Spectral Theory of Linear Operators

by

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Summary

This thesis is concerned with the relationship between spectral decomposition of operators, the functional calculi that operators admit, and Banach space structure. The deep connection between the first two of these concepts has long been known.

The thesis is organised as follows. Chapter 1 is an introduction to the concepts, ideas and constructions that will be used through at this thesis. Particularly we consider numerical range and hermitian operators which have a critical role in all this thesis.

In Chapter 2 we give a brief overview of some of the theory of (strongly) normal (equivalent) operators. Developing the properties of (strongly) normal (equivalent) operators we will show that the possession of a functional calculus on the spectrum of T is equivalent to T' being scalar type prespectral of class X, thus answering a question of Berkson and Gillespie ([12], Remark 1).

The operators considered in Chapter 3 are the well-bounded operators introduced by Smart [62] as a natural analogue of selfadjoint operators on Hilbert space. Well-bounded operators are defined as those which possess a functional calculus for absolutely continuous functions on some compact interval $[a, b]$ of the real line. Their spectral structure was determined by Ringrose [58] and [59]. Well-bounded operators of type (B) were characterised by Berkson, Dowson [10] and by Spain [65] as being those for which the absolutely continuous functional calculus is weakly compact. Every well-bounded operator on a reflexive Banach space is of type (B), and hence has an integral representation with respect to a spectral family of projections. Ringrose showed that the dual of a well-bounded operator can always be written as an integral representation with respect to a family of projections. This family of projections is called a decomposition of the identity. We will show that

if the Banach space X contains a subspace isomorphic to c_0, or a complemented subspace isomorphic to l_1, then there exists a well-bounded operator which is not "decomposable in X" in that the projections in the decomposition of the identity are not the adjoints of projections on X. By applying the results of Chapter 2 we deduce that the set $\{T^n : n \in \mathbb{N}\}$ is hermitian-equivalent if T is well-bounded operator with decomposition of the identity of bounded variation.

The operators considered in Chapter 4 are the AC-operators. Berkson and Gillespie introduced the concept of an AC-operator as an operator which possesses a functional calculus for the absolutely continuous functions on some rectangle in $\mathbb{C}$ [12]. Berkson and Gillespie showed that these operators can be characterised by the fact that they possess a splitting into real and imaginary parts, $T = U + iV$, where U and V are commuting well-bounded operators [12]. They showed [12] that if U and V are well-bounded operators of type (B) this splitting is unique, and that if $S \in L(X)$ commutes with $U + iV$ then S commutes with U and V. It was shown that neither result is guaranteed if the type (B) hypothesis is omitted [11]. We will show that if S commutes with the AC-operator $T = U + iV$ where U and V are well-bounded with decomposition of the identity of bounded variation then S commutes with U and V. It is shown if $T = U + iV$ is an AC-operator where U and V are well-bounded operators with decomposition of the identity of bounded variation, and if either X does not contain a copy of c_0, or if U and V are decomposable in X, then the representation is unique. We also explore some properties of AC-operators by applying the theory of (Foiaş) decomposable operators.

Since 1954 the problem of giving sufficient conditions for the sum and product of two commuting spectral operators to be spectral has attracted attention. The boundedness of the Boolean algebra of projections generated by the two resolutions of the identity is critical. In 1954 Wermer [70] proved an affirmative result on Hilbert space. McCarthy [48] showed that this did not remain true in a general Banach space. In 1964 McCarthy [49] showed that an affirmative result holds on closed linear subspaces of L_p spaces, where $2 \leq p \leq \infty$. Much later, in 1997, Gillespie [33] proved that if $\mathcal{E}$ and $\mathcal{F}$ are two commuting Boolean algebra on X, where X is a Banach lattice or a closed linear subspace of a p-concave Banach lattice ($p < \infty$), then the Boolean algebra of projections generated by $\mathcal{E}$ and $\mathcal{F}$ is also bounded. As a consequence of this he showed that the sum and product of two commuting spectral operators is also spectral in each of the above cases. We will show that the weakly closed algebra generated by the real and imaginary parts of a finite family of commuting scalar-type spectral operators on a Banach lattice not containing c_0, and on a closed linear subspace of a p-concave Banach lattice, where $p < \infty$, is a W^*-algebra, and that every operator in this algebra is a scalar-type spectral operator.

Contents

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Statement

This thesis is submitted in accordance with the regulations for the degree of Doctor of Philosophy in the University of Glasgow. No part of it has been previously submitted by the author for a degree at any university. The material contained within this work is original, except where explicitly mentioned to the contrary. Chapter Five is the joint work of the author, his supervisor and Dr H.R. Dowson. The work presented in Chapter 2 appears in [29].

Chapter 1

Preliminaries

1.1 General notation

$\mathbb{R}$ will denote the set of real numbers, $\mathbb{C}$ the complex numbers, $\mathbb{Z}$ integers, $\mathbb{N}$ the positive integers. Throughout, X denotes a non-zero complex Banach space, otherwise arbitrary unless the contrary is explicitly stated. The dual space of X is denoted by X'. We write $\langle x, x' \rangle$ for the value of the functional x' at the point x in X.

Definition 1.1.1. *Let X be a Banach space. The weakest topology on X which makes each element $x' : X \to \mathbb{C}$ continuous is called the* weak topology *of X. A typical open neighbourhood of $x \in X$ for the weak topology has the form*

$$\{y \in X : |\langle x, x' \rangle| < \epsilon,\ x' \in \mathcal{F}\},$$

for some $\epsilon > 0$ and some finite set $\mathcal{F} \subseteq X'$.

A net of elements $\{x_\alpha\} \subseteq X$ converges to $x \in X$ in the weak topology if and only if

$$\lim_\alpha \langle x_\alpha, x' \rangle = \langle x, x' \rangle, \qquad x' \in X'.$$

The weak topology on X is denoted by $\sigma(X, X')$.

We have the following basic facts about weak topology (see [26], Chapter V):

1. A subset of X is norm bounded if and only if it is weakly bounded,

2. If Y is any convex subset of X, then the closure of Y with respect to the norm topology coincides with the closure of Y with respect to weak topology,

3. A linear functional $\Lambda : X \to \mathbb{C}$ is continuous with respect to the topology $\sigma(X, X')$ on X if and only if $\Lambda \in X'$.

Definition 1.1.2. *Let X be a Banach space and let $\sum_{n=1}^{\infty} x_n$ be a series of elements $x_n \in X$.*

(i) The series is said to be unconditionally norm convergent *if there exists $x \in X$ such that $\sum_{n=1}^{\infty} x_{\pi(n)} = x$, for all bijections $\pi : \mathbb{N} \to \mathbb{N}$.*

(ii) The series is said to be weakly subseries convergent *if each subseries $\sum_{n=1}^{\infty} x_{n_k}$ converges (to some element of X) in the weak topology.*

The series $\sum_{n=1}^{\infty} x_n$ in X is unconditionally norm convergent whenever it is weakly subseries convergent ([19], chapter 1, §4).

Definition 1.1.3. *The weakest topology on X' which makes each of the linear functional $\langle x, \cdot \rangle : X' \to \mathbb{C}$, $x \in X$, defined by $x' \mapsto \langle x, x' \rangle$, for $x' \in X'$, continuous is called the* weak-star topology *of X' and is denoted by $\sigma(X', X)$.*

A typical open neighbourhood of $x' \in X'$ has the form

$$\{y' \in X' : |\langle x, x' \rangle - \langle x, y' \rangle| < \epsilon, \ x \in \mathcal{F}\},$$

for some $\epsilon > 0$ and some finite set $\mathcal{F} \subseteq X$. In particular, a net of elements $\{x'_\alpha\} \subseteq X'$ converges to $x' \in X'$ for the weak-star topology if and only if

$$\lim_{\alpha} \langle x, x'_\alpha \rangle = \langle x, x' \rangle, \qquad x \in X.$$

Definition 1.1.4. *Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on X. They are said to be* equivalent norms *if they define the same topology on X.*

$\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if and only if there are positive constants M_1 and M_2 such that $M_1\|x\|_1 \leq \|x_2\| \leq M_2\|x\|_1$ for all x in X.

Throughout this thesis operator means "bounded linear operator". The collection of all operators on the X is denoted by $L(X)$. If we define

$$\|T\| = \sup\{\|Tx\| : x \in X, \|x\| \leq 1\}$$

then $L(X)$, with this norm, is a Banach algebra.

The three most commonly used topologies in $L(X)$ are the norm topology, the strong operator topology, and weak operator topology.

Definition 1.1.5. *The* strong operator topology *on $L(X)$ is the topology defined by the basic set of neighbourhoods*

$$\{R : R \in L(X), \|(T-R)x\| < \epsilon,\ x \in \mathcal{F}\}$$

where $\epsilon > 0$ is arbitrary, and $\mathcal{F}$ is an arbitrary finite subset of X. In particular a net of element $\{T_\alpha\}$ converges to T if and only if $\{T_\alpha x\}$ converges to Tx for every $x \in X$.

Definition 1.1.6. *The* weak operator topology *on $L(X)$ is the topology defined by the basic set of neighbourhoods*

$$\{R : R \in L(X), |\langle (T-R)x, x'\rangle| < \epsilon,\ \ x' \in \mathcal{F}',\ \ x \in \mathcal{F}\},$$

where $\epsilon > 0$ is arbitrary, and $\mathcal{F}'$ and $\mathcal{F}$ are arbitrary finite sets of elements in X' and X, respectively. In particular a net of elements $\{T_\alpha\}$ converges to T if and only if $\{\langle T_\alpha x,\ x'\rangle\}$ converges to $\langle Tx, x'\rangle$ for every $x \in X$ and and $x' \in X'$.

We shall abbreviate "weak (strong) operator topology" to "weak (strong) topology". When $\mathfrak{A}$ is a subset of $L(X)$ we write $\overline{\mathfrak{A}}^w$ and $\overline{\mathfrak{A}}^s$ for the weak and strong closure of $\mathfrak{A}$, respectively. If $\mathfrak{A}$ is convex then $\overline{\mathfrak{A}}^w = \overline{\mathfrak{A}}^s$ ([26], VI. 1.5).

For any T in $L(X)$, let T' in $L(X')$ be its adjoint:

$$\langle Tx, x'\rangle = \langle x, T'x'\rangle \quad (x \in X,\ x' \in X').$$

Definition 1.1.7. *The resolvent set $\varrho(T)$ of T is the set of complex numbers λ for which $\lambda \mathrm{I} - T$ is invertible in the Banach algebra $L(X)$.*

Definition 1.1.8. *The spectrum $\sigma(T)$ of T is defined to be $\mathbb{C} \setminus \varrho(T)$.*

Definition 1.1.9. *Let $T \in L(X)$. The spectral radius $\nu(T)$ is defined by*

$$\nu(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}.$$

1.2 BV(J) and BV($J \times K$) as Banach Algebras

Much of this thesis is concerned with functions of bounded variation.

1.2.1 BV(J)

Let $J = [a, b]$ be a compact interval contained in the real line. We denote by $\mathfrak{P} = \mathfrak{P}(J)$ the set of all finite partitions $\Lambda = \{a = \lambda_0 < \lambda_1 < \cdots \lambda_n = b\}$ of J. If f is a scalar-valued function on J, then the variation of f over J is defined to be

$$\operatorname{var}_J f = \sup_{\Lambda \in \mathfrak{P}} \sum_{\Lambda} \left| f(\lambda_j) - f(\lambda_{j-1}) \right|.$$

If $\operatorname{var}_J f < \infty$, then f is said to be of *bounded variation* over J. Let BV(J) denote the set of functions of bounded variation over J. If we define

$$\|f\|_J = |f(b)| + \operatorname{var}_J f$$

then $\|\cdot\|_J$ is a norm on BV(J) which makes this space into a Banach algebra [61].

A function $f : J \to \mathbb{C}$ is said to be *absolutely continuous* on J if for all $\epsilon > 0$ there exists a $\delta > 0$ such that $\sum_{j=1}^{n} \left| f(\lambda_j) - f(\mu_j) \right| < \epsilon$ for every finite collection of pairwise disjoint subintervals $(\lambda_j,\ \mu_j)$ of J for which $\sum_{j=1}^{n} |\lambda_j - \mu_j| < \delta$.

Definition 1.2.1. *Let $J = [a, b]$. The Banach subalgebra of* BV(J) *consisting of absolutely continuous functions on J is denoted by $AC(J)$. For f in $AC(J)$*

$$|||f||| = f(b) + \int_a^b |f'(t)|\, dt.$$

1.2.2 BV($J \times K$)

Let $J = [a, b]$ and $K = [c, d]$ be two fixed intervals in $\mathbb{R}$. The notation of variation for a complex-valued function defined on $J \times K$ used here is due to Hardy [36] and Krause. (See the discussion in [37], §254.) Let Λ be a rectangular partition of $J \times K$:

$$a = s_0 < s_1 < \cdots < s_n = b, \qquad c = t_0 < t_1 < \cdots < t_m = d.$$

For a function $f : J \times K \to \mathbb{C}$ define

$$V_\Lambda(f) = \sum_{i=1}^{n}\sum_{j=1}^{m} |f(s_i,t_j) - f(s_i,t_{j-1}) - f(s_{i-1},t_j) + f(s_{i-1},t_{j-1})|$$

and the *variation* of f on $J \times K$ by

$$\mathrm{var}_{J\times K}(f) = \sup\{V_\Lambda(f) : \Lambda \textit{ is a rectangular partition of } J \times K\}.$$

The function f is said to be of *bounded variation* on $J \times K$ (according to Hardy and Krause) if each of the numbers

$$\mathrm{var}_{J\times K} f, \qquad \mathrm{var}_J f(\cdot, d), \qquad \mathrm{var}_K f(b, \cdot)$$

is finite. Denote by $\mathrm{BV}(J \times K)$ the set of all functions $f : J \times K \to \mathbb{C}$ of bounded variation. It is readily verified that with the norm given by

$$|||f||| = |f(b,d)| + \mathrm{var}_J f(\cdot, d) + \mathrm{var}_K f(b, \cdot) + \mathrm{var}_{J\times K} f,$$

$\mathrm{BV}(J \times K)$ is a Banach space.

Remark 1.2.2. *It should be mentioned that, as in the one-dimensional case, we take* $\mathrm{var}_{J\times K} f = 0$ *if either* J *or* K *reduces to a single point.*

It will be shown with pointwise product and norm $|||\cdot|||$, $\mathrm{BV}(J \times K)$ is a Banach algebra. To prove this, it is convenient to note first that $\mathrm{BV}(J \times K)$ may be considered as the l_1-direct sum

$$\mathbb{C} \oplus \mathrm{BV}_0(J) \oplus \mathrm{BV}_0(K) \oplus \mathrm{BV}_0(J \times K). \tag{1.1}$$

Here

$$\mathrm{BV}_0(J) = \{f \in \mathrm{BV}(J) : \ f(b) = 0\} \qquad \mathrm{BV}_0(K) = \{f \in \mathrm{BV}(K) : \ f(d) = 0\}$$

and

$$\mathrm{BV}_0(J \times K) = \{f \in \mathrm{BV}(J \times K) : \ f(s,d) = f(b,t) = 0 \ (s \in J, t \in K)\}.$$

The identification of $\mathrm{BV}(J \times K)$ with (1.1) arises from writing $f \in \mathrm{BV}(J \times K)$ as

$$f(s,t) = f(b,d) + f_1(s) + f_2(t) + f_3(s,t),$$

where

$$f_1(s) = f(s,d) - f(b,d), \qquad f_2(t) = f(b,t) - f(b,d)$$

and

$$f_3(s,t) = f(s,t) - f(s,d) - f(b,t) + f(b,d).$$

Note that

$$\operatorname{var}_J f_1 = \operatorname{var}_J f(\cdot,d), \qquad \operatorname{var}_K f_2 = \operatorname{var}_K f(b,\cdot)$$

and

$$\operatorname{var}_{J\times K} f_3 = \operatorname{var}_{J\times K} f,$$

so that the above algebraic identification is indeed isometric.

The next lemma and theorem are due to Berkson and Gillespie [12].

Lemma 1.2.3. *Let $f \in \mathrm{BV}_0(J)$, $g \in \mathrm{BV}_0(K)$ and $h, k \in \mathrm{BV}_0(J \times K)$. Then*

1. $fg \in \mathrm{BV}_0(J \times K)$ *and* $\operatorname{var}_{J\times K}(fg) \leq \operatorname{var}_J f \ \operatorname{var}_K g$,
2. $fh \in \mathrm{BV}_0(J \times K)$ *and* $\operatorname{var}_{J\times K}(fh) \leq \operatorname{var}_J f \ \operatorname{var}_{J\times K} h$,
3. $gh \in \mathrm{BV}(J \times K)$ *and* $\operatorname{var}_K g \ \operatorname{var}_{J\times K} h$,
4. $hk \in \mathrm{BV}_0(J \times K)$ *and* $\operatorname{var}_{J\times K}(hk) \leq \operatorname{var}_{J\times K} h \ \operatorname{var}_{J\times K} k$.

Proof. ([12], Lemma 1). □

Theorem 1.2.4. *Under the pointwise product and norm $|||\cdot|||$, the space $\mathrm{BV}(J\times K)$ is a unital Banach algebra.*

Proof. This is a consequence of Lemma 1.2.3 and the decomposition (1.1), together with the fact that $\mathrm{BV}_0(J)$ and $\mathrm{BV}_0(K)$ are subalgebras of $\mathrm{BV}(J)$ and $\mathrm{BV}(K)$ respectively. □

1.2.3 Functional Calculus

When Λ is a compact Hausdorff space, we write $C(\Lambda)$ for the Banach algebra of continuous complex-valued functions on Λ with pointwise-defined algebraic operations and the supremum norm. When $J = [a,b]$, we write $\mathrm{BV}(J)$ for the Banach algebra of complex-valued functions of bounded variation on J with norm $|||\cdot|||$ defined by $|||f||| = |f(b)| + \text{var}(f, J)$, and $AC[a,b]$ for the subalgebra of BV(J) consisting of absolutely continuous functions on J. Suppose that $\mathfrak{F}$ is a Banach algebra of scalar-valued functions on some subset $S \subseteq \mathbb{C}$ and that $\mathfrak{F}$ contains polynomials. For $n = 0, 1, \cdots$, let $e_n(z) = z^n$.

Let X be a Banach space. An *$\mathfrak{F}$-functional calculus* for T is a Banach algebra homomorphism $\Theta : \mathfrak{F} \to L(X)$ for which $\Theta(e_n) = T^n\,(n = 0, 1, ...)$. Some authors prefer the term "operational calculus" to "functional calculus". We shall say that a $\mathfrak{F}$-functional calculus Θ is *compact* (respectively *weakly compact*) if, for all $x \in X$, the operator $\Theta_x : \mathfrak{F} \to X$ defined by $\Theta_x(f) = \Theta(f)x$ is compact when X is given its norm (weak) topology. We shall be concerned here with the algebras $C(\sigma(T))$, BV(J), BV($J \times K$), $AC(J)$ and $AC(J \times K)$.

Vector Measures

Let Ω be a non-empty set. A family of subsets Σ of Ω is called an *algebra* (of sets) if

1. $\Omega \in \Sigma$ and $\phi \in \Sigma$ (where ϕ denote the empty set),

2. $\Omega \backslash E$ belongs to Σ whenever $E \in \Sigma$, and

3. $\bigcup_{j \in \mathcal{F}} E_j \in \Sigma$ for every finite collection $\{E_j : j \in \mathcal{F}\} \subseteq \Sigma$.

If Σ is an algebra of sets with the additional property that $\bigcup_{n \in \mathbb{N}} E_n \in \Sigma$ for every sequence $\{E_n\} \subseteq \Sigma$, then it is called a *σ-algebra*. In this case the pair (Ω, E) is called a *measurable space*.

Let (Ω, Σ) be a measurable space. A function $\mu : \Sigma \to \mathbb{C}$ is called a *complex measure* if $\mu(\bigcup_{n \in \mathbb{N}} E_n) = \sum_{n=1}^{\infty} \mu(E_n)$ whenever $\{E_n\} \subseteq \Sigma$ is a sequence of pairwise disjoint sets, meaning that $E_n \cap E_m = \phi$ whenever $n \neq m$. We say that μ is *σ-additive*. In this case triple (Ω, Σ, μ) is called a *measure space*.

By $L_p(\Omega, \Sigma, \mu)$, $1 \leq p \leq \infty$, we denote the Banach spaces of equivalence classes of measurable functions on (Ω, Σ, μ) whose p^{th} power is integrable (rep. are essentially bounded if $p = \infty$). If (Ω, Σ, μ) is the usual Lebesgue measure space on $[0, 1]$

we denote $L_p(\mu)$ by L_p. If (Γ, Σ, μ) is the discrete measure space on a set Γ with $\mu(\{\gamma\}) = 1$ for every $\gamma \in \Gamma$ we denote $L_p(\mu)$ by $l_p(\Gamma)$. If $\Gamma = \{1, 2, \cdots, n\}$, $n < \infty$, we also denote $l_p(\Gamma)$ by l_p^n, while l_p denotes $l_p(\Gamma)$ with $\Gamma = \mathbb{N}$.

Let Σ be a σ-algebra of subsets of an arbitrary set Ω. An additive set function μ defined on Σ with values in the Banach space X is said to be a *vector measure* if

$$\langle \mu(\bigcup(\tau_n)), x' \rangle = \sum_{n=1}^{\infty} \langle \mu(\tau_n), x' \rangle \qquad (x' \in X')$$

for every pairwise disjoint sequence $\{\tau_n\}$ of sets in Σ.

Lemma 1.2.5 (Banach-Orlicz-Pettis). *Any vector measure μ is strongly countably additive. That is, $\mu(\bigcup(\tau_n)) = \sum_{n=1}^{\infty} \mu(\tau_n)$ for each sequence $\{\tau_n\}$ of pairwise disjoint sets in Σ.*

Proof. ([5], Lemma 2.2) or ([26], IV. 10.1). □

The results of ([26], IV. 10) show that every μ-measurable μ-essentially bounded complex-valued function on Ω can be integrated with respect to μ, and that the integral satisfies the dominated convergence theorem.

Lemma 1.2.6. *If $\{f_n\}$ is a sequence of μ-integrable function which converges μ-almost everywhere to f, and if g is a μ-integrable function such that $|f_n(s)| \leq |g(s)|$ μ-almost everywhere $(n = 1, 2, ...)$, then f is μ-integrable and*

$$\int_E f(s)\mu(ds) = \lim_{n \to \infty} \int_E f_n(s)\mu(ds), \qquad (E \in \Sigma).$$

Proof. ([26], IV. 10) □

Lemma 1.2.7. *Let X be a Banach space an $\mu : \Sigma \to X$ be a vector measure. Then its range $\mu(\Sigma)$ is a relatively weakly compact subset of X. That is, the closure in X of $\mu(\Sigma)$ with respect the weak topology $\sigma(X, X')$ is weakly compact.*

Proof. ([57], Proposition I.4) □

When Λ is a compact Hausdorff space, we write $S(\Lambda)$ and $S_0(\Lambda)$ for the σ-algebra of Borel and Baire sets of Λ, respectively and $B(\Lambda)$ and $B_0(\Lambda)$ for the family of Borel and bounded Baire measurable complex-valued functions on Λ respectively. We say that a vector measure μ on $S(\Lambda)$ is regular if $\langle \mu(\cdot), x' \rangle$ is a regular measure for each x' in X'.

Lemma 1.2.8 ([5], Theorem 3.2). *Let Λ be a compact Hausdorff space, let X be a Banach space and let $\Theta : C(\Lambda) \to X$ be a weakly compact map. Then there exists a unique regular vector measure μ on $S(\Lambda)$ with values in X such that*

$$\Theta f = \int_\Lambda f(\lambda)\, \mu(d\lambda) \qquad (f \in C(\Lambda)).$$

Conversely, if μ is a vector measure on $S_0(\Lambda)$ with values in X, and if $\Theta : C(\Lambda) \to X$ is defined by $\Theta f = \int_\Lambda f(\lambda)\, \mu(d\lambda)$, then Θ is a weakly compact operator.

Lemma 1.2.9 ([52], Theorem 5). *If X is a Banach space which does not contain a subspace isomorphic to c_0, in particular if X is weakly complete, then any bounded operator*

$$\Theta : C(\Lambda) \to X$$

is weakly compact.

Boolean algebras of projections

An operator E in $L(X)$ is called a projection if $E^2 = E$. We write $E \leq F$ when E and F are projections and $E = EF = FE$. If E and F are commuting projections, then $E \vee F = E + F - EF$ and $E \wedge F = EF$ are also projections. The ranges of $E \vee F$ and $E \wedge F$ of commuting projections E and F are given by the equations $(E \wedge F)X = EX \cap FX$, $(E \vee F)X = EX + FX = \overline{\text{lin}}(EX, FX)$, where $\overline{\text{lin}}(EX, FX)$ means the closed linear manifold spanned by EX and FX.

