EXTENDING MODULES RELATIVE TO MODULE CLASSES

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A thesis presented to the

University of Glasgow

Faculty of Science

for the degree of

Doctor of Philosophy

To my family

STATEMENT

This thesis is submitted in accordance with the regulations for the degree of Doctor of Philosophy of the University of Glasgow.

Chapter 1, chapter 2, chapter 3 and chapter 4 of this thesis are all original with the exception of the well-known properties in chapter 1 as well as other instances indicated within the text.

SUMMARY

The notion of an extending module can be traced back to work of von Neumann in the 1930s. His interest in quantum mechanics led him to develop "continuous geometry", which we today refer to as upper and lower continuous complete modular lattices.

In a series of papers [34]-[36], von Neumann developed the theory of continuous geometries and their realisation as the lattice of principal left ideals of a von Neumann regular ring (see [33], [37]). Regular rings he called *continuous* if the lattice of principal left ideals is upper and lower continuous. Utumi [53], [54], [56] continued this study. He defined a regular ring to be *left continuous* if its lattice of principal left ideals is upper continuous, and proved that a regular ring R is left continuous if and only if the left R-module R is extending. Utumi [55] also studied rings which need not be regular but for which the left R-module R is continuous or quasi-continuous (i.e. π-injective) and these concepts were carried over to modules by Jeremy [21], [22], Takeuchi [50], and Mohamed-Bouhy [27].

Continuous and quasi-continuous modules were studied by various authors and a theory was developed. For a good account of this see the monographs by Mohamed and Muller [28] and Dung, Huynh, Smith and Wisbauer [5]. Perhaps one ought to mention the major contributions of Muller and his students, in particular Kamal and Rizvi, and also of Harada and his students, particularly Oshiro (see, for example, [16]-[18], [15], [23]-[25], [30], [31], [32], [39], [40], [41]). Of course, one could mention many other names and their articles here.

Seemingly independent of the above development, Goldie [10], [11] considered complements in his study of quotient rings and this was the inspiration for Hajarnavis to consider left CS-rings, i.e. rings R for which R as an R-module is

extending, and to publish [2] with Chatters. Chatters subsequently collaborated with Khuri in [3] to consider endomorphism rings of modules over left CS-rings. In fact complements occupy a very important place in the theory. Our extending definitions are based on it in two different ways.

Because of the disparate nature of the development of the theory, different authors have adopted different terminology. Harada [14], [17] and his school have used term "extending module" as the dual to "lifting module", and this is also used in [5]. Chatters and Hajarnavis use "CS" for "complements are summands" and this terminology has been widely adopted also.

In the development of the theory (with its different origins) it has become increasingly clear that more general modules warranted study because continuous and quasi-continuous modules have a common property, namely the extending property of submodules, i.e. every submodule is essential in a direct summand; equivalently, every closed submodule is a direct summand. Among examples of extending modules, we could point out semisimple modules, uniform modules and quasi-injective modules. Also, any free abelian group of finite rank is an extending module.

The purpose of this study is to give an up-to-date presentation of known and new results on extending modules and related notions with respect to an R-module class $\mathcal{X}$. By assuming basic facts from Module Theory, our treatment is essentially self-contained.

In the first chapter, some background material is given together with the definitions of the two types of extending module with respect to a class of modules. We investigate the extending property with respect to related module classes and direct sum decompositions of extending modules. We also define two types of weak extending module and compare with extending modules both with respect to a class of modules.

The second chapter concerns the structure and properties of extending modules with respect to certain standard classes of modules, namely Goldie torsion modules, nonsingular modules, modules with finite uniform dimension and finitely generated modules. We also investigate the particular case of torsion modules over Dedekind domains.

The importance of injective modules in Module Theory and more generally in Algebra is obvious in the 1960s and 1970s, largely, but not exclusively, through the impact of the publication of the lecture notes of Carl Faith [9]. Since that time there has been continuing interest in such modules and their various generalizations which arose not only directly from the study of injectives but also from the work of John von Neumann mentioned above. Some results obtained for injective modules can be transferred readily to injective modules with respect to R-module classes $\mathcal{X}$.

In chapter three, we investigate the injective and also quasi-injective modules with respect to R-module classes and characterise them.

In chapter four, we study modules with general quasi-continuity property and this notion is related to ideas and results found earlier in the thesis. We also generalize the concept of a quasi-continuous module by means of a property $Q(\mathcal{X})$ relative to a class $\mathcal{X}$ of modules. Modules satisfying $Q(\mathcal{X})$ are studied and the property $Q(\mathcal{X})$ related to the extending properties introduced in the earlier chapters. In addition, we investigate modules with this general quasi-continuity property with respect to related module classes and with respect to certain specific classes of modules.

ACKNOWLEDGEMENT

I wish to give my deepest thanks to my supervisor Prof. P.F. Smith, for his inspiration, guidance, and encouragement during my research and preparation of this Thesis. It is also my pleasure to thank him for his consideration and kindness to my life in Glasgow.

I also wish to express my gratitude to Prof. A. Harmanci at Hacettepe University for first introducing me to the world of ring and module theory.

I am also very grateful to the Mathematics Department of the University of Glasgow for its support in letting me attend so many conferences and seminars in the U.K. and in Europe. In particular, I am deeply grateful to Prof. K. A. Brown for all the assistance that he gave me.

I am also indebted to Afyon Kocatepe University for their financial support from 1993 to 1997, during my thesis.

Finally, special thanks to my family and friends, whose love and support have made this thesis possible.

Contents

TABLE OF SYMBOLS

Let R be any ring. Then we denote various classes of R-modules as follows:

$\mathcal{M}$	:	class of all R-modules
$\mathcal{I}$	:	class of injective R-modules
$\mathcal{P}$	:	class of projective R-modules
$\mathcal{G}$	:	class of finitely generated R-modules
$\mathcal{G}_1$	:	class of cyclic R-modules
$\mathcal{U}$	:	class of R-modules with finite uniform dimension
$\mathcal{U}_1$	:	class of uniform R-modules
$\mathcal{C}$	:	class of semisimple R-modules
$\mathcal{C}_1$	:	class of simple R-modules
$\mathcal{A}$	:	class of Artinian R-modules
$\mathcal{N}$	:	class of Noetherian R-modules
$\mathcal{L}$	:	class of R-modules with finite composition length
$\mathcal{S}$	:	class of singular R-modules
$\mathcal{T}$	:	class of Goldie torsion R-modules
$\mathcal{F}$	:	class of nonsingular R-modules
$\mathcal{Z}$	:	class of zero R-modules
$N \leq_e M$	:	N an essential submodule of M
$E(M)$	:	injective hull of an R-module M
ACC	:	ascending chain condition
DCC	:	descending chain condition
$\mathrm{r}(m)$	:	right annihilator of m in R, for $m \in M$
UC	:	for "unique closure"
$\mathbb{M}$	:	non-empty set of submodules of an R-module
$P \trianglelefteq_p R$	:	P a prime ideal of R

Chapter 1

MODULES WHICH ARE EXTENDING RELATIVE TO MODULE CLASSES

In this chapter we will introduce basic definitions and some properties which will be required in the following chapters.

Given a ring R and a class $\mathcal{X}$ of right R-modules, a right R-module M can be an extending module relative to $\mathcal{X}$ in two different ways. Various general properties of such extending modules are given and, in case of specific classes of R-modules, we characterize them.

1.1 Introduction

Throughout this thesis all rings are associative with identity element and all modules are unital right modules. Let R be a ring and M be an R-module. A nonzero submodule N of a module M is called *essential* in M, written $N \leq_e M$, if it has nonzero intersection with any nonzero submodule of M. For any submodule N of M, a *closure of* N *(in* M*)* is a submodule K of M which is maximal in the collection of submodules H of M which contain N as an essential submodule. For any module M, $E(M)$ will denote its injective hull.

A submodule K of M is called *closed (in* M*)* if K has no proper essential extension in M. By Zorn's Lemma for each submodule N of M there exists a closed submodule K of M such that N is essential in K. Given any submodule N of M, by a *complement of* N *(in* M*)* we mean a submodule L of M which is maximal in the collection of submodules H with the property $H \cap N = 0$. A submodule L is called a *complement (in* M*)* if there exists a submodule N of M such that L is a complement of N. It is well known that a submodule K of M is closed if and only if K is a complement in M. The module M is called an *extending module* if every closed submodule is a direct summand of M.

In view of the above remarks we see that the following statements are equivalent for an R-module M:

(i) M is an extending module;

(ii) For every submodule N of M, every closure of N is a direct summand of M;

(iii) For every submodule N of M, every complement of N in M is a direct summand of M.

For a given submodule N of M, it may or may not be easy to check if every closure or if every complement of N is a direct summand. It all depends on M. Moreover, for any module M, in general there will be submodules which have every closure or every complement a direct summand. This is what motivates the following discussion.

By a *class* $\mathcal{X}$ of R-modules we mean a collection of R-modules containing the zero module and closed under isomorphisms, i.e. any module which is isomorphic to some module in $\mathcal{X}$ also belongs to $\mathcal{X}$. By an *$\mathcal{X}$-module* we mean any member of $\mathcal{X}$. If $\mathcal{X}$ is a class of R-modules and M is an R-module then an *$\mathcal{X}$-submodule* of M will be a submodule N of M such that N belongs to $\mathcal{X}$. For any ring R, any class $\mathcal{X}$ of R-modules is *closed under submodules* if a submodule N of M is an $\mathcal{X}$-module whenever the R-module M is an $\mathcal{X}$-module. Next $\mathcal{X}$ is *closed under factor modules* if $M/N \in \mathcal{X}$ for any submodule N of any $\mathcal{X}$-module M. On the other hand, $\mathcal{X}$ is *closed under extension* if every extension of an $\mathcal{X}$-module by an $\mathcal{X}$-module is also an $\mathcal{X}$-module. (i.e. whenever N is a submodule of a module M such that N and M/N are both $\mathcal{X}$-modules then M is an $\mathcal{X}$-module). Finally, $\mathcal{X}$ is *closed under direct sums* if every direct sum of $\mathcal{X}$-modules is an $\mathcal{X}$-module.

Let $\mathcal{X}$ be a class of R-modules. We shall say that an R-module M is *type 1 $\mathcal{X}$-extending* if for every $\mathcal{X}$-submodule N of M, every complement of N in M is a direct summand of M. On the other hand, an R-module M is *type 2 $\mathcal{X}$-extending* if for every $\mathcal{X}$-submodule N of M, every closure of N in M is a direct summand of M. Note that, every extending module is type 1 $\mathcal{X}$-extending and type 2 $\mathcal{X}$-extending.

In the sequel we shall be interested both in general classes $\mathcal{X}$ and in the particular classes $\mathcal{U}$, $\mathcal{G}$, $\mathcal{T}$ and $\mathcal{F}$ of modules with finite uniform dimension, finitely generated modules, Goldie torsion modules and nonsingular modules, respectively. For the class $\mathcal{U}$ of modules with finite uniform dimension, type 2 $\mathcal{U}$-extending modules are discussed in [28, 38, 39, 46] (where they are called modules with $(1-C_1)$), and in [5] and [7] (where they are called uniform-extending modules). If $\mathcal{C}$ is the class of semisimple modules then type 2 $\mathcal{C}$-extending modules are considered in [46] (where they are called CESS-modules).

1.2 Basic Properties and Examples

In this section, we shall give some examples and basic properties of type 1 and type 2 $\mathcal{X}$-extending modules, where $\mathcal{X}$ is a given class of modules. Throughout R is an arbitrary ring and modules are R-modules. The first result is obvious.

Lemma 1.2.1 *Let $\mathcal{M}$ denote the class of all R-modules. Then the following statements are equivalent for an R-module M:*

(i) *M is extending;*

(ii) *M is type 1 $\mathcal{M}$-extending;*

(iii) *M is type 2 $\mathcal{M}$-extending.*

In this case, M is type 1 and type 2 $\mathcal{X}$-extending for any class $\mathcal{X}$ of R-modules.

The second result is also clear.

Lemma 1.2.2 *Let $\mathcal{X} \subseteq \mathcal{Y}$ be classes of R-modules. Then any type 1 (respectively, type 2) $\mathcal{Y}$-extending R-module is type 1 (respectively, type 2) $\mathcal{X}$-extending.*

Next we give a series of results about closed submodules.

Lemma 1.2.3 *Let K be submodule of M. Then the submodule K is closed in M if and only if whenever Q is essential in M such that $K \subseteq Q$ then Q/K is essential in M/K.*

Proof. Suppose K is closed in M. Let Q be essential in M such that $K \subseteq Q$. Let P be a submodule of M such that $K \subseteq P$ and $(Q/K) \cap (P/K) = 0$. Now $K = Q \cap P$ is essential in P and hence $K = P$. Thus Q/K is essential in M/K.

Conversely, suppose that Q/K is essential in M/K for any essential submodule Q containing K. Suppose that K is essential in L. Let K' be a complement of K in M. Then $K \oplus K'$ is essential in M and hence $(K \oplus K')/K$ is essential in M/K. But $L \cap K' = 0$ gives that $((K \oplus K')/K) \cap (L/K) = 0$ and hence $K = L$. Thus K is closed. □

The next lemma can be found in [12, p.18 Proposition 1.5]. We give the proof for completeness.

Lemma 1.2.4 *Let N be any closed submodule of an R-module M and let K be any closed submodule of N. Then K is a closed submodule of M.*

Proof. Let K' be a complement of K in N and N' be a complement of N in M. Then $N \oplus N'$ is essential in M and hence $(N \oplus N')/N$ is essential in M/N by Lemma 1.2.3. Then $(N \oplus N')/K$ is essential in M/K. Similarly $(K \oplus K')/K$ is essential in N/K. Now $(N \oplus N')/K = (N/K) \oplus (K + N')/K$. Thus $(K + K' + N')/K = ((K \oplus K')/K) \oplus (K + N')/K$ is essential in M/K. Suppose that K is essential in V for a submodule $V \subseteq M$. Then $K \cap (K' + N') = 0$

implies $V \cap (K' + N') = 0$ and hence $(V/K) \cap (K + K' + N')/K = 0$. Thus $K = V$. It follows that K is closed. □

Lemma 1.2.5 *Let $K \subseteq N$ be submodules of an R-module M. Let L be a complement of K in M and let H be a complement of K in N such that $N \cap L \subseteq H$. Then $H \subseteq L$.*

Proof. Let $x \in K \cap (H + L)$. Then $x = y + z$ for some $y \in H$, $z \in L$. Thus $x - y \in N \cap L \subseteq H$ and hence $x = (x - y) + y \in H$. It follows that $K \cap (H + L) \subseteq K \cap H = 0$. Since $L \subseteq H + L$ we must have $L = H + L$ and hence $H \subseteq L$. □

Lemma 1.2.6 *Let $M = M_1 \oplus M_2$ be an R-module and let N, K be submodules of M_1. Then K is a complement of N in M_1 if and only if $K \oplus M_2$ is a complement of N in M.*

Proof. Suppose that K is a complement of N in M_1. Let H be any submodule of M such that $K \oplus M_2 \subseteq H$ and $H \cap N = 0$. Then $H = H \cap (M_1 \oplus M_2) = (H \cap M_1) \oplus M_2$, $H \cap M_1$ is a submodule of M_1, $K \subseteq H \cap M_1$ and $(H \cap M_1) \cap N = 0$. Thus $K = H \cap M_1$ and $H = K \oplus M_2$. It follows that $K \oplus M_2$ is a complement of N in M.

Conversely, suppose that $K \oplus M_2$ is a complement of N in M. Let G be any submodule of M_1 such that $K \subseteq G$ and $G \cap N = 0$. Then $K \oplus M_2 \subseteq G \oplus M_2$ and $(G \oplus M_2) \cap N = (G \oplus M_2) \cap M_1 \cap N = G \cap N = 0$. By hypothesis, $K \oplus M_2 = G \oplus M_2$ and hence $G = K$. It follows that K is a complement of N in M_1. □

Lemma 1.2.7 *Let $\mathcal{X}$ be any class of R-modules. Then any direct summand of a type 1 (respectively, type 2) $\mathcal{X}$-extending R-module is type 1 (respectively, type 2) $\mathcal{X}$-extending.*

Proof. Suppose that $M = M_1 \oplus M_2$ and that N is an $\mathcal{X}$-submodule of M_1. Suppose that M is type 1 $\mathcal{X}$-extending. Let K be a complement of N in M_1. By Lemma 1.2.6, $K \oplus M_2$ is a complement of N in M. Hence, by hypothesis, $K \oplus M_2$ is a direct summand of M and it follows that K is a direct summand of M_1. Thus M_1 is type 1 $\mathcal{X}$-extending.

Now suppose that M is type 2 $\mathcal{X}$-extending. Let L be any closure of N in M_1. By Lemma 1.2.4, L is a closure of N in M. Thus L is a direct summand of M and hence also of M_1. It follows that M_1 is type 2 $\mathcal{X}$-extending. □

Let M be a module. A module X is called *M-injective* if for every submodule N of M and homomorphism $\vartheta : N \longrightarrow X$ there exists a homomorphism $\theta : M \longrightarrow X$ such that $\theta|_N = \vartheta$, i.e. $\theta(n) = \vartheta(n)$ for all n in N. (We say that θ lifts to ϑ or that ϑ lifts to M.) The module X is called *injective* if X is M-injective for every module M.

The following Lemma can be found in [20].

Lemma 1.2.8 *Let a module $M = M_1 \oplus M_2$ be a direct sum of submodules M_1, M_2. Then the following statements are equivalent:*

(i) *M_2 is M_1-injective;*

(ii) *For each submodule N of M with $N \cap M_2 = 0$, there exists a submodule M' of M such that $M = M' \oplus M_2$ and $N \subseteq M'$.*

Proof. (i) $\Rightarrow$ (ii). For $i = 1, 2$, let $\pi_i : M \longrightarrow M_i$ denote the projection mapping. Consider the following diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & N & \xrightarrow{\alpha} & M_1 \quad \text{exact} \\ & & {\scriptstyle\beta}\downarrow & & \\ & & M_2 & & \end{array}$$

where $\alpha = \pi_1|_N$ and $\beta = \pi_2|_N$. By (i), there exists a homomorphism $\phi : M_1 \longrightarrow M_2$ such that $\phi\alpha = \beta$. Let $M' = \{x + \phi(x) : x \in M_1\}$. It easy to check that $M = M' \oplus M_2$ and $N \subseteq M'$.

(ii) $\Rightarrow$ (i). Let K be a submodule of M_1 and $\vartheta : K \longrightarrow M_2$ a homomorphism. Let $L = \{y - \vartheta(y) : y \in K\}$. Then L is a submodule of M and $L \cap M_2 = 0$. By (ii), $M = L' \oplus M_2$ for some submodule L' such that $L \subseteq L'$. Let $\pi : M \longrightarrow M_2$ denote the canonical projection (for the direct sum $M = L' \oplus M_2$). Then $\chi = \pi|_{M_1} : M_1 \longrightarrow M_2$ and, for any $y \in K$, $\chi(y) = \pi\{y - \vartheta(y) + \vartheta(y)\} = \vartheta(y)$. It follows that χ lifts ϑ to M_1. Thus M_2 is M_1-injective. $\square$

Lemma 1.2.9 *Let $\mathcal{I}$ denote the class of injective R-modules. Then any R-module M is type 1 $\mathcal{I}$-extending and type 2 $\mathcal{I}$-extending.*

Proof. It is clear that M is type 2 $\mathcal{I}$-extending. Suppose that N is an injective submodule of M and that K is a complement of N in M. There exists a submodule N' of M such that $M = N \oplus N'$. By Lemma 1.2.8, there exists a submodule N'' of M such that $M = N \oplus N''$ and $K \subseteq N''$. Since $N \oplus K$ is essential in M it follows that $K = (N \oplus K) \cap N''$ is essential in N'' and hence $K = N''$. It follows that M is type 1 $\mathcal{I}$-extending. $\square$

Note that if R is a (right and left) Artinian serial ring with Jacobson radical J and $J^2 = 0$ then every R-module is extending, in other words, type 1 and type

2 $\mathcal{M}$-extending, by [6, Theorem 11]. However, in this case if $J \neq 0$ then $\mathcal{M} \not\subseteq \mathcal{I}$ [1, Corollary 18.8].

Given a class $\mathcal{X}$ of R-modules it is natural to ask whether there is a relationship between type 1 and type 2 $\mathcal{X}$-extending R-modules. We show next that in general no such relationship exists.

Let R be commutative integral domain and let M be a right R-module. The set

$$\tau(M) = \{x \in M : xr = 0 \text{ for some } 0 \neq r \in R\}$$

is a submodule of M and it is called the *torsion submodule* of M. We will say that, M is a *torsion module* if $M = \tau(M)$, and M is *torsion-free* if $\tau(M) = 0$. Note that $\tau(M/\tau(M)) = 0$.

For any ring R and R-module M, $Z(M)$ will denote the singular submodule of M, i.e.

$$Z(M) = \{m \in M : mE = 0 \text{ for some essential right ideal } E \text{ of } R\}.$$

If $Z(M) = M$ then M is called a *singular module* and M is called a *nonsingular module* if $Z(M) = 0$.

If R is a commutative domain, then the essential ideals of R are exactly the nonzero ideals, and so the singular submodule of any R-module is just its torsion submodule. In this case the nonsingular R-modules are exactly the torsion-free R-modules.

Example 1.2.10 Let $\mathcal{F}$ denote the class of torsion-free $\mathbb{Z}$-modules. Let p be any prime. Then the $\mathbb{Z}$-module $M = (\mathbb{Z}/\mathbb{Z}p) \oplus \mathbb{Z}$ is type 1 $\mathcal{F}$-extending but not type 2 $\mathcal{F}$-extending.

Proof. Let N be any nonzero torsion-free submodule of M and let L be any complement of N in M. It is easy to check that $L = (\mathbb{Z}/\mathbb{Z}p) \oplus 0$. Thus M is type 1 $\mathcal{F}$-extending.

Let $K = \mathbb{Z}(1 + \mathbb{Z}p, p)$. Let H be an essential extension of the submodule K in M. Because K is uniform, H is uniform and hence H is cyclic, say $H = \mathbb{Z}(m + \mathbb{Z}p, n)$ for some $m, n \in \mathbb{Z}$. Now $(1 + \mathbb{Z}p, p) = s(m + \mathbb{Z}p, n)$ for some $s \in \mathbb{Z}$ and it is clear that $s = \pm 1$. Thus $K = H$. Therefore K is a torsion-free closed submodule of M but K is not a direct summand of M. It follows that M is not type 2 $\mathcal{F}$-extending. □

Before giving a second example, we give three results.

Lemma 1.2.11 *Let N and K be submodules of an R-module M. Then K is a complement of N in M if and only if K is closed in M, $N \cap K = 0$ and $N \oplus K$ is an essential submodule of M.*

Proof. For the sufficiency, suppose that K is closed in M, $N \cap K = 0$ and $N \oplus K$ is essential in M. Suppose that H is a submodule of M such that $K \subseteq H$ and $H \cap N = 0$. Let T be a submodule of H with $K \cap T = 0$. Then $T \cap (N \oplus K) = 0$. Since $N \oplus K$ is essential in M, $T = 0$, i.e. K is essential in H. This implies that $K = H$. Then K is a complement of N in M.

Conversely, suppose that K is complement of N in M. Let H be a submodule of M with K essential in H. Thus $H \cap N = 0$. Then $K = H$, i.e. K is closed. Now, we show that $N \oplus K$ is essential in M. Let T be a submodule of M and $T \cap (N \oplus K) = 0$. This implies that $N \cap (T \oplus K) = 0$. Then $T \oplus K = K$, i.e. $T = 0$. It follows that $N \oplus K$ is essential in M. □

Theorem 1.2.12 *Let $\mathcal{X}$ be any class of R-modules which is closed under submodules. Then an R-module M is type 1 $\mathcal{X}$-extending if and only if whenever K is a closed submodule of M such that the R-module M/K contains an essential $\mathcal{X}$-submodule then K is a direct summand of M.*

Proof. Suppose first that M is type 1 $\mathcal{X}$-extending. Let K be any closed submodule of M such that M/K contains an essential $\mathcal{X}$-submodule. Let N be a complement of K in M. Then N embeds in M/K and, because $\mathcal{X}$ is closed under submodules, it follows that N contains an essential $\mathcal{X}$-submodule L. Because K is closed in M, K is a complement of N in M, and hence K is a complement of L in M (Lemma 1.2.11). By hypothesis, K is a direct summand of M.

Conversely, suppose that a submodule B of M is a direct summand whenever B is a closed submodule of M and M/B contains an essential $\mathcal{X}$-submodule. Let P be any $\mathcal{X}$-submodule of M and let C be any complement of P in M. Then $P \cap C = 0$ and $P \oplus C$ is essential in M. Since C is closed it follows that $(P \oplus C)/C$ is essential in M/C by Lemma 1.2.3. But $(P \oplus C)/C \cong P$ so that $(P \oplus C)/C \in \mathcal{X}$ and, by hypothesis, C is a direct summand of M. It follows that M is type 1 $\mathcal{X}$-extending. □

For any ring R, a nonzero R-module M is said to be *uniform* if any two nonzero submodules of M have nonzero intersection, i.e. every nonzero submodule is essential in M. We say that M has *finite uniform dimension (or M is Goldie finite)* if M does not contain an infinite direct sum of nonzero submodules. Let M be a nonzero Goldie finite R-module. Then M contains a uniform submodule U. Moreover, there exist a positive integer n and independent uniform submodules U_i $(1 \leq i \leq n)$ of M such that $U_1 \oplus U_2 \oplus ... \oplus U_n$ is an essential submodule of

M. In this case, n is an invariant for M, called the *Goldie dimension* or *uniform dimension*, written $u.dim(M) = n$. If $N_1 \oplus ... \oplus N_k$ is any direct sum of nonzero submodules N_i $(1 \leq i \leq k)$ of M then $k \leq n$.

