Generalisations of the Almost Stability Theorem

by

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Abstract

This thesis is concerned with the actions of groups on trees and their corresponding decompositions. In particular, we generalise the Almost Stability Theorem of Dicks and Dunwoody [12] and an associated application of Kropholler [23] on when a group of finite cohomological dimension splits over a Poincaré duality subgroup.

In Chapter 1 we give a brief overview of this thesis, some historical background information and also mention some recent developments in this area.

Chapter 2 consists mostly of introductory material, covering group actions on trees, commensurability of groups and completions of certain spaces. The chapter concludes with a discussion of a certain completion introduced in [23] and when this has an underlying group structure.

We then introduce the Almost Stability Theorem in Chapter 3 mentioning some possible directions in which the result may be generalised, how these various conjectures are related and some preliminary results suggesting that such generalisations are plausible. We go on to state the most general version of the theorem currently obtained. The proof of this result, Theorem A, takes up the bulk of Chapter 4 which is based on the approach of the book by Dicks and Dunwoody [12]. In removing the finite edge stabiliser condition we place certain restrictions on the groups that are allowed.

Finally, in Chapter 5 we investigate Poincaré duality groups, the connection between outer derivations and almost equality classes and show how to use Theorem A to obtain a more general version of the results of Kropholler. This work culminates in the result that Theorem B is a corollary of Theorem A.

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Statement

This thesis is submitted in accordance with the regulations for the degree of Doctor of Philosophy at the University of Glasgow.

Chapter 1 and sections 2.1-2.3 cover background material and some basic well known results.

The results in later sections are the author's original work with the exception of those results which are explicitly referenced.

Contents

List of Figures

Chapter 1

Introduction

The main goal of this thesis is a generalisation of the Dicks-Dunwoody Almost Stability Theorem (Theorem III.8.5 of [12]). The Almost Stability Theorem asserts the existence of a G-tree with finite edge stabilisers when given a suitable class of functions on a G-set with finite stabilisers. We obtain an analogous result for the infinite stabiliser case. There are known counterexamples in the more general setting and thus further restrictions to the permitted stabiliser groups are necessary. We attempt to capture the key features of the class of finite groups that make such a construction possible. One important feature of finite groups is that whenever they act on a set they do so with finite orbits. In a similar fashion, for an arbitrary group, G, acting on a set we may still ensure that the orbits of this action are finite if the stabiliser groups are subgroups of finite index in G. For this reason we restrict to classes of subgroups that vary up to finite index, so called commensurability classes. We say that two subgroups H and K are commensurable if the intersection is of finite index in both groups. This is an equivalence relation on the collection of subgroups of G and we call an equivalence class a commensurability class. Since we wish to construct a tree together with an action of our group then we must surely require that the family of edge stabilisers be closed under conjugation and for this reason we introduce the commensurator, $\mathrm{Comm}_G(H)$, of H in G. This is the largest subgroup of G in which H is commensurable with all of its conjugates. The notion of commensurability is discussed in greater detail in section 2.7.

A detailed examination of the techniques employed by Dicks and Dunwoody reveal subtle difficulties if no further restrictions are placed on our chosen commensurability class. One issue already identified in later work of Dicks and Dunwoody is that we must try to avoid subgroups which contain a conjugate of themselves as a proper subgroup. In passing from the case that our group G is finitely generated to the countable case we use the fact that our stabiliser groups are all finitely generated. We do not know at this time whether this condition is necessary to prove our result even in the case that G is finitely generated. The final ingredient for the construction is to have an appropriate notion of size. In the finite case the order of a subgroup gives a measure of size. In the more general setting we use a G-map from our family of subgroups to the integers preserving the partial ordering structure in order to have a measure of size. The G-map structure arises from the observation that G acts on our commensurability class of subgroups by conjugation. In fact in our work the function we choose, which is applicable for a large class of groups, is the Euler characteristic, χ. We require that for any stabiliser group H, the Euler characteristic $\chi(H)$ is a non-zero integer. Notice that if the Euler characteristic were zero then the ordering need not be preserved. For example, given a group of Euler characteristic zero, any of its proper subgroups of finite index will also have zero Euler characteristic. This restriction on the Euler characteristic has the added effect that none of our stabiliser subgroups are conjugate to proper subgroups of themselves - a condition we have already mentioned. It is an open question as to whether or not the Euler characteristic restriction is necessary.

We denote by $\mathscr{S}$ the commensurability class containing some finitely generated subgroup H of G. We call a collection of functions V, defined on the disjoint union of copies of G a G-stable $\mathscr{S}$-almost equality class of functions if V is a maximal set of functions that differ from one another on only finitely many cosets of groups in our admissible family, $\mathscr{S}$. A more formal definition may be found in section 3.2. With the hypotheses discussed above we are able to show that there exists a G-tree with vertex set given by a G-stable $\mathscr{S}$-almost equality class of functions. The main result we prove is Theorem A.

Theorem A. *Let H be a finitely generated subgroup of G with $\chi(H)$ a non-zero integer and $\mathrm{Comm}_G(H) = G$. Let $\mathscr{S}$ be the class of subgroups commensurable with H and A and I be non-empty sets. Suppose that V is a G-stable $\mathscr{S}$-almost equality class in $\mathscr{S}(\sqcup_I G, A)$. Then there exists a G-tree with edge stabilisers in $\mathscr{S}$ and vertex set V.*

The Almost Stability Theorem of Dicks and Dunwoody [12] which covers the case that $\mathscr{S}$ is the family of finite subgroups follows from our result in the case that H is chosen to be the trivial group. A full discussion of the connections between our result and that of Dicks and Dunwoody, as well as other conjectures we have made in this area, is given in Chapter 3. It can be seen that the complete graph on V, under the hypotheses of Theorem

A always has commensurable edge stabilisers however, the condition that H be finitely generated allows us to prove that the edge stabilisers are commensurable with H itself. It seems likely that in the infinitely generated case that there may exist examples where the edge stabilisers are 'smaller' than the subgroups in $\mathscr{S}$. Furthermore the finitely generated condition on H is also used in the induction argument for the case when G is countably infinitely generated. It is this induction argument where the non-zero Euler characteristic requirement is used. To obtain the result in the case that G is finitely generated requires only the weaker condition that H may not contain any conjugate of itself as a proper subgroup.

The Almost Stability Theorem generalises the remarkable work of Stallings and Swan [29,32] in the 1960s that a group of cohomological dimension 1 is free. The Almost Stability Theorem refines the work of Dunwoody in the the late 1970s and early 1980s. In particular the work on groups of cohomological dimension 1, the notion of accessibility of a group [15] and the method of obtaining tree sets via cuts [16]. Other work on splittings of groups has been done since furthering the development of this area of mathematics.

In the 1990s the book of Scott and Swarup [27] develops an analogy between the topological JSJ-decomposition [21, 22] and splittings of certain finitely generated groups. Other results in this direction include the thesis of Sageev and his work on CAT(0)-cube complexes [18, 25, 26]. Rather than considering group actions on trees Sageev considers the notion of a higher dimensional cube complex and obtains splitting results in this way. The notion of complexity has been introduced by Bestvina and Feighn [2], developing techniques that place a bound on how groups may decompose without the restrictions required by many existing accessibility arguments. The notion of folding they introduce is further investigated by Dunwoody [17]. Indeed Dicks and Dunwoody have published further results [13] generalising their Almost Stability Theorem and detailing a counterexample to a generalisation of the theorem in one particular direction.

Kropholler has also studied splittings of finitely generated groups of finite cohomological dimension. In [23] Kropholler introduces a new cohomological functor and combines a homological argument with the techniques of Dunwoody to give a sufficient condition for a finitely generated group to split over a Poincaré Duality subgroup. This work influences another area of interest in this thesis. We investigate the completion introduced by Kropholler in Chapter 2 as well as Poincaré duality groups in Chapter 5. Ultimately we use Theorem A to arrive at the following variant of Kropholler's result.

Theorem B. *Let G be a group of cohomological dimension $n < \infty$. Let H be an $(n-1)$-dimensional Poincaré duality subgroup of G such that $\mathrm{Comm}_G(H) = G$ and $\chi(H)$ is a non-zero integer. Then there exists a G-tree T with edge and vertex stabilisers commensurable with H.*

Kropholler's result concerns the case that G is finitely generated but without the restriction on the Euler characteristic. However, our methods require this condition for the induction argument in the G countable case and to modify the vertex set of a G-tree making it suitable for a transfinite induction argument. Hence in the finitely generated case we may recover the result of Kropholler.

Chapter 2

Preliminaries

2.1 G-sets

We begin by introducing some notation. The work in this thesis builds upon that in [12] and thus we adopt much of the notation from that book. One exception is that we shall denote all group actions throughout on the right unless stated otherwise. When we have an action of a group G on a set, we call such a set a G-set.

Definition 2.1.1. Let X be a G-set. For $x \in X$, we denote by G_x the *stabiliser of x in G*, i.e.

$$G_x = \{g \in G \mid xg = x\}.$$

We shall use the notation H^g to denote the conjugate of a subgroup $H \leq G$ by a group element $g \in G$. That is to say that $H^g = \{g^{-1}hg \mid h \in H\}$. Thus with this convention the stabiliser $G_{gx} = G_x^g$.

Definition 2.1.2. Let X be a G-set. We say that a subset $T \subseteq X$ is a *G-transversal* in X if

$$X = \bigsqcup_{t \in T} tG.$$

Notice then that for $t_1, t_2 \in T$ and $g \in G$, $t_1 = t_2 g \implies t_1 = t_2$. We say that the G-set X is *G-finite* if there exists a finite G-transversal in X.

Definition 2.1.3. Suppose that we have a map $\varphi : X \to Y$ between two G-sets. We say that φ is a *G-map* if $\varphi(xg) = \varphi(x)g$ for all $x \in X$ and $g \in G$.

Notice that for a G-map, $\varphi : X \to Y$, to be well defined it is necessary that for all $x \in X$ that $G_x \leq G_{\varphi x}$.

Suppose that X is a G-set and A is some non-empty set. We denote the set of all functions from X to A by (X, A). This set is itself a G-set with the action of G given by

$$(fg)(x) = f(xg^{-1}) \quad \forall\, g \in G, x \in X \text{ and } \; f \in (X, A).$$

For the remainder of this section let G be a group, X be a G-set and A be some non-empty set. Suppose that we have a G-map from some G-set Y to (X, A). Then for each $y \in Y$ we denote its image in (X, A) by $y|X$.

Definition 2.1.4. Let $Y \to (X, A)$ be a G-map. Then the *dual G-map* $X \to (Y, A), x \mapsto x|Y$ is given by $(x|Y)(y) = (y|X)(x)$.

Proposition 2.1.5. *Let $Y \to (X, A)$ be a G-map. Then the dual map is also a G-map.*

Proof. To see that the dual map is indeed a G-map observe that for all $x \in X, y \in Y$ and $g \in G$,

$$\begin{aligned}
(xg|Y)(y) &= (y|X)(xg) \\
&= ((y|X)g^{-1})(x) \\
&= (yg^{-1}|X)(x) \quad \text{since } Y \to (X, A) \text{ is a } G\text{-map.} \\
&= (x|Y)(yg^{-1}) \\
&= ((x|Y)g)(y).
\end{aligned}$$

□

2.1.1 Almost Equality

Definition 2.1.6. Let X be a G-set and A be a set. For y_1 and $y_2 \in (X, A)$, we denote the subset of X on which the two functions differ by $y_1 \bigtriangledown y_2$. If this set is finite then we say that the two functions y_1 and y_2 are *almost equal,* i.e. if

$$y_1 \bigtriangledown y_2 = \{x \in X \mid y_1(x) \neq y_2(x)\}$$

is a finite set. We denote this $y_1 =_a y_2$.

Notice that almost equality is an equivalence relation and partitions the set (X, A) into equivalence classes which we call *almost equality classes.* Further we observe that the action of G preserves almost equality and thus G acts on the set of almost equality classes. If one of these equivalence classes is fixed by the action of G we call such a class a *G-stable almost equality class.*

Remark 2.1.7. It is a common definition in many areas of mathematics to say that two sets are almost equal if there are only a finite number of elements belonging to one of the sets but not the other. Observe that we can identify a subset A of B with its characteristic function $\chi_A : B \to \mathbb{F}_2$, i.e. $\chi_A(b) = 1$ iff $b \in A$. We can then give a similar definition as above to regain the usual meaning of almost equality of sets. Thus we shall often write $A =_a B$ to mean that two sets — without, necessarily, any further structure — are almost equal.

2.2 G-graphs

Definition 2.2.1. A *G-graph* Γ is a quadruple $(V\Gamma, E\Gamma, \iota_\Gamma, \tau_\Gamma)$ where $V\Gamma$ and $E\Gamma$ are G-sets called the vertex set and edge set of Γ respectively and ι_Γ and τ_Γ are G-maps from $E\Gamma$ to $V\Gamma$ called the incidence maps. For $e \in E\Gamma$, we call $\iota_\Gamma(e)$ and $\tau_\Gamma(e)$ the initial and terminal vertices of e respectively.

Remark 2.2.2.

1. Often, when it is clear what G-graph we are working with we omit the subscript Γ and write simply $\iota(e)$ and $\tau(e)$ for the initial and terminal vertices of e.

2. The incidence maps give an orientation to our graph. For each $e \in E\Gamma$ we denote by e^{-1} the edge with the opposite orientation. That is to say that $\iota(e^{-1}) = \tau(e)$ and $\tau(e^{-1}) = \iota(e)$.

Definition 2.2.3. Let Γ_1, Γ_2 be G-graphs. A *map of G-graphs*, φ, is a G-map $\varphi : E\Gamma_1 \sqcup V\Gamma_1 \to E\Gamma_2 \sqcup V\Gamma_2$ with the following properties:

1. $\varphi(E\Gamma_1) \subseteq E\Gamma_2$

2. $\varphi(V\Gamma_1) \subseteq V\Gamma_2$

3. for all $e \in E\Gamma_1$, $\varphi(\iota(e)) = \iota(\varphi(e))$

4. for all $e \in E\Gamma_1$, $\varphi(\tau(e)) = \tau(\varphi(e))$

That is to say that a map of G-graphs is simply a pair of G-maps between the vertex sets and edge sets which respect the incidence maps of the G-graphs. We often wish to extend the definition of our G-graph map to the oppositely oriented edges. We do so by defining $\varphi(e^{-1}) := (\varphi(e))^{-1}$ for all $e \in E\Gamma$.

Definition 2.2.4. Let Γ be a G-graph. A *path* p in Γ is a sequence $(v_0, e_0^{\varepsilon_0}, v_1, e_1^{\varepsilon_1}, \ldots, e_{n-1}^{\varepsilon_{n-1}}, v_n)$, where $v_i \in V\Gamma, e_i \in E\Gamma$ and $\varepsilon_i \in \{\pm 1\}$ for all $0 \leq i \leq n$, with $\iota(e_i^{\varepsilon_i}) = v_i,\ \tau(e_i^{\varepsilon_i}) = v_{i+1}$ for all $0 \leq i \leq n-1$.

We say that the path p joins v_0 to v_n, or is a path between v_0 and v_n.

A path p is said to be a *reduced path* if there exists no i such that both $e_i = e_{i+1}$ and $\varepsilon_i = -\varepsilon_{i+1}$.

Further a reduced path p is a circuit if $v_0 = v_n$ where $n > 0$.

Definition 2.2.5. Let Γ be a G-graph. We say that Γ is *connected* if for any two distinct vertices in Γ there exists a path joining them.

Definition 2.2.6. Let Γ be a G-graph. We say that Γ is a *G-tree* if Γ is a non-empty, connected G-graph containing no circuits. We call a disjoint union of G-trees a *G-forest.*

Remark 2.2.7. Notice that the above definition of a G-tree is equivalent to stating that there exists a unique reduced path between any two distinct vertices of our graph.

2.3 Cayley Graph of a Group

We now give an important example of a G-graph, namely the Cayley graph of a group.

Definition 2.3.1. Let G be a group and $S \subseteq G$. We say that S *generates* G if every group element $g \in G$ may be written as a product of elements in S together with their inverses, i.e. that we may write,

$$g = s_1^{\epsilon_1} \ldots s_n^{\epsilon_n},$$

where the $s_i \in S$ and $\epsilon_i \in \{\pm 1\}$ for all $1 \leq i \leq n$.

Definition 2.3.2. Let $S \subseteq G$. The *Cayley graph,* $\Gamma(G, S)$ of G with respect to S is the graph with,

$$V\Gamma(G,S) = G$$

$$E\Gamma(G,S) = S \times G$$

$$\iota(s,g) = g$$

$$\tau(s,g) = sg.$$

Remark 2.3.3.

- Clearly G acts on the Cayley graph on the right by $g \cdot \gamma = g\gamma$ and $(s, g) \cdot \gamma = (s, g\gamma) \quad \forall s \in S, g, \gamma \in G$.
- If S is finite then the Cayley graph $\Gamma(G, S)$ is locally finite, meaning that for any vertex there exist only finitely many edges having this vertex as their initial or terminal vertex.
- If S generates G then $\Gamma(G, S)$ is connected since if $g = s_1^{\epsilon_1} \dots s_n^{\epsilon_n} \in G$ then $(1, (s_n, s_n^{\frac{\epsilon_n - 1}{2}})^{\epsilon_n}, s_n^{\epsilon_n}, (s_{n-1}, s_{n-1}^{\frac{\epsilon_{n-1}-1}{2}} s_n)^{\epsilon_{n-1}}, \dots, g)$ is a path in $\Gamma(G, S)$ between 1 and g.

2.3.1 The Word Metric

We recall the notion of a metric space.

Definition 2.3.4. A *metric space* (X, d) is a space X together with a map $d : X \times X \to \mathbb{R}$ satisfying the following axioms,

1. $d(x, y) \geq 0$ for all $x, y \in X$, with equality if and only if $x = y$,
2. $d(x, y) = d(y, x)$ for all $x, y \in X$ and
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Similarly, we define a *pseudo-metric space* to be a space X together with a map $d : X \times X \to \mathbb{R}$ satisfying axioms 2 and 3 above and further that $d(x, y) \geq 0$ for all $x, y \in X$. Finally an *ultrametric space* is a space X together with a map $d : X \times X \to \mathbb{R}$ satisfying axioms 1 and 2 above together with

3. $d(x, z) \leq \max(d(x, y), d(y, z))$ for all $x, y, z \in X$.

Thus we see that an ultrametric space is simply a metric space satisfying a stronger version of the triangle inequality.

For a finitely generated group, a choice of generating set determines a metric on the group. The metric is obtained via the notion of the length of a word in the generating set.

Definition 2.3.5. Let G be a finitely generated group with a given finite generating set S. Then we may write any element of G as a finite product, or word, of elements of S together with their inverses. We say that the word $w = s_1^{\epsilon_1} \dots s_n^{\epsilon_n}$ $(s_i \in S, \epsilon_i \in \{\pm 1\})$ is reduced if for all $1 \leq i \leq n-1$, $s_i = s_{i+1} \implies \epsilon_i = \epsilon_{i+1}$. That is to say that there is no trivial way of cancelling terms in the word w. The set of reduced words then forms the free group $F(S)$ on S and the inclusion $S \subseteq G$ induces a homomorphism $\pi : F(S) \to G$. The *length* of a reduced word w is denoted $len(w)$.

Definition 2.3.6. Let S be a generating set for G. We define the *word metric* on G, with respect to S, denoted d_S by

$$d_S(g,h) = \min\left\{len(w) \mid w \in F(S) \text{ and } \pi(w) = gh^{-1}\right\}.$$

Observe then that the word metric d_S with respect to a generating set S is the same as the usual metric in the Cayley graph $\Gamma(G,S)$ obtained by simply counting the minimum number of edges in a path between two points. Notice here that the metric we obtain is almost surely dependent on the choice of generating set for G. For example if $S \cup S^{-1} \neq G$ then there are group elements $g,h \in G$ with $d_S(g,h) > 1$, whilst if $S \cup S^{-1} = G$ then $d_S(g,h) = 0$ or 1 for all $g,h \in G$.

The following proposition follows easily from the fact that $gk(hk)^{-1} = gh^{-1}$.

Proposition 2.3.7. *Let G be a group and S be a generating set for G. Then the action of G on itself by right multiplication is an isometry.*

Notice that we could have defined the word metric with respect to S by,

$$d_S(g,h) = \min\{len(w) \mid w \in F(S) \text{ and } \pi(w) = g^{-1}h\},$$

and we would then have had a metric for which the action of G by left multiplication is an isometry. The question of whether or not both left and right multiplication by G give rise to an isometry, for a single choice of such a metric, is an interesting one that is investigated further in section 2.9.

2.4 Bass-Serre theory

Definition 2.4.1. Let H_1 and H_2 be groups that contain an isomorphic subgroup K. That is to say that there exist injective homomorphisms $\varphi_1 : K \to H_1$ and $\varphi_2 : K \to H_2$. The *free product of H_1 with H_2 amalgamated along K* denoted $H_1 *_K H_2$ is defined to be the group with the following presentation,

$$H_1 *_K H_2 = \langle H_1, H_2 \mid \varphi_1(k) = \varphi_2(k)\ \forall k \in K\rangle.$$

Definition 2.4.2. Let K and L be subgroups of H and $\varphi : K \to L$ be an isomorphism. The *HNN extension of* H *by* φ denoted $H*_{\varphi,t}$ is defined to be the group with the following presentation,

$$H*_{\varphi,t} = \langle H, t \mid tkt^{-1} = \varphi(k)\ \forall k \in K\rangle.$$

Remark 2.4.3.
- The map φ is often supressed in the above notation and once φ has been introduced the HNN extension is commonly referred to as simply $H*_K$ instead of $H*_{\varphi,t}$ unless this would otherwise cause confusion.
- We say that a group G *splits* over a subgroup K if either $G \cong H_1 *_K H_2$ with $H_1 \neq K \neq H_2$ or $G \cong H*_K$.

An important result of Bass-Serre theory (see section I.5.4 of [28]) is that such a decomposition corresponds to an action of our group on a certain G-tree. Thus we obtain a more geometric description of the structures of such groups.

Theorem 2.4.4. *Suppose that G is the free product of H_1 and H_2 amalgamated along K. Then G acts on a G-tree T with one orbit of edges and two orbits of vertices such that some edge of T has stabiliser K and its endpoints have stabilisers H_1 and H_2.*

Suppose that G is the HNN extension of H by $\varphi : K \to H$. Then G acts on a G-tree T with one orbit of edges and one orbit of vertices such that the stabilisers are conjugates of K.

Remark 2.4.5. With the above theorem we see that G *splits* over K if and only if G acts on a tree T with one orbit of edges and for some edge $e \in ET, G_e = K$.

2.5 Ends of a group

Let G be a group generated by $S = \{s_1, s_2, \dots, s_n\}$.

Definition 2.5.1. The number of *ends* of a group G, denoted $e(G)$ is defined to be,

$$e(G) = \sup_X(\text{ number of infinite components of } \Gamma(G,S) \setminus X),$$

where the supremum is taken over all finite subsets $X \subseteq E\Gamma(G,S)$.

It would appear at first that the number of ends of our group depends upon our choice of generating set, however it can be shown that the number of ends is equal to the dimension of $H^0(G, \mathscr{P}G/\mathscr{F}G)$, see Chapter 2 of [11], where $\mathscr{P}G$ denotes the power set of

G and $\mathscr{F}G$ denotes the collection of all finite subsets of G. Thus the number of ends of G is in fact independent of our choice of S.

Example 2.5.1.

- We see that $e(G) = 0$ if and only if G is finite as for a finite group the Cayley graph must also be finite and indeed for any Cayley graph with a positive number of ends, removing any single edge must leave at least one infinite component and so clearly G itself is infinite.

- $e(\mathbb{Z}) = 2$. To see this consider the Cayley graph of $\mathbb{Z}$ with respect to the generating set consisting of a single generator. Then by removing any non-empty finite collection of edges from the graph we leave two infinite components one of which contains all integers greater than N, the other containing all integers less than $-N$ for some $N \in \mathbb{N}$.

- $e(\mathbb{Z} \times \mathbb{Z}) = 1$. This is also clear by considering the Cayley graph with respect to the generating set $\{(1,0),(0,1)\}$. Any finite subset of the edge set is contained within some bounded ball thus leaving only 1 infinite component. In fact $e(\mathbb{Z}^n) = 1$ for all $n > 1$.

- $e(F_n) = \infty$ where F_n is the free group of rank $n > 1$. Let $S = \{x_1, \ldots, x_n\}$ be a generating set for F_n. Then in the Cayley graph $\Gamma(G,S)$ if we remove the edges joining 1 to x_1, x_1 to $x_1^2, \ldots, x_1^{m-2}$ to x_1^{m-1} we see that this leaves m infinite connected components. Since we may break our graph up in this fashion for any $m \in \mathbb{N}$ we see that F_n must have infinitely many ends.

- It can be seen that these are the only possible values that $e(G)$ may take - Theorem 2.11 of [11].

We have seen that $e(\mathbb{Z}) = 2$ and $e(F_n) = \infty$ for $n > 1$, and now we observe that $\mathbb{Z}$ may be thought of as the HNN extension of the trivial group and F_n is a free product of free groups of rank 1. These are in fact special cases of a remarkable result of Stallings shown in the finitely presented case in [29] and modified to the finitely generated case by Bergman [1].

Theorem 2.5.2. *Let G be a finitely generated group. Then G splits as a free product with amalgamation or HNN extension over some finite subgroup if and only if G has more than one end.*

2.6 E_G and V_G

Let G be a group and E be a G-set. Suppose that V is a subset of (E, A) and that we have a fixed element $v_0 \in V$. We introduce the following notation for certain subsets of V and E.

Definition 2.6.1. For any subset $V' \subset V$, we denote by $E(V')$ the subset of E consisting of the elements on which v_0 differs from some function in V',

$$E(V') = \bigcup_{v \in V'} v \bigtriangledown v_0.$$

Also for any subset $E' \subset E$, we denote by $V(E')$ the subset of V consisting of the functions differing from v_0 on some subset of E',

$$V(E') = \{v \in V \mid v \bigtriangledown v_0 \subseteq E'\}.$$

In particular we are interested in such subsets related to a subgroup. We give the following definition for all subgroups H of G.

Definition 2.6.2. Let $H \leq G$. We define E_H by,

$$E_H = E(v_0 H) = \bigcup_{h \in H} v_0 \bigtriangledown v_0 h.$$

Given this we define the set V_H consisting of the functions v which differ from v_0 on a subset of E_H, i.e.

$$V_H = V(E_H) = \{v \in V \mid v \bigtriangledown v_0 \subseteq E_H\}.$$

At this point it may not be clear why such notation is used however later in the proof we shall see that E_H can be considered as the edge set of an H-tree containing $v_0 H$.

We mention here an important property of E_H.

Proposition 2.6.3. *Let $H \leq G$, then $E_H G \subseteq E_G$ and so in particular we have that E_H is an H-set.*

Proof. Let $e \in E_H$. Then there is some $h \in H$ such that

$$v_0 h(e) \neq v_0(e).$$

Now let $g \in G$ and suppose that

$$(v_0 k)(eg) = v_0(eg) \quad \text{for all } k \in G.$$

Then we obtain the following two equalities,

$$\begin{aligned} v_0(e) &= v_0(eg) \quad \text{taking } k = g \\ (v_0h)(e) &= v_0(eg) \quad \text{taking } k = hg. \end{aligned}$$

However this contradicts the definition of e. □

Lemma 2.6.4. *If G is finitely generated over H then $E_G - E_HG$ is G-finite.*

Proof. Suppose that G is generated by H together with $g_1, g_2, \ldots, g_n$. Then we claim that $E' := E_HG \cup (\cup_{i=1}^n v_0 \bigtriangledown v_0g_i) \subseteq E_G$. Now by the definition of E_H we see that $v_0|(E - E') = v_0g|(E - E')$ for all $g \in H \cup \{g_1, g_2, \ldots, g_n\}$ and hence for all $g \in G$. It follows that $E_G \subseteq E'$ and the result holds. □

2.6.1 Coboundary of a function

The notion of the coboundary of a function gives a measure of how the function partitions the vertex set. The coboundary of the characteristic function of a subset in the Cayley graph provides a means of testing whether a given subset of a group G is almost equal to its translates under the action of G. We shall see that section 2.5 then tells us that the existence of a splitting of a group corresponds to the existence of a function on G with certain conditions on its coboundary.

Definition 2.6.5. Let Γ be a G-graph and A a non-empty set. Let $s : V\Gamma \to A$ be a function on the vertex set of our graph. We denote by δs the *coboundary of s*,

$$\delta s = \{e \in E\Gamma \mid s(\iota e) \neq s(\tau e)\}.$$

We make the observation here that the coboundary map is a G-map.

Proposition 2.6.6. *Let $s : V\Gamma \to A$ be a funcion with the notation as above. Then $\delta(sg) = \delta(s)g$ for all $g \in G$.*

Proof. Let $e \in E\Gamma$. Then

$$\begin{array}{rrcll} & e & \in & \delta s & \\ \Longleftrightarrow & s(\iota e) & \neq & s(\tau e) & \\ \Longleftrightarrow & s(\iota(egg^{-1})) & \neq & s(\tau(egg^{-1})) & \\ \Longleftrightarrow & s(\iota(eg)g^{-1}) & \neq & s(\tau(eg)g^{-1}) & \text{since } \iota, \tau \text{ are } G\text{-maps} \\ \Longleftrightarrow & sg(\iota(eg)) & \neq & sg(\tau(eg)) & \\ \Longleftrightarrow & eg & \in & \delta(sg) & \end{array}$$

□

There is a result of Cohen (section 2 of [11]) that a group has more than one end if and only if there exists a function $s \in (G, \mathbb{Z}_2)$ on the vertex set of the Cayley graph of G such that $|\delta s|$ is finite. Now by section 2.5 we have that G-splits over a finite subgroup if and only if there exists a function in $(G, \mathbb{Z}_2)$ with finite coboundary. This is the motivation for considering the coboundary of functions on other G-sets when we investigate splittings over infinite groups.

2.7 Commensurability

In this section we introduce the notion of commensurable subgroups and investigate some of their basic properties and those of their automorphism groups.

Definition 2.7.1. Let G be a group and H and K subgroups of G. We say that H and K are *commensurable* if $H \cap K$ is a finite index subgroup of both H and of K.

Remark 2.7.2.

- We mention here that the term commensurability is sometimes used to describe a slightly different notion. Some authors for example Bridson and Haefliger ([9] I.8.21) use the term commensurable to mean that H and K have isomorphic finite index subgroups. This is not the same as our notion, for example take $G = \mathbb{Z} \times \mathbb{Z}$, and the two subgroups H and K to be the two factors of $\mathbb{Z}$. Then H and K are commensurable in this alternative definition but not in the one introduced above. In particular, it should be noted that $H \cong K$ is not enough to show that H and K are commensurable.

- It should be noted that commensurability is an equivalence relation. We call the equivalence classes *commensurability classes.*

Example 2.7.1. Let G be a group. Then the class of all finite subgroups of G is a commensurability class.

This example of a commensurability class is what motivates our use of this concept. The Almost Stability Theorem gives the existence of a G-tree with edge stabilisers all belonging to this special commensurability class of finite subgroups. We shall show that an analogous result holds for many other examples of commensurability classes. In the case that a group acts on a tree - or indeed on any set - with commensurable edge stabilisers it follows that the stabilisers must be commensurable with all of their conjugates. Normal subgroups are obvious examples of groups with this property, however there are less trivial examples. We shall use the following notation.

Definition 2.7.3. Let G be a group and $H \leq G$. We define the *commensurator* of H in $G, \mathrm{Comm}_G(H)$ to be,

$$\mathrm{Comm}_G(H) = \{g \in G \mid H^g \text{ is commensurable with } H\}.$$

If $\mathrm{Comm}_G(H) = G$ then we say that H is *near-normal* in G.

Remark 2.7.4. The commensurator of H in G is a subgroup of G containing $N_G(H)$.

We now give some examples of groups with this property.

