

Lagrangian Modelling of nonlinear waves in optical fibres

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The usefulness of this work is left to the reader to decide. If it is indeed useful this is not only due to myself but to all the people that have contributed in this direction through these years and should be acknowledged in this paragraph. In this spirit I would like to thank, first of all, my supervisor Prof. John M. Arnold for his constant support through these years. Furthermore, I would like to express my gratitude to my parents for all the moral backing, domain in which Paras Kommata helped considerably. Computing support of the electrical and electronic engineering department is also acknowledged for their valuable suggestions, and similarly for fellow students of the university. Finally, I gratefully acknowledge the generous financial support of the Hellenic state scholarship foundation.

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ABSTRACT

The Lagrangian perturbation method for the NLS is revisited in the form of an equivalent direct problem. The analogy can be extended to arbitrarily perturbed systems. It is then possible to provide first order perturbation expansions for the fundamental soliton. The case of the damped NLS is considered and shown to fully comply with IST predictions.

Subsequently the problem of NLS initial condition not corresponding to an exact soliton is examined. There are two issues that need to be considered. The location of the soliton solution and the modelling of the continuum.

The location of the soliton solution is handled by considering the integrals of motion of the NLS. The improvement arises by the inclusion of the contributions due to the continuum. The results are compared with numerical calculations and are proved to be satisfactory provided that the initial pulse shape does not depart greatly from the $A\text{sech}(z)$ functional form.

The propagation problem is handled by considering the evolution of the soliton and the continuum separately and recombining them at the required time. Two cases are considered: the far field pattern and the position where the peak of the soliton lies. For the former the recombination of continuum with the soliton is achieved with the help of the inverse part of the IST. For the peak position a Bäcklund transform is considered. Results from both regimes are compared with numerical results and shown to agree satisfactorily.

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1. INTRODUCTION

The past two decades have been marked by an unprecedented increase in the capacity demand of telecommunication channels. This demand, driven mainly by the growth of computer networking and the need for the transmission of video information over public telephony channels, has led to the adoption of optical fibre as the propagation medium. This process has been facilitated by the advent of the low loss fibre in the late seventies. Nowadays the inter-metropolitan network being currently laid is solely fibre-optic.

The fibre as a medium provides three low loss windows situated at $0.85\ \mu m$, $1.3\ \mu m$ and $1.55\ \mu m$. Modern telecommunication systems employ only the latter two windows. Apart from loss, the other limiting factor in optical fibre links is dispersion. The geometry of the waveguide, as well as the intrinsic properties of its bulk material, impose a frequency dependence on the group velocity. This leads to the broadening of transmitted pulses, with obvious effects on the quality of the transmitted pulse-train.

The $1.55\ \mu m$ transmission window, in addition to the fact that it presents the lowest losses of all three, exhibits anomalous group velocity dispersion. This means that the group velocity decreases with increasing wavelength. An effect that can be used to limit the dispersion induced broadening, for this type of dispersion, is the Kerr effect, ie the intensity induced change of the refractive index. Given sufficiently high optical powers, it is possible to produce pulses having these two effects balancing each other. Under these condition pulse broadening does not occur for large propagation distances. This type of behaviour, in optical fibres, was predicted by Hasegawa and Tappert in 1973 [HT73].

The experimental demonstration of this theoretical prediction came much later in 1980 [MSG80]. This experimental demonstration has prompted more research into the possibility of achieving ever larger channel capacities benefiting from the Kerr effect. Nowadays, repeated experiments point to the feasibility of transmission rates of the order of 20 Gb/s over transatlantic distances [N⁺95a] [N⁺95b] [LBR⁺95] [RCD⁺95] -extensive review in [HW96]. During the last few years an increase in the experimental activity motivated by the interest of the major telecommunication companies indicates that nonlinear optical communications are on the verge of being applied commercially.

But let's return to the 1973 paper. The main result of Hasegawa and Tappert is that the envelope of a wave propagating in an optical fibre obeys the NonLinear Schrödinger (NLS) equation. The latter, belongs to a class of nonlinear integrable equations solvable by the inverse scattering method [ZS72]. This type of equation corresponds to a conservative, Hamiltonian system.

A particularly interesting property is the exhibition of soliton solutions. Solitons are robust, localised wave-packets which tend to conserve their properties (energy, momentum etc.) regardless of the presence of radiation in their vicinity. This property, attributed to the integrability of the governing nonlinear differential equation, makes them particularly attractive to optical communication systems. It is possible to prove, that starting from an arbitrary pulse shape, whose area is larger than a certain minimum, and given sufficient propagation distance, after shedding some of its radiation away (continuum) the emerging pulse is a soliton. In this context these localised waves emerge as “natural” carriers of information in the optical fibre.

From the mathematical point of view, the initial value problem corresponding to the NLS can be completely solved with the inverse scattering transform. This fact does not mean that modeling propagation in an optical fibre is a trivial problem. First, even though the solution method for the NLS exists, the result comes in closed form only in very few cases. Second, the NLS is an idealisation. In real fibres a multitude of phenomena, other than second order dispersion and Kerr nonlinearity, are present. The result is that propagation equations, in the regimes of interest, are perturbed versions of the NLS. In general, it is not possible to produce an inverse scattering type solution to the initial value problem in these cases. It is for these reasons that a multitude of perturbation and numerical schemes have emerged during the last twenty years.

These approximation schemes are required to tackle three types of problems: that of non-soliton initial conditions, that of perturbed evolution equation or a combination of both. The central problem here is to predict the key features of emerging pulses given an injected pulse-train.

In contrast with linear fibres, a propagating wavepacket will not only be affected by the characteristics of the link but by radiation in its vicinity. A famous example of this is Gordon-Haus noise [GH86], whereby an amplified soliton incorporates some of the spontaneous emission due to the amplifier. The soliton central frequency shifts, thus affecting its group velocity and eventually its position.

Analytic perturbation techniques have mainly focused in the description of soliton characteristics under the influence of a governing perturbed equations. Relatively little work has been done in the field of describing the main characteristics of the continuum.

This type of radiation is not important on its own, but in conjunction with nonlinearity can affect the characteristics of propagating soliton pulses. The changes brought about are small, but given the fact that the main field of application of solitons is ultralong links, their accumulation can lead to serious degradation of the signal quality.

The objective of this thesis is to address the continuum problem. Of course one can always resort to numerical simulation in order to address the propagation issues in the presence of radiation. It is not our purpose here to make statements as to the comparative use of analytical calculations and numerical computing. The purpose of this choice can be summarised by the commonplace statement that analytical calculation allow for the interpretation of physical processes occurring during propagation and their usefulness is more in the direction of providing insight rather than exact figures. The description of the continuum is not simple. Unlike the soliton whose evolution can be described with a limited number of dynamical variables, dispersive radiation has an infinite number of such degrees. Furthermore, in order to achieve a successful perturbation expansion we need to know the contribution of the discrete (soliton) modes as well as the continuous ones (radiation). This problem, again, is not a trivial one to solve even though mathematical tools for its solution exist (IST).

As far as organisation of the thesis is concerned, it is divided in 6 chapters and an introduction. In the *second* chapter we will be introducing some definitions used later. The derivative expansion method will be described and so will the variational method in the solution of differential equations.

The derivative expansion method will be used in chapter *three* where the governing equation for light propagation in a fibre will be derived from Maxwell's equations. In the process we will be examining the main physical phenomena involved in optical waveguiding.

Having derived the equations describing the evolution of the propagating pulse envelope we will review some of the available solution methods in the *fourth* chapter. Namely we will be looking at the inverse scattering transform solution of the NLS and some of the mathematical properties associated with it. The main perturbation methods will be outlined and compared.

The *fifth* chapter will be devoted to the Lagrangian perturbation method in the form suggested by Anderson and Lisak, namely by predetermining the solution. We will give a description of the method and look for links with other mainstream perturbation methods. The shortcomings of the method will be pointed out.

In the *sixth* chapter an attempt is made to combine this method with structural information of the NLS system to provide information on the evolution of the charac-

teristics of a pulse under non-soliton initial conditions.

Finally the results will be reviewed and suggestions for improvements and further work will be made in the *seventh* and final chapter.

2. MATHEMATICAL METHODS

2.1 Introduction

In this chapter we will be addressing some of the mathematical methods that are going to be used throughout this work. Mathematical problems do not always give exact solutions in closed forms, or, even if this occurs, the final outcome is extremely complicated and does not allow physical interpretation or simple calculations. In these cases we are forced to use either approximations or numerical methods. The latter, although offering nowadays a very efficient analysis tool, do not allow for the derivation of analytic formulae. The only alternative open to us then, should we be looking for analytic results, is that of approximations, among which asymptotic expansions are predominant.

Asymptotic methods are first of all perturbation methods. This means that they can be applied in the specific case where an equation is prevented from being solved by the presence of a “small” term. The “smallness” of the term is usually quantified by the presence of an appropriate parameter. This area of mathematics has been extensively studied during the past few decades leading to numerous techniques [Nay73] [JK82] [BMS76] [Whi74].

The technique that we will be mainly using is called the *derivative expansion method*, and belongs to a greater class of *multiple scales expansions*. In the next section we will be discussing the use of asymptotic analysis. Some definitions that are needed will be introduced at the beginning.

The other part of this chapter will be a discussion of the variational method. According to the latter it is possible to give the solution of a problem by optimising an appropriate quantity. The basic formulas will be given in section 2.3 . Furthermore we will make the appropriate generalisations so that we will be able to combine the derivative expansion procedure with the variational method. Lastly a brief description of the approximate method due to Rayleigh and Ritz will be given, although we will be dealing with this method in details at a later stage. [MF53], [Sag61], [Sta68].

2.2 Asymptotic methods

2.2.1 Some definitions

In order to allow for simpler expressions we will be using the Landau symbols O and o .

Definition 1: [Sir71] Let ϕ and ψ be defined in a common region S of the complex plane. Then:

$$\phi = O(\psi) \quad \text{if} \quad \exists A > 0 : \left| \frac{\phi}{\psi} \right| \leq A \quad \forall z \in S$$

In practice the symbol O will be used here in a much less restricted fashion as the following definition will indicate.

Definition 2: [Sir71] If the functions $\phi(x)$ and $\psi(x)$ are defined in a common domain S and $x_0 \in \bar{S}$, where the bar denotes the closure then we write:

$$\phi(x) = O(\psi(x)) \quad \text{as} \quad x \rightarrow x_0 \quad (2.1)$$

if

$$\lim_{x \rightarrow x_0} \lim_{x \in S} \left| \frac{\phi(x)}{\psi(x)} \right| < \infty$$

Accordingly the symbol o is defined.

Definition 3: [Sir71] If the functions $\phi(x)$ and $\psi(x)$ are defined in a common domain S and $x_0 \in \bar{S}$, where the bar denotes the closure then we write:

$$\phi(x) = o(\psi(x)) \quad \text{as} \quad x \rightarrow x_0 \quad (2.2)$$

if

$$\lim_{x \rightarrow x_0} \lim_{x \in S} \left| \frac{\phi(x)}{\psi(x)} \right| = 0$$

These definitions can be readily extended to functions of many variables. In this case if the prerequisites of relations (2.1, 2.2) are fulfilled, irrespective of the values of the remaining independent variables, then the corresponding definitions are said to be holding *uniformly*.

During the rest of this work we will be dealing with differential problems where a perturbation term is present. This term will be quantified by a small parameter, represented for clarity, by a lowercase greek letter. In order to emphasize the difference between this parameter (ϵ usually) and the other independent variables of the functions that we will encounter, it will appear explicitly, while the others may not.

Definition 4: [JK82] A sequence of functions $\{\phi_n(\epsilon)\}$ involving a parameter ϵ is called an asymptotic sequence as $\epsilon \rightarrow \epsilon_0$ if $\phi_{n+1} = o(\phi_n)$ as $\epsilon \rightarrow \epsilon_0$ for every $n \geq 0$

Definition 5: [JK82] Assume that we have an asymptotic sequence $\{\phi_n(\epsilon)\}$ and a function $f(x; \epsilon)$ which is approximated by the series:

$$\sum_{n=0}^N \phi_n(\epsilon) f_n(x) \quad \text{as } \epsilon \rightarrow \epsilon_0$$

such that:

$$f(x; \epsilon) = \sum_{n=0}^N \phi_n(\epsilon) f_n(x) + o(\phi_N(\epsilon)) \quad \text{as } \epsilon \rightarrow \epsilon_0 \quad (2.3)$$

Then this expression is called an asymptotic approximation to $(N+1)$ terms of the function $f(x; \epsilon)$ as $\epsilon \rightarrow \epsilon_0$ with respect to the asymptotic sequence $\{\phi_n(\epsilon)\}$

If the relation above holds for every $N \geq 0$ the series in (2.3) is called an asymptotic expansion of $f(x; \epsilon)$ as $\epsilon \rightarrow \epsilon_0$. We will be concerned mainly with a type of asymptotic expansions called *asymptotic power series* occurring when $\phi_n(\epsilon) = (\epsilon - \epsilon_0)^n$. For simplicity, problems will be scaled so that $\epsilon_0 = 0$. Expansion (2.3) becomes then:

$$f(x; \epsilon) = \sum_{n=0}^N \epsilon^n f_n(x) + o(\phi_N(\epsilon)) \quad \text{as } \epsilon \rightarrow 0 \quad (2.4)$$

Relations (2.3, 2.4) can be easily generalised for functions depending on more than two variables. Contrary to the usual convergent series, asymptotic expansions are usually divergent and are not generally unique.

Operations on asymptotic series [Geo95] : The operations of addition, multiplication by a scalar and integration of asymptotic series can be performed on a term by term basis. It is not always possible to differentiate asymptotic series or multiply them term wise, but these are possible for asymptotic power series, where the two operations are executed as if we had a usual convergent series. In the analysis of problems with asymptotic series we do not need to consider an infinite amount of terms, we usually limit ourselves to the consideration of the properties of the first few. The main consideration is to determine the regions where expansion (2.4) converges uniformly in x . Expansions where we have uniform convergence throughout the domain of the independent variables are called *regular*. We will be more interested in expansions where the above statement does not hold. This type of asymptotic expansion is called *singular*.

To conclude this part we introduce the notion of the *variational derivative*. Let us consider a function u analytical in its domain of definition which is the real axis. We consider a *functional* F depending on this function and expressed as the integral

of a quantity F_1 over the x axis. We now have for the variation of F with respect to changes in u :

$$\delta F(u) = F(u + \delta u) - F(u) = \int_{-\infty}^{\infty} \frac{\delta F_1}{\delta u(x)} \delta u(x) dx + O(\delta u^2) \quad (2.5)$$

The functional $\frac{\delta F_1}{\delta u(x)}$ is the variational derivative of F with respect to u . If we assume that the function and all its derivatives vanish whenever $|x| \rightarrow \infty$ we can perform integration by parts on (2.5) by which:

$$\frac{\delta F}{\delta u(x)} = \frac{\partial F}{\partial u(x)} - \frac{d}{dx} \frac{\partial F}{\partial (u(x))_x} + \frac{d^2}{dx^2} \frac{\partial F}{\partial (u(x))_{xx}} + \dots \quad (2.6)$$

The x subscripts indicate differentiation with respect to x .

2.2.2 The derivative expansion method

Let us start by considering a simple example [San65]. Suppose that we have a physical quantity, given by a function f whose temporal evolution is described by the differential problem:

$$\frac{d}{dt} f(t) = -\epsilon f(t) \quad , \quad f(0) = 1 \quad , \quad \epsilon > 1 \quad (2.7)$$

where $\epsilon \ll 1$. This problem has an exact solution $f(t) = e^{-\epsilon t}$. Let us now assume that an observer whose clock runs relatively “fast” monitors our physical quantity f . The presence of the small parameter ϵ in the exponent will mean that, for this observer, the function will change very slowly. His first approximation will thus come naturally in the form:

$$f(t) \sim 1 + \dots$$

However, if he keeps track of the function for a sufficiently large interval, he will notice a small decrease. Subsequently he will change from the original approximation to a more “accurate” one:

$$f(t) \sim 1 - \epsilon t + \dots$$

The new approximation will yield better results for short time periods. Nevertheless for $t = \frac{1}{\epsilon}$ the approximated function becomes zero and for a much larger time it becomes negative, which, as we know from the exact solution, is erroneous. The problem with the linear approximation is that given a sufficient time interval it will lead to large deviations from the function it is approximating. The same will happen to the next order correction ($\epsilon^2 \frac{t^2}{2}$) although at a somewhat later time.

Mathematically speaking, the mistake that our observer made was to assume that the solution of (2.7) can be expressed as a power series in ϵ . Let us now suppose

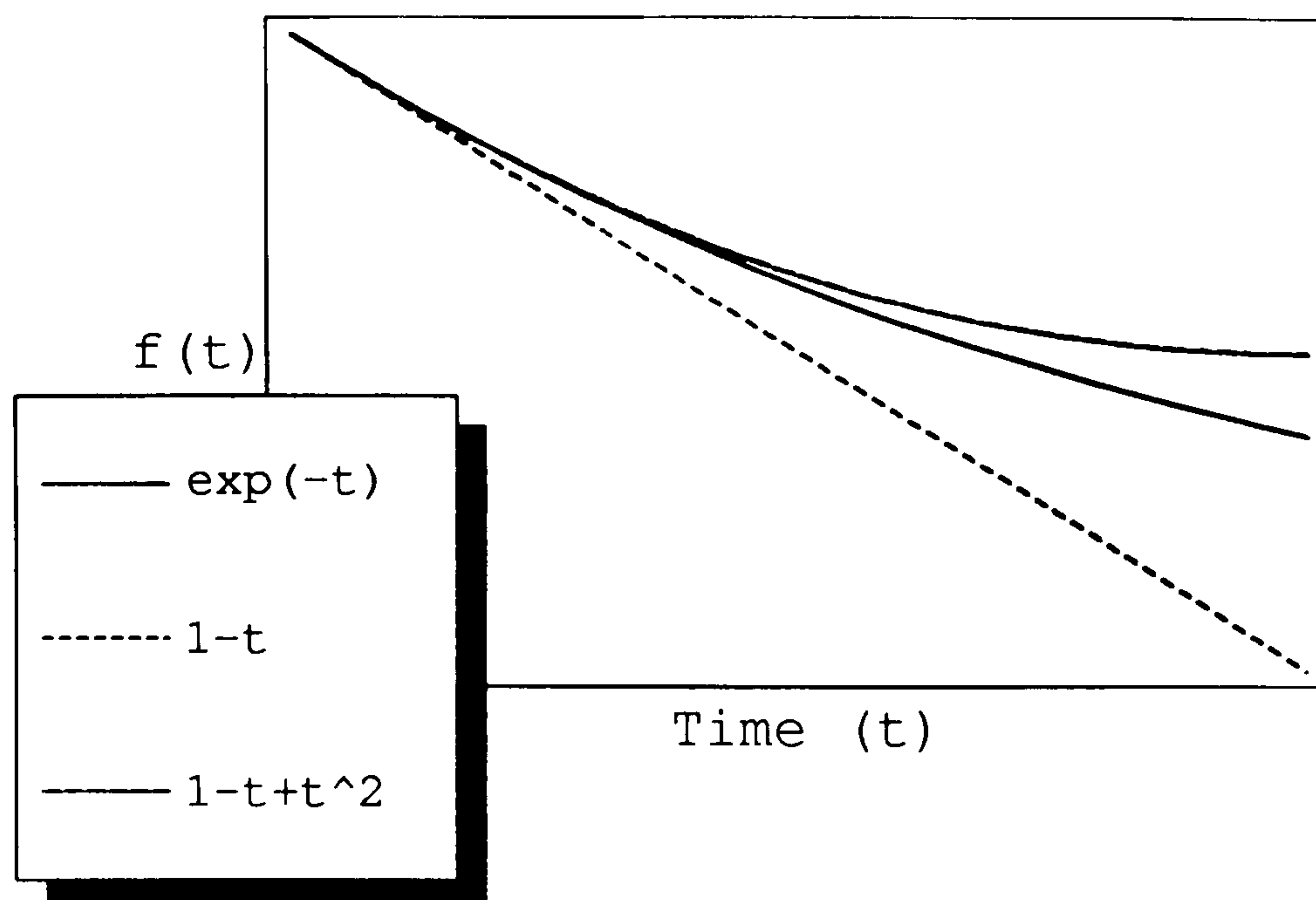


Fig. 2.1: Exponentially decaying function, approximated linearly and quadratically

that the same problem was viewed from the point of view of a second observer whose clock is running at a much “slower” time $t' = \epsilon t$. The function f will now look like $f(t') = e^{-t'}$. For this observer the rate of change of the function is large enough not to allow approximation. Thus for large period no expansions would be allowed. However if he was to monitor the function for a very short period he would be approximating it by a Taylor expansion in powers of $t' = \epsilon t$:

$$f(t) = 1 - \epsilon t + \frac{1}{2} \epsilon^2 t^2 + \dots$$

In the problem described above there were two time scales. It is possible, for a more complicated problem, to encounter N time scales. The idea behind the so called *multiple scales expansions* is to combine the readings of n independent observers in order to come up with a more suggestive description than the one offered by the previously outlined procedure. The derivative expansion method is a systematic way of dealing with this specific problem. In order to understand the mechanisms leading to the extraction of asymptotic solutions we will follow the customary method of using examples, as it is much more suggestive and does not require heavy algebraic calculations. We will be

using the equation of the damped linear harmonic oscillator [Nay73]:

$$\frac{d^2}{dt^2} f(t) + 2\epsilon \frac{d}{dt} f(t) + f(t) = 0 \quad (2.8)$$

complemented by a set of initial conditions and with $\epsilon \ll 1$. The exact solution of this equation is:

$$f(t) = a e^{-\epsilon t} \cos(\sqrt{1 - \epsilon^2} t + \phi) \quad (2.9)$$

Let us proceed in the same way as with the previous example. If we had an observer measuring the solution (2.9) he would be able to adjust his clock either by referencing it to the oscillation, or to the much smaller rate of decay of the amplitude of the oscillation.

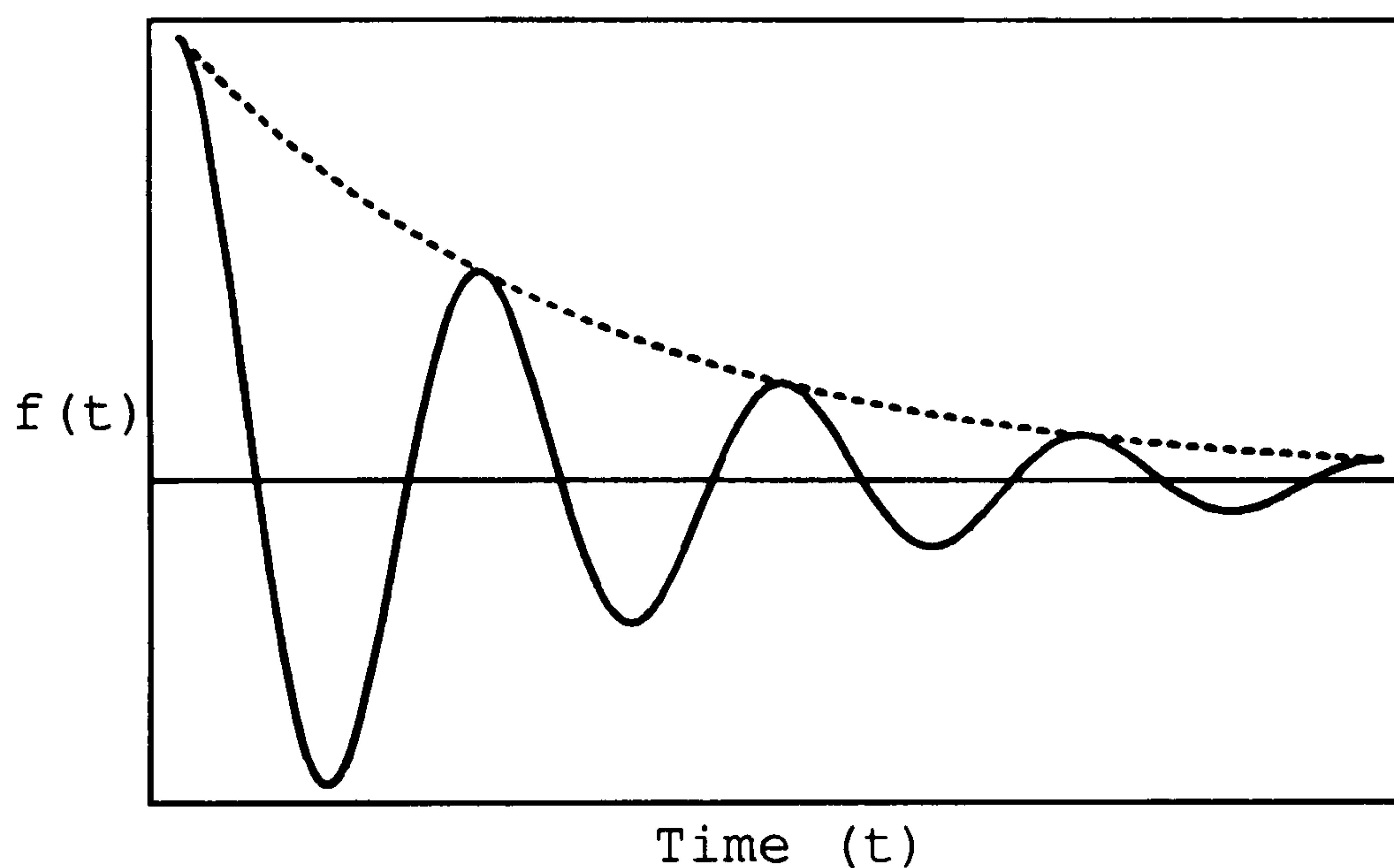


Fig. 2.2: Exponential decaying oscillation, the dashed line represents the amplitude

When attempting to find asymptotic solutions to (2.8) the first step is to assume that the solution $f(t)$ can be expressed as a power series in ϵ :

$$f(t) = f_0(t) + \epsilon f_1(t) + \epsilon^2 f_2(t) + \dots \quad (2.10)$$

Straightforward perturbation theory would require substitution of (2.10) into (2.8), collection of terms in powers of ϵ and equating the coefficients of the expansion with zero. However the derivative expansion method introduces one more expansion in the

problem. It is assumed that the independent variable, t here, is, in its turn, a function of N other independent variables t_n . This effectively extends the domain over which the problem (2.8) is defined to an N -Dimensional real space. For the purpose of the solution only one trajectory in this space needs to be considered, the one where:

$$t_n = \delta^n t \quad (2.11)$$

The value of δ should be derived from the characteristics of the problem in hand. Here the choice is $\delta = \epsilon$.

The introduction of the new variables requires a change in the differential operators:

$$\frac{d}{dt} = \frac{\partial}{\partial t_0} + \epsilon \frac{\partial}{\partial t_1} + \epsilon^2 \frac{\partial}{\partial t_2} + \dots \quad (2.12)$$

Relations (2.10, 2.12) are substituted into (2.8) and the resulting expansion is rewritten as a power series in ϵ . By expanding the exact solution in Taylor series, we can see, that in order to include all main tendencies of the solution, we need expansions at least to $O(\epsilon^2)$. Accordingly, we will be limiting the analysis to the first three terms of the expansion above. The requirement that all coefficients are equal to zero, independently, leads to the following system of equations:

$$\begin{aligned} &O(1) : \\ &\frac{\partial^2}{\partial t_0^2} f_0 + f_0 = 0 \end{aligned} \quad (2.13)$$

$$\begin{aligned} &O(\epsilon) : \\ &\frac{\partial^2}{\partial t_0^2} f_1 + f_1 = -2 \frac{\partial}{\partial t_0} f_0 - 2 \frac{\partial^2}{\partial t_0 \partial t_1} f_0 \end{aligned} \quad (2.14)$$

$$\begin{aligned} &O(\epsilon^2) : \\ &\frac{\partial^2}{\partial t_0^2} f_2 + f_2 = -2 \frac{\partial}{\partial t_0} f_1 - 2 \frac{\partial^2}{\partial t_0 \partial t_1} f_1 - \frac{\partial^2}{\partial t_1^2} f_0 - 2 \frac{\partial^2}{\partial t_0 \partial t_2} f_0 - 2 \frac{\partial}{\partial t_1} f_0 \end{aligned} \quad (2.15)$$

The general solution of (2.13) is:

$$f_0(t_0, t_1, t_2) = f_{00}(t_1, t_2) e^{i t_0} + \bar{f}_{00}(t_1, t_2) e^{-i t_0} \quad (2.16)$$

where the bar denotes complex conjugation. (2.16) is substituted into (eq. 2.14) to give:

$$\frac{\partial^2}{\partial t_0^2} f_1 + f_1 = -2i \left(f_{00} + \frac{\partial}{\partial t_1} f_{00} \right) e^{i t_0} + 2i \left(\bar{f}_{00} + \frac{\partial}{\partial t_1} \bar{f}_{00} \right) e^{-i t_0}$$

The qualitative difference due to the adoption of the derivative expansion is the presence of the time derivative terms on the right hand side of the equation above. The

general solution is:

$$f_1(t_0, t_1, t_2) = f_{11}(t_1, t_2) e^{i t_0} + \bar{f}_{11}(t_1, t_2) e^{-i t_0} - t_0 \left(f_{00} + \frac{\partial}{\partial t_1} f_{00} \right) e^{i t_0} - t_0 \left(\bar{f}_{00} + \frac{\partial}{\partial t_1} \bar{f}_{00} \right) e^{-i t_0} \quad (2.17)$$

Particular attention should be paid to the last two terms on the right hand side of (2.17): they increase with t_0 , leading to the breakdown of the approximation for t_0 sufficiently large. This type of term is referred to as *secular* [JK82]. Their suppression is essential in order to construct estimates valid for large times.

We now notice that f_0 appears in (2.13) differentiated with respect to t_0 and in (2.14) differentiated with respect to both t_0 and t_1 . It is clear that these equations must be compatible in the sense that when we take time derivatives of f_0 we would be able to change the order of differentiation with respect to the two variables: t_0 and t_1 .

Based on the arguments of these two paragraphs, we may prove [San65] that suppression of the secular term can only occur if:

$$f_{00} + \frac{\partial}{\partial t_1} f_{00} = 0 \quad (2.18)$$

which results in:

$$f_1(t_0, t_1, t_2) = f_{11}(t_1, t_2) e^{i t_0} + \bar{f}_{11}(t_1, t_2) e^{-i t_0} \quad (2.19)$$

$$f_{00}(t_1, t_2) = f_{000}(t_2) e^{-t_1} \quad (2.20)$$

These two are, in their turn, substituted in (2.15). The general solution of the equation that will be generated will contain a secular term (linearly increasing with t_0). The requirement that this term should be suppressed leads to:

$$2i \left(f_{11} + \frac{\partial}{\partial t_1} f_{11} \right) = \left(f_{000} - 2i \frac{\partial}{\partial t_2} f_{000} \right) e^{-t_1} \quad (2.21)$$

The general solution of (2.21) for f_{11} will contain a secular term arising from the term on the right hand side. This is clear by a mere inspection. The coefficient of e^{-t_1} is a constant with respect to the variable t_1 . In the general solution this will mean the presence of a term of the type $t_1 e^{-t_1}$. Although this term is bounded in the limit $t_1 \rightarrow \infty$ it will exhibit near linear growth up to time $\frac{1}{\epsilon}$ which is an undesired feature. Hence we will require:

$$\left(f_{000} - 2i \frac{\partial}{\partial t_2} f_{000} \right) = 0$$

giving, together with (2.21):

$$f_2(t_0, t_1, t_2) = f_{22}(t_1, t_2) e^{i t_0} + \bar{f}_{22}(t_1, t_2) e^{-i t_0} \quad (2.22)$$

$$f_{11}(t_1, t_2) = f_{111}(t_2) e^{-t_2} \quad (2.23)$$

$$f_{000}(t_2) = c_0 e^{-i t_2/2} \quad (2.24)$$

This completes the expansion part of the problem. We now have to construct the solution. To this end (2.16, 2.20, 2.24) are combined to yield:

$$f_0(t_0, t_1, t_2) = \left(c_0 e^{i(t_0 - \frac{t_2}{2})} + \bar{c}_0 e^{-i(t_0 - \frac{t_2}{2})} \right) e^{-t_1}$$

By comparing (2.19, 2.23) with their counterparts for f_0 we detect that the solutions for f_1 follow the same pattern as those of f_0 . The findings of the third order equations would have consolidated this observation which will allow us to absorb the contribution from f_1, f_2 into the constant that appears in the zero order solution. This analogy in the solutions of f_0, f_1 and f_2 is not an intrinsic property of the method. It just occurs here because the problem that we are dealing with is linear. Consequently after reinstating the initial time scale with the help of $t_n = \epsilon^n t$ the solution of (2.8) can be written as:

$$f(t) = \left(c e^{i(t - \epsilon^2 \frac{t}{2})} + \bar{c} e^{-i(t - \epsilon^2 \frac{t}{2})} \right) e^{-\epsilon t} + O(\epsilon^3)$$

which by substituting $c = c_0 e^{i\phi_0}$ will give:

$$f(t) = 2 c_0 e^{-\epsilon t} \cos \left(\left(1 - \frac{\epsilon^2}{2} \right) t + \phi_0 \right) + O(\epsilon^3) \quad (2.25)$$

To derive this solution we have taken the expansion up to order ϵ^2 . We therefore expect it to hold for time scales up to $O(\frac{1}{\epsilon^2})$. By comparing (2.25) with the exact solution (2.9) we will notice that the frequency of the oscillation has been expanded up to the ϵ^2 term while the exponential term has been left untouched. We notice thus a selectivity in the parameters that are expanded. This is due to the introduction of additional degrees of freedom in the form of the 2 scales t_1 and t_2 .

In the example analysed here we introduced “stretched” time scales. There is nothing that would have prevented us from using “compressed” scales: $t_{-n} = t^{-n}$ with $n \in \mathbb{Z}$ should we need to do so. Similarly, choices of the type: $t_n = \epsilon^{n/N} t$ where N is a rational are valid.

The point made in the previous paragraph is that the derivative expansion method imposes no restriction as to the choice of the parameter with respect to which the independent variable is expanded. It is the problem that should dictate which scales are to be introduced. As a matter of fact the problem dictates much more than this:

the method employed. Procedures such as the one exposed above cannot be applied to any problem with equally good results. It will fail in some cases [Rub78]. This does not diminish its usefulness, it merely indicates that the application of asymptotic methods, however general they might be, calls for caution.

Let us consider the problem [JK82]:

$$(t + \epsilon f(t)) \frac{d}{dt} f(t) + f(t) = 0 \quad (2.26)$$

whose solution is: $f(t) = -\frac{t}{\epsilon} \pm \left(\left(\frac{t}{\epsilon} \right)^2 + \left(\frac{2}{\epsilon} + f(1) \right) f(1) \right)^{\frac{1}{2}}$. Attempting to solve this by using the derivative expansion method is futile. We do not have a rule allowing us to assign a particular scale to the t variable appearing and even if we could bypass this problem it would be impossible to remove the singularity at $t = 0$.

Let us now try and generalise:

- We have considered ordinary differential equations. Partial differential equations do not present any problem: the other independent variables are treated in the same way as time in the examples above.
- More than one physical phenomenon. We have considered normalised differential equations where the small parameter characterises one of the “stretched” scales. It is possible to have more than one “small” parameter. In this case it is important to know the order relation between them before proceeding to the explicit expansions.
- Degrees of freedom introduced. In (2.11) the parameter δ was used. This happened in order to indicate that for the problem (2.8) we have two degrees of freedom regarding the expansion. In general the number of degrees of freedom in the expansion exceeds the number of independent variables by one. To picture the implications of the last two statements consider the problem:

$$\frac{d^2}{dt^2} f(t) + 2\beta \frac{d}{dt} f(t) + f(t) + \gamma f(t)^2 = 0 \quad (2.27)$$

with $\beta, \gamma \ll 1$. This is a nonlinear counterpart of (2.8). From inspection we notice that damping effects are important on a time scale $O(\frac{1}{\beta})$ while nonlinear effects will be important on the time scale $O(\frac{1}{\gamma})$ (provided that the initial condition is of order $O(1)$ magnitude). Order relations between β and γ depend on the physical instance of the problem and we cannot intervene. Our choice for ϵ and δ on the other hand will depend on the situation that we wish to approximate. For example weak nonlinearity would indicate a choice $\frac{\delta}{\beta} = O\left(\left(\frac{\epsilon}{\gamma}\right)^n\right)$, n being rational smaller than unity, while a balance would call for $n = 1$.

To conclude the section let us outline the derivative expansion procedure as applied to differential problems:

- The equation describing the physical problem is normalised and the perturbing terms identified.
- The number of independent coordinates is increased via definitions of the type (2.11).
- Expansions of the type (2.10, 2.12) are introduced. The initial or the boundary conditions have to be altered in order to fit into the new extended domain of definition. The expansions defined above are substituted into the initial problem and the resulting equation has its terms rearranged in powers of the small expansion parameter. Each of the coefficients of this expansion must be equal to zero independently. This is a consequence of the independence of the original equation on the specific choice of the expansion parameter.
- Equations arising from the previous step are solved starting from the one at lowest order. It is important, during this solution process, to suppress all secular terms. This supplies further conditions, which allow the calculation of the integration constants.
- The initial independent variables are restored into the solutions resulting from the execution of the previous step.

2.3 Variational methods

2.3.1 General

We will now discuss the variational approach to the solution of differential equations [LL76] [Wan95] [MS91]. To introduce the principles of the approach let us take a simple example [LL76]. Consider an unknown function $f(t)$ which together with its first order time derivatives defines the integral :

$$S = \int_{t_a}^{t_b} dt L(f(t), \frac{df(t)}{dt}, t) \quad (2.28)$$

It is customary, in mechanics, to call this integral *action* and the function L *Lagrangian*. We will assume that the form of the Lagrangian function is given. For simplicity, we have assumed that the Lagrangian depends on the function f and its first order time derivative. Generalisations will be treated later. We now try to find the function $f(t)$

which will make the action an extremum (minimum or maximum) within the closed interval $[t_a, t_b]$. Let us assume, without affecting the generality of the problem, that for the function sought the action is minimal. Thus a change from $f(t)$ to:

$$f(t) + \delta f(t) \quad (2.29)$$

by a function $\delta f(t)$ which remains small within the interval $[t_1, t_2]$ will lead to an increase in the action integral. At the endpoints, the value of the function $f(t)$ is fixed. So:

$$\delta f(t_a) = \delta f(t_b) = 0 \quad (2.30)$$

The substitution of the changed function (2.29) in the action integral will lead to its increase from the minimum value by the quantity:

$$\int_{t_a}^{t_b} dt L(f(t) + \delta f(t), \frac{d(f(t) + \delta f(t))}{dt}, t) - \int_{t_a}^{t_b} dt L(f(t), \frac{df(t)}{dt}, t)$$

The first term is expanded in Taylor series with respect to δf . The requisite for the action to be minimal is that the first order terms, in this expansion, are zero. After performing integration by parts we obtain:

$$\delta S = \left[\frac{\partial L}{\partial \dot{f}} \delta f \right]_{t_a}^{t_b} + \int_{t_a}^{t_b} dt \delta f \left(\frac{\partial L}{\partial f} - \frac{d}{dt} \frac{\partial L}{\partial \dot{f}} \right) \quad (2.31)$$

The dot is denoting differentiation with respect to time. The first term of the right hand side of (2.31) is zero by virtue of (2.30) and the second must vanish identically for all values of δf . This can be achieved only if the integrand vanishes. We therefore conclude that the requirement for the minimisation (or maximisation) of the action is equivalent to:

$$\frac{\partial L}{\partial f} - \frac{d}{dt} \frac{\partial L}{\partial \dot{f}} = 0 \quad (2.32)$$

Equations of this type are referred to as *Euler-Lagrange equations*. Let us now recapitulate. We started with two assumptions:

1. The action integral of a function (Lagrangian), which depends on another function(f) is minimum (or maximum).
2. The function f is fixed at the two endpoints of the action integral

These two are equivalent to (2.32). The latter is a differential equation. This equivalence suggests an alternative method in solving a boundary value problem: to associate with it an action optimisation problem. Its solution will be the solution of the initial boundary value problem. Unfortunately, a major difficulty here lies in finding the

Lagrangian function corresponding to the equation that needs to be solved. For this reason, it is very scarce to try to solve a boundary value problem by transforming it into an action optimisation one. Nevertheless the equivalence between these two is a very useful tool in the analysis of physical systems.

Three possible generalisations are of interest to us:

- *The Lagrangian depends on higher order derivatives.*

The dependence of the Lagrangian on the n th order derivative will lead to the presence of a term:

$$\frac{\partial L}{\partial f^{(n)}} \frac{d^n \delta f}{dt^n} \quad \text{where} \quad f^{(n)} = \frac{d^n f(t)}{dt^n}$$

in the first order expansion of the action integral. In this case the integration by parts that was used in the derivation of (2.31) has to be repeated n times. By making use of (2.30) the final outcome is a term

$$(-1)^n \frac{d^n}{dt^n} \frac{\partial L}{\partial f^{(n)}} \delta f$$

in the integrand of (2.31). Equivalently it is possible to rewrite the Euler–Lagrange equation in the form:

$$\sum_{j=0}^N (-1)^j \frac{d^j}{dt^j} \frac{\partial L}{\partial f^{(j)}} = 0 \quad (2.33)$$

where N denotes the maximum derivative order, of the function f , present in the Lagrangian.

- *Dependence of the Lagrangian on more than one function.*

In this case an Euler–Lagrange equation has to be derived for every function $f_k(t)$ that appears in the Lagrangian. The outcome is a system of equations:

$$\sum_{j=0}^{N_k} (-1)^j \frac{d^j}{dt^j} \frac{\partial L}{\partial f_k^{(j)}} = 0 \quad \text{for } k = 1 \dots K \quad (2.34)$$

where K denotes the number of independent functions that appear in the Lagrangian and N_k the maximum derivative order of f_k .

- *The functions f_j that appear in the Lagrangian depend on more than one variable t_1, t_2, \dots, t_n*

In this case the definition of the action has to be altered:

$$S = \int_{(t_1)_{a_1}}^{(t_1)_{b_1}} \int_{(t_2)_{a_2}}^{(t_2)_{b_2}} \dots \int_{(t_n)_{a_n}}^{(t_n)_{b_n}} dt_n \dots dt_2 dt_1 L$$

by introducing a multiple integral that ranges through all the independent variables. Similar alterations occur throughout the analysis to lead to Euler–Lagrange equations of the form:

$$\sum_{(j_1, j_2, \dots, j_n) = (0, 0, \dots, 0)}^{(N_1, N_2, \dots, N_n)} (-1)^{(j_1 + j_2 + \dots + j_n)} \prod_{s=1}^n \frac{\partial^{j_s}}{\partial t_s^{j_s}} \frac{\partial L}{\partial f_k^{(j_s)}} = 0 \quad \text{for } k = 1 \dots K \quad (2.35)$$

with the summation ranging all combinations of the indices j_s . K and N_s have the same significance as in (2.34).

2.3.2 Rayleigh–Ritz method

As explained in the previous section solving a boundary value problem using the variational approach is not often feasible. However there exists an approximate solution method applicable to this case exactly: the *Rayleigh–Ritz method* [Sag61]. Let us assume that we have to solve a differential equation of the type (2.33). As explained in the previous section, this is equivalent to minimising the action that corresponds to the Lagrangian that appears in (2.33). Instead of looking for the minimum of the action over all possible functions the search is limited into a space defined by a set of functions q_i which must comply with the boundary conditions. This space is known as the *Ritz manifold*.

The type of solution that we are seeking can be expressed as a linear combination of the base functions q_i

$$f = \sum_{i=1}^n c_i q_i \quad (2.36)$$

with c_i parameters that are to be calculated. The consequence of (2.36) is that after performing the integration the definition of the action involves, we end up with a function of the n parameters c_i :

$$S(c_1, c_2, \dots, c_n)$$

Then the Euler–Lagrange equations take the form:

$$\frac{\partial}{\partial c_s} S(c_1, c_2, \dots, c_n) = 0 \quad \text{for } s = 1 \dots n \quad (2.37)$$

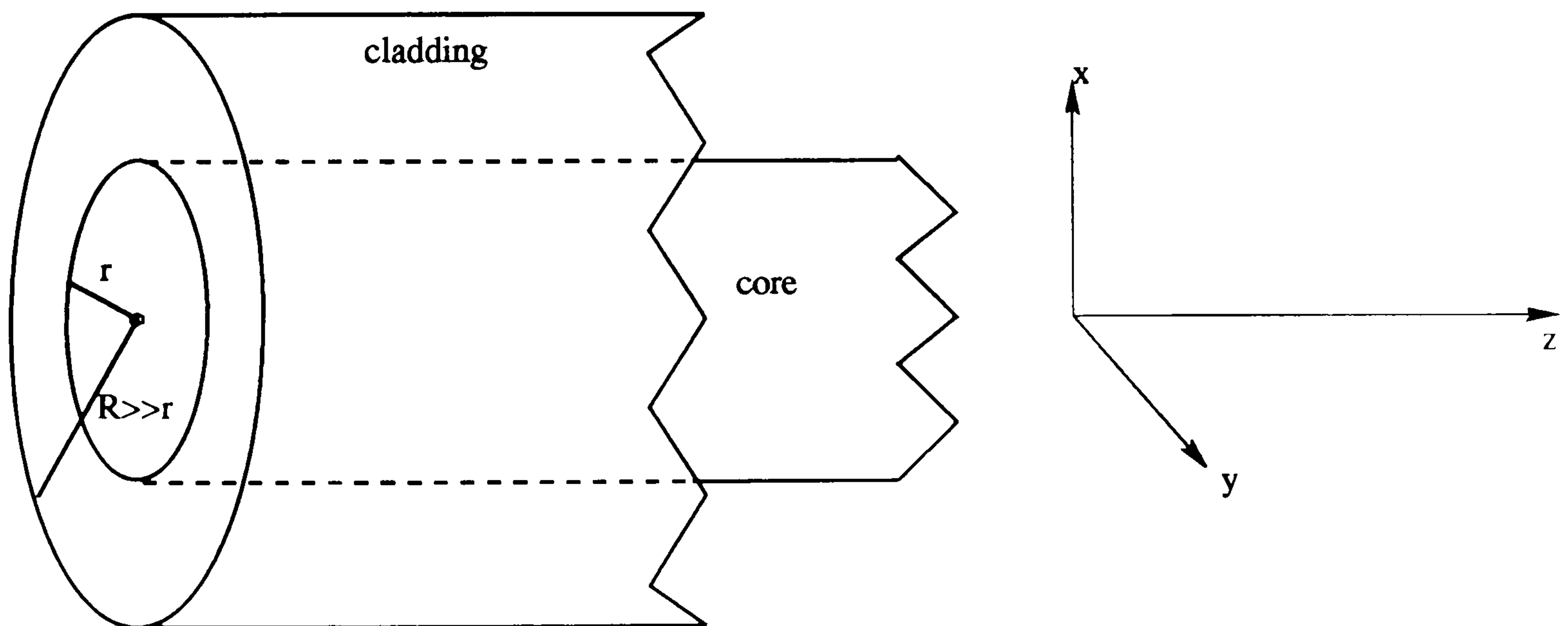
which is the algebraic minimisation problem in n dimensions. It can be demonstrated [Sta68] that solving this problem is equivalent to projecting eq. (2.33) into the Ritz manifold and looking for solutions belonging to it.

In the traditional application of the method an infinite number of base functions is introduced aiming at determining the solution of the initial problem with arbitrary precision. This is not that easy when dealing with nonlinear differential equations. The Ritz method will be used on a different basis explained at a later stage. However one point must be emphasised in the case where the solution is sought in a finite-dimensional Ritz manifold. There is a degree of arbitrariness in the choice of the space in question. Although the solution will be the best one in the given manifold, there is no guarantee that the solution in the reduced space is a good representation of the reality.

3. OPTICAL FIBRES

3.1 Propagation medium description

The propagation medium that we are interested in is step-index fibres. This is the simplest form of fibre and it consists [Gow84] of a cylindrical central region of radius r called *core* surrounded by a layer, called *cladding*. The radius of the outer surface of the cladding is much larger than r (typically $60\mu m$) and for simplicity we will take it to be infinite. The refractive index of the cladding (n_2) is manufactured to be smaller than that of the core (n_1) [SL91]. This allows for rays propagating inside the core to be totally reflected by the core-cladding interface. This *total internal reflection* is the physical mechanism that enables guiding of optical waves within a fibre. To describe



fibre properties with respect to guiding, two parameters are used [Agr89] the relative difference in refractive indices between core(n_1) and cladding(n_2):

$$\Delta = \frac{n_1 - n_2}{n_1}$$

which is typically of the order of 10^{-3} and the normalised frequency:

$$V = \frac{2\pi r (n_1^2 - n_2^2)^{\frac{1}{2}}}{\lambda_0}$$

with λ_0 being the free space wavelength of the propagating wave. V gives a measure of the number of guided modes within the fibre. For $V < 2.405$ only one guided mode

is allowed [Gow84]. In this study we are interested in single mode fibres. A typical commercial core radius for such a fibre is $2\text{--}4\mu\text{m}$ [Agr89] (common wavelengths between 1.3 and $1.6\mu\text{m}$).

