Quasideterminant solutions of noncommutative integrable systems

by

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Keep calm and carry on

Abstract

Quasideterminants are a relatively new addition to the field of integrable systems. Their simple structure disguises a wealth of interesting and useful properties, enabling solutions of noncommutative integrable equations to be expressed in a straightforward and aesthetically pleasing manner. This thesis investigates the derivation and quasideterminant solutions of two noncommutative integrable equations - the Davey-Stewartson (DS) and Sasa-Satsuma nonlinear Schrödinger (SSNLS) equations.

Chapter 1 provides a brief overview of the various concepts to which we will refer during the course of the thesis. We begin by explaining the notion of an integrable system, although no concrete definition has ever been explicitly stated. We then move on to discuss Lax pairs, and also introduce the Hirota bilinear form of an integrable equation, looking at the Kadomtsev-Petviashvili (KP) equation as an example. Wronskian and Grammian determinants will play an important role in later chapters, albeit in a noncommutative setting, and, as such, we give an account of their widespread use in integrable systems.

Chapter 2 provides further background information, now focusing on noncommutativity. We explain how noncommutativity can be defined and implemented, both specifically using a star product formalism, and also in a more general manner. It is this general definition to which we will allude in the remainder of the thesis. We then give the definition of a quasideterminant, introduced by Gel'fand and Retakh in 1991, and provide some examples and properties of these noncommutative determinantal analogues. We also explain how to calculate the derivative of a quasideterminant. The chapter concludes by outlining the motivation for studying our particular choice of noncommutative integrable equations and their quasideterminant solutions.

We begin with the DS equations in Chapter 3, and derive a noncommutative version of this integrable system using a Lax pair approach. Quasideterminant solutions arise in a natural way by the implementation of Darboux and binary Darboux transformations, and,

after describing these transformations in detail, we obtain two types of quasideterminant solution to our system of noncommutative DS equations - a quasi-Wronskian solution from the application of the ordinary Darboux transformation, and a quasi-Grammian solution by applying the binary transformation. After verification of these solutions, in Chapter 4 we select the quasi-Grammian solution to allow us to determine a particular class of solution to our noncommutative DS equations. These solutions, termed *dromions*, are lump-like objects decaying exponentially in all directions, and are found at the intersection of two perpendicular plane waves. We extend earlier work of Gilson and Nimmo by obtaining plots of these dromion solutions in a noncommutative setting. The work on the noncommutative DS equations and their dromion solutions constitutes our paper published in 2009 [34].

Chapter 5 describes how the well-known Darboux and binary Darboux transformations in (2+1)-dimensions discussed in the previous chapter can be dimensionally-reduced to enable their application to $(1+1)$-dimensional integrable equations. This reduction was discussed briefly by Gilson, Nimmo and Ohta in reference to the self-dual Yang-Mills (SDYM) equations, however we explain these results in more detail, using a reduction from the DS to the nonlinear Schrödinger (NLS) equation as a specific example. Results stated here are utilised in Chapter 6, where we consider higher-order NLS equations in $(1+1)$-dimension. We choose to focus on one particular equation, the SSNLS equation, and, after deriving a noncommutative version of this equation in a similar manner to the derivation of our noncommutative DS system in Chapter 3, we apply the dimensionally-reduced Darboux transformation to the noncommutative SSNLS equation. We see that this ordinary Darboux transformation does not preserve the properties of the equation and its Lax pair, and we must therefore look to the dimensionally-reduced binary Darboux transformation to obtain a quasi-Grammian solution. After calculating some essential conditions on various terms appearing in our solution, we are then able to determine and obtain plots of soliton solutions in a noncommutative setting.

Chapter 7 seeks to bring together the various results obtained in earlier chapters, and also discusses some open questions arising from our work.

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Statement

This thesis is submitted in accordance with the regulations for the degree of Doctor of Philosophy at the University of Glasgow.

Chapters 1 and 2 introduce some of the main concepts used throughout the thesis.

Chapters 3 and 4 are the author's own work in collaboration with Claire Gilson, with the exception of those results explicitly referenced. An abridged version of this work appears in [34].

Chapter 5 gives a detailed explanation of the dimensional reduction of a Darboux transformation using the results of Gilson, Nimmo and Ohta in [73] and, as such, this chapter does not contain any new results.

Chapter 6 is the author's own work with Claire Gilson, with any results taken from the literature referenced therein.

The final chapter is devoted to concluding remarks.

Contents

Chapter 1

Introduction

1.1 Integrable systems

To those not associated with the field, the idea of an integrable system is likely to be unfamiliar, and even to those actively involved in the area, the precise definition is at times difficult to pinpoint. There is no generally accepted definition of integrability, however systems in possession of this property share a number of distinguishing features.

The term *integrability*, coming from *integrable*, brings to mind differential equations, and indeed, an integrable system is a system of differential, difference or integro-differential equations for which solutions can be obtained in terms of known functions or integrals. All such systems possess characteristics such as a complete set of conservation laws, exact and rigorous solution methods, for example the Inverse Scattering Transform, and a wealth of explicit non-trivial solutions. Specifically, all systems known to be integrable have one or more of the following [89]:

- solutions that can be expressed in terms of known functions or integrals
- a complete set of conservation laws
- a Lax representation (Lax pair)
- a hierarchy of commuting Hamiltonian flows
- soliton solutions
- a bi-Hamiltonian structure
- a Hirota bilinear form

- the Painlevé property
- Backlünd transformations.

We discuss two of the above properties later in the chapter. The Lax representation, one of the central themes of integrable systems, exploits the fact that the time evolution of a Lax operator (usually a differential operator) is equivalent to a given nonlinear integrable system. This will be discussed further in Section 1.2.1. We will also discuss the Hirota bilinear form in Section 1.2.2.

1.2 Preliminaries

We now introduce some preliminary details and concepts used throughout the thesis.

1.2.1 Lax's method and Lax pairs

In later chapters, we make much use of the *Lax pair* in order to generate noncommutative versions of various integrable equations. Indeed, the point where most discussions of integrability begin is with the idea of a system of differential equations that can be put into Lax pair form.

The theory of Lax pairs, a pair of linear operators depending on x and possibly t (or x, y and possibly t in the two-dimensional case) operating on elements of a Hilbert space, was developed by Peter Lax [55] in 1968 as a way of generalising earlier work by Gardner, Greene, Kruskal and Miura [27] on the application of the so-called inverse scattering method to the initial value problem of the Korteweg-de Vries (KdV) equation. After this introductory work on the KdV equation, the question of whether the method could be extended to other nonlinear evolution equations arose, and shortly afterwards, Zakharov and Shabat [93] proved that the nonlinear Schrödinger (NLS) equation was one such example. In the same year, Wadati [85] provided a method of solution for the modified KdV (mKdV) equation, before Ablowitz, Kaup, Newell and Segur [4] developed the method for the sine-Gordon equation. Ablowitz *et al.* were influential in the development of the theory, showing that a surprisingly large number of nonlinear evolution equations could be solved using this method. Because of the similarity between the Fourier transform method used to solve initial value problems for linear evolution equations and the inverse scattering method for solving initial value problems of nonlinear evolution equations, they coined the phrase Inverse Scattering Transform (IST), the name now used throughout the field. The IST has come to be known as one of the most important developments in mathematical physics in the past forty years.

Examples of other such nonlinear equations that can be re-expressed in terms of a Lax pair and hence solved by the IST include both forms of the Kadomtsev-Petviashvili (KP) equation [18,19,63], [1], the Davey-Stewartson (DS) equations [3], along with many others. A common feature of these equations is the existence of *soliton* solutions, which have no linear analogue.

Solitons

The discovery of the soliton (or solitary wave) was a very significant one and has allowed much progress to be made in the field of integrable systems.

Observation of a solitary wave was first made by John Scott Russell, an engineer, in 1834 while on the bank of the Union canal near Edinburgh. As noted in his submission to the British Association entitled 'Report on Waves' ten years later, Russell was watching the motion of a boat being pulled along the canal by a pair of horses. The boat stopped suddenly, causing a large mass of water to accumulate at its prow. This mass of water took the form of a smooth, rounded wave and continued to travel along the canal and, to Russell's surprise, did not change in shape or speed as it progressed. This is the defining characteristic of a solitary wave. Another important feature of solitary waves is that they remain unchanged after collision or interaction with another wave of the same type, although the waves undergo a phase shift, highlighting their nonlinear nature. This can be seen after interaction - the two waves are not in their expected positions had they moved at a constant speed throughout the collision. The wave of larger amplitude is moved forward and the wave of shorter amplitude moved backward relative to their positions had the collision been linear. This behaviour, i.e. the fact that solitary waves can interact and remain unchanged with the exception of a phase shift, is more reminiscent of particle than of wave behaviour, and thus led Zabusky and Kruskal [90] to term these waves 'solitons', the '-on' suffix used in the same vein as proton, photon and so on.

Lax's generalisation

Here we focus on one of the simplest integrable equations, the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{1.1}$$

for $u = u(x,t)$. This equation arises as the compatibility condition of two linear operators, the first (commonly denoted L), determined from the well-known time-dependent Schrödinger scattering problem with eigenvalue λ, namely [2]

$$L\psi := \psi_{xx} + u(x,t)\psi = \lambda\psi, \tag{1.2}$$

an equation widely studied in the fields of mathematics and physics, with the second operator M governing the associated time evolution of the eigenfunctions ψ [2],

$$\psi_t := M\psi. \tag{1.3}$$

Lax [55] showed that the results obtained for the KdV equation via Inverse Scattering could be generalised and applied to many other nonlinear partial differential equations.

Differentiating (1.2) with respect to t and using (1.3) gives

$$L_t\psi + LM\psi = \lambda_t\psi + \lambda M\psi, \tag{1.4}$$

so that, by (1.2),

$$\left(L_t + (LM - ML)\right)\psi = \lambda_t\psi. \tag{1.5}$$

Thus $L_t + (LM - ML) = 0$ if and only if $\lambda_t = 0$. We define

$$[L, M] := LM - ML \tag{1.6}$$

and call this the *commutator* of the operators L and M. Hence we have

$$L_t + [L, M] = 0 \tag{1.7}$$

if and only if $\lambda_t = 0$. Equation (1.7) is known as Lax's equation. For a suitable choice of L and M, Lax's equation generates a nonlinear evolution equation. For example, defining [84]

$$L = \partial_x^2 + u, \tag{1.8a}$$

$$M = -4\partial_x^3 - 3u\partial_x - 3u_x \tag{1.8b}$$

where $\partial_x = \dfrac{\partial}{\partial x}$ and so on, we can show that L and M satisfy (1.7) so long as $u = u(x,t)$ satisfies the KdV equation (1.1), and thus the KdV equation can be thought of as the compatibility condition of the two linear differential operators L, M given by (1.8). (It should be mentioned here that in practise, the differential operator ∂_t is often included in one of L or M, and we then consider Lax's equation as simply $[L, M] = 0$. This can be seen when we consider the DS equation in Chapter 3).

If we are able to generate a nonlinear evolution equation such as the KdV equation from the compatibility of two linear operators L and M, then equation (1.7) is called the *Lax representation* of the evolution equation, while the pair of operators L, M is known as a *Lax pair*.

In his paper of 1968, Lax [55] indicated how, given a linear differential operator L, a corresponding operator M can be constructed. However, there is no guarantee that a particular nonlinear evolution equation will have a Lax representation, nor no set method to determine the operators L, M if such a representation does exist. Advancement in this area has resulted either from choosing a rather arbitrary form of L and M and investigating the equation that results from their compatibility, or else by considering a particular equation and attempting to devise the corresponding linear operators, both approaches being highly non-trivial.

In the following chapters, we make much use of the notion of a Lax pair in order to generate noncommutative counterparts of some well-known evolution equations.

1.2.2 Hirota bilinear form

As alluded to above, the Inverse Scattering Transform is a powerful tool used to solve a range of initial value problems for nonlinear evolution equations. Although this method is recognised as being one of the major advances in the field of integrable systems, the transformation is far from trivial, requiring sophisticated analytical methods and making strong assumptions regarding the equation under consideration. In an attempt to devise a solution method needing far fewer assumptions and hence applicable to a wider class of equations, Hirota devised his so-called 'bilinear' or 'direct' method which has become a highly regarded solution mechanism leading to multi-soliton solutions and a detailed understanding of soliton scattering. Indeed, the method is the most efficient known for finding soliton and multi-soliton solutions of integrable equations [39]. We give only a brief account of Hirota's method here - a far more detailed discussion can be found in the book [48] devoted to the subject.

Linearisation of a particular nonlinear partial differential equation enables an exact solution to be found with relative ease. However, not all such equations can be linearised, leading Hirota to establish a method to 'bilinearise' nonlinear evolution equations. The bilinear form is written in terms of a new dependent variable and of Hirota's bilinear differential operators. Once in this bilinear form, a perturbation method can be employed in order to find an exact solution. This is the essence of Hirota's direct method. Multi-soliton solutions can easily be obtained by combining soliton solutions. It should be noted here that the fact that an equation can be written in a bilinear form does not by itself imply the equation is integrable.

Example - the Kadomtsev-Petviashvili (KP) equation

As an example we consider the KPII equation, a two-dimensional generalisation of the KdV equation, namely, for $u = u(x, y, t)$,

$$(u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0. \tag{1.9}$$

By employing the dependent variable transformation [48]

$$u = 2(\log f)_{xx}, \tag{1.10}$$

known as a Cole-Hopf transformation, where $f = f(x, y, t)$ is a new dependent variable, substituting in (1.9), integrating twice and choosing constants of integration to be zero leaves

$$ff_{xt} - f_x f_t + ff_{xxxx} + 3f_{xx}^2 - 4f_x f_{xxx} + 3ff_{yy} - 3f_y^2 = 0. \tag{1.11}$$

(We note that transformation (1.10) is the same as that used in the case of the KdV equation, although in the KdV case, f is a function of x and t only).

Hirota noticed that the terms appearing on the left-hand side of (1.11) can be written in a more compact way: he introduced a new binary differential operator, commonly known as a D-operator which, when acting on a pair of functions a, b, is defined by

$$D_x^m D_y^n D_t^p (a \cdot b) = (\partial_x - \partial_{x'})^m \left(\partial_y - \partial_{y'}\right)^n (\partial_t - \partial_{t'})^p \, a(x, y, t) b(x', y', t') \Big|_{\substack{x'=x \\ y'=y \\ t'=t}} \tag{1.12}$$

for non-negative integers m, n and p, where ∂_x denotes $\dfrac{\partial}{\partial x}$ etc. It is then easy to show that (1.11) can be written in terms of these D-operators in the form

$$\left(D_x(D_t + D_x^3) + 3D_y^2\right) f \cdot f = 0. \tag{1.13}$$

This is the Hirota bilinear form of the KP equation.

In order to find soliton solutions of this bilinear equation, we introduce an arbitrary small parameter ϵ and assume that the function f may be expanded in integral powers of ϵ. We let

$$\begin{aligned} f &= 1 + \sum_{n=1}^{\infty} \epsilon^n f_n(x,y,t) \\ &= 1 + \epsilon f_1 + \epsilon^2 f_2 + \epsilon^3 f_3 + \ldots . \end{aligned} \tag{1.14}$$

Clearly this expansion would have to be truncated at some point and would thus give only an approximate solution. However, when considering a bilinear equation, an appropriate choice of f_1 is made so that the infinite expansion truncates with only a finite number of terms, and hence gives an exact solution.

Substituting into (1.13) and collecting like powers of ϵ, we have

$$\mathcal{O}(\epsilon): \quad B(f_1 \cdot 1 + 1 \cdot f_1) = 0, \tag{1.15a}$$

$$\mathcal{O}(\epsilon^2): \quad B(f_2 \cdot 1 + f_1 \cdot f_1 + 1 \cdot f_2) = 0, \tag{1.15b}$$

$$\mathcal{O}(\epsilon^3): \quad B(f_3 \cdot 1 + f_2 \cdot f_1 + f_1 \cdot f_2 + 1 \cdot f_3) = 0, \tag{1.15c}$$

$$\vdots$$

$$\mathcal{O}(\epsilon^r): \quad B\left(\sum_{m=0}^{r} f_{r-m} \cdot f_m\right) = 0 \tag{1.15d}$$

for some positive integer r, where $f_0 = 1$ and B denotes the bilinear operator

$$B = D_x(D_t + D_x^3) + 3D_y^2. \tag{1.16}$$

The coefficient of ϵ gives $B(f_1 \cdot 1) + B(1 \cdot f_1) = 0$ using the property of the bilinear operator such that $B(a \cdot b + c \cdot d) = B(a \cdot b) + B(c \cdot d)$. Thus

$$\left(\partial_x \left(\partial_t + \partial_x^3\right) + 3\partial_y^2\right) f_1 = 0 \tag{1.17}$$

by (1.12), since, for example, $D_x D_t(f_1 \cdot 1) = (f_1)_{xt}$, and so on. Here we assume that $(f_1)_x, (f_1)_y, (f_1)_t, \ldots \to 0$ as $x \to \infty$ [17]. It can be shown that, if f_1 takes the form

$$f_1 = \exp \eta_1, \tag{1.18}$$

where $\eta_1 = P_1x + Q_1y + \Omega_1t + \eta_1^0$ and $\Omega_1P_1 + P_1^4 + 3Q_1^2 = 0$ [17], then f_1 satisfies (1.17). Here, P_1, Q_1 and Ω_1 are constants, and η_1^0 denotes a phase constant.
The coefficient of ϵ^2 gives, after rearranging as before,

$$2\left(\partial_x\left(\partial_t + \partial_x^3\right) + 3\partial_y^2\right) f_2 = -B(f_1 \cdot f_1). \tag{1.19}$$

It can be shown that substitution of $f_1 = \exp\eta_1$, with η_1 defined as above, into the right-hand side of (1.19) gives zero, and hence we can choose $f_2 = 0$. Thus the infinite series (1.14) can be truncated as the finite sum

$$f = 1 + \epsilon f_1, \tag{1.20}$$

and by combining the parameter ϵ and the phase constant η_1^0, we have an exact solution of the bilinear form of the KP equation (1.13), namely

$$f = 1 + \exp\eta_1. \tag{1.21}$$

By substituting in (1.10), we obtain the 1-soliton solution, that is

$$u = \frac{P_1^2}{2}\mathrm{sech}^2\frac{\eta_1}{2}. \tag{1.22}$$

In order to obtain the 2-soliton solution, we use the linear superposition principle and choose

$$f_1 = \exp\eta_1 + \exp\eta_2, \tag{1.23}$$

where $\eta_i = P_ix + Q_iy + \Omega_it + \eta_i^0$ and $\Omega_iP_i + P_i^4 + 3Q_i^2 = 0$ for $i = 1, 2$. We then continue in the same manner as above to obtain a solution describing the interaction of two solitons, namely

$$f = 1 + \exp\eta_1 + \exp\eta_2 + \exp(\eta_1 + \eta_2 + A_{12}), \tag{1.24}$$

where the parameter A_{12} is connected to the phase-shift after the soliton collision.
In principle, we can obtain a solution describing the interaction of any number of solitons by continuing the perturbation calculation to higher orders. We call the solution describing the interaction of n solitons ($n \geq 1$) the n-soliton solution.

Noncommutative case

The noncommutative KP equation, namely

$$(v_t + v_{xxx} + 3v_xv_x)_x + 3v_{yy} - 3[v_x, v_y] = 0, \tag{1.25}$$

where $v = v(x, y, t)$ and $[v_x, v_y] = v_x v_y - v_y v_x$, was considered by Gilson and Nimmo in [36] and is obtained via the compatibility of the same Lax pair

$$L = \partial_x^2 + v_x - \partial_y, \tag{1.26a}$$

$$M = 4\partial_x^3 + 6v_x\partial_x + 3v_{xx} + 3v_y + \partial_t \tag{1.26b}$$

as is used in the commutative case, however the assumption that v and its derivatives commute is relaxed. (Note that, as mentioned on page 4, we have included the operator ∂_t in M above). For instance, v could be thought of as a matrix, in which case multiplication is the usual matrix multiplication. (In the case that variables do commute, differentiation of (1.25) with respect to x and setting $v_x = u$ (and $[v_x, v_y] = 0$) leads to the familiar commutative KP equation (1.9)). A more detailed introduction to noncommutativity will be given in the next chapter. Here we only wish to point out that in the noncommutative case, it is thought not possible to obtain a bilinear form of a nonlinear evolution equation such as the KP equation in a similar manner to the commutative case. We can attempt to use the same Cole-Hopf transformation, i.e. $v = 2(\log f)_{xx}$ for some new dependent variable $f = f(x, y, t)$, where f is a noncommutative variable, for example a matrix. In this case, we write

$$\begin{aligned}\log f &= \log\left(1 + (f-1)\right)\\ &= \log(1+g), \text{ say, where } g = f-1\\ &= g - \frac{g^2}{2} + \frac{g^3}{3} - \dots,\end{aligned} \tag{1.27}$$

so that

$$(\log f)_x = g_x - \frac{1}{2}(g_x g + g g_x) + \frac{1}{3}(g_x g^2 + g g_x g + g^2 g_x) - \dots. \tag{1.28}$$

However, we see that, with each subsequent differentiation, the resulting expressions will become increasingly complicated, and thus it is not possible to obtain a compact form similar to (1.11) expressible in terms of D-operators as in (1.13). Consequently, it is not appropriate to try to obtain a bilinear form of a noncommutative integrable equation in this manner. We are, however, able to obtain a noncommutative analogue of a bilinear form as we shall see in later chapters.

1.2.3 Wronskian and Grammian determinants

Wronskian and Grammian determinants will play a major role in later chapters when we derive solutions of noncommutative integrable equations. Computing solutions of nonlinear evolution equations in terms of Wronskian or Grammian determinants, whether in the commutative or noncommutative case, is highly advantageous as they allow solution verification to be carried out in a fairly straightforward manner and enable the asymptotic properties of a solution to be analysed with relative ease.

In the noncommutative case, we introduce the notion of 'quasi-Wronskians' and 'quasi-Grammians' by extending the familiar definitions of Wronskians and Grammians in the commutative case. These commutative definitions are described below.

Wronskian determinants

We have seen that solutions obtained using Hirota's direct method can be written in the form of exponential functions, and it then follows that the n-soliton solution to the evolution equation under consideration can be expressed as an n^{th}-order polynomial in n exponentials. As we have discussed, the IST is a somewhat more complicated method used to solve nonlinear integrable equations, where an n-soliton solution is obtained in the form of some function of an $n \times n$ determinant [26].

However, although both methods give the required n-soliton solution, verification of this solution by direct substitution is far from easy, as the derivatives of the soliton solution cannot be expressed in a simple, compact manner. For example, differentiating an $n \times n$ determinant gives rise to a *sum* of n determinants, which, in the case of large n, will be a complicated expression on which to work. To overcome this problem, the notion of a *Wronskian* determinant (or more often simply a 'Wronskian') is often introduced, and solution verification by direct substitution can then be implemented easily.

For functions $\phi_i = \phi_i(x, \ldots)$ $(i = 1, 2, \ldots, n)$ of x and possibly infinitely many other variables, the n^{th}-order Wronskian of $\phi_1, \ldots, \phi_n$, commonly denoted $W(\phi_1, \ldots, \phi_n)$, is an $n \times n$ determinant defined by [48]

$$W(\phi_1, \ldots, \phi_n) := \det\left(\partial_x^{j-1}\phi_i\right)_{1 \le i,j \le n} = \begin{vmatrix} \phi_1^{(0)} & \ldots & \phi_n^{(0)} \\ \vdots & & \vdots \\ \phi_1^{(n-1)} & \ldots & \phi_n^{(n-1)} \end{vmatrix}, \tag{1.29}$$

where $\phi_i^{(k)}$ denotes the k^{th} x-derivative of $\phi_i(x)$, i.e.

$$\phi_i^{(k)} = \partial_x^k \phi_i \qquad (k = 0, 1, \ldots, n-1). \tag{1.30}$$

The Wronskian (1.29) is frequently written in the more compact notation introduced by Freeman and Nimmo [25] as

$$W(\phi_1, \ldots, \phi_n) = (\widehat{n-1}), \tag{1.31}$$

where the 'hat' indicates the presence of consecutive derivatives up to order $n-1$.

One of the main advantages of Wronskian determinants comes to light when we consider their derivative: taking the Wronskian $W(\phi_1, \ldots, \phi_n)$ above and differentiating once with respect to x gives

$$\partial_x W(\phi_1, \ldots, \phi_n) = \begin{vmatrix} \phi_1^{(0)} & \ldots & \phi_n^{(0)} \\ \vdots & & \vdots \\ \phi_1^{(n-2)} & \ldots & \phi_n^{(n-2)} \\ \phi_1^{(n)} & \ldots & \phi_n^{(n)} \end{vmatrix}, \tag{1.32}$$

where we have used the fact that a determinant with two identical rows is zero, and hence the only contribution to the derivative of $W(\phi_1, \ldots, \phi_n)$ comes from differentiating the final row of the Wronskian. Thus the derivative of a Wronskian is a *single* determinant. Further differentiations lead to a sum of determinants, however the length of the sum depends not on the order of the determinant but on the number of differentiations carried out. For example, considering the same Wronskian as above, we have

$$\begin{aligned} \partial_x^2 W(\phi_1, \ldots, \phi_n) &= \begin{vmatrix} \phi_1^{(0)} & \ldots & \phi_n^{(0)} \\ \vdots & & \vdots \\ \phi_1^{(n-3)} & \ldots & \phi_n^{(n-3)} \\ \phi_1^{(n-1)} & \ldots & \phi_n^{(n-1)} \\ \phi_1^{(n)} & \ldots & \phi_n^{(n)} \end{vmatrix} + \begin{vmatrix} \phi_1^{(0)} & \ldots & \phi_n^{(0)} \\ \vdots & & \vdots \\ \phi_1^{(n-2)} & \ldots & \phi_n^{(n-2)} \\ \phi_1^{(n+1)} & \ldots & \phi_n^{(n+1)} \end{vmatrix} \\ &= (\widehat{n-3}, n-1, n) + (\widehat{n-2}, n+1), \end{aligned} \tag{1.33}$$

where we have used an extension of the compact notation introduced previously, namely [25]

$$(\widehat{n-i}, n-k_1, n-k_2, \ldots, n-k_{i-1}),$$

with the 'hat' defined as before, and the $n-k_j$ $(j = 1, 2, \ldots, i-1)$ denoting the $(n-k_j)^{\text{th}}$ derivative of the row $(\phi_1, \ldots, \phi_n)$. The constant k_j is not one of the integers $0, \ldots, i$.

The fact that differentiation of a Wronskian leads to a single determinant and not a sum of determinants as is obtained on differentiation of an ordinary determinant therefore results in a far more straightforward manipulation of Wronskians.

In the general theory of partial differential equations, the functions ϕ_i, and also their corresponding Wronskian, will depend on x and other variables y, t etc., and will satisfy linear partial differential equations with constant coefficients in the variables x, y, t and so on. Such linear equations can be obtained from the Lax pair of the nonlinear evolution equation under consideration and are called the *dispersion relations* for the system. Thus we are always able to relate derivatives of ϕ_i with respect to y, t etc. to the derivatives with respect to x, therefore enabling us to express, for example, $\partial_y W(\phi_1, \dots, \phi_n)$, $\partial_t W(\phi_1, \dots, \phi_n)$, and so on, in terms of $W(\phi_1, \dots, \phi_n)$, where $W(\phi_1, \dots, \phi_n)$ consists of the functions ϕ_i and their x-derivatives only.

Example - the KP equation

To understand how the notion of a Wronskian can be put into practise, we introduce the (commutative) KP equation (1.9) as an example. Wronskian solutions of this equation were obtained by Freeman and Nimmo in 1983 [25], however several years previously, Satsuma [80] had noted that the n-soliton solution of the KdV and mKdV equations could be expressed in Wronskian form, and Freeman, Horrocks and Wilkinson had also made advances in the area [23]. Following these notable achievements, Wronskian solutions of other equations, for example the Boussinesq [24], sine-Gordon [26], nonlinear Schrödinger [22] and Davey-Stewartson [22] equations were subsequently obtained.

We have already discussed, in Section 1.2.2, the solution of the KP equation (1.9) obtained from Hirota's direct method, which can be conveniently written in Hirota's notation as

$$\left(D_x(D_t + D_x^3) + 3D_y^2\right) f \cdot f = 0, \tag{1.34}$$

with the D-operators defined as in (1.12). Freeman and Nimmo conjectured that f could be expressed in the form of a Wronskian, namely, for functions $\phi_i = \phi_i(x, y, t)$ $(i = 1, \dots, n)$,

$$f = W(\phi_1, \dots, \phi_n), \tag{1.35}$$

with the Wronskian determinant defined as in (1.29) and written more compactly as

$$f = (\widehat{n-1}). \tag{1.36}$$

Here, the functions ϕ_i satisfy the linear partial differential equations

$$\partial_x^2\phi_i - \partial_y\phi_i = 0, \tag{1.37a}$$

$$4\partial_x^3\phi_i + \partial_t\phi_i = 0, \tag{1.37b}$$

the dispersion relations for the system. These come from considering the Lax pair (1.26) for the KP equation in the trivial vacuum case, that is, when $u = 0$. We calculate the x-derivatives of $f = (\widehat{n-1})$ as before, so that

$$f_x = (\widehat{n-2}, n), \tag{1.38a}$$

$$f_{xx} = (\widehat{n-3}, n-1, n) + (\widehat{n-2}, n+1), \tag{1.38b}$$

and similarly for further x-derivatives, and use the dispersion relations (1.37) to determine the y- and t-derivatives, namely

$$f_y = \left(-(\widehat{n-3}, n-1, n) + (\widehat{n-2}, n+1)\right), \tag{1.39}$$

$$f_t = 4\left(-(\widehat{n-4}, n-2, n-1, n) + (\widehat{n-3}, n-1, n+1) - (\widehat{n-2}, n+2)\right) \tag{1.40}$$

and so on. Substitution of the appropriate derivatives into (1.11), the expanded form of (1.34), gives

$$6\Big((\widehat{n-3}, n-2, n-1)(\widehat{n-3}, n, n+1) - (\widehat{n-3}, n-2, n)(\widehat{n-3}, n-1, n+1) + (\widehat{n-3}, n-2, n+1)(\widehat{n-3}, n-1, n)\Big) = 0. \tag{1.41}$$

The left-hand side can be shown to be the Laplace expansion of a $2n \times 2n$ determinant which is equal to zero, thus verifying the Wronskian solution (1.35). Appropriate forms of the ϕ_i can be chosen in order to generate soliton solutions, see, for example, [25] for details.

The above procedure highlights the simplicity of solution verification in the Wronskian case, thus explaining their widespread use as a solution-generating technique.

Grammian determinants

A Grammian determinant, the determinant of a Gram or Grammian matrix (often written 'Gramian') and named after the Danish actuary Jørgen Pedersen Gram, is one whose elements are in integral form. Specifically, a Grammian determinant $G := \det(g_{ij})_{1\le i,j\le n}$ is the determinant of a matrix with entries [48]

$$g_{ij} := \int_a^b f_i f_j \, \mathrm{d}x \tag{1.42}$$

for real-valued functions f_i, f_j defined on the closed interval $[a, b]$.

As in the Wronskian case, the n-soliton solutions of various nonlinear evolution equations can be expressed in Grammian form. Although we have seen the advantage of the Wronskian technique in that it possesses a simpler form of derivative than an ordinary determinant, the Grammian method of solution is far more powerful. Whereas an n^{th}-order Wronskian (i.e. an $n \times n$ determinant taking the form of a Wronskian) consists of $n-1$ derivatives of a function, say ϕ_i $(i = 1, \ldots, n)$ (thus requiring us to carry out each of these differentiations in turn), in contrast, an n^{th}-order Grammian determinant requires only a single integration. Also, as we shall see later in the noncommutative case, verification of the reality of an obtained solution is far easier in the Grammian than in the Wronskian case.

Example - the KP equation

We again look at the example of the KP equation and show how the n-soliton solution can be expressed in Grammian form.

Nakamura [67] was the first to consider soliton solutions of the KP equation in Grammian form. He noted that the Grammian determinant is related to the determinant with integral entries often used in the IST. However, Nakamura's Grammian approach avoids the need to utilise the Gel'fand-Levitan-Marchenko integral equation of inverse scattering and instead alludes to a Jacobi identity of linear matrix algebra. This will be shown in more detail later when we consider the DS equations and compare our work in the noncommutative case with earlier work by Gilson and Nimmo in the commutative case. We will see that verification of Gilson and Nimmo's Grammian solution in the commutative case is done using a Jacobi identity. Details of such identities will be discussed in the next chapter.

We once again consider the bilinear form of the KP equation (1.13), namely

$$\left(D_x(D_t + D_x^3) + 3D_y^2\right) f \cdot f = 0. \tag{1.43}$$

(Note that the bilinear form considered by Nakamura is a scaled version of the one above). We do not give details here, however Nakamura showed that the n-soliton solution could be expressed in the Grammian form

$$f = \det(h_{ij})_{1\leq i,j\leq n}, \tag{1.44}$$

where

$$h_{ij} = c_{ij} + \int_{-\infty}^{x} \phi_i \psi_j \, \mathrm{d}x, \tag{1.45}$$

with c_{ij} arbitrary constants and $\phi_i = \phi_i(x,y,t)$, $\psi_j = \psi_j(x,y,t)$ satisfying the linear partial differential equations

$$\left(4\partial_x^3 + \partial_t\right)\phi_i = 0, \qquad \left(4\partial_x^3 + \partial_t\right)\psi_j = 0, \tag{1.46a}$$

$$\left(\partial_x^2 - \partial_y\right)\phi_i = 0, \qquad \left(\partial_x^2 + \partial_y\right)\psi_j = 0. \tag{1.46b}$$

A rather lengthy calculation and use of a Jacobi identity then proves that f given by (1.44) is indeed a solution to the bilinear KP equation (1.43).

In contrast, in the noncommutative case, we find that verification of our 'quasi-Grammian' solution to the noncommutative DS equations does not require use of an identity - by direct substitution into a noncommutative analogue of the bilinear form of the equations, the solution is verified immediately. This is also the case when we verify our quasi-Wronskian solution, so that, as was also noted by Gilson and Nimmo in [36], in some sense solution verification is actually easier in the noncommutative than in the commutative case. This will become apparent in Chapter 3.

1.3 Thesis outline

This thesis is organised as follows. Chapter 2 provides background information to noncommutativity and noncommutative integrable systems, and introduces the definition of noncommutativity that will be used throughout the thesis. We also introduce the definition of a quasideterminant, one of the major tools in the study of noncommutative integrable systems, and provide some important properties of quasideterminants. The chapter concludes with some motivation as to why we have chosen to study our particular choice of noncommutative integrable equations and their quasideterminant solutions.

Chapters 3 and 4 concern the Davey-Stewartson (DS) equations in a noncommutative setting. In Chapter 3, we begin by briefly detailing the physical background, and how we can transform between two equivalent forms of the equations. Section 3.3 shows the procedure used to derive a noncommutative version of the commutative DS equations considered by Ablowitz and Schultz.

In Section 3.4, we introduce the concept of a Darboux transformation, an iterative process enabling us to generate quasideterminant solutions of our system of noncommutative DS equations. We explain how an ordinary Darboux transformation can be used in the generation of quasi-Wronskian solutions, while a binary Darboux transformation leads to quasi-Grammian solutions. The section that follows gives a direct verification that the obtained quasideterminant solutions are indeed solutions of our system of noncommutative DS equations.

We move on in Section 3.6 to compare our solution method with that of Gilson and Nimmo, who obtain Grammian solutions of a system of commutative DS equations. We emphasise that aspects of solution verification in the noncommutative case are surprisingly more straightforward than in the commutative case.

Chapter 4 is devoted to the calculation and depiction of a special kind of solution to the noncommutative DS equations, namely dromions. We discover some of the complexities of obtaining such solutions in a noncommutative setting, however the more simple dromion solutions can be calculated and plotted with relative ease.

In Chapter 5, we detail the procedure used to carry out the dimensional reduction of a Darboux transformation, from $(2 + 1)$-dimensions (two spatial and one time dimension) to $(1 + 1)$-dimensions (one spatial and one time dimension). We recall the $(2 + 1)$-dimensional ordinary Darboux transformation in Section 5.1, and, in Section 5.2, show how to reduce this to a $(1 + 1)$-dimensional transformation using the DS and nonlinear Schrödinger (NLS) equations as examples. Section 5.3 details the dimensional reduction of the $(2 + 1)$-dimensional binary Darboux transformation. We note in Section 5.5 that the dimensionally-reduced Darboux and binary Darboux transformations must be modified slightly in order to be applicable to the Sasa-Satsuma NLS (SSNLS) equation studied in the next chapter.

Chapter 6 focuses on higher-order NLS equations, $(1+1)$-dimensional integrable equations based on the simple NLS equation but with higher-order terms. The first section of the chapter provides background information, and also details the higher-order NLS equations known to be integrable. Section 6.2 explains how one such equation, namely the SSNLS equation, can be obtained from a reduction of the 3-component KP hierarchy.

We then proceed to discuss noncommutative versions of various integrable higher-order NLS equations, deriving two of these via the same Lax pair approach as for the noncommutative DS equations in Chapter 3. We choose to focus our attention on the noncommutative SSNLS equation and, in Section 6.4, apply the dimensionally-reduced Darboux transformation obtained in Chapter 5 to this noncommutative equation. We find that certain properties are not preserved by the reduced ordinary Darboux transformation, and we must therefore allude to the reduced binary Darboux transformation to obtain a quasi-Grammian solution which can once again be directly verified. Section 6.5 details the procedure used to obtain soliton solutions in both the commutative and noncommutative settings.

We conclude in Chapter 7 by summarising our findings and discussing some open problems that could be investigated in future work.

Chapter 2

Noncommutative integrable systems

2.1 Introduction

This thesis is concerned with various integrable equations in a noncommutative setting. In this chapter, we introduce the idea of noncommutativity and some of the many ways in which noncommutativity can be defined. As shall be explained below, we choose to define noncommutativity in a very general manner, and only specify the nature of the noncommutativity under consideration when we calculate particular solutions of the equation of interest.

We also introduce the idea of a quasideterminant, a representation of a determinant in a noncommutative setting. Quasideterminants will be used extensively throughout the thesis.

2.2 Definitions of noncommutativity

In simple terms, we say that a binary operation $*$ on a set S is *commutative* if

$$x * y = y * x \tag{2.1}$$

for all $x, y \in S$. In other words, the order of the terms does not affect the final result. Any operation that does not satisfy this property, for example matrix multiplication, is said to be *noncommutative*.

An example of noncommutativity arises in terms of a star product, an associative but noncommutative product with a Poisson bracket. In this case, noncommutativity is defined in terms of the coordinates over which the given integrable equation is specified, rather than the functions present in the equation. In recent years, Hamanaka and Toda [42] have derived a number of noncommutative integrable equations with noncommutativity defined by a Moyal star product, for example the noncommutative KdV equation,

$$u_t + \frac{3}{4}(u_x \star u + u \star u_x) + \frac{1}{4}u_{xxx} = 0, \tag{2.2}$$

where $u = u(x,t)$, and the noncommutative KP equation,

$$u_t + \frac{1}{4}u_{xxx} + \frac{3}{4}(u_x \star u + u \star u_x) + \frac{3}{4}\partial_x^{-1}u_{yy} + \frac{3}{4}\left[u, \partial_x^{-1}u_y\right]_\star = 0 \tag{2.3}$$

for $u = u(x,y,t)$, $\partial_x^{-1} f(x) = \int^x dx'\, f(x')$ and $u_{xxx} = \dfrac{\partial^3 u}{\partial x^3}$. This noncommutativity of coordinates, in either the spatial coordinates or the spatial and time coordinates, is realised by replacing the ordinary products of the fields with star products. Equations (2.2) and (2.3) can be seen, up to suitable scaling, to reduce back to the commutative KdV (1.1) and KP (1.9) equations respectively when $\star$ is standard multiplication and we assume commutativity, i.e. we assume that $u \star u_x = uu_x = u_x \star u = u_x u$. We introduce the definition of the star product via an example.

Example

In general, for noncommutative coordinates x^k and x^l, we define

$$[x^k, x^l]_\star := x^k \star x^l - x^l \star x^k = \mathrm{i}\theta^{kl} \tag{2.4}$$

for some nonzero real constant θ^{kl} and $\mathrm{i} = \sqrt{-1}$. It then follows that

$$\theta^{kl} = -\theta^{lk}. \tag{2.5}$$

We consider a particular case where f and g are arbitrary functions of three variables (coordinates) x^1, x^2, x^3, so that $f = f(\mathbf{x})$, $g = g(\mathbf{x})$ for $\mathbf{x} = (x^1, x^2, x^3)$. We suppose that two of the coordinates, say x^1 and x^2, are noncommutative, so that

$$[x^1, x^2]_\star := x^1 \star x^2 - x^2 \star x^1 = \mathrm{i}\theta^{12} \tag{2.6}$$

and $\theta^{11} = \theta^{22} = \theta^{33} = 0$ by (2.5). The star product of f and g is then given by [41]

$$\begin{aligned} f \star g(\mathbf{x}) &:= \exp\left(\sum_{k,l=1}^{3} \frac{\mathrm{i}}{2}\theta^{kl}\partial_k^{(\mathbf{x}')}\partial_l^{(\mathbf{x}'')}\right) f(\mathbf{x}')g(\mathbf{x}'')|_{\mathbf{x}'=\mathbf{x}''=\mathbf{x}} \\ &= \exp\left(\frac{\mathrm{i}}{2}\theta^{12}\partial_1^{(\mathbf{x}')}\partial_2^{(\mathbf{x}'')} + \frac{\mathrm{i}}{2}\theta^{21}\partial_2^{(\mathbf{x}')}\partial_1^{(\mathbf{x}'')}\right) f(\mathbf{x}')g(\mathbf{x}'')|_{\mathbf{x}'=\mathbf{x}''=\mathbf{x}} \\ &= f(\mathbf{x})g(\mathbf{x}) + \frac{\mathrm{i}}{2}\theta^{12}\big(\partial_1 f(\mathbf{x})\partial_2 g(\mathbf{x}) - \partial_2 f(\mathbf{x})\partial_1 g(\mathbf{x})\big) + \mathcal{O}\left((\theta^{12})^2\right), \end{aligned} \tag{2.7}$$

where $\partial_k^{(\mathbf{x}')} = \dfrac{\partial}{\partial x^{k'}}$. (In the above definition, the sum over k, l tends to be omitted from papers). This explicit representation is known as the Groenewold-Moyal product and, in the commutative limit $\theta^{12} \to 0$, reduces to the ordinary product fg.

In this thesis, we follow the same approach as that used by, for example, Gilson and Nimmo in [36] (noncommutative KP equation), Gilson, Nimmo and Sooman in [38] (noncommutative mKP equation) - we adopt a very general approach and do not initially specify the nature of the noncommutativity under consideration; that is, we simply assume that, for functions f, g, multiplication of f by g is noncommutative: $fg \neq gf$, i.e. $[f, g] \neq 0$. (Here we are assuming noncommutativity of *functions*, rather than coordinates as in the star product above. We could think of f, g as matrices for example). To derive a particular integrable equation in this manner, we utilise the same Lax pair, L, M, say, as in the commutative case but assume no commutativity when calculating the commutator $[L, M]$. This results in a noncommutative version of the integrable equation under consideration, which reduces back to the commutative equation when we relax the noncommutative condition.

For example, as we saw in Section 1.2.2, in their work on the noncommutative KP equation [36], Gilson and Nimmo consider the Lax pair

$$L = \partial_x^2 + v_x - \partial_y, \tag{2.8a}$$

$$M = 4\partial_x^3 + 6v_x\partial_x + 3v_{xx} + 3v_y + \partial_t \tag{2.8b}$$

for $v = v(x, y, t)$, and set the commutator $[L, M] = 0$. Expanding and assuming no commutativity of variables (so that $v_x v_{xx} \neq v_{xx} v_x$ and so on) gives a noncommutative KP equation, namely

$$(v_t + v_{xxx} + 3v_x v_x)_x + 3v_{yy} - 3[v_x, v_y] = 0, \tag{2.9}$$

which, by relaxing the noncommutativity condition, can be differentiatied with respect to x to give, with v_x set equal to u, the well-known commutative KP equation (1.9). We adopt this approach in Chapter 3, where we derive a noncommutative version of the DS equations, and in Chapter 6 when deriving noncommutative Hirota and Sasa-Satsuma NLS equations. Although initially we choose not to specify the nature of our noncommutativity, in Chapter 4 and the later sections of Chapter 6 we make the noncommutativity more explicit by choosing the functions in our noncommutative equations to be of matrix form. This then enables solutions to be calculated and plotted for this particular case of noncommutativity.

2.3 Quasideterminants

Here we briefly recall some of the properties of quasideterminants. A more detailed analysis can be found in the original papers [28, 29].

The notion of a *quasideterminant* was first introduced by Gel'fand and Retakh in [29] as a straightforward way to define the determinant of a matrix with noncommutative entries. Many equivalent definitions of quasideterminants exist, the simplest involving inverse minors. Let $A = (a_{ij})$ be an $n \times n$ matrix with entries over a usually noncommutative unitary ring $\mathcal{R}$. We denote the $(i,j)^{\text{th}}$ quasideterminant by $|\,A\,|_{ij}$, where

$$|\,A\,|_{ij} = a_{ij} - r_i^j (A^{ij})^{-1} s_j^i. \tag{2.10}$$

Here, A^{ij} is the $(n-1)\times(n-1)$ minor matrix obtained from A by deleting the i^{th} row and j^{th} column (note that this matrix must be invertible), r_i^j is the row vector obtained from the i^{th} row of A by deleting the j^{th} entry, and s_j^i is the column vector obtained from the j^{th} column of A by deleting the i^{th} entry.

A common notation employed when discussing quasideterminants is to 'box' the expansion element, i.e. we write

$$|\,A\,|_{11} = \begin{vmatrix} \boxed{a_{11}} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} - a_{12}{a_{22}}^{-1}a_{21} \tag{2.11}$$

to denote the $(1,1)^{\text{th}}$ quasideterminant of a 2×2 matrix $A = (a_{ij})$ $(i,j = 1,2)$. It should be noted that the above expansion formula is also valid in the case of block matrices, provided the matrix to be inverted is square.

Quasideterminants also provide a useful formula for the inverse of a matrix: for an invertible $n \times n$ matrix $A = (a_{ij})\,(i,j = 1,\dots,n)$, the $(i,j)^{\text{th}}$ entry of A^{-1} is given by

$$(A^{-1})_{ij} = \left(|\,A\,|_{ji}\right)^{-1}, \tag{2.12}$$

so that, for $n = 2$ and $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$,

$$A^{-1} = \begin{pmatrix} (a - bd^{-1}c)^{-1} & (c - db^{-1}a)^{-1} \\ (b - ac^{-1}d)^{-1} & (d - ca^{-1}b)^{-1} \end{pmatrix}. \tag{2.13}$$

This formula is required when finding quasideterminants of larger matrices ($n > 2$).

When the elements of A commute, the quasideterminant $|A|_{ij}$ is not simply the determinant of A, but rather a *ratio* of determinants: it is well-known that, for A invertible, the $(j, i)^{\text{th}}$ entry of A^{-1} is

$$(-1)^{i+j} \frac{\det A^{ij}}{\det A}.$$

Then, by (2.12), we can easily see that

$$|A|_{ij} = (-1)^{i+j} \frac{\det A}{\det A^{ij}} \tag{2.14}$$

in the commutative case.

Quasideterminants possess certain row and column multiplication properties. In short, if the i^{th} row of a quasideterminant is left- (right)-multiplied by some element $\lambda \in \mathcal{R}$ and all other rows remain unchanged, this has the effect of left- (right)-multiplying the quasideterminant by λ. A similar result holds for columns. A more detailed explanation of these results, along with many other results relating to quasideterminants, can be found in [28]. Before moving on, we do however detail several important quasideterminant identities which will be useful in later chapters.

2.3.1 Quasideterminant identities

We begin by stating the Jacobi identity for commutative determinants, which is a powerful tool used to verify Grammian-type solutions of an integrable equation. We follow the notation given in [48], where the Jacobi identity stated is identical to the commutative version of the Sylvester identity given in, for example [28]. Hence we could also refer to the Jacobi identity as a (commutative) Sylvester identity.

For $i, j = 1, \ldots, n$, consider an $n \times n$ determinant $D = \det(a_{i,j})$. We denote the $(j, k)^{\text{th}}$ *minor* of D, that is, the $(n-1) \times (n-1)$ determinant obtained by eliminating the j^{th} row and k^{th} column of D, by $D\begin{bmatrix} j \\ k \end{bmatrix}$. Similarly, the $(n-2) \times (n-2)$ determinant obtained by eliminating both the j^{th} and k^{th} rows and the l^{th} and m^{th} columns from D is denoted

$D\begin{bmatrix} j & k \\ l & m \end{bmatrix}$. Then the Jacobi identity states that [48]

$$D\begin{bmatrix} i & j \\ i & j \end{bmatrix} D = D\begin{bmatrix} i \\ i \end{bmatrix} D\begin{bmatrix} j \\ j \end{bmatrix} - D\begin{bmatrix} i \\ j \end{bmatrix} D\begin{bmatrix} j \\ i \end{bmatrix}. \tag{2.15}$$

For example, suppose $n = 3$, so that

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}. \tag{2.16}$$

Taking, for instance, $i = 2$, $j = 3$ in (2.15) gives

$$a_{11}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}. \tag{2.17}$$

This can easily be verified by direct calculation, as can results for larger n, using a computer package if necessary. A proof that the Jacobi identity (2.15) holds for all n can be found in [48].

Noncommutative case

In their paper of 1991 [29], Gel'fand and Retakh define a noncommutative version of the Sylvester identity, valid in the case of quasideterminants. We describe this result below and shall then see that, if we reduce to the commutative case by writing each quasideterminant as a ratio of determinants, we arrive at the commutative Jacobi identity described above. We will see that, perhaps surprisingly, the Sylvester identity for quasideterminants in the noncommutative case appears more straightforward than the Jacobi identity for determinants in the commutative case [30].

For $i, j = 1, \dots, n$, let $A = (a_{ij})$ be a matrix over a (not necessarily commutative) ring $\mathcal{R}$ and, for $i, j = 1, \dots, k$ $(k \leq n)$, let $A_0 = (a_{ij})$ be a $k \times k$ submatrix of A assumed to be invertible over $\mathcal{R}$. We define, for $p, q = k + 1, \dots, n$, quasideterminants

$$c_{pq} = \begin{vmatrix} & A_0 & & a_{1q} \\ & & & \vdots \\ & & & a_{kq} \\ a_{p1} & \dots & a_{pk} & \boxed{a_{pq}} \end{vmatrix}. \tag{2.18}$$

Now consider the $(n-k)\times(n-k)$ matrix $C=(c_{pq})$, a matrix whose entries consist of quasideterminants. Then [29]

Theorem 1 *For* $i,j=k+1,\dots,n$,

$$|\,A\,|_{ij}=|\,C\,|_{ij}\ . \tag{2.19}$$

We do not give details of the proof, but outline an example. Suppose $n=3$, with $A=(a_{ij})$ $(i,j=1,2,3)$, and take the submatrix $A_0=(a_{11})$, so that $k=1$. We assume that A_0 is invertible. Then

$$\begin{aligned} c_{22}&=\begin{vmatrix} a_{11} & a_{12}\\ a_{21} & \boxed{a_{22}}\end{vmatrix}, & c_{23}&=\begin{vmatrix} a_{11} & a_{13}\\ a_{21} & \boxed{a_{23}}\end{vmatrix},\\ c_{32}&=\begin{vmatrix} a_{11} & a_{12}\\ a_{31} & \boxed{a_{32}}\end{vmatrix}, & c_{33}&=\begin{vmatrix} a_{11} & a_{13}\\ a_{31} & \boxed{a_{33}}\end{vmatrix}, \end{aligned} \tag{2.20}$$

so that

$$C=\begin{pmatrix} c_{22} & c_{23}\\ c_{32} & c_{33}\end{pmatrix}=\begin{pmatrix} \begin{vmatrix} a_{11} & a_{12}\\ a_{21} & \boxed{a_{22}}\end{vmatrix} & \begin{vmatrix} a_{11} & a_{13}\\ a_{21} & \boxed{a_{23}}\end{vmatrix}\\ \\ \begin{vmatrix} a_{11} & a_{12}\\ a_{31} & \boxed{a_{32}}\end{vmatrix} & \begin{vmatrix} a_{11} & a_{13}\\ a_{31} & \boxed{a_{33}}\end{vmatrix}\end{pmatrix}. \tag{2.21}$$

Then, for example, by Theorem 1, we see that

$$|\,A\,|_{33}=|\,C\,|_{33}, \tag{2.22}$$

i.e.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & \boxed{a_{33}}\end{vmatrix}=\begin{vmatrix} a_{11} & a_{13}\\ a_{31} & \boxed{a_{33}}\end{vmatrix}-\begin{vmatrix} a_{11} & a_{12}\\ a_{31} & \boxed{a_{32}}\end{vmatrix}\begin{vmatrix} a_{11} & a_{12}\\ a_{21} & \boxed{a_{22}}\end{vmatrix}^{-1}\begin{vmatrix} a_{11} & a_{13}\\ a_{21} & \boxed{a_{23}}\end{vmatrix}. \tag{2.23}$$

(It should be noted here that the notation $|\,C\,|_{33}$ means that we expand C as given in (2.21) about the *entry* c_{33}, not, as in the case of $|\,A\,|_{33}$, the entry in *position* $(3,3)$ (such an entry does not exist in C, being only of size 2×2)).

From (2.23), we see that

$$\begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & \boxed{a_{33}}\end{vmatrix}\begin{vmatrix} a_{11} & a_{13}\\ a_{21} & \boxed{a_{23}}\end{vmatrix}^{-1}=\begin{vmatrix} a_{11} & a_{13}\\ a_{31} & \boxed{a_{33}}\end{vmatrix}\begin{vmatrix} a_{11} & a_{13}\\ a_{21} & \boxed{a_{23}}\end{vmatrix}^{-1}-\begin{vmatrix} a_{11} & a_{12}\\ a_{31} & \boxed{a_{32}}\end{vmatrix}\begin{vmatrix} a_{11} & a_{12}\\ a_{21} & \boxed{a_{22}}\end{vmatrix}^{-1}. \tag{2.24}$$

Also, choosing $i = 3$ and $j = 2$ in Theorem 1 and $A_0 = (a_{11})$ gives

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & \boxed{a_{32}} & a_{33} \end{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & \boxed{a_{22}} \end{vmatrix}^{-1} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & \boxed{a_{32}} \end{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & \boxed{a_{22}} \end{vmatrix}^{-1} - \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & \boxed{a_{33}} \end{vmatrix} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & \boxed{a_{23}} \end{vmatrix}^{-1} . \tag{2.25}$$

Thus, by comparing (2.24) and (2.25), we have the relation

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & \boxed{a_{33}} \end{vmatrix} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & \boxed{a_{23}} \end{vmatrix}^{-1} = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & \boxed{a_{32}} & a_{33} \end{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & \boxed{a_{22}} \end{vmatrix}^{-1} , \tag{2.26}$$

with others following in the same way using an appropriate choice of submatrix A_0. We will see shortly that the row and column homological relations for quasideterminants are identical to relations of the form (2.26).

We can express (2.23) in a more useful form as

$$\begin{vmatrix} A & B & C \\ D & f & g \\ E & h & \boxed{i} \end{vmatrix} = \begin{vmatrix} A & C \\ E & \boxed{i} \end{vmatrix} - \begin{vmatrix} A & B \\ E & \boxed{h} \end{vmatrix} \begin{vmatrix} A & B \\ D & \boxed{f} \end{vmatrix}^{-1} \begin{vmatrix} A & C \\ D & \boxed{g} \end{vmatrix} , \tag{2.27}$$

where A is an $M \times M$ matrix (say), B, C are $M \times 1$ columns, D, E $1 \times M$ rows, and f, g, h, i single entries. Taking C to be the $M \times 1$ zero column, $g = 1$ and $i = 0$ gives

$$\begin{aligned} \begin{vmatrix} A & B & 0 \\ D & f & 1 \\ E & h & \boxed{0} \end{vmatrix} &= \begin{vmatrix} A & 0 \\ E & \boxed{0} \end{vmatrix} - \begin{vmatrix} A & B \\ E & \boxed{h} \end{vmatrix} \begin{vmatrix} A & B \\ D & \boxed{f} \end{vmatrix}^{-1} \begin{vmatrix} A & 0 \\ D & \boxed{1} \end{vmatrix} \\ &= - \begin{vmatrix} A & B \\ E & \boxed{h} \end{vmatrix} \begin{vmatrix} A & B \\ D & \boxed{f} \end{vmatrix}^{-1} . \end{aligned} \tag{2.28}$$

In the commutative case, we write each quasideterminant in (2.23) as a ratio of determinants using (2.14), so that

$$a_{11} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} , \tag{2.29}$$

which we see matches the result of the Sylvester identity in (2.17).