A set $\mathfrak{B}$ of commuting projections on X is called a *Boolean algebra of projections* if $\mathfrak{B}$ contains 0 and I and is a Boolean algebra under $\vee$ and $\wedge$. A Boolean algebra of projections $\mathfrak{B}$ is *bounded* if there is a constant M such that $\|E\| \leq M$, $E \in \mathfrak{B}$.

We will be dealing exclusively with bounded Boolean algebras of projections. The following example shows that not all Boolean algebras of projections are bounded.

Example 1.2.10. *Let $X = L_p(\mathbb{R})$, for some $p \in (1,2)$. For each $t \in \mathbb{R}$, define the translation operator $T_t \in L(X)$ by $T_t f = f_t$, for $f \in X$, where $f_t(s) = f(t+s)$ for a.e. $s \in \mathbb{R}$. A projection $E \in L(X)$ is called a p-multiplier if $ET_t = T_t E$, for all $t \in \mathbb{R}$. It is a known fact from harmonic analysis that the family $\mathfrak{B}_p$ of all p-multiplier projections is a Boolean algebra of projections for which* $\sup\{\|E\| : E \in \mathfrak{B}_p\} = \infty$.

Following [26] we say that an abstract Boolean algebra $\mathcal{E}$ is *(σ-)complete* if each (countable) subset of $\mathcal{E}$ has a supremum and infimum in $\mathcal{E}$.

$\mathcal{E}$, a Boolean algebra of projections on X, is *(σ-)complete on X* if each (countable) subset $\mathcal{F}$ of $\mathcal{E}$ has a supremum and infimum in $\mathcal{E}$ such that

$$\left(\bigvee \mathcal{F}\right) X = \overline{\text{lin}}\{FX : F \in \mathcal{F}\}, \qquad \left(\bigwedge \mathcal{F}\right) X = \bigcap_{F \in \mathcal{F}} FX.$$

Lemma 1.2.11. *If a Boolean algebra of projections is σ-complete as an abstract Boolean algebra, then it is bounded.*

Proof. ([3], Theorem 2.2) or ([57], Theorem III.1). □

1.3 Single-valued extension property

Definition 1.3.1. *An operator $T \in L(X)$ is said to have the* single-valued extension property *if whenever $f : D_f \to X$ is analytic in an open set $D_f \subseteq \mathbb{C}$ and satisfies*

$$(\lambda I - T)f(\lambda) = 0, \qquad (\lambda \in D_f)$$

it follows that $f = 0$ in D_f.

Let $T \in L(X)$ have the single-valued extension property. For each $x \in X$ we denote by $\varrho_T(x)$ the set of elements $\alpha \in \mathbb{C}$ such that there exists an X-valued function $x(\cdot)$ analytic in a neighbourhood of V_α of α, such that

$$(\lambda I - T)x(\lambda) = x \qquad (\lambda \in V_\alpha).$$

In particular, $\varrho(T) \subseteq \varrho_T(x)$. The complement $\sigma_T(x) = \mathbb{C} \backslash \varrho_T(x)$ is the *local spectrum* of T at x; it is a compact subset of $\sigma(T)$, which is non-empty for $x \neq 0$.

For $F \subseteq \mathbb{C}$, let $X_T(F) = \{x \in X : \sigma_T(x) \subseteq F\}$. This is a T-invariant manifold.

Lemma 1.3.2. *Let $T \in L(X)$ have the single-valued extension property. Then*

1. *$F_1 \subseteq F_2$ implies $X_T(F_1) \subseteq X_T(F_2)$,*

2. *$X_T(F)$ is a linear subspace of X,*

3. *$\sigma_T(x) = \phi$ if and only if $x = 0$,*

4. *$\sigma_T(Sx) \subseteq \sigma_T(x)$ for every $S \in L(X)$ with $ST = TS$,*

5. *$\sigma_T(x(\lambda)) = \sigma_T(x)$ for every $x \in X$ and $\lambda \in \varrho_T(x)$.*

Lemma 1.3.3 ([22], Proposition 5.28). *Let $T \in L(X)$. Suppose that $\sigma(T)$ is nowhere dense. Then T has the single-valued extension property.*

Lemma 1.3.4 ([17], Theorem 2.4). *Let $T_1, T_2 \in L(X)$. If T_1 has the single-valued extension property and $T_1 - T_2$ quasinilpotent, then*

$$\sigma_{T_1}(x) = \sigma_{T_2}(x) \qquad (x \in X).$$

Let X and Y be two Banach spaces. $L(X,Y)$ will denote the collection of all bounded linear mappings of X into Y. For $S \in L(X)$ and $T \in L(Y)$ we define $L(T)$, $R(S)$, $C(T,S) : L(X,Y) \to L(X,Y)$ by

$$L(T)A = TA,$$

$$R(S)A = AS,$$

and

$$C(T,S)A = TA - AS,$$

respectively (where $A \in L(X,Y)$). For every $n \geq 1$ we put

$$\begin{aligned} C^n(T,S)(A) &= [L(T) - R(S)]^n(A) \\ &= \sum_{k=0}^{n} \binom{n}{k} (-1)^k L(T)^{n-k} R(S)^k(A) \\ &= \sum_{k=0}^{n} \binom{n}{k} (-1)^k T^{n-k} A S^k. \end{aligned}$$

Definition 1.3.5. *We say that $T, S \in L(X)$ are quasinilpotent equivalent if*

$$\lim_{n\to\infty} \|C(T,S)^n(I)\|^{1/n} = 0 \qquad \& \qquad \lim_{n\to\infty} \|C(S,T)^n(I)\|^{1/n} = 0.$$

This relation is reflexive, symmetric and transitive ([17], p. 11).

Definition 1.3.6. *Let X be a Banach space and $T \in L(X)$. A closed linear subspace $\mathcal{Y}$ of X is called a* spectral maximal space *of T if*

1. *$\mathcal{Y}$ is invariant under T,*

2. *if $\mathcal{Z}$ is another closed linear subspace of X, invariant under T, such that if $\sigma(T|\mathcal{Z}) \subseteq \sigma(T|\mathcal{Y})$, then $\mathcal{Z} \subseteq \mathcal{Y}$.*

A spectral maximal space of $T \in L(X)$ is ultra-invariant under T; that is, invariant under any operator A commuting with T.

Definition 1.3.7. *An operator $T \in L(X)$ is called* decomposable *if any open cover $\mathbb{C} = G_1 \cup G_2$ of the complex plane $\mathbb{C}$ by two open sets G_1 and G_2 yields a splitting of the spectrum $\sigma(T)$ and of the space X in the sense that there exist closed T-invariant linear subspaces Y and Z of X for which $\sigma(T|Y) \subseteq G_1$, $\sigma(T|Z) \subseteq G_2$, and $X = Y + Z$.*

The definition of decomposability has been simplified considerably since Colojoară and Foiaş wrote their book [17]. The original definition of a decomposable operator as developed by Foiaş was somewhat complicated and involved the notion of a spectral maximal space. See [43], [17] and [69] for an account of the classical theory of decomposable operators.

The next three lemmas give some basic facts about (Foiaş) decomposable operators (see [17], chapters 2 and 4).

Lemma 1.3.8. *Let T be (Foiaş) decomposable and F a closed subset of $\sigma(T)$. Then $X_T(F)$ is a spectral maximal space of T, and $\sigma(T|X_T(F)) \subseteq F$. Conversely, if $\mathcal{Y}$ is a spectral maximal space of T, then*

$$\mathcal{Y} = X_T(\sigma(T|\mathcal{Y})).$$

Proof. ([17], Theorem 2.1.5). □

Lemma 1.3.9. *Let $S \in L(X), T \in L(Y)$ be two (Foiaş) decomposable operators and suppose $A \in L(X,Y)$. Then the following assertions are equivalent:*

1. $AX_S(F) \subseteq X_T(F)$ *for every closed set* $F \subseteq \mathbb{C}$.
2. $\lim_{n\to\infty} \|C^n(T,S)(A)\|^{1/n} = 0$

Proof. ([17], Theorem 2.3.3). □

Lemma 1.3.10. *If $T \in L(X)$ is a (Foiaş) decomposable operator then*

$$X_T(\{0\}) = \{x \in X : \lim_{n\to\infty} \|T^n x\|^{1/n} = 0\}.$$

Proof. ([17], Lemma 4.4.4). □

1.4 Hermitian operators

In this section we consider those properties of hermitian operators which are important in the theory of prespectral operators.

Definition 1.4.1. *A* semi-inner-product *on a complex linear space Y is a function* $[\cdot,\cdot] : Y \times Y \to \mathbb{C}$ *satisfying*

1. $[x+y,z] = [x,z] + [y,z]$,
   $[\lambda x, y] = \lambda[x,y], \qquad x,y,z \in Y,\ \lambda \in \mathbb{C}$,
2. $[x,x] > 0, \qquad x \neq 0$,
3. $|[x,y]|^2 \leq [x,x]\,[y,y], \qquad x,y \in Y.$

A space Y with a semi-inner-product is called a *semi-inner-product space.*

Lemma 1.4.2 ([47], Theorem 2). *Let Y be a semi-inner-product space. Then $\|\cdot\| : Y \to \mathbb{C}$ defined by $\|x\| = [x,x]^{1/2}$ is a norm on Y. Conversely, let Y be a normed complex linear space and for each x in Y let $\tilde{x}$ be a bounded functional on Y such that $\langle x, \tilde{x}\rangle = \|x\|^2 = \|\tilde{x}\|^2$. Then $[\cdot,\cdot] : Y \times Y \to \mathbb{C}$ defined by $[x,y] = \langle x, \tilde{y}\rangle$ is a semi-inner-product on Y.*

In general, a normed linear space Y admits infinitely many semi-inner-products compatible with its norm (in the sense that $[x,x] = \|x\|^2,\ x \in Y$). However, a pre-Hilbert space admits a *unique* compatible semi-inner-product ([47], Theorem 3).

When Y is a semi-inner-product space and T is an operator (not necessarily bounded) on Y we define $W(T)$, the numerical range of T, to be the set

$$W(T) = \{[Tx,x] : [x,x] = 1\}.$$

Definition 1.4.3. *An operator $T \in L(X)$ is* hermitian *if $W(T)$ is real.*

We now collect a few basic facts about hermitian operators needed later.

Lemma 1.4.4. *(i) Let $\alpha_1, \cdots, \alpha_n \in \mathbb{R}$ and let $A_1, \cdots, A_n$ be hermitian operators on X. Then $\sum_{r=1}^{n} \alpha_r A_r$ is a hermitian operator.*

(ii) Let A_α be a net of hermitian operators on X. If $\{A_\alpha\}$ converges to A in the weak operator topology of $L(X)$ then A is a hermitian operator.

Proof. ([22], Theorem 4.4). □

Lemma 1.4.5. *Let $A \in L(X)$. The following statements are equivalent:*
(i) A is hermitian.
(ii) $lim_{\alpha\to 0}\ \alpha^{-1}\{\|I - i\alpha A\| - 1\} = 0 \qquad (\alpha \in \mathbb{R})$.
(iii) $\|\exp(i\alpha A)\| = 1 \qquad (\alpha \in \mathbb{R})$.
(iv) $\|\exp(i\alpha A)\| \leqq 1 \qquad (\alpha \in \mathbb{R})$.

Proof. ([22], Thorem 4.7). □

Lemma 1.4.6. *If $A \in L(X)$ is a hermitian operator then $\sigma(A) \subseteq \mathbb{R}$.*

Proof. ([22], Theorem 4.8). □

Theorem 1.4.7 (Sinclair). *Let A, in $L(X)$, be a hermitian operator. Then*

$$\|A\| = \nu(A).$$

Proof. ([22], Theorem 4.10) □

Lemma 1.4.8. *$T \in L(X)$ is hermitian if and only if T' is hermitian.*

Proof.

$$\|\exp(i\alpha T)\| = \|\exp(i\alpha T')\| \qquad (\alpha \in \mathbb{R}).$$

□

By Theorem 1.4.4(i) the sum of two hermitian operators is hermitian. However, powers of hermitian operators need not be hermitian, as the following example, due to M.J. Crabb, shows. See ([15], p. 57, 58).

Example 1.4.9. *Let p defined on $\mathbb{C}^3$ by*

$$p(\alpha,\beta,\gamma) = \sup\{|\lambda^{-1}\alpha + \beta + \lambda\gamma| : \lambda \in \mathbb{C}, |\lambda| = 1\}.$$

Then p is a Banach space norm on $\mathbb{C}^3$. Now $\mathbb{C}^3$ is an algebra under pointwise multiplication. If $a \in \mathbb{C}^3$, let A be the operator defined by

$$Ax = ax \qquad (x \in \mathbb{C}^3).$$

Observe that the operator norm of A is given by

$$\|A\| = \sup\{p(xa) : x \in \mathbb{C}^3, p(x) = 1\}.$$

Now let $a = (-1, 0, 1)$. Then A is hermitian but A^2 is not hermitian. For: given t in $\mathbb{R}$ and $x = (\alpha, \beta, \gamma)$ in $\mathbb{C}^3$ we have

$$p(x\ \exp(ita)) = p(\alpha \exp(-it), \beta, \gamma \exp(it)) = p(\alpha, \beta, \gamma)$$

so that $\|\exp(itA)\| = 1$. Therefore A is hermitian by Lemma 1.4.5. Let $s = -\pi/2$ and $y = (i, 1, i)$. Then

$$p(y)\|\exp(isA^2)\| \geqq p(y\ \exp(isa^2)) = p(1,1,1).$$

Since $p(y) = 5^{1/2}$ and $p(1,1,1) = 3$ it follows that

$$\|\exp(isA^2)\| > 1.$$

Hence by Lemma 1.4.5, A^2 is not hermitian.

Chapter 2

Normal-equivalent and prespectral operators

The class of scalar-type spectral operators on a Banach space was introduced by Dunford [19] as a natural analogue of the normal operators on Hilbert space. They can be characterised by their possession of a weakly compact functional calculus for continuous functions on the spectrum ([41], Corollary 1) or ([63], Theorem). The more general class of scalar-type prespectral operators of class Γ was introduced by Berkson and Dowson [9]. They proved that if $T \in L(X)$ admits a $C(\sigma(T))$ functional calculus, then T' is scalar-type prespectral of class X. The converse implication is immediate if X is reflexive ([22], Theorem 6.17) or $\sigma(T) \subseteq \mathbb{R}$ ([22], Theorem 16.15 and the proof of Theorem 16.16). The question raised by Berkson and Gillespie ([12], Remark 1) has remained open for some time. The problem amounts to finding a decomposition for T with commuting real and imaginary parts, given that T' has such a decomposition. We show that this can always be done, developing the properties of (strongly) normal (equivalent) operators for this purpose.

2.1 Hermitian-equivalent operators

Definition 2.1.1. *An operator $T \in L(X)$ is* hermitian-equivalent *if and only if there is an equivalent norm on X with respect to which T is hermitian.*

Theorem 2.1.2. *T is hermitian-equivalent if and only if there is an $M\,(\geq 1)$ such* *that*

$$\|\exp(itT)\| \leq M \qquad (t \in \mathbb{R}).$$

If this condition is satisfied, then

$$|x| = \sup\{\|\exp(itT)x\| : t \in \mathbb{R}\}$$

defines a norm on X, equivalent to $\|\cdot\|$, with respect to which T is hermitian.

Definition 2.1.3. *Let $\Lambda \subseteq L(X)$. Then Λ is said to be* hermitian-equivalent *if and only if there is an equivalent norm on X with respect to which every operator in Λ is hermitian.*

Lemma 2.1.4 ([22], Theorem 4.17). *Let Λ be a commutative subset of $L(X)$. Then Λ is hermitian-equivalent if and only if each operator in the closed real linear span of Λ is hermitian-equivalent.*

Lemma 2.1.5. *Let $\mathcal{E}$ be a bounded Boolean algebra of projections on X. Then there is an equivalent norm on X with respect to which every operator on $\mathcal{E}$ is hermitian.*

Lemma 2.1.5 is due to Berkson ([7]).

2.2 Normal operators, Strongly normal operators

Definition 2.2.1. *An operator $T \in L(X)$ is* normal *if $T = R + iJ$ where R and J are commuting hermitian operators.*

We shall need the following Fuglede-type result ([24], Lemma 3), and generalisations of it.

Lemma 2.2.2 ([24], Lemma 3). *If $T = R + iJ$ is a normal operator and if $A \in L(X)$ is such that $AT = TA$, then $AR = RA$, $AJ = JA$.*

Remark 2.2.3. *If $T \in L(X)$ is normal, then the operators R and J are determined uniquely by T and we write*

$$T^* = R - iJ.$$

Uniqueness follows from Lemma 2.2.2.

Definition 2.2.4. *An operator* $T \in L(X)$ *is* normal-equivalent *if* $T = R + iJ$ *where* $RJ = JR$ *and* $\{R, J\}$ *is hermitian-equivalent.*

The following result is an immediate consequence of Lemma 2.2.2

Lemma 2.2.5. *If* $T = R + iJ$ *is normal-equivalent, and if* $A \in L(X)$ *is such that* $AT = TA$*, then* $AR = RA$*,* $AJ = JA$*.*

Remark 2.2.6. *The operator* $T = R + iJ$ *is normal-equivalent if and only if* $RJ = JR$ *and*

$$\|\exp(isR + itJ)\| \leq M$$

for some M *and all real* s, t*.*

Lemma 2.2.7. *If* $T \in L(X)$ *is normal-equivalent then* T *can be expressed uniquely in the form* $R + iJ$*, with* $RJ = JR$ *and* $\{R, J\}$ *hermitian-equivalent.*

Proof. If $T = R + iJ = R_1 + iJ_1$, where $RJ = JR$, $R_1 J_1 = J_1 R_1$, and $\{R, J\}$ and $\{R_1, J_1\}$ are hermitian-equivalent, then by Lemma 2.2.5, $\{R, J, R_1, J_1\}$ is a commuting hermitian-equivalent set: by Lemma 2.1.4 we can renorm X to make them simultaneously hermitian. Since $R - R_1 = i(J_1 - J)$ we have

$$\sigma(R - R_1) = \sigma(J_1 - J) = \{0\}:$$

by Sinclair's theorem $R = R_1$, $J = J_1$. □

The next result is more general, relying on more intricate consideration of local theory.

Theorem 2.2.8. *Let* $T_k = R_k + iJ_k \, (k = 1, 2)$ *be two normal-equivalent operators, and suppose that* T_1 *and* T_2 *are quasinilpotent equivalent. Then*

$$R_1 = R_2, \qquad J_1 = J_2.$$

Proof. ([2], Theorem 2). □

If $T \in L(X)$ is normal-equivalent then clearly $T' \in L(X')$ is normal-equivalent. The converse also holds: but is not obvious. We model our proof on that of

Behrends [6]. It depends on Lemma 2.2.9 which is essentially due to Behrends [6]: for completeness we include a proof.

In the following lemma we shall make use of the canonical projection on the third dual of X. If $i_X : X \to X''$ is the canonical injection, then $P = i_{X'}(i_X)'$ is a projection on X''' whose range is $i_{X'}(X')$ and whose kernel is $(i_X(X))^{\perp}$. We have the following facts about i_X, $i_{X'}$, $(i_X)'$ and P :

1. $(i_X)'i_{X'} = (identity)_{X'}$,
2. $Pi_{X'} = i_{X'}$,
3. $(i_X)'P = (i_X)'$,
4. $\|P\| = 1$,
5. $\langle i_X x, Py'''\rangle = \langle x, (i_X)'Py'''\rangle = \langle x, (i_X)'y'''\rangle = \langle i_X x, y'''\rangle$ for each x in X and y''' in X'''.

Lemma 2.2.9. *An operator $T \in L(X')$ is of the form S' (for some $S \in L(X)$) if and only if T'' commutes with the projection $P = i_{X'}(i_X)' : X''' \to X'''$.*

Proof. First note that if $S \in L(X)$ then

$$S''i_X = i_X S.$$

If now $T = S'$ for some $S \in L(X)$ then

$$T'i_X = i_X S$$

so

$$(i_X)'T'' = S'(i_X)' = T(i_X)'$$

and

$$PT'' = i_{X'}(i_X)'T'' = i_{X'}S'(i_X)' = i_{X'}T(i_X)'.$$

Next note that

$$T''i_{X'} = i_{X'}T$$

from which

$$T''P = T''i_{X'}(i_X)' = i_{X'}T(i_X)' :$$

so

$$PT'' = T''P.$$

Conversely suppose $T''P = PT''$. If $y''' \perp i_X(X)$, that is, $Py''' = 0$, then

$$\begin{aligned}\langle T' i_X x, y''' \rangle &= \langle i_X x, T'' y''' \rangle \\ &= \langle i_X x, PT'' y''' \rangle \qquad (\textit{by } 5 \textit{ above}) \\ &= \langle i_X x, T'' P y''' \rangle \\ &= 0\end{aligned}$$

i.e. $y''' \perp T' i_X(X)$. It follows that $T' i_X(X) \subseteq i_X(X)$ so that

$$S = (i_X)^{-1} T' i_X : X \to X$$

is well-defined: and then $T = S'$. □

We can now prove the following theorem, which generalises that of Behrends ([6], Theorem 1).

Theorem 2.2.10. *If $T' \in L(X')$ is normal-equivalent then $T \in L(X)$ is normal-equivalent.*

Proof. If $T' \in L(X')$ is normal-equivalent then $T' = R + iJ$ where $R, J \in L(X')$, and R, J commute, and $\| \exp(isR + itJ) \| \leq M$ for some M and all real s, t. Also $T''' = R'' + iJ''$ is normal-equivalent. By Lemma 2.2.9 we have $T'''P = PT'''$; by Lemma 2.2.5 we get $R''P = PR''$ and $PJ'' = JP''$: hence, by Lemma 2.2.9. there are $H,\ K \in L(X)$ such that $H' = R, K' = J$. So $T = H + iK$; now

$$\| \exp(isH + itK) \| = \| \exp(isR + itJ) \| \leq M$$

for all real s, t, so T is normal-equivalent (Remark 2.2.6). □

Definition 2.2.11. *An operator $T \in L(X)$ is* strongly normal *if $T = R + iJ$ where $RJ = JR$ and the set*

$$\{R^m J^n : m, n = 0, 1, 2, \ldots\}$$

is hermitian.

Remark 2.2.12. *If $T \in L(X)$ is strongly normal, $T = R + iJ$ as above, then the set $\{g_1(R,J) + ig_2(R,J) : g_1, g_2 \in C_{\mathbb{R}}(\sigma(T))\}$ is a commutative C^*-algebra under the operator norm and the natural involution $(g_1(R,J) + ig_2(R,J))^* = g_1(R,J) - ig_2(R,J)$, where $C_{\mathbb{R}}(\sigma(T))$ is the Banach algebra of continuous real -valued functions in two variables on $\sigma(T)$ ([15], §38).*

Definition 2.2.13. *An operator $T \in L(X)$ is* strongly normal-equivalent *if $T = R + iJ$ where $RJ = JR$ and the set*

$$\{R^m J^n : m, n = 0, 1, 2, \dots\}$$

is hermitian-equivalent.

Remark 2.2.14. *If $T \in L(X)$ is strongly normal-equivalent then $T' \in L(X')$ is strongly normal-equivalent.*

The next result is a refinement of Theorem 2.2.10.

Theorem 2.2.15. *If $T' \in L(X')$ is strongly normal-equivalent then $T \in L(X)$ is strongly normal-equivalent.*

Proof. Suppose that there exist operators R and J such that $T' = R + iJ$ and there is an equivalent norm $|\cdot|$ on X' with respect to which the set

$$\{R^m J^n : m, n = 0, 1, 2, \dots\}$$

is hermitian. Since T' is normal-equivalent, by Theorem 2.2.10 there exist H, K such that $T = H + iK$ where $HK = KH$ and H, K are hermitian-equivalent. The set

$$\{R^m J^n : m, n = 0, 1, 2, \dots\}$$

is hermitian-equivalent. So there is an M (≥ 1) such that

$$\|\exp(itR^m J^n)\| \leq M \qquad (t \in \mathbb{R},\ m, n = 0, 1, 2 \dots)$$

and we have

$$\|\exp(itH^m K^n)\| = \|\exp(itR^m J^n)\| \leq M \qquad (t \in \mathbb{R},\ m, n = 0, 1, 2, \dots).$$

If we define

$$|||x||| = \sup\{\|\exp(itH^mK^n)x\| : t \in \mathbb{R},\ m,n = 0,1,2,\dots\}$$

then $|||\cdot|||$ is a norm on X, equivalent to the original norm, and for each $t \in \mathbb{R}$ we have

$$|||\exp(itH^mK^n)||| = 1 \qquad (m,n = 0,1,2,\dots).$$

Therefore with this norm the set $\{H^mK^n : m,n = 0,1,2,\dots\}$ is hermitian: hence T is strongly normal-equivalent. $\square$

Note that if T is strongly normal-equivalent then the closed linear span of

$$\{R^mJ^n : m,n = 0,1,2,\dots\}$$

is a hermitian-equivalent set Lemma 2.1.4: equivalently,

$$\{f(R,J) : f \in C_{\mathbb{R}}(\sigma(T))\}$$

is hermitian-equivalent. We may therefore introduce yet another norm, γ, on X, with respect to which T will be strongly normal:

$$\gamma(x) = \sup\{\|\exp(if(R,J))x\| : f \in C_{\mathbb{R}}(\sigma(T))\}\,.$$

Then $\gamma(x) \geq |||x|||$: so $|||x'||| \leq \gamma(x')$ for $x' \in X'$.

