Essays on Macroeconomic Policy

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Abstract

This thesis contains three chapters. All chapters study different aspects of macroeconomic policy.

The first chapter studies discretionary monetary policy in an economy where economic agents have quasi-hyperbolic discounting. It demonstrates that a benevolent central bank is able to keep inflation under control for a wide range of discount factors. If the central bank, however, does not adopt the household's time preferences and tries to discourage early-consumption and delayed-saving, then a marginal increase in steady state output is achieved at the cost of a much higher average inflation rate. Indeed, it shows that it is desirable from a welfare perspective for the central bank to quasi-hyperbolically discount by more than households do. Welfare is improved because this discount structure emphasizes the current-period cost of price changes and leads to lower average inflation. It contrasts the results with those obtained when policy is conducted according to a Taylor-type rule.

The second chapter analyses the effect of endogenous discounting on wealth inequality in an endowment economy with heterogeneous agents, subject to occasionally binding borrowing constraint. It demonstrates that introduction of Uzawa-type preferences may launch a strong redistribution mechanism leading to high equilibrium real interest rate and a more dispersed wealth distribution in comparison to the model with standard preferences.

The third chapter studies macroprudential policy in a macro-model with a heterogeneous banking sector, prone to asymmetric information and moral hazard a la Boissay et.al. (2016). This model is shown to generate financial crises when a sequence of small positive technology shocks can lead to an increase in lending, as well as to a reduction in all market rates. This paper investigates a scope for a macropridential policy that would reduce probability of a financial crisis, but not lead to a too sharp reduction in a social welfare. It demonstrates that the introduction of a direct proportional tax on interbank lending can substantially reduce the amount of credit and reduce probability of a financial crisis.

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I declare that, except where explicit reference is made to the contribution of others, that this dissertation is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

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Extended Abstract

This thesis contains three chapters. All chapters study different aspects of macroeconomic policy.

The first chapter studies monetary policy in an economy where households have quasi-geometric preferences. In such an economy consumers require instantaneous gratification and will delay decisions that seem inferior from today's perspective. Although they value saving for the future income, they cannot resist to keep consuming today and plan to start saving adequately from tomorrow. This type of preferences has been extensively studies in behavioral economics, but an investigation of its implications for the conduct of monetary policy is novel.

This chapter examines quasi-geometric discounting in a New Keynesian business cycle model. Although the model is standard in many respects, quasi-geometric discounting introduces important complications because the household's decision problem is no longer time-consistent. Unlike previous studies that have focused largely on the effect that quasi-hyperbolic discounting has on consumption, saving, and labour supply, it focuses on its implications for how the central bank should conduct monetary policy. In the absence of an efficient subsidy to offset the monopolistic distortion, discretionary monetary policy gives rise to both an inflation bias and a stabilization bias. It quantifies the impact that the household's quasi-hyperbolic discounting has on how monetary policy is conducted and quantify the magnitude of the discretionary inflation bias. Next, it allows the central bank to also have quasi-hyperbolic discounting and examine the implications the central bank's discounting has for monetary policy. Lastly, it asks whether it is desirable for the central bank to be benevolent, i.e., whether it is desirable for the central bank to quasi-hyperbolically discount the future by more, less, or at the same rate as households.

The chapter has three main results. First, it finds that quasi-hyperbolic households over-consume and under-save in equilibrium, leading to a capital stock that is smaller

than it would be if households discounted geometrically. Second, although discretionary monetary policy continues to result in positive average inflation, because the central bank tries to use inflation surprises to raise output (discretionary inflation bias), the size of the discretionary inflation bias is somewhat smaller when households have quasi-hyperbolic discounting. This result emerges because firms make their pricing and production decisions to maximize their equity-value. Because it is costly to change prices and their equity-holders have quasi-hyperbolic discounting, firms choose to make smaller price changes in response to shocks and to spread price-changes out over time. Allowing the central bank to have quasi-hyperbolic discounting operates in a qualitatively similar way, and also leads to a smaller inflation bias. Third, it shows that not only is it desirable for the central bank to have quasi-hyperbolic discounting, but that it should discount by more than households do. By doing so average inflation is lowered and becomes closer to the Ramsey optimal rate of zero, raising household welfare. This result parallels Rogoff (1985), who showed that discretionary outcomes could be improved by appointing an optimally conservative central banker that cares more about stabilizing inflation than society. With quasi-hyperbolic discounting the central bank cares relatively more about costly prices changes (inflation) in the present, leading it to behave as if it cares more about stabilizing inflation than society does.

The second chapter studies a different type of unconventional household preferences and their implications for dynamics of the economy and for a fiscal policy. The economic environment is very different from the one in the first chapter as this chapter considers a model with heterogenous agents and studies distributional effects. The household preferences are of Uzawa-type, they are characterized by the endogenous rate of time preference that depends on the consumption path. These preferences imply that households with relatively high income and so high consumption discount future by more than households with less resources.

Higher discounting of relatively rich households should make them to consume more and save less and should imply less unequal wealth distribution than would be observed in an economy with constant time-preferences of households. In contrast, this chapter shows that Uzawa-type preferences with CRRA utility can generate a long-run wealth distribution that is substantially more dispersed than the one in an economy with standard preferences.

This chapter analyses a continuous-time economy with incomplete markets and stochastic shocks to exogenous incomes following a two-state Markov process, as in Huggett (1993) but with an exogenous constraint on borrowing. There is unique stationary distribution of wealth, with individuals mobile across consumption, income, and wealth levels. The chapter investigates the effect of endogenous discounting on wealth inequality and compares two models. In the first model, the instantaneous discount rate is a locally linear function of consumption, as in Wang (2007). It yields more concentrated wealth distribution than the model with standard preferences. In the second model – which is the model of main interest – the instantaneous discount rate is an S-shaped function of consumption, meaning that the agents in the tails of the wealth distribution have diminishing marginal discount rate. This model is characterized by a substantially higher equilibrium interest rate and a more dispersed wealth distribution, relative to the model with locally-linear discount rate.

To get intuition for these results it is helpful to trace the differences in the behavior of the negative-wealth agents, who demand loans, and the positive-wealth agents, who supply loans, across three economies: an economy with the standard, constant discount rate (CDR), an economy with locally linear discount rate (LDR), and an economy with S-shaped discount rate (SDR), – when all three economies have the same average discount rate.

In the LDR economy, the densely populated small-positive-wealth group has consumption above the benchmark level and so discounts future by more than their CDR counterpart. It therefore offers a lower supply of loans and thus creates an upward pressure on the equilibrium interest rate relative to the CDR. This effect is substantially amplified by the behavior of wealthy agents. The negative-wealth LDR agents, however, become more patient than their CDR counterparts and thus have a lower demand for loans, leading to a downward pressure on the equilibrium interest rate relative to the CDR. The two opposite effects on interest rate coming from the two tail groups nearly offset each other, and the moderate increase in equilibrium interest rate is implied by the behavior of central group of agents who have only slightly higher impatience than in the CDR economy. With equilibrium interest rate only slightly higher than under CDR, the demand for loans is still higher, as the negative-wealth agents need to refinance their loans, but there is also an income effect forcing them to reduce consumption. The opposite is true for the high-positive-wealth agents and so the aggregate effect is moderate.

The balance of these opposite pressures on interest rate changes significantly if the discount rate of the two tail groups changes only moderately, as in the case with the SDR, despite the fact that the central group with small positive wealth behaves in the same way as in the LDR economy. Given the same interest rate, smaller reduction in patience of the negative-wealth agents implies less reduction in the loan demand and, therefore, weaker downward pressure on the equilibrium interest rate. Similarly, the high-positive-wealth group produces only moderate reduction in loan supply and less upward pressure on the interest rate. However, the reduction in supply is greater than the reduction in demand. Higher interest rate results in a substantially higher debt of the negative-wealth group, and in a higher proportion of population in the left-hand tail. Self-reinforcing mechanism of higher demand for loans to refinance the existing debt and stronger upward pressure on the interest rate produced by this group results in a higher equilibrium interest rate.

The central group with moderate positive wealth generates a moderate increase in general equilibrium interest rate, while the interactions of the two tail groups limit interest rate increase in the LDR economy, but amplify its increase substantially in the SDR economy. The corresponding redistribution of population across wealth levels accompanies the latter process and reinforces the equilibrium effect. Thus, the UP can generate a more dispersed wealth distribution, with fatter tails, than the standard preferences, under a plausible assumption of declining marginal impatience above and below the average discount rate.

The chapter demonstrates that a consumption tax reduces the welfare inequality. Capital income tax increases current consumption, reduces the current saving but does not affect future output. In this economy life-time welfare unambiguously rises because of higher current consumption.

Finally, we demonstrate that in production economy the redistribution effect of Uzawa-type preferences is mitigated. In contrast to the endowment economy with zero total wealth movements of population between borrower and lender positions generated and skews population distribution towards its higher end. No large movements between borrowers and savers are longer possible and the redistribution mechanism does not engage with a shift in preferences. The mechanism discussed in this chapter may help to understand additional reasons leading to high observed income inequality in developing countries with little production possibilities and binding borrowing constraints.

The third chapter studies macroprudential policy in a macro-model with heterogeneous banking sector subject to asymmetric information and moral hazard a la Boissay et.al. (2016). This model is shown to generate financial crises when a sequence of small positive technology shocks can lead to an increase in lending, but also to a reduction in all market rates. Lower interest rates aggravate the agency problem of banks and the interbank market shut. This leads to sharp reduction in lending, financial crisis and a recession. The model is non-linear, with an occasionally binding constraint, but it allows numerical analysis of the implied probability of a financial crisis.

This chapter investigates a scope for a macroprudential policy that would reduce probability of a financial crisis, while not resulting in too sharp reduction of a social welfare. It demonstrates that an introduction of a direct proportional tax on interbank lending can substantially reduce the amount of credit and result in smaller probability of a financial crisis. The tax affects this probability via two main channels. First, it shifts the 'crisis threshold boundary' up - the interbank market will withstand greater 'overlending' so that economy would need to accumulate greater amount of assets to reduce interest rate sufficiently low for the interbank market to freeze. Second, the level of steady state 'overlending' will reduce: the stochastic steady state level of assets is lower with higher rate of the interbank lending tax. As a result, the 'distance' between the state where the economy spends most of the time to the boundary of a rare event unambiguously rises. Although the speed at which the assets accumulate – the interest rate – rises, this effect is relatively small and the overall effect of the macroprudential policy on the economy is positive. Although one expects that a macroprudential policy may create higher costs for financial intermediation, the chapter finds that in our environment a moderate increase in the tax rate results in higher social welfare in the stochastic steady state.

In this model, higher tax rate on interbank lending lowers both, supply and demand for funds at the interbank market. More banks will leave the interbank market to lend directly to firms, reducing supply of funds. In a model with asymmetric information and moral hazard demand for loans may rise with higher interbank rate, as each bank is able to borrow more due to incentive participation constraint. With higher taxes on the interbank lending, however, the incentive participation constraint for lenders is tightened, so the market funding ratio falls. This effect dominates the overall effect on demand, and demand for funds falls with higher tax on lending. The equilibrium interbank rate increases, and so all other interest rates. The efficiency of the marginal bank rises as more banks switch to finance firms directly. In stochastic steady state the total amount of lending and capital falls, and so does output and labour. Consumption falls as well but the social welfare rises as the disutility of labour dominates the effect on period utility.

Chapter 1

Monetary Policy when Preferences are Quasi-Hyperbolic

Based on joined work with Richard Dennis

Abstract

We study discretionary monetary policy in an economy where economic agents have quasi-hyperbolic discounting. We demonstrate that a benevolent central bank is able to keep inflation under control for a wide range of discount factors. If the central bank, however, does not adopt the household's time preferences and tries to discourage early-consumption and delayed-saving, then a marginal increase in steady state output is achieved at the cost of a much higher average inflation rate. Indeed, we show that it is desirable from a welfare perspective for the central bank to quasi-hyperbolically discount by more than households do. Welfare is improved because this discount structure emphasizes the current-period cost of price changes and leads to lower average inflation. We contrast our results with those obtained when policy is conducted according to a Taylor-type rule.

Keywords: Monetary policy, zero lower bound.

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1.1 Introduction

In a seminal paper, Strotz (1956) suggested that people discount short run and long run with different discounts, they are more impatient in the short run than in the long run. A common example to illustrate it is that someone may prefer £110 in 31 days over £100 in 30 days, but prefer £100 now over £110 tomorrow. Such 'preferences reversals' have been extensively tested in experimental settings and well documented (de Villiers, P. A., & Herrnstein, R. J. (1976), Green et al., 1994; Kirby and Herrnstein, 1995, Ainslie, 1992, 2001).

The indirect evidence on preference reversals, or the 'present bias', has also been presented. Fischer (1999) and O'Donoghue and Rabin (1999c, 2001) describe the problem of procrastination with these preferences. An important application of such preferences is a problem of pension finance, when people do not save enough for their retirement (O'Donoghue and Rabin (1999d). Such behavior has been extensively documented in surveys (Farkas and Johnson, 1997, Bernheim, 1995). Addiction and overconsumption of harmful products is another problem that can be described well by a model with preference reversals (O'Donoghue and Rabin, 1999a, Gruber and Koszegi, 2000, Carillo, 1999).

These reversals are inconsistent with the 'standard discounted utility' model which relies on exponential discounting with constant intertemporal discount rate. Both psychologists and behavioral economists suggested that this evidence is consistent with the rate of time preference which declines with time, or, in other words, it can be captured by the notion that households have hyperbolic discounting. Consumers desire instant gratification (Harris and Laibson, 2001) and they value mechanisms that enable them to better exercise self-control and/or to constrain their future selves (Strotz 1956; Laibson 1997). When they discount the future hyperbolically, households value savings for the future income and insurance that they provide, yet cannot resist splurging a little on consumption today while planning to save for the future tomorrow. If they recognize that this behavior will repeat itself day after day, leading them to over-consume and under-save, then today's household will have an incentive to purchase illiquid assets in order to constrain themselves from over-consuming tomorrow. In principle, the same time-inconsistent behavior applies to other intertemporal decisions, such as the purchase of durable goods, and it can be applied to price-setting, capital accumulation, and inventory management decisions, where the firm is operating for the benefit of its hyperbolic equity-holders.

Although hyperbolic discounting features importantly in behavioral economics (Wilkinson and Klaes, 2017), there are relatively few instances of hyperbolic discounting appearing in general equilibrium macroeconomic contexts. Where hyperbolic discounting is considered it invariably appears in the form of quasi-hyperbolic discounting, which combines the usual geometric discounting with a separate factor that discounts all future periods relative to today (Phelps and Pollak, 1968; Laibson, 1997). Studies that have considered quasi-hyperbolic discounting in macroeconomic models have largely concentrated on the stochastic growth model and focused on the possibility of multiple equilibria arising through strategic interaction between the household and its future self (Krusell and Smith, 2003; Maliar and Maliar, 2005, 2006a). Applications of quasi-hyperbolic discounting include Krusell, Kuruşçu, and Smith (2002), who show that the solution to the planner's problem delivers lower welfare than the competitive equilibrium when households have quasi-hyperbolic discounting, and Graham and Snower (2013), who examine a sticky-wage New Keynesian model and demonstrate that quasi-hyperbolic discounting can overturn the Friedman rule. In Graham and Snower's model households prefer positive inflation because it erodes the real wage over time, leading them to work relatively less today and relatively more in the (quasi-hyperbolically discounted) future. Maliar and Maliar (2006b) build on Angeletos, Laibson, Repetto, Tobacman, and Weinberg (2001) and study a neoclassical growth model with heterogeneous households facing idiosyncratic labor productivity shocks and a borrowing constraint. They find that quasi-hyperbolic discounting has a large impact on the income distribution. Maeda (2018) extends Krusell and Smith (2002) to a monetary economy with a cash-in-advance constraint and shows that this constraint on cash-holdings prevents households from over-consuming in equilibrium and, when the government can only control money growth and not taxes, leads to the Friedman rule holding.

In this paper we examine quasi-hyperbolic discounting in a New Keynesian business cycle model and we explore the implications this form of discounting has for how the central bank should conduct monetary policy. The model is one in which monopolistically competitive firms employ capital and labor to produce goods and who set prices subject to Rotemberg (1982) adjustment costs. Households consume goods and supply labor and they have a portfolio of bonds and equities in which to save. In our benchmark scenario, the central bank conducts monetary policy optimally under discretion. Although the model is standard in many respects, quasi-hyperbolic discounting introduces important complications because the household's decision problem is no longer time-consistent. These complications are compounded by the fact that monetary policy is conducted with discretion. Most dynamic stochastic general equilibrium models with quasi-hyperbolic discounting must be solved numerically, which can be challenging because the strategic interactions between households and their future-selves can give rise to multiple equilibria (Krusell and Smith, 2003). We avoid the indeterminacy associated with log-linearization (Maliar and Maliar, 2006a) by solving our nonlinear model using a global solution method and we obtain a unique stable equilibrium by computing the interior solution to a system of generalized Euler equations (as recommended in Maliar and Maliar, 2005). Although the presence of sticky prices and optimal policymaking greatly complicates our model, we obtain considerable simplification by imposing symmetry on household and firm behavior in equilibrium, thereby precluding equilibria that exhibit heterogeneity.

Unlike previous studies that have focused largely on the effect that quasi-hyperbolic discounting has on consumption, saving, and labour supply, we focus on its implications for how the central bank should conduct monetary policy. In the absence of an efficient subsidy to offset the monopolistic distortion, discretionary monetary policy gives rise to both an inflation bias and a stabilization bias. We quantify the impact that the household's quasi-hyperbolic discounting has on how monetary policy is conducted and quantify the magnitude of the discretionary inflation bias. Next, we allow the central bank to also have quasi-hyperbolic discounting and examine the implications the central bank's discounting has for monetary policy. Lastly, we ask whether it is desirable for the central bank to be benevolent, i.e., whether it is desirable for the central bank to quasi-hyperbolically discount the future by more, less, or at the same rate as households. We contrast our results for discretionary policymaking with those from a Taylor-type rule.

We obtain five main results. First, consistent with previous studies, we find that quasi-hyperbolic households over-consume and under-save in equilibrium, leading to a capital stock that is smaller than it would be if households discounted geometrically.

Second, although discretionary monetary policy continues to result in positive average inflation, because the central bank tries to use inflation surprises to raise output (discretionary inflation bias), the size of the discretionary inflation bias is somewhat smaller when households have quasi-hyperbolic discounting. This result emerges because firms make their pricing and production decisions to maximize their equity-value. Because it is costly to change prices and their equity-holders have quasi-hyperbolic discounting, firms choose to make smaller price changes in response to shocks and to spread price-changes out over time. Allowing the central bank to have quasi-hyperbolic discounting operates in a qualitatively similar way, and also leads to a smaller inflation bias. Third, we show that not only is it desirable for the central bank to have quasi-hyperbolic discounting, but that it should discount by more than households do. By doing so average inflation is lowered and becomes closer to the Ramsey optimal rate of zero, raising household welfare. This result parallels Rogoff (1985), who showed that discretionary outcomes could be improved by appointing an optimally conservative central banker that cares more about stabilizing inflation than society. With quasi-hyperbolic discounting the central bank cares relatively more about costly prices changes (inflation) in the present, leading it to behave as if it cares more about stabilizing inflation than society does. Fourth, with quasi-hyperbolic discounting households receive a pecuniary and a non-pecuniary return to owning stocks (or capital). For even small amounts of quasi-hyperbolic discounting the non-pecuniary component can be big, leading to a large total return that spills over to the return on bonds. Fifth, outcomes generated by the Taylor rule often differ greatly from the optimal discretionary policy. From a welfare perspective, greater hyperbolic discounting by households leads to greater inefficiency of the Taylor rule.

The remainder of the paper is organized as follows. In the following section we present our model, outline the decision problems for households and firms and discuss the first-order conditions that emerge in a symmetric equilibrium. Section 1.3 describes the central bank's decision problem and presents the generalized Euler equations associated with optimal discretionary policy. Section 4 focuses on interest rates and asset prices, illustrating how these are determined when agents have quasi-hyperbolic discounting. Section 5 presents the model's benchmark parameterization. Section 1.6 presents our main simulation results. Section 7 looks at policy delegation, examining the relationship between the household's and the central bank's discount rates. Section 1.7 concludes. Appendices contain derivations of the model's equilibrium conditions under different assumptions regarding capital's ownership, illustrate the solution strategy, and present results on numerical accuracy.

1.2 The model

The economy is populated by households, firms, and a government. Households supply labor and consume a bundle of differentiated goods. Households can save through purchasing (risk-free one-period nominal) bonds and stocks, earning income from their wealth and from working. Unlike many business cycle models, the households in our model have hyperbolic preferences (Laibson, 1997)—applying different discount factors at different points in time. Drawing on Phelps and Pollak (1968) and Laibson (1997), we approximate hyperbolic discounting by the quasi-hyperbolic discounting sequence $\{1, \beta\theta, \beta\theta^2, \beta\theta^3, ...\}$, where $\theta \in (0, 1)$ reflects the usual geometric discounting and β allows short-term payoffs to be discounted more or less heavily relative to geometric discounting. If $\beta \in (0, 1)$, then the short-run discount rate is higher than the long-run discount rate; the opposite is true if $\beta > 1$.

We assume that firms own the capital stock—whose initial level was financed through a stock issuance—and that firms finance capital's accumulation over time through retained earnings. The labor market is perfectly competitive, however firms produce differentiated goods that are aggregated and sold to households. Constraining a firm's pricing decision is a Rotemberg-style (Rotemberg, 1982) quadratic cost to changing prices. The government consists primarily of a central bank that is assumed to conduct policy under discretion by setting the nominal return on the bond in order to maximize household welfare. We also consider the case where monetary policy is conducted according to a Taylor-type rule.

Although our main analysis is conducted on the basis that firms own the capital stock, we could alternatively have assumed that households own the capital stock and that they rent it to firms in a perfectly competitive rental market. We show in Appendices A and B that both ownership structures are equivalent, even when households quasi-hyperbolically discount the future.

1.2.1 Households

There is a unit-measure of identical infinitely-lived households who derive utility from consumption and leisure. The representative household's expected discounted lifetime utility from period t onward is given by

$$\mathcal{U}_t = \mathrm{E}_t \left[u_t + \beta \left(\theta u_{t+1} + \theta^2 u_{t+2} + \theta^3 u_{t+3} + ... \right) \right], \tag{1.1}$$

where u_t represents the instantaneous, or momentary, utility obtained in period t, E_t denotes the mathematical expectation operator conditional upon period-t information, and the parameters satisfy $\theta \in (0,1)$ and $\beta > 0$. Equation (1.1) distinguishes between the rate at which households discount the utility obtained in period $t+1$ relative to period t, which is given by $\beta\theta$, from the rate at which they discount the utility obtained in period $t+k$ relative to period $t+k-1$ ($k > 1$), which is given by θ. Following (Krusell and Smith, 2003), equation (1.1) represents a form of quasi-hyperbolic, or quasi-geometric, discounting. Notice that when $\beta = 1$ the standard case of geometric discounting is restored while when $\beta \neq 1$ there is Strotz-style (Strotz, 1956) time inconsistency embedded in household preferences. In the case that $\beta < 1$, households are more impatient today than they are in the future and vice-versa when $\beta > 1$.

We assume that momentary utility is described by the additively-separable function

$$u_t = u(c_t, h_t) = \frac{c_t^{1-\sigma} - 1}{1-\sigma} - \chi \frac{h_t^{1+\upsilon}}{1+\upsilon}, \tag{1.2}$$

where h_t represents hours worked and c_t is an aggregate good formed as a Dixit-Stiglitz bundle (Dixit and Stiglitz, 1977) of differentiated goods

$$c_t = \left[\int_0^1 c_t(j)^{\frac{\varepsilon_t - 1}{\varepsilon_t}} dj \right]^{\frac{\varepsilon_t}{\varepsilon_t - 1}}, \tag{1.3}$$

where $c_t(j)$ denotes goods purchased from the j'th firm and the elasticity of substitution between goods satisfies $\varepsilon_t > 1, \forall\, t$. In equation (1.2), the parameters are assumed to satisfy $\sigma > 0$, $\upsilon > 0$, and $\chi > 0$.

Expressed in terms of aggregate goods, the household's real flow-budget-constraint is

$$c_t + \frac{b_{t+1}}{1+R_t} + Q_t s_{t+1} = w_t h_t + \frac{b_t}{1+\pi_t} + Q_t s_t \left(1 + r_t^s\right),$$

where R_t is the net nominal interest rate, w_t is the real wage rate, π_t is the aggregate good's inflation rate, Q_t is the relative price of stocks, b_t is the real value of non-state-contingent nominal bonds, s_t is the number of stocks, and r_t^s is the dividend yield. With the aggregate consumption good produced according to equation (1.3), the demand for the j'th firm's good, $j \in [0,1]$, is

$$c_t(j) = \left(\frac{P_t(j)}{P_t} \right)^{-\varepsilon_t} c_t,$$

with the price of the aggregate good given by

$$P_t = \left[\int_0^1 P_t(j)^{1-\varepsilon_t} dj \right]^{\frac{1}{1-\varepsilon_t}}.$$

We assume that the representative household cannot precommit to future plans. With the economy's state vector summarized by the vector $\mathbf{Z}_t$, the state variables for the household's problem are b_t, s_t, and $\mathbf{Z}_t$. Adopting the apparatus of a recursive competitive equilibrium, we formulate the household's decision problem through the following Lagrangian, which will be extremized with respect to $\{c_t, h_t, b_{t+1}, s_{t+1}, \lambda_t\}$,

$$\mathcal{U}\left(b_t, s_t, \mathbf{Z}_t\right) = \left[\begin{array}{c} \frac{c_t^{1-\sigma}-1}{1-\sigma} - \chi \frac{h_t^{1+\upsilon}}{1+\upsilon} + \beta\theta \mathrm{E}_t\left[U\left(b_{t+1}, s_{t+1}, \mathbf{Z}_{t+1}\right)\right] \\ +\lambda_t \left(\begin{array}{c} w\left(\mathbf{Z}_t\right) h_t + \frac{b_t}{1+\pi(\mathbf{Z}_t)} + Q\left(\mathbf{Z}_t\right) s_t \left(1 + r^s\left(\mathbf{Z}_t\right)\right) \\ -c_t - \frac{b_{t+1}}{1+R(\mathbf{Z}_t)} - Q\left(\mathbf{Z}_t\right) s_{t+1} \end{array} \right) \end{array} \right], \tag{1.4}$$

taking the equilibrium law-of-motion for $\mathbf{Z}_t$ as given. In equation (1.4) the continuation value $U\left(b_{t+1}, s_{t+1}, \mathbf{Z}_{t+1}\right)$ satisfies the recursion

$$U\left(b_t, s_t, \mathbf{Z}_t\right) = \left[\begin{array}{c} \frac{c_t^{1-\sigma}-1}{1-\sigma} - \chi \frac{h_t^{1+\upsilon}}{1+\upsilon} + \theta \mathrm{E}_t\left[U\left(b_{t+1}, s_{t+1}, \mathbf{Z}_{t+1}\right)\right] \\ +\lambda_t \left(\begin{array}{c} w\left(\mathbf{Z}_t\right) h_t + \frac{b_t}{1+\pi(\mathbf{Z}_t)} + Q\left(\mathbf{Z}_t\right) s_t \left(1 + r^s\left(\mathbf{Z}_t\right)\right) \\ -c_t - \frac{b_{t+1}}{1+R(\mathbf{Z}_t)} - Q\left(\mathbf{Z}_t\right) s_{t+1} \end{array} \right) \end{array} \right].$$

We close our description of the household's problem by noting that the elasticity of substitution between goods is stochastic, with $\varepsilon_t = \varepsilon e^{\zeta_t}$ and ζ_t obeying

$$\zeta_{t+1} = \rho_\zeta \zeta_t + \epsilon_{\zeta t+1},$$

with $\rho_\zeta \in (0,1)$ and $\epsilon_{\zeta t} \sim i.i.d.\ N\left(0, \sigma_\zeta^2\right)$. The elasticity shock, ζ_t, is common to all firms and forms one element in the economy's state vector, $\mathbf{Z}_t$.

1.2.2 Firms

There is a unit-continuum of monopolistically competitive firms. The j'th firm, $j \in [0,1]$, owns capital, $k_t(j)$, and employs labour, $h_t(j)$, using both inputs to produce their output, $y_t(j)$, according to the Cobb-Douglas production function

$$y_t(j) = e^{a_t} k_t(j)^\alpha h_t(j)^{1-\alpha}, \tag{1.5}$$

where $\alpha \in (0,1)$ and a_t is an aggregate technology shock that obeys

$$a_{t+1} = \rho_a a_t + \epsilon_{at+1},$$

with $\rho_a \in (0,1)$ and $\epsilon_{at} \sim i.i.d.\ N(0, \sigma_a^2)$. The aggregate technology, a_t, is another element in the economy's state vector, $\mathbf{Z}_t$.

The firm's capital evolves over time according to the law-of-motion

$$k_{t+1}(j) = (1-\delta) k_t(j) + i_t(j),$$

where the depreciation rate, $\delta \in [0,1]$, is common to all firms. The aggregate capital stock, K_t, is the final element in the economy's state vector, $\mathbf{Z}_t$.

Firms face a Rotemberg-style (Rotemberg, 1982) price adjustment cost, where the adjustment-cost is governed by $\omega \geq 0$. Each period every firm chooses how much labor to employ, how much investment to undertake, and the price at which to sell their good in order to maximize its equity-value. Profits are paid to the firm's equity-holders (households) in the form of a dividend.

After substituting the production function (equation 1.5) into the profit function (and dropping the j-index for notational convenience), the decision problem for the representative firm can be written recursively in the form

$$\mathcal{W}(k_t, p_{t-1}, \mathbf{Z}_t) = \max_{\{p_t, k_{t+1}\}} \left[\begin{array}{c} p_t^{1-\varepsilon_t} Y(\mathbf{Z}_t) - w(\mathbf{Z}_t) \left(\frac{p_t^{-\varepsilon_t} Y(\mathbf{Z}_t)}{e^{a_t} k_t{}^{\alpha}} \right)^{\frac{1}{1-\alpha}} - (k_{t+1} - (1-\delta) k_t) \\ -\frac{\omega}{2} \left(\frac{p_t}{p_{t-1}} (1 + \pi(\mathbf{Z}_t)) - 1 \right)^2 Y(\mathbf{Z}_t) \\ +\beta\theta \mathrm{E}_t \left[\frac{C(\mathbf{Z}_{t+1})^{-\sigma}}{C(\mathbf{Z}_t)^{-\sigma}} W(k_{t+1}, p_t, \mathbf{Z}_{t+1}) \right] \end{array} \right], \tag{1.6}$$

taking the equilibrium law-of-motion for $\mathbf{Z}_t$ as given, where $C(\mathbf{Z}_t)$ denotes aggregate consumption, $Y(\mathbf{Z}_t)$ denotes aggregate output, and p_t denotes the firm's price relative to the aggregate good's price. Complementing equation (1.6) is the following recursive expression for the firm's continuation value

$$W(k_t, p_{t-1}, \mathbf{Z}_t) = \left[\begin{array}{c} p_t^{1-\varepsilon_t} Y(\mathbf{Z}_t) - w(\mathbf{Z}_t) \left(\frac{p_t^{-\varepsilon_t} Y(\mathbf{Z}_t)}{e^{a_t} k_t{}^{\alpha}} \right)^{\frac{1}{1-\alpha}} - (k_{t+1} - (1-\delta) k_t) \\ -\frac{\omega}{2} \left(\frac{p_t}{p_{t-1}} (1 + \pi(\mathbf{Z}_t)) - 1 \right)^2 Y(\mathbf{Z}_t) \\ +\theta \mathrm{E}_t \left[\frac{C(\mathbf{Z}_{t+1})^{-\sigma}}{C(\mathbf{Z}_t)^{-\sigma}} W(k_{t+1}, p_t, \mathbf{Z}_{t+1}) \right] \end{array} \right].$$

1.2.3 Equilibrium conditions and aggregation

In our model all households and all firms are identical and they are of unit mass. We focus our attention on symmetric equilibria for which aggregation across agents implies $k_t = K_t$, $c_t = C_t$, $h_t = H_t$, $b_t = B_t$, and $s_t = S_t$, where capital letters indicate aggregate quantities. The bonds and stocks that are traded among households are assumed to be in zero-net-supply and fixed-net-supply, respectively, so we have $B_t = 0$, $\forall\ t$ and $S_t = 1$, $\forall\ t$, where our normalization that stocks equal 1 is without loss of generality.

We examine the household's decision problem in Appendix A.1. There we show that after aggregating across households the first-order conditions for a symmetric equilibrium from the household's problem can be written as

$$C_t^{-\sigma} w_t = \chi H_t^{\upsilon}, \tag{1.7}$$

$$\frac{C_t^{-\sigma}}{1+R_t} = \beta\theta \mathrm{E}_t \left[\frac{C_{t+1}^{-\sigma}}{1+\pi_{t+1}} \right], \tag{1.8}$$

$$Q_t C_t^{-\sigma} = \beta\theta \mathrm{E}_t \left[C_{t+1}^{-\sigma} Q_{t+1} \left(1 + r_{t+1}^{s}\right) \right]. \tag{1.9}$$

Equation (1.7) is an intra-temporal optimality condition for which the quasi-hyperbolic discounting parameter does not enter. Which is to say that the household's quasi-hyperbolic discounting does not change the trade-off that it faces when making its labor-leisure choice. The same cannot be said for equations (1.8) and (1.9), which are intertemporal optimality conditions associated with saving through purchasing bonds and stocks, respectively. For these saving-decisions, the quasi-hyperbolic discounting alters the rate at which household's discount the future relative to today. To the extent that $\beta < 1$, quasi-hyperbolic discounting serves to increase the compensation that households require in order to defer consumption.

Turning to the firm's decision problem, we show in Appendix A.2 that after aggregating across firms the first-order conditions for a symmetric equilibrium can be expressed as

$$C_t^{-\sigma} = \beta\theta \mathrm{E}_t \left[C_{t+1}^{-\sigma} \left(r_{t+1}^k + 1 - \delta + \frac{(1-\beta)}{\beta} \mathcal{K}_K(\mathbf{Z}_{t+1}) \right) \right], \tag{1.10}$$

$$\pi_t (1+\pi_t) = \frac{(1-\varepsilon_t)(1-\tau)}{\omega} + \frac{\varepsilon_t x_t}{\omega} + \beta\theta \mathrm{E}_t \left[\frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}} \frac{Y_{t+1}}{Y_t} \pi_{t+1} (1+\pi_{t+1}) \right], \tag{1.11}$$

$$r_t^k = \alpha x_t \frac{Y_t}{K_t}, \tag{1.12}$$

$$w_t = (1-\alpha) x_t \frac{Y_t}{H_t}, \tag{1.13}$$

where x_t represents real marginal costs, r_t^k represents the shadow real rental rate of capital, and $\mathcal{K}_K(\mathbf{Z}_t)$ is the derivative of the decision rule for next-period's capital, $K_{t+1} = \mathcal{K}(\mathbf{Z}_t)$, with respect to K_t. Equations (1.12) and (1.13) are intra-temporal conditions that simply define capital's shadow rental rate and the real wage and do not depend on the household's quasi-hyperbolic discounting. Equation (1.11) is the economy's Phillips curve. The structure of the Phillips curve is affected by the quasi-hyperbolic discounting, but only to the extent that it changes the rate at which next-period's outcomes are discounted relative to today. The household's quasi-hyperbolic discounting does not have a larger effect on the Phillips curve's structure because we are focusing on a symmetric equilibrium in which all firms set the same price, which means that in equilibrium the relative goods-price for all firms always equals one.

The household's quasi-hyperbolic discounting does, however, impact equation (1.10), which characterizes the firm's intertemporal decision about capital accumulation and takes the form of a consumption-Euler equation, much like equations (1.8) and (1.9). Interestingly, in equation (1.10) quasi-hyperbolic discounting manifests itself in two ways. First, quasi-hyperbolic discounting changes the rate at which firms discount next-period relative to today, changing the compensation that the firm requires to be enticed to purchase an additional unit of capital rather than pay households a higher dividend. Second, quasi-hyperbolic discounting adds a term involving the derivative $\mathcal{K}_K(\mathbf{Z}_{t+1})$. This additional term, which disappears when $\beta = 1$, says that when making its capital decision, the firm takes into account how the acquisition of an additional unit of capital today changes next-period's capital-acquisition decision, an effect that arises because the firm's equity holders do not have time-invariant preferences. If the household owns the capital stock, then this term, $\mathcal{K}_K(\mathbf{Z}_{t+1})$, arises in the consumption-Euler equation for the capital decision as households use capital accumulation to constrain their future-selves. While the (shadow) rental rate represents a pecuniary return that households receive through owning stocks the derivative term, $\mathcal{K}_K(\mathbf{Z}_{t+1})$, represents a non-pecuniary return.

In addition to these first-order conditions, aggregating across firms and households gives us the aggregate production function

$$Y_t = e^{a_t} K_t^{\alpha} H_t^{1-\alpha},$$

the resource constraint

$$K_{t+1} = (1-\delta) K_t - C_t + \left(1 - \frac{\omega}{2}\pi_t^2\right) Y_t,$$

and the following expression for the dividend yield, r_t^s, which accounts for the pecuniary return on owning stocks

$$Q_t r_t^s = \left(1 - x_t - \frac{\omega}{2}\pi_t^2\right) Y_t + r_t^k K_t - \left(K_{t+1} - (1-\delta) K_t\right). \tag{1.14}$$

Equation (1.14) says that the dividend yield rises with an increase in the shadow rental rate of capital, r_t^k, and with a reduction in real marginal costs, x_t, or inflation, π_t.

1.3 Central bank

We assume that the central bank shares the household's momentary utility function and that it also has quasi-hyperbolic preferences, which is to say that we allow the central bank's discount factors, γ and ξ, to potentially differ from the household's, β and θ. Further, we assume that the central bank does not have access to a commitment technology and that it conducts policy under discretion. With monetary policy conducted under discretion, and with the central bank possessing quasi-hyperbolic preferences, the central bank's decision problem can be summarized by the Bellman equation

$$\mathcal{V}(\mathbf{Z}_t) = \max_{\{\pi_t\}} \left(\frac{C_t^{1-\sigma} - 1}{1-\sigma} - \chi \frac{H_t^{1+\upsilon}}{1+\upsilon} + \gamma\xi \mathrm{E}_t \left[V(\mathbf{Z}_{t+1}) \right] \right),$$

where the continuation value can be expressed recursively in the form

$$V(\mathbf{Z}_t) = \frac{C_t^{1-\sigma} - 1}{1-\sigma} - \frac{\chi}{1+\upsilon} H_t^{1+\upsilon} + \xi \mathrm{E}_t \left[V(\mathbf{Z}_{t+1}) \right],$$

subject to the constraints

$$C_t^{-\sigma} = \theta \mathrm{E}_t \left[L(\mathbf{Z}_{t+1}) \right], \tag{1.15}$$

$$\pi_t (1+\pi_t) C_t^{-\sigma} Y_t = \frac{\varepsilon_t}{\omega} \left(x_t + \frac{(1-\varepsilon_t)(1-\tau)}{\varepsilon_t} \right) C_t^{-\sigma} Y_t + \theta \mathrm{E}_t \left[M(\mathbf{Z}_{t+1}) \right], \tag{1.16}$$

$$\left(1 - \frac{\omega}{2} \pi_t^2 \right) Y_t = C_t + K_{t+1} - (1-\delta) K_t, \tag{1.17}$$

$$Y_t = e^{a_t} K_t^{\alpha} H_t^{1-\alpha}. \tag{1.18}$$

Among these four constraints, two are forward-looking: equations (1.15) and (1.16). In each of these forward-looking constraints we have introduced an auxiliary variable, $L(\mathbf{Z}_t)$ and $M(\mathbf{Z}_t)$, respectively, which are defined according to

$$\begin{aligned} L(\mathbf{Z}_t) &= C_t^{-\sigma} \left(\beta \left(\alpha x_t \frac{Y_t}{K_t} + 1 - \delta \right) + (1-\beta) \mathcal{K}_K(\mathbf{Z}_t) \right), \\ M(\mathbf{Z}_t) &= \beta \pi_t (1+\pi_t) C_t^{-\sigma} Y_t. \end{aligned}$$

Making these auxiliary variables functions of the economy's state in the central bank's decision problem reflects the assumption that policy is set with discretion. Specifically, while able to influence the economy's aggregate state, the discretionary central bank is unable to use policy to influence the process by which private-agents form expectations and must take the functions $L(\mathbf{Z}_t)$ and $M(\mathbf{Z}_t)$ as given when formulating policy.

It is notable from equations (1.15)—(1.18) that the key constraints on the central bank's policy decision are the production technology, the resource constraint, the Phillips curve, and the consumption-Euler equation associated with the optimal capital decision. The consumption-Euler equations associated with bonds (equation 1.8) and stocks (equation 1.9) are not binding constraints, but simply serve to determine equilibrium outcomes for R_t and Q_t, with r^s_t determined by equation (1.14).

The central bank's decision problem is treated in Appendix C, where we show that the first-order conditions for the optimal discretionary policy are

$$\begin{aligned}\frac{\partial}{\partial C_t} \; : \; & C_t^{-\sigma} + \frac{\sigma\chi}{\upsilon+\alpha}\frac{H_t^{1+\upsilon}}{C_t} - \phi_{1t}\left(1+\sigma\frac{1-\alpha}{\upsilon+\alpha}\left(1-\frac{\omega}{2}\pi_t^2\right)\frac{Y_t}{C_t}\right) - \phi_{2t}\sigma C_t^{-\sigma-1} \\ & -\sigma\frac{1+\upsilon}{\alpha+\upsilon}\phi_{3t}\left(\frac{(1-\varepsilon_t)(1-\tau)+\varepsilon_t x_t}{\omega} - \pi_t(1+\pi_t)\right)C_t^{-\sigma-1}Y_t = 0, \qquad (1.19)\end{aligned}$$

$$\frac{\partial}{\partial \pi_t} : -\phi_{3t}(1+2\pi_t)C_t^{-\sigma} - \phi_{1t}\omega\pi_t = 0, \qquad (1.20)$$

$$\begin{aligned}\frac{\partial}{\partial x_t} \; : \; & -\frac{\chi}{\upsilon+\alpha}\frac{H_t^{1+\upsilon}}{x_t} + \phi_{1t}\frac{1-\alpha}{\upsilon+\alpha}\left(1-\frac{\omega}{2}\pi_t^2\right)\frac{Y_t}{x_t} \qquad (1.21) \\ & +\phi_{3t}\left(\frac{\varepsilon_t x_t}{\omega} + \frac{1-\alpha}{\upsilon+\alpha}\frac{(1-\varepsilon_t)(1-\tau)+\varepsilon_t x_t}{\omega} - \frac{1-\alpha}{\upsilon+\alpha}\pi_t(1+\pi_t)\right)C_t^{-\sigma}\frac{Y_t}{x_t} = 0,\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial K_{t+1}} \; : \; & -\frac{\gamma\xi\alpha\chi}{\upsilon+\alpha}\mathrm{E}_t\left[\frac{H_{t+1}^{1+\upsilon}}{K_{t+1}}\right] + \xi\mathrm{E}_t\left[\phi_{1t+1}\left(\alpha\frac{1+\upsilon}{\upsilon+\alpha}\left(1-\frac{\omega}{2}\pi_{t+1}^2\right)\frac{Y_{t+1}}{K_{t+1}} + 1 - \delta\right)\right] \\ & +\xi\alpha\frac{1+\upsilon}{\upsilon+\alpha}\mathrm{E}_t\left[\phi_{3t+1}\left(\frac{(1-\varepsilon_t)(1-\tau)+\varepsilon_t x_{t+1}}{\omega} - \pi_{t+1}(1+\pi_{t+1})\right)\frac{Y_{t+1}}{K_{t+1}}C_{t+1}^{-\sigma}\right] \\ & -\xi(1-\gamma)\mathrm{E}_t\left[\left(C_{t+1}^{-\sigma} + \frac{\sigma\chi}{\upsilon+\alpha}\frac{H_{t+1}^{1+\upsilon}}{C_{t+1}}\right)\mathcal{C}_K(\mathbf{Z}_{t+1})\right] \\ & +\frac{\xi(1-\gamma)\chi}{\upsilon+\alpha}\mathrm{E}_t\left[\frac{H_{t+1}^{1+\upsilon}}{x_{t+1}}\mathcal{X}_K(\mathbf{Z}_{t+1})\right] \\ & -\phi_{2t}\theta\mathrm{E}_t\left[L_K(\mathbf{Z}_{t+1})\right] + \phi_{3t}\theta\mathrm{E}_t\left[M_K(\mathbf{Z}_{t+1})\right] - \phi_{1t} = 0. \qquad (1.22)\end{aligned}$$

where

$$H_t = \left(\left(\frac{1-\alpha}{\chi}\right)e^{a_t}x_t K_t^\alpha C_t^{-\sigma}\right)^{\frac{1}{\upsilon+\alpha}}, \qquad (1.23)$$

$$Y_t = \left(\left(\frac{1-\alpha}{\chi}\right)^{1-\alpha} e^{(1+v)a_t} x_t^{1-\alpha} K_t^{\alpha(1+v)} C_t^{-\sigma(1-\alpha)}\right)^{\frac{1}{v+\alpha}}, \tag{1.24}$$

$$L(\mathbf{Z}_t) = C_t^{-\sigma}\left(\beta\left(\alpha e^{a_t} x_t \frac{Y_t}{K_t} + 1 - \delta\right) + (1-\beta)\mathcal{K}_K(\mathbf{Z}_t)\right), \tag{1.25}$$

$$M(\mathbf{Z}_t) = \beta\pi_t(1+\pi_t)C_t^{-\sigma}Y_t. \tag{1.26}$$

As a counterpoint, we also solve the model for the case where monetary policy is conducted according to the following Taylor-type rule

$$1 + R_t = \frac{1+\overline{\pi}}{\beta\theta}\left(\frac{1+\pi_t}{1+\overline{\pi}}\right)^{\phi_\pi}\left(\frac{Y_t}{Y_{t-1}}\right)^{\phi_y}, \tag{1.27}$$

where $\overline{\pi}$ represents the inflation target, $\phi_\pi > 1$ and $\phi_y > 0$. Following Fernández-Villaverde, et al. (2015) and Dennis (2018), this Taylor rule has the central back setting the nominal interest rate in response to movements in inflation and real output growth.

1.4 Interest rates and asset prices

From the household's optimal bond-holding decision, the net nominal interest rate, R_t, is governed by the Euler equation

$$\frac{1}{1+R_t} = \beta\theta \mathrm{E}_t\left[\frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}}\frac{1}{1+\pi_{t+1}}\right],$$

where the effect of the household's quasi-hyperbolic discounting is seen to cause the future to be discounted more sharply, raising the equilibrium interest rate on average. We can also compute the shadow return on a risk-free real bond, r_t, which must satisfy

$$\frac{1}{1+r_t} = \beta\theta \mathrm{E}_t\left[\frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}}\right],$$

From the firm's decision problem the shadow rental rate of capital is given by

$$r_t^k = \alpha x_t \frac{Y_t}{K_t},$$

where household's quasi-hyperbolic discounting has indirect-effects through the economy's real allocation. Where the shadow rental rate of capital represents the pecuniary return that households receive from owning stocks, the total net return they receive, r_t^{cap}, satisfies

$$1 + r_t^{cap} = r_t^k + 1 - \delta + \frac{(1-\beta)}{\beta}\mathcal{K}_K(\mathbf{Z}_t). \tag{1.28}$$

According to equation (1.28), the total gross return on capital, $1 + r_t^{cap}$, is the sum of two components: the gross pecuniary return, $r_t^k + 1 - \delta$, and the gross non-pecuniary return, $\frac{(1-\beta)}{\beta}\mathcal{K}_K(\mathbf{Z}_t)$. As we will see below, even for relatively small amounts of quasi-hyperbolic discounting the non-pecuniary component can be large.

1.5 Parameterization

We assume that a period in the model corresponds to one quarter of a year and parameterize the model to this frequency. We set the household's (geometric) discount factor, θ, to 0.99, which in the absence of quasi-hyperbolic discounting implies a steady state annual real interest rate of about 4 percent. As is common, we assume log-utility with respect to consumption, i.e. $\sigma = 1$, and we set the relative weight on the disutility of labor, χ, equal to 1. The Frisch labor supply elasticity, ν, is set equal to 1, which is consistent with a host of studies, including Fernández-Villarerde, Guerrón-Quintana, Kuester, and Rubio-Ramírez (2015), Guerrieri and Iacoviello (2017), and Chetty, Guren, Manoli, and Weber (2011), but smaller than Gust, Herbst, López-Salido, and Smith (2017) and Gavin, Keen, Richter, and Throckmorton (2015), who set this elasticity to 2 and 3, respectively. As Fernández-Villarerde, et al, (2015) comment, a lower value for ν (implying a higher labor-supply elasticity) is generally more appropriate for models that do not differentiate between the intensive and extensive margins.

In the production technology, values for α generally range from about 0.3 (Guerrieri and Iacoviello, 2017) to 0.40 (Cooley and Prescott, 1995). We set α equal to 0.33, in line with Gavin, et al, (2015) and Sala, Söderström, and Trigari (2008). In the capital accumulation equation, we set the depreciation rate, δ, to 0.025, which implies that capital depreciates at a 10 percent annualized rate. We set the steady state elasticity of substitution between goods, ε, to 11, implying a steady state mark-up of 10 percent. This value for ε has been used previously in a range of studies, including Krause, López-Salido, and Lubik (2008a) and Dennis (2018), and is consistent with the findings of Basu and Fernald (1997). Other recent studies have set ε to 6 (Christiano, Eichenbaum, and Evans, 2005) or 21 (Fernández-Villarerde et al, 2015; Krause, López-Salido, and Lubik, 2008b), implying much larger and much smaller steady-state markups, respectively, however we found that these values gave implausible values for steady state inflation. Turning to the price adjustment parameter, ω, we set it to 100, consistent with Gust, et al, (2017). In a log-linearized environment, this value for ω makes the Rotemberg model quantitatively similar to a Calvo model where the average frequency of price adjustment equals one year. Elsewhere in the literature, Gavin, et al, (2015) estimate ω to be 59.1, Ireland (2001) estimates it to be about 80, while the estimates in Gertler, Sala, and Trigari (2008) and Sala, Söderström, and Trigari (2008) imply a value closer to 150.

There are two shocks in the model, those to aggregate technology, a_t, and the elasticity of substitution among goods, ζ_t. As is common, these shocks are assumed to follow AR(1) processes:

$$a_{t+1} = \rho_a a_t + \epsilon_{at+1}, \quad \epsilon_{at} \sim i.i.d.\ N\left(0, \sigma_a^2\right),$$

$$\zeta_{t+1} = \rho_\zeta \zeta_t + \epsilon_{\zeta t+1}, \quad \epsilon_{\zeta t} \sim i.i.d. N(0, \sigma_\zeta^2).$$

For the aggregate technology shock, we follow convention (see Faia (2009) and the references therein) and set the persistence parameter, ρ_a, to 0.95 and the standard deviation for the technology innovation, σ_a, to 0.008. For the elasticity of substitution shock, the estimates vary across the literature. Gertler, Sala, and Trigari (2008) estimate ρ_ζ and σ_ζ to be 0.81 and 0.008, Smets and Wouters (2007) estimate them to be 0.89 and 0.1, while Ichiue, Kurozumi, and Sunakawa (2013) estimate them to be 0.7 and 0.05. We set ρ_ζ and

Table 1.1: Benchmark Parameterization

Parameter	Value	Parameter	Value	Parameter	Value
θ	0.99	α	0.33	ρ_a	0.95
σ	1.0	ε	11.0	ρ_ζ	0.85
χ	1.0	ω	100.0	σ_a	0.008
ν	1.0	δ	0.0025	σ_ζ	0.06
β	1.0				

σ_ζ to 0.85 and 0.06, respectively, implying that 90 percent of the distribution for ε_t lies in the interval $[9.1, 13.3]$.

In our benchmark model the central bank's discount factor and its quasi-hyperbolic discount factor are assumed to be the same as for the household, implying that the central bank is benevolent. We summarize our benchmark parameterization in Table 1.1.

For the simulations based on the Taylor-type rule, equation (1.27), we assume $\pi = 2.5$, $\phi_\pi = 1.5$, and $\phi_y = 0.5/4$.

1.6 Results

In this section we present simulation results for a range of different model specifications. We begin with the benchmark model in section 6.1 in which households have quasi-geometric discounting and the central bank is benevolent, sharing the household's discount factors. Section 6.2 treats the case where households and the central bank have quasi-hyperbolic discounting and their discount factors are not equal.

1.6.1 Quasi-hyperbolic discounting

In this section we impose $\theta = \xi$ and $\gamma = \beta$ (so that household's and the central bank discount symmetrically) to maintain the assumption that the central bank is benevolent.

In line with the standard mechanism discussed in Krusell and Smith (2003) and elsewhere, greater quasi-hyperbolic discounting results in households increasing their current consumption and reducing their current saving. As a result, less capital accumulation takes place and output, capital, and consumption are all lower on average. The effect is quantitatively substantial, as illustrated in Table 1.2.

Focusing on the model's stationary distribution, Table 1.2 shows the mean (standard deviations in parentheses) outcomes for the model's key macroeconomic and financial variables for different values of $\beta = \gamma$. Columns (1) and (6) contain results for the benchmark case with the standard geometric discounting.[1] Looking at average outcomes, the table shows that as greater quasi-hyperbolic discounting takes place ($\beta = \gamma$ get smaller)—biasing household and central bank decision-making toward the present—output falls. Specifically, lowering $\beta = \gamma$ from 1.0 to 0.9 causes output to decline by approximately 10 percent.[2] Although greater quasi-hyperbolic discounting causes output, capital, consumption, labour, and the real wage to fall there are important differences in how each of these variables is affected. For example, although lowering $\beta = \gamma$ from 1.0 to 0.9 causes output to fall by 10.02 percent, capital falls by much more (24.55 percent) and labour falls by much less (1.84 percent). Labour does not decline to the same extent as output because households sacrifice some leisure in order to prevent a large decline in consumption. As a consequence, consumption falls by 6.02 percent, considerably less than output. The large decline in capital combined with a smaller decline in labour

[1] These cases are thoroughly discussed in Appendix D to this chapter.

[2] Cutting β from 1.0 to 0.7 causes output to fall by about 30 percent, suggesting a linear relationship between the percent by which β falls and the percent by which output falls.

means that the capital-labour ratio goes down, and with relatively less capital, labour's productivity diminishes and real wages go down (by 7.77 percent).

Looking at real marginal costs, Table 1.2 shows that greater quasi-hyperbolic discounting causes real marginal costs to rise slightly. The effect that quasi-hyperbolic discounting has on real marginal costs is related to the decline that firms face in the demand for their good, which causes them to lower their price markup. To understand the impact quasi-hyperbolic discounting has on inflation, note that quasi-hyperbolic discounting implies that costs to changing prices today are weighted more heavily than those to changing prices in the future. As a consequence, when responding to shocks firms find it beneficial to spread price changes out over time, making smaller price changes in the current period and deferring the remaining price change (and its associated cost) to the future. With smaller price changes taking place today, greater quasi-hyperbolic discounting acts somewhat like an increase in price rigidity. From the central bank's perspective, with quasi-hyperbolic discounting operating similarly to an increase in price rigidity, it calculates that smaller inflation surprises are sufficient to boost output to the efficient level. In equilibrium, then, greater quasi-hyperbolic discounting leads to less inflation.

Turning to the financial variables, the most pronounced and obvious effect of quasi-hyperbolic discounting is to raise the real returns on capital and bonds. With quasi-hyperbolic discounting shifting demand from future- to current-consumption the relative price of current-consumption rises causing the pecuniary return on capital, as reflected in the (shadow) rental rate of capital, to rise. In addition, greater quasi-hyperbolic discounting increases greatly the non-pecuniary return on capital, which causes the (net) total return on capital, r^{cap}, to balloon. With households substituting between stocks and bonds (which do not offer a non-pecuniary return because they are in zero-net-supply) based on their total return, the rise in r^{cap} leads to a commensurate rise in the real interest rate.

Although it is clear from Table 1.2 that quasi-hyperbolic discounting has an important

Table 1.2: Stationary Distribution as a Function of the Quasi-Hyperbolic Discount factor

		Discretion				Taylor rule			
Discounting	$\beta = \gamma$	1.00	0.99	0.95	0.90	1.00	0.99	0.95	0.90
		(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Output	Y	2.540 [0.104]	2.516 [0.103]	2.415 [0.099]	2.286 [0.094]	2.539 [0.103]	2.509 [0.102]	2.387 [0.097]	2.231 [0.091]
Capital	K	21.734 [0.936]	21.181 [0.916]	19.009 [0.837]	16.398 [0.740]	21.722 [0.929]	21.049 [0.903]	18.437 [0.799]	15.367 [0.674]
Consumption	C	1.992 [0.062]	1.981 [0.062]	1.935 [0.062]	1.872 [0.061]	1.991 [0.062]	1.978 [0.062]	1.922 [0.061]	1.842 [0.060]
Investment	I	0.543 [0.053]	0.530 [0.052]	0.475 [0.048]	0.410 [0.044]	0.543 [0.053]	0.526 [0.051]	0.461 [0.047]	0.384 [0.041]
Labour	H	0.881 [0.010]	0.880 [0.009]	0.873 [0.009]	0.865 [0.008]	0.881 [0.010]	0.879 [0.010]	0.871 [0.010]	0.867 [0.010]
Real wage	w	1.756 [0.063]	1.743 [0.063]	1.690 [0.062]	1.619 [0.060]	1.755 [0.063]	1.739 [0.063]	1.674 [0.061]	1.587 [0.059]
Real marginal cost	x	0.909 [0.008]	0.910 [0.008]	0.912 [0.008]	0.915 [0.007]	0.909 [0.011]	0.910 [0.011]	0.912 [0.011]	0.915 [0.011]
Annualized inflation	π	2.580 [0.527]	2.559 [0.520]	2.478 [0.493]	2.385 [0.462]	2.519 [0.253]	2.519 [0.254]	2.520 [0.258]	2.522 [0.267]
Household Welfare	$\mathcal{U}$	29.957 [0.989]	29.270 [0.980]	26.432 [0.942]	22.685 [0.895]	29.960 [0.992]	29.180 [0.981]	25.924 [0.937]	21.457 [0.881]
Nominal interest rate	R	6.782 [0.497]	11.139 [0.516]	30.969 [0.608]	62.443 [0.760]	6.720 [0.660]	11.098 [0.692]	31.026 [0.847]	62.664 [1.111]
Real interest rate	r	4.097 [0.384]	8.367 [0.403]	27.804 [0.489]	58.660 [0.634]	4.098 [0.411]	8.368 [0.432]	27.804 [0.531]	58.660 [0.700]
Rental rate	r^k	4.098 [0.427]	4.342 [0.433]	5.413 [0.460]	7.019 [0.502]	4.098 [0.431]	4.392 [0.440]	5.713 [0.480]	7.756 [0.542]
Return on capital	r^{cap}	4.098 [0.427]	8.369 [0.447]	27.805 [0.543]	58.662 [0.704]	4.098 [0.431]	8.368 [0.453]	27.805 [0.558]	58.618 [0.740]

Note: Statistics calculated using 10^6 simulated observations; standard deviations in brackets.

impact on average outcomes, it also has an effect on dynamics. We compute impulse response functions for the discretionary response to technology shocks (Figure 1.1) and price-elasticity shocks (Figure 1.2) while allowing the extent of the quasi-hyperbolic discounting to vary.