For any ring R, $\mathcal{U}$ will denote the class of R-modules with finite uniform dimension. Recall that $\mathcal{U}$ consists of all R-modules M which do not contain a direct sum of an infinite number of nonzero submodules. Note that $\mathcal{U}$ is closed under submodules.

Corollary 1.2.13 *With the above notation, an R-module M is type 1 $\mathcal{U}$-extending if and only if whenever K is a closed submodule of M such that the R-module M/K has finite uniform dimension then K is a direct summand of M.*

Proof. By Theorem 1.2.12. □

Example 1.2.14 Let $\mathcal{U}$ denote the class of $\mathbb{Z}$-modules with finite uniform dimension. Then any free $\mathbb{Z}$-module of infinite rank is type 2 $\mathcal{U}$-extending but not type 1 $\mathcal{U}$-extending.

Proof. Let M be any free $\mathbb{Z}$-module of infinite rank and let $\{m_i : i \in I\}$ be a basis of M. Let U be any $\mathcal{U}$-submodule of M. Then U contains a finitely generated essential submodule L. There exists a finite subset J of I such that if $N = \oplus_J m_i\mathbb{Z}$ then $L \subseteq N$. Since M/N is torsion-free and $(U+N)/N \cong U/(U \cap N)$ is torsion, it follows that $U \subseteq N$. Let V be any closure of U in M. Since $(V + N)/N \cong V/(V \cap N)$ is torsion, we have $V \subseteq N$ and hence N/V is finitely generated torsion-free. Thus V is a direct summand of N, and hence also of M. It follows that M is type 2 $\mathcal{U}$-extending.

There exists a submodule K of M such that $M/K \cong \mathbb{Q}$. Since M/K is torsion-free it follows that K is closed in M. Note also that M/K is uniform and K is not a direct summand of M. By Corollary 1.2.13, M is not type 1 $\mathcal{U}$-extending. □

1.3 General Classes of Modules

Classes of modules can be combined in different ways to give other classes and we examine how our extending properties behave under these constructions. Again R is an arbitrary ring.

Let $\mathcal{X}$ be any class of R-modules. Then $\mathcal{X}^e$ will denote the class of R-modules which contain an essential $\mathcal{X}$-submodule. Note that $\mathcal{X} \subseteq \mathcal{X}^e$.

Let n be a positive integer and let $\mathcal{X}_i$ $(1 \leq i \leq n)$ be classes of R-modules. Then $\mathcal{X}_1 \oplus ... \oplus \mathcal{X}_n$ is defined to be the class of R-modules M such that $M = M_1 \oplus ... \oplus M_n$ is a direct sum of $\mathcal{X}_i$-submodules M_i $(1 \leq i \leq n)$, and $\mathcal{X}_1 ... \mathcal{X}_n$ will denote the class of R-modules M such that there exist a chain of submodules

$$0 = M_0 \subseteq M_1 \subseteq ... \subseteq M_n = M$$

with the factor module $M_i/M_{i-1} \in \mathcal{X}_i$ $(1 \leq i \leq n)$. In particular, if $\mathcal{X}_i = \mathcal{X}$ $(1 \leq i \leq n)$ then we shall denote the class $\mathcal{X}_1 ... \mathcal{X}_n$ by $\mathcal{X}^n$. We set $\mathcal{X}^\omega = \cup_{n \geq 1} \mathcal{X}^n$.

Also $\mathcal{X}_1 + ... + \mathcal{X}_n$ will denote the class of R-modules M such that there exist $\mathcal{X}_i$-submodules L_i $(1 \leq i \leq n)$ with $M = L_1 + ... + L_n$. If $\mathcal{X}_1 = ... = \mathcal{X}_n = \mathcal{X}$ then we denote $\mathcal{X}_1 + ... + \mathcal{X}_n$ by $n\mathcal{X}$ and define $\mathcal{X}^+ = \cup_{n \geq 1}(n\mathcal{X})$.

A module M is called a *UC-module* if for each submodule N of M there exists a unique closed submodule K of M such that N is essential in K i.e. every submodule has a unique closure in M. For example, semisimple modules, uniform modules and nonsingular modules are all examples of UC-modules. The $\mathbb{Z}$-module $(\mathbb{Z}/\mathbb{Z}p) \oplus (\mathbb{Z}/\mathbb{Z}p^3)$ is not UC (see [47]).

The first result of this section is elementary but we give its proof for completenes.

Proposition 1.3.1 *Let n be a positive integer and let $\mathcal{X}_i$ $(1 \leq i \leq n)$ be classes of R-modules. Then*

(i) $\mathcal{X}_1 \oplus ... \oplus \mathcal{X}_n \subseteq \mathcal{X}_1 ... \mathcal{X}_n$.

(ii) $\mathcal{X}_1 \oplus ... \oplus \mathcal{X}_n \subseteq \mathcal{X}_1 + ... + \mathcal{X}_n$.

(iii) $\mathcal{X}_1 ... \mathcal{X}_n \subseteq (\mathcal{X}_1 \oplus ... \oplus \mathcal{X}_n)^e$ *if $\mathcal{X}_i$ is closed under submodules for $2 \leq i \leq n$.*

(iv) $\mathcal{X}_1 + ... + \mathcal{X}_n \subseteq \mathcal{X}_1 ... \mathcal{X}_n$ *if $\mathcal{X}_i$ is closed under factor modules for $2 \leq i \leq n$.*

(v) $\mathcal{X}_1 + ... + \mathcal{X}_n \subseteq (\mathcal{X}_1 \oplus ... \oplus \mathcal{X}_n)^e$ *if $\mathcal{X}_i$ is closed under factor modules and submodules for $2 \leq i \leq n$.*

(vi) *If $\mathcal{X}_i$ is closed under submodules for $2 \leq i \leq n$, then any $(\mathcal{X}_1 + ... + \mathcal{X}_n)$-module which is also a UC-module belongs to $(\mathcal{X}_1 \oplus ... \oplus \mathcal{X}_n)^e$.*

Proof. (i), (ii) clear.

(iii). By induction on n. If $n = 1$ then there is nothing to prove. Suppose that $n > 1$. Let $M \in \mathcal{X}_1 ... \mathcal{X}_n$. Then there exists a submodule N of M such that $N \in \mathcal{X}_1 ... \mathcal{X}_{n-1}$ and $M/N \in \mathcal{X}_n$. By induction on n, $N \in (\mathcal{X}_1 \oplus ... \oplus \mathcal{X}_{n-1})^e$. Let L be a complement of N in M. Because $N \cap L = 0$, we have $L \cong (L + N)/N \in \mathcal{X}_n$. Moreover, $N \oplus L$ is an essential submodule of M. Thus $M \in (\mathcal{X}_1 \oplus ... \oplus \mathcal{X}_n)^e$.

(iv). Let $H \in \mathcal{X}_1 + ... + \mathcal{X}_n$. Then $H = H_1 + ... + H_n$ for some $\mathcal{X}_i$-submodules H_i $(1 \leq i \leq n)$. By induction $G = H_1 + ... + H_{n-1} \in \mathcal{X}_1 ... \mathcal{X}_{n-1}$. Now

$$H/G \cong (H_n + G)/G \cong H_n/(H_n \cap G) \in \mathcal{X}_n.$$

Thus $H \in \mathcal{X}_1 ... \mathcal{X}_n$.

(v). By (iii), (iv).

(vi). By induction on n. If $n = 1$, then there is nothing to prove. Suppose that $n > 1$. Let $M \in \mathcal{X}_1 + ... + \mathcal{X}_n$. Then $M = M_1 + M_2$ for some $M_1 \in \mathcal{X}_1 + ... + \mathcal{X}_{n-1}$, $M_2 \in \mathcal{X}_n$.

Let K be a complement of $M_1 \cap M_2$ in M_2. Then $(M_1 \cap M_2) \oplus K$ is essential in M_2. Then $M_1 \oplus K$ is essential in M by [47]. Now $K \in \mathcal{X}_n$. Thus $M \in (\mathcal{X}_1 \oplus ... \oplus \mathcal{X}_n)^e$. □

Proposition 1.3.2 *For any class $\mathcal{X}$ of R-modules, an R-module M is type 1 (respectively, type 2) $\mathcal{X}$-extending if and only if M is type 1 (type 2) $\mathcal{X}^e$-extending.*

Proof. The sufficiency follows by Lemma 1.2.2. Now suppose that N is an $\mathcal{X}^e$-submodule of M. There exists an $\mathcal{X}$-submodule L such that L is essential in N. Clearly any closure of N is a closure of L. On the other hand, any complement of N is a complement of L by Lemma 1.2.11. The necessity follows. □

Theorem 1.3.3 *With the above notation, an R-module M is type 1 (respectively, type 2) $(\mathcal{X}_1 \oplus ... \oplus \mathcal{X}_n)$-extending if and only if M is type 1 (type 2) $\mathcal{X}_i$-extending for all $1 \leq i \leq n$.*

Proof. The necessity follows by Lemma 1.2.2. To prove the two converses, in each case we can suppose by induction that $n = 2$. Suppose that M is type 1 $\mathcal{X}_i$-extending for $i = 1, 2$. Let N_i $(i = 1, 2)$ be $\mathcal{X}_i$-submodules of M such that $N_1 \cap N_2 = 0$. Let $N = N_1 \oplus N_2$ and let K be a complement of N in M. Note that $K \cap N = 0$ gives $(K \oplus N_2) \cap N_1 = 0$. By Zorn's Lemma there exists a complement L of N_1 in M such that $K \oplus N_2 \subseteq L$. By hypothesis, L is a direct summand of M, i.e. $M = L \oplus L'$ for some submodule L'. Next note that $K \oplus N_2 = (K \oplus N) \cap L$, by the Modular Law , so that $K \oplus N_2$ is essential in L and hence K is a complement of N_2 in L (Lemma 1.2.11). By Lemma 1.2.6, $K \oplus L'$ is a complement of N_2 in M and by hypothesis $K \oplus L'$, and hence K, is a direct summand of M. It follows that M is type 1 $(\mathcal{X}_1 \oplus \mathcal{X}_2)$-extending, as required.

Now suppose that M is type 2 $\mathcal{X}_i$-extending for $i = 1, 2$. Let N_i $(i = 1, 2)$ be $\mathcal{X}_i$-submodules of M such that $N_1 \cap N_2 = 0$, let $N = N_1 \oplus N_2$ and let H be any closure of N in M. Let L be a closure of N_1 in H. Then L is a closure of N_1 in M by Lemma 1.2.4. By hypothesis L is a direct summand of M. Thus $M = L \oplus L'$ for some submodule L'. Also $H = L \oplus (H \cap L')$. Now $H \cap L'$ is closed in M by Lemma 1.2.4.

Let $0 \neq h \in H \cap L'$. Then there exist $r \in R$, $n_i \in N_i$ for $(i = 1, 2)$ with $0 \neq hr = n_1 + n_2$. Let $\pi' : H \longrightarrow H \cap L'$ denote the canonical projection. Then $hr = \pi'(n_1 + n_2) = \pi'(n_2) \in \pi'(N_2)$. Thus $\pi'(N_2)$ is essential in $H \cap L'$. But $N_1 \cap N_2 = 0$ implies $N_2 \cap L = 0$ and thus $N_2 \cong \pi'(N_2)$. Hence $\pi'(N_2) \in \mathcal{X}_2$. By hypothesis $H \cap L'$ is a direct summand of M. Therefore $H \cap L'$ is a direct summand of L' and hence H is a direct summand of M. □

Corollary 1.3.4 *Let $\mathcal{X}$ and $\mathcal{Y}$ be classes of R-modules such that $\mathcal{X} \subseteq \mathcal{I}$. Then an R-module M is type 1 (respectively, type 2) $\mathcal{Y}$-extending if and only if M is type 1 (respectively, type 2) $(\mathcal{X} \oplus \mathcal{Y})$-extending.*

Proof. By Lemmas 1.2.2 and 1.2.9 and Theorem 1.3.3. □

Given any class $\mathcal{X}$ of R-modules, $\mathcal{X}^{\oplus}$ will denote the class of R-modules which are direct sums of a finite number of $\mathcal{X}$-submodules. Theorem 1.3.3 also has the following immediate consequence.

Corollary 1.3.5 *Given any class $\mathcal{X}$ of R-modules, an R-module M is type 1 (respectively, type 2) $\mathcal{X}$-extending if and only if M is type 1 (type 2) $\mathcal{X}^{\oplus}$-extending.*

Theorem 1.3.6 *Let n be a positive integer and let $\mathcal{X}_i$ $(1 \leq i \leq n)$ be classes of R-modules such that $\mathcal{X}_i$ is closed under submodules for all $2 \leq i \leq n$ or $\mathcal{X}_i$ is closed under factor modules for all $2 \leq i \leq n$. Then an R-module M is type 2 $(\mathcal{X}_1 ... \mathcal{X}_n)$-extending if and only if M is type 2 $\mathcal{X}_i$-extending for all $1 \leq i \leq n$.*

Proof. Note that $\mathcal{X}_i \subseteq \mathcal{X}_1 ... \mathcal{X}_n$ for all $1 \leq i \leq n$. Thus, the necessity follows by Lemma 1.2.2.

Conversely, suppose that M is type 2 $\mathcal{X}_i$-extending for all $1 \leq i \leq n$. If $\mathcal{X}_i$ is closed under submodules for all $2 \leq i \leq n$, then by Proposition 1.3.1 (*iii*), $\mathcal{X}_1 ... \mathcal{X}_n \subseteq (\mathcal{X}_1 \oplus ... \oplus \mathcal{X}_n)^e$. Apply Theorem 1.3.3, Proposition 1.3.2 and Lemma 1.2.2.

Suppose that $\mathcal{X}_i$ is closed under factor modules for all $2 \leq i \leq n$. We will show that M is type 2 $(\mathcal{X}_1 ... \mathcal{X}_n)$-extending by induction on n. If $n = 1$ then there is nothing to prove. Suppose $n > 1$. Let $\mathcal{Y} = \mathcal{X}_1 ... \mathcal{X}_{n-1}$. Then M is type 2 $\mathcal{Y}$-extending by induction on n. Thus it is sufficient to prove the result when $n = 2$.

Let K be a closed submodule of M such that there exist submodules $L \subseteq N$ of K with $L \in \mathcal{X}_1$, $N/L \in \mathcal{X}_2$ and N is essential in K. There exist a closed submodule K' of K such that L is essential in K'. Then K' is a closed submodule of M (Lemma 1.2.4) and hence K' is a direct summand of M, because M is type 2 $\mathcal{X}_1$-extending. There exists a submodule K'' of M such that $M = K' \oplus K''$ and hence $K = K' \oplus (K \cap K'')$. Now again by the Modular Law

$N + K' = K' \oplus ((N + K') \cap K'')$ and

$$(N + K') \cap K'' \cong (N + K')/K' \cong N/(N \cap K') \in \mathcal{X}_2$$

since $L \subseteq N \cap K'$. Moreover, N is essential in K gives $N \cap K''$ essential in $K \cap K''$ and hence $(N + K') \cap K''$ essential in $K \cap K''$. Since $K \cap K''$ is a closed submodule of M (Lemma 1.2.4) it follows that $K \cap K''$ is a direct summand of M and hence $K \cap K''$ is a direct summand of K''. Thus K is a direct summand of M. Consequently, M is type 2 $(\mathcal{X}_1 ... \mathcal{X}_n)$-extending. □

Corollary 1.3.7 *Let $\mathcal{X}$ be any class of R-modules which is closed under submodules or under factor modules. Then an R-module M is type 2 $\mathcal{X}$-extending if and only if M is type 2 $\mathcal{X}^{\omega}$-extending.*

Proof. By Theorem 1.3.6. □

Theorem 1.3.8 *Let R be any ring, let n be a positive integer and let $\mathcal{X}_i$ $(1 \leq i \leq n)$ be classes of R-modules.*

(i) *Suppose that $\mathcal{X}_i$ is closed under factor modules for all $2 \leq i \leq n$. Then an R-module M is type 2 $(\mathcal{X}_1 + ... + \mathcal{X}_n)$-extending if and only if M is type 2 $\mathcal{X}_i$-extending for all $1 \leq i \leq n$.*

(ii) *Suppose that $\mathcal{X}_i$ is closed under submodules for all $2 \leq i \leq n$. Then a UC R-module M is type 2 $(\mathcal{X}_1 + ... + \mathcal{X}_n)$-extending if and only if M is type 2 $\mathcal{X}_i$-extending for all $1 \leq i \leq n$.*

Proof. (i). Let M be type 2 $(\mathcal{X}_1 + ... + \mathcal{X}_n)$-extending. Since $\mathcal{X}_i \subseteq (\mathcal{X}_1 + ... + \mathcal{X}_n)$, M is type 2 $\mathcal{X}_i$-extending for all $1 \leq i \leq n$ by Lemma 1.2.2.

Conversely, suppose that M is type 2 $\mathcal{X}_i$-extending R-module for all $1 \leq i \leq n$. Then M is type 2 $(\mathcal{X}_1 + ... + \mathcal{X}_n)$-extending by Proposition 1.3.1 (iv), Theorem 1.3.6 and Lemma 1.2.2.

(ii). The necessity follows by Lemma 1.2.2 and the sufficiency by Propositions 1.3.1 (vi) and 1.3.2, Theorem 1.3.3 and Lemma 1.2.2. □

Corollary 1.3.9 *Let R be any ring and let $\mathcal{X}$ be class of R-modules which is closed under factor modules. Then an R-module M is type 2 $\mathcal{X}$-extending if and only if M is type 2 $\mathcal{X}^+$-extending.*

Proof. By Theorem 1.3.8 □

Now, we have the following partial analogue for Theorem 1.3.6.

Theorem 1.3.10 *Let n be a positive integer and let $\mathcal{X}_i$ $(1 \leq i \leq n)$ be classes of R-modules such that $\mathcal{X}_i$ is closed under submodules for all $2 \leq i \leq n$. Then an R-module M is type 1 $(\mathcal{X}_1 ... \mathcal{X}_n)$-extending if and only if M is type 1 $\mathcal{X}_i$-extending for all $1 \leq i \leq n$.*

Proof. The necessity follows by Lemma 1.2.2. Conversely, suppose that M is type 1 $\mathcal{X}_i$-extending for all $1 \leq i \leq n$. By induction on n it is sufficient to prove that M is type 1 $(\mathcal{X}_1 ... \mathcal{X}_n)$-extending in case $n = 2$.

Let N be an $(\mathcal{X}_1\mathcal{X}_2)$-submodule of M. Then there exists a submodule $N_1 \subseteq N$ with $N_1 \in \mathcal{X}_1$ and $N/N_1 \in \mathcal{X}_2$. Assume that K is a complement of N in M. Let L be a complement of N_1 in N. Then $N_1 \cap L = 0$ implies that L embeds in $N/N_1 \in \mathcal{X}_2$. Thus $L \in \mathcal{X}_2$. Also $L \oplus N_1$ is essential in N and hence, by Lemma 1.2.11, K is a complement of $L \oplus N_1$. Since M is type 1 $(\mathcal{X}_1 \oplus \mathcal{X}_2)^e$-extending, by Proposition 1.3.2 and Theorem 1.3.3, K is a direct summand of M. □

Question 1.3.11 Is Theorem 1.3.6 true in the type 1 case, i.e. if $\mathcal{X}_i$ $(1 \leq i \leq n)$ is a finite collection of R-module classes, each closed under either submodules or factor modules and the R-module M is type 1 $\mathcal{X}_i$-extending for all $1 \leq i \leq n$, is M type 1 $(\mathcal{X}_1 ... \mathcal{X}_n)$-extending?

1.4 Decomposition of $\mathcal{X}$-extending Modules

Let R be a ring and let $\mathcal{X}$ be a class of R-modules. We call an R-module M *$\mathcal{X}$-free* if M contains no nonzero $\mathcal{X}$-submodule. Clearly any submodule of an $\mathcal{X}$-free module is itself $\mathcal{X}$-free.

Theorem 1.4.1 *Let $\mathcal{X}$ be any class of R-modules which is closed under submodules. Then any direct sum of $\mathcal{X}$-free modules is $\mathcal{X}$-free.*

Proof. Let $M = \oplus_\Lambda M_\lambda$ and let M_λ be an $\mathcal{X}$-free module for all $\lambda \in \Lambda$. Suppose M is not $\mathcal{X}$-free. Then there exists $0 \neq N \subseteq M$, $N \in \mathcal{X}$. Let $0 \neq x \in N$. Note that $0 \neq xR \subseteq N$, $xR \in \mathcal{X}$. There exists a finite subset Λ' of Λ such that $x \in \oplus_{\Lambda'} M_\lambda$. Without loss of generality, $\Lambda' = \{1, ..., n\}$. If $xR \cap (M_1 \oplus ... \oplus M_{n-1}) = 0$ then xR embeds in M_n, a contradiction. Thus $xR \cap (M_1 \oplus ... \oplus M_{n-1}) \neq 0$. But by induction on n, $M_1 \oplus ... \oplus M_{n-1}$ is $\mathcal{X}$-free, a contradiction. Thus M is $\mathcal{X}$-free. □

In Theorem 1.4.1 we need the R-module class $\mathcal{X}$ to be closed under submodules as the following example shows.

Example 1.4.2 Let $\mathcal{X}$ be the class of modules which are not finitely generated or are zero. Let M be a Noetherian module. Then M is $\mathcal{X}$-free but any direct sum of an infinite number of copies of M is not $\mathcal{X}$-free.

Theorem 1.4.3 *Let R be any ring and let $\mathcal{X}$ be a class of R-modules which is closed under direct sums. Let M be a type 2 $\mathcal{X}$-extending R-module. Then $M = M_1 \oplus M_2$ for some extending module M_1 and module M_2 such that there exists an essential submodule N of M_2 with N a direct sum of $\mathcal{X}$-free modules.*

Proof. Suppose first that M has no nonzero $\mathcal{X}$-free submodules. Let K be a closed submodule of M. Let $\{L_i : i \in I\}$ be a maximal collection of independent $\mathcal{X}$-submodules of K. Let $L = \oplus_{i\in I} L_i$. Then $L \in \mathcal{X}$ since $\mathcal{X}$ is closed under direct sums.

Let H be a submodule of K and suppose that $L \cap H = 0$. If $H \neq 0$ then there exists a nonzero $\mathcal{X}$-submodule H' of H and $\{L_i : i \in I\} \cup \{H'\}$ is a collection of independent $\mathcal{X}$-submodules, a contradiction. Thus L is essential in K. By hypothesis K is a direct summand of M. Thus M is an extending module.

Now suppose that M has nonzero $\mathcal{X}$-free submodules. Let $\{Q_j : j \in J\}$ be a maximal collection of nonzero independent $\mathcal{X}$-free submodules of M. Let $Q = \oplus_{j\in J} Q_j$. Let M_1 be a complement of Q in M. Then M_1 is a closed submodule of M. By the choice of Q, M_1 contains no nonzero $\mathcal{X}$-free submodule and hence, by the first part of the proof, M_1 contains an essential $\mathcal{X}$-submodule. Since M is type 2 $\mathcal{X}$-extending $M = M_1 \oplus M_2$ for some submodule M_2 of M. Also M_1 is extending.

Since $Q \cap M_1 = 0$ it follows that $Q \cong \pi_2(Q)$, where $\pi_2 : M \longrightarrow M_2$ is the canonical projection. Let $0 \neq m \in M_2$. Then $0 \neq mr = m_1 + q$ for some $r \in R$,

$m_1 \in M_1$, $q \in Q$. Now

$$mr = \pi_2(mr) = \pi_2(m_1 + q) = \pi_2(q).$$

Thus $\pi_2(Q)$ is essential in M_2 and so the proof is complete. □

Corollary 1.4.4 *Let $\mathcal{X}$ be any class of R-modules which is closed under submodules and direct sums. Let M be a type 2 $\mathcal{X}$-extending module. Then there exists an extending submodule M_1 of M and $\mathcal{X}$-free submodule M_2 of M such that $M = M_1 \oplus M_2$.*

Proof. By Theorems 1.4.1 and 1.4.3. □

Question 1.4.5 Is there a class $\mathcal{X}$ of modules such that $\mathcal{X}$ is not closed under both direct sums and submodules and a type 2 $\mathcal{X}$-extending module M such that M is not a direct sum of an extending module and an $\mathcal{X}$-free module?

Question 1.4.6 Is Corollary 1.4.4 true for type 1 $\mathcal{X}$-extending modules?

1.5 Uniform Decompositions

To consider decomposition properties, we need the following definitions.

