Finiteness Conditions in Group Cohomology

by

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Abstract

In this thesis we investigate groups whose nth cohomology functors commute with filtered colimits for all sufficiently large n. In Chapter 1 we introduce some basic definitions and important background material. We make the definition that a group G has *cohomology almost everywhere finitary* if and only if the set

$$\mathscr{F}(G) := \{n \in \mathbb{N} : H^n(G, -) \text{ commutes with filtered colimits}\}$$

is cofinite in $\mathbb{N}$. We also introduce Kropholler's class $\mathbf{LH}\mathfrak{F}$ of locally hierarchically decomposable groups. We then state a key result of Kropholler (Theorem 2.1 in [30]), which establishes a dichotomy for this class: If G is an $\mathbf{LH}\mathfrak{F}$-group, then the set $\mathscr{F}(G)$ is either finite or cofinite. Kropholler's theorem does not, however, give a characterisation of the $\mathbf{LH}\mathfrak{F}$-groups with cohomology almost everywhere finitary, and this is precisely the problem that we are interested in.

In Chapter 2 we investigate algebraic characterisations of certain classes of $\mathbf{LH}\mathfrak{F}$-groups with cohomology almost everywhere finitary. In particular, we establish sufficient conditions for a group in the class $\mathbf{H}_1\mathfrak{F}$ to have cohomology almost everywhere finitary. We prove a stronger result for the class of groups of finite virtual cohomological dimension over a ring R of prime characteristic p, and use this result to answer an open question of Leary and Nucinkis (Question 1 in [34]). We also consider the class of locally (polycyclic-by-finite) groups, and show that such a group G has cohomology almost everywhere finitary if and only if G has finite virtual cohomological dimension and the normalizer of every non-trivial finite subgroup of G is finitely generated.

We then change direction in Chapter 3, and show an interesting connection between this problem and the problem of group actions on spheres. In particular, we show that if G is an infinitely generated locally (polycyclic-by-finite) group with cohomology almost everywhere finitary, then every finite subgroup of G acts freely and orthogonally on some sphere.

Finally, in Chapter 4 we provide a topological characterisation of the **LH**$\mathfrak{F}$-groups with cohomology almost everywhere finitary. In particular, we show that if G is an **LH**$\mathfrak{F}$-group with cohomology almost everywhere finitary, then $G \times \mathbb{Z}$ has an Eilenberg–Mac Lane space $K(G \times \mathbb{Z}, 1)$ with finitely many n-cells for all sufficiently large n. It is an open question as to whether the **LH**$\mathfrak{F}$ restriction can be dropped here. We also show that the converse statement holds for arbitrary G.

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Statement

This thesis is submitted in accordance with the regulations for the degree of Doctor of Philosophy at the University of Glasgow.

Chapter 1 covers definitions and basic results.

Section 1.4 and Chapters 2, 3 and 4 are the author's original work with the exception of those results which are explicitly referenced. Some of the results therein will be published in [19], [20] and [21].

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Introduction

Groups of type FP_n have been the subject of much interest. This property is usually described in terms of projective resolutions: A group G is said to be of type FP_n, $0 \leq n \leq \infty$, if and only if there is a projective resolution

$$\cdots \to P_k \to P_{k-1} \to \cdots \to P_2 \to P_1 \to P_0 \to \mathbb{Z} \to 0$$

of the trivial $\mathbb{Z}G$-module such that the projective modules P_k are finitely generated for all $k \leq n$. It is easy to see that every group is of type FP_0, and that a group is of type FP_1 if and only if it is finitely generated. Furthermore, for finitely presented groups, type FP_n is equivalent to the existence of an Eilenberg–Mac Lane space with finite n-skeleton. We can also describe this property in terms of finitary functors: If G is a group, and n is a natural number, then the nth cohomology of G is defined as

$$H^n(G, -) := \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, -).$$

This is a functor from the category of $\mathbb{Z}G$-modules to the category of $\mathbb{Z}$-modules, and it is said to be *finitary* if and only if it commutes with filtered colimits (see §3.18 of [1]; also §6.5 of [35]). Bieri [6] (see also Brown [8]) has shown that a group G is of type FP_n if and only if $H^k(G, -)$ is finitary for all $k < n$. Therefore, the property of type FP_n depends upon the low-dimensional cohomology functors of a group behaving well with respect to filtered colimits. However, in this thesis we are interested in groups whose high-dimensional cohomology functors behave well with respect to filtered colimits, even though their low-dimensional cohomology functors may not.

Let G be a group. The *finitary set* $\mathscr{F}(G)$ of G is defined as

$$\mathscr{F}(G) := \{n \in \mathbb{N} : H^n(G, -) \text{ is finitary}\},$$

and we say that G has *cohomology almost everywhere finitary* if and only if $\mathscr{F}(G)$ is cofinite in $\mathbb{N}$.

We are particularly interested in Kropholler's class $\mathbf{LH}\mathfrak{F}$ of locally hierarchically decomposable groups. Briefly, the class $\mathbf{H}\mathfrak{F}$ is the smallest class of groups that contains all finite groups and which contains a group G whenever there is an admissible action of G on a finite-dimensional contractible cell complex for which all isotropy groups already belong to $\mathbf{H}\mathfrak{F}$. The class $\mathbf{LH}\mathfrak{F}$ then comprises those groups whose finitely generated subgroups are in $\mathbf{H}\mathfrak{F}$. For a more detailed description of this class, see §1.3. Kropholler has shown in Theorem 2.1 of [30] that if G belongs to this class $\mathbf{LH}\mathfrak{F}$, then the finitary set $\mathscr{F}(G)$ of G is either finite or cofinite in $\mathbb{N}$, which establishes a dichotomy for $\mathbf{LH}\mathfrak{F}$-groups. However, Kropholler's theorem does not give a characterisation of the $\mathbf{LH}\mathfrak{F}$-groups with cohomology almost everywhere finitary.

In Chapter 2 we investigate algebraic characterisations of certain classes of $\mathbf{LH}\mathfrak{F}$-groups with cohomology almost everywhere finitary. We begin by looking at the class of locally (polycyclic-by-finite) groups, and show in §2.1 that such a group G has cohomology almost everywhere finitary if and only if

(i) G has finite virtual cohomological dimension; and

(ii) The normalizer of every non-trivial finite subgroup of G is finitely generated.

We then use this characterisation to show that if G is a locally (polycyclic-by-finite) group with cohomology almost everywhere finitary, then *every* subgroup of G also has cohomology almost everywhere finitary (see Corollary B below). However, this is not true in general, as can be seen from Proposition 2.1.22 below. We are then led in §2.2 to investigate some closure properties of the class of groups with cohomology almost everywhere finitary. We show that this class is not closed under taking subgroups, extensions or quotients, but it is closed under forming certain free products with amalgamation and HNN-extensions.

Next, in §2.3 we work over a ring R of prime characteristic p, instead of over $\mathbb{Z}$, by defining the nth cohomology of a group G to be

$$H^n(G, -) := \mathrm{Ext}^n_{RG}(R, -).$$

In order to make it clear that we are now working over R, we say that $H^n(G, -)$ is *finitary over R* whenever the functor $\mathrm{Ext}^n_{RG}(R, -)$ is finitary. We can similarly define the notion of a group having cohomology almost everywhere finitary over R. We then prove that a group G of finite virtual cohomological dimension over R has cohomology almost everywhere finitary over R if and only if

(i) G has finitely many conjugacy classes of elementary abelian p-subgroups, and

(ii) The normalizer of every non-trivial elementary abelian p-subgroup of G has cohomology almost everywhere finitary over R.

Now, in [34], Leary and Nucinkis posed the following question: Let G be a group of type VFP over $\mathbb{F}_p$, and P be a p-subgroup of G. Is the centralizer $C_G(P)$ of P necessarily of type VFP over $\mathbb{F}_p$? We use the above result to give a positive answer to this question.

We then return to working over $\mathbb{Z}$, and in §2.4 we consider the class $\mathbf{H}_1\mathfrak{F}$. This is a generalisation of the class of groups of finite virtual cohomological dimension which includes certain groups that are not virtually torsion-free. We show that if G is an $\mathbf{H}_1\mathfrak{F}$-group such that

(i) G has finitely many conjugacy classes of finite subgroups, and

(ii) The normalizer of every non-trivial finite subgroup of G has cohomology almost everywhere finitary,

then G itself has cohomology almost everywhere finitary. However, the converse of this result is false, due to examples of Leary [33]. In fact, these counter-examples show that the converse is false even for the subclass of groups of finite virtual cohomological dimension.

In Chapter 3 we change direction and show an interesting connection between this problem and the problem of group actions on spheres. Recall from §VI.9 of [9] that a *free orthogonal* action of a finite group K on an n-sphere S^n is one which is induced from a linear action of K on $\mathbb{R}^{n+1}$ that is free except at the origin. It is well-known (see, for example, Thomas and Wall [47]) that a finite group K acts freely and orthogonally on some sphere if and only if it satisfies the so-called "pq-conditions"; that is, every subgroup of K of order pq, where p and q are prime, is cyclic. We show that if G is an infinitely generated locally (polycyclic-by-finite) group with cohomology almost everywhere finitary, then every finite subgroup K of G satisfies these pq-conditions, and hence acts freely and orthogonally on some sphere. We also give examples to show that we cannot drop the "infinitely generated" restriction.

Finally, in Chapter 4 we provide a topological characterisation of the $\mathbf{LH}\mathfrak{F}$-groups with cohomology almost everywhere finitary. We begin by introducing in §4.1 the important notion of complete cohomology, which we shall use throughout this chapter. Then in §4.2 we show that if G is an $\mathbf{LH}\mathfrak{F}$-group with cohomology almost everywhere finitary, then $G \times \mathbb{Z}$ has an Eilenberg–Mac Lane space $K(G \times \mathbb{Z}, 1)$ with finitely many n-cells for all sufficiently large n. We do not know whether the $\mathbf{LH}\mathfrak{F}$ restriction can be dropped here. However, we show in §4.3 and §4.4 that the converse holds for any group G.

Chapter 1

Preliminaries

Note that throughout this thesis, all rings are assumed to be associative with identity, and unembellished tensors mean $\otimes_{\mathbb{Z}}$.

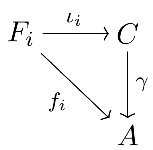
1.1 Categorical Notions

We begin by recalling some important notions from category theory (see the appendix of [51]).

Definition 1.1.1. Let $\mathbb{I}$ be a small category, $\mathfrak{C}$ be any category and $F : \mathbb{I} \to \mathfrak{C}$ be a functor. Set $F_i := F(i)$. Then the *colimit* of F (if it exists) is an object

$$C := \operatorname{colim}_{i \in \mathbb{I}} F_i$$

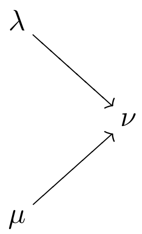
of $\mathfrak{C}$, together with maps $\iota_i : F_i \to C$ in $\mathfrak{C}$ that are "compatible" in the sense that for every $\alpha : j \to i$ in $\mathbb{I}$, the map ι_j factors as $\iota_i F_\alpha : F_j \to F_i \to C$, and that satisfies the following universal property: For every $A \in \mathfrak{C}$ and every system of "compatible" maps $f_i : F_i \to A$, there is a unique $\gamma : C \to A$ such that, for each i, the following diagram commutes:



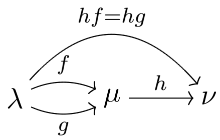
Remark 1.1.2. A category $\mathfrak{C}$ is said to be *cocomplete* if and only if for all small categories $\mathbb{I}$ and all functors $F : \mathbb{I} \to \mathfrak{C}$, $\operatorname{colim}_{i \in \mathbb{I}} F_i$ exists. For example, if R is a ring, then the category of R-modules is cocomplete.

Definition 1.1.3. A small category Λ is said to be *filtered* if and only if

(i) for any two objects $\lambda, \mu \in \Lambda$ there exists an object $\nu \in \Lambda$ and a pair of morphisms



(ii) and given a pair of morphisms $f, g : \lambda \to \mu$, there exists a morphism h with domain μ such that $hf = hg$:



Definition 1.1.4. A *filtered colimit* in a category $\mathfrak{C}$ is simply the colimit of a functor $F : \Lambda \to \mathfrak{C}$, in which Λ is a filtered category. It is denoted by $\varinjlim_{\lambda\in\Lambda} F_\lambda$. The family $(F_\lambda)_{\lambda\in\Lambda}$ of objects of $\mathfrak{C}$, together with the maps $F_\lambda \to F_\mu$ corresponding to morphisms $\lambda \to \mu$ in Λ, is called a *filtered colimit system*.

Remark 1.1.5. In this thesis, we mainly work with filtered colimits in the category of R-modules. Let Λ be a filtered category, and (M_λ) be a filtered colimit system of R-modules. We construct the colimit $\varinjlim_\lambda M_\lambda$ by taking the disjoint union of the modules M_λ and factoring out a certain equivalence relation $\sim$:

$$\varinjlim_{\lambda} M_\lambda := \left(\coprod_{\lambda\in\Lambda} M_\lambda\right) \Big/ \sim .$$

The equivalence relation $\sim$ is generated by the relation $\rightsquigarrow$ which is defined by $m \rightsquigarrow m'$ if and only if there is a morphism $f : \lambda \to \mu$ in Λ such that $m \in M_\lambda$, $m' \in M_\mu$ and $m \mapsto m'$ under the map $M_\lambda \to M_\mu$ determined by f.

Now, let $F : \mathfrak{Mod}_R \to \mathfrak{Mod}_S$ be a functor, and (M_λ) be a filtered colimit system of R-modules. Then, for each λ, we have a natural map

$$M_\lambda \to \varinjlim_{\lambda} M_\lambda,$$

which gives rise to the map

$$F(M_\lambda) \to F(\varinjlim_\lambda M_\lambda).$$

These maps form a compatible system, so by the definition of colimit there is a unique map

$$\varinjlim_\lambda F(M_\lambda) \to F(\varinjlim_\lambda M_\lambda)$$

such that, for each λ, the following diagram commutes:

$$\begin{array}{ccc} F(M_\lambda) & \longrightarrow & \varinjlim_\lambda F(M_\lambda) \\ & \searrow & \downarrow \\ & & F(\varinjlim_\lambda M_\lambda) \end{array}$$

We can then make the following definition (see §3.18 of [1]; also §6.5 of [35]):

Definition 1.1.6. A functor $F : \mathfrak{Mod}_R \to \mathfrak{Mod}_S$ is said to be *finitary* if and only if the natural map

$$\varinjlim_\lambda F(M_\lambda) \to F(\varinjlim_\lambda M_\lambda)$$

is an isomorphism for all filtered colimit systems (M_λ).

Next, recall from Definition 7.5 in [31] that if

$$E \to F \to G$$

is a sequence of functors and natural transformations from $\mathfrak{Mod}_R$ to $\mathfrak{Mod}_S$, then we say that this sequence is *exact at* F if and only if for all R-modules M, the sequence

$$E(M) \to F(M) \to G(M)$$

is exact at $F(M)$. We also say that a longer sequence of functors is *exact* if and only if it is exact at each place where this makes sense. We can now prove the following two important lemmas, which shall be needed later on:

Lemma 1.1.7. *Let*

$$F_1 \to F_2 \to F_3 \to F_4 \to F_5$$

be an exact sequence of functors from $\mathfrak{Mod}_R$ *to* $\mathfrak{Mod}_S$*. If* F_1, F_2, F_4 *and* F_5 *are finitary, then so is* F_3.

Proof. Let (M_λ) be a filtered colimit system of R-modules, so for each λ we have the following exact sequence:

$$F_1(M_\lambda) \to F_2(M_\lambda) \to F_3(M_\lambda) \to F_4(M_\lambda) \to F_5(M_\lambda).$$

Then, as the process of taking filtered colimits of S-modules is exact (Theorem 2.6.15 in [51]), we have the following exact sequence:

$$\varinjlim_\lambda F_1(M_\lambda) \to \varinjlim_\lambda F_2(M_\lambda) \to \varinjlim_\lambda F_3(M_\lambda) \to \varinjlim_\lambda F_4(M_\lambda) \to \varinjlim_\lambda F_5(M_\lambda).$$

We then obtain the following commutative diagram, where the vertical maps are the natural maps coming from the definition of colimit:

$$\begin{array}{ccccccccc}
\varinjlim_\lambda F_1(M_\lambda) & \longrightarrow & \varinjlim_\lambda F_2(M_\lambda) & \longrightarrow & \varinjlim_\lambda F_3(M_\lambda) & \longrightarrow & \varinjlim_\lambda F_4(M_\lambda) & \longrightarrow & \varinjlim_\lambda F_5(M_\lambda) \\
{\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\
F_1(\varinjlim_\lambda M_\lambda) & \longrightarrow & F_2(\varinjlim_\lambda M_\lambda) & \longrightarrow & F_3(\varinjlim_\lambda M_\lambda) & \longrightarrow & F_4(\varinjlim_\lambda M_\lambda) & \longrightarrow & F_5(\varinjlim_\lambda M_\lambda)
\end{array}$$

Now, as F_1, F_2, F_4 and F_5 are finitary, the maps f_1, f_2, f_4 and f_5 are isomorphisms. It then follows from the Five Lemma (see page 13 of [51]) that f_3 is an isomorphism, and we conclude that F_3 is finitary. □

Lemma 1.1.8. *Suppose that* $F : \mathfrak{Mod}_R \to \mathfrak{Mod}_S$ *can be expressed as a direct sum of functors* $F_1, F_2 : \mathfrak{Mod}_R \to \mathfrak{Mod}_S$. *If* F *is finitary, then so are* F_1 *and* F_2.

Proof. As F is the direct sum of F_1 and F_2, we have the following exact sequence of functors:

$$0 \to F_1 \to F \to F_2 \to 0.$$

Let (M_λ) be a filtered colimit system of R-modules. By an argument similar to that in Lemma 1.1.7, we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \varinjlim_\lambda F_1(M_\lambda) & \longrightarrow & \varinjlim_\lambda F(M_\lambda) & \longrightarrow & \varinjlim_\lambda F_2(M_\lambda) & \longrightarrow & 0 \\
& & {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_2} & & \\
0 & \longrightarrow & F_1(\varinjlim_\lambda M_\lambda) & \longrightarrow & F(\varinjlim_\lambda M_\lambda) & \longrightarrow & F_2(\varinjlim_\lambda M_\lambda) & \longrightarrow & 0
\end{array}$$

As F is finitary, we see that f is an isomorphism. It then follows from the Snake Lemma (see page 11 of [51]) that f_1 is a monomorphism and f_2 is an epimorphism.

Now, as F is the direct sum of F_1 and F_2, we also have the following exact sequence of functors:

$$0 \to F_2 \to F \to F_1 \to 0,$$

and hence the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \varinjlim_\lambda F_2(M_\lambda) & \longrightarrow & \varinjlim_\lambda F(M_\lambda) & \longrightarrow & \varinjlim_\lambda F_1(M_\lambda) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_1} & & \\ 0 & \longrightarrow & F_2(\varinjlim_\lambda M_\lambda) & \longrightarrow & F(\varinjlim_\lambda M_\lambda) & \longrightarrow & F_1(\varinjlim_\lambda M_\lambda) & \longrightarrow & 0 \end{array}$$

and a similar argument to above shows that f_2 is a monomorphism and f_1 is an epimorphism. The result now follows. □

1.2 Finiteness Conditions

We begin this section by looking at some finiteness conditions of R-modules (see §VIII.4 of [9]).

Definition 1.2.1. Let M be an R-module. A *projective resolution* $P_* \twoheadrightarrow M$ of M is an exact sequence of R-modules:

$$\cdots \to P_k \to P_{k-1} \to \cdots \to P_2 \to P_1 \to P_0 \to M \to 0,$$

where each P_k is projective.

Remark 1.2.2. Every R-module M has a projective resolution (see §I.1 of [9]).

Definition 1.2.3. The *projective dimension* $\operatorname{proj.dim}_R M$ of an R-module M is the minimum integer n (if it exists) such that there is a projective resolution of M of length n:

$$0 \to P_n \to \cdots \to P_0 \to M \to 0.$$

If no such finite resolution exists, we set $\operatorname{proj.dim}_R M = \infty$.

Definition 1.2.4. An R-module M is said to be of type FP_n, $0 \leq n \leq \infty$, if and only if there is a projective resolution

$$\cdots \to P_k \to P_{k-1} \to \cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

of M such that the projective modules P_k are finitely generated for all $k \leq n$.

Remark 1.2.5. Notice that M is of type FP_0 if and only if it is finitely generated, and is of type FP_1 if and only if it is finitely presented.

Definition 1.2.6. An R-module M is said to be of type FP if and only if there is a projective resolution of M of finite length:

$$0 \to P_n \to \cdots \to P_0 \to M \to 0$$

such that all of the projective modules P_k are finitely generated.

Proposition 1.2.7. *An R-module M is of type* FP *if and only if* $\operatorname{proj.dim}_R M < \infty$ *and M is of type* FP_∞*.*

Proof. This is Proposition 6.1 §VIII in [9]. □

We can similarly define the notions of type FL_n and type FL by using free modules in place of projectives. It is easy to see that if M is an R-module of type FL_∞, then M is of type FP_∞. The converse of this statement is also true, as we shall now show. First, we need the following generalised version of Schanuel's lemma:

Lemma 1.2.8. *Given two partial projective resolutions of an R-module M:*

$$0 \to K_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

$$0 \to L_n \to Q_{n-1} \to \cdots \to Q_0 \to M \to 0$$

(P_i and Q_i projective), then $K_n \oplus Q_{n-1} \oplus P_{n-2} \oplus \cdots$ is isomorphic to $L_n \oplus P_{n-1} \oplus Q_{n-2} \oplus \cdots$.

Proof. This is a slight variation on Lemma 4.4 §VIII in [9]. □

Corollary 1.2.9. *Let M be an R-module of type* FP_∞*. Then M is of type* FL_∞*.*

Proof. Let $P_* \twoheadrightarrow M$ be a projective resolution of M such that P_k is finitely generated for all k.

As M is finitely generated, we can choose a finitely generated free module $F_0 \twoheadrightarrow M$.

Now suppose that we have constructed the partial free resolution

$$F_{n-1} \to \cdots \to F_0 \to M \to 0$$

of length $n-1$, with each F_i finitely generated. Applying Lemma 1.2.8 to the two partial resolutions:

$$0 \to L_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0$$

$$0 \to K_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

(where L_n and K_n denote the respective nth kernels), we obtain the following isomorphism:

$$L_n \oplus P_{n-1} \oplus F_{n-2} \oplus \cdots \cong K_n \oplus F_{n-1} \oplus P_{n-2} \oplus \cdots$$

Then, as K_n is finitely generated, we see that L_n is finitely generated, and hence we can choose a finitely generated free module $F_n \twoheadrightarrow L_n$. This gives the following partial free resolution

$$F_n \to \cdots \to F_0 \to M \to 0$$

of M of length n, with each F_i finitely generated.

The result now follows by induction. □

Next, we have the following characterisation of the R-modules of type FP_n in terms of finitary functors:

Proposition 1.2.10. *Let M be an R-module. Then M is of type FP_n, $0 \leq n \leq \infty$, if and only if the functors $\mathrm{Ext}^k_R(M, -)$ are finitary for all $k < n$.*

Proof. This is Theorem 1.3 in [6] (see also Theorem 4.8 §VIII in [9]). □

Next, we look at some finiteness conditions of groups (see §VIII.5 of [9]).

Definition 1.2.11. Let G be a group. Then G is said to be of type FP_n if and only if the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ is of type FP_n.

Remark 1.2.12. We can similarly define the properties of type FP, type FL_n, etc.

Definition 1.2.13. Let G be a group. The *cohomological dimension* of G is the projective dimension of the trivial $\mathbb{Z}G$-module,

$$\mathrm{cd}\, G := \mathrm{proj.\,dim}_{\mathbb{Z}G}\, \mathbb{Z}.$$

Next, we shall introduce the notion of virtual cohomological dimension. We begin by recalling Serre's Theorem [45]:

Theorem 1.2.14. *If G is a torsion-free group and H is a subgroup of finite index, then*

$$\mathrm{cd}\, H = \mathrm{cd}\, G.$$

Definition 1.2.15. The *virtual cohomological dimension* of a group G is defined as

$$\operatorname{vcd} G := \begin{cases} \operatorname{cd} H, & \text{if } G \text{ contains a finite-index torsion-free subgroup } H \\ \infty, & \text{otherwise} \end{cases}$$

It follows from Serre's Theorem that this definition is independent of the choice of H; for if H and H' are two torsion-free subgroups of finite index, then

$$|H : H \cap H'| = |HH' : H'| \leq |G : H'| < \infty,$$

and similarly $|H' : H \cap H'| < \infty$. Hence, Serre's Theorem gives

$$\operatorname{cd} H = \operatorname{cd}(H \cap H') = \operatorname{cd} H'.$$

Remark 1.2.16. We can also make these definitions over rings R other than $\mathbb{Z}$, by considering the trivial RG-module R. In order to make it clear that we are now working over R, we say that our group G is of type FP_n over R, or that G has finite cohomological dimension over R, etc.