Looking first at the responses to technology shocks, Figure 1.1 reveals that it is the financial variables that quasi-hyperbolic discounting affects most. The solid lines in Figure 1 correspond to $\beta = \gamma = 1$, which is conventional (baseline) geometric discounting. With quasi-hyperbolic discounting causing households to discount the entire future relative to today, increased quasi-hyperbolic discounting leads to an increased focus on today's consumption and leisure. Accordingly, relative to the baseline case, consumption (panel C) rises by more and labour (panel E) rises by less in response to the technology shock. Higher technology boosts the demand for labour, and with the supply of labour increasing by less relative to the baseline case, the real wage (panel F) rises by more, which pushes up real marginal costs (panel G). Because real marginal costs increase by more with quasi-hyperbolic discounting than they do for the baseline case, the firm's production costs are higher and their profitability is lower. At the same time, the real interest rate (panel J) rises by more than the baseline case, due to the increased demand for current consumption relative to future consumption. The discretionary policy response is to increase the nominal interest rate (panel I) by more than the baseline case, primarily due to the higher real interest rate.

Figure 1.2 shows how quasi-hyperbolic discounting alters the model's dynamic behavior following price-elasticity shocks. Relative to the baseline case in which $\beta = \gamma = 1$ (solid lines), with quasi-hyperbolic discounting the impulse responses are (generally) a little more muted. Households value leisure and consumption more today relative to the future, so labour (panel E) rises by less following the shock and consumption rises by more (panel C). Similarly, quasi-hyperbolic discounting makes firms want to defer costly price changes, so inflation (panel H) falls by less than the baseline case. The variables for which the effects of quasi-hyperbolic discounting are most pronounced are the real

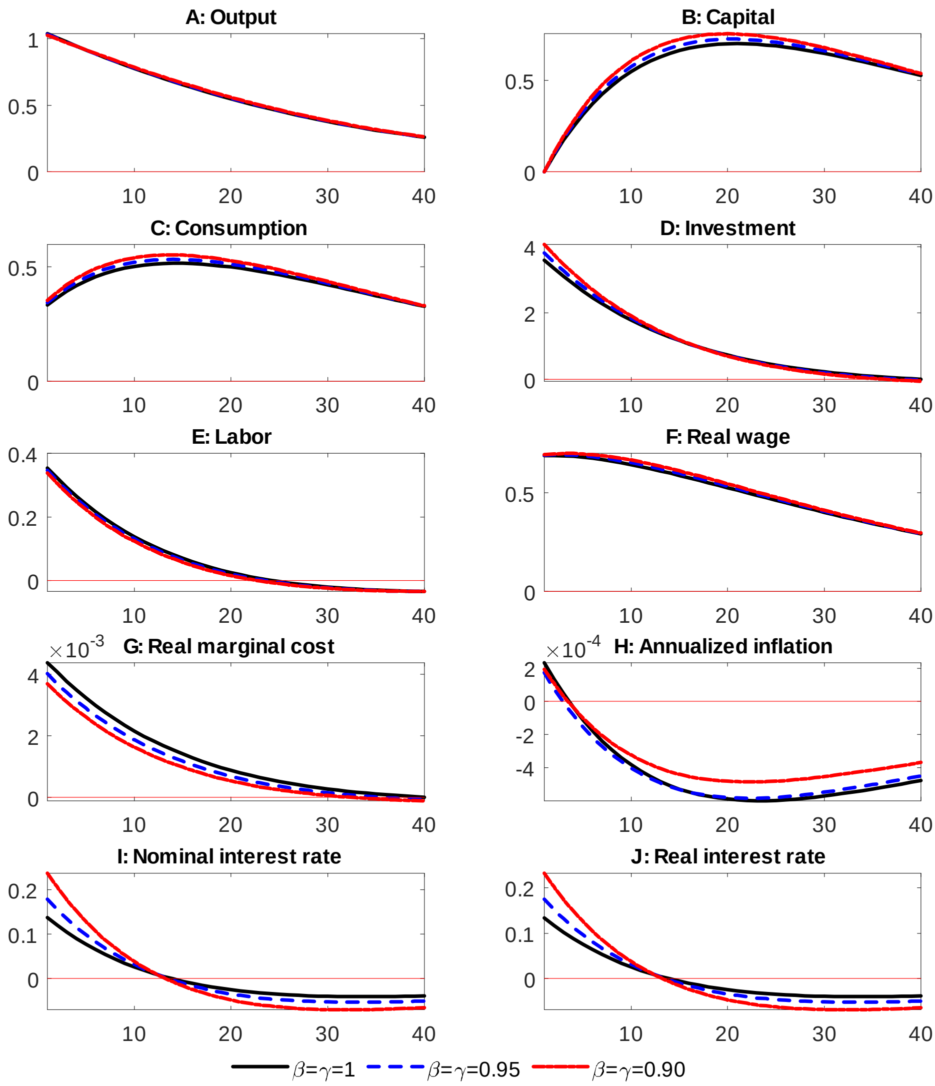

Figure 1.1: Responses to a technology shock with quasi-geometric discounting, $\beta = \gamma$, and discretionary policy

interest rate (panel J) and the nominal return on bonds (panel I). Quasi-hyperbolic discounting makes all of these variables more sensitive to the price-elasticity shock because it changes the relative demand for current consumption such that a bigger change in the relative price of consumption (the real interest rate) is required to induce households to defer consumption.

Although quasi-hyperbolic discounting affects the dynamic behavior of the macroeconomic variables, Figures 1.1 and 1.1 reveals that its greatest impact is on asset returns. This finding is consistent with the simulation results in Table 1.2, which show that the volatilities of asset returns and asset prices rise importantly as quasi-hyperbolic discounting increases.

1.6.2 Policy delegation

In the previous section we allowed the central bank to have quasi-hyperbolic preferences, but we restricted its discount factors to equal those of the representative household. This restriction forced the discretionary central bank to be benevolent, i.e., to conduct policy under discretion in order to maximize the welfare of the representative household. Here, we allow the central bank's quasi-hyperbolic discounting to differ from the representative household. We do this exercise for two reasons. First, by allowing the central bank's discounting to differ from the household's we can assess the degree to which the central bank's quasi-hyperbolic discounting affects economic outcomes. Second, because policy is being conducted under discretion and discretion is suboptimal, it is possible that the government should optimally delegate monetary policy to a central banker whose discounting differs from the household. If this is the case, then a related question is whether the central bank should discount the future by more or less than the household.

Table 1.3 summarizes equilibrium outcomes when the household and the central bank have different discount factors. Comparing columns 1 and 2 column 1 we see that the

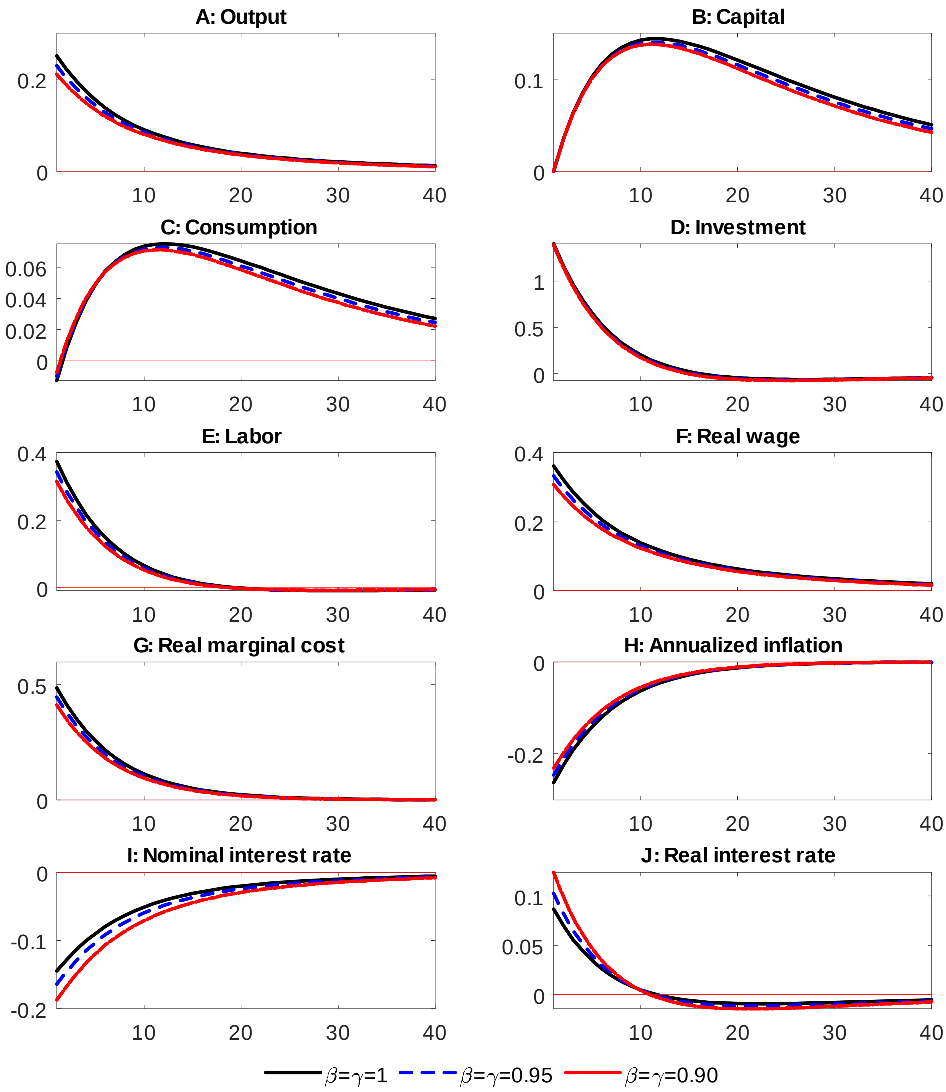

Figure 1.2: Responses to a price-elasticity shock with quasi-geometric discounting, $\beta = \gamma$.

Table 1.3: The Effect of the Central Bank's Quasi-Hyperbolic Discounting

		Discretion			
Household	β	1.00	1.00	0.90	0.90
Central bank	γ	1.00	0.90	0.90	1.00
		(1)	(2)	(3)	(4)
Output	Y	2.540 [0.104]	2.536 [0.104]	2.286 [0.094]	2.302 [0.095]
Capital	K	21.734 [0.936]	21.681 [0.934]	16.398 [0.740]	16.592 [0.746]
Consumption	C	1.992 [0.062]	1.993 [0.062]	1.872 [0.061]	1.876 [0.061]
Investment	I	0.543 [0.053]	0.542 [0.053]	0.410 [0.044]	0.415 [0.044]
Labour	H	0.881 [0.010]	0.880 [0.010]	0.865 [0.008]	0.869 [0.008]
Real wage	w	1.756 [0.063]	1.754 [0.064]	1.619 [0.060]	1.630 [0.060]
Real marginal cost	x	0.909 [0.008]	0.909 [0.009]	0.915 [0.007]	0.919 [0.006]
Inflation	π	2.580 [0.527]	0.703 [0.312]	2.385 [0.462]	4.015 [0.633]
Household Welfare	$\mathcal{U}$	29.957 [0.989]	30.155 [0.988]	22.685 [0.895]	22.555 [0.895]
Nominal interest rate	R	6.782 [0.497]	4.828 [0.388]	62.443 [0.760]	65.029 [0.947]
Real interest rate	r	4.097 [0.384]	4.097 [0.390]	58.660 [0.634]	58.660 [0.622]
Rental rate	r^k	4.098 [0.427]	4.098 [0.435]	7.019 [0.502]	7.016 [0.490]
Return on capital	r^{cap}	4.098 [0.427]	4.098 [0.435]	58.662 [0.704]	58.662 [0.687]

Note: Statistics calculated using 10^6 simulated observations; standard deviations in brackets.

central bank's quasi-hyperbolic discounting ($\gamma = 0.9$) causes it to conduct monetary policy in order to encourage greater consumption and leisure and discourage inflation. Because the central bank places greater emphasis on the present relative to the future, monetary policy is used to encourage households to bring consumption and leisure forward in time while also shifting price changes (which are costly) to the future, where they are discounted more heavily. With greater consumption and leisure taking place, investment and capital fall slightly, which leads to a decline in output. For the financial variables, the real returns on assets are barely affected while the stock price rises.

Where columns (1) and (2) allow us to identify what happens when the central bank has greater quasi-hyperbolic discounting than households (which are not quasi-hyperbolic discounters for that comparison), columns (3) and (4) relate to the opposite comparison: in column (3) both households and the central bank have quasi-hyperbolic discounting whereas in column (4) only the household does. As a consequence, relative to column (3), in column (4) the central bank uses monetary policy to encourage households to defer consumption and leisure, while bringing forward price changes, which raises inflation. In this particular case, the household's labour supply response is large, which increases output and permits consumption to actually rise. Although allowing the central bank's discounting to differ from the household's has effects on real variables, these effects are relatively small. However, the effects on nominal variables are larger and quantitatively significant.

Importantly, one consequence of the central bank's quasi-hyperbolic discounting is to raise household welfare. Household welfare is higher in column (2) than in column (1) and in column (3) than in column (4). In other words, it is desirable from a welfare perspective for the central bank to have quasi-hyperbolic discounting even if household's do not. This finding parallels other situations where distorting the central bank's objectives can raise welfare when monetary policy is conducted under discretion. For example, Dennis (2014) showed that having monetary policy conducted by a discretionary central bank with risk-sensitive preferences could improve welfare (lower loss) because the risk-sensitivity rendered feasible policies that were otherwise infeasible. Here, the central reason why the central bank's quasi-hyperbolic discounting raises household welfare is that it emphasizes the current-period cost of changing prices, in much the same way as greater price rigidity or greater concern for price changes. The outcome is less volatile inflation and an average inflation rate that is lower, closer to zero. Due to the greater emphasis placed on inflation appointing a central banker with quasi-hyperbolic discounting is similar to appointing a conservative central banker (Rogoff, 1985).

To explore more fully whether it is desirable for the central bank to discount the future

Table 1.4: Impact of Quasi-Geometric Discounting on Household Welfare

β	γ^*	Welfare
0.90	0.865	22.6818 [0.8910]
0.92	0.865	24.2281 [0.9098]
0.94	0.865	25.7494 [0.9285]
0.96	0.866	27.2448 [0.9472]
0.98	0.868	28.7130 [0.9660]
1.00	0.873	30.1468 [0.9846]

at a rate that differs from households, Table 1.4 reports the optimal value for the central bank's quasi-hyperbolic discount factor γ^* and the household's welfare level at this point (with the standard deviation for welfare given in square brackets) as a function of the household quasi-hyperbolic discount factor, β. Several interesting and important results are apparent from Table 1.4. First, it is desirable for the central bank's quasi-hyperbolic discounting to be stronger than that of the household ($\gamma^* < \beta$). Second, even when the household does not have quasi-hyperbolic discounting ($\beta = 1$), the central bank should ($\gamma^* < 1$). Third, the optimal value for γ^* is relatively insensitive to changes to β.

1.7 Conclusion

In this paper we study the conduct of discretionary monetary policy in an economy where economic agents have quasi-hyperbolic discounting. Households gain utility through consumption and leisure and save by purchasing bonds and equities. With the exception of the goods market, which is characterized by monopolistic competition and Rotemberg-prices, all other markets are assumed to be perfectly competitive. As is well-known, by weighting the present more than the future, relative to geometric discounting, quasi-hyperbolic discounting has important implications for the equilibrium return on savings, and hence on the capital stock and the level of production. However, in a model where there are costs to changing prices, quasi-hyperbolic discounting also has important consequences for the inflation rate.

With the central bank conducting monetary policy optimally under discretion, we show that the household's quasi-hyperbolic discounting changes the economy's average inflation rate, with greater quasi-hyperbolic discounting giving rise to lower average inflation. The economy's average inflation rate declines because the household's greater emphasis on the present (relative to the future) strengthens the incentive for firms to spread price-changes out over time, benefiting their equity-holders by making smaller price changes today and shifting the remaining price-change to the future (when it is discounted more heavily).

Our model also allows the central bank to have quasi-hyperbolic discounting, and for its discounting to differ from the household. We show that a benevolent central bank—one that shares household's preferences—is able to keep steady state inflation under control for a wide range of discount factors. If the central bank, however, does not adopt the household's time discounting and tries to discourage early consumption and delayed saving, then the resulting equilibrium produces only a small increase in output while generating a substantial rise in inflation. Indeed, we show that it is optimal for the central bank to (quasi-hyperbolically) discount the future more heavily than the household, and that doing so decreases inflation and increases welfare.

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1.A Appendix: The model where firms own capital

This section presents our benchmark model in which firms own the capital stock. We present an alternative formulation in which household's own the capital stock in Appendix B. The equivalence of these two formulations can be seen by comparing Appendices A.3. and B.3. The central bank's decision problem is presented in Appendix C.

1.A.1 Household's problem

The household's decision problem is described by the Lagrangian

$$\begin{aligned}
\mathcal{U}(b_t, s_t, \mathbf{Z}_t) = \; & \min_{\{\lambda_t\}} \max_{\{c_t, h_t, b_{t+1}, s_{t+1}\}} \left(\frac{c_t^{1-\sigma} - 1}{1-\sigma} - \chi \frac{h_t^{1+\upsilon}}{1+\upsilon} \right. \\
& + \beta\theta \mathrm{E}_t \left[U(b_{t+1}, s_{t+1}, \mathbf{Z}_{t+1}) \right] \\
& + \lambda_t \left(w(\mathbf{Z}_t) h_t + \frac{b_t}{1+\pi(\mathbf{Z}_t)} + Q(\mathbf{Z}_t) s_t (1 + r^s(\mathbf{Z}_t)) \right. \\
& \left. \left. + T_t - c_t - \frac{b_{t+1}}{1+R(\mathbf{Z}_t)} - Q(\mathbf{Z}_t) s_{t+1} \right) \right),
\end{aligned}$$

where

$$U(b_t, s_t, \mathbf{Z}_t) = \frac{c_t^{1-\sigma} - 1}{1-\sigma} - \chi \frac{h_t^{1+\upsilon}}{1+\upsilon} + \theta \mathrm{E}_t \left[U(b_{t+1}, s_{t+1}, \mathbf{Z}_{t+1}) \right] \tag{1.29}$$

$$+\lambda_t \Bigg(w(\mathbf{Z}_t) h_t + \frac{b_t}{1+\pi(\mathbf{Z}_t)} + Q(\mathbf{Z}_t) s_t (1 + r^s(\mathbf{Z}_t)) \tag{1.30}$$

$$+T_t - c_t - \frac{b_{t+1}}{1+R(\mathbf{Z}_t)} - Q(\mathbf{Z}_t) s_{t+1} \Bigg). \tag{1.31}$$

The aggregate state vector, $\mathbf{Z}_t$, contains ζ_t, a_t, and K_t, and its equilibrium law-of-motion is taken as given. The first-order conditions with respect to c_t, h_t, b_{t+1}, and s_{t+1} are

$$\frac{\partial \mathcal{U}(b_t, s_t, \mathbf{Z}_t)}{\partial c_t} : c_t^{-\sigma} - \lambda_t = 0, \tag{1.32}$$

$$\frac{\partial \mathcal{U}(b_t, s_t, \mathbf{Z}_t)}{\partial h_t} : -\chi h_t^{\upsilon} + \lambda_t w_t = 0, \tag{1.33}$$

$$\frac{\partial \mathcal{U}(b_t, s_t, \mathbf{Z}_t)}{\partial b_{t+1}} : -\frac{\lambda_t}{1+R_t} + \beta\theta \mathrm{E}_t \left[U_b(b_{t+1}, s_{t+1}, \mathbf{Z}_{t+1}) \right] = 0, \tag{1.34}$$

$$\frac{\partial \mathcal{U}(b_t, s_t, \mathbf{Z}_t)}{\partial s_{t+1}} : -\lambda_t Q_t + \beta\theta \mathrm{E}_t \left[U_s(b_{t+1}, s_{t+1}, \mathbf{Z}_{t+1}) \right] = 0. \tag{1.35}$$

In equilibrium, the decision rules for bonds, stocks, labor and consumption take the form

$$b_{t+1} = \mathcal{B}(b_t, s_t, \mathbf{Z}_t), \tag{1.36}$$

$$s_{t+1} = \mathcal{S}(b_t, s_t, \mathbf{Z}_t), \tag{1.37}$$

$$h_t = \mathcal{H}(b_t, s_t, \mathbf{Z}_t), \tag{1.38}$$

$$c_t = \mathcal{C}(b_t, s_t, \mathbf{Z}_t). \tag{1.39}$$

We now substitute equations (1.36)—(1.39) into equation (1.29) and differentiate the resulting identity with respect to b_t and s_t to get

$$U_b(b_t, s_t, \mathbf{Z}_t) = c_t^{-\sigma} \left(\frac{1}{1+\pi_t} + \frac{1-\beta}{\beta} \left(\frac{\mathcal{B}_b(b_t, s_t, \mathbf{Z}_t)}{1+R_t} + Q_t \mathcal{S}_b(b_t, s_t, \mathbf{Z}_t) \right) \right), \tag{1.40}$$

$$U_s(b_t, s_t, \mathbf{Z}_t) = c_t^{-\sigma} \left(Q_t (1 + r^s{}_t) + \frac{1-\beta}{\beta} \left(\frac{\mathcal{B}_s(b_t, s_t, \mathbf{Z}_t)}{1+R_t} + Q_t \mathcal{S}_s(b_t, s_t, \mathbf{Z}_t) \right) \right). \tag{1.41}$$

Substituting equations (1.40) and (1.41) into equations (1.33)—(1.35), using equation (1.32) to eliminate the Lagrange multiplier, and aggregating across the unit-mass of identical households gives

$$C_t^{-\sigma} w_t = -\chi H_t^{\upsilon}, \tag{1.42}$$

$$\begin{aligned} \frac{C_t^{-\sigma}}{R_t} &= \theta \mathrm{E}_t \Bigg[C_{t+1}^{-\sigma} \Bigg(\frac{\beta}{1+\pi_{t+1}} \\ &\quad + (1-\beta) \Bigg(\frac{\mathcal{B}_B\left(B_{t+1}, S_{t+1}, \mathbf{Z}_{t+1}\right)}{1+R_{t+1}} \\ &\quad + Q_{t+1} \mathcal{S}_B\left(B_{t+1}, S_{t+1}, \mathbf{Z}_{t+1}\right)\Bigg)\Bigg)\Bigg] \end{aligned} \tag{1.43}$$

$$\begin{aligned} C_t^{-\sigma} Q_t &= \theta \mathrm{E}_t \left[C_{t+1}^{-\sigma} \left(\beta Q_{t+1} \left(1 + r_{t+1}^s\right) \right.\right. \\ &\quad + (1-\beta) \Bigg(\frac{\mathcal{B}_S\left(B_{t+1}, S_{t+1}, \mathbf{Z}_{t+1}\right)}{1+R_{t+1}} \\ &\quad \left.\left. + Q_{t+1} \mathcal{S}_S\left(B_{t+1}, S_{t+1}, \mathbf{Z}_{t+1}\right)\Bigg)\right)\right], \end{aligned} \tag{1.44}$$

where C_t and H_t represent aggregate consumption and labor, respectively. Finally, with bonds in zero-net-supply ($B_t = 0 \ \forall \ t$) and stocks in fixed-net-supply ($S_t = 1 \ \forall \ t$, where this normalization is without loss of generality), we have that $\mathcal{B}_B\left(B_{t+1}, S_{t+1}, \mathbf{Z}_{t+1}\right) = \mathcal{B}_S\left(B_{t+1}, S_{t+1}, \mathbf{Z}_{t+1}\right) = \mathcal{S}_B\left(B_{t+1}, S_{t+1}, \mathbf{Z}_{t+1}\right) = \mathcal{S}_S\left(B_{t+1}, S_{t+1}, \mathbf{Z}_{t+1}\right) = 0$, and equations (1.42)—(1.44) simplify to equations (1.7)—(1.9) in the main text. The fact that these derivatives all equal zero simply means that households cannot use the accumulation of bonds and/or stocks to constrain their future selves.

1.A.2 Firm's problem

To formulate the representative firm's decision problem, we first substitute the production function and the demand function for the firm's good into its profit function. With these substitutions, the firm's decision problem takes the form

$$\begin{aligned}\mathcal{W}\left(k_t, p_{t-1}, \mathbf{Z}_t\right) &= \max_{\{p_t, k_{t+1}\}} \left[p_t^{1-\varepsilon_t} Y\left(\mathbf{Z}_t\right)(1-\tau) - w\left(\mathbf{Z}_t\right)\left(p_t^{-\varepsilon_t} Y\left(\mathbf{Z}_t\right) e^{-a_t} k_t^{-\alpha}\right)^{\frac{1}{1-\alpha}}\right. \\ &\quad - \left(k_{t+1} - (1-\delta) k_t\right) - \frac{\omega}{2}\left(\frac{p_t}{p_{t-1}}\left(1+\pi\left(\mathbf{Z}_t\right)\right) - 1\right)^2 Y\left(\mathbf{Z}_t\right) \\ &\quad \left. + \beta\theta \mathrm{E}_t\left[\frac{C\left(\mathbf{Z}_{t+1}\right)^{-\sigma}}{C\left(\mathbf{Z}_t\right)^{-\sigma}} W\left(k_{t+1}, p_t, \mathbf{Z}_{t+1}\right)\right]\right],\end{aligned}$$

where

$$\begin{aligned}W\left(k_t, p_{t-1}, \mathbf{Z}_t\right) &= p_t^{1-\varepsilon_t} Y\left(\mathbf{Z}_t\right)(1-\tau) - w\left(\mathbf{Z}_t\right)\left(p_t^{-\varepsilon_t} Y\left(\mathbf{Z}_t\right) e^{-a_t} k_t^{-\alpha}\right)^{\frac{1}{1-\alpha}} \\ &\quad - \left(k_{t+1} - (1-\delta) k_t\right) - \frac{\omega}{2}\left(\frac{p_t}{p_{t-1}}\left(1+\pi\left(\mathbf{Z}_t\right)\right) - 1\right)^2 Y\left(\mathbf{Z}_t\right) \\ &\quad + \theta \mathrm{E}_t\left[\frac{C\left(\mathbf{Z}_{t+1}\right)^{-\sigma}}{C\left(\mathbf{Z}_t\right)^{-\sigma}} W\left(k_{t+1}, p_t, \mathbf{Z}_{t+1}\right)\right],\end{aligned} \tag{1.45}$$

and where the aggregate state is $\mathbf{Z}_t = \begin{bmatrix} \zeta_t & a_t & K_t \end{bmatrix}'$ and its equilibrium law-of-motion is taken as given.

The first-order conditions can be written as

$$\frac{\partial \mathcal{W}\left(k_t, p_{t-1}, \mathbf{Z}_t\right)}{\partial k_{t+1}} : -1 + \beta\theta \mathrm{E}_t\left[\frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}} W_k\left(k_{t+1}, p_t, \mathbf{Z}_{t+1}\right)\right] = 0, \tag{1.46}$$

$$\begin{aligned}\frac{\partial \mathcal{W}\left(k_t, p_{t-1}, \mathbf{Z}_t\right)}{\partial p_t} &: \left(1-\varepsilon_t\right) p_t^{-\varepsilon_t} Y_t (1-\tau) + \frac{\varepsilon_t}{1-\alpha} w_t p_t^{-\varepsilon_t\left(\frac{\alpha}{1-\alpha}\right)}\left(Y_t e^{-a_t} k_t^{-\alpha}\right)^{\frac{1}{1-\alpha}} \\ &\quad -\omega\left(\frac{p_t}{p_{t-1}}\left(1+\pi_t\right) - 1\right) Y_t \frac{1+\pi_t}{p_{t-1}} + \beta\theta \mathrm{E}_t\left[\frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}} W_p\left(k_{t+1}, p_t, \mathbf{Z}_{t+1}\right)\right] = 0.\end{aligned} \tag{1.47}$$

In order to find $W_k\left(k_t, p_{t-1}, \mathbf{Z}_t\right)$ and $W_p\left(k_t, p_{t-1}, \mathbf{Z}_t\right)$ we substitute the solution

$$k_{t+1} = \mathcal{K}\left(k_t, p_{t-1}, \mathbf{Z}_t\right),$$

$$p_t = \mathcal{P}\left(k_t, p_{t-1}, \mathbf{Z}_t\right),$$

into equation (1.45) and differentiate the resulting identity with respect to k_t and p_{t-1}. From the first-order conditions we obtain

$$\begin{aligned} W_k\left(k_t, p_{t-1}, \mathbf{Z}_t\right) =& \frac{\alpha}{1-\alpha} w_t \frac{h_t}{k_t} + 1 - \delta + \frac{1-\beta}{\beta} \mathcal{K}_k\left(k_t, p_{t-1}, \mathbf{Z}_t\right) \\ &+ \frac{1-\beta}{\beta}\left(\omega\left(\frac{p_t}{p_{t-1}}\left(1+\pi_t\right)-1\right) Y_t \frac{1+\pi_t}{p_{t-1}}\right. \\ &\left. -\left(1-\varepsilon_t\right)\left(1-\tau\right) p_t^{-\varepsilon_t} Y_t - \frac{\varepsilon_t}{1-\alpha} \frac{w_t}{p_t} h_t\right) \mathcal{P}_k\left(k_t, p_{t-1}, \mathbf{Z}_t\right), \end{aligned} \tag{1.48}$$

$$\begin{aligned} W_p\left(k_t, p_{t-1}, \mathbf{Z}_t\right) =& \omega\left(\frac{p_t}{p_{t-1}}\left(1+\pi_t\right)-1\right) \frac{p_t}{p_{t-1}^2}\left(1+\pi_t\right) Y_t + \frac{1-\beta}{\beta} \mathcal{K}_p\left(k_t, p_{t-1}, \mathbf{Z}_t\right) \\ &+ \frac{1-\beta}{\beta}\left(-\left(1-\varepsilon_t\right) p_t^{-\varepsilon_t} Y_t\left(1-\tau\right) - \frac{\varepsilon_t}{1-\alpha} \frac{w_t}{p_t} h_t\right. \\ &\left. +\omega\left(\frac{p_t}{p_{t-1}}\left(1+\pi_t\right)-1\right) Y_t \frac{1+\pi_t}{p_{t-1}}\right) \mathcal{P}_p\left(k_t, p_{t-1}, \mathbf{Z}_t\right). \end{aligned} \tag{1.49}$$

We next substitute equations (1.48) and (1.49) into equation (1.46) and (1.47), and aggregate across firms. In a symmetric equilibrium in which all firms set the same price, so that the price of their good relative to that of the aggregate goods always equals one, this aggregation implies $\mathcal{P}_P\left(\mathbf{Z}_t\right) = \mathcal{P}_K\left(\mathbf{Z}_t\right) = \mathcal{K}_P\left(\mathbf{Z}_t\right) = 0$. To understand why aggregation implies $\mathcal{P}_P\left(\mathbf{Z}_t\right) = \mathcal{P}_K\left(\mathbf{Z}_t\right) = \mathcal{K}_P\left(\mathbf{Z}_t\right) = 0$, notice that if one firm sets the individual price above (below) the aggregate price so that $\mathcal{P}_P\left(\mathbf{Z}_t\right) \neq 0$ then all firms would do the same and the relative price would not equal one, which is inconsistent with the definition of the economy's aggregate price. Further, because the optimal relative price equals to one, it does not vary with the level of aggregate capital, so $\mathcal{P}_K\left(\mathbf{Z}_t\right) = 0$. Lastly, the fact that the optimal relative price always equals one means that $\mathcal{K}_P\left(\mathbf{Z}_t\right) = 0$. As a consequence, after aggregation we get

$$C_t^{-\sigma} = \beta\theta \mathrm{E}_t\left[C_{t+1}^{-\sigma}\left(\frac{\alpha}{1-\alpha} w_{t+1} \frac{H_{t+1}}{K_{t+1}} + 1 - \delta + \frac{1-\beta}{\beta} \mathcal{K}_K\left(\mathbf{Z}_{t+1}\right)\right)\right], \tag{1.50}$$

and

$$\pi_t\left(1+\pi_t\right) = \frac{\left(1-\varepsilon_t\right)\left(1-\tau\right)}{\omega} + \frac{\varepsilon_t}{\omega\left(1-\alpha\right)} \frac{w_t H_t}{Y_t} + \beta\theta \mathrm{E}_t\left[\frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}} \frac{\pi_{t+1}\left(1+\pi_{t+1}\right) Y_{t+1}}{Y_t}\right], \tag{1.51}$$

respectively.

Now, let us define real marginal costs, x_t, the shadow real rental rate of capital, r_t^k, and the real wage, w_t, according to

$$x_t = \frac{1}{1-\alpha}\frac{w_t H_t}{Y_t},$$

$$r_t^k = \alpha x_t \frac{Y_t}{K_t}, \tag{1.52}$$

$$w_t = (1-\alpha)\, x_t \frac{Y_t}{H_t}, \tag{1.53}$$

then equations (1.50) and (1.51) become

$$1 = \beta\theta \mathrm{E}_t \left[\frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}} \left(\alpha \frac{x_{t+1} Y_{t+1}}{K_{t+1}} + 1 - \delta + \frac{1-\beta}{\beta} \mathcal{K}_K \left(\mathbf{Z}_{t+1} \right) \right) \right], \tag{1.54}$$

$$\pi_t \left(1+\pi_t\right) = \frac{\varepsilon_t}{\omega} \left(x_t + \frac{\left(1-\varepsilon_t\right)\left(1-\tau\right)}{\varepsilon_t} \right) + \beta\theta \mathrm{E}_t \left[\frac{C_{t+1}^{-\sigma} Y_{t+1}}{C_t^{-\sigma} Y_t} \pi_{t+1} \left(1+\pi_{t+1}\right) \right]. \tag{1.55}$$

Equations (1.52) and (1.53) correspond to equations (1.12) and (1.13) in the main text and equations (1.54) and (1.55) correspond to equations (1.10) and (1.11) in the main text.

Finally, we note that aggregate profits distributed to households through dividends are given by

$$Q_t r_t^s = \left(1 - \tau - x_t - \frac{\omega}{2}\pi_t^2\right) Y_t + r_t^k K_t - \left(K_{t+1} - \left(1-\delta\right) K_t\right),$$

which corresponds to equation (1.14) in the main text.

1.A.3 Private sector equations

Collecting all of the first-order conditions from Appendices A.1 and A.2 together, and rearranging, we get

$$C_t^{-\sigma} w_t = \chi H_t^{\upsilon},$$

$$\frac{C_t^{-\sigma}}{1+R_t} = \beta\theta \mathrm{E}_t \left[\frac{C_{t+1}^{-\sigma}}{1+\pi_{t+1}} \right],$$

$$C_t^{-\sigma} Q_t = \beta\theta \mathrm{E}_t \left[C_{t+1}^{-\sigma} \left(Q_{t+1} + \left(1 - \tau - x_{t+1} - \frac{\omega}{2}\pi_{t+1}^2\right) Y_{t+1} + r_{t+1}^k K_{t+1} - I_{t+1} \right) \right],$$

$$I_t = K_{t+1} - (1-\delta) K_t,$$

$$C_t + K_{t+1} = Y_t + (1-\delta) K_t - \frac{\omega}{2}\pi_t^2 Y_t,$$

$$C_t^{-\sigma} = \beta\theta \mathrm{E}_t \left[C_{t+1}^{-\sigma} \left(r_{t+1}^k + 1 - \delta + \frac{1-\beta}{\beta} \mathcal{K}_K (\mathbf{Z}_{t+1}) \right) \right],$$

$$\pi_t (1+\pi_t) = \frac{(\varepsilon_t - 1)(1-\tau) - \varepsilon_t x_t}{\omega} + \beta\theta \mathrm{E}_t \left[\frac{C_{t+1}^{-\sigma} Y_{t+1}}{C_t^{-\sigma} Y_t} \pi_{t+1} (1+\pi_{t+1}) \right],$$

$$Y_t = e^{a_t} K_t^{\alpha} H_t^{1-\alpha},$$

$$r_t^k = \frac{\alpha}{1-\alpha} w_t \frac{H_t}{K_t},$$

$$w_t = (1-\alpha) x_t \frac{Y_t}{H_t}.$$

1.B Appendix: The model where household's own capital

Here we consider an alternative version of the model in which household's rather than firms own the capital stock. With households owning the capital stock we assume that there is a perfectly competitive market in which firms can rent the capital from households.

1.B.1 Household's problem

With household's owning the capital stock their optimization problem becomes

$$\begin{aligned}\mathcal{U}\left(k_t, b_t, s_t, \mathbf{Z}_t\right) &= \min_{\{\lambda_t\}} \max_{\{c_t, h_t, k_{t+1t}, b_{t+1}, s_{t+1}\}} \left[\frac{c_t^{1-\sigma}-1}{1-\sigma} - \chi \frac{h_t^{1+\upsilon}}{1+\upsilon}\right. \\ &+\beta\theta \mathrm{E}_t\left[U\left(k_{t+1}, b_{t+1}, s_{t+1}, \mathbf{Z}_{t+1}\right)\right] \\ &+\lambda_t\left(\left(1-\delta+r^k\left(\mathbf{Z}_t\right)\right) k_t+\frac{b_t}{1+\pi\left(\mathbf{Z}_t\right)}+Q\left(\mathbf{Z}_t\right) s_t\left(1+r^s\left(\mathbf{Z}_t\right)\right)\right. \\ &\left.\left.+w\left(\mathbf{z}_t\right) h_t+T_t-c_t-\frac{b_{t+1}}{1+R\left(\mathbf{Z}_t\right)}-k_{t+1}-Q\left(\mathbf{Z}_t\right) s_{t+1}\right)\right],\end{aligned}$$

with the continuation value given recursively by

$$\begin{aligned}U\left(k_t, b_t, s_t, \mathbf{Z}_t\right) &= \frac{c_t^{1-\sigma}-1}{1-\sigma}-\chi \frac{h_t^{1+\upsilon}}{1+\upsilon}+\theta \mathrm{E}_t\left[U\left(k_{t+1}, b_{t+1}, s_{t+1}, \mathbf{Z}_{t+1}\right)\right] \\ &+\lambda_t\left(\left(1-\delta+r^k\left(\mathbf{Z}_t\right)\right) k_t+\frac{b_t}{1+\pi\left(\mathbf{Z}_t\right)}\right. \\ &+w\left(\mathbf{Z}_t\right) h_t+Q\left(\mathbf{Z}_t\right) s_t\left(1+r^s\left(\mathbf{Z}_t\right)\right) \\ &\left.+T_t-c_t-\frac{b_{t+1}}{1+R\left(\mathbf{Z}_t\right)}-k_{t+1}-Q\left(\mathbf{Z}_t\right) s_{t+1}\right).\end{aligned} \tag{1.56}$$

The first-order conditions with respect to c_t, h_t, k_{t+1}, b_{t+1}, and s_{t+1} can be written as

$$\frac{\partial \mathcal{U}\left(k_t, b_t, s_t, \mathbf{Z}_t\right)}{\partial c_t}: c_t^{-\sigma}-\lambda_t=0, \tag{1.57}$$

$$\frac{\partial \mathcal{U}\left(k_t, b_t, s_t, \mathbf{Z}_t\right)}{\partial h_t}: -\chi h_t^{\upsilon}+\lambda_t w_t=0, \tag{1.58}$$

$$\frac{\partial \mathcal{U}\left(k_t, b_t, s_t, \mathbf{Z}_t\right)}{\partial k_{t+1}}: -c_t^{-\sigma}+\beta\theta \mathrm{E}_t\left[U_k\left(k_{t+1}, b_{t+1}, s_{t+1}, \mathbf{Z}_{t+1}\right)\right]=0, \tag{1.59}$$

$$\frac{\partial \mathcal{U}\left(k_t, b_t, s_t, \mathbf{Z}_t\right)}{\partial b_{t+1}}: -\frac{c_t^{-\sigma}}{1+R_t}+\beta\theta \mathrm{E}_t\left[U_b\left(k_{t+1,} b_{t+1}, s_{t+1}, \mathbf{Z}_{t+1}\right)\right]=0, \tag{1.60}$$

$$\frac{\partial \mathcal{U}\left(k_t, b_t, s_t, \mathbf{Z}_t\right)}{\partial s_{t+1}}: -c_t^{-\sigma} Q_t+\beta\theta \mathrm{E}_t\left[U_s\left(k_{t+1,} b_{t+1}, s_{t+1}, \mathbf{Z}_{t+1}\right)\right]=0. \tag{1.61}$$

In order to find $U_k(k_t, b_t, s_t, \mathbf{Z}_t)$, $U_b(k_t, b_t, s_t, \mathbf{Z}_t)$, $U_s(k_t, b_t, s_t, \mathbf{Z}_t)$, we note that the solution we seek will give us the decision rules

$$k_{t+1} = \mathcal{K}(k_t, b_t, s_t, \mathbf{Z}_t),$$

$$b_{t+1} = \mathcal{B}(k_t, b_t, s_t, \mathbf{Z}_t),$$

$$s_{t+1} = \mathcal{S}(k_t, b_t, s_t, \mathbf{Z}_t),$$

$$h_t = \mathcal{H}(k_t, b_t, s_t, \mathbf{Z}_t),$$

which we substitute into equation (1.56) and differentiate the resulting identity with respect to k_t, b_t, and s_t. From the resulting derivatives, and employing equations (1.57)—(1.61), we obtain

$$\begin{aligned} U_k(k_t, b_t, s_t, \mathbf{Z}_t) &= c_t^{-\sigma}\Bigg(1-\delta+r_t^k+\frac{1-\beta}{\beta}\Big(\mathcal{K}_k(k_t, b_t, s_t, \mathbf{Z}_t) \\ &\quad +\frac{\mathcal{B}_k(k_t, b_t, s_t, \mathbf{Z}_t)}{1+R_t}+Q_t\mathcal{S}_k(k_t, b_t, s_t, \mathbf{Z}_t)\Big)\Bigg), \end{aligned} \tag{1.62}$$

$$\begin{aligned} U_b(k_t, b_t, s_t, \mathbf{Z}_t) &= c_t^{-\sigma}\Bigg(\frac{1}{1+\pi_t}+\frac{1-\beta}{\beta}\Big(\mathcal{K}_b(k_t, b_t, s_t, \mathbf{Z}_t) \\ &\quad +\frac{\mathcal{B}_b(k_t, b_t, s_t, \mathbf{Z}_t)}{1+R_t}+Q_t\mathcal{S}_b(k_t, b_t, s_t, \mathbf{Z}_t)\Big)\Bigg), \end{aligned} \tag{1.63}$$

$$\begin{aligned} U_s(k_t, b_t, s_t, \mathbf{Z}_t) &= c_t^{-\sigma}\Bigg(Q_t(1+r_t^s)+\frac{1-\beta}{\beta}\Big(\mathcal{K}_s(k_t, b_t, s_t, \mathbf{Z}_t) \\ &\quad +\frac{\mathcal{B}_s(k_t, b_t, s_t, \mathbf{Z}_t)}{1+R_t}+Q_t\mathcal{S}_s(k_t, b_t, s_t, \mathbf{Z}_t)\Big)\Bigg). \end{aligned} \tag{1.64}$$

With bonds in zero-net-supply ($B_t = 0\ \forall\ t$) and stocks in fixed-net-supply ($S_t = 1\ \forall\ t$), we have $\mathcal{B}_B(K_t, B_t, S_t, \mathbf{Z}_t) = \mathcal{B}_S(K_t, B_t, S_t, \mathbf{Z}_t) = \mathcal{S}_B(K_t, B_t, S_t, \mathbf{Z}_t) = \mathcal{S}_S(K_t, B_t, S_t, \mathbf{Z}_t) = \mathcal{K}_B(K_t, B_t, S_t, \mathbf{Z}_t) = \mathcal{K}_S(K_t, B_t, S_t, \mathbf{Z}_t) = 0$, so substituting equations (1.62)—(1.64) into equations (1.58)—(1.61), aggregating across households, and using equation (1.57) to eliminate the Lagrange multiplier gives

$$C_t^{-\sigma}w_t = \chi H_t^{\upsilon},$$

$$C_t^{-\sigma} = \beta\theta \mathrm{E}_t \left[C_{t+1}^{-\sigma} \left(r_{t+1}^k + 1 - \delta + \frac{1-\beta}{\beta} \mathcal{K}_K \left(K_{t+1}, B_{t+1}, S_{t+1}, \mathbf{Z}_{t+1} \right) \right) \right],$$

$$\frac{C_t^{-\sigma}}{1+R_t} = \beta\theta \mathrm{E}_t \left[\frac{C_{t+1}^{-\sigma}}{1+\pi_{t+1}} \right],$$

$$C_t^{-\sigma} Q_t = \beta\theta \mathrm{E}_t \left[C_{t+1}^{-\sigma} Q_{t+1} \left(1 + r_{t+1}^s \right) \right].$$

1.B.2 Firm's problem

The firm's decision problem takes the form

$$\begin{aligned} \mathcal{W}\left(p_{t-1}, \mathbf{Z}_t\right) &= \max_{\{p_t, k_t\}} \Big[p_t^{1-\varepsilon_t} Y\left(\mathbf{Z}_t\right)(1-\tau) - w\left(\mathbf{Z}_t\right) \left(p_t^{-\varepsilon_t} Y\left(\mathbf{Z}_t\right) e^{-a_t} k_t^{-\alpha} \right)^{\frac{1}{1-\alpha}} \\ &\quad - r^k\left(\mathbf{Z}_t\right) k_t - \frac{\omega}{2} \left(\frac{p_t}{p_{t-1}} \left(1 + \pi\left(\mathbf{Z}_t\right)\right) - 1 \right)^2 Y\left(\mathbf{Z}_t\right) \\ &\quad + \beta\theta \mathrm{E}_t \left[\frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}} W_{t+1}\left(p_t, \mathbf{Z}_{t+1}\right) \right] \Big], \end{aligned}$$

where the firm's continuation value satisfies

$$\begin{aligned} W\left(p_{t-1}, \mathbf{Z}_t\right) &= p_t^{1-\varepsilon_t} Y\left(\mathbf{Z}_t\right)(1-\tau) - w\left(\mathbf{Z}_t\right) \left(p_t^{-\varepsilon_t} Y\left(\mathbf{Z}_t\right) e^{-a_t} k_t^{-\alpha} \right)^{\frac{1}{1-\alpha}} \\ &\quad - r^k\left(\mathbf{Z}_t\right) k_t - \frac{\omega}{2} \left(\frac{p_t}{p_{t-1}} \left(1 + \pi\left(\mathbf{Z}_t\right)\right) - 1 \right)^2 Y\left(\mathbf{Z}_t\right) \\ &\quad + \beta\theta \mathrm{E}_t \left[\frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}} W_{t+1}\left(p_t, \mathbf{Z}_{t+1}\right) \right]. \end{aligned} \tag{1.65}$$

The first-order conditions can be written as

$$\frac{\partial \mathcal{W}\left(p_{t-1}, \mathbf{Z}_t\right)}{\partial k_t} : \frac{\alpha}{1-\alpha} w_t \frac{h_t}{k_t} - r_t^k = 0, \tag{1.66}$$

$$\begin{aligned} \frac{\partial \mathcal{W}\left(p_{t-1}, \mathbf{Z}_t\right)}{\partial p_t} &: (1-\varepsilon_t) p_t^{-\varepsilon_t} Y_t (1-\tau) + \frac{\varepsilon_t}{1-\alpha} w_t p_t^{-\varepsilon_t\left(\frac{\alpha}{1-\alpha}\right)} \left(Y_t e^{-a_t} k_t^{-\alpha} \right)^{\frac{1}{1-\alpha}} \\ &- \omega \left(\frac{p_t}{p_{t-1}} (1+\pi_t) - 1 \right) Y_t \frac{(1+\pi_t)}{p_{t-1}} + \beta\theta \mathrm{E}_t \left[\frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}} W_p\left(p_t, \mathbf{Z}_{t+1}\right) \right]. \end{aligned} \tag{1.67}$$

In order to find $W_p(p_{t-1}, \mathbf{Z}_t)$ we substitute the decision rules

$$k_t = \mathcal{K}(p_{t-1}, \mathbf{Z}_t),$$

$$p_t = \mathcal{P}(p_{t-1}, \mathbf{Z}_t),$$

into (1.65) and differentiate the resulting identity with respect to p_{t-1}. We then use the first-order conditions, equations (1.66) and (1.67), to obtain

$$\begin{aligned}
W_p(p_{t-1}, \mathbf{Z}_t) &= \omega\left(\frac{p_t}{p_{t-1}}(1+\pi_t)-1\right)\frac{p_t}{p_{t-1}^2}(1+\pi_t)Y_t - \frac{1-\beta}{\beta}\frac{\varepsilon_t}{1-\alpha}\frac{w_t}{p_t}h_t\mathcal{P}_p(p_{t-1}, \mathbf{Z}_t) \\
&- \frac{1-\beta}{\beta}(1-\varepsilon_t)(1-\tau)p_t^{-\varepsilon_t}Y_t\mathcal{P}_p(p_{t-1}, \mathbf{Z}_t) \\
&+ \frac{1-\beta}{\beta}\omega\left(\frac{p_t}{p_{t-1}}(1+\pi_t)-1\right)Y_t\frac{(1+\pi_t)}{p_{t-1}}\mathcal{P}_p(p_{t-1}, \mathbf{Z}_t). \qquad (1.68)
\end{aligned}$$

Substituting equation (1.68) into equations (1.66) and (1.65) and aggregating across firms, which in a symmetric equilibrium where all firms set the same price, implies $\mathcal{P}_p(p_t, \mathbf{Z}_{t+1}) = 0$, yields

$$\pi_t(1+\pi_t) = \frac{(1-\varepsilon_t)(1-\tau)}{\omega} + \frac{\varepsilon_t}{\omega}x_t + \beta\theta\mathrm{E}_t\left[\frac{C_{t+1}^{-\sigma}Y_{t+1}}{C_t^{-\sigma}Y_t}(\pi_{t+1}(1+\pi_{t+1}))\right],$$

where

$$w_t = (1-\alpha)x_t\frac{Y_t}{H_t}.$$

Finally, the dividends distributed to households are given by

$$Q_t r_t^s = Y_t(1-\tau) - w_t H_t - r_t^k K_t - \frac{\omega}{2}\pi_t^2 Y_t.$$

1.B.3 Private sector equations

Collecting all of the first-order conditions together, and rearranging, we get

$$C_t^{-\sigma}w_t = \chi H_t^{\upsilon},$$

$$\frac{C_t^{-\sigma}}{1+R_t} = \beta\theta \mathrm{E}_t\left[\frac{C_{t+1}^{-\sigma}}{1+\pi_{t+1}}\right],$$

$$C_t^{-\sigma}Q_t = \beta\theta \mathrm{E}_t\left[C_{t+1}^{-\sigma}\left(Q_{t+1} + \left(1-\tau-x_{t+1}-\frac{\omega}{2}\pi_{t+1}^2\right)Y_{t+1}\right)\right],$$

$$C_t + K_{t+1} = Y_t + (1-\delta)K_t - \frac{\omega}{2}\pi_t^2 Y_t,$$

$$C_t^{-\sigma} = \beta\theta \mathrm{E}_t\left[C_{t+1}^{-\sigma}\left(r_{t+1}^k + 1 - \delta + \frac{1-\beta}{\beta}\mathcal{K}_K(\mathbf{Z}_{t+1})\right)\right],$$

$$\pi_t(1+\pi_t) = \frac{(\varepsilon_t - 1)(1-\tau) - \varepsilon_t x_t}{\omega} + \beta\theta \mathrm{E}_t\left[\frac{C_{t+1}^{-\sigma}Y_{t+1}}{C_t^{-\sigma}Y_t}\pi_{t+1}(1+\pi_{t+1})\right],$$

$$Y_t = e^{a_t}K_t^{\alpha}H_t^{1-\alpha},$$

$$r_t^k = \frac{\alpha}{1-\alpha}w_t\frac{H_t}{K_t},$$

$$w_t = (1-\alpha)\frac{x_tY_t}{H_t},$$

which are equivalent to the equations reported in Appendix A.3 that were obtained under the assumption that firm's own the capital stock.

