Geometric Structures on the Target Space of Hamiltonian Evolution Equations

by

James Ferguson

A thesis submitted to the

Faculty of Information and Mathematical Sciences

at the University of Glasgow

for the degree of

Doctor of Philosophy

February 2008

Abstract

This thesis is concerned with the relationship between integrable Hamiltonian partial differential equations and geometric structures on the manifold in which the dependent variables take their values.

Chapters 1 and 2 are introductory chapters, and as such contains no original material. Chapter 1 covers some basic material from the theory of integrable systems, including the Hamiltonian formalism for PDE's, the concept of a bi-Hamiltonian system, and the dispersionless Lax equation. Chapter 2 is about Frobenius manifolds. It explains their relationship to the WDVV equations of topological quantum field theory, and how they form part of the theory of integrable systems via both the deformed Levi-Civita connection and a flat pencil of metrics.

Chapter 3 is based on [39], which is to appear in the Journal of Geometry and Physics. It is original, except for the background material in Section 3.1. In it we explain the (almost) symplectic geometry associated to Hamiltonian operators of degree 2, and use it to formulate the geometric conditions for two such operators to constitute a bi-Hamiltonian structure. In the case that these operators are associated to symplectic forms, these conditions are expressed as algebraic constraints on a multiplication of one-forms. We also express conditions for a Hamiltonian operator of degree two to be compatible with a hydrodynamic type Hamiltonian operator.

Chapter 4 is based upon [40], which was joint work with Ian Strachan. It is to appear in Communications in Mathematical Physics, and is original except for Section 4.1. It is concerned with the construction of new solutions to the WDVV equations which arise by analogy with the so-called waterbag reductions of the dispersionless KP hierarchy. Superpotentials of existing Frobenius manifolds are deformed by the addition of logarithmic terms, and this results in new WDVV solutions which deform existing ones, including a new class of polynomial solutions which deform solutions associated to the A_N Coxeter

group.

Chapter 5 follows on from Chapter 4, and considers in detail two integrable hierarchies which arise from the WDVV solutions studied there. It is particularly concerned with the Hamiltonian structures of these hierarchies. Appendix A attempts to incorporate some of the features of one of these hierarchies into a construction of a Frobenius structure from a bi-Hamiltonian structure.

Acknowledgements

I would like to begin by thanking my supervisor Ian Strachan for the advice and encouragement he has offered to me throughout the preparation of this thesis. Thanks are also due to Jonathan Nimmo who was also responsible for part of my supervision, and whose undergraduate course was one of the first things to interest me in this area of research. I would also like to thank the Carnegie Trust for the Universities of Scotland, and the Department of Mathematics of the University of Glasgow for the financial support without which the production of this thesis would have been impossible.

I should also thank my various office mates over the years, both of Team 309 and of Team 428, for making this period of study more enjoyable than perhaps it should have been. Similarly, I would like to thank Paul Melvin for many useful conversations. Finally, I am grateful for the kindness and support which has been given to me over the years by my parents and my brother.

Contents

Chapter 1

Integrable Systems

1.1 Introduction

The theme of this thesis is the use of finite-dimensional differential geometry to study partial differential equations, and especially evolutionary ones. Since such equations involve flows on spaces of functions, the natural setting for a geometric interpretation of them is infinite-dimensional; however, by considering special subclasses one can uncover underlying finite-dimensional differential geometry.

For instance, consider the general evolution equation of hydrodynamic type

$$u^i_t = A^i_j(u)u^j_x\,.$$

Under a change of dependent variables $\tilde{u}^i = \tilde{u}^i(u)$, the equation becomes

$$\frac{\partial u^i}{\partial \tilde{u}^r}\tilde{u}^r_t = A^i_j\frac{\partial u^j}{\partial \tilde{u}^s}\tilde{u}^s_x\,,$$

or

$$\tilde{u}^i = \frac{\partial \tilde{u}^i}{\partial u^r}A^r_s\frac{\partial u^s}{\partial \tilde{u}^j}\tilde{u}^j_x\,.$$

We regard the change of dependent variables as a coordinate transform on the space spanned by the u^i, i.e. on the manifold in which the function $u(x,t) = (u^1(x,t),\ldots,u^n(x,t))$ takes its values. From this perspective, we see that the coefficient functions $A^i_j(u)$ have transformed as the components of a $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$-tensor.

In such papers as [61] Novikov put forward a programme in which the infinite-dimensional Poisson geometry of Hamiltonian PDE's is understood in terms of finite-dimensional geometry on the space spanned by their dependent variables. For example, as is discussed in greater detail in Section 1.4, a class of Poisson brackets appropriate to certain equations of hydrodynamic type are the so-called Dubrovin-Novikov brackets (or hydrodynamic type Poisson brackets), which are specified by an operator of the form

$$P^{ij} = g^{ir}\left(\delta^j_r \frac{d}{dx} - \Gamma^j_{rk} u^k_x\right) ,$$

in which the coefficients g^{ij} and Γ^k_{ij} are the components of a flat metric and the Christoffel symbols of its Levi-Civita connection respectively. This enables many results from the theory of these geometric objects, such as the construction of special coordinate systems, to be applied to such PDE's.

Dubrovin has studied in detail the problem of constructing pairs of Poisson brackets satisfying a compatibility relation described in Section 1.5. Underlying this relationship is the notion of a flat pencil of metrics, but further still, one may, if the pair of Poisson brackets possesses some further symmetry, recover the geometry of Frobenius manifolds.

In this chapter, we begin with a discussion of the differential equations which have motivated the Novikov programme, and then introduce the Hamiltonian and bi-Hamiltonian formalism which may be used to describe them. Frobenius manifolds will be dealt with in Chapter 2.

1.2 Soliton Equations

Throughout this chapter, we shall take as our main example the Korteweg-de Vries, or KdV, equation [22]:

$$u_t = 6uu_x + u_{xxx} \,. \tag{1.1}$$

This was derived as a model of the one-dimensional propagation of waves in shallow water. In this context, x is the direction of propagation, t is time, and $u(x,t)$ is the height of the water above its equilibrium level. A relevant solution is

$$u(x,t) = 2\kappa^2 \mathrm{sech}^2(\kappa(x + 4\kappa^2 t)) \,, \tag{1.2}$$

which describes a large hump of displaced water moving leftwards along the channel, without changing shape or speed, quite contrary to the dispersive behaviour associated with low amplitude waves.

Further results, for example [41, 73], established the existence of solutions describing the interaction of several such solitary waves, in which the various 'humps' pass through one another, temporarily merging but eventually separating and recovering their original shapes and speeds, the only effect of their interaction being an advancement or retardation of their position compared to where they would have been had they continued moving at constant speed.

1.2.1 Inverse Scattering and Lax Pairs

The initial value problem for the KdV equation can be solved by first considering the function $u(x,t)$ solving (1.1) as the potential in the time-independent Schrödinger equation

$$\psi_{xx} + u(x)\psi = \lambda\psi\,. \tag{1.3}$$

When the potential u is allowed to evolve in time according to the KdV equation, the resultant time evolution of this eigenvalue problem enjoys some special properties. In particular, it is isospectral, which is to say that the spectrum of eigenvalues λ for which (1.3) has a non-zero solution ψ does not change with time. Since the spectrum is constant, one can consider the time evolution of the eigenfunctions and, most importantly, this time evolution is linear. Specifically, we have

$$\psi_t = \left(4\partial^3 + 6u\partial + 3u_x + \alpha(t)\right)\psi\,, \tag{1.4}$$

where α is some arbitrary function of t alone, and $\partial = \frac{\partial}{\partial x}$.

Further analysis of (1.3) is required if (1.4) is to be used to solve the KdV equation. Typically, one takes the view in (1.3) that $u(x)$ is given, and then one attempts to solve for λ and ψ. In an experimental set-up, however, it is more likely that one would have knowledge about ψ and λ, from which one desires to reconstruct u. In the context of quantum mechanics, $u(x)$ would be the potential well of some object such as an atom, and the eigenfunctions ψ would be the wave functions of electrons passing through it. Owing to the small scale of the considered objects, an experimenter can only reasonably expect to have information about the eigenfunctions as $x \to \pm\infty$. It can also be reasonably supposed that $u(x)$ falls rapidly to zero as x approaches $\pm\infty$; this assumption is sensible in the context of the KdV equation if one wishes to consider localised disturbances.

The eigenvalues of $\partial^2 + u(x)$ fall into two categories. First, all $\lambda < 0$ are eigenvalues, and one can choose eigenfunctions with asymptotic behaviour

$$\psi(x;k) \sim \begin{cases} e^{ikx} + b(k)e^{-ikx} & x \to +\infty \\ a(k)e^{ikx} & x \to -\infty \end{cases}\,, \tag{1.5}$$

where $\lambda = -k^2$.

For $\lambda > 0$, one has a discrete set $\{\lambda_1, \ldots, \lambda_n\}$ of eigenvalues, and the associated eigenfunctions can be chosen to have the following asymptotic behaviour:

$$\psi_i(x) \sim \begin{cases} e^{-\kappa_i x} & x \to +\infty \\ a_i e^{\kappa_i x} & x \to -\infty \end{cases} , \tag{1.6}$$

where $\lambda_i = \kappa_i^2$. Because of the exponential decay, these eigenfunctions represent bound states of the wave function, and thus a relevant quantity to ψ_i is the normalisation constant $c_i > 0$ such that

$$\int_{-\infty}^{\infty} c_i^2 \, |\psi_i|^2 \, dx = 1 \, .$$

The important point of this is that knowledge of the discrete spectrum, $\{\kappa_1, \ldots, \kappa_n\}$, the normalisation constants $\{c_1, \ldots, c_n\}$, and the reflection coefficients $b(k)$ is sufficient to reconstruct the potential $u(x)$. The data $\{\kappa_i\}$, $\{c_i\}$ and $b(k)$ are referred to as the scattering data.

By considering (1.4) applied to the asymptotics of the eigenfunctions (1.5) and (1.6), remembering that $u(x) \to 0$ as $x \to \pm\infty$ and making judicious choices of the function $\alpha(t)$, one has that the scattering data satisfy the system of linear ordinary differential equations

$$\begin{aligned} \frac{db(k)}{dt} &= 8ik^3 b(k) \, , \\ \frac{dc_j}{dt} &= 4\kappa_j^3 c_j \, , \\ \frac{d\kappa_j}{dt} &= 0 \, . \end{aligned}$$

Thus, the Korteweg-de Vries equation can be transformed into this set of linear equations.

The crucial point in this linearisation was the time evolution equation (1.4) satisfied by the eigenfunctions. It was this result which Lax [54] generalised. If one has a spectral problem

$$L\psi = \lambda\psi \tag{1.7}$$

for some differential operator L, and then asks that the eigenfunctions ψ evolve in time according to

$$\psi_t = M\psi \, , \tag{1.8}$$

where M is some other differential operator, then one has that the operators L and M must satisfy the relation

$$L_t + [L, M] = 0 \, , \tag{1.9}$$

called the Lax equation. In the case above, (1.9) gives the KdV equation when L and M are taken to be the operators implicit in (1.3) and (1.4) respectively.

1.2.2 The Kadomstev-Petviashvii Hierarchy

If we take as our spectral operator the pseudo-differential operator

$$L = \partial + a_1\partial^{-1} + a_2\partial^{-2} + \dots\,, \tag{1.10}$$

where the coefficients $a_1, a_2, \dots$ are functions of x, and define the operators M_i for $i = 1, 2, 3, \dots$ by

$$M_i = \left[L^i\right]_+\,, \tag{1.11}$$

where $[\,\cdot\,]_+$ denotes the projection of a pseudo differential operator onto the non-negative powers of ∂, and L^i indicates the composition of i copies of L, then the flows of the Kadomstev-Petviashvii (KP) hierarchy are given by the Lax equations

$$L_{t_i} + [L, M_i] = 0\,. \tag{1.12}$$

The term hierarchy is used here to encapsulate two aspects of the set of equations (1.12): firstly, that they are ordered, in this case by the index i; secondly, that they commute. This latter property means, when one uses (1.12) to define operators $\frac{\partial}{\partial t_i}$ on (functions of) the coefficient functions $a_1, a_2, \dots$, that these operators commute. This commutativity is expressed via the zero-curvature condition

$$\frac{\partial M_j}{\partial t_i} - \frac{\partial M_i}{\partial t_j} + [M_i, M_j] = 0\,.$$

This means that the functions $a_1, a_2, \dots$, as well as being functions of x, are also functions of an infinite set $\{t_1, t_2, \dots\}$ of time variables. The commutativity of the operators $\frac{\partial}{\partial t_i}$ means that the flows associated to them commute, so that one may allow a solution to evolve in t_i for a period δ_i, and then evolve in t_j for a period δ_j, and this is the same as if one had first allowed it to evolve in t_j for time δ_j and then in t_i for δ_i.

If one has a solution $(a_1(x, t_i, t_j), a_2(x, t_i, t_j), \dots)$ of (1.12) for some pair of time variables t_i, t_j, one may regard this as a one-parameter family of solutions of (1.12) for t_i parameterised by the value of t_j.

One of the most important properties of the KP hierarchy is that it contains many other hierarchies as reductions. For instance, the KdV equation is obtained by imposing the restriction

$$L^2 = [L^2]_+ = \partial^2 + u$$

on L. Then the KdV equation is equivalent to (1.12) with $i = 3$. Note that the commutativity property of the hierarchy is preserved in this reduction process, and so the KdV equation is also contained within a hierarchy of commuting flows.

More generally, the Gel'fand-Dikiĭ, or generalised KdV, hierarchies [42] are obtained by imposing

$$L^{n+1} = [L^{n+1}]_+ = \partial^{n+1} + u_1\partial^{n-1} + u_2\partial^{n-2} + \cdots + u_n \,,$$

for $n \geq 1$. The flows for the n^{th} KdV hierarchy are given by (1.12) where L satisfies this restriction. Equivalently, one may define them as

$$\frac{\partial L^{n+1}}{\partial t_i} + [L^{n+1}, M_i] = 0 \,,$$

in which, since $M_i = [L^i]_+$, one deals only with differential operators.

1.2.3 Conservation Laws

A conservation law for a partial differential equation with independent variables x and t is an equation

$$\frac{\partial T}{\partial t} = \frac{\partial X}{\partial x} \tag{1.13}$$

where T and X are expressions in the independent variables, and in the dependent variables and their derivatives. We are particularly interested in conservation laws in which the values of the functional densities T and X at a point x depend only on the fields u and their x-derivatives at the point x, and have no explicit dependence on the independent variables.

The meaning of the term conservation law comes from consideration of the integral

$$I = \int_N T dx \,. \tag{1.14}$$

Here the set N is the range of values the independent variable x may take in the problem. For instance it may be that $N = (-\infty, \infty)$, in which case one supposes that the dependent variables and the expressions T and X vanish rapidly as $x \to \pm\infty$; alternatively, it may be that the equation is defined for $x \in S^1$, as in Section 1.4.

From (1.13) and (1.14) we have

$$\begin{aligned} \frac{dI}{dt} &= \frac{d}{dt}\int_N T dx \,, \\ &= \int_N \frac{\partial T}{\partial t} dx \,, \\ &= \int_N \frac{\partial X}{\partial x} dx \,, \\ &= [X]_{\partial N} \,. \end{aligned}$$

In the two situations described above, X will vanish on the boundary ∂N of N. In the first case, $N = (-\infty, \infty)$, this is because X vanishes as $x \to \pm\infty$; whilst in the second case, $N = S^1$, it is because ∂N is empty. Hence

$$\frac{dI}{dt} = 0 \,.$$

So I is a conserved quantity of the differential equation. One refers to the density T as a conserved density.

The KdV equation can be written in the form of a conservation law via

$$\begin{aligned} u_t &= 6uu_x + u_{xxx} \,, \\ &= \frac{\partial}{\partial x}\left(3u^2 + u_{xx}\right) \,, \end{aligned}$$

which demonstrates that

$$I = \int u dx$$

is a conserved quantity.

One also has

$$\begin{aligned} \frac{\partial}{\partial t}(u^2) &= 2uu_t \,, \\ &= 12u^2u_x + 2uu_{xxx} \,, \\ &= \frac{\partial}{\partial x}\left(4u^3 + 2uu_{xx} - u_x^2\right) \,, \end{aligned}$$

so

$$I = \int u^2 dx$$

is also conserved.

In the case of the KdV equation, one may, in principle, continue this process, finding a conserved density with a highest order term u^n for all n; however, the computations become unmanageable very quickly. The approach taken to demonstrate that there exists an infinite chain of conserved quantities for the KdV equation in [58] is to first consider the change of dependent variables

$$u = w - \hbar w_x - \hbar^2 w^2 \tag{1.15}$$

in (1.1). With this the KdV equation is equivalent to

$$\left(1 - 2\hbar^2 w - \hbar\partial\right) w_t = \left(1 - 2\hbar^2 w - \hbar\partial\right)\left\{6ww_x + w_{xxx} - 6\hbar^2 w^2 w_x\right\} \,, \tag{1.16}$$

so, if w satisfies

$$w_t = 6ww_x + w_{xxx} - 6\hbar^2 w^2 w_x \,, \tag{1.17}$$

then u satisfies the KdV equation.

If we expand w as a formal power series in $\hbar$ as

$$w = \sum_{r\geq 0} \hbar^n w_n \tag{1.18}$$

then we can invert the transformation (1.15) and write the coefficients w_n in terms of u and its x-derivatives. On the ring of formal power series in $\hbar$, the operator $\left(1 - 2\hbar^2 w - \hbar\partial\right)$ has kernel $\{0\}$, so if w satisfies (1.16), then it must satisfy (1.17). (1.17) is in the form of a conservation law

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left(3w^2 + w_{xx} - 2\hbar^2 w^3\right) \,,$$

so

$$I_\hbar = \int w dx$$

is a conserved quantity for (1.17), and consequently for the KdV equation. Since this is true for all $\hbar$, then each of the terms w_n in (1.18) is conserved.

For n odd, one finds that w_n is a full x-derivative, and therefore a trivial conserved quantity; however, each w_{2n} is a non-trivial conserved quantity, and contains, as its highest power in u, a term proportional to u^{n+1}. This therefore establishes the result that the KdV equation has an infinite chain of conserved quantities [58].

So there are two infinite chains associated with the KdV equation: that of its commuting flows, and that of its conserved quantities. These are the aspects of integrability which will be of interest in this thesis.

As one may expect from Noether's theorem in classical mechanics, there is a link between these aspects. In the KP hierarchy, for instance, one has, from (1.12),

$$\frac{\partial L^j}{\partial t_i} = [M_i, L^j].$$

If we define the residue $\langle\psi\rangle$ of a pseudo-differential operator ψ as the coefficient f_{-1} of ∂^{-1} when it is written in the form

$$\psi = \sum_{r=-\infty}^{N} f_r \partial^r \,,$$

then

$$\frac{\partial \langle L^j\rangle}{\partial t_i} = \left\langle \frac{\partial L^j}{\partial t_i} \right\rangle = \left\langle [M_i, L^j] \right\rangle \,. \tag{1.19}$$

Since the coefficient of ∂^{-1} in a commutator is always a full x-derivative [1], (1.19) is a conservation law.

If the system being studied can be put into a Hamiltonian form, then this link between commuting flows and conserved quantities becomes clearer; further, if the system is bi-Hamiltonian, then one is provided with a systematic means of generating these conserved quantities and commuting flows.

1.3 Finite-Dimensional Hamiltonian Systems

Hamilton's equations (see e.g. [11, 70]) for a finite-dimensional system with position coordinates q^i and associated momenta p_i, for $i = 1, \ldots, n$, are

$$\begin{aligned} \frac{dq^i}{dt} &= \frac{\partial H}{\partial p_i}\,, \\ \frac{dp_i}{dt} &= -\frac{\partial H}{\partial q^i}\,, \end{aligned}$$

where $H = H(q^1, \ldots, q^n, p_1, \ldots, p_n)$ is the Hamiltonian function of the system. One can understand these equations geometrically by introducing the Poisson bracket $\{\cdot, \cdot\}$ of two functions f and g:

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} \right) . \tag{1.20}$$

Hamilton's equations are then the equations describing the flow of the vector field X_H defined by $X_H(f) = \{f, H\}$ for all functions f.

One may introduce Hamilton's equations in the more general setting of Poisson geometry [12, 69].

Definition 1.3.1. *A Poisson bracket on an n-dimensional manifold M is a map $\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$, $(f, g) \mapsto \{f, g\}$, which satisfies, for any functions f, g, h on M, the four conditions:*

- *antisymmetry:* $\{f, g\} = -\{g, f\}$,
- *linearity:* $\{af + bg, h\} = a\{f, h\} + b\{g, h\}$ *for any constants* a, b ,
- *product rule:* $\{fg, h\} = f\{g, h\} + g\{f, h\}$,
- *Jacobi identity:* $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$.

A Manifold equipped with a Poisson bracket is called a Poisson manifold.

A vector field X on M is called a Hamiltonian vector field if it is of the form $X = \{\cdot, H\}$, i.e. if X acts on functions f as $X(f) = \{f, H\}$.

The first three conditions identify $\{\cdot,\cdot\}$ as a bivector: a rank two, antisymmetric, contravariant tensor field ω on M. It can therefore be represented, by introducing coordinates $\{u^i\}$ on M, as a matrix of coefficients ω^{ij}, giving

$$\omega = \omega^{ij}\frac{\partial}{\partial u^i}\otimes\frac{\partial}{\partial u^j} = \frac{1}{2}\omega^{ij}\frac{\partial}{\partial u^i}\wedge\frac{\partial}{\partial u^j}\,,$$

and

$$\{f,g\} = \omega^{ij}\frac{\partial f}{\partial u^i}\frac{\partial g}{\partial u^j}\,. \tag{1.21}$$

The Jacobi identity is then equivalent to the following constraint on the components of ω:

$$\omega^{ir}\frac{\partial\omega^{jk}}{\partial u^r} + \omega^{jr}\frac{\partial\omega^{ki}}{\partial u^r} + \omega^{kr}\frac{\partial\omega^{ij}}{\partial u^r} = 0\,. \tag{1.22}$$

The Poisson bracket establishes a Lie algebra structure on $C^\infty(M)$, which has the further property that the map $H \mapsto X_H$ is a Lie algebra anti-homomorphsim into the space of vector fields on M, that is

$$X_{\{G,H\}} = [X_H, X_G]\,.$$

So in particular, two Hamiltonian vector fields X_G and X_H commute if and only if their Hamiltonians are in involution, i.e. $\{G,H\} = 0$; this is equivalent to saying that the function G is a conserved quantity for X_H.

If, as is the case for the Poisson bracket (1.20), the matrix ω^{ij} is non-degenerate, we may introduce its inverse ω_{ij} defined by $\omega_{ir}\omega^{rj} = \delta_i^j$. We thus have a 2-form

$$\omega^{-1} = \omega_{ij}du^i\otimes du^j = \frac{1}{2}\omega_{ij}du^i\otimes du^j\,.$$

The Jacobi identity for $\{\cdot,\cdot\}$ is equivalent to the closedness of ω^{-1}. For this reason, Hamilton's equations are often presented in the context of symplectic geometry.

Definition 1.3.2. *A symplectic form on a manifold is a closed, non-degenerate two-form, and a manifold equipped with one is called a symplectic manifold.*

The Hamiltonian vector field of a function H on a symplectic manifold M with symplectic form α is the unique vector field X_H satisfying $\langle X_H|\,\alpha\rangle = -dH$, where $\langle\cdot|\cdot\rangle$ is the usual paring of vector fields and one-forms.

The Poisson bracket inverse to α can be defined invariantly as $\{F,G\} = X_G(F)$.

Darboux's theorem (see, e.g., [70]) states that around any point in a $2n$-dimensional symplectic manifold (M, α) there exist coordinates $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ in which the symplectic form is written

$$\alpha = \sum_{i=1}^{n} dp_i \wedge dq^i \,,$$

and so the Poisson bracket takes the canonical form (1.20).

One may still find Darboux coordinates around a point in a Poisson manifold (M, ω) provided ω has constant rank in some neighbourhood of that point. Say the rank of the matrix ω^{ij} is $2n$, and the dimension of M is $2n + m$. Then one can find coordinates $(q^1, \ldots, q^n, p_1, \ldots, p_n, c^1, \ldots, c^m)$ in which the Poisson bracket is again given by the expression (1.20). The functions $c^1, \ldots, c^m$ are Casimirs, which is to say that they have the property that $\{f, c^i\} = 0$ for all functions f. This property means that the Hamiltonian vector field associated with any Casimir, or function of Casimirs, is zero.

1.4 Infinite-Dimensional Hamiltonian Systems

One may also introduce Poisson brackets on infinite-dimensional manifolds in order to exhibit partial differential equations as Hamiltonian systems.

Let M be a finite-dimensional manifold, of dimension n, say, which we shall call the target space. Then the loop space of M, denoted by $L(M)$, is the space of smooth maps $u : S^1 \to M$. If we use x as a coordinate on S^1 and $u^1, \ldots, u^n$ as coordinates on M then elements of $L(M)$ are represented by collections of functions $(u^1, \ldots, u^n) : x \mapsto (u^1(x), \ldots, u^n(x))$. We therefore have natural coordinates $u^i(x)$ on $L(M)$, where the index i runs from 1 to n, and the argument x takes values in S^1. It is also necessary to utilise coordinates representing the derivative fields, u^i_x, u^i_{xx} and so on. However, since we want the behaviour of our systems to be local, in the sense that the time evolution of the fields u^i at a point $x \in S^1$ depends only only the value of the fields and their derivatives at that point, and translation invariant in that this dependence does not vary from one point of S^1 to another, we will normally suppress the x-dependence in expressions. Thus a vector field on $L(M)$ has the form[1]

$$X = X^{i,0}(u, u_x, \ldots) \frac{\partial}{\partial u^i} + X^{i,1}(u, u_x, \ldots) \frac{\partial}{\partial u^i_x} + \ldots \,, \tag{1.23}$$

[1]a vector field on $L(M)$ could also include a component in the $\frac{\partial}{\partial x}$ direction; we neglect this possibility in this discussion since none of our objects depend explicitly on x.

or, more compactly,

$$X = X^{i,r} \frac{\partial}{\partial u^{i,r}} ,$$

where $i \in \{1, \ldots, n\}$, $r \in \{1, 2, \ldots\}$ and the notation $u^{i,r}$ is used to denote the r^{th} x-derivative of u^i.