We end this section by giving a characterisation of the groups of type FP_n in terms of finitary functors. Recall that if G is a group and $n \in \mathbb{N}$, then the nth cohomology of G,

$$H^n(G, -) := \operatorname{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, -),$$

is a functor from the category of $\mathbb{Z}G$-modules to the category of $\mathbb{Z}$-modules. Using Proposition 1.2.10, we obtain the following:

Corollary 1.2.17. *Let G be a group. Then G is of type* FP_n*, $0 \leq n \leq \infty$, if and only if $H^k(G, -)$ is finitary for all $k < n$.*

Remark 1.2.18. Again, we can work over rings R other than $\mathbb{Z}$ by defining the nth cohomology of G as

$$H^n(G, -) := \operatorname{Ext}^n_{RG}(R, -).$$

To make it clear that we are now working over R, we say that $H^n(G, -)$ is *finitary over R* if and only if the functor $\operatorname{Ext}^n_{RG}(R, -)$ is finitary. We then obtain a similar result to Corollary 1.2.17, characterising the groups of type FP_n over R.

1.3 Kropholler's Class $\mathrm{LH}\mathfrak{F}$

We begin this section by introducing the notion of a G-CW-complex (see §3 of [32]).

Definition 1.3.1. Let G be a discrete group. A *G-CW-complex* is a G-space X with a filtration

$$X^0 \subseteq X^1 \subseteq X^2 \subseteq \cdots \subseteq X$$

by G-subspaces such that the following axioms hold:

(i) Each X^n is closed in X;

(ii) $\bigcup_{n\in\mathbb{N}} X^n = X$;

(iii) X^0 is a discrete subspace of X;

(iv) For each $n \geq 1$ there is a discrete G-set Δ_n, together with G-maps

$$f : S^{n-1} \times \Delta_n \to X^{n-1} \quad \text{and} \quad \widehat{f} : B^n \times \Delta_n \to X^n$$

such that the following diagram is a pushout:

$$\begin{array}{ccc} S^{n-1} \times \Delta_n & \xrightarrow{f} & X^{n-1} \\ \downarrow & & \downarrow \\ B^n \times \Delta_n & \xrightarrow[\widehat{f}]{} & X^n \end{array}$$

(v) A subspace Y of X is closed if and only if $Y \cap X^n$ is closed for all $n \geq 0$.

Here, we write S^{n-1} and B^n for the standard unit sphere and unit ball in Euclidean n-space, and the vertical maps in the diagram are inclusions. It is useful to adopt the conventions $X^{-1} = \varnothing$ and $\Delta_0 = X^0$. Then for all $n \geq 0$, the nth *cellular chain group* $C_n(X)$ can be defined to be the nth singular homology of the pair X^n, X^{n-1}, and it follows from the Eilenberg–Steenrod Axioms that this is isomorphic as a $\mathbb{Z}G$-module to the permutation module $\mathbb{Z}\Delta_n$ determined by Δ_n. In effect, $C_n(X)$ is the free abelian group on the G-set of n-cells in X.

A G-CW-complex X is said to be *finite-dimensional* if and only if $X^n = X$ for some n, in which case the dimension is the least n for which this happens. For the finite-dimensional case, Axiom (v) is redundant.

Next, we introduce the notions of classes of groups and closure operations (see §1.1 of [41]).

Definition 1.3.2. A *group theoretical class*, or *class of groups*, $\mathfrak{X}$ is a class in the usual sense, consisting of groups, with two additional properties:

(i) If G belongs to $\mathfrak{X}$ and G' is isomorphic to G, then G' belongs to $\mathfrak{X}$; and

(ii) The trivial group belongs to $\mathfrak{X}$.

The group theoretical classes are partially ordered by inclusion, and the notation $\mathfrak{X} \leq \mathfrak{Y}$ is used to denote the fact that $\mathfrak{X}$ is a group theoretical subclass of the group theoretical class $\mathfrak{Y}$, not merely that $\mathfrak{X}$ is a subclass of $\mathfrak{Y}$.

Remark 1.3.3. Some examples of classes of groups include the class $\mathfrak{F}$ of finite groups, and the class $\mathfrak{C}$ of cyclic groups.

Definition 1.3.4. An *operation* is a function $\mathbf{A}$ assigning to each class of groups $\mathfrak{X}$ a class of groups $\mathbf{A}\mathfrak{X}$, subject to the following two conditions:

$$\mathbf{A}\mathfrak{I} = \mathfrak{I},$$

where $\mathfrak{I}$ denotes the trivial class, and

$$\mathfrak{X} \leq \mathbf{A}\mathfrak{X} \leq \mathbf{A}\mathfrak{Y}$$

whenever $\mathfrak{X} \leq \mathfrak{Y}$.

If $\mathfrak{X} = \mathbf{A}\mathfrak{X}$, then the class $\mathfrak{X}$ is said to be $\mathbf{A}$-*closed.*

An operation $\mathbf{A}$ is called a *closure operation* if it is idempotent; that is, if

$$\mathbf{A} = \mathbf{A}^2.$$

If $\mathbf{A}$ is a closure operation, it then follows that $\mathbf{A}\mathfrak{X}$ is the uniquely determined, smallest $\mathbf{A}$-closed class of groups that contains $\mathfrak{X}$.

We can now recall Kropholler's class $\mathbf{LH}\mathfrak{F}$ of locally hierarchically decomposable groups (see [27]).

Definition 1.3.5. Let $\mathfrak{X}$ be a class of groups. A new class $\mathbf{H}_1\mathfrak{X}$ is constructed as follows: A group G belongs to $\mathbf{H}_1\mathfrak{X}$ if and only if there is a finite-dimensional contractible G-CW-complex X with cell stabilizers in $\mathfrak{X}$.

Now define a hierarchy of classes $\mathbf{H}_\alpha\mathfrak{X}$ for each ordinal α by transfinite recursion:

- If $\alpha = 0$, then $\mathbf{H}_\alpha\mathfrak{X} = \mathfrak{X}$;
- If α is a successor ordinal, then $\mathbf{H}_\alpha\mathfrak{X} = \mathbf{H}_1(\mathbf{H}_{\alpha-1}\mathfrak{X})$; and
- If α is a limit ordinal, then $\mathbf{H}_\alpha\mathfrak{X} = \bigcup_{\beta<\alpha} \mathbf{H}_\beta\mathfrak{X}$.

We then define a closure operation $\mathbf{H}$ on classes of groups as follows: A group G belongs to $\mathbf{H}\mathfrak{X}$ if and only if G belongs to $\mathbf{H}_\alpha\mathfrak{X}$ for some ordinal α.

There is also the classically defined closure operation $\mathbf{L}$ on classes of groups, which is defined as follows: A group G belongs to $\mathbf{L}\mathfrak{X}$ if and only if every finite subset F of G is contained in a subgroup of G which belongs to $\mathfrak{X}$. Groups in the class $\mathbf{L}\mathfrak{X}$ are called *locally-*$\mathfrak{X}$ groups.

Now let $\mathfrak{F}$ denote the class of finite groups. The class $\mathbf{LH}\mathfrak{F}$ is now defined.

Remark 1.3.6. Briefly, the class $\mathbf{LH}\mathfrak{F}$ is the smallest $\mathbf{H}$-closed $\mathbf{L}$-closed class containing the class of finite groups. It contains many commonly found groups, including:

- All soluble-by-finite groups, and more generally all elementary amenable groups;
- All linear groups (that is, all subgroups of $\mathrm{GL}_n(k)$, where k is a field), and more generally all groups of automorphisms of a Noetherian module over a commutative ring;
- All groups of finite cohomological dimension, and more generally all groups of finite virtual cohomological dimension; and
- For all sufficiently large e, the free Burnside groups of exponent e.

The class $\mathbf{LH}\mathfrak{F}$ also satisfies a number of closure properties: It is closed under taking subgroups, extensions, directed unions, free products with amalgamation and HNN-extensions. However, it is distinct from the class of all groups:

Proposition 1.3.7. *Thompson's group F with presentation*

$$\langle x_0, x_1, x_2, \ldots : x_i^{-1} x_j x_i = x_{j+1} \textit{ for } i < j \rangle$$

does not belong to $\mathbf{LH}\mathfrak{F}$

Proof. Benson has shown (Theorem 3.3 in [3]) that if G is a torsion-free $\mathbf{LH}\mathfrak{F}$-group of type FP_∞, then G has finite cohomological dimension. However, Brown and Geoghegan have shown in [10] that Thompson's group F is a torsion-free group of type FP_∞ which has infinite cohomological dimension. Therefore, F cannot belong to $\mathbf{LH}\mathfrak{F}$. □

Next, we introduce the important notion of a group having cohomology almost everywhere finitary:

Definition 1.3.8. Let G be a group. The *finitary set* $\mathscr{F}(G)$ of G is defined as

$$\mathscr{F}(G) := \{n \in \mathbb{N} : H^n(G, -) \text{ is finitary}\}.$$

If G is a group such that $\mathscr{F}(G)$ is cofinite in $\mathbb{N}$, then we say that G has *cohomology almost everywhere finitary.*

Remark 1.3.9. It follows from Corollary 1.2.17 that every group of type FP_∞ has cohomology almost everywhere finitary. However, there are a great many examples of groups with cohomology almost everywhere finitary that are not of type FP_∞; for example, the group of rationals $\mathbb{Q}$ is infinitely generated, so cannot be of type FP_∞, but as $\mathrm{cd}\,\mathbb{Q} = 2$, we see that $H^n(\mathbb{Q}, -) = 0$, and hence is finitary, for all $n \geq 3$.

We now introduce the notion of classifying spaces for proper group actions (see §5 of [34]):

Definition 1.3.10. Let G be a discrete group. A *classifying space for proper G-actions*, denoted by $\underline{E}G$, is a G-CW-complex with finite cell stabilizers such that for each finite $K \leq G$, the K-fixed point set $(\underline{E}G)^K$ is contractible.

We can now state the following key theorem of Kropholler (Theorem 2.1 in [30]):

Theorem 1.3.11. *Let G be an $\mathbf{LH}\mathfrak{F}$-group for which $\mathscr{F}(G)$ is infinite. Then:*

(i) $\mathscr{F}(G)$ is cofinite in $\mathbb{N}$;

(ii) There is a bound on the orders of the finite subgroups of G; and

(iii) There is a finite-dimensional model for the classifying space $\underline{E}G$ for proper G-actions.

Remark 1.3.12. Kropholler's theorem is originally stated in terms of the set $\mathscr{F}_0(G)$ of natural numbers n such that the functor $H^n(G, -)$ is 0-*finitary*; that is,

$$\varinjlim_{\lambda} H^n(G, M_\lambda) = 0$$

whenever (M_λ) is a filtered colimit system satisfying $\varinjlim_\lambda M_\lambda = 0$. However, the proof of Kropholler's theorem establishes Theorem 1.3.11 above.

We end this section with the following useful result:

Proposition 1.3.13. *Let G be an $\mathbf{LH}\mathfrak{F}$-group with cohomology almost everywhere finitary. Then G belongs to $\mathbf{H}_1\mathfrak{F}$ and there is a bound on the orders of the finite subgroups of G.*

Proof. The fact that there is a bound on the orders of the finite subgroups of G follows immediately from Theorem 1.3.11. We also see that there is a finite-dimensional model, say X, for the classifying space $\underline{E}G$ for proper G-actions. We know that X^K is contractible for every finite subgroup K of G, so in particular X itself is contractible. Hence, X is a finite-dimensional contractible G-CW-complex with finite cell stabilizers. It then follows that G belongs to $\mathbf{H}_1\mathfrak{F}$. □

1.4 Some Useful Results

In this final section, we introduce some results which are needed later on, but to include them elsewhere would disrupt the flow of the thesis.

We begin with the following result which shall be needed in §2.1.3.

Lemma 1.4.1. *Let G be a group. If we have an exact sequence of RG-modules*

$$0 \to A_r \to A_{r-1} \to \cdots \to A_0 \to R \to 0$$

such that, for each $i = 0, \ldots, r$, the functor $\mathrm{Ext}^*_{RG}(A_i, -)$ is finitary in all sufficiently high dimensions, then G has cohomology almost everywhere finitary over R.

Proof. The case $r = 0$ is immediate. For $r = 1$, there is a short exact sequence

$$0 \to A_1 \to A_0 \to R \to 0,$$

and hence a long exact sequence:

$$\cdots \to \mathrm{Ext}^j_{RG}(A_0, -) \to \mathrm{Ext}^j_{RG}(A_1, -) \to H^{j+1}(G, -) \to$$

$$\to \mathrm{Ext}^{j+1}_{RG}(A_0, -) \to \mathrm{Ext}^{j+1}_{RG}(A_1, -) \to \cdots .$$

Both $\mathrm{Ext}^*_{RG}(A_0, -)$ and $\mathrm{Ext}^*_{RG}(A_1, -)$ are finitary in all sufficiently high dimensions, so G has cohomology almost everywhere finitary over R by Lemma 1.1.7.

For $r \geq 2$, assume by induction that the result is true for sequences of length less than r. We have an exact sequence

$$0 \to A_r \to A_{r-1} \to \cdots \to A_0 \to R \to 0$$

such that, for each $i = 0, \ldots, r$, the functor $\mathrm{Ext}^*_{RG}(A_i, -)$ is finitary in all sufficiently high dimensions. Let $K := \mathrm{Im}(A_{r-1} \to A_{r-2})$, so we have the short exact sequence

$$0 \to A_r \to A_{r-1} \to K \to 0,$$

and an argument similar to the above shows that $\mathrm{Ext}^*_{RG}(K, -)$ is finitary in all sufficiently high dimensions. We then have the following exact sequence:

$$0 \to K \to A_{r-2} \to \cdots \to A_0 \to R \to 0,$$

and the result now follows by induction. □

We now have a change of rings result:

Lemma 1.4.2. *Let G be a group, and let $R_1 \to R_2$ be a ring homomorphism. If $H^n(G, -)$ is finitary over R_1, then $H^n(G, -)$ is finitary over R_2.*

Proof. We see from Chapter 0 of [6] that for any R_2G-module M, we have the following isomorphism:

$$\mathrm{Ext}^n_{R_2G}(R_2, M) \cong \mathrm{Ext}^n_{R_1G}(R_1, M),$$

where M is viewed as an R_1G-module using the homomorphism $R_1 \to R_2$. The result now follows. □

Next, we have the following result which shall be needed in §2.3:

Lemma 1.4.3. *Let G be a group, and R be a ring of prime characteristic p. Then $H^n(G,-)$ is finitary over R if and only if $H^n(G,-)$ is finitary over $\mathbb{F}_p$.*

Proof. If $H^n(G,-)$ is finitary over $\mathbb{F}_p$, then it follows from Lemma 1.4.2 that $H^n(G,-)$ is finitary over R.

Conversely, suppose that $H^n(G,-)$ is finitary over R; that is, the functor $\operatorname{Ext}^n_{RG}(R,-)$ is finitary. Let (M_λ) be a filtered colimit system of $\mathbb{F}_pG$-modules. Then $(M_\lambda \otimes_{\mathbb{F}_p} R)$ is a filtered colimit system of RG-modules, and so the natural map

$$\varinjlim_\lambda \operatorname{Ext}^n_{RG}(R, M_\lambda \otimes_{\mathbb{F}_p} R) \to \operatorname{Ext}^n_{RG}(R, \varinjlim_\lambda M_\lambda \otimes_{\mathbb{F}_p} R)$$

is an isomorphism. Now, as an $\mathbb{F}_p$-vector space, $R \cong \mathbb{F}_p \oplus V$ for some $\mathbb{F}_p$-vector space V. Therefore, for each λ,

$$M_\lambda \otimes_{\mathbb{F}_p} R \cong M_\lambda \oplus M_\lambda \otimes_{\mathbb{F}_p} V,$$

and so

$$\operatorname{Ext}^n_{RG}(R, M_\lambda \otimes_{\mathbb{F}_p} R) \cong \operatorname{Ext}^n_{RG}(R, M_\lambda) \oplus \operatorname{Ext}^n_{RG}(R, M_\lambda \otimes_{\mathbb{F}_p} V).$$

It then follows from Lemma 1.1.8 that the natural map

$$\varinjlim_\lambda \operatorname{Ext}^n_{RG}(R, M_\lambda) \to \operatorname{Ext}^n_{RG}(R, \varinjlim_\lambda M_\lambda)$$

is an isomorphism. Now, we see from Chapter 0 of [6] that

$$\operatorname{Ext}^n_{RG}(R,-) \cong \operatorname{Ext}^n_{\mathbb{F}_pG}(\mathbb{F}_p,-)$$

on RG-modules, so it follows that the natural map

$$\varinjlim_\lambda \operatorname{Ext}^n_{\mathbb{F}_pG}(\mathbb{F}_p, M_\lambda) \to \operatorname{Ext}^n_{\mathbb{F}_pG}(\mathbb{F}_p, \varinjlim_\lambda M_\lambda)$$

is an isomorphism, and hence that $H^n(G,-)$ is finitary over $\mathbb{F}_p$. □

Remark 1.4.4. Note that if G is a group and R is a ring of prime characteristic p, then the isomorphism

$$\operatorname{Ext}^n_{\mathbb{F}_pG}(\mathbb{F}_p,-) \cong \operatorname{Ext}^n_{RG}(R,-)$$

on RG-modules, and the isomorphism

$$\operatorname{Ext}^n_{RG}(R, - \otimes_{\mathbb{F}_p} R) \cong \operatorname{Ext}^n_{\mathbb{F}_pG}(\mathbb{F}_p,-) \oplus \operatorname{Ext}^n_{RG}(R, - \otimes_{\mathbb{F}_p} V)$$

on $\mathbb{F}_pG$-modules together show that $\operatorname{vcd}_R G = \operatorname{vcd}_{\mathbb{F}_p} G$, a fact that is needed in §2.3.

We finish this section by showing that the property of having cohomology almost everywhere finitary passes to subgroups of finite index. We begin by recalling the notions of induction and coinduction (see §III.5 of [9]).

Definition 1.4.5. Let H be a subgroup of G, and M be an RH-module. Then the *induced RG-module* is defined as

$$\mathrm{Ind}_H^G(M) := M \otimes_{RH} RG,$$

and the *coinduced RG-module* is defined as

$$\mathrm{Coind}_H^G(M) := \mathrm{Hom}_{RH}(RG, M).$$

Proposition 1.4.6. *Let H be a subgroup of G of finite index. If $H^n(G, -)$ is finitary over R, then $H^n(H, -)$ is also finitary over R.*

Proof. Suppose that $H^n(G, -)$ is finitary over R. From Shapiro's Lemma (see Proposition 6.2 §III in [9]) we know that

$$H^n(H, -) \cong H^n(G, \mathrm{Coind}_H^G -).$$

Then, as H has finite index in G, it follows from Lemma 6.3.4 in [51] that $\mathrm{Ind}_H^G(-) \cong \mathrm{Coind}_H^G(-)$. Therefore,

$$H^n(H, -) \cong H^n(G, \mathrm{Ind}_H^G -).$$

Now, as tensor products commute with filtered colimits (see §1.2 of [6]), it follows that $\mathrm{Ind}_H^G(-)$ is a finitary functor. Therefore $H^n(H, -)$ is the composite of two finitary functors, and the result now follows. □

Chapter 2

Algebraic Characterisations

In this chapter we investigate algebraic characterisations of certain classes of $\mathbf{LH}\mathfrak{F}$-groups with cohomology almost everywhere finitary. We begin by looking at the class of locally (polycyclic-by-finite) groups in §2.1, then the class of groups of finite virtual cohomological dimension over a ring R of prime characteristic p in §2.3, and finally the class $\mathbf{H}_1\mathfrak{F}$ in §2.4. We also look at some closure properties of the class of groups with cohomology almost everywhere finitary in §2.2.

2.1 Locally (Polycyclic-by-Finite) Groups

We begin by recalling the definition of a locally (polycyclic-by-finite) group. We take the following two definitions from §1.1 of [41]:

Definition 2.1.1. The closure operation $\mathbf{P}$ on classes of groups is defined as follows: Let $\mathfrak{X}$ be a class of groups. Then a group G belongs to $\mathbf{P}\mathfrak{X}$ if and only if there is a series of finite length

$$1 = G_0 \lhd G_1 \lhd \cdots \lhd G_{n-1} \lhd G_n = G$$

in which each factor G_i/G_{i-1} belongs to $\mathfrak{X}$. Groups in the class $\mathbf{P}\mathfrak{X}$ are called *poly-*$\mathfrak{X}$ groups.

Definition 2.1.2. If $\mathfrak{X}$ and $\mathfrak{Y}$ are two classes of groups, then the *extension* or *product class* $\mathfrak{X}\mathfrak{Y}$ is defined as follows: A group G belongs to $\mathfrak{X}\mathfrak{Y}$ if and only if there is a normal subgroup N of G such that $N \in \mathfrak{X}$ and $G/N \in \mathfrak{Y}$. Groups in the class $\mathfrak{X}\mathfrak{Y}$ are called $\mathfrak{X}$*-by-*$\mathfrak{Y}$ groups.

Recall from §1.3 the definition of the closure operation $\mathbf{L}$. Then the class of locally (polycyclic-by-finite) groups is simply the class $\mathbf{L}([\mathbf{P}\mathfrak{C}]\mathfrak{F})$, where $\mathfrak{C}$ denotes the class of cyclic groups, and $\mathfrak{F}$

denotes the class of finite groups.

Remark 2.1.3. The class of locally (polycyclic-by-finite) groups contains all abelian-by-finite groups, and, more generally, all nilpotent-by-finite groups.

The goal of this section is to prove the following characterisation of the locally (polycyclic-by-finite) groups with cohomology almost everywhere finitary:

Theorem A. *Let G be a locally (polycyclic-by-finite) group. Then G has cohomology almost everywhere finitary if and only if*

(i) G has finite virtual cohomological dimension; and

(ii) The normalizer of every non-trivial finite subgroup of G is finitely generated.

Let G be a locally (polycyclic-by-finite) group with cohomology almost everywhere finitary. As the class of locally (polycyclic-by-finite) groups is a subclass of $\mathbf{LH}\mathfrak{F}$ (see [27]), it follows from Proposition 1.3.13 that G belongs to $\mathbf{H}_1\mathfrak{F}$ and there is a bound on the orders of its finite subgroups. In §2.1.1 we shall prove the following lemma:

Lemma 2.1.4. *Let G be an elementary amenable group. Then the following are equivalent:*

(i) G belongs to $\mathbf{H}_1\mathfrak{F}$ and there is a bound on the orders of its finite subgroups;

(ii) G has finite virtual cohomological dimension; and

(iii) G belongs to $\mathbf{H}_1\mathfrak{F}$ and there are finitely many conjugacy classes of finite subgroups.

As every locally (polycyclic-by-finite) group is elementary amenable, we then see that locally (polycyclic-by-finite) groups with cohomology almost everywhere finitary must have finite virtual cohomological dimension. Hence, in order to prove Theorem A it is enough to prove the following:

Theorem 2.1.5. *Let G be a locally (polycyclic-by-finite) group of finite virtual cohomological dimension. Then G has cohomology almost everywhere finitary if and only if the normalizer of every non-trivial finite subgroup of G is finitely generated.*

Suppose that G is a locally (polycyclic-by-finite) group of finite virtual cohomological dimension, so we have the following short exact sequence:

$$0 \to N \to G \to Q \to 0,$$

where N is a torsion-free, locally (polycyclic-by-finite) group of finite cohomological dimension, and Q is a finite group. In order to prove Theorem 2.1.5 we must consider three cases: The first case to consider is when G is torsion-free. In this case, G has finite cohomological dimension, so $H^n(G, -) = 0$, and hence is finitary, for all sufficiently large n. The next simplest case, when G is the direct product $N \times Q$, is treated in §2.1.2, and the general case is then proved in §2.1.3.