1.C Appendix: Discretionary policy

The decision problem facing the discretionary policymaker is summarized by the Bellman equation

$$\mathcal{V}(\mathbf{Z}_t) = \max_{\{C_t,H_t,Y_t,x_t,K_{t+1},\pi_t\}}\left(\frac{C_t^{1-\sigma}-1}{1-\sigma} - \chi\frac{H_t^{1+\upsilon}}{1+\upsilon} + \gamma\xi \mathrm{E}_t\left[V(\mathbf{Z}_{t+1})\right]\right), \tag{1.69}$$

which is subject to the constraints

$$C_t^{-\sigma} = \theta \mathrm{E}_t\left[L(\mathbf{Z}_{t+1})\right], \tag{1.70}$$

$$\left(\pi_t(1+\pi_t) + \frac{(\varepsilon_t-1)(1-\tau)-\varepsilon_t x_t}{\omega}\right)Y_tC_t^{-\sigma} = \theta \mathrm{E}_t\left[M(\mathbf{Z}_{t+1})\right], \tag{1.71}$$

$$\left(1-\frac{\omega}{2}\pi_t^2\right)Y_t = C_t + K_{t+1} - (1-\delta)K_t, \tag{1.72}$$

$$H_t = \left(\frac{1-\alpha}{\chi} x_t Y_t C_t^{-\sigma}\right)^{\frac{1}{1+\upsilon}}, \tag{1.73}$$

$$Y_t = e^{a_t} K_t^{\alpha} H_t^{1-\alpha}, \tag{1.74}$$

with the continuation value satisfying the recursion

$$V(\mathbf{Z}_t) = \frac{C_t^{1-\sigma} - 1}{1-\sigma} - \frac{\chi}{1+\upsilon} H_t^{1+\upsilon} + \xi \mathrm{E}_t \left[V(\mathbf{Z}_{t+1})\right].$$

Because monetary policy is conducted under discretion, the central bank cannot influence how private sector expectations are formed, a restriction imposed by introducing the auxiliary variables $L(\mathbf{Z}_t)$ and $M(\mathbf{Z}_t)$, which are defined according to

$$L(\mathbf{Z}_t) = C_t^{-\sigma} \left[\beta \left(\alpha x_t \frac{Y_t}{K_t} + 1 - \delta\right) + (1-\beta) \mathcal{K}_K(\mathbf{Z}_t)\right],$$

$$M(\mathbf{Z}_t) = \beta C_t^{-\sigma} Y_t \pi_t (1+\pi_t).$$

After substituting equations (1.73) and (1.74) into equations (1.69)—(1.72), the central bank's decision problem can be expressed in terms of the Lagrangian

$$\begin{aligned}
\mathcal{V}(\mathbf{Z}_t) \;=\;& \frac{C_t^{1-\sigma} - 1}{1-\sigma} - \chi \frac{\left(\frac{1-\alpha}{\chi} x_t e^{a_t} K_t^{\alpha} C_t^{-\sigma}\right)^{\frac{1+\upsilon}{\upsilon+\alpha}}}{1+\upsilon} + \gamma \xi \mathrm{E}_t \left[V(\mathbf{Z}_{t+1})\right] \\
& -\phi_{1t} \left(C_t + K_{t+1} - (1-\delta) K_t - \left(1 - \frac{\omega}{2}\pi_t^2\right) e^{a_t} K_t^{\alpha} \left(\frac{1-\alpha}{\chi} x_t e^{a_t} K_t^{\alpha} C_t^{-\sigma}\right)^{\frac{1-\alpha}{\upsilon+\alpha}}\right) \\
& -\phi_{2t} \left(\theta \mathrm{E}_t \left[L(\mathbf{Z}_{t+1})\right] - C_t^{-\sigma}\right) \\
& +\phi_{3t} \left(\frac{(1-\varepsilon_t)(1-\tau) + \varepsilon_t x_t}{\omega} C_t^{-\sigma} e^{a_t} K_t^{\alpha} \left(\frac{1-\alpha}{\chi} x_t e^{a_t} K_t^{\alpha} C_t^{-\sigma}\right)^{\frac{1-\alpha}{\upsilon+\alpha}}\right. \\
& \left. +\theta \mathrm{E}_t \left[M(\mathbf{Z}_{t+1})\right] - \pi_t (1+\pi_t) C_t^{-\sigma} e^{a_t} K_t^{\alpha} \left(\frac{1-\alpha}{\chi} x_t e^{a_t} K_t^{\alpha} C_t^{-\sigma}\right)^{\frac{1-\alpha}{\upsilon+\alpha}}\right),
\end{aligned} \tag{1.75}$$

where ϕ_{1t}, ϕ_{2t}, and ϕ_{3t}, represent the Lagrange multipliers on the three remaining constraints. Now, differentiating equation (1.75) with respect to K_{t+1}, C_t, π_t, and x_t, the first-order conditions are

$$\frac{\partial \mathcal{V}(\mathbf{Z}_t)}{\partial K_{t+1}} : \gamma \xi \mathrm{E}_t \left[V_K(\mathbf{Z}_{t+1})\right] - \phi_{2t} \theta \mathrm{E}_t \left[L_K(\mathbf{Z}_{t+1})\right] + \phi_{3t} \theta \mathrm{E}_t \left[M_K(\mathbf{Z}_{t+1})\right] - \phi_{1t} = 0, \tag{1.76}$$

$$\frac{\partial \mathcal{V}(\mathbf{Z}_t)}{\partial C_t} \quad : \quad C_t^{-\sigma} + \frac{\sigma\chi}{\upsilon+\alpha}\frac{H_t^{1+\upsilon}}{C_t} - \phi_{1t}\left(1+\sigma\frac{1-\alpha}{\upsilon+\alpha}\left(1-\frac{\omega}{2}\pi_t^2\right)\frac{Y_t}{C_t}\right) - \phi_{2t}\sigma C_t^{-\sigma-1}$$
$$- \quad \sigma\frac{1+\upsilon}{\alpha+\upsilon}\phi_{3t}\left(\frac{(1-\varepsilon_t)(1-\tau)+\varepsilon_t x_t}{\omega} - \pi_t(1+\pi_t)\right)C_t^{-\sigma-1}Y_t = 0,$$

$$\frac{\partial \mathcal{V}(\mathbf{Z}_t)}{\partial \pi_t} : -\phi_{3t}(1+2\pi_t)C_t^{-\sigma} - \phi_{1t}\omega\pi_t = 0,$$

$$\frac{\partial \mathcal{V}(\mathbf{Z}_t)}{\partial x_t} \quad : \quad -\frac{\chi}{\upsilon+\alpha}H_t^{1+\upsilon}x_t^{-1} + \phi_{1t}\left(\frac{1-\alpha}{\upsilon+\alpha}\left(1-\frac{\omega}{2}\pi_t^2\right)\frac{Y_t}{x_t}\right)$$
$$+ \quad \phi_{3t}\left(\frac{\varepsilon_t x_t}{\omega} + \frac{1-\alpha}{\upsilon+\alpha}\frac{(1-\varepsilon_t)(1-\tau)+\varepsilon_t x_t}{\omega} - \frac{1-\alpha}{\upsilon+\alpha}\pi_t(1+\pi_t)\right)C_t^{-\sigma}Y_t x_t^{-1} = 0.$$

To progress further we must find $V_K(\mathbf{Z}_t)$. The solution provides the decision rules

$$x_t = \mathcal{X}(\mathbf{Z}_t),$$

$$C_t = \mathcal{C}(\mathbf{Z}_t),$$

$$\pi_t = \pi(\mathbf{Z}_t),$$

$$K_{t+1} = \mathcal{K}(\mathbf{Z}_t),$$

which we substitute into equation (1.75) giving the identity

$$\begin{aligned} V(\mathbf{Z}_t) \quad = \quad & \frac{\mathcal{C}(\mathbf{Z}_t)^{1-\sigma}-1}{1-\sigma} - \chi\frac{\left(\frac{1-\alpha}{\chi}e^{a_t}\mathcal{X}(\mathbf{Z}_t)K_t^{\alpha}\mathcal{C}(\mathbf{Z}_t)^{-\sigma}\right)^{\frac{1+\upsilon}{\upsilon+\alpha}}}{1+\upsilon} \\ & +\xi \mathrm{E}_t\left[V\left(\zeta_{t+1},a_{t+1},\mathcal{K}(\mathbf{Z}_t)\right)\right] - \phi_{1t}\Bigg(\mathcal{C}(\mathbf{Z}_t)+\mathcal{K}(\mathbf{Z}_t)-(1-\delta)K_t \\ & -\left(1-\frac{\omega}{2}\pi(\mathbf{Z}_t)^2\right)\left(\frac{1-\alpha}{\chi}e^{\frac{1+\upsilon}{1-\alpha}a_t}\mathcal{X}(\mathbf{Z}_t)K_t^{\alpha\frac{1+\upsilon}{1-\alpha}}\mathcal{C}(\mathbf{Z}_t)^{-\sigma}\right)^{\frac{1-\alpha}{\upsilon+\alpha}}\Bigg) \\ & -\phi_{2t}\left(\theta \mathrm{E}_t\left[L\left(\zeta_{t+1},a_{t+1},\mathcal{K}(\mathbf{Z}_t)\right)\right]-\mathcal{C}(\mathbf{Z}_t)^{-\sigma}\right) \\ & +\phi_{3t}\Bigg(\frac{(1-\varepsilon_t)(1-\tau)+\varepsilon_t\mathcal{X}(\mathbf{Z}_t)}{\omega}\mathcal{C}(\mathbf{Z}_t)^{-\sigma}\left(\frac{1-\alpha}{\chi}e^{\frac{1+\upsilon}{1-\alpha}a_t}\mathcal{X}(\mathbf{Z}_t)K_t^{\alpha\frac{1+\upsilon}{1-\alpha}}\mathcal{C}(\mathbf{Z}_t)^{-\sigma}\right)^{\frac{1-\alpha}{\upsilon+\alpha}} \\ & +\theta \mathrm{E}_t\left[M\left(\zeta_{t+1},z_{t+1},\mathcal{K}(\mathbf{Z}_t)\right)\right] \\ & -\pi(\mathbf{Z}_t)(1+\pi(\mathbf{Z}_t))\left(\frac{1-\alpha}{\chi}e^{\frac{1+\upsilon}{1-\alpha}a_t}\mathcal{X}(\mathbf{Z}_t)K_t^{\alpha\frac{1+\upsilon}{1-\alpha}}\mathcal{C}(\mathbf{Z}_t)^{-\sigma}\right)^{\frac{1-\alpha}{\upsilon+\alpha}}\Bigg). \end{aligned} \tag{1.77}$$

Then, differentiating equation (1.77) with respect to K_t yields

$$\begin{aligned}
V_K(\mathbf{Z}_t) &= \left(C_t^{-\sigma} + \frac{\sigma\chi}{\upsilon+\alpha}\frac{H_t^{1+\upsilon}}{C_t}\right)\mathcal{C}_K(\mathbf{Z}_t) - \frac{\chi}{\upsilon+\alpha}\frac{H_t^{1+\upsilon}}{x_t}\mathcal{X}_K(\mathbf{Z}_t) \qquad (1.78)\\
&- \frac{\alpha\chi}{\upsilon+\alpha}\frac{H_t^{1+\upsilon}}{K_t} + \xi \mathrm{E}_t\left[V_K(\mathbf{Z}_{t+1})\right]\mathcal{K}_K(\mathbf{Z}_t)\\
&- \phi_{1t}\left(\mathcal{K}_K(\mathbf{Z}_t) + \mathcal{C}_K(\mathbf{Z}_t) - (1-\delta) + \sigma\frac{1-\alpha}{\upsilon+\alpha}\left(1-\frac{\omega}{2}\pi_t^2\right)\frac{Y_t}{C_t}\mathcal{C}_K(\mathbf{Z}_t)\right.\\
&\left. +\omega\pi_t Y_t \pi_K(\mathbf{Z}_t) - \frac{1-\alpha}{\upsilon+\alpha}\left(1-\frac{\omega}{2}\pi_t^2\right)\frac{Y_t}{x_t}\mathcal{X}_K(\mathbf{Z}_t) - \alpha\frac{\upsilon+1}{\upsilon+\alpha}\left(1-\frac{\omega}{2}\pi_t^2\right)\frac{Y_t}{K_t}\right)\\
&- \phi_{2t}\left(\theta\mathrm{E}_t\left[L_K(\mathbf{Z}_{t+1})\right]\mathcal{K}_K(\mathbf{Z}_t) + \sigma C_t^{-\sigma-1}\mathcal{C}_K(\mathbf{Z}_t)\right)\\
&+ \phi_{3t}\left(\theta\mathrm{E}_t\left[M_K(\mathbf{Z}_{t+1})\right]\mathcal{K}_K(\mathbf{Z}_t) - (1+2\pi_t)C_t^{-\sigma}Y_t\pi_K(\mathbf{Z}_t)\right.\\
&-\alpha\frac{\upsilon+1}{\upsilon+\alpha}\left(\pi_t(1+\pi_t) - \frac{(1-\varepsilon_t)(1-\tau)+\varepsilon_t x_t}{\omega}\right)\frac{Y_t}{K_t}C_t^{-\sigma}\\
&-\sigma\frac{\upsilon+1}{\alpha+\upsilon}\left(\frac{(1-\varepsilon_t)(1-\tau)+\varepsilon_t x_t}{\omega} - \pi_t(1+\pi_t)\right)C_t^{-\sigma-1}Y_t\mathcal{C}_K(\mathbf{Z}_t)\\
&\left.-\left(\frac{1-\alpha}{\upsilon+\alpha}\left(\pi_t(1+\pi_t) - \frac{(1-\varepsilon_t)(1-\tau)+\varepsilon_t x_t}{\omega}\right) - \frac{\varepsilon_t x_t}{\omega}\right)\frac{Y_t}{x_t}C_t^{-\sigma}\mathcal{X}_K(\mathbf{Z}_t)\right),
\end{aligned}$$

and using equations (1.76)—(1.78) to simplify we get

$$\begin{aligned}
V_K(\mathbf{Z}_t) &= \left(1-\frac{1}{\gamma}\right)\left(C_t^{-\sigma} + \frac{\sigma\chi}{\upsilon+\alpha}\frac{H_t^{1+\upsilon}}{C_t}\right)\mathcal{C}_K(\mathbf{Z}_t) - \left(1-\frac{1}{\gamma}\right)\frac{\chi}{\upsilon+\alpha}\frac{H_t^{1+\upsilon}}{x_t}\mathcal{X}_K(\mathbf{Z}_t)\\
&- \frac{\alpha\chi}{\upsilon+\alpha}\frac{H_t^{1+\upsilon}}{K_t} + \frac{1}{\gamma}\phi_{1t}\left(\alpha\frac{1+\upsilon}{\upsilon+\alpha}\left(1-\frac{\omega}{2}\pi_t^2\right)\frac{Y_t}{K_t} + 1 - \delta\right)\\
&- \frac{\alpha}{\gamma}\frac{1+\upsilon}{\upsilon+\alpha}\phi_{3t}\left(\pi_t(1+\pi_t) - \frac{(1-\varepsilon_t)(1-\tau)+\varepsilon_t x_t}{\omega}\right)\frac{Y_t}{K_t}C_t^{-\sigma}. \qquad (1.79)
\end{aligned}$$

After substituting equation (1.79) back into equation (1.76), the system of first-order conditions for the discretionary optimization problem can be written as

$$\begin{aligned}
\frac{\partial}{\partial C_t} \; &: \; C_t^{-\sigma} + \frac{\sigma\chi}{\upsilon+\alpha}\frac{H_t^{1+\upsilon}}{C_t} - \phi_{1t}\left(1 + \sigma\frac{1-\alpha}{\upsilon+\alpha}\left(1-\frac{\omega}{2}\pi_t^2\right)\frac{Y_t}{C_t}\right) - \phi_{2t}\sigma C_t^{-\sigma-1}\\
&- \sigma\frac{1+\upsilon}{\alpha+\upsilon}\phi_{3t}\left(\frac{(1-\varepsilon_t)(1-\tau)+\varepsilon_t x_t}{\omega} - \pi_t(1+\pi_t)\right)C_t^{-\sigma-1}Y_t = 0, \qquad (1.80)
\end{aligned}$$

$$\frac{\partial}{\partial \pi_t} \quad : \quad -\phi_{3t}\left(1+2\pi_t\right)C_t^{-\sigma} - \phi_{1t}\omega\pi_t = 0, \tag{1.81}$$

$$- \quad \phi_{2t}\theta \mathrm{E}_t\left[L_K(\mathbf{Z}_{t+1})\right] + \phi_{3t}\theta \mathrm{E}_t\left[M_K(\mathbf{Z}_{t+1})\right] - \phi_{1t} = 0. \tag{1.82}$$

$$\begin{aligned}
\frac{\partial}{\partial x_t} \quad &: \quad -\frac{\chi}{\upsilon+\alpha}\frac{H_t^{1+\upsilon}}{x_t} + \phi_{1t}\frac{1-\alpha}{\upsilon+\alpha}\left(1-\frac{\omega}{2}\pi_t^2\right)\frac{Y_t}{x_t} \\
&+ \quad \phi_{3t}\left(\frac{\varepsilon_t x_t}{\omega} + \frac{1-\alpha}{\upsilon+\alpha}\frac{\left(1-\varepsilon_t\right)\left(1-\tau\right)+\varepsilon_t x_t}{\omega} - \frac{1-\alpha}{\upsilon+\alpha}\pi_t\left(1+\pi_t\right)\right)C_t^{-\sigma}\frac{Y_t}{x_t} = 0,
\end{aligned} \tag{1.83}$$

$$\begin{aligned}
\frac{\partial}{\partial K_{t+1}} \quad &: \quad -\frac{\gamma\xi\alpha\chi}{\upsilon+\alpha}\mathrm{E}_t\left[\frac{H_{t+1}^{1+\upsilon}}{K_{t+1}}\right] + \xi\mathrm{E}_t\left[\phi_{1t+1}\left(\alpha\frac{1+\upsilon}{\upsilon+\alpha}\left(1-\frac{\omega}{2}\pi_{t+1}^2\right)\frac{Y_{t+1}}{K_{t+1}} + 1 - \delta\right)\right] \\
&+ \quad \xi\alpha\frac{1+\upsilon}{\upsilon+\alpha}\mathrm{E}_t\left[\phi_{3t+1}\left(\frac{\left(1-\varepsilon_t\right)\left(1-\tau\right)+\varepsilon_t x_{t+1}}{\omega} - \pi_{t+1}\left(1+\pi_{t+1}\right)\right)\frac{Y_{t+1}}{K_{t+1}}C_{t+1}^{-\sigma}\right] \\
&- \quad \xi\left(1-\gamma\right)\mathrm{E}_t\left[\left(C_{t+1}^{-\sigma} + \frac{\sigma\chi}{\upsilon+\alpha}\frac{H_{t+1}^{1+\upsilon}}{C_{t+1}}\right)\mathcal{C}_K(\mathbf{Z}_{t+1})\right] \\
&+ \quad \frac{\xi\left(1-\gamma\right)\chi}{\upsilon+\alpha}\mathrm{E}_t\left[\frac{H_{t+1}^{1+\upsilon}}{x_{t+1}}\mathcal{X}_K(\mathbf{Z}_{t+1})\right] \\
&- \quad \phi_{2t}\theta\mathrm{E}_t\left[L_K(\mathbf{Z}_{t+1})\right] + \phi_{3t}\theta\mathrm{E}_t\left[M_K(\mathbf{Z}_{t+1})\right] - \phi_{1t} = 0.
\end{aligned} \tag{1.84}$$

where

$$H_t = \left(\left(\frac{1-\alpha}{\chi}\right)e^{a_t}x_t K_t^{\alpha}C_t^{-\sigma}\right)^{\frac{1}{\upsilon+\alpha}}, \tag{1.85}$$

$$Y_t = \left(\left(\frac{1-\alpha}{\chi}\right)^{1-\alpha}e^{(1+\upsilon)a_t}x_t^{1-\alpha}K_t^{\alpha(1+\upsilon)}C_t^{-\sigma(1-\alpha)}\right)^{\frac{1}{\upsilon+\alpha}}, \tag{1.86}$$

$$L(\mathbf{Z}_t) = C_t^{-\sigma}\left(\beta\left(e^{a_t}x_t\alpha K_t^{\alpha-1}H_t^{1-\alpha} + 1 - \delta\right) + \left(1-\beta\right)\mathcal{K}_K(\mathbf{Z}_t)\right), \tag{1.87}$$

$$M(\mathbf{Z}_t) = \beta\pi_t\left(1+\pi_t\right)C_t^{-\sigma}Y_t. \tag{1.88}$$

Equations (1.80)—(1.88) correspond to equations (1.19)—(1.26) in the main text.

1.D Appendix: The benchmark model

In this section we examine the effect that monopolistic competition has on production, consumption, and inflation, in a deterministic environment. Switching the model's stochastic elements off, the effect of varying ε on the model's non-stochastic steady state outcomes, through its consequences for the price markup, are presented in Figure 1.3. To better interpret the effects of monopolistic competition, we also report in Figure 1.3 the steady state results for the flex-price version ($\omega = 0$) of the model. For this exercise, we assume monetary policy is conducted under discretion.

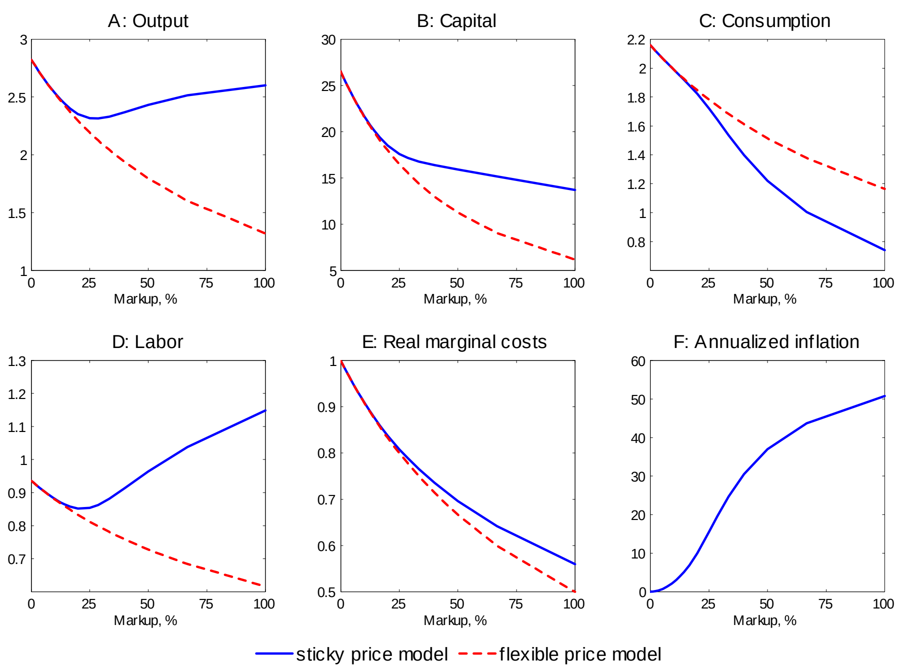

Figure 1.3: Steady state as function of the price markup.

The effect that the price markup has on steady state inflation is shown in Figure 1.3, panel F. When prices are costly to change and there is no production subsidy in place to offset the monopolistic distortion a higher markup leads to higher inflation, with annualized inflation reaching exceedingly high levels as the markup approaches 100 percent. The inflation that occurs as the markup rises is a product of the discretionary central bank's behavior. With monopolistic competition generating inefficiently low output, the central bank lowers the nominal interest rate in order to stimulate demand and raise output. But to meet higher demand for their good firms need to employ more workers, which boosts the demand for labour and pushes up the nominal wage and nominal marginal costs. Facing higher nominal marginal costs firms raise prices, causing inflation. As the markup gets bigger the central bank's efforts to stimulate aggregate demand intensify, giving rise to higher steady state inflation.

Because there are costs to changing prices, non-zero inflation has real costs. These real costs are illustrated in panels A—E through the difference between the solid line, representing the sticky-price model, and the dashed line, representing the flex-price model. Looking at the behavior of the flex-price model, as the markup increases output (panel A), capital (panel B), consumption (panel C), labour (panel D), and real marginal costs (panel E) all decrease monotonically. The higher markup is associated with firms having greater market power and leads to lower production. Lower production means less demand for capital and labour and also leads to declines in consumption and investment. The fact that real marginal costs decrease as the markup increases (panel E) simply reflects the increase in profits associated with firms having greater market power.

When the price markup is not too large, the steady state behavior of the sticky price model is similar to that for the flex-price model. However, as the markup becomes increasingly large important differences between the two models emerge. These differences are driven by the magnitude of inflation and with the output lost due to price-adjustment costs. Specifically, as the markup gets bigger, in order to partly offset the output lost due to price-adjustment costs, firms in the sticky-price model increase their production levels in order to maintain their profitability. As a result, the demand for capital and labour rises in the sticky-price model relative to the flex-price model.[3] Thus, unlike for the flex-price model, where output and labour decline monotonically, in the sticky-price

[3]See also the discussion in Ascari and Rossi (2012).

model an increase in the markup causes output and labour to rise, following an initial fall. For a given markup, steady state output, capital, and labour are all higher in the sticky-price model than in the flex-price model, but this is not the case for consumption, which suffers as goods are devoted to covering price-adjustment costs and to supporting the capital stock. It is also worth noting that real marginal costs are higher in the sticky-price model than the flex-price model, indicating that inflation and the resulting price-adjustment costs have an adverse impact on profits.

Turning to the stochastic model, Table 1.5 reports the mean (standard deviation in brackets) of the stationary distributions for the sticky-price economy under both discretion (column 1) and the Taylor rule (column 2) to those for the flex-price economy (column 3). Comparing the sticky-price and flex-price economies, the main effect of sticky prices is to generate a positive inflation rate (an inflation bias) when policy is conducted under discretion, consistent with Figure 1. With the (stochastic) price markup averaging just over 10 percent, the discretionary central bank's efforts to offset the monopolistic distortion results in higher inflation and a higher nominal interest rate.

Figure 1.4 plots impulse responses for technology shocks under both discretionary policy (solid lines) and the Taylor-rule policy (dashed lines) in the model with sticky prices. Looking at the responses under discretion, a positive technology shock raises the productivity of capital and labour, causing firms to demand more of these inputs, which raises the quantities of capital (panel B) and labour (panel E) traded and increases the real wage (panel F) and the real interest rate (panel J). With more capital and labour employed for production, real output rises (panel A) and the resulting increase in households' real income boosts consumption (panel C). Real marginal costs (panel G) are little-changed by the shock because the productivity increase is captured by higher factor prices. Because real marginal costs are little-affected, firms face minimal pressure to change prices, so inflation too is little-changed by the shock (panel H). As a consequence, monetary policy responds to the shock largely by accommodating it. The higher real return on capital boosts the real return on bonds and the central bank responds by allowing the nominal

Table 1.5: Characteristics of the Stationary Distribution

		Sticky prices		Flexible prices
		Discretion	Taylor rule	
		(1)	(2)	(3)
Output	Y	2.540 [0.104]	2.539 [0.103]	2.534 [0.104]
Capital	K	21.734 [0.936]	21.722 [0.929]	21.664 [0.935]
Consumption	C	1.992 [0.062]	1.991 [0.062]	1.993 [0.062]
Investment	I	0.543 [0.053]	0.543 [0.053]	0.542 [0.053]
Labour	H	0.881 [0.010]	0.881 [0.010]	0.880 [0.011]
Real wage	w	1.756 [0.063]	1.755 [0.063]	1.753 [0.064]
Real marginal cost	x	0.909 [0.008]	0.909 [0.011]	0.909 [0.010]
Annualized inflation	π	2.580 [0.527]	2.519 [0.253]	–
Household welfare	$\mathcal{U}$	29.957 [0.989]	29.960 [0.992]	30.163 [0.988]
Nominal interest rate	R	6.782 [0.497]	6.720 [0.660]	4.097 [0.399]
Real interest rate	r	4.097 [0.384]	4.098 [0.411]	4.097 [0.399]
Rental rate	r^k	14.793 [0.460]	14.793 [0.464]	14.793 [0.482]
Return on capital	r^{cap}	4.098 [0.427]	4.098 [0.431]	4.098 [0.447]

Note: Statistics calculated using 10^6 simulated observations; standard deviations in brackets.

interest rate to rise in line with the higher real interest rate. Qualitatively, the results for the Taylor-rule policy are very similar to the discretionary policy, however it is noticeable that the discretionary policy leads to a much smaller inflation response, at the cost of greater movement in labour (panel E), the real wage (panel F) and output (panel A) when the shock hits.

Turning to Figure 1.5, under discretionary policymaking, a positive shock to the elasticity of substitution among goods leads to a decline in the markup, which has a direct negative impact on inflation (panel H). Greater competition among firms causes output to rise (panel A) and leads to greater demand for capital (panel B) and labour (panel

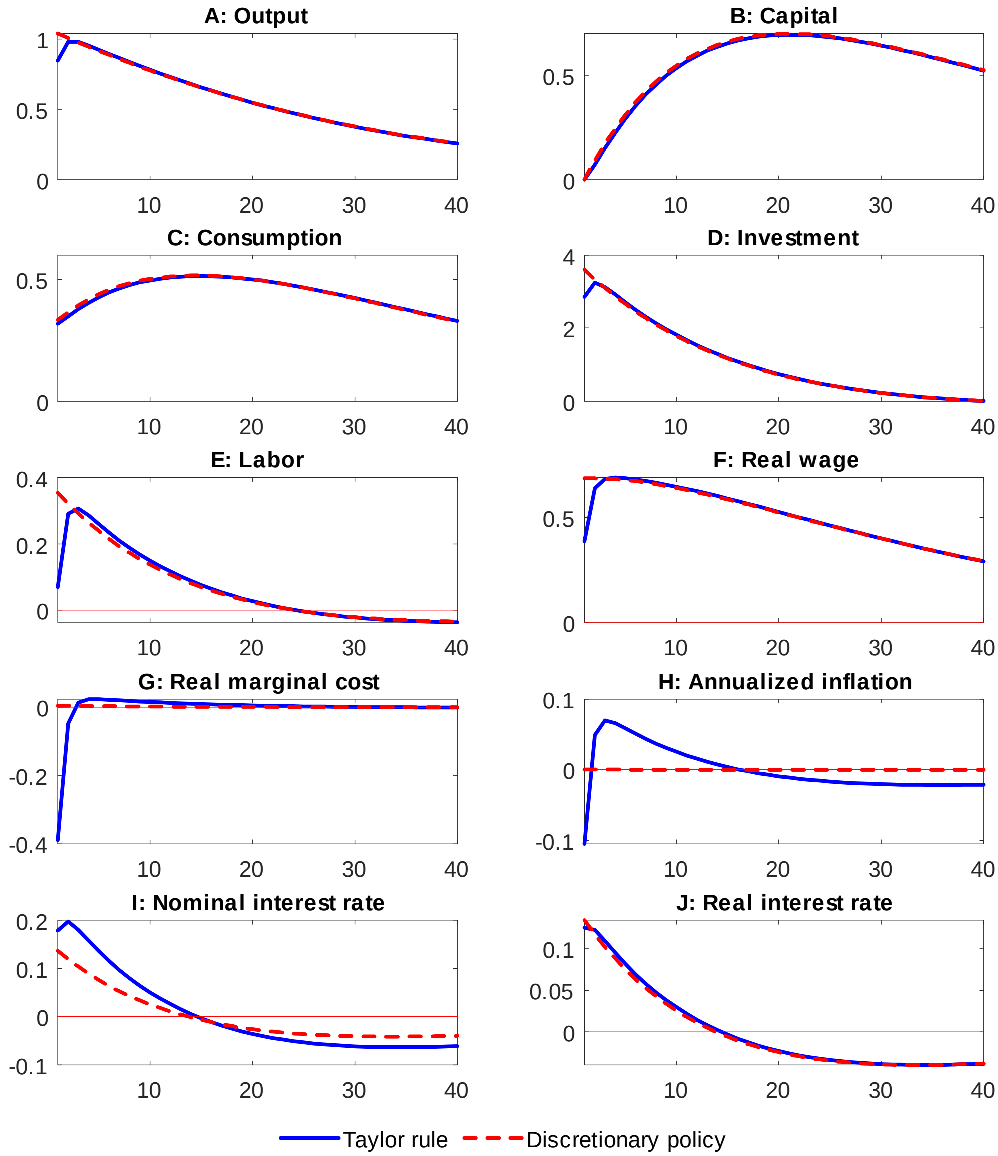

Figure 1.4: Responses to a technology shock under discretion and the Taylor-rule

E). Consumption rises (panel C) as a consequence of higher real income. Increased demand for capital and labour causes the real wage to rise (panel F) and this in turn causes real marginal costs to increase (panel G). Although real marginal costs have gone up, because there is greater competition among firms prices actually fall and inflation goes down (panel H). However, with greater costs and lower prices, firms profitability is adversely affected, which is reflected in a lower stock price. The central bank responds to the shock by lowering the nominal interest rate, but by less than the decline in inflation, allowing the real interest rate to rise and bring the real return on bonds into line with the higher real return on capital. The greatest differences between the discretionary policy and the Taylor rule policy can be seen in the behavior of inflation (panel H), which falls under discretion and rises under the Taylor rule. But this differences in behavior translates into a relatively small difference in the real interest rate (panel J) and the behavior of the real economy is qualitatively similar for the two policies.

1.E Appendix: Numerical solution

To solve the central bank's optimal policy problem, described by equations (1.15)—(1.26) in the main text, it is convenient to rewrite them more compactly as

$$\begin{aligned} 0 &= C_t^{-\sigma} + \frac{\sigma\chi}{\upsilon+\alpha}\frac{H_t^{1+\upsilon}}{C_t} - \phi_{1t}\left(1+\sigma\frac{1-\alpha}{\upsilon+\alpha}\left(1-\frac{\omega}{2}\pi_t^2\right)\frac{Y_t}{C_t}\right) - \phi_{2t}\sigma C_t^{-\sigma-1} \\ & -\sigma\frac{1+\upsilon}{\alpha+\upsilon}\phi_{3t}\left(\frac{\left(1-\varepsilon e^{\zeta_t}\right)(1-\tau)+\varepsilon e^{\zeta_t}x_t}{\omega} - \pi_t\left(1+\pi_t\right)\right)C_t^{-\sigma-1}Y_t, \end{aligned} \tag{1.89}$$

$$0 = -\phi_{3t}\left(1+2\pi_t\right)C_t^{-\sigma} - \phi_{1t}\omega\pi_t, \tag{1.90}$$

$$\begin{aligned} 0 &= -\frac{\chi}{\upsilon+\alpha}\frac{H_t^{1+\upsilon}}{x_t} + \phi_{1t}\frac{1-\alpha}{\upsilon+\alpha}\left(1-\frac{\omega}{2}\pi_t^2\right)\frac{Y_t}{x_t} \\ & +\phi_{3t}\left(\frac{\varepsilon e^{\zeta_t}x_t}{\omega} + \frac{1-\alpha}{\upsilon+\alpha}\frac{\left(1-\varepsilon e^{\zeta_t}\right)(1-\tau)+\varepsilon e^{\zeta_t}x_t}{\omega} - \frac{1-\alpha}{\upsilon+\alpha}\pi_t\left(1+\pi_t\right)\right)C_t^{-\sigma}\frac{Y_t}{x_t} \end{aligned} \tag{1.91}$$

$$0 = \mathcal{D}_{t+1} - \phi_{2t}\theta\mathcal{L}_{K,t+1} + \phi_{3t}\theta\mathcal{M}_{K,t+1} - \phi_{1t} \tag{1.92}$$

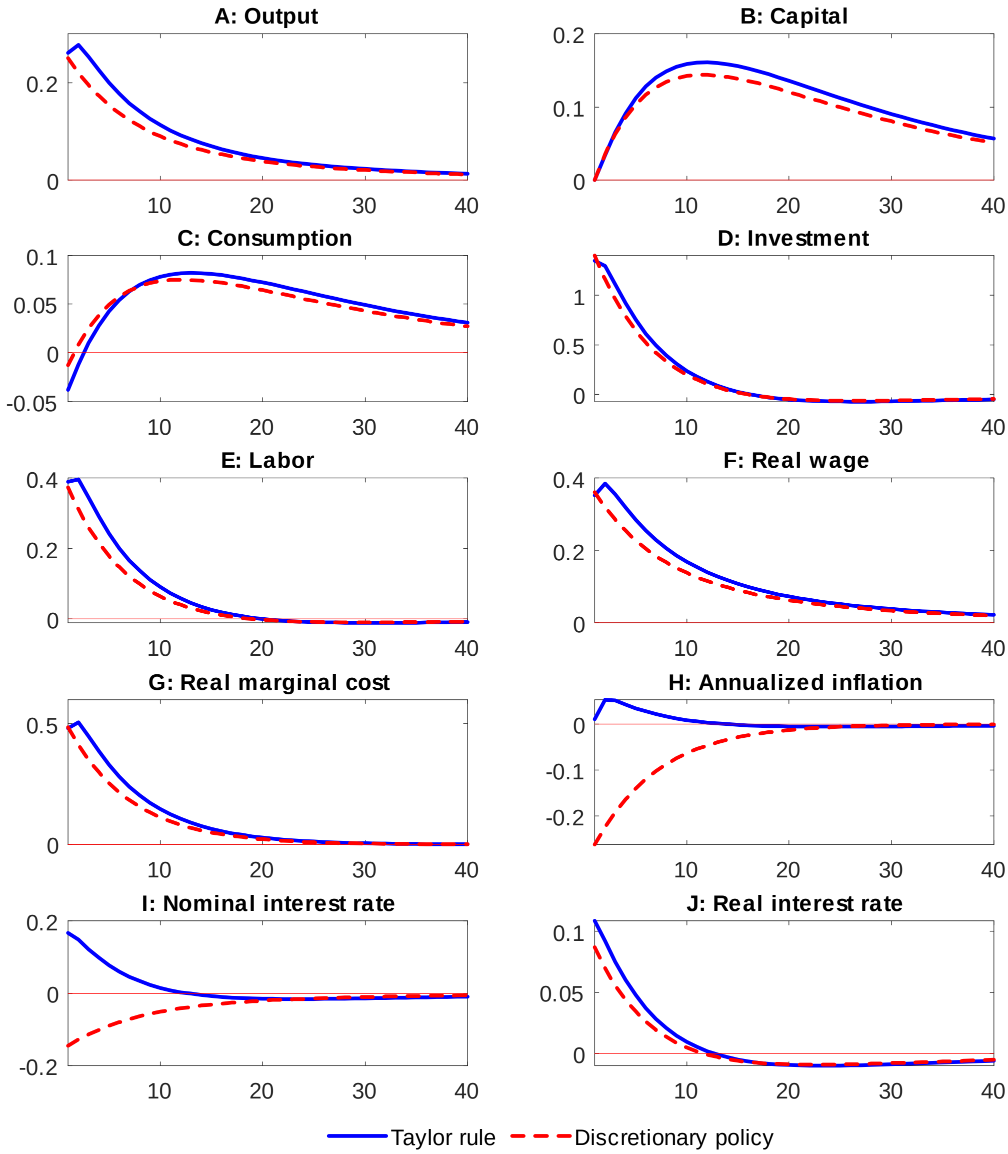

Figure 1.5: Responses to a price-elasticity shock under discretion and the Taylor rule

$$0 = C_t^{-\sigma} - \theta \mathcal{L}_{t+1},$$

$$0 = \left(\pi_t (1 + \pi_t) + \frac{\left(\varepsilon e^{\zeta_t} - 1\right)(1 - \tau) - \varepsilon e^{\zeta_t} x_t}{\omega} \right) Y_t C_t^{-\sigma} - \theta \mathcal{M}_{t+1}, \tag{1.93}$$

$$0 = \left(1 - \frac{\omega}{2}\pi_t^2\right) Y_t - C_t - K_{t+1} + (1 - \delta) K_t, \tag{1.94}$$

where

$$H_t = \left(\left(\frac{1 - \alpha}{\chi} \right) e^{a_t} x_t K_t^{\alpha} C_t^{-\sigma} \right)^{\frac{1}{\upsilon + \alpha}}, \tag{1.95}$$

$$Y_t = \left(\left(\frac{1 - \alpha}{\chi} \right)^{1 - \alpha} e^{(1+\upsilon) a_t} x_t^{1-\alpha} K_t^{\alpha(1+\upsilon)} C_t^{-\sigma(1-\alpha)} \right)^{\frac{1}{\upsilon + \alpha}}, \tag{1.96}$$

and

$$\mathcal{L}_{t+1} = \mathrm{E}_t \left[L((\zeta_{t+1}, a_{t+1}, K_{t+1}) \right], \tag{1.97}$$

$$\mathcal{M}_{t+1} = \mathrm{E}_t \left[M(\zeta_{t+1}, a_{t+1}, K_{t+1}) \right], \tag{1.98}$$

$$\mathcal{D}_{t+1} = \mathrm{E}_t \left[D(\zeta_{t+1}, a_{t+1}, K_{t+1}) \right], \tag{1.99}$$

$$\mathcal{L}_{K,t+1} = \mathrm{E}_t \left[L_K(\zeta_{t+1}, a_{t+1}, K_{t+1}) \right],$$

$$\mathcal{M}_{Kt+1} = \mathrm{E}_t \left[M_K(\zeta_{t+1}, a_{t+1}, K_{t+1}) \right],$$

with the definitions

$$L(\zeta_t, a_t, K_t) \equiv C_t^{-\sigma} \left(\beta \left(e^{a_t} x_t \alpha K_t^{\alpha - 1} H_t^{1-\alpha} + 1 - \delta \right) + (1 - \beta) \mathcal{K}_K(\zeta_t, a_t, K_t) \right),$$

$$M(\zeta_t, a_t, K_t) \equiv \beta \pi_t (1 + \pi_t) C_t^{-\sigma} Y_t,$$

$$\begin{aligned} D(\zeta_t, a_t, K_t) \equiv\ & \xi \phi_{1t} \left(\alpha \frac{1 + \upsilon}{\upsilon + \alpha} \left(1 - \frac{\omega}{2}\pi_t^2\right) \frac{Y_t}{K_t} + 1 - \delta \right) - \frac{\gamma \xi \alpha \chi}{\upsilon + \alpha} \frac{H_t^{1+\upsilon}}{K_t} \\ & + \xi \alpha \frac{1 + \upsilon}{\upsilon + \alpha} \phi_{3t} \left(\frac{\left(1 - \varepsilon e^{\zeta_t}\right)(1 - \tau) + \varepsilon e^{\zeta_t} x_t}{\omega} - \pi_t (1 + \pi_t) \right) \frac{Y_t}{K_t} C_t^{-\sigma} \\ & - \xi (1 - \gamma) \left(C_t^{-\sigma} + \frac{\sigma \chi}{\upsilon + \alpha} \frac{H_t^{1+\upsilon}}{C_t} \right) \mathcal{C}_K(\zeta_t, a_t, K_t) \\ & + \frac{\xi (1 - \gamma) \chi}{\upsilon + \alpha} \frac{H_t^{1+\upsilon}}{x_t} \mathcal{X}_K(\zeta_t, a_t, K_t). \end{aligned}$$

Equations (1.89)—(1.96) are a system of seven equations containing seven unknowns: six control variables, C_t, π_t, x_t, ϕ_{1t}, ϕ_{2t}, and ϕ_{3t}, and one future state variable, K_{t+1}. We solve this nonlinear system on a set of nodes constructed for the state variables whose domain given by $\zeta \in [\zeta_{\min}, \zeta_{\max}]$, $a \in [a_{\min}, a_{\max}]$, and $K \in [K_{\min}, K_{\max}]$. We compute a set of Gauss-Chebyshev nodes, $\mathbf{Z} = \{\zeta_k, a_j, K_i; k = 1...N_\zeta, j = 1...N_a, i = 1...N_K\}$, for the state space $[\zeta_{\min}, \zeta_{\max}] \times [a_{\min}, a_{\max}] \times [K_{\min}, K_{\max}]$ and use a three-dimensional Chebyshev polynomial to approximate the unknown functions.[4]

Using $\mathbf{Z}_{k,j,i} \in \mathbf{Z}$ to denote a particular grid point, our solution algorithm can be summarized as follows:

Step 1. Initialize arrays for $H_t^{(0)}$, $Y_t^{(0)}$, $\pi_t^{(0)}$, $\phi_{1t}^{(0)}$, $\phi_{2t}^{(0)}$, and $\phi_{3t}^{(0)}$, to store solution outcomes.

Step 2. Conjecture initial state-contingent functions for $K_{t+1}^{(0)} = \mathcal{K}^{(0)}(\mathbf{Z}_{k,j,i})$, $C_t^{(0)} = \mathcal{C}^{(0)}(\mathbf{Z}_{k,j,i})$, $x_t^{(0)} = \mathcal{X}^{(0)}(\mathbf{Z}_{k,j,i})$, $L_t^{(0)} = L^{(0)}(\mathbf{Z}_{k,j,i})$, $M_t^{(0)} = M^{(0)}(\mathbf{Z}_{k,j,i})$, and $D_t^{(0)} = D^{(0)}(\mathbf{Z}_{k,j,i})$ at each grid point $\mathbf{Z}_{k,j,i} \in \mathbf{Z}$.

Step 3. At iteration n, approximate the functions $\mathcal{K}^{(n)}$, $\mathcal{C}^{(n)}$, $\mathcal{X}^{(n)}$, $L^{(n)}$, $M^{(n)}$, and $D^{(n)}$ using three-dimensional Chebyshev polynomials whose weights are computed using Chebyshev-regression. Approximate the derivatives $L_K^{(n)}$, $M_K^{(n)}$, $\mathcal{K}_K^{(n)}$, $\mathcal{X}_K^{(n)}$, and $\mathcal{C}_K^{(n)}$ by differentiating the corresponding polynomial.

Step 4. At each grid point, $\mathbf{Z}_{k,j,i} \in \mathbf{Z}$:

Step 4.1. Compute the conditional expectations: $\mathcal{L}_{t+1}^{(n)}$, $\mathcal{M}_{t+1}^{(n)}$, $\mathcal{D}_{t+1}^{(n)}$, $\mathcal{L}_{K,t+1}^{(n)}$, and $\mathcal{M}_{K,t+1}^{(n)}$ using Gauss-Hermite quadrature.

Step 4.2. Solve equations (1.89)—(1.96) using a nonlinear solver and use the solution to update $K_{t+1}^{(n+1)}(\mathbf{Z}_{k,j,i})$, $C_t^{(n+1)}(\mathbf{Z}_{k,j,i})$, $x_t^{(n+1)}(\mathbf{Z}_{k,j,i})$, $\pi_t^{(n+1)}(\mathbf{Z}_{k,j,i})$, $\phi_{1t}^{(n+1)}(\mathbf{Z}_{k,j,i})$, $\phi_{2t}^{(n+1)}(\mathbf{Z}_{k,j,i})$, and $\phi_{3t}^{(n+1)}(\mathbf{Z}_{k,j,i})$.

[4]A similar approach is discussed in Maliar and Maliar (2006), Anderson, Kim, and Yun, 2010), and Maliar and Maliar (2005).

Step 4.3. Update $H_t^{(n+1)}(\mathbf{Z}_{k,j,i})$, $Y_t^{(n+1)}(\mathbf{Z}_{k,j,i})$, $L_t^{(n+1)}(\mathbf{Z}_{k,j,i})$, $M_t^{(n+1)}(\mathbf{Z}_{k,j,i})$, and $D_t^{(n+1)}(\mathbf{Z}_{k,j,i})$ using equations (1.95)—(1.99).

Step 5. Compute the distance

$$\begin{aligned}\Upsilon &= \left\|K_{t+1}^{(n+1)} - K_{t+1}^{(n)}\right\|_\infty + \left\|C_t^{(n+1)} - C_t^{(n)}\right\|_\infty \\ &+ \left\|x_t^{(n+1)} - x_t^{(n)}\right\|_\infty + \left\|\pi_t^{(n+1)} - \pi_t^{(n)}\right\|_\infty \\ &+ \left\|\phi_{1t}^{(n+1)} - \phi_{1t}^{(n)}\right\|_\infty + \left\|\phi_{2t}^{(n+1)} - \phi_{2t}^{(n)}\right\|_\infty + \left\|\phi_{3t}^{(n+1)} - \phi_{3t}^{(n)}\right\|_\infty \\ &+ \left\|L_t^{(n+1)} - L_t^{(n)}\right\|_\infty + \left\|M_t^{(n+1)} - M_t^{(n)}\right\|_\infty + \left\|D_t^{(n+1)} - D_t^{(n)}\right\|_\infty .\end{aligned}$$

If Υ is greater than the given tolerance (we use $1e-6$), then increment the iteration counter, n, and return to Step 3. Otherwise, stop.

We used the following parameters in this algorithm. For the state space, we set the domain $\zeta \in [-3\sigma_\zeta, 3\sigma_\zeta]$, $a \in [-3\sigma_z, 3\sigma_z]$, and $K \in [5, 35]$. We used a grid with 15 nodes for capital and 7 nodes each for technology and the elasticity of substitution. Each function was approximated with a Chebyshev polynomial of order 4 for ζ, 4 for a, and 14 for capital. Conditional expectations were computed using Gauss-Hermite quadrature with 5 points for each shock.

The same algorithm was used to compute Taylor rule policy. We set capital domain $K \in [15, 30]$, and output domain $Y \in [1.5, 3.2]$. We used a grid on output with 9 nodes and a Chebyshev polynomial of order 4 in output to approximate functions. All other parameters were identical to those in the model of discretionary policy.

Table D1 reports the Euler-equation residuals for certain combinations of β and γ. To compute them we split the domain for capital into 200 uniform points and those for technology and the elasticity of substitution into 50 uniform points, and computed the residuals of the consumption Euler equation at each point on this grid. We found the key determinant for accuracy to be the order of the Chebyshev polynomial for capital. When this order was below 14 there was a noticeable decline in accuracy.

Table D1: Numerical accuracy: Consumption-Euler residuals

		Discretionary policy			Taylor-type rule	
Discount factor HH	β	1.00	0.90	0.90	1.00	0.90
Discount factor CB	γ	1.00	0.90	1.00	–	–
Maximum		1.5e-06	1.3e-06	1.1e-06	4.2e-07	5.4e-07
Mean		4.6e-07	4.0e-07	3.9e-07	1.4e-07	1.9e-07
Median		4.3e-07	3.5e-07	3.5e-07	1.2e-07	1.6e-07

To compute the stochastic steady state we used 10^6 random draws. We followed Potter (2000) to compute the nonlinear impulse responses.

1.F Appendix: Other results

Focusing on the model's stationary distribution, Table 1.7 shows the mean (standard deviations in parentheses) outcomes for the model's key macroeconomic and financial variables for different values of $\beta = \gamma$, allowing the steady state to be either inefficient (columns (1)—(4)) or efficient (columns (5)—(8)). Looking at average outcomes, the table shows that as greater quasi-hyperbolic discounting takes place ($\beta = \gamma$ get smaller)—biasing household and central bank decision-making toward the present—output falls. Specifically, lowering $\beta = \gamma$ from 1.0 to 0.9 causes output to decline by approximately 10 percent.[5] Although greater quasi-hyperbolic discounting causes output, capital, consumption, labour, and the real wage to fall there are important differences in how each of these variables is affected. For example, although lowering $\beta = \gamma$ from 1.0 to 0.9 causes output to fall by 10.02 percent, capital falls by much more (24.55 percent) and labour falls by much less (1.84 percent). Labour does not decline to the same extent

[5]Cutting β from 1.0 to 0.7 causes output to fall by about 30 percent, suggesting a linear relationship between the percent by which β falls and the percent by which output falls.

as output because households sacrifice some leisure in order to prevent a large decline in consumption. As a consequence, consumption falls by 6.02 percent, considerably less than output. The large decline in capital combined with a smaller decline in labour means that the capital-labour ratio goes down, and with relatively less capital, labour's productivity diminishes and real wages go down (by 7.77 percent).

Looking at real marginal costs, Table 1.7 shows that greater quasi-hyperbolic discounting causes real marginal costs to rise slightly when the steady state is inefficient (and to have no discernible effect on real marginal costs when the steady state is efficient). The effect that quasi-hyperbolic discounting has on real marginal costs is related to the decline that firms face in the demand for their good, which causes them to lower their price markup. To understand the impact quasi-hyperbolic discounting has on inflation, note that quasi-hyperbolic discounting implies that costs to changing prices today are weighted more heavily than those to changing prices in the future. As a consequence, when responding to shocks firms find it beneficial to spread price changes out over time, making smaller price changes in the current period and deferring the remaining price change (and its associated cost) to the future. With smaller price changes taking place today, greater quasi-hyperbolic discounting acts somewhat like an increase in price rigidity. From the central bank's perspective, with quasi-hyperbolic discounting operating similarly to an increase in price rigidity, it calculates that smaller inflation surprises are sufficient to boost output to the efficient level. In equilibrium, then, greater quasi-hyperbolic discounting leads to less inflation.

Finally, we note from Table 1.8 that the finding that the central bank's quasi-hyperbolic discounting can raise household welfare relies on the economy's steady state being inefficient. In the case where the economy's steady state is efficient there is no discretionary inflation bias. As a consequence, the decline in inflation generated by the central bank's quasi-hyperbolic discounting serves to drive inflation further from zero, which lowers household welfare.

Table 1.6: Characteristics of the Stationary Distribution

		Inefficient s.s., $\tau = 0$		Efficient s.s., $\tau = (1-\varepsilon)^{-1}$	
		Sticky prices	Flexible prices	Sticky prices	Flexible prices
		(1)	(2)	(3)	(4)
Output	Y	2.540 [0.104]	2.534 [0.104]	2.825 [0.115]	2.824 [0.116]
Capital	K	21.734 [0.936]	21.664 [0.935]	26.561 [1.106]	26.556 [1.108]
Consumption	C	1.992 [0.062]	1.993 [0.062]	2.161 [0.066]	2.161 [0.066]
Investment	I	0.543 [0.053]	0.542 [0.053]	0.664 [0.062]	0.664 [0.063]
Labour	H	0.881 [0.010]	0.880 [0.011]	0.936 [0.011]	0.935 [0.012]
Real wage	w	1.756 [0.063]	1.753 [0.064]	2.022 [0.072]	2.021 [0.073]
Real marginal cost	x	0.909 [0.008]	0.909 [0.010]	0.999 [0.011]	0.999 [0.011]
Annualized inflation	π	2.580 [0.527]	–	0.040 [0.220]	–
Household Welfare	$\mathcal{U}$	29.957 [0.989]	30.163 [0.988]	33.190 [0.989]	33.189 [0.989]
Nominal interest rate	R	6.782 [0.497]	4.097 [0.399]	4.139 [0.361]	4.098 [0.400]
Real interest rate	r	4.097 [0.384]	4.097 [0.399]	4.098 [0.394]	4.098 [0.400]
Return on capital	r^k	4.098 [0.427]	4.098 [0.447]	4.099 [0.440]	4.099 [0.449]
Return on capital	r^{cap}	4.098 [0.427]	4.098 [0.447]	4.099 [0.440]	4.099 [0.449]
Dividend yield	r^s	4.101 [0.305]	4.100 [0.332]	4.101 [0.296]	4.101 [0.309]
Asset price	Q	43.966 [1.588]	44.602 [1.607]	54.651 [1.908]	54.672 [1.911]

Note: Statistics calculated using 10^6 simulated observations; standard deviations in brackets.

Table 1.7: Stationary Distribution as a Function of the Quasi-Hyperbolic Discount factor

		Inefficient s.s., $\tau = 0$				Efficient s.s., $\tau = (1-\varepsilon)^{-1}$			
Discounting	$\beta = \gamma$	1.00	0.99	0.95	0.90	1.00	0.99	0.95	0.90
		(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Output	Y	2.540 [0.104]	2.516 [0.103]	2.415 [0.099]	2.286 [0.094]	2.825 [0.115]	2.795 [0.114]	2.676 [0.110]	2.524 [0.104]
Capital	K	21.734 [0.936]	21.181 [0.916]	19.009 [0.837]	16.398 [0.740]	26.561 [1.106]	25.857 [1.082]	23.110 [0.987]	19.847 [0.871]
Consumption	C	1.992 [0.062]	1.981 [0.062]	1.935 [0.062]	1.872 [0.061]	2.161 [0.066]	2.149 [0.066]	2.098 [0.066]	2.028 [0.065]
Investment	I	0.543 [0.053]	0.530 [0.052]	0.475 [0.048]	0.410 [0.044]	0.664 [0.062]	0.646 [0.061]	0.578 [0.057]	0.496 [0.051]
Labour	H	0.881 [0.010]	0.880 [0.009]	0.873 [0.009]	0.865 [0.008]	0.936 [0.011]	0.933 [0.011]	0.924 [0.011]	0.913 [0.010]
Real wage	w	1.756 [0.063]	1.743 [0.063]	1.690 [0.062]	1.619 [0.060]	2.022 [0.072]	2.005 [0.072]	1.939 [0.070]	1.851 [0.068]
Real marginal cost	x	0.909 [0.008]	0.910 [0.008]	0.912 [0.008]	0.915 [0.007]	0.999 [0.011]	0.999 [0.011]	1.000 [0.010]	1.000 [0.010]
Annualized inflation	π	2.580 [0.527]	2.559 [0.520]	2.478 [0.493]	2.385 [0.462]	0.040 [0.220]	0.039 [0.219]	0.038 [0.216]	0.036 [0.211]
Household Welfare	$\mathcal{U}$	29.957 [0.989]	29.270 [0.980]	26.432 [0.942]	22.685 [0.895]	33.190 [0.989]	32.525 [0.979]	29.757 [0.942]	26.064 [0.895]
Nominal interest rate	R	6.782 [0.497]	11.139 [0.516]	30.969 [0.608]	62.443 [0.760]	4.139 [0.361]	8.410 [0.379]	27.852 [0.464]	58.717 [0.606]
Real interest rate	r	4.097 [0.384]	8.367 [0.403]	27.804 [0.489]	58.660 [0.634]	4.098 [0.394]	8.368 [0.413]	27.804 [0.503]	58.661 [0.654]
Return on capital	r^k	4.098 [0.427]	4.342 [0.433]	5.413 [0.460]	7.019 [0.502]	4.099 [0.440]	4.338 [0.447]	5.390 [0.479]	6.964 [0.526]
Return on capital	r^{cap}	4.098 [0.427]	8.369 [0.447]	27.805 [0.543]	58.662 [0.704]	4.099 [0.440]	8.369 [0.462]	27.806 [0.564]	58.662 [0.734]
Dividend yield	r^s	4.101 [0.305]	8.368 [0.588]	27.785 [1.523]	58.617 [2.439]	4.101 [0.296]	8.368 [0.573]	27.787 [1.515]	58.618 [2.500]
Asset price	Q	43.966 [1.589]	22.054 [0.839]	7.267 [0.289]	3.852 [0.155]	54.651 [1.908]	24.429 [1.007]	9.074 [0.349]	4.824 [0.190]

Note: Statistics calculated using 10^6 simulated observations; standard deviations in brackets.

Table 1.8: The Effect of the Central Bank's Quasi-Hyperbolic Discounting

		Inefficient s.s., $\tau = 0$				Efficient s.s., $\tau = (1-\varepsilon)^{-1}$			
Household	β	1.00	1.00	0.90	0.90	1.00	1.00	0.90	0.90
Central bank	γ	1.00	0.90	0.90	1.00	1.00	0.90	0.90	1.00
		(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Output	Y	2.540 [0.104]	2.536 [0.104]	2.286 [0.094]	2.302 [0.095]	2.825 [0.115]	2.825 [0.115]	2.524 [0.104]	2.537 [0.104]
Capital	K	21.734 [0.936]	21.681 [0.934]	16.398 [0.740]	16.592 [0.746]	26.561 [1.106]	26.551 [1.106]	19.847 [0.871]	20.035 [0.877]
Consumption	C	1.992 [0.062]	1.993 [0.062]	1.872 [0.061]	1.876 [0.061]	2.161 [0.066]	2.158 [0.066]	2.028 [0.065]	2.034 [0.065]
Investment	I	0.543 [0.053]	0.542 [0.053]	0.410 [0.044]	0.415 [0.044]	0.664 [0.062]	0.664 [0.063]	0.496 [0.051]	0.501 [0.051]
Labour	H	0.881 [0.010]	0.880 [0.010]	0.865 [0.008]	0.869 [0.008]	0.936 [0.011]	0.936 [0.012]	0.913 [0.010]	0.916 [0.010]
Real wage	w	1.756 [0.063]	1.754 [0.064]	1.619 [0.060]	1.630 [0.060]	2.022 [0.072]	2.020 [0.073]	1.851 [0.068]	1.863 [0.068]
Real marginal cost	x	0.909 [0.008]	0.909 [0.009]	0.915 [0.007]	0.919 [0.006]	0.999 [0.011]	0.999 [0.011]	1.000 [0.010]	1.004 [0.009]
Annualized inflation	π	2.580 [0.527]	0.703 [0.312]	2.385 [0.462]	4.015 [0.633]	0.040 [0.220]	−1.795 [0.039]	0.036 [0.211]	1.658 [0.376]
Household Welfare	$\mathcal{U}$	29.957 [0.989]	30.155 [0.988]	22.685 [0.895]	22.555 [0.895]	33.190 [0.989]	33.053 [0.988]	26.064 [0.895]	26.107 [0.895]
Nominal interest rate	R	6.782 [0.497]	4.828 [0.388]	62.443 [0.760]	65.029 [0.947]	4.139 [0.361]	2.229 [0.387]	58.717 [0.606]	61.291 [0.674]
Real interest rate	r	4.097 [0.384]	4.097 [0.390]	58.660 [0.634]	58.660 [0.622]	4.098 [0.394]	4.098 [0.398]	58.661 [0.654]	58.661 [0.640]
Return on capital	r^k	4.098 [0.427]	4.098 [0.435]	7.019 [0.502]	7.016 [0.490]	4.099 [0.440]	4.099 [0.447]	6.964 [0.526]	6.962 [0.511]
Return on capital	r^{cap}	4.098 [0.427]	4.098 [0.435]	58.662 [0.704]	58.662 [0.687]	4.099 [0.440]	4.099 [0.447]	58.662 [0.734]	58.663 [0.714]
Dividend yield	r^s	4.101 [0.305]	4.101 [0.314]	58.617 [2.439]	58.618 [2.320]	4.101 [0.296]	4.101 [0.307]	58.618 [2.500]	58.621 [2.353]
Asset price	Q	43.966 [1.588]	44.514 [1.603]	3.852 [0.155]	3.751 [0.151]	54.651 [1.908]	54.499 [1.904]	4.824 [0.190]	4.756 [0.185]

Note: Statistics calculated using 10^6 simulated observations; standard deviations in brackets.

Chapter 2

Distributional Effects of Endogenous Discounting

Based on joint work with Nigar Hashimzade and Tatiana Kirsanova

Abstract

This paper analyses the effect of endogenous discounting on wealth inequality in an endowment economy with heterogeneous agents, subject to occasionally binding borrowing constraint. We demonstrate that introduction of Uzawa-type preferences may launch strong redistribution mechanism leading to high equilibrium real interest rate and a more dispersed wealth distribution in comparison to the model with standard preferences.