The material used to fabricate optical fibres is silica glass (amorphous SiO_2) appropriately doped in order to generate the refractive index differences [Gow84] needed for guidance. Ideally, a fibre is a cylindrical structure which remains unchanged as we move along its axis. We will assume that the fibres we consider exhibit both rotational and translational symmetry.

The geometry of the device indicates that either Cartesian or cylindrical coordinates have to be used. It is customary [Agr89] [Gow84] to take the z axis of the coordinate system to be parallel to the axis of the fibre.

Apart from the preferred direction, indicated by the geometry no other direction should be singled out. This means that we will assume that the bulk media is isotropic. The implications of this assumption will be seen later, when considering the response of the media to the Electric field.

3.2 Maxwell's equations

The propagation of radiation in an optical waveguide from a macroscopic point of view is described by the Maxwell equations:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (3.1)$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \quad (3.2)$$

$$\nabla \cdot \mathbf{D} = \rho \quad (3.3)$$

$$\nabla \cdot \mathbf{H} = 0 \quad (3.4)$$

where boldface characters designate vectorial quantities, $\mathbf{E}(\mathbf{r}, t)$ is the electric field, $\mathbf{H}(\mathbf{r}, t)$ the magnetic field, $\mathbf{D}(\mathbf{r}, t)$ the electric displacement, $\mathbf{B}(\mathbf{r}, t)$ the magnetic displacement, $\mathbf{J}(\mathbf{r}, t)$ the current density and $\rho(\mathbf{r}, t)$ is the charge density. The functions depend on four independent variables: time (t) and three space coordinates designated for simplicity by the vector \mathbf{r} . For the case of fibre propagation no free charges are present and we set $\mathbf{J} = 0$, $\rho = 0$ [Agr89].

The electric and magnetic displacements are vectors describing macroscopic response of the material to the electric and magnetic fields respectively. Since in optics we are concerned with non magnetic media [BC90], we set:

$$\mathbf{B} = \mu_0 \mathbf{H}$$

where μ_0 is the free space permeability. The electric displacement is rewritten with the help of the constitutive relation as [Kar94a]:

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \quad (3.5)$$

where $\mathbf{P}(\mathbf{r}, t)$ is the media polarisation and ϵ_0 is the free space permitivity. By taking the external product of ∇ with (3.1), and using (3.2), together with the two definitions above we have:

$$\nabla \times \nabla \times \mathbf{E} = -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{E} - \mu_0 \frac{\partial^2}{\partial t^2} \mathbf{P} \quad (3.6)$$

where c is the speed of light in the vacuum. This equation comprises both electric field and polarisation. The latter originates because of the former [She84] and describes the response of the media fully. Unfortunately (3.6) can seldom be solved exactly and we have to resort to simplifications.

3.3 Polarisation

The first aim, in the task of simplifying (3.6), is to express the polarisation in terms of the electric field. When the latter is sufficiently weak the polarisation can be expressed as [Blo65], [BC90]:

$$\mathbf{P} = \mathbf{P}^{(0)} + \mathbf{P}^{(1)} + \mathbf{P}^{(2)} + \dots \quad (3.7)$$

The superscript denotes the order of the term with respect to the electric field. Thus $\mathbf{P}^{(0)}$ is a static polarisation of the media, $\mathbf{P}^{(1)}$ depends on the electric field linearly, $\mathbf{P}^{(2)}$ quadratically and so on. The zero order term would imply the presence of charges in the media and is neglected forthwith.

As already mentioned the polarisation is a macroscopic response of the medium to the existing electric field. When looking at the medium microscopically we have an electric field that induces responses from individual atoms. The response of the medium at a certain point, will come from the superposition of the responses of the neighbouring atoms. In that sense polarisation, at a certain point in space, will depend on the electric field of the surrounding volume. For our analysis we will assume that the size of this region is minute. Effectively this amounts to assuming that the polarisation at a certain point in space depends on the electric field at the same point only.

By introducing the standard exponential Fourier transform to the problem [Erd54]:

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \\ f(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega \end{aligned}$$

the n th order polarisation appearing in (3.7) can be expressed as [BC90]:

$$\mathbf{P}^{(n)}(t) = \epsilon_0 \int_{-\infty}^{\infty} d\omega_1 \cdots \int_{-\infty}^{\infty} d\omega_n \chi^{(n)}(-\omega_\sigma; \omega_1, \dots, \omega_n) | \mathbf{E}(\omega_1) \cdots \mathbf{E}(\omega_n) e^{i\omega_\sigma t} \quad (3.8)$$

where $\omega_\sigma = \omega_1 + \dots + \omega_n$. $\chi^{(n)}$ is a tensor of order $n + 1$ and is called the n th-order susceptibility. The operator $|$ denotes the tensorial product between the n th order susceptibility tensor and the Fourier transformed 3-dimensional electric field vectors $\mathbf{E}(\omega)$. The n th-order susceptibility is defined as:

$$\chi^{(n)}(-\omega_\sigma; \omega_1, \dots, \omega_n) = \int_{-\infty}^{\infty} d\tau_1 \cdots \int_{-\infty}^{\infty} d\tau_n \mathbf{R}^{(n)}(\tau_1, \dots, \tau_n) \exp \left(-i \sum_{j=1}^n \omega_j \tau_j \right) \quad (3.9)$$

with $\mathbf{R}^{(n)}(\tau_1, \dots, \tau_n)$ a real tensor of rank $n + 1$ called n th-order polarisation response function. The dimensionality of the susceptibility and the polarisation response function become obvious when one considers the vectors that they relate. As an example, the n th order polarisation, which has 3 components, is given in relation to n electric fields, which have 3 components each. Thus each component of the polarisation can be affected by n^3 permutations of the different electric field components. The total number of components that the n th order response function must have is thus 3^{n+1} .

Let us now consider how this applies to the case of the isotropic material that the fibre is made of. First of all the linear susceptibility tensor will have all its off-diagonal elements equal to zero and all the diagonal equal [Kar94a]. It can thus be expressed as:

$$\chi_{ij}^{(1)}(-\omega; \omega) = \delta_{ij} \chi^{(1)}(-\omega; \omega)$$

Isotropy imposes the requirement that all even order susceptibilities vanish [BC90] $\chi^{(2)}$ will be equal to zero. The lowest order nonlinear susceptibility will be the third order one $\chi^{(3)}$. From the 81 components of the $\chi^{(3)}$ tensor it can be proved that only 21 are nonzero [BC90] [She84]. The next nonzero susceptibility will be the fifth order one which will not be considered here.

Under these conditions, (3.7) will be rewritten for the case of the optical fibre:

$$\begin{aligned} \mathbf{P}(t) &= \epsilon_0 \int_{-\infty}^{\infty} d\omega \chi^{(1)}(-\omega; \omega) \mathbf{E}(\omega) \exp(i\omega t) + \\ &\quad \epsilon_0 \int_{-\infty}^{\infty} d\omega_1 \int_{-\infty}^{\infty} d\omega_2 \int_{-\infty}^{\infty} d\omega_3 \chi^{(3)}(-\omega_\sigma; \omega_1, \omega_2, \omega_3) : \mathbf{E}(\omega_1) \mathbf{E}(\omega_2) \mathbf{E}(\omega_3) \exp(i\omega_\sigma t) \\ &= \mathbf{P}_l(t) + \mathbf{P}_{nl}(t) \end{aligned} \quad (3.10)$$

The first term of the right hand side of the first equality is the linear part of the polarisation denoted by $\mathbf{P}_l(t)$ and the other term is the nonlinear polarisation symbolised by $\mathbf{P}_{nl}(t)$.

3.3.1 Refractive index

The refractive index is a quantity widely used to describe optical properties of materials. For the linear case its definition is:

$$n_l(\omega) = \text{Re} \left(1 + \chi^{(1)}(-\omega; \omega) \right)^{\frac{1}{2}}$$

with $\chi^{(1)}$ the first order susceptibility and Re a function returning the real part of its argument. In the general case it is a tensor with the same dimensionality as the linear susceptibility and depending on the frequency of the propagating wave.

Extension to nonlinear cases is not really that easy since the refractive index involves only one wave, while the nonlinear susceptibilities address the interaction of more than one wave. For some limited cases though, such as self phase modulation, cross phase modulation or stimulated Raman scattering [BC90], it is possible to use the refractive index formalism by introducing a nonlinear refraction coefficient. This is of course done on a case-specific basis. As an example, the self phase modulation phenomenon in a bulk isotropic medium would allow the introduction of a quantity δn altering the linear refractive index and defined as:

$$\delta n = \sqrt{\text{Re} \epsilon(\omega)} - n_l(\omega) \quad (3.11)$$

where $\epsilon(\omega)$ is defined as:

$$\epsilon(\omega) = 1 + \chi^{(1)}(-\omega; \omega) + \frac{3}{4} \chi^{(3)}(-\omega; \omega, -\omega, \omega) |E|^2$$

The fraction in front of the nonlinear susceptibility is there to account for degeneracy introduced when dropping the vector equations [BC90].

3.4 Quasi-monochromatic condition

Equation (3.10) in combination with (3.6) gives a complete description of the propagation of the electric field within the fibre. However the presence of the integration operators in (3.10) makes them difficult to use. The next step in the simplification procedure will come from the adoption of the quasi-monochromatic description of the electric field. According to this description [BC90] [KH87], the electric field is rewritten as a superposition of quasi-monochromatic wave-packets:

$$\mathbf{E}(t) = \sum_{\omega_j} \mathbf{E}_{\omega_j}(t) e^{-i(\omega_j t - \mathbf{k}_j \cdot \mathbf{r})} \quad (3.12)$$

where \mathbf{k}_j is the wave-vector corresponding to ω_j . It is assumed that $\mathbf{E}_{\omega_j}(t)$ varies slowly with time:

$$\left| \frac{\partial \mathbf{E}_{\omega_j}(t)}{\partial t} \right| \ll \omega_j \quad (3.13)$$

Condition (3.13) implies that the frequency spread of the wave-packet around its central frequency is small. For a pulse at around $1.5 \mu m$, and assuming a relatively smooth temporal profile, it is possible to use the quasi-monochromatic approach for time widths of the range of 100 fs for the envelope [Agr89].

3.4.1 Linear polarisation

The adoption of the quasi-monochromatic approach suggests a way to achieve simplification of (3.10). Since the electric field will be a superposition of wave-packets each of which has a low spread around a central frequency $-\omega_j$ it is possible to expand the susceptibilities that appear in the integrals of (3.10) in Taylor series around this specific frequency. In the rest of this thesis we will denote as linear polarisation the part of the polarisation that depends linearly on the electric field ($\mathbf{P}^{(1)}$ in 3.7) and as nonlinear the part of the polarisation which exhibits a nonlinear dependence on the electric field ($\mathbf{P}^{(2)}$ and higher in 3.7). For the linear polarisation we will have:

$$\begin{aligned} \mathbf{P}_l(t) = & \chi^{(1)}(-\omega_j; \omega_j) \int_{-\infty}^{\infty} \mathbf{E}(\omega) e^{i\omega t} d\omega + \\ & \left[\frac{\partial \chi^{(1)}(-\omega; \omega)}{\partial \omega} \right]_{\omega=-\omega_j} \int_{-\infty}^{\infty} (\omega + \omega_j) \mathbf{E}(\omega) e^{i\omega t} d\omega + \\ & \frac{1}{2} \left[\frac{\partial^2 \chi^{(1)}(-\omega; \omega)}{\partial \omega^2} \right]_{\omega=-\omega_j} \int_{-\infty}^{\infty} (\omega + \omega_j)^2 \mathbf{E}(\omega) e^{i\omega t} d\omega + \dots \end{aligned} \quad (3.14)$$

which by making use of the properties of the exponential Fourier transform [Erd54] together with (3.12) will give:

$$\begin{aligned} \mathbf{P}_l(t) = & \chi^{(1)}(-\omega_j; \omega_j) \mathbf{E}_{\omega_j}(t) e^{-i\omega_j t} - i \left[\frac{\partial \chi^{(1)}(-\omega; \omega)}{\partial \omega} \right]_{\omega=-\omega_j} \frac{\partial \mathbf{E}_{\omega_j}(t)}{\partial t} e^{-i\omega_j t} - \\ & \frac{1}{2} \left[\frac{\partial^2 \chi^{(1)}(-\omega; \omega)}{\partial \omega^2} \right]_{\omega=-\omega_j} \frac{\partial^2 \mathbf{E}_{\omega_j}(t)}{\partial t^2} e^{-i\omega_j t} + \dots \end{aligned} \quad (3.15)$$

where the spatial dependence has been omitted for simplicity. It is for the same reasons that only one quasi-monochromatic component of the electric field has been considered.

3.4.2 Nonlinear polarisation

It is possible to proceed in a similar way in the treatment of the nonlinear polarisation in (3.10). The nonlinear polarisation at frequency $\omega_\sigma = \omega_1 + \omega_2 + \omega_3$ is determined by the tensorial product of the third order susceptibility with the three electric fields at frequencies ω_1 , ω_2 and ω_3 . Within the frame of the quasi-monochromatic approach

and in order to give a suggestive formula rather than an exact one we will take three electric wave-packets centered at frequencies $-(\omega_1)_0$, $-(\omega_2)_0$ and $-(\omega_3)_0$ respectively. Furthermore, we will assume that $\omega_j = (\omega_1)_0 + (\omega_2)_0 + (\omega_3)_0$ and introduce the notation [Kod85b] [KH87]:

$$\begin{aligned}
 \chi^{(3)}(-\omega_j; (\omega_1)_0, (\omega_2)_0, (\omega_3)_0) &= \chi_{(1\dot{2}3)}^{(3)} \\
 \frac{\partial}{\partial \omega_1} \chi^{(3)}(-\omega_j; \omega_1, (\omega_2)_0, (\omega_3)_0) \Big|_{\omega_1 = -(\omega_1)_0} &= \chi_{(i\dot{2}3)}^{(3)} \\
 \frac{\partial^2}{\partial \omega_1^2} \chi^{(3)}(-\omega_j; \omega_1, (\omega_2)_0, (\omega_3)_0) \Big|_{\omega_1 = -(\omega_1)_0} &= \chi_{(i\ddot{2}3)}^{(3)} \\
 \frac{\partial^2}{\partial \omega_1 \partial \omega_2} \chi^{(3)}(-\omega_j; \omega_1, \omega_2, (\omega_3)_0) \Big|_{\omega_1 = -(\omega_1)_0, \omega_2 = -(\omega_2)_0} &= \chi_{(i\dot{2}\dot{3})}^{(3)} \\
 &\dots
 \end{aligned}$$

The corresponding component of the nonlinear polarisation becomes, in this case:

$$\begin{aligned}
 \mathbf{P}_{nl}(\omega_0) &= \chi_{(1\dot{2}3)}^{(3)} \dot{\mathbf{E}}_{(\omega_1)_0}(t) \mathbf{E}_{(\omega_2)_0}(t) \mathbf{E}_{(\omega_3)_0}(t) \exp(-i\omega_0 t) - \\
 &\quad i \chi_{(i\dot{2}3)}^{(3)} \dot{\mathbf{E}}_{(\omega_1)_0}(t) \frac{\partial \mathbf{E}_{(\omega_2)_0}(t)}{\partial t} \mathbf{E}_{(\omega_3)_0}(t) \exp(-i\omega_0 t) - \\
 &\quad i \chi_{(1\dot{2}\dot{3})}^{(3)} \mathbf{E}_{(\omega_1)_0}(t) \frac{\partial \mathbf{E}_{(\omega_2)_0}(t)}{\partial t} \mathbf{E}_{(\omega_3)_0}(t) \exp(-i\omega_0 t) - \\
 &\quad i \chi_{(1\dot{2}3)}^{(3)} \mathbf{E}_{(\omega_1)_0}(t) \mathbf{E}_{(\omega_2)_0}(t) \frac{\partial \mathbf{E}_{(\omega_3)_0}(t)}{\partial t} \exp(-i\omega_0 t) - \\
 &\quad \frac{1}{2} \chi_{(i\ddot{2}3)}^{(3)} \dot{\mathbf{E}}_{(\omega_1)_0}(t) \frac{\partial^2 \mathbf{E}_{(\omega_2)_0}(t)}{\partial t^2} \mathbf{E}_{(\omega_3)_0}(t) \exp(-i\omega_0 t) - \\
 &\quad \chi_{(i\dot{2}3)}^{(3)} \dot{\mathbf{E}}_{(\omega_2)_0}(t) \frac{\partial \mathbf{E}_{(\omega_2)_0}(t)}{\partial t} \mathbf{E}_{(\omega_3)_0}(t) \exp(-i\omega_0 t) + \\
 &\quad \dots
 \end{aligned} \tag{3.16}$$

It is important, before moving on, to comment on the expansions (3.15) and (3.16), with reference to the discussion on expansions in the previous chapter. A Taylor series was taken of the susceptibilities around a central frequency. This was done in the hope that we would be able to use a limited number of terms to give an accurate description of the polarisations. The prerequisite for such an approach to be valid is that the function expanded does not exhibit very fast variations. This is true only when we are looking at the function in a spectral area where no resonances are present. This statement is true for the linear polarisation but does not always hold for the nonlinear polarisation. In the latter case, the polarisation has to be substituted directly from (3.10) [BDOS92] [BW89] [MC90] [Gor86]. For the moment we will retain the Taylor series approach for the derivation of the equations describing the electric field propagation, but will consider the issue again after the derivation of a first set of equations.

3.5 Linear propagation of light in optical fibres

At this point we will make a digression to consider the linear propagation problem, which although not relevant to our purposes, is useful in introducing quantities and notions used later, in the nonlinear case.

The vector operator identity:

$$\nabla \times \nabla \times = \nabla (\nabla \cdot) - \nabla^2$$

will be used in (3.6) to give:

$$\nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{E} - \mu_0 \frac{\partial^2}{\partial t^2} \mathbf{P}_l \quad (3.17)$$

We will be considering a planar wave propagation. In this case the wave does not extend spectrally beyond its central frequency. We can thus neglect all but the first term in (3.15):

$$\mathbf{P}_l(t) = \chi^{(1)}(-\omega_j; \omega_j) \mathbf{E}(t) \quad (3.18)$$

where the electric field has been restored to its form before the adoption of the quasi-monochromatic condition. This does not affect the equation since the plane wave is a stricter condition than the quasi-monochromatic one.

Substitution of the constitutive relation (3.5) along with (3.18) into the electric displacement equation (3.3) will give:

$$\epsilon_0 \left(\chi^{(1)}(-\omega_j; \omega_j) \right) \nabla \cdot \mathbf{E} + \epsilon_0 \left(\nabla \chi^{(1)}(-\omega_j; \omega_j) \right) \cdot \mathbf{E} = 0$$

The second term in the left hand side of this equation is neglected. This is called the *weak guidance* approximation [SL91] [Mar91] and is the rule when treating the linear propagation in fibres. The weak guidance approximation is justified when the Δ parameter, introduced in section 3.1, is very small, an assumption which is true in most applications. The weak guidance condition leads thus to

$$\nabla \cdot \mathbf{E} = 0,$$

and by virtue of the latter relation to the vanishing of the second term in the left hand side of (3.17). From the physical point of view the previous approximation means that the longitudinal component of the electric field is being neglected [SL91].

The longitudinal translation invariance of the fibre allows us to look for solutions in the form of a product of two functions, one of which depends on the longitudinal coordinate (z) and the other on the transverse ones:

$$\mathbf{E}(\mathbf{r}, t) = q(z, t) \mathbf{U}_T(t)$$

where the T subscripts denotes the dependence on the transverse coordinates.

For plane waves the form of q is the well known exponential:

$$q(z, t) = e^{i\beta z - i\omega t}$$

with ω the circular frequency and β the propagation constant. The substitution of the last two equations into (3.17) will give [Kar94a]:

$$\nabla^2 e^{i\beta z - i\omega t} \mathbf{U}_T + \frac{\omega^2}{c^2} \left(1 + \chi^{(1)}(-\omega_j; \omega_j)\right) e^{i\beta z - i\omega t} \mathbf{U}_T = 0 \quad (3.19)$$

It is this equation that demonstrates how the weak guidance condition has simplified matters. When the ∇^2 operator in the left hand side is put into either its Cartesian or its cylindrical form, (3.19) will be decoupled into two equations: one for the longitudinal components and one for the transverse. In the case of single mode fibres only one solution with β real will be allowed by the boundary conditions at the core-cladding interface. This is the fundamental guided mode. In reality if we take a perfect cylindrical fibre this mode has associated with it a second order degeneracy with respect to polarisation of the electric field. For the rest of our discussion we will assume that this degeneracy is lifted. In practice this can be achieved with the introduction of ellipticity in the core [Kod85b] and the fibres are called polarisation preserving. The velocity of a wavefront will be equal to the phase velocity of the mode:

$$v_{ph} = \frac{\omega}{\beta} \quad (3.20)$$

A more useful quantity is the group velocity which gives the velocity at which energy is traveling within the waveguide. The formula giving the group velocity is [Gow84]:

$$v_g = \frac{d\omega}{d\beta} = v_{ph} \left(1 - \frac{\omega}{v_{ph}} \frac{dv_{ph}}{d\omega}\right)^{-1} \quad (3.21)$$

The group velocity depends generally on the circular frequency and thus the wavelength in the vacuum, which is the traditional measure used in optics. This dependence is due to two reasons: the dependence of the linear susceptibility on the frequency, and the dependence of the guided wave wavenumber on the geometry of the fibre and thus on the frequency of the light propagating in it. The first phenomenon is called chromatic or material dispersion and the second intra-mode or waveguide dispersion [Gow84]. Dispersion effects in fibres are quantified by the group-velocity dispersion parameter:

$$\beta_2 = \frac{d^2}{d\omega^2} \beta = \frac{d}{d\omega} \frac{1}{v_g} = -\frac{1}{v_g^2} \frac{dv_g}{d\omega}$$

in which the major contribution comes from the waveguide dispersion [Agr89]. β_2 becomes zero at a certain wavelength λ_D called the zero dispersion wavelength. For

longer wavelengths than λ_D , β_2 will become negative and light is said to propagate in the anomalous dispersion regime. By this we mean that shorter wavelengths travel faster than longer ones. The contrary occurs in the normal dispersion regime occurring for wavelengths shorter than λ_D . In the case of a pulse that is made of components at different wavelengths a walk-off between them will be observed due to the presence of dispersion. This basically is due to the difference in the group velocities of the components. When considering pulse trains the effect is detrimental in the sense that pulses tend to broaden thus overlapping with adjacent ones.

The function \mathbf{U}_T , which describes the transverse properties of the electric field on the frequency, is only slightly perturbed by the frequency dependence of the susceptibility. Thus for the case of the propagating pulse used above, \mathbf{U}_T is taken to be the same for all components, as they are assumed to be very close in the frequency domain.

To conclude this section let us consider the amount of power that is driven by a mode. To this end the Poynting vector \mathbf{P} is introduced and its projection along the propagation direction z is taken [SL91]:

$$P = \frac{1}{2} \operatorname{Re} \left(\int_{A_\infty} \mathbf{E} \times \bar{\mathbf{H}} \cdot \hat{\mathbf{z}} dA \right) \quad (3.22)$$

where the bar above a quantity denotes complex conjugation, $\hat{\mathbf{z}}$ is the unit vector along the z axis, A_∞ is the infinite cross section along which we are measuring the power flow, P is the power going through the cross section in question and Re indicates that only the real part of the integral in (3.22) is considered. The magnetic field is calculated from the electric field with the help of Maxwell's equations. The power inside the core of the fibre can be calculated by integrating over the cross section of the core. The form of the magnetic field, by the translation invariance argument will be in the form:

$$\mathbf{H}(\mathbf{r}, t) = e^{i\beta z - i\omega t} \mathbf{W}_T(t)$$

where $\mathbf{W}_T(t)$ depends on the transverse coordinates. Within the frame of the weak guidance approximation, use of the Maxwell equations will yield:

$$\mathbf{W}_T(t) = n_1 \sqrt{\left(\frac{\epsilon_0}{\mu_0} \right)} \mathbf{U}_T(t)$$

where n_1 is the refractive index of the core [SL91].

Substitution of this relation in the power flow definition (3.22), together with the use of the vector identity:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C}) \mathbf{B} - (\mathbf{A} \cdot \mathbf{B}) \mathbf{C}$$

will give:

$$P_u = \frac{1}{2} n_1 \sqrt{\left(\frac{\epsilon_0}{\mu_0}\right)} \left(\int_{A_\infty} \mathbf{U}_T \cdot \bar{\mathbf{U}}_T dA \right) \quad (3.23)$$

It should be recalled that \mathbf{U}_T is a vector function that has transverse components only. The dot product in the right hand side provides a useful way for normalising propagation related quantities and will be used in the nonlinear analysis.

3.6 Multiple scales expansion: Considerations

We will now proceed to the consideration of the nonlinear problem. The first step is to identify our problem correctly. We are interested in describing the propagation of a short pulse through a fibre. Using the same argument as in the linear case we will assume that the field can be modeled as a sum of components each of which is written as a product of two functions, one depending on the transverse coordinates and the other on the longitudinal ones:

$$\mathbf{E}^{(\omega_j)} = Q^{(\omega_j)}(z, t) \mathbf{U}_T^{(\omega_j)}$$

The function $Q(z, t)$ will contain the information regarding the envelope of the wave and its temporal evolution, and the function \mathbf{U}_T the transverse profile of the electric field across the fibre cross section. The central frequency of the wave-packet under consideration is $-\omega_j$ ¹. The presence of the frequency parameter, as a superscript, singles out the waves with different central frequencies. The quasi-monochromatic condition imposes a specific form for the electric field (3.12) as a plane wave with a slowly varying envelope. Taking this into account, the electric field can be written in the form of an expansion:

$$\mathbf{E}(\mathbf{r}, t) = \sum_{\mu=1}^{\infty} \epsilon^\mu \mathbf{E}_\mu(\mathbf{r}, t) = \sum_{\mu=1}^{\infty} \epsilon^\mu \left(\sum_j q^{(\omega_j)}(z, t) \mathbf{U}_T^{(\omega_j)} e^{i k_j z - i \omega_j t} \right)_{(\mu)} \quad (3.24)$$

ϵ is a small parameter characterising the asymptotic expansion and needs to be connected to the physics of the problem. The problem that we will be treating will regard the description of the propagation of a single wave-packet at a certain central frequency. Since, however, nonlinearities are present, we expect that generation of higher harmonics will occur. In that sense the presence of the central frequency as a parameter helps

¹ The Fourier transform definitions that we have used have exponents of different signs from the ones used in optics [BC90]. The implication of this definition is that the plane wave $\exp(i \mathbf{k} \cdot \mathbf{x} - i \omega t)$ has central frequency $-\omega$, in our formalism, instead of ω , which would have been the case following the usual conventions.

in the identification of each of the components. The adoption of the derivative expansion method calls for the expansion in asymptotic series of the independent parameters. The independent parameters in the fibre propagation problem are four: time and the three space coordinates. Since we focus on the description of propagation phenomena we will be considering the introduction of multiple coordinates in the manner explained in the previous chapter, for time and the longitudinal space coordinate. Subsequently it is assumed that instead of time the coordinates $t_n = \delta^n t$ are introduced and instead of z the coordinates $z_n = \gamma^n z$ are introduced. The parameters γ and δ will be left undefined at this stage. The consequence of these extensions is that the time and z -derivatives must be substituted by:

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \delta \frac{\partial}{\partial t_1} + \delta^2 \frac{\partial}{\partial t_2} + \dots \quad (3.25)$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial z_0} + \gamma \frac{\partial}{\partial z_1} + \gamma^2 \frac{\partial}{\partial z_2} + \dots \quad (3.26)$$

The three expansions that we have adopted have led us to the introduction of three small parameters (γ , δ and ϵ) that need to be connected between them and with the physical instance with order relations. As explained we will be considering short pulses. The shortness of the pulse can be quantified by a non-dimensional parameter of the type:

$$\frac{T_c}{\tau} = O\left(\left|\frac{\Delta\omega}{\omega_j}\right|\right)$$

where T_c is the period of the carrier wave and τ is some measure of the time duration of the pulse such as the FWHM. The second equality involves the frequency extent of the wave-packet $\Delta\omega$ and its central frequency ω_j . The second parameter can be easily associated with the expansions (3.15) and (3.16) for the polarisations as well as with the expansion of all quantities, slowly varying in the frequency (like $U_T^{(\omega)}$).

Other parameters can be introduced in the description of the system in terms of asymptotic series by considering the physical effects that arise during propagation. First of all, associated with the linear polarisation, is the absorption. In general, the low absorption window of a fibre is used. This means wavelengths around $1.55 \mu m$ where the loss coefficient is in the region of $0.2 dB/km$ [Gow84]. Considering the low value of the absorption coefficient we will neglect absorption in deriving the equations describing propagation. The effect of loss will be introduced *a posteriori* in the final equations.

The second effect associated with the linear polarisation is material dispersion: the change of the real part of the refractive index with the wavelength. As explained in section 3.5 this is due to the combined optical properties of the bulk material as well

as the waveguide. In the discussion in section 3.5 only second order dispersion was considered. This means that a change in wavelength was assumed to affect the velocity of the corresponding wave in a linear way. It is possible, under certain conditions, to have quadratic dependence of the velocity on the wavelength. In this case the dispersion is third order [WCL90] [WMCL86] [WMCL87] [Kar93], [KRWM94] [YW94] [DDI94] [AK95] [Elg93] [Elg92] [KZ90]. Third order dispersion is described by the parameter:

$$\beta_3 = \frac{d^3}{d\omega^3} \beta$$

In the same fashion it is possible to proceed to higher order dispersion [HK90] [KH94b] [Kar94b]. In the analysis here, we will assume that the dispersion properties of the fibre and the spectral extent of the propagating pulse are selected in such a fashion that all dispersion effects can be scaled with the same parameter which we denote as ϵ . In this fashion, the second order dispersion is $O(\epsilon^2)$, the third order dispersion is $O(\epsilon^3)$ and accordingly for higher orders.

Turning to effects due to the nonlinear susceptibility, a small parameter is introduced by considering the nonlinearity. The $\chi^{(3)}$ nonlinear susceptibility gives rise to a multitude of nonlinear phenomena. We are interested here in phenomena that involve the self-interaction of a wave-packet. Such phenomena are by definition phase matched [BC90]. This means that the wavefronts of all waves involved in the interaction will travel at the same velocity. Such a phenomenon is, first of all, self-phase modulation [BC90] [SC78] [FT93] [Agr89] [WLHA94] [DB83]. Here the refractive index of the propagation medium is altered proportionally to the intensity of the field that propagates in it. The phenomenon is usually quantified by rewriting the refractive index of the medium as the sum of a linear refractive index and a part that depends on the square of the propagating electric field:

$$n = n_0 + n_2 |E|^2, \quad (3.27)$$

where the parameter n_2 , called the Kerr coefficient, arises by taking the change in the refractive index as discussed in section 3.3.1 and expanding in Maclaurin series when the nonlinearity is small [BC90]. A parameter quantifying the process can be easily obtained now as the ratio:

$$\frac{n_2 |E|^2}{n_0} \quad (3.28)$$

where n_0 is the linear refractive index of the fibre. We will be considering weak nonlinearity. To make this assumption explicit we take nonlinear effects to be important at scale $O(\epsilon^2)$.

In addition to self phase modulation we expect that the nonlinear response of the medium on the wave will not be instantaneous. For sufficiently long pulses this effect can be neglected, but for short ones we expect the appearance of new nonlinear effects due to the delay in the response. In the frequency domain this translates to the statement that we expect that there is a dependence of the nonlinear susceptibility on the frequency. As far as the real part of the susceptibility is concerned, we will assume that the first order nonlinear dispersion term will appear at order $O(\epsilon^3)$, the second order nonlinear dispersion at order $O(\epsilon^4)$ and accordingly for higher order dispersion terms.

The treatment of the imaginary part of the susceptibility will be deferred for discussion after derivation of the basic set of equations. The effect that is connected to it is the stimulated Raman scattering, which is initiated by the pulse itself [LB92] [LBS92] [MC90] [BW89] [KN91a] [KN92] [NKKY92] [KN91b] [BDOS92] [GDK⁺87] [Gor86] [HN94] [DK92] [MM86].

Except for the effects due to the wave itself or the fibre additional physical effects can be introduced either by considering interactions with other co-propagating electro-magnetic waves [LHD90] [KKN92a] [KKN92b], or acoustical waves [DLPP92] [DVPP92]. Finally, the introduction of more complex propagation devices will, of course, affect the equations [Kiv93] [KW94] [BDW88].

Before proceeding to substitutions in the wave equation we need to establish relations between the different expansions introduced so far.

First of all let us deal with the parameters γ and δ appearing in the series expansions of the independent variables z and t . Since we have taken as a comparison measure the linear dispersive effects we anticipate a choice:

$$O(\delta) = O(\gamma) = O(\epsilon)$$

leading to the adoption of the expansions:

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \epsilon \frac{\partial}{\partial t_1} + \epsilon^2 \frac{\partial}{\partial t_2} + \dots \quad (3.29)$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial z_0} + \epsilon \frac{\partial}{\partial z_1} + \epsilon^2 \frac{\partial}{\partial z_2} + \dots \quad (3.30)$$

The multiple scales are now introduced in the electric field expansion(3.24), where by changing the order of the summation we obtain:

$$\mathbf{E}(\mathbf{r}, t) = \sum_j \mathbf{E}^{[j]} e^{i k_j z_0 - i \omega_j t_0} \quad \text{where} \quad (3.31)$$

$$\mathbf{E}^{[j]} = \sum_{\mu=1}^{\infty} \epsilon^{\mu} \left(q^{[j]}(z_1, z_2, \dots, t_1, t_2, \dots) \mathbf{U}_T^{[j]} \right)_{\mu} \quad (3.32)$$

The slow variation of the envelope function is clearly incorporated in that $q^{[j]}$ depends on the slow scales. The exponent $[j]$ has substituted ω_j for reasons of compactness. The function $U_T^{[j]}$ is independent of z and t . Nevertheless it depends on the central frequency of the electromagnetic wave. This indicates that it should be treated in the same fashion as the others dependent on the frequency variables: by expansion in Taylor series around the central frequency. In addition to this we have to allow for the effect of the nonlinearity on the modes. Thus, we allow one more degree of freedom for the mode profile function to account for this dependency. Bearing in mind these remarks and the independence of $q^{[j]}$ from the frequency, the form of the electric field becomes:

$$\mathbf{E}(\mathbf{r}, t) = \sum_j \mathbf{E}^{[j]} e^{i k_j z_0 - i \omega_j t_0} \quad (3.33)$$

with:

$$\mathbf{E}^{[j]} = \left(1 - i \frac{\partial}{\partial t} \frac{\partial}{\partial \omega} \Big|^{[j]} - \frac{1}{2} \frac{\partial^2}{\partial t^2} \frac{\partial^2}{\partial^2 \omega} \Big|^{[j]} + \dots \right) \sum_{\mu=1}^{\infty} \epsilon^{\mu} \left(q^{[j]}(z_1, z_2, \dots, t_1, t_2, \dots) \right)_{\mu} \mathbf{U}_{T,\mu}^{[j]}$$

The superscript $[j]$ indicates that the differentials are evaluated at frequency $\omega = -\omega_j$ and the partial derivatives in time are defined in (3.29) above.

3.7 Multiple scales expansion: Equations

Although the weak guidance approximation will be used in the nonlinear analysis we will refrain from using it explicitly yet. The operator ∇ can be written as:

$$\nabla = \nabla_{\mathbf{T}} + \hat{z} \frac{\partial}{\partial z}$$

where the first term of the right hand side comprises all transverse components of the differential operator and \hat{z} is the unit vector along the z axis. By making use of this expansion we will have:

$$\nabla \times \nabla \times = \nabla_{\mathbf{T}} \times \nabla_{\mathbf{T}} \times + \hat{z} \times \nabla_{\mathbf{T}} \times \frac{\partial}{\partial z} + \nabla_{\mathbf{T}} \times \hat{z} \times \frac{\partial}{\partial z} + \hat{z} \times \hat{z} \times \frac{\partial^2}{\partial z^2} \quad (3.34)$$

Now we rewrite (3.15) in the same notation as that used in the case of the nonlinear polarisation and substitute it along with (3.16), (3.34) and (3.33) into (3.6). The wave equation for the wave-packet at central frequency $-\omega_j$ will be:

$$\begin{aligned}
& \nabla_{\mathbf{T}} \times \nabla_{\mathbf{T}} \times \mathbf{E}^{[j]} e^{i k_j z_0 - i \omega_j t_0} + \hat{z} \times \nabla_{\mathbf{T}} \times \frac{\partial}{\partial z} \mathbf{E}^{[j]} e^{i k_j z_0 - i \omega_j t_0} + \\
& \nabla_{\mathbf{T}} \times \hat{z} \times \frac{\partial}{\partial z} \mathbf{E}^{[j]} e^{i k_j z_0 - i \omega_j t_0} + \hat{z} \times \hat{z} \times \frac{\partial^2}{\partial z^2} \mathbf{E}^{[j]} e^{i k_j z_0 - i \omega_j t_0} = \\
& - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \left(\left(1 + \chi_{(j)}^{(1)} \right) \mathbf{E}^{[j]} e^{i k_j z_0 - i \omega_j t_0} - \right. \\
& i \chi_{(j)}^{(1)} \frac{\partial \mathbf{E}^{[j]}(t)}{\partial t} e^{i k_j z_0 - i \omega_j t_0} - \\
& \frac{1}{2} \chi_{(j)}^{(1)} \frac{\partial^2 \mathbf{E}^{[j]}(t)}{\partial t^2} e^{i k_j z_0 - i \omega_j t_0} + \dots \\
& \chi_{(123)}^{(3)} \mathbf{E}^{[10]} \mathbf{E}^{[20]} \mathbf{E}^{[30]} e^{-i \omega_j t_0 + i (k_j + \Delta k_j) z_0} - \\
& i \chi_{(123)}^{(3)} \frac{\partial \mathbf{E}^{[10]}}{\partial t} \mathbf{E}^{[20]} \mathbf{E}^{[30]} e^{-i \omega_j t_0 + i (k_j + \Delta k_j) z_0} - \\
& i \chi_{(123)}^{(3)} \mathbf{E}^{[10]} \frac{\partial \mathbf{E}^{[20]}}{\partial t} \mathbf{E}^{[30]} e^{-i \omega_j t_0 + i (k_j + \Delta k_j) z_0} - \\
& i \chi_{(123)}^{(3)} \mathbf{E}^{[10]} \mathbf{E}^{[20]} \frac{\partial \mathbf{E}^{[30]}}{\partial t} e^{-i \omega_j t_0 + i (k_j + \Delta k_j) z_0} - \\
& \frac{1}{2} \chi_{(123)}^{(3)} \frac{\partial^2 \mathbf{E}^{[10]}}{\partial t^2} \mathbf{E}^{[20]} \mathbf{E}^{[30]} e^{-i \omega_j t_0 + i (k_j + \Delta k_j) z_0} - \\
& \chi_{(123)}^{(3)} \frac{\partial \mathbf{E}^{[10]}}{\partial t} \frac{\partial \mathbf{E}^{[20]}}{\partial t} \mathbf{E}^{[30]} e^{-i \omega_j t_0 + i (k_j + \Delta k_j) z_0} + \\
& \left. \dots \right) \tag{3.35}
\end{aligned}$$

where the nonlinear terms ² span all possible combinations that will yield a sum frequency $\omega_j = (\omega_1)_0 + (\omega_2)_0 + (\omega_3)_0$ and $\Delta k_j = (k_1)_0 + (k_2)_0 + (k_3)_0 - k_j$ is a phase-mismatch term [BC90] due to the nonlinear dependence of the wavenumber on the frequency. For the nonlinear phenomena that we are considering Δk_j is zero.

In order to proceed from (3.35) we need to introduce some type of expansion of the independent parameters. The usual procedure is to adopt the reductive perturbation method [Kod85b] [KH92] [KH87] [HF93] [Pot89] [KH94a]. In this analysis the derivative expansion method will be used as explained in the previous section, however due to the complexity of (3.35) a minor alteration to the standard procedure will be introduced. Namely, when it comes to higher order temporal or spatial derivatives instead of introducing the expansions (3.29) and (3.30) explicitly the expansion will be implicit. As an example the second order time derivative will be:

$$\begin{aligned}
\frac{d^2}{dt^2} &= \frac{\partial^2}{\partial t_0^2} + 2 \epsilon \frac{\partial^2}{\partial t_0 \partial t_1} + \epsilon^2 \left(\frac{\partial^2}{\partial t_1^2} + 2 \frac{\partial^2}{\partial t_0 \partial t_2} \right) + \dots \\
&= \frac{\partial^2}{\partial t^2_{(0)}} + \epsilon \frac{\partial^2}{\partial t^2_{(1)}} + \epsilon^2 \frac{\partial^2}{\partial t^2_{(2)}} + \dots
\end{aligned}$$

² Nonlinear here indicates terms where $\chi^{(3)}$ and its derivatives appear.

Let us now focus on the linear part of (3.35). We define the vector operators:

$$\mathbf{L}_{(0)}^{[j]} = \left\{ \nabla_T \times \nabla_T \times + i k_j [\hat{\mathbf{z}} \times \nabla_T \times + \nabla_T \times \hat{\mathbf{z}}] - k_j^2 \hat{\mathbf{z}} \times \hat{\mathbf{z}} \times - \frac{\omega_j^2}{c^2} \left(1 + \chi_{(j)}^{(1)} \right) \right\} \quad (3.36)$$

$$\mathbf{L}_{n(l)}^{[j]} = \frac{(-i)^n}{n!} \left(\frac{\partial^n}{\partial \omega^n} \Big|^{[j]} \frac{\partial^n}{\partial t^n} + \frac{\partial^n}{\partial k^n} \Big|^{[j]} \frac{\partial^n}{\partial z^n} \right) \mathbf{L}_{(0)}^{[j]} \quad (3.37)$$

The superscripts $[j]$ are indicative of the central frequency of the electric field on which the operators are applied, and the subscripts (l) denote which order expansions in terms of the derivative expansion method are considered. The operator $\mathbf{L}_{(0)}^{[j]}$ is of course zero order. Further to these operators we assign a name to the differential operator appearing due to the quasi-monochromatic condition:

$$D_\omega^{[j]} = \left(1 - i \frac{\partial}{\partial t} \frac{\partial}{\partial \omega} \Big|^{[j]} - \frac{1}{2} \frac{\partial^2}{\partial t^2} \frac{\partial^2}{\partial \omega^2} \Big|^{[j]} + \dots \right) \quad (3.38)$$

with the derivatives being evaluated at the central frequency $-\omega_j$.

With the help of the operators just defined it is possible to rewrite (3.35) in the shape:

$$\begin{aligned} & \left(\mathbf{L}_{(0)}^{[j]} + \epsilon \mathbf{L}_{1(1)}^{[j]} + \epsilon^2 \left(\mathbf{L}_{1(2)}^{[j]} + \mathbf{L}_{2(1)}^{[j]} \right) + \dots \right) \left(\epsilon q_{(1)}^{[j]} + \epsilon^2 q_{(2)}^{[j]} + \dots \right) \left(\mathbf{U}_{T,0}^{[j]}, \mathbf{U}_{T,1}^{[j]} + \dots \right) = \\ & \sum_{1,2,3} D_{\omega_1}^{[1_0]} D_{\omega_2}^{[2_0]} D_{\omega_3}^{[3_0]} \frac{(\omega_{1_0} + \omega_{2_0} + \omega_{3_0})^2}{c^2} \\ & \chi_{(123)}^{(3)} \mathbf{U}_T^{[1_0]} \mathbf{U}_T^{[2_0]} \mathbf{U}_T^{[3_0]} \left(\epsilon q_{(1)}^{[1_0]} + \dots \right) \left(\epsilon q_{(1)}^{[2_0]} + \dots \right) \left(\epsilon q_{(1)}^{[3_0]} + \dots \right) \end{aligned} \quad (3.39)$$

The summation in the right hand side ranges over all possible combinations of frequencies that will give a sum equal to ω_j . The next step is to equate all coefficients of the expansion of (3.39) in powers of ϵ to zero. For the first order we have:

$$\mathbf{L}_{(0)}^{[j]} q_{(1)}^{[j]} \mathbf{U}_{T,0}^{[j]} = 0 \quad (3.40)$$

This equation is the linear wave equation for the quasi-monochromatic wave. We have already discussed it under the assumptions of monomodality, polarisation preservation in the fibre and weak guidance. The solutions of (3.40) may be forward or backward propagating within the fibre, depending on the sign of the phase velocity. In the case of non absorbing waveguides it is customary to select modes in such a way that a forward propagating mode at frequency $-\omega_j$ is equal to the complex conjugate of a backward propagating mode at the same frequency [SL91]. This convention will be kept here. The next order equation will be linear as well and will have the form:

$$\mathbf{L}_{(0)}^{[j]} q_{(2)}^{[j]} \mathbf{U}_{T,0}^{[j]} + \mathbf{L}_{(0)}^{[j]} q_{(1)}^{[j]} \mathbf{U}_{T,1}^{[j]} + \mathbf{L}_{1(1)}^{[j]} q_{(1)}^{[j]} \mathbf{U}_{T,0}^{[j]} = 0 \quad (3.41)$$

The first term of the left hand side is, similarly to (3.40), algebraic. Since $\mathbf{U}_{T,0}^{[j]}$ is a solution of the wave equation, the first term is removed. The third term is differential with respect to t_1 and z_1 . Thus:

$$\mathbf{L}_{(0)}^{[j]} q_{(2)}^{[j]} \mathbf{U}_{T,0}^{[j]} = 0 \quad (3.42)$$

$$-i \left(\frac{\partial}{\partial \omega} \right)^{[j]} \mathbf{L}_{(0)}^{[j]} \mathbf{U}_{T,0}^{[j]} \frac{\partial q_{(1)}^{[j]}}{\partial t_{(1)}} - i \left(\frac{\partial}{\partial k} \right)^{[j]} \mathbf{L}_{(0)}^{[j]} \mathbf{U}_{T,0}^{[j]} \frac{\partial q_{(1)}^{[j]}}{\partial z_{(1)}} + \mathbf{L}_{(0)}^{[j]} q_{(1)}^{[j]} \mathbf{U}_{T,1}^{[j]} = 0 \quad (3.43)$$

where use was made of the definition of the operators \mathbf{L} (3.37). If the inner product defined in (3.23) is introduced and symbolised with:

$$\int_A \mathbf{V} \cdot \mathbf{W} dA = (\mathbf{V}, \mathbf{W})$$

then multiplication from left of (3.43) with $\bar{\mathbf{U}}_{T,0}^{[j]}$ together with (3.40) will yield:

$$\left(\frac{\partial}{\partial t_{(1)}} + v_g \frac{\partial}{\partial z_{(1)}} \right) q_{(1)}^{[j]} = 0 \quad (3.44)$$

where v_g is the group velocity defined as:

$$v_g = - \left(\frac{\partial (\bar{\mathbf{U}}_{T,0}^{[j]}, \mathbf{L}_{(0)}^{[j]} \mathbf{U}_{T,0}^{[j]})}{\partial k} \right) \left(\frac{\partial (\bar{\mathbf{U}}_{T,0}^{[j]}, \mathbf{L}_{(0)}^{[j]} \mathbf{U}_{T,0}^{[j]})}{\partial \omega} \right)^{-1} \quad (3.45)$$

The derivative with respect to k is taken at the point k_j and the derivative with respect to ω at the point $-\omega_j$. The term $\mathbf{L}_{(0)}^{[j]} q_{(1)}^{[j]} \mathbf{U}_{T,1}^{[j]}$ must vanish with the multiplication. If this does not happen the resulting differential equation will be of the form:

$$\left(\frac{\partial}{\partial t_{(1)}} + v_g \frac{\partial}{\partial z_{(1)}} + ct \right) q_{(1)}^{[j]}$$

with ct a real constant. The solution of this equation is an exponential. This of course, will lead to instability. Thus we require ct to vanish. The implication is that $\mathbf{U}_{T,1}^{[j]}$ should be a mode. However, since the fibre is single mode than it cannot be a guided one.