A result such as that obtained in (2.26) can be seen to be a *homological relation* as derived by Gel'fand and Retakh in [29]. An $n \times n$ square matrix $A = (a_{ij})$ is considered, along with an $(n-1) \times (n-1)$ minor matrix A^{kl} $(k, l = 1, \ldots, n)$ obtained from A by removing the k^{th} row and l^{th} column. Then Gel'fand and Retakh showed that quasideterminants of the matrix A and its minors are connected by the following homological relations:

Theorem 2 *(i) Row homological relations*

$$-\,|A|_{ij} \cdot |A^{il}|_{sj}^{-1} = |A|_{il} \cdot |A^{ij}|_{sl}^{-1}, \quad s \neq i. \tag{2.30}$$

(ii) Column homological relations

$$-\,|A^{kj}|_{it}^{-1} \cdot |A|_{ij} = |A^{ij}|_{kt}^{-1} \cdot |A|_{kj}, \quad t \neq j. \tag{2.31}$$

For example, choosing A to be a 3×3 matrix with $l = s = 2$ and $i = j = 3$ in (2.30) gives the same result as that obtained from the noncommutative Sylvester identity in (2.26).

Quasideterminant invariance properties

We now show that, similar to determinants, quasideterminants possess properties invariant under elementary row and column operations. We will see in particular that we can subtract rows from the expansion row and leave the quasideterminant unchanged.

Consider, for $(n-1) \times (n-1)$ matrices A, E, $1 \times (n-1)$ row vectors F, C, $(n-1) \times 1$ column vectors O, B, where O denotes the zero column, and single entries g, d, the following quasideterminant [36]:

$$\left| \begin{pmatrix} E & O \\ F & g \end{pmatrix} \begin{pmatrix} A & B \\ C & d \end{pmatrix} \right|_{nn} = \begin{vmatrix} EA & EB \\ FA + gC & \boxed{FB + gd} \end{vmatrix} = g(d - CA^{-1}B) = g \begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix}. \tag{2.32}$$

Thus we see that premultiplying the expansion row of a quasideterminant by g has the effect of premultiplying the whole quasideterminant by g. All other operations leave the quasideterminant unchanged. A similar invariance property exists for column operations involving postmultiplication.

Choosing $g = 1$ in the above, we have

$$\left| \begin{pmatrix} E & O \\ F & 1 \end{pmatrix} \begin{pmatrix} A & B \\ C & d \end{pmatrix} \right|_{nn} = \begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix}. \tag{2.33}$$

Now choose E to be the $(n-1)\times(n-1)$ identity matrix, F the $1\times(n-1)$ row vector $\begin{pmatrix}0 & \dots & 0 & -1\end{pmatrix}$ and O the $(n-1)\times 1$ zero column vector. Consider the $n\times n$ matrix

$$T=\begin{pmatrix}P & Q\\ R & s\\ R & t\end{pmatrix}, \tag{2.34}$$

where P is an $(n-2)\times(n-1)$ matrix, R a $1\times(n-1)$ row vector, Q an $(n-2)\times 1$ column vector and s,t single entries. Then

$$\left|\begin{pmatrix}E & O\\ F & 1\end{pmatrix}\begin{pmatrix}P & Q\\ R & s\\ R & t\end{pmatrix}\right|_{nn}=\left|\begin{pmatrix}1 & 0 & \dots & 0 & 0 & 0\\ 0 & 1 & \dots & 0 & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & 0 & 1 & 0\\ 0 & 0 & \dots & 0 & -1 & 1\end{pmatrix}\begin{pmatrix}P & Q\\ R & s\\ R & t\end{pmatrix}\right|_{nn}=\begin{vmatrix}P & Q\\ R & s\\ 0\dots 0 & \boxed{t-s}\end{vmatrix}. \tag{2.35}$$

However, we know by (2.33) that, for E,F of size as detailed above,

$$\left|\begin{pmatrix}E & O\\ F & 1\end{pmatrix}\begin{pmatrix}P & Q\\ R & s\\ R & t\end{pmatrix}\right|_{nn}=\begin{vmatrix}P & Q\\ R & s\\ R & \boxed{t}\end{vmatrix}. \tag{2.36}$$

Hence

$$\begin{vmatrix}P & Q\\ R & s\\ R & \boxed{t}\end{vmatrix}=\begin{vmatrix}P & Q\\ R & s\\ 0\dots 0 & \boxed{t-s}\end{vmatrix}, \tag{2.37}$$

so that the value of a quasideterminant is unaffected by subtracting rows from the expansion row, a property analogous to that of determinants. We exploit this property in Section 3.5.1 of the next chapter.

2.3.2 Derivatives of a quasideterminant

Here we detail the method used to compute the derivative of a quasideterminant. The results obtained will be utilised later when verifying quasideterminant solutions of the DS equations.

We modify the approach of [36] to derive a formula for the derivative of a general quasideterminant of the form

$$\Xi=\begin{vmatrix}A & B\\ C & \boxed{D}\end{vmatrix}, \tag{2.38}$$

where A, B, C and D are matrices of size $2n \times 2n$, $2n \times 2$, $2 \times 2n$ and 2×2 respectively. (The reasoning behind choosing matrices of this size, rather than the more simple choice of $n \times n$, $n \times 1$, $1 \times n$ and 1×1 matrices, is to allow us to apply the formulae obtained to quasideterminant solutions of the DS equations in Chapter 3. The matrix sizes can easily be modified to ensure the results are applicable to, for example, the 3-component SSNLS equation in Chapter 6). Using the product rule for derivatives, we see that

$$\Xi' = D' - C'A^{-1}B + CA^{-1}A'A^{-1}B - CA^{-1}B'. \tag{2.39}$$

Suppose firstly that A is a Grammian-like matrix with derivative

$$A' = \sum_{i=1}^{k} E_i F_i \tag{2.40}$$

for some integer k, where E_i (F_i) are column (row) vectors of comparable lengths. Then (2.39) becomes

$$\begin{aligned} \Xi' &= D' - C'A^{-1}B - CA^{-1}B' + \sum_{i=1}^{k}(CA^{-1}E_i)(F_iA^{-1}B) \\ &= \begin{vmatrix} A & B \\ C' & \boxed{D'} \end{vmatrix} + \begin{vmatrix} A & B' \\ C & \boxed{O_2} \end{vmatrix} + \sum_{i=1}^{k} \begin{vmatrix} A & E_i \\ C & \boxed{O_2} \end{vmatrix} \begin{vmatrix} A & B \\ F_i & \boxed{O_2} \end{vmatrix}, \end{aligned} \tag{2.41}$$

where O_2 denotes the 2×2 zero matrix $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. If, on the other hand, the matrix A does not have a Grammian-like structure, we can once again write the derivative Ξ' as a product of quasideterminants as above by inserting the $2n \times 2n$ identity matrix in the form

$$I = \sum_{k=0}^{n-1}(f_k\, e_k)(f_k\, e_k)^T, \tag{2.42}$$

where e_k, f_k denote the $2n \times 1$ column vectors with a 1 in the $(2n-2k-1)^{\text{th}}$ and $(2n-2k)^{\text{th}}$ row respectively and zeros elsewhere, so that $(f_k\, e_k)$ denotes the $2n \times 2$ matrix with the $(2n-2k)^{\text{th}}$ and $(2n-2k-1)^{\text{th}}$ entries equal to 1 and every other entry zero.

Inserting the identity in the form (2.42), we find that, using (2.39),

$$\Xi' = D' - C'A^{-1}B + \sum_{k=0}^{n-1} \left(CA^{-1}f_ke_k(f_ke_k)^TA'A^{-1}B - CA^{-1}f_ke_k(f_ke_k)^TB'\right). \tag{2.43}$$

To simplify the above expression, note that

$$(f_ke_k)^TA' = \begin{pmatrix} (A^{2n-2k-1})' \\ (A^{2n-2k})' \end{pmatrix}, \tag{2.44}$$

where A^k denotes the k^{th} row of the matrix A, and similarly for $(f_k e_k)^T B'$. Hence

$$\begin{aligned}&\sum_{k=0}^{n-1}\left(CA^{-1}f_ke_k(f_ke_k)^TA'A^{-1}B - CA^{-1}f_ke_k(f_ke_k)^TB'\right)\\ =&\sum_{k=0}^{n-1}(CA^{-1}f_ke_k)\left(\begin{pmatrix}(A^{2n-2k-1})'\\(A^{2n-2k})'\end{pmatrix}A^{-1}B - \begin{pmatrix}(B^{2n-2k-1})'\\(B^{2n-2k})'\end{pmatrix}\right),\end{aligned} \tag{2.45}$$

and thus (2.43) gives

$$\Xi' = \begin{vmatrix}A & B\\ C' & \boxed{D'}\end{vmatrix} + \sum_{k=0}^{n-1}\begin{vmatrix}A & f_k \quad e_k\\ C & \boxed{O_2}\end{vmatrix}\cdot\begin{vmatrix}A & B\\ \begin{matrix}(A^{2n-2k-1})'\\(A^{2n-2k})'\end{matrix} & \boxed{\begin{matrix}(B^{2n-2k-1})'\\(B^{2n-2k})'\end{matrix}}\end{vmatrix}. \tag{2.46}$$

We can also obtain a column version of the derivative formula by inserting the identity in a different position, i.e. (2.43) is now written as

$$\Xi' = D' - C'A^{-1}B' + \sum_{k=0}^{n-1}\left(CA^{-1}A'f_ke_k(f_ke_k)^TA^{-1}B - C'f_ke_k(f_ke_k)^TA^{-1}B\right). \tag{2.47}$$

We calculate that

$$A'(f_ke_k) = \begin{pmatrix}(A_{2n-2k-1})' & (A_{2n-2k})'\end{pmatrix}, \tag{2.48}$$

where A_k denotes the k^{th} column of the matrix A, and similarly for $C'(f_ke_k)$. Thus we can express Ξ' as

$$\Xi' = \begin{vmatrix}A & B'\\ C' & \boxed{D'}\end{vmatrix} + \sum_{k=0}^{n-1}\begin{vmatrix}A & (A_{2n-2k-1})' \quad (A_{2n-2k})'\\ C & \boxed{(C_{2n-2k-1})' \quad (C_{2n-2k})'}\end{vmatrix}\cdot\begin{vmatrix}A & B\\ (f_ke_k)^T & \boxed{O_2}\end{vmatrix}. \tag{2.49}$$

2.4 Motivation

As mentioned in earlier sections, noncommutative integrable systems have attracted considerable attention in recent years. Much work has been carried out on the derivation of such equations with noncommutativity defined in terms of a star product, for example by Dimakis and Müller-Hoissen [14], Wang and Wadati [87], Paniak [75], Hamanaka and Toda [42], Hamanaka [41], and others. However, the idea of taking the definition of noncommutativity to be very general at first is a relatively new one, with Gilson, Nimmo and their collaborators the major players in this field, beginning with work on the noncommutative KP equation in 2007. They also obtained quasideterminant solutions of the equation via Darboux and binary Darboux transformations. Similar work soon followed, with Gilson, Nimmo and Sooman carrying out an identical procedure on a noncommutative modified KP equation [38], and Li and Nimmo on a non-abelian Toda lattice [59] and a noncommutative semi-discrete Toda equation [60].

It therefore seems natural to continue in this vein and attempt to obtain similar noncommutative generalisations and quasideterminant solutions of other, somewhat more complex, integrable equations. The $(2+1)$-dimensional DS equations are a natural place to begin, as results obtained can be compared and contrasted to those of Gilson and Nimmo in the commutative case in [35], and can also be reduced to results valid for the noncommutative NLS equation in $(1+1)$-dimension. The challenges of extending to a noncommutative setting soon become apparent, particularly when attempting to plot solutions such as dromions - the choice of parameters is highly non-trivial.

Continuing on a similar theme, another natural step is to revert to a $(1+1)$-dimensional situation and, rather than considering the relatively simple NLS equation, whose quasideterminant solutions can easily be obtained by reduction of those solutions obtained for the DS equations, we consider higher-order NLS equations in a noncommutative setting, the Sasa-Satsuma equation being one particular example. Gilson, Hietarinta, Nimmo and Ohta have considered soliton solutions of this equation in the commutative case [33], and hence we take their work one step further by extending to a noncommutative situation.

Chapter 3

The Davey-Stewartson equations

3.1 Background

The Davey-Stewartson (DS) equations have become a topic of much interest in recent years. Derived by Davey and Stewartson in 1974 [13], the system is nonlinear in two spatial and one time dimension, and describes the evolution of a three-dimensional wave-packet with slowly varying amplitude on water of finite depth. In his series of papers beginning in 1965, Whitham [88] began to develop a theory to model such an evolution, describing the motion in terms of a phase and an amplitude variable, with further extensions to the theory subsequently proposed by Lighthill [61] and Hayes [45].

Whitham's work indicated that the evolution of these wave-packets is determined by either a hyperbolic or an elliptic equation. However, his theory does not give any indication as to what exactly is meant by a slow variation in amplitude; it is usual in such a case to introduce a small parameter, say ϵ, with the idea of a slow amplitude variation being explicitly governed by this parameter. Whitham's work does not allude to a parameter of this kind, which led Davey and Stewartson to focus on the so-called method of multiple scales, whereby a small parameter ϵ is included in the expansion. In short, this multiple scales method is used to find an approximate solution to a perturbation problem for both small and large values of the independent variables. The technique replaces the independent variables with fast-scale and slow-scale variables and then considers these new variables to be independent.

In their paper of 1974, Davey and Stewartson showed that the motion of the wave-packet is described by two partial differential equations (although if the wave-packet takes the form of an oblique plane wave (a wave in a number of planes), the two equations can be

converted into a single equation). They considered an area of water with depth h, and used a standard Cartesian coordinate system (x, y, z), with the origin O on the surface of the water and the z-axis pointing vertically upwards, so that the 'bed' of the water area is given by $z = -h$ and the xy-plane lies on the water surface. At time $t = 0$, a progressive wave is formed such that the height of the water's surface is raised to $z = \zeta$, say.
At later times $t > 0$, a velocity potential $\phi(x, y, z, t)$ is such that

$$\partial_x^2\phi + \partial_y^2\phi + \partial_z^2\phi = 0 \text{ for } -h < z < \zeta, \tag{3.1}$$

with suitable boundary conditions. The progressive wave can be assumed to have a solution of the form

$$\phi = \sum_{n=-\infty}^{\infty} \phi_n E^n, \qquad \zeta = \sum_{n=-\infty}^{\infty} \zeta_n E^n, \tag{3.2}$$

where $E = \exp[\mathrm{i}(kx - \omega t)]$, with k denoting the wavenumber and ω the frequency of the progressive wave. The parameters ϕ_n and ζ_n are defined by

$$\phi_n = \sum_{j=n}^{\infty} \epsilon^j \phi_{nj}, \qquad \zeta_n = \sum_{j=n}^{\infty} \epsilon^j \zeta_{nj}, \tag{3.3}$$

and are such that, for $n \geq 0$, $\phi_{-n} = \phi_n^*$, $\zeta_{-n} = \zeta_n^*$, with the ϕ_{nj} functions of x, y, z, t and ζ_{nj} functions of x, y, t. Substituting the expansion (3.2) for ϕ into the partial differential equation (3.1) with appropriate boundary conditions (the expression for ζ in (3.2) is needed to satisfy the boundary conditions) and utilising the method of multiple scales, the well-known Davey-Stewartson equation is obtained, which can be written as

$$\mathrm{i}\partial_\tau A + \lambda\partial_\xi^2 A + \mu\partial_\eta^2 A = \nu\,|\,A\,|^2 A + \nu_1 AQ, \tag{3.4a}$$

$$\lambda_1\partial_\xi^2 Q + \mu_1\partial_\eta^2 Q = \kappa_1\partial_\eta^2\,|\,A\,|^2\,. \tag{3.4b}$$

Here, A and Q are functions of ξ, η and τ, which in turn are functions of x, y and t, while λ, λ_1, μ, μ_1, ν, ν_1, κ_1 depend on various factors such as the wavenumber k, frequency ω, acceleration due to gravity, etc. A full account of the derivation can be found in the original paper [13] by Davey and Stewartson.
Various equivalent forms of the above system of equations are known, and can easily be obtained by suitable scaling of variables. In the next section, we show how to carry out a transformation between two such systems.

3.2 Transformation between equations

Shortly we will derive a noncommutative version of the DS equations by considering the Lax pair of a system of commutative DS equations given by Ablowitz and Schultz in [7], namely, for functions $q = q(x, y, t)$, $r = r(x, y, t)$,

$$\mathrm{i}q_t = -\frac{1}{2\sigma^2}\left(q_{xx} + \sigma^2 q_{yy}\right) + \mathrm{i}q(A_1 - A_2), \tag{3.5a}$$

$$\mathrm{i}r_t = \frac{1}{2\sigma^2}\left(r_{xx} + \sigma^2 r_{yy}\right) - \mathrm{i}r(A_1 - A_2), \tag{3.5b}$$

where A_1, A_2 are such that

$$(\partial_x + \sigma\partial_y)A_1 = -\frac{\mathrm{i}}{2\sigma^2}(\partial_x - \sigma\partial_y)(qr), \tag{3.6a}$$

$$(\partial_x - \sigma\partial_y)A_2 = \frac{\mathrm{i}}{2\sigma^2}(\partial_x + \sigma\partial_y)(qr) \tag{3.6b}$$

and $r = \pm q^*$ (q^* denotes the complex conjugate of q). The constant σ is chosen to be -1 or i for the DSI (hyperbolic case) and DSII (elliptic case) equations respectively. Only the DSI equations give rise to the dromion solutions studied in the next chapter. Before discussing these equations in a noncommutative setting, we firstly highlight a correspondence between the above commutative DSI equations ($\sigma = -1$) and the commutative DSI equations considered by Gilson and Nimmo in [35], namely

$$\mathrm{i}u_t + u_{XX} + u_{YY} - 4u|u|^2 - 2uv = 0, \tag{3.7a}$$

$$v_{XY} + (\partial_X + \partial_Y)^2|u|^2 = 0, \tag{3.7b}$$

for functions $u = u(X, Y, t)$, $v = v(X, Y, t)$. (Gilson and Nimmo investigate only the DSI case as this leads to the dromion solutions considered later in their paper). We define a variable transformation from x, y to X, Y by

$$x = \frac{1}{2}(X + Y), \tag{3.8a}$$

$$y = \frac{1}{2}(X - Y), \tag{3.8b}$$

so that

$$\partial_x = \partial_X + \partial_Y, \tag{3.9a}$$

$$\partial_y = \partial_Y - \partial_X. \tag{3.9b}$$

Then, in the DSI case, (3.5a) and (3.5b) give, on setting $q = u$,

$$\mathrm{i}u_t + u_{XX} + u_{YY} - \mathrm{i}u(A_1 - A_2) = 0 \tag{3.10}$$

and its corresponding complex conjugate. By comparing with (3.7a), we require

$$A_1 - A_2 = -2\mathrm{i}(2|u|^2 + v), \tag{3.11}$$

which transforms (3.10) to the first of Gilson and Nimmo's equations (3.7a).

We now apply the operation $-\frac{1}{4}(\partial_x + \partial_y)$ to (3.6a) and $-\frac{1}{4}(\partial_x - \partial_y)$ to (3.6b). Subtracting the resulting equations and implementing the same variable transformation (3.8) as before, with $q = u$ and $A_1 - A_2$ defined as in (3.11), gives the second of Gilson and Nimmo's equations (3.7b).

In this chapter, we have chosen to work with the DS equations and corresponding Lax pair as given by Ablowitz and Schultz in [7]. Our noncommutative DS system obtained from this Lax pair was found to match exactly the quantum version of the DS equations considered in a later paper by Ablowitz, Schultz and Bar Yaacov [81].

3.3 Noncommutative Davey-Stewartson equations

We now derive a system of noncommutative DS equations, a topic of considerable interest in recent years. Hamanaka [40] derived a system with noncommutativity defined in terms of the Moyal star product, while more recently, Dimakis and Müller-Hoissen [16] determined a similar system from a multicomponent KP hierarchy.

The strategy that we employ here, whereby we introduce noncommutativity into an integrable nonlinear wave equation without destroying the solvability, has previously been considered by others in the field, for example by Lechtenfeld and Popov [57], and by Lechtenfeld, Popov *et al.* [56], where a noncommutative version of the sine-Gordon equation is discussed.

As mentioned in the previous chapter, we are not concerned with the nature of the noncommutativity, and derive a system of noncommutative DS equations in the most general manner by utilising the same Lax pair as in the commutative case but assuming no commutativity of the dependent variables. This method has also been employed by Gilson and Nimmo in [36] for the case of the noncommutative KP equation.

The Lax pair for the (commutative) DS equations (3.5) is given by [7]

$$\varphi_x = \Lambda\varphi - \sigma J\varphi_y, \tag{3.12a}$$

$$\varphi_t = A\varphi - \frac{\mathrm{i}}{\sigma}\Lambda\varphi_y + \mathrm{i}J\varphi_{yy}, \tag{3.12b}$$

where $\varphi = \varphi(x, y, t)$ is a 2×2 solution matrix, J and Λ are the matrices

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} \tag{3.13}$$

for q, r functions of x, y and t such that $r = \pm q^*$, and A is a 2×2 matrix given by

$$A = \begin{pmatrix} A_1 & \frac{\mathrm{i}}{2\sigma^2}(q_x - \sigma q_y) \\ -\frac{\mathrm{i}}{2\sigma^2}(r_x + \sigma r_y) & A_2 \end{pmatrix}, \tag{3.14}$$

with $\sigma = -1$ or $\sigma = \mathrm{i}$ for the DSI and DSII equations respectively. The Lax pair (3.12) can be expressed in an equivalent form as

$$L = \partial_x - \Lambda + \sigma J\partial_y, \tag{3.15a}$$

$$M = \partial_t - A + \frac{\mathrm{i}}{\sigma}\Lambda\partial_y - \mathrm{i}J\partial_{yy}. \tag{3.15b}$$

By setting the commutator $[L, M]$ of this Lax pair equal to zero and assuming no commutativity of variables, we obtain the compatibility condition

$$-A_x + [\Lambda, A] + \Lambda_t - \sigma JA_y + \frac{\mathrm{i}}{\sigma}\Lambda\Lambda_y - \mathrm{i}J\Lambda_{yy} = 0, \tag{3.16a}$$

$$\frac{\mathrm{i}}{\sigma}\Lambda_x + \sigma[A, J] - \mathrm{i}J\Lambda_y = 0. \tag{3.16b}$$

The second equation (3.16b) is satisfied automatically on equating matrix entries, while from (3.16a), we generate a system of noncommutative Davey-Stewartson (ncDS) equations,

$$\mathrm{i}q_t = -\frac{1}{2\sigma^2}\left(q_{xx} + \sigma^2 q_{yy}\right) + \mathrm{i}(A_1 q - qA_2), \tag{3.17a}$$

$$\mathrm{i}r_t = \frac{1}{2\sigma^2}\left(r_{xx} + \sigma^2 r_{yy}\right) - \mathrm{i}(rA_1 - A_2 r), \tag{3.17b}$$

$$(\partial_x + \sigma\partial_y)A_1 = -\frac{\mathrm{i}}{2\sigma^2}(\partial_x - \sigma\partial_y)(qr), \tag{3.17c}$$

$$(\partial_x - \sigma\partial_y)A_2 = \frac{\mathrm{i}}{2\sigma^2}(\partial_x + \sigma\partial_y)(rq). \tag{3.17d}$$

(Note that, taking a dimensional reduction $\partial_y = 0$ gives

$$\pm 2\mathrm{i}q_t = -q_{xx} + 2qrq, \tag{3.18}$$

which, on scaling, matches the noncommutative NLS equation obtained in [15] (where the * multiplication is taken to be ordinary multiplication). To revert to the commutative case, we write $qrq = \pm qq^*q$ as $\pm \mid q \mid^2 q^*$). Our results in this chapter and the next can easily be reduced to those valid for the noncommutative NLS equation by removing all dependence on one of the spatial variables x or y.

There is one important factor that we must consider in the noncommutative case, namely our definition of the matrix Λ in (3.13). In the commutative case, where we consider q and r to be scalar objects, we see that

$$\Lambda^\dagger = \begin{cases} \Lambda & \text{for } r = q^*, \\ -\Lambda & \text{for } r = -q^*, \end{cases} \tag{3.19}$$

where $\Lambda^\dagger$ denotes the Hermitian conjugate (conjugate transpose) of Λ. In the noncommutative case, the conditions (3.19) must be preserved. If we were to take the matrix Λ as in (3.13) and think of q, r as noncommutative objects, for example matrices, with $r = \pm q^*$, we find that (3.19) does not hold. Instead, we define Λ to be the same matrix as in (3.13), however we now define $r = \pm q^\dagger$. (Note that we could equally well have used this definition of r in the commutative case, since, for scalar q, $q^\dagger = q^*$). This does not affect our system of ncDS equations (3.17), so long as we bear in mind that $r = \pm q^\dagger$ in this case.

For notational convenience, and to avoid the use of identities later when verifying solutions, we introduce a 2×2 matrix $S = (s_{ij})$ $(i, j = 1, 2)$ such that $\Lambda = [J, \sigma S]$ [58], thus

$$S = \begin{pmatrix} s_{11} & \dfrac{q}{2\sigma} \\ -\dfrac{r}{2\sigma} & s_{22} \end{pmatrix}. \tag{3.20}$$

Additionally, by setting

$$A = \frac{\mathrm{i}}{\sigma} S_x - \mathrm{i} J S_y, \tag{3.21}$$

(3.16b) is automatically satisfied, and (3.16a) becomes

$$\begin{aligned} &-\frac{\mathrm{i}}{\sigma} S_{xx} + \mathrm{i}JSS_x - \mathrm{i}SJS_x - \mathrm{i}S_xJS + \mathrm{i}S_xSJ + \mathrm{i}\sigma JS_yJS \\ &\qquad - \mathrm{i}\sigma JS_ySJ + \sigma JS_t - \sigma S_tJ - \mathrm{i}\sigma JSS_yJ + \mathrm{i}\sigma SJS_yJ + \mathrm{i}\sigma JS_{yy}J = 0. \end{aligned} \tag{3.22}$$

Note that this is essentially the noncommutative analogue of the Hirota bilinear form (see for example [48]) of the DS equations.

3.4 Darboux transformations

3.4.1 Ordinary Darboux transformations

A Darboux transformation, named after the French mathematician Jean-Gaston Darboux, is an iterative process and a way of generating new solutions of a given integrable equation. One of the major attractions of Darboux transformations as a solution-generating technique is that such a transformation can be applied even to the trivial solution of the equation under consideration and a new non-trivial solution generated. By repeating this process, a whole family of new non-trivial solutions can be constructed.

We look at the particular example of the system of ncDS equations (3.17) and outline how such a Darboux transformation can be implemented.

Consider the Lax operators L, M defined as in (3.15), and suppose that T is the set of zero-eigenvalue, eigenfunctions of L, M, i.e.

$$T = \{\theta : L(\theta) = M(\theta) = 0\}. \tag{3.23}$$

As discussed previously, setting the commutator $[L, M]$ equal to zero and assuming no commutativity of variables generates the system of ncDS equations (3.17). We take a known solution q of this system, for example $q = 0$, and apply a Darboux transformation, to be defined later. This transformation maps the original Lax operators L, M to some new operators $\tilde{L}$, $\tilde{M}$, with eigenfunctions

$$\tilde{T} = \{\tilde{\theta} : \tilde{L}(\tilde{\theta}) = \tilde{M}(\tilde{\theta}) = 0\}. \tag{3.24}$$

These operators are identical to the operators L, M in (3.15) but with Λ replaced by $\tilde{\Lambda}$ and A by $\tilde{A}$, where

$$\tilde{\Lambda} = \begin{pmatrix} 0 & \tilde{q} \\ \tilde{r} & 0 \end{pmatrix} \qquad \text{and} \qquad \tilde{A} = \begin{pmatrix} \tilde{A}_1 & \frac{\mathrm{i}}{2\sigma^2}(\tilde{q}_x - \sigma\tilde{q}_y) \\ -\frac{\mathrm{i}}{2\sigma^2}(\tilde{r}_x + \sigma\tilde{r}_y) & \tilde{A}_2 \end{pmatrix}. \tag{3.25}$$

(We define $\tilde{J} = J$ since J is a constant matrix). The operators L, M are covariant under the action of the Darboux transformation, so that, by setting the commutator of $\tilde{L}$ and $\tilde{M}$ equal to zero and assuming no commutativity, we generate the same system (3.17) as before, but with q replaced by $\tilde{q}$ and so on. This new system of equations has solution $\tilde{q}$, so that, by applying a Darboux transformation, we have generated a new solution of our original system of ncDS equations (3.17), i.e. $\tilde{q}$ satisfies the system (3.17) whenever q does. It therefore follows that the Darboux transformation induces an auto-Bäcklund transformation $q \to \tilde{q}$ for the original system (3.17). Clearly the above process can be repeated many times, allowing us to obtain a family of solutions of this original system of equations.

The Darboux transformation, G, is defined in terms of θ as

$$G_\theta := \partial_y - \theta_y \theta^{-1} = \theta \partial_y \theta^{-1}, \tag{3.26}$$

and if ϕ is a generic eigenfunction of the original Lax operators L, M, then $\tilde{\phi} = G_\theta(\phi)$ is a generic eigenfunction of the transformed operators $\tilde{L}, \tilde{M}$.

We now show how this transformation can be iterated in a natural way. We mainly follow the notation of [70], and denote L by $L_{[1]}$, M by $M_{[1]}$ to indicate the starting levels.

Iteration

Step 1 Let $\theta_1, \ldots, \theta_n$ be eigenfunctions of the original Lax operators $L_{[1]} = L$, $M_{[1]} = M$, and define $\phi_{[1]} = \phi$ to be a generic (arbitrary) eigenfunction of $L_{[1]}$, $M_{[1]}$. We choose one eigenfunction, say θ_1, to define a Darboux transformation to a pair of new Lax operators $L_{[2]}$, $M_{[2]}$, and relabel this eigenfunction as $\theta_{1[1]}$ to indicate that it is an eigenfunction of the original Lax operators $L_{[1]}$, $M_{[1]}$. Our assumption is that $\theta_{1[1]}$ is invertible. Then a Darboux transformation is defined as

$$G_{\theta_{1[1]}} = \partial_y - \theta^{(1)}_{1[1]} \theta^{-1}_{1[1]}, \tag{3.27}$$

where $\theta^{(1)}_{1[1]}$ denotes one differentiation of $\theta_{1[1]}$ with respect to y, and the new Lax pair $L_{[2]} = G_{\theta_{1[1]}} L_{[1]} G^{-1}_{\theta_{1[1]}}$, $M_{[2]} = G_{\theta_{1[1]}} M_{[1]} G^{-1}_{\theta_{1[1]}}$ has generic eigenfunction

$$\phi_{[2]} := G_{\theta_{1[1]}}(\phi) = \phi^{(1)} - \theta^{(1)}_{1[1]} \theta^{-1}_{1[1]} \phi. \tag{3.28}$$

(Note here that since ϕ is an eigenfunction of $L_{[1]} = L$, $M_{[1]} = M$, we think of ϕ as a 2×2 matrix. In a similar manner, each θ_i ($i = 1, \ldots, n$) is considered to be a 2×2 matrix. Later, when we investigate dromion solutions in a noncommutative (matrix) setting, we think of each *entry* of θ_i as having matrix form). In particular, this Darboux transformation maps the eigenfunction $\theta_1 = \theta_{1[1]}$ to $G_{\theta_{1[1]}}\left(\theta_{1[1]}\right) := \theta_{1[2]}$, which is zero (since $G_{\theta_{1[1]}}\left(\theta_{1[1]}\right) = 0$ by definition), and $\theta_2, \ldots, \theta_n$ to $\theta_{2[2]}, \ldots, \theta_{n[2]}$ respectively, where, for $i = 2, \ldots, n$,

$$\theta_{i[2]} = \phi_{[2]} \mid_{\phi \to \theta_i}, \tag{3.29}$$

i.e. we define the transformed functions $\theta_{2[2]}, \ldots, \theta_{n[2]}$ in terms of the original (known) eigenfunctions $\theta_2, \ldots, \theta_n$ of $L_{[1]}, M_{[1]}$. It is easy to show that these transformed functions are *eigenfunctions* of the transformed operators $L_{[2]}, M_{[2]}$.

Step 2 We now select the eigenfunction $\theta_{2[2]}$ to define a Darboux transformation from the Lax operators $L_{[2]}, M_{[2]}$ to some new operators $L_{[3]} = G_{\theta_{2[2]}} L_{[2]} G^{-1}_{\theta_{2[2]}}$ and $M_{[3]} = G_{\theta_{2[2]}} M_{[2]} G^{-1}_{\theta_{2[2]}}$ with generic eigenfunctions

$$\phi_{[3]} = G_{\theta_{2[2]}}\left(\phi_{[2]}\right) = \phi^{(1)}_{[2]} - \theta^{(1)}_{2[2]} \theta^{-1}_{2[2]} \phi_{[2]}. \tag{3.30}$$

In particular, this Darboux transformation maps the eigenfunction $\theta_{2[2]}$ to $G_{\theta_{2[2]}}\left(\theta_{2[2]}\right) := \theta_{2[3]}$, which is zero (since $G_{\theta_{2[2]}}\left(\theta_{2[2]}\right) = 0$ by definition), and $\theta_{3[2]}, \ldots, \theta_{n[2]}$ to $\theta_{3[3]}, \ldots, \theta_{n[3]}$ respectively, where, for $i = 3, \ldots, n$,

$$\theta_{i[3]} = \phi_{[3]} \mid_{\phi \to \theta_i} . \tag{3.31}$$

⋮

Step n $(n \geq 1)$ We select the eigenfunction $\theta_{n[n]}$ to define a Darboux transformation from the Lax operators $L_{[n]}, M_{[n]}$ to some new operators $L_{[n+1]} = G_{\theta_{n[n]}} L_{[n]} G^{-1}_{\theta_{n[n]}}$ and $M_{[n+1]} = G_{\theta_{n[n]}} M_{[n]} G^{-1}_{\theta_{n[n]}}$ with generic eigenfunctions

$$\phi_{[n+1]} = G_{\theta_{n[n]}}\left(\phi_{[n]}\right) = \phi^{(1)}_{[n]} - \theta^{(1)}_{n[n]} \theta^{-1}_{n[n]} \phi_{[n]}. \tag{3.32}$$

In particular, this Darboux transformation maps the eigenfunction $\theta_{n[n]}$ to $G_{\theta_{n[n]}}\left(\theta_{n[n]}\right)$, which is zero as before.

From now on, we choose to simplify our notation slightly, and denote $\theta_{1[1]}$ by θ, and, in general, $\theta_{k[k]}$ by $\theta_{[k]}$ $(k = 2, \ldots, n)$. (Note that we have introduced a shorthand notation only for $\theta_{k[k]}$ since only terms of this form appear in our iterated expressions - terms of the form $\theta_{i[k]}$, say $(k \neq i)$, do not appear). We thus illustrate the Darboux transformation pictorially as

$$\underset{\theta}{L_{[1]}} \xrightarrow{G_\theta} L_{[2]} \xrightarrow{G_{\theta_{[2]}}} L_{[3]} \longrightarrow \ldots \longrightarrow L_{[n]} \xrightarrow{G_{\theta_{[n]}}} L_{[n+1]} \, ,$$

where $L_{[2]}$ corresponds to $\tilde{L}$. Darboux summed up one step of this iterative process in a theorem, applicable to a broad class of operators in a noncommutative setting:

Theorem 3 (Darboux [12], Matveev [64]) *Let*

$$L_{[1]} = \partial_x + \sum_{j=0}^{N} b_j \partial_y^j, \tag{3.33}$$

where $b_j \in R$*, a ring. Let* θ *be an invertible eigenfunction of* $L_{[1]}$*, so that* $L_{[1]}(\theta) = 0$*, and define* $G_\theta = \partial_y - \theta_y \theta^{-1}$*. Then*

$$L_{[2]} = G_\theta L_{[1]} G_\theta^{-1} \tag{3.34}$$

has the same form as $L_{[1]}$*. In particular, for any eigenfunction* $\phi_{[1]} = \phi$ *of* $L_{[1]}$*,*

$$\phi_{[2]} = G_\theta(\phi) \tag{3.35}$$

is an eigenfunction of $L_{[2]}$*.*

The proof of this theorem is straightforward, if a little tedious - we write G_θ in the form $\theta \partial_y \theta^{-1}$ and substitute this, along with the definition of $L_{[1]}$, into $L_{[2]} = G_\theta L_{[1]} G_\theta^{-1}$. Completion of the proof requires us to use the fact that θ is an eigenfunction of $L_{[1]}$, i.e. $L_{[1]}(\theta) = 0$.

In the case of the DS Lax operator $L_{[1]} = L$, we take the b_j to be 2×2 matrices, so that $b_0 = -\Lambda$ and $b_1 = \sigma J$. A similar theorem holds for the operator $M_{[1]} = M$ - with a general operator M of the form $M = \partial_t + \sum_{j=0}^{N} c_j \partial_y^j$, we take $c_0 = -A$, $c_1 = \frac{\mathrm{i}}{\sigma} \Lambda$ and $c_2 = -\mathrm{i}J$ in the DS case.

Although Darboux's theorem gives a connection between the eigenfunctions of the transformed Lax operators and those of the original operators, it does not attempt to give a connection between the coefficients of the transformed equation and those of the original equation. This requires use of the formulae $L_{[2]} = G_\theta L_{[1]} G_\theta^{-1}$, i.e. $L_{[2]} G_\theta = G_\theta L_{[1]}$, and similarly $M_{[2]} G_\theta = G_\theta M_{[1]}$ which will be outlined in the next section.

Quasi-Wronskian form

By defining $\Theta = (\theta_1 \ \dots \ \theta_n)$ and recalling that $\phi_{[1]} = \phi$, it can be shown that the expression for $\phi_{[n+1]}$ in (3.32) can be written as a quasideterminant, namely

$$\phi_{[n+1]} = \begin{vmatrix} \Theta & \phi \\ \vdots & \vdots \\ \Theta^{(n-1)} & \phi^{(n-1)} \\ \Theta^{(n)} & \boxed{\phi^{(n)}} \end{vmatrix}, \tag{3.36}$$

where $\Theta^{(k)}$, $\phi^{(k)}$ denote the k^{th} y-derivatives of Θ and ϕ respectively ($k = 0, \ldots, n$). (Recall that ϕ and each θ_i ($i = 1, \ldots, n$) are 2×2 matrices). Thus we have expressed the formula for the $(n+1)^{\text{th}}$ generic eigenfunction $\phi_{[n+1]}$ in terms of only the known eigenfunctions $\theta_1, \ldots, \theta_n$ and ϕ of the 'seed' Lax pair $L_{[1]} = L$, $M_{[1]} = M$. The Wronskian-like quasideterminant in (3.36) is termed a *quasi-Wronskian*, see [36], and the proof of (3.36) is by induction on n as follows.

The expression is clearly true for $n = 1$, since

$$\begin{aligned} \phi_{[2]} &= \begin{vmatrix} \theta_1 & \phi \\ \theta_1^{(1)} & \boxed{\phi^{(1)}} \end{vmatrix} \\ &= \phi^{(1)} - \theta_1^{(1)} \theta_1^{-1} \phi, \end{aligned} \tag{3.37}$$

which holds by (3.28) since $\theta_{1[1]} = \theta_1$. Now assume that (3.36) is true for some fixed $n \geq 1$. We prove the expression is also true for $n+1$, i.e. we prove that

$$\phi_{[n+1+1]} = \begin{vmatrix} \Theta & \theta_{n+1} & \phi \\ \vdots & \vdots & \vdots \\ \Theta^{(n-1)} & \theta_{n+1}^{(n-1)} & \phi^{(n-1)} \\ \Theta^{(n)} & \theta_{n+1}^{(n)} & \phi^{(n)} \\ \Theta^{(n+1)} & \theta_{n+1}^{(n+1)} & \boxed{\phi^{(n+1)}} \end{vmatrix}. \tag{3.38}$$

We have

$$\begin{aligned} \phi_{[n+1+1]} &= \phi_{[n+1]}^{(1)} - \theta_{n+1[n+1]}^{(1)} \theta_{n+1[n+1]}^{-1} \phi_{[n+1]} \\ &\equiv \phi_{[n+1]}^{(1)} - \theta_{[n+1]}^{(1)} \theta_{[n+1]}^{-1} \phi_{[n+1]} \end{aligned} \tag{3.39}$$

by (3.32) on replacing n by $n+1$. Using the formula for the derivative of a quasideterminant (2.46), it can be seen that

$$\phi_{[n+1]}^{(1)} = \begin{vmatrix} \Theta & \phi \\ \vdots & \vdots \\ \Theta^{(n-2)} & \phi^{(n-2)} \\ \Theta^{(n-1)} & \phi^{(n-1)} \\ \Theta^{(n+1)} & \boxed{\phi^{(n+1)}} \end{vmatrix} + V \phi_{[n+1]}, \tag{3.40}$$

where

$$V = \begin{vmatrix} \Theta & O_2 \\ \vdots & \vdots \\ \Theta^{(n-2)} & O_2 \\ \Theta^{(n-1)} & I_2 \\ \Theta^{(n)} & \boxed{O_2} \end{vmatrix} \tag{3.41}$$

and O_2, I_2 denote the 2×2 zero and identity matrices respectively. In a similar manner to the above, since $\theta_{k[k]} = \phi_{[k]}|_{\phi \to \theta_k}$ for $k = 2, \ldots, n$, it follows that

$$\theta^{(1)}_{n+1[n+1]} \equiv \theta^{(1)}_{[n+1]} = \begin{vmatrix} \Theta & \theta_{n+1} \\ \vdots & \vdots \\ \Theta^{(n-1)} & \theta^{(n-1)}_{n+1} \\ \Theta^{(n+1)} & \boxed{\theta^{(n+1)}_{n+1}} \end{vmatrix} + V\theta_{[n+1]} \tag{3.42}$$

on replacing $\phi_{[n+1]}$ by $\theta_{n+1[n+1]} \equiv \theta_{[n+1]}$ in (3.40). Thus (3.39) is

$$\phi_{[n+1+1]} = \begin{vmatrix} \Theta & \phi \\ \vdots & \vdots \\ \Theta^{(n-1)} & \phi^{(n-1)} \\ \Theta^{(n+1)} & \boxed{\phi^{(n+1)}} \end{vmatrix} + V\phi_{[n+1]} - \left(\begin{vmatrix} \Theta & \theta_{n+1} \\ \vdots & \vdots \\ \Theta^{(n-1)} & \theta^{(n-1)}_{n+1} \\ \Theta^{(n+1)} & \boxed{\theta^{(n+1)}_{n+1}} \end{vmatrix} + V\theta_{[n+1]} \right) \theta^{-1}_{[n+1]}\phi_{[n+1]}$$

$$= \begin{vmatrix} \Theta & \phi \\ \vdots & \vdots \\ \Theta^{(n-1)} & \phi^{(n-1)} \\ \Theta^{(n+1)} & \boxed{\phi^{(n+1)}} \end{vmatrix} - \begin{vmatrix} \Theta & \theta_{n+1} \\ \vdots & \vdots \\ \Theta^{(n-1)} & \theta^{(n-1)}_{n+1} \\ \Theta^{(n+1)} & \boxed{\theta^{(n+1)}_{n+1}} \end{vmatrix} \cdot \begin{vmatrix} \Theta & \theta_{n+1} \\ \vdots & \vdots \\ \Theta^{(n-1)} & \theta^{(n-1)}_{n+1} \\ \Theta^{(n)} & \boxed{\theta^{(n)}_{n+1}} \end{vmatrix}^{-1} \cdot \begin{vmatrix} \Theta & \phi \\ \vdots & \vdots \\ \Theta^{(n-1)} & \phi^{(n-1)} \\ \Theta^{(n)} & \boxed{\phi^{(n)}} \end{vmatrix}$$

by the inductive hypothesis and the fact that $\theta_{k[k]} = \phi_{[k]}|_{\phi \to \theta_k}$

$$= \begin{vmatrix} \Theta & \theta_{n+1} & \phi \\ \vdots & \vdots & \vdots \\ \Theta^{(n-1)} & \theta^{(n-1)}_{n+1} & \phi^{(n-1)} \\ \Theta^{(n)} & \theta^{(n)}_{n+1} & \phi^{(n)} \\ \Theta^{(n+1)} & \theta^{(n+1)}_{n+1} & \boxed{\phi^{(n+1)}} \end{vmatrix} \tag{3.43}$$

by the noncommutative Sylvester identity (2.27). This completes the proof.

3.4.2 Quasi-Wronskian solution of ncDS using Darboux transformations

We now determine the effect of the Darboux transformation $G_{\theta_{1[1]}} = \partial_y - \theta^{(1)}_{1[1]}\theta^{-1}_{1[1]}$ on the Lax operator $L_{[1]} = L$ given by (3.15a), with $\theta_1, \ldots, \theta_n = \theta_{1[1]}, \ldots, \theta_{n[1]}$ eigenfunctions of

L, and $\theta_{1[1]}$ chosen to iterate the Darboux transformation. As before, we denote $\theta_{1[1]}$ by θ, and, in general, $\theta_{k[k]}$ by $\theta_{[k]}$ $(k = 2, \ldots, n)$. Corresponding results hold for the operator $M_{[1]} = M$ given by (3.15b). The operator $L_{[1]} = L$ is transformed to a new operator $L_{[2]}$, say, where

$$L_{[2]} G_\theta = G_\theta L, \tag{3.44}$$

giving

$$-\theta_{xy}\theta^{-1} + \theta_y\theta^{-1}\theta_x\theta^{-1} - \sigma J\theta_{yy}\theta^{-1} + \sigma J\theta_y\theta^{-1}\theta_y\theta^{-1} + \Lambda_{[2]}\theta_y\theta^{-1} + \Lambda_y - \theta_y\theta^{-1}\Lambda = 0 \tag{3.45a}$$

and

$$-\sigma J\theta_y\theta^{-1} - \Lambda_{[2]} + \Lambda + \theta_y\theta^{-1}\sigma J = 0. \tag{3.45b}$$

From (3.45b), we see that $\Lambda_{[2]} = \Lambda - \sigma[J, \theta_y\theta^{-1}]$. Substituting for $\Lambda_{[2]}$ in the left-hand side of (3.45a) gives

$$\begin{aligned} &-\Big(\theta_{xy} - \theta_y\theta^{-1}\theta_x + \sigma J\theta_{yy} - \Lambda\theta_y - \sigma\theta_y\theta^{-1}J\theta_y - \Lambda_y\theta + \theta_y\theta^{-1}\Lambda\theta\Big)\theta^{-1} \\ &= -\Big((\partial_y - \theta_y\theta^{-1})(\theta_x - \Lambda\theta + \sigma J\theta_y)\Big)\theta^{-1} \\ &= -(G_\theta L\theta)\theta^{-1}, \end{aligned} \tag{3.46}$$

which is clearly equal to zero since θ is an eigenfunction of L. Thus (3.45a), (3.45b) are satisfied with $\Lambda_{[2]} = \Lambda - \sigma[J, \theta_y\theta^{-1}]$. Since $\Lambda = [J, \sigma S]$ from (3.20) and therefore $\Lambda_{[2]} = [J, \sigma S_{[2]}]$, we have

$$S_{[2]} = S - \theta_y\theta^{-1}. \tag{3.47}$$

After n repeated applications of the Darboux transformation G_θ,

$$S_{[n+1]} = S_{[n]} - \theta^{(1)}_{[n]}\theta^{-1}_{[n]}, \tag{3.48}$$

where $S_{[1]} = S$, $\theta_{[1]} = \theta = \theta_{1[1]}$ and $\theta_{[k]} = \theta_{k[k]}$ $(k = 2, \ldots, n)$, that is

$$S_{[n+1]} = S - \sum_{i=1}^{n} \theta^{(1)}_{[i]}\theta^{-1}_{[i]}. \tag{3.49}$$

We express $S_{[n+1]}$ in quasideterminant form as

$$\begin{aligned} S_{[n+1]} &= S + \begin{vmatrix} \Theta & O_2 \\ \vdots & \vdots \\ \Theta^{(n-2)} & O_2 \\ \Theta^{(n-1)} & I_2 \\ \Theta^{(n)} & \boxed{O_2} \end{vmatrix} \\ &= S + V, \end{aligned} \tag{3.50}$$

where $\Theta^{(k)}$ denotes the k^{th} y-derivative of Θ $(k = 0, \ldots, n)$ and V is defined as in (3.41). It should be noted here that each of the entries in the above quasideterminant solution is not a single entry but a 2×2 matrix (since the θ_i are eigenfunctions of L, M).

The proof that $S_{[n+1]}$ can be expressed in the quasideterminant form (3.50) is by induction on n: for $n = 1$, we have

$$\begin{aligned} S_{[2]} &= S + \begin{vmatrix} \theta_1 & I_2 \\ \theta_1^{(1)} & \boxed{O_2} \end{vmatrix} \\ &= S - \theta_y \theta^{-1} \text{ since } \theta_1 = \theta_{1[1]} = \theta, \end{aligned} \tag{3.51}$$

which is true by (3.47). We now make the assumption that (3.50) is true for some fixed $n \geq 1$, and prove that

$$S_{[n+1+1]} = S + \begin{vmatrix} \Theta & \theta_{n+1} & O_2 \\ \vdots & \vdots & \vdots \\ \Theta^{(n-1)} & \theta_{n+1}^{(n-1)} & O_2 \\ \Theta^{(n)} & \theta_{n+1}^{(n)} & I_2 \\ \Theta^{(n+1)} & \theta_{n+1}^{(n+1)} & \boxed{O_2} \end{vmatrix}. \tag{3.52}$$

We have

$$S_{[n+1+1]} = S_{[n+1]} - \theta_{[n+1]}^{(1)} \theta_{[n+1]}^{-1} \tag{3.53}$$

by (3.48) on replacing n by $n+1$. Using the formula for the derivative of $\theta_{[n+1]} \equiv \theta_{n+1[n+1]}$ obtained in (3.42), it follows that

$$S_{[n+1+1]} = S_{[n+1]} - \left(\begin{vmatrix} \Theta & \theta_{n+1} \\ \vdots & \vdots \\ \Theta^{(n-1)} & \theta_{n+1}^{(n-1)} \\ \Theta^{(n+1)} & \boxed{\theta_{n+1}^{(n+1)}} \end{vmatrix} + V\theta_{[n+1]} \right) \theta_{[n+1]}^{-1}$$

$$= S + V - \begin{vmatrix} \Theta & \theta_{n+1} \\ \vdots & \vdots \\ \Theta^{(n-1)} & \theta_{n+1}^{(n-1)} \\ \Theta^{(n+1)} & \boxed{\theta_{n+1}^{(n+1)}} \end{vmatrix} \theta_{[n+1]}^{-1} - V \tag{3.54}$$

by the inductive hypothesis. Then, using the fact that $\theta_{[k]} \equiv \theta_{k[k]} = \phi_{[k]}|_{\phi \to \theta_k}$, we have

$$S_{[n+1+1]} = S - \begin{vmatrix} \Theta & \theta_{n+1} \\ \vdots & \vdots \\ \Theta^{(n-1)} & \theta_{n+1}^{(n-1)} \\ \Theta^{(n+1)} & \boxed{\theta_{n+1}^{(n+1)}} \end{vmatrix} \cdot \begin{vmatrix} \Theta & \theta_{n+1} \\ \vdots & \vdots \\ \Theta^{(n-1)} & \theta_{n+1}^{(n-1)} \\ \Theta^{(n)} & \boxed{\theta_{n+1}^{(n)}} \end{vmatrix}^{-1}$$

$$= S + \begin{vmatrix} \Theta & \theta_{n+1} & O_2 \\ \vdots & \vdots & \vdots \\ \Theta^{(n-1)} & \theta_{n+1}^{(n-1)} & O_2 \\ \Theta^{(n)} & \theta_{n+1}^{(n)} & I_2 \\ \Theta^{(n+1)} & \theta_{n+1}^{(n+1)} & \boxed{O_2} \end{vmatrix}, \tag{3.55}$$

where we have used the Sylvester identity (2.28), replacing 0 and 1 by the matrices O_2 and I_2 respectively. Thus by induction, (3.50) is true for all n.

For ease of notation, for integers $i, j = 1, \ldots, n$, we denote by $Q(i,j)$ the quasi-Wronskian [36]

$$Q(i,j) = \begin{vmatrix} \widehat{\Theta} & f_j \quad e_j \\ \Theta^{(n+i)} & \boxed{O_2} \end{vmatrix}, \tag{3.56}$$

where $\widehat{\Theta} = \left(\theta_j^{(i-1)}\right)_{i,j=1,\ldots,n}$ is the $n \times n$ Wronskian matrix of $\theta_1, \ldots, \theta_n$ and $\Theta^{(k)}$ denotes the k^{th} y-derivative of Θ ($k = 0, \ldots, n-1, n+i$), Θ is the row vector $(\theta_1 \ \ldots \ \theta_n)$ of length n, and f_j and e_j are $2n \times 1$ column vectors with a 1 in the $(2n-2j-1)^{\text{th}}$ and $(2n-2j)^{\text{th}}$ row respectively and zeros elsewhere. Again each θ_i is a 2×2 matrix. In this definition of $Q(i,j)$, we allow i,j to take any integer values subject to the convention that if either $2n-2j$ or $2n-2j-1$ lies outside the range $1,2,\dots,2n$, then $e_j = f_j = 0$ and so $Q(i,j)=0$. Hence (3.50) is given by

$$S_{[n+1]} = S + Q(0,0). \tag{3.57}$$

Alternatively, we can express the solution S of the ncDS equations (3.17) such that $\Lambda = [J, \sigma S]$ as

$$S = S_0 + Q(0,0), \tag{3.58}$$

where S_0 is any given solution of the ncDS equations.