There is a distinguished vector field on $L(M)$ which is the infinitesimal generator of spatial translations:

$$\frac{d}{dx} = u^i_x \frac{\partial}{\partial u^i} + u^i_{xx} \frac{\partial}{\partial u^i_x} + u^i_{xxx} \frac{\partial}{\partial u^i_{xx}} + \ldots . \tag{1.24}$$

Since the flow of X is found by solving the equations

$$\begin{aligned} \frac{du^i}{dt} &= X^{i,0} , \\ \frac{du^i_x}{dt} &= X^{i,1} , \\ \vdots &= \vdots \quad , \end{aligned}$$

it is necessary to impose the relations

$$X^{i,p} = \left(\frac{d}{dx} \right)^p X^{i,0} \tag{1.25}$$

on the coefficients of X, which will ensure that the acts of differentiation with respect to x and t commute. In fact, the conditions in (1.25) can be written in the single expression

$$\left[X, \frac{d}{dx} \right] = 0 .$$

Vector fields satisfying this condition are called evolutionary.

The analogues of functions in this infinite-dimensional system are functionals mapping functions $u \in L(M)$ to real (or complex) numbers via integration of some functional density:

$$f(u) = \int_{x \in S^1} F(u(x), u_x(x), u_{xx}(x), \ldots) dx . \tag{1.26}$$

Since the integral of a full x-derivative around a circle is zero, the functionals represented by two functional densities which differ by a full derivative are the same. Thus, if we denote by $\mathcal{A}$ some suitable ring of functions in variables $u^i, u^i_x, \ldots$, then we may identify

$$\{\text{functional densities}\} = \mathcal{A} \Big/ \frac{d}{dx} \mathcal{A} \ .$$

It is sufficient for the material presented here to let $\mathcal{A}$ be the set of functions which depend smoothly on u, and polynomially on $u_x, u_{xx}, \ldots$, and, further, depend upon only finitely many of these latter variables; we shall refer to such functions as differential polynomials.

The action of a vector field on a functional can be calculated as

$$\begin{aligned}
X\left(\int_{x\in S^1} F(u(x),u_x(x),\dots)dx\right) &= \int_{x\in S^1}\left(X^{i,0}\frac{\partial F}{\partial u^i}+X^{i,1}\frac{\partial F}{\partial u^i_x}+\dots\right)dx\,,\\
&= \int_{x\in S^1}\left(X^{i,0}\frac{\partial F}{\partial u^i}+\frac{dX^{i,0}}{dx}\frac{\partial F}{\partial u^i_x}+\dots\right)dx\,,\\
&= \int_{x\in S^1}\left(X^{i,0}\frac{\partial F}{\partial u^i}-X^{i,0}\frac{d}{dx}\left(\frac{\partial F}{\partial u^i_x}\right)+\dots\right)dx\,,\\
&= \int_{x\in S^1} X^{i,0}\frac{\delta F}{\delta u}dx\,. \qquad (1.27)
\end{aligned}$$

So all of the necessary arithmetic can be performed using only the functional density and the first components of the vector field.

It is also useful to introduce 1-forms, or rather 1-form densities, on $L(M)$, the space of which we denote by Λ_1. These are sums of terms

$$\theta=\sum_{i=1}^{n}\sum_{r\geq 0}\theta_{i,r}(u,u_x,\dots)\delta u^{i,r}\,,$$

where the 1-forms $\delta u^{i,r}$ form a basis for Λ_1 dual to $\frac{\partial}{\partial u^{i,r}}$. The Lie derivative of a 1-form θ along a vector field X is calculated as

$$\mathcal{L}_X\theta_{j,q}=X^{r,p}\frac{\partial\theta_{j,q}}{\partial u^{r,p}}+\theta_{r,p}\frac{\partial X^{r,p}}{\partial u^{j,q}}\,,$$

and in particular

$$\begin{aligned}
\mathcal{L}_{\frac{d}{dx}}\theta_{j,q} &= u^{r,p+1}\frac{\partial\theta_{j,q}}{\partial u^{r,p}}+\theta_{r,p}\frac{\partial u^{r,p+1}}{\partial u^{j,q}}\,,\\
&= \frac{d}{dx}(\theta_{j,q})+\theta_{j,q-1}\,.
\end{aligned}$$

As with functional densities, two 1-forms are identified if they differ by a 1-form in the image of $\mathcal{L}_{\frac{d}{dx}}$, the reason being that the action of any two such 1-forms on an evolutionary vector field will be the same. This means that any 1-form $\theta=\theta_{j,q}\delta u^{j,q}$ can be written uniquely in the form $\theta=\theta_j\delta u^j$, where $\delta u^j=\delta u^{j,0}$ and

$$\theta_j=\sum_{q\geq 0}(-1)^q\left(\frac{d}{dx}\right)^q\theta_{j,q}\,.$$

We call the θ_j the reduced components of θ. In particular, for the exterior derivative of a functional we have

$$\begin{aligned}
\delta F &= \frac{\partial F}{\partial u^{j,q}}\delta u^{j,q}\,,\\
&= \sum_{q\geq 0}(-1)^q\left(\frac{d}{dx}\right)^q\left(\frac{\partial F}{\partial u^{j,q}}\right)\delta u^j\,,\\
&= \frac{\delta F}{\delta u^j}\delta u^j\,.
\end{aligned}$$

In [28, 29] Dubrovin and Novikov studied the so-called Poisson brackets of differential-geometric type, which are of the form

$$\{f,g\} = \int \frac{\delta f}{\delta u^i} P^{ij} \left(\frac{\delta g}{\delta u^j} \right) dx \tag{1.28}$$

where P^{ij} is a matrix differential operator (in $\frac{d}{dx}$), with coefficients which are differential polynomials.

Definition 1.4.1. *If $\{\cdot,\cdot\}$ satisfies the conditions of Definition 1.3.1 and thus defines a Poisson bracket on the loop space, then P is referred to as a Hamiltonian operator.*

The evolution equations generated by (1.28) and a Hamiltonian density H are

$$u^i_t = P^{ij} \left(\frac{\delta H}{\delta u^j} \right) . \tag{1.29}$$

An important class of Hamiltonian operators are the hydrodynamic type Poisson brackets, which are of the form:

$$P^{ij} = g^{ij}(u) \frac{d}{dx} + \Gamma^{ij}_k(u) u^k_x \,, \tag{1.30}$$

the significance of which is that they are homogeneous of degree 1 in the grading to be described in Section 1.5.

According to the programme set out by Novikov [61], differential-geometric type Poisson brackets on $L(M)$ should be studied in terms of finite-dimensional differential geometry on the target space M. When expanded as polynomial in $\frac{d}{dx}$ and the derivative fields, the coefficients, which are functions of the fields u^i alone, can often be naturally related to known objects of differential geometry, or else used to define new ones. This identification is made by considering how the Hamiltonian equation (1.29) behaves under a change of variables $\tilde{u}^i = \tilde{u}^i(u^1, \ldots, u^n)$, which allows one to deduce how the coefficients in P behave under a change of coordinates on M.

In the hydrodynamic case, for instance, one has the following transformation rules:

$$\begin{aligned} \tilde{g}^{ij} &= \frac{\partial \tilde{u}^i}{\partial u^p} \frac{\partial \tilde{u}^j}{\partial u^q} g^{pq} \,, \\ \tilde{\Gamma}^{ij}_k &= \frac{\partial \tilde{u}^i}{\partial u^p} \frac{\partial \tilde{u}^j}{\partial u^q} \frac{\partial u^r}{\partial \tilde{u}^k} \Gamma^{pq}_r - \frac{\partial \tilde{u}^i}{\partial u^p} \frac{\partial \tilde{u}^s}{\partial u^q} \frac{\partial \tilde{u}^j}{\partial u^r} \frac{\partial^2 u^r}{\partial \tilde{u}^k \partial \tilde{u}^s} g^{pq} \,, \end{aligned} \tag{1.31}$$

which show that g^{ij} are the components of a rank 2 contravariant tensor on M, whilst the Γ^{ij}_k are related to the Christoffel symbols Γ^i_{jk} of an affine connection by $\Gamma^{ij}_k = -g^{ir}\Gamma^j_{rk}$. The constraints placed on (1.30) by demanding that (1.28) be Poisson can now be converted into geometric statements about g^{ij} and Γ^{ij}_{k}, namely: if g^{ij} is non-degenerate then P is Hamiltonian if and only if g^{ij} is the inverse of a flat metric, and Γ^{i}_{jk} are the Christoffel symbols of its Levi-Civita connection.

This provides us with a Darboux theorem for hydrodynamic operators. Since the metric g is flat, there exists a set of coordinates, referred to as flat coordinates for g, in which the components g_{ij}, or equivalently g^{ij}, are constant, and so in which the Christoffel symbols of the Levi-Civita connection vanish.

The simplest example of such an operator is the constant one-dimensional operator

$$P_1 = \frac{d}{dx}\,. \tag{1.32}$$

Using the Hamiltonian density

$$H_2 = u^3 - \frac{1}{2}u_x^2 \tag{1.33}$$

with this, we obtain the Korteweg-de Vries equation (1.1).

1.5 Poisson Cohomology

The KdV equation is also Hamiltonian with respect to the operator

$$P_2 = 4u\frac{d}{dx} + 2u_x + \left(\frac{d}{dx}\right)^3 \tag{1.34}$$

and the Hamiltonian density

$$H_1 = \frac{1}{2}u^2\,. \tag{1.35}$$

As observed in [55], the Hamiltonian operators P_1 and P_2 of the KdV equation have the property that the operator $P_\lambda = P_1 + \lambda P_2$ is Hamiltonian for all values of the scalar λ. This provided the motivating example for the following definition:

Definition 1.5.1. *Two Poisson brackets $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$ are said to be compatible, or to constitute a bi-Hamiltonian structure, if any linear combination of them is also a Poisson bracket. We may also say two Hamiltonian operators P_1 and P_2 are compatible if their associated Poisson brackets are compatible.*

Poisson cohomology [12, 69] provides a framework in which to study bi-Hamiltonian structures. We consider first the finite dimensional case.

Let M be a finite-dimensional manifold, of dimension n. We denote by Λ^k the space of k-vectors on M; that is: of rank k contravariant tensor fields which are antisymmetric under the permutation of any two indices. So an element $\varpi \in \Lambda^k$ is represented by components $\varpi^{i_1 \cdots i_k}$ such that

$$\varpi = \frac{1}{k!}\varpi^{i_1 \ldots, i_k}\frac{\partial}{\partial u^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial u^{i_k}} = \varpi^{i_1 \ldots i_k}\frac{\partial}{\partial u^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial u^{i_k}}\,.$$

We call elements of $\Lambda^\bullet = \bigcup_{k=0}^n \Lambda^k$ multi-vectors. Elements of Λ^0 are functions on M and elements of Λ^1 are vector fields. Poisson bivectors on M form the subset of elements of Λ^2 which also satisfy the Jacobi identity (1.22).

The Lie derivative provides a map $\mathcal{L} : \Lambda^1 \times \Lambda^k \to \Lambda^k$, $(X, \varpi) \mapsto \mathcal{L}_X \varpi$. The Schouten-Nijenhuis bracket is the unique $\mathbb{R}$-linear (or $\mathbb{C}$-linear) extension of this to a map $[\cdot,\cdot] : \Lambda^k \times \Lambda^l \to \Lambda^{k+l-1}$ which satisfies, for multivectors $X \in \Lambda^1$, $P \in \Lambda^p$, $Q \in \Lambda^q$, $R \in \Lambda^r$:

$$[X, Q] = \mathcal{L}_X Q\,, \tag{1.36}$$

$$[P, Q] = (-1)^{pq}[Q, P]\,, \tag{1.37}$$

$$[P, Q \wedge R] = [P, Q] \wedge R + (-1)^{pq+q} Q \wedge [P, R]\,. \tag{1.38}$$

Two consequences of these conditions are the formulas

$$[X_1 \wedge \cdots \wedge X_p, Q] = \sum_{i=1}^{p} (-1)^{i+1} X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_p \wedge [X_i, Q]\,, \tag{1.39}$$

and

$$(-1)^{p(r-1)}[P, [Q, R]] + (-1)^{q(p-1)}[Q, [R, P]] + (-1)^{r(q-1)}[R, [P, Q]] = 0\,, \tag{1.40}$$

in which $X_1, \ldots, X_p \in \Lambda^1$ and P, Q and R are as before.

Example 1.5.2. *For $f, g \in \Lambda^0$, $X, Y \in \Lambda^1$ and $\omega, \pi \in \Lambda^2$ we have*

$$\begin{aligned}
[f, g] &= 0\,,\\
[X, f] &= X(f)\,,\\
[X, Y] &= \mathcal{L}_X Y\,,\\
[\omega, f] &= \sum_{i,j=1}^{n} \omega^{ij}\frac{\partial f}{\partial u^i}\frac{\partial}{\partial u^j}\,,\\
[\omega, X] &= \mathcal{L}_X \omega\,,\\
[\omega, \pi] &= \frac{1}{6}\sum_{i,j,k=1}^{n} \Big\{ \omega^{ri}\pi^{jk}{}_{,r} + \omega^{rj}\pi^{ki}{}_{,r} + \omega^{rk}\pi^{ij}{}_{,r}\\
&\qquad + \pi^{ri}\omega^{jk}{}_{,r} + \pi^{rj}\omega^{ki}{}_{,r} + \pi^{rk}\omega^{ij}{}_{,r} \Big\}\, \partial_i \wedge \partial_j \wedge \partial_k\,.
\end{aligned}$$

So, in particular, $[\omega, \omega] = 0$ if and only if ω is a Poisson bivector, in which case we also have $[\omega, f] = -X_f = \{f, \cdot\}$.

If we are given a fixed $\omega \in \Lambda^2$, then the operator $\sigma = [\omega, \cdot]$ satisfies $\sigma^2 = 0$ if and only if ω is Poisson. Thus, on a Poisson manifold (M, ω) we have a complex

$$\Lambda^0 \xrightarrow{\sigma} \Lambda^1 \xrightarrow{\sigma} \Lambda^2 \xrightarrow{\sigma} \cdots \xrightarrow{\sigma} \Lambda^k \xrightarrow{\sigma} \Lambda^{k+1} \xrightarrow{\sigma} \ldots ,$$

called the Poisson-Lichnerowicz complex. The associated cohomology groups are the quotient spaces

$$H^i(\sigma) = \frac{\ker(\sigma : \Lambda^i \to \Lambda^{i+1})}{\operatorname{im}(\sigma : \Lambda^{i-1} \to \Lambda^i)} .$$

Example 1.5.3. $H^0(\sigma) = \ker(\sigma : \Lambda^0 \to \Lambda^1) = \{f \in \Lambda^0 : X_f = 0\}$*, so* H^0 *is the set of all Casimir functions.*

Example 1.5.4.

$$H^1(\sigma) = \frac{\{X \in \Lambda^1 : \mathcal{L}_X \omega = 0\}}{\{X_H : H \in \Lambda^0\}} ,$$

so H^1 *is the set of all symmetries of* ω*, modulo Hamiltonian vector fields.*

Example 1.5.5.

$$H^2(\sigma) = \frac{\{\pi \in \Lambda^2 : [\omega, \pi] = 0\}}{\{\pi \in \Lambda^2 : \pi = \mathcal{L}_X \omega \textit{ for some } X \in \Lambda^1\}} .$$

The meaning of this is obtained by considering the Jacobi identity for the bivector $\omega_\varepsilon = \omega + \varepsilon\pi$*, since*

$$\begin{aligned} [\omega_\varepsilon, \omega_\varepsilon] &= [\omega + \varepsilon\pi, \omega + \varepsilon\pi] , \\ &= [\omega, \omega] + \varepsilon[\omega, \pi] + \varepsilon[\pi, \omega] + \varepsilon^2[\pi, \pi] , \\ &= 2\varepsilon[\omega, \pi] + \varepsilon^2[\pi, \pi] . \end{aligned}$$

Thus elements of $\ker(\sigma : \Lambda^2 \to \Lambda^3)$ *are bivectors* π *such that* $\omega + \varepsilon\pi$ *satisfies the Jacobi identity to order* ε*. Such elements are called infinitesimal deformations of* ω*.*

If an infinitesimal deformation π *can be written as* $\pi = \mathcal{L}_X \omega$*, then* $\omega + \varepsilon\pi$ *can be obtained from* ω^{ij} *from the infinitesimal transformation* $u \mapsto \tilde{u}$ *given by* $\tilde{u}^i = u^i + \varepsilon X^i + O(\varepsilon^2)$*.* H^2 *is therefore comprised of infinitesimal deformations of* ω*, modulo these trivial deformations.*

The only result we shall need from finite-dimensional Poisson cohomology is that for a symplectic manifold the Poisson cohomology is isomorphic to the de Rham cohomology, and in particular is trivial for topologically trivial manifolds. So, for the canonical Poisson bracket (1.20) on $\mathbb{R}^{2n}$, all symmetries of the bracket are Hamiltonian vector fields, and all infinitesimal deformations are trivial.

In order to be able to define the cohomology groups H^0, H^1 and H^2 it is only necessary to work with the spaces Λ^0, Λ^1 and Λ^2, and the Schouten-Nijenhuis brackets between objects in them; on the loop space $L(M)$ of a finite-dimensional manifold M, these spaces take the following form:

Λ^0 is the space of functionals, which are represented by their densities;

Λ^1 is the space of evolutionary vector fields;

Λ^2 is the space of local evolutionary bivectors, which are skew-symmetric maps $\Lambda_1 \times \Lambda_1 \to \Lambda^0$ of the form

$$\pi(\theta,\phi) = \int_{x\in S^1} \theta_i \pi^{ij}\,(\phi_j)\,dx\,,$$

where, as in (1.28), π^{ij} is a matrix of differential operators. Here local means that the density of $\pi(\theta,\phi)$ at $x \in S^1$ depends only on the two 1-form densities θ and ϕ at x, and the evolutionary property means that it only depends upon the reduced components (so when it acts on functionals it depends only upon their variational derivatives, as does (1.28)).

As before, the Schouten-Nijenhuis bracket of two functionals is zero. For $X \in \Lambda^1$, the Schouten-Nijenhuis bracket coincides with the Lie derivative, hence, for $F \in \Lambda^0$, $Y \in \Lambda^1$, $\pi \in \Lambda^{ij}$,

$$\begin{aligned}
[X,F] &= X(F) = X^r \frac{\delta F}{\delta u^r}\,,\\
[X,Y]^i &= X^{r,p}\frac{\partial Y^i}{\partial u^{r,p}} - Y^{r,p}\frac{\partial X^i}{\partial u^{r,p}}\,,\\
[X,\pi]^{ij} &= \mathcal{L}_X \pi^{ij}\,,\\
&= X^{k,t}\circ\frac{\partial \pi^{ij}}{\partial u^{k,t}} - \frac{\partial X^i}{\partial u^{k,t}}\circ\left(\frac{d}{dx}\right)^t\circ\pi^{kj} - \pi^{ik}\circ\left(-\frac{d}{dx}\right)^t\circ\frac{\partial X^j}{\partial u^{k,t}}\,,
\end{aligned}$$

where $\circ$ is composition of differential operators.

Given $\phi \in \Lambda_0$, $\pi \in \Lambda^2$, we denote by $\pi(\phi)$ the evolutionary vector field defined by its pairing with 1-forms being $\theta \llcorner \pi(\phi) = \pi(\theta,\phi)$ for all 1-forms θ. Then the Schouten-Nijenhuis bracket of $\pi,\omega \in \Lambda^2$ is given by

$$\begin{aligned}
[\pi,\omega](\phi_1,\phi_2,\phi_3) &= \phi_3\llcorner\pi(\mathcal{L}_{\omega(\phi_1)}\phi_2) + \phi_3\llcorner\omega(\mathcal{L}_{\pi(\phi_1)}\phi_2)\\
&\quad + \phi_1\llcorner\pi(\mathcal{L}_{\omega(\phi_2)}\phi_3) + \phi_1\llcorner\omega(\mathcal{L}_{\pi(\phi_2)}\phi_3)\\
&\quad + \phi_2\llcorner\pi(\mathcal{L}_{\omega(\phi_3)}\phi_1) + \phi_2\llcorner\omega(\mathcal{L}_{\pi(\phi_3)}\phi_1)
\end{aligned}$$

Theorem 5.1 of [20] provides a useful method for determining if a Schouten-Nijenhuis bracket $[\pi,\pi]$ is zero when π is an operator of differential-geometric type:

Theorem 1.5.6. *Given a bivector P with components*

$$P^{ij} = \sum_{r\geq 0} P_r^{ij} \left(\frac{d}{dx}\right)^r ,$$

where $P_r^{ij} \in \mathcal{A}$, and a function $q \in \mathcal{A}$ we define the operators $D_{P_k^{ij}} q$ by

$$D_{P_k^{ij}} q = \sum_{r,s\geq 0} q^{(r)} \frac{\partial P_r^{ij}}{\partial u^{k,s}} \left(\frac{d}{dx}\right)^s ,$$

where $q^{(r)}$ is the r^{th} x-derivative of q.

Then P is Hamiltonian if and only if

$$t_{ijkrs} = t_{ikjsr}$$

for $i, j, k \in \{1, \ldots, n\}$, $r, s \geq 0$, where t_{ijkrs} is the coefficient of $q_1^{(r)} q_2^{(s)}$ in

$$T_{ijk}(q_1, q_2) = \sum_{l=1}^{n} \left((D_{P_l^{ik}} q_1) P^{lj} q_2 + \frac{1}{2} P^{il} (D_{P_l^{jk}} q_1)^* q_2 \right) .$$

By taking the linear terms in T_{ijk} for $P_\lambda = P_1 + \lambda P_2$ this can be used to calculate necessary and sufficient conditions for $[P_1, P_2] = 0$.

Example 1.5.7. *A multiplicative operator*

$$P^{ij} = \omega^{ij} , \tag{1.41}$$

on $L(M)$ is Hamiltonian if and only ω^{ij} is a Poisson bivector on M. It is called the ultralocal Poisson bracket. If ω^{ij} is non-degenerate then the Poisson cohomology of P is trivial.

1.6 Bi-Hamiltonian Systems

The small dispersion expansion of an evolution equation

$$u_t^i = f^i(u, u_x, u_{xx}, \ldots)$$

is obtained from the substitution

$$x \mapsto \varepsilon x , \; t \mapsto \varepsilon t , \tag{1.42}$$

which sends the equation to

$$u_t^i = \frac{1}{\varepsilon} f^i(u, \varepsilon u_x, \varepsilon^2 u_{xx}) .$$

For instance, the KdV equation (1.1) becomes

$$u^i_t = 6uu_x + \varepsilon^2 u_{xxx}\,. \tag{1.43}$$

This induces transformations on other objects on $L(M)$. In particular, for a Hamiltonian operator P with entries

$$P^{ij} = P^{ij}_r(u, u_x, \dots)\left(\frac{d}{dx}\right)^r,$$

this substitution into the Hamiltonian equation (1.29) sends P^{ij} to

$$\frac{1}{\varepsilon}P^{ij}_r(u, \varepsilon u_x, \dots)\left(\varepsilon\frac{d}{dx}\right)^r.$$

Thus, the power of ε in front of a term in P^{ij} counts the total number of derivatives introduced by that term minus one, including both multiplication by derivative fields and differentiation by x, and hence introduces a grading on such operators.

For instance, the ultralocal operator (1.41) has differential degree zero, and hence picks up a factor of ε^{-1}, whilst the hydrodynamic type Hamiltonian operators (1.30) are precisely those of degree 1, and hence have a factor of ε^0.

This enables us to state some results [31, 44] on the Poisson cohomology of hydrodynamic operators.

Theorem 1.6.1. *Let*

$$P^{ij}_{(1)} = g^{ij}\frac{d}{dx} + \Gamma^{ij}_k u^k_x$$

be a hydrodynamic type Hamiltonian operator with g^{ij} non-degenerate. Then any infinitesimal deformation of $P_{(1)}$ of degree greater than or equal to 2 is trivial.

This means that given an expansion

$$P^{ij} = P^{ij}_{(1)} + \varepsilon P^{ij}_{(2)} + \varepsilon^2 P^{ij}_{(3)} + \dots, \tag{1.44}$$

where $P^{ij}_{(1)}$ is as in Theorem 1.6.1 and $P^{ij}_{(r)}$ is of degree r, such that P is Hamiltonian identically in ε, then (cf. Example 1.5.5) there exists an evolutionary vector field X such that $P^{ij} = \mathcal{L}_X P^{ij}_{(1)}$. Note that asking (1.44) is Hamiltonian for all ε is no stronger than asking that

$$P^{ij} = P^{ij}_{(1)} + P^{ij}_{(2)} + P^{ij}_{(3)} + \dots \tag{1.45}$$

is Hamiltonian, since the former is obtained from the latter by taking the small dispersion expansion (1.42).