Questions 2.2.16. *(I) Do γ and $|||\cdot|||$ coincide ?*
(II) Does the norm $|\cdot|$ (on X') coincide with either the dual of of γ or the dual of $|||\cdot|||$?
(III) Is $|\cdot|$ (on X') automatically a dual norm? that is, does there exist an equivalent norm η on X such that $|x'| = sup\{|\langle x,x'\rangle| : \eta(x) = 1\}$?

2.3 Prespectral operators

A family $\Gamma \subseteq X'$ is called *total* if and only if $y \in X$ and $\langle y, f\rangle = 0$, for all $f \in \Gamma$ together imply that $y = 0$. Let Σ be a σ-algebra of subsets of an arbitrary set Ω. Suppose that a mapping $E(\cdot)$ from Σ into the Boolean algebra of projections on X satisfies the following conditions:

1. $E(\delta_1) + E(\delta_2) - E(\delta_1)E(\delta_2) = E(\delta_1 \cup \delta_2)$,

2. $E(\delta_1)E(\delta_2) = E(\delta_1 \cap \delta_2)$ $(\delta_1, \delta_2 \in \Sigma)$,

3. $E(\Omega - \delta) = I - E(\delta)$ $(\delta \in \Sigma)$,

4. $E(\Omega) = I$,

5. there is a real constant M such that $||E(\delta)|| \leq M$ for all $\delta \in \Sigma$,

6. there is a total linear subspace Γ of X^* such that $\langle E(\cdot)x, y\rangle$ is countably additive on Σ, for each x in X and each y in Γ.

Then $E(\cdot)$ is called a *spectral measure* of class (Σ, Γ).

In the following Σ_p denotes the σ-algebra of Borel subsets of the complex plane.

Definition 2.3.1. *An operator S in $L(X)$ is called a* prespectral operator of class Γ *if there is a spectral measure $E(\cdot)$ of class (Σ_p, Γ) on X such that for all $\delta \in \Sigma_p$*

1. $SE(\delta) = E(\delta)\, S \qquad (\delta \in \Sigma_p)$

2. $\sigma(S|E(\delta)X) \subseteq \bar{\delta} \qquad (\delta \in \Sigma_p)$.

The spectral measure $E(\cdot)$ is called a resolution of the identity of class Γ *for S. If, in addition, $S = \int_{\sigma(S)} \lambda E(d\lambda)$ then S is said to be a* scalar-type *operator of class* Γ.

The basic decomposition theorem for prespectral operators is the following result due to Dunford [25].

Lemma 2.3.2. *Let T be a prespectral operator on X with a resolution of the identity $E(\cdot)$ of class Γ. Define*

$$S = \int_{\sigma(S)} \lambda E(d\lambda), \qquad N = T - S.$$

Then S is a scalar-type operator with a resolution of the identity of class Γ, and N is quasinilpotent.

Definition 2.3.3. *Let T be a prespectral operator on X with a resolution of the identity E of class Γ. Define*

$$S = \int_{\sigma(S)} \lambda E(d\lambda), \qquad N = T - S.$$

Then $S + N$ is called the Jordan decomposition of T corresponding to the resolution of the identity $E(\cdot)$. S is called the scalar part and N the radical part of the decomposition.

Lemmas 2.3.4, 2.3.5, 2.3.6, 2.3.7 and 2.3.8 are some basic facts needed later; these lemmas can be found in ([22], §5).

Lemma 2.3.4. *Let T be a prespectral operator on X of class Γ.*
(I) T has a unique resolution of the identity of class Γ.
(II) T has a unique Jordan decomposition for the resolution of the identity of all classes.

Lemma 2.3.5. *Let K be a compact Hausdorff space, and let Θ be continuous algebra homomorphism of $C(K)$ into $L(X)$ with $\Theta(z \mapsto 1) = I, \Theta(z \mapsto z) = S$. Then there is a spectral measure $E(\cdot)$ of class (Σ_K, X) with values in $L(X')$ such that*

$$\Theta(f)' = \int_K f(\lambda)\, E(d\lambda) \qquad (f \in C(K)).$$

Lemma 2.3.6. *Let T be a prespectral operator on X with resolution of the identity $E(\cdot)$ of class Γ. Then T' is prespectral on X' with resolution of the identity $F(\cdot)$of class X such that*

$$\left(\int_K f(\lambda)\, E(d\lambda)\right)' = \int_K f(\lambda)\, F(d\lambda) \qquad (f \in C(\sigma(T)).$$

Moreover if

$$S = \int_{\sigma(T)} \lambda\, E(d\lambda), N = T - S$$

then $S' + N'$ is the Jordan decomposition of T'.

Lemma 2.3.7. *If S be a scalar-type operator of class Γ then S is strongly normal-equivalent.*

Proof. If we define

$$R = \int_{\sigma(T)} \operatorname{Re}\lambda\, E(d\lambda),\ \ J = \int_{\sigma(T)} \operatorname{Im}\lambda\, E(d\lambda)$$

then by Lemma 2.1.5 there is an equivalent norm on X with respect to which $\{E(\tau) : \tau \in \Sigma_p\}$ are all hermitian. Let $p(x, y)$ be a polynomial with real coefficients in the two real variables $x = \operatorname{Re}\lambda$ and $y = \operatorname{Im}\lambda$. There is a sequence of real-valued simple Borel measurable function of λ converging uniformly on $\sigma(S)$ to $p(x, y)$. It follows from Lemma 1.4.4 that $p(R, J)$ is hermitian and so the

$$\{R^m, J^n : m, n = 0, 1, 2, \cdots\}$$

are all hermitian. □

Lemma 2.3.8. *If S is strongly normal-equivalent then S' is a scalar-type operator of class X.*

Proof. Let $\mathcal{A}$ be the closed subalgebra of $L(X)$ generated by $I, R,$ and J. Define $\Omega = \{x + iy : x \in \sigma(R), y \in \sigma(J)\}$. Observe that Ω is compact. Let $p(x, y)$ be a polynomial in the two variables x and y. We define

$$\|p\| = \sup\{|p(x, y)| : x + iy \in \Omega\},$$

$$\Theta(p) = p(R, J).$$

When p has a real coefficients, $\Theta(p)$ is hermitian, and so by Sinclair's theorem the norm and spectral radius of $\Theta(p)$ are equal. Let $\mathcal{M}$ denote the set of multiplicative linear functionals on $\mathcal{A}$. It follows that

$$\begin{aligned}\|\Theta(p)\| &= \sup\{|\phi[p(R, J)| : \phi \in \mathcal{M}\} \\ &= \sup\{|p(\phi(R), \phi(J))| : \phi \in \mathcal{M}\} \\ &\leq \|p\|_\infty.\end{aligned}$$

If p has complex coefficients we can express p in the form $p_1 + ip_2$, where p_1 and p_2 are polynomials in two real variables with real coefficients.Hence

$$\|\Theta(p_1)\| \leq \|p_1\| \leq \|p\|_\infty,$$

$$\|\Theta(p_2)\| \leq \|p\|_\infty,$$

and so

$$\|\Theta(p)\| \leq 2\|p\|_\infty.$$

It follows from the last inequality and the Stone-Weierstrass theorem that Θ can be extended to a continuous, identity preserving algebra homomorphism of $\mathbb{C}(\Omega)$ into $L(X)$. If $f_0(\lambda) = \lambda\,(\lambda \in \Omega)$ then

$$\Theta(f_0) = R + iJ = S$$

and by Lemma 2.3.5 S' is scalar-type of class X. □

The converse of Theorem 2.3.5 and Theorem 2.3.8 also holds (Theorem 2.3.9). Theorem 2.3.9 extends that of Berkson and Gillespie ([12], Theorem 8) and answers the question of ([12], Remark 1 on Theorem 9) affirmatively.

Theorem 2.3.9. *Let $S \in L(X)$. Then the following conditions are equivalent:*

1. *$S' \in L(X')$ is a scalar-type operator of class X,*

2. *$S \in L(X)$ is strongly normal-equivalent,*

3. *there exist a compact subset* Ω *of* $\mathbb{C}$ *and a norm continuous representation* $\Theta : C(\Omega) \mapsto X$ *such that* $\Theta(z \mapsto z) = S$, $\Theta(z \mapsto 1) = I$.

Proof. 1 $\Rightarrow$ 2. Suppose that $S' \in L(X')$ is scalar-type of class X with spectral measure $E(\cdot)$. There is a norm $|\cdot|$ on X', equivalent to the original norm $\|\cdot\|$, for which the values of $E(\cdot)$ are simultaneously hermitian Lemma 2.1.5. Then, putting $R = \int_{\sigma(S)} Re\lambda\, E(d\lambda)$ and $J = \int_{\sigma(S)} Im\lambda\, E(d\lambda)$, we see that $S' = R + iJ$, $RJ = JR$ and $\{R^m J^n : m, n = 0, 1, 2, 3, \ldots\}$ is $|\cdot|$–hermitian (proof of Lemma 2.3.7): so S' is strongly normal-equivalent. Hence, by Theorem 2.2.15, S is strongly normal-equivalent.

2 $\Rightarrow$ 3. If $|\cdot|$ is a norm equivalent to the original norm on X such that $S = H + iK$ where $HK = KH$ and

$$\{H^m K^n : m, n = 0, 1, 2, 3, \ldots\}$$

is $|\cdot|$–hermitian, then, using Sinclair's theorem as in the proof of Lemma 2.3.8, we have

$$|p(H, K)| \leq 2 \sup \{|p(Re\lambda, Im\lambda)| : \lambda \in \sigma(S)\}$$

for all polynomials $p(x, y)$ with complex coefficients. The Stone-Weierstrass theorem ensures the existence of the functional calculus Θ as claimed.

3 $\Rightarrow$ 1. This is immediate from Lemma 2.3.5. □

Theorem 2.3.10. *Let* S_1 *and* S_2 *be two operators with adjoint of scalar-type prespectral of class* X *and suppose* S_1 *and* S_2 *are quasinilpotent equivalent. Then* $S_1 = S_2$.

Proof. By Theorem 2.3.9 S_1 and S_2 are strongly normal-equivalent. Hence, by Theorem 2.2.8 $S_1 = S_2$. □

2.4 Spectral operators

In this section we consider spectral operators, a very important subclass of the prespectral operators.

Definition 2.4.1. *An operator $S \in L(X)$ is a* spectral operator *if there is a spectral measure $E(\cdot)$ defined on Σ_p with values in $L(X)$ such that*

1. *$E(\cdot)$ is countably additive on Σ_p in the strong operator topology,*
2. *$SE(\tau) = E(\tau)\, S \qquad (\tau \in \Sigma_p)$,*
3. *$\sigma(S|E(\tau)X) \subseteq \overline{\tau} \qquad (\tau \in \Sigma_p)$.*

Lemma 2.4.2 lists some of the important properties of resolutions of the identity of spectral operators ([22], Theorems 6.5, 6.6).

Lemma 2.4.2. *(i) $T \in L(X)$ is spectral if and only if it is prespectral of class X'.*
(ii) Let T be a spectral operator on X and let $E(\cdot)$ be the resolution of the identity of class X' for T. Let $A \in L(X)$ and $AT = TA$. Then

$$AE(\tau) = E(\tau)A \qquad (\tau \in \Sigma_p).$$

By Lemma 2.3.6 the natural class of operators to which a $C(\sigma(T))$-functional calculus leads is the scalar-type operators. It has been shown that we can characterise the scalar-type spectral operators by their functional calculus ([41], Corollary 1) or ([63], Theorem).

Lemma 2.4.3. *$S \in L(X)$ is a scalar-type spectral operator if and only if S has a weakly compact functional calculus.*

Theorem 2.4.4. *Let X be a Banach space which does not contain a subspace isomorphic to c_0. Then $S \in L(X)$ is scalar-type spectral if and only if S satisfies any (and hence all) of the conditions in Theorem 2.3.9.*

If $S \in L(X)$ is scalar-type prespectral, then S is strongly normal-equivalent (Lemma 2.3.7) and so S' is scalar-type prespectral of class X (Lemma 2.3.8).

Proof. Suppose $S' \in L(X')$ is scalar-type of class X. Then by Theorem 2.3.9 the operator $S \in L(X)$ has a $C(\sigma(S))$ functional calculus. Since X does not contain a subspace isomorphic to c_0 the $C(\sigma(S))$ functional calculus is weakly compact, (by Lemma 1.2.9). By Lemma 2.4.3, S is scalar-type spectral. □

The next lemma, due to Doust and deLaubenfels ([21], Theorem 3.2), gives us a source of examples of operators which have a $C(\sigma(T))$ functional calculus but are not scalar-type spectral.

Lemma 2.4.5. *Let X be a Banach space which contains a copy isomorphic to c_0. Then there exists an operator $T \in L(X)$ which is not scalar-type spectral, yet admits a $C(\sigma(T))$ functional calculus.*

The next result is immediate corollary of of Theorem 2.3.9 and Lemma 2.4.5.

Corollary 2.4.6. *Let X be a Banach space which contains a subspace isomorphic to c_0. Then there exists an operator which satisfies the three condition of Theorem 2.3.9, but which is not scalar-type spectral.*

Chapter 3

Well-bounded operators

The class of well-bounded operators on a Banach space was introduced by Smart [62] as a natural analogue of selfadjoint operators on Hilbert space, and was first studied by Smart and Ringrose ([58], [59], [62]). Well-bounded operators are defined as those which possess a functional calculus for absolutely continuous functions on some compact interval $[a, b]$ of the real line. Smart and Ringrose [58] proved that on a reflexive Banach space a well-bounded operator can always be written as an integral with respect to a spectral family of projections. Ringrose showed that the dual of a well-bounded operator always admits an integral representation with respect to a family of projections. This family of projections was called a decomposition of the identity (a definition will be given in section 2). Well-bounded operators of type (B) were characterised by Berkson and Dowson [10] and by Spain [65] as being those for which the absolutely continuous functional calculus is weakly compact. Berkson and Dowson [10] introduced the class of well-bounded operators with decomposition of the identity of bounded variation. They showed that there exists a well-bounded operator with decomposition of the identity of bounded variation which is not "decomposable in X" in that the projections in the decomposition of the identity are not the adjoints of projections on X. Doust and deLaubenfels [21] showed that if the Banach space X contain a subspace isomorphic to c_0, or a complemented subspace isomorphic to l_1, then there exists a well-bounded operator on X which is not of type (B). In this chapter it is shown that if a Banach space X contains a subspace isomorphic to c_0, or a complemented subspace isomorphic to l_1, then there exists a well-bounded operator on X which is not decomposable in X.

3.1 Well-bounded operators of types (A) and (B)

Let $\mathcal{P}(J)$ be the subalgebra of $AC(J)$ consisting of the polynomials on J. $\mathcal{P}(J)$ is norm dense in $AC(J)$. Let $T \in L(X)$. we define $p(T)$ in the natural way by setting

$$p(T) = \sum_{n=0}^{k} a_n T^n,$$

where $p(\lambda) = \sum_{n=0}^{k} a_n \lambda^n$. The map $p \mapsto p(T)$ is an algebra homomorphism.

We shall give some of the basic definitions regarding well-bounded operators.

Definition 3.1.1. *An operator $T \in L(X)$ is said to be* well-bounded *if there exist a constant K and a closed interval $J = [a, b] \subseteq \mathbb{R}$ such that*

$$||p(T)|| \leq K \left\{ |p(a)| + \int_a^b |p'(t)|\, dt \right\} \qquad (p \in \mathcal{P}(J).$$

Smart [62] introduced this definition and proved the following fundamental result.

Lemma 3.1.2. *Let T be a well-bounded operator on X with natural algebra homomorphism $\Theta : p \mapsto p(T)$ from $\mathcal{P}(J)$ into $L(X)$. Let K and J be as in Definition 3.1.1. Then Θ has a unique extension to an algebra homomorphism $\Theta : f \to f(T)$ from $AC(J)$ into $L(X)$ such that*

1. $\|f(T)\| \leq K|||f||| \qquad (f \in AC(J)),$
2. *if $S \in L(X)$ and $ST = TS$ then*
   $Sf(T) = f(T)S \qquad (f \in AC(J)),$
3. $f(T') = f(T)' \qquad (f \in AC(J)).$

We refer the reader to ([22], Lemma 15.2) for a proof.

Definition 3.1.3. *A function $u \in L_1(a, b)$ is* C-limitable on the right *at a point s of $[a, b)$ if the indefinite integral of u is differentiable on the right at s.*

Definition 3.1.4. *A function $u \in L_1(a,b)$ is* C-continuous on the right *at a point $s \in [a,b)$ if it is C-limitable on the right at s and if the derivative of the indefinite integral of u at the point s is equal to $u(s)$.*

Lemma 3.1.5. *Given any $x \in X, \phi \in X'$, there exists a function $\omega_{x,\phi}$ in $L_\infty(a,b)$, uniquely determined to within a null function, such that*

$$\langle f(T')\phi, x\rangle = f(b)\langle \phi, x\rangle - \int_a^b \omega_{x,\phi}(\lambda)df(\lambda) \qquad (f \in AC(J),$$

The function $\omega_{x,\phi}$ satisfies

$$\|\omega_{x,\phi}\| \leq \|\phi\|\|x\|,$$

and its equivalence class (modulo null functions) depends linearly on both x and ϕ.

Proof. ([59], Lemma 3). □

The notation of a decomposition of the identity was introduced by Ringrose in [59].

Definition 3.1.6. *A* decomposition of the identity *for X (on J) is a family*

$$\{E(s) : s \in \mathbb{R}\}$$

of projections on X' such that

1. *$E(s) = 0$ for $s < a$ and $E(s) = I$ for $s \geq b$,*
2. *$E(s)E(t) = E(t)E(s) = E(s)$ for $s \leq t$,*
3. *There is a real constant K such that $\|E(s)\| \leq K$ for $s \in \mathbb{R}$,*
4. *The function $s \mapsto \langle x, E(s)y\rangle$ is Lebesgue measurable for $x \in X$, $y \in X'$,*
5. *For each $x \in X$, the map $y \mapsto \langle x, E(s)y\rangle$ from X' into $L_\infty[a,b]$ is continuous when X' and $L_\infty[a,b]$ are given their weak* topologies as the duals of X and $L_1[a,b]$ respectively,*
6. *If $x \in X$, $y \in X'$, $s \in [a,b)$, and if the function $t \mapsto \int_a^t \langle x, E(u)y\rangle\, du$ is right differentiable at s, then the right derivative at s is $\langle x, E(s)y\rangle$.*

If there is a decomposition of the identity for X such that

$$\langle Tx, y\rangle = b\langle x, y\rangle - \int_a^b \langle x, E(u)y\rangle\, du \qquad (x \in X,\ y \in X').$$

We say that the family $\{E(s) : s \in \mathbb{R}\}$ is a decomposition of the identity for T.

The following lemma is due to Ringrose [59]; we shall use this lemma several times.

Lemma 3.1.7. *Let $T \in L(X)$ be a well-bounded operator on X and $\{E(\lambda) : \lambda \in \mathbb{R}\}$ denote the decomposition of the identity constructed in ([22], Lemma 15.17). If $S \in L(X)$ and $ST = TS$, then $S'E(\lambda) = E(\lambda)S'$.*

Proof. ([22], Theorem 15.19). □

Given a decomposition of the identity $\{E(s) : s \in \mathbb{R}\}$ there exists a unique well-bounded operator $T \in L(X)$ such that

$$\langle Tx, y\rangle = b\langle x, y\rangle - \int_a^b \langle x, E(u)y\rangle\, du$$

([22], Theorem 15.6). Every well-bounded operator has such a representation, but in general the decomposition of the identity is not uniquely determined by T ([22], Example 15.25).

Definition 3.1.8. *Let T be a well-bounded operator and let $\{E(s) : s \in \mathbb{R}\}$ be a decomposition of the identity for T. Then T is* decomposable in X *if there exists a family of projections $F(s)$ on X such that $F(s)' = E(s)$ for all $s \in \mathbb{R}$.*
T is well-bounded of type(A) *if it is decomposable in X and the function $\lambda \to F(\lambda)x$ is continuous on the right for every $x \in X$.*

It has recently been shown ([56], Theorem 3.2) that an operator T is well-bounded of type (A) on a Banach space X if and only if T is a well-bounded operator decomposable in X.

Definition 3.1.9. *T is* well-bounded of type (B) *if T well-bounded of type (A) and, in addition, for each real s, $\lim_{t\to s^-} F(t)x$ exists for every $x \in X$.*

The following lemma is due to Ringrose [59]. We use this lemma several times in this chapter and in Chapter 4.

Lemma 3.1.10. *Let T be a well-bounded operator decomposable in X and $\{F(\lambda) : \lambda \in \mathbb{R}\}$ be a family of projections on X whose adjoints form a decomposition of the identity for X. Then*
(i) T is uniquely decomposable;
(ii) if $S \in L(X)$ and $TS = ST$, then $F(\lambda)S = SF(\lambda)$ $(\lambda \in \mathbb{R})$;
(iii) given any x in X and $y' \in X'$, the function $\lambda \to \langle F(\lambda)x, y'\rangle$ is everywhere C-continuous on the right.

Proof. ([22], Theorem 16.3) □

Lemma 3.1.11 is due to Berkson and Dowson [10].

Lemma 3.1.11. *Let T be a well-bounded operator on X, and let $E(\cdot)$ be a decomposition of the identity for T. Then*

$$E(\lambda)X' = \{x' \in X' : \sigma_{T'}(x') \subseteq (-\infty, \lambda]\} \qquad (\lambda \in \mathbb{R}).$$

Proof. ([10], Theorem 5.6). □

Theorem 3.1.12. *Let U and U_1 be commuting well-bounded operators on X, where U is well-bounded of type (A) and $U - U_1$ is quasinilpotent. Then $U = U_1$.*

Proof. Let $E(\cdot)$ be the family of projections on X whose adjoints are a decomposition of the identity of U, By lemmas 3.1.7 and 3.1.10, there exist a decomposition of the identity $\{F(s) : s \in \mathbb{R}\}$ for U_1 such that $E(s)'F(s) = F(s)E(s)'$ for all $s \in \mathbb{R}$. Observe that U' and U_1' have the single valued extension property (Lemma 1.3.3). Given $x' \in X'$ denote by $\sigma_{U_1'}(x')$ the local spectrum of x' relative to U_1'. Since $U - U_1$ is quasinilpotent, so also is $U' - U_1'$. Hence $\sigma_{U_1'}(x')$ is equal to the local spectrum x' relative to U' by Lemma 1.3.4. Hence by Lemma 3.1.11 the projections $E(s)'$ and $F(s)$ have the same range, and hence are equal, since they commute. This is true for all $s \in \mathbb{R}$, and so $U = U_1$. □

Corollary 3.1.13. *If U is a quasinilpotent well-bounded operator, then $U = 0$.*

The integrals described here are based on the modified Stieltjes integral of Krabbe [42]. Spain applied this integration theory to establish various characterisation of well-bounded operators of type (B) ([65], Theorem 5).

Let $\mathcal{E}_J(J = [a, b])$ be the family of functions $E : \mathbb{R} \to L(X)$ satisfying

(*i*) $E(s) = E(s^+) = st \lim_{t \to s^+} E(t)$ $(s \in \mathbb{R})$

(*ii*) $E(s^-) = st \lim_{t \to s^-} E(t)$ exist $(s \in \mathbb{R})$

(*iii*) $E(s) = 0$ $(s < a)$

(*iv*) $E(s) = E(b)$ $(s \geq b)$.

Then $\sup_R \|E(s)\| = \sup_J \|E(s)\| < \infty$ for $E \in \mathcal{E}_J$ ([22], Lemma 17.1).

We say that a sequence $u = (u_k : 0 \leq k \leq m)$ is a *subdivision* of J if $a = u_0 < u_1 < \cdots, u_m = b$. The set U_J of all subdivisions of J admits a partial order $\geq$ defined by refinement: we write

$$u = (u_k : 0 \leq k \leq m) \leq v = (v_j : 0 \leq j \leq n)$$

when u refines v; that is, when each $[u_{k-1}, u_k]\,(1 \leq k \leq m)$ is contained in some $[v_{j-1}, v_j]\,(1 \leq j \leq n)$.

Let $M(u)$ be the family of sequences $u^* = (u_k{}^* : 1 \leq k \leq m)$ such that

$$u_{k-1} < u_k^* < u_k, \qquad (1 \leq k \leq m)$$

for each u in U_J.