A non-empty set $\mathbb{M}$ of submodules of an R-module is called *Noetherian* if it satisfies the *ascending chain condition (ACC)*, i.e. if every ascending chain

$$M_1 \subset M_2 \subset \ldots \text{ of modules in } \mathbb{M}$$

becomes stationary after finitely many steps.

$\mathbb{M}$ is called *Artinian* if it satisfies the *descending chain condition (DCC)*, i.e. every descending chain

$$M_1 \supset M_2 \supset \ldots \text{ of modules in } \mathbb{M}$$

becomes stationary after finitely many steps.

An R-module M is called *Noetherian (Artinian)* if the set of all submodules of M is Noetherian(Artinian).

By definition, R is a *right Noetherian (Artinian)* ring if and only if the module R_R is Noetherian (Artinian).

Let M be an R-module and let $m \in M$. Then we set

$$\mathrm{r}(m) = \{r \in R : mr = 0\}.$$

$\mathrm{r}(m)$ is a right ideal of R, called the *annihilator* of m in R.

A family $\{X_\lambda : \lambda \in \Lambda\}$ of submodules of a module M is called a *local summand* of M, if $\sum_{\lambda \in \Lambda} X_\lambda$ is direct and $\sum_{\lambda \in F} X_\lambda$ is a summand of M for every finite subset $F \subseteq \Lambda$ (see [28]).

Lemma 1.5.1 *Let M be an R-module such that R satisfies ACC on right ideals of the form $\mathrm{r}(m)$, $m \in M$. Then every local direct summand of M is closed in M.*

Proof. Let $N = \oplus_I N_i$ be any local direct summand of M. let L be a submodule of M such that N is essential in L. Suppose that $N \neq L$.

Choose $m \in L \setminus N$ such that $\mathrm{r}(m)$ is maximal in $\{\mathrm{r}(x) : x \in L \setminus N\}$. Clearly $m \neq 0$ and there exists $r \in R$ such that $0 \neq mr \in N$. There exists a finite subset

I' of I such that $mr \in K = \oplus_{I'} N_i$. Now $M = K \oplus K'$ for some submodule K' of M.

There exist $y \in K$, $y' \in K'$ such that $m = y + y'$. Now $mr = yr + y'r$ implies $y'r = 0$. Hence $\mathrm{r}(m)$ is a proper subset of $\mathrm{r}(y')$. But $y' = m - y \in L \setminus N$, contradicting the choice of m. Thus $N = L$ and it follows that N is closed. □

The next result should be compared to [38, Lemma 3].

Theorem 1.5.2 *Let $\mathcal{X}$ be a class of R-modules and let M be a type 1 $\mathcal{X}$-extending R-module such that every nonzero submodule contains a nonzero $\mathcal{X}$-submodule. If R satisfies ACC on right ideals of the form $\mathrm{r}(m)$, where $m \in M$, then M is a direct sum of $\mathcal{X}^e$-submodules.*

Proof. Let $M \neq 0$ and let U be a nonzero $\mathcal{X}$-submodule of M. Let W be a complement of U in M. Since M is type 1 $\mathcal{U}$-extending, it follows that $M = W \oplus W'$ for some submodule W'. Now $U \oplus W$ is essential in M and W is closed in M so that $(U \oplus W)/W$ is an essential submodule of M/W by Lemma 1.2.3. But $U \cong (U \oplus W)/W$ and $W' \cong M/W$. Thus W' is an $\mathcal{X}^e$-submodule of M and is also a direct summand of M.

By Zorn's Lemma, M contains a maximal local summand $\{M_i : i \in I\}$ of M with each submodule M_i $(i \in I)$ an $\mathcal{X}^e$-submodule of M. Let $N = \oplus_I M_i$. By Lemma 1.5.1, N is a closed submodule of M. Suppose that $N \neq M$. Then N is not essential in M. There exists a nonzero $\mathcal{X}$-submodule C of M such that $N \cap C = 0$. Let B be a complement of C in M such that $N \subseteq B$. Since M is type 1 $\mathcal{X}$-extending, it follows that $M = B \oplus B'$ for some submodule B'. The above argument shows that $B' \in \mathcal{X}^e$. Thus $\{M_i : i \in I\} \cup \{B'\}$ is a local summand of M, a contradiction. Thus $M = N = \oplus_I M_i$, as required. □

Note. M. Okado [38] proved Theorem 1.5.2 for the case of an extending module.

We call M *locally Noetherian* if every finitely generated submodule of M is Noetherian.

Corollary 1.5.3 *Let $\mathcal{X}$ be any class of R-modules. Then any locally Noetherian type 1 $\mathcal{X}$-extending module M is a direct sum of $\mathcal{X}^e$-submodules provided every nonzero submodule of M contains a nonzero $\mathcal{X}$-submodule.*

Proof. For each $m \in M$, mR is Noetherian and hence $R/\mathrm{r}(m) \cong mR$ is Noetherian. Thus R satisfies ACC on right ideals of the form $\mathrm{r}(m)$, where $m \in M$. Apply Theorem 1.5.2. □

Corollary 1.5.4 *Let $\mathcal{X}$ be a class of R-modules which is closed under submodules and let M be a type 1 $\mathcal{X}$-extending R-module such that R satisfies ACC on right ideals of the form $\mathrm{r}(m)$, where $m \in M$. Then M is a direct sum of $\mathcal{X}^e$-submodules if and only if every nonzero submodule of M contains a nonzero $\mathcal{X}$-submodule.*

Proof. The sufficiency is proved in Theorem 1.5.2. Conversely, suppose that $M = \oplus_I M_i$ where M_i is an $\mathcal{X}^e$-submodule of M for each $i \in I$. Let N be a nonzero submodule of M. Let $0 \neq m \in N$. There exists a finite subset J of I such that $m \in \oplus_J M_i$. If $|J| = 1$ then $m \in M_i$ for some $i \in I$ and M_i contains an essential $\mathcal{X}$-submodule L. In this case, $mR \cap L$ is a nonzero $\mathcal{X}$-submodule of N. Now suppose that $|J| > 1$. Let $j \in J$ and let $J' = J \setminus \{j\}$. If $mR \cap (\oplus_{J'} M_i) = 0$, then mR embeds in M_j and hence mR contains a nonzero $\mathcal{X}$-submodule. If

$mR \cap (\oplus_{J'} M_i) \neq 0$, then, by induction on $|J|$, $mR \cap (\oplus_{J'} M_i)$, and hence N, contains a nonzero $\mathcal{X}$-submodule. □

1.6 Weak Extending Modules

In this section we give the definition of weak extending modules with respect to a general class of modules and some basic properties.

Let $\mathcal{X}$ be a class of R-modules, for a given general ring R. An R-module M is called *weak type 1 $\mathcal{X}$-extending* if for every $\mathcal{X}$-submodule N of M there exists a complement K of N in M such that K is a direct summand of M. On the other hand, M is called *weak type 2 $\mathcal{X}$-extending* if every $\mathcal{X}$-submodule of M is essential in a direct summand of M, equivalently for every $\mathcal{X}$-submodule N of M there exists a closure L of N in M such that L is a direct summand of M.

Proposition 1.6.1 *Let $\mathcal{X}$ be any class of R-modules. Then any weak type 2 $\mathcal{X}$-extending R-module is weak type 1 $\mathcal{X}$-extending.*

Proof. Let M be any weak type 2 $\mathcal{X}$-extending R-module. Let N be any $\mathcal{X}$-submodule of M. Then there exist submodules K and K' of M such that $M = K \oplus K'$ and N is essential in K. It follows that the direct summand K' is a complement of N. Hence M is weak type 1 $\mathcal{X}$-extending. □

The next result is clear.

Proposition 1.6.2 *Let $\mathcal{X} \subseteq \mathcal{Y}$ be classes of R-modules. Then any weak type 1 (respectively, weak type 2) $\mathcal{Y}$-extending module is weak type 1 (weak type 2) $\mathcal{X}$-extending.*

Proposition 1.6.3 *For any class $\mathcal{X}$ of R-modules, an R-module M is weak type 1 $\mathcal{X}$-extending if and only if M is weak type 1 $\mathcal{X}^e$-extending.*

Proof. Since $\mathcal{X} \subseteq \mathcal{X}^e$, the sufficiency follows by Lemma 1.6.2. For the necessity, the proof of Proposition 1.3.2 can be adapted. □

Since $\mathcal{U} \subseteq \mathcal{G}^e$, Proposition 1.6.2 and 1.6.3 give at once:

Corollary 1.6.4 *Any weak type 1 $\mathcal{G}$-extending module is weak type 1 $\mathcal{U}$-extending. Moreover, the converse holds if R is right Noetherian.*

Proposition 1.6.5 *Let $\mathcal{X}$ be any class of R-modules. Then any type 1 (respectively, type 2) $\mathcal{X}$-extending module is weak type 1 (type 2) $\mathcal{X}$-extending.*

Proof. Clear. □

Lemma 1.6.6 *Let $\mathcal{X}$ be any class of R-modules and M be an R-module. Then M is type 2 $\mathcal{X}^e$-extending if and only if M is weak type 2 $\mathcal{X}^e$-extending.*

Proof. The necessity is clear by Proposition 1.6.5. Conversely, let $N \in \mathcal{X}^e$ and K be a closure of N in M. Then N is essential in K and also there exists an $\mathcal{X}$-submodule N_1 of N such that N_1 is essential in N. Thus $K \in \mathcal{X}^e$. By assumption K is essential in a direct summand of M. Therefore K is a direct summand and M is type 2 $\mathcal{X}^e$-extending. □

Theorem 1.6.7 *Let $\mathcal{X}$ be any class of right R-modules. Then an R-module M is type 2 $\mathcal{X}$-extending if and only if M is weak type 2 $\mathcal{X}^e$-extending.*

Proof. By Proposition 1.3.2 and Lemma 1.6.6. □

The above theorem can also be proved directly.

A class $\mathcal{X}$ of right R-modules will be called *essentially closed* if $\mathcal{X}$ is closed under essential extensions, i.e. a right R-module M belongs to $\mathcal{X}$ if M contains an essential $\mathcal{X}$-submodule. Note that if $\mathcal{X}$ is any class of right R-modules then $\mathcal{X} \subseteq \mathcal{X}^e$, and $\mathcal{X}$ is essentially closed if and only if $\mathcal{X} = \mathcal{X}^e$.

Corollary 1.6.8 *Let $\mathcal{X}$ be any essentially closed class of right R-modules. Then an R-module M is type 2 $\mathcal{X}$-extending if and only if M is weak type 2 $\mathcal{X}$-extending.*

Proof. By Theorem 1.6.7. □

Corollary 1.6.9 *Let R be any ring. Then a right R-module M is type 2 $\mathcal{U}$-extending (respectively, type 2 $\mathcal{T}$-extending) if and only if M is weak type 2 $\mathcal{U}$-extending (respectively, weak type 2 $\mathcal{T}$-extending).*

Proof. Since the class $\mathcal{U}$ of right R-modules with finite uniform dimension and the class $\mathcal{T}$ of Goldie-torsion right R-modules is essentially closed, apply Corollary 1.6.8. □

Lemma 1.6.10 *Let $\mathcal{X}$ be any class of R-modules. Then any weak type 2 $\mathcal{X}$-extending UC-module is type 2 $\mathcal{X}$-extending.*

Proof. Clear. □

Note that, Lemma 1.6.10 is not true for weak type 1 $\mathcal{X}$-extending UC-modules even when they are nonsingular. First we prove the following result.

Proposition 1.6.11 *Let R be a Dedekind domain. Then any free R-module is weak type 1 $\mathcal{G}$-extending.*

Proof. Let F be a free R-module with basis $\{f_i : i \in I\}$. Let N be a finitely generated submodule of F. There exists a finite subset J of I such that $N \subseteq \oplus_{i \in J} f_i R$. Let $G = \oplus_{i \in J} f_i R$. Let K be any complement of N in G. Then K is a direct summand of G because G/K is a finitely generated torsion-free R-module, so that G/K is projective. Let $G' = \oplus_{i \in I \setminus J} f_i R$. Then $K \oplus G'$ is a complement of N in F by Lemma 1.2.6. Clearly $K \oplus G'$ is a direct summand of F. □

Proposition 1.6.12 *Any free $\mathbb{Z}$-module of infinite rank is a torsion-free weak type 1 $\mathcal{U}$-extending R-module which is not type 1 $\mathcal{U}$-extending.*

Proof. Let F be a free $\mathbb{Z}$-module of infinite rank. By Corollary 1.6.4 and Proposition 1.6.11, F is weak type 1 $\mathcal{U}$-extending. There exists a submodule K of F such that $F/K \cong \mathbb{Q}$. Since F/K is torsion-free it follows that K is a closed submodule of F. Moreover, F/K is uniform and K is not a direct summand of F. By Theorem 1.2.12 F is not type 1 $\mathcal{U}$-extending. □

Recall that $\mathcal{M}$ is the class of all R-modules. We can give the following lemma:

Lemma 1.6.13 *Let R be any ring and $\mathcal{M}$ be the class of all R-modules. Let $M = S \oplus U$ where S is a simple and U a uniform R-module. Then M is weak type 1 $\mathcal{M}$-extending. Moreover, M is extending if and only if S is $(U/Soc(U))$-injective.*

Proof. Let N be a submodule of M. If $N \cap S \neq 0$, then $N \cap S = S$ and $S \subseteq N$. Thus $N = S \oplus (N \cap U)$. If $N \cap U = 0$, then $N = S$ with a complement U which is a direct summand of M. Otherwise, $N \cap U \neq 0$ and N is essential in M with a complement 0, also a direct summand of M.

Now suppose that $N \cap S = 0$. If $N \neq 0$, then S is a complement of N since $N \oplus S$ is essential in M. If $N = 0$, then M is a complement. Therefore M is weak type 1 $\mathcal{M}$-extending. For the last part see [57, Proposition 4.3]. □

Consider the following example which shows that Lemma 1.2.1 is not true for weak type 1 $\mathcal{M}$-extending modules and that the converse of Proposition 1.6.1 is false in general.

Example 1.6.14 Let p be any prime. Then the $\mathbb{Z}$-module $M = (\mathbb{Z}/\mathbb{Z}p) \oplus (\mathbb{Z}/\mathbb{Z}p^3)$ is a weak type 1 $\mathcal{M}$-extending module which is not weak type 2 $\mathcal{U}$-extending (and hence not extending).

Proof. Since $\mathbb{Z}/\mathbb{Z}p$ is simple and $(\mathbb{Z}/\mathbb{Z}p^3)$ is a uniform $\mathbb{Z}$-module, by Lemma 1.6.13, $M = (\mathbb{Z}/\mathbb{Z}p) \oplus (\mathbb{Z}/\mathbb{Z}p^3)$ is weak type 1 $\mathcal{M}$-extending. On the other hand, by [57, Proposition 4.3], M is not extending since $(\mathbb{Z}/\mathbb{Z}p)$ is not $(\mathbb{Z}/\mathbb{Z}p^2)$-injective. (Note that $(\mathbb{Z}p^2/\mathbb{Z}p^3)$ is the socle of $(\mathbb{Z}/\mathbb{Z}p^3)$). Now apply Corollary 1.6.9. □

Tercan and Smith [49] call a weak type 1 $\mathcal{M}$-extending module a "module with (C_{11})" and prove that any direct sum of uniform modules satisfies (C_{11}), i.e. is weak type 1 $\mathcal{M}$-extending.

Proposition 1.6.15 *Let $\mathcal{X}$ be a class of R-modules such that $\mathcal{X}$ is closed under submodules. Let M_1 be a weak type 1 $\mathcal{X}$-extending R-module and let M_2 be an injective R-module. Then the R-module $M = M_1 \oplus M_2$ is weak type 1 $\mathcal{X}$-extending.*

Proof. Let L be any $\mathcal{X}$-submodule of M. Let N be a complement of $L \cap M_2$ in L. Then $N \cap M_2 = 0$ and $(L \cap M_2) \oplus N$ is essential in L. By Lemma 1.2.8, we can suppose without loss of generality that $N \subseteq M_1$. Note that N is an $\mathcal{X}$-submodule of M_1. There exists a direct summand N' of M_1 such that N' is a complement of N in M_1. Moreover, because M_2 is injective, there exists a direct summand M' of M_2 such that M' is a complement of $L \cap M_2$ in M_2. Let $L' = N' \oplus M'$. Then L' is a direct summand of M, $[(L \cap M_2) \oplus N] \cap L' = 0$ and $(L \cap M_2) \oplus (N \oplus L')$ is essential in M. By Lemma 1.2.11 it follows that L' is a complement of $(L \cap M_2) \oplus N$ in M. But $(L \cap M_2) \oplus N$ is essential in L. Thus L' is a complement of L in M. It follows that M is weak type 1 $\mathcal{X}$-extending. □

Remark 1.6.16 There is no analogue of Proposition 1.6.15 for the type 2 case. For any prime p, the $\mathbb{Z}$-module $\mathbb{Z}/\mathbb{Z}p$ is simple, and so an extending module, and $\mathbb{Q}$ is an injective $\mathbb{Z}$-module but the $\mathbb{Z}$-module $(\mathbb{Z}/\mathbb{Z}p) \oplus \mathbb{Q}$ is not extending and hence not weak type 2 $\mathcal{M}$-extending by [23, Theorem 1].

Weak type 2 $\mathcal{X}$-extending modules for the class of semisimple right R-modules $\mathcal{C}$ have been studied by Smith [46].

Lemma 1.6.17 Let R be any ring. Then the R-module M is type 2 $\mathcal{C}$-extending if and only if every complement with essential socle is a direct summand.

Proof. For the necessity, let M be a type 2 $\mathcal{C}$-extending module and K be a complement in M such that $socK$ is essential in K. K is a closed submodule in M. Then K is a closure of $socK$ and $socK \in \mathcal{C}$. By assumption K is a direct summand of M.

Conversely, let N be a semisimple submodule of M and K be a closure of N in M. Since $socN = N$ and N is essential in K, $N = socK$ by [1, 9.Exercise 10]. Thus $socK$ is essential in K. By assumption K is a direct summand of M. Then M is type 2 $\mathcal{C}$-extending. □

The next result is taken from [46].

Proposition 1.6.18 *Let R be a Dedekind domain and M be an R-module with finite uniform dimension. Then M is a weak type 2 $\mathcal{C}$-extending module.*

Note that, weak type 2 $\mathcal{X}$-extending modules need not be type 2 $\mathcal{X}$-extending. Let p be any rational prime number and M the $\mathbb{Z}$-module $(\mathbb{Z}/\mathbb{Z}p)\oplus(\mathbb{Z}/\mathbb{Z}p^3)$. Then the $\mathbb{Z}$-module M is weak type 2 $\mathcal{C}$-extending by Proposition 1.6.18, but not type 2 $\mathcal{C}$-extending by Proposition 1.3.2 and Example 1.6.14, because every non-zero submodule has essential Socle.

Weak type 2 $\mathcal{C}$-extending modules share some of the properties of extending modules. For example, Smith [46] proved that if M is a weak type 2 $\mathcal{C}$-extending module which satisfies the ascending chain condition on essential submodules then M is a direct sum of a semisimple module (i.e. $\mathcal{C}$-module) and a Noetherian module.

Question 1.6.19 Do the results of [46] for weak type 2 $\mathcal{C}$-extending modules apply to weak type 1 $\mathcal{C}$-extending modules?

Chapter 2

SOME PARTICULAR CLASSES OF MODULES

Having considered type 1 and type 2 $\mathcal{X}$-extending modules for a general class $\mathcal{X}$ in chapter 1, we now consider particular classes $\mathcal{X}$. In section 2.1 we consider type 1 and type 2 $\mathcal{X}$-extending modules when $\mathcal{X} = \mathcal{T}$ or $\mathcal{F}$. In section 2.2, we investigate the case when $\mathcal{X} = \mathcal{U}$ or $\mathcal{G}$. For example, we show that if R is a commutative domain then any torsion-free type 1 $\mathcal{U}$-extending module is a finite direct sum of injective modules and uniform modules, and is extending. In section 2.3, we prove that if R is Dedekind domain, then any type 2 $\mathcal{C}$-extending torsion R-module is extending. Finally in section 2.4, it is proved that for any ring R, any countably generated type 2 $\mathcal{G}_1$-extending module is a direct sum of submodules each containing a cyclic essential submodule.

2.1 Singular and Nonsingular Modules

We begin this section by considering the classes $\mathcal{S}$ and $\mathcal{T}$ of singular modules and Goldie torsion modules, respectively, over a general ring R. Note that $\mathcal{S} \subseteq \mathcal{T} \subseteq \mathcal{S}^e$, so an R-module M is type 1 (respectively, type 2) $\mathcal{S}$-extending if and only if M is type 1 (type 2) $\mathcal{T}$-extending, by Lemma 1.2.2 and Proposition 1.3.2.

For any module M, the *second singular submodule* $Z_2(M)$ of M is the submodule containing $Z(M)$ such that $Z_2(M)/Z(M)$ is the singular submodule of $M/Z(M)$, i.e., $Z_2(M)/Z(M) = Z(M/Z(M))$. It is well known that $Z_2(M)$ is a closed submodule of M.

Moreover $M \in \mathcal{T}$ if and only if $Z_2(M) = M$. The next result should be compared with [23, Theorem 1].

Theorem 2.1.1 (i) *An R-module M is type 1 $\mathcal{T}$-extending if and only if $M = Z_2(M) \oplus M'$ for some submodule M' of M such that $Z_2(M)$ is extending and $Z_2(M)$ is M'-injective.*

(ii) *An R-module M is type 2 $\mathcal{T}$-extending if and only if $Z_2(M)$ is extending and is a direct summand of M.*

Proof. (i) Suppose first that $M = Z_2(M) \oplus M'$ for some submodule M' of M such that $Z_2(M)$ is extending and $Z_2(M)$ is M'-injective. Let N be a $\mathcal{T}$-submodule of M and let K be any complement of N in M. Note that N is a submodule of $Z_2(M)$ and $N \oplus K$ is essential in M. Now $N \oplus (K \cap Z_2(M)) = (N \oplus K) \cap Z_2(M)$ is essential in $Z_2(M)$. Let L be a complement of $K \cap Z_2(M)$ in K. Then $(K \cap Z_2(M)) \oplus L$ is essential in K

and hence $N \oplus (K \cap Z_2(M)) \oplus L$ is essential in M. We have $Z_2(M) \cap L = 0$. Without loss of generality, $L \subseteq M'$ because $Z_2(M)$ is M'-injective (Lemma 1.2.8). Moreover, $Z_2(M) \oplus L$ is essential in M, so that L is essential in M', because $L = (Z_2(M) \oplus L) \cap M'$. But L is closed in K and K is closed in M, so that L is closed in M (Lemma 1.2.4) and hence $L = M'$. Thus M' is a submodule of K.

Note further that $K = (Z_2(M) \oplus M') \cap K = (Z_2(M) \cap K) \oplus M'$. Thus $Z_2(M) \cap K$ is closed in K so that, by Lemma 1.2.4, $Z_2(M) \cap K$ is closed in M and hence also in $Z_2(M)$. Because $Z_2(M)$ is extending, $Z_2(M) \cap K$ is a direct summand of $Z_2(M)$ and hence K is a direct summand of M. It follows that M is type 1 $\mathcal{T}$-extending.

Conversely, suppose that M is type 1 $\mathcal{T}$-extending. Let K be a complement of the submodule $Z_2(M)$ in M. By assumption K is a direct summand of M. Write $M = K \oplus K'$ for some submodule K'. Now

$$Z_2(M) = Z_2(K) \oplus Z_2(K') = 0 \oplus Z_2(K') \subseteq K'.$$

Note that $K \oplus Z_2(M)$ is essential in $M = K \oplus K'$, and

$$Z_2(M) = Z_2(M) \oplus (K \cap K') = K' \cap (K \oplus Z_2(M)),$$

which is essential in K'. But $Z_2(M)$ is closed in M. Thus $Z_2(M) = K'$ and $M = Z_2(M) \oplus K$.

Note that $Z_2(M)$ is type 1 $\mathcal{T}$-extending by Lemma 1.2.7, so that $Z_2(M)$ is extending. Now we show that $Z_2(M)$ is K-injective. Suppose N is a submodule of M and $N \cap Z_2(M) = 0$. Then N is a submodule of a complement submodule L of $Z_2(M)$ in M. By hypothesis, L is a direct summand of M. Write $M = L \oplus L'$ for some submodule L' of M. Next $Z_2(M) = Z_2(L) \oplus Z_2(L') = Z_2(L') \subseteq L'$. It follows that $L' = Z_2(M) \oplus (L' \cap K)$ and $M = L \oplus L' = L \oplus Z_2(M) \oplus (L' \cap K)$.

Finally $Z_2(M)$ is K-injective by Lemma 1.2.8.

(ii) Suppose first that M is type 2 $\mathcal{T}$-extending. We know that $Z_2(M) \in \mathcal{T}$ and $Z_2(M)$ is closed in M. By assumption $Z_2(M)$ is a direct summand of M. Then $M = Z_2(M) \oplus M'$ for some submodule M' of M.

Let N be a closed submodule of $Z_2(M)$. Then $N \in \mathcal{T}$ and hence N is a direct summand of $Z_2(M)$ by Lemma 1.2.7. Thus $Z_2(M)$ is extending.