With this, the Darboux theorem for hydrodynamic operators can be extended to encompass operators of the form (1.44), which are called (0,n)-brackets in [31]. If we extend our allowed changes of variables to include not only coordinate transformations on M, but also Miura transformations, which are formal power series in ε of the form

$$u^i \mapsto \tilde{u}^i = u^i + \sum_{r\geq 1} \varepsilon^r f^i_r(u, u_x, \ldots, u^{(r)}) ,$$

where f^i_r is a polynomial of degree r in the derivative fields, counting $u^{i,r}$ as degree r, then the existence of the vector field X is equivalent to there existing a Miura transformation which takes the Hamiltonian operator P to its leading term $P_{(1)}$. Then, because $P_{(1)}$ is specified by a flat metric and its Levi-Civita connection, one may combine this with a coordinate transform on M to put P into constant form.

A corollary of this is that there exists n independent Casimirs of P, where n is the dimension of M, which are the coordinates u^i in which $P^{ij} = \eta^{ij}\frac{d}{dx}$, with η constant.

Example 1.6.2. *The first Hamiltonian operator of the KdV equation,*

$$P_1 = \frac{d}{dx},$$

is already in this form in the standard coordinate, u, used in (1.1), whilst the second Hamiltonian operator, now written

$$P_2 = 4u\frac{d}{dx} + 2u_x + \varepsilon^2 \left(\frac{d}{dx}\right)^3 ,$$

takes the constant form

$$P_2 = \frac{d}{dx}$$

in the coordinate v, where v and u are related by $u = v^2 + i\varepsilon v_x$.

Additionally, since the operators (1.32) and (1.34) of the KdV equation are compatible, then for all λ there exists a coordinate $w(u, u_x, \ldots; \lambda)$ on $L(M)$ in which the combination

$$P_\lambda = P_1 - \lambda P_2$$

is constant. It is related to u by the equation

$$u = w - \lambda w^2 - \varepsilon\lambda^{\frac{1}{2}} w_x \, . \tag{1.46}$$

If w is expanded as

$$w = \sum_{r\geq 0} \lambda^{\frac{r}{2}} w_r \tag{1.47}$$

then the first few coefficients are

$$\begin{aligned} w_0 &= u\,, \\ w_1 &= \varepsilon u_x\,, \\ w_2 &= u^2 + \varepsilon^2 u_{xx}\,, \\ w_3 &= 4\varepsilon u u_x + \varepsilon^3 u_{xxx}\,, \\ w_4 &= 2u^3 + 5\varepsilon^2 u_x^2 + 6\varepsilon^2 u u_{xx} + \varepsilon^4 u_{xxxx}\,. \end{aligned} \tag{1.48}$$

In particular, if these coefficients are interpreted as functional densities on $L(M)$ then w_2 and w_4 are proportional to the Hamiltonian densities (1.35) and (1.33) generating the KdV equation, once full derivatives are factored out. For all $r, s \in \{0, 1, 2, \dots\}$ these coefficients satisfy

$$\{w_r, w_s\}_1 = \{w_r, w_s\}_2 = 0\,,$$

and are thus seen to provide a family of commuting integrals for the KdV flow. They also satisfy the recurrence relations

$$\{\cdot, w_{r+2}\}_1 = \{\cdot, w_r\}_2\,.$$

The general result, demonstrating the existence of a family of commuting integrals for a pair of compatible $(0, n)$-brackets is established in via the following two results [31, 55]:

Lemma 1.6.3. *Let $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$ be two Poisson brackets, not necessarily compatible, and let $\{H_p\}$ be a family of Hamiltonians satisfying*

$$\{\cdot, H_p\}_2 = \{\cdot, H_{p+1}\}_1\,.$$

Then $\{H_p, H_q\}_1 = \{H_p, H_q\}_2 = 0$.

Lemma 1.6.4. *Let P_1 and P_2 be a pair of compatible Hamiltonian operators. Suppose that, for all λ, $P_\lambda = P_1 - \lambda P_2$ has Casimirs c^α_λ each of which can be expanded as*

$$c^\alpha_\lambda = c^\alpha_0 + \lambda c^\alpha_1 + \lambda^2 c^\alpha_2 + \dots\,.$$

Then $\{\cdot, c^\alpha_r\}_2 = \{\cdot, c^\alpha_{r+1}\}_1$ and $\{c^\alpha_r, c^\beta_s\}_1 = \{c^\alpha_r, c^\beta_s\}_2 = 0$ for all α, β, r, s.

In general one builds up a Casimir $c_\lambda = \sum_{r\geq 0} \lambda^r c_r$ of $P_\lambda = P_1 - \lambda P_2$ starting from a Casimir c_0 of P_1 by solving the recurrence relation

$$P_1^{ij}\left(\frac{\delta c_{r+1}}{\delta u^j}\right) = P_2^{ij}\left(\frac{\delta c_r}{\delta u^j}\right)\,. \tag{1.49}$$

The obstruction to the solvability of this recursion relation lies in the cohomology group $H^1(\sigma_1)$ of P_1. For we have, from the graded Jacobi identity for the Schouten bracket,

$$[P_1,[P_2,H]]+[P_2,[H,P_1]]+[H,[P_1,P_2]]=0\,,$$

for any functional density H, and hence, since $[P_1,P_2]=0$,

$$\sigma_1\sigma_2 H=-\sigma_2\sigma_1 H\,,$$

where σ_1 and σ_2 are the Poisson-differentials $[P_1,\cdot]$ and $[P_2,\cdot]$ respectively.

Now suppose we have functionals $\{c_r\}$ satisfying, for $r\leq n-1$, the recurrence relation (1.49), which we may write as $\sigma_1 c_{r+1}=\sigma_2 c_r$. Then

$$\begin{aligned}\sigma_1(\sigma_2 c_n) &= -\sigma_2(\sigma_1 c_n)\,,\\ &= -\sigma_2(\sigma_2 c_{n-1})\,,\\ &= 0\,.\end{aligned}$$

So $\sigma_2 c_n$ is σ_1-cocycle. In order to be able to find c_{n+1} satisfying $\sigma_1 c_{n+1}=\sigma_2 c_n$, we need it to be a σ_1-coboundary, i.e. to be trivial in the cohomology group $H^1(\sigma_1)$. For hydrodynamic operators (and hence for $(0,n)$-brackets by Theorem 1.6.1 and the subsequent discussion) we have the following [31]:

Theorem 1.6.5. *Let P be a Hamiltonian operator of hydrodynamic type, and let $X\in H^1([P,\cdot])$ be such that X can be represented by an evolutionary vector field with components of the form*

$$X^i=a^i_r(u)u^r_x+\varepsilon\left(b^i_r(u)u^r_{xx}+b^i_{rs}(u)u^r_x u^s_x\right)+O(\varepsilon^2)\,. \tag{1.50}$$

Then X is trivial in $H^1([P,\cdot])$.

Consequently, if P_1 is hydrodynamic and P_2 is a $(0,n)$-bracket, then the recursion relations (1.49) can be solved. The inductive argument is set in motion using the trivial Casimir $c_{-1}=0$ and a Casimir c_0 of P_1.

Example 1.6.6. *Continuing Example 1.6.2, the Casimir c_λ of $P_\lambda=P_1-\lambda P_2$ is found by solving*

$$C'=\lambda\left(4uC'+2u_xC+\varepsilon^2C'''\right)\,,$$

where $C=\frac{\delta c}{\delta u}$. Writing $c_\lambda=\sum_{r\geq 0}\lambda^r c_r$ and $C_r=\frac{\delta c_r}{\delta u}$ this gives the recurrence relation

$$C'_{r+1}=4uC'_r+2u_xC_r+\varepsilon^2C'''_r\,,$$

from which the Casimir of P_λ can be found, starting with $c_0 = u$, the Casimir of P_1. This gives, for the first few terms:

$$\begin{aligned} c_0 &= u\,, \\ c_1 &= u^2\,, \\ c_2 &= 2u^3 - \varepsilon^2 u_x^2\,, \\ c_3 &= 5u^4 - 10\varepsilon^2 u u_x^2 + \varepsilon^4 u_{xx}^2\,. \end{aligned} \tag{1.51}$$

Notes 1.6.7.

1. *The powers of $\lambda^{\frac{r}{2}}$ in the flat coordinates (1.47) for odd values of r are all full x-derivatives. Thus, when the flat coordinates are interpreted as functional densities these powers vanish, and the densities satisfy the conditions of Lemma 1.6.4.*

2. *The grading of the vector field (1.50) is different from that for functionals. This is to ensure the correct form of the evolution equation $X = \frac{d}{dt}$.*

3. *Theorem 1.6.5 does not say the cohomology group $H^1([P,\cdot])$ is trivial. Any evolutionary vector field X on $L(M)$ with components $X^i = \frac{1}{\varepsilon}a^i(u)$ is a symmetry of (1.30) if and only if the vector field on M with components a^i is a Killing vector of the associated metric g^{ij}; however, no such X can be Hamiltonian with respect to this type of Hamiltonian operator. Thus $H^1([P,\cdot])$ consists precisely of the lifts of the Killing vectors of g^{ij} to the loop space.*

1.7 Dispersionless Integrable Systems

In the small dispersion expansion (1.42) it was assumed that the parameter ε was greater than zero. However, it is possible to consider the limit $\varepsilon \to 0$, called the dispersionless limit. Of course, not all evolution equations will admit this limit. The Sine-Gordon equation, for instance, has as its small dispersion expansion

$$u_{tt} = u_{xx} + \frac{1}{\varepsilon}\sin u\,,$$

which is singular as $\varepsilon \to 0$. Those equations which do possess a dispersionless limit will often change behaviour at $\varepsilon = 0$. For instance the KdV equation (1.43) becomes

$$u_t = 6uu_x\,.$$

Although this no longer admits soliton solutions such as (1.2), much of the discussion above still applies in the dispersionless limit. For instance, the dispersionless KdV equation is Hamiltonian with respect to the two operators

$$P_1 = \frac{d}{dx}$$

and

$$P_2 = 4u\frac{d}{dx} + 2u_x$$

which are the dispersionless limits of (1.32) and (1.34). The relevant Hamiltonians are $H_2 = u^3$ and $H_1 = \frac{1}{2}u^2$. Thus, both the Hamiltonian operators are of hydrodynamic type, and the Hamiltonians do not depend on the derivative fields.

Since the Hamiltonian and bi-Hamiltonian properties of Hamiltonian operators are satisfied identically in ε, these properties are preserved as $\varepsilon \to 0$. Conservation laws are also preserved, as is the commutativity of flows in the hierarchy.

The dispersionless KP hierarchy [3] may be defined analogously to Section 1.2.2. Namely, let λ be the Laurent series

$$\lambda(p) = p + \sum_{r=1}^{\infty} \frac{a_r}{p^r}\,, \tag{1.52}$$

where the coefficients a_r are functions of x. Then for $i \geq 1$ one defines the polynomials μ_i by

$$\mu_i(p) = \left[(\lambda(p))^i\right]_+\,,$$

where $[\,\cdot\,]_+$ means projection onto the polynomial part of p. One then uses the canonical Poisson bracket

$$\{\cdot,\cdot\} = \frac{\partial}{\partial p} \wedge \frac{\partial}{\partial x}$$

on $\mathbb{R}^2 = \{(x,p)\}$ to define the dispersionless Lax equations

$$\frac{\partial \lambda}{\partial t_i} + \{\lambda, \mu_i\} = 0\,. \tag{1.53}$$

As before, one may consider reductions [46] of the dispersionless KP hierarchy by imposing restrictions upon the coefficients in λ. For instance, the dispersionless limits of the Gel'fand-Dikiĭ hierarchies may obtained from (1.53) by the restriction

$$(\lambda(p))^{N+1} = \left[(\lambda(p))^{N+1}\right]_+ = p^{N+1} + u_1 p^{N-1} + u_2 p^{N-2} + \cdots + u_N\,. \tag{1.54}$$

Other reductions are possible, including some, such as the waterbag reductions which are considered in Chapter 4, for which there are no obvious analogous reductions of the dispersive KP hierarchy.

Chapter 2

Frobenius Manifolds

2.1 Introduction

2.1.1 The WDVV Equations and Frobenius Manifolds

Frobenius manifolds [25, 48] were introduced by Dubrovin as a coordinate-free rendition of the Witten-Dijkgraff-Verlinde-Verlinde, or WDVV, equations of topological field theory [18, 19]. These are the following set of conditions on a function F of variables $t^1, \ldots, t^n$:

1. The matrix η defined by
$$\eta_{ij} = \frac{\partial^3 F}{\partial t^1 \partial t^i \partial t^j} \tag{2.1}$$
is constant and non-degenerate. We denote its inverse by η^{ij}.

2. The third derivatives satisfy
$$\frac{\partial^3 F}{\partial t^i \partial t^l \partial t^r} \eta^{rs} \frac{\partial^3 F}{\partial t^j \partial t^k \partial t^s} = \frac{\partial^3 F}{\partial t^i \partial t^j \partial t^s} \eta^{rs} \frac{\partial^3 F}{\partial t^r \partial t^k \partial t^l} \,. \tag{2.2}$$

3. The function F is quasihomogeneous, in that there exists scalars $d_1, \ldots, d_n, d_F$ such that, modulo quadratic expressions in $t^1, \ldots, t^n$,
$$F(\lambda^{d_1} t^1, \ldots, \lambda^{d_n} t^n) = \lambda^{d_F} F(t^1, \ldots, t^n) \tag{2.3}$$
for all values of the scalar λ.

The main feature of a Frobenius manifold is a Frobenius algebra structure on each tangent space, to which are added some conditions describing how these separate algebras patch together.

Definition 2.1.1. *A (commutative, associative) Frobenius algebra, $(A, \circ, \langle \cdot, \cdot \rangle)$, is a (commutative, associative) algebra $(A, \circ)$ together with an non-degenerate symmetric bilinear pairing $\langle \cdot, \cdot \rangle$ such that*

$$\langle X, Y \circ Z \rangle = \langle X \circ Y, Z \rangle . \tag{2.4}$$

Example 2.1.2. *Let $G = \{g_1, g_2, \ldots, g_n\}$ be a finite (commutative) group with identity g_1, and let $\mathbb{C}G$ be the group algebra over G. Then the symmetric bilinear pairing $\langle \cdot, \cdot \rangle$ which has as its action on the basis elements*

$$\langle g_i, g_j \rangle = \begin{cases} 1 & \text{if } g_i g_j = g_1 \\ 0 & \text{otherwise} \end{cases}$$

endows $\mathbb{C}G$ with the structure of a (commutative) associative Frobenius algebra.

Definition 2.1.3. *A Frobenius manifold is a manifold M equipped with the following extra data:*

1. *a flat metric η with Levi-Civita connection ∇;*
2. *a tensorial multiplication of vectors $\circ$, such that*
   - *(a) the inner products defined by η and the multiplications defined by $\circ$ endow each tangent space with the structure of a commutative, associative Frobenius algebra,*
   - *(b) defining c by $c(X, Y, Z) = \eta(X, Y \circ Z)$, which is a symmetric $\left[\begin{smallmatrix}0\\3\end{smallmatrix}\right]$-tensor by the above properties, we have that ∇c is a completely symmetric $\left[\begin{smallmatrix}0\\4\end{smallmatrix}\right]$-tensor;*
3. *a vector field e such that*
   - *(a) $\nabla e = 0$,*
   - *(b) e is the identity element for $\circ$;*
4. *a vector field E such that*
   - *(a) $\nabla\nabla E = 0$,*
   - *(b) $\mathcal{L}_E \eta = D\eta$, for some constant D,*
   - *(c) $\mathcal{L}_E e = -e$,*
   - *(d) $\mathcal{L}_E \circ = \circ$.*

e is called the identity vector field, and E the Euler vector field.

Note: Some authors omit the Euler vector field from their definition of a Frobenius manifold, calling those with one conformal. Here, we refer to manifolds equipped only with the structures specified in 1-3 as non-conformal Frobenius manifolds.

The equivalence between Frobenius manifolds and solutions of the WDVV equations is established as follows.

One introduces flat coordinates $(t^1, \ldots, t^n)$ for the metric η. Since $\nabla e = 0$, we can choose these coordinates so that $e = \frac{\partial}{\partial t^1}$. In these coordinates, c has components c_{ijk}, and the symmetry of ∇c means $c_{ijk,l} = c_{ijl,k}$. Thus we introduce a potential A_{ij} such that

$$c_{ijk} = \frac{\partial A_{ij}}{\partial t^k}\,.$$

Using the symmetry of c_{ijk} we may integrate twice more to obtain a function F, called the free energy of the Frobenius manifold, such that

$$c_{ijk} = \frac{\partial^3 F}{\partial t^i \partial t^j \partial t^k}\,.$$

Because $\eta(X,Y) = \eta(X, Y \circ e) = c(X,Y,e)$ we have

$$\eta_{ij} = c_{1ij} = \frac{\partial^3 F}{\partial t^1 \partial t^i \partial t^j}\,.$$

The components of the multiplication $\circ$ are $c^i_{jk} = \eta^{ir} c_{rjk}$, and the associativity of $\circ$ is equivalent to the set of equations (2.2) for F.

The quasihomogeneity of F is a slight generalisation of (2.3). The condition $\nabla\nabla E = 0$ gives that

$$E = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} q^i_j t^j + r^i \right) \frac{\partial}{\partial t^i}$$

in the flat coordinates.

Since $\mathcal{L}_E \eta_{ij} = D\eta_{ij}$ we have $\mathcal{L}_E \eta^{ij} = -D\eta^{ij}$. Combining this with $\mathcal{L}_E \circ = \circ$ gives $\mathcal{L}_E c_{ijk} = (1+D) c_{ijk}$, which, in terms of the free energy, is $\partial_i \partial_j \partial_k \{E(F) - (1+D)F\} = 0$, and so

$$E(F) = (1+D)F + \text{quadratic terms}\,.$$

If q^i_j can be diagonalised then, because $\mathcal{L}_E e = -e$ implies $q^i_1 = \delta^i_1$, we may redefine the flat coordinates such that

$$E = \sum_{i=1}^{n} \left(d_i t^i + r^i \right) \frac{\partial}{\partial t^i}\,,$$

and still retain $e = \dfrac{\partial}{\partial t^1}$. Further, by translating t^i, we may assume $r^i = 0$ if $d_i \neq 0$. $\mathcal{L}_E e = -e$ gives $d_1 = 1$.

We now give an example from [25] to which we shall return often, in this chapter and in Chapters 4 and 5.

Example 2.1.4. *The Free energy*

$$F = \frac{1}{2}t_1^2 t_2 + \frac{1}{4}t_2^2 \log t_2^2$$

and Euler vector field

$$E = t_1 \frac{\partial}{\partial t_1} + 2t_2 \frac{\partial}{\partial t_2}$$

satisfy the WDVV equations[1]*. The metric is*

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.5}$$

and the multiplication table is

$$\begin{aligned} \frac{\partial}{\partial t_1} \circ \frac{\partial}{\partial t_1} &= \frac{\partial}{\partial t_1}, \\ \frac{\partial}{\partial t_1} \circ \frac{\partial}{\partial t_2} &= \frac{\partial}{\partial t_2}, \\ \frac{\partial}{\partial t_2} \circ \frac{\partial}{\partial t_2} &= \frac{1}{t_2}\frac{\partial}{\partial t_1}. \end{aligned}$$

This Frobenius manifold structure will later be understood as being on the space of rational expressions of the form

$$\lambda(p) = p + \frac{t_2}{p - t_1} \tag{2.6}$$

via the expressions (4.4) and (4.5). The Euler vector field is a result of the invariance of the function (2.6) under the vector field

$$p\frac{\partial}{\partial p} + t_1 \frac{\partial}{\partial t_1} + 2t_2 \frac{\partial}{\partial t_2}.$$

2.1.2 Canonical Coordinates

On most Frobenius manifolds which arise in applications, particularly all of those constructed in Section 2.4 and the non-conformal Frobenius manifolds of Chapter 4, there is another distinguished set of coordinates, called the canonical coordinates, which embody the algebraic structure of the Frobenius algebras. First, we define

[1]For clarity, subscripts are used for variable indices in explicit formulas.

Definition 2.1.5. *An n-dimensional Frobenius algebra $(A, \circ, \langle\cdot,\cdot\rangle)$ is called semisimple if it is the direct sum of n one-dimensional Frobenius algebras. That is, if there exists a basis $\{a_1, \dots, a_n\}$ for A such that*

$$a_i \circ a_j = \delta_{ij} a_j$$

and

$$\langle a_i, a_j \rangle = c_i \delta_{ij} \,,$$

for some constants c_i.

A point p in a Frobenius manifold Frobenius manifold M is called semisimple if the Frobenius algebra structure on T_pM is semisimple. A semisimple (or massive) Frobenius manifold is one in which a generic point is semisimple.

Note that in Definition 2.1.5 and in any discussion involving canonical coordinates, we suspend the summation convention.

Theorem 2.1.6. *In a neighbourhood of a generic point p in a semisimple Frobenius manifold M of dimension n, there exists coordinates $(u_1, \dots, u_n)$, called canonical coordinates, such that*

$$\frac{\partial}{\partial u_i} \circ \frac{\partial}{\partial u_j} = \delta_{ij} \frac{\partial}{\partial u_j}$$

and

$$\eta\left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}\right) = \mu_i \delta_{ij} \,,$$

for some functions μ_i.

Further, the metric η is Egorov; that is there exists a potential function Φ such that

$$\mu_i = \frac{\partial \Phi}{\partial u_i} \,.$$

Proof. This is essentially the proof of Theorem 3.1 of [48] written in terms of vector fields rather than one-forms. In a neighbourhood of p we may construct a basis $(X_1, \dots, X_n)$ of vector fields which satisfy the conditions

$$X_i \circ X_j = \delta_{ij} X_j$$

and

$$\eta(X_i, X_j) = \Phi_i \delta_{ij} \,.$$

Coordinates u_i exist satisfying

$$X_i = \frac{\partial}{\partial u_i}$$

if the vector fields X_i satisfy, for all i, j,

$$[X_i, X_j] = \nabla_{X_i} X_j - \nabla_{X_j} X_i = 0\,.$$

We define functions γ_{ij}^k by

$$\nabla_{X_i} X_j = \sum_{k=1}^{n} \gamma_{ij}^k X_k\,.$$

Since $c(X_i, X_j, X_k) = \Phi_i \delta_{ij} \delta_{jk}$, we have that

$$\begin{aligned}
(\nabla_{X_i} c)(X_j, X_k, X_l) &= X_i(c(X_j, X_k, X_l) - c(\nabla_{X_i} X_j, X_k, X_l) \\
&\quad -c(X_j, \nabla_{X_i} X_k, X_l) - c(X_j, X_k, \nabla_{X_i} X_l)\,, \\
&= X_i(\Phi_j)\delta_{jk}\delta_{kl} - \sum_{r=1}^{n} \left\{ \gamma_{ij}^r c(X_r, X_k, X_l) + \gamma_{ik}^r c(X_j, X_r, X_l) \right. \\
&\quad \left. +\gamma_{il}^t c(X_j, X_k, X_r) \right\}\,, \\
&= X_i(\Phi_j)\delta_{jk}\delta_{kl} - \Phi_l \gamma_{ij}^l \delta_{kl} - \Phi_j \gamma_{ik}^j \delta_{jl} - \Phi_k \gamma_{il}^k \delta_{jk}\,.
\end{aligned}$$

This expression must be symmetric in i, j, k, l. We have, from $(\nabla_{X_i} c)(X_j, X_k, X_k) = (\nabla_{X_k} c)(X_j, X_k, X_i)$,

$$\gamma_{ij}^k = 0\,, \tag{2.7}$$

for i, j, k distinct, whilst $(\nabla_{X_i} c)(X_i, X_j, X_j) = (\nabla_{X_j} c)(X_i, X_i, X_j)$ gives

$$\Phi_j \gamma_{ii}^j = \Phi_i \gamma_{jj}^i\,. \tag{2.8}$$

Thus far we have $\nabla_{X_i} X_j = \gamma_{ij}^i X_i + \gamma_{ij}^j X_j$, and hence

$$[X_i, X_j] = (\gamma_{ij}^i - \gamma_{ji}^i) X_i + (\gamma_{ij}^j - \gamma_{ji}^j) X_j\,,$$

so it is our goal to show that $\gamma_{ij}^i = \gamma_{ji}^i$ for all i, j.

The identity vector field is $e = \sum_{r=1}^n X_r$, and so we have

$$\nabla_{X_i} e = \sum_{r=1}^{n} \nabla_{X_i} X_r = \sum_{r,s=1}^{n} \gamma_{ir}^s X_s\,.$$

Thus we have $\sum_{r=1}^n \gamma_{ir}^j = 0$ for all i, j, since $\nabla e = 0$. For $i \neq j$ this gives

$$\gamma_{ii}^j + \gamma_{ij}^j = 0\,. \tag{2.9}$$

Meanwhile,

$$\begin{aligned}
(\nabla_{X_i} \eta)(X_j, X_k) &= X_i(\eta(X_j, X_k)) - \eta(\nabla_{X_i} X_j, X_k) - \eta(X_j, \nabla_{X_i} X_k)\,, \\
&= X_i(\Phi_j)\delta_{jk} - \Phi_k \gamma_{ij}^k - \Phi_j \gamma_{ik}^j\,,
\end{aligned}$$

and so $\Phi_k \gamma^k_{ij} = -\Phi_j \gamma^j_{ik}$ for all i, j, k. In particular, we have

$$\Phi_j \gamma^j_{ii} = -\Phi_i \gamma^i_{ij} \,. \tag{2.10}$$

Hence, we have

$$\begin{aligned} \Phi_j \gamma^j_{ij} &= -\Phi_j \gamma^j_{ii} \quad \text{by (2.9)}\,, \\ &= -\Phi_i \gamma^i_{jj} \quad \text{by (2.8)}\,, \\ &= \Phi_j \gamma^j_{ji} \quad \text{by (2.10)}\,, \end{aligned}$$

and so $\gamma^j_{ij} = \gamma^j_{ji}$, which proves $[X_i, X_j] = 0$, and hence the existence of the coordinates u_i such that $X_i = \dfrac{\partial}{\partial u_i}$.