Example 2.7.2. Let $G = \langle x, y | x^y = x^2 \rangle$ and $H = \langle x \rangle$. Then H is near-normal in G.

Remark 2.7.5. Indeed G above is just one member of a family of groups known as Baumslag-Solitar groups which have this property.

Example 2.7.3. Let $n \geq 1$. Then $\mathrm{SL}_n(\mathbb{Z})$ is near-normal in $\mathrm{SL}_n(\mathbb{Q})$.

Proof. $\mathrm{GL}_n(\mathbb{Q})$ acts on $\mathbb{Q}^n$ by right multiplication and we consider $\mathbb{Z}^n \subset \mathbb{Q}^n$. Let $X \in \mathrm{SL}_n(\mathbb{Q})$. We aim to show that $\mid \mathrm{SL}_n(\mathbb{Z}) : \mathrm{SL}_n(\mathbb{Z}) \cap X^{-1}\mathrm{SL}_n(\mathbb{Z})X \mid < \infty$ and also that $\mid X^{-1}\mathrm{SL}_n(\mathbb{Z})X : \mathrm{SL}_n(\mathbb{Z}) \cap X^{-1}\mathrm{SL}_n(\mathbb{Z})X| < \infty$. Now since $X \in \mathrm{SL}_n(\mathbb{Q})$, it follows that there exists some natural number α such that $X^{-1}\mathrm{SL}_n(\mathbb{Z})X \subseteq \frac{1}{\alpha}\mathrm{GL}_n(\mathbb{Z})$. For example we may take α to be the least common multiple of the denominators of the entries of X. Thus $\mathbb{Z}^n \cdot X\mathrm{SL}_n(\mathbb{Z})X^{-1} \subseteq \frac{1}{\alpha}\mathbb{Z}^n$. Thus we also have that $\alpha\mathbb{Z}^n \subseteq \mathbb{Z}^n \cdot X\mathrm{SL}_n(\mathbb{Z})X^{-1}$. Now since $\mathrm{SL}_n(\mathbb{Z})$ acts on $\mathbb{Z}^n$ there is an induced action of $\mathrm{SL}_n(\mathbb{Z})$ on the collection of subgroups of $\frac{1}{\alpha}\mathbb{Z}^n$ containing $\alpha\mathbb{Z}^n$. By the classification of finitely generated abelian groups there exist only finitely many such groups and therefore $\mid \mathrm{SL}_n(\mathbb{Z}) : \mathrm{SL}_n(\mathbb{Z}) \cap X\mathrm{SL}_n(\mathbb{Z})X^{-1} \mid < \infty$. Since this holds for all $X \in \mathrm{SL}_n(\mathbb{Q})$ and indices are preserved by conjugation we also see that $\mid X\mathrm{SL}_n(\mathbb{Z})X^{-1} : \mathrm{SL}_n(\mathbb{Z}) \cap X\mathrm{SL}_n(\mathbb{Z})X^{-1} \mid < \infty$ and the result follows. $\square$

For near-normal subgroups, H of G, it is particularly useful to observe that the subsets of G given by either finitely many right or left cosets of a group commensurable with H are the same. We have the following result.

Lemma 2.7.6. *Let H be a near-normal subgroup of G. Denote by $\mathscr{S}$ the collection of all subgroups of G commensurable with H. Then any subset of G can be written as a finite union of left cosets of some subgroup in $\mathscr{S}$ if and only if it can be written as a finite union of right cosets of some subgroup in $\mathscr{S}$.*

Proof. Suppose that X is a finite union of right cosets of some subgroup $L \in \mathscr{S}$. Then we may write,

$$\begin{aligned} X &= \bigsqcup_{i=1}^{n} Lg_i \\ &= \bigsqcup_{i=1}^{n} g_i L^{g_i} \\ &= \bigsqcup_{i=1}^{n} g_i \left(\bigsqcup_{j} \gamma_{i,j} \left(\bigcap_{i=1}^{n} L^{g_i} \right) \right) \\ &= \bigsqcup_{i,j} g_i \gamma_{i,j} \left(\bigcap_{i=1}^{n} L^{g_i} \right), \end{aligned}$$

where the $\gamma_{i,j}$ are a transversal for $\cap_{i=1}^{n} L^{g_i}$ in L^{g_i}. Notice that such a transversal is finite since $L^{g_i} \in \mathscr{S}$ as L is near-normal and $\mathscr{S}$ is closed under taking finite intersections. Thus X has been written as a finite union of left cosets of $\cap_{i=1}^{n} L^{g_i} \in \mathscr{S}$ and by a similar argument any finite union of left cosets of L may be written as a finite union of right cosets of the intersection of conjugates of L. □

Thus we are able to switch between finite unions of left cosets and finite unions of right cosets whenever this is convenient.

We also recall at this point a definition introduced in [13].

Definition 2.7.7. Let $H \leq G$. We say that H is *G-conjugate incomparable* if for all $g \in G,\ H^g \leq H \implies H^g = H$.

In particular, normal groups are G-conjugate incomparable as are all finite groups. In general however this need not be the case. For instance there are examples of non co-Hopfian groups that do not satisfy this condition. A non co-Hopfian group is a group which is isomorphic to a proper subgroup of itself. In the example of the Baumslag Solitar group given above $G = \langle x, y | x^y = x^2 \rangle$, the subgroup $H = \langle x \rangle$ is not G-conjugate incomparable as H^y is a proper subgroup of H. We now have the following lemma asserting that for a G-tree T, to show that the vertex stabilisers are G-conjugate incomparable it is enough to check that the edge stabilisers of T satisfy this property.

Lemma 2.7.8. *Let T be a G-tree. If the edge stabilisers of T are all G-conjugate incomparable then the vertex stabilisers are also G-conjugate incomparable.*

Proof. Let us suppose that the edge stabilisers of T are G-conjugate incomparable and assume that the vertex stabilisers are not. That is to say that there exists a $v \in VT$ and $g \in G$ such that $G_v < G_v^g$. Now since G_v is a proper subgroup of G_v^g it follows that $v \neq vg$ and so we may now consider the first edge, e say, in the path in T from v to vg. Clearly $G_e = G_v$ and thus the edge stabilisers of T are not G-conjugate incomparable. This is our desired contradiction. $\square$

We now introduce the notion of an admissible family of subgroups giving particular attention to the case that the admissible family consists of commensurable subgroups. The following definition can be found in [23].

Definition 2.7.9. An *admissible family* of subgroups is a family of subgroups of G that is closed under conjugation by G and is downwardly directed, i.e. the intersection of any finite collection of subgroups in the family contains another element of the family.

We say that the admissible family, $\mathscr{S}$, is *stable* if for all H and $K \in \mathscr{S}$ such that $K \leq H$ there exists an $L \in \mathscr{S}$ such that $L \leq K$ and $L \lhd H$.

Example 2.7.4. One could simply take the admissible family $\{N\}$ consisting of a single normal subgroup $N \lhd G$. More interestingly, and in most of the cases considered later, for a near-normal subgroup $H \leq G$ we could take the family of subgroups of G commensurable with H. That such a family is downwardly directed follows from the commensurability condition and that it is closed under conjugation follows from the fact that H is commensurable with all of its conjugates in G. This gives a host of non-trivial examples of admissible families.

Lemma 2.7.10. *Let $\mathscr{S}$ be an admissible family of subgroups of G. If $\mathscr{S}$ contains a minimal member $N \in \mathscr{S}$ then N is the unique minimal member and $N \lhd G$.*

Proof. Let $N \in \mathscr{S}$ be minimal. Suppose that $M \in \mathscr{S}$ were also minimal. Then since $\mathscr{S}$ is an admissible family there exists some subgroup L of $M \cap N$ belonging to $\mathscr{S}$. By minimality it follows that $M = N$ and thus N is the unique such minimal subgroup. Let $g \in G$. Now since $\mathscr{S}$ is an admissible family it follows that $N^g \in \mathscr{S}$ and further that some subgroup of $N \cap N^g$ belongs to $\mathscr{S}$. Thus $N^g \cap N = N$. Now suppose that $N < N^g$, we would then have that $N^{g^{-1}} < N$ contradicting the minimality of $N \in \mathscr{S}$. Therefore $N^g = N$ and so $N \lhd G$. $\square$

Lemma 2.7.11. *Let $\mathscr{S}$ be an admissible family of commensurable subgroups of G. Suppose that $\mathscr{S}$ contains a minimal member. Then the groups in $\mathscr{S}$ are all G-conjugate incomparable.*

Proof. Let N be the minimal member of $\mathscr{S}$. Suppose to the contrary that there exists a subgroup $H \in \mathscr{S}$ and an element $g \in G$ such that $H^g < H$. Since $\mathscr{S}$ is an admissible family we have that $H^{g^i} \in \mathscr{S}$ for all $i \in \mathbb{Z}$. By the minimality of N we see that $N \leq \bigcap_{i \geq 0} H^{g^i}$. However, $|H : \bigcap_{i \geq 0} H^{g^i}| = \infty$, contradicting the fact that $N \in \mathscr{S}$ is commensurable with H. $\square$

2.8 Completions of G

In [23] Kropholler obtains splittings of certain finitely generated groups introducing a certain completion of a group G denoted $\widehat{G}_{\mathscr{S}}$ associated to an admissible family of subgroups, $\mathscr{S}$ of G. Kropholler notes that this generalises the profinite completion of a residually finite group and asks the question whether this new completion is always a group. In this section we investigate this completion considering when it is itself a group and also when we simply obtain the well known notion of the metric completion of the group. We state the following definition from section 6 of [23].

Definition 2.8.1. Let $\mathscr{S}$ be an admissible family of subgroups of G. We define the *completion of G (with respect to $\mathscr{S}$)*, $\widehat{G}_{\mathscr{S}}$ to be the set of all functions $f : \mathscr{S} \to \mathscr{P}(G)$ satisfying,

- $f(H) \in H \backslash G$ for all $H \in \mathscr{S}$,
- $f(K) \subseteq f(H)$ whenever $K \subseteq H$ are members of $\mathscr{S}$.

Notice that we may then define a product on $\widehat{G}_{\mathscr{S}}$ allowing us to consider the completion as a monoid. We define the product in the following way. Firstly for all $f \in \widehat{G}_{\mathscr{S}}$ and $H \in \mathscr{S}$

denote by H^f the stabiliser of the right coset $f(H)$ under right multiplication by G. Then for all $f, f' \in \widehat{G}_{\mathscr{S}}$ define the product,

$$f \cdot f'(H) = f(H)f'(H^f).$$

It can be shown that this product is associative and that the function $e : H \mapsto H$ is the identity element of this monoid. This construction works for any group G together with some admissible family $\mathscr{S}$. Kropholler has shown that in the case $\mathscr{S}$ is a stable admissible family then this completion is a group. It is also an interesting question to consider whether the use of right cosets plays any role in this construction. We deal with this question in the following section.

2.8.1 Anti-isomorphism

One might initially suspect that the completion of G should not depend upon whether left or right cosets are used in the definition above. We aim to show now that the two different constructions give anti-isomorphic monoids, thus in particular when the completion is in fact a group, and thus anti-isomorphic to itself, we have that the constructions give isomorphic monoids or indeed isomorphic groups.

To avoid confusion we denote the completions of G via left/right cosets as $\widehat{G}_{\mathscr{S},L}/\widehat{G}_{\mathscr{S},R}$ respectively. We define a map $\varphi_L : \widehat{G}_{\mathscr{S},R} \to \widehat{G}_{\mathscr{S},L}, f \mapsto f_L$ where for all $H \in \mathscr{S}, f_L(H) = gH$ where $f(H) = Hg^{-1}$. The conditions for a function to belong to $\widehat{G}_{\mathscr{S},R}$ corresponds to the conditions for its image under this map belonging to $\widehat{G}_{\mathscr{S},L}$. Thus our map is well-defined and we check that this is an anti-isomorphism of monoids. It is worth observing at this point that the products in the two monoids are not the same but must be adjusted in the obvious way to ensure that the image of the product is a coset. Multiplication in $\widehat{G}_{\mathscr{S},L}$ is given by the following formula,

$$f \cdot f'(H) = f({}^{f'}H)f'(H),$$

where ${}^{f'}H$ denotes the stabiliser of the left coset $f'(H)$ under left multiplication. This makes φ_L an anti-homomorphism of monoids. To see this let $f_1, f_2 \in \widehat{G}_{\mathscr{S},R}$ and $H \in \mathscr{S}$. Suppose that $f_1(H) = Hg_1$ and $f_1(H^{g_1}) = H^{g_1}g_2$, then we see that $(f_1)_L(H) = g_1^{-1}H$ and $(f_2)_L(H^{g_1}) = g_2^{-1}H^{g_1}$. So now $(f_2)_L \cdot (f_1)_L(H) = (f_2)_L({}^{(f_1)_L}H)(f_1)_L(H) = g_2^{-1}g_1^{-1}H$ and also that $f_1 \cdot f_2(H) = f_1(H)f_2(H^{f_1}) = Hg_1g_2$. Hence $(f_1 \cdot f_2)_L = (f_2)_L \cdot (f_1)_L$ and φ_L is an anti-homomorphism. That this is an isomorphism is more straightforward to see and the inverse map is $\varphi_R : f \mapsto f_R$ where $f_R(H) = Hg$ whenever $f(H) = g^{-1}H$. Thus our constructions are isomorphic in the case that they are both groups. However we now show that it is not always the case that $\widehat{G}_{\mathscr{S},R}$ or $\widehat{G}_{\mathscr{S},L}$ is a group. In particular for $G = \mathrm{Symm}(\mathbb{N})$ with the admissible family $\mathscr{S} = \{\ G_X \mid X \subseteq \mathbb{N} \text{ is finite }\}$, where G_X denotes the pointwise stabiliser of X, the monoid $\widehat{G}_{\mathscr{S}}$ is not a group but in fact isomorphic to the monoid of injective maps from $\mathbb{N}$ to itself.

2.8.2 Injective endomorphisms of the natural numbers

We denote the set of all maps from the natural numbers to themselves by $(\mathbb{N}, \mathbb{N})$, and all such injective maps by $\mathrm{Inj}(\mathbb{N}, \mathbb{N})$. We also write all of these maps on the right.

We define a map from the completion of G to $(\mathbb{N}, \mathbb{N})$ as follows.

$$\begin{array}{ccc} \widehat{G}_{\mathscr{S}} & \xrightarrow{\sim} & (\mathbb{N},\mathbb{N}) \\ \\ f & \longmapsto & \tilde{f} \end{array}$$

where $n\tilde{f} = ng$ where $H \in \mathscr{S}$ fixes n and $f(H) = Hg$. We must now verify that this map is well-defined. Suppose now that $Hg = Hg'$ for some choice of $g' \in G$. Then $gg'^{-1} \in H$, it follows that $ngg'^{-1} = n$ and thus $ng = ng'$. Next suppose that we had chosen a different subgroup $K \in \mathscr{S}$ such that K fixes n and that $f(K) = Kt$, where $t \in G$. Then since $\mathscr{S}$ is an admissible family, there exists an $L \leq H \cap K$ in $\mathscr{S}$. It is clear that L also fixes n and since $f \in \widehat{G}_{\mathscr{S}}, f(L) \subseteq f(H) \cap f(K) = Hg \cap Kt$. Let us assume that $f(L) = La$, say, then $ag^{-1} \in H$ and $at^{-1} \in K$ and we have that $ng = na = nt$. Hence we observe that the map $f \mapsto \tilde{f}$ is well-defined.

Now we must check that this a map of monoids. Let $f, g \in \widehat{G}_{\mathscr{S}}$. Then $n\tilde{f} = n\alpha$ where $L \in \mathscr{S}$ fixes n and $f(L) = L\alpha$. Now $n\tilde{f}\tilde{g} = n\alpha\tilde{g} = n\alpha\beta$ where $g(L^\alpha) = L\alpha\beta$ and of course $L^\alpha \in \mathscr{S}$ fixes $n\alpha$. Notice that $f \cdot g(L) = L\alpha\beta$ and it follows that $\tilde{f} \circ \tilde{g} = \widetilde{f \cdot g}$.

Remark 2.8.2. At this point we notice that in a more general context the map $f \mapsto \tilde{f}$ is well-defined if every point of our domain is fixed by some subgroup in our admissible family. Further this map is always a map of monoids via the closed under conjugation condition of our admissible family. Indeed this condition also gives that the image of $f \mapsto \tilde{f}$ lies in the injective maps of our object.

We now claim that $\widehat{G}_{\mathscr{S}}$ is isomorphic as a monoid to the set of injective functions from $\mathbb{N}$ to $\mathbb{N}$. We proceed to show that $\tilde{f}$ is always an injective function and that there exists an inverse map of monoids from $\mathrm{Inj}(\mathbb{N},\mathbb{N})$ to $\widehat{G}_{\mathscr{S}}$.

Let $f \in \widehat{G}_{\mathscr{S}}$. Suppose that $n\tilde{f} = m\tilde{f}$. Then $ng = mg'$ where H and $K \in \mathscr{S}$ fix n and m respectively, $f(H) = Hg$ and $f(K) = Kg'$. Then there exists $L \in \mathscr{S}$ contained in $H \cap K$ such that $f(L) \subseteq Hg \cap Kg'$. By a similar argument as above $na = ng = mg' = ma$, where $f(L) = La$ and so $n = m$ since $a \in G$. It follows that the image of the map $f \mapsto \tilde{f}$ lies in $\mathrm{Inj}(\mathbb{N},\mathbb{N})$.

We now construct the inverse map. Let $f \in \mathrm{Inj}(\mathbb{N},\mathbb{N})$ and $X \subseteq \mathbb{N}$ be a finite set. Choose $\sigma \in \mathrm{Symm}(X \cup (X)f) \subseteq G$ such that $x\sigma = (x)f$ for all $x \in X$. Notice that we may choose such a permutation only when f is an injective map. Now for $H = G_X \in \mathscr{S}$ we define,

$$\overline{f}(H) = H\sigma.$$

It remains to show that $\overline{f} \in \widehat{G}_{\mathscr{S}}$ and that $f \mapsto \overline{f}$ is both a monoid map and inverse to $f \mapsto \tilde{f}$. To see that $\overline{f} \in \widehat{G}_{\mathscr{S}}$ let $f \in \mathrm{Inj}(\mathbb{N},\mathbb{N})$ and $K \leq H$ belonging to $\mathscr{S}$. Then $K = G_Y$ where $Y \supseteq X$ is a finite subset of $\mathbb{N}$, and $\overline{f}(K) = K\sigma'$ say, where σ' is such that $x\sigma' = x\sigma$ for all $x \in X$. Thus we see that $\sigma'\sigma^{-1} \in H$ and hence $K\sigma' \subseteq H\sigma$ which gives that $\overline{f} \in \widehat{G}_{\mathscr{S}}$.

Let $f, g \in \mathrm{Inj}(\mathbb{N},\mathbb{N})$. Then

$$\begin{aligned}
\overline{f} \cdot \overline{g}(H) &= \overline{f}(H)\overline{g}(H^{\overline{f}}) \\
&= H\sigma\overline{g}(H^{\sigma}) \qquad \text{where } x\sigma = xf \quad \forall x \in \mathbb{N}^H \\
&= H\sigma\tau \qquad \text{where } x\sigma\tau = x\sigma g \quad \forall x \in \mathbb{N}^H \\
&= H\sigma\tau \qquad \text{where } x\sigma\tau = xfg \quad \forall x \in \mathbb{N}^H \\
&= \overline{fg}(H).
\end{aligned}$$

Thus we observe that this is in fact a map of monoids. It is now clear from the definitions that these maps are mutually inverse and it follows that $\widehat{G}_{\mathscr{S}} \cong \mathrm{Inj}(\mathbb{N},\mathbb{N})$ as monoids. Therefore in this example $\widehat{G}_{\mathscr{S}}$ is not a group. It should be noted that the above proof does not use the structure of $\mathbb{N}$ and holds indeed if we replace $\mathbb{N}$ with any infinite set. We now investigate some other natural instances of this completion.

2.8.3 Vector spaces over a field

Let V be a vector space over a field k, G be the group of automorphisms of V and $\mathscr{S} = \{G_W \mid W \text{ is a f.d. subspace of } V\}$ be the collection of pointwise stabilisers of finite dimensional subspaces of V. We now show that the image of the map $f \mapsto \tilde{f}$ defined previously lies in the injective linear maps from V to itself. To check that the map $\tilde{f}$ is linear let $k_1, k_2 \in k$ and $v_1, v_2 \in V$. We choose two subgroups H_1, H_2 that fix v_1, v_2 respectively. Then we may choose an $H_3 \in \mathscr{S}$ contained in the intersection of the two subgroups and thus find a linear automorphism $\phi \in G$ such that $\tilde{f}$ acts on both v_1, v_2 as ϕ. To construct an inverse map, thus showing that $\widehat{G}_{\mathscr{S}}$ is isomorphic to the injective linear endomorphisms of V, we must observe that every injective linear map on a finite dimensional subspace of V to V may be extended to an automorphism of V. We may then construct the analagous inverse map as in the case above.

2.8.4 Topological Spaces

Proposition 2.8.3. *Let X be a topological space, G be the group of homeomorphisms of X to itself and let $\mathscr{S}$ be the collection of subgroups fixing (pointwise) some open subset of X. Then the completion of G, $\widehat{G}_{\mathscr{S}}$, is isomorphic to G.*

Remark 2.8.4. It is to be expected that the completion in this case should be a group since the whole space is itself an open set and therefore the trivial group is in the family making $\mathscr{S}$ a stable admissible family of subgroups. That the completion is a group in this case is proved in [23].

Proof. We first observe that the family of subgroups $\mathscr{S}$ is an admissible family since open sets remain open after conjugation by some homeomorphism and since finite unions of open sets are also open. We may define a map $\widehat{G}_{\mathscr{S}} \to G$ where $f \mapsto \tilde{f}$ in an analogous way to the above argument. We define $\tilde{f}(x) = x\varphi$ where $x \in U$ is an open subset of X and $f(G_U) = G_U\varphi$. Notice that we may always choose such a U since the whole space itself is an open set. It is this observation that allows us to see that $\tilde{f} \in G$. Again we observe that this definition is independent of the choice of coset representative since G_U fixes x and also it is independent of the choice of $U \subseteq X$ by the very definition of admissible family. That this map is a map of monoids follows from the definition of an admissible family in the same way as the previous example. We recall that the second axiom for a function belonging to $\widehat{G}_{\mathscr{S}}$ is,

$$f(K) \subseteq f(H) \text{ whenever } K \subseteq H \text{ are members of } \mathscr{S}.$$

Thus it is clear that whenever the trivial group belongs to $\mathscr{S}$, the function $f \in \widehat{G}_{\mathscr{S}}$ is uniquely determined by $f(1)$, i.e. by a single element of G. Hence we may now define a map $G \to \widehat{G}_{\mathscr{S}}$ with $\varphi \mapsto f_\varphi$ where $f_\varphi(H) = H\varphi$ for all $H \in \mathscr{S}$. It is clear that the image of this map lies in $\widehat{G}_{\mathscr{S}}$ and further this is a map of monoids since for all $\varphi, \psi \in G$ and $H \in \mathscr{S}$ we have that,

$$\begin{aligned} (f_\varphi \cdot f_\psi)(H) &= f_\varphi(H) f_\psi(H^{f_\varphi}) \\ &= H\varphi H^\varphi \psi \\ &= H\varphi\psi \\ &= f_{\varphi\psi}(H). \end{aligned}$$

It is now straightforward to verify that for all $g \in \widehat{G}_{\mathscr{S}}$ and $\varphi \in G$ that $f_{\tilde{g}} = g$ and $\widetilde{f_\varphi} = \varphi$. Therefore we have that $G \cong \widehat{G}_{\mathscr{S}}$ as claimed. □

Remark 2.8.5. Notice that in the above argument we could have chosen to consider the family of stabilisers of closed subsets rather than open subsets. We would still obtain an admissible family since finite unions of closed sets are closed and closedness is preserved under conjugation by homeomorphisms. Again we could note that the whole space X is closed and thus the trivial subgroup belongs to our family. This would allow an identical construction of an inverse map $f \mapsto \tilde{f}$. Thus again we would observe that the completion of G is G itself. This however is not so surprising, for example in the case $X = \mathbb{R}^n$ the pointwise stabiliser of an open set stabilises its closure and so in fact both families of subgroups are the same.

2.8.5 Compact subsets

For a topological space X we could also choose our admissible family of subgroups to be the pointwise stabilisers of a given compact subset of X, since compact subsets are preserved by homeomorphisms and closed under finite unions. Unlike the open/closed case the whole space X need not itself be compact and therefore the above construction of the inverse map may no longer be performed.

Let $M_{\mathrm{comp}}(X)$ be the collection of injective maps φ from X to itself such that for any compact subset $C \subseteq X$ the restriction $\varphi|_C$ extends to a homeomorphism of X. We aim to show that $\widehat{G}_{\mathscr{S}} \cong M_{\mathrm{comp}}$ for $G = \mathrm{Homeo}(X)$ and $\mathscr{S}$ the family of subgroups fixing compact subsets of X.

We define a map $\widehat{G}_{\mathscr{S}} \to M_{\mathrm{comp}}, f \mapsto \tilde{f}$ as follows. Let $x \in X$ and let $C \subseteq X$ be a compact subset containing x. Such a set exists since $\{x\}$ itself is compact. Then

$G_C \in \mathscr{S}$ and $f(G_C) = G_C\varphi$ say. Then we define $x\tilde{f} = x\varphi$. Again this is well defined and independent of the choice of C since G_C fixes x and the family $\mathscr{S}$ is admissible. It is clear that $\tilde{f} \in M_{\text{comp}}$ since the finite union of compact sets is compact and there is an obvious choice of homeomorphism extending $\tilde{f}$. For example, φ extends $\tilde{f}|_C$. This is a map of monoids again by the properties of an admissible family.

We require an inverse map $M_{\text{comp}} \to \widehat{G}_{\mathscr{S}},\ \varphi \mapsto \overline{\varphi}$. Define $\overline{\varphi}(H) = H\psi_C$ where $H = G_C$ and ψ_C is a homeomorphism of X extending $\varphi|_C$. This is well defined since for any other possible extending homeomorphism θ_C the restriction of $\varphi_C \circ \theta_C^{-1}$ to C is the identity. We check that $\overline{\varphi} \in \widehat{G}_{\mathscr{S}}$, let $G_C \subseteq G_{C'} \in \mathscr{S}$. Then any extension of $\varphi|_C$ is also an extension of $\varphi|_{C'}$ and so $\overline{\varphi}(G_C) \subseteq \overline{\varphi}(G_{C'})$.

Now observe that for $f \in \widehat{G}_{\mathscr{S}}, G_C \in \mathscr{S}$, that

$$\begin{aligned} \overline{\tilde{f}}(G_C) &= G_C\psi \text{ where } \psi \text{ extends } \tilde{f}|_C \\ &= f(G_C) \text{ by definition of } \tilde{f}. \end{aligned}$$

Also for $\varphi \in M_{\text{comp}}(X), x \in X$,

$$\begin{aligned} x\tilde{\overline{\varphi}} &= x\psi \text{ where } x \in C \text{ compact and } \overline{\varphi}(H_C) = H_C\psi \\ &= x\varphi \text{ by definition of } \overline{\varphi}. \end{aligned}$$

2.9 Metric Spaces

In this section we recall the notions of metric spaces and groups that have a metric space structure. We investigate properties of the metric completion of such spaces and compare these completions to the other completions defined in section 6 of [23] when they exist. We shall also show sufficient conditions for these completions to be homeomorphic.

We begin by recalling some basic definitions from the theory of metric spaces. The following definitions are treated in more detail in for example [9, 14, 31].

Definition 2.9.1. Let (X, d) be a metric space. We say that a sequence of elements $(x_i)_{i\in\mathbb{N}}$ of X is a *Cauchy sequence* in X if for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $i, j > N \implies d(x_i, x_j) < \epsilon$.

We say that a sequence $(x_i)_{i\in\mathbb{N}}$ converges to $x \in X$ if for all $\epsilon > 0$ there exists some $N > 0$ such that $m > N \implies d(x_m, x) < \epsilon$. We say that a metric space (X, d) is *complete* if every Cauchy sequence in X converges to some element of X.

Remark 2.9.2. It can be shown that any Cauchy sequence in $\mathbb{R}$ with respect to the usual metric converges to some element of $\mathbb{R}$ and thus $\mathbb{R}$ is complete. However, even when a metric space (X, d) is not complete we may always find a complete metric space in which (X, d) is dense, namely the so called metric completion of X, denoted $(\overline{X}, \overline{d})$. This can be shown to be unique up to unique isometry commuting with the inclusions of X into the respective completions. One construction is given below, for full details see for example section 11.2 of [31].

Definition 2.9.3. Given a metric space (X, d) we may form the *metric completion* $(\overline{X}, \overline{d})$ in the following way. Let $\mathcal{C}X$ be the set of Cauchy sequences in X. Then we may define an equivalence relation $\sim$ on $\mathcal{C}X$ by declaring that $(x_i)_{i\in\mathbb{N}} \sim (y_i)_{i\in\mathbb{N}}$ if and only if $\lim_{n\to\infty} d(x_i, y_i) = 0$. Now $\overline{X}$ is defined to be the collection of equivalence classes of Cauchy sequences in X. There is then an induced metric $\overline{d} : \overline{X} \times \overline{X} \to \mathbb{R}$ defined by $\overline{d}((x_i), (y_i)) = \lim_{n\to\infty} d(x_i, y_i)$. That this limit exists follows from the completeness of $\mathbb{R}$ and this metric is well defined by the definition of the equivalence relation on $\mathcal{C}X$. We may think of X as being contained in $\overline{X}$ as the subspace of equivalence classes of the constant sequences. It is easily seen that on this subspace $\overline{d}$ agrees with the original metric d and it can also be checked that X is dense in $\overline{X}$.

Definition 2.9.4. Let G be a group. We say that G is a *metric group* if it is endowed with a map $d : G \times G \to \mathbb{R}$ such that (G, d) is a metric space.

Lemma 2.9.5. *Let G be a metric group such that G acts on itself by left and right isometries. Then the completion of G is a group with group multiplication given by pointwise multiplication of Cauchy sequences.*

Proof. We define a product $\star$ on the completion of (G, d) as follows. Let $[(g_i)], [(h_i)]$ be equivalence classes of Cauchy sequences in G. Then we define $[(g_i)] \star [(h_i)] = [(g_i h_i)]$. It can be seen that this gives another Cauchy sequence as for all $\epsilon > 0$ there exists an $N > 1$ such that for all $m, n > N$ we have that $d(g_m, g_n), d(h_m, h_n) < \epsilon$. Then $d(g_m h_m, g_n h_n) \leq d(g_m h_m, g_m h_n) + d(g_m h_n, g_n h_n) \leq 2\epsilon$. Not only does this show that $(g_i h_i)$ is a Cauchy sequence but the same observation gives that the product is well defined. The other properties required follow easily from the fact that G is a group and it is easily seen that the identity element of the completion is the equivalence class of the trivial sequence $[(e)]$. $\square$

Remark 2.9.6. Observe that the above proof uses the fact that G acts on itself on both the left and right by isometries. We must insist on both these conditions as there exist groups with a metric space structure that act only on one side by isometries as can be seen from the following examples.

Example 2.9.1. It is well known that the dihedral group D_{2n} acts on the regular n-gon by rotations and reflections. However, this theory may be adjusted to give an action on a prism with cross section the regular n-gon. We shall describe this action and then verify that this induces a metric on D_{2n} having the property that the group acts on itself by isometries on only one side. First we fix a presentation for our group, $D_{2n} = \langle a, b \mid a^2, b^2, (ab)^n \rangle$. We construct a polyhedron having all sides of length 1 say. For the base of the prism we take a regular n-gon and label the vertices $1, ab, (ab)^2, \dots (ab)^{n-1}$ in a clockwise fashion. For the top of the prism label the vertices $a, aba, (ab)^2a, \dots (ab)^{n-1}a$ again in a clockwise fashion. This construction induces a metric on D_{2n} given by the usual distance between two points in $\mathbb{R}^3$. Now D_{2n} acts on the labels of the vertices by both left and right multiplication. We notice that the generators a, b act on the left as rotations about the axes indicated in Figure 2.1. Thus D_{2n} acts on the left by isometries with respect to this metric. Now a also happens to act on the right by isometries, in particular it acts by swapping the top and bottom faces of our prism, but b does not act on the right by isometries. The element b acts on the right by first swapping the top and bottom faces and then turning the top and bottom faces by $2\pi/n$ in opposite directions. It is clear that this is not an isometry.

