the required result is - taking the periodic source of strength qe^{ikt} per unit area-

$$v = \int_0^\infty \frac{q\,2\pi r\,dr}{4\pi K\rho}\, e^{ikt-\rho i\lambda} \quad ; \quad \rho^2 = r^2 + x^2$$

$$= \frac{qe^{ikt}}{2K}\int_x^\infty e^{-\rho i\lambda}\,d\rho$$

$$= \frac{q}{2Ki\lambda}\, e^{ikt-ix\lambda}$$ as required.

In the same way from (126) we could deduce the result for the instantaneous plane surface source.

With the various spherical results so far obtained (120),(123),(126),(129), we have all the material for the building up of solutions of problems in which the isothermals are concentric spherical surfaces. It will be found that when the boundary conditions do not involve the temperature gradient, the analysis in any particular problem is practically the same as that for the corresponding plane flow case. Thus e.g. the case of a finite sphere cooling from a given symmetrical temperature state owing to its surface being kept at zero temperature corresponds in all respects to that of a finite rod both ends of which are kept at zero temperature. Fundamentally this is due to two considerations already commented on ; (1). the fact that the simple factor $\frac{1}{r}$ converts a plane wave train into a spherical train[*] and (2). the fact that when a train is reflected at the centre of a sphere the reflection takes place under the condition which when analytically expressed is the same as the expression for the zero temperature condition when ~~when~~ the trains in question are plane.[+]

Consider e.g. the case of the periodic source of strength qe^{ikt} per unit area at the surface $r=r_1$ within a sphere of radius a . If we start with the fundamental converging and diverging trains as given by (123) + (123′) , emanating from this source, and follow out a course like that adopted in the compilation of the table on p.32 , we find that the effect at any point, the resultant of the original trains and all their continuations by reflection at thesurface $r=a$ and at the centre is given by[§]

$$\left.\begin{aligned} r<r_1 \qquad & v=\frac{qr_1e^{ikt}}{2Kri\lambda}\cdot 2i\sin\lambda r\,\frac{e^{-i\lambda r_1}+Ae^{-(2a-r_1)i\lambda}}{1+Ae^{-2ai\lambda}} \\ r>r_1 \qquad & v=\frac{qr_1e^{ikt}}{2Kri\lambda}\left\{\text{same expression, } r, r_1 \text{ interchanged.}\right\} \end{aligned}\right\} \quad (130).$$

where A is the reflection constant for the surface $r=a$. If we take as a typical incident-reflected pair

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[*] See p 62.

[+] p 64.

[§] cf the corresponding plane flow results, p 34.

$$v = \frac{q r_1 e^{ikt}}{2Kri\lambda}\left\{e^{-(r-r_1)i\lambda} + Ae^{-(2a-r-r_1)i\lambda}\right\}$$

we find (1) for the surface $r=a$ at zero temperature.

$$A = -1.$$

(2) for the case of radiation at the surface $r=a$ to a medimm at zero temperature.

$$-K\frac{\partial v}{\partial r_a} = hv\;; \qquad A = \frac{i\lambda + \left(\frac{1}{a} - \frac{h}{K}\right)}{i\lambda - \left(\frac{1}{a} - \frac{h}{K}\right)}.$$

As previously, the solution given by $R\frac{1}{\pi}\int_0^\infty v\,dk$ is that for the instantaneous ($t=0$) surface source of strength q per unit area at $r=r_1$. The result for the zero temperature surface condition need not detain us. Clearly it is obtained at once from (53) by multiplying that result by $\frac{r_1}{r}$.

With the second form of A shown above the required form is

$$v_1 = R\frac{q\kappa r_1}{\pi K_1 r}\int_{OB} e^{-\kappa\lambda^2 t}\, 2i\sin\lambda r \quad \frac{\lambda\cos(a-r_1)\lambda - \left(\frac{1}{a} - \frac{h}{K}\right)\sin(a-r_1)\lambda}{\lambda\cos a\lambda - \left(\frac{1}{a} - \frac{h}{K}\right)\sin a\lambda}\,d\lambda\,, \qquad (131)$$

the path of integration again being the line OB of fig. 3 . The evaluation is effected by contour integration taking the closed path of fig. 3, and regarding the indents as being round the points given by the roots of the equation

$$\lambda\cos a\lambda - \left(\frac{1}{a} - \frac{h}{K}\right)\sin a\lambda = 0. \qquad (132).$$

The real part of the result is that which arises from the integral round the indents and is exhibited in the final form

$$v = \frac{2q\kappa r_1}{Kra^2}\sum_1^\infty e^{-\kappa\lambda^2 t}\sin\lambda r_1 \sin\lambda r \quad \frac{a^2K^2\lambda^2+(K-ah)^2}{a^2K^2\lambda^2-h(K-ah)} \qquad (133)$$

where the summation is with regard to all the positive roots of the equation (132).[†]

From this result that corresponding to any arbitrary initial heat distribution prescribed throughout the sphere can be built up. If e.g. we have $t=0$, $v=f(r)$, all we require to do is to replace q in (133) by $\frac{K}{\kappa}f(r_1)dr_1$ and integrate with regard to r_1 from 0 to a.

The sphere of one material surrounded by a concentric sheath of different conductivity.

This problem has been fully solved by Green for the case where the outer sheath is of finite extent.[*] The treatment need not be reproduced here. It is sufficient to remark that the procedure to be adopted resembles very closely that given in Chapter II where we have dealt with the corresponding plane-flow problem. There is no fundamental difference between the two cases. The conditions at the surface of separation in the

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[*] G. III p 248 et. seq.

[†] The solution in this form is given by Carslaw. ibid. p 139.

spherical case , involving as they do the temperature gradient, are rather more complicated (see p. 73) but the correspondence between the results obtained in the two cases at every stage is unmistakable. We therefore confine our attention to the special case where the outer medium is of infinite extent.

The problem is fully represented by the system

$$0<r<a, \quad \frac{\partial v_1}{\partial t} = \kappa_1 \nabla^2 v_1 \quad ; \quad r>a, \quad \frac{\partial v_2}{\partial t} = \kappa_2 \nabla^2 v_2 \qquad \cdots (134)$$

$$r=a, \quad v_1 = v_2 \quad \text{and} \quad K_1 \frac{\partial v_1}{\partial r} = K_2 \frac{\partial v_2}{\partial r}. \qquad \cdots (135)$$

We begin by writing down the results for the case of the periodic source of strength qe^{ikt} per unit area at the surface $r=r_1$. It will be realised from what has already been said that these results can be inferred at once from the corresponding set relating to the plane analogue of the present problem. (see p.58 -)

Thus we have

$$\begin{array}{l}\text{Source} \\ \text{in} \\ \text{Medium I.}\end{array} \left\{ \begin{array}{ll} r<r_1 , \quad v_{1i} = \dfrac{q r_1 e^{ikt}}{2K_1 r i\lambda}\, 2i \sin\lambda r \cdot \dfrac{e^{-i\lambda r_1} + Ae^{-(2a-r_1)i\lambda}}{1+Ae^{-2ai\lambda}} & (136) \\ \\ r<r_1 , \quad v_{1o} = \dfrac{q r_1 e^{ikt}}{2K_1 r i\lambda} \left\{ \text{same expression}, \ r, r_1 \ \text{interchanged} \right\} & \\ \\ v_2 = \dfrac{A' q r_1 e^{ikt}}{2K_2 \mu r i\lambda}\, 2i \sin\lambda r_1 \, \dfrac{e^{-ai\lambda-(r-a)\mu i\lambda}}{1+Ae^{-2ai\lambda}} & (137) \end{array} \right.$$

where the various constants and operators have the meanings previously assigned to them.

$$\begin{array}{l} \text{Source} \\ \text{in} \\ \text{Medium II} \end{array} \left\{ \begin{array}{l} v_1 = \dfrac{Cqr_1}{2K_1ri\lambda} e^{ikt-(r_1-a)\mu i\lambda - ai\lambda} \dfrac{2i \sin r\lambda}{1+Ae^{-2ai\lambda}} \qquad (138) \\[2ex] a<r<r_1, \quad v_2 = \dfrac{qr_1e^{ikt}}{2K_2\mu ri\lambda}\left\{ e^{-(r_1-r)\mu i\lambda} + C'e^{-(r_1+r-2a)\mu i\lambda} \right. \\[2ex] \qquad\qquad\qquad \left. - \dfrac{A'Ce^{-2ai\lambda-(r+r_1-2a)\mu i\lambda}}{1+Ae^{-2ai\lambda}} \right\} \qquad (139) \\[2ex] r>r_1 \quad v_2 = \dfrac{qr_1e^{ikt}}{2K_2\mu ri\lambda}\left\{ \text{same expression, } r, r_1 \text{ interchanged} \right\} \end{array} \right.$$

We have now to investigate the forms of the reflection and transmission coefficients A, A', C, C'. Taking as typical incident-reflected pair and transmitted continuation v_1 and v_2 respectively where

$$v_1 = \frac{qr_1}{2K_1ri\lambda} e^{ikt-(r-r_1)i\lambda} + A\frac{qr_1}{2K_1ri\lambda} e^{ikt-(2a-r-r_1)i\lambda}$$

$$v_2 = A'\frac{qr_1}{2K_2\mu ri\lambda} e^{ikt-(a-r_1)i\lambda-\mu(r-a)i\lambda}$$

the source being inthe inner medium, we find that the conditions (135) yield

$$A = \frac{K_1 - K_2\mu + \frac{1}{ai\lambda}(K_1-K_2)}{d} \;; \quad A' = \frac{2K_2\mu}{d} \qquad (140)$$

where $\quad d = K_1 + K_2\mu - \frac{1}{ai\lambda}(K_1-K_2).$

Likewise when the source is in the outer medium the typical solutions taken are

$$v_2 = \frac{qr_1e^{ikt}}{2K_2\mu ri\lambda}\left\{ e^{-(r_1-r)\mu i\lambda} + C'e^{-(r_1+r-2a)\mu i\lambda} \right\}$$

$$v_1 = \frac{Cqr_1}{2K_1ri\lambda} e^{ikt-(r_1-a)\mu i\lambda-(a-r)i\lambda}$$

and the conditions (135) yield

$$C' = \frac{-K_1 + K_2\mu + \frac{1}{ai\lambda}(K_1 - K_2)}{d} \quad ; \quad C = \frac{2K_1}{d} \qquad (141)$$

From the results (136)—(139), representing the effects in the two media due to the postulated periodic sources we obtain at once a further set to represent the corresponding instantaneous sources by taking solutions of the type

$$v = R\frac{1}{\pi}\int_0^\infty v_1 dk = R\frac{2i\kappa}{\pi}\int_{OB} v_1\lambda d\lambda \; *$$

Thus from (136) we have

$$v_1 = \frac{q r_1 \kappa_1}{\pi K r}\int_{OB} e^{-\kappa_1\lambda^2 t}\, 2i\sin\lambda r\; \frac{e^{-i\lambda r_1} + Ae^{-(2a-r_1)i\lambda}}{1 + Ae^{-2ai\lambda}}\, d\lambda$$

and this result becomes, when the above form of A is inserted

$$v_1 = \frac{2 r_1 \kappa_1}{\pi K_1 r}\int_{OB} e^{-\kappa\lambda^2 t}\, 2i\sin\lambda r\; \frac{K_1\cos(a-r_1)\lambda - (K_1 - K_2)\frac{\sin(a-r_1)\lambda}{a\lambda} + iK_2\mu\sin(a-r_1)\lambda}{d'}\, d\lambda \qquad \cdot\cdot(142)$$

where

$$d' = K_1\cos a\lambda - (K_1 - K_2)\frac{\sin a\lambda}{a\lambda} + iK_2\mu\sin a\lambda \qquad \cdot\cdot\cdot(143)$$