This means that the functions γ^k_{ij} are in fact the Christoffel symbols Γ^k_{ij} of ∇ in canonical coordinates. Writing $\mu_i = H_i^2$ and introducing the rotation coefficients

$$\beta_{ij} = \frac{H_{j,i}}{H_i} \,,$$

we may write the Christoffel symbols as

$$\begin{aligned} \Gamma^k_{ij} &= 0 \qquad \text{for } i, j, k \text{ distinct}\,, \\ \Gamma^i_{ij} &= \frac{H_j}{H_i}\beta_{ji}\,, \\ \Gamma^j_{ii} &= -\frac{H_i}{H_j}\beta_{ji} \qquad \text{for } i \neq j\,, \\ \Gamma^i_{ii} &= \beta_{ii}\,. \end{aligned}$$

Using these to express $\nabla_i c_{jkl}$, we find that the potentiality condition on c is equivalent to the symmetry of the rotation coefficients, i.e.

$$\nabla_i c_{jkl} = \nabla_j c_{ikl} \text{ for all } i, j, k, l \iff \beta_{ij} = \beta_{ji} \text{ for all } i, j\,.$$

This is readily seen to be equivalent to the existence of the potential Φ.

□

As stated in the above proof, the identity vector field in canonical coordinates is

$$e = \frac{\partial}{\partial u_1} + \cdots + \frac{\partial}{\partial u_n}\,.$$

It follows immediately from $\mathcal{L}_E \circ = \circ$ that the Euler vector field is

$$E = u_1 \frac{\partial}{\partial u_1} + \cdots + u_n \frac{\partial}{\partial u_n}\,,$$

after a suitable translation of the u_i, if necessary.

Example 2.1.7. *For the Frobenius manifold of Example 2.1.4, the canonical coordinates are*

$$\begin{aligned} u_1 &= t_1 + 2\sqrt{t_2}\,, \\ u_2 &= t_1 - 2\sqrt{t_2}\,, \end{aligned}$$

(the values of the function λ at its critical points). The metric in these coordinates is

$$\eta = \begin{pmatrix} \frac{1}{8}(u_1 - u_2) & 0 \\ 0 & -\frac{1}{8}(u_1 - u_2) \end{pmatrix},$$

from which we can see that the Egorov potential is

$$\Phi = t_2 = \frac{1}{16}(u_1 - u_2)^2\,.$$

2.1.3 Legendre Transformations

In this section we briefly discuss a symmetry of the WDVV equations which will have some applications in Chapter 5. By a symmetry we mean a map

$$\begin{aligned} t^\alpha &\mapsto \hat{t}^\alpha\,, \\ F(t^1, \ldots, t^n) &\mapsto \hat{F}(\hat{t}^1, \ldots, \hat{t}^n)\,, \\ \eta_{\alpha\beta} &\mapsto \hat{\eta}_{\alpha\beta} \end{aligned}$$

which takes one solution of the WDVV equations in variables t^α to another in variables $\hat{t}^\alpha$.

Following the notation of [25], we denote by S_κ, for $\kappa = 1, \ldots, n$, the Legendre transformation specified by

$$\begin{aligned} \hat{t}^\alpha &= \eta^{\alpha\beta} \frac{\partial^2 F}{\partial t^\beta \partial t^\kappa}\,, \\ \frac{\partial^2 \hat{F}}{\partial \hat{t}^\alpha \partial \hat{t}^\beta} &= \frac{\partial^2 F}{\partial t^\alpha \partial t^\beta}\,, \\ \hat{\eta}_{\alpha\beta} &= \eta_{\alpha\beta}\,. \end{aligned}$$

Note that after the Legendre transformation S_κ, the identity vector field is

$$\hat{e} = \frac{\partial}{\partial \hat{t}^\kappa}\,,$$

not

$$\frac{\partial}{\partial \hat{t}^1}\,.$$

Example 2.1.8. *Applying the transformation S_2 to the WDVV solution of Example 2.1.4 (and then translating $\hat{t}^1 \mapsto \hat{t}^1 - \frac{3}{2}$) one obtains $\hat{t}^1 = \log t^2$ and $\hat{t}^2 = t^1$, and the new free energy is*

$$\hat{F} = \frac{1}{2}\hat{t}^1(\hat{t}^2)^2 + e^{\hat{t}^1}\,.$$

The Euler vector field in these coordinates is

$$\hat{E} = \hat{t}^2\frac{\partial}{\partial \hat{t}^2} + 2\frac{\partial}{\partial \hat{t}^1}\,.$$

2.2 Deformed Flat Connection

One of the main objects on a Frobenius manifold is the deformed flat connection (see e.g. [24, 25]). This is a family of affine connections ∇^z, parameterised by a scalar z, defined by

$$\nabla^z_X Y = \nabla_X Y + zX \circ Y\,. \tag{2.11}$$

Theorem 2.2.1. *For an arbitrary flat connection ∇ and $\left[{1 \atop 2}\right]$-tensor $\circ$, the connection (2.11) is torsion-free for all z if and only if*

$$X \circ Y = Y \circ X \tag{2.12}$$

for all vector fields X, Y. It has vanishing curvature tensor if and only if

$$(\nabla_X \circ)(Y, Z) = (\nabla_Y \circ)(X, Z) \tag{2.13}$$

and

$$X \circ (Y \circ Z) = Y \circ (X \circ Z) \tag{2.14}$$

for all vector fields X, Y, Z. Here, $\nabla_X \circ$ is meant as the covariant derivative of a $\left[{1 \atop 2}\right]$-tensor, and may be evaluated as

$$(\nabla_X \circ)(Y, Z) = \nabla_X(Y \circ Z) - (\nabla_X Y) \circ Z - Y \circ (\nabla_X Z)\,.$$

Given (2.12), equation (2.14) is equivalent to the associativity condition $X \circ (Z \circ Y) = (X \circ Z) \circ Y$, thus on a Frobenius manifold the connection ∇^z is flat for all z.

This means that for all z there exist flat coordinates $\tilde{t}^i(t^1, \ldots, t^n, z)$ on M satisfying $\nabla^z d\tilde{t}^i = 0$ which deform the flat coordinates t^i of ∇. In the coordinates t^i this condition is

$$\frac{\partial^2 \tilde{t}^k}{\partial t^i \partial t^j} = z c^r_{ij} \frac{\partial \tilde{t}^k}{\partial t^r}\,,$$

or, if $\tilde{t}^k$ is expanded is as $\tilde{t}^k(t,z) = \sum_{r=0}^{\infty} z^r \tilde{t}^k_r(t)$, where $\tilde{t}^k_0 = t^k$,

$$\frac{\partial^2 \tilde{t}^k_n}{\partial t^i \partial t^j} = c^r_{ij} \frac{\partial \tilde{t}^k_{n-1}}{\partial t^r} \,. \tag{2.15}$$

The importance of the coefficients $\tilde{t}^k_n$ is that they provide a family of conserved quantities for an evolutionary system. This is established by considering the hydrodynamic type Hamiltonian operator associated to the metric η, which in the flat coordinates for η is

$$P_1^{ij} = \eta^{ij} \frac{d}{dx} \,. \tag{2.16}$$

The functions $\tilde{t}^k_n$ on M, when interpreted as the functional densities on $L(M)$, satisfy

$$\{\tilde{t}^i_n, \tilde{t}^j_m\}_1 = 0 \,,$$

where $\{\cdot,\cdot\}_1$ is the Poisson bracket (1.28) generated by P_1.

This only uses parts 1 and 2 of Definition 2.1.3 which are concerned with the metric and the multiplication; the two vector fields e and E are not used.

The identity vector field $e = \frac{\partial}{\partial t^1}$ gives the simplified recurrence relation:

$$\frac{\partial^2 \tilde{t}^k_n}{\partial t^1 \partial t^j} = \frac{\partial \tilde{t}^k_{n-1}}{\partial t^j} \,.$$

The Euler vector field can be used, on most Frobenius manifolds, to fix the coefficients t^i_n. This is done by asking that the flat coordinates $t^i(t,z)$ be homogeneous with respect to the extended Euler vector field

$$\tilde{E} = E - z\frac{\partial}{\partial z} \,,$$

i.e. that they satisfy $\tilde{E}(\tilde{t}^i) = d_i \tilde{t}^i$, or, equivalently, $E(\tilde{t}^i_n) = (d^i + n)\tilde{t}^i_n$. This fixes the $\tilde{t}^i_n$ uniquely, provided the differences $d_i - d_j$ are not integer for any $i \neq j$.

Example 2.2.2. *In the simplest case, for a one-dimensional Frobenius manifold with coordinate $u = t^1$, and metric and multiplication given by $\eta_{11} = c^1_{11} = 1$, and Euler vector field $E = u\frac{\partial}{\partial u}$, the homogeneous flat coordinate for ∇^z is*

$$\tilde{u} = \sum_{r \geq 0} z^r \frac{1}{(r+1)!} u^{r+1} \,.$$

In the coefficients of this, we recognise multiples of the dispersive limits of the conserved quantities (1.51) of the KdV hierarchy.

Example 2.2.3. *For the Frobenius manifold of Example 2.1.4, the recursion relations for the flat coordinates of the deformed connection are*

$$\begin{aligned}\frac{\partial^2 \tilde{t}_i}{\partial t_1^2} &= z\frac{\partial \tilde{t}_i}{\partial t_1}\,,\\ \frac{\partial^2 \tilde{t}_i}{\partial t_1 \partial t_2} &= z\frac{\partial \tilde{t}_i}{\partial t_2}\,,\\ \frac{\partial^2 \tilde{t}_i}{\partial t_2^2} &= z\frac{1}{t_2}\frac{\partial \tilde{t}_i}{\partial t_1}\,.\end{aligned}$$

The flat coordinates are then

$$\begin{aligned}\tilde{t}_1 &= \left\{\frac{1}{z}+\sum_{n=0}^{\infty} z^{2n+1}\frac{1}{n!(n+1)!}t_2^{n+1}\left(\log t_2 - H_n - H_{n+1}\right)\right\}e^{zt_1}-\frac{1}{z}\,,\\ \tilde{t}_2 &= \left\{\sum_{n=0}^{\infty} z^{2n}\frac{1}{n!(n+1)!}t_2^{n+1}\right\}e^{zt_1}\,,\end{aligned}$$

where H_n is the n^{th} harmonic number:

$$H_n = \sum_{r=1}^{n}\frac{1}{r}\,.$$

Expanding these series, one finds

$$\begin{aligned}\tilde{t}_1 = {} & t_1\\ & +z\left\{\frac{1}{2}t_1^2 + t_2(\log t_2 - \frac{5}{2})\right\}\\ & +z^2\left\{\frac{1}{6}t_1^3 + t_1 t_2(\log t_2 - \frac{5}{2})\right\}\\ & +z^3\left\{\frac{1}{24}t_1^4 + \frac{1}{2}t_1^2 t_2(\log t_2 - \frac{5}{2}) + \frac{1}{2}t_2^2(\log t_2 - \frac{10}{3})\right\}\\ & +O(z^4)\,,\end{aligned}$$

and

$$\begin{aligned}\tilde{t}_2 = {} & t_2\\ & +z\left\{t_1 t_2\right\}\\ & +z^2\left\{\frac{1}{2}t_1^2 t_2 + \frac{1}{2}t_2^2\right\}\\ & +z^3\left\{\frac{1}{6}t_1^3 t_2 + \frac{1}{2}t_1 t_2^2\right\}\\ & +O(z^4)\,.\end{aligned}$$

Taking the z^2 term from the expansion of $\tilde{t}_2$, one obtains as the associated Hamiltonian flow

$$\frac{d}{d\tau_{2,2}}\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} t_1 t_{1,x} + t_{2,x} \\ t_2 t_{1,x} + t_1 t_{2,x} \end{pmatrix}\,, \tag{2.17}$$

or the one-dimensional long wave system [53].

The z^1 term in the expansion of $\tilde{t}_1$ gives the flow

$$\frac{d}{d\tau_{1,1}} \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} t_{2,x}/t_2 \\ t_{1,x} \end{pmatrix}, \tag{2.18}$$

or, as an equation in one variable

$$\frac{d^2 t_2}{d\tau_{1,1}^2} = (\log t_2)_{xx}\,.$$

If, as in Example 2.1.8, we apply the Legendre transformation S_2 to this system and expand the coordinate of the deformed flat connection which comes from $\hat{t}^2$, we obtain the flow

$$\frac{d}{d\tau_{2,1}} \begin{pmatrix} \hat{t}^1 \\ \hat{t}^2 \end{pmatrix} = \begin{pmatrix} \hat{t}^2_x \\ \hat{t}^1_x e^{\hat{t}^1} \end{pmatrix} \tag{2.19}$$

from the coefficient of z^1. As an equation in one variable ($b = \hat{t}^2\,, t = \tau_{2,1}$), this is

$$b_{tt} = \left(e^b\right)_{xx}, \tag{2.20}$$

the continuous Toda lattice.

2.3 Relation to Bi-Hamiltonian Systems

In the previous section a relationship was exhibited between the deformed flat connection and a set of commuting evolution equations. This therefore installs Frobenius manifolds in the general programme of studying partial differential in terms of associated finite-dimensional geometric objects. Another means of understanding their place in such a programme was demonstrated by Dubrovin in [26], which showed that much of the geometry of a Frobenius manifold can be related to a certain bi-Hamiltonian structure on its loop space. We have already introduced the hydrodynamic type Hamiltonian operator (2.16) associated to the constant metric η_{ij}; the second Hamiltonian operator is also of hydrodynamic type, and thus related to a second metric on the Frobenius manifold, which is called the intersection form.

Before introducing this, we first consider more carefully the geometry of Hydrodynamic operators.

As said in Section 1.4, the operator

$$P^{ij} = g^{ij}\frac{d}{dx} + \Gamma^{ij}_k u^k_x\,,$$

where g^{ij} is non-degenerate, is Hamiltonian if and only if g^{ij} is the inverse of a flat metric on M and the Christoffel symbols of its Levi-Civita connection, ∇, are $\Gamma^k_{ij} = -g_{ir}\Gamma^{rk}_j$.

The four conditions on g_{ij} and ∇, namely that $g_{ij} = g_{ji}$, $\nabla_k g_{ij} = 0$ and that ∇ is torsion-free and has zero curvature, can be expressed in terms of the contravariant quantities g^{ij} and Γ^{ij}_k as

$$\begin{aligned} g^{ij} &= g^{ji}\,, && (2.21)\\ \frac{\partial g^{ij}}{\partial u^k} &= \Gamma^{ij}_k + \Gamma^{ji}_k\,, && (2.22)\\ g^{ir}\Gamma^{jk}_r &= g^{jr}\Gamma^{ik}_r\,, && (2.23) \end{aligned}$$

and

$$g^{ir}\left(\partial_l \Gamma^{jk}_r - \partial_r \Gamma^{jk}_l\right) + \Gamma^{ij}_r\Gamma^{rk}_l - \Gamma^{ik}_r\Gamma^{rj}_l = 0\,, \tag{2.24}$$

respectively.

It is now easy to find the conditions on a pair of hydrodynamic type Hamiltonian operators

$$P^{ij}_1 = g^{ij}_1 \frac{d}{dx} + \Gamma_1{}^{ij}_k u^k_x$$

and

$$P^{ij}_2 = g^{ij}_2 \frac{d}{dx} + \Gamma_2{}^{ij}_k u^k_x$$

equivalent to their being compatible; specifically, we substitute $g^{ij} = g^{ij}_\lambda := g^{ij}_1 + \lambda g^{ij}_2$ and $\Gamma^{ij}_k = \Gamma_\lambda{}^{ij}_k := \Gamma_1{}^{ij}_k + \lambda \Gamma_2{}^{ij}_k$ into equations (2.21)-(2.24).

Thus, one may state geometric conditions on g^{ij}_1 and g^{ij}_2 to ensure that P_1 and P_2 are compatible. We have: P_1 and P_2 are compatible if and only if the linear combination g^{ij}_λ of inverse metrics is also the inverse of a flat metric, and the linear combinations $\Gamma_\lambda{}^{ij}_k$ of contravariant Christoffel symbols are the contravariant Christoffel symbols of g^{ij}_λ. If this is the case, we say the metrics g_1 and g_2 form a flat pencil of metrics; alternatively, we say that g_1 and g_2 are compatible.

On a Frobenius manifold, we introduce the second metric, g, by first using the isomorphism defined by η between vector and covectors to induce a multiplication, also denoted $\circ$, of covectors. Then we define g as the metric inverse to the paring of covectors given by

$$(\alpha, \beta)_g = \langle E|\, \alpha \circ \beta \rangle\,.$$

In coordinates, this definition reads

$$g^{ij} = E^r c^{ij}_r\,, \tag{2.25}$$

where $c^{ij}_k = \eta^{ir} c^j_{rk}$. As is shown in [26], the metrics η and g form a flat pencil, and thus determine a bi-Hamiltonian structure on the loop space of a Frobenius manifold. However, additional structure is required on a flat pencil in order for it to determine the structure of a Frobenius manifold.

Definition 2.3.1. *Two compatible flat metrics, g_1 and g_2, on a manifold M are said to form a quasihomogeneous flat pencil if there exists a function τ and a constant d such that the vector fields E and e defined by $E^i = g_1^{ir}\tau_{,r}$ and $e^i = g_2^{ir}\tau_{,r}$ satisfy*

$$\begin{aligned} [e, E] &= e\,, \\ \mathcal{L}_E g_1^{ij} &= (d-1) g_1^{ij}\,, \\ \mathcal{L}_e g_1^{ij} &= g_2^{ij}\,, \\ \mathcal{L}_e g_2^{ij} &= 0\,. \end{aligned}$$

The pencil is called regular if, in addition, the tensor

$$T^i_j = \frac{d-1}{2}\delta^i_j + \nabla^2_j E^i$$

is non-degenerate, where ∇^2 is the Levi-Civita connection of g_2.

Then one has [26]

Theorem 2.3.2. *Every Frobenius manifold carries a natural quasihomogeneous flat pencil; conversely, every regular quasihomogeneous flat pencil on a manifold endows it with the structure of a Frobenius manifold.*

In the correspondence, E and e are the Euler and identity vector fields respectively, g_2 is the metric η, g_1 is the intersection form, and the function τ is $t_1 := \sum_\alpha \eta_{1\alpha} t^\alpha$.

The construction of the commutative associative multiplication of vector fields on the Frobenius manifold is achieved by first introducing a more primitive multiplication of covectors which characterises the compatibility of the metrics. As before, g_1 and g_2 are two flat metrics, and ${\Gamma_1}^{ij}_k$ and ${\Gamma_2}^{ij}_k$ are the contravariant Christoffel symbols of their respective Levi-Civita connections. With these, we first define the tensors

$$\begin{aligned} \Delta^{sjk} &= g_2^{jr} {\Gamma_1}^{sk}_r - g_1^{sr} {\Gamma_2}^{jk}_r\,, \\ \Delta^{jk}_i &= g_{2is} \Delta^{sjk}\,, \end{aligned}$$

from which we then define a multiplication of covectors by

$$(\alpha \diamond \beta)_i = \alpha_j \beta_k \Delta^{jk}_i\,.$$

Conditions (2.21)-(2.24) for g_λ^{ij} and ${\Gamma_\lambda}_k^{ij}$ can now be converted into algebraic statements about the multiplication $\diamond$. Since (2.21) and (2.22) are linear, they follow immediately from the flatness of g_1 and g_2. (2.23) and (2.24) are quadratic in λ, and hence split into three equations, from considering the coefficients of 1, λ and λ^2. The coefficients of 1 and λ^2 simply express the relevant equation for P_1 and P_2 respectively, so it is the coefficients of λ which express the compatibility of the two operators.

From these we see that g_1 and g_2 form a flat pencil of metrics if and only if

$$(\alpha \diamond \beta, \gamma)_1 = (\alpha, \gamma \diamond \beta)_1 \,, \tag{2.26}$$

and

$$(\alpha \diamond \beta) \diamond \gamma = (\alpha \diamond \gamma) \diamond \beta \,, \tag{2.27}$$

for all covectors α, β, γ. Here $(\cdot,\cdot)_1$ is the pairing of covectors defined by g_1^{ij}, i.e. $(\alpha,\beta)_1 = g_1^{ij}\alpha_i\beta_j$. Two consequences of the flatness of g_1 and g_2 together with equations (2.26) and (2.27) are

$$(\alpha \diamond \beta, \gamma)_2 = (\alpha, \gamma \diamond \beta)_2 \,, \tag{2.28}$$

and

$$\nabla_l^2 \Delta_k^{ij} = \nabla_k^2 \Delta_l^{ij} \,. \tag{2.29}$$

If g_1 and g_2 form a regular quasihomogeneous flat pencil, then we define a new multiplication $\circ$ by

$$\alpha \circ \beta = \alpha \diamond T^{-1}(\beta) \,,$$

for covectors α and β. Here T^{-1} is the inverse of the operation $T : \Gamma(T^*M) \to \Gamma(T^*M)$ obtained from regarding the tensor T_j^i as an automorphism of the cotangent bundle. That $\circ$ is commutative and associative follows from the relation $T(\alpha) = d\tau \diamond \alpha$ and the properties of the multiplication $\diamond$. One then uses the isomorphism between TM and T^*M induced by the metric g_2 to obtain a multiplication of vector fields from $\circ$.

Example 2.3.3. *For the Frobenius manifold of Examples 2.1.4 and 2.2.3, equation (2.25) gives the inverse of the intersection form as*

$$g^{-1} = \begin{pmatrix} 2 & t_1 \\ t_1 & 2t_2 \end{pmatrix} . \tag{2.30}$$

The operator T in this case is degenerate.

2.4 Polynomial Frobenius Manifolds

An important class of Frobenius manifolds is comprised of those defined in a natural manner on the orbit spaces of finite Coxeter groups [23, 25]. The significance of such structures is that the associated free energies provide polynomial solutions to the WDVV equations, and, as was later shown by Hertling [47], all polynomial solutions arise in this way.

A Coxeter group, W, is a group of linear transformations of an n-dimensional Euclidean vector space, V, generated by reflections. The ring $S(V)$ of polynomial functions on V (i.e. polynomial in a coordinate system for V determined by a choice of basis) inherits an action of W, and this distinguishes a subring $R = S(V)^W$ of W-invariant polynomials.

By the Chevalley theorem [16], one can choose n independent homogeneous polynomials, $y_1, \ldots, y_n$, which generate R; their degrees, $d_1, \ldots, d_n$, are fixed by the Coxeter group, and can be arranged such that $d_1 = h > d_2 \geq \cdots \geq d_{n-1} > d_n = 2$. h is called the Coxeter number of W. This means that the polynomial y_1 and also the vector field

$$e = \frac{\partial}{\partial y_1}$$

are fixed up to scalar multiples. Every Coxeter group possesses a degree 2 invariant, y_n, which is a multiple of the distance from the origin, i.e.

$$y_n = \alpha \sum_{r=1}^{n} x_r^2 \,,$$

where $(x_1, \ldots, x_n)$ is an orthogonal coordinate system for V.

Typically [25, 47], one extends the action of W to the complexified vector space $V \otimes \mathbb{C}$. The ring $R \otimes \mathbb{C}$ is the ring of polynomial functions on $M = V \otimes \mathbb{C}/W$, and the polynomials $y_1, \ldots, y_n$ form a coordinate system on this manifold. Since the Euclidean metric is invariant under W, it induces a metric on M, whose contravariant components in this coordinate system are

$$g^{ij} = (dy_i, dy_j)^{-1} = \sum_{r=1}^{n} \frac{\partial y_i}{\partial x_r} \frac{\partial y_j}{\partial x_r} \,,$$

where $(\cdot, \cdot)^{-1}$ is the contravariant metric on T^*V determined by the Euclidean metric on V. The contravariant Christoffel symbols of its Levi-Civita connection are determined by

$$\sum_{k=1}^{n} \Gamma^{ij}_k dy_k = \sum_{a,b=1}^{n} \frac{\partial y_i}{\partial x_a} \frac{\partial^2 y_j}{\partial x_a \partial x_b} dx_b \,.$$

From the dilation vector field

$$E^V = \frac{1}{h}\sum_{r=1}^{n} x_r \frac{\partial}{\partial x_r}$$

on V, one obtains the vector field

$$E = \frac{1}{h}\sum_{r=1}^{b}\left(d_1 y_1 \frac{\partial}{\partial y_1} + \cdots + d_n y_n \frac{\partial}{\partial y_n}\right)$$

encoding the degrees of $y_1, \ldots, y_n$, and which is a special conformal symmetry of g.

The components g^{ij} and Γ^{ij}_k are each homogeneous polynomials in $y_1, \ldots, y_n$, and, as can be determined from their degrees, depend at most linearly on y_1. A theorem of Saito [65] establishes that one may define a second metric η on M by

$$\eta^{ij} = \frac{\partial g^{ij}}{\partial y_1}\,,$$

or, invariantly, by

$$\eta^{ij} = \mathcal{L}_e g^{ij}\,, \tag{2.31}$$

and that this second metric is, at a generic point, non-degenerate, and flat. The Christoffel symbols of its Levi-Civita connection are given by

$$\Gamma_{\eta}{}^{ij}_{k} = \frac{\partial \Gamma^{ij}_k}{\partial y_1}\,,$$

or

$$\Gamma_{\eta}{}^{ij}_{k} = \mathcal{L}_e \Gamma^{ij}_k\,. \tag{2.32}$$

(It follows from the transformation rules (1.31) and equation (2.31) that both sides of this equation are of the same differential-geometric type.)