Remark 2.9.7. The above examples demonstrate the existence of one-sided metric groups, however the examples stated still satisfy Lemma 2.9.5 since they are finite groups and thus they are their own metric completion as any Cauchy sequence must eventually be constant. There are however also examples of infinite groups that act on themselves on only one side by isometries. To give such examples we return to the notion of the word metric.

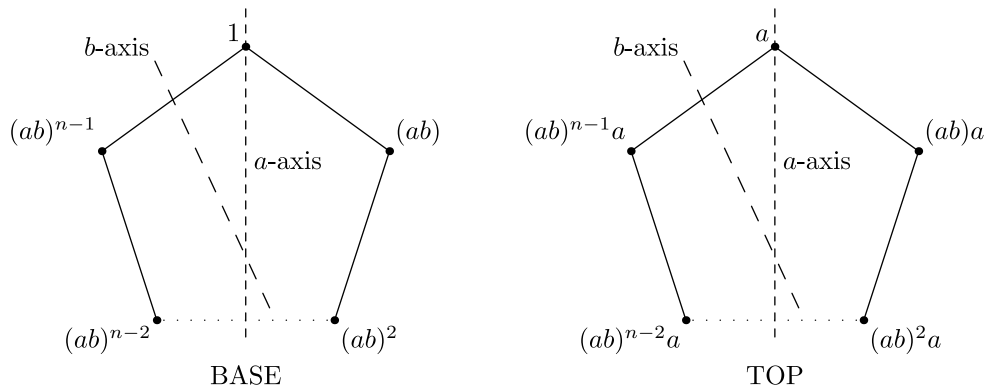


Figure 2.1: Action of D_{2n} on a regular n-gonal prism

2.9.1 When is left action an isometry?

We have already seen that the action of a group on itself by right multiplication is an isometry (Proposition 2.3.7). It is known however that left multiplication is not in general an isometry. One example of such a group for which this is the case is the free group on two generators F_2 with presentation $F_2 = \langle a, b\rangle$. Then $d_S(a, b) = 2 \neq 4 = d_S(ba, b^2)$. It can be seen that a group element g acts on the left by isometry if $g \in \zeta(G)$ as right multiplication by a central element is the same as left multiplication. Thus if G is abelian then both left and right multiplication is by isometries, however the converse is not true. Bridson and Haefliger remark in [9] page 139 that the action given by left multiplication by $g \in G$ is an isometry only if $g \in \zeta(G)$. However, this claim, as stated, is not true. For example, for any non-abelian finite group G we may take G as a finite generating set and then d_G gives the metric on G where $d_G(g, h) = 1 \iff g \neq h$ and is thus preserved by both right and left multiplication. More can be said about which finitely generated groups act on themselves by left multiplication. We state the following two results which appear in [24] as 10.1.3 and 10.1.4.

Proposition 2.9.8. *Let H be a central subgroup of G such that $|G : H| = n < \infty$. Then the map $\varphi : G \to G, x \mapsto x^n$ is a group homomorphism with image in H.*

Lemma 2.9.9. *(Schur) Let G be a group such that the centre $\zeta(G)$ is of finite index in G. Then the derived subgroup $G' = [G, G]$ is finite.*

With these results we may now prove the following lemma.

Lemma 2.9.10. *Let G be a group generated by a finite set $S = \{s_1, s_2, \dots, s_k\}$. Then if*

G acts on the left by isometries with respect to the word metric, d_S, then G' is finite, the order of every element of G' is bounded by $(2k)!$ and G' can be generated by $\binom{2k}{2}$ elements.

Proof. If $\zeta(G)$ is of finite index in G then it follows from Proposition 2.9.8 that $x^{|G:\zeta(G)|} = 1$ for all $x \in G'$ since G' lies in the kernel of φ as $G/\ker\varphi$ is isomorphic to a subgroup of $\zeta(G)$ and thus abelian. So it is now enough to show that $|G : \zeta(G)| \leq (2k)!$. Now if G acts on the left by isometries then it follows that $d_S(1, s_i) = d_S(s_j, s_j s_i)(= 1)$ for all $1 \leq i, j \leq k$. That is to say that either $s_j s_i^{-1} s_j^{-1} = s_t$ or $s_j s_i^{-1} s_j^{-1} = s_t^{-1}$ for some $1 \leq t \leq k$. Hence s_j acts on $S \cup S^{-1}$ by conjugation for all j and thus G acts on $S \cup S^{-1}$. The pointwise stabiliser of $S \cup S^{-1}$ is therefore $\zeta(G)$ and it follows that $|G : \zeta(G)| \leq (2k)!$. That G' can be generated by $\binom{2k}{2}$ elements follows from the identities that $[xy, z] = [y^{x^{-1}}, z^{x^{-1}}][x, z]$ and $[x, yz] = [x, y][x^{y^{-1}}, z^{y^{-1}}]$ for all $x, y, z \in G$ together with the fact that G acts on $S \cup S^{-1}$ by conjugation. $\square$

We now have the following corollary answering when a finitely generated torsion-free group acts on itself on the left by isometries. In particular we now see that any infinite finitely generated torsion-free non-abelian group is an example of a group which acts on itself on the right by isometries but the action by left multiplication is not an isometry.

Corollary 2.9.11. *Let G be a torsion-free group generated by a finite set S. Then G acts on the left and right by isometries with respect to d_S if and only if G is abelian.*

Once again however, the metric spaces we obtain are complete since the word metric takes values in $\mathbb{N}$ and so there are no non-trivial Cauchy sequences. Thus the completion of these spaces are the groups themselves and trivially have an underlying group structure. An interesting question then is whether there exist examples for which G acts on only one side by isometries and the completion of G does not have a group structure given by the pointwise multiplication of the Cauchy sequences. It is worth noting at this point that Lemma 2.9.5 may be refined in the following way.

Definition 2.9.12. Let $(X, d), (Y, d')$ be metric spaces. A map $f : X \to Y$ is said to be *Lipschitz continuous* if there exists some $\lambda > 0$ such that for all $x, y \in X$,

$$d'(fx, fy) \leq \lambda d(x, y).$$

Lemma 2.9.13. *Let G be a metric group such that for each $g \in G$ left and right action by g is a Lipschitz continuous map. Then the completion of G is a group with group multiplication given by pointwise multiplication of Cauchy sequences.*

Proof. The proof of Lemma 2.9.5 is easily modified to obtain the result. □

2.9.2 Comparing the completions

We proceed to show that under certain conditions the completion $\widehat{G}_{\mathscr{S}}$ is not only a group but is homeomorphic to the metric completion of G with respect to the metric obtained via the inverse limit structure of $\widehat{G}_{\mathscr{S}}$.

Definition 2.9.14. Let G be a group and let $\mathcal{S}$ denote a collection of subgroups of G. We call a function $\rho_{\mathcal{S}}^{G} : \mathcal{S} \to \mathbb{N}$ a *depth* in G if it satisfies the following condition,

$$\forall\ H < K \leq G, \quad \rho_{\mathcal{S}}^{G}(H) > \rho_{\mathcal{S}}^{G}(K).$$

Example 2.9.2. Let G be a group and let $\mathcal{S}$ denote the collection of all finite index subgroups of G. Then an example of a depth in G would be the index of a subgroup in G, i.e. $H \mapsto |G : H|$.

Let G be a group, $\mathcal{S}$ a collection of subgroups of G and suppose that ρ is a depth in G with respect to $\mathcal{S}$. Then we may define a metric on G in the following way.

Suppose that there exists a countable descending chain of subgroups $\mathcal{A} = (A_i)_{i \in \mathbb{N}}$

$$A_1 \geq A_2 \geq A_3 \geq \cdots ,$$

in $\mathcal{S}$ with the property that the intersection of all the groups in the chain is trivial. Then we may define a metric $d_{\mathcal{A}}$ on G in the following way.

$$d_{\mathcal{A}}(g_1, g_2) = \begin{cases} 0 & \text{if } g_1 g_2^{-1} \in A_i \quad \forall i \\ 1 & \text{if } g_1 g_2^{-1} \notin A_1 \\ \frac{1}{\rho(A_s)} & \text{if } s = \min_{n \in \mathbb{N}} (g_1 g_2^{-1} \notin A_n) \end{cases}$$

Lemma 2.9.15. *With the definition given above $d_{\mathcal{A}}$ is a metric on G.*

Proof. It is clear that for all $g \in G, d_{\mathcal{A}}(g, g) = 0$ since all the A_is are groups. Now suppose that for $g_1, g_2 \in G$ we have that $d_{\mathcal{A}}(g_1, g_2) = 0$. Then we see that $g_1 g_2^{-1} \in \cap_{i=1}^{\infty} A_i = \{e\}$. That is to say that $g_1 = g_2$. Further since the A_i's are groups it is clear that $g_1 g_2^{-1} \in A_i \iff g_2 g_1^{-1} \in A_i$ and it follows that for all $g_1, g_2 \in G, d_{\mathcal{A}}(g_1, g_2) = d_{\mathcal{A}}(g_2, g_1)$. It remains to show that the triangle inequality holds. Let $g_1, g_2, g_3 \in G$. We may assume without loss of generality that $g_1 \neq g_2 \neq g_3$. Then if $d_{\mathcal{A}}(g_1, g_3) = 0$ the triangle inequality holds and we are done. Suppose now that $d_{\mathcal{A}}(g_1, g_3) > 0$. Thus we may denote by a, b and c

the minima of the $n \in \mathbb{N}$ such that A_n does not contain $g_1g_3^{-1}, g_1g_2^{-1}$ and $g_2g_3^{-1}$ respectively. To verify the triangle inequality we require the following observation. For $i, j, k \in \{1, 2, 3\}$ distinct, $g_ig_j^{-1}, g_jg_k^{-1}, g_ig_k^{-1} \in A_i$ if and only if any two of them belong to A_i. This is true since we may write for example $g_ig_j^{-1} = (g_ig_k^{-1})(g_jg_k^{-1})^{-1}$ and A_i is a group and thus closed under both taking of inverses and products. With this result it is now clear that $a \geq \min(b, c)$. We may assume without loss of generality that $\min(b, c) = b$. Then since ρ is a depth function we have that $\rho(A_a) \geq \rho(A_b)$. Thus we have that $d_{\mathcal{A}}(g_1, g_3) \leq d_{\mathcal{A}}(g_1, g_2)$ and we are done. □

Remark 2.9.16.

- It is clear from the proof that we require the chains to have trivial intersection. Otherwise the function $d_{\mathcal{A}}$ may only be a pseudo-metric.
- Indeed we may see immediately from the last line of the above proof that in fact $d_{\mathcal{A}}(g_1, g_3) \leq \max(d_{\mathcal{A}}(g_1, g_2), d_{\mathcal{A}}(g_2, g_3))$ and so in fact $d_{\mathcal{A}}$ is an ultrametric on G.
- We observe that the proof holds for any function $\rho : \mathcal{S} \to \mathbb{R}^+$ satisfying the condition that
$$\forall\ H < K \leq G, \quad \rho_{\mathcal{S}}(H) > \rho_{\mathcal{S}}(K).$$

We now appeal to a standard result that for a given metric space, X, the supremum of a family of metrics on X that are bounded by some real number, N say, is itself a metric. For the metric just described the maximum distance between any two points is 1 and thus in the above circumstances we may define a metric on G, $d_{\mathcal{S}'} = \sup_{\mathcal{A}} d_{\mathcal{A}}$ where $\mathcal{A}$ runs through any family, $\mathcal{S}'$, of countable descending chains of subgroups in $\mathcal{S}$. In this way we may obtain a metric that does not require a particular choice of descending chain.

Proposition 2.9.17. *Let X be a metric space, $N > 0$ and for all i in some indexing set I, d_i a metric on X such that for all $x, y \in X$, $d_i(x, y) < N$. Then the map $d_{\sup} : X \times X \to \mathbb{R}$ defined by,*

$$d_{\sup}(x, y) = \sup_{i \in I} d_i(x, y),$$

is a metric on X.

Proof. Firstly notice that since all the metrics are bounded above by N it follows that $d_{\sup}(x, y) \leq N$ for all $x, y \in X$ and so the values of $d_{\sup}$ certainly lie in $\mathbb{R}$. That $d_{\sup}(x, y) = 0 \iff x = y$ is clear from the definition since each d_i is a metric on X. It also follows easily that $d_{\sup}(x, y) = d_{\sup}(y, x)$ for all $x, y \in X$. All that remains is to check the triangle inequality. Let $x, y, z \in X$. Then $d_{\sup}(x,z) \leq \sup_{i\in\mathbb{N}} d_i(x,y) + \sup_{i\in\mathbb{N}} d_i(y,z) = d_{\sup}(x,y) + d_{\sup}(y,z)$ and the proof is complete. $\square$

Example 2.9.3.

- Take $G = \mathbb{Z}$ with $\mathcal{S}$ the family of all subgroups $p^n\mathbb{Z}$ for a fixed prime p and any natural number n. Taking ρ to be the map giving the index of the given subgroup in G then we get the metric $d(a,b) = 1/$ (the smallest power of p not dividing $a-b$).

- Similarly consider $G = \mathbb{Z}$ with $\mathcal{S}$ the family of all subgroups of G. Then the metric we obtain is given by $d(a,b) = 1/$ (the smallest natural number not dividing $a-b$).

- G a countable group having a finitely generated subgroup H commensurable with all its conjugates and $\mathscr{S}$ the family of all subgroups of H commensurable with H. Then taking ρ to be the map giving the index of the given subgroup in H the metric obtained is $d(g_1, g_2) = 1/|H : L|$ where L is a finite index subgroup of H not containing $g_1 g_2^{-1}$ of minimal index in H.

Connection with the metric space completion

We now have enough of the theory in place to begin to show how to construct a homeomorphism between $\widehat{G}_{\mathscr{S}}$ and the metric completion of G with respect to the metric on G defined by a descending chain of subgroups. We begin by stating the topology we put on $\widehat{G}_{\mathscr{S}}$. Firstly notice that we may consider $\widehat{G}_{\mathscr{S}}$ as the inverse limit of the coset spaces $H\backslash G$ as H runs through the members of $\mathscr{S}$. We then consider the coset spaces as discrete topological spaces and put the inverse limit topology on $\widehat{G}_{\mathscr{S}}$. Thus the basic open sets in $\widehat{G}_{\mathscr{S}}$ correspond to unions of finitely many cosets of a given group in $\mathscr{S}$.

We give some examples first to keep in mind throughout the following proofs. These examples correspond to the metric completions of Examples 2.9.3, and all these facts shall follow from Theorem 2.9.20.

Example 2.9.4.

- $G = \mathbb{Z}$ and $\mathscr{S}$ the family of subgroups of the form $p^n\mathbb{Z}$. This has metric completion $\varprojlim \mathbb{Z}/p^n\mathbb{Z}$.

- $G = \mathbb{Z}$ and $\mathscr{S}$ the family of all non-trivial subgroups gives rise to the metric completion $\varprojlim_{m\in\mathbb{Z}} \mathbb{Z}/m\mathbb{Z}$.

- G with $\mathscr{S}$ the family of all subgroups commensurable with a given finitely generated subgroup H such that $\mathrm{Comm}_G(H) = G$ has metric completion homeomorphic to

$\widehat{G}_{\mathscr{S}}$.

Lemma 2.9.18. *Let G be a group, $\mathcal{S}$ any collection of subgroups of G with trivial intersection, $\mathscr{S}$ an admissible family of subgroups of G and $\rho : \mathcal{S} \to \mathbb{N}$ a depth function for G. We define the metric $d_{\mathcal{S}}$ on G to be the supremum of the metrics obtained for all countable descending chains in $\mathcal{S}$ with trivial intersection. If for each $H \in \mathscr{S}$ there exists a countable descending chain of subgroups, $\mathcal{A} = (A_n)_{n\in\mathbb{N}}$ in $\mathcal{S}$ such that for some $m \in \mathbb{N}$ we have that $A_m \leq H$ then there exists a continuous map $\varphi : \overline{G} \to \widehat{G}_{\mathscr{S}}$.*

Proof. We construct the map $\varphi : \overline{G} \to \widehat{G}_{\mathscr{S}}$ as follows. Let $x = (x_i)_{i\in\mathbb{N}}$ be a Cauchy sequence in $(G, d_{\mathcal{S}})$. That is to say that,

$$\forall \epsilon > 0, \quad \exists N > 0 \quad \text{such that } m, n > N \implies d_{\mathcal{S}}(x_m, x_n) < \epsilon.$$

Fix a descending chain,

$$\mathcal{K} : \quad \cdots < K_3 < K_2 < K_1 = K,$$

of subgroups in $\mathcal{S}$ with trivial intersection. Then for any K_i in the chain there exists $i' > 0$ such that for all $m, n > i'$ we have that $d_{\mathcal{K}}(x_m, x_n) < \frac{1}{\rho(K_i)}$. Thus for $m, n > i'$ we have $x_m x_n^{-1} \in K_i$ or in other words $K_i x_m = K_i x_n$. Thus the Cauchy sequence x eventually gives a choice of coset of K_i.

Now for any subgroup $K_i \leq H$, we observe that the Cauchy sequence eventually lies in a unique coset of H. We now have a choice of coset for all groups in $\mathscr{S}$ given by x since every group in $\mathscr{S}$ contains an A_m from some chain, $\mathcal{A}$, in $\mathcal{S}$ by hypothesis. For each $H \in \mathscr{S}$ we denote this coset by Hx, observing that x is not an element of G but a Cauchy sequence of elements belonging to G.

We check that for $H \leq L \in \mathscr{S}$ we have that $Hx \subseteq Lx$. Observe that the definition of Hx is that coset of H to which eventually all elements of x belong. Thus it is clear that $Hx \subseteq Lx$ and thus we have a map $\overline{G} \to \widehat{G}_{\mathscr{S}}, x \mapsto (Hx)_{H\in\mathscr{S}}$. Observe that this is well defined if we identify x with its equivalence class given by the relation $x \sim y \iff \lim_{n\to\infty} d_{\mathcal{S}}(x_n, y_n) = 0$. That is to say that two sequences are equivalent if they eventually give the same cosets upon passing to sufficiently small subgroups.

It remains to show that this map is continuous. Suppose now that V is an open subset in $\widehat{G}_{\mathscr{S}}$. Recall that a basic open set in $\widehat{G}_{\mathscr{S}}$ is given by restricting the permitted cosets of H_i for finitely many subgroups $H_i \in \mathscr{S}$. By the downward directed property of $\mathscr{S}$ together with the definition of an element of $\widehat{G}_{\mathscr{S}}$ we see that such a basic open set is defined by restricting the choice of coset at a single $H \in \mathscr{S}$. Let $f \in V$ and y be a Cauchy sequence in the pre-image of f. Then the set of Cauchy sequences z in $(G, d_{\mathcal{S}})$ such that $\lim_{n\to\infty} d_{\mathcal{S}}(y_n, z_n) < \frac{1}{\rho(H)}$ is an open set in $\overline{G}$ containing y whose image lies in V. Thus we see that the map $\overline{G} \to \widehat{G}_{\mathscr{S}}$ is continuous. □

Lemma 2.9.19. *Let G be a group, $\mathcal{S}$ any collection of subgroups of G with trivial intersection, $\mathscr{S}$ an admissible family of subgroups of G and $\rho : \mathcal{S} \to \mathbb{N}$ a depth function for G. We define the metric $d_{\mathcal{S}}$ on G to be the supremum of the metrics obtained for all countable descending chains in $\mathcal{S}$ with trivial intersection. If there exists a descending chain $(H_i)_{i\in\mathbb{N}} \in \mathscr{S}$ such that for all $A \in \mathcal{S}$ with $\rho(A) \leq m$, $H_m \leq A$, then there exists a continuous map $\psi : \widehat{G}_{\mathscr{S}} \to \overline{G}$.*

Proof. Let $f \in \widehat{G}_{\mathscr{S}}$. Then by the properties of $\widehat{G}_{\mathscr{S}}$ we obtain a descending chain of cosets,

$$f(H_1) \supset f(H_2) \supset f(H_3) \supset \cdots .$$

We now choose a sequence of coset representatives, $x_i \in f(H_i)$. Then the sequence (x_i) shall be a Cauchy sequence in G with respect to the metric induced by the descending chain of the H_is since for $i, j > N$ it follows that $x_i x_j^{-1} \in H_N$. That the equivalence class of (x_i) is independent of the choice of descending chain in $\mathscr{S}$ follows by passing to the chain given by the intersection of two such chains. Further the sequence (x_i) is in fact Cauchy with respect to the supremum metric $d_{\mathcal{S}}$ since,

$$\begin{aligned} i, j > N &\implies x_i x_j^{-1} \in H_N \\ &\implies x_i x_j^{-1} \in K \quad \forall\, K \in \mathscr{S} \text{ s.t. } \rho(K) < N \\ &\implies d_{\mathcal{S}}(x_i, x_j) < \frac{1}{N}. \end{aligned}$$

We define the map $\psi : \widehat{G}_{\mathscr{S}} \to \overline{G}$ to be the map sending f to the equivalence class of (x_i), obtained above, in $\overline{G}$. It remains to show that the map ψ is continuous. To see this let $f \in \widehat{G}_{\mathscr{S}}$ and let U_m denote the open ball of radius $\frac{1}{\rho(H_m)}$ in $\overline{G}$ about f. Let $y \in \psi^{-1}(U_m)$ then the set $V_y = \{g \in \widehat{G}_{\mathscr{S}} | g(H_m) = y(H_m)\}$ is an open set in $\widehat{G}_{\mathscr{S}}$ containing y and $\psi(V_y) \subseteq U_m$. □

Theorem 2.9.20. *Let G be a group, $\mathcal{S}$ any collection of subgroups of G with trivial intersection, $\mathscr{S}$ an admissible family of subgroups of G and $\rho : \mathcal{S} \to \mathbb{N}$ a depth function for G. We define a metric, $d_{\mathcal{S}}$, on G given by the supremum of the metrics obtained for all countable descending chains in $\mathcal{S}$ with trivial intersection. If there exist $H_i \in \mathscr{S}, A_i \in \mathcal{S}$*

such that $H_1 \geq A_1 \geq H_2 \geq A_2 \geq \cdots$ then there exist continuous maps $\varphi : \overline{G} \to \widehat{G}_{\mathscr{S}}$ and $\psi : \widehat{G}_{\mathscr{S}} \to \overline{G}$ as defined in Lemmas 2.9.18 and 2.9.19 which are in fact homeomorphisms.

Proof. It is clear that the hypotheses of Lemmas 2.9.18 and 2.9.19 are satisfied and thus the continuous maps φ, and ψ exist. It is enough now to verify that the two compositions are the corresponding identity maps. That $\varphi \circ \psi$ is the identity on $\widehat{G}_{\mathscr{S}}$ is clear. The map ψ gives a Cauchy sequence of coset representatives of the H_i and then φ returns the function picking out the cosets containing the x_i together with a uniquely induced choice of cosets elsewhere. The other composition requires only slightly more thought. We begin with a Cauchy sequence (x_i) and the map φ gives the function that identifies the cosets eventually containing the x_i. The map ψ now gives a collection of coset representatives for these cosets. Notice that this need not return our original x_i, instead we may obtain a sequence (z_i), however, by the above construction it is clear that $\lim_{n\to\infty} d_{\mathcal{S}}(x_i, z_i) = 0$. Therefore both sequences represent the same element of the metric completion of G. □

Corollary 2.9.21. *Let G be a countable group, $H < G$ a finitely generated subgroup such that $\mathrm{Comm}_G(H) = G$ and $\mathscr{S}$ the admissible family of subgroups of G commensurable with H. There is a metric, d, on G obtained by taking the supremum of all metrics obtained via countably infinite descending chains of subgroups of G starting at H with trivial intersection. Then if there exists a descending chain in $\mathscr{S}$ with trivial intersection then the completion of (G, d) is homeomorphic to $\widehat{G}_{\mathscr{S}}$.*

Proof. Apply Theorem 2.9.20 taking $\mathcal{S}$ to be the family of all finite index subgroups of H. The required descending chain is given by taking $H_0 = H$ and for all $i \in \mathbb{N}, H_i$ is chosen to be the intersection of all subgroups of H of index no greater than i. Notice that for this to be a chain of subgroups in $\mathscr{S}$ we need that the group H is finitely generated. Thus there are only finitely many subgroups of H of a given finite index and we know that $\mathscr{S}$ is closed under finite intersections. □

Remark 2.9.22. Notice that the homeomorphisms mentioned in the other examples in the above section may be obtained from Theorem 2.9.20. For example,

- $G = \mathbb{Z}$ and $\mathscr{S}$ the family of subgroups of the form $p^n\mathbb{Z}$. This has the particularly nice property that the family of subgroups $\mathscr{S}$ itself forms a descending chain and thus the supremum metric can be thought of in terms of a metric defined by a single descending chain of subgroups.

- $G = \mathbb{Z}$ and $\mathscr{S}$ the family of all non-trivial subgroups. In this example a suitable choice of descending chain would be

$$\mathbb{Z} \geq 2!\mathbb{Z} \geq 3!\mathbb{Z} \geq \cdots .$$

Chapter 3

Generalisations of the Almost Stability Theorem

In Chapter 4 we shall prove the following theorem. The notation used shall be explained fully in section 3.2.

Theorem A. *Let H be a finitely generated subgroup of G with $\chi(H)$ a non-zero integer and $\mathrm{Comm}_G(H) = G$. Let $\mathscr{S}$ be the admissible family of subgroups commensurable with H and A and I be non-empty sets. Suppose that V is a G-stable $\mathscr{S}$-almost equality class in $\mathscr{S}(\sqcup_I G, A)$. Then there exists a G-tree with edge stabilisers in $\mathscr{S}$ and vertex set V.*

Then in the final chapter we use Theorem A to obtain the following result.

Theorem B. *Let G be a group of cohomological dimension $n < \infty$. Let H be a PD^{n-1} subgroup of G such that $\mathrm{Comm}_G(H) = G$ and $\chi(H)$ is a non-zero integer. Then there exists a G-tree T with edge and vertex stabilisers commensurable with H.*

In this chapter we introduce the Almost Stability Theorem and make some basic observations about when this theorem is straightforward to check. We then mention some conjectures we have made, how they relate to the Almost Stability Theorem, when these results are equivalent and the open questions concerning these conjectures that remain.

3.1 The Almost Stability Theorem

We now state the Almost Stability Theorem.

The Almost Stability Theorem (Dicks-Dunwoody)**.** *Let G be a group, E a G-set with finite stabilisers and A some non-empty set. Let V be a G-stable almost equality class in (E, A). Then there exists a G-tree T with finite edge stabilisers and vertex set V.*

Remark 3.1.1. We first note some special cases of the Almost Stability Theorem for which the proof is trivial. Observe, firstly that if V contains a fixed point, v_0 say, then we may construct a G-tree with an edge joining every point of $V \setminus v_0$ to v_0. It is easy to see that G acts on this tree. In order to show that the edge stabilisers are finite as required we make the assumption here, which shall follow from Proposition 3.7.1, that the complete graph on V has finite edge stabilisers. We can use this argument to obtain a straightforward proof in the following cases:

- V contains some constant function. Clearly the constant functions are fixed by G.
- $|E| < \infty$. Then $V = (E, A)$ and so clearly contains the constant functions.
- The action of G on E is trivial. Here every element of V is fixed by G.
- $|A| < 2$, since V consists of a single function and our tree is a single vertex.

We now give some justification of the statement by showing that the vertex set of a G-tree can be thought of as a subset of a G-stable almost equality class of $(ET, \mathbb{Z}_2)$. The following example demonstrates a key technique in obtaining a class of functions from a G-graph.

Example 3.1.1. For a G-tree T, the structure map $ET \to (VT, \mathbb{Z}_2)$ is given by sending each edge, e, of T to the set of vertices which it points towards, i.e. the set of vertices in the component of $T - \{e\}$ which contains τe. The dual map, also known as the costructure map, then is that which sends each vertex to the set of edges which point towards that vertex. It then follows that for any two vertices $v_1, v_2 \in VT$ the set $v_1|ET \bigtriangledown v_2|ET$ is just the set of edges in the path in T joining v_1 to v_2 and thus $v_1|ET =_a v_2|ET$. Thus we have that each vertex may be considered as a function from the edge set to $\mathbb{Z}_2$ and that these functions lie in an almost equality class. To see that this class is in fact G-stable we observe that for all $v \in VT,\ g \in G$ we have that $vg \in VT$ and thus $v|ET =_a vg|ET$.

3.2 Some Conjectures

We now wish to introduce some potential generalisations of the Almost Stability Theorem. However, before we may discuss these conjectures it is necessary to introduce some new terminology.

Definition 3.2.1. Let $\mathscr{F}$ denote a family of subgroups of G. We denote by $\mathscr{F}(G, A)$ all functions from G to A that are constant on the left cosets, gK, of some group $K \in \mathscr{F}$. In a similar fashion we denote by $\mathscr{F}(\sqcup_I G, A)$ all functions from $\sqcup_I G$ to A, where I is an arbitrary indexing set, that are for each factor G_i constant on the cosets of some group $K_i \in \mathscr{F}$.

Definition 3.2.2. Let $\mathscr{S}$ be an admissible family of commensurable subgroups of G. We say that two functions $f, g \in (G, A)$ are *$\mathscr{S}$-almost equal*, denoted $f =_{\mathscr{S}} g$ if the set $\{x \in G | f(x) \neq g(x)\}$ is contained in finitely many left cosets of some group in $\mathscr{S}$. More generally for an arbitrary indexing set I and two functions $f, g \in (\sqcup_I G, A)$, we say that f and g are *$\mathscr{S}$-almost equal*, denoted $f =_{\mathscr{S}} g$, if for all but finitely many $i \in I$ the restriction of the two functions to G_i, the i-th copy of G, are equal and for the finitely many such exceptions the functions $f|_{G_i}$ and $g|_{G_i}$ are $\mathscr{S}$-almost equal in the original sense.

Remark 3.2.3.

1. Observe that $\mathscr{S}$-almost equality is an equivalence relation and partitions the set of functions from G to a non-empty set A into $\mathscr{S}$-almost equality classes. If a function f from G to A satisfies $f \cdot g =_{\mathscr{S}} f$ for all $g \in G$ then we say that the $\mathscr{S}$-almost equality class containing f is a *G stable $\mathscr{S}$-almost equality class.*

2. Notice that we use here that $\mathscr{S}$ is an admissible family to show that $\mathscr{S}$-almost equality is an equivalence relation. For example for $f_1, f_2, f_3 \in (G, A)$ if $f_1 \bigtriangledown f_2$ lies in a finite union of cosets of H_1 and $f_2 \bigtriangledown f_3$ lies in a finite union of cosets of some subgroup H_2 then $f_1 \bigtriangledown f_3$ lies in a union of cosets of $H_1 \cap H_2$. That this final union is finite follows from the fact that $H_1 \cap H_2$ is of finite index in both H_1 and H_2 as the subgroups in $\mathscr{S}$ are commensurable. That a finite index subgroup of $H_1 \cap H_2$ belongs to $\mathscr{S}$ follows from the fact that $\mathscr{S}$ is an admissible family.

3. Notice that in the case that $\mathscr{S}$ consists of the family of all finite subgroups we retrieve the definition of almost equality.

The following lemma concerning when the properties of a G-stable $\mathscr{S}$-almost equality class passes down to a subgroup will be of particular use in our later induction arguments.

