Boundary-value problems for transversely isotropic hyperelastic solids

by

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Statement of originality

This thesis is submitted in accordance with regulations for the degree of Doctor of Philosophy in the University of Glasgow. It is the result of research and outcome of my original work, unless otherwise explicitly stated, carried out at the University of Glasgow between September 2003 and September 2006, under the direction of Professor R. W. Ogden. No part of thesis has previously been submitted by the author for a degree at this or any other university.

Summary

In this thesis we examine three boundary-value problems combined with the presence of dead-load tractions in respect of transversely isotropic elastic materials.

In particular, Chapter 1 mainly consists of existing preliminary remarks on the continuum (phenomenological) approach used here to study the mechanical response of elastic materials under large strains. More specifically, we discuss, always within the continuous framework, basic kinematical concepts, fundamental stress principles as well as balance laws; those also being appropriately specialized for material bodies under the state of equilibrium, i.e. for static problems. Description of the governing constitutive theory for Cauchy elastic isotropic and transversely isotropic solids follows, with reference to which, the notion of a hyperelastic solid is then prescribed. Further, the necessary connections with the classical linear theory of transversely isotropic solids are generated and finally some typical constitutive inequalities are summarized.

Then, in Chapter 2, we examine the classical problem of finite bending of a rectangular block of elastic material into a sector of a circular cylindrical tube in respect of compressible transversely isotropic elastic materials. More specifically, we consider the possible existence of isochoric solutions. In contrast to the corresponding problem for isotropic materials, for which such solutions do not exist for a compressible material [38], we determine conditions on the form of the strain-energy function for which isochoric solutions are possible. Based on those conditions, some general forms of strain-energy functions that admit isochoric bending are derived. We also, for the considered geometry and deformation, examine aspects of stability predicated on the notion of strong ellipticity. Expressly, for plane strain, we provide necessary and sufficient conditions for strong ellipticity to hold. The material incorporated in this chapter has been accepted for publication in [42].

In Chapter 3 we study the problem of (plane strain) azimuthal shear of a circular cylin-

drical tube of incompressible transversely isotropic elastic material subject to finite deformation. The preferred direction associated with the transverse isotropy lies in the planes normal to the tube axis and is disposed so as to preserve the cylindrical symmetry. For a general form of strain-energy function the considered deformation yields simple expressions for the azimuthal shear stress and the associated strong ellipticity condition in terms of the azimuthal shear strain. These apply for a sense of shear that is either 'with' or 'against' the preferred direction (anti-clockwise and clockwise, respectively), so that material line elements locally in the preferred direction either extend or (at least initially) contract, respectively. For some specific strain-energy functions we then examine local loss of uniqueness of the shear stress-strain relationship and failure of ellipticity for the case of contraction and the dependence on the geometry of the preferred direction. In particular, for a widely used reinforced neo-Hookean material (see, e.g., [77, 63, 62, 47, 48]), we obtain closed-form solutions that determine the domain of strong ellipticity in terms of the relationship between the shear strain and the angle (in general, a function of the radius) between the tangent to the preferred direction and the undeformed radial direction. It is shown, in particular, that as the magnitude of the applied shear stress increases then, after loss of ellipticity, there are two admissible values for the shear strain at certain radial locations. Absolutely stable deformations involve the lower magnitude value outside a certain radius and the higher magnitude value within this radius. The radius that separates the two values increases with increasing magnitude of the shear stress. The results are illustrated graphically for two specific forms of energy function. The work of this chapter has been accepted for publication and will appear in [41]. Also, parts of this work have already been presented in SES-Penn State (2006) by the third author.

In Chapter 4 we are concerned with circular cylindrical tubes composed of incompressible transversely isotropic elastic material subject to simultaneous finite axial extension, inflation and torsion. Here, a great deal of attention is given to the actual kinematics of the problem. Due to the incompressibility constraint, three independent deformations quantities associated with each one of the processes comprising the combined deformation are identified. These serve, in essence, to measure stretch in the axial and azimuthal direction of the body as well as the amount of shear in the planes normal to its radial direction and hence they suffice to fully characterize the resulting strain. Analogously to the azimuthal shear problem examined in the previous chapter, the preferred direction associated with the transverse isotropy is distributed in the planes normal to the tube axis and is disposed so as, in any case, to preserve the cylindrical symmetry. For the considered geometry, the material line elements in the preferred direction always contract when axial extension of the tube is applied. Assuming that the body is held fixed in that extended state, inflation of the tube may be responsible for either further contraction (at least in early stages of the process) or relaxation of the preferred direction. In this situation, the sense of shear is of no importance since the torsional aspect of the deformation has no actual impact on the length of line aliments in that direction. The cylindrical polar components of the Cauchy stress tensor are written down by means of a general form of strain-energy function and then a new universal relation applying for the considered geometry and deformation is generated. In the special situation where the preferred direction lies along, in the undeformed configuration, the radial direction of the body, coaxiality between the Cauchy stress and the left stretch tensors is accomplished and the latter constitutive relation, under appropriate specialization, recovers a well known result holding in the corresponding isotropic theory (see, e.g., [32]). Finally, based on the governing equilibrium equations and in conjunction with the kinematics of the problem, we provide general formulas for the applied loads necessary to support the combined deformation. These are found to apply for a wide range of transversely isotropic materials as well as for isotropic materials. Analogous remarks are briefly made with respect to a specific class of cylindrically orthotropic tubes.

To my beloved family:

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Chapter 1

Continuum basis and elasticity

1.1 Kinematics

1.1.1 Principles of a continuous media

Within the framework of continuum mechanics, a body $\mathcal{B}$ is defined as a set of elements projected upon a three dimensional *Euclidean point space* $\mathcal{E}$. Each element of $\mathcal{B}$, referred to as *particle* or *material point*, has a unique representation in $\mathcal{E}$ and the region $B \subset \mathcal{E}$, say, occupied by the images of the particles is called *configuration* of $\mathcal{B}$.

We assume that the body $\mathcal{B}$ moves (continuously) with time t, occupying various regions in $\mathcal{E}$. Then, if for each t we associate a unique configuration B_t of $\mathcal{B}$ the family of configurations $\{B_t : t \in I \subset \mathbb{R}\}$ prescribes a *motion* of $\mathcal{B}$. In practice, physical observations of the body $\mathcal{B}$ are made in specific configurations and it is therefore convenient to identify a *reference configuration*, namely B_r, which is a fixed, yet arbitrarily chosen, configuration of $\mathcal{B}$. In this way, any particle P of $\mathcal{B}$ may be labelled by its position vector $\mathbf{X}$ relative to some origin O.

As the body moves with time the configuration and hence the position of P changes. Thus, by denoting B_t the configuration of $\mathcal{B}$ at time t, we similarly reference the new place of P in that configuration with a vector $\mathbf{x}$ relative to an origin o which does not necessarily coincide with O. We refer to B_t as the *current configuration.*

The physical properties of a material body $\mathcal{B}$ may equivalently be qualified on referring to either B_r or B_t. In fact, the terminology *referential* (or *Lagrangian*) description is often used in the literature (see, e.g., Ogden [56]) when all the quantities, which may be scalar, vector or tensor fields, that serve to represent the physical status of $\mathcal{B}$ are strictly referred to variables in respect of B_r. This description implies no dependence between $\mathbf{X}$ and t. On the other hand, a *spatial* (or *Eulerian*) description involves determination of all the relevant fields in respect of B_t followed by dependence of $\mathbf{x}$ on t.

1.1.2 Deformation, velocity and acceleration

The path followed in order for P to move from $\mathbf{X}$ to the position $\mathbf{x}$ through time t is defined by the invertible scalar mapping $\boldsymbol{\chi} : B_r \rightarrow B_t$ such that

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t) \quad \text{for all} \quad \mathbf{X} \in B_r, \quad t \in I, \tag{1.1}$$

where $I \subset \mathbb{R}$. In essence, the mapping $\boldsymbol{\chi}$ describes the motion of each particle P, characterised by $\mathbf{X}$, as it proceeds with time and it is called a *deformation* of the body *from* B_r *to* B_t. We mention that since P is generic $\boldsymbol{\chi}$ is, consequently, meant to describe the motion of $\mathcal{B}$ with t as a parameter. It is worth clarifying that, as discussed in Truesdell and Noll [79], the notion of 'deformation' has a general sense including changes both in the shape and the orientation of $\mathcal{B}$ while, for consistency with the definition (1.1), a reference configuration always needs to be introduced [56, 79].

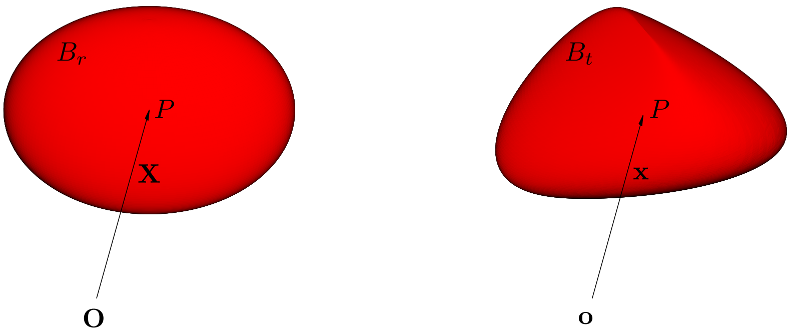

Figure 1.1: Position of a material particle P of $\mathcal{B}$ labelled with $\mathbf{X}$ and $\mathbf{x}$ in B_r and B_t, respectively.

In the referential description, the rate of change of position of P, also referred to as the *velocity* $\mathbf{v}$ of P is defined via the vector field

$$\mathbf{v} = \dot{\mathbf{x}} \equiv \frac{\partial \boldsymbol{\chi}(\mathbf{X}, t)}{\partial t} \tag{1.2}$$

for a given $\mathbf{X}$ and the acceleration $\mathbf{a}$ of P is then given by

$$\mathbf{a} = \dot{\mathbf{v}} \equiv \frac{\partial^2 \boldsymbol{\chi}(\mathbf{X}, t)}{\partial t^2}. \tag{1.3}$$

Similarly, by denoting $\boldsymbol{\chi}_t^{-1} : B_t \rightarrow B_r$ the inverse mapping of $\boldsymbol{\chi}$, obeying

$$\mathbf{X} = \boldsymbol{\chi}_t^{-1}(\mathbf{x}) \quad \text{for all} \quad \mathbf{x} \in B_t, \quad t \in I \tag{1.4}$$

with $I \subset \mathbb{R}$, we may assign a new function , Ψ say, such that

$$\Psi(\mathbf{x}, t) \equiv \boldsymbol{\chi}(\boldsymbol{\chi}_t^{-1}(\mathbf{x}), t), \tag{1.5}$$

to define both $\mathbf{v}$ and $\mathbf{a}$ in spatial description as

$$\mathbf{v} = \dot{\mathbf{x}} = \frac{\partial \Psi(\mathbf{x}, t)}{\partial t} + \frac{\partial \mathbf{x}}{\partial t} \cdot \nabla \Psi(\mathbf{x}, t), \tag{1.6}$$

$$\mathbf{a} = \dot{\mathbf{v}} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}, \tag{1.7}$$

respectively. Here, the symbol ∇ denotes the gradient operator with respect to $\mathbf{x}$.

1.1.3 The deformation gradient tensor and its implications

The discussion provided in the foregoing section clearly suggests that the mapping $\boldsymbol{\chi}$ may assumed to be at least differentiable with respect to $\mathbf{X}$. In the literature, $\boldsymbol{\chi}$ is often regarded as twice-continuously differentiable both with resect to $\mathbf{X}$ and t, yet, under specific circumstances this assertion needs to be suppressed. More details concerning nonsmooth modes of deformation will be given in later chapters.

Accordingly, the differential of (1.1) yields the connection

$$d\mathbf{x} = \mathrm{Grad}\,\boldsymbol{\chi}(\mathbf{X}, t) d\mathbf{X}, \tag{1.8}$$

where Grad is the gradient operator in the reference configuration, i.e. with respect to $\mathbf{X}$, being defined as

$$\mathrm{Grad}\,\boldsymbol{\chi}(\mathbf{X}, t) \equiv \frac{\partial \mathbf{x}}{\partial \mathbf{X}}. \tag{1.9}$$

The entry (1.9) states that the quantity $\mathrm{Grad}\,\boldsymbol{\chi}(\mathbf{X}, t)$ is a second-order tensor field known as the *deformation gradient tensor* depending, in general, both on $\mathbf{X}$ and t. In the special case where the deformation gradient tensor is constant, i.e. $\mathrm{Grad}\,\boldsymbol{\chi}(\mathbf{X}, t)$ does not depend on $\mathbf{X}$ for a fixed t, the deformation is referred to as *homogeneous*. Henceforth, the standard notation $\mathbf{F} = \partial \mathbf{x}/\partial \mathbf{X}$ is adopted for compactness.

It is usually convenient to express the deformation gradient in components form. For this, let X_j and x_i be the components of the position vectors of the particle P with respect to rectangular Cartesian bases $\{\mathbf{E}_j\}$ and $\{\mathbf{e}_i\}$, $(j, i = 1, 2, 3)$ in the reference and current configuration, respectively. Then, if F_{ij} denotes the associated components of $\mathbf{F}$, the argument (1.9) determines $F_{ij} = \partial x_i/\partial X_j$ and

$$\mathbf{F} = F_{ij} \mathbf{e}_i \otimes \mathbf{E}_j, \tag{1.10}$$

where summation over the repeated indices i and j is implied. When the basis $\{\mathbf{E}_j\}$ and/or $\{\mathbf{e}_i\}$ is taken other than Cartesian, the latter can be easily rearranged analogously. Note that, in general, the tensor $\mathbf{F}$ is not symmetric, in other words $F_{ij} \neq F_{ji}$ for $i \neq j$.

In the principles of non-linear field theories and especially in that of continuum solid mechanics the deformation gradient tensor $\mathbf{F}$ has a significant role to play since, amongst others, it describes how small line elements, infinitesimal surface areas and infinitesimal volumes change from the reference to the current configuration during the deformation process. Indeed, as shown in (1.8), $\mathbf{F}$ operates on a small line element $d\mathbf{X}$ at $\mathbf{X} \in B_r$ to linearly transform it into $d\mathbf{x}$ at $\mathbf{x} \in B_t$. Similarly, supposing a surface $\mathcal{S}_r$ in B_r deforms into $\mathcal{S}_t$ in B_t, the deformation gradient tensor establishes the relation

$$\mathbf{n}da = \det(\mathbf{F})\mathbf{F}^{-\mathrm{T}}\mathbf{N}dA \tag{1.11}$$

known as *Nanson's formula*, where dA and da are infinitesimally small area elements on $\mathcal{S}_r$ and $\mathcal{S}_t$ respectively, and $\mathbf{N}$ and $\mathbf{n}$ are their associated unit normals at $\mathbf{X} \in \mathcal{S}_r$ and $\mathbf{x} \in \mathcal{S}_t$. Further, the connection

$$dv = \det(\mathbf{F})dV \tag{1.12}$$

prescribes how infinitesimal volume elements dV in B_r are transformed by $\mathbf{F}$ into dv in B_t. Although the entries (1.11) and (1.12) are purely local, they can easily be put in global forms. Specifically, by letting ∂B_r denote the boundary surface of B_r and ∂B_t the corresponding boundary associated with B_t, the linkages (1.11) and (1.12) may be alternatively be written as

$$\int_{\partial B_t} \mathbf{n}da = \int_{\partial B_r} \det(\mathbf{F})\mathbf{F}^{-\mathrm{T}}\mathbf{N}dA, \tag{1.13}$$

$$\int_{B_t} dv = \int_{B_r} \det(\mathbf{F})dV, \tag{1.14}$$

respectively, describing how the area and the volume of a body change when this undergoes deformation from B_r to B_t. Then, following Ogden [56], the *divergence theorem* is used to provide that (1.13) is equivalent to

$$\mathbf{0} = \int_{B_r} \mathrm{Div}\,(\det(\mathbf{F})\mathbf{F}^{-1})dV, \quad \text{for all} \quad \mathbf{X} \in B_r, \tag{1.15}$$

which, due to regularity assumed here for $\boldsymbol{\chi}$, yields the additional local result

$$\mathrm{Div}\,(\det(\mathbf{F})\mathbf{F}^{-1}) = \mathbf{0}, \quad \text{for all} \quad \mathbf{X} \in B_r. \tag{1.16}$$

Here the symbol Div denotes the divergence operator with respect to the reference configuration.

Another important issue that should be clarified at this point is, in view of (1.8)-(1.12), the qualities of $\mathbf{F}$ and especially that of $\det(\mathbf{F})$. Starting from (1.8), the *conventional requirement* $\det(\mathbf{F}) \neq 0$ needs to be imposed on $\mathbf{F}$ since, otherwise, non-zero line elements in the reference configuration are mapped into line elements with zero length in the current configuration. In other words, taking $\det(\mathbf{F}) \neq 0$ disqualifies the possibility of annihilation of a line element by the deformation process. This assertion is further justified through (1.11) whereas the existence of the inverse mapping $\mathbf{F}^{-1}$ involved in the latter necessitates *non-singular* deformation gradients (i.e. $\det(\mathbf{F}) \neq 0$) in the first place, while, the case of a vanishing surface area element due to deformation is also excluded. Nevertheless, this assumption is not sufficient to ensure a sustainable deformation from a physical perspective. Truly, seeing (1.12), the term $\det(\mathbf{F})$ is clearly understood to be a measure of the change in volume under the deformation and if, by convention, we define volume elements to be positive the more restrictive constraint $\det(\mathbf{F}) > 0$ needs to be adopted. For future reference we introduce the standard notation and convention

$$J = \det(\mathbf{F}) > 0. \tag{1.17}$$

In that respect, if the volume is not to change due to deformation, then

$$J = \det(\mathbf{F}) = 1 \tag{1.18}$$

and the deformation is said to be *isochoric*. In reality, materials capable of preserving their volume under (large) deformations do not exist˙ however, for a number of deformations (1.18) is found to provide a good approximation to material behaviour and it is adopted as an idealization. A material which (ideally) satisfies (1.18) for *all* deformations is called *incompressible* otherwise it is called *compressible*. Probably, the most representative example of an incompressible material is natural rubber. In this thesis, both compressible and incompressible materials are under consideration.

1.1.4 Resolution of stretch and shear

The local attributes of $\mathbf{F}$ may further be considered to provide results of fundamental importance in the development of the theory of continuum mechanics.

First, bearing (1.17) in mind, the property (1.8) is used to establish the relation

$$|d\mathbf{x}|^2 = d\mathbf{X} \cdot (\mathbf{F}^{\mathrm{T}}\mathbf{F})d\mathbf{X}, \tag{1.19}$$

which serves to measure the change in length of line elements from the reference to the current configuration. In a similar manner, by letting $d\mathbf{X}, d\mathbf{X}'$ denote a pair of line elements based at $\mathbf{X} \in B_r$ and $d\mathbf{x}, d\mathbf{x}'$ their corresponding images at $\mathbf{x} \in B_t$, the inner product

$$d\mathbf{x} \cdot d\mathbf{x}' = d\mathbf{X} \cdot (\mathbf{F}^{\mathrm{T}}\mathbf{F})d\mathbf{X}', \tag{1.20}$$

measures changes in angle between two line elements due to deformation.

Given the direction of $d\mathbf{X}$ and $d\mathbf{X}'$ in B_r, both (1.19) and (1.20) are properly reformulated, yielding more transparent and exploitable conclusions. Expressly, if $d\mathbf{X}$ is taken to lie along the unit vector $\mathbf{i}_1$, say, then $d\mathbf{X} = \mathbf{i_1}|d\mathbf{X}|, d\mathbf{x} = (\mathbf{F}\mathbf{i}_1)|d\mathbf{X}|$ and (1.19) becomes

$$|d\mathbf{x}|^2 = \mathbf{i}_1 \cdot (\mathbf{F}^{\mathrm{T}}\mathbf{F}\mathbf{i}_1)|d\mathbf{X}|^2. \tag{1.21}$$

On using (1.21), we may therefore introduce the scalar field $\lambda(\mathbf{i}_1) \in (0, \infty)$, defined by

$$\lambda(\mathbf{i}_1) \equiv |\mathbf{F}\mathbf{i}_1| = [\mathbf{i}_1 \cdot (\mathbf{F}^{\mathrm{T}}\mathbf{F}\mathbf{i}_1)]^{1/2} = \frac{|d\mathbf{x}|}{|d\mathbf{X}|}, \tag{1.22}$$

which is called the *stretch in the direction* $\mathbf{i}_1$ at $\mathbf{X}$. Accordingly, the unit vector $\mathbf{i}_2$ is chosen as $d\mathbf{X}' = \mathbf{i}_2|d\mathbf{X}'|$, so that $d\mathbf{x}' = (\mathbf{F}\mathbf{i}_2)|d\mathbf{X}'|$ and then (1.20) gives

$$d\mathbf{x} \cdot d\mathbf{x}' = [\mathbf{i}_1 \cdot (\mathbf{F}^{\mathrm{T}}\mathbf{F}\mathbf{i}_2)]|d\mathbf{X}||d\mathbf{X}'|. \tag{1.23}$$

Supposing that Φ is the angle between $\mathbf{i}_1$ and $\mathbf{i}_2$ in the reference configuration, i.e. $\mathbf{i}_1 \cdot \mathbf{i}_2 = \cos\Phi$, and ϕ that after deformation, the entry (1.23) in conjunction with (1.22) contributes

$$\cos\phi = \frac{d\mathbf{x} \cdot d\mathbf{x}'}{|d\mathbf{x}||d\mathbf{x}'|} = \frac{\mathbf{i}_1 \cdot (\mathbf{F}^{\mathrm{T}}\mathbf{F}\mathbf{i}_2)}{\lambda(\mathbf{i}_1)\lambda(\mathbf{i}_2)}. \tag{1.24}$$

The change in angle $\Phi - \phi > 0$ (< 0) is called the *shear* of the directions $\mathbf{i}_1$ and $\mathbf{i}_2$ in the *plane of shear* defined by the pair $(\mathbf{i}_1, \mathbf{i}_2)$.

1.1.5 Measurement of deformation: Strain

In practice, experimental tests carried out on material specimens enable us to achieve local measurements in stretch and shear in order to have a better understanding of the mechanical response of solids under deformation. As a matter of fact, significant conclusions regarding the behaviour and the status of a material body undergoing deformation are based on reckoning the difference between the squared lengths of a line element in the reference and current configuration. Translated into mathematical language, this is taken to be given as

$$|d\mathbf{x}|^2 - |d\mathbf{X}|^2 = d\mathbf{X} \cdot (\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{I})d\mathbf{X}, \tag{1.25}$$

arising directly from (1.21) after a simple rearrangement. Here, $\mathbf{I}$ is the identity tensor. Local measurements of the change in $|d\mathbf{x}|^2 - |d\mathbf{X}|^2$ prescribe the *strain* on a material as a result of deformation and hence, through (1.25), the tensor $\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{I}$ constitutes a measure of strain. This prompts us to introduce the so-called *Green* (or *Lagrangian*) *strain tensor*, namely $\mathbf{E}$, defined by

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{I}). \tag{1.26}$$

In components form, the latter is given as

$$E_{ij} = \frac{1}{2}(F_{pi}F_{pj} - \delta_{ij}) = \frac{1}{2}\left(\frac{\partial x_p}{\partial X_i}\frac{\partial x_p}{\partial X_j} - \delta_{ij}\right), \tag{1.27}$$

where the symbol δ_{ij} denotes *Kronecker's delta*.

Strain, however, can also be measured in the current configuration whereas the assumption (1.17) validates the formula $d\mathbf{X} = \mathbf{F}^{-1}d\mathbf{x}$ on use of which (1.25) is alternatively written

$$|d\mathbf{x}|^2 - |d\mathbf{X}|^2 = d\mathbf{x} \cdot (\mathbf{I} - \mathbf{F}^{-\mathrm{T}}\mathbf{F}^{-1})d\mathbf{x}. \tag{1.28}$$

Thus, the counterpart of $\mathbf{E}$ in the current configuration is being defined by the *Eulerian strain tensor* $\mathbf{e}$, namely

$$\mathbf{e} = \frac{1}{2}(\mathbf{I} - \mathbf{F}^{-\mathrm{T}}\mathbf{F}^{-1}), \tag{1.29}$$

with components

$$e_{ij} = \frac{1}{2}\left[\delta_{ij} - (\mathbf{F}^{-\mathrm{T}}\mathbf{F}^{-1})_{ij}\right] = \frac{1}{2}\left(\delta_{ij} - \frac{\partial X_p}{\partial x_i}\frac{\partial X_p}{\partial x_j}\right). \tag{1.30}$$

The relation between $\mathbf{E}$ and $\mathbf{e}$ will be established shortly. We observe that, unlike $\mathbf{F}$, the strain tensors $\mathbf{E}$ and $\mathbf{e}$ are *symmetric* since, from (1.27) and (1.30) we easily obtain $E_{ij} = E_{ji}$ and $e_{ij} = e_{ji}$ respectively.

A material is said to be *unstrained* at $\mathbf{X}$ if the lengths of all line elements $d\mathbf{X}$ based at $\mathbf{X}$ remain unchanged. Obviously, this implies $|d\mathbf{x}| = |d\mathbf{X}|$ for all $d\mathbf{X}$ or, equivalently

$$d\mathbf{X} \cdot (\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{I})d\mathbf{X} = 0 \quad \text{for all } d\mathbf{X}, \tag{1.31}$$

and therefore $\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{I} = \mathbf{0}$ being the zero tensor. Hence, through (1.22), we may also easily derive that the material is unstrained at $\mathbf{X}$ when

$$\lambda(\mathbf{i}_1) = 1 \quad \text{for all unit vectors } \mathbf{i}_1, \tag{1.32}$$

which, in accordance with (1.26), holds if and only if

$$\mathbf{E} = \mathbf{0}. \tag{1.33}$$

It is worth noting that in the situation where the deformation is just a rotation, i.e. $\mathbf{F}$ is simply a *proper orthogonal* tensor independent of the position $\mathbf{X}$, the conditions (1.18) and $\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{I} = \mathbf{0}$ are automatically satisfied, yielding (1.33). Actually, it can be shown that for such deformations the latter is satisfied for all $\mathbf{X}$ in B_r in which case the body is said to be unstrained. For more details we referred to the work of Ogden [56].

In addition, in the case where $\lambda(\mathbf{i}_1) > 1$ the material is said to be *extended* at $\mathbf{X}$, or undergoes *extension* at $\mathbf{X}$, in the direction $\mathbf{i}_1$. Analogously, when $\lambda(\mathbf{i}_1) < 1$ we say that the material is *contracted* at $\mathbf{X}$, or undergoes *contraction* at $\mathbf{X}$, in the direction $\mathbf{i}_1$.

An alternative way to measure strain is through the spatial derivatives of the displacement $\mathbf{u}$ of a particle $\mathbf{X}$ from the reference to the current configuration. This is defined as

$$\mathbf{u} = \mathbf{x} - \mathbf{X}, \tag{1.34}$$

from which follows

$$\mathbf{x} = \mathbf{u} + \mathbf{X}. \tag{1.35}$$

Thus, if we denote by $\mathbf{D}$ the *displacement gradient tensor*, namely $\mathrm{Grad}\,\mathbf{u} \equiv \partial\mathbf{u}/\partial\mathbf{X}$, the definitions (1.35) and (1.9) are combined to give

$$\mathbf{F} = \mathbf{I} + \mathbf{D}. \tag{1.36}$$

Then, the tensor (1.26) can be expressed as

$$\mathbf{E} = \frac{1}{2}(\mathbf{D} + \mathbf{D}^{\mathrm{T}} + \mathbf{D}^{\mathrm{T}}\mathbf{D}) \tag{1.37}$$

and, equivalently, in component form

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_p}{\partial X_i}\frac{\partial u_p}{\partial X_j}\right), \tag{1.38}$$

with u_i being the components of $\mathbf{u}$ with respect to the reference basis $\{\mathbf{E}_i\}$.

In the same spirit, the definition (1.34) can be used to obtain

$$\mathbf{F}^{-1} = \mathbf{I} - \mathbf{d}, \tag{1.39}$$

where $\mathbf{d}$ is the gradient operator $\mathrm{grad}\,\mathbf{u} \equiv \partial\mathbf{u}/\partial\mathbf{x}$ and the Eulerian strain tensor (1.29) becomes

$$\mathbf{e} = \frac{1}{2}(\mathbf{d} + \mathbf{d}^{\mathrm{T}} - \mathbf{d}^{\mathrm{T}}\mathbf{d}). \tag{1.40}$$

If now $\tilde{u}_i$ denotes the components of $\mathbf{u}$ with respect to the current basis $\{\mathbf{e}_i\}$, the counterparts of (1.30) are

$$e_{ij} = \frac{1}{2}\left(\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i} - \frac{\partial \tilde{u}_p}{\partial x_i}\frac{\partial \tilde{u}_p}{\partial x_j}\right). \tag{1.41}$$

1.1.6 Polar decomposition of the deformation gradient

Two important theorems, namely the *square root* and the *polar decomposition* theorem, predicated on the algebra of second-order symmetric tensors are now demonstrated to highlight a few other elements of the local nature of the deformation and especially that of its geometrical interpretation. Expressly, the result of the first, in the order given, theorem is used to establish the arguments of the second˙ however, no details concerning their proofs are provided here since these can widely be found in the literature. We cite here, for example, the books by Ogden [56, 57] and Jaunzemis [36].

The square root theorem

For any positive definite, symmetric second-order tensor, $\mathbf{S}$ *say, there exists a unique, positive definite, symmetric second-order tensor denoted* $\mathbf{T}$*, such that* $\mathbf{T}^2 = \mathbf{S}$.

The polar decomposition theorem

Consider a second-order Cartesian tensor $\mathbf{F}$ *embodying the property* $\det(\mathbf{F}) > 0$. *Then, there exist unique, positive definite, symmetric second-order tensors,* $\mathbf{U}$ *and* $\mathbf{V}$*, as well as a proper orthogonal tensor* $\mathbf{R}$*, such that*

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR}. \tag{1.42}$$

Clearly, due to the conventional assumption (1.17), the latter may directly be applied to the deformation gradient and then the tensors $\mathbf{U}$ and $\mathbf{V}$ involved in (1.42) respectively are called the *right* and *left stretch tensors*. In fact, the tensor $\mathbf{U}$ lies in the reference configuration and $\mathbf{V}$ in the current configuration, yet, both of them serve to measure stretch along their *principal directions* and hence the terminology stretch tensors, while the proper orthogonal tensor $\mathbf{R}$ is employed to rotate their principal axes from one configuration to the other.

To demonstrate this, we first avail of the square root theorem to establish the relations

$$\mathbf{U}^2 = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \mathbf{C}, \quad \mathbf{V}^2 = \mathbf{F}\mathbf{F}^{\mathrm{T}} = \mathbf{B}, \tag{1.43}$$

where now the tensors $\mathbf{C}$ and $\mathbf{B}$ are referred to as the *right* and *left Cauchy-Green deformation tensors*, respectively, both of which serve to measure deformation. The role of $\mathbf{C}$ is undoubtedly justified via (1.25), implying that the Lagrangian strain tensor becomes

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) = \frac{1}{2}(\mathbf{U}^2 - \mathbf{I}). \tag{1.44}$$

In a like manner, the involvement of $\mathbf{B}$ in (1.28) comprehends that the Eulerian strain tensor may be conveyed to

$$\mathbf{e} = \frac{1}{2}(\mathbf{I} - \mathbf{B}^{-1}) = \frac{1}{2}(\mathbf{I} - \mathbf{V}^{-2}). \tag{1.45}$$

Undoubtedly, since $\mathbf{C}^{\mathrm{T}} = (\mathbf{F}^{\mathrm{T}}\mathbf{F})^{\mathrm{T}} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \mathbf{C}$ and in accordance with (1.19), $(1.43)_1$, the tensor $\mathbf{C}$ is, similar to $\mathbf{U}$, symmetric and positive definite. In fact, the aforementioned properties of $\mathbf{U}$ lead to the conclusion that the latter can be written in the spectral form

$$\mathbf{U} = \sum_{i=1}^{3} \lambda_i \mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)}, \tag{1.46}$$

where the scalar quantities $\lambda_i > 0$ are the eigenvalues of $\mathbf{U}$ and $\mathbf{u}^{(i)}$ are the associated (unit) eigenvectors. In other words λ_i signify the principal values of $\mathbf{U}$ with principal directions $\mathbf{u}^{(i)}$ such that $\mathbf{U}\mathbf{u}^{(i)} = \lambda_i \mathbf{u}^{(i)}$. This, in conjunction with $(1.43)_1$ and (1.22) results in

$$\begin{aligned}\lambda(\mathbf{u}^{(i)}) &= [\mathbf{u}^{(i)} \cdot (\mathbf{F}^{\mathrm{T}}\mathbf{F}\mathbf{u}^{(i)})]^{1/2} = [\mathbf{u}^{(i)} \cdot (\mathbf{U}^2\mathbf{u}^{(i)})] \\ &= [(\mathbf{U}\mathbf{u}^{(i)}) \cdot (\mathbf{U}\mathbf{u}^{(i)})]^{1/2} = [(\lambda_i\mathbf{u}^{(i)}) \cdot (\lambda_i\mathbf{u}^{(i)})]^{1/2} = \lambda_i\end{aligned}$$

and hence the eigenvalues of the right stretch tensor indeed measure stretch in the directions $\mathbf{u}^{(i)}$. Accordingly, we refer to λ_i as the *principal stretches* of the deformation while the principal axes $\mathbf{u}^{(i)}$ of $\mathbf{U}$ are often referred to as the *Lagrangian principal axes*.

Next, from (1.42), we have

$$\mathbf{V}(\mathbf{R}\mathbf{u}^{(i)}) = \mathbf{V}\mathbf{R}\mathbf{u}^{(i)} = \mathbf{R}\mathbf{U}\mathbf{u}^{(i)} = \mathbf{R}(\lambda_i\mathbf{u}^{(i)}) = \lambda_i(\mathbf{R}\mathbf{u}^{(i)}),$$

which shows that λ_i are also the principal values of $\mathbf{V}$ relative to the principal directions $\mathbf{R}\mathbf{u}^{(i)}$. Thus, by denoting $\mathbf{v}^{(i)} = \mathbf{R}\mathbf{u}^{(i)}$, we may analogously obtain the expression

$$\mathbf{V} = \sum_{i=1}^{3} \lambda_i \mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)}, \tag{1.47}$$

where again $\mathbf{V}$ is symmetric and positive definite and $\mathbf{v}^{(i)}$ are called the *Eulerian principal axes*.

Consequently, the polar decomposition theorem states that the deformation gradient $\mathbf{F}$ can sufficiently be regarded as a two-phase process consisting of a rotation and a stretching. In any case, the order in which the rotation and the stretch take place does not matter. Truly, while $\mathbf{F} = \mathbf{R}\mathbf{U}$ comprehends stretch of line elements along the $\mathbf{u}^{(i)}$ directions and then rotation of those to the positions $\mathbf{v}^{(i)}$, the equivalent formula $\mathbf{F} = \mathbf{VR}$ implies transition from $\mathbf{u}^{(i)}$ to $\mathbf{v}^{(i)}$ which is followed by a stretch in these final directions.

We remark that connection of $\mathbf{C}$ with $\mathbf{U}$ and of $\mathbf{B}$ with $\mathbf{V}$ readily provide the formulas

$$\mathbf{C} = \sum_{i=1}^{3} \lambda_i^2 \mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)}, \tag{1.48}$$

$$\mathbf{B} = \sum_{i=1}^{3} \lambda_i^2 \mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)}. \tag{1.49}$$

In this connection, the strain tensors (1.44) and (1.45) can also be expressed in terms of the principal stretches, as

$$\mathbf{E} = \frac{1}{2}\sum_{i=1}^{3}(\lambda_i^2 - 1)\mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)}, \quad \mathbf{e} = \frac{1}{2}\sum_{i=1}^{3}(1 - \lambda_i^{-2})\mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)}, \tag{1.50}$$

respectively.

Another direct consequence of the polar decomposition theorem is that the determinant J, given in (1.17), can equivalently be defined by

$$J = \det(\mathbf{U}) = \sqrt{\det(\mathbf{C})} = \det(\mathbf{V}) = \sqrt{\det(\mathbf{B})} = \lambda_1\lambda_2\lambda_3, \tag{1.51}$$

while the linkages (1.42) are also used to derive

$$\mathbf{U} = \mathbf{R}^{\mathrm{T}}\mathbf{VR}, \quad \mathbf{V} = \mathbf{RUR}^{\mathrm{T}}. \tag{1.52}$$

These are now combined with (1.43) to obtain

$$\mathbf{C} = \mathbf{R}^{\mathrm{T}}\mathbf{BR}, \quad \mathbf{B} = \mathbf{RCR}^{\mathrm{T}}, \tag{1.53}$$

and due to (1.44) and (1.45) the tensors $\mathbf{E}$ and $\mathbf{e}$ are then found to satisfy

$$\mathbf{E} = \frac{1}{2}[\mathbf{R}^{\mathrm{T}}(\mathbf{I} - 2\mathbf{e})^{-1}\mathbf{R} - \mathbf{I}], \quad \mathbf{e} = \frac{1}{2}[\mathbf{I} - \mathbf{R}(\mathbf{I} + 2\mathbf{E})^{-1}\mathbf{R}^{\mathrm{T}}]. \tag{1.54}$$

Finally in this section it is useful to note that if no rotation of the principal axes occurs during the deformation i.e. $\mathbf{R} = \mathbf{I}$ we have $\mathbf{F} = \mathbf{U} = \mathbf{V}, \mathbf{C} = \mathbf{B}$ as well as $\mathbf{E} = \mathbf{e}$ and the deformation is known as *pure strain*.

1.2 Stress principles, balance laws and equilibrium

1.2.1 Applied forces and torques

We consider a motion of a deforming material body $\mathcal{B}$ caused by the action of a system of *applied forces*. Applied forces are, by definition, measured in the current configuration B_t and they are considered as the resultant force of *body* and *contact forces*.

The body forces, locally measured *per unit mass*, are distributed throughout the continuum i.e. act directly on each particle of the body in B_t and are (locally) described by a vector field $\mathbf{b}$. Such forces may be associated with gravity or the existence of magnetic fields, for example. The concept of contact forces, on the other hand, demonstrates the effect of the mutual actions of parts of the body which is assumed to (locally) occur across a separating closed surface embedded in $\mathcal{B}$. We mention, for example, pressure.

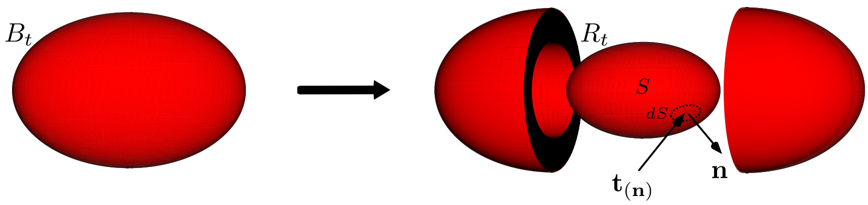

Figure 1.2: Traction $\mathbf{t}_{(\mathbf{n})}$ exerted on material $R_t \subset B_t$ at a point of a surface element $dS \subset S \equiv \partial R_t$, with unit normal $\mathbf{n}$.

In this respect, we let S denote a surface enclosing material occupying $R_t \subset B_t$ with mass

$$m(R_t) = \int_{R_t} \rho dv, \tag{1.55}$$

where here the volume element dv is relevant to R_t while the quantity ρ is an Eulerian scalar field referred to as the *mass density* of the material composing $\mathcal{B}$. Then, as shown in Figure 1.2, the contact force exerted by material $B_t \setminus R_t$ on the material R_t at a fixed point $\mathbf{x} \in S \equiv \partial R_t$ is defined by a vector field $\mathbf{t}_{(\mathbf{n})}$ called the *traction* or *stress vector field per unit surface area*. Obviously, the nature of $\mathbf{t}_{(\mathbf{n})}$ is purely local and applies on an infinitesimal surface area element dS of S. It is also apparent that one may consider infinite surfaces with the above mentioned properties. Thus, depending on the orientation of each surface, the traction $\mathbf{t}_{(\mathbf{n})}$ acting on $\mathbf{x}$ will, in general, vary. Hence, it is assumed to depend continuously on $\mathbf{n}$ with the latter being the outward unit normal to S (and dS).

Accordingly, the contribution of the body and contact forces to the applied force, $\mathcal{T}(R_t)$ say, acting on the material occupying R_t are given by

$$\int_{R_t} \rho \mathbf{b} dv, \quad \int_{\partial R_t} \mathbf{t}_{(\mathbf{n})} dS$$

respectively, and hence

$$\mathcal{T}(R_t) = \int_{R_t} \rho \mathbf{b} dv + \int_{\partial R_t} \mathbf{t}_{(\mathbf{n})} dS. \tag{1.56}$$

The action of body forces on R_t justifies the presence of *body torques*

$$\int_{R_t} \rho (\mathbf{x} \times \mathbf{b}) dv, \tag{1.57}$$

while contact forces yield *contact torques*

$$\int_{\partial R_t} \mathbf{x} \times \mathbf{t}_{(\mathbf{n})} dS. \tag{1.58}$$

Thus, analogously to the resultant force, the resultant torque, denoted $\mathcal{L}(R_t; o)$, is formulated as

$$\mathcal{L}(R_t; o) = \int_{R_t} \rho(\mathbf{x} \times \mathbf{b}) dv + \int_{\partial R_t} \mathbf{x} \times \mathbf{t}_{(\mathbf{n})} dS. \tag{1.59}$$

We remark, however, that due to the rotational nature of (1.57) and (1.58) the vector field $\mathcal{L}$, unlike $\mathcal{T}$, now depends on the choice of an origin o relevant to R_t.

1.2.2 Balance of mass, linear and angular momentum

For a moving material body the mass density is, in general, assumed to depend continuously both on position and time, i.e. $\rho = \rho(\mathbf{x}, t)$ and therefore, referring to the foregoing paragraph, on R_t. By contrast, the mass $m(R_t)$ itself, and despite the notation adopted here, is presumed to be strictly independent of configuration. In other words, the region R_t changes during the motion of the body, yet, still comprises of the same material. This conveys that the mass remains unchanged as the body moves, which, in mathematical terms, corresponds with the *mass conservation* or *mass balance equation*, given by

$$\frac{d}{dt} m(R_t) = 0. \tag{1.60}$$

For a smooth ρ the latter may be combined with (1.55) to give the local formula

$$\frac{d\rho}{dt} + \rho \operatorname{div} \mathbf{v} = 0, \tag{1.61}$$

known as the *continuity equation*, where $\mathbf{v}$ is the velocity as defined in (1.2), div is the divergence operator with respect to $\mathbf{x} \in R_t$ and $d\rho/dt = (\partial \rho/\partial t) + \mathbf{v} \cdot \nabla \rho$; the symbol ∇ denoting the gradient operator with respect to $\mathbf{x}$. In fact, validity of the continuity equation requires continuity of the term $\partial(\rho J)/\partial t$ that, indeed, coincides with the left-hand side of (1.61). This assertion arises from the property

$$J \operatorname{div} \mathbf{v} = \frac{\partial J}{\partial t}, \tag{1.62}$$

on the use of which (1.61) is integrated to give the alternative local connection

$$J\rho = \rho_0, \tag{1.63}$$

with $\rho_0 = \rho_0(\mathbf{X})$ being the counterpart of ρ in the reference configuration. Then, we recall (1.17) to deduce $\rho_0/\rho > 0$.

In view of (1.63) we further obtain that for an isochoric deformation, where $J = 1$, the mass density embodies the property $\rho = \rho_0$ while, from (1.62), we deduce $\operatorname{div}\mathbf{v} = \partial J/\partial t = 0$. Thus, by virtue of (1.51), this can be rearranged in the form

$$\frac{\partial J}{\partial t} \equiv \lambda_1^{-1}\frac{\partial \lambda_1}{\partial t} + \lambda_2^{-1}\frac{\partial \lambda_2}{\partial t} + \lambda_3^{-1}\frac{\partial \lambda_3}{\partial t} = 0. \tag{1.64}$$

Two basic dynamical measures of motion are the *linear* and *angular momentum.* For a material occupying $R_t \subset B_t$, these are taken as

$$\mathbf{M}(R_t) = \int_{R_t} \rho\mathbf{v}dv, \quad \mathbf{H}(R_t; o) = \int_{R_t} \mathbf{x} \times (\rho\mathbf{v})dv, \tag{1.65}$$

respectively, where, again, the symbol o involved in $\mathbf{H}$ signifies dependence with respect to an origin in R_t.

Based on Newton's second law of motion, Euler proposed two fundamental principles of mechanics applicable to all finitely deformable bodies, namely the *balance of linear momentum*

$$\frac{d\mathbf{M}}{dt} = \mathcal{T}, \tag{1.66}$$

and the *balance of angular momentum*

$$\frac{d\mathbf{H}}{dt} = \mathcal{L}, \tag{1.67}$$

together referred to as *Euler's laws of motion.* Coherently, the formulas (1.66) and (1.67) are appropriate to moving material bodies in respect of a system of applied forces as prescribed in Section 1.2.1.

Accordingly, substitution of (1.56) and $(1.65)_1$ into (1.66) and, in like manner, of (1.59) and $(1.65)_2$ into (1.67) enables us to respectively obtain Euler's laws of motion in the full forms

$$\int_{R_t} \rho(\mathbf{a} - \mathbf{b})dv = \int_{\partial R_t} \mathbf{t}_{(\mathbf{n})}dS, \tag{1.68}$$

$$\int_{R_t} \rho\mathbf{x} \times (\mathbf{a} - \mathbf{b})dv = \int_{\partial R_t} \mathbf{x} \times \mathbf{t}_{(\mathbf{n})}dS. \tag{1.69}$$

Recall that $\mathbf{a}$ is the acceleration of a particle with position $\mathbf{x}$ now associated with the region R_t.

1.2.3 Cauchy's theory of stress

Following Cauchy's theorem, the linear dependence of the stress vector field $\mathbf{t}_{(\mathbf{n})}$ on $\mathbf{n}$, assumed so far, justifies the existence of a second-order tensor field $\boldsymbol{\sigma}$ that, for each unit vector $\mathbf{n}$ and for all $\mathbf{x}$ in R_t, satisfies

$$\mathbf{t}_{(\mathbf{n})} = \boldsymbol{\sigma}^{\mathrm{T}}\mathbf{n}. \tag{1.70}$$

The tensor $\boldsymbol{\sigma}$, measuring *force per deformed area*, is called the *Cauchy stress tensor* and it strictly depends on the point $\mathbf{x}$ at which $\mathbf{t}_{(\mathbf{n})}$ acts but not on $\mathbf{n}$ i.e. $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{x})$.

Since the Cauchy stress tensor $\boldsymbol{\sigma}$, in essence, arises from the presence of contact forces, the formula (1.70) has to satisfy Euler's laws of motion. Expressly, under suitable continuity conditions placed upon $\rho, \mathbf{b}$ and $\mathbf{a}$, equation (1.70) is used to establish that (1.68) and (1.69) hold if and only if

$$\mathrm{div}\boldsymbol{\sigma}^{\mathrm{T}} + \rho\mathbf{b} = \rho\mathbf{a}, \tag{1.71}$$

$$\boldsymbol{\sigma}^{\mathrm{T}} = \boldsymbol{\sigma}. \tag{1.72}$$

Equation (1.71) is known as *Cauchy's first law of motion* and is the necessary and sufficient condition for the linear momentum balance to be in place. The angular momentum balance, on the other hand, is justified by the *Cauchy's second law of motion*, namely (1.72), which necessitates symmetry in $\boldsymbol{\sigma}$.

We emphasise, however, that the nature of problems that we will be dealing with in the forthcoming chapters are purely *static* while body forces are considered to be negligible. In this regard, we take $\mathbf{a} = \mathbf{b} = \mathbf{0}$ for all $\mathbf{x} \in R_t$ and the entries (1.71) and (1.72) are written together in the compact form

$$\mathrm{div}\boldsymbol{\sigma} = \mathbf{0}. \tag{1.73}$$

This is known as the *equilibrium equation*.

It is worth noting that the stress tensor $\boldsymbol{\sigma}$ incorporates, in general, nine components. Precisely, as demonstrated in Figure 1.3 when these are referred to rectangular Cartesian coordinates the notation σ_{ij} $(i, j = 1, 2, 3)$ is used to indicate the force acting in the direction i at a point $\mathbf{x}$ in the current configuration on a surface whose normal lies along the direction j. Therefore, the diagonal components, namely σ_{ii} (no sum over i), of $\boldsymbol{\sigma}$ are parallel to the normal of the surface on which they act and are called *normal stresses* while the remaining components σ_{ij} $(i \neq j)$, called the *shear stresses*, are distributed along the planes of action i.e. are tangent to these surfaces. Notwithstanding, in accordance with (1.72), only six components of $\boldsymbol{\sigma}$ are independent and sufficient to interpret the contact forces associated with a motion of a material body.

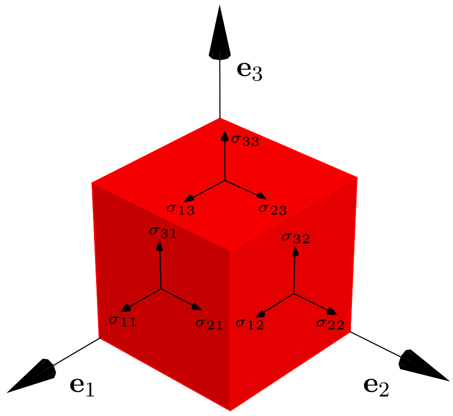

Figure 1.3: Components of the Cauchy stress tensor $\boldsymbol{\sigma}$ acting on three elementary material surfaces normal to $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$ in the current configuration.

Of course, analogous conclusions hold on using different coordinate systems to describe the position of each particle in the current configuration.

With reference to rectangular Cartesian coordinates, the equilibrium equation (4.3.3) may also be written down in components form as

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0, \quad i \in \{1,2,3\} \quad (\text{sum over } j) \tag{1.74}$$

which can be expanded to give

$$\begin{aligned} \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} &= 0, \\ \frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} &= 0, \\ \frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} &= 0. \end{aligned} \tag{1.75}$$

Alternatively, in terms of cylindrical polar coordinates (r, θ, z) the counterparts of (1.75) become

$$\begin{aligned} \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) &= 0, \\ \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2}{r}\sigma_{r\theta} &= 0, \\ \frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{1}{r}\sigma_{rz} &= 0. \end{aligned} \tag{1.76}$$

1.2.4 Energy balance

Based on Cauchy's first and second laws of motion, additional conclusions, now regarding the energy of a deforming body material, can be derived. Actually, as in the classical theory, where the rate of change of the *kinetic energy* of a moving body is equal to the *rate of working* of the applied forces, the continuum approach adopted here is used to establish parallel results. In this case, however, another form of energy, referred to as *elastic strain energy*, is also taken into account.

In point of fact, by denoting

$$E_{kin}(R_t) = \int_{R_t} \frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v}dv,$$

the kinetic energy of the material occupying the region R_t, and

$$P(R_t) = \int_{R_t} \rho\mathbf{b}\cdot\mathbf{v}dv + \int_{\partial R_t} \mathbf{t}_{(\mathbf{n})}\cdot\mathbf{v}dS,$$

the rate of working of the forces acting on this part of material, equations (1.71), (1.72) are combined with (1.12), (1.63) and (1.70) to provide the relation

$$P(R_t) = \frac{d}{dt}E_{kin}(R_t) + \int_{R_t} \mathrm{tr}\,(\boldsymbol{\sigma}\mathbf{D})dv, \tag{1.77}$$

where now the term $\mathbf{D}$ is a symmetric second order tensor called the (*Eulerian*) *strain-rate or rate of stretching tensor*. This is defined via the *velocity gradient tensor*, namely $\mathbf{L} = \mathrm{grad}\,\mathbf{v}$, such that

$$\mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^{\mathrm{T}}). \tag{1.78}$$

Now, the integral

$$\int_{R_t} \mathrm{tr}\,(\boldsymbol{\sigma}\mathbf{D})dv, \tag{1.79}$$

appearing in the right hand side of (1.77) is the *rate of working of the stresses* (or *stress power*) on the region R_t of the body and it states that, apart from kinetic energy, the work done by body and surfaces forces is also converted into an internal potential energy which, in later sections, we refer to as elastic strain energy. The latter argument is in reality a special case and it is substantiated when the mechanical behaviour of the material under examination is considered to be *elastic*. Then, equation (1.77) constitutes the *mechanical energy balance equation*. For materials whose nature is not as such, however, the quantity (1.79) can be taken to incorporated heat dissipation phenomena. For a more detailed discussion on the latter approach we refer to the work of Truesdell and Noll [79] and Coleman and Noll [13].

1.2.5 The nominal stress tensor

In some situations it is more convenient to measure the contact forces acting on a deforming body material with respect to the reference configuration. For this, we employ Nanson's formula (1.11) and (1.17), (1.72) to rearrange (1.70) as

$$\mathbf{t}_{(\mathbf{n})}dS = \boldsymbol{\sigma}\mathbf{n}dS = \mathbf{S}^{\mathrm{T}}\mathbf{N}dS_0, \tag{1.80}$$

where now dS_0 is the projection of $dS \subset S \equiv \partial R_t$ in the reference configuration and

$$\mathbf{S} = J\mathbf{F}^{-1}\boldsymbol{\sigma}, \tag{1.81}$$

represents the so-called *nominal or engineering stress tensor*. The tensor $\mathbf{S}$, and in particular its transpose $\mathbf{S}^{\mathrm{T}}$, introduces the notion of *force per unit reference area*, and unlike $\boldsymbol{\sigma}$, is not in general symmetric. However, it satisfies the connection

$$\mathbf{F}\mathbf{S} = \mathbf{S}^{\mathrm{T}}\mathbf{F}^{\mathrm{T}},$$

arising from the symmetry (1.72) of $\boldsymbol{\sigma}$. The transpose of $\mathbf{S}$ is sometimes referred to as the first Piola-Kirchoff stress tensor but we do not use this terminology here.

Further, in accordance with the discussion given in Section 1.2.3 for $\boldsymbol{\sigma}$, the components S_{ij} of $\mathbf{S}$ represent the force (per unit reference area) acting in the direction j at a point $\mathbf{x}$ in the current configuration on a surface whose normal was in the direction i in the reference configuration.

On use of (1.80), Cauchy's first law of motion (1.68), and hence the equilibrium equation (4.3.3), can be expressed in terms of the nominal stress tensor $\mathbf{S}$. Precisely, by denoting R_0 the counterpart of R_t in the reference configuration, equation (1.80) is applied on the right hand-side of (1.68) to interpret the contact forces over the boundary $S_0 \equiv \partial R_0$ as

$$\int_{\partial R_t} \mathbf{t}_{(\mathbf{n})}dS = \int_{\partial R_t} \boldsymbol{\sigma}\mathbf{n}dS = \int_{\partial R_0} \mathbf{S}^{\mathrm{T}}\mathbf{N}dS_0.$$

At the same time, the entry (1.12) is combined with (1.63) to obtain $\rho dv = \rho_0 dV$. This enables the transformation

$$\int_{R_t} \rho(\mathbf{a}-\mathbf{b})dv = \int_{R_0} \rho_0(\mathbf{a}_0-\mathbf{b}_0)dV,$$

with $\mathbf{a}_0, \mathbf{b}_0$ being the counterparts of $\mathbf{a}$ and $\mathbf{b}$ in the reference configuration, respectively. As a result, equation (1.68) may equivalently be written in its Lagrangian description as

$$\int_{R_0} \rho_0(\mathbf{a}_0-\mathbf{b}_0)dV = \int_{\partial R_0} \mathbf{S}^{\mathrm{T}}\mathbf{N}dS_0 \tag{1.82}$$

and, by the divergence theorem, the analogues of (1.71) and (4.3.3) become

$$\mathrm{Div}\,\mathbf{S} + \rho_0\mathbf{b}_0 = \rho_0\mathbf{a}_0, \tag{1.83}$$

$$\mathrm{Div}\,\mathbf{S} = \mathbf{0}, \tag{1.84}$$

respectively.

Thereafter, similarly to (1.74), the latter may be written in components notation as

$$\frac{\partial S_{ij}}{\partial X_i} = 0, \quad j \in \{1, 2, 3\} \quad \text{(sum over } i\text{)} \tag{1.85}$$

provided the indices i and j are both related to rectangular Cartesian bases or, in full form,

$$\begin{aligned} \frac{\partial S_{11}}{\partial X_1} + \frac{\partial S_{21}}{\partial X_2} + \frac{\partial S_{31}}{\partial X_3} &= 0, \\ \frac{\partial S_{12}}{\partial X_1} + \frac{\partial S_{22}}{\partial X_2} + \frac{\partial S_{32}}{\partial X_3} &= 0, \\ \frac{\partial S_{13}}{\partial X_1} + \frac{\partial S_{23}}{\partial X_2} + \frac{\partial S_{33}}{\partial X_3} &= 0. \end{aligned} \tag{1.86}$$

It is worth noting that if cylindrical polar coordinates (R, Θ, Z) and (r, θ, z), respectively associated with the basis $\{\mathbf{E}_R, \mathbf{E}_\Theta, \mathbf{E}_Z\}$ in the reference configuration and the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ in the current configuration, are used to characterize the status of the deforming body under examination, the connections (see also Kirkinis [43])

$$\frac{\partial \mathbf{e}_r}{\partial R} = \frac{\partial \theta}{\partial R}\mathbf{e}_\theta, \quad \frac{\partial \mathbf{e}_r}{\partial \Theta} = \frac{\partial \theta}{\partial \Theta}\mathbf{e}_\theta, \quad \frac{\partial \mathbf{e}_r}{\partial Z} = \frac{\partial \theta}{\partial Z}\mathbf{e}_\theta, \tag{1.87}$$

$$\frac{\partial \mathbf{e}_\theta}{\partial R} = -\frac{\partial \theta}{\partial R}\mathbf{e}_r, \quad \frac{\partial \mathbf{e}_\theta}{\partial \Theta} = -\frac{\partial \theta}{\partial \Theta}\mathbf{e}_r, \quad \frac{\partial \mathbf{e}_\theta}{\partial Z} = -\frac{\partial \theta}{\partial Z}\mathbf{e}_r, \tag{1.88}$$

constitute that (1.84) may be decomposed as

$$\begin{aligned} \frac{\partial S_{Rr}}{\partial R} + \frac{1}{R}\frac{\partial S_{\Theta r}}{\partial \Theta} + \frac{\partial S_{Zr}}{\partial Z} + \frac{1}{R}(S_{Rr} - S_{\Theta\theta}\frac{\partial \theta}{\partial \Theta}) - (S_{R\theta}\frac{\partial \theta}{\partial R} + S_{Z\theta}\frac{\partial \theta}{\partial R}) &= 0, \\ \frac{\partial S_{R\theta}}{\partial R} + \frac{1}{R}\frac{\partial S_{\Theta\theta}}{\partial \Theta} + \frac{\partial S_{Z\theta}}{\partial Z} + \frac{1}{R}(S_{R\theta} + S_{\Theta r}\frac{\partial \theta}{\partial \Theta}) + S_{Rr}\frac{\partial \theta}{\partial R} + S_{Zr}\frac{\partial \theta}{\partial Z} &= 0, \\ \frac{\partial S_{Rz}}{\partial R} + \frac{1}{R}\frac{\partial S_{\Theta z}}{\partial \Theta} + \frac{\partial S_{Zz}}{\partial Z} + \frac{1}{R}S_{Rz} &= 0. \end{aligned} \tag{1.89}$$

Undoubtedly, on deriving (1.87)-(1.89) we have assumed the general situation where $\theta = \theta(R, \Theta, Z)$, yet, depending on the geometry of the deformation for each particular problem this might not be the case and these should be rearranged and/or simplified analogously.

We also mention that (1.83) can be used to deliver an alternative, yet equivalent Lagrangian formulation of the mechanical energy balance equation (1.77). This is written as

$$\int_{R_0} \rho_0 \mathbf{b}_0 \cdot \dot{\boldsymbol{\chi}} dV + \int_{\partial R_0} (\mathbf{S}^{\mathrm{T}}\mathbf{N}) \cdot \dot{\boldsymbol{\chi}} dS_0 = \frac{d}{dt}\int_{R_0} \frac{1}{2}\rho_0 \dot{\boldsymbol{\chi}} \cdot \dot{\boldsymbol{\chi}} dV + \int_{R_0} \mathrm{tr}\,(\mathbf{S}\dot{\mathbf{F}}) dV, \tag{1.90}$$

where the notations $\dot{\boldsymbol{\chi}} \equiv \partial\boldsymbol{\chi}(\mathbf{X}, t)/\partial t$ and $\dot{\mathbf{F}} \equiv \partial\mathbf{F}/\partial t$ have been adopted for compactness. Now, by noting the connection $\dot{\mathbf{F}} = \mathbf{LF}$, both the symmetry of $\boldsymbol{\sigma}$ as well as (1.78) and (1.81) are engaged to obtain the identities

$$J\mathrm{tr}\,(\boldsymbol{\sigma}\mathbf{D}) = J\mathrm{tr}\,(\boldsymbol{\sigma}\mathbf{L}) = \mathrm{tr}\,(\mathbf{FSL}) = \mathrm{tr}\,(\mathbf{SLF}) = \mathrm{tr}\,(\mathbf{S}\dot{\mathbf{F}}), \tag{1.91}$$

and hence, through (1.12) with $\det(\mathbf{F}) = J$, the stress power (1.79) can alternatively be written over the (referential) volume of R_0 in the form

$$\int_{R_t} \mathrm{tr}\,(\boldsymbol{\sigma}\mathbf{D})dv = \int_{R_0} J\mathrm{tr}\,(\boldsymbol{\sigma}\mathbf{D})dV = \int_{R_0} J\mathrm{tr}\,(\boldsymbol{\sigma}\mathbf{L})dV = \int_{R_0} \mathrm{tr}\,(\mathbf{S}\dot{\mathbf{F}})dV. \tag{1.92}$$

Due to (1.92), the equivalence between (1.77) and (1.90) is well established. Note that the integrand $\mathrm{tr}\,(\boldsymbol{\sigma}\mathbf{D})$ appearing in the left-hand side of (1.92) expresses the power of stress per unit deformed volume, otherwise called the power density, and accordingly $J\mathrm{tr}\,(\boldsymbol{\sigma}\mathbf{D})(= J\mathrm{tr}\,(\boldsymbol{\sigma}\mathbf{L}) = \mathrm{tr}\,(\mathbf{S}\dot{\mathbf{F}}))$ that per unit reference volume.

1.3 Constitutive laws for Cauchy elastic materials

1.3.1 Definition of a Cauchy elastic material

We now introduce the general considerations for a special class of ideal materials called *Cauchy elastic materials*. The basic principle that characterizes the nature of these materials is the strict dependence of their local state of stress in the current configuration on the state of deformation in that configuration.

In mathematical terms, the mechanical behaviour of such materials is described by the constitutive equation

$$\boldsymbol{\sigma} = \mathbf{g}(\mathbf{F}), \tag{1.93}$$

where $\mathbf{g}$ is a *symmetric tensor-valued function*, which is referred to as the *response function of the material*. According to the definition (1.93) the stress $\boldsymbol{\sigma}$ at a point $\mathbf{x}$ in the current configuration of the body is solely determined by the deformation gradient $\mathbf{F}$ at $\mathbf{X}$. However, this *stress relation* does not account for intermediate stages of deformation; in other words, the path of deformation that $\mathbf{X}$ follows in order to reach $\mathbf{x}$ is not to be considered. On the other hand, the connection between $\mathbf{g}$ and $\mathbf{F}$ does indeed imply dependence of the response of the body on the choice of reference configuration. Thus, the form of $\mathbf{g}$ is, in general, defined appropriate to B_r.

When the stress is removed the body returns to its *natural state* i.e. that in the reference configuration, so that $\mathbf{F} = \mathbf{I}$ and $\mathbf{g}(\mathbf{I}) = \mathbf{0}$. Although the latter assertion cannot be taken as a generalization since *residual stresses* may be present.

1.3.2 The principle of material frame-indifference

In this section we discuss the *principle of material frame-indifference* [55, 79] according to which the mechanical behaviour of a material obeying (1.93) remains invariant to superimposed *rigid-body motions* and/or to changes in the external frame of reference used to describe the constitutive equation (1.93).

To demonstrate this, let us assume that a body is deformed from its current configuration B_r to B_t, so that $\mathbf{x} = \chi(\mathbf{X}, t)$. Then, another deformation is superimposed on the body and suppose this occupies the configuration B_t^*, so that

$$\mathbf{x}^* = \mathbf{Q}(t)\mathbf{x} + \mathbf{c}(t). \tag{1.94}$$

Here, the quantity $\mathbf{Q}(t)$ is a proper orthogonal second-order tensor which represents a rotation while $\mathbf{c}(t)$ is a vector field that describes a translation. Note that (1.94) may equivalently be understood to describe an observer transformation.

Following the present notation, we denote $\mathbf{F}$ the deformation gradient from B_r to B_t and similarly $\mathbf{F}^*$, that from B_r to B_t^*. Thus,

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}, \quad \mathbf{F}^* = \frac{\partial \mathbf{x}^*}{\partial \mathbf{X}}, \tag{1.95}$$

and due to (1.94) we derive

$$\mathbf{F}^* = \mathbf{Q}\mathbf{F}. \tag{1.96}$$

Now, if $\mathbf{g}$ is the response function of the material relative to B_r and $\boldsymbol{\sigma}, \boldsymbol{\sigma}^*$ the stresses associated with the deformations $\mathbf{F}$ and $\mathbf{F}^*$, respectively, the constitutive law (1.93) provides

$$\boldsymbol{\sigma} = \mathbf{g}(\mathbf{F}), \quad \boldsymbol{\sigma}^* = \mathbf{g}(\mathbf{F}^*). \tag{1.97}$$

Accordingly, if $\mathbf{n}$ represents the unit normal on the boundary ∂B_t, the transition from B_t to B_t^* yields the transformation $\mathbf{n}^* = \mathbf{Q}\mathbf{n}$ while, the traction $\mathbf{t}_{(\mathbf{n})}$ similarly becomes $\mathbf{t}_{(\mathbf{n}^*)} = \mathbf{Q}\mathbf{t}_{(\mathbf{n})}$. Bearing in mind that $\mathbf{t}_{(\mathbf{n})} = \boldsymbol{\sigma}\mathbf{n}$ and $\mathbf{t}_{(\mathbf{n}^*)} = \boldsymbol{\sigma}^*\mathbf{n}^*$, we therefore conclude the relation

$$(\mathbf{Q}\boldsymbol{\sigma} - \boldsymbol{\sigma}^*\mathbf{Q})\mathbf{n} = \mathbf{0},$$

being in place for all arbitrary unit vectors $\mathbf{n}$, and hence

$$\boldsymbol{\sigma}^* = \mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^{\mathrm{T}}. \tag{1.98}$$

Clearly, the entries (1.96)-(1.98) deliver the required result

$$\mathbf{g}(\mathbf{F}^*) \equiv \mathbf{g}(\mathbf{QF}) = \mathbf{Q}\mathbf{g}(\mathbf{F})\mathbf{Q}^{\mathrm{T}}, \tag{1.99}$$

which states that the constitutive law for elastic materials is *objective.*

1.3.3 Material symmetry

Whereas material frame-indifference requires that the mechanical response of an elastic body remains unchanged due to rotations in the current configuration the actual form of $\mathbf{g}$ and hence of $\boldsymbol{\sigma}$ relies on the symmetries that a material might exhibit relative to some reference configuration. As already mentioned in Section 1.3.1, this implies that a change in the reference configuration affects, in general, the status of stress in the current configuration.