It will be useful to express the quasi-Wronskian solution (3.58) in terms of q and r, the variables in which the ncDS equations (3.17) are expressed. Taking the trivial vacuum solution $S_0 = 0$, we have

$$S = Q(0,0), \tag{3.59}$$

which gives, by applying the quasideterminant expansion formula (2.10),

$$S = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \theta_1^{(n)} & \dots & \theta_n^{(n)} \end{pmatrix} \widehat{\Theta}^{-1} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.60}$$

We now express each θ_i $(i = 1,\dots,n)$ as an appropriate 2×2 matrix

$$\theta_i = \begin{pmatrix} \phi_{2i-1} & \phi_{2i} \\ \psi_{2i-1} & \psi_{2i} \end{pmatrix} \tag{3.61}$$

for $\phi = \phi(x,y,t)$, $\psi = \psi(x,y,t)$, so that

$$\theta_1 = \begin{pmatrix} \phi_1 & \phi_2 \\ \psi_1 & \psi_2 \end{pmatrix}, \tag{3.62a}$$

$$\theta_2 = \begin{pmatrix} \phi_3 & \phi_4 \\ \psi_3 & \psi_4 \end{pmatrix}, \tag{3.62b}$$

$$\vdots$$

$$\theta_n = \begin{pmatrix} \phi_{2n-1} & \phi_{2n} \\ \psi_{2n-1} & \psi_{2n} \end{pmatrix}. \tag{3.62c}$$

Continuing with the expansion (3.60), we can express S as

$$S = \begin{pmatrix} \begin{vmatrix} \widehat{\Theta} & f_0 \\ \phi^{(n)} & \boxed{0} \end{vmatrix} & \begin{vmatrix} \widehat{\Theta} & e_0 \\ \phi^{(n)} & \boxed{0} \end{vmatrix} \\ \\ \begin{vmatrix} \widehat{\Theta} & f_0 \\ \psi^{(n)} & \boxed{0} \end{vmatrix} & \begin{vmatrix} \widehat{\Theta} & e_0 \\ \psi^{(n)} & \boxed{0} \end{vmatrix} \end{pmatrix}, \tag{3.63}$$

where $\phi^{(n)}$, $\psi^{(n)}$ denote the row vectors $\left(\phi_1^{(n)} \ \dots \ \phi_{2n}^{(n)}\right)$, $\left(\psi_1^{(n)} \ \dots \ \psi_{2n}^{(n)}\right)$ respectively, and f_0, e_0 are defined as on page 45. By comparing with (3.20), we immediately see that q, r can be expressed as quasi-Wronskians, namely

$$q = 2\sigma \begin{vmatrix} \widehat{\Theta} & e_0 \\ \phi^{(n)} & \boxed{0} \end{vmatrix}, \qquad r = -2\sigma \begin{vmatrix} \widehat{\Theta} & f_0 \\ \psi^{(n)} & \boxed{0} \end{vmatrix}. \tag{3.64}$$

3.4.3 Binary Darboux transformations

We give an outline of the construction of a binary Darboux transformation, following the notation given in [71]. Consider an operator L and let T be the set of eigenfunctions of L, i.e. $T = \{\theta : L(\theta) = 0\}$. Similarly, let $\tilde{L}, \hat{L}$ be operators with sets of eigenfunctions $\tilde{T}, \hat{T}$ respectively, so that $\tilde{T} = \{\tilde{\theta} : \tilde{L}(\tilde{\theta}) = 0\}$ and $\hat{T} = \{\hat{\theta} : \hat{L}(\hat{\theta}) = 0\}$. The Darboux transformation $G_\theta = \partial_y - \theta_y\theta^{-1} = \theta\partial_y\theta^{-1}$ maps eigenfunctions of L to eigenfunctions of $\tilde{L}$, where $\theta \in T$ and $\tilde{L} = G_\theta L G_\theta^{-1}$. Similarly, define $G_{\hat{\theta}} = \hat{\theta}\partial_y\hat{\theta}^{-1}$ to be the Darboux transformation mapping eigenfunctions of $\hat{L}$ to eigenfunctions of $\tilde{L} = G_{\hat{\theta}}\hat{L}G_{\hat{\theta}}^{-1}$, i.e.

$$T \xrightarrow{G_\theta} \tilde{T} \xleftarrow{G_{\hat{\theta}}} \hat{T},$$

where $\hat{\theta} \in \hat{T}$. The aim of the binary Darboux transformation is to define a mapping from T to $\hat{T}$ in order to determine $\hat{\theta}$. Clearly this is given by $(G_{\hat{\theta}})^{-1}G_\theta$, and thus the binary Darboux transformation is a composition of two ordinary Darboux transformations. However, by the definition of $G_{\hat{\theta}}$ above, this transformation requires knowledge of $\hat{\theta}$, the eigenfunctions to be determined. To find an expression for $\hat{\theta}$, the *adjoint* operator $G_\theta^\dagger$ must be considered. The notion of adjoint can be easily extended from the well-known matrix situation to any ring $\mathcal{R}$. An element $a \in \mathcal{R}$ has adjoint $a^\dagger$, where the adjoint has the following properties: if ∂ is a derivative acting on $\mathcal{R}$,

$$\partial^\dagger = -\partial, \tag{3.65}$$

and for any A, B elements of, or operators on, the ring $\mathcal{R}$,

$$(A+B)^\dagger = A^\dagger + B^\dagger, \tag{3.66a}$$

$$(AB)^\dagger = B^\dagger A^\dagger. \tag{3.66b}$$

We also introduce the notation $A^{-\dagger}$ to denote $\left(A^\dagger\right)^{-1} = \left(A^{-1}\right)^\dagger$. Using the above relations, it can be seen that

$$\begin{aligned}\tilde{L}^\dagger &= \left(G_{\hat\theta}\hat{L}G_{\hat\theta}^{-1}\right)^\dagger \\ &= G_{\hat\theta}^{-\dagger}\hat{L}^\dagger G_{\hat\theta}^\dagger,\end{aligned} \tag{3.67}$$

so that $G_{\hat\theta}^\dagger$ is a Darboux transformation mapping the eigenfunctions of $\tilde{L}^\dagger$ to those of $\hat{L}^\dagger$, i.e. from $\tilde{T}^\dagger$ to $\hat{T}^\dagger$, where

$$\begin{aligned}G_{\hat\theta}^\dagger &= \left(\hat\theta\partial_y\hat\theta^{-1}\right)^\dagger \\ &= -\hat\theta^{-\dagger}\partial_y\hat\theta^\dagger\end{aligned} \tag{3.68}$$

using properties (3.65) and (3.66b). Clearly, $G_{\hat\theta}^\dagger\left(\hat\theta^{-\dagger}\right) = 0$ by definition, so that $\hat\theta^{-\dagger}$ is an eigenfunction of $\tilde{L}^\dagger$, i.e. $\hat\theta^{-\dagger} \in \tilde{S}^\dagger$. Also, since $\tilde{L} = G_\theta L G_\theta^{-1}$, so that $L = G_\theta^{-1}\tilde{L}G_\theta$, we have

$$\begin{aligned}L^\dagger &= \left(G_\theta^{-1}\tilde{L}G_\theta\right)^\dagger \\ &= G_\theta^\dagger\tilde{L}^\dagger G_\theta^{-\dagger},\end{aligned} \tag{3.69}$$

so that $G_\theta^\dagger$ is a Darboux transformation mapping the eigenfunctions of $\tilde{L}^\dagger$ to those of $L^\dagger$, i.e. from $\tilde{T}^\dagger$ to $T^\dagger$. We see that the adjoint Darboux transformations $G_\theta^\dagger$ and $G_{\hat\theta}^\dagger$ act in the opposite directions to their corresponding 'non-adjoint' versions. Thus

$$\rho := G_\theta^\dagger\left(\hat\theta^{-\dagger}\right) \tag{3.70}$$

is an eigenfunction of the adjoint operator $L^\dagger$, i.e. $\rho \in T^\dagger$. We therefore have the following situation:

$$\begin{array}{ccccc} L & \xrightarrow{G_\theta} & \tilde{L} & \xleftarrow{G_{\hat\theta}} & \hat{L} \\ \theta & & & & \hat\theta \\ L^\dagger & \xleftarrow{G_\theta^\dagger} & \tilde{L}^\dagger & \xrightarrow{G_{\hat\theta}^\dagger} & \hat{L}^\dagger\ , \\ \rho & & \hat\theta^{-\dagger} & & \end{array}$$

with the binary Darboux transformation given by

$$L \xrightarrow{G_{\hat\theta}^{-1}G_\theta} \hat{L}\ ,$$

and the corresponding transformation between the adjoint eigenfunctions defined as

$$L^\dagger \xrightarrow{G_{\hat{\theta}}^\dagger G_\theta^{-\dagger}} \hat{L}^\dagger \ .$$

We call this the 'adjoint binary Darboux transformation' to indicate a transformation between adjoint operators (i.e. a transformation between adjoint eigenfunctions), although this transformation is actually the *adjoint inverse* of the binary Darboux transformation $G_{\hat{\theta}}^{-1}G_\theta$.

We have the adjoint eigenfunction ρ defined as

$$\rho = G_\theta^\dagger \left(\hat{\theta}^{-\dagger}\right). \tag{3.71}$$

Substituting for $G_\theta^\dagger = -\theta^{-\dagger}\partial_y\theta^\dagger$ gives

$$\rho = -\theta^{-\dagger}\partial_y \left(\theta^\dagger\hat{\theta}^{-\dagger}\right), \tag{3.72}$$

where $\hat{\theta}$ is the unknown eigenfunction of $\hat{L}$ to be determined. Thus

$$\partial_y^{-1}(\theta^\dagger\rho) = -\theta^\dagger\hat{\theta}^{-\dagger}, \tag{3.73}$$

and hence

$$\begin{aligned}\hat{\theta} &= -\theta\left(\partial_y^{-1}(\rho^\dagger\theta)\right)^{-1} \\ &= -\theta\Omega(\theta,\rho)^{-1}, \end{aligned} \tag{3.74}$$

where we define $\Omega(\theta,\rho) = \partial_y^{-1}(\rho^\dagger\theta)$, i.e. $\Omega(\theta,\rho)_y = \rho^\dagger\theta$. Thus the binary Darboux transformation $G_{\theta,\rho} = G_{\hat{\theta}}^{-1}G_\theta$ is given by

$$\begin{aligned} G_{\theta,\rho} &= \left(\hat{\theta}\partial_y\hat{\theta}^{-1}\right)^{-1}\theta\partial_y\theta^{-1} \text{ by definition of } G_\theta \text{ and } G_{\hat{\theta}} \\ &= \theta\Omega^{-1}\partial_y^{-1}\Omega\partial_y\theta^{-1} \text{ by (3.74)} \\ &= I - \theta\Omega^{-1}\partial_y^{-1}\rho^\dagger, \end{aligned} \tag{3.75}$$

where $\Omega = \Omega(\theta,\rho)$. The above calculation uses the fact that $\partial_y\Omega = \Omega_y + \Omega\partial_y$, giving $\Omega\partial_y = \partial_y\Omega - \Omega_y$. The transformation between the adjoint eigenfunctions, $G_{\hat{\theta}}^\dagger G_\theta^{-\dagger} = \left(G_{\hat{\theta}}^{-1}G_\theta\right)^{-\dagger} = G_{\theta,\rho}^{-\dagger}$, is given by

$$\begin{aligned} G_{\theta,\rho}^{-\dagger} &= \hat{\theta}^{-\dagger}\partial_y\hat{\theta}^\dagger\theta^{-\dagger}\partial_y^{-1}\theta^\dagger \\ &= \theta^{-\dagger}\Omega^\dagger\partial_y\Omega^{-\dagger}\partial_y^{-1}\theta^\dagger \text{ by (3.74)} \\ &= \theta^{-\dagger}\Omega^\dagger\left(\Omega_y^{-\dagger} + \Omega^{-\dagger}\partial_y\right)\partial_y^{-1}\theta^\dagger \\ &= I - \rho\Omega^{-\dagger}\partial_y^{-1}\theta^\dagger, \end{aligned} \tag{3.76}$$

where we note that $\Omega_y^{-\dagger} = -\Omega^{-\dagger}\theta^\dagger\rho\Omega^{-\dagger}$ by differentiating both sides of the equation $\Omega^\dagger\Omega^{-\dagger} = I$ with respect to y and use the fact that $\Omega_y^\dagger = \theta^\dagger\rho$ for $\Omega = \Omega(\theta,\rho)$. Notice that the eigenfunctions and adjoint eigenfunctions have interchanged roles in the adjoint transformation.

In summary, we have found that a binary Darboux transformation $G_{\theta,\rho}$ transforming L, M to some new operators $\hat{L}$, $\hat{M}$ is given by

$$G_{\theta,\rho} = I - \theta\Omega(\theta,\rho)^{-1}\partial_y^{-1}\rho^\dagger, \tag{3.77a}$$

with adjoint transformation

$$G_{\theta,\rho}^{-\dagger} = I - \rho\Omega(\theta,\rho)^{-\dagger}\partial_y^{-1}\theta^\dagger \tag{3.77b}$$

for eigenfunctions θ of L, M and adjoint eigenfunctions ρ of $L^\dagger$, $M^\dagger$. Here I denotes the 2×2 identity matrix. The adjoint Lax pair for the ncDS system is given by

$$L^\dagger = -\partial_x - \mathit{\Lambda}^\dagger - \frac{1}{\sigma}J\partial_y, \tag{3.78a}$$

$$M^\dagger = -\partial_t - A^\dagger + \mathrm{i}\sigma(\mathit{\Lambda}_y^\dagger + \mathit{\Lambda}^\dagger\partial_y) + \mathrm{i}J\partial_{yy}. \tag{3.78b}$$

(Note that $\sigma^\dagger = \dfrac{1}{\sigma}$ for $\sigma = -1, \mathrm{i}$). Let ϕ be an eigenfunction of L, M and ψ an eigenfunction of the corresponding adjoint operators $L^\dagger$, $M^\dagger$, so that $L(\phi) = M(\phi) = 0$ and $L^\dagger(\psi) = M^\dagger(\psi) = 0$. We calculate the dispersion relations for the ncDS equations in the trivial vacuum case, i.e. for $\mathit{\Lambda}, A \equiv 0$ and find that

$$\phi_x = -\sigma J\phi_y, \qquad \phi_t = \mathrm{i}J\phi_{yy}, \tag{3.79a}$$

$$\psi_x = -\frac{1}{\sigma}J\psi_y, \qquad \psi_t = \mathrm{i}J\psi_{yy}. \tag{3.79b}$$

The potential $\Omega(\phi,\psi)$ satisfies

$$\Omega(\phi,\psi)_y = \psi^\dagger\phi, \tag{3.80}$$

from which it follows that

$$\begin{aligned}\Omega(\phi,\psi)_{xy} &= \psi_x^\dagger\phi + \psi^\dagger\phi_x \\ &= -\sigma\left(\psi_y^\dagger J\phi + \psi^\dagger J\phi_y\right)\end{aligned} \tag{3.81}$$

by (3.79a), (3.79b). Thus

$$\begin{aligned}\Omega(\phi,\psi)_x &= -\sigma\left(\psi^\dagger J\phi - \int\psi^\dagger J\phi_y\,\mathrm{d}y + \int\psi^\dagger J\phi_y\,\mathrm{d}y\right) \\ &= -\sigma\psi^\dagger J\phi.\end{aligned} \tag{3.82}$$

Similarly, using the Lax operators M, $M^\dagger$, we deduce that

$$\Omega(\phi, \psi)_t = \mathrm{i}(\psi^\dagger J \phi_y - \psi_y^\dagger J \phi). \tag{3.83}$$

These definitions are also valid for non-scalar eigenfunctions: if Φ is an n-vector and Ψ an m-vector, then $\Omega(\Phi, \Psi)$ is an $m \times n$ matrix.

We now detail the procedure used to perform the iteration of the binary Darboux transformation.

Iteration

We relabel L, $L^\dagger$ as $L_{[1]}$, $L_{[1]}^\dagger$ respectively, and similarly for M, $M^\dagger$, to indicate the starting levels.

Step 1 Let $\theta_1, \ldots, \theta_n$ be eigenfunctions of the original Lax pair $L_{[1]} = L$, $M_{[1]} = M$, and $\rho_1, \ldots, \rho_n$ eigenfunctions of the adjoint Lax operators $L_{[1]}^\dagger = L^\dagger$, $M_{[1]}^\dagger = M^\dagger$. Suppose $\phi_{[1]} = \phi$ is a generic eigenfunction of $L_{[1]}, M_{[1]}$ and $\psi_{[1]} = \psi$ a generic eigenfunction of $L_{[1]}^\dagger, M_{[1]}^\dagger$. We choose $\theta_1 := \theta_{1[1]}$ to be the eigenfunction defining a binary Darboux transformation from $L_{[1]}, M_{[1]}$ to a new Lax pair $L_{[2]}, M_{[2]}$, and similarly $\rho_1 := \rho_{1[1]}$ the eigenfunction defining the adjoint binary Darboux transformation from $L_{[1]}^\dagger, M_{[1]}^\dagger$ to a new adjoint Lax pair $L_{[2]}^\dagger, M_{[2]}^\dagger$. Then the operators $L_{[1]}, M_{[1]}$ are covariant under the action of the binary Darboux transformation

$$G_{\theta_{1[1]},\rho_{1[1]}} = I - \theta_{1[1]} \Omega(\theta_{1[1]}, \rho_{1[1]})^{-1} \partial_y^{-1} \rho_{1[1]}^\dagger, \tag{3.84}$$

while the adjoint operators $L_{[1]}^\dagger, M_{[1]}^\dagger$ are covariant under the adjoint binary Darboux transformation

$$G_{\theta_{1[1]},\rho_{1[1]}}^{-\dagger} = I - \rho_{1[1]} \Omega(\theta_{1[1]}, \rho_{1[1]})^{-\dagger} \partial_y^{-1} \theta_{1[1]}^\dagger. \tag{3.85}$$

The transformed operators

$$L_{[2]} = G_{\theta_{1[1]},\rho_{1[1]}} L_{[1]} G_{\theta_{1[1]},\rho_{1[1]}}^{-1}, \tag{3.86a}$$

$$M_{[2]} = G_{\theta_{1[1]},\rho_{1[1]}} M_{[1]} G_{\theta_{1[1]},\rho_{1[1]}}^{-1} \tag{3.86b}$$

have generic eigenfunctions

$$\phi_{[2]} := G_{\theta_{1[1]},\rho_{1[1]}}(\phi) = \phi - \theta_{1[1]} \Omega(\theta_{1[1]}, \rho_{1[1]})^{-1} \Omega(\phi, \rho_{1[1]}), \tag{3.87}$$

and generic adjoint eigenfunctions

$$\psi_{[2]} := G^{-\dagger}_{\theta_{1[1]},\rho_{1[1]}}(\psi) = \psi - \rho_{1[1]}\Omega(\theta_{1[1]},\rho_{1[1]})^{-\dagger}\Omega(\theta_{1[1]},\psi)^{\dagger}. \tag{3.88}$$

The eigenfunction $\theta_{1[1]}$ is mapped to zero by (3.84), and the adjoint eigenfunction $\rho_{1[1]}$ to zero by (3.85). The remaining eigenfunctions $\theta_2,\dots,\theta_n$ and adjoint eigenfunctions $\rho_2,\dots,\rho_n$ are mapped to $\theta_{2[2]},\dots,\theta_{n[2]}$ and $\rho_{2[2]},\dots,\rho_{n[2]}$ respectively, where, for $i = 2,\dots,n$,

$$\theta_{i[2]} = \phi_{[2]}|_{\phi\to\theta_i}, \qquad \rho_{i[2]} = \psi_{[2]}|_{\psi\to\rho_i}. \tag{3.89}$$

These transformed functions can easily be shown to be eigenfunctions and adjoint eigenfunctions respectively of the transformed operators $L_{[2]}, M_{[2]}$ and $L^{\dagger}_{[2]}, M^{\dagger}_{[2]}$.

$\vdots$

Step n $(n \geq 1)$ To perform the n^{th} iteration of the binary Darboux transformation, we choose the eigenfunction $\theta_{n[n]}$ to define a binary Darboux transformation from the Lax operators $L_{[n]}, M_{[n]}$ to some new Lax operators $L_{[n+1]}, M_{[n+1]}$, and similarly $\rho_{n[n]}$ the adjoint eigenfunction defining the adjoint binary Darboux transformation from $L^{\dagger}_{[n]}, M^{\dagger}_{[n]}$ to $L^{\dagger}_{[n+1]}, M^{\dagger}_{[n+1]}$. The operators $L_{[n]}, M_{[n]}$ are covariant under the action of the binary Darboux transformation

$$G_{\theta_{n[n]},\rho_{n[n]}} = I - \theta_{n[n]}\Omega(\theta_{n[n]},\rho_{n[n]})^{-1}\partial_y^{-1}\rho^{\dagger}_{n[n]}, \tag{3.90}$$

while the adjoint operators $L^{\dagger}_{[n]}, M^{\dagger}_{[n]}$ are covariant under the adjoint binary Darboux transformation

$$G^{-\dagger}_{\theta_{n[n]},\rho_{n[n]}} = I - \rho_{n[n]}\Omega(\theta_{n[n]},\rho_{n[n]})^{-\dagger}\partial_y^{-1}\theta^{\dagger}_{n[n]}. \tag{3.91}$$

The transformed operators

$$L_{[n+1]} = G_{\theta_{n[n]},\rho_{n[n]}} L_{[n]} G^{-1}_{\theta_{n[n]},\rho_{n[n]}}, \tag{3.92a}$$

$$M_{[n+1]} = G_{\theta_{n[n]},\rho_{n[n]}} M_{[n]} G^{-1}_{\theta_{n[n]},\rho_{n[n]}} \tag{3.92b}$$

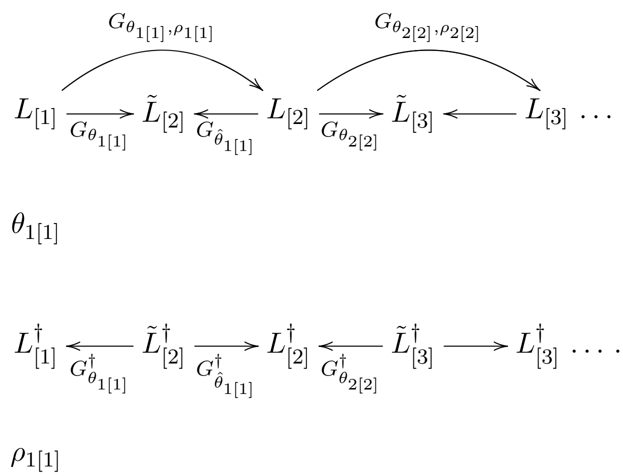
have generic eigenfunctions

$$\phi_{[n+1]} = G_{\theta_{n[n]},\rho_{n[n]}}(\phi_{[n]}) = \phi_{[n]} - \theta_{n[n]}\Omega(\theta_{n[n]},\rho_{n[n]})^{-1}\Omega(\phi_{[n]},\rho_{n[n]}), \tag{3.93}$$

and generic adjoint eigenfunctions

$$\psi_{[n+1]} = G^{-\dagger}_{\theta_{n[n]},\rho_{n[n]}}(\psi_{[n]}) = \psi_{[n]} - \rho_{n[n]}\Omega(\theta_{n[n]},\rho_{n[n]})^{-\dagger}\Omega(\theta_{n[n]},\psi_{[n]})^{\dagger}. \tag{3.94}$$

In particular, the eigenfunction $\theta_{n[n]}$ is mapped to zero by (3.90), and the adjoint eigenfunction $\rho_{n[n]}$ to zero by (3.91). We illustrate the iteration pictorially below, where we have distinguished between the operators corresponding to ordinary Darboux transformations by the use of tilde. In addition, $L_{[2]}$ corresponds to $\hat{L}$ and $\tilde{L}_{[2]}$ to $\tilde{L}$.



Quasi-Grammian form

Defining $\Theta = (\theta_1 \dots \theta_n)$ and $P = (\rho_1 \dots \rho_n)$, we express $\phi_{[n+1]}$ and $\psi_{[n+1]}$ in *quasi-Grammian* form [36] as

$$\phi_{[n+1]} = \begin{vmatrix} \Omega(\Theta, P) & \Omega(\phi, P) \\ \Theta & \boxed{\phi} \end{vmatrix}, \qquad \psi_{[n+1]} = \begin{vmatrix} \Omega(\Theta, P)^\dagger & \Omega(\Theta, \psi)^\dagger \\ P & \boxed{\psi} \end{vmatrix}, \tag{3.95}$$

with

$$\Omega(\phi_{[n+1]}, \psi_{[n+1]}) = \begin{vmatrix} \Omega(\Theta, P) & \Omega(\phi, P) \\ \Omega(\Theta, \psi) & \boxed{\Omega(\phi, \psi)} \end{vmatrix}. \tag{3.96}$$

Note that (3.96) follows from (3.95) using the formula for the derivative of a quasi-Grammian, (2.41), and the usual expansion of a quasideterminant: consider

$$\begin{aligned} \begin{vmatrix} \Omega(\Theta, P) & \Omega(\phi, P) \\ \Omega(\Theta, \psi) & \boxed{\Omega(\phi, \psi)} \end{vmatrix}_y &= \Omega(\phi, \psi)_y + \begin{vmatrix} \Omega(\Theta, P) & \Omega(\phi, P) \\ \Omega(\Theta, \psi)_y & \boxed{O_2} \end{vmatrix} + \begin{vmatrix} \Omega(\Theta, P) & \Omega(\phi, P)_y \\ \Omega(\Theta, \psi) & \boxed{O_2} \end{vmatrix} \\ &\quad + \begin{vmatrix} \Omega(\Theta, P) & P^\dagger \\ \Omega(\Theta, \psi) & \boxed{O_2} \end{vmatrix} \cdot \begin{vmatrix} \Omega(\Theta, P) & \Omega(\phi, P) \\ \Theta & \boxed{O_2} \end{vmatrix} \quad \text{by (2.41)} \\ &= \psi^\dagger \phi - \psi^\dagger \Theta \Omega(\Theta, P)^{-1} \Omega(\phi, P) - \Omega(\Theta, \psi) \Omega(\Theta, P)^{-1} P^\dagger \phi \\ &\quad + \Omega(\Theta, \psi) \Omega(\Theta, P)^{-1} P^\dagger \Theta \Omega(\Theta, P)^{-1} \Omega(\phi, P), \end{aligned} \tag{3.97}$$

where we have used the fact that $\Omega(a,b)_y = b^\dagger a$, and, in (2.41), $\sum_{i=1}^k E_iF_i = P^\dagger\Theta$. We therefore have

$$\begin{aligned}
\begin{vmatrix} \Omega(\Theta,P) & \Omega(\phi,P) \\ \Omega(\Theta,\psi) & \boxed{\Omega(\phi,\psi)} \end{vmatrix}_y &= (\psi^\dagger - \Omega(\Theta,\psi)\Omega(\Theta,P)^{-1}P^\dagger).(\phi - \Theta\Omega(\Theta,P)^{-1}\Omega(\phi,P)) \\
&= \begin{vmatrix} \Omega(\Theta,P)^\dagger & \Omega(\Theta,\psi)^\dagger \\ P & \boxed{\psi} \end{vmatrix}^\dagger . \begin{vmatrix} \Omega(\Theta,P) & \Omega(\phi,P) \\ \Theta & \boxed{\phi} \end{vmatrix} \\
&= \psi^\dagger_{[n+1]}\phi_{[n+1]} \\
&= \Omega(\phi_{[n+1]},\psi_{[n+1]})_y. \qquad (3.98)
\end{aligned}$$

Thus (3.96) follows on integration (where we assume constants of integration are equal to zero).

We now prove the result for $\phi_{[n+1]}$ in (3.95) by induction; the proof for $\psi_{[n+1]}$ arises in a similar manner. We therefore prove that

$$\phi_{[n+1]} = \begin{vmatrix} \Omega(\Theta,P) & \Omega(\phi,P) \\ \Theta & \boxed{\phi} \end{vmatrix}. \qquad (3.99)$$

Observe that the result is true for $n = 1$, since

$$\begin{aligned}
\phi_{[2]} &= \phi - \theta_1\Omega(\theta_1,\rho_1)^{-1}\Omega(\phi,\rho_1) \\
&= \phi - \theta_{1[1]}\Omega(\theta_{1[1]},\rho_{1[1]})^{-1}\Omega(\phi,\rho_{1[1]}), \qquad (3.100)
\end{aligned}$$

which holds by (3.87). Now suppose that (3.99) is true for some fixed $n \geq 1$. We will show that

$$\phi_{[n+1+1]} = \begin{vmatrix} \Omega(\Theta,P) & \Omega(\theta_{n+1},P) & \Omega(\phi,P) \\ \Omega(\Theta,\rho_{n+1}) & \Omega(\theta_{n+1},\rho_{n+1}) & \Omega(\phi,\rho_{n+1}) \\ \Theta & \theta_{n+1} & \boxed{\phi} \end{vmatrix}. \qquad (3.101)$$

We have

$$\phi_{[n+1+1]} = \phi_{[n+1]} - \theta_{n+1[n+1]}\Omega(\theta_{n+1[n+1]},\rho_{n+1[n+1]})^{-1}\Omega(\phi_{[n+1]},\rho_{n+1[n+1]})$$

from (3.93), replacing n by $n+1$

$$= \begin{vmatrix} \Omega(\Theta,P) & \Omega(\phi,P) \\ \Theta & \boxed{\phi} \end{vmatrix} - \begin{vmatrix} \Omega(\Theta,P) & \Omega(\theta_{n+1},P) \\ \Theta & \boxed{\theta_{n+1}} \end{vmatrix} .\Omega(\theta_{n+1[n+1]},\rho_{n+1[n+1]})^{-1}\Omega(\phi_{[n+1]},\rho_{n+1[n+1]}) \qquad (3.102)$$

by the inductive hypothesis (3.99) and the fact that $\theta_{n[n]} = \phi_{[n]}|_{\phi\to\theta_n}$, i.e. $\theta_{n+1[n+1]} = \phi_{[n+1]}|_{\phi\to\theta_{n+1}}$. In order to obtain a quasideterminant expression for $\Omega(\theta_{n+1[n+1]}, \rho_{n+1[n+1]})$, we look to (3.96) and replace $\phi_{[n+1]}$ by $\theta_{n+1[n+1]}$, i.e. ϕ by θ_{n+1} (since $\theta_{n[n]} = \phi_{[n]}|_{\phi\to\theta_n}$) and $\psi_{[n+1]}$ by $\rho_{n+1[n+1]}$, i.e. ψ by ρ_{n+1} (since $\rho_{n[n]} = \psi_{[n]}|_{\psi\to\rho_n}$). An expression for $\Omega(\phi_{[n+1]}, \rho_{n+1[n+1]})$ can be obtained in a similar manner. Thus

$$\phi_{n+1[n+1]} = \begin{vmatrix} \Omega(\Theta,P) & \Omega(\phi,P) \\ \Theta & \boxed{\phi} \end{vmatrix} - \begin{vmatrix} \Omega(\Theta,P) & \Omega(\theta_{n+1},P) \\ \Theta & \boxed{\theta_{n+1}} \end{vmatrix} \cdot \begin{vmatrix} \Omega(\Theta,P) & \Omega(\theta_{n+1},P) \\ \Omega(\Theta,\rho_{n+1}) & \boxed{\Omega(\theta_{n+1},\rho_{n+1})} \end{vmatrix}^{-1} \cdot \begin{vmatrix} \Omega(\Theta,P) & \Omega(\phi,P) \\ \Omega(\Theta,\rho_{n+1}) & \boxed{\Omega(\phi,\rho_{n+1})} \end{vmatrix}, \tag{3.103}$$

which can be seen to equal the right-hand side of (3.101) using the noncommutative Sylvester identity (2.27). This completes the inductive proof.

3.4.4 Quasi-Grammian solution of ncDS using binary Darboux transformations

We now determine the effect of the binary Darboux transformation $G_{\theta_{1[1]},\rho_{1[1]}}$ on the Lax operator $L_{[1]} = L$ given by (3.15a), with $\theta_1, \dots, \theta_n = \theta_{1[1]}, \dots, \theta_{n[1]}$ eigenfunctions of L and $\theta_{1[1]}$ chosen to iterate the Darboux transformation. To simplify our notation slightly, we will denote $\theta_{1[1]}$ by θ, $\hat\theta_{1[1]}$ by $\hat\theta$ and, in general, $\theta_{k[k]}$ by $\theta_{[k]}$ ($k = 2, \dots, n$). Similarly, we denote $\rho_{1[1]}$ by ρ, and, in general, $\rho_{k[k]}$ by $\rho_{[k]}$. Corresponding results hold for the operator M given by (3.15b). Since the binary Darboux transformation $G_{\theta_{1[1]},\rho_{1[1]}} = G_{\theta,\rho}$ is a composition of the two ordinary Darboux transformations $G_{\theta_{1[1]}} \equiv G_\theta$ and $G_{\hat\theta_{1[1]}} \equiv G_{\hat\theta}$ (see the diagram on page 53), we have

$$\tilde L_{[2]} = G_\theta L_{[1]} G_\theta^{-1}, \tag{3.104}$$

giving

$$\tilde\Lambda_{[2]} = \Lambda - \sigma[J, \theta_y\theta^{-1}] \tag{3.105}$$

as in Section 3.4.2, where $\Lambda = \Lambda_{[1]}$, and

$$\tilde L_{[2]} = G_{\hat\theta} L_{[2]} G_{\hat\theta}^{-1}, \tag{3.106}$$

so that

$$\tilde\Lambda_{[2]} = \Lambda_{[2]} - \sigma[J, \hat\theta_y\hat\theta^{-1}] \tag{3.107}$$

with $\hat{\theta} \equiv \hat{\theta}_{1[1]} = -\theta_{1[1]}\Omega(\theta_{1[1]}, \rho_{1[1]})^{-1}$ as in (3.74). (On substituting in (3.106), we obtain two equations, one of which can be solved for $\tilde{\varLambda}_{[2]}$, giving the expression above, and the other simplified to zero as in Section 3.4.2 using the fact that $L_{[2]}(\hat{\theta}) \equiv L_{[2]}(\hat{\theta}_{1[1]}) = 0$). Comparing (3.105) and (3.107), we see that

$$\varLambda - \sigma[J, \theta_y\theta^{-1}] = \varLambda_{[2]} - \sigma[J, \hat{\theta}_y\hat{\theta}^{-1}], \tag{3.108}$$

so that

$$\varLambda_{[2]} = \varLambda - \sigma[J, \theta_y\theta^{-1}] + \sigma[J, \hat{\theta}_y\hat{\theta}^{-1}]. \tag{3.109}$$

Since $\varLambda = [J, \sigma S]$ from page 36 and therefore $\varLambda_{[2]} = [J, \sigma S_{[2]}]$, we have

$$\begin{aligned} S_{[2]} &= S - \theta_y\theta^{-1} + \hat{\theta}_y\hat{\theta}^{-1} \\ &= S - \theta\Omega(\theta, \rho)^{-1}\rho^{\dagger} \end{aligned} \tag{3.110}$$

as $\hat{\theta} = -\theta\Omega(\theta, \rho)^{-1}$ from before. After n repeated applications of the binary Darboux transformation $G_{\theta,\rho}$, we obtain

$$S_{[n+1]} = S_{[n]} - \theta_{[n]}\Omega(\theta_{[n]}, \rho_{[n]})^{-1}\rho_{[n]}^{\dagger}, \tag{3.111}$$

i.e.

$$S_{[n+1]} = S - \sum_{i=1}^{n} \theta_{[i]}\Omega(\theta_{[i]}, \rho_{[i]})^{-1}\rho_{[k]}^{\dagger}, \tag{3.112}$$

where $S_{[1]} = S$, $\theta_{[1]} = \theta_{1[1]} = \theta$ and $\theta_{[k]} = \theta_{k[k]}$ $(k = 2, \ldots, n)$, and similarly $\rho_{[1]} = \rho_{1[1]} = \rho$ and $\rho_{[k]} = \rho_{k[k]}$. By once again defining $\Theta = (\theta_1 \ldots \theta_n)$ and $P = (\rho_1 \ldots \rho_n)$, we express $S_{[n+1]}$ in quasi-Grammian form as

$$S_{[n+1]} = S + \begin{vmatrix} \Omega(\Theta, P) & P^{\dagger} \\ \Theta & \boxed{O_2} \end{vmatrix}, \tag{3.113}$$

where $\Omega(\Theta, P)$ is the Grammian-like matrix defined by (3.80), (3.82) and (3.83). (Note that, for $i = 1, \ldots, n$, each θ_i, ρ_i is a 2×2 matrix (since the θ_i, ρ_i are eigenfunctions of L, M and $L^{\dagger}, M^{\dagger}$ respectively)). The proof is by induction on n as follows.

For $n = 1$, we have

$$\begin{aligned} S_{[2]} &= S + \begin{vmatrix} \Omega(\theta_1, \rho_1) & \rho_1^{\dagger} \\ \theta_1 & \boxed{O_2} \end{vmatrix} \\ &= S - \theta\Omega(\theta, \rho)^{-1}\rho^{\dagger}, \end{aligned} \tag{3.114}$$

which holds by (3.110), since $\theta_1 = \theta_{1[1]} = \theta, \rho_1 = \rho_{1[1]} = \rho$. Now suppose that (3.113) is true for some fixed $n \geq 1$. We prove that the result is also true for $n+1$, i.e. we prove that

$$S_{[n+1+1]} = S + \begin{vmatrix} \Omega(\Theta, P) & \Omega(\theta_{n+1}, P) & P^\dagger \\ \Omega(\Theta, \rho_{n+1}) & \Omega(\theta_{n+1}, \rho_{n+1}) & \rho_{n+1}^\dagger \\ \Theta & \theta_{n+1} & \boxed{O_2} \end{vmatrix}. \tag{3.115}$$

We have

$$\begin{aligned} S_{[n+1+1]} &= S_{[n+1]} - \theta_{[n+1]}\Omega(\theta_{[n+1]}, \rho_{[n+1]})^{-1}\rho_{[n+1]}^\dagger \text{ by (3.111), replacing } n \text{ by } n+1 \\ &= S + \begin{vmatrix} \Omega(\Theta, P) & P^\dagger \\ \Theta & \boxed{O_2} \end{vmatrix} \\ &- \begin{vmatrix} \Omega(\Theta, P) & \Omega(\theta_{n+1}, P) \\ \Theta & \boxed{\theta_{n+1}} \end{vmatrix} \cdot \begin{vmatrix} \Omega(\Theta, P) & \Omega(\theta_{n+1}, P) \\ \Omega(\Theta, \rho_{n+1}) & \boxed{\Omega(\theta_{n+1}, \rho_{n+1})} \end{vmatrix}^{-1} \begin{vmatrix} \Omega(\Theta, P) & P^\dagger \\ \Omega(\Theta, \rho_{n+1}) & \boxed{\rho_{n+1}^\dagger} \end{vmatrix} \end{aligned}$$

by the inductive hypothesis (3.113) and (3.95), (3.96), where we have used the fact that $\theta_{[n]} \equiv \theta_{n[n]} = \phi_{[n]}|_{\phi \to \theta_n}$, $\rho_{[n]} \equiv \rho_{n[n]} = \psi_{[n]}|_{\psi \to \rho_n}$,

$$= S + \begin{vmatrix} \Omega(\Theta, P) & \Omega(\theta_{n+1}, P) & P^\dagger \\ \Omega(\Theta, \rho_{n+1}) & \Omega(\theta_{n+1}, \rho_{n+1}) & \rho_{n+1}^\dagger \\ \Theta & \theta_{n+1} & \boxed{O_2} \end{vmatrix} \tag{3.116}$$

by the noncommutative Sylvester identity (2.27). This completes the proof.

For ease of notation, for integers $i, j = 1, \ldots, n$, we denote by $R(i,j)$ the quasi-Grammian [36]

$$R(i,j) = (-1)^j \begin{vmatrix} \Omega(\Theta, P) & P^{\dagger\,(j)} \\ \Theta^{(i)} & \boxed{O_2} \end{vmatrix}, \tag{3.117}$$

so that (3.113) is given by

$$S_{[n+1]} = S + R(0,0). \tag{3.118}$$

(The reason for the inclusion of the factor $(-1)^j$ in (3.117) will be made clear later). As before, we can express the solution S of the ncDS equations (3.17) such that $\Lambda = [J, \sigma S]$ as

$$S = S_0 + R(0,0), \tag{3.119}$$

where S_0 is any given solution of the ncDS equations and $R(0,0)$ is a Grammian-like quasideterminant. We then take the trivial vacuum solution $S_0 = 0$ and apply (2.10) to give

$$S = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} - \Theta\,\Omega(\Theta, P)^{-1} P^{\dagger}. \tag{3.120}$$

In order to express the quasi-Grammian solution (3.120) in terms of the variables q, r, we define the matrices θ_i as in the quasi-Wronskian case (3.61), so that

$$\Theta = (\theta_1 \dots \theta_n) = \begin{pmatrix} \phi_1 & \dots & \phi_{2n} \\ \psi_1 & \dots & \psi_{2n} \end{pmatrix}. \tag{3.121}$$

We also take

$$P = \Theta H^{\dagger} \tag{3.122}$$

where H is a constant $2n \times 2n$ matrix which we assume to be invertible, and $H^{\dagger}$ denotes the Hermitian conjugate of H. (The reason for choosing P in this form will become apparent - it will be shown that Θ, P satisfy the same dispersion relations in the case $\sigma = -1$, which are unaffected when multiplied by a constant matrix (independent of x, y, t). Although we could conceivably make the simpler choice $P = \Theta$, this does not produce the desired dromion solutions obtained in the next chapter. This has been explained in more detail in Section 4.1.4).

Thus, from (3.120), we obtain

$$S = \begin{pmatrix} \begin{vmatrix} \Omega(\Theta, P) & H\phi^{\dagger} \\ \phi & \boxed{0} \end{vmatrix} & \begin{vmatrix} \Omega(\Theta, P) & H\psi^{\dagger} \\ \phi & \boxed{0} \end{vmatrix} \\ \\ \begin{vmatrix} \Omega(\Theta, P) & H\phi^{\dagger} \\ \psi & \boxed{0} \end{vmatrix} & \begin{vmatrix} \Omega(\Theta, P) & H\psi^{\dagger} \\ \psi & \boxed{0} \end{vmatrix} \end{pmatrix}, \tag{3.123}$$

where ϕ, ψ denote the row vectors $(\phi_1 \dots \phi_{2n})$, $(\psi_1 \dots \psi_{2n})$ respectively, which gives, by comparing the above matrix with (3.20), quasi-Grammian expressions for q, r, namely

$$q = 2\sigma \begin{vmatrix} \Omega(\Theta, P) & H\psi^{\dagger} \\ \phi & \boxed{0} \end{vmatrix}, \qquad r = -2\sigma \begin{vmatrix} \Omega(\Theta, P) & H\phi^{\dagger} \\ \psi & \boxed{0} \end{vmatrix}. \tag{3.124}$$

Hence we have obtained, in (3.64), expressions for q, r in terms of quasi-Wronskians, and in (3.124), expressions in terms of quasi-Grammians. In the next chapter, when we investigate dromion solutions of our system of ncDS equations, we will derive a condition to ensure that, with q, r quasi-Grammians as above, the relation $r = \pm q^{\dagger}$ holds. Although we could also check this relation for our quasi-Wronskian solutions (3.64), the calculation in this case is more difficult, and hence we choose to focus primarily on our quasi-Grammian form of solution.

3.5 Direct verification of quasi-Wronskian and quasi-Grammian solutions

We now show how our obtained quasideterminant solutions can be verified by direct substitution by firstly detailing the methods used to calculate the derivatives of the quasi-Wronskian $Q(i,j)$ and quasi-Grammian $R(i,j)$ using results from Section 2.3.2.

3.5.1 Derivatives of quasi-Wronskians

We saw in Section 3.4.2 that the quasi-Wronskian solution of the ncDS equations (3.17) is given by $S = Q(0,0)$, where S is the 2×2 matrix such that $\varLambda = [J, \sigma S]$, and $Q(i,j)$ the quasi-Wronskian

$$Q(i,j) = \begin{vmatrix} \widehat{\Theta} & f_j \quad e_j \\ \Theta^{(n+i)} & \boxed{\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}} \end{vmatrix}. \tag{3.125}$$

There are two important special cases [36]: when $n+i = n-j-1 \in [0, n-1]$ (i.e. when $i+j+1=0$ and $-n \le i < 0$),

$$Q(i,j) = \begin{vmatrix} \Theta & O_2 \\ \vdots & \vdots \\ \Theta^{(n-j-1)} & I_2 \\ \vdots & \vdots \\ \Theta^{(n-1)} & O_2 \\ \Theta^{(n+i)} & \boxed{O_2} \end{vmatrix} = \begin{vmatrix} \Theta & O_2 \\ \vdots & \vdots \\ \Theta^{(n-j-1)} & I_2 \\ \vdots & \vdots \\ \Theta^{(n-1)} & O_2 \\ O_2 & \boxed{-I_2} \end{vmatrix} = -I_2, \tag{3.126}$$

where we have used the invariance properties of quasideterminants (2.37) to subtract the row containing I_2 from the last row of the quasi-Wronskian. As before, I_2 and O_2 denote the 2×2 identity and zero matrices respectively. Similarly, when $n+i \in [0, n-1]$ but $n+i \neq n-j-1$, we find that $Q(i,j) = O_2$. For arbitrarily large n, we therefore have

$$Q(i,j) = \begin{cases} -I_2 & i+j+1=0, \\ O_2 & (i<0 \text{ or } j<0) \text{ and } i+j+1 \neq 0. \end{cases} \tag{3.127}$$

The dispersion relations (3.79a) for the ncDS system (3.17) will also be utilised, namely, for θ an eigenfunction of L, M,

$$\theta_x = -\sigma J \theta_y, \tag{3.128a}$$

$$\theta_t = \mathrm{i} J \theta_{yy}, \tag{3.128b}$$

and, since $\Theta = (\theta_1 \ldots \theta_n)$, it follows that

$$\Theta_x = -\sigma J \Theta_y, \tag{3.129a}$$

$$\Theta_t = \mathrm{i} J \Theta_{yy}. \tag{3.129b}$$

Thus, using (2.46), we have

$$\begin{aligned} Q(i,j)_y &= \begin{vmatrix} \widehat{\Theta} & f_j \quad e_j \\ \Theta^{(n+i+1)} & \boxed{O_2} \end{vmatrix} + \sum_{k=0}^{n-1} \begin{vmatrix} \widehat{\Theta} & f_k \quad e_k \\ \Theta^{(n+i)} & \boxed{O_2} \end{vmatrix} \cdot \begin{vmatrix} \widehat{\Theta} & f_j \quad e_j \\ \Theta^{(n-k)} & \boxed{O_2} \end{vmatrix} \\ &= Q(i+1,j) + \sum_{k=0}^{n-1} Q(i,k)Q(-k,j), \end{aligned} \tag{3.130a}$$

$$\begin{aligned} Q(i,j)_x &= \begin{vmatrix} \widehat{\Theta} & f_j \quad e_j \\ -\sigma J\Theta^{(n+i+1)} & \boxed{O_2} \end{vmatrix} + \sum_{k=0}^{n-1} \begin{vmatrix} \widehat{\Theta} & f_k \quad e_k \\ \Theta^{(n+i)} & \boxed{O_2} \end{vmatrix} \cdot \begin{vmatrix} \widehat{\Theta} & f_j \quad e_j \\ -\sigma J\Theta^{(n-k)} & \boxed{O_2} \end{vmatrix} \\ &= -\sigma \left(JQ(i+1,j) + \sum_{k=0}^{n-1} Q(i,k)JQ(-k,j) \right), \end{aligned} \tag{3.130b}$$

$$\begin{aligned} Q(i,j)_t &= \begin{vmatrix} \widehat{\Theta} & f_j \quad e_j \\ \mathrm{i}J\Theta^{(n+i+2)} & \boxed{O_2} \end{vmatrix} + \sum_{k=0}^{n-1} \begin{vmatrix} \widehat{\Theta} & f_k \quad e_k \\ \Theta^{(n+i)} & \boxed{O_2} \end{vmatrix} \cdot \begin{vmatrix} \widehat{\Theta} & f_j \quad e_j \\ \mathrm{i}J\Theta^{(n-k+1)} & \boxed{O_2} \end{vmatrix} \\ &= \mathrm{i} \left(JQ(i+2,j) + \sum_{k=0}^{n-1} Q(i,k)JQ(1-k,j) \right). \end{aligned} \tag{3.130c}$$

These can be simplified using (3.127), leaving

$$Q(i,j)_y = Q(i+1,j) - Q(i,j+1) + Q(i,0)Q(0,j), \tag{3.131a}$$

$$Q(i,j)_x = -\sigma(JQ(i+1,j) - Q(i,j+1)J + Q(i,0)JQ(0,j)), \tag{3.131b}$$

$$Q(i,j)_t = \mathrm{i}(JQ(i+2,j) - Q(i,j+2)J + Q(i,1)JQ(0,j) + Q(i,0)JQ(1,j)). \tag{3.131c}$$

3.5.2 Derivatives of quasi-Grammians

From Section 3.4.4, the quasi-Grammian solution of the ncDS equations (3.17) with trivial vacuum is given by $S = R(0,0)$, where $\Lambda = [J, \sigma S]$ for a 2×2 matrix S, and $R(i,j)$ is the quasi-Grammian defined by

$$R(i,j) = (-1)^j \begin{vmatrix} \Omega(\Theta, P) & P^{\dagger\,(j)} \\ \Theta^{(i)} & \boxed{\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}} \end{vmatrix} \tag{3.132}$$

for $i, j = 1, \ldots, n$. Again we utilise the dispersion relations for the system, this time relations in ρ, where ρ is an eigenfunction of $L^\dagger$, $M^\dagger$. We find that, from (3.79b),

$$\rho_x = -\frac{1}{\sigma} J \rho_y, \tag{3.133a}$$

$$\rho_t = \mathrm{i} J \rho_{yy}, \tag{3.133b}$$

and since $P = (\rho_1 \ldots \rho_n)$, it follows that

$$P_x = -\frac{1}{\sigma} J P_y, \tag{3.134a}$$

$$P_t = \mathrm{i} J P_{yy}. \tag{3.134b}$$

(Note that, for $\sigma = -1$, P satisfies the same relations as Θ in (3.129), explaining our choice of relation between P and Θ in (3.122)).

We now calculate the derivatives of $R(i,j)$ using (2.41). From our construction of the binary Darboux transformation in Section 3.4.3, the potential $\Omega(\phi, \psi)$ satisfies

$$\Omega(\phi, \psi)_y = \psi^\dagger \phi, \tag{3.135a}$$

$$\Omega(\phi, \psi)_x = -\sigma \psi^\dagger J \phi, \tag{3.135b}$$

$$\Omega(\phi, \psi)_t = \mathrm{i}(\psi^\dagger J \phi_y - \psi_y^\dagger J \phi). \tag{3.135c}$$

Thus, since $\theta_{[n]} = \phi_{[n]}|_{\phi \to \theta_n}$, $\rho_{[n]} = \psi_{[n]}|_{\psi \to \rho_n}$ and $\Theta = (\theta_1 \ldots \theta_n)$, $P = (\rho_1 \ldots \rho_n)$, it follows that

$$\Omega(\Theta, P)_y = P^\dagger \Theta, \tag{3.136a}$$

$$\Omega(\Theta, P)_x = -\sigma P^\dagger J \Theta, \tag{3.136b}$$

$$\Omega(\Theta, P)_t = \mathrm{i}\left(P^\dagger J \Theta^{(1)} - P^{\dagger(1)} J \Theta\right), \tag{3.136c}$$

where $\Theta^{(1)}$, ${P^\dagger}^{(1)}$ denote one differentiation with respect to y of Θ and $P^\dagger$ respectively. Thus, using (2.41), we have the derivatives

$$\begin{aligned} R(i,j)_y &= (-1)^j \begin{vmatrix} \Omega & P^{\dagger(j)} \\ \Theta^{(i+1)} & \boxed{O_2} \end{vmatrix} + (-1)^j \begin{vmatrix} \Omega & P^{\dagger(j+1)} \\ \Theta^{(i)} & \boxed{O_2} \end{vmatrix} + (-1)^j \begin{vmatrix} \Omega & P^{\dagger} \\ \Theta^{(i)} & \boxed{O_2} \end{vmatrix} \begin{vmatrix} \Omega & P^{\dagger(j)} \\ \Theta & \boxed{O_2} \end{vmatrix} \\ &= R(i+1,j) - R(i,j+1) + R(i,0)R(0,j), \end{aligned} \tag{3.137a}$$

$$\begin{aligned} R(i,j)_x &= (-1)^j \begin{vmatrix} \Omega & P^{\dagger(j)} \\ -\sigma J\Theta^{(i+1)} & \boxed{O_2} \end{vmatrix} + (-1)^j \begin{vmatrix} \Omega & -\sigma P^{\dagger(j+1)}J \\ \Theta^{(i)} & \boxed{O_2} \end{vmatrix} \\ &\quad + (-1)^j \begin{vmatrix} \Omega & -\sigma P^{\dagger} \\ \Theta^{(i)} & \boxed{O_2} \end{vmatrix} \begin{vmatrix} \Omega & P^{\dagger(j)} \\ J\Theta & \boxed{O_2} \end{vmatrix} \\ &= -\sigma(JR(i+1,j) - R(i,j+1)J + R(i,0)JR(0,j)), \end{aligned} \tag{3.137b}$$

$$\begin{aligned} R(i,j)_t &= (-1)^j \begin{vmatrix} \Omega & P^{\dagger(j)} \\ \mathrm{i}J\Theta^{(i+2)} & \boxed{O_2} \end{vmatrix} + (-1)^j \begin{vmatrix} \Omega & -\mathrm{i}P^{\dagger(j+2)}J \\ \Theta^{(i)} & \boxed{O_2} \end{vmatrix} \\ &\quad + (-1)^j \begin{vmatrix} \Omega & -\mathrm{i}P^{\dagger(1)} \\ \Theta^{(i)} & \boxed{O_2} \end{vmatrix} \begin{vmatrix} \Omega & P^{\dagger(j)} \\ J\Theta & \boxed{O_2} \end{vmatrix} + (-1)^j \begin{vmatrix} \Omega & \mathrm{i}P^{\dagger} \\ \Theta^{(i)} & \boxed{O_2} \end{vmatrix} \begin{vmatrix} \Omega & P^{\dagger(j)} \\ J\Theta^{(1)} & \boxed{O_2} \end{vmatrix} \\ &= \mathrm{i}(JR(i+2,j) - R(i,j+2)J + R(i,1)JR(0,j) + R(i,0)JR(1,j)), \end{aligned} \tag{3.137c}$$

where again, $\Omega = \Omega(\Theta, P)$, and we have used O_2 to denote the 2×2 zero matrix. Notice that the above derivatives of $R(i,j)$ match exactly the derivatives of $Q(i,j)$ in (3.131). This explains our definition of $R(i,j)$ given by (3.117) - the coefficient $(-1)^j$ is included to ensure that the derivatives match those of $Q(i,j)$. Thus subsequent calculations to verify the quasi-Wronskian solution of the ncDS equations will also be valid in the quasi-Grammian case, meaning that we need only verify one case.

3.5.3 Quasideterminant solution verification

We now show that

$$S = Q(0,0) \qquad \text{and} \qquad S = R(0,0) \tag{3.138}$$

are solutions of the ncDS equations (3.17), where S is the 2×2 matrix given by (3.20) and $\mathit{\Lambda} = [J, \sigma S]$. Using the derivatives of $Q(i,j)$ obtained in Section 3.5.1, we have, on setting

$i = j = 0$,

$$S_y = Q(0,0)_y = Q(1,0) - Q(0,1) + Q(0,0)^2, \tag{3.139a}$$

$$S_x = Q(0,0)_x = -\sigma\big(JQ(1,0) - Q(0,1)J + Q(0,0)JQ(0,0)\big), \tag{3.139b}$$

$$S_t = Q(0,0)_t = \mathrm{i}\big(JQ(2,0) - Q(0,2)J + Q(0,1)JQ(0,0) + Q(0,0)JQ(1,0)\big), \tag{3.139c}$$

$$\begin{aligned} S_{yy} &= Q(2,0) + Q(0,2) - 2Q(1,1) - Q(0,1)Q(0,0) + Q(0,0)Q(1,0) \\ &\quad + 2\big(Q(1,0)Q(0,0) - Q(0,0)Q(0,1)\big) + 2Q(0,0)^3, \end{aligned} \tag{3.139d}$$

$$\begin{aligned} S_{xx} &= \sigma^2\Big(Q(2,0) + Q(0,2) - 2JQ(1,1)J - Q(0,1)Q(0,0) + Q(0,0)Q(1,0) \\ &\quad - 2\big(Q(0,0)JQ(0,1)J - JQ(1,0)JQ(0,0)\big) + 2Q(0,0)JQ(0,0)JQ(0,0)\Big). \end{aligned} \tag{3.139e}$$

Substituting the above in (3.22), all terms cancel exactly and thus the quasi-Wronskian solution $S = Q(0,0)$ is verified. As mentioned previously, we obtain the same derivative formulae whether we use the quasi-Wronskian or quasi-Grammian formulation, and hence the above calculation also confirms the validity of the quasi-Grammian solution $S = R(0,0)$.

3.6 Comparison of solutions

3.6.1 Comparison with the bilinear (commutative) approach

We have used a direct approach to obtain quasideterminant solutions of a system of noncommutative DS equations. This method of solution has been widely studied in the commutative case, for example by Freeman and Nimmo in 1983 [25], who were the first to use a direct approach to obtain Wronskian solutions of the KdV and KP equations. The same approach was continued by Freeman in 1984 [22], who extended the idea to the NLS and DS equations; and by Hietarinta and Hirota in 1990 [46], who obtained 'double Wronskian' solutions of the DS equations using a direct method. Many other examples of this approach are contained in [48]. In all instances, a change of dependent variable is required, converting the nonlinear equation to Hirota bilinear form. For example, as mentioned earlier, in the case of the KP equation

$$(u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0, \tag{3.140}$$

we apply the Cole-Hopf transformation

$$u = 2(\log f)_{xx} \tag{3.141}$$

and find that f satisfies the bilinear relation

$$\left(D_x(D_t + D_x^3) + 3D_y^2\right) f \cdot f = 0, \tag{3.142}$$

where the D-operators are defined as before (1.12). A possible solution f in the form of a Wronskian or Grammian determinant is then substituted into the bilinear relation and verified using appropriate determinant identities.

For the case of the DS equation, we compare the method of solution used here to obtain (quasi-)Grammian solutions in the noncommutative case with the method of Gilson and Nimmo in [35], who use the direct approach discussed above to verify Grammian solutions in the commutative case. We find that the introduction of the 2×2 matrix S defined in (3.20) provides a noncommutative analogue of the Hirota bilinear form (in the noncommutative case, a *true* bilinear form is believed not to exist), thus avoiding the need for quasideterminant identities when verifying the solution. In contrast, we require certain Jacobi-type identities in order to verify the solution in the commutative case.