We then conclude this section by proving the following corollary in §2.1.4:

Corollary B. *Let G be a locally (polycyclic-by-finite) group. If G has cohomology almost everywhere finitary, then every subgroup of G has cohomology almost everywhere finitary.*

We also give examples to show that Corollary B does not hold in general.

2.1.1 Proof of Lemma 2.1.4

We begin by recalling the definition of elementary amenable groups and their Hirsch length (see [25]):

Definition 2.1.6. Let $\mathfrak{X}_0 = \{1\}$ and let $\mathfrak{X}_1$ be the class of finitely generated abelian-by-finite groups. If $\mathfrak{X}_\alpha$ has been defined for some ordinal α, let $\mathfrak{X}_{\alpha+1} = (\mathbf{L}\mathfrak{X}_\alpha)\mathfrak{X}_1$ be the class of (locally $\mathfrak{X}_\alpha$)-by-$\mathfrak{X}_1$ groups, and if $\mathfrak{X}_\alpha$ has been defined for all ordinals α less than some limit ordinal β, let $\mathfrak{X}_\beta = \bigcup_{\alpha<\beta} \mathfrak{X}_\alpha$. Then the class of elementary amenable groups is $\bigcup_\alpha \mathfrak{X}_\alpha$, where the union is taken over all ordinals α.

Remark 2.1.7. We see that the class of elementary amenable groups is simply the class of groups generated from finite groups and $\mathbb{Z}$ by the operations of extension and increasing union. In particular, every locally (polycyclic-by-finite) group, and, more generally, every soluble-by-finite group belongs to this class.

Definition 2.1.8. The *Hirsch length* $h(G)$ of an elementary amenable group G is defined as follows: If G belongs to $\mathfrak{X}_1$, let $h(G)$ be the torsion-free rank of an abelian subgroup of finite index in G. If $h(G)$ has been defined for all groups G in $\mathfrak{X}_\alpha$ and H belongs to $\mathbf{L}\mathfrak{X}_\alpha$, let

$$h(H) = \sup\{h(F) : F \text{ is an } \mathfrak{X}_\alpha\text{-subgroup of } H\}.$$

Finally, if G belongs to $\mathfrak{X}_{\alpha+1}$, then it has a normal subgroup H in $\mathbf{L}\mathfrak{X}_\alpha$ with quotient in $\mathfrak{X}_1$, so let

$$h(G) = h(H) + h(G/H).$$

Transfinite induction on

$$\alpha(G) := \min\{\alpha : G \in \mathfrak{X}_\alpha\}$$

may be used to prove that $h(G)$ is well-defined; that if H is a subgroup of G, then $h(H) \leq h(G)$; that if H is a normal subgroup of G, then $h(G) = h(H) + h(G/H)$; and that

$$h(G) = \sup\{h(F) : F \text{ is a finitely generated subgroup of } G\}.$$

We can now prove Lemma 2.1.4.

Proof of Lemma 2.1.4 (i) $\Rightarrow$ (ii)

Let G be an elementary amenable group such that G belongs to $\mathbf{H}_1\mathfrak{F}$ and there is a bound on the orders of its finite subgroups.

As G belongs to $\mathbf{H}_1\mathfrak{F}$, there exists a finite-dimensional contractible G-CW-complex X with finite cell stabilizers. Let $n = \dim X$, and consider the augmented cellular chain complex of X:

$$0 \to C_n(X) \to \cdots \to C_0(X) \to \mathbb{Z} \to 0.$$

As X is contractible, this is an exact sequence. Now, for each k, $C_k(X)$ is the free abelian group on the set Δ_k of k-cells in X, and as G permutes these cells, we can split Δ_k up into its G-orbits:

$$\Delta_k = \coprod_{\sigma \in \Sigma_k} G_\sigma \backslash G,$$

where Σ_k is a set of G-orbit representatives of k-cells in X, and G_σ is the stabilizer of σ. Then

$$C_k(X) \cong \mathbb{Z}\Delta_k \cong \bigoplus_{\sigma \in \Sigma_k} \mathbb{Z}[G_\sigma \backslash G].$$

Now, the augmented cellular chain complex of X is $\mathbb{Z}$-split, so tensoring with $\mathbb{Q}$ gives the following exact sequence:

$$0 \to C_n(X) \otimes \mathbb{Q} \to \cdots \to C_0(X) \otimes \mathbb{Q} \to \mathbb{Q} \to 0,$$

and for each k, we have

$$\begin{aligned} C_k(X) \otimes \mathbb{Q} &\cong \textstyle\bigoplus_{\sigma \in \Sigma_k} \mathbb{Z}[G_\sigma \backslash G] \otimes \mathbb{Q} \\ &\cong \textstyle\bigoplus_{\sigma \in \Sigma_k} \mathbb{Q}[G_\sigma \backslash G] \\ &\cong \textstyle\bigoplus_{\sigma \in \Sigma_k} \mathbb{Q} \otimes_{\mathbb{Q}G_\sigma} \mathbb{Q}G, \end{aligned}$$

where the last isomorphism follows from Lemma 2.9 in [6]. Now, as X has finite cell stabilizers, each G_σ is finite, and so by Maschke's Theorem (Theorem 4.2 in [39]) the group ring $\mathbb{Q}G_\sigma$ is semisimple. Every module over a semisimple ring is projective (Corollary 8.2.2(e) in [26]), so $\mathbb{Q}$ is a projective $\mathbb{Q}G_\sigma$-module. Hence, $\mathbb{Q} \otimes_{\mathbb{Q}G_\sigma} \mathbb{Q}G$ is a projective $\mathbb{Q}G$-module, and so $C_k(X) \otimes \mathbb{Q}$ is a projective $\mathbb{Q}G$-module. We then conclude that

$$0 \to C_n(X) \otimes \mathbb{Q} \to \cdots \to C_0(X) \otimes \mathbb{Q} \to \mathbb{Q} \to 0$$

is a projective resolution of the trivial $\mathbb{Q}G$-module, and so the rational cohomological dimension of G is at most n. Now, according to Hillman [24], the Hirsch length of an elementary amenable group is bounded above by its rational cohomological dimension, so we conclude that G has finite Hirsch length.

Next, let $\tau(G)$ denote the unique largest locally finite normal subgroup of G (see page 418 of [43]). It is easy to see that an infinite locally finite group has arbitrarily large finite subgroups. Therefore, as G has a bound on the orders of its finite subgroups, it follows that $\tau(G)$ must be finite.

Now, as G is an elementary amenable group of finite Hirsch length, it follows from a result of Wehrfritz [50] that $G/\tau(G)$ has a poly-(torsion-free abelian) characteristic subgroup of finite index. Hence, G is finite-by-poly-(torsion-free abelian)-by-finite. Then, as finite-by-(torsion-free abelian) groups are (torsion-free abelian)-by-finite (part (b) of [50]), it follows that G has a poly-(torsion-free abelian) characteristic subgroup S of finite index.

Now torsion-free abelian groups of finite Hirsch length have finite cohomological dimension (Theorem 5 §8.8 in [16]), and the extension of two groups of finite cohomological dimension also has finite cohomological dimension (Proposition 9 §8.7 in [16]), so it follows that S has finite cohomological dimension. Therefore, we conclude that G has finite virtual cohomological dimension, as required.

Proof of Lemma 2.1.4 (ii) $\Rightarrow$ (iii)

Let G be an elementary amenable group of finite virtual cohomological dimension. We see from Theorem 11.1 §VIII in [9] that all groups of finite virtual cohomological dimension belong to $\mathbf{H}_1\mathfrak{F}$, so therefore we only need to show that G has finitely many conjugacy classes of finite subgroups.

As G has finite virtual cohomological dimension, it must have a bound on the orders of its finite subgroups. Therefore, the same argument as in the proof of (i) $\Rightarrow$ (ii) shows that G is a poly-(torsion-free abelian)-by-finite group of finite Hirsch length. We proceed by induction on the Hirsch length $h(G)$ of G.

If $h(G)$=1, then G has a torsion-free abelian normal subgroup A of finite Hirsch length such that $G/A = Q$ is finite. We see from Result 11.1.3 in [43] that there is a 1-1 correspondence between the conjugacy classes of complements to A in G and $H^1(Q,A)$, so it is enough to prove that $H^1(Q,A)$ is finite. Since Q is finite, $H^1(Q,A)$ has exponent dividing the order of Q (Corollary 10.2 §III in [9]). We have the following short exact sequence:

$$0 \to A \stackrel{|Q|}{\to} A \stackrel{\pi}{\to} A/|Q|A \to 0.$$

Passing to the long exact sequence in cohomology, we obtain the following monomorphism:

$$0 \to H^1(Q,A) \stackrel{\pi^*}{\to} H^1(Q,A/|Q|A).$$

Now, as $A/|Q|A$ has finite exponent and finite Hirsch length, it is finite. It then follows that $H^1(Q,A)$ is finite, as required.

Suppose $h(G) > 1$. We know that G has a torsion-free abelian normal subgroup A of finite Hirsch length. As $h(G) > h(G/A)$, we see by induction that G/A has finitely many conjugacy classes of finite subgroups. Let F be a finite subgroup of G, so AF lies in one of finitely many conjugacy classes, say those represented by $AK_1, \ldots, AK_m$. Then, as each $H^1(K_i, A)$ is finite, there are only finitely many conjugacy classes of complements to A in AK_i, and F must lie in one of those.

Proof of Lemma 2.1.4 (iii) $\Rightarrow$ (i)

Let $G \in \mathbf{H}_1\mathfrak{F}$ have finitely many conjugacy classes of finite subgroups. Then it is clear that there must be a bound on the orders of its finite subgroups.

2.1.2 The Direct Product Case

Suppose that $G = N \times Q$, where N is a torsion-free, locally (polycyclic-by-finite) group of finite cohomological dimension, and Q is a non-trivial finite group. The goal of this section is to show that G has cohomology almost everywhere finitary if and only if the normalizer of every non-trivial finite subgroup of G is finitely generated. Note that if F is a non-trivial finite subgroup of G, then F must be a subgroup of Q, and so N is a subgroup of $N_G(F)$ of finite index. Hence, $N_G(F)$ is finitely generated if and only if N is. It is therefore enough to prove that G has cohomology almost everywhere finitary if and only if N is finitely generated.

We begin by assuming that N is finitely generated. Therefore N is polycyclic-by-finite, and hence of type FP_∞ (Examples 2.6 in [6]). The property of type FP_∞ is inherited by supergroups of finite index (Proposition 5.1 §VIII in [9]), so G is also of type FP_∞. Then, by Corollary 1.2.17, we see that G has cohomology almost everywhere finitary.

For the converse, we shall prove a more general result which does not place any restrictions on the group N. Firstly, recall the Künneth Theorem (Theorem 3.5.6 in [2]):

Theorem 2.1.9. *Suppose R is a hereditary ring of coefficients (that is, every submodule of a free module is projective). If G_1 and G_2 are groups, M_1 is an RG_1-module and M_2 is an RG_2-module, then for each n we have the following short exact sequence:*

$$\begin{aligned} 0 \to \quad \textstyle\bigoplus_{i+j=n} H^i(G_1, M_1) \otimes_R H^j(G_2, M_2) \quad &\to \quad H^n(G_1 \times G_2, M_1 \otimes_R M_2) \\ &\to \quad \textstyle\bigoplus_{i+j=n-1} \mathrm{Tor}_1^R(H^i(G_1, M_1), H^j(G_2, M_2)) \quad \to 0. \end{aligned}$$

Here, $M_1 \otimes_R M_2$ is regarded as an $R(G_1 \times G_2)$-module via $(m_1 \otimes m_2)(g_1, g_2) = m_1 g_1 \otimes m_2 g_2$.

Proposition 2.1.10. *Let Q be a non-trivial finite group, and N be any group. If there is some natural number k such that $H^k(N \times Q, -)$ is finitary, then N is finitely generated.*

Proof. Suppose that $H^k(N \times Q, -)$ is finitary. As Q is a non-trivial finite group, we can choose a subgroup E of Q of order p, for some prime p, so $N \times E$ is a subgroup of $N \times Q$ of finite index. It then follows from Proposition 1.4.6 that $H^k(N \times E, -)$ is also finitary. Then, by Lemma 1.4.2, we see that $H^k(N \times E, -)$ is finitary over $\mathbb{F}_p$.

Let M be any $\mathbb{F}_p N$-module, and $\mathbb{F}_p$ be the trivial $\mathbb{F}_p E$-module. As $\mathbb{F}_p$ is a field, it is clearly hereditary, and so we can apply the Künneth Theorem. Notice that as every $\mathbb{F}_p$-module is flat, the Tor-term vanishes. We therefore have the following isomorphism:

$$H^k(N \times E, M) \cong \bigoplus_{i+j=k} H^i(N, M) \otimes_{\mathbb{F}_p} H^j(E, \mathbb{F}_p).$$

A simple calculation (Theorem 7.1, Chapter IV in [36]) shows that $H^n(E, \mathbb{F}_p) \cong \mathbb{F}_p$ for all n. Hence, we have

$$H^k(N \times E, M) \cong \bigoplus_{i=0}^{k} H^i(N, M),$$

and as this holds for any $\mathbb{F}_p N$-module M, we have an isomorphism of functors for modules on which E acts trivially. Then, as $H^k(N \times E, -)$ is finitary over $\mathbb{F}_p$, it follows from Lemma 1.1.8 that $H^0(N, -)$ is also finitary over $\mathbb{F}_p$. Then, by Corollary 1.2.17, we conclude that N is of type FP_1 over $\mathbb{F}_p$, and hence is finitely generated, as required. □

The converse of the direct product case now follows immediately.

2.1.3 The General Case

Let G be a locally (polycyclic-by-finite) group of finite virtual cohomological dimension. In this subsection we show that G has cohomology almost everywhere finitary if and only if the normalizer of every non-trivial finite subgroup of G is finitely generated.

We begin with the following useful result:

Proposition 2.1.11. *Let R be a ring, G be an $\mathbf{H}_1\mathfrak{F}$-group and M be an RG-module. Then M has finite projective dimension over RG if and only if M has finite projective dimension over RK for all finite subgroups K of G.*

Proof. This is a special case of Theorem A in [12]. □

Next, recall the Eckmann–Shapiro Lemma (Corollary 2.8.4 in [2]):

Lemma 2.1.12. *Suppose that Γ is a subring of Λ, and that Λ is projective as a Γ-module. If M is a Γ-module and N is a Λ-module, then for each n we have the following isomorphism:*

$$\mathrm{Ext}^n_\Gamma(M, N) \cong \mathrm{Ext}^n_\Lambda(M \otimes_\Gamma \Lambda, N).$$

Remark 2.1.13. Note that if H is a subgroup of G, and M is the trivial RH-module, then the Eckmann–Shapiro Lemma gives the following isomorphism of functors:

$$H^n(H, -) \cong \mathrm{Ext}^n_{RH}(R, -) \cong \mathrm{Ext}^n_{RG}(R \otimes_{RH} RG, -) \cong \mathrm{Ext}^n_{RG}(R[H\backslash G], -).$$

Next, we have the following straightforward lemma:

Lemma 2.1.14. *Let G be a group of finite virtual cohomological dimension. Then G has a torsion-free normal subgroup of finite index.*

Proof. As G has finite virtual cohomological dimension, it has a torsion-free subgroup H of finite index. Let

$$N = \bigcap_{g \in G} H^g.$$

Clearly N is torsion-free and normal, and as there are only finitely many conjugates of H in G, we see that N has finite index in G. □

We can now prove the following:

Theorem 2.1.15. *Let G be a locally (polycyclic-by-finite) group of finite virtual cohomological dimension. If G has cohomology almost everywhere finitary, then the normalizer of every non-trivial finite subgroup of G is finitely generated.*

Proof. Let F be a non-trivial finite subgroup of G, so we can choose a subgroup E of F of order p, for some prime p. As G has finite virtual cohomological dimension, it has a torsion-free normal subgroup N of finite index. Let $H := NE$, so it follows from Proposition 1.4.6 that H has cohomology almost everywhere finitary.

Let Λ denote the set of non-trivial finite subgroups of H, so Λ consists of subgroups of order p. Now H acts on this set by conjugation, so the stabilizer of any $K \in \Lambda$ is $N_H(K)$. Also, for each $K \in \Lambda$, we see that the set of K-fixed points Λ^K is simply the set $\{K\}$, because if K fixed some $K' \neq K$, then KK' would be a subgroup of H of order p^2, which is a contradiction.

We have the following short exact sequence:

$$0 \to J \to \mathbb{Z}\Lambda \xrightarrow{\varepsilon} \mathbb{Z} \to 0,$$

where ε denotes the augmentation map. For each $K \in \Lambda$, we see that J is free as a $\mathbb{Z}K$-module with basis $\{K' - K : K' \in \Lambda\}$. Now, as H belongs to $\mathbf{H}_1\mathfrak{F}$, it follows from Proposition 2.1.11 that J has finite projective dimension over $\mathbb{Z}H$. Now, the short exact sequence $0 \to J \to \mathbb{Z}\Lambda \to \mathbb{Z} \to 0$ gives rise to the following long exact sequence:

$$\cdots \to \mathrm{Ext}^{n-1}_{\mathbb{Z}H}(J, -) \to \mathrm{Ext}^{n}_{\mathbb{Z}H}(\mathbb{Z}, -) \to \mathrm{Ext}^{n}_{\mathbb{Z}H}(\mathbb{Z}\Lambda, -) \to \mathrm{Ext}^{n}_{\mathbb{Z}H}(J, -) \to \cdots,$$

so we conclude that for all sufficiently large n we have the following isomorphism:

$$H^n(H, -) \cong \mathrm{Ext}^{n}_{\mathbb{Z}H}(\mathbb{Z}\Lambda, -).$$

Next, as H acts on Λ, we can split Λ up into its H-orbits, so

$$\Lambda = \coprod_{K \in \mathscr{C}} H_K \backslash H = \coprod_{K \in \mathscr{C}} N_H(K) \backslash H,$$

where K runs through a set $\mathscr{C}$ of representatives of conjugacy classes of non-trivial finite subgroups of H. This gives the following isomorphism:

$$\begin{aligned} H^n(H, -) &\cong \textstyle\prod_{K \in \mathscr{C}} \operatorname{Ext}^n_{\mathbb{Z}H}(\mathbb{Z}[N_H(K) \backslash H], -) \\ &\cong \textstyle\prod_{K \in \mathscr{C}} H^n(N_H(K), -), \end{aligned}$$

where the last isomorphism follows from the Eckmann–Shapiro Lemma. Therefore, if $H^n(H, -)$ is finitary, it follows from Lemma 1.1.8 that $H^n(N_H(E), -)$ is also finitary. Hence, as H has cohomology almost everywhere finitary, we conclude that $N_H(E)$ also has cohomology almost everywhere finitary.

Now, as E is a finite group,

$$|N_H(E) : C_H(E)| < \infty,$$

and so by Proposition 1.4.6 we see that

$$C_H(E) \cong E \times C_N(E)$$

has cohomology almost everywhere finitary. It then follows from Proposition 2.1.10 that $C_N(E)$ is finitely generated, and hence polycyclic-by-finite.

Now, as $E \leq F$, it follows that $C_N(F) \leq C_N(E)$ and as every subgroup of a polycyclic-by-finite group is finitely generated (Chapter 1 in [44]), we see that $C_N(F)$ is finitely generated.

Finally, as N is a subgroup of G of finite index, it follows that

$$|C_G(F) : C_N(F)| < \infty,$$

and so $C_G(F)$ is finitely generated. Hence $N_G(F)$ is finitely generated, as required. □

We have the following corollary:

Corollary 2.1.16. *Let G be an elementary amenable group with cohomology almost everywhere finitary. Then G has finitely many conjugacy classes of finite subgroups, and $C_G(E)$ is finitely generated for every $E \leq G$ of order p.*

Proof. As the class of elementary amenable groups is a subclass of $\mathbf{LH}\mathfrak{F}$, we see from Proposition 1.3.13 that $G \in \mathbf{H}_1\mathfrak{F}$ and there is a bound on the orders of its finite subgroups. It then follows from Lemma 2.1.4 that G has finitely many conjugacy classes of finite subgroups, and furthermore that G has finite virtual cohomological dimension. Therefore, we can choose a torsion-free normal subgroup N of G of finite index.

Let E be any subgroup of G of order p, and let $H := NE$. Following the proof of Theorem 2.1.15, we see that $N_H(E)$ has cohomology almost everywhere finitary. Hence,

$$C_H(E) \cong E \times C_N(E)$$

also has cohomology almost everywhere finitary, and so by Proposition 2.1.10 we see that $C_N(E)$ is finitely generated. The result now follows. □

In the remainder of this subsection we shall prove the converse of Theorem 2.1.15. First, recall the notion of a join of CW-complexes (see [32]).

Definition 2.1.17. For each $n \geq 0$, let Δ^n denote the standard n-simplex in $\mathbb{R}^{n+1}$:

$$\Delta^n := \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i\}.$$

Then, given CW-complexes $X_0, \dots, X_n$, the *join* $X_0 * \cdots * X_n$ is the identification space

$$(\Delta^n \times X_0 \times \cdots \times X_n)/ \sim,$$

where the relation $\sim$ is defined by

$$(t_0, \dots, t_n; x_0, \dots, x_n) \sim (t'_0, \dots, t'_n; x'_0, \dots, x'_n)$$

if and only if, for each i, either $(t_i, x_i) = (t'_i, x'_i)$ or $t_i = t'_i = 0$.

If, in addition, G is a group and the X_i are G-CW-complexes, then the join inherits a G-CW-structure via

$$(t_0, \dots, t_n; x_0, \dots, x_n)g = (t_0, \dots, t_n; x_0 g, \dots, x_n g).$$

The following construction, discussed in [32], will be crucial to our proof:

Definition 2.1.18. Let G be a group, and let $\Lambda(G)$ denote the poset of the non-trivial finite subgroups of G. We can view this poset as a G-simplicial complex $|\Lambda(G)|$ as follows: An n-simplex in $|\Lambda(G)|$ is determined by each strictly increasing chain

$$H_0 < H_1 < \cdots < H_n$$

of $n+1$ non-trivial finite subgroups of G. The action of G on the set of non-trivial finite subgroups induces an action of G on $|\Lambda(G)|$, so that the stabilizer of a simplex is an intersection of normalizers; in the case of the simplex determined by the chain of subgroups above, the stabilizer is

$$\bigcap_{i=0}^{n} N_G(H_i).$$

This complex has the property that, for any non-trivial finite subgroup K of G, the K-fixed point complex $|\Lambda(G)|^K$ is contractible (for a proof of this, see Lemma 2.1 in [32]).

Next, we require the following two results:

Proposition 2.1.19. *Let Y be a G-CW-complex of finite dimension n. Then Y can be embedded into an n-dimensional G-CW-complex $\widetilde{Y}$ which is $(n-1)$-connected in such a way that G acts freely outside Y.*

Proof. This is Lemma 4.4 of [32]. We can take $\widetilde{Y}$ to be the n-skeleton of the join

$$Y * \underbrace{G * \cdots * G}_{n},$$

where G is viewed as a discrete G-space. □

Proposition 2.1.20. *Let Y be an n-dimensional G-CW-complex which is $(n-1)$-connected, for some $n \geq 0$. Suppose that Y^K is contractible for all non-trivial finite subgroups K of G. Then the nth reduced homology group $\widetilde{H_n}(Y)$ is projective as a $\mathbb{Z}K$-module for all finite subgroups K of G.*

Proof. This is Proposition 6.2 of [32]. □

Finally, we can now prove the converse:

Theorem 2.1.21. *Let G be a locally (polycyclic-by-finite) group of finite virtual cohomological dimension. If the normalizer of every non-trivial finite subgroup of G is finitely generated, then G has cohomology almost everywhere finitary.*

Proof. Let $\Lambda(G)$ be the poset of all non-trivial finite subgroups of G, and let $|\Lambda(G)|$ denote its realisation as a G-simplicial complex. Since G has finite virtual cohomological dimension, there is a bound on the orders of its finite subgroups, so $|\Lambda(G)|$ is finite-dimensional, say $\dim |\Lambda(G)| = r$. From Proposition 2.1.19, we can embed $|\Lambda(G)|$ into an r-dimensional G-CW-complex Y which is $(r-1)$-connected, such that G acts freely outside $|\Lambda(G)|$. Consider the augmented cellular chain complex of Y. As Y is $(r-1)$-connected, it has trivial homology except in dimension r, giving the following exact sequence:

$$0 \to \widetilde{H}_r(Y) \to C_r(Y) \to \cdots \to C_0(Y) \to \mathbb{Z} \to 0.$$

By Lemma 1.4.1, it remains to show that the functors $\mathrm{Ext}^*_{\mathbb{Z}G}(\widetilde{H}_r(Y), -)$ and $\mathrm{Ext}^*_{\mathbb{Z}G}(C_l(Y), -)$, $0 \leq l \leq r$, are finitary in all sufficiently high dimensions.

