Some generalizations of Injectivity

by

Catarina Araújo de Santa Clara Gomes

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Statement

This thesis is submitted in accordance with the regulations for the degree of Doctor of Philosophy at the University of Glasgow.

Chapter 1 covers known results; Chapters 2 and 3 are the author's own work, with the exception of some instances indicated within the text.

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Contents

Abstract

Divisible groups, and then injective modules, were introduced by Baer [3], Eckmann and Schopf [14].

Injective objects play an important role in other categories than the category of modules. In the category of Banach spaces, the Hahn-Banach Theorem states that the field of real numbers is injective; in the category of Boolean algebras, a complete Boolean algebra is injective; in the category of normal topological spaces, the closed interval $[0, 1]$ is injective, by Tietze's Theorem; in the category of partially ordered sets, the injective envelope of a partially ordered set is its MacNeille completion. Some of the results of injectives in the category of modules can be carried over into these other categories; namely the result that a direct product of injectives is injective.

The concept of injectivity and some of its generalizations has attracted much interest over the years.

Quasi-injective modules were first defined by Johnson and Wong [29]. Jeremy [27, 28] considered continuous and quasi-continuous modules, following work of Utumi [60], on rings, and earlier work of Von Neumann [44, 45, 46], on continuous geometries. Continuous and quasi-continuous modules were studied by various authors. For a good account of this theory,

see the monographs by Mohamed and Müller [40], and Dung, Huynh, Smith and Wisbauer [13].

Goldie [17, 18] considered complements in his study of quotient rings. Following the work of Goldie, Chatters and Hajarnavis [8] studied extending modules. Independently, extending modules also arose in the work of Harada and his collaborators [22, 23, 49, 50]. Kamal and Muller [31, 32, 33] developed the theory and are responsible for discovering a number of interesting properties.

Some generalizations of extending modules appear in [7, 57], for example.

Extending modules have been studied extensively in recent years and it appears that several classical theorems on injective modules have natural generalizations for extending modules. However, in some sense, the extending property is quite far from injectivity and several questions on extending modules still remain unsolved. A very intriguing question is to find necessary and sufficient conditions for a direct sum of extending modules to be extending. We obtain answers for this problem, in some special cases, and also consider the same problem for generalizations of extending modules.

Chapter 1 covers the background necessary for what follows. In particular, general properties of injectivity and some of its well-known generalizations are stated.

Chapter 2 is concerned with two generalizations of injectivity, namely near and essential injectivity. These concepts, together with the notion of the exchange property, prove to be a key tool in obtaining characterizations of when the direct sum of extending modules is extending.

We find sufficient conditions for a direct sum of two extending modules to be extending, generalizing several known results. We characterize when the direct sum of an extending module and an injective module is extending and when the direct sum of an extending module with the finite exchange property and a semisimple module is extending. We also characterize when the direct sum of a uniform-extending module and a semisimple module is uniform-extending and, in consequence, we prove that, for a right Noetherian ring R, an extending right R-module M_1 and a semisimple right R-module M_2, the right R-module $M_1 \oplus M_2$ is extending if and only if M_2 is $M_1/\mathrm{Soc}(M_1)$-injective.

Chapter 3 deals with the class of self-c-injective modules, that can be characterized by the lifting of homomorphisms from closed submodules to the module itself.

We prove general properties of self-c-injective modules and find sufficient conditions for a direct sum of two self-c-injective to be self-c-injective. We also look at self-cu-injective modules, i.e., modules M such that every homomorphism from a closed uniform submodule to M can be lifted to M itself.

We prove that every self-c-injective free module over a commutative domain that is not a field is finitely generated and then proceed to consider torsion-free modules over commutative domains, as was done for extending modules in [31].

We also characterize when, over a principal ideal domain, the direct sum of a torsion-free injective module and a cyclic torsion module is self-cu-injective.

Chapter 1

Background

In this preliminary chapter, we will fix some notation and state a few well-known results that will be used in the sequel. For other basic definitions, results and notations, we refer the reader to [2, 13, 15, 36, 40, 65] as background references.

1.1 Preliminaries

Notation

Throughout this dissertation, let R be a ring with identity and let all modules be unitary right R-modules.

If N is a submodule of M, we write $N \leq M$; if N is a direct summand of M, we write $N \leq_d M$.

For right modules M and N, $\text{Hom}_R(M, N)$ will denote the set of R-module homomorphisms from M to N. The kernel of any $\alpha \in \text{Hom}_R(M, N)$ is denoted by $\ker \alpha$ and its image by $\alpha(M)$. $\text{End}_R(M)$ will denote the set of endomorphisms of M.

Given a family of modules $\{ M_i \mid i \in I \}$, for each $j \in I$, $\pi_j : \oplus_{i\in I} M_i \to M_j$ denotes the canonical projection with kernel $\oplus_{i\in I\setminus\{j\}} M_i$.

Essential submodules

Let M be any module. A submodule N of M is called *essential*, or *large*, in M if $N \cap K \neq 0$, for every $0 \neq K \leq M$. If N is essential in M, we write $N \leq_e M$.

Some basic facts about essential submodules are stated below.

Proposition 1.1.1 [13, 1.5] *Let K and N be submodules of a module M.*

(i) If $K \leq N$, then $K \leq_e M$ if and only if $K \leq_e N$ and $N \leq_e M$.

(ii) If $N \leq_e M$, then $N \cap K \leq_e K$.

(iii) If $N, K \leq_e M$, then $N \cap K \leq_e M$.

(iv) If $K \leq N$ and $N/K \leq_e M/K$, then $N \leq_e M$.

(v) If $M = \oplus_{i\in I} M_i$ and $N_i \leq_e M_i$, for every $i \in I$, then $\oplus_{i\in I} N_i \leq_e M$.

Complements

Let M be any module. A submodule K of M is called *closed* in M provided K has no proper essential extensions in M, i.e., whenever N is a submodule of M such that $K \leq_e N$, then $K = N$. If K is closed in M, we write $K \leq_c M$.

Given a submodule N of M, a submodule K of M is called a *complement of N* in M if K is maximal in the collection of submodules L of M such that

$L \cap N = 0$. A submodule K of M is called a *complement* in M, if there exists a submodule N of M such that K is a complement of N in M.

An easy application of Zorn's Lemma guarantees the existence of complements. In fact, we can prove the following facts.

Lemma 1.1.2 [13, 1.10] *Let L and N be submodules of a module M such that $L \cap N = 0$.*

(i) There exists a complement K of N such that $L \leq K$.

(ii) $K \oplus N \leq_e M$.

(iii) $K \leq_c M$.

It turns out that a submodule of a module M is closed in M if and only if it is a complement in M. This is a consequence of the following.

Lemma 1.1.3 [13, 1.10] *Let K be a submodule of a module M and let L be a complement of K. Then K is closed in M if and only if K is a complement of L in M.*

We now list some basic properties of complements.

Proposition 1.1.4 *Let L and K be submodules of a module M, with $K \leq L$.*

(i) For every $N \leq M$, there exists $H \leq_c M$ such that $N \leq_e H$.

(ii) $K \leq_c M$ if and only if, whenever $N \leq_e M$ is such that $K \leq N$, then $N/K \leq_e M/K$.

(iii) *If* $K \leq_c L$ *and* $L \leq_c M$*, then* $K \leq_c M$.

(iv) *If* $L \leq_c M$*, then* $L/K \leq_c M/K$.

(v) *If* $K \leq_c M$*, then the closed submodules of* M/K *are of the form* H/K*, where* $H \leq_c M$ *and* $K \leq H$.

Proof. A proof for (i)–(iv) can be found in [13, 1.10].

Suppose that $K \leq_c M$ and let us prove (v). By (iv), $H/K \leq_c M/K$, for every $H \leq_c M$ such that $K \leq H$. Assume now that $H \leq M$ is such that $K \leq H$ and $H/K \leq_c M/K$, and let us prove that $H \leq_c M$. If $N \leq M$ is such that $H \leq_e N$, then, by (ii), $H/K \leq_e N/K$. Because $H/K \leq_c M/K$, we can conclude that $H = N$ and that $H \leq_c M$. □

Uniform submodules

A non-zero module U is said to be *uniform* if any two non-zero submodules of U have non-zero intersection, i.e., if every non-zero submodule of U is essential in U.

Examples of uniform modules are, for an arbitrary ring, simple modules and non-zero submodules of uniform modules. If R is a commutative ring and P is a prime ideal of R, the R-module R/P is uniform. If R is a commutative domain, then its field of fractions is a uniform R-module.

Moreover, for the ring $\mathbb{Z}$, the following are examples of uniform $\mathbb{Z}$-modules.

(i) Cyclic groups $\mathbb{Z}/\mathbb{Z}p^n$ of order p^n, for any prime p and $n \in \mathbb{N}$.

(ii) Prüfer groups $\mathbb{Z}(p^\infty)$ of type p^∞, for any prime p.

(iii) Submodules of the additive group $(\mathbb{Q}, +)$ of rational numbers.

Uniform dimension

Let M be a non-zero module which does not contain a direct sum of an infinite number of non-zero submodules. Then M contains a uniform submodule. Moreover, there exist a positive integer n and independent uniform submodules $U_1, \ldots, U_n$ of M such that $U_1 \oplus \cdots \oplus U_n$ is an essential submodule of M. This positive integer n is an invariant of M, i.e., if k is a positive integer and $V_1, \ldots, V_k$ are independent uniform submodules of M such that $V_1 \oplus \cdots \oplus V_k$ is an essential submodule of M, then $n = k$. We shall call n the *uniform dimension*, or *Goldie dimension*, of M and shall denote it by $u.dim(M)$. The uniform dimension of the zero module is, by definition, 0. If M contains a direct sum of an infinite number of non-zero submodules, then we set the uniform dimension of M to be ∞. For more details, see [20], for example.

Annihilators

Let M be a module and let X be a subset of M. The *right annihilator* of X in R will be denoted by $r(X)$, *i.e.*, $r(X) := \{\, r \in R \mid xr = 0, \text{ for all } x \in X \,\}$.

Given $a \in M$, let $r(a) := r(\{a\})$, and let $(X : a)$ denote the set $\{\, r \in R \mid ar \in X \,\}$. Clearly, if X is a submodule of M, then $(X : a)$ is the right annihilator of $\{a + X\}$ in R, for every $a \in M$.

It is a simple observation that $R/r(a)$ is isomorphic to aR, for every $a \in M$. Also, if $X \leq_e M$, then $(X : a) \leq_e R_R$. These facts will be repeatedly used in the sequel.

Singular and nonsingular modules

For a module M, the *singular submodule* of M will be denoted by $Z(M)$, i.e.,

$$Z(M) := \{\, x \in M \mid xE = 0, \text{ for some } E \leq_e R_R \,\}.$$

The *second singular submodule* of M, denoted by $Z_2(M)$, is the submodule containing $Z(M)$ such that $Z_2(M)/Z(M)$ is the singular submodule of the factor module $M/Z(M)$.

Recall that the module M is called *singular* if $M = Z(M)$ and is called *nonsingular* if $Z(M) = 0$.

Clearly, $Z(M)$ is singular; in fact, it is the largest singular submodule of M. Moreover, $M/Z_2(M)$ is nonsingular and $Z(M) \leq_e Z_2(M) \leq_c M$ (see, for example, [58]).

Proposition 1.1.5 [20, Proposition 3.26] *A module A is singular if and only if it is isomorphic to B/C, for some module B and essential submodule C of B.*

Proposition 1.1.6 [20, Proposition 3.27] *Let B be a submodule of a nonsingular module A. Then A/B is singular if and only if $B \leq_e A$.*

Proposition 1.1.7 [20, Proposition 3.28]

(i) All submodules, factor modules, and sums (direct or not) of singular modules are singular.

(ii) All submodules, direct products, and essential extensions of nonsingular modules are nonsingular.

(iii) Let B be a submodule of a module A. If B and A/B are both nonsingular, then A is nonsingular.

Socle

Recall that the *socle* of a module M is defined to be the sum of all simple submodules of M, or to be the zero submodule, in case in case M has no simple submodules. The socle of M will be denoted by $\mathrm{Soc}(M)$. In the following result we gather some basic facts about $\mathrm{Soc}(M)$ (see [2, Section 9]).

Lemma 1.1.8 *Let M be a module.*

(i) $\mathrm{Soc}(M)$ *is semisimple (i.e., is a direct sum of simple submodules).*

(ii) $\mathrm{Soc}(M) = \cap\{ L \mid L \leq_e M \}$.

(iii) $\mathrm{Soc}(N) = N \cap \mathrm{Soc}(M)$*, for every submodule N of M.*

(iv) $\varphi(\mathrm{Soc}(M)) \leq \mathrm{Soc}(M')$*, for every module M' and $\varphi \in \mathrm{Hom}(M, M')$.*

(iv) If $M = \oplus_{i \in I} M_i$, for some submodules M_i, $i \in I$, of M, then $\mathrm{Soc}(M) = \oplus_{i \in I}\mathrm{Soc}(M_i)$.

Noetherian modules

A module M is called *Noetherian* if it satisfies the ascending chain condition (ACC) on submodules, or, equivalently, if every submodule of M is finitely generated.

The ring R is *right Noetherian* if the module R_R is Noetherian.

A module M is said to be *locally Noetherian* if every finitely generated submodule of M is Noetherian. Any module over a right Noetherian ring is locally Noetherian.

V-modules

A module M is called a *V-module* if every submodule is the intersection of maximal submodules, or, equivalently, if every simple module is M-injective (see [13] or [65]). The ring R is said to be a *right V-ring* if R_R is a V-module.

Projective and hereditary modules

Let M_1 and M_2 be modules. The module M_2 is M_1-*projective* in case for each epimorphism $\alpha : M_1 \to A$ and each homomorphism $\beta : M_2 \to A$, where A is any module, there exists a homomorphism $\gamma : M_2 \to M_1$ such that $\beta = \alpha\gamma$.

A module M is called *hereditary* if every submodule of M is projective.

1.2 Injectivity

Let M_1 and M_2 be modules. The module M_2 is M_1-*injective* if every homomorphism $\alpha : A \to M_2$, where A is a submodule of M_1, can be extended to a homomorphism $\beta : M_1 \to M_2$.

A family of modules $\{ M_i \mid i \in I \}$ is *relatively injective* if M_i is M_j-injective, for every $i, j \in I$, $i \neq j$.

A module M is called *injective* when it is N-injective, for every module N.

The following result is known as Baer's Criterion.

Theorem 1.2.1 [2, 18.3] *The following conditions are equivalent for a module M.*

(i) M is injective.

(ii) M is R_R-injective.

(iii) For every right ideal $I \leq R_R$ and every homomorphism $\alpha : I \to M$, there exists $a \in M$ such that $\alpha(r) = ar$, for every $r \in I$.

Some basic properties of injectivity follow below.

Proposition 1.2.2 [40, Proposition 1.3] *Let M_1 and M_2 be modules. If M_2 is M_1-injective, then, for every submodule N of M_1, M_2 is N-injective and (M_1/N)-injective.*

Proposition 1.2.3 [40, Proposition 1.5] *Let $\{ M_i \mid i \in I \}$ and N be modules. Then N is $(\oplus_{i \in I} M_i)$-injective if and only if N is M_i-injective, for every $i \in I$.*

Proposition 1.2.4 [40, Proposition 1.6] *Let M and $\{ N_i \mid i \in I \}$ be modules. Then $\Pi_{i \in I} N_i$ is M-injective if and only if N_i is M-injective, for every $i \in I$.*

The following result is a generalization of Baer's Criterion.

Proposition 1.2.5 [40, Proposition 1.4] *Let M_1 and M_2 be modules. Then M_2 is M_1-injective if and only if M_2 is aR-injective, for every $a \in M_1$.*

Theorem 1.2.6 [40, Theorem 1.7] *Let $\{ N_i \mid i \in I \}$ be a family of modules. For a module M, the following conditions are equivalent.*

(i) $\oplus_{i \in I} N_i$ *is M-injective.*

(ii) $\oplus_{i \in J} N_i$ *is M-injective, for every countable subset J of I.*

(iii) N_i *is M-injective, for every $i \in I$, and for any choice of $x_n \in N_{i_n}$, with $n \in \mathbb{N}$ and distinct $i_n \in I$, such that $\cap_{n=1}^{\infty} r(x_n) \supseteq r(a)$, for some $a \in M$, the ascending chain*

$$\bigcap_{n=1}^{\infty} r(x_n) \subseteq \bigcap_{n=2}^{\infty} r(x_n) \subseteq \cdots \subseteq \bigcap_{n=k}^{\infty} r(x_n) \subseteq \cdots$$

becomes stationary.

Motivated by these results, the following chain conditions on the ring R, relative to a given family $\{ M_i \mid i \in I \}$ of R-modules, were introduced in [40, page 4]. Here we will follow their notation.

(A_1) For any choice of $x_n \in M_{i_n}$, with $n \in \mathbb{N}$ and distinct $i_n \in I$, the ascending chain

$$\bigcap_{n=1}^{\infty} r(x_n) \subseteq \bigcap_{n=2}^{\infty} r(x_n) \subseteq \cdots \subseteq \bigcap_{n=k}^{\infty} r(x_n) \subseteq \cdots$$

becomes stationary.

(A_2) For any choice of $x_n \in M_{i_n}$, with $n \in \mathbb{N}$ and distinct $i_n \in I$, such that $\cap_{n=1}^{\infty} r(x_n) \supseteq r(y)$, for some $y \in M_j$ $(j \in I)$, the ascending chain

$$\bigcap_{n=1}^{\infty} r(x_n) \subseteq \bigcap_{n=2}^{\infty} r(x_n) \subseteq \cdots \subseteq \bigcap_{n=k}^{\infty} r(x_n) \subseteq \cdots$$

becomes stationary.

Clearly, (A_1) implies (A_2). Examples showing the converse does not hold are discussed in [40, Examples 1.12].

If R is right Noetherian, then (A_1) holds. Suppose that, for each $i \in I$, M_i is a locally Noetherian module. Then, for every $y \in M_i$ $(i \in I)$, $R/r(y)$ is Noetherian, as it is isomorphic to yR. Consequently, condition (A_2) is satisfied.

As an immediate consequence of Theorem 1.2.6, we have the following result.

Proposition 1.2.7 [40, Proposition 1.9] *Let $\{ M_i \mid i \in I \}$ be a family of modules. Then $\oplus_{i \in I\setminus\{j\}} M_i$ is M_j-injective, for every $j \in I$, if and only if the modules $\{ M_i \mid i \in I \}$ are relatively injective and condition (A_2) holds.*

By Proposition 1.2.4, a direct product of injective modules, and hence a finite direct sum of injective modules, is injective. The following result characterizes the injectivity of arbitrary direct sums of modules and is a consequence of Theorem 1.2.6.

Theorem 1.2.8 [40, Proposition 1.10] *Let $\{ M_i \mid i \in I \}$ be a family of modules. The following conditions are equivalent.*

(i) $\oplus_{i \in I} M_i$ is injective.

(ii) $\oplus_{i \in J} M_i$ is injective, for every countable subset J of I.

(iii) M_i is injective, for every $i \in I$, and condition (A_1) holds.

Theorem 1.2.9 [40, Theorem 1.11] *For a module M, the direct sum of any family of M-injective modules is M-injective if and only if M is locally*

Noetherian. In particular, the direct sum of every family of injective R-modules is injective if and only if R is right Noetherian.

Every module M has a minimal injective extension, which is at the same time a maximal essential extension. Such an extension is unique up to isomorphism, is called the *injective hull*, or *injective envelope*, of M and is denoted by $E(M)$.

1.3 Quasi-injectivity

A module M is called *quasi-injective*, or *self-injective*, when it is M-injective.

For example, injective modules and semisimple modules are quasi-injective and direct summands of quasi-injective modules are also quasi-injective.

Some known properties of quasi-injective modules are listed below.

Lemma 1.3.1 [40, Corollary 1.14] *A module M is quasi-injective if and only if $\varphi(M) \leq M$, for every endomorphism φ of $E(M)$.*

Theorem 1.3.2 [40, Theorem 1.18] *Let $\{ M_i \mid i \in I \}$ be a family of modules. The following conditions are equivalent.*

(i) $\oplus_{i\in I} M_i$ *is quasi-injective.*

(ii) $\oplus_{i\in J} M_i$ *is quasi-injective, for every countable subset J of I.*

(iii) M_j *is quasi-injective and* $\oplus_{i\in I\setminus\{j\}} M_i$ *is M_j-injective, for every $j \in I$.*

(iv) $\{ M_i \mid i \in I \}$ *are relatively injective quasi-injective modules and condition (A_2) holds.*

1.4 Quasi-continuous modules

A module M is called a *quasi-continuous module* if $\epsilon(M) \leq M$, for every idempotent endomorphism ϵ of $E(M)$.

Clearly, quasi-injective modules are examples of quasi-continuous modules, and so are uniform modules.

The following gives some equivalent characterizations of quasi-continuous modules, that can be found, for example, in [13, 40, 55].

Theorem 1.4.1 *The following conditions are equivalent for a module M with injective hull E.*

(i) *M is quasi-continuous.*

(ii) *Whenever $E = \oplus_{i \in I} E_i$, for submodules E_i $(i \in I)$ of E, then $M = \oplus_{i \in I}(M \cap E_i)$.*

(iii) *Whenever $E = E_1 \oplus E_2$, for submodules E_1, E_2 of E, then $M = (M \cap E_1) \oplus (M \cap E_2)$.*

(iv) *Every submodule of M is essential in a direct summand of M and, for any direct summands K and L of M with $K \cap L = 0$, the submodule $K \oplus L$ is also a direct summand of M.*

(v) *Whenever L_1 and L_2 are submodules of M with $L_1 \cap L_2 = 0$, then there exist submodules M_1 and M_2 of M such that $M = M_1 \oplus M_2$ and $L_i \leq M_i$, $i = 1, 2$.*

(vi) *Whenever L_1 and L_2 are closed submodules of M with $L_1 \cap L_2 = 0$, then the submodule $L_1 \oplus L_2$ is also a closed submodule of M.*

(vii) *Whenever* L_1 *and* L_2 *are closed submodules of* M *with* $L_1 \cap L_2 = 0$, *then every homomorphism* $\varphi : L_1 \oplus L_2 \to M$ *can be lifted to a homomorphism* $M \to M$.