3.6.2 Comparison of Grammian solutions

We discuss the approach of Gilson and Nimmo in [35], who consider the DS equations in the form

$$\mathrm{i}\partial_t u + \partial_{xx} u + \partial_{yy} u - 4u|u|^2 - 2uv = 0, \tag{3.143a}$$

$$\partial_{xy} v + (\partial_x + \partial_y)^2 |u|^2 = 0, \tag{3.143b}$$

for functions $u = u(x, y, t)$, $v = v(x, y, t)$. By introducing new dependent variables F which is real and G which is complex such that

$$u = G/F, \tag{3.144a}$$

$$v = -(\partial_x + \partial_y)^2 \ln F + a(x, t) + b(y, t), \tag{3.144b}$$

the Hirota bilinear form of the DS equations (3.143) is given by

$$(\mathrm{i}D_t + D_x^2 + D_y^2) G \cdot F = 2(a(x, t) + b(y, t))GF, \tag{3.145a}$$

$$D_x D_y F \cdot F = 2GG^*, \tag{3.145b}$$

where G^* denotes the complex conjugate of G, and $a(x, t)$, $b(y, t)$ correspond to the nontrivial boundary conditions on v. Following the usual method in the commutative case, an ansatz is made for a solution F expressed in terms of a Grammian determinant, namely

$$F = |\mathcal{F}| = |I + H\Phi|, \tag{3.146}$$

where Φ is an $(M+N) \times (M+N)$ matrix of the form

$$\Phi = \begin{pmatrix} \int_{-\infty}^{x} \phi_i {\phi_j}^* \, \mathrm{d}x & 0 \\ 0 & \int_{y}^{\infty} \psi_k {\psi_l}^* \, \mathrm{d}y \end{pmatrix} \tag{3.147}$$

for functions $\phi_i(x,t)$, $\phi_j(x,t)$, $\psi_k(y,t)$, $\psi_l(y,t)$, with $i, j \in \{1, \ldots, M\}$, $k, l \in \{1, \ldots, N\}$. Here, H is a constant Hermitian matrix of size $(M+N) \times (M+N)$ and ϕ, ψ satisfy the time-dependent Schrödinger equations

$$\mathrm{i}\phi_t + \phi_{xx} - 2a(x,t)\phi = 0, \tag{3.148a}$$

$$\mathrm{i}\psi_t - \psi_{yy} + 2b(y,t)\psi = 0. \tag{3.148b}$$

In order to obtain a Grammian expression for G, the derivatives of F are calculated in terms of bordered determinants and substituted into the left-hand side of (3.145b), giving

$$D_x D_y F \cdot F = -2 \begin{vmatrix} 0 & 0 & l^{*T} \\ 0 & 0 & m^{*T} \\ Hl & Hm & \mathcal{F} \end{vmatrix} |\mathcal{F}| + 2 \begin{vmatrix} 0 & l^{*T} \\ Hl & \mathcal{F} \end{vmatrix} \begin{vmatrix} 0 & m^{*T} \\ Hm & \mathcal{F} \end{vmatrix}, \tag{3.149}$$

where l and m are $M+N$ column vectors defined by

$$l = (\phi_1, \ldots, \phi_M \,; 0, \ldots, 0)^T, \tag{3.150a}$$

$$m = (0, \ldots, 0 \,; \psi_1, \ldots, \psi_N)^T. \tag{3.150b}$$

A Jacobi identity is also utilised, namely

$$|A| A_{k,l}^{i,j} = \begin{vmatrix} A_k^i & A_k^j \\ A_l^i & A_l^j \end{vmatrix} \tag{3.151}$$

(compare (2.15)), where $A_{k,\ldots,l}^{i,\ldots,j}$ denotes the minor matrix obtained from the $n \times n$ matrix A by deleting the $i^{\text{th}}, \ldots, j^{\text{th}}$ rows and $k^{\text{th}}, \ldots, l^{\text{th}}$ columns. Choosing

$$A = \begin{pmatrix} 0 & 0 & l^{*T} \\ 0 & 0 & m^{*T} \\ Hl & Hm & \mathcal{F} \end{pmatrix} \tag{3.152}$$

with $\{i,j\}=\{k,l\}=\{1,2\}$, the Jacobi identity gives

$$\begin{vmatrix} 0 & 0 & l^{*T} \\ 0 & 0 & m^{*T} \\ Hl & Hm & \mathcal{F} \end{vmatrix} |\mathcal{F}| - \begin{vmatrix} 0 & l^{*T} \\ Hl & \mathcal{F} \end{vmatrix} \begin{vmatrix} 0 & m^{*T} \\ Hm & \mathcal{F} \end{vmatrix} + \begin{vmatrix} 0 & l^{*T} \\ Hm & \mathcal{F} \end{vmatrix} \begin{vmatrix} 0 & m^{*T} \\ Hl & \mathcal{F} \end{vmatrix} = 0. \quad (3.153)$$

Rearranging and substituting for the first product of determinants in (3.149) gives

$$D_x D_y F \cdot F = 2 \begin{vmatrix} 0 & l^{*T} \\ Hm & \mathcal{F} \end{vmatrix} \begin{vmatrix} 0 & m^{*T} \\ Hl & \mathcal{F} \end{vmatrix}, \quad (3.154)$$

and, by comparing with (3.145b), it follows that

$$GG^* = \begin{vmatrix} 0 & l^{*T} \\ Hm & \mathcal{F} \end{vmatrix} \begin{vmatrix} 0 & m^{*T} \\ Hl & \mathcal{F} \end{vmatrix}. \quad (3.155)$$

Taking

$$G = \begin{vmatrix} 0 & m^{*T} \\ Hl & \mathcal{F} \end{vmatrix}, \quad (3.156)$$

it can be shown that the first determinant on the right-hand side of (3.155) is the complex conjugate of G. It is also straightforward to calculate the derivatives of G in terms of bordered determinants using the same procedure as for F above. These derivatives can then be substituted into the left-hand side of (3.145a) and a pair of Jacobi identities utilised to show that this expression is zero, thus verifying the Grammian solutions for F and G.

In our method used in the noncommutative case, we have avoided the need to propose an ansatz for the solution, requiring a certain degree of intuition and experience, by deriving a quasi-Grammian solution directly via a binary Darboux transformation. Also, the introduction of the 2×2 matrix S in (3.20) to obtain a noncommutative analogue of the Hirota bilinear form of the DS equations avoids the need for identities when verifying the solution.

3.6.3 Comparison of solutions of noncommutative integrable equations

We have explained above how direct verification of quasi-Wronskian and quasi-Grammian solutions of the ncDS equations requires either the introduction of a 2×2 matrix S or the use of quasideterminant identities. In contrast, in the case of the noncommutative KP (ncKP) equation studied by Gilson and Nimmo in [36], it was found that, unlike in the commutative case, where an identity, namely the Laplace expansion of a determinant, is necessary in order to verify the Wronskian solution, verification of the quasi-Wronskian and quasi-Grammian solutions in the noncommutative case (obtained from Darboux and binary Darboux transformations) is automatic without the use of identities, and without the introduction of a matrix in a similar manner to that used here for the ncDS equations. Gilson and Nimmo point out that with all other noncommutative equations they have studied (noncommutatuve Hirota-Miwa [37,72], noncommutative modified KP [38]), direct verification of quasi-Wronskian and quasi-Grammian solutions *does* require the use of quasideterminant identities, and hence the ncKP equation is thought to be exceptional in this respect. The ncDS equations join the list of noncommutative integrable equations requiring quasideterminant identities for solution verification (without the introduction of an appropriate matrix as in our work), thus further confirming the suspicions of Gilson and Nimmo.

3.7 Conclusions

In this chapter we obtained a noncommutative version of the Davey-Stewartson system by utilising the same Lax pair as in the commutative case but relaxing the assumption that the dependent variables commute. We applied a Darboux and binary Darboux transformation to this noncommutative system in order to generate quasi-Wronskian and quasi-Grammian solutions, which were verified by direct substitution. In the next chapter, we look at a particular type of solution to the noncommutative system and, by choosing the dependent variables to be of matrix rather than scalar form, are able to obtain plots of these solutions in a noncommutative setting.

Chapter 4

Solutions of the noncommutative Davey-Stewartson equations

We now consider a particular type of solution to our system of noncommutative Davey-Stewartson equations derived in the previous chapter, namely dromions. As we shall discuss below, dromion solutions of the DS equations have been considered by a number of authors, however only in the commutative case. We show here that results can be extended to the noncommutative case.

4.1 Dromion solutions

4.1.1 Background - dromions

A major development in the understanding of the DS equations came in 1988, when Boiti *et al.* [10] discovered a class of localised solutions (two-dimensional solitons) decaying to zero exponentially in all directions as spatial variables tend to infinity. These solutions undergo a phase shift, and, unlike the $(1+1)$-dimensional case, a possible amplitude change on interaction with other solitons. These new solutions were later termed *dromions* by Fokas and Santini [20], derived from the Greek *dromos* meaning *tracks*, to highlight that the dromions lie at the intersection of perpendicular track-like plane waves. The DS equations were the first found to posses this wider class of solution.

Until the discovery of dromion solutions by Boiti *et al.*, the only known localised exact solutions of the DS equations were the so-called 'lump' solutions conceived by Ablowitz and Satsuma in 1979 [6]. Lump solutions, unlike dromions, decay algebraically in all directions and undergo no change in form upon interaction with similar waves.

Multidromion solutions to the DS system have been obtained using a variety of approaches. The initial discovery by Boiti *et al.* was made using Bäcklund transformations, with further methods being utilised later to obtain the same type of solution - the inverse scattering method [20], Hirota's direct method [46] and others. Multidromion solutions have been determined both in terms of Wronskian [46] and Grammian [35] determinants.

In this section we wish to obtain dromion solutions to our system of ncDS equations (3.17), which we find correspond to those found by Gilson and Nimmo in terms of Grammian determinants in the commutative case [35]. We are then able to plot both dromion and multidromion solutions using our new results.

4.1.2 Dromion solutions - quasi-Wronskian versus quasi-Grammian

In order to obtain dromion solutions, we choose to work with the quasi-Grammian rather than the quasi-Wronskian solution of our system of ncDS equations. The reasoning behind this choice concerns verification of the reality of our solution - verification is far more straightforward in the quasi-Grammian case. This can be seen by alluding to the commutative case, where, by studying the work of Hietarinta and Hirota [46], we see that checking the reality of a solution expressed in terms of a Wronskian determinant can be rather complex. This point was explicitly made by Hietarinta and Hirota.

In their paper of 1990, Hietarinta and Hirota construct an (N, N)-dromion solution (that is, a solution consisting of N plane waves in one direction and N in the perpendicular direction, with each point of overlap of these waves giving rise to a dromion) to a system of DS equations using double Wronskians. By taking a system of commutative DS equations in terms of variables u and v, both functions of x, y and t, and expressing u, v in terms of new dependent variables F and G (with F real), the Hirota bilinear form of the DS equations is obtained, namely

$$(\mathrm{i}D_t + D_x^2 + D_y^2)G \cdot F = 0, \tag{4.1a}$$

$$D_x D_y F \cdot F = 2\,|G|^2. \tag{4.1b}$$

Hietarinta and Hirota then introduce eigenfunctions $\phi(x,t)$, $\psi(y,t)$ satisfying the dispersion relations for the system and use these to construct a double Wronskian τ_n, namely

$$\tau_n = \begin{vmatrix} \phi_1 & \phi_1^{(1)} & \dots & \phi_1^{(n-1)} & \psi_1 & \psi_1^{(1)} & \dots & \psi_1^{(2N-n-1)} \\ \phi_2 & \phi_2^{(1)} & \dots & \phi_2^{(n-1)} & \psi_2 & \psi_2^{(1)} & \dots & \psi_2^{(2N-n-1)} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \phi_{2N} & \phi_{2N}^{(1)} & \dots & \phi_{2N}^{(n-1)} & \psi_{2N} & \psi_{2N}^{(1)} & \dots & \psi_{2N}^{(2N-n-1)} \end{vmatrix}, \tag{4.2}$$

where $n = 1, 2, \dots, 2N-1$. Choosing

$$\phi_i = \sum_{j=1}^{2N} a_{ij} e^{\xi_j}, \tag{4.3a}$$

$$\psi_i = \sum_{j=1}^{2N} b_{ij} e^{\zeta_j} \tag{4.3b}$$

for $i = 1, \dots, 2N$, where $\xi_j = P_j x + \mathrm{i} P_j^2 t$ and $\zeta_j = Q_j y - \mathrm{i} Q_j^2 t$ for complex constants P_j, Q_j, with a_{ij}, b_{ij} the $(i,j)^{\text{th}}$ entries of $2N \times 2N$ matrices A, B respectively, they then conjecture that F, G are such that

$$F = c\tilde{\tau}_N, \qquad G = c\tilde{\tau}_{N+1}, \qquad G^* = c\tilde{\tau}_{N-1} \tag{4.4}$$

for some constant c, where $\tilde{\tau}_n = \tau_n \exp\left(-\sum_{i=1}^{N} \xi_{i+N} + \zeta_i\right)$.

In order to check reality, it must be shown that F is real, i.e. that $c\tilde{\tau}_n$ is real, and also that the expression for G^* is the complex conjugate of the expression for G, i.e. that $c\tilde{\tau}_{N-1} = (c\tilde{\tau}_{N+1})^*$.

In the case of a single dromion solution, the conditions required for this solution to be real are fairly simple to obtain - by writing the columns of the matrices A, B in terms of the columns of two new matrices α, β respectively and of P_j, Q_j $(j = 1, 2)$, Hietarinta and Hirota show that a real solution amounts to ensuring that the matrix $\alpha^{-1}\beta$ is real with unit determinant.

However, for a general (N, N)-dromion solution, these reality conditions are more difficult to obtain. A lengthy calculation (see [46] for full details) leads to the requirement that the matrix $\alpha^{-1}\beta$ must be symplectic. To enable such a choice to be made, Hietarinta and Hirota express α and β in terms of an arbitrary nonsingular matrix and two Hermitian matrices, all of size $N \times N$. Although the entries of these matrices may be easy to choose, to obtain such a condition in order to ensure a real solution is a lengthy and difficult process. As we shall see later, the Grammian approach leads to a much simpler calculation to verify reality. This can be seen in, for example, [78], where Grammian solutions of a system of commutative DS equations are obtained, although the assumption that the system has zero asymptotic state is removed. The method used in [78] to check reality of a solution is the one that we exploit in the next section.

4.1.3 (n, n)-dromion solution - noncommutative (matrix) case

We modify the approach of [35], where dromion solutions of a system of commutative DS equations are determined. We consider the ncDS system (3.17) and, by specifying that certain parameters in the quasi-Grammian are of matrix rather than scalar form, we are able to obtain dromion solutions valid in the noncommutative case. Due to the complexity of this solution compared to the scalar case considered in [35], we look in some detail only at the simplest cases of the $(1, 1)$- and $(2, 2)$-dromion solutions. We do however verify reality for the general case. Note here that to obtain dromion solutions, we consider the DSI case, and hence choose $\sigma = -1$.

Recall the expressions for q, r obtained in terms of quasi-Grammians in (3.124), namely

$$q = -2 \begin{vmatrix} \Omega(\Theta, P) & H\psi^\dagger \\ \phi & \boxed{0} \end{vmatrix}, \qquad r = 2 \begin{vmatrix} \Omega(\Theta, P) & H\phi^\dagger \\ \psi & \boxed{0} \end{vmatrix}, \tag{4.5}$$

where ϕ, ψ denote the row vectors $(\phi_1 \dots \phi_{2n})$, $(\psi_1 \dots \psi_{2n})$ respectively and $H = (h_{ij})$ is a constant invertible square matrix, with † denoting conjugate transpose (Hermitian conjugate). By once again considering the dispersion relations for the system, we are able to choose expressions for ϕ, ψ corresponding to dromion solutions. From (3.129) and the definition of $\Theta = (\theta_1, \dots, \theta_n)$, where θ_i is given by (3.61), it follows that ϕ, ψ satisfy the relations

$$(\phi_j)_x = (\phi_j)_y, \qquad (\phi_j)_t = \mathrm{i}(\phi_j)_{yy}, \tag{4.6a}$$

$$(\psi_j)_x = -(\psi_j)_y, \qquad (\psi_j)_t = -\mathrm{i}(\psi_j)_{yy}. \tag{4.6b}$$

(Since the dispersion relations for P are the same as those for Θ when $\sigma = -1$ (see (3.129), (3.134)), considering P rather than Θ and recalling that $P = \Theta H^\dagger$ will give the same relations (4.6)).

So far we have not specified the nature of the noncommutativity we are considering. One of the more straightforward cases to consider is to take the fields q and r to be 2×2

matrices. Thus for dromion solutions in the noncommutative case, we choose ϕ_j, ψ_j to be 2×2 matrices, so that

$$\phi_j = \alpha_j I_2, \tag{4.7a}$$

$$\psi_j = \beta_j I_2, \tag{4.7b}$$

where I_2 denotes the 2×2 identity matrix, and α_j, β_j the exponentials [35, 46]

$$\alpha_j = \exp(p_j x + \mathrm{i}p_j^2 t + p_j y + \alpha_{j0}), \tag{4.8a}$$

$$\beta_j = \exp(q_j x - \mathrm{i}q_j^2 t - q_j y + \beta_{j0}), \tag{4.8b}$$

for $j = 1, \ldots, 2n$, suitable phase constants α_{j0}, β_{j0} and constants p_j, q_j, whose real parts are taken to be positive in order to give the correct asymptotic behaviour. The matrix H can be assumed to have unit diagonal since we are free to choose the phase constants α_{j0}, β_{j0} arbitrarily [35]. Using the coordinate transformation $X = x + y$, $Y = -(x - y)$, we have

$$\alpha_j = \exp(p_j X + \mathrm{i}p_j^2 t + \alpha_{j0}), \tag{4.9a}$$

$$\beta_j = \exp(-q_j Y - \mathrm{i}q_j^2 t + \beta_{j0}), \tag{4.9b}$$

so that $\alpha_j = \alpha_j(X, t)$, $\beta_j = \beta_j(Y, t)$.

We now choose to simplify our notation so that we are working with only $\phi_1, \ldots, \phi_n$ and $\psi_1, \ldots, \psi_n$ by relabeling ϕ_j as $\phi_{\frac{j+1}{2}}$ for odd j (i.e. $j = 1, 3, \ldots, 2n - 1$) and setting $\phi_j = 0$ for even j ($j = 0, 2, \ldots, 2n$), and similarly relabeling ψ_j as $\psi_{\frac{j}{2}}$ for even j and setting $\psi_j = 0$ for odd j, so that $\theta_j = \mathrm{diag}(\phi_j, \psi_j)$ ($j = 1, \ldots, n$) and

$$\phi = \begin{pmatrix} \phi_1 & 0 & \phi_2 & 0 & \ldots & \phi_n & 0 \end{pmatrix}, \tag{4.10a}$$

$$\psi = \begin{pmatrix} 0 & \psi_1 & 0 & \psi_2 & \ldots & 0 & \psi_n \end{pmatrix}, \tag{4.10b}$$

where each ϕ_j, ψ_j is a 2×2 matrix as defined in (4.7) above. Thus, for $n = 1$, q (which we henceforth denote by q^1 for the $(1,1)$-dromion case and q^n for the (n,n)-dromion case) can be expressed in quasi-Grammian form as

$$q^1 = -2 \begin{vmatrix} \Omega(\Theta, P) \qquad H & \begin{matrix} 0 & 0 \\ 0 & 0 \\ \beta_1^* & 0 \\ 0 & \beta_1^* \end{matrix} \\ \begin{matrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_1 & 0 & 0 \end{matrix} & \boxed{\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}} \end{vmatrix}, \tag{4.11}$$

where $H = (h_{ij})$ is a constant invertible 4×4 matrix. (Note that H is of size $2n\times 2n = 2\times 2$, but we now assume each entry of H takes the form of a 2×2 matrix, meaning H is $4n\times 4n$). Applying the quasideterminant expansion formula (2.10) allows us to express q^1 as a 2×2 matrix, where each entry is a quasi-Grammian, namely

$$q^1 = -2\begin{pmatrix} \begin{vmatrix} & & & & h_{13}\beta_1^* \\ & \Omega(\Theta,P) & & & \vdots \\ & & & & h_{43}\beta_1^* \\ \alpha_1 & 0 & 0 & 0 & \boxed{0} \end{vmatrix} & \begin{vmatrix} & & & & h_{14}\beta_1^* \\ & \Omega(\Theta,P) & & & \vdots \\ & & & & h_{44}\beta_1^* \\ \alpha_1 & 0 & 0 & 0 & \boxed{0} \end{vmatrix} \\ \\ \begin{vmatrix} & & & & h_{13}\beta_1^* \\ & \Omega(\Theta,P) & & & \vdots \\ & & & & h_{43}\beta_1^* \\ 0 & \alpha_1 & 0 & 0 & \boxed{0} \end{vmatrix} & \begin{vmatrix} & & & & h_{14}\beta_1^* \\ & \Omega(\Theta,P) & & & \vdots \\ & & & & h_{44}\beta_1^* \\ 0 & \alpha_1 & 0 & 0 & \boxed{0} \end{vmatrix} \end{pmatrix}$$

$$= -2\begin{pmatrix} q_{11}^1 & q_{12}^1 \\ q_{21}^1 & q_{22}^1 \end{pmatrix}, \text{ say.} \tag{4.12}$$

We consider each quasi-Grammian in turn, but as an example we will look at the quasi-Grammian q_{11}^1. We apply (2.14) to express q_{11}^1 as a ratio of determinants, namely

$$q_{11}^1 = -2\frac{\begin{vmatrix} & & & & h_{13}\beta_1^* \\ & \Omega(\Theta,P) & & & \vdots \\ & & & & h_{43}\beta_1^* \\ \alpha_1 & 0 & 0 & 0 & 0 \end{vmatrix}}{\left|\Omega(\Theta,P)\right|} = -2\frac{G_{11}^1}{F}, \text{ say.} \tag{4.13}$$

(We have introduced the notation G_{vw}^n and q_{vw}^n $(v, w = 1, 2)$ to emphasise that we are considering the $(v,w)^{\text{th}}$ entry of the expansion of q^n in the (n,n)-dromion case). Although (2.14) is valid only in the commutative case, our assumption here is that the variables q, r in our system of DS equations, and also the parameters ϕ_j, ψ_j, are noncommutative. The exponentials α_j, β_j given by (4.9) are clearly commutative by definition, hence we are free to use the result (2.14).

By expanding the quasi-Grammian r in (4.5) in a similar manner and extracting the $(1,1)^{\text{th}}$

entry, we obtain

$$r_{11}^1 = 2\frac{\begin{vmatrix} & & & & h_{11}\alpha_1^* \\ & \Omega(\Theta,P) & & & \vdots \\ & & & & h_{41}\alpha_1^* \\ 0 & 0 & \beta_1 & 0 & 0 \end{vmatrix}}{\left|\Omega(\Theta,P)\right|} = 2\frac{K_{11}^1}{F}, \text{ say,} \tag{4.14}$$

with similar results for q_{12}^1, r_{12}^1 etc.

4.1.4 Reality conditions

In the (n,n)-dromion case, we have

$$q_{vw}^n = -2\frac{G_{vw}^n}{F}, \tag{4.15a}$$

$$r_{vw}^n = 2\frac{K_{vw}^n}{F}, \tag{4.15b}$$

for $v, w = 1, 2$. To verify reality, we must check that $r_{vw}^n = \pm(q_{vw}^n)^\dagger$, i.e. that F is self-adjoint, and $(G_{vw}^n)^\dagger = \pm K_{vw}^n$, where $(G_{vw}^n)^\dagger$ denotes the Hermitian conjugate of G_{vw}^n. Consider

$$\begin{aligned} \mathcal{F} &= \Big(\Omega(\Theta,P)\Big) \\ &= \Big(\textstyle\int P^\dagger\Theta\,\mathrm{d}y + C\Big) \end{aligned} \tag{4.16}$$

since $\Omega_y = P^\dagger\Theta$ for $\Theta = (\theta_1\ \ldots\ \theta_n)$, $P = (\rho_1\ \ldots\ \rho_n)$, so that $F = \det\mathcal{F}$. Here, $C = (c_{ij})$ $(i,j = 1,\ldots,2n)$ is a $2n \times 2n$ (constant) matrix denoting a constant of integration, where each c_{ij} is a 2×2 block. From before, $P = \Theta H^\dagger$, so that

$$\mathcal{F} = \Big(H\textstyle\int \Theta^\dagger\Theta\,\mathrm{d}y + C\Big). \tag{4.17}$$

Thus, since $\Theta = (\theta_1\ \ldots\ \theta_n)$ for $\theta_j = \mathrm{diag}(\phi_j, \psi_j)$ $(j = 1,\ldots,n)$, this gives

$$\mathcal{F} = H\begin{pmatrix} \int_{-\infty}^X \phi_1^*\phi_1\,\mathrm{d}X & O_2 & \ldots & \int_{-\infty}^X \phi_1^*\phi_n\,\mathrm{d}X & O_2 \\ O_2 & \int_Y^\infty \psi_1^*\psi_1\,\mathrm{d}Y & \ldots & O_2 & \int_Y^\infty \psi_1^*\psi_n\,\mathrm{d}Y \\ \vdots & \vdots & & \vdots & \vdots \\ \int_{-\infty}^X \phi_n^*\phi_1\,\mathrm{d}X & O_2 & \ldots & \int_{-\infty}^X \phi_n^*\phi_n\,\mathrm{d}X & O_2 \\ O_2 & \int_Y^\infty \psi_n^*\psi_1\,\mathrm{d}Y & \ldots & O_2 & \int_Y^\infty \psi_n^*\psi_n\,\mathrm{d}Y \end{pmatrix} + C, \tag{4.18}$$

remembering that each ϕ_j, ψ_j is a 2×2 matrix and O_2 denotes the 2×2 zero matrix. The limits of integration are determined from the definitions of ϕ_j, ψ_j $(j = 1,\ldots,n)$ in (4.7), (4.9). We choose the entries of the matrix C so that

$$c_{ij} = \begin{cases} I_2 & \text{for } i = j, \\ O_2 & \text{otherwise.} \end{cases} \tag{4.19}$$

Thus

$$\mathcal{F} = I_{4n} + H\Phi, \tag{4.20}$$

where

$$\Phi = \begin{pmatrix} \int_{-\infty}^{X} \phi_1^* \phi_1 \,\mathrm{d}X & O_2 & \dots & \int_{-\infty}^{X} \phi_1^* \phi_n \,\mathrm{d}X & O_2 \\ O_2 & \int_{Y}^{\infty} \psi_1^* \psi_1 \,\mathrm{d}Y & \dots & O_2 & \int_{Y}^{\infty} \psi_1^* \psi_n \,\mathrm{d}Y \\ \vdots & \vdots & & \vdots & \vdots \\ \int_{-\infty}^{X} \phi_n^* \phi_1 \,\mathrm{d}X & O_2 & \dots & \int_{-\infty}^{X} \phi_n^* \phi_n \,\mathrm{d}X & O_2 \\ O_2 & \int_{Y}^{\infty} \psi_n^* \psi_1 \,\mathrm{d}Y & \dots & O_2 & \int_{Y}^{\infty} \psi_n^* \psi_n \,\mathrm{d}Y \end{pmatrix}, \tag{4.21}$$

and hence

$$F = \det \mathcal{F} = \mid I_{4n} + H\Phi \mid . \tag{4.22}$$

We firstly prove F is self-adjoint, i.e. we prove $F^\dagger = F$. We have

$$\begin{aligned} F &= \mid I_{4n} + H\Phi \mid \\ &= \mid H \mid . \mid H^{-1} + \Phi \mid \end{aligned} \tag{4.23}$$

since we assume H to be invertible. Then

$$\begin{aligned} F^\dagger &= \mid H \mid^\dagger . \mid H^{-1} + \Phi \mid^\dagger \\ &= \mid H^\dagger \mid . \mid (H^\dagger)^{-1} + \Phi \mid \\ &= F, \end{aligned} \tag{4.24}$$

provided $H^\dagger = H$, where we have used the fact that $\left(H^\dagger\right)^{-1} = \left(H^{-1}\right)^\dagger$, and also that $\Phi^\dagger = \Phi$ (which is clear from the definition of Φ above). Thus for F self-adjoint, we require that H is a constant $4n \times 4n$ Hermitian matrix.

Here we explain why we made the choice $P = \Theta H^\dagger$ in (3.122), rather than the conceivably more simple choice of $P = \Theta$. We take our lead from Gilson and Nimmo in the commutative case with F of the form (3.146), i.e.

$$F = |I + H\Phi|, \tag{4.25}$$

where Φ satisfies (3.147), and H and Φ are of compatible sizes. Taking H to be the identity matrix, we can factorise F so that

$$F(x, y, t) = F_1(x, t)F_2(y, t) \tag{4.26}$$

for some functions F_1 and F_2. Then, from (3.144b), it follows that

$$v = -(\ln F_1)_{xx} - (\ln F_2)_{yy} + a(x, t) + b(y, t), \tag{4.27}$$

and, from the bilinear form (3.145b),

$$|u|^2 \equiv 0. \tag{4.28}$$

To see this, we calculate from the definition of the Hirota derivative (1.12) that

$$D_x D_y(F \cdot F) = 2\left(F F_{xy} - F_x F_y\right). \tag{4.29}$$

With F as in (4.26) above, we find that $F_{xy} = F_{1x}F_{2y}$ and $F_xF_y = F_1F_2F_{1x}F_{2y}$. Then

$$D_x D_y(F \cdot F) = 2\left(F_1F_2F_{1x}F_{2y} - F_1F_2F_{1x}F_{2y}\right) = 0, \tag{4.30}$$

so that, by (3.145b),

$$GG^* = 0. \tag{4.31}$$

Since $u = \dfrac{G}{F}$ by (3.144a), it follows that

$$|u|^2 = uu^* = \frac{GG^*}{F^2} = 0. \tag{4.32}$$

In Gilson and Nimmo's notation, the variable v governs the plane waves, and thus, by (4.27), we have plane waves parallel to the x- and y-axes which interact in a linear manner. As a result, these waves do not have any effect on each other, and hence we see no dromions in the u-plane, i.e. $|u|^2 = 0$, where $|u|^2$ governs the dromion height. We must therefore choose a more general H to allow for a nonlinear interaction of the plane waves in the v-plane. This nonlinear interaction leads to dromion solutions.

We follow the same approach in the noncommutative case - although we do not have a bilinear form as in (3.145), nor a concrete simple expression for our plane waves in terms of F as in (3.144b), we consider F to have the same structure as in the commutative case (4.25). We have found that the matrix H must be Hermitian, and, as this matches with the condition required on H in Gilson and Nimmo's work, our assumption for the form of F is a sensible one.

We now derive the condition so that $(G^n_{11})^\dagger = \pm K^n_{11}$. We have

$$G^n_{11} = \begin{vmatrix} \mathcal{F} & H\beta^\dagger \\ \alpha & 0 \end{vmatrix}, \tag{4.33}$$

where α, β are row vectors defined by

$$\alpha = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 & \alpha_2 & 0 & 0 & 0 & \dots & \alpha_n & 0 & 0 & 0 \end{pmatrix}, \tag{4.34a}$$

$$\beta = \begin{pmatrix} 0 & 0 & \beta_1 & 0 & 0 & 0 & \beta_2 & 0 & \dots & 0 & 0 & \beta_n & 0 \end{pmatrix}, \tag{4.34b}$$

while

$$K^n_{11} = \begin{vmatrix} \mathcal{F} & H\alpha^\dagger \\ \beta & 0 \end{vmatrix}. \tag{4.35}$$

Since $\mathcal{F} = H(H^{-1} + \Phi)$, it can be seen that

$$G^n_{11} = \left| \begin{pmatrix} H & O^T \\ O & 1 \end{pmatrix} \begin{pmatrix} H^{-1} + \Phi & \beta^\dagger \\ \alpha & 0 \end{pmatrix} \right|, \tag{4.36}$$

where O denotes the 1×4 row vector $\begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}$, so that

$$\begin{aligned} (G^n_{11})^\dagger &= \begin{vmatrix} H^\dagger & O^T \\ O & 1 \end{vmatrix} \cdot \begin{vmatrix} (H^{-1})^\dagger + \Phi^\dagger & \alpha^\dagger \\ \beta & 0 \end{vmatrix} \\ &= \left| \begin{pmatrix} H & O^T \\ O & 1 \end{pmatrix} \begin{pmatrix} H^{-1} + \Phi & \alpha^\dagger \\ \beta & 0 \end{pmatrix} \right| \\ &= K^n_{11}, \end{aligned} \tag{4.37}$$

provided $H^\dagger = H$, i.e. H Hermitian. (Again we have used the fact that $\Phi^\dagger = \Phi$). Thus we have $r^n_{11} = -(q^n_{11})^\dagger$ so long as H is a constant Hermitian matrix, agreeing with the work of Gilson and Nimmo in [35]. We have shown that, by imposing the relation $P = \Theta H^\dagger$ for a Hermitian matrix H, the condition $r^n_{11} = -(q^n_{11})^\dagger$ holds and thus, for a dromion solution ($\sigma = -1$), with Λ defined as in (3.13), we must have $\Lambda^\dagger = -\Lambda$, i.e. $S^\dagger = -S$ since $\Lambda = [J, \sigma S]$. The binary Darboux transformation preserves this skew-adjoint condition, so that after n iterations, $S^\dagger_{[n+1]} = -S_{[n+1]}$, where we use the fact that $P = \Theta H^\dagger$.

4.1.5 $(1,1)$-dromion solution - matrix case

We now show computer plots of the $(1,1)$-dromion solution in the noncommutative case, where we choose ϕ_1, ψ_1 to be 2×2 matrices as in (4.7). A suitable choice of the parameters

p_1, q_1 and of the 4×4 Hermitian matrix H allows us to obtain plots of the four quasi-Grammian solutions q^1_{11}, q^1_{12}, q^1_{21} and q^1_{22} as detailed in (4.12).

We are restricted in our choice of p_1, q_1 and H in that we require $F\neq 0$. In particular, we derive conditions so that $F>0$. The determinant F can be expanded in terms of minor matrices of $H=(h_{ij})$ $(i,j=1,\dots,4)$, giving

$$\begin{aligned}F = 1 + P_1\left(h^{234}_{234}+h^{134}_{134}\right)e^{2\eta} + Q_1\left(h^{124}_{124}+h^{123}_{123}\right)e^{-2\xi} + P_1^2 h^{34}_{34}e^{4\eta}\\ + Q_1^2 h^{12}_{12}e^{-4\xi} + P_1Q_1\left(h^{24}_{24}+h^{14}_{14}+h^{23}_{23}+h^{13}_{13}\right)e^{2\eta-2\xi}\\ + P_1^2Q_1\left(h^4_4+h^3_3\right)e^{4\eta-2\xi} + P_1Q_1^2\left(h^2_2+h^1_1\right)e^{2\eta-4\xi} + P_1^2Q_1^2 h e^{4\eta-4\xi},\end{aligned} \tag{4.38}$$

where we define $P_1 = 1/(2\Re(p_1))$, $Q_1 = 1/(2\Re(q_1))$, $\eta = \Re(p_1)(X-2\Im(p_1)t)$ and $\xi = \Re(q_1)(Y-2\Im(q_1)t)$, with $h^{rs\dots}_{ij\dots}$ denoting the minor matrix obtained by removing rows $i,j,\dots$ and columns $r,s,\dots$ of H, where $i,j,\dots,r,s,\dots\in\{1,2,3,4\}$. We have used '$h$' to indicate that no rows or columns have been removed, i.e. $h=\det H$. We can also obtain expressions for each G^1_{vw} $(v,w=1,2)$, for instance

$$G^1_{11} = \alpha_1\beta_1^*\left(h^{124}_{234} - P_1h^{14}_{34}e^{2\eta} - Q_1h^{12}_{23}e^{-2\xi} - P_1Q_1h^1_3e^{2\eta-2\xi}\right), \tag{4.39}$$

with α_1,β_1 defined as in (4.9). Similar expansions can be obtained for G^1_{12}, G^1_{21} and G^1_{22}. Thus, it can be seen that for $F>0$, we require $\Re(p_1)$, $\Re(q_1)$ and each of the minor matrices in the expansion of F to be greater than zero. Suitable choices of p_1, q_1 and H have been used to obtain the dromion plots shown later. In addition to the matrix-valued fields q and r, there are also matrix-valued fields A_1 and A_2 in the ncDS system (3.17). Plotting the derivatives of these fields gives plane waves as follows.

From (3.123), we have an expression for the 2×2 matrix S in terms of quasi-Grammians. By (3.21), $A=-\mathrm{i}S_x-\mathrm{i}JS_y$, therefore substituting for S using (3.123) and equating matrix entries gives quasi-Grammian expressions for A_1, A_2, namely

$$A_1 = -\mathrm{i}\begin{vmatrix}\Omega & H\phi^\dagger\\ \phi & \boxed{0}\end{vmatrix}_x - \mathrm{i}\begin{vmatrix}\Omega & H\phi^\dagger\\ \phi & \boxed{0}\end{vmatrix}_y, \tag{4.40a}$$

$$A_2 = -\mathrm{i}\begin{vmatrix}\Omega & H\psi^\dagger\\ \psi & \boxed{0}\end{vmatrix}_x + \mathrm{i}\begin{vmatrix}\Omega & H\psi^\dagger\\ \psi & \boxed{0}\end{vmatrix}_y. \tag{4.40b}$$

We choose ϕ_1, ψ_1 to be 2×2 matrices as before so that the boxed expansion element '0' in each quasi-Grammian is the 2×2 matrix $\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$. Thus, expanding each quasi-Grammian in the usual manner gives a 2×2 matrix where each entry is a quasi-Grammian, and hence we have distinct expressions for A_1, A_2 corresponding to each of the four dromions $q^1_{11}, q^1_{12}, q^1_{21}$ and q^1_{22}. Considering (3.17c)-(3.17d), we find that plotting the combination $(\partial_x + \sigma\partial_y)A_1$ for $\sigma = -1$ gives a plane wave travelling in the X direction, while the combination $(\partial_x - \sigma\partial_y)A_2$ gives a plane wave in the Y direction. These, along with the dromions corresponding to each plane wave, have been plotted at time $t = 0$ in Figures 4.1 and 4.2, where the Hermitian matrix H has been chosen to have unit diagonal. As can be seen

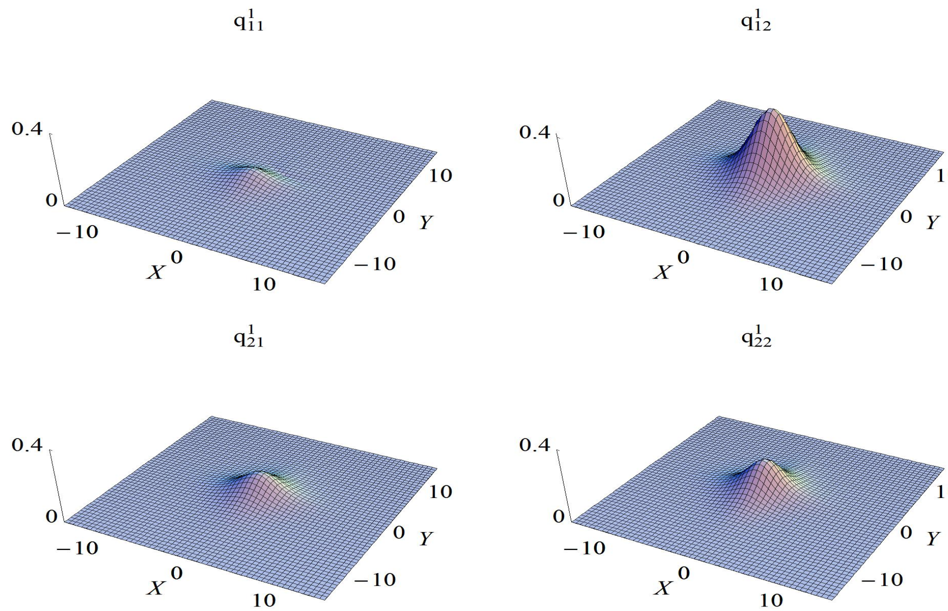


Figure 4.1: $(1,1)$-dromion plots with $p_1 = \frac{1}{2} + \mathrm{i}$, $q_1 = \frac{1}{2} - \mathrm{i}$ and $h_{12} = \frac{1}{2}$, $h_{13} = \frac{1}{4}$, $h_{14} = \frac{3}{4}$, $h_{23} = \frac{1}{3}$, $h_{24} = \frac{1}{2}$, $h_{34} = \frac{1}{3}$.

from Figure 4.1, single dromions of differing heights occur in each of the fields q_{11}, q_{12}, q_{21} and q_{22}. If we were to plot the $(1,1)$-dromion solution in the commutative (scalar) case (that is, if we were to choose q and its Hermitian conjugate to be of scalar rather than matrix form), we would obtain only one dromion in the single field q. This dromion and its plane waves would have the same basic structure as those above, and thus there would be no marked difference in the appearance of the dromions in the commutative and noncommutative cases. The main difference between the two situations concerns the number of parameters - a far greater number in the noncommutative case gives us more freedom to control the heights of the dromions, however some extra care has to be taken in choosing the parameters so that no singularities occur in the solution.

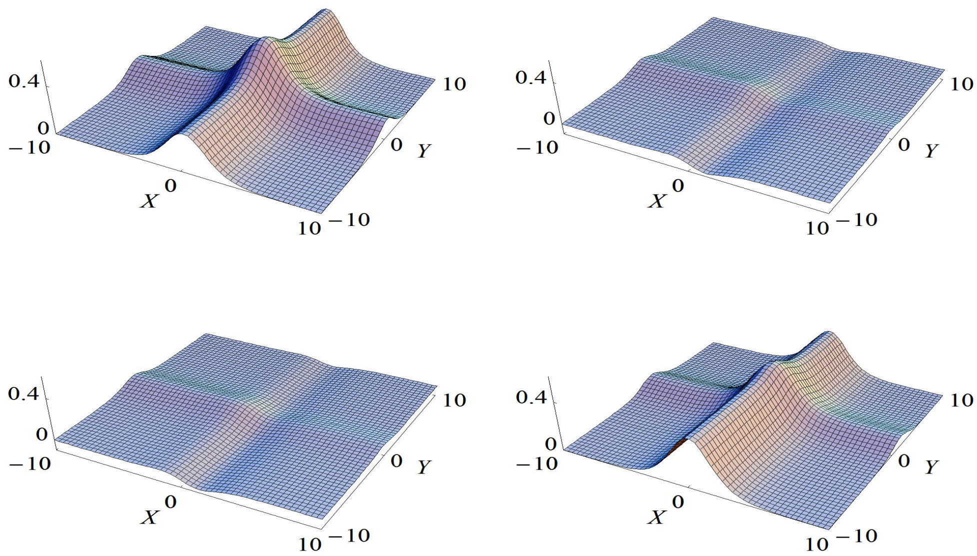


Figure 4.2: Plane waves corresponding to, clockwise from top left, q^1_{11}, q^1_{12}, q^1_{22}, q^1_{21}.

With the aid of a computer package, we can easily determine the amplitude of each of the dromions depicted above. We firstly consider $q^1_{11} = -2G^1_{11}/F$, with F and G^1_{11} given by (4.38) and (4.39) respectively. Calculating the numerical values of F and G^1_{11} at time $t = 0$ and position $(X, Y) = (0, 0)$, we find that $F = 11.9184$ and $G^1_{11} = -0.3125$, giving q^1_{11} an amplitude of 0.0524 (to four decimal places). Similar calculations can be carried out for the remaining three dromions, giving amplitudes for q^1_{12}, q^1_{21} and q^1_{22} of 0.3939, 0.1311 and 0.1847 respectively. (Note that the maximum amplitude of each dromion can be seen to occur at position $(X, Y) = (0, 0)$ as deviations from this position result in a decrease in amplitude).

Clearly this is a very naïve approach to utilise in order to determine dromion amplitudes, and gives no information as to the factors governing these amplitudes. In the scalar (commutative) case, Gilson and Nimmo [35] are able to calculate dromion amplitudes in a relatively straightforward manner by employing the Hirota bilinear form (3.145) of their DS equations (3.143), with the variables u, v defined in (3.144). From this bilinear form, they are able to obtain a simple expression for the dromion amplitudes, namely $\partial_x \partial_y (\ln F)$. Since F in this case takes the compact form

$$F = 1 + \alpha e^{2\eta} + \beta e^{-2\rho} + \gamma e^{2\eta - 2\rho}, \tag{4.41}$$

with $\eta = \Re(p_1)(x - 2\Im(p_1)t)$, $\rho = \Re(q_1)(y - 2\Im(q_1)t)$, p_1, q_1 complex parameters and α, β, γ positive, an easy calculation yields an expression for the dromion amplitudes.
In the noncommutative case, we have no bilinear form as in the commutative case above, and hence calculations of this type are not possible. (An attempt was made to carry out a calculation in the same manner to that done by Gilson and Nimmo, however the resulting expression could not be simplified due to its complexity). This highlights a limitation of the matrix case - although we have succeeded in plotting the $(1, 1)$-dromion solution with relative ease, an analysis of the obtained plots and their properties proves to be rather difficult.

4.1.6 $(2, 2)$-dromion solution - matrix case

In the scalar case [35], Gilson and Nimmo carry out a detailed asymptotic analysis of their (M, N)-dromion solution, and are able to obtain compact expressions for the phase-shifts and changes in amplitude that occur due to dromion interactions. They then use the results of this analysis to study a class of $(2, 2)$-dromions with scattering-type interaction properties. The Hermitian matrix H can be chosen in such a way so that some of the dromions have zero amplitude either as $t \to -\infty$ or as $t \to +\infty$.

For the $(2, 2)$-dromion solution in the matrix case, detailed calculations of this type are more complicated due to the large number of terms involved. However, we can adopt the same approach to carry out some of the more straightforward calculations. In particular, we obtain plots of the situation in which the $(1, 1)^{\text{th}}$ dromion in each of the solutions q^2_{11}, q^2_{12}, q^2_{21} and q^2_{22} does not appear as $t \to -\infty$. These are depicted in Figures 4.3-4.5.

To analyse this situation, we focus our attention on q^2_{11} and consider G^2_{11} in a frame moving with the $(1, 1)^{\text{th}}$ dromion. We define

$$\hat{X} = X - 2\Im(p_1)t, \tag{4.42a}$$

$$\hat{Y} = Y - 2\Im(q_1)t, \tag{4.42b}$$

and consider the limits of G^2_{11} as $t \to -\infty$. Let

$$\begin{aligned} \eta_2 &= \Re(p_2)\,(X - 2\Im(p_2)t) \\ &= \Re(p_2)\left(\hat{X} - 2(\Im(p_2) - \Im(p_1))t\right), \end{aligned} \tag{4.43a}$$

and similarly

$$\begin{aligned}\xi_2 &= \Re(q_2)\left(Y - 2\Im(q_2)t\right)\\ &= \Re(q_2)\left(\hat{Y} - 2(\Im(q_2) - \Im(q_1))t\right). \end{aligned} \tag{4.43b}$$

We choose to order the p_i, q_i $(i = 1, 2)$ by means of their imaginary parts, so that $\Im(p_1) > \Im(p_2)$ and $\Im(q_1) < \Im(q_2)$. Thus, as $t \to -\infty$, $\eta_2 \to -\infty$ and $\xi_2 \to +\infty$. It can easily be shown that η_2, $-\xi_2$ determine the real parts of the exponents in α_2, β_2 respectively, where α_2, β_2 are defined as in (4.9), so that, as $t \to -\infty$, α_2, $\beta_2 \to 0$ (and hence α_2^*, $\beta_2^* \to 0$ also). Therefore, by setting $\alpha_2, \alpha_2^*, \beta_2, \beta_2^* \to 0$ in G_{11}^2 and expanding the resulting determinant, we obtain a compact expression for G_{11}^2 as $t \to -\infty$, namely

$$G_{11}^2 = -\alpha_1\beta_1^*\left(h_{2345678}^{1245678} - P_1 h_{345678}^{145678} e^{2\eta} - Q_1 h_{235678}^{125678} e^{-2\xi} + P_1 Q_1 h_{35678}^{15678} e^{2\eta - 2\xi}\right). \tag{4.44}$$

Similar expressions can be obtained for the other three determinants G_{12}^2, G_{21}^2 and G_{22}^2 by considering an extension of (4.12) to the $(2,2)$-dromion case and interchanging columns appropriately: for example, we firstly interchange columns 3 and 4, and 7 and 8, of H, before using the same expansion as in (4.44), to obtain an analogous expression for G_{12}^2. Thus we have, as $t \to -\infty$, compact expressions for the minors of H governing the $(1,1)^{\text{th}}$ dromion in each of G_{11}^2, G_{12}^2, G_{21}^2, G_{22}^2, namely

$$\begin{pmatrix} G_{11}^2 & G_{12}^2 \\ G_{21}^2 & G_{22}^2 \end{pmatrix} = -\alpha_1\beta_1^* \left\{ \begin{pmatrix} |h_{13}| & |h_{14}| \\ |h_{23}| & |h_{24}| \end{pmatrix} + \hat{P} \begin{pmatrix} \begin{vmatrix} h_{12} & h_{13} \\ h_{22} & h_{23} \end{vmatrix} & \begin{vmatrix} h_{12} & h_{14} \\ h_{22} & h_{24} \end{vmatrix} \\ \begin{vmatrix} h_{11} & h_{13} \\ h_{21} & h_{23} \end{vmatrix} & \begin{vmatrix} h_{11} & h_{14} \\ h_{21} & h_{24} \end{vmatrix} \end{pmatrix} e^{2\eta} \right. \tag{4.45}$$

$$\left. + \hat{Q} \begin{pmatrix} \begin{vmatrix} h_{13} & h_{14} \\ h_{43} & h_{44} \end{vmatrix} & \begin{vmatrix} h_{13} & h_{14} \\ h_{33} & h_{34} \end{vmatrix} \\ \begin{vmatrix} h_{23} & h_{24} \\ h_{43} & h_{44} \end{vmatrix} & \begin{vmatrix} h_{23} & h_{24} \\ h_{33} & h_{34} \end{vmatrix} \end{pmatrix} e^{-2\xi} + P_1\hat{Q} \begin{pmatrix} \begin{vmatrix} h_{12} & h_{13} & h_{14} \\ h_{22} & h_{23} & h_{24} \\ h_{42} & h_{43} & h_{44} \end{vmatrix} & \begin{vmatrix} h_{12} & h_{13} & h_{14} \\ h_{22} & h_{23} & h_{24} \\ h_{32} & h_{33} & h_{34} \end{vmatrix} \\ \begin{vmatrix} h_{11} & h_{13} & h_{14} \\ h_{21} & h_{23} & h_{24} \\ h_{41} & h_{43} & h_{44} \end{vmatrix} & \begin{vmatrix} h_{11} & h_{13} & h_{14} \\ h_{21} & h_{23} & h_{24} \\ h_{31} & h_{33} & h_{34} \end{vmatrix} \end{pmatrix} T e^{2\eta - 2\xi} \right\},$$

where $T = \text{diag}(-1, 1)$, $\hat{P} = P_1 T$ and $\hat{Q} = Q_1 T$. Since we have written out each minor matrix explicitly, rather than using the abbreviated notation as in (4.44), it can easily be seen that, by setting h_{13}, h_{14}, h_{23} and h_{24} equal to zero, the $(1,1)^{\text{th}}$ dromion in each of $q^2_{11}, q^2_{12}, q^2_{21}$ and q^2_{22} will vanish as $t \to -\infty$. This is shown in Figures 4.3-4.5 below. (Note that in the plots of q^2_{11} and q^2_{12} in Figure 4.3, two of the dromions have very small amplitude). In Figure 4.6, we have focused on the dromion solutions of q^2_{22}, plotting these in smaller time increments to highlight how the dromions approach each other and move apart, while in Figure 4.7, we have shown a close-up of the q^2_{22} dromion interaction at $t = 0$. In each of these plots, we choose $p_1 = q_2 = \frac{1}{2} + \mathrm{i}$, $p_2 = q_1 = \frac{1}{2} - \mathrm{i}$, and the Hermitian matrix H such that

$$H = \begin{pmatrix} \frac{81}{16} & 1 & 0 & 0 & 4 & 1 & \frac{1}{16} & \frac{1}{16} \\ 1 & \frac{121}{16} & 0 & 0 & 1 & 4 & \frac{81}{64} & 1 \\ 0 & 0 & \frac{81}{16} & \frac{1}{4} & \frac{1}{16} & 1 & 4 & 1 \\ 0 & 0 & \frac{1}{4} & \frac{1369}{256} & \frac{1}{16} & \frac{81}{64} & \frac{1}{4} & 4 \\ 4 & 1 & \frac{1}{16} & \frac{1}{16} & \frac{81}{16} & 1 & \frac{1}{4} & 1 \\ 1 & 4 & 1 & \frac{81}{64} & 1 & \frac{121}{16} & 1 & 1 \\ \frac{1}{16} & \frac{81}{64} & 4 & \frac{1}{4} & \frac{1}{4} & 1 & \frac{1369}{256} & \frac{1}{4} \\ \frac{1}{16} & 1 & 1 & 4 & 1 & 1 & \frac{1}{4} & \frac{81}{16} \end{pmatrix}. \tag{4.46}$$

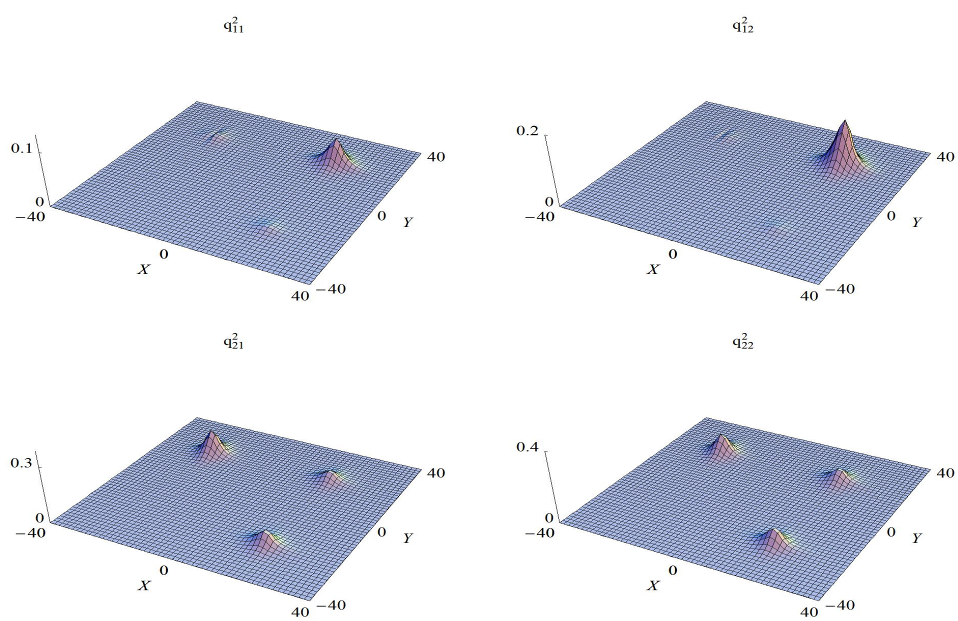


Figure 4.3: $(2, 2)$-dromion plots at $t = -10$.

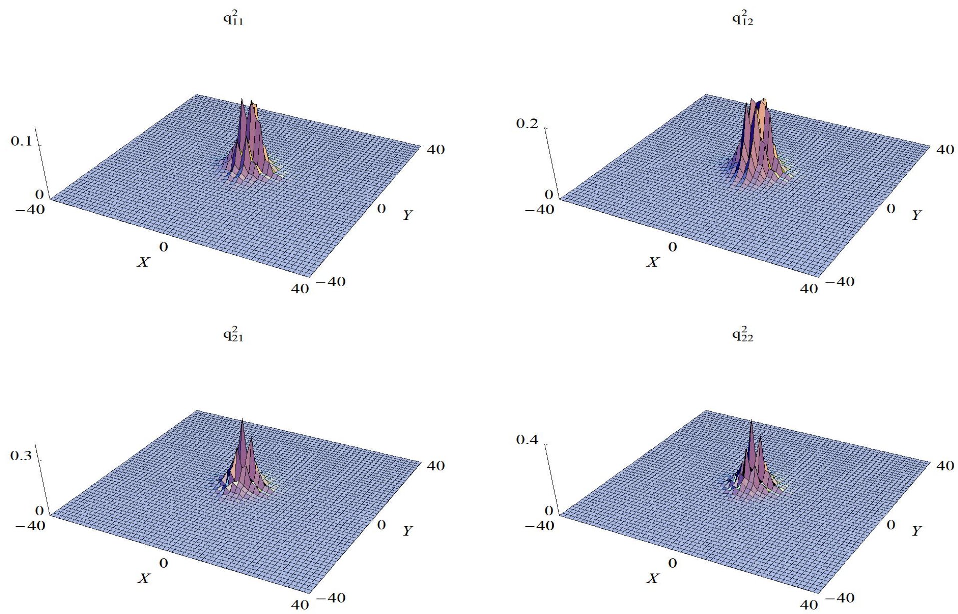


Figure 4.4: $(2, 2)$-dromion plots at $t = 0$.

