OPERATOR INCLUSIONS AND OPERATOR-DIFFERENTIAL INCLUSIONS

by

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To my wife and son

Summary

In this thesis, we study the solvability of some operator inclusions and some evolution inclusions with pseudo-monotone mappings.

In Chapter 1, we mainly recall some known concepts and results related to set-valued mappings and differential equations, which will be used in other chapters.

In Chapter 2, we first introduce a generalized inverse differentiability for set-valued mappings and consider some of its properties. Then, we use this differentiability, Ekeland's Variational Principle and some fixed point theorems to consider constrained implicit function and open mapping theorems and surjectivity problems of set-valued mappings. The mapping considered is of the form $F(x,u)+G(x,u)$. The inverse derivative condition is only imposed on the mapping $x \mapsto F(x,u)$, and the mapping $x \mapsto G(x,u)$ is supposed to be Lipschitz. The constraint made to the variable x is a closed convex cone if $x \mapsto F(x,u)$ is only a closed mapping, and in case $x \mapsto F(x,u)$ is also Lipschitz, the constraint needs only to be a closed subset. We obtain some constrained implicit function theorems and open mapping theorems. Pseudo-Lipschitz property and surjectivity of the implicit functions are also obtained. As applications of the obtained results, we also consider both local constrained controllability of nonlinear systems and constrained global controllability of semilinear systems The constraint made to the control is a time-dependent closed convex cone with possibly empty interior. Our results show that the controllability will be realized if some suitable associated linear systems are constrained controllable.

In Chapter 3, without defining topological degree for set-valued mappings of monotone type, we consider the solvability of the operator inclusion $y_0 \in N_1(x)+N_2(x)$ on bounded subsets in Banach spaces with N_1 a demicontinuous set-valued mapping which is either of

class (S_+) or pseudo-monotone or quasi-monotone, and N_2 is a set-valued quasi-monotone mapping. Conclusions similar to the invariance under admissible homotopy of topological degree are obtained. Some concrete existence results and applications to some boundary value problems, integral inclusions and controllability of a nonlinear system are also given.

In Chapter 4, we will suppose $u \mapsto A(t,u)$ is a set-valued pseudo-monotone mapping and consider the evolution inclusions

$$x'(t) + A(t, x((t)) \ni f(t) \text{ a.e. and } \frac{\mathrm{d}}{\mathrm{d}t}(Bx(t)) + A(t, x((t)) \ni f(t) \text{ a.e.}$$

in an evolution triple (V, H, V^*), as well as perturbation problems of those two inclusions. We first prove the pseudo-monotonicity of $L\hat{A}L^*$ in functional spaces with $\hat{A}$ the realization of A and $(Lx)(t) = \int_0^t x(s)ds$, $(L^*x)(t) = \int_t^T x(s)ds$ and, then, by this result and a variational inequality theorem of Browder, we obtain existence theorems for these two inclusions. Continuity of solutions depending on the function f are also given. For the perturbation problem of the explicit inclusion, we suppose $A(t,\cdot)$ is also accretive as a mapping on V^*, and consider the case when the perturbation is a Lipschitz mapping. For the perturbation problem of the implicit inclusion, we consider the case when it is perturbed by an u.s.c. and uniformly bounded (in $L^q(H)$) mapping.

In Chapter 5, we will study the second order differential evolution inclusions

$$x''(t) + A(t, x'(t)) + Bx(t) \ni f(t),$$

$$x''(t) + A(t, x'(t)) + Bx(t) - F(t, x(t), x'(t)) \ni 0$$

and some related implicit inclusions in an evolution triple (V, H, V^*) with $v \mapsto A(t,v)$ pseudo-monotone from V to V^*, B a symmetric, linear positive operator from V to V^* and F a set-valued mapping from $[0,T] \times H \times H$ to H. Solvability for the first inclusion and the related implicit problems are obtained by transforming them into some first order problems and applying our theorems given in Chapter 4. For the second inclusion, if F satisfies a certain growth condition, we obtain global solutions; if F is only bounded (maps bounded subsets into bounded subsets), then we have the existence of local solutions. A new extension of Wirtinger's inequality is also established in this chapter.

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Statement

This thesis is submitted according to the regulations for the degree of Doctor of Philosophy in the University of Glasgow. It presents part of research results carried out by the author during the academic years 1995-1998.

All the results of this thesis are the original work of the author except for the instances indicated within the text. Some results in §2.1, §2.3, §2.5, §4.1, §4.2 and §4.4 are obtained jointly with Professor J. R. L. Webb.

Contents

Introduction

The main purpose of this work is to present some new results concerning the solutions of some operator inclusions and some differential evolution inclusions.

Operator inclusions and operator-differential inclusions are two of the most general models of many practical problems, and therefore, studying the solutions to those inclusions is very important (sometimes, the study of operator inclusion is made in order to study differential inclusions). The existence and properties of the solutions are usually the main objectives.

Among the classes of operators in nonlinear functional analysis, those that are differentiable are important. Interesting problems for operator equations or inclusions involving such operators are the implicit function problems and open mapping problems, which are closely related to each other, and their applications. Recall that the implicit function problem is : for a two-variable mapping $(x, u) \mapsto F(x, u)$ with $F(x_0, u_0) = 0$, is it possible to solve for x locally, that is, is there $f(u)$ near x_0 such that $F(f(u), u) = 0$ for each u near u_0? How does $f(u)$ depend on u? What property does $u \mapsto f(u)$ have? The open mapping problem is: for a mapping F and an open subset U, is $F(U)$ open or does it contain an open subset?

In the classical implicit function and open mapping theorems, the mapping (single-valued) needs to be Fréchet differentiable and the derivative needs to be surjective. Although Fréchet differentiability is easy to verify in some applications and has received much attention (see Craven and Nashed [30], Frankowska [40] and Klamka [52]), those results cannot be applied if the function is not Fréchet differentiable or is set-valued. So, recently, there have been many publications concerning the non-Fréchet differentiable problems in which several substitutions are used in place of Fréchet differentiability. The first one is naturally Gâteaux differentiability. It was used by Roy etc. in [68], [69] for the openness and surjectivity of nonlinear operators. Some implicit function theorems involving Gâteaux differentiability can be found in Dontchev [33], Kuntz [53], Robinson [70] and the references therein. The second substitution is consideration of set-valued derivatives, including the notion of strict prederivative due to Ioffe [48] (when the mapping is single-valued, some people called it shield or upper derivative, see Chow and Lasota [27], Milójevic [56], Páles [61] and Penot [67]) and a rather complicated higher order set-valued derivative introduced by Frankowska [38] for the study of set-valued open mappings. We note that, in [56], some implicit functions without differentiability were also given, but as compensation, the function was supposed to have a non-zero topological degree. Another replacement is the weak Gâteaux inverse derivative defined by Welsh [78] which was used in the same paper to obtain an open mapping theorem via Brézis and Browder's ordering principle. This notion is similar to the directional contractor of Altman [3], the difference is that the derivative in [3] needs to be linear and that in [78] may be nonlinear. We also remark that the problems considered in each of the papers cited above are unconstrained problems, except one result in each of [67] and [56], and the mapping is single-valued except [33] and [39].

Apart from the pure mathematical interest, in applications, constrained problems and set-valued problems are most important. For example, constrained controllability, which is one application area of the implicit function and open mapping theories, has been widely considered for linear systems since this notion was introduced in [19], see Sö [73] and references therein. However, there exist relatively few papers concerning the constrained controllability of nonlinear systems, particularly in infinite dimensional spaces, see Achhab [1] (for finite dimensional systems), Chukwu and Lenhart [28], Klamka [52] and Papageorgiou [63]. As open mapping theorem is an important tool in such kinds of work (see [28], [52] and Frankowska [38]), new constrained implicit function theorems or open mapping theorems can be used to derive new results on constrained controllability of nonlinear systems. On the other hand, the solution map of nonlinear systems, which is key in the study on controllability, is usually set-valued and, therefore, the study of set-valued problems is necessary. In fact, a nonlinear system can be regarded as a differential inclusion of which the controllability has been studied by many authors (see [5], [38] and the references therein). We also note that the constraint imposed on the control in [27] is the unit ball, in [52] it is a closed convex cone with nonempty interior and in [63], where the nonlinear term is independent of the state and the admissible set is the space of measurable functions instead of the usually desired integrable functions, it is a time dependent cone.

The above indicates that the following question is of interest.

Question 1. Is it possible to define a new concept of differentiability, weaker than each of the four mentioned above and, therefore, to present some new constrained implicit function, open mapping theorems and some new constrained controllability results with a general constraint?

Another class of operators in nonlinear functional analysis is the (set-valued) mappings of monotone type and studies of this kind of equations concentrate on the solvability of

$$N(x) = 0 \tag{1}$$

in a reflexive Banach space X with $N : X \to X^*$ a nonlinear operator of monotone type. We remark that mappings of monotone type includes the classes of compact mappings, mappings of class (S_+), monotone, pseudo-monotone (briefly (PM)) and quasi-monotone (briefly (QM)) mappings, and the relations between them are

$$(S_+) \Longrightarrow (PM) \Longrightarrow (QM), \quad \text{monotone} \Longrightarrow (PM), \quad \text{compact} \Longrightarrow (QM)$$

provided the mappings involved are bounded and demicontinuous.

Problem (1) is important, particularly, in ordinary and partial differential equation theory, and has been widely considered. Much theory has been developed. An important one is topological degree theory, as an extension of the Leray-Schauder degree, developed by Browder [21] with N pseudo-monotone. His interesting work stimulated much further research about the degree of general operators, see Berkovits and Mustonen [12], [13], Browder [22], [23], Skrypnik [71] and references therein. The important property of topological degree, invariance under admissible homotopy, was the main result in each of these papers, and using this property, some related existence results for the above equation (1) were obtained. Solvability of equations involving quasi-monotone operators can be found in [26].

The case when N is set-valued is also significant, for example, in variational inequality theory and differential inclusion theory etc. and also has received much attention recently. Surjectivity of a set-valued mapping $L+N$, with L maximal monotone and N pseudo-monotone defined on the whole space, was considered by Browder and Hess [25]. In order to obtain existence theorem for set-valued problems on a given bounded subset, extensions of topological degrees are often defined so as to use the invariance under admissible homotopy. In [46], Hu and Papageorgiou generalized Browder's degree to the case when there is a set-valued compact perturbation. In [51], Kittilä gave a degree for certain set-valued mappings which can be approximated by sequences of single-valued operators of class (S_+). It seems that the solvability of inclusions involving general set-valued mappings of class (S_+) and quasi-monotone mappings have until now not been considered.

This leads to our second question.

Question 2. Is it possible to consider the solvability, on a bounded subset, of the general inclusion problem

$$y_0 \in N_1(x) + N_2(x)$$

with N_1 a set-valued mapping of monotone type and N_2, as the perturbation, a set-valued quasi-monotone mapping?

The theory of monotone mappings has been widely exploited in the study of differential evolution equations and inclusions with nonlinear operators. To be a little more precise, suppose (V, H, V^*) is an evolution triple, $A : V \to V^*$ is a nonlinear operator and $B \in \mathbf{L}(V, V^*)$. Variants of boundary value problems lead to study of the solvability of the first order differential evolution equation (see Showalter [72] and Zeidler [80])

$$x'(t) + A(x(t)) = f(t) \quad \text{a.e.}, \tag{2}$$

or the second order equation

$$x''(t) + A(x'(t)) + Bx(t) = f(t) \quad \text{a.e.}, \tag{3}$$

and the corresponding implicit equations (see Showalter [72]), for example the equation

$$\frac{\mathrm{d}}{\mathrm{d}\,t}(Bx(t)) + A(t, x(t)) = f(t) \text{ a.e..} \tag{4}$$

A well known theorem given in the late 60's shows that if A is hemicontinuous, coercive and monotone with some growth conditions, then (2) admits a unique solution (see, Proposition III 5.1 in [72]). If, in addition, B is positive and symmetric, then both (3) and (4) admit solutions (see Theorem 33.A in [80] and Corollary III 6.3 in [72]). For both theory and applications, we need to consider the perturbation problems and seek other kinds of assumptions on A instead of the monotonicity condition. Many authors have contributed to these problems and we only review the recent development.

We first recall Hirano's work [43] in which a global existence result for the perturbation problem

$$x'(t) + A(x(t)) + G(x(t)) = f(t) \tag{5}$$

was given, but the assumptions made are strict. For example, it was supposed that

(Si) the embedding, $V \hookrightarrow H$, is compact;

(Sii) $G : V \to H$ is continuous and weakly continuous, and $\langle G(v), v\rangle \geq -c$.

Recently, more general problems were considered by Migórski [55] (and some other authors, see the references in [55]) who considered the global existence of the evolution inclusion

$$x'(t) + A(x(t)) \in F(x(t))$$

without (Sii) and with F a set-valued mapping into H. Also, in Ahmed and Xiang [2], (Si) was dropped and the range of G was extended to V^*, but another strong assumption was imposed, namely

(Siii) $x_n \rightharpoonup x$ implies $(G(x_n), x_n - x) \to 0$.

For example, even the identify operator, $G(v) = v$ for all v, need not satisfy (Siii) even though it is weakly continuous. A recent result was given by Berkovits and Mustonen in [11] who considered equation (2) with A pseudo-monotone instead of monotone. In case the Nemytski operator corresponding to A is pseudo-monotone and coercive, the solvability of equation (2) can be guaranteed by Theorem 32.D in [80], but, we are not sure what assumptions on A, except monotonicity, can ensure the pseudo-monotonicity of the corresponding Nemytski operator.

The second order problem (3) is equivlent to

$$y'(t) + Ay(t) + BLy(t) = f(t)$$

with $Ly(t) = \int_0^t y(s)ds$. So, under suitable assumptions, existence results for (3) could be derived from some existence theorems of (2) as shown in Theorem 33.A of [80]. But, for perturbation problems of second order equations, this is not easy to carry through, special techniques need to be investigated. Under (Si) and using Kakutani's fixed point theorem, Papageorgiou [64] obtained two (global) existence results for the problems

$$x''(t) + A(x'(t)) + Bx(t) \in F(t, x(t))$$

with A single-valued and monotone, $F : [0,T] \times H \to 2^H$ set-valued, u.s.c. or l.s.c. and having linear growth.

For the implicit problem (4), there are also many recent publications. Some authors consider the case when both B and A are the subdifferentials of convex functions (see Barbu and Favini [9] and Colli and Visintin [29]), others consider the case when A is the sum of a maximal monotone mapping and a Lipschitz-like operator, and B is the composition of the injection of $V \hookrightarrow H$ (supposed to be compact) and the subdifferential of a time-dependent convex function (see Hokkanen [44] and [45]). In the very recent works of Andrews etc.[4] and Barbu and Favini [8], some new existence theorems were given for the general operator equation

$$(Bx(t))' + Ax(t) = f(t)$$

with B linear, A monotone, and for some related implicit second order equations in an evolution triple of Hilbert spaces. The significance of these two papers is that the coercivity condition and strong monotonicity (for uniqueness) were imposed to $A + \lambda B (\lambda > 0)$ instead of A as is usual.

We note that (Siii) and the weak continuity imply that G is pseudo-monotone, monotone and hemicontinuous implies maximum monotone and, therefore, pseudo-monotone. This leads to our third question.

Question 3. Is it possible to consider solutions of general implicit or explicit, first order or second inclusions involving set-valued pseudo-monotone mappings instead of monotone mappings, as well as related perturbations problems ?

In this work, we will concentrate on these three questions.

In Chapter 1, we mainly give some preliminaries, including the basic theory of set-valued mappings, differential equations (inclusions), differentiability etc.. Only a few results in this chapter are new.

In Chapter 2, we introduce a generalized γ-inverse derivative for set-valued mapping, use Ekeland's Variational Principle and some fixed point theorems to consider the constrained implicit function, open mapping and surjectivity problems of set-valued mappings. Our derivative, even in the single-valued case, relaxes Welsh's notion, and more general than the notions of Gâteaux derivative and strict prederivative in the situation when they are used for implicit function or open mapping problems, and our results suggest that our concept is more useful. In particular, it allows the mapping to be perturbed by a small set-valued Lipschitz mapping. So, we suppose the mapping considered is of the form $F(x,u)+G(x,u)$. The inverse derivative condition is only imposed on the mapping $x \mapsto F(x,u)$, while the mapping $x \mapsto G(x,u)$ is supposed to be Lipschitz. Continuity with respect to the variable u is not necessary for the existence of the implicit function. The constraint made to the variable x is a closed convex cone if $x \mapsto F(x,u)$ is only a closed mapping, and in case $x \mapsto F(x,u)$ is also Lipschitz, the constraint needs only to be a closed subset. Some constrained implicit function and open mapping theorems are obtained, and pseudo-Lipschitz property and surjectivity of the implicit functions are also proved. Our conclusions generalize the corresponding results of [6], [31], [53], [61], [67], [68], [69] and [78] in several aspects, our conditions are weaker, a constraint is presented, Lipschitz perturbations are considered and the mapping is allowed to be set-valued.

As applications, we use our results to consider constrained controllability problems of nonlinear systems. We suppose that the constraint made on the control is a time-dependent closed convex cone with possibly empty interior and the admissible set is the space of all essentially bounded functions. We consider both local constrained controllability of nonlinear systems and constrained global controllability of semilinear systems. Our results show that the controllability will be realized if some suitable associated linear systems are constrained controllable. For the nonlinear systems, the associated linear system is constructed by the derivatives of the function, therefore, Gâteaux or Fréchet differentiability assumption is needed. For semilinear systems, the associated linear system is given by the linear part.

Some conclusions of this chapter are to appear in two papers that have been accepted for publication (see [15] and [17]) and some of them were obtained jointly with Professor J. R. L. Webb.

In Chapter 3, without defining topological degree for set-valued mappings of monotone type, we consider the solvability of the general operator inclusion problems

$$y_0 \in N_1(x) + N_2(x)$$

on bounded subsets in Banach spaces. Here, N_1 is a demicontinuous set-valued mapping which is either of class (S_+) or pseudo-monotone or quasi-monotone, the perturbation N_2 is always a set-valued quasi-monotone mapping. By using the known degrees for some single-valued operators given in [12] and [71], we obtain conclusions similar to the invariance under admissible homotopy of topological degree. Some concrete existence results are, therefore, obtained which generalize and improve the corresponding ones in [22], [25], [26], [51] and [71]. As applications, we obtain the solvability of some boundary value problems, integral inclusions and a controllability result of a nonlinear system.

In Chapter 4, we will suppose $u \mapsto A(t, u)$ is a set-valued pseudo-monotone mapping and consider the more general evolution inclusion

$$x'(t) + A(t, x((t)) \ni f(t) \quad \text{a.e..} \tag{6}$$

in a general evolution triple $V, H, V^*)$ and the implicit inclusion

$$\frac{\mathrm{d}}{\mathrm{d}\, t}(Bx(t)) + A(t, x((t)) \ni f(t) \quad \text{a.e..} \tag{7}$$

in an evolution triple of Hilbert spaces, as well as perturbation of these problems. We first prove the pseudo-monotonicity of $L\hat{A}L^*$ in functional spaces with $\hat{A}$ the realization of A and $(Lx)(t) = \int_0^t x(s)ds, (L^*x)(t) = \int_t^T x(s)ds$ and, then, by this result and a variational inequality theorem of Browder, we obtain the (global) existence theorems for these two inclusions which generalizes the corresponding ones in [2], [4], [8], [11] and [43]. Continuity of solutions depending on the function f are also given. For the perturbation problem of the explicit inclusion (6), we suppose $A(t, \cdot)$ is also accretive as a mapping on V^*, and consider the case when the perturbation is a Lipschitz mapping. An existence result is obtained and, if the perturbation is single-valued, the solution is unique. For the perturbation problem of the implicit inclusion (7), we consider the case when it is perturbed by an u.s.c. and uniformly bounded (in $L^q(H)$) mapping. In this case, the embedding of V into H needs to be compact.

Some conclusions of this chapter are obtained jointly with J. R. L. Webb and some of them are to appear in a paper that has been accepted for publication, see [18]

In Chapter 5, we will study the second order differential evolution inclusions

$$x''(t) + A(t, x'(t)) + Bx(t) \ni f(t), \tag{8}$$

$$x''(t) + A(t, x'(t)) + Bx(t) - F(t, x(t), x'(t)) \ni 0 \tag{9}$$

and some related implicit inclusions in an evolution triple (V, H, V^*) with $v \mapsto A(t, v)$ pseudo-monotone from V to V^*, B a symmetric, linear positive operator from V to V^* and F a set-valued mapping from $[0, T] \times H \times H$ to H. For inclusion (8) and the related implicit problems, we will derive the existence results by transforming them into some first order problems and applying our theorems obtained in Chapter 4. For inclusion (9), we shall consider both global and local solutions. To obtain the global solutions, we suppose that F satisfies a certain growth condition. Under some other conditions weaker than those in [64], and by a new extension of Wirtinger's inequality established here, we prove existence of solutions and compactness of the solution set. To obtain local solutions, we suppose that F is only bounded instead of the growth condition, and prove the existence of solutions.

The main results of this chapter have been accepted for publication, see [16].

Chapter 1

Preliminaries

In this Chapter, we give some notation, notions and theorems (some are new) related to set-valued mappings, topological degree theory, differentiability and evolution equations, on which the main part of this thesis is based.

1.1 Notation

In this section, we give some notation that will be used throughout this thesis.

(i) Let X be a metric space with the metric d, $D \subset X$. The closure, boundary and interior of D are denoted by $\overline{D}, \partial D$ and $\mathrm{int}(D)$ respectively. We also write

$$B_X = \{x \in X : d(x, 0) < 1\} \text{ when } 0 \in X,$$

$$B_X(x_0, \alpha) = \{x \in X : d(x, x_0) < \alpha\}, \text{ for } x_0 \in X, \alpha \geq 0,$$

$$B_X(D, \alpha) = \{x \in X : d(x, D) < \alpha\}, \text{ for } D \subset X, \alpha \geq 0,$$

$$\mathcal{H}_X(C, D) = \max\left\{\sup_{x \in C} d(x, D), \sup_{x \in D} d(x, C)\right\}, \text{ for } C, D \subset X.$$

$$2^X = \{D : D \text{ is a subset of } X\},$$

$$\mathcal{P}_c(X) = \{D \subset X : D \text{ is closed and bounded}\},$$

$$\mathcal{P}_{cv}(X) = \{D \subset X : D \text{ is closed, bounded and convex}\}.$$

It is known that $\mathcal{H}_X$ is a metric on $\mathcal{P}_c(X)$ and is called the *Hausdorff metric.*

(ii) Let X, Y be two normed spaces with the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ respectively (if there is no confusion, write $\|\cdot\|$ instead of $\|\cdot\|_X$ or $\|\cdot\|_Y$). We denote by $\mathbf{L}(X, Y)$ the space of all bounded linear operators from X to Y,

by X^* the dual space of X with $(\cdot, \cdot)$ the duality between X and X^*;

by X_w the space X endowed with the weak topology;

by $x_n \rightharpoonup x$ the convergence of $x_n \in X$ in the sense $(x_n, x^*) \to (x, x^*)$ for all $x^* \in X^*$.

We write

$\overline{\mathrm{co}}C = \overline{\mathrm{co}\, C}$ with $\mathrm{co}\, C$ the convex hull of C for $C \subset X$,

$C + \lambda D = \{x + \lambda z : x \in C, z \in D\}$ for $C, D \subset X$ and $\lambda \in \mathbb{R} := (-\infty, +\infty)$,

$\|D\|_X = \max\{\|x\|_X : x \in D\}$ for $D \subset X$.

For a sequence of subsets $\{D_n\} \subset X$, we denote by

$$\limsup_{n\to\infty} D_n = \{x \in X : x_{n_k} \to x \text{ for some subsequence } \{n_k\} \text{ with } x_{n_k} \in D_{n_k}\},$$

$$w\text{-}\limsup_{n\to\infty} D_n = \{x \in X : x_{n_k} \rightharpoonup x \text{ for some subsequence } \{n_k\} \text{ with } x_{n_k} \in D_{n_k}\}.$$

(iii) Let $X \neq \emptyset, \Omega \neq \emptyset$ be two sets, $F : \Omega \to 2^X$ be a set-valued mapping. The *domain, range, graph* and *inverse* of F are denoted by $\mathrm{Dom}(F), \mathrm{range}(F), \mathrm{Graph}(F)$ and F^{-1} respectively. Recall that $\mathrm{Dom}(F) = \{\omega \in \Omega : F(\omega) \text{ exists and is not empty}\}$, $\mathrm{range}(F) = F(\Omega) = \cup_{\omega\in\Omega} F(\omega)$, $\mathrm{Graph}(F) = \{(\omega, x) \in \Omega \times X : \omega \in \Omega, x \in F(\omega)\}$ and $F^{-1}(x) = \{\omega \in \Omega : x \in F(\omega)\}$ for all $x \in X$. We also denote by

$$F^{-1}(D) = \{\omega \in \Omega : F(\omega) \cap D \neq \emptyset\} \text{ for a subset } D \subset X.$$

1.2 Continuity of set-valued mappings

In this section, we give some definitions and results related to the continuity of set-valued mappings.

Definition 1.2.1. Suppose X, Y are Hausdorff topological spaces, $F : X \to 2^Y$ is a set-valued mapping. F is said to be

(i) *upper semicontinuous* (abbreviately *u.s.c.*) at $x \in \mathrm{Dom}(F)$ if for each neighbourhood $\mathcal{U}$ of $F(x)$, there exists a neighbourhood $\mathcal{V}$ of x such that

$$F(x') \subset \mathcal{U} \text{ for each } x' \in \mathcal{V};$$

If F is u.s.c. at each $x \in \mathrm{Dom}(F)$, we say F is a *u.s.c. mapping*

(ii) *lower semicontinuous* (abbreviately *l.s.c.*) at $x \in \mathrm{Dom}(F)$ if for each $y \in F(x)$ and each neighbourhood $\mathcal{U}$ of y, there exists a neighbourhood $\mathcal{V}$ of x such that

$$F(x') \cap \mathcal{U} \neq \emptyset \text{ for each } x' \in \mathcal{V};$$

If F is l.s.c. at each $x \in \mathrm{Dom}(F)$, we say F is a *l.s.c. mapping*

(iii) *continuous* (at $x \in \mathrm{Dom}(F)$) if it is both u.s.c. and l.s.c. (at x).

Recall that a single-valued function $\varphi : X \to \mathbb{R}$ is said to be *upper semicontinuous* at x if $x_n \in \mathrm{Dom}(\varphi)$ with $x_n \to x$ implies $\limsup_{n\to\infty} \varphi(x_n) \leq \varphi(x)$, and *lower semicontinuous* at x if $x_n \to x$ implies $\liminf_{n\to\infty} \varphi(x_n) \geq \varphi(x)$. It can be verified that the semicontinuity of the above function φ is equivalent to the same semicontinuity of the set-valued mapping $F(x) := \varphi(x) - \mathbb{R}^+$.

Obviously, for a single-valued mapping, both upper and lower semicontinuity given in Definition 1.2.1 are equivalent to the usual continuity. But, for set-valued mapping, they are not equivalent.

Example 1.2.2. Let $\Omega \neq \emptyset$ be a subset of a Banach space, $\varphi_i, \psi_i : \Omega \to \mathbb{R}(i = 1, \cdots, m)$ be single-valued functions such that $\varphi_i(\omega) \leq \psi_i(\omega)$ for all $\omega \in \Omega$. Define a set-valued mapping $F : \Omega \to 2^{\mathbb{R}^m}$ by

$$F(\omega) = \{(f_1, \cdots, f_m) : f_i \in [\varphi_i(\omega), \psi_i(\omega)]\}.$$

Then

(i) F is u.s.c. iff each φ_i is lower semicontinuous and each ψ_i is upper semicontinuous;

(ii) F is l.s.c. iff each φ_i is upper semicontinuous and each ψ_i is lower semicontinuous.

Proof. (i) Suppose F is u.s.c. at x. If some φ_i, say $\varphi_{i_1}, \cdots, \varphi_{i_k}$, are not lower semicontinuous (if some ψ_i are not upper semicontinuous, the proof is similar), then there exist $x_n \to x$ such that

$$\varphi_{i_j}(x) > \lim_{n\to\infty} \varphi_{i_j}(x_n) =: a_{i_j}, \ j = 1, \cdots, k.$$

Let $b_{i_j} \in (a_{i_j}, \varphi_{i_j}(x))$ and $b_i < \varphi_i(x)$ for those $i \notin \{i_j\}$. Then $\mathcal{U} := \prod_{i=1}^m (b_i, \infty)$ is a neighbourhood of $F(x)$. From the assumption, it follows that $F(x_n) \subset \mathcal{U}$ whenever n is large enough, that is, $\varphi_i(x_n) > b$ and, therefore, $a_{i_j} = \lim_{n\to\infty}\varphi_{i_j}(x_n) \geq b_{i_j}$ which is a contradiction and implies that each φ_i is lower semicontinuous.

Conversely, if each φ_i, ψ_i is lower and upper semicontinuous at x respectively, then, for each neighbourhood $\mathcal{U}$ of $F(x)$, say $\mathcal{U} = \prod_{i=1}^m(\varphi_i(x) - \varepsilon, \psi_i(x) + \varepsilon)$, it is clear that $\mathcal{U}_i := (\varphi_i(x) - \varepsilon, \psi_i(x) + \varepsilon)$ is also a neighbourhood of $\varphi_i(x)$ and $\psi_i(x)$. Therefore there exists $\eta > 0$ such that

$$\varphi_i(x) - \varepsilon \leq \varphi_i(x') \leq \psi_i(x') \leq \psi_i(x) + \varepsilon, \quad \text{for each } x' \in B_X(x,\eta), i = 1, \cdots, m,$$

that is, $F(B_X(x,\eta)) \subset \mathcal{U}$ and, therefore, F is u.s.c. at x.

(ii) Suppose F is l.s.c. at x. If some φ_i, say $\varphi_{i_1}, \cdots, \varphi_{i_k}$, are not upper semicontinuous (resp. ψ_{i_j} are not lower semicontinuous), then there exist $x_n \to x$ such that

$$\varphi_{i_j}(x) < \lim_{n\to\infty}\varphi_{i_j}(x_n) \quad (\text{resp.}\psi_{i_j}(x) > \lim_{n\to\infty}\psi_{i_j}(x_n)),\ j = 1, \cdots, k.$$

Since $\big(\varphi_1(x), \cdots, \varphi_m(x)\big) \in F(x)$ $\big(\text{resp. } \big(\psi_1(x), \cdots, \psi_m(x)\big) \in F(x)\big)$, there exist $y_n = \big(y_{n_1}, \cdots, y_{n_m}\big) \in F(x_n)$ with $y_{n_i} \to \varphi_i(x)$ (resp. $\psi_i(x)$). This yields that, for each $j = 1, \cdots, k$,

$$\varphi_{i_j}(x) = \lim_{n\to\infty} y_{n_{i_j}} \geq \limsup_{n\to\infty}\varphi_{i_j}(x_n) > \varphi_{i_j}(x) \quad \big(\text{resp. } \psi_{i_j}(x) = \lim_{n\to\infty} y_{n_{i_j}} < \psi_{i_j}(x)\big)$$

which is a contradiction.

Conversely, we suppose each φ_i is upper semicontinuous and each ψ_i is lower semicontinuous at x. If F is not l.s.c., then there exists $\mathcal{U} = \prod_{i=1}^m(\varphi_i(x) - \varepsilon, \psi_i(x) + \varepsilon)$ such that $F(x') \cap \mathcal{U} = \emptyset$ for each n large enough and each $x' \in B_X(x, 1/n)$. Let $x_n \in B_X(x, 1/n)$, $y_n = (y_{n_i})_{i=1}^m \in F(x_n)$. Then $x_n \to x$, and $y_{n_i} \leq \varphi_i(x) - \varepsilon$ (or $y_{n_i} \geq \psi_i(x) + \varepsilon$). Since $\varphi_i(x_n) \leq y_{n_i} \leq \psi_i(x_n)$, we see that $\limsup \varphi_i(x_n) \leq \varphi_i(x) - \varepsilon$ (or $\liminf \psi_i(x_n) \geq \psi_i(x) + \varepsilon$) which contradicts the semicontinuity of φ_i (or ψ_i). □

Remark 1.2.3. In [32], only the sufficiency of the above two conclusions are given for the case when $m = 1$.

Like the situation of single-valued mapping, the continuity of set-valued mapping can be also characterized by the inverse image of closed and open subsets. Here we list them in a theorem and give the proof because the author has not seen it.

Theorem 1.2.4. *If $F : X \to 2^Y$ is a set-valued mapping with X, Y Hausdorff topological spaces, then*

(i) F is u.s.c. iff $F^{-1}(D)$ is a closed subset in Y whenever $D \subset X$ is closed;

(ii) F is l.s.c. iff $F^{-1}(D)$ is open in Y whenever $D \subset X$ is open.

Proof. (i) Suppose F is u.s.c., $D \subset Y$ is a closed subset. If $F^{-1}(D)$ is not closed, then there exist a net $\{x_\alpha\} \subset F^{-1}(D)$ with $x_\alpha \to x_0$ in X and $x_0 \notin F^{-1}(D)$. Therefore, $F(x_0) \subset X \backslash D$. Since $X \backslash D$ is open, the upper semicontinuity of F implies that $F(x_\alpha) \subset X \backslash D$ eventually. That is $F(x_\alpha) \cap D = \emptyset$ eventually which contradicts $x_\alpha \in F^{-1}(D)$ and implies that $F^{-1}(D)$ is closed.

Conversely, if F is not upper semicontinuous at x_0, then there exist an open set $\mathcal{U} \supset F(x_0)$, a net x_α with $x_\alpha \to x_0$ and $y_\alpha \in F(x_\alpha) \backslash \mathcal{U}$. So $x_\alpha \in F^{-1}(X \backslash \mathcal{U})$. By our assumptions, $F^{-1}(X \backslash \mathcal{U})$ is closed and, therefore, $x_0 \in F^{-1}(X \backslash \mathcal{U})$ which contradicts $F(x_0) \subset \mathcal{U}$.

(ii) Suppose F is l.s.c., $D \subset Y$ is open. To prove $F^{-1}(D)$ is open, we let $x \in F^{-1}(D)$. Then there exists $y \in F(x) \cap D$ and a neighbourhood $\mathcal{U}$ of y such that $\mathcal{U} \subset D$. The lower semicontinuity of F implies the existence of a neighbourhood $\mathcal{V}$ of x with

$$F(x') \cap \mathcal{U} \neq \emptyset \quad \text{for each} \quad x' \in \mathcal{V}.$$

Since $\mathcal{U} \subset D$, we see that $\mathcal{V} \subset F^{-1}(D)$ which means that $F^{-1}(D)$ is open.

Conversely, suppose $F^{-1}(D)$ is open for each open set D. Let $x \in \mathrm{Dom}(F), y \in F(x)$ and let $\mathcal{U}$ be a neighbourhood of y. We may suppose $\mathcal{U}$ is open. Then $F^{-1}(\mathcal{U})$ is open and, therefore, is a neighbourhood of x. Since $x' \in F^{-1}(\mathcal{U})$ means $F(x') \cap \mathcal{U} \neq \emptyset$, we see that F is l.s.c.. □

The following definitions give some other continuity for set-valued mappings.

Definition 1.2.5. Let X, Y be metric spaces. $F : X \to 2^Y$ is said to be

(i) *ε-δ-upper semicontinuous* (abbreviately ε-δ-u.s.c.) at $x \in \mathrm{Dom}(F)$ if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $F(B_X(x, \delta)) \subset B_Y(F(x), \varepsilon)$;

(ii) *ε-δ-lower semicontinuous* (abbreviately ε-δ-l.s.c.) at $x \in \mathrm{Dom}(F)$ if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $F(x) \subset B_Y(F(x'), \varepsilon)$ for each $x' \in B_X(x, \delta)$;

(iii) ε-δ-*continuous* (at x) if F is both ε-δ-u.s.c. and ε-δ-l.s.c. (at x).

Definition 1.2.6. Suppose X, Y are Banach spaces. $F : X \to 2^Y$ is said to be

(i) *hemicontinuous* if $t \mapsto F(x+ty)$ is u.s.c. from $[0,1]$ to Y_w for all $x, y \in X$;

(ii) *demicontinuous* if F is (sequentially) u.s.c. as a mapping from X to Y_w;

(iii) *finitely continuous* if F is u.s.c. from each finite dimensional subspace of X to Y_w.

Now we give some results regarding the continuity of set-valued mappings.

Theorem 1.2.7. *([74], Proposition 1)*
Let X, Y be Hausdorff topological spaces, $F : X \to \mathcal{P}_{cv}(Y)$ be an u.s.c. mapping with compact values. Suppose $\{x_\alpha\}$ is a net in X and $x_\alpha \to x_0, y_\alpha \in F(x_\alpha)$. Then there exist $y_0 \in F(x_0)$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ such that $y_\beta \to y_0$.

Corollary 1.2.8. *Let X, Y, F be as above. If $x_n \to x_0$ in X, then* $\limsup_{n\to\infty} F(x_n) = w$-$\limsup_{n\to\infty} F(x_n)$.

Proof. Obviously, we need only prove w-$\limsup_{n\to\infty} F(x_n) \subset \limsup_{n\to\infty} F(x_n)$.

Let $y \in w$-$\limsup_{n\to\infty} F(x_n)$. Then there exist $y_{n_k} \in F(x_{n_k})$ with $y_{n_k} \rightharpoonup y$. By Theorem 1.2.7, there exists a subsequence $\{y_{n_{k_j}}\}$ of $\{y_{n_k}\}$ and $y_0 \in F(x_0)$ such that $y_{n_{k_j}} \to y_0$ which implies that $y_0 = y \in \limsup_{n\to\infty} F(x_n)$. This completes the proof. $\square$

Theorem 1.2.9. *([32], Propositions 1.1, 1.2 and 2.1)*
Suppose $F : X \to \mathcal{P}_c(Y)$ is a set-valued mapping with X, Y metric spaces.

(i) If F is u.s.c., then F is ε-δ-u.s.c.; If F is ε-δ-l.s.c., then F is l.s.c.; If, in addition, the values of F are compact, then the converses are also true.

(ii) If F is ε-δ-u.s.c., then the function $x \mapsto d(z, F(x))$ is l.s.c. for each $z \in X$ and the converse is also true if $\overline{F(\mathrm{Dom}(F))}$ is compact.

(iii) If F is ε-δ-u.s.c. and Dom(F) *is closed, then* Graph(F) *is closed.*

We remark that, in [32], Theorem 1.2.9 was proved in the case X, Y are Banach spaces, but one can use the same method to prove the case when X, Y are metric spaces. We also note that (i) remains true even the values of F are not bounded or closed.

Theorem 1.2.10. *([5], Corollary 1.1.1)*

Suppose $F : X \to \mathcal{P}_c(Y)$ is a set-valued mapping with X, Y Hausdorff topological spaces. If $\mathrm{Graph}(F)$ *is closed and Y is compact, then F is u.s.c..*

Theorem 1.2.11. *Suppose X is a metric space, Y is a normed space, $F, G : X \to 2^Y$ are two ε-δ-u.s.c. mappings. Then $F + G$ is ε-δ-u.s.c..*

Proof. Let $\varepsilon > 0, x \in X$. Then there exists $\delta > 0$ such that $F(B_X(x, \delta)) \subset B_Y(F(x), \varepsilon/2)$ and $G(B_X(x, \delta)) \subset B_Y(G(x), \varepsilon/2)$. Therefore,

$$\begin{aligned}(F+G)(B_X(x,\delta)) &\subset F(B_X(x,\delta)) + G(B_X(x,\delta)) \subset B_Y(F(x),\varepsilon/2) + B_Y(G(x),\varepsilon/2) \\ &\subset B_Y(F(x)+G(x),\varepsilon),\end{aligned}$$

that is, $F + G$ is ε-δ-u.s.c.. □

Theorem 1.2.12. *([6], Proposition 1.5.1)*

Suppose X is a metric space, Y, Z are two normed spaces, $F : X \to 2^Y$ and $G : X \to 2^Z$ are two set-valued l.s.c. mappings with convex values, and $h : X \times Z \to Y$ is a continuous operator such that $z \mapsto h(x, z)$ is affine for each $x \in X$. If for each $x \in X$, there exist $\gamma > 0, \delta > 0$ and $r > 0$ such that

$$\gamma B_Y \subset h(x', G(x') \cap rB_Z) - F(x') \quad \text{for each } x' \in B_X(x, \delta),$$

then the set-valued mapping $R : X \to 2^Z$ defined by

$$R(x) := \{z \in G(x) : h(x, z) \in F(x)\}$$

is l.s.c. with nonempty convex values.

1.3 Measurability, selections and Lipschitz mappings

Definition 1.3.1. Suppose $(\Omega, \mathcal{A})$ is a measurable space, X is a complete metric space and $F : \Omega \to 2^X$ is a set-valued mapping with closed values.

(i) F is said to be *measurable* if the inverse image $F^{-1}(O)$ of each open subset $O \subset X$ is measurable.

(ii) A *selection* of F is a single-valued mapping f such that $f(\omega) \in F(\omega)$ for all $\omega \in \Omega$.

Theorem 1.3.2. *([6], Proposition 8.2.1, Theorem 8.2.5)*
Suppose that $(\Omega, \mathcal{A}, \mu)$ is a complete σ-finite measure space, X is a complete separable metric space and $F : \Omega \to 2^X$ is a set-valued mapping with nonempty closed values.

(i) If $\mathcal{A}$ contains all open subset of Ω and F is u.s.c. or l.s.c., then F is measurable.

(ii) If $F_n : \Omega \to 2^X$ are measurable set-valued mappings with closed values, then $\omega \mapsto \limsup_{n\to\infty} F_n(\omega)$ is measurable.

The first selection theorem is

Theorem 1.3.3 (Neumann). *([6], Theorem 8.1.3)*
Under the assumptions of Theorem 1.3.2, let F be a measurable set-valued mapping from Ω to X with nonempty closed values. Then F has a measurable selection.

For a set-valued mapping $F : \Omega \to \mathcal{P}_c(X)$ and a number $p \geq 1$, we always denote by

$$S_F^p = \big\{ f \in L^p(\Omega; X, \mu) : \ f(\omega) \in F(\omega) \text{ a.e. in } \Omega \big\},$$

which is called the *realization* of F, or Nemytski operator corresponding to F in case F is single-valued, in $L^p(\Omega; X, \mu)$. Obviously, S_F^p is closed and $S_F^p \neq \emptyset$ if and only if $\inf\big\{ \|x\| : x \in F(\omega) \big\} \in L^p(\Omega; \mathbb{R}, \mu)$. Moreover, we have

Theorem 1.3.4. *(Papageorgiou [65], Theorems 4.1 and 4.2)*
Suppose $F_n : \Omega \to \mathcal{P}_c(X)$ are measurable set-valued mappings.

(i) If $\big\{ \|F_n(\cdot)\| \big\}$ is uniformly integrable and $\liminf F_n(\omega) \neq \emptyset$ a.e., *then*

$$S^1_{\liminf F_n} \subset \liminf S^1_{F_n}.$$

(ii) Let, additionally, $F_n(\omega) \in \mathcal{P}_{cv}(X), F_n(\omega) \subset G(\omega)$ a.e. with $G : \Omega \to \mathcal{P}_{cv}(X)$. If the values of G are weakly compact and $\omega \mapsto w\text{-}\limsup F_n(\omega)$ is measurable, then

$$w\text{-}\limsup S^1_{F_n} \subset S^1_{w\text{-}\limsup F_n}.$$

Next, we consider the existence of continuous selections. The first one is known as Michael's Theorem.

Theorem 1.3.5 (Michael's Theorem). *([32], Lemma 1.2.1)*
Suppose X, Y are Banach spaces, $\Omega \subset X$ is a nonempty subset, $F : \Omega \to 2^Y$ is a l.s.c. mapping with closed convex values. Then for each $(x_0, y_0) \in \mathrm{Graph}(F)$, F has a continuous selection f with $f(x_0) = y_0$.

To present a selection theorem of non-convex valued mappings, we need

Definition 1.3.6. Suppose X is a separable Banach space. A set $D \subset L^1(\Omega; X)$ is said to be *decomposable* if for all $u, v \in D$ and all open subset $A \subset \Omega$, we have

$$\chi_A \cdot u + \chi_{\Omega \setminus A} \cdot v \in D.$$

Here χ_A is the characteristic function of set A.

Example 1.3.7. Under the assumptions of Theorem 1.3.2, S_F^q is decomposable for each $q \geq 1$. In fact, let $u, v \in S_F^q$ and $A \subset [0, T]$ be an open subset, Since $u, v \in L^q(\Omega; X, \mu)$, we see that $\chi_A u + \chi_{\Omega\setminus A} v \in L^q(\Omega; X, \mu)$. Since $u(\omega), v(\omega) \in F(\omega)$ a.e., we obtain $\chi_A(\omega)u(\omega) + \chi_{\Omega\setminus A}(\omega)v(\omega) \in F(\omega)$ a.e.. Hence, $\chi_A u + \chi_{\Omega\setminus A} v \in S_F^q$.

Theorem 1.3.8. *(Fryszkowski [41], Theorem 3.1)*
Suppose Ω is a topological space, X is a separable Banach space, $F : \Omega \to \mathcal{P}_c(L^1(\Omega; X))$ is a l.s.c. set-valued mapping with decomposable values. Then F has a continuous selection.

Now, we begin to consider Lipschitz mappings.

Definition 1.3.9. Let X be a metric space, Y a normed space, $F : X \to 2^Y \setminus \{\emptyset\}$ be a set-valued mapping. F is said to be

(i) *Lipschitz* of constant l at $x \in X$ if there exists a neighbourhood $\mathcal{U}$ of x such that

$$\text{for all } x_1, x_2 \in \mathcal{U}, \quad F(x_1) \subset F(x_2) + l d(x_1, x_2)\overline{B}_Y.$$

(ii) *pseudo-Lipschitz* of constant l around $(x, y) \in \mathrm{Graph}(F)$ if there exist neighbourhoods $\mathcal{U}$ of x and $\mathcal{V}$ of y such that

$$\text{for all } x_1, x_2 \in \mathcal{U}, \quad F(x_1) \cap \mathcal{V} \subset F(x_2) + l d(x_1, x_2)\overline{B}_Y.$$

Moreover, the number

$$\sup\{r > 0 : F(x_1) \cap B_Y(y,r) \subset F(x_2) + ld(x_1,x_2)\overline{B}_Y, \text{ for } x_1, x_2 \in \mathcal{U}\}$$

is called the *pseudo-Lipschitz modulus* of F related to $\mathcal{U}$.

Obviously, F is (locally) Lipschitz at x is equivalent to that there exists a neighbourhood $\mathcal{U}$ of x such that

$$\text{for all } x_1, x_2 \in \mathcal{U}, \quad \mathcal{H}_Y(F(x_1), F(x_2)) \le ld(x_1, x_2).$$

So, we have

Proposition 1.3.10. *A Lipschitz mapping is both ε-δ-u.s.c. and ε-δ-l.s.c..*

Before giving a general result about the Lipschitz property of the inverse mapping of a surjective linear operator, we recall the notion of cone.

Definition 1.3.11. A subset K of a linear space X is said to be a *cone* if $x \in K, \lambda \ge 0$ imply $\lambda x \in K$, that is, $\lambda K \subset K$ for all $\lambda \ge 0$.

Obviously, the whole space X is a cone. Two important cones in $\mathbb{R}^n$ are

$$\{(x_1, \cdots, x_n) : x_i > 0 \text{ for all } i\} \cup \{0\}.$$

and

$$\{(x_1, \cdots, x_n) : x_i \ge 0 \text{ for all } i \text{ and } x_j > 0 \text{ for some } j\} \cup \{0\}.$$

Example 1.3.12. Suppose X is a Banach space, $K \subset X$ is a cone, $1 \le p \le \infty$. Then

$$K_p := \{x \in L^p(t_0, T; X) : x(t) \in K \ a.\ e.\}$$

is a cone in the space $L^p(t_0, T; X)$. If, in addition, $\text{int}(K) \ne \emptyset$, then $\text{int}(K_\infty) \ne \emptyset$ in $L^\infty(t_0, T; X)$ (see [31]).

Definition 1.3.13. Let X be a normed space, $x \in D \subset X$. The *contingent cone* of D at $x \in \overline{D}$ is the subset

$$T_D(x) = \{v \in X : \liminf_{h \to 0^+} h^{-1} d(x + hv, D) = 0\}.$$

Note, if $x \in \mathrm{int}(D)$ or $\overline{D} = X$, then $T_D(x) = X$.

Theorem 1.3.14. *([6], p.121-122; [32], Proposition 4.1)*

(i) $T_D(x)$ is a closed cone.

(ii) $v \in T_D(x)$ if and only if there exist $v_n \in X$ and positive real numbers $\lambda_n \to 0^+$ such that $v_n \to v$ and $x + \lambda_n v_n \in D$.