More significantly, the two flat metrics η and g are compatible, and, taken with the vector fields e and E, form a regular quasihomogeneous flat pencil (in which $\eta = g_2$, $g = g_1$), with $d = \frac{h-2}{h}$. Thus the construction of Section 2.3 allows us to introduce a Frobenius manifold structure on M. Flat coordinates $t^1, \ldots, t^n$ for η can be chosen such that t^i is a homogeneous polynomial of degree d_i in the y_i coordinates, and the free energy is a homogeneous polynomial in these coordinates.

Example 2.4.1. *The Coxeter group $I_2(k)$ is the set of isometries which preserve the regular k-gon in the Euclidean plane. It is convenient to consider the polygon as sitting in the complex plane, arranged symmetrically about the real axis, with its vertices equidistant from the origin. $I_2(k)$ is then generated the the reflection*

$$\rho : z \mapsto \bar{z}$$

and the rotation

$$s : z \mapsto e^{\frac{2\pi i}{k}} z\,.$$

The invariant polynomials generating R are

$$y_1 = z^k + \bar{z}^k$$

and

$$y_2 = \frac{1}{k} z\bar{z}\,,$$

which we take as our coordinates on the orbit space. (We eventually allow y_1 and y_2 to become complex variables to work with $\mathbb{R}^2 \otimes \mathbb{C}/I_2(k) \cong \mathbb{C}^2/I_2(k)$.)

The contravariant Euclidean metric is given by

$$g^{-1} = \frac{\partial}{\partial z} \otimes \frac{\partial}{\partial \bar{z}} + \frac{\partial}{\partial \bar{z}} \otimes \frac{\partial}{\partial z}$$

and induces the metric

$$g^{ij} = \begin{pmatrix} 2k^{k+1} y_2^{k-1} & y_1 \\ y_1 & \frac{2}{k} y_2 \end{pmatrix}$$

on the orbit space. The Saito metric is therefore

$$\eta^{ij} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,,$$

so the coordinates y_1 and y_2 are flat coordinates for η.

The Euler and identity vector fields in this case are

$$e = \frac{\partial}{\partial y_1}$$

and

$$E = y_1 \frac{\partial}{\partial y_1} + \frac{2}{k} y_2 \frac{\partial}{\partial y_2}\,,$$

which can be seen to satisfy the conditions of Definition 2.3.1, with $d = \frac{k-2}{k}$ and $\tau = y_2$. The tensor T is

$$T = \frac{k-1}{k} dy_1 \otimes \frac{\partial}{\partial y_1} + \frac{1}{k} dy_2 \otimes \frac{\partial}{\partial y_2}\,,$$

which is clearly invertible.

The non-zero Christoffel symbols of the Levi-Civita connection of g^{ij} are

$$\begin{aligned} \Gamma_2^{11} &= k^{k+1}(k-1) y_2^{k-2}\,, \\ \Gamma_1^{12} &= \frac{1}{k}\,, \\ \Gamma_1^{21} &= \frac{k-1}{k}\,, \\ \Gamma_2^{22} &= \frac{1}{k}\,. \end{aligned}$$

From this data, the construction of Section 2.3 gives the multiplication table

$$\begin{aligned}
\frac{\partial}{\partial y_1} \circ \frac{\partial}{\partial y_1} &= \frac{\partial}{\partial y_1}\,, \\
\frac{\partial}{\partial y_1} \circ \frac{\partial}{\partial y_2} &= \frac{\partial}{\partial y_2}\,, \\
\frac{\partial}{\partial y_2} \circ \frac{\partial}{\partial y_2} &= k^{k+2} y_2^{k-2} \frac{\partial}{\partial y_1}\,,
\end{aligned}$$

and the consequent free energy is

$$F = \frac{1}{2} y_1^2 y_2 + \frac{(k y_2)^{k+1}}{k^2 - 1}\,,$$

which satisfies

$$E(F) = \frac{2k+2}{k} F\,.$$

In the above example, the Saito metric η was already found to be constant in the coordinates (y_1, y_2); in general this is not the case, and one must construct the flat coordinates, called the Saito flat coordinates, as functions of the coordinates y_i derived from the invariants of the Coxeter group. See, for instance, Example 4.1.1 which calculates the Frobenius structure on the orbit space of A_3.

The idea of constructing a Frobenius manifold structure on the orbit space of a Coxeter group can be extended to other orbit spaces, like those of complex crystallographic groups [25] or Jacobi groups [7, 8].

Chapter 3

Hamiltonian Operators of Degree 2 in Bi-Hamiltonian Structures

This chapter is concerned with the geometry associated to Hamiltonian operators of differential-geometric type which are homogeneous of degree 2. That is, operators of the form

$$P^{ij} = a^{ij}\left(\frac{d}{dx}\right)^2 + b^{ij}_k u^k_x \frac{d}{dx} + c^{ij}_{kl} u^k_x u^l_x + c^{ij}_k u^k_{xx}\,, \tag{3.1}$$

in which the matrix a^{ij} is non-degenerate. In particular, we are interested in the symplectic or almost symplectic geometry of such operators, and in understanding the behaviour of this geometry when such an operators are included as one or both of the Hamiltonian operators in a bi-Hamiltonian structure.

In Section 3.1 we review the differential geometry of such operators, and in particular relate the subclass which can be put into a constant form by a change of coordinates on M to symplectic connections. Section 3.2 then considers pairs of operators from this subclass, and formulates their compatibility in terms of algebraic constraints on a multiplication which may be defined from the pair. Section 3.3 presents the compatibility of two operators of the form (3.1) without the assumption that they lie in this special class. Then, in Section 3.4, inhomogeneous bi-Hamiltonian structures consisting of a degree 1 and a degree 2 operator are studied.

3.1 Hamiltonian Operators of Degree 2

We begin with a review of known results on Hamiltonian operators of the form (3.1). Such operators have been considered already in, for example, [21, 59, 61, 64]. Amongst the material discussed in these papers there is a conditional Darboux theorem stipulating under what circumstances (3.1) may be put into constant form by a coordinate transformation. The approach taken in these papers is to consider the operator in a special coordinate system in which the coefficients c^{ij}_k and c^{ij}_{kl} both vanish; this is unsuitable for a discussion of bi-Hamiltonian structures, and as such we present the results of this section without the use of special coordinates.

Under the change of coordinates $\tilde{u}^i = \tilde{u}^i(u^p)$ the coefficients in P^{ij} transform as

$$\begin{aligned}
\tilde{a}^{ij} &= \frac{\partial \tilde{u}^i}{\partial u^p}\frac{\partial \tilde{u}^j}{\partial u^q} a^{pq}\,,\\
\tilde{b}^{ij}_k &= \frac{\partial \tilde{u}^i}{\partial u^p}\frac{\partial \tilde{u}^j}{\partial u^q}\frac{\partial u^r}{\partial \tilde{u}^k} b^{pq}_r - 2\frac{\partial \tilde{u}^i}{\partial u^p}\frac{\partial \tilde{u}^s}{\partial u^q}\frac{\partial \tilde{u}^j}{\partial u^r}\frac{\partial^2 u^r}{\partial \tilde{u}^k \partial \tilde{u}^s} a^{pq}\,,\\
\tilde{c}^{ij}_k &= \frac{\partial \tilde{u}^i}{\partial u^p}\frac{\partial \tilde{u}^j}{\partial u^q}\frac{\partial u^r}{\partial \tilde{u}^k} c^{pq}_r - \frac{\partial \tilde{u}^i}{\partial u^p}\frac{\partial \tilde{u}^s}{\partial u^q}\frac{\partial \tilde{u}^j}{\partial u^r}\frac{\partial^2 u^r}{\partial \tilde{u}^k \partial \tilde{u}^s} a^{pq}\,,\\
\tilde{c}^{ij}_{kl} &= \frac{\partial \tilde{u}^i}{\partial u^p}\frac{\partial \tilde{u}^j}{\partial u^q}\frac{\partial u^r}{\partial \tilde{u}^k}\frac{\partial u^s}{\partial \tilde{u}^l} c^{pq}_{rs} + \frac{\partial \tilde{u}^i}{\partial u^p}\frac{\partial \tilde{u}^j}{\partial u^q}\frac{\partial^2 u^r}{\partial \tilde{u}^k \partial \tilde{u}^l} c^{pq}_r\\
&\quad + \frac{\partial \tilde{u}^i}{\partial u^p}\frac{\partial^2 \tilde{u}^j}{\partial u^q \partial u^s}\frac{\partial u^r}{\partial \tilde{u}^{(k}}\frac{\partial u^s}{\partial \tilde{u}^{l)}} b^{pq}_r + \frac{\partial \tilde{u}^i}{\partial u^p}\frac{\partial^3 \tilde{u}^j}{\partial u^q \partial u^r \partial u^s}\frac{\partial u^r}{\partial \tilde{u}^k}\frac{\partial u^s}{\partial \tilde{u}^l} a^{pq}\\
&\quad + \frac{\partial \tilde{u}^i}{\partial u^p}\frac{\partial^2 \tilde{u}^j}{\partial u^q \partial u^r}\frac{\partial^2 u^r}{\partial \tilde{u}^k \partial \tilde{u}^l} a^{pq}\,,
\end{aligned} \tag{3.2}$$

where the brackets denote symmetrisation. So in particular a^{ij} transforms as a rank 2 contravariant tensor on the target space and b^{ij}_k and c^{ij}_k are related to Christoffel symbols of connections by $b^{ij}_k = -2a^{ir}\bar{\Gamma}^j_{rk}$ and $c^{ij}_k = -a^{ir}\Gamma^j_{rk}$. Call these connections $\bar{\nabla}$ and ∇ respectively.

The transformation rules for c^{ij}_{kl} are not determined uniquely by those for P, since (3.1) sees only the part symmetric in k and l. To fix c^{ij}_{kl}, we always assume the antisymmetric part is zero. We denote by a_{ij} the inverse of a^{ij} defined by $a_{ir}a^{rj} = \delta^j_i$.

The condition that the operation defined in (1.28) is skew-symmetric and satisfies the Jacobi identity places constraints on the coefficients appearing in (3.1).

Theorem 3.1.1. *The operator P in equation (3.1) defines a Poisson bracket by equation (1.28) if and only if*

(A) $a^{ij} = -a^{ji}$,

(B) $\nabla_k a^{ij} = b^{ij}_k - 2c^{ij}_k$,

(C) $a^{ir}\left(b^{jk}_r - 2c^{jk}_r\right) = a^{kr}\left(b^{ij}_r - 2c^{ij}_r\right)$,

(D) ∇ *is flat (zero torsion, zero curvature),*

(E) $c^{ij}_{kl} = c^{ij}_{(k,l)} - a_{pr} c^{ri}_{(k} c^{pj}_{l)}$.

Note: In the above theorem we have chosen to state conditions in terms of covariant derivatives with respect to ∇. This is because the Christoffel symbols $\bar{\Gamma}^k_{ij}$ defined by b^{ij}_k are not symmetric in general, making the definition of $\bar{\nabla}$ from them ambiguous.

Proof. [59] states that, by virtue of being Hamiltonian, the operator (3.1) can be put in the form

$$P^{ij} = a^{ij} \left(\frac{d}{dx}\right)^2 + b^{ij}_k u^k_x \frac{d}{dx} \,, \tag{3.3}$$

by a change of coordinates $u^i = u^i(\tilde{u})$, and that for an operator of this shorter form to be Hamiltonian is equivalent to the three conditions

(a) $a^{ij} = -a^{ji}$,

(b) $a^{ij}{}_{,k} = b^{ij}_k$,

(c) $a^{ir} b^{jk}_r = a^{kr} b^{ij}_r$.

We first assume that P is a Poisson bracket, so there exists the special coordinates in which P takes the form (3.3) and (a)-(c) hold. By reversing the change of variables as $\tilde{u}^i = \tilde{u}^i(u)$, conditions (A)-(C) of Theorem 3.1.1 are Mokhov's three conditions converted to tensorial identities. That ∇ is flat follows from its Christoffel symbols, $\Gamma^k_{ij} = -a_{ir} c^{rk}_j$, being zero in the u coordinates.

The formula in condition (E) is derived from the transformation rules above. In changing from flat coordinates u^i to coordinates $\tilde{u}^i$ they give:

$$\begin{aligned} \tilde{c}^{ij}_{kl} &= \frac{\partial \tilde{u}^i}{\partial u^p} \frac{\partial^2 \tilde{u}^j}{\partial u^q \partial u^s} \frac{\partial u^s}{\partial \tilde{u}^{(k}} \frac{\partial u^s}{\partial \tilde{u}^{l)}} b^{pq}_r + \frac{\partial \tilde{u}^i}{\partial u^p} \frac{\partial^3 \tilde{u}^j}{\partial u^q \partial u^r \partial u^s} \frac{\partial u^s}{\partial \tilde{u}^k} \frac{\partial u^s}{\partial \tilde{u}^l} a^{pq} \\ &\quad + \frac{\partial \tilde{u}^i}{\partial u^p} \frac{\partial^2 \tilde{u}^j}{\partial u^q \partial u^r} \frac{\partial^2 u^r}{\partial \tilde{u}^k \partial \tilde{u}^l} a^{pq} \,, \end{aligned}$$

and

$$\begin{aligned} \tilde{c}^{ij}_k &= -\frac{\partial \tilde{u}^i}{\partial u^p} \frac{\partial \tilde{u}^s}{\partial u^q} \frac{\partial \tilde{u}^j}{\partial u^r} \frac{\partial^2 u^r}{\partial \tilde{u}^k \partial \tilde{u}^s} a^{pq} \,, \\ &= \frac{\partial \tilde{u}^i}{\partial u^p} \frac{\partial^2 \tilde{u}^j}{\partial u^q \partial u^r} \frac{\partial u^r}{\partial \tilde{u}^k} a^{pq} \,, \end{aligned}$$

where the last line has used the identity

$$\frac{\partial^2 \tilde{u}^i}{\partial u^r \partial u^s} \frac{\partial u^r}{\partial \tilde{u}^j} \frac{\partial u^s}{\partial \tilde{u}^k} + \frac{\partial \tilde{u}^i}{\partial u^r} \frac{\partial^2 u^r}{\partial \tilde{u}^j \partial \tilde{u}^k} = 0 \,,$$

which is a differential consequence of $\frac{\partial \tilde{u}^i}{\partial u^r}\frac{\partial u^r}{\partial \tilde{u}^j} = \delta^i_j$.

$$\begin{aligned}
\tilde{c}^{ij}_{k,l} &= \frac{\partial \tilde{c}^{ij}_k}{\partial \tilde{u}_l} \\
&= \frac{\partial^2 \tilde{u}^i}{\partial u^p \partial u^s}\frac{\partial u^s}{\partial \tilde{u}^l}\frac{\partial^2 \tilde{u}^j}{\partial u^r \partial u^q}\frac{\partial u^r}{\partial \tilde{u}^k} a^{pq} \\
&\quad + \frac{\partial \tilde{u}^i}{\partial u^p}\frac{\partial^3 \tilde{u}^j}{\partial u^q \partial u^r \partial u^s}\frac{\partial u^r}{\partial \tilde{u}^k}\frac{\partial u^s}{\partial \tilde{u}^l} a^{pq} \\
&\quad + \frac{\partial \tilde{u}^i}{\partial u^p}\frac{\partial^2 \tilde{u}^j}{\partial u^q \partial u^r}\frac{\partial^2 u^r}{\partial \tilde{u}^k \partial \tilde{u}^l} a^{pq} \\
&\quad + \frac{\partial \tilde{u}^i}{\partial u^p}\frac{\partial^2 \tilde{u}^j}{\partial u^q \partial u^r}\frac{\partial u^r}{\partial \tilde{u}^k}\frac{\partial u^s}{\partial \tilde{u}^l} b^{pq}_s ,
\end{aligned}$$

from which we see

$$\tilde{c}^{ij}_{kl} = \tilde{c}^{ij}_{(k,l)} - \frac{\partial^2 \tilde{u}^i}{\partial u^p \partial u^s}\frac{\partial^2 \tilde{u}^j}{\partial u^r \partial u^q}\frac{\partial u^s}{\partial \tilde{u}^{(l}}\frac{\partial u^r}{\partial \tilde{u}^{k)}} a^{pq} .$$

This last term can be seen to be

$$\tilde{a}_{pr}\tilde{c}^{ri}_{(k}\tilde{c}^{pj}_{l)} .$$

Conversely, if (A)-(E) hold, the flatness of ∇ asserts the existence of coordinates in which $c^{ij}_k = 0$, and condition (E) then asserts that $c^{ij}_{kl} = 0$ in these coordinates. □

With the Hamiltonian condition expressed in arbitrary coordinates, we may now emulate the approach to hydrodynamic type Poisson brackets taken by Balinskiĭ and Novikov in [5] and study Hamiltonian operators of degree 2 which depend linearly on the coordinates u^i by relating them to algebraic structures. In fact, we relax the linearity condition slightly, and consider an operator P as in (3.1) with $b^{ij}_k = 2c^{ij}_k$ constants and use condition (E) to define c^{ij}_{kl}, which may therefore depend nonlinearly on the u^i. Then P is Hamiltonian if and only if $a^{ij} = A^{ij}_k u^k + A^{ij}_0$ where A^{ij}_k, A^{ij}_0 are constants with

$$\begin{aligned}
A^{ij}_k &= c^{ij}_k - c^{ji}_k , \\
A^{ir}_l c^{jk}_r &= A^{jr}_l c^{ik}_r , \\
A^{ir}_0 c^{jk}_r &= A^{jr}_0 c^{ik}_r
\end{aligned}$$

and

$$c^{ij}_r c^{rk}_l + c^{ik}_r c^{rj}_l = 0 .$$

If we take an algebra $\mathcal{A}$ with basis $\{e^1, \ldots, e^n\}$, $n = \dim M$, and use c^{ij}_k and A^{ij}_0 to define a multiplication, $\diamond$, and skew-symmetric bilinear form, $\langle \cdot, \cdot \rangle$, by

$$e^i \diamond e^j = c^{ij}_r e^r$$

and

$$\langle e^i, e^j \rangle = A_0^{ij} ,$$

then we may rewrite these conditions as

$$\begin{aligned} e^i \diamond e^j - e^j \diamond e^i &= A_r^{ij} e^r , \\ (I \diamond J) \diamond K &= -(I \diamond K) \diamond J , \qquad (3.4) \\ \Lambda(I, J, K) &= \Lambda(J, I, K) , \qquad (3.5) \\ \text{and} \qquad \langle I, J \diamond K \rangle &= \langle J, I \diamond K \rangle , \end{aligned}$$

for all $I, J, K \in \mathcal{A}$, where Λ is the associator of $\diamond$: $\Lambda(I, J, K) = (I \diamond J) \diamond K - I \diamond (J \diamond K)$.

Algebras satisfying conditions (3.4) and (3.5) have appeared before in [71], in the context of linear hydrodynamic Hamiltonian operators taking values in a completely odd superspace, where the following definition was proposed:

Definition 3.1.2. *An algebra $(\mathcal{A}, \diamond)$ satisfying conditions (3.4) and (3.5) is called a Fermionic Novikov algebra.*

In [4] Fermionic Novikov algebras in dimensions 2-5 were studied, and the listing therein provides a source of examples of Hamiltonian operators of degree two.

Example 3.1.3.

$$\begin{aligned} P \;=\; & \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & -a & -b-(t-1)u^1 \\ 0 & a & 0 & c-u^2 \\ -a & b+(t-1)u^1 & -c+u^2 & 0 \end{pmatrix} \left(\frac{d}{dx}\right)^2 \\ & +2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u^1_x \\ 0 & 0 & -u^1_x & 0 \\ 0 & \tau u^1_x & u^2_x & u^3_x \end{pmatrix} \left(\frac{d}{dx}\right) \\ & + \left(\frac{1}{a}\right) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (u^1_x)^2 \\ 0 & 0 & -(u^1_x)^2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u^1_x \\ 0 & 0 & -u^1_{xx} & 0 \\ 0 & \tau u^1_{xx} & u^2_{xx} & u^3_{xx} \end{pmatrix} \end{aligned}$$

is Hamiltonian for all values of the constants a, b, c and τ with $a \neq 0$. This is the most general Hamiltonian operator associated in the manner discussed above to the algebra designated $(44)_\tau$ in [4].

Returning to the general Hamiltonian operator (3.1), it can be seen from conditions (B) and (E) in Theorem 3.1.1 that the coefficients b^{ij}_k and c^{ij}_{kl} in (3.1) are completely determined by a^{ij} and c^{ij}_k. Thus the Hamiltonian operator on $L(M)$ is represented uniquely on M by only these latter two objects.

Theorem 3.1.4. *There is a one-to-one correspondence between Hamiltonian operators of the form (3.1) on $L(M)$ and pairs (a, ∇) on M consisting of a non-degenerate bivector a^{ij} and a torsion-free connection ∇ satisfying two conditions: firstly, that the curvature of ∇ vanishes, and secondly,*

$$a^{ir}\nabla_r a^{jk} = a^{jr}\nabla_r a^{ki} \,. \tag{3.6}$$

The Christoffel symbols, Γ^k_{ij}, of ∇ are related to c^{ij}_k by $c^{ij}_k = -a^{ir}\Gamma^j_{rk}$. We then have

$$b^{ij}_k = \nabla_k a^{ij} + 2c^{ij}_k \,, \tag{3.7}$$

$$c^{ij}_{kl} = c^{ij}_{k,l} - a_{pr}c^{ri}_{(k}c^{pj}_{l)} \,. \tag{3.8}$$

With this, we may verify the following facts:

Corollary 3.1.5. *[21, 59, 64] For P in (3.1) a Hamiltonian operator we have*

1. *Γ is the symmetric part of $\bar{\Gamma}$,*

2. *Let $\bar{T}^k_{ij} = \bar{\Gamma}^k_{ij} - \bar{\Gamma}^k_{ji}$ be the torsion of $\bar{\nabla}$. Then $\bar{T}_{ijk} = a_{ir}\bar{T}^r_{jk}$ is skew symmetric and the forms $\bar{T} = \frac{1}{6}\bar{T}_{ijk}du^i \wedge du^j \wedge du^k$ and $a = \frac{1}{2}a_{ij}du^i \wedge du^j$ are related by $3\bar{T} = da$.*

3. *$\nabla_i\nabla_j a_{kl} = 0$, that is, a_{ij} is linear in the flat coordinates for ∇.*

Proof. We begin by noting that equation (3.6) is equivalent to the condition

$$\nabla_k a_{ij} = \nabla_i a_{jk} \tag{3.9}$$

on the two-form a_{ij}.

In terms of covariant Christoffel symbols, Theorem 3.1.4 gives

$$\bar{\Gamma}^k_{ij} = \frac{1}{2}a^{kr}\nabla_r a_{ij} + \Gamma^k_{ij} \,, \tag{3.10}$$

from which it is clear that $\bar{\Gamma}^k_{(ij)} = \Gamma^k_{ij}$.

We therefore also have

$$\frac{1}{2}\nabla_k a_{ij} = \bar{\Gamma}_{ijk} - \Gamma_{ijk} \,,$$

where $\bar{\Gamma}_{ijk} = a_{ir}\bar{\Gamma}^r_{jk}$ and $\Gamma_{ijk} = a_{ir}\Gamma^r_{jk}$. Because ∇ is torsion-free we have

$$\begin{aligned} \bar{T}_{ijk} &= \bar{\Gamma}_{ijk} - \bar{\Gamma}_{ikj}\,, \\ &= \bar{\Gamma}_{ijk} - \Gamma_{ijk} - \bar{\Gamma}_{ikj} + \Gamma_{ikj}\,, \\ &= \frac{1}{2}\nabla_k a_{ij} - \frac{1}{2}\nabla_j a_{ik}\,, \\ &= \nabla_k a_{ij}\,, \\ &= \nabla_{[k} a_{ij]}\,, \\ &= \frac{1}{3}(da)_{ijk}\,. \end{aligned}$$

Since ∇ is flat, $\nabla_i\nabla_j a_{kl} = \nabla_j\nabla_i a_{kl}$. By (3.9),

$$\begin{aligned} \nabla_i\nabla_j a_{kl} &= \nabla_i\nabla_k a_{lj}\,, \\ &= \nabla_k\nabla_i a_{lj} \\ &= \nabla_k\nabla_l a_{ij}\,. \end{aligned}$$

So $\nabla_i\nabla_j a_{kl}$ is both symmetric and anti-symmetric in i and j, and hence is zero. □

Lemma 3.1.6. *For a Hamiltonian operator of the form (3.1), the following three statements, presented in both covariant and contravariant forms, are equivalent:*

1. *The 2-form a is closed (and so symplectic), or equivalently a^{ij} satisfies equation (1.22) (and so defines a Poisson bracket on M by equation (1.21));*
2. $\nabla_k a^{ij} = 0$, *i.e.* $\nabla_k a_{ij} = 0$;
3. $b^{ij}_k = 2c^{ij}_k$, *i.e.* $\Gamma^k_{ij} = \bar{\Gamma}^k_{ij}$.