Lemma 3.2.4. *Let $\mathscr{S}$ be a commensurability class of finitely generated near-normal subgroups of G. Suppose that $H \leq G$ contains a member of $\mathscr{S}$. Let $\mathscr{T} = \{H \cap K | K \in \mathscr{S}\}$ and let $V \subseteq \mathscr{S}(\bigsqcup_I G, A)$ be an H-stable $\mathscr{S}$-almost equality class. Then $\mathscr{T}$ is also a* *commensurability class of finitely generated near-normal subgroups of H. Suppose that $G = \bigsqcup_{j\in J} x_j H$. Then V is isomorphic as an H-set to an H-stable $\mathscr{T}$-almost equality class in $\mathscr{T}(\bigsqcup_{I\times J} H, A)$.*

Proof. We first observe that $\mathscr{T}$ is a commensurability class of finitely generated near-normal subgroups of H. It is clear that the groups in $\mathscr{T}$ are commensurable with one another as $\mathscr{S}$ is a commensurability class. Since $\mathscr{T}$ contains a member of $\mathscr{S}$ it follows that the members of $\mathscr{T}$ are commensurable with the subgroups contained in $\mathscr{S}$ and thus are also finitely generated. In fact, $\mathscr{T} = \{K \mid K \in \mathscr{S},\ K \subseteq H\}$.

We define a map $V \to (\bigsqcup_{I\times J} H, A), v \mapsto \overline{v}$ by,

$$\overline{v}(h^{(i,j)}) = v((x_j h)^{(i)}).$$

To see that this is an H-map, let $h_1, h_2 \in H$,

$$\begin{aligned}(\overline{v}\cdot h_2)(h_1^{(i,j)}) &= \overline{v}((h_1 h_2^{-1})^{(i,j)}) \\ &= v((x_j h_1 h_2^{-1})^{(i)}) \\ &= (v\cdot h_2)(x_j h_1^{(i)}) \\ &= \left(\overline{v\cdot h_2}\right)(h_1^{(i,j)}).\end{aligned}$$

This is then an injective H-map and since V is an $\mathscr{S}$-almost equality class it follows that the image lies in a $\mathscr{T}$-almost equality class. It remains to show that the image lies in $\mathscr{T}(\bigsqcup_{I\times J} H, A)$. Suppose that our function v is constant on the cosets of $L \in \mathscr{S}$ in the ith factor. Then we claim that $\overline{v}$ is constant on the cosets of $L \cap H$ in the (i,j)th factors for all j. By assumption then for all $l \in L$ and $g \in G$, we have that $v(gl) = v(g)$. Now for all $l_2 \in L \cap H, h \in H$,

$$\begin{aligned}\overline{v}(hl_2^{(i,j)}) &= v(x_j h l_2^{(i)}) \\ &= v(x_j h^{(i)}) \\ &= \overline{v}(h^{(i,j)}).\end{aligned}$$

Hence we obtain our result. □

We may now state the conjectures under consideration. This first conjecture seems the most natural of our generalisations of the Almost Stability Theorem.

Conjecture A. *Let H be a finitely generated subgroup of G such that* $\mathrm{Comm}_G(H) = G$. *Let E be a G-set with stabilisers commensurable with H and A be a non-empty set. Let*

V be a G-stable almost equality class in (E, A). Then there exists a G-tree T with edge stabilisers commensurable with H and vertex set V.

We have simply replaced the finite stabilisers condition with a finitely generated and commensurable with all of its conjugates condition. It should be straightforward then to notice that this generalises the Almost Stability Theorem as this is simply the case that H is taken to be the trivial group. However, when we consider our attempt to prove Theorem B in Chapter 5 it can be seen that we deal with a system of sets of functions and we wish to allow functions that differ on finitely many cosets of arbitrarily small subgroups commensurable with H. We do not necessarily have a fixed G-set, E in this setting. It is for this very application that we arrive at the following conjecture.

Conjecture A*. *Let H be a finitely generated subgroup of G such that $\mathrm{Comm}_G(H) = G$. Let $\mathscr{S}$ be the admissible family of subgroups of G commensurable with H and A be a non-empty set. Suppose that V is a G-stable $\mathscr{S}$-almost equality class in $\mathscr{S}(G, A)$. Then there exists a G-tree T with edge stabilisers in $\mathscr{S}$ and vertex set V.*

It will be shown in Chapter 5 that Conjecture A* is sufficient to prove Theorem B. However, what is not so clear any more is whether the Almost Stability Theorem may be recovered from this conjecture. This is because there is the following method for identifying a G-stable almost equality class in (E, A) with a G-stable $\mathscr{S}$-almost equality class in $\mathscr{S}(\sqcup G, A)$ whenever the G-set E has stabilisers in $\mathscr{S}$.

Definition 3.2.5. Let $W \subseteq (G/H, A)$ for some subgroup $H \leq G$ and non-empty set A. Then there is an injective map $\iota : (G/H, A) \hookrightarrow \{H\}(G, A)$ where $(\iota(\phi))(g) = \phi(gH)$. This injection allows us to consider W as a subset of $\{H\}(G, A)$.

Throughout this chapter we shall often use this map to pass from a function on the set of cosets of a subgroup H to a function with domain G and constant on the cosets of H without further mention. However, to relate conjectures concerning functions with differing domains it would be helpful to be able to go the other way. That is to say that given a function v in $\mathscr{S}(G, A)$ is there an appropriate choice of G-set, E, such that v arises from a function in (E, A). We can show that there is such a suitable choice in the case that the admissible family contains a minimal, and thus normal by Lemma 2.7.10, subgroup N.

Lemma 3.2.6. *Let $\mathscr{S}$ be an admissible family of commensurable subgroups containing a minimal element N, say. Suppose that V is a G-stable $\mathscr{S}$-almost equality class in*

$\mathscr{S}(\sqcup_I G, A)$. Then there is a G-stable almost equality class, W, in $(\sqcup_I G/N, A)$ such that $V \cong W$ as G-sets.

Proof. Observe that if N is a minimal element of $\mathscr{S}$ then $\mathscr{S}(\sqcup G, A) = \{N\}(\sqcup G, A)$ which may clearly be identified with $(\sqcup G/N, A)$. Almost equality in $(\sqcup G/N, A)$ corresponds to $\mathscr{S}$-almost equality in $\mathscr{S}(\sqcup G, A)$. We already have a map in one direction cf. Definition 3.2.5. Now there is a map $\mathscr{S}(\sqcup G, A) \to (\sqcup G/N, A) : v \mapsto \overline{v}$ where $\overline{v}(gN) = v(g)$. That this map is a G-map follows from the fact that $N \lhd G$ being a minimal element of $\mathscr{S}$. For $\gamma, g \in G$,

$$\begin{aligned}
\overline{v} \cdot g(\gamma N) &= \overline{v}(\gamma N g^{-1}) \\
&= \overline{v}(\gamma g^{-1} N) \qquad \text{as } N \lhd G \\
&= v(\gamma g^{-1}) \\
&= (v \cdot g)(\gamma) \\
&= \overline{v \cdot g}(\gamma N).
\end{aligned}$$

□

In particular the admissible family of finite subgroups contains a minimal element, namely the trivial group. Thus we see that in the case H is taken to be the trivial group Conjecture A* is simply a restatement of the Almost Stability Theorem in the case that E is a transitive G-finite G-set. Thus we make the following minor modification to obtain another conjecture.

Conjecture A.** *Let H be a finitely generated subgroup of G such that* $\mathrm{Comm}_G(H) = G$. *Let $\mathscr{S}$ be the admissible family of subgroups of G commensurable with H and A and I be non-empty sets. Suppose that V is a G-stable $\mathscr{S}$-almost equality class in $\mathscr{S}(\sqcup_I G, A)$. Then there exists a G-tree T with edge stabilisers in $\mathscr{S}$ and vertex set V.*

We will see via Corollary 3.3.9 that Conjecture A** then implies the Almost Stability Theorem. That Conjecture A** implies Conjecture A* is clear as it is the special case where the disjoint union is of a single copy of G, i.e. $|I| = 1$. It is not known whether Conjecture A* is in fact equivalent to Conjecture A**. However in the proof of 3.3.9 we show that the vertex set in Conjecture A is a G-retract of the vertex set in Conjecture A** and so whenever we have the additional condition that the edge stabilisers are G-conjugate incomparable then we may use Theorem 3.3.4 to show that these two conjectures are equivalent.

We also mention at this stage an interesting potential application of any such a generalisation of the Almost Stability Theorem and some variants on this application that may be proved under certain additional hypotheses. These conjectures are known to be true in the case that G is finitely generated by a result of Kropholler [23].

Conjecture B. *Let G be a group of cohomological dimension $n < \infty$. Let H be a PD^{n-1} subgroup of G such that $\mathrm{Comm}_G(H) = G$. Then there exists a G-tree T with edge and vertex stabilisers commensurable with H.*

Conjecture B′. *Let G be a group of cohomological dimension $n < \infty$. Let H be a PD^{n-1} subgroup of G such that $\mathrm{Comm}_G(H) = G$. Suppose that the subgroups of G commensurable with H are G-conjugate incomparable. Then there exists a G-tree T with edge and vertex stabilisers commensurable with H.*

Again, it is straightforward to see that Conjecture B' is a special case of Conjecture B. The motivation for the second conjecture is that the G-conjugate incomparability condition seems to be the minimum restriction for which our techniques used in the proof of the A conjectures are applicable. In fact we can prove that this is true in the case G is finitely generated by Theorem 4.2.16 together with the fact that Conjecture A implies Conjecture B shown in Chapter 5. This retrieves the main result of Kropholler in [23]. In the following section we investigate the relations between the conjectures and discuss the cases in which they are known to be true.

3.3 Connecting the conjectures

In order to prove the more interesting connections between these conjectures we must first utilise a theory that allows us to manipulate the vertex sets of G-trees. To this end we introduce the notion of a G-retract which will allow us to extend and contract G-trees.

Definition 3.3.1. Let V be a G-set. We say that a G-set U is a *G-retract* of V if there exist G-maps $\iota : U \to V$ and $\pi : V \to U$ such that ι is an injective map.

Remark 3.3.2.

- Observe that if U is a G-retract of V we may always choose π such that the composition $\pi \circ \iota$ is the identity on U. This follows from the fact that ιU is a G-subset of V and since ι is injective it follows that the G-map $\pi' : V \to U$,

$$\pi' v = \begin{cases} \pi v & \text{if } v \in V - \iota U \\ u & \text{if } v = \iota u, \end{cases}$$

is well defined.

- If T is a G-tree and T' is a G-subtree then VT' is a G-retract of VT. The required maps being the inclusion map and the G-map sending each vertex in T to the unique closest vertex of T'.

Having noticed the connection between G-retracts and subtrees an important result is that we may always extend a G-tree whenever the vertex set is a G-retract of some other G-set.

Theorem 3.3.3. *Let T be a G-tree. Suppose that VT is a G-retract of a G-set V'. Then the tree T may be extended to a G-tree with vertex set V'.*

Proof. For each vertex $v \in V' \setminus VT$ we add an edge e_v with initial vertex v and terminal vertex $\pi(v)$. Clearly this new graph is connected and closed under the action of G. It follows that it is a tree since we have only added vertices of valency 1 and so any circuits give rise to closed reduced paths in our original tree. □

It is not however the case that every G-retract of the vertex set of a G-tree is itself the vertex set of a G-tree. A partial result in this direction has however been proved by Dicks and Dunwoody in [13].

Theorem 3.3.4. *Let T be a G-tree and let U be a G-retract of VT. Suppose that the edge stabilisers of T are G-conjugate incomparable. Then there exists a G-tree with vertex set U.*

Remark 3.3.5. In fact the result in [13] is obtained in the more general case that no stabiliser of a vertex in $VT - U$ is G-conjugate incomparable. Thus, provided we do not remove any of the vertices with stabilisers conjugate to proper subgroups of themselves, we may extend and contract G-trees with vertex sets that are G-retracts of each other. Further Dicks and Dunwoody give an example of a G-retract of the vertex set of a G-tree that is not the vertex set of any G-tree. The original tree in their example does not have commensurable edge stabilisers.

We make the following observations concerning the conjectures. We would like to show that Conjecture A** $\implies$ the Almost Stability Theorem. Firstly we recall that given an almost equality class, V say, in (E, A) for some G-set E with stabilisers in some admissible family $\mathscr{S}$, we obtain an $\mathscr{S}$-almost equality class in $(\sqcup G, A)$ by identifying a function $\varphi : \sqcup G/G_e \to A$ with the function $\widehat{\varphi} : \sqcup G \to A$ where $\widehat{\varphi}(g^{(i)}) = \varphi(gG_{e_i})$. This is a more general version of the map ι from Definition 3.2.5. We have already dealt with the case that $\mathscr{S}$ has a minimal element (Lemma 3.2.6), however in the more general setting the $\mathscr{S}$-almost equality class in $(\sqcup G, A)$ may be larger than our original almost equality class. Now if we can show that the $\mathscr{S}$-almost equality class, $\widetilde{V}$, that we obtain contains our original V as a G-retract then by Theorem 3.3.4 we see that the tree obtained via Conjecture A** implies the existence of a tree with vertex set V and thus we are done. The difficulty lies in constructing a G-map $\psi : \widetilde{V} \to V$ and also in choosing a suitable V for a given $\widetilde{V}$.

Further if such a retraction map exists then we would have that a G-tree with vertex set V extends to a G-tree with vertex set $\widetilde{V}$. This together with the fact that the complete graph on $\widetilde{V}$ has edge stabilisers in $\mathscr{S}$ whenever the groups in $\mathscr{S}$ are finitely generated (Proposition 3.7.1) gives that Conjecture A implies Conjecture A*. Indeed by the same argument we see that Conjecture A implies Conjecture A**.

Lemma 3.3.6. *Let $\mathscr{S}$ be an admissible family of commensurable subgroups of G containing $H \leq G$ and let A be a non-empty set. Suppose that $V \subset (G/H, A)$ and $\widetilde{V} \subset \mathscr{S}(G, A)$ are a G-stable almost equality class and G-stable $\mathscr{S}$-almost equality class respectively, such that $V \subset \widetilde{V}$ when identifying $(G/H, A)$ with its inclusion in $\mathscr{S}(G, A)$. Then V is a G-retract of $\widetilde{V}$.*

Proof. By hypothesis we already have an injective G-map $V \to \widetilde{V}$. Thus it is enough to show the existence of a G-map $\Psi : \widetilde{V} \to V$. For this it is sufficient to show that for each $\varphi \in \widetilde{V}$ there exists an $f \in V$ such that $G_\varphi \leq G_f$.

Let $\varphi \in \widetilde{V}$. Then since $V \subset \widetilde{V}$ we have that there exists an $f' \in V$ such that $f' =_{\mathscr{S}} \varphi$. We denote the finitely many cosets of H on which f' and φ differ by $Hg_1, \ldots, Hg_n$. Fix some $a_0 \in A$. Let f be the function in V that agrees with f' apart from, at most, on the finitely many cosets Hg_i where f takes the value a_0.

We claim that $G_\varphi \leq G_f$. This follows from the fact that G_φ permutes the cosets $Hg_1, \ldots, Hg_n$. This is true since φ must be constant on the cosets of H apart from the Hg_i since it agrees with f and so for all $x \in G_\varphi, \varphi \cdot x$ must be constant on the cosets of H apart from $Hg_1x, \ldots Hg_nx$ as $\varphi \cdot x(g) = \varphi(gx^{-1})$. To now prove the claim let $x \in G_\varphi$ and suppose $g \in G \setminus \bigsqcup_{i=1}^{n} Hg_i$. Then

$$\begin{aligned}(f \cdot x)(g) &= f(gx^{-1}) \\ &= \varphi(gx^{-1}) && \text{since } G_\varphi \text{ permutes the } Hg_i \\ &= (\varphi \cdot x)(g) = \varphi(g) = f(g) && \text{since } g \notin \bigsqcup_{i=1}^{n} Hg_i.\end{aligned}$$

On the other hand if $g \in \bigsqcup_{i=1}^{n} Hg_i$ then $g \in Hg_i$ for some $1 \leq i \leq n$, then we have that,

$$\begin{aligned}(f \cdot x)(g) &= f(gx^{-1}) \\ &= f(g_i) = a_0 \quad \text{since } G_\varphi \text{ permutes the } Hg_i \text{ and } f \text{ is constant on the } Hg_i.\end{aligned}$$

Hence $G_\varphi \leq G_f$ and the result is clear. □

Lemma 3.3.7. *Let $\mathscr{S}$ be an admissible family of commensurable subgroups of G, E be a G-set with stabilisers in $\mathscr{S}$ and A be a non-empty set. Then we may write $E \cong \sqcup_I G/G_{e_i}$. Suppose that $V \subset (E, A)$ and $\widetilde{V} \subset \mathscr{S}(\sqcup_I G, A)$ are a G-stable almost equality class and G-stable $\mathscr{S}$-almost equality class respectively, such that $V \subset \widetilde{V}$ when considering (E, A) as a subset of $\mathscr{S}(\sqcup_I G, A)$. Then V is a G-retract of $\widetilde{V}$.*

Proof. The proof of Lemma 3.3.6 may easily be modified to obtain the result. Notice that in the notation of the proof of 3.3.6 we have that f' and φ differ on only finitely many $G^{(i)}$ and for each factor on only finitely many cosets of the corresponding G_{e_i}. Set f to be a_0 on all such cosets and by the same argument $G_\varphi \leq G_f$. □

Corollary 3.3.8. *Conjecture A implies Conjecture A**.*

Proof. The difficulty in proving that Conjecture A implies Conjecture A** lies in choosing a suitable G-set E and G-stable almost equality class, $\overline{V}$ say, in (E, A) lying in the preimage of our G-stable $\mathscr{S}$-almost equality class, V, under the map ι from Definition 3.2.5. Let $v \in V$. Then by definition $v \in (\sqcup G/K_i, A)$ for some $K_i \in \mathscr{S}$. Now let $E = \sqcup G/K_i$ and $\overline{V}$ be the almost equality class in (E, A) generated by v. It follows that $\overline{V}$ is G-stable as V itself is G-stable. Now we have that $\overline{V}$ is in fact a G-retract of V and thus the G-tree obtained via Conjecture A may be extended to have vertex set V. □

Corollary 3.3.9. *Conjecture A** implies the original Almost Stability Theorem.*

Proof. That Conjecture A** implies the Almost Stability Theorem follows from Theorem 3.3.4. This is since any function in (E, A) corresponds to a function in $\mathscr{S}(\sqcup G, A)$ and so generates an $\mathscr{S}$-almost equality class containing the image of our original almost equality class. Theorem 3.3.4 applies since finite groups are G-conjugate incomparable. $\square$

Indeed this proves the following more general result.

Corollary 3.3.10. *Suppose that $\mathscr{S}$ consists of G-conjugate incomparable subgroups. Then Conjecture A** implies Conjecture A.*

Proof. The proof is exactly the same as for the previous corollary. We obtain a G-tree from Conjecture A** that has vertex set containing our G-stable almost equality class as a G-retract. The condition on $\mathscr{S}$ then ensures that we may apply Theorem 3.3.4. $\square$

Thus we observe that in the special case that we are able to prove our main result, i.e. that $\mathscr{S}$ consists of subgroups with Euler characteristic a non-zero integer, Conjecture A and Conjecture A** are equivalent. Thus we could have chosen to prove our main result in either context. We give our proof of Conjecture A** however as it remains an open question whether our techniques may be modified to prove the result in the case G is finitely generated without the condition that the subgroups in $\mathscr{S}$ are G-conjugate incomparable. We mention this possibility at the end of the proof of Lemma 4.2.16. A summary of how the conjectures are related is displayed in figure 3.1, where the unlabelled implications are obvious special cases. The most general versions of these results that are known to be true are the consequences of Theorem A, this is simply Conjecture A** with the Euler characteristic restriction on the stabilisers.

$$\begin{tikzcd}
\text{Conjecture A*} \arrow[r, Rightarrow, "5.5"] & \text{Conjecture B} \arrow[r, Rightarrow] & \text{Conjecture B}' \arrow[r, Rightarrow, "3.4.4"] & \text{Theorem B} \\
\text{Conjecture A**} \arrow[u, Rightarrow] \arrow[rr, Rightarrow, "3.3.9"] \arrow[d, Rightarrow, shift left, "\text{3.3.10-When } \mathscr{S} \text{ are } G\text{-conjugate incomparable}"] & & \text{Almost Stability Theorem} & \\
\text{Conjecture A} \arrow[u, Rightarrow, shift left, "3.3.8"] & & &
\end{tikzcd}$$

Figure 3.1: Known relations between the conjectures

3.4 The Euler characteristic of a group

The hypotheses of Theorem A include a condition on the Euler characteristic of the stabiliser groups. Before we may show that Theorem B is a corollary of Conjecture B' it is necessary to recall the notion of the Euler characteristic of a group. The following definitions may be found in, for example [10] Section IX.6.

Definition 3.4.1. We say that a group G is of *finite homological type* if G has finite virtual cohomological dimension and for every G-module M that is finitely generated as an abelian group, $H_i(G, M)$ is finitely generated for all i.

The Euler characteristic is then defined firstly for torsion-free groups and extended.

Definition 3.4.2. Let G be a torsion-free group of finite homological type. Then we define the *Euler characteristic* of G to be

$$\chi(G) = \sum_i (-1)^i \mathrm{rank}_{\mathbb{Z}}(H_i(G, \mathbb{Z})).$$

Let G be a group of finite homological type. Then since G has finite virtual cohomological dimension we may choose a torsion-free subgroup H such that $|G : H| < \infty$ and we define the Euler characteristic of G to be

$$\chi(G) = \frac{\chi(H)}{|G : H|}.$$

Thus in general, the Euler characteristic of a group of finite homological type is a rational number and need not be an integer. That the two definitions above agree and that the second is independent of the choice of H is dealt with in Section IX.7 of [10].

Example 3.4.1. The fundamental group of a closed orientable manifold of even dimension and genus not equal to 1 has Euler characteristic a non-zero integer - Section 2.2 of [20].

We now make a brief observation connecting the B conjectures. We first recall a result of Strebel [30] regarding Poincaré duality groups.

Theorem 3.4.3 (Strebel)**.** *Let H be a PD^n subgroup of G. Suppose that K is a subgroup of H of cohomological dimension n. Then $|H : K| < \infty$.*

Theorem 3.4.4. *Conjecture B' $\implies$ Theorem B*

Proof. Suppose that there exists a group L commensurable with H that is not G-conjugate incomparable. Then there exists an element $g \in G$ such that $L^g < L$. Then by a standard result that appears as Proposition IX.7.3 in [10] we have that $\chi(L^g) = |L : L^g|\chi(L)$. However, since $L \cong L^g$ it follows that $\chi(L) = \chi(L^g)$ and therefore $\chi(L) = 0$ as $|L : L^g| \neq 0$. Now since L is commensurable with H the same argument gives that $\chi(H) = 0$. Thus if $\chi(H)$ were non-zero, it follows that every subgroup commensurable with H is G-conjugate incomparable and thus the hypotheses of Conjecture B′ are satisfied. □

Next we shall introduce some of the key tools required in the proof of the Almost Stability Theorem. In order to construct a tree we have two useful methods. The first involves the introduction of tree G-sets and the second is the construction of a fibred G-tree which can be used to build a G-tree given some other trees on which subgroups of G act.

3.5 Tree G-sets

We are interested now in the construction of G-trees from certain subsets of functions in (E, A). At this point we restrict to the case $A = \mathbb{Z}_2$ since little generality is lost over the case that $\mathbb{Z}_2 \subset A$ as we shall observe later and we have already seen that in the case $|A| < 2$ the proof of the Almost Stability Theorem is trivial. Thus functions in $(E, \mathbb{Z}_2)$ can be thought of as subsets of E and often we shall refer to subsets and their corresponding functions as if they were the same. Notice that intersection of sets is equivalent to the product of their corresponding functions.

Before we say what we mean by a tree set we must first introduce some additional notation.

Definition 3.5.1. Given two subsets U, V of a larger set X we denote by $U\Box V$ the following four sets,

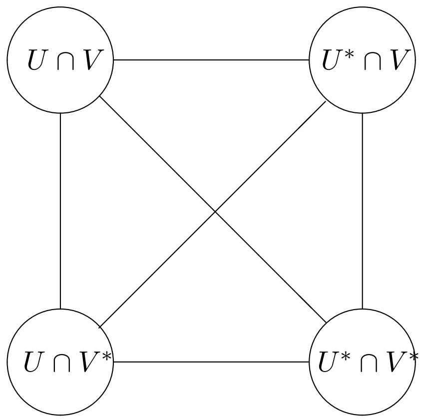
$$U\Box V = \{U \cap V, U \cap V^*, U^* \cap V, U^* \cap V^*\},$$

where by U^* we denote the complement of U in X.

3.5.1 An alternative to the Venn diagram

Often when we think of these four sets a Venn diagram springs to mind. However, when we are interested in the coboundaries of functions which give rise to these sets this particular picture is not always helpful. For example, an edge in the coboundary of one of these four sets has endpoints in two distinct sets and thus crosses one of the boundary lines.

Typically a Venn diagram collapses the boundary between $U \cap V^*$ and $V \cap U^*$ as well as the boundary between $U \cap V$ and $U^* \cap V^*$ to a pair of points which are easily overlooked. A perhaps more helpful style of diagram is used in [12] which we mention now. Consider the following diagram,



The four circles represent the four sets and the lines joining them represent the sets of edges with one endpoint in each of the relevant sets. We have that the union of the two sets on the left are U and the union of the top sets are V thus we have that the coboundary of U is just the union of the four lines between the left and right of the page and the coboundary of V is simply the union of the four lines between the top and bottom of the diagram. This style of diagram will prove useful later.

3.5.2 Nested sets

Definition 3.5.2. We say that two subsets U and V of X are *nested* if one of the four sets in $U \square V$ is the empty set. A collection of subsets of X is said to be nested if its elements are pairwise nested.

Observe that were we to think of these subsets as functions then the nested condition is simply that the zero function must be contained in $U \square V$. Our interest in nested sets lies in the fact that they may in certain circumstances be used to construct trees.

Example 3.5.1. Let T be a G-tree. Then for each edge $e \in ET$ recall that the structure map sends e to $e|VT : VT \to \mathbb{Z}_2$ the function that corresponds to the vertex set of the component of $T \setminus \{e\}$ containing τe. Then the collection of all such functions $ET|VT$ is a nested subset of $(VT, \mathbb{Z}_2)$.

We notice however, that a nested subset obtained from a G-tree as in the above example has a number of additional properties and so we introduce the following definition.

Definition 3.5.3. Let U be a G-set. A subset, E, of $(U, \mathbb{Z}_2)$ is a *tree G-set* if it is a nested G-subset of $(U, \mathbb{Z}_2)$ which contains no constant functions and such that the image of the dual map $U \to (E, \mathbb{Z}_2)$ lies in an almost equality class.

The reason for this terminology should become clear from the following result, Theorem II.1.5 from [12].

Theorem 3.5.4. *Let $E \subset (U, \mathbb{Z}_2)$ be a tree G-set. Then there exists a G-tree, $T(E)$, with edge set E and vertex set a subset of $(E, \mathbb{Z}_2)$. Furthermore, $U|E \subseteq VT(E)$.*

Sketch proof. The proof of this theorem can be found in [12], however the details of the proof are elementary and so we try to outline here only the key steps.

We define a graph with edge set E and declare the initial vertex of an edge e to be the set of all subsets in E strictly containing e or strictly containing its complement and that the terminal vertex of this edge should be $\iota e \cup \{e\}$. The proof then proceeds to show that this graph is in fact a G-tree.

The fact that the image of the dual lies in an almost equality class gives us that there exists a finite path between any two vertices and furthermore the set $v_1 \Box v_2$ tells us the edges and their orientations in the path from v_1 to v_2. We also require the other conditions to show that the graph is connected and contains no simple closed paths. $\square$

3.6 Fibred G-trees

The second important construction we have available to us is that of a fibred G-tree. The idea is to take a G-tree T as our base and take for each $v \in VT$ G_v-trees as fibres. If for each edge incident to a vertex, the edge stabiliser fixes some point in the corresponding fibre then we may attach this edge in our base tree to a vertex in the fibre in a G-invariant fashion. We must introduce one further piece of notation before we may give the formal definition.

Definition 3.6.1. Let H be a subgroup of G and let U be a right H-set. We denote by $U \otimes_H G$ the quotient of $U \times G$ given by identifying (uh, g) with (u, hg) for all $h \in H, g \in G$ and $u \in U$. We denote the image of (u, g) in this quotient by $u \otimes g$.

We now state the following definition taken from [12].

Definition 3.6.2. Given a G-tree T, a G-transversal U for VT and for each $u \in U$ a G_u-tree, T_u we may form the G-forest $Z = \cup_{u \in U} T_u \otimes_{G_u} G$. Let $\psi : VZ \to VT$ be the

G-map sending the fibre VT_u to its corresponding vertex u for each $u \in U$. Suppose that there exist G-maps $\phi_\iota, \phi_\tau : ET \to VZ$ such that $\psi \circ \phi_\iota(e) = \iota(e),\ \psi \circ \phi_\tau(e) = \tau(e)$. Then we may form the G-tree $\widetilde{T}$ consisting of Z together with the edges ET where the initial and terminal vertices of ET are given by the maps ϕ_ι and ϕ_τ. We call $\widetilde{T}$ the *fibred G-tree with base T, fibre T_u over $u \in U$ and attaching maps ϕ_ι and ϕ_τ*. We sometimes simply refer to $\widetilde{T}$ as the fibred G-tree with base T and fibres T_u when the additional information is clear from the context.

Remark 3.6.3. That $\widetilde{T}$ is in fact a tree is shown in [12]. By construction, we may obtain the base tree T from $\widetilde{T}$ by contracting the edges of the fibres.

This is one of the reasons for the requirement of finite stabilisers since for this construction it is necessary only that the stabiliser of each edge in T starting at u stabilises some vertex of T_u to which the initial vertex of T may then be connected. In the finite stabiliser case we can always find such a vertex, however, it can be shown that this is a special case of the following more general result.

Lemma 3.6.4. *Let T be a G-tree and $v \in VT$. Let K be a subgroup of G commensurable with some subgroup L, say, of G_v. Then K fixes some vertex of T.*

Proof. Since K is commensurable with L it follows that

$$K = \bigsqcup_{i=1}^{n} (K \cap L)k_i \quad \text{for some } n \in \mathbb{N}, k_i \in K.$$

Thus we see that,

$$\begin{aligned} vK &= \bigsqcup_{i=1}^{n} v(K \cap L)k_i \\ &= \bigsqcup_{i=1}^{n} vk_i \quad \text{since } L \leq G_v \\ &= \{vk_1, vk_2, \ldots, vk_n\}. \end{aligned}$$

Thus vK is a finite orbit. To see that this implies that K fixes a vertex of T, consider the finite K-tree generated by vK. If this is a single vertex or edge fixed by K then we are done. Otherwise K acts on the subtree obtained by removing the vertices of valency 1 and their incident edges. This gives a tree with strictly fewer edges. Continuing inductively in this way we arrive at a vertex or edge (and hence a pair of vertices) fixed by K.

□

Corollary 3.6.5. *Let T be a G-tree with commensurable edge stabilisers. Let $H \leq G$ be a subgroup commensurable with the edge stabilisers of T. Then H stabilises some vertex of T.*

This result allows us to construct the fibred tree described above whenever the edge stabilisers of both the base and the fibres are commensurable with each other.

Remark 3.6.6. Notice that an important particular case of the above lemma is when the subgroup L is the trivial group. Here we recover the well known result that all finite groups stabilise a vertex.

It should be noted that for the resulting tree to have commensurable edge stabilisers it is enough that the edge stabilisers of the base and fibres are commensurable with each other.

3.7 The complete graph on V

The first step in proving the Almost Stability Theorem is observing that the complete graph on V, our G-stable almost equality class has finite edge stabilisers. From here our task is to find some maximal subgraph which is a tree. We do not give the proof from [12] here, instead we show the more general result that the edge stabilisers are commensurable with one another which shall be a key starting point in the generalisation and go on to show that in the finite stabiliser case we have that the complete graph on V has finite edge stabilisers.