Now, our intention is to identify the appropriate mathematical restrictions that should be placed upon the response function $\mathbf{g}$ so that the choice of reference configuration has no influence on the state of $\boldsymbol{\sigma}$.

In order to succeed that we consider a material body which is deformed from two distinct configurations, B_r and $\bar{B}_r$ say, to the same current configuration B_t and under the assumption that, for both deformations, the stress is equally given by $\boldsymbol{\sigma}$. In this case, if $\mathbf{F}, \mathbf{g}$ respectively denote the deformation gradient and the response function of the material relative to B_r and analogously $\bar{\mathbf{F}}, \bar{\mathbf{g}}$ the deformation gradient and response function relative to $\bar{B}_r$, the constitutive law (1.93) requires

$$\boldsymbol{\sigma} = \mathbf{g}(\mathbf{F}) = \bar{\mathbf{g}}(\bar{\mathbf{F}}), \tag{1.100}$$

where

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}, \quad \bar{\mathbf{F}} = \frac{\partial \mathbf{x}}{\partial \bar{\mathbf{X}}}, \tag{1.101}$$

given that $\mathbf{X} \in B_r, \bar{\mathbf{X}} \in \bar{B}_r$ and $\mathbf{x} \in B_t$.

The transition from the configuration B_r to $\bar{B}_r$ may then, in a similar manner, be defined through the deformation gradient

$$\mathbf{H} = \frac{\partial \mathbf{X}}{\partial \bar{\mathbf{X}}}, \tag{1.102}$$

which, in view of (1.101), is found to satisfy

$$\mathbf{F} = \bar{\mathbf{F}}\mathbf{H}. \tag{1.103}$$

Thus, using (1.103), the requirement (1.100) becomes

$$\mathbf{g}(\bar{\mathbf{F}}\mathbf{H}) = \bar{\mathbf{g}}(\bar{\mathbf{F}}). \tag{1.104}$$

Since the response functions $\mathbf{g}$ and $\bar{\mathbf{g}}$ are defined with respect to different reference configurations occupied by the same body, we expect to have $\mathbf{g} \neq \bar{\mathbf{g}}$ in general. In some situations, however, it is possible to determine suitable deformations $\mathbf{H}$ such that $\mathbf{g} = \bar{\mathbf{g}}$. By putting this argument in fact, equation (1.104) suggests the relation

$$\mathbf{g}(\bar{\mathbf{F}}\mathbf{H}) = \mathbf{g}(\bar{\mathbf{F}}), \tag{1.105}$$

for all deformations $\bar{\mathbf{F}}$ and for all such $\mathbf{H}$. Subsequently, the stress (1.100) is finally given by

$$\boldsymbol{\sigma} = \mathbf{g}(\mathbf{F}) = \mathbf{g}(\bar{\mathbf{F}}). \tag{1.106}$$

From the discussion provided so far, we understand that the role of the tensor $\mathbf{H}$ is to identify all the possible deformations for which the mechanical response of the body is compatible with (1.105) and therefore with (1.106). In this respect, the set of such tensors substantiating the aforementioned properties serves to characterize all the possible reference configurations that are, in terms of the response of the body, indistinguishable.

Accordingly, we may introduce the set $\mathcal{H}$ whose elements $\mathbf{H}$ satisfy

$$\mathbf{g}(\mathbf{F}\mathbf{H}) = \mathbf{g}(\mathbf{F}) \tag{1.107}$$

for all deformation gradients $\mathbf{F}$, which defines the *symmetry of the material relative to B_r*. In particular, by recalling the conventional assumption (1.17) i.e. $\det(\mathbf{H}) > 0$, it is easy to show that $\mathcal{H}$ is a multiplicative group called the *symmetry group of the material relative to B_r*.

It should be emphasized that since the symmetry group of a material consists of deformation gradients a change in the reference configuration corresponds, in general, to a different symmetry group of the same material. It is interesting, however, that in the present notation the structure of the symmetry group, $\bar{\mathcal{H}}$ say, of the material relative to $\bar{B}_r$ can be characterized by means of $\mathcal{H}$ according to *Noll's rule*. This states that if $\mathbf{K}$ is the deformation gradient from B_r to $\bar{B}_r$, then

$$\bar{\mathcal{H}} = \mathbf{K}\mathcal{H}\mathbf{K}^{-1}. \tag{1.108}$$

The latter connection can be obtained via (1.104) after some manipulation (see, e.g., [57]) but we do not pursue any details here. Seeing (1.108), it is straightforward to derive that in the special cases where either $\mathbf{K} \in \mathcal{H}$ or, $\mathbf{K} = K\mathbf{I}$ with $K > 0$ (i.e. $\mathbf{K}$ is a *pure dilatation*) the symmetry group of the material cannot be affected due to change of reference configuration since we then have $\bar{\mathcal{H}} = \mathcal{H}$.

1.3.4 Material specification: isotropic and anisotropic elastic solids

At this point, and with reference to the symmetry operations that constitute the symmetry group of a Cauchy elastic material relative to some reference configuration, two important classes of real entities commonly understood as solids are introduced.

Isotropic elastic solids

A Cauchy elastic material whose symmetry group $\mathcal{H}$ (relative to B_r) coincides with the *proper orthogonal group* is said to be an *isotropic elastic solid relative to* B_r. Accordingly, for such materials all the elements $\mathbf{H} \equiv \mathbf{Q}$ of $\mathcal{H}$ are proper orthogonal second order tensors which satisfy (1.107) for every deformation gradient $\mathbf{F}$. Thus, due to the rotational nature of $\mathbf{Q}$, the (local) mechanical response of an isotropic elastic solid and hence the stress appears to have no awareness of the orientation of the body in B_r.

Further justification for the latter argument may easily be established by following Noll's rule. In this case, if the deformation gradient $\mathbf{K}$ accounts for a rotation $\mathbf{Q}$, the formula (1.108) readily suggests $\bar{\mathcal{H}} = \mathcal{H}$.

Isotropy applies to a wide range of solid materials and especially to those known as rubberlike solids. Obviously, the symmetry operations that these materials exhibit are not perfectly isotropic but their mechanical behaviour can be very well approximated under such an assumption.

Anisotropic elastic solids

On the other hand, not all solid materials can be sufficiently characterized by isotropy. Both in nature as well in industrial applications several materials whose symmetry operations relative to specific reference configurations cannot construct the proper orthogonal group itself, but only strict subgroups of it, are met. We mention, for instance, some biological soft tissues such as arteries and fibre-reinforced composites. These are called *anisotropic elastic solid* materials which, unlike isotropic elastic solids, display significantly different (local) mechanical behaviour in certain directions; the latter referred to as *preferred directions.*

In general, depending on the structure of the body under examination, the number of preferred directions that should be taken into account varies. This fact is actually the criterion that defines the *anisotropy symmetry group* of such a material relative to some reference configuration and is used to provide a more precise identity to anisotropic elastic solids. For example, anisotropic solids featuring a single preferred direction, either locally or from a macroscopic point of view, are known as *transversely isotropic solids* since they follow for the case of transverse isotropy with respect to this direction. Accordingly, a Cauchy elastic material with a preferred direction $\mathbf{M}$ defines a transversely isotropic elastic solid if its symmetry group, $\mathcal{G}$ say, relative to B_r consists of proper orthogonal second order tensors $\mathbf{Q}$, such that

$$\mathbf{QM} = \pm\mathbf{M}. \tag{1.109}$$

Therefore, under transverse isotropy the mechanical response of the body may only remain unaffected due to arbitrary rotations about the direction of $\mathbf{M}$ and due to reflections about the planes lying perpendicular to this direction. Obviously, combination of the two cannot be excluded. Rotational deformations $\mathbf{Q}$ substantiating (1.109) are said to describe the *isotropic planes* for a transversely isotropic solid. Hence, if $\mathcal{H}$ corresponds to isotropy, we easily understand that $\mathcal{G} \subset \mathcal{H}$. It is further apparent that in the present situation any transition from a reference configuration B_r to $\bar{B}_r$, now associated with the symmetry group $\bar{\mathcal{G}}$, yields via (1.108) the connection $\bar{\mathcal{G}} = \mathcal{G}$ for all rotational deformations $\mathbf{K} \equiv \mathbf{Q}$ being consistent with (1.109).

It is also worthwhile noting that the symmetry group $\mathcal{G}$ relative to B_r can essentially be established by two general classes of rotational deformations. Let $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ define an arbitrary orthogonal vector basis relative to some origin in B_r, with $\mathbf{q}_1 \times \mathbf{q}_2 = \mathbf{q}_3$. In addition assume that, without loss of generality, the basis consists of unit vectors and that the preferred direction $\mathbf{M}$ at this point of the body lies along $\mathbf{q}_3$. Then, any element $\mathbf{Q}$ of $\mathcal{G}$ that satisfies $\mathbf{QM} = \mathbf{M}$ transforms $\mathbf{q}_1, \mathbf{q}_2$ into $\bar{\mathbf{q}}_1, \bar{\mathbf{q}}_2$, respectively, while leaving $\mathbf{M} \equiv \mathbf{q}_3$ unchanged. Thus, if ϕ is the angle between $\mathbf{q}_1$ and $\bar{\mathbf{q}}_1$ (and between $\mathbf{q}_2$ and $\bar{\mathbf{q}}_2$), all the possible rotations $\mathbf{Q}$ about $\mathbf{M}$ may be generated by the class of second order proper orthogonal tensors denoted $\mathbf{Q}^{(1)}_{\mathbf{M}}(\phi)$, with matrix representation

$$Q^{(1)}_{\mathbf{M}}(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $\phi \in [0, 2\pi)$. Now, supposing that a reflection about the plane normal to $\mathbf{M}$ is (locally) superimposed on the body, the vector $\bar{\mathbf{q}}_1$ becomes $-\bar{\mathbf{q}}_1$ (i.e. $\mathbf{q}_1$ rotates through $\phi+\pi$), $\bar{\mathbf{q}}_2$ remains unchanged and the preferred direction is given by $-\mathbf{M}(\equiv -\mathbf{q}_3)$. As a result, all the tensors $\mathbf{Q}$ signifying $\mathbf{QM} = -\mathbf{M}$ can analogously be prescribed by the class $\mathbf{Q}^{(2)}_{\mathbf{M}}(\phi)$, with matrix representation

$$Q^{(2)}_{\mathbf{M}}(\phi) = \begin{pmatrix} \cos(\phi+\pi) & \sin(\phi+\pi) & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -\cos\phi & -\sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

where again $\phi \in [0, 2\pi)$. Undoubtedly, for a pure reflection we have $\phi = 0$ and the latter are rearranged analogously. Consequently, the anisotropy symmetry group $\mathcal{G}$ may sufficiently be generated by $\mathbf{Q}^{(1)}_{\mathbf{M}}(\phi)$ and $\mathbf{Q}^{(2)}_{\mathbf{M}}(\phi)$ so, we may write

$$\mathcal{G} = \{\mathbf{Q}^{(1)}_{\mathbf{M}}(\phi), \mathbf{Q}^{(2)}_{\mathbf{M}}(\phi)\}, \quad \phi \in [0, 2\pi). \tag{1.110}$$

A schematic of the above mentioned facts is now given in Figure 1.4.

In the same spirit, anisotropic solids with two preferred directions are said to be *orthotropic solids* while the possibility of having even more preferred directions is also feasible. Here, however, we are mainly concerned with transversely isotropic solids and some further remarks on the case of orthotropy are only made in the later chapters.

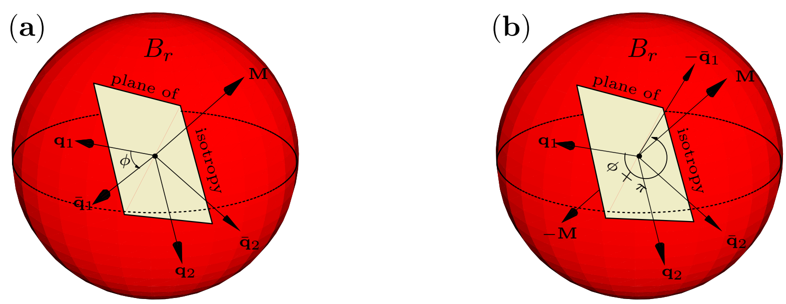

Figure 1.4: Geometrical interpretation of the operations of the tensor $\mathbf{Q}$ when this is regarded as (a) a member of the class $\mathbf{Q}^{(1)}_{\mathbf{M}}(\phi)$ and (b) as member of $\mathbf{Q}^{(2)}_{\mathbf{M}}(\phi)$.

Undistorted configurations

For an isotropic elastic solid the reference configurations relative to which the symmetry operations satisfying (1.107) are capable of establishing the symmetry group $\mathcal{H}$ (i.e. the proper orthogonal group) are called *undistorted states* or *undistorted configurations.* In parallel, under transverse isotropy, undistorted states are characterized by reference configurations which comply with the symmetry group $\mathcal{G}$. In that sense, starting from a reference configuration B_r with the above mentioned properties, we are interested in determining those deformations that bring the body into the state $\bar{B}_r$ without changing its initial 'physical' symmetry operations.

Both for isotropic and transversely isotropic solids, Truesdell and Noll [79] have shown that (1.108) may be combined with the formula $\mathbf{K} = \mathbf{RU}$ (as prescribed by the polar decomposition theorem) to deduce that the entry

$$\mathbf{U} = \mathbf{Q}^{\mathrm{T}}\mathbf{U}\mathbf{Q}, \tag{1.111}$$

constitutes the appropriate restrictions that should be placed between the reference configurations B_r and $\bar{B}_r$ so as to ensure that every element of $\bar{\mathcal{H}}$ and/or $\bar{\mathcal{G}}$ is a proper orthogonal tensor. In addition, the same authors derived that (1.111) is in place if and only if the stretch tensor $\mathbf{U}$ that relates the configurations B_r and $\bar{B}_r$ is of the form

$$\mathbf{U} = \alpha\mathbf{I}, \tag{1.112}$$

for the case of isotropy, and

$$\mathbf{U} = \alpha\mathbf{I} + \beta\mathbf{M}\otimes\mathbf{M}, \tag{1.113}$$

under transverse isotropy. Note that the parameters α and β are scalar fields. Identical restrictions also apply on the response function $\mathbf{g}$ when this is evaluated for $\mathbf{F} \equiv \mathbf{K} = \mathbf{I}$ (i.e. in the configuration B_r), in which case (1.111) recasts to

$$\mathbf{g}(\mathbf{I}) = \mathbf{Q}^{\mathrm{T}}\mathbf{g}(\mathbf{I})\mathbf{Q}. \tag{1.114}$$

The latter suggests

$$\mathbf{g}(\mathbf{I}) = \tilde{\alpha}\mathbf{I}, \tag{1.115}$$

for isotropic solids, and

$$\mathbf{g}(\mathbf{I}) = \tilde{\alpha}\mathbf{I} + \tilde{\beta}\mathbf{M}\otimes\mathbf{M}, \tag{1.116}$$

for transversely isotropic. Analogously to α and β, the parameters $\tilde{\alpha}$ and $\tilde{\beta}$ are scalars. A detailed exposition of the methodology followed in order to obtain (1.112) and (1.115) is given in [56]. Here, we mention in passing that the expression (1.113) and its counterpart (1.116) arise on choosing this member of the class $\mathbf{Q}_{\mathbf{M}}^{(1)}(\phi)$ associated with the value $\phi = \pi/2$ and a second member from $\mathbf{Q}_{\mathbf{M}}^{(2)}(\phi)$ corresponding to the angle $\phi = 0$. These two rotations are sufficient to establish biuniqueness between (1.111), (1.113) and between (1.114), (1.116) under transverse isotropy.

It is straight forward from (1.46) that (1.112) is substantiated if and only if the principal stretches satisfy $\lambda_1 = \lambda_2 = \lambda_3 = \alpha$ (> 0). As a result and bearing in mind the decomposition (1.42), the deformation gradient connecting B_r with $\bar{B}_r$ is written down as

$$\mathbf{K} = \mathbf{RU} = \alpha\mathbf{R}, \tag{1.117}$$

signifying a pure dilatation $\alpha\mathbf{I}$ combined, in general, with a rotation $\mathbf{R}$. Here, the stress (1.115) associated with B_r is hydrostatic. On the other hand, validity of (1.113) requires $\mathbf{M}$ to be a principal direction of $\mathbf{U}$ and the principal stretches in the two remaining principal directions of the same tensor to coincide. Thus, by taking $\mathbf{M} \equiv \mathbf{u}^{(3)}$ for example, the expression (1.113) incorporates $\lambda_1 = \lambda_2 = \alpha$ together with $\lambda_3 = \alpha + \beta$ which, in fact, requires both α and $\alpha + \beta$ to be positive. Hence, for a transversely isotropic solid, two arbitrary undistorted states are connected with a deformation comprising of a pure dilatation, an extension along the preferred direction $\mathbf{M}$ and, due to (1.42), a rotation $\mathbf{R}$. In line with (1.117) we now write

$$\mathbf{K} = \mathbf{RU} = \alpha\mathbf{R} + \beta(\mathbf{RM}) \otimes \mathbf{M}. \tag{1.118}$$

Once more, the stress in B_r needs to be hydrostatic. Physically, the connection (1.116) is often used to describe the existence of (local) residual stress in unloaded equilibrium configurations of transversely isotropic solids with zero traction on the boundary of the body (see, e.g., [14, 30, 31]). This is not, however, the case for (1.115) since isotropic elastic solids obeying such boundary conditions cannot support constitutive equations with residual stress [57].

As recorded in the book by Ogden [56], the deformation gradient (1.117) which connects two undistorted states $B_r, \bar{B}_r$ of an isotropic solid with symmetry groups $\mathcal{H}$ and $\bar{\mathcal{H}}$, respectively, may further be combined with (1.108) to assign the relation

$$\bar{\mathcal{H}} = \mathbf{R}\mathcal{H}\mathbf{R}^{\mathrm{T}}, \tag{1.119}$$

where $\mathbf{R}$ is an arbitrary rotation. If now the body under examination features transverse isotropy and the symmetry groups $\mathcal{G}, \bar{\mathcal{G}}$ are related to the undistorted states B_r and $\bar{B}_r$, respectively,the inverse of (1.118) can be arranged in the form

$$\mathbf{K}^{-1} = \frac{1}{\alpha}\mathbf{R}^{\mathrm{T}} - \left(\frac{\beta}{\alpha+\beta}\right)\mathbf{M}\otimes(\mathbf{RM}), \tag{1.120}$$

in which case, through (1.118) and (1.120), the entry (1.108) contributes

$$\bar{\mathcal{G}} = \mathbf{R}\mathcal{G}\mathbf{R}^{\mathrm{T}}. \tag{1.121}$$

It is interesting that the transformation of $\mathcal{G}$ into $\bar{\mathcal{G}}$ resulting from the deformation (1.118) is in full analogy with that of $\mathcal{H}$ into $\bar{\mathcal{H}}$ as prescribed via (1.119). Again, the rotational tensor $\mathbf{R}$ involved in (1.121) needs to be proper orthogonal but, apart from that, no other restriction is imposed.

1.3.5 Further remarks on the response of elastic solids

It is evident that the notions of material frame-indifference and material symmetry have a role to play in the mechanics of elastic deforming solids since they, in essence, impose particular restrictions on the constitutive law; recall that material frame-indifference qualifies the invariance character of the mechanical response of elastic materials due to arbitrary rotations in the current configuration while, analogously, material symmetry identifies those reference configurations which should be regarded as indistinguishable when measuring the stress the current configuration.

In this connection, it is useful to note that for an isotropic elastic solid the implications of these principles are manifest through the identities

$$\boldsymbol{\sigma} = \mathbf{g}(\mathbf{F}) = \mathbf{g}(\mathbf{V}) = \mathbf{g}(\mathbf{V}\mathbf{Q}^{\mathrm{T}}), \tag{1.122}$$

$$\mathbf{g}(\mathbf{Q}\mathbf{V}\mathbf{Q}^{\mathrm{T}}) = \mathbf{Q}\mathbf{g}(\mathbf{V})\mathbf{Q}^{\mathrm{T}}, \tag{1.123}$$

that hold for all proper orthogonal $\mathbf{Q}$ and for any (arbitrary) positive definite symmetric $\mathbf{V}$. Constructively, the stress (1.122) depends strictly on $\mathbf{V}$ and, in parallel with (1.47), the mechanical response of an isotropic material may solely be determined by measuring the stretches along the Eulerian principal axes. In addition, (1.123) states that the response function of the considered materials is an isotropic tensor function, in which case it follows that $\mathbf{g}$ is coaxial with $\mathbf{V}$. It is important to note that, using (1.42) and (1.43), both (1.122) and (1.123) may equivalently be rearranged in various, yet, equivalent forms [25, 54, 71] where instead of the argument $\mathbf{V}$ we can either have $\mathbf{U}$ or even $\mathbf{B}$ or $\mathbf{C}$. Overall, the

Cauchy stress admits the representation

$$\boldsymbol{\sigma} = \sum_{i=1}^{3} \sigma_i \mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)}, \tag{1.124}$$

where $\sigma_i \equiv \sigma_i(\lambda_1, \lambda_2, \lambda_3)$ are called the *principal stresses* being the components of $\boldsymbol{\sigma}$ along the Eulerian principal axes $\mathbf{v}^{(i)}$. For a detailed discussion on deriving (1.124) we refer to the work of Ogden [56].

A transversely isotropic solid, on the other hand, does not in general support the previous assertions. A simple way to clarify this point is by noting that (1.122) arise from (1.107) on substituting $\mathbf{F} = \mathbf{VR}, \mathbf{H} = \mathbf{R}^{\mathrm{T}}$ and $\mathbf{H} = \mathbf{R}^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}}$ respectively, while (1.123) emerges from (1.122) through (1.103) and (1.104). It is therefore understood that, relative to an undistorted configuration B_r with symmetry group $\mathcal{G}$, the stress $\boldsymbol{\sigma} = \mathbf{g}(\mathbf{F})$ meets (1.122) and its subsequent (1.123) only for these rotations $\mathbf{Q}$ which comply with (1.109), i.e. for all $\mathbf{Q} \in \mathcal{G}$. Correspondingly, under transverse isotropy the Cauchy stress cannot, in general, be sufficiently determined by measuring the stretches along the principal directions of $\mathbf{V}$ alone, but measurements of deformation related to the *deformed preferred direction*, namely

$$\mathbf{m} = \mathbf{FM}, \tag{1.125}$$

are also required [18]. It is evident that here the response function is not an isotropic tensor function and not, in general, coaxial with $\mathbf{V}$. Thus, the spectral decomposition (1.124) does not apply for transversely isotropic solids. Actually, as we will shortly discuss in the coming section, the stress-strain coaxiality criterion (1.124) may only apply for particular deformations.

Owing to the latter properties of the Cauchy stress it is appropriate to adopt the notation

$$\boldsymbol{\sigma} = \mathbf{g}(\mathbf{F}, \mathbf{M} \otimes \mathbf{M}), \tag{1.126}$$

when referring to transversely isotropic elastic solids. Here, the tensors $\mathbf{F}$ and $\mathbf{M} \otimes \mathbf{M}$ should be treated as independent quantities whereas the second in the order given term does not have a deformation character but serves to qualify the dependance between the status of stress in the current configuration and the presence of a preferred direction in the associated reference configuration. For convenience, and without losing generality, $\mathbf{M}$ can always be considered to be a unit vector which is not in general the case for its counterpart $\mathbf{m}$, given by (1.125), in the current configuration.

Finally, it is important to clarify that, in arrangement with (1.93), isotropic elastic solids can be characterized as *homogeneous materials*. This reflects the fact that, despite the local nature of stress, the response function $\mathbf{g}$ depends exclusively on $\mathbf{F}$, yet not on $\mathbf{X}$. On the other hand, by noting that the vector field $\mathbf{M}$ involved in (1.126) depends, in principle, on the position $\mathbf{X}$, transversely isotropic solids and more generally anisotropic solids (i.e. featuring more than one preferred direction) are characterized as *inhomogeneous materials*. Notwithstanding, the possibility of having $\mathbf{M}$ independent of $\mathbf{X}$ cannot be eliminated in which case the constituent mechanical properties of these materials are consider to be homogeneous. Particular examples of transversely isotropic solids with both homogeneous and inhomogeneous mechanical properties are given in Chapters 2,3,4.

1.4 Hyperelastic materials

1.4.1 Constitutive formulation in terms of the strain-energy function

Henceforth, we focus our analysis on a special category of Cauchy elastic materials referred to as *hyperelastic* or *Green elastic materials*. The notion of a hyperelastic material involves an 'energy-based' approach for determining the mechanical response of a Cauchy elastic material under large deformations. In reality, this is a relatively abstract and yet practically useful concept which implicates the existence of a scalar-valued function of tensorial variables commonly known as the *strain-energy function*.

Specifically, by initially assuming that the materials under consideration are perfectly elastic and homogeneous, we recall (1.91) to rewrite the energy balance equation (1.77) in the full form

$$\int_{R_t} \rho\mathbf{b}\cdot\mathbf{v}dv + \int_{\partial R_t} \mathbf{t}_{(\mathbf{n})}\cdot\mathbf{v}dS = \frac{d}{dt}\int_{R_t}\frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v}dv + \int_{R_t}\mathrm{tr}(\boldsymbol{\sigma}\mathbf{L})dv. \qquad (1.127)$$

Now, in line with (1.90)-(1.92) and (1.127), we are urged to introduce a scalar function of $\mathbf{F}$, namely $W(\mathbf{F})$, in such a way that satisfies the relation [56, 57]

$$\frac{\partial}{\partial t}W(\mathbf{F}) = J\mathrm{tr}(\boldsymbol{\sigma}\mathbf{L}) = \mathrm{tr}(\mathbf{S}\dot{\mathbf{F}}). \qquad (1.128)$$

Then, from (1.128) and on using once more (1.12) with $\det(\mathbf{F}) = J$, we write

$$\int_{R_t}\mathrm{tr}(\boldsymbol{\sigma}\mathbf{L})dv = \int_{R_0}\frac{\partial}{\partial t}W(\mathbf{F})dV = \frac{d}{dt}\int_{R_0}W(\mathbf{F})dV = \frac{d}{dt}\int_{R_t}J^{-1}W(\mathbf{F})dv. \qquad (1.129)$$

Thus, (1.90) and (1.127) become

$$\int_{R_0} \rho_0 \mathbf{b}_0 \cdot \dot{\boldsymbol{\chi}} dV + \int_{\partial R_0} (\mathbf{S}^{\mathrm{T}}\mathbf{N}) \cdot \dot{\boldsymbol{\chi}} dS_0 = \frac{d}{dt}\int_{R_0} [\frac{1}{2}\rho_0 \dot{\boldsymbol{\chi}} \cdot \dot{\boldsymbol{\chi}} + W(\mathbf{F})] dV, \tag{1.130}$$

$$\int_{R_t} \rho \mathbf{b} \cdot \mathbf{v} dv + \int_{\partial R_t} \mathbf{t}_{(\mathbf{n})} \cdot \mathbf{v} dS = \frac{d}{dt}\int_{R_t} [\frac{1}{2}\rho \mathbf{v} \cdot \mathbf{v} + J^{-1} W(\mathbf{F})] dv, \tag{1.131}$$

respectively. As a result, the rate of working of the applied force, measured either in $R_0 \subset B_r$ or in $R_t \subset B_t$ as respectively expounded in the left-hand side of (1.130) and (1.131), is possible to be determined through the rate of change of the total mechanical energy of the body. Precisely, the right-hand side of (1.130) states clearly that, apart from kinetic energy, this energy also incorporates potential energy $W(\mathbf{F})$ per unit volume in B_r. In particular, bearing in mind that $W(\mathbf{F})$ is founded on the basis of a purely elastic mechanical behaviour it is understood to interpret a measure of stored energy in the material resulting from the deformation process. As further shown in (1.128) the rate of change of $W(\mathbf{F})$ measures the stress power per unit reference volume.

In this connection the function $W(\mathbf{F})$ is referred to as the *elastic stored energy* per unit reference volume or simply strain-energy function. Of course, the nature of $W(\mathbf{F})$ is strictly local and thus the integrals

$$\int_{R_0} W(\mathbf{F}) dV = \int_{R_t} J^{-1} W(\mathbf{F}) dv, \tag{1.132}$$

are accordingly employed to prescribe the total elastic stored energy in the regions R_0 and R_t.

At this point, it is very important to note that equations (1.128) can be solved completely to deliver explicit formulas both for $\boldsymbol{\sigma}$ and $\mathbf{S}$ in terms of the strain-energy function. Expressly, due to the strict dependence of W on $\mathbf{F}$, the rate $\partial W(\mathbf{F})/\partial t$ can be suitably manipulated to provide the relation

$$\frac{\partial W}{\partial t} = \mathrm{tr}\left(\mathbf{F}\frac{\partial W}{\partial \mathbf{F}}\mathbf{L}\right),$$

in which case comparison with (1.128) yields the expression

$$\boldsymbol{\sigma} = J^{-1}\mathbf{F}\frac{\partial W}{\partial \mathbf{F}} \tag{1.133}$$

for the Cauchy stress tensor. In a similar manner, the nominal stress tensor is found to satisfy

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{F}}. \tag{1.134}$$

As expected, the forms of $\boldsymbol{\sigma}$ and $\mathbf{S}$ are now in full consistency with (1.81) and vice versa.

It should also be emphasized that in this treatise the components of the second-order tensor $\partial W/\partial \mathbf{F}$ involved in both (1.133) and (1.134) are taken to be given in accordance with the convention

$$\left(\frac{\partial W}{\partial \mathbf{F}}\right)_{ij} = \frac{\partial W}{\partial F_{ji}}. \tag{1.135}$$

This implies the notations

$$\sigma_{ij} = J^{-1} F_{ik} \frac{\partial W}{\partial F_{jk}}, \qquad S_{ij} = \frac{\partial W}{\partial F_{ji}}. \tag{1.136}$$

Since a hyperelastic material is in essence a Cauchy elastic material endowed with a strain-energy energy function W, the principle of material frame-indifference (1.99) is now conveyed by means of (1.133) to contribute

$$\frac{\partial W^*}{\partial \mathbf{F}^*} = \frac{\partial W}{\partial \mathbf{F}} \mathbf{Q}^{\mathrm{T}}, \tag{1.137}$$

where $\mathbf{F}^*$ is being given by (1.96) and $W^* \equiv W(\mathbf{F}^*) = W(\mathbf{QF})$. Thus, on noting that $\partial \mathbf{F}/\partial \mathbf{F}^* = \mathbf{Q}^{\mathrm{T}}$, application of the chain rule in the left-hand side of (1.137) gives

$$\frac{\partial W^*}{\partial \mathbf{F}} = \frac{\partial W}{\partial \mathbf{F}}. \tag{1.138}$$

Recall that the objective character of $\mathbf{g}$ is effected for any deformation gradient $\mathbf{F}$ and for all rotations $\mathbf{Q}$ so long as these follow $\det(\mathbf{F}) > 0$ and $\mathbf{QQ}^{\mathrm{T}} = \mathbf{Q}^{\mathrm{T}}\mathbf{Q} = \mathbf{I}$ with $\det(\mathbf{Q}) = 1$. In that respect, (1.138), being equivalent to (1.99), is integrated to give $W^* = W$, i.e.

$$W(\mathbf{QF}) = W(\mathbf{F}), \tag{1.139}$$

for all $\mathbf{F}$ and any $\mathbf{Q}$ with the above mentioned properties. It is then shown that the invariance nature of $\mathbf{g}$ due to arbitrary rotations in the current configuration is directly carried over to W. Note, however, that (1.139) is based on the premise that the constitutive law satisfies $\mathbf{g}(\mathbf{I}) = \mathbf{0}$ or, equivalently, via (1.139) $W(\mathbf{Q}) = W(\mathbf{I}) = 0$.

On the grounds of material symmetry, the strain-energy function W once more appears to follow a similar pattern to that of $\mathbf{g}$. Truly, if apart from homogeneous and elastic, the material under consideration is also taken to be isotropic relative to a reference configuration B_r with symmetry group $\mathcal{H}$, then

$$W(\mathbf{FQ}) = W(\mathbf{F}), \tag{1.140}$$

for any deformation gradient $\mathbf{F}$ and for all proper orthogonal $\mathbf{Q}$ in $\mathcal{H}$. Parallel to (1.122), (1.123), here (1.139) and (1.140) yield

$$W(\mathbf{F}) = W(\mathbf{V}) = W(\mathbf{QVQ}^{\mathrm{T}}), \tag{1.141}$$

while, also

$$W(\mathbf{F}) = W(\mathbf{U}) = W(\mathbf{Q}\mathbf{U}\mathbf{Q}^{\mathrm{T}}). \tag{1.142}$$

Thus, analogously to $\mathbf{g}$, the strain-energy function is an isotropic (scalar) function both by means of $\mathbf{V}$ and $\mathbf{U}$ while, in view of (1.43)-(1.45), the same conclusion applies in terms of $\mathbf{B}, \mathbf{C}, \mathbf{E}$ and $\mathbf{e}$. It then follows that W admits the representations (see, e.g., [57])

$$W = \breve{w}(i_1, i_2, i_3) = \bar{w}(I_1, I_2, I_3), \tag{1.143}$$

where i_1, i_2 and i_3 are the principal invariants of $\mathbf{U}$ (and also of $\mathbf{V}$), namely

$$i_1 = \mathrm{tr}\,(\mathbf{U}), \qquad i_2 = \frac{1}{2}[\mathrm{tr}\,(\mathbf{U})^2 - \mathrm{tr}\,(\mathbf{U}^2)], \qquad i_3 = \det(\mathbf{U}) \equiv J \tag{1.144}$$

and I_1, I_2, I_3 those of $\mathbf{C}$ (and also of $\mathbf{B}$), given by

$$I_1 = \mathrm{tr}\,(\mathbf{C}), \qquad I_2 = \frac{1}{2}[\mathrm{tr}\,(\mathbf{C})^2 - \mathrm{tr}\,(\mathbf{C}^2)], \qquad I_3 = \det(\mathbf{C}) \equiv J^2. \tag{1.145}$$

Using the representation (1.46) for $\mathbf{U}$ the invariants (1.144) can be put in the forms

$$i_1 = \lambda_1 + \lambda_2 + \lambda_3, \qquad i_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3, \qquad i_3 = \lambda_1\lambda_2\lambda_3, \tag{1.146}$$

while, from (1.49), the invariants (1.145) become

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \qquad I_2 = \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2, \qquad I_3 = \lambda_1^2\lambda_2^2\lambda_3^2. \tag{1.147}$$

It follows that (see, e.g., [74, 76])

$$I_1 = i_1^2 - 2i_2, \qquad I_2 = i_2^2 - 2i_1 i_3, \qquad I_3 = i_3^2. \tag{1.148}$$

It is now apparent from (1.143), (1.146) and (1.147) that the strain-energy function W may equivalently be expressed as a symmetric function of the principal stretches, $\hat{w}(\lambda_1, \lambda_2, \lambda_3)$ say, such that

$$W = \hat{w}(\lambda_1, \lambda_2, \lambda_3) = \hat{w}(\lambda_1, \lambda_3, \lambda_2) = \hat{w}(\lambda_3, \lambda_1, \lambda_2), \tag{1.149}$$

for all $\lambda_1, \lambda_2, \lambda_3 \in (0, \infty)$.

In the same spirit, inhomogeneous materials accounting for (1.126) satisfy again the requirements (1.90)-(1.92) and (1.127), hence, the hypothesis (1.128) substantiating the existence of a strain-energy function W can also be adopted here. Thus, the identities (1.129) and consequently (1.130)-(1.139) are further validated for the case of a transversely isotropic solid. As for the response function $\mathbf{g}$, the strain-energy function now has to remain invariant strictly for those rotations $\mathbf{Q}$ (in the reference configuration) for which $\mathbf{QM} = \pm\mathbf{M}$. This implies the dependence $W(\mathbf{F}, \mathbf{M}\otimes\mathbf{M})$ and therefore

$$W(\mathbf{FQ}, \mathbf{QM}\otimes\mathbf{QM}) = W(\mathbf{F}, \mathbf{M}\otimes\mathbf{M}), \tag{1.150}$$

for any deformation gradient $\mathbf{F}$ and for all proper orthogonal tensors $\mathbf{Q}$ being elements of the symmetry group $\mathcal{G}$ relative to an undistorted configuration B_r. Based on the argument (1.142), and since $\mathbf{U}^2 = \mathbf{C}$, the explicit dependence of W on $\mathbf{F}$ and $\mathbf{M}\otimes\mathbf{M}$ is now found to equivalently be expressed as

$$W(\mathbf{F}, \mathbf{M}\otimes\mathbf{M}) = W(\mathbf{C}, \mathbf{M}\otimes\mathbf{M}) = W(\mathbf{QCQ}^{\mathrm{T}}, \mathbf{QM}\otimes\mathbf{QM}), \tag{1.151}$$

for proper orthogonal $\mathbf{Q} \in \mathcal{G}$. This states that W is required to be an isotropic invariant of both $\mathbf{C}$ and $\mathbf{M}\otimes\mathbf{M}$ jointly while it is also apparent that, as for the isotropic theory, the same conclusions apply by interchanging $\mathbf{C}$ with $\mathbf{E}$.

According to Spencer [73] (see also references therein), (1.151) now implies that W can be expressed as a function of either (i_1, i_2, i_3) or (I_1, I_2, I_3) together with two more invariants, namely I_4 and I_5, which depend on $\mathbf{M}$ and embody the anisotropic attribute of the material. These are taken to be respectively given by

$$I_4 = \mathbf{M}\cdot(\mathbf{CM}), \qquad I_5 = \mathbf{M}\cdot(\mathbf{C}^2\mathbf{M}). \tag{1.152}$$

Bearing in mind (1.48), the latter may be read off as

$$I_4 = \lambda_1^2 M_1^2 + \lambda_2^2 M_2^2 + \lambda_3^2 M_3^2, \qquad I_5 = \lambda_1^4 M_1^2 + \lambda_2^4 M_2^2 + \lambda_3^4 M_3^2, \tag{1.153}$$

respectively where here the notations (M_1, M_2, M_3) are used to represent the components of $\mathbf{M}$ referred to the Lagrangian principal axes. We also note that, through (1.43),(1.125) and (1.152), I_4 and I_5 can alternatively be written as

$$I_4 = \mathbf{m}\cdot\mathbf{m}, \qquad I_5 = \mathbf{m}\cdot(\mathbf{Bm}). \tag{1.154}$$

In this case, if (m_1, m_2, m_3) denote the components of $\mathbf{m}$ with respect to the Eulerian principal axes, (1.49) and (1.154) are combined to give

$$I_4 = m_1^2 + m_2^2 + m_3^2, \qquad I_5 = \lambda_1^2 m_1^2 + \lambda_2^2 m_2^2 + \lambda_3^2 m_3^2. \tag{1.155}$$

Note that in line with the definition (1.22) the kinematical interpretation of I_4 is none other than measuring stretch along the preferred direction $\mathbf{M}$. Indeed, since $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$ and $\lambda(\mathbf{M}) \equiv |\mathbf{FM}|$ we readily derive

$$\lambda(\mathbf{M}) \equiv |\mathbf{FM}| = [(\mathbf{FM})\cdot(\mathbf{FM})]^{1/2} = [\mathbf{M}\cdot(\mathbf{CM})]^{1/2} = \sqrt{I_4}. \tag{1.156}$$

We therefore deduce that the material undergoes extension at $\mathbf{X}$ along the direction of $\mathbf{M}$ when $I_4 > 1$ while, for $I_4 < 1$, we have contraction at the associated material points towards the preferred direction.

The exact role of I_5 on the other hand cannot be completely comprehended directly from $(1.152)_2$ and/or $(1.154)_2$. Notwithstanding, Merodio and Ogden [48] have shown that in some situations I_5 may be perceived to register shearing deformation on the preferred direction through the non-diagonal components (i.e. the shear components) of $\mathbf{C}$. Truly, if C_{ij} $(i, j = 1, 2, 3)$ denote the components of $\mathbf{C}$ relative to a rectangular Cartesian basis $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ and the preferred direction is taken to be, for example, $\mathbf{M} \equiv \mathbf{E}_1$, then (1.152) specialize to $I_4 = C_{11}$ and $I_5 = I_4^2 + C_{12}^2 + C_{13}^2$ respectively. Thus, the previous assertion is justified implicitly through the difference $I_5 - I_4^2$ but only under the premise that $\mathbf{M}$ is not an eigenvector of $\mathbf{C}$. If $\mathbf{M}$ is an eigenvector of $\mathbf{C}$ then $C_{12} = C_{13} = 0$. Clearly, the analogous inference can be drawn by choosing $\mathbf{M} = \mathbf{E}_2, \mathbf{E}_3$. The same authors have also demonstrated [47, 48] the involvement of I_5 to the way that surface elements transform under deformation. We now illustrate this matter by writing down Nanson's formula (1.11) as

$$\mathbf{n}da = \sqrt{I_3}\mathbf{F}^{-\mathrm{T}}\mathbf{M}dA, \tag{1.157}$$

implying that an infinitesimal area element dA originated at $\mathbf{X}$ with unit normal $\mathbf{M}$ becomes da with unit normal $\mathbf{n}$ based at $\mathbf{x}$, and the *Cayley-Hamilton theorem* (see, e.g., [36]) appropriated for $\mathbf{C}$, namely

$$\mathbf{C}^3 - I_1\mathbf{C}^2 + I_2\mathbf{C} - I_3\mathbf{I} = \mathbf{0}. \tag{1.158}$$

Since $\mathbf{n}, \mathbf{M}$ are unit vectors and $\mathbf{C}^{-1} = \mathbf{F}^{-1}\mathbf{F}^{-\mathrm{T}}$ it is straightforward from (1.157) that

$$da = \sqrt{I_3}|\mathbf{F}^{-\mathrm{T}}\mathbf{M}|dA = \sqrt{I_3}[(\mathbf{F}^{-\mathrm{T}}\mathbf{M}) \cdot (\mathbf{F}^{-\mathrm{T}}\mathbf{M})]^{1/2}dA = \sqrt{I_3}[\mathbf{M} \cdot (\mathbf{C}^{-1}\mathbf{M})]^{1/2}dA,$$

thus,

$$\left(\frac{da}{dA}\right)^2 = I_3\mathbf{M} \cdot (\mathbf{C}^{-1}\mathbf{M}). \tag{1.159}$$

In line with the definitions (1.152), the entry (1.158) is now suitably manipulated to deliver

$$I_5 = I_1I_4 - I_2 + I_3\mathbf{M} \cdot (\mathbf{C}^{-1}\mathbf{M}). \tag{1.160}$$

Hence, combination of (1.159) and (1.160) yields directly to

$$I_5 = I_1I_4 - I_2 + \left(\frac{da}{dA}\right)^2. \tag{1.161}$$

We distinguish that here the unit vector $\mathbf{n}$ does not serve to describe the orientation of the deformed preferred direction $\mathbf{m}$ defined in (1.125). Indeed, by using (1.161), we express the ratio da/dA by means of the invariants I_1, I_2, I_4 and I_5 which is then substituted into (1.157) to define

$$\mathbf{n} = \sqrt{\frac{I_3}{I_5 + I_2 - I_1 I_4}}\mathbf{B}^{-1}\mathbf{m} = \sqrt{\frac{I_3 I_4}{(I_5 + I_2 - I_1 I_4)}}\mathbf{B}^{-1}\mathbf{m}_0, \tag{1.162}$$

where $\mathbf{m}_0 = |\mathbf{FM}|^{-1}\mathbf{m} \equiv I_4^{-1/2}\mathbf{m}$. An expression similar to (1.162) has also been obtained by Peng *et. al.* [19]. Based on that we may also take the inner product

$$\mathbf{n} \cdot \mathbf{m}_0 = \sqrt{\frac{I_3}{(I_5 + I_2 - I_1 I_4) I_4}},$$

which actually suggests that the area element, $d\alpha$ say, whose unit normal $\mathbf{m}_0$ is based at the same point $\mathbf{x}$ as $\mathbf{n}$, satisfies the connection

$$d\alpha = (\mathbf{n} \cdot \mathbf{m}_0) da = \sqrt{\frac{I_3}{(I_5 + I_2 - I_1 I_4) I_4}} da$$

and hence, from (1.161)

$$\left(\frac{d\alpha}{dA}\right)^2 = \frac{I_3}{I_4}. \tag{1.163}$$

With reference to the work of Spencer [72] we now illustrate that by assigning a positive scalar function, namely $\delta_0 = \delta_0(\mathbf{X})$, to represent the mass density per unit reference area distributed normal to the direction of $\mathbf{M}$ and $\delta = \delta(\mathbf{x}, t)$ the density per unit deformed area lying normal to $\mathbf{m}_0$ then, since the mass is to be conserved, $\delta_0 dA = \delta d\alpha$ which through (1.161) enables the interpretation

$$\left(\frac{\delta_0}{\delta}\right)^2 = \frac{I_3}{I_4}. \tag{1.164}$$

In view of the above discussion, the elastic stored energy (per unit reference volume) of a transversely isotropic elastic solid is represented either in terms of the invariants based on $\mathbf{V}$ and $\mathbf{B}$ or, equivalently based on $\mathbf{U}$ and $\mathbf{C}$, such that

$$W = \breve{W}(i_1, i_2, i_3, I_4, I_5) = \bar{W}(I_1, I_2, I_3, I_4, I_5), \tag{1.165}$$

including the two additional invariants to express the contribution due to the presence of the preferred direction.

It should be emphasized that in the isotropic theory the invariants (i_1, i_2, i_3) defined above are treated, in general, as independent (positive) scalar quantities and so are (I_1, I_2, I_3) and $(\lambda_1, \lambda_2, \lambda_3)$. Under transverse isotropy, we similarly have that all the relevant arguments used to express the elastic stored energy, i.e. $(i_1, i_2, i_3, I_4, I_5)$ or $(I_1, I_2, I_3, I_4, I_5)$, are again taken to vary independently. We note, however, that in this case W cannot sufficiently be prescribed in terms of the principal stretches alone. As clearly shown in (1.153), this is only possible in the special case where $\mathbf{M}$ is independent of $\mathbf{X}$. Even then, W cannot in general be regarded as a symmetric function of λ_1, λ_2 and λ_3 apart from when $M_1 = M_2 = M_3 (= \pm\sqrt{3}/3)$. Following that, we easily obtain the connections $I_4 = I_1/3, I_5 = (I_1^2 - 2I_2)/3$ or, using (1.148), $I_4 = (i_1^2 - 2i_2)^2/3$ and $I_5 = (i_1^4 - 4i_1^2 i_2 + 2i_2^2 + 4i_1 i_3)/3$ requiring only three out of the five invariants expressing the elastic stored energy to remain independent.

In concluding this section it is also useful to note that, both for isotropic as well as for transversely isotropic materials, several other invariant-based representations of W have been reported in the literature but it is not of our purpose to record them here. We exceptionally refer to an interesting recent formulation presented by Steigmann [75] who, in particular, determined the elastic stored energy of a transversely isotropic solid as a function of invariants based exclusively on $\mathbf{U}$, namely $W = \breve{w}(i_1, i_2, i_3, i_4, i_5)$, where now $i_4 = |\mathbf{UM}|$ and $i_5 = i_3|\mathbf{U}^{-1}\mathbf{M}|$. Nonetheless, we only make formal use of (1.165) whereas Steigmann's approach appears to be much more complicated regarding the scope of this treatise. Indeed, following (1.46) and since $I_3 = i_3^2 = \lambda_1^2\lambda_2^2\lambda_3^2$, i_4 and i_5 may be decomposed as

$$i_4 = \sqrt{\lambda_1^2 M_1^2 + \lambda_2^2 M_2^2 + \lambda_3^2 M_3^2}, \qquad i_5 = \sqrt{\lambda_2^2\lambda_3^2 M_1^2 + \lambda_1^2\lambda_3^2 M_2^2 + \lambda_1^2\lambda_2^2 M_3^2},$$

respectively, where again (M_1, M_2, M_3) are the components of $\mathbf{M}$ associated with the the Lagrangian principal axes. In addition, incorporation of i_4, i_5 instead of I_4 and I_5 into W does not provide any further information regarding the response of the body under deformation. To justify this matter we recall (1.43) to obtain $i_4 = [\mathbf{M} \cdot (\mathbf{CM})]^{1/2}$ and $i_5 = [I_3\mathbf{M} \cdot (\mathbf{C}^{-1}\mathbf{M})]^{1/2}$ (with $I_3 = i_3^2$). Hence, comparison of the latter expressions with (1.156) and (1.160) respectively, yields

$$i_4 = \sqrt{I_4} \equiv \lambda(\mathbf{M}), \qquad i_5 = \sqrt{I_5 + I_2 - I_1 I_4} \equiv \frac{da}{dA},$$

from which the kinematical interpretations of i_4 and i_5 are completely understood.

1.4.2 Stress-deformation relations

We now illustrate that, using the definitions $(1.43)_2$, the principal invariants (1.145) may equivalently be written in terms of the components of the deformation gradient tensor $\mathbf{F}$, namely F_{ij}, in the form

$$I_1 = F_{ij}F_{ij}, \qquad I_2 = \frac{1}{2}[(F_{ij}F_{ij})^2 - F_{ij}F_{kj}F_{kp}F_{ip}], \qquad I_3 = (\varepsilon_{ijk}F_{1i}F_{2j}F_{3k})^2. \quad (1.166)$$

Note that the above notations imply summation over the indices $i, j, k, p \in \{1, 2, 3\}$ while ε_{ijk} denotes the *alternating symbol.* Thus, in consistency with the convention (1.135), expressions (1.166) enable us to calculate the derivatives

$$\frac{\partial I_1}{\partial \mathbf{F}} = 2\mathbf{F}^{\mathrm{T}}, \quad \frac{\partial I_2}{\partial \mathbf{F}} = 2I_1\mathbf{F}^{\mathrm{T}} - 2\mathbf{F}^{\mathrm{T}}\mathbf{F}\mathbf{F}^{\mathrm{T}}, \quad \frac{\partial I_3}{\partial \mathbf{F}} = 2I_3\mathbf{F}^{-1}, \qquad (1.167)$$

and hence, for an isotropic hyperelastic solid possessing a strain-energy function $W = \bar{w}(I_1, I_2, I_3)$, the stress tensors (1.133) and (1.134) can be defined explicitly through

$$\boldsymbol{\sigma} = J^{-1}[2\bar{w}_1\mathbf{B} + 2\bar{w}_2(I_1\mathbf{I} - \mathbf{B})\mathbf{B} + 2I_3\bar{w}_3\mathbf{I}], \qquad (1.168)$$

$$\mathbf{S} = 2\bar{w}_1\mathbf{F}^{\mathrm{T}} + 2\bar{w}_2(I_1\mathbf{I} - \mathbf{C})\mathbf{F}^{\mathrm{T}} + 2I_3\bar{w}_3\mathbf{F}^{-1}, \qquad (1.169)$$

respectively. Here, the indices $1, 2, 3$ on $\bar{w}$ indicate differentiation with respect to I_1, I_2 and I_3 respectively.

If, on the other hand, the representation $W = \breve{w}(i_1, i_2, i_3)$ is used instead, the counterparts of (1.167) required in this case to express the stresses by means of the strain-energy function become

$$\frac{\partial i_1}{\partial \mathbf{F}} = \mathbf{R}^{\mathrm{T}}, \quad \frac{\partial i_2}{\partial \mathbf{F}} = i_1\mathbf{R}^{\mathrm{T}} - \mathbf{F}^{\mathrm{T}}, \quad \frac{\partial i_3}{\partial \mathbf{F}} = i_3\mathbf{F}^{-1}. \qquad (1.170)$$

Then

$$\boldsymbol{\sigma} = i_3^{-1}[\breve{w}_1\mathbf{V} + \breve{w}_2(i_1\mathbf{I} - \mathbf{V})\mathbf{V} + i_3\breve{w}_3\mathbf{I}], \qquad (1.171)$$

$$\mathbf{S} = \breve{w}_1\mathbf{R}^{\mathrm{T}} + \breve{w}_2(i_1\mathbf{R}^{\mathrm{T}} - \mathbf{F}^{\mathrm{T}}) + i_3\breve{w}_3\mathbf{F}^{-1}, \qquad (1.172)$$

where now $\breve{w}_p = \partial\breve{w}/\partial i_p$ for $p \in \{1, 2, 3\}$. Note, however, that the manipulations needed in order to derive (1.170) and hence (1.171) and (1.172) are very much involved. In fact, these require resolution of the non-linear algebraic system (1.148) in the first place which, as shown by Sawyers [70], possesses a unique solution by means of i_1, i_2, i_3. An alternative derivation of the above results has also been given by Steigmann [74] recently but we do not pursue any details here.

Since in the isotropic theory the elastic stored energy can also be defined by $W = \hat{w}(\lambda_1, \lambda_2, \lambda_3)$, it is straight forward to show

$$\lambda_i\frac{\partial \hat{w}}{\partial \lambda_i} = \lambda_i\breve{w}_1 + (\lambda_i i_1 - \lambda_i^2)\breve{w}_2 + i_3\breve{w}_3 = 2\lambda_i^2\bar{w}_1 + 2(I_2 - \lambda_i^{-2}I_3)\bar{w}_2 + 2I_3\bar{w}_3, \qquad (1.173)$$

for all $i \in \{1,2,3\}$. Based on the latter, comparison of (1.124) with (1.168) and (1.171) leads to the representations

$$J\sigma_i = \lambda_i \frac{\partial \hat{w}}{\partial \lambda_i}, \quad i \in \{1,2,3\} \tag{1.174}$$

for the principal components of the Cauchy stress tensor. We clarify that the repeated index i appearing in (1.173) and (1.174) does not imply summation.

Evidently, for a transversely isotropic solid whose elastic stored energy is defined through (1.165), determination of the stresses requires the derivatives (1.167) and/or (1.170) together with $\partial I_4/\partial \mathbf{F}$ and $\partial I_5/\partial \mathbf{F}$. Thus, alike (1.166) and due to $(1.43)_1$ the invariants (1.152) are respectively written as

$$I_4 = F_{ij}F_{ik}M_jM_k, \qquad I_5 = F_{pi}F_{pj}F_{si}F_{sk}M_jM_k, \tag{1.175}$$

from which follows

$$\frac{\partial I_4}{\partial \mathbf{F}} = 2\mathbf{M} \otimes \mathbf{FM}, \qquad \frac{\partial I_5}{\partial \mathbf{F}} = 2(\mathbf{M} \otimes \mathbf{FCM} + \mathbf{CM} \otimes \mathbf{FM}), \tag{1.176}$$

always in consistency with (1.135). Hence, the stress tensors (1.133) and (1.134) are respectively read off as

$$\begin{aligned} \boldsymbol{\sigma} =& J^{-1}[2\bar{W}_1\mathbf{B} + 2\bar{W}_2(I_1\mathbf{I} - \mathbf{B})\mathbf{B} + 2I_3\bar{W}_3\mathbf{I} \\ &+ 2\bar{W}_4\mathbf{m} \otimes \mathbf{m} + 2\bar{W}_5(\mathbf{m} \otimes \mathbf{Bm} + \mathbf{Bm} \otimes \mathbf{m})], \end{aligned} \tag{1.177}$$

$$\begin{aligned} \mathbf{S} =& 2\bar{W}_1\mathbf{F}^{\mathrm{T}} + 2\bar{W}_2(I_1\mathbf{I} - \mathbf{C})\mathbf{F}^{\mathrm{T}} + 2I_3\bar{W}_3\mathbf{F}^{-1} \\ &+ 2\bar{W}_4\mathbf{M} \otimes \mathbf{FM} + 2\bar{W}_5(\mathbf{M} \otimes \mathbf{FCM} + \mathbf{CM} \otimes \mathbf{FM}), \end{aligned} \tag{1.178}$$

in the case where the strain-energy function is taken to be given by $W = \bar{W}(I_1, I_2, I_3, I_4, I_5)$, and

$$\begin{aligned} \boldsymbol{\sigma} =& \breve{W}_1\mathbf{V} + \breve{W}_2(i_1\mathbf{I} - \mathbf{V})\mathbf{V} + i_3\breve{W}_3\mathbf{I} \\ &+ 2\breve{W}_4\mathbf{m} \otimes \mathbf{m} + 2\breve{W}_5(\mathbf{m} \otimes \mathbf{Bm} + \mathbf{Bm} \otimes \mathbf{m}), \end{aligned} \tag{1.179}$$

$$\begin{aligned} \mathbf{S} =& \breve{W}_1\mathbf{R}^{\mathrm{T}} + \breve{W}_2(i_1\mathbf{R}^{\mathrm{T}} - \mathbf{F}^{\mathrm{T}}) + i_3\breve{W}_3\mathbf{F}^{-1} \\ &+ 2\breve{W}_4\mathbf{M} \otimes \mathbf{FM} + 2\breve{W}_5(\mathbf{M} \otimes \mathbf{FCM} + \mathbf{CM} \otimes \mathbf{FM}), \end{aligned} \tag{1.180}$$

for $W = \breve{W}(i_1, i_2, i_3, I_4, I_5)$. As before, we adopt the notations $W_p = \partial\bar{W}/\partial I_p$ for $p \in \{1,2,3,4,5\}$, while $\breve{W}_q = \partial\breve{W}/\partial i_q$ for $q \in \{1,2,3\}$ and $\breve{W}_s = \partial\breve{W}/\partial I_s$ for $s \in \{4,5\}$. We note that the results (1.177), (1.178) are due to Ericksen and Rivlin [18] and Spencer [73] while the formulations (1.179), (1.180) essentially arise by simply combining the work of the same authors together with the aforementioned results obtained by Steigmann [74].

As already mentioned in earlier sections, for a generic Cauchy elastic solid material featuring a preferred direction $\mathbf{M}$ the Cauchy stress tensor $\boldsymbol{\sigma}$ is not, in principle and in contrast to the isotropic theory, coaxial with $\mathbf{V}$. Especially for the case of a hyperelastic transversely isotropic solid this fact is clearly reflected through the entries (1.177) and (1.179). From the same expressions, however, we deduce that coaxiality between $\boldsymbol{\sigma}$ and $\mathbf{V}$ is established if and only if the deformed preferred direction $\mathbf{m}$ is an eigenvector of $\mathbf{B}$ or, equivalently, through (1.53) and (1.125), $\mathbf{M}$ is an eigenvector of $\mathbf{C}$. Thus, if for a fixed $i \in \{1,2,3\}$ we take $\mathbf{M} \equiv \mathbf{u}^{(i)}$, the definition (1.48) yields $\mathbf{CM} = \lambda_i \mathbf{M}$ for all relevant $\lambda_i > 0$. Accordingly, the formulas (1.152) read

$$I_4 = \lambda_i^2, \qquad I_5 = I_4^2, \tag{1.181}$$

for all $\lambda_i > 0$ and therefore W can be represented as a (non-symmetric) function of $\lambda_1, \lambda_2, \lambda_3$. We now write

$$W = \breve{W}(i_1, i_2, i_3, I_4, I_5) = \bar{W}(I_1, I_2, I_3, I_4, I_5) = \hat{W}(\lambda_1, \lambda_2, \lambda_3), \tag{1.182}$$

wherein the notation $\hat{W}$ has been introduced. Analogously to (1.173), the connections (1.181) and (1.182) deliver

$$\begin{aligned} \lambda_i \frac{\partial \hat{W}}{\partial \lambda_i} =& \lambda_i \breve{W}_1 + (\lambda_i i_1 - \lambda_i^2)\breve{W}_2 + i_3 \breve{W}_3 + 2(\lambda_j^2 \breve{W}_4 + 2\lambda_j^4 \breve{W}_5)\delta_{ij} \\ =& 2\lambda_i^2 \bar{W}_1 + 2(I_2 - \lambda_i^{-2} I_3)\bar{W}_2 + 2I_3 \bar{W}_3 + 2(\lambda_j^2 \bar{W}_4 + 2\lambda_j^4 \bar{W}_5)\delta_{ij}, \end{aligned} \tag{1.183}$$

where δ_{ij} is the Kronecker delta. Now, comparison of (1.183) with (1.177) and (1.179) illustrates that the Cauchy stress tensor $\boldsymbol{\sigma}$ admits the spectral decomposition (1.124) in which case the principal stresses, namely $\sigma_1, \sigma_2, \sigma_3$, are defined through (1.174) by substituting $\hat{W}$ in the place of $\hat{w}$. We emphasize, however, that this conclusion applies only when $\mathbf{M}$ is directed along a principal direction $\mathbf{u}^{(i)}$ corresponding to a fixed $i \in \{1,2,3\}$ and it cannot be considered as a generalization. A further discussion on this matter including additional results associated with the linear theory of transversely isotropic solids is provided in Chapter 2.

1.4.3 Internal material constraints

Until now the materials under consideration were silently assumed to posses no limitations in their mechanical behaviour other than those imposed by the constitutive law. In particular, the restrictions of material frame-indifference and material symmetry forced into the response functions $\mathbf{g}$ in previous sections were directly reflected to the case of a hyperelastic material yielding the elastic stored energy to sufficiently be determined through specific (independent) scalar deformation variables. Seeing, however, that both for isotropic and transversely isotropic solids the strain-energy function is taken to be dependent on the invariant I_3 $(= i_3^2 \equiv J^2)$ it is clearly understood that our previous assertions account for materials admitting change in volume as a result of deformation, i.e. compressible materials. From that perspective, a slight modification of the constitutive theory described in Sections 1.4.1 and 1.4.2 needs to be done for incompressible materials where we ideally adopt the kinematical constraint (1.18) and hence $i_3 = I_3 = 1$, for all deformations and at each point of the material.

As a matter of fact, in order to represent the elastic stored energy (per unit volume in B_r) for an incompressible hyperelastic solid material, we write the strain energy function in the updated general form

$$W(\mathbf{F}) - p(J-1). \tag{1.184}$$

Here p is a *Lagrange multiplier*, i.e. an arbitrary scalar parameter associated with the incompressibility constraint which cannot be determined *a priori* by the deformation. In physical terms, p is acknowledged as the hydrostatic pressure necessary to conserve the volume of the deforming body. Other than that, the role of p is insignificant in the actual mechanical behaviour of a deforming body, whereas, using (1.184), the rate of working of the stresses remains unaffected from the presence of this parameter. Truly, for an incompressible solid where any deformation is assumed isochoric ($J = 1$), the linkages (1.128) alter to

$$\frac{\partial}{\partial t}[W(\mathbf{F}) - p(J-1)]\bigg|_{J=1} = \mathrm{tr}\,(\boldsymbol{\sigma}\mathbf{L}) = \mathrm{tr}\,(\mathbf{S}\dot{\mathbf{F}}), \tag{1.185}$$

while through (1.64) the latter directly becomes

$$\frac{\partial}{\partial t}W(\mathbf{F})\bigg|_{J=1} = \mathrm{tr}\,(\boldsymbol{\sigma}\mathbf{L}) = \mathrm{tr}\,(\mathbf{S}\dot{\mathbf{F}}),$$

providing justification of our assertion.

Special attention should now be given to the interpretation of (1.184) since the function $W(\mathbf{F})$ involved therein is generic and does not serve to incorporate materials which display particular symmetries in the reference configuration. In that sense, (1.184) is applicable for both isotropic and anisotropic solids.

Now, similarly to the compressible theory, equation (1.185) may be resolved to contribute the expressions

$$\boldsymbol{\sigma} = \mathbf{F}\frac{\partial W}{\partial \mathbf{F}} - p\mathbf{I}, \qquad \mathbf{S} = \frac{\partial W}{\partial \mathbf{F}} - p\mathbf{F}^{-1}, \tag{1.186}$$

being, in the order given, the counterparts of (1.133) and (1.134) appropriated for the incompressible case.

Evidently, by virtue of (1.18) and (1.51), the incompressibility constraint $i_3 = I_3 = 1$ may equivalently be expressed through the connection

$$J \equiv \lambda_1\lambda_2\lambda_3 = 1, \tag{1.187}$$

holding at each point of the material. From this follows that for an incompressible material only two out of the three principal stretches of the deformation are in fact independent.

Thus, when specializing to incompressible isotropic solids and in line with the discussion provided in Section 1.4.1, the elastic stored energy can, in general, be determined by expressing the strain-energy function in terms of two independent scalar deformation quantities. Expressly, here we extend the notations (1.143) to write W as

$$W = \breve{w}(i_1, i_2) = \bar{w}(I_1, I_2), \tag{1.188}$$

arising from elimination of the variables i_3 and I_3 respectively. The latter equalities in conjunction with (1.170), (1.167) (evaluated for $i_3 = I_3 = 1$) are now employed to express the stresses (1.186) in the forms

$$\begin{aligned} \boldsymbol{\sigma} &= \breve{w}_1\mathbf{V} + \breve{w}_2(i_1\mathbf{I} - \mathbf{V})\mathbf{V} - \breve{p}\mathbf{I} \\ &= 2\bar{w}_1\mathbf{B} + 2\bar{w}_2(I_1\mathbf{I} - \mathbf{B})\mathbf{B} - \bar{p}\mathbf{I}, \end{aligned} \tag{1.189}$$

$$\begin{aligned} \mathbf{S} &= \breve{w}_1\mathbf{R}^{\mathrm{T}} + \breve{w}_2(i_1\mathbf{R}^{\mathrm{T}} - \mathbf{F}^{\mathrm{T}}) - \breve{p}\mathbf{F}^{-1} \\ &= 2\bar{w}_1\mathbf{F}^{\mathrm{T}} + 2\bar{w}_2(I_1\mathbf{I} - \mathbf{C})\mathbf{F}^{\mathrm{T}} - \bar{p}\mathbf{F}^{-1}, \end{aligned} \tag{1.190}$$

where the notations $\breve{p}$ and $\bar{p}$ are implemented to interpret p with respect to the pairs of invariants (i_1, i_2) and (I_1, I_2) respectively since, in general, we have $\breve{p} \neq \bar{p}$.

Finally, in view of (1.149) and (1.184) the internal constraint (1.184) is equivalently expressed by $\hat{w}(\lambda_1, \lambda_2, \lambda_3) - p(\lambda_1\lambda_2\lambda_3 - 1)$ and in this connection we write (1.174) as

$$\sigma_i = \lambda_i\frac{\partial}{\partial\lambda_i}[\hat{w}(\lambda_1, \lambda_2, \lambda_3) - p(\lambda_1\lambda_2\lambda_3 - 1)] = \lambda_i\frac{\partial\hat{w}}{\partial\lambda_i} - p, \quad i \in \{1, 2, 3\}, \tag{1.191}$$

where p may be either $\breve{p}$ or $\bar{p}$. A much more detailed and thorough discussion on the way that (1.189)-(1.191) are derived is demonstrated in [57]. The reader is also referred to [56] for an extensive analysis on the constitutive formulation of the mechanics of incompressible isotropic solids.