Theorem 1.3.15. *([6], Corollary 2.2.5; [73], Proposition 2.4)*
Suppose X, Y are Banach spaces, $L \in \mathbf{L}(X, Y)$ and $K \subset X$ is a closed convex cone such that $L(K) = Y$. Then there exists $c > 0$ such that $B_Y \subset L(cB_X \cap K)$ and $y \mapsto L^{-1}(y) \cap K$ (possibly set-valued) is Lipschitz from Y to X.

1.4 Differentiability and distributions

In this section, we give the definitions of some differentiability of operators or functions and some of their properties. The traditional ones are the Gâteaux and Fréchet differentiability.

Definition 1.4.1. Suppose X, Y are normed spaces, $\Omega \subset X$ is an open subset with $x \in \Omega$, $A : \Omega \to Y$ is an operator.

(i) If there exists $A'(x) \in \mathbf{L}(X, Y)$ such that

$$\lim_{||u|| \to 0} \frac{||A(x+u) - A(x) - A'(x)u||}{||u||} = 0,$$

then we say A is *Fréchet differentiable* at x and the bounded linear operator $A'(x)$ is said to be the *Fréchet derivative* of A at x. If

$$\lim_{u,v \to x} \frac{||A(u) - A(v) - A'(x)(u-v)||}{||u-v||} = 0,$$

then we say $A'(x)$ is the *strict Fréchet derivative* of A at x.

(ii) If, for each $u \in X$, the limit

$$\mathrm{D}A(x)u := \lim_{t \to 0} \frac{A(x+tu) - A(x)}{t}$$

exists, then A is said to be *Gâteaux differentiable* at x and the operator $u \mapsto \mathrm{D}A(x)u$ is said to be the *Gâteaux derivative* of A at x. In other words, $\mathrm{D}A(x)u = \frac{\mathrm{d}}{\mathrm{d}t}A(x+tu)|_{t=0}$.

(iii) If there exists $A'(x) \in \mathbf{L}(X, Y)$ such that

$$\lim_{t \to 0} \left(t^{-1}[A(x+tu) - A(x)] - A'(x)u, y^* \right) = 0, \quad \text{for all } u \in X, y^* \in Y^*,$$

then A is said to be *weakly Gâteaux differentiable* at x with the *weak Gâteaux derivative* $A'(x)$.

Theorem 1.4.2. *([75]) (i)* $\mathrm{D}A(x)(ku) = k\,\mathrm{D}A(x)u$ *for all* $k \in \mathbb{R}$.

(ii) If A is Fréchet differentiable at x, then A is Gâteaux differentiable and $\mathrm{D}A(x) = A'(x)$.

(iii) If A is Gâteaux differentiable near x_0, $u \mapsto \mathrm{D}A(x)u$ is linear bounded and $x \mapsto \mathrm{D}A(x)$ is continuous at x_0, then A is Fréchet differentiable at x_0.

(iv) If A is Gâteaux differentiable near x_0, $u \mapsto \mathrm{D}A(x_0)u$ is continuous at $u = 0$ and $x \mapsto \mathrm{D}A(x)$ is continuous at x_0, then $\mathrm{D}A(x_0)$ is linear and bounded.

(v) If $A_1 : X \to Y$ is Gâteaux differentiable at x and $A_2 : Y \to Z$ (another normed space) is Fréchet differentiable at $A_1(x)$, then $A_3 := A_2 \circ A_1$ is Gâteaux differentiable at x and

$$\mathrm{D}A_3(x) = A_2'(A_1(x))\,\mathrm{D}A_1(x).$$

Remark 1.4.3. Recall that a mapping T is said to be *(positively) homogeneous* if

$$T(\lambda x) = \lambda T(x), \quad \text{for all } x \in \mathrm{Dom}(T) \text{ and all (nonnegative) number } \lambda.$$

So, $x \mapsto \mathrm{D}A(x)$ is homogeneous.

Recently, some weaker notions of differentiability have been introduced for solving more general problems (see [3], [6], [39], [48], [67] and [78]), some are defined for set-valued mappings. Among them, we recall the following two.

Definition 1.4.4. (Ioffe [48]) Suppose X, Y are normed spaces. A homogeneous set-valued mapping $\mathcal{A} : X \to 2^Y$ is called a *strict prederivative* of a mapping $F : \mathrm{Dom}(F) \subset X \to 2^Y$ at $x_0 \in \mathrm{Dom}(F)$ if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$F(x_1) \subset F(x_2) + \mathcal{A}(x_1 - x_2) + \varepsilon \|x_1 - x_2\| B_Y$$

for all $x_1, x_2 \in \mathrm{Dom}(F) \cap B_X(x_0, \delta)$.

Clearly, if both F and A are single-valued, the strict prederivative is nothing but the strict Fréchet derivative.

Definition 1.4.5. (Altman [3], Welsh [78]) Let X, Y be normed spaces, $A : X \to Y, \Gamma(x) : Y \to X$ be operators with $x \in X$. If $\Gamma(x)$ is bounded and

$$A(x + \lambda\Gamma(x)y) = A(x) + \lambda y + o(\lambda), \text{ for each } y \in Y,$$

then, $\Gamma(x)$ is said to be the *weak Gâteaux inverse derivative* of A at x. If, in addition, $\Gamma(x)$ is linear, then $\Gamma(x)$ is said to be the *directional contractor* of A at x.

Now, we turn our attention to the derivative of vector valued distributions.

Definition 1.4.6. Suppose X is a Banach space. Let $\mathfrak{D}(a,b)$ be the space of all infinitely differentiable real valued functions on $[a,b]$ with compact support in (a,b). The space $\mathfrak{D}(a,b)$ is topologized as an inductive limit of $\mathfrak{D}_K(a,b)$ where K ranges over all compact subsets and $\mathfrak{D}_K(a,b) = \{\phi \in \mathfrak{D}(a,b) : \text{ supp } \phi \subset K\}$. A linear continuous operator from $\mathfrak{D}(a,b)$ to X is said to be a X-valued *distribution* on (a,b), the set of all those distributions is denote by $\mathfrak{D}'(a,b;X)$. If $u \in \mathfrak{D}'(a,b;X)$, then the distribution $u^{(j)}$ defined by

$$u^{(j)}(\phi) = (-1)^i u(\phi^{(j)}) \text{ for each } \phi \in \mathfrak{D}(a,b)$$

is said to be the *derivative* of order j of u in the sense of distributions. We write $u' := u^{(1)}, u'' := u^{(2)}$.

Suppose $u \in L^1(a,b;X)$ is given. Then a unique X-valued distribution $\hat{u}$ can be defined by

$$\hat{u}(\phi) = \int_a^b u(t)\phi(t)dt \quad \text{for each } \phi \in \mathfrak{D}(a,b).$$

We can identify $\hat{u}$ with u and, therefore

$$L^1(a,b;X) \subset \mathfrak{D}'(a,b;X).$$

Moreover, if $u \in L^1(a,b;X)$ and $v(t) = \int_0^t u(s)ds$, then $v'(t) = u(t)$ a.e..

Theorem 1.4.7. *([7], Theorems 1.2.1 and 1.2.2)* *(i)* $u \in \mathfrak{D}'(a,b;X)$ *with* $u' \in L^p(a,b;X)$ $(p \geq 1)$ *if and only if there exists an absolutely continuous,* a.e.*differentiable function* u_0 *with* $u(t) = u_0(t)$ a.e. *in* (a,b) *and*

$$u'(t) = \lim_{s \to 0} \frac{u_0(t+s) - u_0(s)}{s} =: \frac{\mathrm{d}}{\mathrm{d}t} u_0(t) \text{ a.e. } in\ (a,b).$$

(ii) If X *is reflexive, then each absolutely continuous function* u *is* a.e. *differentiable with* $\mathrm{d}u/\mathrm{d}t \in L^1(0,T;X)$ *and*

$$u(t) = u(a) + \int_a^t \frac{\mathrm{d}}{\mathrm{d}t} u(s) ds \text{ a.e. } \quad on\ [a,b].$$

Since this theorem, we will not distinguish the notions u' and $\mathrm{d}u/\mathrm{d}t$.

In the theory of evolution equations, it is more interesting to consider the space X with $X \subset X^*$ and the function u with $u'(t) \in X^*$. In this case, in order to calculate $u'(t)$ and $(u(t), u'(t))$, a Hilbert space between X and X^* is necessary. This leads to the following definition.

Definition 1.4.8. Let V be a reflexive Banach space with the dual V^*, H be a Hilbert space such that $V \hookrightarrow H \hookrightarrow V^*$ densely and continuously. Then (V, H, V^*) is called an *evolution triple* or a *Gelfand triple.* In case V is also a Hilbert space, we say (V, H, V^*) is an *evolution triple of Hilbert spaces.*

Example 1.4.9. Suppose Ω is a bounded domain in $\mathbb{R}^n, p \geq 2$. Then the Sobolev space $V := W_0^{m,p}(\Omega)$ and the Hilbert space $H := L^2(\Omega)$ form an evolution triple (V, H, V^*) ([80], p.416).

In the sequel, we always suppose (V, H, V^*) is an evolution triple and denote by $(\cdot,\cdot)$ the duality pairing between V and V^*, by $\langle \cdot,\cdot \rangle$ the inner product on H. It is known that

$$(u, v) = \langle u, v \rangle, \text{ for all } u \in V, v \in H$$

and there exist constants $\beta_1, \beta_2 > 0$ such that

$$\|v\|_{V^*} \leq \beta_1 \|v\|_H, \quad \|u\|_H \leq \beta_2 \|u\|_V, \text{ for all } u \in V, v \in H.$$

For $p \geq 2 \geq q > 1$ and $1/p + 1/q = 1$, we write

$$W(a,b) \equiv W(a,b;V,H) := \left\{ u \in L^p(a,b;V) : u' \in L^q(a,b;V^*) \right\}.$$

Clearly, $W(a,b)$ is a linear space. Endowed with the norm $\|u\|_W = \|u\|_{L^p(V)} + \|u'\|_{L^q(V^*)}$, $W(a,b)$ has the following properies.

Theorem 1.4.10. *([72], Propositions III.1.2 and III.1.3)*

(i) $W(a,b)$ is a Banach space and $W(a,b) \hookrightarrow C(a,b;H)$ continuously.

(ii) For all $u, v \in W(a,b)$, the function $t \mapsto \langle u(t), v(t)\rangle$ is absolutely continuous and

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\langle u(t), v(t)\rangle = (u'(t), v(t)) + (u(t), v'(t)), \text{ a.e..}$$

(iii) If, in addition, $V \hookrightarrow H$ compactly, then $W(a,b) \hookrightarrow L^p(a,b;H)$ compactly.

Example 1.4.11. If $B \in \mathrm{L}(V, V^*)$ is *symmetric* (i.e. $(Bu, v) = (u, Bv)$ for all $u, v \in V$) and $u \in W(a,b)$, then it follows from the product rule of differentiation that

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\big(Bu(t), u(t)\big) = \big((Bu(t))', u(t)\big) + \big(Bu(t), u'(t)\big) = 2\big((Bu(t))', u(t)\big).$$

See also [72] Proposition III 3.1.

To close this section, we give the weak closedness of the operator $u \mapsto u^{(k)}$.

Theorem 1.4.12. *([80], Proposition 23.19) If $u_n \rightharpoonup u$ in $L^p(a,b;V)$ and $u_n^{(k)} \rightharpoonup v$ in $L^q(a,b;V^*)$ as $n \to \infty$, then $v(t) = u^{(k)}(t)$ a.e. on (a,b).*

Remark 1.4.13. In [80], the result is proved for the generalized derivative. But the method remains valid in our situation. In fact, our assumptions imply that $u_n\phi^k \rightharpoonup u\phi^k$ in $L^q(a,b;V^*)$ for each $\phi \in \mathfrak{D}(a,b)$. Therefore, we have $\int_a^b u_n(t)\phi^k(t)dt \rightharpoonup \int_a^b u(t)\phi^k(t)dt$. Similarly, for each $\phi \in \mathfrak{D}(a,b)$, we have $\int_a^b u_n^{(k)}(t)\phi(t)dt \rightharpoonup \int_a^b v(t)\phi(t)dt$. So,

$$\int_a^b v(t)\phi(t)dt = \int_a^b u(t)\phi^k(t)dt,$$

which means that $v = u^{(k)}$.

1.5 Topological degree and fixed point theorems

In this section, we recall the Leray-Schauder topological degree for set-valued mappings and some fixed point theorems.

Definition 1.5.1. A set-valued mapping F between two topological spaces is said to be *compact*, if F is u.s.c. and $F(D)$ is relatively compact for each bounded subset $D \subset \text{Dom}(F)$. If a mapping G is of the form $I - F$ with F compact, then G is said to be of *(LS) type (Leray-Schauder type)*..

Clearly, in finite dimensional linear spaces, the classes of compact mappings and mappings of (LS) type are equivalent.

Theorem 1.5.2. *[31] Suppose X is a Banach space, $\Omega \subset X$ is a bounded open subset. Let $(\mathcal{F}, \Omega)$ be the set of all those compact mapping $F : \overline{\Omega} \to \mathcal{P}_{cv}(X)$ with $0 \notin (I - F)(\partial\Omega)$. Then, for each $F \in (\mathcal{F}, \Omega)$, there exists an integer (topological degree) $\deg(I - F, \Omega, 0)$ such that*

(i) (solvability) $\deg(I - F, \Omega, 0) \neq 0$ implies that there exists $x \in \Omega$ with $x \in F(x)$;

(ii) (invariance under admissible homotopy) if $H : [0,1] \times \overline{\Omega} \to \mathcal{P}_{cv}(X)$ is compact and $H(t, \cdot) \in (\mathcal{F}, \Omega)$ for all $t \in [0,1]$, then

$$\deg(I - H(0, \cdot), \Omega, 0) = \deg(I - H(1, \cdot), \Omega, 0);$$

(iii) (Borsuk property) if, in addition, Ω is a symmetric neighbourhood of the origin, and F is odd, then $\deg(I - F, \Omega, 0)$ is odd.

(iv) (Normalization property) $\deg(I - y, \Omega, 0) = 1$ for each $y \in \Omega$.

Remark 1.5.3. The mapping $H(t, x)$ in (ii) is called an *admissible homotopy of compact mapping* and $I - H$ is called an *admissible homotopy of (LS) type mappings*. They are often used, together with (i), to obtain fixed point theorems.

If the domain Ω is not open, the following three fixed point theorems may be applicable.

Theorem 1.5.4 (Kakutani). *([31], Theorem 24.4; [49], Corollary 10.3.10) Suppose X is a locally convex topological space, $D \subset X$ is a bounded, closed and convex subset, $F : D \to \mathcal{P}_{cv}(D)$ is compact, then F has fixed point in D.*

Theorem 1.5.5 (Leray-Schauder Alternative). *([34]; [60], Theorem 2.6)*
Suppose X is a Banach space, $K \subset X$ is a convex subset, and $\Omega \subset X$ is a bounded open subset with $p \in \Omega \cap K$. If $N : \overline{\Omega} \cap K \to \mathcal{P}_{cv}(K)$ is an u.s.c. compact set-valued mapping, then either N has a fixed point in $\overline{\Omega} \cap K$, or there exist $x \in \partial\Omega \cap K$ and $\lambda \in (0,1)$ such that $x \in \lambda N(x) + (1-\lambda)p$.

Theorem 1.5.6 (Banach Contraction Principle). *([49], Theorems 3.1.2 and 10.4.7)*
Suppose X is a complete metric space, $F : X \to \mathcal{P}_c(X)$. If $k \in [0,1)$ is such that

$$\mathcal{H}_X(F(x), F(y)) \leq kd(x,y) \quad \textit{for all } x, y \in X$$

then F has at least one fixed point. If, in addition, F is single-valued, then the fixed point is unique.

1.6 Mappings of monotone type

In this section, we always suppose X is a reflexive Banach space with dual X^* and the duality pairing $(\cdot,\cdot)$.

Definition 1.6.1. An operator $A : \mathrm{Dom}(A) \subset X \to 2^{X^*}$ is said to be (i) of *class (S_+)* (write $A \in (S_+)$), if $x_n \in \mathrm{Dom}(A)$ with $x_n \rightharpoonup x$ in X, $u_n \in Ax_n$ and $\limsup_{n\to\infty}(u_n, x_n - x) \leq 0$ imply $x_n \to x$;

(ii) *monotone*, if $(u - v, x - y) \geq 0$ for all $x, y \in \mathrm{Dom}(A), u \in Ax, v \in Ay$;

(iii) *maximal monotone* if $(u - v, x - y) \geq 0$ for all $x \in \mathrm{Dom}(A), u \in Ax$ imply that $y \in \mathrm{Dom}(A)$ and $v \in Ay$;

(iv) *pseudo-monotone* (write $A \in (PM)$), if $x_n \in \mathrm{Dom}(A)$ with $x_n \rightharpoonup x$ in $X, u_n \in Ax_n$ and $\limsup_{n\to\infty}(u_n, x_n - x) \leq 0$ imply $x \in \mathrm{Dom}(A)$ and, for each $y \in X$, there exists $u(y) \in Ax$ such that

$$(u(y), x - y) \leq \liminf_{n\to\infty}(u_n, x_n - y);$$

(v) *quasi-monotone* (write $A \in (QM)$), if $x_n \in \mathrm{Dom}(A)$ with $x_n \rightharpoonup x$ in X, $u_n \in Ax_n$ imply

$$\limsup_{n\to\infty}(u_n, x_n - x) \geq 0 \text{ (equivalently } \liminf_{n\to\infty}(u_n, x_n - x) \geq 0).$$

Single-valued operators of each of the above monotone types and set-valued monotone mappings are well known, see, for example, [7], [20]–[24], [26] and [31]. Set-valued pseudo-monotone mappings were extensively considered in [25] by Browder and Hess where finite continuity was also imposed in the definition and the domain was supposed to be the whole space. The definitions (only) for set-valued quasi-monotone mappings and mappings of class (S_+) can be found in [51] where the author considered some special subclasses. In Chapter 3 of this thesis, we will discuss the general cases.

It is known that (see, for example, [12]) if the operators involved are all single-valued, bounded and demicontinuous, then we have the following relations.

$$(S_+) \Longrightarrow (PM) \Longrightarrow (QM), \quad \text{monotone} \Longrightarrow (PM), \quad \text{compact} \Longrightarrow (QM).$$

Moreover, each class of operators has a conical structure (that is if A_1, A_2 are same kind of operators, then $A_1 + \lambda A_2$ remains in the same class for all $\lambda > 0$). In Chapter 3, we shall prove that these properties are preserved in the set-valued case.

Example 1.6.2. Each mapping in finite dimensional spaces is of class (S_+) and, therefore, pseudo-monotone if it is also bounded and demicontinuous. So the class of pseudo-monotone mappings is much wider than that of monotone mappings

Example 1.6.3. Suppose (V, H, V^*) is an evolution triple, $B \in \mathbf{L}(V, V^*)$ is a *positive* (i.e. $(Bu, u) \geq 0$ for all $u \in V$) and symmetric operator. For each $x \in L^1(0, T; V)$, write

$$(\hat{B}x)(t) = Bx(t), \quad (Lx)(t) = \int_0^t x(s)ds, \quad (L^*x)(t) = \int_t^T x(s)ds.$$

Then the operators L, L^* and $L^*\hat{B}, \hat{B}L$ are positive and, therefore, monotone from $L^p(0, T; V)$ to $L^q(0, T; V^*)$, where $p \geq q > 1, 1/p + 1/q = 1$.

In fact, it is easy to see that $L, L^*, L^*\hat{B}, \hat{B}L \in \mathbf{L}\big(L^p(0, T; V), L^q(0, T; V^*)\big)$ as we now show. It is known that L^* is the adjoint operator of L if we take L as an operator on $L^p(0, T; V)$ and L^* as an operator on $L^q(0, T; V^*)$. Denote by $(\cdot, \cdot)$ the duality pairing between V and V^* and by $((\cdot,\cdot))$ the duality pairing between $L^p(0,T;V)$ and $L^q(0,T;V^*)$. Let $x \in L^p(0,T;V)$ and write $y(t) = Lx(t)$. Then $y'(t) = x(t)$ a.e., $y(0) = 0$ and

$$((Lx, x)) = \int_0^T (Lx(t), x(t))dt = \int_0^T (y(t), y'(t))dt = \frac{1}{2}\|y(T)\|_H \geq 0.$$

That is L is positive. Similarly, L^* is positive. Noting that $y(0) = 0$, by Example 1.4.11

$$\begin{aligned} ((L^*\hat{B}x, x)) &= \int_0^T (L^*Bx(t), x(t))dt = \int_0^T (Bx(t), Lx(t))dt = \int_0^T (By'(t), y(t))dt \\ &= \frac{1}{2}\int_0^T \frac{\mathrm{d}}{\mathrm{d}t}(By(t), y(t))dt = \frac{1}{2}(By(T), y(T)) \geq 0 \end{aligned}$$

and

$$((\hat{B}Lx, x)) = \int_0^T (BLx(t), x(t))dt = \int_0^T (By(t), y'(t))dt = \frac{1}{2}(By(T), y(T)) \geq 0,$$

that is $L^*\hat{B}$ and $\hat{B}L$ are positive (see also the proof of Theorem 32.E in [80]).

Remark 1.6.4. The operator L defined above has the following extra property.

> *If x_n, x are functions from $[0,T]$ into V with $Lx_n, Lx \in L^p(V)$, $\{Lx_n(t)\}$ bounded in V and $x_n \rightharpoonup x$ in $L^q(V^*)$, then $Lx_{n_j}(t) \rightharpoonup Lx(t)$ in V for a subsequence.*

In fact, we may suppose $Lx_{n_j}(t) \rightharpoonup z$ in V. Noting the fact that $x_n \in L^1(0,t;V)$, for each $v \in V$ $(\subset V^*)$, we have

$$(Lx_{n_j}(t), v) = \int_0^t \langle x_{n_j}(s), v\rangle ds \to \langle z, v\rangle = (z, v).$$

Since $x_n(t) \in V$, for each $y \in L^p(V)$, $x_n \rightharpoonup x$ in $L^q(V^*)$ implies

$$((x_{n_j} - x, y)) = \int_0^T \langle x_{n_j}(s) - x(s), y(s)\rangle ds \to 0.$$

Let $v \in V, y(s) = v$ for $s \leq t$ and $y(s) = 0$ for $s > t$. Then we obtain

$$\int_0^t \langle x_{n_j}(s) - x(s), v\rangle ds \to 0, \quad \text{i.e.} \int_0^t \langle x_{n_j}(s), v\rangle ds \to \langle Lx(t), v\rangle = (Lx(t), v).$$

Hence $(Lx(t), v) = (z, v)$ for all $v \in V$. The density of the embedding of V into H implies $z = Lx(t)$.

The following result involving pseudo-monotone mappings was given by Browder.

Theorem 1.6.5. *(Browder [20], Theorem7.8)*
Let X be a reflexive Banach space, $D \subset X$ be a closed convex subset, N, N_0 be two mappings of D into 2^{X^*} with N monotone and N_0 pseudo-monotone. Suppose $(0,0) \in \mathrm{Graph}(N)$*, N_0 is bounded and finitely continuous, and there exist $R > 0$ and $w_0 \in X^*$ such that*

$$(u, x) > (w_0, x), \quad \text{for all } (x, u) \in \mathrm{Graph}(N_0) \text{ with } \|x\| > R.$$

Then there exists $(x_0, u_0) \in \mathrm{Graph}(N_0)$ such that

$$(y + u_0 - w_0, x - x_0) \geq 0, \quad \text{for all } (x, y) \in \mathrm{Graph}(N).$$

Now we recall a special monotone mapping and some of its properties.

Definition 1.6.6. The mapping defined by

$$J(x) = \{x^* \in X^* : (x, x^*) = \|x\|^2 = \|x^*\|^2\}.$$

is called the *duality mapping* of X.

Theorem 1.6.7. *(i)* $\mathrm{Dom}(J) = X$. *(ii) If X^* is strictly convex, then J is single-valued, demicontinuous, monotone and is of class (S_+). (iii) X is reflexive if and only if J is onto, X^* is strictly convex if and only if J is single-valued. (iv) If $u : \mathbb{R} \to X$ has a weak derivative $u'(s)$ at s and $t \mapsto \|u(t)\|$ is differentiable at s, then*

$$(u'(s), x^*) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\,t}\|u(t)\|^2\Big|_{t=s}, \quad \text{for each } x^* \in J(u(s)).$$

Remark 1.6.8. In Theorem 1.6.7, (i), (ii) can be found in Proposition 8 of Browder [22], see also Proposition 1.1.3 in [7]. (iii) is Proposition 2.16 in [10]. (iv) is Lemma 3.1.2 in [7] which was given by Kato in 1967.

Theorem 1.6.9 (Renorming Theorem). *([7], Theorem 1.1.1)*
For a reflexive Banach space X, there exists an equivalent norm $\|\cdot\|_0$ on X under which X is strictly convex, X^* is also strictly convex under the corresponding dual norm $\|\cdot\|_0^*$.

Because of the above two theorems, we may always suppose the duality mapping between reflexive Banach spaces is single-valued.

1.7 Nonlinear differential evolution equations

In this section, we recall some notions and known results related to differential evolution equations in Banach spaces.

Definition 1.7.1. Let X be a Banach space, $\triangle_T = \{(t,s) : 0 \leq s \leq t \leq T\}$ and $A(t) : \mathrm{Dom}(A(t)) \subset X \to X$ be a linear operator for each $t \in [0,T]$. Suppose there exists a two-variables family of bounded linear operators $\{E(t,s)\} := \{E(t,s) : (t,s) \in \triangle_T\}$ on X such that

(i) $E(t,t) = I$ (I is the identity on X) for all $t \in [0,T]$,

(ii) $E(t,s) = E(t,r)E(r,s)$ for all $0 \leq s \leq r \leq t \leq T$,

(iii) $\partial E(t,s)/\partial t = A(t)E(t,s)$, $\partial E(t,s)/\partial s = -E(t,s)A(s)$ on $\triangle_T$.

Then $\{E(t,s)\}$ is said to be a *evolution system* on X generated by $A(t)$. In addition, if $(t,s) \mapsto E(t,s)x$ is continuous as a mapping into X for each $x \in X$, we say $\{E(t,s)\}$ is *strongly continuous*; If $(t,s) \mapsto E(t,s)$ is continuous as a mapping into $\mathbf{L}(X,X)$, we say $\{E(t,s)\}$ is *uniformly continuous*; If $E(t,s)$ is compact for each $0 \leq s < t \leq T$, we say $\{E(t,s)\}$ is *compact.*

If $\{S(t) : t \in [0,T]\} \subset \mathbf{L}(X)$ is a family of bounded linear operators with $S(0) = I, S(t+s) = S(t)S(s)$ for every $t,s \in [0,T]$, then $S(t)$ is said to be a *semigroup of bounded linear operators* generated by the operator $A : \mathrm{Dom}(A) \to X$

$$Ax = \lim_{t \to 0} \frac{S(t)x - x}{t}.$$

Here $\mathrm{Dom}(A)$ is the set of all $x \in X$ such that the above limit exists. Obviously, $\{E(t,s)\} = \{S(t-s)\}$ is an evolution system generated by A.

By the uniform boundedness theorem, we see that a strongly continuous evolution system is uniformly bounded. It is also known that a compact evolution system is uniformly continuous but the converse is not true.

The following theorem is "well known", but for completeness, we give its proof.

Theorem 1.7.2. *Suppose X is a Banach space, $\{E(t,s)\}$ is a strongly continuous and compact evolution system. For each $x \in L^2(0,T;X)$ and each $t \in [0,t]$, let*

$$(Lx)(t) = \int_0^t E(t,s)x(s)ds.$$

Then both $x \mapsto Lx(t)$ and $x \mapsto Lx$ are are compact linear continuous operators from $L^2(0,T;X)$ to X or $C(0,T;X)$, respectively.

Proof. Obviously, L is linear and continuous.

Let $t \in (0,T]$ and let $D \subset L^2(t,T;X)$ be a bounded subset. Choose $\varepsilon > 0$ such that $t-\varepsilon > 0, t_1 \in (t-\varepsilon, t)$. Then

$$L(D)(t-\varepsilon) = \left\{\int_0^{t-\varepsilon} E(t,s)x(s)ds : x \in D\right\} = E(t,t_1)\left\{\int_0^{t-\varepsilon} E(t_1,s)x(s)ds : x \in D\right\}.$$

Since $E(t,t_1)$ is compact and D is bounded, $L(D)(t-\varepsilon)$ is precompact. Now suppose $x \in D$. Then

$$\left\|\int_0^t E(t,s)x(s)ds - \int_0^{t-\varepsilon} E(t,s)x(s)ds\right\| \le \left\|\int_{t-\varepsilon}^t E(t,s)x(s)ds\right\|$$

$$\le \int_{t-\varepsilon}^t \|E(t,s)\|\|x(s)\|ds \le \left(\int_{t-\varepsilon}^t \|E(t,s)\|^2 ds\right)^{\frac{1}{2}} \left(\int_{t-\varepsilon}^t \|x(s)\|^2 ds\right)^{\frac{1}{2}} \le \varepsilon^{\frac{1}{2}} N\|D\|.$$

Here, $N = \max_{t,s}\|E(t,s)\|$. So $L(D)(t)$ is totally bounded and, therefore, precompact. So L is compact as an operator from $L^2(0,T;X)$ to X.

Secondly, in the usual way we can prove that $\{Lx : x \in D\}$ is equicontinuous and bounded, so the Arzela-Ascoli Theorem implies $L(D)$ is precompact in $C(0,T;X)$. This completes the proof. □

Now we consider the evolution equation

$$\begin{aligned} x'(t) &= A(t)x(t) + f(t,x(t)), \text{ a.e. } t \in [0,T], \\ x(0) &= x_0 \in X, \end{aligned} \tag{1.1}$$

in a Banach space X with $A(t)$ a linear operator on X for each $t \in [0,T]$ and $f : [0,T] \times X \to X$ a nonlinear function.

A function $x : [0,T] \to X$ is said to be a *strong solution* of (1.1) if x is differentiable almost everywhere on $(0,T), x' \in L^1(0,T;X)$, $x(t) \in \mathrm{Dom}(A(t))$ a.e., $x(0) = x_0$ and (1.1) is satisfied.

It is known that if $A(t)$ generates an evolution system $\{E(t,s)\}$, then each strong solution x of (1.1) has the expression

$$x(t) = E(t,0)x_0 + \int_0^t E(t,s)f(s,x(s))ds, \quad t \in [0,T], \tag{1.2}$$

but the converse is not true. So we have the following definition.

Definition 1.7.3. Suppose $A(t)$ is linear and generates an evolution system $\{E(t,s)\}$. A function $x : [0,T] \to X$ is said to be a *mild solution* of (1.1) if x is continuous on $[0,T]$ and satisfies (1.2).

There are some conditions to ensure that a differential equation (even inclusion) has mild solutions. For example, one can find some of those conditions in [14] or [66] and the references therein. To save pages, we will not list them here.

We are more interested in the case when $A(t)$ is also nonlinear with values in X^*, the dual of X, which is the model for many boundary value problems. For example, suppose we are in the position to consider the problem

$$\begin{aligned} &x_t(z,t) + \textstyle\sum_{|\alpha|\le m}(-1)^{|\alpha|}D^\alpha A_\alpha(z, Dx(z,t),t) = g(z,t) \quad \text{on } \Omega \times [0,T], \\ &D^\beta x(z,t) = 0 \text{ on } \partial\Omega \times [0,T] \text{ for } |\beta| \le m-1,\ x(z,0) = x_0(z) \text{ on } \Omega \end{aligned} \tag{1.3}$$

with $\Omega \subset \mathbb{R}^n$ a bounded domain. If we let $V = W_0^{m,p}(\Omega), H = L^2(\Omega), V^* = W^{-m,p}(\Omega)$ and write

$$a(t,u,v) = \int_\Omega \sum_{|\alpha|\le m} A_\alpha(z, Du(z,t),t)D^\alpha v(z)dz, \quad b(t,v) = \int_\Omega g(z,t)v(z)dz$$

for all $u, v \in V$, then problem (1.3) is equivalent to

$$x(0) = x_0 \quad \frac{\mathrm{d}}{\mathrm{d}t}\langle x(t), v\rangle + a(t,x,v) = b(t,v), \text{for all } v \in V$$

with $\langle\cdot,\cdot\rangle$ the inner product on H. Since $\frac{\mathrm{d}}{\mathrm{d}t}\langle x(t),v\rangle = (x'(t),v)$ (see [80] p.420), (1.3) becomes problem (1.1) with $A(t,u), f(t) \in V^*$ given, respectively, by

$$(A(t,u),v) = a(t,u,v), \quad (f(t),v) = b(t,v) \text{ for } v \in V.$$

In general, we suppose (V,H,V^*) is an evolution triple, $A : [0,T] \times V \to V^*$ is a nonlinear operator, $f : [0,T] \to V^*$ is a function and rewrite (1.1) as

$$\begin{aligned} &x'(t) + A(t,x(t)) = f(t), \ \text{ a.e. } t \in [0,T], \\ &x(0) = x_0 \in V, \end{aligned} \tag{1.4}$$

Here, the differentiability is understood in the vector distribution sense.

Definition 1.7.4. Suppose $p \geq 2, q = p/(p-1)$. A function $x \in L^p(0,T;V)$ is said to be a *solution* of (1.4) if x is differentiable almost everywhere in the vector distribution sense, $x' \in L^q(0,T;V^*)$ and satisfies (1.4).

A well-known existence result for (1.4) is

Theorem 1.7.5. *([7], Theorem 3.4.2)*
Let (V, H, V^*) be an evolution triple. Suppose $v \mapsto A(t,v)$ is hemicontinuous and monotone for a.e. $t \in (0,T), t \mapsto A(t,u)$ *is measurable for each $u \in V$. If there exist constants $a_1, a_2, a_3 \geq 0, a_4 > 0$ such that*

$$\|A(t)u\| \leq a_1\|u\|^{p-1} + a_2, \quad (A(t)u, u) \geq a_4\|u\|^p - a_3, \text{ for all } u \in V,$$

then for each $x_0 \in V, f \in L^q(0,T;V^*)$, (1.4) has a unique solution.

Remark 1.7.6. When A is set-valued, this theorem remains valid, see [5] or [7].

Finally, we give a generalization of the well known *Gronwall's inequality* with proof for completeness, of which a more general extension can be found in Webb [77] and further special cases can be found in [57].

Theorem 1.7.7 (Extended Gronwall's Inequality).
Let $h_i \in L^1(0,T), i = 1,2, \phi \in L^\infty(0,T)$ be non-negative functions. Let $c \geq 0$ be a constant such that

$$\phi^2(t) \leq c^2 + 2\int_0^t \left[h_1(s)\phi(s) + h_2(s)\phi^2(s)\right] ds.$$

Then

$$\phi(t) \leq \int_0^t h_1(s) \exp\left(\int_s^t h_2(\tau)d\tau\right) ds + c\exp\left(\int_0^t h_2(s)ds\right).$$

Proof. We may suppose $c > 0$. Otherwise, let $c = 1/n$ and, then, let $n \to \infty$. Write

$$u^2 = c^2 + 2\int_0^t \left[h_1(s)\phi(s) + h_2(s)\phi^2(s)\right] ds.$$

Then $u(t) \geq 0, u(0) = c, \phi(t) \leq u(t)$ and u^2 is absolutely continuous, therefore

$$\frac{d}{dt}u^2(t) = 2h_1(t)\phi(t) + 2h_2(s)\phi^2(t) \leq 2h_1(t)u(t) + 2h_2(s)u^2(t), \text{ a.e..}$$

Since $u(t) \geq c$, u is absolutely continuous and, therefore, $\left(u^2(t)\right)' = 2u(t)u'(t)$. Hence

$$u'(t) \leq h_1(t) + h_2(t)u(t), \quad \text{a.e..}$$

By Gronwall's inequality, we have

$$\phi(t) \leq u(t) \leq \int_0^t h_1(s) \exp\left(\int_s^t h_2(\tau)d\tau\right) ds + c \exp\left(\int_0^t h_2(s)ds\right).$$

This completes the proof. $\square$

Chapter 2

Solvability of Operator Inclusions under Derivative Conditions and Applications in Control Theory

In this chapter, we study the local solvability of operator inclusions by giving some constrained implicit function theorems and open mapping theorems of set-valued mappings. We also study the surjectivity of certain (set-valued) implicit mappings and the applications of the above problems to constrained controllability theory of nonlinear systems. According to the tradition in the study of implicit function, a single-valued mapping will be called a function instead of an operator.

It is known that classical implicit function theorems and open mapping theorems need the function (single-valued) to be Fréchet differentiable and the derivative to be surjective. Although Fréchet differentiability is readily verified in some applications and has received much attention (see [30], [40] and [52]), those results cannot be applied if the function is not Fréchet differentiable or is set-valued. Recently, there have been many publications concerning non-Fréchet differentiable problems in which several substitutions of Fréchet differentiability are used. The first one is naturally Gâteaux differentiability. It was used in [68], [69] for the openness and surjectivity study of nonlinear operators. Some implicit function theorems involving Gâteaux differentiability can be found in [33], [53], [70] and references therein. The second substitution is the use of set-valued derivatives, including Ioffe's strict prederivative [48] or shield, upper derivative as different terminologies for single-valued mappings (see [27], [56], [61] and [67]) and a rather completed high order set-valued derivative introduced by Frankowska for open mapping study of set-valued mapping (see [38]). We note that, in [56], some implicit functions without differentiability were also given, but to compensate, the function was supposed to have a non-zero topological degree. Another replacement is the weak Gâteaux inverse derivative defined by Welsh [78] and was used in the same paper to obtain an open mapping theorem via Brézis and Browder's ordering principle. This notion is similar to Altman's directional contractor [3], the difference is that the derivative needs to be linear in [3] and may be nonlinear in [78].

We also remark that the problems considered in each of the papers cited above are unconstrained problems and the mapping is usually single-valued except for [33] and [39]. For the purpose of applications, constrained problems are most important, see [6], [27] and [52] where some known constrained open mapping theorems can be found under Fréchet differentiability assumption.

In this chapter, we introduce a generalized γ-inverse derivative for set-valued mapping, use Ekeland's Variational Principle and some fixed point theorems to consider the constrained implicit function, open mapping and surjectivity problems of set-valued mappings. Our derivative, even in the single-valued case, relaxes Welsh's notion, Gâteaux derivative and strict prederivative in the situation when they are used for implicit function or open mapping problems and our results suggests that our concept is more useful. In particular, it allows the mapping to be perturbed by a small set-valued Lipschitz mapping. So, we suppose the mapping considered is of the form $F(x,u) + G(x,u)$. The inverse derivative condition is only imposed on the mapping $x \mapsto F(x,u)$, while the mapping $x \mapsto G(x,u)$ is always supposed to be Lipschitz. Continuity with respect to the variable u is not necessary for the existence of an implicit function. The constraint made to the variable x is a closed convex cone if $x \mapsto F(x,u)$ is only a closed mapping, and in case $x \mapsto F(x,u)$ is also Lipschitz, the constraint need only be a closed subset. Some constrained implicit function and open mapping theorems are obtained. Pseudo-Lipschitz property and surjectivity of the implicit functions are also proved. Our conclusions generalize the corresponding results of [6], [31], [53], [61], [67], [68], [69] and [78] in several ways, our conditions are weaker, we allow the presence of constraint, Lipschitz perturbations are considered and the mapping is allowed to be set-valued.

As applications, we use our results to consider constrained controllability problems of nonlinear systems. The notion of constrained controllability was introduced in [19] and has been widely considered for linear systems, see [73] and references therein. However, there exist relatively few papers concerning constrained controllability of nonlinear systems, particularly in infinite dimensional spaces, see [1], [27], [52] and [63]. Moreover, these are restricted to local controllability. The constraint made to the control in [1] is a compact convex subset (for finite dimensional systems), in [27] it is the unit ball, in [52] it is a closed convex cone with nonempty interior and in [63] it is a time dependent cone. We note that, in [63], the nonlinear term is independent of the state and the admissible set is the space of measurable functions.

In our considerations, we suppose that the constraint made on control is a time-dependent closed convex cone with possibly empty interior and the admissible set is the space of all essentially bounded functions. We consider both local constrained controllability of nonlinear systems and constrained global controllability of semilinear systems. Our results shows that the controllability will be realized if some suitable associated linear systems are constrained controllable. For the nonlinear systems, the associated linear system is constructed by the derivatives of the function, therefore, Gâteaux or Fréchet differentiability assumption is needed. For semilinear systems, the associated linear system is given by the linear part.

In Section 2.1, we give our new concept of γ-Gâteaux inverse differentiability and discuss its properties. In Section 2.2, some new constrained implicit function theorems are given, and, as corollaries, we derive some constrained open mapping theorems in Section 2.3. In Section 2.4, we present a theorem about the surjectivity of a kind of (set-valued) implicit mapping. Applications will be arranged in Section 2.5.

2.1 γ-Gâteaux inverse differentiability

In this section, we introduce a new inverse derivative — γ-Gâteaux inverse differentiability for set-valued mappings and give some properties which provide sufficient conditions for a mapping to have a γ-Gâteaux inverse derivative.

Definition 2.1.1. Let X, Y be normed spaces, $F : \mathrm{Dom}(F) \subset X \to 2^Y$ be a set-valued mapping and $x_0 \in \mathrm{Dom}(F), \gamma \geq 0$. We say F possesses a *γ-Gâteaux inverse derivative* $\Gamma(x_0) : Y \to X$ at x_0 if for every $y \in Y$ and $h > 0$ with $x_0 + h\Gamma(x_0)y \in \mathrm{Dom}(F)$, we have

$$F(x_0 + h\Gamma(x_0)y) + h\gamma\|y\|\overline{B}_Y \supset F(x_0) + hy + o(h). \tag{2.1}$$

Here, $o(\cdot) : (0, \infty) \mapsto Y$ is a single-valued mapping and $\|o(h)\|/h \to 0$ as $h \to 0$. Briefly, $\Gamma(x_0)$ is said to be the *γ-G inverse derivative* of F at x_0. In this case, we say F is *γ-G inverse differentiable.*

Remark 2.1.2. If $\gamma \geq 1$, it is easy to see that (2.1) is satisfied for every mapping with $\Gamma \equiv 0$. So we are usually interested in the case $\gamma < 1$, but, in the rest of *this section*, we consider the general case.

Remark 2.1.3. If $\Gamma(x)$ is a γ-G inverse derivative of F, then it is easy to see that $\Gamma(x)$ is also a $(\lambda + \gamma)$-G inverse derivative of F at x for each $\lambda \geq 0$, and $k\Gamma(x)$ is a γ-G inverse derivative of $(1/k)F$ at x for all $k > 0$.

If F is single-valued and $\gamma = 0$, Definition 2.1.1 coincides with the notion "weak Gâteaux inverse derivative" defined in [78] and the notion "directional contractor" defined in [3] (see Definition 1.4.5). The following examples and propositions show that our notion is more general and many mappings are γ-G inverse differentiable with $\gamma \in (0, 1)$, but are not inverse differentiable in the sense of [3] or [78].

Example 2.1.4. Suppose $f : X \to Y$ is a single-valued function, Gâteaux differentiable at x and $\mathrm{D}f(x)$ is surjective. Then f has a 0-G inverse derivative at x given by

$$\Gamma(x)y = x_y$$

with x_y an arbitrary point of X satisfying $y = \mathrm{D}f(x)x_y$. If $\mathrm{D}f(x)$ is bounded, linear and onto, X, Y are Banach spaces, then, by Banach's open mapping theorem, we can choose $\Gamma(x)$ to be bounded.

Example 2.1.5. The function $f(x) = |x|$ on the real line has a γ-G inverse derivative $\Gamma(0)y \equiv 0$ at $x = 0$ for all $\gamma \geq 1$, but has no 0-G inverse derivative.

Before the third example, we show that our inverse differentiability is preserved by small Lipschitz perturbations which property is obviously not valid for other differentiability mentioned above. Therefore, our notion can be used to consider some problems involving the sum of a γ-G inverse differentiable mapping and a Lipschitz mapping.

Proposition 2.1.6. *Let X, Y be normed spaces, $F : D \subset X \to 2^Y$ possess a γ-G inverse derivative $\Gamma(x)$ at $x \in X$ such that*

$$\|\Gamma(x)y\| \leq M\|y\| \quad \textit{with } M > 0, \quad \textit{for all } y \in B_Y. \tag{2.2}$$

Suppose $G : D \to 2^Y$ is Lipschitz with constant k in D. Then $F + G$ has a $(\gamma + kM)$-G inverse derivative $\Gamma_1(x)$ at x such that

$$\mathrm{range}(\Gamma_1(x)) \subset \bigcup_{\lambda>0} \lambda\, \mathrm{range}(\Gamma(x)) \quad \textit{and} \quad \|\Gamma_1(x)y\| \leq M\|y\| \quad \textit{for all } y \in Y. \tag{2.3}$$

If, in addition, (2.2) *is satisfied for all $y \in Y$, then we can let $\Gamma_1(x) = \Gamma(x)$.*

Proof. Let $c > 1$ be given and let

$$\Gamma_1(x)y = \|cy\|\Gamma(x)\Big(\frac{y}{\|cy\|}\Big), \quad \text{for all } y \in Y.$$

Then (2.3) holds and, for each $y \in Y$ and each $h > 0$ with $x + h\Gamma_1(x)y \in D$, there exists $u_y \in \overline{B}_Y$ such that

$$\begin{aligned} F(x + h\Gamma_1(x)y) + h\gamma\|y\|\overline{B}_Y &= F\Big(x + h\|cy\|\Gamma(x)\frac{y}{\|cy\|}\Big) + (h\|cy\|)\gamma\Big\|\frac{y}{\|cy\|}\Big\|\overline{B}_Y \\ &\supset F(x) + hy + o(h). \end{aligned}$$

The Lipschitz property of G and (2.2) implies that

$$G(x) \subset G(x + h\Gamma_1(x)y) + kh\|\Gamma_1(x)y\|\overline{B}_Y \subset G(x + h\Gamma_1(x)y) + hkM\|y\|\overline{B}_Y.$$

So we have

$$F(x + h\Gamma_1(x)y) + G(x + h\Gamma_1(x)y) + h(\gamma + kM)\|y\|\overline{B}_Y \supset F(x) + G(x) + hy + o(h)$$

which implies that $\Gamma_1(x)$ is a $(\gamma + kM)$-G inverse derivative of $F + G$ at x.

If, in addition, (2.2) is satisfied for all $y \in Y$, we can prove by the same method that $\Gamma(x)$ is a $(\gamma + kM)$-G inverse derivative of $F + G$.

This completes the proof. □

Example 2.1.7. Suppose X, Y are Banach spaces, $L : X \to Y$ is a continuous, one-to-one and onto linear operator, $f : X \to Y$ is a function such that $f - L$ is Lipschitz with the constant k. Then f is $k/\|L^{-1}\|$-G inverse differentiable. In fact, by Example 2.1.4, L is 0-G inverse differentiable with the inverse derivative L^{-1}. So, Proposition 2.1.6, implies that f has a $k\|L^{-1}\|$-G inverse derivative.

Next, we give some sufficient conditions for a mapping to have a bounded γ-G inverse derivative with a given γ (< 1).

Proposition 2.1.8. *Let X, Y be normed spaces. Suppose $f : \mathrm{Dom}(f) \subset X \to Y$ is a single-valued function, Gâteaux differentiable at x_0 with Gâteaux derivative $\mathrm{D}f(x_0)$. If there exist $c > 0, \gamma \geq 0$ and a cone $P \subset X$ such that*

$$B_Y \subset \mathrm{D}f(x_0)(P \cap cB_X) + \gamma B_Y, \tag{2.4}$$

then, for each $\lambda > 1$, f possesses a $\lambda\gamma$-G inverse derivative $\Gamma(x_0)$ at x_0 with the properties

$$\Gamma(x_0)B_Y \subset P \quad \textit{and} \quad \|\Gamma(x_0)y\| \leq \lambda c\|y\| \quad \textit{for all } y \in Y.$$

Proof. Let $\lambda > 1$ be given. From (2.4), it follows that, for each $y \in Y$, there exists $v_y \in B_X \cap P, w_y \in B_Y$ such that

$$\frac{y}{\lambda\|y\|} = \mathrm{D}f(x_0)(cv_y) + \gamma w_y, \quad \text{or} \quad y = \mathrm{D}f(x_0)(c\lambda\|y\|v_y) + \gamma\lambda\|y\|w_y. \tag{2.5}$$

Define a mapping $\Gamma(x_0) : Y \to X$ by

$$\Gamma(x_0)0 = 0, \text{ and } \Gamma(x_0)y = c\lambda\|y\|v_y \quad \text{for all } y \in Y, y \neq 0.$$

where for each $y \in Y$, we fix a point v_y in $B_X \cap P$ satisfying (2.5) so that $\Gamma(x_0)$ is well defined. By the definition of $\mathrm{D}f(x_0)$ and (2.5), we see that

$$f(x_0 + h\Gamma(x_0)y) = f(x_0 + hc\lambda\|y\|v_y) = f(x_0) + h\,\mathrm{D}f(x_0)(c\lambda\|y\|v_y) + o(h)$$
$$= f(x_0) + hy + h\lambda\gamma\|y\|(-w_y) + o(h).$$

That is, $\Gamma(x_0)$ is a $\lambda\gamma$-G inverse derivative of f at x_0. Obviously,

$$\|\Gamma(x_0)y\| \leq c\lambda\|y\| \quad \text{for all } y \in Y.$$

Since P is a cone, $v_y \in P$ for every $y \in Y$, so we have $\Gamma(x_0)y = c\lambda\|y\|v_y \in P$ which implies that $\Gamma(x_0)B_Y \subset P$. This completes the proof. □

Remark 2.1.9. If $\mathrm{D}f(x_0)$ is bounded, linear and surjective, then, by Theorem 1.3.15, (2.4) holds with $\gamma = 0$. But the converse is not true.