At order ϵ^3 (3.39) will give:

$$\begin{aligned} & \mathbf{L}_{(0)}^{[j]} q_{(3)}^{[j]} \mathbf{U}_{T,0}^{[j]} + \mathbf{L}_{(0)}^{[j]} q_{(2)}^{[j]} \mathbf{U}_{T,1}^{[j]} + \mathbf{L}_{(0)}^{[j]} q_{(1)}^{[j]} \mathbf{U}_{T,2}^{[j]} + \mathbf{L}_{1(1)}^{[j]} q_{(1)}^{[j]} \mathbf{U}_{T,1}^{[j]} + \mathbf{L}_{1(1)}^{[j]} q_{(2)}^{[j]} \mathbf{U}_{T,0}^{[j]} + \\ & \mathbf{L}_{1(2)}^{[j]} q_{(1)}^{[j]} \mathbf{U}_{T,0}^{[j]} + \mathbf{L}_{2(2)}^{[j]} q_{(1)}^{[j]} \mathbf{U}_{T,0}^{[j]} = \sum_{1,2,3} \chi_{(123)}^{(3)} : \mathbf{U}_{T,0}^{[1_0]} \mathbf{U}_{T,0}^{[2_0]} \mathbf{U}_{T,0}^{[3_0]} q_{(1)}^{[1_0]} q_{(1)}^{[2_0]} q_{(1)}^{[3_0]} \end{aligned} \quad (3.46)$$

The summation has the usual significance ($\omega_j = \omega_{1_0} + \omega_{2_0} + \omega_{3_0}$). The new element in (3.46) is the presence of the nonlinear term. This presence exactly introduces coupling

between different harmonics. For example the choice $\omega_{1_0} = \omega_{2_0} = \omega_{3_0} = \omega_j$ would mean that the nonlinear term would produce the third harmonic of the signals. However, in (3.46) we are interested in the contribution of the nonlinearity at $-\omega_j$. This can be achieved by taking the combinations $(\omega_{1_0}, \omega_{2_0}, \omega_{3_0})$ to be:

$$\begin{aligned} (-\omega_j, -\omega_j, \omega_j) &\rightarrow (j, j, -j) \\ (\omega_j, -\omega_j, -\omega_j) &\rightarrow (-j, j, j) \\ (-\omega_j, \omega_j, -\omega_j) &\rightarrow (j, -j, j) \end{aligned}$$

In the case of the fibre without optical losses, that we are considering, the electric field of a guided mode at frequency ω is the complex conjugate of that at $-\omega$. The contribution of the nonlinear part at frequency $-\omega_j$ will thus become:

$$\frac{\omega_j^2}{c^2} |q_{(1)}^{[j]}|^2 q_{(1)}^{[j]} \mathbf{W}_0$$

where

$$\mathbf{W}_0 = \chi_{(-j,j,j)}^{(3)} \bar{\mathbf{U}}_{T,0}^{[j]} \mathbf{U}_{T,0}^{[j]} \mathbf{U}_{T,0}^{[j]} + \chi_{(j,-j,j)}^{(3)} \mathbf{U}_{T,0}^{[j]} \bar{\mathbf{U}}_{T,0}^{[j]} \mathbf{U}_{T,0}^{[j]} + \chi_{(j,j,-j)}^{(3)} \mathbf{U}_{T,0}^{[j]} \mathbf{U}_{T,0}^{[j]} \bar{\mathbf{U}}_{T,0}^{[j]}$$

with the bar denoting complex conjugation of a quantity.

The secularity condition for (3.46)³ will lead to a decomposition giving the following set of conditions:

$$\mathbf{L}_{(0)}^{[j]} q_{(3)}^{[j]} \mathbf{U}_{T,0}^{[j]} = 0, \quad \mathbf{L}_{(0)}^{[j]} q_{(2)}^{[j]} \mathbf{U}_{T,1}^{[j]} = 0 \quad \mathbf{L}_{(0)}^{[j]} q_{(1)}^{[j]} \mathbf{U}_{T,2}^{[j]} = 0 \quad (3.47)$$

$$\mathbf{L}_{1(1)}^{[j]} q_{(2)}^{[j]} \mathbf{U}_{T,0}^{[j]} = 0 \quad (3.48)$$

$$\mathbf{L}_{1(2)}^{[j]} q_{(1)}^{[j]} \mathbf{U}_{T,0}^{[j]} + \mathbf{L}_{2(2)}^{[j]} q_{(1)}^{[j]} \mathbf{U}_{T,0}^{[j]} + \mathbf{L}_{1(1)}^{[j]} q_{(1)}^{[j]} \mathbf{U}_{T,1}^{[j]} = \frac{\omega_j^2}{c^2} |q_{(1)}^{[j]}|^2 q_{(1)}^{[j]} \mathbf{W}_0 \quad (3.49)$$

The simplification of (3.49) occurs in four steps:

- The equation is left-multiplied with $\bar{\mathbf{U}}_{T,0}^{[j]}$ (or $\bar{\mathbf{U}}_{T,1}^{[j]}$) and integrated over the cross section of the fibre.

³ Implementation of secularity condition occurs here in as follows: The first step is to single out the two terms in (3.46) where $\mathbf{L}_{(0)}^{[j]}$ acts on the zero and first order mode profile function. These are equal to zero according to the arguments laid in the previous order analysis. In a similar fashion we require $\mathbf{U}_{T,0}^{[j]}$ to be a mode in order to avoid instabilities arising from its presence. The remaining terms have both low order corrections and their dependence on higher order time and space coordinates. The terms comprising $q_{(1)}^{[j]}$ can be viewed as driving fields for the second order correction, independent of the coordinates on which the latter depends and must be zero to avoid secularities. Thus, we equate all terms containing the first order correction to zero producing (3.49). Equation (3.48) follows.

- The operators \mathbf{L} are substituted from their definition (3.37).
- The equation that will arise from the previous step will involve the term

$$\frac{\partial^2}{\partial t^2_{(2)}}$$

The time variable that appears here is the one appearing in (3.44). It is eliminated with the help of (3.44) in favour of the space derivative at the same order.

- The relation

$$\frac{d}{dk} = \frac{\partial}{\partial k} + \frac{d\omega}{dk} \frac{\partial}{\partial \omega} = \frac{\partial}{\partial k} + v_g \frac{\partial}{\partial \omega}$$

is used with $\frac{d\omega}{dk}$ being the group velocity defined in (3.45).

After following this procedure the projection of equation (3.49) on $\bar{\mathbf{U}}_{T,0}^{[j]}$ takes the form:

$$\begin{aligned} -\frac{\partial}{\partial \omega} \left(\bar{\mathbf{U}}_{T,0}^{[j]}, \mathbf{L}_{(0)}^{[j]} q_{(1)}^{[j]} \mathbf{U}_{T,0}^{[j]} \right) \left(i \frac{\partial}{\partial t_{(2)}} + v'_g \frac{\partial}{\partial z_{(2)}} + \frac{1}{2} \frac{dv_g}{dk} \frac{\partial^2}{\partial z_{(2)}^2} \right) = \\ = \left(\bar{\mathbf{U}}_{T,0}^{[j]}, \mathbf{W}_0 \right) \frac{\omega_j^2}{c^2} |q_{(1)}^{[j]}|^2 q_{(1)}^{[j]} \end{aligned} \quad (3.50)$$

where

$$\begin{aligned} v'_g = v_g + \left(\frac{\partial}{\partial \omega} \left(\bar{\mathbf{U}}_{T,0}^{[j]}, \mathbf{L}_{(0)}^{[j]} q_{(1)}^{[j]} \mathbf{U}_{T,0}^{[j]} \right) \right)^{-1} \\ \times \left[- \left(\frac{\partial \left(\bar{\mathbf{U}}_{T,0}^{[j]}, \mathbf{L}_{(0)}^{[j]} \mathbf{U}_{T,1}^{[j]} \right)}{\partial k} \right) \left(\frac{\partial \left(\bar{\mathbf{U}}_{T,0}^{[j]}, \mathbf{L}_{(0)}^{[j]} \mathbf{U}_{T,1}^{[j]} \right)}{\partial \omega} \right)^{-1} - v_g \right] \end{aligned}$$

This equation can be rewritten as:

$$i \frac{\partial q_{(1)}^{[j]}}{\partial \xi} + \frac{1}{2} h_1 \frac{\partial^2 q_{(1)}^{[j]}}{\partial z_{(2)}^2} + n_0 |q_{(1)}^{[j]}|^2 q_{(1)}^{[j]} = 0 \quad (3.51)$$

where

$$\xi = t_2 - \frac{z_2}{v'_g}, \quad h_1 = \frac{dv_g}{dk}, \quad n_0 = \left(\bar{\mathbf{U}}_{T,0}^{[j]}, \mathbf{W}_0 \right) \frac{\omega_j^2}{c^2} \left(\bar{\mathbf{U}}_{T,0}^{[j]}, \mathbf{L}_{(0)}^{[j]} \mathbf{U}_{T,0}^{[j]} \right)^{-1}$$

h_1 quantifies the second order dispersion acting on the wave envelope and n_0 the nonlinearity. The inner product between $\bar{\mathbf{U}}_{T,0}^{[j]}$ and the term comprising the higher order mode profile appears as a perturbation of the group velocity, where use is made of the group velocity relation derived at the previous order. The only coupling allowed between modes will be through the nonlinear term. The coupling to this mode will be calculated by taking the inner product of (3.49) with $\bar{\mathbf{U}}_{T,1}^{[j]}$. Equation (3.51) is a generic equation most commonly seen in its normalised form ($h_1 = 1$ and $n_0 = 1$)

[FT87] [Agr89] [AS81]. It is an exactly solvable nonlinear equation called Non Linear Schrödinger Equation (NLS).

At order $O(\epsilon^4)$ we will have:

$$\begin{aligned}
 & \mathbf{L}_{(0)}^{[j]} q_{(3)}^{[j]} \mathbf{U}_{T,1}^{[j]} + \mathbf{L}_{(0)}^{[j]} q_{(2)}^{[j]} \mathbf{U}_{T,2}^{[j]} + \mathbf{L}_{(0)}^{[j]} q_{(1)}^{[j]} \mathbf{U}_{T,3}^{[j]} + \mathbf{L}_{(1)}^{[j]} q_{(3)}^{[j]} \mathbf{U}_{T,0}^{[j]} \\
 & + \mathbf{L}_{(1)}^{[j]} q_{(2)}^{[j]} \mathbf{U}_{T,1}^{[j]} + \mathbf{L}_{(1)}^{[j]} q_{(1)}^{[j]} \mathbf{U}_{T,2}^{[j]} + \mathbf{L}_{(2)}^{[j]} q_{(1)}^{[j]} \mathbf{U}_{T,1}^{[j]} \\
 & \mathbf{L}_{(0)}^{[j]} q_{(4)}^{[j]} \mathbf{U}_{T,0}^{[j]} + \mathbf{L}_{1(1)}^{[j]} q_{(3)}^{[j]} \mathbf{U}_{T,0}^{[j]} + \mathbf{L}_{1(2)}^{[j]} q_{(2)}^{[j]} \mathbf{U}_{T,0}^{[j]} + \mathbf{L}_{2(2)}^{[j]} q_{(2)}^{[j]} \mathbf{U}_{T,0}^{[j]} + \\
 & \mathbf{L}_{1(3)}^{[j]} q_{(1)}^{[j]} \mathbf{U}_{T,0}^{[j]} + \mathbf{L}_{2(3)}^{[j]} q_{(1)}^{[j]} \mathbf{U}_{T,0}^{[j]} + \mathbf{L}_{3(3)}^{[j]} q_{(1)}^{[j]} \mathbf{U}_{T,0}^{[j]} = \\
 & \sum_{s_1+s_2+s_3=4} \sum_{1,2,3} \chi_{(123)}^{(3)} : \mathbf{U}_{T,0}^{[10]} \mathbf{U}_{T,0}^{[20]} \mathbf{U}_{T,0}^{[30]} q_{(s_1)}^{[10]} q_{(s_2)}^{[20]} q_{(s_3)}^{[30]} - \\
 & i \sum_{n=1}^3 \sum_{1,2,3} \frac{\partial}{\partial \omega_n} \left[\frac{(\omega_1 + \omega_2 + \omega_3)^2}{c^2} \chi_{(123)}^{(3)} : \mathbf{U}_{T,0}^{[10]} \mathbf{U}_{T,0}^{[20]} \mathbf{U}_{T,0}^{[30]} \frac{\partial}{\partial t_n} q_{(1)}^{[10]} q_{(1)}^{[20]} q_{(1)}^{[30]} \right] \quad (3.52)
 \end{aligned}$$

The variable t_n appearing in the derivative of the last term of the right hand side of this equation is not related to the multiple scales introduced earlier. It merely denotes that the time derivative operator has to correspond to the electric field whose ω derivative is taken at the earlier part of this term. The analysis for the fourth order equation (3.52) is similar to that for the previous orders. The secularity conditions for the higher order corrections $q_{(m)}^{[j]}$ with $m > 1$ will give the equations like the ones derived at previous orders. The condition for $q_{(1)}^{[j]}$ will be:

$$\begin{aligned}
 & \mathbf{L}_{1(3)}^{[j]} q_{(1)}^{[j]} \mathbf{U}_{T,0}^{[j]} + \mathbf{L}_{2(3)}^{[j]} q_{(1)}^{[j]} \mathbf{U}_{T,0}^{[j]} + \mathbf{L}_{3(3)}^{[j]} q_{(1)}^{[j]} \mathbf{U}_{T,0}^{[j]} + \mathbf{L}_{(2)}^{[j]} q_{(1)}^{[j]} \mathbf{U}_{T,1}^{[j]} = \\
 & -i \sum_{n=1}^3 \sum_{1,2,3} \frac{\partial}{\partial \omega_n} \left[\frac{(\omega_1 + \omega_2 + \omega_3)^2}{c^2} \chi_{(123)}^{(3)} : \mathbf{U}_{T,0}^{[10]} \mathbf{U}_{T,0}^{[20]} \mathbf{U}_{T,0}^{[30]} \frac{\partial}{\partial t_n} q_{(1)}^{[10]} q_{(1)}^{[20]} q_{(1)}^{[30]} \right] \quad (3.53)
 \end{aligned}$$

In order to bring this equation to a simpler form the same procedure as the one used to derive the NLS will be used, with the additional use of (3.50) to tackle the time derivatives at order $O(\epsilon^2)$. The final outcome will be:

$$i \frac{\partial q_{(1)}^{[j]}}{\partial \xi} + \frac{1}{2} h_1 \frac{\partial^2 q_{(1)}^{[j]}}{\partial z^2} + \frac{i}{3!} h_2 \frac{\partial^3 q_{(1)}^{[j]}}{\partial z^3} + n_0 |q_{(1)}^{[j]}|^2 q_{(1)}^{[j]} + i n_1 \left(q_{(1)}^{[j]} \right)^2 \frac{\partial \bar{q}_{(1)}^{[j]}}{\partial z} + i n_2 |q_{(1)}^{[j]}|^2 \frac{\partial q_{(1)}^{[j]}}{\partial z} = 0 \quad (3.54)$$

where ξ , h_1 and n_0 have been defined previously when the NLS was introduced. The remaining constants are defined as:

$$h_2 = \frac{d^2 v'_g}{dk^2}$$

$$n_1 = \left(\bar{\mathbf{U}}_{T,0}^{[j]}, (\mathcal{D}_n)_{k_j} \left[\frac{(\omega_j)^2}{c^2} \mathbf{W}_0 \left(\frac{\partial}{\partial \omega} \left(\bar{\mathbf{U}}_{T,0}^{[j]}, \mathbf{L}_{(0)}^{[j]} \mathbf{U}_{T,0}^{[j]} \right) \right)^{-1} \right] \right)$$

and

$$n_2 = \left(\bar{\mathbf{U}}_{T,0}^{[j]}, (\mathcal{D}_p)_{k_j} \left[\frac{(\omega_j)^2}{c^2} \mathbf{W}_0 \left(\frac{\partial}{\partial \omega} \left(\bar{\mathbf{U}}_{T,0}^{[j]}, \mathbf{L}_{(0)}^{[j]} \mathbf{U}_{T,0}^{[j]} \right) \right)^{-1} \right] \right)$$

To simplify the notation the operators $(\mathcal{D}_n)_{k_j}$ and $(\mathcal{D}_p)_{k_j}$ had to be defined. These are the usual total derivative operators with respect to k_j , acting to their right but they are selective with respect to which quantities they affect when they act on the nonlinear term. Subscript p indicates a derivative acting on quantities dependent on $-k_j$ and the subscript n the same derivative acting on quantities dependent on k_j . Equation (3.54) appears in [Kod85b] for the first time and also in [Pot89]. It may be referred to as a perturbed NLS, although this name is much more general, used in all equations of the form of a small perturbing term added to the NLS. Three new terms make their appearance in (3.54) with reference to (3.51). One of them is linear and the other two nonlinear. The linear one describes the effect of the third order linear dispersion. The other two are nonlinear dispersions arising, partly, from the frequency dependence of the third order susceptibility (shock terms). It is possible, following this line of approach, to generate formal expressions for even higher order equations. However the P-NLS and its extensions, discussed in the next section are sufficient in most cases.

3.8 Perturbed NLS

Having derived a basic set of equations describing the propagation of pulses in a non-linear fibre to different accuracies we now wish to generalise these equations in order to describe phenomena that have been omitted, as discussed in section 3.6. The first generalisation will have to address the problem of the imaginary parts of the susceptibilities. This matter is relatively straightforward in that the exact forms of the susceptibilities can be substituted in the equations derived so far, and the appropriate terms kept.

As far as the absorption is concerned, modern fibres exhibit a spectral window of approximate width $0.1 \mu m$ with relatively flat absorption, centered at $1.55 \mu m$ [Lin89]. For our discussion it is sufficient to consider absorption to be constant. The contribution to the equations would originate from the first order equation in the form of a term:

$$i \left(\frac{\partial}{\partial \omega} \left(\bar{\mathbf{U}}_{T,0}^{[j]}, \mathbf{L}_{(0)}^{[j]} q_{(1)}^{[j]} \right) \right)^{-1} \left(\bar{\mathbf{U}}_{T,0}^{[j]}, \mathbf{U}_{T,0}^{[j]} \right) \frac{\omega_j^2}{c^2} \text{Im} \left(\chi_{(j)}^{(1)} \right)$$

appearing in the left hand side of the equations (3.51) or (3.54) (as in [HK95]). The function Im gives the imaginary part of its argument. The introduction of devices

that would affect the waveguiding properties of the fibre such as frequency dependent filters can be modeled by altering the form of the susceptibility to emulate the newly introduced devices. The constants with which the linear terms would be multiplied would depend on the device properties.

Higher order linear derivative terms arise by extending the multiple scales expansion to the next orders. Nevertheless the functional form of these terms can be inferred by an inspection of the NLS (3.51) and perturbed NLS (3.54) introduced in the previous section. The n th order dispersion term appearing in the left hand side of equations normalised as (3.54) is:

$$-\frac{(-i)^n}{n!} \frac{d^{n-1} v_g}{dk^{n-1}} \frac{\partial^n q_{(1)}^{[j]}}{\partial z^n}$$

For the imaginary part of the nonlinear susceptibility the situation is more complicated than its linear counterpart. It is possible to proceed to a straightforward substitution of the nonlinear susceptibility in its complex form. The perturbed NLS (3.54) would then remain exactly in the same functional form, with the constants n_1 and n_2 in:

$$i n_1 \left(q_{(1)}^{[j]} \right)^2 \frac{\partial \bar{q}_{(1)}^{[j]}}{\partial z} + i n_2 |q_{(1)}^{[j]}|^2 \frac{\partial q_{(1)}^{[j]}}{\partial z}$$

becoming complex quantities instead of real. The imaginary parts of these two terms could be re-expressed in the form of a term modeling the stimulated Raman scattering mentioned in section 3.6 as in [GB91] [KH92] [Pot89] [HK95] having a

$$\left(\frac{\partial |q_{(1)}^{[j]}|^2}{\partial z} \right) q_{(1)}^{[j]}$$

dependence on the envelope function and a self-steepening term [CJ84] [GB91]:

$$\frac{\partial \left(|q_{(1)}^{[j]}|^2 q_{(1)}^{[j]} \right)}{\partial z}$$

However this description is not the best for the Raman scattering which arises from resonant interaction between the electromagnetic field and the propagation medium [BC90] In order to provide a better description the alternative approach is to assume that the imaginary part of the nonlinear susceptibility is not expanded according to the quasi-monochromatic expansions but is modeled by a delayed response function whose parameters are defined in order to fit the Raman gain spectrum [LB92] [HN94] [MC90]. The functional form of the resulting term is:

$$q_{(1)}^{[j]} \int_{-\infty}^t dt' f(t-t') |q_{(1)}^{[j]}|^2(t')$$

Other higher order nonlinear terms can be introduced by proceeding in the multiple scales analysis, done in the previous chapter. The fifth order would introduce a quintic term as well as a term arising from the cascading of the third order nonlinearities. Further to the expansion higher order terms can arise from the adoption of more complex structures for the guiding media [CC94] [DA94] [Her92] or interactions [Moo93] [SAA96] [AAS96].

A multitude of other physical situations can be incorporated within the framework just defined. It is not our task here to cover them all. We will only comment that the common denominator of almost all approaches is the use of the NLS equation together with some perturbing terms added to it. In the next chapter we will examine the solution methods that are available to us when dealing with the initial value problem of the general perturbed NLS

4. THE EVOLUTION EQUATIONS

4.1 *Introduction*

In the previous chapter a set of equations was introduced to describe the evolution of the Electric field of a traveling Electromagnetic wave within a fibre. The present chapter will be devoted to examining the solution methods for these equations. Broadly speaking we can classify solution methods in two categories:

- Numerical methods
- Analytical methods

In this study we will be only considering analytical solution methods. The motivation behind this selection is, as mentioned earlier, that although analytical methods tend to be more difficult to implement, if they can be implemented, and are less exact, they provide information regarding the dynamics of the systems in a clear fashion.

Apart from the low order approximations to the temporal evolution of the electric field, discussed in the previous chapter, all equations of order equal to or higher than the NLS, are nonlinear partial differential equations (PDE). The propagation problem that we wish to solve is, in mathematical terminology, the initial value problem for these equations. Unfortunately this problem is, generally, insoluble for the class of nonlinear PDE's. However there exist a number of exceptions to this rule and the NLS is one of them.

In this chapter we will be discussing the basic properties of the NLS as well as its solution method. Based on this knowledge we will then review some of the perturbation techniques pertinent to our problem.

4.2 *Non Linear Schrödinger equation*

4.2.1 *Solutions of the NLS*

The NLS equation (3.51) describes the evolution of a pulse envelope under the influence of dispersion and weak nonlinearity. The coefficient of the dispersion term in (3.51)

can take positive or negative values depending on the group velocity dispersion regime in the spectral region that we are considering.

The NLS equation is special in that it has associated with it some extraordinary properties. Firstly it can support *soliton* solutions. The latter concept, introduced by Zabusky and Kruskal [ZD65], designates solitary waves which have the property of passing through one another without suffering deformation due to the collisions. This particle-like behaviour prompted the use of the word soliton.

Depending on the dispersion regime two types of solitons can be supported: *bright* solitons occurring when the coefficient of the dispersion term is positive or *dark* solitons when the coefficient is negative [Has89]. We will be treating only the first type of solitons which have the form of a bright pulse in a dark background. For these pulses to survive we require positive coefficient for the second order derivative term in the NLS. This translates into anomalous dispersion for the propagation medium.

Subsequently we renormalise equation (3.51) to the form that we will be using henceforth:

$$i \frac{\partial}{\partial t} q + \frac{1}{2} \frac{\partial^2}{\partial z^2} q + |q|^2 q = 0 \quad (4.1)$$

with $q = q(t, z)$ a complex function that depends on space (z) and the delayed time variable (t). To formulate the initial value problem we need to supplement equation (4.1) with an initial condition:

$$q(t, z)|_{t=0} = q_0(z)$$

Further to the definition of the initial condition we supply a set of boundary conditions:

$$q(t, z) \rightarrow 0 \text{ as } |z| \rightarrow \infty$$

sufficiently rapidly. This is referred to as the *rapidly decreasing* [FT87] set of boundary conditions and is the type of problem that we will be considering.

In addition to the soliton solutions the NLS can support solutions in the form of dispersive waves. The interaction of these solutions with the soliton solutions results in phase changes and spatial translations only. The simplest soliton solution of the NLS has the form:

$$q(t, z) = \nu \operatorname{sech}(\sigma) \exp(i \mu z + i \theta(t)) \quad (4.2)$$

with $\sigma = \nu(z - \xi(t))$, $\xi(t) = \mu t$ and $\theta(t) = \frac{\nu^2 - \mu^2}{2} t$ and will be referred to as the *fundamental soliton solution*.

This extraordinary type of solution prompted more research on the NLS and equations with similar properties [Lax68] [AKNS73] [ZS72] and its mathematical properties

which led to the implementation of a solution method called *Inverse scattering transform method*.

The NLS equation has an infinite number of conserved quantities [ZS72]. This property is referred to as *integrability* [HK95]. The form of these quantities can be derived from the implementation of the inverse scattering transform and will be seen in the next section.

Furthermore the NLS has associated with it a Hamiltonian structure [FT87] [NMPZ84]. An elementary discussion of this matter can start along the lines of the discussion in section 2.3. Our starting point is the usual NLS (4.1). A Lagrangian density function, corresponding to the NLS, is then defined in the form:

$$\mathcal{L}(q, \bar{q}) = i \frac{(\bar{q} q_t - q \bar{q}_t)}{2} - \frac{q_z \bar{q}_z}{2} + \frac{q^2 \bar{q}}{2} \quad (4.3)$$

The subscripts t and z denote partial differentiation with respect to time and space respectively. The Euler-Lagrange equations that can be derived from this Lagrangian density are the normalised NLS equation and its conjugate. The next step is to associate to the solution q a momentum, defined by the formula:

$$p^{[q]} = \frac{\delta}{\delta q_t} \mathcal{L} = i \bar{q} \quad (4.4)$$

with the derivation appearing in the footnote ¹ The reader is reminded that the action is the integral of \mathcal{L} from the initial to the final time. Further to this we introduce a Hamiltonian function depending on the q and p pair and its complex conjugates [NMPZ84] [HK95]:

$$\mathcal{H} = p^{[q]} q_t - \mathcal{L} = \frac{q_z \bar{q}_z}{2} - \frac{q^2 \bar{q}^2}{2} \quad (4.5)$$

The Hamiltonian will generally depend on both coordinate (q) and momentum(p). Let us now consider the space where q and p are coordinates. The true world coordinate z becomes now labels, i.e $q(z_1)$ is a certain coordinate and so is $q(z_2)$ and so on. The newly defined space is infinite dimensional. Each point of this space corresponds to a state of the physical system. The temporal evolution of the system will be described by a certain curve in this space. We need the equations describing the evolution of

¹ We assume that the configuration space is spanned by two sets of coordinates $q(z, t) = q$ and $\bar{q}(z, t) = \bar{q}$. The momentum, as defined above will have to depend on q and \bar{q} and not their time derivatives. To remove this dependence the first term of the Lagrangian is rewritten as $\mathcal{L}(q, \bar{q}) = i\bar{q}q_t - i \frac{(|q|^2)_t}{2} - \frac{q_z \bar{q}_z}{2} + \frac{q^2 \bar{q}}{2}$. The integral of $(|q|^2)_t$ in space (z) is 0 as will be seen later in the section where the integrals of motion are defined. From this and the expression for the Lagrangian both the Hamiltonian and the definition of the momentum follow easily.

these variables along this curve. The latter can be found in numerous mechanics books [LL76]. In the case in hand the equations of motion are:

$$p_t^{[q]} = -\frac{\delta\mathcal{H}}{\delta q} \Rightarrow \bar{q}_t = i \frac{\delta\mathcal{H}}{\delta q} \quad (4.6)$$

$$q_t = \frac{\delta\mathcal{H}}{\delta p^{[q]}} \Rightarrow q_t = -i \frac{\delta\mathcal{H}}{\delta \bar{q}} \quad (4.7)$$

And similarly for the complex conjugate coordinates. The variational derivative appearing above is:

$$\frac{\delta\mathcal{H}}{\delta u} = \sum_{n=0}^{\infty} (-1)^n \frac{d^n}{dz^n} \frac{\partial\mathcal{H}}{\partial u_{nz}} \text{ where } u_{nz} = \frac{\partial^n u}{\partial z^n} \text{ and } u = q, p$$

It can be easily seen that the equations of motion are a restatement of the NLS. The *Poisson bracket* [FT87] between two functionals of q and \bar{q} is introduced in the following form:

$$\{A, B\} = i \int_{-\infty}^{\infty} \left(\frac{\delta A}{\delta q} \frac{\delta B}{\delta \bar{q}} - \frac{\delta A}{\delta \bar{q}} \frac{\delta B}{\delta q} \right) dz \quad (4.8)$$

This structure is non-degenerate and can help establish an algebra of the observables A and B . Furthermore it can be proved that the temporal variation of observable A is:

$$\frac{dA}{dt} = \{\mathcal{H}, A\} \quad (4.9)$$

The Hamiltonian picture however goes much beyond these simple relations. As mentioned above the Hamiltonian will generally depend on both q and p . However it is possible to introduce an invertible transform of these coordinates into a new set (P and Q). This particular set has the following properties:

- Equations (4.6) and (4.7) remain invariant under the substitution $p \rightarrow P, q \rightarrow Q$ and
- The Hamiltonian acquires a simple form depending on half the coordinates only

The latter has the effect of simplifying greatly the equations of motion, since half of the coordinates do not exhibit temporal variation.

Furthermore it turns out that these new coordinates are closely related to the scattering data which we will introduce in the next section [FT87].

4.2.2 The Inverse scattering transform

We will now discuss the application of the Inverse Scattering Transform (IST) in the solution of the initial value problem for the NLS (4.1) introduced by Zakharov and

Shabat [ZS72]. The discussion here has the purpose of outlining the method and fixing the notation needed for the perturbation theory. The IST theory in full detail can be found elsewhere [Lam80] [FT87] [NMPZ84]. The solution of the NLS begins with the observation that (4.1) is the compatibility condition for the following *linear* system of equations [KH92] [FT87]:

$$\frac{\partial}{\partial z} F = U(z, t, \zeta) F \quad (4.10)$$

$$\frac{\partial}{\partial t} F = V(z, t, \zeta) F \quad (4.11)$$

where

$$F = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad U = \begin{pmatrix} -i\zeta & iq \\ i\bar{q} & i\zeta \end{pmatrix} \text{ and } V = \begin{pmatrix} -i\zeta^2 + \frac{i}{2}|q|^2 & i\zeta q - \frac{1}{2}q_z \\ i\zeta\bar{q} + \frac{1}{2}\bar{q}_z & i\zeta^2 - \frac{i}{2}|q|^2 \end{pmatrix}$$

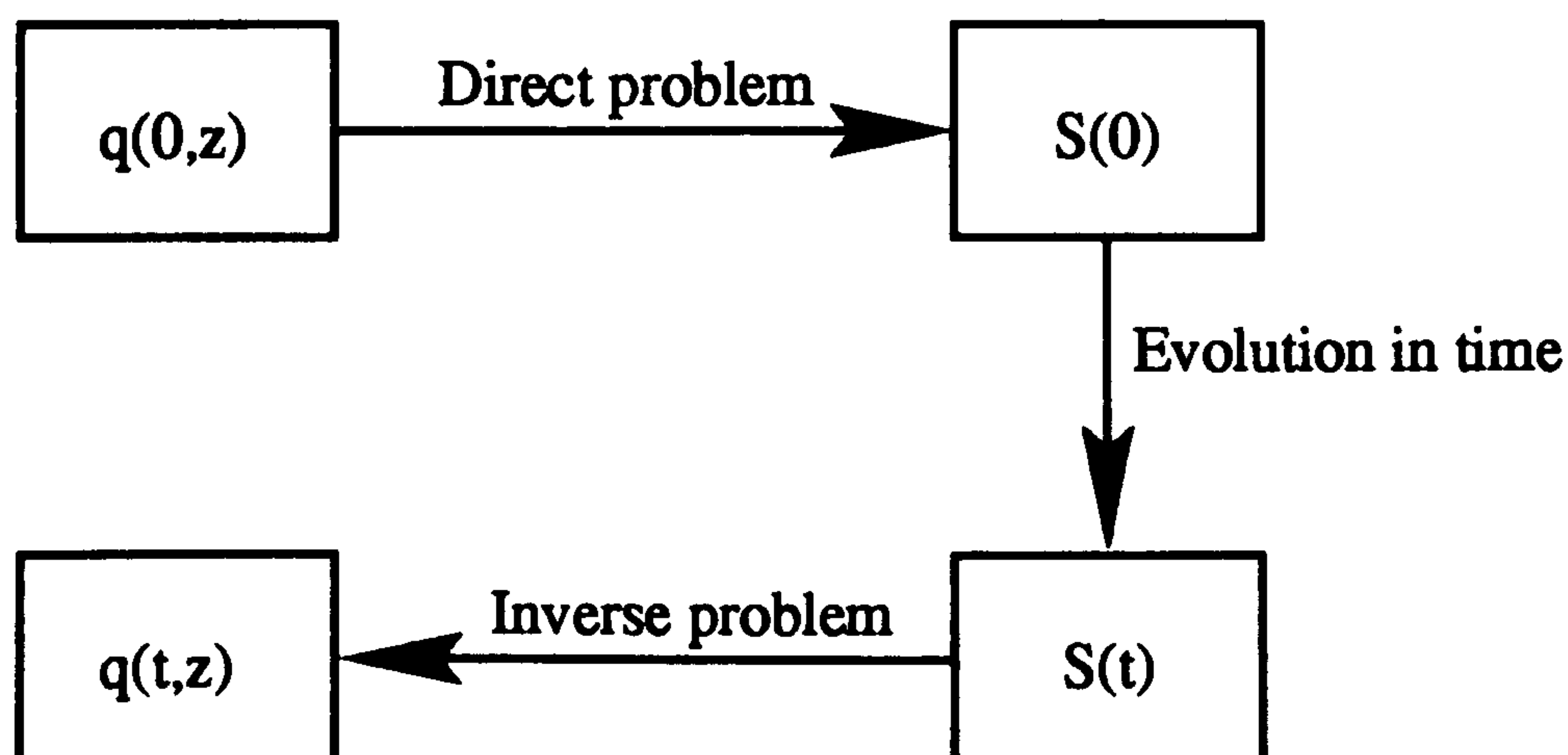
By compatibility condition we mean that

$$\frac{\partial^2 F}{\partial t \partial z} = \frac{\partial^2 F}{\partial z \partial t}$$

should hold for any value of the eigenvalue ζ .

This observation suggests the following way of solving the NLS:

1. *Direct problem*: The problem (4.10) is solved for $q(0, z)$ yielding a set of data that fully describes the solutions of the equation for the given q .
2. *Time evolution*: This set of data is propagated through time using information arising from (4.11).
3. *Inverse problem*: The time propagated set of data is used to reconstruct the function q . This is the inverse of the problem encountered in the first step.



Let us now consider these steps one by one.

Direct transform

Two solutions of (4.10) are defined $\Phi = (\phi_1, \phi_2)^t$ and $\Psi = (\psi_1, \psi_2)^t$, where the superscript t denotes the transpose matrix. These solutions have the following asymptotic values for real $\zeta = \xi$:

$$\Phi(z; \xi) \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\xi z} \text{ as } z \rightarrow -\infty \quad (4.12)$$

$$\Psi(z; \xi) \rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\xi z} \text{ as } z \rightarrow \infty \quad (4.13)$$

Along with this pair of solutions a second pair is defined:

$$\tilde{\Phi} = \begin{pmatrix} \bar{\phi}_2 \\ -\bar{\phi}_1 \end{pmatrix} \text{ and } \tilde{\Psi} = \begin{pmatrix} \bar{\psi}_2 \\ -\bar{\psi}_1 \end{pmatrix}$$

These four solutions are called *Jost solutions* [DEGM82]. The solution space of problem (4.10) is two-dimensional. We take the pair Ψ and $\tilde{\Psi}$ to be the base for the solution space. It is now possible to express the solutions Φ and $\tilde{\Phi}$ in terms of these two:

$$\Phi(z; \xi) = a(\xi) \tilde{\Psi}(z; \xi) + b(\xi) \Psi(z; \xi) \quad (4.14)$$

$$\tilde{\Phi}(z; \xi) = \bar{b}(\xi) \tilde{\Psi}(z; \xi) - \bar{a}(\xi) \Psi(z; \xi) \quad (4.15)$$

The coefficients $a(\xi)$ and $b(\xi)$ are given:

$$a(\xi) = W(\Phi(z; \xi), \Psi(z; \xi)) = (\phi_1 \psi_2 - \psi_1 \phi_2)(\xi) \quad (4.16)$$

$$b(\xi) = -W(\Phi(z; \xi), \tilde{\Psi}(z; \xi)) = (\phi_1 \bar{\psi}_1 + \phi_2 \bar{\psi}_2)(\xi) \quad (4.17)$$

The function W is called the Wronskian and is defined as the determinant of its arguments. It is easy to prove that the Wronskian of two solutions of (4.10) corresponding to the same eigenvalue is independent of x .

Before proceeding let us have a closer look at (4.14). While Φ is defined asymptotically at $-\infty$ the other two are defined at ∞ and moreover $\tilde{\Psi}$ is a plane wave moving in the same direction as Φ while Ψ moves in the other direction. We have thus a situation closely reminiscent of the scattering problem in quantum mechanics. By dividing with $a(\xi)$ everywhere in equation (4.14) and rearranging we have:

$$\tilde{\Psi}(z; \xi) = -\frac{b(\xi)}{a(\xi)} \Psi(z; \xi) + \frac{1}{a(\xi)} \Phi(z; \xi) \quad (4.18)$$

The coefficient $b(\xi)/a(\xi)$ is the reflection coefficient in the scattering problem and the coefficient $1/a(\xi)$ is the transmission coefficient. The knowledge of the reflection coefficient describes the scattering problem (4.10) fully on the real line [DEGM82].

We next look for the bound states of the problem which are associated with soliton solutions. For this purpose the functions Φ , Ψ and a are analytically extended into the upper half plane in ζ ($\text{Im}\zeta > 0$). As $\zeta \rightarrow \infty$ they tend to $(1, 0)^t \exp(-i\zeta x)$, $(0, 1)^t \exp(i\zeta x)$ and 1 respectively.

We are interested in locating the discrete set of values of ζ in the upper half plane, that make the coefficient $a(\zeta)$ equal to 0. For these points (4.14) is rewritten as:

$$\Phi(z; \zeta_n) = b_n \Psi(z; \zeta_n) \text{ with } n = 1 \dots N \quad (4.19)$$

N is the number of such points. This equation indicates that Φ will decay in both limits of positive and negative infinity. The values ζ_n with $n = 1 \dots N$ are called *bound state eigenvalues* [New85]

We now define the set of *scattering data* [KH92] for time $t = 0$ to be the set made of the variables:

$$S(0) = \left[\frac{b(\xi)}{a(\xi)} \text{ for } \xi \text{ real}, \left\{ \zeta_n, \frac{b_n}{\dot{a}_n} \right\} \text{ for } n = 1, 2, \dots N \right] \quad (4.20)$$

with \dot{a}_n denoting the derivative of $a(\zeta)$ with respect to ζ at the point ζ_n . The definition of the scattering data at the initial point in time completes the forward scattering transform.

Time evolution of the scattering data

The scattering transform solution method relies on the fact that the connection between the function q and the scattering data is a one-to-one mapping. In the previous step we defined this data. In order to reconstruct the function q at a certain time we need to propagate the scattering data to that point in time. To achieve this use is made of (4.11) for large $|z|$ to derive the particularly simple equations [New85]:

$$\frac{b(\xi, t)}{a(\xi, t)} = \frac{b(\xi, 0)}{a(\xi, 0)} e^{2i\xi^2 t} \quad (4.21)$$

$$\frac{b_n(t)}{\dot{a}_n(t)} = \frac{b_n(0)}{\dot{a}_n(0)} e^{2i\zeta_n^2 t} \quad (4.22)$$

$$\zeta_n(t) = \zeta_n(0) \quad (4.23)$$

With the help of these relations the scattering data are propagated in time to give the set:

$$S(t) = \left[\frac{b(\xi, t)}{a(\xi, t)} \text{ for } \xi \text{ real}, \left\{ \zeta_n, \frac{b_n(t)}{\dot{a}_n(t)} \right\} \text{ for } n = 1, 2, \dots N \right] \quad (4.24)$$

We denote the analytic continuation of the coefficient $a(\xi)$ in the upper half plane by $a(\zeta)$. It can be proved that $a(\zeta)$ is independent of time [ZS72]. This observation

leads to the discovery of an infinite set of conserved quantities C_n for the NLS [HK95]:

$$(2i)^n C_n = \int_{-\infty}^{\infty} \mathcal{F}_n(z) dz \quad (4.25)$$

with \mathcal{F}_n given by the recursion relation:

$$\mathcal{F}_{n+1} = q \frac{\partial}{\partial z} \left(\frac{1}{q} \mathcal{F}_n \right) + \sum_{l=1}^{n-1} \mathcal{F}_l \mathcal{F}_{n-l}, \quad \mathcal{F}_1 = |q|^2$$

and the integration being along the z axis. The first three conserved quantities can be brought into a symmetrical form. Their physical significance can be easily deduced:

$$\begin{array}{ll} \text{mass} & \int_{-\infty}^{\infty} |q|^2 dz \\ \text{momentum} & \int_{-\infty}^{\infty} \frac{i}{2} \left(q \frac{d\bar{q}}{dz} - \bar{q} \frac{dq}{dz} \right) dz \\ \text{energy} & \int_{-\infty}^{\infty} \left(\left| \frac{dq}{dz} \right|^2 - |q|^4 \right) dz \end{array}$$

Mass corresponds, in physical terms, to the number of photons. Relevant to the discussion regarding the Hamiltonian reformulation of the NLS model is the energy which is exactly the Hamiltonian function introduced there (section 4.2.1).

Inverse Scattering Transform

The final step in the solution method is the inversion of the mapping introduced in the first step. Namely, we wish to use the set of scattering data at time t (4.24) to reconstruct the function $q(x, t)$. To achieve this we consider equation (4.18). The functions Φ and a extend analytically into the upper half ζ plane ($\text{Im}(\zeta) > 0$), the function $\tilde{\Psi}$ extends analytically to the lower half plane ($\text{Im}(\zeta) < 0$) [New85] and the reflection coefficient can only be defined on the real line ($\text{Im}(\zeta) = 0$). After analytically extending and rearranging (4.18) we have:

$$\frac{1}{a(\zeta)} \Phi(z; \zeta) = \tilde{\Psi}(z; \zeta) + \frac{b(\zeta)}{a(\zeta)} \Psi(z; \zeta) \quad (4.26)$$

which indicates that a function (Φa^{-1}) , analytical in the upper half plane, except for a number of poles at $\zeta = \zeta_n$, is constructed from a function which is analytic at the lower half plane ($\tilde{\Psi}$) and a “smooth” function defined on the real axis. This is the *Riemann-Hilbert problem* [New85] [FT87].

Equation (4.26) is divided by $\zeta' - \zeta$ and integrated, leading to the following set of linear integral equations [KH92]:

$$\begin{aligned}\tilde{\Psi}(z; \xi) &= \begin{pmatrix} \bar{\psi}_2 \\ -\bar{\psi}_1 \end{pmatrix}(z; \xi) \\ &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\xi z} + \sum_{n=1}^N \frac{\frac{b_n}{\dot{a}_n} e^{i(\zeta_n - \xi)t}}{\xi - \zeta_n} \Psi(z; \zeta_n) \\ &\quad + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\frac{b(\xi')}{a(\xi')} e^{i(\xi' - \xi)z}}{\xi' - \xi + i0} \Psi(z; \xi') d\xi'\end{aligned}\quad (4.27)$$

$$\begin{aligned}\tilde{\Psi}(z; \zeta_n) &= \begin{pmatrix} \bar{\psi}_2 \\ -\bar{\psi}_1 \end{pmatrix}(z; \zeta_n) \\ &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\bar{\zeta}_n z} + \sum_{m=1}^N \frac{\frac{b_m}{\dot{a}_m} e^{i(\zeta_m - \bar{\zeta}_n)t}}{\bar{\zeta}_n - \zeta_m} \Psi(z; \zeta_m) \\ &\quad + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\frac{b(\xi)}{a(\xi)} e^{i(\xi - \bar{\zeta}_n)z}}{\xi - \bar{\zeta}_n} \Psi(z; \xi) d\xi\end{aligned}\quad (4.28)$$

which are both linear integral equations. The function q is then recovered in the form:

$$q(z) = -2 \sum_{n=1}^N \frac{b_n}{\dot{a}_n} e^{-i\bar{\zeta}_n z} \bar{\psi}_2(z; \zeta_n) - \frac{1}{i\pi} \int_{-\infty}^{\infty} \frac{\bar{b}(\xi)}{\bar{a}(\xi)} e^{-i\xi z} \bar{\psi}_2(z; \xi) d\xi \quad (4.29)$$

During derivation it was assumed that the zeros a_n are simple. The solution at time t consists of two parts. The first term on the right hand side of (4.29) is the discrete part. It is associated with the soliton solutions to the NLS. The second term is a background radiation term corresponding to the dispersive radiation that is allowed to propagate.

4.3 Perturbation methods

4.3.1 Adiabatic perturbation theory

In the previous section we introduced the exact solution method for the NLS using the inverse scattering transform. However, we are interested in a much broader set of perturbed NLS equations which are not exactly solvable. We thus need to introduce some form of perturbation theory. The equation that we want to solve has the general form:

$$i \frac{\partial}{\partial t} q + \frac{1}{2} \frac{\partial^2}{\partial z^2} q + |q|^2 q = \epsilon R(q, \bar{q}) \quad (4.30)$$

The operator R appearing in the right hand side is, in the general case, a nonlinear differential operator acting on q and its complex conjugate.

The simplest situation that we are required to describe is that of the single soliton (4.2), propagating under perturbation.

A first approach is to start with an assumption that the solution of (4.30) can be represented in the form:

$$q(t, z) = q_s(t, z) + \Delta q(t, z)$$

where q_s is the one soliton solution "content" of the propagating field and Δq is a deviation from q_s . This form of solution when substituted in equation (4.30) (4.1) will give the following relation for Δq :

$$-i \frac{\partial}{\partial t} \Delta q = \frac{1}{2} \frac{\partial^2}{\partial z^2} \Delta q + 2|q_s|^2 \Delta q + q_s^2 \overline{\Delta q} - \epsilon R(q_s, \bar{q}_s) \quad (4.31)$$

with the bar denoting complex conjugation.

To solve this equation an approach would be to introduce an adjoint problem which would help projecting out excitations [GH86] [HL90] [HW96]. Let us try to define this problem based on the unperturbed form of (4.31). The presence of the two excitation terms containing q_s will lead to the violation of the conservation of mass requirement. Mass here denotes the integral over z of the square of the absolute value of the solution (see section 4.2.2). In order to rectify this we introduce the adjoint problem of (4.31) setting the condition that its solution, $\widetilde{\Delta q}$, corresponding to Δq , satisfies mass conservation. The mathematical expression for this comes in the form:

$$\frac{d}{dt} \operatorname{Re} \left[\int_{-\infty}^{\infty} dz \overline{\widetilde{\Delta q}} \Delta q \right] = 0$$

The next step is to make an estimation of Δq . We notice that the soliton q_s is expressed by an equation of the type (4.2). We can now incorporate into Δq the variation of the constant parameters ν, μ, ξ, θ . By taking the appropriate Taylor expansions we obtain up to first order:

$$\Delta q = \Delta \nu f_\nu + \Delta \mu f_\mu + \Delta \theta f_\theta + \Delta \xi f_\xi + \Delta q_c \quad (4.32)$$

Δq_c is the higher order correction. The functions $f_\nu, f_\mu, f_\xi, f_\theta$ are derivatives of the soliton solution q_s with respect to the corresponding subscript at time $t = 0$. Clearly this expression can be seen as the projection on the space with coordinates the four functions $f_\nu, f_\mu, f_\xi, f_\theta$. Furthermore we have the inner product defined by the conservation of mass. We only need to determine the projection coefficients.