The evaluation of the integral appearing here, and the three others like it arising from (137)—(139) demands an investigation of the roots of the equation $d' = 0$. If we rewrite the equation in the form

$$\cot a\lambda = \frac{K_1 - K_2}{K_1}\frac{1}{a\lambda} - \frac{iK_2\mu}{K_1} \qquad (143')$$

and put $a\lambda = x + iy$ where x and y are real we find, on equating real and imaginary parts

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* the path of integration being as usual the infinite radius $\theta = -\frac{\pi}{4}$ in the λ plane.

$$\left.\begin{aligned} \frac{\sin 2x}{\cosh 2y-\cos 2x} &= \frac{K_1-K_2}{K_1}\,\frac{x}{x^2+y^2} \\ \frac{\sinh 2y}{\cosh 2y-\cos 2x} &= \frac{K_1-K_2}{K_1}\,\frac{y}{x^2+y^2}+\frac{K_2\mu}{K_1} \\ &\text{and by combining these} \\ \frac{y\sin 2x - x\sinh 2y}{\cosh 2y-\cos 2x} &= -\frac{K_2\mu}{K_1}x \end{aligned}\right\} \qquad (144)$$

Corresponding to each solution (x,y) of the first two of these equations regarded as a simultaneous set we obtain the complex solution $\alpha\lambda=x+iy$ of the equation $(143')$. The original form of α' makes it plain that $x=y=0$ is inadmissible. Also the second equation shows that $y=0$ is inadmissible and thereby dismisses the possibility of $(143')$ having real roots. On the other hand the first equation is satisfied by $x=0$ and consequently suggests the possibility of pure imaginary roots. Such a root of $(143')$ would be of the form iy where y satisfies

$$\coth y = \frac{K_1-K_2}{K_1}\,\frac{1}{y}+\frac{K_2\mu}{K_1} \quad ;$$

the roots of this equation depend on the magnitude of the constants involved and presumably could be found by a graphical method. We are only concerned however with such roots of $(143')$ as may lie within the closed contour formed by the real axis, the infinite radius $\theta=-\frac{\pi}{4}$ and the arc of infinite radius, i.e. the contour of fig. 1 . The real part x of any such root is positive. It is apparent therefore from the third of the equations (144), since

$\cosh 2y - \cos 2x \geqq 0$ for all values of x and y,

the expression

$$2xy\left(\frac{\sin 2x}{2x} - \frac{\sinh 2y}{2y}\right) \text{ is negative.}$$

Thus since x is positive, the factors

$$y \quad \text{and} \quad \left(\frac{\sin 2x}{2x} - \frac{\sinh 2y}{2y}\right)$$

are of opposite sign. It is clear that the bracketed term is negative for all values of x and y and consequently y must be positive. In this way we have shown that all the roots of the equation (143') lie in the upper half of the λ plane.

The integral in (142) taken along the line OB, if we assume convergence along the arc of infinite radius, may be replaced by the integral along the real axis. Thus (142) yields

$$v_1 = \frac{q\kappa_1 r_1}{\pi K_1 r}\int_0^\infty e^{-\kappa_1 t\xi^2}\, 2i\sin r\xi \; \frac{K_1\cos(a-r_1)\xi - (K_1-K_2)\frac{\sin(a-r_1)\xi}{a\xi} + iK_2\mu\sin(a-r_1)\xi}{d'(\xi)}\, d\xi. \quad (145)$$

Taking only the real part of this result as required to represent the instantaneous source at the surface $r = r_1$, we obtain

$$v_1 = \frac{2qr_1\kappa_1 K_2\mu}{\pi r}\int_0^\infty e^{-\kappa t\xi^2}\frac{\sin r_1\xi \sin r\xi}{\left\{K_1\cos a\xi - (K_1-K_2)\frac{\sin a\xi}{a\xi}\right\}^2 + K_2^2\mu^2\sin^2 a\xi}\, d\xi. \quad \ldots(146)$$

In the same way from the results (137)–(139) we obtain, by repeating the argument used above, and after necessary reduction in each case

$$v_2 = \frac{2q\kappa_1 r_1}{\pi r}\int_0^\infty e^{-\kappa_1 t\xi^2} \sin r_1\xi \frac{\left\{K_1\cos a\xi - (K_1-K_2)\frac{\sin a\xi}{a\xi}\right\}\sin\mu(r-a)\xi + K_2\mu\sin a\xi\cos\mu(r-a)\xi}{\left\{K_1\cos a\xi - (K_1-K_2)\frac{\sin a\xi}{a\xi}\right\}^2 + K_2^2\mu^2\sin^2 a\xi}\,d\xi. \quad (147)$$

$$= \frac{2q\kappa_1 r_1}{\pi r}\int_0^\infty e^{-\kappa_1 t\xi^2} \frac{\sin r_1\xi \sin\{\mu(r-a)\xi+\phi\}}{R}\,d\xi \quad (147')$$

where $R\cos\phi = K_1\cos a\xi - (K_1-K_2)\frac{\sin a\xi}{a\xi}$; $R\sin\phi = K_2\mu\sin a\xi$.

The pair of results (146) and (147) apply when the source is in the inner medium. In the same way we obtain, when the source is in the outer medium - adopting the notation just introduced

$$v_1 = \frac{2q\kappa_1 r_1}{\pi r}\int_0^\infty e^{-\kappa_1 t\xi^2} \frac{\sin r\xi \sin\{\mu(r_1-a)\xi+\phi\}}{R}\,d\xi. \quad (148)$$

$$v_2 = \frac{2q\kappa_1 r_1}{\pi K_2\mu r}\int_0^\infty e^{-\kappa_1 t\xi^2} \sin\{(r_1-a)\mu\xi+\phi\}\sin\{(r-a)\mu\xi+\phi\}\,d\xi \quad (149)$$

The full investigation of the integrals appearing in (146)-(149) is not in the meantime attempted. Some useful idea of the kind of results to be expected, however, is obtained if we consider their approximate evaluation. Of various possible approaches we take that suggested by the special case $K_1 = K_2.\ \mu = 1$. It is readily verified that in this case all four results reduce to the form

$$v = \frac{2q\kappa_1 r_1}{\pi K_1 r}\int_0^\infty e^{-\kappa t\xi^2} \sin r_1\xi \sin r\xi\,d\xi. \quad (150).$$

which is the correct form of solution for the case of the source at the surface $r=r_1$ in the infinite medium.

If e.g. there is an initial heat distribution between the surfaces $r_1=a$ and $r_1=b$ given by $q=\frac{1}{r_1}$, the temperature at any subsequent time is given by

$$v=\frac{2\kappa_1}{\pi K_1 r}\int_a^b dr_1\int_0^\infty e^{-\kappa_1\lambda^2 t}\sin\lambda r_1\sin\lambda r\,d\lambda \quad . \quad . \quad . \quad (151)$$

Following from the special case represented by the result (150), it seems natural to take next that in which there is a small difference between the conductivities of the two media. If, accordingly, we write

$$\sigma=\frac{K_1}{K_2\mu} \; ; \quad \sigma'=\frac{K_1-K_2}{K_2\mu} \; ; \quad \sigma=1+\epsilon ,$$

where σ'/σ and ϵ are small quantities, we find e.g. that the result (146) becomes

$$v_1=\frac{2q\kappa r_1}{\pi K_1 r}\int_0^\infty e^{-\kappa_1\lambda^2 t}\sin\lambda r_1\sin\lambda r \times\left\{1-\epsilon\cos 2a\lambda+\sigma'\frac{\sin 2a\lambda}{a\lambda}\right\}d\lambda , \quad . \quad . \quad (152)$$

to the first order of small quantities. To interpret the various parts of this result we rewrite the second term as

$$-\frac{\epsilon q\kappa_1 r_1}{\pi K_1 r}\int_0^\infty e^{-\kappa_1\lambda^2 t}\sin\lambda r\left\{\sin(2a+r_1)\lambda-\sin(2a-r_1)\lambda\right\}d\lambda$$

and the third term as

$$\frac{\sigma' q\kappa_1 r_1}{\pi K_1 r}\int_0^\infty e^{-\kappa_1\lambda^2 t}\sin\lambda r\left\{\frac{\cos(2a-r_1)\lambda-\cos(2a+r_1)\lambda}{a\lambda}\right\}d\lambda \cdot$$

This term can again be rewritten

$$\frac{\sigma' q \kappa_1 r_1}{\pi K_1 r a}\int_0^\infty e^{-\kappa_1\lambda^2 t}\sin\lambda r\int_{2a-r_1}^{2a+r_1}\sin\lambda\xi\, d\xi . d\lambda .$$

$$=\frac{\sigma' q \kappa_1 r_1}{\pi K_1 r a}\int_{2a-r_1}^{2a+r_1} d\xi\int_0^\infty e^{-\kappa_1\lambda^2 t}\sin\lambda\xi\sin\lambda r\, d\lambda .$$

Thus, to the order of small quantities adopted, the effect at any point in the inner medium, on the understanding that there is now no discontinuity of medium at $r=a$, is the same as that due to the following system of spherical sources in infinite continuous media.

(i). The original surface source of strength q in the infinite medium of conductivity K_1;

(ii). a surface sink of strength $\frac{1}{2}\epsilon q r_1/(2a+r_1)$ at the surface $r=2a+r_1$, together with a surface source of strength $\frac{1}{2}\epsilon q r_1/(2a-r_1)$ at the surface $r=(2a-r_1)$, both in the infinite medium of conductivity K_1;

(iii). a continuous distribution of surface sources between the surfaces $r=2a-r_1$ and $r=2a+r_1$ of strength varying inversely as the distance from the center.

In the same way we could treat the results (147),(148),(149), with a view to finding the various solutions in terms of elementary sources. As we are only concerned with the general character of the solutions at present, the details for the other cases need not be reproduced.

With a view to obtaining partial verification of the general results (136)-(139) or (146)-(149), obtained as the solution of our problem, the forms taken by these results for such cases as (a) $K_2 = 0$, $r_1 < a$, (b) $K_1 = 0$, $r_1 > a$, should be considered, and the results compared with those obtained by the direct application of first principles. It will be found that the results deduced from the general ones here obtained reduce to the required form in every case.

The same problem treated in terms of Bessel Functions.

Reverting again to the results (146)-(149), and having regard to certain analogies existing between the solutions of spherical problems and the corresponding cylindrical ones, pointed out by Green in his fourth paper*, it might be useful to show that the results we have obtained here are confirmed when we adopt the special method and notation of that paper. The notation referred to is that of Bessel's Functions of half-odd-integral order. It is found that when the fundamental spherical wave trains are expressed in terms of these functions the summation processes required when we use circular functions are no longer necessary.

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* G. IV p 241.