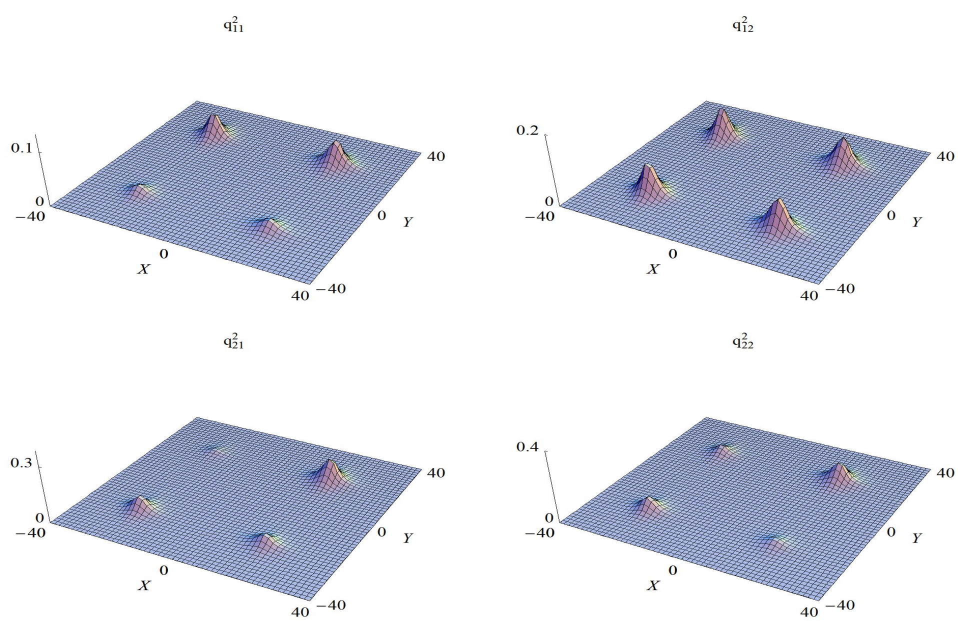


Figure 4.5: $(2, 2)$-dromion plots at $t = 10$.

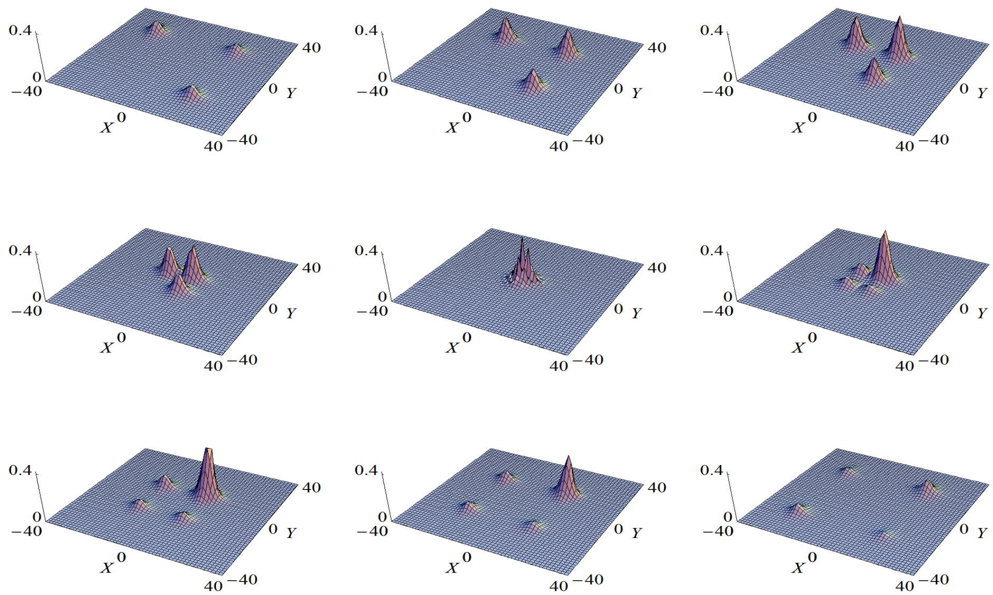


Figure 4.6: q_{22}^2 dromion plots at (top row, left to right) $t = -10, -7.5, -5$, middle row (left to right) $t = -2.5, 0, 2.5$ and bottom row (left to right) $t = 5, 7.5, 10$.

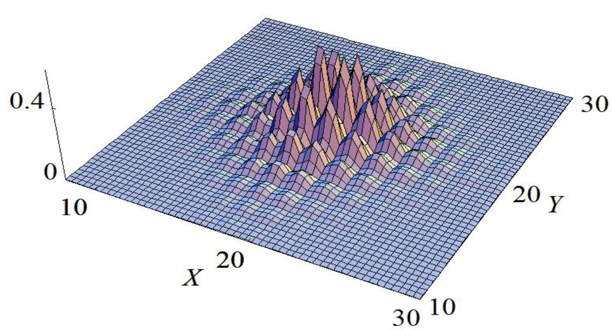


Figure 4.7: Detail of the q_{22}^2 dromion interaction shown in Figure 4.4.

The choice of H given in (4.46) seems rather complex; indeed, we have not been able to find a systematic way to determine appropriate entries of this matrix, and hence have chosen the entries in a rather random manner. However, one way to ensure that our choice of H generates suitable dromion plots is to refer to the work of Gilson and Nimmo in [35] and take H to be a positive definite matrix. In order to choose such an H, we choose any Hermitian matrix, say J, such that the $(1,3)^{\text{th}}$, $(1,4)^{\text{th}}$, $(2,3)^{\text{th}}$ and $(2,4)^{\text{th}}$ entries are zero and multiply this matrix by its conjugate transpose $J^\dagger$. We make the choice

$$J = \begin{pmatrix} \frac{9}{4} & 1 & 0 & 0 & 2 & 1 & \frac{1}{4} & \frac{1}{4} \\ 1 & \frac{11}{4} & 0 & 0 & 1 & 2 & \frac{9}{8} & 1 \\ 0 & 0 & \frac{9}{4} & \frac{1}{2} & \frac{1}{4} & 1 & 2 & 1 \\ 0 & 0 & \frac{1}{2} & \frac{37}{16} & \frac{1}{4} & \frac{9}{8} & \frac{1}{2} & 2 \\ 2 & 1 & \frac{1}{4} & \frac{1}{4} & \frac{9}{4} & 1 & \frac{1}{2} & 1 \\ 1 & 2 & 1 & \frac{9}{8} & 1 & \frac{11}{4} & 1 & 1 \\ \frac{1}{4} & \frac{9}{8} & 2 & \frac{1}{2} & \frac{1}{2} & 1 & \frac{37}{16} & \frac{1}{2} \\ \frac{1}{4} & 1 & 1 & 2 & 1 & 1 & \frac{1}{2} & \frac{9}{4} \end{pmatrix}, \tag{4.47}$$

so that

$$H = JJ^\dagger \tag{4.48}$$

is positive definite and Hermitian, with zeros in the correct positions so that the $(1,1)^{\text{th}}$ dromion in each of q_{11}^2 etc. will vanish as $t \to -\infty$. This highlights one of the main difficulties of obtaining dromion plots in the noncommutative case - we believe that it is not possible to carry out a detailed analysis as in the work of Gilson and Nimmo, and, as a result, our choice of the matrix H is somewhat arbitrary. Nevertheless, we have succeeded in obtaining a range of plots of both the $(1,1)$- and $(2,2)$-dromion solutions in the noncommutative case.

4.2 Conclusions

We have derived and plotted a particular type of solution to our system of noncommutative Davey-Stewartson equations, namely dromions. By choosing the dependent variable q and its Hermitian conjugate $q^\dagger$ to be of matrix rather than scalar form, plots of the $(1,1)$- and $(2,2)$-dromion solutions were obtained in the noncommutative case. In theory, this procedure could be extended to include the $(3,3)$-, $(4,4)$-dromion cases and so on, however the difficulties that we faced in the $(2,2)$-dromion case with regards to choosing the entries of the matrix H indicate that this would not be an easy task. It does not seem possible to generalise results to the (n, n)-dromion case in order that such calculations could be carried out routinely. We have, however, succeeded in extending results for the $(1, 1)$- and $(2, 2)$-dromion situation to the noncommutative case and have obtained new plots of single dromions and dromion interactions.

Chapter 5

Dimensional reduction of Darboux transformations

We have already described in some detail in Chapter 3 the application of both Darboux and binary Darboux transformations to a system of Davey-Stewartson equations. Here we review this procedure and show how the results can be generalised to include a wider class of Lax operator. We then describe a reduction of these Darboux transformations from $(2+1)$- to $(1+1)$-dimensions and indicate their application to the nonlinear Schrödinger equation, a dimensional reduction of the Davey-Stewartson system. This then equips us with the necessary tools to apply, in the next chapter, the reduced Darboux and binary Darboux transformations to the Sasa-Satsuma NLS equation in $(1+1)$-dimension, which can be considered as a dimensional reduction of the 3-component KP hierarchy.

5.1 $(2+1)$-dimensional Darboux transformations

5.1.1 Standard Darboux transformation and its application to the DS system

We saw in Chapter 3 that the so-called 'standard' Darboux transformation, that is

$$L \to \tilde{L} = G_\theta L G_\theta^{-1}, \tag{5.1a}$$

$$M \to \tilde{M} = G_\theta M G_\theta^{-1}, \tag{5.1b}$$

where

$$G_\theta = \partial_y - \theta_y \theta^{-1} = \theta \partial_y \theta^{-1}, \tag{5.2}$$

with θ an eigenfunction of the Lax operators L, M, can be applied to a system of noncommutative DS equations to generate quasi-Wronskian solutions. This Darboux transformation can be applied to other noncommutative nonlinear integrable equations in $(2+1)$-dimensions, for example the KP equation as detailed by Gilson and Nimmo in [36]. In fact, it was shown in [65] that the above Darboux transformation can be applied to Lax operators from a rather general class. We state this result below and show its applicability to the Lax operators of the DS system. The following theorem is due to Matveev [65] and is also stated in [73].

Theorem 4 (Part I) *Let*

$$L = \sum_{j=0}^{N} a_j \partial_y^j, \tag{5.3}$$

where $N \in \mathbb{Z}^+$, a_j are operators (possibly matrices) independent of ∂_y, and let θ be an invertible matrix such that $L(\theta) = \theta C$ for some matrix C such that $\partial_y(C) = 0$. Defining $G_\theta = \theta \partial_y \theta^{-1}$, L is form invariant under the Darboux transformation

$$L \to \tilde{L} = G_\theta L G_\theta^{-1} \tag{5.4}$$

if and only if

$$a_0 = \alpha \partial_x + m_0, \tag{5.5a}$$

$$a_j = m_j \quad j > 0, \tag{5.5b}$$

where α is a constant scalar (i.e. $\partial_y(\alpha) = 0$), m_j $(j \geq 0)$ are matrices and ∂_x is a differential operator independent of ∂_y.

For example, in the case of the DS system, we have

$$L = \partial_x - \varLambda + \sigma J \partial_y, \tag{5.6}$$

with $\varLambda$, J defined as in (3.13) and $\sigma = \mathrm{i}$ or -1. Taking

$$a_0 = \partial_x - \varLambda, \tag{5.7a}$$

$$a_1 = \sigma J, \tag{5.7b}$$

i.e. $\alpha = 1$, $m_0 = -\varLambda$ and $m_1 = \sigma J$, we can see that L is of the required form (5.3), with $N = 1$. In the above example and those that follow, N denotes the order of the Lax operator.

We can adapt Theorem 4 for the second operator M in the Lax pair as follows:

Theorem 4 (Part II) *Let*

$$M = \sum_{j=0}^{N} b_j \partial_y^j, \tag{5.8}$$

where $N \in \mathbb{Z}^+$, b_j are operators (possibly matrices) independent of ∂_y, and let θ be an invertible matrix such that $M(\theta) = \theta D$ for some matrix D such that $\partial_y(D) = 0$. Defining $G_\theta = \theta \partial_y \theta^{-1}$, M is form invariant under the Darboux transformation

$$M \to \tilde{M} = G_\theta M G_\theta^{-1} \tag{5.9}$$

if and only if

$$b_0 = \beta \partial_t + n_0, \tag{5.10a}$$

$$b_j = n_j \quad j > 0, \tag{5.10b}$$

where β is a constant scalar (i.e. $\partial_y(\beta) = 0$), n_j $(j \geq 0)$ are matrices and ∂_x is a differential operator independent of ∂_y.

Thus in the case of the DS system, where

$$M = \partial_t - A + \frac{\mathrm{i}}{\sigma}\Lambda \partial_y - \mathrm{i} J \partial_{yy}, \tag{5.11}$$

with A defined as in (3.14) , we take

$$b_0 = \partial_t - A, \tag{5.12a}$$

$$b_1 = \frac{\mathrm{i}}{\sigma}\Lambda, \tag{5.12b}$$

$$b_2 = -\mathrm{i} J, \tag{5.12c}$$

i.e. $\beta = 1$, $n_0 = -A$, $n_1 = \frac{\mathrm{i}}{\sigma}\Lambda$ and $n_2 = -\mathrm{i}J$. Then M is of the form (5.8) with $N = 2$. Note that both C in Part I and D in Part II of the theorem can be taken to be the zero matrix without loss of generality - in fact, we make this assumption in the DS case in Chapter 3 when we assume $L(\theta) = M(\theta) = 0$.

As was described in Section 3.4.1, Theorem 4 implies that if ϕ is an eigenfunction of L, so that $L(\phi) = 0$, then $\tilde{\phi} := G_\theta(\phi)$ is an eigenfunction of $\tilde{L}$, i.e. a solution of $\tilde{L}(\tilde{\phi}) = 0$. Similar results hold for eigenfunctions of the Lax operator M.

5.2 $(1+1)$-dimensional Darboux transformations

5.2.1 Reduction of standard Darboux transformation and its application to the NLS equation

Theorem 4 has a natural dimensional reduction from (2+1)- to (1+1)-dimensions obtained by making either the x- or y-dependence explicit in the solutions (depending on whether we choose to reduce to a system in x and t or one in y and t). As an example to motivate this reduction, we take the standard $(2+1)$-dimensional Darboux transformation applicable to the DS system (outlined above) and consider a reduction to the NLS equation.

Example: Davey-Stewartson reduction

In the case of the DS system, the Lax operator L is given by (5.6), with L invariant under the standard Darboux transformation $L \to \tilde{L} = G_\theta L G_\theta^{-1}$, where $\theta = \theta(x, y, t)$ satisfies $L(\theta) = \theta C$ for some 2×2 matrix C such that $\partial_y(C) = 0$. (Of course we are free to choose $C = 0$ as explained above). We also define $\phi = \phi(x, y, t)$ to be an eigenfunction of L, so that $L(\phi) = 0$.

There are two possible routes that we can take in order to carry out a reduction of the above procedure from $(2+1)$- to $(1+1)$-dimensions. We either choose to eliminate all y-dependence from the DS system and subsequently obtain an NLS equation in x and t, otherwise we eliminate all x-dependence and obtain an NLS equation in y and t. For simplicity, we consider only one case, the elimination of y-dependence. The elimination of x-dependence arises in a similar manner.

Elimination of y-dependence

We choose to make the y-dependence explicit in the solutions by employing a 'separation of variables' technique. To do so, we define

$$\theta = \theta(x, y, t) := \theta^r(x, t) e^{\Pi y}, \tag{5.13a}$$

$$\phi = \phi(x, y, t) := \phi^r(x, t) e^{\lambda y}, \tag{5.13b}$$

where θ^r and ϕ^r are independent of y, Π is a constant matrix and λ a constant scalar. Here, the superscript r stands for functions in the reduced case. We have thus effectively 'split' θ and ϕ into two parts, one dependent on y, the other independent of y.

We must now determine the effect that this reduction has on G_θ given by (5.2). Note that, with θ defined as in (5.13a) and $j \in \mathbb{Z}^+$,

$$\begin{aligned}\partial_y^j\left(\theta(x,y,t)\right) &= \theta^r(x,t)e^{\Pi y}\Pi^j \\ &= \theta(x,y,t)\Pi^j,\end{aligned} \tag{5.14}$$

so that the dimensional reduction replaces $\partial_y^j(\theta)$ with $\theta\Pi^j$. Further, with ϕ defined as in (5.13b) and $j \in \mathbb{Z}^+$,

$$\begin{aligned}\partial_y^j\left(\phi(x,y,t)\right) &= \phi^r(x,t)e^{\lambda y}\lambda^j \\ &= \lambda^j\phi(x,y,t),\end{aligned} \tag{5.15}$$

i.e. $\partial_y^j(\cdot) = \lambda^j(\cdot)$, so that the dimensional reduction replaces ∂_y^j with λ^j. Thus we have, from Theorem 4,

$$\begin{aligned}G_\theta &= \partial_y - \theta_y\theta^{-1} \\ &= \partial_y - \partial_y(\theta)\theta^{-1},\end{aligned} \tag{5.16}$$

and hence, in the reduced case, replacing ∂_y with λ, $\partial_y(\theta)$ with $\theta\Pi$ and θ with its reduced counterpart defined in (5.13a),

$$\begin{aligned}G_{\theta^r} &= \lambda - \theta\Pi(\theta^r e^{\Pi y})^{-1} \\ &= \lambda - \theta^r e^{\Pi y}\Pi e^{-\Pi y}\theta^{-r} \\ &= \theta^r(\lambda I - \Pi)\theta^{-r},\end{aligned} \tag{5.17}$$

where θ^{-r} denotes $(\theta^r)^{-1}$. We use the notation G_{θ^r} to indicate dependence on θ^r rather than θ. Notice that G_{θ^r} is now independent of y.

From Theorem 4 Part I, we have the condition

$$L(\theta) = \theta C \tag{5.18}$$

and similarly for M in Theorem 4 Part II. Substituting the expression for L given by (5.3), it follows that

$$\sum_{j=0}^{N} a_j\partial_y^j(\theta) = \theta C, \tag{5.19}$$

so that, when we apply the dimensional reduction as above, we obtain

$$\sum_{j=0}^{N} a_j(\theta)\Pi^j = \theta C. \tag{5.20}$$

Replacing θ with its reduced form (5.13a) gives

$$\sum_{j=0}^{N} a_j(\theta^r)e^{\Pi y}\Pi^j = \theta^r e^{\Pi y} C, \tag{5.21}$$

i.e.

$$\sum_{j=0}^{N} a_j(\theta^r)\Pi^j = \theta^r C. \tag{5.22}$$

We are free to choose $C = 0$ without loss of generality as explained previously, hence

$$\sum_{j=0}^{N} a_j(\theta^r)\Pi^j = 0 \tag{5.23}$$

in the dimensionally-reduced case.

Carrying out the same reduction process on our Lax operator L defined in (5.3) gives

$$L^r = \sum_{j=0}^{N} a_j \lambda^j, \tag{5.24}$$

where a_j are operators independent of λ. We thus have the following corollary of Theorem 4 Part I [73]:

Corollary 4 (Part I) *Let*

$$L^r = \sum_{j=0}^{N} a_j \lambda^j, \tag{5.25}$$

where $N \in \mathbb{Z}^+$, a_j are operators independent of λ, and let θ^r be an invertible matrix such that

$$\sum_{j=0}^{N} a_j(\theta^r)\Pi^j = 0 \tag{5.26}$$

for a constant matrix Π. Defining $G_{\theta^r} = \theta^r(\lambda I - \Pi)\theta^{-r}$, L^r is form invariant under the Darboux transformation

$$L^r \to \tilde{L}^r = G_{\theta^r} L^r (G_{\theta^r})^{-1} \tag{5.27}$$

if and only if

$$a_0 = \alpha\partial_x + m_0, \tag{5.28a}$$

$$a_j = m_j \quad j > 0, \tag{5.28b}$$

where α is a constant scalar and m_j $(j \geq 0)$ are matrices.

Thus, with a_0 and a_1 defined as in (5.7), we have

$$L^r = \partial_x - \Lambda + \sigma\lambda J. \tag{5.29}$$

This is one part of the Lax pair for the NLS equation in x and t.

We can easily adapt the above corollary to the case of the reduced form of the Lax operator M of the DS system (i.e. the operator M^r of the NLS equation) as follows:

Corollary 4 (Part II) *Let*

$$M^r = \sum_{j=0}^{N} b_j\lambda^j, \tag{5.30}$$

where $N \in \mathbb{Z}^+$, b_j are operators (possibly matrices) independent of λ, and let θ^r be an invertible matrix such that

$$\sum_{j=0}^{N} b_j(\theta^r)\Pi^j = 0 \tag{5.31}$$

for a constant matrix Π. Defining $G_{\theta^r} = \theta^r(\lambda I - \Pi)\theta^{-r}$, M^r is form invariant under the Darboux transformation

$$M^r \to \tilde{M}^r = G_{\theta^r} M^r (G_{\theta^r})^{-1} \tag{5.32}$$

if and only if

$$b_0 = \beta\partial_t + n_0, \tag{5.33a}$$

$$b_j = n_j \quad j > 0, \tag{5.33b}$$

where β is a constant scalar and n_j are matrices.

Thus, choosing b_0, b_1 and b_2 as in (5.12), we have the Lax operator of the NLS equation in x and t defined as

$$M^r = \partial_t - A + \frac{\mathrm{i}}{\sigma}\lambda\Lambda - \mathrm{i}\lambda^2 J. \tag{5.34}$$

(Although we do not go into details here, setting the commutator of L^r in (5.29) and M^r in (5.34) equal to zero generates the equation

$$q_{xx} + 2qrq \pm 2\mathrm{i}q_t = 0, \tag{5.35}$$

which, on scaling, matches the noncommutative NLS equation (3.18). In this commutator calculation, we take Λ as in (3.13), where q, r are now functions of x and t only, and

$$A = \begin{pmatrix} A_1 & \frac{\mathrm{i}}{2\sigma^2}q_x \\ -\frac{\mathrm{i}}{2\sigma^2}r_x & A_2 \end{pmatrix}, \tag{5.36}$$

for $A_1 = A_1(x,t)$, $A_2 = A_2(x,t)$ (compare (3.14) in Chapter 3), since $q_y = 0$ for $q = q(x,t)$).

We note here that in the work of Gilson, Nimmo and Ohta on the self-dual Yang-Mills equations [73], a generalisation of Corollary 4 must be obtained in order that the Lax operators associated with the self-dual Yang-Mills equations are encompassed. However, as we shall discover in the next chapter, the corollary that we have stated here *does* include Lax operators of the form used in the Sasa-Satsuma NLS equation. Since it is this equation to which we wish to apply the results of the corollary, no such generalisation is needed.

5.3 Dimensional reduction of binary Darboux transformations

5.3.1 Reduction of standard binary Darboux transformation

Corresponding to the reduction of the standard Darboux transformation described above, there also exists a reduction of the standard binary Darboux transformation. This binary Darboux transformation was described in detail in Section 3.4.3, along with its application to the DS system in Section 3.4.4. Here, it was found that, for an eigenfunction θ of the Lax operators L, M and an eigenfunction ρ of the adjoint operators $L^\dagger$, $M^\dagger$, a binary Darboux transformation $G_{\hat{\theta}}^{-1} G_\theta \equiv G_{\theta,\rho}$ transforming L, M to some new operators $\hat{L}$, $\hat{M}$ is defined by

$$G_{\theta,\rho} = I - \theta\Omega(\theta,\rho)^{-1}\partial_y^{-1}\rho^\dagger, \tag{5.37}$$

i.e.

$$G_{\theta,\rho}(\cdot) = (\cdot) - \theta\Omega(\theta,\rho)^{-1}\Omega(\cdot,\rho), \tag{5.38}$$

with the adjoint transformation defined by

$$G_{\theta,\rho}^{-\dagger}(\cdot) = (\cdot) - \rho\Omega(\theta,\rho)^{-\dagger}\Omega(\theta,\cdot)^\dagger, \tag{5.39}$$

where $\cdot$ denotes a function in x, y and t on which $G_{\theta,\rho}$ acts, and the potential Ω satisfies $\Omega(\theta,\rho) = \partial_y^{-1}(\rho^\dagger\theta)$. The transformed operators $\hat{L}, \hat{M}$ have generic eigenfunctions

$$\hat{\phi} = G_{\theta,\rho}(\phi) = \phi - \theta\Omega(\theta,\rho)^{-1}\Omega(\phi,\rho) \tag{5.40}$$

and generic adjoint eigenfunctions

$$\hat{\psi} = G_{\theta,\rho}^{-\dagger}(\psi) = \psi - \rho\Omega(\theta,\rho)^{-\dagger}\Omega(\theta,\psi)^\dagger. \tag{5.41}$$

For simplicity we once again consider only one form of the reduction, from x, y and t to x and t. As in the reduction of the standard Darboux transformation, we make the y-dependence explicit in the solutions by defining

$$\theta = \theta(x, y, t) := \theta^r(x, t)e^{\Pi y}, \tag{5.42a}$$

$$\phi = \phi(x, y, t) := \phi^r(x, t)e^{\lambda y}, \tag{5.42b}$$

with the adjoint eigenfunctions ρ, ψ such that

$$\rho = \rho(x, y, t) := \rho^r(x, t)e^{\Gamma y}, \tag{5.42c}$$

$$\psi = \psi(x, y, t) := \psi^r(x, t)e^{\nu y}. \tag{5.42d}$$

Here, θ^r, ϕ^r, ρ^r and ψ^r are independent of y, Π and Γ are constant matrices, while λ, ν are constant scalars. It then follows that the y-dependence of the potential Ω can also be made explicit by setting

$$\Omega(\theta, \rho) := e^{\Gamma^\dagger y}\Omega^r(\theta^r, \rho^r)e^{\Pi y}. \tag{5.43}$$

By definition (see Section 3.4.3), in $(2+1)$-dimensions we require

$$\Omega(\theta, \rho)_y = \rho^\dagger \theta, \tag{5.44}$$

i.e.

$$\left(e^{\Gamma^\dagger y}\Omega^r(\theta^r, \rho^r)e^{\Pi y}\right)_y = e^{\Gamma^\dagger y}(\rho^r)^\dagger \theta^r e^{\Pi y} \tag{5.45}$$

using (5.42a), (5.42c) and (5.43). This gives

$$\Gamma^\dagger e^{\Gamma^\dagger y}\Omega^r(\theta^r, \rho^r)e^{\Pi y} + e^{\Gamma^\dagger y}\Omega^r(\theta^r, \rho^r)e^{\Pi y}\Pi = e^{\Gamma^\dagger y}(\rho^r)^\dagger \theta^r e^{\Pi y}, \tag{5.46}$$

thus

$$\Gamma^\dagger \Omega^r(\theta^r, \rho^r) + \Omega^r(\theta^r, \rho^r)\Pi = (\rho^r)^\dagger \theta^r. \tag{5.47}$$

We also take

$$\Omega(\phi, \rho) := e^{(\Gamma^\dagger + \lambda I)y}\Omega^r(\phi^r, \rho^r). \tag{5.48}$$

Again, in $(2+1)$-dimensions, we require

$$\Omega(\phi, \rho)_y = \rho^\dagger \phi \tag{5.49}$$

and thus, by (5.42b), (5.42c) and (5.48),

$$\left(e^{(\Gamma^\dagger + \lambda I)y}\Omega^r(\phi^r, \rho^r)\right)_y = e^{\Gamma^\dagger y}(\rho^r)^\dagger \phi^r e^{\lambda y}, \tag{5.50}$$

so that

$$(\Gamma^\dagger + \lambda I)\Omega^r(\phi^r, \rho^r) = (\rho^r)^\dagger \phi^r. \tag{5.51}$$

In addition, we define

$$\Omega(\theta, \psi) := \Omega^r(\theta^r, \psi^r) e^{(\nu^\dagger I + \Pi)y}, \tag{5.52}$$

so that

$$\Omega^r(\theta^r, \psi^r)(\nu^\dagger I + \Pi) = (\psi^r)^\dagger \theta^r. \tag{5.53}$$

We are now in a position to determine the form of the dimensionally-reduced binary Darboux transformation. We have

$$G_{\theta,\rho} = I - \theta\Omega(\theta, \rho)^{-1}\partial_y^{-1}\rho^\dagger \tag{5.54}$$

and hence

$$\begin{aligned} G_{\theta^r,\rho^r} &= I - \theta^r e^{\Pi y}\left(e^{\Gamma^\dagger y}\Omega^r(\theta^r, \rho^r)e^{\Pi y}\right)^{-1}\partial_y^{-1}e^{\Gamma^\dagger y}(\rho^r)^\dagger \\ &= I - \theta^r\Omega^r(\theta^r, \rho^r)^{-1}e^{-\Gamma^\dagger y}\partial_y^{-1}e^{\Gamma^\dagger y}(\rho^r)^\dagger \\ &= I - \theta^r\Omega^r(\theta^r, \rho^r)^{-1}\Gamma^{-\dagger}(\rho^r)^\dagger, \end{aligned} \tag{5.55}$$

where θ^r, ρ^r are independent of y and we assume Γ to be invertible. Thus our expression for G_{θ^r,ρ^r} is a function of x and t only, confirming that we have carried out a dimensional reduction.

Note that, using (5.47), we can rewrite our expression for the reduced binary Darboux transformation G_{θ^r,ρ^r} as

$$\begin{aligned} G_{\theta^r,\rho^r} &= I - \theta^r\Omega^r(\theta^r, \rho^r)^{-1}\Gamma^{-\dagger}(\rho^r)^\dagger\theta^r\theta^{-r} \\ &= I - \theta^r\Omega^r(\theta^r, \rho^r)^{-1}\Gamma^{-\dagger}\left(\Gamma^\dagger\Omega^r(\theta^r, \rho^r) + \Omega^r(\theta^r, \rho^r)\Pi\right)\theta^{-r} \\ &= -\theta^r\Omega^r(\theta^r, \rho^r)^{-1}\Gamma^{-\dagger}\Omega^r(\theta^r, \rho^r)\Pi\theta^{-r}. \end{aligned} \tag{5.56}$$

However, formula (5.55) has the advantage that it is still applicable even when the inverse of θ^r is not defined. In particular, it is valid when θ and ρ are chosen to be of matrix form [73].

We also determine the adjoint reduced binary Darboux transformation $G_{\theta,\rho}^{-\dagger}$: in the (2+1)-dimensional case, we have, from the calculation carried out in (3.75),

$$G_{\theta,\rho} = \theta\Omega(\theta, \rho)^{-1}\partial_y^{-1}\Omega(\theta, \rho)\partial_y\theta^{-1}, \tag{5.57}$$

so that

$$\begin{aligned} G_{\theta,\rho}^{-\dagger} &= (\theta\Omega(\theta,\rho)^{-1}\partial_y^{-1}\Omega(\theta,\rho)\partial_y\theta^{-1})^{-\dagger} \\ &= \theta^{-\dagger}\Omega(\theta,\rho)^{\dagger}\partial_y\Omega(\theta,\rho)^{-\dagger}\partial_y^{-1}\theta^{\dagger} \end{aligned} \tag{5.58}$$

since $\partial^{\dagger} = -\partial$. Thus, in the dimensionally-reduced case,

$$\begin{aligned} G_{\theta^r,\rho^r}^{-\dagger} &= (\theta^r e^{\Pi y})^{-\dagger}(e^{\Gamma^{\dagger} y}\Omega^r(\theta^r,\rho^r)e^{\Pi y})^{\dagger}\partial_y(e^{\Gamma^{\dagger} y}\Omega^r(\theta^r,\rho^r)e^{\Pi y})^{-\dagger}\partial_y^{-1}(\theta^r e^{\Pi y})^{\dagger} \\ &= (\theta^r)^{-\dagger}e^{-\Pi^{\dagger} y}e^{\Pi^{\dagger} y}\Omega^r(\theta^r,\rho^r)^{\dagger}e^{\Gamma y}\partial_y e^{-\Gamma y}\Omega^r(\theta^r,\rho^r)^{-\dagger}e^{-\Pi^{\dagger} y}\partial_y^{-1}e^{\Pi^{\dagger} y}\theta^{r\dagger} \\ &= (\theta^r)^{-\dagger}\Omega^r(\theta^r,\rho^r)^{\dagger}e^{\Gamma y}(-\Gamma)e^{-\Gamma y}\Omega^r(\theta^r,\rho^r)^{-\dagger}e^{-\Pi^{\dagger} y}\Pi^{-\dagger}e^{\Pi^{\dagger} y}\theta^{r\dagger} \\ &= -(\theta^r)^{-\dagger}\Omega^r(\theta^r,\rho^r)^{\dagger}\Gamma\Omega^r(\theta^r,\rho^r)^{-\dagger}\Pi^{-\dagger}\theta^{r\dagger}. \end{aligned} \tag{5.59}$$

In the $(2+1)$-dimensional case,

$$\Omega(\theta,\rho)_y = \rho^{\dagger}\theta, \tag{5.60}$$

so that

$$\Omega(\theta,\rho)_y^{\dagger} = \theta^{\dagger}\rho. \tag{5.61}$$

Then, replacing each term by its reduced counterpart as before gives

$$\left(e^{\Gamma^{\dagger} y}\Omega^r(\theta^r,\rho^r)e^{\Pi y}\right)_y^{\dagger} = (\theta^r e^{\Pi y})^{\dagger}\rho^r e^{\Gamma y}, \tag{5.62}$$

and hence

$$\Pi^{\dagger}\Omega^r(\theta^r,\rho^r)^{\dagger} + \Omega^r(\theta^r,\rho^r)^{\dagger}\Gamma = \theta^{r\dagger}\rho^r. \tag{5.63}$$

We have the adjoint reduced binary Darboux transformation defined in (5.59) as

$$G_{\theta^r,\rho^r}^{-\dagger} = -(\theta^r)^{-\dagger}\Omega^r(\theta^r,\rho^r)^{\dagger}\Gamma\Omega^r(\theta^r,\rho^r)^{-\dagger}\Pi^{-\dagger}\theta^{r\dagger} \tag{5.64}$$

and thus, replacing $\Omega^r(\theta^r,\rho^r)^{\dagger}\Gamma$ with $\theta^{r\dagger}\rho^r - \Pi^{\dagger}\Omega^r(\theta^r,\rho^r)^{\dagger}$ as in (5.63) gives

$$G_{\theta^r,\rho^r}^{-\dagger} = I - \rho^r\Omega^r(\theta^r,\rho^r)^{-\dagger}\Pi^{-\dagger}\theta^{r\dagger}. \tag{5.65}$$

It is often more computationally straightforward to consider the reduced binary transformation as a composition of the two reduced ordinary Darboux transformations G_{θ^r} and $G_{\hat{\theta}^r}$ (as was done in the non-reduced case for the DS equations in Section 3.4.4). We have found the reduced form of the ordinary Darboux transformation G_{θ} in (5.17), and we now obtain a dimensional reduction of the Darboux transformation $G_{\hat{\theta}}$ in a similar manner as follows.

In $(2+1)$-dimensions, we have

$$\hat{\theta} = -\theta\Omega(\theta, \rho)^{-1} \tag{5.66}$$

as in (3.74). Replacing θ and $\Omega(\theta, \rho)$ by their reduced counterparts as defined in (5.13a), (5.43) gives

$$\hat{\theta} = -\theta^r e^{\Pi y} \left(e^{\Gamma^\dagger y} \Omega^r(\theta^r, \rho^r) e^{\Pi y} \right)^{-1}, \tag{5.67}$$

i.e.

$$\begin{aligned} \hat{\theta}(x, y, t) &= -\theta^r(x, t)\Omega^r(\theta^r, \rho^r)^{-1} e^{-\Gamma^\dagger y} \\ &= \hat{\theta}^r(x, t) e^{-\Gamma^\dagger y}, \text{ say,} \end{aligned} \tag{5.68}$$

where $\hat{\theta}^r(x, t) = -\theta^r(x, t)\Omega^r(\theta^r, \rho^r)^{-1}$, so that the dimensional reduction replaces $\partial_y^j(\hat{\theta})$ with $(-1)^j \hat{\theta}(\Gamma^\dagger)^j$ in a similar manner to (5.14).

We also have, in $(2+1)$-dimensions,

$$G_{\hat{\theta}} = \partial_y - \partial_y(\hat{\theta})\hat{\theta}^{-1} \tag{5.69}$$

from (5.16), replacing θ by $\hat{\theta}$. Then

$$\begin{aligned} G_{\hat{\theta}^r} &= \lambda + \hat{\theta}\Gamma^\dagger \left(\hat{\theta}^r e^{-\Gamma^\dagger y} \right)^{-1} \\ &= \lambda + \hat{\theta}^r \Gamma^\dagger \hat{\theta}^{-r}, \end{aligned} \tag{5.70}$$

where $\hat{\theta} = \hat{\theta}^r(x, t)$ is defined as above. This is the dimensionally-reduced version of the Darboux transformation $G_{\hat{\theta}}$.

5.4 A note on dimensions of matrices

In our reduction from Darboux transformations applicable to the $(2+1)$-dimensional DS equation to the dimensionally-reduced Darboux transformations of the $(1+1)$-dimensional NLS equation, we have stated that θ^r is an eigenfunction of the Lax operators L^r, M^r defined in (5.29), (5.34) of the NLS equation, while ρ^r is an eigenfunction of the adjoint Lax operators $L^{r\dagger}$, $M^{r\dagger}$. We have also introduced constant matrices Π in (5.13a) and Γ in (5.42c). However, we have neglected to mention the dimensions of these matrices. In what follows, we choose to omit superscripts.

Since $\theta = \theta(x, t)$ is an eigenfunction of the NLS Lax operators L, M, so that $L(\theta) =$

$M(\theta) = 0$, where L, M involve the 2×2 matrices J and Λ, θ must have 2 rows, however can be chosen to have an arbitrary number of columns. After n iterations of the reduced ordinary Darboux transformation G_θ, we introduce $\Theta = (\theta_1 \ldots \theta_n)$, where each θ_i $(i = 1, \ldots, n)$ is an eigenfunction of L, M. Defining each θ_i to be of size $2 \times s$ for some arbitrary s, we see that Θ has dimension $2 \times ns = 2 \times N$, say, where $N = ns$. Similarly, we define each adjoint eigenfunction ρ_i $(i = 1, \ldots, n)$ to be of size $2 \times s$, so that $P = (\rho_1 \ldots \rho_n)$ has dimension $2 \times N$ also.

We choose the constant matrices Π and Γ each to be of size $s \times s$, so that, for example, the expression for G_θ (i.e. G_{θ^r}) in (5.17) makes sense.

We will shortly move on to consider various noncommutative versions of higher-order NLS equations. We will see that, when we look at the case of the Sasa-Satsuma NLS (SSNLS) equation, a 3-component higher-order NLS equation, the Lax operators now involve 3×3 matrices, and hence the θ_i, ρ_i must be chosen to be of dimension $3 \times s$.

5.5 Darboux transformations applicable to the Sasa-Satuma NLS equation

In the next chapter, we will apply dimensionally-reduced Darboux transformations to the $(1+1)$-dimensional SSNLS equation. For reasons that will be explained in due course, we define the dimensionally-reduced Darboux and binary Darboux transformations in a slightly different manner to those for the DS to NLS reduction described above. We include the complex constant i by setting

$$\theta = \theta(x, y, t) := \theta^r(x, t)e^{-\mathrm{i}\Pi y}, \tag{5.71a}$$

$$\phi = \phi(x, y, t) := \phi^r(x, t)e^{-\mathrm{i}\lambda y}, \tag{5.71b}$$

with Π, λ as before. Then the dimensional reduction replaces $\partial_y^j(\theta)$ with $(-1)^j \mathrm{i}^j \theta \Pi^j$ and ∂_y^j with $(-1)^j \mathrm{i}^j \lambda^j$, so that the dimensionally-reduced Darboux transformation G_{θ^r} is given by

$$G_{\theta^r} = -\mathrm{i}\theta^r(\lambda I - \Pi)\theta^{-r}. \tag{5.72}$$

Similarly, in order to carry out a dimensional reduction of the $(2+1)$-dimensional binary Darboux transformation, we define the adjoint eigenfunctions ρ, ψ such that

$$\rho = \rho(x, y, t) := \rho^r(x, t)e^{\mathrm{i}\Gamma y}, \tag{5.73a}$$

$$\psi = \psi(x, y, t) := \psi^r(x, t)e^{\mathrm{i}\nu y}, \tag{5.73b}$$

and take

$$\Omega(\theta,\rho) := e^{-\mathrm{i}\Gamma^\dagger y}\Omega^r(\theta^r,\rho^r)e^{-\mathrm{i}\Pi y}. \tag{5.74}$$

In addition,

$$\Omega(\phi,\rho) := e^{-\mathrm{i}(\Gamma^\dagger+\lambda I)y}\Omega^r(\phi^r,\rho^r), \tag{5.75}$$

while

$$\Omega(\theta,\psi) := \Omega^r(\theta^r,\psi^r)e^{-\mathrm{i}(\nu^\dagger I+\Pi)y}. \tag{5.76}$$

Then the dimensionally-reduced binary Darboux transformation in (5.55) is given by

$$G_{\theta^r,\rho^r} = I - \mathrm{i}\theta^r\Omega^r(\theta^r,\rho^r)^{-1}\Gamma^{-\dagger}(\rho^r)^\dagger, \tag{5.77}$$

with adjoint

$$G^{-\dagger}_{\theta^r,\rho^r} = I - \mathrm{i}\rho^r\Omega^r(\theta^r,\rho^r)^{-\dagger}\Pi^{-\dagger}(\theta^r)^\dagger. \tag{5.78}$$

Also, the modified versions of (5.51) and (5.53) are

$$-\mathrm{i}(\Gamma^\dagger+\lambda I)\Omega^r(\phi^r,\rho^r) = (\rho^r)^\dagger\phi^r \tag{5.79}$$

and

$$-\mathrm{i}\Omega^r(\theta^r,\psi^r)(\nu^\dagger I+\Pi) = (\psi^r)^\dagger\theta^r \tag{5.80}$$

respectively, while, in a similar manner to (5.70),

$$G_{\hat\theta^r} = -\mathrm{i}(\lambda + \hat\theta^r\Gamma^\dagger\hat\theta^{-r}). \tag{5.81}$$

It is these dimensionally-reduced Darboux transformations and associated results that we apply to the SSNLS equation in the next chapter.

Chapter 6

Higher-order nonlinear Schrödinger equations

6.1 Introduction

6.1.1 Background

The nonlinear Schrödinger equation

The celebrated nonlinear Schrödinger (NLS) equation [5],

$$\mathrm{i}q_t = q_{xx} \pm 2|q|^2 q, \tag{6.1}$$

arises from a coupled pair of nonlinear evolution equations,

$$\mathrm{i}q_t = q_{xx} - 2rq^2, \tag{6.2a}$$

$$-\mathrm{i}r_t = r_{xx} - 2qr^2, \tag{6.2b}$$

setting $r = \mp q^*$, with q, r complex functions of the real variables x, t. Proved integrable via the inverse scattering transform in 1971 [93], the NLS equation has a number of important applications in both mathematics and physics. Benney and Newell [8] indicated how the equation can be used as a model for the evolution of slowly varying small amplitude wave packets in a nonlinear dispersive media. Indeed, the NLS equation has applications in a wide variety of physical systems - water waves [9, 91], plasma physics [92], nonlinear optics [43, 44], and many others. Mathematically, the equation is considered to be one of the fundamental integrable equations admitting an n-soliton solution.

Perhaps one of the most interesting and successful applications of the NLS equation concerns the propagation of short-wave soliton pulses in optical fibres. The field of optical solitons, devised by Hasegawa and Tappert in 1973 [43, 44], has quickly become an area of much research in modern science. For further background on the subject, see, for example, [51] and the references therein.

The Kodama-Hasegawa equation - a higher-order NLS equation

The NLS equation can be used to model short soliton pulses in optical fibres, however, as the pulses become increasingly short, various effects (such as short-frequency shift, third-order dispersion and Kerr dispersion [69, 76]) become apparent and the NLS model is no longer appropriate. In light of this fact, Hasegawa, along with Kodama, developed a suitable higher-order NLS equation to take account of these additional effects, consisting of the NLS equation itself along with perturbative correction terms. We do not give details of the derivation here - a thorough explanation can be found in the original papers [52, 53]. Their equation, which we hereafter refer to as the Kodama-Hasegawa higher-order NLS equation, takes the form

$$\mathrm{i}q_t + \alpha_1 q_{xx} + \alpha_2 |\, q \,|^2 q + \mathrm{i}\epsilon \left(\beta_1 q_{xxx} + \beta_2 |\, q \,|^2 q_x + \beta_3 q (|\, q \,|^2)_x \right) = 0, \tag{6.3}$$

where again q is a complex function of x and t, the α_i $(i = 1, 2)$ and β_j $(j = 1, 2, 3)$ are real constants and ϵ is a real spectral parameter. Note that the independent variables and the parameters differ from those in the original equation of Kodama and Hasegawa. Setting $\epsilon = 0$ gives the standard NLS equation (which can easily be scaled to match (6.1)), while the β_j $(j = 1, 2, 3)$ terms are perturbative corrections.

6.1.2 Integrable higher-order NLS equations

The Kodama-Hasegawa higher-order NLS equation (6.3) need not be integrable unless some restrictions are imposed on the parameters $\beta_1, \beta_2, \beta_3$. With appropriate choices of these real constants, the inverse scattering transform can be applied to verify integrability of the resulting equation. It is known that, along with the NLS equation itself, there are four cases in which integrability can be proven via inverse scattering [79]. These are described below.

The Chen-Lee-Liu (CLL) derivative NLS equation ($\beta_1 : \beta_2 : \beta_3 = 0 : 1 : 0$)

Setting $\beta_1 = \beta_3 = 0$ and $\beta_2 = 1$ in (6.3) gives

$$\mathrm{i}q_t + \alpha_1 q_{xx} + \alpha_2 |q|^2 q + \mathrm{i}\epsilon |q|^2 q_x = 0. \tag{6.4}$$

By choosing $\alpha_1 = 1$ and $\alpha_2 = 0$, we obtain the derivative NLS equation as derived by Chen, Lee and Liu in [11], namely

$$\mathrm{i}q_t + q_{xx} + \mathrm{i}\epsilon |q|^2 q_x = 0. \tag{6.5}$$

This equation was proved integrable by Chen, Lee and Liu in 1979 and confirmed by Nakamura and Chen the following year using a different approach [68].

The Kaup-Newell (KN) derivative NLS equation ($\beta_1 : \beta_2 : \beta_3 = 0 : 1 : 1$)

Setting $\beta_1 = 0$ and $\beta_2 = \beta_3 = 1$ in (6.3) gives

$$\mathrm{i}q_t + \alpha_1 q_{xx} + \alpha_2 |q|^2 q + \mathrm{i}\epsilon \left(|q|^2 q_x + q(|q|^2)_x \right) = 0. \tag{6.6}$$

Choosing $\alpha_1 = 1$ and $\alpha_2 = 0$, we obtain the derivative NLS equation as derived by Kaup and Newell in [50], namely

$$\mathrm{i}q_t + q_{xx} + \mathrm{i}\epsilon \left(|q|^2 q_x + q(|q|^2)_x \right) = 0. \tag{6.7}$$

Wadati and Sogo [86] carry out a dependent variable transformation between the KN derivative NLS equation (referred to by Wadati and Sogo as simply the derivative NLS equation) and the CLL derivative NLS equation. The transformation between these equations has also been discussed by Kundu in [54].

The Hirota NLS (HNLS) equation ($\beta_1 : \beta_2 : \beta_3 = 1 : 6 : 0$)

We set $\beta_1 = 1$, $\beta_2 = 6$ and $\beta_3 = 0$ in (6.3), so that

$$\mathrm{i}q_t + \alpha_1 q_{xx} + \alpha_2 |q|^2 q + \mathrm{i}\epsilon \left(q_{xxx} + 6|q|^2 q_x \right) = 0, \tag{6.8}$$

which gives, on setting $\alpha_1 = \alpha_2 = 0$, an equation known as the Hirota NLS (HNLS) equation [47], that is

$$\mathrm{i}q_t + \mathrm{i}\epsilon \left(q_{xxx} + 6|q|^2 q_x \right) = 0, \tag{6.9}$$

i.e.

$$q_t + \epsilon \left(q_{xxx} + 6|q|^2 q_x \right) = 0, \tag{6.10}$$

which is a complex modified KdV equation and reduces to the modified KdV equation for real q.

The Sasa-Satsuma NLS (SSNLS) equation ($\beta_1 : \beta_2 : \beta_3 = 1 : 6 : 3$)

We finally set $\beta_1 = 1$, $\beta_2 = 6$ and $\beta_3 = 3$ in (6.3) to obtain

$$\mathrm{i}q_t + \alpha_1 q_{xx} + \alpha_2 |q|^2 q + \mathrm{i}\epsilon \left(q_{xxx} + 6|q|^2 q_x + 3q(|q|^2)_x \right) = 0. \tag{6.11}$$

Sasa and Satsuma [79] consider the case where $\alpha_1 = \frac{1}{2}$ and $\alpha_2 = 1$, that is

$$\mathrm{i}q_t + \frac{1}{2} q_{xx} + |q|^2 q + \mathrm{i}\epsilon \left(q_{xxx} + 6|q|^2 q_x + 3q(|q|^2)_x \right) = 0. \tag{6.12}$$

Since $3\beta_1\alpha_2 = \beta_2\alpha_1$, a result by Gilson *et al.* [33] shows that, for $\beta_1 \neq 0$, a gauge transformation can now be implemented to set $\alpha_1 = \alpha_2 = 0$. Sasa and Satsuma apply this gauge transformation and focus on an equivalent version of (6.12), namely

$$q_t + \epsilon \left(q_{xxx} + 6|q|^2 q_x + 3q(|q|^2)_x \right) = 0, \tag{6.13}$$

also proving its integrability. Equation (6.13) is commonly known as the Sasa-Satsuma NLS (SSNLS) equation, and we will denote it as such from now on. It is natural to refer to the SSNLS equation as the complex modified KdV II (mKdV II) equation, since, on scaling ϵ to 1, (6.13) is one of the two integrable complexifications of the mKdV equation [82], the other being, as mentioned above, the Hirota NLS equation (6.10), an integrable complexification of the mKdV I equation.

Note that in the recent paper of Gilson, Hietarinta, Nimmo and Ohta [33], it is assumed that $\beta_1 \neq 0$, hence only the Hirota and Sasa-Satsuma NLS equations are defined as being integrable in this case.

6.2 Reduction from 3-component KP hierarchy

Our understanding of the structure of soliton equations was greatly enhanced by Sato, who, in 1981, made the remarkable discovery that such equations can be organised into infinite hierarchies, and particular equations can be obtained from such hierarchies by suitable reductions. Sato studied the KP hierarchy in particular, where the KP equation (1.9) is a key one. He showed that the KdV and Boussinesq equations arise as reductions of equations in this hierarchy. Other hierarchies have been proposed and studied since Sato's initial work, for example the mKP and Dym hierarchies [74].

Sato's work on the KP hierarchy can be extended to multi-component hierarchies. The standard KP equation (1.9) is a 1-component equation, with the operators in the Lax pair being scalars (i.e. 1×1 matrices) in the commutative case. Thus Sato's studies concerned the 1-component KP hierarchy, where the simplest non-trivial nonlinear equation is the KP equation.

In their paper of 1990 [49], Kajiwara *et al.* consider an extension of Sato's KP hierarchy to two components, and show that the DS equation is the simplest non-trivial nonlinear equation in this new hierarchy. Via suitable reductions, the NLS equation and its higher order extensions, along with the mKdV equation, are obtained.

The SSNLS equation, with Lax pair given in terms of 3×3 matrices as we shall see shortly, arises as a dimensional reduction of the 3-component KP hierarchy. Gilson *et al.* [33] give the equations in this hierarchy in bilinear form, while Kajiwara *et al.* provide an explicit way of constructing the 2-component KP hierarchy in a 'non-bilinear form'. Although we could in theory extend their idea to construct the 3-component hierarchy, the calculation is far from straightforward, and is not necessary for our work.

6.3 Derivation of noncommutative equations

In this section, we discuss noncommutative versions of the integrable equations mentioned above, namely the Chen-Lee-Liu and Kaup-Newell derivative NLS equations and the Hirota and Sasa-Satsuma NLS equations. The Chen-Lee-Liu equation has been considered by Tsuchida and Wadati in the noncommutative (matrix) case [83], and so we simply state their result. In the same paper, a vector generalisation of the Kaup-Newell equation is obtained, and once again we state this equation. We do not believe that the Hirota and Sasa-Satsuma NLS equations have been considered in a noncommutative setting, therefore this is a topic that we wish to discuss in detail.

6.3.1 Chen-Lee Liu derivative NLS equation - matrix case

As stated above, Tsuchida and Wadati [83] considered the CLL derivative NLS equation, namely

$$\mathrm{i}q_t + q_{xx} - \mathrm{i}qrq_x = 0, \tag{6.14a}$$

$$\mathrm{i}r_t - r_{xx} - \mathrm{i}rqr_x = 0, \tag{6.14b}$$

where q, r are complex functions of x, t. (Specifying that $r = \pm q^*$ reduces this coupled system of equations to a single one). By generalising the Lax pair of this set of equations to matrix form, they were able to obtain an integrable matrix version of Chen, Lee and Liu's equation, that is

$$\mathrm{i}Q_t + Q_{xx} - \mathrm{i}QRQ_x = 0, \tag{6.15a}$$

$$\mathrm{i}R_t - R_{xx} - \mathrm{i}R_xQR = 0, \tag{6.15b}$$

where Q, R are matrices. Again, this system can be reduced to a single equation by stipulating that Q and R are complex conjugates. Tsuchida and Wadati also noted that there is no restriction on the size of the matrices Q and R. Thus, choosing, for example, Q and R to be a row and column vector respectively, so that

$$Q = (q_1 \; q_2 \; \ldots \; q_m)\,, \tag{6.16a}$$

$$R = (r_1 \; r_2 \; \ldots \; r_m)^T \tag{6.16b}$$

for some $m \in \mathbb{N}$ gives a coupled version of the CLL derivative NLS equation which, on defining $r_j = \pm q_j^*$ $(j = 1, 2, \ldots, m)$, reduces to a vector version of the equation, that is

$$\mathrm{i}\mathbf{q}_t + \mathbf{q}_{xx} \mp \mathrm{i}\,|\,\mathbf{q}\,|^2\mathbf{q}_x = 0, \tag{6.17}$$

where $\mathbf{q}$ is the m-component vector $\mathbf{q} = (q_1 \; q_2 \; \ldots \; q_m)$.

6.3.2 Kaup-Newell derivative NLS equation - matrix case

In their paper of 1990, Tsuchida and Wadati also state a vector generalisation of the KN derivative NLS equation, namely [83]

$$\mathrm{i}\mathbf{q}_t + \mathbf{q}_{xx} \mp \mathrm{i}(|\,\mathbf{q}\,|^2\,\mathbf{q})_x = 0, \tag{6.18}$$

which has been shown to be integrable, and has previously been studied by Morris and Dodd [66] and Fordy [21].

6.3.3 Hirota and Sasa-Satsuma NLS equations - general noncommutative case

In order to derive noncommutative versions of the Sasa-Satsuma and Hirota NLS equations, we make use of the general m-dimensional Lax operators defined by Ghosh and Nandy [31, 32] as

$$L = \partial_x - \Phi, \tag{6.19a}$$

$$M = \partial_t - \Psi, \tag{6.19b}$$

where

$$\Phi = -\mathrm{i}\lambda J + A, \tag{6.20a}$$

$$\Psi = -4\mathrm{i}\epsilon\lambda^3 J + 4\epsilon\lambda^2 A + 2\epsilon A^3 - 2\mathrm{i}\epsilon\lambda J A^2 + 2\mathrm{i}\epsilon\lambda J A_x - \epsilon A_{xx} + \epsilon A_x A - \epsilon A A_x, \tag{6.20b}$$

with ϵ a real parameter and λ a spectral parameter,

$$J = \left(\sum_{i=1}^{m-1} e_{ii}\right) - e_{mm} \tag{6.21a}$$

and

$$A(x,t) = \sum_{i=1}^{m-1} \alpha_i(x,t) e_{im} - \sum_{i=1}^{m-1} \alpha_i^\dagger(x,t) e_{mi}, \tag{6.21b}$$

where e_{ij} is an $m \times m$ matrix with the ij^{th} entry equal to one and every other entry zero. Note that we have included the term $\alpha_i^\dagger$ in the above definition of A, where † denotes the adjoint (Hermitian conjugate), rather than α_i^* as in the original papers by Ghosh and Nandy. The reasoning behind this is so that we can encompass the noncommutative case in our definition of A: when we treat the α_i as noncommutative objects, for example matrices, we find that we must replace α_i^* by $\alpha_i^\dagger$ so that properties of A which hold in the commutative case also hold when we extend to the noncommutative case. This will become clear later when we study the particular case of the noncommutative SSNLS equation for $m = 3$. In the commutative case, when the α_i are thought of as being scalar objects, $\alpha_i^\dagger = (\alpha_i^*)^T = \alpha_i^*$, since transpose has no effect on a scalar, and hence (6.21b) reduces back to the same definition of A as in [31, 32]. We mention here that the matrix A is skew-Hermitian (or anti-Hermitian), so that

$$A^\dagger = -A. \tag{6.22}$$

There is a further symmetry property of the matrix A valid in the case of the SSNLS equation - this will be detailed in Section 6.3.5.