Keywords: heterogeneous agents; recursive utility; endogenous discounting; wealth distribution; precautionary savings

JEL codes: D91, E21

2.1 Introduction

The standard discounted utility model assumes preferences over time and over payoffs, attaching probabilities to all possible histories. But there are alternatives. Some of them were developed to account for anomalous predictions, others arose from advances in the pure theory of intertemporal choice, see Backus, Routledge and Zin (2005) for a discussion.

Uzawa-type preferences have been born mostly from such advances. The idea that consumers have a preference for advancing the time of future satisfaction has been around at least since Fisher (1930) where the notion of impatience was introduced. How to describe such a preference remained an issue, with one appealing idea to try to define preferences for timing advances entirely in terms of utility function (Koopmans, 1960). Unlike the standard discounted utility approach with constant time preference and the intra-temporal utility, Koopmans (1960) introduced recursive representation of the intertemporal utility, allowing for endogenous time preferences.

To advance the time of future satisfaction requires an assumption of *increasing marginal impatience*, this was identified by Koopmans (1960) and Uzawa (1968). This assumption implies that richer households tend to discount future consumption more heavily. Although a bit counter-intuitive, as we discuss later, the recursive nature of these preferences became very appealing, and they appeared in many models in the 1970-80s. The original application of Uzawa-Koopmans preferences was to growth theory (Uzawa, 1968), to explain the fact that poorer countries tend to delay consumption relatively to richer countries, so they grow faster and can catch up. Uzawa's preference structure has also been used in trade theory by Kouri (1980), Obstfeld (1981) and others. One appealing feature of these preferences is that, dispensing with constant time-preference rates, they generate intertemporal interdependence, with applications to business cycles (Kydland and Prescott, 1982), high excess return to US equity investments (Constantinides, 1990) and drug addiction (Becker and Murphy, 1988) among others. As a bonus, these preferences naturally rule out the random walk behavior of the real exchange rate in open economy models (Schmitt-Grohe and Uribe, 2003). Furthermore, Epstein (1983, 1987) noted that in the stochastic version of the general equilibrium model with Uzawa preferences in discrete time, an *increasing marginal impatience* implies an individual's aversion to correlation between random consumption levels in any two periods. Empirical evidence of intertemporal correlation aversion, consistent with such increasing marginal impatience, was reported by Cheung (2015) and Andersen et al. (2018).

Direct tests of whether richer households tend to discount future consumption more heavily are difficult to design. The literature which compares countries often shows completely different results: poor countries' households seem to discount future more heavily than households in rich countries, see Atkeson and Ogaki (1996) and Lawrance (1991) empirical work. This empirical work, however, does not dismiss the underlying assumption that individuals are impatient and would like to advance the time of future events with high utility.[1] Inherent difficulties and lack of consensus over empirical approaches and

[1] Moreover, the reverse assumption of *decreasing marginal impatience* would result in instability of a representative agent dynamic economy, see Obstfeld (1990). In an economy with permanently patient

use of theoretical models in measuring individual time preferences were noted by Cohen et al. (2020) in a recent survey of a large body of literature in this field.

The assumption of impatience, combined with heterogeneity of agents and incomplete markets, can create rich dynamics and a non-trivial wealth distribution (Epstein and Hynes 1983, Lucas and Stokey 1984, and Farmer and Lahiri 2005). Recently, Wang (2007) demonstrates that a model with heterogeneous-income agents (*a la* Huggett 1993) is able to generate stationary wealth distribution with *relatively* fat tails.

Intuitively, with Uzawa preferences (henceforth UP) the rich, being less patient than the poor, have incentives to consume more and save less. Therefore, in the long run the distribution of wealth with UP should be less dispersed than in the standard case. The model analyzed in Wang (2007) does not allow direct comparison, because it employs CARA exponential utility function for which stationary wealth distribution under constant discounting does not exist. The stationary wealth distribution obtained under UP in Wang (2007) has fat tails *relative to the underlying income distribution*. A similar analysis, based on the discount rate locally linear in consumption, as in Wang (2007), but with the CRRA utility, allows direct comparison between two stationary wealth distributions. Indeed, this framework generates a *more concentrated wealth distribution* under UP than under standard preferences, although with still fatter tails relative to the underlying income distribution. This is one of the results shown in our paper.

Even more interestingly, at odds with the above intuition, we show in this paper that UP with CRRA utility can also generate a long-run wealth distribution that is substantially more dispersed than the one in an economy with standard preferences, when the discount rate is non-linear in consumption. With an S-shaped discount rate, reflecting diminishing marginal impatience in the tails, the wealth distribution in a model with UP

and impatient consumers the patient consumer will have all the wealth. This property is often used as a trick to model a permanently binding borrowing constraint in representative agents literature, see e.g. Iacoviello (2005).

is more skewed and has fatter tails than under the standard preferences. It is, thus, more 'unequal', – and, thus, more empirically relevant, – than the distribution obtained in the standard model.

In this paper we analyze a continuous-time economy with incomplete markets and stochastic shocks to exogenous incomes following a two-state Markov process, as in Huggett (1993) but with an exogenous constraint on borrowing. Achdou et al. (2017) showed that in such an economy with constant discount rate there is a unique stationary distribution of wealth, with individuals mobile across consumption, income, and wealth levels. We assume CRRA instantaneous utility and an endogenous instantaneous discount rate, which depends on the instantaneous consumption. We investigate the effect of endogenous discounting on wealth inequality and compare two models. In the first model, the instantaneous discount rate is a locally linear function of consumption, as in Wang (2007). We demonstrate that it yields more concentrated wealth distribution than the model with standard preferences. In the second model – which is the model of our main interest – the instantaneous discount rate is an S-shaped function of consumption, meaning that the agents in the tails of the wealth distribution have diminishing marginal discount rate. This model is characterizes by a substantially higher equilibrium interest rate and a more dispersed wealth distribution, relative to the model with locally-linear discount rate.

To get intuition for these results it is helpful to trace the differences in the behavior of the negative-wealth agents, who demand loans, and the positive-wealth agents, who supply loans, across three economies: an economy with the standard, constant discount rate (CDR), an economy with locally linear discount rate (LDR), and an economy with S-shaped discount rate (SDR), – when all three economies have the same long-run population average discount rate. Putting together these differences suggests how the stationary distributions would differ across these economies.

In the LDR economy, the densely populated small-positive-wealth group has consumption above the benchmark level and so discounts future by more than their CDR counterpart. It therefore offers a lower supply of loans and thus creates an *upward* pressure on the equilibrium interest rate relative to the CDR. This effect is substantially amplified by the behavior of wealthy agents. The negative-wealth LDR agents, however, become more patient than their CDR counterparts and thus have a lower demand for loans, leading to a *downward* pressure on the equilibrium interest rate relative to the CDR. The two opposite effects on interest rate coming from the two tail groups nearly offset each other, and the moderate increase in equilibrium interest rate is implied by the behavior of central group of agents who have only slightly higher impatience than in the CDR economy. With equilibrium interest rate only slightly higher than under CDR, the demand for loans is still higher, as the negative-wealth agents need to refinance their loans, but there is also an income effect forcing them to reduce consumption. The opposite is true for the high-positive-wealth agents and so the aggregate effect is moderate.

The balance of these opposite pressures on interest rate changes significantly if the discount rate of the two tail groups changes only moderately, as in the case with the SDR, despite the fact that the central group with small positive wealth behaves in the same way as in the LDR economy. Given the same interest rate, smaller reduction in patience of the negative-wealth agents implies less reduction in the loan demand and, therefore, weaker downward pressure on the equilibrium interest rate. Similarly, the high-positive-wealth group produces only moderate reduction in loan supply and less upward pressure on the interest rate. However, the reduction in supply is greater than the reduction in demand. Higher interest rate results in a substantially higher debt of the negative-wealth group, and in a higher proportion of population in the left-hand tail. Self-reinforcing mechanism of higher demand for loans to refinance the existing debt and stronger upward pressure on the interest rate produced by this group results in a higher equilibrium interest rate.

The central group with moderate positive wealth generates a moderate increase in general equilibrium interest rate, while the interactions of the two tail groups limit interest rate increase in the LDR economy, but amplify its increase substantially in the SDR economy. The corresponding redistribution of population across wealth levels accompanies the latter process and reinforces the equilibrium effect. Thus, the UP can generate a more dispersed wealth distribution, with fatter tails, than the standard preferences, under a plausible assumption of declining marginal impatience above and below the average discount rate.

We next investigate the distributional effects of taxes. We find that a consumption tax reduces the welfare inequality. Capital income tax increases current consumption, reduces the current saving but does not affect future output. As a result, in this economy life-time welfare unambiguously rises because of higher current consumption.

Finally, we demonstrate that in production economy the redistribution effect of Uzawa-type preferences is mitigated. In contrast to the endowment economy with zero total wealth, movements of population between borrower and lender positions skew the population distribution towards its higher end. Large movements between borrowers and savers are no longer possible, and the redistribution mechanism does not engage with a shift in preferences. The mechanism discussed in this paper may help to understand additional reasons leading to high observed income inequality in developing countries with little production opportunities and binding borrowing constraints.

The chapter is organized as follows. The next section presents the model. Section 2 describes the functional form of preferences. First order conditions describing stationary equilibrium are obtained in Section 3 and parameterization of the model is given in Section 4. The results are discussed in Section 5 and Section 6 concludes. All proofs are gathered in the Technical Appendix.

2.2 The Model

The economy is populated by a continuum of infinitely-lived individuals with different levels of wealth a and income y. Individual preferences are described by the following life-time utility function:

$$\int_0^\infty \exp\left(-\int_0^s \rho\left(c\left(\tau\right)\right) d\tau\right) u\left(c\left(s\right)\right) ds \tag{2.1}$$

where the discount rate ρ depends on consumption c. The flow utility function $u(c)$ is strictly increasing and strictly concave. An individual receives income y in the form of an endowment of the economy's consumption good. An individual's income follows a two-state Poisson process, $y \in \{y_1, y_2\}$ with $y_2 > y_1$, describing large 'surprise' changes in income. The process jumps from state 1 to state 2 with intensity λ_1 and from state 2 to state 1 with intensity λ_2. Parameters λ_1 and λ_2 are exogenous constants. The income process is uninsurable, and individuals can only lend or borrow in the form of non-contingent private bonds at interest rate r determined in equilibrium.

The wealth accumulation equation takes the form

$$\dot{a} = ra + y - c \tag{2.2}$$

where a is the holding of private bonds. We assume that individuals face a borrowing limit

$$a \geq \underline{a} \tag{2.3}$$

with $-\infty < \underline{a} < 0$.

Individuals choose consumption path c to maximize utility (2.1) subject to the budget constraint (2.2), the borrowing limit (2.3) and an exogenously specified labor income process, taking interest rate as given. We denote the joint probability distribution of income y_j and wealth a by $G_j\left(a, t\right)$, with the corresponding density function $g_j\left(a, t\right)$, $j = 1, 2$.

We assume that private bonds are in zero net supply, so in general equilibrium, at every instant t

$$\int_{\underline{a}}^{\infty} a g_1(a)\, da + \int_{\underline{a}}^{\infty} a g_2(a)\, da = 0. \tag{2.4}$$

Finally, we assume that the flow utility function takes the CRRA form,

$$u(c) = \frac{c^{1-\gamma}-1}{1-\gamma}, \quad \gamma > 0. \tag{2.5}$$

It remains to define the functional form of discount rate $\rho(c)$.

2.3 Uzawa preferences

We assume that individual time preferences are endogenous and depend on the consumption path. The functional form of the instantaneous discount rate $\rho(c)$ must satisfy the following properties: (i) $\rho(c) > 0$, (ii) $\rho_c(c) > 0$, (iii) $\rho(c) < \bar{R} < \infty$. The first property simply states that individuals discount future utility, and the second property is required to ensure existence of a stationary equilibrium (see, for example, Obstfeld 1990).

Figure 2.1 illustrates the shape of the discount rate and its derivative in three cases: standard preferences model with constant discount rate (CDR)

$$\rho(c) = \bar{\rho}, \tag{2.6}$$

locally linear discount rate (LDR)

$$\rho(c) = \bar{\rho} + \Theta(c - c_0). \tag{2.7}$$

and S-shaped discount rate (SDR)

$$\rho(c) = \bar{\rho} + \frac{\Theta}{d} \arctan(d(c - c_0)), \tag{2.8}$$

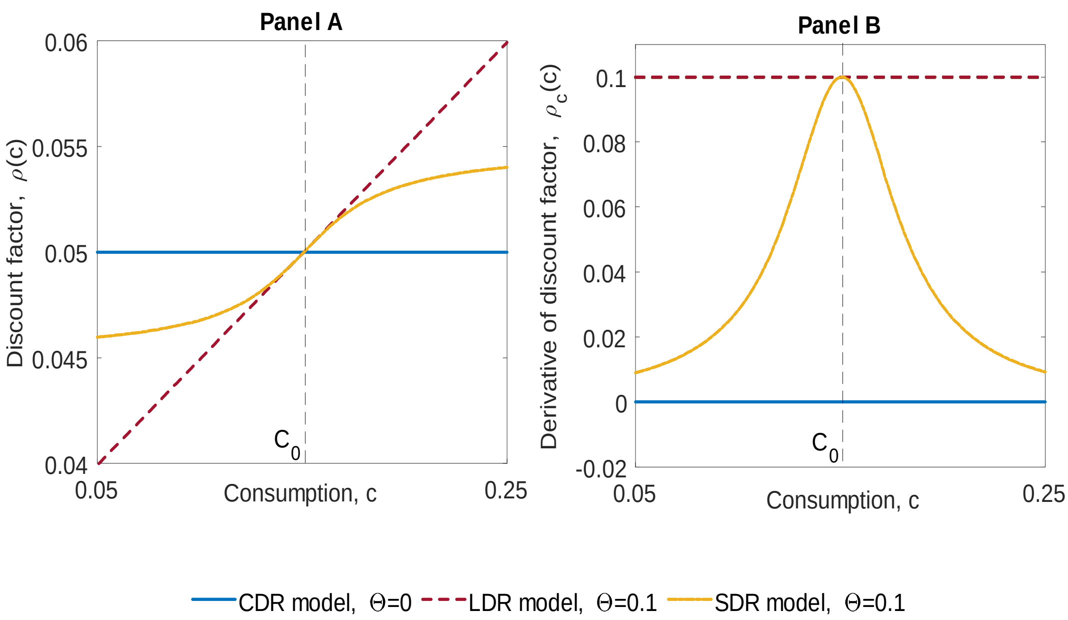

Figure 2.1: Discount rate as a function of consumption.

where parameters $\{\bar{\rho}, \Theta, d, c_0\}$ are all positive. In the consumption domain shown in Figure 2.1 the discount rate varies around the benchmark value of $\bar{\rho} = 0.05$ within about 10%. This consumption domain, as we shall see later, is sufficiently wide and covers nearly entire distribution of consumption for the benchmark parameterization.[2]

In the LDR and SDR specifications parameter $\bar{\rho}$ is the discount rate at a certain benchmark level of consumption c_0. Parameter

$$\Theta \equiv \rho_c(c_0)$$

determines the slope of the discount rate function at the benchmark consumption level. Parameter d affects the curvature of the discount rate in the SDR case. Note that with

[2]We refer to the LDR case as "locally linear" because it may be non-linear outside this consumption domain. In particular, it can be S-shaped, but as long as the tails with diminishing marginal impatience are outside the relevant consumption domain, these tails have no effect.

d tending to zero the discount rate tends to a locally linear function with slope Θ so the LDR is a limiting case of the SDR.

Panel B illustrates that $0 \equiv \left(\frac{\partial\rho(c)}{\partial c}\right)_{CDR} < \left(\frac{\partial\rho(c)}{\partial c}\right)_{SDR} \leq \left(\frac{\partial\rho(c)}{\partial c}\right)_{LDR}$ for any a. Derivative $\left(\frac{\partial\rho(c)}{\partial c}\right)_{LDR} \equiv \Theta$, while derivative $\left(\frac{\partial\rho(c)}{\partial c}\right)_{SDR} = \frac{\Theta}{1+d^2(c-c_0)^2}$ is hump-shaped and achieves maximum $\max\left(\frac{\partial\rho(c)}{\partial c}\right)_{SDR} = \Theta$ at $c = c_0$.

2.4 Stationary Equilibrium

Individuals' consumption and saving decisions and the evolution of their income and wealth can be summarized with Hamilton-Jacobi-Bellman equation

$$0 = \max_c \left(u(c) - \rho(c) V_j(a) + (ra + y_j - c)\frac{\partial V_j(a)}{\partial a} + \lambda_j (V_{-j}(a) - V_j(a))\right), \tag{2.9}$$

and Kolmogorov Forward equation[3]

$$0 = -\frac{d}{da}(s_j(a) g_j(a)) - \lambda_j g_j(a) + \lambda_{-j} g_{-j}(a), \tag{2.10}$$

where $j = 1, 2$ and index $-j$ denotes 'other than j'. Here the saving policy function is

$$s_j(a) = ra + y_j - c_j(a)$$

and the value function is denoted $V_j(a)$.The derivation of these two equations is given in the Technical Appendix.

Maximization of (2.9) yields the following first order condition:

$$\frac{\partial u(c)}{\partial c} = \frac{\partial\rho(c)}{\partial c} V_j(a) + \frac{\partial V_j(a)}{\partial a}. \tag{2.11}$$

[3]This is also known in the literature as the Fokker-Planck equation.

Compared with the CDR model with constant $\rho = \bar{\rho}$ there is an additional term $\frac{\partial \rho(c)}{\partial c} V_j(a)$ in the right-hand side of this equation. This term captures the effect of endogenous discounting on the agent's trade-off. With greater past consumption the agent has incentives to consume more. For this model $V_j(a) < 0$ and, therefore, $\frac{\partial u(c)}{\partial c} < \frac{\partial V_j(a)}{\partial a}$ so that the marginal utility of consumption is less than the marginal value of saving $\frac{\partial V_j(a)}{\partial a}$ at the point of optimality.

Together with market clearing condition (2.4), boundary condition,

$$\frac{\partial V_j(\underline{a})}{\partial a} + \frac{\partial \rho(y_j + r\underline{a})}{\partial c} V_j(\underline{a}) \geq \frac{\partial u(y_j + r\underline{a})}{\partial c} \tag{2.12}$$

and the normalization of the joint distribution,

$$\int_{\underline{a}}^{\infty} g_1(a)\, da + \int_{\underline{a}}^{\infty} g_2(a)\, da = 1 \tag{2.13}$$

the system of equations (2.9) – (2.11) describes the stationary equilibrium.

Lemma 1 *The consumption and saving functions $c_j(a)$ and $s_j(a)$ for $j = 1, 2$ corresponding to the HJB equation (2.9) satisfy the Euler equation,*

$$0 = (r - \rho(c))\, \mu_j(a) + \frac{\partial \mu_j(a)}{\partial a} s_j(a) + \lambda_j \left(\mu_{-j}(a) - \mu_j(a)\right) \tag{2.14}$$

where

$$\mu_j(a) = \frac{\partial V_j(a)}{\partial a} = \frac{\partial u(c)}{\partial c} - \frac{\partial \rho(c)}{\partial c} V_j(a)$$
$$s_j(a) = ra + y_j - c_j(a)$$

The next proposition describes asymptotic saving behavior of the low income types.

Assumption 1 *Discount rate $\rho(c)$ satisfies the following inequality*

$$\frac{\partial^2 u(c)}{\partial c^2} \frac{\partial c_j(a)}{\partial a} < \left(\frac{\partial^2 \rho(c)}{\partial c^2} \frac{\partial c_j(a)}{\partial a} + \frac{\partial \rho(c)}{\partial c}\right) \mu_j(a).$$

Proposition 1 *Suppose that Assumption 1 is satisfied, $r < \rho(c)$, $y_1 < y_2$ and the coefficient of risk aversion $R(c) = -\lim_{a \to \underline{a}} \frac{\partial^2 u(c)}{\partial c^2} / \frac{\partial u(c)}{\partial c} < \infty$. Then the solution to the HJB equation (2.9) and the corresponding policy functions have the following properties:*

1. $s_1(\underline{a}) = 0$ but $s_1(a) < 0$ for $a > \underline{a}$. That is, among those with current low income draw, only individuals exactly at the borrowing constraint are constrained, while those with wealth $a > \underline{a}$ are unconstrained and decumulate assets.

2. as $a \to \underline{a}$ the saving and consumption policy function of the low income type and the corresponding instantaneous marginal propensity to consume satisfy

$$s_1(a) \sim -\sqrt{2\nu_1}\sqrt{a - \underline{a}}$$

$$c_1(a) \sim ra + y_1 - \sqrt{2\nu_1}\sqrt{a - \underline{a}}$$

$$\frac{\partial c_1(a)}{\partial a} =\sim r - \sqrt{\frac{\nu_1}{2(a - \underline{a})}}$$

where

$$\nu_1 = \frac{(r - \rho(c_1(\underline{a})))\,\mu_1(\underline{a}) + \lambda_1(\mu_2(\underline{a}) - \mu_1(\underline{a}))}{\frac{\partial^2 u(c)}{\partial c^2} - V_1(a)\left.\frac{\partial^2 \rho(c_1)}{\partial c_1^2}\right|_{a=\underline{a}}}$$

and

$$V_2(\underline{a}) = \frac{\left(u(c_2(\underline{a})) + s_2(\underline{a})\left.\frac{\partial u(c)}{\partial c}\right|_{c_2(\underline{a})}\right)(\rho(c_1(\underline{a})) + \lambda_1) + \lambda_2 u(c_1(\underline{a}))}{\left(\rho(c_2(\underline{a})) + s_2(\underline{a})\left.\frac{\partial \rho(c)}{\partial c}\right|_{c_2(\underline{a})}\right)(\rho(c_1(\underline{a})) + \lambda_1) + \lambda_2 \rho(c_1(\underline{a}))}$$

$$V_1(\underline{a}) = \frac{u(c_1(\underline{a})) + \lambda_1 V_2(\underline{a})}{\rho(c_1(\underline{a})) + \lambda_1}$$

$$\mu_1(\underline{a}) = \left.\frac{\partial u(c)}{\partial c}\right|_{c_1(\underline{a})} - V_1(\underline{a})\left.\frac{\partial \rho(c)}{\partial c}\right|_{c_1(\underline{a})}$$

$$\mu_2(\underline{a}) = \left.\frac{\partial u(c)}{\partial c}\right|_{c_2(\underline{a})} - V_2(\underline{a})\left.\frac{\partial \rho(c)}{\partial c}\right|_{c_2(\underline{a})}$$

Corollary 2 *Under assumptions of Proposition 1 the wealth of an individual with initial wealth a_0 and successive low income draws y_1 converges to the borrowing constraint in finite time at speed governed by ν_1 :*

$$a(t) - a_0 \sim \frac{\nu_1}{2}(T-t)^2, \; 0 \leq t \leq T$$

where T is the 'hitting time'.

Wealth distribution of the low income group has a point mass at the borrowing constraint, as shown in Achdou et al. (2018); their results are directly transferable to our case. In the next Proposition we borrow their results relevant for our investigation, and also derive a compact expression relating the point mass on the constraint to the savings function.

The cumulative probability function $G_j(a)$ satisfies

$$\int_{\underline{a}}^{\infty} dG_j(a) = \frac{\lambda_{-j}}{\lambda_j + \lambda_{-j}}.$$

Proposition 2 *Assume that $r < \rho(c)$, $y_1 < y_2$, the relative risk aversion $-c\frac{\frac{\partial^2 u(c)}{\partial c^2}}{\frac{\partial u(c)}{\partial c}} < \infty$ and the coefficient of risk aversion $R(c) = -\lim_{a \to \underline{a}} \frac{\frac{\partial^2 u(c)}{\partial c^2}}{\frac{\partial u(c)}{\partial c}} < \infty$. Then there is exist a unique stationary distribution given by*

$$g_j(a) = \frac{\kappa_j}{s_j(a)} \exp\left(-\int_{\underline{a}}^{a}\left(\frac{\lambda_j}{s_j(x)} + \frac{\lambda_{-j}}{s_{-j}(x)}\right)dx\right)$$

for some constants of integration $\kappa_1 < 0$ and $\kappa_2 = -\kappa_1$. The stationary wealth distribution of low income group $g_1(a)$ has a point mass of m_1 at the borrowing constraint $\underline{a}$. The cumulative distribution function has the following asymptotic property

$$G_1(a) \sim m_1 \exp\left(\lambda_1 \sqrt{\frac{2(a-\underline{a})}{\nu_1}}\right)$$

with

$$m_1 = \frac{1}{\lambda_1} s_2\left(\underline{a}\right) g_2\left(\underline{a}\right).$$

The stationary distribution of high income types at the borrowing constraint is bounded, $g_2\left(\underline{a}\right) < \infty$*..*

The saving behavior of the high income group is the same as formulated in Achdou et al. (2017) in the standard case with constant discount ρ.

Proposition 3 *Assume that* $r < \rho(c)$, $y_1 < y_2$ *and relative risk aversion* $-c\frac{\partial^2 u(c)}{\partial c^2}/\frac{\partial u(c)}{\partial c} < \infty$*. Then there exists* $a_{\max} < \infty$ *such as* $s_j(a) < 0$ *for all* $a \geq a_{\max}$*, j=1,2, and* $s_2(a) \sim \zeta_2(a_{\max} - a)$ *as* $a \to a_{\max}$ *for some constant* ζ_2*. The wealth of an individual with initial wealth* a_0 *and successive high income draws* y_2 *converges to* $a_{\max}$ *asymptotically:* $a - a_{\max} \sim e^{-\zeta_2 t}(a_0 - a_{\max})$*.*

Therefore, the support for wealth is limited: $a \in [\underline{a}, a_{\max})$.

2.5 Parameterization

We adopt parameterization from Achdou et al. (2018) and set $y_1 = 0.1, y_2 = 0.2, \lambda_1 = \lambda_2 = 1.2$. In the numerical example discussed in this paper we set $\Theta = 0.1$, and $\kappa = \frac{\pi\Theta}{d} = 0.01$.

It remains to calibrate the benchmark level of consumption. To make different models comparable one plausible choice is to require that the average discount rate in all three equals $\bar{\rho}$. For the SDR model with $\Theta = 0.1$ and $\kappa = 0.01$ this condition pins down $\bar{c}_0^{SDR} = 0.1508$ which is different from $\bar{c}_0^{LDR} = 0.1500$. We can alternatively choose to set

$\tilde{c}_0^{SDR} = \tilde{c}_0^{LDR} = 0.1500$ which is the mean consumption in the CDR model. It turns out that our results are only marginally quantitatively different with $c_0 \in [0.1500, 0.1508]$ and all our conclusions remain unchanged. In tables below we use $\bar{c}_0$ based on equating the average discount rates to $\bar{\rho}$.

2.6 Stationary Distributions and Equilibrium Interest Rate

2.6.1 Constant Discount Rate

The benchmark CDR case with $\Theta = 0$ is thoroughly discussed in Achdou et al. (2017). The model is solved for stationary distributions for each of the two current-income groups: the low-income group (those with current realization of $y = y_1$) and the high-income group ($y = y_2$). Table 2.1 presents numerical characteristics of wealth distributions by different groups and for the entire population, and Figures 2.2 and 2.3 present these distributions labelled as $\Theta = 0$. Comparison of statistics in columns (1) and (2) of Panel III of Table 2.1 suggests that the CDR wealth distribution is substantially more dispersed than the underlying income distribution.

The key properties of the solution, as illustrated in these figures, are the following.

The high-income group are period-savers, while the low-income group are period-dissavers (see Panel A of Figure 2.2). Consumption of both groups rises with wealth (see Panel B).

Both low-income and high-income groups have a bounded wealth support: $\underline{a} \leq a \leq a_{\max}$, but only the low-income group has positive mass on constraint $\underline{a}$. Once individuals

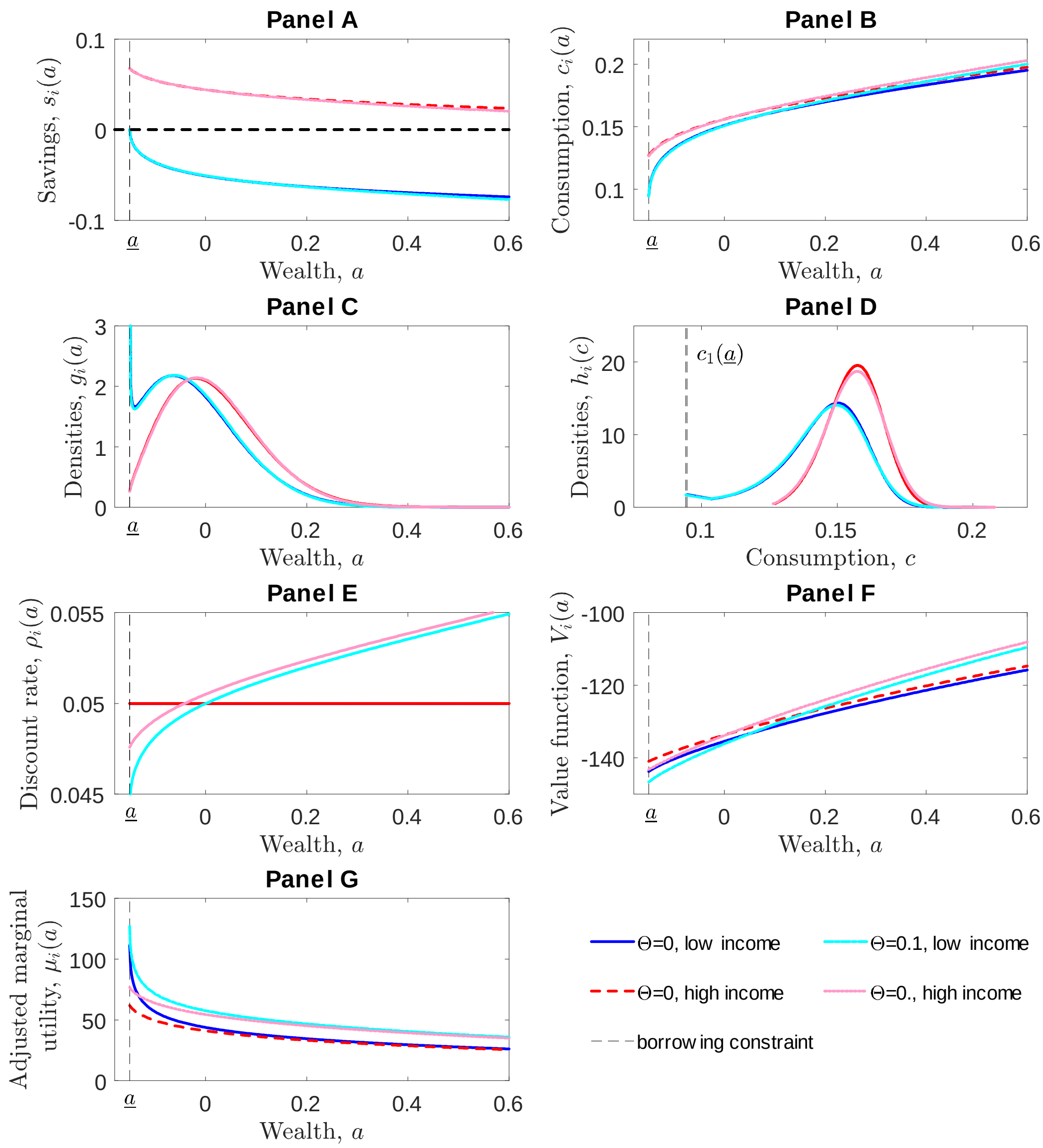

Figure 2.2: Stationary distributions in the LDR and CDR cases

Table 2.1: Numerical Characteristics of the Stationary Distribution

	CDR $\Theta = 0.0$	LDR $\Theta = 0.1$	SDR $\Theta = 0.1$
	(1)	(2)	(3)
Panel I: General equilibrium interest rate and tails of the wealth distribution			
Interest rate r	0.0352	0.0385	0.0392
Asymptotic parameter, ν_1	0.0260	0.0256	0.0216
Mass at the borrowing constraint, m_1	0.0151	0.0146	0.0156
Top 10% by wealth, low income group, $\hat{a}_{L,90}$	0.1074	0.1054	0.1199
Top 10% by wealth, high income group, $\hat{a}_{H,90}$	0.1448	0.1427	0.1593
Panel II: Statistical characteristics of the total wealth distributions			
mean	0.0000	0.0000	0.0000
standard deviation	0.0939	0.0937	0.1053
skewness	0.6697	0.6736	1.0366
kurtosis	3.2674	3.2810	4.3323
Panel III: Statistical characteristics of consumption distributions by current income group			
standard deviation, low income group	0.0168	0.0169	0.0179
standard deviation, high income group	0.0102	0.0106	0.0096
standard deviation, total	0.0196	0.0200	0.0204
Panel IV: Net supply of loans, $L^S - L^D$			
The poor, $a < 0$	-0.0382	-0.0378	-0.0414
The middle class, $a \gtrsim 0$	0.0227	0.0230	0.0203
The rich, $a \gg 0$	0.0155	0.0148	0.0211

hit this constraint, they remain there for a finite time period as long as they subsequently draw low income. This is illustrated in Panel C.

Consumption densities of both groups are plotted in Panel D of Figure 2.2. One can see that the high-income group has almost symmetric distribution of consumption, while the density of the low-income group has a pronounced long tail on the left-hand side and a spike on the right-hand side corresponding to the point-mass on borrowing constraint, where the individuals consume their total income net of interest payments.

Panel E of Figure 2.2 illustrates the distribution of the individual discount rate for each income groups in the LDR case, and Panel F plots the value function for each group.

The level of welfare for the high-income group is higher than for the low-income group for all wealth levels.

For the further discussion, it is helpful to split the total distribution by wealth into three groups: those with negative wealth ($\underline{a} < a < 0$), those with moderate positive wealth ($0 < a < \tilde{a}$) and those with high positive wealth ($\tilde{a} < a$). We will label them as 'the poor', 'the middle class' and 'the rich' respectively.[4] We also refer to these three groups as having wealth of $a < 0$, $a \gtrsim 0$ and $a \gg 0$, respectively. For our numerical example we use $\tilde{a} = |\underline{a}|$.

At every moment of time, the poor agents have negative wealth and thus are net debtors. The low-income poor agents are period-borrowers ($\dot{a} < 0$): they consume more than they earn and finance consumption expenses and interest payments by borrowing the difference. The high-income poor agents are period-savers ($\dot{a} > 0$): they consume less than they earn, use their income to finance consumption and interest payments, and save the difference. However, as both income groups are net debtors ($a < 0$), their need to refinance (roll-over) this debt generates demand for loans. The numerical values are reported in Panel II of Table 2.1.

The middle-class and the rich have positive wealth and thus are net creditors. Among them, the low-income agents are period-borrowers ($\dot{a} < 0$): they consume more than they earn in income and interest on loans, and so they borrow the difference to finance their consumption. The high-income agents are period-savers ($\dot{a} > 0$): they consume less than they earn in income and interest, use their income to finance consumption, and save the rest. These groups generate supply of loans. The numerical values are given in Panel II in Table 2.1.

Interest rate adjusts to ensure that the net supply of loans in the economy is zero.

[4] Note that this classification refers to the stock of accumulated wealth, rather than the flow of income. Thus, a poor individual can receive high income in any given moment t, or a rich individual can receive low income.

Next, we analyze the LDR case as the second convenient benchmark that helps to explain the SDR results.

2.6.2 Linear Discount Rate Model

The LDR model was investigated in Wang (2007), in a setting with the CARA utility function, Gaussian diffusion income process but without a borrowing constraint. In this framework Wang (2007) has demonstrated that the wealth distribution is more dispersed than the underlying income distribution. The CARA utility and unconstrained borrowing allow to derive a closed-form solution for the distribution of wealth, but a stationary distribution only exists for $\Theta > 0$. Lack of the the natural benchmark of CDR ($\Theta = 0$) precludes direct comparison of the LDR and CDR wealth distributions.

In contrast to Wang (2007), our model, with CRRA utility along with the borrowing constraint, does not have a closed-form solution, so we have to resort to numerical computations. However, we can compare stationary wealth distribution to the benchmark CDR case discussed above (see Table 2.1).

Let us start with the comparison of the benchmark CDR case reported in column (1) with the LDR case in column (2). Panel I suggests that the effect of lower discount rate in the left tail of the wealth distribution under LDR is numerically small and ambiguous: although the point mass on the borrowing limit is lower, parameter ν_1 – which is inversely proportional to the slope of the cdf function $G_1(\underline{a})$ – is also lower, so $G_1(a)$ grows faster with a at $a = \underline{a}$ under LDR In addition, top 10% of both income types have less wealth under LDR, so overall the LDR wealth distribution must be slightly more concentrated. Overall, Panel II confirms that the wealth distribution reported in column (3) for the LDR case is more concentrated and less skewed, although quantitatively the effect is not very large. This appears consistent with Uzawa-type preferences: with increasing marginal impatience the rich save less, while the poor dissave less.

Note that the equilibrium interest rate is higher in LDR than in the CDR case. To see whether higher interest rate is consistent with more concentrated distribution of wealth, we look at the three wealth groups described above. It is helpful to describe the comparison of the CDR and LDR economies as a thought experiment of changing the marginal discount rate from $\Theta = 0.0$ to $\Theta = 0.1$. Table 2.2 documents partial and general equilibrium effects on consumption c, saving s, demand and supply of loans L^D and L^S, the interest rate r, the discount rate ρ and the proportion of population n, as we describe next.

Consider the effect of change in the discount rate recorded in columns (1)-(3). A change in the discount rate has substitution effect on consumption and saving decisions. Consumption of the poor ($a < 0$) falls below the benchmark level, so that in the LDR economy their discount rates are lower than the (benchmark) constant discount rate of $\bar{\rho}$ in the CDR economy. Because of these lower discount rates they consume less and borrow less, as recorded in columns (1) and (2). As a result, for a given interest rate, the demand for loans among these individuals falls below that in the CDR case, and this exerts a downward pressure on interest rate, recorded in column (3). Meanwhile, consumption of the middle class ($a \gtrsim 0$) and the rich ($a \gg 0$) is, on average, above the benchmark level, and so their discount rates are higher than the benchmark CDR. With higher discount rate they consume more and save less than they would under the CDR economy. The supply of loans goes down, which has an upward pressure on interest rate compared to the CDR case, this is recorded in column (3).

Panel IV in Table 2.1 shows that, as a result of such changes in the discount rate, in the stationary equilibrium the total supply and total demand for loans are lower in the LDR economy than in the CDR economy. The reduction in supply dominates the reduction in demand, and so the equilibrium interest rate is higher under the LDR compared to the CDR, as recorded in column (4).

Next, consider the effect of higher interest rate on these three groups, recorded in columns (4)-(6). Consider the income effect of higher interest rate on consumption and saving decisions of each group of agents. First, to service their debt, the poor ($a < 0$) increase borrowing and simultaneously reduce their current consumption, as recorded in column (5). Lower current consumption further reduces their discount rate, and higher borrowing increases the demand for loans thus exerting *upward* pressure on the interest rate, as shown in column (6).

Second, with higher interest rate the middle class ($a \gtrsim 0$) and the rich ($a \gg 0$) receive higher return on their saving and increase both saving and current consumption. Higher current consumption further increases their discount rate. At the same time, higher savings mean higher supply of loans thus creating *downward* pressure on the interest rate, as recorded in columns (5)-(6).

Table 2.1 reports the general equilibrium outcomes, also summarized in column (7) of Table 2.2. In general equilibrium, the substitution effect dominates the income effect and determines the outcome for the two tail groups, the poor and the rich. The poor ($a < 0$) in the CDR economy have lower debt thus generating weaker demand for loans, compared to the LDR economy. Furthermore, the point mass on the borrowing constraint under the LDR is smaller, and the relative population share of this group falls as some of its members accumulate positive wealth and move up into the middle class. The rich in the LDR economy have lower wealth thus generating weaker supply of loans, compared to the CDR economy. In addition, the relative population share (denoted by n) of this group falls as some of its members decumulate wealth and move down into the middle class, as shown in Table 2.2.

As a result, the middle class ($a \gtrsim 0$) in the LDR economy is larger than in the CDR economy. However, while it becomes the most populous, it has only moderate consumption and savings, so the equilibrium interest rates do not differ very much between the two economies.

Table 2.2: General Equilibrium Effects of Linear Discount Factor

Wealth Group	Dis-count Rate	Substi-tution Effect	Partial Eqm. Effect	Eqm. Interest Rate	Income Effect	Partial Eq. Effect	LDR General Equilibrium
	(1)	(2)	(3)	(4)	(5)	(6)	(7)
$a < 0$	$\rho\downarrow$	$c\downarrow, s\uparrow$	$L^D\downarrow, r\downarrow$		$s\downarrow, c\downarrow$	$L^D\uparrow, r\uparrow, \rho\downarrow$	$L^D\downarrow, a\uparrow, n\downarrow$
$a \gtrsim 0$	$\rho\uparrow$	$c\uparrow, s\downarrow$	$L^S\downarrow, r\uparrow$	$r\uparrow$	$s\uparrow, c\uparrow$	$L^S\uparrow, r\downarrow, \rho\uparrow$	$L^S\uparrow, a\uparrow, n\uparrow$
$a \gg 0$	$\rho\uparrow$	$c\uparrow, s\downarrow$	$L^S\downarrow, r\uparrow$		$s\uparrow, c\uparrow$	$L^S\uparrow, r\downarrow, \rho\uparrow$	$L^S\downarrow, a\downarrow, n\downarrow$

To summarize, the LDR economy is characterized by a slightly more concentrated wealth distribution and, in this sense, exhibits *lower wealth inequality* than the CDR economy. The distribution of consumption under LDR is more dispersed for both income groups, although the effect is small. Similar to the wealth distribution, the consumption distribution of the low-income group has a point mass at the level of consumption corresponding to the borrowing limit. This boundary is lower in the LDR economy than in the CDR economy. Consumption smoothing leads to the distribution of consumption that is less dispersed than the distribution of wealth for each group for all three types of preferences, but with the LDR the distribution of consumption has lower peaks and fatter tails for both income groups, compared to the CDR. That is, the LDR economy exhibits *higher consumption inequality* for both income groups.

2.6.3 S-Shaped Discount Factor

The result on wealth distribution is completely reversed in the SDR case described by (2.8).[5] Compared to the LDR case, the discount rate $\rho(c)$ deviates less from constant $\bar{\rho}$, and it has diminishing sensitivity to consumption relative to the benchmark. Column (3) in Table 2.1 shows that in the SDR case the equilibrium interest rate rises over the CDR case by more than in the LDR case.

[5] We do not have a proof of uniqueness of the stationary equilibrium in this setting. Instead, we have computed stationary equilibrium using widely different initializations. In our simulations the algorithm converged to the same distribution in all cases. The details are available upon request.

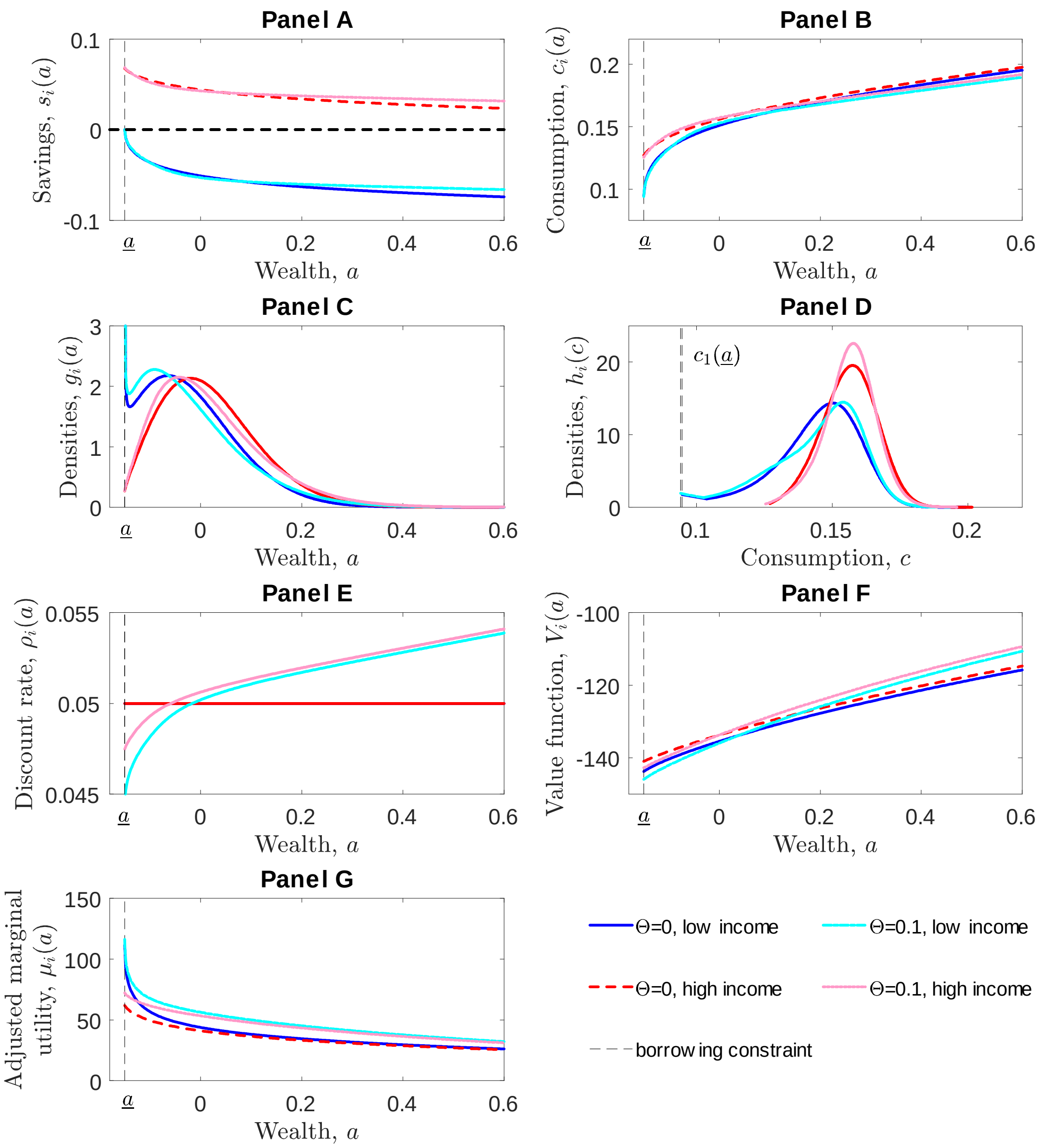

Figure 2.3: Stationary distributions, non-linear discount factor function

Figure 2.3 reports stationary distributions in the SDR case. The quantitative effects are significantly larger than in the LDR case. First, from Panels I and II one can see that the dispersion of consumption and wealth distributions is noticeably larger: there are more agents in the tails of the distribution (the poor and the rich). In particular, the point mass on the borrowing constraint is larger, and the distribution function $G_1(\underline{a})$ at the constraint, is steeper (because ν_1 is smaller). The top 10% of population by wealth have higher level of wealth under the SDR compared to the CDR. In other words, in an SDR economy the consumption and wealth distributions have fatter tails compared to those in a CDR economy. Second, Table 2.1 reports that the equilibrium interest rate is much higher under SDR than under CDR. It is not surprising that a higher interest rate results in fatter tails, as this comes from the substantial income effects. However, the reason for why the interest rate in the SDR economy is even higher than in the LDR economy and why the income effects dominate is more subtle.

Table 2.3: General Equilibrium Effects of Non-Linear Discount Factor

Wealth Group	Discount Rate	Substitution Effect	Partial Eqm. Effect	Eqm. Interest Rate	Income Effect	Partial Eqm. Effect	SDR General Equilibrium
	(1)	(2)	(3)	(4)	(5)	(6)	(7)
$a < 0$	$\rho \downharpoonleft$	$c \downharpoonleft, s \upharpoonright$	$L^D \downharpoonleft, r \downharpoonleft$		$s \downdownarrows, c \downdownarrows$	$L^D \upuparrows, r \upuparrows, \rho \downdownarrows$	$L^D \uparrow\upharpoonright, a \downarrow, n \uparrow$
$a \gtrsim 0$	$\rho \uparrow$	$c \uparrow, s \downarrow$	$L^S \downarrow, r \uparrow$	$r \upuparrows$	$s \upuparrows, c \upuparrows$	$L^S \upuparrows, r \downdownarrows, \rho \upuparrows$	$L^S \downarrow, a \downarrow, n \downarrow$
$a \gg 0$	$\rho \upharpoonright$	$c \upharpoonright, s \downharpoonleft$	$L^S \downharpoonleft, r \upharpoonright$		$s \upuparrows, c \upuparrows$	$L^S \upuparrows, r \downdownarrows, \rho \upuparrows$	$L^S \uparrow\upharpoonright, a \uparrow, n \uparrow$

As before, consider the same three wealth groups: the poor ($a < 0$), the middle class ($a \gtrsim 0$) and the rich ($a \gg 0$). As discussed above, the first group generates the demand for loans, while the other two groups generate the supply of loans. Table 2.3 shows the same directions of effects recorded in columns (1)-(6) as Table 2.2. However, the general equilibrium effects described in column (7) have different directions.

Similar to the LDR economy, in the SDR case the poor agents have lower discount rate than in the benchmark CDR case, and thus have lower current consumption. As a result, for a given interest rate, these individuals have lower demand for loans than in the CDR case, which puts a downward pressure on the interest rate. Importantly, the poor in the SDR case are more impatient than in the LDR case, as one can see in Figure 2.3. They consume more and dissave or borrow more to finance consumption and refinance debt. Their demand for loans does not fall as much as under LDR, and there is *less downward pressure* on equilibrium real interest rate coming from this group, see column (3) in Tables 2.2 and 2.3. In the same vein, the other two groups have higher discount rate than in the CDR case, but they are more patient than in the LDR case. The consumption does not go up as much and the saving does not reduce as much as they do in the LDR case. The reduction in the supply of loans is smaller than in the LDR case; hence there is *less upward pressure* on interest rate.

Although there is less upward and downward pressure on equilibrium interest rate, the balance of these pressures is shifted compared to the LDR case. In the stationary distribution in an SDR economy a larger proportion, – about 50 percent – of population are poor, so that (i) the aggregate demand for loans is higher, (ii) the upward pressure on interest rate is stronger than the downward pressure and (iii) the equilibrium real interest rate is higher than in the LDR case.

A higher interest rate has an income effect on the consumption and saving behavior of each group of agents. The poor increase borrowing to service the debt and, at the same time, reduce the current consumption. This further reduces their discount rates, but limits the reduction in the demand for loans which weakens the downward pressure on the interest rate. Table 2.1 reports that in general equilibrium the effect of higher interest rate dominates, so that the demand for loans is higher in comparison to the CDR case. In general equilibrium, therefore, the poor have higher debt, the point mass on the borrowing constraint is larger, and the relative population share of this group is bigger, as some middle-class agents decumulate wealth and become net borrowers. As the poor constitute about a half of the population, the self-reinforcing mechanism of higher interest rate and higher demand for loans to service previous loans results in higher interest rate in general equilibrium.[6]

A significantly high interest rate has strong income effect on the net creditors, who increase both current consumption and savings. Although the increased supply of loans creates a downward pressure on interest rate, this effect is generated by a relatively small proportion of the overall population, so the general equilibrium interest rate, mainly determined by the net borrowers' behavior, remains high. The rich group accumulate more wealth at the higher rate (because of the higher return on loans) and so their relative share in the population is higher in the SDR economy, while the population share of the middle class is lower, compared to the CDR and LDR economies.

This behavior of different wealth groups bends, not tilts, consumption and saving profiles as shown in Figure 2.3. The stronger income effect of a high interest rate affects both tails of the distribution and leads to greater wealth inequality.

Note that this reinforcing mechanism can only be triggered if $\Theta > 0$. Only then, relative impatience of the poor and the relative patience of the rich generate significant increase in equilibrium interest rate with the poor and rich – the tails of the wealth distribution – wanting to increase borrowing and saving, respectively. Therefore, only LDR and SDR models are comparable, with less (im)patience of the two tail groups resulting in higher interest rate and higher equilibrium demand for loans. These two models are not directly comparable to the CDR model.

To summarize, the distribution of wealth in the SDR economy exhibits a larger inequality in the aggregate and within each group relative to the CDR and LDR economies. The wealth distribution of the low-income group has a larger mass at the borrowing constraint (see Panel I in Table 2.1). While the effect of the SDR on the mean wealth is negligible in this parameterization, the standard deviation, skewness, and kurtosis of the

[6] An exogenous borrowing constraint prevents a Ponzi-game behaviour of accumulating ever growing debt.

wealth distribution are all higher for both the low-income and the high-income groups, as shown in Panel IV in Table 2.1 (see also the bottom quantiles, the medians, and the upper deciles of the wealth distribution for three types of preferences reported in this table). All of these suggests that the wealth distribution under the SDR is more dispersed and more skewed, and in this sense is more unequal than in the CDR case.

The effect on consumption distribution is less clear. The higher interest rate reduces consumption on the borrowing limit. For both groups, under SDR the distribution of consumption has higher peak relative to the CDR. At the same time, it has fatter tails for the low-income group under SDR, while the opposite is true for the high-income group. That is, the SDR reduces consumption inequality for the high-income group and exacerbates it for the low-income group, as shown in Panel III of Table 2.1.

2.7 Transition Dynamics

To explore the transition dynamics in this economy, we conduct the following experiment. We start with stationary distribution generated by the standard CDR model and at some point instantaneously change the discount factor to become dependent on consumption and record the transition dynamics towards the new stationary equilibrium under LDF or SDR assumption in Figures 2.4-2.5.

In the LDR model interest rate rises quickly to its new steady-state level. It then fluctuates around this level for about 3 years, but its deviations are of about 1% of the new level. In contrast, in the SDR model interest rate immediately rises by much more than the new-steady state level and then converges to it with deviations of about 10% from this new level. The much higher interest rate consistently observed after one year from the 'switch to new preferences' generate a noticeable income effect clearly affecting the poor types with negative wealth, as discussed above. There is a clear concentration of

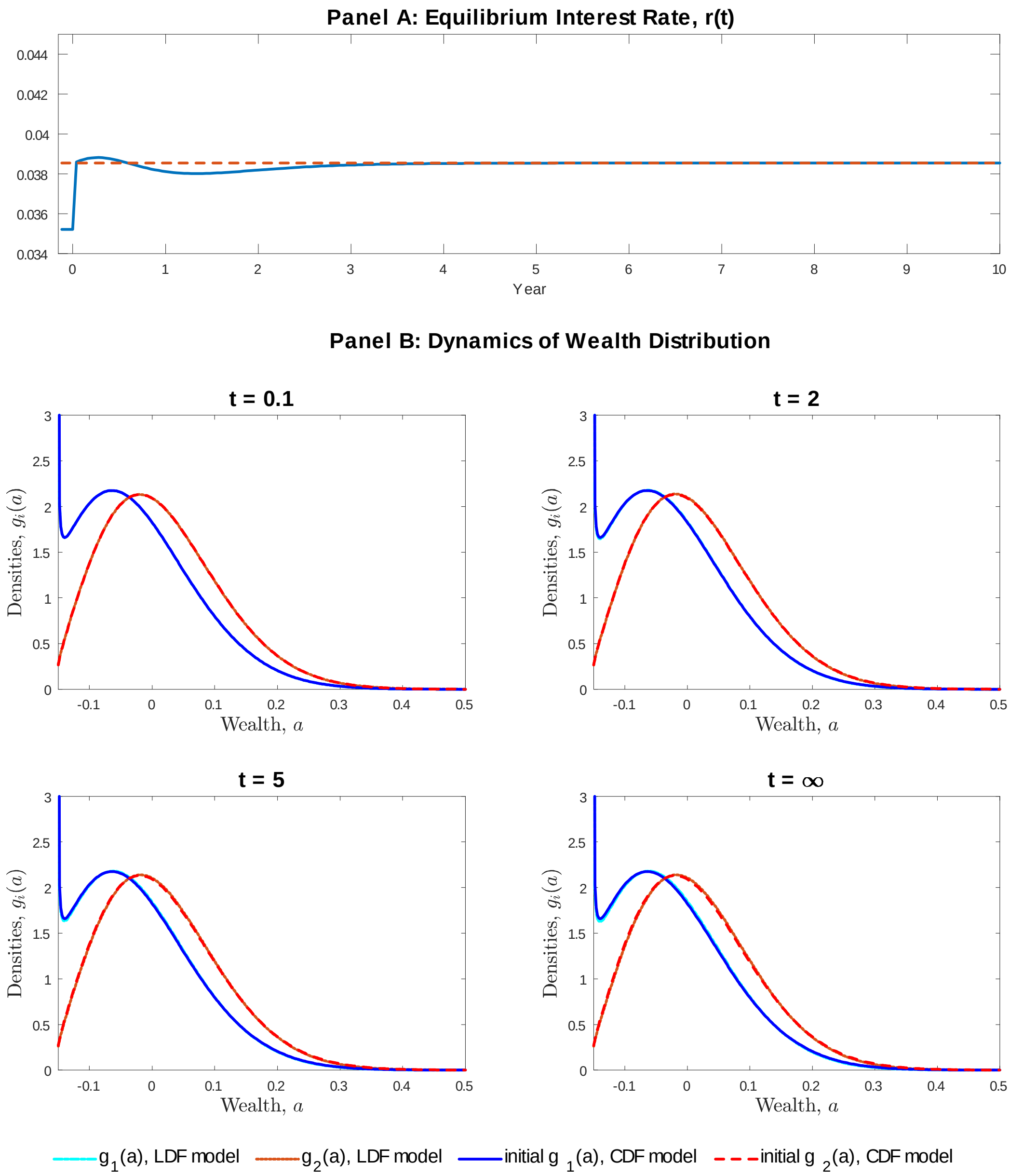

Figure 2.4: Transition dymnamics, LDF model

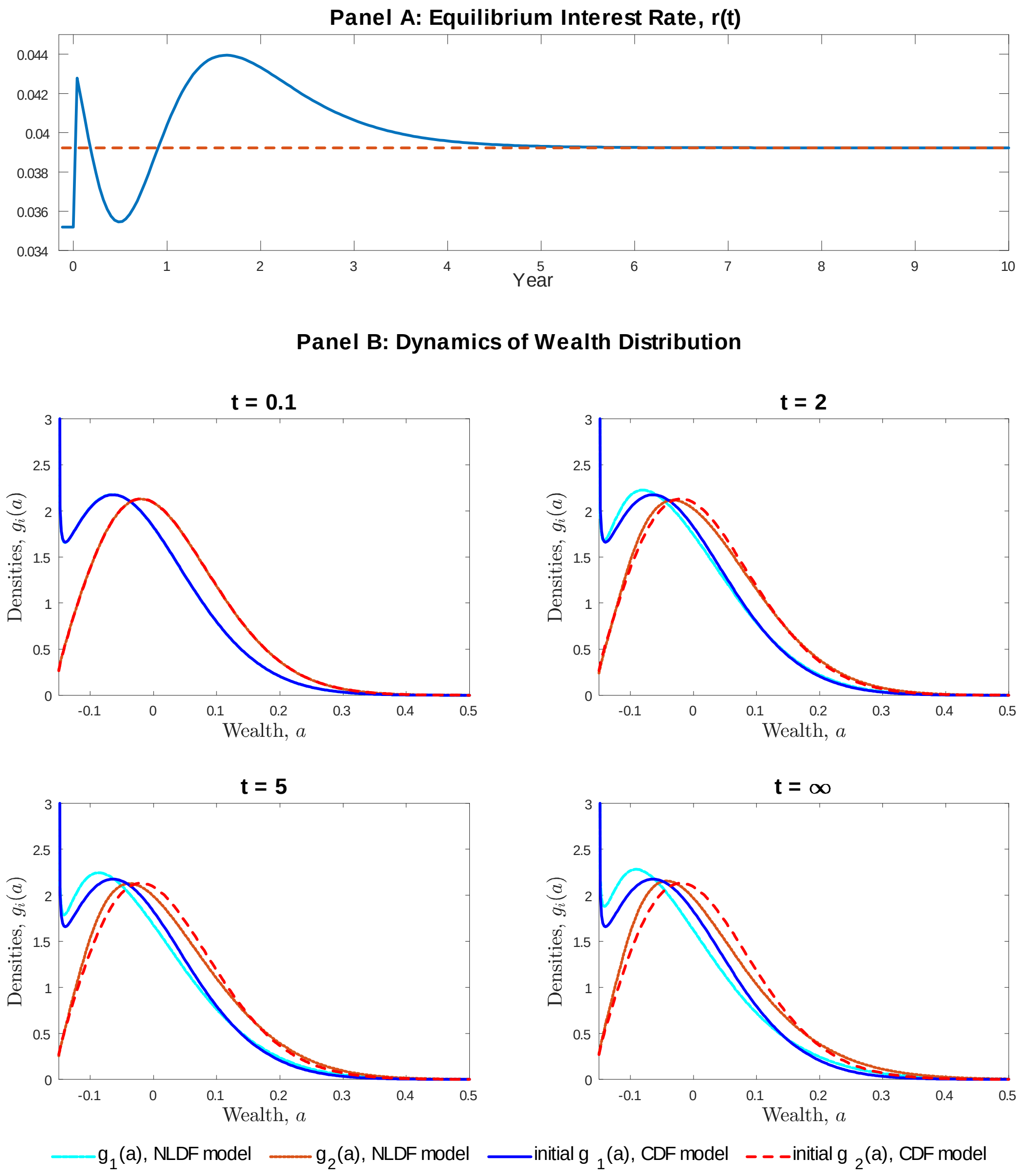

Figure 2.5: Transition Dynamics, NLDF model

population of debtors causing stronger upward pressure on equilibrium interest rate due to the increased need to refinance the existing debt. With strong income effect on the high positive-wealth group the supply of loans increases and the equilibrium interest rate falls to its new steady-state level. The dynamics takes about five years to reach distribution which is reasonably close to the stationary distribution.