Theorem 1.4.2 [40, Theorem 2.13] *Let* $\{ M_i \mid i \in I \}$ *be a family of modules. The following conditions are equivalent.*

(i) $\oplus_{i \in I} M_i$ *is quasi-continuous.*

(ii) $\oplus_{i \in J} M_i$ *is quasi-continuous, for every countable subset* J *of* I.

(iii) M_j *is quasi-continuous and* $\oplus_{i \in I \setminus \{j\}} M_i$ *is* M_j*-injective, for every* $j \in I$.

(iv) $\{ M_i \mid i \in I \}$ *are relatively injective quasi-continuous modules and condition* (A_2) *holds.*

1.5 Continuous modules

There is a class of modules intermediate to the class of quasi-injective modules and the class of quasi-continuous modules, namely the class of continuous modules.

A module M is called a *continuous module* if it has the following two properties:

(C_1) Every submodule of M is essential in a direct summand.

(C_2) Every submodule isomorphic to a direct summand of M is also a direct summand of M.

It is not hard to prove that every quasi-injective module is continuous.

Theorem 1.5.1 [40, Theorem 3.16] *Let* $\{ M_i \mid i \in I \}$ *be a family of modules. The following conditions are equivalent.*

(i) $\oplus_{i \in I} M_i$ *is continuous.*

(ii) $\oplus_{i \in J} M_i$ *is continuous, for every countable subset* J *of* I*.*

(iii) $\oplus_{i \in I} M_i$ *is quasi-continuous and* M_i *is continuous, for every* $i \in I$*.*

(iv) $\{ M_i \mid i \in I \}$ *are relatively injective continuous modules and condition* (A_2) *holds.*

1.6 Extending modules

A module M is called an *extending module*, or a *CS module*, if every submodule of M is essential in a direct summand, or, equivalently, if every closed submodule of M is a direct summand.

By Theorem 1.4.1, quasi-continuous modules are extending.

It is obvious that an indecomposable module is extending if and only if it is uniform.

1.7 Uniform-extending modules

A module M is called a *uniform-extending module* if every closed uniform (i.e., maximal uniform) submodule of M is a direct summand.

Lemma 1.7.1 [13, Lemma 7.7] *Let* M *be a uniform-extending module and let* K *be a closed submodule of* M *with finite uniform dimension. Then* K *is a direct summand of* M*.*

Corollary 1.7.2 [13, Corollary 7.8] *A module with finite uniform dimension is extending if and only if it is uniform-extending.*

1.8 Some examples

To illustrate the hierarchy of the concepts introduced in the previous Sections (injective, quasi-injective, continuous, quasi-continous, extending modules), and at the same time demonstrate that they are all distinct, in [40, page 19], are listed all abelian groups with these properties, as well as regular rings, as right modules over themselves, with these properties.

Let us look now at examples of uniform-extending modules that are not extending.

Let us start by proving that a free $\mathbb{Z}$-module M is extending if and only if M has finite uniform dimension. If M has infinite uniform dimension, then there exists an epimorphism $\alpha : M \rightarrow \mathbb{Q}$. It is not hard to see that $K := \ker\alpha$ is a closed submodule that is not a direct summand, and hence M is not extending. On the other hand, if M has finite uniform dimension, then any submodule N of M is essential in the submodule L such that L/N is the torsion submodule of M/N; and L is a direct summand of M, because the module M/L is finitely generated and torsion-free.

It is not hard to prove that any free $\mathbb{Z}$-module of infinite uniform dimension is uniform-extending (but not extending).

1.9 The exchange property

A module M is said to have the *(finite) exchange property* if, for every (finite) index set I, whenever $M \oplus N = \oplus_{i \in I} A_i$ for modules N and A_i, $i \in I$, then $M \oplus N = M \oplus (\oplus_{i \in I} B_i)$ for submodules B_i of A_i, $i \in I$ (see [13] or [40]).

The exchange property was introduced in [9] and was established for injective modules in [63], for quasi-injective modules in [16] and for continuous modules in [39] (cf. [40, Theorem 3.24]). In [42], it was proved that, for quasi-continuous modules, the finite exchange property implies the exchange property. But, in general, quasi-continuous modules do not have this property, and the ones that have were characterized in [41].

Modules with decompositions into indecomposable summands which enjoy the exchange property were described, for example, in [21, 68]. Also in [68], and among other examples, it was proved that Artinian modules over commutative rings have the exchange property.

A ring R is a *P-exchange ring* if every projective right R-module satisfies the exchange property. Perfect rings, for instance, are a well known example of P-exchange rings. For other examples and results, see [24, 34, 37, 47, 48, 59, 66, 68].

A ring R is an *exchange ring* if R_R satisfies the finite exchange property. This definition is left-right symmetric, as was shown in [64]. In that paper, it was also proved that a right R-module M has the finite exchange property if and only if the endomorphism ring of M is an exchange ring. Examples of exchange rings include von Neumann regular rings and left or right continuous rings. Characterizations of exchange rings were obtained in [5, 43, 47], for example.

The following properties will be repeatedly used in the sequel.

Proposition 1.9.1 [62, Proposition 1] *An indecomposable module has the exchange property if and only if its endomorphism ring is local.*

Lemma 1.9.2 [40, Lemma 3.20] *If $M = M_1 \oplus M_2$, then M has the (finite) exchange property if and only if M_1 and M_2 have the (finite) exchange property.*

1.10 Indecomposable decompositions of modules

Generalizing a fundamental property of semisimple modules, Anderson and Fuller [1] (cf. [2, page 141]) introduced the following important concept for direct decompositions of modules. A decomposition $M = \oplus_{i \in I} M_i$ is said to *complement direct summands* in case for each direct summand A of M there is a subset J of I such that $M = A \oplus (\oplus_{i \in J} M_i)$. Such a decomposition is necessarily an indecomposable decomposition (see [2], for example).

A decomposition $M = \oplus_{i \in I} M_i$ is said to *complement maximal direct summands* if, whenever $M = A \oplus X$, with X an indecomposable summand, there is $i \in I$ such that $M = A \oplus M_i$. Obviously, every decomposition that complements direct summands also complements maximal direct summands.

Every decomposition of a module into summands with local endomorphism rings complements maximal direct summands [2, Theorem 12.6], but the local endomorphism ring hypothesis is not necessary for a decomposition to complement direct summands (see, for example, [2, Exercises 12.5 and 12.6]).

If $M = \oplus_{i\in I} M_i$ is an indecomposable decomposition that complements maximal direct summands, then the conclusion of the Krull-Schmidt Theorem holds, i.e., an indecomposable decomposition of M is unique up to isomorphism [2, Theorem 12.4].

A family $\{ N_i \mid i \in I \}$ of independent submodules of a module M is said to be a *local direct summand* of M if, for any finite subset F of I, $\oplus_{i\in F} N_i$ is a direct summand of M. If, furthermore, $\oplus_{i\in I} N_i$ is also a direct summand of M, then we say that *the local direct summand is a summand.*

A family of modules $\{ M_i \mid i \in I \}$ is called *locally semi-T-nilpotent* if, for any countable set of non-isomorphisms $\{ f_n : M_{i_n} \to M_{i_{n+1}} \}$, with all i_n distinct in I, and for any $x \in M_{i_1}$, there exists k (depending on x) such that $f_k \cdots f_1(x) = 0$.

Lemma 1.10.1 *Let $M = \oplus_{i\in I} M_i$ be a direct sum of modules with local endomorphism rings. The following conditions are equivalent.*

(i) $S := \mathrm{End}(M)$ is a semi-regular ring; that is $S/J(S)$ is von Neumann regular and the idempotents in $S/J(S)$ can be lifted over $J(S)$, where $J(S)$ is the Jacobson radical of S.

(ii) Every local direct summand of M is a summand.

(iii) The decomposition $M = \oplus_{i\in I} M_i$ complements direct summands.

(iv) The family $\{ M_i \mid i \in I \}$ is locally semi-T-nilpotent.

(v) M has the exchange property.

Proof. The equivalence of (i)–(iv) is due to Harada [21]. The equivalenece of (iv) and (v) is due to Zimmermann-Huisgen and Zimmermann [68]. □

N. V. Dung proved the following result, that generalizes [21, Theorems 7.3.15 and 8.2.1].

Theorem 1.10.2 [12, Theorem 3.4] *Let $M = \oplus_{i\in I} M_i$ be an indecomposable decomposition that complements maximal direct summands. The following conditions are equivalent.*

(i) The decomposition $M = \oplus_{i\in I} M_i$ complements direct summands.

(ii) Every non-zero direct summand of M contains an indecomposable direct summand, and the family $\{ M_i \mid i \in I \}$ is locally semi-T-nilpotent.

(iii) Every local direct summand of M is a summand.

By [40, Theorem 2.22], an indecomposable decomposition of a quasi-continuous module always complements direct summands, and every local direct summand is also a direct summand. However, it is still an open question to characterize extending modules which admit indecomposable decompositions (cf. [40, Open problem 8, page 106]). N. V. Dung gives a complete characterization of extending modules which have a decomposition that complements maximal direct summands, as a Corollary of [12, Theorem 3.4].

Theorem 1.10.3 [12, Theorem 4.4] *Let $M = \oplus_{i \in I} M_i$ be a direct sum of uniform submodules and suppose that this decomposition complements maximal direct summands. The following conditions are equivalent.*

(i) M is extending.

(ii) $\oplus_{i \in J} M_i$ is extending, for every countable subset J of I.

(iii) $M_i \oplus M_j$ is extending, for every $i, j \in I$, $i \neq j$, and the family $\{ M_i \mid i \in I \}$ is locally semi-T-nilpotent and satisfies condition (A_2).

Furthermore, if M satisfies any of the above equivalent conditions, then the decomposition $M = \oplus_{i \in I} M_i$ complements direct summands, and any local direct summand of M is also a direct summand.

Sufficient conditions for an extending module to admit an indecomposable decomposition follow below.

Lemma 1.10.4 [13, 8.1] *Let M be a module. If R satisfies ACC on right ideals of the form $r(x)$, $x \in M$, then every local direct summand of M is closed in M.*

Theorem 1.10.5 [13, 8.2] *Let M be an extending module. If R satisfies ACC on right ideals of the form $r(x)$, $x \in M$, then M is a direct sum of uniform submodules.*

Corollary 1.10.6 [13, 8.3] *Any locally Noetherian extending module is a direct sum of uniform submodules.*

In particular, over a right Noetherian ring, any extending module is a direct sum of uniform submodules.

Corollary 1.10.7 [13, 8.4] *Let M be a nonsingular extending module. Then M is a direct sum of uniform submodules if and only if R satisfies ACC on right ideals of the form $r(x)$, $x \in M$.*

Proposition 1.10.8 [13, 8.6] *A locally Noetherian module M is extending if and only if it is uniform-extending and every local direct summand of M is a direct summand.*

Chapter 2

Near and essential injectivity

In recent years, extending modules have been studied extensively and a question that has attracted much attention is when the direct sum of extending modules is extending (see, for example, [10, 11, 12, 25, 26, 31, 61]).

In Section 2.2, we find sufficient conditions for a direct sum of two extending modules to be extending, generalizing several known results.

Trying to get partial converses for the results in Section 2.2, we look at modules with summands with the finite exchange property, in Section 2.3. We characterize when the direct sum of an extending module and an injective module is extending and when the direct sum of an extending module with the finite exchange property and a semisimple module is extending. We also characterize when the direct sum of a uniform-extending module and a semisimple module is uniform-extending and, in consequence, we prove that, for a right Noetherian ring R, an extending right R-module M_1 and a semisimple right R-module M_2, the right R-module $M_1 \oplus M_2$ is extending if and only if M_2 is $M_1/\text{Soc}(M_1)$-injective. Finally, we prove that a ring R is such that every direct sum of an extending (injective) R-module and a semisimple R-module is extending if and only if $R/\mathrm{Soc}(R_R)$ is a right Noetherian right V-ring.

To achieve this, the concepts of near and essential injectivity seem to play a key role. So, in Section 2.1, we start by introducing these notions and proving some criteria for a module to be essentially (nearly) injective, and follow with some examples and general properties, establishing a parallel with what is known for injectivity.

In Section 2.4, we consider direct sums of uniform-extending modules. Let R be a ring and let $\{ M_i \mid i \in I \}$ be a family of R-modules. In case, for every $i \in I$, M_i is a uniform module with local endomorphism ring, N. V. Dung [11] proved that $\oplus_{i\in I} M_i$ is uniform-extending if and only if $M_i \oplus M_j$ is extending, for every $i, j \in I$, $i \neq j$, and condition (A_2) holds. Considering that an indecomposable module has the exchange property if and only if its endomorphism ring is local, it is natural to try to generalize this result to direct sums of modules with the (finite) exchange property. Suppose that, for any $i \in I$, M_i has the finite exchange property. We prove that $\oplus_{i\in I} M_i$ is uniform-extending if and only if $M_i \oplus M_j$ is uniform-extending, for every $i, j \in I$, $i \neq j$, and, for any choice of $x_n \in M_{i_n}$, with $n \in \mathbb{N}$ and distinct $i_n \in I$, such that $\cap_{n=1}^{\infty} r(x_n) \supseteq r(y)$, for some $y \in M_j$ such that yR is uniform ($j \in I$), the ascending chain

$$\bigcap_{n=1}^{\infty} r(x_n) \subseteq \bigcap_{n=2}^{\infty} r(x_n) \subseteq \cdots \subseteq \bigcap_{n=k}^{\infty} r(x_n) \subseteq \cdots$$

becomes stationary. This Section owes a good deal to [10, 11, 23, 32], where most of the ideas and techniques originate (see also [30], for close results).

Part of Section 2.1 and most of Sections 2.2 and 2.3 appeared in [53] (see also [52]).

2.1 Near and essential injectivity

Let M_1 and M_2 be modules. The module M_2 is *nearly M_1-injective* (resp., *essentially M_1-injective*) if every homomorphism $\alpha : A \longrightarrow M_2$, where A is a submodule of M_1 and $\ker\alpha \neq 0$ (resp., $\ker\alpha \leq_e A$), can be extended to a homomorphism $\beta : M_1 \longrightarrow M_2$ (see [13, 2.14]). Obviously, if M_2 is nearly M_1-injective, then M_2 is essentially M_1-injective and, for a uniform module M_1, the two notions coincide.

Observe that a module M is nearly N-injective, for every module N, if and only if it is injective. To see this, let M_1 be any module, let A be a submodule of M_1 and let $\alpha : A \longrightarrow M_2$ be a homomorphism such that $\ker\alpha = 0$. Let B be any nonzero module and consider the homomorphism $\alpha' : A \oplus B \longrightarrow M_2$ such that $\alpha'|_A = \alpha$ and $\alpha'|_B = 0$. As $\ker\alpha' \neq 0$, if M_2 is nearly $(M_1 \oplus B)$-injective, there exists a homomorphism $\beta : M_1 \oplus B \longrightarrow M_2$ that extends α'. Then, clearly, the restriction of β to M_1 extends α.

A module M is *essentially injective* if it is essentially N-injective, for every module N.

For example, every nonsingular module is essentially injective. In fact, for any modules M_1 and M_2, a homomorphism $\alpha : A \longrightarrow M_2$, where A is a submodule of M_1 and $\ker\alpha \leq_e A$, is such that $A/\ker\alpha$ is singular, and therefore $\alpha(A) \leq Z(M_2)$. So, if M_2 is nonsingular, $\alpha(A) \leq Z(M_2) = 0$ and α is the zero homomorphism.

For any prime p, consider the (uniform) $\mathbb{Z}$-modules $\mathbb{Z}/p\mathbb{Z}$, $\mathbb{Z}/p^2\mathbb{Z}$ and

$\mathbb{Z}/p^3\mathbb{Z}$. The module $\mathbb{Z}/p\mathbb{Z}$ is essentially $(\mathbb{Z}/p^2\mathbb{Z})$-injective, but is not essentially $(\mathbb{Z}/p^3\mathbb{Z})$-injective, as it fails to be $(\mathbb{Z}/p^2\mathbb{Z})$-injective (cf. Lemma 2.1.5 below).

In order to obtain characterizations of near and essential injectivity, we need the following Lemma, that generalizes [13, Lemma 7.5].

Lemma 2.1.1 *Let M_1 and M_2 be modules, let X be a submodule of M_1 and let $M := M_1 \oplus M_2$. The following conditions are equivalent.*

(i) M_2 is (M_1/X)-injective.

(ii) For every (closed) submodule N of M such that $N \cap M_2 = 0$ and $\pi_1(N) \cap X \leq N$, there exists a submodule N' of M such that $N \leq N'$ and $M = N' \oplus M_2$.

(iii) For every (closed) submodule N of M such that $N \cap M_2 = 0$ and $X \leq N$, there exists a submodule N' of M such that $N \leq N'$ and $M = N' \oplus M_2$.

Proof. Obviously, (ii) implies (iii).

Let us prove that (i) implies (ii). Suppose that M_2 is (M_1/X)-injective and let N be a submodule of M such that $N \cap M_2 = 0$ and $\pi_1(N) \cap X \leq N$.

Consider the maps $\alpha_0 : N \to M_2$, $a \mapsto \pi_2(a)$, and $\beta_0 : N \to M_1/X$, $a \mapsto \pi_1(a) + X$. As $\pi_1(N) \cap X \leq N \cap M_1 = \ker \alpha_0$, the map $\alpha : N/(\pi_1(N) \cap X) \to M_2$, $a + \pi_1(N) \cap X \mapsto \pi_2(a)$, is a homomorphism. Obviously, $\pi_1(N) \cap X \leq \ker \beta_0$. In order to prove that $\pi_1(N) \cap X = \ker \beta_0$, let $a \in \ker \beta_0$. Then $\pi_1(a) \in X$ and, consequently, $\pi_1(a) \in \pi_1(N) \cap X \leq N$. As $N \cap M_2 = 0$ and $a - \pi_1(a) = \pi_2(a)$, it follows that $a = \pi_1(a) \in \pi_1(N) \cap X$. Therefore,

$\pi_1(N) \cap X = \ker \beta_0$ and the map $\beta : N/(\pi_1(N) \cap X) \to M_1/X$, $a + \pi_1(N) \cap X \mapsto \pi_1(a) + X$, is a monomorphism. Then, by hypothesis, there exists a map $\varphi : M_1/X \to M_2$ such that $\varphi\beta = \alpha$.

Define $N' := \{ a + \varphi(a + X) \mid a \in M_1 \}$. Clearly, N' is a submodule of M and $N' \cap M_2 = 0$. For every $a \in M$, $a = [\pi_1(a) + \varphi(\pi_1(a) + X)] + [\pi_2(a) - \varphi(\pi_1(a) + X)] \in N' + M_2$. So, $M = N' \oplus M_2$. Also, if $a \in N$, then $\pi_2(a) = \alpha(a + \pi_1(N) \cap X) = \varphi\beta(a + \pi_1(N) \cap X) = \varphi(\pi_1(a) + X)$ and $a = \pi_1(a) + \pi_2(a) = \pi_1(a) + \varphi(\pi_1(a) + X) \in N'$. Thus, $N \leq N'$.

Let us prove, now, that (iii) implies (i). Suppose that condition (iii) is valid. In order to prove that M_2 is (M_1/X)-injective, let L be a submodule of M_1 such that $X \leq L$ and let $\alpha : L/X \to M_2$ be any homomorphism. Define $N := \{ a - \alpha(a + X) \mid a \in L \}$. Clearly, N is a submodule of M, $N \cap M_2 = 0$ and $X \leq N$. Then, by hypothesis, there exists a submodule N' of M such that $N \leq N'$ and $M = N' \oplus M_2$. Let $\pi : M \to M_2$ denote the canonical projection with kernel N'. Then, as $X \leq N' = \ker \pi$, the map $\varphi : M_1/X \to M_2$, $a + X \mapsto \pi(a)$, is a homomorphism. For every $a \in L$, $\varphi(a + X) = \pi(a) = \pi[(a - \alpha(a + X)) + \alpha(a + X)] = \alpha(a + X)$. Thus, M_2 is (M_1/X)-injective. □

Lemma 2.1.1 has the following immediate consequences.

Corollary 2.1.2 [13, Lemma 7.5] *Let M_1 and M_2 be modules and let $M := M_1 \oplus M_2$. The following conditions are equivalent.*

(i) M_2 is M_1-injective.

(ii) For every (closed) submodule N of M such that $N \cap M_2 = 0$, there exists a submodule N' of M such that $N \leq N'$ and $M = N' \oplus M_2$.

Proof. This is a consequence of Lemma 2.1.1, when $X = 0$. □

Corollary 2.1.3 *Let M_1 and M_2 be modules and let $M := M_1 \oplus M_2$. The following conditions are equivalent.*

(i) M_2 is $(M_1/\mathrm{Soc}(M_1))$-injective.

(ii) For every (closed) submodule N of M such that $N \cap M_2 = 0$ and $\mathrm{Soc}(\pi_1(N)) \leq N$, there exists a submodule N' of M such that $N \leq N'$ and $M = N' \oplus M_2$.

(iii) For every (closed) submodule N of M such that $N \cap M_2 = 0$ and $\mathrm{Soc}(N) = \mathrm{Soc}(M_1)$, there exists a submodule N' of M such that $N \leq N'$ and $M = N' \oplus M_2$.

Proof. This is a consequence of Lemma 2.1.1, when $X = \mathrm{Soc}(M_1)$. □

We can now characterize near and essential injectivity.