The following proposition is enough to show the result.

Proposition 3.7.1. *Let $V \subseteq \mathscr{S}(\sqcup_I G, A)$ be a G-stable $\mathscr{S}$-almost equality class. Suppose that $\mathscr{S}$ consists of a commensurability class of finitely generated subgroups of G. Then the complete graph on V has edge stabilisers in $\mathscr{S}$.*

Proof. Let $v_1 \neq v_2 \in V$. Then G_{v_1,v_2} acts on the set $v_1 \triangledown v_2$. This follows since for $g \in G_{v_1,v_2}$ and $x \in v_1 \triangledown v_2$,

$$v_1(xg) = v_1 g^{-1}(x) = v_1(x) \neq v_2(x) = v_2 g^{-1}(x) = v_2(xg).$$

Now we may write $v_1 \triangledown v_2$ as a finite union of cosets of some subgroup, K say, in $\mathscr{S}$. Indeed, we may choose such a K so that the restriction of both v_1 and v_2 to $v_1 \triangledown v_2$ is constant on the right cosets of K. Then $|G_{v_1,v_2} : G_{v_1,v_2,Kg}| < \infty$, where by G_{Kg} we denote the coset-wise stabiliser (in our case K^g) and not the point-wise stabiliser. Next we claim that $G_{v_1,v_1 \triangledown v_2} = G_{v_1,v_2,v_1 \triangledown v_2} = G_{v_2,v_1 \triangledown v_2}$ where $G_{v_1 \triangledown v_2}$ is the coset stabiliser of the collection of cosets of K contained in $v_1 \triangledown v_2$.

Let $g \in G_{v_1,v_1 \triangledown v_2}$. Let $x \in v_1 \triangledown v_2$. Then we have that

$$v_2 g(x) = v_2(xg^{-1}) = v_2(x).$$

Further for $x \notin v_1 \triangledown v_2$, noticing then that by the above $xg^{-1} \notin v_1 \triangledown v_2$ we have

$$v_2 g(x) = v_2(xg^{-1}) = v_1(xg^{-1}) = v_1 g(x) = v_1(x) = v_2(x).$$

Thus we have that the edge stabilisers of the complete graph on V are commensurable with each other since our functions lie in an $\mathscr{S}$-almost equality class. It remains to show that the edge stabilisers lie in $\mathscr{S}$. For this we use the fact that $\mathscr{S}$ consists of finitely generated groups.

By the above it is enough to show that for some coset Kx of K in $\mathscr{S}$, and for all $v \in V$ we have that

$$|G_{Kx} : G_{Kx} \cap G_v| < \infty.$$

Now for each $H \in \mathscr{S}$, we have that $KxH = LF$ where $L \in \mathscr{S}$ and F is a finite subset of G. Let $h_1, \ldots, h_m$ be a generating set for H. Then for $1 \leq i \leq m,\ v \triangledown vh_i$ is a finite union of cosets of some group in $\mathscr{S}$ and thus $\bigcup(v \triangledown vg_i)H$ is also a finite union of such cosets by the following result.

$$(v_1 \triangledown v_2)g = v_1 g \triangledown v_2 g,$$

and

$$v_1 \triangledown v_3 = v_1 \triangledown v_2 + v_2 \triangledown v_3,$$

taking $A = \mathbb{F}_2$, or indeed more generally that

$$v_1 \triangledown v_3 \subseteq v_1 \triangledown v_2 \cup v_2 \triangledown v_3.$$

Hence, we observe that for $h_i \in H$ $(1 \leq i \leq m)$,

$$
\begin{aligned}
& v \bigtriangledown vh_1h_2 \ldots h_m \\
\subseteq\; & v \bigtriangledown vh_m \cup vh_m \bigtriangledown vh_1h_2 \ldots h_m \\
=\; & v \bigtriangledown vh_m \cup (v \bigtriangledown vh_1 \ldots h_{m-1})h_m \\
\subseteq\; & v \bigtriangledown vh_m \cup (v \bigtriangledown vh_{m-1} \cup vh_{m-1} \bigtriangledown vh_1 \ldots h_{m-1})h_m \\
& \quad \vdots \\
\subseteq\; & v \bigtriangledown vh_m \cup (v \bigtriangledown vh_{m-1})h_m \cup (v \bigtriangledown vh_{m-2})h_{m-1}h_m \cup \ldots \cup (v \bigtriangledown vh_1)h_2 \ldots h_m \\
\subseteq\; & \bigcup_{i=1}^{m} (v \bigtriangledown vh_i)H
\end{aligned}
$$

and we have the desired result. $\square$

Remark 3.7.2. An important point to note is that if K is any subgroup of G then we may consider the complete graph on V to be a K-graph by restricting the G-action. Then the above result shows that the edge stabilisers of this graph when considered to be a K-graph are commensurable by Proposition 4.1.9, this is despite the fact that the stabilisers $H \cap K$ need no longer be finitely generated.

3.8 Some technical lemmas

We proceed to discuss some important lemmas required in the remainder of the proof. On their own these results do not appear to be very enlightening but they are necessary at many points in our later work.

We begin by proving the following generalisation of Lemma III.5.3 from [12].

Lemma 3.8.1. *Suppose that X is a G-graph with edge stabilisers in some commensurability class $\mathscr{S}$. Let I be an arbitrary indexing set and let $\varphi : \sqcup_I G \to (VX, A)$ be a G-map such that the image of the dual map $VX \to (\sqcup_I G, A)$ lies in an $\mathscr{S}$-almost equality class. Then for any G-transversal S in $\sqcup_I G$ and any G-finite G-subset F of EX,*

$$\bigsqcup_{s \in S} F \cap \delta(s|VX) \text{ is finite.}$$

Proof. Fix an edge e of X. Then $\iota e| \sqcup G =_{\mathscr{S}} \tau e| \sqcup G$ so the set,

$$\{g \in \sqcup G | \{e\} \cap \delta(g|VX) \neq \varnothing\} = s_1x_1H \sqcup \ldots \sqcup s_nx_nH$$

is finite where $s_i \in S(1 \leq i \leq n),\ x_i \in G,\ H \in \mathscr{S}$.

We proceed to show that the set

$$\{s \in S : eG \cap \delta(s|VX) \neq \varnothing\}$$

is finite.

Suppose there exists some $eg \in eG \cap \delta(s|VX)$ for some $g \in G, s \in S$. Then

$$\begin{aligned} & e \in \{e\} \cap \delta(sg^{-1}|VX) \\ \implies \quad & sg^{-1} \in s_1x_1H \sqcup \ldots \sqcup s_nx_nH \\ \implies \quad & s = s_i \text{ for some } 1 \leq i \leq n \text{ since } S \text{ is a } G\text{-transversal.} \end{aligned}$$

Thus

$$\{s \in S : eG \cap \delta(s|VX) \neq \varnothing\}$$

is finite as required.

Since F is a G-finite G-subset of EX it follows that

$$\{s \in S : F \cap \delta(s|VX) \neq \varnothing\}$$

is finite.

It remains to show that

$$\bigcup_{s \in S} F \cap \delta(s|VX)$$

is finite.

It is now enough to show that $eG \cap \delta(s|VX)$ is finite for all $e \in EX, s \in S$.

We aim to show that

$$eG \cap \delta(s|VX) \subseteq \bigcup_{i=1}^{n} eHx_i^{-1} \quad (s \in S)$$

and the result follows from the hypothesis that $H \in \mathscr{S}$ is commensurable with the stabiliser of e.

Suppose $eg \in eG \cap \delta(s|VX)$ where $s \in S$.

By the argument above we observe that $sg^{-1} = s_ix_ih$ for some $1 \leq i \leq n$ and further that $s = s_i$ since S is a G-transversal. Thus $x_ihg \in G_s = \{1\}$. Therefore $hg = x_i^{-1}$ and so $eg = eh^{-1}x_i^{-1} \in eHx_i^{-1}$. It follows that,

$$eG \cap \delta(s|VX) \subseteq \bigcup_{i=1}^{n} eHx_i^{-1} \quad (s \in S).$$

$\square$

3.9 The Boolean ring of a graph

Finally in this chapter we state an important theorem of Dicks and Dunwoody that shall be crucial in our generalisation.

Definition 3.9.1. Let X be a connected G-graph. The *Boolean ring of* X, $\mathscr{B}X$ is the set of all $s \in (VX, \mathbb{Z}_2)$ such that their coboundary δs is finite. Considering the elements of this set as subsets of VX the ring operations are symmetric difference and intersection.

We also define an interesting family of subrings.

Definition 3.9.2. For $n \in \mathbb{N}$ let $\mathscr{B}_n X$ denote the subring of $\mathscr{B}X$ that is generated by the elements of $s \in \mathscr{B}X$ with $|\delta s| \leq n$. An element $s \in \mathscr{B}X$ is *n-thin* if $|\delta s| = n$ and $s \notin \mathscr{B}_{n-1}X$.

Example 3.9.1. Let T be a G-tree. Then $\mathscr{B}_1 T = \mathscr{B}T$. To see this notice that every edge of T determines a subset of VT, namely the set of vertices in the same component of $T - \{e\}$ as τe. Now for any subset $s \in (VT, \mathbb{Z}_2)$ with finite coboundary, the subset may be obtained by taking the intersection of all such subsets obtained from the edges of the coboundary δs with the appropriate choice of orientations of those edges.

The following result is Theorem II.2.20 in [12]. This theorem allows us to construct trees for any connected G-graph retaining information about how our original graph was connected.

Theorem 3.9.3. *If X is a connected G-graph then there exists an ascending chain $E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots$ of tree G-sets, in $(VX, \mathbb{Z}_2)$, of thin elements such that for each $i \in \mathbb{N}$ the set E_i generates $\mathscr{B}_i X$ as a Boolean ring.*

Chapter 4

Proof of Theorem A

We prove the theorem using an induction argument with three main steps. In the first step we show that whenever G is finitely generated over H and we have a given H-tree T_H with vertex set V_H (and a few other technical restrictions) we can embed the G-forest $T_H G$ into a G-tree. We then investigate the notion of incompressibility which is crucial to the second step - manipulating the G-tree in such a way that it now has vertex set V_G. The final step is then a transfinite induction argument that obtains the result from our earlier work. Our approach follows that of Dicks and Dunwoody in Chapter III of [12]. We fix the following notation for the rest of this chapter. Let $\mathscr{S}$ be an admissible family of finitely generated commensurable subgroups of G. Let I be an arbitrary indexing set and let V be a G-stable $\mathscr{S}$-almost equality class in $(\sqcup_I G, A)$, for some non-empty set A, as introduced in Definition 3.2.2. To emphasize the relation to the work of [12] we define $E = \sqcup_I G$ so that $V \subseteq (E, A)$. Notice here that in fact G acts freely on E and so will not in general have stabilisers in $\mathscr{S}$ however this allows us to retain the notation E_G from Definition 2.6.2.

4.1 Step 1 - The finitely generated case

The setup for this section is as follows. We assume that our group G is finitely generated over some subgroup H. That is to say that $H \cup \{g_1, \ldots, g_b\}$ generates G. We fix a specified element $v_0 \in V$. We are given an H-tree T_H with vertex set V_H and assume that $E_H g \cap E_H = \varnothing$ for all $g \in G - H$. Our aim is to show that the H-tree T_H may be extended to a G-tree with vertex set V_G. The proof of this shall take up the first half of this chapter. We require first some preliminary results and to this end we introduce some additional notation.

Let W be a G-finite G-subset of V_G, with G-transversal $\{w_1, \ldots, w_a\}$. We define X to be the G-subgraph of the complete graph on V consisting of $T_H G \cup W$ with the G-set generated by the edges joining v_0 to $w_1, \ldots w_a, v_0 g_1, \ldots, v_0 g_b$ with loops omitted. We abbreviate $T_H G$ to Y and write F for $X - (Y \cup W)$.

Let ST_H be an H-transversal in ET_H, we shall later see that this turns out to be a G-transversal for EY and let S_H be an H-transversal in E_H. By the above hypothesis that $E_H \cap E_H g = \varnothing$ for all $g \in G - H$ this may be extended to a G-transversal S_G in E_G.

Notice that similarly to Remark 3.1.1, it is easy to see that Theorem A is true in the case that $|A| < 2$. Thus we may assume that $\mathbb{Z}_2 \subseteq A$. Now in an analogous fashion to the structure and costructure maps associated with a G-tree, cf. Example 3.1.1, it is desirable for us to think of the edges of this graph as being functions in (VX, A). We do so as follows. Since $VX \subseteq V_G \subseteq (E_G, A)$ we have the dual G-map $E_G \to (VX, A)$. Further we use our H-tree T_H to get that $(V_H, \mathbb{Z}_2) \subseteq (VX, A)$ thinking of A as containing a copy of $\mathbb{Z}_2$. Composing with the structure map for T_H gives an H-map $ET_H \to (VX, A)$ that we aim to show in Corollary 4.1.5 extends to a G-map $EY \to (VX, A)$. This now allows us to identify elements of $EY \cup E_G$ with functions in (VX, A).

The first step towards Theorem A is to construct a G-tree extending T_H, that contains W and for which there is a G-map to V_G. That there is a G-map to V_G shall allow us to find such a G-tree with precisely vertex set V_G using the results of section 4.2. The importance of the G-set W is in dealing with the troublesome vertices which do not have stabilisers in $\mathscr{S}$. We construct a fibred G-tree for this purpose, the base for this tree is obtained via the following theorem.

Theorem 4.1.1. *There exists a G-tree T_Y having a map of G-graphs $Y \cup W \to T_Y$ which is bijective on edge sets.*

We require some preliminary results before we can prove this theorem. The full proof shall be given once we have Lemma 4.1.19 in place.

4.1.1 Preliminaries

Before we discuss the proof there are a few technical points we must note about the setup introduced above.

Lemma 4.1.2. *X is a connected G-graph.*

Proof. Contract all the edges of X and denote the image of v_0 by $\overline{v_0}$. We are left with a G-graph which consists of a set of vertices, one for each connected component of X, and no edges. Notice that $Y = T_H G$ is contracted into $\overline{v_0}G$ since T_H is a tree and thus connected. Furthermore W is contracted to $\overline{v_0}$ since we have edges attaching each element to v_0. Thus the whole of X is contracted to $\overline{v_0}G$. Since H acts on T_H we see that $\overline{v_0}$ is fixed by H, since W is fixed under the action of $\{g_1, g_2, \ldots, g_a\}$ we see that $\overline{v_0}$ is fixed by $\{g_1, g_2, \ldots, g_a\}$ and hence $\overline{v_0}$ is fixed by the whole of G, i.e. $\overline{v_0}G = \overline{v_0}$ and our graph X is connected. □

Lemma 4.1.3. *The map $VX \to V_H : v \mapsto v|E_H$ is an H-retraction.*

Proof. There is an obvious inclusion map $V_H \hookrightarrow VX \subseteq V_G$. This is the required H-map in the opposite direction. □

Lemma 4.1.4. *For $g \in G - H$ the H-retraction $VX \to V_H : v \mapsto v|E_H$ sends $V_H g$ to a single vertex*

Proof. Let $g \in G - H$. Recall that part of our hypothesis above was that $E_H g \cap E_H = \varnothing$. Therefore we have that $E_H g^{-1} \subseteq E_G - E_H$. Since $v \in V_H$ we have that v and v_0 agree on $E_G - E_H$ and so agree on $E_H g^{-1}$. Thus we observe that $vg|E_H = v_0 g|E_H$ and the result holds. □

Corollary 4.1.5. *For all $g \in G - H, ET_H g \cap ET_H = \varnothing$. Hence ST_H is a G-transversal in $EY = ET_H \otimes_H G$, and thus the H-map $ET_H \to (VX, A)$ extends to a G-map $EY \to (VX, A), s \mapsto s|VX$.*

Proof. Let $g \in G - H$ and suppose that $e \in ET_H \cap ET_H g$. Then by Lemma 4.1.4 this edge is both left fixed and collapsed to a single vertex by the H-retraction $VX \to VT_H$. Clearly then $ET_H \cap ET_H g = \varnothing$ and the result follows. □

We define F_0 as follows,

$$F_0 = \bigcup_{s \in S_H} (F \cap \delta s),$$

where we recall that F is the G-finite set of edges joining W and Y, and S_H was an H-transversal for E_H.

Lemma 4.1.6. *$HF_0 = \bigcup_{s \in E_H} (F \cap \delta s)$ and $F_0 = \bigcup_{s \in S_H} (F \cap \delta s)$ is finite.*

Proof. By definition we have that $E_H = S_H H$. Furthermore since F is a G-set and $H \leq G$ it follows that $FH = F$, and so $F_0 H = \bigcup_{s \in E_H}(F \cap \delta s)$. Here we are using Proposition 2.6.6 to see that $(\delta s) \cdot g = \delta(s)g$.

We wish to obtain the result by applying Lemma 3.8.1. By definition X is a G-subgraph of the complete graph on V_G and thus is a G-graph with commensurable edge stabilisers by Lemma 3.7.1. $S_G \subseteq E$ is a G-transversal for E_G and VX can be thought of as a subset of $\mathscr{S}(E_G, A)$ by restriction since $VX \subseteq V_G$. By our definition of the set E it follows that $E_G = \sqcup_J G$ where $J \subseteq I$ and we have a dual G-map $\sqcup_J G \to (VX, A)$. Finally since $V_G \subseteq V$ lies in an $\mathscr{S}$-almost equality class it follows that the double dual G-map $VX \to (\sqcup_J G, A)$ has image lying in an $\mathscr{S}$-almost equality class and so Lemma 3.8.1 applies to give that $\bigcup_{s \in S_G}(F \cap \delta s)$ is finite and therefore any subset, in particular F_0 is also finite. □

Lemma 4.1.7. *For all $s \in ST_H$, we have that $s \in \delta s \subseteq \{s\} \cup F_0 H$.*

Proof. Let $s \in ST_H$. Notice that the edges of X are simply the edges of $Y = T_H G$ together with the G-finite edge set F.

We first consider those edges of T_H lying in δs. Let $e \in ET_H$. Since $VT_H = V_H$, the retraction onto V_H preserves the endpoints of e. That is to say that $\iota e|E_H = \iota e$ and $\tau e|E_H = \tau e$. Therefore we see that $\delta s \cap ET_H = \{s\}$.

Now since T_H is an H-tree, the only edges of Y left to consider are those belonging to $ET_H g$ for $g \in G - H$. However, Lemma 4.1.4 tells us that in this case all of $V_H g$ is sent to a single vertex and thus s is constant on this component. It follows that $\delta s \cap ET_H g = \varnothing$ for all $g \in G - H$.

It remains only to consider the edges in F. Let $f \in F$. Suppose that $f \in \delta s$. Then s must lie in the path in T_H between $\iota f|E_H$ and $\tau f|E_H$. In particular we have that $\iota f|E_H$ and $\tau f|E_H$ are distinct and so $f \in \delta e$ for some $e \in E_H$. That is to say that $f \in F_0 H$ and we have the desired result. □

Lemma 4.1.8. *Suppose that a group G acts on two sets, E and V say, in such a way that the stabilisers are commensurable as subgroups of G. Then for any subgroup $H \leq G$, the G-action restricts to an H-action on E and V and the stabilisers remain commensurable as subgroups of H.*

Proof. This follows from an elementary property of indices, i.e.

Proposition 4.1.9. *Let $K, L \leq G$ such that $|G : K| < \infty$. Then $|L : K \cap L| \leq |G : K|$.*

Proof. Let T be a transversal to $K \cap L$ in L.

$\forall t, t' \in T$

$$\begin{aligned} & Kt = Kt' \\ \implies \quad & t't^{-1} \in K \cap L \\ \implies \quad & (K \cap L)t = (K \cap L)t' \\ \implies \quad & t = t' \end{aligned}$$

□

Lemma 4.1.10. $\bigsqcup_{s \in ST_H} (F_0 H \cap \delta s)$ *is finite.*

Proof. We have an H-map from ET_H to (VX, A). Namely retraction onto V_H followed by the structure map. Again this determines a map $\sqcup H \to (VX, A)$ via the isomorphism $ET_H \cong \sqcup H \backslash H_e$. The dual of this map $VX \to (\sqcup H, A)$ is given by the costructure map of the retraction of the vertex onto V_H. In particular, if we denote by $\mathscr{T}$ the admissible family of subgroups $H \cap K$ where $K \in \mathscr{S}$ then the image of the dual lies in a $\mathscr{T}$-almost equality class since it arises from the costructure map for T_H. Furthermore the stabilisers of ET_H are commensurable with the edge stabilisers of X considered as an H-graph by 4.1.8. This together with Lemma 4.1.6 gives us our result by Lemma 3.8.1. □

Corollary 4.1.11. *The set* $\{e \in EY | \delta e \neq \{e\}\}$ *is G-finite, and for all* $e \in EY, \delta e$ *is finite and* $\delta e \cap EY = \{e\}$. *Hence there exists some integer n such that* $|\delta e| \leq n$ *for all* $e \in EY$.

Proof. We notice from Lemma 4.1.7 that, $s \in \delta s \subseteq \{s\} \cup F_0 H$ for all $s \in ST_H$. Therefore by Lemma 4.1.10 we immediately see that δs is finite for all $s \in ST_H$ and for almost all $s \in ST_H, \delta s$ is simply $\{s\}$. The corollary now follows from the fact that ST_H is a G-transversal for EY, by Corollary 4.1.5, and that $\delta(eg) = (\delta e) \cdot g$. □

We would like that $EY|VX$ is nested which would allow us to construct a G-tree since this set contains no constant functions (it has already been shown that $\{e\} \subseteq \delta e$) and the image of the dual lies in an almost equality class by definition. The resulting tree would have precisely edge set EY and the vertex set would contain VX when identified with the double dual $VX|(EY|VX)$. However this set will not in general be nested and so we consider the obstruction to this nesting, construct a tree using the dual of this set then show that we may construct a second tree from what remains and that this tree is suitable for a fibre over $v_0|R$ in our first tree. In the remainder of this section we sketch the proof. For the following definition recall the use of the square notation introduced in Definition 3.5.1.

Definition 4.1.12. We define the *obstruction to the nesting*, R, in the following way,

$$R = \{r \in (VX, \mathbb{Z}_2) \mid \exists\ e, e' \in EY, r \in (e|VX)\square(e'|VX), \delta r \cap EY = \varnothing\}.$$

This set is the obstruction to $EY|VX$ being nested as will be seen later in the proof. At this moment however we focus on using this set to construct the base of our fibred tree before concerning ourselves with how this set arises.

Lemma 4.1.13. *The set R is a G-set.*

Proof. Let $r \in R, g \in G$. Then

$$\begin{aligned} \delta(rg) \cap EY &= g\delta r \cap EY \\ &= (\delta r \cap EYg^{-1})g \\ &= (\delta r \cap EY)g \quad \text{since } EY \text{ a } G\text{-set.} \\ &= \varnothing. \end{aligned}$$

Since $r \in R$ we know that

$$r = (e|VX)^{\varepsilon_1} \cap (e'|VX)^{\varepsilon_2},$$

for some $e, e' \in EY, \varepsilon_1, \varepsilon_2 \in \{1, *\}$.

Hence, for $g \in G$,

$$\begin{aligned} rg &= (e|VX)^{\varepsilon_1} g \cap (e'|VX)^{\varepsilon_2} g \\ &= (eg|VX)^{\varepsilon_1} \cap (e'g|VX)^{\varepsilon_2}, \end{aligned}$$

since $EY \to (VX, A)$ is a G-map. Further $eg, e'g \in EY$ as EY is a G-set. Thus R is a G-subset of $(VX, \mathbb{Z}_2)$. $\square$

We note further that for each $r \in R$, $|\delta r| \leq |\delta e \cup \delta e'| \leq 2n$, where the last inequality is obtained from Lemma 4.1.11.

4.1.2 The graph $X|R$

The graph $X|R$ has vertex set $VX|R$ and edge set EX. The incidence maps and other properties are given below.

The coboundary map

Recall the double dual map $R \to (VX|R, \mathbb{Z}_2)$ sends $r \mapsto r^*$ where

$$r^*(v^*) = v^*(r) = r(v).$$

We denote the incidence maps for the graph $X|R$ by $\iota_{X|R}$ and $\tau_{X|R}$.

The incidence maps are defined as the composition,

$$EX \xrightarrow{\iota,\tau} VX \to VX|R.$$

Notice that δr has the same meaning as it did for X,

$$\begin{aligned}
\delta r &= \{e \in EX \mid r(\iota e) \neq r(\tau e)\}, \\
\delta r^* &= \{e \in E(X|R) \mid r^*(\iota_{X|R} e) \neq r^*(\tau_{X|R} e)\} \\
&= \{e \in EX \mid r^*((\iota e)^*) \neq r^*((\tau e)^*)\} \\
&= \{e \in EX \mid r(\iota e) \neq r(\tau e)\} \\
&= \delta r.
\end{aligned}$$

Lemma 4.1.14. *$X|R$ is a connected G-graph.*

Proof. Since R is a G-set, the identity map from $R \to (VX, \mathbb{Z}_2)$ is a G-map. Thus the dual map $VX \to (R, \mathbb{Z}_2)$ is a G-map by Proposition 2.1.5.

Now for $g \in G, e \in EX$,

$$\begin{aligned}
\iota_{X|R}(eg) &= (\iota(eg))|R \\
&= (\iota e)g|R \quad \text{since } X \text{ is a } G\text{-graph and so } \iota \text{ is a } G\text{-map} \\
&= ((\iota e)|R)g \quad \text{since the dual map } VX \to (R, \mathbb{Z}_2) \text{ is a } G\text{-map.}
\end{aligned}$$

To see that $X|R$ is connected suppose that we have two vertices $v_1, v_2 \in VX$ connected by an edge e. Then it is clear that the points $v_1|R, v_2|R$ are joined in $X|R$ by the edge e. It follows that $X|R$ is connected since X is connected. $\square$

The existence of a tree G-subset E_R

Since $X|R$ is a connnected G-graph it follows from Theorem 3.9.3 that $\mathscr{B}_{2n}(X|R)$ contains a tree G-subset, E_R which generates $\mathscr{B}_{2n}(X|R)$ as a Boolean ring, where n is the integer obtained from Corollary 4.1.11. Notice that

$$\mathscr{B}_m(X) \subseteq \mathscr{B}(X) \subseteq (VX, \mathbb{Z}_2),$$

and so in the case of interest E_R is contained in $(VX|R, \mathbb{Z}_2)$.

Lemma 4.1.15. *Since $E_R \subseteq (VX|R, \mathbb{Z}_2)$ there exists a dual map $VX|R \to (E_R, \mathbb{Z}_2)$. This map is injective.*

Proof. Let $v_1|R, v_2|R$ be distinct elements of $VX|R$.

Thus there exists some $r \in R$ such that $v_1^*(r) \neq v_2^*(r)$, i.e. $r^*(v_1^*) \neq r^*(v_2^*)$.

Now $R|(VX|R)$ lies in the Boolean ring generated by E_R since it is a collection of subsets, r, of $VX|R$ with $|\delta r| \leq 2n$.

Thus r^* belongs to the Boolean ring generated by E_R. Therefore there exists an element of E_R containing one of v_1^* and v_2^* but not the other (i.e. if E_R does not distinguish between v_1^* and v_2^* then the Boolean ring generated by E_R cannot distinguish them either).

$\therefore \quad VX|R \to (E_R, \mathbb{Z}_2)$ is injective. $\square$

We are now able to form a G-tree T_R from the tree G-set E_R which contains VX as a subset of its vertex set since by the construction of the tree the image of the double dual is contained in the vertex set and so we obtain the inclusion $VX|R \subseteq VT_R$. This tree shall be used as the base for our fibred G-tree. We proceed to construct the required fibres.

Notice now that for any edge $e \in EY$, then for all $r \in R$ we have that $\delta r \cap EY = \varnothing$ and so we see that the components of Y map to single vertices of the graph $X|R$. We now denote the stabiliser of $v_0|R$ by G_0 and the G_0-subgraph of Y of components mapped to $v_0|R$ by Y_0. We observe the following fact about Y_0.

Proposition 4.1.16. $Y = Y_0 \otimes_{G_0} G$

Proof. Denote the map, $Y \to X|R$ discussed above, by ϕ. Then $\phi^{-1}(v_0|R) = Y_0$. Recall that $Y = T_H G = Y_0 G$ ($T_H \subseteq Y_0$ and Y_0 is a subgraph of the G-graph Y) where Y_0 is the set of components sent to $v_0|R$ by the map $e \mapsto e|R$. Thus $\mathrm{Im}\phi = Y|R = Y_0G|R = (v_0|R)G$.

The result follows by Lemma III.3.4 from [12], namely,

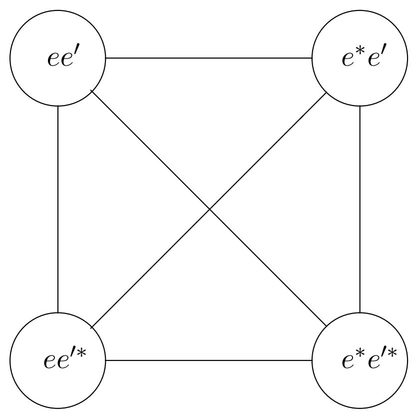
Lemma 4.1.17. *If U, V are G-sets, $\phi : V \to U$ a G-map and U_0 a G-transversal in U then there is a natural identification of G-sets $V = \bigcup_{u \in U_0} \phi^{-1}(u) \otimes_{G_u} G$.*

$\square$

Next, we denote by VX_0 the subset of VX which is sent to $v_0|R$. We aim to show that $EY_0|VX_0$ is always nested and so we can use this set to form the fibres for our G-tree.

Lemma 4.1.18. *$EY_0|VX_0$ is a nested set.*

Proof. Let e, e' be distinct elements of EY_0. By Corollary 6.8, we have that $\delta e \cap EY = \{e\}$ and $\delta e' \cap EY = \{e'\}$. Consider the diagram below which was introduced in section 3.5.1.



It is easily seen that e and e' meet at a corner. The element at the opposite corner, r say, then belongs to $(e|VX)\square(e'|VX)$. It also has the property that $e, e' \notin \delta r$ and so $\delta r \cap EY = \varnothing$ or else $\delta e($ or $\delta e') \cap EY$ would contain more than one element. Thus we see that $r \in R$. It is now that we see how the definition of the set R, the obstruction to the nesting, was obtained.

We next recall that VX_0 is the set of vertices sent to $v_0|R$. Hence for all $v \in VX_0, r \in R$ we have that

$$\begin{aligned} v^*(r) &= v_0^*(r) \\ \Longrightarrow r(v) &= r(v_0) \end{aligned}$$

Thus $r|VX_0$ is constant for all $r \in R$.

Furthermore we notice that in our case above neither $e|VX_0$ nor $e'|VX_0$ are constant since both have coboundary in EY_0. Thus it follows that $r|VX_0$ is zero by considering the two cases (one of which is impossible) and looking at each of the four sets. A perhaps simpler way of observing this is that in order for the function to be constantly 1 it would be necessary for both $e|VX_0$ and $e'|VX_0$ to be constantly 1 which would contradict the fact that neither was constant. Thus $EY_0|VX_0$ is a nested G_0-subset of $(VX_0, \mathbb{Z}_2)$. □

Lemma 4.1.19. *$EY_0|VX_0$ is a tree G_0-subset of* $(VX_0, \mathbb{Z}_2)$.

Proof. At this point we should note that since $VY_0 \subseteq VX_0$, the set $EY_0|VX_0$ contains no constant functions thus to prove that it is a tree G_0-subset it is enough to show that $VX_0|EY_0$ the image of the dual lies in an almost equality class. The argument is identical to the finite stabiliser case in III.6.9 of [12]. Observe that by Lemma 4.1.7 we have the following inclusion,

$$\bigsqcup_{s\in ST_H}(eG\cap\delta s)\subseteq\left(\bigsqcup_{s\in ST_H}(eG\cap\{s\})\right)\bigsqcup\left(\bigsqcup_{s\in ST_H}(F_0H\cap\delta s)\right),$$

for any $e \in EX$.