We expand our discussion to the case of transversely isotropic solids subject to the same constraint and we modify (1.165) in an analogous manner, namely

$$W = \breve{W}(i_1, i_2, I_4, I_5) = \bar{W}(I_1, I_2, I_4, I_5). \tag{1.192}$$

With reference to (1.192), the derivatives (1.170), (1.167) accompanied by (1.176) are now exploited to particularize (1.186) such that (see, also [72])

$$\begin{aligned}\boldsymbol{\sigma} =& \breve{W}_1\mathbf{V} + \breve{W}_2(i_1\mathbf{I} - \mathbf{V})\mathbf{V} + 2\breve{W}_4\mathbf{m}\otimes\mathbf{m} \\ &+ 2\breve{W}_5(\mathbf{m}\otimes\mathbf{Bm} + \mathbf{Bm}\otimes\mathbf{m}) - \breve{p}\mathbf{I} \\ =& 2\bar{W}_1\mathbf{B} + 2\bar{W}_2(I_1\mathbf{I} - \mathbf{B})\mathbf{B} + 2\bar{W}_4\mathbf{m}\otimes\mathbf{m} \\ &+ 2\bar{W}_5(\mathbf{m}\otimes\mathbf{Bm} + \mathbf{Bm}\otimes\mathbf{m}) - \bar{p}\mathbf{I},\end{aligned} \tag{1.193}$$

$$\begin{aligned}\mathbf{S} =& \breve{W}_1\mathbf{R}^{\mathrm{T}} + \breve{W}_2(i_1\mathbf{R}^{\mathrm{T}} - \mathbf{F}^{\mathrm{T}}) + 2\breve{W}_4\mathbf{M}\otimes\mathbf{FM} \\ &+ 2\breve{W}_5(\mathbf{M}\otimes\mathbf{FCM} + \mathbf{CM}\otimes\mathbf{FM}) - \breve{p}\mathbf{F}^{-1} \\ =& 2\bar{W}_1\mathbf{F}^{\mathrm{T}} + 2\bar{W}_2(I_1\mathbf{I} - \mathbf{C})\mathbf{F}^{\mathrm{T}} + 2\bar{W}_4\mathbf{M}\otimes\mathbf{FM} \\ &+ 2\bar{W}_5(\mathbf{M}\otimes\mathbf{FCM} + \mathbf{CM}\otimes\mathbf{FM}) - \bar{p}\mathbf{F}^{-1},\end{aligned} \tag{1.194}$$

where, similar to the isotropic theory, $\breve{p}$ and $\bar{p}$ are associated with the manifolds (i_1, i_2, I_4, I_5) and (I_1, I_2, I_4, I_5) respectively.

We note in passing that, apart from the incompressibility constraint, other kinematical restrictions serving to encapsulate the nature of the mechanical response of particular classes of hyperelastic materials may be imposed into the constitutive law. A typical constraint is that of *inextensibility*, often met in studies concerned with highly anisotropic solids and especially with composite structures incorporating a single or multiple fibre families of extremely high stiffness. We cite for instance the work of Adkins and Rivlin [5] and Green and Adkins [24] who analyzed the response of hyperelastic sheets reinforced by a network of high strength cords (fibres). Further aspects of elastic structures featuring inextensibility as a result of (strong) fibre reinforcement were investigated by Beatty [11] while several contributions on this matter are due to Spencer (see., e.g., [72, 73]).

Although the above mentioned materials are likely capable of undergoing large deformations, such a possibility is disqualified along the direction of each fibre family associated with a unit vector field, namely the preferred direction, which is regarded as modelling the fibres as a continuous distribution. In that sense, if $\mathbf{M}$ is a preferred direction of the material, inextensibility can be ideally expressed as

$$\ell(\mathbf{F}) \equiv |\mathbf{FM}|^2 - 1 = 0, \tag{1.195}$$

for all deformation gradients $\mathbf{F}$ and then the material is said to be *inextensible* in the direction of $\mathbf{M}$. In order to incorporate the inextensibility constraint in terms of the elastic stored energy we write W in the modified form

$$W(\mathbf{F}) - q\ell(\mathbf{F}), \tag{1.196}$$

where again q is a Lagrange multiplier. If the material is transversely isotropic, this leads directly to $I_4 = 1$ and then W may be represented by means of the remaining invariants discussed in Section 1.4.1.

1.5 Constitutive restrictions under transverse isotropy

Henceforth, we mainly focus our analysis on transversely isotropic solids. Our first imperative is to introduce the appropriate restrictions on the strain-energy function which provide consistency with the classical linear theory of transversely isotropic elasticity. The issue of physically admissible modes of deformation is then of our concern. In the context of a realistic material behaviour we summarize a number of mathematical statements, previously proposed by various other authors, which aim to establish compatibility between the nature of the deformation and the actual response of the body predicated by the constitutive laws.

1.5.1 Connection to the linear theory

For a purely elastic body material undergoing ‘small’ deformations the strain tensors $\mathbf{E}$ and $\mathbf{e}$ are quantitatively indistinguishable since all the non-linear terms appearing in both (1.38) and (1.41) are at our disposal, i.e.

$$E_{ij} \cong \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) \qquad e_{ij} \cong \frac{1}{2}\left(\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i}\right), \tag{1.197}$$

while, in the same grounds, we have

$$\frac{\partial u_i}{\partial X_j} = \frac{\partial \tilde{u}_i}{\partial x_p}\frac{\partial u_p}{\partial X_j} + \frac{\partial \tilde{u}_i}{\partial x_p}\delta_{pj} \cong \frac{\partial \tilde{u}_i}{\partial x_j}. \tag{1.198}$$

For the purposes of the current section, it is therefore convenient to determine the strain through the symmetric second order tensor $\boldsymbol{\varepsilon}$ whose components, namely

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) = \frac{1}{2}\left(\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_i}{\partial x_j}\right), \tag{1.199}$$

may equivalently be perceived as either being Lagrangian or Eulerian.

Accordingly, whereas in the non-linear approach we have $W(\mathbf{F}) = W(\mathbf{E}) = W(\mathbf{e})$, here the elastic stored energy will, in general, be treated as a function depending exclusively on $\boldsymbol{\varepsilon}$, i.e. $W = W(\boldsymbol{\varepsilon})$. Following that, we write

$$\frac{\partial}{\partial t}W(\boldsymbol{\varepsilon}) = \frac{\partial W}{\partial \varepsilon_{ij}}\frac{\partial \varepsilon_{ij}}{\partial t} \equiv \operatorname{tr}\left(\frac{\partial W}{\partial \boldsymbol{\varepsilon}}\mathbf{L}\right), \tag{1.200}$$

together with $J = \det(\mathbf{F}) = \det(\mathbf{I} + \mathbf{L}) \cong 1 + \operatorname{tr}(\mathbf{L})$ and $\operatorname{tr}(\mathbf{L}) \equiv \operatorname{tr}(\boldsymbol{\varepsilon})$. This in turn suggests $J\operatorname{tr}(\boldsymbol{\sigma}\mathbf{L}) \cong \operatorname{tr}(\boldsymbol{\sigma}\mathbf{L})$ and, as a result, the first equation in (1.128) is now written in the approximate form

$$\operatorname{tr}\left(\frac{\partial W}{\partial \boldsymbol{\varepsilon}}\mathbf{L}\right) = \operatorname{tr}(\boldsymbol{\sigma}\mathbf{L}), \tag{1.201}$$

implying that, for a compressible hyperelastic solid, the associated linearized Cauchy stress tensor can be determined through

$$\boldsymbol{\sigma} = \frac{\partial W}{\partial \boldsymbol{\varepsilon}},$$

or, equivalently, in index form

$$\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}}. \tag{1.202}$$

Within the linear regime, however, *Hooke's law* prescribes that, for deformations taking place in one dimension, the stress has always to remain proportional to the strain. For general three-dimensional deformations, this is analogously expressed by the constitutive relation

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}, \tag{1.203}$$

where the parameters $C_{ijkl}, (i, j, k, l) \in \{1, 2, 3\}$, are the components of the so-called *stiffness matrix* which, by arguments similar to those stated in Section 1.3.2 and 1.3.3, appear to possess the symmetries

$$C_{ijkl} = C_{klij} = C_{jikl} = C_{ijlk}. \tag{1.204}$$

Thus, from (1.202) and (1.203) we easily deduce that $\partial W/\partial \varepsilon_{ij} = C_{ijkl}\varepsilon_{kl}$ and hence, given that $W = 0$ for $\boldsymbol{\varepsilon} = \mathbf{0}$, integration of the latter equation yields

$$W = \frac{1}{2}C_{ijkl}\varepsilon_{ij}\varepsilon_{kl}. \tag{1.205}$$

We clarify that the parameters C_{ijkl} are associated with *extension* and *shear moduli* as well as with *Poisson's ratios* and hence they serve to characterize the elastic properties of each hyperelastic material. Thus, for a transversely isotropic solid, they are expected to distinguish position and orientation. In other words, the distinctive directional qualities of materials of this kind are now incorporated through the parameters C_{ijkl} which, in particular, depend on the distribution of the preferred direction $\mathbf{M}$ at each material particle in the reference configuration and hence, in general, on $\mathbf{X}$.

Evidently, when we are concerned with small deformations the invariant based representations of the elastic stored energy discussed in Section 1.4.1 are required to be in full agreement with the one adopted here. This readily suggests that, in order to establish compatibility between (1.165) and (1.205), we have to express $\breve{W}$ and $\bar{W}$ by means of the components of $\boldsymbol{\varepsilon}$.Starting from the case where $W = \bar{W}(I_1, I_2, I_3, I_4, I_5)$ we first clarify that for any arbitrarily chosen fixed point $\mathbf{X}$ in the reference configuration we have $\mathbf{F} = \mathbf{I}$, which, through (1.22), provides $\lambda_1 = \lambda_2 = \lambda_3 = 1$, and hence, from (1.147), (1.153), we obtain $I_1 = I_2 = 3$ and $I_3 = I_4 = I_5 = 1$. Thus, assuming that suitable continuity conditions are placed upon $W = \bar{W}$, we are prompted to express the elastic stored energy as a *Taylor series* about an arbitrary fixed point $\mathbf{X}$ in B_r, i.e. in a neighborhood of $(I_1, I_2, I_3, I_4, I_5) = (3, 3, 1, 1, 1)$, such as

$$W = \left\{ \sum_{k=0}^{\infty} \frac{1}{k!} \left[\sum_{p=1}^{5} (I_p - \bar{I}_p) \frac{\partial}{\partial \bar{I}_p} \right]^k \bar{W}(\bar{I}_1, \bar{I}_2, \bar{I}_3, \bar{I}_4, \bar{I}_5) \right\}_{\bar{I}_1=\bar{I}_2=3, \bar{I}_3=\bar{I}_4=\bar{I}_5=1} . \qquad (1.206)$$

Seeing, however, the form of (1.205) and bearing in mind the entries (1.44), (1.145), (1.152), $(1.197)_1$ and (1.199) we deduce that the linear counterpart of (1.206) can only incorporate those terms corresponding to $\kappa \leqslant 2$. In particular, now we are making use of the linkages

$$\begin{aligned}
& I_1 - 3 = 2\mathrm{tr}\,(\boldsymbol{\varepsilon}), \quad I_2 - 3 = 2[2\mathrm{tr}\,(\boldsymbol{\varepsilon}) + \mathrm{tr}\,(\boldsymbol{\varepsilon})^2 - \mathrm{tr}\,(\boldsymbol{\varepsilon}^2)], \\
& I_3 - 1 = 2[\mathrm{tr}\,(\boldsymbol{\varepsilon}) + \mathrm{tr}\,(\boldsymbol{\varepsilon})^2 - \mathrm{tr}\,(\boldsymbol{\varepsilon}^2)], \quad I_4 - 1 = 2\mathbf{M} \cdot (\boldsymbol{\varepsilon}\mathbf{M}), \\
& I_5 - 1 = 4[\mathbf{M} \cdot (\boldsymbol{\varepsilon}\mathbf{M}) + \mathbf{M} \cdot (\boldsymbol{\varepsilon}^2\mathbf{M})], \quad (I_1 - 3)^2 = 4\mathrm{tr}\,(\boldsymbol{\varepsilon})^2, \\
& (I_2 - 3)^2 = 16\mathrm{tr}\,(\boldsymbol{\varepsilon})^2, \quad (I_3 - 1)^2 = 4\mathrm{tr}\,(\boldsymbol{\varepsilon})^2, \\
& (I_4 - 1)^2 = 4[\mathbf{M} \cdot (\boldsymbol{\varepsilon}\mathbf{M})]^2, \quad (I_5 - 1)^2 = 16[\mathbf{M} \cdot (\boldsymbol{\varepsilon}\mathbf{M})]^2, \\
& (I_1 - 3)(I_2 - 3) = 8\mathrm{tr}\,(\boldsymbol{\varepsilon})^2, \quad (I_1 - 3)(I_3 - 1) = 4\mathrm{tr}\,(\boldsymbol{\varepsilon})^2, \\
& (I_1 - 3)(I_4 - 1) = 4\mathrm{tr}\,(\boldsymbol{\varepsilon})\mathbf{M} \cdot (\boldsymbol{\varepsilon}\mathbf{M}), \quad (I_1 - 3)(I_5 - 1) = 8\mathrm{tr}\,(\boldsymbol{\varepsilon})\mathbf{M} \cdot (\boldsymbol{\varepsilon}\mathbf{M}), \\
& (I_2 - 3)(I_3 - 1) = 8\mathrm{tr}\,(\boldsymbol{\varepsilon})^2, \quad (I_2 - 3)(I_4 - 1) = 8\mathrm{tr}\,(\boldsymbol{\varepsilon})\mathbf{M} \cdot (\boldsymbol{\varepsilon}\mathbf{M}), \\
& (I_2 - 3)(I_5 - 1) = 16\mathrm{tr}\,(\boldsymbol{\varepsilon})\mathbf{M} \cdot (\boldsymbol{\varepsilon}\mathbf{M}), \quad (I_3 - 1)(I_4 - 1) = 4\mathrm{tr}\,(\boldsymbol{\varepsilon})\mathbf{M} \cdot (\boldsymbol{\varepsilon}\mathbf{M}), \\
& (I_3 - 1)(I_5 - 1) = 8\mathrm{tr}\,(\boldsymbol{\varepsilon})\mathbf{M} \cdot (\boldsymbol{\varepsilon}\mathbf{M}), \quad (I_4 - 1)(I_5 - 1) = 8[\mathbf{M} \cdot (\boldsymbol{\varepsilon}\mathbf{M})]^2,
\end{aligned} \qquad (1.207)$$

to recast W in the form

$$\begin{aligned} W = \bar{W} &+ 2\mathrm{tr}\,(\boldsymbol{\varepsilon})(\bar{W}_1 + 2\bar{W}_2 + \bar{W}_3) - 2\mathrm{tr}\,(\boldsymbol{\varepsilon}^2)(\bar{W}_2 + \bar{W}_3) \\ &+ 2\mathrm{tr}\,(\boldsymbol{\varepsilon})^2(\bar{W}_2 + \bar{W}_3 + \bar{W}_{11} + 4\bar{W}_{22} + \bar{W}_{33} + 2\bar{W}_{13} + 4\bar{W}_{12} + 4\bar{W}_{23}) \\ &+ 4\mathrm{tr}\,(\boldsymbol{\varepsilon})\mathbf{M}\cdot(\boldsymbol{\varepsilon}\mathbf{M})(\bar{W}_{14} + 2\bar{W}_{15} + 2\bar{W}_{24} + 4\bar{W}_{25} + \bar{W}_{34} + 2\bar{W}_{35}) \\ &+ 2\mathbf{M}\cdot(\boldsymbol{\varepsilon}\mathbf{M})(\bar{W}_4 + 2\bar{W}_5) + 4\mathbf{M}\cdot(\boldsymbol{\varepsilon}^2\mathbf{M})\bar{W}_5 \\ &+ 2[\mathbf{M}\cdot(\boldsymbol{\varepsilon}\mathbf{M})]^2(\bar{W}_{44} + 4\bar{W}_{55} + 4\bar{W}_{45}), \end{aligned} \tag{1.208}$$

where the subscripts $1, ..., 5$ on $\bar{W}$ indicate differentiation with respect to $I_1, ..., I_5$, respectively, each one of which derivatives has been evaluated in the reference configuration (where $I_1 = I_2 = 3$ and $I_3 = I_4 = I_5 = 1$). Hence, by writing

$$\mathrm{tr}\,(\boldsymbol{\varepsilon}) = \varepsilon_{ii}, \quad \mathrm{tr}\,(\boldsymbol{\varepsilon}^2) = \varepsilon_{ij}\varepsilon_{ji}, \quad \mathbf{M}\cdot(\boldsymbol{\varepsilon}\mathbf{M}) = \varepsilon_{ij}M_iM_j, \quad \mathbf{M}\cdot(\boldsymbol{\varepsilon}^2\mathbf{M}) = \varepsilon_{ik}\varepsilon_{kj}M_iM_j, \tag{1.209}$$

comparison of (1.205) with (1.208) leads directly to the requirements

$$\bar{W} = 0, \quad \bar{W}_1 + 2\bar{W}_2 + \bar{W}_3 = 0, \quad \bar{W}_4 + 2\bar{W}_5 = 0, \tag{1.210}$$

while also

$$\begin{aligned} C_{ijkl} =& 2(\bar{W}_1 + \bar{W}_2)(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}) + 4(\bar{W}_{44} + 4\bar{W}_{55} + 4\bar{W}_{45})M_iM_jM_kM_l \\ &+ 4(\bar{W}_2 + \bar{W}_3 + \bar{W}_{11} + 4\bar{W}_{22} + \bar{W}_{33} + 2\bar{W}_{13} + 4\bar{W}_{12} + 4\bar{W}_{23})\delta_{ij}\delta_{kl} \\ &+ 4(\bar{W}_{14} + 2\bar{W}_{15} + 2\bar{W}_{24} + 4\bar{W}_{25} + \bar{W}_{34} + 2\bar{W}_{35})(M_kM_l\delta_{ij} + M_iM_j\delta_{kl}) \\ &+ 2\bar{W}_5(M_iM_k\delta_{jl} + M_iM_l\delta_{jk} + M_jM_k\delta_{il} + M_jM_l\delta_{ik}). \end{aligned} \tag{1.211}$$

We remark that the connections (1.211) are tantamount to those presented by Spencer [73]. In essence, the only difference is that here the parameters C_{ijkl} are directly related to five independent combinations in terms of the derivatives of $\bar{W}$ instead of five, in general undetermined, independent material constants as suggested by Spencer.

On the other hand, when the formulation $W = \breve{W}(i_1, i_2, i_3, I_4, I_5)$ is used, we recall the identities (1.148) to write

$$\breve{W}(i_1, i_2, i_3, I_4, I_5) = \bar{W}(i_1^2 - 2i_2, i_2^2 - 2i_1i_3, i_3^2, I_4, I_5). \tag{1.212}$$

Now, since for $\lambda_1 = \lambda_2 = \lambda_3 = 1$ the invariants (1.146) reduce to $i_1 = i_2 = 3, i_3 = 1$, it easy to show that the analogues of (1.210) become

$$\breve{W} = 0, \quad \breve{W}_1 + 2\breve{W}_2 + \breve{W}_3 = 0, \quad \breve{W}_4 + 2\breve{W}_5 = 0, \tag{1.213}$$

respectively, while (1.211) transform into

$$\begin{aligned}C_{ijkl} =&(\breve{W}_1+\breve{W}_2)(\delta_{ik}\delta_{jl}+\delta_{jk}\delta_{il})/2+4(\breve{W}_{44}+4\breve{W}_{55}+4\breve{W}_{45})M_iM_jM_kM_l\\&+(\breve{W}_2+\breve{W}_3+\breve{W}_{11}+4\breve{W}_{22}+\breve{W}_{33}+2\breve{W}_{13}+4\breve{W}_{12}+4\breve{W}_{23})\delta_{ij}\delta_{kl}\\&+2(\breve{W}_{14}+2\breve{W}_{15}+2\breve{W}_{24}+4\breve{W}_{25}+\breve{W}_{34}+2\breve{W}_{35})(M_kM_l\delta_{ij}+M_iM_j\delta_{kl})\\&+2\breve{W}_5(M_iM_k\delta_{jl}+M_iM_l\delta_{jk}+M_jM_k\delta_{il}+M_jM_l\delta_{ik}).\end{aligned} \tag{1.214}$$

It is apparent that here the subscripts $1, 2$ and 3 on $\breve{W}$ are used to denote differentiation with respect to the arguments i_1, i_2 and i_3, respectively, whereas 4 and 5 are associated with the invariants I_4 and I_5 as before.

At this point it is important to note that (1.210) and (1.213) serve to stipulate the necessity of a material body whose stress and energy vanish in the reference configuration. These assertions may well be established by recalling the discussion provided in Section 1.4.1 in order to signify (1.139) together with the definitions (1.165) and the stress-strain relationships (1.177), (1.179) (accounting for $\breve{W}$ and $\bar{W}$ respectively) when these are evaluated for $\lambda_1=\lambda_2=\lambda_3=1$ and hence, from (1.47) and (1.49), for $\mathbf{V}=\mathbf{B}=\mathbf{I}$. At the same time, the conditions (1.211) and/or (1.214) result in additional restrictions on the strain-energy function when this is evaluated in the reference configuration so as to preserve its invariant (isotropic) character with respect to rotations and reflections about the preferred direction $\mathbf{M}$.

Parallel to (1.184), the elastic stored energy of an incompressible material undergoing small deformations is expressed through

$$W=\frac{1}{2}C_{ijkl}\varepsilon_{ij}\varepsilon_{kl}-p\varepsilon_{ss}, \tag{1.215}$$

with $J-1\cong \operatorname{tr}(\boldsymbol{\varepsilon})=\varepsilon_{ss}$ while, analogously, the components (1.202) of $\boldsymbol{\sigma}$ are given by

$$\sigma_{ij}=\frac{\partial W}{\partial \varepsilon_{ij}}=C_{ijkl}\varepsilon_{kl}-p\delta_{ij}. \tag{1.216}$$

Further, since, for an isochoric deformation, we have $I_3=1$ at each point of the body (either in B_r or B_t), the difference I_3-1 listed in (1.207) prescribes the identity

$$\operatorname{tr}(\boldsymbol{\varepsilon})+\operatorname{tr}(\boldsymbol{\varepsilon})^2-\operatorname{tr}(\boldsymbol{\varepsilon}^2)=0. \tag{1.217}$$

This can be expanded in components form to give

$$\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}+2(\varepsilon_{11}\varepsilon_{22}+\varepsilon_{11}\varepsilon_{33}+\varepsilon_{22}\varepsilon_{33}-\varepsilon_{12}^2-\varepsilon_{23}^2-\varepsilon_{13}^2)=0, \tag{1.218}$$

wherefrom, after some simple manipulation, we obtain

$$\mathrm{tr}\,(\boldsymbol{\varepsilon})^2 = \mathrm{tr}\,(\boldsymbol{\varepsilon})\mathbf{M}\cdot(\boldsymbol{\varepsilon}\mathbf{M}) = 0, \tag{1.219}$$

followed by

$$\begin{aligned}\mathrm{tr}\,(\boldsymbol{\varepsilon})^2 - \mathrm{tr}\,(\boldsymbol{\varepsilon}^2) &= -2(\varepsilon_{11}^2 + \varepsilon_{22}^2 + \varepsilon_{11}\varepsilon_{22} + \varepsilon_{12}^2 + \varepsilon_{23}^2 + \varepsilon_{13}^2)\\ &= -2(\varepsilon_{11}^2 + \varepsilon_{33}^2 + \varepsilon_{11}\varepsilon_{33} + \varepsilon_{12}^2 + \varepsilon_{23}^2 + \varepsilon_{13}^2)\\ &= -2(\varepsilon_{22}^2 + \varepsilon_{33}^2 + \varepsilon_{22}\varepsilon_{33} + \varepsilon_{12}^2 + \varepsilon_{23}^2 + \varepsilon_{13}^2).\end{aligned} \tag{1.220}$$

In view of (1.218) and (1.220) we conclude that, due to the incompressibility constraint, only two of the principal components of $\boldsymbol{\varepsilon}$ remain independent. Hence, by following the same methodology used to derive (1.208) the corresponding expression, now appropriated for $W = \bar{W}(I_1, I_2, I_4, I_5)$, cannot be defined in a unique manner. Consequently, depending on the component of $\boldsymbol{\varepsilon}$ that we choose to eliminate each time, the connections between the parameters C_{ijkl} and the derivatives of $\bar{W}$, these evaluated in the reference configuration, will vary analogously. It is apparent that the same conclusion applies when the the elastic energy is taken to be determined as a function of the principal invariants (i_1, i_2, I_4, I_4) as prescribed in (1.192), where, in this case, the counterpart of (1.212) specializes to

$$\breve{W}(i_1, i_2, I_4, I_5) = \bar{W}(i_1^2 - 2i_2, i_2^2 - 2i_1, I_4, I_5). \tag{1.221}$$

The same issue actually arises from the point at which the incompressibility condition is used. We clarify this argument by once more referring to the work of Spencer [73] who, in particular, defined the elastic stored energy of an incompressible transversely isotropic hyperelastic solid as a function of the non-zero terms $\mathrm{tr}\,(\boldsymbol{\varepsilon}^2), [\mathbf{M}\cdot(\boldsymbol{\varepsilon}\mathbf{M})]^2$ and $\mathbf{M}\cdot(\boldsymbol{\varepsilon}^2\mathbf{M})$ including an arbitrary multiple of $\mathrm{tr}\,(\boldsymbol{\varepsilon})$, namely $-p\mathrm{tr}\,(\boldsymbol{\varepsilon})$, in order to incorporate the hydrostatic pressure. Spencer's approach may be declared directly from (1.208) by making instant use of the properties (1.217), (1.219) and by also omitting the term $\mathbf{M}\cdot(\boldsymbol{\varepsilon}\mathbf{M})$ which, unlike (1.215), is clearly linear in terms of the components of $\boldsymbol{\varepsilon}$ since $J - 1 \cong \mathrm{tr}\,(\boldsymbol{\varepsilon})$ is negligible anyway. By these means, the associated stress tensor $\boldsymbol{\sigma}$ is easily found to be a function of $\boldsymbol{\varepsilon}$, $[\mathbf{M}\cdot(\boldsymbol{\varepsilon}\mathbf{M})]\mathbf{M}\otimes\mathbf{M}, \mathbf{M}\otimes(\boldsymbol{\varepsilon}\mathbf{M})$ and $(\boldsymbol{\varepsilon}\mathbf{M})\otimes\mathbf{M}$ where, for obvious reasons, the term $-p\mathbf{I}$ is also included. Notwithstanding, if the incompressibility constraint is applied after the stress $\boldsymbol{\sigma}$ is calculated, the latter appears to be a function of the aforementioned quantities but now conjoined with $[\mathbf{M}\cdot(\boldsymbol{\varepsilon}\mathbf{M})]\mathbf{I}$. Thus, in line with (1.216) and despite the fact the additional argument can be absorbed into $-p\mathbf{I}$ (i.e. p is arbitrary), the parameters C_{ijkl} will be of a different form compared to those arising from Spencer's design.

In any case, from the above discussion it can easily be drawn that now, unlike the compressible theory, the parameters C_{ijkl} can sufficiently be associated, and yet not in a unique manner, with only three independent combinations of derivatives of $\bar{W}$ and/or $\breve{W}$ all of which are evaluated in the reference configuration. One of these possible formulations will be presented in Chapter 2 for more particular geometries of the preferred direction $\mathbf{M}$.

Independently of the above mentioned facts we will, however, assume that in the reference configuration the conditions

$$\bar{W} = 0, \quad 2\bar{W}_1 + 4\bar{W}_2 = \bar{p}_0, \quad \bar{W}_4 + 2\bar{W}_5 = 0, \tag{1.222}$$

$$\breve{W} = 0, \quad 2\breve{W}_1 + 4\breve{W}_2 = \breve{p}_0, \quad \breve{W}_4 + 2\breve{W}_5 = 0, \tag{1.223}$$

are always in place so as to ensure that the body remains stress and energy free. In particular, the first equalities in both (1.222) and (1.223) are justified in the same way as $(1.210)_1$ and $(1.213)_1$, respectively, while also, given that $\boldsymbol{\sigma} = \mathbf{0}$ in B_r, the remaining requirements are taken in consistency with the expressions (1.193) when these are evaluated for $\lambda_1 = \lambda_2 = \lambda_3 = 1$, implying, as already mentioned, that $\mathbf{V} = \mathbf{B} = \mathbf{I}$. In that sense, here the symbols $\bar{p}_0$ and $\breve{p}_0$ have been introduced to characterise the values of $\bar{p}$ and $\breve{p}$, respectively, in B_r.

1.5.2 Constitutive inequalities

In this section we are concerned with boundary-value problems of transversely isotropic hyperelastic solids undergoing non-homogeneous deformations. In particular, we consider deforming material bodies in the absence of body forces whose state of stress is determined on resolving the equilibrium equation, as prescribed from either (4.3.3) or (1.84), now coupled with appropriate boundary conditions.

Since the nature of the materials under examination is assumed to be hyperelastic, the entry (1.134) accounting for a general (i.e. not necessarily transversely isotropic) unconstrained material is employed to rewrite the equilibrium equation, namely $\operatorname{Div}\mathbf{S} = \mathbf{0}$, into the equivalent component form

$$\mathcal{A}_{\alpha i \beta j} x_{j,\alpha\beta} = 0, \tag{1.224}$$

while for the incompressible case the formula (1.186) analogously delivers

$$\mathcal{A}_{\alpha i \beta j} x_{j,\alpha\beta} - p_{,i} = 0. \tag{1.225}$$

Here the notations

$$\mathcal{A}_{\alpha i\beta j} = \frac{\partial^2 W}{\partial F_{i\alpha}\partial F_{j\beta}}, \quad x_{j,\alpha\beta} = \frac{\partial^2 x_j}{\partial X_\alpha \partial X_\beta}, \quad p_{,i} = \frac{\partial p}{\partial x_i}, \tag{1.226}$$

have been adopted for convenience with the Greek and Latin indices being used in association with the reference and current configuration respectively.

In the light of (1.224) and (1.225) it is understood that the equilibrium equation, this combined with appropriate boundary conditions, forms a coupled system of three highly nonlinear second-order partial differential equations for the components of the position vector $\mathbf{x} \in B_t$. On this basis, the form of the strain-energy function used to represent each hyperelastic solid has an important role to play regarding the resolution and hence the implications, in the sense of a realistic material behaviour, of (1.224) and/or (1.225).

From that perspective, the issues of existence, prescribed by the *principle of stationary potential energy*† [56], and uniqueness of solution of the equilibrium equation, both in a local as well in a global scale, have been a subject of research. In fact, these aspects in conjunction with the notion of a *stable solution*, the latter tending to render equilibrium configurations associated with minimum potential energy while compatible with the boundary conditions (see, e.g., [29]), have been, amongst other, the main motivating factors that led the researchers to impose various limitations on the form of the strain-energy function. It is important to distinguish that the above mentioned postulates can be used as the main criteria for identification of the class of deformations that can be admitted by a particular strain-energy function W. Conversely, given the deformation is known, they can either yield certain limitations on an assigned strain-energy function or even suggest appropriate forms for W (i.e. by means of *constitutive modelling*) that are consistent with the nature of the deformation under investigation. In what follows we are engrossed in such considerations but mainly within the scope of how these aim to restrict the constitutive law, whether for all or for some subset of deformations, so as to deliver physically realistic solutions of the equilibrium equation. Restrictions of this nature that function in the form of inequalities are known as *constitutive inequalities* and these are of particular interest here.

A typical example of a restriction of this kind was introduced by Baker and Ericksen

†The same principle has been recorded in a variety of textbooks sometimes under a different name. We mention, for example, the book by Truesdell and Noll [79] where there the authors used the terminology *first variational theorem*.

[8] who, in particular, suggested the inequalities

$$(\sigma_i - \sigma_j)(\lambda_i - \lambda_j) > 0, \qquad \lambda_i \neq \lambda_j \tag{1.227}$$

implying that the principal stretches are in the same ascending numerical order as that of the associated principal stresses. Expressed otherwise, the inequalities (1.227) assert that the greater principal stress always occurs in the same direction with the greater principal stretch and vice versa. In the case of transverse isotropy, and especially when this is associated with fibre reinforcement, the inequalities (1.227) do not, in general, establish physically realistic material behaviour. Truly, assume that a fibre-reinforced hyperelastic composite, this now characterised by a single family of fibres, is deformed in such a way that the left Cauchy-Green deformation tensor $\mathbf{B}$ is coaxial with the deformed preferred direction $\mathbf{m}$. As already mentioned in Section 1.4.2 this establishes coaxiality between the Cauchy stress tensor $\boldsymbol{\sigma}$ and left stretch tensor $\mathbf{V}$. Accordingly, all the shear components of $\boldsymbol{\sigma}$ vanish while it is diagonal with respect to the Eulerian principal axes, i.e. $\sigma_i = \sigma_{ii}$ (no sum over i). The inequalities (1.227) then require that if $\sigma_{11} > \sigma_{22}$ the resulting strain and hence the stretch λ_1 in the direction $\mathbf{v}^{(1)}$ has, in any case, to be greater compared to the stretch λ_2 in the direction $\mathbf{v}^{(2)}$. In other words, even if $|\sigma_{11}|$ is infinitesimally greater than $|\sigma_{22}|$ the material undergoes larger deformation in the planes parallel to $\mathbf{v}^{(1)}$. However, this conflicts with the strong directional mechanical properties that such materials normally embody; the later incorporating stiffening response in the fibre direction. The same restrictions, on the other hand, appear to be physically meaningful in the context of isotropic materials but we do not pursue this design idea here. We only illustrate that, using either (1.174) or (1.191), these can be written in the equivalent compact form

$$\frac{\lambda_i \hat{w}_i - \lambda_j \hat{w}_j}{\lambda_i^2 - \lambda_j^2} > 0, \qquad \lambda_i \neq \lambda_j \tag{1.228}$$

in terms of the strain-energy function $\hat{w}(\lambda_1, \lambda_2, \lambda_3)$ where now $\hat{w}_p = \partial\hat{w}/\partial\lambda_p$. Similarly, bearing in mind (1.173), the inequalities (1.227) recast to

$$\breve{w}_1 + \breve{w}_2\lambda_i > 0, \qquad i = 1, 2, 3 \tag{1.229}$$

$$\bar{w}_1 + \bar{w}_2\lambda_i^2 > 0, \qquad i = 1, 2, 3 \tag{1.230}$$

by means of the functions $\breve{w}(i_1, i_2, i_3)$ and $\bar{w}(I_1, I_2, I_3)$ respectively. In a like manner, (1.229) and (1.230) are easily found apply for the case where the material is taken to be incompressible, yet, now appropriated for $\breve{w}(i_1, i_2)$ and $\bar{w}(I_1, I_2)$ respectively. For a detailed discussion regarding the implications of (1.227) in respect of both compressible and incompressible isotropic hyperelastic solids we refer to the work of Truesdell and Toupin [80] and Truesdell [78].

Another widely adopted constitutive inequality relies on the hypothesis that the strain-energy function W should be restricted within the class of *strictly convex* functions of the deformation gradient tensor $\mathbf{F}$. Given that W is twice-continuously differentiable with respect to $\mathbf{F}$ and the latter is defined on a *convex* (open) *subset* of the Euclidian vector-space, this constitutive assumption is assigned though the requirement

$$\operatorname{tr}\left\{(\frac{\partial^2 W}{\partial \mathbf{F} \partial \mathbf{F}}\mathbf{A})\mathbf{A}\right\} > 0, \qquad \mathbf{A} \neq \mathbf{0} \tag{1.231}$$

holding for each such $\mathbf{F}$. The implementation of the (strict) convexity constraint as a constitutive requirement placed upon W is sufficient to ensure uniqueness of solution of the equilibrium equation for boundary value problems combined with the presence of *dead-load tractions* [29], these being surface tractions independent of the deformation gradient $\mathbf{F}$ which act on part of the boundary ∂Br. Precisely, in the case of dead loading, the inequality (1.231) further suffices for uniqueness of solution in respect of *incremental* boundary-value problems [56]. Accordingly, strict convexity of W supports the argument that if a stable solution for such problems exists then this is unique, while, obviously, smoothness of the strain energy (i.e to the degree desired by the equilibrium equation) and hence of the stress is ensured a priori. Constructively, in respect of real material response, the possibility of buckling and the occurrence of *instability phenomena*, such as formulations of *kink-bands* in fibre-reinforced composites (see, e.g. [82]), associated in general with discontinuous modes of deformation is excluded. In addition, as first noted by Coleman and Noll [13] and then by Truesdell and Noll [79], the statement (1.231) can, in some situations, be inconsistent with the principle of material frame indifference, namely (1.139) when applied to W, while it cannot either preclude that finite amount of energy is required to reduce the volume of an infinitesimal material specimen into zero. As clearly explained by Ball [9] the latter contradicts such natural conditions as

$$W(\mathbf{F}) \to +\infty \qquad \text{as} \qquad \det(\mathbf{F}) \to 0^+, \tag{1.232}$$

which, amongst others, is normally expected to be satisfied by the constitutive law (i.e. for unconstrained materials). Similarly, strict convexity appears to be problematic for incompressible materials where $\det(\mathbf{F}) = 1$. We remark that (1.232) essentially assigns a singularity to W under severe strains and it can also be interpreted to state that: infinite stress should be accompanied by infinite strains.

It is therefore evident that the concept of strict convexity of W turns out to be too restrictive since it fails to incorporate certain behavioural material aspects which are indeed likely to occur. We mention, however, that this is not the case for the weaker and yet invariant constitutive hypothesis known as the *strong ellipticity condition*. This is expressed through the requirement

$$\operatorname{tr}\left\{[\frac{\partial^2 W}{\partial \mathbf{F} \partial \mathbf{F}}(\mathbf{s} \otimes \mathbf{n})](\mathbf{s} \otimes \mathbf{n})\right\} > 0, \tag{1.233}$$

for any pair of arbitrary non-vanishing vectors $\mathbf{s}$ and $\mathbf{n}$. Provided the form of the strain-energy function is given, a deformation gradient $\mathbf{F}$ satisfying (1.233) for all $\mathbf{s} \otimes \mathbf{n} \neq \mathbf{0}$ is said to be a *strongly elliptic deformation* for that W. If, also, all the deformations that can be supported by a particularised W are strongly elliptic, then the material itself is called a *strongly elliptic material*.

In view of (1.231), it is apparent that (1.233) is a rank-one convexity condition for W. Thus, strict convexity implies strong ellipticity but the opposite is not necessarily true. Accordingly, by means of the solutions to the equilibrium equation, strong ellipticity cannot in general guarantee either uniqueness or stability of solution for dead-load traction boundary-value problems. Notwithstanding, this according to the incremental stability criterion, every stable solution for such problems (which, as noted in [79], need not be unique) satisfies the strong ellipticity condition.

Despite these matters, the notion of strong ellipticity has been associated with a number of interesting mechanical issues in regard to the response of hyperelastic materials. For example, substantiation of strong ellipticity ensures smoothness of the solutions of the governing equilibrium equations while, in the context of isotropic materials, it implies other physically meaningful constitutive inequalities (see, e.g. [80, 79]) amongst which the requirements (1.227) introduced earlier in this section. Strong ellipticity is also the necessary and sufficient condition for travelling waves to be propagated within a medium with real and positive speeds [78]. In an analogous manner, the concept of failure of strong ellipticity can be put into parallelism with various material instabilities. Indeed, breakdown of strong ellipticity entails the possibility of the emergence of singular surfaces across which either the second derivative of the deformation field becomes discontinuous or the deformation gradient $\mathbf{F}$ suffers a finite jump. Such surfaces are known as surfaces of weak and strong discontinuity, respectively. Conversely, the existence of surfaces of weak or strong discontinuity typify loss of strong ellipticity. In these grounds, given the form of W, the equation that defines the onset of ellipticity failure determines both the deformation associated with such surfaces and the direction of the normal to that surfaces. In the case of fibre reinforced hyperelastic solids the angle between the deformed preferred direction and the normal to the direction of the above mentioned surfaces may then be used as a criterion for identification of particular failure mechanisms taking place during the deformation processes. For an extensive discussion and further references regarding the interpretation of non-elliptic modes of deformation with respect to instability phenomena in fibre reinforced materials we refer to the work of Merodio and Ogden [47, 48].

The issue of strong ellipticity, including particular references in association with the linear theory presented in Section 1.5.1, is also discussed in Chapter 2 in respect of bending deformations of transversely isotropic elastic blocks. It also receives a great deal of attention in Chapter 3 where we study the problem of azimuthal shear of a circular cylindrical tube of incompressible transversely isotropic elastic material subject to finite deformation.

Chapter 2

Bending of transversely isotropic blocks

2.1 Introduction

The problem of finite bending of a rectangular elastic block into a sector of a circular cylindrical tube has been examined by many researchers, almost exclusively for isotropic materials. First, in Rivlin [66], necessary and sufficient conditions for the solution of this problem in terms of the boundary data were derived for incompressible Mooney-Rivlin and neo-Hookean materials by assuming that the block remains in its deformed state in the absence of applied tractions on its curved surfaces but with appropriate tractions applied on its other surfaces. Corresponding results for a general incompressible isotropic material were given by Rivlin [67]. A similar analysis was presented by Green and Zerna [26] and by Green and Adkins [24]. Green and Adkins also examined the problem for incompressible transversely isotropic, initially curved incompressible isotropic and compressible isotropic rectangular blocks. Formulation of the governing equilibrium equations in respect of compressible isotropic materials and the derivation of closed-form solutions for the general class of the so-called harmonic materials were given in Ogden [56], wherein the incompressible case is also discussed. Furthermore, several classes of compressible isotropic materials were investigated by Jiang [38], in which it was shown that finite *isochoric* bending of a block is only sustainable for incompressible materials. The problem of bending in the compressible theory was also discussed by Aron and Wang [7], who used constant modified stretches to express the total energy as a function of the deformed volume V and to deduce that (under plane strain) it attains a minimum at a certain value V_0 of V. In addition, a stability analysis for semi-linear harmonic materials (under plane strain) was discussed by the same authors in [6]. In a recent paper, Bruhns et al. [12] examined the same problem for compressible and incompressible isotropic Hencky materials.

In the present analysis, we consider the problem of bending for transversely isotropic elastic materials. In Section 2.2, the bending deformation is formulated and, under the appropriate specialization for particular directions of the axis of transverse isotropy, the governing differential equations are derived. As expected for the considered deformation, the expressions obtained have the same structure as for the case of compressible isotropic materials given in [56], except that material properties, expressed in terms of a strain-energy function, are different.

Specialization to isochoric bending is then discussed in Section 2.3, and attention is confined mainly to the case of plane strain. The remaining equilibrium equation identifies necessary and sufficient conditions on the energy function for the considered deformation to be sustainable, and, in particular, restrictions on the classical (linear) elastic constants are imposed. In this connection it is interesting to examine the status of so-called *reinforcing models*, for which an isotropic energy function is augmented by an added function that reflects the transverse isotropy as a basic model representing the influence of reinforcing fibres. The linear specialization of the strong ellipticity inequalities (see, for example, Payton [60] and Merodio and Ogden [49]) shows that the considered bending deformation cannot, in general, be achieved for such materials for realistic forms of the reinforcement model. Along the lines of the work of Jiang and Ogden [39, 40], some general forms of strain-energy functions that admit isochoric bending are derived. Some specific forms of these strain energies are then chosen to illustrate the results, and some closed-form solutions are obtained. Numerical calculations are used to demonstrate the stress distributions in the deformed block for two specific energy functions.

In Section 2.4 we examine aspects of the stability of the block for the considered deformation as embodied in the notion of strong ellipticity. For plane strain we provide necessary and sufficient conditions for strong ellipticity to hold.

Finally, Section 2.5 contains a brief discussion of the incompressible counterpart of the analysis presented here.

2.2 Formulation of the problem

2.2.1 Definition and kinematics of the bending deformation

Consider a (stress-free) rectangular block defined, in its reference configuration B_r, by

$$-A \leqslant X_1 \leqslant A, \quad -B \leqslant X_2 \leqslant B, \quad -C \leqslant X_3 \leqslant C, \tag{2.1}$$

where (X_1, X_2, X_3) are rectangular Cartesian coordinates relative to the basis vectors $\{\mathbf{E}_i\}, i \in \{1, 2, 3\}$. Suppose that the body is deformed so that the planes $X_1 =$ constant become sectors of the cylindrical surface $r =$ constant, planes $X_2 =$ constant become planes $\theta =$ constant and planes $X_3 =$ constant become planes $z =$ constant, where (r, θ, z) are cylindrical polar coordinates.

The equations describing the deformation may be written

$$r = f(X_1), \quad \theta = g(X_2), \quad z = \lambda X_3, \tag{2.2}$$

where λ is a constant and the functions f and g are to be determined. We assume that the deformation is symmetric about the X_1 axis, so that $g(-X_2) = -g(X_2)$, and, for definiteness, we take $f(A) > f(-A)$. For convenience we set the notations

$$f(-A) = a_1, \quad f(A) = a_2, \quad g(B) = \alpha, \tag{2.3}$$

so that $a_2 > a_1$.

If we let $\{\mathbf{e}_a\}, a \in \{r, \theta, z\}$ be the cylindrical polar basis vectors in the deformed configuration B_t, then the position vector of a particle in this configuration is given by $\mathbf{x} = r\mathbf{e}_r + z\mathbf{e}_z$, and the deformation gradient tensor $\mathbf{F}$ takes the form

$$\mathbf{F} = f'(X_1)\mathbf{e}_r \otimes \mathbf{E}_1 + f(X_1)g'(X_2)\mathbf{e}_\theta \otimes \mathbf{E}_2 + \lambda \mathbf{e}_z \otimes \mathbf{E}_3. \tag{2.4}$$

Equivalently, as discussed in Section 1.1.6, the deformation gradient can be decomposed as $\mathbf{F} = \mathbf{RU} = \mathbf{VR}$, where now

$$\mathbf{U} = f'\mathbf{E}_1 \otimes \mathbf{E}_1 + fg'\mathbf{E}_2 \otimes \mathbf{E}_2 + \lambda \mathbf{E}_3 \otimes \mathbf{E}_3, \tag{2.5}$$

$$\mathbf{V} = f'\mathbf{e}_r \otimes \mathbf{e}_r + fg'\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \lambda \mathbf{e}_z \otimes \mathbf{e}_z, \tag{2.6}$$

$$\mathbf{R} = \mathbf{e}_r \otimes \mathbf{E}_1 + \mathbf{e}_\theta \otimes \mathbf{E}_2 + \mathbf{e}_z \otimes \mathbf{E}_3. \tag{2.7}$$

From equations (2.5)–(2.7) we deduce that the Lagrangian principal axes coincide with the Cartesian basis vectors $\{\mathbf{E}_i\}$, while the Eulerian principal axes are aligned with the cylindrical polar basis vectors $\{\mathbf{e}_i\}$. Seeing (1.46) and (1.47), the associated principal stretches can therefore be identified as

$$\lambda_1 = f'(X_1), \quad \lambda_2 = f(X_1)g'(X_2), \quad \lambda_3 = \lambda. \tag{2.8}$$

Accordingly, the right and left Cauchy-Green deformation tensors (1.43) (see also (1.48), (1.49)) are then given by

$$\mathbf{C} = \lambda_1^2\mathbf{E}_1 \otimes \mathbf{E}_1 + \lambda_2^2\mathbf{E}_2 \otimes \mathbf{E}_2 + \lambda_3^2\mathbf{E}_3 \otimes \mathbf{E}_3, \tag{2.9}$$

$$\mathbf{B} = \lambda_1^2\mathbf{e}_r \otimes \mathbf{e}_r + \lambda_2^2\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \lambda_3^2\mathbf{e}_z \otimes \mathbf{e}_z, \tag{2.10}$$

respectively.

2.2.2 Some restrictions on the constitutive law

We now assume that the body material features a preferred direction which is locally prescribed by a unit vector $\mathbf{M}$. In particular, we let $\mathbf{M}$ lie in the (X_1, X_2)-plane. As illustrated in Section 1.4.2, for a transversely isotropic elastic solid the Cauchy stress $\boldsymbol{\sigma}$ is coaxial with $\mathbf{V}$ if and only if the deformed preferred direction, namely $\mathbf{m} = \mathbf{FM}$, is an eigenvector of $\mathbf{B}$ or, equivalently, $\mathbf{M}$ is an eigenvector of $\mathbf{C}$.

Thus, if $\mathbf{M}$ is directed along the X_1 axis, the expression (2.9) yields $\mathbf{CM} = \lambda_1^2\mathbf{M}$ for all $\lambda_1 > 0$. The invariants (1.152) then specialize parallel to (1.181), and in particular

$$I_4 = \lambda_1^2, \quad I_5 = I_4^2. \tag{2.11}$$

As a result, the Cauchy stress tensor can be written in the spectral form

$$\boldsymbol{\sigma} = \sigma_1\mathbf{e}_r \otimes \mathbf{e}_r + \sigma_2\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \sigma_3\mathbf{e}_z \otimes \mathbf{e}_z, \tag{2.12}$$

with

$$\sigma_i = J^{-1}\lambda_i\frac{\partial W}{\partial \lambda_i}, \quad i \in \{1, 2, 3\}, \tag{2.13}$$

and now we may represent W as a function of $\lambda_1, \lambda_2, \lambda_3$. Bearing in mind (1.182), we write

$$W = \bar{W}(I_1, I_2, I_3, I_4, I_5) = \hat{W}(\lambda_1, \lambda_2, \lambda_3), \tag{2.14}$$

but now the representation $\breve{W}(i_1, i_2, i_3, I_4, I_5)$ is excluded from consideration. We once more emphasize that, in contrast to $\bar{W}(I_1, I_2, I_3, I_4, I_5)$, the formulation $\hat{W}(\lambda_1, \lambda_2, \lambda_3)$ applies for specific deformations such as the one considered here and is not in general valid. Note that shear stresses are not required to achieve the considered deformation.

Since the body under examination is taken to be initially unstrained, the elastic stored energy and the stress should vanish in B_r. For this we require the properties (1.210), now written together in the compact form as

$$\bar{W} = \bar{W}_1 + 2\bar{W}_2 + \bar{W}_3 = \bar{W}_4 + 2\bar{W}_5 = 0. \tag{2.15}$$

Further, when $\mathbf{M}$ is chosen as above, then, for consistency with (1.211), the conditions

$$\bar{W}_{11} + 4\bar{W}_{12} + 4\bar{W}_{22} + 4\bar{W}_{23} + 2\bar{W}_{13} + \bar{W}_{33} = c_{22}/4, \tag{2.16}$$

$$\bar{W}_{14} + 2\bar{W}_{15} + 2\bar{W}_{24} + 4\bar{W}_{25} + \bar{W}_{34} + 2\bar{W}_{35} = (c_{12} - c_{23})/4, \tag{2.17}$$

$$\bar{W}_{44} + 4\bar{W}_{55} + 4\bar{W}_{45} + 2\bar{W}_5 = (c_{11} - c_{22} + 2c_{23} - 2c_{12})/4, \tag{2.18}$$

$$\bar{W}_1 + \bar{W}_2 + \bar{W}_5 = c_{55}/2, \tag{2.19}$$

$$\bar{W}_2 + \bar{W}_3 = (c_{23} - c_{22})/4, \tag{2.20}$$

should also be satisfied in the reference configuration, where now all the non-zero material parameters C_{ijkl} are expressed via the connections

$$\begin{aligned} &c_{ij} = C_{iijj}, \quad i, j \in \{1, 2, 3\}, \quad c_{22} = c_{33}, \quad c_{12} = c_{13}, \\ &c_{44} = c_{55} = C_{1212} = C_{1313}, \quad c_{66} = C_{2323} = (c_{22} - c_{23})/2, \end{aligned} \tag{2.21}$$

which constitute the standard notation for the elastic constants used in the classical theory of transverse isotropy for the case in which $\mathbf{E}_1$ is the direction of transverse isotropy [46]. We mention that the counterparts of (2.16)–(2.20) with $\mathbf{E}_3$ as the direction of transverse isotropy were given in [48]. It is also worth mentioning that when the special strain-energy function $W = \hat{W}(\lambda_1, \lambda_2, \lambda_3)$ is used, the restrictions (2.15) become

$$\hat{W} = \hat{W}_1 = \hat{W}_2 = \hat{W}_3 = 0, \tag{2.22}$$

while the analogues of (2.16)–(2.20) recast into

$$\begin{aligned} &\hat{W}_{11} = c_{11}, \quad \hat{W}_{12} = \hat{W}_{13} = c_{12}, \\ &\hat{W}_{23} = c_{23}, \quad \hat{W}_{22} = \hat{W}_{33} = c_{22}. \end{aligned}$$

In this case, the subscripts $1, 2$ and 3 on $\hat{W}$ imply differentiation with respect to the principal stretches λ_1, λ_2 and λ_3, respectively. Note that here the elastic constant c_{55} is not involved.

Finally, we illustrate that, by virtue of (1.205), the strong ellipticity condition (1.233) suggests that the elastic constants $c_{11}, ..., c_{55}$ prescribed in (2.21) should always be taken to be consistent with

$$C_{ijkl} n_i n_k s_j s_l > 0, \tag{2.23}$$

for all non-zero vectors $\mathbf{n} = (n_1, n_2, n_3)$ and $\mathbf{s} = (s_1, s_2, s_3)$. We remark, however, that the strong ellipticity condition may in general be analyzed in terms of the so-called *acoustic tensor* (see, e.g., [78, 79]), denoted $\mathbf{Q}(\mathbf{n})$, whose components are (in the linear theory) defined by

$$Q_{jl}(\mathbf{n}) = C_{ijkl} n_i n_k. \tag{2.24}$$

Seeing (2.24), the inequality (2.23) may then be established if and only if $\mathbf{Q}(\mathbf{n})$ is *positive define* for each non-vanishing vector $\mathbf{n}$. Note that, without loss of generality, $\mathbf{n}$ can be regarded as a unit vector.

Following that, we require the inequalities

$$Q_{ii}(\mathbf{n}) > 0, \quad i \in \{1, 2, 3\} \tag{2.25}$$

together with

$$Q_{ii}(\mathbf{n}) Q_{jj}(\mathbf{n}) - Q_{ij}(\mathbf{n})^2 > 0, \quad i \neq j \in \{1, 2, 3\} \tag{2.26}$$

and

$$\det[\mathbf{Q}(\mathbf{n})] > 0, \tag{2.27}$$

to hold jointly for all unit vectors $\mathbf{n}$. Thus, for the particular choice of $\mathbf{M}$, the components (2.24) of the tensor $\mathbf{Q}(\mathbf{n})$ may be written in the explicit forms

$$Q_{11}(\mathbf{n}) = c_{11} n_1^2 + c_{55}(n_2^2 + n_3^2), \quad Q_{22}(\mathbf{n}) = c_{22} n_2^2 + c_{55} n_1 n_2 + (c_{22} - c_{23}) n_3^2/2, \tag{2.28}$$

$$Q_{33}(\mathbf{n}) = c_{55} n_1^2 + (c_{22} - c_{23}) n_2^2/2 + c_{22} n_3^2, \quad Q_{12}(\mathbf{n}) = (c_{12} + c_{55}) n_1 n_2, \tag{2.29}$$

$$Q_{13}(\mathbf{n}) = (c_{12} + c_{55}) n_1 n_3, \quad Q_{23}(\mathbf{n}) = (c_{22} + c_{23}) n_2 n_3/2, \tag{2.30}$$

which are then, as appropriate, substituted into (2.25)–(2.27) to identify that the inequalities

$$c_{11} > 0, \quad c_{22} > 0, \quad c_{55} > 0, \quad c_{22} > c_{23} \tag{2.31}$$

and

$$\mid c_{12} + c_{55} \mid < c_{55} + \sqrt{c_{11} c_{22}}, \tag{2.32}$$

are necessary and sufficient conditions for strong ellipticity to hold in the classical theory (see, e.g., [49, 60]).

If we consider that the preferred direction is parallel to the X_2 axis then $\mathbf{CM} = \lambda_2^2\mathbf{M}$ for all $\lambda_2 > 0$, while equations (2.11) are replaced by

$$I_4 = \lambda_2^2, \quad I_5 = I_4^2, \tag{2.33}$$

and appropriate changes are needed in the subscripts in (2.31) and (2.32) for this case.

2.2.3 Reduction of the equilibrium equations

As discussed in Section 1.2.3, when cylindrical polar coordinates are used to characterize the status of a deforming material body in some current configuration $\mathcal{B}_t$, then the equilibrium equation div$\boldsymbol{\sigma} = \mathbf{0}$ (in the absence of body forces) yields the three scalar equations (1.76). For the considered deformation, where no shear stress is required and also $\sigma_{rr} = \sigma_1, \sigma_{\theta\theta} = \sigma_2, \sigma_{zz} = \sigma_3$, the latter reduce to

$$\frac{\partial\sigma_1}{\partial r} + \frac{1}{r}(\sigma_1 - \sigma_2) = 0, \quad \frac{\partial\sigma_2}{\partial\theta} = 0. \tag{2.34}$$

Since λ_3 is a constant and λ_1 depends only on X_1 it follows from equations $(2.34)_2$ and (2.8) that

$$\frac{\partial\sigma_2}{\partial\lambda_2}g''(X_2) = 0. \tag{2.35}$$

Hence, assuming that $\partial\sigma_2/\partial\lambda_2 \neq 0$, which, in view of the above inequalities, certainly holds in the reference configuration, we deduce that

$$g(X_2) = \beta X_2, \tag{2.36}$$

where β is a constant, which will be determined through the boundary conditions $(2.3)_3$ such that $\beta = \alpha/B > 0$. (Note that if, instead of the X_1 axis, the X_2 axis is chosen to identify the direction of transverse isotropy then g has the same form.)

A combination of (2.36) and (2.8) leads to

$$\lambda_2 = \beta f(X_1), \quad \frac{\mathrm{d}\lambda_2}{\mathrm{d}X_1} = \beta\lambda_1. \tag{2.37}$$

Hence, each of λ_1 and λ_2 depends only on X_1. Then, through use of $(2.8)_1$, (2.13), (2.37) and some manipulation, equation $(2.34)_1$ simplifies to

$$\frac{\mathrm{d}\hat{W}_1}{\mathrm{d}X_1} = \beta\hat{W}_2, \tag{2.38}$$

where we are using the representation $W = \hat{W}(\lambda_1, \lambda_2, \lambda_3)$ and the subscripts 1 and 2 on $\hat{W}$ signify differentiation with respect to λ_1 and λ_2, respectively. It follows, on use of $(2.37)_2$, that

$$\frac{\mathrm{d}}{\mathrm{d}X_1}(\lambda_1 \hat{W}_1) = \frac{\mathrm{d}}{\mathrm{d}X_1}(\hat{W}). \tag{2.39}$$

Since, from (2.22), we have $\hat{W} = \hat{W}_1 = 0$ in the reference configuration and the result of integrating (2.39) must hold for all deformations of the considered form, we obtain

$$\hat{W} = \lambda_1 \hat{W}_1, \tag{2.40}$$

which is an (implicit) first-order differential equation for $f(X_1)$ for any given form of strain-energy function. This equation is the same as that arising for an isotropic material except that here $\hat{W}$ does not in general possess the symmetry in $(\lambda_1, \lambda_2, \lambda_3)$ that holds in the isotropic situation (see, e.g., [56, 38]).

As we have already mentioned, it was first shown by Rivlin [66, 67] and then by several other authors (see, e.g., [24, 26, 56]) that it is possible to hold the body in its current configuration even if there are no tractions on the curved surfaces $r = a_1, a_2$ of the deformed block. This requires $\sigma_1 = 0$ on $X_1 = \pm A$, which, because of (2.13) and (2.40), can be expressed in terms of the strain-energy function as

$$W = 0 \quad \text{on} \quad X_1 = \pm A, \tag{2.41}$$

where W is either $\bar{W}$ or $\hat{W}$, as appropriate.

2.3 Isochoric specialization

If the deformation is considered to be isochoric then $\lambda_1\lambda_2\lambda_3 = 1$ and from (2.8) we therefore have

$$f'(X_1)f(X_1)g'(X_2)\lambda_3 = 1. \tag{2.42}$$

As discussed by Rivlin [66, 67], solution of the preceding equation leads to

$$f(X_1)^2 = 2X_1/(\beta\lambda_3) + a, \quad g(X_2) = \beta X_2,$$

where the constant $\beta > 0$ is again given via $(2.3)_3$ and

$$a = (a_1^2 + a_2^2)/2, \quad a_2^2 = 4A/(\beta\lambda_3) + a_1^2. \tag{2.43}$$

It then follows that the deformation can be described by

$$r = \sqrt{a + \frac{2X_1}{\beta\lambda_3}}, \quad \theta = \beta X_2, \quad z = \lambda_3 X_3, \tag{2.44}$$

and we deduce that the principal stretches can be written as

$$\lambda_1 = 1/\beta\lambda_3 r, \quad \lambda_2 = \beta r, \quad \lambda_3 = \lambda. \tag{2.45}$$

Application of (2.42) and (2.45) to (2.13) shows that σ_1 and σ_2 depend only on X_1 while θ depends only on X_2. Therefore, for an isochoric deformation, $(2.34)_2$ is satisfied identically while $(2.34)_1$ again leads to the equation $\hat{W} = \lambda_1 \hat{W}_1$. We emphasize that we are considering here an isochoric deformation in a compressible material, not an incompressible material.

Furthermore, we note that the solutions (2.44) arising from the kinematical restriction (2.42) are universal solutions since they apply independently of the constitutive law. Thus, in this case, the radial equilibrium equation (2.40) serves to identify the possible forms of W that admit the isochoric bending deformation.

2.3.1 Plane strain specialization

Henceforth, we confine our analysis to the plane strain specialization. We consider that the deformation is in the (X_1, X_2) coordinate plane, such that $z = X_3$, with (r, θ) being independent of X_3, and the direction $\mathbf{M}$ is parallel to the considered plane. The components of $\mathbf{F}$ and $\mathbf{C}$ satisfy $F_{33} = C_{33} = 1$, and the out-of-plane principal stretch is now $\lambda_3 = 1$. The principal invariants (1.147) reduce to

$$I_1 = \lambda_1^2 + \lambda_2^2 + 1, \quad I_2 = \lambda_1^2\lambda_2^2 + \lambda_1^2 + \lambda_2^2, \quad I_3 = \lambda_1^2\lambda_2^2, \tag{2.46}$$

and we rewrite (2.11) and (2.33) together compactly as

$$I_4 = \lambda_\kappa^2, \quad I_5 = I_4^2, \quad \kappa \in \{1, 2\}, \tag{2.47}$$

wherein the subscript κ has been introduced to identify the orientation of the unit vector field $\mathbf{M}$ in the undeformed configuration.

For either $\kappa = 1$ or 2 we deduce from (2.46) and (2.47) the connections

$$I_2 = I_1 + I_3 - 1, \quad I_5 = (I_1 - 1)I_4 - I_3, \tag{2.48}$$

and it follows that, for plane strain, the function $\bar{W}$ introduced in (2.14) depends only on the invariants I_1, I_3 and I_4. Accordingly, we may introduce a reduced strain-energy function, denoted $\bar{\bar{W}}$ and defined by

$$\bar{\bar{W}}(I_1, I_3, I_4) = \bar{W}(I_1, I_1 + I_3 - 1, I_3, I_4, (I_1 - 1)I_4 - I_3), \tag{2.49}$$

for either value of κ.

Now, if $\mathbf{F}$ denotes the corresponding in-plane deformation gradient, the associated restricted expressions for the nominal and the Cauchy stress tensors are given by

$$\boldsymbol{\sigma} = J^{-1}[2\bar{\bar{W}}_1\mathbf{B} + 2I_3\bar{\bar{W}}_3\mathbf{I} + 2\bar{\bar{W}}_4\mathbf{m}\otimes\mathbf{m}], \tag{2.50}$$

$$\mathbf{S} = 2\bar{\bar{W}}_1\mathbf{F}^{\mathrm{T}} + 2I_3\bar{\bar{W}}_3\mathbf{F}^{-1} + 2\bar{\bar{W}}_4\mathbf{M}\otimes\mathbf{FM}. \tag{2.51}$$

Note that in order to maintain the plane strain deformation the out-of-plane stress components σ_{33} and S_{33} are in general non-zero. These are not given by (2.50), (2.51), but, if needed, they may be calculated from (1.177) and (1.178), respectively, evaluated for the considered plane strain specialization.

The counterparts of (2.15) for $\bar{\bar{W}}$ are

$$\bar{\bar{W}} = \bar{\bar{W}}_1 + \bar{\bar{W}}_3 = \bar{\bar{W}}_4 = 0 \tag{2.52}$$

for $\kappa = 1, 2$, while (2.16)–(2.20) specialize to

$$\bar{\bar{W}}_{11} + 2\bar{\bar{W}}_{13} + \bar{\bar{W}}_{33} = c_{22}/4, \quad \bar{\bar{W}}_{44} - 2\bar{\bar{W}}_3 = (c_{11} + c_{22} - 2c_{12})/4, \tag{2.53}$$

$$2\bar{\bar{W}}_{14} + 2\bar{\bar{W}}_{34} + \bar{\bar{W}}_{44} = (c_{11} - c_{22})/4, \quad \bar{\bar{W}}_3 = -c_{55}/2, \tag{2.54}$$

with the derivatives of $\bar{\bar{W}}$ being evaluated for $(I_1, I_3, I_4) = (3, 1, 1)$ (see also [48]). Finally, the properties (2.31) that the elastic constants should satisfy are reduced to

$$c_{11} > 0, \quad c_{22} > 0, \quad c_{55} > 0, \tag{2.55}$$

while (2.32) is still in place. We recall that the properties (2.53)–(2.55) and (2.32) correspond to $\kappa = 1$.

2.3.2 Certain classes of materials

As discussed in Jiang [38], a finite isochoric bending deformation of a rectangular block is not sustainable for compressible isotropic materials. Here, we present two specific classes of transversely isotropic compressible materials, depending on the choice of κ, for which the considered isochoric bending deformation can be achieved.

For this purpose, we substitute (2.49) into (2.40) to obtain

$$\bar{\bar{W}}(I_1, I_3, I_4) = 2\lambda_1^2\bar{\bar{W}}_1(I_1, I_3, I_4) + 2I_3\bar{\bar{W}}_3(I_1, I_3, I_4) + 2\lambda_1^2\bar{\bar{W}}_4(I_1, I_3, I_4)\delta_{1\kappa}, \tag{2.56}$$

where $\delta_{1\kappa}$ is the Kronecker delta, with $\kappa \in \{1, 2\}$.

Since the deformation is considered to be isochoric, equation (2.56) for $\kappa = 1$ applies for

$$I_1 = \lambda_1^2 + \lambda_1^{-2} + 1, \quad I_3 = 1, \quad I_4 = \lambda_1^2 \tag{2.57}$$

for all $\lambda_1 > 0$. Differentiation of (2.56) with respect to λ_1 yields

$$\begin{aligned} &2(\lambda_1 - \lambda_1^{-3})(2\lambda_1^2\bar{\bar{W}}_{11} + 2\bar{\bar{W}}_{13} + 2\lambda_1^2\bar{\bar{W}}_{14} - \bar{\bar{W}}_1) \\ &\quad + 4\lambda_1^3(\bar{\bar{W}}_{14} + \bar{\bar{W}}_{44}) + 2\lambda_1(2\bar{\bar{W}}_1 + \bar{\bar{W}}_4 + 2\bar{\bar{W}}_{34}) = 0, \end{aligned} \tag{2.58}$$

for all $\lambda_1 > 0$.

On use of (2.52)–(2.54), we see that in the limit $\lambda_1 \to 1$ equation (2.58) holds if and only if

$$c_{11} = c_{12}. \tag{2.59}$$

This is a necessary restriction on the class of compressible transversely isotropic materials for which an isochoric bending deformation of a rectangular block can be achieved. We note that for this class of materials the conditions (2.55) and (2.32) can be reduced to

$$c_{22} > c_{11} > 0, \quad c_{55} > 0. \tag{2.60}$$

For $\kappa = 2$, equation (2.56) with $I_4 = \lambda_1^{-2}$ gives, analogously to (2.58), the connection

$$2\lambda_1^{-3}(\bar{\bar{W}}_4 - 2\bar{\bar{W}}_{34}) + 4\lambda_1(\bar{\bar{W}}_1 - \lambda_1^{-2}\bar{\bar{W}}_{14}) + 2(\lambda_1 - \lambda_1^{-3})(2\lambda_1^2\bar{\bar{W}}_{11} + 2\bar{\bar{W}}_{13} - \bar{\bar{W}}_1) = 0, \tag{2.61}$$

for all $\lambda_1 > 0$. This leads again to the restriction (2.59), and the corresponding inequalities that the elastic constants satisfy are

$$c_{22} > c_{11} > 0, \quad c_{44} > 0. \tag{2.62}$$

It is also worth noting that the inequalities $(2.60)_1$ show that for $\kappa = 1$ the materials have "low" anisotropy, since the elastic modulus c_{11}, which is directly related to the stiffness in the X_1 direction, is less than c_{22}, which is defined with respect to the isotropic planes of the body.