When there is symmetry about a point, the only functions required are those of the $K_{\frac{1}{2}}$ and $I_{\frac{1}{2}}$ type and their derivatives. Thus

$$K_{\frac{1}{2}}(z) = \frac{\pi e^{-z}}{\sqrt{2\pi z}} \quad ; \quad I_{\frac{1}{2}}(z) = \frac{1}{\sqrt{2\pi z}}(e^{z} - e^{-z}).$$

With this notation we find that the two fundamental wave trains emanating from the periodic source at the surface $r = r_1$ in the inner medium are given by[*]

$$\left.\begin{aligned} r > r_1, \quad v_o &= \frac{q r_1}{K_1}\left(\frac{r_1}{r}\right)^{\frac{1}{2}} e^{ikt} I_{\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa_1}}\, r_1\right) K_{\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa_1}}\, r\right), \\ r < r_1, \quad v_i &= \frac{q r_1}{K_1}\left(\frac{r_1}{r}\right)^{\frac{1}{2}} e^{ikt} K_{\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa_1}}\, r_1\right) I_{\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\, r\right), \end{aligned}\right\} \quad . \quad . \quad . \quad (150)$$

Taking the first of these as $\rho r^{-\frac{1}{2}} K_{\frac{1}{2}}(i\lambda r)$, the meaning of ρ being apparent, and representing the continuations by reflection and transmission respectively by

$$A \rho r^{-\frac{1}{2}} I_{\frac{1}{2}}(i\lambda r) \quad ; \quad A' \rho r^{-\frac{1}{2}} K_{\frac{1}{2}}(\mu i \lambda r),$$

we remark that the first of these, travelling towards the centre of the sphere, is reflected there, setting up the corresponding reflected train of the $K_{\frac{1}{2}}$ type. As has already been pointed out,[†] however, the converging train of the $I_{\frac{1}{2}}$ type includes as part of itself a train of $K_{\frac{1}{2}}$ type of just the amount required to satisfy the condition at $r = 0$. We can, therefore, represent the temperature in the inner medium completely by

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[*] G. IV p. 236.

[†] ibid

$$r > r_1 , \qquad v_1 = p r^{-\frac{1}{2}} K_{\frac{1}{2}}(i\lambda r) + A p r^{-\frac{1}{2}} I_{\frac{1}{2}}(i\lambda r) \tag{151}$$

and that in the outer medium by

$$v_2 = A' p\, r^{-\frac{1}{2}} K_{\frac{1}{2}}(\mu i \lambda r). \tag{152}$$

We have now to find A and A' by taking into account the conditions at the surface $r = a$,stated in (135) above. Noting that

$$\frac{d}{dr}\left\{ r^{-\frac{1}{2}} K_{\frac{1}{2}}(i\lambda r)\right\} = -i\lambda r^{-\frac{1}{2}} K_{\frac{3}{2}}(i\lambda r),$$

$$\frac{d}{dr}\left\{ r^{-\frac{1}{2}} I_{\frac{1}{2}}(i\lambda r)\right\} = i\lambda r^{-\frac{1}{2}} I_{\frac{3}{2}}(i\lambda r),$$

we find that A and A' are given by

$$\left.\begin{aligned} A &= \frac{K_1 . K_{\frac{3}{2}}(i\lambda a) . K_{\frac{1}{2}}(\mu i \lambda a) - K_2 \mu . K_{\frac{1}{2}}(i\lambda a) K_{\frac{3}{2}}(\mu i\lambda a)}{d'} = \frac{\alpha}{d'}, \text{ say,} \\ A' &= \frac{K_1 \left\{ K_{\frac{1}{2}}(i\lambda a) I_{\frac{3}{2}}(i\lambda a) + K_{\frac{3}{2}}(i\lambda a) I_{\frac{1}{2}}(i\lambda a)\right\}}{d'} = \frac{K_1}{i\lambda a d'} \end{aligned}\right\} \quad \ldots \tag{153}$$

where

$$d' = K_1 I_{\frac{3}{2}}(i\lambda a) K_{\frac{1}{2}}(\mu i\lambda a) + K_2 \mu . I_{\frac{1}{2}}(i\lambda a) K_{\frac{3}{2}}(\mu i \lambda a). \quad \ldots \tag{154}$$

From these observations we have e.g.

$$v_1 = \frac{q r_1}{K_1}\left(\frac{r_1}{r}\right)^{\frac{1}{2}} e^{ikt} I_{\frac{1}{2}}(i\lambda r_1)\left\{ K_{\frac{1}{2}}(i\lambda r) + \frac{\alpha}{d'} I_{\frac{1}{2}}(i\lambda r)\right\} \quad \ldots \tag{155}$$

If now we use the forms

$$I_{\frac{3}{2}}(z) = \sqrt{\frac{2}{\pi z}}\left(\cosh z - \frac{\sinh z}{z}\right) ; \quad K_{\frac{3}{2}}(z) = \sqrt{\frac{2}{\pi z}}\, e^{-z}\left(1 + \frac{1}{z}\right)$$

we find , after the necessary reduction

$$d' = \frac{e^{-\mu i \lambda a}}{i\lambda a\sqrt{\mu}}\left\{K_1 \cos a\lambda - (K_1 - K_2)\frac{\sin a\lambda}{a\lambda} + iK_2\mu \sin a\lambda\right\},$$

$$d' K_{\frac{1}{2}}(i\lambda r) + \alpha I_{\frac{1}{2}}(i\lambda r) = \sqrt{\frac{\pi}{2}}\,\frac{e^{-\mu i \lambda a}}{i\lambda a\sqrt{\mu i \lambda r}}\left\{K_1 \cos(a-r)\lambda\right.$$

$$\left. - (K_1 - K_2)\frac{\sin(a-r)\lambda}{a\lambda} + iK_2\mu \sin(a-r)\lambda\right\}.$$

When these forms are inserted in (155) and when the subsequent integration with respect to λ is performed we are at once led to the result (146) above. In the same way we could verify the remaining results (147)-(149) .

This analysis, together with the results of Green's investigations of the analogies already referred to, suggests that ultimately evaluation of the integrals in (146) -(149) may be effected by working out in full the corresponding cylindrical problem and then replacing the Bessel Functions of integral order that appear in the solution by those of the half-odd-integral order as required for the spherical case. The essential part of the analysis for the cylindrical problem is given in the next chapter.

CHAPTER IV.

PROBLEMS INVOLVING CYLINDRICAL FLOW.

We confine our attention in the first instance to problems where the effects are symmetrical about an axis ($r=0$). In the circumstances the equation of heat conduction becomes

$$\frac{\partial v}{\partial t} = \kappa\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r}\right) \tag{156}$$

Testing this equation for solutions of the type $e^{ikt}R$, where R is a function of r alone we find that R must satisfy

$$\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r} - \frac{ik}{\kappa}R = 0. \tag{157}$$

The solution of this equation is

$$R = AI_0\left(\sqrt{\frac{ik}{\kappa}}\,r\right) + BK_0\left(\sqrt{\frac{ik}{\kappa}}\,r\right), \tag{158}$$

where the I_0 and K_0 functions are the modified Bessel Functions of zero order of the first and second kinds respectively.*

Thus

$$I_0(z) = 1 + \frac{z^2}{2^2} + \frac{z^4}{2^2 4^2} + \frac{z^6}{2^2.4^2.6^2} + \cdots \tag{159}$$

Gray, Mathews and MacRobert, Bessel Functions (1922) p 20 et seq.
In the sequel this work is denoted by GMM.

$$K_0(z) = (\log 2 - \gamma) I_0(z) - \log z I_0(z) + \frac{z^2}{2^2} + \frac{z^4}{2^2 4^2}(1+\tfrac{1}{2}) + \frac{z^6}{2^2 4^2 6^2}(1+\tfrac{1}{2}+\tfrac{1}{3}) + \cdots \quad (160)$$

where γ is Euler's constant. When the variable z is large, these functions are sufficiently indicated by the first few terms of their asymptotic expansions. Thus

$$I_0(z) = \frac{e^z}{\sqrt{2\pi z}}\left\{1 + \frac{1^2}{(8z)} + \frac{1^2 3^2}{2!(8z)^2} + \cdots \right\}$$

$$K_0(z) = \sqrt{\frac{\pi}{2z}}\, e^{-z}\left\{1 - \frac{1^2}{(8z)} + \frac{1^2 3^2}{2!(8z)^2} - \cdots \right\}$$

It is apparent from these expansions that the $K_0(z)$ function is infinite like $-\log z$ at $z=0$ and has the value zero when z is infinite. On the other hand the I_0 function is finite at $z=0$ and infinite at $z=\infty$.

From these remarks we conclude that the two fundamental diverging cylindrical trains are the real and imaginary parts of

$$e^{ikt} K_0\left(\sqrt{\frac{ik}{\kappa}}\, r\right) ,$$

and the two fundamental converging trains the real and imaginary parts of

$$e^{ikt} I_0\left(\sqrt{\frac{ik}{\kappa}}\, r\right) .$$

The Line Source in the Infinite Medium.

We might show e.g. how the well-known solutions for the periodic and instantaneous line sources can be expressed in terms of the fundamental train solutions we have just found.

Suppose that along the axis $r=0$ heat is being generated at the rate of qe^{ikt} units per unit length. Clearly only the diverging trains are involved in this case. The required solution is given by

$$v = Ae^{ikt}K_0(i\lambda r) \quad , \quad \sqrt{\frac{ik}{\kappa}} = i\lambda$$

provided A is chosen to satisfy

$$\mathop{\mathcal{L}}_{r\to 0}\left(-K2\pi r\frac{\partial v}{\partial r}\right) = qe^{ikt}$$

By actually differentiating $K_0(z)$ as given in (160), we find that $\mathcal{L}\,zK_0'(z) = -1$. Hence clearly $A = \frac{q}{2\pi K}$ and the solution required to represent the periodic line source is

$$v = \frac{q}{2\pi K}e^{ikt}K_0\left(\sqrt{\frac{ik}{\kappa}}\,r\right). \qquad (161)$$

The instantaneous line source.

Consider the solution

$$v = \frac{q}{2\pi^2 K}\int_0^\infty e^{ikt}K_0\left(\sqrt{\frac{ik}{\kappa}}\,r\right)dk \qquad (162)$$

$$= \frac{iq\kappa}{\pi^2 K}\int_{OB} e^{-\kappa\lambda^2 t}K_0(i\lambda r)\lambda\, d\lambda$$

the path of integration being the infinite radius OB of fig. 1 Introducing the J_0, G_0 functions,* we have

$$K_0(i\lambda r) = G_0(-\lambda r) = G_0(\lambda r) - \pi i J_0(\lambda r)$$

so that the solution in question is

$$v = \frac{iq\kappa}{\pi^2 K}\int_{OB} e^{-\kappa\lambda^2 t}\left\{G_0(\lambda r) - \pi i J_0(\lambda r)\right\}\lambda\, d\lambda \qquad (163)$$

The integrand has no singularity within the closed contour of

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* G.M.M p. 23.

fig. 1 so that if we assume that the integral is zero along the arc of infinite radius, we have

$$v = \frac{iq\kappa}{\pi^2 K}\int_0^\infty e^{-\kappa t\xi^2}\left\{G_0(r\xi) - \pi i J_0(r\xi)\right\}\xi d\xi. \tag{164}$$

The imaginary part of $G_0(r\xi)$ is $\frac{i\pi}{2}J_0(r\xi)$ so that when we take the real part only of the last result we obtain

$$v = \frac{q\kappa}{2\pi K}\int_0^\infty e^{-\kappa t\xi^2} J_0(r\xi)\xi d\xi. \tag{165}$$

$$= \frac{q}{4\pi K t} e^{-\frac{r^2}{4\kappa t}} \tag{166}$$

by an adaptation of a well-known theorem.* This result is at once recognised as the solution for the instantaneous line source in the infinite medium. The definite integral form of solution as given in (162) shows the infinite combination of fundamental cylindrical trains required to represent such a source.

The cylindrical surface source in the infinite medium.