Proof. We see, from the characterisation of Hamiltonian operators given in Theorem 3.1.4,

$$\begin{aligned} a^{ij} \text{ is Poisson} &\iff a^{ir}a^{jk}_{,r} + a^{jr}a^{ki}_{,r} + a^{kr}a^{ij}_{,r} = 0 \\ &\iff a^{ir}\nabla_r a^{jk} + a^{jr}\nabla_r a^{ki} + a^{kr}\nabla_r a^{ij} = 0 \\ &\iff 3a^{kr}\nabla_r a^{ij} = 0 \\ &\iff \nabla_k a^{ij} = 0\,, \\ &\iff b^{ij}_k = 2c^{ij}_k\,. \end{aligned}$$

□

Lemma 3.1.6 therefore tells us that in the special case where the leading coefficient in P is the inverse of a symplectic form, the pair (a, ∇) defining P can be thought of as containing the symplectic form a_{ij}, and a torsionless connection compatible with it (in the sense that $\nabla a = 0$); that is, a symplectic connection. More precisely (see e.g. [9]):

Definition 3.1.7. *A symplectic connection on a symplectic manifold (M, ω) is a smooth connection ∇ which is torsion-free and compatible with the symplectic form ω, i.e.*

$$\nabla_X Y - \nabla_Y X - [X, Y] = 0$$

and

$$(\nabla\omega)(X, Y, Z) = X(\omega(Y, Z)) - \omega(\nabla_X Y, Z) - \omega(Y, \nabla_X Z) = 0\,,$$

where X, Y and Z are vector fields on M.

In local coordinates $\{x^i\}$, introducing Christoffel symbols Γ^k_{ij} for ∇ and writing $\omega = \frac{1}{2}\omega_{ij} dx^i \wedge dx^j$, the conditions for ∇ to be a symplectic connection read $\Gamma^k_{ij} = \Gamma^k_{ji}$, as usual, and

$$\nabla_k \omega_{ij} = \frac{\partial \omega_{ij}}{\partial x^r} - \Gamma^r_{ki}\omega_{rj} - \Gamma^r_{kj}\omega_{ir} = 0\,. \tag{3.11}$$

This definition is analogous to that of the Levi-Civita connection of a pseudo-Riemannian metric. However there is an important difference in that the Levi-Civita connection is uniquely specified by its metric. From the compatibility condition (3.11) it can be seen that if Γ^k_{ij} are the Christoffel symbols of a symplectic connection for ω, then the connection with Christoffel symbols $\tilde{\Gamma}^k_{ij} = \Gamma^k_{ij} + \omega^{kr} S_{rij}$ is a symplectic connection if and only if the tensor S_{ijk} is completely symmetric. In [43] a symplectic manifold with a specified symplectic connection is called, in light of [36], a Fedosov manifold. Here we call the pair (ω, ∇) of a symplectic form and a symplectic connection a Fedosov structure on M, and call the structure flat if ∇ is flat.

In the discussion of Hamiltonian operators it is convenient to work with contravariant quantities. We call

$$\Gamma^{ij}_k = -\omega^{ir}\Gamma^j_{rk}$$

the contravariant Christoffel symbols of the symplectic connection.

Result 3.1.8. *The compatibility of ∇ and ω is equivalent to*

$$\frac{\partial \omega^{ij}}{\partial x^k} = \Gamma^{ij}_k - \Gamma^{ji}_k\,.$$

Result 3.1.9. ∇ *being torsion-free is equivalent to* $\omega^{ir}\Gamma^{jk}_r = \omega^{jr}\Gamma^{ik}_r$.

The curvature of ∇,

$$R^k_{slt} = \partial_s\Gamma^k_{lt} - \partial_l\Gamma^k_{st} + \Gamma^k_{sr}\Gamma^r_{lt} - \Gamma^k_{lr}\Gamma^r_{st} \,,$$

can be expressed in terms of contravariant quantities by raising indices as

$$R^{ijk}_l = \omega^{is}\omega^{jt}R^k_{slt} \,.$$

This gives

Result 3.1.10.

$$R^{ijk}_l = \omega^{ir}\left(\partial_l\Gamma^{jk}_r - \partial_r\Gamma^{jk}_l\right) + \Gamma^{ij}_r\Gamma^{rk}_l + \Gamma^{ik}_r\Gamma^{rj}_l \,.$$

Having introduced symplectic connections, we are now in a position to interpret the following Darboux theorem [64] for Hamiltonian operators of degree 2:

Theorem 3.1.11. *Given a Hamiltonian operator*

$$P^{ij} = a^{ij}\left(\frac{d}{dx}\right)^2 + b^{ij}_k u^k_x \frac{d}{dx} + c^{ij}_{kl}u^k_x u^l_x + c^{ij}_k u^k_{xx}$$

where a^{ij} *is non-degenerate, then* P *can be put in the constant form* $P^{ij} = \omega^{ij}\left(\frac{d}{dx}\right)^2$ *(where* ω *is a constant matrix) by a change of target space coordinates* $\{u^i\}$ *if and only if* a_{ij} *is closed. The coordinates in which this happens are flat coordinates for the connection* $\Gamma^k_{ij} = -g_{ir}c^{rk}_j$ *which can be chosen, using a linear substitution, to be canonical coordinates for the symplectic form* $a_{ij} = \omega_{ij}$.

In arbitrary coordinates operators satisfying the conditions of Theorem 3.1.11 have the form

$$P^{ij} = \omega^{ij}\left(\frac{d}{dx}\right)^2 + 2\Gamma^{ij}_k u^k_x \frac{d}{dx} + c^{ij}_{kl}u^k_x u^l_x + \Gamma^{ij}_k u^k_{xx} \qquad (3.12)$$

where ω^{ij} is the inverse of a symplectic form, $c^{ij}_{kl} = \Gamma^{ij}_{(k,l)} - \omega_{pr}\Gamma^{ri}_{(k}\Gamma^{pj}_{l)}$, and Γ^{ij}_k are the contravariant Christoffel symbols of a flat symplectic connection compatible with ω. This class of operators on $L(M)$ is therefore in one-to-one correspondence with flat Fedosov structures on M.

3.2 Flat Pencils of Fedosov Structures

In this section we consider pairs of Hamiltonian operators of the form (3.12):

$$\begin{aligned} P_1^{ij} &= \omega_1^{ij}\left(\frac{d}{dx}\right)^2 + 2\Gamma_1{}_k^{ij}u_x^k\frac{d}{dx} + c_1{}_{kl}^{ij}u_x^k u_x^l + \Gamma_1{}_k^{ij}u_{xx}^k\,, \\ P_2^{ij} &= \omega_2^{ij}\left(\frac{d}{dx}\right)^2 + 2\Gamma_2{}_k^{ij}u_x^k\frac{d}{dx} + c_2{}_{kl}^{ij}u_x^k u_x^l + \Gamma_2{}_k^{ij}u_{xx}^k\,. \end{aligned}$$

The first fact to establish is that if P_1 and P_2 are compatible then all elements of the pencil, $P_\lambda = P_1 + \lambda P_2$, remain in the class (3.12).

Theorem 3.2.1. *If P_1 and P_2 are compatible then ω_1^{ij} and ω_2^{ij} form a finite-dimensional bi-Hamiltonian structure on the target space.*

Proof. P_λ could have the general form

$$P_\lambda^{ij} = a_\lambda^{ij}\left(\frac{d}{dx}\right)^2 + b_\lambda{}_k^{ij}u_x^k\frac{d}{dx} + c_\lambda{}_{kl}^{ij}u_x^k u_x^l + c_\lambda{}_k^{ij}u_{xx}^k\,,$$

but clearly $b_\lambda{}_k^{ij} = 2\Gamma_1{}_k^{ij} + 2\lambda\Gamma_2{}_k^{ij}$ and $c_\lambda{}_k^{ij} = \Gamma_1{}_k^{ij} + \lambda\Gamma_2{}_k^{ij}$, so $b_\lambda{}_k^{ij} = 2c_\lambda{}_k^{ij}$, and hence, by Lemma 3.1.6, a_λ^{ij} satisfies the Jacobi identity (1.22) for all λ. □

So we write

$$P_\lambda^{ij} = \omega_\lambda^{ij}\left(\frac{d}{dx}\right)^2 + 2\Gamma_\lambda{}_k^{ij}u_x^k\frac{d}{dx} + c_\lambda{}_{kl}^{ij}u_x^k u_x^l + \Gamma_\lambda{}_k^{ij}u_{xx}^k\,.$$

Given a $\left[{1 \atop 1}\right]$-tensor, L, its Nijenhuis torsion is (see, e.g., [20]) the $\left[{1 \atop 2}\right]$-tensor, N, specified by

$$N(X,Y) = L^2\,[X,Y] + [LX, LY] - L\,[LX,Y] - L\,[X,LY]$$

for any two vector fields X and Y. Its components are therefore

$$N_{jk}^i = L_j^s L_{k,s}^i - L_k^s L_{j,s}^i + L_s^i L_{j,k}^s - L_r^i L_{k,j}^r\,. \tag{3.13}$$

As is demonstrated in [38], a necessary condition for the compatibility of the hydrodynamic type Poisson brackets associated with two flat metrics η and g is the vanishing of the Nijenhuis torsion of $L_j^i = g^{ir}\eta_{rj}$. An immediate corollary [35] of Theorem 3.2.1 is that the vanishing of the Nijenhuis torsion of $L_j^i = \omega_1^{ir}\omega_{2rj}$ is a necessary condition for the compatibility of P_1 and P_2.

3.2.1 Multiplication of Covectors

As in [26], we proceed to understand the compatibility conditions on P_1 and P_2 in terms of the algebraic properties of a tensorial multiplication of covectors on M.

Definition 3.2.2. *Using the tensors*

$$\begin{aligned} \Delta^{sjk} &= \omega_2^{jr}{\Gamma_1}_r^{sk} - \omega_1^{sr}{\Gamma_2}_r^{jk}\,, \\ \Delta_i^{jk} &= \omega_{2is}\Delta^{sjk}\,, \end{aligned}$$

we define a multiplication $\diamond$ of covectors on M by

$$(\alpha \diamond \beta)_i = \alpha_j \beta_k \Delta_i^{jk}\,.$$

Theorem 3.2.3. *The compatibility of P_1 and P_2 is equivalent to*

$$(I, J \diamond K)_2 = (J, I \diamond K)_2\,, \tag{3.14}$$

$$\text{and} \quad (I \diamond J) \diamond K = 0\,, \tag{3.15}$$

for all covectors I, J, K on M. Here $(\cdot,\cdot)_2$ is the skew-symmetric bilinear form on T^*M induced by ω_2^{ij}, i.e. $(I, J)_2 = I_r J_s \omega_2^{rs}$. The compatibility also implies

$$\nabla_l^2 \Delta_k^{ij} = \nabla_k^2 \Delta_l^{ij}\,. \tag{3.16}$$

Because of Theorem 3.2.1, we phrase the compatibility of P_1 and P_2 in terms of Fedosov structures on M, and break the above theorem into stages:

Definition 3.2.4. *Two flat Fedosov structures (ω_1, ∇^1) and (ω_2, ∇^2), where ∇^1 and ∇^2 have contravariant Christoffel symbols ${\Gamma_1}_k^{ij}$ and ${\Gamma_2}_k^{ij}$ respectively, are said to be*

(i) almost compatible *if and only if $(\omega_\lambda, \nabla^\lambda)$ is a Fedosov structure for all λ, where the connection ∇^λ is given by ${\Gamma_\lambda}_k^{ij} = {\Gamma_1}_k^{ij} + \lambda {\Gamma_2}_k^{ij}$.*

(ii) almost compatible and flat *if and only if they are almost compatible, and in addition the curvature of ∇^λ vanishes for all λ.*

(iii) compatible *if and only if they are almost compatible and flat, and ${c_\lambda}_{kl}^{ij} = {\Gamma_\lambda}_{(k,l)}^{ij} - \omega_{\lambda pr}{\Gamma_\lambda}_{(k}^{ri}{\Gamma_\lambda}_{l)}^{pj}$ satisfies ${c_\lambda}_{kl}^{ij} = {c_1}_{kl}^{ij} + \lambda {c_2}_{kl}^{ij}$ for all λ.*

The compatibility of two flat Fedosov structures on M is equivalent to the compatibility of the associated Poisson brackets on $L(M)$.

We now turn to the two Fedosov structures defined by P_1 and P_2, and to the pair $(\omega_\lambda, \nabla^\lambda)$ defined by P_λ. From the linearity of Result 3.1.8 in the contravariant symbols it can be seen that ω_λ is automatically ∇^λ-constant, so the almost compatibility of (ω_1, ∇^1) and (ω_2, ∇^2) is equivalent to ∇^λ being torsion free, i.e. to

$$\omega_\lambda^{ir} {\Gamma_\lambda}_l^{jk} = \omega_\lambda^{jr} {\Gamma_\lambda}_l^{ik} .$$

In flat coordinates for ∇^2, this condition reduces to

$$\omega_2^{ir} {\Gamma_1}_r^{jk} = \omega_2^{jr} {\Gamma_1}_r^{ik} . \tag{3.17}$$

Note that we already have

$$\omega_1^{ir} {\Gamma_1}_r^{jk} = \omega_1^{jr} {\Gamma_1}_r^{ik} . \tag{3.18}$$

Lemma 3.2.5. *If (ω_1, ∇^1) and (ω_2, ∇^2) are almost compatible, then the flatness of ∇^λ is equivalent to either, and hence both, of*

$$\begin{aligned} \partial_l {\Gamma_1}_s^{jk} - \partial_s {\Gamma_1}_l^{jk} &= 0 & (3.19)\\ \text{and} \quad {\Gamma_1}_r^{ij} {\Gamma_1}_l^{rk} + {\Gamma_1}_r^{ik} {\Gamma_1}_l^{rj} &= 0 & (3.20) \end{aligned}$$

in the flat coordinates for ∇^2.

Proof. The contravariant curvature of Γ_λ is

$$\begin{aligned} {R_\lambda}_l^{ijk} &= \omega_\lambda^{ir} \left(\partial_l {\Gamma_\lambda}_r^{jk} - \partial_s {\Gamma_\lambda}_l^{jk} \right) + {\Gamma_\lambda}_r^{ij} {\Gamma_\lambda}_l^{rk} + {\Gamma_\lambda}_r^{ik} {\Gamma_\lambda}_l^{rj} \\ &= {R_1}_l^{ijk} \\ &\quad + \lambda \Big\{ \omega_2^{is} \left(\partial_l {\Gamma_1}_s^{jk} - \partial_s {\Gamma_1}_l^{jk} \right) + \omega_1^{is} \left(\partial_l {\Gamma_2}_s^{jk} - \partial_s {\Gamma_2}_l^{jk} \right) \\ &\qquad + {\Gamma_2}_r^{ij} {\Gamma_1}_l^{rk} + {\Gamma_1}_r^{ij} {\Gamma_2}_l^{rk} + {\Gamma_1}_r^{ik} {\Gamma_2}_l^{rj} + {\Gamma_2}_r^{ik} {\Gamma_1}_l^{rj} \Big\} \\ &\quad + \lambda^2 {R_2}_l^{ijk} , \end{aligned}$$

which in flat coordinates for ${\Gamma_2}_k^{ij}$ reads

$$\begin{aligned} {R_\lambda}_l^{ijk} &= \omega_1^{ir} \left(\partial_l {\Gamma_1}_r^{jk} - \partial_r {\Gamma_1}_l^{jk} \right) + {\Gamma_1}_r^{ij} {\Gamma_1}_l^{rk} + {\Gamma_1}_r^{ik} {\Gamma_1}_l^{rj} \\ &\quad + \lambda \omega_2^{is} \left(\partial_l {\Gamma_1}_s^{jk} - \partial_s {\Gamma_1}_l^{jk} \right) . \end{aligned}$$

The vanishing of the order λ term is equivalent to equation (3.19), and with this the vanishing of the λ-independent term is equivalent to (3.20). □

Lemma 3.2.6. *If* (ω_1, ∇^1) *and* (ω_2, ∇^2) *are almost compatible then the condition* $c_{\lambda}{}^{ij}_{kl} = \Gamma_{\lambda}{}^{ij}_{(k,l)} - \omega_{\lambda pr}\Gamma_{\lambda}{}^{ri}_{(k}\Gamma_{\lambda}{}^{pj}_{l)}$ *reads, in the flat coordinates for* ∇^2,

$$\Gamma_1{}^{ij}_{r}\Gamma_1{}^{rk}_{l} - \Gamma_1{}^{ik}_{r}\Gamma_1{}^{rj}_{l} = 0\,. \tag{3.21}$$

Proof. For an arbitrary Fedosov structure (ω, ∇) the object $c^{ij}_{kl} = \Gamma^{ij}_{(k,l)} - \omega_{pr}\Gamma^{ri}_{(k}\Gamma^{pj}_{l)}$ can be converted into a quadratic expression in contravariant quantities as

$$\omega^{sk}c^{ij}_{kl} = \omega^{sk}\Gamma^{ij}_{(k,l)} - \frac{1}{2}\Gamma^{si}_{p}\Gamma^{pj}_{l} + \frac{1}{2}\Gamma^{pi}_{l}\Gamma^{sj}_{p}\,. \tag{3.22}$$

This has similarities to the formula for covariant curvature obtained in Result 3.1.10; only certain signs have changed. Indeed, if we define a quantity c^{j}_{rkl} by

$$c^{j}_{rkl}dx^r = \frac{1}{2}\left(\nabla_{\partial_k}\nabla_{\partial_l} + \nabla_{\partial_l}\nabla_{\partial_k}\right)dx^j\,, \tag{3.23}$$

then $c^{ij}_{kl} = \omega^{ir}c^{j}_{rkl}$.

We have two ways of expanding $\omega_{\lambda}^{sk}c_{\lambda}{}^{ij}_{kl}$, corresponding to whether we choose first to substitute it into equation (3.22), or to expand the pencil quantities. We work in flat coordinates for ∇^2; in these, $c_2{}^{ij}_{kl}$ also vanishes. First expanding the pencil we have

$$\begin{aligned} \omega_{\lambda}^{sk}c_{\lambda}{}^{ij}_{kl} &= \left(\omega_1^{sk} + \lambda\omega_2^{sk}\right)c_1{}^{ij}_{kl}\,, \\ &= \omega_1^{sk}c_1{}^{ij}_{kl} + \lambda\omega_2^{sk}c_1{}^{ij}_{kl}\,, \end{aligned}$$

whilst (3.22) gives

$$\begin{aligned} \omega_{\lambda}^{sk}c_{\lambda}{}^{ij}_{kl} &= \omega_{\lambda}^{sk}\Gamma_{\lambda}{}^{ij}_{(k,l)} - \frac{1}{2}\Gamma_{\lambda}{}^{si}_{p}\Gamma_{\lambda}{}^{pj}_{l} + \frac{1}{2}\Gamma_{\lambda}{}^{pi}_{l}\Gamma_{\lambda}{}^{sj}_{p}\,, \\ &= \left(\omega_1^{sk} + \lambda\omega_2^{sk}\right)\Gamma_1{}^{ij}_{(k,l)} - \frac{1}{2}\Gamma_1{}^{si}_{p}\Gamma_1{}^{pj}_{l} + \frac{1}{2}\Gamma_1{}^{pi}_{l}\Gamma_1{}^{sj}_{p}\,. \end{aligned}$$

The order 1 terms merely express equation (3.22) for P_1. Equality of the order λ terms is equivalent to $\Gamma_1{}^{ij}_{(k,l)} = c_1{}^{ij}_{kl}$ and so to

$$\begin{aligned} \omega_1^{sk}\Gamma_1{}^{ij}_{(k,l)} &= \omega_1^{sk}c_1{}^{ij}_{kl}\,, \\ &= \omega_1^{sk}\Gamma_1{}^{ij}_{(k,l)} - \frac{1}{2}\Gamma_1{}^{si}_{p}\Gamma_1{}^{pj}_{l} + \frac{1}{2}\Gamma_1{}^{pi}_{l}\Gamma_1{}^{sj}_{p}\,. \end{aligned}$$

□

Proof of Theorem 3.2.3. Using equation (3.17) in Definition 3.2.2 it can be seen that in the flat coordinates for ∇^2 we have $\Delta^{ij}_{k} = \Gamma_1{}^{ij}_{k}$. Thus we may regard equations (3.17),(3.19),(3.20) and (3.21) as identities on Δ^{ij}_{k}; the result is Theorem 3.2.3. □

The condition imposed by equation (3.20) for an almost compatible and flat pair of Fedosov structures on the multiplication $\diamond$ is $(I \diamond J) \diamond K = -(I \diamond K) \diamond J$, i.e. the first condition (3.4) satisfied by the multiplication of a Fermionic Novikov algebra. In general (3.5) is not satisfied even for compatible Fedosov structures, however we do have, for two flat Fedosov structures, (ω_1, ∇^1), (ω_2, ∇^2), which are almost compatible,

$$\begin{aligned} &\omega_1^{ir}\nabla^2_r\Delta^{jk}_l - \omega_1^{jr}\nabla^2_r\Delta^{ik}_l \\ &\quad = \Delta^{ij}_r\Delta^{rk}_l - \Delta^{ir}_l\Delta^{jk}_r - \Delta^{ji}_r\Delta^{rk}_l + \Delta^{jr}_k\Delta^{ik}_r \,. \end{aligned}$$

So, in particular, if Δ^{ij}_k is constant in the flat coordinates for ∇^2, almost compatible and flat Fedosov structures will define a Fermionic Novikov algebra structure on the covectors of M.

In [4] it emerged that examples of such algebras which do not also satisfy the 'Bosonic' relation $(I \diamond J) \diamond K = (I \diamond K) \diamond J$, and hence $(I \diamond J) \diamond K = 0$, are relatively rare. ∇^2-constant multiplications arising from pairs of Fedosov structures which are almost compatible and flat, but not compatible, such as that given in Example 3.2.10 below, are in this class.

3.2.2 The Pencil in Flat Coordinates

We now turn our consideration to the form the pencil takes in the flat coordinates for ∇^2. From the elements of the proof of Theorem 3.2.3 we have

$$P^{ij}_\lambda = \left(\omega_1^{ij} + \lambda\omega_2^{ij}\right)\left(\frac{d}{dx}\right)^2 + 2{\Gamma_1}^{ij}_k u^k_x \frac{d}{dx} + {\Gamma_1}^{ij}_{k,l} u^k_x u^l_x + {\Gamma_1}^{ij}_k u^k_{xx} \,. \tag{3.24}$$

The Jacobi identity for P_λ (without assuming P_1 and P_2 are Hamiltonian themselves) is equivalent to the constraints

(i) ω_2^{ij} is constant and antisymmetric,

(ii) ω_1^{ij} is antisymmetric,

(iii) $\omega_1^{ir}{\Gamma_1}^{jk}_r = \omega_1^{jr}{\Gamma_1}^{ik}_r$,

(iv) $\omega_1^{ij}{}_{,k} = {\Gamma_1}^{ij}_k - {\Gamma_1}^{ji}_k$,

(v) $\omega_2^{ir}{\Gamma_1}^{jk}_r = \omega_2^{jr}{\Gamma_1}^{ik}_r$,

(vi) ${\Gamma_1}^{ij}_{k,l} = {\Gamma_1}^{ij}_{l,k}$

(vii) ${\Gamma_1}^{ij}_r{\Gamma_1}^{rk}_l = 0$.

Proposition 3.2.7. *In a fixed coordinate system* $\{u^i\}$ *(the flat coordinates for* Γ_2*), given a constant non-degenerate 2-form* ω_2^{ij} *and a vector field* $B = B^r\partial_r$ *satisfying*

$$\left(\omega_2^{is}B^r_{,s} - \omega_2^{rs}B^i_{,s}\right)\omega_2^{jp}B^k_{,pr} = \left(\omega_2^{js}B^r_{,s} - \omega_2^{rs}B^j_{,s}\right)\omega_2^{ip}B^k_{,pr} \tag{3.25}$$

and

$$B^j_{,ir}\omega_2^{rs}B^k_{,sl} = 0 \tag{3.26}$$

then the prescription

$$\begin{aligned} \omega_1^{ij} &= -(\mathcal{L}_B\omega_2)^{ij} = \omega_2^{ir}B^j_{,r} - \omega_2^{jr}B^i_{,r}\,, \\ \Gamma_1{}^{ij}_k &= \omega_2^{ir}B^j_{,rk} \end{aligned}$$

satisfies the constraints (i)-(vii)*. Further, all solutions of* (i)-(vii) *have this form.*

Proof. Equations (3.25) and (3.26) are the quadratic constraints, $\omega_1^{ir}\Gamma_1{}^{jk}_r = \omega_1^{jr}\Gamma_1{}^{ik}_r$ and $\Gamma_1{}^{ij}_r\Gamma_1{}^{rk}_l = 0$ respectively. That ω_1 and Γ_1 satisfy the (linear) constraints (iv), (v) and (vi) is an immediate consequence of their definition.

Using the Poincare lemma together with the symmetries expressed in conditions (vi) and (v), we have the existence of a vector field satisfying $\Gamma_1{}^{ij}_k = \omega_2^{ir}A^j{}_{,rk}$. With this condition (iv) gives $\omega_1^{ij} = -(\mathcal{L}_A\omega_2)^{ij} + c^{ij}$, where c^{ij} is a constant antisymmetric matrix. We may now introduce a vector field B with $B^i = A^i + \frac{1}{2}x^s w_{2sr}c^{ri}$ which satisfies $\omega_1^{ij} = -\mathcal{L}_B\omega_2^{ij}$ and $\Gamma_1{}^{ij}_k = \omega_2^{ir}B^j_{,rk}$. □

Since ω_2 is a symplectic form, its symmetries are precisely (locally) Hamiltonian vector fields. Therefore, if ω_2 and ω_1 are given, the requirement that $\omega_1^{ij} = -\mathcal{L}_B\omega_2^{ij}$ fixes the non-Hamiltonian part of B. Then the condition $\Gamma_1{}^{ij}_k = \omega_2^{ir}B^j_{,rk}$ fixes the Hamiltonian to within a quadratic function. From the point of view of the multiplication of covectors from Section 3.2.1, the Hamiltonian affects only the commutative part of $\diamond$, thus the anti-commutative part is fixed by ω_1^{ij} and ω_2^{ij}.