2.3.3 A note on reinforcing models

At this point, it is useful to remark that several authors (see, e.g., [77, 62, 63, 47, 48]) have considered a decomposition of the strain-energy function of the form

$$\bar{\bar{W}} = \bar{\bar{W}}_{\text{iso}}(I_1, I_3) + \bar{\bar{W}}_{\text{fib}}(I_4), \tag{2.63}$$

in which the first term $\bar{\bar{W}}_{\text{iso}}$ represents the isotropic base material, while the additional term $\bar{\bar{W}}_{\text{fib}}$ represents the reinforcement associated with a family of fibres whose referential direction is the preferred direction $\mathbf{M}$.

The properties $(2.53)_2$ and (2.54) show that the class of strain-energy functions (2.63) must satisfy the connection

$$c_{22} - c_{12} = 2c_{55} \tag{2.64}$$

when $\kappa = 1$, and

$$c_{11} - c_{12} = 2c_{44} \tag{2.65}$$

for $\kappa = 2$. The first of these requirements is consistent with (2.59) and the inequalities (2.60), but the second is not consistent with (2.59) and (2.62). However, the conditions $(2.53)_2$, (2.54) and $(2.60)_3$ show that the class of materials (2.64) can admit the isochoric bending deformation only for $\kappa = 1$ and then such that

$$\bar{\bar{W}}''_{\text{fib}}(1) < 0, \tag{2.66}$$

implying non-convexity of $\bar{\bar{W}}_{\text{fib}}$ in a neighbourhood of $I_4 = 1$.

This is not consistent with the models adopted in the above-cited papers, where the anisotropic part of the strain-energy has been taken to satisfy

$$\bar{\bar{W}}_{\text{fib}}(1) = \bar{\bar{W}}'_{\text{fib}}(1) = 0, \quad \bar{\bar{W}}''_{\text{fib}}(1) > 0, \tag{2.67}$$

for $\kappa \in \{1, 2\}$. However, in such cases the models used provide reinforcement, i.e. the stiffness in the preferred direction is larger than in the transverse directions, in contrast to the situation here. It can therefore be concluded that the considered deformation is not possible for strictly reinforcing models. Such models are studied in detail in Chapters 3 and 4 by means of incompressible transversely isotropic solids undergoing shearing deformations.

2.3.4 Some specific strain-energy functions

In this section we present two examples of strain-energy functions that can admit an isochoric bending deformation. The relevant necessary and sufficient condition is obtained from (2.56) in each case, with I_1, I_3 and I_4 given by

$$I_1 = \lambda_1^2 + \lambda_1^{-2} + 1, \quad I_3 = 1, \quad I_4 = \lambda_\kappa^2. \tag{2.68}$$

Since $I_3 = 1$, we may write $I_1 = I_4 + I_4^{-1} + 1$, for $\kappa \in \{1, 2\}$, although in the present analysis we do not make formal use of this connection. The specializations of (2.56) for $\kappa = 1$ and $\kappa = 2$ may be written as

$$\bar{\bar{W}} = 2I_4\bar{\bar{W}}_1 + 2\bar{\bar{W}}_3 + 2I_4\bar{\bar{W}}_4 \tag{2.69}$$

and

$$I_4\bar{\bar{W}} = 2\bar{\bar{W}}_1 + 2I_4\bar{\bar{W}}_3, \tag{2.70}$$

respectively, evaluated for $I_3 = 1$.

In the following we adopt an approach towards the construction of forms of W used by Jiang and Ogden [39, 40], and the particular forms of energy function considered are motivated by those examined in these references.

Case (1): $\kappa = 1$.

First, we consider the class of strain-energy functions for which $\bar{\bar{W}}$ has the form

$$\bar{\bar{W}}(I_1, I_3, I_4) = h_1(I_1 - I_4 + I_3)g_0(I_3) + h_2(I_1 - I_4)\sqrt{I_3} + C_0 I_4(\sqrt{I_3})^{-1}, \tag{2.71}$$

where h_1 is a function to be determined, C_0 is a material constant and the functions h_2 and g_0 are to be consistent with the requirements (2.52)–(2.54).

If, without loss of generality, we set $g_0(1) = 1$ then substitution of (2.71) into (2.69) leads to the differential equation

$$h_1'(\bar{I}) + qh_1(\bar{I}) = 0, \tag{2.72}$$

where q and $\bar{I}$ are defined by

$$2q = 2g_0'(1) - 1, \quad \bar{I} = I_1 - I_4 + 1. \tag{2.73}$$

The general solution of (2.72) is

$$h_1(\bar{I}) = C_1 \mathrm{e}^{-q\bar{I}}, \tag{2.74}$$

where C_1 is a constant. In respect of (2.71) the requirements (2.52)–(2.54) give $C_0 = c_{55}/2$, together with

$$h_2(2) + h_1(3) = -c_{55}/2, \quad h_2'(2) - qh_1(3) = c_{55}/2, \tag{2.75}$$

$$h_2''(2) + q^2 h_1(3) = (c_{22} - c_{11} - 4c_{55})/4, \tag{2.76}$$

$$4h_2''(2) - h_2(2) + 4h_2'(2) - 8qh_1(3) + 4h_1(3)g_0''(1) = c_{22} - 3c_{55}/2. \tag{2.77}$$

The class of strain-energy functions (2.71) admitting isochoric bending deformation is now specialized to

$$\bar{\bar{W}}(I_1, I_3, I_4) = C_1 \mathrm{e}^{-q(I_1 - I_4 + I_3)} g_0(I_3) + h_2(I_1 - I_4)\sqrt{I_3} + (c_{55}/2) I_4 (\sqrt{I_3})^{-1} \tag{2.78}$$

for any choice of the parameter q and non-zero C_1, and for any functions g_0 and h_2 that satisfy (2.75)–(2.77). In particular, for any given C_1 and q, which can in general be chosen independently, equations (2.75)–(2.77) simply serve to identify the properties that g_0 and h_2 should satisfy in the reference configuration, but no restriction otherwise on the forms of these functions is imposed. The expression (2.78), together with (2.75)–(2.77), represents a large class of functions admitting isochoric bending deformation. Finally, we note that for $C_1 = 0$ the requirements (2.77) and (2.76) lead to $c_{11} = -4c_{55}$, in which case the properties (2.60) are violated. For this reason the possibility $C_1 = 0$ is excluded from consideration.

Note, however, that the specialization $q = 0$ is admissible, in which case the strain-energy function (2.78) reduces to

$$\bar{\bar{W}}(I_1, I_3, I_4) = C_1 g_0(I_3) + h_2(I_1 - I_4)\sqrt{I_3} + (c_{55}/2) I_4 (\sqrt{I_3})^{-1}, \tag{2.79}$$

which is valid for all non-zero disposable parameters C_1 and all functions h_2 and g_0 that satisfy the appropriate specializations of (2.75)–(2.77).

Case (2): $\kappa = 2$.

Next, we examine the class of strain-energy functions of the form

$$\bar{\bar{W}}(I_1, I_3, I_4) = h_3(I_1 I_4) g_1(I_3) + h_4(I_4)\sqrt{I_3}. \tag{2.80}$$

Similarly to the previous case, by taking $g_1(1) = 1$ and setting $2\eta = 2g_1'(1) - 1$, we follow the same procedure in respect of (2.70) to particularize $h_3(I_1 I_4)$. This leads to

$$h_3(I_1 I_4) = C_2 \mathrm{e}^{-\eta I_1 I_4}, \tag{2.81}$$

where again, C_2 is a material parameter and h_3, h_4, g_1 satisfy

$$h_3(3) = -c_{44}/2\eta, \quad h_3(3) + h_4(1) = 0, \quad h_4'(1) = -3c_{44}/2, \tag{2.82}$$

$$h_4''(1) + 9\eta^2 h_3(3) = (c_{22} - c_{11} - 4c_{44})/4, \tag{2.83}$$

$$4\eta(\eta + 1) h_3(3) - 4h_3(3) g_1''(1) = -c_{11} - c_{44}/2\eta. \tag{2.84}$$

Hence, in this case, equation (2.80) is replaced by

$$\bar{\bar{W}}(I_1, I_3, I_4) = C_2 \mathrm{e}^{-\eta I_1 I_4} g_1(I_3) + h_4(I_4)\sqrt{I_3} \tag{2.85}$$

for all non-zero parameters η and C_2 that satisfy (2.82)-(2.84) in respect of (2.81). As for the case of the functions (2.78), these serve to determine the conditions that g_1 and h_4 should satisfy in the reference configuration. In addition, we emphasize that the class of strain-energy functions (2.80) fails to admit isochoric bending deformation for $C_2 = 0$ and/or $\eta = 0$ since in either case we deduce that $c_{44} = 0$ and the strong ellipticity condition $(2.62)_3$ is then violated.

2.3.5 Application of the boundary conditions

As we have already mentioned, the solutions (2.78), (2.79), (2.85) derived in the previous section correspond to large classes of transversely isotropic materials admitting the considered isochoric bending deformation under plane strain. However, these solutions are not necessarily compatible with the boundary conditions (2.41) imposed on our problem. In this respect, the arbitrary functions h_2 and h_4 involved need to be properly specified to ensure that the deformed body is traction free on the boundaries $X_1 = \pm A$.

For illustration, we now examine the strain-energy functions (2.79) by taking

$$h_2(I_1 - I_4) = -(c_{55}/8)\,(I_1 - I_4 - 4)^2 - C_1, \tag{2.86}$$

noting that this is compatible with (2.75) and (2.76) for $q = 0$, and the nonlinear algebraic system (2.41) can be solved analytically in respect of the data a and β. The equilibrium equation (2.71) is then satisfied identically with

$$\sigma_1 = -(c_{55}/8)[(\lambda_1^{-2} - 2)^2 - 2(\lambda_1^{-2} + 2\lambda_1^2) + 5], \tag{2.87}$$

while σ_2 and σ_3 take the forms

$$\begin{aligned} \sigma_2 &= -(c_{55}/8)(4\lambda_1^2 - 18\lambda_1^{-2} + 5\lambda_1^{-4} + 9), \\ \sigma_3 &= -(c_{55}/8)(4\lambda_1^2 - 2\lambda_1^{-2} + \lambda_1^{-4} - 3). \end{aligned} \tag{2.88}$$

It should be noted, however, that the properties (2.75) and (2.76) impose the further restriction $3c_{55} = c_{22} - c_{11}$, which is compatible with (2.60). From (2.77) a condition on $g_0''(1)$ may also be derived but we do not need it here.

By recalling the expressions $(2.44)_1$ and $(2.45)_1$ (now with $\lambda_3 = 1$), the system (2.41) is solved to give

$$a = (40/9)A^2, \quad \beta = 3/(4A), \tag{2.89}$$

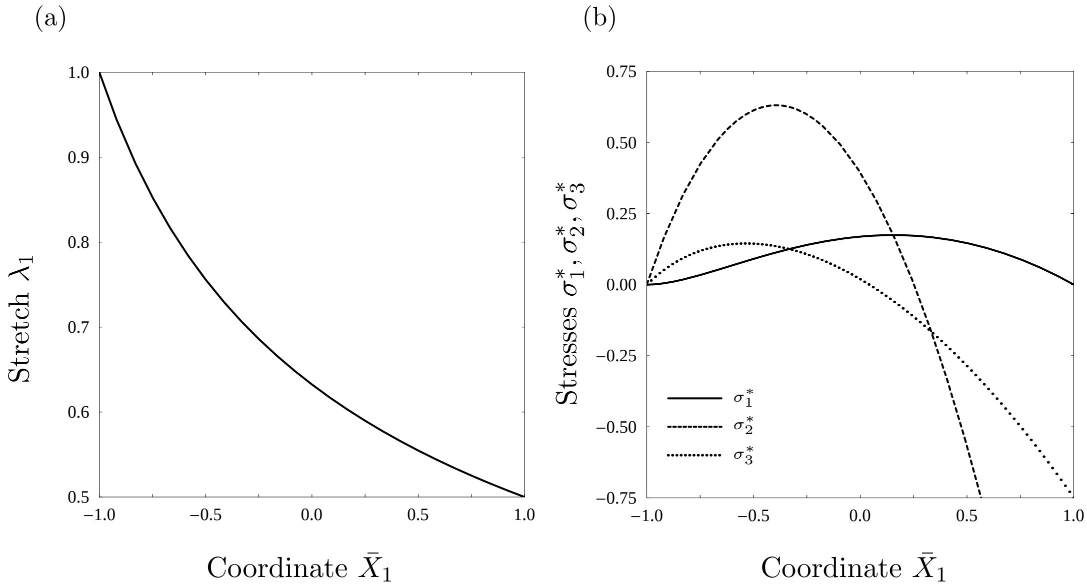

Figure 2.1: Plots of (a) the stretch λ_1 and (b) the dimensionless stress components σ_1^*, σ_2^* and σ_3^* vs $\bar{X}_1$.

and from $(2.3)_{1,2}$ we obtain

$$a_1 = 2a_2 = (8/3)A. \tag{2.90}$$

Moreover, the range of $\lambda_1 = \lambda_1(X_1)$ for which such a deformation is sustainable may also be identified via (2.89), $(2.44)_1$ and $(2.45)_1$ as

$$\lambda_1(A) = 0.5 \leqslant \lambda_1(X_1) \leqslant 1 = \lambda_1(-A), \tag{2.91}$$

for all $A > 0$ and $-A \leqslant X_1 \leqslant A$. Consequently, from $(2.47)_1$ (i.e. for $\kappa = 1$) and (2.91) it follows that the material is compressed in the X_1 direction for $X_1 > -A$. The resulting stretch distribution as a function of the dimensionless coordinate $\bar{X}_1 = X_1/A$ is depicted in Figure 2.1(a). We observe that the inequalities (2.91) hold independently of the value of A.

In addition, the stress components σ_1, σ_2 and σ_3 are plotted in Figure 2.1(b) as functions of $\bar{X}$ in dimensionless form $\sigma_i^* = \sigma_i/c_{55}$, $i \in \{1, 2, 3\}$. The non-monotonic nature of $\sigma_1, \sigma_2, \sigma_3$ is now evident. We note that σ_1 vanishes for $\bar{X}_1 = \pm 1$, as prescribed, and takes its maximum value for $\lambda_1 \approx 0.605$ (equivalently, for $\bar{X}_1 \approx 0.155$). Also, σ_2 and σ_3 vanish on the boundary $\bar{X}_1 = -1$ of the block, where $\lambda_1 = 1$, while additionally $\sigma_2 = 0$ for $\lambda_1 \approx 0.589$ ($\bar{X}_1 \approx 0.252$) and $\sigma_3 = 0$ at $\lambda_1 \approx 0.625$ ($\bar{X}_1 \approx 0.041$).

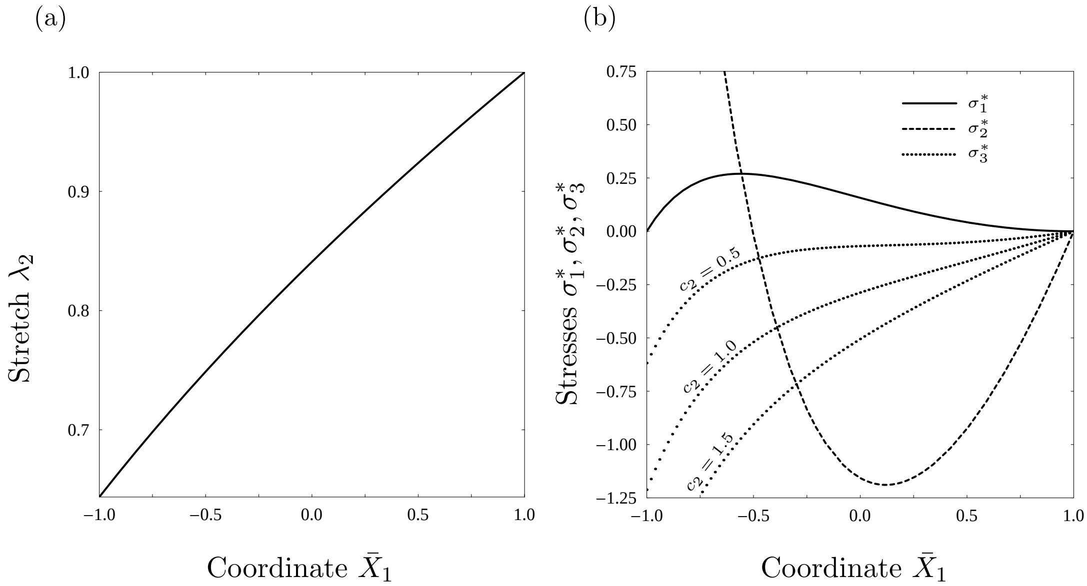

Figure 2.2: Plots of (a) the stretch λ_2 and (b) the dimensionless stress components σ_1^*, σ_2^* and σ_3^* vs $\bar{X}_1$.

Finally, the moment of the stress σ_2 (about the origin $r = \theta = 0$) that maintain the material in its deformed state is now calculated from the formula

$$M = 2C \int_{-A}^{A} r\lambda_1\sigma_2 dX_1, \tag{2.92}$$

which can be evaluated explicitly to give $M \approx -(0.318/\beta^2)Cc_{55} \approx -0.566A^2Cc_{55}$.

Next we consider the class of materials (2.85) for the case in which the function h_4 is chosen as

$$h_4(I_4) = (c_{44}/2\eta)\mathrm{e}^{\eta(2-I_4-I_4^2)} - c_1(I_4^{-1} - 2)^2 + c_1 I_4^2. \tag{2.93}$$

The form of (2.93) satisfies the required restrictions, and the counterparts of (2.87) and (2.88) are

$$\sigma_1 = -c_1\left[(\lambda_1^2 - 2)^2 - \lambda_1^{-4}\right], \tag{2.94}$$

and

$$\sigma_2 = c_1(3\lambda_1^4 - 4\lambda_1^2 + 5\lambda_1^{-4} - 4), \tag{2.95}$$

$$\sigma_3 = c_{44}\mathrm{e}^{\eta(2-\lambda_1^{-2}-\lambda_1^{-4})}(\lambda_1^{-2} - 1) - c_1(\lambda_1^4 - 4\lambda_1^2 - \lambda_1^{-4} + 4), \tag{2.96}$$

wherein the notation $c_1 = (c_{22} - c_{11} + 3c_{44})/16$ has been introduced.

On use of (2.94), solution of the system (2.41) yields

$$a = 4(4 + 3\sqrt{2})A^2, \quad \beta = (2 - \sqrt{2})/(4A), \tag{2.97}$$

with

$$a_1 = \sqrt{2 + 2\sqrt{2}a_2 - 2\sqrt{2}A} = 2(2 + \sqrt{2})A. \tag{2.98}$$

We observe that for the particular choice of h_3 the deformation is sustainable only within the range

$$\lambda_1(A) = 1 \leqslant \lambda_1(X_1) \leqslant 1.554 \approx \lambda_1(-A), \tag{2.99}$$

for all $A > 0$ and $-A \leqslant X_1 \leqslant A$. We recall, however, that since in this case the direction of transverse isotropy is in the X_2 direction, we have $I_4 = \lambda_2^2 = \lambda_1^{-2}$. The obvious inference is that there is contraction in the X_2 direction for all values of X_1 except $X_1 = A$. The distribution of the stretch $\lambda_2 = \lambda_1^{-1}$ as a function of $\bar{X}_1$ is plotted in Figure 2.2(a).

Corresponding plots of the stress components $\sigma_1, \sigma_2, \sigma_3$ are given in Figure 2.2(b) where, analogously to the previous case, we use the dimensionless forms $\sigma_i^* = \sigma_i/c_1$, $i \in \{1, 2, 3\}$. It can now easily be derived from (2.94) and (2.95) that σ_1 and σ_2 are non-monotonic as functions of λ_2 or, equivalently, of X_1. Indeed, σ_1 reaches a maximum value at $\lambda_2 \approx 0.737$, corresponding to $\bar{X}_1 \approx -0.558$ while, σ_2 has a minimum at $\lambda_2 \approx 0.861$, corresponding to $\bar{X}_1 \approx 0.119$. Furthermore, we notice that σ_2 vanishes for the values $\lambda_2 \approx 0.748$ and 1 and hence for $\bar{X}_1 \approx -0.504$ and $\bar{X}_1 = 1$.

We now observe that σ_3 is the only principal stress component that depends on the three parameters c_{44}, η and c_1. Essentially, both the nature and the magnitude of this component is adjusting due to different classes of strain energies and with respect to various extension and shear moduli so that the body can undergo an isochoric deformation while, at the same time, the boundary conditions (2.41) are satisfied. For illustration, the curves $(\sigma_3^*, \bar{X}_1)$ are presented here for $\eta = 0.5$ and $c_2 = 0.5, 1, 1.5$, where c_2 is defined as $c_2 = c_{44}/c_1$.

Finally, the moment M is in this case calculated as

$$M = 2C \int_{-A}^{A} r\lambda_1\sigma_2 dX_1 \approx -(0.078/\beta^2)Cc_1 \approx -3.627A^2Cc_1. \tag{2.100}$$

2.4 Strongly elliptic modes of deformation

2.4.1 Necessary and sufficient conditions for strong ellipticity

The issue of stability of modes of deformation such as that considered in the foregoing sections is an important one, and, in particular, the notion of loss of strong ellipticity has a role to play in this regard. In this section we examine the strong ellipticity condition for the considered deformation. For transversely isotropic compressible elastic solids this has been discussed by Merodio and Ogden [48] and, in particular, they gave a general expression for the strong ellipticity condition for plane strain. As for the linear theory discussed earlier in Section 2.2.2, the strong ellipticity condition may be analyzed in terms of the acoustic tensor $\mathbf{Q}(\mathbf{n})$, whose components are quadratic in the components of the unit vector $\mathbf{n}$ (for a general discussion we refer to the work of Truesdell and Noll [78]). Analogously to (2.25)–(2.27), but now confined to two dimensions, expressly in the $(1,2)$ plane with $\mathbf{n}$ lying in that plane, necessary and sufficient conditions for strong ellipticity are

$$Q_{11}(\mathbf{n}) > 0, \quad Q_{11}(\mathbf{n})Q_{22}(\mathbf{n}) - [Q_{12}(\mathbf{n})]^2 > 0 \tag{2.101}$$

for all unit vectors $\mathbf{n} = (n_1, n_2, 0)$.

For a compressible material the components of $\mathbf{Q}(\mathbf{n})$ for plane strain ($\lambda_3 = 1$) are, from Merodio and Ogden [48] but in the present notation, given by

$$\begin{aligned} Q_{ij} =& 4\bar{\bar{W}}_{11}\lambda_i^2\lambda_j^2 n_i n_j + 4I_3\bar{\bar{W}}_{13}(\lambda_i^2 + \lambda_j^2)n_i n_j + 4I_3^2\bar{\bar{W}}_{33}n_i n_j \\ &+ 4I_3\bar{\bar{W}}_{34}(\mathbf{n}\cdot\mathbf{m})(n_i m_j + n_j m_i) + 4\bar{\bar{W}}_{14}(\mathbf{n}\cdot\mathbf{m})(\lambda_i^2 n_i m_j + \lambda_j^2 n_j m_i) \\ &+ 4\bar{\bar{W}}_{44}(\mathbf{n}\cdot\mathbf{m})^2 m_i m_j + 2\bar{\bar{W}}_1\delta_{ij}(\lambda_1^2 n_1^2 + \lambda_2^2 n_2^2) + 2I_3\bar{\bar{W}}_3 n_i n_j + 2\bar{\bar{W}}_4\delta_{ij}(\mathbf{n}\cdot\mathbf{m})^2, \end{aligned} \tag{2.102}$$

for $i, j \in \{1, 2\}$. When specialized to the considered deformation and on use of the (plane strain) energy function defined by $\hat{\hat{W}}(\lambda_1, \lambda_2) = \hat{W}(\lambda_1, \lambda_2, 1)$, we obtain simply

$$Q_{11} = \lambda_1^2\hat{\hat{W}}_{11}n_1^2 + 2\bar{\bar{W}}_1\lambda_2^2 n_2^2, \tag{2.103}$$

$$Q_{22} = \lambda_2^2\hat{\hat{W}}_{22}n_2^2 + 2(\bar{\bar{W}}_1 + \bar{\bar{W}}_4)\lambda_1^2 n_1^2, \tag{2.104}$$

$$Q_{12} = \lambda_1\lambda_2\hat{\hat{W}}_{12}n_1 n_2 - 2I_3\bar{\bar{W}}_3 n_1 n_2, \tag{2.105}$$

where $\hat{\hat{W}}_{ij} = \partial^2\hat{\hat{W}}/\partial\lambda_i\partial\lambda_j$.

After a little manipulation using (2.105) it can be shown that the inequalities (2.101) lead to

$$\hat{\hat{W}}_{11} > 0, \quad \hat{\hat{W}}_{22} > 0, \quad \bar{\bar{W}}_1 > 0, \quad \bar{\bar{W}}_1 + \bar{\bar{W}}_4 > 0, \tag{2.106}$$

$$\sqrt{\hat{\hat{W}}_{11}\hat{\hat{W}}_{22}} - \hat{\hat{W}}_{12} + 2\sqrt{\bar{\bar{W}}_1(\bar{\bar{W}}_1 + \bar{\bar{W}}_4)} + 2\sqrt{I_3}\bar{\bar{W}}_3 > 0, \tag{2.107}$$

$$\sqrt{\hat{\hat{W}}_{11}\hat{\hat{W}}_{22}} + \hat{\hat{W}}_{12} + 2\sqrt{\bar{\bar{W}}_1(\bar{\bar{W}}_1 + \bar{\bar{W}}_4)} - 2\sqrt{I_3}\bar{\bar{W}}_3 > 0, \tag{2.108}$$

which, jointly, are necessary and sufficient conditions on the material properties for strong ellipticity to hold for the considered deformation. Note that both $\hat{\hat{W}}$ and $\bar{\bar{W}}$ are used here since the expressions are simpler in this form.

It is worth noting in passing that for an isotropic material the above inequalities, when expressed entirely in terms of $\hat{\hat{W}}$, reduce to

$$\hat{\hat{W}}_{11} > 0, \quad \hat{\hat{W}}_{22} > 0, \quad \frac{\lambda_1 \hat{\hat{W}}_1 - \lambda_2 \hat{\hat{W}}_2}{\lambda_1^2 - \lambda_2^2} > 0, \tag{2.109}$$

$$\sqrt{\hat{\hat{W}}_{11}\hat{\hat{W}}_{22}} - \hat{\hat{W}}_{12} + \frac{\hat{\hat{W}}_1 + \hat{\hat{W}}_2}{\lambda_1 + \lambda_2} > 0, \tag{2.110}$$

$$\sqrt{\hat{\hat{W}}_{11}\hat{\hat{W}}_{22}} + \hat{\hat{W}}_{12} - \frac{\hat{\hat{W}}_1 + \hat{\hat{W}}_2}{\lambda_1 + \lambda_2} > 0, \tag{2.111}$$

as obtained by [16].

2.4.2 Numerical illustration and additional remarks

For illustration, the ellipticity status of the strain-energy function (2.79) undergoing isochoric bending, with h_2 being given by (2.86), is now discussed. For the considered materials, we deduce via $(2.60)_1$ that the first and the fourth requirements (2.106) are automatically satisfied within the range of admissible values of λ_1, as defined in (2.91). On the other hand, the second of these inequalities fails if and only if the dimensionless quantity $c_3 = (c_{11}/c_{55}) > 0$ does not exceed the approximate value 21.75. In this connection, the inequality $(2.106)_2$ fails earlier (for values of λ_1 closer to 1) when c_3 is close to zero, corresponding to $\lambda_1 \approx 0.724$ ($\bar{X}_1 \approx -0.394$). Note that an increase in the ratio c_3 amounts to a decrease in the value of λ_1 for which $(2.106)_2$ first fails. In the same spirit, it can easily be shown that breakdown of $(2.106)_3$ occurs when λ_1 reaches the value $\sqrt{3}/3$, independently of the magnitude of the associated elastic material parameters.

It is now interesting that the status of (2.107) depends on c_3 in a similar way as for $(2.106)_2$. Once more, small values of c_3 correspond to larger values of λ_1 for which (2.107) is violated. We emphasize, however, that if c_3 is taken close to zero, (2.107) fails instantly for λ_1 close to 1 ($\bar{X} \approx -1$) while, also, the restriction $c_3 \geqslant 21.75$ is not in this case influential. We further observe that for any fixed value of c_3, violation of (2.107) occurs for values of λ_1 closer to 1 than for those associated with the failure of $(2.106)_2$ or $(2.106)_3$. Finally, bearing in mind (2.91) we readily deduce that (2.108) always holds.

Therefore, for a deformation with the considered properties, the inequality (2.107) alone is sufficient to asses the failure of ellipticity. In that respect, the influence of c_3 on the onset of loss of strong ellipticity is exemplified in Figure 2.3(a) in terms of the coordinate $\bar{X}_1$. It is worth noting that when c_3 exceeds the approximate value 1.785 loss of strong ellipticity is always expected close to $\lambda_1 \approx 0.75$, or, equivalently at $\bar{X}_1 \approx -0.481$.

It is now evident that, in terms of the components of the acoustic tensor, the onset of failure of ellipticity is strictly associated with breakdown of $(2.101)_2$. For the considered strain-energy function this gives explicitly

$$\begin{aligned}&4c_3\lambda_1^{10}n_1^4+(3\lambda_1^2-1)(2\lambda_1^6+2c_3\lambda_1^4+9\lambda_1^2-5)n_2^4\\&+[4c_3\lambda_1^{10}+6\lambda_1^8+(6c_3-2)\lambda_1^6-3(2c_3+3)\lambda_1^4+6\lambda_1^2-1]n_1^2n_2^2=0.\end{aligned} \tag{2.112}$$

The implications of (2.112) are illustrated in Figure 2.3(b), in which n_1^2 is plotted against $\bar{X}_1$ for two distinct values of c_3. This then identifies the direction of the unit vector $\mathbf{n}$ for which ellipticity fails. Clearly, decrease in the value of the ratio c_3 induces ellipticity to fail first for $\bar{X}_1$ closer to -1 also corresponding to values of n_1 closer to 1. If c_3 exceeds the value 1.785 this fact is only consequential regarding those solutions of (2.112) lying close to $n_1 = 0$, in which case the smaller c_3 is the closer to $\bar{X}_1 = -1$ ellipticity fails initially. However, it appears that this assertion is valid only for a relatively small range of $c_3 \geqslant 1.785$ since, as c_3 increases, the solutions of (2.112) in terms of n_1^2 tend to stabilize. Specifically, in the limit $n_1 \to 0$, (2.112) reduces to

$$\bar{\bar{W}}_1\hat{\hat{W}}_{22} \equiv \frac{1}{8}c_{55}^2\lambda_2^4(3\lambda_1^2-1)(2\lambda_1^6+2c_3\lambda_1^4+9\lambda_1^2-5)=0 \tag{2.113}$$

and hence, for $0 < c_3 < 21.75$ loss of ellipticity is initiated from $\lambda_1 \approx 0.724$ ($\bar{X}_1 \approx -0.394$) when $c_3 \to 0$, and, for $c_3 \geqslant 21.75$, from $\lambda_1 = \sqrt{3}/3$ ($\bar{X}_1 \approx 0.333$) independently of the value of c_3.

At this point, it should be clarified that the necessary and sufficient conditions (2.106)-(2.108) adopted here for strong ellipticity to hold, and therefore the analysis provided so far for a specific class of the strain-energies (2.79), are local. Nevertheless, for the considered deformation, the geometry of the deformed body furnishes that any surface $r =$ constant, across which ellipticity is lost, should be circular cylindrical. This fact now necessitates $\mathbf{n} = \mathbf{e}_r$ and it is therefore apparent that strongly elliptic modes of bending deformation are sustainable if and only if the simple requirements

$$\hat{\hat{W}}_{11} > 0, \quad \bar{\bar{W}}_1 + \bar{\bar{W}}_4 > 0, \tag{2.114}$$

hold simultaneously.

Accordingly, for the material model examined above we have the interesting conclusion that ellipticity failure is possible only in the case where the deformed body becomes non-symmetric, since, as already mentioned, the latter inequalities hold automatically. It is

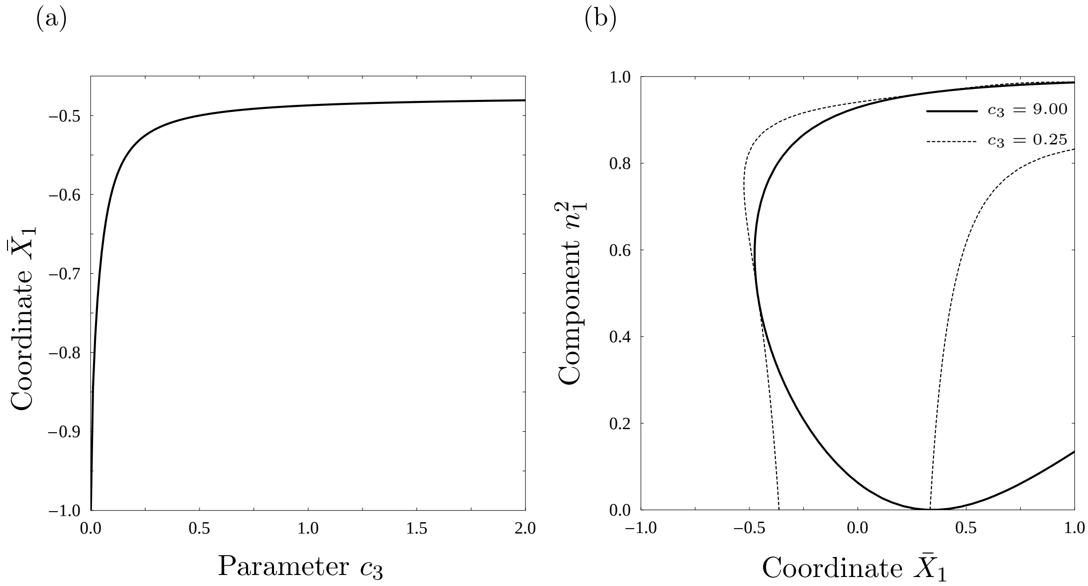

Figure 2.3: Plots of (a) the dimensionless coordinate $\bar{X}_1$ at which ellipticity is lost as a function of the dimensionless material parameter c_3 and (b) the corresponding value of n_1^2 as a function of $\bar{X}_1$.

worth noting, however, that if $\mathbf{n}$ is taken not strictly radial but very close to the direction of $\mathbf{e}_r$, failure of ellipticity is likely to occur. This fact is illustrated in Figure 2.4 where we plot the solutions $\bar{X}_1$ of (2.112) by means of the variation of c_3 for three fixed values of n_1 close to 1.

2.5 Incompressible materials

If the considered material is incompressible then the deformation gradient must satisfy the internal constraint $\det \mathbf{F} \equiv \lambda_1\lambda_2\lambda_3 = 1$ at each point of the material. Thus, given the preferred direction $\mathbf{M}$ is an eigenvector of $\mathbf{C}$, i.e. the connections (2.47) are established for each $\kappa \in \{1, 2\}$, the Cauchy stress follows the spectral decomposition (2.12) with the only difference that here its principal components are defined by

$$\sigma_i = \lambda_i \frac{\partial W}{\partial \lambda_i} - p, \quad i \in \{1, 2, 3\}. \tag{2.115}$$

As before, p is a Lagrange multiplier associated with the incompressibility constraint. In fact, the latter can be either $\bar{p}$ or $\hat{p}$ whereas, analogously to the compressible theory (in three dimensions), the elastic stored energy may now be represented through

$$W = \bar{W}(I_1, I_2, I_4, I_5) = \hat{W}(\lambda_1, \lambda_2, \lambda_3). \tag{2.116}$$

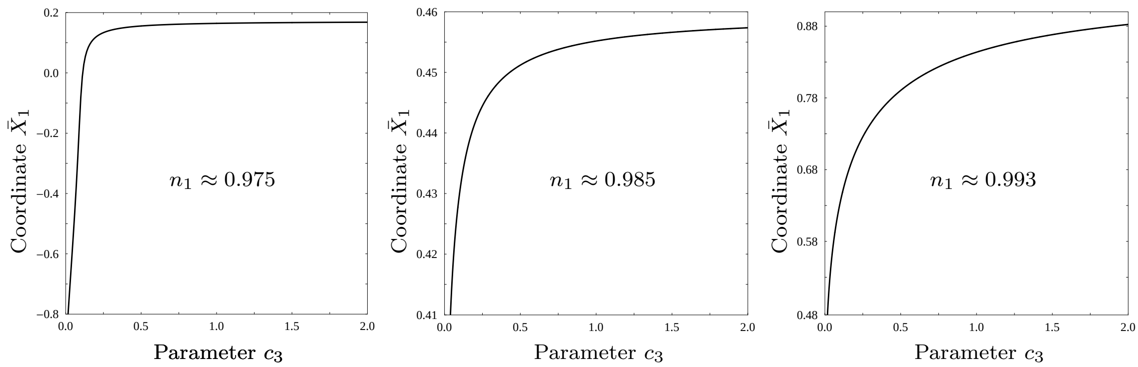

Figure 2.4: Plots of the dimensionless coordinate $\bar{X}_1$ at which ellipticity is lost as a function of the dimensionless material parameter c_3 for three fixed values of n_1 close to 1.

In this case we recall that the connections (1.222), namely $\bar{W} = \bar{W}_4 + 2\bar{W}_5 = 0$ and $2\bar{W}_1 + 4\bar{W}_2 = \bar{p}_0$ this being the value of $\bar{p}$ in B_r, justify the argument of an initially energy and stress-free material while, using (1.219) and (1.220), the corresponding properties of (2.16)–(2.20) are reduced to (see, also, [50])

$$\bar{W}_1 + \bar{W}_2 = (c_{22} - c_{23})/4, \tag{2.117}$$

$$\bar{W}_1 + \bar{W}_2 + \bar{W}_5 = c_{55}/2, \tag{2.118}$$

$$\bar{W}_{44} + 4\bar{W}_{45} + 4\bar{W}_{55} = (c_{11} + c_{22} - 2c_{12} - 4c_{55})/4, \tag{2.119}$$

with the derivatives of $\bar{W}$ being evaluated for $(I_1, I_2, I_4, I_5) = (3, 3, 1, 1)$. Based on the grounds of (1.219) and (1.220), we also employ (1.215) and we follow the same procedure as the one discussed in Section 2.2.2 to deduce that now the inequalities

$$c_{22} > c_{23}, \quad c_{55} > 0, \quad c_{11} + c_{22} - 2c_{12} > 0, \tag{2.120}$$

are necessary and sufficient conditions for strong ellipticity to hold in the classical incompressible theory when $\mathbf{M}$ is directed along the X_1 axis.

Due to the incompressibility constraint it is now evident that the connections (2.42)–(2.45), obtained in Section 2.3, are still valid. Thus, through (2.115), the equilibrium equation $(2.34)_2$ provides that the Lagrange multiplier should, for each choice of $\kappa \in \{1, 2\}$ and regardless the representation of the strain energy, depend strictly on X_1 while the first of these equations yields again (2.38). Accordingly, when $W = \hat{W}(\lambda_1, \lambda_2, \lambda_3)$, combination of the latter with (2.44), (2.45) and (2.115) delivers

$$\frac{d\hat{W}}{dX_1} = \lambda_1(\sigma_2 - \sigma_1)/r = \frac{d\sigma_1}{dX_1},$$

which is then integrated to give

$$\hat{W} = \sigma_1,$$

or, equivalently

$$\hat{W} = \lambda_1 \hat{W}_1 - \hat{p}. \tag{2.121}$$

for each $\kappa \in \{1, 2\}$. We note that the integration constant which should normally appear in the latter connection is omitted since it is trivial to show that this takes the value zero. As expected, equation (2.121) is of the same form as the one arising in the incompressible isotropic case (see, e.g., [56]) only now $\hat{W}$ is not symmetric in $(\lambda_1, \lambda_2, \lambda_3)$.

It is apparent that here the equilibrium equation imposes no restriction on the strain-energy function but simply serves to determine the Lagrange multiplier prescribed either by $\bar{p}$ or $\hat{p}$. Hence, for the incompressible theory and with reference to the work of Jiang and Ogden [39, 40], general forms of strain-energy functions in respect of bending deformation (under plane strain) for transversely isotropic elastic materials may be identified. For this purpose and bearing in mind (2.48), (2.49), now both evaluated for $I_3 = 1$, we define the strain-energy functions $\hat{\bar{W}}(I_1, I_4) = \bar{\bar{W}}(I_1, 1, I_4)$ and $\hat{W}(\lambda_1) = \hat{\hat{W}}(\lambda_1, \lambda_2)$, where $\hat{\bar{W}}$ can, for example, be one of the functions discussed in Section 4.2 or any other function satisfying the required conditions. By these means, when specializing to plane strain, equation (2.121) is appropriately rearranged to identify the form of the Lagrange multiplier, which along with $\hat{\bar{W}}$ is denoted $\hat{\bar{p}}$, involved in the counterpart of (2.50) and/or (2.51) given by

$$\boldsymbol{\sigma} = 2\hat{\bar{W}}_1 \mathbf{B} + 2\hat{\bar{W}}_4 \mathbf{m} \otimes \mathbf{m} - \hat{\bar{p}} \mathbf{I}, \tag{2.122}$$

$$\mathbf{S} = 2\hat{\bar{W}}_1 \mathbf{F}^{\mathrm{T}} + 2\hat{\bar{W}}_4 \mathbf{M} \otimes \mathbf{F}\mathbf{M} - \hat{\bar{p}} \mathbf{F}^{-1}, \tag{2.123}$$

respectively, where $\mathbf{I}$ is the (two-dimensional) identity tensor.

As a result the specialization (2.63) discussed in the Section 2.3.3 is also now admissible and may be written as

$$\hat{\bar{W}} = \hat{\bar{W}}_{\mathrm{iso}}(I_1) + \hat{\bar{W}}_{\mathrm{fib}}(I_4). \tag{2.124}$$

This is one possibility amongst within a very wide class of incompressible transversely materials that may be examined under the considered bending deformation.

Chapter 3

Azimuthal shear of a transversely isotropic tube

3.1 Introduction

The problem of azimuthal shear of a circular cylindrical tube composed of elastic material has been discussed in many publications since the pioneering work of Rivlin [67], primarily for isotropic elastic solids, compressible or incompressible. A review of the literature is provided by Jiang and Ogden [39], to which reference can be made for detailed citations. To the best of our knowledge, relatively little has been done for anisotropic bodies undergoing azimuthal shear deformation, although Jiang and Beatty [37] examined the helical shear problem (of which the azimuthal shear problem is a special case) for transversely elastic materials whose direction of transverse isotropy is either axial, circumferential or helical; also, for transversely isotropic materials, Tsai and Fan [81] analyzed the anti-plane shear problem. In both cases the attention of these authors was focussed mainly on constructing classes of strain-energy functions capable of undergoing the considered deformations. See also the recent contribution by Merodio et al. [35] concerned with the rectilinear shear of a slab of fiber-reinforced elastic material.

In the work of Abeyaratne [2] the azimuthal shear problem has been studied in detail for incompressible, isotropic elastic materials from the point of view of loss of ellipticity. Specifically, loss of ellipticity, at intermediate ranges of loading applied at the boundaries of the tube heralds the emergence of certain non-smooth solutions. The existence of such solutions requires that the strain energy be non-convex as a function of the shear strain. In this chapter we examine the problem of azimuthal shear for a circular cylindrical tube of transversely isotropic elastic material in terms of loss of ellipticity, which requires loss of strict convexity of the strain energy as a function of the shear strain. The direction of transverse isotropy (the preferred direction) is taken to lie in planes normal to the axis of the tube so that the problem has a plane strain character. Moreover, this direction depends (in general) only on the radius through the material so that circular symmetry is maintained.

In Section 3.2, the geometry of the problem and the kinematics associated with the azimuthal shear deformation are introduced, while the form of the strain-energy function for a transversely isotropic material with the restriction to plane strain is given together with the (in-plane) Cauchy stress tensor and its polar components in Section 3.3. The components of the equilibrium equation are then summarized in Section 3.4. In Section 3.5 the form of the strong ellipticity condition appropriate for the considered specialization is stated. We consider a special class of material models consisting of an isotropic base material augmented by a reinforcement dependent on the preferred direction. As is known from the isotropic problem [2], loss of ellipticity requires loss of monotonicity of the shear stress versus shear strain relationship; in other words, a strain energy that is a non-convex function of the amount of shear. This is also the case here although the chosen energy function is non-convex only for negative shear strain.

The well-known neo-Hookean model augmented with the so-called *standard reinforcing model* (see, e.g., [77, 63, 62, 47, 48]) is then the focus of attention in Section 3.6. The notion of strong ellipticity is studied in terms of the magnitude and direction of the applied (azimuthal shear) loading and the resulting shear strain in the material. Closed-form solutions are derived that determine the domain of strong ellipticity, on the boundaries of which ellipticity is lost. Analysis of the azimuthal governing equation yields conclusions relating loss of strict convexity of the considered strain-energy function to the existence of multiple solutions. In particular, there are in some circumstances three choices for the shear strain, only two of which are admissible. The degree of anisotropy and the geometry of the preferred direction at each point of the body serve to characterize the nature of the surfaces of discontinuity (strong or weak) emerging from the failure of strong ellipticity, which may only happen when the preferred direction undergoes contraction. The surfaces of discontinuity are circular cylinders concentric with the tube. In the special case in which the preferred direction is taken to be radial the azimuthal shear causes extension of the preferred direction for either sense of the shear (and no loss of ellipticity). More generally, we consider a preferred direction that depends on the radius in such a fashion that it extends for positive (anticlockwise) shear, but for which in negative (clockwise) shear it may either extend or contract. In the case of negative shear, the distinction between extension and contraction is dependent on the radial position, the precise disposition of the fibers, the degree of anisotropy and the magnitude of the applied shear stress.

For the same reinforcement, the Varga model is then, in Section 3.7, chosen to represent the isotropic base material. In this case closed-form solutions are not readily obtainable, and we therefore present numerical results that are parallel to those for the neo-Hookean material. In particular, we again determine a relationship between loss of ellipticity and the existence of non-smooth and multiple solutions that turns out to be very similar to that obtained for the reinforced neo-Hookean model. Unlike the previous case, however, negative shear always leads to ellipticity failure regardless the degree of anisotropy of the considered material.

Finally, several numerical examples are used in Section 3.8 to illustrate some of the aspects discussed in the foregoing paragraphs, and the overall response of a body undergoing such a deformation is also highlighted. As in the isotropic material case, a unique energy minimizing deformation can be associated with each value of applied shear stress (or twist angle), and deformations containing a surface of discontinuity are confined to a particular interval of this shear stress. However, in contrast to the case of loss of ellipticity in isotropic tubes [2], certain radial variations of the preferred direction give loss of ellipticity that is always confined to a small region of the tube. This includes cases in which ellipticity can be lost at only a single internal radius and cases in which loss of ellipticity occurs over an interval of internal radii. In the latter case, a surface of discontinuity emerges in the interior of the tube, increases its radius under increasing twist, and then disappears while still strictly interior to the tube.

3.2 The azimuthal shear deformation

3.2.1 Definition of the deformation

Consider a circular cylindrical tube composed of incompressible hyperelastic material with reference geometry defined by

$$A \leqslant R \leqslant B, \quad 0 \leqslant \Theta \leqslant 2\pi, \quad 0 \leqslant Z \leqslant L, \tag{3.1}$$

where (R, Θ, Z) are cylindrical polar coordinates in the reference configuration (assumed free of stress) relative to a cylindrical polar basis $\{\mathbf{E}_I\}, I \in \{R, \Theta, Z\}$.

The deformation of *pure azimuthal shear* is defined by

$$r = R, \quad \theta = \Theta + g(R), \quad z = Z, \tag{3.2}$$

where (r, θ, z) are cylindrical polar coordinates in the deformed configuration associated with the cylindrical polar basis $\{\mathbf{e}_i\}, i \in \{r, \theta, z\}$, and $g(R) = g(r)$ is a function to be determined. We suppose that

$$g(a) = 0, \quad g(b) = \psi, \tag{3.3}$$

where ψ, which may be positive or negative, is the angle of rotation of the outer boundary $r = b = B$ relative to $r = a = A$.

3.2.2 Kinematics of the problem

For the considered material geometry (3.1) and deformation (3.2), the deformation gradient tensor $\mathbf{F}$ takes the form

$$\mathbf{F} = \mathbf{R} + \gamma \mathbf{e}_\theta \otimes \mathbf{E}_R, \tag{3.4}$$

where $\mathbf{R} = \mathbf{e}_r \otimes \mathbf{E}_R + \mathbf{e}_\theta \otimes \mathbf{E}_\Theta + \mathbf{e}_z \otimes \mathbf{E}_Z$, while the corresponding left Cauchy-Green tensor, introduced in $(1.43)_2$, is

$$\mathbf{B} = \mathbf{I} + \gamma(\mathbf{e}_r \otimes \mathbf{e}_\theta + \mathbf{e}_\theta \otimes \mathbf{e}_r) + \gamma^2 \mathbf{e}_\theta \otimes \mathbf{e}_\theta, \tag{3.5}$$

where $\mathbf{I}$ is the identity tensor, $\gamma = rg'(r)$ is the amount of shear (locally a simple shear in the planes normal to $\mathbf{e}_z$) and the prime on g indicates differentiation with respect to $r = R$.

Let $\mathbf{M}$ be a unit vector, defined in the reference configuration, that identifies locally a preferred direction. In particular, we suppose that $\mathbf{M}$ lies in (R, Θ)-planes, so that we may write

$$\mathbf{M} = M_R \mathbf{E}_R + M_\Theta \mathbf{E}_\Theta, \quad M_R^2 + M_\Theta^2 = 1. \tag{3.6}$$

The geometrical nature of the preferred direction may be characterized in terms of the *scalar bijection* mapping $G : [A, B] \longrightarrow [0, \Theta_1 - \Theta_0]$ via the equation

$$\Theta = G(R) + \Theta_0, \quad G(A) = 0, \quad G(B) = \Theta_1 - \Theta_0, \tag{3.7}$$

where $\Theta_0 \in [0, 2\pi]$ and $\Theta_1 - \Theta_0 > 0$ is fixed independently of Θ_0.

Then,

$$M_R = \frac{1}{\sqrt{(RG'(R))^2+1}}, \quad M_\Theta = \frac{RG'(R)}{\sqrt{(RG'(R))^2+1}}, \tag{3.8}$$

and $G'(R) = dG(R)/dR$. We assume here that $G'(R) \geqslant 0$. A schematic of a possible geometrical arrangement of the preferred direction is depicted in Figure 3.1. Note that M_R and M_Θ are functions of the radius R only. It will sometimes be convenient to identify the preferred direction in terms of the angle $\alpha = \alpha(R)$, with $\alpha \in [0, \pi/2]$ defined by

$$\tan\alpha = M_\Theta/M_R = RG'(R). \tag{3.9}$$

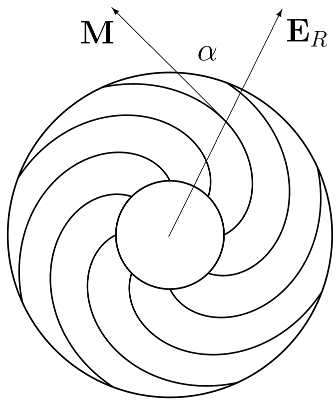

Figure 3.1: Illustration of a possible arrangement of the preferred direction in the reference configuration according to equation (3.7). The angle α is identified by noting that $M_R = \cos\alpha$.

Under the considered deformation $\mathbf{M}$ becomes $\mathbf{m}$ which, according to (1.125), (3.4) and $(3.6)_1$, is now given by

$$\mathbf{m} = M_R\mathbf{e}_r + (M_\Theta + \gamma M_R)\mathbf{e}_\theta. \tag{3.10}$$

Further, the kinematic invariants of interest identified with the considered deformation are now prescribed by I_1 and I_4, these being calculated as

$$I_1 = \operatorname{tr}\mathbf{B} = 3+\gamma^2, \quad I_4 = \mathbf{m}\cdot\mathbf{m} = 1 + 2\gamma M_R M_\Theta + \gamma^2 M_R^2, \tag{3.11}$$

respectively. Note that $(3.11)_2$ reduces to $I_4 = 1+\gamma^2$ when $\mathbf{M} = \mathbf{E}_R$ ($\alpha = 0$), while $I_4 = 1$ if $\mathbf{M} = \mathbf{E}_\Theta$ ($\alpha = \pi/2$). In the latter case the deformation corresponds to simple shear in the preferred direction. We recall that, in general, for an incompressible transversely isotropic material (in three dimensions) there are four independent invariants associated with the deformation and the preferred direction $\mathbf{M}$. As noted in Section 2.5, however, when those invariants are chosen to be represented by the manifold (I_1, I_2, I_4, I_5) and the deformation is restricted to plane strain, it is not necessary to consider I_2 and I_5 separately from I_1 and I_4. Indeed, it is easy to show that in plane strain, and wether or not $\mathbf{M}$ is an eigenvector of $\mathbf{C}$ (and equivalently $\mathbf{m}$ an eigenvector of $\mathbf{B}$), the connections (2.48) derived earlier for the associated compressible are always validated. Following that, we now incorporate the incompressibility constraint ($I_3 = 1$) in (2.48) and we write

$$I_2 = I_1, \qquad I_5 = (I_1 - 1)I_4 - 1. \tag{3.12}$$

Since $M_R \geqslant 0$ (with equality for $G'(R) \to \infty$) and $M_\Theta \geqslant 0$ then for positive shear ($\gamma > 0$) we have $I_4 \geqslant 1$ for $R \in [A, B]$. On the other hand, for negative shear ($\gamma < 0$) and $\alpha \neq 0,\ \pi/2$,

$$I_4 \gtreqless 1 \quad \text{according as} \quad \gamma(\gamma + 2\tan\alpha) \gtreqless 0. \tag{3.13}$$

Now, for $\gamma < 0$, with $\alpha \neq 0$, it follows from (3.13) that $I_4 < 1$ for $\gamma > -2\tan\alpha$ and $I_4 > 1$ for $\gamma < -2\tan\alpha$. Note that α may be a constant, a monotonic increasing or decreasing function of R or none of these. If α is a constant then $RG'(R)$ is a constant and the resulting curve is a logarithmic spiral, with $G(R)$ given by

$$G(R) = (\Theta_1 - \Theta_0)\log(R/A)/\log(B/A). \tag{3.14}$$

Suppose, for illustration, that α is either a constant or an increasing function of $r = R$ and that $|\gamma|$ is a decreasing function of r. In fact, it is shown in Section 5 that if α is constant then the latter condition holds whenever the strong ellipticity condition holds (but otherwise it may or may not hold depending on the material properties and the dependence of α on r). Then, at the inner boundary $r = a$, as γ decreases from zero, I_4 steadily decreases from 1 until it reaches its minimum value at $\gamma = -\tan\alpha$ and then starts to increase. Therefore, due to monotonicity of $|\gamma|$, we have $I_4 < 1$ for $a \leqslant r \leqslant b$ but when $\gamma = -2\tan\alpha$ at the inner boundary, I_4 returns to the value 1, but remains less than 1 for $a < r \leqslant b$. Thereafter, I_4 is greater than 1 at the inner boundary and there is a value of $r \in (a, b)$, denoted r_α, at which $\gamma = -2\tan\alpha$, so that $I_4 > 1$ for $a \leqslant r < r_\alpha$ and $I_4 < 1$ for $r_\alpha < r \leqslant b$. The radial location r_α increases with $|\gamma|$. At sufficiently large $|\gamma|$, r_α reaches the value b and thereafter I_4 is greater than 1 for $a \leqslant r \leqslant b$.

3.3 Constitutive law: transverse isotropy

Owing to the linkages (3.12), it is evident that for a transversely isotropic incompressible elastic solid, when restricted to the plane strain specialization, the strain-energy function

W may be treated in general as a function of I_1 and I_4 alone. In particular, bearing in mind the last expression in (1.192) and following closely the methodology provided in Chapter 2, we are prompted to write

$$W = \bar{\bar{W}}(I_1, I_4), \tag{3.15}$$

such that $\bar{\bar{W}}(I_1, I_4) = \bar{W}(I_1, I_1, I_4, (I_1 - 1)I_4 - 1)$. Due to (3.15), the associated in-plane Cauchy and nominal stress tensors $\boldsymbol{\sigma}$ and $\mathbf{S}$ are then of the form (2.122) and (2.123), respectively, but in the present notation are written down as

$$\boldsymbol{\sigma} = 2\bar{\bar{W}}_1\mathbf{B} + 2\bar{\bar{W}}_4\mathbf{m} \otimes \mathbf{m} - \bar{\bar{p}}\mathbf{I}, \tag{3.16}$$

$$\mathbf{S} = 2\bar{\bar{W}}_1\mathbf{F}^{\mathrm{T}} + 2\bar{\bar{W}}_4\mathbf{M} \otimes \mathbf{FM} - \bar{\bar{p}}\mathbf{F}^{-1}. \tag{3.17}$$

As before, $\mathbf{I}$ and $\mathbf{B}$ are the in-plane identity and left Cauchy-Green tensors, respectively, $\bar{\bar{p}}$ is the corresponding Lagrange multiplier associated with the incompressibility constraint and $\bar{\bar{W}}_1 = \partial\bar{\bar{W}}/\partial I_1, \bar{\bar{W}}_4 = \partial\bar{\bar{W}}/\partial I_4$.

Accordingly, for a material body whose energy and stress vanish in B_r (where $I_1 = 3$ and $I_4 = 1$), we require

$$\bar{\bar{W}} = \bar{\bar{W}}_4 = 0, \qquad \bar{\bar{W}}_1 = \bar{\bar{p}}_0, \tag{3.18}$$

where $\bar{\bar{p}}_0$ is the value of $\bar{\bar{p}}$ in that configuration. It is also worthwhile noting that, when $\mathbf{M}$ is defined by (3.6), consistency with the classical linear incompressible theory discussed in Section 1.5.1 can be established through the connections

$$\bar{\bar{W}}_1 = [c_{13} - c_{12} + c_{23} - c_{33} + c_{44} + 2(c_{55} + c_{66})]/6, \tag{3.19}$$

$$\bar{\bar{W}}_{44} = [3(c_{11} + c_{22}) - 8(c_{13} + c_{23} - c_{33}) - 16(c_{55} + c_{66}) + 2c_{12} + 4c_{44}]/12, \tag{3.20}$$

with the derivatives of $\bar{\bar{W}}$ evaluated for $(I_1, I_4) = (3, 1)$. Now, the correlation between $(3.18)_1$ and (3.19) is apparent. We emphasize that here the parameters $c_{11}, ..., c_{66}$ are, in general, functions of the angle $\alpha = \alpha(R)$ and are defined similarly to (2.21), yet do not necessarily embody the symmetries presented there. Precisely, assuming that M_R and M_Θ are strictly positive, the aforemention material quantities are identified as

$$\begin{aligned} &c_{ij} = C_{iijj}, \quad i, j \in \{1, 2, 3\}, \\ &c_{44} = C_{1212} \quad c_{55} = C_{1313}, \quad c_{66} = C_{2323}, \end{aligned} \tag{3.21}$$

where now, due to the symmetries of the body under examination and the dependence of $\mathbf{M}$ on $\alpha = \alpha(R)$, the subscripts $1, 2$ and 3 on the parameters C_{ijkl} are associated with the cylindrical polar coordinates (R, Θ, Z) respectively. We remark that the quantities (3.21) may be found to satisfy particular symmetries which may lead to a different set of parameters, including, for example, the non-vanishing terms $c_{77} = C_{1112}, c_{88} = C_{1222}, c_{99} = C_{1323}$, appearing in the right-hand side of (3.19) and (3.20). Although these symmetries can be written down explicitly, they are too complicated since they require the involvement of the components of M_R and M_Θ of $\mathbf{M}$ and for this reason they are not presented here. However, they can be used to identify that, in the special case where $M_R = 1$ (i.e. $\alpha = 0$), the connections (2.21) are re-established and hence (3.19) and (3.20) specialize to $\bar{\bar{W}}_1 = c_{55}/2$ and $\bar{\bar{W}}_{44} = (c_{11} + c_{22} - 2c_{12} - 4c_{55})/4$, respectively. In fact, by substituting $\bar{\bar{W}}$ with $\hat{\bar{W}}$, the two latter conditions may be regarded as the specializations of (2.117)–(2.119) introduced earlier in Section 2.5 for the bending problem when the considered deformation is restricted to plane strain. Analogous conclusions can easily be drawn for the case where $M_\Theta = 1$ (i.e. $\alpha = \pi/2$).

The (in-plane) cylindrical polar components of $\boldsymbol{\sigma}$ are read off as

$$\sigma_{rr} = 2\bar{\bar{W}}_1 + 2\bar{\bar{W}}_4 M_R^2 - \bar{\bar{p}}, \tag{3.22}$$

$$\sigma_{\theta\theta} = 2\bar{\bar{W}}_1(1+\gamma^2) + 2\bar{\bar{W}}_4(M_\Theta + \gamma M_R)^2 - \bar{\bar{p}}, \tag{3.23}$$

$$\sigma_{r\theta} = 2\bar{\bar{W}}_1\gamma + 2\bar{\bar{W}}_4 M_R(M_\Theta + \gamma M_R), \tag{3.24}$$

from which it is easy to show that

$$\sigma_{\theta\theta} - \sigma_{rr} = \gamma\sigma_{r\theta} + 2\bar{\bar{W}}_4(\gamma M_R M_\Theta + M_\Theta^2 - M_R^2). \tag{3.25}$$

In view of (3.11) and (3.9) we may now introduce a new function, denoted $\tilde{W}$, such that

$$\tilde{W}(\gamma, \alpha) = \bar{\bar{W}}(I_1, I_4), \tag{3.26}$$

and differentiation of (3.26) yields the simple formula

$$\sigma_{r\theta} = \frac{\partial\tilde{W}}{\partial\gamma} \tag{3.27}$$

for the shear stress. We note in passing that for any elastic material for which the strain energy can be regarded as a function of the single deformation variable γ for the considered deformation the formula (3.27) applies. In (3.26) we should remark that α is a material parameter, not a deformation variable, and its inclusion reflects the fact that the material properties are inhomogeneous if α depends on R. With this in mind we note that the connection (3.25) can be rewritten in the form

$$\frac{\partial \tilde{W}}{\partial \alpha} = \gamma^2 \sigma_{r\theta} + \gamma(\sigma_{rr} - \sigma_{\theta\theta}). \tag{3.28}$$

3.4 Equilibrium equations

For the deformation and constitutive law discussed in the foregoing sections the equilibrium equation $\mathrm{div}\boldsymbol{\sigma} = \mathbf{0}$ (in the absence of body forces) has just two non-trivial components, namely the radial equation

$$\frac{\mathrm{d}\sigma_{rr}}{\mathrm{d}r} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0 \tag{3.29}$$

and the azimuthal equation

$$\frac{\mathrm{d}}{\mathrm{d}r}(r^2 \sigma_{r\theta}) = 0 \tag{3.30}$$

both of which arise on specializing the expressions (1.76) demonstrated in Section 1.2.3.

The azimuthal equation (3.30) integrates to give, in conjunction with (3.27),

$$\sigma_{r\theta} \equiv \frac{\partial \tilde{W}}{\partial \gamma} = \frac{\tau_\theta b^2}{r^2}, \tag{3.31}$$

which, for any given form of $\tilde{W}$, serves to determine γ as a function of r, and hence, via $\gamma = rg'(r)$, the deformation function $g(r)$, subject to the boundary conditions (3.3). We emphasize that the form of the function $g(r)$ depends on the form of the strain-energy function. The parameter τ_θ in (3.31) is a constant, representing the value of the azimuthal stress component on the boundary $r = b$. In addition, the quantity $2\pi\tau_\theta b^2$ represents the resultant torque (twisting moment). Either τ_θ or ψ in $(3.3)_2$ (but not both) can be regarded as providing the boundary condition on $r = b$.

For the isotropic theory, as discussed by Jiang and Ogden [39], positive γ (i.e. positive ψ) is associated with $\tau_\theta > 0$ while $\gamma < 0$ is related to $\tau_\theta < 0$. This assertion, in conjunction with (3.31), clearly suggests that

$$\sigma_{r\theta} \equiv \frac{\partial \tilde{W}}{\partial \gamma} \gtreqless 0 \quad \text{according as} \quad \gamma \gtreqless 0. \tag{3.32}$$

We adopt these restrictions in what follows, and more detailed commentary on them is provided after equation (3.50).

Once γ is determined, the role of the radial equation (3.29) is to determine σ_{rr}, or equivalently p. Integration of equation (3.29), on use of (3.28) and (3.31), yields

$$\sigma_{rr}(r) = \sigma_{rr}(a) + \int_a^r \left(g'\tau_\theta b^2 - (g')^{-1}\frac{\partial \tilde{W}}{\partial \alpha} \right) \frac{\mathrm{d}R}{R^2}, \tag{3.33}$$

in which R is used as the integration variable. Because of the incompressibility constraint the value of $\sigma_{rr}(a)$ is at our disposal. If required, the normal component $\sigma_{\theta\theta}$ can now be obtained from (3.29).

At this point, it is worth mentioning that, for a transversely isotropic material, unlike the situation in the isotropic theory, equation (3.31) might have an alternative role. Specifically, if a certain shear-stress-strain response and distribution is required, then equation (3.31) can be used to identify the preferred directions, in other words the components (3.8) or $G(R)$ itself, for which such a deformation is sustainable. We do not pursue this design idea in the present analysis, however.

3.5 Strong ellipticity and a class of reinforcing models

3.5.1 Local and global considerations for strong ellipticity

We now discuss the strong ellipticity condition for the considered azimuthal shear deformation and constitutive law. For this purpose we draw on the general plane strain strong ellipticity condition for transversely isotropic materials given by Merodio and Ogden [47] for a strain-energy function of the form $\bar{\bar{W}}(I_1, I_4)$. In the present notation, this may be written

$$2\bar{\bar{W}}_{11}[\mathbf{n}\cdot(\mathbf{Ba})]^2 + 4\bar{\bar{W}}_{14}\mathbf{n}\cdot(\mathbf{Ba})(\mathbf{n}\cdot\mathbf{m})(\mathbf{n}\times\mathbf{m})_3$$
$$+ 2\bar{\bar{W}}_{44}(\mathbf{n}\cdot\mathbf{m})^2(\mathbf{n}\times\mathbf{m})_3^2 + \bar{\bar{W}}_1\mathbf{n}\cdot(\mathbf{Bn}) + \bar{\bar{W}}_4(\mathbf{n}\cdot\mathbf{m})^2 > 0 \qquad (3.34)$$

for all in-plane unit vectors $\mathbf{n}$ and $\mathbf{a}$ satisfying $\mathbf{a}\cdot\mathbf{n} = 0$, where $(\mathbf{n}\times\mathbf{m})_3 = n_1m_2 - n_2m_1$, (n_1, n_2) and (m_1, m_2) being the components of $\mathbf{n}$ and $\mathbf{m}$, respectively.