Conversely, suppose that $M = Z_2(M) \oplus M'$ for some submodule M' of M and $Z_2(M)$ is extending. Let N be a submodule of M with $N \in \mathcal{T}$. Then N is a submodule of $Z_2(M)$. Suppose that K is any closure of N in M. Then K is a submodule of $Z_2(M)$. Thus K is a direct summand of $Z_2(M)$, so that K is a direct summand of M. Thus M is type 2 $\mathcal{T}$-extending. □

Corollary 2.1.2 *With the above notation, any type 1 $\mathcal{T}$-extending R-module is type 2 $\mathcal{T}$-extending.*

Proof. By Theorem 2.1.1. □

The converse of Corollary 2.1.2 is not true in general, as the following example shows:

Example 2.1.3 For any prime p, the $\mathbb{Z}$-module $M = (\mathbb{Z}/\mathbb{Z}p) \oplus \mathbb{Z}$ is type 2 $\mathcal{T}$-extending but not type 1 $\mathcal{T}$-extending.

Proof. By Theorem 2.1.1. □

In the Goldie torsion theory for mod-R, $\mathcal{T}$ is the class of torsion R-modules and the class $\mathcal{F}$ of torsion-free modules is the class of nonsingular R-modules.

Now we consider $\mathcal{F}$-extending modules of the two types. The next result can also be compared with [23, Theorem 1].

Theorem 2.1.4 (i) *An R-module M is type 1 $\mathcal{F}$-extending if and only if $M = Z_2(M) \oplus M'$ for some extending submodule M' of M.*

(ii) *An R-module M is type 2 $\mathcal{F}$-extending if and only if $M = Z_2(M) \oplus M'$ for some extending submodule M' of M such that $Z_2(M)$ is M'-injective.*

Proof. (i) Suppose first that M is type 1 $\mathcal{F}$-extending. Let N be a complement of $Z_2(M)$ in M. $Z_2(M)$ is closed in M and $N \cap Z_2(M) = 0$, $N \oplus Z_2(M)$ is essential in M. By Lemma 1.2.11, $Z_2(M)$ is a complement of N and $N \in \mathcal{F}$. By assumption $M = Z_2(M) \oplus M'$ for some submodule M'. Thus M' is type 1 $\mathcal{F}$-extending and so is extending since $M' \in \mathcal{F}$.

Conversely, let N be an $\mathcal{F}$-submodule of M. Suppose that K is a complement of N in M. We have $K + Z_2(M) = Z_2(M) \oplus ((K + Z_2(M)) \cap M')$ by the Modular Law. Suppose that $N \cap (K + Z_2(M)) \neq 0$. Let $0 \neq x = y + z$ where $y \in K$, $z \in Z_2(M)$, $x \in N$. Then there exists an essential right ideal E of R such that $zE \subseteq Z(M)$. Note $xE \neq 0$. Thus there exists $e \in E$ such that $xe \neq 0$. There exists an essential right ideal F of R such that $zeF = 0$. But $xeF \neq 0$, so that $0 \neq xef = yef + zef = yef + 0 \in N \cap K = 0$, for some $f \in F$, a contradiction. Thus $N \cap (K + Z_2(M)) = 0$. But K is maximal with respect to $K \cap N = 0$. It follows that $K + Z_2(M) = K$, i.e. $Z_2(M) \subseteq K$. Thus $K = Z_2(M) \oplus (K \cap M')$. Since K is closed in M, $K \cap M'$ is closed in M by Lemma 1.2.4. Thus $K \cap M'$ is closed in M' and hence $K \cap M'$ is a direct summand of M'. It follows that K is a direct summand of M. Therefore M is type 1 $\mathcal{F}$-extending.

(ii). Suppose that $M = Z_2(M) \oplus M'$ for some extending submodule M' of M such that $Z_2(M)$ is M'-injective. Let N be an $\mathcal{F}$-submodule of M. Let K be any closure of N in M. Since $N \cap Z_2(M) = 0$, we have $K \cap Z_2(M) = 0$. There exists a submodule M'' of M such that $M = Z_2(M) \oplus M''$ and $K \subseteq M''$ by Lemma 1.2.8. Clearly $M'' \cong M'$, so that M'' is extending because M' is extending. But K is a closure of N in M'' and hence K is a direct summand of M'', hence also of M. Therefore M is type 2 $\mathcal{F}$-extending.

Conversely, suppose that M is type 2 $\mathcal{F}$-extending. Let K be a complement of the submodule $Z_2(M)$ in M. Clearly $K \cap Z(M) = 0$. Thus $K \in \mathcal{F}$. By assumption K is a direct summand of M. Write $M = K \oplus K'$ for some submodule K'. Now

$$Z_2(M) = Z_2(K) \oplus Z_2(K') = 0 \oplus Z_2(K') \subseteq K'.$$

But $K \oplus Z_2(M)$ is essential in $M = K \oplus K'$ since K is a complement of $Z_2(M)$ in M. Next $Z_2(M) = Z_2(M) \oplus (K \cap K') = K' \cap (K \oplus Z_2(M))$ and hence $Z_2(M)$ is essential in K'. But $Z_2(M)$ is closed in K'. Thus $Z_2(M) = K'$ and $M = K \oplus Z_2(M)$.

Note that K is type 2 $\mathcal{F}$-extending by Lemma 1.2.7 and $K \in \mathcal{F}$. Hence K is extending.

Now we show that $Z_2(M)$ is K-injective. Let L be any closed submodule of M such that $L \cap Z_2(M) = 0$. Then L is a direct summand of M and hence $M = L \oplus L'$ for some submodule L'. Moreover $Z_2(M) = Z_2(L) \oplus Z_2(L') \subseteq L'$ gives that $L' = Z_2(M) \oplus (L' \cap K)$ and hence $M = L \oplus Z_2(M) \oplus (L' \cap K)$. By Lemma 1.2.8, $Z_2(M)$ is K-injective. □

In contrast to Corollary 2.1.2 we have the following result.

Corollary 2.1.5 *With the above notation, any type 2 $\mathcal{F}$-extending R-module is type 1 $\mathcal{F}$-extending.*

Proof. By Theorem 2.1.4. □

The converse of Corollary 2.1.5 is not true in general as the following example shows.

Example 2.1.6 For any prime p, the $\mathbb{Z}$-module $M = (\mathbb{Z}/\mathbb{Z}p) \oplus \mathbb{Z}$ is type 1 $\mathcal{F}$-extending but not type 2 $\mathcal{F}$-extending.

Proof. By Theorem 2.1.4. □

In view of Theorem 1.3.10, it is no surprise to learn from Theorem 2.1.1 and 2.1.4 that an R-module M is extending if and only if M is type 1 (respectively, type 2) $\mathcal{T}$-extending and type 1 (type 2) $\mathcal{F}$-extending. This fact has an obvious generalization to an arbitrary torsion theory.

2.2 Classes $\mathcal{U}$ and $\mathcal{G}$

In general, it is not the case that if R is a ring, $\mathcal{X}$ a class of R-modules and M a type 2 $\mathcal{X}$-extending R-module then every closure of a direct sum of an infinite number of $\mathcal{X}$-submodules is a direct summand, as the following example shows:

Example 2.2.1 Let F be any free $\mathbb{Z}$-module of infinite rank. Then F is type 2 $\mathcal{G}$-extending but there exist a free submodule H of F such that H is closed in F and H is not a direct summand of F.

Proof. Let N be any finitely generated submodule of F and let K be any closure of N in F. There exists a finitely generated free direct summand F' of F such that $N \subseteq F'$. Since K/N is torsion, it follows that $K \subseteq F'$. Now F'/K is finitely generated torsion-free, so that K is a direct summand of F', and hence also of F. Thus F is type 2 $\mathcal{G}$-extending. There exists a $\mathbb{Z}$-epimorphism $\varphi : F \longrightarrow \mathbb{Q}$. Let H=Kerφ. Then F/H is a torsion-free $\mathbb{Z}$-module, so that H is closed in F. Clearly H is free and hence H is a direct sum of cyclic submodules. But H is not a direct summand of F because $F/H \cong \mathbb{Q}$ □

Note that in Example 2.2.1, if the free $\mathbb{Z}$-module F has countably infinite rank, then so too does H and hence there exists a chain of finitely generated submodules

$$0 = H_0 \subseteq H_1 \subseteq H_2 \subseteq ... \subseteq \cup_{n\geq 1} H_n = H$$

such that H_i/H_{i-1} is cyclic for all $i \geq 1$. However H is closed in F but H is not a direct summand of F. Thus Corollary 1.3.7 cannot be extended.

Corollary 1.3.5 can be applied to particular classes of modules, as we now demonstrate:

Proposition 2.2.2 *Let $\mathcal{U}_1$ and $\mathcal{U}$ denote the classes of uniform R-modules and of R-modules with finite uniform dimension, respectively. Then an R-module M is type 1 (respectively, type 2) $\mathcal{U}$-extending if and only if M is type 1 (type 2) $\mathcal{U}_1$-extending.*

Proof. Note that $\mathcal{U}_1 \subseteq \mathcal{U}$ and that every module in $\mathcal{U}$ contains an essential submodule which is a finite direct sum of uniform modules. Apply Lemma 1.2.2, Proposition 1.3.2 and Corollary 1.3.5. □

A module M is *uniform-extending* if every uniform submodule is essential in a direct summand of M. Note that uniform-extending is the same as weak type 2 $\mathcal{U}_1$-extending. Since $\mathcal{U}_1$ is essentially closed, by Corrolary 1.6.8 and Proposition 2.2.2, uniform-extending is the same as type 2 $\mathcal{U}$-extending.

The argument of Proposition 2.2.2 gives the next result immediately.

Proposition 2.2.3 *Let $\mathcal{C}_1$ and $\mathcal{A}$ denote the classes of simple R-modules and of Artinian R-modules, respectively. Then an R-module M is type 1 (respectively, type 2) $\mathcal{A}$-extending if and only if M is type 1 (type 2) $\mathcal{C}_1$-extending.*

Note that in Proposition 2.2.3 we can replace $\mathcal{A}$ by the class $\mathcal{A}^c$ of finitely cogenerated modules. It is not the case that any type 2 $\mathcal{C}_1$-extending module is type 2 $\mathcal{C}$-extending, as the following example shows:

Example 2.2.4 Let K be any field and let $K_n = K$ $(n \geqslant 1)$. Let $S = \prod_{n \geqslant 1} K_n$ and let R denote the subring of S consisting of all sequences $\{k_n\}$ with $k_n \in K$ $(n \geqslant 1)$ and $k_m = k_{m+1} = ...$ for some $m \geqslant 1$. Then every R-module is type 1 and type 2 $\mathcal{C}_1$-extending but R_R is not type 1 nor type 2 $\mathcal{C}$-extending.

Proof. The ring R is a commutative von Neumann regular ring and hence every simple R-module is injective (see [44, Theorem 6]). Thus every R-module is type 2 $\mathcal{C}_1$-extending.

Let M be any R-module and S be a simple submodule of M. Let K be a complement of S in M. By [44, Theorem 6], S is injective. Then $M = S \oplus S'$ for some submodule S'. By Lemma 1.2.8 without loss of generality $K \subseteq S'$. Thus $K = S'$. Then M is type 1 $\mathcal{C}_1$-extending.

On the other hand, let I be the ideal of R consisting of all sequences $\{k_n\}$ such that $k_{2n} = 0$ for all $n \geq 1$ and $k_i = 0$ for all $i \geq m$, for some $m \geq 1$, i.e. $I = K \oplus 0 \oplus K \oplus 0 \oplus K \oplus \ldots$. Let x be any element of R such that I is essential in $I + xR$. Then $xE \subseteq I$ for some essential ideal E of R. If $T = \oplus_{n \geq 1} K_n$, then T is the socle of R and hence $T \subseteq E$. Thus $xT \subseteq I$ and $x \in I$. Thus I is a closed semisimple submodule of R_R but I is not a direct summand, i.e. R_R is not type 2 $\mathcal{C}$-extending.

Let $J = 0 \oplus K \oplus 0 \oplus K \oplus \ldots$. Then $I \cap J = 0$ and $I \oplus J$ is essential in R. Since I is closed, I is a complement of the semisimple submodule J of R by Lemma 1.2.11. Then $I \neq Re$ for $e = e^2 \in R$. Thus I not a direct summand of R, so R_R is not type 1 $\mathcal{C}$-extending. □

We now consider the classes $\mathcal{G}_1$ and $\mathcal{G}$ of cyclic and finitely generated modules, respectively. Note that the class $\mathcal{G}_1$ and $\mathcal{G}$ are both closed under factor modules and that, $\mathcal{G}_1 \subseteq (\mathcal{G}_1)^+ = \mathcal{G} = (\mathcal{G}_1)^\omega$. Lemma 1.2.2 and Corollary 1.3.7 give the following result:

Proposition 2.2.5 *With the above notation, an R-module M is type 2 $\mathcal{G}$-extending if and only if M is type 2 $\mathcal{G}_1$-extending.*

Proposition 2.2.6 *Let R be any ring. Then any type 1 (respectively, type 2) $\mathcal{G}$-extending module is type 1 (respectively, type 2) $\mathcal{U}$-extending.*

Proof. By Lemma 1.2.2 and Proposition 1.3.2, because $\mathcal{U} \subseteq \mathcal{G}^e$. □

Not every type 1 (type 2) $\mathcal{U}$-extending module is type 1 (type 2) $\mathcal{G}$-extending as we show next. First we prove a general result:

Proposition 2.2.7 *The following statements are equivalent for an indecomposable R-module M:*

(i) *M is uniform;*

(ii) *M is type 1 $\mathcal{G}$-extending;*

(iii) *M is type 2 $\mathcal{G}$-extending.*

Proof. (i) $\Rightarrow$ (ii), (iii) clear.

(ii) $\Rightarrow$ (i). Assume that M is type 1 $\mathcal{G}$-extending. Let $0 \neq m \in M$. Let K be a complement of mR in M. By hypothesis K is a direct summand of M. Then $K = 0$ or $K = M$. Thus $K = 0$. Hence $mR \cap B \neq 0$ for every nonzero submodule B of M, i.e. mR is essential in M for all $0 \neq m \in M$. Thus M is uniform.

(iii) $\Rightarrow$ (i). Assume that M is type 2 $\mathcal{G}$-extending. Let $0 \neq m \in M$. Let K be a closure of mR in M. By hypothesis K, is a direct summand of M. Then $K = 0$ or $K = M$. Thus $K = M$. Hence mR is essential in M for all $0 \neq m \in M$. Therefore M is uniform. □

A right *Öre domain* is any domain R in which every two nonzero elements have a nonzero common right multiple, i.e., for each nonzero $x, y \in R$ there exist $r, s \in R$ such that $xr = ys \neq 0$.

For example, every commutative domain is right Öre.

Corollary 2.2.8 *Let R be any domain. Then*

(i) *the R-module R is type 1 and type 2 $\mathcal{U}$-extending.*

(ii) *The following statements are equivalent:*

(a) R is right Öre;

(b) the R-module R is type 1 $\mathcal{G}$-extending;

(c) the R-module R is type 2 $\mathcal{G}$-extending.

Proof. (i) If R has no uniform right ideals then R_R is both type 1 and type 2 $\mathcal{U}$-extending. Suppose that R has a uniform right ideal U. Let $0 \neq u \in U$. Define $\varphi : R_R \longrightarrow U$ by $\varphi(r) = ur$ $(r \in R)$. Since R is a domain, φ is an R-monomorphism. Then $R_R \cong im\varphi$ is a submodule of U. Thus R_R is uniform. Then R_R is extending, so it is type 1 and type 2 $\mathcal{U}$-extending.

(ii) By Proposition 2.2.7. □

If R is a right Noetherian ring then $\mathcal{G} \subseteq \mathcal{U}$. Hence Lemma 1.2.2 gives that an R-module M is type 1 (respectively, type 2) $\mathcal{U}$-extending if and only if M is type 1 (type 2) $\mathcal{G}$-extending. In particular, Example 1.2.14 shows that any free $\mathbb{Z}$-module of infinite rank is type 2 $\mathcal{G}$-extending but not type 1 $\mathcal{G}$-extending.

Any direct summand of an injective module is injective but, in general, direct sums of injective modules are not necessarily injective, although any finite direct sum of injective modules is injective. H. Bass proved in his Ph.D. (1956) that, every direct sum of injective R-modules is injective if and only if R is right Noetherian. The following example shows that a direct sum of injective modules also is not extending in general (see [4]).

Example 2.2.9 Let K be a field and let V be any infinite dimensional vector space over K. Let $R = End(V_K)$, the ring of all linear mappings $\sigma : V \rightarrow V$, operating on the left. Then R is a right self-injective von Neumann regular ring which is not left self-injective by [12, Proposition 2.23]. Thus by [4, Proposition 3] every direct sum of countably many copies of R_R is extending but not every direct sum of copies of R_R is extending.

Theorem 2.2.10 *Let M_i ($i \in I$) be any collection of injective R-modules and let $M = \oplus_{i \in I} M_i$. Then M is type 1 $\mathcal{G}$-extending (and type 1 $\mathcal{U}$-extending).*

Proof. Let N be a finitely generated submodule of M and let K be a complement of N in M. Then $N \subseteq \oplus_{i \in J} M_i$ for some finite subset J of I. Hence $E(N) \subseteq M$ and $M = E(N) \oplus N'$ for some submodule N'. Since $E(N) \cap K = 0$, it follows that $M = E(N) \oplus M''$ for some submodule M'' with $K \subseteq M''$, by Lemma 1.2.8. But $N \cap M'' = 0$ gives $K = M''$ and hence K is a direct summand of M. Thus M is type 1 $\mathcal{G}$-extending. (Note that M is type 1 $\mathcal{U}$-extending by Proposition 2.2.6). □

Before giving an example related to Theorem 2.2.10 some of the following properties can be found in [7, 8, 28]

Proposition 2.2.11 *Let M_i ($i \in I$) be indecomposable injective R-modules and let $M = \oplus_{i \in I} M_i$. Then M is type 2 $\mathcal{U}$-extending if and only if M is quasi-injective.*

Proof. See [7, Corollary 3.6]. □

Given a module N, a module U is *essentially N-injective* if for every submodule L of N and homomorphism $\varphi : L \longrightarrow U$, with $\ker\varphi$ essential in L, there exists a homomorphism $\theta : N \longrightarrow U$ such that $\varphi = \theta|_L$.

The modules M_i ($i \in I$) *satisfy* (A_2) provided for all distinct $i(n) \in I$ ($n \in \mathbb{N} \cup \{0\}$) and elements $x_n \in M_{i(n)}$ ($n \in \mathbb{N} \cup \{0\}$) such that $\mathrm{r}(x_0) \subseteq \mathrm{r}(x_n)$ ($n \in \mathbb{N}$), the ascending chain

$$\cap_{n \geq 1} \mathrm{r}(x_n) \subseteq \cap_{n \geq 2} \mathrm{r}(x_n) \subseteq \cap_{n \geq 3} \mathrm{r}(x_n) \subseteq \ldots$$

becomes stationary.

Proposition 2.2.12 *Let $M = \oplus_{i\in I}M_i$. Then $\oplus_{I\setminus\{j\}}M_i$ is M_j-injective for every $j \in I$ if and only if M_j is M_k-injective for all $j \neq k \in I$ and (A_2) holds.*

Proof. See [28, Proposition 1.9]. □

Lemma 2.2.13 *Let M_i $(i \in I)$ be uniform R-modules with local endomorphism rings and $M = \oplus_{i\in I}M_i$. Then M is type 2 $\mathcal{U}$-extending if and only if $M_i \oplus M_j$ is extending for all i, j in I and the modules M_i $(i \in I)$ satisfy (A_2). In this case, for each $j \in J$ the module $\oplus_{i\in I\setminus\{j\}}M_i$ is essentially M_j-injective.*

Proof. See [8, Lemma 2.2 and Lemma 2.3]. □

A module M is said to have the *exchange property* if for any index set I, whenever $M \oplus N = \oplus_{i\in I}A_i$ for modules N and A_i $(i \in I)$, then $M \oplus N = M \oplus (\oplus_{i\in I}B_i)$ for submodules $B_i \subseteq A_i$ $(i \in I)$.

Lemma 2.2.14 *Let M_i $(i \in I)$ be uniform R-modules with local endomorphism rings and $M = \oplus_{i\in I}M_i$. Then M is extending if and only if M is type 2 $\mathcal{U}$-extending and there does not exist a sequence $\{i(n)\}_{n\in \mathbb{N}}$ of distinct elements of I and non-isomorphic monomorphisms $\varphi_n : M_{i(n)} \longrightarrow M_{i(n+1)}$ $(n \in \mathbb{N})$. In this case M has the exchange property.*

Proof. See [7, Theorem 3.4 and Corollary 3.5]. □

Lemma 2.2.15 *Let M_i $(i \in I)$ be indecomposable injective R-modules. Then the following statements are equivalent for the module $M = \oplus_{i\in I}M_i$:*

(i) *M is type 2 $\mathcal{U}$-extending;*

(ii) *M is extending;*

(iii) *M is quasi-continuous;*

(iv) *M is quasi-injective;*

(v) *The modules M_i $(i \in I)$ satisfy (A_2).*

Proof. (iv) $\Longrightarrow$ (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) clear.

(i) $\Longrightarrow$ (v) By Lemma 2.2.13.

(v) $\Longrightarrow$ (iv) Clearly M_i is M_j-injective for all $i, j \in I$ and, by assumption, M_i $(i \in I)$ satisfy (A_2). Then fix $i \neq j$ in I. Note that $N = \oplus_{k \in I \setminus \{j\}} M_k$ is M_j-injective by Proposition 2.2.12. Thus $N' = \oplus_{k \in I \setminus \{i,j\}} M_k$ is M_i-injective by Proposition 2.2.12. $N = M_i \oplus N'$, M_i is M_i-injective and N' is M_i-injective, so N is M_i-injective. Then N is M_k-injective for every $k \in I$. By [28, Proposition 1.5] N is M-injective. Then $M = N \oplus M_j$ is M-injective. □

Example 2.2.16 There exists a commutative subdirectly irreducible ring R and indecomposable injective R-modules M_n $(n \in \mathbb{N})$ such that $M = \oplus_{n \in \mathbb{N}} M_n$ is not type 2 $\mathcal{U}$-extending and (not type 2 $\mathcal{G}$-extending).

Proof. Let p be any prime, let $\mathbb{Z}(p^\infty)$ denote the Prufer p-group and let $R = \begin{bmatrix} \mathbb{Z} & \mathbb{Z}(p^\infty) \\ 0 & \mathbb{Z} \end{bmatrix}$. That is, R is the commutative ring whose elements are the "matrices"

$\begin{bmatrix} a & x \\ 0 & a \end{bmatrix}$ where $a \in \mathbb{Z}$, $x \in \mathbb{Z}(p^\infty)$ and addition and multiplication are the usual matrix addition and multiplication. If A is the (unique) subgroup of $\mathbb{Z}(p^\infty)$ of order p then it is easy to check that:

$\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}$ is the intersection of all nonzero ideals of R, i.e R is subdirectly irreducible.

Since $\mathbb{Z}(p^\infty)$ is not a finitely generated $\mathbb{Z}$-module, it follows that R is not a Noetherian ring. By [45, Theorem 4.1] there exist simple modules S_n ($n \in \mathbb{N}$) such that $\oplus_{n\in\mathbb{N}} E(S_n)$ is not an injective module. Let $M_1 = E(R)$ and $M_{n+1} = E(S_n)$ ($n \in \mathbb{N}$). Then M_n is indecomposable injective for all $n \in \mathbb{N}$. Let $M = \oplus_{n\in\mathbb{N}} M_n$. By Proposition 2.2.12 and Lemma 2.2.13, M is not type 2 $\mathcal{U}$-extending because $\oplus_{n\geq 2} M_n$ is not M_1-injective. Since every type 2 $\mathcal{G}$-extending module is type 2 $\mathcal{U}$-extending (see Proposition 2.2.6), it follows that M is not type 2 $\mathcal{G}$-extending. □

Theorem 2.2.17 *Let M_i $(i \in I)$ be nonsingular injective modules and let $M = \oplus_{i\in I} M_i$. Then M is type 2 $\mathcal{G}$-extending.*

Proof. let K be any closed submodule of M such that K contains a finitely generated essential submodule N. There exists a finite subset J of I such that $N \subseteq \oplus_{i\in J} M_i$. Since M is nonsingular it follows that $K \subseteq \oplus_{i\in J} M_i$. But $\oplus_{i\in J} M_i$ is injective, so that K is a direct summand of $\oplus_{i\in J} M_i$, and hence also of M. □

Proposition 2.2.18 *Let M be a type 1 $\mathcal{U}$-extending R-module such that R satisfies ACC on right ideals of the form $\mathrm{r}(m)$, where $m \in M$. Then M is a direct sum of uniform modules if and only if every nonzero submodule of M contains a uniform submodule.*

Proof. By Corollary 1.5.4. □

Corollary 2.2.19 *Any locally Noetherian type 1 $\mathcal{U}$-extending module is a direct sum of uniform modules.*

Proof. Let M be a locally Noetherian type 1 $\mathcal{U}$-extending R-module. Let N be a nonzero submodule of M. Let $0 \neq m \in N$. Then mR is Noetherian and hence mR contains a uniform submodule and $R/\mathrm{r}(m)$ is Noetherian. By Proposition 2.2.18, M is a direct sum of uniform submodules. □

Note that we do not have the same result for the type 2 $\mathcal{U}$-extending case, as the following example shows:

Example 2.2.20 Let $M_n = \mathbb{Z}$ $(n \in \mathbb{N})$ and let M be the $\mathbb{Z}$-module $\prod_{n \in \mathbb{N}} M_n$. Then M is a locally Noetherian type 2 $\mathcal{U}$-extending $\mathbb{Z}$-module but M is not a direct sum of uniform modules.