Similarly the solutions to the adjoint problem are found, and turn out to be orthonormal to the set of functions $f_\nu, f_\mu, f_\xi, f_\theta$. In order to model the effect of background radiation, if this is wished, a noise term $s(t, z)$ is added to the right hand side of (4.31). The perturbation (4.32) is now substituted into the governing equation (4.31) with the noise term. The equation that arises is left-multiplied by each of the adjoint

functions $\widetilde{f}_\nu, \widetilde{f}_\mu, \widetilde{f}_\xi, \widetilde{f}_\theta$ to give the following equations of motion [HW96]:

$$\frac{d}{dt} \Delta\nu = S_\nu(t) \quad (4.33)$$

$$\frac{d}{dt} \Delta\mu = S_\mu(t) \quad (4.34)$$

$$\frac{d}{dt} \Delta\xi = \Delta\mu + S_\xi(t) \quad (4.35)$$

$$\frac{d}{dt} \Delta\theta = \Delta\nu + S_\theta(t) \quad (4.36)$$

$$(4.37)$$

for the case of $\nu = 1, \mu = 0, \xi(0) = 0, \theta(0) = 0$. The source terms in the right hand sides are defined as:

$$S_p(t) = \text{Re} \left[\int_{-\infty}^{\infty} dz \widetilde{\widetilde{f}}_p e^{-it/2} (s(t, z) - \epsilon R(q_s, \bar{q}_s)) \right]$$

Under the assumption that dispersive radiation generation is not excessive and the change in the soliton parameters is gradual, large changes in the soliton parameters are allowed.

This first perturbation expansion is based on the assumption that the distortion of the soliton shape can be modeled by altering the soliton parameters at the slower time scale. The generic name for this kind of perturbation expansion is *adiabatic perturbation theory*.

The region of validity of the adiabatic perturbation theory is of the order $t = O(\epsilon^{-1})$. In order to proceed to better approximations we need to calculate the first order corrections to the fundamental solution. To achieve this two possible alternatives are open to us:

- Direct perturbation methods or
- IST-based perturbation theory.

4.3.2 Direct perturbation method

There are several versions of the direct perturbation problem [GO81] [OP71] [KA81]. The starting point is the perturbed NLS (4.30). It is assumed that the perturbation introduces a slow time scale in the problem. This slow scale is symbolised by $\tau = \epsilon t$. The solution to this perturbed NLS is assumed to be in the form:

$$q = q_0 + \epsilon q_1 + \dots$$

where each of the functions q_i , $i = 1, \dots$ depends on a set of “fast” variables $\theta_1, \theta_2, \dots$ and a set of “slow” variables p_1, p_2, \dots . The adjectives fast and slow denote dependence

on t and τ respectively. To picture this allocation of variables, in the example of the single soliton solution (4.2) a possible choice would be: $\theta_1 = \sigma$, $\theta_2 = \mu z + \theta$ and $p_1 = \nu$, $p_2 = \mu$

Subsequently the solution is substituted in the equation to be solved. To leading order the result is the NLS equation for q_0 . At the next order we have:

$$\mathcal{L}(\partial_{\theta_i}, q_0) q_1 = R(q_0, \bar{q}_0) - i \frac{\partial}{\partial \tau} q_0 \quad (4.38)$$

The left hand side of this equation is the linearised NLS. The first term in the right hand side is the perturbation term while the second term arises from the fact that we allowed the parameters of the soliton to be modulated at the slow time scale. The next step is to assume that the function q_1 can be decomposed into:

$$q_1 = (\phi_1 + i \psi_1) \exp(i \theta + i \mu z)$$

where ϕ_1 and ψ_1 are real functions. This solution is substituted into (4.38) and after the separation of real and imaginary parts we will have two equations:

$$\mathcal{L}_R \phi_1 = \operatorname{Re} \left[R(q_0, \bar{q}_0) - i \frac{\partial}{\partial \tau} q_0 \right] \quad (4.39)$$

$$\mathcal{L}_I \psi_1 = \operatorname{Im} \left[R(q_0, \bar{q}_0) - i \frac{\partial}{\partial \tau} q_0 \right] \quad (4.40)$$

where the left hand sides of these two equations are the real and the imaginary part of the linearised equation, respectively. The solution of the homogeneous part of the above system of equations comes from the differentiation of the fundamental soliton with respect to the two fast variables θ_1 and θ_2 defined earlier. The final step before solving the inhomogeneous system is to introduce secularity conditions. The latter appear as orthogonality conditions between the solution of the homogeneous system and the inhomogeneous term of the full equation:

$$\int_{-\infty}^{\infty} \frac{\partial q_0}{\partial \sigma} \operatorname{Re} \left[R(q_0, \bar{q}_0) - i \frac{\partial}{\partial \tau} q_0 \right] dz = 0 \quad (4.41)$$

$$\int_{-\infty}^{\infty} q_0 \operatorname{Im} \left[R(q_0, \bar{q}_0) - i \frac{\partial}{\partial \tau} q_0 \right] dz = 0 \quad (4.42)$$

with the coordinates defined at the beginning of this section.

Extension of this approach into higher orders is straightforward [GO81]. However a major drawback of the theory is that the formulas that arise are not uniformly valid throughout space z . To construct uniformly valid expansions one needs to consider radiative effects as well. The latter do not give localised solutions, like the ones that the secularity condition forces us to adopt.

A way to include radiation into the picture [KA81] is to take matched expansions at first order. This means that the solutions obtained previously are expanded for large σ . Subsequently equations (4.39) and (4.40) are taken at large σ , where all localised terms vanish. The equations arising from the previous step are solved with the boundary conditions found earlier. Alternatively, matching can be achieved by just retaining the solutions that would yield functional forms compatible to the original localised solutions expanded for large σ [WCL90].

A different approach is to map the problem defined by the linearised NLS into a linear dispersive problem [Gor92], using a Bäcklund type transform. Although this result arises from the IST, it is self contained in that IST is not used in the perturbation calculations. The idea itself is that equation:

$$i \frac{\partial q_r}{\partial t} + \frac{1}{2} \frac{\partial^2 q_r}{\partial z^2} + 2|\text{sech}(\sigma)|^2 q_r + \text{sech}^2(\sigma) e^{2i\mu z + i\theta} \bar{q}_r = 0 \quad (4.43)$$

is solved for:

$$q_r = -\frac{\partial^2 f}{\partial z^2} + 2\gamma \frac{\partial f}{\partial z} - \gamma^2 f + \text{sech}^2(\sigma) e^{2i\mu z + i\theta} \bar{f} \quad (4.44)$$

The function γ is defined as $-\overline{[q_0^{-1}(\partial q_0)/(\partial z)]}$ and q_0 is the fundamental soliton solution to the NLS. The function f is the solution for the linear dispersive problem:

$$i \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial z^2} = 0 \quad (4.45)$$

This approach was later on generalised [Elg93] [EBK95] for perturbed NLS problems. The problem in adopting such an approach is that finding the direct transform for the initial radiation is not trivial.

Having seen the direct method approach to the perturbation problem, we will now attempt to see things from the IST point of view.

4.3.3 IST perturbation theory

IST theory provides a natural way of solving the initial value problem of nonlinear PDE's. It is thus natural to seek solutions of the perturbed NLS within its framework. There are two possibilities. A limited number of equations can be solved by deriving a new set of matrices U and V satisfying (4.10), (4.11) and whose compatibility condition:

$$U_t - U_z + [U, V] = 0 \quad (4.46)$$

(*zero curvature representation* [FT87]) is equivalent to the equation in hand. This approach has been successfully applied to a number of problems in the past ([MT⁺93]

[SS91] [KC78] [CHL91] for example). However these are isolated cases where the coefficients of the terms involved into the equations have certain ratios. The global problem is dealt with by a perturbation expansion of the IST. This approach was first introduced in the seventies [Kau76] [Kau90] [Kau91] [KM78a], [Kar79] [KM78b] and extensively reviewed in [KM89].

Let us start with a brief description of how the method works. First of all the presence of the perturbing term will modify the zero curvature representation to:

$$U_t - U_z + [U, V] = G = \epsilon R + i \zeta_t \sigma_3 \quad (4.47)$$

The subscripts t and z in the expression denote partial differentiation with respect to the variable while

$$R = \begin{pmatrix} 0 & R(q, \bar{q}) \\ -\bar{R}(q, \bar{q}) & 0 \end{pmatrix} \text{ and } \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

It is assumed that from the pair of equations (4.10), (4.11) the first one remains valid while the second no longer stands. This implies that we can still use the existing set of Jost functions to describe evolution in the scattering space, although this evolution will differ from the trivial NLS one. To formulate IST perturbation theory we need the evolution of the spectral parameters. This can be calculated through differentiation of (4.10) with respect to time, use of the condition (4.47) and solution of the resulting problem.

We will not go into the details of this particular calculations (they can be found in the literature [Lam80], [KM78a], [DP91]). Instead we will use the Hamiltonian formalism to express the particular problem. The purpose is twofold: First we demonstrate that the Hamiltonian formalism can help removing much of the bulk of calculations needed in IST related theories and secondly to create the background for the introduction of an “averaged Hamiltonian” perturbation method in the next chapter.

Our starting point is again equation (4.30). We will take the Hamiltonian of the unperturbed problem to be H_0 . Let us now assume that it is possible to write a Hamiltonian functional for the perturbed problem in the form:

$$H = H_0 + \Delta H$$

Equations (4.6) and (4.7) is now rewritten in the form:

$$\bar{q}_t = i \frac{\delta H_0}{\delta q} + i \frac{\delta \Delta H}{\delta q} \quad (4.48)$$

$$q_t = i \frac{\delta H_0}{\delta \bar{q}} + i \frac{\delta \Delta H}{\delta \bar{q}} \quad (4.49)$$

A direct comparison with (4.30) will give the following functional relations:

$$\frac{\delta\Delta H}{\delta q} = \epsilon \bar{R}, \quad \frac{\delta\Delta H}{\delta \bar{q}} = \epsilon R \quad (4.50)$$

As mentioned earlier it is possible to determine the temporal behaviour of observables by considering their Poisson bracket with the Hamiltonian (eq. 4.9). In order to fully describe the time propagation of the scattering data we need the evolution of the coefficients $a(\zeta)$ and $b(\zeta)$ connecting the Jost functions.

Let us now calculate the variation of $s(\zeta)$ where s can be either a or b :

$$\frac{d}{dt} s(\zeta) = \{H_0, s(\zeta)\} + \{\Delta H, s(\zeta)\} \quad (4.51)$$

The contribution of the perturbation is the second term of the right hand side. We now use the definition of the Poisson bracket (4.8) to get:

$$\frac{d}{dt} s(\zeta) = \{H_0, s(\zeta)\} + i \int_{-\infty}^{\infty} \left(\frac{\delta\Delta H}{\delta q(z)} \frac{\delta s(\zeta)}{\delta \bar{q}(z)} - \frac{\delta\Delta H}{\delta \bar{q}(z)} \frac{\delta s(\zeta)}{\delta q(z)} \right) dz \quad (4.52)$$

At this point the z coordinate is introduced explicitly in order to avoid confusion for the remaining analysis. To simplify this expression we have to make use of the set of equations (4.50). Furthermore we have to calculate the variational derivative of $s(\zeta)$. The latter is calculated by making use of formulas (4.16) and (4.17). Let us start with $a(\zeta)$:

$$\frac{\delta a(\zeta)}{\delta q(z)} = W \left(\frac{\delta \Phi(y; \zeta)}{\delta q(z)}, \Psi(y; \zeta) \right) + W \left(\Phi(y; \zeta), \frac{\delta \Psi(y; \zeta)}{\delta q(z)} \right) \quad (4.53)$$

In order to calculate the variational derivative of Φ and Ψ we will take the variational derivative of equation (4.10) with respect to $q(z)$. Thus:

$$\frac{d}{dy} \frac{\delta F(y)}{\delta q(z)} = U(y, t, \zeta) \frac{\delta F(y)}{\delta q(z)} + \frac{\delta U(y, t, \zeta)}{\delta q(z)} F(y) \quad (4.54)$$

with:

$$\frac{\delta U(y, t, \zeta)}{\delta q(z)} = \begin{pmatrix} 0 & i \delta(y - z) \\ 0 & 0 \end{pmatrix}$$

For the Jost function Φ which is defined through its asymptotic behaviour in the vicinity of negative infinity we expect to have one more condition for the variational derivative.

Namely:

$$\frac{\delta \Phi(y)}{\delta q(z)} = 0 \text{ for } y < z \quad (4.55)$$

Similarly for Ψ we have the condition:

$$\frac{\delta \Psi(y)}{\delta q(z)} = 0 \text{ for } z < y \quad (4.56)$$

The solution can be constructed by taking variation of constants. Its substitution into (4.53) will lead to:

$$\frac{\delta a(\zeta)}{\delta q(z)} = i \phi_2(z; \zeta) \psi_2(z; \zeta) \quad (4.57)$$

The subscripts correspond to the component of the Jost solution. A similar line of thought for the remaining variational derivatives will give:

$$\frac{\delta a(\zeta)}{\delta \bar{q}(z)} = i \phi_1(z; \zeta) \psi_1(z; \zeta) \quad (4.58)$$

$$\frac{\delta b(\zeta)}{\delta q(z)} = -i \phi_2(z; \zeta) \tilde{\psi}_2(z; \zeta) \quad (4.59)$$

$$\frac{\delta b(\zeta)}{\delta \bar{q}(z)} = i \phi_1(z; \zeta) \tilde{\psi}_1(z; \zeta) \quad (4.60)$$

These formulas are used in order to evaluate the Poisson brackets (4.52). The resulting time derivatives are exactly as in [KM78a]:

$$\frac{\partial}{\partial t} a(\zeta) = i \epsilon \left[a(\zeta) \alpha^+(\zeta, \zeta) + b(\zeta) \alpha(\zeta, \zeta) \right] \quad (4.61)$$

$$\frac{\partial}{\partial t} b(\zeta) = -2i \zeta^2 b(\zeta) + i \epsilon \left[a(\zeta) \bar{\alpha}(\zeta, \zeta) - b(\zeta) \alpha^+(\zeta, \zeta) \right] \quad (4.62)$$

where ζ is complex, the bar denotes complex conjugation and:

$$\alpha(\zeta, \zeta') = \left(\widetilde{\Psi(\zeta)}, R \Psi(\zeta') \right) \quad (4.63)$$

$$\alpha^+(\zeta, \zeta') = - \left(\bar{\Psi}(\zeta), \sigma_3 R \Psi(\zeta') \right) \quad (4.64)$$

and R the matrix introduced in equation (4.47). The analysis from this point onward is straightforward. The changes in the discrete spectrum are described by alterations of the discrete eigenvalues as well as the normalisation coefficients. To calculate the change in the eigenvalues the total differential of $a(\zeta)$ is considered around its zero. For the normalisation coefficient the difficulty arising from the appearance of $\widetilde{\Psi(\zeta)}$ is overcome by the consideration of the ζ derivatives instead [Kau76]. The variations of the discrete spectrum correspond to the adiabatic perturbation theory. For the continuous spectrum the eigenvalues are constant with time and the variation of the reflection coefficient is directly calculated from (4.61) and (4.62) [Lam80].

In order to reconstruct the solution equations (4.27) and (4.28) need to be expanded with the introduction of higher orders in ϵ for both scattering data and Jost functions [DP91].

Extension to higher orders in ϵ is achieved by considering the changes of the Jost functions as well [DP91].

It can be proved [NMPZ84], that the pairs of functions:

$$n(\lambda) = \frac{1}{\pi} \ln \left(\frac{1}{|a(\lambda)|^2} \right), \quad \phi(\lambda) = \arg(b(\lambda)) \quad (4.65)$$

for the continuous spectrum and

$$N_k = 2 \lambda_k, \quad \Phi_k = \ln(1/b_k), \quad k = 1, 2, \dots \quad (4.66)$$

for the discrete constitute canonical pairs of action-angle type variables. The Hamiltonian of the NLS when transformed in these variables is expressed as:

$$H_0 = \frac{i}{3} \sum_k (\bar{N}_k^3 - N_k^3) + 4 \int_{-\infty}^{\infty} \lambda^2 n(\lambda) d\lambda \quad (4.67)$$

Unfortunately perturbation terms will involve all coordinates, thus destroying the action angle character of the coordinate above. From this point the choice of the optimal coordinates in the scattering phase space depends solely on the functional form of the perturbation.

The use of the Hamiltonian formalism in the expression of an IST perturbation theory does not add any new element. It is not expected to do so. The only benefit that it provides is more transparency into the calculations related to the temporal variation of the spectral parameters. The use of Hamiltonians would seem to be reducing the spectrum of perturbations envisaged to purely Hamiltonian ones. At first order, however we do not envisage alterations from the analysis which has just preceded. This is because equations (4.50) are valid to first order, regardless of the character of the perturbation.

The use of the Riemann formalism for the inverse problem ([DP91]) instead of the traditional Gelfand-Levitan-Marchenko ([KM78a]) approach provides more simplification since integral equations are substituted by purely algebraic ones.

4.3.4 Alternative approaches

In addition to the well established analyses above, there are several other approaches. Most prominent among them is the Lagrangian method, which is going to be analysed extensively in the next chapter. Perturbation techniques can also be based on the Hamiltonian, in the time-space domain [BA94] [Bla91], at least to the adiabatic level.

A different approach, applicable to conservative systems, is to try and transform the perturbed equation into a solvable one [HK95] [Kod85a]. The main point is that if we have an equation perturbed from an integrable one to order $O(\epsilon)$, which can be expressed in the Hamiltonian formalism, it is possible to find a transform that connects

the solution of the equation in question to an integrable one. The idea was suggested by Kodama. The details can be found elsewhere [Kod85a]. Here we will only give the results of the application of the theory to the perturbed equation (3.54) written here:

$$i q_t + \frac{1}{2} q_{zz} + |q|^2 q = i \epsilon \left(\beta_1 q_{zzz} + \beta_2 (|q|^2 q)_z + \beta_3 q (|q|^2)_z \right) \quad (4.68)$$

If q is a solution then it is possible to define a function ϕ_1 with the following property: When the function $v(t, z)$ is a solution of the equation

$$i v_t + \frac{1}{2} v_{zz} + |v|^2 v = \beta_1 (v_{zzz} + 6|v|^2 v_z), \quad (4.69)$$

then the function

$$q = \exp(\Phi \cdot \nabla) v = v + \epsilon \phi_1[v, \bar{v}] + O(\epsilon^2) \quad (4.70)$$

with

$$\phi_1[v, \bar{v}] = i \left(3\beta_1 - \frac{1}{2}\beta_2 \right) v_z + i (6\beta_1 - 2\beta_2 - \beta_3) v \int_{-\infty}^z |v(z')|^2 dz'$$

is a solution of (4.68). The equation (4.69) is integrable with the means of the IST (Hirota equation [Hir73], [MT⁺93])

Regarding the radiation emitted due to possible resonances further to the IST and the approach due to Gordon, both explained earlier, it is possible to use linearisation in the frequency domain [KZ90] [Kar93]. An alternative technique, valid for the fundamental soliton at least, was developed by Karpman [Kar93] [Kar94b]. In this approach the space variable is extended to the complex domain. The solution of the fundamental unperturbed equation is assumed to be defined over the complex plane. The remaining steps are:

- The linearised perturbed equation is taken and solved with a WKB approximation, leading to the first order radiative solution.
- The fundamental solution is rewritten in a fashion that resembles the WKB solutions functionally. It is then possible to locate a quantity of the soliton solution that is analogous to the wavelength of the radiation. The set of the points at which these two are equal is identified.
- The full solution (fundamental and first order) is integrated over a contour that passes above the point of the set just defined that is nearest to the real axis. By making use of the transformation of the fundamental solution into radiation it is possible to obtain expressions for the radiation similar, up to a constant, with the corresponding quasi-numerical solutions of the problem in hand [KZ90] [WMCL86].

4.4 Considerations

Having introduced some of the perturbation schemes available we should now establish some connection of the mathematical notation to the physical reality. In the previous chapter (3.6) a parameter ϵ was introduced as the ratio of the propagating pulse bandwidth to the central frequency. Subsequently the set of equations describing pulse propagation was derived as a power series in ϵ . The n th order equation in that power series would be valid up to distances $O(\lambda_0 \epsilon^{-n}) = O(\epsilon^{-n})$. In real applications the ϵ parameter is approximately 10^{-4} [HW96] [HW96], giving a region of the order of thousands of kilometers where the NLS is valid. However, there is a tendency for the narrowing of the used pulses. A reduction by one order in epsilon (10^{-4}) will lead to the reduction of the region of validity of the NLS to kilometers. It is for such cases that the use of perturbed NLS equations is needed.

By far the most popular procedure is the adiabatic perturbation. It has two main advantages over the other procedures:

First it is easy to implement. The equations describing adiabatic propagation can be derived by the consideration of the Integrals of motion, or by the Hamiltonian picture or from the procedures outlined in section 4.3. No knowledge of IST is required and they can be applied in almost any situation.

Secondly, when the correct choice of time varying parameters is made, it provides a clear, identifiable picture of the motion, since as was explained earlier the individual solitons are treated as particles with a position and a momentum to characterise them.

However, one should always keep in mind that the adiabatic theory is only a first approach, valid for small time intervals and most important, neglecting the effects of radiation due to the perturbing term. It is thus important that adiabatic theory is used with caution, in this respect.

Regarding higher order perturbation methods, the direct method has the advantage of not introducing complex mathematical problems into the dynamics of the perturbed equation like the IST does. Secondly it is better than IST in describing localised solution deformations in that the amount of calculations is small and relatively easy to perform. By contrast the solution of the same problem with IST perturbation theory involves lengthy integrations even for functionally simple perturbations. On the other hand IST gives a full picture of radiation unlike direct methods. To describe radiation with direct methods, specific considerations have to be introduced for each spectral area where dispersive radiation can arise. The subsequent calculations are all but simple [KZ90] [WMCL86]. Furthermore the application of the direct method is

obscure when the initial conditions do not correspond to a unique asymptotic solution. For example the treatment of the NLS with initial condition $A \operatorname{sech}(z)$, $A \neq 1$ is relatively easy with IST since we know the parameters of the asymptotic solution while being highly problematic for the any direct method unless some way is devised to predict the asymptotic solution [KS95].

The only reservations regarding IST are the following: IST perturbation “attaches” a perturbed problem to a specific equation, the NLS. However there is no guarantee that the perturbed equation has the same dynamics as the NLS. The latter is associated with a compatibility problem for two-dimensional functions. It is possible to define compatibility problems in more dimensions and generate higher order integrable equations. An IST perturbation scheme can be defined for each of these equations. It is not possible to determine beforehand which of them has dynamics most closely related to the equation examined so as to use the corresponding perturbation scheme.

Another point that should be made is that the IST perturbation procedure introduced earlier does not predict the generation of new discrete eigenvalues or the destruction of existing ones. We cannot therefore predict whether a new soliton is formed (or one is destroyed).

We have seen, in this chapter, a brief description of some of the analytical tools in the description of pulse propagation. We have seen an exact method and some approximate ones, their advantages and disadvantages. We left out an alternative method based on the Lagrangian of the equation to be solved. This method will be covered in detail in the next chapter.

5. THE LAGRANGIAN PERTURBATION METHOD

5.1 Introduction

The Lagrangian method was introduced [BLA79] [AL83] to provide a simple alternative in the determination of the equations for the soliton parameters in the adiabatic approximation. The starting point, of the initial version, is the Lagrangian density (4.3) associated with the NLS. For the perturbed problem one has to produce a suitable Lagrangian density [AL86] [H⁺93] [Des94] or transform the existing one in order to include the perturbation in the constants associated with the already existing terms [And83]. Let us assume that the differential equation to be solved has the form:

$$\mathcal{M}(q, \bar{q}) = 0 \quad (5.1)$$

where \mathcal{M} is one of the nonlinear differential operators appearing in chapter 3. The nonlinear differential operator \mathcal{M} is acting on a function q and its complex conjugate. The function q depends on time (propagation variable) and space. Following the scheme of section 2.3, the PDE (5.1) is expressed as a Lagrangian optimisation problem:

$$S = \int_0^t dt' \int_{b_-}^{b_+} dz \mathcal{L}(q, \bar{q}) \quad (5.2)$$

For the rapidly decreasing case, that we have been dealing with, the points b_- , b_+ correspond to $-\infty$ and ∞ respectively. Nevertheless we leave them here undefined for reasons that will be explained later. In the case of the NLS the Lagrangian density \mathcal{L} is given by equation (4.3).

The next step, in this analysis, is to select a trial function (ansatz) for the solution of the problem and substitute it in the Lagrangian density. This trial function will have a predetermined z dependence. The only flexibility that we allow is through some, suitably chosen parameters A_i , $i = 1 \dots N$ which are assumed to be functions of time. The determination of the temporal evolution of these parameters is the final target. This step is the most crucial in the whole method. The choice of a specific profile makes it impossible to predict changes in the pulse shape (or the phase dependence on z).

From this point onwards the analysis is very similar to the traditional Rayleigh-Ritz method outlined in section 2.3.2. Namely after substitution of the trial function in the Lagrangian a reduced problem is defined:

$$\int_0^t dt' \mathbf{L}(A_i(t'), t') \quad (5.3)$$

where the function \mathbf{L} comes about from integration throughout space of the original Lagrangian:

$$\mathbf{L}(A_i(t), t) = \int_{b_-}^{b_+} dz L(q(z; A_i(t)), \bar{u}(z; A_i(t))) \quad (5.4)$$

The function $\mathbf{L}(A_i(t'), t')$ is a reduced Lagrangian depending on time only. The final step is to find the Euler-Lagrange equations corresponding to problem (5.3). This will lead to a set of quasi-linear ordinary differential equations, whose solution describes adiabatic propagation. By the term quasi-linear it is meant that the differential equation will be linear with respect to the derivatives. However the coefficients of the equations need not be linear in the unknown functions.

The most important shortcoming of the method is, as described earlier, the commitment to a specific shape for the solution. It is exactly for this reason that the method should be applied with extreme care as pointed out in [AAPC93] [KMS95]. This is particularly true when the perturbation theory is only applied up to the adiabatic level as will be seen from the example that follows.

We will attempt to consider, using the variational method, the NLS initial value problem where the initial pulse profile is of the form $A \operatorname{sech}(x)$ with $A < 1.5$. We will take a trial function [ALR88a]:

$$q = a(t) \operatorname{sech}\left(\frac{z}{w(t)}\right) e^{i z^2 b(t) + i \theta(t)} \quad (5.5)$$

which is substituted into the Lagrangian (4.3). After carrying out the integration (5.4) we will have the optimisation problem (5.3) with \mathbf{L} given by:

$$\mathbf{L} = \frac{-a(t)^2}{3 w(t)} + \frac{2 a(t)^4 w(t)}{3} - \frac{\pi^2 a(t)^2 b(t)^2 w(t)^3}{3} - \frac{\pi^2 a(t)^2 w(t)^3 b'(t)}{6} - 2 a(t)^2 w(t) \theta'(t) \quad (5.6)$$

Subsequently, the Euler-Lagrange equations of this Lagrangian are calculated. If we define the conserved quantity “mass” (see section 4.2.2) to be equal to $2M$ then we can prove that the Euler equations can be reduced to a Newton equation for the motion of a particle with coordinate $y(t)$ in a potential well $U(y(t))$:

$$\frac{d^2}{dt^2} y(t) + \frac{d}{dy} U = 0 \quad (5.7)$$

$$\text{with } U = \frac{2(1 - 2M^2 y(t))}{M^4 \pi^2 y(t)^2} \text{ and } y(t) = a(t)^{-2}$$

For initial values at the bottom of the potential well the one soliton solution is recovered. However if the initial point is different from the potential minimum oscillation of the pulse parameters will occur. The circular frequency of this oscillation, for small deviations from the potential minimum is approximately $\omega_0 = 2M/\pi$. By taking the exact theoretical predictions [SY74] it can be found that both frequency and mean values of the oscillation are wrong [KS95] [KMS95]. Furthermore the chirp parameter b will take positive values in one half period and negative for the other, indicating alternation of the direction of motion of the dispersive radiation.

To understand the reason for this collapse we have to look at the physical picture. The initial condition in hand corresponds to one soliton and an amount of background radiation. From IST theory these two are decoupled and propagate independently. The trial function is a single pulse, localised in space. We thus expect a good agreement between the exact picture and the Lagrangian approximation method when some degrees of freedom are introduced to account for the non localised part of the initial radiation. However the emerging adiabatic solution is built on the implicit assumption that all initial radiation is reshaped and transformed into a propagating soliton. This type of solution, cannot exist as we can easily see (by direct substitution).

5.2 Euler-Lagrange Equations

Having stressed the importance of the correct selection of an ansatz we return to the set of equations (5.3) and (5.4). There are two main difficulties in reaching the form (5.3). The first one is to formulate the Lagrangian function that corresponds to equation (5.1). The second problem is of computational nature: It is often not possible to derive exact analytical expressions for the integral (5.4).

The purpose of this section is to tackle the first problem. Namely we will see that, in the rapidly decreasing case, the formulation of a Lagrangian function is not a prerequisite for the derivation of the Euler-Lagrange equations. To achieve this we need formulas for the variational derivatives of $L(A_i(t), t)$ in (5.3) with respect to the functions A_i , $i = 1 \dots N$.

Before doing this let us introduce some notation. According to the analysis in section 2.3, equation (5.1) can be re-expressed as a variational derivative of the Lagrangian L with respect to the function \bar{q} :

$$\frac{\delta L}{\delta \bar{q}} = \sum_{i=0} \sum_{j=0} (-1)^{i+j} \frac{d^{i+j}}{dz^i dt^j} \frac{\partial L}{\partial \bar{q}_{(iz,jt)}} = 0 \quad (5.8)$$

where

$$\bar{q}_{(nz,mt)} = \frac{\partial^{n+m}}{\partial z^n \partial t^m} \bar{q}$$

This variational derivative involves derivation in both time and space. Since our problem involves functions A_i that depend on time solely, we will need a second variational derivative operator, defined in the z direction:

$$\frac{\delta^{(z)}}{\delta f} = \sum_{j=0} (-1)^j \frac{d^j}{dz^j} \frac{\partial}{\partial f_{jz}} \quad (5.9)$$

where f an arbitrary function and the subscript jz denotes derivation with respect to z , j times. Making use of this definition it is possible to write (5.8) in the form:

$$\sum_{j=0} (-1)^j \frac{d^j}{dt^j} \left(\frac{\delta L^{(z)}}{\delta \bar{q}_{jt}} \right) = 0 \quad (5.10)$$

The functions q and \bar{q} depend, in their turn, on the functions A_i . The differentiation of functionals of the former two functions with respect to A_i will involve application of the chain rule. In order to make the formulas used more compact we introduce a vector \hat{q} defined as:

$$\hat{q} = (q, q_z, q_{zz}, \dots, \bar{q}, \bar{q}_z, \bar{q}_{zz}, \dots) \quad (5.11)$$

as well as a corresponding vector derivative:

$$\nabla_q = \left(\frac{\partial}{\partial q}, \frac{\partial}{\partial q_z}, \frac{\partial}{\partial q_{zz}}, \dots, \frac{\partial}{\partial \bar{q}}, \frac{\partial}{\partial \bar{q}_z}, \frac{\partial}{\partial \bar{q}_{zz}}, \dots \right) \quad (5.12)$$

To complete the algebra needed, we will introduce a dot product between vectors in the usual form:

$$(a_1, a_2, \dots) \cdot (b_1, b_2, \dots) = \sum_{i=1} a_i b_i \quad (5.13)$$

In order to calculate the Euler-Lagrange equations we need the variational derivative of \mathbf{L} in (5.3) with respect to the function A_j :

$$\frac{\delta \mathbf{L}}{\delta A_j} = \sum_{\kappa=0}^{\kappa_m} (-1)^\kappa \frac{d^\kappa}{dt^\kappa} \mathbf{L}_{(A_j)_{\kappa t}}$$

where the subscript κt denotes differentiation κ times with respect to time. The index κ takes all values between 0 and the maximum time derivative order in \mathbf{L} , designated by κ_m . Next, use is made of the definition of \mathbf{L} in (5.4). After switching the order between summation and integration we have:

$$\frac{\delta \mathbf{L}}{\delta A_j} = \int_{b_-}^{b_+} dz \sum_{\kappa=0}^{\kappa_m} (-1)^\kappa \frac{d^\kappa}{dt^\kappa} L_{(A_j)_{\kappa t}}$$

But we already know that the Lagrangian L depends on $(A_j)_{\kappa t}$ through the function q and its complex conjugate. By making use of definitions (5.11), (5.12) and (5.13) the relation above is rewritten as:

$$\frac{\delta \mathbf{L}}{\delta A_j} = \int_{b_-}^{b_+} dz \sum_{\kappa=0}^{\kappa_m} \sum_{\lambda=0}^{\kappa_m} \frac{d^\kappa}{dt^\kappa} (-1)^\kappa \left[(\hat{q}_{\lambda t})_{(A_j)_{\kappa t}} \right] \cdot \nabla_{q_{\lambda t}} L \quad (5.14)$$

The limits of the λ summation should in fact be reduced to $\kappa \leq \lambda \leq \kappa_m$, since for $\lambda < \kappa$, the terms in the λ series are zero. However this choice of limits will not have any effect as these terms will be eliminated by the identity:

$$\left[(\hat{q}_{\lambda t})_{(A_j)_{\kappa t}} \right] = \binom{\lambda}{\kappa} \left[(\hat{q}_{(\lambda-\kappa)t})_{(A_j)} \right] \quad (5.15)$$

proved in the appendix at the end of this chapter. This identity and the product rule formula for the k th order derivative:

$$\frac{d^k}{dt^k} f(t) g(t) = \sum_{j=0}^k \binom{k}{j} f_j t g_{(k-j)t} \quad (5.16)$$

will be substituted into (5.14) in order to get:

$$\frac{\delta \mathbf{L}}{\delta A_j} = \int_{b_-}^{b_+} dz \sum_{\kappa=0}^{\kappa_m} \sum_{\lambda=0}^{\kappa_m} \sum_{\mu=0}^{\kappa} (-1)^\kappa \binom{\lambda}{\kappa} \binom{\kappa}{\mu} \left[(\hat{q}_{(\lambda-\mu)t})_{(A_j)} \right] \cdot \nabla_{q_{\lambda t}} L_{\mu t} \quad (5.17)$$

In order to simplify this equation we must use the identity:

$$\sum_{\mu=\nu}^{\kappa} (-1)^\mu \binom{\kappa}{\mu} \binom{\mu}{\nu} = (-1)^\kappa \delta_{\kappa\nu} \quad \text{with } \nu \leq \kappa \quad (5.18)$$

also proven in the appendix. The symbol $\delta_{\kappa\nu}$ is the Kronecker delta. After substitution of (5.18) in (5.17) and the interchange between integration and summation we will have:

$$\frac{\delta \mathbf{L}}{\delta A_j} = \sum_{\lambda=0}^{\kappa_m} (-1)^\lambda \int_{b_-}^{b_+} dz \left(\hat{q}_{(A_j)} \right) \cdot \nabla_{q_{\lambda t}} L_{\lambda t} \quad (5.19)$$

The final step will come from the explicit substitution of the components of the dot product in the integrand of (5.19) and the use of integration by parts on the resulting formula. By taking these steps and performing some straightforward algebra we reach the following result:

$$\begin{aligned} \frac{\delta \mathbf{L}}{\delta A_j} = & \sum_{\lambda=0}^{\kappa_m} (-1)^\lambda \sum_{i=1}^{i_m} \left[(q_{A_j})_{(i-1)z} \frac{\delta L_{\lambda t}}{\delta q_{(iz, \lambda t)}} + (\bar{q}_{A_j})_{(i-1)z} \frac{\delta L_{\lambda t}}{\delta \bar{q}_{(iz, \lambda t)}} \right]_{b_-}^{b_+} + \\ & \sum_{\lambda=0}^{\kappa_m} (-1)^\lambda \int_{b_-}^{b_+} dz \left(q_{A_j} \frac{\delta L_{\lambda t}^{(z)}}{\delta q_{\lambda t}} + \bar{q}_{A_j} \frac{\delta L_{\lambda t}^{(z)}}{\delta \bar{q}_{\lambda t}} \right) \end{aligned} \quad (5.20)$$

where i_m is the order of highest z derivative in the Lagrangian function L .

For the rapidly decreasing case the limits b_- and b_+ tend to negative and positive infinity, respectively. Furthermore the solution, at these points, vanishes together with all its z derivatives. we thus expect the first term on the right hand side of (5.20) to vanish. To simplify things further equation (5.10) and its complex conjugate are used in (5.20) to give:

$$\int_{-\infty}^{\infty} [\mathcal{M}(q, \bar{q}) \bar{q}_{A,j} + \overline{\mathcal{M}}(\bar{q}, q) q_{A,j}] dz = 0 \quad (5.21)$$

Equation (5.21) completes the proof of the statement at the beginning of this section. Before using it in the perturbation theory of (5.1) a couple of comments must be made.

In the “orthodox” Rayleigh-Ritz method, briefly introduced in section 2.3.2, the idea was to introduce a sufficient number of functions - solutions to the problem - whose superposition would lead to a sufficiently accurate description of the exact solution. This, of course, cannot be achieved exactly in nonlinear problems as we can no longer rely on the superposition principle. However equation (5.21) indicates that we are assuming that our solution lives in a manifold with coordinates the functions $q_{A,j}$. In order to define it we fix the free varying parameters in a way that makes the result of the substitution of the trial function into equation (5.1) orthogonal to this space. We can now see why the choice of the parameters is crucial to the whole approximation process: missing out degrees of freedom that are important means not only that the solution is inaccurate, at best, but we do not have any means of quantifying the size of the error we are making. A possible solution to this problem will come by introducing progressively more degrees of freedom as we will attempt to do in the next section.

Secondly we have seen that no assumptions have been made as to the form of the Lagrangian. Thus, the need for the explicit formulation of the Lagrangian density function is substituted by an existence condition. The usefulness of this lies in problems where the definition of a Lagrangian is not as straightforward. As such we can classify, first of all, nonconservative problems, where alternative methods have to be used ([And83]). In cases like this instead of going through the Lagrangian approach, as described in the previous section it is possible to adopt the direct version expressed by (5.21).

5.3 Direct perturbation expansion

The appearance of equation (5.1) and its complex conjugate, in (5.21) suggests that we can implement a Rayleigh-Ritz version the use of of the direct perturbation method. The standard solution method is to solve the zero order problem, then substitute the solutions to the first order problem and determine the corrections and work recursively

for higher orders. Instead of doing this, the set of functions is predetermined and substituted into the equations. The task we have to fulfill is to find the conditions under which this set of functions will be a solution to the system under consideration.

Let us see how this statement applies. We will first assume that the left hand side of equation (5.1) corresponds to the superposition of an operator that we can solve exactly and a small perturbation characterised by a parameter ϵ :

$$\mathcal{M}_{(0)}(q, \bar{q}) + \epsilon \mathcal{R}(q, \bar{q}) \quad (5.22)$$

The perturbation method will be applied in the usual fashion:

First we introduce a trial function which, we assume, can be expanded in powers of ϵ :

$$q = q_0(z, t) + \epsilon q_1(z, t) + \epsilon^2 q_2(z, t) + \dots \quad (5.23)$$

The functions q_i , $i = 1, 2, \dots$ are trial functions that depend on time through sets of time dependent functions $A_{\nu_i}^{(i)}$ with $i = 1, 2, \dots$ and $\nu_i = 1, 2, \dots, N_i$ and N_i is the number of new functions of this type for each q_i . Since the trial functions at a certain order are determined from the ones at the previous order it is assumed that the q_1 will depend on the set of $A^{(1)}$ as well as from the set made of $A^{(0)}$. Similar considerations hold for higher orders. Substitution of the trial functions into the original operator will lead to an expansion of the type:

$$\begin{aligned} &\mathcal{M}_{(0)}(q_0, \bar{q}_0) + \epsilon \left(\hat{q}_1 \cdot \nabla_{q_0} \mathcal{M}_{(0)} + ((\hat{q}_1)_{t_0} + (\hat{q}_0)_{t_1}) \cdot \nabla_{q_0 t_0} \mathcal{M}_{(0)} + \right. \\ &\left. ((\hat{q}_1)_{t_0 t_0} + 2(\hat{q}_0)_{t_1 t_1}) \cdot \nabla_{q_0 t_0 t_0} \mathcal{M}_{(0)} + \dots + \mathcal{R}(q_0, \bar{q}_0) \right) + O(\epsilon^2) \end{aligned} \quad (5.24)$$

The dot product and the vectors associated with it are defined in equations (5.11), (5.12) and (5.13). For reasons of simplicity we will avoid doing the full scale expansion but will limit the study in the first order.

The second step in the solution is to introduce multiple scaling in the time domain. The slow scales introduced are designated by $t_i = \epsilon^i t$ and their introduction alters the form of temporal derivatives as described by equation (2.12). The time dependent functions $A_j^{(m)}$ are now assumed to depend on multiple variables t_i :

$$A_j^{(m)} = A_j^{(m)}(t_0, t_1, \dots)$$

A further degree of freedom is introduced by expanding the functions $A_j^{(m)}$ in multiple orders:

$$A_j^{(m)} = A_j^{(m,0)} + \epsilon A_j^{(m,1)} + \epsilon^2 A_j^{(m,2)} + \dots$$

In order to proceed further we need to incorporate these two assumptions into the operator (5.24). The only alterations will occur in the time derivative terms.

There we must take under consideration the dependence of the trial function q on time through the trial functions $A_j^{(m)}$. Thus the derivative operator with respect to t_n will be rewritten as:

$$\frac{\partial}{\partial t_n} = \sum_{k=0}^m \sum_{j=1}^N \sum_{l=0} \epsilon^l \frac{\partial A_j^{(k,l)}}{\partial t_n} \frac{\partial}{\partial A_j^{(k,0)}} \quad (5.25)$$

with $n = 0, 1, \dots$ and m defined by the function q_m on which the differentiation operator is acting. Similar expressions can be derived for the higher order derivatives although they are not useful in the NLS case. In order to derive the differential equations describing the temporal evolution we need to use (5.21) together with the equations derived above. The term $(q_m)A_j^{(m)}$ appearing in (5.21) should be expanded as:

$$(q_0)_{A_j^{(0,0)}} + \epsilon \left[(q_0)_{A_j^{(0,1)}} + (q_1)_{A_j^{(1,0)}} \right] + O(\epsilon^2)$$

The appearance of the first term in the square bracket stems from the application of the variational derivation on the action. The minimisation condition appears as the product of the Euler Equation with the variation of the function with respect to which the variational derivative is taken. This is introduced here by the presence of the first term of the square bracket in that order instead of the fundamental.

After straightforward substitution into (5.21) the fundamental order Euler equation for $A_j^{(0)}$ will be:

$$\int_{-\infty}^{\infty} \left[\left(\mathcal{M}_{(0)}(q_0, \bar{q}_0) \right) (q_0)_{A_j^{(0,0)}} + \left(\overline{\mathcal{M}}_{(0)}(\bar{q}_0, q_0) \right) (\bar{q}_0)_{A_j^{(0,0)}} \right] dz = 0 \quad (5.26)$$

After considering all possible j 's a system of N_0 ordinary differential equations will be formed. Its solution will determine the temporal behaviour of the trial function q_0 . What should be pointed out at this point is that the solution of the system (5.26) will not necessarily make the trial function q_0 a solution of the unperturbed equation corresponding to (5.22). In the next order we have the system of equations:

$$\begin{aligned} & \int_{-\infty}^{\infty} \left[(\hat{q}_1) \cdot \nabla_{q_0} \mathcal{M}_{(0)} + (\hat{q}_1)_{t_0} \cdot \nabla_{(q_0)t_0} \mathcal{M}_{(0)} + \dots \right] (q_0)_{A_j^{(s,1)}} dz + \\ & \int_{-\infty}^{\infty} \left[(\mathcal{R}(q_0, \bar{q}_0)) + (\hat{q}_0)_{t_1} \cdot \nabla_{(q_0)t_0} \mathcal{M}_{(0)} + \dots \right] (q_0)_{A_j^{(s,1)}} dz + \\ & \int_{-\infty}^{\infty} \left[\mathcal{M}_{(0)}(q_0, \bar{q}_0) \right] \left[(q_1)_{A_j^{(s,1)}} + (q_0)_{A_j^{(s,1)}} \right] dz + c.c. = 0 \end{aligned} \quad (5.27)$$

where *c.c.* stands for complex conjugate and s takes the values 0 and 1. The separation was made here in such a way so that the first term includes the linearisation of the operator \mathcal{M} and the second includes the remaining perturbation effects. The third term comprises driving effects arising from the fact that the ansatz that we have considered may not be a solution of the unperturbed system in hand. In the system of equations (5.27) two types of A_j 's appear:

1. First those that are introduced for the first time ($s = 1$). To derive the Euler equations corresponding to these functions only the third integral and its complex conjugate needs to be used since (q_0) is independent of $A_j^{(1,k)}$. The Euler equations will be conditions on the functions derived in the previous order. They are trivially valid only if the trial function q_0 is a solution to the unperturbed problem corresponding to (5.22).
2. The remaining A_j 's ($s = 0$) have been introduced in the previous order. Their Euler equations will involve all terms above except possibly the last integral. Their behaviour at time scale t_0 is known from the previous order. The task here is to determine their behaviour in time scale t_1 as well as imposing compatibility conditions on the A_j 's of the previous category.

The Euler-Lagrange equations associated with variational derivation of the $s = 1$ functions are:

$$\int_{-\infty}^{\infty} \left[\mathcal{M}_{(0)}(q_0, \bar{q}_0) (q_1)_{A_j^{(1,0)}} + \overline{\mathcal{M}}_{(0)}(\bar{q}_0, q_0) (\bar{q}_1)_{A_j^{(1,0)}} \right] dz = 0 \quad (5.28)$$

These are just N_1 algebraic equations imposed on the functions $A_j^{(0,0)}$ and $A_j^{(1,0)}$.

The equations arising from variation of the $s = 0$ functions will comprise secular terms. These, basically are all terms that are completely independent of the parameters $A_j^{(1,0)}$. Such terms come about from:

$$\int_{-\infty}^{\infty} \left[\left[(\mathcal{R}(q_0, \bar{q}_0)) + (\hat{q}_0)_{t_1} \cdot \nabla_{(q_0)t_0} \mathcal{M}_{(0)} + \dots \right] (q_0)_{A_j^{(0,0)}} + \mathcal{M}_{(0)}(q_0, \bar{q}_0) (q_0)_{A_j^{(0,1)}} \right] dz \quad (5.29)$$

and its complex conjugate. In order to ensure that no secularities will occur they have to be equated to zero independently. This gives us a set of N_0 secularity conditions. We can easily see that these equations determine the behaviour of the constants appearing in the fundamental solution in time scales t_1 . In fact these equations together with the fundamental order ones form the adiabatic approximation.

Bearing in mind (5.29) and (5.27) we are able to define a third system of N_0 differential equations which will yield the behaviour of the $A_j^{(1,0)}$ parameters in time scales t_0 :

$$\begin{aligned} & \int_{-\infty}^{\infty} \left[(\hat{q}_1) \cdot \nabla_{q_0} \mathcal{M}_{(0)} + (\hat{q}_1)_{t_0} \cdot \nabla_{(q_0)t_0} \mathcal{M}_{(0)} + \dots \right] (q_0)_{A_j^{(0,0)}} dz + \\ & \int_{-\infty}^{\infty} \left[\mathcal{M}_{(0)}(q_0, \bar{q}_0) \right] (q_1)_{A_j^{(0,0)}} dz + c.c. = 0 \end{aligned} \quad (5.30)$$

This step completes the analysis in the first order. If the trial functions q_0 and q_1 are selected to be exact solutions to the fundamental and linearised equation respectively

this last system of equations is true trivially. Extension to higher orders follows along the same principles.

The last question that has remained unanswered is that of the trial functions. As in the version described at the beginning of this chapter there are no universal guidelines. However by examining equation (5.28) we notice that unless the first order trial function is a superposition of the derivatives of q_0 with respect to the $A_j^{(0,0)}$ functions new orthogonality conditions will be introduced. This will lead to the creation of an overdetermined system of equations relating different $A_j^{(0,0)}$ and thus convergence to a solution at the fundamental order.

Secondly the equations at higher orders (5.30) are linearised. This, of course means that we can adopt a trial function in the form of a superposition of solutions of the linearised equations. But even if we do not so we will be limited by the practical need to be able to perform the integrations analytically. Classes of functions that allow such manipulation are usually derivatives of the fundamental solution and products of the function with powers of z (The reader should bear in mind that these approaches describe perturbations of the fundamental single soliton).