With consideration of the transformation rules (3.2), one can phrase Proposition 3.2.7 as the existence of a vector field B such that

$$\begin{aligned} \omega_1^{ij} &= -\mathcal{L}_B\omega_2^{ij}\,, \\ \Gamma_1{}^{ij}_k &= -\mathcal{L}_B\Gamma_2{}^{ij}_k\,. \end{aligned} \tag{3.27}$$

We can also calculate from (3.2) the correct interpretation of the Lie derivative for an object of type c^{ij}_{kl}, namely:

$$\begin{aligned}\mathcal{L}_X c^{ij}_{kl} &= X^r c^{ij}_{kl,r} - X^i_{,r} c^{rj}_{kl} - X^j_{,r} c^{ir}_{kl} + X^r_{,k} c^{ij}_{rl} + X^r_{,l} c^{ij}_{kr} \\ &\quad + X^r_{,kl} c^{ij}_r - \frac{1}{2} X^j_{rl} b^{ir}_k - \frac{1}{2} X^j_{,rk} b^{ir}_l - X^j_{,rkl} a^{ir} .\end{aligned}$$

If we work in the flat coordinates for Γ_2, so that the components ${c_2}^{ij}_{kl} = 0$, we have for our pencil

$$\begin{aligned}-\mathcal{L}_B {c_2}^{ij}_{kl} &= +\omega_2^{ir} B^j_{,rkl} , \\ &= (\omega_2^{ir} B^j_{,rk})_{,l} , \\ &= {\Gamma_1}^{ij}_{k,l} .\end{aligned}$$

Now, in the flat coordinates for ∇^2 we have the relation ${c_1}^{ij}_{kl} = {\Gamma_1}^{ij}_{k,l}$. The linearity of the transformation rules shows that the Lie derivative of ${c_2}^{ij}_{kl}$ should be an object of the same type as ${c_1}^{ij}_{kl}$. Thus we have, in addition to (3.27),

$${c_1}^{ij}_{kl} = -\mathcal{L}_B {c_2}^{ij}_{kl} .$$

One may understand these three infinitesimal relations between the coefficients of P_1 and P_2 as averring the existence on $L(M)$ of an evolutionary vector field

$$\hat{B} = B^i(u(x)) \frac{\partial}{\partial u^i(x)} + \dots$$

such that

$$P_1^{ij} = -\mathcal{L}_{\hat{B}} P_2^{ij} .$$

The approach to finding pairs of compatible Poisson brackets by expressing the second bracket as a Lie derivative of the first along some vector field and solving constraints on this vector field was taken in [66].

We now turn our attention to some examples of pairs of Fedosov structures, using the framework of Proposition 3.2.7.

Example 3.2.8. Two-dimensional pencils. *Without loss of generality we take*

$$\omega_2 = \frac{\partial}{\partial u^1} \wedge \frac{\partial}{\partial u^2} ,$$

where u^1 and u^2 are a flat coordinate system for ∇^2.

We take

$$B = f(u^1, u^2) \frac{\partial}{\partial u^1} + g(u^1, u^2) \frac{\partial}{\partial u^2}$$

and from it calculate ω_1 and Γ_1 according to (3.27). In particular

$$\omega_1 = (f_{,1} + g_{,2})\omega_2 \, ,$$

from which it follows immediately that (ω_1, ∇^1) and (ω_2, ∇^2) are almost compatible.

They are almost compatible and flat if and only if $h = f + \lambda g$ satisfies the homogeneous Monge-Ampere Equation $h_{12}^2 - h_{11}h_{22} = 0$ for all λ.

They are compatible if and only if $a = f + \lambda g$ and $b = f + \mu g$ satisfy

$$a_{12}b_{12} - a_{11}b_{22} = 0$$

for all λ, μ.

For instance, one may recover the three two-dimensional Fermionic Novikov algebras of [4] as constant multiplications via

(T1) $f = u^1$, $g = 0$,

(T2) $f = u^1$, $g = (u^1)^2$,

(T3) $f = (u^1)^2$, $g = 0$.

Example 3.2.9. Commutative algebras. *In the case in which ω_1 is constant in the flat coordinates for ∇^2, we have, by condition (iv),*

$$\Gamma_1{}^{ij}_k = \Gamma_1{}^{ji}_k \, ,$$

so that the multiplication $\diamond$ is commutative.

In particular if

$$\omega_1 = \omega_2 = \omega = \sum_{i=1}^n \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} \, ,$$

then the non-Hamiltonian part of B is

$$\sum_{i=1}^n q^i \frac{\partial}{\partial q^i} \, .$$

To this we may add a Hamiltonian vector field, giving

$$B = \sum_{i=1}^n \left(\left[q^i + \frac{\partial H}{\partial p_i} \right] \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i} \right) .$$

Since $\omega_1 = \omega_2$, equation (3.25) is immediate. Equation (3.26) becomes

$$H_{,ijr}\, \omega^{rs} H_{,skl} = 0 \, ,$$

where the indices i, j, k, l, r, s account for both q and p variables.

A solution to this is $H = f(x^1, x^2, \ldots, x^n)$, where each x^i is either p_i or q^i; only one from each pair of conjugate variables features in H.

It is not hard to see that Proposition 3.2.7 can be modified to describe almost compatible and flat pairs of Fedosov structures. Specifically, we replace equation (3.26) by the expression corresponding to ${\Gamma_1}^{ij}_r {\Gamma_1}^{rk}_l = {\Gamma_1}^{ik}_r {\Gamma_1}^{rj}_l$, namely:

$$B^j_{,ir}\omega_2^{rs}B^k_{,sl} = B^j_{,lr}\omega_2^{rs}B^k_{,si}\,. \tag{3.28}$$

Example 3.2.10. *The Fedosov structures specified by*

$$\begin{aligned} \omega_2 &= \frac{\partial}{\partial q_1}\wedge\frac{\partial}{\partial p_1}+\frac{\partial}{\partial q_2}\wedge\frac{\partial}{\partial p_2}\,, \\ {\Gamma_2}^{ij}_k &= 0\,, \\ B &= \frac{3}{2}q_1^2\frac{\partial}{\partial q_1}+2q_1q_2\frac{\partial}{\partial q_2}+q_1p_2\frac{\partial}{\partial p_2}\,, \end{aligned}$$

and $\omega_1^{ij} = -\mathcal{L}_B\omega_2^{ij}$ *and* ${\Gamma_1}^{ij}_k = -\mathcal{L}_B{\Gamma_2}^{ij}_k$ *are almost compatible and flat, but not compatible.*

The non-zero components of ω_1 *and* $\diamond$ *are*

$$\begin{aligned} \{q_1,p_1\}_1 = \{q_2,p_2\}_1 &= 3q_1\,, \\ \{q_2,p_1\}_1 &= 2q_2\,, \\ \{p_2,p_1\}_1 &= p_2\,, \end{aligned}$$

and

$$\begin{aligned} dq_2\diamond dp_2 &= dq_1\,, \\ dp_1\diamond dq_1 &= -3dq_1\,, \\ dp_1\diamond dq_2 &= -2dq_2\,, \\ dp_1\diamond dp_2 &= -dp_2\,, \\ dp_2\diamond dq_2 &= -2dq_1\,. \end{aligned}$$

Thus, the products

$$\begin{aligned} (dp_1\diamond dq_2)\diamond dp_2 &= -2dq_1 \\ \text{and}\qquad (dp_1\diamond dp_2)\diamond dq_2 &= 2dq_1 \end{aligned}$$

violate equation (3.15) but not (3.4). Note that $\diamond$ *also satisfies (3.5) and thus defines a Fermionic Novikov algebra which is not 'Bosonic'.*

3.2.3 ωN Manifold with Potential

The cotangent bundle T^*Q of a manifold Q is naturally equipped with a symplectic form, and thus cotangent bundles form the basic set of examples of symplectic manifolds. One may hope to find examples of finite-dimensional bi-Hamiltonian structures on cotangent bundles by exploiting the existence of additional structures on the underlying manifolds. The main object used to do this is a $\left[{1 \atop 1}\right]$-tensor L^i_j on Q whose Nijenhuis torsion is zero. Such an object was utilised by Benenti [6] to demonstrate the separability of the geodesic equations on a class of Riemannian manifolds. This result was later interpreted in [49] in terms of a bi-Hamiltonian structure on T^*Q which was extended to a degenerate Poisson pencil on $T^*Q \times \mathbb{R}$.

To obtain Fedosov structures we require more than just a tensor L^i_j on Q with vanishing Nijenhuis torsion; we also need a means of specifying the connections. If Q is equipped with a torsion-free connection $\widetilde{\nabla}$, then the Nijenhuis torsion of a $\left[{1 \atop 1}\right]$-tensor L^i_j can be written as

$$N^i_{jk} = L^s_j \widetilde{\nabla}_s L^i_k - L^s_k \widetilde{\nabla}_s L^i_j - L^i_s \widetilde{\nabla}_j L^s_k + L^i_s \widetilde{\nabla}_k L^s_j \,.$$

If there exists a vector field, A, on Q such that $L^i_j = \widetilde{\nabla}_j A^i$ then

$$N^i_{jk} = (\widetilde{\nabla}_j A^s)(\widetilde{\nabla}_s \widetilde{\nabla}_k A^i) - (\widetilde{\nabla}_k A^s)(\widetilde{\nabla}_s \widetilde{\nabla}_j A^i) - (\widetilde{\nabla}_s A^i)(R^s_{jkr} A^r) \,,$$

where R^i_{jkl} is the curvature tensor of $\widetilde{\nabla}$.

So, if $\widetilde{\nabla}$ is flat then the vanishing of the Nijenhuis tensor of $L = \widetilde{\nabla} A$ is equivalent to the identity

$$(\widetilde{\nabla}_j A^s)(\widetilde{\nabla}_s \widetilde{\nabla}_k A^i) = (\widetilde{\nabla}_k A^s)(\widetilde{\nabla}_s \widetilde{\nabla}_j A^i) \,. \tag{3.29}$$

Proposition 3.2.11. *Given a manifold Q endowed with a flat connection $\widetilde{\nabla}$ and a vector field A satisfying (3.29), the cotangent bundle T^*Q is endowed with a compatible pair of Fedosov structures, (ω_1, ∇^1) and (ω_2, ∇^2), as follows:*

ω_2 is the canonical Poisson bracket on T^*Q.

The connection ∇^2 on T^*Q is the horizontal lift [72] of the connection $\widetilde{\nabla}$ on Q; i.e. the Christoffel symbols ${\Gamma_2}^k_{ij}$ of ∇^2 are zero in the coordinates induced on T^*Q by the flat coordinates for $\widetilde{\nabla}$.

(ω_1, ∇^1) is calculated from (ω_2, ∇^2) according to the prescription of Proposition 3.2.7, where the vector field B is the horizontal lift of A to T^*Q.

Proof. Let $\{q^1, \ldots, q^n\}$ be flat coordinates for $\widetilde{\nabla}$ on Q, and $\mathcal{C} = \{q^1, \ldots, q^n, p_1, \ldots, p_n\}$ be the induced coordinates on T^*Q. Then

$$\omega_2 = \sum_{r=1}^{n} \frac{\partial}{\partial q^r} \wedge \frac{\partial}{\partial p_r}$$

and

$$B = \sum_{r=1}^{n} A^i \frac{\partial}{\partial q^i} \,.$$

The space of sections of the cotangent bundle of T^*Q, Ω, naturally splits into $\mathcal{P} = \text{span}\{dp_i\}$ and $\mathcal{Q} = \text{span}\{dq^i\}$. For ${\Gamma_1}^{ij}_{k} = \omega_2^{ir} B^j_{,rk}$ to be non-zero requires k to represent a variable q^k, and i to represent a p_i variable. Thus $\Omega \diamond \Omega \subseteq \mathcal{Q}$ and $\mathcal{Q} \diamond \Omega = \{0\}$, meaning that $(\Omega \diamond \Omega) \diamond \Omega = \{0\}$. So the relation (3.26), ${\Gamma_1}^{ij}_r {\Gamma_1}^{rk}_l = 0$, is satisfied.

ω_1^{ij} has only one kind of non-zero component, $\omega^{p_i q^j} = A^j_{,i}$, so the expression $\omega_1^{ir} {\Gamma_1}^{jk}_r$ has only one non-zero case:

$$\sum_{x^r \in \mathcal{C}} \omega_1^{p_i x^r} {\Gamma_1}^{p_j q^k}_{x^r} = \sum_{r=1}^{n} \omega_1^{p_i q^r} {\Gamma_1}^{p_j q^k}_{q^r} = A^r{}_{,i}\, A^k{}_{,rj} \,,$$

which is seen to be symmetric in i and j by condition (3.29), which in the flat coordinates q^i reads

$$A^s{}_{,j}\, A^i{}_{,sk} = A^s{}_{,k}\, A^i{}_{,sj} \,.$$

□

Example 3.2.12. *If the eigenvalues of $L : TQ \to TQ$ are functionally independent in some neighbourhood then they may be used as coordinates, and L takes the form*

$$L = \sum_{i=1}^{n} u^i \frac{\partial}{\partial u^i} \otimes du^i \,.$$

In this case we may set $A = \sum_{i=1}^{n} \frac{1}{2}(u^i)^2 \frac{\partial}{\partial u^i}$, and have $\widetilde{\nabla}$ defined by vanishing Christoffel symbols in these coordinates.

This gives, writing v_i as the conjugate coordinate to u^i on T^*Q,

$$\begin{aligned} \omega_2 &= \sum_{i=1}^{n} \frac{\partial}{\partial u^i} \wedge \frac{\partial}{\partial v_i} \,, \\ \omega_1 &= \sum_{i=1}^{n} u^i \frac{\partial}{\partial u^i} \wedge \frac{\partial}{\partial v_i} \,, \\ {\Gamma_2}^{ij}_{k} &= 0 \\ {\Gamma_1}^{v_i u^i}_{u^i} &= -1 \,, \end{aligned}$$

and all other Christoffel symbols zero.

3.3 Compatibility of Almost Symplectic Connections

In this section we consider the compatibility of two Hamiltonian operators of the form (3.1) without the assumption that the leading coefficient is the inverse of a symplectic form. Since this leading coefficient is still non-degenerate, we are in the context of almost symplectic geometry.

As is noted in [43], if a torsion-free connection ∇ is compatible with a two-form a in the sense that $\nabla a = 0$, then that two-form must be closed. So, in almost symplectic geometry, if one wishes to have a connection such that the covariant derivative of the two-form is zero, one must allow that connection to have torsion. [43] demonstrates that on an almost symplectic manifold there is a one-to-one correspondence between such connections $\bar{\nabla}$ and connections ∇ which are torsion-free, which in one direction is simply given by letting ∇ be the symmetric part of $\bar{\nabla}$. Equation (3.7) of Theorem 3.1.4, or equivalently equation (3.10) of Corollary 3.1.5, describes this correspondence in the special case that the torsion of $\bar{\nabla}$ is skew-symmetric. This condition of skew-torsion on a connection satisfying $\bar{\nabla} a = 0$ is equivalent to the equation (3.6) (i.e. to $\nabla_i a_{jk} = \nabla_k a_{ij}$) on its symmetric part.

Thus, the geometry of a Hamiltonian operator

$$P^{ij} = a^{ij}\left(\frac{d}{dx}\right)^2 + b^{ij}_k u^k_x \frac{d}{dx} + c^{ij}_k u^k_{xx} + c^{ij}_{kl} u^k_x u^l_x$$

can be said to be specified either by a pair (a, ∇) consisting of a bivector and a torsion-free connection satisfying the conditions specified in Theorem 3.1.4, or by a pair $(a, \bar{\nabla})$ consisting of an almost symplectic form a and a connection $\bar{\nabla}$ such that

$$\bar{\nabla}_k a_{ij} = a_{ij,k} - \bar{\Gamma}^r_{ik} a_{rj} - \bar{\Gamma}^r_{jk} a_{ir} = 0\,,$$

and that

$$\bar{T}_{ijk} = a_{kr}\left(\bar{\Gamma}^r_{ij} - \bar{\Gamma}^r_{ji}\right)$$

is a completely skew-symmetric tensor, and that the symmetric part of $\bar{\nabla}$ is flat.

We choose to work in terms of the torsion-free connection ∇ as in Theorem 3.1.4. We call a pair (a, ∇) consisting of an almost symplectic form a and a torsion-free connection ∇ satisfying

$$\nabla_i a_{jk} = \nabla_k a_{ij}$$

an almost Fedosov structure, and call the structure flat if ∇ is. We shall always associate to an almost Fedosov structure the inverse of a_{ij} denoted by a^{ij} satisfying $a^{ir} a_{rj} = \delta^i_j$, and the quantities b^{ij}_k, c^{ij}_k and c^{ij}_{kl} as defined in Theorem 3.1.4. Lemma 3.1.6 shows that almost Fedosov structures in which a is a symplectic form are automatically Fedosov structures.

We can thus define the following generalisation of Definition 3.2.4:

Definition 3.3.1. *Two flat almost Fedosov structures (a_1, ∇^1) and (a_2, ∇^2), where ∇^1 and ∇^2 have contravariant Christoffel symbols ${c_1}^{ij}_k$ and ${c_2}^{ij}_k$ respectively, are said to be*

(i) almost compatible *if and only if $(a_\lambda, \nabla^\lambda)$ is an almost Fedosov structure for all λ, where $a^{ij}_\lambda = a^{ij}_1 + \lambda a^{ij}_2$ and the connection ∇^λ is given by ${c_\lambda}^{ij}_k = {c_1}^{ij}_k + \lambda {c_2}^{ij}_k$.*

(ii) almost compatible and flat *if and only if they are almost compatible, and in addition the curvature of ∇^λ vanishes for all λ.*

(iii) compatible *if and only if they are almost compatible and flat, and ${c_\lambda}^{ij}_{kl} = {c_\lambda}^{ij}_{(k,l)} - a_{\lambda pr}{c_\lambda}^{ri}_{(k}{c_\lambda}^{pj}_{l)}$ satisfies ${c_\lambda}^{ij}_{kl} = {c_1}^{ij}_{kl} + \lambda {c_2}^{ij}_{kl}$ for all λ.*

As before, the compatibility of the two flat almost Fedosov structures is equivalent to the compatibility of their associated Poisson brackets. Note that if two Fedosov structures are almost compatible in the sense of Definition 3.3.1, that is as almost Fedosov structures, then they are almost compatible in the sense of Definition 3.2.4, that is as Fedosov structures.

In order to be able to use this definition, it is necessary to express everything in terms of contravariant quantites.

Results 3.3.2. *Let (a, ∇) be a pair consisting of a non-degenerate bivector a^{ij} and a connection ∇, and let b^{ij}_k, c^{ij}_k, c^{ij}_{kl} be defined from it as in Theorem 3.1.4. Then:*

∇ is torsion-free if and only if

$$a^{ir}c^{jk}_r = a^{jr}c^{ik}_r\,.$$

$a^{ir}\nabla_r a^{jk} = a^{jr}\nabla_r a^{ki}$ *if and only if*

$$a^{ir}\left(a^{jk}_{,r} - c^{jk}_r + c^{kj}_r\right) = a^{jr}\left(a^{ki}_{,r} - c^{ki}_r + c^{ik}_r\right)\,.$$

Now let (a, ∇) be an almost Fedosov structure. We define the contravariant curvature tensor as

$$R^{ijk}_l = a^{is}a^{jt}R^k_{slt}\,.$$

This gives

$$R^{ijk}_l \;=\; a^{ir}(c^{jk}_{r,l} - c^{jk}_{l,r}) + c^{ij}_r c^{rk}_l + c^{ik}_r c^{rj}_l - (b^{ij}_r - 2c^{ij}_r)c^{rk}_l + c^{ik}_r(b^{rj}_l - 2c^{rj}_l)\,,$$

or, eliminating b^{ij}_k,

$$
\begin{aligned}
R^{ijk}_l &= a^{ir}(c^{jk}_{r,l} - c^{jk}_{l,r}) + c^{ij}_r c^{rk}_l + c^{ik}_r c^{rj}_l - (\nabla_r a^{ij})c^{rk}_l + c^{ik}_r(\nabla_l a^{rj})\,,\\
&= a^{ir}(c^{jk}_{r,l} - c^{jk}_{l,r}) + c^{ij}_r c^{rk}_l + c^{ik}_r c^{rj}_l - (a^{ij}_{,r} - c^{ij}_r + c^{ji}_r)c^{rk}_l + c^{ik}_r(a^{rj}_{,l} - c^{rj}_l + c^{jr}_l)\,.
\end{aligned}
$$

Finally,

$$
a^{sk}c^{ij}_{kl} = a^{sk}c^{ij}_{(k,l)} - \frac{1}{2}c^{si}_p c^{pj}_l + \frac{1}{2}c^{pi}_l c^{sj}_p\,,
$$

which is the same expression as in (3.22).

Note: although the expression for c^{ij}_{kl} is the same as in the symplectic case, the interpretation of the quantities given in Equation (3.23) does not apply.

For the rest of this section we shall consider two Hamiltonian operators of degree two, P_1^{ij} and P_2^{ij} identified with flat almost Fedosov structures (a_1, ∇^1) and (a_2, ∇^2) respectively. We shall work in flat coordinates u^i for ∇^2, so that

$$
P_1^{ij} = a_1^{ij}\left(\frac{d}{dx}\right)^2 + {b_1}^{ij}_k u^k_x \frac{d}{dx} + {c_1}^{ij}_k u^k_{xx} + {c_1}^{ij}_{kl} u^k_x u^l_x \tag{3.30}
$$

and

$$
P_2^{ij} = a_2^{ij}\left(\frac{d}{dx}\right)^2 + {b_2}^{ij}_k u^k_x \frac{d}{dx}\,, \tag{3.31}
$$

where the coefficients ${b_1}^{ij}_k$, ${b_2}^{ij}_k$, and ${c_1}^{ij}_{kl}$ can be expressed in terms of the other coefficients via equations (3.7) and (3.8) of Theorem 3.1.4. In particular we have

$$
{b_2}^{ij}_k = {a_2}^{ij}_{,k}
$$

(Mokhov's condition (b)) in these coordinates.

Theorem 3.3.3. *The Hamiltonian operators (3.30) and (3.31) are compatible if and only if*

$$
\begin{aligned}
a_2^{ir}{c_1}^{jk}_r &= a_2^{jr}{c_1}^{ik}_r\,, && (3.32)\\
a_1^{ir}{a_2}^{jk}_{,r} + a_2^{ir}\left({a_1}^{jk}_{,r} - {c_1}^{jk}_r + {c_1}^{kj}_r\right) &= a_1^{jr}{a_2}^{ki}_{,r} + a_2^{jr}\left({a_1}^{ki}_{,r} - {c_1}^{ki}_r + {c_1}^{ik}_r\right)\,, && (3.33)\\
a_2^{ir}\left({c_1}^{jk}_{r,l} - {c_1}^{jk}_{l,r}\right) - {a_2}^{ij}_{,r}{c_1}^{rk}_l + {c_1}^{ik}_r{a_2}^{rj}_{,l} &= 0\,, && (3.34)\\
{c_1}^{ij}_{kl} &= {c_1}^{ij}_{(k,l)}\,. && (3.35)
\end{aligned}
$$

Corollary 3.3.4. *If P_1 and P_1 are compatible, then there exists a set of functions X^i such that*

$$
{c_1}^{ij}_k = a_2^{ir}X^j_{,rk}\,. \tag{3.36}
$$

Proof. The existence of the X^i is, in fact, equivalent to equations (3.32) and (3.34) which express the flatness of ∇^λ to order λ.

The torsion-free property $a_2^{ir} {c_1}_k^{ij} = a_2^{jr} {c_1}_r^{ik}$ of the pencil, and the cyclic symmetry condition $a_{2ij,k} = a_{2jk,i}$ allow equation (3.34) to be written as

$$(a_{2jr} {c_1}_i^{rk})_{,l} = (a_{2jr} {c_1}_l^{rk})_{,i} \,,$$

which is equivalent to the local existence of a set of function B_j^k such that

$${c_1}_k^{ij} = a_2^{ir} B_{r,k}^j \,.$$

Equation (3.32) is then

$$a_2^{ir} a_2^{js} B_{s,r}^k = a_2^{jr} a_2^{is} B_{s,r}^k \,,$$

or

$$B_{i,j}^k = B_{j,i}^k \,,$$

which is locally equivalent to the existence of function X^i such that $B_j^i = X_{,j}^i$. □

Given functions X^i satisfying ${c_1}_k^{ij} = a_2^{ir} X_{,rk}^j$, equation (3.33) is equivalent to

$$\left(a_{2jr} a_{2ks} \left(a_1^{rs} + \mathcal{L}_X a_2^{rs}\right)\right)_{,i} = \left(a_{2kr} a_{2is} \left(a_1^{rs} + \mathcal{L}_X a_2^{rs}\right)\right)_{,j} \,, \tag{3.37}$$

where we have chosen to interpret the X^i as the components of a vector field X to simplify the formula. Not that equation (3.37) is not a potentiality condition on a_1^{ij}, since it is of the form $\partial_i F_j = -\partial_j F_i$, not the required $\partial_i F_j = \partial_j F_i$. It is not clear what conditions are necessary to ensure the existence of an evolutionary vector field X such that $\mathcal{L}_X P_2^{ij} = P_1^{ij}$.