Proof. Because the ring $\mathbb{Z}$ is Noetherian, it is clear that M is locally Noetherian. Let U be a maximal uniform submodule of M. Because M is a torsion-free module it follows that M/U is torsion-free (if $U \subseteq V \subseteq M$ and V/U is torsion then U is essential in V hence $U = V$). Let $0 \neq \{a_n\} \in U$. Let d be the greatest common divisor of the elements $\{a_n : n \in \mathbb{N}\}$. There exist $b_n \in \mathbb{Z}$ $(n \in \mathbb{N})$ such that $a_n = db_n$ $(n \in \mathbb{N})$. Then $d\{b_n\} = \{a_n\} \in U$ and $\{b_n\} \in U$.

For each $k \in \mathbb{N}$, let d_k be the greatest common divisor of the elements b_n $(1 \leq n \leq k)$. Note that d_{k+1} divides d_k for all $k \in \mathbb{N}$. Thus we have the following ascending chain of ideals in $\mathbb{Z}$:

$$\mathbb{Z}d_1 \subseteq \mathbb{Z}d_2 \subseteq \mathbb{Z}d_3 \subseteq \ldots \; .$$

There exists a positive integer t such that $\mathbb{Z}d_t = \mathbb{Z}d_{t+1} = \mathbb{Z}d_{t+2} = \ldots$. Hence d_t is the greatest common divisor of the elements b_n $(n \in \mathbb{N})$, i.e. $d_t = \pm 1$. Also we can suppose without loss of generality that $b_n \neq 0$ for some $1 \leq n \leq t$.

Let V be the submodule $\mathbb{Z}(b_1, ..., b_t)$ of the free $\mathbb{Z}$-module $F = \mathbb{Z}^{(t)}$. If $c_i \in \mathbb{Z}$ $(1 \leq i \leq t)$ and $f(c_1, ..., c_t) \in V$ for some $0 \neq f \in \mathbb{Z}$, then there exists $g \in \mathbb{Z}$ such that $fc_i = gb_i$ $(1 \leq i \leq t)$. Suppose that $f \neq \pm 1$. Let p be any prime divisor of f. Then p divides gb_i for each $1 \leq i \leq t$. Since the elements b_i $(1 \leq i \leq t)$ have greatest common divisor ± 1, it follows that p does not divide b_j for some $1 \leq j \leq t$. Hence p divides g. It is now clear that f divides g and hence $(c_1, ..., c_t) = (g/f)(b_1, ..., b_t) \in V$. Thus F/V is finitely generated torsion-free, so that F/V is a free $\mathbb{Z}$-module and V is a direct summand of F. Let V' be a submodule of F such that $F = V \oplus V'$.

Let us return to U. Let $0 \neq \{e_n\} \in U$. There exists $0 \neq h \in \mathbb{Z}$ such that $h\{e_n\} \in \mathbb{Z}\{b_n\}$. By the argument we used in the previous paragraph we have $\{e_n\} \in \mathbb{Z}\{b_n\}$. Thus $U = \mathbb{Z}\{b_n\}$. Let

$$W = \{\{q_n\} \in M : (q_1, ..., q_t) \in V'\}.$$

Clearly W is a submodule of M. Suppose that $z\{b_n\} \in W$ for some $z \in \mathbb{Z}$. Then $z(b_1, ..., b_t) \in V'$ and hence $z(b_1, ..., b_t) = 0$, i.e. $zb_i = 0$ $(1 \leq i \leq t)$. Since $b_i \neq 0$ for some $1 \leq i \leq t$ it follows that $z = 0$. Thus $U \cap W = 0$. Now let $\{m_n\} \in M$. There exists $x \in \mathbb{Z}$ such that

$$(m_1, ..., m_t) = x(b_1, ..., b_t) + (y_1, ..., y_t)$$

for some $(y_1, ..., y_t) \in V'$. Then $\{m_n\} = x\{b_n\} + \{y_n\}$, where

$$y_n = m_n - xb_n \quad (n \geq t + 1).$$

Clearly $\{y_n\} \in W$ and hence $\{m_n\} \in U + W$. It follows that $M = U \oplus W$. We have proved that M is type 2 $\mathcal{U}$-extending (Proposition 2.2.2).

Moreover, we have seen that if U is any maximal uniform submodule of M then $U = \mathbb{Z}\{b_n\}$ for some non-zero element $\{b_n\}$. In particular, this means that $U \cong \mathbb{Z}$. If $M = \oplus_{i \in I} U_i$ where U_i is a uniform submodule of M for each $i \in I$, then U_i is a maximal uniform submodule and hence $U_i \cong \mathbb{Z}$ for each $i \in I$. In this case, M is a free $\mathbb{Z}$-module, contradicting [1, p.202 Ex.3]. Thus M is not a direct sum of uniform modules. □

Remark 2.2.21 Note that in [5, 8.5] we have the following (see also [5, 8.1 (1)]). Suppose that M is a module such that

(i) every nonzero submodule contains a uniform submodule,

(ii) every local direct summand is a summand, and

(iii) M is type 2 $\mathcal{U}$-extending.

Then M is extending.

In [23] Kamal and Müller proved that if R is a commutative domain and M is a torsion-free extending R-module then $M = M_1 \oplus ... \oplus M_n$ for some positive integer n, injective R-module M_1 (possibly zero) and uniform R-modules M_i $(2 \leq i \leq n)$. We now extend this result to type 1 $\mathcal{U}$-extending modules. First we note the following result:

Lemma 2.2.22 *The following statements are equivalent for an R-module M with finite uniform dimension:*

(i) *M is extending;*

(ii) *M is type 1 $\mathcal{U}$-extending;*

(iii) *M is type 2 $\mathcal{U}$-extending.*

Proof. Clear. □

The next result generalises [23, Theorem 5].

Theorem 2.2.23 *Let R be a commutative domain. Let M be a torsion-free type 1 $\mathcal{U}$-extending R-module which contains no nonzero injective submodule. Then M has finite uniform dimension.*

Proof. Suppose $M \neq 0$. For each $0 \neq m \in M$, $mR \cong R$ which is a uniform R-module. Thus every nonzero submodule of M contains a uniform submodule. Obviously R satisfies ACC on ideals of the form $\mathrm{r}(m)$ $(m \in M)$. By Proposition 2.2.18, M is a direct sum of uniform submodules. Suppose that M does not have finite uniform dimension. We can suppose without loss of generality that $M = M_1 \oplus M_2 \oplus M_3 \oplus \ldots$, where M_i is a uniform module for each $i \geq 1$.

Let $i \geq 1$. Let $0 \neq x \in M_i$. Then $xR \cong R$ and hence $E(xR) \cong E(R) = F$ where F is the field of fractions of R. Since xR is essential in M_i we can suppose that $M_i \subseteq E(xR)$ and hence, without loss of generality, that $R \subseteq M_i \subseteq F$.

Let $0 \neq c \in R$. Define $\varphi : M \longrightarrow F$ by

$$\varphi(m_1, \ldots, m_n, 0, 0, \ldots) = m_1 + c^{-1}m_2 + \ldots + c^{-(n-1)}m_n$$

for any positive integer n and elements $m_i \in M_i$ $(1 \leq i \leq n)$. Clearly φ is an R-homomorphism. Let K be the kernel of φ. Then $M/K \cong \varphi(M)$ is a submodule of F, so that M/K is torsion-free and uniform. Thus K is closed in M. By Corollary 1.2.13, K is a direct summand of M, say $M = K \oplus K'$ for some submodule K' of M. Note that $K' \cong M/K$ so that K' is uniform. Let $0 \neq x \in K'$. Then $x \in M_1 \oplus \ldots \oplus M_k$ for some positive integer k. Since K'/xR is torsion, it follows that $K' \subseteq M_1 \oplus \ldots \oplus M_k$.

Since $R \subseteq M_{k+1}$, the element $m = (0, ..., 0, 1, 0, 0, ...) \in M$, where 1 is the $(k+1)$th component. There exist $y \in K$, $z \in K'$ such that $m = y + z$ and hence $\varphi(m) = \varphi(z)$. Now $z = (z_1, ..., z_k, 0, 0, ...)$ for some $z_i \in M_i$ $(1 \leq i \leq k)$ and hence

$$c^{-k} = \varphi(z) = z_1 + c^{-1}z_2 + ... + c^{-(k-1)}z_k.$$

It follows that $c^{-1} = c^{k-1}z_1 + c^{k-2}z_2 + ... + cz_{k-1} + z_k \in V = M_1 + M_2 + M_3 + ... \subseteq F$. Therefore $c^{-1} \in V$ for all $0 \neq c \in R$. Hence $F = V$.

Define $\theta : M \longrightarrow F$ by $\theta(m_1, ...m_n, 0, 0, ...) = m_1 + ... + m_n$ for all positive integers n and elements $m_i \in M_i$ $(1 \leq i \leq n)$. In viewing the above, it follows that θ is an epimorphism. Let $L = ker\theta$. The argument given for K shows that $M = L \oplus L'$ for some submodule L' of M. But $L' \cong M/L \cong F$, which is an injective R-module, a contradiction. Therefore M has finite uniform dimension. □

Theorem 2.2.24 *Let R be a commutative domain. Then a torsion-free R-module M is extending if and only if M is type 1 $\mathcal{U}$-extending. In this case, M is a finite direct sum of injective modules and uniform modules.*

Proof. The necessity is clear. Conversely, suppose that M is type 1 $\mathcal{U}$-extending. There exist an injective submodule M_1 and a submodule M_2 such that $M = M_1 \oplus M_2$ and M_2 contains no nonzero injective submodules. By Lemma 1.2.7, M_2 is type 1 $\mathcal{U}$-extending and by Theorem 2.2.23 M_2 has finite uniform dimension. But this implies M_2 is extending by Lemma 2.2.22. Finally, [20, Theorem 4] gives that M is extending. The last part of the proof follows by Proposition 2.2.18. □

Theorem 2.2.24 is not true for the case of type 2 $\mathcal{U}$-extending modules. For example, if M is a free $\mathbb{Z}$-module of infinite rank then M is type 2 $\mathcal{U}$-extending but not extending (see Example 1.2.14), and is not a finite direct sum of injective modules and uniform modules.

2.3 Modules over Dedekind Domains

We begin this section with the following general result.

Proposition 2.3.1 *Let R be any ring and let $M = \oplus_{i\in I}M_i$ be the direct sum of nonsingular R-modules $M_i (i \in I)$ such that $\oplus_{j\in J}M_j$ is type 2 $\mathcal{G}$-extending (respectively, type 2 $\mathcal{U}$-extending) for every finite subset J of I. Then M is type 2 $\mathcal{G}$-extending (respectively, type 2 $\mathcal{U}$-extending.)*

Proof. Suppose first that $\oplus_{j\in J}M_j$ is type 2 $\mathcal{G}$-extending for all finite $J \subseteq I$. Let N be a submodule of $M = \oplus_{i\in I}M_i$ with $N \in \mathcal{G}$. Since N is finitely generated and $N \subseteq \oplus_{i\in I}M_i$, there exists a finite subset J of I such that $N \subseteq \oplus_{j\in J}M_j$. Let K be any closure of N in M. Note that, for $k \in K$, $k = \{m_i\}$, there exists E which is essential in R_R such that $kE \subseteq N \subseteq \oplus_{j\in J}M_j$. Then for all $i \notin J$, $m_iE = 0$ implies $m_i = 0$, because of all M_i being nonsingular R-modules ($i \in I$). Thus $k \in \oplus_{j\in J}M_j$ and $K \subseteq \oplus_{j\in J}M_j$. By assumption K is a direct summand of $\oplus_{j\in J}M_j$. Consequently, K is a direct summand of $\oplus_{i\in I}M_i = M$. Thus M is type 2 $\mathcal{G}$-extending.

Now suppose that $\oplus_{j\in J}M_j$ is type 2 $\mathcal{U}$-extending for all finite $J \subseteq I$. Let N be a submodule of $\oplus_{i\in I}M_i$ with $N \in \mathcal{U}$. Thus $u.dim(N) < \infty$. There exists an essential submodule N' of N, N' finitely generated. Now $N' \subseteq \oplus_{j \in J} M_j$ for some finite subset J of I. Let K be any closure of N in M. Note that N' is essential in K and N' finitely generated. By the same argument as above, $K \subseteq \oplus_{j \in J} M_j$. Thus K is a direct summand of $\oplus_{j \in J} M_j$ since $\oplus_{j \in J} M_j$ is type 2 $\mathcal{U}$-extending. Then K is a direct summand of $\oplus_{i \in I} M_i$. Consequently, M is type 2 $\mathcal{U}$-extending. □

Corollary 2.3.2 *Let R be a right nonsingular ring such that every finitely generated nonsingular right R-module is projective. Then every free right R-module is type 2 $\mathcal{G}$-extending.*

Proof. Let M be a free right R-module. Then $M = (R_R)^{(\wedge)}$ for some set $\wedge$. Since R is a right nonsingular ring, M_R is nonsingular.

Assume that F is any finitely generated free module. Let N be a submodule of F and let K be a closure of N in F. Then F/K is finitely generated nonsingular, so F/K is a projective module by hypothesis. Thus K is a direct summand of F. So F is extending. Then M is type 2 $\mathcal{G}$-extending. □

Corollary 2.3.3 *Let R be a Prüfer domain. Then every free R-module is type 2 $\mathcal{G}$-extending.*

Proof. Every finitely generated torsion-free R-module is projective by [26, Theorem 1]. Apply Corollary 2.3.2. □

Remark 2.3.4 Let R be a Prüfer domain. Let M be a free R-module. Then M is extending if and only if M is finitely generated.

Proof. The necessity follows by Theorem 2.2.23. Conversely, let K be a closed submodule of the finitely generated free R-module M. Then M/K is finitely generated torsion-free and hence projective. Thus K is a direct summand of M. □

Proposition 2.3.5 *Let R be a Dedekind domain and let M be a torsion R-module. Then the following statements are equivalent:*

(i) *M is extending;*

(ii) *M is type 2 $\mathcal{C}$-extending;*

(iii) *M is type 2 $\mathcal{C}_1$-extending and M is a direct sum of uniform modules.*

Proof. (i) ⇔ (ii). Note that $\mathcal{C} \subseteq \mathcal{T} \subseteq \mathcal{C}^e$. Apply Lemma 1.2.2 and Proposition 1.3.2.

(i) ⇒ (iii). Clearly M is type 2 $\mathcal{C}_1$-extending. Any Dedekind domain is a Noetherian domain. Then M is direct sum of uniform modules by Corollary 2.2.19.

(iii) ⇒ (i). Now let M be a type 2 $\mathcal{C}_1$-extending module such that $M = \oplus_{i \in I} M_i$ with M_i uniform for every $i \in I$. Let U_R be a uniform module. Let

$$P = \{r \in R : ur = 0 \text{ for some } 0 \neq u \in U\}$$

Then $P \trianglelefteq_p R$ and P is the associated prime ideal. Also there exist $0 \neq v \in U$ such that $vP = 0$. Now

$$U \subseteq E(R/P).$$

Since R is a Dedekind domain, $U \cong E(R/P)$ or $U \cong R/P^n$ for some $n \geq 1$. Note that there exist an index set Λ, independent submodules N_λ $(\lambda \in \Lambda)$ of M and distinct prime ideals P_λ $(\lambda \in \Lambda)$ of R such that

(i) $M = \oplus_{\lambda\in\Lambda} N_\lambda$,

(ii) for every $\lambda \in \Lambda$ and $m \in N_\lambda$, there exist $n \geq 1$ such that $P_\lambda^n m = 0$ (i.e $M(P_\lambda) = N_\lambda$), and

(iii) for every $\lambda \in \Lambda$, $N_\lambda = \oplus_{i\in I(\lambda)} M_i$ for some non-empty subset $I(\lambda)$ of I.

If $i \neq j \in I$, then $M_i \oplus M_j$ is type 2 $\mathcal{C}_1$-extending and hence type 2 $\mathcal{A}$-extending (Proposition 2.2.3). Since M_i and M_j are both uniform, $M_i \oplus M_j$ is extending. By [24, Corollary 23] M is extending. □

Question 2.3.6 Suppose that $\mathcal{X}$ is any class of R-modules and an R-module $M \in \mathcal{X}^e$. If M is type 2 $\mathcal{X}$-extending, is M a direct sum of uniform modules?

2.4 Countably generated or projective Modules

Theorem 2.4.1 *Let R be any ring and let M be a countably generated type 2 $\mathcal{G}_1$-extending R-module. Then $M = M_1 \oplus M_2 \oplus M_3 \oplus \ldots$ where each M_i has a cyclic essential submodule.*

Proof. Let $M = m_1R + m_2R + m_3R + \ldots$. There exists a closed submodule K_1 in M such that m_1R is essential in K_1. Then K_1 is a direct summand of M since M is type 2 $\mathcal{G}_1$-extending. Thus $M = K_1 \oplus K_1'$ for some submodule K_1'. Note that $m_2 = k_2 + k_2'$ for some $k_2 \in K_1$, $k_2' \in K_1'$.

Now there exists a closed submodule K_2 of K_1' such that $k_2'R$ is essential in K_2. Next K_2 is closed in M (Lemma 1.2.4) and hence K_2 is a direct summand of M. Thus K_2 is a direct summand of K_1'. Therefore $K_1' = K_2 \oplus K_2'$ for some submodule K_2' of K_1', $M = K_1 \oplus K_2 \oplus K_2'$ and $m_1R + m_2R \subseteq K_1 \oplus K_2$. Take $m_3 = k_3 + l_3 + l_3'$ for some $k_3 \in K_1$, $l_3 \in K_2$, $l_3' \in K_2'$.

Now there exists a closed submodule K_3 of K_2' such that $l_3'R$ is essential in K_3 and K_3 is a direct summand of K_2' by the same argument as above.

By repeating this argument, for every $n \geq 1$ we have

$M = K_1 \oplus ... \oplus K_n \oplus K_n'$ and $m_1R + m_2R + ... + m_nR \subseteq K_1 \oplus ... \oplus K_n$. Thus

$$\begin{aligned} M &= m_1R + m_2R + ... \\ &\subseteq K_1 + K_2 + K_3 + ... \subseteq M \end{aligned}$$

Also $K_1 + K_2 + K_3 + ...$ is a direct sum. Then $M = K_1 \oplus K_2 \oplus K_3 \oplus ...$ and moreover for every $n \geq 1$, K_n has a cyclic essential submodule. □

Corollary 2.4.2 *Let R be any ring and let M be a projective type 2 $\mathcal{G}$-extending R-module. Then M is a direct sum of modules with cyclic essential submodule.*

Proof. By Proposition 2.4.1 and [1, Corollary 26.2]. □

Corollary 2.4.3 *Let R be a right Noetherian ring and let M be a type 2 $\mathcal{U}$-extending R-module which is either countably generated or projective. Then M is a direct sum of uniform modules.*

Proof. For right Noetherian rings, any type 2 $\mathcal{U}$-extending module is type 2 $\mathcal{G}$-extending, so type 2 $\mathcal{G}_1$-extending. By Theorem 2.4.1 and Corollary 2.4.2, $M = \oplus_{\lambda \in \Lambda} M_\lambda$, where M_λ has a cyclic essential submodule for every $\lambda \in \Lambda$. Let $\lambda \in \Lambda$; then there exist $m \in M_\lambda$ such that mR is essential in M_λ. Since mR is Noetherian, M_λ is type 2 $\mathcal{U}$-extending with finite uniform dimension. Thus M_λ is a finite direct sum of uniform submodules. Hence M is a direct sum of uniform submodules. □

Chapter 3

INJECTIVE MODULES RELATIVE TO MODULE CLASSES

For any prime p, the $\mathbb{Z}$-module $(\mathbb{Z}/\mathbb{Z}p) \oplus (\mathbb{Z}/\mathbb{Z}p^3)$ is not extending because the submodule $K = \mathbb{Z}(1 + \mathbb{Z}p, p + \mathbb{Z}p^3)$ is closed, but cannot be a direct summand, since it has order p^2. There has been a lot of interest in recent years in determining under which conditions a direct sum of extending modules is extending (see, for example [5, 17, 20, 23, 24, 25, 28, 38, 51]). Also note that $(\mathbb{Z}/\mathbb{Z}p^3)$ is $(\mathbb{Z}/\mathbb{Z}p)$-injective but $(\mathbb{Z}/\mathbb{Z}p)$ is not $(\mathbb{Z}/\mathbb{Z}p^3)$-injective. In this chapter we investigate relative injectivity with respect not only to modules but also module classes. In section 3.1 for a module M and a class $\mathcal{X}$ of R-modules we define what we mean by an $(M, \mathcal{X})$-injective module and give some basic properties. In section 3.2 we give two different characterisations of when a module is $(M, \mathcal{X})$-injective. In section 3.3, we investigate what happens when $\mathcal{X}$ is a particular class of modules, for example $\mathcal{X} = \mathcal{G}_1$.

3.1 $\mathcal{X}$-injective Modules

The definition of M-injective module has been given in Chapter 1. Let R be a ring with identity. Recall that all modules are unital right R-modules. Let $\mathcal{X}$ be a class of R-modules. Let U, M be R-modules. We say that U is $(M, \mathcal{X})$-*injective* if for every $\mathcal{X}$-submodule L of M, every homomorphism $\varphi : L \longrightarrow U$ can be lifted to M, i.e. there exists a homomorphism $\theta : M \longrightarrow U$ such that $\theta(x) = \varphi(x)$ for all $x \in L$. Thus U is $(M, \mathcal{M})$-injective if and only if U is M-injective. On the other hand, every R-module U is $(M, \mathcal{I})$-injective for any module M.

Lemma 3.1.1 *Let $\mathcal{X}$ be a class of R-modules and let U, M be R-modules. Then U is $(M, \mathcal{X})$-injective if and only if given any $\mathcal{X}$-module N and diagram*

$$\begin{array}{ccccc} 0 & \longrightarrow & N & \xrightarrow{\alpha} & M \\ & & \beta\big\downarrow & & \\ & & U & & \end{array}$$

with exact row, there exists a homomorphism $\theta : M \longrightarrow U$ such that $\theta\alpha = \beta$.

Proof. The sufficiency part is clear.

Conversely, consider the diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & N & \xrightarrow{\mu} & \alpha(N) & \xrightarrow{\iota} & M \\ & & \beta\big\downarrow & & & & \\ & & U & & & & \end{array}$$

where $\mu : N \longrightarrow \alpha(N)$ is given by $\mu(x) = \alpha(x)$ $(x \in N)$ and $\iota : \alpha(N) \longrightarrow M$ is inclusion. Note that μ is an isomorphism and $\alpha = \iota\mu$. Let $\varphi = \beta\mu^{-1} : \alpha(N) \longrightarrow U$. By hypothesis, there exist a homomorphism $\theta : M \longrightarrow U$ such that $\theta\iota = \varphi$. Thus $\beta = \varphi\mu = \theta\iota\mu = \theta\alpha$, as required. □

Lemma 3.1.2 *Let $\mathcal{X}$ be a class of R-modules and let U, M be R-modules such that U is $(M, \mathcal{X})$-injective. Then U is $(N, \mathcal{X})$-injective for any submodule N of M.*

Proof. Clear. □

Lemma 3.1.3 *Let M_1 and M_2 be R-modules, let $M = M_1 \oplus M_2$ and let N be a submodule of M. Then $(N + M_1) \cap M_2 = \pi_2(N)$ where $\pi_2 : M \longrightarrow M_2$ is the canonical projection.*

Proof. Let $y \in \pi_2(N)$. Then there exists $x \in N$ such that $y = \pi_2(x)$, i.e $x - y \in M_1$. Now $y = x - (x - y) \in (N + M_1) \cap M_2$. Thus $\pi_2(N) \subseteq (N + M_1) \cap M_2$. Conversely, let $u \in (N + M_1) \cap M_2$. Then $u = v + w$ for some $v \in N$, $w \in M_1$, and hence

$$u = \pi_2(u) = \pi_2(v + w) = \pi_2(v) + 0 \in \pi_2(N).$$

Thus $(N + M_1) \cap M_2 \subseteq \pi_2(N)$. It follows that $\pi_2(N) = (N + M_1) \cap M_2$. □

Proposition 3.1.4 *Let $\mathcal{X}$ be a class of R-modules which is closed under submodules and factor modules and let M_i $(i \in I)$ be R-modules. Then an R-module U is $(\oplus_I M_i, \mathcal{X})$-injective if and only if U is $(M_i, \mathcal{X})$-injective for all $i \in I$.*

Proof. The necessity is clear by Lemma 3.1.2. Conversely, suppose that U is $(M_i, \mathcal{X})$-injective for all $i \in I$. Let L be an $\mathcal{X}$-submodule of $M = \oplus_I M_i$ and let $\varphi : L \longrightarrow U$ be a homomorphism. Let $i \in I$. Since $\mathcal{X}$ is closed under submodules, $L \cap M_i$ is an $\mathcal{X}$-submodule of M_i. By hypothesis the homomorphism $\varphi|_{L\cap M_i} : L \cap M_i \longrightarrow U$ lifts to M_i. There exists $\beta_i : M_i \longrightarrow U$ such that $\beta_i(x) = \varphi(x)$ $(x \in L \cap M_i)$. Define $\theta_i : L + M_i \longrightarrow U$ by $\theta_i(x+m) = \varphi(x) + \beta_i(m)$ for all $x \in L$, $m \in M_i$. To check that θ_i is well-defined, suppose that $x \in L$, $m \in M_i$ and $x + m = 0$. Then $x = -m \in L \cap M_i$ and hence

$$\theta_i(x+m) = \varphi(x) + \beta_i(m) = \varphi(x) - \beta_i(x) = 0.$$

Thus θ_i is well-defined. Moreover θ_i is a homomorphism, and for all $x \in L$, $\theta_i(x) = \varphi(x)$. Thus φ can be lifted to $L + M_i$.