Notice that for every non-trivial finite subgroup K of G, $Y^K = |\Lambda(G)|^K$, as the copies of G that we have added in the construction of Y have free orbits, and so have no fixed points under K. Thus, Y is an r-dimensional G-CW-complex which is $(r-1)$-connected, such that Y^K is contractible for every non-trivial finite subgroup K of G. It follows from Proposition 2.1.20 that $\widetilde{H}_r(Y)$ is projective as a $\mathbb{Z}K$-module for all finite subgroups K of G. Then by Proposition 2.1.11, $\widetilde{H}_r(Y)$ has finite projective dimension over $\mathbb{Z}G$, and so $\mathrm{Ext}^n_{\mathbb{Z}G}(\widetilde{H}_r(Y), -) = 0$, and thus is finitary, for all sufficiently large n.

Next, for each $0 \leq l \leq r$, consider the functor $\mathrm{Ext}^*_{\mathbb{Z}G}(C_l(Y), -)$. Provided that $n \geq 1$, we see that

$$\mathrm{Ext}^n_{\mathbb{Z}G}(C_l(Y), -) \cong \mathrm{Ext}^n_{\mathbb{Z}G}(C_l(|\Lambda(G)|), -)$$

as the copies of G that we have added in the construction of Y have free orbits, and so the free-abelian group on them is a free module. Now,

$$\mathrm{Ext}^n_{\mathbb{Z}G}(C_l(|\Lambda(G)|), -) \cong \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}|\Lambda(G)|_l, -),$$

where $|\Lambda(G)|_l$ consists of all the l-simplicies

$$K_0 < K_1 < \cdots < K_l$$

in $|\Lambda(G)|$. As G acts on $|\Lambda(G)|_l$, we can therefore split $|\Lambda(G)|_l$ up into its G-orbits, where the stabilizer of such a simplex is $\bigcap_{i=0}^{l} N_G(K_i)$. We then obtain the following isomorphism:

$$\begin{aligned}
\operatorname{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}|\Lambda(G)|_l, -) &\cong \operatorname{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}[\textstyle\coprod_{\mathscr{C}} \bigcap_{i=0}^l N_G(K_i)\backslash G], -) \\
&\cong \textstyle\prod_{\mathscr{C}} \operatorname{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}[\bigcap_{i=0}^l N_G(K_i)\backslash G], -) \\
&\cong \textstyle\prod_{\mathscr{C}} H^n(\bigcap_{i=0}^l N_G(K_i), -),
\end{aligned}$$

where the product is taken over a set $\mathscr{C}$ of representatives of conjugacy classes of non-trivial finite subgroups of G. Now, as G has finite virtual cohomological dimension, it follows from Lemma 2.1.4 that there are only finitely many conjugacy classes of finite subgroups, and so this product is finite.

Now, for each l-simplex $K_0 < \cdots < K_l$ we have

$$\bigcap_{i=0}^{l} N_G(K_i) \leq N_G(K_l).$$

Then, as $N_G(K_l)$ is finitely generated, it follows that $\bigcap_{i=0}^l N_G(K_i)$ is also finitely generated, and hence polycyclic-by-finite. Therefore, $\bigcap_{i=0}^l N_G(K_i)$ is of type FP_∞, and so by Corollary 1.2.17, $H^n(\bigcap_{i=0}^l N_G(K_i), -)$ is finitary. Thus $\operatorname{Ext}^n_{\mathbb{Z}G}(C_l(Y), -)$ is isomorphic to a finite product of finitary functors, and hence by Lemma 1.1.7 is finitary. As this holds for all $n \geq 1$, we see that $\operatorname{Ext}^*_{\mathbb{Z}G}(C_l(Y), -)$ is finitary in all sufficiently high dimensions.

We then conclude from Lemma 1.4.1 that G has cohomology almost everywhere finitary. □

This completes our proof of Theorem 2.1.5, and hence of Theorem A.

2.1.4 Proof of Corollary B

Corollary B. *Let G be a locally (polycyclic-by-finite) group. If G has cohomology almost everywhere finitary, then every subgroup of G has cohomology almost everywhere finitary.*

Proof. As G has cohomology almost everywhere finitary, it follows from Theorem A that G has finite virtual cohomological dimension and the normalizer of every non-trivial finite subgroup of G is finitely generated.

Let H be any subgroup of G, so

$$\operatorname{vcd} H \leq \operatorname{vcd} G < \infty.$$

Also, let F be a non-trivial finite subgroup of H. Then $N_G(F)$ is finitely generated, hence polycyclic-by-finite, and as

$$N_H(F) \leq N_G(F),$$

we see that $N_H(F)$ is also finitely generated. Therefore, we conclude from Theorem A that H has cohomology almost everywhere finitary. □

This result does not hold in general, however, as the following proposition shows:

Proposition 2.1.22. *Let G be a group of type* FP_∞ *which has an infinitely generated subgroup H, and let Q be a non-trivial finite group. Then $G \times Q$ has cohomology almost everywhere finitary, but $H \times Q$ does not.*

Proof. As G is of type FP_∞, it follows that $G \times Q$ is also of type FP_∞, and so has cohomology almost everywhere finitary. However, as H is infinitely generated, it follows from Proposition 2.1.10 that $H^n(H \times Q, -)$ is not finitary for any n. □

Remark 2.1.23. Let G be the free group on two generators x, y, so G is of type FP_∞ (Example 2.6 in [6]), and let H be the subgroup of G generated by $y^n x y^{-n}$ for all n. We then have a counter-example showing that Corollary B does not hold in general.

2.2 Closure Properties

In this section we investigate some closure properties of the class of groups with cohomology almost everywhere finitary. We have already seen in §2.1.4 that this class is not closed under taking subgroups. Next, we give an example to show that it is not closed under extensions:

Example 2.2.1. The group of rationals $\mathbb{Q}$ has finite cohomological dimension, and so has cohomology almost everywhere finitary. Also, the group C_2 is finite, hence of type FP_∞, and so has cohomology almost everywhere finitary. However, as $\mathbb{Q}$ is infinitely generated, we see from Proposition 2.1.10 that $\mathbb{Q} \times C_2$ does not have cohomology almost everywhere finitary.

We now give an example to show that this class is not closed under taking quotients:

Example 2.2.2. The group $\mathbb{Q} \times \mathbb{Z}$ has finite cohomological dimension, and so has cohomology almost everywhere finitary. However, we have the epimorphism:

$$\mathbb{Q} \times \mathbb{Z} \twoheadrightarrow \mathbb{Q} \times (\mathbb{Z}/2\mathbb{Z}),$$

and again we see from Proposition 2.1.10 that $\mathbb{Q} \times (\mathbb{Z}/2\mathbb{Z})$ does not have cohomology almost everywhere finitary.

However, the class of groups with cohomology almost everywhere finitary is closed under forming certain free products with amalgamation and HNN-extensions, as we shall now show. Firstly, recall the following definition from §2.4 of [6].

Definition 2.2.3. Let G_1 and G_2 be groups with subgroups $S_1 \leq G_1$ and $S_2 \leq G_2$. Assume that S_1 and S_2 are isomorphic via an isomorphism $\theta : S_1 \to S_2$. Then the *amalgamated free product of G_1 and G_2 with amalgamated subgroups S_α* is defined to be

$$G = G_1 *_{S_1=S_2} G_2 := \langle G_1, G_2 | \operatorname{rel} G_1, \operatorname{rel} G_2, s = \theta(s) \text{ for all } s \in S_1 \rangle,$$

where $\operatorname{rel} G_\alpha$ denotes the relations in some chosen presentation of G_α.

Remark 2.2.4. There are obvious homomorphisms $j_\alpha : G_\alpha \to G$, $\alpha = 1, 2$, induced by the identity on G_α. From §2.4 of [6], we see that these maps are monomorphisms, and that

$$j_1(G_1) \cap j_2(G_2) = j_1(S_1) = j_2(S_2).$$

Therefore, we shall use the j_α as identifications; that is, we consider G_1 and G_2 as being subgroups of G, with

$$S = G_1 \cap G_2 = S_1 = S_2.$$

Proposition 2.2.5. *Let $G := G_1 *_S G_2$. If G_1, G_2 and S have cohomology almost everywhere finitary, then so does G.*

Proof. Theorem 2.10 in [6] gives the following long exact sequence of functors:

$$\cdots \to H^k(G, -) \to H^k(G_1, -) \oplus H^k(G_2, -) \to H^k(S, -) \to H^{k+1}(G, -) \to \cdots,$$

and as G_1, G_2 and S have cohomology almost everywhere finitary, the result now follows from Lemma 1.1.7. □

Next, recall the following definition from §2.5 of [6]:

Definition 2.2.6. Let G be a group with isomorphic subgroups S, T and let $\sigma : S \to T$ be a given isomorphism. The *HNN-group $G^* = G*_{S,\sigma}$ over the base group G with associated subgroups S, T and stable letter p* is defined to be

$$G^* := \langle G, p | \operatorname{rel} G, psp^{-1} = \sigma(s) \text{ for all } s \in S \rangle.$$

Remark 2.2.7. From §2.5 of [6], we see that the obvious homomorphism $j : G \to G^*$ is a monomorphism. We can therefore consider G as a subgroup of G^*.

Proposition 2.2.8. *Let $G^* = G*_{S,\sigma}$. If G and S have cohomology almost everywhere finitary, then so does G^*.*

Proof. Theorem 2.12 in [6] gives the following long exact sequence of functors:

$$\cdots \to H^{k-1}(S, -) \to H^k(G^*, -) \to H^k(G, -) \to H^k(S, -) \to \cdots ,$$

and as G and S have cohomology almost everywhere finitary, the result now follows from Lemma 1.1.7. □

2.3 Groups of Finite Virtual Cohomological Dimension Over a Ring of Prime Characteristic

In this section we shall work over a ring R of prime characteristic p, instead of over $\mathbb{Z}$. The goal is to prove the following characterisation of the groups of finite virtual cohomological dimension over R with cohomology almost everywhere finitary over R:

Theorem C. *Let R be a ring of prime characteristic p, and let G be a group of finite virtual cohomological dimension over R. Then the following are equivalent:*

(i) G has cohomology almost everywhere finitary over R;

(ii) G has finitely many conjugacy classes of elementary abelian p-subgroups, and the normalizer of every non-trivial elementary abelian p-subgroup of G is of type FP_∞ *over R; and*

(iii) G has finitely many conjugacy classes of elementary abelian p-subgroups, and the normalizer of every non-trivial elementary abelian p-subgroup of G has cohomology almost everywhere finitary over R.

Recall from [23] that an *elementary abelian p-group* is one which is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^n$ for some natural number n.

Remark 2.3.1. Note that it suffices to prove Theorem C for the case $R = \mathbb{F}_p$, by Lemma 1.4.3 and Remark 1.4.4.

2.3.1 Proof of Theorem C (i) $\Rightarrow$ (ii)

Let G be a group of finite virtual cohomological dimension over $\mathbb{F}_p$ with cohomology almost everywhere finitary over $\mathbb{F}_p$. We begin by showing that the normalizer of every non-trivial elementary abelian p-subgroup of G is of type FP_∞ over $\mathbb{F}_p$. In fact, we shall show that the normalizer of every non-trivial p-subgroup of G is of type FP_∞ over $\mathbb{F}_p$. We begin with the following straightforward lemma:

Lemma 2.3.2. *Let N be any group, and Q be a non-trivial finite group whose order is divisible by p. If $N \times Q$ has cohomology almost everywhere finitary over $\mathbb{F}_p$, then $N \times Q$ is of type FP_∞ over $\mathbb{F}_p$.*

Proof. Suppose that $N \times Q$ is not of type FP_∞ over $\mathbb{F}_p$, so N is not of type FP_∞ over $\mathbb{F}_p$. Therefore, there is some n such that $H^n(N, -)$ is not finitary over $\mathbb{F}_p$.

Let E be a subgroup of Q of order p, so by an argument similar to the proof of Proposition 2.1.10 we obtain, for each m, the following isomorphism of functors:

$$H^m(N \times E, -) \cong \bigoplus_{i=0}^{m} H^i(N, -),$$

for modules on which E acts trivially.

As $H^n(N, -)$ is not finitary over $\mathbb{F}_p$, it follows from Lemma 1.1.8 that $H^m(N \times E, -)$ is not finitary over $\mathbb{F}_p$ for all $m \geq n$. Therefore, by Proposition 1.4.6, we see that $H^m(N \times Q, -)$ is not finitary over $\mathbb{F}_p$ for all $m \geq n$, which is a contradiction. $\square$

Now, in order to show that the normalizer of every non-trivial p-subgroup of G is of type FP_∞ over $\mathbb{F}_p$, we require an analogue of Proposition 2.1.11. To prove such a result, we need to construct a G-CW-complex with certain properties. We begin by recalling from §I.4 in [9] that if Y is a CW-complex, and $p : X \to Y$ is the universal cover of Y, then X inherits a CW-structure together with an action of $\pi_1(Y)$ which permutes the cells. Explicitly, the open cells of X lying over an open cell σ of Y are simply the connected components of $p^{-1}\sigma$; these cells are permuted freely and transitively by $\pi_1(Y)$, and each is mapped homeomorphically onto σ under p. Thus X is a free $\pi_1(Y)$-CW-complex and, for each $n \geq 0$, the nth cellular chain group $C_n(X)$ is a free $\mathbb{Z}[\pi_1(Y)]$-module with one basis element for each n-cell of Y. Also, recall from §1.2 of [22] that the fundamental group of a CW-complex Y depends only on its 2-skeleton.

Proposition 2.3.3. *Let G be a group such that $\operatorname{cd}_{\mathbb{F}_p} G < \infty$, and let $n := \max\{\operatorname{cd}_{\mathbb{F}_p} G, 3\}$. Then there exists an n-dimensional free G-CW-complex X such that*

$$0 \to C_n(X) \otimes \mathbb{F}_p \to \cdots \to C_0(X) \otimes \mathbb{F}_p \to \mathbb{F}_p \to 0$$

is an exact sequence of $\mathbb{F}_pG$-modules.

Proof. This is a generalisation of the proof of Theorem 7.1 §VIII in [9]:

We construct the skeleta of the desired complex X inductively. Firstly, let Y^2 be the 2-complex associated to some presentation of G (see Exercise 2 §II.5 in [9]), so $\pi_1(Y^2) \cong G$. Also, let X^2 denote the universal cover of Y^2, so X^2 is simply-connected, and so by the Hurewicz Theorem (Theorem 4.32 in [22]) has reduced homology equal to zero in dimensions 0 and 1.

Now assume inductively that Y^{k-1} has been constructed, and that its universal cover X^{k-1} has reduced homology equal to zero in dimensions $0, \dots, k-2$. Choose a set of generators (z_α) for the $\mathbb{Z}G$-module $\widetilde{H}_{k-1}(X^{k-1})$. Again by the Hurewicz Theorem, we can find for each α a map

$$f_\alpha : S^{k-1} \to X^{k-1}$$

which represents z_α in the sense that

$$\widetilde{H}_{k-1}(f_\alpha) : \widetilde{H}_{k-1}(S^{k-1}) \to \widetilde{H}_{k-1}(X^{k-1})$$

sends a generator of $\widetilde{H}_{k-1}(S^{k-1})$ to z_α. We now set

$$Y^k := Y^{k-1} \cup \bigcup_\alpha e^k_\alpha,$$

where each k-cell e^k_α is attached to Y^{k-1} via the composite

$$S^{k-1} \xrightarrow{f_\alpha} X^{k-1} \to Y^{k-1}.$$

Let X^k be the universal cover of Y^k. We can view X^{k-1} as the $(k-1)$-skeleton of X^k; indeed, X^k is obtained from X^{k-1} by attaching k-cells via the maps f_α and their transforms under the action of G on X^{k-1}. Following the proof of Theorem 7.1 §VIII in [9], we see that X^k has reduced homology equal to zero in dimensions $0, \dots, k-1$.

Next, let $n := \max\{\operatorname{cd}_{\mathbb{F}_p} G, 3\}$, and consider X^{n-1}. By induction, X^{n-1} has reduced homology equal to zero in dimensions $0, \dots, n-2$, and so the augmented cellular chain complex of X^{n-1} gives the following exact sequence of $\mathbb{Z}G$-modules:

$$0 \to \widetilde{H}_{n-1}(X^{n-1}) \to C_{n-1}(X^{n-1}) \to \cdots \to C_0(X^{n-1}) \to \mathbb{Z} \to 0.$$

As this sequence is $\mathbb{Z}$-split, tensoring with $\mathbb{F}_p$ gives the following exact sequence of $\mathbb{F}_pG$-modules:

$$0 \to \widetilde{H}_{n-1}(X^{n-1}) \otimes \mathbb{F}_p \to C_{n-1}(X^{n-1}) \otimes \mathbb{F}_p \to \cdots \to C_0(X^{n-1}) \otimes \mathbb{F}_p \to \mathbb{F}_p \to 0,$$

and as $\operatorname{cd}_{\mathbb{F}_p} G \leq n$, it follows from Lemma 2.1 §VIII in [9] that $\widetilde{H}_{n-1}(X^{n-1}) \otimes \mathbb{F}_p$ is a projective $\mathbb{F}_pG$-module. Therefore, by the Eilenberg trick (Lemma 2.7 §VIII in [9]) we can choose a free $\mathbb{F}_pG$-module $(\mathbb{F}_pG)^\beta$ such that

$$(\widetilde{H}_{n-1}(X^{n-1}) \otimes \mathbb{F}_p) \oplus (\mathbb{F}_pG)^\beta \cong (\mathbb{F}_pG)^\beta.$$

We then replace Y^{n-1} by

$$\overline{Y}^{n-1} := Y^{n-1} \vee S^{n-1} \vee S^{n-1} \vee \cdots ,$$

where there is one copy of S^{n-1} for each basis element of $(\mathbb{F}_pG)^\beta$. The effect of this on the augmented cellular chain complex $\widetilde{C}_*(X^{n-1})$ is simply to add the free $\mathbb{Z}G$-module $(\mathbb{Z}G)^\beta$ to $C_{n-1}(X^{n-1})$ with $\partial|_{(\mathbb{Z}G)^\beta} = 0$.

The universal cover $\overline{X}^{n-1}$ now has

$$\widetilde{H}_{n-1}(\overline{X}^{n-1}) \cong \widetilde{H}_{n-1}(X^{n-1}) \oplus (\mathbb{Z}G)^\beta,$$

and so

$$\widetilde{H}_{n-1}(\overline{X}^{n-1}) \otimes \mathbb{F}_p \cong (\widetilde{H}_{n-1}(X^{n-1}) \otimes \mathbb{F}_p) \oplus (\mathbb{F}_pG)^\beta \cong (\mathbb{F}_pG)^\beta.$$

We can then attach n-cells e^n_β to $\overline{Y}^{n-1}$, corresponding to basis elements z_β of $(\mathbb{Z}G)^\beta$, to obtain $\overline{Y}^n$. Let $\overline{X}^n$ denote the universal cover, so we see that

$$C_n(\overline{X}^n) \cong (\mathbb{Z}G)^\beta,$$

and hence that

$$C_n(\overline{X}^n) \otimes \mathbb{F}_p \cong (\mathbb{F}_pG)^\beta \cong \widetilde{H}_{n-1}(\overline{X}^{n-1}) \otimes \mathbb{F}_p.$$

This gives the following exact sequence of $\mathbb{F}_pG$-modules:

$$0 \to C_n(\overline{X}^n) \otimes \mathbb{F}_p \to \cdots \to C_0(\overline{X}^n) \otimes \mathbb{F}_p \to \mathbb{F}_p \to 0,$$

and we conclude that $X := \overline{X}^n$ is an n-dimensional free G-CW-complex with the required properties. $\square$

Next, recall from §I of [9] that if (C, d) (resp. (C', d')) is a chain complex of right (resp. left) R-modules, then their *tensor product* $C \otimes_R C'$ is defined as

$$(C \otimes_R C')_n := \bigoplus_{p+q=n} C_p \otimes_R C'_q,$$

with differential D given by

$$D(c \otimes c') := dc \otimes c' + (-1)^{\deg c} c \otimes d'c' \quad \text{for } c \in C,\, c' \in C'.$$

We can now prove the following:

Proposition 2.3.4. *Suppose that* $\operatorname{vcd}_{\mathbb{F}_p} G < \infty$*. Then there exists a finite dimensional G-CW-complex X with cell stabilizers finite of bounded order, such that*

$$0 \to C_n(X) \otimes \mathbb{F}_p \to \cdots \to C_0(X) \otimes \mathbb{F}_p \to \mathbb{F}_p \to 0$$

is an exact sequence of $\mathbb{F}_p G$*-modules.*

Proof. This is a generalisation of the proof of Theorem 3.1 §VIII in [9]:

As $\operatorname{vcd}_{\mathbb{F}_p} G < \infty$, it has a subgroup H of finite index such that $\operatorname{cd}_{\mathbb{F}_p} H < \infty$. It then follows from Proposition 2.3.3 that there exists a finite-dimensional free H-CW-complex X' such that $\widetilde{C}_*(X') \otimes \mathbb{F}_p$ is exact. Let $X := \operatorname{Hom}_H(G, X')$, where $\operatorname{Hom}_H(-,-)$ denotes maps in the category of H-sets, so we have an induced action of G on X. If we choose a set of coset representatives $g_1, \ldots, g_m$ for $H \backslash G$, we then obtain a bijection

$$\phi : X \to \prod_{i=1}^{m} X',$$

given by evaluation at $g_1, \ldots, g_m$. Since the product on the right has a natural CW-structure, we can use ϕ to give X a topology and a CW-structure. Following the proof of Theorem 3.1 §VIII in [9], we see that this structure is independent of the choice of coset representatives, and also of the ordering of the cosets. Furthermore, we see that the action of G on X preserves the CW-structure, and so X is a well-defined finite-dimensional G-CW-complex.

Next, notice that

$$\widetilde{C}_*(X) \cong \widetilde{C}_*(X' \times \cdots \times X') \cong \widetilde{C}_*(X') \otimes \cdots \otimes \widetilde{C}_*(X').$$

(see §3.B of [22]). Therefore,

$$\begin{aligned}\widetilde{C}_*(X)\otimes\mathbb{F}_p &\cong \widetilde{C}_*(X)\otimes(\mathbb{F}_p\otimes\cdots\otimes\mathbb{F}_p)\\ &\cong (\widetilde{C}_*(X')\otimes\cdots\otimes\widetilde{C}_*(X'))\otimes(\mathbb{F}_p\otimes\cdots\otimes\mathbb{F}_p)\\ &\cong (\widetilde{C}_*(X')\otimes\mathbb{F}_p)\otimes\cdots\otimes(\widetilde{C}_*(X')\otimes\mathbb{F}_p)\end{aligned}$$

is the tensor product of finitely many exact chain complexes, and hence is exact (Proposition 0.8 §I in [9]).