2.8 Distributional Effects of Taxes

In this section we introduce production and compare the equilibrium outcomes with the endowment economy. We investigate how the distributional consequences of endogenous discounting are affected by redistributive policies. In this model consumption and asset holding are the only decision variables of individual agents, therefore taxes on consumption and on capital income are the natural choice.

Consumption and capital income taxes are redistributed as a uniform lump-sum payment to everyone. The individual budget constraint can be written as

$$\dot{a} = y_j + \left(1 - t_a \times 1_{[a \geq 0]}\right) ra - (1 + t_c)\, c + T$$

where t_c is consumption tax rate, t_a is capital income tax rate and T is a lump-sum government transfer. We assume that the government balances budget at every instant so the government budget constraint can be written as

$$t_c \int_{\underline{a}}^{a_{\max}} \left[c_1(a)\, g_1(a) + c_2(a)\, g_2(a)\right] da + t_a r \int_0^{a_{\max}} a \left[g_1(a; r) + g_2(a; r)\right] da = T.$$

To account for taxes we modify the HJB equation

$$\begin{aligned} 0 \;=\; & \max_c \left(u(c) - \rho(c)\, V_j(a)\right. \\ & + \left(\left(1 - t_a \times 1_{[a \geq 0]}\right) ra + y_j - (1 + t_c)\, c + T\right) \frac{\partial V_j(a)}{\partial a} \\ & \left. + \lambda_j \left(V_{-j}(a) - V_j(a)\right)\right) \end{aligned} \tag{2.15}$$

while in Kolmogorov Forward equation (2.10) the saving policy function accounts for taxation:

$$s_j(a) = \left(1 - t_a \times 1_{[a \geq 0]}\right) ra + y_j - (1 + t_c)\, c_j(a) + T.$$

Maximization of (2.15) yields the following first order condition

$$\frac{\partial u(c)}{\partial c} = \frac{\partial \rho(c)}{\partial c} V_j(a) + (1 + t_c) \frac{\partial V_j(a)}{\partial a} \tag{2.16}$$

The boundary condition becomes

$$(1 + t_c) \frac{\partial V_j(\underline{a})}{\partial a} + \frac{\partial \rho\left((y_j + r\underline{a} + T) / (1 + t_c)\right)}{\partial c} V_j(\underline{a}) \geq \frac{\partial u\left((y_j + r\underline{a} + T) / (1 + t_c)\right)}{\partial c} \tag{2.17}$$

as no capital income tax is imposed on borrowers with wealth $\underline{a}$.

The adjusted marginal utility in the Euler equation (2.14) can be written as

$$\mu_j(a) = \frac{\partial V_j(a)}{\partial a} = \frac{1}{(1 + t_c)} \left(\frac{\partial u(c)}{\partial c} - \frac{\partial \rho(c)}{\partial c} V_j(a) \right).$$

Consider consumption tax first. As this tax is applied to all consumers, we can prove the following proposition which describes asymptotic saving behavior of the low income types.

Proposition 4 *Assume that Assumption 1 is satisfied, $r < \rho(c)$, $y_1 < y_2$, $t_c > 0$ and the coefficient of risk aversion $R(c) = -\lim_{a \to \underline{a}} \frac{\partial^2 u(c)}{\partial c^2} / \frac{\partial u(c)}{\partial c} < \infty$. Then the solution to the HJB equation (2.9) and the corresponding policy functions have the following properties:*

1. $s_1(\underline{a}) = 0$ but $s_1(a) < 0$ for $a > \underline{a}$. That is, only individuals exactly at the borrowing constraint are constrained, while those with wealth $a > \underline{a}$ are unconstrained and decumulate assets.

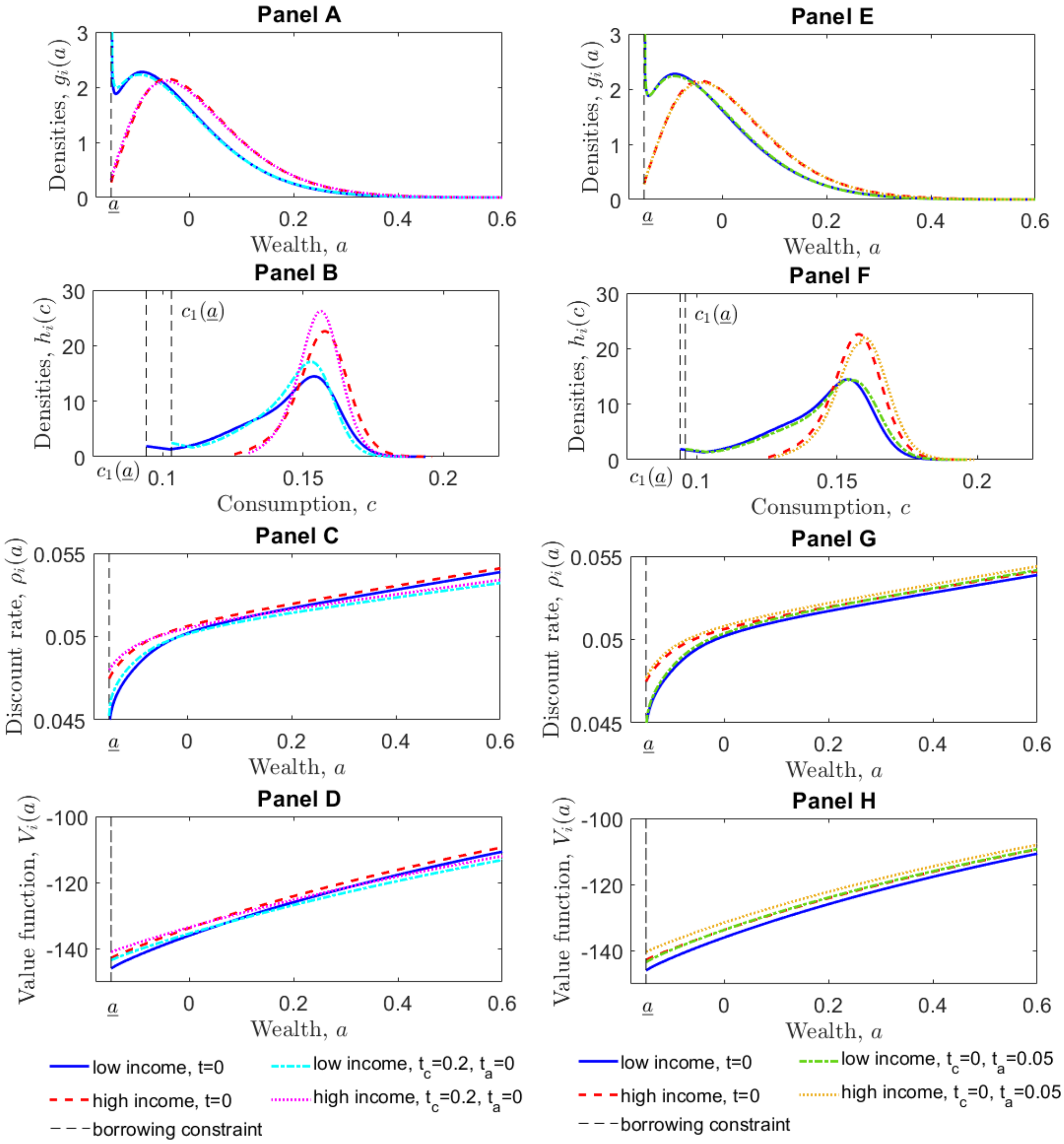

Figure 2.6: Distributional effects of taxes, SDR model

Table 2.4: Distributional effects of taxes

	CDR $\Theta = 0.0$ No tax	LDR $\Theta = 0.1$ No tax	t_c 20%	t_a 5%	NLDR $\Theta = 0.1$ No tax	t_c 20%	t_a 5%
	(1)	(2)	(3)	(4)	(5)	(6)	(7)
Panel I: General equilibrium interest rate and behavior on the borrowing constraint							
Interest rate r	0.0352	0.0385	0.0408	0.0401	0.0392	0.0416	0.0405
Parameter, ν_1^{tax}	0.0260	0.0256	0.0286	na	0.0216	0.0223	na
Mass at m_1	0.0151	0.0146	0.0161	0.0142	0.0156	0.0177	0.0157
Panel II: Statistical characteristics of total wealth distributions							
standard deviation	0.0939	0.0937	0.0959	0.0925	0.1053	0.1077	0.1042
skewness	0.6697	0.6736	0.7064	0.6343	1.0366	1.0716	0.9780
kurtosis	3.2674	3.2810	3.3501	3.2000	4.3323	4.4725	4.1152
Panel III: Standard deviation of consumption distributions by income group							
low income group	0.0168	0.0169	0.0144	0.0169	0.0179	0.0152	0.0179
high income group	0.0102	0.0106	0.0088	0.0108	0.0096	0.0081	0.0097
total	0.0196	0.0200	0.0169	0.0200	0.0204	0.0173	0.0203
Panel IV: Net supply of loans, $L^S - L^D$							
The poor, $a < 0$	-0.0382	-0.0378	-0.0385	-0.0373	-0.0414	-0.0423	-0.0412
The middle class, $a \gtrsim 0$	0.0227	0.0230	0.0119	0.0233	0.0203	0.0201	0.0207
The rich, $a \gg 0$	0.0155	0.0148	0.0228	0.0139	0.0211	0.0222	0.0205
Panel IV: Statistical characteristics of wealth distributions by current income group							
Bottom 20%, $\hat{a}_{L,20}$	-0.1043	-0.1043	-0.1064	-0.1023	-0.1106	-0.1126	-0.1106
Median, $\hat{a}_{L,50}$	-0.0337	-0.0337	-0.0337	-0.0317	-0.0420	-0.0420	-0.0400
Top 10%, $\hat{a}_{L,90}$	0.1074	0.1054	0.1074	0.1033	0.1199	0.1220	0.1178
Bottom 20%, $\hat{a}_{H,20}$	-0.0670	-0.0670	-0.0690	-0.0649	-0.0732	-0.0753	-0.0732
Median, $\hat{a}_{H,50}$	0.0057	0.0057	0.0036	0.0057	-0.0026	-0.0026	-0.0005
Top 10%, $\hat{a}_{H,90}$	0.1448	0.1427	0.1469	0.1406	0.1593	0.1614	0.1573
Median Welfare, $V_{L,50}$	-137.01	-138.15	-137.16	-135.87	-138.39	-137.40	-135.93
Median Welfare, $V_{H,50}$	-133.51	-133.55	-133.43	-131.48	-133.85	-133.60	-131.50

2. as $a \to \underline{a}$ the saving and consumption policy function of the low-income type and the corresponding instantaneous marginal propensity to consume satisfy

$$s_1(a) \sim -\sqrt{2\nu_1^{tax}}\sqrt{a - \underline{a}}$$

$$c_1(a) \sim \frac{1}{(1+t_c)}\left(ra + y_1 - \sqrt{2\nu_1^{tax}}\sqrt{a - \underline{a}}\right)$$

$$\frac{\partial c_1(a)}{\partial a} =\sim \frac{1}{(1+t_c)}\left(r - \sqrt{\frac{\nu_1^{tax}}{2(a-\underline{a})}}\right)$$

where

$$\nu_1^{tax} = \frac{(1+t_c)^2\left((r-\rho(\underline{c}))\mu_1(\underline{a}) + \lambda_1(\mu_2(\underline{a}) - \mu_1(\underline{a}))\right)}{\frac{\partial^2 u(\underline{c}_1)}{\partial c^2} - V_1(\underline{a})\frac{\partial^2 \rho(\underline{c}_1)}{\partial c^2}}$$

and

$$V_2(\underline{a}) = \frac{\left(u(\underline{c}_2) + \frac{s_2(\underline{a})}{(1+t_c)}\frac{\partial u(\underline{c}_2)}{\partial c}\right)(\rho(\underline{c}_1)+\lambda_1) + \lambda_2 u(\underline{c}_1)}{\left(\rho(\underline{c}_2) + \frac{s_2(\underline{a})}{(1+t_c)}\frac{\partial \rho(\underline{c}_2)}{\partial c}\right)(\rho(\underline{c}_1)+\lambda_1) + \lambda_2 \rho(\underline{c}_1)}$$

$$V_1(\underline{a}) = \frac{u(\underline{c}_1) + \lambda_1 V_2(\underline{a})}{\rho(\underline{c}_1) + \lambda_1}$$

$$\mu_1(\underline{a}) = \frac{1}{(1+t_c)}\left(\frac{\partial u(\underline{c}_1)}{\partial c} - \frac{\partial \rho(\underline{c}_1)}{\partial c}V_1(\underline{a})\right)$$

$$\mu_2(\underline{a}) = \frac{1}{(1+t_c)}\left(\frac{\partial u(\underline{c}_1)}{\partial c} - \frac{\partial \rho(\underline{c}_1)}{\partial c}V_2(\underline{a})\right)$$

The effect of consumption tax on consumption and savings is non-monotone: both high and low income groups with sufficiently large positive wealth reduce consumption and increase savings. The negative-wealth group increases consumption. The point mass on the borrowing constraint is larger, but the cdf function $G_1(\underline{a})$ is flatter at the constraint, as parameter ν_1 is higher. The distributional effect of the consumption tax can be described as higher inequality, if measured by the moments: standard deviation, skewness, and kurtosis are all higher with tax both for current poor and current rich, see columns (2), (3), (5), (6) in Table 2.4 and Panels A-D in Figure 2.6 for SDR model. In this sense, the consumption tax appears to be regressive. However, the welfare effect is similar to the effect on discount factor, see panels C and D in Figure 2.6. Both low current income and high current income groups with sufficiently negative wealth gain from the consumption tax, whereas those with sufficiently large positive wealth lose. Consumption tax results in flatter $V_j(a)$ curves and in this sense reduces the welfare inequality. For LDR and SDR the standard deviation of consumption distribution falls with higher tax rate, see Table 2.4.

Capital income tax is only levied on individuals with positive wealth (there is no subsidy to borrowers for interest payments) so it is not possible to obtain analytical results similar to those for consumption tax. We report and briefly discuss the numerical results only.

Capital income tax results in uniform increase of welfare for both current income groups, as one can see from Table 2.4 and panels E-H in Figure 2.6. Consumption and discount rates rise as well as interest rate. As this is endowment economy with exogenous level of output (streams of y_1 and y_2) , increase in current consumption reduces the current saving but does not affect future output. In a production economy a reduction in saving would have negative effect on the future output and therefore on future earnings and consumption, and the overall effect on welfare may not be unambiguously positive. In contrast, in the endowment economy life-time welfare unambiguously rises because of higher current consumption.

2.9 Distributional Effects of Uzawa-type Preferences in A Production Economy

As noted in the previous section, an unambiguously positive effect of capital income tax on the aggregate welfare is fundamentally the property of an endowment economy. To investigate further the differences between an endowment economy and a production economy in the equilibrium outcomes of Uzawa-type recursive preferences we now analyze an economy where wealth is used as a productive capital.

We assume that the production sector can be described by a representative firm hiring labour and renting capital from perfectly competitive markets. An individual's income

y is determined by an economy-wide wage w and idiosyncratic labour productivity $z \in \{z_1, z_2\}$, with $z_2 > z_1$, as

$$y_j = wz_j$$

The wealth accumulation equation takes the form

$$\dot{a} = ra + wz - c \tag{2.18}$$

where a is the holding of wealth in the form of capital. We assume that individuals face an exogenous borrowing limit

$$a \geq \underline{a} \tag{2.19}$$

with $-\infty < \underline{a} < 0$.

As in the Huggett's (1993) model, individuals choose consumption path c to maximize utility (2.1) subject to the budget constraint (2.2), the borrowing limit (2.3), and an exogenously specified labor productivity process, taking interest rate as given. We denote the joint probability distribution of productivity z_j and wealth a by $G_j(a, t)$, with the corresponding density function $g_j(a, t)$, $j = 1, 2$.

In a stationary equilibrium, at every instant t

$$\int_{\underline{a}}^{\infty} ag_1(a)\, da + \int_{\underline{a}}^{\infty} ag_2(a)\, da = K. \tag{2.20}$$

Capital depreciates at rate δ. We assume Cobb-Douglas production function

$$Y = \Phi K^{\alpha} L^{1-\alpha}$$

so that from the firm's profit maximization it follows

$$r = \alpha\Phi \left(\frac{K}{L}\right)^{\alpha-1} \tag{2.21}$$

$$w = (1-\alpha)\,\Phi \left(\frac{K}{L}\right)^{\alpha} \tag{2.22}$$

and we normalize $L = 1$.

Finally, we assume that the flow utility function takes the same CRRA form (2.5).

When we set full depreciation of capital $\delta = 1$ and the capital share $\alpha = 0$ we obtain the endowment economy, with results discussed in the previous chapter. Important characteristics of this economy with constant discount rate are presented in column (1) of Table 2.5 and in Figures 2.7 and 2.8 using solid lines.

We now increase capital share while keeping full depreciation, see column (2) in Table 2.5. This is equivalent to injecting wealth into an endowment economy. Indeed, equation (2.22) implies

$$w = (1-\alpha)\,\Phi K^{\alpha}$$

and we can choose $\Phi = \frac{1}{(1-\alpha)K^{\alpha}}$ to normalize $w = 1$ in general equilibrium. The model with normalized wage is isomorphic to the endowment economy with $A > 0$.

The results in the previous section were shown to originate from movements of population between borrower and lender positions, and the main quantitative requirement was that the number of lenders was comparable to the number of borrowers, so that the redistribution mechanism could interact with a shift in preferences.

This requirement is no longer necessary. With positive total wealth the exogenous borrowing limit $\underline{a} = -0.15$ imposes weaker constraint on wealthier population. In particular, the point mass at this constraint goes down substantially. A possible way to see if distributional mechanisms remain operational is to make the borrowing limit less negative, so that it constrains households in a quantitatively similar way.

Column (2) in Table 2.5, therefore, imposes much smaller borrowing limit, to (partly) compensate for higher total wealth. The new borrowing limit ensures a quantitatively similar value for the point mass at the constraint.

Table 2.5: Numerical Characteristics of the Stationary Distribution

Type of Economy	CDR Endowment	CDR Production	SDR	CDR	SDR
	(1)	(2)	(3)	(4)	(5)
Capital share, α	0.00	0.30	0.30	0.3	0.3
Depreciation rate, δ	1.00	1.00	1.00	0.10	0.10
Borrowing limit, $\underline{a}$	-0.15	-0.04	-0.04	-0.04	-0.04
Mass at the borrowing constraint, m_1	0.015	0.015	0.015	0.002	0.001
Standard deviation of the total wealth distributions	0.094	0.044	0.044	0.298	0.283
Interest rate, r	0.035	0.037	0.039	0.048	0.049

Table 2.5 and Figures 2.7 and 2.8 demonstrate that the effect of wealth injection although with tighter borrowing constraint is more concentrated wealth distributions. The average wealth is positive, and there are more lenders then borrowers. Consumption distribution is also more concentrated around lower average consumption level.

Column (3) demonstrates that with these initial conditions, a switch to the Uzawa-type variable discounting rate does not result in any notable changes in wealth distribution. Dashed and dash-dotted lines in Figure 2.8 are virtually indistinguishable, as columns (2) and (3) in Table 2.5 confirm.

Reducing depreciation rate allows reusing capital. This has positive effect on the average level of aggregate wealth, and reduces the impact of the borrowing limit. Comparison of columns (2) and (4) in Table 2.5 shows that, keeping the borrowing limit constant, a reduction in depreciation rate in CDR economy results in sharp reduction of the point mass at the constraint, strong increase in interest rate and a substantially more dispersed wealth distribution. This is also illustrated in Figure 2.7.

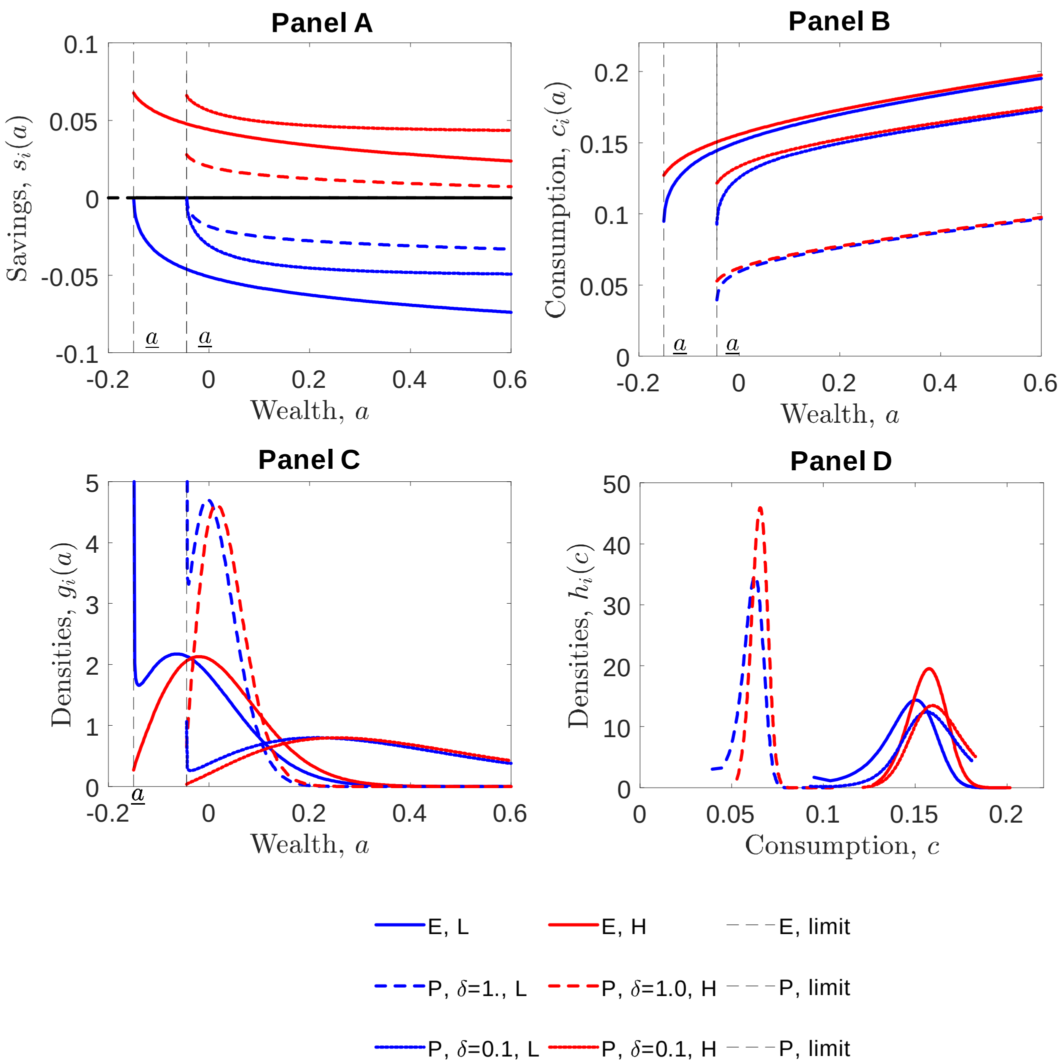

Figure 2.7: Constant Discount Rate Model, Comparison of Distributions in Endowment and Production Economics.

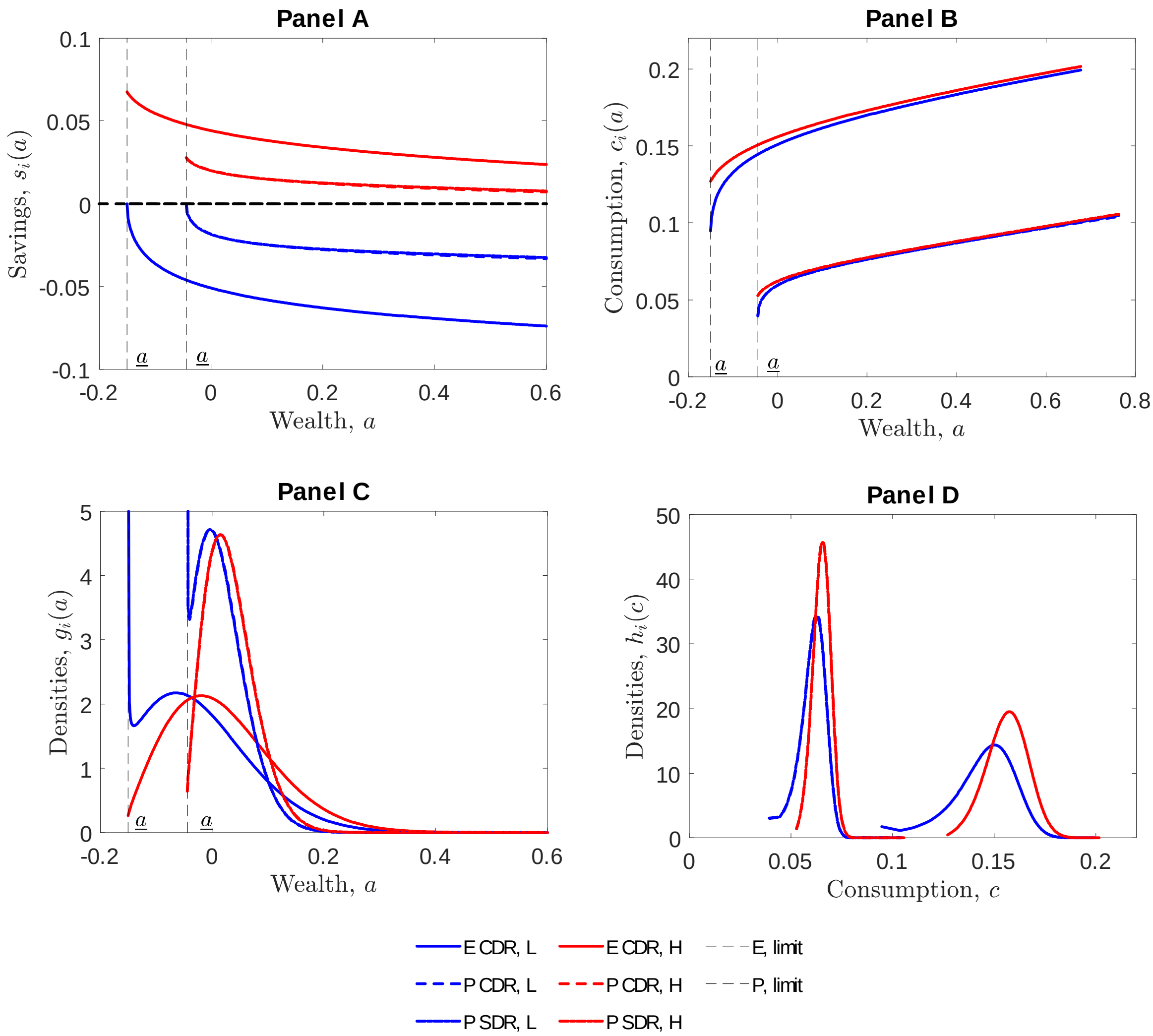

Figure 2.8: Comparison of Distributions for Constant and Variable Discounting Models

Columns (4) and (5) in Table 2.5 demonstrate that a shift to variable Uzawa-type discounting has virtually no effect on wealth inequality. Introducing production sector is likely to lead to lower inequality even with Uzawa-type preferences.

2.10 Conclusion

In this paper we demonstrate that with Uzawa-type preferences, a simple model with uninsurable risk can generate more unequal wealth distribution than the standard model with constant discount rate. With Uzawa-type preferences there is substantial income effect which results in higher consumption and saving by high-wealth households. We also demonstrate that consumption tax can reduce wealth inequality.

Finally, endogenous Uzawa-type discounting by households may have substantial effect on wealth and consumption inequality in an endowment economy with borrowing constraints. However, with production possibilities resulting in endogenous creation of positive wealth, the effect of discounting on wealth inequality is greatly reduced. The mechanism discussed in this chapter may help to understand additional reasons leading to high observed income inequality in developing countries with little production possibilities and binding borrowing constraints.

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2.A Derivation of HJB, KF and Euler equations with endogenous discount rate

2.A.1 HJB equation

Household maximizes the following value function:

$$V\left(a_t, y_t; t\right) = \max_{c_t} \mathbb{E}_t \left[\int_t^T \exp\left(-\int_t^s \rho\left(c_\tau\right) d\tau\right) u\left(c_s\right) ds\right] + g\left(a_T\right)$$

subject to

$$da_t = \left(r_t a_t + y_t - c_t\right) dt$$

$$dy_t = \mu_t dt + \sigma_t dW + \lambda_t dN_t$$

and given state a_t.

Consider interval $[t, t+h]$. We can rewrite

$$
\begin{aligned}
V(a_t, y_t; t) &= \max_{c_t} \mathbb{E}_t \left[\int_t^{t+h} \exp\left(-\int_t^s \rho(c_\tau)\, d\tau \right) u(c_s)\, ds \right. \\
&\quad \left. + \int_{t+h}^T \exp\left(-\int_t^s \rho(c_\tau)\, d\tau \right) u(c_s)\, ds \right] \\
&= \max_{c_t} \mathbb{E}_t \left[\int_t^{t+h} \exp\left(-\int_t^s \rho(c_\tau)\, d\tau \right) u(c_s)\, ds \right. \\
&\quad \left. + \exp\left(-\int_t^{t+h} \rho(c_\tau)\, d\tau \right) \int_{t+h}^T \exp\left(-\int_{t+h}^s \rho(c_\tau)\, d\tau \right) u(c_s)\, ds \right] \\
&= \max_{c_t} \mathbb{E}_t \left[\int_t^{t+h} \exp\left(-\int_t^s \rho(c_\tau)\, d\tau \right) u(c_s)\, ds \right. \\
&\quad \left. + \exp\left(-\int_t^{t+h} \rho(c_\tau)\, d\tau \right) V(a_{t+h}, y_{t+h}; t+h) \right],
\end{aligned}
$$

with terminal condition

$$
V(a_T, y_T; T) = g(a_T, y_T).
$$

It follows

$$
\begin{aligned}
0 &= \max_{c_t} \mathbb{E}_t \left[\int_t^{t+h} \exp\left(-\int_t^s \rho(c_\tau)\, d\tau \right) u(c_s)\, ds \right. \\
&\quad \left. + \exp\left(-\int_t^{t+h} \rho(c_\tau)\, d\tau \right) V(a_{t+h}, y_{t+h}; t+h) - V(a_t, y_t; t) \right] \\
&= \max_{c_t} \mathbb{E}_t \left[\frac{1}{h} \int_t^{t+h} \exp\left(-\int_t^s \rho(c_\tau)\, d\tau \right) u(c_s)\, ds \right. \\
&\quad \left. + \frac{1}{h} \left(\exp\left(-\int_t^{t+h} \rho(c_\tau)\, d\tau \right) V(a_{t+h}, y_{t+1}; t+h) - V(a_t, y_t; t) \right) \right]
\end{aligned}
$$

Take the limit as $h \to 0$:

$$
\begin{aligned}
& \lim_{h \to 0} \frac{1}{h} \int_t^{t+h} \exp\left(-\int_t^s \rho(c_\tau)\, d\tau \right) u(c_s)\, ds \\
\approx\ & \lim_{h \to 0} \frac{1}{h} \int_t^{t+h} \left(1 - \int_t^s \rho(c_\tau)\, d\tau \right) u(c_s)\, ds = u(c_t)
\end{aligned}
$$

$$\begin{aligned}
& \lim_{h\to 0}\frac{1}{h}\left[\exp\left(-\int_t^{t+h}\rho\left(c_\tau\right)d\tau\right)V\left(a_{t+h},y_{t+h};t+h\right)-V\left(a_t,y_t;t\right)\right] \\
\approx\ & \lim_{h\to 0}\frac{1}{h}\left[\left(1-\int_t^{t+h}\rho\left(c_\tau\right)d\tau\right)V\left(a_{t+h},y_{t+h};t+h\right)-V\left(a_t,y_t;t\right)\right] \\
=\ & \lim_{h\to 0}\frac{1}{h}\left[\left(V\left(a_{t+h},y_{t+h};t+h\right)-V\left(a_t,y_t;t\right)\right)-V\left(a_{t+h},y_{t+1};t+h\right)\int_t^{t+h}\rho\left(c_\tau\right)d\tau\right] \\
=\ & V_t\left(a_t,y_t;t\right)+\Delta V\left(a_t,y_t;t\right)-\rho\left(c_t\right)V\left(a_t,y_t;t\right)
\end{aligned}$$

where

$$\begin{aligned}
\Delta V\left(a_t,y_t;t\right) &= V_a\left(a_t,y_t;t\right)\left[r_t a_t+y_t-c_t\right]+\mu_t V_y\left(a_t,y_t;t\right) \\
&\quad +\frac{\sigma_t^2}{2}V_{yy}\left(a_t,y_t;t\right)+\lambda_t E\left[V\left(a_t,y_t';t\right)-V\left(a_t,y_t;t\right)\right].
\end{aligned} \tag{2.23}$$

Finally,

$$0=\max_{c_t}E_t\left[u\left(c_t\right)+V_t\left(a_t,y_t;t\right)+\Delta V\left(a_t,y_t;t\right)-\rho\left(c_t\right)V\left(a_t,y_t;t\right)\right]$$

The first order condition (Euler equation) can be written as:

$$0=u'\left(c_t\right)-\rho'\left(c_t\right)V-V_a$$

Assume the stochastic income process y follows Poisson processes, as in Achdou et al. (2017). If we define

$$\begin{aligned}
V_j\left(a_t\right) &\equiv \left.V\left(a_t,y_t\right)\right|_{y_t=y_j}, \\
V_j'\left(a_t\right) &\equiv \left.V_a\left(a_t,y_t\right)\right|_{y_t=y_j},
\end{aligned}$$

then

$$\Delta V_j\left(a_t,y_j\right)\equiv\left.\Delta\ V\left(a_t,y_t\right)\right|_{y_t=y_j}=V_j'\left(a_t\right)\left[r_t a_t+y_j-c_t\right]+\lambda_j\left[V_{-j}\left(a_t\right)-V_j\left(a_t\right)\right]$$

and the HJB equation becomes

$$0=\max_{c_t}E_t\left[\left[u\left(c_t\right)-\rho\left(c_t\right)V_j\left(a_t\right)+\Delta V_j\left(a_t\right)\right]\right]$$

The time-dependent evolution of the economy is described by the same equation with an additional term:

$$0 = \max_{c_t} E_t \left[\left[u\left(c_t\right) - \rho\left(c_t\right) V_j\left(a_t, t\right) + \Delta V_j\left(a_t, t\right) \right] \right] + \frac{\partial V_j\left(a_t, t\right)}{\partial t}.$$

We can introduce the local rate of time preference (Epstein and Hynes 1983, Obstfeld 1990):

$$\widetilde{\rho}\left(c, V_j; t\right) \equiv -\frac{d}{dt} \log V_j\left(a_t, t\right) \Big|_{\dot{c}(t)=0}, \tag{2.24}$$

which is the continuous-time analogue of the marginal rate of substitution between consumption at t and consumption at $t - dt$. At $t = 0$ present value of (2.24) is given by

$$\widetilde{\rho}\left(c, V_j\right) \equiv \widetilde{\rho}\left(c, V_j; t\right) \exp\left(-\int_0^t \rho\left(c_\tau\right) d\tau \right) = -\frac{d}{dt} \log V_j\left(a_0, y_0; 0\right) \Big|_{\dot{c}(t)=0}.$$

There is no steady state in this economy in the sense that $\dot{c}(t) = 0$ for all agents at the same time, but we can define the local rate of time preference for a given individual by calculating the log-derivative along his consumption path at the point where $\dot{c}(t) = 0$ for this agent.

The derivations in Obstfeld (1990) are reproduced below. For our economy

$$\widetilde{\rho}\left(c, \phi\right) = \rho\left(c\right) + \frac{\rho'\left(c\right)}{u'\left(c\right) - \rho'\left(c\right) V_j} \frac{\partial V_j\left(a_t, t\right)}{\partial t} \Big|_{\dot{c}(t)=0}.$$

Thus, the local rate of time preference is higher than the instantaneous time discount rate, $\widetilde{\rho}\left(c, \phi\right) > \rho\left(c\right)$,when

$$\frac{\rho'\left(c\right)}{u'\left(c\right) - \rho'\left(c\right) V_j} \frac{\partial V_j\left(a_t, t\right)}{\partial t} \Big|_{\dot{c}(t)=0} > 0$$

Since $\rho'\left(c\right) > 0$ and $u'\left(c\right) - \rho'\left(c\right) V_j\big|_{\dot{c}(t)=0} = V_{j,a} > 0$, the sign is determined by

$$\frac{\partial V_j\left(a_t, t\right)}{\partial t} = -\max_{c_t} E_t \left[\left[u\left(c_t\right) - \rho\left(c_t\right) V_j\left(a_t, t\right) + \Delta V_j\left(a_t, t\right) \right] \right].$$

This is positive when c_t is below its level in the stationary equilibrium (where $\frac{\partial V_j(a_t,t)}{\partial t} = 0$). Therefore, when c_t is growing towards its stationary equilibrium level from below the agent puts lower weight on the future utility, meaning higher impatience at lower consumption levels. The opposite is true for c_t falling towards stationary equilibrium from above. Impatience is measured by the difference between the rate of time preference and the instantaneous discount factor.

2.A.2 KF equation

Suppose individual wealth of j-type evolves

$$d\tilde{a}_t = s_j\left(\tilde{a}_t, t\right) dt$$

there are two densities $g_j\left(a, t\right), j = 1, 2$.

We can discretize evolution of wealth:

$$\tilde{a}_{t+\Delta} = \tilde{a}_t + s_j\left(\tilde{a}_t, t\right) \times \Delta \tag{2.25}$$

this is if a j-type person had $\tilde{a}_t$ and makes per period saving decision $s_j\left(\tilde{a}_t, t\right)$ then after Δ being in state j, the wealth is $\tilde{a}_{t+\Delta}$.

After saving decision is made, next period income is realized for a j-state is $\tilde{y}_{t+\Delta}$. Define

$$G_j\left(a, t\right) = \Pr\left(\tilde{a}_t \leq a, \tilde{y}_t = y_j\right)$$

is fraction of population with income y_j with wealth below a. Here

$$G_1\left(\underline{a}, t\right) + G_2\left(\underline{a}, t\right) = 0$$

$$\lim_{a \to \infty} \left(G_1\left(a, t\right) + G_2\left(a, t\right)\right) = 1$$

Define density

$$g_j(a,t) = \frac{\partial G_j(a,t)}{\partial a}$$

From (2.25) it follows

$$\tilde{a}_t = \tilde{a}_{t+\Delta} - s_j(\tilde{a}_{t+\Delta}, t+\Delta) \times \Delta$$

which can be interpreted as if j-type individual at time $t+\Delta$ has $\tilde{a}_{t+\Delta}$ then his wealth a moment before was $\tilde{a}_t = \tilde{a}_{t+\Delta} - s_j(\tilde{a}_{t+\Delta}, t+\Delta) \times \Delta$.

Consider fraction of individuals with wealth below a at $t+\Delta$. Let $s_j(\tilde{a}_{t+\Delta}, t+\Delta) < 0$.

$$\begin{aligned}
\Pr(\tilde{a}_{t+\Delta} \leq a) &= \Pr(\tilde{a}_t \leq a) + \Pr(a \leq \tilde{a}_t \leq a - s_j(a, t+\Delta) \times \Delta) \\
&= \Pr(\tilde{a}_t \leq a - s_j(a, t+\Delta) \times \Delta)
\end{aligned}$$

Then

$$\begin{aligned}
\Pr(\tilde{a}_{t+\Delta} \leq a, \tilde{y}_{t+\Delta} = y_j) &= p_j(\Delta) \Pr(\tilde{a}_{t+\Delta} \leq a, \tilde{y}_t = y_j) \\
&\quad + (1 - p_{-j}(\Delta)) \Pr(\tilde{a}_{t+\Delta} \leq a, \tilde{y}_t = y_{-j}) \\
&= p_j(\Delta) \Pr(\tilde{a}_t \leq a - s_j(a, t+\Delta) \times \Delta, \tilde{y}_t = y_j) \\
&\quad + (1 - p_{-j}(\Delta)) \Pr(\tilde{a}_t \leq a - s_{-j}(a, t+\Delta) \times \Delta, \tilde{y}_t = y_{-j}) \\
&= (1 - \lambda_j \Delta) \Pr(\tilde{a}_t \leq a - s_j(a, t+\Delta) \times \Delta, \tilde{y}_t = y_j) \\
&\quad + \lambda_{-j} \Delta \Pr(a_t \leq a - s_{-j}(a, t+\Delta) \times \Delta, \tilde{y}_t = y_{-j})
\end{aligned}$$

so that

$$\begin{aligned}
G_j(a, t+\Delta) &= (1 - \lambda_j \Delta) G_j(a - s_j(a, t+\Delta) \times \Delta, t) \\
&\quad + \lambda_{-j} \Delta G_{-j}(a - s_{-j}(a, t+\Delta) \times \Delta, t)
\end{aligned}$$

subtract $G_j(a,t)$ from both sides

$$\begin{aligned} G_j(a,t+\Delta) - G_j(a,t) &= (1-\lambda_j\Delta) G_j(a - s_j(a,t+\Delta)\times\Delta, t) \\ &\quad + \lambda_{-j}\Delta G_{-j}(a - s_{-j}(a,t+\Delta)\times\Delta, t) - G_j(a,t) \end{aligned}$$

and divide by Δ :

$$\begin{aligned} \frac{G_j(a,t+\Delta) - G_j(a,t)}{\Delta} &= \frac{G_j(a - s_j(a,t+\Delta)\times\Delta, t) - G_j(a,t)}{\Delta} \\ &\quad - \lambda_j G_j(a - s_j(a,t+\Delta)\times\Delta, t) \\ &\quad + \lambda_{-j} G_{-j}(a - s_{-j}(a,t+\Delta)\times\Delta, t). \end{aligned} \tag{2.26}$$

Linear approximation at a yields

$$G_j(a - s_j(a,t+\Delta)\times\Delta, t) \simeq G_j(a,t) - s_j(a,t+\Delta)\times\Delta\times\frac{\partial G_j(a,t)}{\partial a}$$

so that (2.26) can be written as

$$\begin{aligned} \frac{G_j(a,t+\Delta) - G_j(a,t)}{\Delta} &= -s_j(a,t+\Delta)\frac{\partial G_j(a,t)}{\partial a} - \lambda_j G_j(a - s_j(a,t+\Delta)\times\Delta, t) \\ &\quad + \lambda_{-j} G_{-j}(a - s_{-j}(a,t+\Delta)\times\Delta, t) \end{aligned}$$

Take limit $\Delta \to 0$ to obtain

$$\frac{\partial G_j(a,t)}{\partial t} = -s_j(a,t)\frac{\partial G_j(a,t)}{\partial a} - \lambda_j G_j(a,t) + \lambda_{-j} G_{-j}(a,t)$$

which is equivalent to

$$\frac{\partial G_j(a,t)}{\partial t} = -s_j(a,t)\, g_j(a,t) - \lambda_j G_j(a,t) + \lambda_{-j} G_{-j}(a,t).$$

Differentiate it with respect to a to yield

$$\frac{\partial}{\partial t} g_j(a,t) = -\frac{\partial}{\partial a}\left[s_j(a,t)\, g_j(a,t)\right] - \lambda_j g_j(a,t) + \lambda_{-j} g_{-j}(a,t).$$

Finally, in a stationary equilibrium we have

$$0 = -\frac{d}{da}\left[s_j(a)\, g_j(a)\right] - \lambda_j g_j(a) + \lambda_{-j} g_{-j}(a)$$

2.A.3 Euler Equation

HJB equation can be written as

$$0 = \max_{c} \left(u(c) - \rho(c) V_j(a) + (ra + y_j - c) \frac{\partial V_j(a)}{\partial a} + \lambda_j (V_{-j}(a) - V_j(a)) \right),$$

Differentiate it with respect to c to yield

$$\frac{\partial u(c)}{\partial c} = \frac{\partial \rho(c)}{\partial c} V_j(a) + \frac{\partial V_j(a)}{\partial a}. \tag{2.27}$$

In equilibrium, all decision variables are functions of the state, a, so that

$$c = C(a) \tag{2.28}$$

substitute it back to the Bellman equation, it becomes an identity

$$0 = u(C(a)) - \rho(C(a)) V_j(a) + (ra + y_j - C(a)) \frac{\partial V_j(a)}{\partial a} + \lambda_j (V_{-j}(a) - V_j(a)), \tag{2.29}$$

We can differentiate it with respect to a :

$$\begin{aligned} 0 \;=\; & \left(\frac{\partial u(c)}{\partial c} - \frac{\partial \rho(c)}{\partial c} V_j(a) \right) \frac{\partial C(a)}{\partial a} - \rho(c) \frac{\partial V_j(a)}{\partial a} \\ & + \left(r - \frac{\partial C(a)}{\partial a} \right) \frac{\partial V_j(a)}{\partial a} + s_j(a) \frac{\partial^2 V_j(a)}{\partial a^2} + \lambda_j \left(\frac{\partial V_{-j}(a)}{\partial a} - \frac{\partial V_j(a)}{\partial a} \right) \end{aligned} \tag{2.30}$$

where

$$s_j(a) = ra + y_j - C(a).$$

Introduce new variable

$$\mu_j(a) = \frac{\partial V_j(a)}{\partial a} = \frac{\partial u(c)}{\partial c} - \frac{\partial \rho(c)}{\partial c} V_j(a)$$

where the last equality follows from equation (2.27).

Substitute $\mu_j(a)$ and $\frac{\partial \mu_j(a)}{\partial a} = \frac{\partial^2 V_j(a)}{\partial a^2}$ into (2.30) to yield

$$0 = \mu_j(a) \frac{\partial C(a)}{\partial a} - \rho(c) \mu_j(a) + \left(r - \frac{\partial C(a)}{\partial a} \right) \mu_j(a) + s_j(a) \frac{\partial \mu_j(a)}{\partial a} + \lambda_j \left(\mu_{-j}(a) - \mu_j(a) \right)$$

which, after rearrangements yields (2.14).

2.B Proposition 1

Part 1.

Consider $\underline{a} < a < a_0$ where a_0 corresponds to the reference consumption point, c_0. In this domain $\rho(c)$ is a convex function, i.e. $0 < \frac{\partial\rho(c_1)}{\partial c} \leq \frac{\partial\rho(c_2)}{\partial c}$. We know that $0 < \frac{\partial u(c_2)}{\partial c} < \frac{\partial u(c_1)}{\partial c}$ and $V_1(a) < V_2(a) < 0$ for any a. Consider

$$\mu_1(a) = \frac{\partial u(c_1)}{\partial c} - V_1(a)\frac{\partial\rho(c_1)}{\partial c}$$

$$\mu_2(a) = \frac{\partial u(c_2)}{\partial c} - V_2(a)\frac{\partial\rho(c_2)}{\partial c}$$

Then $0 < \mu_2(a) < \mu_1(a)$ as soon as $\rho(c)$ is not too convex, i.e. it satisfies Assumption 1. If $a \geq a_0$ then $\rho(c)$ is a concave function so that $0 < \mu_2(a) < \mu_1(a)$. In this case the right hand side of Euler equation

$$\frac{\partial\mu_1(a)}{\partial a}\frac{s_1(a)}{\mu_1(a)} = \left(\rho(c) - r - \lambda_1\left(\frac{\mu_2(a)}{\mu_1(a)} - 1\right)\right)$$

is strictly positive but $\frac{\partial\mu_1(a)}{\partial a}$ is negative. Therefore, $s_1(a) \leq 0$ for $\underline{a} < a < a_0$. As on the borrowing constraint wealth cannot fall, then $s_1(\underline{a}) \geq 0$. It follows that $s_1(\underline{a}) = 0$.

Part 2.

On the borrowing constraint $s_1(\underline{a}) = 0$ but $\mu_1(\underline{a}) < \infty$. HJB equations, evaluated at $\underline{a}$ yield

$$0 = u(\underline{c}_1) - \rho(\underline{c}_1)V_1(\underline{a}) + \lambda_1(V_2(\underline{a}) - V_1(\underline{a}))$$

$$0 = u(\underline{c}_2) - \rho(\underline{c}_2)V_2(\underline{a}) + s_2(\underline{a})\mu_2(\underline{a}) + \lambda_2(V_1(\underline{a}) - V_2(\underline{a}))$$

where

$$\mu_2(\underline{a}) = \frac{\partial u(c)}{\partial c} - V_2(\underline{a})\frac{\partial\rho(c)}{\partial c}$$

From these three equations we obtain

$$V_2(\underline{a}) = \frac{s_2(\underline{a})\frac{\partial u(c)}{\partial c} + u(\underline{c}_2) + \lambda_2 \frac{u(\underline{c}_1)}{(\rho(\underline{c}_1)+\lambda_1)}}{\left(\frac{\lambda_2 \rho(\underline{c}_1)}{(\rho(\underline{c}_1)+\lambda_1)} + \rho(\underline{c}_2) + s_2(\underline{a})\frac{\partial \rho(c)}{\partial c}\right)}$$

$$V_1(\underline{a}) = \frac{u(\underline{c}_1)}{(\rho(\underline{c}_1)+\lambda_1)} + \frac{\lambda_1}{(\rho(\underline{c}_1)+\lambda_1)} V_2(\underline{a})$$

$$\mu_2(\underline{a}) = \frac{\partial u(c)}{\partial c} - \frac{\partial \rho(c)}{\partial c} V_2(\underline{a})$$

so $V_2(\underline{a}), V_1(\underline{a})$ and $\mu_2(\underline{a})$ are all finite at $\underline{a}$.

From Euler equation (2.14) for agents type $j = 1$

$$\frac{\partial \mu_1(a)}{\partial a}\frac{s_1(a)}{\mu_1(a)} = -\left(r - \rho(c) + \lambda_1\left(\frac{\mu_2(a)}{\mu_1(a)} - 1\right)\right) \tag{2.31}$$

it follows that

$$\left.\frac{\partial \mu_1(a)}{\partial a}\right|_{a=\underline{a}} = \infty$$

as $s_1(\underline{a}) = 0$ and all other variables in the RHS of (2.31) are finite.

Therefore, $\left.\frac{\partial C_1(a)}{\partial a}\right|_{a=\underline{a}} = \infty$ as $\left.\frac{\partial V_1(a)}{\partial a}\right|_{a=\underline{a}} = \mu_1(\underline{a}) < \infty$. Indeed,

$$\left.\frac{\partial \mu_1(a)}{\partial a}\right|_{a=\underline{a}} = \infty = \left(\frac{\partial^2 u(c)}{\partial c^2} - V_1(\underline{a})\frac{\partial^2 \rho(c)}{\partial c^2}\right)\left.\frac{\partial C_1(a)}{\partial a}\right|_{a=\underline{a}} - \mu_1(\underline{a})\frac{\partial \rho(c)}{\partial c}$$

Here, in the RHS all variables except $\left.\frac{\partial C_1(a)}{\partial a}\right|_{a=\underline{a}}$ are finite. So $\left.\frac{\partial C_1(a)}{\partial a}\right|_{a=\underline{a}}$ must be infinite.