Suppose that at the surface $r = r_1$ heat is being emitted at the rate of qe^{ikt} per unit area. To represent the effect at any point we require both the diverging and the converging fundamental trains. Thus we take the solution given by

$$\left.\begin{array}{ll} r < r_1, & v_i = Aqe^{ikt} I_0(i\lambda r) \\ r > r_1, & v_o = A'qe^{ikt} K_0(i\lambda r) \end{array}\right\} \tag{167}$$

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* G.M.M p. 69.

where the constants A and A' are chosen to satisfy the conditions at the surface $r=r_1$ viz.

$$v_i = v_o \quad \text{and} \quad -K\left(\frac{\partial v_o}{\partial r} - \frac{\partial v_i}{\partial r}\right) = qe^{ikt}$$

These give

$$AI_0(i\lambda r_1) = A' K_0(i\lambda r_1)$$

$$Ki\lambda\{A' K_1(i\lambda r_1) + AI_1(i\lambda r_1)\} = 1$$

Whence $\dfrac{A}{K_0(i\lambda r_1)} = \dfrac{A'}{I_0(i\lambda r_1)} = \dfrac{AI_1(i\lambda r_1)+A'K_1(i\lambda r_1)}{(K_0I_1 + I_0K_1)_{i\lambda r_1}} = \dfrac{1/Ki\lambda}{1/i\lambda r_1} = \dfrac{r_1}{K}$

where we have made use of such well known Bessel Function properties as

$$-K_0' = K_1 \quad ; \quad I_0' = I_1 \quad ; \quad K_0(x)I_1(x) + I_0(x)K_1(x) = 1/x.$$

Thus the required solution for the periodic surface soutce at $r=r_1$ is

$$\left.\begin{aligned} r<r_1 \; ; \quad v_i &= \frac{qr_1}{K} e^{ikt} K_0(i\lambda r_1)I_0(i\lambda r) \\ r>r_1 \; ; \quad v_o &= \frac{qr_1}{K} e^{ikt} I_0(i\lambda r_1)K_0(i\lambda r). \end{aligned}\right\} \qquad (168)$$

It is worthy of note that the expression for v_i violates no condition at the axis $r=0$; (compare the analysis for the corresponding spherical case p. 64) and that v_o reduces to correct form in the limiting case when $r_1 \to 0$. If we suppose that this limit is reached in such a way that $2\pi r_1 q$ tends to a definite limit q_0 we find that v_o then takes the form required to represent the axial line source as shown in (161) above.

It is also of interest to show that when r and r_1 are very great with $(r-r_1)$ finite, the above expressions for the converging and diverging trains representing the cylindrical periodic surface source reduce to the correct forms for the corresponding plane trains. In the circumstances the Bessel Functions are sufficiently represented by the first terms of their asymptotic expansions. Thus

$$I_0(z) \doteqdot \frac{e^{z}}{\sqrt{2\pi z}} \quad ; \qquad K_0(z) \doteqdot \sqrt{\frac{\pi}{2z}}\, e^{-z}$$

so that the above expression for v_i, e.g. becomes

$$\frac{qr_1}{K} e^{ikt} \pi \frac{e^{-i\lambda r_1}}{\sqrt{2\pi i\lambda r_1}} \cdot \frac{e^{i\lambda r}}{\sqrt{2\pi i\lambda r}}$$

$$= \frac{q}{2Ki\lambda} e^{ikt-(r_1-r)i\lambda} \qquad \text{as required.}$$

The instantaneous surface source.

The solution in this case is obtained from (168) by taking

$$r < r_1 \qquad v = \frac{qr_1}{\pi K}\int_0^{\infty} e^{ikt} K_0(i\lambda r_1) I_0(i\lambda r)\, dk$$

$$= \frac{2iq\kappa r_1}{\pi K}\int_{OB} e^{-\kappa\lambda^2 t} K_0(i\lambda r_1) I_0(i\lambda r)\, \lambda\, d\lambda \qquad (169)$$

the path of integration again being the infinite radius OB of fig. 1.

Making use of the identity

$$I_0(i\lambda r_1)K_0(i\lambda r) = J_0(\lambda r_1)G_0(-\lambda r)$$

and using the same argument as in the evaluation of the integral in (162) we have

$$v = \frac{2iq\kappa r_1}{\pi K}\int_0^\infty e^{-\kappa t\xi^2} J_0(r_1\xi)\left\{G_0(r\xi) - \pi i J_0(r\xi)\right\}\xi d\xi, \qquad (170).$$

and, on taking the real part of this result

$$v = \frac{q\kappa r_1}{K}\int_0^\infty e^{-\kappa t\xi^2} J_0(r_1\xi)J_0(r\xi)\xi d\xi. \qquad (171).$$

$$= \frac{qr_1}{2Kt} e^{-\frac{r^2+r_1^2}{4\kappa t}} I_0\left(\frac{rr_1}{2\kappa t}\right) \qquad (172).$$

by an adaptation of a standard theorem.*

The result obtained is in agreement with the known solution of this problem. It has been obtained by Green using the wave train theory but a different analysis.+

It is of interest to show that the result can also be obtained by integrating the instantaneous line source solution over the cylindrical surface of radius r_1. Using fig.4, on the understanding that the circle represents a section of the cylinder of radius r_1, we see that the effect at **P** at distance r of the instantaneous line source of strength $q r_1 d\theta$ per unit length on the surface $r = r_1$ is

$$\frac{q r_1 d\theta}{4\pi K t} e^{-\frac{\rho^2}{4\kappa t}}$$

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* G.M.M. p. 69.

+ G.II p. 707.

where $$\rho^2 = r^2 + r_1^2 - 2 r r_1 \cos\theta.$$

Thus the effect required is

$$\frac{q r_1}{4\pi K t}\int_0^{2\pi} e^{-\frac{\rho^2}{4\kappa t}}\, d\theta$$

$$= \frac{q r_1}{2\pi K t}\, e^{-\frac{r^2+r_1^2}{4\kappa t}} \int_0^{\pi} e^{\frac{r r_1}{2\kappa t}\cos\theta}\, d\theta.$$

$$= \frac{q r_1}{2 K t}\, e^{-\frac{r^2+r_1^2}{4\kappa t}}\, I_0\left(\frac{r r_1}{2\kappa t}\right)^{*}$$

in agreement with (172) above.

In the present connection we might go a step further and show that the line source solution can itself be obtained by integrating the point source solution along an infinite line; thus ultimately the surface source solution is expressible in terms of the point source. The necessary demonstration is given by Green[+] and need not be reproduced here.

Reflection of cylindrical trains.

Consider the case of a periodic surface source of strength $q e^{ikt}$ per unit area at the surface $r = r_1$ within the solid cylinder of radius a. First effects within the cylinder are given by the solution shown in (168). It is clear however that the arrival of the train v_0 at the surface $r = a$ will in general violate the temperature condition at this surface. We must

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[*] G.M.M p. 46.

[+] G. IV p 239.

therefore suppose that the train is reflected at this surface, initiating a reflected converging train, the total effect in the region $r > r_1$ being obtained by superposing this reflected train on the original diverging train v_0. Clearly also, as has been indicated in corresponding plane and spherical cases this train must also be added to v_i so that the source condition at the surface $r = r_1$ may be maintained.

A suitable form of reflected train is

$$v_r = A\frac{q r_1}{K} e^{ikt} I_0(i\lambda r)$$

where the constant A depends on the condition at the surface $r = a$.

<u>(α). The surface $r = a$ kept at zero temperature.</u>

If the reflection takes place under this condition we require

$$I_0(i\lambda r_1) K_0(i\lambda a) + A I_0(i\lambda a) = 0 \;; \qquad \text{whence} \quad A = -I_0(i\lambda r_1)\frac{K_0(i\lambda a)}{I_0(i\lambda a)} \qquad (173)$$

With this form we find that (v_i) becomes

$$r < r_1 , \qquad v_i = \frac{q r_1}{K} e^{ikt} \frac{I_0(i\lambda r)}{I_0(i\lambda a)} \left\{ K_0(i\lambda r_1) I_0(i\lambda a) - I_0(i\lambda r_1) K_0(i\lambda a) \right\} \qquad (174)$$

Introducing the notation

$$\begin{aligned} f_0(\lambda r, \lambda a) &= K_0(i\lambda r) I_0(i\lambda a) - I_0(i\lambda r) K_0(i\lambda a) \\ &= G_0(\lambda r) J_0(\lambda a) - J_0(\lambda r) G_0(\lambda a) = -f_0(\lambda a, \lambda r), \qquad (175) \end{aligned}$$

the above result becomes

$$r < r_1 , \qquad v_i = \frac{q r_1}{K} e^{ikt} \frac{J_0(\lambda r)}{J_0(\lambda a)} f_0(\lambda r_1, \lambda a) \qquad (176)$$

The corresponding result for the region $r>r_1$ is obtained by interchanging r and r_1 in the various Bessel Functions.

(β). Radiation at the surface $r=a$.

If the train v_0 on arrival at the surface $r=a$ is reflected under the condition that there is radiation to a medium at zero temperature we have

$$r=a, \qquad -K\frac{\partial v}{\partial r}=hv$$

Applying this to v given by

$$v= I_0(i\lambda r_1)K_0(i\lambda r)+AI_0(i\lambda r)$$

we readily find

$$A=I_0(i\lambda r_1)\,\frac{Ki\lambda.\,K_1(i\lambda a)-hK_0(i\lambda a)}{Ki\lambda\,I_1(i\lambda a)+hI_0(i\lambda a)} \qquad (177)$$

and with this form of $\mathbf{A}$,(v_i) becomes

$$r<r_1\,, \quad v_i=\frac{qr_1}{K}e^{ikt}I_0(i\lambda r)\left\{\frac{Ki\lambda\left[K_0(i\lambda r_1)I_1(i\lambda a)+I_0(i\lambda r_1)K_1(i\lambda a)\right]+h\left[K_0(i\lambda r_1)I_0(i\lambda a)-I_0(i\lambda r_1)K_0(i\lambda a)\right]}{Ki\lambda\,I_1(i\lambda a)+hI_0(i\lambda a)}\right]$$

Introducing the notation

$$f_1(\lambda a,\lambda r_1)=i\left\{K_1(i\lambda a)I_0(i\lambda r_1)+I_1(i\lambda a)K_0(i\lambda r_1)\right\}^{*} \qquad (178)$$

$$=G_1(\lambda a)J_0(\lambda r_1)-J_1(\lambda a)G_0(\lambda r_1),$$

the result v_i assumes the form

$$r<r_1\,, \quad v_i=-\frac{qr_1}{K}e^{ikt}I_0(i\lambda r).\,\frac{K\lambda f_1(\lambda a,\lambda r_1)-hf_0(\lambda a,\lambda r_1)}{K\lambda J_1(\lambda a)-hJ_0(\lambda a)} \qquad (179)$$

with, as formerly a corresponding result in the region $r>r_1$ got by interchanging r and r_1 in the Bessel Functions.

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* $I_1(it)=iJ_1(t)$; $K_1(it)=iG_1(-t)=-\pi J_1(t)-iG_1(t)$

Instantaneous surface source at $r=r_1$.

(α). In the case of the zero temperature surface condition, the required solution is that given by the real part of v where

$$r<r_1, \qquad v=\frac{qr_1}{\pi K}\int_0^\infty e^{ikt}\frac{J_0(\lambda r)}{J_0(\lambda a)}f_0(\lambda r_1,\lambda a)\,dk \qquad (180)$$

$$=\frac{2i\kappa q r_1}{\pi K}\int_{OB} e^{-\kappa\lambda^2 t}\frac{J_0(\lambda r)}{J_0(\lambda a)}f_0(\lambda r_1,\lambda a).\lambda\,d\lambda\,, \qquad (181)$$

the path of integration being the line OB of fig. 1 .