Finally, notice that there is a canonical map $X \to X'$, given by evaluation at $1 \in G$. This map is H-equivariant and takes cells to cells. Since H acts freely on X', it follows that H acts freely on X. Thus for any cell σ of X, we then have $G_\sigma \cap H = \{1\}$, and hence

$$|G_\sigma| = |G_\sigma : G_\sigma \cap H| = |HG_\sigma : H| \leq |G : H| < \infty,$$

so each G_σ is finite with order bounded by $|G : H|$. □

We are now able to prove the required analogue of Proposition 2.1.11:

Lemma 2.3.5. *Let G be a group such that $\mathrm{vcd}_{\mathbb{F}_p} G < \infty$, and M be an $\mathbb{F}_pG$-module. If M is projective as an $\mathbb{F}_pK$-module for all finite subgroups K of G, then M has finite projective dimension over $\mathbb{F}_pG$.*

Proof. We see from Proposition 2.3.4 that there exists a finite-dimensional G-CW-complex X with finite cell stabilizers, such that

$$0 \to C_n(X)\otimes\mathbb{F}_p \to \cdots \to C_0(X)\otimes\mathbb{F}_p \to \mathbb{F}_p \to 0$$

is exact. Now, for each k, $C_k(X)$ is a permutation module,

$$C_k(X) \cong \bigoplus_{\sigma\in\Sigma_k} \mathbb{Z}[G_\sigma\backslash G],$$

where Σ_k is a set of G-orbit representatives of k-cells in X, and G_σ is the stabilizer of σ. Therefore,

$$\begin{aligned}C_k(X)\otimes\mathbb{F}_p &\cong \textstyle\bigoplus_{\sigma\in\Sigma_k}\mathbb{F}_p[G_\sigma\backslash G]\\ &\cong \textstyle\bigoplus_{\sigma\in\Sigma_k}\mathbb{F}_p\otimes_{\mathbb{F}_pG_\sigma}\mathbb{F}_pG.\end{aligned}$$

As this sequence is $\mathbb{F}_p$-split, tensoring with M gives the following exact sequence:

$$0 \to M\otimes_{\mathbb{F}_p}(C_n(X)\otimes\mathbb{F}_p) \to \cdots \to M\otimes_{\mathbb{F}_p}(C_0(X)\otimes\mathbb{F}_p) \to M \to 0.$$

Then, for each k, we have

$$M \otimes_{\mathbb{F}_p} (C_k(X) \otimes \mathbb{F}_p) \cong \bigoplus_{\sigma \in \Sigma_k} M \otimes_{\mathbb{F}_p G_\sigma} \mathbb{F}_p G,$$

and as M is projective as an $\mathbb{F}_p G_\sigma$-module, we see that $M \otimes_{\mathbb{F}_p G_\sigma} \mathbb{F}_p G$ is projective as an $\mathbb{F}_p G$-module. Therefore, $M \otimes_{\mathbb{F}_p} (C_k(X) \otimes \mathbb{F}_p)$ is projective as an $\mathbb{F}_p G$-module, and so the above exact sequence is a projective resolution of M, and we then conclude that M has finite projective dimension over $\mathbb{F}_p G$. □

Next, recall (see §3 of [30]) that two subgroups H and K of a group G are said to be *commensurable* if and only if $H \cap K$ has finite index in both H and K.

Lemma 2.3.6. *If H and K are commensurable subgroups of a group G, then H is of type* FP_∞ *over $\mathbb{F}_p$ if and only if K is.*

Proof. This follows from the fact that the property of type FP_∞ passes to subgroups and supergroups of finite index. □

In particular, if F is a finite subgroup of G, then $C_G(F)$ and $N_G(F)$ are commensurable, and so we can make use of the fact that $C_G(F)$ is of type FP_∞ over $\mathbb{F}_p$ if and only if $N_G(F)$ is.

We are now almost ready to prove that if G is a group of finite virtual cohomological dimension over $\mathbb{F}_p$ with cohomology almost everywhere finitary over $\mathbb{F}_p$, then the normalizer of every non-trivial p-subgroup of G is of type FP_∞ over $\mathbb{F}_p$. We first need to recall Chouinard's Theorem [11]:

Proposition 2.3.7. *Let G be a finite group, k be a field of characteristic $p > 0$, and M be a kG-module. Then M is projective as a kG-module if and only if M is projective as a kE-module for all elementary abelian p-subgroups E of G.*

We can now prove our result for subgroups of order p:

Lemma 2.3.8. *Let G be a group of finite virtual cohomological dimension over $\mathbb{F}_p$ with cohomology almost everywhere finitary over $\mathbb{F}_p$, and let P be a subgroup of order p. Then every subgroup of G commensurable with $N_G(P)$ is of type* FP_∞ *over $\mathbb{F}_p$.*

Proof. As G has finite virtual cohomological dimension over $\mathbb{F}_p$, we can choose a normal subgroup N of finite index such that $\mathrm{cd}_{\mathbb{F}_p} N < \infty$. It follows from Proposition 4.11 in [6] that N has no p-torsion. Let $H := NP$, so H also has cohomology almost everywhere finitary over $\mathbb{F}_p$.

Let $\mathcal{A}_p(H)$ denote the set of all non-trivial elementary abelian p-subgroups of H, so $\mathcal{A}_p(H)$ consists of subgroups of order p. Now H acts on this set by conjugation, so the stabilizer of any $E \in \mathcal{A}_p(H)$ is simply $N_H(E)$. Also, for each $E \in \mathcal{A}_p(H)$, we see that the set of E-fixed points $\mathcal{A}_p(H)^E$ is simply the set $\{E\}$.

We then have the following short exact sequence of $\mathbb{F}_pH$-modules:

$$0 \to J \to \mathbb{F}_p\mathcal{A}_p(H) \xrightarrow{\varepsilon} \mathbb{F}_p \to 0,$$

where ε denotes the augmentation map, and we see for each $E \in \mathcal{A}_p(H)$, J is free as an $\mathbb{F}_pE$-module with basis $\{E' - E : E' \in \mathcal{A}_p(H)\}$. Therefore, if K is any finite subgroup of H, we see that J restricted to K is an $\mathbb{F}_pK$-module such that its restriction to every elementary abelian p-subgroup of K is free. It then follows from Chouinard's Theorem that J is projective as an $\mathbb{F}_pK$-module. As this holds for every finite subgroup K of H, it follows from Lemma 2.3.5 that J has finite projective dimension over $\mathbb{F}_pH$.

An argument similar to the proof of Theorem 2.1.15 then shows that $N_H(P)$ has cohomology almost everywhere finitary over $\mathbb{F}_p$. Hence,

$$C_H(P) \cong P \times C_N(P)$$

has cohomology almost everywhere finitary over $\mathbb{F}_p$, and by Lemma 2.3.2, $C_N(P)$ is of type FP_∞ over $\mathbb{F}_p$. Thus, $N_G(P)$ is of type FP_∞ over $\mathbb{F}_p$, and the result now follows. □

Next, we need the following result:

Proposition 2.3.9. *Let*

$$0 \to N \to G \to Q \to 0$$

be a short exact sequence of groups, such that N is of type FP_∞ *over $\mathbb{F}_p$. Then G is of type* FP_n *over $\mathbb{F}_p$, $0 \le n \le \infty$, if and only if Q is.*

Proof. This is Proposition 2.7 in [6]. □

We can finally prove our key result:

Theorem 2.3.10. *Let G be a group of finite virtual cohomological dimension over $\mathbb{F}_p$ with cohomology almost everywhere finitary over $\mathbb{F}_p$, and let P be a non-trivial p-subgroup. Then every subgroup of G commensurable with $N_G(P)$ is of type* FP_∞ *over $\mathbb{F}_p$.*

Proof. Suppose that P has order p^k, where $k \geq 1$. We proceed by induction on k.

If $k = 1$, then the result follows from Lemma 2.3.8.

Suppose now that $k \geq 2$. As the centre $\zeta(P)$ of P is non-trivial (Theorem 4.3.1 in [18]), we can choose a subgroup $E \leq \zeta(P)$ of order p. Then $C_G(E)$ is of type FP_∞ over $\mathbb{F}_p$ by Lemma 2.3.8, and Proposition 2.3.9 shows that $C_G(E)/E$ is also of type FP_∞ over $\mathbb{F}_p$. By induction, the normalizer of P/E in $C_G(E)/E$, which is

$$(N_G(P) \cap C_G(E))/E,$$

is of type FP_∞ over $\mathbb{F}_p$. Another application of Proposition 2.3.9 shows that $N_G(P) \cap C_G(E)$ is of type FP_∞ over $\mathbb{F}_p$, and as

$$C_G(P) \leq N_G(P) \cap C_G(E) \leq N_G(P),$$

we see that $N_G(P)$ is of type FP_∞ over $\mathbb{F}_p$. The result now follows. □

In [34], Leary and Nucinkis posed the following question: "If G is a group of type VFP over $\mathbb{F}_p$, and P is a p-subgroup of G, is the centralizer $C_G(P)$ of P necessarily of type VFP over $\mathbb{F}_p$?" Recall (see §2 of [34]) that a group G is said to be of type VFP over $\mathbb{F}_p$ if and only if it has a subgroup of finite index which is of type FP over $\mathbb{F}_p$. We are now in a position to answer their question.

Corollary 2.3.11. *Let G be a group of type* VFP *over $\mathbb{F}_p$, and let P be a p-subgroup of G. Then $C_G(P)$ is also of type* VFP *over $\mathbb{F}_p$.*

Proof. If P is trivial, then the result is immediate. Assume, therefore, that P is non-trivial. As $\mathrm{vcd}_{\mathbb{F}_p} G < \infty$, we see from Theorem 2.3.10 that $C_G(P)$ is of type FP_∞ over $\mathbb{F}_p$. Then, as G has a subgroup H of finite index such that $\mathrm{cd}_{\mathbb{F}_p} H < \infty$, we see that $C_H(P)$ is of type FP_∞ over $\mathbb{F}_p$, and that

$$\mathrm{cd}_{\mathbb{F}_p} C_H(P) \leq \mathrm{cd}_{\mathbb{F}_p} H < \infty.$$

Hence, by Proposition 1.2.7 we see that $C_H(P)$ is of type FP over $\mathbb{F}_p$, and so $C_G(P)$ is of type VFP over $\mathbb{F}_p$, as required. □

In the remainder of this section, we shall show that if G is a group of finite virtual cohomological dimension over $\mathbb{F}_p$ with cohomology almost everywhere finitary over $\mathbb{F}_p$, then G has finitely many conjugacy classes of elementary abelian p-subgroups. Firstly, recall the Universal Coefficient Theorem (see [17]):

Theorem 2.3.12. *Let R be a hereditary ring. Then, for all $n \geq 1$ and all trivial RG-modules T, we have the following isomorphism:*

$$H^n(G, T) \cong \mathrm{Hom}_R(H_n(G, R), T) \oplus \mathrm{Ext}^1_R(H_{n-1}(G, R), T).$$

Lemma 2.3.13. *Let G be a group. If $H^n(G, -)$ is finitary over $\mathbb{F}_p$, then $H^n(G, \mathbb{F}_p)$ is finite-dimensional as an $\mathbb{F}_p$-vector space.*

Proof. Suppose that $H^n(G, \mathbb{F}_p)$ is infinite-dimensional as an $\mathbb{F}_p$-vector space. As $\mathbb{F}_p$ is hereditary, we can apply the Universal Coefficient Theorem, and as every $\mathbb{F}_p$-module is free, the Ext-term vanishes to give the following isomorphism:

$$H^n(G, \mathbb{F}_p) \cong \mathrm{Hom}_{\mathbb{F}_p}(H_n(G, \mathbb{F}_p), \mathbb{F}_p),$$

so $H^n(G, \mathbb{F}_p)$ is the dual vector space of $H_n(G, \mathbb{F}_p)$. Hence, $H_n(G, \mathbb{F}_p)$ is also infinite-dimensional, with basis $\{e_i : i \in I\}$, say. It then follows that

$$H^n(G, \mathbb{F}_p) \cong \prod_I \mathbb{F}_p.$$

Next, let $\bigoplus_J \mathbb{F}_p$ be an infinite direct sum of copies of $\mathbb{F}_p$. As G acts trivially on $\bigoplus_J \mathbb{F}_p$, it follows from the Universal Coefficient Theorem that

$$H^n(G, \bigoplus_J \mathbb{F}_p) \cong \mathrm{Hom}_{\mathbb{F}_p}(H_n(G, \mathbb{F}_p), \bigoplus_J \mathbb{F}_p) \cong \prod_I \bigoplus_J \mathbb{F}_p.$$

Finally, as $H^n(G, -)$ is finitary over $\mathbb{F}_p$, the natural map

$$\bigoplus_J H^n(G, \mathbb{F}_p) \to H^n(G, \bigoplus_J \mathbb{F}_p)$$

is an isomorphism; that is,

$$\bigoplus_J \prod_I \mathbb{F}_p \cong \prod_I \bigoplus_J \mathbb{F}_p,$$

which is clearly a contradiction. □

Henn has shown in Theorem A.8 of [23] that if G is an $\mathbf{H}_1\mathfrak{F}$-group of type FP_∞, then there are only finitely many conjugacy classes of elementary abelian p-subgroups. We shall adapt Henn's argument to our case. We begin by recalling the definition of a uniform F-isomorphism from [23]:

Definition 2.3.14. A homomorphism $\phi : A \to B$ of $\mathbb{F}_p$-algebras is called a *uniform F-isomorphism* if and only if there exists a natural number n such that:

- If $x \in \operatorname{Ker}\phi$, then $x^{p^n} = 0$; and
- If $y \in B$, then y^{p^n} is in the image of ϕ.

Proposition 2.3.15. *If G is a discrete group such that there exists a finite-dimensional G-CW-complex X with all cell stabilizers finite of bounded order, then there exists a uniform F-isomorphism*

$$\phi : H^*(G, \mathbb{F}_p) \to \lim_{\mathcal{A}_p(G)^{\mathrm{op}}} H^*(E, \mathbb{F}_p),$$

where $\mathcal{A}_p(G)$ is the category with objects the elementary abelian p-subgroups E of G, and morphisms the group homomorphisms which can be induced by conjugation by an element of G.

Proof. This is a special case of Theorem A.4 in [23]. □

We can now prove the following:

Proposition 2.3.16. *Let G be a group of finite virtual cohomological dimension over $\mathbb{F}_p$ with cohomology almost everywhere finitary over $\mathbb{F}_p$. Then G has finitely many conjugacy classes of elementary abelian p-subgroups.*

Proof. This is a generalisation of Theorem A.8 in [23]:

As $\operatorname{vcd}_{\mathbb{F}_p} G < \infty$, we see from Proposition 2.3.4 that there exists a finite-dimensional G-CW-complex X with cell stabilizers finite of bounded order. It then follows from Proposition 2.3.15 that there is a uniform F-isomorphism

$$\phi : H^*(G, \mathbb{F}_p) \to \lim_{\mathcal{A}_p(G)^{\mathrm{op}}} H^*(E, \mathbb{F}_p).$$

Now assume that there are infinitely many conjugacy classes of elementary abelian p-subgroups of G. As $\operatorname{vcd}_{\mathbb{F}_p} G < \infty$, there is a bound on the orders of the p-subgroups, and so there must be infinitely many maximal elementary abelian p-subgroups of G of the same rank k (although k itself need not necessarily be maximal). Following Henn's argument, we can use this fact to construct infinitely many linearly independent non-nilpotent classes in the inverse limit in some degree (for the details, see the proof of Theorem A.8 in [23]). Now, raising these to a large enough power and using the fact that ϕ is a uniform F-isomorphism, we see that $H^*(G, \mathbb{F}_p)$ is infinite-dimensional as an $\mathbb{F}_p$-vector space in some degree m such that $H^m(G, -)$ is finitary over $\mathbb{F}_p$. This gives a contradiction to Lemma 2.3.13. □

2.3.2 Proof of Theorem C (ii) ⇒ (iii)

This is immediate.

2.3.3 Proof of Theorem C (iii) ⇒ (i)

Let G be a group of finite virtual cohomological dimension over $\mathbb{F}_p$, such that G has finitely many conjugacy classes of elementary abelian p-subgroups and the normalizer of every non-trivial elementary abelian p-subgroup of G has cohomology almost everywhere finitary over $\mathbb{F}_p$. We shall show that G has cohomology almost everywhere finitary over $\mathbb{F}_p$. We begin by recalling the notion of a simplicial complex $|X|$ associated to a poset X (see [46]). This is a generalisation of the construction discussed in Definition 2.1.18.

Definition 2.3.17. If X is a poset, then we denote by $|X|$ the simplicial complex whose n-simplicies are chains

$$x_0 < x_1 < \cdots < x_n$$

in X. If X is a G-poset (that is, a poset together with an order-preserving action of G), then $|X|$ inherits the structure of a G-simplicial complex.

Remark 2.3.18. We are mainly interested in the following posets: If G is a group, let $\mathcal{A}_p(G)$ denote the poset of all non-trivial elementary abelian p-subgroups of G, and let $\mathcal{S}_p(G)$ denote the poset of all non-trivial finite p-subgroups of G. We see from Remark 2.3(i) in [46] that the inclusion of posets $\mathcal{A}_p(G) \hookrightarrow \mathcal{S}_p(G)$ induces a G-homotopy equivalence

$$|\mathcal{A}_p(G)| \simeq_G |\mathcal{S}_p(G)|$$

between the G-simplicial complexes.

Proposition 2.3.19. *Let X and Y be G-CW-complexes, and $\phi : X \to Y$ be a G-equivariant cellular map. Then ϕ is a G-homotopy equivalence if and only if $\phi^H : X^H \to Y^H$ is a homotopy equivalence for all subgroups H of G.*

Proof. This is Proposition 2.7 §II in [7]. □

Next, we have the following definition of Quillen (see [40]):

Definition 2.3.20. A poset X is said to be *conically contractible* if and only if there is some $x_0 \in X$ and a poset map $f : X \to X$ such that $x \leq f(x) \geq x_0$ for all $x \in X$.

Lemma 2.3.21. *Let X be a poset. If X is conically contractible, then the simplicial complex $|X|$ is contractible.*

Proof. Suppose that X is conically contractible, so there is an $x_0 \in X$ and a poset map $f : X \to X$ such that $x \leq f(x) \geq x_0$ for all $x \in X$. From [40], we see that if $\phi, \psi : Y \to Y'$ are poset maps such that $\phi(y) \leq \psi(y)$ for all $y \in Y$, then $|\phi|$ and $|\psi|$ are homotopic, where $|\phi|$ denotes the simplicial map induced by ϕ. It therefore follows that the maps $|\operatorname{id}_x|$, $|f|$ and the constant map with value x_0 from $|X|$ to itself are homotopic. Hence $|X|$ is contractible. □

As discussed in [40], if a group G acts on a poset X, then we have the formula

$$|X|^G = |X^G|$$

relating the fixed-points for the action of G on X and the action of G on the simplicial complex $|X|$. This is because a simplex of $|X|$ is carried onto itself by an element g if and only if all the vertices of the simplex are fixed under g. We can now prove the following key lemma:

Lemma 2.3.22. *The complex $|\mathcal{A}_p(G)|^E$ is contractible for all $E \in \mathcal{A}_p(G)$.*

Proof. We follow an argument similar to the proof of Lemma 2.1 in [32]:

If $H \in \mathcal{S}_p(G)^E$, then EH is a p-subgroup of G. We can therefore define a function

$$f : \mathcal{S}_p(G)^E \to \mathcal{S}_p(G)^E$$

by $f(H) = EH$, so for all $H \in \mathcal{S}_p(G)^E$ we have:

$$H \leq f(H) \geq E.$$

We then see that $\mathcal{S}_p(G)^E$ is conically contractible, and so, by Lemma 2.3.21, the complex $|\mathcal{S}_p(G)^E|$ is contractible. Finally, by Proposition 2.3.19, we see that

$$|\mathcal{A}_p(G)|^E \simeq |\mathcal{S}_p(G)|^E = |\mathcal{S}_p(G)^E|,$$

and the result now follows. ☐

We also require the following result:

Proposition 2.3.23. *Let Y be an n-dimensional G-CW-complex which is $(n-1)$-connected, for some $n \geq 0$. Suppose that Y^E is contractible for all non-trivial elementary abelian p-subgroups E of G. Then $\widetilde{H}_n(Y) \otimes \mathbb{F}_p$ is projective as an $\mathbb{F}_pE$-module for all elementary abelian p-subgroups E of G.*

Proof. This is a straightforward generalisation of Proposition 6.2 in [32]. ☐

We can now prove the following:

Theorem 2.3.24. *Let G be a group of finite virtual cohomological dimension over $\mathbb{F}_p$. If G has finitely many conjugacy classes of elementary abelian p-subgroups, and the normalizer of every non-trivial elementary abelian p-subgroup of G has cohomology almost everywhere finitary over $\mathbb{F}_p$, then G has cohomology almost everywhere finitary over $\mathbb{F}_p$.*

Proof. Let $\mathcal{A}_p(G)$ be the poset of all non-trivial elementary abelian p-subgroups of G, and let $|\mathcal{A}_p(G)|$ denote its realisation as a G-simplicial complex. As $\mathrm{vcd}_{\mathbb{F}_p} G < \infty$, there is a bound on the orders of the p-subgroups, and so $|\mathcal{A}_p(G)|$ is finite-dimensional, say $\dim |\mathcal{A}_p(G)| = r$. By Proposition 2.1.19, we can embed $|\mathcal{A}_p(G)|$ into an r-dimensional G-CW-complex Y which is $(r-1)$-connected, such that G acts freely outside $|\mathcal{A}_p(G)|$. The augmented cellular chain complex of Y then gives the following exact sequence of $\mathbb{Z}G$-modules:

$$0 \to \widetilde{H}_r(Y) \to C_r(Y) \to \cdots \to C_0(Y) \to \mathbb{Z} \to 0,$$

and as this sequence is $\mathbb{Z}$-split, tensoring with $\mathbb{F}_p$ gives the following exact sequence of $\mathbb{F}_pG$-modules:

$$0 \to \widetilde{H}_r(Y) \otimes \mathbb{F}_p \to C_r(Y) \otimes \mathbb{F}_p \to \cdots \to C_0(Y) \otimes \mathbb{F}_p \to \mathbb{F}_p \to 0.$$

By Lemma 1.4.1, it remains to show that the functors $\mathrm{Ext}^*_{\mathbb{F}_pG}(\widetilde{H}_r(Y) \otimes \mathbb{F}_p, -)$ and $\mathrm{Ext}^*_{\mathbb{F}_pG}(C_l(Y) \otimes \mathbb{F}_p, -)$, $0 \leq l \leq r$, are finitary in all sufficiently high dimensions.

Firstly, notice that for every $E \in \mathcal{A}_p(G)$, $Y^E = |\mathcal{A}_p(G)|^E$, and hence is contractible, as the copies of G we have added in the construction of Y have free orbits, and so have no fixed points under E. Therefore, Proposition 2.3.23 shows that $\widetilde{H}_r(Y) \otimes \mathbb{F}_p$ is projective as an $\mathbb{F}_pE$-module for all elementary abelian p-subgroups E of G.

Let K be a finite subgroup of G, so $\widetilde{H}_r(Y) \otimes \mathbb{F}_p$ restricted to K is an $\mathbb{F}_pK$-module with the property that its restriction to every elementary abelian p-subgroup of K is projective. It then follows from Chouinard's Theorem that $\widetilde{H}_r(Y) \otimes \mathbb{F}_p$ is projective as an $\mathbb{F}_pK$-module. As this holds for every finite subgroup K of G, it then follows from Proposition 2.3.5 that $\widetilde{H}_r(Y) \otimes \mathbb{F}_p$ has finite projective dimension over $\mathbb{F}_pG$. Hence $\mathrm{Ext}^n_{\mathbb{F}_pG}(\widetilde{H}_r(Y) \otimes \mathbb{F}_p, -) = 0$, and thus is finitary, for all sufficiently large n.

Next, for each $0 \leq l \leq r$, consider the functor $\mathrm{Ext}^*_{\mathbb{F}_pG}(C_l(Y) \otimes \mathbb{F}_p, -)$. Provided that $n \geq 1$, we see that

$$\begin{aligned} \mathrm{Ext}^n_{\mathbb{F}_pG}(C_l(Y) \otimes \mathbb{F}_p, -) &\cong \mathrm{Ext}^n_{\mathbb{F}_pG}(C_l(|\mathcal{A}_p(G)|) \otimes \mathbb{F}_p, -) \\ &\cong \mathrm{Ext}^n_{\mathbb{F}_pG}(\mathbb{F}_p|\mathcal{A}_p(G)|_l, -), \end{aligned}$$

where $|\mathcal{A}_p(G)|_l$ consists of all the l-simplicies

$$E_0 < E_1 < \cdots < E_l$$

in $|\mathcal{A}_p(G)|$. As G acts on $|\mathcal{A}_p(G)|_l$, we can therefore split $|\mathcal{A}_p(G)|_l$ up into its G-orbits, where the stabilizer of such a simplex is $\bigcap_{i=0}^l N_G(E_i)$. We then obtain the following isomorphism:

$$\begin{aligned} \mathrm{Ext}^n_{\mathbb{F}_pG}(\mathbb{F}_p|\mathcal{A}_p(G)|_l, -) &\cong \mathrm{Ext}^n_{\mathbb{F}_pG}(\mathbb{F}_p[\textstyle\coprod_{\mathscr{C}} \bigcap_{i=0}^l N_G(E_i)\backslash G], -) \\ &\cong \textstyle\prod_{\mathscr{C}} \mathrm{Ext}^n_{\mathbb{F}_pG}(\mathbb{F}_p[\bigcap_{i=0}^l N_G(E_i)\backslash G], -) \\ &\cong \textstyle\prod_{\mathscr{C}} H^n(\bigcap_{i=0}^l N_G(E_i), -), \end{aligned}$$

where the product is taken over a set $\mathscr{C}$ of representatives of conjugacy classes of non-trivial elementary abelian p-subgroups of G. As we are assuming that G has only finitely many such conjugacy classes, this product is finite.