A pair $\bar{u} = (u, u^*)$ with $u \in U_J$ and $u^* \in M(u)$ is called a *marked partition* of J. We write π_J for the family of marked partition of J and define the pre-order $\geq$ on π_J by setting $(u, u^*) \geq (v, v^*)$ if and only if $u \geq v$.

Let $\pi_J^i = \{\bar{u} = (u, u^*) \in \pi_J : u_{k-1} < u_k{}^* < u_k, 1 \leq k \leq m\}$ and let

$$\pi_J^r = \{\bar{u} = (u, u^*) \in \pi_J : u_k^* = u_k, 1 \leq k \leq m\}.$$

The sets U_j, π_J, π_J^i and π_J^r are directed by $\geq$. Also, π_J^i and π_J^r are cofinal in π_J.

Let Φ and Ψ be functions on J, one taking values in $\mathbb{C}$, the other taking values in $L(X)$. When $\bar{u} \in \pi_J$, we define

$$\sum \Phi(\Psi \Delta \bar{u}) = \sum_{k=1}^{m} \Phi(u_k^*)(\Psi(u_k) - \Psi(u_{k-1})).$$

The following integrals are defined as net limits in the strong operator topology of $L(X)$ when they exist. We write $st\lim$ for a limit in the strong operator topology. Then

$$\int_J \Phi\, d\Psi = st\lim_{\pi_J} \sum \Phi(\Psi\Delta\bar{u}),$$

an ordinary Stieltjes refinement integral.

$$\int_J^r \Phi\, d\Psi = st\lim_{\pi_J^r} \sum \Phi(\Psi\, \Delta\bar{u}),$$

a right Cauchy integral.

$$\int_J^i \Phi\, d\Psi = st\lim_{\pi_J^i} \sum \Phi(\Psi\Delta\bar{u}),$$

a modified Stieltjes integral.

Let $\mathcal{N}_J$ be the Banach subalgebra of $\mathrm{BV}(J)$ consisting of the functions in $\mathrm{BV}(J)$ which are left continuous on $(a, b]$. We define π_J^g for each g in $\mathrm{BV}(J)$ thus:

$$\pi_J^g = \begin{cases} \pi_J, & g \in \mathcal{N}_J, \\ \pi_J^i, & g \in \mathrm{BV}(J)\backslash\mathcal{N}_J. \end{cases}$$

Let $E_u = \sum_1^m E(u_{k-1})\chi_{[u_{k-1},u_k)} + E(b)\chi_{[b,\infty)}$ when $E \in \mathcal{E}_J$ and $u \in U_J$. The following integral is defined as a net limit in strong operator topology when it exists.

$$\oint_J E\, dg = st\lim_{\pi_J^g} \sum E(g\Delta\bar{u}) \qquad (g \in \mathrm{BV}(J),\ E \in \mathcal{E}(J)).$$

It is easy to verify that if $\oint E_1\, dg$ and $\oint E_2\, dg$ exist, then $\oint (E_1 + E_2)\, dg$ also exists and

$$\oint (E_1 + E_2) dg = \oint E_1\, dg + \oint E_2\, dg.$$

We have the following basic facts about the integral $\oint E\, dg$.

Lemma 3.1.14. *Let $g \in \mathrm{BV}(J)$ and $E \in \mathcal{E}(J)$. Then $\oint_J E\, dg$ exists and*

$$\oint_J E\, dg = st\lim_{U_j} \oint E_u\, dg.$$

Also

$$\left\| \oint_J E\, dg \right\| \le \mathrm{var}(g, J) \sup_J \|E(s)\|,$$

and

$$\left\| \oint_J E\, dg\, x \right\| \le \mathrm{var}(g, J) \sup_J \|E(s)x\| \qquad (x \in X).$$

Proof. ([22], Theorem 17.4). □

For g in $\mathrm{BV}(J)$ and $E \in \mathcal{E}(J)$, we define

$$S(g,E) = g(b)E(b) - \oint_J E\,dg.$$

Lemma 3.1.15. *Let $g \in \mathrm{BV}(J)$, $E \in \mathcal{E}(J)$, $T \in L(X)$ and $s \in J$. Then*

1. $S(g\chi_{[s,\infty)}T) = g(s)T$,
2. $\|S(g,E)\| \leq |||g||| \sup_J \|E(s)\|$,
3. $\|S(g,E)x\| \leq |||g||| \sup_J \|E(s)x\| \qquad (x \in X)$,
4. $S(\chi_{[a,s]}, E) = E(s)$.
5. $$S(g,E) = \begin{cases} g(a)E(a) + \oint_J g\,dE, & g \in \mathcal{N}_J, \\ g(a)E(a) + \oint_J^r g\,dE, & g \in \mathrm{BV}(J)\backslash\mathcal{N}_J. \end{cases}$$

Proof. ([65], Lemma 6 and Theorem 3). □

We shall write $\int_J^{\oplus} g\,dE$ instead of $S(g,E)$ when $g \in \mathrm{BV}(G)$ and $E \in \mathcal{E}(J)$.

Lemma 3.1.16. *Let T be a well-bounded operator on X and let $J = [a,b]$ and K be chosen so that for every complex polynomial p we have*

$$\|p(T)\| \leq K\{|p(b)| + \mathrm{var}_J(p)\}.$$

Let $\Theta : AC(J) \to L(X)$ be the $AC(J)$-functional calculus discussed in Lemma 3.1.2. Then the following conditions are equivalent.

(i) T is of type (B).

(ii) For every $x \in X$, $\Theta_x : AC(J) \to X$ $(\Theta_x(f) = \Theta(f)x)$ is a compact linear map of $AC(J)$ into $L(X)$.

(iii) For every $x \in X$, $\Theta_x : AC(J) \to X$ $(\Theta_x(f) = \Theta(f)x)$ is a weakly compact linear map of $AC(J)$ into $L(X)$.

(iv) $T = \int_J^{\oplus} \lambda\, dE(\lambda)$ where $E(\cdot)$ is the decomposition of the identity of T.

Proof. ([65], Theorem 5). □

If X is a reflexive Banach space then every well-bounded operator on X is of type (B). If X contains either a subspace isomorphic to c_0, or else a complemented subspace isomorphic to l_1, then there exists a well-bounded operator on the Banach space X which is not of type (B) ([21], Theorem 4.4).

It has been shown if X contains either a subspace isomorphic to c_0, or else a complemented subspace isomorphic to l_1, then there exists a well-bounded operator which is not decomposable in X. The following result is useful and does not appear to have been published before.

Theorem 3.1.17. *Let $X = Y \oplus Z$ where X and Z are closed subspaces of X and let $P \in L(X)$ be the projection onto Y with kernel Z. Let $T \in L(X)$ be a well-bounded operator of type (A) with the decomposition of the identity $E(s)$ $(s \in \mathbb{R})$ and let $PT = TP$. Then setting $G(s)y' = E(s)(y', 0)$ $(s \in \mathbb{R})$ gives a decomposition of the identity on $Y' = P'X'$. Also, Y is invariant subspace for T and the restriction $T|_Y$ on Y is well-bounded of type (A).*

Proof. Let $F(\cdot)$ be the family of projections such that $E(s) = F(s)'$. Since

$$PT = TP$$

we have $PF(s) = F(s)P$ by Lemma 3.1.10. If we define

$$\begin{aligned} G_1(s)y &= F(s)(y,0) \\ G(s) &= G_1(s)' \qquad\qquad (s \in \mathbb{R}) \end{aligned}$$

then we have

$$\begin{aligned} \langle y, G(s)y' \rangle &= \langle G_1(s)y, y' \rangle \\ &= \langle F(s)(y,0), (y',0) \rangle \\ &= \langle (y,0), E(s)(y',0) \rangle. \end{aligned}$$

By Theorem 3.1 of [54], $G(s)$ is a decomposition of the identity on the dual $Y' = P'X'$. Also, Y is an invariant subspace for T and $G(\cdot)$ is a decomposition of the identity for $T|_Y$.

Since $G(s) = G_1(s)'$ $(s \in \mathbb{R})$, we see that $T|_Y$ is decomposable in Y. □

Our main result relies on the following construction from [54], Example 4.13. Note that c is the linear space of all convergent sequences of scalars, and c_0 is the linear space of all sequences converging to zero.

Example 3.1.18 ([54], Example 4.13). *On the Banach space c_0 define projections Q_n $(n \in \mathbb{N})$ by*

$$Q_n(x_1, x_2, x_3, \dots) = (\underbrace{x_n, \dots, x_n}_{n\ times}, x_{n+1}, x_{n+2}, \dots).$$

For $n \geq 1$, let $P_n = Q_n - Q_{n+1}$. Define $T = \sum_{n=1}^{\infty} \dfrac{1}{n} P_n$. Then T is well-bounded of type (B) but T' is not decomposable in X'.

For the convenience of the reader we reproduce the details as given in [54].

Let $P_n = Q_n - Q_{n+1}$. Then $\{P_n\}$ forms a sequence of disjoint finite-rank projections so Theorem 2.3 [53], shows that $T = \sum_{n=1}^{\infty} \dfrac{1}{n} P_n$ is well-bounded. Indeed, since $Q_n \to 0$ in the strong operator topology, the operator T is well-bounded of type (B). The unique decomposition of the identity for T is given by

$$F(s) = \begin{cases} 0 & \text{if } s \leq 0, \\ Q_n' & \text{if } s \in \left[\frac{1}{n}, \frac{1}{n-1}\right) \text{ for } n \geq 2, \\ I & \text{if } s \geq 1. \end{cases}$$

T is well-bounded and hence T' is well-bounded. However, T' is not well-bounded of type (A). Suppose that T is decomposable in X' and let $\{E(s) : s \in \mathbb{R}\} \subseteq L(l_\infty)$ denote any decomposition of the identity for T'. If $s \neq 0$ then $E(s) = F(s)'$. But there does not exist $S \in L(l_1)$ such that $S' = E(0)$. If $y = (y_n) \in c$ then $E(0)y = l(y)u$ where $u = (1, 1, 1, \dots) \in c$ and $l(y) = \lim_n y_n$. Fix $x = (x_n) \in l_1$ and $y = (y_n) \in c \subseteq l_\infty$. Define

$$G(t) = \int_0^t \langle x, E(s)y \rangle \, ds.$$

For $t \in (0, 1)$, let N_t be the unique integer such that $1 - t < tN_t \leq 1$. Then

$$\begin{aligned} \frac{G(t) - G(0)}{t} &= \frac{1}{t} \int_0^t \langle x, E(s)y \rangle \, ds \\ &= \frac{1}{t} \int_0^{1/N_t} \langle x, E(s)y \rangle \, ds + \frac{1}{t} \int_{1/N_t}^t \langle x, E(s)y \rangle \, ds. \end{aligned}$$

Now
$$\left|\frac{1}{t}\int_{1/N_t}^{t}\langle x, E(s)y\rangle\,du\right| \le \frac{|t-1/N_t|}{t}K\|x\|\|y\| \to 0,$$
as $t \to 0^+$. On the other hand
$$\begin{aligned}\frac{1}{t}\int_0^{1/N_t}\langle x, E(s)y\rangle\,ds \\ &= \frac{1}{t}\sum_{n=N_t+1}^{\infty}\frac{1}{n(n-1)}\langle Q_n'x, y\rangle \\ &= \frac{1}{t}\sum_{n=N_t+1}^{\infty}\frac{1}{n(n-1)}\left(y_n\sum_{i=1}^{n}x_i + \sum_{i=n+1}^{\infty}x_iy_i\right).\end{aligned}$$
Let
$$\epsilon_n = \left(y_n\sum_{i=1}^{n}x_i + \sum_{i=n+1}^{\infty}x_iy_i\right) - l(y)\sum_{i=1}^{\infty}x_i.$$
Now, given any $\epsilon > 0$ there exists $t_\epsilon > 0$ such that for all $n > N_{t_\epsilon}$, $|\epsilon_n| < \epsilon$. Thus,
$$\begin{aligned}\frac{1}{t}\int_0^{1/N_t}\langle x, E(s)y\rangle\,ds \\ &= \frac{1}{t}\sum_{n=N_t+1}^{\infty}\frac{1}{n(n-1)}\left(l(y)\sum_{i=1}^{\infty}x_i + \epsilon_n\right) \\ &= \frac{1}{tN_t}l(y)\sum_{i=1}^{\infty}x_i + \frac{1}{t}\sum_{n=N_t+1}^{\infty}\frac{\epsilon_n}{n(n-1)}.\end{aligned}$$
For any $t < t_\epsilon$,
$$\left|\frac{1}{t}\sum_{n=N_t+1}^{\infty}\frac{\epsilon_n}{n(n-1)}\right| < \frac{\epsilon}{t}\sum_{n=N_t+1}^{\infty}\frac{1}{n(n-1)} \le \epsilon.$$
On the other hand, $(tN_t)^{-1} \to 1$ as $t \to 0^+$. It follows that G is right differentiable at 0 and that
$$\lim_{t\to 0^+}\frac{G(t)-G(0)}{t} = l(y)\sum_{i=1}^{\infty}x_i = \langle x, l(y)u\rangle.$$
By condition (6) for a decomposition of the identity, we must have that $E(0)y = l(y)u$. If $y = (y_n) \in l_\infty$, then y is the weak* limit of the sequence $\{\omega_n\} \subseteq c$ where $\omega_n = (y_1, y_2, \ldots, y_n, y_n, \ldots)$. Suppose that there exists $S \in L(l_1)$ satisfying $S' =$

$E(0)$. Then, for any $x \in l_1$,

$$\begin{aligned}\langle x, E(0)y\rangle &= \langle Sx, y\rangle \\ &= \lim_n \langle Sx, \omega_n\rangle = \lim_n \langle x, E(0)\,\omega_n\rangle \\ &= \lim_n \langle x, l(\omega_n)\,v\rangle = \lim_n y_n \sum_{i=1}^{\infty} x_i.\end{aligned}$$

But the last limit may not exist. It follows that no such operator S can exist. Thus T' is not decomposable in l_1.

A Banach space X contains a complemented subspace isomorphic to l_1 if X' contains a copy of c_0 ([45], Proposition 2.e.8).

We can now prove the main result of this section.

Theorem 3.1.19. *Let X be a Banach space which contains a complemented subspace isomorphic to l_1. Then there exists a well-bounded operator $S \in L(X)$ which is not decomposable in X.*

Proof. Let $X = Y \oplus Z$ where Y is a space isomorphic to l_1. By Example 3.1.18 there exists an operator T which is well-bounded but not decomposable in Y. We define the operator $S = T \oplus 0$ on $X = Y \oplus Z$. By Theorem 4.3 of [55], S is well-bounded. We define the projection Π of X onto Y by $\Pi(y, z) = (y, 0)$. Then $\Pi S = S\Pi$. If S decomposable in X, then by Lemma 3.1.7, we have

$$\Pi' E(s) = E(s)\Pi' \qquad (s \in \mathbb{R})$$

where $E(s)$ is the decomposition of the identity for S. By Theorem 3.1.17, the operator $S|_Y = T$ is decomposable in Y. This gives us a contradiction. □

The next example shows "X does not contain a copy of c_0" is not sufficient for every well-bounded operator to be decomposable in X. This example is due to Ringrose.

Example 3.1.20. *Let $X = L_1(0, 1)$, and let $T \in L(X)$ be defined by the equation*

$$Tx(t) = tx(t) + \int_0^t x(u)\,du \qquad (1 \le t \le 1).$$

For any polynomial p, it follows easily that

$$(p(t)x)(t) = p(t)x(t) + p'(t)\int_0^t x(u)du.$$

Hence

$$p(T) \leq \sup_{t\in[0,1]} |p(T)| + \operatorname*{var}_{[0,1]} p \leq 2\{p(1) + \operatorname*{var}_{[0,1]} p\}.$$

Thus T is a well-bounded operator. It is easily seen (by its uniqueness) that the homomorphism of Lemma 3.1.2 is determined by the equation

$$(f(T)x)(t) = f(t)x(t) + f'(t)\int_0^t x(u)\,du \qquad (x \in X, f \in AC(J)).$$

We shall make the customary identification of X' with $L_\infty(0,1)$. When $\phi \in X'$, $x \in X$, $f \in AC(J)$, we have

$$\begin{aligned}\langle \phi, f(T)x\rangle &= \int_0^1 \phi(t)f(T)x(t)\,dt + \int_0^1 \phi(t)f'(t)\left(\int_0^t x(u)\,du\right)dt \\ &= f(1)\langle\phi,x\rangle - \int_0^1 \{\int_0^t \phi(u)x(u)\,du - \phi(t)\int_0^t x(u)\,du\}f'(t)\,dt.\end{aligned}$$

Hence the functions $\omega_{x,\phi}$ of Lemma 3.1.5 are given by

$$\omega_{x,\phi}(t) = \int_0^t \phi(u)x(u)du - \phi(t)\int_0^t x(u)du.$$

The first term on the right- hand side of this equation is absolutely continuous, and therefore $C-$limitable on the right throughout $[0,1)$. However, for suitably chosen x and ϕ, the second term will not have this property. For example, define

$$x(t) = \begin{cases} 1 & \text{if } 0 < t < 1/2, \\ 0 & \text{if } 1/2 \leq t < 1. \end{cases}$$

$$\phi(t) = 2 + \sin(\log|t - 1/2|) \qquad (0 < t < 1, t \neq 1/2)$$

$$\phi(1/2) = 0.$$

It is easily verified that $\phi \in L_\infty(0,1)$and ϕ is not C-limitable on the right at $t = 1/2$. It follows that $\omega_{x,\phi}$ is not C-limitable on the right throughout $[0,1)$. Now, by Lemma 3.1.10, T is not decomposable in X. (Note that the space X is weakly complete.)

An interesting question is whether there exists a non-reflexive Banach space X on which every well-bounded operator is decomposable in X.

3.2 Well-bounded operators with decomposition of the identity of bounded variation

Definition 3.2.1. *Let $T \in L(X)$ be a well-bounded operator on X. A decomposition of the identity $\{E(s) : s \in \mathbb{R}\}$ for T is said to be of* bounded variation *if the function $s \to \langle x, E(s)y' \rangle$ is of bounded variation on $\mathbb{R}$ for every $x \in X$ and $y' \in X'$.*

Example 3.2.2 ([22], Example 16.19). *Let X be the Banach space of all convergent sequences $w = \{\beta_n\}$ of complex numbers under the norm $\|w\| = \sup_n|\beta_n|$. The pairing of X' with l_1 given by*

$$\langle w, f\rangle = \lambda_1 \lim_{n\to\infty} \beta_n + \sum_{n=1}^{\infty} \beta_n \lambda_{n+1},$$

where $f = \{\lambda_n\} \in l_1$, induces an isometric isomorphism of l_1 onto X'. Define $T \in L(X)$, by

$$T\{\beta_n\} = \left\{-\frac{\beta_n}{n}\right\}.$$

T is a well-bounded operator decomposable in X with decomposition of the identity of bounded variation but T is not a well-bounded operator of type (B).

Example 3.2.3 was introduced by Gillespie in [30] to show that the sum of two commuting well-bounded operators is not necessarily well-bounded. This example also shows that there exists a well-bounded operator of type (B) which is not a well-bounded operator with decomposition of the identity of bounded variation.

Example 3.2.3. *Let e_n denote the element of l_2 with 1 in the n^{th} place and 0 elsewhere. Then $\{e_n\}$ constitutes an orthonormal basis for l_2. Define $\alpha_1 = 0$ and*

$$\alpha_n = \frac{1}{n \log n} \qquad (n = 2, 3, 4, \cdots).$$

Observe that $\alpha_n \geq 0$ $(n = 1, 2, 3, \cdots)$, $\sum_{j=1}^{\infty} j{\alpha_j}^2 < \infty$ and $\sum_{j=1}^{\infty} \alpha_j = \infty$. Let $\{x_n\}$ and $\{y_n\}$ be defined by

$$x_{2n-1} = e_{2n-1} + \sum_{i=1}^{n} \alpha_{n-i+1} e_{2i}, \qquad x_{2n} = e_{2n} \qquad (n = 1, 2, 3, \cdots),$$

$$y_{2n-1} = e_{2n-1}, \qquad y_{2n} = \sum_{i=1}^{n} (-\alpha_{n-i+1}) e_{2i-1} + e_{2n} \qquad (n = 1, 2, 3, \cdots).$$

Then

1. $\{x_n\}$ *is a basis of* l_2,

2. $\langle x_n, y_m\rangle = \delta_{nm}$ *for* $n, m = 1, 2, 3, \cdots$,

3. $\langle e_{2n-1}, y_{2m-1}\rangle = \delta_{nm}$ *for* $n, m = 1, 2, 3, \cdots$,

4. $\|\sum_{j=1}^{n} n^{-1/2} x_{2j-1}\| \to \infty$ *as* $n \to \infty$.

If we define p_n *in* $L(l_2)$ *by*

$$p_n x = \langle x, y_n\rangle x_n \qquad x \in l_2$$

for $n = 1, 2, 3, \cdots$ *we see that* p_n *is a projection,* $p_n p_m = 0$ $(n \neq m)$, *and* $I = st\lim\sum_{n=1}^{\infty} p_n$. *(This notation means that the series converges in the the strong operator topology of* $L(l_2)$.*) By ([22], Theorem 18.5) the series* $\displaystyle\sum_{n=1}^{\infty}\frac{n+1}{n}p_n$ *converges strongly in* $L(l_2)$ *and its sum is a well-bounded operator of type (B). Let* $T = st\lim\displaystyle\sum_{n=1}^{\infty}\frac{n+1}{n}p_n$. *If* T *is a well-bounded operator with decomposition of the identity of bounded variation then* T' *is scalar-type of class* l_2. l_2 *is a Hilbert space and hence* T *is scalar-type spectral. But* T *is not scalar-type spectral (a spectral operator has an unconditionally convergent expansion while* T *has a conditionally convergent expansion). It follows that* T *is not well-bounded with decomposition of the identity of bounded variation.*

Examples 3.2.2 and 3.2.3 show that neither of the classes of well-bounded operators with decomposition of the identity of bounded variation and of well-bounded operators of type (B) includes the other.

The next lemma, due to Berkson and Dowson, relates well-bounded operators and scalar-type operators ([10], Theorem 5.2).

Lemma 3.2.4. *Suppose that* $T \in L(X)$ *and* $\sigma(T) \subseteq \mathbb{R}$. *Then the following conditions are equivalent.*

(i) T *is a well-bounded operator with a decomposition of the identity of bounded variation.*

(ii) There is a compact interval J *and a constant* M *such that*

$$\|p(T)\| \leq 4M \sup_{t\in J} |p(t)|$$

for all complex polynomials p.

(iii) T' is a scalar-type operator of class X.

If (i) holds, then T is a uniquely decomposable well-bounded operator.

Proof. ([10], Theorem 5.2). □

Theorem 3.2.5. *If T is a well-bounded operator with decomposition of the identity of bounded variation then the set*

$$\{T^n : n = 0, 1, 2, \cdots\}$$

is hermitian-equivalent.

Proof. If T is a well-bounded operator with decomposition of the identity of bounded variation, then, by Lemma 3.2.4, T' is real scalar-type of class X with resolution of the identity $E(\cdot)$. Then by Theorem 2.3.9 there are operators $H, K \in L(X)$ such that $T = H + iK$, $\{H^n K^m : n, m = 0, 1, 2, \cdots\}$ is hermitian-equivalent and $H' = \int_{\sigma(S)} \operatorname{Re}\lambda\, E(d\lambda)$ and $K' = \int_{\sigma(S)} \operatorname{Im}\lambda\, E(d\lambda)$. Note that $\sigma(T) \subseteq \mathbb{R}$ so that $\sigma(T') \subseteq \mathbb{R}$: then $K' = 0$: so that $T = H$. □

Corollary 3.2.6. *Let $T \in L(X)$ be well-bounded with decomposition of the identity of bounded variation, $x_0 \in X$ and*

$$\lim_{n\to\infty} \|T^n x_0\|^{1/n} = 0.$$

Then $Tx_0 = 0$.

Proof. By Theorem 3.2.5 T is hermitian-equivalent. The result now follows from ([2], Proposition 1). □

We generalise this last result to apply to all well-bounded operators in the next chapter (Theorem 4.2.10), relying on more intricate considerations of local theory.

Corollary 3.2.7. *Let $T_1 \in L(X)$ and $T_2 \in L(X)$ be well-bounded operators with decomposition of the identity of bounded variation, $S \in L(X,Y)$, and suppose that* $\lim_{n\to\infty} \|C(T_1,T_2)^n S\| = 0$. *Then $T_1 S = S T_2$.*

Recall that two operators $T, S \in L(X)$ are said to be *quasinilpotent equivalent*, $T \overset{q}{\sim} S$ ([17], Definition 1.2.1), if and only if

$$\lim_{n\to\infty} \|C(T,S)^n I\|^{1/n} = 0 = \lim_{n\to\infty} \|C(S,T)^n I\|^{1/n}$$

where I is the identity operator on X.