The evaluation is effected by integrating the function

$$F(\lambda)=\frac{2i\kappa q r_1}{\pi K}e^{-\kappa\lambda^2 t}\frac{J_0(\lambda r)}{J_0(\lambda a)}f_0(\lambda r_1,\lambda a)\lambda$$

round the closed contour of fig. 3 ,on the understanding that the indents are round the points given by the roots of the equation

$$J_0(\lambda a)=0. \qquad (182)$$

Assuming that the integral along the arc of infinite radius vanishes, we find that the real part of the required integral comprises two parts, (1), that arising from the integrals round the indents and (2), that arising from the integral along the real axis of the imaginary part of $F(\lambda)$. Disposing of the latter part first, we observe that if λ is real,as it is along the real ξ axis we have

$$f_0(r_1\xi,a\xi)=G_0(r_1\xi)J_0(a\xi)-J_0(r_1\xi)G_0(a\xi)$$

of which the imaginary part is $i\frac{\pi}{2}J_0(r_1\xi)J_0(a\xi)-\frac{i\pi}{2}J_0(r_1\xi)J_0(a\xi)=0.$

Accordingly the real part of the result arising from the source (2) is zero.

The integral round an indent at λ where $J_0(\lambda a)=0$ is

$$-\pi i \frac{2iq\kappa r_1}{\pi K} e^{-\kappa\lambda^2 t} \frac{J_0(\lambda r) f_0(\lambda r_1, \lambda a).\lambda}{a\, J_1(\lambda a)}$$

By using the general theorem

$$G_1(\lambda a) J_0(\lambda a) - J_1(\lambda a) G_0(\lambda a) = \frac{1}{\lambda a}$$

we see that at a root of (182), $G_0(\lambda a) = -\dfrac{1}{\lambda a J_1(\lambda a)}$,

and $f_0(\lambda r_1, \lambda a)$ becomes $-J_0(\lambda r_1) G_0(\lambda a) = \dfrac{J_0(\lambda r_1)}{\lambda a\, J_1(\lambda a)}$;

and the evaluation round the indent

$$\frac{2q\kappa r_1}{K} e^{-\kappa\lambda^2 t} \frac{J_0(\lambda r_1) J_0(\lambda r)}{a^2 J_1^2(\lambda a)}$$

Thus the final result is

$$v = \frac{2q\kappa r_1}{K} \sum_1^\infty e^{-\kappa\lambda^2 t} \frac{J_0(\lambda r_1) J_0(\lambda r)}{a^2 J_1^2(\lambda a)}. \qquad (183)$$

where the summation is with regard to all the positive roots of the equation (182).

(β). In the case of the radiation condition at the surface $r=a$ the result required is

$$v = -\frac{2iq\kappa r_1}{\pi K} \int_{OB} e^{-\kappa\lambda^2 t} J_0(\lambda r) \frac{K\lambda f_1(\lambda a, \lambda r_1) - h f_0(\lambda a, \lambda r_1)}{K\lambda\, J_1(\lambda a) - h\, J_0(\lambda a)} \lambda d\lambda. \qquad (184)$$

In this case we must understand that the indents in fig. 3 are round the points given by the roots of the equation

$$K\lambda J_1(\lambda a) - h J_0(\lambda a) = 0. \qquad (185)$$

The argument used in the evaluation is the same as that given in connection with (181) above. It is readily shown that if ξ is real, the imaginary part of

$$K\xi f_1(a\xi, r_1\xi) - h f_0(a\xi, r_1\xi) \quad \text{is zero.}$$

The part of v arising from the integral round an indent is

$$-\pi i \, \frac{2iqkr_1}{\pi K} e^{-k\lambda^2 t} J_0(\lambda r) \, \frac{K\lambda f_1(\lambda a, \lambda r_1) - h f_0(\lambda a, \lambda r_1)}{\frac{d}{d\lambda}\left\{K\lambda J_1(\lambda a) - h J_0(\lambda a)\right\}} \lambda.$$

Also, at a root of (185), we have

$$\frac{d}{d\lambda}\left\{K\lambda J_1(\lambda a) - h J_0(\lambda a)\right\} \equiv Ka\lambda J_0(\lambda a) + haJ_1(\lambda a)$$

$$= \frac{a}{K\lambda}(K^2\lambda^2 + h^2) J_0(\lambda a)$$

also $K\lambda f_1(\lambda a, \lambda r_1) - h f_0(\lambda a, \lambda r_1)$ becomes

$$J_0(\lambda r_1)\left\{K\lambda G_1(\lambda a) - h G_0(\lambda a)\right\} = \frac{K J_0(\lambda r_1)}{a J_0(\lambda a)}$$

With these simplifications we find that the required result is

$$v = \frac{2qkr_1}{K} \sum_1^\infty e^{-k\lambda^2 t} \frac{K^2\lambda^2}{K^2\lambda^2 + h^2} \, \frac{J_0(\lambda r_1) J_0(\lambda r)}{a^2 J_0^2(\lambda a)} \qquad (186)$$

where the summation is with regard to all the positive roots of the equation (185).

Effects due to a prescribed initial heat distribution.

If e.g. we have $t = 0$, $v = f(r)$, $0 \leqq r \leqq a$, the results are (a). The exposed surface at zero temperature.

$$v = 2\sum_{1}^{\infty} e^{-\kappa\lambda^2 t}\frac{J_0(\lambda r)}{a^2 J_1^2(\lambda a)}\int_0^a f(r_1)J_0(\lambda r_1)r_1\,dr_1. \qquad (187)$$

the summation being with regard to all the positive roots of the equation (182).

(b). Radiation at the exposed surface.

$$v = 2\sum_{1}^{\infty} e^{-\kappa\lambda^2 t}\frac{K^2\lambda^2}{K^2\lambda^2+h^2}\,\frac{J_0(\lambda r)}{J_0^2(\lambda a)}\int_0^a f(r_1)J_0(\lambda r_1)r_1\,dr_1 \qquad (188)$$

the summation being with regard to all the positive roots of the equation (185).

The results (187) and (188) are known from other considerations to be correct.[*] The special forms they take when $t=0$ give well known theorems in the expansion of functions in special types of Bessel's Series.

The surface of the cylinder kept at a prescribed temperature.

It is of interest to investigate this case from first principles, beginning with the train converging inwards from the surface maintained at the periodic temperature $\theta_0 e^{ikt}$ and following out an argument similar to that adopted in connection with the corresponding plane problem, but the solution can be obtained at once from that given in (187).

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[*] See e.g. Carslaw p.117

Consider V given by

$$V = f(r) - 2\sum_1^\infty e^{-\kappa\lambda^2 t}\frac{J_0(\lambda r)}{a^2 J_1^2(\lambda a)}\int_0^a f(r_1)J_0(\lambda r_1)r_1\,dr_1 \qquad (189)$$

This solution satisfies the heat conduction equation provided

$$\nabla^2 f(r) = 0$$

also $V=0,\ t=0$; $V=f(a),\ r=a$

$V=f(r),\ t=\infty$; and $V \neq \infty,\ r=0.$

$\nabla^2 f(r) = 0$ gives $f(r) = A\log r + v_0$, say,

The only way of satisfying these conditions is to take $f(r)=v_0$ so that the required solution is

$$V = v_0\left\{1 - 2\sum_1^\infty e^{-\kappa\lambda^2 t}\frac{J_0(\lambda r)}{a J_1(\lambda a)}\right\} \qquad (190)$$

in agreement with the known result of this problem.

Problems involving heat flow in the infinitely long hollow cylinder.

The internal and external radii being a, b respectively we consider the two following cases.

(α). An initial temperature prescribed throughout the cylinder; the exposed surfaces $r=a$ and $r=b$ kept at zero temperature.

(β). The inner surface $r=a$ maintained at a prescribed temperature; radiation at the surface $r=b$.

We begin again by considering the fundamental trains emanating from a periodic surface source of strength qe^{ikt} at the surface $r=r_1$. Representing these by

$$r<r_1, \quad \rho I_0(i\lambda r) \quad \text{and} \quad r>r_1, \quad \rho' K_0(i\lambda r)$$

where

$$\rho, \rho' = \frac{qr_1}{K} e^{ikt} \{K_0(i\lambda r_1), I_0(i\lambda r_1)\} \quad ;$$

we next write down the various contributions to v the resultant effect at any point, made by the complete system of trains set up by the fundamentals, by their repeated reflections at the surfaces $r=a$ and $r=b$. In this way we have, following up the original converging train $\rho I_0(i\lambda r)$,

$r<r_1$		$r>r_1$	
	$\rho I_0(i\lambda r)$		
$A\rho K_0(i\lambda r)$		$A\rho K_0(i\lambda r)$	
	$AB\rho I_0(i\lambda r)$		$AB\rho I_0(i\lambda r)$
$A^2B\rho K_0(i\lambda r)$		$A^2B\rho K_0(i\lambda r)$	
	$A^2B^2\rho I_0(i\lambda r)$		$A^2B^2\rho I_0(i\lambda r)$
$A^3B^2\rho K_0(i\lambda r)$		. . .	

where A and B are the reflection coefficients for the surfaces $r=a$ and $r=b$. Thus the total partial effect due to the original converging train is

$$r<r_1, \quad \rho\frac{I_0(i\lambda r)+AK_0(i\lambda r)}{1-AB} \quad \text{and} \quad r>r_1, \quad A\rho\frac{K_0(i\lambda r)+BI_0(i\lambda r)}{1-AB}.$$

In the same way if we tabulate the effects in both parts of the field due to the original diverging train and perform the summations we get

$$r < r_1 , \qquad B\rho' \frac{I_0(i\lambda r) + AK_0(i\lambda r)}{1 - AB}$$

$$r > r_1 , \qquad \rho' \frac{K_0(i\lambda r) + BI_0(i\lambda r)}{1 - AB}$$

Thus we obtain the total effects due to the <u>periodic</u> source at $r = r_1$ as

$$\left. \begin{aligned} r < r_1 , \qquad v &= \frac{I_0(i\lambda r) + AK_0(i\lambda r)}{1 - AB}(\rho + B\rho') \\ r > r_1 , \qquad v &= \frac{K_0(i\lambda r) + BI_0(i\lambda r)}{1 - AB}(A\rho + \rho') \end{aligned} \right\} \qquad (191)$$

With the surfaces $r = a$, $r = b$ at zero temperature we find

$$A = -\frac{I_0(i\lambda a)}{K_0(i\lambda a)} \quad ; \qquad B = -\frac{K_0(i\lambda b)}{I_0(i\lambda b)}.$$

Thus the first of (191) becomes, after necessary reduction

$$r < r_1, \qquad v = \frac{q r_1}{K} e^{ikt} \frac{f_0(\lambda a, \lambda r). f_0(\lambda r_1, \lambda b)}{f_0(\lambda a, \lambda b)}. \qquad (192)$$

the corresponding result for the region $r > r_1$ being obtained by interchanging r and r_1 in the f_0 functions.

The result corresponding to (192) for the case of the <u>instantaneous</u> source of strength q per unit area at the surface $r = r_1$, obtained as in various previous cases is

$$v = \frac{2iq\kappa r_1}{\pi K} \int_{OB} e^{-\kappa\lambda^2 t} \frac{f_0(\lambda a, \lambda r). f_0(\lambda r_1, \lambda b)}{f_0(\lambda a, \lambda b)} \lambda d\lambda. \qquad (193)$$

The integration is effected by using the contour of fig. 3, the indents being understood to be at the positive roots of the equation

$$f_0(\lambda a, \lambda b) \equiv G_0(\lambda a) J_0(\lambda b) - J_0(\lambda a) G_0(\lambda b) = 0^{*} \qquad (194)$$

The theory of the evaluation of the integral appearing in (193) is the same as that given in previous cases. At a root of (194) we have

$$\frac{d}{d\lambda} f_0(\lambda a, \lambda b) \equiv -a\, G_1(\lambda a) J_0(\lambda b) - b\, G_0(\lambda a) J_1(\lambda b) + a J_1(\lambda \overset{a}{b}) G_0(\lambda b) + b J_0(\lambda a) G_1(\lambda b)$$

$$= \frac{J_0^2(\lambda a) - J_0^2(\lambda b)}{\lambda J_0(\lambda a) J_0(\lambda b)},$$

where we have made use of known relations between the functions. Further at a root of (194)

$$f_0(\lambda r_1, \lambda b) = \frac{J_0(\lambda b)}{J_0(\lambda a)} f_0(\lambda r_1, \lambda a)$$

Thus ultimately the result required is

$$v = \frac{2 q \kappa r_1}{K} \sum_{1}^{\infty} e^{-\kappa \lambda^2 t} \frac{\lambda^2 J_0^2(\lambda b)}{J_0^2(\lambda a) - J_0^2(\lambda b)} f_0(\lambda a, \lambda r) f_0(\lambda r_1, \lambda a)^{+} \qquad (195)$$

where the summation is with regard to all the positive roots of the equation (194).