Now, for each l-simplex $E_0 < E_1 < \cdots < E_l$ we have

$$C_G(E_l) \leq \bigcap_{i=0}^{l} N_G(E_i) \leq N_G(E_l),$$

and so

$$|N_G(E_l) : \bigcap_{i=0}^{l} N_G(E_i)| < \infty.$$

Then, as $N_G(E_l)$ has cohomology almost everywhere finitary over $\mathbb{F}_p$, we see from Proposition 1.4.6 that $\bigcap_{i=0}^{l} N_G(E_i)$ has cohomology almost everywhere finitary over $\mathbb{F}_p$, and so for all sufficiently large n, $H^n(\bigcap_{i=0}^{l} N_G(E_i), -)$ is finitary over $\mathbb{F}_p$. Therefore, for all sufficiently large n, $\mathrm{Ext}^n_{\mathbb{F}_pG}(C_l(Y) \otimes \mathbb{F}_p, -)$ is isomorphic to a finite product of finitary functors, and hence is finitary.

We then conclude from Lemma 1.4.1 that G has cohomology almost everywhere finitary over $\mathbb{F}_p$, as required. □

This completes our proof of Theorem C.

2.4 Groups in the Class $\mathbf{H}_1\mathfrak{F}$

In this short section we consider groups in the class $\mathbf{H}_1\mathfrak{F}$ with cohomology almost everywhere finitary. The proof of Theorem 2.1.21 generalizes immediately to give us the following:

Theorem 2.4.1. *Let G be a group in the class $\mathbf{H}_1\mathfrak{F}$ such that*

(i) G has finitely many conjugacy classes of finite subgroups; and

(ii) The normalizer of every non-trivial finite subgroup of G has cohomology almost everywhere finitary.

Then G has cohomology almost everywhere finitary.

However, the converse is false. In fact, it is false even for the subclass of groups of finite virtual cohomological dimension, as we shall now show. First, recall the following definition from §I.4 of [9]:

Definition 2.4.2. A CW-complex Y is called an *Eilenberg–Mac Lane space* $K(G, 1)$ if and only if

(i) Y is connected;

(ii) $\pi_1(Y) \cong G$; and

(iii) The universal cover of Y is contractible.

Next, recall (see [34]) that a group G is said to be of type F if and only if there is a finite Eilenberg–Mac Lane space $K(G, 1)$; that is, a finite-dimensional Eilenberg–Mac Lane space with only finitely many cells in each dimension.

We now have the following result of Leary (Theorem 20 in [33]):

Proposition 2.4.3. *Let Q be a finite group not of prime power order. Then there is a group H of type* F *and a group $G = H \rtimes Q$ such that G contains infinitely many conjugacy classes of subgroups isomorphic to Q and finitely many conjugacy classes of other finite subgroups.*

As H is of type F, it has a finite Eilenberg–Mac Lane space, say Y. As the universal cover X of Y is contractible, its augmented cellular chain complex is an exact sequence of $\mathbb{Z}H$-modules:

$$0 \to C_n(X) \to \cdots \to C_0(X) \to \mathbb{Z} \to 0,$$

and as Y has only finitely many cells in each dimension, we see that each $C_k(X)$ is finitely generated.

Hence, we see that H has finite cohomological dimension, and is of type FP_∞. Therefore, G is a group of finite virtual cohomological dimension which is of type FP_∞, and hence has cohomology almost everywhere finitary, but G does *not* have finitely many conjugacy classes of finite subgroups, which gives us a counter-example to the converse of Theorem 2.4.1 above.

Chapter 3

Group Actions on Spheres

In this chapter we change direction slightly, and show an interesting connection to the problem of group actions on spheres. Recall from [47] that a finite group acts freely and orthogonally on some sphere if and only if every subgroup of order pq, where p and q are prime, is cyclic. The goal of this section is to prove the following result:

Theorem D. *Let G be an infinitely generated locally (polycyclic-by-finite) group with cohomology almost everywhere finitary. Then every finite subgroup of G acts freely and orthogonally on some sphere.*

Notice that we cannot drop the "infinitely generated" restriction, as every finite group is of type FP_∞ and so by Corollary 1.2.17 has cohomology almost everywhere finitary.

We begin with the following key lemma:

Lemma 3.0.4. *Let Q be a non-cyclic group of order pq, where p and q are prime, and let A be a $\mathbb{Z}$-torsion-free $\mathbb{Z}Q$-module such that the group $A \rtimes Q$ has cohomology almost everywhere finitary. Then A is finitely generated.*

Proof. Let $G := A \rtimes Q$. For any $K \leq Q$, we write $\widehat{K}$ for the element of $\mathbb{Z}Q$ given by

$$\widehat{K} := \sum_{k \in K} k.$$

Notice that $\widehat{K}.A$ is contained in the set of K-invariant elements A^K of A.

There are two cases to consider:

If Q is abelian, then $p = q$ and Q has $p + 1$ subgroups $E_0, \dots, E_p$ of order p (see §4.4 of [18]). We then have the following equation in $\mathbb{Z}Q$:

$$\sum_{i=0}^{p} \widehat{E_i} = \widehat{Q} + p.1,$$

so it follows that, for any $a \in A$,

$$p.a = \sum_{i=0}^{p} \widehat{E_i}.a - \widehat{Q}.a \in \sum_{i=0}^{p} A^{E_i} + A^Q$$

and hence

$$p.A \subseteq \sum_{i=0}^{p} A^{E_i} + A^Q.$$

Now, if $K \leq Q$ is non-trivial, then, as G has cohomology almost everywhere finitary, it follows from Theorem A that $N_G(K)$ is finitely generated. Therefore, as $A^K \leq N_G(K)$, we see that A^K is finitely generated. Hence, it follows that $p.A$ is finitely generated, and as A is torsion-free, we then conclude that A is finitely generated also.

On the other hand, if Q is non-abelian, then $p \neq q$, and without loss of generality we may assume that $p < q$. Then Q has one subgroup F of order q and q subgroups $H_0, \dots, H_{q-1}$ of order p (see §4.4 of [18]). We then have the following equation in $\mathbb{Z}Q$:

$$\sum_{i=0}^{q-1} \widehat{H_i} + \widehat{F} = \widehat{Q} + q.1,$$

and the proof continues as above. □

Next, recall from §10.4 of [42] that a group G is said to be *upper-finite* if and only if every finitely generated homomorphic image of G is finite. We see from Lemma 10.42 in [42] that the class of upper-finite groups is closed under taking extensions and homomorphic images. Also recall that the *upper-finite radical* $\sigma(G)$ of any group G is the product of all of its upper-finite normal subgroups. We see from the Corollary to Lemma 10.42 in [42] that $\sigma(G)$ is itself upper-finite.

Lemma 3.0.5. *Let A and B be abelian groups. If A is upper-finite, then $A \otimes B$ is upper-finite.*

Proof. If $b \in B$, then $A \otimes b$ is a homomorphic image of A and hence is upper-finite. Then, as $A \otimes B$ is generated by all the $A \otimes b$, it is also upper-finite. □

We now recall the definition of a nilpotent group (see §1.B of [44]):

Definition 3.0.6. Let G be a group. The *lower central series* of G,

$$G = \gamma_1(G) \geq \gamma_2(G) \geq \cdots \geq \gamma_n(G) \geq \gamma_{n+1}(G) \geq \cdots ,$$

is defined by setting $\gamma_1(G) := G$ and, for $n \geq 1$,

$$\gamma_{n+1}(G) := [\gamma_n(G), G] = \langle [x, g] : x \in \gamma_n(G), g \in G \rangle.$$

Then G is said to be *nilpotent* if and only if this series eventually terminates; that is, there is some k such that $\gamma_k(G) = 1$.

Remark 3.0.7. We usually denote $\gamma_2(G)$ by G', and call it the *derived subgroup* of G.

Lemma 3.0.8. *Let G be an upper-finite nilpotent group. Then its derived subgroup G' is also upper-finite.*

Proof. As G is upper-finite, it follows that G/G' is also upper-finite. Also, as G is nilpotent, it has a finite lower central series

$$G = \gamma_1(G) \geq \gamma_2(G) \geq \cdots \geq \gamma_k(G) = 1,$$

where $\gamma_2(G) = G'$.

From Proposition 12 in [44], we see that, for each i, there is an epimorphism

$$\underbrace{G/G' \otimes \cdots \otimes G/G'}_{i} \twoheadrightarrow \gamma_i(G)/\gamma_{i+1}(G).$$

From Lemma 3.0.5, we see that $\underbrace{G/G' \otimes \cdots \otimes G/G'}_{i}$ is upper-finite, and so it follows that $\gamma_i(G)/\gamma_{i+1}(G)$ is upper-finite. Then, as the class of upper-finite groups is closed under extensions, we conclude that G' is also upper-finite. □

Lemma 3.0.9. *Let G be a torsion-free nilpotent group of finite Hirsch length. If the centre of G is finitely generated, then G is finitely generated.*

Proof. Let $\sigma(G)$ denote the upper-finite radical of G. As G is nilpotent of finite Hirsch length, it is a special case of Lemma 10.45 in [42] that $G/\sigma(G)$ is finitely generated. Suppose that $\sigma(G) \neq 1$.

Following an argument of Robinson (Lemma 10.44 in [42]), we see that for each $g \in G$,

$$[\sigma(G), g]\sigma(G)'/\sigma(G)'$$

is a homomorphic image of $\sigma(G)$, and so is upper-finite. Thus $[\sigma(G), G]/\sigma(G)'$ is upper-finite. Now $\sigma(G)$ is an upper-finite nilpotent group, so by Lemma 3.0.8 we see that its derived subgroup $\sigma(G)'$ is also upper-finite. It then follows that $[\sigma(G), G]$ is upper-finite. Similarly, we see by induction that $[\sigma(G), {}^m G] := [\sigma(G), \underbrace{G, \ldots, G}_{m}]$ is upper-finite.

Choose the largest m such that $[\sigma(G), {}^m G] \neq 1$. Then $[\sigma(G), {}^m G] \subseteq \zeta(G)$, so $[\sigma(G), {}^m G]$ is finitely generated, and hence finite. Then, as G is torsion-free, we see that $[\sigma(G), {}^m G] = 1$, which is a contradiction. Therefore, $\sigma(G) = 1$, and so G is finitely generated. □

Finally, recall from §1.3 of [41] that the *Fitting subgroup* $\mathrm{Fitt}(G)$ of a group G is the product of all of its nilpotent normal subgroups. We can now prove Theorem D:

Proof of Theorem D. Let G be an infinitely generated locally (polycyclic-by-finite) group with cohomology almost everywhere finitary. As the class of locally (polycyclic-by-finite) groups is a subclass of $\mathbf{LH}\mathfrak{F}$, it follows from Proposition 1.3.13 that G belongs to the class $\mathbf{H}_1\mathfrak{F}$, and there is a bound on the orders of its finite subgroups. It then follows from the proof of Lemma 2.1.4(i) $\Rightarrow$ (ii) that G has a characteristic subgroup S of finite index, such that S is torsion-free soluble of finite Hirsch length. (Recall from §2.1 in [41] that the class of soluble groups is simply the class $\mathbf{P}\mathfrak{A}$ of poly-abelian groups).

Now suppose that not every finite subgroup of G acts freely and orthogonally on some sphere, so by [47] we see that G has a non-cyclic subgroup Q of order pq, where p and q are prime.

As S is a torsion-free soluble group of finite Hirsch length, it follows from a result of Wehrfritz (Corollary 1.2 in [49]) that S is linear over the rationals. It then follows from a result of Gruenberg (Theorem 8.2(ii) in [48]) that the Fitting subgroup $F := \mathrm{Fitt}(S)$ of S is nilpotent. The centre $\zeta(F)$ of F is a characteristic subgroup of G, so we can consider the subgroup $\zeta(F)Q = \zeta(F) \rtimes Q$ of G. As G has cohomology almost everywhere finitary, it follows from Corollary B that $\zeta(F) \rtimes Q$ also has cohomology almost everywhere finitary, and as $\zeta(F)$ is a $\mathbb{Z}$-torsion-free $\mathbb{Z}Q$-module, we see from Lemma 3.0.4 that $\zeta(F)$ is finitely generated. Therefore, F is a torsion-free nilpotent group of finite Hirsch length with finitely generated centre. It then follows from Lemma 3.0.9 that F is finitely generated. Finally, as S is torsion-free soluble of finite Hirsch length, it follows from Theorem 10.33 in [42] that S/F is polycyclic, and hence finitely generated. We then conclude that S is finitely generated, which is a contradiction. □

Chapter 4

A Topological Characterisation

In this chapter we prove the following topological characterisation of the $\mathbf{LH}\mathfrak{F}$-groups with cohomology almost everywhere finitary:

Theorem E. *Let G be a group in the class $\mathbf{LH}\mathfrak{F}$. Then the following are equivalent:*

(i) G has cohomology almost everywhere finitary;

(ii) $G \times \mathbb{Z}$ has an Eilenberg–Mac Lane space $K(G \times \mathbb{Z}, 1)$ with finitely many n-cells for all sufficiently large n; and

(iii) G has an Eilenberg–Mac Lane space $K(G, 1)$ which is dominated by a CW-complex with finitely many n-cells for all sufficiently large n.

We begin by introducing the notion of complete cohomology in §4.1. We then prove the implication (i) ⇒ (ii) in §4.2; this is the only part of the proof where we use the assumption that G belongs to $\mathbf{LH}\mathfrak{F}$. We do not know whether (i) ⇒ (ii) holds for arbitrary G. Finally, we prove the implications (ii) ⇒ (iii) in §4.3, and (iii) ⇒ (i) in §4.4. These hold for any G.

4.1 Complete Cohomology

We begin by introducing Benson and Carlson's definition of complete cohomology. We take the following from §4.2 of [28]:

Definition 4.1.1. Let R be a ring, and M and N be R-modules. Let $\operatorname{Phom}_R(M,N)$ denote the group of all R-module homomorphisms $M \to N$ which factor through a projective module, and let

$$\underline{\operatorname{Hom}}_R(M,N) := \operatorname{Hom}_R(M,N)/\operatorname{Phom}_R(M,N).$$

Now, for any R-module M, let FM denote the free module on the underlying set of M, so F is a functor from the category of R-modules to itself. There is an obvious natural map $FM \twoheadrightarrow M$ determined by sending each free generator of FM to the corresponding element of M. We write ΩM for the kernel of this map, so Ω is also a functor from the category of R-modules to itself. Therefore, for any pair of modules M, N, there is a map

$$\Omega : \operatorname{Hom}_R(M,N) \to \operatorname{Hom}_R(\Omega M, \Omega N),$$

but unfortunately this is not a homomorphism. However, the induced map

$$\Omega : \underline{\operatorname{Hom}}_R(M,N) \to \underline{\operatorname{Hom}}_R(\Omega M, \Omega N)$$

is an additive group homomorphism.

This process can now be iterated to produce the following colimit system:

$$\underline{\operatorname{Hom}}_R(M,N) \to \underline{\operatorname{Hom}}_R(\Omega M, \Omega N) \to \underline{\operatorname{Hom}}_R(\Omega^2 M, \Omega^2 N) \to \cdots$$

We then define the zeroth complete cohomology group $\widehat{\operatorname{Ext}}^0_R(M,N)$ as

$$\widehat{\operatorname{Ext}}^0_R(M,N) := \varinjlim_i \underline{\operatorname{Hom}}_R(\Omega^i M, \Omega^i N).$$

Similarly, we define the nth complete cohomology group $\widehat{\operatorname{Ext}}^n_R(M,N)$ as

$$\widehat{\operatorname{Ext}}^n_R(M,N) := \widehat{\operatorname{Ext}}^0_R(M, \Omega^n N) = \varinjlim_i \underline{\operatorname{Hom}}_R(\Omega^i M, \Omega^{i+n} N).$$

Remark 4.1.2. Note that, although $\Omega^n N$ has no meaning for $n < 0$, the above definition of $\widehat{\operatorname{Ext}}^n_R(M,N)$ makes sense for any integer n, because in the colimit only a finite number of initial terms are undefined.

We work mainly with Benson and Carlson's definition, but at certain points in this chapter we shall use an equivalent definition due to Vogel (see §2 of [5]):

Definition 4.1.3. Let R be a ring, and M and N be R-modules. Also, let $P_* \twoheadrightarrow M$ and $Q_* \twoheadrightarrow N$ be projective resolutions of M and N respectively. An *almost-chain map* of degree n from P_* to Q_* is a set of functions $\mu = \{\mu_i\}$ such that, for each i, $\mu_i : P_{i+n} \to Q_i$ is an R-homomorphism, and for all but finitely many i the diagram

$$\begin{array}{ccc} P_{i+n} & \xrightarrow{\partial} & P_{i+n-1} \\ \downarrow{\scriptstyle \mu_i} & & \downarrow{\scriptstyle \mu_{i-1}} \\ Q_i & \xrightarrow[\partial]{} & Q_{i-1} \end{array}$$

commutes. We assume that $P_i = \{0\} = Q_i$ if $i < 0$, so that the definition makes sense even if i or n is negative. Two almost-chain maps $\mu, \nu : P_* \to Q_*$ of degree n are *almost-chain homotopic* if there exists a set $\sigma = \{\sigma_i\}$ of R-homomorphisms $\sigma_i : P_{i+n-1} \to Q_i$ such that, for all but finitely many i,

$$\mu_i - \nu_i = \sigma_i \partial + \partial \sigma_{i+1}.$$

It can be checked that the composition of two almost-chain maps is an almost-chain map, that the degrees are additive, and that almost-chain homotopy is an equivalence relation.

We then define $\widehat{\operatorname{Ext}}^n_R(M, N)$ to be the group of almost-chain homotopy equivalence classes of almost-chain maps of degree n from P_* to Q_*. It can be shown that this definition is independent of the choice of resolutions.

Finally, note that there is a third equivalent definition of complete cohomology due to Mislin [38], which is stated in terms of satellites of functors.

4.2 Proof of Theorem E (i) $\Rightarrow$ (ii)

The key to proving (i) $\Rightarrow$ (ii) is to show that if G is an $\mathbf{LH}\mathfrak{F}$-group with cohomology almost everywhere finitary and $P_* \twoheadrightarrow \mathbb{Z}$ is a projective resolution of the trivial $\mathbb{Z}G$-module, then there is some kernel M of this resolution which is isomorphic to a direct summand of a $\mathbb{Z}G$-module with a projective resolution that is eventually finitely generated. In order to prove this result we need to use complete cohomology. In particular, we make the following two definitions:

Definition 4.2.1. Let R be a ring. An R-module M is said to be *completely finitary* (over R) if and only if the functor

$$\widehat{\operatorname{Ext}}^n_R(M, -)$$

is finitary for all integers n.

Remark 4.2.2. We see from 4.1(ii) in [27] that if M is an R-module such that $\operatorname{Ext}^n_R(M,-)$ is finitary for infinitely many n, then M is completely finitary. In particular, every R-module of type FP_∞ is completely finitary.

Definition 4.2.3. Let R be a ring. An R-module N is said to be *completely flat* (over R) if and only if

$$\widehat{\operatorname{Ext}}^0_R(M,N)=0$$

for all completely finitary R-modules M.

The next step in the proof is to show that if G is an $\mathbf{LH}\mathfrak{F}$-group and N is a $\mathbb{Z}G$-module, then N is completely flat as a $\mathbb{Z}G$-module if and only if N is completely flat as a $\mathbb{Z}K$-module for all finite subgroups K of G. We begin with the following technical propositions:

Proposition 4.2.4. *Let (V_λ) be a filtered colimit system of $\mathbb{Z}G$-modules, and let $N := \varinjlim_\lambda V_\lambda$. If each of the natural maps $V_\lambda \to N$ represents zero in $\widehat{\operatorname{Ext}}^0_{\mathbb{Z}G}(V_\lambda, N)$, then N is completely flat.*

Proof. Let M be a completely finitary $\mathbb{Z}G$-module, and $\phi \in \widehat{\operatorname{Ext}}^0_{\mathbb{Z}G}(M,N)$. Since M is completely finitary, we see that the natural map

$$\varinjlim_\lambda \widehat{\operatorname{Ext}}^0_{\mathbb{Z}G}(M,V_\lambda) \to \widehat{\operatorname{Ext}}^0_{\mathbb{Z}G}(M,N)$$

is an isomorphism. Therefore, we can view ϕ as an element of $\varinjlim_\lambda \widehat{\operatorname{Ext}}^0_{\mathbb{Z}G}(M,V_\lambda)$, and so ϕ is represented by some $\widetilde{\phi} \in \widehat{\operatorname{Ext}}^0_{\mathbb{Z}G}(M,V_\lambda)$ for some λ. Now, as the following diagram commutes:

$$\begin{array}{ccc} \widehat{\operatorname{Ext}}^0_{\mathbb{Z}G}(M,V_\lambda) & \longrightarrow & \varinjlim_\lambda \widehat{\operatorname{Ext}}^0_{\mathbb{Z}G}(M,V_\lambda) \\ & \searrow & \downarrow \\ & & \widehat{\operatorname{Ext}}^0_{\mathbb{Z}G}(M,N) \end{array}$$

we see that ϕ is in fact the image of $\widetilde{\phi}$ under the map

$$\widehat{\operatorname{Ext}}^0_{\mathbb{Z}G}(M,\iota) : \widehat{\operatorname{Ext}}^0_{\mathbb{Z}G}(M,V_\lambda) \to \widehat{\operatorname{Ext}}^0_{\mathbb{Z}G}(M,N)$$

induced by the natural map $\iota : V_\lambda \to N$.

The image $\widehat{\operatorname{Ext}}^0_{\mathbb{Z}G}(M,\iota)(\widetilde{\phi})$ is defined as follows: Since

$$\widetilde{\phi} \in \widehat{\operatorname{Ext}}^0_{\mathbb{Z}G}(M,V_\lambda) = \varinjlim_i \underline{\operatorname{Hom}}_{\mathbb{Z}G}(\Omega^i M, \Omega^i V_\lambda),$$

we see that $\widetilde{\phi}$ is represented by a map

$$\alpha : \Omega^k M \to \Omega^k V_\lambda$$

for some k. We can then consider the map

$$f : \Omega^k M \xrightarrow{\alpha} \Omega^k V_\lambda \xrightarrow{\Omega^k \iota} \Omega^k N.$$

Let $\overline{f}$ denote the image of f in $\underline{\operatorname{Hom}}_{\mathbb{Z}G}(\Omega^k M, \Omega^k N)$, and let $[\overline{f}]$ denote the image of $\overline{f}$ in the colimit $\varinjlim_i \underline{\operatorname{Hom}}_{\mathbb{Z}G}(\Omega^i M, \Omega^i N)$. Then

$$\widehat{\operatorname{Ext}}^0_{\mathbb{Z}G}(M,\iota)(\widetilde{\phi}) := [\overline{f}].$$

Now, as ι represents zero in $\widehat{\operatorname{Ext}}^0_{\mathbb{Z}G}(V_\lambda, N)$, we see that the map $\Omega^i \iota$ factors through a projective module for all sufficiently large i. Hence, the map $\Omega^i f$ also factors through a projective for all sufficiently large i, and it then follows that $[\overline{f}] = 0$.

We then conclude that $\widehat{\operatorname{Ext}}^0_{\mathbb{Z}G}(M,N) = 0$, and as this holds for all completely finitary modules M, the result then follows. □

Proposition 4.2.5. *Let R be a ring, and let M be an R-module. Then M can be expressed as a filtered colimit of finitely presented modules.*

Proof. Consider the pointed category whose objects are ordered pairs (U,ϕ), where U is a finitely presented R-module and ϕ is a homomorphism from U to M, and whose morphisms are the obvious commutative triangles. Choose a skeletal small subcategory Λ, and define a functor $F : \Lambda \to \mathfrak{Mod}_R$ by sending an object (U,ϕ) of Λ to U and a morphism

$$\begin{array}{ccc} U & \xrightarrow{\phi} & M \\ {\scriptstyle f}\downarrow & \nearrow_{\phi'} & \\ U' & & \end{array}$$

to $U \xrightarrow{f} U'$. Then the colimit of F is M, which gives the desired result. □

Using the previous two propositions, we can now prove the following important lemma:

Lemma 4.2.6. *Let G be a group, and N be a $\mathbb{Z}G$-module. If K is a finite subgroup of G such that N is completely flat over $\mathbb{Z}K$, then $N \otimes_{\mathbb{Z}K} \mathbb{Z}G$ is completely flat over $\mathbb{Z}G$.*

Proof. This is a slight generalisation of the proof of Lemma 6.4 in [29]:

Viewed as a $\mathbb{Z}K$-module, we may write N as a filtered colimit of finitely presented modules,

$$N = \varinjlim_{\lambda} V_\lambda,$$

and as K is finite, the group ring $\mathbb{Z}K$ is Noetherian and so every finitely presented $\mathbb{Z}K$-module is of type FP_∞ (see §2 of [15]). Hence, each V_λ is completely finitary over $\mathbb{Z}K$. Then, as N is completely flat over $\mathbb{Z}K$, each of the natural maps $V_\lambda \to N$ represents zero in $\widehat{\mathrm{Ext}}^0_{\mathbb{Z}K}(V_\lambda, N)$.