Lemma 2.1.4 *Let M_1 and M_2 be modules and let $M := M_1 \oplus M_2$. The following conditions are equivalent.*

(i) M_2 is nearly M_1-injective.

(ii) M_2 is (M_1/X)-injective, for every nonzero submodule X of M_1.

(iii) For every (closed) submodule N of M such that $N \cap M_1 \neq 0$ and $N \cap M_2 = 0$, there exists a submodule N' of M such that $N \leq N'$ and $M = N' \oplus M_2$.

Proof. It is not hard to prove the equivalence of (i) and (ii) and Lemma 2.1.1 gives the equivalence of (ii) and (iii). □

Lemma 2.1.5 *Let M_1 and M_2 be modules and let $M := M_1 \oplus M_2$. The following conditions are equivalent.*

(i) M_2 is essentially M_1-injective.

(ii) M_2 is (M_1/X)-injective, for every essential submodule X of M_1.

(iii) For every submodule N of M such that $N \cap M_1 \leq_e M_1$ and $N \cap M_2 = 0$, there exists a submodule N' of M such that $N \leq N'$ and $M = N' \oplus M_2$.

(iv) For every closed submodule N of M such that $N \cap M_1 \leq_e M_1$ and $N \cap M_2 = 0$, $M = N \oplus M_2$.

(v) For every (closed) submodule N of M such that $N \cap M_1 \leq_e N$, there exists a submodule N' of M such that $N \leq N'$ and $M = N' \oplus M_2$.

Proof. It is not hard to prove that (i) implies (ii). Let us prove the converse. Suppose that condition (ii) holds, let A be a submodule of M_1, let $\alpha : A \to M_2$ be a homomorphism such that $\ker \alpha \leq_e A$ and consider the homomorphism $\bar{\alpha} : A/\ker \alpha \to M_2$, $a + \ker \alpha \mapsto \alpha(a)$. Let B be a complement of A in M_1. Then $X := \ker \alpha \oplus B \leq_e M_1$. Consider the homomorphism $\varphi : A/\ker \alpha \to M_1/X$, $a + \ker \alpha \mapsto a + X$. Because $A \cap X = \ker \alpha \oplus (A \cap B) =$

$\ker\alpha$, φ is a monomorphism. On the other hand, by hypothesis, M_2 is (M_1/X)-injective. Then, there exists a map $\bar{\beta} : M_1/X \longrightarrow M_2$ such that $\bar{\alpha}(a+\ker\alpha) = \bar{\beta}\varphi(a+\ker\alpha) = \bar{\beta}(a+X)$, for every $a \in A$. Let $\beta : M_1 \longrightarrow M_2$, $a \mapsto \bar{\beta}(a + X)$. Then, $\beta(a) = \alpha(a)$, for every $a \in A$. Therefore, M_2 is essentially M_1-injective.

Lemma 2.1.1 gives the equivalence of (ii) and (iii) and, obviously, (iv) implies (iii).

Let us prove, now, that (iii) implies (v). Suppose that condition (iii) holds and let N be a submodule of M such that $N \cap M_1 \leq_e N$. Let L be a complement of $N \cap M_1$ in M_1. Then, $(N \oplus L) \cap M_1 = (N \cap M_1) \oplus L \leq_e M_1$. Also, $(N \cap M_1) \cap [N \cap (L \oplus M_2)] = N \cap [L \oplus (M_1 \cap M_2)] = N \cap L = 0$. As $N \cap M_1 \leq_e N$, $N \cap (L \oplus M_2) = 0$ and, consequently, $(N \oplus L) \cap M_2 = 0$. By hypothesis, there exists a submodule N' of M such that $N \oplus L \leq N'$ and $M = N' \oplus M_2$.

To conclude the proof, let us show that (v) implies (iv). Suppose that condition (v) holds and let K be a closed submodule of M such that $K \cap M_1 \leq_e M_1$ and $K \cap M_2 = 0$. Let us remark that $K \cap M_1 \leq_e K$, because $(K \cap M_1) \oplus M_2 \leq_e M$ and $K \cap M_1 = K \cap [(K \cap M_1) \oplus M_2]$. Then, by hypothesis, there exists a submodule K' of M such that $K \leq K'$ and $M = K' \oplus M_2$. We have $K \leq_e K'$, because $K \oplus M_2 \leq_e M$ and $K = K' \cap (K \oplus M_2)$. Since K is closed in M, we can conclude that $K = K'$ and $M = K \oplus M_2$. □

Corollary 2.1.6 *Let M_1 and M_2 be modules. If M_2 is $(M_1/\mathrm{Soc}(M_1))$-injective, then M_2 is essentially M_1-injective.*

Proof. The result follows easily from Lemma 2.1.5, bearing in mind that, for every $X \leq_e M_1$, $\mathrm{Soc}(M_1) \leq X$. □

In general, the converse for the last result does not hold, though Corollary 2.3.13 gives a partial converse for it. The $\mathbb{Z}$-module $\mathbb{Z}$ is essentially $\mathbb{Z}$-injective, but is not $(\mathbb{Z}/\mathrm{Soc}(\mathbb{Z}))$-injective (observe that $\mathrm{Soc}(\mathbb{Z}) = 0$).

In what follows, we will look at some basic properties of near and essential injectivity.

Proposition 2.1.7 *Let M_1 and M_2 be modules. If M_2 is nearly (resp., essentially) M_1-injective, then, for every submodule N of M_1, M_2 is nearly (resp., essentially) N-injective and nearly (resp., essentially) (M_1/N)-injective.*

Proof. For near injectivity, the result is an easy consequence of Proposition 1.2.2 and Lemma 2.1.4.

Let us prove the result for essential injectivity. Suppose that M_2 is essentially M_1-injective and let $N \leq M_1$.

By definition of essential injectivity, it is easy to see that M_2 is also essentially N-injective.

Let $X \leq M_1$ be such that $N \leq X$ and $X/N \leq_e M_1/N$. By Proposition 1.1.1(iv), $X \leq_e M_1$. Thus, by assumption and Lemma 2.1.5, M_2 is (M_1/X)-injective. Therefore, it is also $[(M_1/N)/(X/N)]$-injective, since these two modules are isomorphic. So, M_2 is essentially (M_1/N)-injective. □

Proposition 2.1.8 *Let M and $\{ N_i \mid i \in I \}$ be modules. Then $\Pi_{i \in I} N_i$ is nearly (resp., essentially) M-injective if and only if N_i is nearly (resp., essentially) M-injective, for every $i \in I$.*

Proof. This is an obvious consequence of Proposition 1.2.4 and Lemma 2.1.4 (resp., Lemma 2.1.5). □

Proposition 2.1.9 *Let $\{ M_i \mid i \in I \}$ and N be modules. Then N is essentially $(\oplus_{i \in I} M_i)$-injective if and only if N is essentially M_i-injective, for every $i \in I$.*

Proof. The necessity follows from Proposition 2.1.7.

Conversely, suppose that N is essentially M_i-injective, for every $i \in I$, and let $X \leq_e \oplus_{i \in I} M_i$. Then, for every $i \in I$, $X \cap M_i \leq_e M_i$ and, by hypothesis, together with Lemma 2.1.5, N is $[M_i/(X \cap M_i)]$-injective. From Proposition 1.2.3, we can conclude that N is $\{\oplus_{i \in I}[M_i/(X \cap M_i)]\}$-injective, so that N is also $\{[\oplus_{i \in I} M_i]/[\oplus_{i \in I}(X \cap M_i)]\}$-injective. By Proposition 1.2.2, N is $[(\oplus_{i \in I} M_i)/X]$-injective. Finally, by Lemma 2.1.5, we can conclude that N is essentially $(\oplus_{i \in I} M_i)$-injective. □

The $\mathbb{Z}$-module $\mathbb{Z}$ is essentially (nearly) $\mathbb{Z}$-injective, so that it is also essentially $(\mathbb{Z} \oplus \mathbb{Z})$-injective, by Proposition 2.1.9. But it fails to be nearly $(\mathbb{Z} \oplus \mathbb{Z})$-injective, as it is not $[(\mathbb{Z} \oplus \mathbb{Z})/(\mathbb{Z} \oplus 0)]$-injective, i.e., it is not self-injective.

The modules M_1 and M_2 are *relatively essentially injective* if M_i is essentially M_j-injective, for every $i, j \in \{1, 2\}$, $i \neq j$.

Compare the following result with Theorem 1.4.1(v).

Lemma 2.1.10 *Let M_1 and M_2 be modules and let $M := M_1 \oplus M_2$. Then M_1 and M_2 are relatively essentially injective if and only if, for all (closed)* *submodules K and L of M such that $K \cap M_1 \leq_e K$ and $L \cap M_2 \leq_e L$, there exist submodules K' and L' of M such that $K \leq K'$, $L \leq L'$ and $M = K' \oplus L'$.*

Proof. Suppose, firstly, that M_1 and M_2 are relatively essentially injective and let K and L be (closed) submodules of M such that $K \cap M_1 \leq_e K$ and $L \cap M_2 \leq_e L$. The fact that M_2 is essentially M_1-injective guarantees, by Lemma 2.1.5, that there exists a submodule K' of M such that $K \leq K'$ and $M = K' \oplus M_2$. Then M_1 and K' are isomorphic and, therefore, K' is essentially M_2-injective. Again by Lemma 2.1.5, and because $L \cap M_2 \leq_e L$, there exists a submodule L' of M such that $L \leq L'$ and $M = K' \oplus L'$.

Let us prove the converse. Let K be any submodule of M such that $K \cap M_1 \leq_e K$ and let $L := M_2$. By hypothesis, there exist submodules K' and L' of M such that $K \leq K'$, $L \leq L'$ and $M = K' \oplus L'$. Then $L' = (M_1 \oplus M_2) \cap L' = (M_1 \cap L') \oplus M_2$ and $M = K' \oplus (M_1 \cap L') \oplus M_2$. By Lemma 2.1.5, we can conclude that M_2 is essentially M_1-injective. Analogously, we can prove that M_1 is essentially M_2-injective. □

The following result is a version of Baer's Criterion for essential injectivity.

Proposition 2.1.11 *Let M_1 and M_2 be modules. Then M_2 is essentially M_1-injective if and only if M_2 is essentially aR-injective, for every $a \in M_1$. Moreover, a module is essentially injective if and only if it is essentially R_R-injective.*

Proof. The necessity is given at once by Proposition 2.1.7. Conversely, suppose that M_2 is essentially aR-injective, for every $a \in M_1$, and let $X \leq_e M_1$. For $a \in M_1$, $aR \cap X \leq_e aR$. By hypothesis and Lemma 2.1.5, taking in account that the submodules $(aR+X)/X$ and $aR/(aR\cap X)$ are isomorphic, we can conclude that M_2 is $[(aR+X)/X]$-injective, for every $a \in M_1$. It follows, by Proposition 1.2.5, that M_2 is (M_1/X)-injective. Thus, again by Lemma 2.1.5, M_2 is essentially M_1-injective

The last statement follows easily. □

Let us introduce another generalization of injectivity.

Let M_1 and M_2 be modules. The module M_2 is *u-essentially M_1-injective* if every homomorphism $\alpha : U \to M_2$, where U is a uniform submodule of M_1 and $\ker\alpha \neq 0$ (i.e., $\ker\alpha \leq_e U$), can be extended to a homomorphism $\beta : M_1 \to M_2$.

Clearly, if M_2 is essentially M_1-injective, then M_2 is u-essentially M_1-injective. In what follows, we can see that these two notions coincide when M_1 is a direct sum of uniform modules. We also prove some basic properties of u-essential injectivity.

An example of a module M_2 that is u-essentially M_1-injective but not essentially M_1-injective, for some module M_1, is provided in the end of this Section.

Lemma 2.1.12 *Let M_1 and M_2 be modules and let $M := M_1 \oplus M_2$. Then M_2 is u-essentially M_1-injective if and only if, for every (closed) uniform submodule N of M such that $N \cap M_1 \neq 0$, there exists a submodule N' of M such that $N \leq N'$ and $M = N' \oplus M_2$.*

Proof. Assume that M_2 is u-essentially M_1-injective and let N be a uniform submodule of M such that $N \cap M_1 \neq 0$. As $N \cap M_2 = 0$, the restriction of π_1 to N is an isomorphism between N and $\pi_1(N)$, so that $\pi_1(N)$ is also uniform. Consider the homomorphism $\alpha : \pi_1(N) \to M_2$, $x \mapsto \pi_2(\pi_1|_N)^{-1}(x)$. The map α can be extended to a homomorphism $\beta : M_1 \to M_2$, since M_2 is u-essentially M_1-injective and $\ker\alpha = N \cap M_1 \neq 0$. Define $N' := \{\, x + \beta(x) \mid x \in M_1 \,\}$. Clearly, N' is a submodule of M and $M = N' \oplus M_2$. For every $x \in N$, $\beta\pi_1(x) = \alpha\pi_1(x) = \pi_2(x)$ and hence $x = \pi_1(x) + \beta\pi_1(x) \in N'$. Thus, $N \leq N'$.

Conversely, assume that, for every uniform submodule N of M such that $N \cap M_1 \neq 0$, there exists a submodule N' of M such that $N \leq N'$ and $M = N' \oplus M_2$. Let K be a closed uniform submodule of M_1 and let $\alpha : K \to M_2$ be a homomorphism such that $\ker\alpha \neq 0$. Define $N := \{\, x - \alpha(x) \mid x \in K \,\}$. Clearly, N is a uniform submodule of M such that $N \cap M_1 = \ker\alpha \neq 0$. Then, by hypothesis, there exists a submodule N' of M such that $N \leq N'$ and $M = N' \oplus M_2$. Let $\pi : M \to M_2$ denote the projection with kernel N' and let $\beta : M_1 \to M_2$ be the restriction of π to M_1. For every $x \in K$, $\beta(x) = \pi(x) = \pi((x - \alpha(x)) + \alpha(x)) = \alpha(x)$ and, therefore, β extends α. Thus, M_2 is u-essentially M_1-injective. □

Lemma 2.1.13 *Let M_1 and M_2 be modules. If M_2 is u-essentially M_1-injective, then M_2 is u-essentially N-injective, for every submodule N of M_1.*

Proof. Clear, by definition. □

Corollary 2.1.14 *Let M_1 be a direct sum of uniform modules. A module M_2 is u-essentially M_1-injective if and only if it is essentially M_1-injective.*

Proof. The sufficiency is obvious.

Suppose that M_2 is u-essentially M_1-injective and that $M_1 = \oplus_{i \in I} M_{1i}$, where M_{1i} is uniform, for every $i \in I$. By Lemma 2.1.13, M_2 is u-essentially (i.e., essentially) M_{1i}-injective, for every $i \in I$. Using Proposition 2.1.9, we can conclude that M_2 is essentially M_1-injective. □

Next we characterize essential (resp., u-essential) injectivity over an extending (resp., uniform-extending) module.

Lemma 2.1.15 *Let M_1 be an extending module, let M_2 be any module and let $M := M_1 \oplus M_2$. Then M_2 is essentially M_1-injective if and only if the following condition holds.*

(∗) *For every closed submodule K of M such that $K \cap M_1 \leq_e K$, there exists a submodule M_{11} of M_1 such that $M = K \oplus M_{11} \oplus M_2$.*

In particular, if M_2 is essentially M_1-injective, then every closed submodule K of M such that $K \cap M_1 \leq_e K$ is a direct summand of M.

Proof. It is obvious that condition (∗) implies that M_2 is essentially M_1-injective, by Lemma 2.1.5.

Suppose now that M_2 is essentially M_1-injective and let K be a closed submodule of M such that $K \cap M_1 \leq_e K$. As $K \cap M_2 = 0$, the restriction of π_1 to K is an isomorphism between K and $\pi_1(K)$. Then, from $K \cap M_1 \leq_e K$, we can conclude that $K \cap M_1 \leq_e \pi_1(K)$. Since M_1 is extending, there exist $M_{11}, M_{12} \leq M_1$ such that $M = M_{11} \oplus M_{12}$ and $\pi_1(K) \leq_e M_{12}$. Hence $K \cap M_1 \leq_e M_{12}$. Observe that $K \leq \pi_1(K) \oplus \pi_2(K) \leq M_{12} \oplus M_2$, $K \cap M_{12} = K \cap M_1 \leq_e M_{12}$ and $K \cap M_2 = 0$. On the other hand, Proposition 2.1.7

guarantees that M_2 is essentially M_{12}-injective. Thus, by Lemma 2.1.5, $M_{12} \oplus M_2 = K \oplus M_2$, so that $M = K \oplus M_{11} \oplus M_2$. □

Lemma 2.1.16 *Let M_1 be a uniform-extending module, let M_2 be any module and let $M := M_1 \oplus M_2$. The following conditions are equivalent.*

(i) M_2 is u-essentially M_1-injective.

(ii) For every closed uniform submodule K of M such that $K \cap M_1 \neq 0$, there exists a submodule M_{11} of M_1 such that $M = K \oplus M_{11} \oplus M_2$.

(iii) M_2 is essentially (nearly) U-injective, for every uniform submodule U of M_1.

In particular, if M_2 is u-essentially M_1-injective, then every closed uniform submodule K of M such that $K \cap M_1 \neq 0$ is a direct summand of M.

Proof. By Lemma 2.1.12, (ii) implies (i); that (i) implies (iii) follows by Lemma 2.1.13. Let us prove that (iii) implies (ii).

Suppose that M_2 is essentially U-injective, for every uniform submodule U of M_1. Let K be a closed uniform submodule of M such that $K \cap M_1 \neq 0$. Then, as K is uniform, $K \cap M_2 = 0$ and K is isomorphic to $\pi_1(K)$. Consequently, $\pi_1(K)$ is also uniform and, because M_1 is uniform-extending, $\pi_1(K)$ is essential in a direct summand of M_1. Suppose that $M_1 = M_{11} \oplus M_{12}$, where $\pi_1(K) \leq_e M_{12}$. Obviously, M_{12} is also uniform and, by hypothesis, M_2 is essentially M_{12}-injective. On the other hand, $K \leq \pi_1(K) \oplus \pi_2(K) \leq M_{12} \oplus M_2$ and $K \cap M_{12} \leq_e M_{12}$. Then, by Lemma 2.1.5, $M_{12} \oplus M_2 = K \oplus M_2$. Thus, $M = M_{11} \oplus M_{12} \oplus M_2 = K \oplus M_{11} \oplus M_2$. Therefore, condition (ii) is satisfied. □

The following results are versions of Theorem 1.2.6 for near, essential and u-essential injectivity.

Proposition 2.1.17 *Let $\{ M_i \mid i \in I \}$ be a family of modules. For a module A, the following conditions are equivalent.*

(i) $\oplus_{i\in I} M_i$ *is nearly A-injective.*

(ii) $\oplus_{i\in J} M_i$ *is nearly A-injective, for every countable subset J of I.*

(iii) M_i *is nearly A-injective, for every $i \in I$, and for any choice of $x_n \in M_{i_n}$, with $n \in \mathbb{N}$ and distinct $i_n \in I$, such that $\cap_{n=1}^{\infty} r(x_n) \supseteq (X : a)$, for some $a \in A$ and some nonzero submodule X of A, the ascending chain*

$$\bigcap_{n=1}^{\infty} r(x_n) \subseteq \bigcap_{n=2}^{\infty} r(x_n) \subseteq \cdots \subseteq \bigcap_{n=k}^{\infty} r(x_n) \subseteq \cdots$$

becomes stationary.

(iv) M_i *is nearly A-injective, for every $i \in I$, and for any choice of $x_n \in M_{i_n}$, with $n \in \mathbb{N}$ and distinct $i_n \in I$, such that $\cap_{n=1}^{\infty} r(x_n) \supseteq r(a)$, for some $a \in A$, the ascending chain*

$$\bigcap_{n=1}^{\infty} r(x_n) \subseteq \bigcap_{n=2}^{\infty} r(x_n) \subseteq \cdots \subseteq \bigcap_{n=k}^{\infty} r(x_n) \subseteq \cdots$$

becomes stationary.

Proof. The equivalence of (i), (ii) and (iii) follows by Theorem 1.2.6 and Lemma 2.1.4, bearing in mind that, for every $a \in A$ and every submodule X of A, $(X : a)$ is the right annihilator of $\{a + X\}$ in R.

As $r(a) \subseteq (X : a)$, for every $a \in A$ and every submodule X of A, (iv) implies (iii). Let us prove the converse. Assume that condition (iii) holds and let $x_n \in M_{i_n}$, with $n \in \mathbb{N}$ and distinct $i_n \in I$, be such that $J := \cap_{n=1}^{\infty} r(x_n) \supseteq r(a)$, for some $a \in A$. Without loss of generality, assume that $J \supset r(a)$. Then $aJ \neq 0$ and $(aJ : a) \subseteq J = \cap_{n=1}^{\infty} r(x_n)$. By hypothesis, the ascending chain

$$\bigcap_{n=1}^{\infty} r(x_n) \subseteq \bigcap_{n=2}^{\infty} r(x_n) \subseteq \cdots \subseteq \bigcap_{n=k}^{\infty} r(x_n) \subseteq \cdots$$

becomes stationary. □

Proposition 2.1.18 *Let $\{ M_i \mid i \in I \}$ be a family of modules. For a module A, the following conditions are equivalent.*

(i) $\oplus_{i \in I} M_i$ is essentially A-injective.

(ii) $\oplus_{i \in J} M_i$ is essentially A-injective, for every countable subset J of I.

(iii) M_i is essentially A-injective, for every $i \in I$, and for any choice of $x_n \in M_{i_n}$, with $n \in \mathbb{N}$ and distinct $i_n \in I$, such that $\cap_{n=1}^{\infty} r(x_n) \supseteq (X : a)$, for some $a \in A$ and some essential submodule X of A, the ascending chain

$$\bigcap_{n=1}^{\infty} r(x_n) \subseteq \bigcap_{n=2}^{\infty} r(x_n) \subseteq \cdots \subseteq \bigcap_{n=k}^{\infty} r(x_n) \subseteq \cdots$$

becomes stationary.