Corollary 3.2.8. *Let T_1 and T_2 be well-bounded operators with decomposition of the identity of bounded variation. Suppose $T_1 \overset{q}{\sim} T_2$. Then $T_1 = T_2$.*

Proof. T_1 and T_2 are hermitian-equivalent by Theorem 3.2.5. Now the result follows from ([2], Corollary 2). □

Recall that two operators $T, S \in L(X)$ are said to be *quasinilpotent equivalent*, $T \overset{q}{\sim} S$ ([17], Definition 1.2.1), if and only if

$$\lim_{n\to\infty} \|C(T,S)^n \mathrm{I}\|^{1/n} = 0 = \lim_{n\to\infty} \|C(S,T)^n \mathrm{I}\|^{1/n}$$

where I is the identity operator on X.

Corollary 3.2.8. *Let T_1 and T_2 be well-bounded operators with decomposition of the identity of bounded variation. Suppose $T_1 \overset{q}{\sim} T_2$. Then $T_1 = T_2$.*

Proof. T_1 and T_2 are hermitian-equivalent by Theorem 3.2.5. Now the result follows from ([2], Corollary 2). □

Chapter 4

AC-operators

4.1 AC-operators and well-bounded operators with dual of scalar-type

In [12] Berkson and Gillespie introduced the concept of an AC-operator as an operator which possesses a functional calculus for the absolutely continuous functions on some rectangle in $\mathbb{C}$ (more detailed definitions are given below). Berkson and Gillespie showed that these operators can be characterised by the fact they possess a splitting into real and imaginary parts, $T = U + iV$, where U and V are commuting well-bounded operators. They showed [12] that if U and V are well-bounded of type (B) this splitting is unique, and that if $S \in L(X)$ commutes with $U + iV$ then S commutes with U and V. Berkson, Gillespie and Doust later showed that neither result is guaranteed if the type (B) hypothesis is omitted [11].

In this chapter the AC-operators $U + iV$, where U and V are commuting well-bounded with decomposition of the identity of bounded variation are studied. In this case if $S \in L(X)$ commutes $U + iV$ then S commutes with U and V. It is shown if $T = U + iV$, where U and V are commuting well-bounded operators with decomposition of the identity of bounded variation, and if either X does not contain a copy of c_0, or if U and V are decomposable in X, then this representation is unique.

One of the major complications one encounters when trying to extend this theory to operators with complex spectra is deciding upon the correct concept of an *absolutely continuous* function of two variables to use. In the discussion that follows we shall identify the subsets of $\mathbb{R}^2$ with subsets of $\mathbb{C}$ in the usual way. Let m denote Lebesgue measure on $\mathbb{R}^2$. Recall that if $J = [a, b]$ and $K = [c, d]$ are two compact intervals in $\mathbb{R}$, and if Λ is a rectangular partition of $J \times K$:

$$a = s_0 < s_1 < \cdots < s_n = b, \qquad c = t_0 < t_1 < \cdots < t_m = d,$$

then for a function $f : J \times K \to \mathbb{C}$, we define

$$V_\Lambda(f) = \sum_{i=1}^{n}\sum_{j=1}^{m} |f(s_i, t_j) - f(s_i, t_{j-1}) - f(s_{i-1}, t_j) + f(s_{i-1}, t_{j-1})|$$

and

$$\mathrm{var}_{J\times K}(f) = \sup\{V_\Lambda(f) : \Lambda \text{ is a rectangular partition of } J \times K\}.$$

A function f is of bounded variation if $\mathrm{var}_{J\times K} f$, $\mathrm{var}_J f(\cdot, d)$ and $\mathrm{var}_K f(b, \cdot)$ are all finite. By Theorem 1.2.4 the set $\mathrm{BV}(J \times K)$ of all functions $f : J \times K \to \mathbb{C}$ of bounded variation is a Banach algebra under the norm

$$|||f||| = |f(b, d)| + \mathrm{var}_J f(\cdot, d) + \mathrm{var}_K f(b, \cdot) + \mathrm{var}_{J\times K} f.$$

A function $f : J \times K \to \mathbb{C}$ is said to be *absolutely continuous* if

1. For all $\epsilon > 0$, there exists $\delta > 0$ such that

   $$\sum_{R\in\mathfrak{R}} \mathrm{var}_R f < \epsilon$$

   whenever $\mathfrak{R}$ is a finite collection of non-overlapping subrectangles of $J \times K$ with $\sum_{R\in\mathfrak{R}} m(R) < \delta$;

2. The marginal functions $f(\cdot, d)$and $f(b, \cdot)$ are absolutely continuous functions on J and K respectively.

The set $AC(J \times K)$ of all absolutely continuous functions $f : J \times K \to \mathbb{C}$ is a Banach subalgebra of $\mathrm{BV}(J \times K)$, and is the closure in $\mathrm{BV}(J \times K)$ of the polynomials in two real variables on $J \times K$. Equivalently, one can consider $AC(J \times K)$ to be the closure in $\mathrm{BV}(J \times K)$ of the polynomial functions $p(z, \bar{z})$ on $J \times K \subset \mathbb{C}$.
Define the functions $u, v \in AC(J \times K)$ by $u(x, y) = x$ and $v(x, y) = y$.

Definition 4.1.1. *An operator $T \in L(X)$ is said to be an* AC-operator *if there exists a continuous unital Banach algebra homomorphism* $\Theta : AC(J \times K) \to L(X)$ *for which* $\Theta(u+iv) = T$.

Berkson and Gillespie ([12], Theorem 5) proved that this is equivalent to the condition that T can be written as $T = U+iV$ where U and V are commuting well-bounded operators on X. They showed that if U and V are well-bounded of type (B) the representation in the form $T = U+iV$ is unique and if $S \in L(X)$ commutes with T then S commutes with U and V. However, neither result is guaranteed if the type (B) hypothesis is omitted as is shown by ([11], Examples 3.1 and 3.4), reproduced below.

The next lemma, due to Berkson and Gillespie [12], links absolutely continuous functions of two variables and operators of the form $U+iV$, with U and V commuting and well-bounded.

Lemma 4.1.2. *Let $T \in L(X)$. Then the following conditions are equivalent:*

1. *There exist commuting well-bounded operators U and V on X such that $T = U+iV$,*
2. *There exist compact intervals J and K in $\mathbb{R}$ and a norm-continuous representation $\Theta : AC(J \times K) \to L(X)$ such that $\Theta(u+iv) = T$.*

Proof. ([12], Theorem 5). □

Operators of the form $U+iV$ with U and V commuting well-bounded operators and of type (B) were characterised by Berkson and Gillespie in terms of weakly compact representation of algebras of the form $AC(J \times K)$.

Lemma 4.1.3. *Let $T \in L(X)$. Then the following conditions are equivalent:*

1. *There exist commuting type (B) well-bounded operators U and V on X such that $T = U+iV$,*
2. *There exist compact intervals J and K in $\mathbb{R}$ and a strongly-compact representation $\Theta : AC(J \times K) \to L(X)$ such that $\Theta(u+iv) = T$,*

3. *There exist compact intervals J and K in $\mathbb{R}$ and a weakly-compact representation Θ of $AC(J \times K)$ on X such that $\Theta(u+iv) = T$.*

Proof. ([12], Theorem 6). ☐

The next two lemmas are due to Berkson and Gillespie; we shall generalise these lemmas in §2.

Lemma 4.1.4. *Let U and V be commuting well-bounded operators of type (B) on X and let $S \in L(X)$ commute with $U + iV$. Then S commutes with U and V.*

Proof. ([12], Lemma 4). ☐

Lemma 4.1.5. *Let*

$$T = U + iV = U_1 + iV_1$$

where

1. *U and V are commuting well-bounded operators of type (B) on X,*

2. *U_1 and V_1 are commuting well-bounded operators on X.*

Then $U = U_1$ and $V = V_1$.

Proof. ([12], Theorem 7). ☐

Corollary 4.1.6. *An AC-operator on a reflexive space can be expressed uniquely in the form $U + iV$, with U and V commuting well-bounded operators of type (B).*

Example 4.1.7 ([11], Example 3.1). *Let $X = L_\infty[0,1] \oplus L_1[0,1]$, with norm*

$$\|(f,g)\| = \|f\|_\infty + \|g\|_1.$$

Define the operator $U \in L(X)$ by $U(f,g) = (hf, hg)$, where h is the function $h(t) = t$, $t \in [0,1]$ Then U is well-bounded, and so $T = U + iU$ is an AC-operator. Consider *now the operator $Q \in L(X)$ given by $Q(f,g) = (0,f)$. For any $\alpha \in \mathbb{C}$ and any non-negative integer n, a simple induction proof shows that $(U+\alpha Q)^n = U^n + nU^{n-1}\alpha Q$. Thus for any polynomial p, $p(U+\alpha Q) = p(U) + p'(U)\alpha Q$. If $(f,g) \in X$ then*

$$\begin{aligned}\|p(U,\alpha Q)(f,g)\| &= \|pf\|_\infty + \|\alpha p' f + pg\|_1 \\ &\leq \|p\|_\infty \|f\|_\infty + |\alpha| \|p'\|_1 \|f\|_\infty + \|p\|_\infty \|g\|_1 \\ &\leq (1+|\alpha|)\|p\|_{BV[0,1]} \|(f,g)\|,\end{aligned}$$

and so $U+\alpha Q$ is well-bounded.

Let $A = U+Q$ and let $B = U+iQ$. Then A and B are well-bounded, and since U and Q commute, A and B also commute. Now $A+iB = U+iU = T$.

Example 4.1.8 ([11], Example 3.4). *Let T be the operator defined in Example 4.1.7. The operator $S(f,g) = (f,0)$ commutes with T, but it does not commute with A or B.*

We do have however, the following positive results.

Theorem 4.1.9. *Let U and V be commuting well-bounded operators with decompositions of the identity of bounded variation on X and let $S \in L(X)$ commute with $U+iV$. Then S commutes with U and V.*

Proof. By Theorem 3.2.5 the operators U, V are hermitian-equivalent. Since $UV = VU$ it follows that $U+iV$ is normal-equivalent. By 2.2.5 we have

$$SU = US, \qquad SV = VS.$$

□

Theorem 4.1.10. *Let*

$$T = U + iV = U_1 + iV_1$$

where

1. *U and V are commuting well-bounded operators of type (A) on X, with decompositions of the identity of bounded variation,*

2. *U_1 and V_1 are commuting well-bounded operators on X.*

Then $U = U_1$ and $V = V_1$.

Proof. U_1 commutes with $U_1 + iV_1$ and hence U_1 commutes with $U + iV$. By Theorem 4.1.9, U_1 commutes with U and V. Similarly V_1 commutes with U and V. Hence the set $\{U, V, U_1, V_1\}$ is commutative. Since well-bounded operators have real spectra, we can apply standard Gelfand theory to deduce that $U - U_1$ and $V - V_1$ are quasinilpotent. Now, by Theorem 3.1.12, $U = U_1$ and $V = V_1$. □

When X does not contain a copy of c_0 we need not assume that the real and imaginary parts are decomposable in X.

Theorem 4.1.11. *Suppose that X does not contain a copy of c_0. Let*

$$T = U + iV = U_1 + iV_1$$

where

1. *U and V are commuting well-bounded operators on X with decomposition of the identity of bounded variation,*

2. *U_1 and V_1 are commuting well-bounded operators on X.*

Then $U = U_1$ and $V = V_1$.

Proof. U and V are real scalar-type spectral operators ([20], Theorem 2). Now by ([22], Theorem 16.17) U and V are well-bounded and decomposable in X. The result follows from Theorem 4.1.10. □

4.2 AC-operators and (Foiaş) decomposable operators

Let Ω be a subset of the complex plane. An algebra $\mathcal{A}$ of complex functions defined on Ω is called *normal* if for every open finite covering $\{G_i\}_{1\leq i\leq n}$ of $\bar{\Omega}$ there exist functions $f_i \in \mathcal{A}$ such that

1. $f_i(\Omega) \subseteq [0,1]$, $\quad (1 \leq i \leq n)$,
2. $\operatorname{supp}(f_i) \subseteq G_i$, $\quad (1 \leq i \leq n)$ where $\operatorname{supp}(f_i) = \overline{\{\lambda \in \Omega | f(\lambda \neq 0)\}}$
3. $\sum_{i=1}^{n} f_i = 1 \quad$ *on* Ω.

Definition 4.2.1. *An algebra $\mathcal{A}$ of complex functions defined on the set $\Omega \subseteq \mathbb{C}$ is called* admissible *if*

1. $(\lambda \mapsto \lambda) \in \mathcal{A}$, $(\lambda \mapsto 1) \in \mathcal{A}$,

2. $\mathcal{A}$ is normal,

3. for every $f \in \mathcal{A}$ and every $\xi \notin$ $\operatorname{supp}(f)$, *the function*

$$f_\xi(\lambda) = \begin{cases} \dfrac{f(\lambda)}{\xi - \lambda} & \text{if } \lambda \in \Omega \backslash \{\xi\}, \\ \\ 0 & \text{if } \lambda \in \Omega \cap \{\xi\}. \end{cases}$$

belongs to $\mathcal{A}$.

Definition 4.2.2. *Let $\mathcal{A}$ be an admissible algebra. A mapping $\Theta : \mathcal{A} \to L(X)$ is called a $\mathcal{A}$-spectral function if*

1. $\Theta : \mathcal{A} \to L(X)$ is an algebraic homomorphism and $\Theta(\lambda \mapsto 1) = \mathrm{I}$,

2. The $L(X)$-valued function $\xi \mapsto \Theta(f_\xi)$ is analytic on $\mathbb{C}\backslash \operatorname{supp}(f)$.

Definition 4.2.3. *Let $\mathcal{A}$ be an algebra of complex functions defined on the closed set $\Omega \subseteq \mathbb{C}$. $\mathcal{A}$ will be called* topologically admissible *if*

1. $(\lambda \mapsto \lambda) \in \mathcal{A}$, $(\lambda \mapsto 1) \in \mathcal{A}$,

2. $\mathcal{A}$ *is normal,*

3. $\mathcal{A}$ *is endowed with a locally convex topology* τ *such that if* $\{f_n\} \subseteq \mathcal{A}$ *is a Cauchy sequence in* τ *and* $f_n(\lambda) \to 0$ *for every* $\lambda \in \Omega$, *then* $f_n \to 0$ *in* τ,

4. *for every* $f \in \mathcal{A}$ *and every* $\xi \notin \operatorname{supp}(f)$, *the function*

$$f_\xi(\lambda) = \begin{cases} \dfrac{f(\lambda)}{\xi - \lambda} & \text{if } \lambda \in \Omega \backslash \{\xi\}, \\ \\ 0 & \text{if } \lambda \in \Omega \cap \{\xi\}. \end{cases}$$

belongs to $\mathcal{A}$, *and the mapping* $\xi \to f_\xi$ *of* $\mathbb{C} \backslash \operatorname{supp}(f)$ *into* $\mathcal{A}$ *is continuous.*

Lemma 4.2.4. *Let* $J = [a,b]$, $K = [c,d]$ *and* $\Omega = J \times K$, *let* $\mathcal{A} = AC(\Omega)$ *and let* $f \in \mathcal{A}$, *and* f_ξ *be as in Definition 4.2.3. Then*

1. $f_\xi \in \mathcal{A}$,

2. *the mapping* $\xi \to f_\xi$ *of* $\mathbb{C} \backslash \operatorname{supp}(f)$ *into* $\mathcal{A}$ *is continuous.*

Proof. Let $\xi \notin \operatorname{supp}(f)$. We can find a C^∞ function $\kappa_{f,\xi}$ and a closed disc D_ξ containing ξ such that $\kappa_{f,\xi}|_{\operatorname{supp}(f)} = 1$ and $\kappa_{f,\xi}|_{D_\xi} = 0$; then
$\lambda \mapsto \dfrac{\kappa_{f,\xi}}{\xi - \lambda} \in C^\infty$ and $f_\xi(\lambda) = \dfrac{f(\lambda)\kappa_{f,\xi}(\lambda)}{\xi - \lambda}$, so is in $\mathcal{A}$.
The mapping $\xi \mapsto f_\xi$ is clearly continuous. □

Corollary 4.2.5. $AC(J)$ *and* $AC(J \times K)$ *are topologically admissible algebras.*

Definition 4.2.6. *Let* $\mathcal{A}$ *be a topologically admissible algebra. A mapping* $\Theta : \mathcal{A} \to L(X)$ *is called a* continuous $\mathcal{A}$-spectral function *if*

1. $\Theta : \mathcal{A} \to L(X)$ *is an algebraic homomorphism, and* $\Theta(\lambda \mapsto 1) = I$,

2. $\Theta : \mathcal{A} \to L(X)$ *is continuous ([17], Definition 3.5.3).*

Remark 4.2.7. *By ([17], Theorem 3.5.4) every continuous $\mathcal{A}$-spectral function is an $\mathcal{A}$-spectral function.*

Definition 4.2.8. *An operator $S \in L(X)$ is called* $\mathcal{A}$-scalar *if there exists an $\mathcal{A}$-spectral function Θ such that $\Theta(\lambda) = S$. Such an $\mathcal{A}$-spectral function will be called an $\mathcal{A}$-spectral function of S.*

Theorem 4.2.9. *Well-bounded operators and AC-operators are (Foiaş) decomposable.*

Proof. By Corollary 4.2.5 well-bounded operators and AC-operators are $\mathcal{A}$-scalar and hence by ([17], Theorem 3.1.16) they are (Foiaş) decomposable operators. □

If $T \in L(X)$ is a (Foiaş) decomposable operator then for any closed subset F of $\sigma(T)$ the subspace $X_T(F)$ is a spectral maximal space of T (Lemma 1.3.8).

Theorem 4.2.10. *Let $T \in L(X)$ be a well-bounded operator, $x_0 \in X$ and*

$$\lim_{n\to\infty} \|T^n x_0\|^{1/n} = 0.$$

Then $Tx_0 = 0$.

Proof. T is (Foiaş) decomposable hence $X_T(\{0\})$ is a closed subspace of X which is invariant for T and satisfies $\sigma(T|X_T(\{0\})) = \{0\} \cap \sigma(T)$ (Lemma 1.3.8). Therefore, $T|X_T(\{0\})$ is a quasinilpotent well-bounded operator. Hence by Corollary 3.1.13 $T|X_T(\{0\} = 0$. Now by Lemma 1.3.10 we have

$$X_T(\{0\}) = \{x \in X : \lim_{n\to\infty} \|T^n x\|^{1/n} = 0\}.$$

Thus $x_0 \in X_T(\{0\})$ and $Tx_0 = 0$. □

Corollary 4.2.11. *Let $T = U + iV$ where U, V are commuting well-bounded operators. If*

$$\lim_{n\to\infty} \|T^n x\|^{1/n} = 0$$

for some $x \in X$, then $Ux = Vx = 0$.

Proof. There is a continuous homomorphism $\Theta : AC(J \times K) \to L(X)$ such that $\Theta(u) = U$, $\Theta(v) = V$ and $\Theta(u + iv) = U + iV$.
Now $\lim_{n\to\infty} \|T^n x\|^{1/n} = 0$ and hence $x \in X_T(\{0\})$ (Lemma 1.3.10). This gives

$$x \in X_T(\{0\}) \subseteq X_T(u^{-1}(\{0\}) = X_{\Theta(u)}(\{0\}) = X_U(\{0\}),$$

where the inclusion is by Lemma 1.3.2 and the first equality is by ([17], Theorem 3.2.4). Hence

$$\lim_{n\to\infty} \|U^n x\|^{1/n} = 0.$$

By Theorem 4.2.10, $Ux = 0$. Similarly we can show that $Vx = 0$. □

Lemma 4.2.12 is due to Gillespie (private communication).

Lemma 4.2.12. *Let $T = U + iV$, where U, V are commuting type (B) well-bounded operators on a Banach space X. Fix $a, b \in \mathbb{R}$ with $a < b$ and let $F = \{z \in \mathbb{C} : a \le Rez \le b\}$. Then*

$$X_T(F) = [E(b) - E(a^-)]\, X$$

where $E(\cdot)$ is the spectral family of U.

Proof. Standard Gelfand theory shows that the spectrum of the restriction of T to $[E(b) - E(a)]\, X$ is contained in F. Hence $[E(a) - E(b)]\, X \subseteq X_T(F)$. Now U and V commute with T, and hence $X_T(F)$ is invariant under both U and V (Lemma 1.3.9). Again, standard Gelfand theory implies that the spectrum of the restriction of U to $X_T(F)$ is contained in $[a, b]$. It now follows from ([22], Theorem 19.3) that $X_T(F) \subseteq [E(b) - E(a^-)]\, X$. □

We can now prove the following theorem, which generalises that of Berkson and Gillespie ([12], Lemma 4).

Theorem 4.2.13. *Suppose X and Y are Banach spaces. Let $T_1 = U_1 + iV_1 \in L(X)$ and $T_2 = U_2 + iV_2 \in L(Y)$ be AC-operators where $U_i, V_i, (i = 1, 2)$ are commuting type (B) well-bounded operators. Let $S \in L(X, Y)$ be an operator such that*

$$\lim_{n\to\infty} \|C(T_2, T_1)^n S\|^{1/n} = 0.$$

Then $U_2 S = SU_1$, $V_2 S = SV_1$ and $T_2 S = ST_1$.

Proof. It is sufficient to show that $E_2(a)S = SE_1(a)$, $(a \in \mathbb{R})$ where $\{E_1(s) : s \in \mathbb{R}\}$ and $\{E_2(s) : s \in \mathbb{R}\}$ are the spectral families of U_1 and U_2 respectively. Let $a, b \in \mathbb{R}$, $a < b$ and $F = \{z \in \mathbb{C} : a \leq Re\, z \leq b\}$. By Lemma 1.3.9 we have $SX_{T_1}(F) \subseteq Y_{T_2}(F)$ and hence $S\,[E_1(b) - E_1(a^-)]\,X \subseteq [E_2(b) - E_2(a^-)]\,Y$ (Lemma 4.2.12). If $\{a_n\}$ is a sequence decreasing to a, then $E_i(a_n) \to E_i(a)$ $(i = 1, 2)$ strongly; but $S\,[E_1(b) - E_1(a_n{}^-)]\,X \subseteq [E_2(b) - E_2(a_n{}^-)]\,Y$ so

$$S\,[E_1(b) - E_1(a)]\,X \subseteq [E_2(b) - E_2(a)]\,Y.$$

Taking b sufficiently large and positive we get $S\,[\mathrm{I} - E_1(a)]\,X \subseteq [\mathrm{I} - E_2(a)]\,Y$ and taking a sufficiently large and negative we get $SE_1(b)X \subseteq E_2(b)Y$. Hence for $a \in \mathbb{R}$ we have $SE_1(a)X \subseteq E_2(a)Y$, $S\,[\mathrm{I} - E_1(a)]\,X \subseteq [\mathrm{I} - E_2(a)]\,Y$ and therefore $E_2(a)S = SE_1(a)$. □

Corollary 4.2.14. *Let $T_1 = U_1 + iV_1 \in L(X)$ and $T_2 = U_2 + iV_2 \in L(X)$ be AC-operators where $U_i, V_i, (i = 1, 2)$ are commuting type (B) well-bounded operators. Suppose $T_1 \overset{q}{\sim} T_2$. Then $U_1 = U_2$, $V_1 = V_2$ and $T_1 = T_2$.*

Proof. With $Y = X$ and $S = \mathrm{I}$ in Theorem 4.2.13 we obtain $U_1 = U_2$, $V_1 = V_2$ and $T_1 = T_2$. □

The following results are immediate corollaries of Theorem 3.2.5 and ([2], Theorem 1).

Corollary 4.2.15. *Suppose X and Y are Banach spaces. Let $T_1 = U_1 + iV_1 \in L(X)$ and $T_2 = U_2 + iV_2 \in L(Y)$ be AC-operators, where U_i, V_i $(i = 1, 2)$ are commuting well-bounded operators with decompositions of the identity of bounded variation on X, Y. Suppose further that*

$$\lim_{n \to \infty} \|C(T_2, T_1)^n S\|^{1/n} = 0.$$

Then $U_2S = SU_1$, $V_2S = SV_1$, and $T_2S = ST_1$.

Corollary 4.2.16. *Let X be a Banach space and let $T_i = U_1 + iV_1 \in L(X)$ ($i = 1, 2$) be AC-operators with $U_i, V_i, (i = 1, 2)$ commuting well-bounded operators with decompositions of the identity of bounded variation. Suppose $T_1 \overset{q}{\sim} T_2$. Then $U_1 = U_2$, $V_1 = V_2$ and $T_1 = T_2$.*

4.3 Examples

In this section we shall present some examples of AC-operators. The next Theorem due to Berkson and Gillespie [12] gives us a source of examples of AC-operators. For completeness we include a proof.