In (6.21b) above, the α_i can be chosen as $q(x,t)$ or $q^*(x,t)$ in the commutative (scalar) case, and as $q(x,t)$, $q^*(x,t)$, $q^T(x,t)$ or $q^\dagger(x,t)$ in the noncommutative case, where $q(x,t)$ is a noncommutative object. However, we do not specify the nature of the noncommutativity at this stage. Later in the chapter, we explicitly choose q to take the form of a 2×2 matrix. The entries of A will also be of this form, and the entries of J are then replaced by the 2×2 zero and identity matrices.

Clearly, from the definitions of J and A in (6.21a), (6.21b) above, it can easily be seen that

$$J^2 = I_m \tag{6.23a}$$

where I_m denotes the $m \times m$ identity matrix, and J and A are anti-commutative, i.e.

$$JA + AJ = 0. \tag{6.23b}$$

The resulting equation generated from the Lax operators L, M depends on the choice of entries in the matrix A.

Setting the commutator of L and M equal to zero and equating powers of λ results in three equations, namely

$$A_t + \epsilon\left(A_{xxx} - 2(A^2)_x A - 3A^2 A_x + 2AA_x A - A_x A^2\right) = 0, \tag{6.24a}$$

$$\begin{aligned}3JAA_x - JA_{xx} - 2JA^2A - JA_xA - 2AJA^2 + 2AJA_x \\ + 2A^2AJ + 2JA^3 - A_{xx}J + A_xAJ - AA_xJ = 0,\end{aligned} \tag{6.24b}$$

and

$$-2A^2 - 2A_x + 2JA^2J - 2JA_xJ = 0. \tag{6.24c}$$

It is straightforward to show that (6.24b) and (6.24c) are automatically satisfied using only properties (6.23a), (6.23b) and simple differentiation and algebraic manipulation.

Using the same conditions (6.23a), (6.23b), we can simplify (6.24a): by manipulating (6.23b), it can be shown that

$$(A^2)_x A = A_x A^2 + AA_x A, \tag{6.25}$$

and hence (6.24a) simplifies to

$$A_t + \epsilon A_{xxx} - 3\epsilon(A^2 A_x + A_x A^2) = 0 \tag{6.26}$$

as in [31]. It is this equation from which we are able to generate higher-order NLS equations with appropriate choices of A.

Before moving on to generate noncommutative versions of the Sasa-Satsuma and Hirota NLS equations, we note that, as in the DS case, we can introduce, for notational convenience, an $m \times m$ matrix satisfying (6.23b). We let

$$A = [J, T] \tag{6.27}$$

for an $m \times m$ matrix $T = (t_{ij})$ $(i, j = 1, \ldots, m)$. This then enables T to be determined explicitly for a suitable choice of A, as will be shown later. Replacing A by $[J, T]$ means that (6.24b), (6.24c) are satisfied automatically, while (6.26) becomes

$$\begin{aligned} &JT_t - T_tJ + \epsilon(JT_{xxx} - T_{xxx}J) \\ &\quad - 3\epsilon(JTJTJT_x - JT^2T_x - T^2JT_x + TJTT_x - JTJTT_xJ + JT^2JT_xJ + T^2T_xJ \\ &\qquad - TJTJT_xJ + JT_xJTJT - JT_xJT^2J - JT_xT^2 + JT_xTJTJ \\ &\qquad\qquad - T_xTJT + T_xT^2J + T_xJT^2 - T_xJTJTJ) = 0, \end{aligned} \tag{6.28}$$

which we can rewrite in commutator form as

$$\begin{aligned} &[J, T_t + \epsilon T_{xxx} - 3\epsilon(TJTJT_x + JTJTT_x + T_xTJTJ + T^2JT_xJ \\ &\qquad - T^2T_x + T_xJTJT - T_xJT^2J - T_xT^2)] = 0. \end{aligned} \tag{6.29}$$

This is the noncommutative analogue of the Hirota bilinear form of the higher-order NLS equations generated by the Lax pair (6.19). We now move on to generate noncommutative Hirota and Sasa-Satsuma NLS (ncHNLS and ncSSNLS) equations for various values of m.

6.3.4 $m = 2$

Commutative case

We firstly take $m = 2$ and consider the commutative case, so that, from (6.21a),

$$\begin{aligned} J &= e_{11} - e_{22} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \end{aligned} \tag{6.30}$$

while from (6.21b),

$$A = \alpha_1 e_{12} - \alpha_1^* e_{21} \\ = \begin{pmatrix} 0 & \alpha_1 \\ -\alpha_1^* & 0 \end{pmatrix}, \tag{6.31}$$

remembering that $\alpha_1^\dagger = \alpha_1^*$ in the commutative case, that is, when we consider α_1 to be a scalar object.

We choose $\alpha_1 = q$ (and hence $\alpha_1^* = q^*$), where q is a scalar function of x and t and, by equating the $(1,2)^{\text{th}}$ entries, (6.26) reduces to a commutative 2-component Hirota NLS equation [47]:

$$q_t + \epsilon q_{xxx} + 6\epsilon |\, q\, |^2 q_x = 0, \tag{6.32}$$

where $|\, q\, |^2$ denotes the product qq^*. Choosing $\alpha_1 = q^*$ (and hence $\alpha_1^* = q$) and equating $(1,2)^{\text{th}}$ entries gives the complex conjugate of (6.32).

Noncommutative case

In the noncommutative case, where $A = \alpha_1 e_{12} - \alpha_1^\dagger e_{21}$, we choose $\alpha_1 = q$ and hence $\alpha_1^\dagger = q^\dagger$, but we now consider q to be a noncommutative object. This gives, again using (6.26) and equating $(1,2)^{\text{th}}$ entries, a noncommutative version of the Hirota NLS equation, namely

$$q_t + \epsilon q_{xxx} + 3\epsilon(qq^\dagger q_x + q_x q^\dagger q) = 0. \tag{6.33}$$

Choosing $\alpha_1 = q^\dagger$, and hence $\alpha_1^\dagger = q$, and equating $(1,2)^{\text{th}}$ entries gives the corresponding adjoint (Hermitian conjugate) equation of (6.33).

6.3.5 $m = 3$

The case $m = 3$ is more interesting, as we are able to generate both the HNLS and SSNLS equations as follows.

Commutative case

In the 3-component commutative case, (6.21a) gives

$$\begin{aligned} J &= e_{11} + e_{22} - e_{33} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \end{aligned} \tag{6.34}$$

while (6.21b) gives

$$\begin{aligned} A &= \alpha_1 e_{13} + \alpha_2 e_{23} - \alpha_1^* e_{31} - \alpha_2^* e_{32} \\ &= \begin{pmatrix} 0 & 0 & \alpha_1 \\ 0 & 0 & \alpha_2 \\ -\alpha_1^* & -\alpha_2^* & 0 \end{pmatrix}, \end{aligned} \tag{6.35}$$

where $\alpha_1^\dagger = \alpha_1^*$, $\alpha_2^\dagger = \alpha_2^*$ when we think of α_1, α_2 as commutative scalar objects. Note that in the case $m = 3$, the Lax pair (6.19) agrees with that given by Sasa and Satsuma in [79], which they show generates a gauge equivalent version of the (commutative) SSNLS equation. It was in this paper that the SSNLS equation was first proposed.

We firstly choose $\alpha_1 = \alpha_2 = q$ (so that $\alpha_1^* = \alpha_2^* = q^*$), where again q is a scalar function of x and t. Then, by equating $(1,3)^{\text{th}}$ entries, (6.26) reduces to a commutative HNLS equation

$$q_t + \epsilon q_{xxx} + 12\epsilon |\, q\, |^2 q_x = 0 \tag{6.36}$$

and its corresponding complex conjugate, which can be made to match the HNLS equation obtained in the case $m = 2$ by appropriate scaling of the fields q and q^*.

Alternatively, we can choose $\alpha_1 = q$, $\alpha_2 = q^*$ (so that $\alpha_1^* = q^*, \alpha_2^* = q$). Then (6.26) reduces to a commutative SSNLS equation, namely

$$q_t + \epsilon q_{xxx} + 6\epsilon |\, q\, |^2 q_x + 3\epsilon (|\, q\, |^2)_x q = 0 \tag{6.37}$$

and its corresponding complex conjugate. In this case,

$$A = \begin{pmatrix} 0 & 0 & q \\ 0 & 0 & q^* \\ -q^* & -q & 0 \end{pmatrix}, \tag{6.38}$$

and we see that A satisfies

$$A = SA^*S \tag{6.39}$$

for

$$S = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6.40}$$

(Premultiplication of an arbitrary 3×3 matrix by S permutes the first two rows of the matrix and leaves the third row unchanged, while postmultiplication by S permutes the first two columns, leaving the third column unchanged). Note that $S^{-1} = S$, since $S.S = I_3$, the 3×3 identity matrix. Properties (6.22) and (6.39) will be exploited later when we discuss the application of Darboux transformations to the SSNLS equation.

Note that these two equations, namely the HNLS equation (6.36) and the SSNLS equation (6.37), are the only two possible equations in the commutative case that can be obtained from the compatibility condition (6.26). Setting $\alpha_1 = q^*$, $\alpha_2 = q$, or $\alpha_1 = \alpha_2 = q^*$ again give equations (6.36) and (6.37), along with their corresponding complex conjugates.

Noncommutative case

In this case, $A = \alpha_1 e_{13} + \alpha_2 e_{23} - \alpha_1^\dagger e_{31} - \alpha_2^\dagger e_{32}$. We now assume q is a noncommutative object, and set $\alpha_1 = \alpha_2 = q$ (so $\alpha_1^\dagger = \alpha_2^\dagger = q^\dagger$). This gives another ncHNLS equation,

$$q_t + \epsilon q_{xxx} + 6\epsilon(qq^\dagger q_x + q_x q^\dagger q) = 0, \tag{6.41}$$

and the corresponding adjoint equation can be obtained by setting $\alpha_1 = \alpha_2 = q^\dagger$. Note that (6.41) can easily be scaled to match (6.33).
Secondly, we set $\alpha_1 = q$ and $\alpha_2 = q^*$ (so that $\alpha_1^\dagger = q^\dagger, \alpha_2^\dagger = q^{*\dagger} = q^T$). We then obtain, from (6.26), a noncommutative version of the SSNLS equation, namely

$$q_t + \epsilon q_{xxx} + 3\epsilon(qq^\dagger q_x + qq^{*\dagger} q_x^* + q_x q^\dagger q + q_x q^{*\dagger} q^*) = 0, \tag{6.42}$$

with the corresponding adjoint equation obtained by setting $\alpha_1 = q^\dagger$, $\alpha_2 = q^T$.

We have described only two possible cases above, namely the choices $\alpha_1 = \alpha_2 = q$ (giving a ncHNLS equation), and $\alpha_1 = q, \alpha_2 = q^*$ (giving rise to a ncSSNLS equation). Many more possibilities exist, for example $\alpha_1 = q, \alpha_2 = q^\dagger$, or $\alpha_1 = q^*, \alpha_2 = q^{\dagger *} = q^T$, and so on.

However, as mentioned above, the matrix A satisfies condition (6.39) in the commutative (scalar) case which must be preserved when we extend to the noncommutative case. We find that only the choice $\alpha_1 = q, \alpha_2 = q^*$ (or $\alpha_1 = q^*, \alpha_2 = q$) preserves this special condition, and so it is this one on which we focus from now on.

Clearly the process of generating both commutative and noncommutative higher-order NLS equations can be continued for $m = 4, 5$ and so on. For example, in the case $m = 4$, we can generate an HNLS equation equivalent to those obtained in the cases $m = 2, 3$ (up to scaling). We can also obtain new equations (i.e. ones with different ratios of coefficients which do not reduce to the standard HNLS or SSNLS equations). In the $m = 4$ case, two such (commutative) equations are

$$q_t + \epsilon q_{xxx} + 12\epsilon|q|^2 q_x + 3\epsilon(|q|^2)_x q = 0 \tag{6.43}$$

and

$$q_t + \epsilon q_{xxx} + 6\epsilon|q|^2 q_x + 6\epsilon(|q|^2)_x q = 0. \tag{6.44}$$

These equations have coefficients in the ratio 1 : 12 : 3 and 1 : 6 : 6 respectively. However, they are not necessarily integrable; in fact, it is known (see [33]) that (6.44) is not integrable, but has been investigated in the literature, for example in [62, 77]. We do not consider these new equations but instead focus our attention only on the noncommutative integrable SSNLS equation obtained in the case $m = 3$ above.

6.3.6 General case

Before discussing quasideterminant solutions of the noncommutative version of the SSNLS equation, we note that it is possible to derive both a commutative and a noncommutative higher-order NLS equation for a general m. The commutative case has been considered by Ghosh and Nandy [31, 32], however the noncommutative case gives us a new result.

Commutative case

Suppose that our Lax operators L, M in (6.19) are m-dimensional. Then, from (6.21b), we can see that we have $(m-1)$ α_is $(i = 1, \ldots, m-1)$. We choose to identify l of these α_is with q (a scalar function of x, t) for some $l = 1, \ldots, m-1$, and the remaining $(m-l-1)$ α_is with q^*. Then (6.26) gives

$$q_t = \epsilon q_{xxx} + 6l\epsilon|q|^2 q_x + 3(m-l-1)\epsilon(|q|^2)_x q, \tag{6.45}$$

with coefficients in the ratio $1 : 6l : 3(m-l-1)$. Of course, there is no guarantee that such an equation will be integrable for all choices of l, m - the well-known cases of the HNLS equation, with coefficient ratios $1 : 6 : 0$, i.e. $l = 1$ and $m = 2$, and the SSNLS equation, with coefficient ratios $1 : 6 : 3$, i.e. $l = 1$ and $m = 3$, are known to be integrable. As we have shown, these two equations can be obtained for other values of m (for example, we obtain the HNLS equation when $m = 3$) and can be reduced to the 'standard' versions of the equations by suitable scalings. However, we choose to focus solely on the SSNLS equation in the case $l = 1$, $m = 3$ (i.e. a 3-component equation), albeit in the noncommutative case.

Noncommutative case

We can show that, by identifying l of the α_is with q, a noncommutative object, and the remaining $(m-l-1)$ of the α_is with q^*, the noncommutative analogue of (6.45) is

$$q_t = \epsilon q_{xxx} + 3l\epsilon(qq^\dagger q_x + q_x q^\dagger q) + 3(m-l-1)\epsilon(qq^{*\dagger} q^*_x + q_x q^{*\dagger} q^*). \tag{6.46}$$

In the remainder of this chapter, we consider only the ncSSNLS equation obtained from (6.46) in the case $l = 1$, $m = 3$, i.e.

$$q_t = \epsilon q_{xxx} + 3\epsilon(qq^\dagger q_x + qq^{*\dagger} q^*_x + q_x q^\dagger q + q_x q^{*\dagger} q^*). \tag{6.47}$$

6.4 Darboux transformations for the SSNLS equation

To begin, we look at the result of application of the reduced ordinary Darboux transformation given in (5.72), namely

$$G_\theta = -\mathrm{i}\theta(\lambda I - \Pi)\theta^{-1}, \tag{6.48}$$

to the Lax operator

$$L = \partial_x + \mathrm{i}\lambda J - A \tag{6.49}$$

of the ncSSNLS equation (6.47). Here, Π is a constant $s \times s$ matrix with each entry taking the form of a noncommutative object, for example a matrix, while A is defined in (6.35) with $\alpha_1 = q$, $\alpha_2 = q^*$ and $q = q(x,t)$ a noncommutative object. We have chosen, for notational convenience, to omit superscripts from now on. Thus θ as given in (6.48) is actually θ^r, a function of x and t only.

It should be noted here that, in the majority of what follows, we will only state results for the Lax operator L of the SSNLS equation, although the same results also hold for the Lax operator M as given in (6.19b).

6.4.1 Application of reduced Darboux transformation to SSNLS equation

We now apply the dimensionally-reduced Darboux transformation (detailed in Corollary 4) to the Lax pair of the SSNLS equation, and thus consider Lax operators of the form

$$L^r = \sum_{j=0}^{N} a_j \lambda^j, \tag{6.50a}$$

$$M^r = \sum_{j=0}^{N} b_j \lambda^j \tag{6.50b}$$

for operators a_j and b_j independent of λ, where $N \in \mathbb{Z}^+$. Then, with the Lax operator L of the SSNLS equation given by (now omitting superscripts)

$$L = \partial_x + \mathrm{i}\lambda J - A, \tag{6.51}$$

we choose

$$a_0 = \partial_x - A, \tag{6.52a}$$

$$a_1 = \mathrm{i}J, \tag{6.52b}$$

i.e., in the notation of the corollary, $\alpha = 1$, $m_0 = -A$ and $m_1 = \mathrm{i}J$. We also have

$$M = \partial_t + \epsilon(A_{xx} - A_x A + AA_x - 2A^3) + 2\mathrm{i}\lambda\epsilon(JA^2 - JA_x) - 4\lambda^2\epsilon A + 4\mathrm{i}\lambda^3\epsilon J, \tag{6.53}$$

and hence we choose, in Part II of the corollary,

$$b_0 = \partial_t + \epsilon(A_{xx} - A_x A + AA_x - 2A^3), \tag{6.54a}$$

$$b_1 = 2\mathrm{i}\epsilon(JA^2 - JA_x), \tag{6.54b}$$

$$b_2 = -4\epsilon A, \tag{6.54c}$$

$$b_3 = 4\mathrm{i}\epsilon J, \tag{6.54d}$$

where, in the notation of the corollary, $\beta = 1$, $n_0 = \epsilon(A_{xx} - A_x A + AA_x - 2A^3)$, $n_1 = 2\mathrm{i}\epsilon(JA^2 - JA_x)$, $n_2 = -4\epsilon A$ and $n_3 = 4\mathrm{i}\epsilon J$.

To determine the effect of the reduced Darboux transformation G_θ in (6.48) on the eigenfunctions of the SSNLS Lax pair, we use the same notation as in Chapter 3, and denote

L by $L_{[1]}$, M by $M_{[1]}$.

Iteration

Step 1 Let $\theta_1, \ldots, \theta_n$ be eigenfunctions of $L_{[1]} = L$, $M_{[1]} = M$, and $\phi_{[1]} = \phi$ a generic eigenfunction of $L_{[1]}$, $M_{[1]}$. Choose $\theta_{1[1]} := \theta_1$ to define a dimensionally-reduced Darboux transformation

$$G_{\theta_{1[1]}} = -\mathrm{i}\left(\lambda - \theta_{1[1]}\Pi\theta_{1[1]}^{-1}\right) \tag{6.55}$$

to a new Lax pair $L_{[2]}, M_{[2]}$, where $\theta_{1[1]}^{(1)} := -\mathrm{i}\theta_{1[1]}\Pi^1$ and $\theta_{1[1]}^{(1)}$ denotes one differentiation of $\theta_{1[1]}$ with respect to x. The new Lax pair

$$L_{[2]} = G_{\theta_{1[1]}} L_{[1]} G_{\theta_{1[1]}}^{-1}, \tag{6.56a}$$

$$M_{[2]} = G_{\theta_{1[1]}} M_{[1]} G_{\theta_{1[1]}}^{-1} \tag{6.56b}$$

has generic eigenfunction

$$\begin{aligned} \phi_{[2]} &:= G_{\theta_{1[1]}}(\phi) \\ &= \phi^{(1)} - \theta_1^{(1)}\theta_1^{-1}\phi \end{aligned} \tag{6.57}$$

since $\theta_{1[1]} = \theta_1$. As in the (2+1)-dimensional case, the eigenfunction $\theta_1 = \theta_{1[1]}$ is mapped to zero, and the remaining eigenfunctions $\theta_2, \ldots, \theta_n$ to $\theta_{2[2]}, \ldots, \theta_{n[2]}$, where, for $i = 2, \ldots, n$,

$$\theta_{i[2]} = \phi_{[2]} \mid_{\phi \to \theta_i} . \tag{6.58}$$

$\vdots$

Step n $(n \geq 1)$ The eigenfunction $\theta_{n[n]}$ defines a dimensionally-reduced Darboux transformation from $L_{[n]}$, $M_{[n]}$ to some new operators

$$L_{[n+1]} = G_{\theta_{n[n]}} L_{[n]} G_{\theta_{n[n]}}^{-1}, \tag{6.59a}$$

$$M_{[n+1]} = G_{\theta_{n[n]}} M_{[n]} G_{\theta_{n[n]}}^{-1} \tag{6.59b}$$

with generic eigenfunctions

$$\begin{aligned} \phi_{[n+1]} &= G_{\theta_{n[n]}}(\phi_{[n]}) \\ &= \phi_{[n]}^{(1)} - \theta_{n[n]}^{(1)}\theta_{n[n]}^{-1}\phi_{[n]}. \end{aligned} \tag{6.60}$$

In particular, this Darboux transformation maps the eigenfunction $\theta_{n[n]}$ to $G_{\theta_{n[n]}}\left(\theta_{n[n]}\right)$, i.e. to zero.

By defining $\Theta = (\theta_1 \ldots \theta_n)$, we can show by induction that (6.60) can be expressed in quasi-Wronskian form as

$$\phi_{[n+1]} = \begin{vmatrix} \Theta & \phi \\ \vdots & \vdots \\ \Theta^{(n-1)} & \phi^{(n-1)} \\ \Theta^{(n)} & \boxed{\phi^{(n)}} \end{vmatrix}, \tag{6.61}$$

where $\Theta^{(n)} := -\mathrm{i}\Theta\hat{\Pi}^n$, and $\hat{\Pi}$ is the $ns \times ns = N \times N$ matrix such that

$$\hat{\Pi} = \mathrm{diag}(\underbrace{\Pi, \Pi, \ldots, \Pi}_{n}), \tag{6.62}$$

where each matrix Π is of size $s \times s$. The inductive proof is carried out in a similar manner to that done in Section 3.4 of Chapter 3.

6.4.2 Quasi-Wronskian solution of ncSSNLS using reduced Darboux transformation

We now determine the effect of the reduced Darboux transformation (6.55) on the Lax operator $L_{[1]} = L$ given by (6.51), with $\theta_1, \ldots, \theta_n = \theta_{1[1]}, \ldots, \theta_{n[1]}$ eigenfunctions of L, and $\theta_{1[1]}$ chosen to iterate the transformation. To simplify our notation slightly, we will denote $\theta_{1[1]}$ by θ, and, in general, $\theta_{k[k]}$ by $\theta_{[k]}$ $(k = 2, \ldots, n)$. Corresponding results hold for the operator M given by (6.53). The operator $L_{[1]} = L$ is transformed to a new operator $L_{[2]}$, say, where

$$L_{[2]} = G_\theta L G_\theta^{-1}, \tag{6.63}$$

giving

$$-\theta_x \Pi \theta^{-1} + \theta \Pi \theta^{-1} \theta_x \theta^{-1} + A_{[2]} \theta \Pi \theta^{-1} - \theta \Pi \theta^{-1} A = 0 \tag{6.64a}$$

and

$$J\theta\Pi\theta^{-1} - \theta\Pi\theta^{-1} J + \mathrm{i}(A - A_{[2]}) = 0. \tag{6.64b}$$

Note that, in carrying out the reduction on the 'standard' (2 + 1)-dimensional Darboux transformation, we have, as indicated previously, replaced ∂_y^j with $(-1)^j \mathrm{i}^j \lambda^j$. Hence λ is effectively playing the role of ∂_y in the reduced case, thus when we apply the reduced Darboux transformation to the SSNLS Lax operator L, we set coefficients of λ^j equal to zero, rather than coefficients of ∂_y^j as was done in, for example, the DS case in Chapter 3. From (6.64b), we see that

$$A_{[2]} = A - \mathrm{i}[J, \theta\Pi\theta^{-1}], \tag{6.65}$$

i.e. by introducing the notation $\theta^{(1)} = -\mathrm{i}\theta\Pi^1$, where $\theta^{(1)}$ denotes one differentiation of θ with respect to x, we have

$$A_{[2]} = A + [J, \theta^{(1)}\theta^{-1}]. \tag{6.66}$$

Substituting for $A_{[2]}$ in (6.64a) using (6.65) gives

$$\left(-\theta_x\Pi + \theta\Pi\theta^{-1}\theta_x + A\theta\Pi + \mathrm{i}\theta\Pi\theta^{-1}J\theta\Pi - \mathrm{i}J\theta\Pi^2 - \theta\Pi\theta^{-1}A\theta\right)\theta^{-1} = 0. \tag{6.67}$$

In order to see that the left-hand side of this equation is zero, we look to (5.26), where we have, omitting superscripts,

$$\sum_{j=0}^{N} a_j(\theta)\Pi^j = 0, \tag{6.68}$$

i.e.

$$\theta_x - A\theta + \mathrm{i}J\theta\Pi = 0 \tag{6.69}$$

using (6.52). This result can be used to confirm the validity of (6.67).

As mentioned in (6.27), we introduce an $m \times m = 3 \times 3$ matrix $T = (t_{ij})$ $(i, j = 1, 2, 3)$ such that $A = [J, T]$. We then have, from (6.66),

$$[J, T_{[2]}] = [J, T] + [J, \theta^{(1)}\theta^{-1}], \tag{6.70}$$

so that

$$T_{[2]} = T + \theta^{(1)}\theta^{-1}. \tag{6.71}$$

After n repeated applications of the reduced Darboux transformation,

$$T_{[n+1]} = T_{[n]} + \theta_{[n]}^{(1)}\theta_{[n]}^{-1}, \tag{6.72}$$

that is

$$T_{[n+1]} = T + \sum_{j=1}^{n} \theta_{[j]}^{(1)}\theta_{[j]}^{-1}, \tag{6.73}$$

where $T_{[1]} = T$, $\theta_{[1]} = \theta$ and $\theta_k = \theta_{k[k]}$. By induction, we express $T_{[n+1]}$ in quasi-Wronskian form as

$$T_{[n+1]} = T - \begin{vmatrix} \Theta & O_3 \\ \vdots & \vdots \\ \Theta^{(n-2)} & O_3 \\ \Theta^{(n-1)} & I_3 \\ \Theta^{(n)} & \boxed{O_3} \end{vmatrix}, \tag{6.74}$$

where O_3 and I_3 denote the 3×3 zero and identity matrices respectively, and $\Theta = (\theta_1 \ldots \theta_n)$. As before, $\Theta^{(k)} = -\mathrm{i}\Theta\hat{\Pi}^k$, where $\Theta^{(k)}$ denotes the k^{th} x-derivative of Θ $(k = 0, \ldots, n)$, and $\hat{\Pi}$ is an $ns \times ns = 3s \times 3s$ matrix defined as in (6.62). The inductive proof of (6.74) is similar to that in Chapter 3, Section 3.4.2.

6.4.3 Invariance of Darboux transformation

Above, we have applied the appropriate $(1+1)$-dimensional Darboux transformation to the Lax operator L of the (noncommutative) SSNLS equation. However, since a Darboux transformation is, by definition, a special type of gauge transformation that keeps the Lax pair associated with the particular equation under consideration *invariant*, we must ensure that our transformed Lax operator $\tilde{L}$ $(\equiv L_{[2]})$ has the same form as our original operator L (and similarly for the Lax operator M and transformed operator $\tilde{M} \equiv M_{[2]}$). Here we mainly focus on one iteration of the transformation $L \to \tilde{L}$, $M \to \tilde{M}$. In particular, since L is of the form

$$L = \partial_x + \mathrm{i}\lambda J - A, \tag{6.75}$$

with

$$A = \begin{pmatrix} 0 & 0 & q \\ 0 & 0 & q^* \\ -q^\dagger & -q^{\dagger *} & 0 \end{pmatrix}, \tag{6.76}$$

the transformed matrix $\tilde{A}$ must be of the form

$$\tilde{A} = \begin{pmatrix} 0 & 0 & \tilde{q} \\ 0 & 0 & \tilde{q}^* \\ -\tilde{q}^\dagger & -\left(\tilde{q}^\dagger\right)^* & 0 \end{pmatrix}, \tag{6.77}$$

and similarly for the transformed matrix $A_{[n+1]}$ after n iterations of the Darboux transformation. In short, the Darboux transformation must preserve properties (6.22) and (6.39),

so that

$$\tilde{A}^{\dagger} = -\tilde{A} \tag{6.78a}$$

and

$$\tilde{A} = S\tilde{A}^{*}S \tag{6.78b}$$

for S defined as in (6.40). We do not go into details here, however we can show that, with an appropriate choice of the matrices Θ and Π, the ordinary Darboux transformation can preserve condition (6.39) (the 'complex conjugacy condition'). In particular, we define

$$\Theta = S\Theta^{*}S_1, \tag{6.79}$$

where S_1 is an $ns \times ns$ matrix with real entries such that $S_1^2 = I_{ns}$. The 'skew-Hermitian condition' (6.22) implies that the transformed Lax operator must satisfy $\tilde{L}^{\dagger} = -\tilde{L}$. In general, a symmetry in a Lax pair is not preserved by an ordinary Darboux transformation but can be preserved by a binary one [73]. In the case of the ncSSNLS equation discussed here, it is possible for the ordinary Darboux transformation to preserve the skew-Hermitian condition, however the resulting requirement that $\theta_x\theta^{-1}$ is Hermitian is difficult to realise. We therefore allude to the reduced binary Darboux transformation noted in Section 5.5.

6.4.4 Application of reduced binary Darboux transformation to SSNLS equation

We now determine the effect of the reduced binary Darboux transformation detailed in Section 5.5 on the Lax operators L, M of the SSNLS equation. We relabel L, $L^{\dagger}$ as $L_{[1]}$, $L_{[1]}^{\dagger}$ respectively, and similarly for M, $M^{\dagger}$, to indicate the starting levels, and omit superscript r from now on.

The adjoint Lax pairs $L^{\dagger}$, $M^{\dagger}$ of the SSNLS equation satsify, from (6.19) and the definition of adjoint given in Section 3.4.3 of Chapter 3,

$$L^{\dagger} = -\partial_x - \Phi^{\dagger}, \tag{6.80a}$$

$$M^{\dagger} = -\partial_t - \Psi^{\dagger}, \tag{6.80b}$$

where

$$\Phi^\dagger = \mathrm{i}\lambda J + A^\dagger, \tag{6.81a}$$

$$\Psi^\dagger = 4\mathrm{i}\epsilon\lambda^3 J + 4\epsilon\lambda^2 A^\dagger + 2\epsilon(A^3)^\dagger + 2\mathrm{i}\epsilon\lambda J(A^2)^\dagger - 2\mathrm{i}\epsilon\lambda J A_x^\dagger - \epsilon A_{xx}^\dagger + \epsilon A^\dagger A_x^\dagger - \epsilon A_x^\dagger A^\dagger. \tag{6.81b}$$

Iteration

Step 1 Let $\theta_1, \dots, \theta_n$ be eigenfunctions of $L_{[1]} = L$, $M_{[1]} = M$, and $\rho_1, \dots, \rho_n$ eigenfunctions of the adjoint Lax operators $L_{[1]}^\dagger = L^\dagger$, $M_{[1]}^\dagger = M^\dagger$. Suppose $\phi_{[1]} = \phi$ is a generic eigenfunction of $L_{[1]}, M_{[1]}$ and $\psi_{[1]} = \psi$ a generic eigenfunction of $L_{[1]}^\dagger, M_{[1]}^\dagger$. We choose $\theta_1 := \theta_{1[1]}$ to be the eigenfunction defining a reduced binary Darboux transformation from $L_{[1]}, M_{[1]}$ to a new Lax pair $L_{[2]}, M_{[2]}$, and similarly $\rho_1 := \rho_{1[1]}$ the eigenfunction defining the adjoint binary Darboux transformation from $L_{[1]}^\dagger, M_{[1]}^\dagger$ to a new adjoint Lax pair $L_{[2]}^\dagger, M_{[2]}^\dagger$. The reduced binary Darboux transformation is defined as

$$G_{\theta_{1[1]},\rho_{1[1]}} = I - \mathrm{i}\theta_{1[1]}\Omega(\theta_{1[1]}, \rho_{1[1]})^{-1}\Gamma^{-\dagger}\rho_{1[1]}^\dagger, \tag{6.82}$$

with adjoint

$$G_{\theta_{1[1]},\rho_{1[1]}}^{-\dagger} = I - \mathrm{i}\rho_{1[1]}\Omega(\theta_{1[1]}, \rho_{1[1]})^{-\dagger}\Pi^{-\dagger}\theta_{1[1]}^\dagger. \tag{6.83}$$

The transformed operators

$$L_{[2]} = G_{\theta_{1[1]},\rho_{1[1]}} L_{[1]} G_{\theta_{1[1]},\rho_{1[1]}}^{-1}, \tag{6.84a}$$

$$M_{[2]} = G_{\theta_{1[1]},\rho_{1[1]}} M_{[1]} G_{\theta_{1[1]},\rho_{1[1]}}^{-1} \tag{6.84b}$$

have generic eigenfunctions

$$\begin{aligned} \phi_{[2]} &:= G_{\theta_{1[1]},\rho_{1[1]}}(\phi) \\ &= \phi - \mathrm{i}\theta_{1[1]}\Omega(\theta_{1[1]}, \rho_{1[1]})^{-1}\Gamma^{-\dagger}\rho_{1[1]}^\dagger\phi \\ &= \phi - \theta_{1[1]}\Omega(\theta_{1[1]}, \rho_{1[1]})^{-1}(I + \lambda I\Gamma^{-\dagger})\Omega(\phi, \rho_{1[1]}) \end{aligned} \tag{6.85}$$

by (5.79), and generic adjoint eigenfunctions

$$\begin{aligned} \psi_{[2]} &:= G_{\theta_{1[1]},\rho_{1[1]}}^{-\dagger}(\psi) \\ &= \psi - \mathrm{i}\rho_{1[1]}\Omega(\theta_{1[1]}, \rho_{1[1]})^{-\dagger}\Pi^{-\dagger}\theta_{1[1]}^\dagger\psi \\ &= \psi - \rho_{1[1]}\Omega(\theta_{1[1]}, \rho_{1[1]})^{-\dagger}(I + \nu I\Pi^{-\dagger})\Omega(\theta_{1[1]}, \psi)^\dagger \end{aligned} \tag{6.86}$$

by (5.80). The eigenfunction $\theta_{1[1]}$ is mapped to zero by (6.82), and the adjoint eigenfunction $\rho_{1[1]}$ to zero by (6.83). The remaining eigenfunctions $\theta_2, \ldots, \theta_n$ and adjoint eigenfunctions $\rho_2, \ldots, \rho_n$ are mapped to $\theta_{2[2]}, \ldots, \theta_{n[2]}$ and $\rho_{2[2]}, \ldots, \rho_{n[2]}$ respectively, where, for $i = 2, \ldots, n$,

$$\theta_{i[2]} = \phi_{[2]}|_{\phi\to\theta_i}, \qquad \rho_{i[2]} = \psi_{[2]}|_{\psi\to\rho_i}. \tag{6.87}$$

$\vdots$

Step n ($n \geq 1$) To perform the n^{th} iteration of the reduced binary Darboux transformation, we choose the eigenfunction $\theta_{n[n]}$ to define a reduced binary Darboux transformation from the Lax operators $L_{[n]}, M_{[n]}$ to some new Lax operators $L_{[n+1]}, M_{[n+1]}$, and similarly $\rho_{n[n]}$ the adjoint eigenfunction defining the adjoint reduced binary Darboux transformation from $L^\dagger_{[n]}, M^\dagger_{[n]}$ to $L^\dagger_{[n+1]}, M^\dagger_{[n+1]}$. The operators $L_{[n]}, M_{[n]}$ are covariant under the action of the reduced binary Darboux transformation

$$G_{\theta_{n[n]},\rho_{n[n]}} = I - \mathrm{i}\theta_{n[n]}\Omega(\theta_{n[n]}, \rho_{n[n]})^{-1}\Gamma^{-\dagger}\rho^\dagger_{n[n]}, \tag{6.88}$$

while the adjoint operators $L^\dagger_{[n]}, M^\dagger_{[n]}$ are covariant under the adjoint binary Darboux transformation

$$G^{-\dagger}_{\theta_{n[n]},\rho_{n[n]}} = I - \mathrm{i}\rho_{n[n]}\Omega(\theta_{n[n]}, \rho_{n[n]})^{-\dagger}\Pi^{-\dagger}\theta^\dagger_{n[n]}. \tag{6.89}$$

The transformed operators

$$L_{[n+1]} = G_{\theta_{n[n]},\rho_{n[n]}} L_{[n]} G^{-1}_{\theta_{n[n]},\rho_{n[n]}}, \tag{6.90a}$$

$$M_{[n+1]} = G_{\theta_{n[n]},\rho_{n[n]}} M_{[n]} G^{-1}_{\theta_{n[n]},\rho_{n[n]}} \tag{6.90b}$$

have generic eigenfunctions

$$\begin{aligned}\phi_{[n+1]} &= \phi_{[n]} - \mathrm{i}\theta_{n[n]}\Omega(\theta_{n[n]}, \rho_{n[n]})^{-1}\Gamma^{-\dagger}_{[n]}\rho^\dagger_{n[n]}\phi_{[n]} \\ &= \phi_{[n]} - \theta_{n[n]}\Omega(\theta_{n[n]}, \rho_{n[n]})^{-1}(I + \lambda I\Gamma^{-\dagger})\Omega(\phi_{[n]}, \rho_{[n]}),\end{aligned} \tag{6.91}$$

and adjoint eigenfunctions

$$\begin{aligned}\psi_{[n+1]} &= \psi_{[n]} - \mathrm{i}\rho_{n[n]}\Omega(\theta_{n[n]}, \rho_{n[n]})^{-\dagger}\Pi^{-\dagger}_{[n]}\theta^\dagger_{n[n]}\psi_{[n]} \\ &= \psi_{[n]} - \rho_{n[n]}\Omega(\theta_{n[n]}, \rho_{n[n]})^{-\dagger}(I + \nu I\Pi^{-\dagger})\Omega(\theta_{n[n]}, \psi_{[n]})^\dagger.\end{aligned} \tag{6.92}$$

The eigenfunction $\theta_{n[n]}$ is mapped to zero by (6.88), and the adjoint eigenfunction $\rho_{n[n]}$ to zero by (6.89).

Quasi-Grammian form

Defining $\Theta = (\theta_1 \ldots \theta_n)$ and $P = (\rho_1 \ldots \rho_n)$, we express $\phi_{[n+1]}$ and $\psi_{[n+1]}$ in quasi-Grammian form as

$$\phi_{[n+1]} = \begin{vmatrix} \Omega(\Theta, P) & (I + \lambda I \hat{\Gamma}^{-\dagger})\Omega(\phi, P) \\ \Theta & \boxed{\phi} \end{vmatrix}, \quad \psi_{[n+1]} = \begin{vmatrix} \Omega(\Theta, P)^\dagger & (I + \nu I \hat{\Pi}^{-\dagger})\Omega(\Theta, \psi)^\dagger \\ P & \boxed{\psi} \end{vmatrix}, \tag{6.93}$$

where $\hat{\Gamma}$ is an $ns \times ns = 3 \times 3$ matrix such that

$$\hat{\Gamma} = \mathrm{diag}(\underbrace{\Gamma, \Gamma, \ldots, \Gamma}_{n}), \tag{6.94}$$

with each matrix Γ of size $s \times s$, and $\hat{\Pi}$ is defined as in (6.62).

6.4.5 Quasi-Grammian solution of ncSSNLS using reduced binary Darboux transformation

In this section, we calculate the effect of the binary Darboux transformation (6.82) on the Lax operator $L_{[1]} = L$ of the SSNLS equation given by (6.51), with $\theta_1, \ldots, \theta_n = \theta_{1[1]}, \ldots, \theta_{n[1]}$ eigenfunctions of L, and $\theta_{1[1]}$ chosen to iterate the Darboux transformation. As in previous sections, we denote $\theta_{1[1]}$ by θ, and, in general, $\theta_{k[k]}$ by $\theta_{[k]}$ $(k = 2, \ldots, n)$, and similarly, $\rho_{1[1]}$ by ρ and $\rho_{k[k]}$ by $\rho_{[k]}$. Corresponding results hold for the operator M given by (6.53). In what follows, we choose to omit superscripts, so that $\theta = \theta^r$ and $\Omega = \Omega^r(\theta^r, \rho^r)$.

As mentioned in the previous chapter, it is convenient to consider the reduced binary Darboux transformation as a composition of the two reduced ordinary Darboux transformations G_θ and $G_{\hat{\theta}}$. We have the same situation as depicted in the diagram on page 53 in Chapter 3, with each term replaced by its reduced counterpart. The operator $L_{[1]} = L$ is transformed to a new operator $\tilde{L}_{[2]}$ by the reduced Darboux transformation $G_{\theta_{1[1]}} \equiv G_\theta$ defined in (5.72), so that $\tilde{L}_{[2]} = G_\theta L G_{\theta^{-1}}$, giving

$$\tilde{A}_{[2]} = A - \mathrm{i}[J, \theta \Pi \theta^{-1}] \tag{6.95}$$

as in (6.65). We also have $\tilde{L}_{[2]} = G_{\hat{\theta}} L_{[2]} G_{\hat{\theta}}^{-1}$, and, using the reduced form of $G_{\hat{\theta}}$ obtained in (5.81) and omitting superscripts, this gives two equations on equating coefficients of λ, namely

$$\theta_x\Omega^{-1}\Gamma^\dagger\Omega\theta^{-1} - \theta\Omega^{-1}\Omega_x\Omega^{-1}\Gamma^\dagger\Omega\theta^{-1} + \theta\Omega^{-1}\Gamma^\dagger\Omega_x\theta^{-1} - \theta\Omega^{-1}\Gamma^\dagger\Omega\theta^{-1}\theta_x\theta^{-1} - \tilde{A}_{[2]}\theta\Omega^{-1}\Gamma^\dagger\Omega\theta^{-1} + \theta\Omega^{-1}\Gamma^\dagger\Omega\theta^{-1}A_{[2]} = 0 \quad (6.96)$$

and

$$J\theta\Omega^{-1}\Gamma^\dagger\Omega\theta^{-1} - \theta\Omega^{-1}\Gamma^\dagger\Omega\theta^{-1}J - \mathrm{i}\left(A_{[2]} - \tilde{A}_{[2]}\right) = 0. \quad (6.97)$$

From (6.97), we have

$$\tilde{A}_{[2]} = A_{[2]} + \mathrm{i}[J, \theta\Omega^{-1}\Gamma^\dagger\Omega\theta^{-1}]. \quad (6.98)$$

Substituting in (6.96),

$$\theta_x\Omega^{-1}\Gamma^\dagger\Omega\theta^{-1} - \theta\Omega^{-1}\Omega_x\Omega^{-1}\Gamma^\dagger\Omega\theta^{-1} + \theta\Omega^{-1}\Gamma^\dagger\Omega_x\theta^{-1} - \theta\Omega^{-1}\Gamma^\dagger\Omega\theta^{-1}\theta_x\theta^{-1} - A_{[2]}\theta\Omega^{-1}\Gamma^\dagger\Omega\theta^{-1} - \mathrm{i}J\theta\Omega^{-1}(\Gamma^\dagger)^2\Omega\theta^{-1} + \mathrm{i}\theta\Omega^{-1}\Gamma^\dagger\Omega\theta^{-1}J\theta\Omega^{-1}\Gamma^\dagger\Omega\theta^{-1} + \theta\Omega^{-1}\Gamma^\dagger\Omega\theta^{-1}A_{[2]} = 0. \quad (6.99)$$

The left-hand side can be shown to be zero in a similar manner to Section 6.4.2 - since $\hat{\theta}(x,y,t) = \hat{\theta}^r(x,t)e^{\mathrm{i}\Gamma^\dagger y}$ from a modified version of (5.68), we see that the dimensional reduction replaces $\partial_y^j(\hat{\theta})$ with $(\mathrm{i}^j)\hat{\theta}(\Gamma^\dagger)^j$. We use Corollary 4 Part I on page 93, but replace L by $\hat{L}$, θ by $\hat{\theta}$ and the constant C by $\hat{C}$.
Comparing (6.95) and (6.98), we see that

$$A_{[2]} = A - \mathrm{i}[J, \theta\Pi\theta^{-1}] - \mathrm{i}[J, \theta\Omega^{-1}\Gamma^\dagger\Omega\theta^{-1}]. \quad (6.100)$$

Although not immediately obvious, by once again introducing a 3×3 matrix $T = (t_{ij})$ $(i, j = 1, 2, 3)$ such that $A = [J, T]$ and performing some simple algebraic manipulation, we can see that this expression for $A_{[2]}$ can be written in quasi-Grammian form: (6.100) is

$$[J, T_{[2]}] = [J, T] - \mathrm{i}[J, \theta\Pi\theta^{-1}] - \mathrm{i}[J, \theta\Omega^{-1}\Gamma^\dagger\Omega\theta^{-1}], \quad (6.101)$$

i.e.

$$T_{[2]} = T - \mathrm{i}\theta\Pi\theta^{-1} - \mathrm{i}\theta\Omega^{-1}\Gamma^\dagger\Omega\theta^{-1}, \quad (6.102)$$

where $\Omega = \Omega(\theta, \rho)$. From a modified version of (5.47), we see that (omitting superscripts), $-\mathrm{i}\Gamma^\dagger\Omega = \rho^\dagger\theta + \mathrm{i}\Omega\Pi$, and hence (6.102) is

$$\begin{aligned} T_{[2]} &= T - \mathrm{i}\theta\Pi\theta^{-1} - \mathrm{i}\theta\Omega^{-1}(\mathrm{i}\rho^\dagger\theta - \Omega\Pi)\theta^{-1} \\ &= T + \theta\Omega^{-1}\rho^\dagger, \end{aligned} \quad (6.103)$$

which we express in quasi-Grammian form as

$$T_{[2]} = T - \begin{vmatrix} \Omega(\theta,\rho) & \rho^\dagger \\ \theta & \boxed{O_3} \end{vmatrix}, \tag{6.104}$$

where O_3 denotes the 3×3 zero matrix. After n repeated applications of the reduced binary Darboux transformation $G_{\theta,\rho}$, we obtain

$$T_{[n+1]} = T_{[n]} + \theta_{[n]}\Omega(\theta_{[n]},\rho_{[n]})^{-1}\rho_{[n]}^\dagger, \tag{6.105}$$

that is

$$T_{[n+1]} = T + \sum_{i=1}^{n} \theta_{[i]}\Omega(\theta_{[i]},\rho_{[i]})^{-1}\rho_{[i]}^\dagger, \tag{6.106}$$

where $T_{[1]} = T$, $\theta_{[1]} = \theta = \theta_{1[1]}$, $\theta_{[k]} = \theta_{k[k]}$ $(k = 2,\dots,n)$, and similarly $\rho_{[1]} = \rho = \rho_{1[1]}$, $\rho_{[k]} = \rho_{k[k]}$. By defining $\Theta = (\theta_1 \dots \theta_n)$ and $P = (\rho_1 \dots \rho_n)$, we can then, by induction, express $T_{[n+1]}$ in quasi-Grammian form as

$$T_{[n+1]} = T - \begin{vmatrix} \Omega(\Theta,P) & P^\dagger \\ \Theta & \boxed{O_3} \end{vmatrix}. \tag{6.107}$$

The inductive proof is carried out as in Chapter 3, Section 3.4.4.

In order to satisfy the skew-Hermitian condition, we require that

$$A_{[n+1]}^\dagger = -A_{[n+1]}, \tag{6.108}$$

so that

$$\left[J, T_{[n+1]}\right]^\dagger = -\left[J, T_{[n+1]}\right], \tag{6.109}$$

and hence

$$T_{[n+1]}^\dagger = T_{[n+1]}. \tag{6.110}$$

Using (6.107), we thus require

$$P\Omega(\Theta,P)^{-\dagger}\Theta^\dagger = \Theta\Omega(\Theta,P)^{-1}P^\dagger \tag{6.111}$$

since $A^\dagger = -A \Rightarrow T^\dagger = T$.

We firstly obtain an explicit expression for $\Omega^r(\Theta^r, P^r)$, the reduced form of $\Omega(\Theta, P)$. (Note that in the above, we have omitted superscripts for convenience, so that $\Omega(\Theta, P)$ is actually $\Omega^r(\Theta^r, P^r)$, a $(1+1)$-dimensional entity, where $\Theta^r = (\theta_1^r \ldots \theta_n^r)$, $P^r = (\rho_1^r \ldots \rho_n^r)$).

Consider the Lax operator L for the SSNLS equation, namely

$$L_{SSNLS} = \partial_x + \mathrm{i}\lambda J - A. \tag{6.112}$$

As described in the previous chapter, the dimensional reduction replaces ∂_y by $-\mathrm{i}\lambda$, so that, by applying this reduction in reverse, the Lax operator (6.112) comes about as a dimensional reduction of

$$L = \partial_x - J\partial_y - A. \tag{6.113}$$

The entries of A are now functions of x, y and t. With $\theta = \theta(x, y, t)$ an eigenfunction of L and $\rho = \rho(x, y, t)$ an eigenfunction of the adjoint operator $L^\dagger$, each of size $3 \times s$ for some s, we have

$$J\theta_y = \theta_x - A\theta \tag{6.114}$$

and

$$J\rho_y = \rho_x + A^\dagger \rho, \tag{6.115}$$

so that

$$\rho_y^\dagger J = \rho_x^\dagger + \rho^\dagger A. \tag{6.116}$$

In $(2+1)$-dimensions, from the definition of $\Omega(\theta, \rho)$ on page 49,

$$\Omega(\theta, \rho) = \int \rho^\dagger \theta \,\mathrm{d}y. \tag{6.117}$$

Now consider

$$\begin{aligned}
\left(\int \rho^\dagger J\theta \,\mathrm{d}x\right)_y &= \int \rho_y^\dagger J\theta \,\mathrm{d}x + \int \rho^\dagger J\theta_y \,\mathrm{d}x \\
&= \int (\rho_x^\dagger + \rho^\dagger A)\theta \,\mathrm{d}x + \int \rho^\dagger(\theta_x - A\theta) \,\mathrm{d}x \text{ by } (6.114), (6.116) \\
&= \int (\rho^\dagger \theta)_x \,\mathrm{d}x \\
&= \Omega(\theta, \rho)_y.
\end{aligned} \tag{6.118}$$

Hence

$$\Omega(\theta,\rho)=\int \rho^{\dagger}J\theta\,\mathrm{d}x+C \tag{6.119}$$

for some constant of integration C. Replacing each term by its reduced counterpart as defined in Section 5.5 gives

$$e^{-\mathrm{i}\Gamma^{\dagger}y}\Omega^{r}(\theta^{r},\rho^{r})e^{-\mathrm{i}\Pi y}=\int e^{-\mathrm{i}\Gamma^{\dagger}y}\rho^{r\dagger}J\theta^{r}e^{-\mathrm{i}\Pi y}\,\mathrm{d}x+C, \tag{6.120}$$

so that

$$\Omega^{r}(\theta^{r},\rho^{r})=\int \rho^{r\dagger}J\theta^{r}\,\mathrm{d}x+C, \tag{6.121}$$

where $\theta^{r}=\theta^{r}(x,t)$, $\rho^{r}=\rho^{r}(x,t)$, and thus we have an explicit expression for $\Omega^{r}(\theta^{r},\rho^{r})$ in $(1+1)$-dimension. Hence

$$\Omega^{r}(\Theta^{r},P^{r})=\int P^{r\dagger}J\Theta^{r}\,\mathrm{d}x+C \tag{6.122}$$

for $\Theta=(\theta_1\ \ldots\ \theta_n)$, $P=(\rho_1\ \ldots\ \rho_n)$. We define $C=(c_{ij})$ $(i,j=1,\ldots,ns)$ to be an $ns\times ns$ constant matrix, where, in the commutative case, each c_{ij} is a scalar (1×1) entry. Choosing C to have entries given by

$$c_{ij}=\begin{cases}1 & \text{for } i=j,\\ 0 & \text{otherwise,}\end{cases} \tag{6.123}$$

it then follows that

$$\Omega^{r}(\Theta^{r},P^{r})=\int P^{r\dagger}J\Theta^{r}\,\mathrm{d}x+I_{ns}. \tag{6.124}$$

We also define

$$P=\Theta S_2, \tag{6.125}$$

where S_2 is an invertible Hermitian $ns\times ns$ matrix with real entries. (We will see later that, as in the DS case, Θ, P satisfy the same dispersion relations, and hence this choice of Θ is a sensible one). Then the left-hand side of (6.111) is

$$\Theta S_2\Omega(\Theta,P)^{-\dagger}S_2{}^{-1}P^{\dagger},$$

which is equal to the right-hand side so long as $\Omega(\Theta,P)=S_2\Omega(\Theta,P)^{\dagger}S_2^{-1}$. This can easily be verified using (6.124) and (6.125), where we use the fact that S_2 is Hermitian. Hence (6.111) holds, and thus the skew-Hermitian condition is satisfied so long as $P=\Theta S_2$.

Looking back to our quasi-Grammian solution (6.107), as in the DS case discussed in Chapter 3, for notational convenience we define a general quasi-Grammian $R(i,j)$ $(i,j = 1,\dots,n)$ to be of the form

$$R(i,j) = (-1)^j \begin{vmatrix} \Omega(\Theta, P) & P^{(j)\dagger} \\ \Theta^{(i)} & \boxed{O_3} \end{vmatrix}, \tag{6.126}$$

so that (6.107) is given by

$$T_{[n+1]} = T - R(0,0), \tag{6.127}$$

which can be expressed in a more simple form as

$$T = T_0 - R(0,0), \tag{6.128}$$

where T_0 is any given solution of the ncSSNLS equation. We choose the trivial vacuum solution $T_0 = 0$ for simplicity.

Expanding (6.128) with $T_0 = 0$ via the usual expansion of a quasideterminant (2.10) gives

$$T = O_3 + \Theta\Omega(\Theta, P)^{-1}P^\dagger. \tag{6.129}$$

We have $\Theta = (\theta_1 \dots \theta_n)$, where each θ_i $(i = 1,\dots,n)$ is an eigenfunction of the Lax operators L, M of size $3 \times s$, and hence Θ is of size $3 \times N$ for some $N = ns$ to be chosen later. Define

$$\Theta = \begin{pmatrix} \boldsymbol{\Theta_1} \\ \boldsymbol{\Theta_2} \\ \boldsymbol{\Theta_3} \end{pmatrix}, \tag{6.130}$$

where the $\boldsymbol{\Theta_i}$ $(i = 1, 2, 3)$ are row vectors of arbitrary length N. Similarly, $P = (\rho_1 \dots \rho_n)$ and hence $P^\dagger = \mathrm{col}(\rho_1^\dagger \dots \rho_n^\dagger)$, where each ρ_i $(i = 1,\dots,n)$ is an eigenfunction of the adjoint Lax operators $L^\dagger$, $M^\dagger$ of size $3 \times s$. Then each $\rho_i^\dagger$ has 3 columns and an arbitrary number s of rows, and $P^\dagger$ is of size $N \times 3$, where $N = ns$ as above. Define

$$P^\dagger = \begin{pmatrix} \mathbf{P_1^\dagger} & \mathbf{P_2^\dagger} & \mathbf{P_3^\dagger} \end{pmatrix}, \tag{6.131}$$

where the $\mathbf{P_i^\dagger}$ $(i = 1, 2, 3)$ are column vectors of arbitrary length N.