Proposition 2.1.10. *Let X, Y be normed spaces, $P \subset X$ is a cone $L : D \subset X \to Y$ be a positively homogeneous operator and $c > 0, \gamma \geq 0$ be such that*

$$B_Y \subset L(P \cap cB_X) + \gamma B_Y. \tag{2.6}$$

Suppose $F : D \to 2^Y$ is a set-valued mapping and $\alpha \geq 0$, $x_0 \in P \cap D$, $\varepsilon_0 > 0$ are such that

$$F(x_0) + L(x - x_0) \subset F(x) + \alpha\|x - x_0\|\overline{B}_Y \quad \textit{for all } x \in D \cap B_X(x_0, \varepsilon_0). \tag{2.7}$$

Then, for each $\lambda > 1$, F possesses a $\lambda(\gamma + \alpha c)$-G inverse derivative $\Gamma(x_0)$ at x_0 such that

$$\Gamma(x_0)B_Y \subset P \quad \textit{and } \|\Gamma(x_0)y\| \leq \lambda c\|y\| \quad \textit{for all } y \in Y.$$

Proof. Let $\lambda > 1$ be given and $y \in Y\backslash\{0\}$. Then $y/(\lambda\|y\|) \in Y$. From (2.6), it follows that there exist $u_y = u(\lambda, y) \in P \cap B_X, v_y = v(\lambda, y) \in B_Y$ such that

$$y = \lambda\|y\|L(cu_y) + \lambda\|y\|\gamma v_y = \lambda c\|y\|L(u_y) + \lambda\gamma\|y\|v_y. \tag{2.8}$$

Define a mapping $\Gamma(x_0) : Y \to X$ by

$$\Gamma(x_0)y = \lambda c\|y\|u_y \quad \text{for all } y \in Y,$$

where $u_y \in P \cap B_X$ satisfies (2.8), and in case there are several such points, we fix one so that $\Gamma(x_0)$ is well defined. Since $u_y \in P \cap B_X$ and P is a cone, we see that

$$\Gamma(x_0)B_Y \subset P, \text{ and } \|\Gamma(x_0)y\| \leq \lambda c\|y\| \text{ for all } y \in Y.$$

Moreover, by (2.8) and the homogeneity of L, we have

$$L(\Gamma(x_0)y) = \lambda c\|y\|L(u_y) = y - \lambda\gamma\|y\|v_y \quad \text{for each } y \in Y.$$

Suppose $h > 0$ is so small that $x_0 + h\Gamma(x_0)y \in D \cap B_X(x_0, \varepsilon_0)$. Then, (2.7) implies that

$$\begin{aligned} F(x_0) + hy &= F(x_0) + L\big(h\Gamma(x_0)y\big) + h\lambda\gamma\|y\|v_y \\ &\subset F\big(x_0 + h\Gamma(x_0)y\big) + h\alpha\|\Gamma(x_0)y\|\overline{B}_y + h\lambda\gamma\|y\|B_Y \\ &\subset F\big(x_0 + h\Gamma(x_0)y\big) + h\lambda(\gamma + \alpha c)\|y\|\overline{B}_Y. \end{aligned}$$

Hence, $\Gamma(x_0)$ is a $\lambda(\gamma + \alpha c)$-G inverse derivative of F at x_0. This completes the proof.

□

From the proof, it can be seen that, if P is a convex cone, (2.7) needs only to be satisfied for all $x \in D \cap P \cap B_X(x_0, \delta)$.

We may call L satisfying (2.7) the α-approximation of F at x_0. Next, we will show that this approximation can be a set-valued mapping if F is single-valued and continuous. To do this, we recall that, for a subset D of a normed space Z, the *Hausdorff measure of noncompactness* of D, denoted by $\chi(D)$, is defined by

$$\chi(D) = \inf\{\varepsilon > 0 : D \subset \cup_{i=1}^{n} B_Z(x_i, \varepsilon) \text{ for some } x_1, \cdots, x_n \in D\}.$$

Proposition 2.1.11. *Let X be a normed space, Y be a Banach space, $P \subset X$ be a closed convex cone and $\mathcal{L} \subset \mathbf{L}(X, Y)$ be a bounded convex subset and $c > 0$ be such that*

$$B_Y \subset L(P \cap cB_X), \quad \text{for each } L \in \mathcal{L}. \tag{2.9}$$

Suppose $f : E \to Y$ is a continuous function and $x_0 \in P, \alpha \geq 0, \varepsilon_0 > 0$ are such that

$$f(x) - f(x_0) \in \mathcal{L}(x - x_0) + \alpha\|x - x_0\|\overline{B}_Y \quad \text{for all } x \in P \cap B_X(x_0, \varepsilon_0). \tag{2.10}$$

Then, for each $\lambda > 1$ and $\gamma > \chi(\mathcal{L}), f$ possesses a $\lambda c(\gamma + \alpha)$-G inverse derivative $\Gamma(x_0)$ at x_0 such that $\Gamma(x_0)B_Y \subset P$ and $\|\Gamma(x_0)y\| \leq \lambda c\|y\|$ for all $y \in Y$.

Proof. Since $\gamma > \chi(\mathcal{L})$, there exist $L_1, \cdots, L_n \in \mathcal{L}$ such that

$$\mathcal{L} \subset \bigcup_{i=1}^{n} B_{\mathbf{L}(X,Y)}(L_i, \gamma).$$

Denote by $\mathcal{T} = \mathrm{co}\{L_1, \cdots, L_n\}$. Then $\mathcal{T} \subset \mathcal{L}$ is a compact convex subset in $\mathbf{L}(X, Y)$ and

$$f(x) - f(x_0) \in \mathcal{T}(x - x_0) + (\alpha + \gamma)\|x - x_0\|B_Y, \quad \text{for all } x \in P \cap B_X(x_0, \varepsilon_0). \quad (2.11)$$

Let $\lambda > 1$ be given and $y \in Y$. From (2.9), it follows that, for each $L \in \mathcal{T}$,

$$R_y(L) := \{x \in B_X(0, \lambda c\|y\|) \cap P : Lx - y = 0\}$$

is nonempty. Obviously, $R_y(L)$ is also closed and convex. By Theorem 1.2.12, $R_y(\cdot)$ is lower semicontinuous as a set-valued mapping. By Theorem 1.3.5, there exists a continuous operator $L \mapsto x(L, y) \in R_y(L)$, that is

$$y = Lx(L, y), \quad x(L, y) \in P \quad \text{and} \quad \|x(L, y)\| \leq \lambda c\|y\|.$$

Since P is a cone, there exists $a > 0$ such that $x_0 + hx(L, y) \in P \cap B_X(x_0, \varepsilon_0)$ for each $h \in [0, a]$ and each $L \in \mathcal{T}$. Let

$$\Phi(L) = \left\{ S \in \mathcal{T} : \begin{array}{c} \|f(x_0 + hx(L, y)) - f(x_0) - hSx(L, y)\| \leq (\alpha + \gamma)\|hx(L, y)\| \\ \text{for all } h \in [0, a] \end{array} \right\}.$$

The compactness of $\mathcal{T}$ and (2.11) shows that Φ is a well defined, compact set-valued mapping on $\mathcal{T}$. To prove Φ is upper semicontinuous, we suppose $D \subset \mathbf{L}(X, Y)$ is a closed subset and $L_n \in \Phi^{-1}(D)$ with $L_n \to L$. Then there exist $S_n \in \mathcal{T} \cap D$ such that

$$\|f(x_0 + hx(L_n, y)) - f(x_0) - hS_n x(L_n, y)\| \leq (\alpha + \gamma)\|hx(L_n, y)\|\}, \text{ for all } h \in [0, a].$$

We may suppose $S_n \to S \in \mathcal{T} \cap D$ due to the compactness of $\mathcal{T}$ and closedness of D. Since $L \mapsto x(L, y)$ is continuous, we have

$$\|f(x_0 + hx(L, y)) - f(x_0) - hSx(L, y)\| \leq (\alpha + \gamma)\|hx(L, y)\|\}, \text{ for all } h \in [0, a].$$

which implies that $L \in \Phi^{-1}(D)$. So $\Phi^{-1}(D)$ is closed and, therefore, Φ is upper semicontinuous. Applying Kakutani's fixed point theorem (Theorem 1.5.4), we see that there exists $L_y \in \mathcal{T}$ such that $L_y \in \Phi(L_y)$, that is

$$\|f(x_0 + hx(L_y, y)) - f(x_0) - hL_y x(L_y, y)\| \leq (\alpha + \gamma)\|hx(L_y, y)\| \leq \lambda c(\alpha + \gamma)h\|y\|$$

for all $h \in [0, a]$. Noting that $L_y x(L_y, y) = y$, we see that

$$f(x_0 + hx(L_y, y)) \subset f(x_0) + hy + h\lambda c(\alpha + \gamma)\|y\|w_y. \tag{2.12}$$

Define a mapping $\Gamma(x_0) : Y \to X$ by

$$\Gamma(x_0)y = x(L_y, y).$$

(If there are several $x(L_y, y)$, we fix one arbitrarily so that $\Gamma(x_0)$ is well defined.) From (2.12), we see that $\Gamma(x_0)$ is a $\lambda c(\alpha + \gamma)$-G inverse derivative of f at x_0. Obviously $\Gamma(x_0)B_Y \subset P$, and $\|\Gamma(x_0)y\| \leq \lambda c\|y\|$ for all $y \in Y$. This completes the proof. □

Remark 2.1.12. We note that if a single-valued map f has a strict prederivative $\mathcal{L} \in \mathrm{L}(X, Y)$ at x_0 (see Definition 1.4.4), then (2.10) is satisfied. This strict prederivative assumption was used in [61] together with condition (2.9) in the case $P = X$ for implicit function problems. It was also used in [67] for open mapping problem of single-valued map together with other kinds of conditions.

2.2 Constrained implicit function theorems

In this section, we use γ-G inverse differentiability to derive some constrained implicit function theorems. The mapping we will consider is of the form $F(x, u) + G(x, u)$ with F, G set-valued mappings, the γ-G inverse differentiability assumption is only imposed on the mapping $x \mapsto F(x, u)$, while $x \mapsto G(x, u)$ is supposed to be locally Lipschitz.

Before discussing our problems, let's recall Ekeland's Variational Principle given by Ekeland in [36] and here we take the version given in [6].

Lemma 2.2.1 (Ekeland's Variational Principle). *([6], Theorem 3.3.1)*

Suppose $V : Z \mapsto \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous nontrivial function defined on a complete metric space (Z, d) and bounded from below. Let $x_0 \in \mathrm{Dom}(V)$ and $\varepsilon > 0$ be fixed. Then there exists $\overline{x} \in Z$ such that

$$V(\overline{x}) + \varepsilon d(x_0, \overline{x}) \leq V(x_0),$$
$$V(\overline{x}) < V(x) + \varepsilon d(x, \overline{x}), \quad \textit{for each } x \neq \overline{x}.$$

We also need the following lemma.

Lemma 2.2.2. *Suppose D_1, D_2, D_3 are three subsets of a normed space in which the Hausdorff distance is denoted by $\mathcal{H}(\cdot, \cdot)$. Then*

$$d(0, D_1 + D_2) \leq d(0, D_1 + D_3) + \mathcal{H}(D_2, D_3). \tag{2.13}$$

Proof. Let $\varepsilon > 0$. Then there exists $y_i \in D_i$ $(i = 1, 2, 3)$ such that

$$\|y_1 + y_3\| \leq d(0, D_1 + D_3) + \varepsilon, \quad \text{and} \quad \|y_2 - y_3\| \leq \mathcal{H}(D_2, D_3) + \varepsilon.$$

So

$$\begin{aligned} d(0, D_1 + D_2) &\leq \|y_1 + y_2\| \leq \|y_1 + y_3\| + \|y_2 - y_3\| \\ &\leq d(0, D_1 + D_3) + \mathcal{H}(D_2, D_3) + 2\varepsilon. \end{aligned}$$

By letting $\varepsilon \to 0$, we obtain (2.13) and complete the proof. □

Now we consider the constrained implicit function problems. For convenience, in the sequel, we always suppose the following conditions hold unless stated otherwise.

(H2.2.1) X, Y are Banach spaces, U is a metric space and $K, \Omega \subset X$ are subsets with $K \subset \Omega$.

(H2.2.2) $F, G : \overline{\Omega} \times U \to 2^Y$ are two set-valued mappings with bounded closed values, $x_0 \in K, u_0 \in U$ are such that

$$\lim_{u \to u_0} d\big(0, F(x_0, u) + G(x_0, u)\big) = 0. \tag{2.14}$$

(H2.2.3) $r, \delta > 0, k \geq 0$ are given such that, for each $u \in B_U(u_0, r), x \mapsto G(x, u)$ is ε-δ-u. s. c. and, restricted to $K \cap B_X(x_0, \delta)$, is Lipschitz with constant k.

First, we consider the case when the constraint K is a closed convex cone and the mapping $x \mapsto F(x, u)$ is only closed or ε-δ-u.s.c.. The case when the constraint is not a cone will be considered later where $x \mapsto F(x, u)$ needs to be locally Lipschitz.

Theorem 2.2.3. *Under (H2.2.1), (H2.2.2) and (H2.2.3), let $\Omega = K$ be a closed convex cone. Suppose that there exist $\gamma \geq 0, M > 0$ with $\gamma + kM < 1$ satisfying the following conditions.*

(F2.2.1) For each $u \in B_U(u_0, r), x \mapsto F(x, u)$ is either a closed single-valued mapping or an ε-δ-u. s. c. set-valued mapping.

(F2.2.2) For each $u \in B_U(u_0, r), x \mapsto F(x, u)$ possesses a γ-G inverse derivative $\Gamma_u(x)$ at each $x \in K \cap B_X(x_0, \delta)$ such that

$$\|\Gamma_u(x)y\| \leq M\|y\|, \quad \text{for all } y \in B_Y, x \in K \cap B_X(x_0, \delta) \tag{2.15}$$

$$\Gamma_u(x)B_Y \subset K, \quad \text{for all } x \in K \cap B_X(x_0, \delta). \tag{2.16}$$

Write

$$\eta = \sup \left\{ r_1 \in (0, r] : \sup_{u \in B_U(u_0, r_1)} d\big(0, F(x_0, u) + G(x_0, u)\big) < \frac{1 - \gamma - kM}{M}\delta \right\}. \tag{2.17}$$

Then for each $u \in B_U(u_0, \eta)$, there is $x_u \in K \cap B_X(x_0, \delta)$ such that

$$0 \in F(x_u, u) + G(x_u, u), \quad \text{and} \quad \lim_{u \to u_0} x_u = x_0.$$

If, in addition, $u \mapsto F(x, u) + G(x, u)$ is locally Lipschitz at u_0 with constant l uniformly in $x \in K \cap B_X(x_0, \delta)$, then the (constrained) implicit mapping

$$W(u) := \big\{x \in K : 0 \in F(x, u) + G(x, u)\big\}$$

is pseudo-Lipschitz around (u_0, x_0) with constant $lM/(1 - \gamma - kM)$.

Proof. From (2.14), it follows that the number η given by (2.17) is positive.

Let $u \in B_U(u_0, \eta)$ be given. Then, by the definition of η, we see that there exists $\varepsilon_0 \in \big(0, (1-\gamma-kM)/M\big)$ such that

$$d\big(0, F(x_0, u) + G(x_0, u)\big) < \varepsilon_0 \delta. \tag{2.18}$$

Define a new metric d_θ on K by

$$d_\theta(x_1, x_2) = \max\big\{\|x_1 - x_2\|, \theta\|F(x_1, u) - F(x_2, u)\|\big\} \text{ for } x_1, x_2 \in K.$$

Here $\theta > 0$ with $\theta(1+\gamma) < M$ if $x \mapsto F(x,u)$ is a closed single-valued mapping and $\theta = 0$ if $x \mapsto F(x,u)$ is an ε-δ-u.s.c. set-valued mapping. By (F2.2.1), the space K endowed with the metric d_θ is complete and is denoted by K_θ in the following. From (H2.2.3), $G(\cdot, u) : K_\theta \to 2^Y$ is ε-δ-u.s.c.. Obviously, $F(\cdot, u) : K_\theta \to Y$ is continuous or ε-δ-u.s.c. according to whether it is single or set-valued. So, by Theorem 1.2.11, $F(\cdot, u) + G(\cdot, u)$ is ε-δ-u.s.c. and, by Theorem 1.2.9, the function

$$V(x) := d\big(0, F(x,u) + G(x,u)\big)$$

is lower semicontinuous. For a given $\varepsilon \in \big(\varepsilon_0, (1-\gamma-kM)/M\big)$, applying Ekeland's Variational Principle (Lemma 2.2.1) to the function $V(x)$ in the metric space K_θ, we obtain $x_u \in K_\theta$ such that

$$V(x_u) \leq V(x) + \varepsilon d_\theta(x, x_u), \quad \text{for all } x \in K_\theta, \tag{2.19}$$

$$d_\theta(x_u, x_0) \leq \big(V(x_0) - V(x_u)\big)/\varepsilon. \tag{2.20}$$

From (2.18) and (2.20), it follows that

$$\|x_u - x_0\| \leq d_\theta(x_u, x_0) \leq \varepsilon_0\delta/\varepsilon < \delta,$$

that is, $x_u \in K \cap B_X(x_0, \delta)$.

Next, we prove $0 \in F(x_u, u) + G(x_u, u)$. If not, then $a := d\big(0, F(x_u, u) + G(x_u, u)\big) > 0$. Let $\beta > 0$ be so small that $kM + \varepsilon M + \gamma + \beta < 1$. Then, for each $h > 0$, there exists $y := y(h) \in F(x_u, u) + G(x_u, u) \backslash \{0\}$ such that

$$\|y\| \leq d\big(0, F(x_u, u) + G(x_u, u)\big) + h\beta = V(x_u) + h\beta.$$

By (2.16), $z = \Gamma_u(x_u)\left(\frac{-y}{\lambda\|y\|}\right)$ with $\lambda > 1$ is well defined and $z \in K, \|z\| \leq M/\lambda$. Since K is a convex cone, $x_u + hz \in K$. Replacing x in (2.19) by $x_u + hz$ and using Lemma 2.2.2, we obtain

$$\begin{aligned} V(x_u) &\leq V(x_u + hz) + \varepsilon d_\theta(x_u + hz, x_u) \\ &\leq d\big(0, F(x_u + hz, u) + G(x_u, u)\big) + \mathcal{H}_Y\big(G(x_u + hz, u), G(x_u, u)\big) \\ &\quad + \varepsilon \max\big\{h\|z\|, \theta\|F(x_u + hz, u) - F(x_u, u)\|\big\} \\ &\leq d\big(0, F(x_u + hz, u) + h(\gamma/\lambda)\overline{B}_Y + G(x_u, u)\big) + \mathcal{H}_Y\big(\{0\}, h(\gamma/\lambda)\overline{B}_Y\big) \\ &\quad + \mathcal{H}_Y\big(G(x_u + hz, u), G(x_u, u)\big) \\ &\quad + \varepsilon \max\big\{h\|z\|, \theta\|F(x_u + hz, u) - F(x_u, u)\|\big\}. \end{aligned} \tag{2.21}$$

By the definition of γ-G inverse derivative, we have

$$F(x_u + hz, u) + h(\gamma/\lambda)B_Y \supset F(x_u, u) + h\frac{-y}{\lambda\|y\|} + o(h).$$

Suppose h is small enough so that $h < \lambda a$ (so $h < \lambda\|y\|$) and $x_u + hz \in B_X(x_0, \delta)$. As G is Lipschitz in $x, \lambda > 1$ and $y \in F(x_u, u) + G(x_u, u)$, from (2.21) it follows that

$$\begin{aligned} V(x_u) &\leq d\big(0, F(x_u, u) - (hy)/(\lambda\|y\|) + o(h) + G(x_u, u)\big) + h\gamma \\ &\quad + kh\|z\| + \varepsilon h \max\left\{\frac{M}{\lambda}, \theta + \frac{\theta\|o(h)\|}{h} + \theta\gamma\right\} \\ &\leq \left\|y - \frac{h}{\lambda\|y\|}y\right\| + \|o(h)\| + h\gamma + kh\frac{M}{\lambda} + \varepsilon h \max\left\{\frac{M}{\lambda}, \theta + \frac{\theta\|o(h)\|}{h} + \theta\gamma\right\} \\ &\leq \|y\| - \frac{h}{\lambda} + \|o(h)\| + h\gamma + kh\frac{M}{\lambda} + \varepsilon h \max\left\{\frac{M}{\lambda}, \theta + \frac{\theta\|o(h)\|}{h} + \theta\gamma\right\} \\ &\leq V(x_u) + h\beta - \frac{h}{\lambda} + \|o(h)\| + h\gamma + kh\frac{M}{\lambda} + \varepsilon h \max\left\{\frac{M}{\lambda}, \theta + \frac{\theta\|o(h)\|}{h} + \theta\gamma\right\}. \end{aligned}$$

That is

$$\frac{1}{\lambda} \leq \beta + \frac{\|o(h)\|}{h} + k\frac{M}{\lambda} + \gamma + \varepsilon \max\left\{\frac{M}{\lambda}, \theta + \frac{\theta\|o(h)\|}{h} + \theta\gamma\right\}.$$

By letting $\lambda \to 1, h \to 0$, we have

$$1 \leq \beta + kM + \gamma + \varepsilon\max\{M, \theta(1 + \gamma)\} = \beta + kM + \gamma + \varepsilon M < 1.$$

This is a contradiction. Therefore, $0 \in F(x_u, u) + G(x_u, u)$. From (2.14) and (2.20), we see also that

$$\|x_u - x_0\| \leq d_\theta(x_u, x_0) \leq d\big(0, F(x_0, u) + G(x_0, u)\big)/\varepsilon \to 0 \quad \text{as} \quad u \to u_0.$$

Now, additionally, we suppose $u \mapsto F(x,u)+G(x,u)$ is locally Lipschitz at u_0 with constant $l > 0$. Then $u \mapsto F(x_0,u)+G(x_0,u)$ is ε-δ-u.s.c. near u_0 and, therefore, is closed restricted to a closed neighbourhood of u_0 (see Theorem 1.2.9). So (2.14) implies that $0 \in F(x_0,u_0)+G(x_0,u_0)$, that is, $(u_0,x_0) \in \text{Graph}(W)$. We fix $r_1 \in (0,\eta)$. Then

$$\sup_{u \in B_U(u_0,r_1)} d\big(0, F(x_0,u)+G(x_0,u)\big) < \frac{1-\gamma-kM}{M}\delta$$

and, therefore, there exists $\varepsilon_0 \in (0,(1-\gamma-kM)/M)$ such that

$$d\big(0, F(x_0,u)+G(x_0,u)\big) < \varepsilon_0\delta, \text{ for all } u \in B_U(u_0,r_1).$$

Let $r_2 \le r_1$ be positive and small enough so that $lr_2 < \delta\varepsilon_0/4$ and $u \mapsto F(x,u)+G(x,u)$ is Lipschitz with constant l restricted to $B_U(u_0,r_2)$. Denote by d_U the metric of the space U. To prove that the implicit mapping W is pseudo-Lipschitz, we need only prove that

$$W(u_1) \cap B_X(x_0,\delta/2) \subset W(u) + \frac{lM}{1-\gamma-kM} d_U(u,u_1)\overline{B}_X \tag{2.22}$$

hold for all $u, u_1 \in B_U(u_0,r_2)$.

In fact, let $u,u_1 \in B_U(u_0,r_2)$ be given. For each $\overline{x} \in W(u_1)\cap B_X(x_0,\delta/2)$ and each $\varepsilon \in \big(\varepsilon_0,(1-\gamma-kM)/M\big)$, applying Ekeland's Variational Principle, we see that there exists $\hat{x} \in K$ satisfying (2.19) (with the same θ) and such that

$$\|\hat{x}-\overline{x}\| \le d_\theta(\hat{x},\overline{x}) \le d\big(0,F(\overline{x},u)+G(\overline{x},u)\big)/\varepsilon.$$

Noting $0 \in F(\overline{x},u_1)+G(\overline{x},u_1)$, we have

$$\begin{aligned} \|\hat{x}-\overline{x}\| &\le \mathcal{H}_Y\big(F(\overline{x},u)+G(\overline{x},u), F(\overline{x},u_1)+G(\overline{x},u_1)\big)/\varepsilon \\ &\le l d_U(u,u_1)/\varepsilon < 2lr_2/\varepsilon. \end{aligned} \tag{2.23}$$

This yields

$$\|\hat{x}-x_0\| \le \|\hat{x}-\overline{x}\| + \|\overline{x}-x_0\| \le 2lr_2/\varepsilon + \delta/2 < \delta.$$

Hence $\hat{x} \in K \cap B_X(x_0,\delta)$. Using the same method as used above, we can prove that $0 \in F(\hat{x},u)+G(\hat{x},u) = 0$, that is, $\hat{x} \in W(u)$. Moreover, from (2.23), it follows that

$$d(\overline{x},W(u)) \le \|\overline{x}-\hat{x}\| \le l d_U(u,u_1)/\varepsilon, \quad \text{or} \quad \overline{x} \in W(u) + \frac{l}{\varepsilon} d_U(u,u_1)\overline{B}_X.$$

Since $\overline{x} \in W(u_1) \cap B_X(x_0, \delta/2)$ is arbitrary, we obtain

$$W(u_1) \cap B_X(x_0, \delta/2) \subset W(u) + \frac{l}{\varepsilon} d_U(u, u_1)\overline{B}_X.$$

Since $\varepsilon \in \big(\varepsilon_0, (1-\gamma-kM)/M\big)$ is arbitrary, it can be replaced by $(1-\gamma-kM)/M$ which gives (2.22). Hence, W is pseudo-Lipschitz around (u_0, x_0) with constant $lM/(1-\gamma-kM)$.

This completes the proof. □

Remark 2.2.4. Here we do not make any explicit continuity condition on the function $u \mapsto F(x,u) + G(x,u)$ for the existence of the implicit function, such a condition has always been required in previous studies, for example, see [56], [61]. If a similar continuity condition is imposed so that $0 \in F(x_0, u_0) + G(x_0, u_0)$, then $\lim_{u \to u_0} x_u = x_0$ implies that the implicit function $u \mapsto x_u$ is continuous at u_0.

Note, (2.16) is always satisfied for unconstrained problems.

Remark 2.2.5. From (2.22), we see that the pseudo-Lipschitz modulus of W only depends on δ. This note will be useful for the proof of some corollaries we give later.

Remark 2.2.6. Noting Proposition 2.1.6 and Theorem 1.2.11, if $F(\cdot, u)$ is ε-δ-upper semicontinuous, the sum $F + G$ can be treated as one mapping instead of a sum. Of course, the results in both cases are equivalent. The same situation also arises in the following result. We consider the sum so that we can easily deduce some corollaries.

Now, we consider the case when the constraint is only a closed subset. In this case, both $x \mapsto F(x,u)$ and $x \mapsto G(x,u)$ should be locally Lipschitz, but a condition made on the γ-G inverse derivative can be a little less strict.

Theorem 2.2.7. *Under (H2.2.1)–(H2.2.3), let $K \subset X$ be a closed subset and Ω be a neighbourhood of K. Suppose F satisfies the following conditions.*

(F2.2.3) For each $u \in B_U(u_0, r)$ and each $x \in K \cap B_X(x_0, \delta)$, $F(x,u) + G(x,u)$ has a minimum point, that is $\|y\| = d\big(0, F(x,u) + G(x,u)\big)$ for some $y \in F(x,u) + G(x,u)$.

(F2.2.4) For each $u \in B_U(u_0, r)$, the mapping $x \mapsto F(x,u)$ is ε-δ-u.s.c. and, restricted to $\Omega \cap B_X(x_0, \delta)$, is Lipschitz with constant k_1.

(F2.2.5) For each $u \in B_U(u_0, r)$, the mapping $x \mapsto F(x,u)$ possesses a γ-G inverse derivative $\Gamma_u(x)$ at each $x \in K \cap B_X(x_0, \delta)$ and there exist $\mu \in [0,1), M > 0$ with $0 \le \mu(k_1 + k)M + kM + \gamma < 1$ such that

$$\begin{aligned} &\|\Gamma_u(x)y\| \le M\|y\|, \quad \textit{for all } y \in B_Y,\ x \in K \cap B_X(x_0, \delta), \\ &\Gamma_u(x)B_Y \subset T_K(x) + \mu\Gamma_u(x)B_Y, \quad \textit{for } x \in K \cap B_X(x_0, \delta). \end{aligned} \tag{2.24}$$

Write

$$\eta = \sup\left\{ r_1 \le r : \sup_{u \in B_U(u_0, r_1)} d\big(0, F(x_0, u) + G(x_0, u)\big) < \frac{1 - \gamma - \mu(k_1 + k)M - kM}{(1 + \mu)M}\delta \right\}.$$

Then for each $u \in B_U(u_0, \eta)$, there is $x_u \in K \cap B_X(x_0, \delta)$ such that

$$0 \in F(x_u, u) + G(x_u, u) \quad \textit{and} \quad \lim_{u \to u_0} x_u = x_0.$$

If, in addition, $u \mapsto F(x,u) + G(x,u)$ is locally Lipschitz at u_0 with constant l uniformly in $x \in K \cap B_X(x_0, \delta)$, then the constrained implicit mapping W defined in Theorem 2.2.3 is pseudo-Lipschitz with constant $\dfrac{l(1+\mu)M}{1 - \mu(k_1 + k)M - kM - \gamma}$.

Proof. Let $u \in B_U(u_0, \eta)$ be given. Then there exists $\varepsilon_0 \in \big(0, (1 - \gamma - kM)/M\big)$ such that

$$d\big(0, F(x_0, u) + G(x_0, u)\big) < \varepsilon_0 \delta. \tag{2.25}$$

Let $\varepsilon \in \big(\varepsilon_0, (1 - \mu(k_1 + k)M - kM - \gamma)/[(1 + \mu)M]\big)$. Then

$$(1 + \mu)\varepsilon M + \mu(k_1 + k)M + kM + \gamma < 1.$$

Our assumption (F2.2.4), Theorem 1.2.11 and Theorem 1.2.9 imply that the function

$$V(x) := d\big(0, F(x,u) + G(x,u)\big)$$

is lower semicontinuous. Applying Ekeland's Variational Principle to $V(\cdot)$ in the metric space K, we see there exists $x_u \in K$ such that

$$V(x_u) \le V(x)\| + \varepsilon\|x - x_u\|, \quad \text{for all } x \in K, \tag{2.26}$$

$$\|x_u - x_0\| \le V(x_0)/\varepsilon \le \varepsilon_0\delta/\varepsilon < \delta. \tag{2.27}$$

Suppose $0 \notin F(x_u, u) + G(x_u, u)$. By assumption (F2.2.3), there exists $y \in F(x_u, u) + G(x_u, u), y \neq 0$ such that

$$\|y\| = d\big(0, F(x_u, u) + G(x_u, u)\big) = V(x_u).$$

Let $z = \Gamma_u(x_u)\left(\frac{-y}{\lambda\|y\|}\right)$ with $\lambda > 1$. Then our assumption (2.24) and (2.27) imply that there exists $y \in B_Y$ such that $z \in T_K(x_u) + \mu\Gamma_u(x_u)y$. Therefore, by Theorem 1.3.14, there exist $v_n \in Y$ and $h_n \to 0$ such that

$$v_n \to z - \mu\Gamma_u(x_u)y, \quad x_u + h_n v_n \in K.$$

Clearly, $\{z\} \cup \{v_n\}$ is bounded. Since Ω is a neighbourhood of K, we can suppose $x_u + h_n z, x_u + h_n v_n \in \Omega \cap B_X(x_0, \delta)$. Substituting $x = x_u + h_n v_n$ into (2.26), using Lemma 2.2.2 and the Lipschitz properties of $F(\cdot, u)$ and $G(\cdot, u)$, we obtain

$$\begin{aligned} V(x_u) &\leq V(x_u) + \varepsilon h_n\|v_n\| \\ &\leq d\big(0, F(x_u + h_n z, u) + G(x_u, u)\big) + \mathcal{H}_Y\big(G(x_u + h_n v_n, u), G(x_u, u)\big) \\ &\quad + \mathcal{H}_Y\big(F(x_u + h_n z, u), F(x_u + h_n v_n, u)\big) + \varepsilon h_n\|v_n\| \\ &\leq d\big(0, F(x_u + h_n z, u) + G(x_u, u)\big) + k_1 h_n\|z - v_n\| + (k + \varepsilon)h_n\|v_n\| \\ &\leq d\big(0, F(x_u + h_n z, u) + G(x_u, u) + h_n(\gamma/\lambda)\overline{B}_Y\big) \\ &\quad h_n(\gamma/\lambda) + k_1 h_n\|z - v_n\| + (k + \varepsilon)h_n\|v_n\|. \end{aligned} \tag{2.28}$$

By the definition of γ-G inverse derivative and using the same method as used in Theorem 2.2.3 (with $\beta = \theta = 0$), we have (if n is sufficiently large)

$$V(x_u) \leq V(x_u) - (h_n/\lambda) + h_n\gamma + k_1 h_n\|z - v_n\| + (k + \varepsilon)h_n\|v_n\| + \|o(h_n)\|,$$

that is

$$\frac{1}{\lambda} \leq k_1\|z - v_n\| + (k + \varepsilon)\|v_n\| + \gamma + \frac{\|o(h_n)\|}{h_n}.$$

Letting $\lambda \to 1, n \to \infty$, we obtain

$$\begin{aligned} 1 &\leq k_1\|\mu\Gamma_u(x_u)y\| + \gamma + (k + \varepsilon)\|z - \mu\Gamma_u(x_u)y\| \\ &\leq \mu k_1 M\|y\| + \gamma + (k + \varepsilon)\|z\| + (k + \varepsilon)\mu M\|y\| \\ &\leq \mu(k_1 + k)M + kM + \varepsilon(1 + \mu)M + \gamma < 1. \end{aligned}$$

This is a contradiction. Therefore $0 \in F(x_u, u) + G(x_u, u)$.

The rest of the proof is the same as that in Theorem 2.2.3.

This completes the proof. $\square$

Remark 2.2.8. Although the assumption $\mu < 1$ is not used in the proof, the other assumptions seem to imply that μ should be less than 1. We can show this in the case F is single-valued below.

In fact, for each $x \in K \cap B_X(x_0, \delta)$, $y \in Y$ and $h > 0$ with $x + h\Gamma(x)y \in K \cap B_X(x_0, \delta)$, from the definition of γ-G inverse derivative, it follows that

$$F(x + h\Gamma(x)y) = F(x) + hy + h\gamma\|y\|w_y + o(h)$$

for some $w_y \in \overline{B_Y}$ (note F is single-valued). So the Lipschitz assumption of F and the boundedness of $\Gamma(x)$ imply that

$$\left\|y + \|y\|\gamma w_y\right\| \leq \frac{1}{h}\left\|F(x + h\Gamma(x)y) - F(x) + o(h)\right\| \leq k_1 M\|y\| + \frac{o(h)}{h}.$$

Therefore,

$$1 - \gamma \leq \left\|\frac{y}{\|y\|} + \gamma w_y\right\| \leq k_1 M + \frac{\|o(h)\|}{h\|y\|} \leq (k_1 + k)M + \frac{\|o(h)\|}{h\|y\|}.$$

Letting $h \to 0$, we have $1 - \gamma \leq (k_1 + k)M$. Since $0 \leq \mu(k_1 + k)M + kM + \gamma < 1$, we obtain

$$\mu < \frac{1 - \gamma - kM}{(k_1 + k)M} \leq 1.$$

Remark 2.2.9. There are some standard conditions that ensure (F2.2.3). For example, we may suppose the values of $F + G$ are compact, or only closed convex in case Y is a reflexive Banach space.

Theorem 2.2.10. *Suppose the condition* (2.24) *in Theorem 2.2.7 is replaced by*

$$\Gamma_u(x)\{y \in Y : \|y\| = 1\} \subset T_K(x) + \mu\Gamma_u(x)\overline{B}_Y, \;\; x \in K \cap B_X(x_0, \delta).$$

Then the conclusion remains valid.

Proof. The proof is similar to that of Theorem 2.2.7, we need only replace λ by 1 and take $y \in \overline{B}_Y$. $\square$

If the constraint K in Theorem 2.2.7 or 2.2.10 is convex, we have a better result — the same conclusion as in Theorem 2.2.3.

Theorem 2.2.11. *Under the conditions of Theorem 2.2.7 or 2.2.10, suppose K is convex. Then the number η in the conclusions is*

$$\eta = \sup\left\{ r_1 \in (0,r] : \sup_{u\in B_U(u_0,r_1)} d\big(0, F(x_0,u)+G(x_0,u)\big) < \frac{1-\gamma-kM}{M}\delta \right\}.$$

and the pseudo-Lipschitz constant of W is $\dfrac{lM}{1-\gamma-kM}$.

Proof. Since K is convex, $T_K(x)$ is a closed convex cone for each $x \in K$, so for each pair of constants $c_1 > 0, c_2 > 0$, we have

$$c_1 T_K(x) + c_2 T_K(x) \subset T_K(x).$$

By (2.24), for each $x \in B_X(x_0, \delta)$, we have

$$\begin{aligned} \Gamma(x)B_Y &\subset T_K(x) + \mu\Gamma_u(x)B_Y \subset T_K(x) + \mu(T_K(x) + \mu\Gamma_u(x)B_Y) \\ &\subset T_K(x) + \mu^2\Gamma_u(x)B_Y \subset \cdots \subset T_K(x) + \mu^n\Gamma_u(x)B_Y. \end{aligned}$$

Since (F2.2.5), $\Gamma_u(x)B_Y$ is bounded. So

$$\Gamma_u(x)B_Y \subset \overline{T_K(x)} = T_K(x).$$

That is, (2.24) holds also with $\mu = 0$ and therefore Theorem 2.2.7 applies with $\mu = 0$ to complete the proof. $\square$

Remark 2.2.12. This corollary means that the Lipschitz constant of F can be arbitrarily large provided the constraint is convex.

Next, we use the above theorems to deduce some conclusions in which the γ-G inverse derivative will not be imposed explicitly. For convenience to consider both cases (cone constraint and non-cone constraint), we denote by

$$\Omega_K = \begin{cases} K & \text{if the constraint } K \text{ is a closed convex cone,} \\ \overline{\Omega} & \text{if } K \text{ is only a closed subset.} \end{cases}$$

$$E_K(x) = \begin{cases} K & \text{if the constraint } K \text{ is a closed convex cone,} \\ T_K(x) & \text{if } K \text{ is only a closed subset.} \end{cases}$$

Obviously, $E_K(x)$ is always a cone, and if K is convex, then $E_K(x)$ is convex.

Corollary 2.2.13. *Under (H2.2.1)–H(2.2.3), Let $L : \Omega_K \times U \to Y$ be a function with $L(\cdot, u)$ positively homogeneous. Suppose there exist $c > 0, \alpha, \beta \geq 0$ with $\beta + \alpha c + kc < 1$ such that, for all $u \in B_U(u_0, r), x \in K \cap B_X(x_0, \delta)$ and all $x' \in \Omega_K \cap B_X(x_0, \delta)$,*

$$F(x,u) + L(x' - x, u) \subset F(x', u) + \alpha \|x - x'\| \overline{B}_Y, \tag{2.29}$$

$$B_Y \subset L(cB_X \cap E_K(x), u) + \beta B_Y. \tag{2.30}$$

Then the conclusions of Theorem 2.2.3 remain true with $\gamma = \beta + \alpha c$ and $M = c$ provided at least one of the following two conditions holds

(i) K is a closed convex cone, $\Omega = K$ and (F2.2.1) is satisfied.

(ii) K is a closed subset, Ω is a neighbourhood of K and (F2.2.3), (F2.2.4) are satisfied.

Proof. Let $\lambda > 1$ be such that $\lambda(\beta + \alpha c + kc) < 1$.

By (2.29), (2.30) and Proposition 2.1.10, we see that, for each $u \in B_U(u_0, r)$, the mapping $x \mapsto F(x, u)$ possesses a $\lambda(\beta + \alpha c)$-G inverse derivative $\Gamma_u(x)$ at each $x \in K \cap B_X(x_0, \delta)$ and

$$\Gamma_u(x) B_Y \subset E_K(x), \quad \text{for all } u \in B_U(u_0, r), x \in K \cap B_X(x_0, \delta), \tag{2.31}$$

$$\|\Gamma_u(x) y\| \leq c\lambda \|y\|, \quad \text{for all } y \in Y. \tag{2.32}$$

By Theorem 2.2.3 in case (i) holds or Theorem 2.2.7 in case (ii) holds, for each $u \in B_U(u_0, \eta_1)$ with

$$\eta_1 = \sup\left\{ r_1 \in (0, r] : \sup_{u \in B_U(u_0, r_1)} d\big(0, F(x_0, u) + G(x_0, u)\big) < \frac{1 - \lambda(\beta + \alpha c) - \lambda kc}{\lambda c} \delta \right\},$$

the inclusion $0 \in F(x, u) + G(x, u)$ has a solution $x_u \in K \cap B_X(x_0, \delta)$ with $\lim_{u \to u_0} x_u = x_0$ and, under the extra assumptions made in Theorem 2.2.3, the constrained implicit mapping W defined in Theorem 2.2.3 is pseudo-Lipschitz around (u_0, x_0) of constant $b(\lambda) := \lambda lc/(1 - (\beta + \alpha c)\lambda - \lambda kc)$. Since $\lambda > 1$ with $(\beta + \alpha c + kc)\lambda < 1$ is arbitrary and noting Remark 2.2.5, we see that the λ in the above constant $b(\lambda)$ and the definition of η_1 can be replaced by 1. This completes the proof. □

Corollary 2.2.14. *Under (H2.2.1)–(H2.2.3), let $F \equiv f : X \times U \to Y$ be single-valued. Suppose, for each $u \in B_U(u_0, r)$, $x \mapsto f(x, u)$ is weakly Gâteaux differentiable with weak derivative $f'_x(x, u)$ and that $f'_x(x_0, u)(K) = Y$, $\|f'_x(x, u) - f'_x(x_0, u)\| \leq \alpha$ with some $\alpha > 0$ for all $x \in K \cap B_X(x_0, \delta)$. Then the conclusions of Corollary 2.2.13 remain true with some new $c > 0$ provided that α and k are sufficiently small.*

Proof. Let $L(\cdot, u) := f'_x(x_0, u)$. Since $L(K, u) = Y$ for each $u \in B_U(u_0, r)$, from Theorem 1.3.15, it follows that (2.30) is satisfied with $\beta = 0$ and some constant $c > 0$.

Let $x_1, x_2 \in K \cap B_X(x_0, \delta)$ and $y^* \in Y^*$. Then

$$\begin{aligned} &y^*(f(x_1, u) - f(x_2, u) - f'_x(x_0, u)(x_1 - x_2)) \\ &\qquad = \int_0^1 \langle [f'_x(x_1 + t(x_2 - x_1), u) - f'_x(x_0, u)](x_1 - x_2), y^* \rangle \, dt. \end{aligned}$$

Choosing y^* with $\|y^*\| = 1$ such that

$$y^*(f(x_1, u) - f(x_2, u) - f'_x(x_0, u)(x_1 - x_2)) = \|f(x_1, u) - f(x_2, u) - f'_x(x_0, u)(x_1 - x_2)\|,$$

we obtain that

$$\|f(x_1, u) - f(x_2, u) - f'_x(x_0, u)(x_1 - x_2)\| \leq \alpha \|x_1 - x_2\|.$$

That is (2.29) is also satisfied. So the conclusion follows from Corollary 2.2.13. $\square$

Corollary 2.2.15. *Under (H2.2.1)–(H2.2.3), let $F := f : \Omega_K \times U \to Y$ be single-valued and $c > 0, \gamma \in [0, 1)$ be constants with $ck + \gamma < 1$. Suppose, for each $u \in B_U(u_0, r)$, the function $x \mapsto f(x, u)$ is Gâteaux differentiable with the derivative $\mathrm{D}f(x, u) := \mathrm{D}_x f(x, u)$ at each $x \in K \cap B_X(x_0, \delta)$ and*

$$B_Y \subset \mathrm{D}f(x, u)(cB_X \cap E_K(x)) + \gamma B_Y \quad \text{for all } x \in B_X(x_0, \delta) \cap K. \tag{2.33}$$

(i) If K is a closed convex cone, $\Omega = K$ and (F2.2.1) is satisfied, then the conclusions of Theorem 2.2.3 remain true with $M = c$.

(ii) If K is only a closed subset, Ω is a neighbourhood of K and (F2.2.3) and (F2.2.4) are satisfied, then the conclusions of Theorem 2.2.3 remain also true with $M = c$.

Proof. From our assumptions and Proposition 2.1.8, it follows that, for each $u \in B_U(u_0, r)$ and $\lambda > 1$ with $\lambda(ck + \gamma) < 1$, $x \mapsto f(x, u)$ possesses a $\lambda\gamma$-G inverse derivative $\Gamma(x)$ at each $x \in K \cap B_X(x_0, \delta)$ satisfying (2.31) and (2.32). Applying Theorem 2.2.3 in case (i) holds or Theorem 2.2.7 (with F replaced by f) in case (ii) holds, we see that, for each $u \in B_U(u_0, \eta_2)$ with

$$\eta_2 = \sup\left\{ r_1 \in (0, r] : \sup_{u \in B_U(u_0, r_1)} d\big(0, f(x_0, u) + G(x_0, u)\big) < \frac{1 - \lambda\gamma - \lambda kc}{\lambda c}\delta \right\},$$

$0 \in f(x, u) + G(x, u)$ admits solution $x_u \in K \cap B_X(x_0, \delta)$ with $\lim_{u \to u_0} x_u = x_0$ and, under the extra assumptions made in Theorem 2.2.3, the constrained implicit mapping W is pseudo-Lipschitz around (u_0, x_0) with constant $b(\lambda) := \lambda lc/(1 - \alpha c\lambda - \lambda kc)$. Since $\lambda > 1$ with $\lambda(\gamma + kc) < 1$ is arbitrary, it can be replaced by 1 in the above constant $b(\lambda)$ and the definition of η_2 according to Remark 2.2.5. This completes the proof. $\square$

Remark 2.2.16. If $\mathrm{D}f(x)$ is linear, then (2.33) implies

$$B_Y \subset \overline{\mathrm{D}f(x)\Big(\frac{c}{1-\gamma} B_X \cap E_K(x)\Big)} \tag{2.34}$$

for all $x \in B_X(x_0, \delta) \cap K$. In fact, from (2.33), it follows that

$$\begin{aligned} B_Y &\subset \mathrm{D}f(x)(cB_X \cap E_K(x)) + \gamma\big(\mathrm{D}f(x)(cB_X \cap E_K(x)) + \gamma B_Y\big) \\ &\subset \mathrm{D}f(x)((1+\gamma)cB_X \cap E_K(x)) + \gamma^2 B_Y \\ &\subset \cdots \subset \mathrm{D}f(x)\big((\Sigma_{i=0}^n \gamma^i c)B_X \cap E_K(x)\big) + \gamma^{n+1} B_Y. \end{aligned}$$

By letting $n \to \infty$, we obtain (2.34).

It is not hard to show that if (2.33) is replaced by (2.34), we have the same conclusion.

Corollary 2.2.17. *Under the conditions of Corollary 2.2.15, suppose, in addition,* $x \mapsto \mathrm{D}f(x, u)v$ *is continuous at* x_0 *uniformly on* $v \in B_X$ *and* (2.33) *is replaced by*

$$B_Y \subset \mathrm{D}f(x_0, u)(cB_X \cap E_K(x)) + \gamma_1 B_Y, \text{ for all } x \in B_X(x_0, \delta) \cap K, \tag{2.35}$$

for some $\gamma_1 < \gamma$. *Then the conclusions remain true with* δ *replaced by* $\delta_1 \le \delta$.