Where ellipticity fails $\mathbf{n}$ defines the normal to the associated (weak or strong) surface of discontinuity. While the inequality (3.34) is local, for the present problem we have to consider the global implications of the constraints imposed by the geometry. In particular, analogously to the situation described in Section 2.4.2 for the case of elastic blocks undergoing bending deformations, if the circular geometry is to be maintained then any surface of discontinuity is necessarily constrained to be circular cylindrical and concentric with the tube. Thus, we may take $\mathbf{n} = \mathbf{e}_r$ and $\mathbf{a} = \mathbf{e}_\theta$. Then, $\mathbf{n}\cdot(\mathbf{Bn}) = 1$, $(\mathbf{n}\times\mathbf{m})_3 = (M_\Theta + \gamma M_R)$, $\mathbf{n}\cdot(\mathbf{Ba}) = \gamma$, $\mathbf{n}\cdot\mathbf{m} = M_R$, and (3.34) therefore reduces to

$$2\bar{\bar{W}}_{11}\gamma^2 + 4\bar{\bar{W}}_{14}\gamma M_R(M_\Theta + \gamma M_R) + 2\bar{\bar{W}}_{44}M_R^2(M_\Theta + \gamma M_R)^2 + \bar{\bar{W}}_1 + \bar{\bar{W}}_4 M_R^2 \equiv \frac{1}{2}\tilde{W}_{\gamma\gamma} > 0,$$

where a subscript γ signifies the partial derivative $\partial/\partial\gamma$. Thus, for the considered problem, strong ellipticity is equivalent to the simple inequality $\tilde{W}_{\gamma\gamma} > 0$, and loss of ellipticity therefore occurs, if at all, at a value of r for which $\tilde{W}_{\gamma\gamma} = 0$. We note in passing that, based on the same arguments, the inequality (3.34) specializes to

$$\bar{\bar{W}}_1 + 2\bar{\bar{W}}_{44} M_R^2 M_\Theta^2 \equiv \tilde{W}_{\gamma\gamma}/2 > 0 \tag{3.35}$$

when evaluated in the reference configuration, where $(I_1, I_4) = (3, 1)$ (i.e. from $(3.18)_2$ $\bar{\bar{W}}_4 = 0$) and/or $\gamma = 0$. Hence, since, in the reference configuration, the terms $\bar{\bar{W}}_1$ and $\bar{\bar{W}}_{44}$ are constants, strong ellipticity is possible to provide restrictions on the geometry of the preferred direction. Specifically, if $\bar{\bar{W}}_1 + 2\bar{\bar{W}}_{44}$ and $\bar{\bar{W}}_{44}$ are positive, the inequality (3.35) then requires

$$M_1^0 < M_R < M_2^0, \tag{3.36}$$

with

$$M_1^0 = \sqrt{\frac{\bar{\bar{W}}_{44} - \sqrt{2\bar{\bar{W}}_1\bar{\bar{W}}_{44} + \bar{\bar{W}}_{44}^2}}{2}}, \quad M_2^0 = \sqrt{\frac{\bar{\bar{W}}_{44} + \sqrt{2\bar{\bar{W}}_1\bar{\bar{W}}_{44} + \bar{\bar{W}}_{44}^2}}{2}},$$

when $\bar{\bar{W}}_1 \leqslant 0$, while for $\bar{\bar{W}}_1 \geqslant 0$, (3.36) becomes

$$M_2^0 < M_R < M_1^0. \tag{3.37}$$

Finally, for either $\bar{\bar{W}}_1 + 2\bar{\bar{W}}_{44}$ or $\bar{\bar{W}}_{44}$ equals zero, (3.35) is established for all $M_R \neq \sqrt{2}/2$. Clearly, by virtue of (3.19) and (3.20), the status of (3.35) can be analyzed in terms of the parameters $c_{11}, ..., c_{66}$ but we do not use this approach here. We only remark that, interestingly but not surprisingly, when $M_R = 1$, i.e. the symmetries (2.21) are validated, (3.35) is satisfied if and only if $c_{11} + c_{22} - 2c_{12} > 0$, the latter being identical with the requirement $(2.120)_3$ given in Section 2.5. Although the implications of (3.35) can be put into parallelism with the results that are presented in Section 3.6 and thereafter, we henceforth deliberately consider (for simplicity) material models which satisfy the strong ellipticity condition in the reference configuration.

Expressed otherwise, strong ellipticity is equivalent to the strain-energy function being a strictly locally convex function of γ. Thus,

$$\tilde{W}_{\gamma\gamma} \equiv \frac{\partial \sigma_{r\theta}}{\partial \gamma} > 0. \tag{3.38}$$

This is easily confirmed in the case of an isotropic material, for which we write (3.15) as $\hat{W} = E(I_1)$. Then, necessary and sufficient conditions for (3.34) are [1, 47]

$$2(I_1 - 3)E''(I_1) + E'(I_1) > 0, \quad E'(I_1) > 0. \tag{3.39}$$

But, since $I_1 = 3 + \gamma^2$, we introduce the notation $\tilde{E}(\gamma)$ defined by $\tilde{E}(\gamma) = E(I_1)$. The inequalities (3.39) then become

$$\tilde{E}''(\gamma) > 0, \quad \tilde{E}'(\gamma) \gtreqless 0 \quad \text{for} \quad \gamma \gtreqless 0. \tag{3.40}$$

Note, however, that the latter is equivalent to the adopted condition (3.32), and hence, for isotropic materials, the remaining strong ellipticity condition is simply $\tilde{E}''(\gamma) > 0$.

It should be emphasized that for the deformation under examination, and as for the isotropic theory [39], the ellipticity requirement (3.38), if it holds for all γ, ensures, for any given τ_θ, uniqueness of the solution of (3.31) for γ provided the growth condition $\tilde{W}_\gamma \to \infty$ as $\gamma \to \infty$ holds. Loss of ellipticity is therefore closely related to loss of uniqueness of the solution for γ.

Finally in this section, we note that

$$\tilde{W}_{\gamma\gamma}\frac{\mathrm{d}\gamma}{\mathrm{d}r} + \tilde{W}_{\gamma\alpha}\frac{\mathrm{d}\alpha}{\mathrm{d}r} = -2\frac{\tau_\theta b^2}{r^3}, \tag{3.41}$$

and recall that by (3.31) and (3.32) τ_θ has the same sign as γ. This shows that if (3.38) holds and if α is independent of r then $\mathrm{d}\gamma/\mathrm{d}r \gtreqless 0$ according as $\tau_\theta \lesseqgtr 0$. Thus, $|\gamma|$ is a decreasing function of r whenever strong ellipticity holds. If α depends on r then, in general, whether or not $|\gamma|$ is monotonic depends both on the nature of this dependence and on how $\tilde{W}$ depends on α. An example of non-monotonicity is illustrated in Section 3.8.2.

3.5.2 Strain-energy functions of separable form

In order to examine loss of ellipticity in detail we focus attention on two particular strain-energy functions within the general class characterized by the separable form

$$\bar{\bar{W}}(I_1, I_4) = E(I_1) + F(I_4), \tag{3.42}$$

in which the first term $E(I_1)$ represents the isotropic base material and the additional term $F(I_4)$ represents the reinforcement associated with the transversely isotropic nature of the considered materials, the invariant I_4 being associated with the preferred direction $\mathbf{M}$. Recall that such specializations of the strain-energy function have been briefly discussed in Chapter 2 but under a different notation. As noted there (see the discussion provided in Sections 2.3.3 and 2.5 and also the references therein), several authors have considered a decomposition of the form (3.42), or specializations thereof, where, in each case, the reinforcing contribution $F(I_4)$ has been taken to satisfy

$$F'(I_4) > 0\,(< 0) \quad \text{for} \quad I_4 > 1\,(< 1), \quad F'(1) = 0. \tag{3.43}$$

These conditions ensure that the contribution of $F(I_4)$ to the component of the Cauchy stress tensor (3.16) in the deformed preferred direction is positive (negative) under extension (contraction) of the preferred direction.

It is useful in what follows to examine certain properties of the energy function $\bar{\bar{W}}(I_1, I_4)$ specified in (3.42) in respect of its dependence on γ and to write

$$\tilde{W}(\gamma) = \tilde{E}(\gamma) + \tilde{F}(\gamma), \tag{3.44}$$

dropping the explicit dependence on α. It is similarly convenient to use the notation $\sigma_{r\theta} = s(r)$ so as to emphasize the dependence of the shear stress on r, whence, from (3.31),

$$\frac{\tau_\theta b^2}{r^2} = s(r) = \tilde{E}'(\gamma) + \tilde{F}'(\gamma). \tag{3.45}$$

We emphasize that for a strain-energy function of the form (3.42) satisfying (3.43), the properties of $\tilde{E}(\gamma)$ have to be consistent with (3.32), where in general $\tilde{E}(\gamma)$ need not be a convex function. Thus, loss of ellipticity of the base material may occur with $E'(I_1) = 0$ for either positive or negative γ whatever the sign of $\tilde{E}''(\gamma)$.

For $\gamma > 0$, and hence $s(r) > 0$, we have $I_4 > 1$ and from (3.43) with $\partial I_4/\partial\gamma = 2\cos\alpha(\sin\alpha + \gamma\cos\alpha)$ we obtain $\tilde{F}'(\gamma) > 0$, and the possibility of $\tilde{E}'(\gamma) \leqslant 0$ is therefore admitted, i.e. the isotropic base material in isolation may lose ellipticity for some value of r.

For $\gamma < 0$, and hence $s(r) < 0$, on the other hand, depending on the magnitude of γ, as we have seen, either extension or contraction of the preferred direction may arise. Specifically, for $-2\tan\alpha < \gamma < 0$ and $\alpha \neq 0$, we recall from (3.13) that $I_4 < 1$ and then $\tilde{F}'(\gamma) < 0$ for $\gamma > -\tan\alpha$, and equation $(3.45)_2$ again admits the possibility that $\tilde{E}'(\gamma)$ may be positive, negative or zero. However, when, with increasing $|\gamma|$, I_4 reaches its minimum value (at $\gamma = -\tan\alpha$) we have $\tilde{F}'(\gamma) = 0$ and hence $s(r) = \tilde{E}'(\gamma)$, which excludes the possibility of $\tilde{E}'(\gamma) \geqslant 0$. If, however, $-2\tan\alpha < \gamma < -\tan\alpha$ then $\tilde{F}'(\gamma) > 0$ and hence we must have $\tilde{E}'(\gamma) < s(r) < 0$, while in the limiting case $\gamma = -2\tan\alpha$, $I_4 = 1$ and $s(r) = \tilde{E}'(\gamma)$. Finally, for $\gamma < -2\tan\alpha$, $I_4 > 1$, $\tilde{F}'(\gamma) < 0$ and $\tilde{E}'(\gamma)$ may take either sign.

For the special case in which $\alpha = \pi/2$ we have $I_4 = 1$ for both positive and negative γ, and then $s(r) = \tilde{E}'(\gamma)$ since $\tilde{F}'(\gamma) = F'(1) = 0$ for any γ. In this case, for the considered deformation to be admissible $\tilde{E}'(\gamma)$ must have the same sign as γ. On the other hand, if $\alpha = 0$ (and the preferred direction is radial) then $I_4 > 1$ and there is no distinction between the effect of positive and negative γ.

In concluding this section, we remark that for the specific class of strain-energy functions (3.42) considered here, the properties of the isotropic base material (elliptic or not) have to be consistent with the requirements discussed above in order for the considered deformation to be admissible. However, we restrict attention henceforth to conventional isotropic base material functions. Much more detail on the possibilities for loss of ellipticity due to the properties of $\tilde{E}(\gamma)$ and $\tilde{F}(\gamma)$ can be gleaned from the work of Merodio and Ogden [47].

3.6 The reinforced neo-Hookean model

In this section we examine in detail the ellipticity status of the neo-Hookean isotropic material augmented with the so-called standard reinforcing model under the pure azimuthal shear deformation. In particular, the breakdown of strong ellipticity and loss of strict local convexity of $\tilde{W} = \bar{\bar{W}}$ are associated here with the existence of non-unique solutions of the boundary-value problem.

Equation (3.42) is now specialized to

$$\bar{\bar{W}}(I_1, I_4) = \frac{1}{2}\mu\left[I_1 - 3 + \rho(I_4 - 1)^2\right], \tag{3.46}$$

so that

$$E(I_1) = \frac{1}{2}\mu(I_1 - 3), \quad F(I_4) = \frac{1}{2}\mu\rho(I_4 - 1)^2, \tag{3.47}$$

where the constant $\mu\,(> 0)$ represents the shear modulus of the isotropic base material and $\rho\,(> 0)$ is a material constant that characterizes the degree of anisotropy associated with the presence of the preferred direction. Clearly, using (3.19) and (3.20), both μ and ρ can be expressed in terms of the parameters $c_{11}, ..., c_{66}$ but we do not use this connections here. For the considered deformation we have

$$I_1 - 3 = \gamma^2, \quad I_4 - 1 = \gamma\cos\alpha(2\sin\alpha + \gamma\cos\alpha). \tag{3.48}$$

It follows that

$$\tilde{W}_\gamma = \mu\gamma\left(2\rho\cos^4\alpha\,\gamma^2 + 6\rho\cos^3\alpha\sin\alpha\,\gamma + 4\rho\cos^2\alpha\sin^2\alpha + 1\right), \tag{3.49}$$

and it is then easy to show that the inequalities (3.32) hold if and only if

$$\rho \sin^2 2\alpha < 8. \tag{3.50}$$

Although the inequalities (3.32) could be relaxed to provide an alternative route to loss of ellipticity, here we assume that they hold and that loss of ellipticity is associated solely with the condition $\tilde{W}_{\gamma\gamma} = 0$. The restriction (3.50) on the parameter ρ and the angle α reflects the very special choice of strain-energy function.

More specifically, as discussed in [63] (see Figure 12 therein), $\rho > 8$ offers the possibility that shearing with respect to a particular range of reinforcing orientations gives a resolved shear stress with opposite sign to that of the amount of shear. Similar phenomena are noted in [50] and would likely render the considered radially symmetric solutions unstable with respect to more general deformations that are beyond the scope of this thesis. Here attention is restricted to (3.50), and more generally (3.32), so as to justify exclusive focus on the pure azimuthal shear deformations (3.2).

We remark that to the extent that the transverse isotropy studied here is associated with fiber reinforcement, constancy of the reinforcing parameter ρ might be regarded as associated with a constant fiber density (independent of r). More generally, however, one could consider ρ to depend on r and the subsequent analysis given in the present chapter for constant ρ could provide a point of entry for the consideration of any such generalization.

3.6.1 Multiple solutions

Here we investigate the existence of multiple solutions of the azimuthal equilibrium equation (3.31) for γ for given values of the applied shear loading τ_θ. For this purpose it is convenient to use the notations defined by

$$\sigma(\gamma) = \tilde{W}_\gamma/\mu, \quad \tau(r) = s(r)/\mu = \tau_\theta b^2/\mu r^2. \tag{3.51}$$

In respect of (3.46), equation (3.31) yields the cubic

$$\sigma(\gamma) \equiv 2\rho M_R^4 \gamma^3 + 6\rho M_R^3 M_\Theta \gamma^2 + (4\rho M_R^2 M_\Theta^2 + 1)\gamma = \tau(r) \tag{3.52}$$

for γ. An immediate useful observation is that both $\gamma = -\tan\alpha$ and $\gamma = -2\tan\alpha$ cause the reinforcing term in (3.52) to vanish, thus yielding $\sigma = \tau = \gamma$ for these two special values of γ. For $\gamma = -2\tan\alpha$, in which case $I_4 = 1$, this correspondence is a consequence of $F'(1) = 0$ in (3.43). For $\gamma = -\tan\alpha$ it is a consequence of $\partial I_4/\partial\gamma = 0$, which in turn renders $\tilde{F}'(\gamma) = 0$.

In the special case $M_R = 0$, equation (3.52) yields $\gamma = \tau(r)$ and the solution is exactly that arising in the isotropic theory, i.e. the anisotropy has no influence, either for positive or negative $\tau(r)$. Henceforth, we assume $M_R \neq 0$ ($\alpha \neq \pi/2$).

If $\tau_\theta > 0$ then $\tau(r) > 0$ and, according to (3.32), we must have $\gamma > 0$, in which case the left-hand side of (3.52) is a monotonic increasing function of γ, $\tilde{W}$ is a locally strictly convex function of γ and the strong ellipticity condition holds. Hence, (3.52) yields a unique value for γ. We shall not pursue discussion of this case.

The situation of particular interest is when $\tau_\theta < 0$ so that $\tau(r) < 0$ and, by (3.32), $\gamma < 0$. First, it is easy to show that if $\rho \sin^2 2\alpha < 2$ then the left-hand side of equation (3.52) is again a monotonic increasing function of γ. Hence (3.52) has a unique negative solution, $\gamma = \gamma_1$ say, defined for all $M_R \in (0, 1]$ (and all $r \in [a, b]$). This solution is given explicitly by

$$\gamma_1 = -\tan\alpha + \frac{6^{-2/3}Q^2 - 6^{-1/3}\rho M_R^4(1 - 2\rho M_R^2 M_\Theta^2)}{\rho M_R^4 Q}, \tag{3.53}$$

for all $a \leqslant r \leqslant b$ with $M_R \in (0, 1]$ and for any value of $\tau = \tau(r) < 0$ with real $Q \equiv Q(r)$ being given by

$$Q^3 = 9\rho^2 M_R^7(\tau M_R + M_\Theta) + 3^{1/2}\rho^{3/2}M_R^6\sqrt{27\rho M_R^2(\tau M_R + M_\Theta)^2 + 2(1 - 2\rho M_R^2 M_\Theta^2)^3}. \tag{3.54}$$

Note that when $\rho = 2$ and $\alpha = \pi/4$ equation (3.52) simplifies to $(\gamma + 1)^3 = \tau + 1$, and hence $\gamma = \gamma_1 = -1 + (\tau + 1)^{1/3}$. This is negative for $\tau < 0$ and yields the same result as the specialization of (3.53) with (3.54).

For $a \leqslant r \leqslant b$, the (unique) deformation function, $g_1(r)$ say, is determined by integration of the equation $rg_1'(r) = \gamma_1$ with the boundary conditions (3.3).

Second, for $\tau < 0$, loss of uniqueness of solution of (3.52) may occur when $\rho \sin^2 2\alpha$ exceeds the value 2. Then, independently of the magnitude of $\tau < 0$, γ is again given by (3.53), but only for values of r for which

$$\text{either} \quad 0 < M_R < M_1 \quad \text{or} \quad M_2 < M_R \leqslant 1, \tag{3.55}$$

where

$$M_1 = \sqrt{\left(\rho - \sqrt{\rho^2 - 2\rho}\right)/2\rho}, \quad M_2 = \sqrt{\left(\rho + \sqrt{\rho^2 - 2\rho}\right)/2\rho}. \tag{3.56}$$

The formula for γ_1 is also valid when $M_R = M_1$ or $M_R = M_2$, including the special case $\tau = -\tan\alpha$, for which $\gamma_1 = -\tan\alpha$ is a triple root.

Non-uniqueness of the roots γ for $\tau < 0$ is possible only if $\rho > 2$ and only for values of r such that $\rho \sin^2 2\alpha > 2$, or equivalently

$$M_1 < M_R < M_2, \tag{3.57}$$

which is, in fact, anticipated on the basis of equation (4.16) in [63].

We now consider the effect of increasing the magnitude of the shear stress τ_θ on the boundary $r = b$, or equivalently of $\tau(b) = \tau_\theta/\mu$. In Figure 3.2 we plot, for a series of given values of $\tau(r)$, the function $\sigma(\gamma) - \tau(r)$ against γ, where $\sigma(\gamma)$ is defined in (3.52). Since $\sigma(0) = 0$ the intercept on the vertical axis is $-\tau(r)$.

For small values of $|\tau(r)|$ the equation $\sigma(\gamma) - \tau(r) = 0$ clearly has a single solution, which is the value γ_1 identified in (3.53). As $|\tau(r)|$ increases a second solution of (3.52) emerges when the curve (the lower dashed curve in Figure 3.2) just touches the horizontal axis. At this point the two roots for γ, denoted γ_2 and γ_3, are given by

$$\gamma_2 = -\tan\alpha + \frac{\sqrt{6}\sqrt{2\rho M_R^2 M_\Theta^2 - 1}}{3\sqrt{\rho} M_R^2}, \tag{3.58}$$

$$\gamma_3 = -\tan\alpha - \frac{\sqrt{6}\sqrt{2\rho M_R^2 M_\Theta^2 - 1}}{6\sqrt{\rho} M_R^2}. \tag{3.59}$$

The value γ_2 is the specialization of γ_1, while γ_3 is the double root associated with the maximum point on the curve.

The corresponding value of $\tau(r)$ at this point is denoted τ_1 and is given by

$$\tau_1 = -\tan\alpha + \frac{\sqrt{6\rho\left(2\rho M_R^2 M_\Theta^2 - 1\right)^3}}{9\rho M_R^2}. \tag{3.60}$$

Note that, in general, α depends on r and so, therefore, does the value τ_1.

As $|\tau(r)|$ increases further then three distinct real roots for γ are obtained. However, Q in (3.54) is now complex and some manipulations are required to rewrite (3.53) in the simplified form

$$\gamma_1 = -\tan\alpha + \frac{6^{-2/3}}{\rho M_R^4}(Q + \bar{Q}), \tag{3.61}$$

where $\bar{Q}$ is the complex conjugate of Q. The other two (real) roots, denoted γ_4 and γ_5, are given similarly by

$$\gamma_4 = -\tan\alpha - \frac{6^{-2/3}}{2\rho M_R^4}[Q + \bar{Q} - \mathrm{i}\sqrt{3}(Q - \bar{Q})], \tag{3.62}$$

$$\gamma_5 = -\tan\alpha - \frac{6^{-2/3}}{2\rho M_R^4}[Q + \bar{Q} + \mathrm{i}\sqrt{3}(Q - \bar{Q})]. \tag{3.63}$$

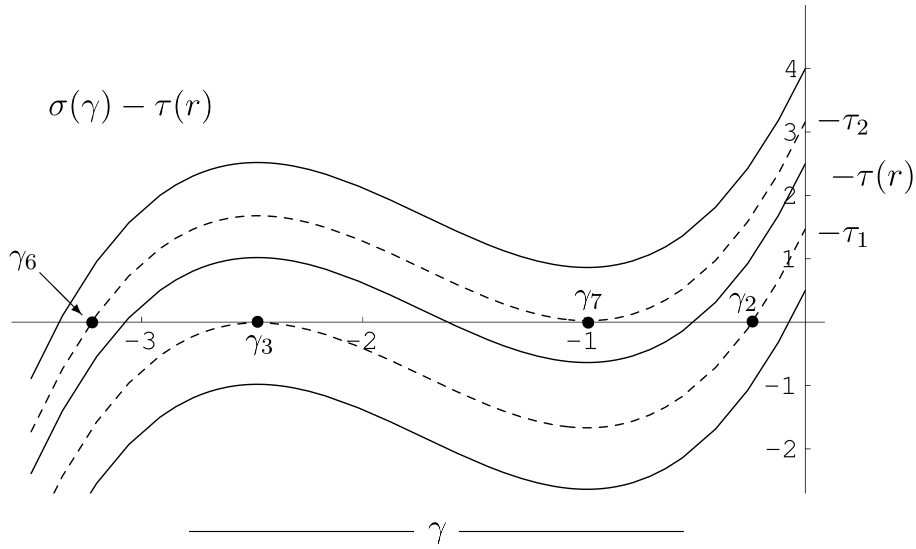

Figure 3.2: Plot of the cubic $\sigma(\gamma) - \tau(r)$ from (3.52) against γ for different values of $\tau(r)$: from bottom to top, $\tau(r) > \tau_1$, $\tau(r) = \tau_1$ (lower dashed curve), $\tau_2 < \tau(r) < \tau_1$, $\tau(r) = \tau_2$ (upper dashed curve), $\tau(r) < \tau_2$. The values $\gamma_2, \gamma_3, \gamma_6, \gamma_7$ are identified by the $\bullet$ symbol, while $\gamma_1, \gamma_4, \gamma_5$ are ordered according to $\gamma_5 \leqslant \gamma_3$, $\gamma_3 \leqslant \gamma_4 \leqslant \gamma_7$, $\gamma_7 \leqslant \gamma_1 \leqslant 0$. Note that $\tilde{W}_{\gamma\gamma} < 0$ for $\gamma_3 < \gamma < \gamma_7$.

It is convenient to label these two roots so that $\gamma_5 \leqslant \gamma_4\,(< 0)$, noting that they are both equal to γ_3 when $\tau(r) = \tau_1$. Then, with reference to Figure 3.2, it is easy to show that for $\tau_2 < \tau(r) < \tau_1$, the following orderings hold: $\gamma_5 < \gamma_3 < -\tan\alpha\,(< 0)$, $\gamma_3 < \gamma_4\,(< 0)$. These imply that $\gamma_5 + \tan\alpha < 0$ (and hence $Q + \bar{Q} > 0$), while $\gamma_4 + \tan\alpha$ may be either positive or negative.

The roots γ_1 and γ_4 merge when $\tau(r)$ reaches the value τ_2 given by

$$\tau_2 = -\tan\alpha - \frac{\sqrt{6\rho(2\rho M_R^2 M_\Theta^2 - 1)^3}}{9\rho M_R^2}, \tag{3.64}$$

which corresponds to the upper dashed curve in Figure 3.2 and depends on r if α does. The two roots in this case, denoted γ_6, γ_7, are given by

$$\begin{aligned} \gamma_6 &= -\tan\alpha - \frac{\sqrt{6}\sqrt{2\rho M_R^2 M_\Theta^2 - 1}}{3\sqrt{\rho} M_R^2}, & (3.65) \\ \gamma_7 &= -\tan\alpha + \frac{\sqrt{6}\sqrt{2\rho M_R^2 M_\Theta^2 - 1}}{6\sqrt{\rho} M_R^2}. & (3.66) \end{aligned}$$

The (double) root γ_7 is the specialization of γ_1 (and γ_4) for this case and γ_6 is the most negative root. Note that for $\tau_2 < \tau(r) < \tau_1$ we have $\gamma_4 < \gamma_7 < \gamma_1\,(< 0)$, $\gamma_6 < \gamma_5$ and $\gamma_7 + \tan\alpha > 0$, while for $\tau(r) < \tau_2$ there is again only one real root for γ, which we label

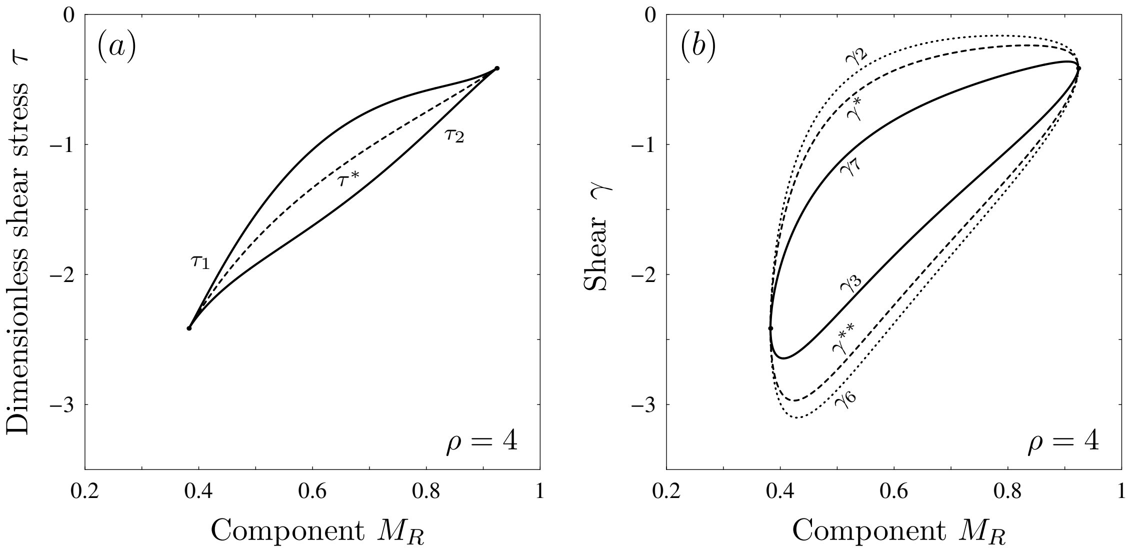

Figure 3.3: (a) Plots of the critical values τ_1 (upper curve) and τ_2 (lower curve) against M_R in (M_R, τ) space for $\rho = 4$; (b) plots of the critical values γ_2, γ_6 (dotted curves) and γ_3, γ_7 (continuous curves) in (M_R, γ) space for $\rho = 4$. The dashed curves and the symbols $\tau^*, \gamma^*, \gamma^{**}$ are identified in Section 3.6.3.

as γ_5 $(< \gamma_6)$. We note in passing that γ_2 and γ_6 correspond to the same value of I_4, which may be written $1 - \gamma_2\gamma_6 \cos^2\alpha$. Similarly, for γ_3 and γ_7 we have $I_4 = 1 - \gamma_3\gamma_7 \cos^2\alpha$. Moreover, I_4 is less for γ_3 than for γ_2. Note that $\gamma_6 > -2\tan\alpha$ provided (3.50) holds.

Since τ and (in general) α depend on r the above results are purely local, i.e. they apply for fixed values of r. While $|\tau|$, as defined in (3.51), is a decreasing function of r, neither τ_1 nor τ_2 is in general a monotonic function of α. Thus, the disposition of possible shear strains for $r \in (a, b)$ can be quite complex. To illustrate to possibilities we show, in Figure 3.3(a), the nature of the values τ_1 and τ_2 defined by (3.60) and (3.64), respectively, as functions of M_R. In Figure 3.3(b) the corresponding values γ_2, γ_3 and γ_6, γ_7 are shown, in each case for $\rho = 4$. If α is a monotonic increasing (decreasing) function of r, and therefore M_R monotonic decreasing (increasing), the curves in Figure 3.3 can be interpreted as illustrating qualitatively the dependence of these values on the radius r.

Clearly, the values τ_1 and τ_2 are critical for determining the existence of multiple values for γ and hence non-unique continuous deformation fields $g(r)$ in (3.2). For $\tau(r)$ between τ_1 and τ_2, non-uniqueness is possible. Subject to the restriction $\rho \sin^2 2\alpha < 8$, an increase in the parameter $\rho > 2$ corresponds to expansion of the domain where non-uniqueness of γ is possible since τ_1 is a monotonic increasing function and τ_2 a monotonic decreasing function of ρ. For comparison with Figure 3.3, Figure 3.4 shows corresponding results for

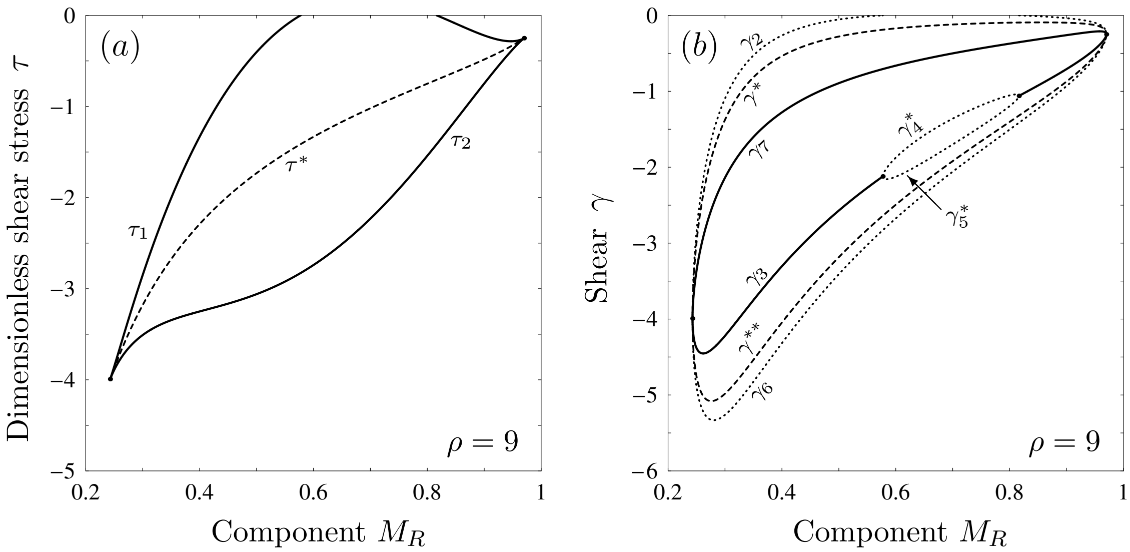

Figure 3.4: (a) Plots of the critical values τ_1 (upper curve) and τ_2 (lower curve) against M_R in (M_R, τ) space for $\rho = 9$; (b) plots of the critical values γ_2, γ_6 (dotted curves) and γ_3, γ_7 (continuous curves) in (M_R, γ) space for $\rho = 9$. The interval $M_R \in [\sqrt{3}/3, \sqrt{6}/3]$ is excluded since the inequality (3.50) is violated. The inadmissible shear strains γ_4^* and γ_5^* are within this interval. The dashed curves and the symbols $\tau^*, \gamma^*, \gamma^{**}$ are identified in Section 3.6.3.

$\rho = 9$. For this value of ρ we have to ensure that the inequality (3.50) is satisfied. It is provided $M_R < \sqrt{3}/3 \approx 0.577$ or $M_R > \sqrt{6}/3 \approx 0.817$. In Figure 3.4(a) the τ_1 curve cuts the axis $\tau = 0$ at the values $M_R = \sqrt{3}/3, \sqrt{6}/3$ and the interval $M_R \in [\sqrt{3}/3, \sqrt{6}/3]$ is therefore excluded from consideration. This applies equally in Figure 3.4(b), which, for the same interval, reveals inadmissible values for γ, denoted γ_4^* and γ_5^*, that are obtained from the formulas (3.62) and (3.63) for γ_4 and γ_5 with (3.54) for $\tau = 0$. Note that τ_1 is not here a monotonic function of M_R.

We remark that, for a certain range of values of $\rho > 2$, τ_1 is monotonically increasing in the component M_R, as is evident in the example in Figure 3.3. This means that loss of uniqueness is initiated for points with M_R close to M_2 since the latter is associated with the minimum possible value of $|\tau_1|$. More specifically, when ρ exceeds the approximate value 5.39, τ_1 loses its monotonic character; however, the least value of $|\tau|$ for which loss of uniqueness first occurs is still close to $M_R = M_2$ until ρ reaches the approximate value 6.19. Thereafter, multiplicity of choices for γ first occurs away from M_2, and, in particular, for $M_R \approx 0.764 < M_2 \approx 0.956$, while any further increase in ρ results in a decrease in the value of M_R for which $|\tau_1|$ is minimized.

The dashed curves in Figures 3.3 and 3.4 have a special significance, which will be discussed in Section 3.6.3.

In order to simplify the discussion of multiplicity, we now focus on the case in which α is independent of r. Then, τ_1 and τ_2 are also independent of r, and, with reference to Figure 3.2, we may determine the effect of increasing $|\tau(r)|$ as follows.

We note that $|\tau(r)|$ has its largest value at $r = a$. For $0 > \tau(a) > \tau_1$ there is a unique value $\gamma = \gamma_1$ that applies for $a \leqslant r \leqslant b$. For $\tau(a) = \tau_1$ this value ($= \gamma_2$) again applies for $a \leqslant r \leqslant b$, but a second value γ_3 becomes possible at $r = a$. For $\tau(b) > \tau_1 > \tau(a) > \tau_2$, three values are possible, corresponding to the intercepts of the central curve in Figure 3.2 with the horizontal axis. These are γ_1 in (3.61) and γ_4 and γ_5 in (3.62) and (3.63), respectively, labelled so that $\gamma_5 < \gamma_4 < \gamma_1$. There is then a value of $r \in (a, b)$, r^* say, such that $\tau(r^*) = \tau_1$. For $r > r^*$ the value γ_1 is the only one possible, but for $r < r^*$ three values, namely γ_1, γ_4 and γ_5, are possible. As we shall discuss further in Section 3.6.3, any use of the choice γ_4 in the construction of the function $g(r)$ gives rise to an energetically unstable deformation by all conventional stability criteria. Hence the choice γ_4 will not be admitted. Thus, for $r < r^*$ only the two values γ_1 and γ_5 are admissible. Even so, the possibility of γ having a jump from γ_1 to γ_5 arises, whereupon the determination of the point or points at which such a jump occurs requires further discussion. This will also be provided in Section 3.6.3 in relation to the stability status of the different values of γ. Such jumps are called *elastostatic shocks* in [2] and, in the context of fiber reinforced materials, are referred to as *kink surfaces* in [52, 53, 20]. With further increase in $|\tau_\theta|$, the possibility of two admissible values of γ is retained provided $\tau(b)$ reaches the value τ_1 before $\tau(a)$ reaches the value τ_2. If, however, $\tau(a)$ reaches τ_2 before $\tau(b)$ reaches τ_1 a further increase in τ_θ will generate two circles, of radii r^* and $r^{**} < r^*$, say, such that there is only a single value (γ_5) for $a < r < r^{**}$, two admissible values (γ_1 and γ_5) for $r^{**} < r < r^*$ and only one (γ_1) for $r^* < r < b$.

The situation described above mirrors that studied by Abeyaratne [2] in the context of the isotropic problem with a non-monotone shear stress response function. As shown for the isotropic material problem in [2], it will generally be the case that smooth solutions will not exist for certain values of ψ, which in turn motivates the explicit need for deformations involving such a discontinuity surface. Once such elastostatic shock solutions are admitted, one then typically obtains a multiplicity of solutions for certain values of ψ whereupon the issue of selecting solutions of physical significance becomes central to further progress. As elaborated in Section 3.6.3 we shall here follow [2] by invoking an absolute stability selection criterion that essentially selects global energy minimizers within the class of azimuthal shear deformations (2). As discussed further in what follows, such solutions have an equivalent interpretation of dissipation free shock motion when the problem is viewed quasi-statically for a time varying ψ. As shown in [2], an immediate consequence is that a unique solution $g(r)$, which may or may not involve an elastostatic shock, follows for each value ψ and the same solution is obtained for the quasi-static interpretation of the problem irrespective of whether ψ is increasing or decreasing. Indeed, Abeyaratne provides full details for associating any boundary value ψ to such a solution for the isotropic problem. In the present analysis we do not provide the same focus on mapping the boundary value ψ onto solutions and instead refer the reader to [2] for more detail on how to treat this aspect of the problem. This allows us to retain our focus on the new issues pertaining to the effect of the fiber reinforcement associated with transverse isotropy.

The possibilities just described are reflected in Figure 3.3, for example. For values of M_R between $M_1 = (\sqrt{2-\sqrt{2}})/2 \approx 0.383$ and $M_2 = (\sqrt{2+\sqrt{2}})/2 \approx 0.924$ (for $\rho = 4$), we see that as τ decreases from zero there is initially one value for γ, namely $\gamma_1 > \gamma_2$. Two values, γ_2 and γ_3, become possible when τ reaches τ_1. As τ decreases further, three values of γ become possible: one (γ_1) is found between γ_2 and γ_7, a second (γ_4) between γ_7 and γ_3 and a third (γ_5) between γ_3 and γ_6. The value γ_4, as mentioned already, is not admissible. After τ reaches τ_2 (and γ reaches γ_7) there is again uniqueness. Thus, with reference to Figure 3.3(b), we see that outside the closed curve defined by γ_2 and γ_6 the value of γ is uniquely determined, while inside this curve two admissible choices (γ_1 and γ_5) are possible.

In constructing any such discontinuity surface across which the shear strain γ jumps between γ_1 and γ_5 it is to be remarked that there is no associated change in the value of τ_θ even though τ_θ appeared originally as an integration constant in (3.31). Continuity of τ_θ is necessary for continuity of $\sigma_{r\theta}$. Continuity of σ_{rr} follows from (3.33), which in turn ensures traction continuity across the discontinuity surface.

The case $\alpha = 0$. Finally in this section, we consider the exceptional case for which the preferred direction is taken to be radial for $r \in [a, b]$, i.e. $\alpha \equiv 0$. Then γ may be computed from (3.53) for any τ (positive or negative) and $\rho > 0$, whilst (3.54) reduces to

$$Q^3 = 9\rho^2\tau + \sqrt{81\rho^4\tau^2 + 6\rho^3}. \tag{3.67}$$

After some manipulation it can be shown that $\gamma = \gamma_1$ has the form

$$\gamma_1 = 3\frac{6^{1/3}\rho\tau}{q_+^{2/3} + q_-^{2/3} + 6^{1/3}\rho}, \tag{3.68}$$

where $q_\pm = \sqrt{81\rho^4\tau^2 + 6\rho^3} \pm 9\rho^2\tau$. The antisymmetry of γ_1 with respect to the change of sign of τ is then apparent. The equation $rg_1'(r) = \gamma_1$ is solved to give

$$g_1(r) = L(r) - L(a) \tag{3.69}$$

for $a \leqslant r \leqslant b$, where

$$\sqrt{2\rho}L(r) = -\frac{\sqrt{3}}{2}(x - x^{-1}) + \tan^{-1}\left(\frac{x - x^{-1}}{\sqrt{3}}\right), \tag{3.70}$$

$x\,(>0)$ is defined by $x = Q/(6\rho^3)^{1/6}$, $Q > 0$, and we recall that $\tau = \tau_\theta b^2/\mu r^2$.

3.6.2 Loss of ellipticity

The loss of uniqueness discussed in the foregoing section is closely related to loss of ellipticity, and the connection will be elaborated in the present section. Strict local convexity of $\tilde{W} \equiv \bar{\bar{W}}$ as a function of γ is equivalent to the strong ellipticity condition $\tilde{W}_{\gamma\gamma} > 0$, and for the considered reinforced neo-Hookean model (3.46) this yields

$$6\rho M_R^4\gamma^2 + 12\rho M_R^3 M_\Theta\gamma + 4\rho M_R^2 M_\Theta^2 + 1 > 0. \tag{3.71}$$

This holds for $\gamma > 0$, while for it to hold for all γ it is necessary and sufficient that

$$\rho \sin^2 2\alpha < 2. \tag{3.72}$$

Thus, bearing in mind the restriction (3.50), failure of ellipticity requires

$$2 \leqslant \rho \sin^2 2\alpha < 8. \tag{3.73}$$

Note, in particular, that the left-hand inequality in (3.73) cannot hold unless $\rho \geqslant 2$. For $\rho = 2$ only equality can hold and it requires that $\alpha = \pi/4$.

Failure of the inequality (3.71) can occur for those r for which $M_1 \leqslant M_R \leqslant M_2$ and only when γ is such that $\gamma_3 \leqslant \gamma \leqslant \gamma_7$. In particular, $\tilde{W}_{\gamma\gamma} = 0$ for $\gamma = \gamma_3$ and $\gamma = \gamma_7$, which correspond to $\tau = \tau_1$ and $\tau = \tau_2$, respectively. Thus, the emergence of a second value for γ when $\tau = \tau_1$ coincides with loss of ellipticity. For the case of constant α, as $|\tau|$ increases ellipticity fails first on $r = a$ and thereafter on a circle of radius $r = r^*$, which increases until $r = b$ is reached. The (unique) value γ_1 applies for $r^* < r < b$, while for

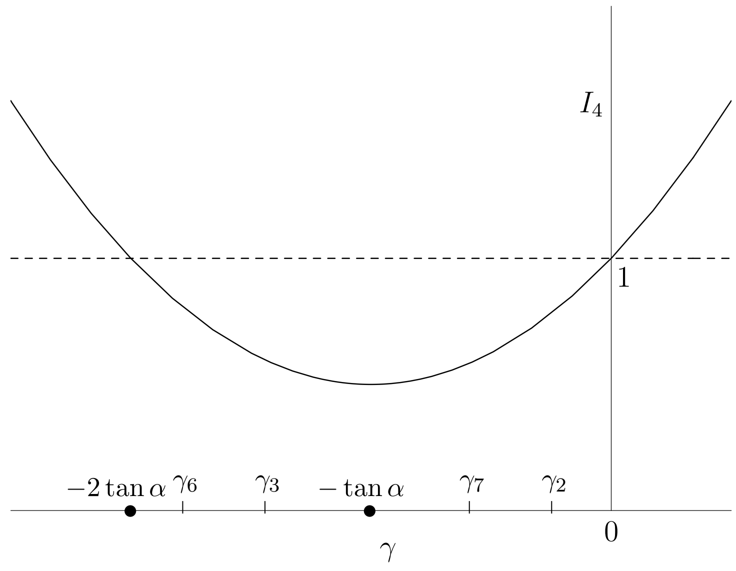

Figure 3.5: Plot of the invariant I_4 against γ showing the relative positions of $\gamma_2, \gamma_3, \gamma_6, \gamma_7$. The value $I_4 = 1$ is shown as the dashed line, which cuts the I_4 curve at $\gamma = -2\tan\alpha$. The minimum of I_4 occurs at $\gamma = -\tan\alpha$. The values $-\tan\alpha$ and $-2\tan\alpha$ are indicated by the symbol $\bullet$ on the γ axis.

$a < r < r^*$ an alternative value is possible, i.e. γ_1 can jump to γ_5. For each of γ_1 and γ_5 strong ellipticity holds (i.e. the slope of the central curve in Figure 3.2 is positive for each of these values). The middle value γ_4 is not admissible since at this point $\tilde{W}_{\gamma\gamma} < 0$, i.e. it is unstable. Indeed, $\tilde{W}_{\gamma\gamma} < 0$ for $\gamma_3 < \gamma < \gamma_7$, as can be seen in Figure 3.2. For the model under examination, loss of uniqueness of the solution of (3.52) for $\gamma < 0$ implies failure of strong ellipticity, but the converse is not true in general since, for $\rho \sin^2 2\alpha = 2$, the roots for γ all coincide at a horizontal point of inflection ($\gamma = -\tan\alpha$). Such a situation corresponds to a weak discontinuity, with γ continuous but $\mathrm{d}\gamma/\mathrm{d}r$ discontinuous at the value of r in question. This can happen only for $M_R = M_1$ or $M_R = M_2$ with $\gamma_3 = \gamma_7 = -\tan\alpha\,(= \tau_1 = \tau_2)$.

We now examine the ellipticity status in terms of the invariant I_4 since it is clear that breakdown of ellipticity is always associated with $I_4 < 1$. The relative placements of the values $\gamma_6, \gamma_3, \gamma_7, \gamma_2$ are shown in Figure 3.5 together with a plot of the invariant I_4 as a function of γ. For the considered material model and deformation, $\tilde{W}_{\gamma\gamma} < 0$ for $\gamma_3 < \gamma < \gamma_7$, and $I_4 < 1$ for all r for which $\gamma_3 \leqslant \gamma \leqslant \gamma_7$ holds. On the other hand, $I_4 < 1$ does not, in general, imply $\tilde{W}_{\gamma\gamma} < 0$. Indeed, $\tilde{W}_{\gamma\gamma} > 0$ for either $-2\tan\alpha < \gamma < \gamma_3$ or $\gamma_7 < \gamma < 0$, for which intervals $I_4 < 1$.

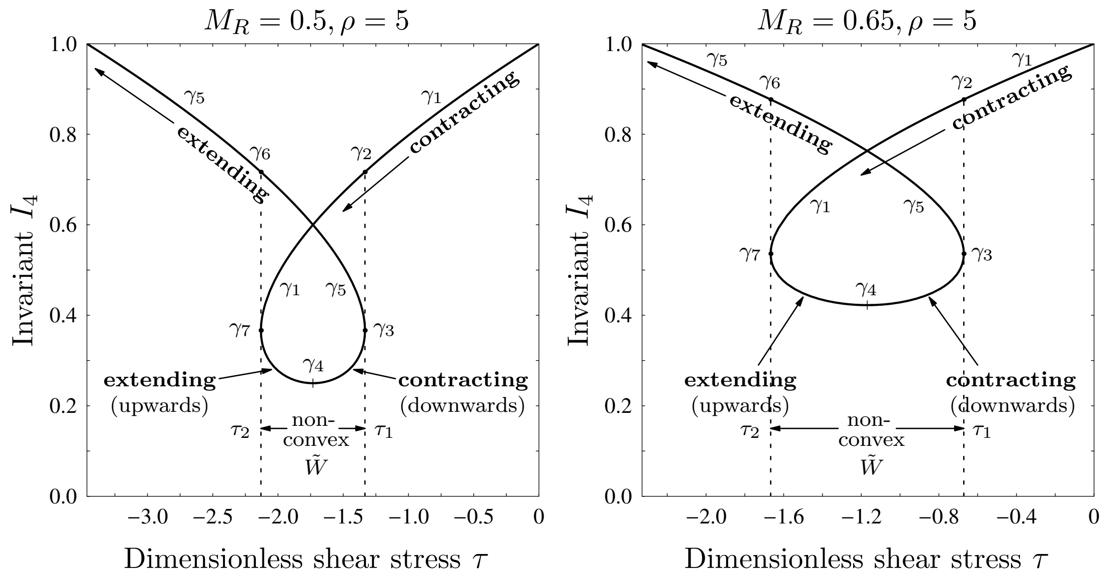

Figure 3.6: Plots of the invariant I_4 against $\tau \leqslant 0$ for $M_R = 0.5, 0.65$ with $\rho = 5$ showing the locations of the different γ values. Note that γ_4 is located in the region of non-convex $\tilde{W}$.

Thus, it is generally the case that if loss of ellipticity takes place it will first occur before I_4 reaches its minimum value. The exception is for the non-generic situation in which $\gamma_7 = \gamma_4 = \gamma_3 = -\tan\alpha$, corresponding to $M_R = M_1 = M_2$. In such a situation, any subsequent change in the boundary condition generally causes I_4 to cease to be at its minimum value, whereupon ellipticity is regained. An example of this transient loss of ellipticity is presented in Section 3.8.2. In Figure 3.6, for $\rho = 5$ and $M_R = 0.5, 0.65$, the dependence of $I_4 \leqslant 1$ on $\tau \leqslant 0$ is plotted, with the locations of the different values for γ identified.

3.6.3 Energy minimal solutions

Consider a programme of loading such that σ and correspondingly γ decreases from zero. For $-\sigma < -\tau_1$, as we have seen, there is only one solution of (3.52) for γ, namely the root γ_1. We focus on the values of σ such that $-\tau_1 < -\sigma < -\tau_2$, when there are two roots, γ_1 and $\gamma_5 < \gamma_1$. With reference to Figure 3.7, let τ^* be the value of σ for which the horizontal line $\sigma = \tau^*$ cuts the σ curve to form two closed regions with equal areas. This is, of course, the well-known Maxwell line. Let γ^* and $\gamma^{**} < \gamma^*$ be the corresponding values of γ_1 and γ_5. The Maxwell line is therefore defined by the equality (viz. equations

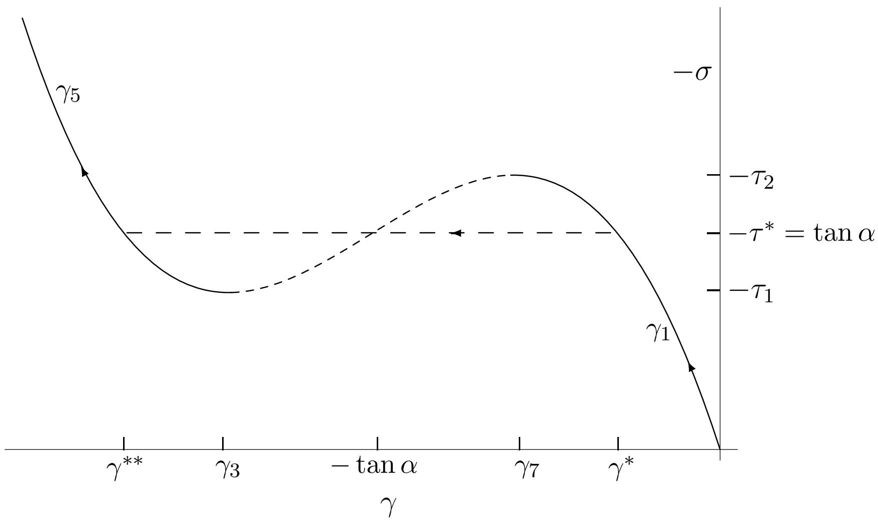

Figure 3.7: Representative plot of $-\sigma(\gamma)$, as given by (3.52), against $\gamma\,(< 0)$. As $-\sigma$ increases from zero the continuous curve is followed until $-\sigma$ reaches the value $-\tau^* > -\tau_1$, at which point this path loses stability and the solution jumps to the left-hand continuous part of the curve, which is stable as $-\sigma$ increases further. The stable path is indicated by the arrows. The horizontal dashed line at $\sigma = \tau^*$ is the Maxwell line, for which the two closed regions cut off the curve have equal areas. The dashed part of the curve and the continuous parts for $\gamma_7 \leqslant \gamma_1 < \gamma^*$ and $\gamma^{**} < \gamma_5 \leqslant \gamma_3$ correspond to unstable solutions.

(3.2)–(3.4) in [17])

$$\tilde{W}(\gamma^{**}) - \tilde{W}(\gamma^*) - \tau^*(\gamma^{**} - \gamma^*) = 0. \tag{3.74}$$

The dashed curves in Figures 3.3(a) and 3.4(a) are plots of the relevant values of τ^* for the examples therein against M_R, and in Figures 3.3(b) and 3.4(b) the associated curves of γ^* and γ^{**} are shown.

For solutions containing a discontinuity surface involving transition between γ_1 and γ_5, the Maxwell stress τ^* provides the only value of σ at which such a surface can be located if the solution is to be stable in an absolute sense. In the event that τ^* is independent of r, it then follows from (3.51) and (3.52) that there is at most one radial location at which $\sigma = \tau^*$, and this location varies with the applied external shear stress $\tau(b)$ or, equivalently, the twist ψ in (3.3). This is the case for homogeneous, isotropic materials, as discussed in Abeyaratne [2], and also for the materials considered here provided that all constitutive parameters (including α) are independent of r. More general possibilities apply if α depends on r.

The sense in which such solutions are absolutely stable is as described in [2], namely,

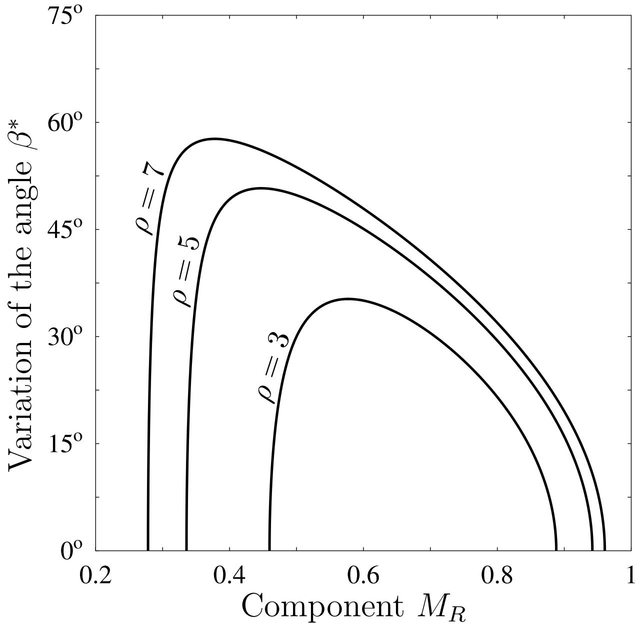

Figure 3.8: Plot of the angle $\beta^*(>0)$ against M_R for $\rho = 3, 5, 7$.

for the same boundary conditions, such a solution minimizes the overall energy with respect to all other deformations, either smooth solutions or those containing one or more discontinuity surfaces. This eliminates consideration of the branch of solutions associated with any descending branch in Figure 3.2, i.e. γ_4, and also eliminates γ_1 if $\gamma_7 < \gamma_1 < \gamma^*$ and γ_5 if $\gamma^{**} < \gamma_5 < \gamma_3$. It is worth observing here that such unstable γ_1 and γ_5 can be regarded as metastable in the sense that solutions involving these values are minimizers with respect to continuously differentiable variations in the twist function $g(r)$ in (3.2). Since, however, such continuously differentiable variations permit neither the formation of new discontinuity surfaces nor the alternative placement of existing discontinuity surfaces, they do not address the absolutely stable solutions that we consider here. Hence the condition for the stability of the shear strain γ_1 (and instability of γ_5) is

$$\tilde{W}(\gamma_5) - \tilde{W}(\gamma_1) - \sigma(\gamma_5 - \gamma_1) > 0, \tag{3.75}$$

where $\sigma = \sigma(\gamma_1) = \sigma(\gamma_5)$. In the context of isotropic elasticity the stability analysis has been discussed in detail by Abeyaratne [2] and the present situation follows closely that analysis.

We remark here that equation (3.74) is the specialization of the admissibility condition $[\![W\mathbf{I} - \mathbf{F}^{\mathrm{T}}\mathbf{S}^{\mathrm{T}}]\!]\mathbf{N} = \mathbf{0}$ for equilibrium shocks [3], where $[\![\ ]\!]$ indicates the jump in the enclosed quantity across the surface with unit normal $\mathbf{N}$ in the reference configuration, $\mathbf{I}$

is the identity tensor and $\mathbf{S} = \mathbf{F}^{-1}\boldsymbol{\sigma}$ is the nominal stress tensor. This guarantees that quasi-static motion of an equilibrium shock (i.e. the surface of strain discontinuity) is dissipation free.

It is easily shown from (3.52) that $\sigma(-\gamma - 2\tan\alpha) + 2\tan\alpha = -\sigma(\gamma)$, which means that $\sigma(\gamma)$ is antisymmetric about the point $\gamma = -\tan\alpha, \sigma = -\tan\alpha$. It follows that (3.74) provides the explicit Maxwell values $\tau^* = -\tan\alpha$ and

$$\gamma^* = -\tan\alpha + \frac{\sqrt{2\rho M_R^2 M_\Theta^2 - 1}}{\sqrt{2\rho}M_R^2}, \quad \gamma^{**} = -\tan\alpha - \frac{\sqrt{2\rho M_R^2 M_\Theta^2 - 1}}{\sqrt{2\rho}M_R^2}. \tag{3.76}$$

Note that γ^* is negative by virtue of (3.73).

Now, from $(3.2)_2$, $(3.7)_1$, (3.9) and (3.10), we find that

$$r\frac{\mathrm{d}\theta}{\mathrm{d}r} = rG'(r) + rg'(r) = \tan\alpha + \gamma = \tan\beta, \tag{3.77}$$

where $\tan\beta = m_\theta/m_r$, i.e. β is the angle (measured counterclockwise) between the deformed fiber direction and the radial direction $\mathbf{e}_r$. Let β^* and β^{**} be the values of β corresponding to the values γ^* and γ^{**} in (3.76). It follows that $\tan\beta^* > 0$, $\tan\beta^{**} < 0$ and $\beta^{**} + \beta^* = 0$. This means that the deformed fiber directions on the two sides of the circle of discontinuity are symmetrically disposed relative to the $\mathbf{e}_r$ direction. Note that $\mathbf{e}_r$ is normal to the direction of shear. A similar symmetry arises in the rectilinear shear problem examined by Merodio et al. [35]. Note that β^* (and hence β^{**}) is non-monotonic as a function of $M_R = \cos\alpha$ and has a maximum (minimum) at $M_R = 1/\sqrt{\rho}\,(\rho > 2)$. Figure 3.8 shows the behaviour of β^* as a function of M_R for three different values of $\rho > 2$.

We close this section with a remark regarding the issue of selecting solutions since it is worth mentioning that other means for selecting solutions involving an elastostatic shock are also possible [53, 4]. Such alternative selection criteria have received a great deal of attention in the recent literature, especially as it relates to the continuum mechanical modelling of solid-solid phase transformations. In these alternative resolutions, it is necessary to ensure that the selection criterion is consistent with the second law of thermodynamics [45]. Furthermore, unlike the absolutely stable solutions considered here, such alternative resolutions would typically provide some hysteresis in the solution dependence on ψ.

3.7 The reinforced Varga model

An alternative representation for the strain-energy function is examined at this point. Instead of I_1, we use the principal invariant i_1 (= tr $\mathbf{V}$) of the stretch tensor $\mathbf{V} = \mathbf{B}^{1/2}$. Analogously to (3.42), we consider the class of strain-energy functions

$$\breve{W}(i_1, I_4) = \breve{E}(i_1) + F(I_4), \tag{3.78}$$

and we note the connection $I_1 = i_1^2 - 2i_1$. Then,

$$\hat{W}(I_1, I_4) = \tilde{W}(\gamma, \alpha) = \breve{W}(i_1, I_4), \quad E(I_1) = \breve{E}(i_1). \tag{3.79}$$

For the problem under examination, we then have, on use of $(3.11)_1$,

$$i_1 = 1 + \sqrt{4 + \gamma^2}. \tag{3.80}$$

As a particular example, we now focus on the so-called Varga model, defined by

$$\breve{E}(i_1) = 2\mu(i_1 - 3) = 2\mu\sqrt{4 + \gamma^2} - 2 = \tilde{E}(\gamma), \tag{3.81}$$

augmented by the same reinforcement $(3.47)_2$ in order to characterize the response of a transversely isotropic circular cylindrical tube under the pure azimuthal shear deformation. As for the neo-Hookean model, the parameter μ involved in (3.81) is the shear modulus of the isotropic base material, and the counterpart of (3.46) is

$$\breve{W}(i_1, I_4) = \frac{1}{2}\mu\left[4(i_1 - 3) + \rho(I_4 - 1)^2\right]. \tag{3.82}$$

Since $\tilde{E}'(\gamma) = 2\mu\gamma/\sqrt{1+\gamma^2}$ the monotonic nature of the function $\sigma(\gamma)$ associated with the Varga base material is apparent. Note, however, that, in contrast to the neo-Hookean base material $\tilde{E}'$ tends to a finite value as $\gamma \to \infty$.

The azimuthal equation (3.31) now specializes to

$$\sigma(\gamma) \equiv 2\gamma(4 + \gamma^2)^{-1/2} + 2\rho M_R^2\gamma(M_R^2\gamma^2 + 3M_R M_\Theta\gamma + 2M_\Theta^2) = \tau(r). \tag{3.83}$$

Note that the finite asymptote for $\sigma(\gamma)$ persists for either $\rho = 0$ or $M_R = 0$. More generally, however, $\sigma(\gamma) \to \pm\infty$ as $\gamma \to \pm\infty$. Unlike its counterpart (3.52), equation (3.83) does not admit explicit expressions for γ, so that necessary and sufficient conditions for the uniqueness of γ are not in general obtainable in closed form. However, for $\gamma > 0$ it is easy to show that $\sigma(\gamma)$ is a monotonic increasing function, and hence γ is determined uniquely for $\tau > 0$. Possible loss of uniqueness is, as for the model (3.46), strictly associated with negative γ. When $\tau(r) < 0$ equation (3.83) may have multiple roots for $\gamma\,(< 0)$ even for very small values of the parameter ρ. As expected, an increase in ρ results in a decrease in the value of τ for which loss of uniqueness is initiated. In other words, we see that the analogues of $\tau_1(r)$ and $|\tau_2(r)|$, denoted $\breve{\tau}_1(r)$ and $|\breve{\tau}_2(r)|$, respectively, are monotonically increasing with ρ, so that larger values of ρ result in an expansion of the region encompassing multiple choices for γ.

It is therefore clear that the general qualitative properties of the boundary $\breve{\tau}_1(r)$ are very similar to those of $\tau_1(r)$. Let us quantify the maximal set of points for which multiplicity of γ occurs, analogously to (3.57), by the inequalities

$$\breve{M}_1 < M_R < \breve{M}_2, \tag{3.84}$$

where $\breve{M}_1 = \breve{M}_1(\rho)$ and $\breve{M}_2 = \breve{M}_2(\rho)$. Then, for any fixed $\rho \geqslant 2$ we obtain the nesting

$$\breve{M}_1 < M_1 \leqslant M_2 < \breve{M}_2, \tag{3.85}$$

which, for all relevant r, leads to

$$\tau_1(r) < \breve{\tau}_1(r) < 0, \tag{3.86}$$

but the relative disposition of $\tau_2(r)$ and $\breve{\tau}_2(r)$ is not immediately clear and depends on r.

In the limiting cases where $\mathbf{M}$ is either radial or circumferential uniqueness of γ is, as for the model (3.46), guaranteed. For $M_R = 0$, (3.83) may be solved to give

$$\gamma = 2\tau(4 - \tau^2)^{-1/2}, \tag{3.87}$$

γ and τ having the same sign, so there is clearly an upper bound on $|\tau|$ for which the considered deformation is admissible, a point observed previously for the purely isotropic Varga model (see, for example, the general discussion in [39], which includes results for the Varga material as a special case). For $M_\Theta = 0$, on the other hand, a closed-form solution of (3.83) is not obtainable.

The variations of the boundaries $\breve{\tau}_1(r)$ and $\breve{\tau}_2(r)$ with respect to the component M_R are illustrated in Figure 3.9 for three fixed values of ρ. Also shown are the plots of the Maxwell stress $\breve{\tau}^*(r)$ against M_R.

The requirement for (3.82) to be strongly elliptic is

$$\rho M_R^2(3M_R^2\gamma^2 + 6M_R M_\Theta \gamma + 2M_\Theta^2)(4+\gamma^2)^{3/2} + 4 > 0, \tag{3.88}$$

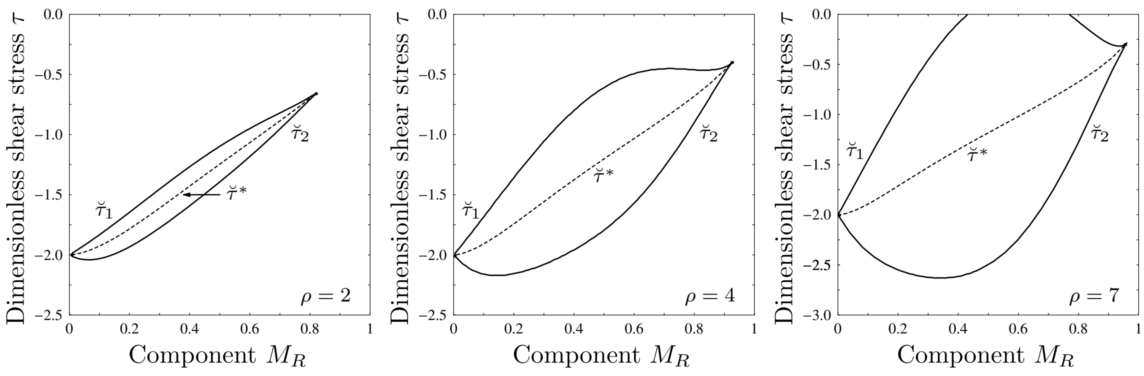

Figure 3.9: Plots of the limiting functions $\breve{\tau}_1$ and $\breve{\tau}_2$ against the component M_R for $\rho = 2, 4, 7$ for the reinforced Varga material. The dashed curves are plots of the Maxwell value $\breve{\tau}^*$.

which is, as for its counterpart (3.71), automatically satisfied throughout the body for positive γ. For negative γ, on the other hand, violation of (3.88) may now occur for any $\rho > 0$, although the associated non-strongly-elliptic domain cannot be identified explicitly. Numerical calculations, however, show that the correlation between ellipticity failure and the existence of multiple values for γ is entirely analogous to that for the reinforced neo-Hookean model. Thus, the properties

$$\breve{\tau}_2(r) \leqslant \tau(r) \leqslant \breve{\tau}_1(r) \tag{3.89}$$

associated with (3.84) serve to characterize the domain for which multiple values of γ arise and on the boundary of which ellipticity fails.