If the initial temperature throughout the cylinder is given by $t=0$, $v=f(r)$, $0 \leqq r \leqq a$, we see at once from this last result that the solution in this case is

$$v = 2 \sum_{1}^{\infty} e^{-\kappa \lambda^2 t} \frac{\lambda^2 J_0^2(\lambda b)}{J_0^2(\lambda a) - J_0^2(\lambda b)} f_0(\lambda a, \lambda r) \int_0^a f(r_1) f_0(\lambda r_1, \lambda a) r_1 \, dr_1 \qquad (196)$$

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[*] For a discussion of the roots of this equation see GMM p. 82.

[+] This result may be exhibited in various forms. We select that which makes it apparent that the result in the region $r > r_1$ is the same.

82.
that
This result may be exhibited ...

(β). Suppose in the first place that the inner surface of the cylinder $r=a$ is maintained at the periodic temperature $\theta_0 e^{ikt}$. The original train emanating from this surface is accordingly represented by

$$\theta_0 e^{ikt} \frac{K_0(i\lambda r)}{K_0(i\lambda a)}.$$

This train and its various continuations by reflection at the surfaces $r=b$ and $r=a$ builds up the complete temperature system at any point. Thus we find

$$v = \theta_0 e^{ikt} \frac{K_0(i\lambda r) + BI_0(i\lambda r)}{1 - AB} \qquad (197)$$

where A and B the reflection coefficients at the surfaces $r=a$ and $r=b$ are given by

$$A = -\frac{I_0(i\lambda a)}{K_0(i\lambda a)} \quad ; \quad B = \frac{Ki\lambda . K_1(i\lambda b) - h K_0(i\lambda b)}{Ki\lambda\, I_1(i\lambda b) + hI_0(i\lambda b)} \qquad (198)$$

With these forms inserted we find that (197) becomes after necessary reduction

$$v = \theta_0 e^{ikt} \frac{K\lambda f_1(\lambda b, \lambda r) - h f_0(\lambda b, \lambda r)}{K\lambda f_1(\lambda b, \lambda a) - h f_0(\lambda b, \lambda a)}$$

$$= \theta_0 e^{ikt} \frac{U(\lambda, r)}{U(\lambda, a)}, \text{ say.} \qquad (199)$$

This result gives the effect due to the periodic temperature $\theta_0 e^{ikt}$ at the surface $r=a$. Thus the effect due to the instantaneous doublet of strength $2K\theta_0$ per unit area over the surface $r=a$ - or, what is the same thing, the effect due to the instantaneous creation of the temperature θ_0 at $r=a$ at $t=0$ - is given by

$$v = \frac{\theta_0}{\pi}\int_0^\infty e^{ikt}\,\frac{U(\lambda, r)}{U(\lambda, a)}\,dk$$

$$= \frac{2\kappa i\theta_0}{\pi}\int_{OB} e^{-\kappa\lambda^2 t}\,\frac{U(\lambda, r)}{U(\lambda, a)}\,\lambda\, d\lambda \qquad (200)$$

the path of integration being that already used on numerous occasions. The integration is effected in the usual manner and yields

$$v = -2\kappa\theta_0 \sum_1^\infty e^{-\kappa\lambda^2 t}\,\frac{U(\lambda, r)\,\lambda}{\frac{d}{d\lambda}\left\{U(\lambda, a)\right\}} \qquad (201)$$

where the summation is with regard to all the positive roots of the equation

$$U(\lambda, a) \equiv K\lambda f_1(\lambda b, \lambda a) - h f_0(\lambda b, \lambda a) = 0. \qquad (202)$$

At a root of this equation we find

$$U(\lambda, r) \equiv K\lambda\, f_1(\lambda b, \lambda r) - h f_0(\lambda b, \lambda r)$$

$$= \frac{K\lambda}{f_0(\lambda b, \lambda a)}\left\{f_1(\lambda b, \lambda r) f_0(\lambda b, \lambda a) - f_0(\lambda b, \lambda r) f_1(\lambda b, \lambda a)\right\} = \frac{K\, f_0(\lambda r, \lambda a)}{b\, f_0(\lambda b, \lambda a)}$$

Thus the above result becomes

$$v = -2\kappa\theta_0 \sum_1^\infty e^{-\kappa\lambda^2 t}\,\frac{K\lambda\, f_0(\lambda r, \lambda a)}{b\, f_0(\lambda b, \lambda a)\,\frac{d}{d\lambda}\left\{U(\lambda, a)\right\}} \qquad (203)$$

From this result we obtain that finally required, the effect when the inner surface $r = a$ of the hollow cylinder is kept at the constant temperature θ_0 from the instant $t = 0$ onwards.

This may be shown as

$$V = -2\theta_0 \sum_{1}^{\infty} (1 - e^{-\kappa\lambda^2 t}) \frac{K f_0(\lambda r, \lambda a)}{b\lambda f_0(\lambda b, \lambda a) \frac{d}{d\lambda} U(\lambda, a)}$$

$$= \theta_0 \frac{h \log \frac{b}{r} + \frac{K}{b}}{h \log \frac{b}{a} + \frac{K}{b}} + 2\theta_0 \sum_{1}^{\infty} e^{-\kappa\lambda^2 t} \frac{K f_0(\lambda r, \lambda a)}{b\lambda f_0(\lambda b, \lambda a) \frac{d}{d\lambda} U(\lambda, a)} \qquad (204)$$

where the first term on the right has been filled in from direct considerations of the steady state ultimately attained.

The implication of this last result is the purely mathematical theorem indicated by

$$\frac{h \log \frac{b}{r} + \frac{K}{b}}{h \log \frac{b}{a} + \frac{K}{b}} = -2 \sum_{1}^{\infty} \frac{K f_0(\lambda r, \lambda a)}{b\lambda f_0(\lambda b, \lambda a) \frac{d}{d\lambda} U(\lambda, a)} \qquad (205)$$

the summation being with regard to all the positive roots of the equation (202). It might be useful to give a direct demonstration of this result.

We assume the possibility of the expansion

$$f(r) = A_1 f_0(\lambda_1 r, \lambda_1 a) + A_2 f_0(\lambda_2 r, \lambda_2 a) + \cdots = \sum_{1}^{\infty} A_m f_0(\lambda_m r, \lambda_m a) \qquad (206)$$

where $\lambda_1, \lambda_2 \ldots$ etc. are the successive roots of the equation (202). From this expansion we have

$$\int_a^b f(r) f_0(\lambda_n r, \lambda_n a) r\, dr = \sum_{1}^{\infty} \int_a^b A_m f_0(\lambda_m r, \lambda_m a) . f_0(\lambda_n r, \lambda_n a) r\, dr \qquad (207)$$

To evaluate the integral on the right of this equation we observe that $u_m \equiv f_0(\lambda_m r, \lambda_m a)$, $u_n = \text{do}$, being solutions of Bessel's equation, satisfy respectively *

$$\frac{d^2u_m}{dr^2} + \frac{1}{r}\frac{du_m}{dr} + \lambda_m^2 u_m = 0.$$

$$\frac{d^2u_n}{dr^2} + \frac{1}{r}\frac{du_n}{dr} + \lambda_n^2 u_n = 0.$$

whence $r(u_n u_m'' - u_m u_n'') + (u_n u_m' - u_m u_n') + (\lambda_m^2 - \lambda_n^2)u_m u_n r = 0.$

so that

$$(\lambda_n^2 - \lambda_m^2)\int_a^b u_m u_n r\,dr = \left[r\left(u_n\frac{du_m}{dr} - u_m\frac{du_n}{dr}\right)\right]_a^b. \qquad (208)$$

At $r = a$, $u_n = u_m = 0,$

so that the value of the form on the right of (208) at $r = a$ is zero.

$$\frac{du_m}{dr} = -\lambda_m G_1(\lambda_m r)J_0(\lambda_m a) + \lambda_m J_1(\lambda_m r)G_0(\lambda_m a)$$

and therefore

$$\left(u_n\frac{du_m}{dr}\right)_b = f_0(\lambda_n b, \lambda_n a)\{-\lambda_m f_1(\lambda_m b, \lambda_m a)\}$$
$$= -\frac{h}{K} f_0(\lambda_n b, \lambda_n a).f_0(\lambda_m b, \lambda_m a)$$

since λ_m is a root of the equation (202).

It is clear from the symmetrical form of this evaluation that $\left(u_m\frac{du_n}{dr}\right)_b$ has the same value and consequently we have

$$\int_a^b u_m u_n r\,dr = 0, \qquad \lambda_n \neq \lambda_m.$$

When $\lambda_n = \lambda_m$ this result is replaced by

$$\int_a^b u_m^2 r\,dr = \mathcal{L}_{\lambda_n \to \lambda_m} \frac{1}{\lambda_n^2 - \lambda_m^2}\left[r\left(u_n\frac{du_m}{dr} - u_m\frac{du_n}{dr}\right)\right]_a^b$$
$$= \frac{1}{2\lambda_m}\left[r\left(\frac{\partial u_m}{\partial\lambda}.\frac{du_m}{dr} - u_m\frac{\partial}{\partial\lambda}\frac{du_m}{dr}\right)\right]_a^b, \ \lambda = \lambda_m. \qquad (209)$$

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* cf. G.M.M pp 69.70.

It is readily verified that

$$\frac{\partial u}{\partial \lambda} = -\, r f_1(\lambda r, \lambda a) + a f_1(\lambda a, \lambda r)$$

Thus $\left(\frac{\partial u}{\partial \lambda}\right)_{r=a} = 0$ and since $(u_m)_a = 0$, the value of the expression in the square bracket at the lower limit is zero.