Proof. Let $\beta \in (0, \gamma - \gamma_1)$. The unform continuity of $\mathrm{D}f(\cdot, u)v$ implies that there exists $\delta_1 \in (0, \delta]$ such that

$$\|\mathrm{D}f(x,u)v - \mathrm{D}f(x_0,u)v\| \leq \beta/c, \quad \text{for all } x \in B_X(x_0, \delta_1), v \in B_X. \tag{2.36}$$

Let $x \in K \cap B_X(x_0, \delta_1)$ and $y \in B_Y$. (2.35) implies that there exist $x_y \in B_X \cap E_K(x)$, $z_y \in B_Y$ such that

$$y = \mathrm{D}f(x_0,u)(cx_y) + \gamma_1 z_y = \mathrm{D}f(x,u)(cx_y) + [\mathrm{D}f(x_0,u) - \mathrm{D}f(x,u)](cx_y) + \gamma_1 z_y.$$

From (2.36), it follows that

$$\|[\mathrm{D}f(x_0,u) - \mathrm{D}f(x,u)](cx_y) + \gamma_1 z_y\| \leq \beta + \gamma_1 < \gamma.$$

Therefore, we have $y = \mathrm{D}f(x,u)(cx_y) + \gamma w_y$ for some $w_y \in B_Y$, which implies (2.33) with δ replaced by δ_1. This completes the proof. □

Remark 2.2.18. If $f(\cdot, u)$ is continuously Fréchet differentiable at x_0, then $\mathrm{D}f(\cdot, u)v$ is uniformly continuous on bounded sets. Moreover, if $\mathrm{D}f(x_0,u)(K) = Y$, then (2.35) is satisfied according to Theorem 1.3.15 and, therefore, we obtain the classical implicit function theorem by letting $K = X, G(x,u) \equiv \{0\}$.

Corollary 2.2.19. *Under (H2.2.1)–H(2.2.3), let $F \equiv f : \Omega_K \times U \to Y$ be a single-valued function with $x \mapsto f(x,u)$ continuous for each $u \in B_U(u_0, r)$. Suppose there exists a convex family of linear operators $\mathcal{L} \subset \mathbf{L}(X,Y)$ and $c, \alpha \geq 0$ with $c(k + \alpha + \chi(\mathcal{L})) < 1$ such that*

$$f(x',u) - f(x,u) \in \mathcal{L}(x' - x) + \alpha\|x' - x\|\overline{B}_Y,$$
$$B_Y \subset L(cB_X \cap E_K(x))$$

for all $x \in K \cap B_X(x_0, \delta), u \in B_U(u_0, r)$, all $x' \in \Omega_K \cap E_K(x) \cap B_X(x_0, \delta)$ and all $L \in \mathcal{L}$. Then the conclusions of Theorem 2.2.3 remain true with $M = c, \gamma = c(\alpha + \chi(\mathcal{L}))$ if at least one of the following two conditions holds.

(i) $K = \Omega$ is a closed convex cone.

(ii) K is a closed convex subset, Ω is a neighbourhood of K and (F2.2.3) and (F2.2.4) are satisfied.

Proof. From our assumptions and proposition 2.1.11, it follows that, for each $u \in B_U(u_0, r)$ and $\lambda > 1, \beta > \chi(\mathcal{L})$ with $\lambda c(k + \beta + \alpha) < 1$, $x \mapsto f(x, u)$ possesses a $\lambda c(\alpha + \beta)$-G inverse derivative $\Gamma(x)$ at each $x \in K \cap B_X(x_0, \delta)$ satisfying (2.31) and (2.32). Applying Theorem 2.2.3 in case (i) holds or Theorem 2.2.7 in case (ii) holds with $M = \lambda c, \gamma = \lambda c(\alpha + \beta)$and $\mu = 0$, we see that, for each $u \in B_U(u_0, \eta_2)$ with

$$\eta_3 = \sup\left\{r_1 \in (0, r] : \sup_{u \in B_U(u_0, r_1)} d\big(0, f(x_0, u) + G(x_0, u)\big) < \frac{1 - \lambda c(\alpha + k + \beta)}{\lambda c}\delta\right\},$$

$0 \in f(x, u) + G(x, u)$ admits solution $x_u \in K \cap B_X(x_0, \delta)$ with $\lim_{u \to u_0} x_u = x_0$ and, under the extra assumptions made in Theorem 2.2.3, the implicit mapping W is pseudo-Lipschitz around (u_0, x_0) with constant $b(\lambda) := l\lambda c/[1 - \lambda c(\alpha + k + \beta)]$. Since $\lambda > 1, \beta > \chi(\mathcal{L})$ with $\lambda c(\alpha + \beta + k) < 1$ are arbitrary, they can be replaced by 1 and $\chi(\mathcal{L})$, respectively, in the above constant $b(\lambda)$ and the definition of η_3. This completes the proof. □

Remark 2.2.20. Corollary 2.2.19 generalizes Theorem 3 of [61] where the unconstrained problem with $G(x, u) \equiv \{0\}$ is considered under the assumption that $\mathcal{L}$ is a uniform strict prederivative and the injectivity of $L \in \mathcal{L}$ is imposed (in Theorem 4 of [61]) for the continuity of the implicit function.

To close this section, we give a theorem in case X is finite dimensional and both F and G are single-valued. For the proof, we need a proposition regarding the differentiability of a function $f : \mathbb{R}^n \to Y$ which is a generalization of a result implicit in Theorem 2.1 of [62] where $Y = \mathbb{R}$.

Proposition 2.2.21. *Suppose function $f : \mathbb{R}^n \to Y$ is locally Lipschitz, Gâteaux differentiable at x_0 and the derivative map $v \mapsto \mathrm{D}f(x_0)v$ is also Lipschitz. Then, for each $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\|f(x_0 + v) - f(x_0) - \mathrm{D}f(x_0)v\| \leq \varepsilon\|v\|, \quad \text{for all } v \in \mathbb{R}^n \text{ with } \|v\| \leq \delta.$$

Proof. Suppose the conclusion is not true. Then there exists $\varepsilon_0 > 0$ and $v_j \in \mathbb{R}^n$ with $\|v_j\| \leq 1/j$ such that

$$\|f(x_0 + v_j) - f(x_0) - \mathrm{D}f(x_0)v_j\| \geq \varepsilon_0\|v_j\|, \quad \text{for each } j = 1, 2, \cdots.$$

Since f is locally Lipschitz, there exist $M > 0, r > 0$ such that

$$\|f(x_0 + x) - f(x_0 + y)\| \le M\|x - y\|, \quad \text{for all } x, y \in r\overline{B} := r\overline{B}_{\mathbb{R}^n}.$$

Let $u_j = r\frac{v_j}{\|v_j\|}, \lambda_j = \|v_j\|/r$. Then $u_j \in r\overline{B}, \lambda_j \to 0$ and $\lambda_j u_j = v_j$. We may suppose $u_j \to u_0 \in \mathbb{R}^n$ and the Lipschitz constant of $v \mapsto \mathrm{D}f(x_0)v$ is d. Then we have

$$\begin{aligned}
\|f(x_0 + \lambda_j u_0) \;&-\; f(x_0) - \lambda_j \mathrm{D}f(x_0)u_0\| \ge \|f(x_0 + \lambda_j u_j) - f(x_0) - \mathrm{D}f(x_0)(\lambda_j u_j)\| \\
&\quad - \|f(x_0 + \lambda_j u_j) - f(x_0 + \lambda_j u_0)\| \\
&\quad - \|\mathrm{D}f(x_0)(\lambda_j u_j) - \mathrm{D}f(x_0)(\lambda_j u_0)\| \\
&\ge \; \varepsilon_0\|v_j\| - M\lambda_j\|u_j - u_0\| - \lambda_j d\|u_j - u_0\| \\
&= \; \lambda_j\big(\varepsilon_0 r - M\|u_j - u_0\| - d\|u_j - u_0\|\big).
\end{aligned}$$

Therefore,

$$\liminf_{j\to\infty} \frac{\|f(x_0 + \lambda_j u_0) - f(x_0) - \lambda_j \mathrm{D}f(x_0)u_0\|}{\lambda_j} \ge \varepsilon_0 r$$

which contradicts the Gâteaux differentiability assumption. This completes the proof.

□

Theorem 2.2.22. *Suppose $f, g : \mathbb{R}^n \times U \to Y$ are continuous functions and satisfy the following conditions.*

(i) $x \mapsto f(x, u_0)$ is locally Lipschitz, Gâteaux differentiable at x_0 with the derivative $\mathrm{D}_1 f(x_0, u_0)$.

(ii) $f(x, u) + g(x, u) \in \mathrm{range}\big(\mathrm{D}_1 f(x_0, u_0)\big)$ for all (x, u) in a neighbourhood $\mathcal{W}$ of (x_0, u_0).

(iii) $v \mapsto \mathrm{D}_1 f(x_0, u_0)v$ is Lipschitz invertible and the Lipschitz constant of the inverse is M.

(iv) There exist $k, k_1 \ge 0$ with $kM < 1$ such that

$$\|g(x, u) - g(x_0, u_0)\| \le k\|x - x_0\| + k_1\|u - u_0\|, \quad \text{for all } (x, u) \in \mathcal{W}.$$

Then, for each neighbourhood $\mathcal{V}$ of x_0, there exists a neighbourhood $\mathcal{U}$ of u_0 and a function $\phi : \mathcal{U} \to \mathcal{V}$ such that

$$f(\phi(u), u) + g(\phi(u), u) = f(x_0, u_0) + g(x_0, u_0), \quad \text{for every } u \in \mathcal{U}.$$

Proof. Without loss of generality, we let $f(x_0, u_0) = g(x_0, u_0) = 0$ (otherwise, we can consider $\hat{f}(x, u) := f(x, u) - f(x_0, u_0)$ and $\hat{g}(x, u) := g(x, u) - g(x_0, u_0)$).

Let $\eta = 1 - kM$ and write $P = \mathrm{D}_1 f(x_0, u_0)$. From Proposition 2.2.21 and our assumptions, it follows that there exists $\alpha > 0$ such that

$$x_0 + \alpha\overline{B} \subset \mathcal{V} \text{ and } \|f(x_0 + x, u_0) - Px\| \leq \frac{\eta}{3M}\alpha, \text{ for each } x \in \alpha\overline{B}. \tag{2.37}$$

The continuity of f and the compactness of $\alpha\overline{B}$ imply that there exists $\beta \in (0, \eta\alpha/(6Mk_1))$ such that

$$\|f(x_0 + x, u) - f(x_0 + x, u_0)\| \leq \frac{\eta}{3M}\alpha, \text{ for all } x \in \alpha\overline{B}, u \in B_U(u_0, \beta). \tag{2.38}$$

By our assumption (iv), for all $x \in \alpha\overline{B}$ and all $u \in B_U(u_0, \beta)$, we have

$$\|g(x_0 + x, u)\| = \|g(x_0 + x, u) - g(x_0, u_0)\| \leq k\alpha + k_1\beta. \tag{2.39}$$

Now, let $u \in B_U(u_0, \beta)$ be given and let

$$N_u(x) = x - P^{-1}\big(f(x_0 + x, u) + g(x_0 + x, u)\big), \text{ for each } x \in \alpha\overline{B}.$$

By the Lipschitz property of P^{-1} and our assumption (ii), N_u is a well-defined continuous operator on $\mathbb{R}^n$. (2.37)–(2.39) imply that, for each $x \in \alpha\overline{B}$, we have

$$\begin{aligned}
\|N_u(x)\| &\leq \|P^{-1}Px - P^{-1}\big(f(x_0 + x, u_0) + g(x_0 + x, u_0)\big)\| \\
&\quad + \|P^{-1}\big(f(x_0 + x, u_0) + g(x_0 + x, u_0)\big) - P^{-1}\big(f(x_0 + x, u) + g(x_0 + x, u)\big)\| \\
&\leq \|P^{-1}Px - P^{-1}f(x_0 + x, u_0)\| \\
&\quad + \|P^{-1}f(x_0 + x, u_0) - P^{-1}\big(f(x_0 + x, u_0) + g(x_0 + x, u_0)\big)\| \\
&\quad + M\|f(x_0 + x, u_0) + g(x_0 + x, u_0) - f(x_0 + x, u) - g(x_0 + x, u)\| \\
&\leq M\|Px - f(x_0 + x, u_0)\| + M\|g(x_0 + x, u)\| \\
&\quad + M\|f(x_0 + x, u_0) - f(x_0 + x, u)\| + M\|g(x_0 + x, u_0) - g(x_0 + x, u)\| \\
&\leq M\frac{\eta}{3M}\alpha + Mk\alpha + Mk_1\beta + M\frac{\eta}{3M}\alpha + Mk_1\beta \leq \alpha(\eta + kM) = \alpha.
\end{aligned}$$

That is N_u maps $\overline{B}$ into itself. Applying Theorem 1.5.4, N_u has fixed point $x_u \in \overline{B}$ and, therefore,

$$P^{-1}\big(f(x_0 + x_u, u) + g(x_0 + x_u, u)\big) = 0.$$

By letting $\phi(u) = x_0 + x_u$, we obtain

$$f(\phi(u), u) + g(\phi(u), u) = P(0) = 0.$$

This completes the proof. $\square$

Remark 2.2.23. If $g(x, u) \equiv 0$ and $U = \mathbb{R}^m, Y = \mathbb{R}^n$, We obtain the main result of [53] where the proof is along the same line. We should remark that, in [53], a extra condition like our assumption (ii) should be imposed so that the operator N_u is well defined.

2.3 Some open mapping theorems

Definition 2.3.1. Suppose X, Y are Banach spaces. A mapping $F : \mathrm{Dom}(F) \subset X \to 2^Y$ is said to be *(locally) open* at $(x_0, y_0) \in \mathrm{Graph}(F)$ if $B_Y(y_0, a) \subset F(B_X(x_0, b))$ for some $a, b > 0$.

From the definitions of controllability and openness, we see that the local controllability of the nonlinear system

$$x'(t) = A(t)x(t) + f(t, x(t), u(t)), \quad x(0) = x_0$$

in time T is equivalent to the local openness of the (set-valued) mapping

$$u \mapsto \{x(T) : x \in S(u)\}$$

with $S(u)$ the solution set of the above equation. So in this section, we give some constrained open mapping theorems for a general set-valued mapping so that we can study constrained controllability problems later.

The mapping we will consider is of the form

$$F + G : \Omega_K \subset X \to \mathcal{P}_c(Y).$$

with X, Y Banach spaces, $K \subset \Omega \subset X$, Ω_K, as well as $E_K(x)$ in the following, are as in the previous section.

In this section, we impose the following basic assumptions.

(H2.3.1) $x_0 \in K$, $\delta > 0$, $G : \Omega_K \to P_c(Y)$ is ε-δ-u.s.c. and, restricted to $\Omega_K \cap B_X(x_0, \delta)$, is Lipschitz with constant $k(\delta) \geq 0$.

(H2.3.2) If K is a closed convex cone, then $\Omega = K$ and $F : K \to P_c(Y)$ is either a closed single-valued function or a ε-δ-u.s.c. set-valued mapping; If K is only a closed subset, then Ω is a neighbourhood of K and $F : \overline{\Omega} \to P_c(Y)$ is a ε-δ-u.s.c. set-valued mapping and, restricted to $K \cap B_X(x_0, \delta)$, is Lipschitz.

(H2.3.3) $y_0 \in F(x_0) + G(x_0)$.

To prove that $F + G$ is open at (x_0, y_0), we need some extra assumptions on F. Each of the following four conditions will be enough.

(F2.3.1) K is a closed convex cone, F possesses a γ-G inverse derivative $\Gamma(x)$ at each $x \in B_X(x_0, \delta) \cap K$ and there exists $c(\delta) > 0$ with $\gamma + k(\delta)c(\delta) < 1$ such that

$$\begin{aligned} &\Gamma(x)B_Y \subset K \quad \text{for each } x \in B_X(x_0, \delta) \cap K, \\ &\|\Gamma(x)y\| \leq c(\delta)\|y\| \quad \text{for all } x \in B_X(x_0, \delta) \cap K, y \in B_Y. \end{aligned} \tag{2.40}$$

(F2.3.2) There exists a positively homogeneous operator $L : \Omega_K \to Y$ and there exist $\alpha \geq 0, c(\delta) > 0, \gamma > 0$ with $\alpha c(\delta) \leq \gamma < 1 - k(\delta)c(\delta)$ such that

$$\begin{aligned} &F(x) + L(x' - x) \subset F(x') + \alpha\|x' - x\|\overline{B}_Y, \\ &B_Y \subset c(\delta)L(B_X \cap E_K(x)) + (\gamma - \alpha c(\delta))B_Y \end{aligned}$$

for all $x \in B_X(x_0, \delta) \cap K$ and all $x' \in \Omega_K \cap B_X(x_0, \delta)$.

(F2.3.3) $F := f$ is single-valued, Gâteaux differentiable at each $x \in B_X(x_0, \delta) \cap K$ with derivative $\mathrm{D}f(x)$, and there exist $\gamma \geq 0, c(\delta) > 0$ with $k(\delta)c(\delta) < 1 - \gamma$ such that

$$B_Y \subset c(\delta)\,\mathrm{D}f(x)(B_X \cap E_K(x)) + \gamma B_Y, \quad \text{for all } x \in B_X(x_0, \delta) \cap K. \tag{2.41}$$

(F2.3.4) K is convex, $F := f$ is single valued, $\mathcal{L} \in \mathbf{L}(X, Y)$ is a convex family of bounded linear operators and there exist constants $c(\delta) \geq 0, \gamma = c(\delta)(\alpha + \chi(\mathcal{L}))$ with

$\alpha \geq 0$ such that

$$c(\delta)\big(\alpha + \chi(\mathcal{L}) + k(\delta)\big) < 1,$$
$$f(x_1) - f(x_2) \in \mathcal{L}(x_1 - x_2) + \alpha\|x_1 - x_2\|\overline{B}_Y, \text{ for all } x_1, x_2 \in K \cap B_X(x_0, \delta),$$
$$B_Y \subset c(\delta)L(B_X \cap E_K(x)), \text{ for each } L \in \mathcal{L},\ x \in B_X(x_0, \delta) \cap K.$$

Theorem 2.3.2. *Under (H2.3.1)-(H2.3.3), suppose at least one of the above four assumptions ((F2.3.1)—F(2.3.4)) is satisfied corresponding to whether K is a closed convex cone or only a closed subset, then*

$$B\left(y_0, \frac{1-\gamma-k(\delta)c(\delta)}{c(\delta)}\delta\right) \subset (F+G)(B_X(x_0,\delta) \cap K) \tag{2.42}$$

and the constrained inverse mapping $(F+G)_K^{-1}$ defined by

$$(F+G)_K^{-1}(y) = \{x \in B_X(x_0,\delta) \cap K : y \in F(x) + G(x)\}$$

is pseudo-Lipschitz around (y_0, x_0) with constant $c(\delta)/\big(1-\gamma-k(\delta)c(\delta)\big)$. If, in addition, the corresponding one of the four assumptions ((F2.3.1)—(F(2.3.4)) is satisfied for all $\delta > 0$ and

$$\lim_{\delta\to\infty} \frac{\delta}{c(\delta)}\big(1-\gamma-k(\delta)c(\delta)\big) = \infty, \tag{2.43}$$

then $(F+G)(K) = Y$.

Proof. Let $U = Y, \hat{F}(x,u) \equiv F(x), \hat{G}(x,u) \equiv G(x) - u$ for all $x \in K, u \in U$, and let $r = \infty, l = 1$. Then the conditions in Theorem 2.2.3 are satisfied if (F2.3.1) holds or the corresponding conditions in Corollary 2.2.13, 2.2.15 or 2.2.19 are satisfied, respectively, if (F2.3.2), (F2.3.3) or (F2.3.4) holds. Since

$$d(0, \hat{F}(x_0,u) + \hat{G}(x_0,u)) = d(u, F(x_0) + G(x_0)) \leq \|u - y_0\|, \text{ for all } u \in Y,$$

we see that the number η in Theorem 2.2.3 or Corollary 2.2.13, 2.2.15 or 2.2.19 is

$$\eta = \frac{1-\gamma-k(\delta)c(\delta)}{c(\delta)}.$$

So (2.42) follows from Theorem 2.2.3 or Corollary 2.2.13, 2.2.15 or 2.2.19.

Now, suppose the additional conditions hold. Let $y \in Y$. (2.43) implies that there exists $\delta > 0$ such that

$$y \in B\left(y_0, \frac{1-\gamma-k(\delta)c(\delta)}{c(\delta)}\delta\right).$$

Therefore, the conclusion obtained in our first step implies $y \in \text{range}(F+G)$.

This completes the proof. $\square$

Remark 2.3.3. In [47], Ioffe defined the surjection modulus and surjection constant of a set-valued mapping F at $(x, y) \in \text{Graph}(F)$ by

$$\begin{aligned} \text{sur}(F, x, y)(\delta) &= \sup\{r \geq 0 : B_Y(y, r) \subset F(B(x, \delta))\}, \\ \text{sur}(F, x, y) &= \liminf_{\delta \to 0} \frac{\text{sur}(F, x, y)(\delta)}{\delta}. \end{aligned}$$

In this sense, under the conditions of Theorem 2.3.2, we have

$$\begin{aligned} \text{sur}(F+G, x_0, y_0)(\delta) &\geq \frac{1-\gamma-k(\delta)c(\delta)}{c(\delta)}\delta, \\ \text{sur}(F+G, x_0, y_0) &\geq \liminf_{\delta \to 0} \frac{1-\gamma-k(\delta)c(\delta)}{c(\delta)}. \end{aligned}$$

Remark 2.3.4. (a) If (F2.3.1) holds, $K = X, G(x) \equiv 0, \gamma = 0$ and F is single-valued, we obtain the main result of [78] and, therefore, the other results of [78] and that of [68] can be also deduced (because the results of [68] are corollaries of [78]). We also remark that, in [78], (2.40) is assumed to be satisfied for every $y \in Y$ and our proof seems to be simpler due to the use of Ekeland's Variational principle instead of Brézis and Browder's Theorem. Moreover, in both [68] and [78], no Lipschitz property of the inverse function F^{-1} is asserted.

(b) If (F2.3.2) holds, Theorem 2.3.6 generalizes Kachurovskii's Open Mapping Theorem (see Theorem15.5 in [31]) where the unconstrained single-valued problem was considered by supposing L is a surjective bounded linear operator. The method used in [31] is different from ours and no surjection modulus is given in [31]. This generalization can not be derived from the theorem in [78].

(c) If (F2.3.3) holds, Theorem 2.3.6 generalizes Theorem 3.4.3 (therefore, Graves' Theorem) in [6] where $G(x) \equiv 0$, f is supposed to be Fréchet differentiable and the surjection modulus is not given.

(d) If (F2.3.4) holds, Theorem 2.3.6 generalizes the corresponding open mapping results in [61] and [67] where the authors considered the cases when $G(x) \equiv 0$ and $\mathcal{L}$ is assumed to be a strict prederivative and, moreover, in [61], the problem was unconstrained, in [67], some additional stricter conditions were imposed due the method used.

(e) Our surjectivity result is motivated by Theorem 3.1 of [69] where the authors only considered the case when $K = X, F$ is single-valued, Gâteaux differentiable, $\gamma = 0, G(x) \equiv 0$ and $s \mapsto M(s)$ was supposed to be continuous with $\int_0^\infty M^{-1}(s)ds = \infty$.

Remark 2.3.5. We can also introduce some other kinds of conditions to ensure Theorem 2.3.2. For example, we may suppose f is continuously Fréchet differentiable and replace (2.41) by

$$B_Y \subset c(\delta)\,\mathrm{D}f(x_0)(B_X \cap E_K(x)) + \gamma B_Y.$$

Of course, in this case, the conclusion should be

$$B_Y\left(y_0, \frac{1-\gamma-k(\delta_1)c(\delta_1)}{c(\delta_1)}\delta_1\right) \subset (F+G)(B_X(x_0,\delta_1)\cap K)$$

with some $\delta_1 \leq \delta$. We can also suppose f is weakly differentiable as we have done in Corollary 2.2.17 and, in this case, the corresponding result extends an open mapping theorem (Corollary 15.2 of [31]) of Browder [22].

Similarly, we give an open mapping theorem corresponding to Theorem 2.2.7.

Theorem 2.3.6. *Under (H2.3.1)–(H2.3.3), let K be a closed subset, let the Lipschitz constant of F, restricted to $B_X(x_0,\delta)\cap\Omega$, be k_1. Suppose $\gamma \geq 0, M > 0$ and $\mu \in [0,1)$ are constants such that*

$$\gamma + kM + \mu(k_1+k)M < 1.$$

If F possesses a γ-G inverse derivative $\Gamma(x)$ at each $x \in B_X(x_0,\delta)\cap K$ such that

$$\Gamma(x)B_Y \subset T_K(x) + \mu\Gamma(x)B_Y \quad \textit{for each } x \in B_X(x_0,\delta)\cap K,$$

$$\|\Gamma(x)y\| \leq M\|y\| \ \textit{with } M > 0, \quad \textit{for all } x \in B_X(x_0,\delta)\cap K, y \in B_Y,$$

then

$$B\left(y_0, \frac{1-\gamma-kM-\mu(k_1+k)M}{(1+\mu)M}\delta\right) \subset (F+G)(B_X(x_0,\delta)\cap K)$$

and the constrained inverse mapping $(F+G)_K^{-1}$ *is pseudo-Lipschitz around* (y_0, x_0) *of constant* $(1+\mu)M/(1-\gamma-kM-\mu(k_1+k)M)$. *If, in addition, K is convex, then* μ *can be replaced by* 0.

Proof. It is the same as in Theorem 2.3.2, we need only apply Theorem 2.2.7 or Theorem 2.2.11 instead of Theorem 2.2.3 and the Corollaries. □

2.4 Surjectivity of certain implicit mappings

Recently, many authors use the controllability of linear system

$$x'(t) = A(t)x(t) + Bu(t), \quad x(0) = x_0$$

to study the controllability of semilinear system

$$x'(t) = A(t)x(t) + f(t, x(t), u(t)) + Bu(t), \quad x(0) = x_0. \tag{2.44}$$

For example, see [42], [54], [58], [59], [79] and the reference therein. In most of these publications, $A(t)$ is supposed to generate an evolution system $\{E(t,s)\}$. In this case, each solution $x(t,u)$ (an implicit mapping) of equation (2.44) has the expression

$$x(t,u) = E(t,0)x_0 + \int_0^t E(t,s)f(s, x(s,u), u(s))ds + \int_0^t E(t,s)Bu(s)ds.$$

If we let $P_T x = x(T)$ for each continuous function x, we see that the controllability of system (2.44) becomes the surjectivity of the mapping $u \mapsto P_T x(\cdot, u)$. So we are motivated to consider the surjectivity of the composition $P_T W$ of a general operator P_T and an implicit mapping W given by

$$W(u) := \{x : 0 \in F(x,u) + G(x,u)\}$$

with F, G given. From Theorem 2.3.2, it follows that if P_T is surjective, $F(x,u) \equiv F(x)$ and $G(x,u) \equiv G(x) - u$, then $P_T W$ is surjective provided the conditions of Theorem 2.3.2 are satisfied.

In this section, we will consider the case when F, G are single-valued and $G(x,u) \equiv Hu - x$ with H linear, and give some relations between the surjectivity of $P_T W$ and $P_T W_1$

with $W_1(u) = \{x : Hu - x = 0\}$ motivated by the above. A special case of this kind of unconstrained problems was treated in [50].

In the following of this section, we always suppose that

(H2.4.1) X, Y, V are Banach spaces, $K \subset V$ is a closed convex cone and $y_0 \in Y$.

(H2.4.2) $H : K \to Y, P_T : Y \to X$ are linear continuous operators and $F : Y \times K \to Y$ is a uniformly bounded nonlinear operator.

Write

$$I_K(F) = \{y \in Y : y = y_0 + F(y, v) + Hv \ \text{ for some } v \in K\},$$
$$I_K(0) = \{y \in Y : y = y_0 + Hv \ \text{ for some } v \in K\}.$$

Theorem 2.4.1. *Under (H2.4.1) and (H2.4.2), we have*

i) $\overline{P_T I_K(F)} = X$ *implies* $\overline{P_T I_K(0)} = X$;

ii) If, in addition, F, H are compact, F is continuous, then $P_T I_K(0) = X$ *implies* $P_T I_K(F) = X$.

Proof. Let $k_1 := \sup\{\|F(y, v)\| : y \in Y, v \in K\} < \infty$.

i) Suppose $\overline{P_T I_K(F)} = X$. If $\overline{P_T I_K(0)} \neq X$, then

$$X = X - P_T y_0 \neq \overline{P_T H(K)} = \overline{P_T I_K(0)}$$

and, therefore, there exists $x_1 \in X \setminus \overline{P_T LH(K)}$ with $d := d(x_1, \overline{P_T H(K)}) > 0$. Since K is a closed convex cone and P_T, H are continuous linear operators, $\overline{P_T H(K)}$ is also a closed convex cone. So, for each $r > 0$, we have

$$\begin{aligned} d(rx_1, \overline{P_T H(K)}) &= \inf\{\|rx_1 - P_T Hv\| : v \in K\} \\ &= r \inf\{\|x_1 - P_T H(v/r)\| : v \in K\} \\ &= rd(x_1, \overline{P_T H(K)}) = rd. \end{aligned}$$

Let $y \in I_K(F)$. Then there exists $v \in K$ such that $y = y_0 + F(y,v) + Hv$. Therefore $\|F(y,v)\| \leq k_1$ and

$$\begin{aligned} \|P_T y - rx_1\| &= \|P_T y_0 + P_T F(y,v) + P_T Hv - rx_1\| \\ &\geq \|P_T Hv - rx_1\| - \|P_T F(y,v) + P_T y_0\| \\ &\geq rd - \big(\|P_T y_0\| + k_1\|P_T\|\big). \end{aligned}$$

We choose r large enough so that

$$\|P_T y - rx_1\| \geq \alpha > 0.$$

Since y is arbitrary in $I_K(F)$, we see that rx_1 is not in $\overline{P_T I_K(F)}$. This is a contradiction.

ii) Suppose F, H are compact, F is continuous and $P_T I_K(0) = X$. Then

$$P_T H(K) = X - P_T y_0 = X.$$

By Theorem 1.3.15, $J(x) := \{v \in K : P_T Hv = x\}$ is a Lipschitz set-valued mapping with closed convex values. From Theorem 1.2.9 and Theorem 1.3.5, it follows that there exists a continuous single-valued operator $j : X \to K$ such that $j(x) \in J(x)$ for all $x \in X$.

Let x_T be an arbitrary point of X. Define an operator Φ on $Y \times K$ by

$$\Phi(y,v) = \big(y_0 + F(y,v) + Hv, j(x_T - P_T y_0 - P_T F(y,v))\big) \text{ for } y \in Y, v \in K.$$

By the compactness of F and H and continuity of each operator involved, we see that Φ is a compact continuous operator. Suppose there exist $\lambda \in [0,1], y_\lambda \in Y, v_\lambda \in K$ such that $(y_\lambda, v_\lambda) = \lambda\Phi(y_\lambda, v_\lambda)$, that is

$$y_\lambda = \lambda y_0 + \lambda F(y_\lambda, v_\lambda) + \lambda H v_\lambda, \tag{2.45}$$

$$v_\lambda = \lambda j(x_T - P_T y_0 - P_T F(y_\lambda, v_\lambda)). \tag{2.46}$$

Then (2.46) implies that $v_\lambda \in K$. Since F is uniformly bounded on $Y \times K$ and compact, the set $\{x_T - P_T y_0 - P_T F(y_\lambda, v_\lambda)\}$ is compact and, therefore, $\{v_\lambda\}$ is bounded, say $\|v_\lambda\| \leq k_2$ with some $k_2 > 0$. Then, by (2.45), we have

$$\|y_\lambda\| \leq \|y_0\| + \|F(y_\lambda, v_\lambda)\| + \|H\|\|v_\lambda\| \leq \|y_0\| + k_1 + k_2\|H\|.$$

This implies that $\{(y_\lambda, v_\lambda) : \lambda \in [0,1]$ and (y_λ, v_λ) satisfies (2.45) – (2.46)$\}$ is bounded. Let

$$R = \sup\{\|w\| : w \in Y \times K, w = \lambda\Phi(w) \text{ for some } \lambda \in [0,1]\}$$

and $\Omega = \{w \in Y \times K : \|w\| < R\}$. Then $\lambda w \neq \Phi(w)$ for all $\lambda \geq 1$ and $w \in \partial\Omega \cap K$. By Theorem 1.5.5, Φ has a fixed point $(y, v) \in Y \times K$. By the definitions of j and Φ, $y \in I_K(F)$ and $P_T y = x_T$. This completes the proof. □

Remark 2.4.2. Since $P_T H$ is linear and continuous, $P_T I_K(0) = P_T H(K) + P_T y_0 = X$ implies that $B_X \subset P_T H(K \cap cB_{U_{ad}})$ for some $c > 0$ (see Theorem 1.3.15). If H is compact, then B_X is compact, that is X is finite dimensional. So conclusion ii) in Theorem 2.4.1 holds only when X is finite dimensional.

2.5 Applications to constrained controllability of nonlinear systems

In this section, we use the results obtained in previous sections to consider the *constrained controllability* (that is, controllability under constraint made to the control) of the nonlinear and semilinear systems.

2.5.1 Constrained local controllability of nonlinear systems

First, we consider the constrained local controllability of a nonlinear system

$$\begin{cases} x'(t) = f(t, x(t), u(t)), & t \in [0, T] \\ x(0) = x_0 \end{cases} \tag{2.47}$$

with the constraint

$$x(t) \in K(t) \quad \text{a.e. on } [0, T].$$

Such problems have been studied by Chukwu and Lenhart [27] with $K(t) \equiv \overline{B_U}$, by Klamka [52] where $K(t) \equiv K$ is a closed convex cone with nonempty interior, and by Papageorggiou [63] with f independent of x and the control only measurable.

Here, we suppose

(H2.5.1) for each t, $K(t)$ is a closed convex cone with possibly empty interior;

(H2.5.2) both the state space X and the control space U are Banach spaces, f : $[0,T] \times X \times U$ is measurable with respect to the first variable, continuous with respect to the last two arguments;

(H2.5.3) for each $u \in L^\infty(0,T;U)$, equation (2.47) admits exactly one solution, which is denoted by $x(t, x_0, u)$, and depends continuously on u.

Let

$$U_{ad} = L^\infty(0,T;U), \quad K_{ad} = \{u \in U_{ad} : u(t) \in K(t) \text{ a.e.}\}.$$

The constrained reachable set of (2.47) is denoted by

$$\mathbf{R}_T(K) = \{x(T, x_0, u) : u \in K_{ad}\}.$$

Definition 2.5.1. System (2.47) is said to be

(i) *K-constrained exactly locally controllable* at $x_T \in X$ if $x_T \in \text{int}(\mathbf{R}_T(K)) \neq \emptyset$;

(ii) *K-constrained exactly controllable* on a subset $C \subset X$ if $C \subset \mathbf{R}_T(K)$; In particular, if $C = X$, we say the system is K-constrained exactly (globally) controllable;

(iii) *K-constrained approximately (globally) controllable* if $X = \overline{\mathbf{R}_T(K)}$.

Let $\mathrm{D}_2 f, \mathrm{D}_3 f$ be the Gâteaux derivatives of f with respect to second and third variables respectively when these exist, and associate (2.47) with the linear system

$$\begin{cases} z'(t) = \mathrm{D}_2 f(t, x(t, x_0, u), u(t))z(t) + \mathrm{D}_3 f(t, x(t, x_0, u), u(t))v(t), & t \in [0,T] \\ z(0) = 0 \end{cases} \tag{2.48}$$

The unique solution of (2.48) (supposed to exist) for given $u, v \in U_{ad}$ will be denoted by $z(t, x_0, u, v)$. K-constrained controllability of (2.48) can be defined similarly.

Lemma 2.5.2. *Suppose X, Y, Z are normed spaces, $f : X \times Y \to Z$ possesses linear Gâteaux derivative $\mathrm{D} f(x,y)$ at (x,y) in each direction. Then f possesses linear partial Gâteaux derivative $\mathrm{D}_1 f(x,y), \mathrm{D}_2 f(x,y)$ at (x,y) in each direction, and*

$$\mathrm{D} f(x,y)(u,v) = \mathrm{D}_1 f(x,y)u + \mathrm{D}_2 f(x,y)v, \quad \textit{for all } u \in X, v \in Y.$$

Proof. Let $u, v \in X \times Y$. The existence of $\mathrm{D} f(x,y)$ implies that

$$\mathrm{D} f(x,y)(u,0) = \lim_{t \to 0} \frac{f(x+tu,y) - f(x,y)}{t}$$

exists. So $\mathrm{D}_1 f(x,y)u$ exists and equals $\mathrm{D} f(x,y)(u,0)$. Similarly, $\mathrm{D}_2 f(x,y)v = \mathrm{D} f(x,y)(0,v)$. Since $\mathrm{D} f(x,y)$ is linear, we have

$$\mathrm{D} f(x,y)(u,v) = \mathrm{D} f(x,y)(u,0) + \mathrm{D} f(x,y)(0,v) = \mathrm{D}_1 f(x,y)u + \mathrm{D}_2 f(x,y)v.$$

This completes the proof. □

Lemma 2.5.3. *Suppose $f(t,x,u)$ possesses a linear Gâteaux derivative in each direction with respect to (x,u), and is Fréchet differentiable with respect to x, and $x(t,x_0,u)$ has a linear Gâteaux derivative with respect to u. Then*

$$\mathrm{D}_u x(t,x_0,u)v = z(t,x_0,u,v).$$

Proof. By the definition of $x(t,x_0,u)$

$$x(t,x_0,u) = x_0 + \int_0^t f(s, x(s,x_0,u), u(s))ds.$$

Applying Lemma 2.5.2 and Theorem 1.4.2 (iv), we obtain

$$\begin{aligned}
\mathrm{D}_u x(t,x_0,u)v &= \int_0^t \mathrm{D}_u f(s,x(s,x_0,u),u(s))v ds \\
&= \int_0^t \mathrm{D}_2 f(s,x(s,x_0,u),u(s)) \, \mathrm{D}_u x(s,x_0,u)v ds \\
&\quad + \int_0^t \mathrm{D}_3 f(s,x(s,x_0,u),u(s))v(s) ds, \\
\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{D}_u x(t,x_0,u)v &= \mathrm{D}_2 f(t,x(t,x_0,u),u(t)) \, \mathrm{D}_u x(t,x_0,u)v \\
&\quad + \mathrm{D}_3 f(t,x(t,x_0,u),u(t))v(t),
\end{aligned}$$

and

$$\mathrm{D}_u x(0, x_0, u)v = 0.$$

This means that $z(t) = D_u x(t, x_0, u)v$ is the solution $z(t, x_0, u, v)$ of the associated linear system (2.48) for the given control v. $\square$

Remark 2.5.4. If the differentiability in Lemma 2.5.3 is in Fréchet sense, then Lemma 2.5.3 coincides with Lemma 3.1 of [27].

Theorem 2.5.5. *Suppose $f(t, x, u)$ has a linear Gâteaux derivative in each direction with respect to (x, u), and is Fréchet derivable with respect to x. Let $K(t) \subset U$ be a closed convex cone for each $t \in [0, T]$, and suppose the mapping $u \mapsto x(t, x_0, u)$ has a linear Gâteaux derivative. If there exist $\delta > 0, c > 0$ such that for each $u \in B_{U_{ad}}(0, \delta)$, the associated linear system (2.48) is $K \cap cB_U$-constrained exactly controllable on B_X, then the nonlinear system (2.47) is K-constrained exactly controllable on $B_X(x(T, x_0, 0), \delta/c)$.*

Proof. By the assumptions and Lemma 2.5.3, for each $u \in B_{U_{ad}}(0, \delta)$, we have

$$\begin{aligned} B_X &\subset \{z(T, x_0, u, v) : v \in cB_{U_{ad}} \cap K_{ad}\} \\ &= z\left(T, x_0, u, cB_{U_{ad}} \cap K_{ad}\right) \\ &= \mathrm{D}_u x(T, x_0, u) \left(cB_{U_{ad}} \cap K_{ad}\right). \end{aligned}$$

It is easy to see that K_{ad} is a closed convex cone and, by our assumption, the mapping $u \mapsto x(T, x_0, u)$ is continuous from U_{ad} to X. By Theorem 2.3.2 (under (F2.3.3)),

$$B_X(x(T, x_0, 0), \delta/c) \subset x(T, x_0, K_{ad}) = \mathbf{R}_T(K),$$

that is (2.47) is K-constrained exactly controllable on $B_X(x(T, x_0, 0), \delta/c)$. $\square$

If f is Fréchet differentiable with respect to (x, u), we need only suppose (2.48) with $u = u_0$ controllable to ensure the controllability of (2.47).

Theorem 2.5.6. *Let $K(t) \subset U$ be a closed convex cone for each $t \in [0, T]$ and let $u_0 \in \cap_{t\in[0,T]} K(t), x_0(t) = x(t, x_0, u_0)$. Suppose f is Fréchet differentiable with respect to $(x, u), \mathrm{D}_3 f$ is continuous and $\mathrm{D}_2 f(t, x, u) = A_1(t) + H(t, x, u)$. Here* *(i) $A_1(t)$ is a linear operator on X for each t and generates an evolution system $\{E(t,s)\}$ with $\|E(t,s)\| \le m$;*

ii) H is measurable with respect to t, continuous with respect to the last two arguments and there exists a neighbourhood $\mathcal{U}$ of u_0 and $w \in L^1(0,T)$ such that

$$\|H(t,x(t),u(t))\| \le w(t) \text{ a.e. for all } u \in \mathcal{U} \text{ and all } x \text{ in a neighbourhood of } x_0(\cdot).$$

If the solution mapping $u \mapsto x(t,x_0,u)$ has linear Gâteaux derivative near $u_0(t) \equiv u_0$ and the system

$$\begin{cases} z'(t) = \mathrm{D}_2 f(t,x(t,x_0,u_0),u_0)z(t) + \mathrm{D}_3 f(t,x(t,x_0,u_0),u_0)v(t), \quad t \in [0,T] \\ z(0) = 0 \end{cases} \tag{2.49}$$

is K-constrained exactly controllable, then the nonlinear system (2.47) is K-constrained exactly locally controllable at $x(T,x_0,u_0)$.

Proof. Let g be the function from U_{ad} to X given by

$$g(u) = x(T,x_0,u), \quad u \in U_{ad}.$$

Then by Lemma 2.5.3

$$\mathrm{D}g(u)v = z(T,x_0,u,v) \text{ for all } v \in U_{ad}.$$

Since (2.49) is K-constrained exactly controllable (that is $z(T,x_0,u_0,K_{ad}) = X$) and $z(T,x_0,u_0,v)$ is bounded linear with respect to v, by Theorem 1.3.15, there exists $c > 0$ such that

$$B_X \subset \left\{z(T,x_0,u_0,v) : v \in K_{ad} \cap cB_{U_{ad}}\right\} = z(T,x_0,u_0,K_{ad} \cap cB_{U_{ad}}). \tag{2.50}$$

Let $\delta \in (0,1)$. Since $u \mapsto x(t,x_0,u)$ is continuous, by our assumptions, there exists a neighbourhood $B_{U_{ad}}(u_0,\delta_1)$ of u_0 such that

$$\begin{aligned} &\|H(t,x(t,x_0,u),u(t))\| \le w(t) \text{ a.e.,} \\ &\|\mathrm{D}_3 f(t,x(t,x_0,u),u(t)) - \mathrm{D}_3 f(t,x(t,x_0,u_0),u_0(t))\| \le \frac{\delta}{M}, \\ &\|H(t,x(t,x_0,u),u(t)) - H(t,x(t,x_0,u_0),u_0(t))\| \le \frac{\delta}{M} \end{aligned}$$

for all $u \in B_{U_{ad}}(u_0, \delta_1)$. Here

$$\begin{aligned} M &= mN\left(1 + \exp\left(\int_0^T w(t)dt\right)\int_0^T w(t)dt\right), \\ N &= \max\left\{\int_0^T \|z(t, x_0, u_0, v_0)\|dt + \int_0^T \|v_0(t)\|dt : v_0 \in K_{ad}, \|v_0\| \le c\right\}. \end{aligned}$$

(Using a similar method to the following, it can be seen that $N < \infty$).

For $v_0 \in K_{ad} \cap cB_{U_{ad}}$, let $z_{u_0}(t) = z(t, x_0, u_0, v_0)$, and consider the function

$$z_u(t) = z(t, x_0, u, v_0).$$

Since

$$z_u(t) = \int_0^t E(t,s)[H(s, x(s, x_0, u), u(s))z_u(s) + \mathrm{D}_3 f(s, x(s, x_0, u), u(s))v_0(s)]ds,$$

we have

$$\begin{aligned} \|z_u(t) - z_{u_0}(t)\| &\le \int_0^t \|E(t,s)\| \|H(s, x_u(s), u(s))\| \|z_u(s) - z_{u_0}(s)\|ds \\ &\quad + \int_0^t \|E(t,s)\| \big[\|H(s, x_u(s), u(s)) - H(s, x_{u_0}(s), u_0(s))\| \|z_{u_0}(s)\| \\ &\quad + \|\mathrm{D}_3 f(s, x_u(s), u(s)) - \mathrm{D}_3 f(s, x_{u_0}(s), u_0(s))\| \|v_0(s)\|\big]ds \\ &\le m\int_0^t w(s)\|z_u(s) - z_{u_0}(s)\|ds + \frac{m\delta}{M}\int_0^T [\|z_{u_0}(s)\| + \|v_0(s)\|]ds \\ &= m\int_0^t w(s)\|z_u(s) - z_{u_0}(s)\|ds + \frac{mN\delta}{M}. \end{aligned}$$

By Gronwall's inequality, we have

$$\begin{aligned} \|z_u(t) - z_{u_0}(t)\| &\le \frac{mN\delta}{M} + \exp\left(m\int_0^T w(s)ds\right)\int_0^t \frac{mN\delta}{M} w(s)ds \\ &\le \frac{mN}{M}\left(1 + \exp\left(m\int_0^T w(s)ds\right)\int_0^T w(s)ds\right)\delta \le \delta, \end{aligned}$$

which implies that, for each $u \in B_{U_{ad}}(u_0, \delta_1)$,

$$\{z(T, x_0, u, v_0) : v_0 \in K_{ad} \cap cB_{U_{ad}}\}$$

is a δ-net of

$$\{z(T, x_0, u_0, v_0) : v_0 \in K_{ad} \cap cB_{U_{ad}}\}.$$

According to (2.50), we have

$$\begin{aligned} B_X &\subset z(T, x_0, u, K_{ad} \cap cB_{U_{ad}}) + \delta B_X \\ &= \mathrm{D}g(u)(K_{ad} \cap cB_{U_{ad}}) + \delta B_X. \end{aligned}$$

Applying Theorem 2.3.2 (under (F2.3.3)), we see $g(u_0) \in \mathrm{int}(g(K_{ad}))$, that is (2.47) is constrained exactly locally controllable at $g(u_0)$. □

Remark 2.5.7. Theorems 2.5.5 and 2.5.6 generalizes the result in [52] where f is supposed to be continuously Fréchet differentiable and the interior of constraint K (independent of t) need be nonempty.

Theorem 2.5.8. *Under the conditions of Theorem 2.5.6, if X is finite dimensional, then the K-constrained approximate controllability of linear system* (2.49) *implies the K-constrained exact local - controllability of nonlinear system* (2.47).

Proof. We claim that there exists $c > 0$ such that

$$B_X \subset \overline{z(T, x_0, u_0, cB_{U_{ad}} \cap K_{ad})} + \frac{1}{4}\overline{B}_X. \tag{2.51}$$

In fact, if the claim is not true, then there exists $x_n \in B_X$ for each n such that

$$d\big(x_n, \overline{z(T, x_0, u_0, nB_{U_{ad}} \cap K_{ad})}\big) \geq \frac{1}{4}.$$

That is

$$E_n := \big\{x \in B_X : d\big(x_n, \overline{z(T, x_0, u_0, nB_{U_{ad}} \cap K_{ad})}\big) \geq \frac{1}{4}\big\} \neq \emptyset.$$

Obviously, E_n is closed and $E_{n+1} \subset E_n$. Since X is finite dimensional, there exists $\overline{x} \in \cap_{n=1}^{\infty} E_n$, that is

$$d\big(\overline{x}, z\big(T, x_0, u_0, nB_{U_{ad}} \cap K_{ad}\big)\big) \geq \frac{1}{4}, \text{ for all } n \geq 1.$$

This contradicts the approximate controllability assumption and proves that (2.51) is true. So we have

$$B_X \subset z(T, x_0, u_0, K_{ad} \cap cB_{U_{ad}}) + \frac{1}{2}B_X.$$

Let $g(u) = x(T, x_0, u)$, $\delta \in (0, 1/2)$. Using the same method as used in the proof of Theorem 2.5.6, we can show that there exists $\delta_1 > 0$ such that, for each $u \in B_{U_{ad}}(u_0, \delta_1)$,

$$z\big(T, x_0, u, K_{ad} \cap cB_{U_{ad}}\big)$$

is a δ-net of

$$z\big(T, x_0, u_0, K_{ad} \cap cB_{U_{ad}}\big).$$

Therefore,

$$B_X \subset z(T, x_0, u, K_{ad} \cap cB_{U_{ad}}) + (\frac{1}{2} + \delta)B_X = \mathrm{D}g(u)(K_{ad} \cap cB_{U_{ad}}) + (\frac{1}{2} + \delta)B_X.$$

Applying Theorem 2.3.2 again, completes the proof. □

Remark 2.5.9. It is known that, in finite dimensional spaces, the approximate and exact controllability of unconstrained linear systems are equivalent. If $f(t, x, u) = Ax + Bu$ with B, U linear bounded, then the corresponding associated linear system remains the same. So Theorem 2.5.8 implies that this equivalence is preserved to the constrained case.