It is worth mentioning here that, as for (3.71), breakdown of (3.88) is also possible at the limiting values $\breve{M}_1$ and $\breve{M}_2$, although there is no associated discontinuity in γ. This again corresponds to the emergence of a weak discontinuity in γ, i.e. a discontinuity in $\mathrm{d}\gamma/\mathrm{d}r$, at the appropriate value of r. Finally, we note that the possibility that (3.83) yields negative, and therefore physically inadmissible, values of γ for positive $\tau(r)$ may also arise here. More specifically, for ρ greater than about 6 there exists a subinterval of (3.84) in which multiplicity of γ, in conjunction with loss of ellipticity, occurs. For this subinterval, which varies with ρ, the boundaries of the domain where positive shear stress results in negative γ are determined via equation (3.83) by resolving the latter in the limit $\tau \to 0^-$. In the case $\rho = 7$, illustrated in the right-hand plot in Figure 3.10, such a subinterval can be observed.

The ellipticity status of the material model (3.82) is illustrated in Figure 3.10 for three values of ρ. The notations $\breve{\gamma}_2, \breve{\gamma}_3, \breve{\gamma}_6, \breve{\gamma}_7$ and $\breve{\gamma}_4^*, \breve{\gamma}_5^*$ have been adopted in parallel with

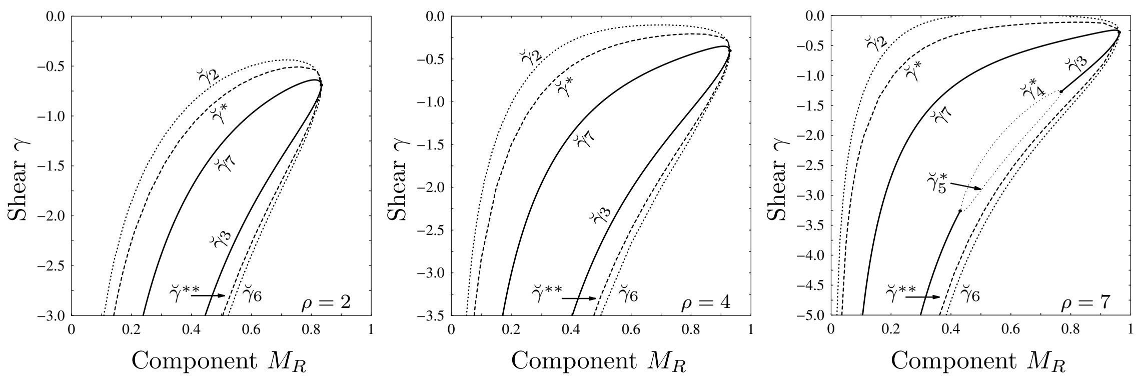

Figure 3.10: Characterization of the boundaries of the strongly elliptic domain (dotted curves) for the material model (3.82) in terms of the shear γ (vertical axis) and M_R (horizontal axis) for $\rho = 2, 4, 7$: the curves are given by $\breve{\gamma}_2$ and $\breve{\gamma}_6$. Also shown (continuous curves) are $\breve{\gamma}_3$ and $\breve{\gamma}_7$, within which the inequality (3.88) is reversed. The dotted loop in the right-hand plot is defined by the inadmissible values $\breve{\gamma}_4^*$ and $\breve{\gamma}_5^*$. Also shown for $\rho = 2, 4, 7$ are the (dashed) curves of the Maxwell values $\breve{\gamma}^*$ and $\breve{\gamma}^{**}$ corresponding to $\breve{\tau}^*$.

those used in Section 3.6.3. In particular, the counterparts of Figures 3.3(b) and 3.4(b) are plotted for $\rho = 2, 4, 7$ in order both to highlight the connection between the existence of multiple values for γ and the notion of strong ellipticity. Plots of the values of $\breve{\gamma}^*, \breve{\gamma}^{**}$ associated with the Maxwell stress $\breve{\tau}^*$ are also shown.

3.8 Numerical examples and discussion

In this section we illustrate some aspects of the response of the model (3.46) under the considered deformation. For numerical purposes we fix the radial dimensions at $a = A = 1$ (units) and $b = B = 6$ (units), while the parameters ρ and $\tau(b)$ are specified separately for each example. Our main aim is to highlight the influence of the anisotropy parameter ρ and the preferred direction $\mathbf{M}$ on the overall response of the body.

3.8.1 Radial reinforcement

First, we examine the simple case in which $\mathbf{M}$ is radial. Then, bearing in mind that (3.46) is convex as a function of γ in this case, the solution for γ is unique and smooth. We observe that $|\gamma|$ is a monotonic decreasing function of r, while, as expected, the corresponding solution $|g(r)|$ increases with r. Furthermore, for fixed $\tau(b)$, an increase in ρ results in a decrease in the value of $|\gamma|$ and hence of $|g(r)|$ at any point of the body,

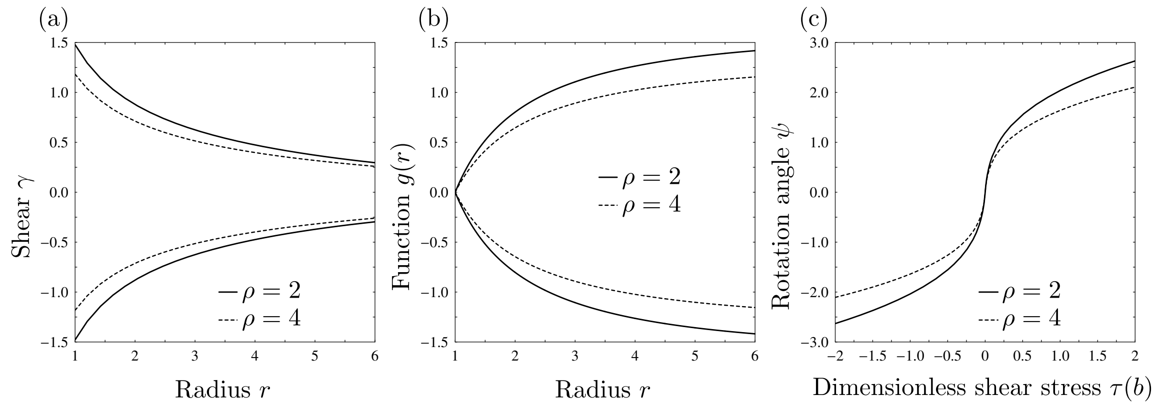

Figure 3.11: Plots of (a) the shear γ, and (b) the function $g(r)$ against the radius r for the reinforced neo-Hookean model (3.46) with $\rho = 2, 4$, $\mathbf{M} = \mathbf{E}_R$ and $\tau(b) = \pm 0.4$. In (c) the resulting rotation angle $\psi = g(b)$ is plotted against $\tau(b)$ for $\rho = 2, 4$.

whilst larger values of $\tau(b)$ yield larger strains. Clearly, because of the nature of the radial anisotropy considered here, the material response is the same for either sense of the shear. In Figure 3.11 we plot (a) the amount of shear γ and (b) the associated rotation function $g(r)$ against the radius r for $\rho = 2, 4$ while in (c) the dependence of the rotation angle ψ on $\tau(b)$ is illustrated, again for $\rho = 2, 4$. Both positive and negative shears are included so as to compare, in subsequent sections, with the unsymmetric situation between positive and negative shears when $\mathbf{M}$ is not radial.

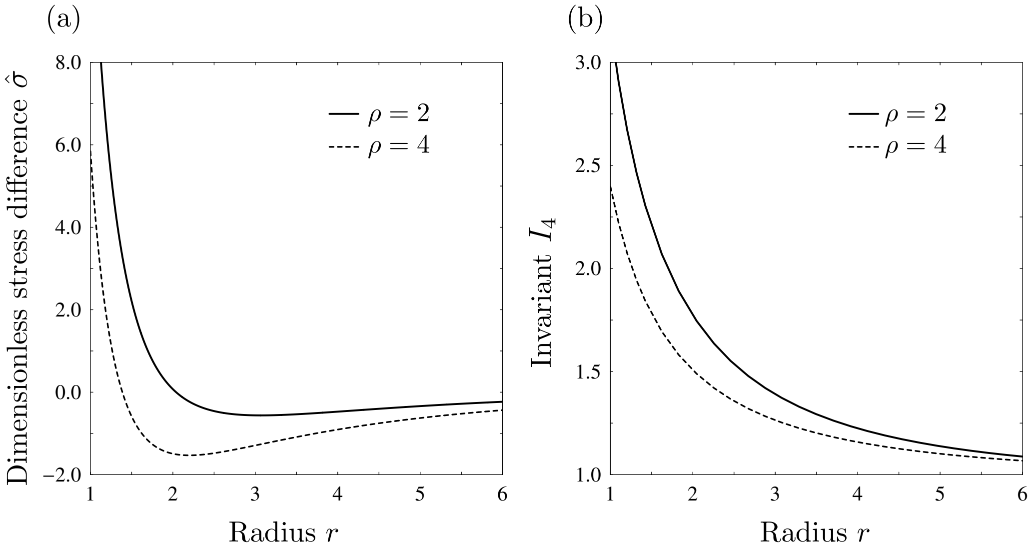

Figure 3.12: Plots of (a) the dimensionless stress difference $\hat{\sigma}$, and (b) the invariant I_4 as functions of the radius r for the model (3.46) with $\rho = 2, 4$, $\mathbf{M} = \mathbf{E}_R$ and $\tau(b) = 0.4$.

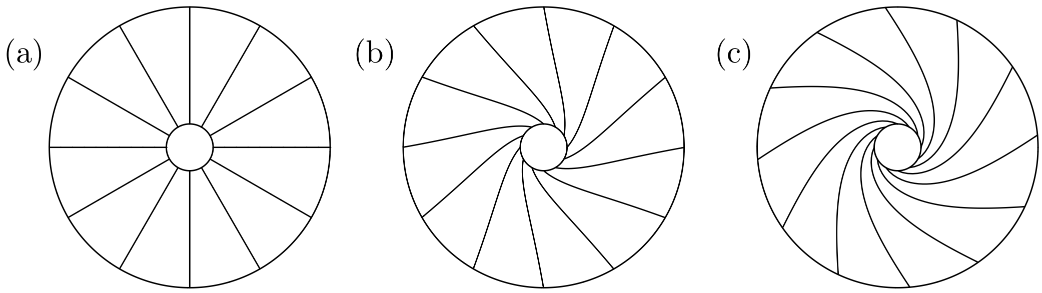

Figure 3.13: Cross-section of a tube undergoing positive (anticlockwise) pure azimuthal shear deformation: (a) the undeformed (stress-free) configuration with $\mathbf{M} = \mathbf{E}_R$; (b) $\rho = 2$, $\tau(b) = 0.2$; (c) $\rho = 2$, $\tau(b) = 1.2$.

In addition, for the same values of $\tau(b)$ and ρ, the dimensionless stress difference $\hat{\sigma} = (\sigma_{\theta\theta} - \sigma_{rr})/\mu \equiv r(\mathrm{d}\sigma_{rr}/\mathrm{d}r)/\mu$ and the invariant I_4 are plotted against r in Figure 3.12. We observe that both $\hat{\sigma}$ and $I_4\,(>1)$ are larger on the inner boundary of the body. Unlike γ, $\hat{\sigma}$ and I_4 are invariant under change of sign of $\tau(b)$. In Figure 3.13 we demonstrate the results of the considered deformation on a cross section of a tube for a fixed value of the parameter $\rho\,(=2)$ and for two values of $\tau(b)\,(>0)$ so as to illustrate how the preferred direction changes under the deformation.

3.8.2 Reinforcement with radially varying α

For definiteness, we now consider the preferred direction to be defined by the family of curves

$$R = c_1(\Theta - \Theta_0) + c_2, \tag{3.90}$$

where c_1 and c_2 are constants. In respect of (3.90) the function $G(R)$, according to the definition (3.7), takes the simple form

$$G(R) = (R - c_2)/c_1. \tag{3.91}$$

We choose $\Theta_0 = 0$ (radians) and, with reference to (3.7), $\Theta_1 = 2$ (rad), and then, for the specific values of A and B adopted here, we obtain $c_1 = 2.5$ and $c_2 = 1$. Thus, from (3.91) the components (3.8) of $\mathbf{M}$ are given by

$$M_R = \frac{1}{\sqrt{(2R/5)^2 + 1}}, \quad M_\Theta = \frac{2R/5}{\sqrt{(2R/5)^2 + 1}}. \tag{3.92}$$

For this geometry, on application of positive shear, the overall response of the body is found to be similar to that for $\mathbf{M} = \mathbf{E}_R$. In fact, for $\tau > 0$, equation (3.52) guarantees

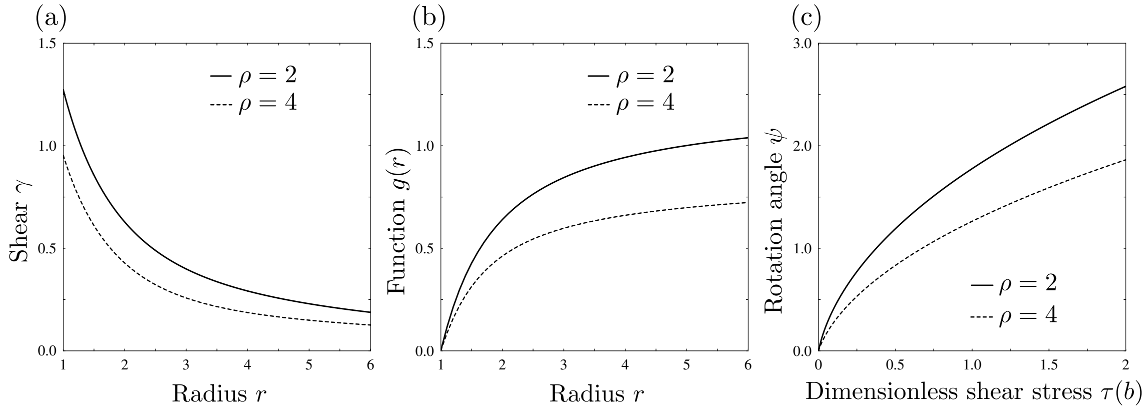

Figure 3.14: Plots of (a) the shear γ and (b) $g(r)$ versus the radius r for the model (3.46), with the anisotropy defined by the geometry (3.92): $\rho = 2, 4, \tau(b) = 0.4$. In (c) the rotation angle $\psi = g(b)$ is plotted as a function of $\tau(b)\,(> 0)$: $\rho = 2, 4$.

smooth and unique values for γ. Moreover, γ and g are also monotonic functions of r, and changes in $\tau(b)\,(> 0)$ and ρ have an analogous impact on the overall response of the body, as for the case of radial reinforcement. However, we remark that the dependence of $\mathbf{M}$ on r leads to stronger reinforcement of the material since the value of γ_1 is smaller at any r than for $\mathbf{M} = \mathbf{E}_R$ at the same shear stress and for the same value of ρ. It is also apparent that the values of $\hat{\sigma}$ and I_4 follow a similar pattern as functions of r. Results for the considered geometry are illustrated in Figure 3.14, in which the amount of shear and the function g are plotted against r for the same values of $\tau(b)\,(> 0)$ and ρ as used in Figure 3.11. The dependence of the rotation angle ψ on $\tau(b)$ is also shown. The results should be compared with those in the upper halves of the plots in Figure 3.11.

The benign response of the material (37) with reinforcement (83) that was found to hold sway for positive shear does not always persist for negative shear. Specifically, as we now show, there is the possibility of loss of ellipticity, multiple values of γ and the emergence of discontinuity surfaces. For $\rho < 2$, the (unique, smooth) root γ_1 is now given by (3.53) with (3.92), but is not necessarily a monotonic function of r. In fact, monotonicity is only to be expected for sufficiently small values of $|\tau(b)|$ and for materials with a small value of ρ. Indeed, as ρ increases, loss of monotonicity of γ, associated with smaller magnitudes of the shear stress, is initiated closer to the inner boundary $r = a$ of the body, while an increase in $|\tau(b)|$ results in translation of such a point closer to $r = b$. Nevertheless, by writing $\gamma = \gamma(r)$ to indicate the dependence of γ on r, the property $|\gamma(a)| > |\gamma(b)|$ always holds. We emphasize, on the other hand, that the possible non-monotonic nature of γ is

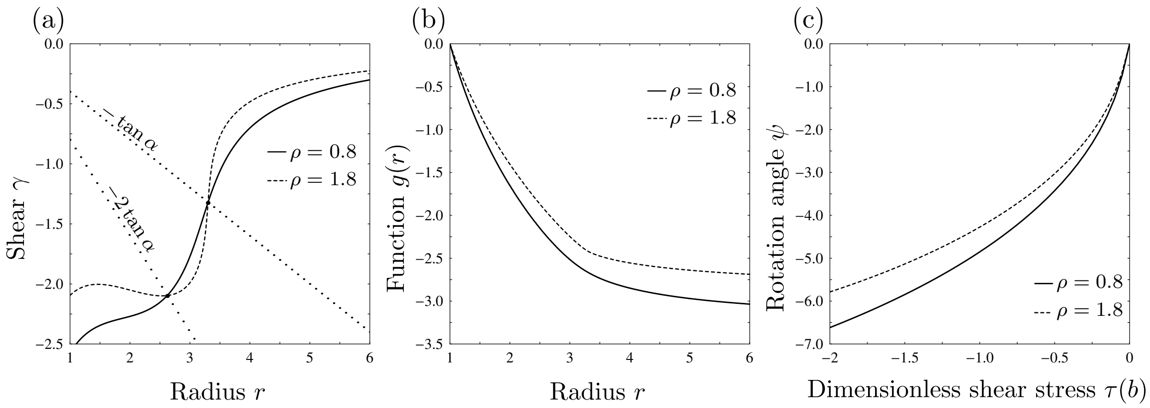

Figure 3.15: Plots of (a) the shear γ and (b) the function $g(r)$ versus the radius r for the model (3.46) with the anisotropy defined by the geometry (3.92): $\rho = 0.8, 1.8, \tau(b) = -0.4$. In (a) the (dotted) lines show the transitional values $-\tan\alpha$ and $-2\tan\alpha$, which depend linearly on r. In (c) the rotation angle $\psi = g(b)$ is plotted against $\tau(b)\,(< 0)$.

not reflected in that of g, which is monotonic in r, while variation of the parameter $\tau(b)$ does not modify these conclusions.

By contrast, the parameter ρ has a very crucial role. For a fixed value of $\tau(b)$, an increase in ρ leads to a decrease in the value of $|\gamma|$, but only for points for which $\tau(r) > -\tan\alpha$. If there is a point for which $\tau(r) = -\tan\alpha$ then γ would also take the value $-\tan\alpha$ there independently of the value of $\rho\,(< 2)$, while, for points for which $-2\tan\alpha < \tau(r) < -\tan\alpha$, larger values of ρ correspond to smaller $|\gamma|$ until $\tau(r)$ reaches the value $-2\tan\alpha$, at which point we have $\gamma = -2\tan\alpha$, again independently of the value of ρ. For $\tau(r) < -2\tan\alpha$, the initial correlation between ρ and γ is recovered for all relevant r. Note that the above analysis applies for any $\mathbf{M}$ that depends on r. The consequences of negative shearing on a material incorporating the properties (3.46) and (3.92) are illustrated in Figure 3.15, in which the curves γ_1 and g are plotted as functions of r for a fixed value $\tau(b)\,(< 0)$ and for two values of $\rho\,(< 2)$. Also plotted, in Figure 3.15(c), are the corresponding results for the rotation angle ψ as a function of $\tau(b)$. These plots should be contrasted with those in the lower halves of the plots in Figure 3.11.

In Figure 3.16, analogously to Figure 3.12, $\hat{\sigma}$ and I_4, are plotted against r for the same values of $\tau(b)$ and ρ used above to illustrate the variation of γ_1, while Figure 3.17 shows the effect of the deformation on the preferred direction for a tube cross-section with material characterized by (3.46) with (3.92). These plots should be contrasted with those in Figures 3.12 and 3.13, respectively. Although, on the scale shown here, there appears to be an abrupt change in the gradient of the deformed preferred direction in Figure 3.17(c) and an associated abrupt change in the corresponding curve for I_4 for $\rho = 1.8$ in Figure 3.16(b), in fact, since $\rho < 2$, the deformation is still smooth (there is no discontinuity in γ). As we show next, however, this smoothness is lost for $\rho > 2$.

Attention is now turned to the case where $\rho \geqslant 2$, for which, as explained previously in Section 3.6, negative shear may be associated with failure of strong ellipticity. For $\rho = 2$, the formulas (3.92) indicate that loss of ellipticity is confined to the surface $r = 2.5$ and only for the value $\tau(b) = -(2.5/6)^2 \approx -0.174$. For $\tau(b)$ both less than or greater than this value the deformation is everywhere elliptic. We recall that for $\rho = 2$ uniqueness and continuity of the shear strain $\gamma = \gamma_1$, as given by (3.53), holds for all r. The value $\tau(b) = -0.174$ gives the temporary appearance of a cusp in the plot of I_4 at $r = 2.5$. These effects are shown in Figure 3.18. It should also be noted that high slopes in the γ_1 and $\hat{\sigma}$ curves are to be expected for points where $\gamma = -\tan\alpha$, or, equivalently, $\tau(r) = -\tan\alpha$, while for the same points the curves of I_4 exhibit rapid change. However, a true kink obtains only for the specific value $\tau(b) = -0.174$, at which ellipticity is temporarily lost.

As ρ increases, the domain $M_1 \leqslant M_R \leqslant M_2$ expands and therefore strong ellipticity can fail both for a wider range of loading values $\tau(b)$ and a wider range of locations r. This is exhibited in Figure 3.19, which shows how the region of non-strong-ellipticity, which is quite narrow, expands with ρ in a plot of $\tau(b)$ against r. The previously discussed case

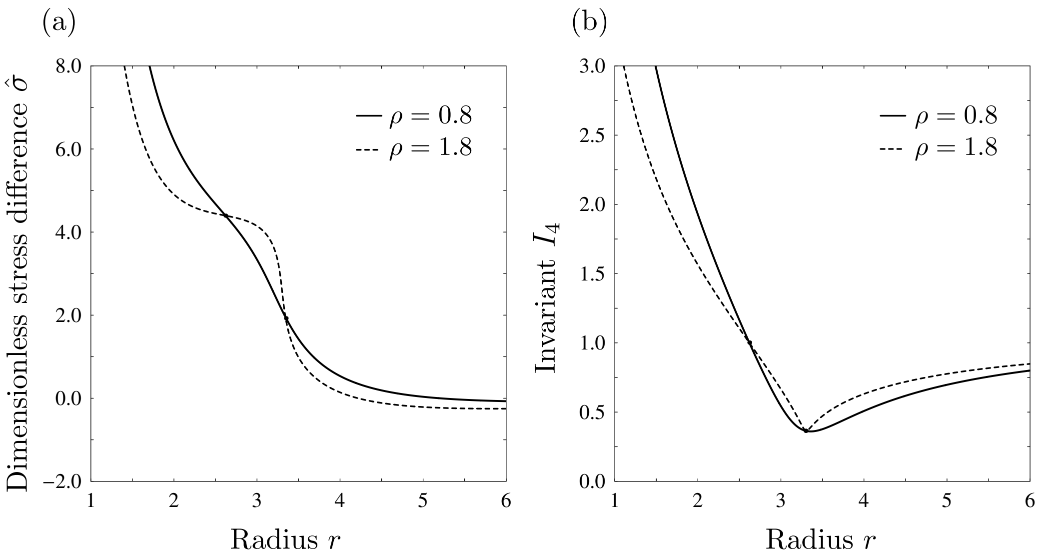

Figure 3.16: Plots of (a) the dimensionless stress difference $\hat{\sigma}$, and (b) the invariant I_4 against the radius r for the model (3.46) with (3.92) for $\tau(b) = -0.4$ and $\rho = 0.8, 1.8$.

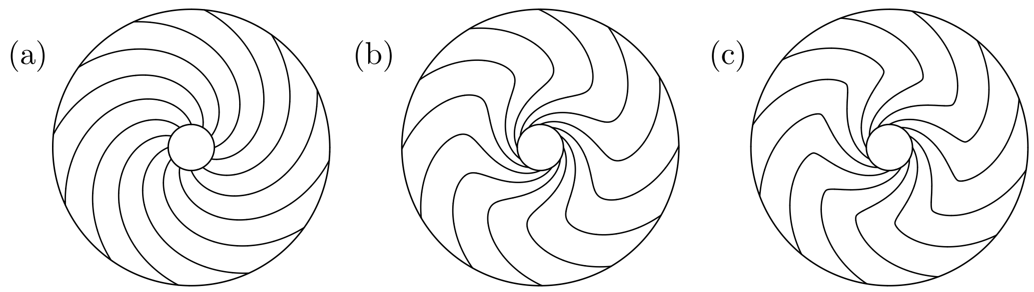

Figure 3.17: Cross-section of a tube undergoing a negative (clockwise) pure azimuthal shear deformation for the model (3.46) with the preferred direction defined by (3.92): (a) undeformed (stress-free) configuration; (b) deformed configuration with (b) $\rho = 0.8$ and $\tau(b) = -0.4$ and (c) $\rho = 1.8$ and $\tau(b) = -0.4$.

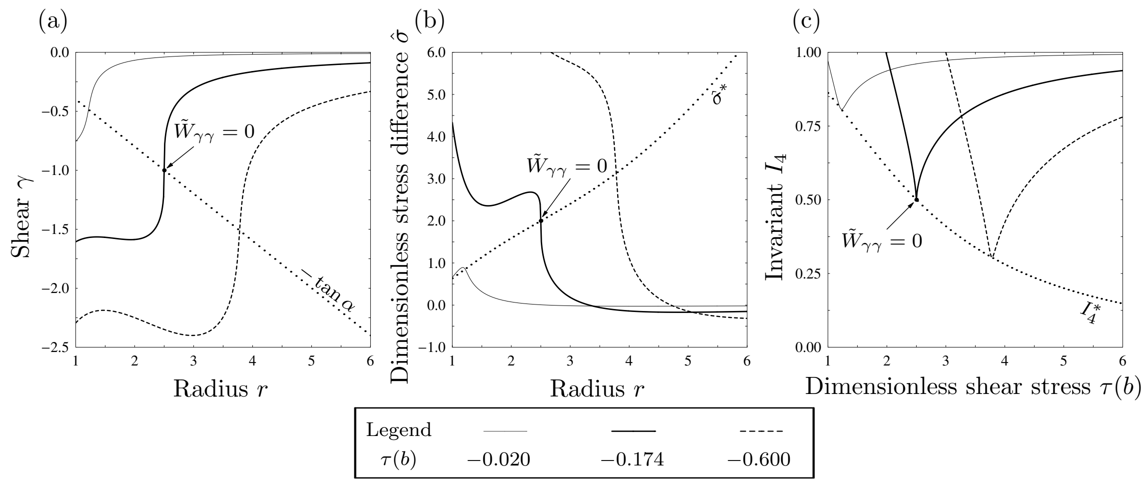

Figure 3.18: Plots of (a) the shear γ, (b) the dimensionless stress difference $\hat{\sigma}$, and (c) the invariant I_4 versus radius r for the model (3.46) with (3.92), for $\rho = 2$ and, in each case, $\tau(b) = -0.02, -0.174, -0.6$. Only $\tau(b) = -0.174$ is associated with loss of ellipticity. In (a) the point of loss of ellipticity on the dotted line $-\tan\alpha$ is shown; the corresponding point in (b) and (c) is shown on the dotted curves indicated by $\hat{\sigma}^*$ and I_4^*, respectively.

of $\rho = 2$ gives the single point $(r, \tau(b)) = (2.5, -0.174)$, whereas the $\rho > 2$ regions are nested within each other. This is not apparent from the figure since in order to distinguish the curves they are shifted vertically by different amounts. At $\rho = 3.9668$ the nonelliptic region first encounters one of the external boundaries, namely $r = 6$, while the boundary $r = 1$ is reached for $\rho = 4.205$. For $\rho > 4.205$ strong ellipticity can fail at any point of the body.

We now focus attention on the case $\rho = 3$ for which we note, as reflected in Figure

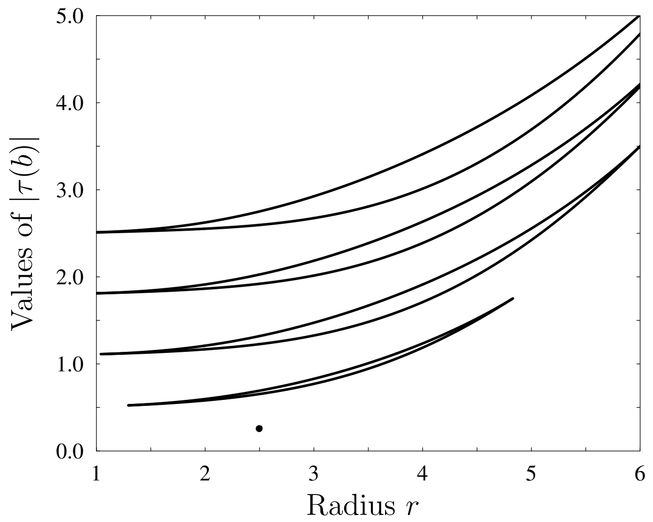

Figure 3.19: Plots of the values of $|\tau(b)|$ corresponding to loss of ellipticity against the radius $r \in [1, 6]$ for $\rho = 2, 3, 3.9668, 4.205, 5$. For $\rho = 2$ there is an isolated point at $r = 2.5, \tau(b) = -0.174$; $\rho = 3.9968$ corresponds to the value at which the curve just reaches the boundary $r = 6$; similarly for $\rho = 4.205$ and the boundary $r = 1$. The plots are nested, with each successive curve (for increasing ρ) enclosing the previous one, but have been shifted vertically, by $0.1, 0.5, 1.1, 1.8, 2.5$, respectively, to enable them to be distinguished.

3.19, that failure of ellipticity is associated with the radial values

$$1.294 \lesssim r \lesssim 4.829, \tag{3.93}$$

and the associated range of loadings

$$0.0241 \lesssim |\tau(b)| \lesssim 1.2517. \tag{3.94}$$

As shown in Figure 3.20(a), in which γ is plotted as a function of r, loss of convexity of the strain energy (3.46) at $r \approx 1.294$, requiring $|\tau(b)| \approx 0.0241$, does not yield a discontinuity in γ. On the other hand, for any $\tau(b)$ such that $0.0241 < |\tau(b)| < 1.2517$ the inequality $\tilde{W}_{\gamma\gamma} \leqslant 0$ always holds within a subinterval of $1.294 < r < 4.829$, leading to multiple values for γ. In each case, however, it is possible to construct a unique non-smooth stable solution through the body by placing an elastostatic shock at the location where σ matches the Maxwell value τ^*. Figure 3.20(b) shows the consequence of the non-smooth solution on the curve $g(r)$ (discontinuity in the tangent), while the rotation (twist) angle ψ is plotted as a function of $\tau(b)$ in Figure 3.20(c). On close inspection a kink in the latter curve at $\tau(b) = -0.0241$ can be discerned.

Clearly, since both $\hat{\sigma}$ and I_4 depend on γ, any discontinuity in γ is reflected in the curves

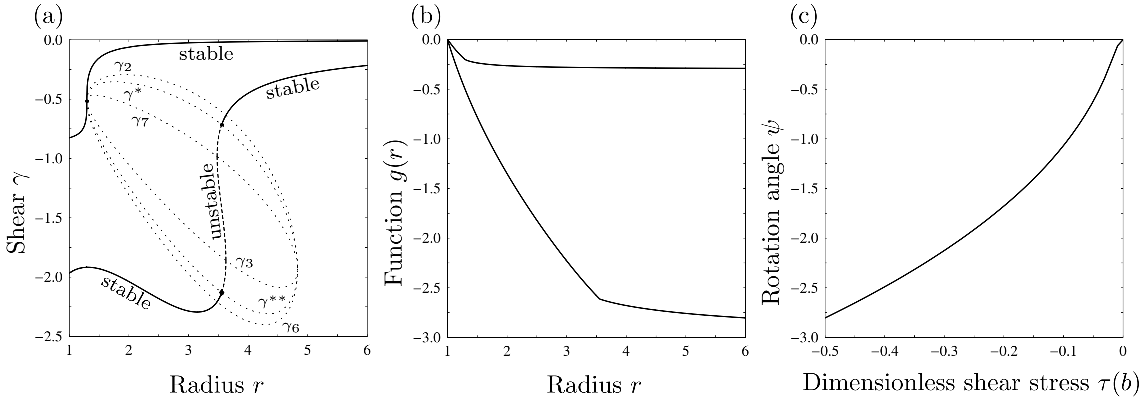

Figure 3.20: Plots of (a) the shear γ and (b) the function $g(r)$ versus the radius r for the model (3.46) with anisotropy defined by the geometry (3.92): $\rho = 3$ and $\tau(b) = -0.0241, -0.5$. In (c) the rotation angle $\psi = g(b)$ is plotted against $\tau(b)\,(< 0)$. The dotted curves in (a) are plots of $\gamma_2, \gamma_3, \gamma_6, \gamma_7, \gamma^*, \gamma^{**}$, as indicated.

$\hat{\sigma}$ and I_4 that are plotted in Figures 3.21(a) and (b), respectively. In (a) the notations $\hat{\sigma}_2, \hat{\sigma}_3, \hat{\sigma}_6, \hat{\sigma}_7, \hat{\sigma}_*, \hat{\sigma}_{**}$ serve to identify the values of $\hat{\sigma}$ associated with $\gamma_2, \gamma_3, \gamma_6, \gamma_7, \gamma^*, \gamma^{**}$, respectively. Similarly, the symbols $I_{4,2}, I_{4,3}, I_{4,6}, I_{4,7}, I_{4,*}, I_{4,**}$ identify the corresponding values of I_4 in (b). Note, however, that a jump in $\hat{\sigma}$ is due to that in $\sigma_{\theta\theta}$ since σ_{rr} is continuous.

For the following discussion we use the notation τ_f and τ_s, respectively, for the values $\tau(b) = -0.0241$ and $\tau(b) = -1.2517$, the subscripts indicating 'first' and 'second'. It follows that absolutely stable solutions contain an equilibrium shock for $\tau_s < \tau < \tau_f < 0$. Imagine therefore that the originally undeformed cylinder is subject to decreasing (dimensionless) shear stress $\tau(b)$ and consider the associated quasi-static progression of absolutely stable solutions. The associated deformations are continuous and smooth until $\tau(b) = \tau_f$ at which point loss of ellipticity takes place at $r = 1.294$. Further decrease in $\tau(b)$ gives rise to an elastostatic shock, initially at $r = 1.294$, which separates the two different elliptic values $\gamma = \gamma_1$ and $\gamma = \gamma_5$. Under continued decrease in $\tau(b)$, this discontinuity surface increases its radial location, before eventually disappearing at $r = 4.829$ when $\tau(b) = \tau_s$. For $\tau(b) < \tau_s$ the deformation is again classically smooth. The connection between the radial location r of the elastostatic shock and $\tau(b)$ is given simply by the formula $\tau(r) = \tau^* = -\tan\alpha$, which, since $\tau(r) = \tau(b)b^2/r^2$ and $\tan\alpha = 2r/5$, yields $r^3 = -90\tau(b)$. The discontinuity is highlighted in Figure 3.22, which shows the deformed cross-section of the tube for two values of $\tau(b)$ compared with the reference configuration.

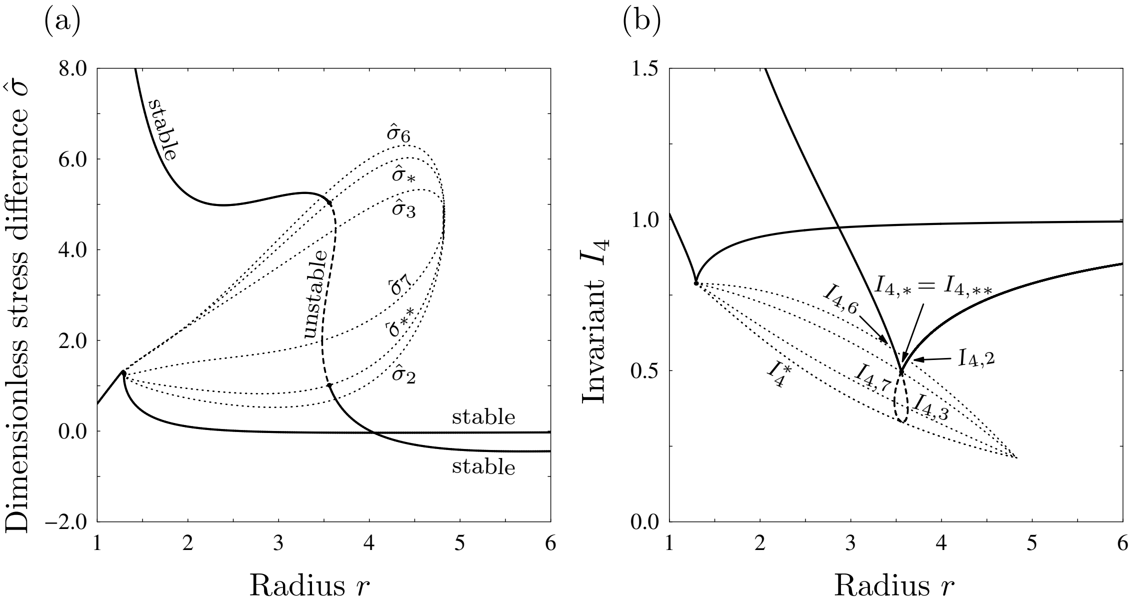

Figure 3.21: Plots of (a) the dimensionless stress difference $\hat{\sigma}$, and (b) the invariant I_4 in terms of the radius r for the model (3.46) with (3.92) and $\rho = 3$ for $\tau(b) = -0.0241, -0.5$. In (a) the symbols $\hat{\sigma}_2, \hat{\sigma}_3, \hat{\sigma}_6, \hat{\sigma}_7, \hat{\sigma}_*, \hat{\sigma}_{**}$ identify the values of $\hat{\sigma}$ associated with $\gamma_2, \gamma_3, \gamma_6, \gamma_7, \gamma^*, \gamma^{**}$; in (b) the corresponding values of I_4 are labelled $I_{4,2}, I_{4,3}, I_{4,6}, I_{4,7}, I_{4,*}, I_{4,**}$.

The first, $\tau(b) = -0.0241$, heralds the emergence of the shock, whereas the shock has both increased its strength and its radial location at the second value $\tau(b) = -0.5$. Note that the associated kink in the case $\tau(b) = -0.0241$ (Figure 3.22(b)) is close to the inner boundary and is hardly noticeable in the figure. Across the shock the angles made by the two deformed preferred directions are symmetrically disposed with respect to the local radial direction in accordance with the discussion in Section 3.6.3. We recall that Figure 3.8 shows the kink angle β^* as a function of M_R, including the case $\rho = 3$ considered here. Since, for the considered geometry, $\tan\alpha = 0.4r$ we have $M_R = 1/\sqrt{1 + 0.16r^2}$, from which the behaviour of β^* as a function of r can be deduced. The preferred direction is found to be maximally kinked at $r \approx 3.536$, corresponding to $\tau(b) \approx -0.491$.

It is worth emphasizing that by taking $\alpha = \alpha(r)$ the tube has been rendered effectively inhomogeneous. This has permitted the quasi-static shock discussed above in connection with Figure 3.21 to remain confined within the tube interior regardless of the magnitude of the twist ψ (equivalently of $\tau(b)$). This contrasts with the situation described by Abeyaratne [2] in which the shock always emerges at the inner radius and travels all the way to the outer radius as the twist magnitude increases. From a practical perspective,

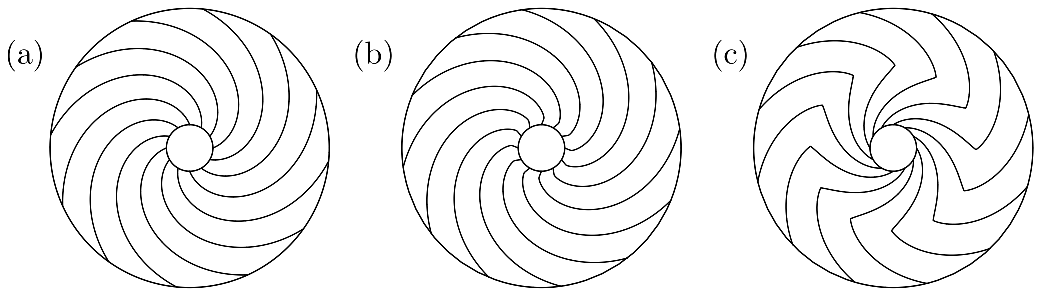

Figure 3.22: Cross-section of a tube undergoing negative (clockwise) pure azimuthal shear deformation for the model (3.46) with the preferred direction defined by (3.92): (a) undeformed (stress-free) configuration; deformed configuration with (b) $\rho = 3$ and $\tau(b) = -0.0241$ and (c) $\rho = 3$ and $\tau(b) = -0.5$.

such internally confined shocks could present challenges for the assessment of any damage associated with shock formation and movement.

Broadly similar results to those described above for the reinforced neo-Hookean model (3.46) have been obtained for the material model (3.82) and we do not report them separately. Generally, the Varga model represents a material whose energy absorption capacity is low, and even for very small values of ρ a negative shear deformation yields almost immediate ellipticity breakdown.

Chapter 4

Combined extension, inflation and torsion of anisotropic elastic tubes

4.1 Introduction

In this chapter we examine the problem of the combined extension, inflation and torsional deformation of a transversely isotropic circular cylindrical tube composed of incompressible elastic material. Several authors have studied the latter deformations, mainly separately, from many different aspects in the past. In brief, torsional deformations for incompressible isotropic materials were first examined by Rivlin in a series of papers [66, 67, 65] while also in [64] and [68] experimental data were provided. Torsional tests as well as theoretical studies for some special classes of incompressible isotropic materials were presented by Green and Adkins [24] and a similar discussion may be found in Green and Zerna [26]. Furthermore, Gent and Rivlin [23] have considered the case of combined uniform extension, uniform inflation and torsion for a class of cylindrically isotropic incompressible circular tubes. It should be mentioned, however, that these results were mainly based on small amounts of torsional strain whereas the presence of the uniform inflation was directed to retain the inner radius of the considered body unchanged.

Comparison of the well know Ogden model for rubber-like solids with the data given in [68] for rigid and tubular cylinders composed of natural rubber under combined extension and torsional deformation has been presented by Ogden and Chadwick [58]. In addition, Haughton and Ogden [27, 28] presented a detailed analysis for combined extension and inflation of such materials including bifurcation into non-circular cylindrical modes of deformation. More recently, Horgan and Saccomandi [69] signified the necessity of more sophisticated material models in order to capture the hardening response (i.e. materials with limiting chain extensibility) of elastic incompressible materials under large strain torsional deformations.

In the compressible isotropic theory a wide class of materials admitting isochoric pure torsion deformation was proposed by Polignone and Horgan [61]. In the same spirit, Kirkinis and Ogden [44] derived analogous solutions but also introduced a methodology to generate their results for the incompressible theory. Different aspects in respect of pure torsion, concerning special classes of compressible materials and loss of ellipticity have also been studied in [10] and [33], respectively, amongst others.

To the best of my knowledge, very few authors have studied analogous problems with respect to incompressible anisotropic (fibre-reinforced) elastic solids. Under the restriction of idealized fibre-reinforcement (i.e. inextensible fibres), Spencer [72] discussed the problem of extension and torsion of rigid elastic cylinders augmented with one and two families of helical fibres. For two symmetric fibre families situated in helical mode, remarks on the extension and inflation of hollow cylinders can also be found in [73], yet the analysis is mainly restricted to the linear theory. The same problem with respect to large deformations and for materials composed of soft tissues (arteries), has been examined by Ogden and Schulze-Bauer [59], where the helical fibre-reinforcement was used to model the anisotropic contribution of the collagen to the overall response of the tissue. In the same spirit, Horgan and Saccomandi [34] discussed the combined extension and inflation problem for soft tissues by taking into account limiting chain extensibility. A thorough analysis, concerning again the mechanical response of arteries, for simultaneous uniform extension, inflation and torsion, is given in Holzapfel *et. al.* [32] and more recently in Gasser *et. al.* [21].

In the present analysis, we consider the problem of combined finite extension, inflation and torsion with special attention to transversely isotropic materials with non-homogeneous elastic properties as those discussed in Chapter 3. Distinctively, in Section 4.2 we introduce and analyze the appropriate kinematics associated with the combined deformation. We establish formulas for the principal stretches of the deformation in terms of the three deformation quantities which (locally) serve to characterize each one of the processes composing the combined deformation. Similarly, we provide an appropriate formula for the angle of shear as a continuous function of the aforementioned quantities and we further substantiate that, within the incompressible framework, those may be treated as independent parameters. At the same time, although the shearing deformation taking place in the planes normal to the radial direction of the deformed body is a result of torsion, we highlight the significant contribution of the axial extension and uniform inflation deformations on the way that the angle of shear changes.

In Section 4.3, we particularize our discussion for the case of transversely isotropic materials. The preferred direction associated with the transverse isotropy lies in the planes normal to the tube axis and is disposed so as to preserve the cylindrical symmetry. Under the premise that the tangent to the preferred direction and the undeformed radial direction form an angle, denoted α (this, in general, being a function of the radius), the influence of the combined deformation on the status of the preferred direction is examined. In any case, axial extension of the tube brings the preferred direction into a compressive mode, the degree of which, however, is independent of the distribution of α. By contrast, the inflation process may be responsible for either further contraction or relaxation of the preferred direction. For a certain range of α, inflation of the tube always causes additional contraction of the preferred direction down to its minimal possible length, at least in early stages of the inflation process, before it reaches its path to relaxation. On the other hand, for a specified range of α, immediate relaxation of the preferred direction due to inflation is guaranteed. Within this class of materials, incorporation of the torsional aspect of the deformation has no actual impact on the length of the preferred direction, yet the status of the latter is related to the degree of influence of the shearing effect in that direction. Note that here, unlike the azimuthal shear problem, the sense of shear (i.e. clockwise and anticlockwise, respectively) is of no importance since the shearing deformation is taking place normal to the planes of distribution of the preferred direction.

Thereafter, using a generic strain-energy function for a transversely isotropic material and always assuming that the preferred direction is distributed as above, we determine the cylindrical polar components of the Cauchy stress tensor identified with the considered combined deformation. With the only exceptions of a radially and circumferentially distributed preferred direction, the considered geometry and deformation necessitates the presence of axial and azimuthal shear stresses with the help of which a universal relation is accomplished. As a matter of fact, the latter universal relation may well be specialized for a body whose preferred direction is radially distributed. In this case coaxiality between the Cauchy stress tensor and the left stretch tensor is established and hence the latter relation is found to hold just as for an isotropic material. Based on the above arguments and in conjunction with the kinematics of the problem, conditions sufficient to establish a sustainable deformation are derived. Resolution of the associated equilibrium equations follows and the axial and azimuthal shear stress components are determined completely as inversely proportional functions of the deformed radius of the tube. The latter fact, enables the exclusion of two subclasses of materials whose preferred direction is locally aligned either with the radial or with the circumferential direction of the undeformed body since in those cases the equilibrium equations are no longer substantiated. Further, we provide expressions for the applied loads, namely for the internal pressure, the axial force and the resultant moment, that are required to support the extension, inflation and torsional deformation. After some simple manipulation, these formulas, also applying for isotropic materials, are found to recover the expressions derived in [57] (see also references therein) when the body is subjected to axial extension and inflation alone. Finally, a brief discussion is presented for separate classes of transversely isotropic materials with the preferred direction taken to lie in the planes parallel to the planes of the shearing deformation while a small reference to orthotropic materials with analogous properties is also made.

4.2 Combined extension, inflation and torsion

4.2.1 Description of the problem

Consider a (stress-free) circular cylindrical tube composed of incompressible elastic material with reference geometry defined by

$$A \leqslant R \leqslant B, \quad 0 \leqslant \Theta \leqslant 2\pi, \quad 0 \leqslant Z \leqslant L, \tag{4.1}$$

where (R, Θ, Z) are cylindrical polar coordinates in the reference configuration relative to a cylindrical polar basis $\{\mathbf{E}_I\}, I \in \{R, \Theta, Z\}$.

We now assume that three independent deformations are imposed on the body. In particular, we initially consider that the tube is uniformly extended to length $l = \lambda_z L$. While the body is held fixed to that state, the plane face $z = l$ is rotated through an angle τz and, at the same time, uniform pressure of amount P $(\geqslant 0)$ is applied on its inner surface. With the twist per unit length of the deformed tube, namely τ, being positive, negative, or zero, the torsional deformation is directed in such a way that the plane face $z = 0$ remains fixed and the plane sections of the tube normal to its axis remain plane. Also, the pressure P is regarded as being applied along the radial direction of the body and it is considered to be of such magnitude that it causes inflation of the tube. On the other hand, in the same direction, the outer surface of the body is assumed to be traction-free. In any case, we adopt the hypothesis that the circular cylindrical geometry of the body is always maintained.

Provided that the deformed body occupies the space

$$a \leqslant r \leqslant b, \quad 0 \leqslant \theta \leqslant 2\pi, \quad 0 \leqslant z \leqslant l, \tag{4.2}$$

this being identified with respect to cylindrical polar coordinates (r, θ, z) associated with the polar basis $\{\mathbf{e}_i\}, i \in \{r, \theta, z\}$, in the current configuration, the combined deformation is defined by

$$r^2 - a^2 = \lambda_z^{-1}(R^2 - A^2), \quad \theta = \Theta + \tau\lambda_z Z, \quad z = \lambda_z Z, \tag{4.3}$$

together with the radial boundary conditions

$$\sigma_{rr} = \begin{cases} -P & \text{on} \quad r = a \\ 0 & \text{on} \quad r = b \end{cases} \tag{4.4}$$

corresponding to pressure P on the inside of the tube and zero traction on the outside. Clearly, here σ_{rr} represents the radial components of the Cauchy stress tensor $\boldsymbol{\sigma}$.

4.2.2 Kinematics and analysis of the combined deformation

The kinematical interpretation of the combined deformation, as this is prescribed in (4.3), now provides that the deformation gradient tensor $\mathbf{F}$ takes the form

$$\mathbf{F} = \lambda_r \mathbf{e}_r \otimes \mathbf{E}_R + \lambda_\theta \mathbf{e}_\theta \otimes \mathbf{E}_\Theta + \lambda_z \mathbf{e}_z \otimes \mathbf{E}_Z + \lambda_z \gamma \mathbf{e}_\theta \otimes \mathbf{E}_Z, \tag{4.5}$$

where the notations

$$\lambda_r = (\lambda_z \lambda_\theta)^{-1}, \quad \lambda_\theta = r/R, \quad \gamma = \tau r, \tag{4.6}$$

have been adopted for compactness. The deformation gradient (4.5) may equivalently be decomposed by means of three second-order tensors, denoted $\mathbf{F}_1, \mathbf{F}_2$ and $\mathbf{F}_3$, such as $\mathbf{F} = \mathbf{F}_3\mathbf{F}_2\mathbf{F}_1$, where, in particular

$$\mathbf{F}_1 = \mathbf{e}_r \otimes \mathbf{E}_R + \mathbf{e}_\theta \otimes \mathbf{E}_\Theta + \mathbf{e}_z \otimes \mathbf{E}_Z, \tag{4.7}$$

accounts for a transition from the basis $\{\mathbf{E}_I\}$ to $\{\mathbf{e}_i\}$ (i.e. it describes a rotation arising from the torsional process), further

$$\mathbf{F}_2 = \lambda_r \mathbf{e}_r \otimes \mathbf{e}_r + \lambda_\theta \mathbf{e}_\theta \otimes \mathbf{e}_\theta + \lambda_z \mathbf{e}_z \otimes \mathbf{e}_z, \tag{4.8}$$

implicates that uniform extension and, as we will shortly clarify, presumably inflation are taking place in the body, while the presence of

$$\mathbf{F}_3 = \mathbf{I} + \gamma \mathbf{e}_\theta \otimes \mathbf{e}_z \tag{4.9}$$

justifies the existence a simple shear deformation of amount γ in the planes normal to $\mathbf{e}_r$. It is then understood that $\mathbf{F}_3$, along with $\mathbf{F}_1$, serve to incorporate the torsional part of the combined deformation into $\mathbf{F}$. Similarly to the discussion provided in Chapter 3, here we illustrate that positive shear ($\gamma > 0$) is associated with an anticlockwise twisting deformation of the tube about its axis. Analogous conclusions hold for negative shear ($\gamma < 0$).

By virtue of (4.5) and (4.6), the right and the left Cauchy-Green deformation tensors, those defined in (1.43), are then easily found to take the forms

$$\begin{aligned} \mathbf{C} = {} & \lambda_r^2 \mathbf{E}_R \otimes \mathbf{E}_R + \lambda_\theta^2 \mathbf{E}_\Theta \otimes \mathbf{E}_\Theta + \lambda_z^2 (1+\gamma^2) \mathbf{E}_Z \otimes \mathbf{E}_Z \\ & + \gamma \lambda_z \lambda_\theta (\mathbf{E}_Z \otimes \mathbf{E}_\Theta + \mathbf{E}_\Theta \otimes \mathbf{E}_Z), \end{aligned} \tag{4.10}$$

$$\begin{aligned} \mathbf{B} = {} & \lambda_r^2 \mathbf{e}_r \otimes \mathbf{e}_r + (\lambda_\theta^2 + \gamma^2 \lambda_z^2) \mathbf{e}_\theta \otimes \mathbf{e}_\theta + \lambda_z^2 \mathbf{e}_z \otimes \mathbf{e}_z \\ & + \gamma \lambda_z^2 (\mathbf{e}_z \otimes \mathbf{e}_\theta + \mathbf{e}_\theta \otimes \mathbf{e}_z), \end{aligned} \tag{4.11}$$

respectively.

It is important to clarify that here the quantities λ_θ and λ_z are not in general the principal stretches of the deformation. In fact, as clearly demonstrated in (4.8), λ_θ and λ_z are strictly associated with the part of the deformation process that is responsible for the uniform extension and inflation of the tube. Indeed, it is apparent that λ_θ and λ_z are measures of stretch in the circumferential (equivalently azimuthal) and axial direction of the body, respectively, while the combination of the two, namely $\lambda_r = (\lambda_z \lambda_\theta)^{-1}$, expresses the corresponding stretch in the radial direction. If now no torsional deformation is involved, then $\tau = 0$ and hence, from $(4.6)_3$, $\gamma = 0$. Following that, (4.9) specializes to $\mathbf{F}_3 = \mathbf{I}$ and also, obviously, (4.5) reduces to $\mathbf{F} = \mathbf{F}_2\mathbf{F}_1$. Note, however, that $\mathbf{F}$ can be identified as being identical to $\mathbf{F}_2$ since, in the absence of shearing deformation and while extending and inflating the tube, equation $(4.3)_2$, i.e. evaluated for $\tau = 0$, ensures that the direction of every (non-zero) line element lying in the planes normal to the axis of the body remains invariant. This, in conjunction with the pronounced connection $\mathbf{e}_z = \mathbf{E}_Z$, suffices for $\mathbf{e}_r = \mathbf{E}_R$ and $\mathbf{e}_\theta = \mathbf{E}_\Theta$, both resulting in $\mathbf{F}_1 = \mathbf{I}$ and $\mathbf{F} = \mathbf{F}_2$. Accordingly, from (4.10) and (4.11), we deduce that the tensors $\mathbf{C}$ and $\mathbf{B}$ coincide and, in particular, that they can be put into the diagonal form

$$\mathbf{C} = \mathbf{B} = \lambda_r^2 \mathbf{e}_r \otimes \mathbf{e}_r + \lambda_\theta^2 \mathbf{e}_\theta \otimes \mathbf{e}_\theta + \lambda_z^2 \mathbf{e}_z \otimes \mathbf{e}_z.$$

It is easily understood from the latter that, in the case where $\gamma = 0$, the quantities $\lambda_r, \lambda_\theta$ and λ_z represent the principal stretches of the deformation, those associated with the principal directions $\mathbf{e}_r, \mathbf{e}_\theta$ and $\mathbf{e}_z$, respectively, which may equivalently be perceived as either being Lagrangian or Eulerian.

It is also very important to distinguish that even when the torsional aspect of the combined deformation is taken into consideration, the involvement of γ ($\neq 0$) in one of the components of $\mathbf{F}$, namely in the $F_{\theta R}$ ($= \lambda_z \gamma$) component, has no actual influence regarding the preservation of the volume of the body; the latter statement is attributed to the incompressible nature of the material in examination. This is easily justified in two ways. First, using (4.5) and along with (4.7)–(4.9), we readily obtain

$$J = \det(\mathbf{F}) = \det(\mathbf{F}_3)\det(\mathbf{F}_2)\det(\mathbf{F}_1) = \det(\mathbf{F}_2) = \lambda_r \lambda_\theta \lambda_z,$$

in which case the incompressibility constraint discussed earlier in Section 1.4.3 asserts that (see also [32])

$$\lambda_r \lambda_\theta \lambda_z = 1. \tag{4.12}$$

Undoubtedly, due to $(4.6)_1$, equation (4.12) holds automatically for all $R \in [A, B]$ or, equivalently, through $(4.2)_1$, $(4.3)_1$ and $(4.6)_2$, for all $r \in [a, b]$. Despite this fact, (4.12) states that the space occupied by the body at each phase of the combined deformation (i.e. independently of the amount of the pressure and of the shearing load applied on the tube) is solely directed by the axial and azimuthal stretch. Exactly analogously conclusions can also be drawn on use of $(4.3)_1$ which simply provides the geometrical interpretation of the incompressibility constraint for the considered body and deformation. Precisely, following the work of Ogden [56], the aforementioned argument is established implicitly through the identities

$$\lambda_z \lambda_a^2 - 1 = \frac{R^2}{A^2}(\lambda_z \lambda_\theta^2 - 1) = \frac{B^2}{A^2}(\lambda_z \lambda_b^2 - 1), \tag{4.13}$$

arising, in the light of $(4.6)_2$, from $(4.3)_1$, where now λ_a and λ_b are defined by

$$\lambda_a = a/A, \qquad \lambda_b = b/B, \tag{4.14}$$

respectively.

We emphasize that once the body is extended to length l and held there fixed, inflation of the tube may only take place if

$$\lambda_z \lambda_a^2 \geqslant 1, \tag{4.15}$$

where, in particular, the equality in (4.15) serves to identify the value of the inner radius of the tube before such a deformation is initiated. With these in mind, it is easily understood that, when extension and inflation occur, λ_θ is necessarily a decreasingly monotonic function of either R or r, i.e.

$$\lambda_a \geqslant \lambda_\theta \geqslant \lambda_b, \tag{4.16}$$

and also, from (4.13) and (4.15), that

$$\lambda_z \lambda_\theta^2 \geqslant 1, \tag{4.17}$$

for all $R \in [A, B]$. From the latter further follows that, if there is no inflation then $\lambda_\theta = \lambda_z^{-1/2}$ for all $R \in [A, B]$ and hence (4.16) become strict equalities. In the situation where the tube is kept to maintain its original length, i.e. that in the reference configuration, the subsequent analysis becomes simply a special case of what we have discussed so far with the only difference that λ_z should be taken equal to unity.

Bearing (4.11) in mind, we now explicate that during the deformation process one of the Eulerian principal axis, $\mathbf{v}^{(1)}$ say, remains aligned with $\mathbf{e}_r$ and corresponds to the principal stretch $\lambda_1 = \lambda_r$. The two remaining principal directions, namely $\mathbf{v}^{(2)}$ and $\mathbf{v}^{(3)}$, are then distributed in the (θ, z) plane and may be related to $\mathbf{e}_\theta, \mathbf{e}_z$ as

$$\mathbf{v}^{(2)} = \cos\phi\, \mathbf{e}_\theta + \sin\phi\, \mathbf{e}_z, \quad \mathbf{v}^{(3)} = -\sin\phi\, \mathbf{e}_\theta + \cos\phi\, \mathbf{e}_z. \tag{4.18}$$

Here, the term ϕ signifies the orientation of $\mathbf{v}^{(2)}, \mathbf{v}^{(3)}$ with respect to $\mathbf{e}_\theta, \mathbf{e}_z$. We also remark that since, as already mentioned, for $\gamma = 0$ both $\mathbf{e}_\theta$ and $\mathbf{e}_z$ are distributed along the principal directions of the deformation (i.e. $\phi = 0$ with $\mathbf{v}^{(2)} = \mathbf{e}_\theta$ and $\mathbf{v}^{(3)} = \mathbf{e}_z$), then, for $\gamma > 0$ (< 0), the quantity ϕ is understood to measure the shear in the (θ, z) planes of the (deformed) body. To avoid any confusion with the terminology used for γ, we henceforth refer to ϕ as the angle of shear.

The entries (4.18) now prompt us to introduce a second-order proper orthogonal tensor

$$\mathbf{Q} = \mathbf{e}_r \otimes \mathbf{e}_r + \cos\phi(\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \mathbf{e}_z \otimes \mathbf{e}_z) + \sin\phi(\mathbf{e}_z \otimes \mathbf{e}_\theta - \mathbf{e}_\theta \otimes \mathbf{e}_z), \tag{4.19}$$

which, in view of (4.11) and due to the fact that the left stretch tensor $\mathbf{V}$ admits the spectral decomposition

$$\mathbf{V} = \sum_{i=1}^{3} \lambda_i \mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)}, \tag{4.20}$$

enables the connections

$$\mathbf{B} = \mathbf{Q}^{\mathrm{T}}\mathbf{V}^2\mathbf{Q}, \qquad \mathbf{V}^2 = \mathbf{Q}\mathbf{B}\mathbf{Q}^{\mathrm{T}}. \tag{4.21}$$

Clearly, the quantities λ_2 and λ_3 incorporated in (4.20) represent the principal stretches of the deformation associated with the directions $\mathbf{v}^{(2)}$ and $\mathbf{v}^{(3)}$, respectively. Accordingly, the first connection in (4.21) yields the conclusion that the components of $\mathbf{B}$ and $\mathbf{V}$ have to satisfy the relations

$$\lambda_2^2 \cos^2\phi + \lambda_3^2 \sin^2\phi = \lambda_\theta^2 + \lambda_z^2\gamma^2, \tag{4.22}$$

$$\lambda_2^2 \sin^2\phi + \lambda_3^2 \cos^2\phi = \lambda_z^2, \tag{4.23}$$

$$(\lambda_2^2 - \lambda_3^2)\sin\phi\cos\phi = \lambda_z^2\gamma, \tag{4.24}$$

from which, after some manipulation, we deduce that

$$(\lambda_2^2 - \lambda_z^2)(\lambda_z^2 - \lambda_3^2) = \lambda_z^4\gamma^2, \quad \lambda_2\lambda_3 = \lambda_z\lambda_\theta, \quad \lambda_2^2 + \lambda_3^2 = \lambda_\theta^2 + \lambda_z^2\gamma^2 + \lambda_z^2, \tag{4.25}$$

together with

$$\sin 2\phi = \frac{2\gamma\lambda_z^2}{\lambda_2^2 - \lambda_3^2}\ , \qquad \cos 2\phi = \frac{\lambda_2^2 + \lambda_3^2 - 2\lambda_z^2}{\lambda_2^2 - \lambda_3^2} \tag{4.26}$$

and hence (see also Holzaphel et al. [32])

$$\tan 2\phi = \frac{2\gamma\lambda_z^2}{\lambda_\theta^2 + \gamma^2\lambda_z^2 - \lambda_z^2}. \tag{4.27}$$

Seeing (4.25), the strict dependence of both λ_2 and λ_3 on r (equivalently on R) is apparent. Note, however, that in the presence of the torsional deformation ($\gamma \neq 0$), the principal stretches λ_2 and λ_3 can never become equal at a fixed $r \in [a, b]$ since this then would imply violation of the requirement $(4.25)_1$.

Formulation of the principal stretches λ_2, λ_3 as explicit functions of $\lambda_\theta, \lambda_z$ and γ may also be succeeded in a similar manner. Truly, it is straightforward to show that, by virtue of (4.11), (4.19) and (4.20), the entry $(4.21)_2$ now submits to

$$\lambda_2 = \sqrt{\lambda_\theta^2\cos^2\phi + \lambda_z^2(\gamma\cos\phi + \sin\phi)^2}\ , \tag{4.28}$$

$$\lambda_3 = \sqrt{\lambda_\theta^2\sin^2\phi + \lambda_z^2(\gamma\sin\phi - \cos\phi)^2}\ . \tag{4.29}$$

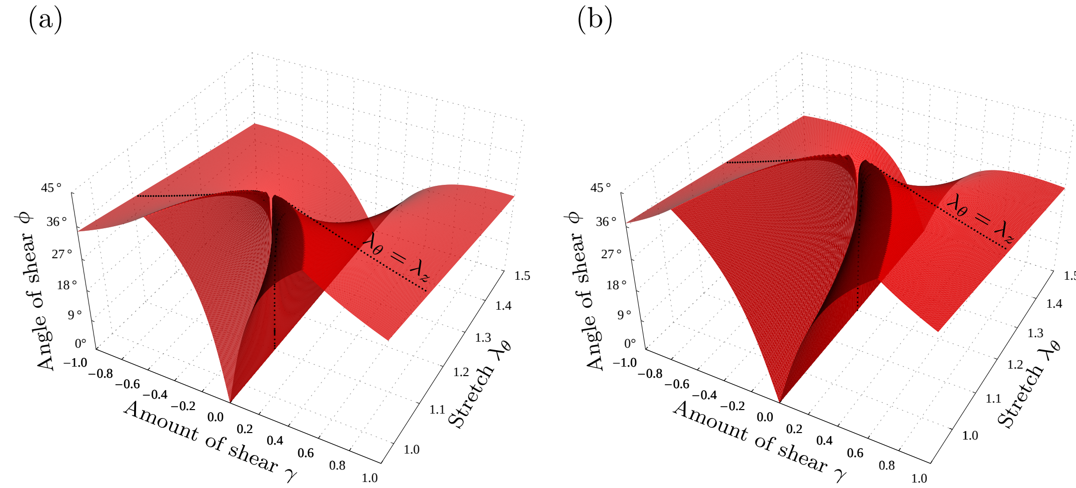

Figure 4.1: Plots of the angle of shear ϕ, as this is prescribed in (4.31), against γ and λ_θ for (a) $\lambda_z = 1.1$ and (b) $\lambda_z = 1.2$. We note that, in both cases, the dotted curve represents the contour $\lambda_\theta = \lambda_z$ of the surface $\phi = \phi(\gamma, \lambda_\theta)$ for which $|\gamma_0| \to 0^+$ and $\phi_{\max} = 45^{\text{o}}$.

Here, ϕ is possible to be defined in many different ways. Notwithstanding, the possibility of representing the angle of shear based on (4.27) alone is excluded from consideration; whereas it is clear that if

$$\lambda_\theta^2 = \lambda_z^2(1 - \gamma^2), \tag{4.30}$$

then ϕ, hence, λ_2 and λ_3, and consequently $\mathbf{F}$, become discontinuous. Thus, to avoid such a situation, we are making use of $(4.26)_2$ together with $(4.25)_{2,3}$ and we write

$$\phi = \frac{1}{4}\cos^{-1}\left[\frac{2(\lambda_\theta^2 + \gamma^2\lambda_z^2 - \lambda_z^2)^2}{\lambda_\theta^4 + \lambda_z^4(\gamma^2+1)^2 + 2\lambda_z^2\lambda_\theta^2(\gamma^2 - 1)} - 1\right]. \tag{4.31}$$

In view of (4.31) it easy to deduce that in the reference configuration, where $\gamma = 0$ and $\lambda_\theta = \lambda_z = 1$, we have $\phi = 0$ for all $R \in [A, B]$ and hence from (4.28) and (4.29) $\lambda_2 = \lambda_3 = 1$.