Again

$$\left(\frac{du}{dr}\right)_b = -\,\lambda f_1(\lambda b, \lambda a)$$

Hence we have

$$\int_a^b u_m^2 \cdot r\,dr = \frac{b}{2\lambda_m}\left[-\lambda f_1(\lambda b,\lambda a)\frac{\partial}{\partial\lambda} f_0(\lambda b,\lambda a) - f_0(\lambda b,\lambda a)\frac{\partial}{\partial\lambda}\left\{-\lambda f_1(\lambda b,\lambda a)\right\}\right]$$

$$= \frac{b f_0(\lambda b,\lambda a)}{2\lambda}\left[-\frac{h}{K}\,\frac{\partial}{\partial\lambda} f_0(\lambda b,\lambda a) + \frac{\partial}{\partial\lambda}\left\{\lambda f_1(\lambda b,\lambda a)\right\}\right]$$

$$= \frac{b f_0(\lambda b,\lambda a)}{2K\lambda}\,\frac{\partial}{\partial\lambda}\left\{K\lambda f_1(\lambda b,\lambda a) - h f_0(\lambda b,\lambda a)\right\}$$

$$= \frac{b f_0(\lambda b,\lambda a)}{2K\lambda}\,\frac{\partial}{\partial\lambda}\left\{U(\lambda,a)\right\} \qquad (210)$$

and finally

$$A_m = \frac{2K\lambda \int_a^b f(r)\,u_m \cdot r\,dr}{b f_0(\lambda b,\lambda a)\,\frac{\partial}{\partial\lambda}\left\{U(\lambda,a)\right\}} \qquad (211)$$

We consider next the special form taken by A_m when $f(r)$ is given by

$$f(r) = \frac{h \log\frac{b}{r} + \frac{K}{b}}{h \log\frac{b}{a} + \frac{K}{b}}$$

It is easily proved that

$$\int_a^b \left(h \log\frac{b}{r} + \frac{K}{b}\right) G_0(\lambda r)\, r\,dr$$

$$= \frac{1}{\lambda^2}\left\{\left[K\lambda G_1(\lambda b) - h G_0(\lambda b)\right] - \left[\frac{K}{b} + h\log\frac{b}{a}\right]\lambda a G_1(\lambda a) + h G_0(\lambda a)\right\}$$

Likewise

$$\int_a^b \left(h \log \frac{b}{r} + \frac{K}{b}\right) J_0(\lambda r)\, r\, dr$$

$$= \frac{1}{\lambda^2}\left\{ \left[K\lambda J_1(\lambda b) - h J_0(\lambda b)\right] - \left[\frac{K}{b} + h \log \frac{b}{a}\right] \lambda a J_1(\lambda a) + h J_0(\lambda a) \right\}$$

Multiplying the first of these results by $J_0(\lambda a)$, the second by $G_0(\lambda a)$, and remembering that λ is a root of the equation (202), we have

$$\int_a^b \frac{h \log \frac{b}{r} + \frac{K}{b}}{h \log \frac{b}{a} + \frac{K}{b}}\, u_m(r)\, r\, dr = -\frac{1}{\lambda^2}$$

and therefore

$$A_m = -\frac{2K}{b\lambda f_0(\lambda b, \lambda a)\frac{\partial}{\partial \lambda}\{U(\lambda, a)\}} \qquad (212)$$

exactly as required by the equation (205) above.

An important verification of the result (204) is obtained if we assume that a, r and b are all very large, with $(b-a)$ finite and $a \leqq r \leqq b$. In these circumstances we can replace each Bessel Function by the first term of its asymptotic expansion and the form then taken by the result should give the solution for the corresponding plane problem. Thus we find

$$f_0(\lambda r, \lambda a) \text{ becomes } \frac{\sin\lambda(a-r)}{\lambda\sqrt{ar}}; \quad f_1(\lambda r, \lambda a) \text{ becomes } \frac{\cos\lambda(a-r)}{\lambda\sqrt{ar}}$$

If these substitutions be made and if the origin is suitably shifted after substitution it will be found that the result (204) reduces to

$$V = \theta_0\left\{\frac{K + h(b-r)}{K + bh} - 2\sum_1^{\infty} e^{-\kappa\lambda^2 t} \frac{h^2 + K^2\lambda^2}{h(K + bh) + bK^2\lambda^2} \frac{\sin\lambda r}{\lambda}\right\} \qquad (213)$$

the summation being with regard to all the positive roots of the equation

$$K\lambda \cos\lambda b + h \sin\lambda b = 0. \qquad (214)$$

The solution (213) thus obtained is that for a finite rod of length b one end of which $r=0$ is kept at constant temperature θ_0 while the other end $r=b$ radiates to a medium at zero temperature. The form of solution is in entire agreement with that obtained directly for this problem in a previous chapter.*

Problems involving the long solid cylinder composed of two materials of different conductivity.

The case contemplated is that in which we have an infinitely long cylindrical core of radius a and conductivity C_1 surrounded by a coaxial layer of thickness $(b-a)$ and conductivity C_2.+

Postulating first a periodic source of strength qe^{ikt} at the surface $r=r_1$ within the inner medium, the surface $r=b$ of the cylinder being kept at zero temperature, we consider the fundamental trains emanating from the source and their continuations by reflection and transmission at the various boundaries; in this way we find that the effects in the various parts of the field can be represented by - the notation being that adopted in previous two-medium problems -

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* R.I p.946. + The notation C_1 &c for conductivity is introduced to avoid possible confusion with K_1 &c used in Bessel Function Theory.

$$\left.\begin{aligned} 0 \leqq r \leqq r_1, \quad & v_{1_i} = \frac{q r_1}{C_1} e^{ikt} I_0(i\lambda r)\left\{K_0(i\lambda r_1) + A I_0(i\lambda r_1)\right\} \\ r_1 \leqq r \leqq a, \quad & v_{1_0} = \frac{q r_1}{C_2} e^{ikt} I_0(i\lambda r_1)\left\{K_0(i\lambda r) + A I_0(i\lambda r)\right\} \end{aligned}\right\} \qquad (215)$$

$$a \leqq r \leqq b, \quad v_2 = \frac{q r_1}{C_2} e^{ikt} \frac{A' I_0(i\lambda r_1)}{I_0(\mu i\lambda b)} \left\{ K_0(\mu i\lambda r) I_0(\mu i\lambda b) - K_0(\mu i\lambda b) I_0(\mu i\lambda r) \right\} \qquad (216)$$

where $\mu = \sqrt{\frac{\kappa_1}{\kappa_2}}$. Here it is to be understood that the second term in each of the first two of these results includes (i) the train reflected from the surface $r=a$ at incidence of the original diverging train and (ii) all the trains partially retransmitted inwards across the surface $r=a$ from the outer medium after one or more reflections at the surface $r=b$. Similarly the expression for v_2 in (216) includes the first transmitted continuation of the original diverging train and all the other trains set up by the reflections of this one at the surfaces $r=b$ and $r=a$. To simplify the analysis we have chosen forms for v_{1_i} and v_{1_0} that at once satisfy the conditions at $r=r_1$, and a form for v_2 that satisfies the condition at $r=b$. It remains to choose A and A' to satisfy the double condition at the surface $r=a$ viz.

$$v_{1_0} = v_2 \ ; \quad C_1 \frac{\partial v_1}{\partial r} = C_2 \frac{\partial v_2}{\partial r}$$

It is found that

with $\quad d = C_1 I_1(i\lambda a) f_0(\mu\lambda a, \mu\lambda b) - i\mu C_2 I_0(i\lambda a) f_1(\mu\lambda a, \mu\lambda b)$

$$\left.\begin{aligned} A &= \frac{C_1 K_1(i\lambda a) f_0(\mu\lambda a, \mu\lambda b) + i\mu C_2 K_0(i\lambda a) f_1(\mu\lambda a, \mu\lambda b)}{d} \\ A' &= \frac{C_2 I_0(\mu i\lambda b)/i\lambda a}{d} \end{aligned}\right\} \qquad (217)$$

With these forms substituted and the necessary reduction performed we find that the results (215),(216) become

$$0 \leqq r \leqq r_1, \quad v_1 = \frac{q r_1}{C_1} e^{ikt} J_0(\lambda r) \frac{-C_1 f_1(\lambda a, \lambda r_1) f_0(\mu\lambda a, \mu\lambda b) + \mu C_2 f_0(\lambda a, \lambda r_1) f_1(\mu\lambda a, \mu\lambda b)}{F(\lambda)}$$

$$r_1 \leqq r \leqq a, \quad v_1 = \frac{q r_1}{C_1} e^{ikt} \left\{ \text{same expression, } r \text{ and } r_1 \text{ interchanged} \right\} \qquad (218)$$

$$a \leqq r \leqq b, \quad v_2 = -\frac{q r_1 e^{ikt} J_0(\lambda r_1)}{\lambda a} \frac{f_0(\mu\lambda r, \mu\lambda b)}{F(\lambda)}. \qquad (219)$$

where $$F(\lambda) = C_1 J_1(\lambda a) f_0(\mu\lambda a, \mu\lambda b) - \mu C_2 J_0(\lambda a) f_1(\mu\lambda a, \mu\lambda b) \qquad (220)$$

These results apply when the periodic source is located within the inner medium. It is of equal importance that we obtain the corresponding results when the source is in the outer medium.

Beginning with the periodic surface source at $r = r_1$, $a \leqq r_1 \leqq b$, we write down tentative solutions in terms of the elementary wave trains as follows

$$0 \leqq r \leqq a, \quad v_1 = \frac{E' q r_1}{C_1} e^{ikt} \frac{I_0(i\lambda r)}{I_0(\mu i\lambda b) + E K_0(\mu i\lambda b)} \qquad (221)$$

$$\left.\begin{aligned} a \leqq r \leqq r_1, \quad v_2 &= \frac{q r_1}{C_2} e^{ikt} \frac{\{I_0(\mu i\lambda r) + E K_0(\mu i\lambda r)\} f_0(\mu\lambda r_1, \mu\lambda b)}{I_0(\mu i\lambda b) + E K_0(\mu i\lambda b)} \\ r_1 \leqq r \leqq b, \quad v_2 &= \frac{q r_1}{C_2} e^{ikt} \left\{ \text{same expression, } r \text{ and } r_1 \text{ interchanged} \right\}. \end{aligned}\right\} (222)$$

In writing down these results we have been guided by the general principle of the interchangeability of r and r_1 in the forms for v_2 in the regions $r \lessgtr r_1$, by the zero temperature condition at $r=b$ and by the dual condition at the surface $r=r_1$. E' is of the nature of transmission coefficient from the outer to the inner medium, E of the nature of reflection coefficient at the surface $r=a$ within the outer medium. When we take into account the conditions at this surface we find

$$\left.\begin{aligned} E &= \frac{-C_1 I_1(i\lambda a) I_o(\mu i\lambda a) + \mu C_2 I_o(i\lambda a) I_1(\mu i\lambda a)}{d'} \\ E' &= \frac{C_1}{i\lambda a d'} \end{aligned}\right\} \qquad (223)$$

where $$d' = C_1 I_1(i\lambda a) K_o(\mu i\lambda a) + \mu C_2 I_o(i\lambda a) K_1(\mu i\lambda a)$$

When these forms are inserted in (221),(222) we obtain

$$0 \leqq r \leqq a, \qquad v_1 = -\frac{q r_1 e^{ikt}}{\lambda a} \frac{J_o(\lambda r) f_o(\mu\lambda r_1, \mu\lambda b)}{F(\lambda)} \qquad (224)$$

$$a \leqq r \leqq r_1, \qquad v_2 = \frac{q r_1}{C_2} e^{ikt} \frac{f_o(\mu\lambda r_1, \mu\lambda b) F(\mu\lambda a, \mu\lambda r)}{F(\lambda)}$$

$$r_1 \leqq r \leqq b, \qquad v_2 = \frac{q r_1}{C_2} e^{ikt} \left\{ \text{same expression, } r \text{ and } r_1 \text{ interchanged} \right\}, \qquad (225)$$

$F(\lambda)$ having the form defined in (220) above*

Proceeding next to solutions of the type $\frac{1}{\pi}\int_o^\infty v\,dk$ corresponding to the existence of instantaneous surface sources within the inner and outer media respectively we find in succession noting considerable simplifications in form when λ is a root of $F(\lambda)=0$.

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* $F(\lambda) \equiv F(\mu\lambda a, \mu\lambda b) = C_1 J_1(\lambda a) f_o(\mu\lambda a, \mu\lambda b) - \mu C_2 J_o(\lambda a) f_1(\mu\lambda a, \mu\lambda b)$.