It follows that

$$\left.\frac{\partial s_1(a)}{\partial a}\right|_{a=\underline{a}} = r - \left.\frac{\partial C_1(a)}{\partial a}\right|_{a=\underline{a}} = \infty$$

We can now rearrange and evaluate (2.31) at $\underline{a}$:

$$\begin{aligned}&\frac{\left(\frac{\partial^2 u(c)}{\partial c^2} - V_1(\underline{a})\frac{\partial^2 \rho(c)}{\partial c^2}\right)}{\mu_1(\underline{a})}\left.\left(s_1(a)\, r - s_1(a)\frac{\partial s_1(a)}{\partial a}\right)\right|_{a=\underline{a}} - s_1(\underline{a})\frac{\partial \rho(c)}{\partial c} \\ = \;& -\left(r - \rho(c) + \lambda_1\left(\frac{\mu_2(\underline{a})}{\mu_1(\underline{a})} - 1\right)\right)\end{aligned}$$

From this equation

$$\lim_{a\to\underline{a}} s_1(a)\frac{\partial s_1(a)}{\partial a} = \frac{(r-\rho(c))\,\mu_1(\underline{a})+\lambda_1(\mu_2(\underline{a})-\mu_1(\underline{a}))}{\frac{\partial^2 u(c)}{\partial c^2}-V_1(\underline{a})\frac{\partial^2\rho(c)}{\partial c^2}} = \nu_1$$

the limit is finite as all variables in RHS are finite. As Achdou et al. (2018) show

$$\lim_{a\to\underline{a}} \frac{(s_1(a))^2}{a-\underline{a}} = 2s_1(a)\frac{\partial s_1(a)}{\partial a} = 2\frac{(r-\rho(c))\,\mu_1(\underline{a})+\lambda_1(\mu_2(\underline{a})-\mu_1(\underline{a}))}{\frac{\partial^2 u(c)}{\partial c^2}-V_1(\underline{a})\frac{\partial^2\rho(c)}{\partial c^2}} = 2\nu_1$$

so that

$$(s_1(a))^2 \sim 2\nu_1(a-\underline{a})$$

and

$$s_1(a) \sim -\sqrt{2\nu_1}\sqrt{a-\underline{a}} \tag{2.32}$$

Therefore

$$\begin{aligned} c_1(a) &= ra+y_1+s_1(a) \sim ra+y_1-\sqrt{2\nu_1}\sqrt{a-\underline{a}} \\ \frac{\partial c_1(a)}{\partial a} &=\sim r-\sqrt{\frac{\nu_1}{2(a-\underline{a})}} \end{aligned}$$

2.C Proposition 2

KF equation for type j can be written as

$$0 = -\frac{d}{da}\left[s_j(a)\,g_j(a)\right]-\lambda_j g_j(a)+\lambda_{-j}g_{-j}(a).$$

Their sum yields

$$\frac{d}{da}\left[s_1(a)\,g_1(a)+s_2(a)\,g_2(a)\right]=0$$

so that

$$s_1(a)\,g_1(a)+s_2(a)\,g_2(a)=A.$$

The support for both distributions is bounded from the left (the support is $a\geq\underline{a}$). This implies $A=0$. Therefore

$$s_1(a)\,g_1(a)=-s_2(a)\,g_2(a). \tag{2.33}$$

Substitute into KF equation

$$\frac{d}{da}\left[g_j(a)\right] = -\left(\frac{\frac{d}{da}\left[s_j(a)\right]}{s_j(a)} + \frac{\lambda_j}{s_j(a)} + \frac{\lambda_{-j}}{s_{-j}(a)}\right) g_j(a)$$

which solution is

$$g_j(a) = \frac{\kappa_j}{s_j(a)} \exp\left(-\int_{\underline{a}}^{a}\left(\frac{\lambda_j}{s_j(x)} + \frac{\lambda_{-j}}{s_{-j}(x)}\right) dx\right)$$

for some constants of integration with $\kappa_1 + \kappa_2 = 0$

$$\begin{aligned} g_1(\underline{a}+\delta) &= \frac{\kappa_1}{s_1(\underline{a}+\delta)} \exp\left(-\int_{\underline{a}}^{\underline{a}+\delta}\left(\frac{\lambda_1}{s_1(x)} + \frac{\lambda_2}{s_2(x)}\right) dx\right) \\ &= \frac{\kappa_1}{s_1(\underline{a}+\delta)} \exp\left(-\int_{\underline{a}}^{\underline{a}+\delta}\left(\frac{\lambda_1}{s_1(x)}\right) dx\right) \exp\left(-\int_{\underline{a}}^{\underline{a}+\delta}\left(\frac{\lambda_2}{s_2(x)}\right) dx\right) \end{aligned}$$

$$\lim_{\delta\to 0} \exp\left(-\int_{\underline{a}}^{\underline{a}+\delta}\left(\frac{\lambda_2}{s_2(x)}\right) dx\right) = 1$$

Use (2.32) to obtain

$$\begin{aligned} \lim_{\delta\to 0} \frac{1}{-s_1(\underline{a}+\delta)} \exp\left(\int_{\underline{a}}^{\underline{a}+\delta}\left(\frac{\lambda_1}{-s_1(x)}\right) dx\right) &= \lim_{\delta\to 0} \frac{1}{\sqrt{2\nu_1}\sqrt{\delta}} \exp\left(-\int_0^{\delta}\left(\frac{\lambda_1}{\sqrt{2\nu_1}\sqrt{x}}\right) dx\right) \\ &= \lim_{\delta\to 0} \frac{1}{\sqrt{2\nu_1}\sqrt{\delta}} \exp\left(-\frac{2\lambda_1\sqrt{\delta}}{\sqrt{2\nu_1}}\right) = +\infty \end{aligned}$$

Therefore, $g_1(a)$ is unbounded at $\underline{a}$:

$$\begin{aligned} \lim_{a\to\underline{a}} g_1(a) &= \lim_{a\to\underline{a}} \frac{\kappa_1}{s_1(\underline{a}+\delta)} \exp\left(-\int_{\underline{a}}^{\underline{a}+\delta}\left(\frac{\lambda_1}{s_1(x)}\right) dx\right) \\ &= \lim_{a\to\underline{a}} \frac{\kappa_1}{-\sqrt{2\nu_1}\sqrt{a-\underline{a}}} \exp\left(-\lambda_1\sqrt{\frac{2(a-\underline{a})}{\nu_1}}\right) = \infty. \end{aligned}$$

There is point mass at $\underline{a}$ for $g_1(a)$. Denote it m_1, then

$$m_1 + \lim_{\delta\to 0}\int_{\underline{a}+\delta}^{\overline{a}} g_1(a)\,da = \frac{\lambda_2}{\lambda_1+\lambda_2} \tag{2.34}$$

$$\int_{\underline{a}}^{\overline{a}} g_2(a)\,da = \frac{\lambda_1}{\lambda_1+\lambda_2} \tag{2.35}$$

Also, integration of KF equation yields

$$\int_{\underline{a}+\delta}^{\overline{a}} d\left[s_1(a)\,g_1(a)\right] = -\int_{\underline{a}+\delta}^{\overline{a}} \lambda_1 g_1(a)\,da + \int_{\underline{a}+\delta}^{\overline{a}} \lambda_2 g_2(a)\,da$$

so that

$$\begin{aligned}
0 &= \lim_{\delta\to 0} s_1(\underline{a}+\delta)\,g_1(\underline{a}+\delta) - \lambda_1 \lim_{\delta\to 0}\int_{\underline{a}+\delta}^{\overline{a}} g_1(a)\,da + \lambda_2 \lim_{\delta\to 0}\int_{\underline{a}+\delta}^{\overline{a}} g_2(a)\,da \\
&= \lim_{\delta\to 0} s_1(\underline{a}+\delta)\,g_1(\underline{a}+\delta) - \lambda_1\left(\frac{\lambda_2}{\lambda_1+\lambda_2} - m_1\right) + \lambda_2\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right) \\
&= \lim_{\delta\to 0} s_1(\underline{a}+\delta)\,g_1(\underline{a}+\delta) + \lambda_1 m_1 \\
&= \lim_{\delta\to 0}\left(-\sqrt{2\nu_1}\sqrt{\delta}\right)\frac{\kappa_1}{-\sqrt{2\nu_1}\sqrt{\delta}}\exp\left(-\lambda_1\sqrt{\frac{2\delta}{\nu_1}}\right) + \lambda_1 m_1 = \kappa_1 + \lambda_1 m_1
\end{aligned}$$

where we used (2.34)-(2.35). We obtain

$$\kappa_1 = -\lambda_1 m_1 = -\kappa_2 < 0$$

and

$$\lim_{a\to\underline{a}} g_1(a) = +\infty$$

Now

$$\begin{aligned}
g_1(a) &= \frac{\kappa_1}{s_1(a)}\exp\left(-\int_{\underline{a}}^{a}\left(\frac{\lambda_1}{s_1(x)} + \frac{\lambda_2}{s_2(x)}\right)dx\right) \\
&= -\frac{\lambda_1 m_1}{s_1(a)}\exp\left(-\int_{\underline{a}}^{a}\left(\frac{\lambda_1}{s_1(x)} + \frac{\lambda_2}{s_2(x)}\right)dx\right)
\end{aligned}$$

$$
\begin{aligned}
g_2(a) &= \frac{\kappa_2}{s_2(a)} \exp\left(-\int_{\underline{a}}^{a} \left(\frac{\lambda_1}{s_1(x)} + \frac{\lambda_2}{s_2(x)}\right) dx\right) \\
&= \frac{\lambda_1 m_1}{s_2(a)} \exp\left(-\int_{\underline{a}}^{a} \left(\frac{\lambda_1}{s_1(x)} + \frac{\lambda_2}{s_2(x)}\right) dx\right)
\end{aligned}
$$

Use (2.35):

$$
\int_{\underline{a}}^{\overline{a}} g_2(a)\, da = \frac{\lambda_1}{\lambda_1 + \lambda_2}
$$

$$
\int_{\underline{a}}^{\overline{a}} \frac{\lambda_1 m_1}{s_2(a)} \exp\left(-\int_{\underline{a}}^{a} \left(\frac{\lambda_1}{s_1(x)} + \frac{\lambda_2}{s_2(x)}\right) dx\right) da = \frac{\lambda_1}{\lambda_1 + \lambda_2}
$$

so that

$$
m_1 = \frac{\lambda_2}{\lambda_1 + \lambda_2} \tilde{m}_1
$$

where

$$
\frac{1}{\tilde{m}_1} = \lambda_2 \int_{\underline{a}}^{\overline{a}} \frac{1}{s_2(a)} \exp\left(-\int_{\underline{a}}^{a} \left(\frac{\lambda_1}{s_1(x)} + \frac{\lambda_2}{s_2(x)}\right) dx\right) da
$$

Equation (2.33) yields

$$
\begin{aligned}
& \frac{\lambda_1}{s_1(x)} + \frac{\lambda_2}{s_2(x)} = \frac{\lambda_1 g_1(x)}{s_1(x)\, g_1(x)} + \frac{\lambda_2 g_2(x)}{s_2(x)\, g_2(x)} \\
&= -\frac{\lambda_1 g_1(x)}{s_2(x)\, g_2(x)} + \frac{\lambda_2 g_2(x)}{s_2(x)\, g_2(x)} = \frac{-\lambda_1 g_1(x) + \lambda_2 g_2(x)}{s_2(x)\, g_2(x)} \\
&= -\frac{1}{s_2(x)\, g_2(x)} \frac{d}{dx} \left(s_2(x)\, g_2(x)\right)
\end{aligned}
$$

where the last equality follows from the KF equation

$$
\frac{d}{da}\left[s_2(a)\, g_2(a)\right] = -\lambda_2 g_2(a) + \lambda_1 g_1(a).
$$

It follows

$$
\begin{aligned}
& -\int_{\underline{a}}^{a} \left(\frac{\lambda_1}{s_1(x)} + \frac{\lambda_2}{s_2(x)}\right) dx \\
&= -\int_{\underline{a}}^{a} \left(-\frac{1}{s_2(x)\, g_2(x)} \frac{d}{dx} \left(s_2(x)\, g_2(x)\right)\right) dx \\
&= \ln \frac{s_2(a)\, g_2(a)}{s_2(\underline{a})\, g_2(\underline{a})}
\end{aligned}
$$

It implies

$$\begin{aligned}
\frac{1}{\tilde{m}_1} &= \lambda_2 \int_{\underline{a}}^{\overline{a}} \frac{1}{s_2(a)} \exp\left(-\int_{\underline{a}}^{a}\left(\frac{\lambda_1}{s_1(x)}+\frac{\lambda_2}{s_2(x)}\right)dx\right)da \\
&= \lambda_2 \int_{\underline{a}}^{\overline{a}} \frac{1}{s_2(a)} \exp\left(\ln\frac{s_2(a)\,g_2(a)}{s_2(\underline{a})\,g_2(\underline{a})}\right)da \\
&= \frac{\lambda_2}{s_2(\underline{a})\,g_2(\underline{a})}\int_{\underline{a}}^{\overline{a}} g_2(a)\,da \\
&= \frac{\lambda_2}{s_2(\underline{a})\,g_2(\underline{a})}\frac{\lambda_1}{\lambda_1+\lambda_2}
\end{aligned}$$

Finally,

$$m_1 = \frac{\lambda_2}{\lambda_1+\lambda_2}\tilde{m}_1 = \frac{1}{\lambda_1}s_2(\underline{a})\,g_2(\underline{a})$$

To derive asymptotic behavior of the cdf, we use KF equation

$$0 = -\frac{d}{da}\left[s_j(a)\,g_j(a)\right] - \lambda_j g_j(a) + \lambda_{-j}g_{-j}(a).$$

and integrate it to obtain

$$0 = -s_1(a)\frac{dG_1(a)}{da} - \lambda_1 G_1(a) + \lambda_2 G_2(a)$$

at the limit,

$$0 = -\lim_{a\to\underline{a}} s_1(a)\frac{dG_1(a)}{da} - \lim_{a\to\underline{a}}\lambda_1 G_1(a)$$

so that

$$\frac{dG_1(a)}{G_1(a)} = \frac{\lambda_1}{-s_1(a)}da$$

which we integrate to obtain

$$\begin{aligned}
G_1(a) &= m_1 \exp\left(\int_{\underline{a}}^{a}\frac{\lambda_1}{-s_1(a)}da\right) \sim m_1 \exp\left(\int_{\underline{a}}^{a}\frac{\lambda_1}{\sqrt{2\nu_1}\sqrt{a-\underline{a}}}da\right) \\
&= m_1 \exp\left(\lambda_1\sqrt{\frac{2(a-\underline{a})}{\nu_1}}\right)
\end{aligned}$$

2.D Proposition 4

Part 1.

Consider $\underline{a} < a < a_0$ where a_0 corresponds to the reference consumption point, c_0. In this domain $\rho(c)$ can be a convex function, i.e. $0 < \frac{\partial \rho(c_1)}{\partial c} \leq \frac{\partial \rho(c_2)}{\partial c}$. We know that $0 < \frac{\partial u(c_2)}{\partial c} < \frac{\partial u(c_1)}{\partial c}$ and $V_1(a) < V_2(a) < 0$ for any a. Consider

$$\mu_1(a) = \frac{1}{(1+t_c)}\left(\frac{\partial u(c_1)}{\partial c} - V_1(a)\frac{\partial \rho(c_1)}{\partial c}\right)$$

$$\mu_2(a) = \frac{1}{(1+t_c)}\left(\frac{\partial u(c_2)}{\partial c} - V_2(a)\frac{\partial \rho(c_2)}{\partial c}\right)$$

Then $0 < \mu_2(a) < \mu_1(a)$ as soon as $\rho(c)$ is not too convex, i.e. it satisfies Assumption 1. If $a \geq a_0$ then $\rho(c)$ is semi-concave and so $0 < \mu_2(a) < \mu_1(a)$. In this case the right hand side of Euler equation

$$\frac{\partial \mu_1(a)}{\partial a}\frac{s_1(a)}{\mu_1(a)} = \left(\rho(c) - r - \lambda_1\left(\frac{\mu_2(a)}{\mu_1(a)} - 1\right)\right)$$

is strictly positive but $\frac{\partial \mu_1(a)}{\partial a}$ is negative. Therefore, $s_1(a) \leq 0$ for $\underline{a} < a < a_0$. As on the borrowing constraint wealth cannot fall, then $s_1(\underline{a}) \geq 0$. It follows that $s_1(\underline{a}) = 0$.

Part 2.

On the borrowing constraint $s_1(\underline{a}) = 0$ but $\mu_1(\underline{a}) < \infty$. HJB equations, evaluated at $\underline{a}$ yield

$$0 = u(\underline{c}_1) - \rho(\underline{c}_1)V_1(\underline{a}) + \lambda_1(V_2(\underline{a}) - V_1(\underline{a}))$$

$$0 = u(\underline{c}_2) - \rho(\underline{c}_2)V_2(\underline{a}) + s_2(\underline{a})\mu_2(\underline{a}) + \lambda_2(V_1(\underline{a}) - V_2(\underline{a}))$$

where

$$\mu_2(\underline{a}) = \frac{1}{(1+t_c)}\left(\frac{\partial u(c)}{\partial c} - V_2(\underline{a})\frac{\partial \rho(c)}{\partial c}\right)$$

From these three equations we obtain

$$V_2(\underline{a}) = \frac{\frac{s_2(\underline{a})}{(1+t_c)}\frac{\partial u(c)}{\partial c} + u(\underline{c}_2) + \frac{\lambda_2 u(\underline{c}_1)}{(\rho(\underline{c}_1)+\lambda_1)}}{\left(\frac{\lambda_2\rho(\underline{c}_1)}{(\rho(\underline{c}_1)+\lambda_1)} + \rho(\underline{c}_2) + \frac{s_2(\underline{a})}{(1+t_c)}\frac{\partial\rho(c)}{\partial c}\right)}$$

$$V_1(\underline{a}) = \frac{u(\underline{c}_1) + \lambda_1 V_2(\underline{a})}{(\rho(\underline{c}_1) + \lambda_1)}$$

$$\mu_2(\underline{a}) = \frac{1}{(1+t_c)}\left(\frac{\partial u(c)}{\partial c} - \frac{\partial\rho(c)}{\partial c}V_2(\underline{a})\right)$$

so $V_2(\underline{a}), V_1(\underline{a})$ and $\mu_2(\underline{a})$ are all finite at $\underline{a}$.

From Euler equation (2.14) for agents type $j = 1$

$$\frac{\partial\mu_1(a)}{\partial a}\frac{s_1(a)}{\mu_1(a)} = -\left(r - \rho(c) + \lambda_1\left(\frac{\mu_2(a)}{\mu_1(a)} - 1\right)\right) \tag{2.36}$$

it follows that

$$\left.\frac{\partial\mu_1(a)}{\partial a}\right|_{a=\underline{a}} = \infty$$

as $s_1(\underline{a}) = 0$ and all other variables in the RHS of (2.36) are finite.

Therefore, $\left.\frac{\partial C_1(a)}{\partial a}\right|_{a=\underline{a}} = \infty$ as $\left.\frac{\partial V_1(a)}{\partial a}\right|_{a=\underline{a}} = \mu_1(\underline{a}) < \infty$. Indeed,

$$\left.\frac{\partial\mu_1(a)}{\partial a}\right|_{a=\underline{a}} = \infty = \frac{1}{(1+t_c)}\left(\frac{\partial^2 u(c)}{\partial c^2} - V_1(\underline{a})\frac{\partial^2\rho(c)}{\partial c^2}\right)\left.\frac{\partial C_1(a)}{\partial a}\right|_{a=\underline{a}} - \frac{1}{(1+t_c)}\mu_1(\underline{a})\frac{\partial\rho(c)}{\partial c}$$

Here, in the RHS all variables except $\left.\frac{\partial C_1(a)}{\partial a}\right|_{a=\underline{a}}$ are finite. So $\left.\frac{\partial C_1(a)}{\partial a}\right|_{a=\underline{a}}$ must be infinite.

It follows that

$$\left.\frac{\partial s_1(a)}{\partial a}\right|_{a=\underline{a}} = r - (1+t_c)\left.\frac{\partial C_1(a)}{\partial a}\right|_{a=\underline{a}} = \infty$$

We can now rearrange and evaluate (2.36) at $\underline{a}$:

$$\begin{aligned}&\left(\frac{1}{(1+t_c)}\left(\frac{\partial^2 u(c)}{\partial c^2} - V_1(\underline{a})\frac{\partial^2\rho(c)}{\partial c^2}\right)\left.\frac{\partial C_1(a)}{\partial a}\right|_{a=\underline{a}} - \frac{1}{(1+t_c)}\mu_1(\underline{a})\frac{\partial\rho(c)}{\partial c}\right)\frac{s_1(\underline{a})}{\mu_1(\underline{a})}\\ = \quad & -\left(r - \rho(c) + \lambda_1\left(\frac{\mu_2(\underline{a})}{\mu_1(\underline{a})} - 1\right)\right)\end{aligned}$$

$$\left.\frac{\partial C_1(a)}{\partial a}\right|_{a=\underline{a}} = \frac{1}{(1+t_c)}\left(r - \left.\frac{\partial s_1(a)}{\partial a}\right|_{a=\underline{a}}\right) = \infty$$

$$\begin{aligned}
&\frac{\left(\frac{\partial^2 u(c)}{\partial c^2} - V_1(\underline{a})\frac{\partial^2 \rho(c)}{\partial c^2}\right)}{(1+t_c)\,\mu_1(a)}\left(s_1(a)\,r - s_1(a)\left.\frac{\partial s_1(a)}{\partial a}\right|_{a=\underline{a}}\right) - s_1(a)\frac{\partial \rho(c)}{\partial c}\\
= \quad & -(1+t_c)\left(r - \rho(c) + \lambda_1\left(\frac{\mu_2(a)}{\mu_1(a)} - 1\right)\right)
\end{aligned}$$

From this equation

$$\begin{aligned}
&s_1(a)\,r - \frac{(1+t_c)\,\mu_1(a)\left(s_1(a)\frac{\partial \rho(c)}{\partial c} - (1+t_c)\left(r - \rho(c) + \lambda_1\left(\frac{\mu_2(a)}{\mu_1(a)} - 1\right)\right)\right)}{\left(\frac{\partial^2 u(c)}{\partial c^2} - V_1(\underline{a})\frac{\partial^2 \rho(c)}{\partial c^2}\right)}\\
= \quad & s_1(a)\frac{\partial s_1(a)}{\partial a}
\end{aligned}$$

we take the limit

$$\lim_{a\to\underline{a}} s_1(a)\frac{\partial s_1(a)}{\partial a} = \frac{(1+t_c)^2\left((r-\rho(c))\,\mu_1(a) + \lambda_1\left(\mu_2(\underline{a}) - \mu_1(\underline{a})\right)\right)}{\frac{\partial^2 u(c)}{\partial c^2} - V_1(\underline{a})\frac{\partial^2 \rho(c)}{\partial c^2}} = \nu_1^{tax}.$$

The limit is finite as all variables in RHS are finite. As Achdou et al. (2018) show

$$\lim_{a\to\underline{a}} \frac{(s_1(a))^2}{a-\underline{a}} = 2s_1(a)\frac{\partial s_1(a)}{\partial a} =$$

$$2\frac{(1+t_c)^2\left((r-\rho(c))\,\mu_1(\underline{a}) + \lambda_1\left(\mu_2(\underline{a}) - \mu_1(\underline{a})\right)\right)}{\frac{\partial^2 u(c)}{\partial c^2} - V_1(\underline{a})\frac{\partial^2 \rho(c)}{\partial c^2}} = 2\nu_1^{tax}$$

so that

$$(s_1(a))^2 \sim 2\nu_1^{tax}(a-\underline{a})$$

and

$$s_1(a) \sim -\sqrt{2\nu_1^{tax}}\sqrt{a-\underline{a}}$$

Therefore

$$c_1(a) = \frac{1}{(1+t_c)}\left(ra + y_1 + s_1(a)\right) \sim \frac{1}{(1+t_c)}\left(ra + y_1 - \sqrt{2\nu_1^{tax}}\sqrt{a-\underline{a}}\right)$$

$$\frac{\partial c_1(a)}{\partial a} =\sim \frac{1}{(1+t_c)}\left(r - \sqrt{\frac{\nu_1^{tax}}{2(a-\underline{a})}}\right)$$

Chapter 3

Macroprudential Policy and Banking Crises

Abstract

This paper studies macroprudential policy in a macro-model with a heterogeneous banking sector, prone to asymmetric information and moral hazard *a la* Boissay et.al. (2016). This model is shown to generate financial crises when a sequence of small positive technology shocks can lead to an increase in lending, as well as to a reduction in all market rates. This paper investigates a scope for a macroprudential policy that would reduce probability of a financial crisis, but not lead to a too sharp reduction in a social welfare. It demonstrates that the introduction of a direct proportional tax on interbank lending can substantially reduce the amount of credit and reduce probability of a financial crisis.

Keywords: Moral Hazard, Asymmetric Information, Lending Boom, Credit Crunch, Banking Crisis, Macroprudential Policy

JEL Reference Number: E32, E44, G01, G21

3.1 Introduction

The financial crisis of 2009 has highlighted the failure of early literature to build quantitatively sound models to demonstrated vulnerability of economies to financial shocks. The last decade of economic research has led the rapid growth of models aiming to explain how a sequence of shocks can lead to a deep and prolong recession combined with a financial crisis.

One strand of the emerging literature is mainly concerned with demonstrating how the financial system can propagate the impact of adverse shock and spread it, leading to a significant decrease in economic activity. This literature begins with Mishkin

(1978), Bernanke (1983) and Gertler (1988), and demonstrates how bank balance sheet and collateral constraints (Bernanke, Gertler, and Gilchrist, 1999) can amplify shocks to have quantitatively important macroeconomic consequences. Post-crisis literature has shifted towards more explicit modelling of financial intermediaries, including their fragility, leading to occasional defaults (crises), see Nolan and Thoenissen (2009), Jermann and Quadrini (2009), Gertler and Kiyotaki (2010, 2015), Akinci and Queralto (2017) and Brunnermeier and Sannikov (2014) to mention only a few.

This literature has shown how a particular shock can cause a financial crisis with a deep recession, but it failed to discuss how such shock may come along. They do not relate to the stylized fact uncovered in Schularik and Taylor (2012) that a financial crisis is a rare event that follows a long period of credit growth. A credit boom can lead to a buildup of fragilities in the financial system (Lorenzoni, 2008) or a high level of accumulated debt can lead to changes in economic behavior around the point of potential instability of debt (Gorton and Ordoñez, 2014).

In a recent paper Boissay, Collard and Smets (2016) demonstrate how a crisis can follow a credit boom caused by a sequence of standard, not necessarily large, technology shocks. In this model, the interbank market intermediates lending to firms. Banks are heterogeneous in their efficiency and are prone to asymmetric information and moral hazard problems. Positive (and persistent) technology shock leads to an increase in lending, as well as to a decrease in all market rates. Lower interest rates exacerbate agency problems of banks and the interbank market closes. This leads to a sharp decline in lending, the financial crisis and recession. The model is non-linear, with an occasionally binding constraint, but it allows a numerical analysis of the implied probability of a financial crisis.

Boissay et.al. (2016) does not have 'overborrowing' in the sense of Lorenzoni (2008) or Bianchi and Mendoza (2012), since a social planner with access to a state-contingent taxation will not reduce the amount of credit, but rather extend it, which will result in higher probability of a financial crisis. The output loss due to the crisis can be mitigated by an appropriate design of taxes and subsidies.

We, therefore, are exploring the possibility for a different type of macroprudential policy, a policy that would reduce the likelihood of a financial crisis, while not lead to a too sharp reduction in social welfare. We demonstrate that the introduction of a direct proportional tax on interbank lending can substantially reduce the volume of credit and reduce the probability of a financial crisis. The tax affects this probability through two main channels. First, it shifts the 'crisis threshold boundary' upward - the interbank market will withstand a greater 'overlending', so the economy would need to accumulate more assets before the interest rate drops low enough for the interbank market to freeze. Second, the level of the steady state 'overlending' will decrease: the level of assets in the stochastic steady state is lower at a higher tax rate on interbank lending. As a result, the 'distance' between the state in which the economy spends most of the time and the boundary of a rare event unambiguously rises. Although the rate of asset accumulation – interest rate – is increasing, this effect is relatively small, and the overall impact of macroprudential policies on the economy is positive. Although we expect that macroprudential policies to create higher costs for financial intermediation, we find that in our environment a moderate increase in tax rates leads to an increase in social welfare in a stochastic steady state.

In this model, increasing the tax rate on interbank lending reduces both supply and demand of funds in the interbank market. More banks will leave the interbank market to directly lend to firms, reducing supply of funds. In a model with asymmetric information and moral hazard, the demand for loans may rise with higher interbank rate, since each bank is able to borrow more due to incentive participation constraint. With higher taxes on interbank lending, however, the incentive participation constraint for lenders is tightened, so the market funding ratio falls. This effect dominates the overall effect on demand, and the demand for funds falls with a higher tax on lending. The equilibrium interbank rate increases, and so all other interest rates. The efficiency of the marginal bank rises as more banks switch to finance firms directly. In stochastic steady state the total amount of lending and capital falls, as well as output and labour. Consumption also falls, but the social welfare rises, as the disutility of labour dominates the period utility.

The chapter is organized as follows. Section 3.2 describes the model. We then briefly discuss how to solve the model in section 3.3. Section 3.4 presents results and section 3.5 concludes.

3.2 The Model

The model follows Boissay et al. (2013). It is based on the standard RBC-type model but with heterogeneous banking sector. Households sector is homogeneous, it consumes and supplies labour to firms, and it saves in the form of deposits. The production sector is standard with continuum of identical perfectly competitive firms which take loans to rent capital, employ labour and produce. The banking sector, however is heterogeneous, with different privately-known efficiency levels of each bank. The banking sector attracts deposits from households, issues loans to firms, but is also engaged into wholesale trade at an interbank market. We consider the detailed set up in this section.

3.2.1 Households

All households are identical and are infinitely lived. Households have preferences over consumption and labour and maximize utility

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u\left(c_t, h_t\right) \tag{3.1}$$

where utility function $u(c_t, h_t)$ satisfies the usual regularity conditions, i.e. $u'_c(c_t, h_t) > 0$, $u''_c(c_t, h_t) < 0$. Parameter $\beta < 1$ is the household discount factor, and $\mathbb{E}_0$ is conditional expectations operator.

At the beginning of each period household has assets a_t, it supplies labour h_t with trending labour productivity Ψ_t and is paid the wage rate w_t. The household consumes c_t. The budget constraint is therefore

$$c_t + a_{t+1} = \tilde{r}_t a_t + w_t \Psi_t h_t + \chi_t + g_t + T_t. \tag{3.2}$$

There are no financial frictions between banks and households and financial wealth and a_t can be thought of either bank deposits or bank equity. Households lend to banks at *gross* deposit rate $r_t > 1$ which will be later endogenously determined. The after-tax real rate is

$$\tilde{r}_t = (1 + (r_t - 1)(1 - \tau_t)).$$

Here χ_t is lump-sum transfer that corresponds to banks' intermediation cost, T_t is transfer which corresponds to tax rebate, $T_t = (r_t - 1)\tau_t a_t$, and g_t collects all other transfers.

Households optimize objective (3.1) subject to constraint (3.2) and the optimization problem is standard and yields the following first order conditions:

$$w_t = -\frac{1}{\Psi_t} \frac{u_h(c_t, h_t)}{u_c(c_t, h_t)}$$

$$u_c(c_t, h_t) = \beta u_c(c_{t+1}, h_{t+1}) \tilde{r}_{t+1} \tag{3.3}$$

$$c_t + a_{t+1} = \tilde{r}_t a_t + w_t \Psi_t h_t + \pi_t + \chi_t + g_t + T_t \tag{3.4}$$

which are the labour supply equation, consumption Euler equation and equation describing accumulation of household wealth.

3.2.2 Firms

All firms are identical and live for one period only. Each firm produces a homogeneous good that can either be consumed or invested, and used factors of production capital k_t and labour h_t. Production technology is given by production function $e^{z_t}F(k_t, h_t)$. The level of total factor productivity is determined by a technology shock which follows an AR(1) process

$$z_t = \rho_z z_{t-1} + \varepsilon_t$$

where $|\rho_z| < 1$ and ε_t is standard normal stochastic innovation. ε_t is realized before the firm decides on the production plan.

Capital depreciates at rate $0 < \delta < 1$. The firm is born at the beginning of the period and needs to borrow capital k_t from the bank at *gross* corporate loan rate $R_t > 1$ which will be determined later. The corporate loan is repaid at the end of the period, $R_t k_t > k_t$. Firms rent labour services at the wage rate w_t.

The profit maximization problem of a firm is standard and can be written as follows

$$\max_{h_t,k_t} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^{\tau} \left(e^{z_t}F(k_t, h_t) + (1-\delta) k_t - w_t \Psi_t h_t - R_t k_t \right).$$

Firms produce, pay wages and pay banks at the end of the period, they also keep $(1-\delta) k_t$ after one period.

The first order conditions to the firm optimization problem are standard, they yield

$$\Psi_t w_t = e^{z_t} F_h(k_t, h_t), \tag{3.5}$$

$$R_t = e^{z_t} F_k(k_t, h_t) + 1 - \delta, \tag{3.6}$$

which determine demand for labour and capital.

3.2.3 Banking sector

The banking sector is heterogeneous. Banks differ in respect to banking (intermediation) technology. They are involved in two types of activity: they do retail banking by attracting household deposits and lending to firms, and they participate in an interbank market and reallocate funds among themselves. There are asymmetric information and moral hazard problems for banks which affect the work of the interbank market. In retail banking business, banks own capital which they rent to firms. They also collect household's deposits and issue loans to firms, that are used to rent capital from banks. The model of the banking sector is one-period.

Interbank rate

There is a continuum of risk-neutral banks which exist for one period between $t-1$ and t and die at the end of period t. When banks are born at the beginning of the period they are identical and each of them attracts the same amount of deposits. Then they draw a random level of intermediation skill, or the level of efficiency p, $0 \leq p \leq 1$.

The individual level of efficiency is privately known, and only the distribution of ps is a public knowledge. The distribution is described by a cumulative density function $\mu(p)$ such that $\mu(0) = 0$, $\mu(1) = 1$, $\mu'(p) > 0$.

Bank p pays an intermediation cost $(1-p)R$ per unit of corporate loan at the end of period t so that its effective gross return on loan is pR. (Most efficient banks will have p close to one.) The intermediation cost is rebated to household in the form of lump-sum transfers χ_t. This intermediation cost reflects monitoring costs that banks have as loan providers and loan services.

Bank indexed p borrows maximum amount x_t from households and can borrow a fraction of x_t, $x_t\phi_t$, at the interbank market at gross interbank lending rate $\rho_t > 1$. It also pays proportional tax on interbank borrowing, regulated by variable $\Xi_t < 1$, so that the regulator gets $x_t\phi_t\Xi_t$ which is rebated to households. If this amount is lent to firms at rate R_t, the revenue is $pR_tx_t(1+\phi_t) - \rho_t\phi_tx_t - x_t\phi_t\Xi_t$ and the after-tax rate of return is $pR_t(1+\phi_t) - \rho_t\phi_t - \phi_t\Xi_t$. Alternatively, the bank can lend everything at the interbank market achieving the rate of return $1 + (\rho_t - 1)\Theta_t$ where $0 \leq 1 - \Theta_t \leq 1$ is an interbank lending tax.

The interbank lending tax is a macroprudential policy tool. It is proportional tax paid on net interest income. Tax receipts are then rebated to households in the form of lump sum transfers. If the interbank interest rate ρ_t falls below one, Θ_t regulates a proportional subsidy paid from lump-sum taxes levied on households.

The bank's return on deposits is r_t is a function of bank's efficiency p :

$$r_t(p) = \max\{pR_t(1+\phi_t) - \rho_t\phi_t - \phi_t\Xi_t, 1 + (\rho_t - 1)\Theta_t\}. \tag{3.7}$$

The marginal banker is indifferent between lending to firms and to the interbank market so that

$$pR_t(1+\phi_t) - \rho_t\phi_t - \phi_t\Xi_t = 1 + (\rho_t - 1)\Theta_t$$

is a participation constraint which determines the efficiency level of the marginal banker $\bar{p}_t$ and can be written as

$$\bar{p}_t = \frac{1 + (\rho_t - 1)\Theta_t + \rho_t\phi_t + \phi_t\Xi_t}{R_t(1+\phi_t)}. \tag{3.8}$$

Banks with $p_t < \bar{p}_t$ do not participate in lending to firms and sell deposits at the interbank market only. The direct effect of higher interbank lending taxes results in a reduction in $\bar{p}_t$ so that more banks choose to lend to firms directly. Higher tax on interbank borrowing increases the number of interbank participating banks and increases the average efficiency of banks that lend directly to firms, as the marginal bank has to have higher efficiency to be indifferent between lending to firms directly and lending at the interbank market with higher costs.

As an outside option, banks can use 'storage facility' under gross rate γ, $\gamma < 1 + (\rho_t - 1)\Theta_t$. Banks can divert the whole amount $(1 + \phi_t)x_t$, at the end of the period get $\gamma(1 + \phi_t)x_t$, and try to keep it. The investments into the storage facility cannot be traced and cannot be seized by creditors. Keeping funds in the storage facility is also costly, so the banks may end up keeping only $\gamma(1 + \theta\phi_t)x_t \leq \gamma(1 + \phi_t)x_t$, as $0 \leq \theta \leq 1$. Here θ is the cost of diversion, when $\theta = 1$ then the bank keeps the whole amount, and when $\theta = 0$ the bank cannot keep the return on the whole amount of the diverted interbank borrowing. The existence of the storage facility gives rise to a *moral hazard* problem, as the gain from diversion increases with the amount diverted, ϕ_t, and the opportunity cost of diversion increases with bank efficiency and the relative return, $R_t - \gamma$. Most efficient banks will prefer to operate 'as normal', but inefficient banks may find attractive to divert funds.

There is an *asymmetric information* problem when banks' efficiency is privately known, and lenders cannot observe and verify it. Therefore, all contracts signed at the interbank market are the same for all banks. The market funding ratio ϕ_t and the interbank rate ρ_t does not depend on the level of bank efficiency p. As lenders want to prevent borrowers from using the storage facility, they impose upper limit on borrowing so that even most inefficient banks with $p < \bar{p}_t$ will not find borrowing and then diverting profitable. In other words, the return on diversion should be less then the return on interbank activity, so the banks do not attempt to divert and instead participate in the 'standard' banking activity. The incentive compatibility constraint can be written as

$$\gamma(1 + \theta\phi_t) \leq 1 + (\rho_t - 1)\Theta_t. \tag{3.9}$$

This constraint always binds, so that it satisfies with equality

$$\gamma(1 + \theta\phi_t) = 1 + (\rho_t - 1)\Theta_t.$$

It follows that the amount of interbank borrowing is determined by:

$$\phi_t = \frac{1-\gamma+(\rho_t-1)\,\Theta_t}{\gamma\theta}. \tag{3.10}$$

Lower return on interbank lending due to taxes results in lower market funding ratio.

Together, constraints (3.10) and (3.8) determine the bank p decision to participate in the interbank market and how much to borrow.

Equilibrium at the interbank market.

Banks with efficiency p are distributed over [0,1], with cumulative distribution $\mu(p)$. The mass of participating banks with ability above p, $m(p)$ is therefore can be written as

$$m(p) = \int_p^1 d\mu = \mu(1) - \mu(p) = 1 - \mu(p)$$

With the efficiency level of the marginal banker $\bar{p}_t$, a mass $\mu(\bar{p}_t)$ lend x_t and the aggregate supply of funds is $\mu(\bar{p}_t)\,x_t$. A mass $1-\mu(\bar{p}_t)$ borrow $\phi_t x_t$ and the aggregate demand is $(1-\mu(\bar{p}_t))\,\phi_t x_t$. The market clears when demand is equal to supply and

$$\mu(\bar{p}_t)\,x_t = (1-\mu(\bar{p}_t))\,\phi_t x_t, \tag{3.11}$$

which can be rewritten as

$$\begin{aligned}\mu\left(\frac{1+(\rho_t-1)\,\Theta_t+\rho_t\phi_t+\phi_t\Xi_t}{R_t\,(1+\phi_t)}\right) &= \left(1-\mu\left(\frac{1+(\rho_t-1)\,\Theta_t+\rho_t\phi_t+\phi_t\Xi_t}{R_t\,(1+\phi_t)}\right)\right)\\ &\quad\times\frac{1+(\rho_t-1)\,\Theta_t-\gamma}{\gamma\theta}.\end{aligned}$$

Using it, we can determine the gross return on loans to firms:

$$R_t = \frac{1 + (\rho_t - 1)\,\Theta_t + \rho_t\phi_t + \phi_t\Xi_t}{(1+\phi_t)\,\mu^{-1}\left(\frac{1+(\rho_t-1)\Theta_t-\gamma}{(\gamma(\theta-1)+1+(\rho_t-1)\Theta_t)}\right)} = \Omega\left(\rho_t, \Theta_t, \phi_t\right), \tag{6}$$

where ϕ_t is determined from equation (3.10).

For given Θ_t and Ξ_t this equation has five solutions. Two do not satisfy $\gamma < 1 + (\rho_t - 1)\,\Theta_t$, one always exists: $R_t = \gamma$ (no trade at the interbank market) and another two which may not exist. There is a threshold $\bar{R}_t = \Omega\left(\bar{\rho}_t; \Theta_t, \Xi_t\right)$ when these two solutions are identical and with further reduction in R_t they disappear and there is no trade at the interbank market. In other words, the line $R_t = \Omega\left(\rho_t, \phi_t; \Theta_t, \Xi_t\right)$ is U-shaped in ρ_t, and there exists a limit below which there is no trade at the interbank market.

Finally, the banking sector's return on equity x_t is given by

$$r_t = \int_0^1 r_t(p)\,d\mu(p) = \begin{cases} \int_{\bar{p}_t}^1 pR_t\,(1+\phi_t)\,d\mu(p) \overset{(3.11)}{=} \int_{\bar{p}_t}^1 \frac{pR_t}{(1-\mu(\bar{p}_t))}d\mu(p)\,, & \text{if trade exists} \\ \int_{\frac{\gamma}{R_t}}^1 pR_t d\mu(p) + \int_0^{\frac{\gamma}{R_t}} \gamma d\mu(p) = R_t \int_{\frac{\gamma}{R_t}}^1 p d\mu(p) + \gamma\mu\left(\frac{\gamma}{R_t}\right), & \text{no trade} \end{cases} \tag{3.12}$$

In the second case the interbank market is shut, so the funding ratio $\phi_t = 0$, but efficient banks with efficiency level $p \geq \frac{\gamma}{R_t}$ still lend to the corporate sector. Inefficient banks use storage technology. In equilibrium no diversion of funds happens, as the participation constraint holds.

Parameterization

In all numerical results we assume the standard Cobb-Douglas functional form for the production function:

$$F\left(k_t, h_t, z_t\right) = e^{z_t} k_t^{\alpha} \left(\Psi_t h_t\right)^{1-\alpha}$$

so that its derivatives are:

$$F_h\left(k_t, h_t, z_t\right) = (1-\alpha)\, \Psi_t e^{z_t} k_t^{\alpha} \left(\Psi_t h_t\right)^{-\alpha}$$

$$F_k\left(k_t, h_t, z_t\right) = \alpha e^{z_t} k_t^{\alpha-1} \left(\Psi_t h_t\right)^{1-\alpha}.$$

We assume the following utility function:

$$u\left(c_t, h_t\right) = \frac{1}{1-\sigma}\left(c_t - \vartheta \Psi_t \frac{h_t^{1+\upsilon}}{1+\upsilon}\right)^{1-\sigma}$$

with derivatives

$$u_c\left(c_t, h_t\right) = \left(c_t - \vartheta \Psi_t \frac{h_t^{1+\upsilon}}{1+\upsilon}\right)^{-\sigma}$$

$$u_h\left(c_t, h_t\right) = -\vartheta \Psi_t h_t^{\upsilon}\left(c_t - \vartheta \Psi_t \frac{h_t^{1+\upsilon}}{1+\upsilon}\right)^{-\sigma}.$$

Finally, the cumulative distribution function of the bank efficiency is determined by

$$\mu\left(p\right) = p^{\lambda}.$$

3.2.4 Aggregation and Equilibrium

In our model all households and all firms are identical, and each of them is of unit mass. As a consequence, aggregation across agents implies: $k_t = \Psi_t K_t$; $c_t = \Psi_t C_t$; $h_t = H_t$; $a_t = \Psi_t A_t$, where capital letters indicate aggregate quantities.

Aggregate supply of corporate loans

If the interbank market is open then all banks finance total capital K_t with A_t, if the market is shut, then only efficient banks supply capital to firms. Therefore, the amount of capital supplied by banks can be written as

$$K_t^s = \left\{ \begin{array}{c} A_t, \text{ if trade exists} \\ \int_{\frac{\gamma}{R_t}}^1 A_t d\mu\left(p\right) = A_t\left(\mu\left(1\right) - \mu\left(\frac{\gamma}{R_t}\right)\right) = A_t\left(1 - \mu\left(\frac{\gamma}{R_t}\right)\right) \end{array} \right.$$

When $R_t = \bar{R}_t$ and the market freezes, then $A_t = \bar{A}_t$ which is the maximum quantity of assets which the bank can reallocate efficiently, called the absorption capacity of the banking sector.

Using parameterization above and solving (3.6) for capital yields

$$K_t^d = \left(\frac{1-\alpha}{\vartheta}\right)^{\frac{1}{\upsilon}} \left(\frac{\alpha}{\bar{R}_t + \delta - 1}\right)^{\frac{\alpha+\upsilon}{\upsilon(1-\alpha)}} e^{\frac{1+\upsilon}{\upsilon(1-\alpha)}z_t} = \Gamma_t e^{\frac{1+\upsilon}{\upsilon(1-\alpha)}z_t} \tag{3.13}$$

where coefficient

$$\Gamma_t = \left(\frac{1-\alpha}{\vartheta}\right)^{\frac{1}{\upsilon}} \left(\frac{\alpha}{\left(\bar{R}_t + \delta - 1\right)}\right)^{\frac{\alpha+\upsilon}{\upsilon(1-\alpha)}}$$

depends on taxation Θ_t, and the volume of trade ϕ_t via $\bar{R}$, where $\bar{R}$ is the 'lowest' point where the interbank market trade exists.

Transfers

Appendix demonstrates that lump-sum transfer that corresponds to banks' intermediation cost χ_t can be written as

$$\chi_t = \left\{ \begin{array}{c} A_t\left(R_t - r_t\right), \text{ trade} \\ A_t\left(R_t - r_t\right) - \left(A_t - K_t\right)\left(R_t - \gamma\right), \text{ no trade} \end{array} \right. ,$$

tax rebate transfers are

$$T_t = (r_t - 1)\tau_t a_t,$$

and the remaining transfers are

$$g_t = \left(\left(\rho_t - 1\right)\left(1 - \Theta_t\right) + \Xi_t\right) a_t \phi_t.$$

3.3 Solving the Decentralized Problem

3.3.1 Equilibrium Conditions

Appendix demonstrates that the aggregation of first order conditions, accounting for transfers and using the parameterization, yields the following system.

<table>
<tr><th></th><th>trade</th><th>no trade</th></tr>
<tr><td>1</td><td colspan="2">$C_t + \psi A_{t+1} = e^{z_t} K_t^{\alpha} h_t^{1-\alpha} + (1-\delta) K_t + \gamma (A_t - K_t) + ((\rho_t - 1)(1 - \Theta_t) + \Xi_t) \phi_t A_t$</td></tr>
<tr><td>2</td><td colspan="2">$R_t = \alpha e^{z_t} K_t^{\alpha-1} H_t^{1-\alpha} + 1 - \delta$</td></tr>
<tr><td>3</td><td colspan="2">$\left(C_t - \vartheta \frac{H_t^{1+\upsilon}}{1+\upsilon}\right)^{-\sigma} = \beta \psi^{-\sigma} \mathbb{E}_t \left(C_{t+1} - \vartheta \frac{H_{t+1}^{1+\upsilon}}{1+\upsilon}\right)^{-\sigma} (1 + (r_{t+1} - 1)(1 - \tau_{t+1}))$</td></tr>
<tr><td>4</td><td colspan="2">$H_t = \left(\frac{(1-\alpha)}{\vartheta} e^{z_t} K_t^{\alpha}\right)^{\frac{1}{\upsilon+\alpha}}$</td></tr>
<tr><td>5</td><td>$K_t = A_t$</td><td>$K_t = A_t \left(1 - \left(\frac{\gamma}{R_t}\right)^{\lambda}\right)$</td></tr>
<tr><td>6</td><td>$r_t = R_t \frac{\lambda}{(\lambda+1)} \frac{\left(1-\bar{p}_t^{\lambda+1}\right)}{\left(1-\bar{p}_t^{\lambda}\right)}$</td><td>$r_t = R_t \left(\frac{\lambda+\bar{p}_t^{\lambda+1}}{\lambda+1}\right)$</td></tr>
<tr><td>7</td><td>$\rho_t = \frac{\bar{p}_t R_t (1+\phi_t) - 1 + \Theta_t - \phi_t \Xi_t}{(\Theta_t + \phi_t)}$</td><td>$\rho_t = \gamma$</td></tr>
<tr><td>8</td><td>$\bar{p}_t^{\lambda} = \frac{1-\gamma+(\rho_t-1)\Theta_t}{(\gamma(\theta-1)+1+(\rho_t-1)\Theta_t)}$</td><td>$\bar{p}_t = \frac{\gamma}{R_t}$</td></tr>
<tr><td>9</td><td>$\phi_t = \frac{\bar{p}_t^{\lambda}}{\left(1-\bar{p}_t^{\lambda}\right)}$</td><td>$\phi_t = 0$</td></tr>
</table>

3.3.2 Occasionally Binding Constraint

Appendix demonstrates that relationships between p_t, ϕ_t, R_t yield the following function of interest rate as a function of interbank rate

$$\begin{aligned} R_t \;=\; & \left(\frac{\theta\gamma}{(1+(\rho_t-1)\,\Theta_t-\gamma)}+1\right)^{\frac{1}{\lambda}}\left(\frac{\theta\gamma\,(1-\Theta_t)}{\theta\gamma+(1+(\rho_t-1)\,\Theta_t-\gamma)}\right. \\ & +\rho_t\left(\frac{\theta\gamma\,(\Theta_t-1)}{\theta\gamma+(1+(\rho_t-1)\,\Theta_t-\gamma)}+1\right) \\ & \left.+\Xi_t\left(1-\frac{\theta\gamma}{\theta\gamma+1-\gamma+(\rho_t-1)\,\Theta_t}\right)\right) \end{aligned} \tag{3.14}$$

which describes a U-shaped function $R_t = R_t(\rho_t)$. We can find its minimum with coordinates $(\bar{\rho}_t, \bar{R}_t)$. Then the efficiency level of the marginal banker $\bar{p}_t$ is determined from equation:

$$p_{\min} = \left(\frac{1+(\bar{\rho}_t-1)\,\Theta_t-\gamma}{(\gamma\,(\theta-1)+1+(\bar{\rho}_t-1)\,\Theta_t)}\right)^{\frac{1}{\lambda}}$$

and equation (3.13) yields the boundary

$$\bar{A}_t = \left(\frac{1-\alpha}{\vartheta}\right)^{\frac{1}{\upsilon}}\left(\frac{\alpha}{\bar{R}_t+\delta-1}\right)^{\frac{\alpha+\upsilon}{\upsilon(1-\alpha)}} e^{\frac{1+\upsilon}{\upsilon(1-\alpha)}z_t} = \Gamma_t e^{\frac{1+\upsilon}{\upsilon(1-\alpha)}z_t}$$

It is clear that the level of interbank taxes Θ_t and Ξ_t affect the boundary trough interest rate $\bar{R}_t$. If there is trade then $\Theta_t < 1, \Xi_t > 0$ and $\Theta_t = 1, \Xi_t = 0$ if the interbank market is shut.

Boundary $\bar{A}_t$ is binding, when interest rate $\bar{R}_t$ occasionally moves down so that the interbank market shuts down. The boundary conditions on the left side of the boundary can be determined from equations

$$\bar{p}_t^{\lambda} = \frac{1+(\rho_t-1)\,\Theta_t-\gamma}{(\gamma\,(\theta-1)+1+(\rho_t-1)\,\Theta_t)},$$

$$\rho_t = \frac{\bar{p}_t R_t\,(1+\phi_t)-1+\Theta_t-\phi_t\Xi_t}{(\Theta_t+\phi_t)},$$

$$\phi_t = \frac{\bar{p}_t^{\lambda}}{\left(1-\bar{p}_t^{\lambda}\right)},$$
$$r_t = R_t \frac{\lambda}{\lambda+1} \frac{\left(1-\bar{p}_t^{\lambda+1}\right)}{\left(1-\bar{p}_t^{\lambda}\right)}$$

that determine $\bar{p}_t, \phi_t, R_t, r_t$ in the 'trade' domain. As we know the limit $\bar{R}$, we can compute other variables.

However, the problem is not symmetric. Equations

$$p_t = \frac{\gamma}{R_t}$$
$$\rho_t = \gamma$$
$$\phi_t = 0$$
$$r_t = R_t \left(\frac{\lambda + p_t^{\lambda+1}}{\lambda+1}\right)$$

determine $\bar{p}_t, \phi_t, R_t, r_t$ in the 'no trade' domain. At the right from boundary $\bar{A}_t$:

$$\begin{aligned}
\check{\rho} &= \gamma \\
\check{p} &= \frac{\gamma}{\check{R}} \\
\check{\phi} &= 0 \\
\check{r} &= \check{R}\left(\frac{\lambda + \check{p}^{\lambda+1}}{\lambda+1}\right)
\end{aligned}$$

However, as we do not know $\check{R}$, we can only compute these values numerically, as a limit of the solution in the 'no trade' area.

3.3.3 Calibration and Numerical Solution

Parameters follows Boissay et.al. (2016) and are given in the following table

adjusted discount factor	$\beta\psi^{-\sigma}$	1/1.03
capital share	α	0.3
elasticity of labour supply	υ	0.5
rate of depreciation	δ	0.1
parameter of bank efficiency function	λ	21.5
parameter in diversion function	θ	0.093
growth rate of the economy	ψ	1.012
relative weight of disutility of labour	ϑ	0.945
intertemporal elasticity of substitution	σ	4.5
return on storage facility	γ	0.9417
persistence of technology shock	ρ_z	0.98
standard deviation of shock innovation	σ_z	0.013

We solve the non-linear problem numerically using projection methods. All unknown function are parameterized as functions of two states, asset position A_t and the level of technology z_t. We then solve generalized Euler equations on the grid. As the model has occasionally binding constraint, we parameterized functions using splines, as they have shown better performance than Chebyshev Polynomials. The solution algorithm is described in Appendix.

3.4 Results

To investigate the role of the proposed macroprudential policy it is instructive to compare the allocation generated by our model to several well known benchmark allocations. Main results are summarized in Tables 3.1 and 3.2. Table 3.1 compares steady state allocation for several important models, that are closely related to the one we consider here. Table 3.2 provides steady state effects of the proposed macroprudential regulation in the stochastic setting.

3.4.1 Real Business Cycles Model

The decentralized model with interbank market collapses to the standard Real Business Cycles (Kydland and Prescott, 1982) model once financial frictions are removed. This version of RBC model can be summarized by the following set of equations

$$A_t = K_t$$

$$r_t = R_t$$

$$H_t = \left(\frac{(1-\alpha)}{\vartheta} e^{z_t} K_t^{\alpha}\right)^{\frac{1}{\upsilon+\alpha}}$$

$$\left(C_t - \vartheta \frac{H_t^{1+\upsilon}}{1+\upsilon}\right)^{-\sigma} = \beta\psi^{-\sigma}\mathbb{E}_t\left(C_{t+1} - \vartheta\frac{H_{t+1}^{1+\upsilon}}{1+\upsilon}\right)^{-\sigma} r_{t+1}$$

$$R_t = \alpha e^{z_t} K_t^{\alpha-1} H_t^{1-\alpha} + 1 - \delta$$

$$C_t + \psi K_{t+1} = e^{z_t} K_t^{\alpha} H_t^{1-\alpha} + (1-\delta) K_t$$

Solution algorithm is discussed in Appendix and the steady state allocation is given in column (1) in Table 3.1. This model generates the highest consumption and utility levels, it also generates steady state level of assets well above the absorption capacity boundary $\bar{A}$.

3.4.2 Constrained Efficient Equilibrium

It is instructive to look at constrained efficient allocation in our model. The central planner chooses assets, consumption, labour and capital to maximize the household utility subject to the household budget constraint. The central planner knows the effect of household consumption and saving decisions on market rates, and therefore on banks' efficiency. In this equilibrium the policymaker solves the following problem:

$$\max \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t u(C_t, H_t) = \max \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \frac{1}{1-\sigma}\left(C_t - \vartheta\frac{H_t^{1+\upsilon}}{1+\upsilon}\right)^{1-\sigma}$$

subject to constraints

	trade	no trade
1	$C_t + \psi A_{t+1} = e^{z_t} K_t^{\alpha} H_t^{1-\alpha} + (1-\delta) K_t + \gamma (A_t - K_t)$	
2	$R_t = \alpha e^{z_t} K_t^{\alpha-1} H_t^{1-\alpha} + (1-\delta)$	
3	$K_t = A_t$	$K_t = A_t \left(1 - \left(\frac{\gamma}{R_t}\right)^{\lambda}\right)$

Solution is given in Appendix. We arrive to the following system of first order conditions.

In trade area:

$$R_t = \alpha e^{z_t} K_t^{\alpha-1} h_t^{1-\alpha} + (1-\delta) \tag{3.15}$$

$$h_t = \left(\frac{(1-\alpha)}{\vartheta} e^{z_t} K_t^{\alpha}\right)^{\frac{1}{\upsilon+\alpha}} \tag{3.16}$$

$$\left(C_t - \frac{\vartheta h_t^{1+\upsilon}}{1+\upsilon}\right)^{-\sigma} = \beta \psi^{-\sigma} \mathbb{E}_t \left(C_{t+1} - \frac{\vartheta h_{t+1}^{1+\upsilon}}{1+\upsilon}\right)^{-\sigma} R_{t+1} \tag{3.17}$$

$$\psi_{t+1} A_{t+1} = e^{z_t} K_t^{\alpha} h_t^{1-\alpha} + (1-\delta) K_t - C_t \tag{3.18}$$

Note that because the labour allocation is the same as in the decentralized equilibrium, the values of $\bar{\rho}, \bar{R}, \bar{A}$ do not change.

In no trade (crisis) area:

$$R_t = \alpha e^{z_t} K_t^{\alpha-1} h_t^{1-\alpha} + (1-\delta) \tag{3.19}$$

$$p_t = \frac{\gamma}{R_t} \tag{3.20}$$

$$h_t = \left(\frac{(1-\alpha)}{\vartheta} e^{z_t} K_t^{\alpha}\right)^{\frac{1}{\upsilon+\alpha}} \left(1 + \alpha\lambda \frac{p_t^{\lambda}}{1-p_t^{\lambda}} \frac{R_t - \gamma}{R_t}\right)^{\frac{1}{\upsilon+\alpha}} \tag{3.21}$$

$$\left(C_t - \frac{\vartheta h_t^{1+\upsilon}}{1+\upsilon}\right)^{-\sigma} = \beta\psi^{-\sigma}\mathbb{E}_t\left(C_{t+1} - \frac{\vartheta h_{t+1}^{1+\upsilon}}{1+\upsilon}\right)^{-\sigma} \tag{3.22}$$

$$\times R_{t+1}\left(1 - \frac{R_{t+1}-\gamma}{R_{t+1}}p_{t+1}^{\lambda}\left(1+\lambda\left(1-\alpha\right)\frac{R_{t+1}-(1-\delta)}{R_{t+1}}\right)\right)$$

$$\psi_{t+1}A_{t+1} = e^{z_t}K_t^{\alpha}h_t^{1-\alpha} + (1-\delta)K_t + \gamma\left(A_t - K_t\right) - C_t \tag{3.23}$$

This nonlinear system can be solved numerically and the result labelled CE is presented in column (2) in Table 3.1. It is apparent that this equilibrium generates lower household welfare than the standard RBC model without financial frictions. Both consumption and labor are reduced and interest rate spread is positive. The steady state level of assets is close to the absorption capacity boundary $\bar{A}$. In a stochastic setting this model has much higher probability of a financial crisis.

3.4.3 Optimal State Contingent Tax on Savings

Given the crisis boundary $\bar{A}$ and high probability of crossing it, the central planner may tax the return on saving with $\tau_t > 0$ and give incentives to households to dissave.

If the interbank market open then the central planner decides on τ_{t+1}, but because it only enters the consumption Euler equation, we assume that the policymaker choose the optimal consumption plan, and then the tax can be chosen consistent with the plan.

We, therefore, form the following Lagrangian:

$$\mathbb{E}_t\sum_{t=0}^{\infty}\beta^t\left(\begin{array}{c}\frac{1}{1-\sigma}\left(C_t - \frac{\vartheta}{1+\upsilon}\left(\frac{(1-\alpha)}{\vartheta}e^{z_t}\left(A_t\right)^{\alpha}\right)^{\frac{1+\upsilon}{\upsilon+\alpha}}\right)^{1-\sigma}\\ +\phi_{1t}\left(e^{\frac{\upsilon+1}{\alpha+\upsilon}z_t}\left(\frac{(1-\alpha)}{\vartheta}\right)^{\frac{1-\alpha}{\upsilon+\alpha}}A_t^{\alpha\frac{\upsilon+1}{\alpha+\upsilon}} + (1-\delta)A_t - C_t - A_{t+1}\psi\right)\end{array}\right)$$

other variables can be found as functions of C_t, A_t.