Theorem 4.3.1. *Let T be a well-bounded operator on X and let $J = [a, b]$ and K_1 be chosen so that for every complex polynomial p*

$$\|p(T)\| \leq K_1\{|p(b)| + \mathrm{var}_J(p)\}.$$

Suppose $f \in AC(J)$ is such that $\mathrm{Re}\, f$ and $\mathrm{Im}\, f$ are piecewise monotonic. Then $f(T)$ is an AC-operator.

Proof. Let $J = [a, b]$ and let

$$a = a_0 < a_1 < \cdots < a_n = b$$

be such that $\mathrm{Re}\, f$ is monotonic on each interval $[a_{j-1}, a_j]$. Let K be an interval containing $\mathrm{Re}\, f(a_j)$ for $j = 0, \cdots, n$. For each polynomial p,

$$\mathrm{var}_J(p \circ \mathrm{Re}\, f) = \sum_{j=1}^{n} \alpha_j(p),$$

where $\alpha_j(p)$ is the variation of p over the interval with endpoints $\mathrm{Re}\, f(a_{j-1})$ and $\mathrm{Re}\, f(a_j)$ and "$\circ$" denotes the composition of functions. Hence

$$\mathrm{var}_J(p \circ \mathrm{Re}\, f) \leq n\, \mathrm{var}_K\, p$$

and so

$$\|p(\mathrm{Re}\, f(T)\| = \|(p \circ \mathrm{Re}\, f)(T)\| \leq K_1 \||p \circ \mathrm{Re}(f)\||_J \leq K_1\{p\, \mathrm{Re}\, f(b) + n\, \mathrm{var}_K\, p\}.$$

Hence $\mathrm{Re}\, f(T)$ is well-bounded. Similarly we can show $\mathrm{Im}\, f(T)$ is well-bounded. □

Corollary 4.3.2. *Let T be a well-bounded operator. Then* $\exp(iT)$ *and* $p(T)$ *(where p is a polynomial) are AC-operators.*

As a consequence of this corollary, a number of naturally occurring operators are AC-operators.

Let G be a locally compact abelian group and suppose that μ is a left Haar measure on G: that is, μ is a regular positive Borel measure on G such that $\mu(s+E) = \mu(E)$ for each $s \in G$ and each Borel set $E \subseteq G$. For $1 \leq p \leq \infty$, let $L_p(G)$ denote the usual Banach space of equivalence classes of Borel measurable complex valued functions on G whose p^{th} powers are integrable with respect to to μ and the norm

$$\|f\|_p = \left[\int_G |f|^p d\mu(t)\right]^{1/p} \qquad (f \in L_p(G)).$$

Here $L_\infty(G)$ denotes the usual Bnach space of equivalence classes of essentially bounded μ-measurable functions on G with the norm $\|f\|_\infty = ess\sup_{t\in G}|f(t)|$. For $1 \leq p \leq \infty$, $f \in L_1(G)$ and $g \in L_p(G)$, we define $f * g$ for almost all $s \in G$ by

$$f * g(s) = \int_G f(s-t)g(t)\, d\mu(t).$$

The element $f * g$, is called the convolution of f and g. If the Haar measure μ is normalized so that $\mu(G) = 1$, then $L_p(G)$ $(1 \leq p \leq \infty)$ is a commutative Banach algebra with convolution as multiplication.

Definition 4.3.3. *Let G be a locally compact abelian group and let $1 \leq p \leq \infty$. For each $x \in G$ the mapping $R_x : L_p(G) \to L_p(G)$ is defined by $(R_x f)(y) = f(y+x)$, where $f \in L_p(G)$ and $y \in G$ a.e. (locally a.e. in the case $p = \infty$).*

Let G be a locally compact abelian group. If $1 \leq p \leq \infty$, then $R_x \in L(L_p(G))$, $\|R_x(f)\| = \|f\|_p$, $(f \in L_p(G),\ x \in G)$ and for each $f \in L_p(G)$ the mapping from G to $L(L_p(G))$, defined by $x \mapsto R_x, (x \in G)$, is uniformly continuous.

Lemma 4.3.4. *Let G be a locally compact group, let $1 < p < \infty$ and, for $x \in G$, let R_x be the translation operator on $L_p(G)$. Then there exists a unique well-bounded operator T_x on $L_p(G)$ such that* $\exp(iT_x) = R_x$.

Proof. ([31], Theorem 1). □

The following example is due to Berkson and Gillespie [12].

Example 4.3.5. *Let G be a locally compact group, let $1 < p < \infty$ and, for $x \in G$, let R_x be the translation operator on $L_p(G)$. Lemma 4.3.4 and Corollary 4.3.2 show that R_x is an AC-operator.*

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, be the unit circle in the complex plane. The unit disc $\{z : |z| < 1\}$ is denoted by $\mathbb{D}$, and the closed unit disc $\{z : |z| \leq 1\}$ by $\bar{\mathbb{D}}$. Let $p > 0$ and let f be holomorphic in $\mathbb{D}$. If

$$h_p(f, r) = \frac{1}{2\pi} \int |f(re^{it})|^p \, dt$$

then $h_p(f, r)$ is a monotone nondecreasing function of r.

Definition 4.3.6. *The space $H^p(\mathbb{D})$, $p > 0$ is the (linear) space of all functions f holomorphic in $\mathbb{D}$, such that*

$$\|f\|_{H^p}^p = \lim_{r \to 1^-} h_p(f, r) = \sup_{0<p<1} h_p(f, r) < \infty.$$

If $f \in H^p(\mathbb{D})$, $p > 0$ then the radial limit $\lim_{r \to 1^-} f(re^{it})$ exists for almost all $t \in \mathbb{T}$ and, denoting it by $f(e^{it})$, we have

$$\|f\|_{H^p}^p = \frac{1}{2\pi} \int |f(e^it)|^p \, dt.$$

For $p \geq 1$, $\|\cdot\|_{H^p}$ is a norm and $H^p(\mathbb{D})$ endowed with this norm can be identified with a closed subspace of $L_p(\mathbb{T})$. See [40] for a fuller account.

Definition 4.3.7. *Let $Aut(G)$ denote the group of conformal maps of $\mathbb{D}$ onto $\mathbb{D}$. A one-parameter group of Möbius transformations of $\mathbb{D}$, $\{\phi_t\}$, is a homomorphism $t \mapsto \phi_t$ of the additive group of of $\mathbb{R}$ into $Aut(\mathbb{D})$ such that for each $z \in \mathbb{D}$, $\phi_t(z)$ is a continuous function of t on $\mathbb{R}$ and, for some $t \in \mathbb{R}$, ϕ_t is not the identity map.*

Lemma 4.3.8. *Let $\{\phi_t\}$ be a one-parameter group of Möbius transformations of $\mathbb{D}$. Then the set of common fixed points in the extended plane must be one of the following:*

1. *a doubleton set consisting of a point of $\mathbb{D}$ and its symmetric image with respect to $\mathbb{T}$ (elliptic case),*

2. *a singleton subset of* $\mathbb{T}$ *(parabolic case),*

3. *a doubleton subset of* $\mathbb{T}$ *(hyperbolic case).*

Moreover, for any $u \in \mathbb{R}$ *such that* ϕ_u *is not the identity map, the fixed points of* ϕ_u *are the common fixed points of the group* $\{\phi_t\}$.

Proof. ([14], Proposition (1.5)). □

Lemma 4.3.9. *Let* $\{\phi_t\}$ *be a one-parameter group of Möbius transformations of* $\mathbb{D}$. *We have: (i) If* $\{\phi_t\}$ *is elliptic, then there are unique constants* $c \in \mathbb{R}$ $c \neq 0$ *and* $\tau \in \mathbb{D}$ *such that* $\phi_t(z) = \gamma_\tau(e^{ict}\gamma_\tau(z))$ *for* $t \in \mathbb{R}$, $z \in \mathbb{D}$, *where* $\gamma_\tau(z) = (z-\tau)/(\bar{\tau}z-1)$. *(ii) If* $\{\phi_t\}$ *is parabolic, there are* $c \in \mathbb{R}$, $c \neq 0$ *and* $\alpha \in \mathbb{T}$ *such that for* $t \in \mathbb{R}$, $z \in \mathbb{D}$,

$$\phi_t(z) = \frac{(1-ict)z + ict\alpha}{-ic\bar{\alpha}tz + (1+ict)}.$$

(iii) If $\{\phi_t\}$ *is hyperbolic, there are unique constants* $c \in \mathbb{R}$, $\alpha, \beta \in \mathbb{T}$, *with* $c > 0$, $\alpha \neq \beta$, *such that for* $t \in \mathbb{R}$, $z \in \mathbb{D}$, $\phi_t(z) = \sigma_{\alpha,\beta}^{-1}(e^{ct}\sigma_{\alpha,\beta}(z))$, *where* $\sigma_{\alpha,\beta}(z) = (z-\alpha)/(z-\beta)$.

Conversely, the equations in (i), (ii), and (iii) above define groups of the respective types.

Proof. ([13], Theorem (1.6)). □

If $\{\phi_t\}$ is a group of Möbius transformations of $\mathbb{D}$, and $1 \leq p < \infty$, one can select in a canonical way a branch of $(\phi_t')^{1/p}$ for $t \in \mathbb{R}$ so that ${\phi'_{s+t}}^{1/p} = [(\phi'_s)^{1/p} \circ \phi_t][\phi'_t]^{1/p}$, for $s, t \in \mathbb{R}$, where ϕ'_t denotes $\frac{d}{dz}\phi_t(z)$ ([13] p. 231). Henceforth the symbol $(\phi'_t)^{1/p}$ will always indicate this special branch.The isometric groups in $H^p(\mathbb{D})$ can now be described.

Lemma 4.3.10. *let* $\{T_t\}$ *be a strongly continuous one-parameter group of isometries of* $H^p(\mathbb{D})$, $1 \leq p < \infty$, $p \neq 2$. *Then*

1. *If* $\{T_t\}$ *is continuous in the uniform operator topology, there is a real constant* ω *such that* $T_t = e^{i\omega t}I$ *for* $t \in \mathbb{R}$, *where* I *is the identity operator.*

2. *If $\{T_t\}$ is not continuous in the uniform operator topology, then $\{T_t\}$ has a unique representation in the form*

$$T_t f = e^{i\omega t}[\phi_t']^{1/p} f(\phi_t) \ (t \in \mathbb{R}, f \in H^p(\mathbb{D})),$$

where ω is a real constant, and $\{\phi_t\}$ is a one-parameter group of Möbius transformations of $\mathbb{D}$.

Conversely, for such ω and $\{\phi_t\}$, if $1 \le p < \infty$, $e^{i\omega t}[\phi_t']^{1/p} f(\phi_t)$ $(t \in \mathbb{R}, f \in H^p(\mathbb{D}))$ defines a one-parameter group of isometries of $H^p(\mathbb{D})$ that is continuous in the strong, but not in the uniform, operator topology.

Proof. ([14], Theorem (2.4); [13], Theorem (2.1)). □

Lemma 4.3.11. *Let $\{T_t\}$ be a strongly continuous one-parameter group of isometries on $H^p(\mathbb{D})$, $1 < p < \infty$. Then there exists unique well-bounded operator A_t on $H^p(\mathbb{D})$ such that* $\exp(iA_t) = T_t$.

Proof. ([8], Proposition 2.6). □

Example 4.3.12. *Let $\{T_t\}$ be a strongly continuous one-parameter group of isometries on $H^p(\mathbb{D})$, $1 < p < \infty$. Lemma 4.3.11 and Corollary 4.3.2 show that T_t is an AC-operator.*

It has been shown that two further properties that one might hope AC-operators to possess also fail. Suppose that T is a normal operator on a Hilbert space. It is immediate from the definition of normality that for any $\alpha \in \mathbb{C}$, αT is also normal. Even on a Hilbert space however, the class of AC-operators fails to be closed under scalar multiplication ([11], Example 4.1).

Example 4.3.13. *Let p_n $(n \in \mathbb{N})$ be projections constructed in Example 3.2.3. If we define $A = st\lim\sum_{n=1}^{\infty} \lambda_n p_n$ and $B = st\lim\sum_{n=1}^{\infty} \mu_n p_n$ where $\lambda_n = (n+1)/n$ $(n \in \mathbb{N})$ and $\mu_{2n-1} = \mu_{2n} = (2n-1)/2n$ $(n \in \mathbb{N})$. By ([22], Theorem 18.5) A and B are well-bounded operators. Let $T = A + iB$ and $\alpha = 1 - i$. Suppose that αT is an AC-operator with representation $\alpha T = C + iD$. Since p_n commutes with A and B,* *it also commutes with* αT. *Then, by Theorem 4.2.13,* p_n *commutes with* C *and* D. *Now for each* n,

$$\begin{aligned}\alpha T p_n &= C p_n + i D p_n \\ &= (A+B)p_n + i(B-A)p_n \\ &= ((\lambda_n + \mu_n) + i(\mu_n - \lambda_n))p_n,\end{aligned}$$

and so $(C+iD)_{|p_n X} = ((\lambda_n + \mu_n) + i(\mu_n - \lambda_n))\,\mathrm{I}$. *Since* $\sigma(C_{|p_n X}) \subseteq \sigma(C) \subseteq \mathbb{R}$ *and the range of* p_n *is one-dimensional, this implies that* $C_{|p_n X} = (\lambda_n + \mu_n)I$ *and* $D_{|p_n X} = (\mu_n - \lambda_n)I$. *Thus* $C = st\lim \sum_{n=1}^{\infty} C p_n = A + B$ *and* $D = B - A$. *By [30]* $A+B$ *is not a well-bounded operator and hence* αT *cannot be an* AC*-operator.*

Chapter 5

Boolean algebras of projections

5.1 V^*-algebras

Definition 5.1.1. *A* C^*-algebra *is a complex Banach algebra* A *with an involution* $*$, *satisfying* $\|x^*x\| = \|x\|^2$ *for all* x *in* A.

Since $\|x^*x\| \leq \|x^*\|\|x\|$ we have $\|x\| \leq \|x^*\|$ for each x in A, whence $\|x\| = \|x^*\|$, so that the involution is isometric.
The Gelfand -Naimark theorem characterises C^*-algebras as the norm closed selfadjoint subalgebras of $L(H)$, where H is a Hilbert space.

Let $\mathfrak{A}$ be a closed subalgebra of $L(X)$ and $\mathfrak{H}$ be the set of hermitian operators in $\mathfrak{A}$.

Definition 5.1.2. $\mathfrak{A}$ *is a* V^*-algebra *if* $I \in \mathfrak{A}$ *and* $\mathfrak{A} = \mathfrak{H} + i\mathfrak{H}$.

By ([16], Theorem 2. 8) $\mathfrak{A}$ is V^*-algebra if and only if $\mathfrak{A}$ is a C^*-algebra under the (Vidav) involution $* : R + iJ \longrightarrow R - iJ$ $(R, J \in \mathfrak{H})$. When $\mathfrak{A}$ is a V^*-algebra, then $\mathfrak{A} = \mathfrak{H} + i\mathfrak{H}$ and if $\mathfrak{A} = \mathfrak{K} + i\mathfrak{K}$ where $\mathfrak{K}$ is a set of hermitian operators in $\mathfrak{A}$, then $\mathfrak{H} = \mathfrak{K}$.

A bounded linear functional ω on $\mathfrak{A}$ is called a *state* if $\omega(I) = \|\omega\|$.

For each $x \in X$, the functional $\omega_x : L(X) \to \mathbf{C} : T \to [Tx, x]$ is a state on every V^*-algebra in $L(X)$ where $[\cdot, \cdot]$ is a semi-inner-product on X. These functional are called *point states*.

The strong operator topology and the weak operator topology on $L(X)$ are of paramount importance: important here too are the BWO topology and BSO topology, the strongest topologies coinciding with the weak and strong topologies on bounded subsets of $L(X)$: see ([26], VI, §9). The BWO topology coincides with the ultraweak topology, the BSO topology with ultrastrong topology, on L(H), when H is a Hilbert space.

The following theorem is a generalisation of the Kaplansky Density Theorem: ([68], Theorem).

Theorem 5.1.8 (BWO Closure Theorem). *Let $\mathfrak{A}$ be a V^*-algebra such that $\overline{(\mathfrak{A}_1)}^w$ is weakly compact. Then $\widetilde{\mathfrak{A}}$, the BWO closure of $\mathfrak{A}$, is a W^*-algebra and*

$$(\widetilde{\mathfrak{A}})_1 = \overline{(\mathfrak{A}_1)}^w.$$

Moreover, any faithful representation of $\widetilde{\mathfrak{A}}$ as von Neumann algebra is BWO bicontinuous.

Proof. ([68], Theorem). □

Remark 5.1.9. *It remains open, in general, to decide whether $\widetilde{\mathfrak{A}} = \overline{\mathfrak{A}}^w$.*

If $\mathfrak{A} \subset L(X)$ and X is reflexive the compactness condition in Theorem 5.1.8 is satisfied. We will prove that the compactness condition is satisfied if X does not contain a copy of c_0 (the Banach space of sequences that converge to zero). For a proof of this fact we need the following generalisation of a theorem of Grothendieck [35] which states that $\Theta : \mathfrak{A} \to X$ is weakly compact whenever $\mathfrak{A}$ is a C^*-algebra and X is weakly sequentially complete. Pełczyński extended this result in [52]; he showed that X need only be assumed not to contain a copy of c_0. Akemann, Dodds and Gamlen [1] extended this theorem yet further and showed that it holds whenever $\mathfrak{A}$ is a C^*-algebra and X does not contain a copy of c_0. Spain's result is even more general and it is proved in a more elementary manner ([66], Theorem 2).

Theorem 5.1.10 (Akemann, Dodds and Gamlen). *If B is a C^*-algebra, if $\Theta : B \to X$ is a bounded operator, and X does not contain an isomorphic copy of c_0, then Θ is a weakly compact mapping.*

Remark 5.1.11. *A stronger version, where $\mathcal{B}$ may be any complete Jordan algebra of operators, not necessarily commutative, can be found in ([66], Theorem 2). That proof relies on James's characterisation of weakly compact sets. Both Akemann, Dodds and Gamlen [1] and Spain [66] use the Bessaga-Pełczyński result that X contains no copy of c_0 if and only if all series $\sum_n x_n$ in X with $\sum_n |\langle x_n, x'\rangle|$ convergent for all $x' \in X'$ are unconditionally norm convergent.*

Our main theorem in this section relies on the following result which was set as an exercise in ([26], VI. 9.1.2).

Theorem 5.1.12. *Let $\mathfrak{B}$ be a subset of $L(X)$. Then $\overline{\mathfrak{B}}^w$ is compact in the weak operator topology if, for each $x \in X$, $(\mathfrak{B}x)^w$ is weakly compact.*

Proof. We define

$$Y = \prod_{x \in X} \{(\mathfrak{B}x)^w : \|x\| \leq 1\}.$$

By Tychonoff's theorem Y is compact. If $A \in \overline{\mathfrak{B}}^w$ let $\tau(A) \in Y$ be defined by $\tau(A)_x = Ax$. It will be shown that τ is a homeomorphism from $\overline{\mathfrak{B}}^w$ onto $\tau(\overline{\mathfrak{B}}^w)$ with the relative topology from Y, and that $\tau(\overline{\mathfrak{B}}^w)$ is closed in Y. Thus $\overline{\mathfrak{B}}^w$ is compact in the weak operator topology.

To see that τ is injective suppose that $\tau(A_1) = \tau(A_2)$. Then for every $x \in X_1$ we have $\tau(A_1)_x = \tau(A_2)_x$. Therefore, for every $x \in X_1$, we have $A_1 x = A_2 x$ and hence $A_1 = A_2$.

The sets

$$\{A \in \overline{\mathfrak{B}}^w : |\langle Ax, x'\rangle| < \epsilon\} \qquad (x \in X_1,\ x' \in X_1')$$

form a subbasis for $\overline{\mathfrak{B}}^w$ in the weak operator topology, while the sets

$$\{\tau(A) : A \in \overline{\mathfrak{B}}^w, |\langle \tau(A)_x, x'\rangle| < \epsilon\} \qquad (x \in X_1,\ x' \in X_1')$$

form a subbasis for $\tau(\overline{\mathfrak{B}}^w)$ in the product topology of Y. It is clear that τ is a homeomorphism.

Let A_α be a net in $\overline{\mathfrak{B}}^w$, and let $f \in Y$ and suppose $\tau(A_\alpha) \to f$ in Y. So, for every $x \in X_1$, $\tau(A_\alpha)_x \to f_x$. If we define A by $Ax = f_x$ then A is linear. Given $x \in X$, $(x \neq 0)$ let $\gamma > 0$ be such that $\|\gamma x\| = 1$. Then we define $Ax = \gamma^{-1}(A\gamma x)$. If also $\beta > 0$ is such that $\|\beta x\| \leq 1$, then

$$\begin{aligned}\langle \gamma^{-1}A(\gamma x), y'\rangle &= \gamma^{-1}\lim\langle A_\alpha(\gamma x), y'\rangle \\ &= \beta^{-1}\lim\langle A_\alpha(\beta x), y'\rangle \\ &= \langle \beta^{-1}A(\beta x), y'\rangle.\end{aligned}$$

So A is well defined. If $\|x\| \leq 1$ then $A_\alpha x \in (\mathfrak{B}x)^w$ and

$$\langle Ax, y'\rangle = \lim\langle A_\alpha x, y'\rangle \qquad (y' \in X')$$

and hence

$$Ax \in (\mathfrak{B}x)^w.$$

So $f = \tau(A) \in \tau(\overline{\mathfrak{B}}^w)$. Therefore $\tau(\overline{\mathfrak{B}}^w)$ is closed in the product topology of Y.

□

Theorem 5.1.13. *Let X be a Banach space which does not contain a copy of c_0. If $\mathfrak{A}$ is a V^*-subalgebra of $L(X)$, then $\widetilde{\mathfrak{A}}$ is a W^*-algebra and*

$$(\widetilde{\mathfrak{A}})_1 = \overline{(\mathfrak{A}_1)}^w.$$

Proof. For each $x \in X$ we define $\tau_x : \mathfrak{A} \to X$ by $\tau_x(A) = Ax$: the map $\tau_x : \mathfrak{A} \to X$ is bounded and linear. By Theorem 5.1.10, τ_x is weakly compact; that is $(\mathfrak{A}_1 x)^w$ is weakly compact. Now by Theorem 5.1.12, $\overline{(\mathfrak{A}_1)}^w$ is compact in the weak operator topology; by Theorem 5.1.8, $\widetilde{\mathfrak{A}}$ is a W^*-algebra and

$$(\widetilde{\mathfrak{A}})_1 = \overline{(\mathfrak{A}_1)}^w.$$

□

The following questions seem still to be open:

I. Is $\widetilde{\mathfrak{A}}$ closed in the weak operator topology?

II. Is $*$ continuous in the weak operator topology?

Commutative C^*-algebras on X

The remaining results in this section apply to any commutative unital C*-subalgebra $\mathcal{B}$ of $L(X)$, and in particular to any algebra generated by a Boolean algebra of (hermitian) projections.

The operators in a commutative C*-subalgebra of $L(X)$ are called *normal* (sometimes *strongly normal*). *Abstractly,* they enjoy all the properties of normal operators on Hilbert spaces.

Let Λ be the maximal ideal space of $\mathcal{B}$ and Θ the *inverse Gelfand map*

$$\Theta : C(\Lambda) \rightarrow \mathcal{B}$$

which is a unital isometric *-isomorphism. (Θ is also called the *functional calculus* for $\mathcal{B}$.)

On restricting Θ to the C*-subalgebra generated by I, T (for any $T \in \mathcal{B}$) we obtain a functional calculus for a (strongly) normal T: a unital isometric *-isomorphism

$$\Theta_T : C(\sigma(T)) \rightarrow \mathcal{B}$$

such that

$$\Theta_T(z \mapsto 1) = I$$

$$\Theta_T(z \mapsto z) = T$$

$$\Theta_T(z \mapsto \bar{z}) = T^*$$

$$\|\Theta_T(f)\| = \|f\|_{\sigma(T)}$$

.

The following two lemmas demonstrate how to some extent the normal operators on a Banach space mimic normal operators on a Hilbert space.

Lemma 5.1.14. *Let $\mathcal{B}$ be a commutative C^*-algebra on X and let $\mathcal{H}$ be the set of hermitian elements of $\mathcal{B}$. Suppose that ${}^{H}_{K} \in \mathcal{H}$ and $0 \leq H \leq K$. Then*

$$\|Hx\| \leq \|Kx\| \qquad (x \in X).$$

Proof. For any $\epsilon > 0$ the operator $L = H(K+\epsilon I)^{-1}$ is defined in $\mathcal{H}$, and, by the functional calculus, $0 \leq L \leq I$; so $\|L\| \leq 1$. It follows that $\|Hx\| = \|L(K+\epsilon I)x\| \leq \|(K+\epsilon I)x\|$: and ϵ is arbitrary (positive). □

The next result, originally due to Palmer [51] Lemma 2.7, helps us extend the C* structure from $\mathcal{B}$ to $\mathcal{C} = \overline{\mathcal{B}}^w$. The following short proof is taken from [18].