Inst. Source in Inner Medium.

$$\left.\begin{aligned} 0 \leqq r \leqq r_1, \quad & \vartheta_1 = \frac{2q\kappa_1 r_1}{a} \sum_1^\infty e^{-\kappa\lambda^2 t} \frac{f_0(\mu\lambda a, \mu\lambda b) J_0(\lambda r_1) J_0(\lambda r)}{J_0(\lambda a)\, F'(\lambda)}. \\ r_1 \leqq r \leqq a, \quad & \vartheta_1 = \text{the same expression.} \end{aligned}\right\} \quad (226b)$$

$$a \leqq r \leqq b, \quad \vartheta_2 = \frac{2q\kappa_1 r_1}{a} \sum_1^\infty e^{-\kappa\lambda^2 t} \frac{J_0(\lambda r_1) f_0(\mu\lambda r, \mu\lambda b)}{F'(\lambda)} \quad (227)$$

Inst. Source in Outer Medium.

$$0 \leqq r \leqq a, \quad \vartheta_1 = \frac{2q\kappa_1 r_1}{a} \sum_1^\infty e^{-\kappa\lambda^2 t} \frac{J_0(\lambda r) f_0(\mu\lambda r_1, \mu\lambda b)}{F'(\lambda)} \quad (228)$$

$$\left.\begin{aligned} a \leqq r \leqq r_1, \quad & \vartheta_2 = \frac{2q\kappa_1 r_1}{a} \sum_1^\infty e^{-\kappa\lambda^2 t} \frac{J_0(\lambda a)}{f_0(\mu\lambda a, \mu\lambda b)}. \\ & \qquad \times \frac{f_0(\mu\lambda r_1, \mu\lambda b) f_0(\mu\lambda r, \mu\lambda b)}{F'(\lambda)} \\ r_1 \leqq r \leqq b, \quad & \vartheta_2 = \text{the same expression.} \end{aligned}\right\} \quad (229)$$

the summation in each case being with regard to all the positive roots of the equation $F(\lambda) = 0$.

The group of results exhibited above is of considerable importance. By using the known volume integral process we can at once obtain the solutions corresponding to any symmetrical initial state prescribed throughout the cylindrical core and surrounding sheath. Thus e.g. if initially we have

$$\vartheta = f_1(r), \quad 0 \leqq r \leqq a \quad ; \quad \vartheta = f_2(r), \quad a \leqq r \leqq b,$$

the effect at any point in the inner medium at any later time is obtained if we replace q in (226) by $\frac{C_1}{\kappa_1} f_1(r_1) dr_1$ and integrate with respect to r_1 from 0 to a ; replace q in (228) by $\frac{C_2}{\kappa_2} f_2(r_1) dr_1$ and integrate from a to b and add these results. In like manner also we could obtain the effect at any later time in the outer medium.

In the above results we have all the material for the investigation e.g. of heat losses from long cylinders surrounded by coaxial layers of lagging. The arithmetical examination of the results obtained for various thicknesses of the lagging and for various values of the relative conductivity might lead to results of interest and of practical value. This aspect of the case is reserved for future consideration.

It will be remembered also that the solutions here obtained only apply on the assumption of the zero temperature surface condition at the outer boundary. Clearly the tentative solutions given in (215-6), (221-2) and all the subsequent results would be of entirely different form were this condition otherwise. The general method of approach to any other problem involving a different boundary condition has however been sufficiently indicated.

Spherical-Cylindrical analogues.

In Green's fourth paper[*] the attention is drawn to striking structure resemblances between the solutions of certain cylindrical problems on the one hand and the solutions of the corresponding spherical flow cases on the other. These

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[*] G.IV p 241.

resemblances might conveniently be illustrated by the following scheme, prepared in connection with the second of the hollow cylinder problems investigated above.

	Cylindrical.	Spherical.
Fundamental train from source $\theta_0 e^{ikt}$ at surface $r=a$.	$\theta_0 e^{ikt} \dfrac{K_0(i\lambda r)}{K_0(i\lambda a)}$.	$\theta_0 e^{ikt}\left(\dfrac{a}{r}\right)^{\frac{1}{2}} \dfrac{K_{\frac{1}{2}}(i\lambda r)}{K_{\frac{1}{2}}(i\lambda a)}$.
Reflection coefficient at surface $r=a$.	$-I_0(i\lambda a) \div K_0(i\lambda a)$.	$-I_{\frac{1}{2}}(i\lambda a) \div K_{\frac{1}{2}}(i\lambda a)$.
Reflection coefficient at surface $r=b$, radiation condition.	$\dfrac{Ki\lambda K_1(i\lambda b) - h K_0(i\lambda b)}{Ki\lambda I_1(i\lambda b) - h I_0(i\lambda b)}$	$\dfrac{Ki\lambda . K_{\frac{3}{2}}(i\lambda b) - h K_{\frac{1}{2}}(i\lambda b)}{Ki\lambda I_{\frac{3}{2}}(i\lambda b) - h I_{\frac{1}{2}}(i\lambda b)}$
Effect of periodic source.	$\theta_0 e^{ikt} \dfrac{u_n(\lambda, r)}{u_n(\lambda, a)}$	$\theta_0 e^{ikt}\left(\dfrac{a}{r}\right)^{\frac{1}{2}} \dfrac{u_{n+\frac{1}{2}}(\lambda . r)}{u_{n+\frac{1}{2}}(\lambda, a)}$

The subscripts n, $n+\frac{1}{2}$ denote that in the passage from the cylindrical case to the spherical any Bessel function of order n is to be replaced by the corresponding function of order $n+\frac{1}{2}$.

It is clear that the analogy persists right up to the final solutions. Hence if we adapt in this way the result (104) we obtain the solution for the case of the hollow sphere, the inner surface $r=a$ of which is kept at the constant temperature θ_0 while the outer surface $r=b$ radiates to a medium at the temperature zero. The adaptation is easily carried out and leads to a result that is readily verified. It is of greater interest however to show the application of the transformation to the group of cylindrical results (226)-(229) and in this way to obtain a set of spherical flow solutions known on other grounds to be correct.

We find e.g. that in the transition the cylindrical functions $f_0(\mu\lambda r_1, \mu\lambda b)$, $f_1(\mu\lambda a, \mu\lambda b)$ are replaced by the spherical functions $f_{\frac{1}{2}}(\mu\lambda r_1, \mu\lambda b)$, $f_{\frac{3}{2}}(\mu\lambda a, \mu\lambda b)$ respectively, where

$$f_{\frac{1}{2}}(\mu\lambda r_1, \mu\lambda b) = G_{\frac{1}{2}}(\mu\lambda r_1) J_{\frac{1}{2}}(\mu\lambda b) - J_{\frac{1}{2}}(\mu\lambda r_1) G_{\frac{1}{2}}(\mu\lambda b).$$

$$= \frac{1}{\mu\lambda\sqrt{br_1}} \sin \mu\lambda(b-r_1)$$

$$f_{\frac{3}{2}}(\mu\lambda a, \mu\lambda b) = G_{\frac{3}{2}}(\mu\lambda a) J_{\frac{1}{2}}(\mu\lambda b) - J_{\frac{3}{2}}(\mu\lambda a) G_{\frac{1}{2}}(\mu\lambda b)$$

$$= -\frac{1}{\mu\lambda\sqrt{ab}} \left\{ \frac{\sin \mu\lambda(b-a)}{\mu\lambda a} + \cos \mu\lambda(b-a) \right\}$$

Thus we find that $F(\lambda)$ as defined in (220) becomes $\frac{1}{\mu\lambda a}\sqrt{\frac{2}{\pi\lambda b}}\, f_2(\lambda)$ where

$$f_2(\lambda) = (C_1 - C_2)\frac{\sin \lambda a}{\lambda a} \sin \mu(b-a)\lambda - C_1 \cos \lambda a . \sin \mu(b-a)\lambda$$
$$- \mu C_2 \sin \lambda a . \cos \mu(b-a)\lambda.$$

Hence finally transformed the result (229) of the set yields

$$v_2 = \frac{2qkr_1}{r} \sum_1^\infty e^{-\kappa\lambda^2 t} \frac{\sin \lambda a . \sin \mu(b-r_1)\lambda . \sin \mu(b-r)\lambda}{\sin \mu(b-a)\lambda\, f_2'(\lambda)} \quad \ldots (230)$$

where the summation is with regard to all the positive roots of the equation $f_2(\lambda) = 0$.

This result is in entire agreement with that obtained by Green in the direct treatment of the spherical problem.* In exactly the same way we could transform the results (226), (227) (228) to obtain the complete set of solutions relating to the sphere surrounded by the concentric sheath of different

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* G III pp 258-9.

conductivity. Likewise by the adaptation here indicated the solution of <u>any</u> spherical flow problem could be obtained by first obtaining that of its ~~spherical~~ cylindrical analogue. In this connection the problem of the finite sphere imbedded in an infinite mass of material of different conductivity, a case discussed in a previous chapter, is noted for further investigation.

Continuous Heat Sources.

In the problems discussed above the various sources postulated have all been of the periodic or instantaneous initial type. It is useful to show how, by direct time integration, solutions corresponding to the existence of continuous or permanent sources may be obtained.

If e.g. in the result (195) - relating to the instantaneous surface source within the material of the hollow cylinder whose surfaces are kept at zero temperature - we replace q by $q\,dt'$, t by $(t-t')$ and integrate with regard to t' from o to t, we obtain the solution for the case of a surface source emitting q units of heat per unit area per second from the instant $t=o$ onwards. The integration is easily effected. Writing down the form the result takes when t becomes indefinitely great, we obtain the " steady state" solution **V** as

$$V = \frac{2qr_1}{K}\sum_{1}^{\infty}\frac{J_0^2(\lambda b)}{J_0^2(\lambda a) - J_0^2(\lambda b)} f_0(\lambda r_1, \lambda a). f_0(\lambda r, \lambda a). \quad . \quad . \quad (231)$$

the summation being with regard to all the positive roots of the equation (194).

This solution however is one that can be obtained directly from the differential equation and boundary conditions of the steady state and may be exhibited as

$$\left.\begin{array}{ll} r \leqq r_1, & V_i = \dfrac{qr_1}{K}\dfrac{\log b - \log r_1}{\log b - \log a}\log\dfrac{r}{a} \\[2ex] r \geqq r_1 & V_o = \dfrac{qr_1}{K}\dfrac{\log a - \log r_1}{\log b - \log a}\log\dfrac{r}{b} \end{array}\right\} \quad . \quad . \quad . \quad . (232)$$

The identity of the solutions (231) and (232) is readily shown. If we attempt the development

$$F(r) \equiv V = \sum A_m f_0(\lambda r, \lambda a), \qquad a \leqq r \leqq b,$$

the summation being with respect to the positive roots of (194), and **V** having the form shown above we find, by the usual method of determining the coefficients, that A_m is given by

$$A_m = \frac{\int_a^{r_1} f_0(\lambda r, \lambda a). V_i r dr + \int_{r_1}^{b} f_0(\lambda r, \lambda a). V_o r dr}{\int_a^b \{f_0(\lambda r, \lambda a)\}^2 r dr}$$

$$= \frac{J_0^2(\lambda b) f_0(\lambda r_1, \lambda a)}{J_0^2(\lambda a) - J_0^2(\lambda b)}, \qquad \text{after necessary reduction.}$$

Clearly this form of A_m demonstrates the equivalence of the two solutions.

It will thus be seen that (231) gives the correct normal- function development corresponding to the existence of a continuous source within the material of a hollow cylinder whose surfaces are kept at zero temperature. If q and K be replaced by the corresponding electrostatic constants the development in question becomes that for the potential between coaxial conducting cylinders kept at zero potential due to a coaxial distribution of electric charge.

Clearly we might apply the process indicated here to any of the other cylindrical, spherical or plane flow solutions already obtained and thereby obtain in developed form the " normal function" expansion of the steady state in each case. The mathematical agreement of the developed form of this state with the undeveloped form obtained from first principles would seem to afford important confirmation of the various results to which our theory has led.