2.5.2 Constrained global controllability of semilinear systems

In this part, we let $X, Y, U, K, U_{ad}, K_{ad}, \mathbf{R}_T(K)$ be the same as in the previous subsection and use our Theorems 2.3.2 and 2.4.1 to deal with the K-constrained global controllability of the semilinear system

$$\begin{cases} \dot{x}(t) = A(t)x(t) + f(t, x(t), u(t)) + Bu(t), \quad t \in [0, T], \\ x(0) = x_0. \end{cases} \tag{2.52}$$

via the controllability of the linear system

$$\begin{cases} \dot{x}(t) = A(t)x(t) + Bu(t), \quad t \in [0, T], \\ x(0) = x_0. \end{cases} \tag{2.53}$$

Here, $A(t)$ is a linear operator on X for each $t \in [0, T]$ and generates an evolution system $E(t, s)$, B is a bounded linear operator from U to X and f is a measurable function from $[0, T] \times X \times U$ to X.

If there is no constraint, such a problem has been widely considered in both finite and infinite dimensional spaces, see [42], [54], [58], [59], [79] and references therein, where the methods used are mainly topological degree and fixed point theorems.

Theorem 2.5.10. *Suppose that there exists $w \in L^\infty(0,T)$ such that $\|f(t,x,u)\| \le w(t)$ for almost all $(t,x,u) \in [t,T]\times X\times K(t)$ and the evolution system $E(t,s)$ generated by $A(t)$ is uniformly bounded. Then the K-constrained approximate controllability of nonlinear system (2.52) implies the same controllability of linear system (2.53); If, in addition, X is finite dimensional, then the K-constrained exact controllability of the linear system (2.53) implies the same controllability of the nonlinear system (2.52).*

Proof. Let $Y = C(0,T;X), V = U_{ad}, \tilde{K} = K_{ad}$ and $y_0 = E(\cdot,0)x_0 \in Y$. Define the linear operators P_T, H and nonlinear operator F by

$$
\begin{aligned}
&P_T y = y(T) \quad \text{for } y \in Y;\\
&(Hv)(t) = \int_0^t E(t,s)Bv(s)ds \quad \text{for } v \in V;\\
&F(y,v)(t) = \int_0^t E(t,s)f(s,y(s),v(s))ds \quad \text{for } y \in Y, v \in V.
\end{aligned}
$$

Then it is easy to see that all the conditions of Theorem 2.4.1 are satisfied and $P_T I_{\tilde{K}}(F) = \mathbf{R}_T(K)$. Moreover, if X is finite dimensional, then $E(t,s)$ is compact and, therefore, H, F are compact due to Theorem 1.7.2. So the proof follows from Theorem 2.4.1. □

Remark 2.5.11. In Theorem 2.5.10, we do not need to suppose a solution of (2.52) exists for every u, but the proof shows that the solution for the appropriate u exists.

Theorem 2.5.12. *Suppose $E(t,s)$ or B is compact, X is infinite dimensional. Then the linear system (2.53) can never be exactly controllable.*

Proof. By Theorem 1.7.2, the linear operator H defined in Theorem 2.5.10 is compact. So, by Remark 2.4.2, the range of $P_T H$ is never be the whole space X, that is (2.53) is never exact controllable. □

Remark 2.5.13. This theorem is the main result of [76], but our proof is simpler.

Now, we consider the case when f is independent of the state x, that is the system

$$\begin{cases} x'(t) = A(t)x(t) + f(t, u(t)) + Bu(t), & t \in [0, T], \\ x(0) = 0. \end{cases} \tag{2.54}$$

Theorem 2.5.14. *Suppose the following conditions are satisfied.*

(i) $\|E(t, s)\| \leq m$, *for all* t, s;

(ii) $\|f(t, u) - f(t, v)\| \leq k\|u - v\|$ *with* $k > 0$, *for all* $u, v \in K$;

(iii) the associated linear system (2.53) *is* K*-constrained exactly controllable.*

Then system (2.54) *is* K*-constrained exactly controllable provided* kmT *is sufficiently small.*

Proof. By assumption (ii), for each $u \in U_{ad}$, each equation (2.54) and (2.53) has exactly one solution $x(t, 0, u)$ and $z(t, 0, u)$ which have the expression, respectively,

$$x(t, 0, u) = \int_0^t E(t, s)[f(s, u(s)) + Bu(s)]ds, \quad \text{for all } t \in [0, T],$$
$$z(t, 0, u) = \int_0^t E(t, s)Bu(s)ds, \quad \text{for all } t \in [0, T].$$

It is easy to show that both $u \mapsto x(T, 0, u)$ and $u \mapsto z(T, 0, u)$ are well defined continuous operators from U_{ad} to X and $z(T, \cdot)$ is bounded linear. Moreover, for all $u, v \in U_{ad}$, we have

$$\begin{aligned} &\|x(T, 0, u) - x(T, 0, v) - [z(T, 0, u) - z(T, 0, v)]\| \\ &\quad \leq \int_0^T \|E(T, s)[f(s, u(s)) - f(s, v(s))]\|ds \\ &\quad \leq mk \int_0^T \|u(s) - v(s)\|ds \leq mkT\|u - v\|. \end{aligned}$$

Our assumption (iii) and Theorem 1.3.15 imply that there exists $c > 0$ such that

$$B_X \subset z(T, 0, cB_{U_{ad}} \cap K_{ad}).$$

Therefore, the assumption (F2.3.2) of Section 2.3 is satisfied for all $\delta > 0$ with $F(\cdot) = x(T, 0, \cdot), L(\cdot) = z(T, 0, \cdot), c(\delta) \equiv c$ and $\gamma = kmTc$ provided $kmT < 1/c$. Obviously, condition (2.43) is also satisfied with $k(\delta) \equiv 0$. Hence, by Theorem 2.3.2, we see $x(T, 0, K_{ad}) = X$, that is system (2.54) is K-constrained exact controllable. □

This theorem means that the constrained controllability of a linear system implies the constrained controllability of a perturbed system if the perturbation is Lipschitz with a sufficiently small constant.

Chapter 3

Solvability of Operator Inclusions of Monotone Type

Consider the following operator equation

$$Nx = y_0 \tag{3.1}$$

in an abstract space X with $N : X \to X$ a nonlinear operator of monotone type.

Such a problem is important, particularly, in ordinary and partial differential equation theory and has been widely considered. Much theory has been developed to treat (3.1). An important one is topological degree theory, as an extension of the Leray-Schauder degree, developed by Browder [21] for the case when N is pseudo-monotone. His interesting work stimulated much further research about the degree of general operators, see [12], [13], [22], [23], [71] and the references therein. The important property of topological degree, invariance under admissible homotopy, was the main result in each of the papers mentioned above, and from this property, some related existence results for (3.1) were obtained.

The case when N is set-valued is also significant, for example, in variational inequality theory and differential inclusion theory etc. and also received much attention recently. When N is pseudo-monotone, surjectivity results can be found in Browder and Hess's paper [25]. In order to obtain existence theorem for set-valued problems on a given bounded subset, some people still define topological degrees so as to use the invariance under admissible homotopy. In [46], Hu and Papageorgiou generalized Browder's degree to the case when there is a set-valued compact perturbation. In [51] Kittilä gave a degree for certain set-valued mappings which can be approximated by sequences of single-valued operators of class (S_+).

In this chapter, without defining topological degree, we consider the solvability (on a bounded subset) of the general inclusion problems

$$y_0 \in N_1(x) + N_2(x) \tag{3.2}$$

in Banach spaces. Here, N_1 is a demicontinuous set-valued mapping which is either of class (S_+) or pseudo-monotone or quasi-monotone, the perturbation N_2 is always a set-valued quasi-monotone mapping. By the known degrees for some single-valued operators given in [71], we obtain conclusions, for our set-valued problems, similar to the invariance under admissible homotopy of topological degree. Some concrete existence results are, therefore, obtained which generalize and improve the corresponding ones in [25], [26] and [51]. Applications to differential, integral inclusions and controllability of nonlinear systems are also given.

We begin this chapter by giving some properties of set-valued mappings of monotone type.

3.1 Properties of set-valued mappings of monotone type

The definitions of set-valued monotone, pseudo-monotone, quasi-monotone mappings and mappings of class (S_+) have been given in §1.6. In this section, we shall give some properties of set-valued mappings of monotone type, most of them are known in the single-valued case. One can find the known results in [24], [25], [31] and [80].

In the following, we always suppose X is a reflexive Banach space with dual X^*, $(\cdot,\cdot)$ denotes the duality between X and X^* unless stated ortherwise. For the meaning of X_w and $x_n \rightharpoonup x$, see §1.1.

Proposition 3.1.1. *A mapping $N : \mathrm{Dom}(N) \subset X \to 2^{X^*}$ is pseudo-monotone if and only if $x_n \in \mathrm{Dom}(N), x_n \rightharpoonup x_0$ in $X, u_n \in N(x_n)$ and $\limsup_{n\to\infty}(u_n, x_n - x_0) \leq 0$ imply that $x_0 \in \mathrm{Dom}(N)$ and, for each $x \in X$, there exists $u = u(x) \in N(x_0)$ and a subsequence $\{n_k\}$ such that*

$$(u, x_0 - x) \leq \liminf_{k\to\infty}(u_{n_k}, x_{n_k} - x).$$

Proof. The necessity is obvious, so we only prove the sufficiency.

If N is not pseudo-monotone, then there exist $x_n, x_0 \in \mathrm{Dom}(N), \hat{x} \in X, x_n \rightharpoonup x_0$ and $u_n \in N(x_n)$ with $\limsup_{n\to\infty}(u_n, x_n - x_0) \leq 0$ such that

$$(u, x_0 - \hat{x}) > \liminf_{n\to\infty}(u_n, x_n - \hat{x}), \quad \text{for each } u \in N(x_0).$$

We may suppose that

$$\liminf_{n\to\infty}(u_n, x_n - \hat{x}) = \lim_{i\to\infty}(u_{n_i}, x_{n_i} - \hat{x}). \tag{3.3}$$

Then, by the assumptions, there exists $\hat{u} \in N(x_0)$ and a subsequence $\{n_{i_k}\}$ such that

$$(\hat{u}, x_0 - \hat{x}) \leq \liminf_{k\to\infty}(u_{n_{i_k}}, x_{n_{i_k}} - \hat{x}).$$

By (3.3)

$$\liminf_{k\to\infty}(u_{n_{i_k}}, x_{n_{i_k}} - \hat{x}) = \lim_{i\to\infty}(u_{n_i}, x_{n_i} - \hat{x}) = \liminf_{n\to\infty}(u_n, x_n - \hat{x}),$$

so, we have

$$(\hat{u}, x_0 - \hat{x}) \leq \liminf_{n\to\infty}(u_n, x_n - \hat{x}).$$

This is a contradiction which completes the proof. □

Proposition 3.1.2. *Let $N : \mathrm{Dom}(N) \subset X \to 2^{X^*}$ be a demicontinuous mapping with bounded values. Then, $N \in (S_+)$ implies $N \in (PM)$; $N \in (PM)$ implies $N \in (QM)$.*

Proof. First, we suppose $N \in (S_+)$, $x_n \rightharpoonup x$ in $X, u_n \in N(x_n)$ and $\limsup\limits_{n\to\infty}(u_n, x_n - x) \leq 0$. Then $x_n \to x$. Since the values of N are bounded subsets and N is demicontinuous, we see that $\{u_n\}$ is bounded. Since X is reflexive, there exists a subsequence $\{u_{n_k}\}$ such that $u_{n_k} \rightharpoonup u$ for some $u \in X^*$ and $(u_{n_k}, x_{n_k} - x) \to 0$. Therefore, for every $y \in X$, we have

$$\liminf_{k\to\infty}(u_{n_k}, x_{n_k} - y) = \lim_{k\to\infty}(u_{n_k}, x - y) = (u, x - y).$$

The demicontinuity of N implies that $u \in Nx$. So, $N \in (PM)$.

The second conclusion follows easily from the definitions. $\square$

Proposition 3.1.3. *Suppose $N_1 \in (S_+), N_2 \in (QM)$. Then $N_1 + N_2 \in (S_+)$.*

Proof. Let $x_n \rightharpoonup x$ in $X, u_n = v_n + w_n \in N_1(x_n) + N_2(x_n)$ with $v_n \in N_1(x_n), w_n \in N_2(x_n)$ such that $\limsup_{n\to\infty}(u_n, x_n - x) \le 0$. Then $N_2 \in (QM)$ implies that $\limsup_{n\to\infty}(w_n, x_n - x) \ge 0$ and, therefore,

$$\limsup_{n\to\infty}(v_n, x_n - x) \le \limsup_{n\to\infty}(v_n + w_n, x_n - x) \le 0.$$

Since $N_1 \in (S_+)$, we have $x_n \to x$ which completes the proof. $\square$

Lemma 3.1.4. *(Browder [20], Proposition 7.1)*

Let C be a closed, bounded and convex subset in a reflexive Banach space Z, $C_1 \subset Z^*$ be convex. Suppose that for each $w \in C_1$, there exists $u \in C$ such that $(w, u) \ge 0$. Then there exists an element $x_0 \in C$ such that $(w, x_0) \ge 0$ for all $w \in C_1$.

Proposition 3.1.5. *Let $N : \mathrm{Dom}(N) \subset X \to P_{cv}(X^*)$ be a pseudo-monotone mapping. Suppose that $x_n \rightharpoonup x_0$ in $X, u_n \in N(x_n)$ and $\limsup_{n\to\infty}(u_n, x_n - x_0) \le 0$. Then*

$$\lim_{n\to\infty}(u_n, x_n - x_0) = 0, \tag{3.4}$$

$$\emptyset \ne w\text{-}\limsup_{n\to\infty}\{u_n\} \subset N(x_0). \tag{3.5}$$

Recall that w-$\limsup_{n\to\infty}\{u_n\} = \{u : u_{n_k} \rightharpoonup u$ for some subsequence $\{n_k\}\}$.

Proof. (3.4) follows easily from the definition.

Since N is pseudo-monotone, for each $y \in X$, there exists $u(y) \in N(x_0)$ such that

$$\begin{aligned}(u(y), x_0 - y) &\le \liminf_{n\to\infty}(u_n, x_n - y) = \liminf_{n\to\infty}[(u_n, x_0 - y) + (u_n, x_n - x_0)] \\ &= \liminf_{n\to\infty}(u_n, x_0 - y).\end{aligned} \tag{3.6}$$

Let $x = x_0 - y$ and $x = y - x_0$ respectively, we obtain that

$$(u(x_0 - x), x) \le \liminf_{n\to\infty}(u_n, x), \quad \limsup_{n\to\infty}(u_n, x) \le (u(x - x_0), x), \quad \text{for all } x \in X.$$

Since $u(x_0-x), u(x-x_0) \in N(x_0)$ and $N(x_0)$ is bounded, we see that $\{(u_n, x)\}$ is bounded for each $x \in X$. By the uniformly boundedness principle, $\{u_n\}$ is bounded. Since X is reflexive, there exist $u_{n_k} \rightharpoonup u$ for some $u \in X$ which implies that w-$\limsup_{n\to\infty}\{u_n\} \neq \emptyset$.

To prove w-$\limsup_{n\to\infty}\{u_n\} \subset N(x_0)$, we need only show the above u is in $N(x_0)$. In fact, from (3.6), it follows that

$$(u(y), x_0 - y) \leq (u, x_0 - y), \quad \text{for all } y \in X.$$

By Lemma 3.1.4, there exists $u_0 \in N(x_0)$ such that

$$(u_0 - u, x_0 - x) \leq 0 \quad \text{for all } \ x \in X,$$

that is $u = u_0 \in N(x_0)$, and $u_{n_k} \rightharpoonup u_0$. This completes the proof. □

Proposition 3.1.6. *Let $N_1, N_2 : D \subset X \to P_{cv}(X^*)$ be pseudo-monotone mappings. Then $N_1 + N_2$ is also pseudo-monotone.*

Proof. It is easy to see that $(N_1 + N_2)(x) \in P_{cv}(X^*)$ for each $x \in D$.

Let $x_n \in D, w_n \in (N_1 + N_2)(x_n)$ with $x_n \rightharpoonup x_0$ and $\limsup_{n\to\infty}(w_n, x_n - x_0) \leq 0$. We may suppose $w_n = u_n + v_n$ with $u_n \in N_1(x_n), v_n \in N_2(x_n)$. Using the same method as used in the proof of Proposition 9 of [25], we can prove that

$$\limsup_{n\to\infty}(u_n, x_n - x_0) \leq 0, \quad \limsup_{n\to\infty}(v_n, x_n - x_0) \leq 0.$$

By the pseudo-monotonicity of N_1 and N_2, we see that $x_0 \in D$ and, for each $y \in X$, there exist $u(y) \in N_1(x_0)$ and $v(y) \in N_2(x_0)$ such that

$$(u(y), x_0 - y) \leq \liminf_{n\to\infty}(u_n, x_n - y), \quad (v(y), x_0 - y) \leq \liminf_{n\to\infty}(v_n, x_n - y).$$

So, $w(y) := u(y) + v(y) \in (N_1 + N_2)x_0$ satisfies

$$\begin{aligned}(w(y), x_0 - y) &\leq \liminf_{n\to\infty}(u_n, x_n - y) + \liminf_{n\to\infty}(v_n, x_n - y)\\ &\leq \liminf_{n\to\infty}(w_n, x_n - y).\end{aligned}$$

This proves the pseudo-monotonicity of $N_1 + N_2$. □

Proposition 3.1.7. *Let $N : \mathrm{Dom}(N) \subset X \to P_{cv}(X^*)$ be a pseudo-monotone mapping.*

(i) If $\mathrm{Dom}(N)$ *is closed, then N is sequentially closed (that is* $\mathrm{Graph}(N)$ *is sequentially closed) in $X \times X_w^*$.*

(ii) If N is locally bounded, then $\mathrm{Dom}(N)$ *is closed and N is demicontinuous.*

(iii) If N is locally bounded on each finite-dimensional subsequence of X and $\mathrm{Dom}(N) = X$, *then N is finite continuous.*

Proof. (i) Suppose $x_n \to x_0, u_n \in N(x_n)$ and $u_n \rightharpoonup u$ for some $u \in X^*$. Then

$$\lim_{n\to\infty} (u_n, x_n - x_0) = 0.$$

Using Proposition 3.1.5, we see $u \in N(x_0)$ and, therefore, N is sequentially closed.

(ii) Suppose N is locally bounded and $x_n \in \mathrm{Dom}(N)$ with $x_n \to x \in X$. Let $u_n \in N(x_n)$. The local boundedness of N implies that $\{u_n\}$ is bounded, therefore, $\lim_{n\to\infty}(u_n, x_n - x) = 0$. By the pseudo-monotonicity of N, we see $x \in \mathrm{Dom}(N)$ which gives the closedness of $\mathrm{Dom}(N)$.

If N is not demicontinuous at x_0, then there exists a neighbourhood V of $N(x_0)$ in X_w^* and $x_n \to x_0, u_n \in N(x_n)$ such that $u_n \notin V$ and $u_n \rightharpoonup u$. The sequential closedness of N proved above implies $u \in N(x_0) \subset V$. This is a contradiction.

(iii) It is actually Proposition 6 in [25].

This completes the proof. □

Proposition 3.1.8. *If $N : \mathrm{Dom}(N) \subset X \to P_{cv}(X^*)$ maps bounded subsets into relatively compact subsets, then N is quasi-monotone.*

Proof. This follows easily from the definitions of quasi-monotonicity and compactness. □

Lemma 3.1.9. *([7], Theorem 2.1.5)*

If $N : \mathrm{Dom}(N) \subset X \to P_{cv}(X^*)$ is monotone, $x_0 \in \mathrm{int}(\mathrm{Dom}(N))$, then N is locally bounded at x_0.

Proposition 3.1.10. *Suppose $N : \mathrm{Dom}(N) \to P_{cv}(X^*)$ is a monotone and hemicontinuous set-valued mapping.*

(i) If either Dom$(N) = X$ *or N is locally bounded, then N is demicontinuous.*

(ii) If either Dom(N) *is closed convex or N is bounded, then N is pseudo-monotone.*

Proof. First, we prove N is demicontinuous. To do this, we need to prove that $N^{-1}(D)$ is closed in X for each closed subset $D \subset X_w^*$.

If $N^{-1}(D)$ is not closed, then there exist $x_n \in N^{-1}(D), x_0 \notin N^{-1}(D)$ with $x_n \to x_0$ in X. Therefore, there exist $u_n \in N(x_n) \cap D$. By Lemma 3.1.9 and our assumption, $\{u_n\}$ is bounded in each case. So we can suppose (by passing to subsequence) that $u_n \rightharpoonup u_0$ for some $u_0 \in X^*$. Since D is closed in $X_w^*, u_0 \in D$. The monotonicity of N implies that

$$(u_n - u, x_n - x) \geq 0, \quad \text{for all } x \in X, u \in N(x).$$

Therefore

$$(u_0 - u, x_0 - x) \geq 0, \quad \text{for all } x \in X, u \in N(x).$$

Let $z \in X, x_{t_n} = x_0 + t_n z$ with $t_n \to 0$ and let $u_{t_n} \in N(x_{t_n})$ with $u_{t_n} \rightharpoonup u(z)$ (passing to a subsequence if necessary). Then $(u_0 - u_{t_n}, z) \leq 0$ and, therefore, $(u_0 - u(z), z) \leq 0$. The hemicontinuity of N implies $u(z) \in N(x_0)$. Since z is arbitrary, by Lemma 3.2.3, we see that $u_0 \in N(x_0)$ which implies $x_0 \in N^{-1}(D)$ and contradicts the assumption. So N is demicontinuous.

Now, we prove the pseudo-monotonicity of N under the corresponding assumptions. In case Dom(N) is closed convex, this is just Proposition 7.4 in [20]. So we suppose N is bounded. Let $x_n \rightharpoonup x_0$ in $X, u_n \in N(x_n)$ and $\limsup_{n\to\infty}(u_n, x_n - x) \leq 0$. The boundedness of N implies $u_{n_k} \rightharpoonup u_0$ for some subsequence $\{n_k\}$ and $u_0 \in X^*$. From the demicontinuity of N proved in the first step, it follows that $u_0 \in N(x_0)$. By the monotonicity of N we see $(u_n - u_0, x_n - x_0) \geq 0$ and, therefore,

$$(u_n, x_n - x_0) \to 0, \text{ as } n \to \infty.$$

Hence, for every $y \in X$, we have

$$\liminf_{n\to\infty}(u_n, x_n - y) = \liminf_{n\to\infty}(u_n, x_0 - y) = (u_0, x_0 - y),$$

which means that N is pseudo-monotone. $\square$

3.2 Solvability of inclusions in Banach space

In this section, we suppose X is a real separable reflexive Banach space and choose n-dimensional subspaces $\{E_n\}$ of X with E_n having basis $\{e_1, \cdots, e_n\}$ such that

$$E_n \subset E_{n+1}, \quad \text{and} \quad \overline{\cup E_n} = X.$$

We will consider the solvability in $\overline{\Omega}$ of the problem

$$y \in N_1(x) + N_2(x) \tag{3.7}$$

under the following basic assumptions.

(H3.2.1) Ω is a bounded open subset of X.

(H3.2.2) $N_1, N_2 : \overline{\Omega} \to P_{cv}(X^*)$ are bounded demicontinuous set-valued mappings.

Because of Proposition 3.1.7, if N_1 (or N_2) is pseudo-monotone, then demicontinuity need not be imposed explicitly.

For a set-valued mapping $N : \mathrm{Dom}(N) \subset X \to 2^{X^*}$, we define an approximate mapping $[N]_n$ by

$$[N]_n(x) = \left\{ \sum_{i=1}^{n} (v, e_i)e_i : v \in N(x) \right\} \quad \text{for } x \in E_n \cap \mathrm{Dom}(N).$$

Similarly, for a point $y \in X^*$ we denote by

$$[y]_n = \sum_{i=1}^{n} (y, e_i)e_i.$$

Obviously, $[y]_n \in E_n$ and $[N]_n$ maps $E_n \cap \Omega$ into E_n as a set-valued mapping for each n.

We first give some concepts related to class (S_+) and quasi-monotone mappings.

Definition 3.2.1. Let $F : [0,1] \times D \to P_{cv}(X^*)$ be a bounded, demicontinuous mapping. F is said to be a

(i) *homotopy of class* (S_+) (on D), if $x_n \in D, x_n \rightharpoonup x$ in $X, t_n \to t$ in $[0,1]$ and $u_n \in F(t_n, x_n)$ with $\limsup_{n\to\infty}(u_n, x_n - x) \le 0$ imply that $x \in D$ and $x_n \to x$;

(ii) *quasi-monotone homotopy* (on D), if $x_n \in D, x_n \rightharpoonup x$ in $X, t_n \to t$ in $[0,1]$ and $u_n \in F(t_n, x_n)$ imply $\liminf_{n\to\infty}(u_n, x_n - x) \ge 0$.

Definition 3.2.2. If $(t,x) \mapsto H(t,x)$ is defined by $H(t,x) = F(t,x) + G(t,x)$ with F and G homotopies (on $\overline{\Omega}$) of class (S_+) and quasi-monotone respectively, then H is said to be an *admissible homotopy* (on $\overline{\Omega}$).

If H is an admissible homotopy, then the mapping $x \mapsto H(t,x)$ for a given t is denoted by H_t.

Lemma 3.2.3. *Let Ω be a bounded open subset of X, H an admissible homotopy on $\overline{\Omega}$, $y : [0,1] \to X^*$ be continuous and $D \subset \overline{\Omega}$ a closed subset. If $y(t) \notin H_t(D)$ for all $t \in [0,1]$, then there exists $n_0 > 0$ such that*

$$[y(t)]_n \notin [H_t]_n(D \cap E_n) \quad \textit{for all } t \in [0,1], n \geq n_0. \tag{3.8}$$

Proof. Without loss of generality, we suppose that $y(t) \equiv 0$ (otherwise, we consider $H_t - y(t)$ instead of H_t).

Suppose (3.8) is not true. Let $H = F + G$ with F and G homotopies of class (S_+) and quasi-monotone respectively. Then there exist $n_k \to \infty, t_k \in [0,1]$ and $x_k \in D \cap E_{n_k}$ such that $0 \in [H_{t_k}]_{n_k}(x_k)$ for all k. That is, there exist $v_k \in F(t_k, x_k)$ and $w_k \in G(t_k, x_k)$ such that

$$\sum_{i=1}^{n_k} (v_k + w_k, e_i)e_i = 0 \quad \text{for } k \geq 1, 1 \leq i \leq n_k.$$

Then, we have

$$(v_k + w_k, e_i) = 0, \text{ and } (v_k + w_k, x_k) = 0 \quad \text{for } k \geq 1, 1 \leq i \leq n_k, \tag{3.9}$$

(Note $x_k \in E_{n_k}$). We may suppose that $x_k \rightharpoonup x$ in $X, t_k \to t \in [0,1]$ and $v_k \rightharpoonup v, w_k \rightharpoonup w$ for some $v, w \in X$ (or pass to subsequences). Then, it follows from (3.9) that $(v+w, e_i) = 0$ for all i which implies $(v+w, x) = 0$ for all $x \in X$ and, therefore, $v + w = 0$. So,

$$\lim_{k\to\infty} (v_k + w_k, x_k - x) = 0.$$

Since G is a quasi-monotone homotopy, $\liminf_{k\to\infty}(w_k, x_k - x) \geq 0$. So, we have

$$\limsup_{k\to\infty} (v_k, x_k - x) \leq 0.$$

Since F is a homotopy of class (S_+) and D is closed, it follows that $x_k \to x \in D$. The demicontinuity of F and G imply that $v \in F(t,x)$ and $w \in G(t,x)$. This contradicts the assumptions and completes the proof. $\square$

Now, under (H3.2.1) and (H3.2.2), let $N = N_1 + N_2$ with $N_1 \in (S_+), N_2 \in (QM), y \in X$ and $y \notin N(\partial\Omega)$. By letting $H_t \equiv N, y(t) \equiv y, D = \partial\Omega$ in Lemma 3.2.3, we see that $[y]_n \notin [N]_n(\partial\Omega \cap E_n)$ whenever $n \geq n_0$ for some $n_0 > 0$. It is easy to show from our assumptions that $[N]_n : E_n \cap \Omega \to 2^{E_n}$ is a compact mapping. So, the topological degree $\deg([N]_n, \Omega \cap E_n, [y]_n)$ is well defined. Moreover, if both N_1 and N_2 are single-valued (in this case, we can suppose $N_2 = \{0\}$), this degree has the following property.

Lemma 3.2.4. *[71] Let $N_1 \in (S_+)$ be single-valued, bounded and demicontinuous, $y \in X$ and $y \notin N_1(\partial\Omega)$. Then there exists n_1 such that*

$$\deg([N_1]_n, \Omega \cap E_n, [y]_n) = constant, \quad \textit{for all } n \geq n_1, \tag{3.10}$$

and this constant is independent of the choice of $\{E_n\}$.

Definition 3.2.5. A bounded demicontinuous operator $T \in (S_+)$ is said to be a *reference mapping* related to $w \in T(\Omega)$ if

$$\lim_{n\to\infty} \deg([T]_n, \Omega \cap E_n, [w]_n) \neq 0.$$

The set of all such reference mapping related to w is denoted by $\mathcal{R}(w)$. That is, $T \in \mathcal{R}(w)$ if there exists $n_1 \geq 1$ such that $\deg([T]_n, \Omega \cap E_n, [w]_n) \neq 0$ for each $n \geq n_1$.

Example 3.2.6. (i) It is easy to see that the duality mapping $J : X \to X^*$ (under an equivalent norm, J is single-valued) is a reference mapping.

(ii) If $T \in (S_+)$ is an odd operator and has no solutions on the boundary of a symmetric Ω, then $[T]_n$ is also odd and, by Lemma 3.2.3, has no solutions on $\partial\Omega \cap E_n$ once n is large enough. So Borsuk Theorem (Theorem 1.5.2 (iii)) and Lemma 3.2.4 imply $T \in \mathcal{R}(0)$.

Now, we use the above to give some existence results for (3.7). For simplicity, we suppose $y = 0$ (otherwise, replace N_1 by $N_1 - y$).

Theorem 3.2.7. *Under (H3.2.1) and (H3.2.2), suppose $T \in \mathcal{R}(w)$, $w \in T(\Omega)$ are such that*

$$(1-t)(T(x)-w) \notin -t(N_1+N_2)(x) \quad \textit{for all } x \in \partial\Omega, t \in [0,1]. \tag{3.11}$$

Then,

(i) if $N_1 \in (S_+), N_2 \in (QM)$, then $0 \in (N_1+N_2)(x)$ admits solutions in $\overline{\Omega}$;

(ii) if $N_1, N_2 \in (QM)$, then $0 \in \overline{(N_1+N_2)(\overline{\Omega})}$;

(iii) if $N_1+N_2 \in (PM)$ and Ω is convex, then $0 \in (N_1+N_2)(x)$ admits solutions in $\overline{\Omega}$.

Proof. Without loss of generality, we suppose $w = 0$ and write $N = N_1 + N_2$.

(i) Define $H : [0,1] \times \overline{\Omega} \to 2^{X^*}$ by

$$H(t,x) = (1-t)T(x) + tN(x) = (1-t)T(x) + tN_1(x) + tN_2(x).$$

Since T is single-valued, we see that H is an admissible homotopy and, therefore, $[H_t]_n$ is a homotopy for compact mappings in finite dimensional space. (3.11) implies that $0 \notin H(t,x)$ for all $t \in [0,1]$ and $x \in \partial\Omega$. So, by Lemmas 3.2.3 and 3.2.4, there exists $n_1 \geq 1$ such that

$$0 \notin [H_t]_n(x) \text{ and } \deg([T]_n, \Omega \cap E_n, 0) \neq 0 \text{ for all } t \in [0,1], x \in \partial\Omega \cap E_n, n \geq n_1.$$

Applying the invariance under homotopy of set-valued compact mappings, we obtain

$$\begin{aligned} \deg([N]_n, \Omega \cap E_n, 0) &= \deg([H_1]_n, \Omega \cap E_n, 0) = \deg([H_0]_n, \Omega \cap E_n, 0) \\ &= \deg([T]_n, \Omega \cap E_n, 0) \neq 0 \end{aligned}$$

for all $n \geq n_1$. Therefore, there exists $x_n \in \overline{\Omega} \cap E_n$ such that $0 \in [N]_n(x_n)$ for all $n \geq n_1$. From Lemma 3.2.3, we see that there exists $x \in \overline{\Omega}$ with $0 \in N(x)$.

(ii) Suppose $0 \notin \overline{N(\partial\Omega)}$ (otherwise, the assertion is true). For each $\varepsilon > 0$, consider the mapping

$$N_\varepsilon := N_1 + \varepsilon T + N_2.$$

By Proposition 3.1.3, $N_1 + \varepsilon T \in (S_+)$, it is also demicontinuous since T is single-valued.

We claim that there exists $\varepsilon_0 > 0$ such that

$$0 \notin (1-t)T(x) + tN_\epsilon(x) \quad \text{for all } x \in \partial\Omega, t \in [0,1] \text{ and } \varepsilon \in (0, \varepsilon_0). \tag{3.12}$$

In fact, if (3.12) is not true, there would exist $t_n \in [0,1], x_n \in \partial\Omega$ and $\varepsilon_n > 0$ with $t_n \to t \in [0,1], \varepsilon_n \to 0, x_n \rightharpoonup x \in X$ such that

$$0 \in (1-t_n)T(x_n) + t_n N_{\varepsilon_n}(x_n) \quad \text{for all } n \geq 1.$$

Therefore there exist $v_n \in N_1(x_n), w_n \in N_2(x_n)$ such that

$$(1 - t_n + t_n\varepsilon_n)T(x_n) + t_n(v_n + w_n) = 0. \tag{3.13}$$

We may suppose $T(x_n) \rightharpoonup u, v_n \rightharpoonup v$ and $w_n \rightharpoonup w$ in X^* for some $u, v, w \in X^*$ (by passing to subsequences), and $1 - t_n + t_n\varepsilon_n > 0$. Then

$$(1-t)T(x) + t(v + w) = 0. \tag{3.14}$$

Since both T (it is demicontinuous) and N_2 are quasi-monotone,

$$\liminf_{n\to\infty}(T(x_n), x_n - x) \geq 0, \ \liminf_{n\to\infty}(w_n, x_n - x) \geq 0.$$

From (3.13), it follows that

$$\begin{aligned} \limsup_{n\to\infty}(v_n, x_n - x) &\leq -\liminf_{n\to\infty}\frac{1-t_n+t_n\varepsilon_n}{t_n}(T(x_n), x_n - x) \\ &\quad -\liminf_{n\to\infty}(w_n, x_n - x) \leq 0. \end{aligned}$$

Since $N_1 \in (S_+)$, we see that $x_n \to x \in \partial\Omega$. Therefore, $u \in T(x)$, $v \in N_1(x), w \in N_2(x)$. By (3.14) and (3.11), a contradiction is obtained. Hence, (3.12) is true.

Applying the conclusion (i) to N_ε, there exists $x_\varepsilon \in \Omega$ such that

$$0 \in N_\varepsilon(x_\varepsilon) = N(x_\varepsilon) - \varepsilon T(x_\varepsilon) \quad \text{for each } \varepsilon \in (0, \varepsilon_0).$$

This implies $0 \in \overline{N(\overline{\Omega})}$.

(iii) We need only verify that $N(\overline{\Omega})$ is a closed subset. To do this, Let $y_n \in N(\overline{\Omega})$ and $y_n \to y$ in X^*. Then there exist $x_n \in \overline{\Omega}$ such that $y_n \in N(x_n)$. We may suppose (by passing to a subsequences) that $x_n \rightharpoonup x$ in X. Then

$$\limsup(u_n, x_n - x) = \limsup(y_n, x_n - x) = 0.$$

By Proposition 3.1.5, $y \in N(x)$. Since Ω is bounded and convex, X is reflexive, we see that $x \in \overline{\Omega}$.

This completes the proof. $\square$

Remark 3.2.8. Condition (3.11) can be replaced by the following more general condition.

There exist an admissible homotopy H and $T \in \mathcal{R}(w)$ such that

$$H(0,\cdot) = T - w,\ H(1,\cdot) = N \ \textit{and}\ 0 \notin H(t,x) \ \textit{for all}\ \in [0,1], x \in \partial\Omega.$$

Theorem 3.2.7 is therefore similar to the invariance under homotopy of topological degree, both can be used to obtain existence results.

Corollary 3.2.9. *Under (H3.2.1), (H3.2.2), let $T \in \mathcal{R}(w)$ be a linear invertible reference operator. If there exist $c_1 \in (0, \|T^{-1}\|^{-1}), c_2 \geq 0$ such that*

$$(1 - c_1\|T^{-1}\|)^{-1}(c_2\|T^{-1}\| + \|T^{-1}w\|) < \inf\{\|x\| : x \in \partial\Omega\},$$

$$\sup\{\|Tx - w - u\| : u \in (N_1 + N_2)(x)\} \leq c_1\|x\| + c_2 \quad \textit{on}\ \partial\Omega,$$

then the conclusions of Theorem 3.2.7 remain true.

Proof. We need only verify (3.11).

In fact, if there exist $t \in [0,1]$ and $x \in \partial\Omega$ such that $(1-t)(Tx - w) = -tu$ with $u \in (N_1 + N_2)(x)$, then we have $Tx = t(Tx - w - u) + w$ and

$$\|x\| \leq \|T^{-1}\|\|Tx - u - w\| + \|T^{-1}w\| \leq \|T^{-1}\|(c_1\|x\| + c_2) + \|T^{-1}w\|.$$

Therefore,

$$\|x\| \leq (1 - c_1\|T^{-1}\|)^{-1}(c_2\|T^{-1}\| + \|T^{-1}w\|).$$

This contradicts the assumption and completes the proof. $\square$

Corollary 3.2.10. *Under (H3.2.1) and (H3.2.2), suppose $0 \in \Omega$. If*

$$(u, x) > -\|u\|\|x\| \quad \textit{for all}\ x \in \partial\Omega\ \textit{and}\ u \in (N_1 + N_2)(x), \tag{3.15}$$

then the conclusions of Theorem 3.2.7 remain true.

Proof. We need only verify (3.11) for $T = J$. In fact, if there are $x \in \partial\Omega$ and $t \in [0,1]$ such that $(1-t)J(x) \in -t(N_1 + N_2)(x)$, then there exists $u \in (N_1 + N_2)(x)$ such that $(1-t)Jx = -tu$. Therefore $(1-t)\|x\| = (1-t)\|Jx\| = t\|u\|$ and $(1-t)\|x\|^2 = (1-t)(Jx, x) = -t(u, x)$. Obviously, $t \neq 0$, and $(u, x) = -\|u\|\|x\|$ which contradicts (3.15). This completes the proof. $\square$

Remark 3.2.11. Corollary 3.2.10 generalizes Theorems 4.2, 4.3 and 4.4 of [51] where $N_2 \equiv \{0\}$ and the conditions on N_1 are stricter.

Noting Proposition 3.1.10, we have an immediate consequence as below.

Corollary 3.2.12. *Suppose Ω is a bounded open subset of X with $0 \in \Omega, N : \overline{\Omega} \to P_{cv}(X^*)$ is a bounded, monotone and hemicontinuous set-valued mapping. If*

$$(u, x) > -\|u\|\|x\| \quad \textit{for all } x \in \partial\Omega \textit{ and } u \in N(x),$$

then $0 \in N(x)$ has solution in $\overline{\Omega}$.

A surjectivity result is

Corollary 3.2.13. *Under (H3.2.1) and (H3.2.2), suppose* $\mathrm{Dom}(N_1) = \mathrm{Dom}(N_2) = X$ *and the following condition is satisfied for every single-valued selection $u(\cdot)$ of $(N_1+N_2)(\cdot)$.*

$$\lim_{\|x\|\to\infty} \left(\frac{(u, x)}{\|x\|} + \|u\| \right) = \infty. \tag{3.16}$$

Then $(N_1 + N_2)(X) = X^*$ provided $(N_1 + N_2) \in (PM)$ or $N_1 \in (S_+), N_2 \in (QM)$ or $N_1 + N_2$ is monotone and hemicontinuous; If $N_1, N_2 \in (QM)$, then $\overline{(N_1 + N_2)(X)} = X^*$.

Proof. Let $y \in X^*$. Replace N_1 by $N_1 - y$ in Corollary 3.2.10, we need only prove (3.15) is satisfied for a suitable $\Omega := \{x \in X : \|x\| < r\}$ with some $r > 0$.

If such an Ω does not exist, then, for each $n \geq 1$, there exist $x_n \in X$ with $\|x_n\| \geq n$ and $u_n \in N(x)$ such that

$$(u_n, x_n) - (y, x_n) \leq -\|u_n - y\|\|x_n\|. \tag{3.17}$$

From (3.16), it follows that whenever n is large enough, we have

$$\frac{(u_n, x_n)}{\|x_n\|} + \|u_n\| > 2\|y\|.$$

Using (3.17), we see that

$$2\|y\| < \|u_n\| + \frac{(y, x_n)}{\|x_n\|} - \|u_n - y\| \leq 2\|y\|.$$

This is a contradiction and completes the proof. □

Remark 3.2.14. Corollary 3.2.13 generalizes the corresponding result in [51], as well as that in [26] where the authors considered the surjectivity of single-valued quasi-monotone operators.

In order to give another result, we need the following definition.

Definition 3.2.15. A mapping $N : X \to 2^{X^*}$ is said to be *asymptotically quasilinear* with the *asymptote* N_∞ if $N_\infty : X \to 2^{X^*}$ is an upper semicontinuous mapping with nonempty closed values and such that

i) $\alpha N_\infty(x) = N_\infty(\alpha x)$ for all $\alpha \geq 0, x \in X$;

ii) $\overline{\cup_{x \in S} N_\infty(x)}$ is compact, where, $S := \{x \in X : \|x\| = 1\}$;

iii) For every $\varepsilon > 0$, there exists $K > 0$ such that whenever $x \in X$ with $\|x\| \geq K$ and $u \in N(x)$, we have $w(x, u) \in N_\infty(x)$ such that $\|u - w(x, u)\| \leq \varepsilon \|x\|$.

A mapping $N : X \to 2^{X^*}$ is said to be *asymptotically linear* with the asymptote N_∞ if $N_\infty : X \to X^*$ is a continuous linear operator and for every $\varepsilon > 0$, there exists $K > 0$ such that whenever $x \in X$ with $\|x\| \geq K$ and $u \in N(x)$, we have $\|u - N_\infty(x)\| \leq \varepsilon \|x\|$.

Corollary 3.2.16. *Let $S = \{x \in X : \|x\| = 1\}$. Suppose N_1 (or N_2 respectively) is asymptotically quasilinear with the asymptote N_∞, N_2 (or N_1 respectively) is asymptotically linear with the asymptote $L_\infty \in (S_+)$. If there exists a homogeneous demicontinuous operator $A : X \to X^*$ with $-A$ quasi-monotone such that*

$$0 \notin L_\infty x - tN_\infty(x) - (1-t)A(x) \text{ for all } t \in [0,1], x \in S, \tag{3.18}$$

then the conclusions of Theorem 3.2.7 remain true with a suitable $\Omega \subset X$.

Proof. We first claim that there exist $\delta > 0$ such that

$$\|L_\infty z - tw - (1-t)A(z)\| \geq \delta \text{ for all } z \in S, w \in N_\infty(z), t \in [0,1]. \tag{3.19}$$

In fact, otherwise there exist $t_i \in [0,1], z_i \in S, w_i \in N_\infty(z_i)$ such that

$$L_\infty z_i - t_i w_i - (1-t_i)A(z_i) \to 0.$$

By passing to subsequences, we may suppose that $t_i \to t \in [0,1], z_i \rightharpoonup z \in X, w_i \to w \in w \in X^*$. Since $-A$ is quasi-monotone, we have

$$\limsup_{i\to\infty}(L_\infty z_i, z_i - z) \le \limsup_{i\to\infty}(L_\infty z_i - t_i w_i - (1-t_i)A(z_i), z_i - z) = 0.$$

This implies $z_i \to z \in S$ because $L_\infty \in (S_+)$. Therefore, $w \in N_\infty(z), A(z_i) \rightharpoonup A(z)$ in X^* due to the continuity of N_∞ and A, and then $L_\infty z - tw - (1-t)A(z) = 0$ which contradicts (3.18) and gives (3.19).

Now, let $x \in X, v_1 \in N_1(x), v_2 \in N_2(x)$. From Definition 3.2.15, there exists $R_0 > 0$ such that

$$\|v_1 - L_\infty(x)\| \le \frac{1}{4}\delta \ \text{ and } \ \|v_2 - w(x,v_2)\| \le \frac{1}{4}\delta \ \text{ for all } x \in \partial\Omega,$$

where $\Omega = \{x \in X : \|x\| < R_0\}, w(x,v_2)$ is from Definition 3.2.15. So, we have

$$\begin{aligned}\|t(v_1+v_2) + (1-t)(L_\infty x - A(x))\| \ge \|x\|\Big\|L_\infty\frac{x}{\|x\|} - t\frac{w(x,v_2)}{\|x\|} - (1-t)A\frac{x}{\|x\|}\Big\| \\ -t\|v_1 - L_\infty(x)\| - t\|v_2 - w(x,v_2)\| \ge \frac{1}{4}\delta R_0,\end{aligned}$$

for all $t \in [0,1], x \in \partial\Omega, v_1 \in N_1(x), v_2 \in N_2(x)$. This implies that

$$(1-t)(L_\infty x - A(x)) \notin -t(N_1(x) + N_2(x)) \ \text{ for all } t \in [0,1], x \in \partial\Omega,$$

and

$$(L_\infty - A)(x) \ne 0 \ \text{ for all } x \in \partial\Omega.$$

Since $L_\infty - A$ is an odd operator of class (S_+), we see $L_\infty - A \in \mathcal{R}(0)$ (see Example 3.2.6). The conclusion follows from Theorem 3.2.7. □

Remark 3.2.17. Corollary 3.2.16 is a generalization of Theorem 2.3 of [37] where N_1, N_2 are supposed to be single-valued, the space is a Hilbert space and the operator A is linear and compact.

Remark 3.2.18. From the proof of Corollary 3.2.16, we see that (3.18) can be replaced by

$$0 \notin L_\infty x + tN_\infty(x) - (1-t)A(x) \quad \text{for all } t \in [0,1], x \in S.$$

If, in addition, $A(x) \in N_\infty(x)$ and $N_\infty(x)$ is convex, (3.18) can be replaced by

$$0 \notin L_\infty x - N_\infty(x) \quad \text{for all } x \in S.$$

3.3 Applications

In this section, we show that the results obtained in §3.2 can be applied to study some boundary value problems, integral inclusions and controllability of some nonlinear systems.

3.3.1 Applications to elliptic boundary value problems

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\Gamma, m \geq 1$. Consider the problem

$$\sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(z, u, \cdots, D^m u) \in -G(z,u), \tag{3.20}$$

$$D^\beta u \mid_\Gamma = 0 \text{ for } |\beta| \leq m-1. \tag{3.21}$$

Let N be the number of multi-indices α with $|\alpha| \leq m$ and for $\xi = \{\xi_\alpha : |\alpha| \leq m\} \in \mathbb{R}^N$, write $\xi = (\eta, \zeta)$, where $\eta = \{\xi_\alpha : |\alpha| \leq m-1\}$ and $\zeta = \{\xi_\alpha : |\alpha| = m\}$. Suppose $p \in [2, \infty)$ and $q = p/(p-1)$. We impose the following hypotheses on A_α and G.