Lemma 5.1.15. *For all $B \in \mathcal{B}$ and $x \in X$*

$$\|Bx\| = \|B^*x\|.$$

Proof. For $\epsilon > 0$ the functional calculus gives

$$\left\|B - B^2(B^*B+\epsilon I)^{-1}B^*\right\| = \left\|\epsilon B(B^*B+\epsilon I)^{-1}\right\| \leq \sqrt{\epsilon}/2,$$

and

$$\left\|B^2(B^*B+\epsilon I)^{-1}\right\| \leq 1.$$

Thus, for any $x \in X$

$$\|Bx\| = \lim_{\epsilon\to 0}\|B^2(B^*B+\epsilon I)^{-1}B^*x\| \leq \|B^*x\|,$$

and then $\|B^*x\| \leq \|B^{**}x\| = \|Bx\|$. □ □

The weak closure of a *commutative* C*-algebra on X is also a C*-algebra on X.

Theorem 5.1.16. *Let $\mathcal{B}$ be a commutative C*-algebra on X and $\mathcal{H}$ the set of hermitian elements of $\mathcal{B}$. Let $\overline{\mathcal{H}}^w$ be the weak operator topology closure of $\mathcal{H}$, and $\overline{\mathcal{B}}^w$ the weak operator topology closure of $\mathcal{B}$. Then*

$$\overline{\mathcal{B}}^w = \overline{\mathcal{H}}^w + i\overline{\mathcal{H}}^w$$

is a C-algebra. Moreover, $\left(\overline{\mathcal{B}}^w\right)_1 = \overline{\mathcal{B}_1}^w$. So $\mathcal{B}^{\sim} = \overline{\mathcal{B}}^w$.*

Proof. First note that the weak and strong closures coincide for $\mathcal{H}$ and $\mathcal{B}$ (they are both convex sets). Now Lemma 5.1.15 shows that $\overline{\mathcal{B}}^s = \overline{\mathcal{H}}^s + i\overline{\mathcal{H}}^s$, so $\overline{\mathcal{B}}^w$ is a C*-algebra.

Consider $H \in (\overline{\mathcal{H}}^w)_1$. Then $K = (I - [I - H^2]^{\frac{1}{2}})/H \in \overline{\mathcal{H}}^w$, and $H = 2K/(I + K^2)$. Take a net K_α in $\mathcal{H}$ converging strongly to K: put $H_\alpha = 2K_\alpha/(I + K_\alpha^2)$. Then

$$H_\alpha - H = 2(I + K_\alpha^2)^{-1}(K_\alpha - K)(I + K^2)^{-1} + \frac{1}{2}H_\alpha(K - K_\alpha)H$$

so $H \in \overline{\mathcal{H}_1}^w$. By the Russo-Dye Theorem [15] §38 we have $(\overline{\mathcal{B}}^w)_1 \subseteq \overline{\mathcal{B}_1}^w$. □

Corollary 5.1.17. *If, further, the unit ball of $\mathcal{B}$ is relatively weakly compact, then $\overline{\mathcal{B}}^w$ is a W^*-algebra and any faithful representation of $\overline{\mathcal{B}}^w$ as a concrete von Neumann algebra on a Hilbert space is BWO bicontinuous (that is, weakly bicontinuous on bounded sets)*

Proof. Use Theorem 5.1.8 □

Remark 5.1.18. *We show later (§5.2) that any such faithful representation is also BSO bicontinuous (that is, strongly bicontinuous on bounded sets). The proof (maybe the result) depends on being able to represent $\overline{\mathcal{B}}^w$ by a spectral measure: and the presence of c_0 as a subspace of X seems to be the natural obstruction to this: see §5.3.*

5.2 Boolean algebras of projections and the algebras they generate

Let X be a complex Banach space, and $\mathcal{E}$ a bounded Boolean algebra of projections on X. Write $\operatorname{aco}\mathcal{E}$ for the absolutely convex hull of $\mathcal{E}$ in $L(X)$.

It is known that X can be renormed so that each element of $\mathcal{E}$ is hermitian (Lemma 2.1.5).

Theorem 5.2.1. *Let $\mathcal{E}$ be a Boolean algebra of hermitian projections on a complex Banach space X. Then $\mathcal{A}$, the linear span of $\mathcal{E}$, is the *-algebra generated by $\mathcal{E}$: $\mathcal{A}$ is a commutative unital algebra, and $\mathcal{A} = \mathcal{H} + i\mathcal{H}$, where $\mathcal{H}$ is the set of hermitian elements of $\mathcal{A}$. So $\mathcal{B} = \overline{\mathcal{A}}$ is a commutative C^*-algebra on X.*

Proof. Immediate from the Vidav-Palmer Theorem. □

Lemma 5.2.2. *Let $S \in \mathcal{A}$ and suppose that $-I \leq S \leq I$. Then*

$$S \in 2\operatorname{aco}\mathcal{E}.$$

Proof. Suppose first that $0 \leq S \leq I$. Write S in *$\mathcal{E}$-step-form* as $S = \sum_{j=1}^{M} \lambda_j E_j$, where the E_j are pairwise disjoint. Then $0 \leq \lambda_j \leq 1$. Arrange the λ_j in descending order: then $\|S\| = \lambda_1$. Define $\lambda_{M+1} = 0$ and use Abel summation —

$$S = \sum_{j=1}^{M} \lambda_j E_j = \sum_{j=1}^{M} (\lambda_j - \lambda_{j+1}) \left(\sum_{k=1}^{j} E_j \right) \in \operatorname{aco}\mathcal{E}.$$

If $-I \leq S \leq I$, split S into its positive and negative parts. □

Theorem 5.2.3. *Let $\mathcal{E}$ be a Boolean algebra of hermitian projections on a complex Banach space X, and let $\mathcal{B}$ be the C^*-algebra it generates: let $\mathcal{B}_1$ be the closed unit ball of $\mathcal{B}$. Then*

$$\mathcal{B}_1 \subseteq 4\,\overline{\operatorname{aco}}\,\mathcal{E}.$$

Proof. Consider an element $B \in \mathcal{B}$ such that $\|B\| < 1$. Given $\epsilon > 0$ we can find $S = R + iJ$ in $\mathcal{A}$ such that $\|B - R - iJ\| \leq \min\{\epsilon, 1 - \|B\|\}$. Now $\frac{\|R\|}{\|J\|} \leq 1$, so by Lemma 5.2.2, $\frac{R}{J} \in 2\operatorname{aco}\mathcal{E}$. □

Corollary 5.2.4. *The following are equivalent:*

(I) $\mathcal{B}_1$ *is relatively weakly compact*

(II) $\operatorname{aco}\mathcal{E}$ *is relatively weakly compact*

(III) $\mathcal{E}$ *is relatively weakly compact.*

Proof. Use the Krein-Šmulian Theorem. □

Theorem 5.2.5. *Let* $\mathcal{E}$ *be a relatively weakly compact Boolean algebra of hermitian projections on a complex Banach space* X, *and let* $\mathcal{B}$ *be the* C^**-algebra generated by* $\mathcal{E}$. *Then* $\overline{\mathcal{B}}^w$ *is a* W^**-algebra and any faithful representation of* $\overline{\mathcal{B}}^w$ *as a concrete von Neumann algebra on a Hilbert space is BWO bicontinuous (that is, weakly bicontinuous on bounded sets).*

Proof. This follows from Corollary 5.2.4 and Corollary 5.1.17. □

5.3 σ-complete Boolean algebras of projections and spectral measures

On Hilbert space

On a Hilbert space $\mathcal{H}$ the following two facts are classical. We sketch their (elementary) proofs for the convenience of the reader.

Lemma 5.3.1. *Any monotone net of hermitian projections on* $\mathcal{H}$ *has a supremum, to which it converges strongly.*

Proof. Let $(E_\alpha)_{\alpha\in A}$ be such a net. The generalized Cauchy-Schwarz inequality $\langle P^2\xi, \xi\rangle \le \langle P\xi, \xi\rangle\langle P^3\xi, \xi\rangle$, which holds for any positive operator P on $\mathcal{H}$ and any element $\xi \in \mathcal{H}$, shows that the net $(E_\alpha)_{\alpha\in A}$ is strongly Cauchy: and its limit must be the supremum. □

Lemma 5.3.2. *Suppose that* $(E_\alpha)_{\alpha\in A}$ *is a net of hermitian projections that converges weakly to a projection* E. *Then it converges strongly.*

Proof. This is immediate from the calculation

$$\begin{aligned}\|(E-E_\alpha)\,\xi\|^2 &= \langle (E-E_\alpha)^2\,\xi,\,\xi\rangle \\ &= \langle E^2\xi,\,\xi\rangle - \langle EE_\alpha\,\xi,\,\xi\rangle - \langle E_\alpha E\,\xi,\,\xi\rangle + \langle E_\alpha^2\xi,\,\xi\rangle \\ &\to \langle (E-E^2)\,\xi,\,\xi\rangle = 0. \qquad \square\end{aligned}$$

□

It follows that *on a Hilbert space* every Boolean algebra $\mathcal{E}$ of hermitian projections can be extended to a *complete* one; that $\overline{\mathcal{E}}^s$ is the smallest such complete extension; and that $\overline{\mathcal{E}}^s = \overline{\mathcal{E}}^w \bigcap \{projections\ on\ \mathcal{H}\}$.

On a Banach space

On a Banach space the situation is more delicate. It was shown by Bade that if $\mathcal{E}$ is σ-complete on X then $\overline{\mathcal{E}}^s$ is complete on X ([28], XVII. 3.23): and that the family of projections in $\overline{\mathcal{E}}^w$ coincides with $\overline{\mathcal{E}}^s$. See Corollary 5.3.9 below for a proof (independent of Bade's original methods).

Let $\mathcal{E}$ be a σ-complete Boolean algebras of projection on X. Then $\mathcal{E}$ is σ-complete on X if and only if every bounded monotone (sequence) net in $\mathcal{E}$ converges strongly to a limit ([28], XVII. 3.4): and then $\mathcal{E}$ must be bounded.

A σ-complete Boolean algebra of projections $\mathcal{E}$ on X can be identified with a spectral measure of class X' on the Borel sets of the Stone space of $\mathcal{E}$ ([19], Chapter I): each vector measure $\mathcal{E}\,x$ is strongly countably additive.

Lemma 5.3.3. *If μ is a strongly countably additive vector measure with values in X then* $\mathrm{aco}\{\mu(\sigma) : \sigma \in \Sigma\}$ *is relatively weakly compact.*

Proof. Essentially this is a result of Bartle, Dunford and Schwartz ([5], Lemma 2.3): see also ([19], I.2.7 & I.5.3). □

Corollary 5.3.4. *If $\mathcal{E}$ is σ-complete then the set $\overline{\mathrm{aco}}^{\,w}(\mathcal{E}x)$ is weakly compact for each $x \in X$.*

Theorem 5.3.5. *Let $\mathcal{E}$ be a (bounded) σ-complete Boolean algebra of projections. Then $\mathcal{C} = \overline{\mathcal{B}}^w$, the commutative C^*-algebra generated by $\mathcal{E}$ in the weak operator topology, is a W^*-algebra, and $\mathcal{C}_1 = \overline{\mathcal{B}_1}^w \subseteq 4\,\overline{\mathrm{aco}}^w\mathcal{E}$. Furthermore, any faithful representation of $\mathcal{C}$ as a von Neumann algebra on a Hilbert space is weakly bicontinuous on bounded sets.*

Proof. Because $\overline{\mathrm{aco}}^w(\mathcal{E}x)$ is weakly compact for each $x \in X$ (Corollary 5.3.4) it follows that $\overline{\mathcal{B}_1}^w x$ is relatively weakly compact in X for each $x \in X$, and so $\overline{\mathcal{B}_1}^w$ is relatively weakly compact, by Theorem 5.1.12.

Theorem 5.2.5 shows that $\mathcal{C}$ is a W*-algebra, and any faithful representation of $\mathcal{C}$ as a von Neumann algebra on a Hilbert space is weakly bicontinuous on bounded sets. □

Theorem 5.3.6. *Let $\mathcal{B}$ be a commutative C^*-algebra on X such that $\mathcal{B}_1$ is relatively weakly compact. Let $\mathcal{B} = \overline{\mathcal{C}}^w$. Then there is a representing spectral measure $E(\cdot)$ defined on the Borel sets of the Gelfand space Λ of $\mathcal{C}$ such that*

$$\Theta(f) = \int_\Lambda f(\lambda)E(d\lambda) \qquad (f \in C(\Lambda)).$$

Proof. Let $\pi : \mathcal{C} \to L(H)$ be BWO continuous representation of $\mathcal{C}$ as a concrete W*-algebra. Let $\tilde{E}(\cdot)$ be a representing spectral measure for $\pi(\mathcal{C})$:

$$\pi \circ \Theta(f) = \int_\Lambda f(\lambda)\tilde{E}(d\lambda) \qquad (f \in C(\Lambda)).$$

Now define $E(\cdot) = \pi^{-1}\tilde{E}(\cdot)$: this yields a spectral measure on X ($E(\cdot)$ is weakly countably additive, hence, by the Banach-Orlicz-Pettis theorem, strongly countably additive): and then

$$\Theta(f) = \int_\Lambda f(\lambda)E(d\lambda) \qquad (f \in C(\Lambda)).$$

□

It is immediate that for a bounded net $(T_\alpha)_{\alpha\in A}$ of operators *on a Hilbert space* we have

$$(T_\alpha)_{\alpha\in A} \to_{strongly} 0 \iff (T_\alpha^* T_\alpha)_{\alpha\in A} \to_{weakly} 0.$$

A similar result for operators on a Banach space seems to be available only for normal operators belonging to a common W*-algebra.

Theorem 5.3.7. *Let C be a commutative W^*-algebra on X. Suppose that $(S_\alpha)_{\alpha\in A}$ is a bounded net in C. Then*

$$(S_\alpha)_{\alpha\in A} \rightarrow_{strongly} 0 \Longleftrightarrow (S_\alpha^* S_\alpha)_{\alpha\in A} \rightarrow_{weakly} 0.$$

Proof. Clearly $S_\alpha \rightarrow_{strongly} 0$ implies that $S_\alpha^* S_\alpha \rightarrow_{strongly} 0$ and so $S_\alpha^* S_\alpha \rightarrow_{weakly} 0$.

Let $E(\cdot)$ be the representing spectral measure for C guaranteed by Theorem 5.3.6.

Suppose that $S_\alpha^* S_\alpha \rightarrow_{weakly} 0$. Let $f_\alpha = \Theta^{-1} S_\alpha$. Then

$$\lim_\alpha \langle S_\alpha^* S_\alpha x,\ x'\rangle = \lim_\alpha \int_\Lambda |f_\alpha|^2 \langle E(d\lambda)x,\ x'\rangle \qquad (x \in X, x' \in X').$$

Therefore $\lim_\alpha f_\alpha = 0$ in var $\langle E(\cdot)x,\ x'\rangle$ measure and

$$\lim_\alpha \int_\Lambda f_\alpha \langle E(d\lambda)x,\ x'\rangle = 0.$$

For fixed $x \in X$ the set $\{\langle E(\cdot)x,\ x'\rangle : \|x'\| \le 1\}$ is a relatively weakly compact set of measures ([26], IV.10.2): hence $\lim_\alpha \int_\Lambda f_\alpha \langle E(d\lambda)x,\ x'\rangle = 0$ uniformly for $\|x'\| \le 1$ ([35],Théorème 2). Therefore $\lim_\alpha \int_\Lambda f_\alpha E(d\lambda)x = 0$ that is, $S_\alpha \rightarrow_{strongly} 0$. □

Corollary 5.3.8. *Let C be a commutative W^*-algebra on X. Then any faithful concrete representation of C as a von Neumann algebra is weakly and strongly bi-continuous on bounded sets.*

Corollary 5.3.9. *Let $\mathcal{E}$ be a σ-complete Boolean algebra of projections, and let $(E_\alpha)_{\alpha\in A}$ be a monotone net of hermitian projections in the commutative W^*-algebra C generated on X by $\mathcal{E}$. Then $(E_\alpha)_{\alpha\in A}$ converges strongly to a projection in C. So $\overline{\mathcal{E}}^s$ is complete on X. What is more, $\overline{\mathcal{E}}^s = \overline{\mathcal{E}}^w \bigcap\{$projections in $C\}$.*

Proof. This follows immediately from the known result on Hilbert spaces and from the strong bicontinuity of faithful representations as guaranteed by the theorem. □

The next corollary complements ([64],Theorem 5) and ([32],Theorems 1, 2).

Corollary 5.3.10. *Let $\mathcal{E}$ be a bounded Boolean algebra of projections on a Banach space X and suppose that $\mathcal{E}$ is relatively weakly compact. Then $\mathcal{E}$ has a (σ-)complete extension contained in $\overline{\mathcal{E}}^s$.*

Remark 5.3.11. *This happens automatically when $X \not\supset c_0$ (see §5.3).*

Corollary 5.3.12 ([28], (XVII. 3.7)). *Let $\mathcal{E}$ be a complete bounded Boolean algebra of projections on a Banach space X. Then $\mathcal{E}$ is strongly closed.*

Remark 5.3.13. *The results of [23] overlap with ours.*

5.4 In the absence of c_0

We can now present a theorem which is stronger than any other known to us in this area.

Theorem 5.4.1. *Let $\mathcal{E}$ be a bounded Boolean algebra of projections on a Banach space X and suppose that X does not contain an isomorphic copy of c_0. Then the weakly closed algebra generated by $\mathcal{E}$ is a W^*-algebra and any faithful representation of $\overline{\mathcal{B}}^w$ as a concrete von Neumann algebra on a Hilbert space is BWO and BSO bicontinuous. Moreover, every operator in $\overline{\mathcal{B}}^w$, the W^*-algebra generated by $\mathcal{E}$ in the weak operator topology, is a scalar-type spectral operator.*

Proof. Theorem 5.1.13 shows that $\mathcal{E}$ is relatively weakly compact. The result follows from Theorem 5.2.5, Corollary 5.3.8 and Theorem 2.4.4. □

Corollary 5.4.2. *Let $\mathcal{T}$ be a commuting finite family of scalar-type spectral operators on a Banach space X that does not contain an isomorphic copy of c_0. Suppose that the Boolean algebra generated by the resolutions of the identity for each $T \in \mathcal{T}$ is uniformly bounded. Then every operator in the weakly closed *-algebra generated by $\mathcal{T}$ is a scalar-type spectral operator.*

Remark 5.4.3. *If X contains c_0 then there is a strongly closed bounded Boolean algebra $\mathcal{F}$ of projections on X which is not complete ([32], Theorem 2). Then the weakly closed algebra generated by $\mathcal{F}$ cannot have relatively weakly compact unit ball, and there can be no BWO bicontinuous faithful representation of this algebra on a Hilbert space.*

5.5 Boolean algebras with countable basis

As remarked above, c_0 seems to be the natural essential obstruction to extending the results of the previous section. It is of course conceivable that a closer analysis will provide a proof that the sum and product of a pair of commuting scalar-type spectral operators must be a scalar-type spectral operator (so long as the Boolean algebra generated by their resolutions of the identity is bounded).

We shall say that a Boolean algebra $\mathcal{E}$ has a *countable basis* if it contains a countable orthogonal subfamily $\mathcal{F} = (F_m)_{m\in\mathrm{N}}$ such that every $E \in \mathcal{E}$ can be written as the strong *sum* of a subset of this family. Note that then $I = \sum_{m=1}^{\infty} F_m$, the sum being strongly convergent.

Lemma 5.5.1. *Let C be a commutative C^*-algebra on X and $(F_m)_{m\in\mathrm{N}}$ a countable family of positive elements of C such that $\sum_{m=1}^{\infty} F_m$ converges in the strong topology. Let C_m be any sequence in C for which $0 \le C_m \le I$ $(\forall m)$. Then*

$$\sum_{m=1}^{\infty} C_m F_m$$

converges strongly.

Proof. Note that $0 \le C_m F_m \le F_m$ $(\forall m)$. Then, for $M < N$,

$$0 \le \sum_{m=M+1}^{N} C_m F_m \le \sum_{m=M+1}^{N} F_m,$$

so, by Lemma 5.1.14, the sequence $(C_m F_m)_{m\in\mathrm{N}}$ is a strongly Cauchy sequence, hence strongly convergent. □

The following theorem generalises ([33], Theorem 3.6).

Theorem 5.5.2. *Suppose that $\mathcal{E}^{(1)}$ and $\mathcal{E}^{(2)}$ are two commuting σ-complete Boolean algebras of projections on X and that the Boolean algebra $\mathcal{E}$ generated by $\mathcal{E}^{(1)}$ and $\mathcal{E}^{(2)}$ is bounded. Assume, further, that $\mathcal{E}^{(2)}$ has a countable basis $\mathcal{F} = (F_m)_{m\in\mathrm{N}}$. Then $\mathcal{E}$ has a σ-complete extension, and hence a complete extension.*

Proof. As remarked in §5.2 we may, and shall, assume that all the elements of $\mathcal{E}^{(1)}$ and $\mathcal{E}^{(2)}$ are hermitian. Let C be the weakly closed C*-algebra generated by $\mathcal{E}$.

For each sequence of projections $(E_m^{(1)})_{m\in\mathbb{N}}$ taken from $\mathcal{E}^{(1)}$ we can, by Lemma 5.5.1, define $E = \sum_{m=1}^{\infty} E_m^{(1)} F_m \in \mathcal{C}$: each such E is a hermitian projection in $\mathcal{C}$ so has norm ≤ 1.

Consider

$$\mathcal{G} \triangleq \left\{ \sum_{m=1}^{\infty} E_m^{(1)} F_m : E_m^{(1)} \in \mathcal{E}^{(1)} \right\}.$$

It is clear that $F_m \in \mathcal{G}$ $(\forall m)$: so $\mathcal{E}^{(2)} \subseteq \mathcal{G}$. Note also that for any $E^{(1)} \in \mathcal{E}^{(1)}$ we have $E^{(1)} = \sum_m E^{(1)} F_m$: so $E^{(1)} \in \mathcal{G}$. Thus $\mathcal{E}^{(1)} \vee \mathcal{E}^{(2)} \subseteq \mathcal{G}$.

It is clear that $\mathcal{G}$ is closed under products. Further, for any $E = \sum_{m=1}^{\infty} E_m^{(1)} F_m \in \mathcal{G}$ we have

$$I - E = \sum_{m=1}^{\infty} \left[I - E_m^{(1)}\right] F_m \in \mathcal{G},$$

so $\mathcal{G}$ is a Boolean algebra of hermitian projections on X.

Note that for any such $E \in \mathcal{G}$ we have $EF_m = E_m^{(1)} F_m$ $(\forall m)$: thus any element of $\mathcal{G}$, which can be written, though not in a unique manner, as an (orthogonal) sum

$$E = \sum_{m=1}^{\infty} E_m^{(1)} F_m,$$

satisfies

$$E = \sum_{m=1}^{\infty} E_m^{(1)} F_m = \sum_{m=1}^{\infty} EF_m.$$

Now consider a sequence $(E_h)_{h\in\mathbb{N}}$ of pairwise orthogonal projections in $\mathcal{G}$:

$$E_h = \sum_{m=1}^{\infty} E_{h,m}^{(1)} F_m = \sum_{m=1}^{\infty} E_h F_m.$$

For each k and m define

$$G_{k,m} \triangleq \bigvee_{h=1}^{k} E_{h,m}^{(1)} \in \mathcal{E}^{(1)}$$

and then define

$$G_m \triangleq \bigvee_{k=1}^{\infty} G_{k,m} = \bigvee_{h=1}^{\infty} E_{h,m}^{(1)} \in \mathcal{E}^{(1)}.$$

Note that for each k and m

$$G_{k,m} F_m = \bigvee_{h=1}^{k} E_{h,m}^{(1)} F_m = \sum_{h=1}^{k} E_{h,m}^{(1)} F_m = \left(\sum_{h=1}^{k} E_h \right) F_m.$$

Suppose given $x \in X$ and $\epsilon > 0$. Then there exists an M such that

$$\left\| x - \sum_{m=1}^{M} F_m x \right\| < \epsilon$$

and then we can find N such that for $1 \leq m \leq M$ and $k \geq N$

$$\|(G_m - G_{k,m})x\| < \epsilon/M.$$

Suppose that $j < k$: then $0 \leq \sum_{h=j+1}^{k} E_h \leq I$, so, by Lemma 5.1.14,

$$\begin{aligned}
\left\| \left(\sum_{h=j+1}^{k} E_h \right) x \right\| &\leq \left\| \left(\sum_{h=j+1}^{k} E_h \right) \left(x - \sum_{m=1}^{M} F_m x \right) \right\| + \sum_{m=1}^{M} \left\| \left(\sum_{h=j+1}^{k} E_h \right) F_m x \right\| \\
&\leq \left\| x - \sum_{m=1}^{M} F_m x \right\| + \sum_{m=1}^{M} \|(G_{k,m} - G_{j,m}) F_m x\| \\
&\leq \left\| x - \sum_{m=1}^{M} F_m x \right\| + \sum_{m=1}^{M} \|(G_{k,m} - G_{j,m}) x\| \\
&\leq \epsilon + \epsilon = 2\epsilon. \qquad (5.1)
\end{aligned}$$

This shows that $\mathcal{G}$ is σ-complete. Then $\overline{\mathcal{G}}^s$ is complete, by Corollary 5.3.9.