Clearly the second term is finite since ST_H is a G-transversal. The third union is finite by application of Lemma 4.1.10 and we see that the first term must also be finite. We notice at this point that the above implies that there exist only finitely many elements y of EY such that $e \in \delta y$ for a fixed edge e.

Thus we have that $\sqcup_{s\in ST_H}(eG \cap \delta s)$ is finite. Now we see that for almost all $s \in ST_H$ that $eG \cap \delta s = \varnothing$. We denote the finitely many elements of ST_H such that $eG \cap \delta s \neq \varnothing$ as $\{s_1, s_2, \ldots, s_t\}$. We denote the finitely many elements in these sets as follows,

$$eG\cap\delta s_i=\{eg_{i1},eg_{i2},\ldots,eg_{im(i)}\}\ \text{ where } m(i)\in\mathbb{N} \text{ and } 1\leq i\leq t.$$

Suppose now that we have an element $y \in EY$ such that $e \in \delta y$. Then since ST_H is a G-transversal in EY we may write $y = s_j g$ for some $1 \leq j \leq t$. Now we have that $eg^{-1} \in \delta s_j$, i.e. $eg^{-1} = eg_{jk}$ for some $1 \leq k \leq m(j)$. Thus we see that $g_{jk}g \in G_e$. It follows that

$$\begin{aligned} s_j g &\in s_j g_{jk}^{-1} G_e \\ &\subseteq \bigcup_{\substack{1\leq j\leq t\\ 1\leq k\leq m(j)}} s_j g_{jk}^{-1} G_e, \end{aligned}$$

which is a finite set since the edge stabilisers of X are commensurable. □

We may now complete the proof of Theorem 4.1.1.

Proof. Let $v, v' \in VX_0$. We may now choose a path p in X joining v to v' and from the above we see that there are only finitely many edges $e \in EY$ such that δe meets p, and so $v|EY_0 =_a v'|EY_0$. Hence we are now able to form a G_0-tree $T_0 = T(EY_0|VX_0)$. By construction this has edge set EY_0 and the double dual $VX_0|(EY_0|VX_0)$ is contained in the vertex set of T_0 giving us a G-map $VX_0 \to VT_0$. Again we can easily check that such a construction connects the vertices with the expected edges and so we obtain a map of G_0-graphs $Y_0 \cup VX_0 \to T_0$ bijective on edge sets.

We now patch together the maps we have formed above to obtain a map of G-graphs as follows,

$$\begin{aligned} Y \cup VX &= (VX - VX_0 \otimes_{G_0} G) \cup ((Y_0 \cup VX_0) \otimes_{G_0} G) \\ &\to (VT_R - (v_0|R)G) \cup (T_0 \otimes_{G_0} G) \subseteq T_Y. \end{aligned}$$

We observe that this map is injective on edge sets with image $ET_Y - ET_R$ thus by contracting all the edges of T_Y which also belong to ET_R we obtain the required tree. $\square$

We now state a technical lemma which shall be used in the proof of Theorem 4.1.21.

Lemma 4.1.20. *Let $K \leq G, S_K$ a K-transversal for E_G, and E' be a G-set with stabilisers in a class, $\mathscr{S}$, of commensurable subgroups of G. Let $\theta : E_G \to \mathscr{P}E'$ be a G-map. Then for $e \in E'$, we have the following result,*

$$\bigsqcup_{f \in E_G} (\{e\} \cap \theta f) \text{ is finite} \implies \bigsqcup_{s \in S_K} (eK \cap \theta s) \text{ is finite.}$$

Proof. Suppose that $\sqcup_{f \in E_G} (\{e\} \cap \theta f)$ is a finite set. Then we have a finite subset $\{s_1k_1, s_2k_2, \ldots, s_nk_n\} \subseteq E_G$ with $k_j \in K, s_j \in S_K$ $(1 \leq j \leq n)$ such that $\{e\} \cap \theta(s_ik_i) \neq \varnothing$. Notice now that,

$$\begin{aligned} ek \in \theta s &\Rightarrow e \in \theta(s)k^{-1} = \theta(sk^{-1}) \\ &\Rightarrow s \in \{s_1, \ldots, s_n\}. \end{aligned}$$

Thus we have that $eK \cap \theta s = \varnothing$ for almost all $s \in S_K$. It remains to show that $eK \cap \theta s$ is finite for all $s \in S_K$. We use a similar argument to the above. Suppose now that $ek \in \theta s$.

Then from the above argument it is clear that $sk^{-1} = s_i k_i$ for some $1 \leq i \leq n$. Since S_K is a K-transversal it follows that $s = s_i$ and $k_i k \in G_{s_i}$. Therefore,

$$\begin{aligned} ek &= ek_i^{-1} k_i k \\ &\in ek_i^{-1} G_{s_i} \\ &\subseteq \bigcup_{1 \leq i \leq n} ek_i^{-1} G_{s_i}. \end{aligned}$$

Here G_{s_i} is trivial and so $ek \in \{ek_i^{-1} \mid 1 \leq i \leq n\}$. $\square$

We now proceed to construct the fibres for our G-tree and at the same time observe the existence of associated G-maps to V_G that shall be pieced together to complete the proof of our first step.

Theorem 4.1.21. *There exists a G-tree T_W with edge stabilisers commensurable with the complete graph on V such that there are G-maps $VY \cup W \to VT_W \to V_G$ whose composite is the inclusion map.*

Proof. Let $E_1 = \{e \in EY | \delta e = \{e\}\}$. We have that $EX - E_1$ is G-finite since both $F = EX - EY$, and $EY - E_1$ are G-finite by definition of the graph X and Corollary 4.1.11 respectively. Now X is a G-graph with commensurable edge stabilisers, and so it follows from Lemma 3.8.1 that $\delta = \sqcup_{s \in S_G}((EX - E_1) \cap \delta s)$ is finite. We set $m = \max\{1, |\delta|\}$. Now we have as before a tree G-subset E_m of $(VX, \mathbb{Z}_2)$ such that E_m generates $\mathscr{B}_m X$. Furthermore, since $m \geq 1$ then by construction we may assume that E_1 belongs to E_m. Immediately then we obtain a G-tree $T_m = T(E_m)$. This is the tree we require and the difficulty in the remainder of the proof is to show the existence of a G-map $VT_m \to V_G$.

Firstly, we notice that the edge stabilisers of T_m are commensurable with the edge stabilisers of the complete graph on V. This follows from the fact that for any element e of E_m the corresponding coboundary contains no more than m elements, and that any group element stabilising e must also stabilise the set δe.

Let $s \in S_G$. We proceed to find a subset $\delta_s \subseteq E_m$ that refines s as a function on VX. Begin by observing that each component of $X - \delta$ has coboundary in δ and so lies in $\mathscr{B}_m X$. Since there are only a finite number of such components it follows that there exists a finite subset δ_m of E_m such that each component of $X - \delta$ belongs to the ring generated by δ_m.

In the remainder of the proof we shall adopt the following notation. Let $S'_G = \{s \in S_G | \delta s \subseteq E_1\}$ and $S''_G = S_G - S'_G$. For $s \in S_G$, we define,

$$\delta'_s = \begin{cases} \{e|VX : e \in \delta s \cap E_1\} & \text{if } s \in S'_G \\ \{e|VX : e \in \delta s \cap E_1\} \cup \delta_m & \text{if } s \in S''_G. \end{cases}$$

Let $v, v' \in VX$ such that $v(e) = v'(e)$ for all $e \in \delta_s$. We claim that v and v' lie in the same component of $X - ((EX - E_1) \cap \delta s)$. Firstly, notice that if $s \in S'_G$, then this claim amounts to saying that both vertices lie in the same component of the connected graph X and there is nothing to prove. Consider now the case where $s \in S''_G$. Since $\delta_s \supseteq \delta_m$, we see that each of the generators of $X - \delta$ agree on v and v' and thus each component of $X - \delta$ agree, that is to say that v and v' lie in the same component as claimed. Hence we may choose a reduced path p between v and v' that does not intersect $(EX - E_1) \cap \delta s$. Further we observe that v and v' agree on $e \in \delta s \cap E_1$ and so p does not meet the edge e either. We have proved that no edge of δs lies on p and thus s is constant on p and v agrees with v' on s also and so δ_s refines s as required.

We have shown the existence of a subset of E_m refining s for each $s \in S_G$ and since S_G is a transversal for E_G we see that the partition of VX induced by deleting all the edges of T_m is finer than the e partition for any $e \in E_G$. Recall that we may identify VX with $VX|E_G$ and so we see that the map $VX \to VT_m$ is injective and treat this map as an inclusion.

Finally, we must construct a G-map $VT_m \to V_G$. Notice that to satisfy the statement of the theorem it is necessary that this map be inclusion on VX. In order to define a G-map on the remainder of VT_m we must find an element of V_G fixed by G_v for all $v \in VT_m$. Let $v \in VT_m$ and henceforth let K denote G_v. Observe that for $e \in E_m$ we have the following inequality,

$$\bigsqcup_{s \in S_G} (eG \cap \delta_s) \subseteq \left(\bigsqcup_{s \in S''_G} (eG \cap \delta_m) \right) \bigsqcup \left(\bigsqcup_{s \in S_G} (eG \cap \delta s \cap E_1) \right),$$

and we show that the right hand side is finite. That the first term is finite follows from the fact that S''_G is finite, as is δ_m. Since the stabilisers of E are commensurable with those of δ_m, together with the fact that $\iota e =_{\mathscr{S}} \tau e$ as functions on E_G we see that the second term on the right hand side is also finite and so therefore the left hand side is also finite.

We now make the following additional definition. For each $f \in E_G$ we let $\delta_f = \cup\{\delta_s g | (s, g) \in S_G \times G,\ sg = f\}$. Thus we easily observe that δ_f refines $f, \delta_f g = \delta_{fg}$ for all $g \in G$ and further, by above, that $\sqcup_{f \in E_G}(\{e\} \cap \delta_f)$ is finite for all $e \in E_m$.

We proceed to construct an element of V_G fixed by K by considering the K-subtree

T_K of T_m generated by v_0K, and the partition of this tree induced by S_K a K-transversal for E_G.

We have by Lemma 3.8.1 that $\sqcup_{s\in S_K}\delta s \cap ET_K$ is finite and by Lemma 4.1.20 that also $\sqcup_{s\in S_K}\delta_s \cap ET_K$ is finite. Therefore we have that for almost all $s \in S_K$ that s is constant on v_0K, and by removing only finitely many branches from ET_K we arrive at a graph $\widetilde{T}_K$ on which s is constant for all $s \in S_K$. Since the edge stabilisers of T_K are commensurable we are able to construct an element of V_G which is fixed by K as follows.

Define $w : E_G = S_KK \to A$ to be,

$$w(sk) = \begin{cases} v_0(sk) & \text{if } s \text{ is constant on } v_0K, \\ v_0(sk') & \text{whenever } v_0k' \in \widetilde{T}_K. \end{cases}$$

Clearly this function is stabilised by K, it remains to show that it belongs to V_G and so we aim to show that w is $\mathscr{S}$-almost equal to v_0. Note firstly that for all $s \in S_K$, if $v_0k \in \widetilde{T}_K$ then $w(sk) = v_0(sk)$ by definition of the graph $\widetilde{T}_K$.

Let $\widetilde{S}_K = \{s_1, s_2, \ldots, s_p\}$ be the set of $s \in S_K$ such that s is not constant on T_K, and $K' = \{k \in K|\ v_0k \notin \widetilde{T}_K\}$. If we denote the edge of T_K which has endpoint v_0 by e_0, then it follows that for all $k \in K',\ e_0k \in \{e_0k_1, e_0k_2, \ldots, e_0k_q\}$ for some $q \in \mathbb{N},\ k_i \in K'$ since we removed only finitely many edges from T_K to obtain $\widetilde{T}_K$. Hence we see that for each $k \in K'$ that $kk_j^{-1} \in G_{e_0}$ for some $1 \le j \le q$. Therefore for $1 \le i \le p$,

$$\begin{aligned} s_ik &= s_ikk_j^{-1}k_j \quad \text{for some } 1 \le j \le q \\ &\in s_iG_{e_0}k_j \\ &\subseteq \bigcup_{\substack{1\le i\le p \\ 1\le j\le q}} s_iG_{e_0}k_j. \end{aligned}$$

Notice that this is a finite collection of cosets of $G_{e_0k_j} \in \mathscr{S}$ as the stabilisers of the edges of T_K are commensurable with the stabilisers of the elements of E as T_K is a K-subtree of T_m. $\square$

We complete the first step of the proof by piecing together the trees obtained in the previous two results.

Theorem 4.1.22. *There exists a G-tree T with edge stabilisers commensurable with those of the complete graph on V which has $Y \cup W$ as a G-subgraph and there exists a G-map $VT \to V_G$.*

Proof. We already have by Theorem 4.1.1, a G-tree T_Y and a map of G-graphs $\psi : Y \cup W \to T_Y$, and by Theorem 4.1.21 a G-tree T_W with an injective G-map $VY \cup W \to VT_W$. Let U be a G-transversal in VT_Y, and $Z = \bigcup_{u \in U} T_W \otimes_{G_u} G$. Then by Lemma 4.1.17 we observe that $VY \cup W = \bigcup_{u \in U} \psi^{-1}(u) \otimes_{G_u} G$ and we have the following injections,

$$VY \cup W \subseteq \bigcup_{u \in U} (VY \cup W) \otimes_{G_u} G \subseteq \bigcup_{u \in U} VT_W \otimes_{G_u} G = VZ.$$

The map from $VZ \to VT_Y$ sending everything in $VT_W \otimes_{G_u} g$ to ug is then a G-map and by the definition of the isomorphism in the proof of Lemma 4.1.17 we also see that this map agrees with the map ψ on $VY \cup W$. Thus we now speak of the map $\psi : VZ \to VT_Y$. Let $\phi : ET_Y \to EY$ be inverse to the map $\psi : EY \to ET_Y$. Then we have that for all $e \in ET_Y$, $\psi(\iota\phi e) = \iota(\psi\phi e) = \iota e$ and $\psi(\tau\phi e) = \tau e$, since ψ is a graph homomorphism.

Next, form the fibred G-tree, T, with base T_Y and for each $u \in U$, fibre T_W over u. We use the maps $\iota\phi, \tau\phi : ET_Y \to VY \subseteq VZ$ as the attaching maps. By Theorem 4.1.1, we may identify ET_Y with EY in T and now we have that $Y \cup W$ is a G-subgraph of the fibred tree T which has commensurable edge stabilisers by construction and we also have a G-map,

$$VT = \bigcup_{u \in U} VT_W \otimes_{G_u} G \to \bigcup_{u \in U} V_G \otimes_{G_u} G \to V_G.$$

$\square$

Thus by combining results 4.1.1, 4.1.21 and 4.1.22 we arrive at the following result.

Theorem 4.1.23. *Suppose that G is finitely generated over H and that $E_H g \cap E_H = \varnothing$ for all $g \in G - H$. For any G-finite G-subset W of V_G and any H-tree T_H with vertex set V_H, the G-graph $W \cup T_H G$ embeds in a G-tree T with edge stabilisers commensurable with those of the complete graph on V and for which there exists a G-map $VT \to V_G$.*

Notice that so far in our proof we have used neither the condition that the groups in $\mathscr{S}$ have non-zero Euler characteristic or even the weaker condition that those groups are G-conjugate incomparable. The latter condition is used in our proof of step 2 in the following section.

4.2 Step 2 - Adjusting the vertex set

In this section we aim to adjust the G-tree obtained in the previous section in order to arrive at a G-tree with vertex set V_G. This shall be crucial in order to use our induction argument in the subsequent section. We manipulate the vertex set by considering the notion of G-incompressibility as introduced in section III.7 of [12].

Definition 4.2.1. Let $V \subseteq W$ be G-sets. We say that W is *G-incompressible over V* if every G-map $W - V \to W$ is an automorphism of $W - V$. Equivalently, for every $w \in W - V$, $w' \in W$, if $G_w \subseteq G_{w'}$ then $wG = w'G$ and $G_w = G_{w'}$.

Proof. Suppose that V is G-incompressible over W. Let $v \in V - W$ and $v' \in V$ such that $G_v \subseteq G_{v'}$. We define the map $\varphi : V - W \to V$ to be the identity map on $(V - W) - vG$ and $\varphi(vg) = v'g$. Clearly this is a G-map and since $G_v \subseteq G_{v'}$ we see that this map is well defined. Since V is G-incompressible over W we have that φ is an automorphism of $V - W$ and so $v' \in V - W$. However, by our definition of φ we see that $\varphi(vG) = vG = v'G$, and also since φ is an automorphism it follows that there is a well defined inverse G-map and so $G_{v'} \subseteq G_v$ as desired.

To see that the converse is true, let $\phi : V - W \to V$ be a G-map and let $v \in V - W$. Then $\phi v \in V$ and $G_v \subseteq G_{\phi v}$. So by our hypothesis we have that $vG = (\phi v)G, G_v = G_{\phi v}$. In particular $\phi v \in V - W$ and since $G_{\phi v} \subseteq G_v$ we may construct a map $\psi : \phi(V - W) \to V - W$ sending ϕv to v. That this is a G-map and that the maps ϕ and ψ are mutually inverse is easily verified. $\square$

Remark 4.2.2. We now observe the fact that the condition that $\forall\, v \in V - W$ and $v' \in V$ we have that $G_v \subseteq G_{v'} \implies vG = v'G$ is equivalent to every G-map from $V - W \to V$ being a surjective endomorphism. The above proof is easily modified to obtain this result. In such a case we shall say that V is G-almost incompressible over W.

Definition 4.2.3. Let T be a G-tree and Y a G-subgraph of T. If T' is a G-tree obtained from T by contracting edges, and VT' is a G-retract of VT containing VY, then we say that T' is obtained by *compressing* T over Y. If the only such tree T' is T itself we say that T is *incompressible over* Y.

Example 4.2.1. Let $G = \mathbb{Z} = \langle x \rangle$. Let T be the tree with vertex set $\mathbb{Z}$ and having for each $i \in \mathbb{Z}$ an edge joining i to $i + 1$. We consider two different actions of G on this tree.

- Suppose that the action of G is given by x sending the vertex i to $i+1$ for all $i \in \mathbb{Z}$. Then there is only one G-orbit of edges and thus the only G-tree T', other than T itself, that may be obtained by contracting edges is the tree consisting of a single vertex. This vertex then has stabiliser G and so there does not exist a G-map from

VT' to VT as G acts freely on T. Thus VT' cannot be a G-retract of VT unless T' is T itself and so T is G-incompressible over any G-subgraph.

- Suppose instead that the action of G is given by x sending the vertex i to $i+2$ for all $i \in \mathbb{Z}$. Now denote by Y the G-subgraph consisting of the vertices labelled by even integers. Let e be the edge of T connecting 0 to 1. Obtain a G-tree T' by contracting the G-orbit of e, collapsing the vertex 1 to 0. Now $VT' = VY$ and there is a G-map $VT \to VT'$ given by the identity on the even integers and sending every odd i to $i-1$. Thus T' is obtained by compressing T over Y. Notice that the tree T' obtained is isomorphic as a G-tree to the tree in our previous example.

Definition 4.2.4. Let $e \in ET$, where T is a G-tree. Then we say that e is *compressible* over a G-subgraph Y if e has a vertex $v \in VT - VY$ and other vertex v' such that $vG \neq v'G$ and $G_v \subseteq G_{v'}$.

If e is not compressible then it is said to be incompressible over Y.

The following result is based on Lemma III.7.2 in [12].

Lemma 4.2.5. *Let T be a G-tree with edge stabilisers that are G-conjugate incomparable and Y be a G-subgraph of T. Then the following are equivalent.*

1. *VT is G-incompressible over VY.*

2. *T is incompressible over Y.*

3. *T has no compressible edges over Y.*

Proof. (1) $\implies$ (2). Suppose that there exists a G-tree $T' \neq T$ which may be obtained by compressing T over Y. Then VT' is a proper G-retract of VT containing VY and thus there is a G-map from $VT - VY \to VT'$ (the restriction of the retraction map) which is clearly not a surjective endomorphism of $VT - VY$ and it follows that VT is G-compressible over VY.

(2) $\implies$ (3). Now suppose that there exists a compressible edge e of T over Y. Then if we contract each edge in the orbit of e we obtain a G-tree T', say. Then there is a well defined G-map $VT \to VT'$ sending v to v' in the notation of 4.2.4. Clearly there is a G-map $VT' \to VT$ since we may identify VT' with a subset of VT as the stabiliser of the vertex obtained via contracting e is $G_{v'}$.

(3) $\implies$ (1). Assume that there are no compressible edges of T over Y. Let $v \in VT - VY$ and $v' \in VT$ such that $G_v \subseteq G_{v'}$. Suppose that the path in T connecting v and v' is a single edge, then it is easy to see that either this edge is compressible or $Gv = Gv'$ and hence $G_v = G_{v'}$ by G-conjugate incomparability. Thus we may assume that the length of the path is strictly greater than 1. Now let e be the first edge in this path and let the vertices be denoted v and v''. Then it is clear that $G_v \subseteq G_{v''}$, $G_e = G_v$ and since there are no compressible edges we have that $vG = v''G$. It is now obvious that $v'' \in VT - VY$ and that $G_v = G_{v'}$ as the edge stabilisers are G-conjugate incomparable. Thus the result holds by induction on the length of the path between v and v'. □

Definition 4.2.6. Let V be a G-set. Then there is a decomposition,

$$V = V_{\text{HNN}} \sqcup V_{\text{comm}},$$

where,

$$V_{\text{HNN}} = \{v \in V \mid G_v \text{ is not } G\text{-conjugate incomparable}\}$$

$$V_{\text{comm}} = \{v \in V \mid G_v \text{ is } G\text{-conjugate incomparable}\}.$$

Notice that both V_{HNN} and V_{comm} are G-sets as G-conjugate incomparability is preserved by conjugation by an element of G.

With the above definition in place we have the following result.

Lemma 4.2.7. *Let T be a G-tree and let Y be a G-subgraph of T. Then the following are equivalent.*

1. *Every G-map $VT \to VT - VY$ restricts to an automorphism of $(VT - VY)_{comm}$.*
2. *T is incompressible over Y.*
3. *T has no compressible edges over Y.*

Proof. Identical to proof of Lemma 4.2.5. □

Remark 4.2.8. We would like to replace (1) in the statement above by the statement VT is G-almost incompressible over VY. However, in the proof of (3) $\implies$ (1) the final induction argument on the length of the path no longer holds since G_v may be a strict subgroup of $G_{v''}$.

Definition 4.2.9. Denote by

$$(V)_\infty = \{v \in V : |G_v : G_{vv'}| = \infty, \quad v \neq v' \in V\}.$$

That is to say that $(V)_\infty$ is the subset of V whose stabilisers in G are not commensurable with the edge stabilisers of the complete graph on V.

The size of an incompressible G-tree

We now introduce a notion of size for G-finite G-incompressible G-trees which includes Dicks and Dunwoody's notion of size from section III.7 of [12] whilst allowing us to bound the length of chains of G-maps between such trees in the more general case that their edge stabilisers have non-zero integral Euler characteristic. We first introduce the notion of order reversing and order preserving functions.

Definition 4.2.10. Let T be a G-finite, G-incompressible G-tree with edge stabilisers in $\mathscr{S}$. We say that a function $\rho : \mathscr{S} \to \mathbb{N}$ is *order reversing* if for all H and $K \in \mathscr{S}$ and $g \in G$ we have that,

$$H < K \implies \rho(H) > \rho(K) \quad \text{and} \quad \rho(H) = \rho(H^g).$$

We say that a function $\pi : \mathscr{S} \to \mathbb{N}$ is *order preserving* if for all H and $K \in \mathscr{S}$ and $g \in G$ we have that,

$$H < K \implies \pi(H) < \pi(K) \quad \text{and} \quad \pi(H) = \pi(H^g).$$

Two useful examples to keep in mind throughout this chapter are given below.

Example 4.2.2.

- Given a family of subgroups $\mathscr{S}$ with Euler characteristic a non-zero integer, an example of an order reversing map would be $\pi : \mathscr{S} \to \mathbb{N}$ given by $\pi(H) = |\chi(H)|$.
- If $\mathscr{S}$ is a family of finite groups then an example of an order preserving map would be the map $\pi : \mathscr{S} \to \mathbb{N}$ where $\pi(H)$ is simply the order of H.

Notice that by definition such a function may only exist when the edge stabilisers are G-conjugate incomparable. We proceed to define the size sequences of a G-tree with respect to such maps.

Definition 4.2.11. Let T be a G-finite G-incompressible G-tree with edge stabiliser in $\mathscr{S}$. Suppose that we are given an order preserving map $\pi : \mathscr{S} \to \mathbb{N}$. We define the π-*size* of T to be the following sequence of natural numbers,

$$\text{size}^{\pi}(T) = (|G\backslash ET| - |G\backslash VT|, |G\backslash E_1|, |G\backslash E_2|, \ldots),$$

where $E_n = \{e \in ET \mid \pi(G_e) = n\}$. Since T is G-finite we see that size(T) is an eventually zero sequence.

We give the usual lexicographical ordering on these π-size sequences. Notice that in the case that π gives the order of the subgroup we retrieve the Dicks-Dunwoody notion of size. We now introduce a different notion of size which will be useful not in the finite stabiliser case but in the case that our stabilisers are of non-zero Euler characteristic. In the transfinite induction argument it is necessary to bound the possible length of a chain of certain G-trees with G-maps from one to the next, this is ensured in the original proof by the finite edge stabiliser condition. The restriction on the Euler characteristic is sufficient to form such a bound even in the more general setting, to see this requires this new notion of size which we define for all order reversing functions. Lemma 4.2.15 demonstrates how this bound is obtained by Dicks and Dunwoody together with how the argument may be modified for the infinite stabiliser case.

Definition 4.2.12. Let T be a G-finite G-incompressible G-tree with commensurable edge stabilisers in $\mathscr{S}$. Suppose that we are given an order reversing map $\rho : \mathscr{S} \to \mathbb{N}$. We define the *$\rho$-size* of T to be the following sequence of natural numbers,

$$\text{size}_{\rho}(T) = (|G\backslash ET| - |G\backslash VT|, |G\backslash E_1|, |G\backslash E_2|, \ldots),$$

where $E_n = \{e \in ET \mid \rho(G_e) = n\}$.

Since ρ reverses the ordering we must place a different ordering on these sequences. We say that $\text{size}_\rho(T_1) < \text{size}_\rho(T_2)$ if $|G\backslash ET_2| - |G\backslash VT_2| < |G\backslash ET_1| - |G\backslash VT_1|$ or if $|G\backslash ET_2| - |G\backslash VT_2| = |G\backslash ET_1| - |G\backslash VT_1|$ and $|G\backslash E_i|_{T_1} = |G\backslash E_i|_{T_2}$ for all $i > N$ and $|G\backslash E_N|_{T_1} > |G\backslash E_N|_{T_2}$ for some $N \in \mathbb{N}$.

The following lemma appears as Lemma III.7.3 in [12].

Lemma 4.2.13. *If T and T' are G-trees and $VT \approx VT'$ as G-sets then $ET \approx ET'$ as G-sets.*

This shall allow us to replace our G-tree in Lemma 4.2.15 with one for which it is easier to study the edge orbits.

Observe the following isomorphisms of G-sets.

Lemma 4.2.14. *Let T_1 and T_2 be incompressible G-finite G-trees, U_1 and U_2 be G-transversals in VT_1 and VT_2 respectively. Then there are the following isomorphisms of G-sets,*

$$\bigcup_{u \in U_2} VT_1 \otimes_{G_u} G \approx \bigcup_{u \in U_2} VT_1 \times G/G_u \approx VT_1 \times VT_2.$$

Proof. We define the following maps, let

$$\varphi : \bigcup_{u \in U_2} VT_1 \otimes_{G_u} G \to \bigcup_{u \in U_2} VT_1 \times G/G_u$$

be defined by

$$\varphi(v \otimes g) = (vg, G_u g),$$

and

$$\psi : \bigcup_{u \in U_2} VT_1 \times G/G_u \to VT_1 \times VT_2$$

be defined by

$$\psi(v, G_u g) = (v, ug).$$

That these are well defined G-set isomorphisms is easily seen. □

The following lemma is both useful in this section as well as being key to the induction argument in the following section. The finite stabiliser version of this result is to be found in [12] as Lemma III.7.5.

Lemma 4.2.15. *If T_1 and T_2 are incompressible G-finite G-trees with edge stabilisers in $\mathscr{S}$ and there exists a G-map $VT_1 \to VT_2$ then $|G\backslash T_1| \geq |G\backslash VT_2|$. Furthermore, if $\rho : \mathscr{S} \to \mathbb{N}$ is an order reversing function (resp. $\pi : \mathscr{S} \to \mathbb{N}$ an order preserving function) then $\mathrm{size}_\rho(T_1) \geq \mathrm{size}_\rho(T_2)$ (resp. $\mathrm{size}^\pi(T_1) \geq \mathrm{size}^\pi(T_2)$) with equality if and only if $VT_1 \to VT_2$ is an isomorphism.*

Proof. Let U_1, U_2 be G-transversals in VT_1, VT_2, respectively. Since the edge stabilisers of T_1 and T_2 are commensurable we may form the fibred G-tree $\widetilde{T}$ with base T_2 and fibre T_1 over u for each $u \in U_2$. By Lemma 4.2.14 we see that the vertex set of $\widetilde{T}$ considered as a G-set is isomorphic to the following,

$$V\widetilde{T} = \bigcup_{u \in U_2} VT_1 \otimes_{G_u} G \approx \bigcup_{u \in U_2} VT_1 \times G/G_u \approx VT_1 \times VT_2.$$

Since we are given a G-map $VT_1 \to VT_2$, call this map φ, say. Then we see that for $v \in VT_1$, the composition of this map with the identity and projection maps, $v \mapsto (v, \varphi v) \mapsto v$, gives the identity G-map on VT_1. Thus we observe that VT_1 is a G-retract of $V\widetilde{T}$. The map $VT_1 \to V\widetilde{T}$ carries edges of T_1 to paths of $\widetilde{T}$ (Notice that it does not necessarily send edges to edges since the G-map need not be a graph map - a simple example to keep in mind is taking the star of a vertex and bisecting each edge). Since our trees are G-finite we may collect these paths to obtain a G-finite G-subtree. By adding only finitely many G-orbits of edges we may include all of ET_2 in this tree. Here we are identifying T_2 with the subtree of $V\widetilde{T}$ obtained by contracting all the edges of the fibres.

We next define the G-tree T obtained from $\widetilde{T}$ by contracting all edge orbits of compressible edges in $E\widetilde{T} - ET_2$. Notice that T is then a G-finite G-tree, all of its compressible edges lie in ET_2 and T contains a copy of T_2. We now compress the tree T to an incompressible G-tree T'. Since we obtained both T and T' from $\widetilde{T}$ by contracting compressible edges it follows that VT' is a G-retract of VT, which is itself a G-retract of $V\widetilde{T}$ and so there exists a G-map $VT' \to V\widetilde{T}$. Further since VT_1 is a G-retract of $V\widetilde{T}$ we also have a G-map $V\widetilde{T} \to VT_1$. We may now compose maps to obtain the following G-maps, $VT_1 \to V\widetilde{T} \to VT \to VT'$ and $VT' \to V\widetilde{T} \to VT_1$. Since both VT_1 and VT' are incompressible it follows that they have no G-maps onto proper G-subsets of themselves and so the compositions $VT_1 \to VT' \to VT_1$ and $VT' \to VT_1 \to VT'$ must be bijections. Thus we see that $VT_1 \approx VT'$ as G-sets and so by Lemma 4.2.13 we have that $ET_1 \approx ET'$ and so for the remainder of the proof we identify T_1 with T', since we are only concerned with the number of orbits and not the structure of how the tree is connected.