H(A): $A_\alpha : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is such that

i) $z \mapsto A_\alpha(z, \xi)$ is measurable for all $\xi \in \mathbb{R}^N$ and $\xi \mapsto A_\alpha(z,\xi)$ is continuous for almost all $z \in \Omega$;

ii) There exist $c_1, c_2 > 0$ and $k_1 \in L^q(\Omega)$ and $k_2 \in L^1(\Omega)$ such that

$$|A_\alpha(z,\xi)| \leq c_1 \|\xi\|^{p-1} + k_1(z) \text{ for all } |\alpha| \leq m, z \in \Omega \text{ and } \xi \in \mathbb{R}^N,$$
$$\sum_{|\alpha|\leq m} A_\alpha(z,\xi)\xi_\alpha \geq c_2\|\xi\|^p - k_2(z) \quad \text{for all } z \in \Omega, \xi \in \mathbb{R}^N; \tag{3.22}$$

iii) for all $(\eta, \zeta), (\eta, \zeta')$ in $\mathbb{R}^N$ with $\zeta \neq \zeta'$, and all $z \in \Omega$,

$$\sum_{|\alpha|=m} [A_\alpha(z, \eta, \zeta) - A_\alpha(z, \eta, \zeta')][\zeta_\alpha - \zeta'_\alpha] > 0.$$

H(G): $G : \Omega \times \mathbb{R} \to 2^{\mathbb{R}} \backslash \{\emptyset\}$ is a set-valued mapping such that

i) $G(z, u) = [g_1(z, u), g_2(z, u)]$ is measurable and $\eta \mapsto G(z, \eta)$ is upper semicontinuous for almost all $z \in \Omega$;

ii) there exist $c_3, c_4 > 0, k_3 \in L^q(\Omega)$ such that

$$|G(z, u)| = \max\{|g_1(z, u)|, |g_2(z, u)|\} \leq c_3|u| + k_3(z) \text{ a.e. on } \Omega, \tag{3.23}$$

$$\inf\{g_1(z, u)u, g_2(z, u)u\} \geq -c_4 \text{ on } \Omega. \tag{3.24}$$

Theorem 3.3.1. *Under the above assumptions, problem* (3.20)-(3.21) *admits solutions.*

To prove Theorem 3.3.1, we let $X = W_0^{m,p}(\Omega)$ with $X^* = W^{-m,q}(\Omega)$ and define mappings $N_1, N_2 : X \to X^*$ by

$$(N_1(u), v) = \int_\Omega \sum_{|\alpha| \leq m} A_\alpha(z, u, \cdots, D^m u) D^\alpha v dz,$$
$$N_2(u) = \{g \in L^2(\Omega) : g(z) \in G(z, u(z)) \; a.e.\}.$$

Then we have

Lemma 3.3.2. *(Browder [24], Theorem 1)*
Under the assumption (HA), N_1 is a bounded and pseudo-monotone operator from X to X^*.

Lemma 3.3.3. *Under H(G) (i) and* (3.23), *N_2 is upper semicontinuous from X_w to X^* and, therefore, pseudo-monotone.*

Proof. By Theorem 1.3.3 and the assumptions on G, N_2 is a well defined measurable bounded set-valued mapping from X to $L^2(\Omega) \subset X^*$.

Since $N_2(u) = S^1_{G(\cdot, u(\cdot))}$, by Theorem 1.3.4, we see that N_2 is u.s.c. from $H := L^2(\Omega)$ to H_w. Since both X embeds into H and H embeds into X^* compactly, we see that N_2 is u.s.c. from X_w to X^*. This completes the proof. □

Proof of Theorem 3.3.1

From the above two lemmas, it follows that $N = N_1 + N_2$ is pseudo-monotone. Moreover, by the assumptions H(A) ii) and H(G) ii), there exist constants $c_5, c_6 > 0$ such that

$$\inf_{w \in N(u)} (w, u) = (N_1(u), u) + \inf_{w \in N_2(u)} (w, u) \geq c_5 \|u\|^p - c_6.$$

So (3.18) is satisfied and, therefore, $0 \in N(u)$ has a solution, that is, problem (3.20)-(3.21) admits solutions.

Remark 3.3.4. In Theorem 28 of [46], p is assumed to be strictly larger than 2, the growth condition on G is linear (without (3.24)) and N_1 must be proved to be (S_+) in $W_0^{m,p}(\Omega)$. Of course, in that case, our Theorem 3.2.7 can be applied to obtain the same conclusion.

3.3.2 Applications to integral inclusions

Consider the implicit integral inclusion

$$p(t, x(t)) \in \int_0^1 k(t, s) F(s, x(s)) ds + f(t), \quad t \in [0, 1], \tag{3.25}$$

where, $p \in C([0,1] \times \mathbb{R}, \mathbb{R})$, $f \in C([0,1], \mathbb{R}), k \in L^2([0,1] \times [0,1], \mathbb{R})$, while $F : [0,1] \times \mathbb{R} \to 2^{\mathbb{R}}$ is a set-valued mapping with closed convex values satisfying the Caratheodory conditions (measurable and upper semicontinuous with respect to the first and second argument respectively). If $p(t, x) \equiv x$, this is a problem considered in [60]; if F is single-valued, it is the problem studied in [37]. It is known that the problems

$$(p(t, x))' \in F(t, x), \quad x(0) = a, \quad \text{and } (p(t, x))'' \in F(t, x), \quad x(0) = x(1) = 0$$

are special cases of (3.25).

We impose the following conditions similar to those in [37].

i) $x \to p(t, x)$ is strongly increasing and $\alpha > 0$ is such that

$$\lim_{|x| \to \infty} |p(t, x) - \alpha x| / |x| = 0 \text{ uniformly in } t \in [0, 1]. \tag{3.26}$$

ii) There exist $\beta_2 \geq \beta_1 \geq 0$ such that

$$\beta_1 \leq \liminf_{|x|\to\infty} m_F(t,x) \leq \limsup_{|x|\to\infty} M_F(t,x) \leq \beta_2 \;\text{ uniformly in } t \in [0,1];$$

here, $m_F(t,x) := \inf\{u/x : u \in F(t,x)\}, M_F(t,x) := \sup\{u/x : u \in F(t,x)\}$.

iii) There exists $\delta > 0$ such that the equation

$$\alpha x(t) = \int_0^1 k(t,s)y(s)x(s)ds$$

has no solution for each $y \in L^2(0,1;\mathbb{R})$ with $\beta_1 - \delta \leq y(t) \leq \beta_2 + \delta$.

We let $X = L^2(0,1;\mathbb{R}) = X^*$ and define mappings on X as below

$$\begin{aligned}
&N_1(x) = p(\cdot,x), \quad L_\infty(x) = \alpha x;\\
&N_2(x) = \left\{\int_0^1 k(\cdot,s)f(s)ds : f \in S^1_{F(\cdot,x(\cdot))}\right\}\\
&\mathcal{M} = \left\{x \to \int_0^1 k(\cdot,s)y(s)x(s)ds : y \in X, \beta_1 - \delta \leq y(t) \leq \beta_2 + \delta\right\};\\
&N_\infty(x) = \{Kx : K \in \mathcal{M}\}.
\end{aligned}$$

Then inclusion (3.25) is equivalent to

$$0 \in (N_1 - N_2)(x) \text{ in } X.$$

In Theorem 3.1 of [37], the following result can be found.

Lemma 3.3.5. *[37] Under the above assumptions, N_∞ is a positively homogeneous upper semicontinuous mapping on X with nonempty, convex and compact values, $0 \notin L_\infty x - N_\infty(x)$ for all $x \in X$ with $\|x\| = 1$ and*

$$\overline{\cup\{N_\infty(x) : x \in X, \|x\| = 1\}}$$

is compact.

Theorem 3.3.6. *Under the above assumptions, problem (3.25) admits solutions.*

Proof. Since $x \mapsto p(t,x)$ is strongly monotone, we see that N_1 is strongly monotone. So $N_1 \in (S_+)$. The continuity of p implies the continuity of N_1. From (3.26), it follows that, for each $\varepsilon > 0$, there exists $n(\varepsilon) > 0$ such that

$$|p(t,x) - \alpha x| < \varepsilon|x|, \quad \text{for all } |x| \geq n(\varepsilon).$$

Let $m = \max\{|p(t,x) - \alpha x| : |x| \leq n(\varepsilon)\}$. Then

$$|p(t,x) - \alpha x| < m + \varepsilon|x|, \quad \text{for all } x \in \mathbb{R}$$

and, therefore,

$$||N_1(x) - L_\infty(x)|| \leq m + \varepsilon||x||, \quad \text{for all } x \in L^2(0,1;\mathbb{R})$$

and

$$\limsup_{||x||\to\infty} \frac{||N_1(x) - L_\infty(x)||}{||x||} \leq \varepsilon.$$

Since ε is arbitrary, we see that N_1 is asymptotically linear with the asymptote L_∞.

By our assumption ii) and the boundedness of F, we see that there exist $r > 0, a \geq 0$ such that

$$F(t,x) \subset \begin{cases} [\beta_1 - \delta, \beta_2 + \delta]x, & \text{for } |x| > r, \\ (-a, a), & \text{for } |x| \leq r. \end{cases}$$

Write $A = \{x : |x| \leq r\}, B = \{x : |x| > r\}$ and use $\chi_A(\cdot)$ to stand for the characteristic function of set A. Then, for each $x \in X$, each $y \in S^1_{F(\cdot,x(\cdot))}$ can be decomposed as

$$y(t) = y_1(t) + y_2(t)x(t) \tag{3.27}$$

with

$$y_1(t) := \chi_A(t)y(t) \in [-a, a], \ y_2(t) := \chi_B(t)y(t)/x(t) \in [\beta_1 - \delta, \beta_2 + \delta]$$

and therefore

$$S^1_{F(\cdot,x(\cdot))} \subset X = L^2(0,1;\mathbb{R}). \tag{3.28}$$

We claim that $\hat{F}(x) := S^1_{F(\cdot,x(\cdot))}$ is demicontinuous from X to X. In fact, let $D \subset X$ be a weakly closed subset and $x_n \in \hat{F}^{-1}(D)$ with $x_n \to x_0$ in X. By passing to a subsequence, we may suppose $x_n(t) \to x_0(t)$ almost everywhere in $[0, 1]$. Then the upper semicontinuity of F implies

$$\limsup_{n\to\infty} F(t, x_n(t)) \subset F(t, x_0(t)) \ a.\ e..$$

$x_n \in \hat{F}^{-1}(D)$ implies that there exist $y_n \in \hat{F}(x_n) \cap D$. By the boundedness of F, we may suppose $y_n \rightharpoonup y_0 \in X$. Then $y_0 \in D$ and, by Theorem 1.3.4,

$$y_0 \in \text{co}\, S^1_{\limsup F(\cdot,x_n(\cdot))} \subset \text{co}\, S^1_{F(\cdot,x_0(\cdot))} = S^1_{F(\cdot,x_0(\cdot))}.$$

So $x_0 \in \hat{F}^{-1}(D)$ and, therefore, $\hat{F}$ is demicontinuous. By (3.27) and the compactness of $x \mapsto \int_0^1 k(\cdot,s)x(s)ds$, we see that N_2 maps bounded subsets into relatively compact subsets and is upper semicontinuous, that is, N_2 is compact. So $N_1 - N_2 \in (PM)$.

Next, we prove that N_2 is asymptotically linear with the asymptote N_∞. By Lemma 3.3.5, we need only verify condition iii) of Definition 3.2.15. Let $\varepsilon > 0$ be given. For each $x \in X$ and each $u \in N_2(x)$, there exists $f \in S^1_{F(\cdot,x(\cdot))}$ such that

$$u = \int_0^1 G(\cdot,s)f(s)ds.$$

By (3.27) and (3.28), $f = f_1 + f_2 x$ with $\|f_1(t)\| \le a$ and $f_2 \in [\beta_1 - \delta, \beta_2 - \delta]$. So $w := \displaystyle\int_0^1 G(\cdot,s)f_2(s)x(s)ds \in N_\infty(x)$ and

$$\|u - w\| = \left\|\int_0^1 G(\cdot,s)f_1(s)ds\right\| \le ab \text{ with } b := \left\|\int_0^1 G(\cdot,s)ds\right\|.$$

Therefore

$$\frac{\|u-w\|}{\|x\|} \le \varepsilon, \quad \text{for all } x \text{ with } \|x\| > ab/\varepsilon.$$

This proves that N_2 is asymptotically linear with the asymptote N_∞ and, therefore, $-N_2$ is asymptotically linear with the asymptote $-N_\infty$.

So, noting Remark 3.2.18 and Lemma 3.26, all the conditions made in Corollary 3.2.16 are satisfied and, therefore, $0 \in N_1(x) - N_2(x)$ or inclusion (3.25) admits solutions. □

3.3.3 Applications to controllability problems

Let $T > t_0$. Consider the controllability of the implicit nonlinear system with delay

$$\begin{aligned} &L(x(t),x'(t)) - A(t)x(t) = f(t,x(t),x(\delta(t)),u(t)) \quad t \in [t_0,T] \\ &x(t) = x_0 \text{ for } t \le t_0. \end{aligned} \tag{3.29}$$

Here $A(t)$ is a linear operator on a separable Hilbert space H for each t with the domain $\text{Dom}(A(t)) \equiv D$ a linear subspace of H, f is a nonlinear function from $[t_0,T] \times H^2$ to H, the delay δ is a continuous function from $[t_0,T]$ to $\mathbb{R}$ and L is an operator from H^2 to H.

Theorem 3.3.7. *Write $I = [t_0, T], I_\delta = I \cup \mathrm{range}(\delta)$. Suppose the following conditions are satisfied.*

(i) $\|f(t,x,y,u)\| \leq a_1 + a_2\|x\| + a_3\|y\| + a_4\|u\|$ for all $(t,x,y,u) \in I \times H^3$ with a_i $(i = 1,2,3,4)$ positive constants.

(ii) For all bounded subsets $D_1, D_2 \subset H$, there exist a constant $b_1 > 0$ and measurable functions $b_2, b_3 \in L^2(I, \mathbb{R})$ (each b_i may depend on D_1, D_2) such that:

$$\langle f(t,x,y,u), u\rangle \geq b_1\|u\|^2 - b_2(t)\|u\| - b_3(t), \ \textit{for all } t \in I, u \in H, x \in D_1, y \in D_2.$$

(iii) $A(\cdot)x \in L^2(I,H)$ for every $x \in D$.

(iv) $L(x(\cdot), x'(\cdot)) \in L^2(I,H)$ for each affine function $x : I_\delta \to D$.

(v) $u \mapsto f(t,x,y,u)$ is monotone for every (t,x,y).

Then $D \subset \mathbf{R}_T(x_0)$ for each $x_0 \in H$ and so system (3.29) is approximately (or exactly) controllable if $\overline{D} = H$ (resp. $D = H$).

Proof. Let $x_T \in D, x(t) = \frac{t-t_0}{T-t_0}(x_T - x_0) + x_0$ for $t \in I$ and $x(t) = x_0$ for $t \leq t_0$. Then $x(t) \in D$ for all $t \in I$ and , by our assumptions (iii) and (iv), $Ax, L(x,x') \in L^2(I,H)$. We define an operator N on $L^2(I,H)$ by

$$\begin{aligned} Nu(t) &= f(t,x(t),x(\delta(t)),u(t)) - L(x(t),x'(t)) + A(t)x(t) \\ &= f(t,x(t),x(\delta(t)),u(t)) - L(x(t),x'(t)) + \frac{t-t_0}{T-t_0}A(t)(x_T - x_0) - A(t)x_0. \end{aligned}$$

Then our assumptions imply that N maps $L^2(I,H)$ into $L^2(I,H)$ and is hemicontinuous since f is continuous in u. By our assumption (v), for all $u, v \in L^2(I,H)$, we have

$$\begin{aligned} \langle Nu - Nv, u - v\rangle_L &= \int_{t_0}^{T} \langle Nu(t) - Nv(t), u(t) - v(t)\rangle dt \\ &= \int_{t_0}^{T} \langle f(t,x(t),x(\delta(t)),u(t)) - f(t,x(t),x(\delta(t)),v(t)), u(t) - v(t)\rangle dt \\ &\geq 0, \end{aligned}$$

that is N is monotone. Here, $\langle\cdot,\cdot\rangle_L$ stands for the inner product in $L^2(I,H)$. The norm in $L^2(I,H)$ will be denoted by $\|\cdot\|_L$. From (ii) and (iii), it follows that

$$\begin{aligned}
\langle Nu,u\rangle_L &= \int_{t_0}^{T}\langle Nu(t),u(t)\rangle dt\\
&= \int_{t_0}^{T}\langle f(t,x(t),x(\delta(t)),u(t)),u(t)\rangle dt-\int_{t_0}^{T}\langle L(x(t),x'(t)),u(t)\rangle dt\\
&\quad+\int_{t_0}^{T}\langle\frac{1}{T-t_0}x_T+\frac{t-t_0}{T-t_0}A(t)(x_T-x_0)+A(t)x_0,u(t)\rangle dt\\
&\geq \int_{t_0}^{T}b_1\|u(t)\|^2dt-\int_{t_0}^{T}b_2(t)\|u(t)\|dt-\int_{t_0}^{T}b_3(t)dt-\|L(x,x')\|_L\|u\|_L\\
&\quad+\int_{t_0}^{T}\langle\frac{1}{T-t_0}x_T+\frac{t-t_0}{T-t_0}A(t)(x_T-x_0)+A(t)x_0,u(t)\rangle dt\\
&\geq b_1\|u\|_L^2-(\|b_2\|_L+\|L(x,x')\|_L)\|u\|_L-\int_{t_0}^{T}|b_3(t)|dt\\
&\quad-\left[\frac{1}{T-t_0}\|x_T\|_L+\|A(\cdot)(x_T-x_0)\|_L+\|A(\cdot)x_0\|_L\right]\|u\|_L. \qquad (3.30)
\end{aligned}$$

Let r be the root of the quadratic equation determined by the right hand of (3.30). Then we have

$$\langle Nu,u\rangle_L\geq 0,\quad \text{for all } u\in L^2(I,H) \text{ with } \|u\|_L=r.$$

By Corollary 3.2.12, $Nu=0$ has solutions in $r\overline{B}_{L^2(I,H)}$, that is there exists $u\in L^2(I,H)$ such that x is a solution of the corresponding equation (3.29). Obviously, $x(T)=x_T, x(t_0)=x_0$. This completes the proof. □

Remark 3.3.8. We can also suppose $u\mapsto -f(t,x,y,u)$ is monotone instead of (v) and replace (ii) by

(ii′) there exist a constant $b_1>0$ and measurable functions $b_2,b_3\in L^2(I,\mathbb{R})$ (each b_i may depend on D_1,D_2) such that:

$$\langle f(t,x,y,u),u\rangle\leq -b_1\|u\|^2+b_2(t)\|u\|+b_3(t),\ \text{for all } t\in I,u\in H,x\in D_1,y\in D_2.$$

Remark 3.3.9. Although Theorem 3.3.7 can be obtained from the corresponding surjectivity result of monotone operator (for example, Theorem 11.2 in [31]), our method can give the approximate position of the desired control u which is in $r\overline{B}_{L^2(I,H)}$. For the detail and other research on the controllability with preassigned responses, see [14].

Chapter 4

Solvability of First Order Nonlinear Evolution Inclusions

Let H be a Hilbert space, $V \subset H$ be a reflexive Banach subspace, $A : [0,T] \times V \to V^*$ be a nonlinear operator. Variants of boundary value problems need the solvability of the evolution equation (see Showalter [72] and Zeidler [80])

$$x'(t) + A(t, x(t)) = f(t) \quad \text{a.e.} \tag{4.1}$$

and the implicit equation (see Showalter [72])

$$\frac{\mathrm{d}}{\mathrm{d}\,t}(Bx(t)) + A(t, x(t)) = f(t) \quad \text{a.e.} \tag{4.2}$$

with $B \in \mathbf{L}(V, V^*)$. It is well known that if $A(\cdot, x)$ is measurable, $A(t, \cdot)$ is hemicontinuous, coercive and monotone with some growth conditions, then (4.1) admits a unique solution for any given function $f \in L^2(0,T; V^*)$ and initial value $x_0 \in V$ (see Theorem 1.7.5). If, in addition, B is positive and symmetric, then (4.2) admits solutions (see Corollary III 6.3 in [72]). For both theory and applications, we need to consider perturbation problems and seek other kinds of assumptions on A instead of the monotonicity condition. Many authors have contributed to these problems.

We first recall Hirano's work [43] in which a global existence result for the perturbation problem

$$x'(t) + A(x(t)) + G(x(t)) = f(t) \tag{4.3}$$

was given, but the assumptions made are strict. For example, it was supposed that

(Si) the embedding, $V \hookrightarrow H$, is compact;

(Sii) $G : V \to H$ is continuous and weakly continuous, and $\langle G(v), v \rangle \geq -c$.

Recently, (Sii) was removed (but the range of perturbation still needs to be in H) and more general problems considered by Migórski [55] who considered the global existence of the evolution inclusion

$$x'(t) + A(x(t)) \in F(x(t))$$

with F a set-valued mapping into H. Also, in Ahmed and Xiang [2], (Si) was dropped and the range of G was extended to V^*, but another strong assumption was imposed, namely

(Siii) $x_n \rightharpoonup x$ implies $(G(x_n), x_n - x) \to 0$.

For example, even the identify operator, $G(v) = v$ for all v, need not satisfy (Siii) (it is weakly continuous). A recent result was given by Berkovits and Mustonen in [11] who considered equation (4.1) in the case when A is only pseudo-monotone with some growth and coercivity conditions. We point out that the assumption of demicontinuity on A in [11] is not necessary because of Proposition 27.7 in [80]. In case the Nemytski operator corresponding to A is pseudo-monotone and coercive, Theorem 32.D gives the solvability of equation (4.1). But, we are not sure what assumptions on A can ensure the pseudo-monotonicity of the corresponding Nemytski operator.

For the implicit problem, some authors consider the case when both B and A are the subdifferentials of convex functions (see Barbu and Favini [9] and Colli and Visintin [29]), others consider the case when A is the sum of a maximal monotone mapping and a Lipschitz-like operator, and B is the composition of the injection of $V \hookrightarrow H$ (supposed to be compact) and the subdifferential of a time-dependent convex function (see Hokkanen [44] and [45]). In very recent works, Andrews etc.[4] and Barbu and Favini [8] considered the operator-equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(Bx(t)) + Ax(t) = f(t)$$

in an evolution triple of Hilbert spaces with B a linear, positive, and symmetric operator and A a monotone operator. The significance of these two papers is that the coercivity condition and strong monotonicity (for uniqueness) were made to $A+\lambda B$ ($\lambda > 0$) instead of A as is usual.

We note that (Siii) and the weak continuity imply that G is pseudo-monotone, monotone and hemicontinuous implies maximum monotone and, therefore, pseudo-monotone (see Proposition 8 of [25]). So, in this chapter, we will suppose $u \mapsto A(t,u)$ is a set-valued pseudo-monotone mapping and consider the more general evolution inclusion

$$x'(t) + A(t, x((t)) \ni f(t) \quad \text{a.e..} \tag{4.4}$$

in a general evolution triple and implicit inclusion

$$\frac{\mathrm{d}}{\mathrm{d}\,t}(Bx(t)) + A(t, x((t)) \ni f(t) \quad \text{a.e..} \tag{4.5}$$

in an evolution triple of Hilbert spaces, as well as their perturbation problems.

Global existence theorems to (4.4) and (4.5) will be given which generalizes the corresponding ones in [2], [4], [8], [11] and [43]. Continuity of solutions depending on the function f are also given. For the perturbation problem of (4.4), we suppose $A(t,\cdot)$ is also accretive as a mapping on V^* and the perturbation is a Lipschitz mapping. An existence result is obtained and, if the perturbation is single-valued, the solution is unique. For the perturbation problem of (4.5), we consider the case when (4.5) is perturbed by an u.s.c. and uniformly bounded (in $L^q(H)$) mapping. In this case, the embedding of V into H needs to be compact.

4.1 Pseudo-monotonicity of a mapping in functional spaces

In this chapter, we always suppose that (V, H, V^*) is an evolution triple, i.e. V is a real separable reflexive Banach space with dual V^*, H is a real separable Hilbert space such that $V \hookrightarrow H \hookrightarrow V^*$ densely and continuously. The inner product on H is denoted by $\langle\cdot,\cdot\rangle$. The duality between V and V^* and that between $L^p(0,T;V)$ and $L^q(0,T;V^*)$ are denoted by $(\cdot,\cdot)$, $(\!(\cdot,\cdot)\!)$ respectively. Here $T > 0, p \geq 2, q > 0$ are fixed real numbers and $1/p + 1/q = 1$. For $r > 1$ and a space X, the functional space $L^r(0,T;X)$ will be abbreviated to $L^r(X)$. For a set-valued mapping $A : X \to 2^{X^*}$, we denote by

$$\|Ax\| = \sup_{y \in Ax} \|y\|, \quad \text{for each } x \in X.$$

In this section, we consider the properties of the mapping $L\hat{A}L^*$ for a given mapping $A : [0,T] \times V \to 2^{V^*}$, where $\hat{A} : L^p(V) \to 2^{L^q(V^*)}$ is the Nemytski mapping corresponding to $A(\cdot, \cdot + x_0)$, that is

$$\hat{A}x = S^q_{A(\cdot,x(\cdot)+x_0)} := \{g \in L^q(V^*) : g(t) \in A(t, x(t) + x_0) \text{ a.e.}\} \quad \text{for } x \in L^p(V),$$

and L, L^* are the linear operators defined by

$$(Lf)(t) = \int_0^t f(s)ds, \quad \text{for each } f \in L^r(X), r \in (1,\infty),$$

$$(L^*f)(t) = \int_t^T f(s)ds, \text{ for each } f \in L^r(X), r \in (1,\infty)$$

related to the corresponding Banach space X. It is known that L^* is the adjoint operator of L, both L and L^* are linear, continuous, and positive (see Example 1.6.3).

We suppose $A : [0,T] \times V \to 2^{V^*}$ satisfies the following assumptions.

(H4.1.1) $(t,v) \mapsto A(t,v)$ is measurable with nonempty closed convex values.

(H4.1.2) There exist $a_1 \geq 0, a_2 \in L^q(\mathbb{R})$ such that

$$\|A(t,v)\|_{V^*} \leq a_1 \|v\|_V^{p-1} + a_2(t), \quad \text{for all } v \in V, t \in [0,T].$$

(H4.1.3) There exist $a_3, a_4 \geq 0, a_5 \in L^1(\mathbb{R})$ and $\alpha \in (0,p)$ such that

$$\inf_{u \in A(t,v)} (u,v) \geq a_3 \|v\|_V^p - a_4 \|v\|_V^\alpha - a_5(t), \quad \text{for all } v \in V, t \in [0,T].$$

(H4.1.4) $v \mapsto A(t,v)$ is pseudo-monotone for every $t \in [0,T]$.

Note, (H4.1.3) is equivalent to the existence of $c_3 > 0, c_4 \in L^1(\mathbb{R})$ such that

$$\inf_{u \in A(t,v)} (u,v) \geq c_3 \|v\|_V^p - c_4(t), \quad \text{for all } v \in V, t \in [0,T]. \tag{4.6}$$

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If there exists $t_0 \in [0,T]$ such that $\liminf_{n\to\infty} h_n(t_0) < 0$, then there exist n_i such that

$$\lim_{n_i\to\infty} (z_{n_i}(t_0), Lx_{n_i}(t_0) - Lx(t_0)) < 0.$$

By (H4.1.2) and (4.6), for all t, we have

$$\begin{aligned} h_n(t) &\geq c_3\|Lx_n(t)\|_V^p - c_4(t) - \|z_n(t)\|_{V^*}\|Lx(t)\|_V \\ &\geq c_3\|Lx_n(t)\|_V^p - a_1\|Lx_n(t)\|^{p-1}\|Lx(t)\|_V - a_2\|Lx(t)\|_V - c_4(t) \qquad (4.9) \\ &\geq -h(t), \qquad (4.10) \end{aligned}$$

where,

$$h(t) := h(t, Lx) = \begin{cases} a_2(t)\|Lx(t)\|_V + c_4(t), & \text{if } c_3\|Lx_n(t)\|_V \geq a_1\|Lx(t)\|_V, \\ (a_1^p/c_3^{p-1})\|Lx(t)\|_V^p & \\ \qquad +a_2(t)\|Lx(t)\|_V + c_4(t), & \text{if } c_3\|Lx_n(t)\|_V < a_1\|Lx(t)\|_V. \end{cases}$$

So $\{Lx_{n_j}(t_0)\}$ is bounded and, therefore, $Lx_{n_i}(t_0) \rightharpoonup Lx(t_0)$ (Remark 1.6.4). By the pseudo-monotonicity of A, we obtain

$$\lim_{n\to\infty} (z_{n_i}(t_0), Lx_{n_i}(t_0) - Lx(t_0)) = 0$$

which is a contradiction. Hence $\liminf_{n\to\infty} h_n(t) \geq 0$ for all $t \in [0,T]$, and therefore

$$\lim_{n\to\infty} h_n^-(t) = 0, \quad \text{for all} \quad t \in [0,T], \qquad (4.11)$$

where $h_n^-(t) = \max\{-h_n(t), 0\} = h_n^+(t) - h_n(t)$ with $h_n^+(t) = \max\{h_n(t), 0\} = |h_n(t)| - h_n^-(t)$ for all $t \in [0,T]$. By Fatou's Lemma,

$$\begin{aligned} 0 \leq \int_0^T \liminf_{n\to\infty} h_n(t)dt \leq \liminf_{n\to\infty} \int_0^T h_n(t)dt = \liminf_{n\to\infty}((z_n, Lx_n - Lx)) \\ \leq \limsup_{n\to\infty}((z_n, Lx_n - Lx)) \leq 0, \end{aligned}$$

which implies that

$$\lim_{n\to\infty} \int_0^T h_n(t)dt = \lim_{n\to\infty} ((z_n, Lx_n - Lx)) = 0. \qquad (4.12)$$

It is easy to see that $h \in L^1(\mathbb{R})$ and $0 \leq h_n^-(t) \leq h(t)$. By the Dominated Convergence Theorem and (4.11), $\lim_{n\to\infty} \int_0^T h_n^-(t)dt = 0$. Together with (4.12), we have

$$\lim_{n\to\infty} \int_0^T h_n^+(t)dt = \lim_{n\to\infty} \int_0^T [h_n(t) + h_n^-(t)]dt = 0,$$

so that

$$\lim_{n\to\infty} \int_0^T |h_n(t)|dt = \lim_{n\to\infty} \int_0^T [h_n^+ + h_n^-(t)]dt = 0.$$

Therefore, there exists a subsequence $\{h_{n_k}\}$ such that

$$h_{n_k}(t) = (z_{n_k}(t), Lx_{n_k}(t) - Lx(t)) \to 0 \quad \text{a.e..} \tag{4.13}$$

By the pseudo-monotonicity of A, for each $y \in L^p(V)$ and each $t \in [0,T]$ satisfying (4.13), there exists $w_t \in A(t, Lx(t))$ such that

$$(w_t, Lx(t) - Ly(t)) \leq \liminf_{k\to\infty}(z_{n_k}(t), Lx_{n_k}(t) - Ly(t)).$$

Let

$$\alpha(t) = \liminf_{k\to\infty}(z_{n_k}(t), Lx_{n_k}(t) - Ly(t)),$$

and

$$F(t) = \{w \in A(t, Lx(t)) : (w, Lx(t) - Ly(t)) \leq \alpha(t))\}.$$

Then, it is easy to see that $F(t)$ is nonempty closed convex and

$$\|F(t)\|_{V^*} \leq \|A(t, Lx(t))\|_{V^*} \leq a_1\|Lx(t)\|_V^{p-1} + a_2(t), \tag{4.14}$$

Since α is measurable, Lx, Ly are continuous, by (H4.1.1), F is measurable. So applying Theorem 1.3.3 on F, we find a measurable selector $z(t) \in F(t)$ a.e. with $z \in L^1(V^*)$. By (4.14), $z \in L^q(V^*)$, Therefore,

$$(z(t), Lx(t) - Ly(t)) \leq \alpha(t) = \liminf_{k\to\infty}(z_{n_k}(t), Lx_{n_k}(t) - Ly(t)) \quad \text{a.e..}$$

As we have proved above, $(z_{n_k}(t), Lx_{n_k}(t) - Ly(t)) \geq h(t, Ly)$ with $h(t, Ly)$ integrable, applying Fatou's Lemma, we have

$$\begin{aligned}((z, Lx - Ly)) &\leq \int_0^T \liminf_{k\to\infty}(z_{n_k}(t), Lx_{n_k}(t) - Ly(t))dt \\ &\leq \liminf_{k\to\infty} \int_0^T (z_{n_k}(t), Lx_{n_k}(t) - Ly(t))dt = \liminf_{k\to\infty}((z_{n_k}, Lx_{n_k} - Ly)).\end{aligned}$$

Therefore L^*z ($\in L^*\hat{A}Lx$) satisfies

$$((L^*z, x-y)) \le \liminf_{k\to\infty}((L^*z_{n_k}, x_{n_k}-y)),$$

which shows that $L^*\hat{A}L$ is pseudo-monotone because of Proposition 3.1.1. $\square$

Remark 4.1.2. In the above lemma, we do not impose any continuity on A. By Proposition 3.1.7, the explicit demicontinuity assumption in [11] is not necessary.

Corollary 4.1.3. *Under (H4.1.1)-(H4.1.3), suppose $v \mapsto A(t,v)$ is demicontinuous and quasi-monotone. Then $L^*\hat{A}L$ is quasi-monotone.*

Proof. Suppose $x_n \rightharpoonup x$ in $L^p(V)$ and $z_n \in L^*\hat{A}Lx_n$. Let $j : V \to V^*$ be the duality mapping with $\hat{j}$ the corresponding Nemytski operator. By Theorem 1.6.7 and Theorem 1.6.9, j is single-valued and of class (S_+). So, for each $\varepsilon > 0$, $v \mapsto \varepsilon j(v) + A(t,v)$ is pseudo-monotone and, therefore, quasi-monotone due to Propositions 3.1.2 and 3.1.3. So $\limsup_{n\to\infty}((\varepsilon L^*\hat{j}Lx_n + z_n, x_n - x)) \ge 0$. Since

$$\varepsilon \limsup_{n\to\infty}((\hat{j}Lx_n, Lx_n - Lx)) + \limsup_{n\to\infty}((z_n, x_n - x)) \ge \limsup_{n\to\infty}((\varepsilon L^*\hat{j}Lx_n + z_n, x_n - x))$$

and ε is arbitrary, we see $\limsup_{n\to\infty}((z_n, x_n - x)) \ge 0$ and, therefore, $L^*\hat{A}L$ is quasi-monotone. $\square$

The following proposition is a generalization of Lemma A of [2] and the corresponding one in [11].

Proposition 4.1.4. *Under (H4.1.1)- (H4.1.4), if x_n, x are functions from $[0,T]$ into V, , $z_n \in \hat{A}Lx_n$ with $Lx_n \rightharpoonup Lx$ in $L^p(V), x_n \rightharpoonup x$ in $L^q(V^*)$ and* $\limsup((z_n, Lx_n - Lx)) \le 0$*, then, there exists $z \in \hat{A}Lx$ and subsequence $\{z_{n_j}\}$ such that $z_{n_j} \rightharpoonup z$, $((z_{n_j}, Lx_{n_j})) \to ((z, Lx))$.*

Proof. Noting Remark 1.6.4 and using almost the same method as we have used in proposition 4.1.1, we see that there exists a subsequence n_j such that for each $y \in L^p(V)$, there exists $z(y) \in \hat{A}Lx$ such that

$$((z(y), Lx - y)) \le \liminf_{j\to\infty}((z_{n_j}, Lx_{n_j} - y)).$$

By taking $y = Lx$, we obtain

$$\liminf_{j\to\infty} ((z_{n_j}, Lx_{n_j} - Lx)) = 0.$$

So $\{z_{n_j}\}$ is bounded in $L^q(V^*)$. We may assume that $z_{n_j} \rightharpoonup z_0$ in $L^q(V^*)$. Hence, for each $y \in L^p(V)$, there exists $z(y) \in \hat{A}Lx$ such that

$$((z(y), Lx - y)) \le ((z_0, Lx - y)).$$

By Lemma 3.1.4, there exists $z \in \hat{A}Lx$ such that

$$((z, Lx - y)) \le ((z_0, Lx - y)) \quad \text{for all } y \in L^p(V),$$

that is $z = z_0$ and, therefore, $z_{n_j} \rightharpoonup z$ in $L^q(V^*)$. □

4.2 Solutions for explicit problems

In this section, under (H4.1.1)–(H4.1.4), we consider the existence and properties of solutions to the explicit evolution inclusion

$$x'(t) + A(t, x(t)) \ni f(t) \quad \text{a.e.} \tag{4.15}$$

$$x(0) = x_0 \in V \tag{4.16}$$

with $f \in L^q(V^*)$.

A function $x \in L^p(V)$ is said to a *solution* of (4.15)-(4.16) if x is differentiable a.e. in the vector distribution sense, $x' \in L^q(V^*)$ and $f(t) - x'(t) \in A(t, x(t))$ a.e..

It is known that problem (4.15)-(4.16) is equivalent to

$$L^*x + L^*\hat{A}Lx \ni L^*f.$$

In fact, x is a solution of the above inclusion if and only if $y = Lx + x_0$ is a solution of (4.15)–(4.16).

First, we give an existence result.

Theorem 4.2.1. *Under (H4.1.1)– (H4.1.4), problem* (4.15)–(4.16) *admits at least one solution x satisfying*

$$\|x'\|_{L^q(V^*)} \le \hat{a}_1 \max\Big\{1, \Big(\frac{\hat{a}_4 + \|\hat{a}_5\| + \|f\|}{\hat{a}_3}\Big)^{\frac{p-1}{p-r}}\Big\} + \|\hat{a}_2\| + \|f\|,$$

$$\|x\|_{L^q(V^*)} \le \|L\| \left[\hat{a}_1 \max\left\{1, \left(\frac{\hat{a}_4 + \|\hat{a}_5\| + \|f\|}{\hat{a}_3}\right)^{\frac{p-1}{p-r}}\right\} + \|\hat{a}_2\| + \|f\|\right] + \|x_0\| T^{1/q}$$

with $\|\hat{a}_5\| := \|\hat{a}_5\|_{L^1(\mathbb{R})}, \|\hat{a}_2\| := \|\hat{a}_2\|_{L^q(\mathbb{R})}, \|f\| := \|f\|_{L^q(V^*)}$.

Proof. Let $J : L^q(V^*) \to L^p(V)$ be the duality mapping. By Theorem 1.6.7 and Theorem 1.6.9, we may suppose J is single valued monotone and demicontinuous.

Suppose $\varepsilon > 0$ is a constant, and let $D_n = \{x \in L^p(V) : \|x\|_{L^p(V)} \le n\}$ for each $n > 0$. Consider the approximating inclusion

$$L^* f \in \varepsilon J x + L^* x + L^* \hat{A} L x. \tag{4.17}$$

Since L^* is continuous and positive, $\varepsilon J + L^* - L^* f$ is pseudo-monotone. By Proposition 4.1.1, $L^* \hat{A} L$ is pseudo-monotone and, therefore, by Proposition 3.1.6, $\varepsilon J + L^* - L^* f + L^* \hat{A} L$ is pseudo-monotone. By (4.7) and the boundedness of J, L and L^*, we see that $\varepsilon J + L^* - L^* f + L^* \hat{A} L$ is bounded and, therefore, finitely continuous according to Proposition 3.1.7. So, by Theorem 1.6.5 , there exist $x_n \in D_n, z_n \in \hat{A} L x_n$ such that

$$((\varepsilon J x_n + L^* x_n - L^* f + L^* z_n, x - x_n)) \ge 0 \quad \text{for all } x \in D_n. \tag{4.18}$$

Letting $x = 0$ in (4.18), we have

$$((\varepsilon J x_n + L^* x_n - L^* f + L^* z_n, x_n)) \le 0.$$

Noting the fact $L^p(V) \subset L^q(V^*)$), (4.8) and the coercivity of J, we see that

$$\begin{aligned} ((\varepsilon J x_n + L^* x_n - L^* f + L^* z_n, x_n)) \ge \;& \varepsilon \|x_n\|^2_{L^q(V^*)} + \hat{a}_3 \|Lx\|^p_{L^p(V)} - \|\hat{a}_5\|_{L^1(\mathbb{R})} \\ & - \hat{a}_4 \|Lx\|^r_{L^p(V)} - ((f, Lx_n)). \end{aligned} \tag{4.19}$$

So both $\{\|x_n\|_{L^q(V^*)}\}$ and $\{\|Lx_n\|_{L^p(V)}\}$ are bounded. We may suppose that $x_n \rightharpoonup \hat{x}$ in $L^q(V^*), Lx_n \rightharpoonup L\hat{x}$ in $L^p(V)$. By taking $x = \hat{x}$ in (4.18), we obtain:

$$((\varepsilon J x_n + L^* x_n - L^* f + L^* z_n, x_n - \hat{x})) \le 0. \tag{4.20}$$

Noting the fact that

$$\liminf_{n \to \infty} ((L^* x_n, x_n - \hat{x})) \ge 0, \quad \liminf_{n \to \infty} ((J x_n, x_n - \hat{x})) \ge 0, \tag{4.21}$$

we have

$$\begin{aligned}\limsup_{n\to\infty}((z_n, Lx_n - L\hat{x})) &= \limsup_{n\to\infty}((L^*z_n, x_n - \hat{x}))\\ &\leq \limsup_{n\to\infty}((L^*x_n - L^*f + \varepsilon Jx_n + L^*z_n, x_n - \hat{x}))\\ &\quad - \liminf_{n\to\infty}((L^*x_n - L^*f + \varepsilon Jx_n, x_n - \hat{x})) \leq 0.\end{aligned}$$

From Proposition 4.1.4, there exists a subsequence $z_{n_j} \rightharpoonup z_0 \in \hat{A}L\hat{x}$ such that

$$\lim_{j\to\infty}((L^*z_{n_j}, x_{n_j})) = ((L^*z_0, \hat{x})).$$

Also, from (4.20) and the pseudo-monotonicity of $\varepsilon J + L^* - L^*f + L^*\hat{A}L$, we may suppose that $\varepsilon Jx_{n_j} + L^*x_{n_j} - L^*f + L^*z_{n_j} \rightharpoonup \varepsilon J\hat{x} + L^*\hat{x} - L^*f + L^*z_0$. By (4.18) and the positivity of L^* and J, for each $x \in D_{n_j}$, we have

$$\begin{aligned}((L^*x_{n_j} - L^*f + \varepsilon Jx_{n_j} + L^*z_{n_j}, x)) &\geq ((L^*x_{n_j} - L^*f + \varepsilon Jx_{n_j} + L^*z_{n_j}, x_{n_j}))\\ &\geq ((L^*z_{n_j} - L^*f, x_{n_j})).\end{aligned}$$

So, we obtain

$$((\varepsilon J\hat{x} + L^*\hat{x} - L^*f + L^*z_0, x)) \geq ((L^*z_0 - L^*f, \hat{x})), \quad \text{for all } \ x \in L^p(V).$$

Let $u \in L^p(V)$ be such that $\|u\|_{L^p(V)} = 1$ and

$$((\varepsilon J\hat{x} + L^*\hat{x} - L^*f + L^*z_0, u)) = \|\varepsilon J\hat{x} + L^*\hat{x} - L^*f + L^*z_0\|_{L^q(V^*)}.$$

Take $x = \hat{x} - \delta u$ with δ real in the last inequality. We obtain

$$((L^*\hat{x} + \varepsilon J\hat{x}, \hat{x})) \geq \delta((\varepsilon J\hat{x} + L^*\hat{x} - L^*f + L^*z_0, u)) = \delta\|\varepsilon J\hat{x} + L^*\hat{x} - L^*f + L^*z_0\|_{L^q(V^*)}.$$

Since δ is arbitrary, we obtain $\varepsilon J\hat{x} + L^*\hat{x} - L^*f + L^*z_0 = 0$, that is (4.17) admits solutions for each $\varepsilon > 0$.

Now, let $\varepsilon = 1/n$ and $x_n \in L^p(V)$ be a corresponding solution of problem (4.17), i.e.

$$\frac{1}{n}Jx_n + L^*x_n + L^*z_n = L^*f \ \text{ with } z_n \in \hat{A}Lx_n \ \ n \geq 1. \tag{4.22}$$

This is equivalent to

$$\frac{1}{n}(L^*)^{-1}Jx_n + z_n - f = -x_n \ \text{ with } \ z_n \in \hat{A}Lx_n, \ \ n \geq 1 \tag{4.23}$$

since L^* is injective. Scalar multiplying (4.22) by x_n, and using (4.19), we have

$$\begin{aligned}\|f\|_{L^q(V^*)}\|Lx_n\|_{L^p(V)} \geq ((L^*f, x_n)) &= (((1/n)Jx_n + L^*x_n + L^*z_n, x_n)) \\ &\geq \frac{1}{n}\|x_n\|^2_{L^q(V^*)} + \hat{a}_3\|Lx_n\|^p_{L^p(V)} - \hat{a}_4\|Lx_n\|^r_{L^p(V)} - \|\hat{a}_5\|.\end{aligned}$$

So we have

$$\|Lx_n\|_{L^p(V)} \leq \max\Big\{1, \Big(\frac{\hat{a}_4 + \|\hat{a}_5\| + \|f\|}{\hat{a}_3}\Big)^{1/(p-r)}\Big\},$$

which implies that $\{Lx_n\}$ is bounded in $L^p(V)$. Scalar multiplying (4.23) by Jx_n, using the fact that $(((L^*)^{-1}Jx_n, Jx_n)) \geq 0$ for each n, and by (4.7), we have

$$\begin{aligned}\|x_n\|^2_{L^q(V^*)} &\leq -((z_n - f, Jx_n)) \leq (\|\hat{A}Lx_n\|_{L^q(V^*)} + \|f\|)\|x_n\|_{L^p(V)} \\ &\leq \left(\hat{a}_1\|Lx_n\|^{p-1}_{L^p(V)} + \|\hat{a}_2\| + \|f\|\right)\|x_n\|_{L^p(V)}.\end{aligned}$$

This gives

$$\|x_n\|_{L^q(V^*)} \leq a_1 \max\Big\{1, \Big(\frac{\hat{a}_4 + \|\hat{a}_5\|}{\hat{a}_3 + \|f\|}\Big)^{(p-1)/(p-r)}\Big\} + \|\hat{a}_2\| + \|f\|$$

and, therefore, $\{x_n\}$ is bounded in $L^q(V^*)$, $\{Lx_n\}$ is bounded in $L^p(V)$. We may suppose, by passing to subsequences if necessary, that $x_n \rightharpoonup x$ in $L^q(V^*)$ and $Lx_n \rightharpoonup Lx$ in $L^p(V)$. By (4.21) and (4.22), we see that

$$\limsup_{n\to\infty}((z_n, Lx_n - Lx)) = -\liminf_{n\to\infty}\big[(1/n)((Jx_n, Lx_n - Lx)) + ((L^*x_n, x_n - x))\big] \leq 0.$$

So, by Proposition 4.1.4, we may suppose $z_n \rightharpoonup z_0$ for some $z_0 \in \hat{A}Lx$. By the boundedness of J, letting $n \to \infty$ in (4.22), we obtain

$$L^*x + L^*z_0 = L^*f \quad \text{or} \quad L^*f \in L^*x + L^*\hat{A}Lx.$$

Hence, $y = Lx + x_0 \in L^p(V)$ is a solution of (4.15)–(4.16). Obviously

$$\|y'\|_{L^q(V^*)} = \|x\|_{L^q(V^*)} \leq \hat{a}_1 \max\Big\{1, \Big(\frac{\hat{a}_4 + \|\hat{a}_5\| + \|f\|}{\hat{a}_3}\Big)^{\frac{p-1}{p-r}}\Big\} + \|\hat{a}_2\| + \|f\|,$$

$$\|y\|_{L^q(V^*)} \leq \|L\|\left[\hat{a}_1 \max\Big\{1, \Big(\frac{\hat{a}_4 + \|\hat{a}_5\| + \|f\|}{\hat{a}_3}\Big)^{\frac{p-1}{p-r}}\Big\} + \|\hat{a}_2\| + \|f\|\right] + \|x_0\|T^{1/q}.$$

□

Remark 4.2.2. Theorem 4.2.1 generalizes the results in [2], [11] and [43]. If A is single valued, we obtain the corresponding result of [11] (note we do not impose demicontinuity); If $A(t,x) = A(t)x + G(x)$ with $A(t)$ being single valued monotone hemicontinuous and $G : V \to V^*$ coercive, weakly continuous and satisfying Siii), we obtain the theorem in [2]; If, alternatively, G satisfies Si) and Sii), we obtain the conclusion of [43]. Since, weak continuity and Si) or Siii) are sufficient to imply the pseudo-monotonicity, the strong continuity assumption in [2] or [43] is not necessary.

By Theorem 1.4.10 , we see that the solution x of (4.15)–(4.16) satisfies the following regularity conditions

$$x \in C(0,T;H), dx/dt \in L^q(V^*).$$

Corollary 4.2.3. *Under (H4.1.1)–(H4.1.3), suppose $v \mapsto A(t,v)$ is quasi-monotone and demicontinuous for each $t \in [0,T]$. Then problem (4.15)–(4.16) is almost solvable in the sense $f \in \overline{\mathrm{range}(L^* + L^*\hat{A}L)}$. Precisely, if we denote by j the duality operator from V to V^*, then for each n, there exists $x_n \in L^p(V)$ with $x(0) = x_0$ such that*

$$x_n'(t) + A(t, x_n(t)) \ni -\frac{1}{n} j(x_n(t)) + f(t), \text{ a.e.} \tag{4.24}$$

and $j(x_n)/n \to 0$ uniformly.