For each λ, let $P_* \twoheadrightarrow V_\lambda$ be a $\mathbb{Z}K$-projective resolution of V_λ, and $Q_* \twoheadrightarrow N$ be a $\mathbb{Z}K$-projective resolution of N. Using Vogel's definition of complete cohomology, we see that the almost-chain map $P_* \to Q_*$ coming from the natural map $V_\lambda \to N$ is almost-chain homotopic to the zero map. Therefore, the induced almost-chain map $P_* \otimes_{\mathbb{Z}K} \mathbb{Z}G \to Q_* \otimes_{\mathbb{Z}K} \mathbb{Z}G$ is also almost-chain homotopic to the zero map.

Hence, each of the induced natural maps $V_\lambda \otimes_{\mathbb{Z}K} \mathbb{Z}G \to N \otimes_{\mathbb{Z}K} \mathbb{Z}G$ represents zero in $\widehat{\mathrm{Ext}}^0_{\mathbb{Z}G}(V_\lambda \otimes_{\mathbb{Z}K} \mathbb{Z}G, N \otimes_{\mathbb{Z}K} \mathbb{Z}G)$, and the result now follows from Proposition 4.2.4. □

Next, we have two straightforward lemmas:

Lemma 4.2.7. *Let M be a completely finitary $\mathbb{Z}G$-module, and N be a completely flat $\mathbb{Z}G$-module. Then $\widehat{\mathrm{Ext}}^n_{\mathbb{Z}G}(M, N) = 0$ for all $n \geq 0$.*

Proof. This is a slight generalisation of Lemma 6.6 in [29]:

The case $n = 0$ follows by definition. Assume, therefore, that $n \geq 1$. Choose a projective resolution $P_* \twoheadrightarrow M$ of M and let M' denote the nth kernel, so there is an exact sequence

$$0 \to M' \to P_{n-1} \to \cdots \to P_0 \to M \to 0.$$

A simple dimension-shifting argument shows that

$$\widehat{\mathrm{Ext}}^k_{\mathbb{Z}G}(M', -) \cong \widehat{\mathrm{Ext}}^{k+n}_{\mathbb{Z}G}(M, -)$$

for all integers k, so M' is also completely finitary. Hence, we see that

$$\widehat{\mathrm{Ext}}^n_{\mathbb{Z}G}(M, N) = \widehat{\mathrm{Ext}}^0_{\mathbb{Z}G}(M', N) = 0.$$

□

Lemma 4.2.8. *Let*

$$0 \to N_m \to \cdots \to N_0 \to N \to 0$$

be an exact sequence of $\mathbb{Z}G$*-modules such that each* N_i *is completely flat. Then* N *is also completely flat.*

Proof. The case $m = 0$ is immediate. For $m = 1$, we have a short exact sequence

$$0 \to N_1 \to N_0 \to N \to 0.$$

Let M be a completely finitary $\mathbb{Z}G$-module. We have the following long exact sequence (see §3.3 of [28]):

$$\cdots \to \widehat{\operatorname{Ext}}^0_{\mathbb{Z}G}(M, N_1) \to \widehat{\operatorname{Ext}}^0_{\mathbb{Z}G}(M, N_0) \to \widehat{\operatorname{Ext}}^0_{\mathbb{Z}G}(M, N) \to \widehat{\operatorname{Ext}}^1_{\mathbb{Z}G}(M, N_1) \to \cdots .$$

Using Lemma 4.2.7, we see that $\widehat{\operatorname{Ext}}^0_{\mathbb{Z}G}(M, N) = 0$ and hence that N is completely flat.

For $m \geq 2$, assume by induction that the result is true for sequences of length less than m. We have an exact sequence

$$0 \to N_m \to \cdots \to N_0 \to N \to 0$$

such that each N_i is completely flat. Let

$$K := \operatorname{Im}(N_{m-1} \to N_{m-2}),$$

so we have the short exact sequence:

$$0 \to N_m \to N_{m-1} \to K \to 0,$$

and an argument similar to the above shows that K is completely flat. We then have the following exact sequence

$$0 \to K \to N_{m-2} \to \cdots \to N_0 \to N \to 0,$$

and the result now follows by induction. □

Next, recall the following version of the Eckmann–Shapiro Lemma for complete cohomology (Lemma 1.3 in [29]):

Lemma 4.2.9. *Let* H *be a subgroup of* G*,* V *be a* $\mathbb{Z}H$*-module and* N *be a* $\mathbb{Z}G$*-module. Then, for all integers* n*, there is a natural isomorphism*

$$\widehat{\operatorname{Ext}}^n_{\mathbb{Z}G}(V \otimes_{\mathbb{Z}H} \mathbb{Z}G, N) \cong \widehat{\operatorname{Ext}}^n_{\mathbb{Z}H}(V, N).$$

We can now prove our key result, which is a slight generalisation of Theorem B in [29]:

Proposition 4.2.10. *Let G be an* $\mathbf{LH}\mathfrak{F}$*-group, and N be a $\mathbb{Z}G$-module. Then the following are equivalent:*

(i) N is completely flat as a $\mathbb{Z}G$-module;

(ii) N is completely flat as a $\mathbb{Z}K$-module for all finite subgroups K of G.

Proof. The case (i) $\Rightarrow$ (ii) follows from Lemma 4.2.9.

For (ii) $\Rightarrow$ (i), it is enough to show that $N \otimes_{\mathbb{Z}H} \mathbb{Z}G$ is completely flat over $\mathbb{Z}G$ for all $\mathbf{LH}\mathfrak{F}$-subgroups H of G. Let $\mathfrak{X}$ be the set of all subgroups H such that $N \otimes_{\mathbb{Z}H} \mathbb{Z}G$ is completely flat over $\mathbb{Z}G$. We see from Lemma 4.2.6 that all finite subgroups belong to $\mathfrak{X}$. The next step is to show that all $\mathbf{H}\mathfrak{F}$-subgroups of G belong to $\mathfrak{X}$:

Let H be a subgroup of G and suppose that there is a finite-dimensional contractible H-CW-complex X with cell stabilizers in $\mathfrak{X}$. The augmented cellular chain complex of X is an exact sequence

$$0 \to C_m \to \cdots \to C_0 \to \mathbb{Z} \to 0$$

of $\mathbb{Z}H$-modules. As this sequence is $\mathbb{Z}$-split, tensoring with N gives the following exact sequence:

$$0 \to N \otimes C_m \to \cdots \to N \otimes C_0 \to N \to 0,$$

and induction to $\mathbb{Z}G$ gives

$$0 \to (N \otimes C_m) \otimes_{\mathbb{Z}H} \mathbb{Z}G \to \cdots \to (N \otimes C_0) \otimes_{\mathbb{Z}H} \mathbb{Z}G \to N \otimes_{\mathbb{Z}H} \mathbb{Z}G \to 0.$$

Now, for $0 \leq k \leq m$, C_k is a permutation module with stabilizers in $\mathfrak{X}$,

$$C_k = \bigoplus_{\sigma \in \Sigma_k} \mathbb{Z}[H_\sigma \backslash H] \cong \bigoplus_{\sigma \in \Sigma_k} (\mathbb{Z} \otimes_{\mathbb{Z}H_\sigma} \mathbb{Z}H),$$

where Σ_k is a set of H-orbit representatives of k-cells in X, and H_σ is the stabilizer of σ. Thus,

$$N \otimes C_k \cong \bigoplus_{\sigma \in \Sigma_k} (N \otimes_{\mathbb{Z}H_\sigma} \mathbb{Z}H),$$

and so

$$(N \otimes C_k) \otimes_{\mathbb{Z}H} \mathbb{Z}G \cong \bigoplus_{\sigma \in \Sigma_k} (N \otimes_{\mathbb{Z}H_\sigma} \mathbb{Z}G).$$

By induction, each $N \otimes_{\mathbb{Z}H_\sigma} \mathbb{Z}G$ is completely flat, and as the class of completely flat modules is closed under direct sums, we see that $(N \otimes C_k) \otimes_{\mathbb{Z}H} \mathbb{Z}G$ is completely flat. It then follows from Lemma 4.2.8 that $N \otimes_{\mathbb{Z}H} \mathbb{Z}G$ is completely flat. We then conclude that all $\mathbf{H}\mathfrak{F}$-subgroups belong to $\mathfrak{X}$.

Finally, if H is a subgroup of G such that every finitely generated subgroup of H belongs to $\mathfrak{X}$, then H also belongs to $\mathfrak{X}$, because $N \otimes_{\mathbb{Z}H} \mathbb{Z}G$ can be written as the colimit

$$N \otimes_{\mathbb{Z}H} \mathbb{Z}G = \varinjlim_{K} N \otimes_{\mathbb{Z}K} \mathbb{Z}G,$$

where K runs through the finitely generated subgroups of H (see Theorem 1.6.3 of [37]), and the class of completely flat modules is closed under filtered colimits. It then follows that $N \otimes_{\mathbb{Z}H} \mathbb{Z}G$ is completely flat for all $\mathbf{LH}\mathfrak{F}$-subgroups H of G, as required. $\square$

Next, recall from [3, 4] that a $\mathbb{Z}G$-module M is said to be *cofibrant* if and only if $M \otimes B$ is projective, where $B := B(G, \mathbb{Z})$ denotes the $\mathbb{Z}G$-module of bounded functions from G to $\mathbb{Z}$. For finite groups, there is a simple characterisation of the cofibrant modules, as the next lemma shows:

Lemma 4.2.11. *Let G be a finite group, and V be a $\mathbb{Z}G$-module. Then V is cofibrant if and only if V is free as a $\mathbb{Z}$-module.*

Proof. First, note that as G is a finite group, $B \cong \mathbb{Z}G$.

Suppose that V is free as a $\mathbb{Z}$-module. Then $V \otimes B \cong V \otimes \mathbb{Z}G$ is free as a $\mathbb{Z}G$-module, and hence V is cofibrant.

Conversely, suppose that V is cofibrant, so $V \otimes B \cong V \otimes \mathbb{Z}G$ is a projective $\mathbb{Z}G$-module. Then $V \otimes \mathbb{Z}G$ is a projective $\mathbb{Z}$-module, but as $\mathbb{Z}$ is a principal ideal domain, every projective $\mathbb{Z}$-module is free (see §5.4 in [26]). Hence, $V \otimes \mathbb{Z}G$ is free as a $\mathbb{Z}$-module, and so it follows that V is free as a $\mathbb{Z}$-module. $\square$

We now make the following definition:

Definition 4.2.12. Let G be a group. A $\mathbb{Z}G$-module is called *basic* if and only if it is of the form $U \otimes_{\mathbb{Z}K} \mathbb{Z}G$, where K is a finite subgroup of G and U is a completely finitary, cofibrant $\mathbb{Z}K$-module.

The next step in our proof is the following construction, which associates to any $\mathbb{Z}G$-module M a module M_∞. This is a variation on the construction found in §4 of [29]:

Definition 4.2.13. Let G be a group, and M be a $\mathbb{Z}G$-module. We construct a chain

$$M = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$$

inductively so that for each $n \geq 0$ there is a short exact sequence

$$0 \to C_n \to M_n \oplus P_n \to M_{n+1} \to 0$$

in which:

(i) C_n is a direct sum of basic modules;

(ii) P_n is projective; and

(iii) every map from a basic module to M_n factors through C_n.

Set $M_0 := M$. Let $n \geq 0$ and suppose that M_n has been constructed. Consider the pointed category whose objects are ordered pairs (C, ϕ), where C is a basic module and ϕ is a homomorphism from C to M_n, and whose morphisms are the obvious commutative triangles. Choose a set $\mathfrak{X}_n$ containing at least one object of this category from each isomorphism class. Set

$$C_n := \bigoplus_{(C,\phi) \in \mathfrak{X}_n} C$$

and use the maps ϕ associated to each object to define a map $C_n \to M_n$. Properties (i) and (iii) are now guaranteed.

Now as C_n is cofibrant, $C_n \otimes B$ is projective and we can set $P_n := C_n \otimes B$. Finally, M_{n+1} can be defined as the cokernel of this inclusion $C_n \to M_n \oplus P_n$, or in other words the pushout, and since the map $C_n \to P_n$ is an inclusion, it follows that the induced map $M_n \to M_{n+1}$ is also injective and we regard M_n as a submodule of M_{n+1}. Finally, let M_∞ be the union of the chain of modules M_n; that is,

$$M_\infty := \varinjlim_n M_n.$$

The following technical proposition is used in the proof of Proposition 4.2.19:

Proposition 4.2.14. *Let G be a group, and M be a $\mathbb{Z}G$-module. Construct the chain*

$$M = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$$

of $\mathbb{Z}G$-modules as in Definition 4.2.13*. Then, for each n, we can express M_{n+1} as a filtered colimit:*

$$M_{n+1} := \varinjlim_{\lambda} \frac{M_n \oplus P_n}{C_{n,\lambda}},$$

where each $C_{n,\lambda}$ is poly-basic.

Proof. As in Definition 4.2.13, we have the following short exact sequence

$$0 \to C_n \to M_n \oplus P_n \to M_{n+1} \to 0,$$

where C_n is a direct sum of basic modules. Write C_n as a filtered colimit of its finitely generated subsums:

$$C_n = \varinjlim_{\lambda} C_{n,\lambda}.$$

The result now follows. □

The next key step in our proof is to use Proposition 4.2.10 to show that this module M_∞ is completely flat. We begin with the following useful result:

Lemma 4.2.15. *Let M and N be $\mathbb{Z}G$-modules. If M is cofibrant, then the natural map*

$$\underline{\mathrm{Hom}}_{\mathbb{Z}G}(M, N) \to \widehat{\mathrm{Ext}}^0_{\mathbb{Z}G}(M, N)$$

is an isomorphism.

Proof. Lemma 3.2.8 in [37] is not stated in quite the right terms for our case. However, its proof establishes the above lemma without needing any changes (see also Lemma 2.3 in [29] and the proof of Lemma 8.5 in [3]). □

Next, we recall the definition of a complete resolution (Definition 1.1 in [14]):

Definition 4.2.16. Let R be a ring and let M be an R-module. Then a *complete resolution* $\mathbf{P}$ of M is an acyclic complex of projective R-modules

$$\cdots \to P_2 \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \cdots,$$

indexed by the integers, such that

(i) $\mathbf{P}$ coincides with a projective resolution of M in all sufficiently high dimensions; and

(ii) $\mathrm{Hom}_R(\mathbf{P}, Q)$ is acyclic for every projective R-module Q.

We can now prove our key result:

Lemma 4.2.17. *Let G be an $\mathbf{LH}\mathfrak{F}$-group, and M be any $\mathbb{Z}G$-module. Then the module M_∞, constructed as in Definition* 4.2.13*, is completely flat.*

Proof. This is a generalisation of Lemma 4.1 in [29]:

As G belongs to $\mathbf{LH}\mathfrak{F}$, we see from Proposition 4.2.10 that it is enough to show that M_∞ is completely flat over $\mathbb{Z}K$ for all finite subgroups K of G. Then by the Eckmann–Shapiro Lemma, it is enough to show that

$$\widehat{\mathrm{Ext}}^0_{\mathbb{Z}G}(U \otimes_{\mathbb{Z}K} \mathbb{Z}G, M_\infty) = 0$$

for every finite subgroup K of G and every completely finitary $\mathbb{Z}K$-module U.

Fix K and U. As K is finite, we see from Theorem 1.5 in [14] that U has a complete resolution, say

$$\cdots \to P_2 \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \cdots .$$

Let V be the zeroth kernel in this resolution, so V is a submodule of the projective $\mathbb{Z}K$-module P_{-1}. Since P_{-1} is free as a $\mathbb{Z}$-module, we see that V, viewed as a $\mathbb{Z}$-module, is a submodule of a free $\mathbb{Z}$-module, and so is free. Hence, by Lemma 4.2.11, we see that V is a cofibrant $\mathbb{Z}K$-module. Now,

$$\cdots \to P_2 \to P_1 \to P_0 \to V \to 0$$

is a projective resolution of V, and by the definition of a complete resolution, we know that this projective resolution coincides with a projective resolution of U in all sufficiently high dimensions. Hence, using Vogel's definition of complete cohomology, we see that it is enough to prove that

$$\widehat{\mathrm{Ext}}^0_{\mathbb{Z}G}(V \otimes_{\mathbb{Z}K} \mathbb{Z}G, M_\infty) = 0.$$

Vogel's definition also shows that V is a completely finitary $\mathbb{Z}K$-module. Therefore, we only need to show that $\widehat{\mathrm{Ext}}^0_{\mathbb{Z}G}(C, M_\infty) = 0$ for all basic $\mathbb{Z}G$-modules C.

Let C be a basic $\mathbb{Z}G$-module. As C is cofibrant, it follows from Lemma 4.2.15 that the natural map

$$\underline{\mathrm{Hom}}_{\mathbb{Z}G}(C, M_\infty) \to \widehat{\mathrm{Ext}}^0_{\mathbb{Z}G}(C, M_\infty)$$

is an isomorphism. Let $\phi \in \underline{\mathrm{Hom}}_{\mathbb{Z}G}(C, M_\infty)$. As C is basic, it is completely finitary, and we see that the natural map

$$\varinjlim_n \underline{\mathrm{Hom}}_{\mathbb{Z}G}(C, M_n) \to \underline{\mathrm{Hom}}_{\mathbb{Z}G}(C, M_\infty)$$

is an isomorphism. Therefore, we can view ϕ as an element of $\varinjlim_n \underline{\mathrm{Hom}}_{\mathbb{Z}G}(C, M_n)$, and so ϕ is represented by some $\widetilde{\phi} \in \underline{\mathrm{Hom}}_{\mathbb{Z}G}(C, M_n)$ for some n. Then, as the following diagram commutes:

$$\begin{array}{ccc} \underline{\mathrm{Hom}}_{\mathbb{Z}G}(C, M_n) & \longrightarrow & \varinjlim_n \underline{\mathrm{Hom}}_{\mathbb{Z}G}(C, M_n) \\ & \searrow & \downarrow \\ & & \underline{\mathrm{Hom}}_{\mathbb{Z}G}(C, M_\infty) \end{array}$$

we see that ϕ is in fact the image of $\widetilde{\phi}$ under the map

$$\underline{\mathrm{Hom}}_{\mathbb{Z}G}(C, \iota) : \underline{\mathrm{Hom}}_{\mathbb{Z}G}(C, M_n) \to \underline{\mathrm{Hom}}_{\mathbb{Z}G}(C, M_\infty)$$

induced by the natural map $\iota : M_n \to M_\infty$.

The image $\underline{\mathrm{Hom}}_{\mathbb{Z}G}(C, \iota)(\widetilde{\phi})$ is defined as follows: As $\widetilde{\phi} \in \underline{\mathrm{Hom}}_{\mathbb{Z}G}(C, M_n)$, it is represented by some map $\alpha : C \to M_n$. We can then consider the map

$$f : C \xrightarrow{\alpha} M_n \xrightarrow{\iota} M_\infty.$$

Let $\overline{f}$ denote the image of f in $\underline{\mathrm{Hom}}_{\mathbb{Z}G}(C, M_\infty)$. Then

$$\underline{\mathrm{Hom}}_{\mathbb{Z}G}(C, \iota)(\widetilde{\phi}) := \overline{f}.$$

Now, by construction, we see that the composite $C \to M_n \hookrightarrow M_{n+1}$ factors through the projective module P_n. Hence, f factors through a projective, and so $\overline{f} = 0$. We then conclude that $\underline{\mathrm{Hom}}_{\mathbb{Z}G}(C, M_\infty) = 0$, and so $\widehat{\mathrm{Ext}}^0_{\mathbb{Z}G}(C, M_\infty) = 0$, and therefore M_∞ is completely flat over $\mathbb{Z}G$, as required. $\square$

We shall now use the fact that M_∞ is completely flat to prove the key result mentioned at the beginning of this section; namely, if G is an $\mathbf{LH}\mathfrak{F}$-group with cohomology almost everywhere finitary and $P_* \twoheadrightarrow \mathbb{Z}$ is a projective resolution of the trivial $\mathbb{Z}G$-module, then there is some kernel of this resolution which is isomorphic to a direct summand of a $\mathbb{Z}G$-module with a projective resolution that is eventually finitely generated. We begin by recalling the following variation on Schanuel's Lemma (Lemma 3.1 in [29]):

Lemma 4.2.18. *Let*

$$0 \to M'' \xrightarrow{\iota} M \xrightarrow{\pi} M' \to 0$$

be any short exact sequence of R-modules in which π factors through a projective module Q. Then M is isomorphic to a direct summand of $Q \oplus M''$.

Proposition 4.2.19. *Let G be an $\mathbf{LH}\mathfrak{F}$-group and M be a completely finitary, cofibrant $\mathbb{Z}G$-module. Then M is isomorphic to a direct summand of the direct sum of a poly-basic module and a projective module.*

Proof. This is a generalisation of an argument found in §4 of [29]:

As in Definition 4.2.13, construct the chain

$$M = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$$

of $\mathbb{Z}G$-modules, and let $M_\infty := \varinjlim_n M_n$. As $G \in \mathbf{LH}\mathfrak{F}$, we see from Lemma 4.2.17 that M_∞ is completely flat, and so

$$\widehat{\operatorname{Ext}}^0_{\mathbb{Z}G}(M, M_\infty) = 0.$$

Also, as M is cofibrant, it follows from Lemma 4.2.15 that

$$\underline{\operatorname{Hom}}_{\mathbb{Z}G}(M, M_\infty) = 0.$$

Then, as M is completely finitary, we see that

$$\varinjlim_n \underline{\operatorname{Hom}}_{\mathbb{Z}G}(M, M_n) = 0.$$

Therefore, there must be some n such that the identity map on M maps to zero in $\underline{\operatorname{Hom}}_{\mathbb{Z}G}(M, M_n)$. Hence, we see that the inclusion $M \hookrightarrow M_n$ factors through a projective module. By Proposition 4.2.14, we can write M_n as a filtered colimit,

$$M_n = \varinjlim_\lambda \frac{M_{n-1} \oplus P_{n-1}}{C_{n-1,\lambda}},$$

where each $C_{n-1,\lambda}$ is poly-basic. As M is completely finitary, it follows that there is some λ such that

$$M \hookrightarrow \frac{M_{n-1} \oplus P_{n-1}}{C_{n-1,\lambda}}$$

factors through a projective module.

Now, we can also write M_{n-1} as a filtered colimit:

$$M_{n-1} = \varinjlim_{\lambda} \frac{M_{n-2} \oplus P_{n-2}}{C_{n-2,\lambda}},$$

so we see that there is some λ such that

$$M \hookrightarrow \frac{(\frac{M_{n-2} \oplus P_{n-2}}{C_{n-2,\lambda}}) \oplus P_{n-1}}{C_{n-1,\lambda}}$$

factors through a projective module.