We consider ET_1, ET_2 to be subsets of ET. Since all the compressible edges of T lie in T_2 we see that all the edges of T not in T_2 must belong to T_1 and so we take $ET = ET_1 \cup ET_2$, and observe that all of the edges $ET_2 - ET_1$ are compressible whilst all of $ET_1 - ET_2$ are incompressible. We now consider the components of the graph $T - ET_2$ to be the vertices of T_2 and we write $VT_2 = VT_{21} \sqcup VT_{22}$ where VT_{21} are the vertices of VT_2 which consist of just a single vertex of T and VT_{22} the remainder of the vertices of T_2, that is to say the vertices which consist of more than one vertex (and thus contain some edge) of T. We now observe that VT_{21} can clearly be identified with a G-subset of VT and thus we obtain an injective G-map $VT_{21} \to VT$. Next we compose this map as follows $VT_{21} \to VT \to VT_1 \to VT \to VT_2$ (this last map is obtained by $VT \to V\widetilde{T} \to VT_1 \to VT_2$). Notice that since VT_2 is incompressible it follows that the image of the above map must be VT_{21} and so in particular the image of the map $VT_1 \to VT_2$ must contain VT_{21}. Since the vertices in VT_{22} must contain an edge in

$ET_1 - ET_2$ it follows that we have a surjective G-map $ET_1 - ET_2 \to VT_{22}$. Combining the two maps constructed above we observe that

$$|G\backslash VT_2| \leq |G\backslash(VT_1 \cup (ET_1 - ET_2))| \leq |G\backslash T_1|.$$

It remains now only to consider the size sequences. We first prove the result in the case of an order preserving function π. Notice first that if $VT_1 \to VT_2$ were an isomorphism then $ET_1 \cong ET_2$ and so $\text{size}^\pi(T_1) = \text{size}^\pi(T_2)$. Thus we must show that $\text{size}^\pi(T_1) < \text{size}^\pi(T_2)$ if the map $VT_1 \to VT_2$ is not an isomorphism. Firstly, observe that T_1 is obtained from T by contracting orbits of compressible edges thus for each orbit contracted both one orbit of edges and one orbit of vertices are lost and so $|G\backslash ET_1| - |G\backslash VT_1| = |G\backslash ET| - |G\backslash VT|$. However T_2 is obtained from T by contracting orbits of edges that needn't be compressible and so when one orbit of edges are removed either one or none of the vertex orbits are lost and so $|G\backslash ET_2| - |G\backslash VT_2| \leq |G\backslash ET| - |G\backslash VT|$. Thus we have that the first term in the size sequences are as desired and we may assume without loss that equality holds. If equality holds then one important observation is that for any edge $f \in ET_1 - ET_2$ we lose one vertex orbit by contracting this edge and so ιf and τf lie in different G-orbits and since the edge is incompressible in T we see that G_f is a proper subgroup of both $G_{\iota f}$ and $G_{\tau f}$.

There are now two cases to consider. Suppose that $ET_2 - ET_1 = \varnothing$. Then $ET_1 = ET$, $T = T_1$ and it is clear that $\text{size}^\pi(T_1) > \text{size}^\pi(T_2)$ unless $T_1 = T_2$ in which case $VT_1 \to VT_2$ must be an isomorphism as $VT_1 \cong VT_2$ are both G-incompressible. Finally suppose that $ET_2 - ET_1 \neq \varnothing$. We choose $e \in ET_2 - ET_1$ with $\pi(G_e)$ minimal. We shall denote the end points of e by u and v, and we assume without loss that e is compressed to u in T_1. Let $\overline{v}$ denote the image of v in T_2. Now if $\overline{v} = \{v\}$ then $G_{\overline{v}} = G_v = G_e$ but e is not compressible in T_2 and so $\overline{u}$ and $\overline{v}$ must belong to the same orbit and therefore $\overline{u} = \{u\}$. Hence u and v belong to the same orbit contradicting the fact that e is compressible in T. It is therefore the case that $\overline{v} \in VT_{22}$. Thus we may find an edge $f \in \overline{v}$ incident to v with $f \in ET_1 - ET_2$. However, by the above argument we have that G_f is a proper subgroup of $G_v = G_e$. Thus we have found an edge stabiliser in ET_1 that is a strict subgroup of G_e and thus $\text{size}^\pi(T_1) > \text{size}^\pi(T_2)$ as desired. Notice that the same proof as above holds in the case ρ is an order reversing map. We instead choose $e \in ET_2 - ET_1$ with $\rho(G_e)$ maximal. $\square$

We may now complete the second step of our argument. The finitely generated case of our main result is given by the following lemma.

Lemma 4.2.16. *Suppose that G is finitely generated over H and that $E_H g \cap E_H = \varnothing$ for all $g \in G - H$. Further suppose that the subgroups in $\mathscr{S}$ are G-conjugate incomparable. Then any H-tree T_H with vertex set V_H extends to a G-tree T_G with vertex set V_G. Further, for any such T_G, the G-tree obtained by contracting T_H to a vertex can be compressed to an incompressible G-finite G-tree.*

Proof. By Theorem 4.1.23, we have that for W any G-finite G-subset of $(V_G - V_H G)_\infty$ there exists a G-tree T with commensurable edge stabilisers containing $W \cup T_H G$ as a G-subgraph and a map $VT \to V_G$. If we assume that G is generated by $H \cup \{g_1, \dots, g_m\}$ and choose a finite subtree X of T containing $\{v_0, v_0 g_1, \dots v_0 g_m\}$ then we see that $XG \cup T_H G$ is connected by contracting all the edges of $XG \cup T_H G$ and observing that the image of v_0 is stabilised by a generating set for G. Since XG is G-finite we may proceed to contract orbits of edges until we arrive at a G-tree denoted T_W that is G-incompressible over $T_H G$. Notice now that $VT_W - T_H G$ is G-finite and there is a G-map $VT_W - T_H G \to V_G$ obtained by collapsing G-orbits.

Since T_W is incompressible over $T_H G$ it follows that the image of this map does not meet $T_H G$ i.e., we have a G-map $VT_W - T_H G \to V_G - T_H G$.

T contains W by definition and so we have an embedding $W \subseteq VT_W - V_H G$. Let $\overline{T}_W$ be the G-finite G-tree obtained by contracting all the edges in $T_H G$, and denote by $\overline{v_0}$ the image of v_0 under this contraction.

Then we have that $V\overline{T}_W = (VT_W - V_H G) \cup \overline{v_0} G$ where W is contained in the first term and so the embedding of W is into $V\overline{T}_W$. Thus we now see that $|G \backslash W| \leq |G \backslash (V\overline{T}_W)_\infty|$ since $W \subseteq (V_G)_\infty$.

In particular, we have $T_\varnothing$ and a map $(VT_\varnothing - V_H G)_\infty \to (V_G - V_H G)_\infty$. let W be a G-finite G-subset of $(V_G - V_H G)_\infty$ which contains the image of $(VT_\varnothing - V_H G)_\infty$. We also have the map $(V_H G)_\infty \to V_H G$. Thus we have a G-map

$$(VT_\varnothing)_\infty \to (V_G - V_H G)_\infty \cup (V_H G)_\infty = (V_G)_\infty.$$

Let $v \in VT_\varnothing - (VT_\varnothing)_\infty$. Then G_v is commensurable with the stabilisers of the edges of T_W and so we have that G_v fixes some vertex of T_W and we obtain a G-map

$$VT_\varnothing - (VT_\varnothing)_\infty \to VT_W.$$

Combining these two maps we arrive at a G-map $VT_\varnothing \to VT_W$.

This induces a map $V\overline{T}_\varnothing \to V\overline{T}_W$. Compressing edges of these trees cannot increase $|G\backslash\overline{T}_\varnothing|$ and does not alter $|G\backslash(V\overline{T}_W)_\infty|$. Thus we may apply Lemma 4.2.15 to obtain $|G\backslash\overline{T}_\varnothing| \geq |G\backslash(V\overline{T}_W)_\infty| \geq |G\backslash W|$. Thus we may take $W = (V_G - V_H G)_\infty$ and set $T_G = T_W$.

This final part of the argument requires that the stabilisers are G-conjugate incomparable. We currently have G-maps $VT_G \to V_G \to VT_G$ which restrict to the identity on $V_H G$. Now since T_G is incompressible over $T_H G$ we have that the composition must be bijective on $VT - V_H G$ and thus in particular the map $VT_G \to V_G$ is injective. Thus VT_G is a G-retract of V_G and we may extend T_G to a G-tree with vertex set V_G and we have our result. □

In the general case where the edge stabilisers needn't be G-conjugate incomparable we observe that the map $VT_G \to V_G$ needn't be injective. However, if it were true that whenever such a G-map exists then there must exist another such G-map that is injective then our result would still hold. Notice that the only difficulty here is in the case that a vertex stabiliser is G-conjugate comparable thanks to Lemma 4.2.7. Thus we arrive at the following question.

Question. *Let V be a G-stable $\mathscr{S}$-almost equality class in $\mathscr{S}(\sqcup G, A)$. Suppose that E is a G-set with stabilisers in $\mathscr{S}$ such that every G-map $E \to E$ restricts to an automorphism on E_{comm}. If there exist G-maps $E \to V_G$ and $V_G \to E$ then does there exist an injective G-map $E \to V_G$?*

This is equivalent to the statement that if for all $e \in E$, there exists $e' \in E$ and $v \in V_G$ with the property that $G_e \leq G_v \leq G_{e'}$ then for all $e \in E$ there exists some $\hat{v} \in V_G$ such that $G_e = G_{\hat{v}}$.

If the answer to the above question is true then we may remove the G-conjugate incomparable condition from the hypotheses of Lemma 4.2.16. It is worth noting however that even without this conjecture in the general case we do obtain a G-tree together with a G-map from its vertex set to V_G. This is sufficient together with the work in Chapter 5 to prove Conjecture B in the case that G is finitely generated. Thus we recover the result of Kropholler [23].

We pause at the end of this section to notice that although we have introduced the size sequence attached to the Euler characteristic function, we have not yet utilised the condition that the subgroups in $\mathscr{S}$ have non-zero Euler characteristic. This shall be used in the following final segment of our proof.

4.3 Step 3 - Transfinite Induction

Having proven the finitely generated case we proceed to the induction argument to complete our proof. In order to repeatedly apply our result for the finitely generated case we require to show that certain groups are finitely generated for which the following lemma shall be utilised. The following result is a modification of Lemma III.8.1 from [12] however the proof goes exactly the same way.

Lemma 4.3.1. *Let T be a G-tree with finitely generated commensurable edge stabilisers and let H stabilise a vertex v_0 of T. If G is finitely generated over H then G_{v_0} is finitely generated over H, and for each $v \in VT - v_0G$, G_v is finitely generated.*

Proof. We follow the construction given in [12]. We let $H \cup \{g_1, g_2, \ldots, g_n\}$ generate G. We have a G-map from $G\backslash H$ to VT which sends H to v_0. We proceed to construct a graph by drawing each vertex of T as a circle and for each edge in the star of that vertex we add a vertex to the boundary of our circle and attach the edge there (Notice that this construction ensures that the endpoints of edges in T now have the same stabilisers in our new graph as the original edges). Inside the circle we add one vertex for each element of $G\backslash H$ which maps there. For each $1 \leq i \leq n$, we add edges to our diagram joining u_0 to $u_0 g_i$ corresponding to the paths in T joining v_0 to $v_0 g_i$. We let X be the G-graph composed of the G-translates of these paths. Notice here that X must be G-finite and further X is connected. Extend the G-map from $G\backslash H \to VT$ above to a G-map $\phi : X \to VT$ as in [12]. Let $v \in VT$. If v does not lie in the image of ϕ, then v is not a vertex of the subtree $\widetilde{T}$ of T generated by v_0G. It follows that G_v acts on the component of $T - \widetilde{T}$. In particular G_v fixes the closest vertex of $\widetilde{T}$ which is at least one edge away, so G_v fixes the path and therefore is commensurable with the edge stabilisers which are by hypothesis finitely generated and it follows that G_v is itself finitely generated. We may now assume that v lies in the image of ϕ. Taking the graph $X_v = \phi^{-1}(v)$, we observe that X_v is a connected G_v-finite G_v-graph. Thus by the structure theorem for groups acting on connected graphs (Theorem I.9.2 [12]) we have that G_v is finitely generated over H. $\square$

4.4 The Induction Argument

Our induction argument follows the proof found in [12] which itself draws on techniques that originally appear in section 6 of [11]. We prove the following result, crucial to the proof of the main theorem in the countable case.

Theorem 4.4.1. *Suppose that H contains a subgroup commensurable with the edge stabilisers of the complete graph on V and that the Euler characteristic of the edge stabiliser is a non-zero integer. Let G be finitely generated over H such that $E_H g \cap E_H = \varnothing$ for all $g \in G - H$. Suppose also that $H \leq K \leq G$ and whenever $K \leq L \leq G$ and L is finitely generated over H and $E_K L = E_L$ then $E_K g \cap E_K = \varnothing$ for all $g \in L - K$.*

If an H-tree T_H with vertex set V_H extends to a K-tree T_K with vertex set V_K, then T_K extends to a G-tree T_G with vertex set V_G.

Proof. The idea of this proof is to construct a chain of subgroups (G_n) of G containing K and finitely generated over H. We then construct a descending chain of G_n-subtrees containing V_H and show that this process must eventually terminate. In this proof we use the following notation:

Let

$$E_0 = E_G, \quad V_0 = V(E_0) = V_G, \quad G_0 = G_{V_0} = G.$$

Now for all $n \geq 1$, let

$$E_n = E_K G_{n-1}, \quad V_n = V(E_n), \quad G_n = G_{V_n} \cap G_{n-1}.$$

We observe then that,

$$E_n \subseteq E_{n-1}, \quad V_n \subseteq V_{n-1}, \quad G_n \leq G_{n-1}, \quad E_n G_n = E_n.$$

Further for $A \leq G$, denote by $\mathscr{S}_A$ the collection of subgroups,

$$\mathscr{S}_A = \{ \ S \in \mathscr{S} \mid S \subseteq A \ \}.$$

Recall that Lemma 3.2.4 gives that $\mathscr{S}_A$ is a commensurability class of finitely generated subgroups whenever $H \leq A$.

Let $n \geq 0$. Assume that G_n contains K and is finitely generated over H. We proceed to show that G_{n+1} is finitely generated over H and contains K.

Recall that $V_n = \{v \in V \mid v \bigtriangledown v_0 \subseteq E_n\}$. Furthermore since V_n is contained in an $\mathscr{S}$-almost equality class, all functions $\mathscr{S}_{G_n}$-almost equal to some $v \in V_n$ are elements of V_n, and by definition E_{n+1} is a G_n-set. From this we obtain G_n-set isomorphisms,

$$V_n \to V_n|E_n \to V_n|(E_n - E_{n+1}) \times V_n|E_{n+1}.$$

In particular we have the map,

$$\phi : V_n \to V_n|(E_n - E_{n+1}),$$

and define $\overline{v}_0 = v_0|(E_n - E_{n+1})$.

Let $\overline{v}_0 \in U$ be a G_n-transversal for $V_n|(E_n - E_{n+1})$. Then for $u \in U$,

$$\phi^{-1}(u) \cong \{u\} \times V_n|E_{n+1} \cong V_n|E_{n+1}$$

as G_{nu}-sets. A special case of note is that $\phi^{-1}(\overline{v}_0) = V_{n+1}$. Clearly $\phi(V_{n+1}) = \overline{v}_0$, and this fact together with that that ϕ is a G_n-map gives us that $G_{n\overline{v}_0} = G_{n+1}$ since for $g \in G_{n\overline{v}_0}, v \in V_{n+1}$ we have that,

$$\phi(vg) = \phi(v)g = \overline{v}_0 g = \overline{v}_0,$$

and so $vg \in V_{n+1}$, and $G_{n\overline{v}_0} \leq G_{V_{n+1}} \cap G_n = G_{n+1}$. To see that the reverse inclusion holds notice firstly that $G_{n+1} \leq G_n$ by definition and also for $g \in G_{n+1},\ v \in V_{n+1}$ we have that,

$$\overline{v}_0 = \phi(vg) = \phi(v)g = \overline{v}_0 g,$$

where the first equality holds since V_{n+1} is a G_{n+1}-set (since E_{n+1} is also) and it follows that $G_{n+1} \leq G_{\overline{v}_0}$. Thus we now have that $G_{n\overline{v}_0} = G_{n+1}$.

We now observe that we have an isomorphism of G_n-sets,

$$V_n \cong \bigsqcup_{u \in U} (V_n|E_{n+1}) \otimes_{G_{nu}} G_n,$$

with the G_n-action on the right on the tensor product. This isomorphism is obtained via Lemma 4.1.17 having observed $\phi^{-1}(u) \cong V_n|E_{n+1}$.

In particular we have the following isomorphisms,

$$\theta : V_n \to \bigsqcup_{u \in U} (V_n|E_{n+1}) \otimes_{G_{nu}} G_n$$

$$(v_2|E_{n+1}, v_1|(E_n - E_{n+1})) = (v_2|E_{n+1}, ug) \mapsto ((v_2|E_{n+1})g^{-1} \otimes g).$$

$$\psi : \bigsqcup_{u \in U} (V_n|E_{n+1}) \otimes_{G_{nu}} G_n \to V_n$$

$$v \otimes g \mapsto (vg, ug).$$

Thus we aim to construct a fibred G-tree with the vertex set of the fibres given isomorphic to $V_n|E_{n+1}$ as G_{nu}-sets for each $u \in U$.

Now $E_K \subseteq E_K G_n = E_{n+1}$. Recall, $E_K = \cup_{k \in K} v_0 \bigtriangledown v_0 k \subseteq E_{n+1}$, and therefore K fixes $\overline{v}_0$. Further, by the induction hypothesis we have that $K \leq G_n$ and so $K \leq G_{n\overline{v}_0} = G_{n+1}$. To see that G_{n+1} is finitely generated over H we construct a G_n-tree and apply Lemma 4.3.1

Let $W = V_n|(E_n - E_{n+1})$. Then W is a G_n-stable $\mathscr{S}_{G_n}$-almost equality class, via Lemma 3.2.4, and since H fixes $\overline{v_0}$ it follows that $W_H = \{\overline{v_0}\}$. We may now apply Theorem 4.2.16 to extend $\{\overline{v_0}\}$ to a G_n-tree with vertex set W_{G_n}. This may in turn be extended to a G_n-tree with vertex set W as W_{G_n} is a G_n-retract of W. We shall use this tree as the base for a fibred G_n-tree. The fibres are obtained as follows. First apply Lemma 4.3.1 to the above tree to see that G_{n+1} is finitely generated over H and that for all $u \in U - \{\overline{v_0}\}$ the group G_{nu} is finitely generated. We then proceed in a similar fashion as to the construction of the base. Let $W' = V_n|E_{n+1}$ be a G_{n+1}-stable $\mathscr{S}_{G_{n+1}}$-almost equality class. Since $E_H \subseteq E_n$ for all $n \in \mathbb{N}$ it follows that $V_H \subseteq V_n$ for all $n \in \mathbb{N}$. Thus we may think of V_H as sitting inside W'. Now apply Theorem 4.2.16 to extend T_H to a G_{n+1}-tree with vertex set $W'_{G_{n+1}}$ which may again be extended to a G_{n+1}-tree T'_{n+1} with vertex set W' as $W'_{G_{n+1}}$ is a G_{n+1}-retract of W'. The tree T'_{n+1} shall be our fibre over $\overline{v_0}$.

Similarly, for each $u \in U - \{v_0\}$ we let $W'' = V_n|E_{n+1}$ be the corresponding G_{nu}-stable $\mathscr{S}_{G_{nu}}$-almost equality class. Since G_{nu} is finitely generated we may take the tree $\{v_0|E_{n+1}\}$ on which the trivial group acts and use Theorem 4.2.16 to extend this to a G_{nu}-tree with vertex set $W''_{G_{nu}}$, again this may be further extended to a G_{nu}-tree with vertex set W''. This shall be our fibre over u. Then we may form the fibred G_n-tree T_n with base and fibres as given above.

From the above identification of V_n, we see that there is a natural identification $VT_n = V_n$ and that we have a G_{n+1}-subtree T'_{n+1} with vertex set V_{n+1} containing T_H.

We next construct a sequence of G-trees denoted $T^{(i)}$ for each $i \in \mathbb{N}$ having vertex set V_G such that $T^{(n)}$ contains T_n as a G_n subtree with vertex set V_n. To begin we take $T^{(0)}$ to be T_0 as above. Recall that T_0 contains a subtree T'_1 with vertex set V_1. To construct $T^{(1)}$ we contract the orbits of edges in T'_1 and use the resulting tree as a base for a fibred tree having fibre T_1 over v_0 and all other fibres trivial. Thus we arrive at another G-tree with vertex set V_G that now contains $T_1 \supseteq T'_2$ as subtrees. We continue this process constructing $T^{(j)}$ by contracting T'_{j+1} to a single vertex and forming the fibred G-tree with this base and fibre T_{j+1} over v_0. Notice that by construction we now have that $T_n g \cap T_n = \varnothing$ for all $g \in G - G_n$. We now claim that for some value of n we have that $G_{n-1} = G_n$. This is easy to see in the case that $G_i \in \mathscr{S}$ for sufficiently large i since each G_i contains H which in turn contains some subgroup in $\mathscr{S}$. Since $\mathscr{S}$ consists of finitely generated subgroups there exists only finitely many subgroups between any two members of $\mathscr{S}$. Hence we may now assume that no G_i belongs to $\mathscr{S}$. In this event we use Theorem 4.2.16 to construct an incompressible G-finite G-tree $\overline{T}^{(\infty)}$ by contracting T_H to a vertex in $T^{(0)}$ and compressing edges. Now in a similar fashion we may for $n \geq 0$ use Theorem 4.2.16 to construct G-finite G-incompressible G-trees $\overline{T}^{(n)}$ from $T^{(n)}$ by contracting T_n and compressing edges. Since $G_n \notin \mathscr{S}$ and compressing edges preserves vertices with stabilisers not in $\mathscr{S}$ we see that G_n fixes a vertex of $\overline{T}^{(n)}$. Since we now have a descending sequence of trees,

$$T_0 \supseteq T_1 \supseteq \cdots \supseteq T_H,$$

we arrive at a sequence of G-maps,

$$V\overline{T}^{(0)} \leftarrow V\overline{T}^{(1)} \leftarrow \cdots \leftarrow V\overline{T}^{(\infty)},$$

and the following information on their $|\chi|$-sizes,

$$\text{size}_{|\chi|}(\overline{T}^{(0)}) \leq \text{size}_{|\chi|}(\overline{T}^{(1)}) \leq \ldots \leq \text{size}_{|\chi|}(\overline{T}^{(\infty)}).$$

We denote by $|\chi|$ in the above the function $|\chi| : \mathscr{S} \to \mathbb{N}, H \mapsto |\chi(H)|$. Notice that this is simply the Euler characteristic in the case that $\mathscr{S}$ contains a subgroup of positive Euler characteristic. It is at this point that we first use the condition that the Euler characteristic is non-zero. This is crucial to our argument, in that it allows us to consider the corresponding size sequences (from Definition 4.2.12) and thus deduce that the above sequence of trees terminates. Since there are only finitely many size sequences between $\text{size}_{|\chi|}(\overline{T}^{(0)})$ and $\text{size}_{|\chi|}(\overline{T}^{(\infty)})$ we have by Lemma 4.2.15 that eventually these G-maps are isomorphisms. Since the vertex with stabiliser G_{n-1} gets mapped to the vertex with stabiliser G_n it follows that $G_{n-1} = G_n$ for some $n \in \mathbb{N}$.

To complete the proof we observe that $v_0 G_{n-1} = v_0 G_n \subseteq V_n$. Hence $E(v_0 G_{n-1}) \subseteq E(V_n) = EV(E_K G_{n-1}) = E_K G_{n-1}$. However, it is clear that $E_K G_{n-1} \subseteq E_{G_{n-1}}$ for all n and so $E_{G_{n-1}} = E_K G_{n-1}$ which gives that $V_{G_{n-1}} = V(E_{G_{n-1}}) = V(E_K G_{n-1}) = V_n$. Now by hypothesis we have that $E_K g \cap E_K = \varnothing$ for all $g \in G_{n-1} - K$ and so by Theorem 4.2.16 we may extend T_K to a G_{n-1}-tree with vertex set V_n. Now if we take this tree as T_n in our earlier construction then the tree $T^{(n)}$ is as required. □

We have now completed the necessary work to generalise the Almost Stability Theorem. The following two results are simply Theorems III.8.4 and III.8.5 of [12] stated in our more general setting. The only change necessary to the proofs are that our more general preliminary results are used instead. We include the full proofs here for completeness.

Theorem 4.4.2. *Suppose that G is countably generated over H, that H contains a subgroup belonging to $\mathscr{S}$, and that $E_H g \cap E_H = \varnothing$ for all $g \in G - H$. Then any H-tree T_H with vertex set V_H extends to a G-tree T_G with vertex set V_G.*

Proof. Let $g_1, g_2, \ldots$ be a countable sequence of elements in G such that G is generated by $H \cup \{g_1, g_2, \ldots\}$. We now construct a chain of subgroups $H = G_0 \leq G_1 \leq \cdots$ which have the following properties, for all $n \geq 0$,

1. $g_n \in G_n$
2. G_n is finitely generated over H
3. If L is finitely generated over H with $G_n \leq L \leq G$, then $E_{G_n} L = E_L$ implies that $E_{G_n} g \cap E_{G_n} = \varnothing$ for all $g \in L - G_n$.

Let $n \geq 1$ and suppose that we have constructed $G_0, \ldots, G_{n-1}$. Let K be a subgroup of G which is finitely generated over H and contains $G_{n-1} \cup \{g_n\}$. We observe from Lemma 2.6.4 that $K \backslash (E_K - E_H K)$ is finite and thus there exists a K with the above properties such that $|K \backslash (E_K - E_H K)|$ is minimal. We take this group K to be our G_n. Clearly properties 1 and 2 hold. Suppose that 3 does not. Then there exists a group $L \leq G$ such that $G_n \leq L$ is finitely generated over $H, E_{G_n} L = E_L$ but $E_{G_n} g \cap E_{G_n} \neq \varnothing$ for some $g \in L - G_n$. The last two conditions above can be seen to be equivalent to the map $E_{G_n} \otimes_{G_n} L \to E_L$ which sends $e \otimes l \mapsto el$ being surjective but not injective. Thus in this case we may remove $E_H L$ from both domain and image to obtain a surjective map $(E_{G_n} - E_H G_n) \otimes_{G_n} L \to E_L - E_H L$. It can be observed from this then that

$$|L \backslash (E_L - E_H L)| < |L \backslash ((E_{G_n} - E_H G_n) \otimes_{G_n} L)| = |G_n \backslash (E_{G_n} - E_H G_n)|.$$

However this contradicts our choice of G_n and thus property 3 is also satisfied and we obtain the chain of groups we desire.

Now let $T_0 = T_H$. We may use Theorem 4.4.1 to show that T_0 extends to a G_1-tree T_1, and inductively for each $n \geq 1$, that T_n extends to a G_{n+1}-tree T_{n+1}. We now take the G-tree $T = \bigcup_{n \geq 0} T_n$ to be our tree T_G and it is clear that the result holds for this tree. $\square$

We now complete the proof of our main theorem, namely Theorem A. With our preliminary results in place the final transfinite induction argument is almost identical to the proof of the Almost Stability Theorem in [12] though we appeal to our more general versions of the previous results.

Proof. (of Theorem A)

Index the elements of G with some ordinal γ. We shall construct a chain of subgroups G_β for $\beta \in [0, \gamma]$. We denote E_{G_β}, V_{G_β} by E_β and V_β respectively and require that our chain of groups satisfy the following properties for all $\beta \in [0, \gamma]$,

- $\{g_\alpha : \alpha \in [0, \beta)\} \subseteq G_\beta$,
- $E_\beta g \cap E_\beta = \varnothing$ for all $g \in G - G_\beta$,
- G_β is countably generated over $G_{\beta-1}$ if β is a successor ordinal,
- $G_\beta = \bigcup_{\alpha<\beta} G_\alpha$ if β is a limit ordinal.

Let $\beta \in [1, \gamma]$ and assume now that we have constructed G_α for all $\alpha \in [0, \beta)$. If β is a limit ordinal then we define $G_\beta = \bigcup_{\alpha<\beta} G_\alpha$ and clearly the four conditions above are satisfied.

We may now assume that β is a successor ordinal. We proceed to construct a second ascending chain of groups, the union of which we shall use as our G_β. The same technique may be utilised to ensure that the group G_1 contains an element of $\mathscr{S}$ (by instead choosing K_0 to be generated by g_1 together with a finitely generated subgroup in $\mathscr{S}$). Let K_0 be the group generated by $G_{\beta-1} \cup \{g_\beta\}$. Suppose that $n \geq 0$ and that we have constructed a subgroup K_n of G which is finitely generated over $G_{\beta-1}$. We define K_{n+1} to be the subgroup generated by $K_n \cup \{g \in G | E_{K_n} g \cap E_{K_n} \neq \varnothing\}$. We claim that K_{n+1} is finitely generated over K_n. To see this let $S_{\beta-1} \subseteq E_{\beta-1}$ and S be K_n-transversals for $E_{\beta-1}K_n$ and $E_{K_n} - E_{\beta-1}K_n$ respectively. We observe from Lemma 2.6.4 that S is finite and write $S = \{s_1, s_2, \ldots s_t\}$. Further we have that $S_{\beta-1}$ is in fact a G-transversal for $E_{\beta-1}G$, since we have constructed $G_{\beta-1}$ satisfying the above hypothesis and $G_{\beta-1} \leq K_n$. It is a trivial observation that every group element $g \in G$ such that $S_{\beta-1}g \cap S_{\beta-1} \neq \varnothing$ belongs to $G_{\beta-1}$. We proceed to show that if $Sg \cap (S \cup S_{\beta-1})$ is non-empty then g belongs to some finitely generated subgroup of G.

Suppose that $e \in Sg \cap (S \cup S_{\beta-1})$. Then $e = s_i g$ for some $1 \leq i \leq t$. Let us first consider the case where $e = s_j$ for some $1 \leq j \leq t$. For each $1 \leq a, b \leq t$, choose a group element g_{ab} such that $s_a g_{ab} = s_b$. If no such element exists then we set $g_{ab} = 1$. Denote by A the set of elements, $A = \{g_{ab} | 1 \leq a, b \leq t\}$. Now it is clear that g belongs to the subgroup generated by $\bigcup_{k=1}^{t} G_{s_k} \cup A$.

The other case which we must consider is where $e \in S_{\beta-1}$. First we observe that there is only a finite set of elements of $S_{\beta-1}$ such that $Sg \cap S_{\beta-1} \neq \varnothing$, since

$$\begin{aligned} & s_i g_1 = s', \;\; s_i g_2 = s'' \\ \Longrightarrow \quad & s'' g_2^{-1} g_1 = s' \\ \Longrightarrow \quad & s' = s'' \end{aligned}$$

for all $g_1, g_2 \in G, s_i \in S, 1 \leq i \leq t, s', s'' \in S_{\beta-1}$. We denote this finite subset of $S_{\beta-1}$ by B.

It is now clear that for $g \in G$,

$$Sg \cap (S \cup S') \neq \varnothing$$

implies that g belongs to the group generated by

$$\bigcup_{i=1}^{t} G_{s_i} \cup A \cup \bigcup_{b \in B} G_b.$$

Since the stabilisers are all finitely generated by hypothesis it follows that the above group is itself finitely generated. Thus we have that K_{n+1} is finitely generated over K_n, and we define K_m inductively for all $m \geq 0$. Define $G_\beta = \cup_{n \geq 0} K_n$. It is easy to check that the conditions for G_β are satisfied and so by transfinite induction we have the desired chain of subgroups.

Next we form the same ascending chain of subtrees of the complete graph on V as in [12]. Our more general version of Theorem III.8.4 can now be applied, namely Theorem 4.4.2 and we now obtain our more generalised version of the Almost Stability Theorem. $\square$

In fact, now combining this result with that of Dicks and Dunwoody [13], stated in this thesis as Theorem 3.3.4 we obtain the following theorem.