To conclude this section, what we demonstrated here is not some new perturbation scheme. This approach is well established and relatively old [OP71] [GO81]. What we saw is that within the framework of the Lagrangian approach to perturbation theory we were able to reconstruct the direct perturbation method. The benefits of starting this are two:

First the secularity conditions arise naturally, once we identify which functions are involved where. This is not true for the conventional direct method, since the identification of the secularity condition comes from the solution of the linearised problem.

Secondly, we do not need to perform a perturbation expansion around the zero order solution. Condition (5.28) together with a selection of a first order approximation in a functional form linearly independent from the first order derivatives of the fundamental function with respect to $A_j^{(0,0)}$ will force a better approximation at the fundamental order. This approach does not constitute a solution of the propagation problem when initial conditions do not correspond to an exact solution. This is obvious since we cannot, without recurring to IST, find the correct initial conditions for radiation and solitary pulse and we have made no provisions for the former since the functions are implicitly considered as localised.

Finally, all the shortcomings of the direct expansion method are present here as well. This is expected from the fact that we merely restated the direct perturbation method in a fashion that makes computations more straightforward

5.4 Example: The damped NLS

In order to illustrate the use of the previous approach we will attempt to apply the principles discussed in the two earlier sections with an example; that of the damped NLS equation:

$$i \frac{\partial}{\partial t} q + \frac{1}{2} \frac{\partial^2}{\partial x^2} q + |q|^2 q + i \gamma q = 0 \quad (5.31)$$

This equation cannot be directly treated with the Lagrangian method since we cannot find any Lagrangian whose minimisation will lead to equation (eq. 5.31). It is possible to treat the problem indirectly by transforming the solution q so that damping becomes incorporated to the coefficient of the nonlinear term of the NLS ([And83]). Let us now try to approach this problem from the viewpoint exposed in the previous two sections. We are looking for the adiabatic changes of the soliton parameters as well as the stationary changes in the shape of the fundamental order solution.

Let us assume that the fundamental order solution q_0 is:

$$q_0 = 2n(t) \operatorname{sech}(2n(t)(x - \xi(t))) e^{i2m(t)(x - \xi(t)) + i\theta(t)} \quad (5.32)$$

and leave q_1, q_2 undefined for the time being.

If we assume that the damping term is a perturbation quantified by the small parameter γ then we have to adopt a power series expansion in γ for all of the parameters of the problem. To apply our method let us introduce multiple scales: t_0, t_1, t_2, \dots where $t_1 = \gamma t_0, \dots$ and expand the governing equation (eq. 5.31) in multiple orders:

$$\begin{aligned} \mathcal{M} = \mathcal{M}_0 + \gamma \mathcal{M}_1 + \dots = \\ i \frac{\partial}{\partial t_0} q_0 + \frac{1}{2} \frac{\partial^2}{\partial x^2} q_0 + |q_0|^2 q_0 + \\ \gamma \left[\left(i \frac{\partial}{\partial t_1} q_0 + i q_0 \right) + \left(i \frac{\partial}{\partial t_0} q_1 + \frac{1}{2} \frac{\partial^2}{\partial x^2} q_1 + 2|q_0|^2 q_1 + q_0^2 \bar{q}_1 \right) \right] + O(\gamma^2) = 0 \end{aligned} \quad (5.33)$$

We will restrict the analysis to the first order correction. For q_1 we assume the form $q_1 = f(t, x) q_0$ with $f(t, x)$ needing to be determined. We will address this problem immediately after solving the zero order equations, which correspond to the NLS equation as can be easily seen from (eq. 5.33). The solution is easily found by using (eq. 5.26) or even the straightforward Lagrangian method. The solution is:

$$\begin{aligned} \frac{d}{dt_0} n &= 0 \\ \frac{d}{dt_0} m &= 0 \end{aligned}$$

$$\begin{aligned}\frac{d}{dt_0} \xi &= 2m \\ \frac{d}{dt_0} \theta &= 2(m^2 + n^2)\end{aligned}\tag{5.34}$$

These relations guarantee that the zero order part of (5.33) is identically zero. Now, since we are looking for changes of shape of the fundamental solution. The choice of q_1 made in the previous paragraph makes this clear. Thus $f(x, t)$ is an envelope modifying function due to the perturbation. It should thus have the same properties as its “generator”, i.e imaginary and symmetric (since iq_0 has an envelope with these properties). From the discussion in the previous section we look for forms for q_1 by differentiating q_0 with respect to n, m, ξ, θ . From the functions arising we deduce that a possible selection could be: $f(t, x) = i [a(t) + 2n(t)b(t)[x - \xi(t)] \tanh(2n(t)(x - \xi(t)))]$ Instead of this, however we will consider a slightly different envelope function:

$$f(t, x) = i [a(t) + 4n(t)^2 b(t)[x - \xi(t)]^2]$$

The reason for this selection is that we wish to compare the results of the application of our formulas to the already existing analytic approach due to Karpman and Maslov ([KM78a]).

Having introduced q_1 we will move to the equations that would have been derived from the first order (in γ) Lagrangian. As explained in the previous section the equation for this order (eq. 5.27) can be decomposed into a set of multiple equations.

Assuming that A represents one of n, m, θ, ξ the first order equations will read:

$$\int_{-\infty}^{\infty} (\bar{q}_0)_A \left[\left(i \frac{\partial}{\partial t_1} q_0 + i q_0 \right) + \left(i \frac{\partial}{\partial t_0} q_1 + \frac{1}{2} \frac{\partial^2}{\partial x^2} q_1 + 2|q_0|^2 q_1 + q_0^2 \bar{q}_1 \right) \right] dx + c.c. = 0\tag{5.35}$$

The term in the square bracket is the linearised equation where the forcing term is the content of the first bracket. The latter is the behaviour of the soliton parameters in time scale t_1 . In order to avoid secularities this behaviour should be decoupled from the linearised first order equation, which is the second parenthesis of the square bracket. To become more concrete let us evaluate the first order equations:

$$\frac{\partial}{\partial t_1} \xi = 0\tag{5.36}$$

$$4 \frac{\partial}{\partial t_1} n + 8n = 0\tag{5.37}$$

$$8 \frac{\partial}{\partial t_1} (mn) + 16mn = 0\tag{5.38}$$

$$8m \frac{\partial}{\partial t_1} \xi - 4 \frac{\partial}{\partial t_1} \theta + \frac{\pi^2}{3} \frac{\partial}{\partial t_0} b - 4 \frac{\partial}{\partial t_0} a = 0\tag{5.39}$$

where we notice that the last equation has mixed time scale derivatives. Secularity will arise if the terms containing t_1 derivatives do not evaluate to zero. Thus, the first two terms of (eq. 5.39) need to be equated to zero independently. Thus we arrive at the following result:

$$\frac{\partial}{\partial t_1} \xi = 0 \quad (5.40)$$

$$\frac{\partial}{\partial t_1} \theta = 0 \quad (5.41)$$

$$\frac{\partial}{\partial t_1} m = 0 \quad (5.42)$$

$$\frac{\partial}{\partial t_1} n = -2n \quad (5.43)$$

$$a(t_0, t_1, \dots) = \frac{\pi^2}{12} b(t_0, t_1, \dots) + c(t_1, t_2, \dots) \quad (5.44)$$

To determine a and b we need to do the calculations at order $O(\gamma^2)$. From zero and first order dependencies we notice that m remains a constant to order $O(\gamma)$. We can thus without affecting approximations to first order set it equal to zero. The result is that ξ is a constant which we choose to be zero. Let us now consider the situation at order $O(\gamma^2)$. The equations at this order will arise from integration of terms of the form:

$$(\bar{q}_0)_A \mathcal{M}_2 + (\bar{q}_1)_A \mathcal{M}_1 + c.c.$$

where c.c. stands for complex conjugate, \mathcal{M}_i stands for the i th order term in expansion (5.33 and A is one of the unknown temporal functions. The first term in the expression above will only lead to setting conditions for the t_0 behaviour of the temporal functions characterising q_2 and the t_1 behaviour of the functions a and b . The complete t_0 behaviour of the latter two can be deduced from considering just the integrals of:

$$(\bar{q}_1)_A \mathcal{M}_1 + c.c.$$

In our case A is one of n, θ, b . Thus, we derive three equations:

$$8n - 32bn^3 = 0 \quad (5.45)$$

$$\frac{2\pi^2}{15} (10c + 3\pi^2 b) n \frac{\partial}{\partial t_0} b = 0 \quad (5.46)$$

$$\frac{2(4bn^2 - 1)(90c - 150b\pi^2 + 30c\pi^2 + 13b\pi^4)}{135} = 0 \quad (5.47)$$

These relations are all valid simultaneously if $b = \frac{1}{4n^2} + c(t_1, \dots)$. Thus the form of q_1 is:

$$q_1 = i\gamma \frac{1}{4n^2} \left[\left(\frac{\pi^2}{12} + 4n^2 x^2 \right) + c \right] q_0 \quad (5.48)$$

The latter, (apart from the presence of c) is exactly the same as that predicted by IST treatment of the problem ([KM78a]).

5.5 Connection of the optimisation problem with observables

Let us now return to equation (5.21) and concentrate on the NLS family of equations. In those equations there is only a first order derivation with respect to the evolution parameter (time). If we separate this term and group the rest in a generic nonlinear differential operator $\mathcal{N}(q, \bar{q})$ then these equations acquire the general form:

$$i q_t + \mathcal{N}(q, \bar{q}) = 0 \quad (5.49)$$

When this formula is substituted into (5.21) and the integrals are regrouped, in a way as to distinguish terms involving time derivatives from the rest we will have:

$$i \int_{-\infty}^{\infty} [q_t \bar{q}_{A_j} - \bar{q}_t q_{A_j}] dz + \int_{-\infty}^{\infty} [\mathcal{N}(q, \bar{q}) \bar{q}_{A_j} + \overline{\mathcal{N}}(\bar{q}, q) q_{A_j}] dz = 0 \quad (5.50)$$

The first term of (5.50) suggests that we can associate with the variable A_j a quantity whose temporal variation is described by (5.50). To see this clearly let us introduce *observables* into the analysis.

By *observables* we designate real valued functionals of the solutions of equation (5.49). For the rapidly decreasing case we can define a class of observables in the form:

$$\mathcal{F} = \int_{-\infty}^{\infty} F dz$$

with F depending on functions q, \bar{q} and their derivatives.

Bearing in mind this dependence on q and \bar{q} the temporal evolution of the observable \mathcal{F} is:

$$\frac{\partial \mathcal{F}}{\partial t} = \int_{-\infty}^{\infty} \frac{\partial \hat{q}}{\partial t} \cdot \nabla_q F dz$$

where equations (5.11), (5.12) and (5.13) were used. By making use of the rapidly decreasing boundary condition defined in section 4.2.1 and integration by parts the above is reduced to:

$$\frac{\partial \mathcal{F}}{\partial t} = \int_{-\infty}^{\infty} \left(\frac{\partial q}{\partial t} \frac{\delta F}{\delta q} + \frac{\partial \bar{q}}{\partial t} \frac{\delta F}{\delta \bar{q}} \right) dz \quad (5.51)$$

The resemblance, in shape, of (5.51) and the first term of (5.50) is obvious. By comparing the two terms it is possible to connect the quantities A_j and F by introducing the set of equations:

$$i q_{A_j} = - \frac{\delta F}{\delta \bar{q}} \quad (5.52)$$

$$i \bar{q}_{A_j} = \frac{\delta F}{\delta q} \quad (5.53)$$

by virtue of which (5.50) acquires the form:

$$\frac{\partial \mathcal{F}}{\partial t} + \int_{-\infty}^{\infty} [\mathcal{N}(q, \bar{q}) \bar{q}_{A_j} + \bar{\mathcal{N}}(\bar{q}, q) q_{A_j}] dz = 0 \quad (5.54)$$

The point made by the introduction of (5.50) is that there is a connection between the variational method and the perturbation approaches involving the study of the temporal propagation of the invariants of the NLS. The equation describing the temporal propagation of an observable \mathcal{F} is the Euler equation corresponding to a parameter A_j connected with the observable through the set of transformations (5.52) and (5.53).

We will return to this type of transform shortly, when developing a Hamiltonian aspect of the Lagrangian method, but before this let us insist on (5.50). This equation corresponds to the integration in space of the following type of equation which is strongly reminiscent of a conservation law [Whi74] [JK82]:

$$\frac{\partial F}{\partial t} + \frac{\partial P}{\partial z} + c(z, t) = 0 \quad (5.55)$$

It is now tempting to continue the analogy by defining F as the density, P the associated flux and $c == [\mathcal{N}(q, \bar{q}) \bar{q}_{A_j} + \bar{\mathcal{N}}(\bar{q}, q) q_{A_j}]$. Because we run the integration from $-\infty$ to $+\infty$ under the rapidly decreasing boundary conditions, the flux P did not appear in (5.54). However the flux term appears explicitly in (5.20):

$$P = \sum_{\lambda=0}^{\kappa_m} (-1)^\lambda \sum_{i=1}^{i_m} \left[\left(q_{A_j} \right)_{(i-1)z} \frac{\delta L_{\lambda t}}{\delta q_{(iz, \lambda t)}} + \left(\bar{q}_{A_j} \right)_{(i-1)z} \frac{\delta L_{\lambda t}}{\delta \bar{q}_{(iz, \lambda t)}} \right]$$

What prevents (5.55) from looking exactly like a conservation law is the presence of the function $c(z, t)$ in the right hand side. On the vast majority of the problems handled we are looking at effects associated with localised pulses. It is then reasonable to assume that we can select some appropriate observables whose Euler-Lagrange equations will have the function $c(z, t)$ localised in space (for example number of photons , energy etc). This assumption implies that $c(z, t)$ is now a source term. Let us assume for example that $c(z, t)$ is considerable in the area surrounding the origin of the z axis. We now introduce a boundary layer at $z = a > 0$ with the requirement that:

$$\frac{\left[\int_a^{+\infty} c(z, t) dz \right]}{\left[\int_{-\infty}^{+\infty} c(z, t) dz \right]} \ll 1 \quad (5.56)$$

In this case equations like (5.55) can be treated as conservation laws in the locality of $z = a$. The error introduced is determined by the ratio (5.56).

5.6 Ritz method and Hamiltonian formalism

In the previous section the main subject was the temporal behaviour of observables and how this is connected to the selection of the trial function. In the process we implicitly introduced a Hamiltonian view of the problem in hand. Let us now clarify this connection.

The starting point is the Lagrangian of our problem. Again we will separate the temporal part from the rest thus rewriting it as:

$$L = i (\bar{q} q_t - q \bar{q}_t) + L_1$$

where L_1 represents the remaining terms. When the trial function is substituted into the Lagrangian the first term is rewritten in terms of the derivatives of the new unknown functions A_j , $j = 1 \dots N$. After the spatial integration step we have:

$$\mathbf{L} = i \sum_{j=1}^N \left[\int_{-\infty}^{\infty} (\bar{q} q_{A_j} - q \bar{q}_{A_j}) dz \right] (A_j)_t + \int_{-\infty}^{\infty} L_1 dz \quad (5.57)$$

In this equation the way to associate a momentum with a certain variable A_j is to consider the definition [LL76]:

$$p_j = \frac{\partial \mathbf{L}}{\partial (A_j)_t} = i \int_{-\infty}^{\infty} (\bar{q} q_{A_j} - q \bar{q}_{A_j}) dz \quad (5.58)$$

The momentum, p_j , just defined obeys equations (5.52) and (5.53), as can be easily verified. It is thus equal, up to a constant to the observable \mathcal{F} defined through them. The idea here is to assume that the newly introduced momenta are independent functions. The total number of coordinates, thus doubles (A_j and p_j). The space spanned by these coordinates is called phase space. The introduction of the trial function, however, imposes an upper limit to the dimensionality of the phase space: the number of independent A_j 's. This means that the momenta defined in (5.58) are, in their turn functions of the variables A_j . The next step, in the Hamiltonian reformulation, is to define the Hamiltonian function. The latter is connected to the Lagrangian with the formula:

$$\mathbf{H} = \sum_j p_j (A_j)_t - \mathbf{L} = - \int_{-\infty}^{\infty} L_1 dz$$

by virtue of which the Lagrangian is rewritten as:

$$\mathbf{L} = \sum_j p_j (A_j)_t - \mathbf{H} \quad (5.59)$$

In the case of the NLS equation the Hamiltonian will be the third invariant (energy). We now need to take the Euler Lagrange equations of this function. However as

mentioned above there can be only N linearly independent functions in the set of $2N$ functions: $\{A_1, A_2, \dots, A_N, p_1, \dots, p_N\}$. Let us assume that a number k , with $k \leq N/2$, of functions A_j will be related to the momenta of an equal number of parameters A_j . For those it is possible to perform a transform and reexpress the Lagrangian (5.59) in terms of the set: $\{A_1, A_2, \dots, A_k, p_1, \dots, p_k, A_{2k+1}, \dots, A_N\}$. From the first $2k$ variables we will have the Euler equations:

$$(p_j)_t = -\frac{\delta H}{\delta A_j} \quad (5.60)$$

$$(A_j)_t = \frac{\delta H}{\delta p_j} \quad (5.61)$$

These are exactly the Hamiltonian equations of motion. For the remaining A_j , however it is not possible to reproduce this elegant representation. The Euler Equation for A_n with $2k + 1 \leq n \leq N$ will be:

$$(p_n)_t - (A_n)_t \frac{\partial p_n}{\partial A_n} = -\frac{\delta H}{\delta A_n} \quad (5.62)$$

These are generally first order quasi linear differential equations. They can be equivalently expressed, though, as a set of algebraic conditions between the unknown functions A_j which ensure solvability. This can be understood from the following argument. Let us suppose that the first term of the reduced Lagrangian (5.57) can be grouped into m having each the time derivative of a different function. There are two possible cases. If $2m \leq N$ then the number of parameters that we can associate with momenta is $k = m$ and the remaining $N - 2k$ parameters appear only in the Hamiltonian. Their Euler equations then are algebraic. If $2m \geq N$, then we have to set $k = N - m$, the limiting factor now being the momenta. In this case the left hand side of the differential equation (5.62) will be a linear combination of the left hand sides of the “Hamiltonian” equations (5.60) and (5.61). This will not be so for the corresponding right hand sides. This will lead again to algebraic equations, in order to ensure compatibility. The presence of the algebraic conditions indicates that we are facing a problem of motion with constraints.

The imposition of a fully “Hamiltonian” picture comes only by introducing enough variables to ensure that the condition $2k = N$ is fulfilled. When this is the case we notice that the algorithm introduced at the beginning of this chapter is fully incorporated into the Hamiltonian formalism. Namely: a trial function is selected having an even number of parameters which are allowed to vary with time. With the help of (5.58) the unknowns A_j are transformed into $A_k - p_k$ pairs. Substitution of the transformed trial function into the Hamiltonian H and integration throughout the z domain will lead to

the definition of a new reduced Hamiltonian \mathbf{H} . The final step is to apply Hamilton's equations of motion with the new Hamiltonian.

5.7 Appendix: proofs of relations used

Relation:

$$\sum_{\mu=\nu}^{\kappa} (-1)^{\mu} \binom{\kappa}{\mu} \binom{\mu}{\nu} = (-1)^{\kappa} \delta_{\kappa\nu} \quad \text{with } \nu \leq \kappa \quad (5.63)$$

is proved in the following way. Since $\nu \leq \kappa$ we rewrite $\mu = \nu + r$ and $\kappa = \nu + s$. Now the combinations appearing in the sum are:

$$\binom{\nu + s}{\nu + r} \binom{\nu + r}{\nu} = \frac{(\nu + s)!}{(s - r)! \nu! r!} = \frac{(\nu + s)!}{s! \nu!} \frac{s!}{(s - r)! r!} = \binom{\nu + s}{\nu} \binom{s}{r}$$

It is then possible to rewrite the sum in (5.63) as:

$$(-1)^{\nu} \binom{\nu + s}{\nu} \sum_{r=0}^s (-1)^s \binom{s}{r} \quad (5.64)$$

To complete the proof we only need to evaluate the sum. Take the binomial expansion:

$$(1 + z)^s = \sum_{r=0}^s \binom{s}{r} (z)^r$$

with $z = -1$. The latter is zero when $s \neq 0$ and equal to 1 when $s = 0$. This completes the proof.

6. SOLITONS UNDER PERTURBATIONS

6.1 *The initial value problem*

6.1.1 *General*

When considering perturbations, there are two main types of problem: that of perturbed equations and that of perturbed initial conditions. The former has an extensive bibliographic covering as seen in the previous chapters. On the other hand very little analytical work has been carried out with respect to the second problem ([MF96] and references therein).

The main concern in NLS related initial values problems is the determination of the emerging soliton. The latter is fully described by the discrete spectrum in the scattering domain. In the IST framework this particular problem amounts to solving the direct scattering problem for the given set of initial conditions. In the general case this can be achieved either by approximation methods [BN94] [Lew85] or by numerical evaluation [BO92] [Kau77].

A different approach in determining the soliton content of an initial condition would involve the use of the integrals of motion associated with the NLS equation. Along those lines are the problem described at the beginning of the previous chapter as well as the approach in [KS95] and [LSH72]. To make the discussion more concrete let us select the following initial value problem:

$$q_0(0, z) = A \operatorname{sech}(z) \quad (6.1)$$

whose direct problem can be solved exactly [SY74] and is functionally close to the envelope of the fundamental soliton. Qualitatively both approaches [AL83] and [KS95] are the same: it is assumed that the radiation content of the initial condition has negligible contribution to the first and third integral of motion of the NLS, respectively (see 4.2.2). Assuming that the emerging fundamental soliton takes the form: $n \operatorname{sech}(nz)$, its amplitude is predicted to be $n = A^2$ and $n = A^{\frac{2}{3}} (-1 + 2A^2)^{\frac{1}{3}}$ respectively. Since both approaches neglect radiation it is expected that they give their worst agreement in the area close to soliton cutoff. This can be seen clearly in (fig. 6.1). A better prediction using the first integral of motion can be achieved by the use of empirical

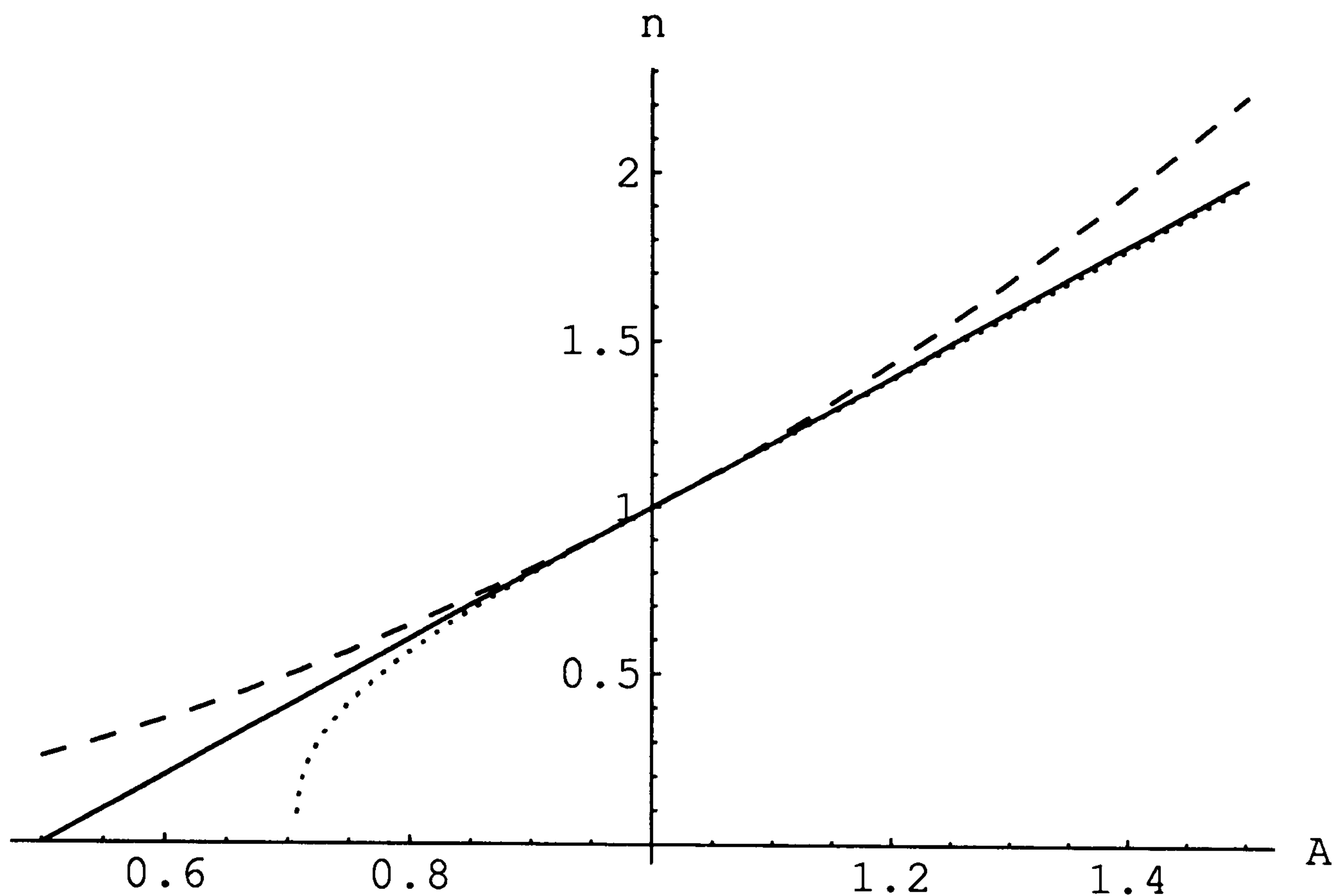


Fig. 6.1: Emerging soliton amplitude (n) vs initial amplitude (A). Solid line is IST prediction [SY74], dotted line is prediction based on energy integral and dashed line according to mass integral.

arguments [ALR88b]. This does not reduce the disagreement close to cutoff. Obviously the situation in this region is unsatisfactory. If the integrals of motion are to be used as estimates the contribution of the dispersive radiation has to be calculated. From numerical simulations it can be seen that the dispersive radiation emanating from the initial condition 6.1 is a smooth pedestal extending away from the soliton core (fig. 6.2) Let us now remove the soliton and propagate just the dispersive radiation part corresponding to this initial condition. It can be noticed that the two shapes are within the limits of perturbation (fig. 6.3) A further comparison of the arguments of the two solutions will reveal that they are both quadratically chirped with coefficients very close to each other. The difference in the phases is again within the perturbation limit (fig. 6.4). This close relation between the radiation parts prompts us to consider connections between the two. This will be achieved with the use of IST.

6.1.2 IST considerations

We have already decided on the methodology that we will be using. Let us see what can be achieved in our problem. We are looking primarily to determine the contribution of the dispersive radiation part to the integrals of motion. From IST theory we know

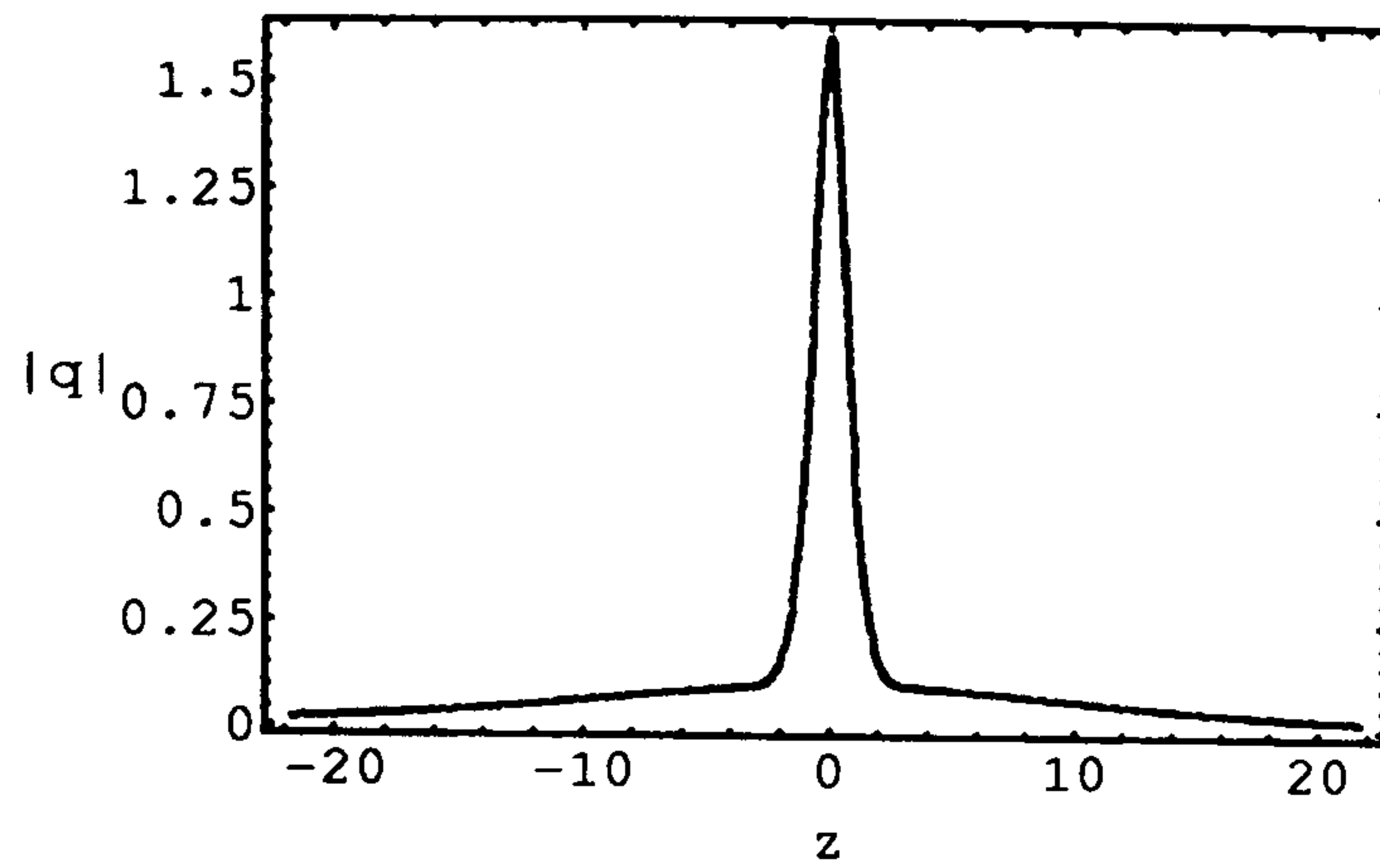


Fig. 6.2: Absolute value of the numerically calculated solution for $A = 1.3$, time is 5π

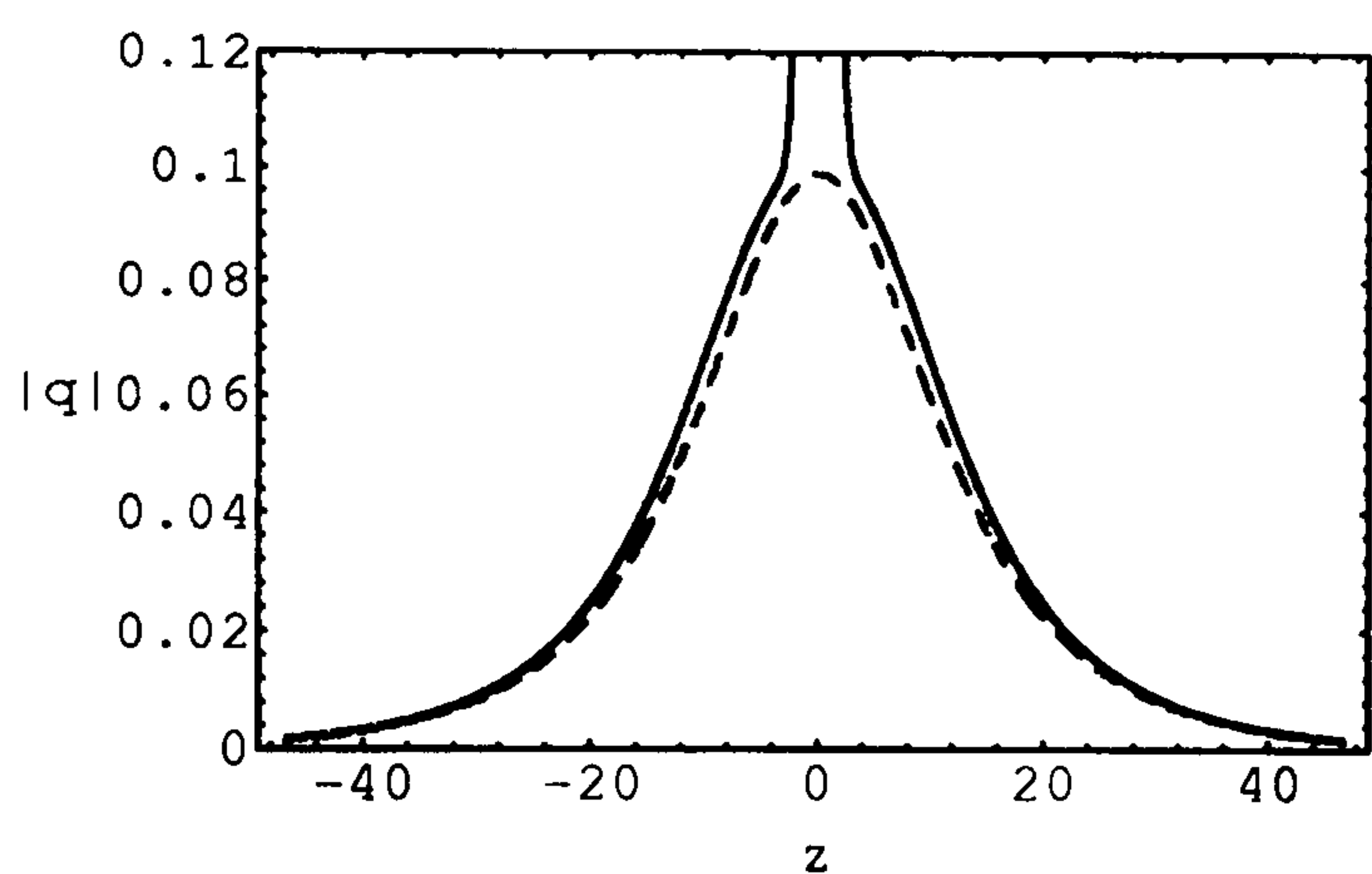


Fig. 6.3: Comparison of the envelopes of the 5π propagated initial conditions $1.3\text{sech}(z)$ (continuous curve) and $0.3\text{sech}(z)$ (dashed curve).

that the contribution of the continuous part of the spectrum is independent from that of the discrete. This contribution to the k th integral of motion is [NMPZ84]:

$$c_{r,k} = \frac{1}{(2i)^{k-1}} \int_{-\infty}^{\infty} \xi^{k-1} n(\xi) d\xi \quad \text{with} \quad (6.2)$$

where ξ is the eigenvalue of the z part of the AKNS pair (4.10) and the canonical variables describing dispersive radiation are:

$$n(\xi) = \frac{1}{\pi} \ln \left(\frac{1}{|a(\xi; A)|^2} \right) \quad \text{and} \quad \phi(\xi; A) = \arg(b(\xi; A)) \quad (6.3)$$

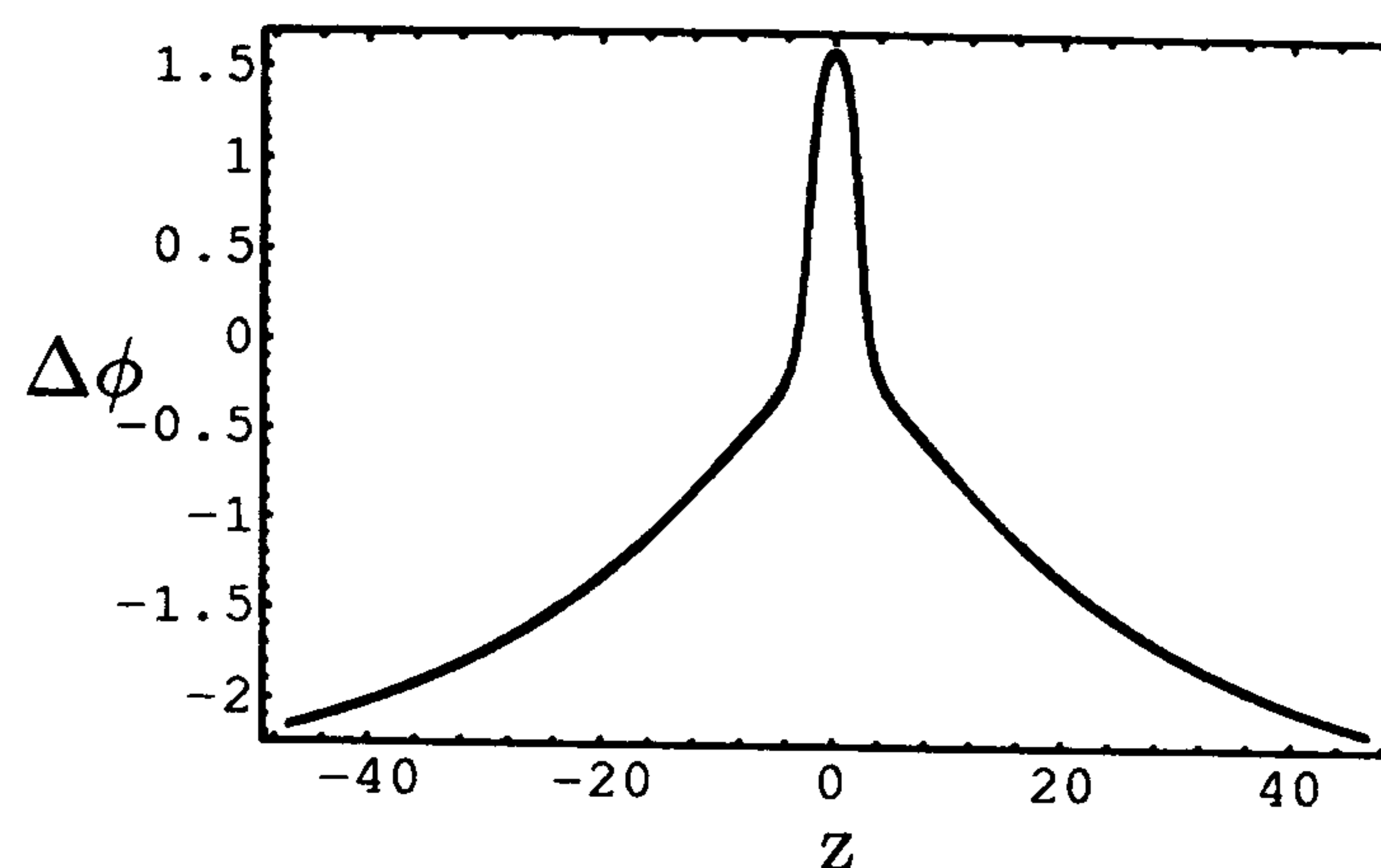


Fig. 6.4: Phase difference between the 5π propagated initial conditions $1.3\text{sech}(z)$ and $0.3\text{sech}(z)$

with a and b the coefficients connecting different Jost solutions (4.14) For the initial value problem (6.1) the coefficients a and b are given by [SY74]:

$$a(\xi; A) = \frac{\Gamma(-i\xi + \frac{1}{2})^2}{\Gamma(-i\xi + A + \frac{1}{2}) \Gamma(-i\xi - A + \frac{1}{2})} \quad (6.4)$$

$$b(\xi; A) = i \sin(\pi A) \text{sech}(\pi \xi) \quad (6.5)$$

By making use of the shift property of the gamma function [WW27]:

$$\Gamma(z+1) = z\Gamma(z)$$

we have:

$$a(\xi; A+1) = -\frac{A + \frac{1}{2} + i\xi}{A + \frac{1}{2} - i\xi} a(\xi; A) \quad (6.6)$$

It is obvious that the increase of A by unity only affects the phase of $a(\xi; A)$ and adds a π phase to $b(\xi; A)$. By inspection we see that the canonical variable $n(\xi)$, appearing in the integrals of motion is insensitive to the increase of A by unity. Thus we can safely conclude that in terms of IST theory and for this specific initial value problem the dispersive radiation part of the initial condition (6.1) when $0.5 < A < 1.5$ contributes to the integrals of motion equivalently to the initial condition $(A-1)\text{sech}(z)$

The next step is to try to connect the two fields in the $z-t$ domain. To this end we need the expression used for the reconstruction of the radiation from the scattering data (4.29). After complex conjugation, the part corresponding to the continuous part of the scattering domain spectrum will be:

$$\bar{q}(z, t; A) = \frac{1}{i\pi} \int_{-\infty}^{\infty} \frac{b(\xi; A)}{a(\xi; A)} e^{i\xi z} \psi_2(z; \xi; A) e^{2i\xi^2 t} d\xi \quad (6.7)$$

where B and a are defined above and:

$$\psi_2(z; \xi; A) = (1 - \tanh(z))^{-i\xi/2} (1 + \tanh(z))^{i\xi/2} F\left(-A, A; -i\xi + \frac{1}{2}; \frac{(1 - \tanh(z))}{2}\right)$$

[SY74] and F is the hypergeometric function. We are interested in the behaviour of this radiation away from the origin, where the soliton lies. Since the NLS equation is invariant with respect to space inversion we only need to consider the positive z semiaxis.

Away from the origin we can do the following approximations:

$$\begin{aligned} (1 - \tanh(z)) &\rightarrow 2e^{-2z} \\ (1 + \tanh(z)) &\rightarrow 2 \end{aligned}$$

and by Taylor expansion around the origin:

$$F\left(-A, A; -i\xi + \frac{1}{2}; \frac{(1 - \tanh(z))}{2}\right) \approx 1 - \frac{A^2}{e^{2z} \left(\frac{1}{2} - i\xi\right)} \approx 1$$

Thus for z away from the origin we have:

$$\bar{q}(z, t; A) \approx \frac{1}{i\pi} \int_{-\infty}^{\infty} \frac{b(\xi; A)}{a(\xi; A)} e^{2i\xi z} e^{2i\xi^2 t} d\xi \quad (6.8)$$

The dependence of this relation on A is purely located at the reflection coefficient b/a . By increasing A by unity and avoiding the poles of the Γ function ($A = \pm \frac{1}{2}$) we have from (6.6) and (6.5):

$$\frac{b(\xi; A+1)}{a(\xi; A+1)} = \frac{b(\xi; A)}{a(\xi; A)} \exp\left(-2i \tan^{-1}\left(\frac{\xi}{A+1/2}\right)\right) \quad (6.9)$$

Now for the initial condition envisaged the reflection coefficient has contribution to the integral (6.8) around the origin of the ξ axis. This fact together with the presence of the quadratic term in ξ in the exponential prompts us to take the Taylor expansion of the inverse tangent function in (6.9). By combining the result of this expansion with (6.9) and (6.8) we have:

$$\bar{q}(z, t; A+1) = \bar{q}\left(z - \left(\frac{1}{2} + A\right)^{-1}, t; A\right) \approx \frac{1}{i\pi} \int_{-\infty}^{\infty} \frac{b(\xi; A)}{a(\xi; A)} e^{2i\xi(z - (\frac{1}{2} + A)^{-1})} e^{2i\xi^2 t} d\xi \quad (6.10)$$

Clearly the difference is a phase translation depending on A . Comparison of numerical results is given in figure (fig. 6.5) where very good agreement can be observed. For the phases of the solution the situations remains in the same order of magnitude as in (fig. 6.4)

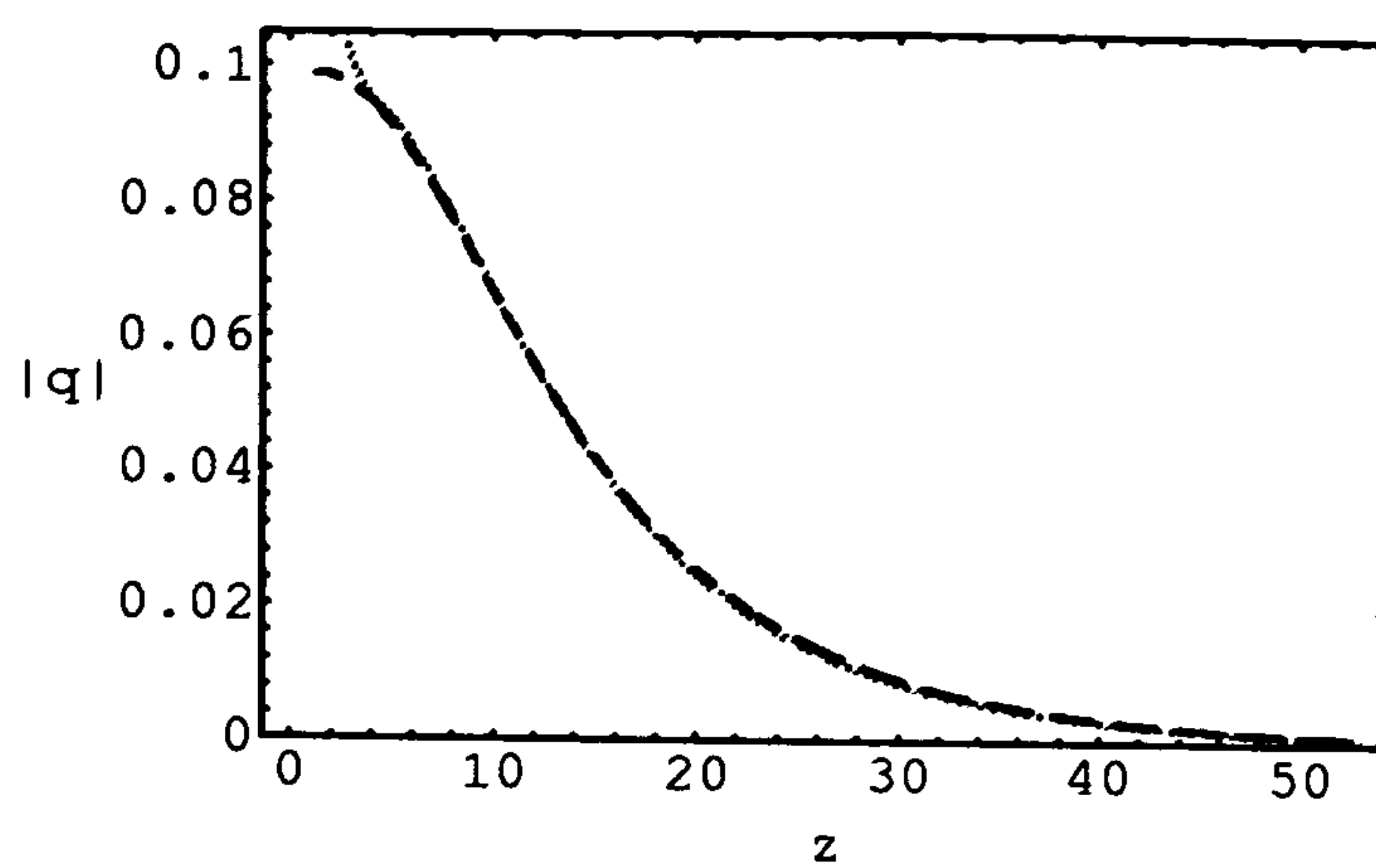


Fig. 6.5: Same as (fig. 6.3) with the data for $0.3\text{sech}(z)$ translated to the right by $(0.3 + 1/2)^{-1} = 1.25$ (dotted line).