Proof. For each n, define a mapping $A_n : [0,T] \times V \to V^*$ by

$$A_n(t,v) = \frac{1}{n} j(v) + A(t,v), \quad \text{for } t \in [0,T], v \in V.$$

Since j is single-valued, of class (S_+) and demicontinuous, A_n satisfies (H4.1.1) and (H4.1.4). Moreover, it is easy to see that

$$\|A_n(t,v)\|_{V^*} \le \big(1 + a_1\big)\|v\|_V^{p-1} + 1 + a_2(t)$$

$$\inf_{u \in A_n(t,v)} (u,v) \ge a_3\|v\|_V^p - a_4\|v\|_V^\alpha - a_5(t)$$

for all $v \in V, t \in [0,T]$ and each $n > 0$. Applying Theorem 4.2.1, there exists $x_n \in L^p(V)$ satisfying (4.24) for each $n > 0$ and $\|x_n\|_{L^q(V^*)} \le C$ for some C independent of n. Since $\|j(x_n(t))\|_{V^*} = \|x_n(t)\|_V$, we see that $\{j(x_n)\}$ is bounded in $L^q(V^*)$ and, therefore, $j(x_n)/n \to 0$ in $L^q(V^*)$. □

Corollary 4.2.4. *Under (H4.1.1)-(H4.1.4), suppose* $C : [0,T] \times V \to 2^{V^*}$ *is measurable with nonempty closed convex values and such that*

(H4.2.1) for all $t \in [0,T], v \mapsto C(t,v)$ *is compact and upper semicontinuous from* V_w *to* V_w^* *as a set-valued mapping;*

(H4.2.2) there exist $d_1 \geq 0, d_2 \in L^q(\mathbb{R}), d_4 \in L^1(\mathbb{R})$ *and either* $d_3 \geq 0, r \in [0,p)$ *or* $d_3 \in [0,a_3)$, $r = p$ *such that*

$$\|C(t,v)\|_{V^*} \leq d_1\|v\|_V^{p-1} + d_2(t), \quad \textit{for all } v \in V, t \in [0,T],$$

$$\inf_{u \in C(t,v)} (u,v) \geq -d_3\|v\|_V^r - d_4(t), \quad \textit{for all } v \in V, t \in [0,T].$$

Then the inclusion

$$x'(t) + A(t,x(t)) + C(t,x(t)) \ni f(t) \text{ a.e.} x(0) = x_0 \in V \tag{4.25}$$

admits solutions for each $f \in L^q(V^*)$.

Proof. Let $t \in [0,T], v_n \rightharpoonup v$ in $V, w_n \in C(t,v_n)$ and $\limsup(w_n, v_n - v) \leq 0$. By (H4.2.1), there exist a subsequence $\{w_{n_k}\}$ and $w \in C(t,v), w_{n_k} \to w$. So for every $u \in V$, we have

$$\liminf_{k\to\infty}(w_{n_k}, v_{n_k} - u) = \liminf_{k\to\infty}[(w_{n_k} - w, v_{n_k} - u) + (w, v_{n_k} - u)] = (w, v-u).$$

By Proposition 3.1.1, $C(t,\cdot)$ is pseudo-monotone. And therefore, $A(t,\cdot)+C(t,\cdot)$ is pseudo-monotone. It is easy to see that $A+C$ satisfies (H4.1.1), (H4.1.2) and (H4.1.3) with new constants and functions. Hence, by Theorem 4.2.1, (4.25) admits solutions. □

Remark 4.2.5. If $v \mapsto A(t,v)$ is of class (S_+) and demicontinuous, then $v \mapsto C(t,v)$ need only be demicontinuous and quasi-monotone, because in this case $v \mapsto A(t,v) + C(t,v)$ is still pseudo-monotone.

Now, we give some properties of the solution set

$$S(f) = \{x_f \in L^p(V) : x_f \text{ is a solution of } (4.15) - (4.16)\}$$

in the space $W(0,T)$. Recall $W(0,T) = \{x \in L^p(V) : x' \in L^q(V^*)\}$.

Theorem 4.2.6. *Under (H4.1.1)-(H4.1.4),* $S(f)$ *is a bounded, weakly closed subset in* $W(0,T)$. *If, in addition,* $V \hookrightarrow H$ *compactly, then the mapping* $f \mapsto S(f)$ *is upper semicontinuous as a set-valued mapping from* $L^q(H)_w$ *to both* $W(0,T)_w$ *and* $L^p(H)$.

Since a similar result will be given in next section for the implicit problems (Theorem 4.3.5) and the proofs are almost the same, we omit the proof here. One can obtain it just replacing the operator B in Theorem 4.3.5 by the identity operator although the triple in Theorem 4.3.5 is of Hilbert spaces, which is not necessary for the explicit problems.

Remark 4.2.7. Suppose $f \mapsto S(f)$ is single-valued and $F : [0,T] \times H \to 2^H$ is a measurable set-valued mapping. If the mapping $f \mapsto S^q_{F(\cdot,S(f))}$ has a fixed point f in $L^q(H)$, then $x = S(f)$ is a solution to the perturbation problem

$$x'(t) + A(t, x(t)) - F(t, x(t)) \ni 0, \text{ a.e. with } x(0) = x_0.$$

Using the same method as used in Theorem 1 of [55], we can prove that a sufficient condition for mapping $f \mapsto S^q_{F(\cdot,S(f))}$ to have fixed points is that $\text{Graph}(F(t,\cdot))$ is sequentially closed in $H \times H_w$ and $\|F(t,u)\|_H \le a(t) + b\|u\|^{2/q}$ a.e. with $a \in L^q(\mathbb{R}), b > 0$.

Theorem 4.2.8. *Under (H4.1.1)-(H4.1.4), suppose $A(t,\cdot)$ is monotone and $V \hookrightarrow H$ compactly, then the mapping $f \mapsto S(f)$ is single-valued and continuous from $L^q(H)_w$ to $C(0,T;H)$, monotone on $L^q(V^*)$ and*

$$\|S(f) - S(g)\|_H \le \int_0^t \|f(s) - g(s)\|_H ds \quad \text{for all } f, g \in L^q(H). \tag{4.26}$$

Proof. From Theorem 1.7.5, $f \mapsto S(f)$ is single-valued.

Let $f_n \rightharpoonup f$ in $L^q(H)$, $x_n = x_{f_n}, x = x_f$. Then $\{f_n\}$ is bounded and, therefore $\{x_n\}$ is bounded in $W(0,T)$. So, by passing to a subsequence, we may assume that $x_n \rightharpoonup y$ in $W(0,T)$. Clearly, there exist $z_n \in S^q_{A(\cdot,x_n(\cdot))}, z \in S^q_{A(\cdot,x(\cdot))}$ such that, for a.e. t

$$(x_n'(t) - x'(t), x_n(t) - x(t)) + (z_n(t) - z(t), x_n(t) - x(t)) = \langle f_n(t) - f(t), x_n(t) - x(t)\rangle.$$

Since $A(t,\cdot)$ is monotone, $2(x_n'(t) - x'(t), x_n(t) - x(t)) = \frac{\mathrm{d}}{\mathrm{d}t}\|x_n(t) - x(t)\|_H^2$, and $x_n(0) = x(0) = x_0$, we have

$$\frac{1}{2}\|x_n(t) - x(t)\|_H^2 \le \int_0^t \langle f_n(s) - f(s), x_n(s) - x(s)\rangle ds \tag{4.27}$$

$$\le \int_0^T |\langle f_n(s) - f(s), x_n(s) - x(s)\rangle| ds. \tag{4.28}$$

Since $W(0,T) \hookrightarrow L^p(H)$ compactly, we may suppose that $x_n \to y$ strongly in $L^p(H)$. If we let $\chi(s) = 1$ for $s \le t$ and $\chi(s) = 0$ for $s > t$, then

$$\int_0^t \langle f_n(s) - f(s), y(s) - x(s)\rangle ds = \int_0^T \langle f_n(s) - f(s), \chi(s)(y(s) - x(s))\rangle ds \to 0. \quad (4.29)$$

By (4.27), we see that $\|x_n(t) - x(t)\|_H \to 0$ for each t. Since $\{x_n\}$ is bounded in $C(0,T;H)$, $x_n \to x$ in $L^p(H)$ and therefore, by (4.28), we see $x_n(t) \to x(t)$ uniformly. This shows that $f \mapsto x_f$ is continuous from $L^q(H)_w$ into $C(0,T;H)$.

It is easy to see that (4.27) is valid if $f_n, f \in L^q(V^*)$. Replacing f_n by $g \in L^q(V^*)$ and letting $t = T$, we obtain that $((x_f - x_g, f - g)) \ge \|x_f(T) - x_g(T)\|_H^2/2 \ge 0$. This proves the monotonicity of $S(\cdot)$.

Again, by (4.27), we have

$$\frac{1}{2}\|x_n - x(t)\|_H^2 \le \int_0^t \|f_n(s) - f(s)\|_H \|x_n(s) - x(s)\|_H ds.$$

Replacing f_n by $g \in L^q(H)$ and using Theorem 1.7.7, we obtain (4.26). □

Remark 4.2.9. The continuity of $f \mapsto S(f)$ is also claimed in [55] where the method is different from ours.

4.3 Solutions for implicit problems

In this section, under (H4.1.1)–(H4.1.4), we suppose both V and H in the evolution triple (V, H, V^*) are Hilbert spaces to study the solvability of the implicit evolution inclusion

$$\frac{\mathrm{d}}{\mathrm{d}t}(Bx(t)) + A(t, x(t)) \ni f(t) \text{ a.e.} \quad (4.30)$$

$$Bx(0) = Bx_0 \text{ with } x_0 \in V. \quad (4.31)$$

Here $B \in \mathbf{L}(V, V^*)$ and is always assumed to be positive and symmetric.

Denote by $\langle \cdot, \cdot \rangle_V$ the inner product on V. Then, for each $u \in V$, a unique element $\Lambda u \in V^*$ is determined by

$$(\Lambda u, v) = \langle u, v \rangle_V, \text{ for all } v \in V.$$

We call $u \mapsto \Lambda u$ the *canonical operator* of V. Obviously, Λ is a bounded, linear, injective and symmetric operator from V to V^* and

$$(\Lambda u, u) = \|u\|_V^2 \text{ for all } u \in V.$$

By Corollary 3.2.13, we see that $\varepsilon\Lambda + B$ is onto V^* for each $\varepsilon > 0$. So $\varepsilon\Lambda + B$ is an isomorphism from V to V^*. It is known that, for all $u, v \in V^*$, the bilinear form

$$\langle u, v\rangle_W := \big((\varepsilon\Lambda + B)^{-1}u, v\big)$$

is an inner product on V^* and the space $W := (V^*; \langle\cdot,\cdot\rangle_W)$ is a Hilbert space. We denote the norm in W by $\|\cdot\|_W$ $\big($i.e. $\|u\|_W = \sqrt{\langle u, u\rangle_W}\big)$.

Proposition 4.3.1. *(i)* $k := \inf\limits_{\varepsilon>0} \inf\limits_{v\in V^*\setminus\{0\}} \dfrac{\|(\varepsilon\Lambda + B)^{-1}v\|_V}{\|v\|_{V^*}} > 0.$

(ii) $\|(\varepsilon\Lambda + B)^{-1}\|^{-1/2}\|v\|_W \le \|v\|_{V^*} \le \|\varepsilon\Lambda + B\|^{1/2}\|v\|_W$ *for each* $v \in V^*$. *Here the norm of linear operators are taken in* $\mathbf{L}(V, V^*)$ *or* $\mathbf{L}(V^*, V)$.

Proof. (i) If $k = 0$, then there exists sequences $\{v_n\} \in V^*$ and $\{\varepsilon_n\}$ such that

$$\|v_n\|_{V^*} = 1, \quad \varepsilon_n \to 0, \quad \|(\varepsilon_n\Lambda + B)^{-1}v_n\|_V \to 0.$$

Writing $u_n = (\varepsilon_n\Lambda + B)^{-1}v_n$, we have

$$1 = \|v_n\|_{V^*} = \|(\varepsilon_n\Lambda + B)u_n\|_{V^*} \le (\varepsilon_n\|\Lambda\| + \|B\|)\|u_n\|_V \to 0$$

which is a contradiction.

(ii) Let $v \in V^*$. Then

$$\|v\|_W^2 = \big((\varepsilon\Lambda + B)^{-1}v, v\big) \le \|(\varepsilon\Lambda + B)^{-1}v\|_V\|v\|_{V^*} \le \|(\varepsilon\Lambda + B)^{-1}\|\|v\|_{V^*}^2, \tag{4.32}$$

which implies the first part of our inequalities. Also, there exists $u \in V, \|u\|_V = 1$ such that $\|v\|_{V^*} = (u, v)$. Write $z = (\varepsilon\Lambda + B)u \in V^*$. Using (4.32), we have

$$\begin{aligned}\|v\|_{V^*} &= ((\varepsilon\Lambda + B)^{-1}z, v) = \langle z, v\rangle_W \le \|v\|_W\|z\|_W \\ &\le \|v\|_W\|(\varepsilon\Lambda + B)^{-1}\|^{1/2}\|z\|_{V^*} \\ &\le \|v\|_W\|(\varepsilon\Lambda + B)^{-1}\|^{1/2}\|\varepsilon\Lambda + B\|\|u\|_V.\end{aligned}$$

Since $\|u\|_V = 1$ and $\|(\varepsilon\Lambda + B)^{-1}\| \le \|\varepsilon\Lambda + B\|^{-1}$, we have $\|v\|_{V^*} \le \|\varepsilon\Lambda + B\|^{1/2}\|v\|_W$.

This completes the proof. □

Theorem 4.3.2. *Under (H4.1.1)–(H4.1.4), suppose $B \in \mathrm{L}(V, V^*)$ is positive and symmetric. Then, for each $f \in L^q(V^*)$, problem* (4.30)-(4.31) *has at least one solution $x \in L^p(V)$ with $x' \in L^q(V^*)$ and there exist constant $k_1, k_2 > 0$ such that*

$$\|x\|_{L^p(V)}, \|x'\|_{L^q(V^*)} \leq k_1 + k_2 \|f\|^s_{L^q(V^*)} \text{ with } s = \max\{(p-1)/(p-\alpha), p\}. \tag{4.33}$$

Proof. Let $\Lambda : V \to V^*$ be the canonical isomorphism of V. Let $\varepsilon \in \big(0, \|B\|/(2\|\Lambda\|)\big)$ and write

$$A_\varepsilon(t, v) := A(t, (\varepsilon\Lambda + B)^{-1} v), \text{ for all } t \in [0, T], v \in W.$$

By Proposition 4.3.1 (ii), we see that $A_\varepsilon : [0, T] \times W \to 2^W$ is a well-defined measurable mapping with closed convex values. We claim that $A_\varepsilon(t, \cdot)$, as a mapping on W, is still pseudo-monotone, coercive and has $(p-1)$-growth.

Suppose $v_n \rightharpoonup v$ in W, $w_n \in A_\varepsilon(t, v_n)$ and $\limsup_{n\to\infty} \langle w_n, v_n - v\rangle_W \leq 0$. Let $x_n = (\varepsilon\Lambda + B)^{-1} v_n, x = (\varepsilon\Lambda + B)^{-1} v$. Then $w_n \in A(t, x_n), x_n \rightharpoonup x$ in V and

$$\begin{aligned} 0 \geq \limsup_{n\to\infty} \langle w_n, v_n - v\rangle_W &= \limsup_{n\to\infty} (w_n, (\varepsilon\Lambda + B)^{-1}(v_n - v)) \\ &= \limsup_{n\to\infty} (w_n, x_n - x). \end{aligned}$$

Since $A(t, \cdot)$ is pseudo-monotone, there exists $w(y) \in A(t, x)$ for each $y \in V^*$ such that

$$\begin{aligned} \langle w(y), v - y\rangle_W = (w(y), x - (\varepsilon\Lambda + B)^{-1} y) &\leq \liminf_{n\to\infty} (w_n, x_n - (\varepsilon\Lambda + B)^{-1} y) \\ &= \liminf_{n\to\infty} \langle w_n, v_n - y\rangle_W. \end{aligned}$$

This means that $A_\varepsilon(t, \cdot)$ is pseudo-monotone on W.

To verify the coercivity and the growth condition, we suppose $v \in W$ and let $y \in A(t, (\varepsilon\Lambda + B)^{-1} v)$. Then

$$\|y\|^2_W = \langle y, y\rangle_W = ((\varepsilon\Lambda + B)^{-1} y, y) \leq \|(\varepsilon\Lambda + B)^{-1}\| \|y\|^2_{V^*}.$$

Since

$$2\|B\| \geq \|\varepsilon\Lambda + B\| \geq \|B\| - \varepsilon\|\Lambda\| \geq \|B\|/2 \text{ and } \|\varepsilon\Lambda + B\| \|(\varepsilon\Lambda + B)^{-1}\| \leq 1, \tag{4.34}$$

by (H4.1.2) and Proposition 4.3.1 (ii), we obtain

$$\begin{aligned} \|y\|_W &\le a_1 \|(\varepsilon\Lambda + B)^{-1}\|^{1/2} \|(\varepsilon\Lambda + B)^{-1})v\|_{V^*}^{p-1} + \|(\varepsilon\Lambda + B)^{-1}\|^{1/2} a_2(t) \\ &\le a_1 \|(\varepsilon\Lambda + B)^{-1}\|^{p} \|v\|_W^{p-1} + \|(\varepsilon\Lambda + B)^{-1}\|^{1/2} a_2(t) \\ &\le \frac{2^p a_1}{\|B\|^p} \|v\|_W^{p-1} + \frac{2^{1/2}}{\|B\|^{1/2}} a_2(t). \end{aligned} \tag{4.35}$$

On the other hand, by (H4.1.3), (4.34) and Proposition 4.3.1, we have

$$\begin{aligned} \langle y, v\rangle_W = \big(y, (\varepsilon\Lambda + B)^{-1}v\big) &\ge a_3 \|(\varepsilon\Lambda + B)^{-1}v\|_V^p - a_4 \|(\varepsilon\Lambda + B)^{-1}v\|_V^\alpha - a_5(t) \\ &\ge a_3 k^p \|v\|_{V^*}^p - a_4 \|(\varepsilon\Lambda + B)^{-1}v\|_V^\alpha - a_5(t) \\ &\ge a_3 k^p \|(\varepsilon\Lambda + B)^{-1}\|^{-p/2} \|v\|_W^p - a_4 \|(\varepsilon\Lambda + B)^{-1}\|^{\alpha} \|v\|_{V^*}^\alpha - a_5(t) \\ &\ge 2^{-p/2} a_3 k^p \|B\|^{p/2} \|v\|_W^p - a_4 (2/\|B\|)^{\alpha/2} \|v\|_W^\alpha - a_5(t). \end{aligned}$$

So, applying Theorem 4.2.1 and noting the fact $\|(\varepsilon\Lambda + B)x_0\|_W \le \big(2\|B\|\big)^{1/2}\|x_0\|_V$, we see that there exist $x_\varepsilon \in L^p(W)$ with $x_\varepsilon' \in L^q(W)$ such that

$$x_\varepsilon'(t) + A_\varepsilon(t, x_\varepsilon(t)) \ni f(t), \text{ a.e.}$$

$$x_\varepsilon(0) = (\varepsilon\Lambda + B)x_0,$$

$$\|x_\varepsilon\|_{L^p(V)}, \|x_\varepsilon'\|_{L^q(W)} \le C_1 + C_2\|f\|_{L^q(V^*)}^s \text{ with } C_1, C_2 > 0 \text{ independent of } \varepsilon.$$

From Proposition 4.3.1 (ii) and (4.34), it follows that there exist $C_3, C_4 > 0$ independent of ε such that

$$|x_\varepsilon'\|_{L^q(V^*)}, \|x_\varepsilon\|_{L^p(V^*)} \le C_3 + C_4\|f\|_{L^q(V^*)}^s.$$

Let n be so large that $1/n < \|B\|/(2\|\Lambda\|)$ and let $\varepsilon = 1/n, y_n = ((1/n)\Lambda + B)^{-1}x_\varepsilon$. Then $y_n \in L^p(V)$ and $y_n' = ((1/n)\Lambda + B)^{-1}x_\varepsilon' \in L^q(V) \subset L^q(V^*)$ satisfy

$$\begin{aligned} &((1/n)\Lambda + B)y_n'(t) + z_n(t) = f(t), \text{a.e. } t \in [0, T], \\ &y_n(0) = x_0 \end{aligned} \tag{4.36}$$

for some $z_n \in L^q(V^*)$ and $z(t) \in A(t, y_n(t))$ a.e. (that is $z_n \in \hat{A}Ly_n')$ for each n. Suppose $\beta > 0$ is the constant satisfying

$$\|u\|_{V^*} \leq \beta \|u\|_V, \quad \text{for all } u \in V.$$

Then there exist constants $C_5, C_6 > 0$ such that

$$\|y_n'\|_{L^q(V^*)} \leq \beta \|y_n'\|_{L^q(V)} \leq \beta \|((1/n)\Lambda + B)^{-1}\| \|x_\varepsilon'\|_{L^q(V^*)} \leq C_5 + C_6 \|f\|^s_{L^q(V^*)},$$
$$\|y_n\|_{L^p(V)} \leq \|((1/n)\Lambda + B)^{-1}\| \|x_\varepsilon\|_{L^p(V^*)} \leq C_5 + C_6 \|f\|^s_{L^q(V^*)}.$$

So we may suppose that

$$\begin{aligned} y_n' &\rightharpoonup y' \ \text{ in } L^q(0,T;V^*), \\ y_n = Ly_n' + x_0 &\rightharpoonup y = Ly' + x_0 \ \text{ in } L^p(0,T;V), \\ z_n &\rightharpoonup z \ \text{ in } L^q(0,T;V^*), \\ By_n &\rightharpoonup By \ \text{ in } L^q(0,T;V^*), \\ ((1/n)\Lambda + B)y_n' &\rightharpoonup (By)' \ \text{ in } L^q(0,T;V^*). \end{aligned}$$

By (4.36), Remark 1.4.11 and noting $y_n(0) - y(0) = x_0 - x_0 = 0$, we have

$$\begin{aligned} \limsup_{n\to\infty} ((z_n, y_n - y)) &= \limsup_{n\to\infty} \int_0^T \big(f(t) - ((1/n)\Lambda - B)y_n'(t), y_n(t) - y(t)\big)\,dt \\ &= -\liminf_{n\to\infty} \int_0^T \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\,t}\big(B(y_n(t) - y(t)), y_n(t) - y(t)\big)\,dt \\ &= -\frac{1}{2}\liminf_{n\to\infty}\big(B(y_n(T) - y(T)), y_n(T) - y(T)\big). \end{aligned}$$

Since B is positive, we see that $\limsup_{n\to\infty}((z_n, Ly_n' - Ly')) \leq 0$. By Proposition 4.1.4, $z_{n_j} \rightharpoonup z \in \hat{A}(Ly')$ for some subsequence $\{n_j\}$. So $(By)' + z = f$, that is

$$f(t) \in (By(t))' + A(t, y(t)) \ \text{ a.e., and } y(0) = x_0.$$

Obviously,

$$\|y'\|_{L^q(V^*)}, \|y\|_{L^p(V)} \leq C_5 + C_6 \|f\|^s_{L^q(V^*)}.$$

This completes the proof. □

Remark 4.3.3. This theorem generalizes the corresponding results in [4] and [8] where $A(t,x) \equiv Ax$ was supposed to be Lipschitz, monotone or maximal monotone, respectively. We note that, in both [4] and [8], the coercivity was imposed on the sum $A+\lambda B$ for some $\lambda > 0$. If $p > 2$ or λ small enough, this is equivalent to our condition (H4.1.3).

Remark 4.3.4. We can also suppose A is only quasi-monotone to give an approximate solvability result for the implicit problems (similar to Corollary 4.2.3). Since the procedure is the same, we omit it here.

The following result concerning the solution set of (4.30)-(4.31) is similar to Theorem 4.2.6.

Theorem 4.3.5. *Let*

$$S(f) = \{x \in W(0,T) : x \text{ is a solution of } (4.30)\text{-}(4.31) \text{ satisfying } (4.33)\}.$$

Under the assumptions of Theorem 4.3.2, $S(f)$ is a bounded weakly closed subset of $W(0,T)$. If, in addition, $V \hookrightarrow H$ compactly, then the mapping $f \mapsto S(f)$ is upper semicontinuous as a set-valued mapping from $L^q(H)_w$ to both $W(0,T)_w$ and $L^p(H)$.

Proof. The boundedness follows directly from Theorem 4.3.2.

Suppose $f \in L^q(V^*)$ and $x_n \in S(f)$ with $x_n \rightharpoonup x$ in $W(0,T)$. Then $x_n \rightharpoonup x$ in $L^p(V), x_n' \rightharpoonup x'$ in $L^q(V^*), Bx_n \rightharpoonup Bx$ and there exist $z_n \in S^q_{A(\cdot,x_n(\cdot))}$ such that

$$(Bx_n(t))' + z_n(t) = f(t) \text{ a.e..}$$

Multiplying both sides by $x_n - x$ and using Remark 1.4.11, we have

$$\begin{aligned}\big((Bx(t))', x_n(t) - x(t)\big) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(Bx_n(t) - Bx(t), x_n(t) - x(t)\big)\\ + \big(z_n(t), x_n(t) - x(t)\big) = \big(f(t), x_n(t) - x(t)\big)\end{aligned}$$

and, therefore

$$\begin{aligned}&\lim_{n\to\infty} ((z_n, x_n - x)) = \limsup_{n\to\infty}((f - (Bx)', x_n - x))\\ &+\frac{1}{2}\limsup_{n\to\infty}\big[-\big(B(x_n(T) - x(T)), x_n(T) - x(T)\big)\big] \le 0.\end{aligned} \tag{4.37}$$

Applying Proposition 4.1.4 to the sequence $\{x_n'\}$, we see that there exist a subsequence $\{z_{n_j}\}$ and a point $z \in S^q_{A(\cdot,x(\cdot))}$ such that $z_{n_j} \rightharpoonup z$ in $L^q(V^*)$. By Theorem 1.4.12, we have

$$(Bx_{n_j})' = f - z_{n_j} \rightharpoonup f - z = (Bx)'.$$

Hence $(Bx)' + z = f$, i.e. $x \in S(f)$. This proves the closedness.

Now, suppose $V \hookrightarrow H$ compactly. If S is not u.s.c. from $L^q(H)_w$ to $W(0,T)_w$ or $L^p(H)$, then there exist $f_n \rightharpoonup f$ in $L^q(H)$, $x_n \in S(f_n)$ and a neighbourhood $\mathcal{V}$ of $S(f)$ in $W(0,T)_w$ or $L^p(H)$ with $x_n \notin \mathcal{V}$ for all $n > 0$. Since $\{f_n\}$ is bounded in $L^q(V^*)$ and x_n satisfy (4.33), $\{x_n\}$ is bounded in $W(0,T)$. We may suppose (by passing to subsequence) that

$$x_n \rightharpoonup x \text{ in } L^p(V), \quad x_n' \rightharpoonup x' \text{ in } L^q(V^*)$$

for some $x \in W(0,T)$ and, therefore, $Bx_n \rightharpoonup Bx$ in $L^q(V^*)$. The continuous embedding of $W(0,T)$ into $C(0,T;H)$ implies $x(0) = x_0$. Since $W(0,T) \hookrightarrow L^p(H)$ compactly (see Theorem 1.4.10), we may suppose $x_n \to x$ in $L^p(H)$. Therefore

$$(\!(f_n, x_n - x)\!) = (\!(f_n, x_n - x)\!)_H \to 0 \quad \text{as} \quad n \to \infty.$$

Here, $(\!(\cdot,\cdot)\!)_H$ stands for the duality pairing between $L^p(H)$ and $L^q(H)$. Let $z_n \in S^q_{A(\cdot,x_n(\cdot))}$ be the functions such that

$$(Bx_n)'(t) + z_n(t) = f_n(t), \quad \text{a.e.}.$$

So, using the same method as used to obtain (4.37), we have

$$\limsup_{n\to\infty}(\!(z_n, x_n - x)\!) = \limsup_{n\to\infty}\Big[(\!(f_n - (Bx)', x_n - x)\!) - \frac{1}{2}\big(Bx_n(T) - Bx(T), x_n(T) - x(T)\big)\Big] \le 0.$$

Applying Proposition 4.1.4 to the sequence $\{x_n'\}$, we see that there exist a subsequence $\{z_{n_j}\}$ and $z \in S^q_{A(\cdot,x(\cdot))}$ such that $z_{n_j} \rightharpoonup z$ in $L^q(V^*)$ and $(\!(z_{n_j}, x_{n_j} - x)\!) \to 0$. From Theorem 1.4.12, we have $\big(Bx(t)\big)' + z(t) = f(t)$ a.e. which implies $x \in S(f) \subset \mathcal{V}$. In case $\mathcal{V}$ is a neighbourhood of $S(f)$ in $W(0,T)_w$, this has contradicted the assumption $x_n \notin \mathcal{V}$ for all n. In case $\mathcal{V}$ is a neighbourhood of $S(f)$ in $L^p(H)$, the compact embedding of $W(0,T)$ into $L^p(H)$ implies that we can suppose (by passing to a further sequence) $x_{n_j} \to x$ in $L^p(H)$ which also contradicts our assumption.

This completes the proof. $\square$

Remark 4.3.6. As shown in Remark 4.2.7, this result is useful in the consideration of the perturbation problem of (4.30)-(4.31) (see next section).

4.4 Perturbation problems

In this section, under (H4.1.1)-(H4.1.3), suppose $G : [0,T] \times V \to V^*$ is a set-valued mapping, we consider the perturbed explicit problem

$$x'(t) + A(t, x(t)) + G(t, x(t)) \ni f(t) \text{ a.e., } x(0) = x_0 \in V \tag{4.38}$$

and the perturbed implicit problem

$$\frac{\mathrm{d}}{\mathrm{d}t}(Bx(t)) + A(t, x(t)) + G(t, x(t)) \ni f(t) \text{ a.e., } Bx(0) = Bx_0. \tag{4.39}$$

First, we treat (4.38) under the assumption that $x \mapsto G(t,x)$ is Lipschitz with range$(G) \subset V^*$. Secondly we consider (4.39) under the assumption that $x \mapsto G(t,x)$ is only upper semicontinuous with range$(G) \subset H$. In each case, some additional assumptions on the mapping A and the space V or V^* are also necessary.

For our first case, we denote by $\mathcal{H}_{V^*}(\cdot,\cdot)$ the Hausdorff distance on V^*, by j_* the duality operator from V^* to V respectively.

Theorem 4.4.1. *Under (H4.1.1)-(H4.1.4), suppose*

(i) The norm $\|\cdot\|_{V^*}$ is differentiable,

(ii) $v \mapsto A(t,v)$ is accretive *as a mapping in V^* (recall $V \subset V^*$),* i.e.

$$\big(\overline{u} - \overline{v}, j_*(u - v)\big) \geq 0 \quad \textit{for all } u, v \in V, \overline{u} \in A(t,u), \overline{v} \in A(t,v), t \in [0,T],$$

(iii) $G : [0,T] \times V \to \mathcal{P}_c(V^*)$ is measurable, $\|G(t,0)\|_{V^*} \in L^q(\mathbb{R})$ and there exists $k > 0$ such that

$$\mathcal{H}_{V^*}(G(t,v_1), G(t,v_2)) \leq k\|v_1 - v_2\|_{V^*}, \quad \textit{for all } t \in [0,T], v_1, v_2 \in V.$$

Then the perturbation problem (4.38) *admits solutions. If, in addition, G is single-valued, then the solution is unique.*

For the proof, we need the following lemma.

Lemma 4.4.2. *Suppose (H4.1.1)–(H4.1.4) and the assumptions (i) (ii) of Theorem 4.4.1 hold. Then, for every $g \in L^q(V^*)$, the problem*

$$x'(t) + A(t, x(t)) + g(t) \ni 0 \ \text{a.e.}, \ x(0) = x_0 \in V$$

admits a unique solution $x_g \in L^q(V^*)$ and the solution map $r : g \mapsto x_g$ is such that

$$\|r(g_1)(t) - r(g_2)(t)\|_{V^*} \leq \int_0^t \|g_1(s) - g_2(s)\|_{V^*} ds, \ \textit{for all}\ t \in [0,T], g_1, g_2 \in L^q(V^*).$$

Proof. The existence is a direct consequence of Theorem 4.2.1. Now we suppose that $g_i \in L^q(V^*)$, x_i is one of the corresponding solutions, $i = 1, 2$. Then there exist $y_i(t) \in A(t, x_i(t))$ such that

$$\begin{aligned}\big(\dot{x}_1(t) - \dot{x}_2(t), j_*(x_1(t) - x_2(t))\big) + \big(y_1(t) - y_2(t), j_*(x_1(t) - x_2(t))\big)\\ = -\big(g_1(t) - g_2(t), j_*(x_1(t) - x_2(t))\big) \ \text{a.e.}.\end{aligned}$$

By Theorem 1.6.7 (iii), $\big(x_1'(t) - x_2'(t), j_*(x_1(t) - x_2(t))\big) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|x_1(t) - x_2(t)\|_{V^*}^2$. Since our assumption (ii) and $x_1(0) = x_2(0) = x_0$, we have

$$\begin{aligned}\frac{1}{2}\|x_1(t) - x_2(t)\|_{V^*}^2 &\leq -\int_0^t \big(g_1(s) - g_2(s), j_*(x_1(s) - x_2(s))\big) ds\\ &\leq \int_0^t \|g_1(s) - g_2(s)\|_{V^*} \|j_*(x_1(s) - x_2(s))\|_V ds\\ &= \int_0^t \|g_1(s) - g_2(s)\|_{V^*} \|x_1(s) - x_2(s)\|_{V^*} ds.\end{aligned}$$

By the generalized Gronwall's inequality (Theorem 1.7.7), we have

$$\|x_1(t) - x_2(t)\|_{V^*} \leq \int_0^t \|g_1(s) - g_2(s)\|_{V^*} ds.$$

This is the claimed inequality and, by taking $g_1 = g_2$, we obtain the uniqueness. This completes the proof. □

Proof of Theorem 4.4.1

Let r be the same operator as in Lemma 4.4.2 and, for $g \in L^q(V^*)$, let

$$F(g) = S^1_{G(\cdot, r(f-g)(\cdot))}.$$

By Theorem 1.3.3, $F(g) \neq \emptyset$ for every $g \in L^q(V^*)$ and $F(g) \subset L^q(V^*)$ because of our assumptions on G. It is easy to show that $F(g)$ is closed and bounded.

Take $g_1, g_2 \in L^q(V^*)$ and let $\varepsilon > 0, x \in F(g_1)$ be given. We may suppose that $x(t) \in G(t, r(f-g_1)(t))$ for each $t \in [0,T]$. Then by the definition of Hausdorff distance, for every t, there exist $y_t \in G(t, r(f-g_2)(t))$ such that

$$\begin{aligned}\|x(t)-y_t\|_{V^*} &\leq \mathcal{H}_{V^*}(G(t,r(f-g_1)(t)), G(t,r(f-g_2)(t)) + \varepsilon \\ &\leq k\|r(f-g_1)(t) - r(f-g_2)(t)\|_{V^*} + \varepsilon =: K(t).\end{aligned}$$

Let

$$D(t) = \{y \in G(t, r(f-g_2)(t)) : \|x(t)-y\|_{V^*} \leq K(t)\}.$$

Then, for each $t \in [0,T]$, $D(t)$ is nonempty, closed, bounded and

$$\|D(t)\|_{V^*} \leq K(t) + \|x(t)\|_{V^*}.$$

Noting that $K(\cdot)$ and $\|\cdot\|$ are continuous, $x(\cdot)$ is measurable, using the same method as used in the proof of Proposition 4.1.1, we have a measurable function $y : [0,T] \to V^*$ such that $y(t) \in D(t) \subset G(t, r(f-g_2)(t))$ almost everywhere. So we have

$$\|x(t)-y(t)\|_{V^*} \leq k\|r(f-g_1)(t) - r(f-g_2)(t)\|_{V^*} + \varepsilon, a.e..$$

Let $l > 0$ be a real number such that

$$kT^{1/q}(2lp)^{-1/p} < 1.$$

For each $x \in L^q(V^*)$, let

$$\|x\|_l = \left(\int_0^T \exp(-2lqt)\|x(t)\|_{V^*}^q dt\right)^{1/q}.$$

Clearly, $\|\cdot\|_l$ is a norm on $L^q(V^*)$ and equivalent to the usual one. Write

$$c = \left(2q^2 l\big(1-\exp(-2q^2 lT)\big)\right)^{1/q}.$$

By Lemma 4.4.2 and Hölder's Inequality, we obtain

$$
\begin{aligned}
\|x-y\|_l &= \left(\int_0^T \exp(-2lqt)\|x(t)-y(t)\|_{V^*}^q dt\right)^{\frac{1}{q}} \\
&\leq k\left(\int_0^T \exp(-2lqt)\left(\int_0^t \|g_1(s)-g_2(s)\|_{V^*} ds\right)^q dt\right)^{\frac{1}{q}} + \varepsilon c \\
&\leq k\left(\int_0^T \exp(-2lqt)\left(\int_0^t \exp(-2ls)\exp(2ls)\|g_1(s)-g_2(s)\|_{V^*} ds\right)^q\right)^{\frac{1}{q}} + \varepsilon c \\
&\leq k\left(\int_0^T \exp(-2lqt)\left(\int_0^t \exp(-2lqs)\|g_1(s)-g_2(s)\|_{V^*}^q ds\right)\right. \\
&\qquad \left.\left(\int_0^t \exp(2lps)ds\right)^{q/p} dt\right)^{\frac{1}{q}} + \varepsilon c \\
&\leq kT^{1/q}(2lp)^{1/p}\|g_1-g_2\|_l + \varepsilon c
\end{aligned}
$$

Since g_1, g_2 are arbitrary, the Hausdorff distance between $F(g_1)$ and $F(g_2)$ in $L^q(V^*)$ endowed with the new norm $\|\cdot\|_l$ is such that

$$\mathcal{H}_{L^q(V^*)}(F(g_1), F(g_2)) \leq kT^{1/q}(2lp)^{1/p}\|g_1-g_2\|_l + \varepsilon c.$$

By letting $\varepsilon \to 0$, we obtain

$$\mathcal{H}_{L^q(V^*)}(F(g_1), F(g_2)) \leq kT^{1/q}(2lp)^{1/p}\|g_1-g_2\|_l.$$

So F is a contraction on $L^q(V^*)$, and therefore F has a fixed point $g$according to Theorem 1.5.6. Obviously, $x = r(f-g)$ is a solution of (4.38).

If, G is single-valued, then F is single-valued and, therefore, the fixed point is unique which implies the uniqueness of the solution. This completes the proof. $\square$

Now, we consider the implicit perturbation problem (4.39). This time, we need to suppose that the non-perturbed problem (4.30)-(4.31) has unique solution for each $f \in L^q(H)$. This can be guaranteed by supposing $\lambda B + A(t,\cdot)$ is strongly monotone for some $\lambda > 0$ (see Theorem 2 in [8]). If $Bx \equiv x$, Lemma 4.4.2 may be applied.

Theorem 4.4.3. *Under (H4.2.1)-(H4.2.4), suppose*

(i) $V \hookrightarrow H$ compactly,

(ii) for each $f \in L^q(H)$, problem (4.30)-(4.31) *has a unique solution,*

(iii) $G : [0,T] \times H \to \mathcal{P}_{cv}(H)$ *is a measurable set-valued mapping,* $v \mapsto G(t,v)$ *is u.s.c. as a mapping from* H *into* H_w *(*i.e. *demicontinuous on* H*) and*

$$\|G(t,v)\|_H \leq d(t) \text{ i.e. } on \ [0,T] \ \ with \ d \in L^q(H).$$

Then problem (4.39) *has solutions.*

Proof. Let x_f be the unique solution of problem (4.30)-(4.31) and let

$$F(g) = S^1_{G(\cdot, x_{f-g}(\cdot))}, \quad D = \{x \in L^q(H) : \|x(t)\|_H \leq d(t)\}.$$

Then, by Theorem 1.3.3 and our assumptions, F is a well-defined mapping from D into itself and, clearly, $F(g)$ is closed convex for each g.

Since D is weakly closed in $L^q(H)$, if F is u.s.c. in $L^q(H)$ under the weak topology, then from Theorem 1.5.4, it follows that F has fixed point g and, therefore, x_{f-g} is a solution of (4.39).

To prove F is u.s.c. in $L^q(H)$ under the weak topology, let $(g_n, z_n) \in \text{Graph}(F)$ and $g_n \rightharpoonup g, z_n \rightharpoonup z$ in $L^q(H)$. By Theorem 4.3.5, $x_{f-g_n} \to x_{f-g}$ in $L^p(H)$ (by passing to a subsequence) and, therefore, $x_{f-g_n}(t) \to x_{f-g}(t)$ in H for a.e. $t \in [0,T]$. By our assumption (iii), $\text{Graph}\big(G(t,\cdot)\big)$ is sequentially closed in $H \times H_w$. So

$$w\text{-}\limsup_{n\to\infty} G(t, x_{f-g_n}(t)) \subset G(t, x_{f-g}(t))$$

for almost all t. Invoking Theorem 1.3.4, we have

$$z \in w\text{-}\limsup_{n\to\infty} F(g_n) \subset S^1_{w\text{-}\limsup_{n\to\infty} G(\cdot, x_{f-g_n}(\cdot))} \subset S^1_{G(\cdot, x_{f-g}(\cdot))} = F(g),$$

that is, $(g,z) \in \text{Graph}\, F$. Since D is weakly compact, F is weakly u.s.c. (see Theorem 1.2.10). □

4.5 Examples

Now, we present an example to illustrate the applicability of our results.

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\Gamma, m \geq 1, T > 0$. Consider the problem

$$\frac{\partial(b(z)u(t,z))}{\partial t} + \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(t,z,u,\cdots,D^m u) \in -G(z,Du,\cdot,D^{m-1}u) \quad (4.40)$$

$$D^\beta u \mid_{[0,T]\times\Gamma} = 0 \text{ for } |\beta| \leq m-1, \ u(0,z) = u_0(z) \text{ for } z \in \Omega. \quad (4.41)$$

Here, $b : \Omega \to \mathbb{R}$ is a nonnegative measurable function and $b \in L^\infty(\Omega), G(z,w) = [g_1(z,w), g_2(z,w)] : \Omega\times\mathbb{R}^{N-1} \to 2^{\mathbb{R}}\backslash\{\emptyset\}$ is a measurable set-valued mapping, $w \mapsto G(z,w)$ is upper semicontinuous for almost all $z \in \Omega$.

Let N be the number of multi-indices α with $|\alpha| \leq m$ and for $\xi = \{\xi_\alpha : |\alpha| \leq m\} \in \mathbb{R}^N$, write $\xi = (\eta, \zeta)$, where $\eta = \{\xi_\alpha : |\alpha| \leq m-1\}$ and $\zeta = \{\xi_\alpha : |\alpha| = m\}$. Denote by $\Omega_T = [0,T] \times \Omega$. We impose the following hypotheses on A_α and G.

H(A): $A_\alpha : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is such that

i) $(t,z) \mapsto A_\alpha(t,z,\xi)$ is measurable for all $\xi \in \mathbb{R}^N$ and $\xi \mapsto A_\alpha(t,z,\xi)$ is continuous for almost all $(t,z) \in \Omega_T$;

ii) there exist $c_1, c_2 > 0$ and $k_1, k_2 \in L^1(\Omega_T)$ such that

$$|A_\alpha(x,\xi)| \leq c_1|\xi| + k_1(x) \text{ for all } |\alpha| \leq m, x \in \Omega_T \text{ and } \xi \in \mathbb{R}^N,$$

$$\sum_{|\alpha|\leq m} A_\alpha(x,\xi)\xi_\alpha \geq c_2|\xi|^2 - k_2(x) \quad \text{for all } x \in \Omega_T, \xi \in \mathbb{R}^N;$$

iii) for all $(\eta,\zeta), (\eta,\zeta')$ in $\mathbb{R}^N$ with $\zeta \neq \zeta'$, and all $x \in \Omega_T$,

$$\sum_{|\alpha|=m} [A_\alpha(x,\eta,\zeta) - A_\alpha(x,\eta,\zeta')][\zeta_\alpha - \zeta'_\alpha] > 0.$$

H(G): There exist $c_3 > 0, r > 0$ and $k_3 \in L^1(\Omega)$ such that

$$|G(z,w)| = \max\{|g_1(z,w)|, |g_2(z,w)|\} \leq c_3|w|^{1-r} + k_3(z),$$

for all $z \in \Omega_T$ and $u \in \mathbb{R}$.

Consider the evolution triple given by $V = W_0^{m,2}(\Omega), H = L^2(\Omega)$ and $V^* = W^{-m,2}(\Omega)$. Define mappings $B : V \to V^*, F : [0,T] \times V \to V^*$ and $C : V \to 2^{V^*}$ by

$$Bu = b(\cdot)u,$$

$$\langle F(t,u), v\rangle = \int_\Omega \sum_{|\alpha|\leq m} A_\alpha(t,z,u,\cdots,D^m u) D^\alpha v dz,$$

$$C(u) = \{g \in V^* : g(z) \in G(z,u(z)) \ a.e.\}.$$

Then, $B \in \mathbf{L}(V, V^*)$ is positive and symmetric. By Lemma 3.3.2, $u \mapsto F(t,u)$ is pseudo-monotone and, clearly, satisfying (A2), (A3). By Lemma 3.3.3, C is weakly upper semi-continuous compact with closed convex values. So $u \mapsto A(t,u) := F(t,u) + C(u)$ is pseudo-monotone. Our assumption H(G) implies that, for each $g \in C(u)$ and $u \in V$

$$\|g\|_{V^*} \leq \hat{c}_3 \|u\|_{V^*}^{1-r} + \hat{c}_4$$

with $\hat{c}_3 \geq 0, \hat{c}_4 \geq 0$. Since $r > 0$, A satisfies all the other conditions made in Theorem 4.3.2. Hence, problem (4.40)-(4.41) admits solutions.

Note, if, in the assumption H(G), $r = 0$, we need an extra condition, say

$$\inf\{g_1(z,w)w, g_2(z,w)w\} \geq -c_4\|w\|^2 - c_5(z)$$

with $c_4 \geq 0$ small enough and $c_5 \in L^2(\Omega)$ to ensure the coercivity of A. If we replace H(A) (ii) by

$$|A_\alpha(x,\xi)| \leq c_1|\xi|^{p-1} + k_1(x) \text{ for all } |\alpha| \leq m, x \in \Omega_T \text{ and } \xi \in \mathbb{R}^N,$$
$$\sum_{|\alpha|\leq m} A_\alpha(x,\xi)\xi_\alpha \geq c_2|\xi|^p - k_2(x) \text{ for all } x \in \Omega_T, \xi \in \mathbb{R}^N$$

with $p > 2$, then the above extra condition is not necessary even when $r = 0$.

Remark 4.5.1. Since our mapping A need not to be monotone, the result in [8] cannot be applied to our example even without the set-valued perturbation G.

We also remark that our results can be applied to the other examples presented in [8], [72] or [80].

Chapter 5

Solvability of Second Order Nonlinear Evolution Inclusions

In this chapter, we will study the second order explicit evolution inclusions

$$x''(t) + A(t, x'(t)) + Bx(t) \ni f(t), \tag{5.1}$$

$$x''(t) + A(t, x'(t)) + Bx(t) - F(t, x(t), x'(t)) \ni 0 \tag{5.2}$$

and some related implicit inclusions in an evolution triple (V, H, V^*) with $v \mapsto A(t, v)$ pseudo-monotone from V to V^*, B a symmetric, linear positive operator from V to V^* and F a set-valued mapping from $[0, T] \times H \times H$ to H.

It is easy to see that problem (5.1) is equivalent to the first order problem

$$y'(t) + A(t, y(t)) + BLy(t) \ni f(t)$$

with $Ly(t) = \int_0^t y(s)ds$. Since the realization of BL is positive and bounded, existence conclusions for (5.1) are not hard to obtain from the known results provided $v \mapsto A(t, v)$ is monotone (see Theorem 33.A in [80]). In the present situation where $v \mapsto A(t, v)$ is pseudo-monotone, we will use the method used in Theorem 4.2.2 to obtain existence results for (5.1) and a property of the solution set, as well as the solvability of an implicit inclusion.

For problem (5.2), the idea used in Theorem 4.4.3 will be used. We note that in case $F(t, x, y) \equiv F(t, x)$, $x \mapsto F(t, x)$ has a linear growth condition and $x \mapsto A(t, x)$ is single-valued monotone, Papageorgiou [64] has obtained two (global) existence results. Here, we shall consider both global and local solutions for the general inclusion (5.2). For global solutions, we suppose that F satisfies a $\frac{2}{q}$-growth condition. Under some other conditions weaker than those in [64], and by a new extension of Wirtinger's inequality established here, we prove the existence of solutions and the compactness of solution set. For the local solutions, we suppose that F is bounded instead of imposing the growth condition, and prove existence of solutions.

5.1 An extension of Wirtinger's inequality

The following inequality is often known as Wirtinger's inequality (see [57]).