The FOCs describe the evolution of the economy *under optimal taxation:*

$$U_t^{-\sigma} = \frac{\beta}{\psi}\mathbb{E}_t U_{t+1}^{-\sigma}\left(\alpha\left(\frac{(1-\alpha)}{\vartheta}\right)^{\frac{1-\alpha}{\upsilon+\alpha}} e^{\frac{\upsilon+1}{\alpha+\upsilon}z_{t+1}} A_{t+1}^{\upsilon\frac{\alpha-1}{\alpha+\upsilon}} + (1-\delta)\right) \tag{3.24}$$

$$U_t = C_t - \frac{\vartheta}{1+\upsilon}\left(\frac{(1-\alpha)}{\vartheta}e^{z_t}A_t^{\alpha}\right)^{\frac{1+\upsilon}{\upsilon+\alpha}} \tag{3.25}$$

$$\psi A_{t+1} = e^{\frac{\upsilon+1}{\alpha+\upsilon}z_t}\left(\frac{(1-\alpha)}{\vartheta}\right)^{\frac{1-\alpha}{\upsilon+\alpha}} A_t^{\alpha\frac{\upsilon+1}{\alpha+\upsilon}} + (1-\delta)A_t - C_t \tag{3.26}$$

We now find the optimal tax rate, consistent with this dynamics of the economy. From equations (3.24) with (3.26) optimal *future* tax is

$$\tau_{t+1} = R_{t+1}\frac{\theta\gamma + \lambda(1-\bar{p}_{t+1})(\gamma - R_{t+1}\bar{p}_{t+1})}{\theta\gamma(\lambda(1-R_{t+1})+1) + R_{t+1}\lambda(1-\bar{p}_{t+1})(\gamma - R_{t+1}\bar{p}_{t+1})}$$

it depends on $\bar{p}_t$.

At the boundary

$$\bar{\tau} = \frac{\bar{R}\left(\theta\gamma + \lambda(1-p_{\min})\left(\gamma - \bar{R}p_{\min}\right)\right)}{\theta\gamma\left(\lambda\left(1-\bar{R}\right)+1\right) + \bar{R}\lambda(1-p_{\min})\left(\gamma - \bar{R}p_{\min}\right)}$$

and the denominator is not zero, which can be checked numerically.

If the interbank market is shut, the decentralized equilibrium can be described by the following system:

$$\begin{aligned} C_t + \psi_{t+1}A_{t+1} &= \left(\left(\frac{(1-\alpha)}{\vartheta}\right)^{\frac{1-\alpha}{\upsilon+1}} e^{z_t}A_t^{\alpha}\left(1-p_t^{\lambda}\right)^{\alpha}\right)^{\frac{\upsilon+1}{\alpha+\upsilon}} \\ &\quad + (1-\delta-\gamma)A_t\left(1-p_t^{\lambda}\right) + \gamma A_t \end{aligned}$$

$$R_t = \alpha\left(\frac{1-\alpha}{\vartheta}\right)^{\frac{1-\alpha}{\upsilon+\alpha}}\left(A_t\left(1-p_t^{\lambda}\right)\right)^{\upsilon\frac{\alpha-1}{\alpha+\upsilon}} e^{\frac{1+\upsilon}{\upsilon+\alpha}z_t} + (1-\delta)$$

$$\left(C_t - \frac{\vartheta}{1+\upsilon}\left(\frac{(1-\alpha)}{\vartheta}e^{z_t}\left(A_t\left(1-\bar{p}_t^{\lambda}\right)\right)^{\alpha}\right)^{\frac{1+\upsilon}{\upsilon+\alpha}}\right)^{-\sigma}$$

$$= \beta\psi_{t+1}^{-\sigma}\left(C_{t+1} - \frac{\vartheta}{1+\upsilon}\left(\frac{(1-\alpha)}{\vartheta}e^{z_{t+1}}\left(A_{t+1}\left(1-\bar{p}_{t+1}^{\lambda}\right)\right)^{\alpha}\right)^{\frac{1+\upsilon}{\upsilon+\alpha}}\right)^{-\sigma}$$

$$\times\left(1+\left(r_{t+1}-1\right)\left(1-\tau_{t+1}\right)\right)$$

$$r_t = R_t\left(\frac{\lambda+\bar{p}_t^{\lambda+1}}{\lambda+1}\right)$$

$$\bar{p}_t = \frac{\gamma}{R_t}$$

As before, the central planner decides on τ_{t+1}, but because it only enters the consumption Euler equation, we assume that the policymaker choose the optimal consumption plan, and then the tax can be chosen consistent with the plan.

We, therefore, form the following Lagrangian:

$$\sum_{t=0}^{\infty}\beta^t\left(\frac{1}{1-\sigma}\left(C_t - \frac{\vartheta}{1+\upsilon}\left(\frac{(1-\alpha)}{\vartheta}e^{z_t}\left[A_t^{\alpha}\left(1-\bar{p}_t^{\lambda}\right)^{\alpha}\right]\right)^{\frac{1+\upsilon}{\upsilon+\alpha}}\right)^{1-\sigma}\right.$$

$$+\phi_{1t}\left(\left(\left(\left(\frac{(1-\alpha)}{\vartheta}\right)^{\frac{1-\alpha}{\upsilon+1}}e^{z_t}A_t^{\alpha}\left(1-\bar{p}_t^{\lambda}\right)^{\alpha}\right)^{\frac{\upsilon+1}{\alpha+\upsilon}} + (1-\delta)A_t\right.\right.$$

$$\left.-\left(1-\delta-\gamma\right)A_t\bar{p}_t^{\lambda} - C_t - A_{t+1}\psi_{t+1}\right)$$

$$\left.+\phi_{2t}\left((1-\delta)+\alpha\left(\frac{1-\alpha}{\vartheta}\right)^{\frac{1-\alpha}{\upsilon+\alpha}}e^{\frac{1+\upsilon}{\upsilon+\alpha}z_t}\left(A_t\left(1-\bar{p}_t^{\lambda}\right)\right)^{\upsilon\frac{\alpha-1}{\alpha+\upsilon}} - R_t\right) + \phi_{4t}\left(R_t\bar{p}_t-\gamma\right)\right)$$

where we do not include equation for r_t as constraints, as it can be found as functions of $C_t, A_t, R_t, \bar{p}_t$.

The first order conditions yield the system which describes the evolution of the economy *under optimal taxation:*

$$h_t = \left(\frac{(1-\alpha)}{\vartheta} e^{z_t} A_t^{\alpha} \left(1-\bar{p}_t^{\lambda}\right)^{\alpha}\right)^{\frac{1}{\upsilon+\alpha}} \tag{3.27}$$

$$U_t^{-\sigma} = \frac{\beta}{\psi} U_{t+1}^{-\sigma} \left(\frac{\left(1-p_{t+1}^{\lambda}\right)\left(R_{t+1}-\gamma\right) R_{t+1}}{\left(R_{t+1} + (1-\alpha)\lambda\left(R_{t+1}-(1-\delta)\right)\frac{p_{t+1}^{\lambda}}{\left(1-p_{t+1}^{\lambda}\right)}\frac{\upsilon}{\alpha+\upsilon}\right)} + \gamma \right) \tag{3.28}$$

$$\begin{aligned} A_{t+1}\psi &= e^{\frac{\upsilon+1}{\alpha+\upsilon}z_t} \left(\frac{(1-\alpha)}{\vartheta}\right)^{\frac{1-\alpha}{\upsilon+\alpha}} A_t^{\alpha\frac{\upsilon+1}{\alpha+\upsilon}} \left(1-p_t^{\lambda}\right)^{\alpha\frac{\upsilon+1}{\alpha+\upsilon}} \\ &\quad + (1-\delta) A_t - (1-\delta-\gamma) A_t p_t^{\lambda} - C_t \end{aligned} \tag{3.29}$$

$$R_t = \alpha e^{z_t} A_t^{\alpha-1} \left(1-p_t^{\lambda}\right)^{\alpha-1} H_t^{1-\alpha} + (1-\delta) \tag{3.30}$$

$$R_t p_t = \gamma \tag{3.31}$$

We now find the optimal tax rate, consistent with this dynamics of the economy. Compare equation (3.24) with (3.28). They must be identical for consistency of the model. This condition gives us the equation for optimal taxes

$$\tau_t = \frac{R_t \frac{\lambda+\bar{p}_t^{\lambda+1}}{\lambda+1} - \gamma}{\left(R_t \frac{\lambda+\bar{p}_t^{\lambda+1}}{\lambda+1} - 1\right)} - \frac{\left(1-p_t^{\lambda}\right)\left(R_t-\gamma\right) R_t}{\left(R_t - \left(R_t-(1-\delta)\right)\lambda \frac{p_t^{\lambda}}{\left(1-p_t^{\lambda}\right)} \upsilon \frac{\alpha-1}{\alpha+\upsilon}\right)\left(R_t \frac{\lambda+\bar{p}_t^{\lambda+1}}{\lambda+1} - 1\right)}$$

If there is trade, then equations (3.15)-(3.18) are equivalent to (3.24)-(3.26).

However, if there is no trade then the system (3.19)-(3.23) is not equivalent to (3.27)-(3.31). Specifically,

$$h_t = \left(\frac{(1-\alpha)}{\vartheta} e^{z_t} A_t^{\alpha} \left(1-\bar{p}_t^{\lambda}\right)^{\alpha} X_t\right)^{\frac{1}{\upsilon+\alpha}}$$

$$X_t = \begin{cases} 1 + \alpha\lambda \frac{\bar{p}_t^\lambda}{\left(1-\bar{p}_t^\lambda\right)} \frac{(R_t - \gamma)}{R_t}, \text{ CE equilibrium} \\ 1, \text{ optimal time-contingent taxes} \end{cases}$$

$$\left(C_t - \frac{\vartheta h_t^{1+\upsilon}}{1+\upsilon}\right)^{-\sigma} = \beta \psi_{t+1}^{-\sigma} \left(C_{t+1} - \frac{\vartheta h_{t+1}^{1+\upsilon}}{1+\upsilon}\right)^{-\sigma} R_{t+1}$$

$$R_{t+1} = \begin{cases} R_{t+1} - (R_{t+1} - \gamma)\, \bar{p}_{t+1}^\lambda \left(1 + \lambda(1-\alpha) \frac{(R_{t+1} - (1-\delta))}{R_{t+1}}\right), \text{ CE equilibrium} \\ \gamma + \frac{\left(1 - p_{t+1}^\lambda\right)(R_{t+1} - \gamma) R_{t+1}}{R_{t+1} + (1-\alpha)\lambda(R_{t+1} - (1-\delta)) \frac{p_{t+1}^\lambda}{\left(1 - p_{t+1}^\lambda\right)} \frac{\upsilon}{\alpha+\upsilon}}, \text{ optimal time-contingent taxes} \end{cases}$$

The social planner creates an additional employment. The return on saving is also different. Under optimal taxes consumption in 'no trade' area is lower than in the efficient equilibrium, so that inability of the policymaker to manipulate labour results in excessive taxation. Lower tax rate would improve the 'no trade' situation but will not bring enough taxes to subsidize the 'trade' situation, where the economy spends most of the time. Manipulating labour supply is more efficient than imposing state-contingent tax on savings.

To summarize, as the 'no trade' allocation is different, the optimal taxation cannot replicate the centralized equilibrium. It is optimal to subsidize households in the trade area and tax in the crisis area. Column (3) in Table 3.1, with solution labelled OT, illustrates it clearly. Given proximity of the steady state level of assets to absorption capacity of the banking sector, this model generates high probability of a financial crisis.

3.4.4 Decentralized Competitive Equilibrium

The benchmark calibration of the decentralized model yields steady state allocation presented in column (4) in Table 3.1 and the stochastic allocation is given in column (1) in Table 3.2. Financial frictions and inability to use state-contingent taxes result in lowest consumption and welfare levels out of the four regimes considered by now.

		RBC	CE	OT	DC
		(1)	(2)	(3)	(4)
assets	a	3.711	3.338	3.379	2.776
consumption	c	1.192	1.148	1.142	1.055
labor	h	1.124	1.101	1.082	1.008
loan rate	R	1.030	1.042	1.036	1.048
deposit rate	r	1.030	1.012	1.012	1.030
spread	$R-r$	0.000	0.030	0.024	0.018
interbank rate	ρ	–	0.968	0.976	1.007
efficiency	$\bar{p}$	–	0.929	0.942	0.961
trade volume	ϕ	–	0.303	0.389	0.745
welfare	W	-284.8	-341.6	-310.6	-347.9
boundary	$\bar{A}$	–	3.390	3.390	3.390
min. loan rate	$\bar{R}$	–	1.035	1.035	1.035
min. interb. rate	$\bar{\rho}$	–	0.975	0.975	0.975
min. bank eff.	$\bar{p}_{\min}$	–	0.942	0.942	0.942

Table 3.1: Steady State Values for Different Equilibria

Tax Rate Θ_t:		0%	5%	10%	15%	20%
		(1)	(2)	(3)	(4)	(5)
assets	a	2.7581	2.7286	2.6974	2.6609	2.6127
consumption	c	1.0496	1.0448	1.0398	1.0341	1.0255
labor	h	1.0024	0.9989	0.9941	0.9893	0.9822
loan rate	R	1.0492	1.0500	1.0508	1.0518	1.0532
deposit rate	r	1.0306	1.0317	1.0328	1.0343	1.0361
spread	$R-r$	0.0186	0.0183	0.0180	0.0175	0.0171
interbank rate	ρ	1.0060	1.0076	1.0093	1.0114	1.0139
efficiency	$\bar{p}$	0.9588	0.9596	0.9605	0.9616	0.9627
trade volume	ϕ	0.7302	0.7483	0.7690	0.7939	0.8232
welfare	W	-363.9	-362.5	-361.3	-359.6	-358.6
probability	P	2.2842	1.9468	1.7120	1.3952	1.1612
boundary	$\bar{A}$	3.390	3.412	3.437	3.466	3.498
min. loan rate	$\bar{R}$	1.035	1.035	1.034	1.034	1.033
min. interb. rate	$\bar{\rho}$	0.975	0.973	0.971	0.970	0.967
min. bank eff.	$\bar{p}_{\min}$	0.942	0.941	0.941	0.941	0.941

Table 3.2: The Effect of Interbank Lending Fee on Stochastic Steady State in Decentralaised Competitive Equilibrium

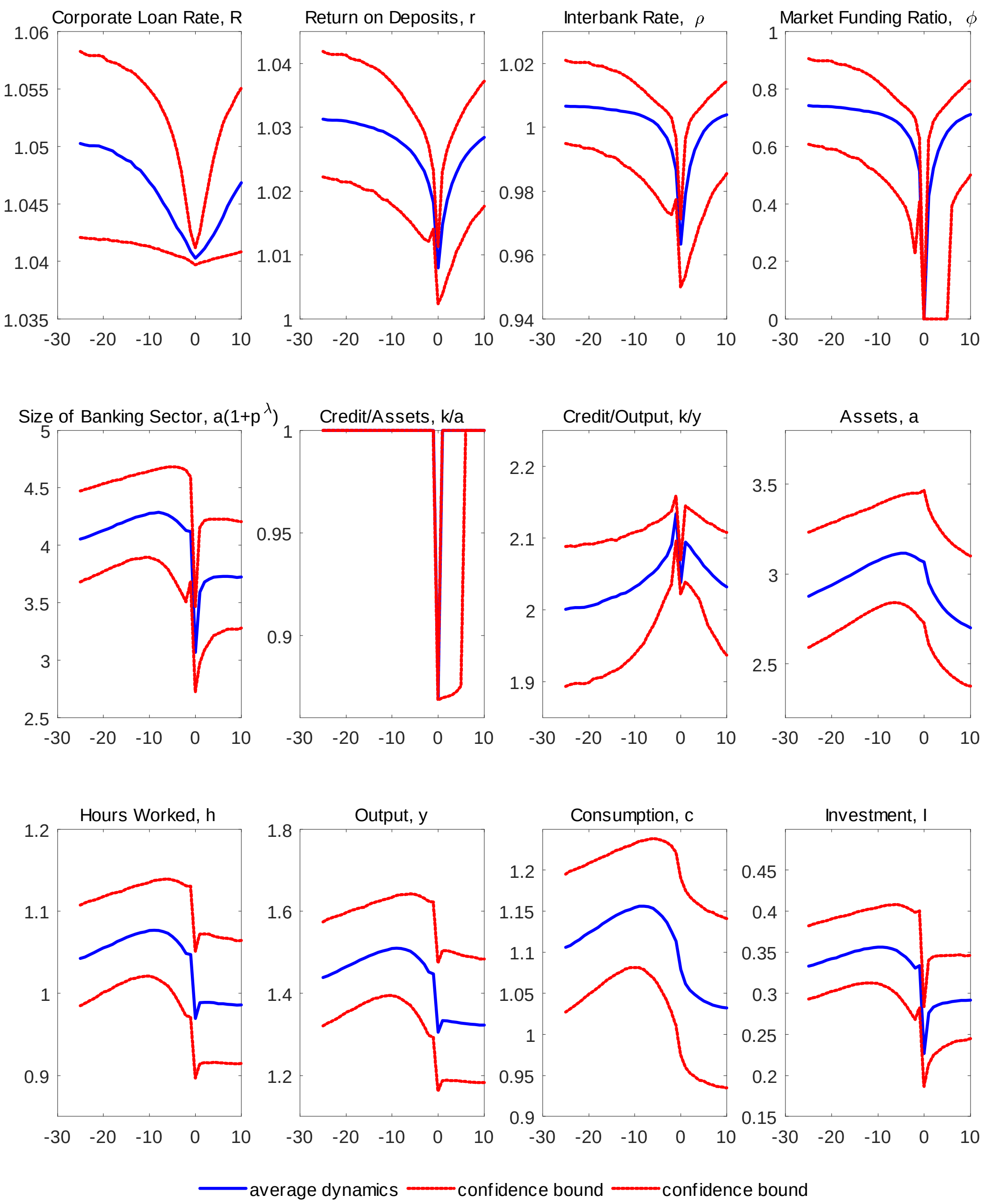

Figure 3.1: Typical dynamic path of the economy around the crisis boundary.

Our calibration was chosen to replicate the probability to hit boundary $\bar{A}_t$ to be 2.3% as reported in BCS.

A sufficiently long sequence of highly persistent technology shock results in low interest rate on corporate loans that falls below the threshold $\bar{R}_t$. Technology shocks triggering crisis are not particularly large, just positive. Interbank market close, the amount of funds channeled to firms contracts and the price of capital rises. Interest rate R_t rises above the threshold and the interbank market reopens. A typical path of key macroeconomic variables to financial recession is plotted in Figure 3.1. The time scale is normalized such that the interbank market closes in period zero.

Monte Carlo simulations show that the market 'feels' the approach of the boundary – the behavior is highly non-linear in all pre-crisis years. A sequence of positive technology shocks drive all interest rates down. Households reduce saving and increase consumption before getting to close to the crisis boundary. Closer to the threshold households realize that the face a large reduction in income should the financial recession materialize and so they reduce consumption to save and hedge themselves against a possible recession. The dis-saving however is not fast enough, it does not lead to an increase of interest rate and does not allow to avoid the crisis. This happens because households do not internalize the effect of their actions on banks and then on the overall economy. The hedging is only partial and consumption falls when the interbank market shuts.

Consider imposing an interbank lending tax which is a proportional tax imposed on the net interest income from lending activity. We describe the tax by a return variable Θ_t. If the interbank market is open then then $\Theta_t < 1$ while if the interbank market is shut then $\Theta_t \equiv 1$.

To understand how the proposed macroprudential policy works, it is instructive to consider the interbank market clearing condition (3.11). Using parameterization this

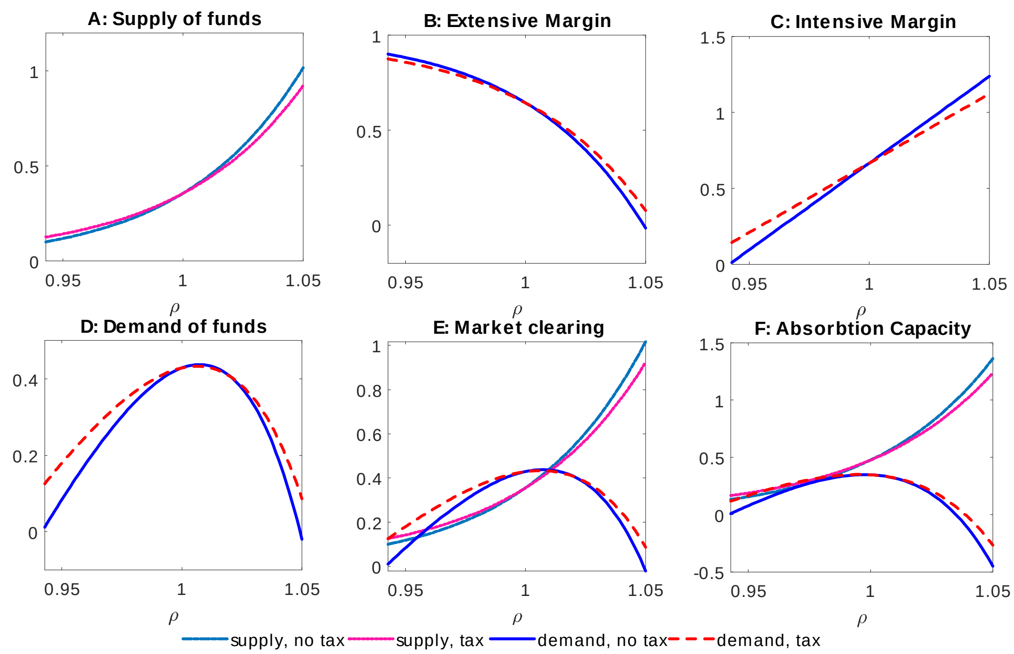

Figure 3.2: Interbank Market Clearing

equation can be rewritten as

$$\underbrace{x_t\left(\frac{1+(\rho_t-1)\,\Theta_t+\rho_t\phi_t+\phi_t\Xi_t}{R_t\,(1+\phi_t)}\right)^\lambda}_{\text{Supply}} = \underbrace{\overbrace{x_t\left(1-\left(\frac{1+(\rho_t-1)\,\Theta_t+\rho_t\phi_t+\phi_t\Xi_t}{R_t\,(1+\phi_t)}\right)^\lambda\right)}^{\text{Extensive Margin}} \times \overbrace{\frac{1+(\rho_t-1)\,\Theta_t-\gamma}{\gamma\theta}}^{\text{Intensive Margin}}}_{\text{Demand}} \tag{3.32}$$

where the expression

$$\left(\frac{1+(\rho_t-1)\,\Theta_t+\rho_t\phi_t+\phi_t\Xi_t}{R_t\,(1+\phi_t)}\right)^\lambda = \left(\frac{\gamma\theta+\rho_t+\Xi_t}{R_t}+\theta\gamma\frac{((1-\theta)\,\gamma-\rho_t)-\Xi_t}{R_t\,(\gamma\theta+1-\gamma+(\rho_t-1)\,\Theta_t)}\right)^\lambda$$

describes the supply of funds given loan rate R_t and where we substituted the market funding ratio from equation (3.10). The supply of funds is an unambiguously increasing function of ρ_t as the first term numerically dominates the second. It is plotted with solid line in Panel A in Figure 3.2. As interbank lending taxes are imposed on net return and become a subsidy when the gross interest rate ρ_t falls below one, lower Θ_t reduces supply for $\rho_t > 1$ and increases supply for $\rho_t < 1$, so that it pivots supply and makes it flatter for a given loan rate R_t.

The aggregate demand is affected by Θ_t via two channels. First, the aggregate demand decreases with the interbank rate as fewer banks will want to borrow, see Panel B in Figure 3.2. Second, the higher interbank rate increases aggregate demand as each borrower is able to borrow more with higher interest rate, see Panel C in Figure 3.2. Lower Θ_t makes both lines flatter, increasing the extensive margin, and reducing intensive margin for $\rho_t > 1$. The total demand for funds is hump-shaped, see Panel D, and higher interbank taxes increase it everywhere except small positive $\rho_t - 1$, where the effect of intensive margin dominates and the volume of trade falls with rising interbank lending taxes (lower Θ_t). Panel E plots demand and supply for equilibrium loan rate $R_t = 1.049$ and demonstrates that there are two equilibria, with $\rho_t > 1$ in the relevant equilibrium. A close inspection shows that, given R_t, both demand and supply fall in the neighborhood of the equilibrium with $\rho_t > 1$ as discussed above. As a result, higher interbank lending taxes are likely to increase the equilibrium interbank rate ρ_t in the steady state, which is indeed the case, see Table 3.2.

Moreover, these two equilibria may not exist if interest rate falls below certain threshold $\bar{R}_t$, as Panel F, which plots demand and supply for $R_t = \bar{R}_t = 1.035$. A close inspection of this panel suggests that demand and supply lines in the economy with interbank lending taxes still intersect, so the threshold interest rate $\bar{R}_t$ should fall with lower Θ_t. This is, indeed, the case, see Table 3.1.

Figure 3.3 summarizes these results. Market clearing condition (3.32) is solved with respect to the loan rate to yield $R_t = \Omega\left(\rho_t, \Theta_t, \phi_t\right)$, which is U-shaped in $\rho_t > \gamma$ area, as plotted with solid line in Panel A in Figure 3.3. With lower Θ_t this line moves down-left, and the minimum $\bar{R}_t$ at which the trade is possible is also going down-left, see the the

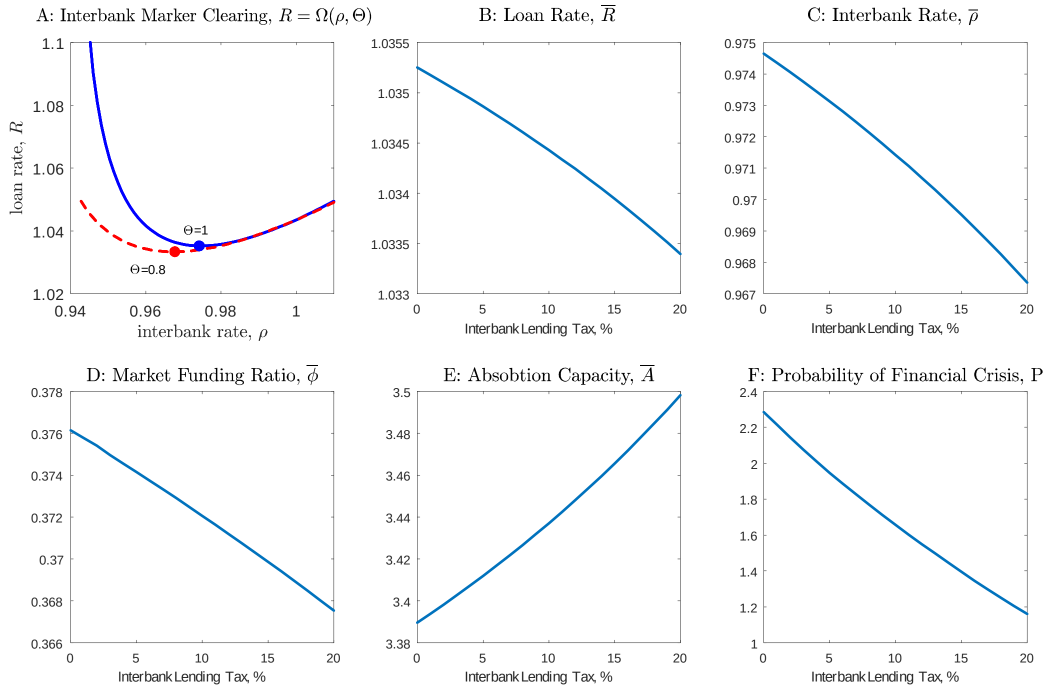

Figure 3.3: The Effect of Interbank Lending Tax on Threshold Characteristics of the Interbank Market, and on Probability of Financial Crises

top left panel in Figure 3.3 which plots the market clearing condition $R_t = \Omega(\rho_t, \Theta_t, \phi_t)$ for two cases $\Theta_t = 1$ and $\Theta_t = 0.8$ with solid and dashed lines respectively. The threshold $\bar{R}_t = \Omega(\bar{\rho}_t; \Theta_t)$ below which the trade cannot happen is shown by large dots on each line in Panel A, and as a function of the interbank tax size in Panel B. The threshold interbank rate $\bar{\rho}_t$ also falls. The efficiency level of the marginal banker $\bar{p}_{\min}$ and the market funding ratio $\bar{\phi}_t$ at the threshold fall. Lower threshold interest rate on loans $\bar{R}_t$ increase the absorption capacity $\bar{A}$ and allows agents to accumulate more assets before the threshold is achieved and the interbank market closes, see Panel E in Figure 3.3. As a result of higher threshold, the boundary $\bar{A}$ it is likely to be achieved less frequently, and the probability of financial crises should go down with Θ_t. Panel F in Figure 3.3 confirms that this is exactly what happens.

However, the frequency at which the absorption capacity $\bar{A}$ is achieved is also affected by the steady state allocation, as it determines the actual 'distance' to the threshold, and how large is the 'speed' of movement of the economy to boundary $\bar{A}$.

It turns out that the second effect works in the same direction as the threshold effect described above. Higher interbank taxes result in higher interbank rate in the stochastic steady state (see Table 3.2), which drives up loan and deposit rates. The higher deposit rate and the intertemporal substitution effect makes households to consume less in equilibrium. Lower current consumption results in lower current demand, and so lower output, capital; and labour demand. At the same time, the efficiency of the marginal bank falls as less efficient banks leave the interbank market and finance firms directly. The overall supply of credit in this economy falls. At the same time, less efficient banks leave the interbank market and finance firms directly. The supply of credit is allocated to its best users. The market funding ratio ϕ (which determines the trade volume) rises and the interest rate spread – a measure of financial frictions – reduces with higher interbank lending tax. As a result, there is no increase in accumulated assets in the stochastic steady state, they decrease with higher taxes and lower Θ_t.

We therefore, obtain that the steady state level of assets – around this state the economy spends most of the time – goes down, while the absorption capacity level goes up. Although all interest rates go up and so the 'speed' to move towards $\bar{A}_t$ increases, this effect is limited, and the overall probability of a financial crisis goes down.

Although it is expected that a macroprudential policy would lead to a deterioration of welfare, as new frictions and constraints are imposed, the results reported in Table 3.2 suggest the opposite. This is explained by greater reduction in hours worked, and so higher utility of leisure outweighs disutility of lower consumption.

3.5 Conclusion

We discuss a regulation problem in a simple RBC-type economy with banking sector and an interbank market, which is capable to tell a coherent story of a run up to the financial crisis of 2009. The unregulated model features a banking sector and a fragile interbank market which closes should interest rate fall sufficiently low. When a sequence of positive technology shocks results in capital accumulation and high volume of lending to firms – either directly or through the interbank market – the interbank interest rate falls below the threshold, the closure of the interbank market results in sharp contraction in real and financial activity. We demonstrate that introducing an interbank lending tax as an instrument of macroprudential policy results in substantial reduction of the probability of a crisis event. Although such measures reduces the aggregate supply of credit, it promotes the allocation of credit to its best users, and leads to higher household welfare.

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3.A Appendix: Interbank Borrowing Tax

Tax on interbank borrowing works in a different way. An interbank borrowing taxes unambiguously raise supply of funds at the interbank market, as the marginal efficiency to lend to firms goes up and more efficient banks decide to lend at the interbank market. The supply relationship shifts to the left with higher Ξ_t. Only the extensive margin is affected by higher borrowing taxes Ξ_t. With higher tax on borrowing the hump-shaped demand schedule shifts down-left, with a very substantial effect on equilibrium: higher interbank borrowing rate reduces both the equilibrium interbank rate ρ_t and the corporate loan rate R_t in the steady state, see Figure3.4. This figure compares the benchmark case of no taxes with tax rate of 5%. Panel E reports that the point of intersection of demand and supply shifts to the left substantially.

Figure 3.5 shows the effect of the borrowing tax on the absorbtion capacity. It is apparent that the effect is likely to increase the probability of crises. With higher tax the interest rate, at which the interbank market closes, is rising. Although additional investigation is needed, it is highly likely that this policy does nor reduce probability of the financial crisis.

3.B Appendix: Decentralized Problem

3.B.1 Solving Decentralized Equilibrium

Collect all FOCs for firms and households, first substitute profit

$$\pi_t = e^{z_t} F\left(k_t, h_t\right) + \left(1-\delta\right) k_t - w_t \Psi_t h_t - R_t k_t$$

into the household budget constraint and collect all together

$$\begin{aligned} c_t + a_{t+1} &= \tilde{r}_t a_t + w_t \Psi_t h_t + \pi_t + \chi_t + T_t + g_t = \tilde{r}_t a_t + w_t \Psi_t h_t \\ &\quad + e^{z_t} F\left(k_t, h_t\right) + \left(1-\delta\right) k_t - w_t \Psi_t h_t - R_t k_t + \chi_t + g_t + T_t \\ &= \tilde{r}_t a_t + e^{z_t} F\left(k_t, h_t\right) + \left(1-\delta\right) k_t - R_t k_t + \chi_t + g_t + T_t \end{aligned}$$

$$\Psi_t w_t = e^{z_t} F_h\left(k_t, h_t\right)$$

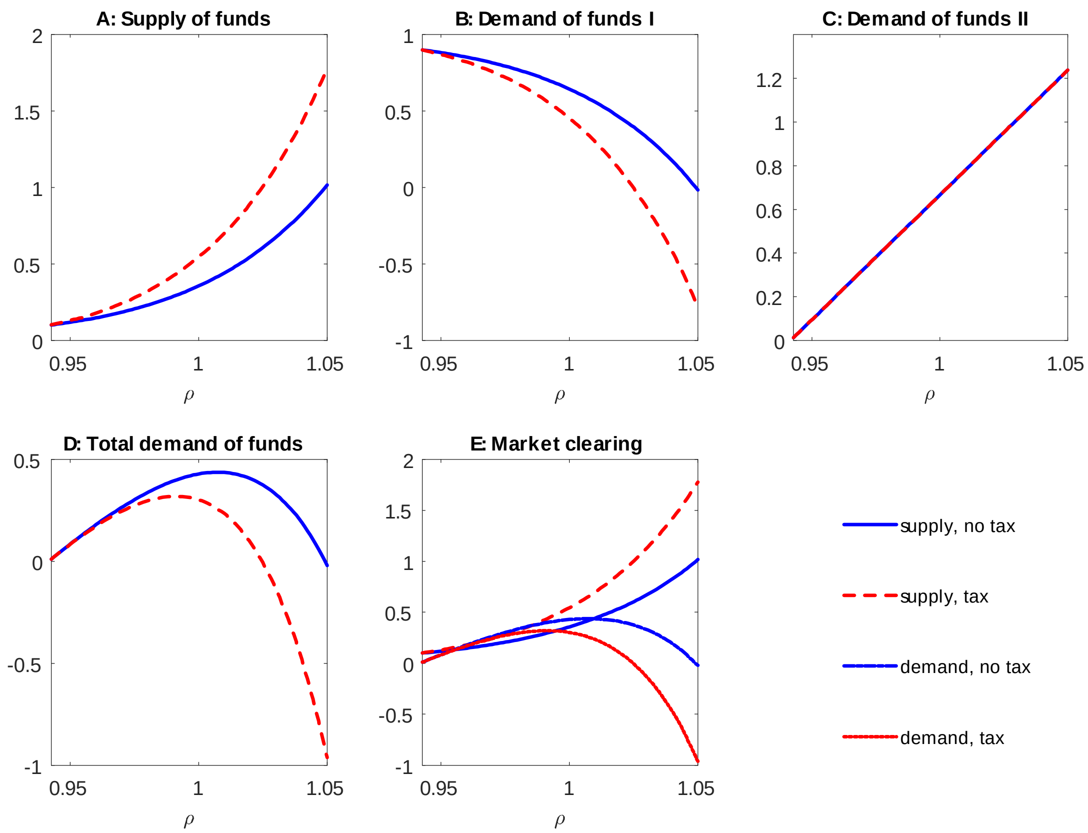

Figure 3.4: Interbank Market Clearing, tax on borrowing

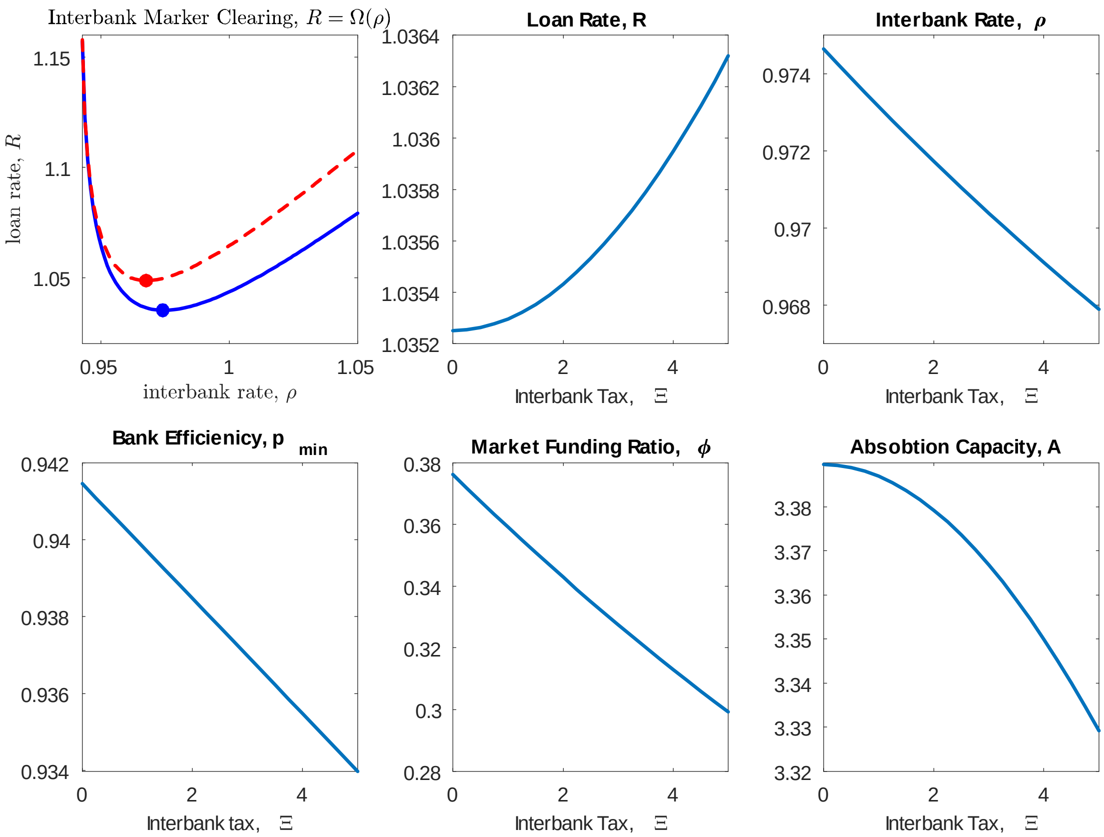

Figure 3.5: The Effect of Interbank Borrowing Tax on Threshold Characteristics of the Interbank Market

$$R_t = e^{z_t} F_k\left(k_t, h_t\right) + (1-\delta)$$
$$\Psi_t w_t = -\frac{u_h\left(c_t, h_t\right)}{u_c\left(c_t, h_t\right)}$$
$$u_c\left(c_t, h_t\right) = \beta u_c\left(c_{t+1}, h_{t+1}\right) \tilde{r}_{t+1}$$
$$c_t + a_{t+1} = \tilde{r}_t a_t + e^{z_t} F\left(k_t, h_t\right) + (1-\delta) k_t - R_t k_t + \chi_t + g_t + T_t$$

substitute utility and production function:

$$\Psi_t w_t = (1-\alpha) \Psi_t e^{z_t} k_t^{\alpha} \left(\Psi_t h_t\right)^{-\alpha}$$

$$R_t = \alpha e^{z_t} k_t^{\alpha-1} \left(\Psi_t h_t\right)^{1-\alpha} + (1-\delta)$$
$$\Psi_t w_t = -\frac{-\vartheta \Psi_t h_t^{\upsilon}\left(c_t - \vartheta \Psi_t \frac{h_t^{1+\upsilon}}{1+\upsilon}\right)^{-\sigma}}{\left(c_t - \vartheta \Psi_t \frac{h_t^{1+\upsilon}}{1+\upsilon}\right)^{-\sigma}} = \vartheta \Psi_t h_t^{\upsilon}$$
$$\left(c_t - \vartheta \Psi_t \frac{h_t^{1+\upsilon}}{1+\upsilon}\right)^{-\sigma} = \beta\left(c_{t+1} - \vartheta \Psi_{t+1} \frac{h_{t+1}^{1+\upsilon}}{1+\upsilon}\right)^{-\sigma} \tilde{r}_{t+1}$$
$$c_t + a_{t+1} = \tilde{r}_t a_t + e^{z_t} k_t^{\alpha} \left(\Psi_t h_t\right)^{1-\alpha} + (1-\delta) k_t - R_t k_t + \chi_t + g_t + T_t$$

substitute out wage, solve for labor

$$h_t = \left(\frac{(1-\alpha)}{\vartheta} e^{z_t} \left(\frac{k_t}{\Psi_t}\right)^{\alpha}\right)^{\frac{1}{\upsilon+\alpha}}$$

solve for the absorption capacity of the banking sector

$$R_t = \alpha e^{z_t} k_t^{\alpha-1} \left(\Psi_t h_t\right)^{1-\alpha} + (1-\delta) = \alpha\left(\frac{1-\alpha}{\vartheta}\right)^{\frac{1-\alpha}{\upsilon+\alpha}} \left(\frac{k_t}{\Psi_t}\right)^{\upsilon\frac{\alpha-1}{\alpha+\upsilon}} e^{\frac{1+\upsilon}{\upsilon+\alpha} z_t} + (1-\delta)$$

which yields demand for capital

$$\begin{aligned} k_t^d &= \left(\frac{R_t - (1-\delta)}{\alpha} e^{-\frac{1+\upsilon}{\upsilon+\alpha} z_t} \left(\frac{1-\alpha}{\vartheta}\right)^{-\frac{1-\alpha}{\upsilon+\alpha}} \Psi_t^{+\upsilon\frac{\alpha-1}{\alpha+\upsilon}}\right)^{\frac{\alpha+\upsilon}{\upsilon(\alpha-1)}} \\ &= \left(\frac{1-\alpha}{\vartheta}\right)^{\frac{1}{\upsilon}} \left(\frac{\alpha}{R_t + \delta - 1}\right)^{\frac{\alpha+\upsilon}{\upsilon(1-\alpha)}} e^{\frac{1+\upsilon}{\upsilon(1-\alpha)} z_t} \Psi_t. \end{aligned}$$

Therefore, if $R_t = \bar{R}_t$ then

$$\frac{k_t^d}{\Psi_t} = \left(\frac{1-\alpha}{\vartheta}\right)^{\frac{1}{\upsilon}} \left(\frac{\alpha}{\bar{R}_t + \delta - 1}\right)^{\frac{\alpha+\upsilon}{\upsilon(1-\alpha)}} e^{\frac{1+\upsilon}{\upsilon(1-\alpha)} z_t} = \Gamma_t e^{\frac{1+\upsilon}{\upsilon(1-\alpha)} z_t}$$

where

$$\Gamma_t = \left(\frac{1-\alpha}{\vartheta}\right)^{\frac{1}{\upsilon}} \left(\frac{\alpha}{\left(\bar{R}_t + \delta - 1\right)}\right)^{\frac{\alpha+\upsilon}{\upsilon(1-\alpha)}}$$

and it depends on taxation Ξ_t, and the volume of trade ϕ_t

Note that by definition $\bar{R}$ is the 'lowest' point where the trade exists, so $\frac{k_t^d}{\Psi_t} = \frac{k_t^s}{\Psi_t} = \frac{a_t}{\Psi_t}$ and

$$\frac{\bar{a}_t}{\Psi_t} = \left(\frac{1-\alpha}{\vartheta}\right)^{\frac{1}{\upsilon}} \left(\frac{\alpha}{\bar{R} + \delta - 1}\right)^{\frac{\alpha+\upsilon}{\upsilon(1-\alpha)}} e^{\frac{1+\upsilon}{\upsilon(1-\alpha)} z_t} = \Gamma_t e^{\frac{1+\upsilon}{\upsilon(1-\alpha)} z_t}$$

We have two situations to describe:

1. trade does not exist

2. trade exists

In the limiting case the trade does exist, $R_t = \bar{R}_t$

$$r_t = \int_0^1 r_t(p)\, d\mu(p) = \left\{ \begin{array}{c} \int_{\bar{p}_t}^1 pR_t (1+\phi_t)\, d\mu(p) \overset{(6a)}{=} \int_{\bar{p}_t}^1 pR_t \left(1 + \frac{\mu(\bar{p}_t)}{(1-\mu(\bar{p}_t))}\right) d\mu(p) \\ = \int_{\bar{p}_t}^1 \frac{pR_t}{(1-\mu(\bar{p}_t))} d\mu(p),\ \text{trade} \\ \int_{\frac{\gamma}{R_t}}^1 pR_t d\mu(p) + \int_0^{\frac{\gamma}{R_t}} \gamma d\mu(p) \\ = R_t \int_{\frac{\gamma}{R_t}}^1 p d\mu(p) + \gamma\mu\left(\frac{\gamma}{R_t}\right),\ \text{no trade} \end{array} \right. \tag{7}$$

Finally, define transfers as

$$
\begin{aligned}
\chi_t &= \left\{ \begin{array}{c} \int_{\bar{p}_t}^{1} (1-p)(1+\phi_t) R_t a_t d\mu(p) = (1+\phi_t) R_t a_t \left(1 - \mu(\bar{p}_t) - \int_{\bar{p}_t}^{1} p d\mu(p)\right) \\ = (1+\phi_t) R_t a_t \left(\frac{1}{(1+\phi_t)} - \int_{\bar{p}_t}^{1} p d\mu(p)\right) \\ = R_t a_t - a_t \int_{\bar{p}_t}^{1} p R_t (1+\phi_t) d\mu(p) = a_t (R_t - r_t), \text{ trade} \\ \int_{\frac{\gamma}{R_t}}^{1} (1-p) R_t a_t d\mu(p) = \int_{\frac{\gamma}{R_t}}^{1} R_t a_t d\mu(p) - \int_{\frac{\gamma}{R_t}}^{1} p R_t a_t d\mu(p) \\ = R_t a_t \left(1 - \mu\left(\frac{\gamma}{R_t}\right)\right) - a_t \int_{\frac{\gamma}{R_t}}^{1} p R_t d\mu(p) \\ = R_t a_t \left(1 - \mu\left(\frac{\gamma}{R_t}\right)\right) - a_t \left(r_t - \gamma\mu\left(\frac{\gamma}{R_t}\right)\right) \\ = a_t (R_t - r_t) - (a_t - k_t)(R_t - \gamma), \text{ no trade} \end{array} \right. \\
T_t &= (r_t - 1)\tau_t a_t \\
g_t &= ((\rho_t - 1)(1 - \Theta_t) + \Xi_t) a_t \phi_t
\end{aligned}
$$

The final system can be written as

$$c_t + a_{t+1} = \tilde{r}_t a_t + e^{z_t} k_t^{\alpha} (\Psi_t h_t)^{1-\alpha} + (1-\delta) k_t - R_t k_t + \chi_t + g_t + T_t$$

$$R_t = \alpha e^{z_t} k_t^{\alpha-1} (\Psi_t h_t)^{1-\alpha} + (1-\delta)$$

$$\left(c_t - \vartheta \Psi_t \frac{h_t^{1+\upsilon}}{1+\upsilon}\right)^{-\sigma} = \beta \left(c_{t+1} - \vartheta \Psi_{t+1} \frac{h_{t+1}^{1+\upsilon}}{1+\upsilon}\right)^{-\sigma} (1 + (r_{t+1} - 1)(1 - \tau_{t+1}))$$

$$h_t = \left(\frac{(1-\alpha)}{\vartheta} e^{z_t} \left(\frac{k_t}{\Psi_t}\right)^{\alpha}\right)^{\frac{1}{\upsilon+\alpha}}$$

$$T_t = (r_t - 1)\tau_t a_t$$

$$\chi_t = \left\{ \begin{array}{c} (R_t - r_t) a_t, \text{ trade} \\ R_t k_t - r_t a_t + \gamma (a_t - k_t), \text{ no trade} \end{array} \right.$$

$$a_t = \left\{ \begin{array}{c} k_t, \text{ trade} \\ \left(1 - \mu\left(\frac{\gamma}{R_t}\right)\right)^{-1} k_t, \text{ no trade} \end{array} \right.$$

$$\frac{r_t}{R_t} = \left\{ \begin{array}{c} \int_{\bar{p}_t}^{1} \frac{p}{(1-\mu(\bar{p}_t))} d\mu(p), \text{ trade} \\ \int_{\frac{\gamma}{R_t}}^{1} p d\mu(p) + \frac{\gamma}{R_t}\mu\left(\frac{\gamma}{R_t}\right), \text{ no trade} \end{array} \right.$$

$$\bar{p}_t = \left\{ \begin{array}{c} \frac{1 + (\rho_t - 1)\Theta_t + \rho_t \phi_t}{R_t(1+\phi_t)}, \text{ trade} \\ \frac{\gamma}{R_t}, \text{ no trade} \end{array} \right.$$

$$g_t = \begin{cases} \left(\left(\rho_t - 1\right)\left(1-\Theta_t\right) + \Xi_t\right)\phi_t a_t, \text{ trade} \\ 0, \text{ no trade} \end{cases}$$

$$\mu\left(\frac{1+\left(\rho_t-1\right)\Theta_t+\rho_t\phi_t+\phi_t\Xi_t}{R_t\left(1+\phi_t\right)}\right) = \frac{1+\left(\rho_t-1\right)\Theta_t-\gamma}{1+\left(\rho_t-1\right)\Theta_t+\gamma\left(\theta-1\right)}$$

$$\phi_t = \frac{1-\gamma+\left(\rho_t-1\right)\Theta_t}{\gamma\theta}$$

then

$$\mu\left(\frac{\gamma\theta+\gamma\theta\left(\rho_t-1\right)\Theta_t+\left(\rho_t+\Xi_t\right)\left(1-\gamma+\left(\rho_t-1\right)\Theta_t\right)}{R_t\left(\gamma\left(\theta-1\right)+1+\left(\rho_t-1\right)\Theta_t\right)}\right) = \frac{1+\left(\rho_t-1\right)\Theta_t-\gamma}{1+\left(\rho_t-1\right)\Theta_t+\gamma\left(\theta-1\right)}$$

$$\rho_t = \begin{cases} RootOf\left(\mu\left(\frac{1+(\rho_t-1)\Theta_t-\gamma}{(\gamma(\theta-1)+1+(\rho_t-1)\Theta_t)}\right) = \frac{1+(\rho_t-1)\Theta_t-\gamma}{(\gamma(\theta-1)+1+(\rho_t-1)\Theta_t)}\right), \text{ trade} \\ \gamma, \text{ no trade} \end{cases}$$

We simplify the system. We relabel variables $A_t = \frac{a_t}{\Psi_t}, K_t = \frac{k_t}{\Psi_t}, C_t = \frac{c_t}{\Psi_t}, X_t = \frac{\chi_t}{\Psi_t}, \Upsilon_t = \frac{T_t}{\Psi_t}, G_t = \frac{g_t}{\Psi_t}, \psi_{t+1} = \frac{\Psi_{t+1}}{\Psi_t}$ and substitute out transfers

$$C_t + \psi_{t+1}A_{t+1} = \left[e^{z_t}K_t^{\alpha}h_t^{1-\alpha} + \left(1-\delta\right)K_t\right] + \gamma\left(A_t - K_t\right) + \left(\rho_t - 1\right)\left(1-\Theta_t\right)\phi_t A_t$$

$$R_t = \alpha e^{z_t}K_t^{\alpha-1}h_t^{1-\alpha} + \left(1-\delta\right)$$

$$\left(C_t - \vartheta\frac{h_t^{1+\upsilon}}{1+\upsilon}\right)^{-\sigma} = \beta\psi_{t+1}^{-\sigma}\left(C_{t+1} - \vartheta\frac{h_{t+1}^{1+\upsilon}}{1+\upsilon}\right)^{-\sigma}\left(1+\left(r_{t+1}-1\right)\left(1-\tau_{t+1}\right)\right)$$

$$h_t = \left(\frac{\left(1-\alpha\right)}{\vartheta}e^{z_t}K_t^{\alpha}\right)^{\frac{1}{\upsilon+\alpha}}$$

$$K_t = \begin{cases} A_t, \text{ trade} \\ A_t\left(1-\left(\frac{\gamma}{R_t}\right)^{\lambda}\right), \text{ no trade} \end{cases}$$

$$r_t = \begin{cases} R_t\frac{\lambda}{(\lambda+1)}\frac{\left(1-\bar{p}_t^{\lambda+1}\right)}{\left(1-\bar{p}_t^{\lambda}\right)}, \text{ trade} \\ R_t\left(\frac{\lambda+\bar{p}_t^{\lambda+1}}{\lambda+1}\right), \text{ no trade} \end{cases}$$

$$\rho_t = \begin{cases} \frac{\bar{p}_tR_t(1+\phi_t)-1+\Theta_t}{(\Theta_t+\phi_t)}, \text{ trade} \\ \gamma, \text{ no trade} \end{cases}$$

$$\bar{p}_t = \begin{cases} \left(\frac{1+(\rho_t-1)\Theta_t-\gamma}{(\gamma(\theta-1)+1+(\rho_t-1)\Theta_t)}\right)^{\frac{1}{\lambda}}, \text{ trade} \\ \frac{\gamma}{R_t}, \text{ no trade} \end{cases}$$

$$\phi_t = \begin{cases} \frac{\bar{p}_t^{\lambda}}{\left(1-\bar{p}_t^{\lambda}\right)}, & \text{trade} \\ 0, & \text{no trade} \end{cases}$$

where unknowns are: $r_t, \rho_t, R_t, h_t, C_t, K_t, A_t, \bar{p}_t, \phi_t$.

Finally, at the boundary

$$\bar{A}_t = \left(\frac{1-\alpha}{\vartheta}\right)^{\frac{1}{\upsilon}} \left(\frac{\alpha}{\bar{R}+\delta-1}\right)^{\frac{\alpha+\upsilon}{\upsilon(1-\alpha)}} e^{\frac{1+\upsilon}{\upsilon(1-\alpha)}z_t} = \Gamma e^{\frac{1+\upsilon}{\upsilon(1-\alpha)}z_t}$$

and if there is trade then $\Theta_t \leq 1$ and $\Theta_t = 1$ if the interbank market is shut.