Continuing in this way, we eventually obtain a map $M \hookrightarrow N$ that factors through some projective module Q, where N has been constructed in such a way that we have a short exact sequence

$$0 \to K \to M \oplus P \to N \to 0,$$

where $P := P_0 \oplus \cdots \oplus P_{n-1}$, and K admits a filtration

$$0 = K_{-1} \leq K_0 \leq \cdots \leq K_{n-1} = K,$$

with each K_i/K_{i-1} isomorphic to $C_{i,\lambda}$. We see that the second map in the above short exact sequence must factor through $P \oplus Q$, and as K is clearly poly-basic, the result now follows from Lemma 4.2.18. $\square$

We can now prove the following:

Proposition 4.2.20. *Let G be an $\mathbf{LH}\mathfrak{F}$-group, and M be a completely finitary, cofibrant $\mathbb{Z}G$-module. Then M is isomorphic to a direct summand of a $\mathbb{Z}G$-module which has a projective resolution that is eventually finitely generated.*

Proof. We begin by showing that basic $\mathbb{Z}G$-modules are direct summands of $\mathbb{Z}G$-modules with projective resolutions that are eventually finitely generated. Recall that basic $\mathbb{Z}G$-modules are of the form $U \otimes_{\mathbb{Z}K} \mathbb{Z}G$, where K is a finite subgroup of G and U is a completely finitary, cofibrant $\mathbb{Z}K$-module. Write U as a filtered colimit of finitely presented modules,

$$U = \varinjlim_{\lambda} U_\lambda.$$

As U is completely finitary and cofibrant, it follows that $\underline{\mathrm{Hom}}_{\mathbb{Z}K}(U, -)$ is finitary, and so the natural map

$$\varinjlim_{\lambda} \underline{\mathrm{Hom}}_{\mathbb{Z}K}(U, U_\lambda) \to \underline{\mathrm{Hom}}_{\mathbb{Z}K}(U, U)$$

is an isomorphism. Therefore, the identity map on U gives rise to a map in $\underline{\mathrm{Hom}}_{\mathbb{Z}K}(U,U)$ which factors through U_λ for some λ. Hence, we see that the identity map $U \to U$ must factor through $Q \oplus U_\lambda$ for some projective $\mathbb{Z}K$-module Q. Therefore, U is a direct summand of $Q \oplus U_\lambda$. Now, as K is finite, every finitely presented $\mathbb{Z}K$-module is of type FP_∞, so in particular U_λ is of type FP_∞. Then, as $U \otimes_{\mathbb{Z}K} \mathbb{Z}G$ is a direct summand of $Q \otimes_{\mathbb{Z}K} \mathbb{Z}G \oplus U_\lambda \otimes_{\mathbb{Z}K} \mathbb{Z}G$, where $Q \otimes_{\mathbb{Z}K} \mathbb{Z}G$ is projective, and $U_\lambda \otimes_{\mathbb{Z}K} \mathbb{Z}G$ is of type FP_∞, we see that $U \otimes_{\mathbb{Z}K} \mathbb{Z}G$ is a direct summand of a $\mathbb{Z}G$-module with a projective resolution that is eventually finitely generated.

Next, as poly-basic modules are built up from basic modules by extensions, we see from the Horseshoe Lemma (see page 37 of [51]) that every poly-basic $\mathbb{Z}G$-module is a direct summand of a $\mathbb{Z}G$-module with a projective resolution that is eventually finitely generated.

Finally, if G is an $\mathbf{LH}\mathfrak{F}$-group, and M is a completely finitary, cofibrant $\mathbb{Z}G$-module, it follows from Proposition 4.2.19 that M is isomorphic to a direct summand of $P \oplus C$, for some projective module P and some poly-basic module C. Then, as C is a direct summand of a $\mathbb{Z}G$-module with a projective resolution that is eventually finitely generated, the result now follows. □

We now have the following technical proposition:

Proposition 4.2.21. *Let G be an $\mathbf{LH}\mathfrak{F}$-group, and M be a completely finitary $\mathbb{Z}G$-module. Then $M \otimes B$ has finite projective dimension over $\mathbb{Z}G$.*

Proof. This is a generalisation of Proposition 9.2 in [12]:

Let K be a finite subgroup of G. We see from Lemma 9.1(2) in [12] that B is free as a $\mathbb{Z}K$-module, so $M \otimes B$ is a direct sum of copies of $M \otimes \mathbb{Z}K$ as a $\mathbb{Z}K$-module. From Lemma 6.3(2) in [12], we see that

$$\mathrm{proj.\,dim}_{\mathbb{Z}K}\, M \otimes \mathbb{Z}K = \mathrm{proj.\,dim}_{\mathbb{Z}}\, M,$$

which is finite, since every $\mathbb{Z}$-module has projective dimension at most 1 (see §4.1 of [51]). Therefore, $M \otimes B$ has finite projective dimension over $\mathbb{Z}K$, and it then follows from Lemma 4.2.3 in [28] that

$$\widehat{\mathrm{Ext}}^0_{\mathbb{Z}K}(A, M \otimes B) = 0$$

for any $\mathbb{Z}K$-module A. In particular, we see that $M \otimes B$ is completely flat over $\mathbb{Z}K$. As this holds for any finite subgroup K of G, we see from Proposition 4.2.10 that $M \otimes B$ is completely flat over $\mathbb{Z}G$. Then, as M is completely finitary over $\mathbb{Z}G$, we see that

$$\widehat{\mathrm{Ext}}^0_{\mathbb{Z}G}(M, M \otimes B) = 0,$$

and it then follows from Lemma 2.2 in [13] that $M \otimes B$ has finite projective dimension over $\mathbb{Z}G$. $\square$

We can now prove our key result:

Lemma 4.2.22. *Let G be an* $\mathbf{LH}\mathfrak{F}$*-group with cohomology almost everywhere finitary, and $P_* \twoheadrightarrow \mathbb{Z}$ be a projective resolution of the trivial $\mathbb{Z}G$-module. Then there is some n such that the nth kernel of this resolution is a completely finitary, cofibrant $\mathbb{Z}G$-module.*

Proof. Choose $n_0 \in \mathbb{N}$ such that $H^n(G, -)$ is finitary for all $n \geq n_0$, and let M be the n_0th kernel of this resolution,

$$M := \operatorname{Ker}(P_{n_0-1} \to P_{n_0-2}).$$

Then $\operatorname{Ext}^i_{\mathbb{Z}G}(M, -)$ is finitary for all $i \geq 1$, and so it follows from 4.1(ii) in [27] that M is completely finitary. It then follows from Proposition 4.2.21 that $M \otimes B$ has finite projective dimension, say

$$\operatorname{proj.dim}_{\mathbb{Z}G} M \otimes B = k.$$

We now have the projective resolution $P_{n_0+*} \twoheadrightarrow M$ of M with kth kernel M' cofibrant, since $P_{n_0+*} \otimes B \twoheadrightarrow M \otimes B$ is a projective resolution of $M \otimes B$ in which the kth kernel is $M' \otimes B$. Also, as $\operatorname{Ext}^i_{\mathbb{Z}G}(M', -)$ is finitary for all $i \geq 1$, we see that M' is completely finitary. Then, as M' is the $(n_0 + k)$th kernel of the resolution $P_* \twoheadrightarrow \mathbb{Z}$, the result now follows. $\square$

Before we can finish the proof of (i) $\Rightarrow$ (ii), we need the following two straightforward results:

Proposition 4.2.23. *Let R be a ring, and suppose that*

$$0 \to N' \to N \to P_n \to \cdots \to P_0 \to M \to 0$$

is an exact sequence of R-modules such that the P_i are projective, and N' and N have projective resolutions that are eventually finitely generated. Then the partial projective resolution

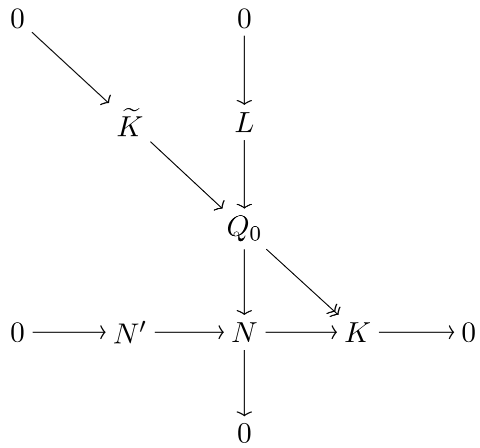
$$P_n \to \cdots \to P_0 \to M \to 0$$

of M can be extended to a projective resolution that is eventually finitely generated.

Proof. Let $K := \operatorname{Ker}(P_n \to P_{n-1})$, so we have the following short exact sequence:

$$0 \to N' \to N \to K \to 0.$$

Next, let $Q_* \twoheadrightarrow N$ be a projective resolution of N that is eventually finitely generated, and let L denote the zeroth kernel. We then have the following diagram:



where $\widetilde{K}$ is an extension of N' by L, and since both N' and L have projective resolutions that are eventually finitely generated, it follows from the Horseshoe Lemma (see page 37 of [51]) that $\widetilde{K}$ also has such a resolution. We then have the following exact sequence:

$$0 \to \widetilde{K} \to Q_0 \to P_n \to \cdots \to P_0 \to M \to 0,$$

and the result now follows. □

Proposition 4.2.24. *Let M be an R-module. If M has a projective resolution that is eventually finitely generated, then M has a free resolution that is eventually finitely generated.*

Proof. Let $P_* \twoheadrightarrow M$ be a projective resolution of M that is eventually finitely generated; say P_j is finitely generated for all $j \geq n$, and let

$$K := \mathrm{Ker}(P_{n-1} \to P_{n-2}).$$

Then K is of type FP_∞, and so by Corollary 1.2.9 is of type FL_∞. We can therefore choose a free resolution $F_{n+*} \twoheadrightarrow K$ of K with all the free modules finitely generated. This gives the following exact sequence:

$$\cdots \to F_{n+1} \to F_n \to P_{n-1} \to \cdots \to P_2 \to P_1 \to P_0 \to M \to 0.$$

Next, recall the Eilenberg trick (Lemma 2.7 §VIII in [9]): For any projective R-module P, we can choose a free R-module F such that $P \oplus F \cong F$. Therefore, using this, we can replace the projective modules P_i in the above exact sequence by free modules F_i, at the expense of changing F_n to a larger free module F'_n. We then have the following free resolution

$$\cdots \to F_{n+2} \to F_{n+1} \to F'_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0$$

of M, with the F_j finitely generated for all $j \geq n+1$. □

We also need the following technical proposition:

Proposition 4.2.25. *Let X^n be an $(n-1)$-connected n-dimensional G-CW-complex, where $n \geq 2$. Let $\phi : F \twoheadrightarrow H_n(X^n)$ be a surjective $\mathbb{Z}G$-module map from a free $\mathbb{Z}G$-module F to the nth homology of X^n. Then X^n can be embedded into an n-connected $(n+1)$-dimensional G-CW-complex X^{n+1} such that G acts freely outside X^n and there is a short exact sequence*

$$0 \to H_{n+1}(X^{n+1}) \to F \to H_n(X^n) \to 0.$$

Proof. This is Proposition 5.1 in [32]: □

Finally, we can now use Lemma 4.2.22 to prove the implication (i) ⇒ (ii) of Theorem E:

Theorem 4.2.26. *Let G be an $\mathbf{LH}\mathfrak{F}$-group with cohomology almost everywhere finitary. Then $G\times\mathbb{Z}$ has an Eilenberg–Mac Lane space $K(G\times\mathbb{Z}, 1)$ with finitely many n-cells for all sufficiently large n.*

Proof. Let Y be the 2-complex associated to some presentation of G (see Exercise 2 §II.5 in [9]), so $\pi_1(Y) \cong G$. Also, let $\widetilde{Y}$ denote the universal cover of Y, so $\widetilde{Y}$ is simply-connected, and so by the Hurewicz Theorem (Theorem 4.32 in [22]) has reduced homology equal to zero in dimensions 0 and 1. Therefore, the augmented cellular chain complex of $\widetilde{Y}$ is a partial free resolution of the trivial $\mathbb{Z}G$-module, which we denote

$$F_2 \to F_1 \to F_0 \to \mathbb{Z} \to 0.$$

We can extend this to a free resolution $F_* \twoheadrightarrow \mathbb{Z}$ of the trivial $\mathbb{Z}G$-module, and as G is an $\mathbf{LH}\mathfrak{F}$-group with cohomology almost everywhere finitary, it follows from Lemma 4.2.22 that there is some $n \geq 3$ such that the nth kernel

$$M := \mathrm{Ker}(F_{n-1} \to F_{n-2})$$

of this resolution is a completely finitary, cofibrant $\mathbb{Z}G$-module. We then have the following exact sequence of $\mathbb{Z}G$-modules:

$$0 \to M \to F_{n-1} \to \cdots \to F_0 \to \mathbb{Z} \to 0.$$

Next, let C_∞ denote the infinite cyclic group. The circle S^1 is an Eilenberg–Mac Lane space $K(C_\infty, 1)$ with universal cover $\mathbb{R}$ (Example 1B.1 in [22]), and the augmented cellular chain complex of $\mathbb{R}$ is the following free resolution of the trivial $\mathbb{Z}C_\infty$-module:

$$0 \to \mathbb{Z}C_\infty \to \mathbb{Z}C_\infty \to \mathbb{Z} \to 0.$$

If we tensor these two exact sequences together, we obtain the following exact sequence of $\mathbb{Z}[G \times C_\infty]$-modules:

$$0 \to M \otimes \mathbb{Z}C_\infty \to M \otimes \mathbb{Z}C_\infty \oplus F_{n-1} \otimes \mathbb{Z}C_\infty \to$$

$$F_{n-1} \otimes \mathbb{Z}C_\infty \oplus F_{n-2} \otimes \mathbb{Z}C_\infty \to \cdots \to F_0 \otimes \mathbb{Z}C_\infty \to \mathbb{Z} \to 0.$$

Now, as M is a completely finitary, cofibrant $\mathbb{Z}G$-module, it follows from Proposition 4.2.20 that M is isomorphic to a direct summand of some $\mathbb{Z}G$-module L which has a projective resolution that is eventually finitely generated.

We then obtain the following exact sequence of $\mathbb{Z}[G \times C_\infty]$-modules:

$$0 \to L \otimes \mathbb{Z}C_\infty \to L \otimes \mathbb{Z}C_\infty \oplus F_{n-1} \otimes \mathbb{Z}C_\infty \to$$

$$F_{n-1} \otimes \mathbb{Z}C_\infty \oplus F_{n-2} \otimes \mathbb{Z}C_\infty \to \cdots \to F_0 \otimes \mathbb{Z}C_\infty \to \mathbb{Z} \to 0.$$

It now follows from Propositions 4.2.23 and 4.2.24 that we can extend the partial free resolution

$$F_{n-1} \otimes \mathbb{Z}C_\infty \oplus F_{n-2} \otimes \mathbb{Z}C_\infty \to \cdots \to F_0 \otimes \mathbb{Z}C_\infty \to \mathbb{Z} \to 0$$

of the trivial $\mathbb{Z}[G \times C_\infty]$-module to a free resolution that is eventually finitely generated. We shall denote this resolution by $F'_* \twoheadrightarrow \mathbb{Z}$.

Next, let X^2 denote the subcomplex of $\widetilde{Y} \times \mathbb{R}$ consisting of the 0, 1 and 2-cells. Then, as

$$\widetilde{C}_*(\widetilde{Y} \times \mathbb{R}) \cong \widetilde{C}_*(\widetilde{Y}) \otimes \widetilde{C}_*(\mathbb{R}),$$

we see that the augmented cellular chain complex of X^2 is the following:

$$F_2' \to F_1' \to F_0' \to \mathbb{Z} \to 0,$$

and, furthermore, that $\widetilde{H}_i(X^2) = 0$ for $i = 0, 1$. We therefore have the following exact sequence:

$$0 \to \widetilde{H}_2(X^2) \to F_2' \to F_1' \to F_0' \to \mathbb{Z} \to 0,$$

and as $F_3' \twoheadrightarrow \widetilde{H}_2(X^2)$, it follows from Proposition 4.2.25 that we can embed X^2 into a 2-connected 3-complex X^3 such that we have the following short exact sequence:

$$0 \to \widetilde{H}_3(X^3) \to F_3' \to \widetilde{H}_2(X^2) \to 0.$$

Then $F_4' \twoheadrightarrow \widetilde{H}_3(X^3)$, and we can continue as before.

By induction, we can then construct a space, which we denote by X, such that $C_n(X) = F_n'$ for all n. Then, as the free resolution $F_*' \twoheadrightarrow \mathbb{Z}$ is eventually finitely generated, it follows that $C_n(X)$ is finitely generated for all sufficiently large n. Also, we see that $\widetilde{H}_i(X) = 0$ for all i, and so X is contractible (see §I.4 in [9]).

We see from Proposition 1.40 in [22] that X is the universal cover for the quotient space $\overline{X} := X/G \times C_\infty$, and furthermore that $\overline{X}$ has fundamental group isomorphic to $G \times C_\infty$. Thus, $\overline{X}$ is an Eilenberg–Mac Lane space $K(G \times C_\infty, 1)$, and as $C_n(X)$ is finitely generated for all sufficiently large n, we conclude that $\overline{X}$ has finitely many n-cells for all sufficiently large n, as required. □

4.3 Proof of Theorem E (ii) ⇒ (iii)

We do not require the assumption that G belongs to $\mathbf{LH}\mathfrak{F}$ for this section.

Firstly, recall from page 528 of [22] that a space Y is said to be *dominated* by a space K if and only if Y is a retract of K in the homotopy category; that is, there are maps $i : Y \to K$ and $r : K \to Y$ such that $ri \simeq \mathrm{id}_Y$.

We now prove the implication (ii) ⇒ (iii) of Theorem E:

Proposition 4.3.1. *Suppose that K is a $K(G \times \mathbb{Z}, 1)$ space with finitely many n-cells for all sufficiently large n. Then G has an Eilenberg–Mac Lane space $K(G, 1)$ which is dominated by K.*

Proof. As every group has an Eilenberg–Mac Lane space (Theorem 7.1 §VIII in [9]), we can choose a $K(G, 1)$ space Y. Then, as S^1 is a $K(\mathbb{Z}, 1)$ space, we see from Example 1B.5 in [22] that $Y \times S^1$ is a $K(G\times\mathbb{Z},1)$ space. Then, as $K(G\times\mathbb{Z},1)$ spaces are unique up to homotopy equivalence (Theorem 1B.8 in [22]), we see that $Y \times S^1 \simeq K$, and hence that Y is dominated by K. $\square$

4.4 Proof of Theorem E (iii) $\Rightarrow$ (i)

Once again, we do not require the assumption that G belongs to $\mathbf{LH}\mathfrak{F}$ for this section.

We thank Philipp Reinhard for explaining the following argument:

Lemma 4.4.1. *Let Y be a $K(G,1)$ space which is dominated by a CW-complex with finitely many cells in all sufficiently high dimensions. Then we may choose this complex to have fundamental group isomorphic to G.*

Proof. Let K be a CW-complex with finitely many cells in all sufficiently high dimensions, such that K dominates Y. We shall construct a CW-complex L, also with finitely many cells in all sufficiently high dimensions, such that Y is dominated by L and L has fundamental group isomorphic to G.

As Y is dominated by K, there are maps

$$Y \xrightarrow{i} K \xrightarrow{r} Y$$

such that $ri \simeq \mathrm{id}_Y$. By replacing K with the connected component of K that i maps into, we may assume that K is connected. Let $y_0 \in Y$ be a basepoint for Y, and let $k_0 := i(y_0)$ be a basepoint for K. Also, let $y_1 := r(k_0)$. Then applying the functor π_1 gives the following maps:

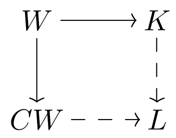
$$\begin{array}{ccc} \pi_1(Y,y_0) & \xrightarrow{\pi_1(i)} & \pi_1(K,k_0) \\ & \searrow^{\cong} & \downarrow{\scriptstyle \pi_1(r)} \\ & & \pi_1(Y,y_1) \end{array}$$

Hence, $\pi_1(r)$ is surjective. Let K' denote the kernel of $\pi_1(r)$, and let W be a bouquet of circles, with one circle for each generator in some chosen presentation of K', so we have a map $W \to K$ which sends a circle in W to a based loop in K which represents the generator of K' associated to that circle.

Next, recall from Chapter 0 of [22] that the *cone* CW on W is

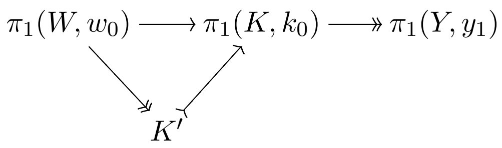
$$CW := (W \times I)/(W \times \{0\}),$$

where I denotes the unit interval $[0,1]$. There is an obvious map $W \to CW$, so we can now form the following pushout:

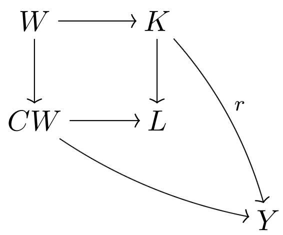


From the definition of pushout, it follows that L is also a CW-complex with finitely many cells in all sufficiently high dimensions.

Next, consider the composite $W \to K \xrightarrow{r} Y$. Applying π_1 gives the following:



so we see that the map $W \to K \xrightarrow{r} Y$ is nullhomotopic, and hence lifts through the cone CW (see page 387 of [22]). This gives us the following:



and so by the definition of pushout, there is an induced map $L \to Y$ making the above diagram commute. If we now compose this with the map $Y \xrightarrow{i} K \to L$, we obtain a map $Y \to L \to Y$ that is homotopic to the identity on Y. Hence, Y is dominated by L.

Finally, by van Kampen's Theorem (Theorem 1.20 in [22]), we see that

$$\begin{aligned} \pi_1(L, l_0) &\cong \pi_1(K, k_0)/\operatorname{Im}(\pi_1(W, w_0) \to \pi_1(K, k_0)) \\ &\cong \pi_1(Y, y_1) \\ &\cong G, \end{aligned}$$

as required. □

Next, recall from [8] that if $P := (P_i)_{i \geq 0}$ is a chain complex of projective $\mathbb{Z}G$-modules, then we define the *cohomology theory* $H^*(P, -)$ *determined by* P as

$$H^n(P, M) := H^n(\operatorname{Hom}_{\mathbb{Z}G}(P_*, M))$$

for every $\mathbb{Z}G$-module M and every $n \in \mathbb{N}$.

Lemma 4.4.2. *Let $P := (P_i)_{i \geq 0}$ be a chain complex of projective $\mathbb{Z}G$-modules. If P_{n-1}, P_n and P_{n+1} are finitely generated, then $H^n(P, -)$ is finitary.*

Proof. Firstly, recall from Lemma 4.7 §VIII in [9] that if Q is a finitely generated projective module, then the functor $\mathrm{Hom}_{\mathbb{Z}G}(Q, -)$ is finitary.

Next, let $M := \mathrm{Coker}(P_{n+1} \to P_n)$, so we have the following exact sequence of modules:

$$P_{n+1} \to P_n \to M \to 0,$$

which gives the following exact sequence of functors:

$$0 \to 0 \to \mathrm{Hom}_{\mathbb{Z}G}(M, -) \to \mathrm{Hom}_{\mathbb{Z}G}(P_n, -) \to \mathrm{Hom}_{\mathbb{Z}G}(P_{n+1}, -).$$

It then follows from Lemma 1.1.7 that $\mathrm{Hom}_{\mathbb{Z}G}(M, -)$ is finitary. Then, as we have the following exact sequence of functors:

$$\mathrm{Hom}_{\mathbb{Z}G}(P_{n-1}, -) \to \mathrm{Hom}_{\mathbb{Z}G}(M, -) \to H^n(P, -) \to 0 \to 0,$$

the result now follows from another application of Lemma 1.1.7. □

Finally, we can now prove the implication (iii) ⇒ (i) of Theorem E:

Proposition 4.4.3. *Suppose that G has an Eilenberg–Mac Lane space $K(G, 1)$ which is dominated by a CW-complex with finitely many n-cells for all sufficiently large n. Then G has cohomology almost everywhere finitary.*

Proof. This is a generalisation of the proof of Proposition 6.4 §VIII in [9]:

Let Y be such a $K(G, 1)$ space. By Lemma 4.4.1, we see that Y is dominated by a CW-complex K with finitely many cells in all sufficiently high dimensions, such that K has fundamental group isomorphic to G. Let $\widetilde{Y}$ and $\widetilde{K}$ denote the respective universal covers. We see that $C_*(\widetilde{Y})$ is a retract of $C_*(\widetilde{K})$ in the homotopy category of chain complexes over $\mathbb{Z}G$. Therefore, we obtain maps giving the following commutative diagram:

$$\begin{array}{ccc} H^*(G,-) & \longrightarrow & H^*(C,-) \\ & \diagdown\!\!\diagdown & \downarrow \\ & & H^*(G,-) \end{array}$$

where $H^*(C,-)$ denotes the cohomology theory determined by $C_*(\widetilde{K})$. We then conclude that $H^*(G,-)$ is a direct summand of $H^*(C,-)$.

Now, as K has finitely many cells in all sufficiently high dimensions, it follows that $C_*(\widetilde{K})$ is eventually finitely generated, and so by Lemma 4.4.2 that $H^k(C,-)$ is finitary for all sufficiently large k. The result then follows from an application of Lemma 1.1.8. □

This completes our proof of Theorem E.

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