□

From this we obtain

Theorem 5.5.3. *Let $\mathcal{E}^{(1)}$ and $\mathcal{E}^{(2)}$ be two σ-complete Boolean algebras of projections on X . Suppose that the Boolean algebra $\mathcal{E}$ generated by $\mathcal{E}^{(1)}$ and $\mathcal{E}^{(2)}$ is bounded, and that $\mathcal{E}^{(2)}$ has a countable basis. Then the weakly closed algebra C generated by $\mathcal{E}$ is a W^*-algebra.*

Corollary 5.5.4. *(Extension of [33], Theorem 3.6) Let X be a Banach space and T_1, T_2 be commuting scalar-type spectral operators on X with resolutions of the identity $\mathcal{E}^{(1)}$, $\mathcal{E}^{(2)}$ such that $\mathcal{E}^{(1)} \vee \mathcal{E}^{(1)}$ is bounded. Suppose further that one of these operators has countable spectrum. Then all operators in the weakly closed *-algebra generated by T_1 and T_2 are scalar-type spectral operators.*

5.6 Boundedness criteria for Boolean algebras of projections

Definition 5.6.1. *A partially ordered Banach space X over the reals is a* Banach lattice *provided*

1. $x \leq y$ *implies* $x + z \leq y + z$, *for every* $x, y, z \in X$,
2. $ax \geq 0$, *for every* $x \geq 0$ *in* X *and every nonnegative real* a,
3. *for all* $x, y \in X$ *there exist a least upper bound (l.u.b.)* $x \vee y$ *and a greatest lower bound (g.l.b.)* $x \wedge y$,
4. $\|x\| \leq \|y\|$ *whenever* $|x| \leq |y|$, *where the absolute value* $|x|$ *of* $x \in X$ *is defined by* $|x| = x \vee (-x)$.

The continuity of the lattice operations implies, in particular, that the set $C = \{x : x \in X, x \geq 0\}$ is norm closed. The set C, which is a convex cone, is called the *positive cone* of X.
For an element x in a Banach lattice X we put $x_+ = x \vee 0$ and $x_- = -(x \wedge 0)$ and $|x| = x_+ + x_-$. Two elements $x, y \in X$ for which $|x| \wedge |y| = 0$ are said to be *disjoint*.

Every Banach lattice X has the *decomposition property*: if x_1, x_2 and y are positive elements in X and $y \leq x_1 + x_2$ then there are $0 \leq y_1 \leq x_1$ and $0 \leq y_2 \leq x_2$ such that $y = y_1 + y_2$.

Definition 5.6.2. *The Banach lattices X and Y are said to be* order isometric *if there exists a linear isometry T from X onto Y which is also an order isomorphism.*

Definition 5.6.3. *A* sublattice *Y of a Banach lattice X is a closed subspace of X such that $x \vee y$ (and thus $x \wedge y = x + y - x \vee y$) belongs to Y whenever $x, y \in Y$.*

The dual X' of Banach lattice X is also a Banach lattice provided that its positive cone is defined by $x' \geq 0$ in X' if $x'(x) \geq 0$, for every $x \geq 0$ in X. It is easily verified that, for any $x', y' \in X'$ and every $x \geq 0$ in X, we have

$$(x' \vee y')(x) = \sup\{x'(u) + y'(x - u) : 0 \leq u \leq x\}$$

and

$$(x' \wedge y')(x) = \inf\{x'(v) + y'(x - v) : 0 \leq v \leq x\}.$$

Let X be a real Banach lattice and let $\widetilde{X}$ be the linear space $X \oplus X$ which is made into a complex linear space by setting

$$(a + ib)(x_1, x_2) = (ax_1 - bx_2, ax_2 + bx_1).$$

We can define an absolute value and norm on $\widetilde{X}$ by putting

$$|(x_1, x_2)| = (|x_1|^2 + |x_2|^2), \qquad \|(x_1, x_2)\|_{\widetilde{X}} = \||(x_1, x_2)|\|_X.$$

The space $(\widetilde{X}, \|, \|_{\widetilde{X}})$ is said to be a *complex Banach lattice* or more precisely, the *complexification* of the real Banach lattice X. As expected, the complex $L_p(\mu)$ or $C(K)$ spaces are the complexifications of the real $L_p(\mu)$ or $C(K)$ with the same μ, respectively K.

Theorem 5.6.4 is due to Gillespie ([33],Theorem 2.5).

Theorem 5.6.4. *Let X be a complex Banach lattice and $\mathcal{E}, \mathcal{F}$ be commuting bounded Boolean algebras of projection on X. Then the Boolean algebra of projections $\mathcal{E} \vee \mathcal{F}$ generated by $\mathcal{E}$ and $\mathcal{F}$ is bounded. Furthermore*

$$\|\mathcal{E} \vee \mathcal{F}\| \leq \frac{(2^6 + 1)}{2} \|\mathcal{E}\|^2 \|\mathcal{F}\|^2.$$

For the convenience of the reader we reproduce the details as given in [33].

Proof. A typical element of $\mathcal{E} \vee \mathcal{F}$ has the form

$$G = \sum_{j=1}^{m} \sum_{k=1}^{n} \alpha_{jk} E_j F_k,$$

where $E_1, \cdots, E_m$ are mutually disjoint elements of $\mathcal{E}$, $F_1, \cdots, F_n$ are mutually disjoint elements of $\mathcal{F}$, $\sum E_j = \sum F_k = I$, and each α_{jk} equals 0 or 1. Then

$$2G - I = \sum_{j=1}^{m} \sum_{k=1}^{n} \beta_{jk} E_j F_k,$$

where each β_{jk} equals ± 1. By ([33], Lemma 2.4) we have

$$\begin{aligned}\|(2G-I)x\| &= 8\|\mathcal{E}\|\|\mathcal{F}\|\left\|\left(\sum_{j=1}^{m}\sum_{k=1}^{n}|E_jF_k(2G-\mathrm{I})x|^2\right)^{1/2}\right\| \\ &= 8\|\mathcal{E}\|\|\mathcal{F}\|\left\|\left(\sum_{j=1}^{m}\sum_{k=1}^{n}|E_jF_kx|^2\right)^{1/2}\right\| \\ &\leq 64\|\mathcal{E}\|^2\|\mathcal{F}\|^2\|x\|\end{aligned}$$

for all $x \in X$ This gives the required result. □

Kakutani [39] gave an example of two commuting bounded Boolean algebras of projections on a Banach space which generate an unbounded Boolean algebra of projections. Since every Banach space is isomorphic to a subspace of $C(K)$ space, it follows that Lemma 5.6.4 does not extend to arbitrary subspaces of Banach lattices. However, it does extend to subspaces of Banach lattices which are p-concave for some p in the range $1 \leq p < \infty$.

Definition 5.6.5. *A Banach lattice W is* p-concave *if there is a constant $M < \infty$ such that*

$$\left(\sum_{k=1}^{n}\|w_j\|^p\right)^{1/p} \leq M\left\|\left(\sum_{k=1}^{n}|w_j|^p\right)\right\|^{1/p}$$

for all finite sequences $w_1, \cdots, w_n$ in W.

Remark 5.6.6. *The least possible constant M is called the* p-concavity constant *of W: we shall denote it by $M_p(w)$. This notation is usually applied to real Banach lattices. It is clear that a complex Banach lattice W is p-concave if and only if* Re W *is p-concave, and when both lattices are p-concave, $M_p(W) = M_p(\mathrm{Re}\, W)$. Also, if W is an L_p space, where $1 \leq p < \infty$, then W is p-concave with p-concavity constant 1.*

Theorems 5.6.7 and 5.6.9 are due to Gillespie.

Theorem 5.6.7. *Let $1 \leq p < \infty$ and let X be a closed subspace of a p-concave complex Banach lattice W. Suppose $\mathcal{E}, \mathcal{F}$ are commuting bounded Boolean algebras* *of projections on X. Then the Boolean algebra of projections $\mathcal{E} \vee \mathcal{F}$ is bounded. Furthermore,*

$$\|\mathcal{E} \vee \mathcal{F}\| \leq \alpha_p M_p(W) \|\mathcal{E}\| \|\mathcal{F}\|,$$

where α_p is a constant which depends on p but not on W.

Proof. ([33], Theorem 2.6). □

Theorems 5.6.4 and 5.6.7 imply similar boundedness results for other related classes of Banach spaces as follows.

Definition 5.6.8. *Let $1 \leq p \leq \infty$ and $1 \leq \lambda < \infty$. A Banach space X is said to be an $\mathcal{L}_{p,\lambda}$ space if for every finite-dimensional subspace B of X there is finite-dimensional subspace C of X such that $C \supseteqq B$ and*
$d(C, l_p^n) = \inf\{\|c - t\| : c \in C, t \in l_p^n\} \leq \lambda$ where $n = dim\, C$. A Banach space is said to be an $\mathcal{L}_p$ space ($1 \leq p \leq \infty$) if it is an $\mathcal{L}_{p,\lambda}$ space for some $\lambda < \infty$.

The basic theory of $\mathcal{L}_p$ spaces can be found in [44].

The *unconditional basis constant* $\mathfrak{X}(E)$ of a given Banach space E is the least constant λ having the following property: There exists a basis $\{e_i\}$ for E such that $\left\|\sum_{i\in I} \epsilon_i x_i e_i\right\| \leq \lambda$ whenever $\sum_{i\in I} x_i e_i \in E$ has norm one and $\epsilon_i = \pm 1\, (i \in I)$, with $\epsilon_1 = 1$ for all but finitely many i. If no such λ exists, set $\mathfrak{X}(E) = \infty$. We do not exclude the case where the index set I is uncountable, in which case all vectors $\sum_{i\in I} x_i e_i$ have $x_i = 0$ for all but countably many indices i. More generally define the *local unconditional constant* of E, $\mathfrak{X}_u(E)$, to be the infimum of all scalars λ having the following property: Given any finite-dimensional subspace $F \subseteq E$, there exist a space U and operators $\alpha \in L(F, U)$, $\beta \in L(U, E)$, such that $\beta\alpha$ is the identity on F and $\|\alpha\| \|\beta\| \mathfrak{X}(u) \leq \lambda$. If no such λ exists, set $\mathfrak{X}_u(E) = \infty$. In case $\mathfrak{X}_u(E) < \infty$, we say that E has *local unconditional structure.*

See [34] for a fuller account.

Theorem 5.6.9. *Let $\mathcal{E}$ and $\mathcal{F}$ be commuting bounded Boolean algebras of projections on a Banach space X. Then the Boolean algebra $\mathcal{E} \vee \mathcal{F}$ is bounded and satisfies*

$$\|\mathcal{E} \vee \mathcal{F}\| \leq \alpha_X \, \|\mathcal{E}\|^2 \|\mathcal{F}\|^2,$$

where α_X is constant depending only on X, in each of the following cases.

1. *X is a subspace of an $\mathcal{L}_p$ space for some p in the range $1 \leq p < \infty$.*

2. *X is a complemented subspace of an $\mathcal{L}_\infty$.*

3. *X has local unconditional structure.*

Proof. ([33], Theorem 2.9). □

Theorem 5.6.10. *Let X be a Banach space and T_1, T_2 be commuting scalar-type spectral operators on X with resolutions of the identity $\mathcal{E}^{(1)}$, $\mathcal{E}^{(2)}$. Then all operators in the weakly closed *-algebra generated by T_1 and T_2 are scalar-type spectral operators in each of the following cases.*

1. *X is a Banach lattice which does not contain a copy of c_0.*

2. *X is a subspace of a p-concave Banach lattice, where $1 \leq p < \infty$.*

3. *X is a subspace of an $\mathcal{L}_p$ space where $1 \leq p < \infty$.*

4. *X is a complemented subspace of an $\mathcal{L}_\infty$ space.*

Proof. (1) is immediate of Corollary 5.4.2 and Theorem 5.6.4. For (2) note that a Banach lattice which is p-concave for some finite p cannot contain a copy of c_0 ([46], p. 52) and then apply Corollary 5.4.2 and Theorem 5.6.4. By ([44], Theorem I) part (3) is a special case of (2). For (4), let $\mathcal{E}^1$, $\mathcal{E}^2$ be the spectral measures of T_1, T_2 By ([50], Theorem 14), the spectral measure $\mathcal{E}^1$ is atomic and for each $x \in X$, the vector measure $\mathcal{E}^1(\cdot)x$ is supported on a countable set. Now the result follows from Theorem 5.5.3. □

Notation

The following list includes notation which is either not defined in the body of the thesis or which is used in a different section to where it is defined.

$\mathbb{R}, \mathbb{C}$	the real and complex scalar fields
$\mathbb{N}, \mathbb{Z}$	the integers and the positive integers
X	a real or complex Banach space
X'	the Banach space of continuous linear functionals on X
$\langle x, x' \rangle$	the linear functional $x' \in X'$ evaluated at $x \in X$
$\sigma(X, X')$	the weak topology on X
$\sigma(X', X)$	the weak-star topology on X'
SOT	the strong operator topology on $L(X)$
WOT	the weak operator topology on $L(X)$
$st \lim$	the limit in the strong operator topology
$w \lim$	the limit in the weak operator topology
$BV(J \times K)$	the space of functions of bounded variation on $J \times K$ where J and K are compact intervals
$AC(J \times K)$	the space of absolutely continuous functions on $J \times K$ where J and K are compact intervals

$C(\Lambda)$	the space of continuous functions on the Hausdorff space Λ
(Ω, Σ, μ)	a positive measure space
$L_p(\Omega, \Sigma, \mu)$	the space of equivalence classes of p-integrable Σ-measurable functions on Ω
L_p	the space $L_p(\Omega, \Sigma, \mu)$ where (Ω, Σ, μ) is Lebesgue measure space on $[0, 1]$
$l_p(\Gamma)$	the space $L_p(\Gamma, \sigma, \mu)$ where (Γ, Σ, μ) is the discrete measure space on a set Γ with $\mu(\{\gamma\}) = 1$ for every $\gamma \in \Gamma$
l_p^n	the space $l_p(\Gamma)$ where $\Gamma = \{1, 2, \cdots, n\}$
l_p	the space $l_p(\Gamma)$ where $\Gamma = \mathbb{N}$
$\|f\|_p$	the norm of f in $L_p(\Omega, \Sigma, \mu)$
$L_\infty(\Omega, \Sigma, \mu)$	the space of equivalence classes of essentially bounded Σ-measurable function on Ω
$\varrho(T)$	the resolvent set of T
$\sigma(T)$	the spectrum of T
$\sigma_T(x)$	the local spectrum of T at x
$[\cdot, \cdot]$	semi-inner-product
$W(T)$	the numerical range of T
$\nu(T)$	the spectral radius of T
$L_p(G)$	see p. 61
$H^p(\mathbb{D})$	see p. 62
$\operatorname{aco} \mathcal{E}$	the absolutely convex hull of $\mathcal{E}$ in $L(X)$
$\overline{\operatorname{aco}}^w \mathcal{E}$	the closure of $\operatorname{aco} \mathcal{E}$ in the weak operator topology of $L(X)$

Bibliography

[1] C.A. Akemann, P.G. Dodds and J.L.B. Gamlen, Weak compactness in the dual space of a C^*-algebra, J. Functional Analysis **10** (1972), 446-450.

[2] E. Albrecht, On some classes of generalized spectral operators, Arch. Math. (Basel) **30** (1978), no. 3, 297-303.

[3] W.G. Bade, On Boolean algebras of projections and algebras of operators, Trans. American Math. Soc. **80** (1955), 345-360.

[4] J.Y. Barry, On the convergence of ordered sets of projections, Proc. American Math. Soc. **5** (1954), 313-314.

[5] R.G. Bartle, N. Dunford and J. Schwartz, Weak compactness and vector measures, Canadian J. Math. **7** (1955), 289-305.

[6] E. Behrends, Normal operators and multipliers on complex Banach spaces and a symmetry property of L_1-predual spaces, Israel Journal of Mathematics **47** (1984), 23-28.

[7] E. Berkson, A characterization of scalar type operators on reflexive Banach spaces, Pacific J. Math. **13** (1963), 365-373.

[8] E. Berkson, Spectral families of projections in Hardy spaces, J. Functional Analysis **60** (1985) (2), 146-167.

[9] E. Berkson and H.R. Dowson, Prespectral operators, Illinois J. Math. **13** (1969), 291-315.

[10] E. Berkson and H.R. Dowson, On uniquely decomposable well-bounded operators, Proc. London Math. Soc. (3), **22** (1971), 339-358.

[11] E. Berkson, I. Doust and T.A. Gillespie, Properties of AC-operators. Acta Sci. Math. (Szeged), **63** (1997), 249-257.

[12] E. Berkson and T.A. Gillespie, Absolutely continuous function of two variables and well-bounded operators, J. London Math. Soc. (2) **30** (1984), 305-321.

[13] E. Berkson, R. Kaufman and H. Porta, Möbius transformations of the disc and one-parameter groups of isometries of H^p, Trans. American Math. Soc. **199** (1974), 223-239.

[14] E. Berkson and H. Porta, Hermitian operators and one-parameter groups of isometries in Hardy spaces, Trans. American Math. Soc. **185** (1973), 331-344.

[15] F.F. Bonsall and J. Duncan, Complete normed algebras, Springer-Verlag (1973).

[16] F.F. Bonsall and J. Duncan, Numerical ranges of operators on normed spaces and of elements of normed algebras, Cambridge University Press (1971).

[17] I. Colojoară and C. Foiaş, Theory of Generalized Spectral Operators, Gordon and Breach, New York (1968).

[18] M.J. Crabb and P.G. Spain, Commutators and normal operators, Glasgow Math. J. **18** (1977), 197-198.

[19] J. Diestel and J.J. Uhl Jr., Vector measures, American Math. Soc., Providence, RI (1977).

[20] I. Doust, Well-bounded and scalar-type spectral operators on spaces not containing c_0, Proc. American Math. Soc. **105** (1989), 367–370.

[21] I. Doust and R. deLaubenfels, Functional calculus, Integral representation, and Banach space geometry, Quaestiones Math. **17** (1994), 161-171.

[22] H.R. Dowson, Spectral Theory of Linear Operators, Academic Press (1978).

[23] H.R. Dowson, T.A. Gillespie, A representation theorem for a complete Boolean algebra of projections, Proc. Roy. Soc. Edin. **83A** (1979), 225-237.

[24] H.R. Dowson, T.A. Gillespie and P.G. Spain, A commutativity theorem for hermitian operators, Math. Ann. **220** (1976), 215-217.

[25] N. Dunford, Spectral operators. Pacific J. Math. **4** (1954), 321-354

[26] N. Dunford and J.T. Schwartz, Linear operators, Part I : General theory, Wiley-Interscience, New York (1958).

[27] N. Dunford and J.T. Schwartz, Linear operators, Part II : Spectral theory, self-adjoint operators in Hilbert space, Wiley-Interscience, New York (1963).

[28] N. Dunford and J.T. Schwartz, Linear operators, Part III : Spectral operators, Wiley-Interscience, New York (1971).

[29] M.B. Ghaemi, Normal-equivalent operators and operators with dual of scalar-type, Proc. Edinburgh Math. Soc. **43** (2000), 261-268.

[30] T.A. Gillespie, Commuting well-bounded operators on Hilbert spaces, Proc. Edinburgh Math. Soc. (2) **20** (1976), 167-172.

[31] T.A. Gillespie, A spectral theorem for L_p translation, J. London Math. Soc. (2), **11** (1975), 499-508.

[32] T.A. Gillespie, Strongly closed bounded Boolean algebras of projections, Glasgow Math. J. **22** (1981), 73-75.

[33] T.A. Gillespie, Boundedness criteria for Boolean algebras of projections, J. Functional Analysis **148** (1997), 70-85.

[34] Y. Gordon and D.R. Lewis, Absolutely summing operators and local unconditional structures, Acta Math. **133** (1974), 27-48.

[35] A. Grothendieck, Sur les applications linéaires faiblement compactes d'espaces du type $C(K)$, Canadian J. Math. **5** (1953), 129-173.

[36] G.H. Hardy, On double Fourier series, Quart. J. Math. **37** (1906), 53-79.

[37] E.W. Hobson, The theory of functions of a real variable and the theory of Fourier series, Vol. 1, third edition (Cambridge University Press, 1927; Dover, New York, 1957).

[38] R.V. Kadison, Operator algebras with a faithful weakly-closed representation, Ann. Math. **64** (1956), 175-181.

[39] S. Kakutani, An example concerning uniform boundedness of spectral measure, Pacific J. Math. **4** (1954), 363-372.

[40] Y. Katznelson, An Introduction to Harmonic Analysis, Dover, New York (1976).

[41] I. Kluvánek, Characterization of scalar-type spectral operators. Arch. Math. (Brno) **2** (1966), 153-156.

[42] G.L. Krabbe, A Helly convergence theorem for Stieltjes integrals, Indagationes Math. **27** (1965), 52-69.

[43] K.B. Laursen and M.M. Neumann, An introduction to local spectral theory, The Clarendon Press, Oxford University Press (2000).

[44] J. Lindenstrauss and H.P. Rosental, The $\mathcal{L}_p$ spaces, Israel J. Math. **7** (1969), 325-349.

[45] J. Lindenstrauss and L. Tzafriri, Classical Banach spaces I: Sequence spaces, Springer-Verlag, Berlin (1977).

[46] J. Lindenstrauss and L. Tzafriri, Classical Banach spaces II: Function Spaces, Springer-Verlag, Berlin (1979).

[47] G. Lumer, Semi-inner-product spaces, Trans. American Math. Soc. **100** (1961), 29-43.

[48] C.A. McCarthy, Commuting Boolean algebras of projections I, Pacific J. Math. **11** (1961), 295-307.

[49] C.A. McCarthy, Commuting Boolean algebras of projections II, Proc. American Math. Soc. **15** (1964), 781-787.

[50] C.A. McCarthy, and L. Tzafriri, Projections in $\mathcal{L}_1$ and $\mathcal{L}_\infty$-spaces, Pacific J. Math. **26** (1968), 529-546.

[51] T.W. Palmer, Unbounded normal operators on Banach spaces, Trans American Math. Soc. **133** (1968), 385-414.

[52] A. Pełczynski, Projections in certain Banach spaces, Studia Math. **19** (1960), 209-228.

[53] C. Qingping and I. Doust, Compact well-bounded operators *(Preprint)*.

[54] C. Qingping and I. Doust, The dual theory of well-bounded operators, J. Operator Theory, **37** (1997), 35-50.

[55] C. Qingping and I. Doust, Well-bounded operators on nonreflexive Banach spaces, Proc. Amer. Math. Soc., **124** (1996), 799-808.

[56] C. Qingping and I. Doust, Some remarks on well-bounded and scalar-type decomposable operators, (to appear).

[57] W. Ricker, Operator algebras generated by commuting projections: a vector measure approach, Lecture Notes in Mathematics **1711**, Springer-Verlag (1999).

[58] J.R. Ringrose, On well-bounded operators, J. Austral. Math. Soc. (Series A) **1** (1960), 334-343.

[59] J.R. Ringrose, On well-bounded operators II, Proc. London Math. Soc. (3) **13** (1963), 613-638.

[60] S. Sakai, C^*-algebras and W^*-algebras, Springer-Verlag, Berlin, Heidelberg, New York (1971).

[61] W.H. Sills, On absolutely continuous function and the well-bounded operator, Pacific J. Math. **17** (1966), 349-366.

[62] D.R. Smart, Conditionally convergent spectral expansions, J. Austral. Math. Soc. (Series A) **1** (1960), 319-333.

[63] P.G. Spain, On scalar-type spectral operators, Proc. Camb. Phil. Soc. **69** (1971), 409-410.

[64] P.G. Spain, On commutative V^*-algebras. II. Glasgow Math. J. **13** (1972), 129-134.

[65] P.G. Spain, On well-bounded operators of type (B), Proc. Edinburgh Math. Soc. (2) **18** (1972), 35-48.

[66] P.G. Spain, A Generalisation of a Theorem of Grothendieck, Quarterly J. Math. **27** (1976), 475-479.

[67] P.G. Spain, V^*-algebras with weakly compact unit spheres, J. London Math. Soc. **4** (1971), 62-64.

[68] P.G. Spain, The W^*-closure of a V^*-algebra, J. London Math. Soc. **7** (1973), 385-386.

[69] F.-H. Vasilescu, Analytic functional calculus and spectral decompositon, Editura Academiei and D. Reidel Publ. Co.

[70] J. Wermer, Commuting spectral operators in Hilbert space, Pacific J. Math. **4** (1954), 355-361.