Thus

$$T = O_3 + \begin{pmatrix} \boldsymbol{\Theta_1} \\ \boldsymbol{\Theta_2} \\ \boldsymbol{\Theta_3} \end{pmatrix} \Omega(\Theta, P)^{-1} \begin{pmatrix} \mathbf{P_1^\dagger} & \mathbf{P_2^\dagger} & \mathbf{P_3^\dagger} \end{pmatrix}$$

$$= - \begin{pmatrix} \begin{vmatrix} \Omega(\Theta, P) & \mathbf{P_1^\dagger} \\ \boldsymbol{\Theta_1} & \boxed{0} \end{vmatrix} & \begin{vmatrix} \Omega(\Theta, P) & \mathbf{P_2^\dagger} \\ \boldsymbol{\Theta_1} & \boxed{0} \end{vmatrix} & \begin{vmatrix} \Omega(\Theta, P) & \mathbf{P_3^\dagger} \\ \boldsymbol{\Theta_1} & \boxed{0} \end{vmatrix} \\ \begin{vmatrix} \Omega(\Theta, P) & \mathbf{P_1^\dagger} \\ \boldsymbol{\Theta_2} & \boxed{0} \end{vmatrix} & \begin{vmatrix} \Omega(\Theta, P) & \mathbf{P_2^\dagger} \\ \boldsymbol{\Theta_2} & \boxed{0} \end{vmatrix} & \begin{vmatrix} \Omega(\Theta, P) & \mathbf{P_3^\dagger} \\ \boldsymbol{\Theta_2} & \boxed{0} \end{vmatrix} \\ \begin{vmatrix} \Omega(\Theta, P) & \mathbf{P_1^\dagger} \\ \boldsymbol{\Theta_3} & \boxed{0} \end{vmatrix} & \begin{vmatrix} \Omega(\Theta, P) & \mathbf{P_2^\dagger} \\ \boldsymbol{\Theta_3} & \boxed{0} \end{vmatrix} & \begin{vmatrix} \Omega(\Theta, P) & \mathbf{P_3^\dagger} \\ \boldsymbol{\Theta_3} & \boxed{0} \end{vmatrix} \end{pmatrix}. \qquad (6.132)$$

Above, we introduced the 3×3 matrix T such that $A = [J, T]$. With $T = (t_{ij})$ $(i, j = 1, 2, 3)$, this gives

$$\begin{pmatrix} 0 & 0 & q \\ 0 & 0 & q^* \\ -q^\dagger & -q^{\dagger *} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2t_{13} \\ 0 & 0 & 2t_{23} \\ -2t_{31} & -2t_{32} & 0 \end{pmatrix}, \qquad (6.133)$$

so that, by comparing with (6.132), we have quasi-Grammian expressions for $q, q^*, q^\dagger$ and $q^{\dagger *}$, namely

$$q = -2 \begin{vmatrix} \Omega(\Theta, P) & \mathbf{P_3^\dagger} \\ \boldsymbol{\Theta_1} & \boxed{0} \end{vmatrix}, \qquad q^* = -2 \begin{vmatrix} \Omega(\Theta, P) & \mathbf{P_3^\dagger} \\ \boldsymbol{\Theta_2} & \boxed{0} \end{vmatrix}, \qquad (6.134a)$$

$$q^\dagger = -2 \begin{vmatrix} \Omega(\Theta, P) & \mathbf{P_1^\dagger} \\ \boldsymbol{\Theta_3} & \boxed{0} \end{vmatrix}, \qquad q^{\dagger *} = -2 \begin{vmatrix} \Omega(\Theta, P) & \mathbf{P_2^\dagger} \\ \boldsymbol{\Theta_3} & \boxed{0} \end{vmatrix}. \qquad (6.134b)$$

We have obtained a quasi-Grammian solution q of the ncSSNLS equation, along with its complex conjugate, adjoint and complex conjugate adjoint. To show that this solution is unique (i.e. to ensure that, for example, the two quasi-Grammians in (6.134a) are indeed complex conjugate, etc.), we utilise the conditions on the $\boldsymbol{\Theta_i}$ and $\mathbf{P_i}$ $(i = 1, 2, 3)$ obtained previously.

6.4.6 Uniqueness of solution

We summarise the conditions that have been determined earlier in the chapter in order to ensure that application of a Darboux transformation gives a transformed matrix with the correct structure. We have, from (6.79), that

$$\Theta = S\Theta^* S_1, \tag{6.135}$$

where S is the permutation matrix defined in (6.40) and S_1 is an $ns \times ns$ matrix such that $S_1^2 = I_{ns}$. In addition,

$$P = \Theta S_2, \tag{6.136}$$

for an $ns \times ns$ Hermitian matrix S_2. We assume both S_1 and S_2 are invertible with real entries (so that $S_2^T = S_2$ since S_2 is Hermitian). With the above conditions, along with our definition of $\Omega(\Theta, P)$ in (6.124), we can prove the uniqueness of our solution in (6.134). To do so, we require some results obtained later in the chapter. The calculations are rather tedious and have thus been detailed in Appendix A. (It is here that we also explain the reasoning behind our choice of dimensional reduction $\partial_y \to -\mathrm{i}\lambda$). We find that we must impose additional conditions on the matrices S_1 and S_2 such that

$$S_1 \text{ is orthogonal, i.e. } S_1 S_1^T = I_{ns}, \text{ so that } S_1^T = S_1^{-1}, \tag{6.137a}$$

$$S_1 \text{ and } S_2 \text{ commute, so that } S_1 S_2 = S_2 S_1. \tag{6.137b}$$

Thus the ncSSNLS equation (6.47) has a unique quasi-Grammian solution, namely

$$q = -2 \begin{vmatrix} \Omega(\Theta, P) & \mathbf{P_3^\dagger} \\ \mathbf{\Theta_1} & \boxed{0} \end{vmatrix}, \tag{6.138}$$

so long as $\Theta = S\Theta^* S_1$ and $P = \Theta S_2$ for S defined in (6.40), where S_2 is an $ns \times ns = N \times N$ Hermitian matrix with real entries, and S_1 and S_2 satisfy (6.137). We do not go into details here, however we can verify that this quasi-Grammian is indeed a solution of the ncSSNLS equation in a similar manner to the DS case in Chapter 3.

6.5 Soliton solutions - commutative case

We are now in a position to determine and plot the soliton solutions of our ncSSNLS equation (6.47). To begin, we firstly choose to revert to the commutative case, where calculations are somewhat more straightforward. Although our main interest lies in solutions in a noncommutative situation, the commutative version of the SSNLS equation and its soliton solutions has received little attention, with only the original paper by Sasa and Satsuma [79] and work by Gilson *et al.* [33] making advances in this area. We therefore choose to focus more on the commutative situation here than we did in the DS case in Chapter 3. We obtain soliton solutions in a commutative setting, which then give us the framework to extend to the noncommutative case and obtain corresponding soliton plots in a noncommutative setting.

6.5.1 n-soliton solution

As in the DS case in Chapter 3, we consider our quasi-Grammian solution

$$q = -2 \begin{vmatrix} \Omega(\Theta, P) & \mathbf{P_3}^\dagger \\ \mathbf{\Theta_1} & \boxed{0} \end{vmatrix}, \tag{6.139}$$

where $\mathbf{\Theta_1}$ denotes the first row of our matrix of eigenfunctions Θ, and similarly $\mathbf{P_3}^\dagger$ is the third column of $P^\dagger$, i.e. the Hermitian conjugate of the third row of P, where P is the matrix of adjoint eigenfunctions. Since we are considering the commutative case, $\Theta = (\theta_1 \, \dots \, \theta_n)$ consists of matrices θ_i $(i = 1, \dots, n)$, each an eigenfunction of the Lax operators L, M, of size $3 \times s$ with scalar (1×1) entries, and in a similar manner, $P = (\rho_1 \, \dots \, \rho_n)$ contains matrices ρ_i $(i = 1, \dots, n)$, each eigenfunctions of $L^\dagger, M^\dagger$, of size $3 \times s$ with scalar entries. Hence $\Omega(\Theta, P) = \int P^\dagger J \Theta \, \mathrm{d}x + I_N$ is an $N \times N$ matrix with scalar entries, where $N = ns$ for some s to be chosen. Later, when we move on to the noncommutative case, we will suppose that each θ_i, ρ_i have entries of matrix form.

In order to obtain a specific quasi-Grammian solution in the commutative case, we must choose the constant s mentioned above. We consider the relatively simple case of $s = 4$, so that each θ_i $(i = 1, \dots, n)$ is a 3×4 matrix, as is each ρ_i $(i = 1, \dots, n)$. The θ_i and ρ_i have scalar entries in the commutative setting.

Define, for $i = 1, \dots, n$,

$$\theta_i = \begin{pmatrix} \phi_{4i-3} & \phi_{4i-2} & \phi_{4i-1} & \phi_{4i} \\ \psi_{4i-3} & \psi_{4i-2} & \psi_{4i-1} & \psi_{4i} \\ \chi_{4i-3} & \chi_{4i-2} & \chi_{4i-1} & \chi_{4i} \end{pmatrix} \tag{6.140}$$

for functions $\phi = \phi(x,t)$, $\psi = \psi(x,t)$, $\chi = \chi(x,t)$, so that

$$\Theta = \begin{pmatrix} \theta_1 & \theta_2 & \dots & \theta_{n-1} & \theta_n \end{pmatrix} = \begin{pmatrix} \phi_1 & \phi_2 & \dots & \phi_{4n-1} & \phi_{4n} \\ \psi_1 & \psi_2 & \dots & \psi_{4n-1} & \psi_{4n} \\ \chi_1 & \chi_2 & \dots & \chi_{4n-1} & \chi_{4n} \end{pmatrix}. \tag{6.141}$$

The entries of Θ satisfy the dispersion relation for the system, calculated as follows. Since θ is an eigenfunction of the Lax operator L (6.19a), $L(\theta) = 0$, and hence $\theta_x = -\mathrm{i}\lambda J\theta$. Then

$$\begin{aligned} \theta_{xx} &= -\mathrm{i}\lambda J\theta_x \\ &= -\lambda^2\theta \end{aligned} \tag{6.142}$$

by the definition of θ_x above, and thus

$$\begin{aligned} \theta_{xxx} &= -\lambda^2(-\mathrm{i}\lambda J\theta) \\ &= \mathrm{i}\lambda^3 J\theta. \end{aligned} \tag{6.143}$$

Then, since θ is also an eigenfunction of the Lax operator M (6.19b), we have

$$\begin{aligned} \theta_t &= -4\mathrm{i}\lambda^3\epsilon J\theta \\ &= -4\epsilon\theta_{xxx}. \end{aligned} \tag{6.144}$$

It then follows that, for $\Theta = (\theta_1 \dots \theta_n)$, $\Theta_t = -4\epsilon\Theta_{xxx}$, hence, for $i = 1, \dots, 4n$, we have

$$(\phi_i)_t = -4\epsilon\,(\phi_i)_{xxx}\,, \tag{6.145a}$$

$$(\psi_i)_t = -4\epsilon\,(\psi_i)_{xxx}\,, \tag{6.145b}$$

$$(\chi_i)_t = -4\epsilon\,(\chi_i)_{xxx}\,. \tag{6.145c}$$

From now on we choose the real constant $\epsilon = 1$ for simplicity.

We know, from (6.137), that S_1 and S_2 are $ns \times ns = 4n \times 4n$ matrices satisfying particular conditions in order to ensure uniqueness of our quasi-Grammian solution, namely S_1 must be an orthogonal matrix whose square is the identity, and S_1 and S_2 must commute. In addition, we assume that both S_1 and S_2 have real entries, with S_2 chosen to be Hermitian. For simplicity, we choose S_1 to be a permutation matrix such that

$$S_1 = \mathrm{diag}(\underbrace{\tilde{S}_1, \tilde{S}_1, \dots, \tilde{S}_1}_{n}), \tag{6.146}$$

where

$$\tilde{S}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{6.147}$$

Then, defining

$$S_2 = \mathrm{diag}(\underbrace{\tilde{S}_2, \tilde{S}_2, \ldots, \tilde{S}_2}_{n}), \tag{6.148}$$

where $\tilde{S}_2 = (s_{ij})$ $(i, j = 1, \ldots, 4)$ is a 4×4 Hermitian matrix, (6.137b) implies that $\tilde{S}_2$ takes the form

$$\tilde{S}_2 = \begin{pmatrix} s_{11} & s_{12} & s_{13} & s_{14} \\ s_{12} & s_{11} & s_{14} & s_{13} \\ s_{13} & s_{14} & s_{33} & s_{34} \\ s_{14} & s_{13} & s_{34} & s_{33} \end{pmatrix}. \tag{6.149}$$

We choose

$$\tilde{S}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{6.150}$$

(Note that although we could have conceivably left $\tilde{S}_2$ as an arbitrary 4×4 matrix for now, when we later express our quasi-Grammian solution as a ratio of determinants and expand the denominator using Mathematica in order to find conditions so that this denominator is non-zero, the obtained expression is extremely unwieldy. We therefore choose our matrix $\tilde{S}_2$ at this stage, and firstly check, using Mathematica, that this gives a non-zero *numerator* in our ratio expression. Having done so, we then find conditions on our parameters in order that the denominator of the ratio is non-zero with this particular choice of $\tilde{S}_2$. The choice of $\tilde{S}_2$ is not unique, however it has the important benefit that the numerator of our ratio is non-trivial - other choices of this matrix, such as $\tilde{S}_2 = I_4$ or $\tilde{S}_2 = \tilde{S}_1$, result in a trivial numerator). It then follows that, since $P = \Theta S_2$, we have, for $i = 1, \ldots, n$,

$$\begin{aligned} \rho_i &= \theta_i \tilde{S}_2 \\ &= \begin{pmatrix} \phi_{4i-1} & \phi_{4i} & \phi_{4i-3} & \phi_{4i-2} \\ \psi_{4i-1} & \psi_{4i} & \psi_{4i-3} & \psi_{4i-2} \\ \chi_{4i-1} & \chi_{4i} & \chi_{4i-3} & \chi_{4i-2} \end{pmatrix}, \end{aligned} \tag{6.151}$$

so that

$$P = (\rho_1, \dots, \rho_n) = \begin{pmatrix} \phi_3 & \phi_4 & \phi_1 & \phi_2 & \dots & \phi_{4n-1} & \phi_{4n} & \phi_{4n-3} & \phi_{4n-2} \\ \psi_3 & \psi_4 & \psi_1 & \psi_2 & \dots & \psi_{4n-1} & \psi_{4n} & \psi_{4n-3} & \psi_{4n-2} \\ \chi_3 & \chi_4 & \chi_1 & \chi_2 & \dots & \chi_{4n-1} & \chi_{4n} & \chi_{4n-3} & \chi_{4n-2} \end{pmatrix}. \tag{6.152}$$

Thus our quasi-Grammian solution (6.139) takes the form

$$q = -2 \begin{vmatrix} \Omega(\Theta, P) & \chi^\dagger \\ \phi & \boxed{0} \end{vmatrix}, \tag{6.153}$$

where ϕ, χ denote the row vectors

$$\phi = \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 & \phi_4 & \dots & \phi_{4n-4} & \phi_{4n-3} & \phi_{4n-1} & \phi_{4n} \end{pmatrix}, \tag{6.154a}$$

$$\chi = \begin{pmatrix} \chi_3 & \chi_4 & \chi_1 & \chi_2 & \dots & \chi_{4n-1} & \chi_{4n} & \chi_{4n-3} & \chi_{4n-2} \end{pmatrix} \tag{6.154b}$$

respectively.

As in our work on the DS equations in Chapter 3, we choose to simplify our notation so that we work with only $\phi_1, \dots, \phi_{2n}$, $\psi_1, \dots, \psi_{2n}$ and $\chi_1, \dots, \chi_{2n}$. For $i = 1, \dots, 4n$, we relabel

$$\phi_j \text{ as } \phi_{2i} \text{ for } j = 4i,$$

$$\phi_j \text{ as } \phi_{2i-1} \text{ for } j = 4i - 1,$$

and set $\phi_j = 0$ for $j = 4i - 2$, $j = 4i - 3$. The same conditions also hold for $\psi_1, \dots, \psi_{4n}$. We also relabel

$$\chi_j \text{ as } \chi_{2i} \text{ for } j = 4i - 2,$$

$$\chi_j \text{ as } \chi_{2i-1} \text{ for } j = 4i - 3,$$

and set $\chi_j = 0$ for $j = 4i, j = 4i-1$. We thus have

$$\begin{aligned}\phi &= \begin{pmatrix}\phi_1 & \phi_2 & \phi_3 & \phi_4 & \phi_5 & \phi_6 & \phi_7 & \phi_8 & \dots & \phi_{4n-3} & \phi_{4n-2} & \phi_{4n-1} & \phi_{4n}\end{pmatrix}\\ &= \begin{pmatrix}0 & 0 & \phi_1 & \phi_2 & 0 & 0 & \phi_3 & \phi_4 & \dots & 0 & 0 & \phi_{2n-1} & \phi_{2n}\end{pmatrix}, \end{aligned} \tag{6.155a}$$

$$\begin{aligned}\psi &= \begin{pmatrix}\psi_1 & \psi_2 & \psi_3 & \psi_4 & \psi_5 & \psi_6 & \psi_7 & \psi_8 & \dots & \psi_{4n-3} & \psi_{4n-2} & \psi_{4n-1} & \psi_{4n}\end{pmatrix}\\ &= \begin{pmatrix}0 & 0 & \psi_1 & \psi_2 & 0 & 0 & \psi_3 & \psi_4 & \dots & 0 & 0 & \psi_{2n-1} & \psi_{2n}\end{pmatrix}, \end{aligned} \tag{6.155b}$$

$$\begin{aligned}\chi &= \begin{pmatrix}\chi_1 & \chi_2 & \chi_3 & \chi_4 & \chi_5 & \chi_6 & \chi_7 & \chi_8 & \dots & \chi_{4n-3} & \chi_{4n-2} & \chi_{4n-1} & \chi_{4n}\end{pmatrix}\\ &= \begin{pmatrix}\chi_1 & \chi_2 & 0 & 0 & \chi_3 & \chi_4 & 0 & 0 & \dots & \chi_{2n-1} & \chi_{2n} & 0 & 0\end{pmatrix}. \end{aligned} \tag{6.155c}$$

Our choice of the matrix $\Theta = \mathrm{col}\begin{pmatrix}\mathbf{\Theta_1} & \mathbf{\Theta_2} & \mathbf{\Theta_3}\end{pmatrix}$ must satisfy the condition (6.135), so that

$$\mathbf{\Theta_1} = \mathbf{\Theta_2}^* S_1, \tag{6.156a}$$

$$\mathbf{\Theta_2} = \mathbf{\Theta_1}^* S_1, \tag{6.156b}$$

$$\mathbf{\Theta_3} = \mathbf{\Theta_3}^* S_1. \tag{6.156c}$$

In order that these conditions are satisfied, we choose

$$\phi = \begin{pmatrix}0 & 0 & \phi_1 & \psi_1^* & 0 & 0 & \phi_2 & \psi_2^* & \dots & 0 & 0 & \phi_{2n-1} & \psi_{2n-1}^*\end{pmatrix}, \tag{6.157a}$$

$$\psi = \begin{pmatrix}0 & 0 & \psi_1 & \phi_1^* & 0 & 0 & \psi_2 & \phi_2^* & \dots & 0 & 0 & \psi_{2n-1} & \phi_{2n-1}^*\end{pmatrix}, \tag{6.157b}$$

$$\chi = \begin{pmatrix}\chi_1 & \chi_1^* & 0 & 0 & \chi_2 & \chi_2^* & 0 & 0 & \dots & \chi_{2n-1} & \chi_{2n-1}^* & 0 & 0\end{pmatrix}, \tag{6.157c}$$

i.e. for even $i = 2, 4, \dots, 2n-2, 2n$, we choose $\phi_i = \psi_{i-1}^*$ and $\chi_i = \chi_{i-1}^*$, so that

$$\Theta = \begin{pmatrix}0 & 0 & \phi_1 & \psi_1^* & \dots & 0 & 0 & \phi_{2n-1} & \psi_{2n-1}^*\\ 0 & 0 & \psi_1 & \phi_1^* & \dots & 0 & 0 & \psi_{2n-1} & \phi_{2n-1}^*\\ \chi_1 & \chi_1^* & 0 & 0 & \dots & \chi_{2n-1} & \chi_{2n-1}^* & 0 & 0\end{pmatrix}. \tag{6.158}$$

It then follows that $P = \Theta S_2$ is given by

$$P = \begin{pmatrix}\phi_1 & \psi_1^* & 0 & 0 & \dots & \phi_{2n-1} & \psi_{2n-1}^* & 0 & 0\\ \psi_1 & \phi_1^* & 0 & 0 & \dots & \psi_{2n-1} & \phi_{2n-1}^* & 0 & 0\\ 0 & 0 & \chi_1 & \chi_1^* & \dots & 0 & 0 & \chi_{2n-1} & \chi_{2n-1}^*\end{pmatrix}. \tag{6.159}$$

We choose the entries of ϕ, ψ, χ to satisfy the dispersion relations for the SSNLS equation as given in (6.145) with $\epsilon = 1$. We therefore choose, for $i = 1, \ldots, 2n$,

$$\phi_i = a\alpha_i, \tag{6.160a}$$

$$\psi_i = a\beta_i, \tag{6.160b}$$

$$\chi_i = \gamma_i, \tag{6.160c}$$

where a is a real constant (to be defined later), and $\alpha_i, \beta_i, \gamma_i$ are exponentials such that

$$\alpha_i = \exp(p_i x - 4p_i^3 t), \tag{6.161a}$$

$$\beta_i = \exp(q_i x - 4q_i^3 t), \tag{6.161b}$$

$$\gamma_i = \exp(r_i x - 4r_i^3 t) \tag{6.161c}$$

for complex constants p_i, q_i, r_i. We will later derive conditions on the p_i, q_i, r_i in order to give the correct asymptotic behaviour. Thus we have the quasi-Grammian solution in the commutative case defined to be

$$q = -2\begin{vmatrix} \Omega(\Theta, P) & \tilde{\chi}^\dagger \\ \phi & \boxed{0} \end{vmatrix}, \tag{6.162}$$

where ϕ, $\tilde{\chi}$ denote the row vectors

$$\phi = \begin{pmatrix} 0 & 0 & \phi_1 & \psi_1^* & \ldots & 0 & 0 & \phi_{2n-1} & \psi_{2n-1}^* \end{pmatrix}, \tag{6.163a}$$

$$\tilde{\chi} = \begin{pmatrix} 0 & 0 & \chi_1 & \chi_1^* & \ldots & 0 & 0 & \chi_{2n-1} & \chi_{2n-1}^* \end{pmatrix} \tag{6.163b}$$

respectively, and the ϕ_i, χ_i ($i = 1, \ldots, 2n$) are as defined as in (6.160), (6.161). Since we are considering a commutative situation, we can express the quasi-Grammian in (6.162) as

$$\begin{aligned} q &= -2\frac{\begin{vmatrix} \Omega(\Theta, P) & \tilde{\chi}^\dagger \\ \phi & 0 \end{vmatrix}}{\begin{vmatrix} \Omega(\Theta, P) \end{vmatrix}} \\ &= -2\frac{G}{F}, \text{ say.} \end{aligned} \tag{6.164}$$

As in the DS case, we obtain an explicit expression for the determinant F by setting

$$\begin{aligned} \mathcal{F} &= \Omega(\Theta, P) \\ &= \int P^\dagger J \Theta \, \mathrm{d}x + I_{4n} \text{ by (6.124), omitting superscripts,} \end{aligned} \tag{6.165}$$

so that $F = \det \mathcal{F}$. Then, since $P = \Theta S_2$, it follows that

$$\mathcal{F} = S_2 \int \Theta^\dagger J \Theta \, \mathrm{d}x + I_{4n}. \tag{6.166}$$

Substituting for Θ using (6.158), we see that

$$\mathcal{F} = S_2 \Psi + I_{4n} \tag{6.167}$$

for

$$\Psi = \begin{pmatrix} -\int^x \chi^{1,1} & O_2 & \dots & -\int^x \chi^{2n-1,1} & O_2 \\ O_2 & \int^x \phi^{1,1}\psi^{1,1} & \dots & O_2 & \int^x \phi^{1,2n-1}\psi^{1,2n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ -\int^x \chi^{2n-1,1} & O_2 & \dots & -\int^x \chi^{2n-1,2n-1} & O_2 \\ O_2 & \int^x \phi^{2n-1,1}\psi^{2n-1,1} & \dots & O_2 & \int^x \phi^{2n-1,2n-1}\psi^{2n-1,2n-1} \end{pmatrix}, \tag{6.168}$$

where all integrals are with respect to x, and the limits of integration are determined from the definitions of ϕ_j, ψ_j, χ_j ($j = 1, \dots, 2n-1$) in (6.160), (6.161). For notational convenience, we have defined, for $i, j = 1, \dots, 2n-1$,

$$\chi^{i,j} := \begin{pmatrix} \chi_i^* \chi_j & \chi_i^* \chi_j^* \\ \chi_i \chi_j & \chi_i \chi_j^* \end{pmatrix}, \qquad \phi^{i,j}\psi^{i,j} := \begin{pmatrix} \phi_i^*\phi_j + \psi_i^*\psi_j & \phi_i^*\psi_j^* + \psi_i^*\phi_j^* \\ \psi_i\phi_j + \phi_i\psi_j & \psi_i\psi_j^* + \phi_i\phi_j^* \end{pmatrix}. \tag{6.169}$$

Thus

$$F = \det \mathcal{F} = | I_{4n} + S_2 \Psi | \, . \tag{6.170}$$

We now consider the simplest example of the 1-soliton solution ($n = 1$) and derive appropriate conditions to ensure that this solution, expressed as a ratio of determinants, has a non-zero denominator.

6.5.2 1-soliton solution

Choosing $n = 1$, we have, from (6.158),

$$\Theta = \theta_1 = \begin{pmatrix} 0 & 0 & \phi_1 & \psi_1^* \\ 0 & 0 & \psi_1 & \phi_1^* \\ \chi_1 & \chi_1^* & 0 & 0 \end{pmatrix}, \tag{6.171}$$

while from (6.159),

$$P = \rho_1 = \begin{pmatrix} \phi_1 & \psi_1^* & 0 & 0 \\ \psi_1 & \phi_1^* & 0 & 0 \\ 0 & 0 & \chi_1 & \chi_1^* \end{pmatrix}, \tag{6.172}$$

with

$$\phi_1 = a \exp\left(p_1 x - 4p_1^3 t\right), \tag{6.173a}$$

$$\psi_1 = a \exp\left(q_1 x - 4q_1^3 t\right) \tag{6.173b}$$

and

$$\chi_1 = \exp\left(r_1 x - 4r_1^3 t\right) \tag{6.173c}$$

for a real constant a. From (6.168), (6.169),

$$\Psi = \begin{pmatrix} -\int^x \chi_1^* \chi_1 \,\mathrm{d}x & -\int^x \chi_1^* \chi_1^* \,\mathrm{d}x & 0 & 0 \\ -\int^x \chi_1 \chi_1 \,\mathrm{d}x & -\int^x \chi_1 \chi_1^* \,\mathrm{d}x & 0 & 0 \\ 0 & 0 & \int^x \phi_1^*\phi_1 + \psi_1^*\psi_1 \,\mathrm{d}x & \int^x \phi_1^*\psi_1^* + \psi_1^*\phi_1^* \,\mathrm{d}x \\ 0 & 0 & \int^x \psi_1\phi_1 + \phi_1\psi_1 \,\mathrm{d}x & \int^x \psi_1\psi_1^* + \phi_1\phi_1^* \,\mathrm{d}x \end{pmatrix}. \tag{6.174}$$

Choosing $p_1 = q_1$ in (6.173) for simplicity, it then follows from (6.170) that

$$F = \begin{vmatrix} 1 & 0 & 2\int^x \phi_1^*\phi_1 \,\mathrm{d}x & 2\int^x \phi_1^*\phi_1^* \,\mathrm{d}x \\ 0 & 1 & 2\int^x \phi_1\phi_1 \,\mathrm{d}x & 2\int^x \phi_1\phi_1^* \,\mathrm{d}x \\ -\int^x \chi_1^*\chi_1 \,\mathrm{d}x & -\int^x \chi_1^*\chi_1^* \,\mathrm{d}x & 1 & 0 \\ -\int^x \chi_1\chi_1 \,\mathrm{d}x & -\int^x \chi_1\chi_1^* \,\mathrm{d}x & 0 & 1 \end{vmatrix}. \tag{6.175}$$

Since our quasi-Grammian solution q is expressed as a ratio of determinants with denominator F as in (6.164), we must ensure that F is non-zero. Expanding gives

$$F = 1 + Pe^{\eta_1+\eta_1^*+\xi_1+\xi_1^*} + Qe^{2(\eta_1+\xi_1^*)} + Q^*e^{2(\eta_1^*+\xi_1)} + Re^{2(\eta_1+\eta_1^*+\xi_1+\xi_1^*)}, \tag{6.176}$$

where

$$P = \frac{1+a^2}{2\Re(p_1)\Re(r_1)}, \tag{6.177a}$$

$$Q = \frac{a}{2p_1 r_1^*}, \tag{6.177b}$$

$$R = \frac{1+2a^2+a^4}{16\left(\Re(p_1)\right)^2}\left(\frac{1}{\left(\Re(r_1)\right)^2} - \frac{1}{r_1 r_1^*}\right) + \frac{a^2}{4p_1 p_1^*}\left(\frac{1}{r_1 r_1^*} - \frac{1}{\left(\Re(r_1)\right)^2}\right), \tag{6.177c}$$

and

$$\eta_1 = p_1 x - 4p_1^3 t, \tag{6.178a}$$

$$\xi_1 = r_1 x - 4r_1^3 t. \tag{6.178b}$$

Removing a factor of $e^{\eta_1+\eta_1^*+\xi_1+\xi_1^*} = e^{2(\Re(\eta_1)+\Re(\xi_1))}$, which we will denote by e^A, say, we have

$$F = e^A \left(P + Qe^B + Q^* e^{-B} + R^{\frac{1}{2}} \left(R^{\frac{1}{2}} e^A + R^{-\frac{1}{2}} e^{-A}\right)\right), \tag{6.179}$$

where $B = \eta_1 - \eta_1^* - (\xi_1 - \xi_1^*) = 2\mathrm{i}(\Im(\eta_1) - \Im(\xi_1))$. It can easily be seen that P, R are purely real. We let

$$Q = Q_1 + \mathrm{i}Q_2 \tag{6.180}$$

for $Q_1, Q_2 \in \mathbb{R}$, so that

$$\begin{aligned} Qe^B + Q^* e^{-B} &= Q_1 \left(e^B + e^{-B}\right) + \mathrm{i}Q_2 \left(e^B - e^{-B}\right) \\ &= 2\left(Q_1 \cos(2\omega) - Q_2 \sin(2\omega)\right), \end{aligned} \tag{6.181}$$

where $\omega = \Im(\eta_1) - \Im(\xi_1) \in \mathbb{R}$. Thus

$$F = e^A \left(P + 2\left(Q_1 \cos(2\omega) - Q_2 \sin(2\omega)\right) + R^{\frac{1}{2}} \left(R^{\frac{1}{2}} e^A + R^{-\frac{1}{2}} e^{-A}\right)\right), \tag{6.182}$$

where $A, P, Q_1, Q_2, R \in \mathbb{R}$. For $F \neq 0$ (in particular $F > 0$), we require that

$$P + 2\left(Q_1 \cos(2\omega) - Q_2 \sin(2\omega)\right) + R^{\frac{1}{2}} \left(R^{\frac{1}{2}} e^A + R^{-\frac{1}{2}} e^{-A}\right) > 0. \tag{6.183}$$

Thus, since $R^{\frac{1}{2}} e^A + R^{-\frac{1}{2}} e^{-A} > 0$, we can take

$$P + R^{\frac{1}{2}} + 2\,|\,Q_1\,| > 2\,|\,Q_2\,| \tag{6.184}$$

for $F > 0$. (There are, however, other possibilities, but the inequality in (6.184) is sufficient to ensure $F > 0$).

We now explain the reason for the inclusion of the real constant a in our definitions of ϕ_i and ψ_i in (6.160). From (6.176), we see that the terms in Q and Q^* are a conjugate pair, hence their sum is real, and every other term in the expansion is real. Thus F is purely real, no matter our choice of a. Expanding G in a similar manner gives

$$\begin{aligned} G = &- \left(a e^{\eta_1+\xi_1^*} + e^{\eta_1^*+\xi_1}\right) \\ &- e^{2(\eta_1+\xi_1^*)+(\eta_1^*+\xi_1)} \left(\frac{a}{2p_1 r_1^*} - \frac{a(1+a^2)}{4\Re(p_1) r_1^*} - \frac{a}{2p_1 \Re(r_1)} + \frac{a(1+a^2)}{4\Re(p_1)\Re(r_1)}\right) \\ &- e^{(\eta_1+\xi_1^*)+2(\eta_1^*+\xi_1)} \left(\frac{a^2}{2p_1^* r_1} - \frac{1+a^2}{4\Re(p_1) r_1} - \frac{a^2}{2p_1^* \Re(r_1)} + \frac{1+a^2}{4\Re(p_1)\Re(r_1)}\right). \end{aligned} \tag{6.185}$$

It is clear that, for $a = 1$, each term in the expansion has a corresponding conjugate partner, and thus G is purely real. Similarly, for $a = -1$, it can be seen that G is purely imaginary. However, for $a \neq \pm 1$, the terms do not combine in conjugate pairs, and hence G consists of a real and imaginary part. Then $q = -2G/F$ is purely real for $a = 1$ (F and G both real), purely imaginary for $a = -1$ (F real, G imaginary), and complex (consisting of a real and imaginary part) for $a \neq \pm 1$. Since we wish our solution q to be complex, we choose $a \neq \pm 1$. In the plots that follow, we choose $a = 2$.

In order to obtain plots of the 1-soliton solution in the commutative case, we choose the complex constants p_1, r_1 so that the condition (6.184) holds. For various values of these constants, we show a two-dimensional plot at one particular time value ($t = 0.1$), along with three-dimensional plots where we see the soliton moving along the x-axis as time progresses, giving rise to a 'train' of solitons. These plots are shown in Figures 6.1-6.3 below. The two-dimensional plots do not take the form of a smooth curve, however this is due to the fact that we are plotting a complex solution with an imaginary part of an oscillatory nature. Figure 6.1 shows the soliton train moving from negative to positive values of x as time progresses, while Figure 6.2 shows the soliton moving in the opposite direction. The soliton is almost stationary in Figure 6.3.

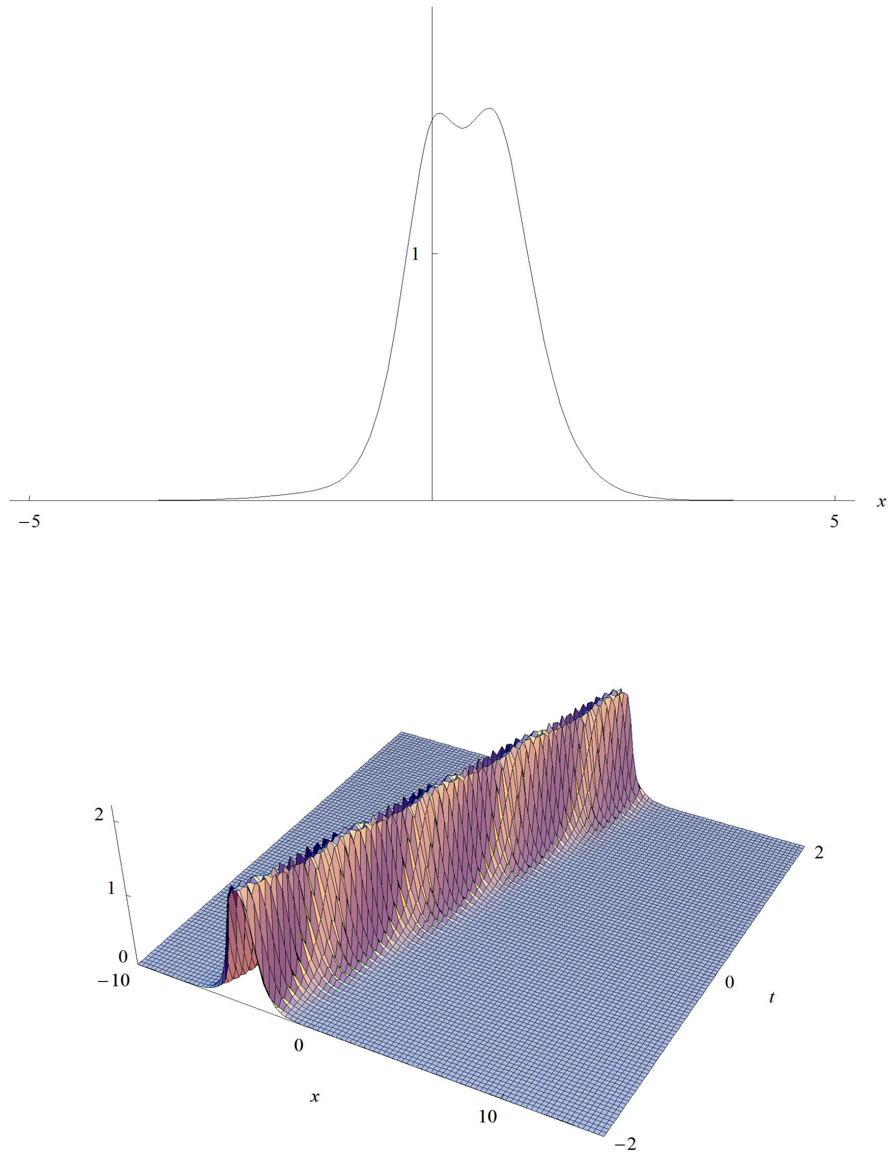


Figure 6.1: $p_1 = 1.9 + \mathrm{i}$, $r_1 = 0.5 + 0.3\mathrm{i}$.

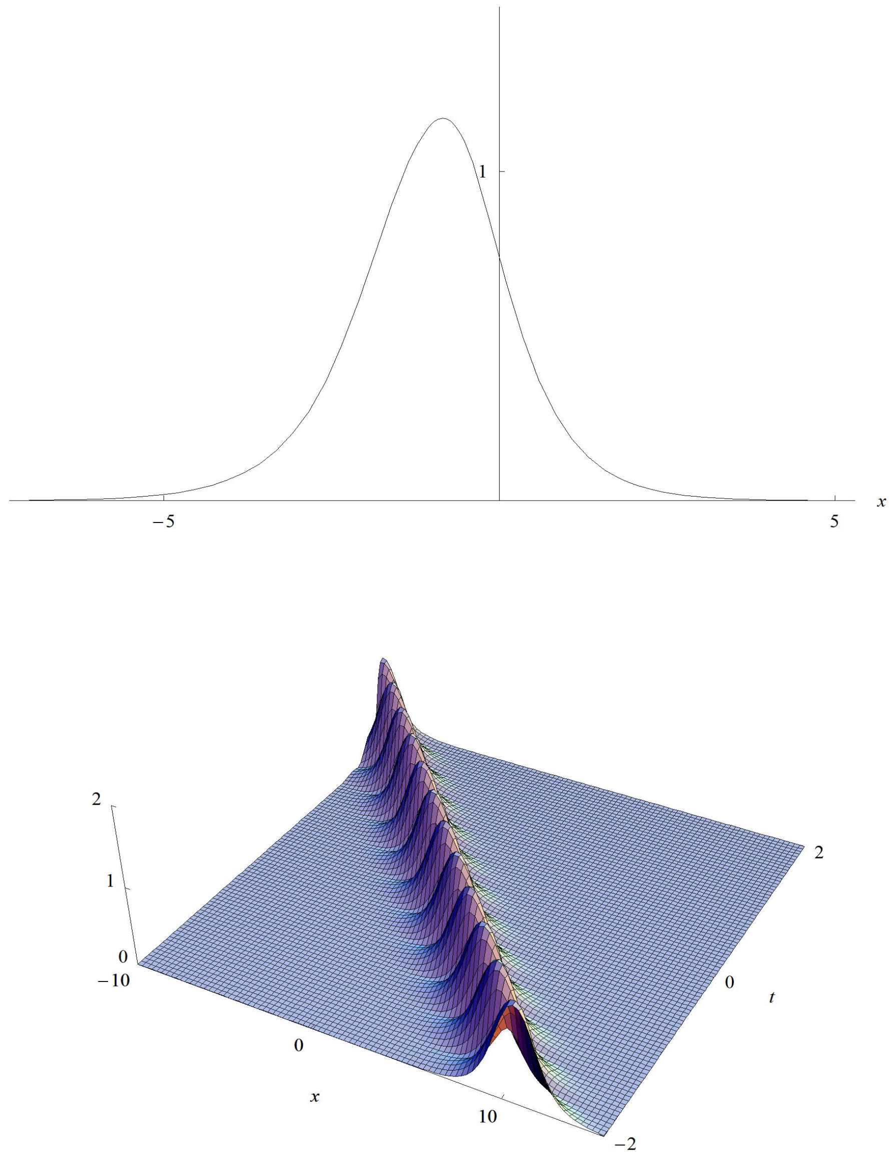


Figure 6.2: $p_1 = 1 - 0.8\mathrm{i}$, $r_1 = 0.3 - \mathrm{i}$.

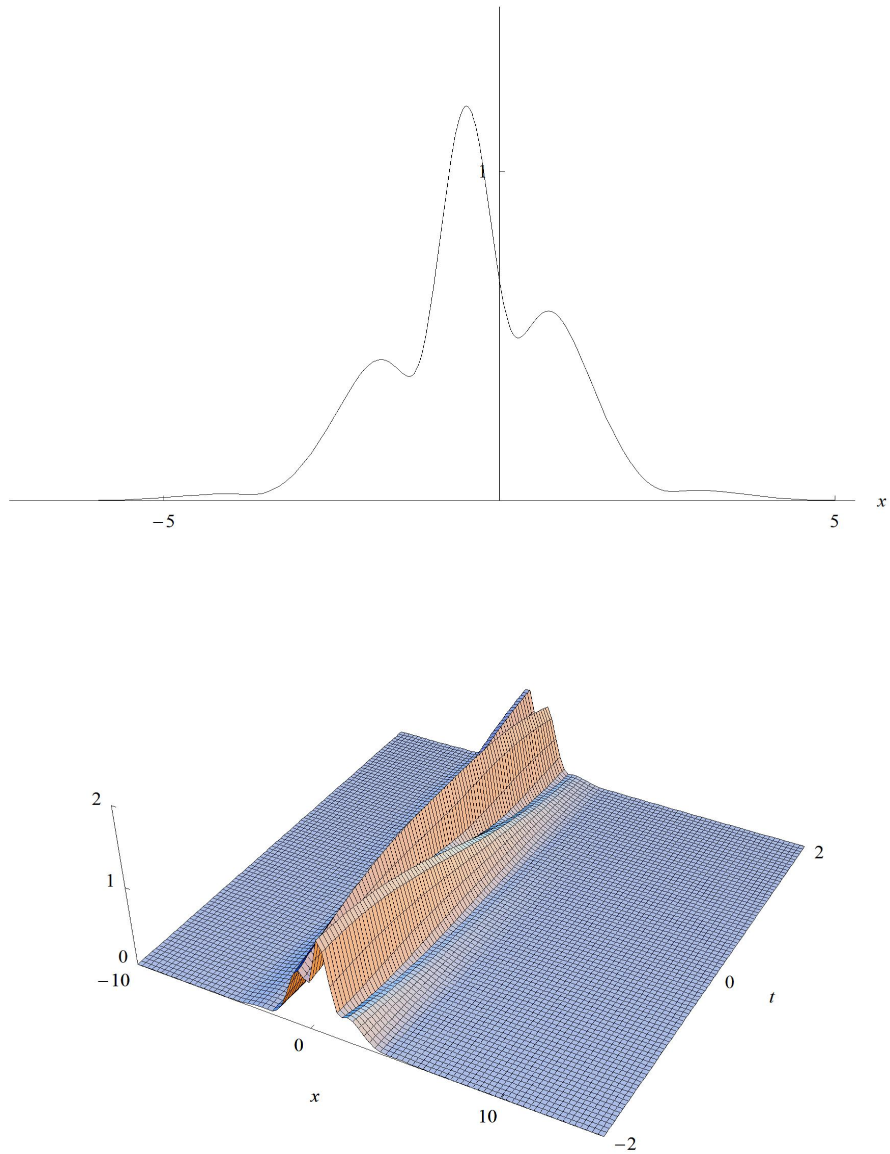


Figure 6.3: $p_1 = 1 + 0.3\mathrm{i}$, $r_1 = 0.3 - \mathrm{i}$.

6.6 Analysis of the 1-soliton solution

6.6.1 Soliton speed

In order to determine the factors governing the speed of the solitons we have plotted, we firstly consider the determinant F as given in (6.175). By removing appropriate factors from the rows and columns of this determinant, we find that

$$F = e^{2(\eta_1+\eta_1^*+\xi_1+\xi_1^*)} \begin{vmatrix} e^{-(\eta_1^*+\xi_1)} & 0 & \frac{a^2}{\Re(p_1)} & \frac{a^2}{p_1^*} \\ 0 & e^{-(\eta_1+\xi_1^*)} & \frac{a^2}{p_1} & \frac{a^2}{\Re(p_1)} \\ -\frac{1}{2\Re(r_1)} & -\frac{1}{2r_1^*} & e^{-(\eta_1+\xi_1^*)} & 0 \\ -\frac{1}{2r_1} & -\frac{1}{2\Re(r_1)} & 0 & e^{-(\eta_1^*+\xi_1)} \end{vmatrix}. \tag{6.186}$$

In a similar manner, we can show that

$$G = e^{2(\eta_1+\eta_1^*+\xi_1+\xi_1^*)} \begin{vmatrix} e^{-(\eta_1^*+\xi_1)} & 0 & \frac{a^2}{\Re(p_1)} & \frac{a^2}{p_1^*} & 0 \\ 0 & e^{-(\eta_1+\xi_1^*)} & \frac{a^2}{p_1} & \frac{a^2}{\Re(p_1)} & 0 \\ -\frac{1}{2\Re(r_1)} & -\frac{1}{2r_1^*} & e^{-(\eta_1+\xi_1^*)} & 0 & 1 \\ -\frac{1}{2r_1} & -\frac{1}{2\Re(r_1)} & 0 & e^{-(\eta_1^*+\xi_1)} & 1 \\ 0 & 0 & 1 & 1 & 0 \end{vmatrix}, \tag{6.187}$$

so that the ratio G/F is given by the ratio of the two determinants above.
Now consider $e^{-(\eta_1+\xi_1^*)}$, where

$$\begin{aligned} -(\eta_1+\xi_1^*) &= -\left((p_1+r_1^*)x - 4(p_1^3+(r_1^*)^3)t\right) \\ &= -(p_1+r_1^*)\left(x - 4\frac{p_1^3+(r_1^*)^3}{p_1+r_1^*}t\right). \end{aligned} \tag{6.188}$$

This expression describes a wave moving with a speed given by

$$4\frac{\Re(p_1^3+(r_1^*)^3)}{\Re(p_1+r_1^*)},$$

enabling us to easily calculate the speeds of the solitons we have plotted above. We find that the speed of the soliton in Figure 6.1 is 1.92 units, while in Figure 6.2, the soliton is moving at a faster rate with a speed of -5.52 units, the negative sign indicating that the soliton travels in the opposite direction. The soliton in Figure 6.3 is almost stationary, with a speed of -0.44 units.

6.6.2 Properties

We now look in more detail at the 1-soliton solution with p_1, r_1 chosen as in Figure 6.2. Plotting our solution q for various values of the coefficient a in (6.173) gives us a better understanding of the soliton structure.

Looking at $\Re(q)$, we see that, beginning at $a = -2$, $|\Re(q)|$ decreases, reaching zero at $a = -1$ (as explained earlier), and increases for $a > -1$ (see Figure 6.4). There is also a change in $|\Im(q)|$ - we see a reduction in amplitude from $a = -1$, with $|\Im(q)|$ equal to zero for $a = 1$. The amplitude then begins to increase again as a increases (Figure 6.5). This is reflected in our plot of $|q| = \left((\Re(q))^2 + (\Im(q))^2\right)^{1/2}$ - we see a marked difference in the plots for $a = -2$, $a = -1$ (where $|\Re(q)| = 0$), $a = 0$ and $a = 1$ (where $|\Im(q)| = 0$). This is depicted in Figure 6.6.

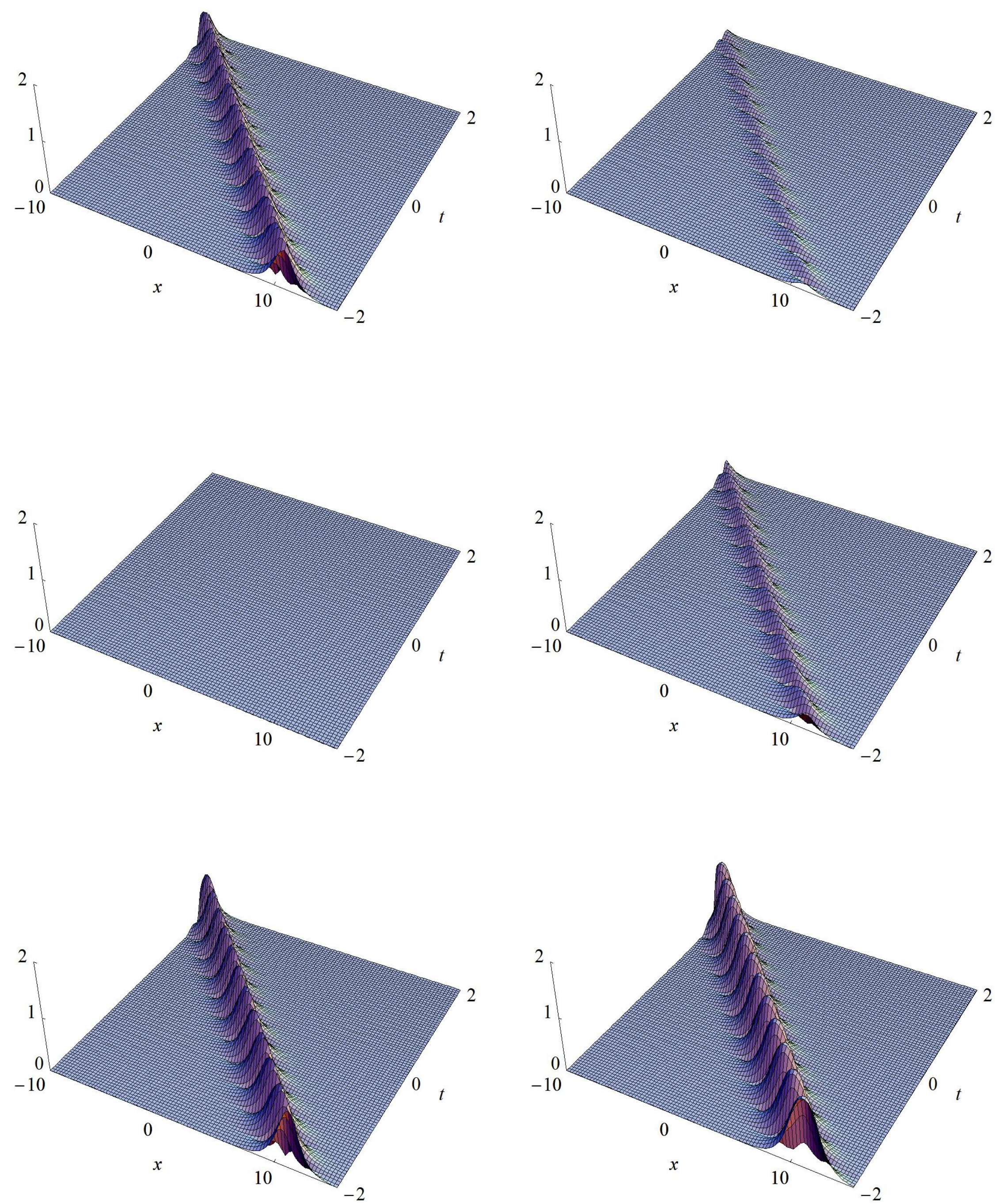


Figure 6.4: Commutative plots of $|\Re(q)|$ for top row (left to right) $a = -2, -1.1$, middle row (left to right) $a = -1, -0.8$ and bottom row (left to right) $a = 0, 1$.

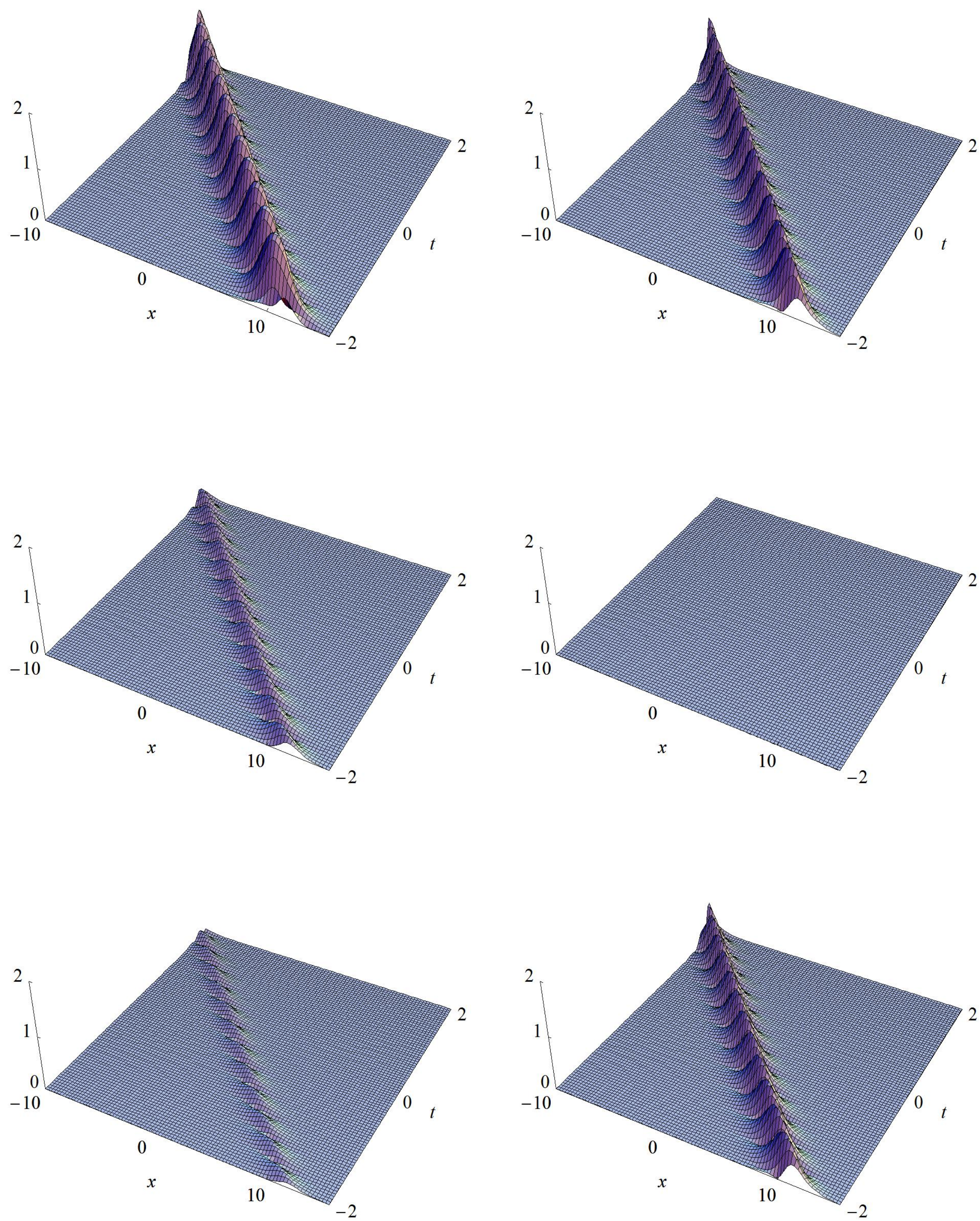


Figure 6.5: Commutative plots of $|\Im(q)|$ for top row (left to right) $a = -1, 0$, middle row (left to right) $a = 0.8, 1$ and bottom row (left to right) $a = 1.1, 2$.

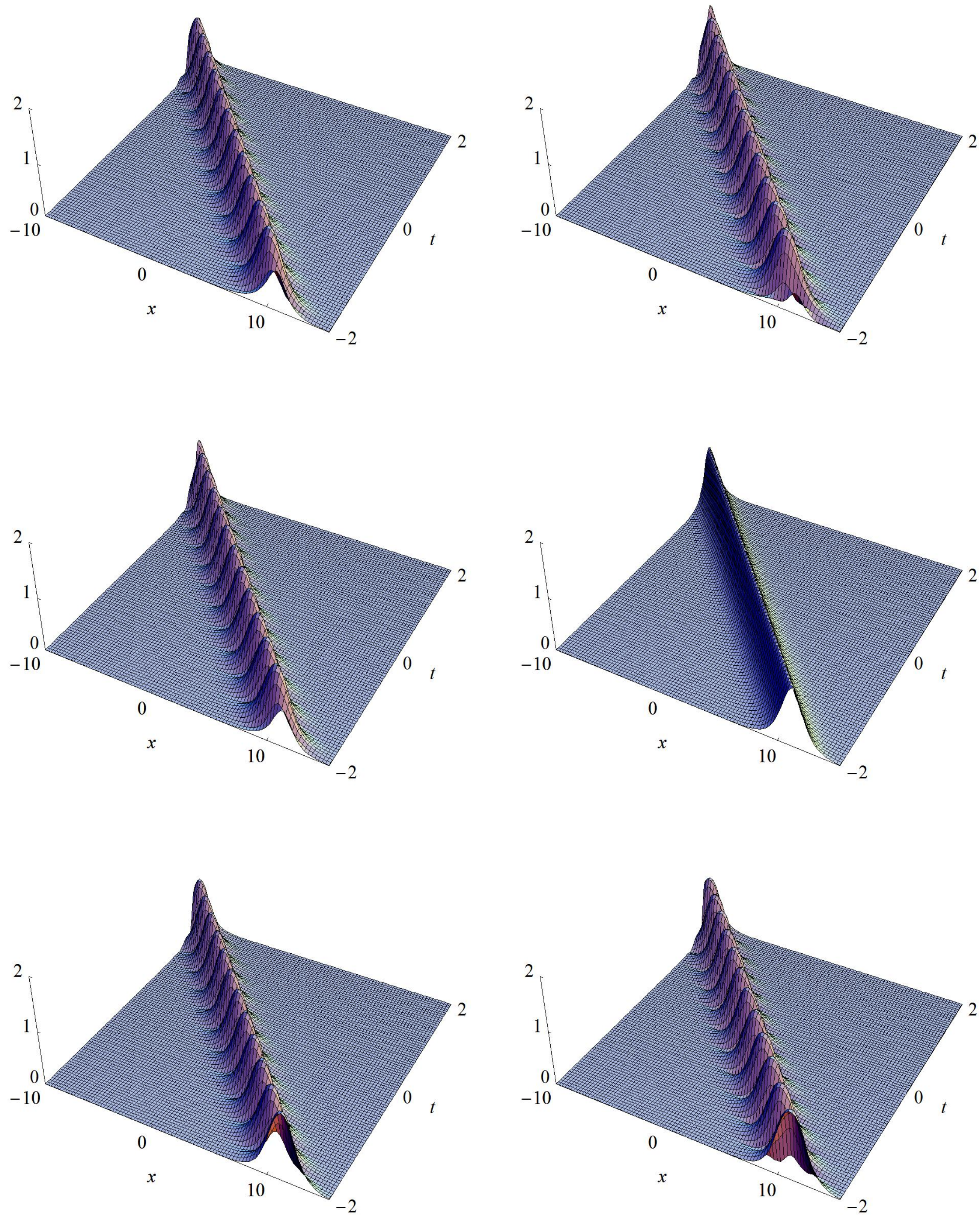


Figure 6.6: Commutative plots of $|\,q\,|$ for top row (left to right) $a = -2, -1$, middle row (left to right) $a = -0.5, 0$ and bottom row (left to right) $a = 0.5, 1$.