Proof. By Theorem 1.2.6 and Lemma 2.1.5. □

Proposition 2.1.19 *Let $\{ M_i \mid i \in I \}$ be a family of modules. For a uniform-extending module A, the following conditions are equivalent.*

(i) $\oplus_{i\in I}M_i$ *is u-essentially A-injective.*

(ii) $\oplus_{i\in J}M_i$ *is u-essentially A-injective, for every countable subset J of I.*

(iii) M_i *is u-essentially A-injective, for every* $i \in I$*, and for any choice of* $x_n \in M_{i_n}$*, with* $n \in \mathbb{N}$ *and distinct* $i_n \in I$*, such that* $\cap_{n=1}^{\infty} r(x_n) \supseteq r(a)$*, for some* $a \in U$*, where U is a uniform submodule of A, the ascending chain*

$$\bigcap_{n=1}^{\infty} r(x_n) \subseteq \bigcap_{n=2}^{\infty} r(x_n) \subseteq \cdots \subseteq \bigcap_{n=k}^{\infty} r(x_n) \subseteq \cdots$$

becomes stationary.

(iv) M_i *is u-essentially A-injective, for every* $i \in I$*, and for any choice of* $x_n \in M_{i_n}$*, with* $n \in \mathbb{N}$ *and distinct* $i_n \in I$*, such that* $\cap_{n=1}^{\infty} r(x_n) \supseteq r(a)$*, for some* $a \in A$ *such that aR is uniform, the ascending chain*

$$\bigcap_{n=1}^{\infty} r(x_n) \subseteq \bigcap_{n=2}^{\infty} r(x_n) \subseteq \cdots \subseteq \bigcap_{n=k}^{\infty} r(x_n) \subseteq \cdots$$

becomes stationary.

Proof. Lemma 2.1.16 and Proposition 2.1.17 give the equivalence of (i), (ii) and (iii) and, obviously, (iii) and (iv) are equivalent. □

Corollary 2.1.20 *Let* $\{ M_i \mid i \in I \}$ *be a family of modules. For a module A, the following conditions are equivalent.*

(i) M_i *is A-injective, for every* $i \in I$*, and* $\oplus_{i\in I}M_i$ *is nearly A-injective.*

(ii) $\oplus_{i\in I}M_i$ *is A-injective.*

Proof. By Proposition 2.1.17 and Theorem 1.2.6. □

Motivated by these results, let us introduce the following chain conditions on the ring R, relative to a given family $\{ M_i \mid i \in I \}$ of R-modules.

(B_1) For any choice of $x_n \in M_{i_n}$, with $n \in \mathbb{N}$ and distinct $i_n \in I$, such that $\cap_{n=1}^{\infty} r(x_n) \leq_e R_R$, the ascending chain

$$\bigcap_{n=1}^{\infty} r(x_n) \subseteq \bigcap_{n=2}^{\infty} r(x_n) \subseteq \cdots \subseteq \bigcap_{n=k}^{\infty} r(x_n) \subseteq \cdots$$

becomes stationary.

(B_2) For any choice of $x_n \in M_{i_n}$, with $n \in \mathbb{N}$ and distinct $i_n \in I$, such that $\cap_{n=1}^{\infty} r(x_n) \supseteq (X : y)$, for some $y \in M_j$ and some essential submodule X of M_j $(j \in I)$, the ascending chain

$$\bigcap_{n=1}^{\infty} r(x_n) \subseteq \bigcap_{n=2}^{\infty} r(x_n) \subseteq \cdots \subseteq \bigcap_{n=k}^{\infty} r(x_n) \subseteq \cdots$$

becomes stationary.

(C) For any choice of $x_n \in M_{i_n}$, with $n \in \mathbb{N}$ and distinct $i_n \in I$, such that $\cap_{n=1}^{\infty} r(x_n) \supseteq r(y)$, for some $y \in M_j$ such that yR is uniform $(j \in I)$, the ascending chain

$$\bigcap_{n=1}^{\infty} r(x_n) \subseteq \bigcap_{n=2}^{\infty} r(x_n) \subseteq \cdots \subseteq \bigcap_{n=k}^{\infty} r(x_n) \subseteq \cdots$$

becomes stationary.

Let us look at some of the relations between these chain conditions. Obviously, (A_2) implies both (B_2) and (C), and (A_1) implies (B_1). Also, (B_1) implies (B_2), since, for every module A, every essential submodule X of A and every $a \in A$, $(X : a) \leq_e R_R$.

For a family $\{ M_i \mid i \in I \}$ of uniform modules, conditions (A_2), (B_2) and (C) are equivalent (we can prove that (B_2) implies (A_2), in these circumstances, using the argument in the proof of Theorem 2.1.17).

A family of modules $\{ M_i \mid i \in I \}$ is *relatively nearly injective* (resp., *relatively essentially injective, relatively u-essentially injective*) if M_i is nearly (resp., essentially, u-essentially) M_j-injective, for every $i, j \in I$, $i \neq j$.

As an immediate consequence of Propositions 2.1.17 and 2.1.18, we have the following result.

Proposition 2.1.21 *Let $\{ M_i \mid i \in I \}$ be a family of modules. Then $\oplus_{i \in I \setminus \{j\}} M_i$ is nearly (resp., essentially) M_j-injective, for every $j \in I$, if and only if the modules $\{ M_i \mid i \in I \}$ are relatively nearly (resp., essentially) injective and condition (A_2) (resp., (B_2)) holds.*

By Proposition 2.1.8, a direct product of nearly (resp., essentially) injective modules, and hence a finite direct sum of nearly (resp., essentially) injective modules, is nearly (resp., essentially) injective. The following result characterizes the near (resp., essential) injectivity of arbitrary direct sums of modules and is a consequence of Lemma 2.1.17 (resp., Lemma 2.1.18).

Theorem 2.1.22 *Let $\{ M_i \mid i \in I \}$ be a family of modules. The following conditions are equivalent.*

(i) $\oplus_{i \in I} M_i$ is nearly (resp., essentially) injective.

(ii) $\oplus_{i \in J} M_i$ is nearly (resp., essentially) injective, for every countable subset J of I.

(iii) M_i *is nearly (resp., essentially) injective, for every* $i \in I$, *and condition* (A_1) *(resp.,* (B_1)*) holds.*

Proof. Lemma 2.1.17 (resp., Lemma 2.1.18) gives at once the equivalence of (i) and (ii) and shows condition (iii) implies the other two. It remains to be proved that (i) implies (iii).

Assuming that $\oplus_{i\in I} M_i$ is nearly (resp., essentially) injective, we know that M_i is nearly (resp., essentially) injective, for every $i \in I$, and we need to show that (A_1) (resp., (B_1)) holds.

Let $x_n \in M_{i_n}$, with $n \in \mathbb{N}$ and distinct $i_n \in I$ be, without loss of generality, such that $J := \cap_{n=1}^{\infty} r(x_n) \neq 0$ (resp., be such that $J := \cap_{n=1}^{\infty} r(x_n) \leq_e R_R$). By hypothesis and Lemma 2.1.4 (resp., Lemma 2.1.5), $\oplus_{i\in I} M_i$ is (R/J)-injective. Observing that $J = r(1 + J)$, Theorem 1.2.6 guarantees that the ascending chain

$$\bigcap_{n=1}^{\infty} r(x_n) \subseteq \bigcap_{n=2}^{\infty} r(x_n) \subseteq \cdots \subseteq \bigcap_{n=k}^{\infty} r(x_n) \subseteq \cdots$$

becomes stationary. Therefore, condition (A_1) (resp., (B_1)) holds. □

The next results are versions of Theorem 1.2.9 for near and essential injectivity.

Theorem 2.1.23 *For a module* A, *the following conditions are equivalent.*

(i) A *is locally Noetherian.*

(ii) The direct sum of any family of injective modules is A*-injective.*

(iii) The direct sum of any family of A-injective modules is A-injective.

(iv) The direct sum of any family of injective modules is nearly A-injective.

(v) The direct sum of any family of nearly A-injective modules is nearly A-injective.

In particular, the direct sum of every family of (nearly) injective R-modules is (nearly) injective if and only if R is right Noetherian.

Proof. It is obvious that (iii) implies (ii) and that (v) implies (iv); Corollary 2.1.20 gives the equivalence of (ii) and (iv); Theorem 1.2.9 shows that (i) is equivalent to (iii).

Let us prove that (ii) implies (i). For $a \in A$, since $R/r(a)$ and aR are isomorphic, we will prove that aR is Noetherian by showing that any ascending chain

$$r(a) = I_0 \leq I_1 \leq \cdots \leq I_n \leq \cdots$$

of right ideals of R is ultimately stationary. For every $i \in \mathbb{N}$, let M_i be the injective hull of R/I_i, i.e., $M_i := E(R/I_i)$. Since each M_i is injective, $\oplus_{i \in \mathbb{N}} M_i$ is A-injective, by assumption. Consider the set of elements $\{ x_i := 1 + I_i \in M_i \mid i \in \mathbb{N} \}$. For every $i \in \mathbb{N}$, as $r(x_i) = I_i$, we also have $\bigcap_{n \geq i}^{\infty} r(x_n) = \bigcap_{n \geq i}^{\infty} I_n = I_i$. The A-injectivity of $\oplus_{i \in \mathbb{N}} M_i$ implies, by Theorem 1.2.6, that the ascending chain

$$r(a) = I_0 \subseteq \bigcap_{n=1}^{\infty} r(x_n) = I_1 \subseteq \bigcap_{n=2}^{\infty} r(x_n) = I_2 \subseteq \cdots \subseteq \bigcap_{n=k}^{\infty} r(x_n) = I_n \subseteq \cdots$$

becomes stationary. Therefore aR is Noetherian and A is locally Noetherian.

Finally, let us show that (i) implies (v). Let $\{ M_i \mid i \in I \}$ be a family of nearly A-injective modules. Let X be a nonzero submodule of A. By Lemma 2.1.4, M_i is (A/X)-injective, for every $i \in I$. On the other hand, A/X, being a quotient of the locally Noetherian module A, is also locally Noetherian. Thus, Theorem 1.2.9 guarantees that $\oplus_{i \in I} M_i$ is (A/X)-injective. Again by Lemma 2.1.4, we can conclude that $\oplus_{i \in I} M_i$ is nearly A-injective. Thus, condition (v) holds.

The last statement of the Theorem is obvious. □

Theorem 2.1.24 *For a module A, the following conditions are equivalent.*

(i) $A/\mathrm{Soc}(A)$ *is locally Noetherian.*

(ii) A/X *is locally Noetherian, for every* $X \leq_e A$.

(iii) The direct sum of any family of injective modules is essentially A-injective.

(iv) The direct sum of any family of essentially A-injective modules is essentially A-injective.

In particular, the direct sum of every family of essentially injective R-modules is essentially injective if and only if $R/\mathrm{Soc}(R_R)$ is right Noetherian.

Proof. Firstly, let us prove the equivalence of conditions (i) and (ii). Since, for every $X \leq_e A$, $\mathrm{Soc}(A) \leq X$, we can conclude that, if the module $A/\mathrm{Soc}(A)$ is locally Noetherian, then A/X is also locally Noetherian.

Conversely, suppose that A/X is locally Noetherian, for every $X \leq_e A$. For every $a \in A$, we want to prove that $(aR + \mathrm{Soc}(A))/\mathrm{Soc}(A)$ is Noetherian, which is equivalent to proving that $aR/\mathrm{Soc}(aR)$ is Noetherian, since $\mathrm{Soc}(aR) = aR \cap \mathrm{Soc}(A)$. By [13, 5.15], $aR/\mathrm{Soc}(aR)$ is Noetherian if and only if aR satisfies ACC on essential submodules. Let B be a complement of aR in A and let

$$X_0 \leq X_1 \leq X_2 \leq \cdots \leq X_n \leq \cdots$$

be an ascending chain of essential submodules of aR. Then

$$X_0 \oplus B \leq X_1 \oplus B \leq X_2 \oplus B \leq \cdots \leq X_n \oplus B \leq \cdots$$

is an ascending chain of essential submodules of A. By hypothesis, the module $(aR \oplus B)/(X_0 \oplus B)$, being a cyclic submodule of $A/(X_0 \oplus B)$ with $X_0 \oplus B \leq_e A$, is Noetherian, so that aR/X_0 is also Noetherian. Therefore, the chain

$$X_0 \leq X_1 \leq X_2 \leq \cdots \leq X_n \leq \cdots$$

is stationary and aR satisfies ACC on essential submodules. Finally, we can *conclude that the module $A/Soc(A)$ is locally Noetherian.*

It is obvious that (iv) implies (iii).

Let us prove that (iii) implies (ii). Let $X \leq_e A$. For every family $\{ M_i \mid i \in I \}$ of injective modules, the hypothesis and Lemma 2.1.5 guarantee that $\oplus_{i \in I} M_i$ is (A/X)-injective. Then, by Theorem 2.1.23, we can conclude that A/X is locally Noetherian.

It remains to be proved that (ii) implies (iv). Let $\{ M_i \mid i \in I \}$ be a family of essentially A-injective modules and let $X \leq_e A$. By Lemma 2.1.5,

M_i is (A/X)-injective, for every $i \in I$. On the other hand, by hypothesis, A/X is locally Noetherian. Thus, Theorem 1.2.9 guarantees that $\oplus_{i\in I}M_i$ is (A/X)-injective. Again by Lemma 2.1.5, we can conclude that $\oplus_{i\in I}M_i$ is essentially A-injective. Thus condition (iv) holds.

The last statement of the Theorem is obvious. □

We will finish this Section with some examples.

Let K be a field and let V be an infinite dimensional vector space over K. The ring

$$R := \begin{bmatrix} K & V \\ 0 & K \end{bmatrix}$$

is such that

$$\mathrm{Soc}(R_R) = \begin{bmatrix} 0 & V \\ 0 & K \end{bmatrix}.$$

Then, $R/\mathrm{Soc}(R_R)$ is isomorphic to K and, therefore, is Noetherian, though R itself is not right Noetherian. By Theorems 2.1.23 and 2.1.24, the direct sum of every family of essentially injective R-modules is essentially injective, but there exists a family of (nearly) injective R-modules that is not (nearly) injective. Theorem 2.1.22 guarantees that this particular family satisfies (B_1) but does not satisfy (A_1).

Let R be a commutative Von Neumann regular ring. Observe that every uniform ideal of R is simple, so that every R-module is trivially u-essentially R_R-injective. As $R/\mathrm{Soc}(R_R)$ need not be Noetherian, not every R-module is essentially R_R-injective (cf. Theorem 2.1.24). Consider, for example, a field K and let $R := \Pi_{n\in\mathbb{N}}K_n$, where $K_n = K$, for every $n \in \mathbb{N}$. Then R is a commutative Von Neumann regular ring such that $\mathrm{Soc}(R_R) = \oplus_{n\in\mathbb{N}}K_n$. Thus $R/\mathrm{Soc}(R_R)$ is not Noetherian.

2.2 Sufficient conditions for a direct sum of two extending modules to be extending

We now look at sufficient conditions for a direct sum of two extending modules to be extending. For this, we will need the following Lemma.

Lemma 2.2.1 *Let M_1 and M_2 be extending modules and let $M := M_1 \oplus M_2$. The following conditions are equivalent.*

(i) M is an extending module.

(ii) Every closed submodule K of M such that $K \cap M_1 = 0$ or $K \cap M_2 = 0$ is a direct summand of M.

(iii) Every closed submodule K of M such that $K \cap M_1 \leq_e K$, $K \cap M_2 \leq_e K$ or $K \cap M_1 = K \cap M_2 = 0$ is a direct summand of M.

Proof. The equivalence of (i) and (ii) is given in [13, Lemma 7.9] and it is obvious that (ii) implies (iii).

Let us prove that (iii) implies (ii). Suppose that condition (iii) is valid and let L be a closed submodule of M such that $L \cap M_2 = 0$, the case $L \cap M_1 = 0$ being analogous. Let K be a closed submodule of L such that $L \cap M_1 \leq_e K$. By Proposition 1.1.4, K is closed in M. Clearly, $K \cap M_1 = L \cap M_1 \leq_e K$ and then, by hypothesis, K is a direct summand of M. Suppose that $M = K \oplus K'$. Then $L = L \cap (K \oplus K') = K \oplus (L \cap K')$, $(L \cap K') \cap M_1 = (L \cap M_1) \cap K' \leq K \cap K' = 0$ and $(L \cap K') \cap M_2 \leq L \cap M_2 = 0$. Again by Proposition 1.1.4, $L \cap K'$ is closed in M. Thus, by assumption, $L \cap K'$ is a direct summand of M and, consequently, is also a direct summand of K'. Therefore, $L = K \oplus (L \cap K')$ is a direct summand of $K \oplus K' = M$. □

Theorem 2.2.2 *Let M_1 and M_2 be extending (resp., uniform-extending) modules and let $M := M_1 \oplus M_2$. If one of the following conditions holds, then M is extending (resp., uniform-extending).*

(i) M_2 is essentially (resp., u-essentially) M_1-injective and every closed (resp., closed uniform) submodule K of M such that $K \cap M_1 = 0$ is a direct summand of M.

(ii) M_1 and M_2 are relatively essentially (resp., u-essentially) injective and every closed (resp., closed uniform) submodule K of M such that $K \cap M_1 = K \cap M_2 = 0$ is a direct summand of M.

(iii) M_1 is M_2-injective and M_2 is essentially (resp., u-essentially) M_1-injective.

Proof. (i) and (ii) follow from Lemmas 2.1.15 and 2.2.1 (resp., Lemmas 2.1.16 and 2.2.1).

Let us prove (iii). Suppose that M_1 is M_2-injective and M_2 is essentially (resp., u-essentially) M_1-injective. Let K be a closed (resp., closed uniform) submodule of M such that $K \cap M_1 = 0$. By Corollary 2.1.2, there exists a submodule K' of M such that $K \leq K'$ and $M = M_1 \oplus K'$. As K' is isomorphic to M_2, K' is extending (resp., uniform-extending) and K, being a closed (resp., closed uniform) submodule of K', is a direct summand of K'. Thus, K is also a direct summand of M. By (i), M is extending. □

We shall prove partial converses for Theorem 2.2.2 and some of the Corollaries below (cf. Section 2.3).

Corollary 2.2.3 [25, Theorem 8] *Let $\{M_1, \ldots, M_n\}$ be a finite family of relatively injective modules. Then $M_1 \oplus \cdots \oplus M_n$ is extending if and only if M_i is extending, for every $i \in \{1, \ldots, n\}$.*

Proof. This is a consequence of Theorem 2.2.2(iii). □

For any prime p, consider the (uniform) $\mathbb{Z}$-modules $\mathbb{Z}/p\mathbb{Z}$, $\mathbb{Z}/p^2\mathbb{Z}$ and $\mathbb{Z}/p^3\mathbb{Z}$. The $\mathbb{Z}$-module $M := \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^3\mathbb{Z}$ is not extending, because $K := (1 + p\mathbb{Z}, p + p^3\mathbb{Z})\mathbb{Z}$ is a closed submodule of M which is not a direct summand. On the other hand, Theorem 2.2.2 guarantees that the $\mathbb{Z}$-module $N := \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z}$ is extending. Recall that, as we have remarked in the beginning of Section 2.1, $\mathbb{Z}/p\mathbb{Z}$ is essentially $(\mathbb{Z}/p^2\mathbb{Z})$-injective, but is neither $(\mathbb{Z}/p^2\mathbb{Z})$-injective, nor essentially $(\mathbb{Z}/p^3\mathbb{Z})$-injective.

Corollary 2.2.4 *Let M_1 be an extending (resp., uniform-extending) module and let M_2 be a semisimple module. If M_2 is essentially (resp., u-essentially) M_1-injective, then $M_1 \oplus M_2$ is extending (resp., uniform-extending).*

Proof. This is a consequence of Theorem 2.2.2(iii), considering that every module is injective over a semisimple module. □

As trivial consequences of Corollary 2.2.4, we get the following known results.

Corollary 2.2.5 *Let M_1 and M_2 be extending modules.*

(i) [61, Proposition 5.8] *If M_1 is a finite direct sum of uniform modules and M_2 is a finitely generated semisimple module that is $(M_1/\mathrm{Soc}(M_1))$-injective, then $M_1 \oplus M_2$ is extending.*

(ii) [26, Theorem 4.4] *If M_2 is semisimple and nearly M_1-injective, then $M_1 \oplus M_2$ is extending.*

Bearing in mind that nonsingular modules are essentially injective, Theorem 2.2.2(iii) has the following immediate consequence.

Corollary 2.2.6 [25, Theorem 4] *Let M_1 and M_2 be extending modules. If M_1 is nonsingular and M_2 is M_1-injective, then $M_1 \oplus M_2$ is extending.*

The equivalence of (i) and (ii) in the next Theorem is the well-known result [31, Theorem 1].

Theorem 2.2.7 *For a module M, the following conditions are equivalent.*

(i) *M is extending.*

(ii) *$M = Z_2(M) \oplus M'$, for some $M' \leq M$ such that both $Z_2(M)$ and M' are extending and $Z_2(M)$ is M'-injective.*

(iii) *$M = Z_2(M) \oplus M'$, for some $M' \leq M$ such that both $Z_2(M)$ and M' are extending and $Z_2(M)$ is essentially M'-injective.*

Proof. Obviously, (ii) implies (iii).

If condition (iii) is satisfied, then clearly $\mathrm{Hom}(A, M') = 0$, for every $A \leq Z_2(M)$, so that M' is $Z_2(M)$-injective. Therefore, by Theorem 2.2.2(iii), M is extending and condition (i) holds.