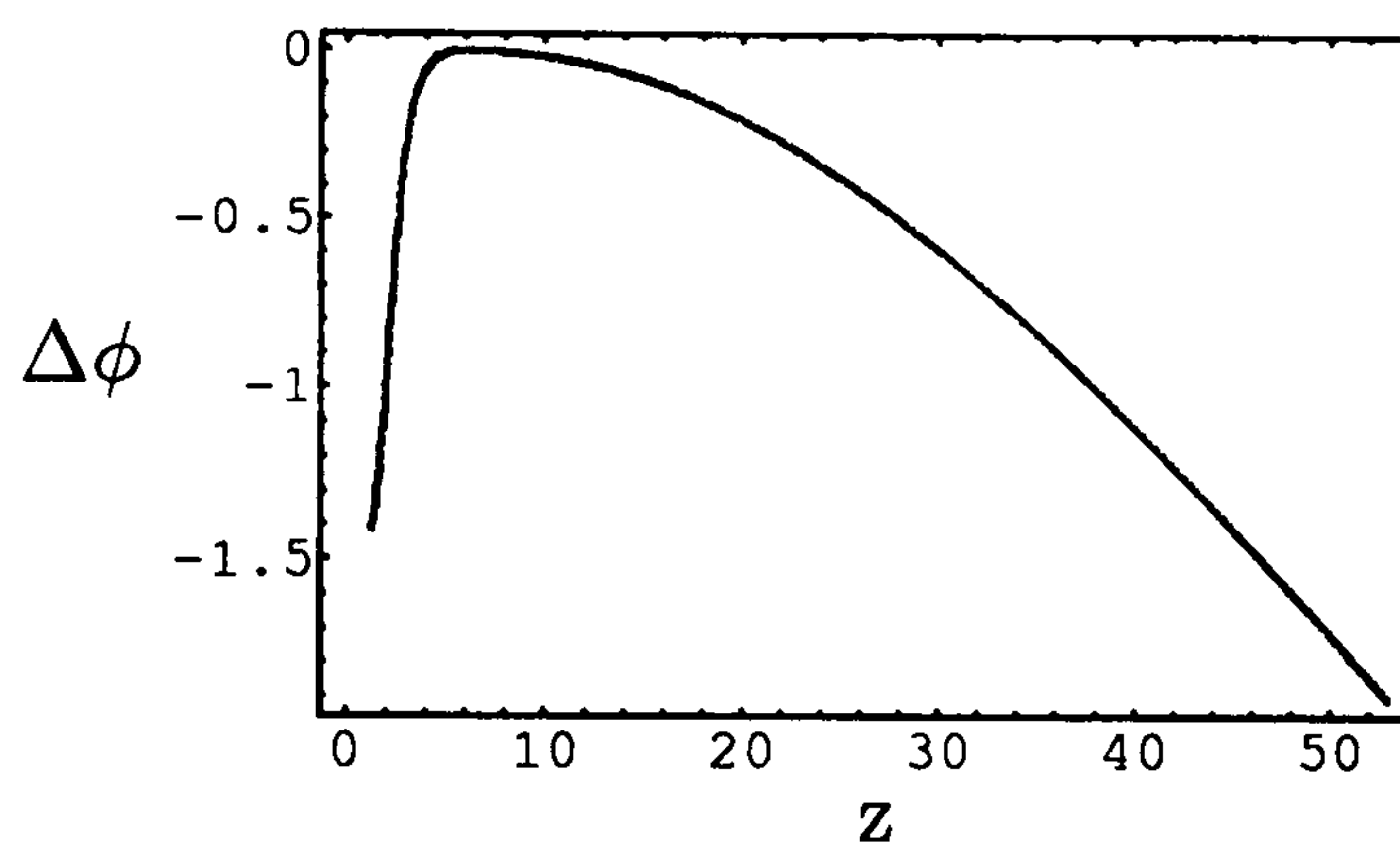


Fig. 6.6: Same as (fig. 6.4) depicting the positive semiaxis only and with the data for $0.3\text{sech}(z)$ translated to the right.

6.1.3 Calculation of the soliton content

Let us now take advantage of the first conclusion of the previous section in order to determine the soliton content of the initial condition $A\text{sech}(z)$. We will base the calculation on the first and the third integral of the NLS. The equivalence of the dispersive radiation part will allow us to determine the contribution of the latter to the integrals of motion using the trial function $r\text{sech}(z)$ while the soliton part takes

the standard form: $n \operatorname{sech}(nz)$. The following two equations will arise:

$$\begin{aligned} A^2 &= n + r^2 \\ A^2 - 2A^4 &= r^2 - 2r^4 - n^3 \end{aligned}$$

leading to the solution $n = 2A - 1$. Surprisingly this is the exact prediction derived from IST considerations [SY74]!

Let us now try to capitalise on this coincidence. Suppose that instead of (6.1) we have an initial condition corresponding to the addition of a small perturbing term of order $O(\epsilon)$. The continuous spectrum then would itself be affected to the same order. The periodicity with respect to the initial amplitude ($A \rightarrow A + 1$) observed for (6.1) would no longer hold in general [BN94]. However, since we are interested in comparing the cases of no asymptotic soliton with first order asymptotic soliton and the departures from the scattering data of the previous example are of $O(\epsilon)$ it is plausible to do calculations assuming that the radiation has the same functional form as the initial condition. To test this statement we consider three types of initial conditions:

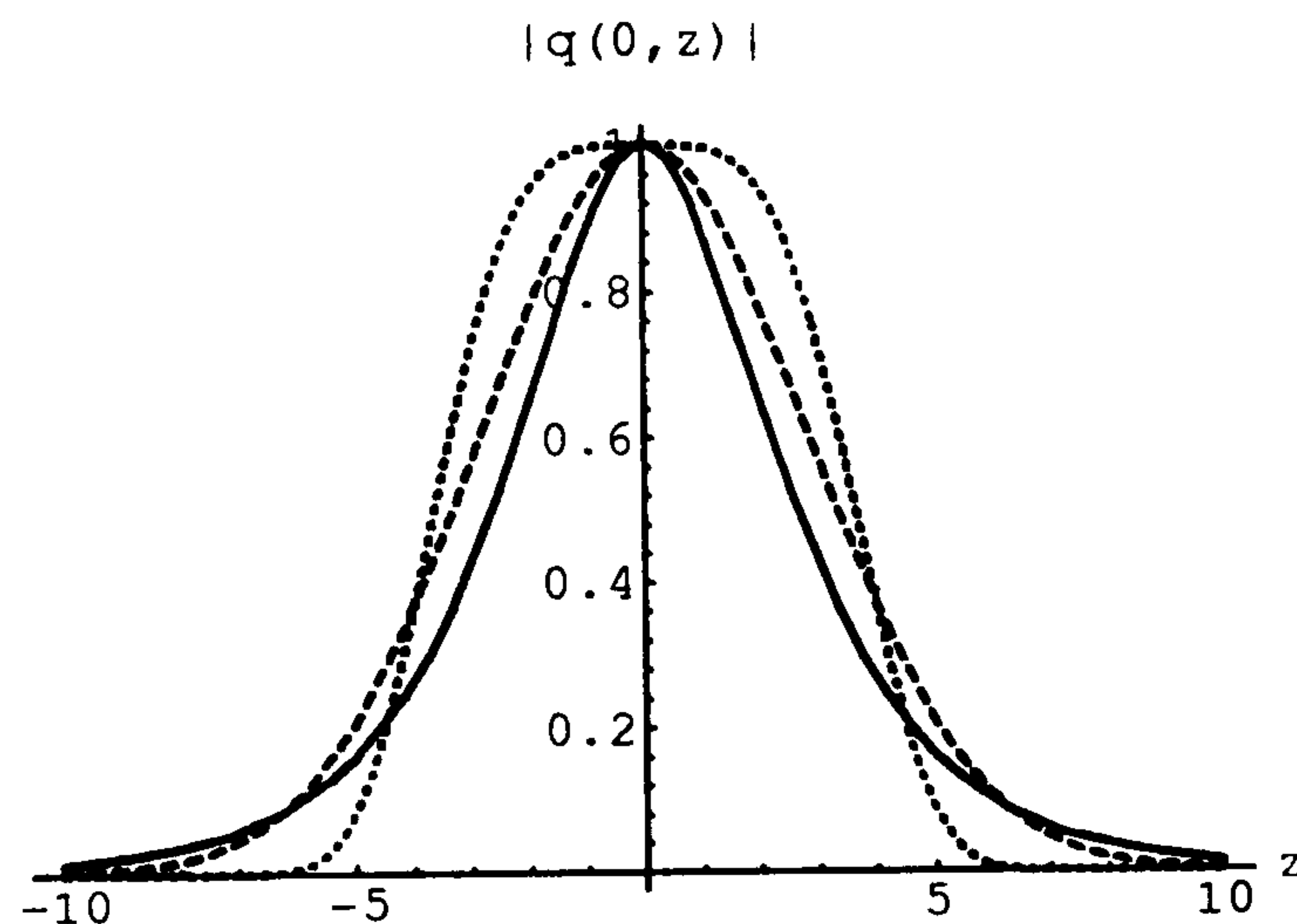


Fig. 6.7: Absolute values of the initial profiles considered. Solid line corresponds to the hyperbolic secant functional form while the dashed corresponds to the gaussian and the dotted to the supergaussian functional form.

- chirped hyperbolic secant: $A \operatorname{sech}(z) \exp(ibz^2)$
- chirped gaussian: $A \exp(ibz^2 - \frac{z^2}{w^2})$ and
- chirped supergaussian pulses: $A \exp\left(ibz^2 - \frac{z^4}{w^4}\right)$

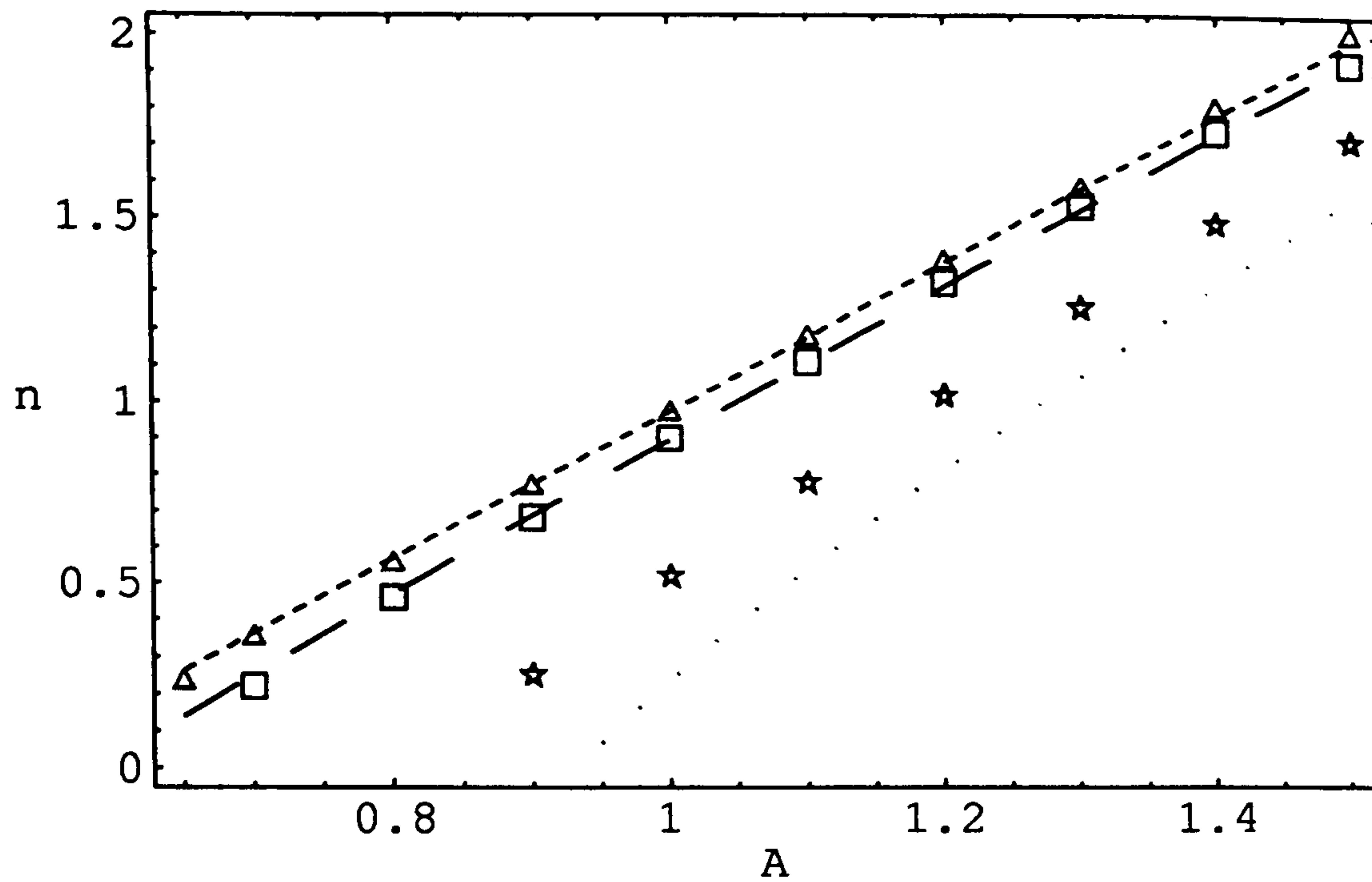


Fig. 6.8: Emerging soliton amplitude (n) vs initial amplitude (A) for the chirped hyperbolic secant initial condition. Points are numerically calculated values with triangles corresponding to $b=0.1$, squares to $b=0.2$ and stars to $b=0.5$. The lines are the predicted amplitudes. Small-dashed corresponds to $b=0.1$, large-dashed to $b=0.2$ and dotted to $b=0.5$. It is reminded that for $b=0$ the correct emerging amplitude is predicted.

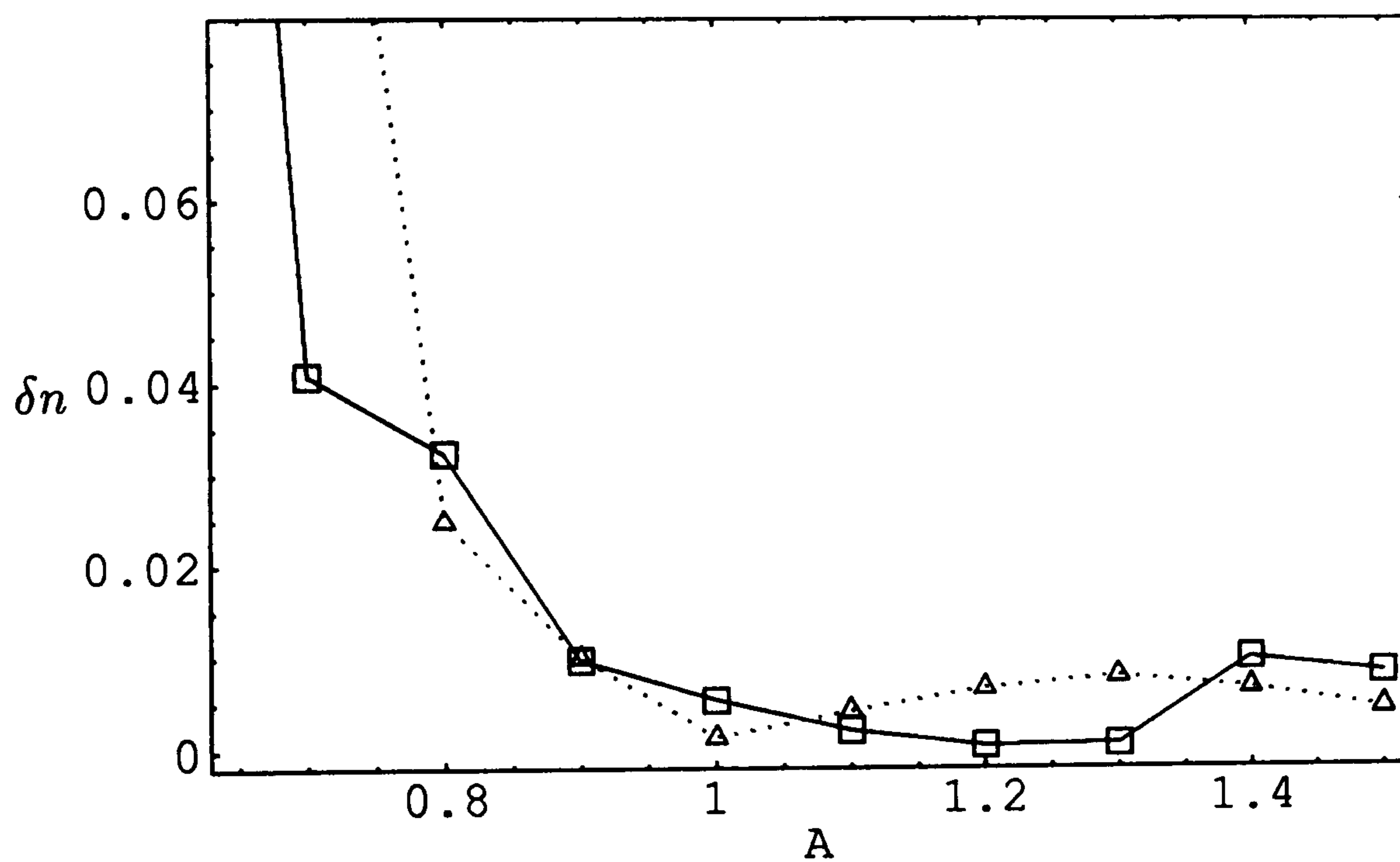


Fig. 6.9: Absolute relative error of the predicted emerging soliton amplitude for the chirped hyperbolic secant waveform vs the initial amplitude of the pulse. x's correspond to $b=0.1$ and triangles to $b=0.2$.

For these three families of initial conditions we calculate the amplitude of the emerging soliton first analytically, using the previously described procedure and then numerically

using by a standard split-step Fourier transform method [Agr89] [YH83].

For the sech initial condition we expect the best agreement since it is functionally the closest to the exact case studied. The emerging soliton can be determined by the formula:

$$n = -1 + \sqrt{4A^2 - b^2\pi^2} \quad (6.11)$$

Comparison with the numerical results is given in (fig. 6.8). Agreement becomes progressively worse as the value of the chirp is increased. The lack of numerical data in the area close to cutoff is due to the computational cost of such calculations, since the emergence of the pure soliton state occurs at extremely large distances.

The gaussian initial profile is quite close to the sech one. The periodicity of the continuous spectrum, invoked earlier is known not to hold here [BN94]. By performing standard calculations we have a predicted asymptotic amplitude:

$$n = \frac{-6w + 2\sqrt{3}\sqrt{w^2(3 - \pi + \sqrt{2}A^2\pi w^2 - b^2\pi w^4)}}{2\sqrt{\pi}w^2} \quad (6.12)$$

For the numerical calculations a value of $w = 2$ was used. As expected the agreement between numerical results and those of formula (6.12) are slightly worse than in the previous case, but still quite accurate provided that the chirp parameter b takes small values. Increase of b leads to a systematic underestimation of the emerging soliton amplitude. The third example is functionally remote from the sech profile (fig. 6.7). The predicted asymptotic amplitude of the soliton is given by:

$$n = \frac{6 \left(-1 + \sqrt{1 + \frac{\Gamma(\frac{1}{4}) \left(2^{\frac{1}{4}} A^2 w^2 \Gamma(\frac{1}{4}) - 2 b^2 w^4 \Gamma(\frac{3}{4}) - 4 \Gamma(\frac{7}{4}) \right)}{6\sqrt{2}}} \right)}{w \Gamma(\frac{1}{4})} \quad (6.13)$$

However we notice that the predictions become completely erroneous as the chirp parameters increases. The relative success for $b = 0$ is coincidental and can be expected for almost any profile by virtue of the area theorem ([Kau77]).

Before concluding this section an overview of the method followed should be included. We have started from a case where it happens that the canonical variables describing the dispersive radiation demonstrate a periodicity with the amplitude of the initial condition. Subsequently it was claimed that although perturbations will destroy this property, we can still, for the case of interest, assume that the contribution of the radiation to the integrals of motion can be evaluated by assuming that its shape is that of the initial pulse profile. This, however does not constitute a proper perturbation procedure. In fact the profile of the radiation should be assumed to be in the form of

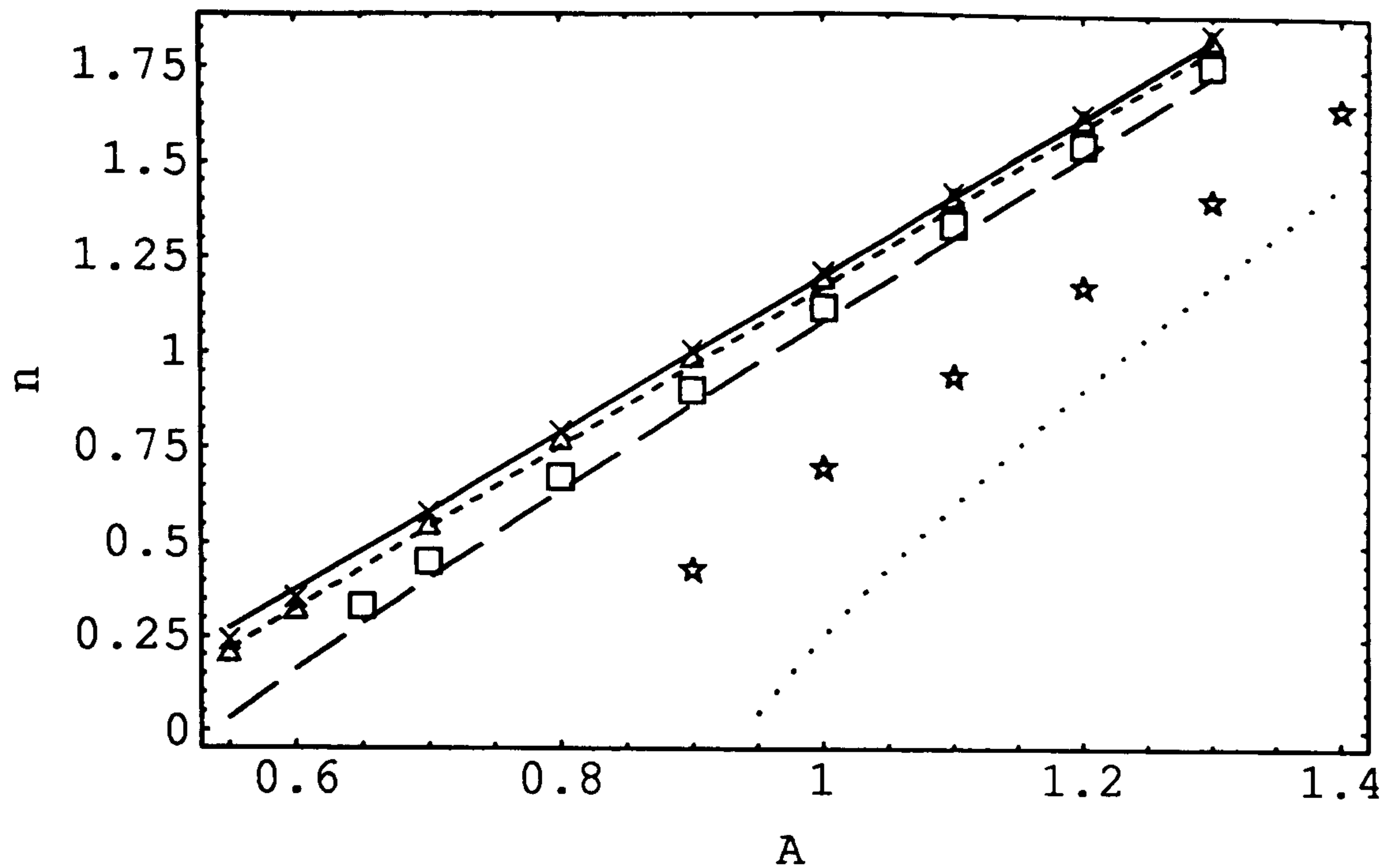


Fig. 6.10: Emerging soliton amplitude (n) vs initial amplitude (A) for the chirped gaussian initial condition. Points are numerically calculated values with x's corresponding to $b=0$, triangles to $b=0.1$, squares to $b=0.2$ and stars to $b=0.5$. The lines are the predicted amplitudes. Solid line corresponds to $b=0$, small-dashed to $b=0.1$, large dashed to $b=0.2$ and dotted line to $b=0.5$.

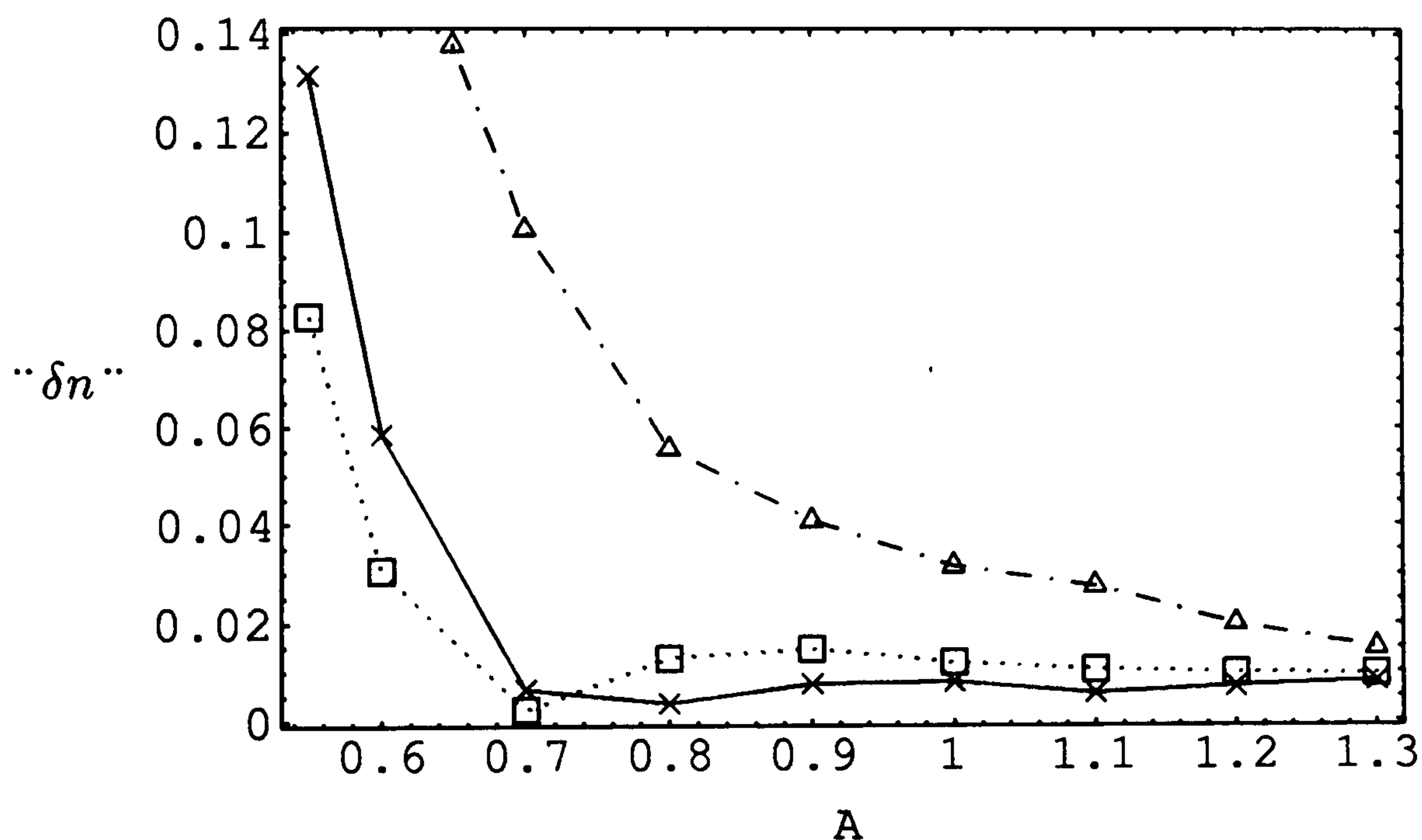


Fig. 6.11: Absolute relative error of the predicted emerging soliton amplitude for the chirped gaussian waveform vs the initial amplitude of the pulse. x's correspond to $b=0$, squares to $b=0.1$ and triangles to $b=0.2$.

the sech pulse that best approaches the profile in hand. The latter, on the other hand, raises the problem of defining the criterion of similarity, which would affect simplicity. We circumvented this problem by the hypothesis used. The price for that is worse

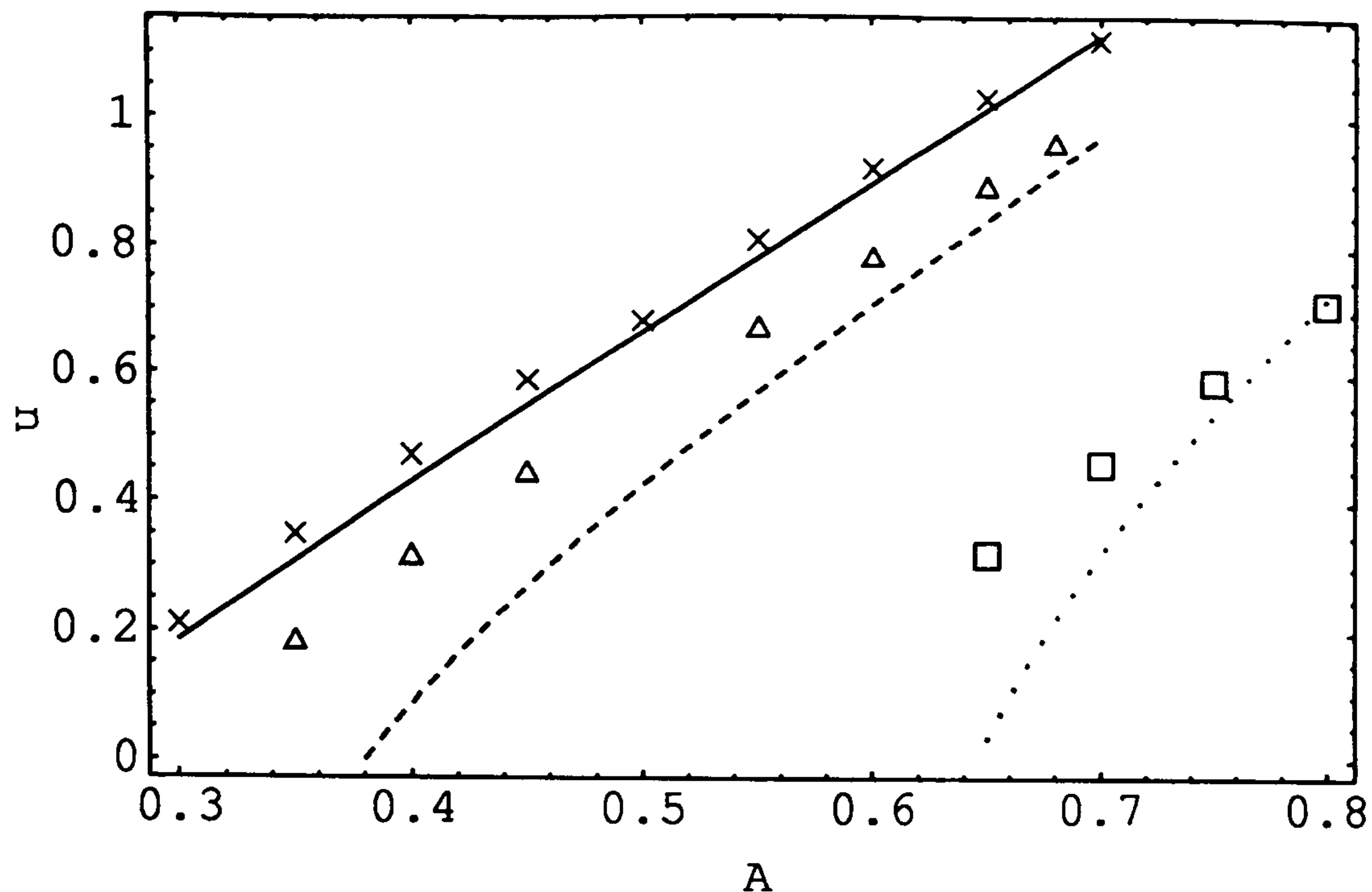


Fig. 6.12: Emerging soliton amplitude (n) vs initial amplitude (A) for the chirped supergaussian initial condition. Points are numerically calculated values with x's corresponding to $b=0$, triangles to $b=0.1$ and squares to $b=0.2$. The lines are the predicted amplitudes. Solid line corresponds to $b=0.1$, small-dashed to $b=0.2$ and dotted to $b=0.5$.

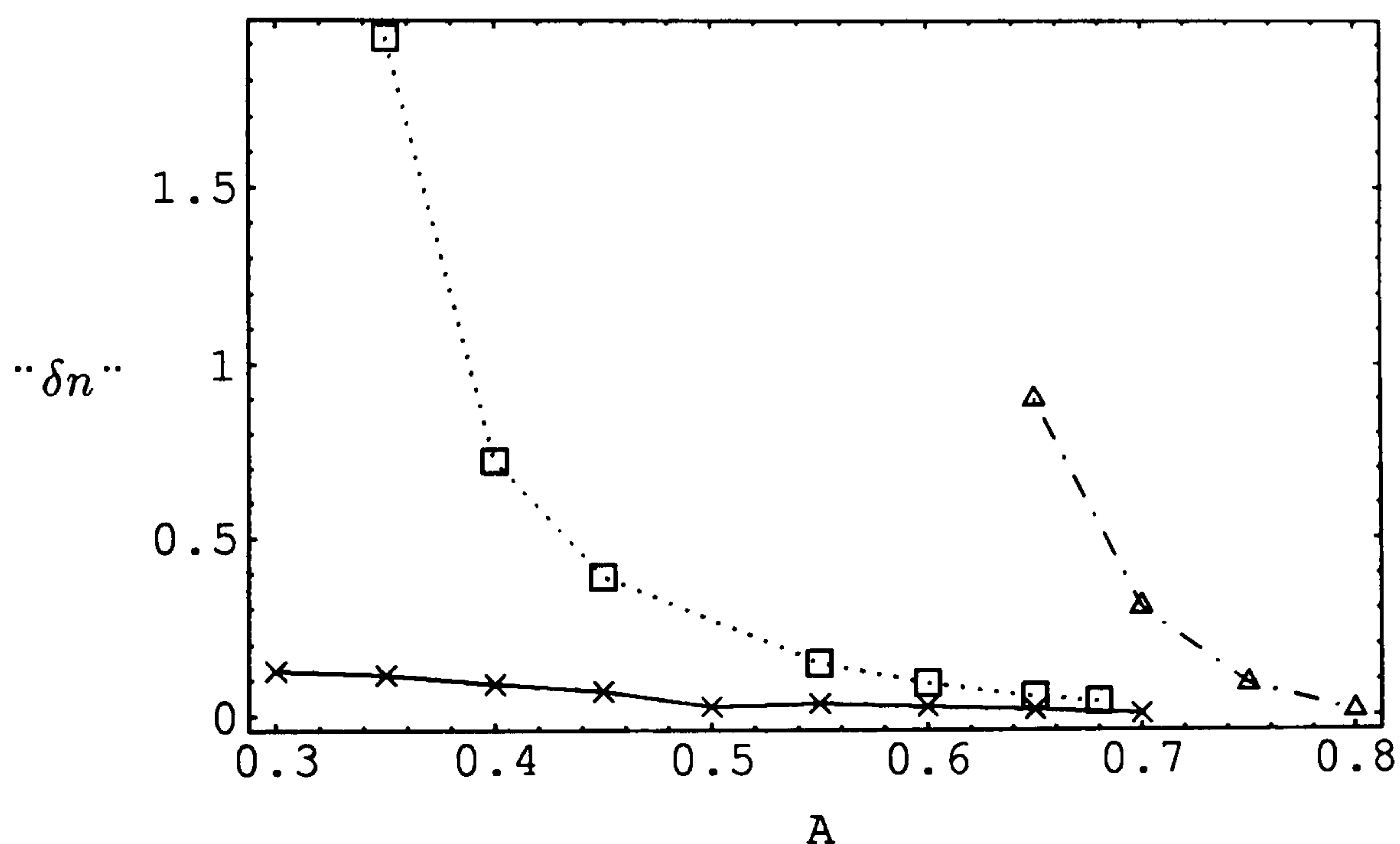


Fig. 6.13: Absolute relative error of the predicted emerging soliton amplitude for the chirped supergaussian waveform vs the initial amplitude of the pulse. x's correspond to $b=0$, squares to $b=0.1$ and triangles to $b=0.2$.

agreement with numerical results, which however seems to be small for the case of sech and gaussian pulses with small chirp. When, however, the departure from those shapes is large, there is no guarantee as to the relevance of the dispersive radiation

term included, thus the results follow.

These can be recapitulated as follows. In the case of unchirped pulses the approach of [KS95] is efficient as long as one stays away from the cutoff region. In that area the quality of the predicted figures deteriorates rapidly. The approach proposed here fills this gap exactly. The comparison with numerical results reveals good agreement and a further comparison with the IST predicted cutoff condition [Kiv89] [Bur88] will give good agreement at least for the cases studied here. The introduction of chirp adds to the perturbation of the continuous spectrum away from the assumed shape and thus makes predictions worse. In this case this is demonstrated as a systematic underestimation of the amplitude of the emerging soliton.

6.1.4 Lagrangian treatment of solitonless radiation in the $A \operatorname{sech}(z)$ case

Based on the IST considerations laid out earlier we will attempt to describe the evolution process for a pulse with the initial condition $A \operatorname{sech}(z)$. Instead however of going straight for the data regarding the core of the pulse we will attempt to describe first the situation away from the origin.

While doing the IST analysis it was observed that the tails (due to dispersive radiation) are slightly perturbed by the presence of a soliton. This radiation behaves essentially as would linear radiation. The presence of the nonlinear terms acts as a perturbation on the phase. This can be confirmed by looking at the asymptotic form of such radiation [KMS95]. But for this type of radiation the ansatz used by Anderson [AL83]:

$$q_r(z, t) = a(t) \operatorname{sech}(w(t) z) \exp(i \theta(t) + i b(t) z^2) \quad (6.14)$$

covers the main features. Our first step towards the description of dispersive radiation is to use this ansatz in conjunction with the variational method for initial conditions below cutoff ($A < 0.5$).

The Lagrangian corresponding to the NLS is:

$$L = i (\bar{q} q_t - q \bar{q}_t) - \bar{q}_z q_z + |q|^4$$

After the substitution of the trial function (6.14) into the Lagrangian the result is integrated throughout space (z). The average Lagrangian is then:

$$\mathbf{L} = \frac{-2 \pi^2 a(t)^2 b(t)^2}{3 w(t)^3} + \frac{4 a(t)^4}{3 w(t)} - \frac{2 a(t)^2 w(t)}{3} - \frac{\pi^2 a(t)^2 b'(t)}{3 w(t)^3} - \frac{4 a(t)^2 \theta'(t)}{w(t)} \quad (6.15)$$

The Euler equations can be extracted easily in the form:

$$a'(t) = -(a(t) b(t)) \quad (6.16)$$

$$b'(t) = -2b(t)^2 - \frac{2a(t)^2 w(t)^2}{\pi^2} + \frac{2w(t)^4}{\pi^2} \quad (6.17)$$

$$\theta'(t) = \frac{5a(t)^2 - 2w(t)^2}{6} \quad (6.18)$$

$$w'(t) = -2b(t)w(t) \quad (6.19)$$

These equations can all be expressed in terms of a Newton type equation [AL83], the asymptotic form of which (for widths much larger than zero) is:

$$\frac{4a(0)^2}{\pi^2} - \frac{2w(0)^2}{\pi^2} + \frac{1}{2} \left(\frac{d}{dt} w(t)^{-1} \right)^2 = 0$$

When this equation is solved for the initial conditions $w(0) = 1$ it recreates the correct qualitative behaviour:

$$a(0) \sqrt{\frac{\pi}{2\sqrt{1+2a(0)^2}t}} \operatorname{sech}\left(\frac{\pi z}{2\sqrt{1+2a(0)^2}t}\right) \exp\left[i\frac{z^2}{2t} + i\left(\frac{\pi^2}{12(1+2a(0)^2)t} + \frac{5a(0)^2\pi \ln(t)}{12\sqrt{1+2a(0)^2}}\right)\right] \quad (6.20)$$

The nonlinear term contributes the logarithm term in the exponent. To assess the accuracy of the variational approach we need to compare with numerical results. However the qualitative features can be inferred.

We know [KS95] that the Lagrangian method with this particular trial function will yield a cutoff amplitude of $\sqrt{2}/2$ which is higher than the predicted one. This indicates that the nonlinear contribution to the evolution is being underestimated. Thus we expect slightly faster decay of the peak of the waveform accompanied by a width that is increasing at a higher rate than the correct. The departure from the correct propagated initial condition should be more pronounced as the initial amplitude $a(0)$ approaches the -correct- cutoff value of $1/2$. The overestimation of the dispersion term leads to overestimation of the chirp parameter $b(t)$ near the centre of the pulse and an underestimation in the regions far from the centre. The differences can be visualised in figures (fig. 6.14) and (fig. 6.16).

The shape of the graph of the phase difference can be easily understood if one considers the fact that the nonlinearity is stronger near the centre of the pulse and weaker as we move along to the wings. From the asymptotic formula (6.20) we notice that the logarithm term in the exponent has a constant coefficient instead of the correct z dependent one [KMS95]. Nevertheless if one considers that the value of the b coefficient at the same time is approximately 0.1 the perturbation is less than 1

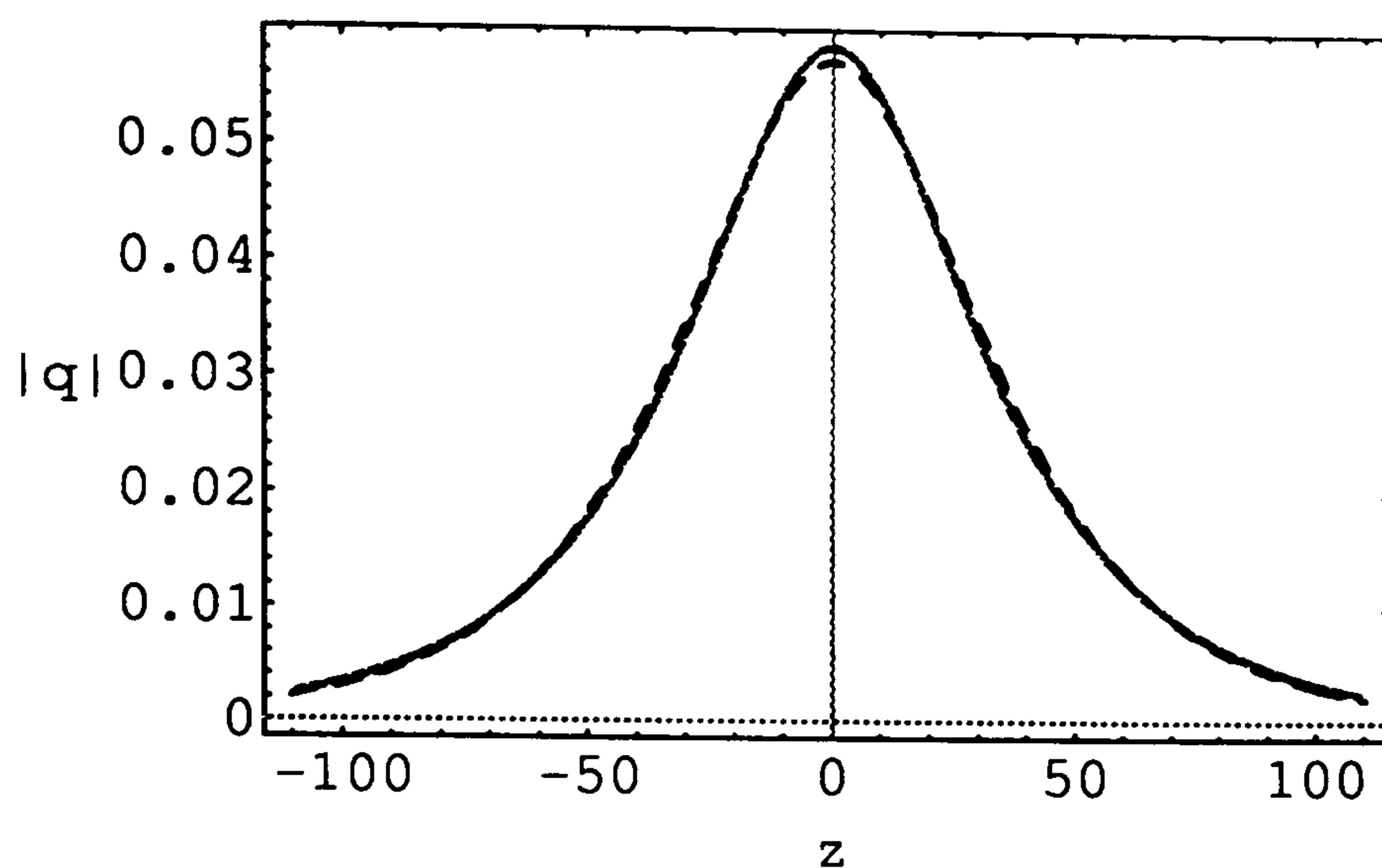


Fig. 6.14: Comparison between the envelopes of dispersive radiation with initial profile $0.3 \operatorname{sech}(z)$ at time 15π . Dotted line is the Lagrangian prediction and continuous the numerically calculated one.

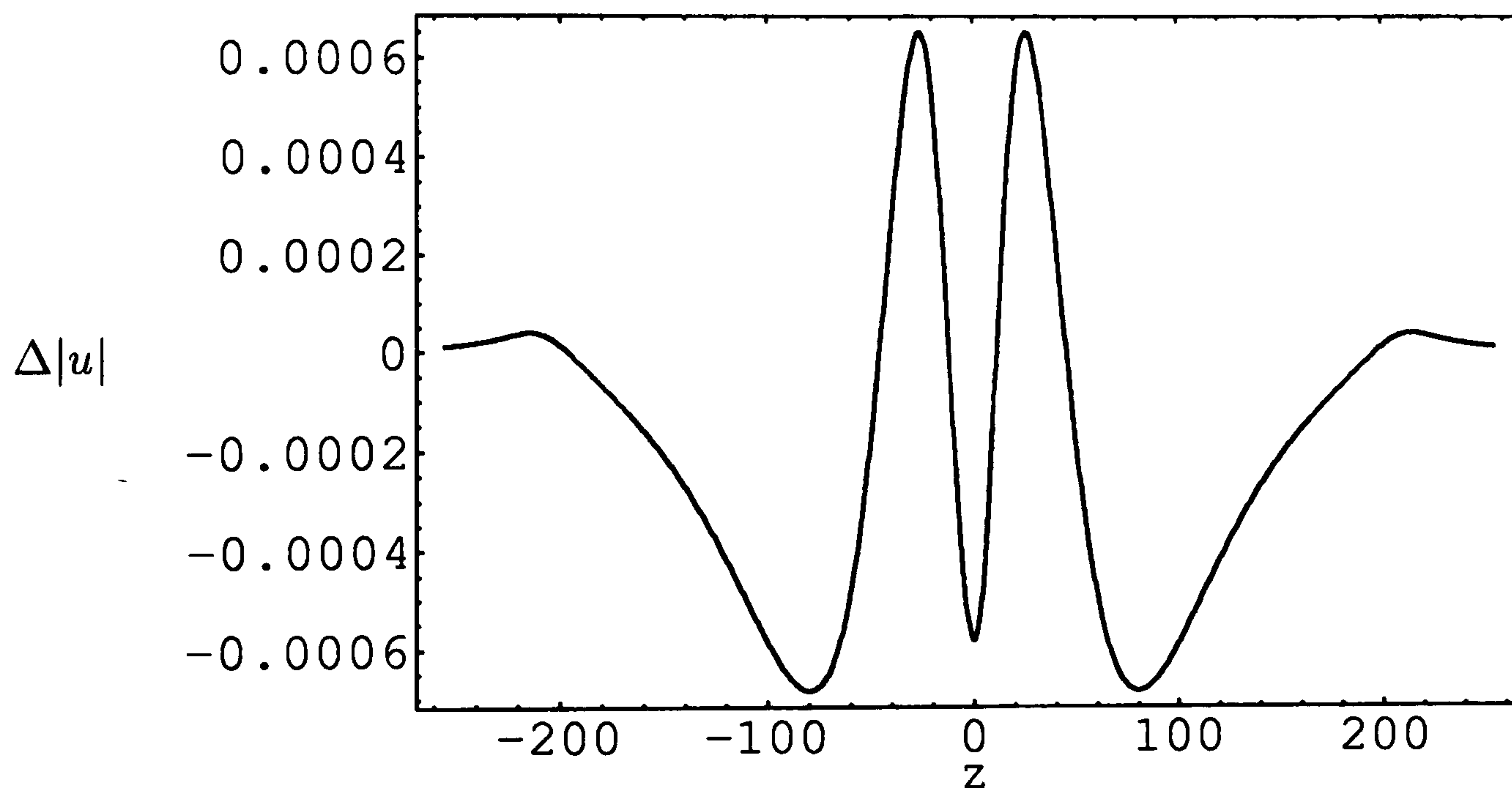


Fig. 6.15: Difference between the two profiles for $0.3 \operatorname{sech}(z)$ appearing in 6.14.