By Zorn's Lemma there exists a maximal subset J of I such that φ can be lifted to $L + (\oplus_J M_j)$. Suppose that $M \neq L + (\oplus_J M_j)$. There exists $k \in I - J$ such that $M_k \not\subseteq L + (\oplus_J M_j)$.

Let $M' = \oplus_J M_j$ and let $\theta_J : L + M' \longrightarrow U$ be a homomorphism such that $\theta_J(x) = \varphi(x)$ for all $x \in L$. Let $W = \oplus_{i \neq k} M_i$ and let $\pi : M \longrightarrow M_k$ be the canonical projection with kernel W. Then Lemma 3.1.3 gives that $\pi(L) = (L + W) \cap M_k$. Hence $(L + W) \cap M_k \in \mathcal{X}$ and hence the submodule $(L + M') \cap M_k \in \mathcal{X}$. By hypothesis, there exists a homomorphism $\mu : M_k \longrightarrow U$ such that $\mu(x) = \theta_J(x)$ for all $x \in (L + M') \cap M_k$. Now define

$$\theta' : L + ((\oplus_J M_j) \oplus M_k) \longrightarrow U$$

by $\theta'(x + y) = \theta_J(x) + \mu(y)$ for all $x \in L + (\oplus_J M_j)$, $y \in M_k$. By the above argument θ' is well-defined. Clearly θ' is a homomorphism and, for all $x \in L$,

$$\theta'(x) = \theta_J(x) = \varphi(x).$$

Thus φ can be lifted to $L + \{(\oplus_J M_j) \oplus M_k\}$, contradicting the choice of J. It follows that $M = L + (\oplus_J M_j)$. Therefore U is $(\oplus_I M_i, \mathcal{X})$-injective. □

Corollary 3.1.5 *Let $\{M_i\}_I$ be any collection of R-modules. Then an R-module N is $(\oplus_{i \in I} M_i)$-injective if and only if N is M_i-injective for every $i \in I$.*

Proof. Take $\mathcal{X} = \mathcal{M}$ in Proposition 3.1.4. □

Proposition 3.1.6 *Let $\mathcal{X}$ be any class of R-modules and let M, U_i $(i \in I)$ be R-modules. Then the direct product $\prod_I U_i$ is $(M, \mathcal{X})$-injective if and only if U_i is $(M, \mathcal{X})$-injective for all $i \in I$.*

Proof. For the necessity, let $j \in I$. Let K be any $\mathcal{X}$-submodule of M and consider

$$\begin{array}{ccc} K & \xrightarrow{\iota} & M \\ {\scriptstyle\varphi}\downarrow & & \\ U_j & & \\ {\scriptstyle\iota}\downarrow & & \\ \prod_I U_i & & \end{array}$$

where ι denotes inclusion. By hypothesis there exists a homomorphism $\theta : M \longrightarrow \prod_I U_i$ such that $\iota\varphi = \theta\iota$. Let $\pi_j : \prod_I U_i \longrightarrow U_j$ denote the canonical projection. Then $\pi_j\theta : M \longrightarrow U_j$ is a homomorphism and $\pi_j\theta(x) = \pi_j\iota\varphi(x) = \varphi(x)$ for all $x \in K$. It follows that U_j is $(M, \mathcal{X})$-injective.

Conversely, suppose that U_i is $(M, \mathcal{X})$-injective for all $i \in I$. Let L be any $\mathcal{X}$-submodule of M and let $\alpha : L \longrightarrow \prod_I U_i$ be a homomorphism. For each $j \in I$, $\pi_j\alpha : L \longrightarrow U_j$ is a homomorphism and hence there exists a homomorphism β_j :

$M \longrightarrow U_j$ such that $\beta_j(x) = \pi_j\alpha(x)$ for all $x \in L$. Now define $\beta : M \longrightarrow \prod_I U_i$ by $\beta(m) = \{\beta_i(m)\}$ $(m \in M)$. Clearly β is a homomorphism and

$$\beta(x) = \{\beta_i(x)\} = \{\pi_i\alpha(x)\} = \alpha(x)$$

for all $x \in L$. Thus $\alpha = \beta|_L$. It follows that $\prod_I U_i$ is $(M, \mathcal{X})$-injective. □

Let $\mathcal{X}$ be any class of R-modules. Let M be an R-module and let N be a submodule of M. Let $\mathcal{X}(M/N)$ denote the class of R-modules A such that $A = 0$ or $A \cong K/N$ for some $\mathcal{X}$-submodule K of M with $N \subseteq K$.

Lemma 3.1.7 *Let $\mathcal{X}$ be any class of R-modules, let M be an R-module and let U be an $(M, \mathcal{X})$-injective R-module. Then U is $(M/N, \mathcal{X}(M/N))$-injective for every submodule N of M.*

Proof. Let K be an $\mathcal{X}$-submodule of M with $N \subseteq K$ and let $\varphi : K/N \longrightarrow U$ be a homomorphism. Consider the diagram,

$$\begin{array}{ccccc} 0 & \longrightarrow & K & \stackrel{\iota}{\longrightarrow} & M \\ & & \pi\downarrow & & \pi\downarrow \\ 0 & \longrightarrow & K/N & \stackrel{\iota}{\longrightarrow} & M/N \\ & & \varphi\downarrow & & \\ & & U & & \end{array}$$

where ι denotes inclusion and π projection. By hypothesis there exists a homomorphism $\theta' : M \longrightarrow U$ such that $\varphi\pi = \theta'\iota$. For any $x \in N$, $\theta'(x) = \varphi\pi(x) = 0$, so $N \subseteq ker\theta'$ and hence θ' induces a homomorphism $\theta : M/N \longrightarrow U$ such that $\theta(x + N) = \theta'(x) = \varphi(x + N)$ for all $x \in K$. Thus $\varphi = \theta|_{K/N}$. By Lemma 3.1.1 the result follows. □

Corollary 3.1.8 *Let $\mathcal{X}$ be a class of R-modules which is closed under extensions. Let M be an R-module and let U be an $(M, \mathcal{X})$-injective R-module. Then U is $(M/N, \mathcal{X})$-injective for every $\mathcal{X}$-submodule N of M.*

Proof. Let $\overline{K}$ be any $\mathcal{X}$-submodule of M/N. Then $\overline{K} = K/N$ for some submodule K of M containing N. Since N and K/N both belong to $\mathcal{X}$, it follows that $K \in \mathcal{X}$. Thus $\overline{K} \in \mathcal{X}(M/N)$. The result now follows by Lemma 3.1.7. □

By a *Serre class* of R-modules we mean a class which is closed under submodules, factor modules and extensions. It is easy to check that $\mathcal{X}$ is a Serre class of R-modules if and only if for every exact sequence of R-modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow O$$

we have $M \in \mathcal{X}$ if and only if $M' \in \mathcal{X}$ and $M'' \in \mathcal{X}$. For example, $\mathcal{N}$, $\mathcal{A}$ and $\mathcal{T}$ are all Serre classes.

Corollary 3.1.9 *Let $\mathcal{X}$ be any Serre class of R-modules, let M be an R-module with submodules M_i $(i \in I)$ such that $M = \sum_I M_i$ and $M \cong (\oplus_I M_i)/N$ for some $\mathcal{X}$-submodule N of $\oplus_I M_i$. Then an R-module U is $(M, \mathcal{X})$-injective if and only if U is $(M_i, \mathcal{X})$-injective for all $i \in I$.*

Proof. By Lemma 3.1.2, Proposition 3.1.4 and Corollary 3.1.8. □

Theorem 3.1.10 *Let $\mathcal{X}$ be any Serre class of R-modules, let M, U be R-modules and let N be an $\mathcal{X}$-submodule of M. Then U is $(M, \mathcal{X})$-injective if and only if*

(i) *U is $(N, \mathcal{X})$-injective,*

(ii) *U is $(M/N, \mathcal{X})$-injective,*

(iii) *every homomorphism $\varphi : N \longrightarrow U$ lifts to M.*

Proof. The necessity follows by Lemmas 3.1.1, 3.1.2 and Corollary 3.1.8.

Conversely, suppose that (i), (ii), (iii) all hold. Let K be any $\mathcal{X}$-submodule of M and let $\varphi : K \longrightarrow U$ be a homomorphism. Then $K \cap N \in \mathcal{X}$ and hence there exists a homomorphism $\alpha : N \longrightarrow U$ such that $\varphi|_{K\cap N} = \alpha|_{K\cap N}$ by (i). By (iii), α can be lifted to $\beta : M \longrightarrow U$. Let $\mu = (\varphi - \beta)|_K$. Then $\mu : K \longrightarrow U$ and $\mu(K \cap N) = 0$. Define $\lambda : (K + N)/N \longrightarrow U$ by $\lambda(k + N) = \mu(k)$ $(k \in K)$. Since $\mu(K \cap N) = 0$ it follows that λ is well-defined and clearly λ is a homomorphism. Also $(K + N)/N \cong K/(K \cap N) \in \mathcal{X}$ so that λ lifts to a homomorphism $\delta : M/N \longrightarrow U$ by (ii). Let $\pi : M \longrightarrow M/N$ denote the projection mapping and let $\theta = \beta + \delta\pi$. Then $\theta : M \longrightarrow U$ is a homomorphism and for all $k \in K$,

$$\theta(k) = \beta(k) + \delta(k + N) = \varphi(k) - \mu(k) + \mu(k) = \varphi(k).$$

Thus $\varphi = \theta|_K$. It follows that U is $(M, \mathcal{X})$-injective. □

Note In Theorem 3.1.10, the necessity requires that $\mathcal{X}$ be closed under extensions and the sufficiency holds if $\mathcal{X}$ is closed under submodules and factor modules.

Proposition 3.1.11 *Let U, M be R-modules such that M is locally Noetherian. Then U is M-injective if and only if U is $(M, \mathcal{G})$-injective.*

Proof. The necessity is clear. Conversely, suppose that U is $(M, \mathcal{G})$-injective. Let $m \in M$. Then U is $(mR, \mathcal{G})$-injective by Lemma 3.1.2. Since mR is Noetherian it follows that U is mR-injective. Let $M' = \oplus_{m\in M} mR$. By Corollary 3.1.5,

U is M'-injective. Finally, Lemma 3.1.7 gives that U is M-injective since M is a homomorphic image of M'. □

Let $\mathcal{X}$ be any class of R-modules. An R-module M is called *$\mathcal{X}$-noetherian* if every ascending chain of $\mathcal{X}$-submodules terminates, i.e. given

$$K_1 \subseteq K_2 \subseteq K_3 \subseteq ...,$$

with K_i an $\mathcal{X}$-submodule of M for each $i \geq 1$, there exists a positive integer n such that $K_n = K_{n+1} = K_{n+2} = ...$ A module M is *locally $\mathcal{X}$-noetherian* if every finitely generated submodule is $\mathcal{X}$-noetherian.

Lemma 3.1.12 *Let $\mathcal{X}$ be any class of R-modules which is closed under submodules. If M is locally $\mathcal{X}$-noetherian then for each $m \in M$, every $\mathcal{X}$-submodule of mR is finitely generated.*

Proof. Let $m \in M$ and let K be an $\mathcal{X}$-submodule of mR. Let $K_1 \leq K_2 \leq K_3 \leq ...$ be any ascending chain of finitely generated submodules of K. Since $\mathcal{X}$ is closed under submodules we have $K_i \in \mathcal{X}$ for all $i \geq 1$ and hence $K_n = K_{n+1} = K_{n+2} = ...$ for some positive integer n. It follows that K is finitely generated. □

Note that, the converse of Lemma 3.1.12 is not true in general because of the following example.

Example 3.1.13 Let $R = \begin{bmatrix} K & V \\ 0 & K \end{bmatrix} = \left\{ \begin{bmatrix} a & v \\ 0 & a \end{bmatrix} : a \in K, \quad v \in V \right\}$ where K is a field and V an infinite dimensional vector space over K. R is a commutative ring.

Let $v_1, v_2, v_3, \ldots$ be an infinite set of linearly independent elements of V. Then

$$\begin{bmatrix} 0 & Kv_1 \\ 0 & 0 \end{bmatrix} \subsetneq \begin{bmatrix} 0 & Kv_1 + Kv_2 \\ 0 & 0 \end{bmatrix} \subsetneq \begin{bmatrix} 0 & Kv_1 + Kv_2 + Kv_3 \\ 0 & 0 \end{bmatrix} \subsetneq \ldots \subseteq R$$

is an ascending chain of Noetherian submodules of the cyclic R-module R. Thus R is not locally $\mathcal{N}$-noetherian.

3.2 Further Characterizations of $\mathcal{X}$-injective Modules

Lemma 1.2.8 can be generalized with respect to R-module classes in the following way.

Lemma 3.2.1 *Let $\mathcal{X}$ be any R-module class. Let a module $M = M_1 \oplus M_2$ be a direct sum of submodules M_1, M_2. Then the following statements are equivalent:*

(i) *M_2 is $(M_1, \mathcal{X})$-injective;*

(ii) *for every $\mathcal{X}$-submodule N of M with $N \cap M_2 = 0$, there exists a submodule M' of M such that $M = M' \oplus M_2$ and $N \subseteq M'$.*

Proof. (i) $\Rightarrow$ (ii). For $i = 1, 2$, let $\pi_i : M \longrightarrow M_i$ denote the projection mapping. Let N be any $\mathcal{X}$-submodule of M with $N \cap M_2 = 0$. Consider the following diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & N & \xrightarrow{\alpha} & M_1 \quad \text{exact} \\ & & \beta \big\downarrow & & \\ & & M_2 & & \end{array}$$

where $\alpha = \pi_1|_N$, $\beta = \pi_2|_N$. By (i), there exists a homomorphism $\phi : M_1 \longrightarrow M_2$ such that $\phi\alpha = \beta$. Let $M' = \{x + \phi(x) : x \in M_1\}$. It can easily be checked that M' is a submodule of M, $M' \cap M_2 = 0$, $M = M' \oplus M_2$ and $N \subseteq M'$.

(ii) $\Rightarrow$ (i). Let N be an $\mathcal{X}$-submodule of M_1 and $\varphi : N \longrightarrow M_2$ be a homomorphism. Let $L = \{n - \varphi(n) : n \in N\}$. Then L is a submodule of M and $L \cap M_2 = 0$. Also $L \in \mathcal{X}$ because $L \cong N$. By hypothesis $M = L' \oplus M_2$ for some submodule L' of M such that $L \subseteq L'$. Let $\pi : M \longrightarrow M_2$ denote the canonical projection for the direct sum $M = L' \oplus M_2$. Then $\chi : \pi|_{M_1} : M_1 \longrightarrow M_2$ is a homomorphism and for any $n \in N$, $\chi(n) = \pi(n - \varphi(n) + \varphi(n)) = \varphi(n)$. It follows that χ lifts φ to M_1. Thus M_2 is $(M_1, \mathcal{X})$-injective. □

Theorem 3.2.2 *Let $\mathcal{X}$ be a Serre class of R-modules. Then the following statements are equivalent for any R-module M:*

(i) *U is $(M, \mathcal{X})$-injective;*

(ii) *for every $K \in \mathcal{X}$, every $\mathcal{X}$-submodule N of M, and all monomorphisms $\varphi : K \longrightarrow M/N$ and $\alpha : K \longrightarrow U$, there exists a homomorphism $\theta : M/N \longrightarrow U$ such that $\theta\varphi = \alpha$.*

Proof. (i) $\Rightarrow$ (ii). Let $K \in \mathcal{X}$ and N be a submodule of M with $N \in \mathcal{X}$. Consider the diagram

$$\begin{array}{ccccc} & & 0 & & \\ & & \downarrow & & \\ 0 & \longrightarrow & K & \xrightarrow{\alpha} & U \text{ exact} \\ & & \varphi\downarrow & & \\ & & M/N & & \\ & & \text{exact} & & \end{array}$$

Let L be the submodule of M containing N with $L/N = \varphi(K)$. Then $\varphi : K \longrightarrow L/N$ is an isomorphism. Notice that $N \in \mathcal{X}$ and $L/N \in \mathcal{X}$ so $L \in \mathcal{X}$ because $\mathcal{X}$ is closed under extensions. Consider the following diagram

$$\begin{array}{ccccccccc} & & & & 0 & \longrightarrow & L & \xrightarrow{\iota} & M \\ & & & & & & {\scriptstyle\pi}\downarrow & & {\scriptstyle\pi}\downarrow \\ 0 & \longrightarrow & K & \xrightarrow{\varphi} & & & L/N & \xrightarrow{\iota} & M/N \\ & & {\scriptstyle\alpha}\downarrow & & & & & & \\ & & U & & & & & & \end{array}$$

where ι denotes inclusion and π projection. By hypothesis, there exists a homomorphism $\beta : M \longrightarrow U$ such that $\beta\iota = \alpha\varphi^{-1}\pi$. Now $\pi(N) = 0$ so that $\beta(N) = 0$ and hence β induces a homomorphism $\theta : M/N \longrightarrow U$ given by $\theta(m+N) = \beta(m)$ $(m \in M)$.

Let $x \in K$. Then $\varphi(x) = y + N$ for some $y \in L$ and

$$\theta\varphi(x) = \theta(y + N) = \beta(y) = \alpha\varphi^{-1}\pi(y) = \alpha\varphi^{-1}(y + N) = \alpha(x).$$

Thus $\theta\varphi = \alpha$.

(ii) $\Rightarrow$ (i). Let K be any $\mathcal{X}$-submodule of M and $\varphi : K \longrightarrow M$ any homomorphism. Consider

$$\begin{array}{ccccc} 0 & \longrightarrow & K & \xrightarrow{\iota} & M \quad \text{exact} \\ & & {\scriptstyle\varphi}\downarrow & & \\ & & U & & \end{array}$$

Now consider

$$\begin{array}{ccccc} & & 0 & & \\ & & \downarrow & & \\ 0 & \longrightarrow & K/ker\varphi & \xrightarrow{\overline{\varphi}} & U \quad \text{exact} \\ & & {\scriptstyle\iota}\downarrow & & \\ & & M/ker\varphi & & \\ & & \text{exact} & & \end{array}$$

where $\overline{\varphi}$ is the induced monomorphism. Note that, because $\mathcal{X}$ is closed under submodules and factor modules, $K \in \mathcal{X}$ gives $ker\varphi \in \mathcal{X}$ and $K/ker\varphi \in \mathcal{X}$. By hypothesis there exists $\beta : M/ker\varphi \longrightarrow U$ such that $\beta\iota = \overline{\varphi}$. Let $\pi : M \longrightarrow M/ker\varphi$ be the canonical projection. Then $\beta\pi : M \longrightarrow U$. Let $x \in K$. Then

$$\beta\pi(x) = \beta(x + ker\varphi) = \beta\iota(x + ker\varphi) = \overline{\varphi}(x + ker\varphi) = \varphi(x).$$

Thus $\beta\pi(x) = \varphi(x)$ for all $x \in K$. It follows that U is $(M, \mathcal{X})$-injective. □

3.3 Injective Modules relative to Different Classes

What happens when U is $(M, \mathcal{X})$-injective and $\mathcal{Y}$ is some other class of modules. Is U also $(M, \mathcal{Y})$-injective? The first result is obvious.

Lemma 3.3.1 *Let U, M be R-modules such that U is $(M, \mathcal{X})$-injective for some class $\mathcal{X}$ of R-modules. Then U is $(M, \mathcal{Y})$-injective for any class $\mathcal{Y} \subseteq \mathcal{X}$.*

Note that if $\mathcal{X}$ is a class of R-modules and U is $(M, \mathcal{X}^e)$-injective then U is $(M, \mathcal{X})$-injective by Lemma 3.3.1 because $\mathcal{X} \subseteq \mathcal{X}^e$. The converse holds true if U is nonsingular.

Proposition 3.3.2 *Let U, M be R-modules such that U is nonsingular and $\mathcal{X}$ be a class of R-modules. Then U is $(M, \mathcal{X})$-injective if and only if U is $(M, \mathcal{X}^e)$-injective.*

Proof. Sufficiency is clear by Lemma 3.3.1. Conversely, let N be any $\mathcal{X}^e$-submodule of M and let $\varphi : N \longrightarrow U$ be a homomorphism. There exists an essential submodule L of N such that $L \in \mathcal{X}$. Now $\varphi|_L : L \longrightarrow U$ is a homomorphism and hence there exists a homomorphism $\theta : M \longrightarrow U$ such that $\theta(m) = \varphi(m)$ for all $m \in L$. Let $n \in N$. There exists an essential right ideal E of R such that $nE \subseteq L$. Let $e \in E$. Then

$$\begin{aligned}(\theta(n) - \varphi(n))e &= \theta(n)e - \varphi(n)e \\ &= \theta(ne) - \varphi(ne) = 0.\end{aligned}$$

Thus $[\theta(n) - \varphi(n)]E = 0$. Because U is nonsingular, $\theta(n) = \varphi(n)$. It follows that $\varphi = \theta|_N$. Thus U is $(M, \mathcal{X}^e)$-injective. □

Proposition 3.3.2 fails in general if U is not nonsingular. In order to produce an example we first consider divisible modules. Let R be a (not necessarily commutative) domain. An R-module U is called *divisible* if $U = Uc = \{uc : u \in U\}$ for all $0 \neq c \in R$.

Lemma 3.3.3 *Let R be a domain. Then an R-module U is divisible if and only if U is $(R_R, \mathcal{G}_1)$-injective.*

Proof. Suppose first that U is divisible. Let $0 \neq c \in R$ and let $\varphi : cR \longrightarrow U$ be a homomorphism. There exists $u \in U$ such that $\varphi(c) = uc$. Define $\theta : R \longrightarrow U$ by $\theta(r) = ur$ $(r \in R)$. Then $\varphi = \theta|_{cR}$. It follows that U is $(R_R, \mathcal{G}_1)$-injective.

Conversely, suppose that U is $(R_R, \mathcal{G}_1)$-injective. Let $0 \neq d \in R$ and let $v \in U$. Define $\alpha : dR \longrightarrow U$ by $\alpha(ds) = vs$ $(s \in R)$. Then α is a homomorphism and hence α lifts to a homomorphism $\beta : R \longrightarrow U$. Now $v = \alpha(d) = \beta(d) = \beta(1)d \in Ud$. It follows that $U = Ud$. Hence U is divisible. □

Lemma 3.3.4 *Let R be a right Öre domain. Then an R-module U is injective if and only if U is $(R_R, \mathcal{G}_1^e)$-injective.*

Proof. The necessity is clear. Conversely, suppose that U is $(R_R, \mathcal{G}_1^e)$-injective. Let E be any nonzero right ideal of R. Let $0 \neq e \in E$. Then eR is an essential submodule of the R-module E. Thus $E \in \mathcal{G}_1^e$. By Baer's Lemma it follows that U is injective. □

Lemma 3.3.5 *A commutative domain R is Dedekind if and only if every $(R_R, \mathcal{G}_1)$-injective R-module is $(R_R, \mathcal{G}_1^e)$-injective.*

Proof. By Lemma 3.3.3 and 3.3.4, every $(R_R, \mathcal{G}_1)$-injective R-module is $(R_R, \mathcal{G}_1^e)$-injective if and only if every divisible R-module is injective. Apply [45, Theorem 2.8] to complete the proof. □

Recall that if $\mathcal{X}$ is a class of R-modules then $\mathcal{X}^+$ denotes the class of R-modules each of which is a sum of a finite number of $\mathcal{X}$-submodules. In particular, $\mathcal{G} = \mathcal{G}_1^+$. Now we show that if U is $(M, \mathcal{X})$-injective then U need not be $(M, \mathcal{X}^+)$-injective.

Proposition 3.3.6 *Let R be a commutative Noetherian domain. Then R is Dedekind if and only if every $(R_R, \mathcal{G}_1)$-injective R-module is $(R_R, \mathcal{G}_1^+)$-injective.*

Proof. In Lemma 3.3.3 we saw that the $(R_R, \mathcal{G}_1)$-injective R-module are precisely the divisible R-modules. Because $\mathcal{G} = \mathcal{G}_1^+$ and R is Noetherian, the $(R_R, \mathcal{G}_1^+)$-injective R-modules are precisely the injective R-modules. Again we apply [45, Theorem 2.8] to complete the proof. □

It is well known that an R-module M is quasi-injective if and only if $\theta(M) \subseteq M$ for every endomorphism θ of $E(M)$. The module M is called *quasi-continuous* if $\theta(M) \subseteq M$ for every idempotent endomorphism θ of $E(M)$. Quasi-continuous modules form an important class of modules which have been extensively studied in recent years (see, for example, [5], [48]). In particular, in [5, 2.10] or [28, Theorem 2.8] we find the following result.