That (i) implies (ii) is given in [31, Theorem 1], but we include a proof for completeness. If M is extending, and because $Z_2(M) \leq_c M$, we have $M = Z_2(M) \oplus M'$, for some submodule M' of M. Both $Z_2(M)$ and M' are extending and it only remains to be proved that $Z_2(M)$ is M'-injective. Let K be a closed submodule of M such that $K \cap Z_2(M) = 0$. Clearly, $Z_2(K) = 0$. As M is extending, K is a direct summand of M and $M = K \oplus K'$, for some submodule K' of M. Then $Z_2(M) = Z_2(K) \oplus Z_2(K') = Z_2(K') \leq K'$, so that $K' = Z_2(M) \oplus (K' \cap M')$ and $M = K \oplus Z_2(M) \oplus (K' \cap M')$. By Corollary 2.1.2, we can conclude that $Z_2(M)$ is M'-injective. □

2.3 Extending modules with summands satisfying the finite exchange property

Trying to get a converse for Theorem 2.2.2 and some of its Corollaries, we consider modules with summands with the finite exchange property and obtain partial converses.

We will start by proving three technical results that will be used in the sequel.

Lemma 2.3.1 *Let M_1 and M_2 be modules, let $M := M_1 \oplus M_2$ and let K be a direct summand of M such that $K \cap M_1 \leq_e K$. If K has the finite exchange property, then $M = K \oplus A \oplus M_2$, for some $A \leq M_1$.*

Proof. Because K has the finite exchange property, $M = K \oplus A \oplus B$, for some $A \leq M_1$ and $B \leq M_2$. As $K \cap M_1 \leq_e K$ and $K \cap M_1 \cap (A \oplus M_2) = K \cap [A \oplus (M_1 \cap M_2)] = K \cap A = 0$, we can conclude that $K \cap (A \oplus M_2) = 0$. Therefore, $(K \oplus A) \cap M_2 = 0$ and, consequently, $M = K \oplus A \oplus M_2$. □

Lemma 2.3.2 *Let K and K' be modules, let $M := K \oplus K'$ and let L be a submodule of M with the finite exchange property. If $M = N' \oplus L$, for some $N' \leq K'$, then K has the finite exchange property.*

Proof. Because $M = N' \oplus L$, for some $N' \leq K'$, then $K' = K' \cap (N' \oplus L) = N' \oplus (K' \cap L)$, $M = K \oplus K' = K \oplus N' \oplus (K' \cap L)$ and $L = L \cap [K \oplus N' \oplus (K' \cap L)] = [(K \oplus N') \cap L] \oplus (K' \cap L)$. Thus, it is easy to see that $M = [(K \oplus N') \cap L] \oplus K'$ and we can conclude that K is isomorphic to $(K \oplus N') \cap L$, which is a direct summand of L. Therefore, K has the finite exchange property. □

Lemma 2.3.3 *Let M_1 be any module, let M_2 be a module with the finite exchange property and let $M := M_1 \oplus M_2$. If K is a uniform direct summand of M, then K has the finite exchange property or there exists a submodule L of M such that $K \leq L$ and $M = L \oplus M_2$.*

Proof. Suppose that $M = K \oplus K'$. Because M_2 has the finite exchange property, $M = N \oplus N' \oplus M_2$, for some $N \leq K$ and $N' \leq K'$. But K is uniform and, so, either $M = K \oplus N' \oplus M_2$ or $M = N' \oplus M_2$. In the first case, $M = L \oplus M_2$, where $L := K \oplus N'$, and, in the second case, K has the finite exchange property, by Lemma 2.3.2. □

At this point, we are able to prove the following key result.

Proposition 2.3.4 *Let M_1 be any module and let M_2 be a module with the finite exchange property. If $M_1 \oplus M_2$ is extending (resp., uniform-extending), then M_1 is essentially (resp., u-essentially) M_2-injective.*

Proof. Suppose that $M := M_1 \oplus M_2$ is extending and let K be a closed submodule of M such that $K \cap M_2 \leq_e K$. As M is extending, K is a direct summand of M. Suppose that $M = K \oplus K'$. Thus, because M_2 has the finite exchange property, $M = N \oplus N' \oplus M_2$, for some $N \leq K$ and $N' \leq K'$. Then, as $(K \cap M_2) \cap N = N \cap M_2 = 0$, $N = 0$ and $M = N' \oplus M_2$. Therefore, by Lemma 2.3.2, K has the finite exchange property and, by Lemma 2.3.1, $M = K \oplus M_1 \oplus B$, for some $B \leq M_2$. By Lemma 2.1.5, M_1 is essentially M_2-injective.

The result for M uniform-extending follows analogously. □

We don't know if, for any modules M_1 and M_2 such that $M_1 \oplus M_2$ is extending, M_1 and M_2 are relatively essentially injective.

Proposition 2.3.4 has several consequences, of which we state a few.

Corollary 2.3.5 *Let M_1 and M_2 be modules with the finite exchange property and let $M := M_1 \oplus M_2$. Then M is extending if and only if M_1 and M_2 are extending and relatively essentially injective and every closed submodule K of M such that $K \cap M_1 = K \cap M_2 = 0$ is a direct summand of M.*

Proof. By Theorem 2.2.2 and Proposition 2.3.4. □

The next result is a partial converse for Corollary 2.2.4.

Theorem 2.3.6 *Let M_1 be a module with the finite exchange property and let M_2 be a semisimple module. Then $M_1 \oplus M_2$ is extending if and only if M_1 is extending and M_2 is essentially M_1-injective.*

Proof. By Corollary 2.2.4 and Proposition 2.3.4. □

In particular, Theorem 2.3.6 characterizes when the direct sum of a continuous module and a semisimple module is extending.

Versions of Corollary 2.3.5 and Theorem 2.3.6 for uniform-extending modules could be given, but we will obtain better results below (cf. Corollary 2.3.10, Theorem 2.3.11).

Theorem 2.3.7 *Let M_1 be any module and let M_2 be an injective module. Then $M_1 \oplus M_2$ is extending (resp., uniform-extending) if and only if M_1 is extending (resp., uniform-extending) and essentially (resp., u-essentially) M_2-injective.*

Proof. By Theorem 2.2.2 and Proposition 2.3.4. □

For uniform-extending modules, these results can be improved, due to the following Proposition (compare with Proposition 2.3.4).

Proposition 2.3.8 *Let M_1 be any module and let M_2 be a module with the finite exchange property. If $M_1 \oplus M_2$ is uniform-extending, then M_2 is u-essentially M_1-injective.*

Proof. Suppose that $M := M_1 \oplus M_2$ is uniform-extending and let K be a closed uniform submodule of M such that $K \cap M_1 \neq 0$. As M is uniform-extending, K is a direct summand of M. By Lemma 2.3.3, K has the finite exchange property or there exists a submodule L of M such that $K \leq L$ and $M = L \oplus M_2$. In the first case, and because $K \cap M_1 \leq_e K$, Lemma 2.3.1 guarantees that $M = K \oplus A \oplus M_2$, for some $A \leq M_1$. Therefore, M_2 is u-essentially M_1-injective. □

Corollary 2.3.9 *Let M_1 be a direct sum of uniform modules and let M_2 be a module with the finite exchange property. If $M_1 \oplus M_2$ is uniform-extending, then M_2 is essentially M_1-injective.*

Proof. By Corollary 2.1.14 and Proposition 2.3.8. □

Versions of the previous results (Corollary 2.3.5 and Theorem 2.3.6), for uniform-extending modules, follow below. Observe that the hypothesis of M_1 having the exchange property was dropped.

Corollary 2.3.10 *Let M_1 be any module, let M_2 be a module with the finite exchange property and let $M := M_1 \oplus M_2$. Then M is uniform-extending if and only if M_1 and M_2 are uniform-extending and relatively u-essentially injective and every closed uniform submodule K of M such that $K \cap M_1 = K \cap M_2 = 0$ is a direct summand of M.*

Proof. By Theorem 2.2.2 and Propositions 2.3.4 and 2.3.8. □

Theorem 2.3.11 *Let M_1 be any module and let M_2 be a semisimple module. Then $M_1 \oplus M_2$ is uniform-extending if and only if M_1 is uniform-extending and M_2 is u-essentially M_1-injective.*

Proof. By Corollary 2.2.4 and Proposition 2.3.8. □

In certain cases, these results can somewhat be improved. We will need the following result, that generalizes [26, Proposition 4.2].

Proposition 2.3.12 *Let M_1 be a module with zero socle and let M_2 be a module with essential socle and the finite exchange property. Then $M_1 \oplus M_2$ is extending if and only if M_1 and M_2 are extending, M_1 is essentially M_2-injective and M_2 is M_1-injective.*

Proof. The sufficiency follows from Theorem 2.2.2.

Conversely, suppose that $M := M_1 \oplus M_2$ is extending. Obviously, M_1 and M_2 are extending and, by Proposition 2.3.4, M_1 is essentially M_2-injective. Let us prove that M_2 is M_1-injective. Let K be a closed submodule of M such that $K \cap M_2 = 0$ and $\mathrm{Soc}(K) = 0$. As M is extending, K is a direct summand of M. Suppose that $M = K \oplus K'$. Then, $\mathrm{Soc}(K') = \mathrm{Soc}(M) = \mathrm{Soc}(M_2) \leq_e M_2$ and $K' \cap M_2 \leq_e M_2$. By Lemma 2.3.1 and because M_2 has the finite exchange property, $M = K \oplus N' \oplus M_2$, for some $N' \leq K'$. Therefore, M_2 is M_1-injective, by Corollary 2.1.3. □

The next result gives a partial converse for Corollary 2.1.6.

Corollary 2.3.13 *Let M_1 be any module and let M_2 be a module with essential socle and the finite exchange property. If $M_1 \oplus M_2$ is extending, then the following conditions are equivalent.*

(i) M_2 is essentially M_1-injective.

(ii) M_2 is $(M_1/\mathrm{Soc}(M_1))$-injective.

Proof. In general, (ii) implies (i) (cf. Corollary 2.3.6).

Suppose that $M_1 \oplus M_2$ is extending and that M_2 is essentially M_1-injective. Being extending, $M_1 = M_{11} \oplus M_{12}$, where $\mathrm{Soc}(M_1) \leq_e M_{11}$. So, $\mathrm{Soc}(M_{11}) \leq_e M_{11}$ and $\mathrm{Soc}(M_{12}) = 0$. Then, M_2 is $(M_{11}/\mathrm{Soc}(M_{11}))$-injective, because it is essentially M_{11}-injective. Also, M_2 is M_{12}-injective, by Proposition 2.3.12. Therefore, M_2 is $(M_1/\mathrm{Soc}(M_1))$-injective. □

In particular, Corollary 2.3.13, together with other previous results, has the following consequences.

Corollary 2.3.14 *Let M_1 be an extending module with the finite exchange property and let M_2 be a semisimple module. The following conditions are equivalent.*

(i) $M_1 \oplus M_2$ is extending.

(ii) M_2 is essentially M_1-injective.

(iii) M_2 is $(M_1/\mathrm{Soc}(M_1))$-injective.

Proof. By Theorem 2.3.6 and Corollary 2.3.13. □

Theorem 2.3.15 *Let M_1 be an extending module that is a direct sum of uniform submodules, let M_2 be a semisimple module and let $M := M_1 \oplus M_2$. The following conditions are equivalent.*

(i) M is extending.

(ii) M *is uniform-extending.*

(iii) M_2 *is u-essentially* M_1*-injective.*

(iv) M_2 *is essentially* M_1*-injective.*

(v) M_2 *is* $(M_1/\mathrm{Soc}(M_1))$*-injective.*

Proof. Obviously, (i) implies (ii) and (v) implies (iv); (ii) implies (iii), by Theorem 2.3.11; (iii) is equivalent to (iv), by Corollary 2.1.14; and (iv) implies (i), by Corollary 2.2.4. Also, by Corollary 2.3.13, if (i) holds, then (iv) implies (v). □

Corollary 2.3.16 *Let* M_1 *be a module such that* R *satisfies ACC on right ideals of the form* $r(x)$, $x \in M_1$, *and let* M_2 *be a semisimple module. Then* $M_1 \oplus M_2$ *is extending if and only if* M_1 *is extending and* M_2 *is* $(M_1/\mathrm{Soc}(M_1))$*-injective.*

Proof. By Theorems 2.3.15 and 1.10.5. □

In particular, Theorem 2.3.15 characterizes when the direct sum of an extending module and a semisimple module is extending, over a right Noetherian ring.

[26, Theorem 4.6], [61, Proposition 5.2] and [52, Theorem 9] are consequences of Theorem 2.3.15. We can also improve [52, Theorems 13 and 18] with the following result.

Theorem 2.3.17 *The following conditions on a ring* R *are equivalent.*

(i) $M_1 \oplus M_2$ *is extending, for every extending R-module* M_1 *and every simple (semisimple) R-module* M_2*.*

(ii) $M_1 \oplus M_2$ *is extending, for every injective R-module* M_1 *and every simple (semisimple) R-module* M_2*.*

(iii) $R/\mathrm{Soc}(R_R)$ *is a (right Noetherian) right V-ring.*

Proof. Obviously, (i) implies (ii).

Let us prove that (ii) implies (iii). Suppose that $M_1 \oplus M_2$ is extending, for every injective R-module M_1 and every simple (semisimple) R-module M_2. Let S be a simple (semisimple) R-module. Then, $E(R_R) \oplus S$ is extending and, by Corollary 2.3.14, S is $(E(R_R)/\mathrm{Soc}(E(R_R)))$-injective. But $\mathrm{Soc}(E(R_R)) = \mathrm{Soc}(R_R)$ and, so, S is $(R_R/\mathrm{Soc}(R_R))$-injective. Therefore, by [13, 2.5], $R/\mathrm{Soc}(R_R)$ is a (right Noetherian) right V-ring.

Let us prove, finally, that (iii) implies (i). Suppose that $R/\mathrm{Soc}(R_R)$ is a (right Noetherian) right V-ring. Let M_1 be an extending R-module and let M_2 be a simple (semisimple) R-module. By Theorem 1.2.9, M_2 is $(R/\mathrm{Soc}(R_R))$-injective. Then, as $M_1/\mathrm{Soc}(M_1)$ is an $(R/\mathrm{Soc}(R_R))$-module, M_2 is also $(M_1/\mathrm{Soc}(M_1))$-injective and, by Corollary 2.2.4, $M_1 \oplus M_2$ is extending. □

Examples of right Noetherian right V-rings are Cozzens domains (cf. [15]). Also, at the end of Section 2.1, there is an example of a ring R such that the ring $R/\mathrm{Soc}(R_R)$ is isomorphic to a field K, and therefore is a right Noetherian right V-ring.

Let M_1 be any module and let M_2 be a module with the finite exchange property (in particular, semisimple). It remains an open problem to determine whether M_2 is essentially M_1-injective, in case $M_1 \oplus M_2$ is extending.

2.4 Direct sums of uniform-extending modules

Let $\{ M_i \mid i \in I \}$ be a family of modules with the finite exchange property. In this section, we give necessary and sufficient conditions for the direct sum $\oplus_{i\in I} M_i$ to be uniform-extending.

We start with some technical Lemmas.

Lemma 2.4.1 *Let $\{ M_i \mid i \in I \}$ be a family of modules, let $M := \oplus_{i\in I} M_i$ and let K be a uniform submodule of M. If J is minimal among the subsets of I such that $K \cap (\oplus_{i\in J} M_i) \neq 0$, then K is isomorphic to $\pi_j(K)$, for every $j \in J$.*

Proof. Let $J' := I \setminus J$ and let $j \in J$. Due to the minimality of J, $K \cap (\oplus_{i\in J\setminus\{j\}} M_i) = 0$ and, consequently, $K \cap (\oplus_{i\in J} M_i) \cap (\oplus_{i\in I\setminus\{j\}} M_i) = K \cap (\oplus_{i\in J\setminus\{j\}} M_i) = 0$. Then, as $K \cap (\oplus_{i\in J} M_i) \leq_e K$, $K \cap (\oplus_{i\in I\setminus\{j\}} M_i) = 0$ and we can conclude that K is isomorphic to $\pi_j(K)$. □

Lemma 2.4.2 *Let $\{ M_i \mid i \in I \}$ be a family of relatively u-essentially injective modules that satisfies condition (C), let $M := \oplus_{i\in I} M_i$ and let K be a closed uniform submodule of M. If there exists a subset J of I such that $\oplus_{i\in J} M_i$ is uniform-extending and $K \cap (\oplus_{i\in J} M_i) \neq 0$, then K is a direct summand of M.*

Proof. If $J = I$, the result is trivial. Suppose that J is a proper subset of I and that J is minimal among the subsets of I such that $K \cap (\oplus_{i \in J} M_i) \neq 0$. Note that J is finite.

For each $i \in J$, by Lemma 2.4.1, K is isomorphic to $\pi_i(K)$ and, so, $\pi_i(K)$ is uniform. As M_i is uniform-extending, $\pi_i(K)$ is essential in a direct summand N_i of M_i, which is also uniform.

Let $J' := I \setminus J$. Let $i \in J$ and $j \in J'$. By hypothesis, M_j is u-essentially M_i-injective. Since condition (C) is satisfied and by Proposition 2.1.19, $\oplus_{j \in J'} M_j$ is u-essentially M_i-injective, and therefore essentially N_i-injective, for every $i \in J$. Then, $\oplus_{j \in J'} M_j$ is also essentially $(\oplus_{i \in J} N_i)$-injective, by Proposition 2.1.9. On the other hand, $K \leq \oplus_{i \in I} \pi_i(K) \leq M' := (\oplus_{i \in J} N_i) \oplus (\oplus_{i \in J'} M_i)$ and $K \cap (\oplus_{i \in J} N_i) = K \cap (\oplus_{i \in J} M_i) \neq 0$, where $\oplus_{i \in J} N_i$ is uniform-extending. Then, Lemma 2.1.16 guarantees that K is a direct summand of M', and also of M. □

Corollary 2.4.3 *Let $\{ M_i \mid i \in I \}$ be a family of relatively u-essentially injective modules that satisfies condition (C). If $\oplus_{i \in F} M_i$ is uniform-extending, for every finite subset F of I, then $\oplus_{i \in I} M_i$ is uniform-extending.*

Proof. If the set I is finite, the result is trivial. If I is infinite, the result follows by Lemma 2.4.2, bearing in mind that, for each submodule N of M, there exists a finite subset F of I such that $N \cap (\oplus_{i \in F} M_i) \neq 0$. □

Corollary 2.4.4 *Let $\{ M_i \mid i \in I \}$ be a family of relatively injective modules that satisfies condition (C). Then M_i is uniform-extending, for every $i \in I$, if and only if $\oplus_{i \in I} M_i$ is uniform-extending.*

Proof. The sufficiency is clear. Conversely, suppose that M_i is uniform-extending, for every $i \in I$. By Theorem 2.2.2, $\oplus_{i \in F} M_i$ is uniform-extending, for every finite subset F of I. Then, by Corollary 2.4.3, the result follows. □

In particular, by Corollary 2.4.4, over a right Noetherian ring, every direct sum of relatively injective uniform-extending modules is uniform-extending.

Compare the following results with Corollaries 2.4.3 and 2.4.4.

Proposition 2.4.5 *Let $\{ M_i \mid i \in I \}$ be a family of relatively essentially injective modules that satisfies condition (B_2) and let $M := \oplus_{i \in I} M_i$. If every local direct summand of M is a summand and $\oplus_{i \in F} M_i$ is extending, for every finite subset F of I, then M is extending.*

Proof. Let K be a closed submodule of M. By Zorn's Lemma, K contains a maximal local direct summand $\{ N_a \mid a \in A \}$ of M. By hypothesis, $N := \oplus_{a \in A} N_a$ is a direct summand of M. So, N is also a direct summand of K. Suppose that $K = N \oplus N'$ and that $N' \neq 0$. Let $x \in N' \setminus \{0\}$. Clearly, there exists a finite subset F of I such that $x \in \oplus_{i \in F} M_i$. The submodule xR is essential in a closed submodule X of N'. Note that X is also closed in M. On the other hand, from $xR \leq_e X$, we can conclude that $(\oplus_{i \in F} M_i) \cap X \leq_e X$. As condition (B_2) holds, Proposition 2.1.18 guarantees that $\oplus_{i \in I \setminus F} M_i$ is essentially $(\oplus_{i \in F} M_i)$-injective. By assumption, $\oplus_{i \in F} M_i$ is extending, so that, by Lemma 2.1.15, X is a direct summand of M. Thus, X is also a direct summand of N' and we can write $N' = X \oplus Y$, for some submodule Y of N'. Now, we can conclude that $K = N \oplus N' = N \oplus X \oplus Y$, with $\{ N_a \mid a \in A \} \cup \{X\}$ a local direct summand, contradicting the maximality of $\{ N_a \mid a \in A \}$. Therefore, $N' = 0$ and $K = N$ is a direct summand of M. We have proved that M is extending. □

Corollary 2.4.6 *Let $\{ M_i \mid i \in I \}$ be a family of relatively injective modules that satisfies condition (B_2) and let $M := \oplus_{i \in I} M_i$ be such that every local summand of M is a summand. Then M_i is extending, for every $i \in I$, if and only if M is extending.*

Proof. The sufficiency is obvious. Conversely, suppose that M_i is extending, for every $i \in I$. By Theorem 2.2.2, $\oplus_{i \in F} M_i$ is extending, for every finite subset F of I. Then, by Proposition 2.4.5, the result follows. □

We also have the following fact.

Corollary 2.4.7 *Let $\{ M_i \mid i \in I \}$ be a family of relatively injective extending modules and let $M := \oplus_{i \in I} M_i$. If M is locally Noetherian, then M is extending if and only if every local direct summand of M is a summand.*

Proof. By Corollary 2.4.4 and Proposition 1.10.8. □

At this point, we need the following Lemma, that is just a reformulation of [4, Lemma 2] (see also [11, Lemma 2.1]).