Theorem 4.4.3. *Let H be a finitely generated subgroup of G with $\chi(H)$ a non-zero integer and $\mathrm{Comm}_G(H) = G$. Let $\mathscr{S}$ be the admissible family of subgroups commensurable with H and A and I be non-empty sets. Suppose that V is a G-retract of a G-stable $\mathscr{S}$-almost equality class in $\mathscr{S}(\sqcup_I G, A)$. Then there exists a G-tree with edge stabilisers in $\mathscr{S}$ and vertex set V.*

Chapter 5

Conjectures concerning duality groups

In this chapter we recall the notion of a duality group, some recent work of Kropholler [23] in this area and how we may use our generalisation of the Almost Stability Theorem to extend these results.

5.1 Duality groups

For compact oriented manifolds there is a well known notion of Poincaré duality arising from the standard homology and cohomology of manifolds. This corresponds to a duality of the fundamental groups of such manifolds in terms of their usual group cohomology. This concept may be generalised to capture a similar notion of duality for other groups and for cohomology computed over some ring other than $\mathbb{Z}$. Thus we introduce the following definitions from section 9.2 of [6].

Definition 5.1.1. A group G is said to be an *n-dimensional duality group over R* if there exist for all $i \in \mathbb{Z}$ and for all right RG-modules M isomorphisms

$$H^i(G, M) \cong H_{n-i}(G, M \otimes_{RG} D)$$

where we call the left RG-module D the dualising module. We often refer to G as simply a duality group when the ring R and dimension n is clear from the context. Notice that in the above definition of duality group that D has a G-module structure and that this may or may not be given by the trivial action of G on D.

In the case that G is a duality group over $\mathbb{Z}$ and $D \cong \mathbb{Z}$ we say that G is an *n-dimensional Poincaré duality group*, written PD^n group.

Remark 5.1.2.

- When D has trivial G-action we say that G is an *orientable* duality group. We call G non-orientable when this is not the case. Notice that in the case of Poincaré duality groups we may always pass to a subgroup of finite index to ensure orientability. This is because the automorphism group of $\mathbb{Z}$ has order 2.
- The notion of a Poincaré Duality group was first introduced in [7] and detailed accounts of such groups and their properties may be found in [3–6, 8, 30].

5.2 Elementary properties of duality groups

We state here some basic properties of such groups that we shall use in the proceeding work. All of the results in this section may be found in for example Section 9 of [6].

Proposition 5.2.1. *The module D is isomorphic to $H^n(G, RG)$ as an abelian group.*

Theorem 5.2.2. *Let G be a group. Then G is an n-dimensional duality group over R if and only if the following 3 conditions hold.*

1. *G is of type FP over R,*
2. *$H^i(G, RG) = 0$ for $i \neq n$,*
3. *$H^n(G, RG)$ is flat as an R-module.*

This classification is particularly useful since the first condition tells us that the group G is necessarily finitely generated.

Proposition 5.2.3. *Let G be an n-dimensional duality group with dualising module D and H a finite index subgroup of G. Then H is also an n-dimensional duality group with dualising module D with the original action restricted to H.*

Notice however from the equivalent definition given above that a duality group must be of type FP and since this property is not preserved by taking finite index supergroups the converse does not hold in general, we do however have the following result.

Proposition 5.2.4. *Let G be a group without R-torsion. Then if G has a finite index subgroup that is an n-dimensional duality group over R then G is also an n-dimensional duality group.*

In particular if G is a group of finite cohomological dimension over R then a subgroup H is a duality group if and only if every subgroup of G commensurable with H is a duality group.

5.3 Conjecture B

The following theorem appears in [23].

Theorem 5.3.1 (Kropholler)**.** *Let G be a finitely generated group of cohomological dimension $n < \infty$. Let H be a PD^{n-1}-subgroup of G such that* $\mathrm{Comm}_G(H) = G$*. Then G splits over a subgroup commensurable with H.*

We wish to remove the condition that G be finitely generated and aim to show that our generalisation of the Almost Stability Theorem can be used to prove a more general version of this result.

In the paper [23] Kropholler generalised the Lyndon-Hochschild-Serre spectral sequence obtained for normal subgroups to admissible families of subgroups. The functors $H^i(\mathscr{S}, -) = \varinjlim_{H \in \mathscr{S}} H^i(H, -)$ are defined for all $i \in \mathbb{Z}$. In particular the first such functor is given by $H^0(\mathscr{S}, M) = \cup_{H \in \mathscr{S}} M^H$. The category Mod$-\mathbb{Z}G/\mathscr{S}$ is then defined to be the full subcategory of right $\mathbb{Z}G$-modules, with objects the modules M that may be written as $M = \cup_{H \in \mathscr{S}} M^H$. The functors $H^i(\mathscr{S}, -)$ are then seen to be functors from Mod$-\mathbb{Z}G$ to Mod$-\mathbb{Z}G/\mathscr{S}$. The functor $H^0(G \backslash \mathscr{S}, -)$ is defined to be the restriction of the G-fixed point functor to the new category Mod$-\mathbb{Z}G/\mathscr{S}$ with right derived functors $H^i(G/\mathscr{S}, -)$. More details on these functors may be found in [23]. We make particular note of the following key result, Theorem A from section 1 of the Kropholler paper that will play an important role in our work.

Theorem 5.3.2. *Let $\mathscr{S}$ be an admissible family of subgroups.There is a spectral sequence*

$$H^p(G/\mathscr{S}, H^q(\mathscr{S}, M)) \implies H^{p+q}(G, M),$$

natural in the G-module M.

In particular we have the following reuslt.

Corollary 5.3.3. *Let $\mathscr{S}$ be an admissible family of PD^{n-1}-subgroups of a group G of cohomological dimension n. Let A be an abelian group. Then there is an isomorphism,*

$$H^1(G/\mathscr{S}, H^{n-1}(\mathscr{S}, AG)) \cong H^n(G, AG).$$

Proof. This follows from Theorem 5.3.2 since $H^i(K, AG) = 0$ for all $K \in \mathscr{S}$ and $i \neq n-1$ as K are PD^{n-1}-groups. $\square$

We now recall the following conjecture from Chapter 3:

Conjecture A*. *Let H be a finitely generated subgroup of G such that $\mathrm{Comm}_G(H) = G$. Let $\mathscr{S}$ be the admissible family of subgroups of G commensurable with H and A be a non-empty set. Suppose that V is a G-stable $\mathscr{S}$-almost equality class in $\mathscr{S}(G, A)$. Then there exists a G-tree T with edge stabilisers in $\mathscr{S}$ and vertex set V.*

Remark 5.3.4. We begin by noticing that in the case that $\mathscr{S}$ is the admissible family of finite subgroups of G that we recover the original Almost Stability Theorem in the special case that the G-set is G-finite. This may not seem to be the most obvious generalisation of the Almost Stability Theorem indeed originally we aimed to prove another conjecture:

Conjecture A. Let H be a finitely generated subgroup of G such that $\mathrm{Comm}_G(H) = G$. Let E be a G-set with stabilisers commensurable with H and A be a non-empty set. Let V be a G-stable almost equality class in (E, A). Then there exists a G-tree T with edge stabilisers commensurable with H and vertex set V.

However, this was modified to become Conjecture A* for the purposes of the application we shall use in this chapter for which it will become apparent that there is no clear choice of G-set E.

We also recall a further conjecture that we shall proceed to show follows from Conjecture A*.

Conjecture B. *Let G be a group of cohomological dimension $n < \infty$. Let H be a PD^{n-1} subgroup of G such that $\mathrm{Comm}_G(H) = G$. Then there exists a G-tree T with edge and vertex stabilisers commensurable with H.*

Before we show that Conjecture A* implies conjecture B we introduce some notation which will simplify our discussion of G-trees arising from derivations.

5.4 Modules and derivations

In the rest of this section we let G denote a group. Our motivation for studying the objects below is that they arise as the vertex sets of G-trees and knowledge of the structure of the derivations, in particular the subgroups on which they restrict to inner derivations, provides information on the stabilisers of the action of the group G on the tree.

Definition 5.4.1. Let Ω be a G-set with an abelian group structure and $\delta : G \to \Omega$ be a function. We say that δ is a *derivation* if for all $g, h \in G$,

$$\delta(gh) = \delta(g)h + \delta(h).$$

Remark 5.4.2. The definition of a derivation typically requires that Ω be a G-module, however in our definition above we do not require that the addition is respected by the group action. In the following two subsections we investigate a certain case in which a G-module structure arises on Ω.

Definition 5.4.3. Let M be a G-module. We denote by $\mathrm{Der}(G, M)$ the collection of all derivations from G to M. We say that a derivation $d \in \mathrm{Der}(G, M)$ is *inner* if there exists an element $m \in M$ such that for all $g \in G,\ d(g) = mg - m$.

Definition 5.4.4. Let M be a G-module and $d \in \mathrm{Der}(G, M)$. We define $(M)_d$ to be the G-set with underlying set M and with G-action defined as follows,

$$m * g = mg + dg \qquad \text{for all } m \in M, g \in G.$$

The following results will answer the question as to when a function $\delta : G \to M$ gives a G-action in the same way as above.

5.4.1 Which functions define actions?

Lemma 5.4.5. *Let M be a G-module and $\delta : G \to M$ be a function. Then the operation $*$ defined in 5.4.4 is a G-action if and only if δ is a derivation in the sense of Definition 5.4.1.*

Proof. For all $m \in M, g, h \in G$,

$$\begin{aligned} & m * (gh) = (m * g) * h \\ \iff \quad & mgh + \delta(gh) = (mg + \delta(g))h + \delta(h) \\ \iff \quad & \delta(gh) = \delta(g)h + \delta(h) \quad \text{since } M \text{ is a } G\text{-module.} \end{aligned}$$

Now suppose that δ is a derivation. Then $\delta(1) = 0$ and the result is clear. □

It is now a natural question to ask whether δ being a derivation from some G-set Ω, and the operation $*$ being a G-action implies that Ω is in fact a G-module. In fact this is not the case as can be seen in the following example.

Example 5.4.1. Let $\mathbb{Z}$ act on $\mathbb{Z}$ by addition. Then we have that $\mathbb{Z}$ is a $\mathbb{Z}$-set but clearly not a $\mathbb{Z}$-module. If we take $\delta : \mathbb{Z} \to \mathbb{Z}$ to be the zero map then we have that δ is a derivation and $*$ is simply our original action. Thus $(\mathbb{Z})_\delta = \mathbb{Z}$, i.e. the identity map is also a map of $\mathbb{Z}$-sets.

Thus we may obtain a G-set from a derivation together with a G-set which was not a G-module to begin with. Indeed this example can be adjusted to give a whole family of examples for all G-sets which are not G-modules but do however have an underlying abelian group structure together with the zero derivation.

We now address the question of which G-module structures arise in general.

5.4.2 Module Structure

In this section we assume that we have a G-set M with underlying abelian group structure and a derivation $d : G \to M$ such that the operation $*$ from Definition 5.4.4 is a G-action. Notice that we do not assume that M is a G-module, we do not require that the G-action distributes over the addition in M. Although M need not be a G-module we aim to show that certain subgroups of M do indeed have a G-module structure.

Since M has an abelian group structure, to prove that a subgroup of M is a G-module it is enough to show that the action of G distributes over addition in the subgroup and that the subgroup is closed under the action of G. This of course is trivially satisfied when M is a G-module. We now state the first result in this direction.

Lemma 5.4.6. *The action of G distributes over addition in the abelian subgroup generated by the image of d.*

Proof. Since $*$ is an action we have for all $m \in M, g, h \in G$,

$$mgh + d(gh) = (mg + dg)h + dh.$$

Now since d is a derivation this simplifies to give,

$$mgh + d(g)h = (mg + dg)h.$$

This holds for all $m \in M$ and so in particular for all m in the image of d. This allows, inductively, to prove that the action of G distributes over finite sums of elements in the image of d. It remains to show that the G-action distributes over additive inverses, however this follows from two simple observations. Firstly that the above formula gives that

$$(m + d(g)h)k = mk + d(g)hk$$

for all $m \in M, g, h, k \in G$. Then secondly, that since d is a derivation we see that for all $g \in G, -d(g) = d(g^{-1})g$. Hence $(m + m')g = mg + m'g$ for all $g \in G, m, m'$ in the abelian group generated by the image of d. □

The following result allows us to show that M contains a G-module.

Theorem 5.4.7. *The abelian subgroup generated by the image of d is invariant under the action of G.*

Proof. Since d is a derivation we have that for all $g, h \in G$,

$$d(gh) = d(g)h + d(h).$$

Thus for $g_1, g_2 \in G$ we have,

$$d(g_1)g_2 = d(g_1g_2) - d(g_2).$$

This gives that the generators of the subgroup generated by d remain in the subgroup under the action by G and the result follows from Lemma 5.4.6. □

Corollary 5.4.8. *The abelian subgroup generated by the image of d is a G-module.*

Thus we see that M must have some subgroup which is a G-module.

Remark 5.4.9. It should be noted that as we stated earlier it is easy to construct examples where $*$ is a G-action if we take d to be the zero derivation. However, in this case the G-module generated by the image of d is the zero module and as uninteresting as we would expect.

Corollary 5.4.10. *If the image of d generates M as an abelian group then M is a G-module.*

5.4.3 G-summands and G-retracts

We investigate the structure of the G-sets $(M)_d$ obtained from G-modules M and derivations $d : G \to M$ and how this corresponds to the structure of the original G-module M.

Definition 5.4.11. Let M be a G-module and N be a G-submodule. We say that N is a *G-summand* of M if there exist G-linear maps $\iota : N \to M, \pi : M \to N$, such that the composition $\pi \circ \iota$ is the identity on N.

Definition 5.4.12. Let U, V be G-sets. We say that U is a *G-retract* of V if there exist G-maps $\iota : U \to V, \pi : V \to U$, such that ι is injective and π is surjective.

Lemma 5.4.13. *Let M be a G-module and suppose that N is a G-summand of M and that we have a derivation $d : G \to N$. Then $(N)_d$ is a G-retract of $(M)_d$.*

Proof. Since N is a G-summand we have G-linear maps $N \to M \to N$ such that the composition is the identity. It is then a straightforward check that these maps are G-maps $(N)_d \to (M)_d \to (N)_d$. Since the maps have not changed as maps of sets it is clear that the composition is still the identity on N and thus $(N)_d$ is a G-retract of $(M)_d$. □

The following result will be of most use to us in the following arguments.

Lemma 5.4.14. *Let M be a G-module and $\delta : G \to M$ be a derivation. Then the stabilisers of $(M)_\delta$ are precisely the subgroups of G on which δ is inner.*

Proof. Suppose that δ is inner on H. Then there is an $m \in M$ such that $\delta h = mh - m$ for all $h \in H$. We claim now that H stabilises $-m$.

$$\begin{aligned}(-m) * h &= -mh + \delta h \\ &= -mh + mh - m \\ &= -m.\end{aligned}$$

Let $m \in M$. We aim to show that δ is inner on G_m. We claim that for all $x \in G_m, \delta x = (-m)x - (-m)$. To see this let $x \in G_m$.

$$\begin{aligned}m * x &= mx + \delta x \\ &= m \text{ since } x \in G_m \\ \text{i.e. } \delta x &= m - mx.\end{aligned}$$

□

5.5 Proof of Theorem A $\implies$ Theorem B

We aim to prove Theorem B by showing that it follows as a consequence of Theorem A. We do so by showing that, more generally Conjecture A* $\implies$ Conjecture B and our result follows as a special case. We introduce the following useful piece of notation.

Definition 5.5.1. We denote by $\mathscr{A}\mathscr{S}(G,A)$ the collection of functions in $\mathscr{S}(G,A)$ that are $\mathscr{S}$-almost equal to their translates by G. That is to say that

$$\mathscr{A}\mathscr{S}(G,A) = \{f \in \mathscr{S}(G,A) \mid f \cdot k =_{\mathscr{S}} f \ \forall k \in G\}.$$

We denote by $\mathscr{A}\mathscr{S}_{\mathscr{F}}(G,A)$ the family of all functions in $\mathscr{A}\mathscr{S}(G,A)$ supported on finitely many cosets of some subgroup in $\mathscr{S}$.

Now we shall first prove that Conjecture B follows from Conjecture A* in the particular case that $n = 2$ and later state a result needed to obtain the more general case from this same argument.

Theorem 5.5.2. *Let G be a group of cohomological dimension n and H a n-dimensional duality group over R such that $\mathrm{Comm}_G(H) = G$. Then there exists a derivation $\delta : G \to H^{n-1}(\mathscr{S}, AG)$ that is outer on all subgroups of cohomological dimension n and restricts to the zero map on some $L \in \mathscr{S}$.*

Proof. Since G has cohomological dimension n then there exists a projective resolution of the trivial module,

$$0 \longrightarrow K \xrightarrow{d_{n-1}} F_{n-1} \xrightarrow{d_{n-2}} \cdots \longrightarrow F_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

such that the modules F_i are free RG-modules for all $0 \leq i \leq n-1$. We may use this resolution to compute the cohomology of G, in particular we obtain the following exact sequence.

$$0 \longrightarrow \mathrm{Hom}_{RG}(F_{n-1}, K) \longrightarrow \mathrm{Hom}_{RG}(K,K) \longrightarrow H^n(G,K) \longrightarrow 0$$

Thus we observe that the identity map on K gives rise to a non-trivial element of $H^n(G,K)$ if and only if the map d_{n-1} does not split as an RG-map. It is clear that this map cannot split since G is of cohomological dimension n. Similarly we may use the resolution to compute the cohomology of any subgroup H of G and hence the identity map on K gives a non-trivial element of $H^n(G,K)$ that restricts to a non-trivial element of $H^n(H,K)$

for all subgroups H of G of cohomological dimension n. Now since K is a projective G-module we may consider this as a non-trivial element in $H^n(G, AG)$ for some abelian group A. Thus by virtue of the isomorphism $H^n(G, AG) \cong H^1(G/\mathscr{S}, H^{n-1}(\mathscr{S}, AG))$ from Corollary 5.3.3 we have an outer derivation $\delta : G \to H^{n-1}(\mathscr{S}, AG)$ that restricts to zero on some subgroup in $\mathscr{S}$ yet remains outer when restricted to any cohomological dimension n subgroup. □

It is necessary at this point then to take a moment to investigate $H^1(\mathscr{S}, AG)$. Indeed we study $H^n(\mathscr{S}, AG)$ in the more general setting that $\mathscr{S}$ is an admissible family of n-dimensional duality groups.

Lemma 5.5.3. *Let K be an orientable duality group of dimension n over R with $H^n(K, RK)$ a free R-module. Then $H^n(K, AG)$ is isomorphic as an RG bimodule to the set of functions from G to a direct sum of copies of A that is supported on finitely many cosets of K and constant on those cosets.*

Proof. Firstly observe that

$$\begin{aligned}
H^n(K, AG) &\cong H_0(K, AG \otimes_{RK} D) \quad \text{by duality} \\
&\cong AG \otimes_{RK} D \otimes_{RK} R \\
&\cong AG \otimes_{RK} D \quad \text{since } D \text{ has trivial } G\text{-action} \\
&\cong AG \otimes_{RK} \oplus R \\
&\cong \oplus\left(AG \otimes_{RK} R\right).
\end{aligned}$$

So it is enough to show that $AG \otimes_{RK} R$ is isomorphic to the functions from G to A constant and non-zero on finitely many cosets of K. We define the following map and check that this is an isomorphism. Let $\varphi : AG \otimes_{RK} R \to \mathscr{A}\mathscr{S}_{\mathscr{F}}(G, A)$ such that $\varphi(ag \otimes 1)(g) = a$. Since $ag \otimes 1 = agk \otimes 1$ for all $k \in K$ (as G acts on R trivially) it follows that any function in the image of φ is constant on the cosets of K and clearly is supported on only finitely many such cosets. That this is a G-map again follows from the fact that the action of G on R is trivial since for $x, g \in G,\ x \cdot (\varphi(ag \otimes 1))(xg) = \varphi(ag \otimes 1)(g) = a = \varphi(axg \otimes 1)(xg) = \varphi(x \cdot (ag \otimes 1))(xg)$.

Now any function $f : G \to A$ that is non-zero and constant on finitely many cosets of K is uniquely determined by a finite list of coset representatives x_i of the support of f together with the corresponding values of f on each of these representatives, α_i say. Then there is a map $\psi : \mathscr{A}\mathscr{S}_{\mathscr{F}}(G, A) \to AG \otimes_{RK} R$ defined by $f \mapsto \sum \alpha_i x_i \otimes 1$. It is clear that ψ is then the inverse of φ. $\square$

Theorem 5.5.4. *Let $\mathscr{S}$ be an admissible family of n-dimensional duality groups over R with dualising modules all free over R. Then $H^n(\mathscr{S}, AG)$ is isomorphic as an RG-bimodule to the functions from G to a direct sum of copies of A supported on finitely many cosets of some subgroup $L \in \mathscr{S}$ and constant on those cosets.*

Proof. Let $K \in \mathscr{S}$ then it can be found in Bieri [6] that the following diagram commutes,

$$\begin{tikzcd} H^n(K, AG) \arrow[r] \arrow[d, "\mathrm{res}"'] & AG \otimes_{RK} D \arrow[d, "\mathrm{Tr}"] \\ H^n(L, AG) \arrow[r] & AG \otimes_{RL} D, \end{tikzcd}$$

where the horizontal maps are duality isomorphisms and L is a finite index subgroup of K. The map Tr is the transfer map given by $\mathrm{Tr}(ag \otimes d) = \sum_{t \in T} agt \otimes t^{-1}d$ where T is a set of right coset representatives of L in K. It can be shown that this is independent of the choice of transversal T. In the case that our groups are orientable duality groups then we have that $\mathrm{Tr}(ag \otimes d) = \sum_{t \in T} agt \otimes d$. Now in the case that D is a free R-module we have that $D \cong \oplus R$ and thus $AG \otimes D \cong \oplus(AG \otimes R)$. Then we have that $\mathrm{Tr}(ag \otimes r) = \sum_T agt \otimes r = \sum_T argt \otimes 1$, and we claim that this preserves the function in $\mathcal{A}\mathscr{S}_{\mathscr{F}}(G, A)$ obtained via the isomorphism defined in Lemma 5.5.3. This can be seen since $ag \otimes 1 \in AG \otimes_{RK} D$ corresponds to the function that evaluates to a on the coset gK and $\sum_T agt \otimes 1$ corresponds to the function that evaluates to a on the cosets gtL and $\sqcup_{t \in T} tL = K$ and thus we have the same functions on G. $\square$

Lemma 5.5.5. *Suppose that $\mathscr{S}$ consists of $n-1$ dimensional duality groups. Then there exists the following exact sequence:*

$$0 \longrightarrow A \longrightarrow \mathscr{A}\mathscr{S}(G, A) \longrightarrow \mathrm{Der}(G, H^{n-1}(\mathscr{S}, AG)) \longrightarrow \mathrm{Der}(\mathscr{S}, H^{n-1}(\mathscr{S}, AG)),$$

where $\mathrm{Der}(\mathscr{S}, H^{n-1}(\mathscr{S}, AG)) = \varinjlim_{H \in \mathscr{S}} \mathrm{Der}(H, H^{n-1}(\mathscr{S}, AG))$.

Proof. We should first make clear what the maps are in this sequence. The first map is that which sends a to the function that is constantly a on G. Clearly this map is injective and so our sequence is exact at A. The final map is the restriction map to $\mathrm{Der}(\mathscr{S}, H^{n-1}(\mathscr{S}, AG))$. The remaining map $\mathscr{A}\mathscr{S}(G, A) \to \mathrm{Der}(G, H^{n-1}(\mathscr{S}, AG))$ is defined by $v \mapsto d_v$ where d_v is given by $d_v(g) = vg - v$. That $vg - v \in H^{n-1}(\mathscr{S}, AG)$ follows from the fact that

$v \in \mathscr{AS}(G, A)$ together with the identification from Theorem 5.5.4. That d_v is a derivation follows from the definition of our map as for $g, h \in G$ we have that,

$$\begin{aligned} d_v(gh) &= vgh - v \\ &= vgh - vh + vh - v \\ &= (vg - v)h + vh - v \\ &= d_v(g)h + d_v h. \end{aligned}$$

If $d_v = 0$ then it follows that for all $g \in G, d_v g = vg - v = 0$. That is to say that v is fixed by G, thus constant and in the image of A. Therefore our sequence is exact at $\mathscr{AS}(G, A)$. We proceed to demonstrate exactness at $\mathrm{Der}(G, H^1(\mathscr{S}, AG))$. The fact that the image of $\mathscr{AS}(G, A)$ lies in the kernel of the following map uses the fact that $v \in \mathscr{AS}(G, A)$ is constant on the cosets of some subgroup, K say, in $\mathscr{S}$. Thus for all $g \in G, k \in K$,

$$\begin{aligned} v \cdot k(g) - v(g) &= v(gk^{-1}) - v(g) \\ &= 0. \end{aligned}$$

Hence the image of v restricts to zero on K. That every element of the kernel lies in the image of $\mathscr{AS}(G, A)$ follows from the fact that the original definition of v given a derivation δ holds and this is constant on the cosets of the group that it restricts to zero on. That is to say that we define $v : G \to A$ by $v(x) = -\delta(x)(x)$. Then as in the original case we observe that this function maps to our original derivation δ since for all $g, x \in G$,

$$\begin{aligned} \delta x &= \delta(xg^{-1}g) \\ &= \delta(xg^{-1}) \cdot g + \delta g, \end{aligned}$$

and so,

$$\begin{aligned} (\delta g)(x) + v(x) &= (\delta g)(x) - \delta(x)(x) \\ &= -((\delta(xg^{-1})) \cdot g)(x) \\ &= -(\delta(xg^{-1}))(xg^{-1}) \\ &= v(xg^{-1}) \\ &= (vg)(x). \end{aligned}$$

Then it follows that $(vg - v)(x) = (\delta g)(x)$. It can be seen that the function v defined as above is then constant on the cosets of K, a subgroup in $\mathscr{S}$ that δ restricts to zero on, since for all $x \in G, k \in K$,

$$\begin{aligned} v(xk) &= -\delta(xk)(xk) \\ &= -(\delta(x) \cdot k + \delta(k))(xk) \\ &= -\delta(x) \cdot k(xk) \\ &= -\delta(x)(x). \end{aligned}$$

Thus $v \in \mathscr{A}\mathscr{S}(G, A)$. $\square$

Thus for every outer derivation δ of G to $H^1(\mathscr{S}, AG)$ that restricts to zero on a subgroup in $\mathscr{S}$ there exists a corresponding G-stable $\mathscr{S}$-almost equality class in $\mathscr{S}(G, A)$. This is particularly useful thanks to the following version of Theorem IV.2.5 from [12].

Theorem 5.5.6. *Suppose that Conjecture A* is true. For an abelian group A and derivation $d : G \to H^{n-1}(\mathscr{S}, AG)$, there exists a G-tree with edge stabilisers in $\mathscr{S}$ and with vertex set given by $(H^{n-1}(\mathscr{S}, AG))_d$.*

Proof. It is clear from Lemma 5.5.5 that such a derivation gives rise to a G-stable $\mathscr{S}$-almost equality class in $\mathscr{S}G, A$ generated by $v \in \mathscr{A}\mathscr{S}(G, A)$ such that $d = d_v$. Now this $\mathscr{S}$-almost equality class $V = v + \mathscr{A}\mathscr{S}_{\mathscr{F}}(G, A)$ is isomorphic as a G-set to $(\mathscr{A}\mathscr{S}_{\mathscr{F}}(G, A))_\delta$ with the isomorphism given by $v + f \mapsto f$. Conjecture A* then gives a G-tree with vertex set $(\mathscr{A}\mathscr{S}_{\mathscr{F}}(G, A))_\delta$. $\square$

It follows that the application of Conjecture A* gives the existence of a G-tree T with edge stabilisers commensurable with $H \in \mathscr{S}$ and all vertex stabilisers of cohomological dimension 1 by the properties of the groups on which δ is inner. It remains to show that the vertex stabilisers are free of rank strictly less than 2. The following theorem is to be found in [19].

Theorem 5.5.7 (M. Hall)**.** *Let G be a free group and H a finitely generated subgroup. Then there exists a subgroup F of finite index in G that contains H as a free factor.*

In light of the above result suppose that one of the vertex stabilisers G_v, is free of rank > 1. Then for any edge e incident to v, G_v contains a subgroup $F = F' * G_e$ where F' is non-trivial. Then clearly for any $x \in F'$ it cannot be true that G_e^x is commensurable with G_e. This contradicts the fact that $\mathscr{S}$ is an admissible family and it follows that the vertex stabilisers must all be infinite cyclic and therefore we have that Conjecture A* implies

Conjecture B in the case that $n = 2$ and $R = \mathbb{Z}$. Now most of the above argument goes through in full generality. What is needed, however is some analogue to the Hall Theorem in the more general case. To this end we have the following results using the spectral sequence argument from section 1 of [23].

Theorem 5.5.8. *Let G be a group of cohomological dimension n over R. Let H be an n-dimensional duality group over R with dualising module free as an R-module. Suppose further that H is near-normal in G. Then $|G : H| < \infty$.*

Proof. By Lemma 5.2.2 we see that $H^i(H, RH) = 0$ for all $i \neq n$. Denote by $\mathscr{S}$ the family of subgroups of G commensurable with H. Now by Theorem 5.3.2 there is a spectral sequence with $H^j(G/\mathscr{S}, H^{i-j}(\mathscr{S}, RG)) \implies H^i(G, RG)$. Since G has cohomological dimension n it follows that $H^n(G, RG) \neq 0$, however for all $i \neq n$ we have that $H^i(H, RG) = 0$ as $RG = RH \otimes_{RH} RG$. Thus we have that $H^n(G, RG) \cong H^0(G/\mathscr{S}, H^n(\mathscr{S}, RG))$, and since $H^0(G/\mathscr{S}, -)$ is simply the restriction of the G-fixed point functor we see that $H^n(G, RG) \cong H^n(\mathscr{S}, RG)^G$. Now by Lemma 5.5.4 we see that this is non-zero if and only if $|G : H| < \infty$ since G acts on those functions by permuting the finitely many cosets of its support. □

With these results our previous proof now gives the following theorem.

Theorem 5.5.9. *Let G be a group of cohomological dimension n over R. Suppose that H is an $n-1$-dimensional duality group over R such that* $\mathrm{Comm}_G(H) = G$ *and the dualising module of H is R-free. Then Conjecture A* implies that there exists a G-tree with edge and vertex stabilisers commensurable with H.*

Proof. Theorem 5.5.2 produces a derivation $\delta : G \to H^{n-1}(\mathscr{S}, AG)$ that is outer on every subgroup of G of cohomological dimension n yet restricts to zero in $\mathrm{Der}(\mathscr{S}, H^{n-1}(\mathscr{S}, AG))$. Lemma 5.5.5 then gives a G-stable $\mathscr{S}$-almost equality class in $\mathscr{S}(G, A)$. Thus Conjecture A* gives a G-tree, T with edges stabilisers in $\mathscr{S}$ and vertex stabilisers all of cohomological dimension $n - 1$. Now Theorem 5.5.8 tells us that no group of cohomological dimension n may contain an infinite index $n - 1$-dimensional duality group and thus the vertex stabilisers are themselves duality groups in $\mathscr{S}$. □

However, in the case that $\mathscr{S}$ consists of subgroups of non-zero Euler characteristic then the statement of Conjecture A* is precisely Theorem A, thus we arrive at the following corollary.

Corollary 5.5.10. *Theorem A implies Theorem B.*

Hence we now have the following theorem generalising the results of Kropholler [23].

Theorem B. *Let G be a group of cohomological dimension $n < \infty$. Let H be a PD^{n-1} subgroup of G such that $\mathrm{Comm}_G(H) = G$ and $\chi(H)$ is a non-zero integer. Then there exists a G-tree T with edge and vertex stabilisers commensurable with H.*

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