Proposition 3.3.7 *The following statements are equivalent for a module M:*

(i) *M is quasi-continuous;*

(ii) *For all submodules N_1, N_2 with $N_1 \cap N_2 = 0$ there exist submodules M_1, M_2 such that $M = M_1 \oplus M_2$ and $N_i \subseteq M_i$ $(i = 1, 2)$;*

(iii) *For any family of independent submodules N_i $(i \in I)$ of M there exist independent submodules M_i $(i \in I)$ such that $M = \oplus_{i \in I} M_i$ and $N_i \subseteq M_i$ for all $i \in I$;*

(iv) (a) *For any submodule N of M there exists a direct summand K of M such that N is essential in K, and*

(b) *for all direct summands K, L of M with $K \cap L = 0$ the submodule $K \oplus L$ is also a direct summand of M.*

Note that, in particular, Proposition 3.3.7 gives that quasi-continuous modules are extending.

Now we do have a positive result for quasi-continuous modules.

Proposition 3.3.8 *Let $\mathcal{X}_i$ $(i \in I)$ be classes of R-modules. Let U, M be R-modules such that M is quasi-continuous. Then U is $(M, \oplus_I \mathcal{X}_i)$-injective if and only if U is $(M, \mathcal{X}_i)$-injective for all $i \in I$.*

Proof. The necessity follows by Lemma 3.3.1. Conversely, suppose that U is $(M, \mathcal{X}_i)$-injective for all $i \in I$. Let L be a $(\oplus_I \mathcal{X}_i)$-submodule of M and let $\varphi : L \longrightarrow U$ be a homomorphism. There exist independent $\mathcal{X}_i$-submodules L_i $(i \in I)$ such that $L = \oplus_I L_i$. By Proposition 3.3.7, $M = \oplus_I M_i$ for some submodules M_i with $L_i \subseteq M_i$ $(i \in I)$. Since U is $(M, \mathcal{X}_i)$-injective and hence $(M_i, \mathcal{X}_i)$-injective (Lemma 3.1.2), it follows that there exists a homomorphism $\theta_i : M_i \longrightarrow U$ such that $\varphi|_{L_i} = \theta_i | L_i$. Define $\theta : M \longrightarrow U$ as follows: for each $m \in M$, $m = \sum_I m_i$ where $m_i \in M_i$ $(i \in I)$ and at most a finite number of m_i are nonzero, so we define $\theta(m) = \sum_I \theta_i(m_i)$. It is clear that θ is a homomorphism and $\varphi = \theta|_L$. It follows that U is $(M, \oplus_I \mathcal{X}_i)$-injective. □

We do not know in general whether for given R-modules U and M such that U is $(M, \mathcal{X}_i)$-injective for some finite collection $\mathcal{X}_i$ $(1 \leq i \leq n)$ of classes of R-modules, then U is $(M, \mathcal{X}_1 \oplus ... \oplus \mathcal{X}_n)$-injective.

Chapter 4

QUASI-CONTINUOUS MODULES RELATIVE TO MODULE CLASSES

Quasi-injective modules are quasi-continuous and it is not hard to prove that direct summands of quasi-continuous modules are quasi-continuous. Since quasi-continuous modules are example of extending modules, we now study quasi-continuous modules. This concept leads us to study quasi-continuous modules with respect to R-module classes. In this chapter, we investigate modules M with the property that for each submodule N_1 in some given class $\mathcal{X}$ of modules and submodule N_2 with $N_1 \cap N_2 = 0$ there exist submodules M_1, M_2 of M such that $M = M_1 \oplus M_2$ and $N_i \subseteq M_i$ $(i = 1, 2)$. Moreover the properties we obtain are essentially self-contained.

One motivation for this investigation is the following simple observation. Let

R be a ring which is not right Noetherian. Then there exist injective R-modules U_n ($n \geq 1$) such that the module $M = \oplus_{n\geq 1} U_n$ is not injective. In fact, it may turn out to be the case that M is not even quasi-continuous (see [28, Proposition 2.10]). Let N be a finitely generated submodule and let L be a submodule of M such that $N \cap L = 0$. There exists a positive integer k such that $N \subseteq \oplus_{n=1}^{k} U_n$. Now $\oplus_{n=1}^{k} U_n$ is an injective module. Let N' denote the injective hull of N contained in $\oplus_{n=1}^{k} U_n$. Then N' is injective and hence $M = N' \oplus N''$ for some submodule N'' of M. Since N is essential in N', it follows that $N' \cap L = 0$. Recall that, Lemma 1.2.8 gives the existence of a submodule L' of $M = \oplus_{n\geq 1} U_n$ such that $M = N' \oplus L'$, $N \subseteq N'$ and $L \subseteq L'$. Thus property (ii) of Proposition 3.3.7 holds for this particular module M in case N_1 or N_2 is finitely generated although it does not hold for general submodules N_1, N_2. This leads us to consider (ii) for a restricted class of submodules of M.

4.1 $(C1)_{\mathcal{X}}$, $(C2)_{\mathcal{X}}$, $(C3)_{\mathcal{X}}$ Conditions

Let $\mathcal{X}$ be any R-module class. We consider the following conditions:

$(C1)_{\mathcal{X}}$: Every $\mathcal{X}$-submodule is essential in a direct summand.

$(C2)_{\mathcal{X}}$: If an $\mathcal{X}$-submodule A of M is isomorphic to a direct summand of M, then A is a direct summand of M.

$(C3)_{\mathcal{X}}$: Let $A \in \mathcal{X}$ and X be a direct summand of M, if A is a direct summand of M and $A \cap X = 0$, then $A \oplus X$ is also a direct summand of M.

A module M is called weak type 2 *$\mathcal{X}$-extending*, *$\mathcal{X}$-continuous*, *$\mathcal{X}$-quasi-continuous*, respectively, if it satisfies condition $(C1)_{\mathcal{X}}$, conditions $(C1)_{\mathcal{X}}$ and $(C2)_{\mathcal{X}}$, conditions $(C1)_{\mathcal{X}}$ and $(C3)_{\mathcal{X}}$.

Lemma 4.1.1 *If M satisfies condition $(Ci)_{\mathcal{X}}$ for $(i = 2, 3)$, then every direct summand of M also satisfies it.*

Proof. Suppose that $M = M_1 \oplus M_2$ for some submodules M_1, M_2. Suppose that M satisfies $(C2)_{\mathcal{X}}$. Let A be any $\mathcal{X}$-submodule of M_1 which is isomorphic to a direct summand X of M_1. Since X is a direct summand of M, A is a direct summand of M because M satisfies $(C2)_{\mathcal{X}}$. Thus $M = A \oplus A'$ for some submodule A'. Then $M_1 = M_1 \cap (A \oplus A') = A \oplus (M_1 \cap A')$ by the Modular Law, i.e., A is a direct summand of M_1. Then M_1 satisfies $(C2)_{\mathcal{X}}$.

Now suppose that M satisfies $(C3)_{\mathcal{X}}$. Let $A_1 \in \mathcal{X}$ and X_1 be a direct summand of M_1. Also let A_1 be a direct summand of M_1 such that $A_1 \cap X_1 = 0$. Since A_1 and X_1 are both direct summands of M and $A_1 \cap X_1 = 0$, by hypothesis $A_1 \oplus X_1$ is a direct summand of M. Then $M = A_1 \oplus X_1 \oplus T$ for some submodule T of M. Thus $M_1 = M_1 \cap (A_1 \oplus X_1 \oplus T) = A_1 \oplus X_1 \oplus (M_1 \cap T)$ by the Modular Law. Then M_1 satisfies $(C3)_{\mathcal{X}}$. □

Lemma 4.1.2 *If a module satisfies condition $(C2)_{\mathcal{X}}$, then it satisfies condition $(C3)_{\mathcal{X}}$.*

Proof. Assume that the module M satisfies $(C2)_{\mathcal{X}}$. Let both M_1 and M_2 be direct summands of M with M_2 an $\mathcal{X}$-submodule such that $M_1 \cap M_2 = 0$. Then $M = M_1 \oplus M_1'$ for some submodule M_1'. Let π denote the canonical projection such that $\pi : M_1 \oplus M_1' \longrightarrow M_1'$. Then $M_1 \oplus M_2 = M_1 \oplus \pi(M_2)$. Since $\pi|_{M_2}$ is monomorphism $M_2 \cong \pi(M_2)$ and $\pi(M_2) \in \mathcal{X}$. By assumption, $\pi(M_2)$ is a direct summand of M. Also $\pi(M_2) \subseteq M_1'$. Thus $\pi(M_2)$ is a direct summand of M_1'. Then $M_1 \oplus M_2 = M_1 \oplus \pi(M_2)$ is a direct summand of $M_1 \oplus M_1' = M$, i.e. M satisfies $(C3)_{\mathcal{X}}$. □

Note that,

$\mathcal{X}$-continuous $\Longrightarrow$ $\mathcal{X}$-quasi-continuous $\Longrightarrow$ weak type 2 $\mathcal{X}$-extending.

Lemma 4.1.3 *The following statements are equivalent for a module M:*

(i) *M satisfies $(C3)_{\mathcal{X}}$;*

(ii) *For all summands P, Q of M such that $P \in \mathcal{X}$ and $P \cap Q = 0$, there exists a submodule P' of M such that $M = P \oplus P'$ and $Q \subseteq P'$.*

Proof. (i) $\Rightarrow$ (ii). Let P and Q be direct summands of M such that $P \in \mathcal{X}$ with $Q \cap P = 0$. Then by hypothesis $Q \oplus P$ is a direct summand of M. Hence $M = P \oplus Q \oplus Q''$ for some submodule Q'' of M. Thus $P' = Q \oplus Q''$ has the requisite properties.

(ii) $\Rightarrow$ (i). Let K and L be direct summands of M such that $K \in \mathcal{X}$ and $K \cap L = 0$. There exists a submodule K' of M such that $M = K \oplus K'$ and $L \subseteq K'$. But $M = L \oplus L'$ for some submodule L'. Hence $K' = L \oplus (K' \cap L')$. Thus $M = K \oplus L \oplus (K' \cap L')$. Then M satisfies $(C3)_{\mathcal{X}}$. □

Proposition 4.1.4 *Let $\mathcal{X}$ be an essentially closed R-module class. A weak type 2 $\mathcal{X}$-extending module M is $\mathcal{X}$-quasi-continuous if and only if whenever $M = M_1 \oplus M_2$ is a direct sum of submodules M_1, M_2, then M_2 is $(M_1, \mathcal{X})$-injective.*

Proof. Suppose first that M is $\mathcal{X}$-quasi-continuous. Suppose $M = M_1 \oplus M_2$. Let N be an $\mathcal{X}$-submoudule of M with $N \cap M_2 = 0$. Since M is weak type 2

$\mathcal{X}$-extending, there exists a direct summand N' of M such that N is essential in N'. Clearly $N' \cap M_2 = 0$. Because $\mathcal{X}$ is essentially closed, $N' \in \mathcal{X}$. By Lemma 4.1.3, $M = M' \oplus M_2$ for some submodule M' of M such that $N' \subseteq M'$. Note that $N \subseteq M'$. By Lemma 3.2.1, M_2 is $(M_1, \mathcal{X})$-injective.

Conversely, suppose that M_2 is $(M_1, \mathcal{X})$-injective whenever $M = M_1 \oplus M_2$. By Lemma 3.2.1 and 4.1.3, M satisfies $(C3)_{\mathcal{X}}$. □

4.2 A Special Property

Given a class $\mathcal{X}$ of R-modules we say that an R-module M *satisfies property* $Q(\mathcal{X})$ ("Q" for quasi-continuous) if for each $\mathcal{X}$-submodule N and submodule L of M with $N \cap L = 0$ there exist submodules N', L' such that $M = N' \oplus L'$, $N \subseteq N'$ and $L \subseteq L'$. For example, our above discussion shows that any direct sum of injective modules satisfies $Q(\mathcal{G})$, where $\mathcal{G}$ is the class of finitely generated modules. Two extremes are given in the next result.

Proposition 4.2.1 (i) *An R-module M is quasi-continuous if and only if M satisfies $Q(\mathcal{M})$.*

(ii) *Every R-module satisfies $Q(\mathcal{I})$.*

Proof. (i) Clear by Proposition 3.3.7.

(ii) Let M be any R-module. Let N be an injective submodule and L be a submodule of M such that $N \cap L = 0$. Then $M = N \oplus N'$ for some submodule N' of M. Because N is N'-injective, Lemma 1.2.8 applies to give a submodule L' of M such that $M = N \oplus L'$ and $L \subseteq L'$. Thus M satisfies $Q(\mathcal{I})$. □

Now we make three elementary introductory observations. The first is the following:

Lemma 4.2.2 *Let $\mathcal{X}$ be any class of R-modules and let M be an R-module which satisfies $Q(\mathcal{X})$. Then any direct summand of M satisfies $Q(\mathcal{X})$.*

Proof. Suppose that M_1 and M_2 are submodules of M such that $M = M_1 \oplus M_2$. Let N be an $\mathcal{X}$-submodule of M_1 and let L be a submodule of M_1 such that $N \cap L = 0$. Consider the submodules N and $L \oplus M_2$ of M. By hypothesis, there exist submodules N' and L' of M such that $M = N' \oplus L'$, $N \subseteq N'$ and $L \oplus M_2 \subseteq L'$. Hence $L' = L' \cap (M_1 \oplus M_2) = M_2 \oplus (L' \cap M_1)$, $M = N' \oplus L' = N' \oplus (L' \cap M_1) \oplus M_2$ and $M_1 = (L' \cap M_1) \oplus [(N' + M_2) \cap M_1]$. Note that $N \subseteq N' \cap M_1 \subseteq (N' + M_2) \cap M_1$ and $L \subseteq L' \cap M_1$. Thus M_1 satisfies $Q(\mathcal{X})$. □

Our second elementary observation is the following:

Lemma 4.2.3 *Let $\mathcal{X}$ be any class of R-modules, let U be an $\mathcal{X}$-module and let M be any R-module such that the R-module $U \oplus M$ satisfies $Q(\mathcal{X})$. Then U is M-injective.*

Proof. Let L be any submodule of the module $X = U \oplus M$ such that $U \cap L = 0$. There exist submodules N' and L' of X such that $X = N' \oplus L'$, $U \subseteq N'$ and $L \subseteq L'$. Clearly $N' = U \oplus (N' \cap M)$ and $X = U \oplus U'$ where $U' = (N' \cap M) \oplus L'$. Note that $L \subseteq U'$. By Lemma 1.2.8, U is M-injective. □

Our third observation is as follows:

Lemma 4.2.4 *Let $\mathcal{X}$ be any class of R-modules and let M be an R-module which satisfies $Q(\mathcal{X})$. Let N be any $\mathcal{X}$-submodule of M and L be any complement of N in M. Then $M = N' \oplus L$ for some closure N' of N in M.*

Proof. Since $N \cap L = 0$, it follows that $M = N' \oplus L'$ for some submodules N', L' such that $N \subseteq N'$ and $L \subseteq L'$. But $L' \cap N = 0$ gives $L = L'$. Moreover, $N \oplus L$ essential in M gives $N = (N \oplus L) \cap N'$ essential in N'. Clearly N' is closed in M, so that N' is a closure of N in M. □

For an essentially closed class $\mathcal{X}$ we have the following corollary:

Corollary 4.2.5 *Let $\mathcal{X}$ be any essentially closed class of R-modules and let M be an R-module which satisfies $Q(\mathcal{X})$. Let N be an $\mathcal{X}$-submodule of M. Then $M = N' \oplus L'$ for any closure N' of N and complement L' of N in M.*

For any R-module class $\mathcal{X}$, a collection $\{M_i : i \in I\}$ of R-modules will be called *relatively $\mathcal{X}$-injective* if M_i is $(M_j, \mathcal{X})$-injective for every $i \neq j$ $(i, j \in I)$. A module M is called *$\mathcal{X}$-quasi-injective* if M is $(M, \mathcal{X})$-injective, i.e for every $\mathcal{X}$-submodule K of M, every $\varphi : K \longrightarrow M$ lifts to M.

Lemma 4.2.6 *Suppose that $\mathcal{X}$ is any class of R-modules. Let $M = M_1 \oplus M_2$ be a direct sum of submodules such that M has property $Q(\mathcal{X})$. Then M_1 and M_2 are relatively $\mathcal{X}$-injective.*

Proof. Let N be any $\mathcal{X}$-submodule of M with $N \cap M_2 = 0$. By hypothesis, there exists submodules M', M'' of M such that $M = M' \oplus M''$, $N \subseteq M'$ and $M_2 \subseteq M''$. Then $M'' = M_2 \oplus (M'' \cap M_1)$, so that

$$M = [M' \oplus (M'' \cap M_1)] \oplus M_2$$

and $N \subseteq M' \oplus (M'' \cap M_1)$. By Lemma 3.2.1, M_2 is $(M_1, \mathcal{X})$-injective. Similarly, M_1 is $(M_2, \mathcal{X})$-injective. □

Relative quasi-continuous modules have been considered by other authors. For example, Page [43] considers quasi-continuous modules relative to a torsion theory τ. Given an R-module M, a submodule N of M is called a *τ-summand* if there exists a submodule L of M such that $N \cap L = 0$ and $N \oplus L$ is a *τ-dense* submodule of M (i.e. $M/(N \oplus L)$ is τ-torsion). Then the module M is called *τ-quasi-continuous* if it satisfies the following properties:

$(C1)_\tau$ for every submodule N of M there exists a submodule K of M such that N is essential in K and K is a τ-summand of M, and

$(C3)_\tau$ if K and L are τ-summands of M with $K \cap L = 0$ then $K \oplus L$ is also a τ-summand of M.

He proves that a module M is τ-quasi-continuous if and only if for all submodules N, L of M with $N \cap L = 0$ there exist submodules N', L' of M such that $N' \cap L' = 0$, $N \subseteq N'$, $L \subseteq L'$ and $N' \oplus L'$ is τ-dense in M.

Oshiro [39] also considers relative quasi-continuous modules but his approach differs from that of Page. Let M be an R-module and let $\mathcal{B}$ be a non-empty collection of submodules of M such that

(a) if $B \in \mathcal{B}$ and A is a submodule of M such that $B \cong A$ then $A \in \mathcal{B}$, and

(b) if $B \in \mathcal{B}$ and A is an essential extension of B in M then $A \in \mathcal{B}$. Then Oshiro defines the module M to be $\mathcal{B}$-quasi-continuous if it has the following properties:

$(C1)_\mathcal{B}$ for any B in $\mathcal{B}$ there exists a direct summand A of M such that B is essential in A, and

$(C3)_{\mathcal{B}}$ for any B in $\mathcal{B}$ with B a direct summand of M, and direct summand K of M such that $B \cap K = 0$, the submodule $B \oplus K$ is also a direct summand of M.

Given such a collection $\mathcal{B}$ of submodules of M we determine that $\mathcal{B}_\circ = \mathcal{B} \cup \{0\}$ and we define $\mathcal{X}$ to be the collection of R-modules which are isomorphic to submodules in $\mathcal{B}_\circ$. Thus we have class of R-modules $\mathcal{X}$. Conversely, given any essentially closed class $\mathcal{X}$ of R-modules, let $\mathcal{B}$ denote the collection of $\mathcal{X}$-submodules of M. Then $\mathcal{B}$ is a non-empty collection of submodules of M satisfying Oshiro's conditions (a) and (b).

Lemma 4.2.7 *Let $\mathcal{X}$ be any class of R-modules and let M be an R-module with $Q(\mathcal{X})$. Then M is type 1 $\mathcal{X}$-extending and weak type 2 $\mathcal{X}$-extending.*

Proof. Let K be any $\mathcal{X}$-submodule of M and let L be a complement of K in M. By hypothesis, there exist submodules M_1, M_2 of M with $M = M_1 \oplus M_2$, $K \subseteq M_1$ and $L \subseteq M_2$. Since $K \cap M_2 = 0$ it follows that $L = M_2$. Thus M is type 1 $\mathcal{X}$-extending. Moreover, $K \oplus L$ is an essential submodule of M, and

$$(K \oplus L) \cap M_1 = K + (L \cap M_1) \subseteq K + (M_2 \cap M_1) = K \subseteq M_1,$$

so that K is essential in M_1. Thus M is weak type 2 $\mathcal{X}$-extending. □

Lemma 4.2.8 *Let M be any module with property $Q(\mathcal{X})$. Then M is $\mathcal{X}$-quasi-continuous.*

Proof. By Lemma 4.2.7, M is weak type 2 $\mathcal{X}$-extending. Now let K and L be direct summands of M such that $K \in \mathcal{X}$ and $K \cap L = 0$. There exist submodules M_1 and M_2 of M such that $M = M_1 \oplus M_2$, $K \subseteq M_1$ and $L \subseteq M_2$.

But $M = K \oplus K' = L \oplus L'$ for some submodules K', L'. Thus $M_1 = K \oplus (M_1 \cap K')$ and $M_2 = L \oplus (M_2 \cap L')$, so that $M = K \oplus L \oplus (M_1 \cap K') \oplus (M_2 \cap L')$. Then M satisfies $(C3)_{\mathcal{X}}$. $\square$

Question 4.2.9 *Is the converse of Lemma* 4.2.8 *true?*

Theorem 4.2.10 *Let $\mathcal{X}$ be any essentially closed class of R-modules. The following statements are equivalent for an R-module M.*

(i) *M satisfies $Q(\mathcal{X})$,*

(ii) *M is $\mathcal{X}$-quasi-continuous and type 1 $\mathcal{X}$-extending,*

(iii) *M is type 1 and type 2 $\mathcal{X}$-extending and M has $(C3)_{\mathcal{X}}$.*

Proof. (i) $\Rightarrow$ (iii). Suppose that M satisfies $Q(\mathcal{X})$. Then M satisfies $(C3)_{\mathcal{X}}$ by Lemma 4.1.3. By Lemma 4.2.7 and Corollary 1.6.8, M is type 2 $\mathcal{X}$-extending and type 1 $\mathcal{X}$-extending.

(ii) $\Leftrightarrow$ (iii). Clear by Corollary 1.6.8 and hypothesis.

(iii) $\Rightarrow$ (i). Let A, B be submodules of M such that $A \in \mathcal{X}$ and $A \cap B = 0$. Let K be a complement of A in M with $B \subseteq K$. Since M is type 1 $\mathcal{X}$-extending it follows that K is a direct summand of M. Because M is type 2 $\mathcal{X}$-extending, there exists a direct summand L of M such that A is essential in L. Because $\mathcal{X}$ is essentially closed, $L \in \mathcal{X}$. Also we have $L \cap K = 0$. Since M has $(C3)_{\mathcal{X}}$, $L \oplus K$ is a direct summand of M. Then $M = L \oplus K \oplus P$ for some submodule P and $A \subseteq L$, $B \subseteq K \oplus P$. Thus M has $Q(\mathcal{X})$. $\square$

4.3 Classes of Modules with Property $Q(\mathcal{X})$

Let R be any ring. The basic question we wish to consider in this section is if $\mathcal{X}$ and $\mathcal{Y}$ are classes of R-modules which are related in some way and M is an R-module which satisfies $Q(\mathcal{X})$, does M also satisfy $Q(\mathcal{Y})$? The first result is clear.

Lemma 4.3.1 *Let $\mathcal{X} \subseteq \mathcal{Y}$ be classes of R-modules. Then every R-module which satisfies $Q(\mathcal{Y})$ also satisfies $Q(\mathcal{X})$.*

Lemma 4.3.2 *Let $\mathcal{X}$ be any class of R-modules. Then a nonsingular R-module M satisfies $Q(\mathcal{X})$ if and only if M satisfies $Q(\mathcal{X}^e)$.*

Proof. Because $\mathcal{X} \subseteq \mathcal{X}^e$, the sufficiency follows by Lemma 4.3.1. Conversely, suppose that M satisfies $Q(\mathcal{X})$, let N be an $\mathcal{X}^e$-submodule of M and let L be a submodule of M with $N \cap L = 0$. There exists an $\mathcal{X}$-submodule K of M such that K is essential in N. Clearly $K \cap L = 0$ and hence $M = M_1 \oplus M_2$ for some submodules M_1, M_2 such that $K \subseteq M_1$ and $L \subseteq M_2$. Since N/K is singular it follows that $N/(N \cap M_1) \cong (N + M_1)/M_1$ is singular. However $M/M_1 \cong M_2$ which is nonsingular. Thus $N = N \cap M_1 \subseteq M_1$. Hence M satisfies $Q(\mathcal{X}^e)$. □

Recall that $\mathcal{U}$ denotes the class of modules of finite uniform dimension and $\mathcal{G}$ the class of finitely generated R-modules. Then $\mathcal{U} \subseteq \mathcal{G}^e$. By Lemma 4.3.1 and 4.3.2 any nonsingular module which satisfies $Q(\mathcal{G})$ also satisfies $Q(\mathcal{U})$. The converse is false. If R is any domain which is not right Öre then the right R-module R is nonsingular and has no uniform submodules, so satisfies $Q(\mathcal{U})$ vacuously, but does not satisfy $Q(\mathcal{G})$. If R is a right Noetherian ring then $\mathcal{G} \subseteq \mathcal{U}$

and in this case any R-module which satisfies $Q(\mathcal{U})$ also satisfies $Q(\mathcal{G})$ by Lemma 4.3.1.