Lemma 2.4.8 *Let M_1 and M_2 be uniform modules with local endomorphism rings such that $M_1 \oplus M_2$ is extending. If $f : A_1 \to A_2$ is an isomorphism, where $A_i \leq M_i$, $i = 1, 2$, then either f can be extended to a monomorphism $M_1 \to M_2$ or f^{-1} can be extended to a monomorphism $M_2 \to M_1$.*

Proof. Let $M := M_1 \oplus M_2$ and consider the submodule $B := \{ x - f(x) \mid x \in A_1 \}$ of M. As M is extending, B is essential in a direct summand C of M. By [2, Corollary 12.7], either $M = C \oplus M_2$ or $M = M_1 \oplus C$.

Suppose firstly that $M = C \oplus M_2$ and let π be the projection of M onto M_2 with kernel C. Let $g : M_1 \to M_2$ be the restriction of π to M_1. It is easy to check that g extends f. Also, $\ker g \cap A_1 = \ker f = 0$ and, because $A_1 \leq_e M_1$, g is a monomorphism $M_1 \to M_2$ that extends f.

Suppose now that $M = M_1 \oplus C$ and let σ be the projection of M onto M_1 with kernel C. As above, it is not hard to see that the restriction of σ to M_2 is a monomorphism that extends f^{-1}. □

Before looking at finite direct sums of uniform-extending modules with the finite exchange property, we need the following Lemma.

Lemma 2.4.9 [2, Proposition 5.5] *Let M_1 and M_2 be modules, let $M := M_1 \oplus M_2$ and let A be any submodule of M. Then $M = A \oplus M_2$ if and only if the restriction of π_1 to A is an isomorphism between A and M_1.*

Lemma 2.4.10 *Let M_1, M_2 and M_3 be modules with the finite exchange property. If $M_1 \oplus M_2$, $M_1 \oplus M_3$ and $M_2 \oplus M_3$ are uniform-extending, then $M_1 \oplus M_2 \oplus M_3$ is also uniform-extending.*

Proof. Because $M_1 \oplus M_2$, $M_1 \oplus M_3$ and $M_2 \oplus M_3$ are uniform-extending, by Proposition 2.3.4, the modules M_1, M_2 and M_3 are relatively u-essentially injective.

Let K be a closed uniform submodule of $M := M_1 \oplus M_2 \oplus M_3$. Let J be minimal among the subsets of $I := \{1, 2, 3\}$ such that $K \cap (\oplus_{i \in J} M_i) \neq 0$. If

J is a proper subset of I, then K is a direct summand of M, by Lemma 2.4.2 (condition (C) is trivially satisfied by a finite family). Suppose that $J = I$. Then, $K \cap (M_1 \oplus M_2) = K \cap (M_1 \oplus M_3) = K \cap (M_2 \oplus M_3) = 0$.

For any $i \in I$, K is isomorphic to $\pi_i(K)$ and, so, $\pi_i(K)$ is uniform. As M_i is uniform-extending, $\pi_i(K)$ is essential in a direct summand N_i of M_i, which is also uniform. Being a direct summand of M_i, N_i has the finite exchange property and, consequently, its endomorphism ring is local.

For every $i, j \in I$, the maps $f_{ij} : \pi_i(K) \to \pi_j(K)$, $\pi_i(a) \mapsto \pi_j(a)$, are isomorphisms. By Lemma 2.4.8, either f_{ij} can be extended to a monomorphism $g_{ij} : N_i \to N_j$ or f_{ji} can be extended to a monomorphism $g_{ji} : N_j \to N_i$, for every $i, j \in I$, $i \neq j$. Considering that $f_{jk}f_{ij} = f_{ik}$, for every $i, j, k \in I$, it is not hard to see that there exists an $i \in I$ such that, for every $j \in J$, f_{ij} can be extended to N_i. Without loss of generality, suppose that $i = 1$.

Clearly, $K = \{x + f_{12}(x) + f_{13}(x) \mid x \in \pi_1(K)\}$. Let $K' := \{x + g_{12}(x) + g_{13}(x) \mid x \in N_1\}$. It can easily be seen that K' is isomorphic to N_1 and, therefore, is uniform. On the other hand, $K \leq K'$. Then, $K \leq_e K'$ and, because K is closed, $K = K'$. Thus, $\pi_1(K) = N_1$ and, by Lemma 2.4.9, $N_1 \oplus N_2 \oplus N_3 = K \oplus N_2 \oplus N_3$. So, K is a direct summand of M.

Therefore, M is uniform-extending. □

Lemma 2.4.11 *Let $\{ M_i \mid i \in I \}$ be a family of modules with the finite exchange property. If $\oplus_{i \in I} M_i$ is uniform-extending, then the family $\{ M_i \mid i \in I \}$ satisfies condition (C).*

Proof. Suppose that $\oplus_{i \in I} M_i$ is uniform-extending. Then, for every $j \in I$,

$\oplus_{i\in I\setminus\{j\}} M_i$ is u-essentially M_j-injective, by Proposition 2.3.4. So, by Proposition 2.1.19, condition (C) is satisfied. $\square$

We can finally prove the main result of this Section.

Theorem 2.4.12 *Let $\{ M_i \mid i \in I \}$ be a family of modules with the finite exchange property. The following conditions are equivalent.*

(i) $\oplus_{i\in I} M_i$ *is uniform-extending.*

(ii) $\oplus_{i\in J} M_i$ *is uniform-extending, for every countable subset J of I.*

(iii) $M_i \oplus M_j$ *is uniform-extending, for every $i, j \in I$, $i \neq j$, and the family $\{ M_i \mid i \in I \}$ satisfies condition (C).*

Proof. Obviously, (i) implies (ii).

That (ii) implies (iii) follows by Lemma 2.4.11 and the fact that $\{ M_i \mid i \in I \}$ satisfies condition (C) if and only if every countable subfamily of this family satisfies condition (C).

Let us prove that (iii) implies (i). Suppose that $M_i \oplus M_j$ is uniform-extending, for every $i, j \in I$, $i \neq j$, and that the family $\{ M_i \mid i \in I \}$ satisfies condition (C). By Proposition 2.3.4, $\{ M_i \mid i \in I \}$ is a family of relatively u-essentially injective modules. On the other hand, by induction and using Lemma 2.4.10, we can prove that $\oplus_{i\in F} M_i$ is uniform-extending, for every finite subset F of I. Therefore, by Corollary 2.4.3, $\oplus_{i\in I} M_i$ is uniform-extending. $\square$

Compare Theorem 2.4.12 with Theorems 1.2.8, 1.3.2, 1.4.2 and 1.5.1.

Corollary 2.4.13 [11, Lemma 2.3] *Let* $\{ M_i \mid i \in I \}$ *be a family of uniform modules with local endomorphism rings. The following conditions are equivalent.*

(i) $\oplus_{i \in I} M_i$ *is uniform-extending.*

(ii) $\oplus_{i \in J} M_i$ *is uniform-extending, for every countable subset* J *of* I.

(iii) $M_i \oplus M_j$ *is extending, for every* $i, j \in I$, $i \neq j$, *and the family* $\{ M_i \mid i \in I \}$ *satisfies condition* (A_2).

Proof. The result is an immediate consequence of Theorem 2.4.12, bearing in mind that, because M_i is uniform, for every $i \in I$, conditions (C) and (A_2) are equivalent. □

Using Corollary 2.4.13, N. V. Dung proceeds to prove the following Theorem, which was later generalized by [12, Theorem 4.4] (cf. Theorem 1.10.3).

Theorem 2.4.14 [11, Theorem 2.4] *Let* $\{ M_i \mid i \in I \}$ *be a family of uniform modules with local endomorphism rings. The following conditions are equivalent.*

(i) $\oplus_{i \in I} M_i$ *is extending.*

(ii) $\oplus_{i \in J} M_i$ *is extending, for every countable subset* J *of* I.

(iii) $M_i \oplus M_j$ *is extending, for every* $i, j \in I$, $i \neq j$; *the family* $\{ M_i \mid i \in I \}$ *satisfies condition* (A_2); *and there does not exist an infinite sequence of monomorphisms that are not isomorphisms*

$$M_{i_1} \xrightarrow{f_1} M_{i_2} \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} M_{i_n} \xrightarrow{f_n} \cdots,$$

with the i_n distinct in I.

Furthermore, if M satisfies either of the above equivalent conditions, then $\{ M_i \mid i \in I \}$ is locally semi-T-nilpotent, and M has the exchange property.

Chapter 3

c-Injectivity

This Chapter is dedicated to another generalization of injectivity, namely c-injectivity.

As we have seen, a module M is quasi-injective if, for any submodule N of M, any homomorphism $\alpha : N \to M$ can be lifted to a homomorphism $\beta : M \to M$. Continuous and quasi-continuous modules are other classes of modules that can be characterized by the lifting of homomorphisms from certain submodules to the module itself, as was shown in [56]. In fact, in this paper, P. F. Smith and A. Tercan studied the following property, for a module M:

(P_n) For every submodule K of M that can be written as a finite direct sum $K_1 \oplus \cdots \oplus K_n$ of complements $K_1, \cdots, K_n$ of M, every homomorphism $\alpha : K \to M$ can be lifted to a homomorphism $\beta : M \to M$.

and proved that a module is quasi-continuous if and only if it satisfies (P_2).

We are now concerned with the study of self-c-injective modules, i.e., modules that satisfy (P_1). Extending modules are an example of modules with this property.

Self-c-injective modules are also a special case of the generalized quasi-injective modules studied by Harada [22]. Recall that a module M is said to be *GQ-injective* (generalized quasi-injective), if, for any submodule N isomorphic to a closed submodule K of M, any homomorphism from N to M can be extended to M.

In Section 3.1, we prove general properties of self-c-injective modules and find sufficient conditions for a direct sum of two self-c-injective modules to be self-c-injective. We also look at self-cu-injective modules, i.e., modules M such that every homomorphism from a closed uniform submodule to M can be lifted to M itself.

Section 3.2 considers self-c-injective modules over commutative domains. We prove that every self-c-injective free module over a commutative domain that is not a field is finitely generated and then proceed to consider torsion-free modules over commutative domains, as was done for extending modules in [31].

Finally, in Section 3.3, we look at self-c-injective modules over principal ideal domains, characterizing when the direct sum of a torsion-free injective module and a cyclic torsion module is self-cu-injective.

For the theory of principal ideal domains and other undefined concepts, we refer the reader to [54, 67], for example.

3.1 c-Injectivity

Let M_1 and M_2 be modules. The module M_2 is *M_1-c-injective* (resp., *M_1-cu-injective*) if every homomorphism $\alpha : K \longrightarrow M_2$, where K is a closed (resp., closed uniform) submodule of M_1, can be extended to a homomorphism $\beta : M_1 \longrightarrow M_2$.

Clearly, if M_2 is M_1-injective, then M_2 is M_1-c-injective.

The modules M_1 and M_2 are *relatively c-injective* (resp., *relatively cu-injective*) if M_i is M_j-c-injective (resp., M_i is M_j-cu-injective), for every $i, j \in \{1, 2\}$, $i \neq j$.

A module M is called *self-c-injective* (resp., *self-cu-injective*) when it is M-c-injective (resp., M-cu-injective); and is called *c-injective* (resp., *cu-injective*) when it is N-c-injective (resp., N-cu-injective), for every module N.

Proposition 3.1.1 *A module M is extending (resp., uniform-extending) if and only if every module is M-c-injective (resp., M-cu-injective).*

Proof. The necessity is clear. Conversely, suppose that every module is M-c-injective and let K be a closed submodule of M. By hypothesis, there exists a homomorphism $\alpha : M \longrightarrow K$ that extends the identity $\iota : K \longrightarrow K$. It is not hard to see that $M = K \oplus \ker \alpha$, so that K is a direct summand of M. Therefore, M is extending.

The proof for M uniform-extending follows analogously. □

In particular, by Proposition 3.1.1, every extending module is self-c-injective. But not every self-c-injective module is extending. Consider, for example, the $\mathbb{Z}$-modules $M_1 := \mathbb{Z}/p\mathbb{Z}$, for a prime p, and $M_2 := \mathbb{Q}$. Let us show that the $\mathbb{Z}$-module $M := M_1 \oplus M_2$ is self-c-injective but it is not extending. Consider the local ring $\mathbb{Z}_p$. It is not hard to see that the closed submodules of M which are not direct summands are of the form $(1+p\mathbb{Z}, q)\mathbb{Z}_p$, for some $q \in \mathbb{Q} \setminus \{0\}$. To show that M is self-c-injective it is sufficient to prove that, for $q \in \mathbb{Q} \setminus \{0\}$, every homomorphism $\alpha : (1+p\mathbb{Z}, q)\mathbb{Z}_p \longrightarrow M$ can be lifted to M. Let $K := (1+p\mathbb{Z}, q)\mathbb{Z}_p$. Suppose that $\alpha(1+p\mathbb{Z}, q) = (a+p\mathbb{Z}, b)$, for some $a \in \mathbb{Z}$ and $b \in \mathbb{Q}$. It is not hard to see that the mapping $\beta : M \longrightarrow M$, defined by $\beta(c+p\mathbb{Z}, d) = (ca+p\mathbb{Z}, db/q)$, for all $c \in \mathbb{Z}$ and $d \in \mathbb{Q}$, is a well-defined homomorphism that extends α. Thus, M is self-c-injective.

The following result characterizes c-injectivity and cu-injectivity (compare with Lemmas 2.1.1, 2.1.4, 2.1.5 and 2.1.12).

Lemma 3.1.2 *Let M_1 and M_2 be modules and let $M := M_1 \oplus M_2$. Then M_2 is M_1-c-injective (resp., M_1-cu-injective) if and only if, for every (closed) submodule (resp., every (closed) uniform submodule) N of M such that $N \cap M_2 = 0$ and $\pi_1(N) \leq_c M_1$, there exists a submodule N' of M such that $N \leq N'$ and $M = N' \oplus M_2$.*

Proof. Assume that M_2 is M_1-c-injective and let N be a submodule of M such that $N \cap M_2 = 0$ and $\pi_1(N) \leq_c M_1$. As $N \cap M_2 = 0$, the restriction of π_1 to N is an isomorphism between N and $\pi_1(N)$. Consider the homomorphism $\alpha : \pi_1(N) \longrightarrow M_2$, $x \mapsto \pi_2(\pi_1|_N)^{-1}(x)$. The map α can be extended to a homomorphism $\beta : M_1 \longrightarrow M_2$, since M_2 is M_1-c-injective and $\pi_1(N) \leq_c M_1$. Define $N' := \{ x + \beta(x) \mid x \in M_1 \}$. Clearly, N' is a submodule of M and

$M = N' \oplus M_2$. For every $x \in N$, $\beta\pi_1(x) = \alpha\pi_1(x) = \pi_2(x)$ and hence $x = \pi_1(x) + \beta\pi_1(x) \in N'$. Thus, $N \leq N'$.

Conversely, assume that, for every submodule N of M such that $N \cap M_2 = 0$ and $\pi_1(N) \leq_c M_1$, there exists a submodule N' of M such that $N \leq N'$ and $M = N' \oplus M_2$. Let K be a closed submodule of M_1 and let $\alpha : K \longrightarrow M_2$ be a homomorphism. Define $N := \{\, x - \alpha(x) \mid x \in K \,\}$. Clearly, N is a submodule of M such that $N \cap M_2 = 0$. It is not hard to prove that $\pi_1(N) = K$ and so $\pi_1(N) \leq_c M_1$. Then, by hypothesis, there exists a submodule N' of M such that $N \leq N'$ and $M = N' \oplus M_2$. Let $\pi : M \longrightarrow M_2$ denote the projection with kernel N' and let $\beta : M_1 \longrightarrow M_2$ be the restriction of π to M_1. For every $x \in K$, $\beta(x) = \pi(x) = \pi((x - \alpha(x)) + \alpha(x)) = \alpha(x)$ and, therefore, β extends α. Thus, M_2 is M_1-c-injective.

Finally, observe that, if N is a submodule of M such that $N \cap M_2 = 0$ and $\pi_1(N) \leq_c M_1$, then $N \leq_c M$. In fact, if $N \leq_e K \leq_c M$, then $K \cap M_2 = 0$ and π_1 gives an isomorphism between K and $\pi_1(K)$. Therefore, from $N \leq_e K$ we can conclude that $\pi_1(N) \leq_e \pi_1(K)$. On the other hand, we have $\pi_1(N) \leq_c M_1$ and so $\pi_1(N) = \pi_1(K)$. Thus, $N = K \leq_c M$.

The proof for cu-injectivity follows similarly. □

Below follow some general properties of c-injectivity and cu-injectivity.

Lemma 3.1.3 *Let M_1 and M_2 be modules. If M_2 is M_1-c-injective, then, for every closed submodule N of M_1, M_2 is N-c-injective and (M_1/N)-c-injective.*

Proof. Let N be a closed submodule of M_1.

As every closed submodule of N is also a closed submodule of M_1, it is obvious that M_2 is N-c-injective.

Let us prove now that M_2 is (M_1/N)-c-injective. Let K/N be a closed submodule of M_1/N and consider a homomorphism $\alpha : K/N \longrightarrow M_2$. By Lemma 1.1.4, $K \leq_c M_1$.

Let $\pi : M_1 \longrightarrow M_1/N$ and $\pi' : K \longrightarrow K/N$ be the canonical epimorphisms. As M_2 is M_1-c-injective, there exists a homomorphism $\beta : M_1 \longrightarrow M_2$ that extends $\alpha\pi'$. Since $N \leq \ker\beta$, the existence of a homomorphism $\gamma : M_1/N \longrightarrow M_2$ such that $\gamma\pi = \beta$ is guaranteed. For every $a \in K$, $\gamma(a+N) = \gamma\pi(a) = \beta(a) = \alpha\pi'(a) = \alpha(a+N)$. Therefore γ extends α and M_2 is (M_1/N)-c-injective. □

Lemma 3.1.4 *Let M_1 and M_2 be modules. If M_2 is M_1-cu-injective, then, for every closed submodule N of M_1, M_2 is N-cu-injective.*

Proof. Clear. □

Lemma 3.1.5 *Let M and $\{ N_i \mid i \in I \}$ be modules. Then $\Pi_{i \in I} N_i$ is M-c-injective (resp., M-cu-injective) if and only if N_i is M-c-injective (resp., M-cu-injective), for every $i \in I$.*

Proof. The proof follows as for injectivity (see, for example, [54, Proposition 2.2]). □

Corollary 3.1.6 *Let M_1 and M_2 be modules and let $M = M_1 \oplus M_2$. If M is self-c-injective (resp., self-cu-injective), then M_1 and M_2 are both self-c-injectives (resp., self-cu-injectives) and are relatively c-injective (resp.,* *relatively cu-injective). In particular, a direct summand of a self-c-injective (resp., self-cu-injective) module is self-c-injective (resp., self-cu-injective).*

Proof. By Lemmas 3.1.3 (resp., 3.1.4) and 3.1.5. □

The converse of Corollary 3.1.6 is not true, in general. Consider, for example, the $\mathbb{Z}$-modules $M_1 := \mathbb{Z}/p\mathbb{Z}$, for a prime p, and $M_2 := \mathbb{Z}$. Both M_1 and M_2 are uniform, so that they are self-c-injectives and relatively c-injective. It will be proved in Section 3.2 (cf. Proposition 3.2.3) that $M_1 \oplus M_2$ is not self-c-injective.

Note that this example also shows that [6, Theorem 2] is not valid. The cited result states that, if M_1 is a quasi-continuous module with finite uniform dimension, M_2 is self-c-injective and M_1-injective, then $M_1 \oplus M_2$ is self-c-injective.

In order to obtain sufficient conditions for a direct sum of two self-c-injective (resp., self-cu-injective) modules to be self-c-injective (resp., self-cu-injective), we need the following Lemmas.

Lemma 3.1.7 *Let M_1 and M_2 be modules such that M_2 is essentially (resp., u-essentially) M_1-injective. If a module is M_1-c-injective (resp., M_1-cu-injective) and M_2-injective, then it is $(M_1 \oplus M_2)$-c-injective (resp., $(M_1 \oplus M_2)$-cu-injective).*

Proof. Let $M := M_1 \oplus M_2$ and suppose that N is a M_1-c-injective and M_2-injective module.

Let $K \leq_c M$ and consider a homomorphism $\alpha : K \longrightarrow N$. Take $H \leq_c K$ such that $K \cap M_1 \leq_c H$. Then $H \cap M_1 = K \cap M_1 \leq_e H$ and, because M_2 is essentially M_1-injective, by Lemma 2.1.5, there exists a submodule H' of M such that $M = H' \oplus M_2$ and $H \leq H'$.

Clearly, $H \leq_c H'$ and, since M_1 and H' are isomorphic, N is H'-c-injective. Thus, there exists a homomorphism $\beta : H' \longrightarrow N$ that extends the restriction of α to H. Obviously, β can be extended by the homomorphism $\beta\pi : M \longrightarrow N$, where $\pi : M \longrightarrow H'$ is the projection of M onto H' with kernel M_2.

Consider the homomorphism $\alpha - \beta\pi : K \longrightarrow N$, $x \mapsto \alpha(x) - \beta\pi(x)$. As $K \cap M_1 \leq H \leq \ker(\alpha - \beta\pi)$, $\alpha - \beta\pi$ can be lifted to a homomorphism $\gamma : K/(K \cap M_1) \longrightarrow N$, $x + K \cap M_1 \mapsto \alpha(x) - \beta\pi(x)$.

The homomorphism $\phi : K/(K \cap M_1) \longrightarrow M_2$, $x + K \cap M_1 \mapsto \pi_2(x)$, is clearly injective. Since N is M_2-injective, there exists $\delta : M_2 \longrightarrow N$ such that $\delta\phi = \gamma$. Clearly, $\delta\pi_2 : M \longrightarrow N$ extends δ.

Consider, finally, the homomorphism $\theta := \beta\pi + \delta\pi_2 : M \longrightarrow N$. For all $x \in K$, $\theta(x) = \beta\pi(x) + \delta\pi_2(x) = \beta\pi(x) + \delta\phi(x + K \cap M_1) = \beta\pi(x) + \gamma(x + K \cap M_1) = \beta\pi(x) + \alpha(x) - \beta\pi(x) = \alpha(x)$. Therefore, θ extends α and N is M-c-injective.