3.B.2 Absorption Capacity

Recall that we have three equations to determine p_t, ϕ_t, R_t

$$\phi_t = \frac{1-\gamma+(\rho_t-1)\,\Theta_t}{\gamma\theta}$$

$$p_t = \frac{1+(\rho_t-1)\,\Theta_t+\rho_t\phi_t+\phi_t\Xi_t}{R_t\,(1+\phi_t)}$$

$$p_t^{\lambda} = \frac{\phi_t}{1+\phi_t}$$

from where

$$\frac{\phi_t}{(1+\phi_t)} = \frac{1+(\rho_t-1)\,\Theta_t-\gamma}{(\gamma\,(\theta-1)+1+(\rho_t-1)\,\Theta_t)}$$

$$p_t = \left(\frac{\phi_t}{1+\phi_t}\right)^{\frac{1}{\lambda}} = \frac{1+(\rho_t-1)\,\Theta_t+\rho_t\phi_t+\phi_t\Xi_t}{R_t\,(1+\phi_t)}$$

and

$$\phi_t = \frac{1-\gamma+(\rho_t-1)\,\Theta_t}{\theta\gamma}$$

$$R_t = \frac{1+(\rho_t-1)\,\Theta_t+\rho_t\phi_t+\phi_t\Xi_t}{\left(\frac{\phi_t}{1+\phi_t}\right)^{\frac{1}{\lambda}}(1+\phi_t)}$$

Substitute out ϕ_t to yield

$$
\begin{aligned}
R_t &= \left(\frac{\theta\gamma}{(1+(\rho_t-1)\,\Theta_t-\gamma)}+1\right)^{\frac{1}{\lambda}}\left(\frac{\theta\gamma\,(1-\Theta_t)}{\theta\gamma+(1+(\rho_t-1)\,\Theta_t-\gamma)}\right. \\
&\quad +\rho_t\left(\frac{\theta\gamma\,(\Theta_t-1)}{\theta\gamma+(1+(\rho_t-1)\,\Theta_t-\gamma)}+1\right) \\
&\quad \left.+\Xi_t\left(1-\frac{\theta\gamma}{\theta\gamma+1-\gamma+(\rho_t-1)\,\Theta_t}\right)\right)
\end{aligned} \tag{3.33}
$$

which is a form of equation (6) but where ϕ_t is substituted out. In what follows we assume $1+(\rho_t-1)\,\Theta_t>\gamma$. This implies that the denominator is positive as $\rho_t>1+\frac{\gamma-1}{\Theta_t}>1+\frac{\gamma-1-\theta\gamma}{\Theta_t}$

We find minimum of (3.33) in analytical form:

$$
\begin{aligned}
R_t' &= -\frac{1}{\lambda}\frac{\theta\gamma\Theta_t}{(1+(\rho_t-1)\,\Theta_t-\gamma)^2}\left(\frac{\theta\gamma}{(1+(\rho_t-1)\,\Theta_t-\gamma)}+1\right)^{\frac{1}{\lambda}-1} \\
&\quad \left(\begin{array}{c}\frac{\theta\gamma(1-\Theta_t)}{\theta\gamma+(1+(\rho_t-1)\Theta_t-\gamma)}+\rho_t\left(\frac{\theta\gamma(\Theta_t-1)}{\theta\gamma+(1+(\rho_t-1)\Theta_t-\gamma)}+1\right)\\ +\Xi_t\left(1-\frac{\theta\gamma}{\theta\gamma+1-\gamma+(\rho_t-1)\Theta_t}\right)\end{array}\right) \\
&\quad +\left(\frac{\theta\gamma}{(1+(\rho_t-1)\,\Theta_t-\gamma)}+1\right)^{\frac{1}{\lambda}} \\
&\quad \left(\begin{array}{c}-\frac{\theta\gamma(1-\Theta_t)\Theta_t}{(\theta\gamma+(1+(\rho_t-1)\Theta_t-\gamma))^2}+\Xi_t\left(\frac{\theta\gamma\Theta_t}{(\theta\gamma+1-\gamma+(\rho_t-1)\Theta_t)^2}\right)\\ +\left(\frac{\theta\gamma(\Theta_t-1)}{\theta\gamma+(1+(\rho_t-1)\Theta_t-\gamma)}+1\right)-\rho_t\left(\frac{\theta\gamma(\Theta_t-1)\Theta_t}{(\theta\gamma+(1+(\rho_t-1)\Theta_t-\gamma))^2}\right)\end{array}\right)
\end{aligned}
$$

$$
\begin{aligned}
R_t' &= -\frac{1}{\lambda}\frac{\theta\gamma\Theta_t}{(1+(\rho_t-1)\,\Theta_t-\gamma)^2}\left(\frac{\theta\gamma}{(1+(\rho_t-1)\,\Theta_t-\gamma)}+1\right)^{\frac{1}{\lambda}-1} \\
&\quad \frac{\left(\begin{array}{c}\theta\gamma\,(1-\Theta_t)+\rho_t\,(\theta\gamma\,(\Theta_t-1)+\theta\gamma+(1+(\rho_t-1)\,\Theta_t-\gamma))\\ +\Xi_t\,(\theta\gamma+1-\gamma+(\rho_t-1)\,\Theta_t-\theta\gamma)\end{array}\right)}{\theta\gamma+(1+(\rho_t-1)\,\Theta_t-\gamma)} \\
&\quad +\left(\frac{\theta\gamma}{(1+(\rho_t-1)\,\Theta_t-\gamma)}+1\right)^{\frac{1}{\lambda}-1}\left(\frac{\theta\gamma}{(1+(\rho_t-1)\,\Theta_t-\gamma)}+1\right) \\
&\quad \frac{\left(\begin{array}{c}-\theta\gamma\,(1-\Theta_t)\,\Theta_t+\Xi_t\theta\gamma\Theta_t+(\theta\gamma\,(\Theta_t-1)+\theta\gamma+(1+(\rho_t-1)\,\Theta_t-\gamma))\\ \times(\theta\gamma+(1+(\rho_t-1)\,\Theta_t-\gamma))-\rho_t\theta\gamma\,(\Theta_t-1)\,\Theta_t\end{array}\right)}{(\theta\gamma+(1+(\rho_t-1)\,\Theta_t-\gamma))^2}
\end{aligned}
$$

If $\Theta_t = 1$ and $\Xi_t = 0$ then solution

$$R'_t = \left(\frac{\theta\gamma}{(1+(\rho_t-1)-\gamma)}+1\right)^{\frac{1}{\lambda}-1} \frac{(1-\theta)\lambda\gamma^2 + (\theta\lambda-\theta-2\lambda)\gamma\rho_t + \lambda\rho_t^2}{\lambda(\gamma-\rho_t)^2}$$

from where:

$$\bar{\rho} = \frac{-(\theta\lambda\gamma-\theta\gamma-2\lambda\gamma)+\sqrt{(\theta\lambda\gamma-\theta\gamma-2\lambda\gamma)^2-4(1-\theta)\lambda^2\gamma^2}}{2\lambda}$$

and

$$\frac{\bar{\rho}}{\bar{R}} = \left(\frac{\bar{\rho}-\gamma}{\bar{\rho}-\gamma+\theta\gamma}\right)^{\frac{1}{\lambda}}$$

In general case

$$R'_t = \frac{\left(\frac{\theta\gamma}{(1+(\rho_t-1)\Theta_t-\gamma)}+1\right)^{\frac{1}{\lambda}-1}}{\lambda(\gamma+\Theta_t-\Theta_t\rho_t-1)^2(-\gamma-\Theta_t+\theta\gamma+\Theta_t\rho_t+1)} P(\rho_t)$$

where

$$\begin{aligned}
P(\rho_t) \;=\;& \left(\lambda\Theta_t^3\right)\rho_t^3 + \Theta_t^2(3\lambda-\theta\gamma-3\lambda\gamma-3\lambda\Theta_t+2\theta\lambda\gamma)\rho_t^2 \\
&+\Theta_t\rho_t(-(\gamma-1)(3\lambda-\theta\gamma-3\lambda\gamma+3\theta\lambda\gamma) \\
&+\Theta_t\left(-6\lambda-\theta^2\gamma^2+\theta\gamma+6\lambda\gamma-\theta\lambda\gamma^2+\theta^2\lambda\gamma^2-3\theta\lambda\gamma+3\lambda\Theta_t\right)) \\
&+\lambda(\gamma-1)^2(-\gamma+\theta\gamma+1)-\Theta_t\left(3\lambda+\theta^2\gamma^2-6\lambda\gamma-3\lambda\Theta_t\right. \\
&+3\lambda\gamma^2-\theta\lambda\gamma^2-\theta\lambda\gamma^3-\theta^2\lambda\gamma^2+\theta^2\lambda\gamma^3+2\theta\lambda\gamma \\
&\left.+\Theta_t\left(-\theta^2\gamma^2+3\lambda\gamma+\lambda\Theta_t-\theta\lambda\gamma^2+\theta^2\lambda\gamma^2-\theta\lambda\gamma\right)\right)
\end{aligned}$$

So that

$$\bar{\rho}_t = \mathrm{RootOf}(P(\rho_t))$$

$$\bar{R}_t = \frac{(1 + (\bar{\rho}_t - 1)\,\Theta_t + \bar{\rho}_t\phi_t)}{(1+\phi_t)} \left(\frac{(\gamma(\theta-1) + 1 + (\bar{\rho}_t - 1)\,\Theta_t)}{1 + (\bar{\rho}_t - 1)\,\Theta_t - \gamma}\right)^{\frac{1}{\lambda}}$$

$$p_{\min} = \left(\frac{1 + (\bar{\rho}_t - 1)\,\Theta_t - \gamma}{(\gamma(\theta-1) + 1 + (\bar{\rho}_t - 1)\,\Theta_t)}\right)^{\frac{1}{\lambda}}$$

$$\bar{A}_t = \left(\frac{1-\alpha}{\vartheta}\right)^{\frac{1}{\upsilon}} \left(\frac{\alpha}{\bar{R}_t + \delta - 1}\right)^{\frac{\alpha+\upsilon}{\upsilon(1-\alpha)}} e^{\frac{1+\upsilon}{\upsilon(1-\alpha)}z_t} = \Gamma_t e^{\frac{1+\upsilon}{\upsilon(1-\alpha)}z_t}$$

and the boundary $\bar{A}_t$ depends on taxes Θ_t.

3.C Appendix: Centralized Equilibrium

We assume that the policymaker solves the following problem:

$$\max_{H_t, C_t} \sum_{t=0}^{\infty} \beta^t u(C_t, H_t) = \max_{H_t, C_t} \sum_{t=0}^{\infty} \beta^t \frac{1}{1-\sigma} \left(C_t - \vartheta \frac{H_t^{1+\upsilon}}{1+\upsilon}\right)^{1-\sigma}$$

subject to constraints

$$C_t + \psi A_{t+1} = \left[e^{z_t} K_t^{\alpha} H_t^{1-\alpha} + (1-\delta) K_t\right] + \gamma (A_t - K_t)$$

$$R_t = \alpha e^{z_t} K_t^{\alpha-1} H_t^{1-\alpha} + (1-\delta)$$

$$K_t = \begin{cases} A_t, \text{ trade} \\ A_t \left(1 - \left(\frac{\gamma}{R_t}\right)^{\lambda}\right), \text{ no trade} \end{cases}$$

3.C.1 Interest rates in Equilibrium with Trade

The Lagrangians can be written

$$\sum_{t=0}^{\infty} \beta^t \left(\frac{\Psi_t^{1-\sigma}}{1-\sigma} \left(C_t - \frac{\vartheta h_t^{1+\upsilon}}{1+\upsilon}\right)^{1-\sigma} + \phi_{1t}\left(\alpha e^{z_t} A_t^{\alpha-1} h_t^{1-\alpha} + (1-\delta) - R_t\right)\right.$$
$$\left. + \phi_{2t}\left(e^{z_t} A_t^{\alpha} h_t^{1-\alpha} + (1-\delta) A_t - C_t - \psi_{t+1} A_{t+1}\right)\right)$$

and the first order conditions are:

$$\frac{\partial}{\partial C_t}:\Psi_t^{1-\sigma}\left(C_t-\frac{\vartheta h_t^{1+\upsilon}}{1+\upsilon}\right)^{-\sigma}-\phi_{2t}$$

$$\frac{\partial}{\partial h_t}:-\vartheta h_t^{\upsilon}\Psi_t^{1-\sigma}\left(C_t-\frac{\vartheta h_t^{1+\upsilon}}{1+\upsilon}\right)^{-\sigma}+\phi_{1t}(1-\alpha)\alpha e^{z_t}A_t^{\alpha-1}h_t^{-\alpha}+\phi_{2t}(1-\alpha)e^{z_t}A_t^{\alpha}h_t^{-\alpha}$$

$$\frac{\partial}{\partial R_t}:+\phi_{1t}$$

$$\frac{\partial}{\partial A_{t+1}}:\beta\phi_{1t+1}\alpha(\alpha-1)e^{z_{t+1}}A_{t+1}^{\alpha-2}h_{t+1}^{1-\alpha}+\beta\phi_{2t+1}\left(\alpha e^{z_{t+1}}A_{t+1}^{\alpha-1}h_{t+1}^{1-\alpha}+(1-\delta)\right)-\phi_{2t}\psi_{t+1}$$

$$\frac{\partial}{\partial\phi_{1t}}:\alpha e^{z_t}A_t^{\alpha-1}h_t^{1-\alpha}+(1-\delta)=R_t$$

$$\frac{\partial}{\partial\phi_{2t}}:e^{z_t}A_t^{\alpha}h_t^{1-\alpha}+(1-\delta)A_t-C_t=\psi_{t+1}A_{t+1}$$

Substitute out Lagrange multipliers to yield

$$h_t=\left(\frac{(1-\alpha)}{\vartheta}e^{z_t}A_t^{\alpha}\right)^{\frac{1}{\upsilon+\alpha}}$$

$$\left(C_t-\frac{\vartheta h_t^{1+\upsilon}}{1+\upsilon}\right)^{-\sigma}=\beta\psi_{t+1}^{-\sigma}\left(C_{t+1}-\frac{\vartheta h_{t+1}^{1+\upsilon}}{1+\upsilon}\right)^{-\sigma}R_{t+1}$$

$$R_t=\alpha e^{z_t}A_t^{\alpha-1}h_t^{1-\alpha}+(1-\delta)$$

$$\psi_{t+1}A_{t+1}=e^{z_t}A_t^{\alpha}h_t^{1-\alpha}+(1-\delta)A_t-C_t$$

Note that because the labour allocation is the same as in the decentralized equilibrium, the values of $\bar{\rho},\bar{R},\bar{A}$ do not change.

3.C.2 Interest rates in Equilibrium with No trade

The Lagrangian can be written as

$$\sum_{t=0}^{\infty}\beta^{t}\left(\frac{\Psi_t^{1-\sigma}}{1-\sigma}\left(C_t-\frac{\vartheta}{1+\upsilon}h_t^{1+\upsilon}\right)^{1-\sigma}\right.$$
$$+\phi_{1t}\left(\alpha e^{z_t}A_t^{\alpha-1}h_t^{1-\alpha}\left(1-\left(\frac{\gamma}{R_t}\right)^{\lambda}\right)^{\alpha-1}+(1-\delta)-R_t\right)$$
$$\left.+\phi_{2t}\left(\begin{array}{c} e^{z_t}A_t^{\alpha}h_t^{1-\alpha}\left(1-\left(\frac{\gamma}{R_t}\right)^{\lambda}\right)^{\alpha}+(1-\delta)A_t \\ +(\gamma+\delta-1)A_t\left(\frac{\gamma}{R_t}\right)^{\lambda}-C_t-\psi_{t+1}A_{t+1}\end{array}\right)\right)$$

The first order conditions are

$$\frac{\partial}{\partial C_t}:\Psi_t^{1-\sigma}\left(C_t-\frac{\vartheta h_t^{1+\upsilon}}{1+\upsilon}\right)^{-\sigma}=\phi_{2t}$$

$$\frac{\partial}{\partial h_t}:h_t^{\upsilon+\alpha}=\frac{(1-\alpha)}{\vartheta}e^{z_t}A_t^{\alpha}\left(1-\left(\frac{\gamma}{R_t}\right)^{\lambda}\right)^{\alpha}\left(1+\alpha\lambda\frac{\left(\frac{\gamma}{R_t}\right)^{\lambda}}{\left(1-\left(\frac{\gamma}{R_t}\right)^{\lambda}\right)}\frac{(R_t-\gamma)}{R_t}\right)$$

$$\frac{\partial}{\partial R_t}:\phi_{1t}=\Psi_t^{1-\sigma}\left(C_t-\frac{\vartheta h_t^{1+\upsilon}}{1+\upsilon}\right)^{-\sigma}(R_t-\gamma)A_t\left(\frac{\gamma}{R_t}\right)^{\lambda}\frac{\lambda}{R_t}$$

$$\frac{\partial}{\partial A_{t+1}}:\beta\phi_{1t+1}(\alpha-1)\left(R_{t+1}-(1-\delta)\right)+\beta A_{t+1}\phi_{2t+1}\left(R_{t+1}+(\gamma-R_{t+1})\left(\frac{\gamma}{R_{t+1}}\right)^{\lambda}\right)-\phi_{2t}A_{t+1}\psi_{t+}$$

$$\frac{\partial}{\partial \phi_{1t}}:\alpha e^{z_t}A_t^{\alpha-1}h_t^{1-\alpha}\left(1-\left(\frac{\gamma}{R_t}\right)^{\lambda}\right)^{\alpha-1}=R_t-(1-\delta)$$

$$\frac{\partial}{\partial \phi_{2t}}:e^{z_t}A_t^{\alpha}h_t^{1-\alpha}\left(1-\left(\frac{\gamma}{R_t}\right)^{\lambda}\right)^{\alpha}+(1-\delta)A_t+(\gamma+\delta-1)A_t\left(\frac{\gamma}{R_t}\right)^{\lambda}-C_t-\psi_{t+1}A_{t+1}$$

They can be simplified to yield

$$h_t=\left(\frac{(1-\alpha)}{\vartheta}e^{z_t}A_t^{\alpha}\left(1-\left(\frac{\gamma}{R_t}\right)^{\lambda}\right)^{\alpha}\right)^{\frac{1}{\upsilon+\alpha}}\left(1+\alpha\lambda\frac{\left(\frac{\gamma}{R_t}\right)^{\lambda}}{\left(1-\left(\frac{\gamma}{R_t}\right)^{\lambda}\right)}\frac{(R_t-\gamma)}{R_t}\right)^{\frac{1}{\upsilon+\alpha}}$$

$$\left(C_t-\frac{\vartheta h_t^{1+\upsilon}}{1+\upsilon}\right)^{-\sigma}=\beta\psi_{t+1}^{-\sigma}\left(C_{t+1}-\frac{\vartheta h_{t+1}^{1+\upsilon}}{1+\upsilon}\right)^{-\sigma}\left(\begin{array}{c}R_{t+1}-(R_{t+1}-\gamma)\left(\frac{\gamma}{R_{t+1}}\right)^{\lambda} \\ \times\left(1+\lambda(1-\alpha)\frac{(R_{t+1}-(1-\delta))}{R_{t+1}}\right)\end{array}\right)$$

$$R_t = \alpha e^{z_t} A_t^{\alpha-1} h_t^{1-\alpha} \left(1 - \left(\frac{\gamma}{R_t}\right)^{\lambda}\right)^{\alpha-1} + (1-\delta)$$

$$\psi_{t+1} A_{t+1} = e^{z_t} A_t^{\alpha} h_t^{1-\alpha} \left(1 - \left(\frac{\gamma}{R_t}\right)^{\lambda}\right)^{\alpha} + (1-\delta) A_t + (\gamma + \delta - 1) A_t \left(\frac{\gamma}{R_t}\right)^{\lambda} - C_t$$

$$h_t = \left(\frac{(1-\alpha)}{\vartheta} e^{z_t} A_t^{\alpha} \left(1 - p_t^{\lambda}\right)^{\alpha}\right)^{\frac{1}{\upsilon+\alpha}} \left(1 + \alpha\lambda \frac{p_t^{\lambda}}{\left(1 - p_t^{\lambda}\right)} \frac{(R_t - \gamma)}{R_t}\right)^{\frac{1}{\upsilon+\alpha}}$$

$$\begin{aligned}
\left(C_t - \frac{\vartheta h_t^{1+\upsilon}}{1+\upsilon}\right)^{-\sigma} &= \beta \psi_{t+1}^{-\sigma} \left(C_{t+1} - \frac{\vartheta h_{t+1}^{1+\upsilon}}{1+\upsilon}\right)^{-\sigma} \\
&\quad \times \left(R_{t+1} - (R_{t+1} - \gamma) p_{t+1}^{\lambda} \left(1 + \lambda (1-\alpha) \frac{(R_{t+1} - (1-\delta))}{R_{t+1}}\right)\right)
\end{aligned}$$

$$R_t = \alpha e^{z_t} A_t^{\alpha-1} h_t^{1-\alpha} \left(1 - p_t^{\lambda}\right)^{\alpha-1} + (1-\delta)$$

$$\psi_{t+1} A_{t+1} = e^{z_t} A_t^{\alpha} h_t^{1-\alpha} \left(1 - p_t^{\lambda}\right)^{\alpha} + (1-\delta) A_t + (\gamma + \delta - 1) A_t p_t^{\lambda} - C_t$$

$$p_t = \frac{\gamma}{R_t}$$

3.D Appendix: Note on Numerical Solution

The standard deterministic RBC model can be written in three equations

$$C_t + \psi A_{t+1} = \left(\frac{(1-\alpha)}{\vartheta}\right)^{\frac{1-\alpha}{\upsilon+\alpha}} A_t^{\alpha \frac{\upsilon+1}{\alpha+\upsilon}} + (1-\delta) A_t$$

$$r_t = \alpha \left(\frac{1-\alpha}{\vartheta}\right)^{\frac{1-\alpha}{\upsilon+\alpha}} A_t^{\upsilon \frac{\alpha-1}{\alpha+\upsilon}} + (1-\delta)$$

$$\left(C_t - \frac{\vartheta}{1+\upsilon} \left(\frac{(1-\alpha)}{\vartheta}\right)^{\frac{1+\upsilon}{\upsilon+\alpha}} A_t^{\alpha \frac{1+\upsilon}{\upsilon+\alpha}}\right)^{-\sigma} = \beta \psi^{-\sigma} \left(C_{t+1} - \frac{\vartheta}{1+\upsilon} \left(\frac{(1-\alpha)}{\vartheta}\right)^{\frac{1+\upsilon}{\upsilon+\alpha}} A_{t+1}^{\alpha \frac{1+\upsilon}{\upsilon+\alpha}}\right)^{-\sigma} r_{t+1}$$

or 2 equations

$$C_t + \psi A_{t+1} = \left(\frac{(1-\alpha)}{\vartheta}\right)^{\frac{1-\alpha}{\upsilon+\alpha}} A_t^{\alpha\frac{\upsilon+1}{\alpha+\upsilon}} + (1-\delta) A_t$$

$$\begin{aligned} & \left(C_t - \frac{\vartheta}{1+\upsilon}\left(\frac{(1-\alpha)}{\vartheta}\right)^{\frac{1+\upsilon}{\upsilon+\alpha}} A_t^{\alpha\frac{1+\upsilon}{\upsilon+\alpha}}\right)^{-\sigma} \\ = \; & \beta\psi^{-\sigma}\left(C_{t+1} - \frac{\vartheta}{1+\upsilon}\left(\frac{(1-\alpha)}{\vartheta}\right)^{\frac{1+\upsilon}{\upsilon+\alpha}} A_{t+1}^{\alpha\frac{1+\upsilon}{\upsilon+\alpha}}\right)^{-\sigma}\left(\alpha\left(\frac{1-\alpha}{\vartheta}\right)^{\frac{1-\alpha}{\upsilon+\alpha}} A_{t+1}^{\upsilon\frac{\alpha-1}{\alpha+\upsilon}} + (1-\delta)\right) \end{aligned}$$

The only state is A_t

Construct grid on $A_t : [A_{\min}, A_{\max}]$

For each node (known) A_t apply the following algorithm

Solution algorithm 1:

Guess solution i

$$C_t^i = c^i(A_t)$$

Compute update

$$A_{t+1}^i = \left(\frac{(1-\alpha)}{\vartheta}\right)^{\frac{1-\alpha}{\upsilon+\alpha}} A_t^{\alpha\frac{\upsilon+1}{\alpha+\upsilon}} + (1-\delta) A_t - C_t^i$$

$$C_{t+1}^i = c^i\left(A_{t+1}^i\right)$$

Use the last equation to update the guess

$$
\begin{aligned}
C_t^{i+1} &= \frac{\vartheta}{1+\upsilon}\left(\frac{(1-\alpha)}{\vartheta}\right)^{\frac{1+\upsilon}{\upsilon+\alpha}}(A_t)^{\alpha\frac{1+\upsilon}{\upsilon+\alpha}} \\
&+\left(C_{t+1}^i-\frac{\vartheta}{1+\upsilon}\left(\frac{(1-\alpha)}{\vartheta}\right)^{\frac{1+\upsilon}{\upsilon+\alpha}}\left(A_{t+1}^i\right)^{\alpha\frac{1+\upsilon}{\upsilon+\alpha}}\right) \\
&\times\left(\beta\psi^{-\sigma}\left(\alpha\left(\frac{1-\alpha}{\vartheta}\right)^{\frac{1-\alpha}{\upsilon+\alpha}}A_{t+1}^{i\upsilon\frac{\alpha-1}{\alpha+\upsilon}}+(1-\delta)\right)\right)^{-\frac{1}{\sigma}}
\end{aligned}
$$

Now, we know C_t^{i+1} and A_t, we can update function $c^{i+1}()$ by approximating C_t^{i+1} by a polynomial to get $C_t^{i+1}=c^{i+1}(A_t)$

We then compute

$$A_{t+1}^{i+1}=\left(\frac{(1-\alpha)}{\vartheta}\right)^{\frac{1-\alpha}{\upsilon+\alpha}}A_t^{\alpha\frac{\upsilon+1}{\alpha+\upsilon}}+(1-\delta)A_t-C_t^{i+1}$$

$$C_{t+1}^{i+1}=c^{i+1}\left(A_{t+1}^{i+1}\right)$$

and so on.

Solution Algorithm 2:

Guess solution i:

$$A_{t+1}^i=a^i(A_t)$$

Use it to find

$$A_{t+2}^i=a^i\left(A_{t+1}^i\right)$$

Compute

$$C_{t+1}^i=\left(\frac{(1-\alpha)}{\vartheta}\right)^{\frac{1-\alpha}{\upsilon+\alpha}}A_{t+1}^{i\alpha\frac{\upsilon+1}{\alpha+\upsilon}}+(1-\delta)A_{t+1}^i-\psi A_{t+2}^i$$

compute

$$
\begin{aligned}
C_t^i &= \frac{\vartheta}{1+\upsilon}\left(\frac{(1-\alpha)}{\vartheta}\right)^{\frac{1+\upsilon}{\upsilon+\alpha}} A_t^{\alpha\frac{1+\upsilon}{\upsilon+\alpha}} \\
&+ \left(C_{t+1}^i - \frac{\vartheta}{1+\upsilon}\left(\frac{(1-\alpha)}{\vartheta}\right)^{\frac{1+\upsilon}{\upsilon+\alpha}} A_{t+1}^{i\alpha\frac{1+\upsilon}{\upsilon+\alpha}}\right) \\
&\times \left(\beta\psi^{-\sigma}\left(\alpha\left(\frac{1-\alpha}{\vartheta}\right)^{\frac{1-\alpha}{\upsilon+\alpha}} A_{t+1}^{i\upsilon\frac{\alpha-1}{\alpha+\upsilon}} + (1-\delta)\right)\right)^{-\frac{1}{\sigma}}
\end{aligned}
$$

Update assets:

$$
A_{t+1}^{i+1} = \frac{1}{\psi}\left(\left(\frac{(1-\alpha)}{\vartheta}\right)^{\frac{1-\alpha}{\upsilon+\alpha}} A_t^{\alpha\frac{\upsilon+1}{\alpha+\upsilon}} + (1-\delta)A_t - C_t^i\right)
$$

So we can update $a^{i+1}()$ by a polynomial to

$$
A_{t+1}^{i+1} = a^{i+1}(A_t)
$$

Figure 3.6 plots deterministic solutions as functions of assets.

3.E Appendix: Optimal State Contingent Tax on Savings

3.E.1 Trade

If the interbank market open then the decentralized equilibrium can be described by the following system ($\Theta_t = 1$):

$$
C_t + A_{t+1}\psi_{t+1} = e^{\frac{\upsilon+1}{\alpha+\upsilon}z_t}\left(\frac{(1-\alpha)}{\vartheta}\right)^{\frac{1-\alpha}{\upsilon+\alpha}} A_t^{\alpha\frac{\upsilon+1}{\alpha+\upsilon}} + (1-\delta)A_t
$$

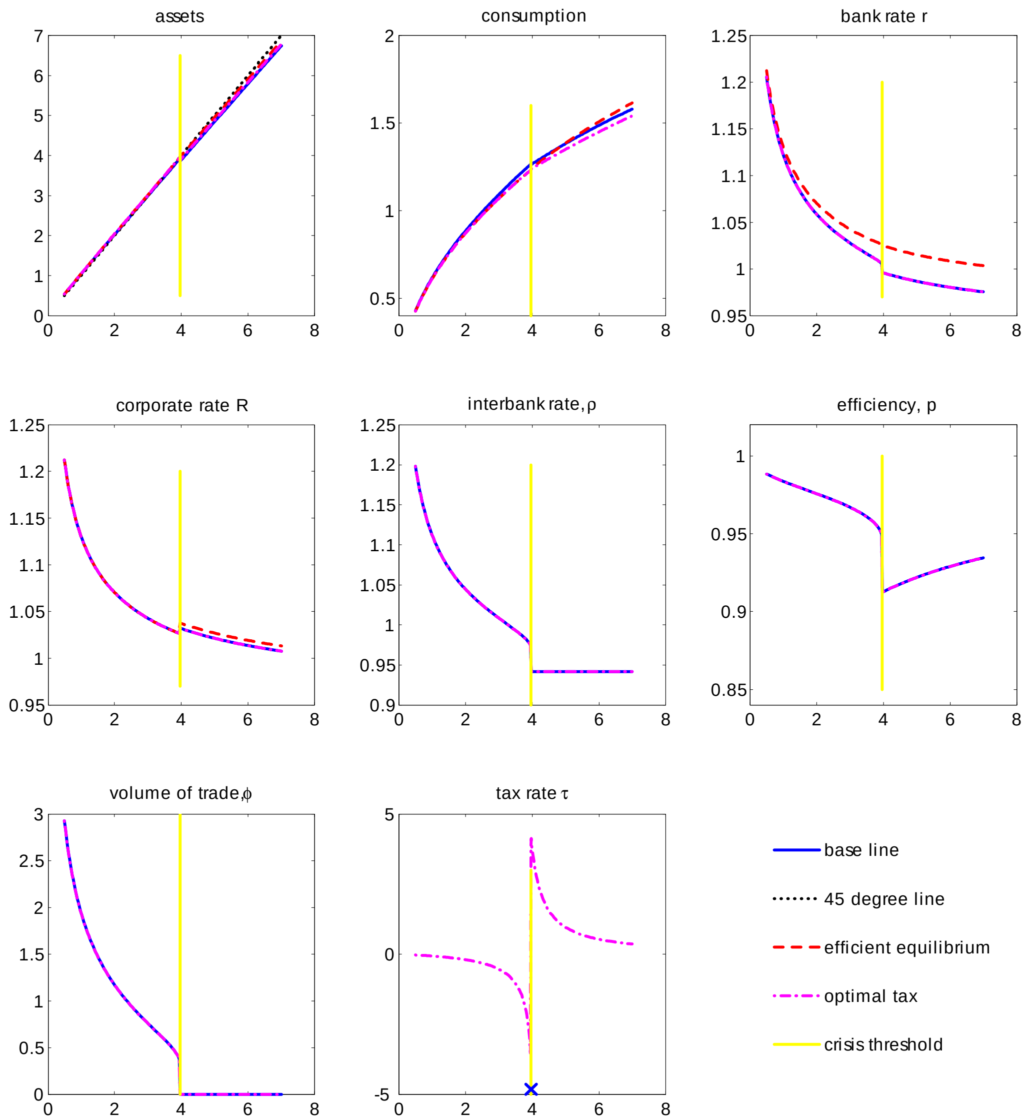

Figure 3.6: Deterministic solution on a grid

$$R_t = \alpha \left(\frac{1-\alpha}{\vartheta} \right)^{\frac{1-\alpha}{\upsilon+\alpha}} A_t^{\upsilon \frac{\alpha-1}{\alpha+\upsilon}} e^{\frac{1+\upsilon}{\upsilon+\alpha} z_t} + (1-\delta)$$

$$U_t = C_t - \frac{\vartheta}{1+\upsilon} \left(\frac{(1-\alpha)}{\vartheta} e^{z_t} A_t^{\alpha} \right)^{\frac{1+\upsilon}{\upsilon+\alpha}}$$

$$U_t^{-\sigma} = \beta \psi_{t+1}^{-\sigma} \left(U_{t+1}\right)^{-\sigma} \left(1 + \left(r_{t+1} - 1\right)\left(1 - \tau_{t+1}\right)\right)$$

$$r_t = R_t \frac{\lambda}{(\lambda+1)} \frac{\left(1-\bar{p}_t^{\lambda+1}\right)}{\left(1-\bar{p}_t^{\lambda}\right)}$$

$$\bar{p}_t R_t - \gamma = \bar{p}_t^{\lambda} \left(\gamma \left(\theta - 1\right) + \bar{p}_t R_t\right)$$

$$\phi_t = \frac{\bar{p}_t^{\lambda}}{\left(1-\bar{p}_t^{\lambda}\right)}$$

$$\rho_t = \bar{p}_t R_t$$

The central planner decides on τ_{t+1}, but because it only enters the consumption Euler equation, we assume that the policymaker choose the optimal consumption plan, and then the tax can be chosen consistent with the plan.

We, therefore, form the following Lagrangian:

$$\sum_{t=0}^{\infty} \beta^t \left(\frac{1}{1-\sigma} \left(C_t - \frac{\vartheta}{1+\upsilon} \left(\frac{(1-\alpha)}{\vartheta} e^{z_t} \left(A_t\right)^{\alpha} \right)^{\frac{1+\upsilon}{\upsilon+\alpha}} \right)^{1-\sigma} \right.$$
$$\left. +\phi_{1t} \left(e^{\frac{\upsilon+1}{\alpha+\upsilon} z_t} \left(\frac{(1-\alpha)}{\vartheta} \right)^{\frac{1-\alpha}{\upsilon+\alpha}} A_t^{\alpha \frac{\upsilon+1}{\alpha+\upsilon}} + (1-\delta) A_t - C_t - A_{t+1} \psi \right) \right)$$

other variables can be found as functions of C_t, A_t.

The FOCs are:

$$
\begin{aligned}
\frac{\partial}{\partial A_{t+1}} \;&:\; \beta\left(C_{t+1}-\frac{\vartheta}{1+\upsilon}\left(\frac{(1-\alpha)}{\vartheta}e^{z_{t+1}}\left(A_{t+1}\right)^{\alpha}\right)^{\frac{1+\upsilon}{\upsilon+\alpha}}\right)^{-\sigma} \\
&:\; \times\left(-\frac{(1-\alpha)}{\upsilon+\alpha}\left(\frac{(1-\alpha)}{\vartheta}e^{z_{t+1}}\left(A_{t+1}\right)^{\alpha}\right)^{\frac{1+\upsilon}{\upsilon+\alpha}-1}\alpha e^{z_{t+1}}\left(A_{t+1}\right)^{\alpha-1}\right) \\
&:\; -\phi_{1t}\psi+\beta\phi_{1t+1}\left(\alpha\frac{\upsilon+1}{\alpha+\upsilon}e^{\frac{\upsilon+1}{\alpha+\upsilon}z_{t+1}}\left(\frac{(1-\alpha)}{\vartheta}\right)^{\frac{1-\alpha}{\upsilon+\alpha}}A_{t+1}^{\alpha\frac{\upsilon+1}{\alpha+\upsilon}-1}+(1-\delta)\right)
\end{aligned}
$$

$$\frac{\partial}{\partial C_t}:\left(C_t-\frac{\vartheta}{1+\upsilon}\left(\frac{(1-\alpha)}{\vartheta}e^{z_t}\left(A_t\right)^{\alpha}\right)^{\frac{1+\upsilon}{\upsilon+\alpha}}\right)^{-\sigma}-\phi_{1t}$$

$$\frac{\partial}{\partial \phi_{1t}}:e^{\frac{\upsilon+1}{\alpha+\upsilon}z_t}\left(\frac{(1-\alpha)}{\vartheta}\right)^{\frac{1-\alpha}{\upsilon+\alpha}}A_t^{\alpha\frac{\upsilon+1}{\alpha+\upsilon}}+(1-\delta)A_t-C_t-A_{t+1}\psi$$

From where, substituting out Lagrange multipliers, we get the system which describes the evolution of the economy *under optimal taxation:*

$$
\begin{aligned}
\left(C_t-\frac{\vartheta}{1+\upsilon}h_t^{1+\upsilon}\right)^{-\sigma} &= \beta\psi_{t+1}^{-\sigma}\left(C_{t+1}-\frac{\vartheta}{1+\upsilon}h_{t+1}^{1+\upsilon}\right)^{-\sigma} \\
&\quad\times\left(\alpha\left(\frac{(1-\alpha)}{\vartheta}\right)^{\frac{1-\alpha}{\upsilon+\alpha}}e^{\frac{\upsilon+1}{\alpha+\upsilon}z_{t+1}}A_{t+1}^{\upsilon\frac{\alpha-1}{\alpha+\upsilon}}+(1-\delta)\right)
\end{aligned}
\tag{3.34}
$$

$$h_t=\left(\frac{(1-\alpha)}{\vartheta}e^{z_t}A_t^{\alpha}\right)^{\frac{1}{\upsilon+\alpha}} \tag{3.35}$$

$$A_{t+1}\psi_{t+1}=e^{\frac{\upsilon+1}{\alpha+\upsilon}z_t}\left(\frac{(1-\alpha)}{\vartheta}\right)^{\frac{1-\alpha}{\upsilon+\alpha}}A_t^{\alpha\frac{\upsilon+1}{\alpha+\upsilon}}+(1-\delta)A_t-C_t \tag{3.36}$$

We now find the optimal tax rate, consistent with this dynamics of the economy. Compare equation (3.34) with (3.28). They must be identical for consistency of the model. This condition gives us the equation for optimal *future* taxes.

$$1+(R_{t+1}\frac{\lambda}{(\lambda+1)}\frac{\left(1-\bar{p}_{t+1}^{\lambda+1}\right)}{\left(1-\bar{p}_{t+1}^{\lambda}\right)}-1)\left(1-\tau_{t+1}\right)=\alpha\left(\frac{(1-\alpha)}{\vartheta}\right)^{\frac{1-\alpha}{\upsilon+\alpha}}e^{\frac{\upsilon+1}{\alpha+\upsilon}z_{t+1}}A_{t+1}^{\upsilon\frac{\alpha-1}{\alpha+\upsilon}}+(1-\delta)=R_{t+1}$$

from where the *future* tax is

$$
\begin{aligned}
\tau_{t+1} &= 1-\frac{\left(R_{t+1}-1\right)}{\left(R_{t+1}\frac{\lambda}{(\lambda+1)}\frac{\left(1-\bar{p}_{t+1}^{\lambda+1}\right)}{\left(1-\bar{p}_{t+1}^{\lambda}\right)}-1\right)} \\
&= 1-\frac{\left(R_{t+1}-1\right)\left(1-\bar{p}_{t+1}^{\lambda}\right)}{\left(R_{t+1}\frac{\lambda}{(\lambda+1)}\left(1-\bar{p}_{t+1}^{\lambda+1}\right)-\left(1-\bar{p}_{t+1}^{\lambda}\right)\right)} \\
&= R_{t+1}\frac{\left(\lambda\left(1-\bar{p}_{t+1}^{\lambda+1}\right)-(\lambda+1)\left(1-\bar{p}_{t+1}^{\lambda}\right)\right)}{\left(R_{t+1}\lambda\left(1-\bar{p}_{t+1}^{\lambda+1}\right)-(\lambda+1)\left(1-\bar{p}_{t+1}^{\lambda}\right)\right)}
\end{aligned}
$$

it depends on $\bar{p}_t$. Recall that $\bar{p}_t^{\lambda}=\frac{R_t\bar{p}_t-\gamma}{(\theta\gamma+R_t\bar{p}_t-\gamma)}$ so we can simplify

$$
\begin{aligned}
\tau_{t+1} &= R_{t+1}\frac{\left(\lambda\left(1-\bar{p}_{t+1}\frac{R_{t+1}\bar{p}_{t+1}-\gamma}{(\theta\gamma+R_{t+1}\bar{p}_{t+1}-\gamma)}\right)-(\lambda+1)\left(1-\frac{R_{t+1}\bar{p}_{t+1}-\gamma}{(\theta\gamma+R_{t+1}\bar{p}_{t+1}-\gamma)}\right)\right)}{\left(R_{t+1}\lambda\left(1-\bar{p}_{t+1}\frac{R_{t+1}\bar{p}_{t+1}-\gamma}{(\theta\gamma+R_{t+1}\bar{p}_{t+1}-\gamma)}\right)-(\lambda+1)\left(1-\frac{R_{t+1}\bar{p}_{t+1}-\gamma}{(\theta\gamma+R_{t+1}\bar{p}_{t+1}-\gamma)}\right)\right)} \\
&= R_{t+1}\frac{\theta\gamma+\lambda\left(1-p_{t+1}\right)\left(\gamma-R_{t+1}p_{t+1}\right)}{\theta\gamma\left(\lambda\left(1-R_{t+1}\right)+1\right)+R_{t+1}\lambda\left(1-p_{t+1}\right)\left(\gamma-R_{t+1}p_{t+1}\right)}
\end{aligned}
$$

At the boundary

$$
\bar{\tau}=\frac{\bar{R}\left(\theta\gamma+\lambda\left(1-p_{\min}\right)\left(\gamma-\bar{R}p_{\min}\right)\right)}{\theta\gamma\left(\lambda\left(1-\bar{R}\right)+1\right)+\bar{R}\lambda\left(1-p_{\min}\right)\left(\gamma-\bar{R}p_{\min}\right)}
$$

and the denominator is not zero, which can be checked numerically.

3.E.2 No Trade

If the interbank market is shut, the decentralized equilibrium can be described by the following system ($\Theta_t=1$):

$$
C_t+\psi_{t+1}A_{t+1}=\left(\left(\frac{(1-\alpha)}{\vartheta}\right)^{\frac{1-\alpha}{\upsilon+1}}e^{z_t}A_t^{\alpha}\left(1-p_t^{\lambda}\right)^{\alpha}\right)^{\frac{\upsilon+1}{\alpha+\upsilon}}+\left(1-\delta-\gamma\right)A_t\left(1-p_t^{\lambda}\right)+\gamma A_t
$$

$$R_t = \alpha\left(\frac{1-\alpha}{\vartheta}\right)^{\frac{1-\alpha}{\upsilon+\alpha}}\left(A_t\left(1-p_t^{\lambda}\right)\right)^{\upsilon\frac{\alpha-1}{\alpha+\upsilon}} e^{\frac{1+\upsilon}{\upsilon+\alpha}z_t} + (1-\delta)$$

$$\begin{aligned}
&\left(C_t - \frac{\vartheta}{1+\upsilon}\left(\frac{(1-\alpha)}{\vartheta}e^{z_t}\left(A_t\left(1-\bar{p}_t^{\lambda}\right)\right)^{\alpha}\right)^{\frac{1+\upsilon}{\upsilon+\alpha}}\right)^{-\sigma} && (3.37)\\
= \;& \beta\psi_{t+1}^{-\sigma}\left(C_{t+1} - \frac{\vartheta}{1+\upsilon}\left(\frac{(1-\alpha)}{\vartheta}e^{z_{t+1}}\left(A_{t+1}\left(1-\bar{p}_{t+1}^{\lambda}\right)\right)^{\alpha}\right)^{\frac{1+\upsilon}{\upsilon+\alpha}}\right)^{-\sigma} && (3.38)\\
&\times\left(1+\left(r_{t+1}-1\right)\left(1-\tau_{t+1}\right)\right)
\end{aligned}$$

$$r_t = R_t\left(\frac{\lambda+\bar{p}_t^{\lambda+1}}{\lambda+1}\right)$$

$$\bar{p}_t = \frac{\gamma}{R_t}$$

As before, the central planner decides on τ_{t+1}, but because it only enters the consumption Euler equation, we assume that the policymaker choose the optimal consumption plan, and then the tax can be chosen consistent with the plan.

We, therefore, form the following Lagrangian:

$$\begin{aligned}
&\sum_{t=0}^{\infty}\beta^t\left(\frac{\Psi_t^{1-\sigma}}{1-\sigma}\left(C_t - \frac{\vartheta}{1+\upsilon}\left(\frac{(1-\alpha)}{\vartheta}e^{z_t}\left[A_t^{\alpha}\left(1-\bar{p}_t^{\lambda}\right)^{\alpha}\right]\right)^{\frac{1+\upsilon}{\upsilon+\alpha}}\right)^{1-\sigma}\right.\\
&+\phi_{1t}\left(\begin{array}{c}\left(\left(\frac{(1-\alpha)}{\vartheta}\right)^{\frac{1-\alpha}{\upsilon+1}}e^{z_t}A_t^{\alpha}\left(1-\bar{p}_t^{\lambda}\right)^{\alpha}\right)^{\frac{\upsilon+1}{\alpha+\upsilon}} + (1-\delta)A_t\\ -(1-\delta-\gamma)A_t\bar{p}_t^{\lambda} - C_t - A_{t+1}\psi_{t+1}\end{array}\right)\\
&\left.+\phi_{2t}\left((1-\delta)+\alpha\left(\frac{1-\alpha}{\vartheta}\right)^{\frac{1-\alpha}{\upsilon+\alpha}}e^{\frac{1+\upsilon}{\upsilon+\alpha}z_t}\left(A_t\left(1-\bar{p}_t^{\lambda}\right)\right)^{\upsilon\frac{\alpha-1}{\alpha+\upsilon}} - R_t\right) + \phi_{4t}\left(R_t\bar{p}_t-\gamma\right)\right)
\end{aligned}$$

where we do not include equation for r_t as constraints, as it can be found as functions of $C_t, A_t, R_t, \bar{p}_t$.

The FOCs are:

$$\frac{\partial}{\partial R_t} : -\phi_{2t} + \phi_{4t} p_t$$

$$\begin{aligned}
\frac{\partial}{\partial A_{t+1}} \quad : \quad & -\phi_{1t}\psi_{t+1} - \frac{1+\upsilon}{\upsilon+\alpha}\frac{\vartheta}{1+\upsilon}\left(\frac{(1-\alpha)}{\vartheta}e^{z_{t+1}}A_{t+1}^{\alpha}\left(1-p_{t+1}^{\lambda}\right)^{\alpha}\right)^{\frac{1+\upsilon}{\upsilon+\alpha}-1} \\
& \times\beta\Psi_{t+1}^{1-\sigma}\left(C_{t+1} - \frac{\vartheta}{1+\upsilon}\left(\frac{(1-\alpha)}{\vartheta}e^{z_{t+1}}A_{t+1}^{\alpha}\left(1-p_{t+1}^{\lambda}\right)^{\alpha}\right)^{\frac{1+\upsilon}{\upsilon+\alpha}}\right)^{-\sigma} \\
& \times\frac{(1-\alpha)\,\alpha}{\vartheta}e^{z_{t+1}}A_{t+1}^{\alpha-1}\left(1-p_{t+1}^{\lambda}\right)^{\alpha} \\
& +\beta\phi_{1t+1}\left(\begin{array}{c} \alpha\frac{\upsilon+1}{\alpha+\upsilon}e^{\frac{\upsilon+1}{\alpha+\upsilon}z_{t+1}}\left(\frac{(1-\alpha)}{\vartheta}\right)^{\frac{1-\alpha}{\upsilon+\alpha}}A_{t+1}^{\alpha\frac{\upsilon+1}{\alpha+\upsilon}-1}\left(1-p_{t+1}^{\lambda}\right)^{\alpha\frac{\upsilon+1}{\alpha+\upsilon}} \\ +(1-\delta)-(1-\delta-\gamma)\,p_{t+1}^{\lambda} \end{array}\right) \\
& +\beta\phi_{2t+1}\left(\alpha\upsilon\frac{\alpha-1}{\alpha+\upsilon}\left(\frac{1-\alpha}{\vartheta}\right)^{\frac{1-\alpha}{\upsilon+\alpha}}e^{\frac{1+\upsilon}{\upsilon+\alpha}z_{t+1}}A_{t+1}^{\upsilon\frac{\alpha-1}{\alpha+\upsilon}-1}\left(1-p_{t+1}^{\lambda}\right)^{\upsilon\frac{\alpha-1}{\alpha+\upsilon}}\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial p_t} \quad : \quad & -\frac{1+\upsilon}{\upsilon+\alpha}\frac{\vartheta}{1+\upsilon}\left(-\lambda p_t^{\lambda-1}\right)\left(\frac{(1-\alpha)}{\vartheta}e^{z_t}A_t^{\alpha}\left(1-p_t^{\lambda}\right)^{\alpha}\right)^{\frac{1+\upsilon}{\upsilon+\alpha}-1}\Psi_t^{1-\sigma}\frac{(1-\alpha)\,\alpha}{\vartheta}e^{z_t} \\
& \times A_t^{\alpha}\left(1-p_t^{\lambda}\right)^{\alpha-1}\left(C_t - \frac{\vartheta}{1+\upsilon}\left(\frac{(1-\alpha)}{\vartheta}e^{z_t}A_t^{\alpha}\left(1-p_t^{\lambda}\right)^{\alpha}\right)^{\frac{1+\upsilon}{\upsilon+\alpha}}\right)^{-\sigma} \\
& +\phi_{1t}\left(\begin{array}{c} \left(-\lambda p_t^{\lambda-1}\right)\alpha\frac{\upsilon+1}{\alpha+\upsilon}e^{\frac{\upsilon+1}{\alpha+\upsilon}z_t}\left(\frac{(1-\alpha)}{\vartheta}\right)^{\frac{1-\alpha}{\upsilon+\alpha}}A_t^{\alpha\frac{\upsilon+1}{\alpha+\upsilon}}\left(1-p_t^{\lambda}\right)^{\alpha\frac{\upsilon+1}{\alpha+\upsilon}-1} \\ -(1-\delta-\gamma)\,\lambda A_t p_t^{\lambda-1} \end{array}\right) \\
& +\phi_{2t}\left(\left(-\lambda p_t^{\lambda-1}\right)\upsilon\frac{\alpha-1}{\alpha+\upsilon}\alpha\left(\frac{1-\alpha}{\vartheta}\right)^{\frac{1-\alpha}{\upsilon+\alpha}}e^{\frac{1+\upsilon}{\upsilon+\alpha}z_t}A_t^{\upsilon\frac{\alpha-1}{\alpha+\upsilon}}\left(1-p_t^{\lambda}\right)^{\upsilon\frac{\alpha-1}{\alpha+\upsilon}-1}\right) + \phi_{4t}R_t
\end{aligned}$$

$$\frac{\partial}{\partial C_t} : \Psi_t^{1-\sigma}\left(C_t - \frac{\vartheta}{1+\upsilon}\left(\frac{(1-\alpha)}{\vartheta}e^{z_t}\left[A_t^{\alpha}\left(1-p_t^{\lambda}\right)^{\alpha}\right]\right)^{\frac{1+\upsilon}{\upsilon+\alpha}}\right)^{-\sigma} - \phi_{1t}$$

$$\frac{\partial}{\partial\phi_{1t}} : e^{\frac{\upsilon+1}{\alpha+\upsilon}z_t}\left(\frac{(1-\alpha)}{\vartheta}\right)^{\frac{1-\alpha}{\upsilon+\alpha}}A_t^{\alpha\frac{\upsilon+1}{\alpha+\upsilon}}\left(1-\bar{p}_t^{\lambda}\right)^{\alpha\frac{\upsilon+1}{\alpha+\upsilon}} + (1-\delta)\,A_t - (1-\delta-\gamma)\,A_t\bar{p}_t^{\lambda} - C_t - A_{t+1}\psi_{t+1}$$

$$\frac{\partial}{\partial\phi_{2t}} : (1-\delta) + \alpha\left(\frac{1-\alpha}{\vartheta}\right)^{\frac{1-\alpha}{\upsilon+\alpha}}e^{\frac{1+\upsilon}{\upsilon+\alpha}z_t}A_t^{\upsilon\frac{\alpha-1}{\alpha+\upsilon}}\left(1-\bar{p}_t^{\lambda}\right)^{\upsilon\frac{\alpha-1}{\alpha+\upsilon}} - R_t$$

$$\frac{\partial}{\partial\phi_{4t}} : R_t\bar{p}_t - \gamma$$

From where, substituting out Lagrange multipliers, we get the system which describes the evolution of the economy *under optimal taxation:*

$$h_t = \left(\frac{(1-\alpha)}{\vartheta} e^{z_t} A_t^{\alpha} \left(1-\bar{p}_t^{\lambda}\right)^{\alpha} \right)^{\frac{1}{v+\alpha}}$$

$$\begin{aligned}
\left(C_t - \frac{\vartheta h_t^{1+v}}{1+v} \right)^{-\sigma} &= \beta \psi_{t+1}^{-\sigma} \left(C_{t+1} - \frac{\vartheta h_{t+1}^{1+v}}{1+v} \right)^{-\sigma} \\
&\times \left(\frac{\left(1-p_{t+1}^{\lambda}\right)\left(R_{t+1}-\gamma\right) R_{t+1}}{\left(R_{t+1} + (1-\alpha)\lambda \left(R_{t+1} - (1-\delta)\right) \frac{p_{t+1}^{\lambda}}{\left(1-p_{t+1}^{\lambda}\right)} \frac{v}{\alpha+v} \right)} + \gamma \right)
\end{aligned}$$

$$A_{t+1}\psi_{t+1} = e^{\frac{v+1}{\alpha+v} z_t} \left(\frac{(1-\alpha)}{\vartheta} \right)^{\frac{1-\alpha}{v+\alpha}} A_t^{\alpha \frac{v+1}{\alpha+v}} \left(1-p_t^{\lambda}\right)^{\alpha \frac{v+1}{\alpha+v}} + (1-\delta) A_t - (1-\delta-\gamma) A_t p_t^{\lambda} - C_t$$

$$R_t = \alpha e^{z_t} A_t^{\alpha-1} \left(1-p_t^{\lambda}\right)^{\alpha-1} h_t^{1-\alpha} + (1-\delta) \qquad (3.39)$$

$$R_t p_t = \gamma \qquad (3.40)$$

We now find the optimal tax rate, consistent with this dynamics of the economy. Compare equation (3.24) with (3.28). They must be identical for consistency of the model. This condition gives us the equation for optimal *future* taxes.

$$1 + \left(R_{t+1} \left(\frac{\lambda + \bar{p}_{t+1}^{\lambda+1}}{\lambda+1} \right) - 1 \right) (1-\tau_{t+1}) = \frac{\left(1-p_{t+1}^{\lambda}\right)\left(R_{t+1}-\gamma\right) R_{t+1}}{\left(R_{t+1} - \left(R_{t+1} - (1-\delta)\right) \lambda \frac{p_{t+1}^{\lambda}}{\left(1-p_{t+1}^{\lambda}\right)} v \frac{\alpha-1}{\alpha+v} \right)} + \gamma$$

from where the future tax is

$$\tau_{t+1} = 1 - \frac{\frac{\left(1-p_{t+1}^{\lambda}\right)\left(R_{t+1}-\gamma\right) R_{t+1}}{\left(R_{t+1} - \left(R_{t+1}-(1-\delta)\right)\lambda \frac{p_{t+1}^{\lambda}}{\left(1-p_{t+1}^{\lambda}\right)} v \frac{\alpha-1}{\alpha+v} \right)} + \gamma - 1}{\left(R_{t+1} \left(\frac{\lambda+\bar{p}_{t+1}^{\lambda+1}}{\lambda+1} \right) - 1 \right)}$$

it depends on $\bar{p}_t$. Here $p_t^\lambda = \left(\frac{\gamma}{R_t}\right)^\lambda$. Simple algebra yields

$$\tau_{t+1} = \frac{R_{t+1}\left(\frac{\lambda+\bar{p}_{t+1}^{\lambda+1}}{\lambda+1}\right)-\gamma}{\left(R_{t+1}\left(\frac{\lambda+\bar{p}_{t+1}^{\lambda+1}}{\lambda+1}\right)-1\right)} - \frac{\left(1-p_{t+1}^\lambda\right)\left(R_{t+1}-\gamma\right)R_{t+1}}{\left(R_{t+1}-\left(R_{t+1}-(1-\delta)\right)\lambda\frac{p_{t+1}^\lambda}{\left(1-p_{t+1}^\lambda\right)}\upsilon\frac{\alpha-1}{\alpha+\upsilon}\right)\left(R_{t+1}\left(\frac{\lambda+\bar{p}_{t+1}^{\lambda+1}}{\lambda+1}\right)-1\right)}$$

3.F Appendix: Approximation of functions by Chebyshev polynomials

3.F.1 Orthogonality

Denote M=nodes, N=degree.

Chebyshev polynomials $T_p(x_i)$ and $T_q(x_i)$ of different polynomial degrees p and q have the following properties:

$$\sum_{i=1}^{M_x} T_p(x_i)\,T_q(x_i) = \begin{cases} 0 & p \neq q \\ \frac{M}{2} & p = q \neq 0 \\ M & p = q = 0 \end{cases}$$

3.F.2 One-dimensional

Suppose we want to find coefficients θ_p in the following representation of function $f(x_i)$

$$f(x_i) = \sum_{p=0}^{N_x} \theta_p T_p(x_i)$$

Note that

$$f(x_i)\,T_s(x_i) = \sum_{p=0}^{N_x} \theta_p T_p(x_i)\,T_s(x_i)$$

take the sum

$$\sum_{i=1}^{M_x} f(x_i)\,T_s(x_i) = \sum_{i=1}^{M_x}\sum_{p=0}^{N_x} \theta_p T_p(x_i)\,T_s(x_i)$$

use the orthogonality property

$$\begin{aligned}\sum_{i=1}^{M_x} f(x_i)\,T_s(x_i) &= \sum_{p=0}^{N_x}\theta_p \sum_{i=1}^{M_x} T_p(x_i)\,T_s(x_i) = \sum_{p=0}^{N_x}\theta_p \sum_{i=1}^{M_x} T_p(x_i)\,T_s(x_i) \\ &= \begin{cases} \frac{M_x}{2}\theta_s & s \neq 0 \\ M_x\theta_0 & s = 0 \end{cases}\end{aligned}$$

we obtain

$$\begin{aligned}\theta_0 &= \frac{1}{M_x}\sum_{i=1}^{M_x} f(x_i)\,T_0(x_i) = \frac{1}{M_x}\sum_{i=1}^{M_x} f(x_i) \\ \theta_s &= \frac{2}{M_x}\sum_{i=1}^{M_x} f(x_i)\,T_s(x_i)\end{aligned}$$

3.F.3 Two-dimensional

Similarly, for a two-dimensional function

$$f(x_i, y_j) = \sum_{p=0}^{N_x}\sum_{q=0}^{N_y} \theta_{pq} T_p(x_i)\,T_q(y_i)$$

we can write

$$
\begin{aligned}
\sum_{j=1}^{M_y}\sum_{i=1}^{M_x} f\left(x_i, y_j\right) T_s\left(x_i\right) T_u\left(y_i\right) &= \sum_{j=1}^{M_y}\sum_{i=1}^{M_x}\sum_{p=0}^{N_x}\sum_{q=0}^{N_y} \theta_{pq} T_p\left(x_i\right) T_q\left(y_i\right) T_s\left(x_i\right) T_u\left(y_i\right) \\
&= \sum_{p=0}^{N_x}\sum_{q=0}^{N_y} \theta_{pq} \sum_{i=1}^{M_x} T_p\left(x_i\right) T_s\left(x_i\right) \sum_{j=1}^{M_y} T_q\left(y_i\right) T_u\left(y_i\right) \\
&= \sum_{p=0}^{N_x}\sum_{q=0}^{N_y} \theta_{pq} \left\{ \begin{array}{cc} 0 & p \neq s \\ \frac{M_x}{2} & p = s \neq 0 \\ M_x & p = s = 0 \end{array} \right. \times \left\{ \begin{array}{cc} 0 & q \neq u \\ \frac{M_y}{2} & q = u \neq 0 \\ M_y & q = u = 0 \end{array} \right. \\
&= \theta_{su} \left\{ \begin{array}{cc} \frac{M_x}{2} & s \neq 0 \\ M_x & s = 0 \end{array} \right. \times \left\{ \begin{array}{cc} \frac{M_y}{2} & u \neq 0 \\ M_y & u = 0 \end{array} \right.
\end{aligned}
$$

from where

$$
\sum_{j=1}^{M_y}\sum_{i=1}^{M_x} f\left(x_i, y_j\right) T_s\left(x_i\right) T_u\left(y_i\right) = \left\{ \begin{array}{cc} \theta_{00} M_x M_y & s = 0, u = 0 \\ \theta_{0u} \frac{M_x M_y}{2} & s = 0, u \neq 0 \\ \theta_{s0} \frac{M_x M_y}{2} & s \neq 0, u = 0 \\ \theta_{su} \frac{M_x M_y}{4} & s \neq 0, u \neq 0 \end{array} \right.
$$