Lemma 5.1.1 (Wirtinger's Inequality). *Let $2k$ be an even positive integer, f be a continuous function from $[0,1]$ to $\mathbb{R}$, $f' \in L^{2k}(0,1)$, $f(0)=0$. Then*

$$\int_0^1 |f(t)|^{2k} dt \le C \int_0^1 |f'(t)|^{2k} dt,$$

for $C = \frac{1}{2k-1}\left(\frac{2k}{\pi}\sin\frac{\pi}{2k}\right)^{2k}$.

In this section, we shall extend this to the case when the function f takes values in an abstract space, which seems to be new in this context and will be used in next section.

Proposition 5.1.2. *Let $(X, \|\cdot\|)$ be a Banach space, $2k$ be an even positive integer, $f:[0,1]\to X$ be continuous, $f' \in L^{2k}(0,1;X)$, $f(0)=0$. Then*

$$\int_0^1 \|f(t)\|^{2k} dt \le C \int_0^1 \|f'(t)\|^{2k} dt$$

with C the same number as above.

Proof. First, we suppose that $f(t) \neq 0$ for each $t \in (0,1)$. In this case, the continuity of f implies that $\alpha_n := \min\{\|f(t)\| : t \in [\frac{1}{n}, 1-\frac{1}{n}]\} > 0$ for each n. Let n be given and take $\varepsilon \in (0, \alpha_n/2)$. By the continuity of f, there exist

$$0 = t_0 < t_1 = \frac{1}{n} < t_2 < \cdots < t_{m-1} < t_m = 1 - \frac{1}{n} < t_{m+1} = 1$$

such that for each $i = 1, 2, \cdots, m-1$,

$$t \in [t_i, t_{i+1}) \text{ implies } \|f(t) - f(t_i)\| < \varepsilon$$

and, therefore,

$$\|f(t)\| \geq \|f(t_i)\| - \varepsilon, \ \|f(t_i)\| \geq \|f(t)\| - \varepsilon.$$

By the Hahn-Banach Theorem, for each $i = 0, 1, \cdots, m$, there exist $x_i^* \in X^*, \|x_i^*\| = 1$, and $x_i^* f(t_i) = \|f(t_i)\|$. So for each $i = 1, \cdots, m-1$, we have

$$|x_i^*(f(t) - f(t_i))| \leq \|x_i^*\| \|(f(t) - f(t_i))\| < \varepsilon, \ t \in [t_i, t_{i+1}),$$

and

$$\begin{aligned} |x_i^* f(t)| &\geq |x_i^* f(t_i)| - |x_i^*(f(t) - f(t_i))| \\ &\geq \|f(t_i)\| - \varepsilon \geq \|f(t)\| - 2\varepsilon > 0, \quad t \in [t_i, t_{i+1}). \end{aligned} \tag{5.3}$$

Let the function $g : [0, 1] \to \mathbb{R}$ be defined by

$$g(t) = \begin{cases} x_i^* f(t) & t \in [t_i, t_{i+1}), i = 1, \cdots, m-1, \\ 0 & t \in [0, t_1) \text{ and } t \in (t_m, 1]. \end{cases}$$

Then the discontinuities of g are at most $t_1, \cdots, t_m$. Also $g(0) = 0$ and $g'(t) = x_i^* f'(t)$ for $t \in (t_i, t_{i+1}), i = 1, \cdots, m-1$, that is $g' \in L^{2k}(0, 1)$. Applying Wirtinger's Inequality, we obtain

$$\int_0^1 |g(t)|^{2k} dt \leq C \int_0^1 |g'(t)|^{2k} dt \leq C \int_{1/n}^{1-1/n} \|f'(t)\|^{2k} dt \leq C \int_0^1 \|f'(t)\|^{2k} dt. \tag{5.4}$$

From (5.3), it follows that

$$\begin{aligned} \int_0^1 |g(t)|^{2k} dt &= \int_{1/n}^{1-1/n} |g(t)|^{2k} dt = \sum_{i=1}^{m-1} \int_{t_i}^{t_{i+1}} |x_i^* f(t)|^{2k} dt \\ &\geq \sum_{i=1}^{m-1} \int_{t_i}^{t_{i+1}} \left(\|f(t)\| - 2\varepsilon\right)^{2k} dt = \int_{1/n}^{1-1/n} \left(\|f(t)\| - 2\varepsilon\right)^{2k} dt. \end{aligned}$$

Letting $\varepsilon \to 0$, we obtain

$$\int_0^1 |g(t)|^{2k} dt \geq \int_{1/n}^{1-1/n} \|f(t)\|^{2k} dt.$$

Substituting this into (5.4), we obtain

$$\int_{1/n}^{1-1/n} \|f(t)\|^{2k} dt \le C \int_0^1 \|f'(t)\|^{2k} dt.$$

We let $n \to \infty$ to yield

$$\int_0^1 \|f(t)\|^{2k} dt \le C \int_0^1 \|f'(t)\|^{2k} dt.$$

Now, we prove the general case. If $f(t) \equiv 0$, the result holds, so we suppose $f(t) \not\equiv 0$ and let

$$U = \{t \in (0,1) : f(t) \neq 0\}.$$

Then U is open and equal to the union of some disjoint open intervals, say

$$U = \bigcup_{i=1}^{\infty} (\alpha_i, \beta_i).$$

Here $0 \le \alpha_i < \beta_i \le \alpha_{i+1} \le 1, f(\alpha_i) = 0, i = 1, 2, \cdots$, and $f(t) \neq 0$ for all $t \in (\alpha_i, \beta_i)$ and each i. Using the result of the first step, we obtain that

$$\begin{aligned}
\int_{\alpha_i}^{\beta_i} \|f(t)\|^{2k} dt &= (\beta_i - \alpha_i) \int_0^1 \|f((\beta_i - \alpha_i)s + \alpha_i)\|^{2k} ds \\
&\le C(\beta_i - \alpha_i)^{2k+1} \int_0^1 \|f'((\beta_i - \alpha_i)s + \alpha_i)\|^{2k} ds \\
&\le C(\beta_i - \alpha_i)^{2k} \int_{\alpha_i}^{\beta_i} \|f'(t)\|^{2k} dt \le C \int_{\alpha_i}^{\beta_i} \|f'(t)\|^{2k} dt.
\end{aligned}$$

Noting $\int_{\beta_i}^{\alpha_{i+1}} \|f(t)\|^{2k} dt = 0 = \int_{\beta_i}^{\alpha_{i+1}} \|f'(t)\|^{2k} dt$ for $i = 0, 1, \cdots, (\beta_0 = 0)$, we have

$$\int_0^1 \|f(t)\|^{2k} dt = \sum_{i=1}^{\infty} \int_{\alpha_i}^{\beta_i} \|f(t)\|^{2k} dt \le C \sum_{i=1}^{\infty} \int_{\alpha_i}^{\beta_i} \|f'(t)\|^{2k} dt = C \int_0^1 \|f'(t)\|^{2k} dt.$$

This completes the proof. □

Corollary 5.1.3. *Suppose $f : [0, T] \to X$ is continuous, $f(0) = \bar{x}, f' \in L^2(0, T)$. Then*

$$\int_0^T \|f(t)\|^2 dt \le \frac{4T^2}{\pi^2} \int_0^T \|f'(t)\|^2 dt + 2\|\bar{x}\| \int_0^T \|f(t)\| dt - T\|\bar{x}\|^2. \tag{5.5}$$

In particular, if X is separable and reflexive and f is absolutely continuous, then

$$\int_0^T \|f(t)\|^2 dt \le \frac{4T^2}{\pi^2} \int_0^T \|f'(t)\|^2 dt + 2T\|\bar{x}\| \int_0^T \|f'(t)\| dt + T\|\bar{x}\|^2. \tag{5.6}$$

Proof. Let

$$f_1(t) = f(Tt) - \bar{x}, \quad t \in [0,1].$$

Then f_1 satisfies the conditions of Proposition 5.1.2. Noting $f_1'(t) = Tf'(Tt)$, we have

$$\int_0^1 \|f_1(t)\|^2 dt \le \frac{4}{\pi^2} \int_0^1 \|f_1'(t)\|^2 dt = \frac{4T}{\pi^2} \int_0^T \|f'(t)\|^2 dt.$$

However,

$$\begin{aligned}\int_0^1 \|f_1(t)\|^2 dt &\ge \int_0^1 \|f(Tt)\|^2 dt + \|\bar{x}\|^2 - 2\|\bar{x}\| \int_0^1 \|f(Tt)\| dt \\ &\ge \frac{1}{T} \int_0^T \|f(t)\|^2 dt + \|\bar{x}\|^2 - \frac{2}{T}\|\bar{x}\| \int_0^T \|f(t)\| dt,\end{aligned}$$

which proves (5.5). If X is separable and reflexive, f is absolutely continuous, then

$$\int_0^T \|f(t)\| dt = \int_0^T \|\bar{x} + \int_0^t f'(s) ds\| dt \le T\|\bar{x}\| + T \int_0^T \|f'(t)\| dt.$$

Substituting this into (5.5), we obtain (5.6). This completes the proof. □

5.2 Existence results for inclusions with only pseudo-monotone mappings

In the remainder of this chapter, we always suppose (V, H, V^*) to be an evolution triple with $V \hookrightarrow H$ compactly. Denote by $(\cdot,\cdot)$ the duality between V and V^* and by $\langle \cdot,\cdot \rangle$ the inner product on H. Suppose $T > 0, p \ge 2$ are fixed numbers and $q = p/(p-1)$. The space $L^r(0,T;X)$ is abbreviated as $L^r(X)$ for any number $r > 0$ and space X.

In this section, we will consider the solvability of the second order explicit evolution inclusion

$$\begin{aligned}&x''(t) + A(t, x'(t)) + Bx(t) \in f(t) \text{ a.e. on } [0,T], \\ &x(0) = x_0 \in V, \ x'(0) = x_1 \in V\end{aligned} \tag{5.7}$$

and the second order implicit evolution inclusion

$$\begin{aligned}&\left((Bx(t))' + m(x(t))\right)' + Qx(t) = f(t) \text{ a.e.} \\ &m(x(t)) \in A(t, x(t)), \text{ a.e.,} \\ &Bx(0) = Bx_0, ((Bx)' + m(x))(0) = Qx_1.\end{aligned} \tag{5.8}$$

under the following basic assumptions

(H5.2.1) $t \mapsto (A(t,u),v)$ is measurable, $u \mapsto A(t,u)$ is pseudo-monotone for all $u, v \in V, t \in [0,T]$;

(H5.2.2) $\|A(t,v)\|_* \le a_1(t) + a_2\|v\|^{p-1}$ for *a.e.* $v \in V$ with $a_1 \in L^q(\mathbb{R}), a_2 > 0$;

(H5.2.3) $(A(t,v),v) \ge a_3\|v\|^p - a_4(t)$ for all $v \in V$ with $a_3 > 0, a_4 \in L^1(\mathbb{R})$;

(H5.2.4) $B \in \mathbf{L}(V,V^*), (Bu,v) = (u,Bv)$ and $(Bu,u) \ge 0$ for all $u, v \in V$;

(H5.2.5) $Q \in \mathbf{L}(V,V^*), (Qu,v) = (u,Qv)$ and $(Qu,u) \ge \omega\|u\|_V^p$ with some $\omega > 0$ for all $u, v \in V$.

As in Chapter 4, we denote by L the linear operator given by

$$(Lx)(t) = \int_0^t x(s)ds.$$

Our first result is

Theorem 5.2.1. *Under (H5.2.1)-(H5.2.4), there exist* $c_1, c_2 > 0$ *such that inclusion* (5.7) *has at least one solution* x *with* $x' \in W(0,T)$ *for each* $f \in L^q(V^*)$ *and*

$$\|x\|_{L^p(V)}, \|x'\|_{L^p(V)}, \|x''\|_{L^q(V^*)} \le c_1 + c_2\|f\|_{L^q(V^*)}. \tag{5.9}$$

If, in addition, $v \mapsto A(t,v)$ *is monotone, then the solution is unique.*

Proof. It is known that (5.7) is equivalent to

$$y'(t) + A(t,y(t)) + BLy(t) \ni f(t), \quad \text{a.e.}, \quad y(0) = x_1,$$

and, therefore, equivalent to the operator inclusion

$$L^*(f - Bx_1) \in L^*y + L^*(\hat{A} + \hat{B}L)Ly = L^*\hat{A}Ly + L^*\hat{B}L^2y \tag{5.10}$$

with $\hat{A}x = S^q_{A(\cdot,x(\cdot)+x_1)}$, $(\hat{B}x)(t) = Bx(t)$. By Example 1.6.3, $\hat{B}L \in \mathbf{L}(L^p(V), L^q(V^*))$ is positive, so is $L^*\hat{B}L^2$. From Proposition 4.1.1, it follows that $L^*(\hat{A} + \hat{B}L)L$ is pseudo-monotone from $L^p(V)$ to $L^*(V^*)$. Since $\hat{B}L$ is bounded, using almost the same procedure as used in Theorem 4.2.1 (just replace $\hat{A}$ there by $\hat{A}+\hat{B}L$), we see that (5.10) has a solution y such that

$$\|y\|_{L^p(V)}, \|y'\|_{L^q(V^*)} \leq k_1 + k_2\|f\|_{L^q(V^*)}$$

with some $k_1, k_2 > 0$ independent of f. So $x = Ly + x_0$ is a solution of (5.7) and

$$\|x\|_{L^p(V)} \leq k_1\|L\| + T^{1/p}\|x_0\|_V + k_2\|L\|\|f\|_{L^q(V^*)},$$

$$\|x'\|_{L^p(V)}, \|x''\|_{L^q(V^*)} \leq k_1 + k_2\|f\|_{L^q(V^*)}.$$

Now suppose, in addition, $v \mapsto A(t,v)$ is monotone and let x_1, x_2 be two solutions to (5.7) corresponding to a given $f \in L^q(V^*)$. Then there exist $z_i \in S^q_{A(\cdot,x_i(\cdot))}$ such that

$$x_i''(t) + z_i(t) + Bx_i(t) = f(t) \;\text{ a.e.}, \; x_i(0) = x_0, \; x_i'(0) = x_1, \; i = 1, 2.$$

This yields

$$x_1''(t) - x_2''(t) + z_1(t) - z_2(t), + B(x_1(t) - x_2(t)) = 0 \;\text{ a.e.}.$$

Multiplying by $x_1'(t) - x_2'(t)$ and noting the monotonicity of A and the facts that

$$\big(x_1''(t) - x_2''(t), x_1'(t) - x_2'(t)\big) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\,t}\|x_1'(t) - x_2'(t)\|_H^2, \tag{5.11}$$

$$\big(B(x_1(t) - x_2(t)), x_1'(t) - x_2'(t)\big) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\,t}\big(B(x_1(t) - x_2(t)), x_1(t) - x_2(t)\big), \tag{5.12}$$

we have

$$\|x_1'(t) - x_2'(t)\|_H^2 + \big(B(x_1(t) - x_2(t)), x_1(t) - x_2(t)\big) \leq 0$$

for all t. Since B is positive, we see $\|x_1'(t) - x_2'(t)\|_H = 0$ for all t and, therefore, $x_1 = x_2$ which proves the uniqueness. □

Remark 5.2.2. In case $A(t,\cdot)$ is monotone, we obtain Theorem 33.A of [80]. Moroever, in [80], B is assumed to be coercive for the uniqueness part.

Similar to Theorems 4.2.6 and 4.2.8, we have

Theorem 5.2.3. *Under (H5.2.1)-(H5.2.4), let x_f be an arbitrary solution of problem* (5.7) *satisfying* (5.9). *Denote by $S(f) = \{x_f'\}$. If $V \hookrightarrow H$ compactly, then the mapping $f \mapsto S(f)$ is upper semicontinuous as a set-valued mapping from $L^q(H)_w$ to both $W(0,T)_w$ and $L^p(H)$. If, in addition, $A(t,\cdot)$ is monotone, then $f \mapsto x_f$ is continuous from $L^q(H)_w$ to $C^1(0,T;H)$.*

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bounded in $W(0,T)$. So, by passing to a subsequence, we may assume that $x'_n \rightharpoonup y$ in $W(0,T)$ and then we have

$$(x''_n(t) - x''(t), x'_n(t) - x'(t)) + (z_n(t) - z(t), x'_n(t) - x'(t))$$
$$+(B(x_n(t) - x(t)), x'_n(t) - x'(t)) = \langle f_n(t) - f(t), x'_n(t) - x'(t)\rangle \text{ a.e.}$$

with some $z_n \in S^q_{A(\cdot,x'_n(\cdot))}, z \in S^q_{A(\cdot,x'(\cdot))}$. Since $A(t,\cdot)$ is monotone, B is positive, $x_n(0) = x(0) = x_0, x'_n(0) = x'(0) = x_1$, using (5.11) and (5.12), we have

$$\frac{1}{2}\|x'_n(t) - x'(t)\|^2_H \leq \int_0^t \langle f_n(s) - f(s), x'_n(s) - x'(s)\rangle ds. \tag{5.13}$$

Since $W(0,T) \hookrightarrow L^p(H)$ compactly, we may suppose that $x'_n \to y$ strongly in $L^p(H)$. So (5.13) gives

$$\|x'_n(t) - x'(t)\|_H \to 0 \quad \text{uniformly,}$$

and therefore

$$\|x_n(t) - x(t)\|_H \to 0 \ \text{ uniformly}$$

due to $\|x_n(t) - x(t)\|_H \leq \int_0^T \|x'_n(t) - x'(t)\|_H dt$, that is $x' = y$. This shows that $f \mapsto x_f$ is continuous from $L^q(H)_w$ into $C^1(0,T;H)$.

This completes the proof. □

Remark 5.2.4. If there exists $b > 0$ such that $(Bu,u) \geq b\|u\|^2$ for all $u \in V$, then it can be proved that $f \mapsto x_f$ is continuous from $L^q(H)_w$ to $C(0,T;V)$.

For problem (5.8), we have the following result.

Theorem 5.2.5. *Under (H5.2.1)–(H5.2.5), suppose V is also a Hilbert space. Then problem* (5.8) *has at least one solution $x \in L^p(V)$ with $Bx' + m(x) \in L^q(V^*)$ for each $f \in L^q(V^*)$.*

Proof. Obviously, (5.8) is equivalent to

$$(\tilde{B}z(t))' + N(t,z(t)) \ni \hat{f}(t) \text{ a.e. and } \tilde{B}z(0) = \tilde{B}z_0$$

in the evolution triple (V^2, H^2, V^{*2}) with

$$\tilde{B} = \begin{pmatrix} B & 0 \\ 0 & Q \end{pmatrix}, \ N(t,\cdot) = \begin{pmatrix} A(t,\cdot) & -Q \\ Q & 0 \end{pmatrix}, \ \hat{f} = \begin{pmatrix} 0 \\ f \end{pmatrix}, \ z_0 = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}.$$

We take the duality pairing between V^2 and V^{*2} as

$$\langle\langle (u,v),(x,y)\rangle\rangle = \langle u,x\rangle + \langle v,y\rangle, \text{ for } u,v \in V^*, x,y \in V.$$

Here, in order to make the duality pairing different from the points-pairing $(u,v) \in V^2$ or V^{*2}, we use $\langle\cdot,\cdot\rangle$ to stand for the duality pairing between V and V^*.

Let $z_n := (x_n, y_n) \in V^2, w_n = (u_n, v_n) \in N(t, z_n)$ such that $z_n \rightharpoonup z = (x,y) \in V^2$ and $\limsup_{n\to\infty} \langle\langle (u_n,v_n),(x_n,y_n)-(x,y)\rangle\rangle \le 0$. Then $u_n \in A(t,x_n) - Qy_n, v_n = Qx_n$ and $x_n \rightharpoonup x$, $y_n \rightharpoonup y$, $Qx_n \rightharpoonup Qx$, $Qy_n \rightharpoonup Qy$. Since Q is symmetric, we see that

$$\begin{aligned}(\liminf)\ \limsup_{n\to\infty} \langle\langle (u_n,v_n),(x_n,y_n)-(x^*,y^*)\rangle\rangle &= (\liminf)\ \limsup_{n\to\infty} \langle u_n + Qy_n, x_n - x^*\rangle \\ &\quad + \langle Qy, x^*\rangle - \langle Qx, y^*\rangle, \text{ for all } x^*, y^* \in V. \end{aligned} \tag{5.14}$$

By taking $x^* = x, y = y^*$ in (5.14), we obtain $\limsup_{n\to\infty}\langle u_n + Qy_n, x_n - x\rangle \le 0$, therefore, the pseudo-monotonicity of A implies that, for each $(\hat{x},\hat{y}) \in V^2$, there exists $u^* \in A(t,x)$ such that

$$\langle u^*, x - \hat{x}\rangle \le \liminf_{n\to\infty} \langle u_n + Qy_n, x_n - \hat{x}\rangle. \tag{5.15}$$

Let $\hat{u} = u^* - Qy, \hat{v} = Qx$. Then $(\hat{u},\hat{v}) \in N(t,(x,y))$. Using (5.14) and (5.15), we have

$$\begin{aligned}\langle\langle (\hat{u},\hat{v}),(x,y)-(\hat{x},\hat{y})\rangle\rangle &= \langle u^*, x-\hat{x}\rangle + \langle Qy,\hat{x}\rangle - \langle Qx,\hat{y}\rangle \\ &\le \liminf_{n\to\infty} \langle\langle (u_n,v_n),(x_n,y_n)-(\hat{x},\hat{y})\rangle\rangle.\end{aligned}$$

That is, $N(t,\cdot)$ is pseudo-monotone. Also, it can be proved easily that the other conditions of Theorem 4.3.2 are satisfied in the present situation. So the conclusion follows. □

5.3 Global existence results for general problems

In this section, under (H5.2.1)-(H5.2.4), we consider the (global) existence of solutions to the evolution inclusion

$$x''(t) + A(t,x'(t)) + Bx(t) - F(t,x(t),x'(t)) \ni 0 \quad \text{a.e.} \tag{5.16}$$

$$x(0) = x_0 \in V,\ x'(0) = x_1 \in V \tag{5.17}$$

with $F : [0,T] \times H \times H \to 2^H$ a set-valued mapping satisfying the following assumptions.

(H5.3.1) F is measurable and there exist $b_1 \in L^q(0,T), b_2 \geq 0, b_3 \geq 0$ such that

$$\|F(t,u,v)\|_H \leq b_1(t) + b_2\|u\|_H^{\frac{2}{q}} + b_3\|v\|_H^{\frac{2}{q}} \text{ for a.e. } t;$$

(H5.3.2) $F(t,u,v) \in \mathcal{P}_{cv}(H)$ and $(u,v) \mapsto F(t,u,v)$ is u.s.c. from $H \times H$ into H_w.

or

(H5.3.2′) $F(t,u,v) \in \mathcal{P}_c(H)$ and $(u,v) \mapsto F(t,u,v)$ is l.s.c. from $H \times H$ into H.

We denote the *solution set* of (5.16)-(5.17) by $S(x_0, x_1)$.

To begin the consideration, we first make *a priori* estimates.

Lemma 5.3.1. *Under (H5.2.2)-(H5.2.4) and (H5.3.1), if the solution set $S(x_0,x_1)$ of (5.16)-(5.17) is nonempty, then the set*

$$S_1(x_0,x_1) := \{x' \in W(0,T) : x \in S(x_0,x_1)\}$$

is bounded and hence relatively weakly compact in $W(0,T)$, both $\{\|x(t)\|_H : x \in S(x_0,x_1)\}$ and $\{\|x'(t)\|_H : x \in S(x_0,x_1)\}$ are (uniformly) bounded in H.

Proof. Without loss of generality, we may suppose $x_0 = 0$.

Let $x \in S(0,x_1)$. Then $x' \in W$ and there exist $f(t) \in F(t,x(t),x'(t))$ and $z(t) \in A(t,x'(t))$ such that

$$(x''(t),x'(t)) + (z(t),x'(t)) + (Bx(t),x'(t)) = \langle f(t),x'(t)\rangle \text{ a.e..}$$

We need to prove that there exist constants $M, \overline{M}$ independent of x such that

$$\|x'\|_{L^p(V)}, \|x''\|_{L^q(V^*)} \leq \overline{M}, \quad \|x(t)\|_H, \|x'(t)\|_H \leq M, \text{ for all } t.$$

Since $(x''(t),x'(t)) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|x'(t)\|_H^2$, and $(Bx(t),x'(t)) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(Bx(t),x(t))$, by the coercivity of A, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|x'(t)\|_H^2 + a_3\|x'(t)\|_V^p + a_4(t) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(Bx(t),x(t)) \leq \langle f(t),x'(t)\rangle \text{ a.e.}$$

Integrating this inequality from 0 to t gives

$$\frac{1}{2}\|x'(t)\|_H^2 + a_3\int_0^t \|x'(t)\|_V^p dt + \frac{1}{2}(Bx(t),x(t)) - M_0 \leq \int_0^t \langle f(s),x'(s)\rangle ds, \qquad (5.18)$$

where $M_0 = \int_0^T |a_4(t)|dt + \frac{1}{2}|x_1|_H^2$. Let $\beta > 0$ be a number such that

$$\|u\|_H \le \beta \|u\|_V, \quad \text{for all } u \in V.$$

By our assumptions (H5.2.4), (H5.3.1) and Young's inequality, we have, for each $\varepsilon > 0$

$$\frac{1}{2}\|x'(t)\|_H^2 + a_3 \int_0^t \|x'(s)\|_V^p ds \le M_0 + \int_0^t \left| b_1(s) + b_2\|x(s)\|_H^{\frac{2}{q}} + b_3\|x'(s)\|_H^{\frac{2}{q}} \right| \|x'(s)\|_H ds \tag{5.19}$$

$$\begin{aligned} &\le M_0 + \frac{\varepsilon^p}{p}\int_0^t \|x'(s)\|_H^p ds + \frac{1}{q\varepsilon^q}\int_0^t \left| b_1(s) + b_2\|x(s)\|_H^{2/q} + b_3\|x'(s)\|_H^{2/q} \right|^q ds \\ &\le M_0 + \frac{\beta^p \varepsilon^p}{p} \int_0^t \|x'(s)\|_V^p ds \\ &\quad + \frac{3^q}{q\varepsilon^q}\int_0^t \left[|b_1(s)|^q + b_2^q \|x(s)\|_H^2 + b_3^q \|x'(s)\|_H^2 \right] ds. \end{aligned} \tag{5.20}$$

Choose $\varepsilon > 0$ so that $a_3 = \beta^p \varepsilon^p / p$. Then by Corollary 5.1.3

$$\frac{1}{2}\|x'(t)\|_H^2 \le M_0 + \frac{3^q}{\varepsilon^q q} M_* + \frac{3^q}{\varepsilon^q q}\left(\frac{4T^2}{\pi^2} b_2^q + b_3^q \right) \int_0^t \|x'(s)\|_H^2 ds,$$

where $M_* = \int_0^T |b_1(s)|^q dt$. Let $N = 3^q(4T^2 b_2^q + \pi^2 b_3^q)/(\pi^2 \varepsilon^q q)$. By the generalized Gronwall's inequality (Theorem 1.7.7), we have

$$\|x'(t)\|_H \le \sqrt{2M_0 + 2\frac{3^q}{\varepsilon^q q} M_* \exp(TN)} =: M_1, \quad t \in [0,T].$$

Since x' is continuous in H, we have

$$\|x(t)\|_H \le \int_0^t \|x'(s)\|_H ds \le TM_1 =: M_2, \text{ for all } t \in [0,T].$$

By (5.19)

$$a_3 \int_0^T \|x'(t)\|_V^p dt \le M_0 + \int_0^T \left(|b_1(t)| + b_2 M_2^{2/q} + b_3 M_1^{2/q} \right) M_1 dt.$$

This implies that

$$\|x'\|_{L^p(V)} \le \left(\frac{M_0}{a_3} + \frac{1}{a_3}\int_0^T \left(|b_1(t)| + b_2 M_2^{2/q} + b_3 M_1^{2/q} \right) M_1 dt \right)^{1/p} =: M_3.$$

Since $x' \in L^p(V)$, by Theorem 1.4.7, there exists an absolutely continuous function $x_1 : [0,T] \to V$ such that $x(t) = x_1(t), x'(t) = x_1'(t)$ a.e.. So we obtain

$$\begin{aligned} \|x(t)\|_V = \|x_1(t)\|_V = \| \int_0^t x_1'(s) ds \| &\le \int_0^t \|x_1'(s)\|_V ds \\ &= \int_0^t \|x'(s)\|_V ds \le \|x'\|_{L^p(V)} T^{1/q} \le M_3 T^{1/q} =: M_4 \quad \text{a.e..} \end{aligned}$$

Since for each $u \in L^p(V)$,

$$\begin{aligned}
\int_0^T (x''(t), u(t))dt &\le \int_0^T |(A(t, x'(t)) + Bx(t), u(t)) - \langle f(t), u(t)\rangle| dt \\
&\le \int_0^T \left[a_1(t) + a_2\|x'(t)\|_V^{p-1} + \|B\|\|x(t)\|_V\right] \|u(t)\|_V dt \\
&\quad + \beta \int_0^T \left[b_1(t) + b_2\|x(t)\|_H^{2/q} + b_3\|x'(t)\|_H^{2/q}\right] \|u(t)\|_V dt \\
&\le \int_0^T \left[a_1(t) + \|B\|M_4 + \beta b_1(t) + \beta b_2 M_2^{2/q} + \beta b_3 M_1^{2/q}\right] \|u(t)\| dt \\
&\quad + a_2\|x'\|_{L^p(V)}^{p/q}\|u\|_{L^p(V)} \\
&\le \left[\|a_1\|_{L^q(\mathbb{R})} + \beta\|b_1\|_{L^q(\mathbb{R})} + \|B\|M_4\right] \|u\|_{L^p(V)} \\
&\quad + \left[\beta\left(b_2 M_2^{2/q} + b_3 M_1^{2/q}\right) T^{1/q} + a_2 M_3^{p/q}\right] \|u\|_{L^p(V)},
\end{aligned}$$

we see that

$$\|x''\|_{L^q(V^*)} \le \|a_1\| + \beta\|b_1\| + \|B\|M_4 + \beta\left(b_2 M_2^{2/q} + b_3 M_1^{2/q}\right) T^{1/q} + a_2 M_3^{p/q} =: M_5$$

This completes the proof by taking $M = \max\{M_2, M_1\}, \overline{M} = \max\{M_3, M_5\}$. □

Remark 5.3.2. Even in case F is independent of x' and $p = q = 2$ as in [64], this a priori estimation improves that in [64] since we do not suppose B to be coercive.

Now we give the conditions for $S(x_0, x_1)$ to be nonempty and compact.

Theorem 5.3.3. *Under (H5.2.1)-(H5.2.4), (H5.3.1) and (H5.3.2), suppose $A(t,\cdot)$ is monotone (note, being monotone and pseudo-monotone is equivalent to being monotone and hemicontinuous). Then* (5.16)-(5.17) *admits solutions and the solution set is relatively weakly compact in $W(0,T)$ and compact in $C^1(0,T;H)$. If, in addition, there exist non-negative functions $k_1, k_2 \in L^1(\mathbb{R})$ such that*

$$\langle w_1 - w_2, v_1 - v_2\rangle \le k_1(t)\|v_1 - v_2\|_H^2 + k_2(t)\|u_1 - u_2\|_H\|v_1 - v_2\|_H, \tag{5.21}$$

for all $t \in [0,T], u_i, v_i \in H, w_i \in F(t, u_i, v_i), i = 1, 2,$ then the solution is unique.

Proof. By Lemma 5.3.1, there exists a constant $M > 0$ such that

$$\|x(t)\|_H \le M, \quad \|x'(t)\|_H \le M, \quad t \in [0,T], \quad x \in S(x_0, x_1).$$

Let

$$\hat{F}(t,u,v)=\begin{cases} F(t,u,v) & \text{if } \|u\|_H\le M, \|v\|_H\le M,\\ F(t,\frac{Mu}{\|u\|_H},v) & \text{if } \|u\|_H> M, \|v\|_H\le M,\\ F(t,u,\frac{Mv}{\|v\|_H}) & \text{if } \|u\|_H\le M, \|v\|_H> M,\\ F(t,\frac{Mu}{\|u\|_H},\frac{Mv}{\|v\|_H}) & \text{if } \|u\|_H> M, \|v\|_H> M. \end{cases}$$

Then it is easy to show that $\hat{F}$ maps $[0,T]\times H\times H$ into $\mathcal{P}_{cv}(H)$ satisfying (H5.3.2) and $|\hat{F}(t,u,v)|\le b_1(t)+b_2M+b_3M=:h(t)$ a.e. with $h\in L^q(\mathbb{R})$. Let

$$B=\{f\in L^q(H):\|f(t)\|_H\le h(t)\ \ a.e.\ \}$$

$$R(f)=S^1_{\hat{F}}:=\{g\in L^1(H):g(t)\in\hat{F}(t,x_f(t),x'_f(t))\ \text{ a.e. }\},\ \text{for all } f\in B,$$

where x_f is the solution of (5.7) for a given $f\in B$. By Theorems 5.2.1 and 5.2.3, x_f is a well-defined continuous operator from B_w to $C^1(0,T;H)$. Since $\hat{F}$ is measurable and L^1-integrally bounded, by Theorem 1.3.3, we have that $R(f)\ne\emptyset$. Moreover since $\hat{F}$ is $\mathcal{P}_{cv}(H)$-valued and bounded by $h(t)$, the set-valued function R has closed and convex values and $\|g(t)\|_H\le h(t)$ for every $g\in R(f)$, that is $R:B\to\mathcal{P}_{cv}(B)$.

Next we prove R is u.s.c. under the weak topology. Since B is weakly compact in $L^q(H)$, by Theorem 1.2.10, we need only prove that Graph(R) is weakly closed in $B\times B$. To do this, let $(f_n,x_n)\in$ Graph$(R), f_n\rightharpoonup f, x_n\rightharpoonup x$ in $L^q(H)$. By Theorem 5.2.3, $x_{f_n}(t)\to x_f(t), x'_{f_n}(t)\to x'_f(t)$ in H for all $t\in[0,T]$. Since F is u.s.c., so is $\hat{F}$, and therefore

$$w\text{-}\limsup_n \hat{F}(t,x_{f_n}(t),x'_{f_n}(t))\subset\hat{F}(t,x_f(t),x'_f(t)), a.e.$$

Using Theorem 1.3.4, we have

$$\begin{aligned} x\in w\text{-}\limsup_n R(f_n)&=w\text{-}\limsup_n S^1_{\hat{F}(\cdot,x_{f_n}(\cdot),x'_{f_n}(\cdot))}\\ &\subset S^1_{w\text{-}\limsup_n \hat{F}(\cdot,x_{f_n}(\cdot),x'_{f_n}(\cdot))}\subset S^1_{\hat{F}(\cdot,x_f(\cdot),x'_f(\cdot))}=R(f). \end{aligned}$$

This means that $(f,x)\in$ Graph(R) and so Graph(R) is weakly closed.

Exploiting Kakutani's fixed point theorem (Theorem 1.5.4), we deduce that there exists $f\in B, f\in R(f)$. Obviously $x=x_f$ is a solution to the inclusion (5.16)-(5.17) with the modified set-valued function $\hat{F}$. Using the same method as that in Lemma

5.3.1, we can show that $\|x(t)\|_H \leq M, \|x'(t)\|_H \leq M$ uniformly, so $\hat{F}(t, x(t), x'(t)) = F(t, x(t), x'(t))$, and therefore x is a solution of (5.16)-(5.17).

Note that $S(x_0, x_1) \subset r(B)$ with r the mapping $f \mapsto x_f$. Since B is bounded and weakly compact in $L^q(H)$, by Theorem 5.2.3, $S(x_0, x_1)$ is relatively compact in $C^1(0, T; H)$ and weakly relatively compact in $W(0, T)$. Now let $x_n \in S(x_0, x_1), x_n \to x$ in $C^1(0, T; H), x_n = x_{f_n}$ with $f_n(t) \in F(t, x_n(t), x_n'(t))$ a.e.. Obviously, $f_n \in B$, so by passing to a subsequence, we may suppose $f_n \rightharpoonup f$ in $L^q(H)$. Using Theorem 5.2.3 once more, we see $x_n(t) \to x_f(t) = x(t), x_n'(t) \to x_f'(t) = x'(t)$. So the continuity of F implies $f(t) \in F(t, x(t), x'(t))$ a.e. and, therefore, $x \in S(x_0, x_1)$, that is $S(x_0, x_1)$ is closed in $C^1(0, T; H)$ which implies the compactness of $S(x_0, x_1)$.

To prove the uniqueness under (5.21), let x, y be two different solution of (5.16)-(5.17). Then there exist $f(t) \in F(t, x(t), x'(t)),\ \ g(t) \in F(t, y(t), y'(t))$ a.e. such that

$$x''(t) + A(t, x'(t)) + Bx(t) \ni f(t), \quad y''(t) + A(t, y'(t)) + By(t) \ni g(t).$$

By the method used for obtaining (5.13) and noting (5.21), we have

$$\begin{aligned}
\frac{1}{2}\|x'(t) - y'(t)\|_H^2 &\leq \int_0^t \langle f(s) - g(s), x'(s) - y'(s) \rangle ds \\
&\leq \int_0^t k_1(s)\|x'(s) - y'(s)\|_H^2 ds \\
&\quad + \int_0^t k_2(s)\|x(s) - y(s)\|_H \|x'(s) - y'(s)\|_H ds \\
&\leq \int_0^t (k_1(s) + Tk_2(s))\|x'(s) - y'(s)\|_H^2 ds.
\end{aligned}$$

So $\|x'(s) - y'(s)\|_H^2 = 0$ according to Gronwall's inequality and, therefore, $x = y$.

This completes the proof. □

Remark 5.3.4. Theorem 5.3.3 generalizes Theorem 3.1 of [64] not only in the form of inclusion but also in the conditions since we do not suppose B to be coercive and (H5.3.2), (H5.2.3) are also weaker than the corresponding hypotheses in [64] where the author only considered the case of $p = q = 2$ and F is independent of x'. Theorem 3.2 in [64] can also be generalized to the following.

Theorem 5.3.5. *Under (H5.2.1)-(H5.2.4) and (H5.3.1), (H5.3.2′), suppose $A(t,\cdot)$ is monotone. Then* (5.16)-(5.17) *admits solutions, and the solution is unique if* (5.21) *is satisfied.*

Proof. Let $\hat{F}, R, h$ be the same as in the proof of Theorem 5.3.3 with B, the domain of R, replaced by

$$K = \{f \in L^1(H) : \|f(t)\|_H \leq h(t)\}.$$

By Lemma 5.3.1, $\hat{F}$ is l.s.c. and takes values in $\mathcal{P}_c(H)$.

It is known that K is weakly compact in $L^1(H)$ (see for example Corollary IV.8.11 in [35]). If $f_n \in K, f_n \rightharpoonup f$ in $L^1(H)$, since K is also weakly compact in $L^q(H)$, we may suppose $f_n \rightharpoonup f$ in $L^q(H)$, which implies (by Theorem 5.2.3) that $f \mapsto x_f$ is continuous from K_w to $C^1(0, T_0; H)$. So by (H5.3.2′) and Theorem 1.3.4, R is l.s.c. from K_w to $L^1(H)$. Applying Theorem 1.3.8, we see that R has a continuous selection $\eta : K_w \to K_w$ and $\eta(f) \in R(f)$. So η admits a fixed point f by the fixed point theorem and $x = x_f$ is a solution of (5.16)-(5.17).

The proof for the uniqueness is the same as that in Theorem 5.3.3. □

5.4 Local existence results for general problems

In this section, instead of the growth condition (H5.3.1), we suppose F satisfies a bounded condition to consider the local existence of inclusion (5.16)-(5.17).

Theorem 5.4.1. *Under (H5.2.1)-(H5.2.4), suppose that $A(t,\cdot)$ is monotone for* a.e. t *and, for any bounded subset $D \subset H$, there exists $M > 0$ such that*

$$\sup\{\|F(t,u,v)\|_H : u, v \in D, t \in [0,T]\} \leq M.$$

If either (H5.3.2) or (H5.3.2′) is satisfied, then (5.16)-(5.17) *admits solutions on* $[0, T_0]$ *for some* $T_0 \in (0, T]$.

Proof. By our assumption on F, there are constants $k > 0, M > 0, T_0 \in (0, T]$ such that

$$\sup\{\|F(t,u,v)\|_H : u, v \in D, t \in [0,T_0]\} \leq M, \tag{5.22}$$

$$T_0 M \leq k, \quad T_0 d + \frac{1}{2} T_0^2 M \leq k,$$

where $D = \{u \in H : |u| \leq \max\{\|x_0\|_H, d\} + k\}$, $\;d = \sqrt{\|x_1\|_H^2 + 2(Bx_0, x_0)}$.

Let

$$D_1 = \{x \in L^q(H) : \|x(t)\|_H \leq M \text{ a.e. on } [0, T_0]\}.$$

Then $D_1 \subset L^\infty(0, T_0; H)$ and by Theorem 5.2.3, $D_2 := \{x_f : f \in D_1\}$ is bounded in $C^1(0, T_0; H)$ (note x_f is the unique solution of (5.7)), and for $f \in D_1, x = x_f$ is such that

$$(x''(t), x'(t)) + (A(t, x'(t)), x'(t)) + (Bx(t), x'(t)) = (f(t), x'(t)) \text{ a.e..}$$

By the same method as used to get (5.18), and using our assumptions, we have

$$\|x'(t)\|_H^2 \leq \|x_1\|_H^2 + 2(Bx_0, x_0) + 2\int_0^t \|f(s)\|_H \|x'(s)\|_H ds.$$

By Theorem 1.7.7, we have

$$\|x'(t)\|_H \leq d + \int_0^t \|f(s)\|_H ds \leq d + tM \leq d + k, \tag{5.23}$$

$$\|x(t)\| \leq \|x_0\|_H + \int_0^t \|x'(s)\|_H ds \leq \|x_0\|_H + td + \frac{1}{2}t^2 M \leq \|x_0\|_H + k \tag{5.24}$$

on $[0, T_0]$, that is $x'(t), x(t) \in D$. For each $x \in D_2$, by Theorem 1.3.3 , $K_0(x) := S^1_{F(\cdot, x(\cdot), x'(\cdot))} \neq \emptyset$. So if $v \in K_0(x)$, then $v(t) \in F(t, x(t), x'(t))$ a.e. and $x = x_f$ with some $f \in D_1$. Therefore (5.23), (5.24) and (5.22) imply that $\|v(t)\|_H \leq M$ a.e., that is $K_0(x) \subset D_1$ is bounded in $L^q(0, T_0; H)$ and $K_0(x) := S^q_{F(\cdot, x(\cdot), x'(\cdot))}$.

Let

$$K = \overline{\text{co}} \bigcup_{x \in D_2} K_0(x),$$

and consider the set-valued mapping

$$R(f) = S^1_{F(\cdot, x_f(\cdot), x_f'(\cdot))} \text{ for } f \in D_1.$$

Since D_1 is convex and closed in every $L^s(H)$ $(s \geq 1)$ space, we have $K \subset D_1$, and R maps K into itself.

If (H5.3.2) holds, we consider K as a subset of $L^q(0, T_0; H)$. Since K is closed, bounded and convex, K is a weakly compact subset of $L^q(0, T_0; H)$. By the same method as that used in Theorem 5.3.3, R is weakly-weakly u.s.c. and admits a fixed point f, and $x = x_f$ is the solution of (5.16)-(5.17) on $[0, T_0]$.

If (H5.3.2$'$) holds, we consider K as a subset of $L^1(0, T_0; H)$. Using the same method as used in Theorem 5.3.5 (replace $[0, T]$ by $[0, T_0]$), we see that R has a continuous selection $\eta : K_w \to K_w$ and η admits a fixed point f and, therefore, $x = x_f$ is a solution of (5.16)-(5.17) on $[0, T_0]$.

This completes the proof. $\square$

Remark 5.4.2. Using the above idea, some similar local existence results can also be obtained for the first order differential evolution inclusion studied in [55], the method is almost the same as the one used above.

5.5 Examples

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega, V = W_0^{m,2}(\Omega), H = L^2(\Omega), V^* = W^{-m,2}(\Omega)$, then (V, H, V^*) is a Gelfand triple and all the imbeddings are compact.

Consider the following inclusion

$$\frac{\partial^2 x(t,z)}{\partial t^2} + a(t,z,x(t,z)) + \triangle x \in b(t,z,x(t,z)) \text{ for } (t,z) \in [0,T] \times \Omega \text{ a.e.,} \tag{5.25}$$

$$D^\beta x(t,z) = 0 \quad \text{on} \quad [0,T] \times \partial\Omega \quad \text{for} \quad |\beta| \le m-1,$$

$$x(0,z) = x_0(z) \in W_0^{m,2}(\Omega), \quad \frac{\partial x(0,z)}{\partial t} = x_1(z) \in L^2(\Omega) \text{ on } \Omega, \tag{5.26}$$

with

$$a(t,z,x(t,z)) = \sum_{|\alpha| \le m} (-1)^{|\alpha|} D^\alpha A_\alpha(t,z,x,\cdots,D^m x),$$

$$b(t,z,x(t,z)) = G\big(t,z,x(t,z),\partial x(t,z)/\partial t\big)$$

and G is a set-valued function.

For each $|\alpha| \le m$, we suppose $A_\alpha : [0,T] \times \Omega \times \mathbb{R}^n \to \mathbb{R}$ satisfies

(i) $(t,z) \to A_\alpha(t,z,\xi)$ is measurable for all $\xi \in \mathbb{R}^n$ and $\xi \to A_\alpha(t,z,\xi)$ is continuous for all $(t,z) \in [0,T] \times \partial\Omega$;

(ii) $\sum_{|\alpha| \le m} (A_\alpha(t,z,\xi) - A_\alpha(t,z,\xi'))(\xi_\alpha - \xi'_\alpha) \ge 0$ for all $\xi', \xi \in \mathbb{R}^n, t \in [0,T], z \in \Omega$;

(iii) there exist $a, c > 0, h \in L^2([0,T] \times \overline{\Omega})$ such that $\sum_{|\alpha| \le m} A_\alpha(t,z,\xi)\xi \ge c|\xi|^2$, $|A_\alpha(t,z,\xi)| \le a|\xi| + h(t,z)$ for all $\xi \in \mathbb{R}^n, t \in [0,T], z \in \overline{\Omega}$

Define the nonlinear mapping $A(t,x) : [0,T] \times V \to V^*$, and the linear mapping operator $B : V \to V^*$ by

$$
\begin{aligned}
(A(t,x),y) &= \sum_{|\alpha| \le m} \int_\Omega A_\alpha(t,z,\xi(x)) D^\alpha y dz, \quad \text{for } x,y \in V,\ t \in [0,T], \\
(Bx,y) &= \int_\Omega DxDydz, \quad \text{for } x,y \in V.
\end{aligned}
$$

Then, by standard arguments (see [80]), the operators A, B satisfy the conditions (H5.2.1)-(H5.2.4) in the previous section respectively with $p = q = 2$ and $A(t,\cdot)$ is monotone. Problem (5.25)-(5.26) can be rewritten as

$$
x''(t) + A(t,x'(t)) + Bx(t) \in F(t,x(t),x'(t)) \tag{5.27}
$$

$$
x(0) = x_0 \in V,\ x'(0) = x_1 \in H, \tag{5.28}
$$

here F is a set-valued function from $[0,T] \times H \times H$ to 2^H corresponding to G. We can impose some suitable assumptions on G to ensure F satisfies the conditions (H5.3.1), (H5.3.2) or (H5.3.2$'$) so that (5.27)-(5.28) (i.e. (5.25)-(5.26)) admits solutions and the solution set is compact in $C^1(0,T;H)$. For example, we may let

$$
G(t,z,x,y) = [f_1(t,z,x,y), f_2(t,z,x,y)]
$$

with $f_1, f_2 : [0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ measurable functions, continuous with respect to the last two variables, $f_1 < f_2$ and $|f_i(t,z,x,y)| \le b_1(t,z) + b_2|x| + b_3|y|, b_1 \in L^2([0,T] \times \Omega), b_2 > 0, b_3 > 0$. Then

$$
F(t,x,y) := \{h \in H : f_1(t,z,x(z),y(z)) \le h(z) \le f_2(t,z,x(z),y(z)) \ \text{ a.e.}\}
$$

satisfies the conditions in Theorem 5.3.3. In particular, if $f_i(t,z,x,y) = f(t,z,x,y)u_i(z), i = 1,2,\ 0 < u_1(z) < u_2(z) < M$ a.e., and f satisfies the assumptions on f_1, then (5.25)-(5.26) corresponds to a control problem with the control constraint

$$
U(t,z) = \{\nu \in \mathbb{R} : u_1(z) \le \nu \le u_2(z)\}.
$$

Additionally, if we introduce a cost function $J(x) = \int_0^T \int_\Omega L(t,z,x(t,z))dzdt$ which is to be minimized over all admissible trajectories, and suppose it is lower semicontinuous in $C(0,T;H)$, then, since the solution set is compact, we see that the distributed parameter optimal control problem has a solution.

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