6.1.5 Dispersive radiation in the presence of fundamental soliton-I

The far field

From the IST analysis between case where no soliton and 1 soliton are present we have seen that the form of the dispersive radiation is similar but for the presence of

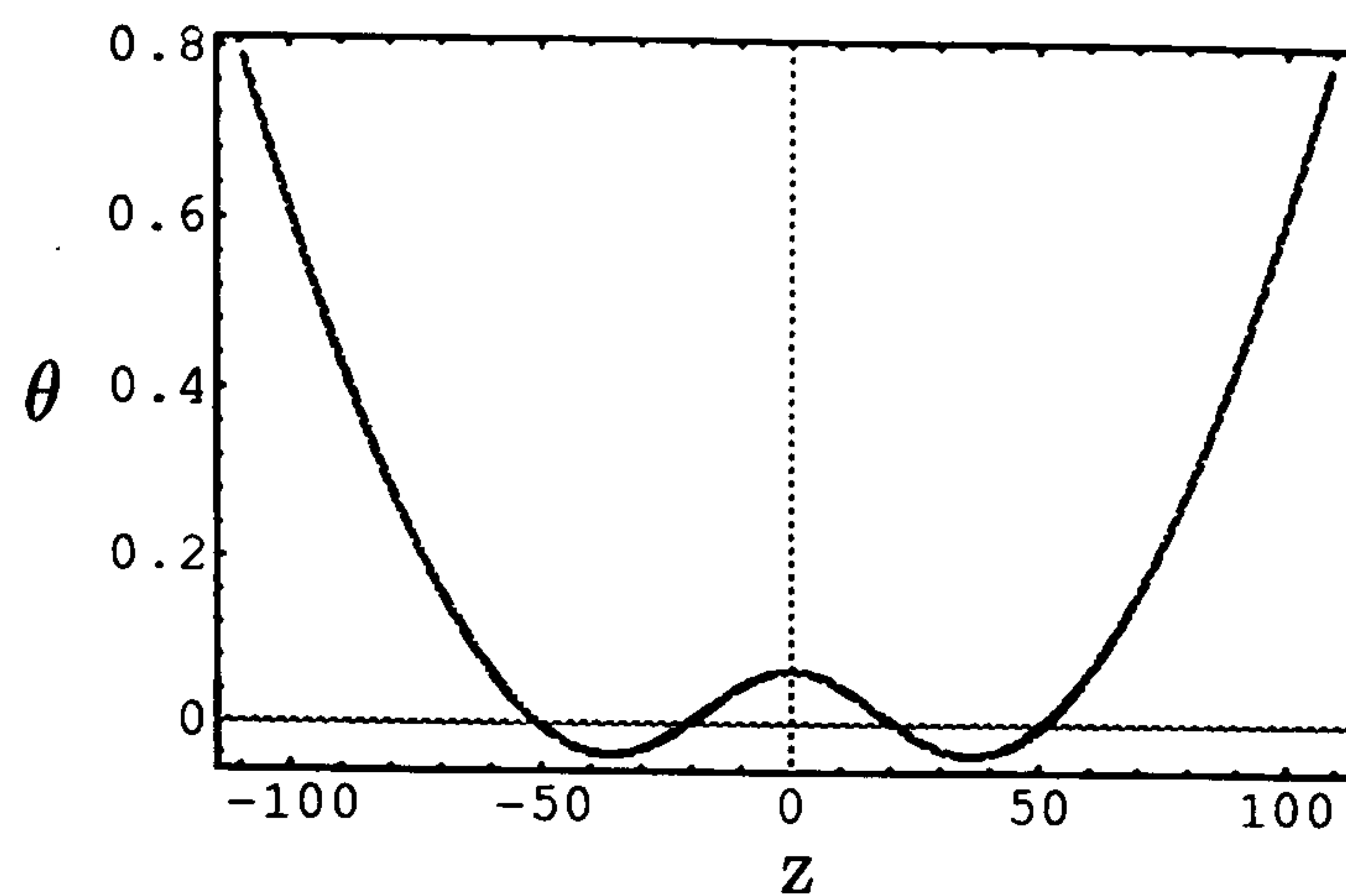


Fig. 6.16: Difference between the phases (in radians) of dispersive radiation with initial profile $0.3 \operatorname{sech}(z)$ at time 15π . Positive value indicates larger phase of the numerically calculated wave.

some spatial displacement which is related to the amount of dispersive radiation. The example that we have considered ($0.3 \operatorname{sech}(z)$) is connected to two initial conditions containing one soliton: $1.3 \operatorname{sech}(z)$ and $0.7 \operatorname{sech}(z)$. Since we have established a good description of the nonsolitonic initial condition with the use of the Lagrangian method we expect equally good results for the two initial conditions just laid. Instead of looking at the shape this time we will be considering the temporal evolution of the flows related to the first three integrals of motion. Unfortunately in order to be able to see the pure dispersive radiation content we have to move relatively far from the soliton centre. In our case the position $z = 25$ was selected. For the mass ($\int_{-\infty}^{\infty} |q|^2 dz$) the corresponding flow can be easily calculated from the NLS:

$$M = \frac{i}{2} (\bar{q} q_z - q \bar{q}_z)$$

The result of the comparison can be viewed in figure (fig. 6.17). The result is quite good and its quality improves when moving away from the transient regime. However the displacement of the evaluation point of the Lagrangian results towards the centre of the pulse by the quantity $(0.3 + 1/2)^{-1} = 1.25$ results in a clear improvement even for the transient (fig. 6.18, 6.19).

An analogous treatment is done for the momentum ($i \int_{-\infty}^{\infty} (\bar{q} q_z - q \bar{q}_z) dz$) where the flow is calculated to be:

$$P = -|q|^4 + q_z \bar{q}_z - \frac{1}{2} (q_{zz} \bar{q} + \bar{q}_{zz} q)$$

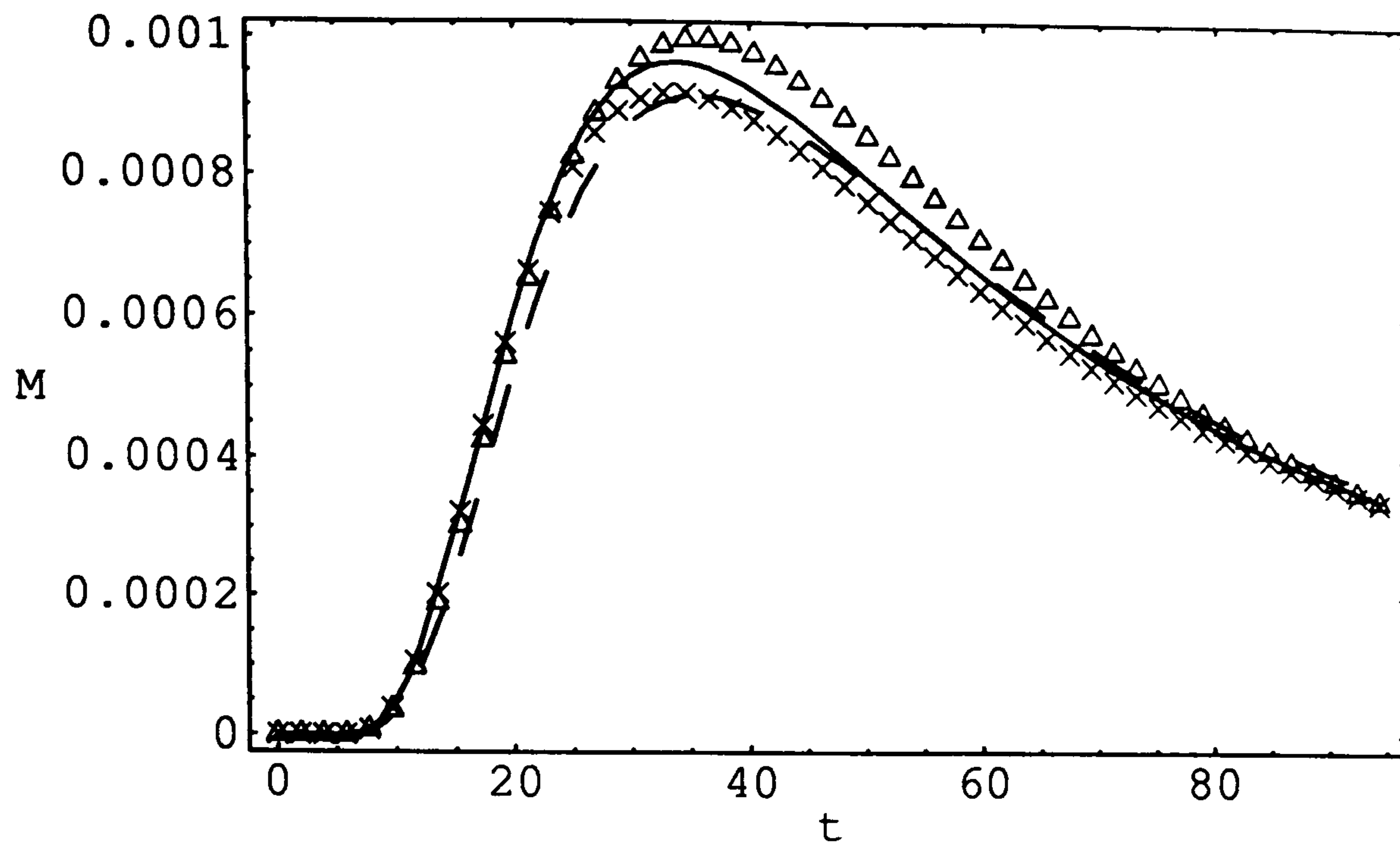


Fig. 6.17: Comparison between mass flows (M) at $z = 25$ for the numerically computed initial condition $0.7 \operatorname{sech}(z)$ (triangles), $1.3 \operatorname{sech}(z)$ (x's) the standard Lagrangian prediction (dashed line) and the corrected Lagrangian prediction (continuous line).

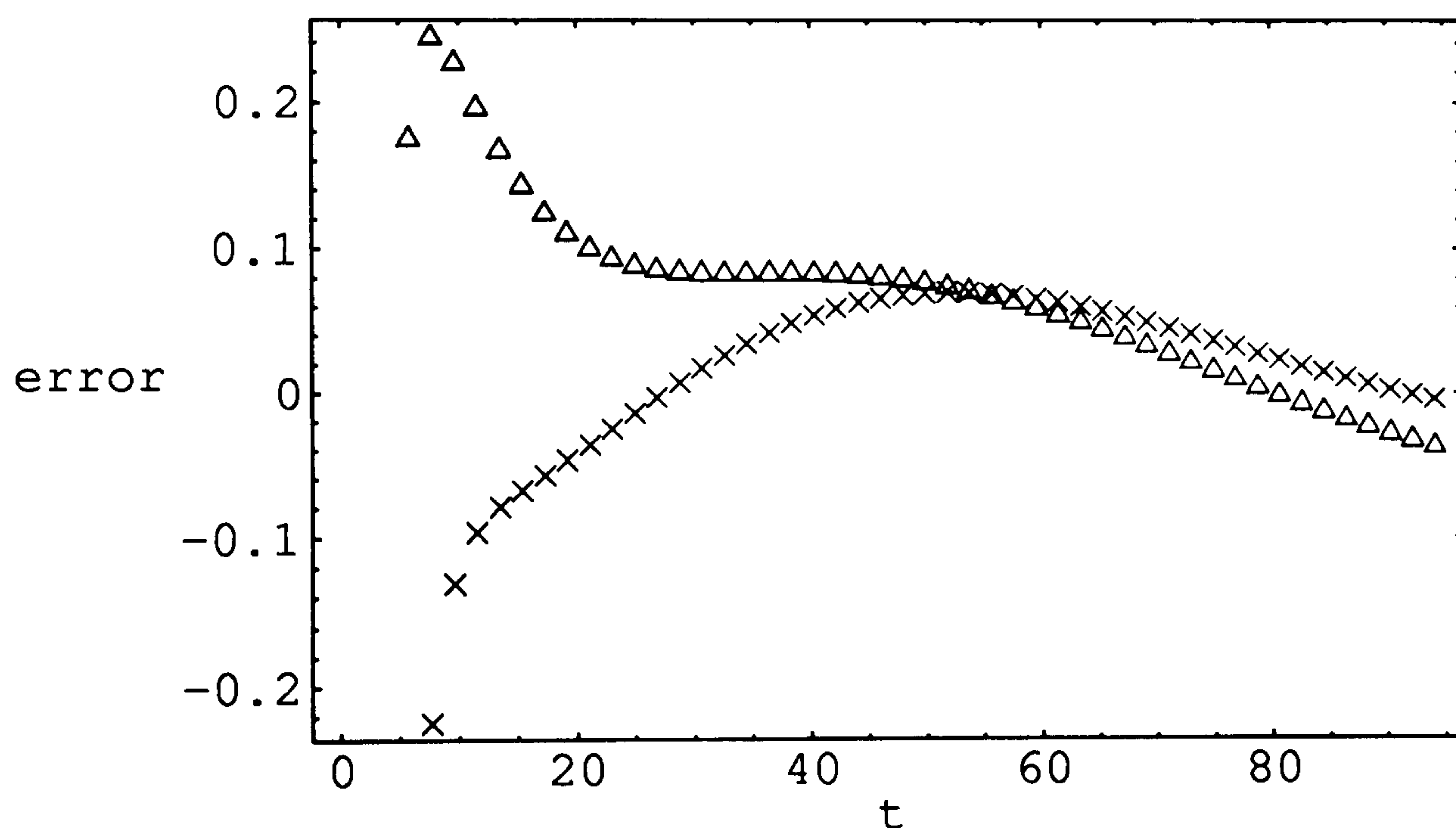


Fig. 6.18: Relative errors for the mass flow corresponding to the initial condition $0.7 \operatorname{sech}(z)$. The triangles correspond to the uncorrected Lagrangian and the x's to the corrected one.

and the comparison between Lagrangian and numerically calculated results can be viewed in figures (fig. 6.20, 6.21, 6.22)

Finally for the Energy ($\int_{-\infty}^{\infty} (q_z \bar{q}_z - |q|^4) dz$) the flow is given by the quantity:

$$E = i |q|^2 (q \bar{q}_z - \bar{q} q_z) + \frac{i}{2} (q_{zz} \bar{q}_z + \bar{q}_{zz} q_z)$$

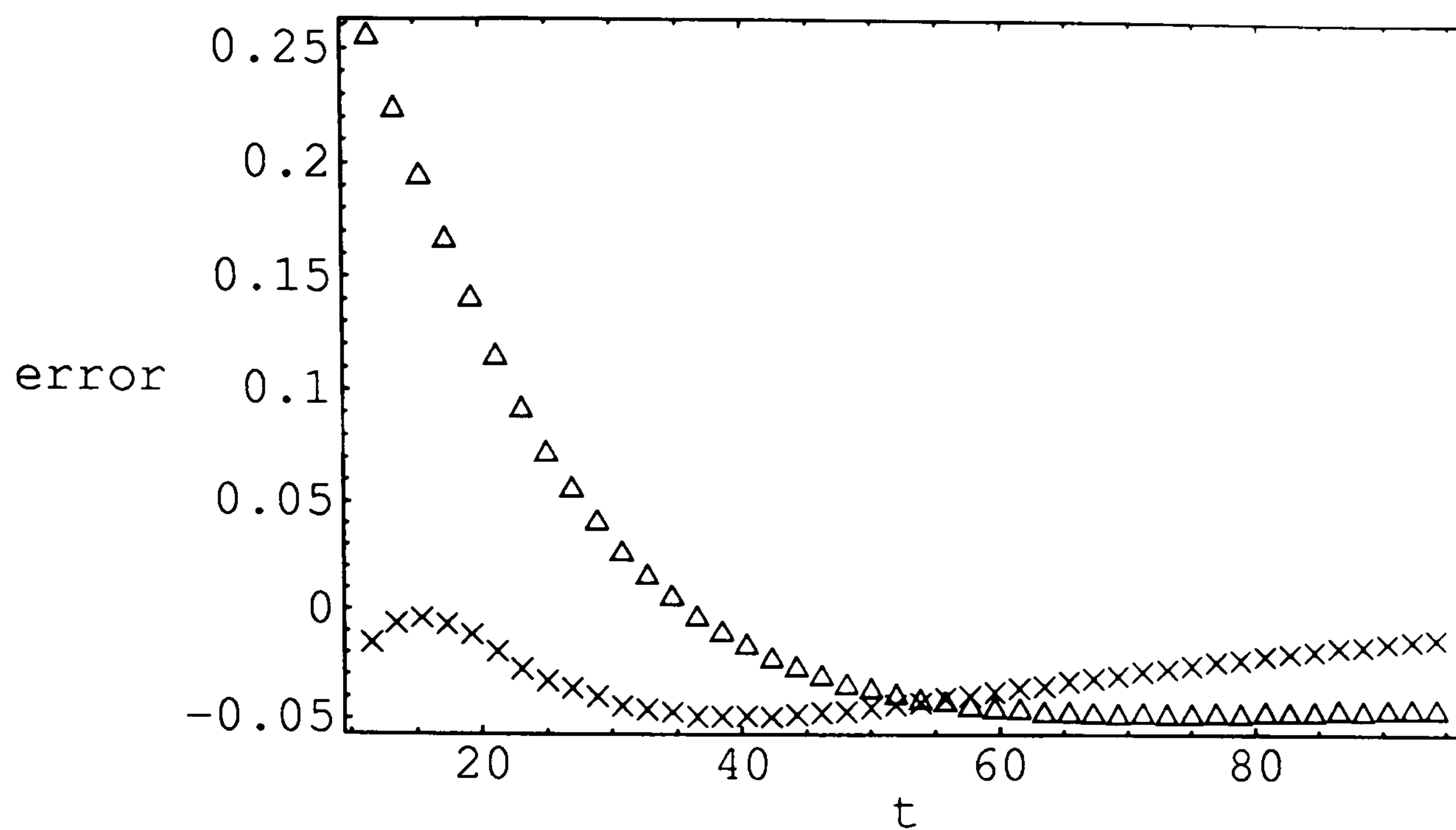


Fig. 6.19: Relative errors for the mass flow corresponding to the initial condition $1.3 \operatorname{sech}(z)$. The triangles correspond to the uncorrected Lagrangian and the x's to the corrected one.

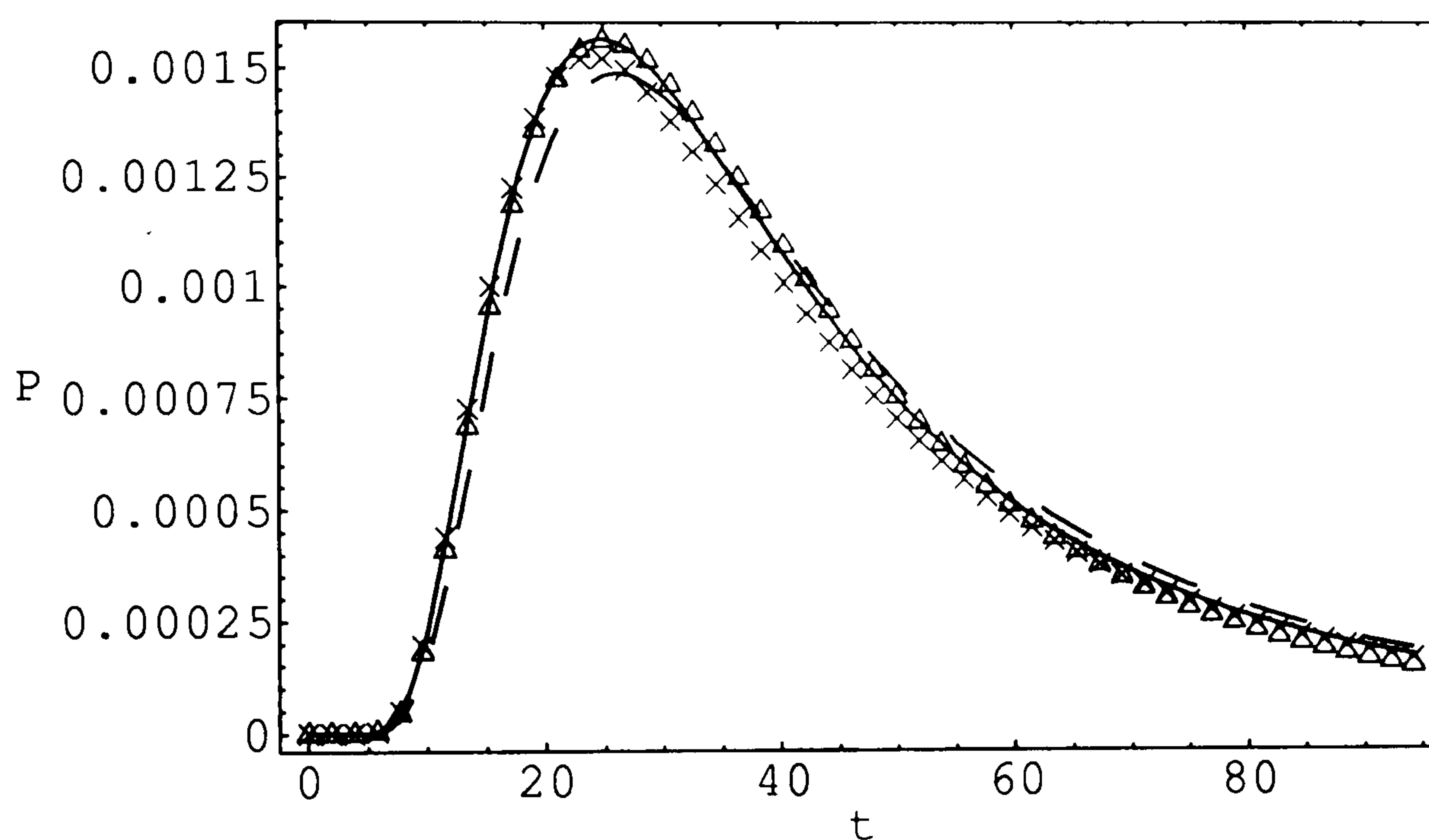


Fig. 6.20: Comparison between momentum flows (P) at $z = 25$ for the numerically computed initial condition $0.7 \operatorname{sech}(z)$ (triangles), $1.3 \operatorname{sech}(z)$ (x's) the standard Lagrangian prediction (dashed line) and the corrected Lagrangian prediction (continuous line).

and the results are displayed in figures (fig. 6.23, 6.24, 6.25).

In the previous paragraph we attempted, with the example taken, to assess the quality of the predictions given by the particular ansatz when applied to the problem of radiation due to initial conditions that are not pure soliton ones. We have seen that the agreement is good and the use of the displacement calculated by IST considerations

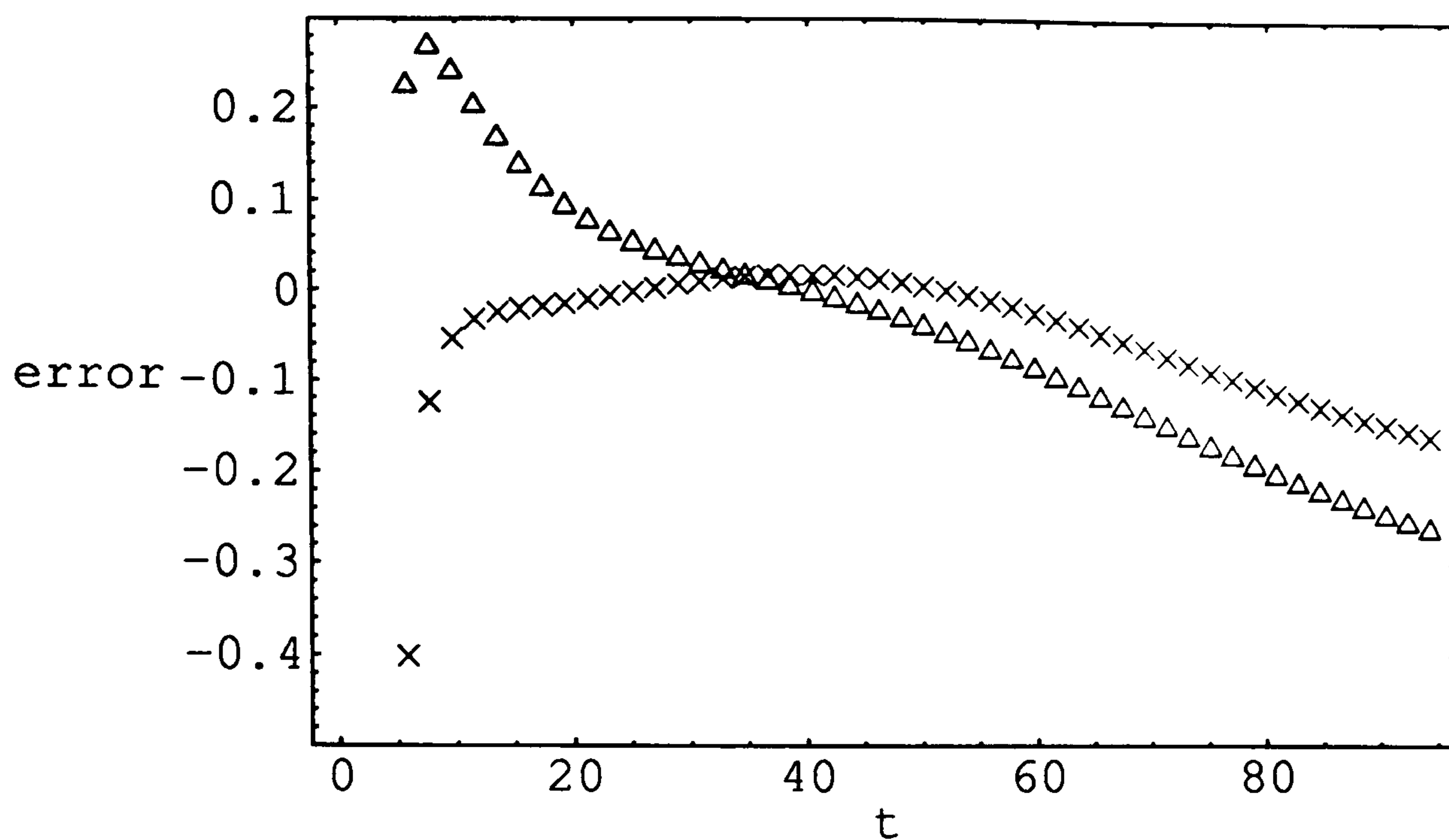


Fig. 6.21: Relative errors for the momentum flow corresponding to the initial condition $0.7 \operatorname{sech}(z)$. The triangles correspond to the uncorrected Lagrangian and the x's to the corrected one.

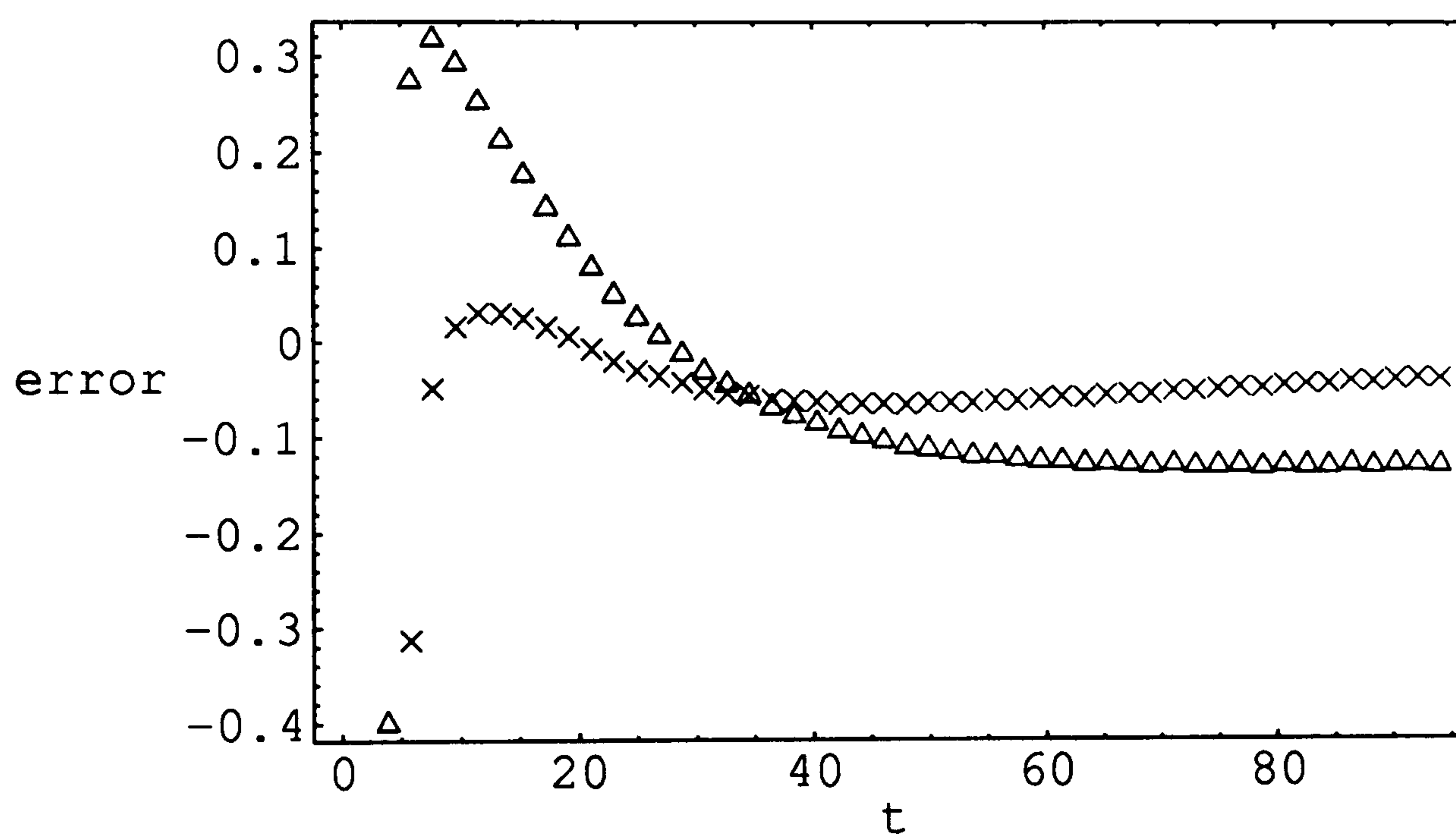


Fig. 6.22: Relative errors for the momentum flow corresponding to the initial condition $1.3 \operatorname{sech}(z)$. The triangles correspond to the uncorrected Lagrangian and the x's to the corrected one.

enhances the agreement. The choice of the particular method of comparison (flows related to integrals of motion) is due to the fact that these quantities are local and not defined with respect to the properties at the peak of the pulse as are quantities such as FWHM, amplitude e.t.c. In this sense they provide a good description away from the soliton location. The distance at which the quantities were sampled is relatively large

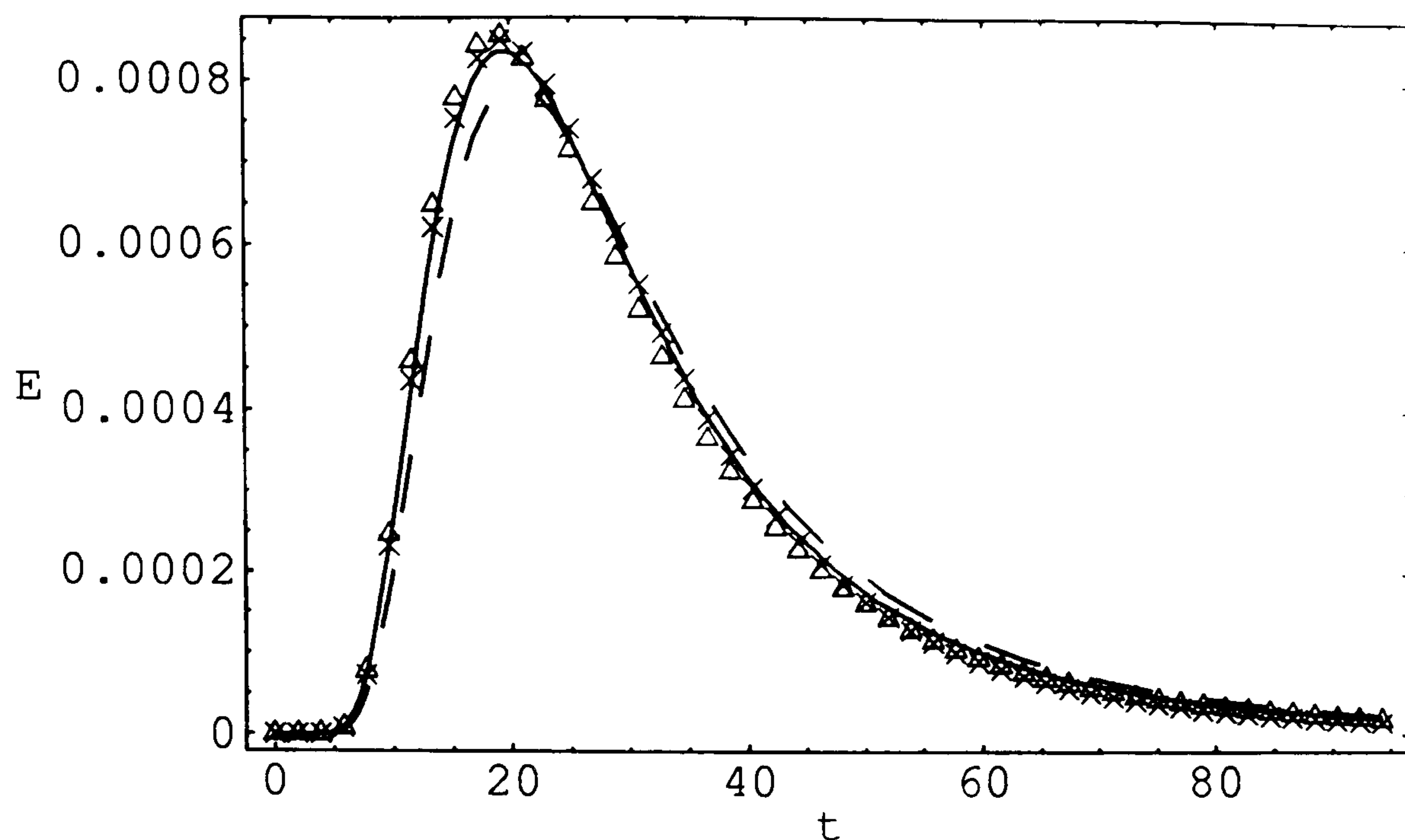


Fig. 6.23: Comparison between energy flows (E) at $z = 25$ for the numerically computed initial condition $0.7 \operatorname{sech}(z)$ (triangles), $1.3 \operatorname{sech}(z)$ (x's) the standard Lagrangian prediction (dashed line) and the corrected Lagrangian prediction (continuous line).

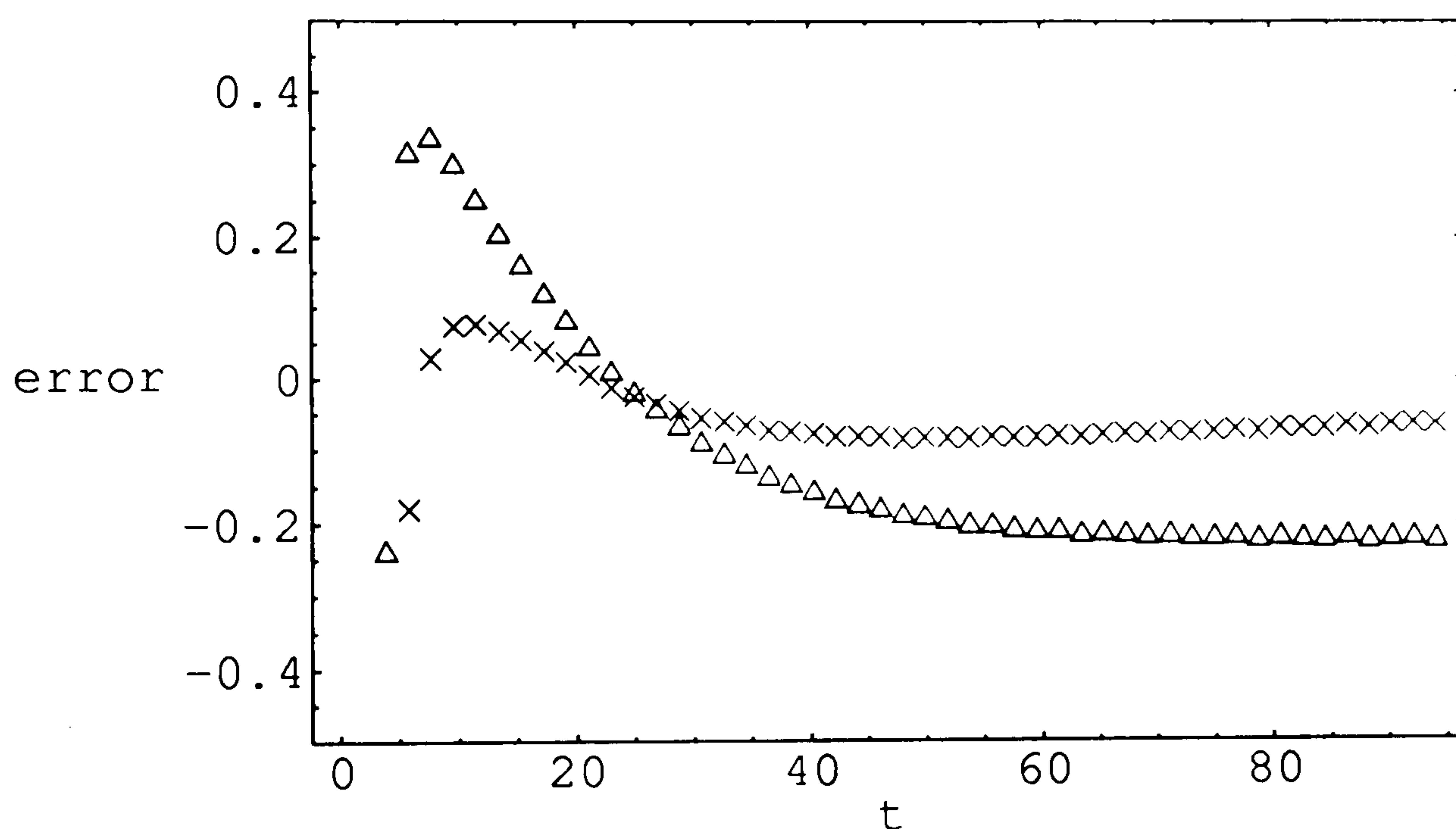


Fig. 6.24: Relative errors for the energy flow corresponding to the initial condition $0.7 \operatorname{sech}(z)$. The triangles correspond to the uncorrected Lagrangian and the x's to the corrected one.

compared to the FWHM of the emerging soliton (7.5 and 35 times larger than the FWHM point of the emerging solitons $0.4 \operatorname{sech}(0.4 z)$ and $1.4 \operatorname{sech}(1.4 z)$ respectively) It is recognised that these distances are abnormally large in real soliton communications situation when one expects soliton separations of the order of 6 times the FWHM. The choice was dictated by the need to isolate the dispersive tail effects from the soliton

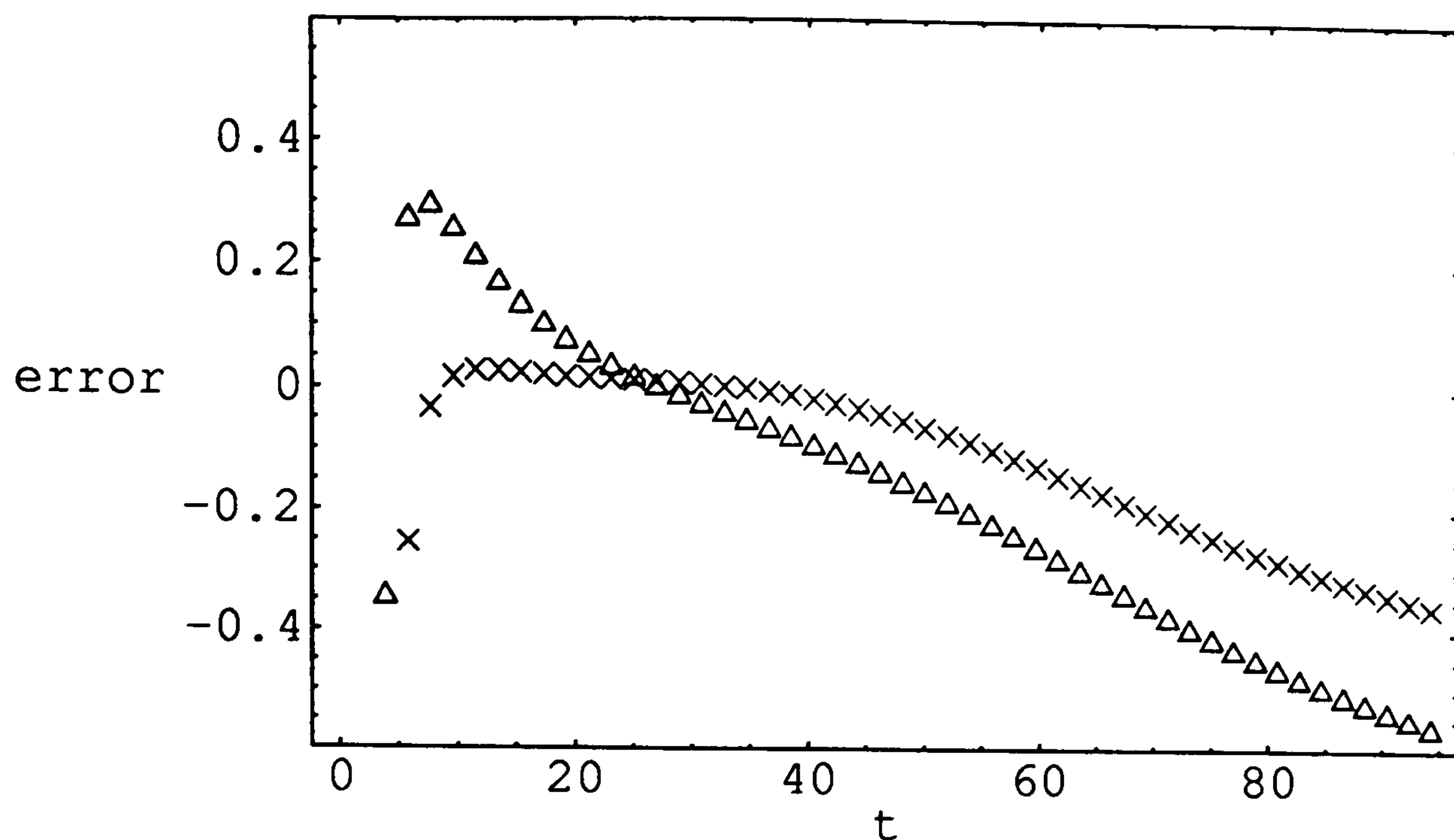


Fig. 6.25: Relative errors for the energy flow corresponding to the initial condition $1.3 \operatorname{sech}(z)$. The triangles correspond to the uncorrected Lagrangian and the x's to the corrected one.

ones as much as possible.

6.1.6 Dispersive radiation in the presence of fundamental soliton -II

Bäcklund transform

When referring to the core area we usually mean the soliton peak (assuming symmetric initial conditions). In order to see how the presence of the soliton affects the radiation at the peak position we will be using the Bäcklund transform for the NLS. The transform in question connects two solutions of the NLS [Che74], [DB77]. They consist of two parts: a temporal and a spatial. Once the spatial is known the temporal can be determined from the NLS itself.

In the case where a soliton ($B \operatorname{sech}(Bz)$) is added to a solution the spatial part of the Bäcklund transform can be derived from the following set of equations [Che74]:

$$2Bu(z, t) + \left(\overline{q(z, t)} \mp q'(z, t) \right) u^2(z, t) + (q(z, t) \mp q'(z, t)) = 0 \quad (6.21)$$

$$2u_z(z, t) = \left(\overline{q(z, t)} \pm q'(z, t) \right) u^2(z, t) + (q(z, t) \pm q'(z, t)) \quad (6.22)$$

where q and q' are solutions of the NLS and u is a functional of the Jost solutions of the associated scattering problem. Let us assume that q' is the continuum radiation part and that q is the dressed solution corresponding to radiation q' . Furthermore we assume that the continuum radiation is a mere perturbation to the soliton solution.

This allows us to perform an expansion of the form:

$$q(z, t) = q^{(0)}(z, t) + \epsilon q^{(1)}(z, t) + \dots \quad (6.23)$$

$$q'(z, t) = \epsilon q'^{(1)}(z, t) + \dots \quad (6.24)$$

$$u(z, t) = u^{(0)}(z, t) + \epsilon u^{(1)}(z, t) + \dots \quad (6.25)$$

where $q^{(0)} = B \operatorname{sech}(Bz) \exp(i B^2 t / 2)$ and substitute into the master equations (eq. 6.21, 6.22). Solution at zero order will yield the form of $u^{(0)} = e^{i B^2 t / 2 - Bz}$ which is then substituted into the first order equations:

$$4n u^{(1)} + 2 \overline{q^{(0)}} u^{(0)} u^{(1)} + (\overline{q^{(1)}} \mp q'^{(1)}) (u^{(0)})^2 + (q^{(1)} \mp q'^{(1)}) = 0 \quad (6.26)$$

$$2(u^{(1)})_z = 2 \overline{q^{(0)}} u^{(0)} u^{(1)} + (\overline{q^{(1)}} \pm q'^{(1)}) (u^{(0)})^2 + (q^{(1)} \pm q'^{(1)}) \quad (6.27)$$

When equation (6.26) is solved for $u^{(1)}$ we get the following solution:

$$u^{(1)} = - \frac{(\overline{q^{(1)}} \mp q'^{(1)}) e^{i B^2 t - 2 B z} + (q^{(1)} \mp q'^{(1)})}{2 B \tanh(Bz)} \quad (6.28)$$

We notice that the denominator becomes zero once z is set to zero. Thus the numerator of the right hand side of (eq. 6.28) becomes zero at the origin and thus $q^{(1)} = \pm q'^{(1)}$ for $z = 0$.

The second equation (6.27) will lead to a differential equation connecting the two expressions for the radiation. Namely we get the following ordinary differential equation:

$$\cosh(Bz)^2 \frac{d}{dz} \left(e^{Bz} \operatorname{sech}(Bz) \operatorname{cosech}(Bz) f(q^{(1)}, \overline{q^{(1)}}) \right) + \frac{d}{dz} \left(e^{Bz} \coth(Bz) f(q'^{(1)}, \overline{q'^{(1)}}) \right) = 0 \quad (6.29)$$

with $f(a, b) = b(u^{(0)})^2 + a$. This is a first order ordinary differential equation and is solved for the boundary condition $q^{(1)} = \pm q'^{(1)}$ for $z = 0$ derived above.

The evolution equations for the two radiation functions can be directly derived from the NLS by substitution of the expansions adopted above. Thus for $q'^{(1)}$ we have:

$$\frac{\partial}{\partial t} q'^{(1)} = i \frac{1}{2} \frac{\partial^2}{\partial z^2} q'^{(1)} \quad (6.30)$$

and for $q^{(1)}$:

$$\frac{\partial}{\partial t} q^{(1)} = i \frac{1}{2} \frac{\partial^2}{\partial z^2} q^{(1)} + 2i |q^{(0)}|^2 q^{(1)} + i (q^{(0)})^2 \overline{q^{(1)}} \quad (6.31)$$

The resemblance of the last two with the equations derived in [Gor92] and [Elg93] may lead to the conclusion that we are looking at the same transform. However, while the equation for $q'^{(1)}$ is exact there, here it is only a first approximation. In fact in the next section we will be using the NLS as the evolution equation for $q'(1)$ rather than the linearised version (eq. 6.30). With these remarks in mind let us proceed to the modelling of the peak behaviour.

Core area

In order to acquire a better picture of the soliton-radiation interaction in the particular area we ran a numerical simulation of the two initial conditions mentioned in the previous subsection ($0.7 \operatorname{sech}(z)$ and $1.3 \operatorname{sech}(z)$). To isolate the radiation, the soliton content of the solution was calculated from IST theory and subtracted from the results of the simulation. The remaining dispersive radiation, was compared against the picture just described. The quasiperiodicity of the modulation in the central area of the pulse was observed indeed. The depth of the modulation was of the order of magnitude of the amplitude at the centre as extrapolated by the far field area.

Furthermore there are instances where the depth of the modulation becomes very small and the envelope of the dispersive radiation resembles that in the solitonless case. The oscillatory nature of the interaction, with frequency depending on the emerging soliton, indicates that the phase of the radiative part (at the centre) is a slowly varying temporal function, exactly as in the solitonless case. Furthermore the fact that the modulation is of the same order of magnitude as the extrapolated amplitude together with the $t^{-1/2}$ evolution of the latter points to the deduction that after some sufficient time the dispersive radiation can be treated as a perturbation on the emerging soliton.