As the deformation process begins, all three deformation quantities, namely γ, λ_θ and λ_z, have a significant contribution to the way that the angle of shear changes. We clarify, however, that no distinction between anticlockwise and clockwise twisting deformation (i.e. between positive and negative γ respectively) has to be made here since, from (4.31), the angle of shear ϕ appears to be an even function of γ. In order to illustrate the range of the effect of each one of those terms on ϕ, we conduct our analysis in a local scale; that is for a fixed $r \in [a, b]$ in the deformed configuration. Let's first assume that the axial stretch λ_z (> 1) is prescribed and that the azimuthal stretch λ_θ associated with an identified fixed point remains unchanged. Expressed otherwise, we initially consider that the body is axially extended and inflated up to a certain degree and while the torsional deformation takes place the applied pressure P (> 0) and the axial load N, this to be discussed in more detail shortly, are directed in such a way that both the radial and the axial dimensions of the tube are held fixed. Then, by taking $|\tau|$ or, equivalently $|\gamma|$, to increase gradually from zero, the angle of shear is analogously increasing from zero until it reaches a maximum point, $\phi_{\max}$ say, associated with the value $|\tau_0|$ and/ or $|\gamma_0|$. Any further increase in $|\tau|$ beyond the value $|\tau_0|$ clearly yields an increase in $|\gamma|$ which exceeds the value $|\gamma_0|$, yet, the angle of shear starts to decrease. As $|\tau| \to \infty$ and hence $|\gamma| \to \infty$, the angle of shear becomes $\phi = 0$. It should be emphasized that the non-monotonic (local) behaviour of ϕ described here may be associated with any $r \in [a, b]$ and hence with any value of λ_θ being consistent with (4.13)–(4.17). In fact, we will shortly demonstrate that there exists only one isolated value of λ_θ where the nature of ϕ is purely decreasingly monotonic in respect of γ. In any case, the range of variation of ϕ depends to a great extent on the disposition of the point r relative to the boundaries a and b, hence, on the disposition of λ_θ relative to λ_a and λ_b. We illustrate this matter by noting that if the point of selection is taken to satisfy

$$\lambda_z^{-1/2} \leqslant \lambda_\theta \leqslant \lambda_z, \tag{4.32}$$

then the greater the value of λ_θ is, i.e. for r closer to the inner boundary a, the smaller amount of shear $|\gamma|$ (but not necessarily smaller twist $|\tau|$) is required to effect an increase in ϕ. Accordingly, as λ_θ increases from the value $\lambda_z^{-\frac{1}{2}}$ to λ_z, the amount of shear, namely $|\gamma_0|$, identified with the maximum angle of shear constantly decreases. In other words, $|\gamma_0|$ can be perceived as a decreasingly monotonic function of λ_θ. In particular, when (4.32) are substantiated, the value $|\gamma_0|$ is always prescribed from (4.30) in which case we readily deduce that

$$|\gamma_0| = \sqrt{1 - \lambda_z^{-2}\lambda_\theta^2}\ . \tag{4.33}$$

Thus, if $\lambda_\theta = \lambda_z^{-\frac{1}{2}}$ we obtain $|\gamma_0| = \sqrt{1 - \lambda_z^{-3}}$ and once λ_θ reaches the upper boundary λ_z, we have $|\gamma_0| \to 0^+$. From the latter follows that, in the special situation where $\lambda_\theta = \lambda_z$, the angle of shear is gradually decreasing from $\phi_{\max}$ to 0 at the same time in which $|\gamma|$ increases from 0 to ∞. The connection (4.33) further yields the interesting conclusion that, for these points $r \in [a, b]$ being consistent with (4.32), the maximum value of ϕ is always found to be $\phi_{\max} = \pi/4$. Based on the same arguments, we also remark that, by keeping λ_θ to be fixed, an increase in λ_z implicates an increase in $|\gamma_0|$ apart from obviously when

$\lambda_\theta = \lambda_z$. It is apparent, however, that depending on the degree of extension and inflation of the body there may or may not exist a point $r \in [a, b]$ for which the corresponding value of λ_θ satisfies (4.32). We therefore clarify that for the latter to be substantiated for at least one $r \in [a, b]$ we must necessarily have $\lambda_b \leqslant \lambda_z$. Analogously, the possibility of having

$$\lambda_\theta > \lambda_z \tag{4.34}$$

for some fixed $r \in [a, b]$ is also feasible and it requires $\lambda_a > \lambda_z$ with λ_b being either greater or smaller than λ_z. In such a situation the non-monotonic behaviour of ϕ with respect to $|\gamma|$ is still preserved for each distinct value of λ_θ (i.e. for all such $r \in [a, b]$ with $\lambda_\theta > \lambda_z$) and λ_z. Notwithstanding, here the counterpart of (4.33) is defined by

$$|\gamma_0| = \sqrt{-1 + \lambda_z^{-2}\lambda_\theta^2} \tag{4.35}$$

the latter implying that, under the assumption that λ_z is held fixed and finite, the angle of shear can never reach the value $\pi/4$ and also that the value of $|\gamma_0|$ increases with λ_θ; yet the corresponding value of $\phi_{\max}$ decreases. By contrast, we detect that, for each r corresponding to a fixed λ_θ ($> \lambda_z$), larger values of λ_z contribute larger $\phi_{\max}$, this accompanied by smaller $|\gamma_0|$. The results of the combined deformation on the actual behaviour of the angle of shear, as this defined in (4.31), are now demonstrated in Figures 4.1 where, in particular, we plot ϕ against γ and λ_θ for two distinct values of the axial stretch, namely $\lambda_z = 1.1, 1.2$. Further, for the same values of λ_z, Figure 4.2 (a) shows the nature of variation of $|\gamma_0|$ and (b) that of $\phi_{\max}$ in terms of the azimuthal stretch λ_θ, where in both cases we are making use of the expressions (4.33), (4.35) associated with (4.32) and (4.34), respectively.

Following the above discussion, a number of conclusions regarding the way that the principal stretches λ_2 and λ_3 change during the combined deformation can be derived. Such an analysis, however, is very much involved and we do not refer to any details here. We only note in passing that, unlike the angle of shear, the stretches λ_2 and λ_3 do depend on the sense of the shearing deformation. Obviously, for an isotropic material, such a distinction (i.e. considering $\gamma > 0$ and $\gamma < 0$ separately) is of no importance, since, in any case, the material response will be identical. As shown in Chapter 3, however, the direction of shear can be consequential in respect of the mechanical response of a transversely isotropic material. A further discussion on this subject is presented in Section 3.3.

In concluding this section we clarify that the expressions (4.22)–(4.35) are likewise

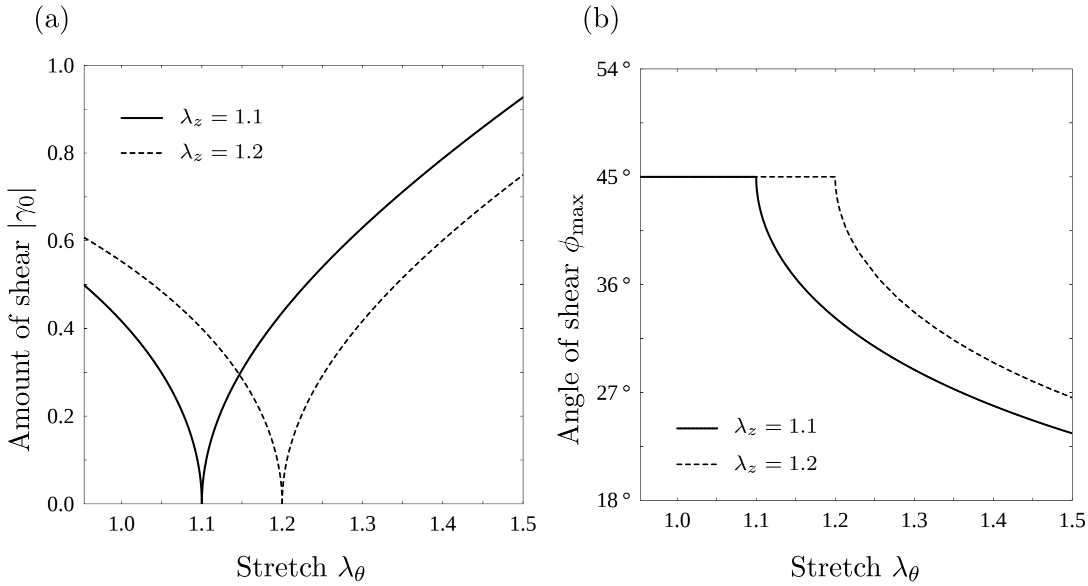

Figure 4.2: Plots of (a) the amount of shear $|\gamma_0|$ and (b) the maximum angle of shear $\phi_{\max}$ against the azimuthal stretch λ_θ for $\lambda_z = 1.1, 1.2$.

validated for the case where the material is taken to be compressible. In such a situation, however, the azimuthal stretch λ_θ, prescribed in (4.6), cannot be expressed a priori as an explicit function of either R or r. Indeed, for unconstrained materials, equation $(4.3)_1$ is replaced by a general expression $r = r(R)$ which simply states the strict dependence of the deformed radius r on R. In this case, the form of r is determined by resolving the associated governing equations; these, assuming that we are concerned with a static problem, obtained from the equilibrium equations discussed in Chapter 1 in conjunction with the boundary conditions (4.4). The subsequent analysis follows closely to the discussion provided by Kirkinis and Ogden [44] to which reference can be made for more details.

4.2.3 Consequences of the incompressibility constraint

In line with the discussion provided in Holzaphel et al. [32], we now illustrate that the combined extension, inflation and torsional deformation can, in general, be represented by four deformation quantities, namely by the cylindrical polar components $E_{RR}, E_{\Theta\Theta}, E_{ZZ}$ and $E_{\Theta Z}$ of the Lagrangian strain tensor $\mathbf{E}$, those, due to (1.44) and (4.10), being determined through

$$\lambda_r^2 = 2E_{RR} + 1, \quad \lambda_\theta^2 = 2E_{\Theta\Theta} + 1, \quad \lambda_z^2(1+\gamma^2) = 2E_{ZZ} + 1, \quad \gamma\lambda_z\lambda_\theta = E_{\Theta Z}, \tag{4.36}$$

respectively. Thus, if, in a like manner, we let E_1, E_2 and E_3 denote the principal components of $\mathbf{E}$, the connections (1.44) and (4.10) can further be used to assign the relations

$$\lambda_i = 2E_i + 1, \quad i \in \{1, 2, 3\} \tag{4.37}$$

and therefore the incompressibility constraint $\lambda_1\lambda_2\lambda_3 = 1$ may equivalently be written as

$$(2E_1 + 1)(2E_2 + 1)(2E_3 + 1) = 1. \tag{4.38}$$

Bearing in mind the connections (4.28), (4.29) derived earlier in Section 4.2.2 for the principal stretches λ_2 and λ_3, respectively, and since, as already mentioned, $\lambda_1 = \lambda_r$, it is now easily understood through (4.36) and (4.37) that connections between the principal and the cylindrical polar components of the strain tensor $\mathbf{E}$ may indeed be established. In fact, it is straightforward to show here that $E_1 = E_{RR}$; the latter also implying that the referential basis vector $\mathbf{E}_R$ is distributed along the Lagriangian principal vector $\mathbf{u}^{(1)}$. Therefore, if, by virtue of (4.38), the component E_{RR} is eliminated, the deformation may be sufficiently prescribed by means of the three remaining independent deformation quantities $E_{\Theta\Theta}, E_{ZZ}$ and $E_{\Theta Z}$ alone, in which case, in view of (4.36), it is apparent that the quantities $\lambda_\theta, \lambda_z$ and γ may be taken to vary independently during the combined deformation.

Thereupon, we cite the properties

$$\lambda_\theta \frac{\partial \lambda_2}{\partial \lambda_\theta} + \gamma \frac{\partial \lambda_2}{\partial \gamma} = \lambda_2 \cos^2 \phi, \quad \lambda_\theta \frac{\partial \lambda_2}{\partial \lambda_\theta} + \lambda_z \frac{\partial \lambda_2}{\partial \lambda_z} = \lambda_2, \tag{4.39}$$

$$\lambda_\theta \frac{\partial \lambda_3}{\partial \lambda_\theta} + \gamma \frac{\partial \lambda_3}{\partial \gamma} = \lambda_3 \sin^2 \phi, \quad \lambda_\theta \frac{\partial \lambda_3}{\partial \lambda_\theta} + \lambda_z \frac{\partial \lambda_3}{\partial \lambda_z} = \lambda_3, \tag{4.40}$$

arising from (4.28), (4.29) and (4.31).

4.3 Transversely isotropic materials

In what follows, the mechanical response of incompressible transversely isotropic elastic solid tubes undergoing a combined extension, inflation and torsional deformation is examined. In particular, following closely the investigation presented in Chapter 3, here we are mainly concerned with material bodies featuring inhomogeneous elastic properties.

4.3.1 Materials with inhomogeneous properties

We begin our analysis by considering a unit vector $\mathbf{M}$, defined in the reference configuration, serving to identify locally a preferred direction. Further, we assume that $\mathbf{M}$ lies in the (R, Θ)-planes of the undeformed body and, in particular, that it is characterized by the properties (3.6)–(3.9).

The nature of the combined deformation, as this is predicated on the deformation gradient tensor (4.5), now furnishes that, when the body is brought to its deformed state, the vector $\mathbf{M}$ becomes $\mathbf{m}$, this defined by

$$\mathbf{m} = \mathbf{F}\mathbf{M} = (\lambda_z \lambda_\theta)^{-1} M_R \mathbf{e}_r + \lambda_\theta M_\Theta \mathbf{e}_\theta. \tag{4.41}$$

The three dimensional character of the problem illustrated in the previous section, in conjunction with the incompressible and anisotropic attribute of the materials under consideration, qualify the presence of four independent invariants, namely

$$I_1 = \mathrm{tr}\,(\mathbf{B}) = (\lambda_z \lambda_\theta)^{-2} + \lambda_\theta^2 + \lambda_z^2(\gamma^2 + 1), \tag{4.42}$$

$$I_2 = \frac{1}{2}[\mathrm{tr}\,(\mathbf{B})^2 - \mathrm{tr}\,(\mathbf{B}^2)] = \lambda_\theta^{-2}(\gamma^2 + 1) + (\lambda_z \lambda_\theta)^2 + \lambda_z^{-2}, \tag{4.43}$$

$$I_4 = \mathbf{m} \cdot \mathbf{m} = (\lambda_z \lambda_\theta)^{-2} M_R^2 + \lambda_\theta^2 M_\Theta^2, \tag{4.44}$$

$$I_5 = \mathbf{m} \cdot (\mathbf{B}\mathbf{m}) = (\lambda_z \lambda_\theta)^{-4} M_R^2 + (\lambda_\theta^4 + \gamma^2 \lambda_z^2 \lambda_\theta^2) M_\Theta^2, \tag{4.45}$$

associated with the deformation and the preferred direction $\mathbf{M}$. We note that the invariants (4.44), (4.45) reduce to $I_4 = \sqrt{I_5} = (\lambda_z \lambda_\theta)^{-2}$ when $\mathbf{M} = \mathbf{E}_R$ (i.e. $\alpha(R) = 0$) and analogously, for $\mathbf{M} = \mathbf{E}_\Theta$ (i.e. $\alpha(R) = \pi/2$), they specialize to $I_4 = \lambda_\theta^2$ and $I_5 = \lambda_\theta^4 + \gamma^2 \lambda_z^2 \lambda_\theta^2$, respectively. In these two cases, the torsional deformation corresponds to a simple shear deformation normal and parallel to the direction of $\mathbf{M}$, respectively. For the time being, however, we adopt the restrictions $M_R, M_\Theta \in (0, 1)$ or, equivalently $\alpha(R) \in (0, \pi/2)$, for all $R \in [A, B]$, since we will shortly clarify that the aforementioned cases may be accountable for unsustainable modes of deformation.

In any case, here we observe that, unlike the pure azimuthal shear problem examined in Chapter 3, the torsional deformation incorporated in our problem has no actual influence on the length of the line elements distributed along the preferred direction of the body. Indeed, it is apparent from (4.44) that the invariant I_4, being responsible for measuring stretch in the direction of $\mathbf{M}$, depends only on λ_z and λ_θ. In other words, only the extension and inflation processes can generate a change in I_4 the status of which, for each distinct point $r \in [\alpha, b]$, depends to a great extend on the orientation of $\mathbf{M}$ at each relevant point $R \in [A, B]$ in the reference configuration.

Let us first assume that the body is subjected to uniaxial extension and held there fixed. In that configuration, where $\lambda_\theta = \lambda_z^{-\frac{1}{2}}$ throughout the body, it is straightforward from (4.41) that $\mathbf{m} = \lambda_z^{-\frac{1}{2}}\mathbf{M}$, thus (4.44) assigns $I_4 = \lambda_z^{-1}$ (< 1) and hence contraction (of the same degree) is taking place in the $\mathbf{M}$ direction at each point of the tube. Accordingly, at this stage of the process the distribution of $\alpha(R)$ is not consequential in I_4. By contrast, when we apply pressure P of such amount that causes inflation of the tube, the angle $\alpha(R)$ plays a critical role in the status of I_4. For this reason it is important to recall the discussion provided in Section 3.2.2 where there it was clearly explained that $\tan\alpha = M_\Theta/M_R$, hence $\alpha(R)$ can be a constant, a monotonic increasing or decreasing function of R or none of these. Nevertheless, we now assume for convenience that $\alpha(R)$ increases with R. Keeping that in mind, we now explicate that, as the inflation begins, the nature of I_4 appears to be non-monotonic in λ_θ, yet only for those points which, in the reference configuration, are associated with $\alpha(R) \in (0, \pi/4)$. Expressly, for such geometries of the angle $\alpha(R)$, it follows from (4.44) that, as λ_θ increases from the value $\lambda_z^{-\frac{1}{2}}$, the resulting value of I_4 is steadily decreasing from λ_z^{-1} until it reaches it minimum value, namely $I_4^{\min} = \lambda_z^{-1}\sin(2\alpha)$, corresponding to the stretch

$$\lambda_{\min} = (\lambda_z \tan\alpha)^{-\frac{1}{2}}. \tag{4.46}$$

Thereafter, for $\lambda_\theta > \lambda_{\min}$, the invariant I_4 becomes monotonic increasing and, in particular, it regains its initial status $I_4 = \lambda_z^{-1}$ (i.e. just before the inflation begins) when the azimuthal stretch attains the value

$$\lambda_* = \sqrt{\frac{1 + \sqrt{1 - 4M_R^2 M_\Theta^2}}{2\lambda_z M_\Theta^2}}\,, \tag{4.47}$$

following the route to $I_4 = 1$ at $\lambda_\theta = \lambda^*$, where

$$\lambda^* = \sqrt{\frac{1 + \sqrt{\lambda_z^2 - 4M_R^2 M_\Theta^2}}{2M_\Theta^2}}\,. \tag{4.48}$$

Finally, once λ_θ exceeds λ^*, extension of the preferred direction is taking place since then I_4 becomes greater than 1. For $\alpha(R) \in [\pi/4, \pi/2)$, on the other hand, inflation of the tube is always associated with a monotonically increasing I_4. More specifically, equation (4.44) now prescribes that I_4 starts from its minimum value, that being $I_4 = \lambda_z^{-1}$ at $\lambda_\theta = \lambda_z^{-\frac{1}{2}}$, steadily increases as λ_θ becomes larger and then, analogously to the previous case, $I_4 \gtrless 1$ according as $\lambda_\theta \gtrless \lambda^*$.

In Figure 4.3, the relative placements of the values $\lambda_{\min}, \lambda_*, \lambda^*$ are shown together with a plot of I_4 as a function of λ_θ. It is important to clarify that, given $\alpha \in (0, \pi/2)$, the formula (4.44) always yields $\partial I_4/\partial\alpha \geqslant 0$ with the equality holding only for $\lambda_\theta = \lambda_z^{-\frac{1}{2}}$.

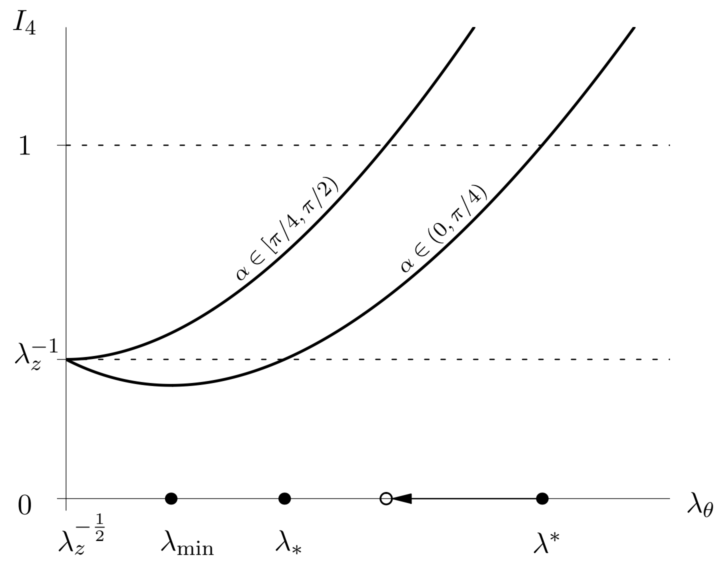

Figure 4.3: Plot of the invariant I_4 against λ_θ showing the relative positions of $\lambda_{\min}, \lambda_*, \lambda^*$. Note that the symbol $\bullet$ represents the above mentioned critical values of the stretch λ_θ associated with those points which, in the reference configuration, correspond to $\alpha(R) \in (0, \pi/4)$. Analogously, for $\alpha \in [\pi/4, \pi/2)$ the only critical value of λ_θ, namely λ^*, is represented by the symbol $\circ$.

Since all the above mentioned results have a purely local character, i.e. they apply only for fixed values of either R or r, the latter remark furnishes that, for a prescribed value of λ_θ corresponding to a point $r \in [a, b]$, the values of I_4 will in general be greater if the point of selection is associated with a larger $\alpha(R)$ in the reference configuration. We emphasize, however, that even under the premise that α is increasing with R, largest values of I_4 are not necessarily to be expected either closer to the boundary $R = B$ (equivalently $r = b$) or closer to $R = A$ (equivalently $r = a$); the is latter always associated with the largest λ_θ. In order to illustrate this matter we first show, in Figure 4.4, the nature of the values $\lambda_{\min}, \lambda_*$ and λ^* defined by (4.46), (4.47) and (4.48), respectively, as functions of M_R which, for a monotonically increasing $\alpha(R)$, may be perceived to decrease with R. It is there clearly demonstrated that when $M_R = \sqrt{2}/2$ (i.e. $\alpha = \pi/4$) we have $\lambda_{\min} = \lambda_* = \lambda_z^{-\frac{1}{2}}$ and as M_R becomes greater, implying that α decreases from $\pi/4$, these quantities gradually increase and always comply with $\lambda_* > \lambda_{\min}$. Similarly, the value of λ^* ($> \lambda_z^{-\frac{1}{2}}$), this being defined for $M_R \in (0, 1)$, has a monotonically increasing character. Note that for any $M_R \geqslant \sqrt{2}/2$ we obtain the nesting $\lambda^* > \lambda_* \geqslant \lambda_{\min}$. It is therefore obvious that, analogously to λ_θ, all the terms $\lambda^*, \lambda_*, \lambda_{\min}$ can be treated as monotonically decreasing functions of R and equivalently of r. Hence, although the azimuthal stretch is always larger for those r closer to the inner boundary a (i.e. for M_R closer to 1), the preferred direction is more likely

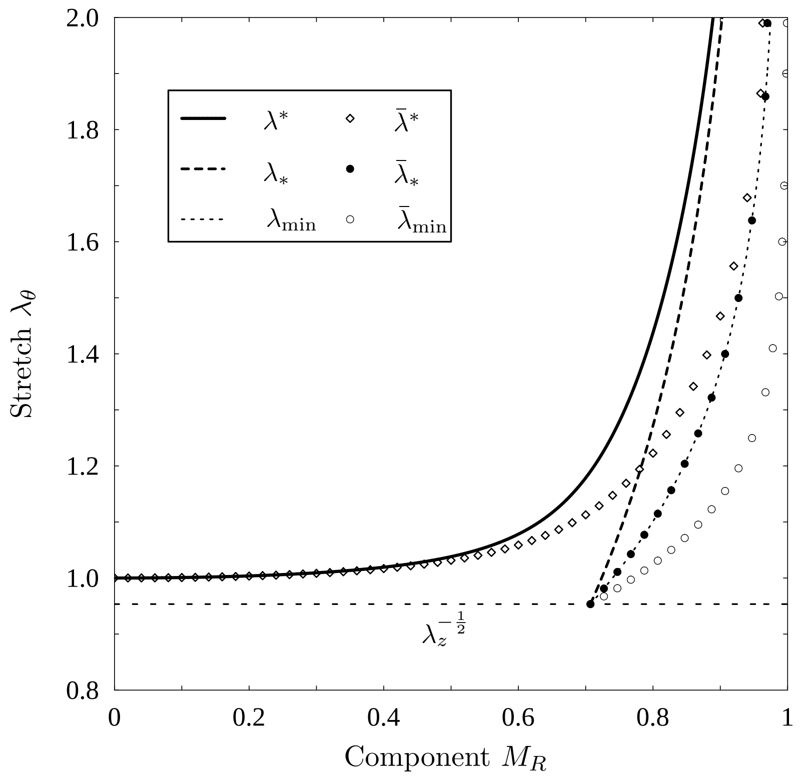

Figure 4.4: Plots of the critical values $\lambda_{\min}, \lambda_*, \lambda^*$ and $\bar{\lambda}_{\min}, \bar{\lambda}_*, \bar{\lambda}^*$, as those prescribed by (4.46)–(4.48) and (4.49)–(4.51), respectively, against the component M_R in (λ_θ, M_R) space for $\lambda_z = 1.1$. Note that in the case where $M_R = \sqrt{2}/2$ (i.e. $\alpha = \pi/4$) we have $\lambda_* = \lambda_{\min} = \bar{\lambda}_* = \bar{\lambda}_{\min} = \lambda_z^{-\frac{1}{2}}$.

to undergo contraction ($I_4 < 1$), if not being $I_4 < \lambda_z^{-1}$, since for such points the values of all $\lambda^*, \lambda_*, \lambda_{\min}$ become extremely large. Especially in the limiting situations where the boundary $r = a$ is taken to be associated with $M_R \to 1^-$ we obtain $\lambda_{\min} \to +\infty$ and hence, for a finite inflation, it is unlikely to have $I_4 > \lambda_z^{-1}$ at the inner surface of the tube. On the hand, as we approach $r = b$, the component M_R decreases and so does the critical value λ^*, where $I_4 = 1$, and eventually, for $M_R \to 0^+$, it becomes $\lambda^* \to 1$. Thus, here relative small stretch is sufficient to generate extension of the preferred direction. We remark however that a necessary condition for $I_4 \geqslant 1$ to be substantiated at some $r \in [a, b]$ is $\lambda_a \geqslant 1$ and yet extension of the preferred direction is not necessarily initiated at the inner boundary $r = a$.

In contrast to the situation described for I_4, the torsional part of the combined deformation is incorporated in the invariant I_5. Indeed, it is apparent through (4.45) that, apart from λ_θ and λ_z, the invariant I_5 also depends on the amount of shear γ. Note, however, that here both positive and negative γ effect on I_5 in exactly the same way. We now remark that, when the body is subjected to uniform axial extension alone, this requiring $\lambda_\theta = \lambda_z^{-\frac{1}{2}}$ and $\gamma = 0$ throughout the body, equation (4.45) reduces to $I_5 = \lambda_z^{-2}$ $(= I_4^2)$ and it is therefore clear that at this stage of the process we have $I_5 < I_4 < 1$. Further, it is a straightforward task to deduce that when the inflation begins, with $\gamma = 0$, the qualities of I_5 are exactly analogous with those provided earlier for I_4. In passing, assuming that $\alpha \in (0, \pi/4)$, we have I_5 to decrease from its starting value, namely $I_5 = \lambda_z^{-2}$, until when λ_θ reaches the value

$$\bar{\lambda}_{\min} = (\lambda_z^2 \tan\alpha)^{-\frac{1}{4}}, \tag{4.49}$$

corresponding to the minimum $I_5^{\min} = \lambda_z^{-2}\sin(2\alpha)$. After that, I_5 gradually increases with an increasing λ_θ $(> \bar{\lambda}_{\min})$ and, in particular, in the case where the azimuthal stretch takes the value

$$\bar{\lambda}_* = \left(\frac{1+\sqrt{1-4M_R^2M_\Theta^2}}{2\lambda_z M_\Theta^2}\right)^{1/4}, \tag{4.50}$$

it regains the value $I_5 = \lambda_z^{-2}$. Finally, when λ_θ reaches the value $\bar{\lambda}^*$, the latter defined by

$$\bar{\lambda}^* = \left(\frac{1+\sqrt{\lambda_z^2-4M_R^2M_\Theta^2}}{2M_\Theta^2}\right)^{1/4}, \tag{4.51}$$

the invariant I_5 becomes equal to 1, while for $\lambda_\theta > \bar{\lambda}^*$ we obtain $I_5 > 1$. On the other hand, and parallel to the behaviour of I_4, whenever $\alpha \in [\pi/4, \pi/2)$ the invariant I_5 $(\geqslant \lambda_z^{-2})$ is always increasing with λ_θ $(\geqslant \lambda_z^{-\frac{1}{2}})$. Note that once more the quantity $\bar{\lambda}^*$ serves to render this value of λ_θ identified with $I_5 = 1$. Analogously to their counterparts, denoted $\lambda_{\min}, \lambda_*, \lambda^*$, the critical values (4.49)–(4.51) of the stretch λ_θ associated with I_5 are found to satisfy $\bar{\lambda}^* < \bar{\lambda}_* \leqslant \bar{\lambda}_{\min}$ with the equality holding strictly for $\alpha = \pi/4$. For illustration, these quantities are also plotted in Figure 4.4 as functions of the component M_R. It is also worth mentioning the connections

$$\bar{\lambda}_{\min} = \sqrt{\lambda_z^{-\frac{1}{2}}\lambda_{\min}}, \quad \bar{\lambda}_* = \sqrt{\lambda_*}, \quad \bar{\lambda}^* = \sqrt{\lambda^*}, \tag{4.52}$$

wherefrom the behaviour of I_5 can be studied in parallel with that of I_4.

At this point, we explicate that in the case where the torsional deformation is also taken into account, the resulting simple shear (locally) taking place normal to the radial direction of the deformed body always entails an increase in I_5; that regardless the nature of the angle $\alpha(R)$. As a matter of fact, a straightforward differentiation of (4.45) with respect to γ shows that, if the azimuthal and the axial stretches are prescribed (i.e. the body is extended and inflated up to a certain degree), the invariant I_5 complies with $\partial I_5/\partial\gamma > 0$ provided that $\alpha(R) \in (0, \pi/2)$ and $\gamma \neq 0$. We remark, however, that, not surprisingly, for those points which, in the reference configuration, are associated with larger α, the shearing effect is more pronounced on I_5 whereas, as for I_4, here we have $\partial I_5/\partial\alpha > 0$. Evidently, during the combined deformation, and given that λ_z is held fixed, the invariant I_5 is apprehended as a function of the deformation quantities λ_θ and γ, those, as explained in Section 4.2.3, taken to vary independently. Despite this matter, it is important to clarify that, depending on the contribution of γ, larger λ_θ may or may not submit larger I_5. As a matter of fact, in line with the results obtained in the previous paragraphs, we now deduce that, no matter what the value of γ might be, the invariant I_5 is always found to increase monotonically with λ_θ, yet strictly for those $r \in [a, b]$ being in conformity with $\alpha(R) \in [\pi/4, \pi/2)$ in the reference configuration. For the points associated with $\alpha(R) \in (0, \pi/4)$, however, the contribution of γ has a significant effect on the way that I_5 changes with λ_θ whereas the presence of the shearing deformation now establishes that

$$\frac{\partial I_5}{\partial\lambda_\theta} \gtreqless 0 \quad \text{according as} \quad |\gamma| \gtreqless \frac{\sqrt{2(1-\lambda_z^4\lambda_\theta^8\tan^2\alpha)}}{\lambda_z^3\lambda_\theta^3\tan\alpha}\,. \tag{4.53}$$

We remark that (4.53) is substantial only for $\lambda_z^{-\frac{1}{2}} \leqslant \lambda_\theta < \bar{\lambda}_{\min}$ since, otherwise, we obtain $\partial I_5/\partial\lambda_\theta \geqslant 0$ with the equality holding strictly for $\lambda_\theta = \bar{\lambda}_{\min}$ which, following $(4.53)_2$, requires $\gamma = 0$ and hence $I_5 = I_5^{\min}$. Accordingly, we deduce that whenever $\lambda_z^{-\frac{1}{2}} \leqslant \lambda_\theta < \bar{\lambda}_{\min}$ with the second part of (4.53) holding as an equality the derivative $\partial I_5/\partial\lambda_\theta$ vanishes but the invariant I_5 does not attain a minimum since, for any $\gamma \neq 0$, we have $\partial I_5/\partial\gamma > 0$. Nevertheless, it is worth mentioning that, the connections $\lambda_z^{-\frac{1}{2}} \leqslant \lambda_\theta < \bar{\lambda}_{\min}$ can be used to identify the range of $r \in [a, b]$ for which I_5 is possible to decrease with λ_θ while, in the same grounds, the condition

$$|\gamma| < \sqrt{\frac{2(\cot^2\alpha - 1)}{\lambda_z^3}},$$

furnishes the maximal range of $|\tau|$ corresponding to each one of those r so as the previous argument is possible to be substantiated.

In concluding this section it is also worth mentioning in passing that, due to the contribution of γ, it is possible to have $I_5 \geqslant 1$ even for very small values of λ_θ. This fact maintains arguably that the condition $I_4 \geqslant 1$ yields $I_5 \geqslant 1$ but the opposite is not necessarily true.

4.3.2 Strain-energy function and Cauchy stress

By recalling the discussion provided in Section 1.4.3, we may now write down the strain-energy function, namely W, in the general form

$$W = \bar{W}(I_1, I_2, I_4, I_5), \tag{4.54}$$

from which follows that the Cauchy stress tensor admits the representation

$$\boldsymbol{\sigma} = 2\bar{W}_1\mathbf{B} + 2\bar{W}_2(I_1\mathbf{I} - \mathbf{B})\mathbf{B} + 2\bar{W}_4\mathbf{m}\otimes\mathbf{m} + 2\bar{W}_5(\mathbf{m}\otimes\mathbf{Bm} + \mathbf{Bm}\otimes\mathbf{m}) - \bar{p}\mathbf{I}, \tag{4.55}$$

with $\bar{p}$ denoting the Lagrange multiplier associated with the incompressible nature of the materials under study, $\mathbf{I}$ being the identity tensor and $\bar{W}_i = \partial\bar{W}/\partial I_i, i \in \{1, 2, 4, 5\}$. Accordingly, as clearly explained in Section 1.5.1, the connections (4.54) and (4.55) then implicate the restrictions

$$\bar{W} = 0, \quad 2\bar{W}_1 + 4\bar{W}_2 = \bar{p}_0, \quad \bar{W}_4 + 2\bar{W}_5 = 0, \tag{4.56}$$

which designate a body whose energy and stress vanish in the reference configuration (i.e. where $I_1 = I_2 = 3$ and $I_4 = I_5 = 1$) with $\bar{p}_0$ now representing the value of $\bar{p}$ in that configuration. Note that the restrictions (4.56) may also accompanied with

$$\begin{aligned}
&\bar{W}_1 + \bar{W}_2 = [c_{12} - c_{13} - c_{23} + c_{33} - c_{44} + c_{55} + c_{66}]/6,\\
&\bar{W}_1 + \bar{W}_2 + \bar{W}_5 = [c_{13} - c_{12} + c_{23} - c_{33} + c_{44} + 2(c_{55} + c_{66})]/6,\\
&\bar{W}_{44} + 4\bar{W}_{45} + 4\bar{W}_{55} = [3(c_{11} + c_{22}) - 8(c_{13} + c_{23} - c_{33}) - 16(c_{55} + c_{66}) + 2c_{12} + 4c_{44}]/12,
\end{aligned} \tag{4.57}$$

which constitute the necessary conditions that W should satisfy having regard to the linear theory of incompressible transversely isotropic materials whose preferred direction is taken to be defined by $\mathbf{M} = M_R\mathbf{E}_R + M_\Theta\mathbf{E}_\Theta$. In particular, the connections (4.57) are apprehended as the counterparts of (3.19), (3.19) for the case where the deformation is considered to take place in three dimensions; this being the situation for the combined deformation examined in this chapter. Analogously to the discussion provided in Section 3.3, the parameters $c_{11}, ..., c_{66}$ are in general functions of the angle $\alpha(R)$ and are identified with the properties (3.21), those, in particular, defined once more in the reference configuration with the cylindrical polar coordinates (R, Θ, Z). It should also be emphasized that, on the basis of the results demonstrated in Section 1.5.1, the derivatives of W may be found to satisfy different and yet equivalent conditions in the reference configuration apart from those derived here. Indeed, appropriate restriction on the strain-energy function may well be imposed by means of a different set of elastic parameters which, nevertheless, due the symmetries of W, are possible to be put into correspondence with the ones appearing in (4.57). In any case, the present approach is mainly used for consistency with the notation adopted in the previous chapters and in particular with the results presented in Section 2.5 since in the special case where $M_R = 1$ (i.e. $\alpha = 0$) the connections (2.21) are again validated and hence (4.57) are easily found to specialize to (2.117)–(2.119).

We now substitute (4.11) and (4.42) into (4.55) to deduce that, for the problem under examination, the cylindrical polar components of $\boldsymbol{\sigma}$ can be read off as

$$\begin{aligned}\sigma_{rr} =& 2\bar{W}_1(\lambda_z\lambda_\theta)^{-2} + 2\bar{W}_2[\lambda_z^{-2} + \lambda_\theta^{-2}(\gamma^2+1)] \\ &+ [2\bar{W}_4(\lambda_z\lambda_\theta)^{-2} + 4\bar{W}_5(\lambda_z\lambda_\theta)^{-4}]M_R^2 - \bar{p}, \end{aligned}\tag{4.58}$$

$$\begin{aligned}\sigma_{\theta\theta} =& 2\bar{W}_1(\lambda_\theta^2 + \lambda_z^2\gamma^2) + 2\bar{W}_2(\lambda_z^{-2} + \gamma^2\lambda_\theta^{-2} + \lambda_z^2\lambda_\theta^2) \\ &+ [2\bar{W}_4\lambda_\theta^2 + 4\bar{W}_5(\lambda_\theta^4 + \gamma^2\lambda_z^2\lambda_\theta^2)]M_\Theta^2 - \bar{p}, \end{aligned}\tag{4.59}$$

$$\sigma_{zz} = 2\bar{W}_1\lambda_z^2 + 2\bar{W}_2(\lambda_\theta^{-2} + \lambda_z^2\lambda_\theta^2) - \bar{p}, \tag{4.60}$$

$$\sigma_{\theta z} = 2\bar{W}_1(\gamma\lambda_z^2) + 2\bar{W}_2(\gamma\lambda_\theta^{-2}) + 2\bar{W}_5(\gamma\lambda_z^2\lambda_\theta^2)M_\Theta^2, \tag{4.61}$$

$$\sigma_{r\theta} = [2\bar{W}_4\lambda_z^{-1} + 2\bar{W}_5(\lambda_z^{-1}\lambda_\theta^2 + \gamma^2\lambda_z + \lambda_z^{-3}\lambda_\theta^{-2})]M_R M_\Theta, \tag{4.62}$$

$$\sigma_{rz} = 2\bar{W}_5(\gamma\lambda_z)M_R M_\Theta. \tag{4.63}$$

In view of the connections (4.42)–(4.45) and whereas $\tan\alpha = M_\Theta/M_R$, we are, analogously to the methodology provided in Chapter 3, prompted to introduce a new function, namely $\tilde{W}$, such that

$$\tilde{W}(\lambda_\theta, \lambda_z, \gamma, \alpha) = \bar{W}(I_1, I_2, I_4, I_5). \tag{4.64}$$

Then, differentiation of (4.64) yields the simple connections

$$\sigma_{\theta\theta} - \sigma_{rr} = \lambda_\theta\frac{\partial\tilde{W}}{\partial\lambda_\theta} + \gamma\frac{\partial\tilde{W}}{\partial\gamma}, \tag{4.65}$$

$$\sigma_{\theta\theta} + \sigma_{zz} - 2\sigma_{rr} = \lambda_\theta\frac{\partial\tilde{W}}{\partial\lambda_\theta} + \lambda_z\frac{\partial\tilde{W}}{\partial\lambda_z}, \tag{4.66}$$

for the normal stresses, while

$$\sigma_{\theta z} = \frac{\partial\tilde{W}}{\partial\gamma}. \tag{4.67}$$

As we will shortly see, the expressions (4.65)–(4.67) are of significant importance for determining the applied forces required to maintain the combined deformation. In what follows, we also establish that for any elastic material for which the strain energy can be regarded as a function of the deformation variables $\lambda_\theta, \lambda_z$ and γ for the considered deformation the formulas (4.65)–(4.67) apply. Clearly, given that the preferred direction

$\mathbf{M}$ varies with R, the quantity α is incorporated in (4.64) as a material parameter, not as a deformation variable, and, as in (3.26), its inclusion reflects the fact that the material properties are inhomogeneous. Following that, we are able to deduce that the explicit dependence of the strain-energy on α enables the linkages

$$\lambda_z\lambda_\theta^2\frac{\partial\tilde{W}}{\partial\alpha} = (\lambda_z^2\lambda_\theta^4 - 1)\sigma_{r\theta} + \gamma\sigma_{rz}, \tag{4.68}$$

$$2\gamma\lambda_z^3\sigma_{\theta z} = \{\gamma\lambda_z^3(\sigma_{\theta\theta} - \sigma_{zz}) - \tan\alpha[\gamma\lambda_z^4\lambda_\theta^2\sigma_{r\theta} + (\lambda_z^4\lambda_\theta^2 - 1)\sigma_{rz}]\}\tan 2\phi, \tag{4.69}$$

with ϕ being given in (4.31). We comment that if the preferred direction $\mathbf{M}$ coincides with the direction of the basis vector $\mathbf{E}_R$ (i.e. $\alpha = 0$) then the connections (4.62) and (4.63) ensure that (4.68) is satisfied identically while, at the same time, the connection (4.69) agrees with the universal relation (see also [32])

$$\tan 2\phi = \frac{2\sigma_{\theta z}}{\sigma_{\theta\theta} - \sigma_{zz}}\ . \tag{4.70}$$

As a matter of fact, it is worth clarifying that, when $\mathbf{M}$ is chosen as above, the explicit dependance of $\tilde{W}$ on α may be dropped and hence the strain energy can sufficiently be prescribed by means of the deformation quantities $\lambda_\theta, \lambda_z$ and γ. This further implies that $\mathbf{M}$ becomes an eigenvector of $\mathbf{C}$ (and $\mathbf{m}$ of $\mathbf{B}$) and that the strain-energy can treated as a function of the principal stretches λ_1, λ_2 and λ_3 or, equivalently, as a function of the principal invariants invariants I_1, I_2 and I_4 alone since, as already mentioned, in such a case we obtain the connection $I_5 = \sqrt{I_4} = (\lambda_z\lambda_\theta)^{-2}$. Thus, the entry (4.64) may now be replaced by

$$W = W(\lambda_1, \lambda_2, \lambda_3) = \tilde{W}(\lambda_\theta, \lambda_z, \gamma) = \bar{W}(I_1, I_2, I_4), \tag{4.71}$$

while the Cauchy stress tensor can be written down in the spectral form

$$\boldsymbol{\sigma} = \sum_{i=1}^{3}\sigma_i\mathbf{v}^{(i)}\otimes\mathbf{v}^{(i)}, \tag{4.72}$$

with the principal stresses, namely σ_i, being defined through

$$\sigma_i = \lambda_i\frac{\partial W}{\partial\lambda_i} - p, \quad i \in \{1, 2, 3\}. \tag{4.73}$$

Note that here p is again a Lagrange multiplier which is, in general, different from the one appearing in (4.55) and/or (4.58)–(4.60). Thus, by virtue of (4.72), it is a straightforward task to deduce that all the non-zero cylindrical polar components of $\boldsymbol{\sigma}$ now satisfy

$$\sigma_{rr} = \sigma_1, \quad \sigma_{\theta z} = (\sigma_2 - \sigma_3)\cos\phi\sin\phi, \tag{4.74}$$

$$\sigma_{\theta\theta} = \sigma_2\cos^2\phi + \sigma_3\sin^2\phi, \quad \sigma_{zz} = \sigma_2\sin^2\phi + \sigma_3\cos^2\phi, \tag{4.75}$$

wherefrom, after some simple manipulation, the relation (4.70) may once more be established. Combination of (4.73)–(4.75) in addition yields

$$\sigma_{\theta\theta} - \sigma_{rr} = \lambda_2 \frac{\partial W}{\partial \lambda_2} \cos^2 \phi + \lambda_3 \frac{\partial W}{\partial \lambda_3} \sin^2 \phi - \lambda_1 \frac{\partial W}{\partial \lambda_1}, \tag{4.76}$$

$$\sigma_{\theta\theta} + \sigma_{zz} - 2\sigma_{rr} = \lambda_2 \frac{\partial W}{\partial \lambda_2} + \lambda_3 \frac{\partial W}{\partial \lambda_3} - 2\lambda_1 \frac{\partial W}{\partial \lambda_1}, \tag{4.77}$$

$$\sigma_{\theta z} = \left(\lambda_2 \frac{\partial W}{\partial \lambda_2} - \lambda_3 \frac{\partial W}{\partial \lambda_3} \right) \cos\phi \sin\phi, \tag{4.78}$$

those, due to (4.71), (4.39) and (4.40), being identical with (4.65)–(4.67) respectively. Evidently, exactly the same conclusions may be derived for the case where the material is taken to be isotropic with the only difference that, unlike the present situation, W then has to be symmetric in λ_1, λ_2 and λ_3 and similarly $\bar{W}$ has to depend strictly on the invariants I_1 and I_2.

If, on the other hand, the preferred direction complies with $\mathbf{M} = \mathbf{E}_\Theta$ (i.e. $\alpha = \pi/2$) then both (4.68) and (4.69) are automatically satisfied. Note, however, that such a specialization does not imply that $\mathbf{M}$ becomes an eigenvector of $\mathbf{C}$ and hence (4.71)–(4.78) are no longer validated. In particular, since the choice $\mathbf{M} = \mathbf{E}_\Theta$ renders a constant α, the strain energy can still be determined by means of the principal stretches of the deformation and yet the invariants I_4 and I_5 remain independent; those being determined through $I_4 = \lambda_\theta^2$ and $I_5 = \lambda_\theta^4 + \gamma^2\lambda_\theta^2\lambda_z^2$, respectively. Even then, the expressions (4.65)–(4.67) are valid but the stress components $\sigma_{\theta\theta}, \sigma_{zz}$ and $\sigma_{\theta z}$ are no longer consistent with (4.70).

It is finally worth mentioning that, in line with (4.13)–(4.17), the connections (4.68) and (4.69) may be further used to identify that, during the combined deformation, the conditions

$$\mathcal{F}_1(r) \geqslant 0, \qquad \mathcal{F}_2(r) \geqslant 0, \tag{4.79}$$

with

$$\mathcal{F}_1(r) = (\lambda_z \lambda_\theta^2 \frac{\partial \tilde{W}}{\partial \alpha} - \gamma \sigma_{rz})\sigma_{r\theta}, \tag{4.80}$$

$$\mathcal{F}_2(r) = \{[\sigma_{\theta\theta} - \sigma_{zz} - 2\sigma_{\theta z} \cot 2\phi - \sigma_{r\theta} \tan\alpha]\gamma \cot\alpha + (\lambda_z^{-3} - 1)\sigma_{rz}\}(\sigma_{rz} + \gamma\sigma_{r\theta}), \tag{4.81}$$

should always hold at each $r \in [a, b]$. This means that, for the deformation to be sustainable, the data λ_z, a and τ which, as shown in (4.3) are necessary to fully characterize the position of each material particle in the deformed configuration (i.e. the deformation itself), have to be chosen in such a way that the constitutive law is consistent with the conditions (4.79).

4.3.3 Resolution of the equilibrium equations and applied forces

Despite the three-dimensional character of the deformation under consideration, the strict dependence of the strain-energy function on either the deformed radial component r or, equivalently, on the referential radial component R of the body, is evident. Owing to that and assuming the absence of body forces, the equilibrium equation $\operatorname{div}\boldsymbol{\sigma} = \mathbf{0}$ has three non-trivial components, namely the radial equation

$$\frac{\mathrm{d}\sigma_{rr}}{\mathrm{d}r} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0, \tag{4.82}$$

conjoined with

$$\frac{\mathrm{d}(r^2\sigma_{r\theta})}{\mathrm{d}r} = 0, \qquad \frac{\mathrm{d}(r\sigma_{rz})}{\mathrm{d}r} = 0, \tag{4.83}$$

referred to as the azimuthal and the axial equation, respectively.

It is then apparent that, using the boundary conditions (4.4) introduced earlier in Section 4.2.1, the radial equation may be integrated to deliver the expression

$$P = \int_a^b (\sigma_{\theta\theta} - \sigma_{rr})\frac{\mathrm{d}r}{r}, \tag{4.84}$$

through which the necessary pressure required to cause inflation of the cylindrical tube can be calculated. In a like manner, integration of the equations (4.83) enable the formulations

$$\sigma_{r\theta} = \frac{\tau_\theta b^2}{r^2}, \qquad \sigma_{rz} = \frac{\tau_z b}{r}, \tag{4.85}$$

for the azimuthal and axial stress respectively. Note that the parameters τ_θ, τ_z in (4.85) are constants representing the values of $\sigma_{r\theta}$ and σ_{rz}, respectively, on the outer boundary $r = b$ of the deformed tube. Depending on the data available, equations (4.85) have a multiple role to play. First, for a body whose preferred direction is specified, those serve to determine the shearing forces required to be applied on the boundary $r = b$, i.e. the forces identified with the values τ_θ and τ_z, so as, for certain stress-strain response and distribution associated with a certain degree of extension, inflation and torsion, the circular cylindrical shape of the body to be maintained. Alternatively, assuming that the shearing response of the body in the planes $\theta =$ constant and $z =$ constant is directed in a specific manner and that the body is subjected to a certain degree of uniform axial extension (i.e. the values of τ_θ, τ_z and λ_z are known), they may be used to determine the pair (λ_θ, γ) and consequently identify the data (a, τ) associated with the response of the body due to inflation and twisting deformation. Analogously, the pair (α, λ_z) or (τ, λ_z) may also be calculated given the data τ or α, respectively. Further to that, if a certain overall stress-strain response and distribution is required during the combined deformation, then equations (4.85) can be used to identify the preferred directions, in other words the components M_R and M_Θ or $G(R)$ itself, for which such a deformation is sustainable. It is important to emphasize, however, that no matter what the implications and/or the interpretations of (4.85) might be, the stress components $\sigma_{r\theta}$ and σ_{rz} have, in any case, to be consistent with (4.79)–(4.81).

From a different point of view, another important result may also be gleaned on the basis of the aforementioned equations. Precisely, in view of (4.62) and (4.63) it is apparent that if the components M_R and M_Θ are taken to vanish locally (i.e. clearly not simultaneously) at some point in the reference configuration then, in the deformed configuration, the corresponding values of both shear stress components $\sigma_{r\theta}$ and σ_{rz} have to vanish as well. Following (4.85), we are then to deduce that $\tau_\theta = \tau_z = 0$ and hence the conditions $\sigma_{r\theta} = \sigma_{rz} = 0$ should be substantiated throughout the deformed body; that is for all $r \in [a, b]$. The latter, however, implicates the requirements

$$\bar{W}_4 = \bar{W}_5 = 0, \qquad \text{for all } r \in [a, b], \tag{4.86}$$

those yielding the false conclusion that, independently of the broad distribution of $\mathbf{M}$ in the reference configuration, the actual contribution of the preferred direction in the overall response of the deforming body has to be negligible, i.e. the anisotropy has no influence during the combined deformation. As a result, we readily infer that the combined deformation *cannot* be sustainable for these materials which *locally* embody the properties $M_R = 0$ or $M_\Theta = 0$ and equivalently $\alpha = 0, \pi/2$.

We now illustrate that, apart from the applied pressure prescribed in (4.84), application of the coupled forces N and M, those respectively defined via

$$N = 2\pi \int_a^b \sigma_{zz} r \mathrm{d}r, \quad M = 2\pi \int_a^b \sigma_{\theta z} r^2 \mathrm{d}r,$$

is also required for the body to be held in its deformed state. In particular, here N expresses the axial load that needs to be applied on the ends of the tube so that this maintains the length $l = \lambda_z L$ and M is the corresponding twisting moment of the stress $\sigma_{\theta z}$ (about the axis $r = 0$) owing exclusively to the torsional part of the combined deformation. After some simple manipulation, the connections (4.65)–(4.67) may be used to contribute the formulas

$$P = \int_a^b \left(\lambda_\theta \frac{\partial \tilde{W}}{\partial \lambda_\theta} + \gamma \frac{\partial \tilde{W}}{\partial \gamma} \right) \frac{\mathrm{d}r}{r}, \tag{4.87}$$

$$N = \pi \int_a^b \left(2\lambda_z \frac{\partial \tilde{W}}{\partial \lambda_z} - \lambda_\theta \frac{\partial \tilde{W}}{\partial \lambda_\theta} - 3\gamma \frac{\partial \tilde{W}}{\partial \gamma} \right) r\mathrm{d}r + Pa^2\pi, \tag{4.88}$$

$$M = 2\pi \int_a^b \frac{\partial \tilde{W}}{\partial \gamma} r^2 \mathrm{d}r, \tag{4.89}$$

for these forces which, according to the discussion provided earlier, apply for any geometry $\alpha(R) \in [0, \pi/2]$. Obviously, the situation in which $\alpha = 0, \pi/2$ implies $M_R = 1, 0$ for all $R \in [A, B]$. Also, by recalling (4.68) and (4.69), it is easily understood that, given $\alpha(R) \in [0, \pi/2)$, the quantities (4.87)–(4.89) are possible to be expressed in various and yet equivalent forms but we do not present any details here.

In view of (4.87)–(4.89) we may further conclude that the necessary loads that should be applied on the body in order to support the combined deformation are found to satisfy

$$N = Pa^2\pi - \frac{3}{2}M\tau + \pi \int_\alpha^b \left(2\lambda_z \frac{\partial \tilde{W}}{\partial \lambda_z} - \lambda_\theta \frac{\partial \tilde{W}}{\partial \lambda_\theta} \right) r\mathrm{d}r. \tag{4.90}$$

Therefore, by denoting $N_{\mathrm{ral}} = N - Pa^2\pi$ the so-called *reduced axial load* describing the actual measured force at the edges of the tube (see, e.g., [57, 32]), the linkage (4.90) assigns the relation

$$N_{\mathrm{ral}} = -\frac{3}{2}M\tau + \pi \int_\alpha^b \left(2\lambda_z \frac{\partial \tilde{W}}{\partial \lambda_z} - \lambda_\theta \frac{\partial \tilde{W}}{\partial \lambda_\theta} \right) r\mathrm{d}r. \tag{4.91}$$

We remark that, by virtue of (4.87)–(4.89) and their subsequent expression (4.90) and/or (4.91), the assertion of a material which is incapable of distinguishing between negative and positive sense of shearing deformation is once more justified. Indeed, for the considered class of transversely isotropic materials, the arguments (4.42)–(4.45), in conjunction with (4.64), readily suggest that both the pressure and the axial load associated with the combined deformation remain invariant when interchanging the direction of the twist, namely τ. In other words the integrals (4.87) and (4.88) appear to be even functions of the parameter τ, i.e. $P(-\tau) = P(\tau)$ and $N(-\tau) = N(\tau)$. On the other hand, the resultant moment M does, not surprisingly, depend on the direction of shear. Note, however, that (4.89) states $M(-\tau) = -M(\tau)$; the latter implying that the force associated exclusively with the torsional part of the combined deformation changes its direction in parallel with the direction of the twist, yet its actual magnitude depends solely on the value of $|\tau|$. Exactly the same conclusion can be gleaned through (4.91) whereas, based on the qualities of P and N, it is apparent that $N_{\mathrm{ral}}(-\tau) = N_{\mathrm{ral}}(\tau)$, from which the previous argument follows as a direct consequence. This fact now establishes that the overall mechanical response of the materials under investigation is, unlike the situation described in Chapter 3 for the case of the pure azimuthal shear deformation, invariant to the direction of the torsional deformation.

From the discussion provided so far it is easily understood that for specific forms of the strain-energy function we may be led to unsustainable modes of deformation. Precisely, we illustrate that the conditions (4.79) introduced in the previous section are not sufficient to establish the requirement $P \geqslant 0$ and hence additional restrictions on the constitutive law may be imposed to do so. Evidently, for the pressure to retain its non-negative nature, the constraint

$$\sigma_{\theta\theta} - \sigma_{rr} \geqslant 0, \quad \text{for all} \quad r \in [a, b], \tag{4.92}$$

would be a legitimate limitation to adopt while, assuming the growth conditions

$$\frac{\partial \tilde{W}}{\partial \gamma} \to \pm\infty, \qquad \text{for} \quad \gamma \to \pm\infty,$$

and bearing in mind the form of M it would also be appropriate to adopt

$$\sigma_{\theta z} \equiv \frac{\partial \tilde{W}}{\partial \gamma} \gtreqless 0 \quad \text{according as} \quad \gamma \gtreqless 0. \tag{4.93}$$

Interestingly, for the case of pure torsion, where from (4.69) and (4.27) it is evident that $\gamma\sigma_{\theta z} = \sigma_{\theta\theta} - \sigma_{zz} - \tan\alpha\,\sigma_{r\theta}$, the restrictions (4.93) analogously imply

$$\sigma_{\theta\theta} - \sigma_{zz} \gtreqless \tan\alpha\,\sigma_{r\theta} \quad \text{according as} \quad \gamma \gtreqless 0.$$

When the axial extension and/or inflation of the tube is taken into account, we may, on the basis of (4.92), (4.93) and in conjunction with (4.68) and (4.69), derive further restrictions on the components of $\boldsymbol{\sigma}$ and/or on the form of the strain-energy function sufficient to ensure a mechanically bearable material response. We remark, however, that the above mentioned requirements are sufficient and not necessary to establish a meaningful deformation and therefore they appear to be too restrictive. Instead, when appropriate, i.e. depending on form of the strain-energy function, suitable restrictions on the data a, λ_z and τ may be imposed, those, in particular, varying with respect to the elastic properties of the materials under investigation.

For the general combined deformation we finally illustrate that, following the work of Ogden [57] (see also references therein), the identities (4.13) and (4.14) may be suitably manipulated to provide the connection

$$r\frac{\mathrm{d}\lambda_\theta}{\mathrm{d}r} = -\lambda_\theta(\lambda_z\lambda_\theta^2 - 1), \tag{4.94}$$

which can then be applied to (4.87)–(4.89) and change the independent variable from r to λ_θ. In particular, if there is no torsion involved (i.e. $\gamma = 0$), equation (4.94) enables us to exactly recover the formulations for both P and N derived in [57] for the case where the tube is subjected to extension and inflation only now the material properties are, in general, considered to be inhomogeneous. In the same spirit, assuming that all three independent deformation processes are taken into account, we can alternatively interchange the variable r incorporated in the integrands of the above mentioned quantities with γ since $\gamma = \tau r$ and hence $\mathrm{d}r = \tau\mathrm{d}\gamma$.

4.3.4 A different class of anisotropic materials

In this section we present a brief discussion for another class of transversely isotropic materials under combined extension, inflation and torsional deformation. Specifically, here we are once more concerned with circular cylindrical structures featuring a single preferred direction which is, in the reference configuration, locally characterized by a unit vector $\mathbf{M}$. This is now defined by

$$\mathbf{M} = M_\Theta\mathbf{E}_\Theta + M_Z\mathbf{E}_Z. \tag{4.95}$$

Note that in (4.95) the components M_Θ and M_Z serve in essence to identify the angle, ψ say, between the preferred direction and the circumferential direction of the tube. Accordingly, we may write $M_\Theta = \cos\psi$ and $M_Z = \sin\psi$ and hence $\tan\psi = M_Z/M_\Theta$. Generally speaking, it is apparent that the angle ψ may be taken to vary with position (i.e. it may depend on one or more of R, Θ and Z), yet here we assume for simplicity that $\psi =$ constant throughout the undeformed tube; this indicating materials whose mechanical properties are homogeneous. We remark that to the extent that the transverse isotropy studied here is associated with fibre reinforcement, the vector field (4.95) may be perceived to model (in a continuous manner) a single family of helical fibres (spirals) situated in a circular cylindrical tube. Analogous fibre geometries (mainly two symmetric spiral families) are often met in the literature [73, 15, 72] in respect of deformable elastic solids and especially lately for the study of the mechanical response of soft tissues (see, e.g., [32, 21]).

In view of the above discussion we may, without loss of generality, adopt the restriction $\psi \in [0, \pi/2]$. In particular, for $\psi = \pi/2$ the preferred direction (4.95) reduces to $\mathbf{M} = \mathbf{E}_Z$

and it can therefore be accounted to represent a family of perfectly straight fibres aligned with the axis of the tube. For $\psi = 0$, on the other hand, (4.95) becomes $\mathbf{M} = \mathbf{E}_\Theta$, this, as for the previously mentioned case associated with $\alpha = \pi/2$, being understood to represent a family of circular fibres distributed on the (R, Θ)-planes of the tube which are, in fact, centered at $(R, \Theta) = (0, 0)$.

For the considered geometry and deformation, the status of the preferred direction in the deformed configuration is now characterized by the vector field

$$\mathbf{m} = (\lambda_\theta \cos\psi + \gamma\lambda_z \sin\psi)\mathbf{e}_\theta + \lambda_z \sin\psi \mathbf{e}_z. \tag{4.96}$$

The actual contribution of $\mathbf{M}$ in the response of the body is, in this case, incorporated through the invariants

$$I_4 = (\lambda_\theta \cos\psi + \gamma\lambda_z \sin\psi)^2 + \lambda_z^2 \sin^2\psi, \tag{4.97}$$

$$\begin{aligned} I_5 =& (\lambda_\theta^2 \cos\psi + \gamma\lambda_z\lambda_\theta \sin\psi)^2 + (\gamma\lambda_z\lambda_\theta \cos\psi + \lambda_z^2 \sin\psi)^2 \\ &+ \lambda_z^4(\gamma^4 + 2\gamma^2)\sin^2\psi + \gamma^3\lambda_z^3\lambda_\theta \sin(2\psi), \end{aligned} \tag{4.98}$$

while the invariants I_1, I_2 are still being given by (4.42) and (4.43), respectively. Not surprisingly, here we observer that, unlike their counterparts (4.44) and (4.45), the invariants I_4 and I_5 do (in general) distinguish between positive and negative sense of shearing deformation. This argument is not, however, factual when $\psi = 0$ since then (4.97) and (4.98) specialize to $I_4 = \lambda_\theta^2$ and $I_5 = \lambda_\theta^4 + \gamma^2\lambda_z^2\lambda_\theta^2$, respectively. Although the formulas (4.97) and (4.98) may be further used to assign the exact (local) effect of the combined deformation on the preferred direction such an investigation is beyond the scopes of this thesis. We remark, however, that many details regarding the variation of I_4 in respect of the considered deformation and geometry may be understood on the basis of the analysis provided by Qiu and Pence [63], while, for the invariant I_5, the present situation follows closely the work presented by Merodio and Ogden [51] to which reference can be made for a more extensive discussion.

In line with (4.54) and (4.64), the strain-energy function may now be expressed through

$$W = \tilde{\tilde{W}}(\lambda_\theta, \lambda_z, \gamma, \psi) = \bar{W}(I_1, I_2, I_4, I_5), \tag{4.99}$$

where again the quantity ψ is incorporated in $\tilde{\tilde{W}}$ as a material parameter and not as a deformation variable. Accordingly, when $\mathbf{M}$ is defined as above, the formula (4.55) once validates the restrictions (4.56) provided that, in the reference configuration, the body is taken to be free of energy and stress, while, for consistency with the linear theory, expressions similar to (4.57) can also be derived but we do not need them here. In any case, we remark that, due to the different qualities of W, the form and/or the value of the Lagrange multiplier (i.e. both in the reference and deformed configurations) incorporated in (4.55) will now be different compared to the one applying for the strain-energies (4.64).

Owing to (4.95)–(4.99), the non-trivial cylindrical polar components of $\boldsymbol{\sigma}$ are now given by

$$\sigma_{rr} = 2\bar{W}_1(\lambda_z\lambda_\theta)^{-2} + 2\bar{W}_2[\lambda_z^{-2} + \lambda_\theta^{-2}(\gamma^2+1)] - \bar{p}, \tag{4.100}$$

$$\begin{aligned}\sigma_{\theta\theta} =& 2\bar{W}_1(\lambda_\theta^2 + \lambda_z^2\gamma^2) + 2\bar{W}_2(\lambda_z^{-2} + \gamma^2\lambda_\theta^{-2} + \lambda_z^2\lambda_\theta^2)\\ &+ (\lambda_\theta\cos\psi + \gamma\lambda_z\sin\psi)\{2\bar{W}_4(\lambda_\theta\cos\psi + \gamma\lambda_z\sin\psi)\\ &+ 4\bar{W}_5[(\lambda_\theta^2 + \gamma^2\lambda_z^2)(\lambda_\theta\cos\psi + \gamma\lambda_z\sin\psi) + \gamma\lambda_z^3\sin\psi\} - \bar{p},\end{aligned} \tag{4.101}$$

$$\begin{aligned}\sigma_{zz} =& 2\bar{W}_1\lambda_z^2 + 2\bar{W}_2(\lambda_\theta^{-2} + \lambda_z^2\lambda_\theta^2) + \lambda_z^2\sin\psi\{2\bar{W}_4\sin\psi\\ &+ 4\bar{W}_5[\gamma\lambda_\theta\lambda_z\cos\psi + \lambda_z^2(\gamma^2+1)\sin\psi]\} - \bar{p},\end{aligned} \tag{4.102}$$

$$\begin{aligned}\sigma_{\theta z} =& 2\bar{W}_1(\gamma\lambda_z^2) + 2\tilde{W}_2(\gamma\lambda_\theta^{-2}) + 2\bar{W}_4[\lambda_z\sin\psi(\lambda_\theta\cos\psi + \gamma\lambda_z\sin\psi)]\\ &+ 2\bar{W}_5\{\lambda_z(\lambda_\theta\cos\psi + \gamma\lambda_z\sin\psi)[(2\gamma^2\lambda_z^2 + \lambda_\theta^2 + \lambda_z^2)\sin\psi\\ &+ \gamma\lambda_z\lambda_\theta\cos\psi] + \gamma\lambda_z^4\sin^2\psi\},\end{aligned} \tag{4.103}$$

from which it is evident that, for the considered class of materials, the presence of the axial and azimuthal shear stress components $\sigma_{r\theta}, \sigma_{rz}$ is unnecessary to support the combined deformation. Despite this fact, the formulas (4.101)–(4.103) are now unable to appoint a connection between $\sigma_{\theta\theta}, \sigma_{zz}$ and $\sigma_{\theta z}$. Such connections may only be derived by involving additional combinations of the partial derivatives $\bar{W}_4$ and $\bar{W}_5$. The same conclusion applies even for the simple cases where $\psi = \pi/2, 0$.