The result for cu-injectivity follows analogously. □

Lemma 3.1.8 *Let M_1 and M_2 be modules such that M_1 is extending (resp., uniform-extending) and M_2-injective and M_2 is essentially (resp., u-essentially) M_1-injective. If a module is M_2-c-injective (resp., M_2-cu-injective), then it is $(M_1 \oplus M_2)$-c-injective (resp., $(M_1 \oplus M_2)$-cu-injective).*

Proof. Let $M := M_1 \oplus M_2$ and suppose that N is a M_2-c-injective module.

Let $K \leq_c M$ and consider a homomorphism $\alpha : K \longrightarrow N$. Take $H_1 \leq_c K$ such that $K \cap M_1 \leq_e H_1$. Then $H_1 \cap M_1 = K \cap M_1 \leq_e H_1$ and, because M_2 is essentially M_1-injective, by Lemma 2.1.5, there exists a submodule H of M such that $M = H \oplus M_2$ and $H_1 \leq H$.

Clearly, $H_1 \leq_c H$ and, since M_1 and H are isomorphic, H is extending. Thus, H_1 is a direct summand of H. Suppose that $H = H_1 \oplus H_2$. Then, $M = H_1 \oplus H_2 \oplus M_2$ and $K = H_1 \oplus L$, where $L := (H_2 \oplus M_2) \cap K$.

Since $L \cap M_1 = (H_2 \oplus M_2) \cap K \cap M_1 \leq (H_2 \oplus M_2) \cap H_1 = 0$ and M_1 is M_2-injective, there exists a submodule L' of M such that $M = M_1 \oplus L'$ and $L \leq L'$.

Clearly, $L \leq_c L'$ and, since M_2 and L' are isomorphic, N is L'-c-injective. Thus, there exists a homomorphism $\beta : L' \longrightarrow N$ that extends the restriction of α to L.

Let θ_1, θ_2 and θ_3 be the projections of $M = H_1 \oplus H_2 \oplus M_2$ onto H_1, H_2 and M_2, respectively. Consider the homomorphism $\gamma : M \longrightarrow N$ such that $\gamma(x) = \alpha\theta_1(x) + \beta\varphi(\theta_2(x) + \theta_3(x))$, where φ is the projection of M onto L' with kernel M_1. For every $x \in K$, we have $\theta_2(x) + \theta_3(x) \in (H_2 \oplus M_2) \cap K = L$ and hence $\gamma(x) = \alpha\theta_1(x) + \beta\varphi(\theta_2(x) + \theta_3(x)) = \alpha\theta_1(x) + \alpha(\theta_2(x) + \theta_3(x)) = \alpha(x)$. Therefore, γ extends α to M and N is M-c-injective.

The result for cu-injectivity follows analogously. □

We can now prove the following.

Theorem 3.1.9 *Let M_1 and M_2 be modules and let $M := M_1 \oplus M_2$. If one of the following conditions holds, then M is self-c-injective (resp., self-cu-injective).*

(i) *M_1 and M_2 are both self-c-injective (resp., self-cu-injective) and are relatively injective.*

(ii) *M_1 is extending (resp., uniform-extending) and M_2-injective, M_2 is self-c-injective (resp., self-cu-injective) and essentially (resp., u-essentially) M_1-injective.*

Proof. By Lemmas 3.1.5, 3.1.7 and 3.1.8. □

Next we will look at further properties of c-injectivity that will be required in the sequel.

Recall that a submodule N of a module M is called *fully invariant* if $\varphi(N) \leq N$, for all $\varphi \in \mathrm{End}(M)$.

Proposition 3.1.10 *Let M be a self-c-injective module. Then every fully invariant closed submodule of M is self-c-injective. In particular, $Z_2(M)$ is a self-c-injective module.*

Proof. Let N be a fully invariant closed submodule of M, let K be a closed submodule of N and let $\alpha : K \to N$ be a homomorphism. Since N is a closed submodule of M, it follows that K is also a closed submodule of M. Then, by hypothesis, there exists a homomorphism $\beta : M \to M$ that extends α. Note that $\beta(N) \leq N$, by hypothesis. Hence $\beta|_N : N \to N$ is a homomorphism and α is the restriction of this homomorphism to K. It follows that N is self-c-injective. □

Lemma 3.1.11 *Let M be a self-c-injective module and let K be a closed submodule of M. If K is isomorphic to M, then K is a direct summand of M.*

Proof. Let $\alpha : K \longrightarrow M$ be an isomorphism. There exists a homomorphism $\beta : M \longrightarrow M$ that extends α, since M is self-c-injective. For any $x \in M$, there exists $y \in K$ such that $\beta(x) = \alpha(y) = \beta(y)$ and hence $x - y \in \ker\beta$. It follows that $x = y + (x - y) \in K + \ker\beta$. Moreover, $K \cap \ker\beta = \ker\alpha = 0$. Thus $M = K \oplus \ker\beta$ and K is a direct summand of M. □

The next results show that, in some cases, the notions of c-injectivity and cu-injectivity coincide.

Proposition 3.1.12 *Let M_1 and M_2 be modules such that $u.dim(M_1) \leq 2$. Then M_2 is M_1-c-injective if and only if M_2 is M_1-cu-injective.*

Proof. Clear. □

Proposition 3.1.13 *Let M_1 be an extending module and let M_2 be a uniform module such that M_2 is essentially M_1-injective. Then $M_1 \oplus M_2$ is self-c-injective if and only if it is self-cu-injective.*

Proof. The necessity is obvious. Let us prove the converse.

Suppose that $M := M_1 \oplus M_2$ is self-cu-injective. Let K be a closed submodule of M and let $\alpha : K \longrightarrow M$ be a homomorphism.

Take $H_1 \leq_c K$ such that $K \cap M_1 \leq_e H_1$. Then $H_1 \cap M_1 = K \cap M_1 \leq_e H_1$ and, because M_2 is essentially M_1-injective, by Lemma 2.1.5, there exists a submodule H of M such that $M = H \oplus M_2$ and $H_1 \leq H$.

Clearly, $H_1 \leq_c H$ and, since M_1 and H are isomorphic, H is extending. Thus, $H = H_1 \oplus H_2$, for some submodule H_2 of H, so that $M = H_1 \oplus H_2 \oplus M_2$ and $K = H_1 \oplus L$, where $L := (H_2 \oplus M_2) \cap K$. Since $L \cap M_1 = (H_2 \oplus M_2) \cap K \cap M_1 \leq (H_2 \oplus M_2) \cap H_1 = 0$, L embeds in M_2 and hence is uniform.

By hypothesis, the restriction of α to L lifts to M and, in particular, there exists a homomorphism $\beta : H_2 \oplus M_2 \to M$ that extends the restriction of α to L. Therefore, it is clear that α can be extended to M. We can conclude that M is self-c-injective. □

3.2 Self-c-injective modules over commutative domains

In this Section, we look at self-c-injective modules over commutative domains.

The following result generalizes [6, Theorem 1].

Theorem 3.2.1 *Suppose that R is a commutative domain and let F be a self-c-injective free module. Then F is finitely generated or R is a field.*

Proof. Suppose that F is not finitely generated and let us prove that R is a field. By Corollary 3.1.6, we can assume, without loss of generality, that $F = R \oplus R \oplus R \oplus \cdots$. Consider the elements $e_n := (0, \ldots, 0, 1, 0, 0, \ldots) \in F$, where 1 is the n-th component of e_n, for each positive integer n.

Let Q denote the field of fractions of R and let $0 \neq c \in R$. We aim to prove that c is a unit in R.

Define a homomorphism $\varphi : F \to Q$ by

$$\varphi(r_1, r_2, r_3, \ldots) := r_1 + c^{-1}r_2 + c^{-2}r_3 + \cdots,$$

for all $(r_1, r_2, r_3, \ldots) \in F$, and let $K := \ker \varphi$. Consider the elements $f_n := (0, \ldots, 0, 1, -c, 0, 0, \ldots) = e_n - ce_{n+1} \in K$, for each positive integer n, and let us prove that K is a free submodule of F with basis $\{f_1, f_2, f_3, \ldots\}$.

Clearly, $L := \sum_{n\in\mathbb{N}} f_nR \leq K$. In order to prove that $L = K$, let $(r_1, r_2, \ldots, r_n, 0, 0, \ldots) \in K$. Then $r_1 + c^{-1}r_2 + \cdots + c^{-n+2}r_{n-1} + c^{-n+1}r_n = 0$ and hence $r_n + cr_{n-1} + \cdots + c^{n-2}r_2 + c^{n-1}r_1 = 0$. Thus

$$\begin{aligned} & (r_1, r_2, \ldots, r_{n-1}, r_n, 0, 0, \ldots) = \\ = \quad & (r_1, r_2, \ldots, r_{n-1}, -cr_{n-1} - \cdots - c^{n-2}r_2 - c^{n-1}r_1, 0, 0, \ldots) \\ = \quad & r_1x_1 + r_2x_2 + \cdots + r_{n-1}x_{n-1}, \end{aligned}$$

where $x_i = e_i - c^{n-i}e_n$, for $1 \leq i \leq n-1$.

Clearly, $x_{n-1} = f_{n-1} \in L$. If, for some $2 \leq i \leq n-1$, $x_i \in L$, then $x_{i-1} = f_{i-1} + cx_i \in L$. By induction, $x_i \in L$, for all $1 \leq i \leq n-1$, and hence $(r_1, r_2, \ldots, r_n, 0, 0, \ldots) \in L$. It follows that $K \leq L$ and hence $K = L$.

Let us prove that the set $\{f_1, f_2, f_3, \ldots\}$ is linearly independent. Suppose that, for a positive integer m and for some $s_i \in R$, $1 \leq i \leq m$, we have $s_1f_1 + \cdots + s_mf_m = 0$, i.e.,

$$s_1(1, -c, 0, 0, \ldots) + s_2(0, 1, -c, 0, 0, \ldots) + \cdots + s_m(0, \ldots, 0, 1, -c, 0, 0, \ldots) = 0,$$

so that $s_1 = 0$, $s_2 - cs_1 = 0$, $s_3 - cs_2 = 0$, $\ldots$, $s_m - cs_{m-1} = 0$, $-cs_m = 0$. Thus, $s_1 = s_2 = \cdots = s_m = 0$.

It follows that K is a free module with basis $\{ f_n \mid n \in \mathbb{N} \}$. Hence K is isomorphic to F. Moreover, F/K, being isomorphic to a submodule of Q_R, is a torsion-free module and hence K is a closed submodule of F. Then, by Lemma 3.1.11, K is a direct summand of F. Suppose that $F = K \oplus K'$.

Now, K' is isomorphic to F/K, which in turn is isomorphic to $\varphi(F)$, so that K' is a uniform submodule of F. Let $0 \neq u \in K'$. Then $u = (u_1, \ldots, u_q, 0, 0, \ldots)$, for some positive integer q and elements $u_i \in R$, $1 \leq$

$i \leq q$. Then $K' \leq \{ (v_1, \ldots, v_q, 0, 0, \ldots) \mid v_i \in R, 1 \leq i \leq q \}$. So, $e_{q+1} = z + (v_1, \ldots, v_q, 0, 0, \ldots)$, for some $z \in K$, $v_i \in R$, $1 \leq i \leq q$. Hence

$$c^{-q} = \varphi(e_{q+1}) = \varphi(v_1, \ldots, v_q, 0, 0, \ldots) = v_1 + c^{-1}v_2 + \cdots + c^{-q+1}v_q,$$

so that $c^{-1} = v_q + cv_{q-1} + \cdots + c^{q-1}v_1 \in R$. Thus c is a unit in R. It follows that R is a field. □

Lemma 3.2.2 *Let M be a module and let N be an essential submodule of M. For every $m \in M$, $\{ (r, mr) \mid r \in R, mr \in N \}$ is a closed submodule of the module $R \oplus N$.*

Proof. Let $m \in M$ and let $V := \{ (r, mr) \mid r \in R, mr \in N \}$. Clearly V is a submodule of $R \oplus N$ and $V \cap (0 \oplus N) = 0$. Let W be a submodule of $R \oplus N$ such that $V \leq W$, $V \neq W$. Then there exist $s \in R$ and $x \in N$ such that $(s, x) \in W$ and $x \neq ms$. Hence $x - ms \neq 0$ and, since $N \leq_e M$, $(x - ms)R \cap N \neq 0$. Therefore, there exists $t \in R$ such that $y := (x - ms)t \in N \setminus \{0\}$. Now $mst = xt - y \in N$ and $(0, y) = (s, x)t - (st, mst) \in W$, so that $W \cap (0 \oplus N) \neq 0$. Thus V, being a complement of $0 \oplus N$ in $R \oplus N$, is closed in $R \oplus N$. □

Proposition 3.2.3 *Suppose that R is a commutative domain and let c be a non-zero non-unit element of R. Then the R-module $R \oplus (R/cR)$ is not self-cu-injective.*

Proof. Let Q denote the field of fractions of R, let $N = R/cR$ and let $M = c^{-1}R/cR$. Then N is a submodule of M.

Let $m \in M$, $m \neq 0$. Then $m = c^{-1}r + cR$, for some $r \in R$. If $r \in cR$, then $m \in N$; if $r \in R \setminus cR$, then $cm = r + cR \in N \setminus \{0\}$. Thus $N \cap mR \neq 0$, for every $m \in M \setminus \{0\}$, and N is essential in M.

Let $X := R \oplus N$, let $m := c^{-1} + cR \in M$ and let $V := \{ (r, mr) \mid r \in R, mr \in N \}$. By Lemma 3.2.2, V is a closed submodule of the module X.

Let $r \in R$ be such that $mr \in N$. Then $c^{-1}r + cR = mr = s + cR$, for some $s \in R$, and $c^{-1}r - s \in cR$. So, $r \in cR$. Hence $cR = \{ r \in R \mid mr \in N \}$ and $V = \{ (cr, r + cR) \mid r \in R \}$.

Define a mapping $\alpha : V \to X$ by $\alpha(cr, r + cR) = (r, cR)$, for every $r \in R$. Clearly, α is a homomorphism. Supppose that α lifts to a homomorphism $\beta : X \to X$, with $\beta(0, 1 + cR) = (a_1, a_2 + cR)$ and $\beta(1, cR) = (b_1, b_2 + cR)$, for some $a_1, a_2, b_1, b_2 \in R$. Then $c(a_1, a_2 + cR) = c\beta(0, 1 + cR) = \beta(0, cR) = (0, cR)$, so that $a_1 = 0$. Now $(1, cR) = \alpha(c, 1 + cR) = \beta(c, 1 + cR) = c\beta(1, cR) + \beta(0, 1 + cR) = c(b_1, b_2 + cR) + (0, a_2 + cR)$. Hence $1 = cb_1$, a contradiction.

It follows that X is not self-c-injective. □

Corollary 3.2.4 [6, Lemma 3] *Let R be a principal ideal domain and let M be a finitely generated self-c-injective module. Then M is free or is a torsion module.*

Proof. By Lemma 3.1.6 and Proposition 3.2.3. □

Now we consider torsion-free modules over commutative domains.

Let us fix the following notation:

(∗) R is a commutative domain with field of fractions Q; M_1 and M_2 are R-submodules of Q such that $R \leq M_1 \cap M_2$; $M := M_1 \oplus M_2$; r and s are non-zero elements of R. For any element $q \in Q$ and R-submodule N of Q, we set $q^{-1}N := \{ x \in Q \mid qx \in N \}$. In case $q \neq 0$, $q^{-1}N = \{ y/q \in Q \mid y \in N \}$. Also, if L and N are R-submodules of Q, we set $(L : N) := \{ q \in Q \mid qN \leq L \}$.

[31] provides information on when M is an extending module (cf. [55, Corollary 2.8]).

Theorem 3.2.5 [31] *Let R be a commutative domain with field of fractions Q and let M_1 and M_2 be R-submodules of Q such that $R \leq M_1 \cap M_2$. Then the R-module $M := M_1 \oplus M_2$ is extending if and only if*

$$R \leq [(M_1 : M_1) \cap (sM_2 : rM_1)] + [(M_2 : M_2) \cap (rM_1 : sM_2)],$$

for all non-zero elements r, s of R.

Let us characterize when M is self-c-injective.

Lemma 3.2.6 *With notation (∗), let $N := r^{-1}M_1 \cap s^{-1}M_2$ and let $K := \{ (rx, sx) \mid x \in N \}$. Then K is a closed submodule of M. Moreover, a mapping $\varphi : K \to M$ is an R-homomorphism if and only if there exist $u \in (M_1 : N)$ and $v \in (M_2 : N)$ such that $\varphi(rx, sx) = (ux, vx)$, for all $x \in N$.*

Proof. Let $q_i \in M_i$, $i = 1, 2$. Suppose that $c(q_1, q_2) \in K$, for some $0 \neq c \in R$. There exists $x \in N$ such that $c(q_1, q_2) = (rx, sx)$, i.e, $cq_1 = rx$

and $cq_2 = sx$. Then $r(x/c) = q_1$ and $s(x/c) = q_2$, so that $x/c \in r^{-1}M_1 \cap s^{-1}M_2 = N$ and $(q_1, q_2) = (r(x/c), s(x/c)) \in K$. It follows that K is a closed submodule of M.

Suppose that $u \in (M_1 : N)$ and $v \in (M_2 : N)$ are such that $\varphi(rx, sx) = (ux, vx)$, for all $x \in N$. It is easy to check that $\varphi : K \longrightarrow M$ is a homomorphism.

Conversely, let $\varphi : K \longrightarrow M$ be a homomorphism. Then $\varphi(r, s) = (u, v)$, for some $u \in M_1$ and $v \in M_2$. Let $x \in N$. Then $x = a/b$, for some $a, b \in R$, $b \neq 0$. Now

$$b\varphi(rx, sx) = \varphi(brx, bsx) = \varphi(ar, as) = a\varphi(r, s) = a(u, v).$$

Suppose that $\varphi(rx, sx) = (p, q)$, where $p \in M_1$ and $q \in M_2$. The fact that $b(p, q) = a(u, v)$ gives $bp = au$ and $bq = av$, so that, in Q, $p = au/b = ux$ and $q = av/b = vx$. Thus $\varphi(rx, sx) = (ux, vx)$. Note that $ux \in M_1$ and $vx \in M_2$. □

Lemma 3.2.7 *With notation (*), a mapping $\theta : M \longrightarrow M$ is an R-homomorphism if and only if there exist elements $a \in (M_1 : M_1)$, $b \in (M_1 : M_2)$, $c \in (M_2 : M_1)$ and $d \in (M_2 : M_2)$ such that $\theta(x, y) = (ax + by, cx + dy)$, for all $x \in M_1$, $y \in M_2$.*

Proof. Suppose that $a \in (M_1 : M_1)$, $b \in (M_1 : M_2)$, $c \in (M_2 : M_1)$ and $d \in (M_2 : M_2)$ are such that $\theta(x, y) = (ax + by, cx + dy)$, for all $x \in M_1$, $y \in M_2$. It is easy to check that $\theta : M \longrightarrow M$ is an R-homomorphism.

Conversely, let $\theta : M \longrightarrow M$ be an R-homomorphism and let $\iota : M \longrightarrow Q \oplus Q$ be the inclusion homomorphism. As $Q \oplus Q$ is an injective R-module, there exists an R-homomorphism $\psi : Q\oplus Q \to Q\oplus Q$ such that $\psi\iota = \iota\theta$. It is easy to check that ψ is a Q-homomorphism. Hence there exist $a, b, c, d \in Q$ such that $\psi(p,q) = (ap + bq, cp + dq)$, for all $p, q \in Q$. Let $x \in M_1$ and $y \in M_2$. Then $\theta(x,0) = \psi(x,0) = (ax, cx)$, so that $ax \in M_1$ and $cx \in M_2$. Also, $\theta(0,y) = \psi(0,y) = (by, dy)$, so that $by \in M_1$ and $dy \in M_2$. It follows that $a \in (M_1 : M_1)$, $b \in (M_1 : M_2)$, $c \in (M_2 : M_1)$ and $d \in (M_2 : M_2)$. Furthermore, we have $\theta(x,y) = \psi(x,y) = (ax + by, cx + dy)$, for all $x \in M_1$, $y \in M_2$. □

Lemma 3.2.8 *With notation* $(*)$, *let* $N := r^{-1}M_1 \cap s^{-1}M_2$ *and let* $K := \{ (rx, sx) \mid x \in N \}$. *Then every homomorphism* $\varphi : K \to M$ *can be lifted to* M *if and only if*

$$(M_1 : N) \leq (M_1 : M_1)r + (M_1 : M_2)s$$

and

$$(M_2 : N) \leq (M_2 : M_1)r + (M_2 : M_2)s.$$

Proof. Suppose, firstly, that every homomorphism $\varphi : K \to M$ can be lifted to M. Let $u \in (M_1 : N)$ and $v \in (M_2 : N)$. Define $\varphi : K \to M$ by $\varphi(rx, sx) = (ux, vx)$, for all $x \in N$. By Lemma 3.2.6, φ is a homomorphism. By Lemma 3.2.7, there exist $a \in (M_1 : M_1)$, $b \in (M_1 : M_2)$, $c \in (M_2 : M_1)$ and $d \in (M_2 : M_2)$ such that, for all $x \in N$, $(ux, vx) = \varphi(rx, sx) = (arx + bsx, crx + dsx)$. Since $R \leq M_1 \cap M_2$, it follows that $R \leq N$ and hence $1 \in N$, so that $(u, v) = (ar + bs, cr + ds)$. Then, $u = ar + bs \in (M_1 : M_1)r + (M_1 : M_2)s$ and $v = cr + ds \in (M_2 : M_1)r + (M_2 : M_2)s$. Thus,

$$(M_1 : N) \leq (M_1 : M_1)r + (M_1 : M_2)s$$

and

$$(M_2 : N) \leq (M_2 : M_1)r + (M_2 : M_2)s.$$

Conversely, suppose that these two inclusions hold. Let $\alpha : K \longrightarrow M$ be any R-homomorphism. By Lemma 3.2.6, there exist $u \in (M_1 : N)$ and $v \in (M_2 : N)$ such that $\alpha(rx, sx) = (ux, vx)$, for all $x \in N$. By hypothesis, there exist $a \in (M_1 : M_1)$, $b \in (M_1 : M_2)$, $c \in (M_2 : M_1)$ and $d \in (M_2 : M_2)$ such that $u = ar+bs$ and $v = cr+ds$. Let $\beta : M \longrightarrow M$ be the mapping defined by $\beta(y, z) = (ay+bz, cy+dz)$, for all $y \in M_1$ and $z \in M_2$. By Lemma 3.2.7, β is an R-homomorphism. For any $x \in N$, $\beta(rx, sx) = (arx+bsx, crx+dsx) = (ux, vx) = \alpha(rx, sx)$. Therefore, α is the restriction of β to K. □

Theorem 3.2.9 *Let R be a commutative domain with field of fractions Q and let M_1 and M_2 be R-submodules of Q such that $R \leq M_1 \cap M_2$. Then the R-module $M := M_1 \oplus M_2$ is self-c-injective if and only if*

$$(M_1 : r^{-1}M_1 \cap s^{-1}M_2) \leq (M_1 : M_1)r + (M_1 : M_2)s$$

and

$$(M_2 : r^{-1}M_1 \cap s^{-1}M_2) \leq (M_2 : M_1)r + (M_2 : M_2)s,$$

for all non-zero elements r, s of R.