As far as the soliton peak is concerned we will rely on the condition derived for the existence of the Bäcklund transform, namely that at that point the radiation is the same regardless of the presence of a soliton. We will make a comparison of the numerical data, at the peak position, and the results arising from the straightforward superposition of the asymptotically emerging soliton and the dispersive radiation, as described by the ansatz (6.14). The time dependent parameters of this pulse are assumed to vary in the same fashion as they would were the soliton not present. The quantities compared will be the peak amplitude and phase. The initial conditions for the radiation are fixed so that the superposition of dispersive radiation and emerging soliton coincides with the parameters of the initial pulse, when z is set equal to zero. The governing equation for the solitonless radiation is the NLS. Adoption of the linear dispersion relation is also possible (no self-phase modulation) but would lead to a divergence of the phase by $\pi/2$.

The results of these comparisons are summarised in graphs (fig. 6.26 - 6.35) It can be seen that the agreement becomes progressively better after the first few periods of the observed oscillation. This can be better observed in the the case where the initial amplitude approaches soliton generation threshold. We notice that the curves describing central amplitude exhibit considerable mismatch at the beginning but improve at large times. As the dispersive radiation part is reduced (cases $0.8 \operatorname{sech}(z)$ and

$1.2 \operatorname{sech}(z)$) then the agreement between exact and approximate description becomes better.

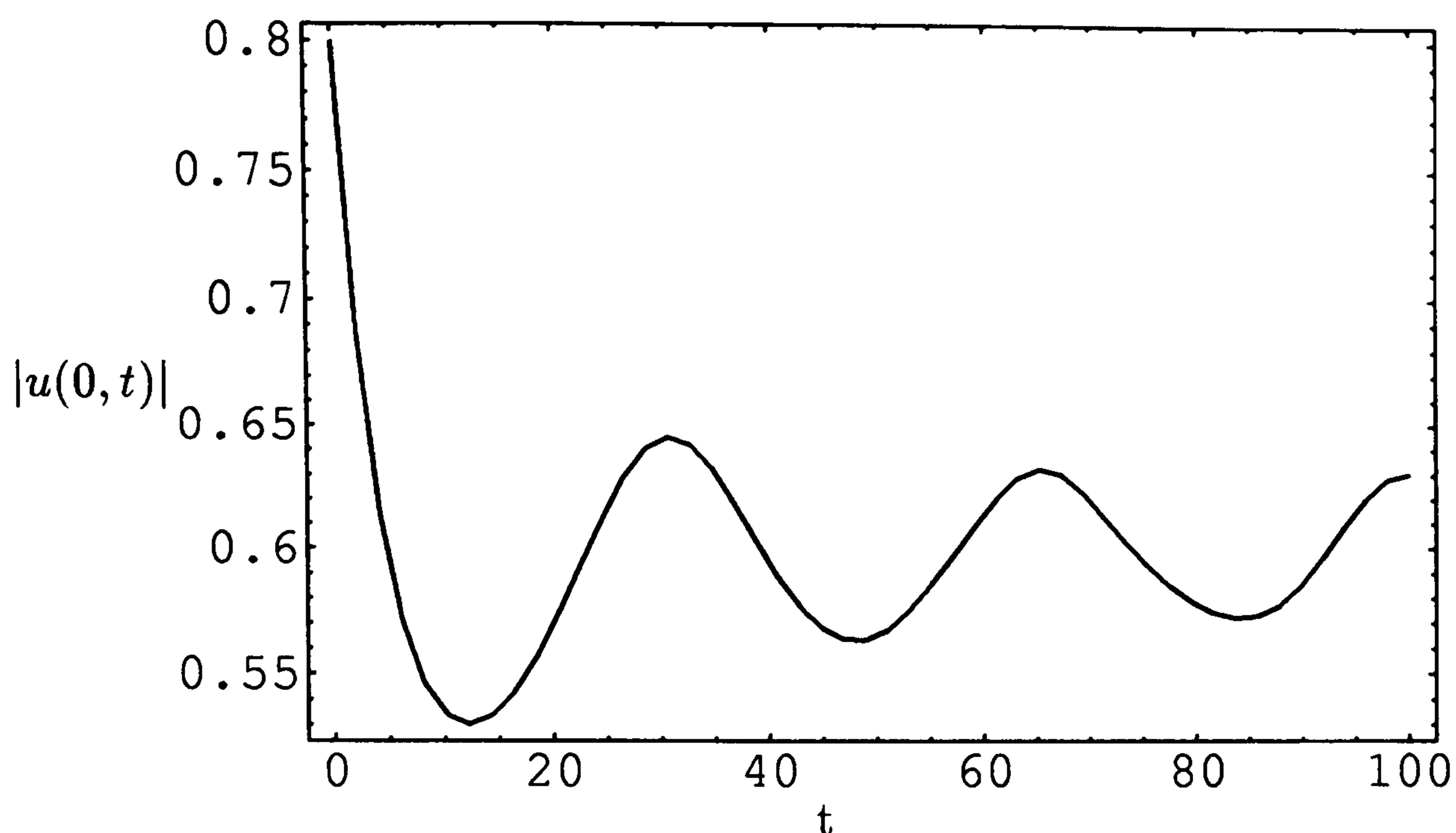


Fig. 6.26: Peak amplitude of the radiation corresponding to the initial profile $0.8 \operatorname{sech}(z)$. Numerical simulation

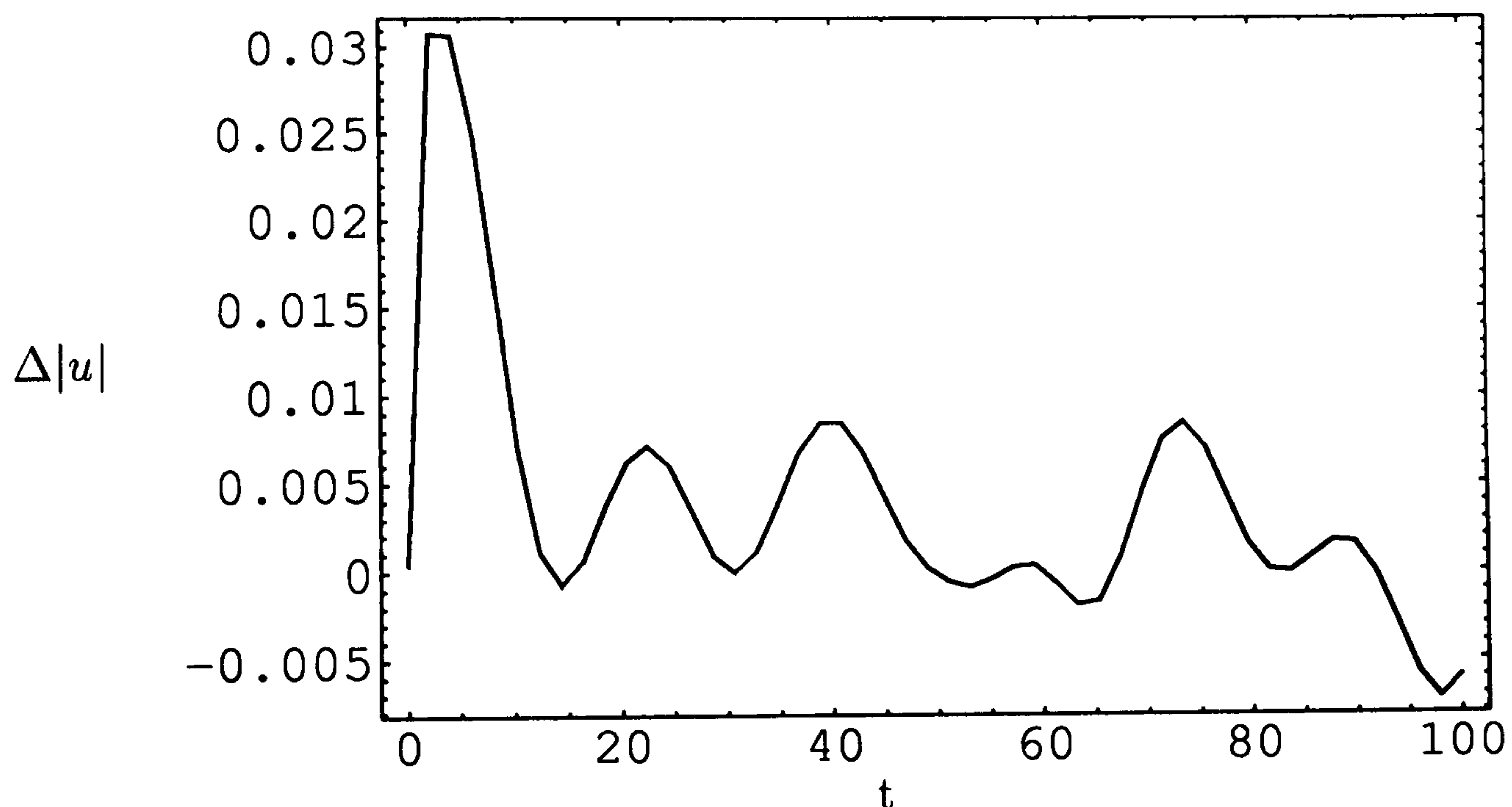


Fig. 6.27: Difference between the absolute peak value of the numerically propagated $0.8 \operatorname{sech}(z)$ initial condition and that predicted by the Lagrangian prediction ($z=0$)

Before concluding this section we add a note: We have used a result arising from a multiple scales expansion. It would seem that this result would hold only for situations where the radiation content is very small. However we have witnessed it holding in the

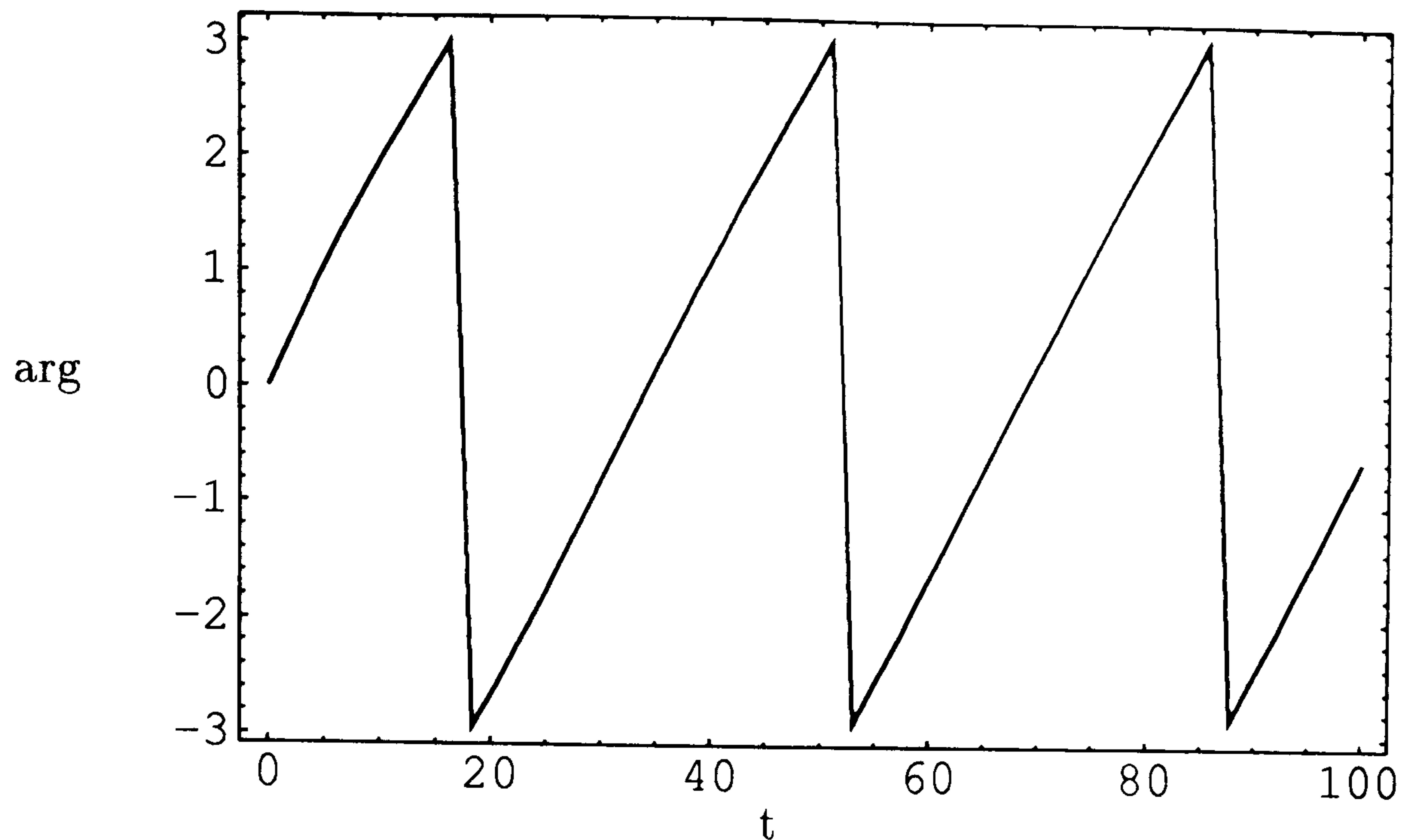


Fig. 6.28: Peak phase of the radiation corresponding to the initial profile $0.8 \operatorname{sech}(z)$. Numerical simulation

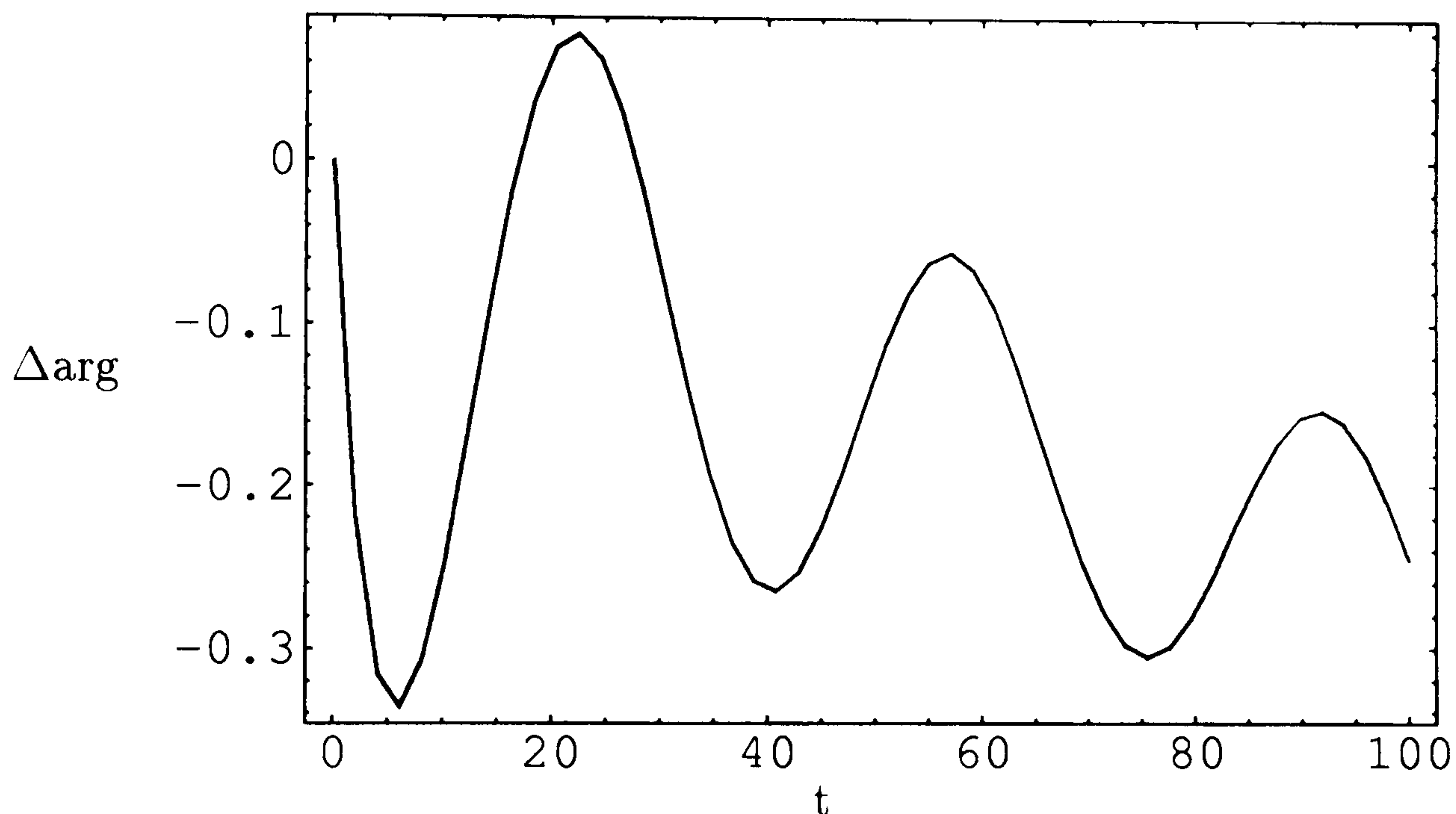


Fig. 6.29: Difference between the phase of the numerically propagated $0.8 \operatorname{sech}(z)$ initial condition and that predicted by the Lagrangian prediction, at the soliton peak.

case where the initial amplitude of the $A \operatorname{sech}$ pulse is 0.65. This is justified considering that after some time the percentage of dispersive radiation in the core area diminishes enough for it to be considered as a perturbation. This validates the multiple scales expansion at those times and explains the improvement of the agreement as we propagate in time. In short the description put forward seems to provide a good description in the general case. However this description is pertinent to this set of data only. For example, the same approach is not going to apply in the case of radiation impacting

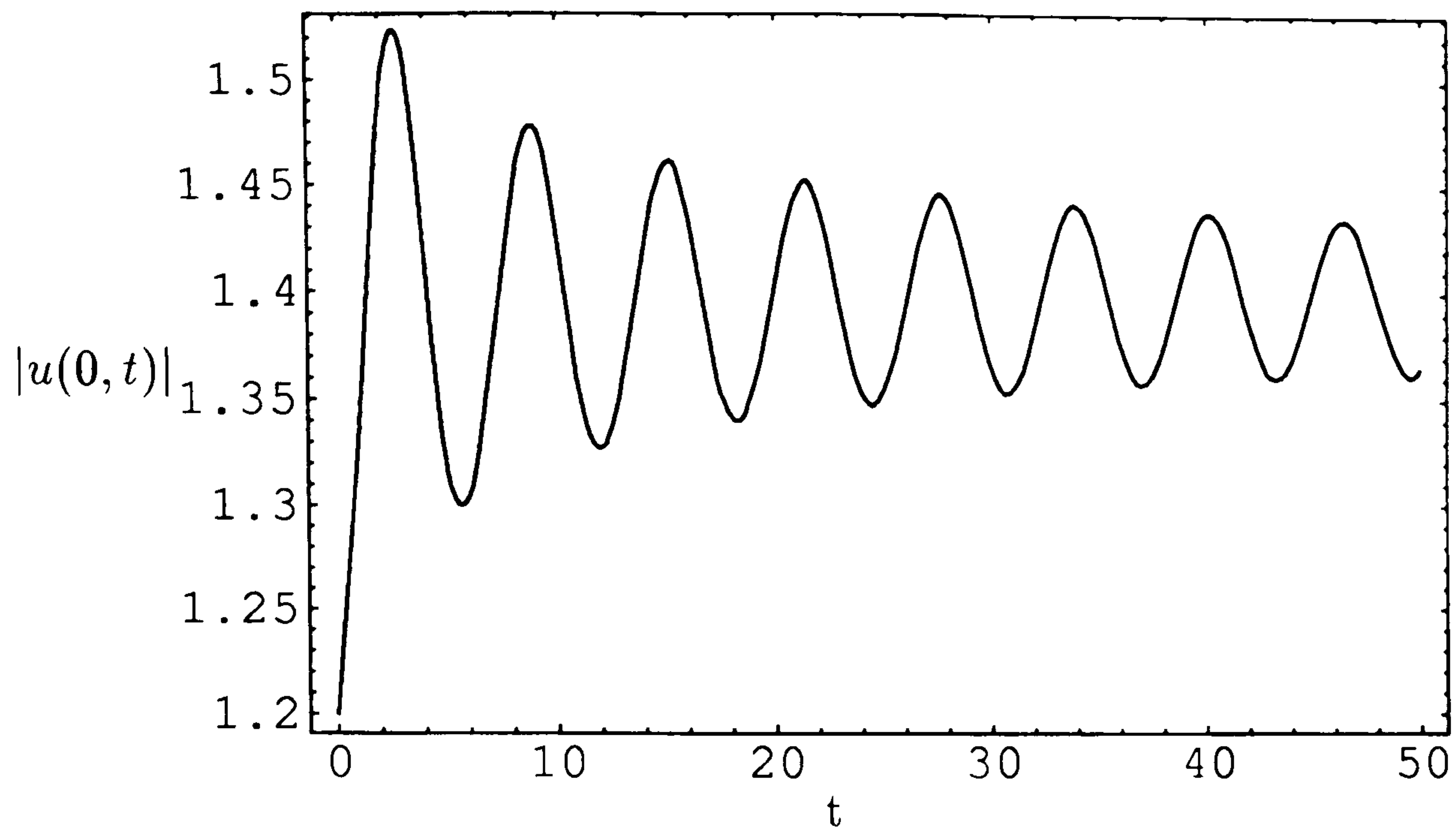


Fig. 6.30: Peak amplitude of the radiation corresponding to the initial profile $1.2\text{sech}(z)$.
Numerical simulation

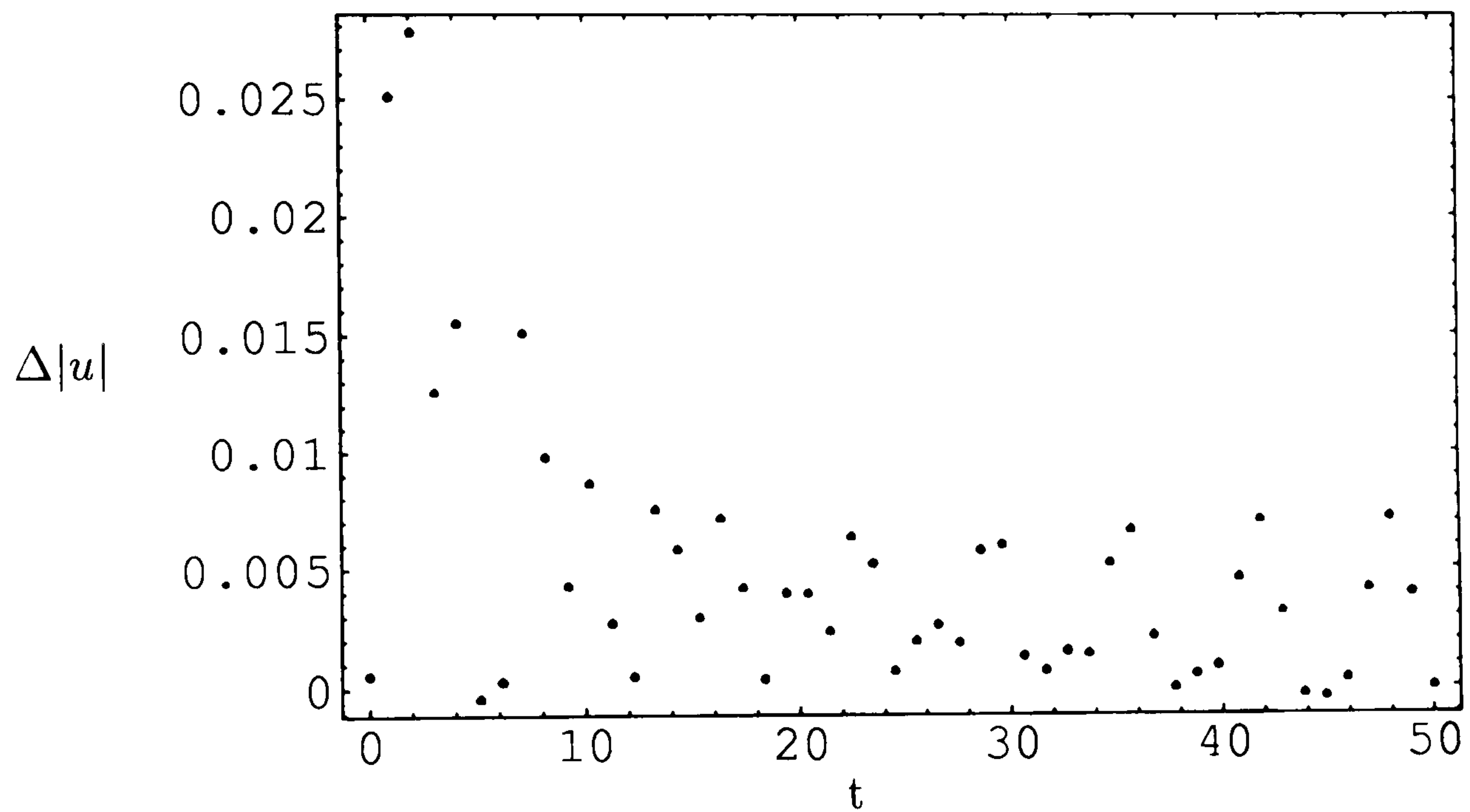


Fig. 6.31: Difference between the absolute peak value of the numerically propagated $1.2\text{sech}(z)$ initial condition and that given by the Lagrangian prediction ($z=0$)

on a soliton from one side, since displacement of the centre occurs [KMS95] and we have no means of calculating it in a simple fashion.

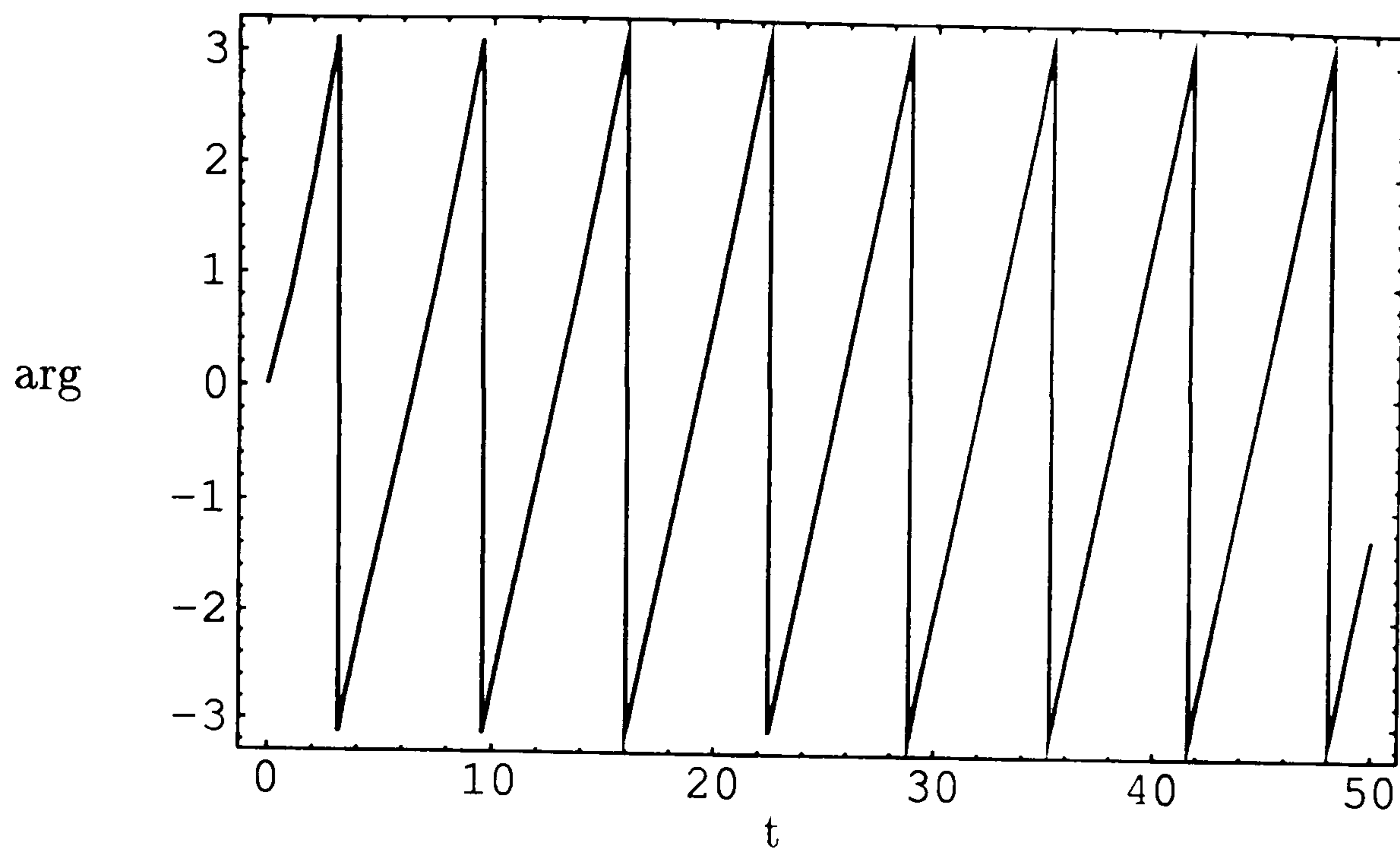


Fig. 6.32: Peak phase of the radiation corresponding to the initial profile $1.2 \operatorname{sech}(z)$. Numerical simulation

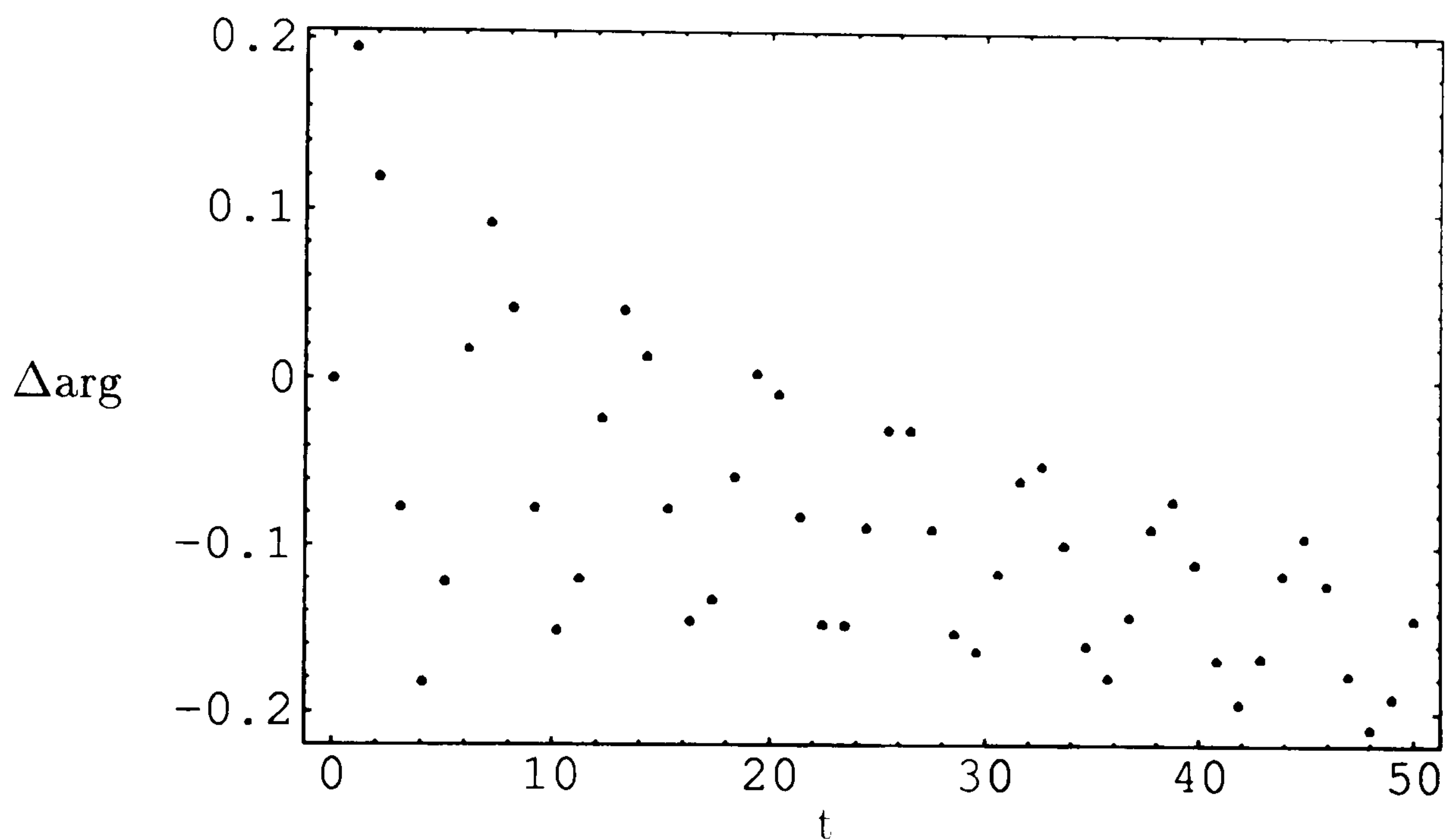


Fig. 6.33: Difference between the phase of the numerically propagated $1.2 \operatorname{sech}(z)$ initial condition and that given by the Lagrangian prediction, at the soliton peak.

6.2 Discussion

We have attempted, in this section, to provide a description of the evolution under initial condition that does not correspond to a pure soliton. The line of thought followed can be summarised as follows.

The concept of independence of propagation between soliton and dispersive radiation is borrowed from IST theory. Using this, in conjunction with the exact properties

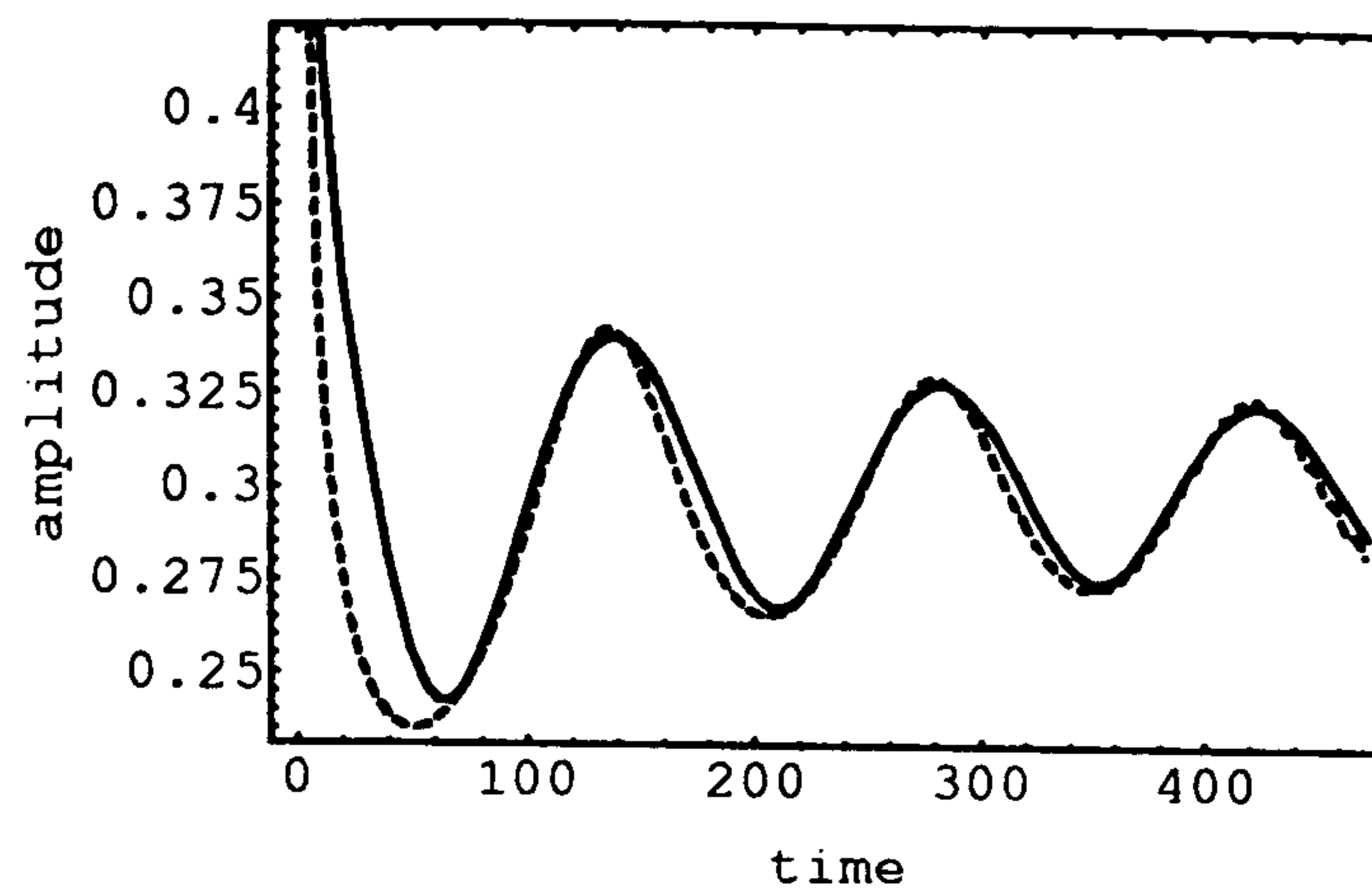


Fig. 6.34: Peak amplitude of the radiation corresponding to the initial profile $0.65 \operatorname{sech}(z)$. Dashed curve arises from numerical simulation. The oscillations at large time are due to reflections from the boundaries.

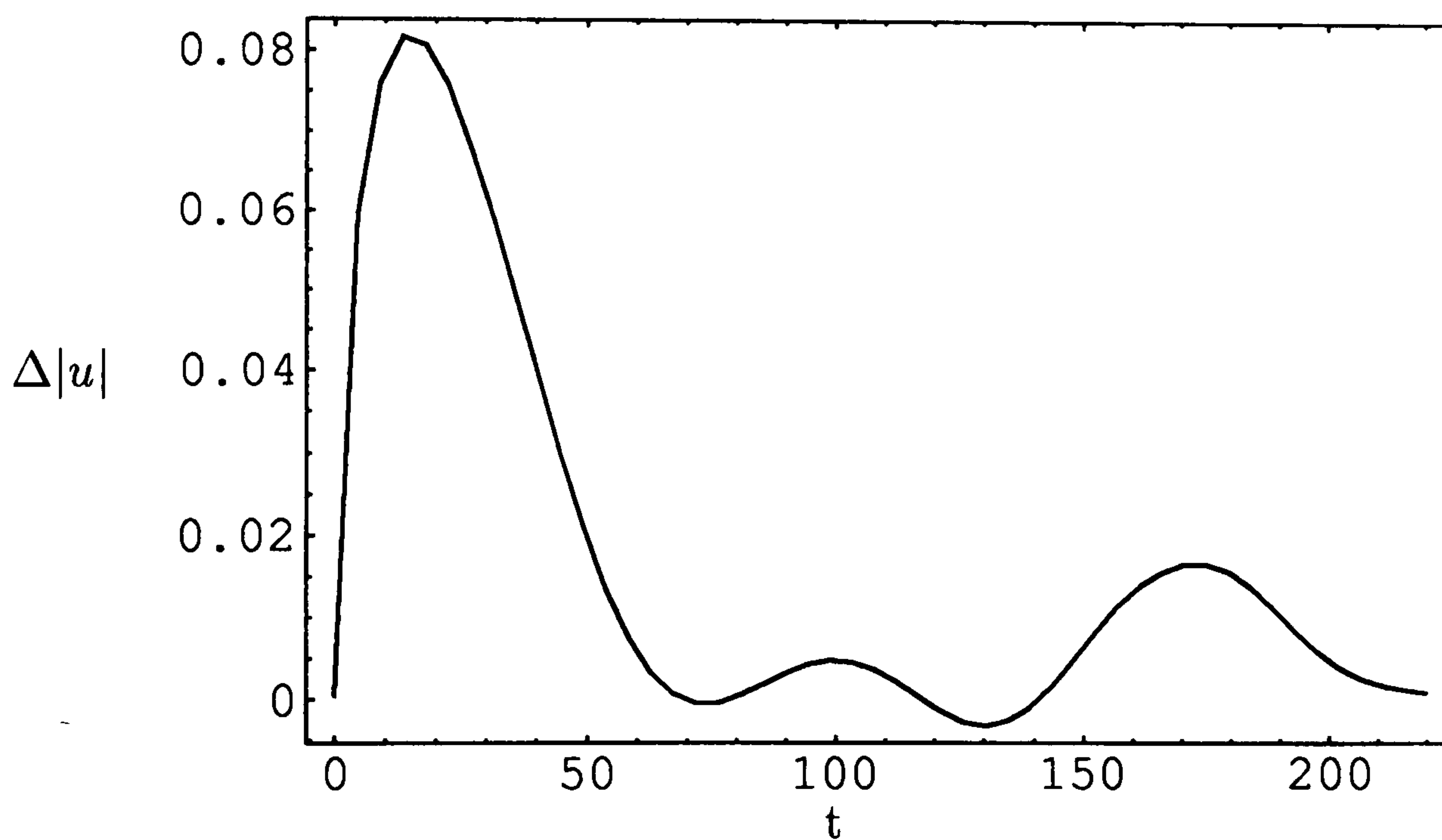


Fig. 6.35: Difference between the absolute peak value of the numerically propagated $0.65 \operatorname{sech}(z)$ initial condition and that given by the Lagrangian prediction ($z=0$).

of the IST spectrum related to the $A \operatorname{sech}(z)$ initial profile, a perturbation expansion is introduced and applied to zero order, in order to calculate the soliton content.

Subsequently, instead of looking at the central area of the pulse we attempt to model the dispersive radiation field, strictly for the case mentioned above. The quality of the modelling is evaluated using the fluxes associated with the integrals of motion.

Finally, the Bäcklund transform of the NLS is introduced in order to connect the

dispersive radiation fields in the presence and absence of the fundamental soliton. The condition for the existence of the transform at the central point indicates that the two fields are equal at that point. An alternative approach, would have been to use a set of relations proposed by Gordon [Gor92], in modelling the radiative field for initial profiles slightly different from the pure soliton [CPI⁺93].

In all our discussion there is one weak point: the absence of a rigorous theoretical justification for the calculation involved in the derivation of the emerging soliton. A zero order approach was adopted, instead of a first order one, which would be expected to yield better results. The reason for that is the following. We need to define similarity between a family of arbitrary initial profile (sufficiently smooth) and a family of sech pulses. Unfortunately similarity would mean that they should have the same emerging soliton, which is our initial problem anyway. It is not clear how this can be achieved without relying heavily on IST.

On the other hand we have provided an alternative approach of the problem of prediction of soliton content and based on that supplied a description of the propagation of the $A \operatorname{sech}(z)$ pulse with results closely matched to the numerically evaluated ones. The extension of this analysis in the case of initial pulse shapes different to the one examined however calls for the introduction of the perturbation theory described above.

7. CONCLUSIONS

7.1 *General discussion*

Throughout the previous chapters we have tried to illuminate some aspects of the perturbative treatment of soliton propagation. We have briefly reviewed the most commonly used perturbation methods. The majority of them deals with the propagation under a perturbed NLS regime of a pulse whose initial shape is a perfect soliton and the target is to describe the characteristics of the asymptotically emerging pulse, if there is one. Very little has been done in the direction of the description of soliton and background radiation interaction. The main reason for this is the difficulty encountered in the description of the propagation of a continuum in the presence of a soliton. The degrees of freedom involved are infinite and it is not an easy task determining which are the important ones. It is for this reason that treatment of radiation can be found in very few references. Even fewer deal with the problem of initial conditions containing soliton and radiation. The treatment of such cases relies heavily on numerical calculations. Although the latter, mainly because of their accuracy are precious for engineering work, they do not provide insight into the physics of the processes involved. It is for this reason, the promotion of the understanding of nonlinear waveguiding, that analytical descriptions of radiation -soliton interactions is needed.

The major obstacle, as explained above, is the number of degrees of freedom that dispersive radiation has. In order to produce reasonably simple descriptions of the dispersive radiation field these have to be reduced. A perturbative method for treating systems with limited numbers of degrees of freedom is the Lagrangian perturbation method outlined in chapter 5. The most crucial problem is the selection of the appropriate degrees of freedom. After this is done a problem arises in the handling of the integrals associated with the evaluation of the reduced Lagrangian. Of course one needs to formulate the Lagrangian density for the problem handled. In that chapter we have seen that in order to produce the Euler equations describing the evolution of the reduced set of variables this is not really needed and the perturbed equation is enough. Furthermore we have seen that Lagrangian method does not constitute a separate perturbation method. It can be adjusted in a way reproducing results ex-

tracted from other perturbation methods. This is assured by the proper selection of the trial functions. Thus adiabatic propagation equations are exactly derived from the Lagrangian perturbation method subject to choosing the fundamental soliton as a trial function. Furthermore it is possible to increase the degrees of freedom thus reproducing the results of direct perturbation theory [OP71]. Although the number of computations increases application is straightforward and the method degenerates into a classical Rayleigh-Ritz method, since the problem is linear at higher orders. In order to look more into the problem of selection of degrees of freedom we have attempted to look into how observables (mass , momentum, etc) can be incorporated into this formalism. The result there was a restatement of the Lagrangian into a Hamiltonian method where the degrees of freedom were observables and associated transforms. Unfortunately we were unable to provide a action-angle formalism similar to the one appearing in the Hamiltonian formulation of the NLS problem.

In the sixth chapter we tried to tackle the non soliton initial value problem for the NLS. The integrability of the equation allows us to consider the soliton and the dispersive radiation as separate entities. The first problem thus is to determine what the soliton content of an initial pulse is. To this end the first and the third integrals of motion were used. The calculation involved was simple and did not involve any numerics. Its justification was based on the independent contributions of the soliton and radiation part to the integrals of motion of the NLS. The agreement with numerical and IST results was checked and found to be good provided that the functional form of the initial condition was within perturbation limits from the $A \operatorname{sech}(z)$ initial condition. No assumptions were made regarding the type of perturbation. When the departure from the sech shape becomes considerable the formulas give increasingly erroneous results. By comparison the asymptotic state can be determined either using WKB method or fully numerical ones. In any case, when it comes to slightly perturbed shapes, this method seems to provide good quality results without involving complex or time consuming calculations as the other methods do.

The next step is to describe the propagation of dispersive radiation in the presence of the soliton. The analysis was confined to the $A \operatorname{sech}(z)$, since the approach followed could be rigorously cross-checked with IST methods. To solve the problem, the emerging soliton corresponding to the initial condition was calculated. The scattering parameters connected with the radiation in the presence of soliton were found to be connected to solitonless radiation parameters in a very simple fashion. This allows us to propagate the solitonless radiation and then dress it with the asymptotically emerging soliton. The effects of the dressing on the far field is a mere spatial displacement

depending on the amplitude of the emerging soliton. The solitonless radiation was propagated using the Lagrangian method. Although the method does not give exact results it provides good agreement as long as the initial condition is away from the soliton formation threshold. This was due to the lack of flexibility from the ansatz chosen, leading to an underestimation of the nonlinearity. However, away from this region we were able to provide a good description of the evolution of the far field in the presence of a first order soliton. As far as the peak characteristics -amplitude and phase- are concerned we added the contributions of both parts to provide a picture that agrees with the numerically propagated exact solution. Overall the semi-analytical description provided was found to give good results for an acceptable range of initial conditions. Furthermore, the Bäcklund transform introduced provides a tool for full description of the dressed radiation step without having to resort to IST (although in this case the transform is derived from the AKNS problem) for computations.

7.2 Further work

Having briefly recapitulated the previous chapters we will conclude with some thoughts regarding future work that needs to be undertaken. The obvious thing to achieve is the generalisation of the results provided for the NLS in the case where a perturbation term is present. Since the whole mathematical structure associated with integrability is no longer present the first problem is to associate the perturbed problem with the appropriate unperturbed one. This, of course, has to be done on the basis of the asymptotically emerging solitons. It is possible that an answer on this problem can be achieved by the introduction of the appropriate alterations to the integrals of motion of the unperturbed equation.

A second problem that needs to be dealt with, is the provision of a full perturbation model based on the Bäcklund transform. Loosely speaking the perturbation procedure would be effectuated in three steps:

1. Stripping of the initial condition from the emerging soliton.
2. Independent propagation of the soliton and the dispersive radiation.
3. recombination of the two.

This procedure is not trivial since the propagation characteristics of the two components of the solution cannot be easily calculated.

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