It is interesting however that, for once more, a set of straightforward differentiations on (4.99) and comparison with the entries (4.100)–(4.103) yields the connections (4.65)–(4.67) with $\tilde{\tilde{W}}$ in the place of $\tilde{W}$. Thence, due to the absence of the shear stresses $\sigma_{r\theta}$ and σ_{rz}, it is apparent that here the equilibrium equation reduces only to (4.82) (i.e. $\tilde{\tilde{W}}$ is strictly a function of r) while it also arises that the applied pressure P, the axial load N and the resultant moment M are formulated again by the analogous of (4.87), (4.88) and (4.89), respectively, wherefrom (4.91) is once more substantiated. It also is worthwhile noting that for pure torsion the homogeneous properties of the materials under consideration enable, analogously to the discussion provided in the previous section, the formulation $P = [\tilde{\tilde{W}}]_{\gamma_a}^{\gamma_b}$. In concluding this section, we further remark that significant qualitative and quantitative differences in P, N and M are to be expected here in respect of positive and negative sense of shearing deformation and especially when the angle ψ is taken to be sufficiently large; in such cases the overall response of the body under deformation is profoundly different by means of positive and negative γ.

Although in this thesis our main focus is on transversely isotropic materials, it is worthwhile noting that by considering incompressible elastic tubes reinforced with two families of helical fibres (i.e. featuring two preferred direction) we are able to derive exactly the analogous formulas as those given in (4.65)–(4.67). It is therefore understood that the expressions (4.87)–(4.89) derived earlier for P, N and M, respectively, further apply for the case of cylindrically orthotropic materials. We mention in passing that, for such materials, we may, apart from (4.95), identify a second preferred direction, defined by the unit vector

$$\mathbf{M}' = M'_\Theta \mathbf{E}_\Theta + M'_Z \mathbf{E}_Z$$

such that $M'_\Theta = \cos\beta$ and $M'_Z = -\sin\beta$. Clearly, here β is the counterpart of ψ serving to designate the angle between the direction of $\mathbf{M}'$ and the circumferential direction of the undeformed body. Accordingly, the strain-energy function can here be expressed as

$$W = \hat{\hat{W}}(\lambda_\theta, \lambda_z, \gamma, \psi, \beta) = \bar{\bar{W}}(I_1, I_2, I_4, I_5, I_6, I_7, I_8, I_9), \tag{4.104}$$

with the invariants I_1, I_2, I_4, I_5 being given by (4.42), (4.43), (4.97), (4.98), respectively, while the remaining invariants are defined through

$$I_6 = \mathbf{M}' \cdot (\mathbf{C}\mathbf{M}'), \quad I_7 = \mathbf{M}' \cdot (\mathbf{C}^2\mathbf{M}'), \quad I_8 = \mathbf{M} \cdot (\mathbf{C}\mathbf{M}'), \quad I_9 = (\mathbf{M} \cdot \mathbf{M}')^2. \tag{4.105}$$

In the light of (4.104) and (4.105) we may therefore use the same methodology provided earlier to justify our assertions concerning the form of P, N and M for the specific class of orthotropic materials.

We summarize our results by noting that the approach used here, this being established on the basis of treating the deformation quantities γ, λ_θ and λ_z independently, enables a unified formulation of the quantities P, N and M in respect of a very large range of incompressible materials (i.e. including isotropic and anisotropic) under combined extension, inflation and torsion. We mention, however, that this may be alternatively be done by means of two, by definition independent, principal stretches, λ_2, λ_3 say, yet use of the quantities $\lambda_\theta, \lambda_z$ and γ allows us to have a better understanding of the degree of the influence of each one of the deformations separately. We finally mention that use of the quantities $\lambda_\theta, \lambda_z$ and γ may possibly provide the way to examine even more complicated reinforcements; by considering, for example, three or four fibre families at the same time. This part is, however, still under examination and will be presented elsewhere.

4.4 Numerical illustrations

In this section we illustrate some of the kinematical aspects of the combined extension, inflation and torsional deformation discussed so far together with their implications in respect of the response of circular cylindrical tubes within the class of materials featuring the qualities described in Section 4.3.1. For numerical purposes we fix the radial dimensions of the undeformed body and in particular, for consistency with Section 3.8, we take $A = 1$ (units) and $B = 6$ (units). In order to keep our calculations simple, we consider that, in any case, the strain-energy function is of the separable form

$$\bar{W}(I_1, I_2, I_4, I_5) = E(I_1) + T(I_5), \tag{4.106}$$

from which it is apparent that the explicit dependence of the strain energy on the invariants I_2 and I_4 is essentially dropped. Precisely, we adopt the formulation introduced in Merodio and Ogden [51] by specializing the latter so that $E(I_1)$ is taken to be represented by the neo-Hookean isotropic material model augmented with $T(I_5)$, this being the counterpart for I_5 of the standard reinforcing model used in Chapter 3. Accordingly, we write

$$E(I_1) = \frac{1}{2}\mu(I_1 - 3), \quad T(I_5) = \frac{1}{2}\mu\rho(I_5 - 1)^2, \tag{4.107}$$

thus, equation (4.106) specializes to

$$\bar{W} = \frac{1}{2}\mu\left[I_1 - 3 + \rho(I_5 - 1)^2\right]. \tag{4.108}$$

We remark once more that the parameter μ (> 0) is a constant which represents the shear modulus of the isotropic base material and ρ (> 0) is a material constant that characterizes the degree of anisotropy associated with the presence of the preferred direction. Depending on way that $\mathbf{M}$ is distributed in the undeformed configuration, both μ and ρ may be appropriately expressed by means of the parameters $c_{11}, ..., c_{66}$, but we do not use this approach here.

4.4.1 Radial reinforcement

First we assume a circular cylindrical tube with a radial $\mathbf{M}$. Here, the qualities of the preferred direction signify the absence of the shear stresses $\sigma_{r\theta}$ and σ_{rz} and hence, according to the analysis provided in Section 4.3.2, there is no need to restrict the strain-energy function as suggested through (4.79)–(4.81) in order to accomplish a sustainable combined deformation. Clearly, the kinematical restrictions (4.16)–(4.17) are still required for the inflation to take place but apart from those the combined deformation is in any case eligible since for the particular geometry of $\mathbf{M}$ the inequalities (4.92) and (4.93) are always satisfied automatically. Recall that, when $\mathbf{M}$ is chosen as above, the shearing deformation has no actual impact on the status of the preferred direction while it is also unnecessary to distinguish between positive and negative γ whereas in both cases the quantities P, N_{ral} and M appear essentially to be identical.

In what follows, the material model (4.108) is used to exemplify the nature of the forces, namely P, N_{ral} and M, necessary to support the combined deformation. Expressly, in the light of (4.87)–(4.91) and since, from $(4.3)_1$, we have $b = [a^2 + \lambda_z^{-1}(B^2 - A^2)]^{1/2}$, it is clear that $\tilde{W}$ $(= \bar{W})$ depends strictly on the deformation variables $\lambda_\theta, \lambda_z$ and γ and obviously that all P, N_{ral} and M may be treated as functions of the parameters a, τ and λ_z. On these grounds, the implications of each one of the aforementioned deformation parameters in the mechanical response of radially reinforced tubes is demonstrated in Figures 4.5–4.9.

In detail, in Figure 4.5(a) the applied pressure, now prescribed in the dimensionless form $P^* = P/\mu$, is plotted in respect of the material model (4.106) against the inner deformed radius a for a fixed amount of axial extension, namely $\lambda_z = 1.15$. There, three separate cases, associated with three values, $\tau = 0, 0.1, 0.15$, of the twist are also considered, while the degree of anisotropy of the materials under examination is taken to be identified with $\rho = 1.5$. Note that the isotropic case is also included for comparison. Bearing in mind the restrictions (4.15)–(4.17), it is then shown that, for the specific choice of the above mentioned parameters and as the inflation process begins, the applied pressure P^* is initially increasing with a $(\geqslant \lambda_z^{-1/2} A \approx 0.9325)$, it reaches a maximum and thereafter gradually decreases. Evidently, for very large a the pressure required to inflate an isotropic and an anisotropic tube appears to be identical. This non-monotonic behaviour of P^* is not, however, surprising, since the particular choice of $\mathbf{M}$ implies that whenever severe contraction is taking place along the radial direction (i.e. this is obviously the case here), the fibre reinforcement, as attributed to (4.107), reaches a minimum at $I_5 = 1/2$. Accordingly, the contribution of $T(I_5)$ to the dimensionless radial stress component σ_{rr}/μ, this predicated on $4I_5\bar{W}_5/\mu = 4I_5T'(I_5)/\mu$, becomes maximum at that stage and this is directly reflected in $P^* = \sigma_{rr}(a)/\mu$. It is important to distinguish that the twist τ has no

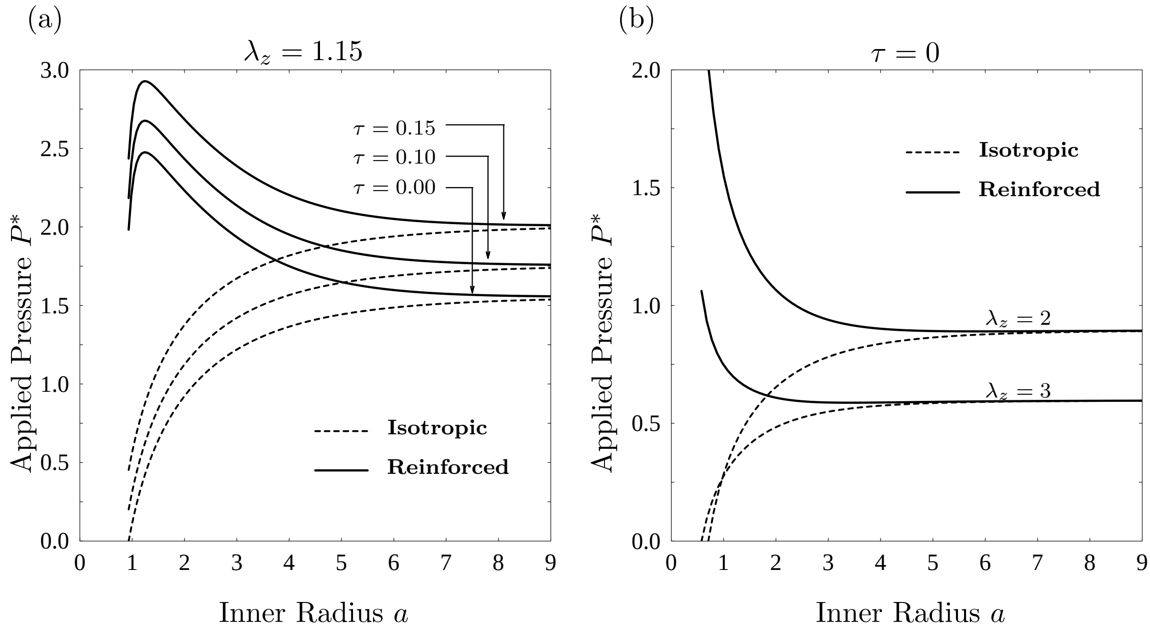

Figure 4.5: Plots of the dimensionless pressure P^* against the deformed inner radius a of the tube for the radially reinforced material model (4.108) as well as for the isotropic neo-Hookean model. In (a) the curves (P^*, a) are identified with $\lambda_z = 1.15$ while the twist is assumed to be prescribed by three distinct values, namely $\tau = 0, 0.1, 0.15$. Similarly, in (b) the (P^*, a) curves are demonstrated for $\tau = 0.1$ and $\lambda_z = 2, 3$.

actual impact on the qualitative properties of the curves (P^*, a). Indeed, for any choice of λ_z and ρ, an increase in τ always results in higher pressure P^* for each a, yet the nature of the curves (P^*, a) remains unchanged and strictly dependent on the axial extension and the degree of anisotropy characterizing the body. We mention, for instance, that given $\lambda_z = 1.15$ and $\rho = 1.5$, the pressure P^* always attains a maximum at $a \approx 1.244$ regardless the value of τ. On the other hand, depending on the amount of axial extension and the degree of anisotropy, the pressure P^* may appear to have a completely different character while changing with an increasing a.

Generally speaking, for any fixed ρ, larger values of λ_z are responsible for reduction in P^* and especially when the inflation process is associated with relatively large a. We clarify this matter by noting that, for each ρ, there exists a certain range of λ_z for which P^* behaves non-monotonically with a. When λ_z and ρ are as such, larger axial extension is initially expected to cause an increase in P^*, i.e. compared to that required to cause the same degree of inflation for smaller λ_z. Thereafter, however, the deformed inner radius reaches a certain value, this varying with ρ, beyond which less pressure is sufficient to

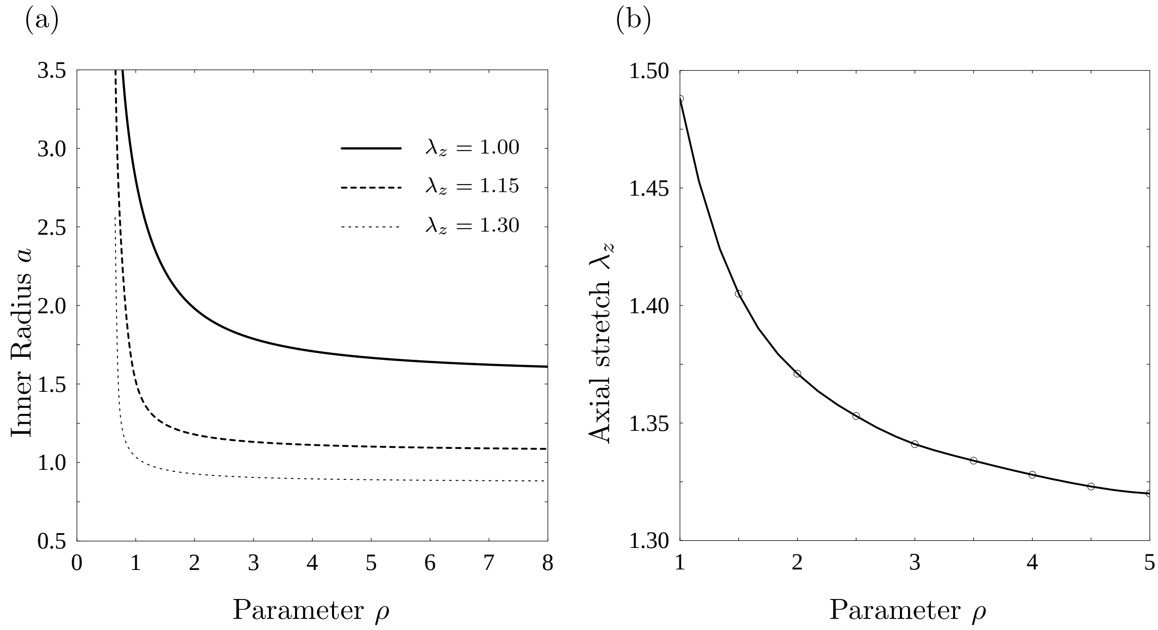

Figure 4.6: Plots of (a) the value of a at which the dimensionless pressure P^* is maximized and (b) the maximal possible axial stretch λ_z establishing a non-monotonic relation of P^* with a as functions of the parameter ρ for the material model (4.106) and provided that $\mathbf{M} = \mathbf{E}_R$. In (a) the curves (a, ρ) are demonstrated for $\lambda_z = 1, 1.15, 1.3$.

prescribe the same inflation deformation compared to that associated with smaller λ_z. Having said that, it is apparent that when, for each distinct ρ, the axial stretch λ_z exceeds that certain value, any further increase of the latter effects a monotonic relation between P^* and a and accordingly larger λ_z are followed by a vertical transition (i.e. downwards) of the curves (P^*, a). Those arguments are now clearly demonstrated in Figure 4.5(b) where P^* is plotted against a for $\rho = 1.5$ and $\lambda_z = 2, 3$. Note that no twist is taken to be incorporated there since, as already explained, the torsional deformation has no influence on the qualities of P^*. We emphasize that for extremely large λ_z the non-monotonic relation of P^* and a is possible to be re-established, yet now associated with the presence of a minimum in P^* and followed by an upward transition of the curve (P^*, a) compared to that corresponding to smaller λ_z. Nevertheless, such situations are not taken into account here since these are most likely to render unstable modes of deformation.

The dependence of the variation of the value of a at which P^* attains a maximum on the parameter ρ is also depicted in Figure 4.6(a) for $\lambda_z = 1, 1.15, 1.3$. We observe that for larger ρ and λ_z the internal pressure P^* is maximized at smaller a. Finally, in Figure 4.6(b) the curve (λ_z, ρ) provides the upper limit of λ_z associated with each ρ for

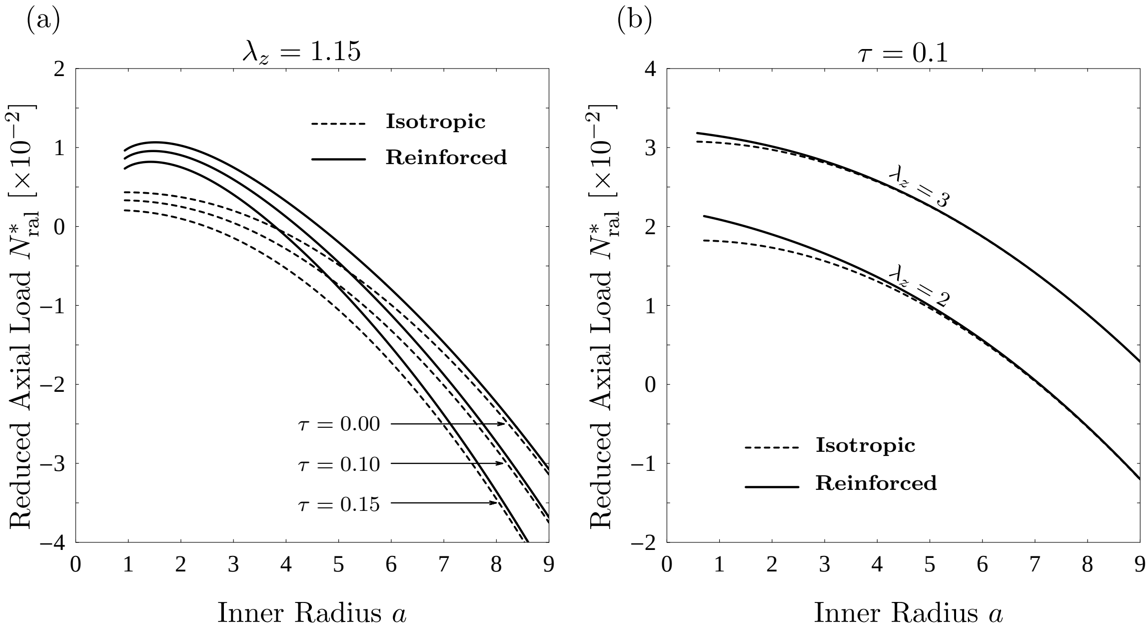

Figure 4.7: Plots of the dimensionless reduced axial load N^*_{ral} versus the deformed inner radius a for the isotropic neo-Hookean as well as for the radially reinforced material model (4.108); this being characterized by $\rho = 1.5$. In (a) the curves (N^*_{ral}, a) are demonstrated for $\lambda_z = 1.15$ and $\tau = 0, 0.1, 0.15$ while in (b) for $\lambda_z = 2, 3$ together with $\tau = 0.1$.

which the pressure P^* exhibits a non-monotonic relation with a as that demonstrated in Figure 4.5(a).

In an analogous manner, the behaviour of the reduced axial for both the radial reinforced as well for the isotropic body under the considered combined deformation is demonstrated in Figures 4.7 and 4.8. Precisely, in 4.7(a), the quantity $N^*_{\mathrm{ral}} = N_{\mathrm{ral}}/\mu$ is plotted against the deformed inner radius a with the parameters of interest being once more specified as $\lambda_z = 1.15$ and $\tau = 0, 0.2, 0.15$ as mentioned above while, for the anisotropic case, we have chosen again $\rho = 1.5$. Accordingly, we notice that the qualitative properties of the pressure P^* are directly reflected upon the reduced axial load N^*_{ral}. Indeed, the non-monotonic relation between N^*_{ral} and a is also detected here for the case of the radial reinforced tube. It should be noted that the value of a for which N^*_{ral} possesses a maximum is, in general, different for that corresponding to maximum pressure P^* and yet again both quantities appear to respond in a very similar manner in respect of the various choices of λ_z and ρ. In Figure 4.7(b), we see, for example, that by keeping ρ to be fixed, larger λ_z constitute a monotonic N^*_{ral} when this is taken to vary with a. We remark, however, that here the twist τ has a significant influence on the qualities of the curves (N^*_{ral}, a), whereas,

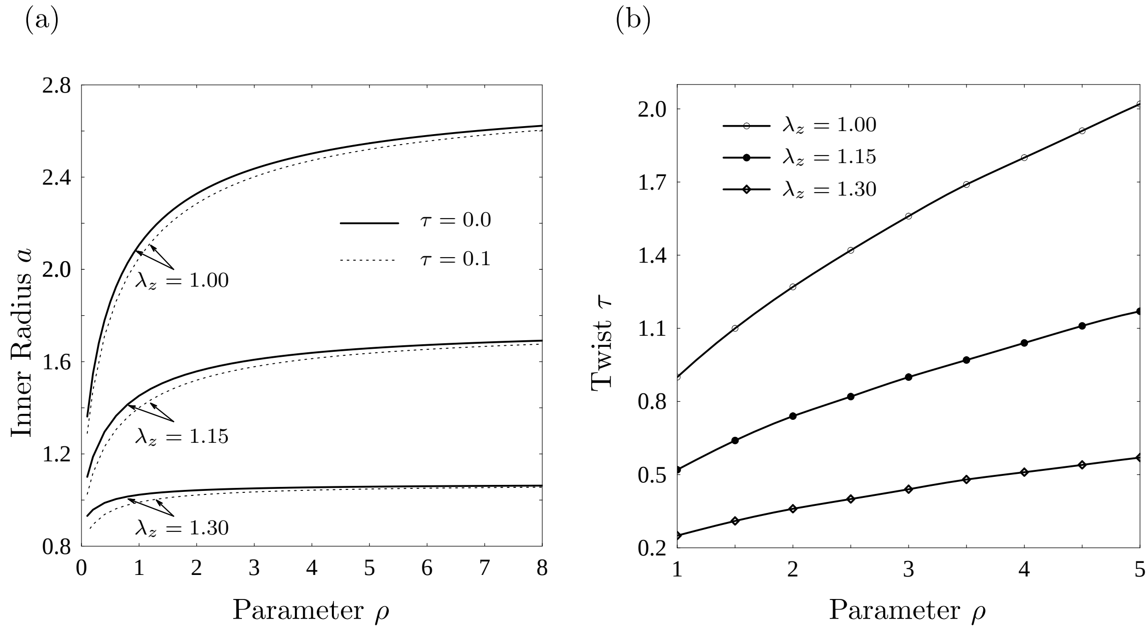

Figure 4.8: Plots of (a) the value of a at which the dimensionless reduced axial load N^*_{ral} is maximized given the twist $\tau = 0, 0.1$ and (b) the maximal possible twist τ establishing a non-monotonic relation of N^*_{ral} with a as functions of the parameter ρ for the material model (4.106) and provided that $\mathbf{M} = \mathbf{E}_R$. Note that in both (a) and (b) the curves (a, ρ) and (τ, ρ) are respectively demonstrated for $\lambda_z = 1, 1.15, 1.3$.

as shown in Figure 4.8(a), for an increasing τ and by keeping λ_z and ρ fixed, the value of a rendering a maximum N^*_{ral} gradually decreases. Similarly, the maximal range of λ_z in which N^*_{ral} changes non-monotonically with a now appears to depend both on τ and ρ. This dependence is demonstrated implicitly in Figure 4.8(b), where, in particular, for three fixed values of λ_z we identify the upper boundary of τ associated with each distinct ρ where a non-monotonic relation between N^*_{ral} and a is to be expected.

Finally, for the geometry, deformation and material model under examination the character of the dimensionless resultant moment $M^* = M/\mu$ is exemplified on the basis of Figures 4.9. It is important to clarify, however, that due to the arrangement of the preferred direction here the moment cannot distinguish between isotropic and anisotropic tubes and hence M^* essentially serves to encapsulate the mechanical response of the isotropic neo-Hookean material. Bearing that in mind, it is then shown in 4.9(a) that M^* may always be seen as an increasing function of both a and τ regardless the choice of λ_z. By contrast, we remark that an intensive axial extension of the tube does not necessarily comply with an increase in M^*. Truly, the implications of the latter argument are demonstrated in 4.9(b)

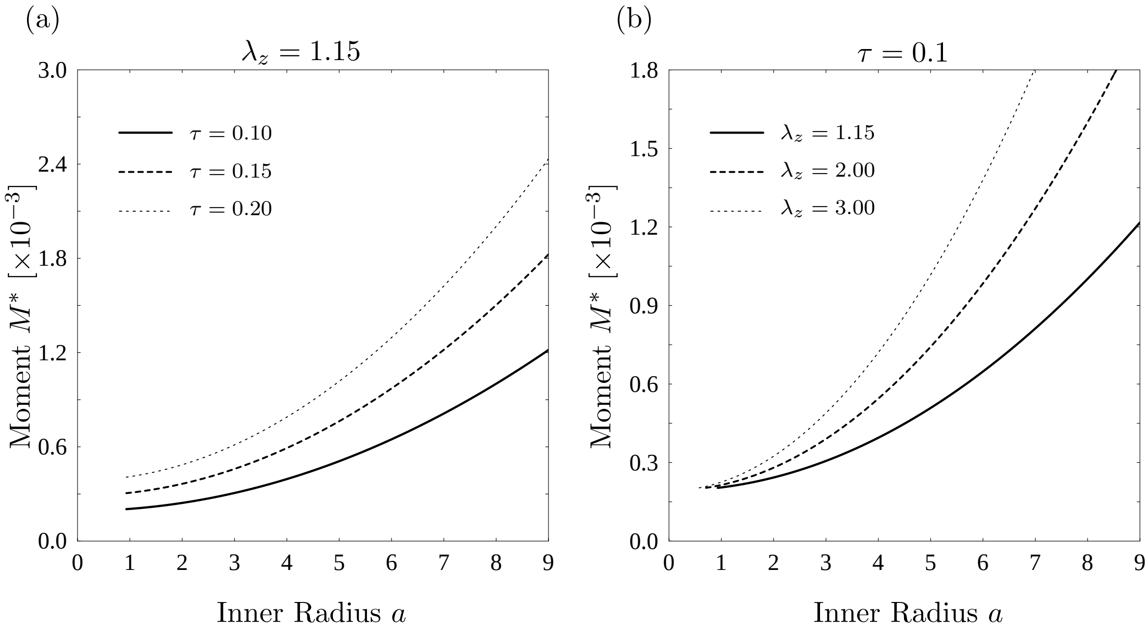

Figure 4.9: Plots of the dimensionless twisting moment M^* against the deformed inner radius a for the neo-Hookean material model. In (a) the parameters of interest are specified as $\lambda_z = 1.15$ and $\tau = 0.1, 0.15, 0.2$ while in (b) we consider $\lambda_z = 1.15, 2, 3$ and $\tau = 0.1$.

where it is observed that, for a prescribed τ, larger λ_z are accompanied with smaller M^* at relatively early stages of inflation, i.e. for relatively small a. Following that, M^* can be regarded as an increasing function of λ_z only under the premise that the body is subjected to severe radial inflation.

4.4.2 Reinforcement with radially varying α

Following the approach introduced in Section 3.8, we once more, for definiteness, consider a class of materials whose preferred direction, in the undeformed configuration, is defined by the family of curves

$$R = c_1(\Theta - \Theta_0) + c_2. \tag{4.109}$$

Recall that the parameters c_1, c_2 are constants and that (4.109) enables, in view of the definitions (3.7), the introduction of the function $G(R) = (R - c_2)/c_1$. Bearing in mind the radial dimensions of the body chosen here and assuming that $\Theta_0 = 0, \Theta_1 = 2$ (radians), the definitions (3.7) leads to $c_1 = 2.5$ and $c_2 = 1$. Hence, from (3.8), the components of $\mathbf{M}$ are

$$M_R = \frac{1}{\sqrt{(2R/5)^2 + 1}}, \quad M_\Theta = \frac{2R/5}{\sqrt{(2R/5)^2 + 1}}. \tag{4.110}$$

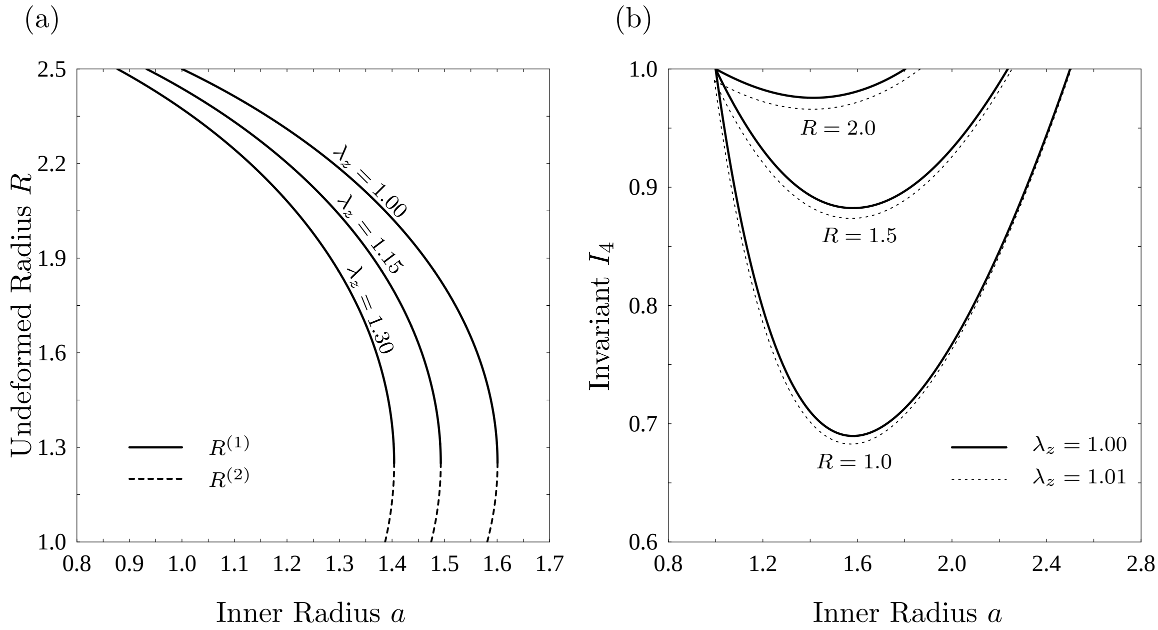

Figure 4.10: (a) Plots of the critical values $R^{(1)}$ and $R^{(2)}$ against a in (R, a) space for $\lambda_z = 1.00, 1.15, 1.30$; (b) plots of the invariant I_4 at the points $R = 1.0, 1.5, 2.0$ against a for $\lambda_z = 1.00, 1.01$.

Note that here we have $\tan\alpha = 2R/5$, so that the angle $\alpha(R)$ does not take the values 0 or $\pi/2$ at any $R \in [A, B]$, a fact which, as explained in Section 4.3.3, is of critical importance.

Clearly, due to the dependence of $\mathbf{M}$ on R, the necessity for the shear stresses $\sigma_{r\theta}$ and σ_{rz} is in this case essential. Nevertheless, it is a straightforward task to show that, for the present choice of $\mathbf{M}$ and given that the kinematical restrictions (4.15)–(4.17) are in place, the requirements are always met. From that perspective, there is no need to impose any restrictions on the data λ_z, a and τ associated with the extension, inflation and torsional deformations, respectively.

Our first imperative is now to identify the effect of each one of the deformation processes on the preferred direction for the particular geometry of $\mathbf{M}$ adopted here. Starting from the invariant I_4, we bring back to mind that only the extension and inflation of the tube is responsible for any change in length along the preferred direction. Precisely, we illustrate that for each $R \in [1, 2.5)$ we expect I_4 to initially decrease starting from the value λ_z^{-1} whereas this set of points is strictly associated with $\alpha(R) \in (0, \pi/4)$. By contrast, for each $R \in [2.5, 6]$, those points corresponding to $\alpha(R) \geqslant \pi/4$ with the equality holding at $R = 2.5$, we have immediate increase of I_4 from the value λ_z^{-1}. In any case, it is clear that the axial extension of the tube brings the body into a compressive mode. When

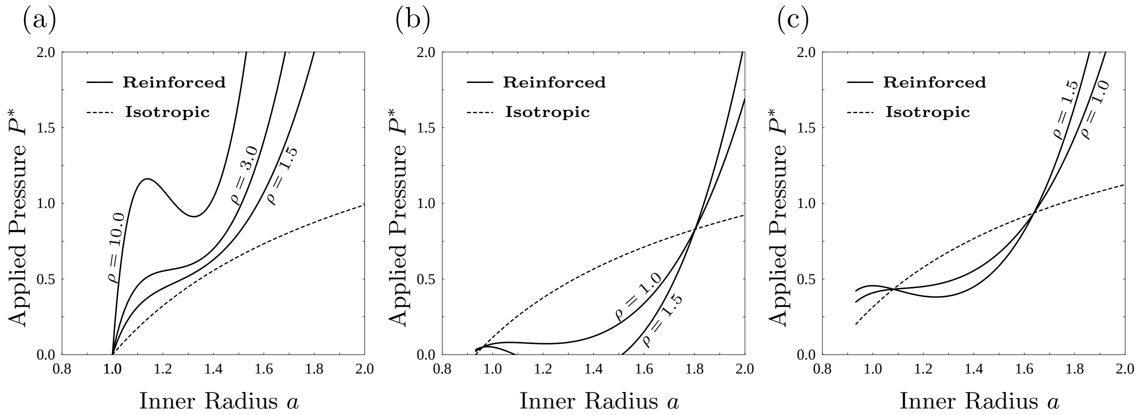

Figure 4.11: Plots of the dimensionless pressure P^* against the deformed inner radius a for the reinforced material model (4.108), with the anisotropy defined by the geometry (4.110), reflecting the case of (a) pure inflation for the values $\rho = 1.5, 3.0, 10.0$ (b) axial extension and inflation for $\lambda_z = 1.15$ and $\rho = 1.0, 1.5$ and (c) combined extension, inflation and torsion for $\lambda_z = 1.15, \tau = 0.1$ and $\rho = 1.0, 1.5$. Note that, in any case, the broken lines are used to prescribe the mechanical behaviour of the associated isotropic neo-Hookean model.

pressure sufficient to cause inflation is applied at the inside of the tube we have further contraction of the body along $\mathbf{M}$ but only within the domain $R \in [1, 2.5)$. In this domain, and for the particular geometry of $\mathbf{M}$, it is important to distinguish that I_4 first reaches its minimum boundary, namely $I_4^{\min} = \lambda_z^{-1}\sin(2\alpha)$, closer to $R = 2.5$. We emphasize that $I_4^{\min}$ is a function of R and it is therefore used to identify the minimum possible value of I_4 associated with each distinct $R \in [1, 2.5)$ and not the minimum of the function $I_4 \equiv I_4(R)$ with respect to $R \in [1, 6]$. Accordingly, resolution of (4.46) enables us to identify the connections

$$R^{(1)} = \frac{5 + \sqrt{25 + 16(A^2 + a^2\lambda_z)}}{4} \quad \text{for} \quad a \in (a_0, a_2], \tag{4.111}$$

$$R^{(2)} = \frac{5 - \sqrt{25 + 16(A^2 + a^2\lambda_z)}}{4} \quad \text{for} \quad a \in [a_1, a_2], \tag{4.112}$$

where, for compactness, we have introduced the notations

$$a_0 = A\lambda_z^{-1/2}, \quad a_1 = \sqrt{\frac{3 + 2A^2}{2\lambda_z}}, \quad a_2 = \frac{1}{4}\sqrt{\frac{25 + 16A^2}{\lambda_z}}.$$

We remark that the symbols $R^{(1)}$ and $R^{(2)}$ used here describe the values of R at which, for each value of λ_z and a, the equation $I_4 = I_4^{\min}$ is satisfied. Conversely, for any given λ_z, the connections may be used to identify the amount of inflation, in other words the value of a, that is required so that, at a specified R, the corresponding value of I_4 takes

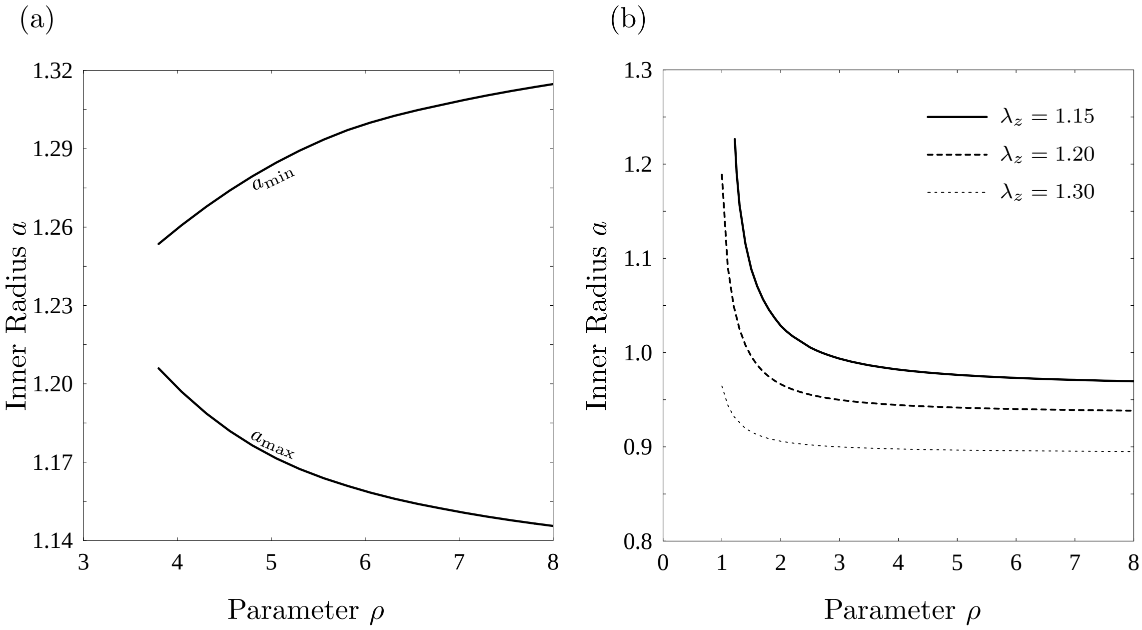

Figure 4.12: (a) Plots of the critical values $a_{\min}, a_{\max}$ against the anisotropic parameter ρ in (a, ρ) space; (b) plots of the inner radius a at which the connection $P^* = 0$ is initially satisfied as a function of the anisotropic parameter ρ for $\lambda_z = 1.15, 1.20, 1.30$.

its minimum possible value. On the grounds of (4.111), (4.112) significant conclusion regarding the status of the preferred direction may be gleaned. Expressly, the formulas derived here for $R^{(1)}$ and $R^{(2)}$ state that when a departs from its initial value, namely $a_0 = A\lambda_z^{-1/2}$, the first point for which the invariant I_4 reaches the lower boundary $I_4^{\min}$ is $R \equiv R^{(1)} \rightarrow 2.5^-$. At the same time, for all $R \in [1, 2.5)$ away from $R^{(1)}$, we have $I_4^{\min} < I_4 < \lambda_z^{-1}$ and assuming that a constantly increases, the value of $R^{(1)}$ associated with $I_4 = I_4^{\min}$ also decreases. It is then understood that, for a fixed λ_z and a $(> a_0)$, the preferred direction is brought to a state of relaxation for those $R > R^{(1)}$, is minimized at $R = R^{(1)}$, while, for $R < R^{(1)}$ is in a compressive mode.

An interesting situation may be detected when a reaches the value a_1 since then $I_4 = I_4^{\min}$ at two points, prescribed by $R^{(1)}$ and $R^{(2)}$, simultaneously. In such a case, and in particular within the range $a_1 \leqslant a < a_2$, I_4 is in a relaxing state for $R > R^{(1)}$, clearly minimized at $R = R^{(1)}, R^{(2)}$, while for $R^{(2)} < R < R^{(1)}$ and $R < R^{(2)}$ the mode of I_4 is again that of compression. In addition, for $a = a_2$ we have $R^{(1)} = R^{(2)}$ and hence $I_4 = I_4^{\min}$ only at one isolated point. Note that the latter value of a identifies the maximum amount of inflation that the body can undergo so that there exists a point R where I_4 attains its minimum value. Expressed otherwise, the point $R = R^{(1)} = R^{(2)}$ is the last point at which

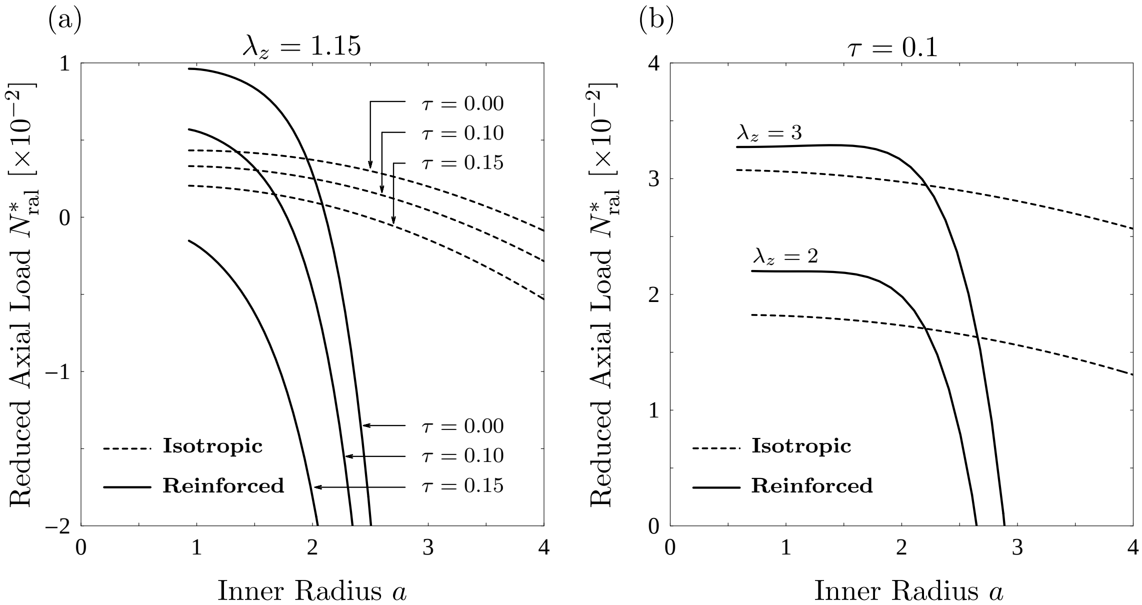

Figure 4.13: Plots of the dimensionless reduced axial load N^*_{ral} versus the deformed inner radius a for the isotropic neo-Hookean and the reinforced material model (4.108) with the anisotropy defined by the geometry (4.110) along with $\rho = 1.5$. In (a) the curves (N^*_{ral}, a) are demonstrated for $\lambda_z = 1.15$ and $\tau = 0, 0.1, 0.15$ while in (b) for $\lambda_z = 2, 3$ together with $\tau = 0.1$.

$I_4 = I_4^{\min}$ since this requires the maximum possible a; that being identified with a_2. This means that, away from $R = R^{(1)} = R^{(2)}$, the invariant I_4 has already reached its minimum for $a < a_2$ and hence, for all those points, I_4 tends to relax. Finally, it is evident that if $a > a_2$ we have $I_4 > I_4^{\min}$ for each $R \in [1, 2.5)$. The above assertions concerning the local behaviour of I_4 for the points $R \in [1, 2.5)$ are now illustrated graphically in Figures 4.10. Expressly, in 4.10(a) the expressions $R^{(1)}$ and $R^{(2)}$ are plotted against the deformed inner radius a for $\lambda_z = 1, 1.15, 1.3$. Further, in 4.10(b), we demonstrate the variation of I_4, this associated with the points $R = 1, 1.5, 2$, with an increasing a for $\lambda_z = 1, 1.01$. In both figures the (compressive) role of λ_z in the actual local behaviour of I_4 can also be detected.

In what follows the implications of the above discussion are parallel to those of the material model (4.108). In particular, analogously to the case where $\mathbf{M}$ has been taken to lie along the radial direction of the body, here we provide a short discussion relative to the effect of each one of the extension, inflation and torsional deformations on the dimensionless quantities P^*, N^*_{ral} and M^*.

We show, in Figure 4.11(a), that when the body is subjected to uniform inflation alone, i.e. $\lambda_z = 1$ and $\tau = 0$, the dimensionless pressure P^* increases monotonically with a, yet

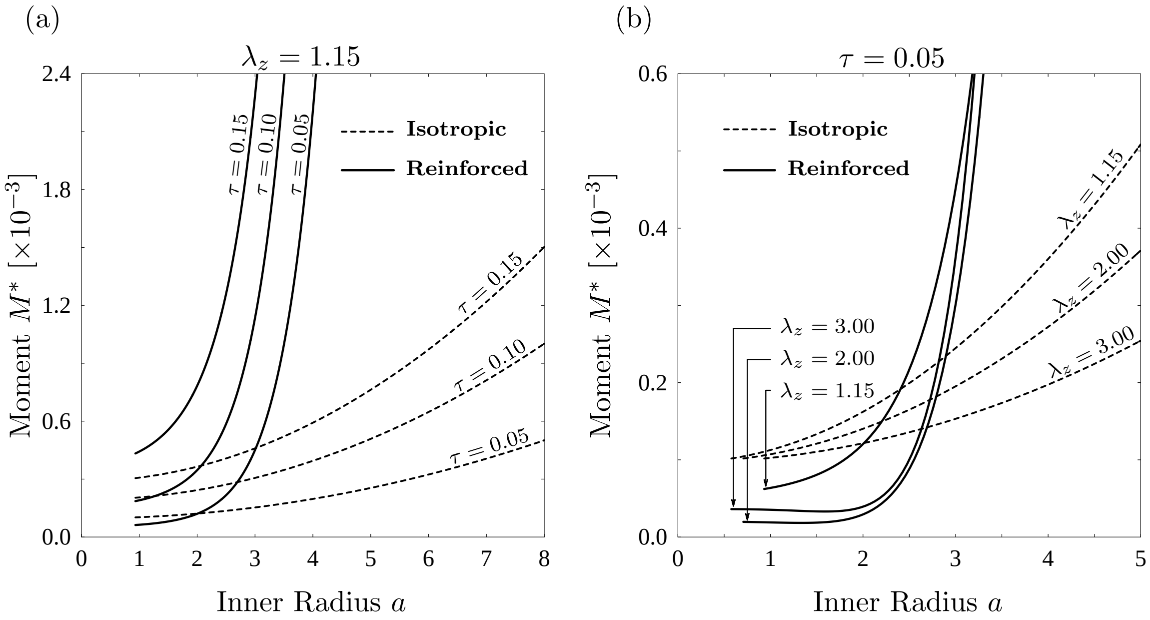

Figure 4.14: Plots of the dimensionless moment M^* against the deformed inner radius a for the isotropic neo-Hookean and the reinforced material model (4.108) with the anisotropy defined by the geometry (4.110) along with $\rho = 1.5$. In (a) the curves (N^*_{ral}, a) are demonstrated for $\lambda_z = 1.15$ and $\tau = 0, 0.1, 0.15$ while in (b) for $\lambda_z = 1.15, 2.00, 3.00$ together with $\tau = 0.05$.

only under the premise that ρ is relatively small. We notice, in particular, that once ρ reaches the approximate value 3.8, the pressure P^* becomes non-monotonic with respect to a and attains a (local) maximum and a (local) minimum. As further demonstrated in Figure 4.12(a), and analogously to the situation described for $\mathbf{M} = \mathbf{E}_R$, the values of a at which P^* is locally maximized and thereafter minimized, now denoted $a_{\max}$ and $a_{\min}$ respectively, both change with ρ. Expressly, we see that for larger ρ the value of $a_{\max}$ decreases whereas that of $a_{\min}$ decreases.

Bearing in mind the connections (4.52) and due to the discussion provided above we are now able to clarify that the non-monotonic relation of P^* with a is strongly related to the compressive effect that the inflation has on the line elements lying along $\mathbf{M}$, at least within a certain domain of R and a. Indeed, owing to this situation, for sufficiently large ρ, the reinforcing term has a negative contribution on the pressure integrand for some r and positive for some others. Evidently, this effect is enhanced for $\lambda_z > 1$ since then the preferred direction is subjected to severer contraction. The additional implications of the axial extension deformation on the behaviour of P^* relevant to a are now demonstrated in Figure 4.11(b) for $\lambda_z = 1.15$ and $\rho = 1, 1.5$. There we see that, even for small λ_z (> 1)

and ρ, the pressure P^* becomes non-monotonic in a almost immediately. Specifically, we clarify that larger λ_z and/or ρ may now be associated with $P^* \leqslant 0$. This clearly means that the combined extension and inflation is no longer sustainable and that stability of deformation has definitely been lost. For illustration, in Figure 4.12(b) we demonstrate the dependence of the value of a at which $P^* = 0$ is initiated with the parameters ρ and λ_z.

The implications of the torsional part of the deformation on the behaviour of the dimensionless pressure are now exhibited in Figure 4.11(c). There, P^* is once more plotted against a for $\rho = 1, 1.5$ with the other parameters of interest being held fixed and specified as $\lambda_z = 1.15$ and $\tau = 0.1$. It is apparent that although the local simple shear deformation is inconsequential in respect of the status of I_4, it has a profound increasing effect on I_5 which is in turn reflected upon the pressure P^*. Truly, when torsion is also incorporated in the combined deformation, the invariant I_5 always increases with γ (i.e. $\partial I_5/\partial\gamma > 0$) and hence the local shear generates a contradictive effect on the integrand of P^* to that caused by the axial extension and initially by the inflation, in other words, by stretches λ_z and λ_θ respectively. In this manner, for any given ρ and λ_z, an increase in τ ($\neq 0$) always results in larger pressure at any stage of the inflation process, while also, based on the same arguments, facilitates the possibility of a monotonic relation between P^* and a.

Without referring to any details, we finally remark that, for the considered geometry and deformation, the dimensionless reduced axial load and resultant moment, denoted N^*_{ral} and M^* respectively, are found to behave in a broadly similar manner to that described earlier for the case where $\mathbf{M} = \mathbf{E}_R$, yet under the premise that the body is subjected to small strains. When, however, large strains are considered, and especially if severe axial extension and/or inflation are taking place, both the quantities N^*_{ral} and M^* are essentially found to reflect, as for the pressure P^*, the contraction-extension progress of the line elements in the $\mathbf{M}$ direction of the tube. A brief illustration of the above mentioned arguments is now provided in Figures 4.13 where, in particular, N^*_{ral} is plotted against a for (a) $\lambda_z = 1.15$ and $\tau = 0, 0.1, 0.15$ and (b) for $\lambda_z = 2, 3$ and $\tau = 0.1$. Similarly, in Figures 4.14 the moment M^* is plotted with respect to a for (a) $\lambda_z = 1.15$ and $\tau = 0.05, 0.1, 0.15$ and (b) for $\lambda_z = 1.15, 2, 3$ and $\tau = 0.05$. Note that in any case the anisotropic parameter is held fixed and specified as $\rho = 1.5$.

Chapter 5

Closure

In this thesis the understanding of the mechanical behaviour of transversely isotropic elastic solids subject to finite deformations has been in focus. For this purpose, assuming simple body geometries, boundary-value problems, considered as classical in the isotropic theory, have been re-examined with reference to the corresponding transversely isotropic principles.

Motivated mainly by the work of Jiang and Ogden [39], Jiang [38] and Kirkinis and Ogden [44], a large part of this work has been concentrated on the possible existence of new classes of compressible transversely isotropic elastic blocks capable to support isochoric bending deformations. The anisotropic character of the materials under question have yield conditions, arising from both the associated linear theory as well as from reduction of the governing equilibrium equations, that provide restrictions on the strain-energy function in order to admit the considered deformation. It was shown that those restrictions do not, in contrast to the associated isotropic theory, disqualify particular classes of transversely isotropic compressible materials to undergo isochoric bending.

Unlike the majority of studies so far presented in the literature, special emphasis has been given in the last two chapters to the investigation of the mechanical response of circular cylindrical structures which are, owing to their anisotropic character, apprehended to embody non-homogeneous elastic properties. There, one of the main tasks was to introduce a simple geometrical approach that enables an explicit representation of the preferred direction associated with the transverse isotropy. Precisely, a new framework capable to describe more general classes of transversely isotropic materials whose preferred direction depends, in particular, on the radius of the body, has been established. Based on that, the important influence of anisotropy in the overall response of circular cylindrical tubes under deformation was highlighted in a variety of contexts.

In conclusion, the results presented in this thesis can be used to provide a point of entry for the consideration of modelling solid materials with more complicated inherent symmetries (in the reference configuration). In addition, they may, under appropriate modification, be related to the study of soft tissue mechanics since resent experimental and theoretical research (see [32, 22] and references therein) has shown that arterial tissues can efficiently be modelled as multilayered thick-walled tubes composed of anisotropic incompressible material.

References

[1] R. C. Abeyaratne. Discontinuous deformation gradients in plane finite elastostatics of incompressible materials. *J. Elasticity*, 10(3):255–293, 1980.

[2] R. C. Abeyaratne. Discontinuous deformation gradients in the finite twisting of an incompressible elastic tube. *J. Elasticity*, 11(1):43–80, 1981.

[3] R. C. Abeyaratne. An admissibility condition for equilibrium shocks in finite elasticity. *J. Elasticity*, 13(2):175–184, 1983.

[4] R. C. Abeyaratne and J. K. Knowles. On the dissipative response due to discontinuous strains in bars of unstable elastic material. *Internat. J. Solids Structures*, 24(10):1021–1044, 1988.

[5] J. E. Adkins and R. S. Rivlin. Large elastic deformations of isotropic materials. X. Reinforcement by inextensible cords. *Philos. Trans. Roy. Soc. London. Ser. A.*, 248:201–223, 1955.

[6] M. Aron and Y. Wang. On deformations with constant modified stretches describing the bending of rectangular blocks. *Quart. J. Mech. Appl. Math.*, 48(3):375–387, 1995.

[7] M. Aron and Y. Wang. Remarks concerning the flexure of a compressible nonlinearly elastic rectangular block. *J. Elasticity*, 40(2):99–106, 1995.

[8] M. Backer and J. L. Ericksen. Inequalities restricting the form of the stress-deformation relations for isotropic elastic solids and reiner-rivlin fluids. *J. Wash. Acad. Sci.*, 44:33–35, 1954.

[9] J. M. Ball. Convexity conditions and existence theorems in nonlinear elasticity. *Arch. Rational Mech. Anal.*, 63(4):337–403, 1976/77.

[10] M. Beatty. *In Nonlinear Effects in FLuids and Solids (M.M Carroll and M.A. Hayes eds.)*. Plenum Press, New York., 1996.

[11] M. F. Beatty. General solutions in the equilibrium theory of inextensible elastic materials. *Acta Mech.*, 29(1-4):119–126, 1978.

[12] O. T. Bruhns, H. Xiao, and A. Meyers. Finite bending of a rectangular block of an elastic Hencky material. *J. Elasticity*, 66(3):237–256, 2002.

[13] B. D. Coleman and W. Noll. On the thermostatics of continuous media. *Arch. Rational Mech. Anal.*, 4:97–128 (1959), 1959.

[14] B. D. Coleman and W. Noll. Material symmetry and thermostatic inequalities in finite elastic deformations. *Archive for Rational Mechanics and Analysis*, 15:87–111, 1964.

[15] J. A. Crossley, A. H. England, and A. J. M. Spencer. Bending and flexure of cylindrically monoclinic elastic cylinders. *Internat. J. Solids Structures*, 40(25):6999–7013, 2003. A special issue in honour of George J. Dvorak (Blacksburg, VA, 2002).

[16] M. A. Dowaikh and R. W. Ogden. On surface waves and deformations in a compressible elastic half-space. *Stability Appl. Anal. Contin. Media*, 1(1):27–45, 1991.

[17] J. L. Ericksen. Equilibrium of bars. *J. Elasticity*, 5(3–4):191–201, 1975. Special issue dedicated to A. E. Green.

[18] J. L. Ericksen and R. S. Rivlin. Large elastic deformations of homogeneous anisotropic materials. *J. Rational Mech. Anal.*, 3:281–301, 1954.

[19] P. J. Flory. An anisotropic hyperelastic constitutive model with fiber-matrix shear interaction for the human annulus fibrosus. *J. Ap. Mech.*, 73:815–824, 2006.

[20] Y. B. Fu and Y. T. Zhang. Continuum-mechanical modelling of kink-band formation in fibre-reinforced composites. *International Journal of Solids and Structures*, 43:3306–3323, 2006.

[21] T. C. Gasser, R. W. Ogden, and G. A. Holzapfel. Hyperelastic modelling of arterial layers with distributed collagen fibre orientations. *submitted to Royal Society.*

[22] T. C. Gasser, R. W. Ogden, and G. A. Holzapfel. Hyperelastic modelling of arterial layers with distributed collagen fibre orientations. *J. R. Soc. InterfaceJ*, 3:15–35, 2006. Soft tissue mechanics.

[23] A. N. Gent and R. S. Rivlin. Experiments on the mechanics of rubber II:. *Proc. Phys. Soc. Lond. B.*, LXV:487, 1952.

[24] A. E. Green and J. E. Adkins. *Large elastic deformations.* Second edition, revised by A. E. Green. Clarendon Press, Oxford, 1970.

[25] A. E. Green and R. S. Rivlin. The mechanics of non-linear materials with memory. *Arch. Rational Mech. Anal.*, 1:1–21, 1957.

[26] A. E. Green and W. Zerna. *Theoretical elasticity.* Second edition. Clarendon Press, Oxford, 1968.

[27] D. M. Haughton and R. W. Ogden. Bifurcation of inflated circular cylinders of elastic material under axial loading. I. Membrane theory for thin-walled tubes. *J. Mech. Phys. Solids*, 27:179–212, 1979.

[28] D. M. Haughton and R. W. Ogden. Bifurcation of inflated circular cylinders of elastic material under axial loading. II. Exact theory for thick-walled tubes. *J. Mech. Phys. Solids*, 27:489–512, 1979.

[29] R. Hill. On uniqueness and stability in the theory of finite elastic strain. *J. Mech. Phys. Solids*, 5:229–241, 1957.

[30] A. Hoger. On the residual stress possible in an elastic body with material symmetry. *Archive for Rational Mechanics and Analysis*, 88:271–290, 1985.

[31] A. Hoger. The constitutive equation for finite deformations of a transversely isotropic hyperelastic material with residual stress. *Journal of Elasticity*, 33:107–118, 1993.

[32] G. A. Holzapfel, T. C. Gasser, and R. W. Ogden. A new constitutive framework for arterial wall mechanics and a comparative study of material models. *J. Elasticity*, 61(1-3):1–48 (2001), 2000. Soft tissue mechanics.

[33] C. O. Horgan and D. A. Polignone. A note on the pure torsion of a circular cylinder for a compressible nonlinearly elastic material with nonconvex strain-energy. *J. Elasticity*, 37(2):167–178, 1994/95.

[34] C. O. Horgan and G. Saccomandi. A description of arterial wall mechanics using limiting chain extensibility constitutive models. *Biomechan. Model Mechanobiol.*, 1:251–266, 2003.

[35] G. S. J. Merodio and I. Sgura. The rectilinear shear of fiber-reinforced incompressible non-linearly elastic solids. *Int. J. Non-Linear Mech.*, in press.

[36] W. Jaunzemis. *Continuum Mechanics.* Macmillan, 1967.

[37] Q. Jiang and M. F. Beatty. On compressible materials capable of sustaining axisymmetric shear deformations. IV. Helical shear of anisotropic hyperelastic materials. *J. Elasticity*, 62(1):47–83, 2001.

[38] X. Jiang. Thesis. On Deformations of Compressible Hyperelastic Material, 1998.

[39] X. Jiang and R. W. Ogden. On azimuthal shear of a circular cylindrical tube of compressible elastic material. *Quart. J. Mech. Appl. Math.*, 51(1):143–158, 1998.

[40] X. Jiang and R. W. Ogden. Some new solutions for the axial shear of a circular cylindrical tube of compressible elastic material. *Internat. J. Non-Linear Mech.*, 35(2):361–369, 2000.

[41] F. Kassianidis, J. Merodio, R. W. Ogden, and T. Pence. Azimuthal shear of a transversely isotropic elastic solid. *Math. Mech. Solids*, in press.

[42] F. Kassianidis and R. W. Ogden. On large bending deformations of transversely isotropic rectangular elastic blocks. *Note di Matematica*, in press.

[43] E. Kirkinis. Shear deformations of compressible isotropic elastic materials. M.Sc Thesis, University of Glasgow, 2005.

[44] E. Kirkinis and R. W. Ogden. On extension and torsion of a compressible elastic circular cylinder. *Math. Mech. Solids*, 7(4):373–392, 2002.

[45] J. K. Knowles. On the dissipation associated with equilibrium shocks in finite elasticity. *J. Elasticity*, 9(2):131–158, 1979.

[46] A. E. H. Love. *A treatise on the Mathematical Theory of Elasticity.* Dover Publications, New York, 1944. Fourth Ed.

[47] J. Merodio and R. W. Ogden. Material instabilities in fiber-reinforced nonlinearly elastic solids under plane deformation. *Arch. Mech. (Arch. Mech. Stos.)*, 54(5-6):525–552, 2002.

[48] J. Merodio and R. W. Ogden. Instabilities and loss of ellipticity in fiber-reinforced compressible non-linearly elastic solids under plane deformation. *Internat. J. Solids Structures*, 40(18):4707–4727, 2003.

[49] J. Merodio and R. W. Ogden. A note on strong ellipticity for transversely isotropic linearly elastic solids. *Quart. J. Mech. Appl. Math.*, 56(4):589–591, 2003.

[50] J. Merodio and R. W. Ogden. Mechanical response of fiber-reinforced incompressible non-linearly elastic solids. *International Journal of Non-Linear Mechanics*, 40:213–227, 2005.

[51] J. Merodio and R. W. Ogden. Mechanical response of fiber-reinforced incompressible non-linearly elastic solids. *International Journal of Non-Linear Mechanics*, 40:213–227, 2005.

[52] J. Merodio and T. J. Pence. Kink surfaces in a directionally reinforced neo-Hookean material under plane deformation. I. Mechanical equilibrium. *J. Elasticity*, 62(2):119–144, 2001.

[53] J. Merodio and T. J. Pence. Kink surfaces in a directionally reinforced neo-Hookean material under plane deformation. II. Kink band stability and maximally dissipative band broadening. *J. Elasticity*, 62(2):145–170, 2001.

[54] W. Noll. A mathematical theory of the mechanical behavior of continuous media. *Arch. Rational Mech. Anal.*, 2:198–226, 1958.

[55] W. Noll. A frame-free formulation of elasticity. *J. Elasticity*, 83(3):291–307, 2006.

[56] R. W. Ogden. *Nonlinear elastic deformations*. Ellis Horwood Series: Mathematics and its Applications. Ellis Horwood Ltd., Chichester, 1984.

[57] R. W. Ogden. *Nonlinear Elasticity with Application to Material Modelling*. 2003. AMAS Lecture Notes No. 6.

[58] R. W. Ogden and P. Chadwick. On the deformation of solid and tubular cylinders of incompressible isotropic elastic materials. *J. Mech. Phys. Solids.*, 20:77–90, 1972.

[59] R. W. Ogden and C. A. J. Schulze-Bauer. Phenomenological and structural aspects of the mechanical response of arteries. In: Casey J. and Bao G. (eds). *Mechanics in Biology*, AMD-242/BDE-46:125–140, 2000.

[60] R. G. Payton. *Elastic Wave Propagation in Transversely Isotropic Media.* (Martinus Nijhoff, The Hague 1983).

[61] D. A. Polignone and C. O. Horgan. Pure torsion of compressible non-linearly elastic circular cylinders. *Quart. Appl. Math.*, 49:591–607, 1991.

[62] G. Y. Qiu and T. J. Pence. Loss of ellipticity in plane deformation of a simple directionally reinforced incompressible nonlinearly elastic solid. *J. Elasticity*, 49(1):31–63, 1997.

[63] G. Y. Qiu and T. J. Pence. Remarks on the behavior of simple directionally reinforced incompressible nonlinearly elastic solids. *J. Elasticity*, 49(1):1–30, 1997.

[64] R. S. Rivlin. Torsion of a rubber cylinder. *J. Appl. Phys*, 18:444–449, 1947.

[65] R. S. Rivlin. A note on the torsion of an incompressible highly elastic cylinder. *Proc. Camb. Phil. Soc*, 45:485–487, 1948.

[66] R. S. Rivlin. Large elastic deformations of isotropic materials. v. the problem of flexure. *Proc. Roy. Soc. London. Ser. A.*, 195:463–473, 1949.

[67] R. S. Rivlin. Large elastic deformations of isotropic materials. VI. further results in the theory of torsion, shear and flexure. *Philos. Trans. Roy. Soc. London. Ser. A.*, 242:173–195, 1949.

[68] R. S. Rivlin and D. Saunders. Large elastic deformations of isotropic materials vii: experiments on the deformation of rubber. *Phil. Trans. Al. Soc. Lond. A.*, 243:251–288, 1951.

[69] G. Saccomandi and C. O. Horgan. Simple torsion of isotropic, hyperelastic, incompressible materials with limiting chain extensibility. *J. Elasticity*, 56:159–170, 1999.

[70] K. Sawyers. Comments on the paper: "Determination of the stretch and rotation in the polar decomposition of the deformation gradient" [Quart. Appl. Math. **42** (1984), no. 1, 113–117; MR0736511 (86a:73004)] by A. Hoger and D. E. Carlson. *Quart. Appl. Math.*, 44(2):309–311, 1986.

[71] A. J. M. Spencer. The static theory of finite elasticity. *J. Inst. Maths Applies*, 6:164–200, 1970.

[72] A. J. M. Spencer. *Deformations of Fibre-reinforced Materials.* Oxford University Press., 1972.

[73] A. J. M. Spencer. *Continuum Theory of the Mechanics of Fibre-Reinforced Composites.* 1984. CISM Courses and Lecture Notes No. 282.

[74] D. J. Steigmann. Invariants of the stretch tensors and their application to finite elasticity theory. *Math. Mech. Solids*, 7(4):393–404, 2002.

[75] D. J. Steigmann. Frame-invariant polyconvex strain-energy functions for some anisotropic solids. *Math. Mech. Solids*, 8(5):497–506, 2003. Dedicated to Raymond W. Ogden on the occasion of his 60th birthday.

[76] D. J. Steigmann. On isotropic, frame-invariant, polyconvex strain-energy functions. *Quart. J. Mech. Appl. Math.*, 56(4):483–491, 2003.

[77] N. Triantafyllidis and R. C. Abeyarante. Instability of a finately deformed fiber-reinforced elastic material. *Journal of Applied Mechanics*, 50(2):149–156, 1983.

[78] C. Truesdell. General and exact theory of waves in finite elastic strain. *Arch. Rational Mech. Anal.*, 8:263–296, 1961.

[79] C. Truesdell and W. Noll. *The non-linear field theories of mechanics.* Springer-Verlag, Berlin, third edition, 2004. Edited and with a preface by Stuart S. Antman.

[80] C. Truesdell and R. Toupin. Static grounds for inequalities in finite strain of elastic materials. *Arch. Rational Mech. Anal.*, 12:1–33, 1963.

[81] H. Tsai and X. Fan. Anti-plane shear deformations in compressible transversely isotropic materials. *J. Elasticity*, 54(1):73–88, 1999.

[82] T. J. Vogler and S. Kyriakides. Initiation and axial propagation of kink bands in fiber composites. *Acta Materialia*, 45(2):2443–2454, 1983.