The following two properties can be found in [28]:

Theorem 4.3.3 *Let $\{M_i : i \in I\}$ be a family of quasi-continuous modules. Then the following are equivalent:*

(i) *$M = \oplus_{i \in I} M_i$ is quasi-continuous,*

(ii) *$\oplus_{I \setminus \{i\}} M_j$ is M_i-injective for every $i \in I$,*

(iii) *M_i is M_k-injective for all $i \neq k \in I$ and (A_2) holds.*

Proof. See [28, Theorem 2.13]. □

Corollary 4.3.4 *Let $\{M_i : i \in I\}$ be any family of R-modules. Then $\oplus_{i=1}^{n} M_i$ is quasi-continuous if and only if each M_i is quasi-continuous and M_j-injective for all $j \neq i$.*

Proof. See [28, Corollary 2.14]. □

Lemma 4.3.5 *Let $M = \oplus_{i \in I} M_i$ where M_i is injective for all $i \in I$. Then M satisfies $Q(\mathcal{G})$.*

Proof. Let N be any finitely generated submodule of M and L be any submodule of M such that $L \cap N = 0$. Then there exists a finite subset J of I such that $N \subseteq \oplus_{j \in J} M_j$ which is injective. Let $\overline{N} = E(N)$ be a maximal essential extension of N in $\oplus_{j \in J} M_j$. Then $\overline{N}$ is a direct summand of $\oplus_{j \in J} M_j$. $\overline{N}$ is also a direct summand of M. Thus $M = \overline{N} \oplus N'$ for some submodule N' of M. Now $\overline{N}$ is N'-injective and $\overline{N} \cap L = 0$. Then by Lemma 1.2.8 there exists a submodule N'' of M such that $M = \overline{N} \oplus N''$ and $L \subseteq N''$. Therefore M satisfies $Q(\mathcal{G})$. □

Question 4.3.6 Is Lemma 4.3.5 true for the case $Q(\mathcal{U})$?

Theorem 4.3.7 *Let R be a right Noetherian ring and M be a right R-module. Then M satisfies $Q(\mathcal{U})$ if and only if M is quasi-continuous. In particular M satisfies $Q(\mathcal{G})$.*

Proof. Suppose first that M satisfies $Q(\mathcal{U})$. Then M is type 1 $\mathcal{U}$-extending by Lemma 4.2.7. By Corollary 2.2.19, $M = \oplus_{i\in I} M_i$, with M_i a uniform R-module for all $i \in I$.

Fix $i \in I$. For all $j \in I \setminus \{i\}$, $M_i \oplus M_j$ satisfies $Q(\mathcal{U})$ and $M_i \oplus M_j \in \mathcal{U}$. Then $M_i \oplus M_j$ is quasi-continuous. Thus M_i is M_j-injective by Corollary 4.3.4. By Theorem 4.3.3, M is quasi-continuous. The converse and the last part are clear. □

For a commutative domain R we also have the following result:

Proposition 4.3.8 *Let R be a commutative domain. Then the following statements are equivalent for a torsion-free R-module M:*

(i) *M is quasi-continuous;*

(ii) *M satisfies $Q(\mathcal{U})$;*

(iii) *M satisfies $Q(\mathcal{G})$.*

Proof. It is obvious that (i) implies both (ii), (iii); (iii) implies (ii) by Lemma 4.3.2 and 4.3.1; finally, we show (ii) implies (i). Let M be a torsion-free R-module with property $Q(\mathcal{U})$. Then M_R is type 1 $\mathcal{U}$-extending by Theorem 4.2.10. Thus M_R is extending by Theorem 2.2.24. By Theorem 2.2.23, $M = M_1 \oplus M_2$ where M_1 is injective and M_2 has finite uniform dimension, and so $M_2 \in \mathcal{U}$. Then M_2

has $Q(\mathcal{U})$ by Lemma 4.2.2, i.e., M_2 is quasi-continuous. Next, $M_1 = \oplus_{i \in I} M_{1i}$ where M_{1i} is indecomposable injective. By assumption for each $i \in I$, $M_{1i} \oplus M_2$ satisfies $Q(\mathcal{U})$ and hence $M_{1i} \oplus M_2$ is quasi-continuous since $M_{1i} \oplus M_2 \in \mathcal{U}$. Thus M_2 is M_{1i}-injective for every $i \in I$ by Corollary 4.3.4. Then M_2 is M_1-injective by Proposition 3.1.5. Therefore M is quasi-continuous by Theorem 4.3.3. □

Proposition 4.3.9 *Let n be a positive integer and let $\mathcal{X}_i$ $(1 \leq i \leq n)$ be classes of R-modules. Then an R-module M satisfies $Q(\mathcal{X}_1 \oplus ... \oplus \mathcal{X}_n)$ if and only if M satisfies $Q(\mathcal{X}_i)$ for all $1 \leq i \leq n$.*

Proof. The necessity follows by Lemma 4.3.1.

Conversely, suppose that M satisfies $Q(\mathcal{X}_i)$ for all $1 \leq i \leq n$. Let N be any $(\mathcal{X}_1 \oplus ... \oplus \mathcal{X}_n)$-submodule of M and let L be a submodule of M such that $N \cap L = 0$. Then $N = N_1 \oplus ... \oplus N_n$ for some $\mathcal{X}_i$-submodules N_i $(1 \leq i \leq n)$ of M. Now $N_1 \cap (N_2 \oplus ... \oplus N_n \oplus L) = 0$ so that $M = M_1 \oplus M_2$ for some submodules M_1, M_2 such that $N_1 \subseteq M_1$ and $N_2 \oplus ... \oplus N_n \oplus L \subseteq M_2$. By Lemma 4.2.2, M_2 satisfies $Q(\mathcal{X}_i)$ for all $2 \leq i \leq n$. By induction on n there exist submodules M_3, M_4 of M_2 such that $M_2 = M_3 \oplus M_4$, $N_2 \oplus ... \oplus N_n \subseteq M_3$ and $L \subseteq M_4$. Hence $M = (M_1 \oplus M_3) \oplus M_4$, $N_1 \oplus ... \oplus N_n \subseteq M_1 \oplus M_3$ and $L \subseteq M_4$. It follows that M satisfies $Q(\mathcal{X}_1 \oplus ... \oplus \mathcal{X}_n)$. □

Proposition 4.3.9 has the following immediate corollary:

Corollary 4.3.10 *Let $\mathcal{X}$ be any class of R-modules. Then an R-module M satisfies $Q(\mathcal{X})$ if and only if M satisfies $Q(\mathcal{X}^{\oplus})$.*

The next result is an analogue of Proposition 4.3.9. Note that if a module M satisfies $Q(\mathcal{X}_i^e)$ $(1 \leq i \leq n)$ then M satisfies $Q(\mathcal{X}_1^e \oplus ... \oplus \mathcal{X}_n^e)$ by Proposition 4.3.9. In fact we can say more.

Theorem 4.3.11 *Let n be a positive integer, let $\mathcal{X}_i$ $(1 \leq i \leq n)$ be classes of R-modules and let $\mathcal{X} = \mathcal{X}_1 \oplus ... \oplus \mathcal{X}_n$. Then an R-module M satisfies $Q(\mathcal{X}^e)$ if and only if M satisfies $Q(\mathcal{X}_i^e)$ for all $1 \leq i \leq n$.*

Proof. Since $\mathcal{X}_i \subseteq \mathcal{X}$ and hence $\mathcal{X}_i^e \subseteq \mathcal{X}^e$ for all $1 \leq i \leq n$, the necessity follows by Lemma 4.3.1.

Conversely, suppose that M satisfies $Q(\mathcal{X}_i^e)$ for all $1 \leq i \leq n$. Let N be an $\mathcal{X}^e$-submodule and let L be a submodule of M such that $N \cap L = 0$. There exists a closed submodule N' of M such that N is essential in N'. Note that $N' \cap L = 0$. There exist $\mathcal{X}_i$-submodules N_i $(1 \leq i \leq n)$ of N such that $N_1 \oplus ... \oplus N_n$ is essential in N. There exists a closed submodule N_1' of N' such that N_1 is essential in N_1'. By Lemma 1.2.4, N_1' is closed in M. By Zorn's Lemma there exists a complement L' of N_1 (or N_1') in M such that $N_2 \oplus ... \oplus N_n \oplus L \subseteq L'$. By Corollary 4.2.5, $M = N_1' \oplus L'$, because M satisfies $Q(\mathcal{X}_1^e)$.

Now $N' = N_1' \oplus (N' \cap L')$ and $(N_1 \oplus ... \oplus N_n) \cap (N' \cap L') = N_2 \oplus ... \oplus N_n$ is essential in $N' \cap L'$. Thus $N' \cap L' \in \mathcal{Y}^e$ where $\mathcal{Y} = \mathcal{X}_2 \oplus ... \oplus \mathcal{X}_n$. But L' satisfies $Q(\mathcal{X}_i^e)$ for all $2 \leq i \leq n$, by Lemma 4.2.2. By induction on n, L' satisfies $Q(\mathcal{Y}^e)$. There exist submodules P, Q of L' such that $L' = P \oplus Q$, $N' \cap L' \subseteq P$ and $L \subseteq Q$. Finally, note that $M = N_1' \oplus P \oplus Q$, $N \subseteq N' \subseteq N_1' \oplus P$ and $L \subseteq Q$. It follows that M satisfies $Q(\mathcal{X}^e)$. □

Corollary 4.3.12 *Any R-module M satisfies $Q(\mathcal{U})$ if and only if M satisfies $Q(\mathcal{U}_1)$.*

Proof. The class $\mathcal{U}_1$ is essentially closed and any module in $\mathcal{U}$ is an essential extension of a finite direct sum of uniform modules. Apply Theorem 4.3.11. □

Theorem 4.3.13 *Let n be a positive integer and let $\mathcal{X}_i$ $(1 \leq i \leq n)$ be essentially closed classes of R-modules such that $\mathcal{X}_i$ is closed under factor modules for all $2 \leq i \leq n$. Then M satisfies $Q(\mathcal{X}_1 + ... + \mathcal{X}_n)$ if and only if M satisfies $Q(\mathcal{X}_i)$ for all $1 \leq i \leq n$.*

Proof. The necessity follows by Lemma 4.3.1. Conversely, suppose that M satisfies $Q(\mathcal{X}_i)$ for all $1 \leq i \leq n$. Let N_i be an $\mathcal{X}_i$-submodule of M for each $1 \leq i \leq n$, let $N = N_1 + ... + N_n$ and let L be a submodule of M such that $N \cap L = 0$. There exists a closed submodule N' of M such that N is essential in N'. Clearly $N' \cap L = 0$. There exists a closed submodule H of N' such that N_1 is essential in H. Clearly $H \cap L = 0$. Let L' be a complement of H in M such that $L \subseteq L'$. Since $H \in \mathcal{X}_1$ there exist submodules M_1, M_2 of M such that $M = M_1 \oplus M_2$, $H \subseteq M_1$ and $L' \subseteq M_2$. Since $H \oplus L'$ is essential in M it follows that H is essential in M_1 and hence $H = M_1$.

Let $\pi : M \longrightarrow M_2$ denote the canonical projection. Then

$$\pi(N) = \pi(N_2) + ... + \pi(N_n) \in \mathcal{X}_2 + ... + \mathcal{X}_n.$$

Also $\pi(N) \subseteq M_1 + N = H + N \subseteq N'$, so that $\pi(N) \cap L = 0$. By Lemma 4.2.2, M_2 satisfies $Q(\mathcal{X}_i)$ for all $2 \leq i \leq n$ and by induction on n, there exist submodules M_3, M_4 of M_2 such that $M_2 = M_3 \oplus M_4$, $\pi(N) \subseteq M_3$ and $L \subseteq M_4$. Then $M = (M_1 \oplus M_3) \oplus M_4$, $N \subseteq M_1 + \pi(N) \subseteq M_1 \oplus M_3$ and $L \subseteq M_4$. It follows that M satisfies $Q(\mathcal{X}_1 + ... + \mathcal{X}_n)$. □

In view of Corollary 4.3.12 it is natural to ask whether any module with $Q(\mathcal{G}_1)$ also satisfies $Q(\mathcal{G})$.

4.4 Direct Sums

Let R be a ring and let M_i $(1 \leq i \leq n)$ be a finite collection of R-modules. We recall that the modules M_i $(1 \leq i \leq n)$ are *relatively injective* if M_i is M_j-injective for all $1 \leq i \neq j \leq n$. It is well known that the module $M = M_1 \oplus ... \oplus M_n$ is quasi-continuous if and only if the modules M_i $(1 \leq i \leq n)$ are quasi-continuous and relatively injective (see,for example, Corollary 4.3.4). We now generalise this fact by proving:

Theorem 4.4.1 *Let $\mathcal{X}$ be an essentially closed class of R-modules such that $\mathcal{X}$ is closed under submodules. Let M_i $(1 \leq i \leq n)$ be a finite collection of relatively injective R-modules. Then the R-module $M = M_1 \oplus ... \oplus M_n$ satisfies $Q(\mathcal{X})$ if and only if M_i satisfies $Q(\mathcal{X})$ for all $1 \leq i \leq n$.*

Proof. The necessity follows by Lemma 4.2.2.

Conversely, suppose that M_i satisfies $Q(\mathcal{X})$ for all $1 \leq i \leq n$. By induction on n, to prove that M satisfies $Q(\mathcal{X})$ we can suppose without loss of generality that $n = 2$. Let N be an $\mathcal{X}$-submodule and let L be a submodule of $M = M_1 \oplus M_2$ such that $N \cap L = 0$. Let N' be a closure of N in M. Because N is essential in N' we have $N' \in \mathcal{X}$ and $N' \cap L = 0$. Thus, without loss of generality, we can infer that $N = N'$, i.e. N is closed in M.

Suppose next that $N \cap M_1 = 0$. Because M_1 is M_2-injective, Lemma 1.2.8 allows us to assume, without loss of generality that $N \subseteq M_2$. Then Corollary 4.2.5 gives $M_2 = N \oplus H$ for any complement H of N in M_2. By Lemma 4.2.3, N is H-injective. But M_2 being M_1-injective implies N is M_1-injective and hence N is $(H \oplus M_1)$-injective see Corollary 3.1.5. But $M = N \oplus (H \oplus M_1)$ and $N \cap L = 0$ so that, applying Lemma 1.2.8 again, there exists a direct summand M' of M such that $M = N \oplus M'$ and $L \subseteq M'$.

In general, $N \cap M_2$ is an $\mathcal{X}$-submodule of M, because $\mathcal{X}$ is closed under submodules, and there exists a closed submodule K of N such that $N \cap M_2$ is essential in K. By Lemma 1.2.4, K is a closed submodule of M. Moreover K is an $\mathcal{X}$-submodule of M, $K \cap M_1 = 0$ and $K \cap L = 0$. By the above argument, $M = K \oplus K'$ for some submodule K' such that $L \subseteq K'$. Note that $N = K \oplus (N \cap K')$, so that $N \cap K'$ is a closed submodule of M by Lemma 1.2.4. Moreover $(N \cap K') \cap M_2 \subseteq K \cap K' = 0$. By the above argument, $M = (N \cap K') \oplus K''$ for some submodule K'' such that $L \subseteq K''$. Hence

$$M = K \oplus K' = K \oplus (N \cap K') \oplus (K' \cap K'') = N \oplus (K' \cap K''),$$

and $L \subseteq K' \cap K''$. It follows that M satisfies $Q(\mathcal{X})$. □

For any ring R, the class $\mathcal{U}$ of R-modules with finite uniform dimension is essentially closed and is also closed under submodules. Thus Theorem 4.4.1 has the following immediate corollary:

Corollary 4.4.2 *Let M_i $(1 \leq i \leq n)$ be a finite collection of relatively injective R-modules. Then the R-module $M = M_1 \oplus \ldots \oplus M_n$ satisfies $Q(\mathcal{U})$ if and only if M_i satisfies $Q(\mathcal{U})$ for all $1 \leq i \leq n$.*

Examples of classes of modules which are both essentially closed and closed under submodules include the class $\mathcal{T}$ of Goldie torsion modules and the class $\mathcal{F}$ of Goldie torsion-free (i.e. nonsingular) modules. As an application of Theorem 4.4.1 we next characterise modules which satisfy $Q(\mathcal{T})$.

Theorem 4.4.3 *An R-module M satisfies $Q(\mathcal{T})$ if and only if $M = Z_2(M) \oplus M'$ for some submodule M' of M such that $Z_2(M)$ is quasi-continuous and M'-injective.*

Proof. Suppose first that M satisfies $Q(\mathcal{T})$. Because $\mathcal{T}$ is essentially closed, $Z_2(M)$ is a closed $\mathcal{T}$-submodule of M and hence $M = Z_2(M) \oplus M'$ for some submodule M' of M by Lemma 4.2.4. By Lemma 4.2.3, $Z_2(M)$ is M'-injective and by Lemma 4.2.2 and Theorem 4.2.10, $Z_2(M)$ is quasi-continuous.

Conversely, suppose that $M = Z_2(M) \oplus M'$, $Z_2(M)$ is quasi-continuous and $Z_2(M)$ is M'-injective. Clearly $Hom(L, M') = 0$ for any submodule L of $Z_2(M)$, and hence M' is $Z_2(M)$-injective. Clearly also M' satisfies $Q(\mathcal{T})$. Moreover by Proposition 3.3.7, $Z_2(M)$ satisfies $Q(\mathcal{T})$. Finally, Theorem 4.4.1 gives that M satisfies $Q(\mathcal{T})$. □

Using Theorem 4.4.3, we can show that for the class $\mathcal{T}$, not every $\mathcal{T}$-quasi-continuous module satisfies $Q(\mathcal{T})$. For example, let S be a simple $\mathbb{Z}$-module and let M denote the $\mathbb{Z}$-module $S \oplus \mathbb{Z}$. Because S is not $\mathbb{Z}$-injective, Theorem 4.4.3 shows that M does not satisfy $Q(\mathcal{T})$. Since the only $\mathcal{T}$-submodules of M are 0 and S, it is easy to check that M satisfies $(C1)_{\mathcal{T}}$ and $(C3)_{\mathcal{T}}$, i.e. M is $\mathcal{T}$-quasi-continuous.

Theorem 4.4.1 for the class $\mathcal{T}$ is as follows:

Theorem 4.4.4 *Let M_i $(1 \leq i \leq n)$ be a finite collection of R-modules and let $M = M_1 \oplus ... \oplus M_n$. Then M satisfies $Q(\mathcal{T})$ if and only if M_i satisfies $Q(\mathcal{T})$ for all $1 \leq i \leq n$ and $Z_2(M_i)$ is M_j-injective for all $1 \leq i \neq j \leq n$.*

Proof. Suppose first that M satisfies $Q(\mathcal{T})$. By Lemma 4.2.2, M_i satisfies $Q(\mathcal{T})$ and hence, by Theorem 4.4.3, $M_i = Z_2(M_i) \oplus M_i'$ for some submodule M_i', for all $1 \leq i \leq n$. Let $1 \leq i \neq j \leq n$. Then $M_i \oplus M_j = Z_2(M_i) \oplus M_i' \oplus M_j$ satisfies $Q(\mathcal{T})$ and hence $Z_2(M_i) \oplus M_j$ satisfies $Q(\mathcal{T})$ by Lemma 4.2.2. By Lemma 4.2.3, $Z_2(M_i)$ is M_j-injective.

Conversely, suppose that M_i satisfies $Q(\mathcal{T})$ for all $1 \leq i \leq n$ and that $Z_2(M_i)$ is M_j-injective for all $1 \leq i \neq j \leq n$. To prove that M satisfies $Q(\mathcal{T})$, we can suppose, without loss of generality, that $n = 2$. By Theorem 4.4.3, for $i = 1, 2$, M_i contains a submodule M_i' such that $M_i = Z_2(M_i) \oplus M_i'$. Then

$$M = M_1 \oplus M_2 = Z_2(M_1) \oplus Z_2(M_2) \oplus M_1' \oplus M_2' = Z_2(M) \oplus M',$$

where $M' = M_1' \oplus M_2'$. By hypothesis, the modules $Z_2(M_1)$ and $Z_2(M_2)$ are relatively injective and satisfy $Q(\mathcal{T})$. Hence $Z_2(M)$ satisfies $Q(\mathcal{T})$, i.e. $Z_2(M)$ is quasi-continuous (see Theorem 4.4.1 and Proposition 3.3.7). Moreover, by hypothesis $Z_2(M_1)$ is M_1'-injective and $Z_2(M_1)$ is M_2'-injective. Thus $Z_2(M_1)$ is M'-injective. Similarly $Z_2(M_2)$ is M'-injective. Thus $Z_2(M)$ is M'-injective. By Theorem 4.4.3, M satisfies $Q(\mathcal{T})$. □

There is an analogue to each of Theorems 4.4.3 and 4.4.4 for the class $\mathcal{F}$ of nonsingular R-modules.

Theorem 4.4.5 *An R-module M satisfies $Q(\mathcal{F})$ if and only if $M = Z_2(M) \oplus M'$ for some quasi-continuous submodule M' of M such that $Z_2(M)$ is M'-*

injective.

Proof. Suppose first that M satisfies $Q(\mathcal{F})$. Let M' be a complement of $Z_2(M)$ in M. Then M' is an $\mathcal{F}$-submodule of M and $Z_2(M)$ is a complement of M' in M. By Corollary 4.2.5, $M = Z_2(M) \oplus M'$. By Proposition 3.3.7 and Lemma 4.2.2, M' is quasi-continuous. Let N be any submodule of M such that $N \cap Z_2(M) = 0$. Clearly N is an $\mathcal{F}$-submodule of M and, by hypothesis, $M = N' \oplus L'$ for some submodules N', L' such that $N \subseteq N'$ and $Z_2(M) \subseteq L'$. It follows that $L' = Z_2(M) \oplus (M' \cap L')$ by the Modular Law and hence $M = Z_2(M) \oplus M''$ where $M'' = N' \oplus (M' \cap L')$ is a submodule of M with $N \subseteq M''$. By Lemma 1.2.8, $Z_2(M)$ is M'-injective.

Conversely, suppose that $M = Z_2(M) \oplus M'$ for some quasi-continuous submodule M' such that $Z_2(M)$ is M'-injective. It is not difficult to see that M' is $Z_2(M)$-injective and that $Z_2(M)$ satisfies $Q(\mathcal{F})$. Finally, by Theorem 4.4.1, M satisfies $Q(\mathcal{F})$. □

Corollary 4.4.6 *Let M_i $(1 \leq i \leq n)$ be a finite collection of R-modules and let $M = M_1 \oplus ... \oplus M_n$. Then M satisfies $Q(\mathcal{F})$ if and only if $M_i = Z_2(M_i) \oplus M_i'$ for some quasi-continuous submodule M_i' such that $Z_2(M_i)$ is M_i'-injective for all $1 \leq i \leq n$ and M_i is M_j'-injective for all $1 \leq i \neq j \leq n$.*

Proof. Suppose first that M satisfies $Q(\mathcal{F})$. By Lemma 4.2.2, M_i satisfies $Q(\mathcal{F})$ and hence, by Theorem 4.4.5, $M_i = Z_2(M_i) \oplus M_i'$ for some quasi-continuous submodule M_i' such that $Z_2(M_i)$ is M_i'-injective, for all $1 \leq i \leq n$. Let $1 \leq i \neq j \leq n$. Then $M_i \oplus M_j = Z_2(M_i) \oplus M_i' \oplus Z_2(M_j) \oplus M_j'$ satisfies $Q(\mathcal{F})$ and hence so too does $Z_2(M_i) \oplus M_j'$, by Lemma 4.2.2. By Theorem 4.4.5, $Z_2(M_i)$ is M_j'-injective.

Moreover, by Lemma 4.2.2, $M_i' \oplus M_j'$ satisfies $Q(\mathcal{F})$ and M_i' is M_j'-injective by Lemma 4.2.3 because M_i' is an $\mathcal{F}$-submodule. Thus M_i is M_j'-injective.

Conversely, suppose that the modules M_i $(1 \leq i \leq n)$ have the stated conditions. To prove that M satisfies $Q(\mathcal{F})$, we can suppose without loss of generality that $n = 2$. Now $M = Z_2(M) \oplus M'$ where $Z_2(M) = Z_2(M_1) \oplus Z_2(M_2)$ and $M' = M_1' \oplus M_2'$. By hypothesis $Z_2(M_1)$ is M'-injective and so too is $Z_2(M_2)$. Thus $Z_2(M)$ is M'-injective. Moreover, by Theorem 4.4.1, M' is a quasi-continuous module. Finally, Theorem 4.4.5 gives that M satisfies $Q(\mathcal{F})$.

□

Note in particular that Theorems 4.4.3 and 4.4.5 together give that a module M is quasi-continuous if and only if M satisfies $Q(\mathcal{T})$ and $Q(\mathcal{F})$ (see Corollary 4.3.4).

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