6.7 Noncommutative (matrix) case

We now move on to the noncommutative case, where, as for the DS case in Chapter 4, we suppose the functions ϕ_i, ψ_i, χ_i defined in (6.160) take the form of 2×2 matrices. In particular, we let

$$\phi_i = a\alpha_i I_2, \tag{6.189a}$$

$$\psi_i = a\beta_i I_2, \tag{6.189b}$$

$$\chi_i = \gamma_i I_2 \tag{6.189c}$$

for a real constant a, where $\alpha_i, \beta_i, \gamma_i$ are exponentials as in (6.161) and I_2 denotes the 2×2 identity matrix.

6.7.1 1-soliton solution

We begin with the 1-soliton solution ($n = 1$) and choose the matrix $\tilde{S}_1$ as in (6.147), but replace the entries 0 and 1 by the 2×2 zero and identity matrices respectively. Then, since S_1 ($= \tilde{S}_1$ for $n = 1$) and S_2 ($= \tilde{S}_2$ for $n = 1$) are required to commute, the Hermitian matrix $S_2 = (s_{ij})$ $(i, j = 1, 2, \dots, 8,\ i \le j)$ must take the form

$$S_2 = \begin{pmatrix} s_{11} & s_{12} & s_{13} & s_{14} & s_{15} & s_{16} & s_{17} & s_{18} \\ s_{12} & s_{22} & s_{14} & s_{24} & s_{25} & s_{26} & s_{27} & s_{28} \\ s_{13} & s_{14} & s_{11} & s_{12} & s_{17} & s_{18} & s_{15} & s_{16} \\ s_{14} & s_{24} & s_{12} & s_{22} & s_{27} & s_{28} & s_{25} & s_{26} \\ s_{15} & s_{25} & s_{17} & s_{27} & s_{55} & s_{56} & s_{57} & s_{58} \\ s_{16} & s_{26} & s_{18} & s_{28} & s_{56} & s_{66} & s_{58} & s_{68} \\ s_{17} & s_{27} & s_{15} & s_{25} & s_{57} & s_{58} & s_{55} & s_{56} \\ s_{18} & s_{28} & s_{16} & s_{26} & s_{58} & s_{68} & s_{56} & s_{66} \end{pmatrix}. \tag{6.190}$$

It then follows that, since $P = \Theta S_2$ (S_2 Hermitian), so that $\rho_1 = \theta_1 S_2$ in the 1-soliton case, we have

$$P = \rho_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & \phi_1 & 0 & \psi_1^* & 0 \\ 0 & 0 & 0 & 0 & 0 & \phi_1 & 0 & \psi_1^* \\ 0 & 0 & 0 & 0 & \psi_1 & 0 & \phi_1^* & 0 \\ 0 & 0 & 0 & 0 & 0 & \psi_1 & 0 & \phi_1^* \\ \chi_1 & 0 & \chi_1^* & 0 & 0 & 0 & 0 & 0 \\ 0 & \chi_1 & 0 & \chi_1^* & 0 & 0 & 0 & 0 \end{pmatrix} .S_2, \tag{6.191}$$

with S_2 defined as above. Thus our quasi-Grammian solution q (which we denote by q^1 for the 1-soliton case), defined for the commutative case in (6.162), is given by

$$q^1 = -2\begin{vmatrix} \Omega(\theta_1,\rho_1) & \gamma^{11} & \gamma^{12} \\ \alpha^{11} & \boxed{0} & 0 \\ \alpha^{12} & 0 & 0 \end{vmatrix}, \tag{6.192}$$

where α^{11}, α^{12} are row vectors

$$\alpha^{11} = \begin{pmatrix} 0 & 0 & 0 & 0 & a\alpha_1 & 0 & a\alpha_1^* & 0 \end{pmatrix}, \tag{6.193a}$$

$$\alpha^{12} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & a\alpha_1 & 0 & a\alpha_1^* \end{pmatrix} \tag{6.193b}$$

(choosing $p_1 = q_1$ and hence $\phi_1 = \psi_1$ as before), and γ^{11}, γ^{12} the column vectors

$$\gamma^{11} = \begin{pmatrix} s_{11}\gamma_1^* + s_{13}\gamma_1 \\ s_{12}\gamma_1^* + s_{14}\gamma_1 \\ s_{13}\gamma_1^* + s_{11}\gamma_1 \\ s_{14}\gamma_1^* + s_{12}\gamma_1 \\ s_{15}\gamma_1^* + s_{17}\gamma_1 \\ s_{16}\gamma_1^* + s_{18}\gamma_1 \\ s_{17}\gamma_1^* + s_{15}\gamma_1 \\ s_{18}\gamma_1^* + s_{16}\gamma_1 \end{pmatrix}, \qquad \gamma^{12} = \begin{pmatrix} s_{12}\gamma_1^* + s_{14}\gamma_1 \\ s_{22}\gamma_1^* + s_{24}\gamma_1 \\ s_{14}\gamma_1^* + s_{12}\gamma_1 \\ s_{24}\gamma_1^* + s_{22}\gamma_1 \\ s_{25}\gamma_1^* + s_{27}\gamma_1 \\ s_{26}\gamma_1^* + s_{28}\gamma_1 \\ s_{27}\gamma_1^* + s_{25}\gamma_1 \\ s_{28}\gamma_1^* + s_{26}\gamma_1 \end{pmatrix}, \tag{6.194}$$

with γ^{11} the adjoint of the penultimate row of P in (6.191), and γ^{12} the adjoint of the final row of P.

Expanding in the usual manner gives

$$\begin{aligned} q^1 &= -2\begin{pmatrix} \begin{vmatrix} \Omega(\theta_1,\rho_1) & \gamma^{11} \\ \alpha^{11} & \boxed{0} \end{vmatrix} & \begin{vmatrix} \Omega(\theta_1,\rho_1) & \gamma^{12} \\ \alpha^{11} & \boxed{0} \end{vmatrix} \\ \begin{vmatrix} \Omega(\theta_1,\rho_1) & \gamma^{11} \\ \alpha^{12} & \boxed{0} \end{vmatrix} & \begin{vmatrix} \Omega(\theta_1,\rho_1) & \gamma^{12} \\ \alpha^{12} & \boxed{0} \end{vmatrix} \end{pmatrix} \\ &= -2\begin{pmatrix} q_{11}^1 & q_{12}^1 \\ q_{21}^1 & q_{22}^1 \end{pmatrix}, \text{ say,} \end{aligned} \tag{6.195}$$

so that, for example,

$$q_{11}^1 = -2\frac{\begin{vmatrix} \Omega(\theta_1,\rho_1) & \gamma^{11} \\ \alpha^{11} & 0 \end{vmatrix}}{\left|\Omega(\theta_1,\rho_1)\right|} = -2\frac{G_{11}^1}{F}. \tag{6.196}$$

We use the same notation as in Chapter 4.

Our main difficulty in the noncommutative case concerned the choice of the matrix S_2. As a first attempt, we replaced the entries 0 and 1 in our choice of $\tilde{S_2} = S_2$ for the commutative case (6.150) by the 2×2 zero and identity matrices respectively, that is, we chose the entries $s_{15} = s_{26} = 1$ and every other entry equal to zero. However, this resulted in $G^1_{12} = G^1_{21} = 0$ and $G^1_{11} = G^1_{22}$, and hence we obtained only a single 1-soliton plot in the 2×2 matrix case, rather than the desired four. We moved on to choose $s_{15} \neq s_{26}$ and all remaining entries of S_2 equal to zero. We obtained non-identical 1-soliton plots for q^1_{11} and q^1_{22}, although the solutions for q^1_{12} and q^1_{21} were still trivial. The next logical step was to choose the entries s_{15}, s_{16}, s_{25} and s_{26} unequal and non-zero, and every other entry zero. This enabled us to obtain four distinct 1-soliton plots in each of the four matrix entries. It is clear that the 1-soliton solutions q^1_{11}, q^1_{12}, q^1_{21} and q^1_{22} are governed by the entries s_{15}, s_{16}, s_{25} and s_{26} respectively. Choosing one of these entries to be zero results in the disappearance of one of the soliton solutions, for example choosing $s_{16} = 0$ means that the 1-soliton solution q^1_{12} is trivial.

Once again, we must ensure that our denominator $F = |\Omega(\Theta, P)|$ is real and non-zero. The determinant in this case is of size 8×8, and thus the expansion is rather lengthy and difficult to work with. We have given the full expansion in Appendix B (for one particular choice of the matrix S_2), and we can clearly see that each term is either purely real or can be paired with its complex conjugate. Thus F is purely real as required. Finding conditions so that F is non-zero is far more difficult in the noncommutative case than in the commutative case described earlier. For this reason, we utilise the same condition (6.184) as in the commutative case and, since we obtain plots of the expected form with no singularities, we believe that these conditions give rise to a non-zero F. As for the DS equation, we have come across a difficulty of extending to the noncommutative case - the obtained expansions of determinants are very cumbersome and require the aid of a computer package. Such expressions are troublesome to analyse, and hence this is why we choose to focus only on the 1-soliton solution in the noncommutative case.

We choose the complex constants p_1, r_1 in order to satisfy condition (6.184), and once again show, for each of $q^1_{11}, q^1_{12}, q^1_{21}$ and q^1_{22}, a two-dimensional plot at $t = 0.1$, and a three-dimensional plot in order to see the solitons progress with time. These plots are shown in Figures 6.7-6.12. In Figures 6.7-6.10, we have chosen $s_{15} = 1$, $s_{16} = \frac{1}{4}$, $s_{25} = \frac{1}{18}$ and $s_{26} = \frac{1}{2}$ in (6.190), with every other entry of S_2 equal to zero, and the constant $a = 2$ in (6.189a), (6.189b). Figures 6.11 and 6.12 show plots with a different choice of the entries s_{15}, s_{16}, s_{25} and s_{26}, and again $a = 2$.

From Figures 6.11 and 6.12, we see that the 1-soliton solution can propogate with two peaks. This phenomenon is discussed by Sasa and Satsuma in [79], where it is shown that whether the 1-soliton solution is single- or double-peaked depends on the value of a constant c in the 1-soliton solution. In our work, comparing Figures 6.7 and 6.8 with Figures 6.11 and 6.12 (where the values of p_1 and r_1 are identical), we can see that the number of peaks of our 1-soliton solution depends on the choice of the matrix entries s_{15}, s_{16}, s_{25} and s_{26}. However, due to the lengthy expansions of the determinants F and G in our noncommutative solution, determining exactly how the choice of these entries gives rise to either single- or double-peaked solutions may not be possible.

6.8 Conclusions

In this chapter, we obtained noncommutative versions of two higher-order NLS equations. We then focused on one of these, the Sasa-Satsuma NLS equation, and derived both quasi-Wronskian and quasi-Grammian solutions via suitable dimensional reductions of Darboux and binary Darboux transformations. We used the quasi-Grammian form of solution to derive and plot soliton solutions in both a commutative and noncommutative setting. As in the Davey-Stewartson case, limitations were encountered on extending to a noncommutative situation. However, we did succeed in obtaining plots of noncommutative soliton solutions, with the speed of the solitons in both the commutative and noncommutative cases determined in a straightforward manner.

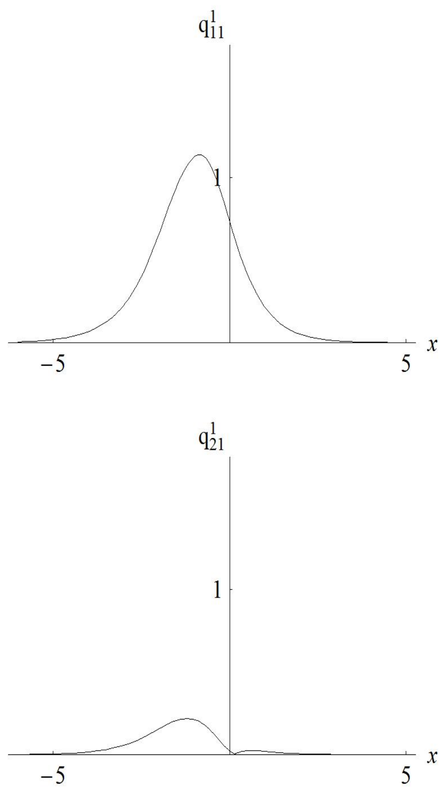


Figure 6.7: Two-dimensional 1-soliton plots at $t = 0.1$, with $p_1 = 1 - 0.8\mathrm{i}$, $r_1 = 0.3 - \mathrm{i}$ and $s_{15} = 1$, $s_{16} = \frac{1}{4}$, $s_{25} = \frac{1}{18}$, $s_{26} = \frac{1}{2}$.

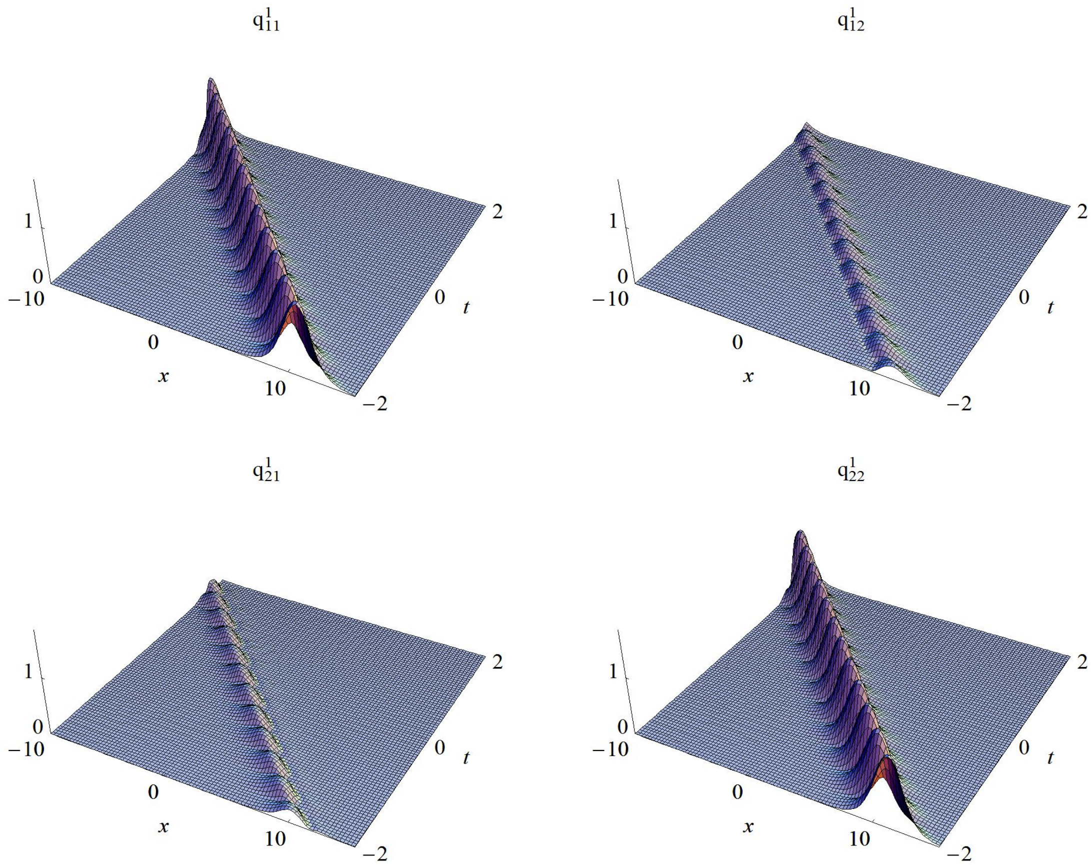


Figure 6.8: Three-dimensional 1-soliton plots with $p_1 = 1 - 0.8\mathrm{i}$, $r_1 = 0.3 - \mathrm{i}$ and $s_{15} = 1$, $s_{16} = \frac{1}{4}$, $s_{25} = \frac{1}{18}$, $s_{26} = \frac{1}{2}$.

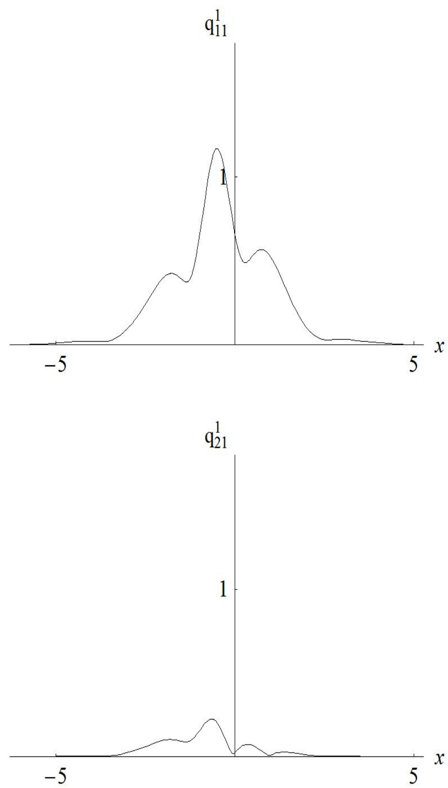


Figure 6.9: Two-dimensional 1-soliton plots at $t = 0.1$, with $p_1 = 1 + 0.3\mathrm{i}$, $r_1 = 0.3 - \mathrm{i}$ and $s_{15} = 1$, $s_{16} = \frac{1}{4}$, $s_{25} = \frac{1}{18}$, $s_{26} = \frac{1}{2}$.

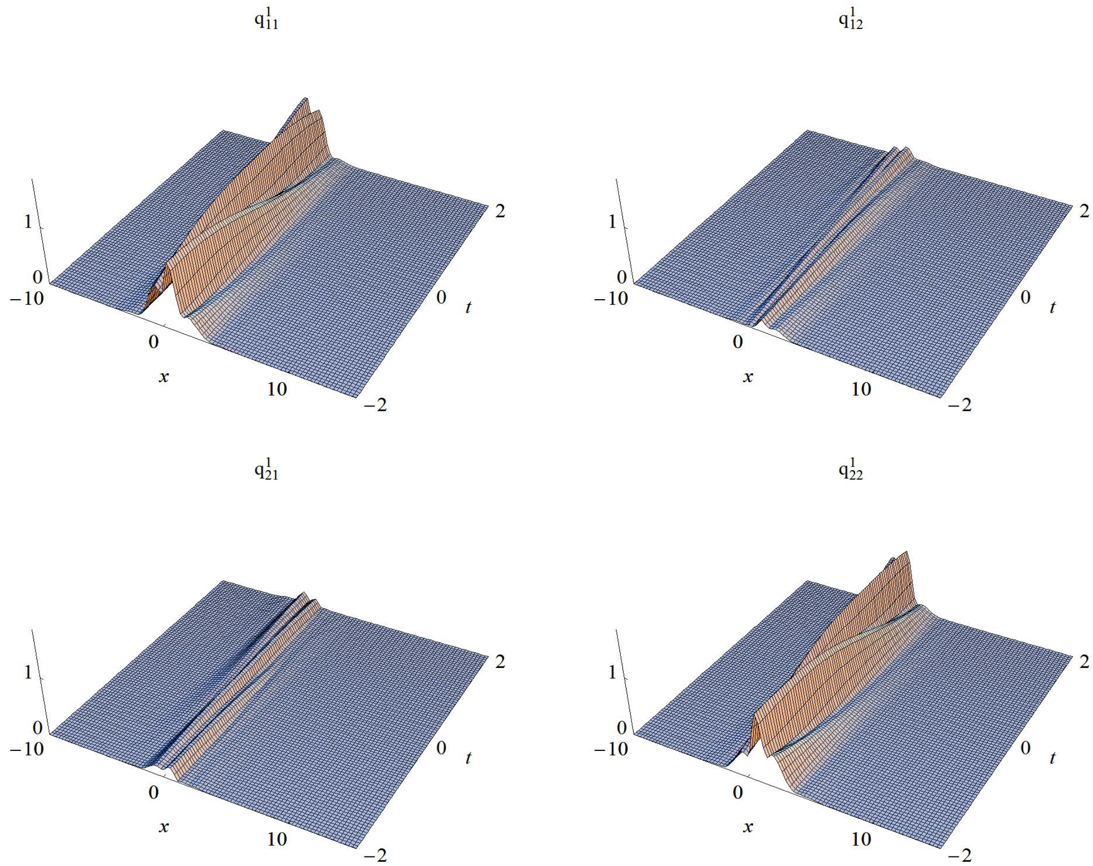


Figure 6.10: Three-dimensional 1-soliton plots with $p_1 = 1 + 0.3\text{i}$, $r_1 = 0.3 - \text{i}$ and $s_{15} = 1$, $s_{16} = \frac{1}{4}$, $s_{25} = \frac{1}{18}$, $s_{26} = \frac{1}{2}$.

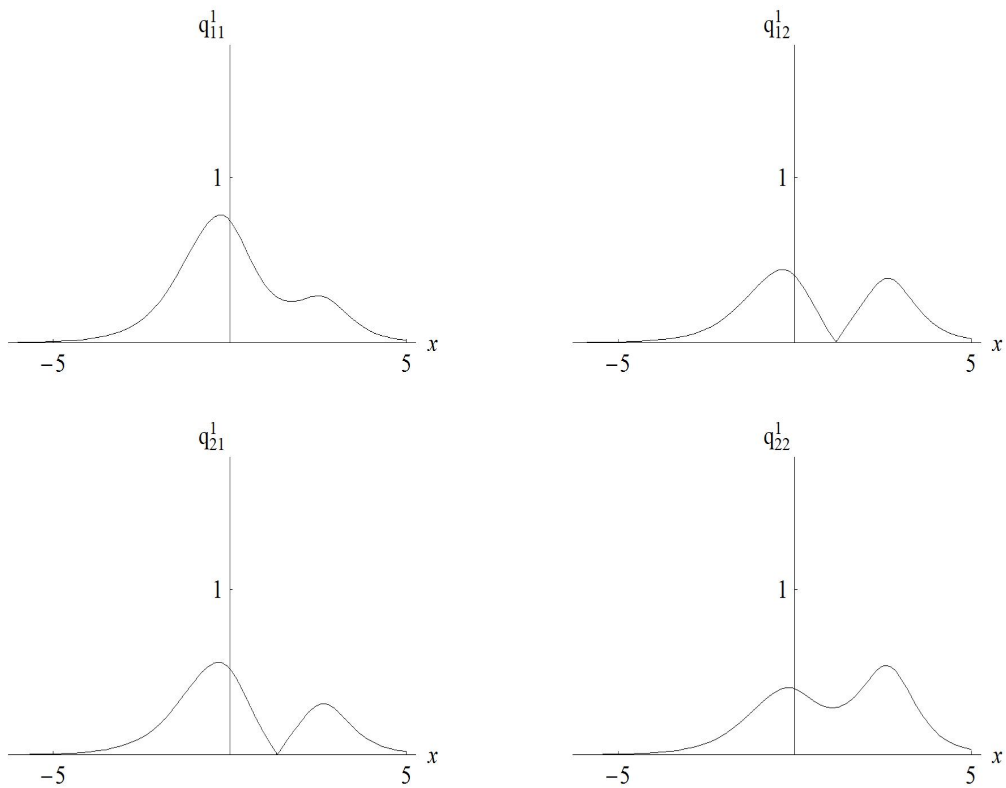


Figure 6.11: Two-dimensional 1-soliton plots at $t = 0.1$, with $p_1 = 1 - 0.8\mathrm{i}$, $r_1 = 0.3 - \mathrm{i}$ and $s_{15} = \frac{1}{3}$, $s_{16} = \frac{1}{4}$, $s_{25} = \frac{1}{5}$, $s_{26} = \frac{1}{6}$.

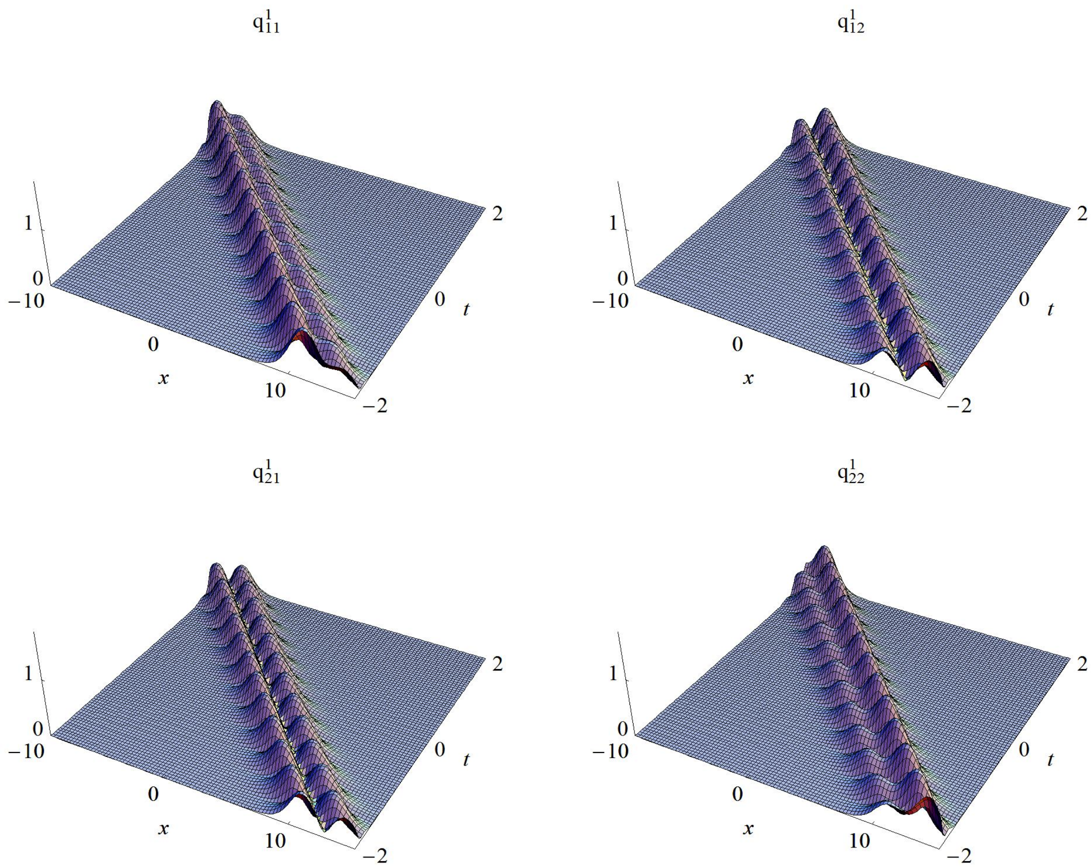


Figure 6.12: Three-dimensional 1-soliton plots with $p_1 = 1 - 0.8\mathrm{i}$, $r_1 = 0.3 - \mathrm{i}$ and $s_{15} = \frac{1}{3}$, $s_{16} = \frac{1}{4}$, $s_{25} = \frac{1}{5}$, $s_{26} = \frac{1}{6}$.

Chapter 7

Conclusions and further work

7.1 Summary

Quasideterminants have proved to be a very useful tool when determining solutions of noncommutative integrable equations, enabling these solutions to be expressed in a simple and compact manner. We derived, in Chapter 3, a noncommutative version of the Davey-Stewartson system via a Lax pair approach and subsequently obtained solutions to this noncommutative system in both quasi-Wronskian and quasi-Grammian form using Darboux and binary Darboux transformations. The simplicity of the quasideterminant structure enabled solution verification to be carried out with relative ease. We then moved on in Chapter 4 to look at one particular class of solution to our system of noncommutative Davey-Stewartson (ncDS) equations, namely dromions, and were able to obtain plots in a noncommutative (matrix) setting. Although we succeeded in plotting the $(1,1)$- and $(2,2)$-dromion solutions in this case, other solutions, for example the $(3,3)$-dromion solution, proved difficult to investigate due to the increasing complexity of the expressions involved. We considered dromion solutions in a 2×2 matrix setting, however our results could, in theory, be extended to an $n \times n$ matrix setting ($n \geq 3$). Again, calculations in such cases will be technically difficult and time-consuming. Implementation of a computer programme may be possible in order to deal with such difficulties.

The aim of Chapter 5 was to describe the reduction of the standard $(2+1)$-dimensional Darboux and binary Darboux transformations, applied in Chapter 3 to the ncDS system, to a $(1+1)$-dimensional situation. Our work here followed that of Gilson, Nimmo and Ohta, who successfully described this dimensional reduction and its application to the self-dual Yang-Mills equations. We were then able, in Chapter 6, to apply a slightly modified version of these dimensionally-reduced Darboux transformations to a noncommutative Sasa-Satsuma nonlinear Schrödinger equation, obtained in the same manner as the ncDS system in Chapter 3. Once again, we found compact quasi-Wronskian and quasi-Grammian expressions for the solution of our noncommutative equation. The quasi-Grammian solution was then used to plot soliton solutions in a noncommutative setting. Although limitations were encountered due to the determinant expressions being very lengthy and difficult to analyse, we did succeed in obtaining plots of the 1-soliton solution in the 2×2 matrix case. Further solutions, for example the 2- and 3-soliton solutions, could, in theory, be determined and plotted in a similar manner, however this procedure would be technically complex to carry out by hand.

7.2 Open questions

A number of continuations of our work are possible. One interesting topic would be to investigate other types of solution to our system of ncDS equations, for example solitoffs, a hybrid of the soliton and dromion. Such a solution was found to exist for a commutative Davey-Stewartson system by Gilson in 1992, and resembles a truncated plane wave soliton tending exponentially to a non-zero value in only one direction. We believe that solitoff solutions in a noncommutative setting have so far not been studied in the literature. Dromion-solitoff interactions would be another possible topic of discussion in the noncommutative case.

The process whereby a noncommutative integrable equation is derived via a Lax pair approach, and quasideterminant solutions obtained by the application of Darboux transformations, has been implemented for other equations, for example the KP equation by Gilson and Nimmo, and the modified KP equation by Gilson, Nimmo and Sooman. However, there are many other equations for which this technique has not been applied - the Boussinesq and Sawada-Kotera equations being two such examples, and it would be beneficial to investigate these in a similar manner. We would then be in a better position to discuss both the similarities of, and differences between, noncommutative integrable equations and their quasideterminant solutions.

Appendix A

A.1 Uniqueness of ncSSNLS quasi-Grammian solution

As detailed in Section 6.4.6, we derive conditions to ensure that our quasi-Grammian solution (6.134) to the ncSSNLS equation is unique. We consider the commutative and noncommutative cases separately.

A.1.1 Commutative case

We take q to be a scalar object, so that $q^\dagger = q^*$ and $q^{\dagger^*} = q$. Utilising (2.14) to express the quasi-Grammians (6.134) as ratios of determinants, we have

$$q = -2\frac{\begin{vmatrix} \Omega(\Theta, P) & \mathbf{P_3}^\dagger \\ \mathbf{\Theta_1} & 0 \end{vmatrix}}{\begin{vmatrix} \Omega(\Theta, P) \end{vmatrix}}, \qquad q^* = -2\frac{\begin{vmatrix} \Omega(\Theta, P) & \mathbf{P_3}^\dagger \\ \mathbf{\Theta_2} & 0 \end{vmatrix}}{\begin{vmatrix} \Omega(\Theta, P) \end{vmatrix}}, \tag{A.1a}$$

$$q^\dagger = -2\frac{\begin{vmatrix} \Omega(\Theta, P) & \mathbf{P_1}^\dagger \\ \mathbf{\Theta_3} & 0 \end{vmatrix}}{\begin{vmatrix} \Omega(\Theta, P) \end{vmatrix}}, \qquad q^{\dagger^*} = -2\frac{\begin{vmatrix} \Omega(\Theta, P) & \mathbf{P_2}^\dagger \\ \mathbf{\Theta_3} & 0 \end{vmatrix}}{\begin{vmatrix} \Omega(\Theta, P) \end{vmatrix}}, \tag{A.1b}$$

and hence we must show that $q^\dagger = q^*$ and $q^{\dagger^*} = q$, i.e.

$$\begin{vmatrix} \Omega(\Theta, P) & \mathbf{P_1}^\dagger \\ \mathbf{\Theta_3} & 0 \end{vmatrix} = \begin{vmatrix} \Omega(\Theta, P) & \mathbf{P_3}^\dagger \\ \mathbf{\Theta_2} & 0 \end{vmatrix} \tag{A.2a}$$

and

$$\begin{vmatrix} \Omega(\Theta, P) & \mathbf{P_2}^\dagger \\ \mathbf{\Theta_3} & 0 \end{vmatrix} = \begin{vmatrix} \Omega(\Theta, P) & \mathbf{P_3}^\dagger \\ \mathbf{\Theta_1} & 0 \end{vmatrix}. \tag{A.2b}$$

We only prove (A.2a), the proof of (A.2b) follows in a similar manner.

Since $P = \Theta S_2$, we have $\mathbf{P_i} = \mathbf{\Theta_i} S_2$ $(i = 1, 2, 3)$, and similarly, condition (6.135) gives

$$\mathbf{\Theta_1} = \mathbf{\Theta_2}^* S_1, \tag{A.3a}$$

$$\mathbf{\Theta_2} = \mathbf{\Theta_1}^* S_1, \tag{A.3b}$$

$$\mathbf{\Theta_3} = \mathbf{\Theta_3}^* S_1. \tag{A.3c}$$

Thus

$$\begin{aligned} \begin{vmatrix} \Omega(\Theta, P) & \mathbf{P_1}^\dagger \\ \mathbf{\Theta_3} & 0 \end{vmatrix} &= \begin{vmatrix} \Omega(\Theta, P) & S_2 S_1^\dagger \mathbf{\Theta_2}^T \\ \mathbf{P_3} S_2^{-1} & 0 \end{vmatrix} \\ &= \begin{vmatrix} S_1^{-\dagger} S_2^{-1} \Omega(\Theta, P) S_2 & \mathbf{\Theta_2}^T \\ \mathbf{P_3} & 0 \end{vmatrix} \\ &= \begin{vmatrix} S_2^* \Omega(\Theta, P)^T (S_2^{-1})^T (S_1^{-\dagger})^T & \mathbf{P_3}^T \\ \mathbf{\Theta_2} & 0 \end{vmatrix} \end{aligned} \tag{A.4}$$

since transposing has no effect on the value of a determinant. Then, since

$$\begin{aligned} P &= \Theta S_2 \\ &= S P^* S_2^{-1} S_1 S_2 \end{aligned} \tag{A.5}$$

by (6.135), (6.136), so that

$$\mathbf{P_3} = \mathbf{P_3}^* S_2^{-1} S_1 S_2 \tag{A.6a}$$

and

$$\mathbf{P_3}^T = S_2^* S_1^T (S_2^{-1})^* \mathbf{P_3}^\dagger, \tag{A.6b}$$

it can be shown that

$$\begin{vmatrix} \Omega(\Theta, P) & \mathbf{P_1}^\dagger \\ \mathbf{\Theta_3} & 0 \end{vmatrix} = \begin{vmatrix} S_2 (S_1^T)^{-1} \Omega(\Theta, P)^T S_2^{-1} S_1^{-1} & \mathbf{P_3}^\dagger \\ \mathbf{\Theta_2} & 0 \end{vmatrix}, \tag{A.7}$$

where we use the fact that $(A^{-1})^T = (A^T)^{-1}$ for an invertible matrix A. We take S_2 to be Hermitian as before, so that $S_2^T = S_2$ since we assume S_2 (and S_1) have real entries. Also,

$$\Omega(\Theta, P) = \int P^\dagger J \Theta \, \mathrm{d}x + I_{ns}, \tag{A.8}$$

and hence

$$\Omega(\Theta, P)^{\dagger} = \int \Theta^{\dagger} J P \, \mathrm{d}x + I_{ns}$$
$$= S_2^{-1} \Omega(\Theta, P) S_2 \text{ by (6.136)}, \tag{A.9}$$

while

$$\Omega(\Theta, P)^{*} = S_2 S_1^{\dagger} S_2^{-1} \left(\int P^{\dagger} J \Theta \, \mathrm{d}x \right) S_1 + I_{ns} \tag{A.10}$$

by (6.135), (6.136) and the fact that $SJS = J$ by definition. Thus

$$\Omega(\Theta, P)^{T} = \Omega(\Theta, P)^{\dagger *} = S_2^{-1} \Omega(\Theta, P)^{*} S_2$$
$$= S_1^{\dagger} S_2^{-1} \left(\int P^{\dagger} J \Theta \, \mathrm{d}x \right) S_1 S_2 + I_{ns} \tag{A.11}$$

by (A.10). It therefore follows that

$$S_2 (S_1^T)^{-1} \Omega(\Theta, P)^{T} S_2^{-1} S_1^{-1} = S_2 (S_1^T)^{-1} S_1^{\dagger} S_2^{-1} \left(\int P^{\dagger} J \Theta \, \mathrm{d}x \right) + S_2 (S_1^T)^{-1} S_2^{-1} S_1^{-1}. \tag{A.12}$$

The right-hand side is equal to $\Omega(\Theta, P)$ if and only if the conditions

$$S_2 (S_1^T)^{-1} S_1^{\dagger} S_2^{-1} = I_{ns} \tag{A.13a}$$

and

$$S_2 (S_1^T)^{-1} S_2^{-1} S_1^{-1} = I_{ns} \tag{A.13b}$$

hold. We impose a relation on S_1 such that

$$S_1 \text{ is orthogonal, i.e. } S_1^T = S_1^{-1}, \tag{A.14}$$

and, since we assume S_1 has real entries, it follows that $S_1^{\dagger} = S_1^{-1}$. Then (A.13a) is satisfied. To satisfy (A.13b), we require

$$S_2 S_1 S_2^{-1} S_1^{-1} = I_{ns} \tag{A.15}$$

using the relation on S_1 above, so that

$$S_1 S_2 = S_2 S_1. \tag{A.16}$$

By imposing conditions (A.14) and (A.16) on the matrices S_1 and S_2, we therefore obtain

$$\begin{vmatrix} \Omega(\Theta, P) & \mathbf{P_1}^{\dagger} \\ \mathbf{\Theta_3} & 0 \end{vmatrix} = \begin{vmatrix} \Omega(\Theta, P) & \mathbf{P_3}^{\dagger} \\ \mathbf{\Theta_2} & 0 \end{vmatrix} \tag{A.17}$$

as required. Thus in the commutative case, we have solutions

$$q = -2\frac{\begin{vmatrix} \Omega(\Theta, P) & \mathbf{P_2}^{\dagger} \\ \mathbf{\Theta_3} & 0 \end{vmatrix}}{\begin{vmatrix} \Omega(\Theta, P) \end{vmatrix}} = -2\frac{G}{F}, \text{ say,} \tag{A.18a}$$

and

$$q^* = -2\frac{\begin{vmatrix} \Omega(\Theta, P) & \mathbf{P_1}^{\dagger} \\ \mathbf{\Theta_3} & 0 \end{vmatrix}}{\begin{vmatrix} \Omega(\Theta, P) \end{vmatrix}} = -2\frac{K}{F}, \text{ say.} \tag{A.18b}$$

It remains to show that these solutions are indeed complex conjugate, i.e. we prove that F is real, and $G^* = K$. From (6.170), the determinant F can be expressed in the form

$$\begin{aligned} F &= | \, \Omega(\Theta, P) \, | \\ &= | \, I_{4n} + S_2 \Psi \, |, \end{aligned} \tag{A.19}$$

where $\Psi = \Theta^{\dagger} J \Theta$, so that $\Psi^{\dagger} = \Psi$. Then

$$F = | \, S_2 \, | \, . \, | \, S_2^{-1} + \Psi \, |, \tag{A.20}$$

so that

$$\begin{aligned} F^* &= | \, S_2^* \, | \, . \, | \, (S_2^{-1})^* + \Psi^* \, | \\ &= | \, S_2^{\dagger} \, | \, . \, | \, S_2^{-\dagger} + \Psi^{\dagger} \, | \\ &= F \end{aligned} \tag{A.21}$$

since S_2, Ψ are Hermitian and transposing has no effect on the value of a determinant. Note that this is the reason for our decision to achieve a dimensionally-reduced Darboux transformation by replacing ∂_y by $-\mathrm{i}\lambda$, rather than simply λ as was done by Gilson, Nimmo and Ohta for the self-dual Yang-Mills equation [73]. If we replace ∂_y by λ, it then follows that

$$F = | \, I_{4n} - \mathrm{i} S_2 \Psi \, |, \tag{A.22}$$

and thus we cannot prove $F^* = F$ as above due to the change of sign when we take the complex conjugate of the complex constant i.

We also show that

$$\begin{vmatrix} \Omega(\Theta, P)^* & \mathbf{P_2}^{\dagger *} \\ \mathbf{\Theta_3}^* & 0 \end{vmatrix} = \begin{vmatrix} \Omega(\Theta, P) & \mathbf{P_1}^{\dagger} \\ \mathbf{\Theta_3} & 0 \end{vmatrix}. \tag{A.23}$$

Note that, by (A.5), we have $\mathbf{P_2} = \mathbf{P_1}^* S_2^{-1} S_1 S_2$, and also $\mathbf{\Theta}_3^* = \mathbf{\Theta_3} S_1$ by (A.3c). Thus

$$\begin{aligned}\begin{vmatrix} \Omega(\Theta,P)^* & \mathbf{P_2}^{\dagger *} \\ \mathbf{\Theta_3}^* & 0 \end{vmatrix} &= \begin{vmatrix} \Omega(\Theta,P)^* & S_2 S_1^\dagger S_2^{-1} \mathbf{P_1}^\dagger \\ \mathbf{\Theta_3} S_1 & 0 \end{vmatrix} \\ &= \begin{vmatrix} S_2 S_1^{-\dagger} S_2^{-1} \Omega(\Theta,P)^* S_1^{-1} & \mathbf{P_1}^\dagger \\ \mathbf{\Theta_3} & 0 \end{vmatrix}. \end{aligned} \tag{A.24}$$

Using our definition of $\Omega(\Theta, P)^*$ in (A.10) and conditions (A.14), (A.16), it can be shown that

$$S_2 S_1^{-\dagger} S_2^{-1} \Omega(\Theta,P)^* S_1^{-1} = \Omega(\Theta,P), \tag{A.25}$$

and hence

$$\begin{vmatrix} \Omega(\Theta,P)^* & \mathbf{P_2}^{\dagger *} \\ \mathbf{\Theta_3}^* & 0 \end{vmatrix} = \begin{vmatrix} \Omega(\Theta,P) & \mathbf{P_1}^\dagger \\ \mathbf{\Theta_3} & 0 \end{vmatrix} \tag{A.26}$$

as required.

A.1.2 Noncommutative case

In the noncommutative case, we have four distinct solutions q, q^*, $q^\dagger$ and $q^{\dagger *}$, each one a 2×2 matrix. We choose the entries of Θ, P to be of size 2×2, so that Θ, P are now of size $6 \times 2N$, i.e.

$$\Theta = \begin{pmatrix} \mathbf{\Theta_1} & \mathbf{\Theta_2} & \dots & \mathbf{\Theta_6} \end{pmatrix}^T, \tag{A.27a}$$

and

$$P^\dagger = \begin{pmatrix} \mathbf{P}_\mathbf{1}^\dagger & \mathbf{P}_\mathbf{2}^\dagger & \dots & \mathbf{P}_\mathbf{6}^\dagger \end{pmatrix}, \tag{A.27b}$$

where the $\mathbf{\Theta_i}$ $(i = 1, 2, \dots, 6)$ are row vectors of arbitrary length $2N$, and the $\mathbf{P_i}^\dagger$ column vectors of the same length. It then follows that

$$q = -2 \begin{pmatrix} \begin{vmatrix} \Omega(\Theta,P) & \mathbf{P}_\mathbf{5}^\dagger \\ \mathbf{\Theta_1} & \boxed{0} \end{vmatrix} & \begin{vmatrix} \Omega(\Theta,P) & \mathbf{P}_\mathbf{6}^\dagger \\ \mathbf{\Theta_1} & \boxed{0} \end{vmatrix} \\ \begin{vmatrix} \Omega(\Theta,P) & \mathbf{P}_\mathbf{5}^\dagger \\ \mathbf{\Theta_2} & \boxed{0} \end{vmatrix} & \begin{vmatrix} \Omega(\Theta,P) & \mathbf{P}_\mathbf{6}^\dagger \\ \mathbf{\Theta_2} & \boxed{0} \end{vmatrix} \end{pmatrix} = -2 \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}, \text{ say.} \tag{A.28}$$

Similarly,

$$q^* = -2 \begin{pmatrix} \begin{vmatrix} \Omega(\Theta, P) & \mathbf{P_5^\dagger} \\ \mathbf{\Theta_3} & \boxed{0} \end{vmatrix} & \begin{vmatrix} \Omega(\Theta, P) & \mathbf{P_6^\dagger} \\ \mathbf{\Theta_3} & \boxed{0} \end{vmatrix} \\ \\ \begin{vmatrix} \Omega(\Theta, P) & \mathbf{P_5^\dagger} \\ \mathbf{\Theta_4} & \boxed{0} \end{vmatrix} & \begin{vmatrix} \Omega(\Theta, P) & \mathbf{P_6^\dagger} \\ \mathbf{\Theta_4} & \boxed{0} \end{vmatrix} \end{pmatrix} = -2 \begin{pmatrix} q_{11}^* & q_{12}^* \\ q_{21}^* & q_{22}^* \end{pmatrix}, \tag{A.29}$$

$$q^\dagger = -2 \begin{pmatrix} \begin{vmatrix} \Omega(\Theta, P) & \mathbf{P_1^\dagger} \\ \mathbf{\Theta_5} & \boxed{0} \end{vmatrix} & \begin{vmatrix} \Omega(\Theta, P) & \mathbf{P_2^\dagger} \\ \mathbf{\Theta_5} & \boxed{0} \end{vmatrix} \\ \\ \begin{vmatrix} \Omega(\Theta, P) & \mathbf{P_1^\dagger} \\ \mathbf{\Theta_6} & \boxed{0} \end{vmatrix} & \begin{vmatrix} \Omega(\Theta, P) & \mathbf{P_2^\dagger} \\ \mathbf{\Theta_6} & \boxed{0} \end{vmatrix} \end{pmatrix} = -2 \begin{pmatrix} q_{11}^\dagger & q_{12}^\dagger \\ q_{21}^\dagger & q_{22}^\dagger \end{pmatrix}, \tag{A.30}$$

and

$$q^{\dagger *} = -2 \begin{pmatrix} \begin{vmatrix} \Omega(\Theta, P) & \mathbf{P_3^\dagger} \\ \mathbf{\Theta_5} & \boxed{0} \end{vmatrix} & \begin{vmatrix} \Omega(\Theta, P) & \mathbf{P_4^\dagger} \\ \mathbf{\Theta_5} & \boxed{0} \end{vmatrix} \\ \\ \begin{vmatrix} \Omega(\Theta, P) & \mathbf{P_3^\dagger} \\ \mathbf{\Theta_6} & \boxed{0} \end{vmatrix} & \begin{vmatrix} \Omega(\Theta, P) & \mathbf{P_4^\dagger} \\ \mathbf{\Theta_6} & \boxed{0} \end{vmatrix} \end{pmatrix} = -2 \begin{pmatrix} {q_{11}^\dagger}^* & {q_{12}^\dagger}^* \\ {q_{21}^\dagger}^* & {q_{22}^\dagger}^* \end{pmatrix}. \tag{A.31}$$

We consider the solutions in the $(1, 1)$ positions only. Since each is a scalar 1×1 entity (as the 'boxed' element in each quasi-Grammian is 1×1), we are free to implement (2.14) to express the $(1, 1)$ solutions as a ratio of determinants, so that

$$q_{11} = -2 \frac{\begin{vmatrix} \Omega(\Theta, P) & \mathbf{P_5^\dagger} \\ \mathbf{\Theta_1} & 0 \end{vmatrix}}{\left|\Omega(\Theta, P)\right|} = -2\frac{G}{F}, \qquad q_{11}^* = -2 \frac{\begin{vmatrix} \Omega(\Theta, P) & \mathbf{P_5^\dagger} \\ \mathbf{\Theta_3} & 0 \end{vmatrix}}{\left|\Omega(\Theta, P)\right|} = -2\frac{H}{F}, \tag{A.32a}$$

$$q_{11}^\dagger = -2 \frac{\begin{vmatrix} \Omega(\Theta, P) & \mathbf{P_1^\dagger} \\ \mathbf{\Theta_5} & 0 \end{vmatrix}}{\left|\Omega(\Theta, P)\right|} = -2\frac{J}{F}, \qquad {q_{11}^\dagger}^* = -2 \frac{\begin{vmatrix} \Omega(\Theta, P) & \mathbf{P_3^\dagger} \\ \mathbf{\Theta_5} & 0 \end{vmatrix}}{\left|\Omega(\Theta, P)\right|} = -2\frac{K}{F}. \tag{A.32b}$$

We firstly show that the solutions for q_{11} and q_{11}^* are complex conjugate, i.e. we show that F is real, and $G^* = H$.

In a similar manner to the commutative case detailed above, we see that F is real since S_2 is a Hermitian matrix. (The matrix Ψ in (A.19) is still Hermitian when we extend to the noncommutative case).

We also prove that $G^* = H$, i.e.

$$\begin{vmatrix} \Omega(\Theta, P)^* & \mathbf{P_5}^{\dagger *} \\ \mathbf{\Theta_1}^* & 0 \end{vmatrix} = \begin{vmatrix} \Omega(\Theta, P) & \mathbf{P_5}^{\dagger} \\ \mathbf{\Theta_3} & 0 \end{vmatrix}. \tag{A.33}$$

In the noncommutative case, each entry of the permutation matrix S defined in (6.40) is of size 2×2, so that

$$S = \begin{pmatrix} O & I & O \\ I & O & O \\ O & O & I \end{pmatrix}, \tag{A.34}$$

where O and I denote the 2×2 zero and identity matrices respectively. By (A.5), we have $\mathbf{P_5} = \mathbf{P_5}^* S_2^{-1} S_1 S_2$, and $\mathbf{\Theta_1}^* = \mathbf{\Theta_3} S_1$ since $\Theta = S\Theta^* S_1$. Then we can easily prove (A.33) in a similar manner to the commutative case above, imposing conditions (A.14), (A.16).

We must also show that the solutions for q_{11} and $q_{11}^\dagger$ in (A.32) are Hermitian conjugate, the solutions for q_{11} and ${q_{11}^\dagger}^*$ are transpose to one another, and so on. We do not detail the calculations here, however each one can be shown in a straightforward manner, once again imposing conditions (A.14), (A.16).

Appendix B

B.1 Noncommutative SSNLS determinant expansion

Here we give the expansion of the denominator F of the noncommutative 1-soliton solution obtained in Section 6.7.1. It can then easily be seen that this expansion is purely real. We have

$$
\begin{aligned}
F = 1 &+ \frac{Aa^2}{B}\left(\frac{1}{p_1^* r_1}e^{2(\eta_1^*+\xi_1)} + \frac{1}{p_1 r_1^*}e^{2(\eta_1+\xi_1^*)}\right) \\
&+ \frac{Ca^4}{D}\left(\frac{1}{(p_1^*)^2 r_1^2}e^{4(\eta_1^*+\xi_1)} + \frac{1}{p_1^2 (r_1^*)^2}e^{4(\eta_1+\xi_1^*)}\right) + \frac{Aa^2}{4Ep_{re}r_{re}}e^{\eta_1+\eta_1^*+\xi_1+\xi_1^*} \\
&+ \left(\frac{a^4}{p_1 p_1^*}\left\{\frac{H}{Jr_1 r_1^*} - \frac{K}{2L(r_{re})^2}\right\} + \frac{a^4}{4(p_{re})^2}\left\{\frac{M}{4N(r_{re})^2} - \frac{K}{Lr_1 r_1^*}\right\}\right)e^{2(\eta_1+\eta_1^*+\xi_1+\xi_1^*)} \\
&+ \frac{Pa^6}{p_1 r_1^*}\left(\frac{1}{Qp_1 p_1^* r_1 r_1^*} - \frac{1}{4R(p_{re})^2 r_1 r_1^*} - \frac{1}{4Rp_1 p_1^*(r_{re})^2}\right)e^{2(2\eta_1+\eta_1^*+\xi_1+2\xi_1^*)} \\
&+ \left(\frac{Pa^6}{p_1^* r_1}\left\{\frac{1}{Qp_1 p_1^* r_1 r_1^*} - \frac{1}{4R(p_{re})^2 r_1 r_1^*} - \frac{1}{4Rp_1 p_1^*(r_{re})^2}\right\}\right. \\
&\left.+ \frac{Pa^6}{16S(p_{re})^2(r_{re})^2}\left\{\frac{1}{p_1^* r_1} + \frac{1}{p_1 r_1^*}\right\}\right)e^{2(\eta_1+2\eta_1^*+2\xi_1+\xi_1^*)} \\
&- Pa^6\left(\frac{1}{4p_1 p_1^* p_{re} r_{re}}\left\{\frac{1}{4L(r_{re})^3} - \frac{1}{Jr_1 r_1^*}\right\}\right. \\
&\left.- \frac{1}{16(p_{re})^3 r_{re}}\left\{\frac{1}{Lr_1 r_1^*} - \frac{1}{4N(r_{re})^2}\right\}\right)e^{3(\eta_1+\eta_1^*+\xi_1+\xi_1^*)} \\
&+ \frac{Ca^4}{4Tp_{re}r_{re}}\left(\frac{1}{p_1^* r_1}e^{\eta_1+3\eta_1^*+3\xi_1+\xi_1^*} + \frac{1}{p_1 r_1^*}e^{3\eta_1+\eta_1^*+\xi_1+3\xi_1^*}\right) \\
&+ Ua^8\left(\frac{1}{p_1^2 (p_1^*)^2}\left\{\frac{1}{2Vr_1^2 (r_1^*)^2} + \frac{1}{32W(r_{re})^4} - \frac{1}{4Qr_1 r_1^*(r_{re})^2}\right\}\right. \\
&+ \frac{1}{4p_1 p_1^*(p_{re})^2}\left\{\frac{1}{4Jr_1 r_1^*(r_{re})^2} - \frac{1}{16S(r_{re})^4} - \frac{1}{Qr_1^2 (r_1^*)^2}\right\} \\
&\left.+ \frac{1}{32(p_{re})^4}\left\{\frac{1}{Wr_1^2 (r_1^*)^2} + \frac{1}{32L(r_{re})^4} - \frac{1}{2Sr_1 r_1^*(r_{re})^2}\right\}\right)e^{4(\eta_1+\eta_1^*+\xi_1+\xi_1^*)},
\end{aligned} \tag{B.1}
$$

where p_{re}, r_{re} denote $\Re(p_1), \Re(r_1)$ respectively, η_1 and ξ_1 are defined in (6.178), and $A, B, C, \ldots$ are constants. We see that each term in the expansion is either purely real, or else has a complex conjugate partner. We have chosen the entries of the matrix S_2 ($= \tilde{S}_2$ for the 1-soliton solution) in (6.190) so that $s_{15} = 1$, $s_{16} = \frac{1}{4}$, $s_{25} = \frac{1}{18}$, $s_{26} = \frac{1}{2}$, and every other entry equal to zero. Corresponding expansions for other choices of the entries of S_2 will also be real in a similar manner. However, the expansion of F for an arbitrary S_2 is too lengthy to write down.

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