Proof. The necessity follows by Lemmas 3.2.6 and 3.2.8. Conversely, suppose that

$$(M_1 : r^{-1}M_1 \cap s^{-1}M_2) \leq (M_1 : M_1)r + (M_1 : M_2)s$$

and

$$(M_2 : r^{-1}M_1 \cap s^{-1}M_2) \leq (M_2 : M_1)r + (M_2 : M_2)s,$$

for all non-zero elements r, s of R.

Let K be a closed submodule of M. If $K \cap (M_1 \oplus 0) \neq 0$, then $K \cap (M_1 \oplus 0)$ is a closed submodule of $M_1 \oplus 0$ and hence $K \cap (M_1 \oplus 0) = M_1 \oplus 0$. Thus, $M_1 \oplus 0 \leq K$ and $K = M_1 \oplus 0$ or $K = M$, so that K is a direct summand of M. Similarly, if $K \cap (0 \oplus M_2) \neq 0$, then K is a direct summand of M. Thus we can suppose that $K \cap (M_1 \oplus 0) = K \cap (0 \oplus M_2) = 0$. In particular, K is uniform.

Let $(q_1, q_2) \in K$, where $0 \neq q_1, q_2 \in Q$. There exist non-zero elements $r, s, c \in R$ such that $q_1 = r/c$ and $q_2 = s/c$. Thus $(r, s) = c(q_1, q_2) \in K$. By Lemma 3.2.6, $K = \{ (rx, sx) \mid x \in N \}$, because K and $\{ (rx, sx) \mid x \in N \}$ are both closures of the submodule $(r, s)R$ and M has unique closures, since it is nonsingular. By Lemma 3.2.8, every homomorphism $\alpha : K \to M$ can be lifted to M. Thus, M is self-c-injective. □

3.3 Self-c-injective modules over principal ideal domains

In order to characterize when, over a principal ideal domain, the direct sum of a torsion-free injective module and a cyclic torsion module is self-cu-injective, we need the following Lemmas.

Lemma 3.3.1 [26, Lemma 2.4] *Let M_1 and M_2 be modules and let $M := M_1 \oplus M_2$. A submodule K of M is a complement of M_2 in M if and only if there exists a homomorphism $\varphi : M_1 \to E(M_2)$ such that $K = \{ x + \varphi(x) \mid x \in \varphi^{-1}(M_2) \}$.*

Given a positive integer n, modules $M_1, \dots, M_n$ are called *compatible* if, for all $1 \le i \le n$ and elements $m_j \in M_j$, $1 \le j \le n$, we have $r(m_i) + r(\{ m_j \mid 1 \le j \le n, j \ne i \}) = R$.

Lemma 3.3.2 *Assume that R is a right hereditary ring and let M be a module such that $M = M_0 \oplus M_1 \oplus \cdots \oplus M_n$ for some positive integer n, nonsingular injective submodule M_0 and singular uniform submodules $M_i = m_iR$, $1 \le i \le n$, with $E(M_1), \dots, E(M_n)$ compatible. Let K be a non-zero closed submodule of M such that $K \cap (M_1 \oplus \cdots \oplus M_n) = 0$. Then $x_0 + x_1 + \cdots + x_n \in K$, for some $0 \ne x_0 \in M_0$ and $x_i \in \{0, m_i\}$, $1 \le i \le n$. Moreover, $K \subseteq M_0 \oplus (\oplus_{i=1}^{n} x_iR)$.*

Proof. There exists $0 \ne m = m' + m'' \in K$, where $m' \in M_0$ and $m'' \in \overline{M} := M_1 \oplus \cdots \oplus M_n$. Since $K \cap \overline{M} = 0$, it follows that $m' \ne 0$. There exists an essential right ideal E of R such that $m''E = 0$. Then $mE = m'E \ne 0$. Thus $K \cap M_0 \ne 0$. There exists a submodule M_0' of M_0 such that $M_0 = E(K \cap M_0) \oplus M_0'$. Note that $K \cap M_0 \cap M_0' = 0$. Since $K \cap \overline{M} = 0$, it follows that K embeds in M_0 and hence K is nonsingular. Suppose that $K \cap (M_0' \oplus \overline{M}) \ne 0$ and let $0 \ne a \in K \cap (M_0' \oplus \overline{M})$. Then $aF \subseteq K \cap M_0' = K \cap M_0 \cap M_0' = 0$, for some essential right ideal F of R. Thus $a = 0$, that is K is a complement of $M_0' \oplus \overline{M}$ in $M = E(K \cap M_0) \oplus M_0' \oplus \overline{M}$.

By Lemma 3.3.1, there exists a homomorphism $\varphi : E(K \cap M_0) \to M_0' \oplus E(M_1) \oplus \cdots \oplus E(M_n)$ such that $K = \{ y + \varphi(y) \mid y \in E(K \cap M_0), \varphi(y) \in M_0' \oplus M_1 \oplus \cdots \oplus M_n \}$. For each $1 \le i \le n$, let $\pi_i : M_0' \oplus E(M_1) \oplus \cdots \oplus E(M_n) \to E(M_i)$ be the canonical projection. Let $1 \le i \le n$ and consider the homomorphism $\pi_i\varphi : E(K \cap M_0) \to E(M_i)$.

Suppose that $\pi_i\varphi \neq 0$. Because R is right hereditary, $\pi_i\varphi(E(K \cap M_0))$ is a non-zero injective submodule of the indecomposable module $E(M_i)$ and hence $\pi_i\varphi(E(K \cap M_0)) = E(M_i)$. In particular, there exists $e_0 \in E(K \cap M_0)$ such that $\pi_i\varphi(e_0) = m_i$. Now $\varphi(e_0) = e' + e_1 + \cdots + e_n$, for some $e' \in M_0'$, $e_j \in E(M_j)$, $1 \leq j \leq n$, and $e_i = m_i$. There exist $s \in r(m_i)$, $t \in r(e_1, \ldots, e_{i-1}, e_{i+1}, \ldots e_n)$ such that $1 = s + t$. Then $\varphi(e_0 t) = e't + m_i t = e't + m_i(1-s) = e't + m_i$. It follows that $e_0 t + e't + m_i \in K$. Let $z_i := e_0 t + e't$. Then $z_i \in M_0$ and $z_i + m_i \in K$. If $\pi_i\varphi = 0$, choose any $z_i \in K \cap M_0$. In any case, $z_i + x_i \in K$, where $x_i \in \{0, m_i\}$.

We have proved that, for each $1 \leq i \leq n$, there exist $z_i \in M_0$ such that $z_i + x_i \in K$, where $x_i = 0$, if $\pi_i\varphi = 0$, and $x_i = m_i$, if $\pi_i\varphi \neq 0$. Then $z + x_1 + \cdots + x_n \in K$, where $z := z_1 + \cdots + z_n \in M_0$. Because $K \cap \overline{M} = 0$, it follows that $z \neq 0$.

Finally, note that $K = \{ y + \varphi(y) \mid y \in E(K \cap M_0), \varphi(y) \in M_0' \oplus M_1 \oplus \cdots \oplus M_n \} \subseteq M_0 \oplus (\oplus_{i=1}^{n} x_i R)$, because $x_i = 0$ if and only if $\pi_i\varphi = 0$. □

Theorem 3.3.3 *Suppose that R is a principal ideal domain and let the module $M := M_1 \oplus M_2$ be the direct sum of a torsion-free injective submodule M_1 and a cyclic torsion submodule M_2. Then M is self-cu-injective.*

Proof. There exists an element $m \in M_2$ such that $M_2 = mR$. Let $I := r(m)$. Then I is a non-zero ideal of R. If $I = R$, then $M_2 = 0$ and there is nothing to prove.

Suppose that $I \neq R$. Note that $I = P_1^{k_1} \cdots P_n^{k_n}$, for some positive integers n, k_i, $1 \leq i \leq n$, and distinct maximal ideals P_i, $1 \leq i \leq n$, of R. It follows that R/I is isomorphic to $(R/P_1^{k_1}) \oplus \cdots \oplus (R/P_n^{k_n})$ and the R-module R/I is extending.

Since M_2 is isomorphic to R/I, we have $M_2 = L_1 \oplus \cdots \oplus L_n$, where $m = l_1 + \cdots + l_n$, $L_i = l_i R$ and $P_i^{k_i} = r(l_i)$, for $1 \leq i \leq n$. For each $1 \leq i \leq n$ and each $w \in E(L_i)$, there exists a positive integer v such that $wP_i^v = 0$, by [54, Proposition 4.23]. It follows that $E(L_1), \ldots, E(L_n)$ are compatible.

Let U be a maximal uniform submodule of M. If $U \cap M_1 = 0$, then $U \subseteq M_2$, because M_2 is the torsion submodule of M. Since M_2 is isomorphic to R/I, it follows that U is a closed submodule of the extending module M_2 and hence U is a direct summand of M_2. In this case, it is clear that any homomorphism $\varphi : U \to M$ can be lifted to M.

Now suppose that $U \subseteq M_1$. Then U is a direct summand of M_1 and any homomorphism $\varphi : U \to M$ can be lifted to M.

Otherwise, $U \cap M_1 \neq 0$ and $U \not\subseteq M_1$. Clearly $U \cap M_2 = 0$ and hence, by rearranging the modules $L_1, \ldots, L_n$, if necessary, Lemma 3.3.2 gives that $(l_0 + l_1 + \cdots + l_k)R \subseteq U \subseteq M_1 \oplus L_1 \oplus \cdots \oplus L_k$, for some $0 \neq l_0 \in M_1$, $1 \leq k \leq n$. Let $X := L_1 \oplus \cdots \oplus L_k$ and $x := l_1 + \cdots + l_k$, so that $X = xR$.

Let $T := \{ c \in R \mid Xc = X \}$. Then T is a multiplicatively closed subset of the domain R and we let S denote the subring $\{ r/t \mid r \in R, t \in T \}$ of Q, the field of fractions of R. Given $a \in M_1$ and $b \in X$, we define $(a + b)(r/t) := ar/t + b'r$, where $b' \in X$ satisfies $b = b't$, for all $r \in R$, $t \in T$. This makes $M_1 \oplus X$ into an S-module. Note that, for each $c \in T$, $x = xrc$, for some $r \in R$, and hence $x(1 - rc) = 0$, i.e., $X(1 - rc) = 0$. It follows that X is T-torsion-free. Also, M_1 is a vector space over Q.

Now, we claim that $U = (l_0 + x)S$. Let $V := \{a \in M_1 \oplus X \mid at \in U, \text{ for some } t \in T\}$. Clearly, V is a submodule of $M_1 \oplus X$ and $U \leq V$. Since $M_1 \oplus X$ is T-torsion-free, it follows that U is essential in V. Thus $U = V$.

Let $r \in R$, $t \in T$. Then $((l_0 + x)(r/t))t = (l_0 + x)r \in U$ and hence $(l_0+x)(r/t) \in U$. Thus $(l_0+x)S \subseteq U$. Let $u \in U$. Then $(l_0+x)R+uR \subseteq U$, so that $(l_0+x)R+uR$ is a finitely generated uniform module over a principal ideal domain and hence is cyclic. Suppose that $(l_0+x)R+uR = (p+xd)R$, for some $p \in M_1$, $d \in R$. There exists $c \in R$ such that $l_0+x = (p+xd)c$ and hence $x = xdc$. It follows that $1-dc \in r(x)$ and hence $X = Xdc \subseteq Xc \subseteq X$, i.e., $X = Xc$ and $c \in T$. Hence $p+xd = (l_0+x)(1/c) \in (l_0+x)S$. It follows that $u \in (l_0+x)S$. Thus $U = (l_0+x)S$.

Let $y := l_{k+1}+\cdots+l_n$, so that $m = x+y$. Because $P_1, \ldots, P_n$ are distinct maximal ideals, $R = (P_1^{k_1} \cap \ldots \cap P_k^{k_k}) + (P_{k+1}^{k_{k+1}} \cap \ldots \cap P_n^{k_n}) = r(x) + r(y)$. Then, there exists $c \in R$ such that $xc = x$ and $yc = 0$. Clearly, $c \in T$.

Let $\varphi : U \to M_2$ be an R-homomorphism. Suppose that $\varphi[(l_0+x)(1/c)] = mf$, for some $f \in R$. Then

$$\varphi(l_0 + x) = \varphi[(l_0 + x)(1/c)]c = mfc = (x + y)fc = xf.$$

A similar argument shows that $\varphi(U) \leq X$. If $r \in R$ and $t \in T$, then $(l_0 + x)(r/t) = l_0(r/t) + x'r$, where $x' \in X$ and $x't = x$. So,

$$(\varphi[(l_0 + x)(r/t)])t = \varphi[(l_0 + x)r] = xfr = x'tfr.$$

Because X is T-torsion-free, it follows that $\varphi[(l_0 + x)(r/t)] = x'fr$.

Note that $M = M_1 \oplus xR \oplus yR$ and let $\pi : M \to xR$ be the canonical projection with kernel $M_1 \oplus yR$. Define $\theta : xR \to M_2$ by $\theta(z) = zf$, for all $z \in xR$. Clearly, θ is an R-homomorphism. Then $\theta\pi : M \to M_2$ is a homomorphism and $\theta\pi(M) \leq xR$. Also, for $r \in R$ and $t \in T$,

$$(\theta\pi[(l_0 + x)(r/t)])t = \theta\pi[(l_0 + x)r] = \theta(xr) = xfr = x'tfr.$$

Again because X is T-torsion-free, it follows that

$$\theta\pi[(l_0 + x)(r/t)] = x'fr = \varphi[(l_0 + x)(r/t)].$$

Thus φ can be lifted to M. We have proved that M_2 is M-c-injective.

Since M_1 is an injective module, it now follows that M is self-cu-injective, by Lemma 3.1.5. □

Combining Theorem 3.3.3 and Proposition 3.1.13, we have the next result without further proof.

Theorem 3.3.4 *Suppose that R is a principal ideal domain and let the module $M := M_1 \oplus M_2$ be the direct sum of a torsion-free indecomposable injective submodule M_1 and a cyclic torsion submodule M_2. Then M is self-c-injective.*

Proposition 3.3.5 *Suppose that R is a principal ideal domain, let p be a prime in R and let M be a p-primary module with uniform dimension 2. Then M is self-c-injective.*

Proof. If M is injective, then there is nothing to prove. Suppose that $M = M_1 \oplus M_2$, where M_1 is indecomposable injective and $M_2 = mR$, where $mp^n = 0$, $mp^{n-1} \neq 0$, for some integer n. Let U be a maximal uniform submodule of M and let $\varphi : U \to M_2$ be a homomorphism.

Either U is isomorphic to M_1 and U is a direct summand of M or U is cyclic. Suppose that U is cyclic. Then $U = (x + ma)R$, for some $x \in M_1$ and $a \in R$ with $ma \neq 0$. Suppose that $a \in pR$. Then $a = pb$, for some $b \in R$, and $x = yp$, for some $y \in M_1$. Let $U' := (y + bm)R$ and note that $(y + mb)p = x + ma$. Then U' is a cyclic p-primary module, so is uniform, and $U \subseteq U'$. It follows that $U = U'$. Hence we can suppose, without loss of generality, that $a \notin pR$, in fact $a = 1$.

Suppose that $\varphi(x + m) = mr$, for some $r \in R$. Define $\theta : M \to M_2$ by $\theta(z + mc) = mcr$, for all $z \in M_1$, $c \in R$. It is clear that θ is well-defined and is a homomorphism. Moreover, for all $s \in R$, $\theta((x + m)s) = \theta(xs + ms) = msr = \varphi((x + m)s)$. Thus φ is the restriction of θ to U.

Hence every homomorphism from U to M_2 can be lifted to M. Since M_1 is injective, it follows that M is self-c-injective, by Lemma 3.1.5.

Now suppose that $M = m_1R \oplus m_2R$, where m_1 has order ideal p^sR and m_2 has order ideal p^tR, for positive integers $s \leq t$. Let U be a maximal uniform submodule of M. Since m_2R is quasi-injective, by [40, page 19], it follows that m_2R is m_1R-injective, by [40, Proposition 1.3], and hence M-injective, by [40, Proposition 1.5]. Thus any homomorphism from U to m_2R can be lifted to M.

Let $\varphi : U \to m_1R$ be a homomorphism. Then $U = (m_1a + m_2b)R$, for some $a, b \in R$. By the above argument, we can suppose without loss of generality that $a = 1$ or $b = 1$. If $b = 1$, then $M = M_1 \oplus U$ and φ lifts to M. Suppose that $a = 1$ and $\varphi(m_1 + m_2b) = m_1r$, for some $r \in R$. Define $\theta : M \to m_1R$ by $\theta(m_1r_1 + m_2r_2) = m_1r_1r$, for all $r_1, r_2 \in R$. Then θ is well defined and is a homomorphism. Moreover, φ is the restriction of θ to U. It follows that any homomorphism from U to m_1R lifts to M.

Therefore, M is self-c-injective. □

Corollary 3.3.6 *Suppose that R is a Dedekind domain, let P be a maximal ideal of R and let M be a P-torsion module with uniform dimension 2. Then M is self-c-injective.*

Proof. Without loss of generality, by localizing at P, we can suppose that R is a local ring with unique maximal ideal P. By [67, Theorem 16, page 278], R is a principal ideal domain. Apply Proposition 3.3.5. □

Contrast Proposition 3.3.5 and Corollary 3.3.6 with the following example.

Let p be any prime in $\mathbb{Z}$ and let M be the $\mathbb{Z}$-module $(\mathbb{Z}/p\mathbb{Z})\oplus(\mathbb{Z}/p^2\mathbb{Z})\oplus(\mathbb{Z}/p^4\mathbb{Z})$. Let us prove that M is not self-cu-injective.

Let U denote the submodule $(1+p\mathbb{Z}, p+p^2\mathbb{Z}, p+p^4\mathbb{Z})\mathbb{Z}$. Then U is a cyclic p-torsion module, and so is uniform. Suppose that U is essential in a submodule V of M. Thus, V is also uniform and $V = (a+p\mathbb{Z}, b+p^2\mathbb{Z}, c+p^4\mathbb{Z})\mathbb{Z}$, for some $a, b, c \in \mathbb{Z}$. There exists $d \in \mathbb{Z}$ such that

$$(1+p\mathbb{Z}, p+p^2\mathbb{Z}, p+p^4\mathbb{Z}) = (a+p\mathbb{Z}, b+p^2\mathbb{Z}, c+p^4\mathbb{Z})d.$$

Hence 1 is congruent to da, modulo p, so that $d \notin p\mathbb{Z}$. Therefore, $U = V$ and we can conclude that U is a closed submodule of M.

Define a homomorphism $\alpha : U \longrightarrow M$ by

$$\alpha[(1+p\mathbb{Z}, p+p^2\mathbb{Z}, p+p^4\mathbb{Z})r] = (p\mathbb{Z}, r+p^2\mathbb{Z}, p^4\mathbb{Z}),$$

for all $r \in \mathbb{Z}$. Suppose that α can be lifted to a homomorphism $\beta : M \to M$. Then

$$\beta(p\mathbb{Z}, p^2\mathbb{Z}, 1 + p^4\mathbb{Z}) = (u + p\mathbb{Z}, v + p^2\mathbb{Z}, w + p^4\mathbb{Z}),$$

for some $u, v, w \in \mathbb{Z}$. Thus

$$\beta(p\mathbb{Z}, p^2\mathbb{Z}, p^2 + p^4\mathbb{Z}) = (p\mathbb{Z}, p^2\mathbb{Z}, p^2w + p^4\mathbb{Z})$$

and

$$\alpha(p\mathbb{Z}, p^2\mathbb{Z}, p^2 + p^4\mathbb{Z}) = \alpha[(1 + p\mathbb{Z}, p + p^2\mathbb{Z}, p + p^4\mathbb{Z})p] = (p\mathbb{Z}, p + p^2\mathbb{Z}, p^4\mathbb{Z}).$$

Therefore, $\alpha(p\mathbb{Z}, p^2\mathbb{Z}, p^2 + p^4\mathbb{Z}) \neq \beta(p\mathbb{Z}, p^2\mathbb{Z}, p^2 + p^4\mathbb{Z})$ and α cannot be lifted to M. It follows that M is not self-cu-injective.

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