Result 3.3.5. *For*

$$P^{ij} = a^{ij} \left(\frac{d}{dx}\right)^2 + b_k^{ij} u_x^k \frac{d}{dx}$$

we have

$$\mathcal{L}_X P^{ij} = \tilde{a}^{ij} \left(\frac{d}{dx}\right)^2 + \tilde{b}_k^{ij} u_x^k \frac{d}{dx} + \tilde{c}_{kl}^{ij} u_x^k u_x^l + \tilde{c}_k^{ij} u_{xx}^k$$

where

$$\begin{aligned}
\tilde{a}^{ij} &= X^r a_{,r}^{ij} - a^{ir} X_{,r}^j - a^{rj} X_{,r}^i \,, \\
\tilde{b}_k^{ij} &= X^r b_{k,r}^{ij} + b_r^{ij} X_{,k}^r - b_k^{rj} X_{,r}^i - b_k^{ir} X_{,r}^j - 2a^{ir} X_{,rk}^j \,, \\
\tilde{c}_k^{ij} &= -a^{ir} X_{,rk}^j \,, \\
\tilde{c}_{kl}^{ij} &= -a^{ir} X_{,rkl}^j - b_{(k}^{ir} X_{,l)r}^j \,.
\end{aligned}$$

In particular, if b_k^{ij} is constant -that is, if P_1^{ij} is of the type considered in Section 3.2- then we have $\tilde{b}_k^{ij} = 2\tilde{c}_k^{ij}$.

One could attempt to interpret the compatibility of the two operators P_1 and P_2 in terms of a multiplication, as in Definition 3.2.2, in which case the Jacobi identity for P_1 could be used to remove the derivatives of ${c_1}^{ij}_k$ appearing in equations (3.34) and (3.35). Of particular interest, is that (3.35) is equivalent to

$$ {c_1}^{ij}_r {c_1}^{rk}_l = {c_1}^{ik}_r {c_1}^{rj}_l $$

given the Jacobi identity for P_1. This means that the algebra with structure constants ${c_1}^{ij}_k$ satisfies the first condition of a 'Boson' Novikov algebra:

$$ (\alpha \diamond \beta) \diamond \gamma = (\alpha \diamond \gamma) \diamond \beta \,. $$

It is the second restriction, that $(\alpha \diamond \beta) \diamond \gamma = -(\alpha \diamond \gamma) \diamond \beta$, which changes. Upon using

$$ \begin{aligned} {R_1}^{ijk}_l &= {a_1}^{ir}({c_1}^{jk}_{r,l} - {c_1}^{jk}_{l,r}) + {c_1}^{ij}_r {c_1}^{rk}_l + {c_1}^{ik}_r {c_1}^{rj}_l \\ &\quad - ({a_1}^{ij}_{,r} - {c_1}^{ij}_r + {c_1}^{ji}_r){c_1}^{rk}_l + {c_1}^{ik}_r({a_1}^{rj}_{,l} - {c_1}^{rj}_l + {c_1}^{jr}_l) \\ &= 0\,, \end{aligned} $$

to substitute for ${c_1}^{jk}_{r,l} - {c_1}^{jk}_{l,r}$, equation (3.34) becomes

$$ {c_1}^{ij}_r {c_1}^{rk}_l + {c_1}^{ik}_r {c_1}^{ij}_l = \left(L^i_s {a_2}^{sj}_{,r} - \nabla^1_r {a_1}^{ij}\right) {c_1}^{rk}_l + {c_1}^{ik}_r \left(\nabla^1_l {a_1}^{rj} - L^r_s {a_2}^{sj}_{,l}\right) \,, $$

which, as a condition on $\diamond$, would be

$$ \begin{aligned} (I \diamond J) \diamond K + (I \diamond K) \diamond J &= \big((\nabla^2 a_2)(L(I), J) - (\nabla^1 a_1)(I, J)\big) \diamond K \\ &\quad + \big((\nabla^1 a_1)(I \diamond K, J) - (\nabla^2 a_2)(L(I \diamond K), J)\big) \,, \end{aligned} $$

in which terms of the form $(\nabla a)(Y, Z)$ are to be understood as one-forms via

$$ \begin{aligned} \langle X | (\nabla a)(Y, Z) \rangle &= (\nabla_X a)(Y, Z)\,, \\ &= X\,(a(Y, Z)) - a(\nabla_X Y, Z) - a(Y, \nabla_X Z)\,. \end{aligned} $$

So, for generic Hamiltonian operators of degree 2, the vanishing of the curvature of the pencil does not lead to purely algebraic constraints on the coefficients, since it involves derivatives of the two bivectors a_1 and a_2. Similarly, the cyclic symmetry condition (3.33) also involves derivatives. This undermines the usefulness of the interpretation of the compatibility of P_1 and P_2 via a multiplicative structure on the space of one-forms, which came from being able to consider the coefficients in the operators as the structure constants of algebraic objects on each tangent plane.

3.4 Bi-Hamiltonian Structures in Degrees 1 and 2

We now consider a pair of operators, P_1 and P_2 in which P_1 is a Hamiltonian operator of hydrodynamic type and P_2 is of second order, i.e.:

$$\begin{aligned} P_1^{ij} &= g^{ij}(u)\frac{d}{dx} + \Gamma^{ij}_k(u)u^k_x\,, \\ P_2^{ij} &= a^{ij}\left(\frac{d}{dx}\right)^2 + b^{ij}_k u^k_x \frac{d}{dx} + c^{ij}_{kl}u^k_x u^l_x + c^{ij}_k u^k_{xx}\,, \end{aligned}$$

where g^{ij} is the inverse of a flat metric g_{ij} on M and $\Gamma^{ij}_k = -g^{ir}\Gamma^j_{rk}$ where the Γ^k_{ij} are the Christoffel symbols of the Levi-Civita connection of g. We also assume that P_2^{ij} is antisymmetric, so that $a^{ij} = -a^{ji}$, $b^{ij}_k = a^{ij}_{,k} + c^{ij}_k + c^{ji}_k$ and $c^{(ij)}_{kl} = c^{(ij)}_{(k,l)}$.

The motivation [31] for studying such pairs of operators comes not from regarding them as separate Hamiltonian operators, but from thinking of P_2^{ij} as a first order (dispersive) deformation of P_1^{ij} into some non-homogeneous Hamiltonian operator $P^{ij} = P_1^{ij} + \varepsilon P_2^{ij} + O(\varepsilon^2)$. Thus, in such a pair, it is sensible to regard the geometry of P_1^{ij} as being more intrinsic than any associated to P_2^{ij}.

We choose to work in flat coordinates for g so that g^{ij} is constant and $\Gamma^{ij}_k = 0$. Theorem 1.5.6, calculated in these coordinates, gives

Theorem 3.4.1. *P_2 is an infinitesimal deformation of P_1, i.e. $P^{ij} = P_1^{ij} + \varepsilon P_2^{ij} + O(\varepsilon^2)$ satisfies the Jacobi identity to order ε, if and only if*

(I) $g^{ir}c^{jk}_r + g^{jr}c^{ik}_r = 0\,,$

(II) $c^{ij}_{kl} = c^{ij}_{(k,l)}\,,$

(III) $g^{ir}c^{jk}_{l,r} = g^{jr}(c^{ik}_{l,r} - c^{ik}_{r,l})\,,$

(IV) $g^{ir}(a^{jk}_{,r} - c^{jk}_r) + g^{jr}(a^{ki}_{,r} - c^{ki}_r) + g^{kr}(a^{ij}_{,r} - c^{ij}_r) = 0$

in the flat coordinates for g^{ij}.

By introducing the tensor $T^{ij}_k = a^{ir}\Gamma^j_{rk} + c^{ij}_k$ it is easy to convert conditions (I), (III) and (IV) to arbitrary coordinates, whilst condition (II) becomes

$$2c^{ij}_{kl} = c^{ij}_{k,l} + c^{ij}_{l,k} - c^{ri}_k\Gamma^j_{rl} - c^{ri}_l\Gamma^j_{rk} + T^{ij}_r\Gamma^r_{kl} + T^{rj}_k\Gamma^i_{rl} + T^{rj}_l\Gamma^i_{rk}\,.$$

To consider a bi-Hamiltonian structure involving operators P_1^{ij} and P_2^{ij} one need only add conditions (C), (D) and (E) of Theorem 3.1.1 to Theorem 3.4.1, however, condition (II) above allows (E) to be replaced by $c^{ij}_r c^{rk}_l = c^{ik}_r c^{rj}_l$.

Example 3.4.2. *As discussed in section 3.1, P_2 with $b^{ij}_k = 2c^{ij}_k$ constant and a^{ij} non-degenerate is Hamiltonian if and only if $a^{ij} = A^{ij}_k u^k + A^{ij}_0$ with $A^{ij}_k = c^{ij}_k - c^{ji}_k$, A^{ij}_0 is constant, c^{ij}_k are the structure constants of a Fermionic Novikov algebra $(\mathcal{A}, \diamond)$, and A^{ij}_0 defines a skew-symmetric bilinear form on $\mathcal{A}$ satisfying $\langle I, J \diamond K\rangle = \langle J, I \diamond K\rangle$.*

If we ask that P_2 satisfies the above constancy conditions in the flat coordinates for g^{ij}, then, defining an inner product on $\mathcal{A}$ by $(e^i, e^j) = g^{ij}$, we have that the compatibility of P_1 and P_2 is equivalent to the additional constraints:

$$\begin{aligned}(I \diamond J) \diamond K &= (I \diamond K) \diamond J\,,\\ (I, J \diamond K) &= -(J, I \diamond K)\end{aligned}$$

and

$$(I, [J, K]) + (J, [K, I]) + (K, [I, J]) = 0\,,$$

where $[I, J] = I \diamond J - J \diamond I$ is the commutator of $\diamond$, which is a Lie bracket by equation (3.5).

For example, if we take the algebra $(\mathcal{A} = \mathrm{span}\{e^1, e^2, e^3, e^4\}, \diamond)$ where the only non-zero products are $e^3 \diamond e^3 = e^1$ and $e^4 \diamond e^3 = e^2$ then we may take as our symplectic form and metric

$$[\omega^{ij}] = \begin{pmatrix} 0 & 0 & a & b\\ 0 & 0 & b & c\\ -a & -b & 0 & d - u^2\\ -b & -c & -d + u^2 & 0 \end{pmatrix}$$

and

$$[g^{ij}] = \begin{pmatrix} 0 & 0 & 0 & e\\ 0 & 0 & -e & 0\\ 0 & -e & f & g\\ e & 0 & g & h \end{pmatrix},$$

for any choice of the constants a, b, c, d, e, f, g, h such that $e \neq 0$ and $b^2 \neq ac$.

This algebra, essentially $(57)_{-1}$, is the only algebra in [4] of dimension 2 or 4 which admits non-degenerate forms $(\cdot, \cdot)$ and $\langle \cdot, \cdot \rangle$ satisfying the above compatibility conditions with $\diamond$, other than the trivial case in which all products are zero, i.e. in which the Hamiltonian operators share the same flat connection, and so are simultaneously constant.

Proposition 3.4.3. *If P_2 is an infinitesimal deformation of P_1 then there exists a tensor field A^i_j such that*

$$\begin{aligned} a^{ij} &= g^{ir}A^j_r - g^{jr}A^i_r\,, \\ b^{ij}_k &= 2g^{is}A^j_{s,k} - g^{jr}A^i_{k,r} - g^{is}A^j_{k,s}\,, \\ c^{ij}_{kl} &= g^{is}A^j_{s,kl} - g^{is}A^j_{(k,l)s}\,, \\ c^{ij}_k &= g^{is}A^j_{s,k} - g^{is}A^j_{k,s} \end{aligned} \tag{3.38}$$

in flat coordinates for g^{ij}. Further, any $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$-tensor field A^i_j produces an infinitesimal deformation of P_1 by the above formulae.

Proof. Using the non-degeneracy of g^{ij}, we introduce objects θ^k_{ij} and ϕ_{ij} by

$$\begin{aligned} c^{ij}_k &= g^{ir}\theta^j_{rk}\,, \\ a^{ij} &= g^{ir}g^{js}\phi_{rs}\,. \end{aligned}$$

Then condition (I) of Theorem 3.4.1 is equivalent to $\theta^k_{ij} = -\theta^k_{ji}$, and so we regard θ^k_{ij} as a family of 2-forms θ^k indexed by k.

Condition (III) is equivalent to $\theta^k_{jl,i} = \theta^k_{il,j} - \theta^k_{ij,l}$, so that $d\theta^k = 0$ for each k. This allows us to introduce a family of 1-forms ψ^k such that

$$\theta^k_{ij} = (d\psi^k)_{ij} = \psi^k_{i,j} - \psi^k_{j,i}\,.$$

Each ψ^k can be adjusted by the addition of the exterior derivative, df^k, of some function f^k without affecting the value of θ^k_{ij}.

Writing $\alpha_{ij} = \phi_{ij} - g_{jr}\psi^r_i + g_{jr}\psi^r_k$, we find that condition (IV) is equivalent to the closedness of the 2-form α_{ij}, upon substituting ϕ_{ij} and ψ^i_j for a^{ij} and c^{ij}_k. Thus we may introduce a 1-form h with components h_i such that $\alpha_{ij} = h_{i,j} - h_{j,i}$, and so

$$\phi_{ij} = g_{jr}\psi^r_i - g_{jr}\psi^r_j + h_{i,j} - h_{j,i}\,.$$

If we now let $A^i_j = \psi^i_j + (g^{ir}h_r)_{,j}$ then we have $\theta^k_{ij} = A^k_{i,j} - A^k_{j,i}$ and $\phi_{ij} = g_{jr}\psi^r_i - g_{ir}\psi^r_j$, so that the two equations $a^{ij} = g^{ir}A^j_r - g^{jr}A^i_r$ and $c^{ij}_k = g^{ir}A^j_{r,k} - g^{jr}A^j_{k,r}$ are satisfied. The remaining two equations follow easily from $c^{ij}_{kl} = c^{ij}_{k,l}$ and $b^{ij}_k = a^{ij}_k + c^{ij}_k + c^{ji}_k$.

For the converse, it is easy to check that conditions (I)-(IV) of Theorem 3.4.1 follow from (3.38) for any tensor field A^i_j. □

As with Proposition 3.2.7, Proposition 3.4.3 may be understood as asserting the existence of an evolutionary vector field

$$e = A^i_j\,(u(x))\,u^j_x(x)\frac{\partial}{\partial u^i(x)} + \dots$$

satisfying $P_2 = -\mathcal{L}_e P_1$ whenever P_2 is an infinitesimal deformation of P_1. The existence of this vector field is guaranteed by Theorem 1.6.1, since P_2 is an infinitesimal deformation of P_1, and therefore trivial. As such, Proposition 3.4.3 can serve as an alternative proof of Theorem 3.4.1.

There is a freedom in A^i_j of $A^i_j \mapsto A^i_j + g^{ir}f_{,rj}$ for some function f, which does not affect the coefficients of P_2. This corresponds to adjusting e by a Hamiltonian vector field, $e \mapsto e + P_1(\delta f)$.

If, with reference to Lemma 3.1.6, we impose the additional constraint on (3.38) that $b^{ij}_k = 2c^{ij}_k$ then we have the potentiality condition $g_{jr}A^r_{k,i} = g_{ir}A^r_{k,j}$, so that there exists a 1-form B_k such that

$$A^i_j = g^{ir}B_{j,r}\,. \tag{3.39}$$

In this case $a^{ij} = g^{ir}g^{jr}(B_{r,s} - B_{s,r}) = g^{ir}g^{jr}(dB)_{rs}$ and the freedom $A^i_j \mapsto A^i_j + g^{ir}f_{,rj}$ is $B \mapsto B + df$. This means that B can be determined purely from g^{ij} and a^{ij}, and thus there is no freedom in the choice of c^{ij}_k and c^{ij}_{kl}. In fact we may write explicitly

$$c^{ij}_k = g^{js}g_{kr}\frac{\partial a^{ir}}{\partial u^s}\,, \quad c^{ij}_{kl} = c^{ij}_{(k,l)}\,, \tag{3.40}$$

and with this, P_2 is an infinitesimal deformation of P_1 if and only if

$$g^{ir}a^{jk}_{,r} + g^{jr}a^{ki}_{,r} + g^{kr}a^{ij}_{,r} = 0\,, \tag{3.41}$$

which is equivalent to the closedness of the two-form

$$\phi_{ij} = g_{ir}g_{js}a^{rs}\,.$$

Corollary 3.4.4. *Given a flat metric g and a symplectic form ω, there is at most one choice of flat symplectic connection ∇ such that the degree 2 Hamiltonian operator specified by (ω, ∇) is compatible with the hydrodynamic operator specified by g.*

Clearly, if this connection exists it is given by (3.40), so this definition must be checked against Theorem 3.1.1 to verify

$$P_2^{ij} = \omega^{ij}\left(\frac{d}{dx}\right)^2 + 2c^{ij}_k u^k_x\frac{d}{dx} + c^{ij}_{kl}u^k_x u^l_x + c^{ij}_k u^k_{xx}$$

is Hamiltonian. Since equation (3.41) is a consequence of the antisymmetry of P_2, compatibility with the Hydrodynamic operator follows immediately.

We conclude this section with an example of this type.

Example 3.4.5. *The Kaup-Broer system [62],*

$$\begin{pmatrix} u^1_t \\ u^2_t \end{pmatrix} = \begin{pmatrix} u^1_{xx} + 2u^2_x + 2u^1u^1_x \\ -u^2_{xx} + 2(u^1u^2)_x \end{pmatrix},$$

is described by the pair of compatible Hamiltonian operators

$$\begin{aligned} P_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{d}{dx}, \\ P_2 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \left(\frac{d}{dx}\right)^2 + \begin{pmatrix} 2 & u^1 \\ u^1 & 2u^2 \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} 0 & u^1_x \\ 0 & u^2_x \end{pmatrix}. \end{aligned}$$

Scaling $x \mapsto \varepsilon x$, $t \mapsto \varepsilon t$ splits P_2 into $P_2^{(1)} + \varepsilon P_2^{(2)}$ where

$$\begin{aligned} P_2^{(1)} &= \begin{pmatrix} 2 & u^1 \\ u^1 & 2u^2 \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} 0 & u^1_x \\ 0 & u^2_x \end{pmatrix}, \\ P_2^{(2)} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \left(\frac{d}{dx}\right)^2. \end{aligned}$$

Since $P_2 = P_2^{(1)} + \varepsilon P_2^{(2)}$ is Hamiltonian for all ε, $P_2^{(1)}$ and $P_2^{(2)}$ constitute a bi-Hamiltonian structure of the type considered above. A set of flat coordinates for the metric in $P_2^{(1)}$ is

$$\begin{aligned} \tilde{u}^1 &= u^1, \\ \tilde{u}^2 &= \sqrt{4u^2 - (u^1)^2}, \end{aligned}$$

in which

$$\begin{aligned} \tilde{P}_2^{(1)} &= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \frac{d}{dx}, \\ \tilde{P}_2^{(2)} &= \frac{2}{\tilde{u}^2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \left(\frac{d}{dx}\right)^2 + \frac{4}{(\tilde{u}^2)^2} \begin{pmatrix} 0 & -\tilde{u}^2_x \\ 0 & \tilde{u}^1_x \end{pmatrix} \frac{d}{dx} \\ &\quad + \frac{4}{(\tilde{u}^2)^3} \begin{pmatrix} 0 & (\tilde{u}^2_x)^2 \\ 0 & -\tilde{u}^1_x \tilde{u}^2_x \end{pmatrix} + \frac{2}{(\tilde{u}^2)^2} \begin{pmatrix} 0 & -\tilde{u}^2_{xx} \\ 0 & \tilde{u}^1_{xx} \end{pmatrix}. \end{aligned}$$

So in this situation we have, for the 1-form in (3.39),

$$B = \frac{\tilde{u}^1}{2\tilde{u}^2} d\tilde{u}^2.$$

Note: The dispersionless limit of the Kaup-Broer system is the long wave system (2.17) discussed in Example 2.2.3. One can see, in the dispersionless limit of the bi-Hamiltonian structure above, the hydrodynamic type Poisson brackets determined by the metric (2.5) and intersection form (2.25) of the associated Frobenius manifold.

3.5 Conclusions

In section 3.2 an approach was taken based upon the methods of [26] to study compatible pairs of Hamiltonian operators of degree 2 which satisfy the conditions of the relevant Darboux theorem, Theorem 3.1.11. As for Hydrodynamic Poisson pencils, the compatibility could be reduced to algebraic constraints on a multiplication of covectors. Driving this was the ability to reduce a given Hamiltonian operator on $L(M)$ to a flat Fedosov structure (ω, ∇) on M, which are natural symplectic analogues of the pair consisting of a flat metric and its Levi-Civita connection which determines a Hydrodynamic Poisson bracket.

In section 3.3, compatible pairs of hamiltonian operators of degree 2 were studied, without the assumption that either possessed Darboux coordinates. The form of the compatibility conditions suggests that an approach based upon algebraic structures is less relevant here, since the derivatives of the structure functions of the algebras play a role.

Proposition 3.2.7 can easily be extended to confirm the existence of a vector field B realising $P_1 = -\mathcal{L}_B P_2$ whenever P_1, of the form (3.1) is an infinitesimal deformation of P_2 as a Hamiltonian operator, provided ${b_1}^{ij}_k = 2{c_1}^{ij}_k$. Result 3.3.5 shows that ${b_1}^{ij}_k = 2{c_1}^{ij}_k$ is also a necessary condition. Thus we have determined the trivial deformations of a degree 2 Hamiltonian operator admitting a constant form, which are themselves of degree 2. Clearly, whether the operator can be put in a constant form or not, a different approach is necessary to understand deformations of higher degrees.

Corollary 3.3.4 fails to provide conditions under which an infinitesimal deformation of a generic Hamiltonian operator of degree 2 is trivial. Clearly, if a vector field Y exists satisfying $P_1 = -\mathcal{L}_{\tilde{Y}} P_2$ (where $\tilde{Y}$ is the lift of Y to the loop space), then it differs from the vector field X of Corollary 3.3.4 by at most a linear vector field (in the flat coordinates for ∇^2). Since ${b_1}^{ij}_k$ is defined from ${c_1}^{ij}_k$ and a_1^{ij}, it is only necessary to check this, and that

$$a_{2ir} a_{2js} a_1^{rs} + \mathcal{L}_Y a_{2ij} = 0\,. \tag{3.42}$$

Since, by Corollary 3.1.5, a_{2ij} is linear in the flat coordinates for ∇^2, a necessary condition to be able to find a Y satisfying (3.42) is that

$$\Xi_{ij} = a_{2ir}a_{2js}a_1^{rs} + \mathcal{L}_X a_{2ij}$$

is linear. It is not clear whether this is sufficient.

Finally, there is a certain artificiality to the examples of compatible Fedosov structures presented in section 3.2. Given Theorem 3.2.1's assertion that underlying a pair of compatible Fedosov structures is a finite-dimensional bi-Hamiltonian structure, the question is raised asking which finite-dimensional bi-Hamiltonian structures admit symplectic connections forming almost compatible, almost compatible and flat, or compatible Fedosov structures? It would be interesting to exhibit a pair of compatible Fedosov structures in which the flat coordinates for one of the connections are in some sense physical.

Chapter 4

Logarithmic Deformations of WDVV Solutions

This chapter is concerned with constructing deformations of known solutions of the WDVV equations via so-called waterbag deformations of an associated object known as the superpotential. Amongst the solutions deformed in this manner are those related to the Coxeter groups A_N, B_N and D_N, as well as those coming from classes of rational reductions of the dispersionless KP hierarchy. The deformations are such that they destroy the quasi-homogeneity of the free energies, resulting in non-conformal Frobenius manifolds; however, if one imagines that the deformation parameters introduced in the construction have a degree of their own, some form of homogeneity remains in the deformed free energies.

Of particular interest is a subclass of these deformations which provides new polynomial solutions to the equations of associativity. These deform the A_N-polynomial solutions, in the sense that the free energy takes the form

$$F(t^1, \dots, t^N, b) = F^{(0)}(t^1, \dots, t^N) + kF^{(1)}(t^1, \dots, t^N, b) \tag{4.1}$$

where $F^{(0)}$ is the polynomial solution defining the Frobenius manifold structure on the space $\mathbb{C}^N/A_N$ and k is some deformation parameter. Such solutions satisfy a pseudo-quasi-homogeneity condition. With the Euler vector field

$$E = \sum_{i=1}^{N} \frac{(N+2-i)}{N+1} t^i \frac{\partial}{\partial t^i} + \frac{b}{N+1}\frac{\partial}{\partial b} \tag{4.2}$$

we define the degree of an invariant function f, denoted $\deg(f)$, by

$$(N+1)\mathcal{L}_E f = \deg(f) f \,.$$

With this, each part of F is separately quasi-homogeneous:

$$\begin{aligned}\deg\left(F^{(0)}\right) &= (2N+4)F^{(0)}\,,\\ \deg\left(F^{(1)}\right) &= (N+3)F^{(1)}\,.\end{aligned}$$

So by assigning a fictitious scaling degree of $(N+1)$ to the deformation parameter k the full solution may be thought of as pseudo-quasi-homogeneous.

We begin by describing the Landau-Ginzburg constructions by which the Frobenius manifold is constructed from a given superpotential function, illustrating how this works for the Frobenius structure on the orbit space $\mathbb{C}^n/A_N$. We then explain the origin of the logarithmic deformations in terms of the dKP hierarchy. In Section 4.2 the construction of the A_N deformations is demonstrated in detail, as illustrative of the other deformations which are summarised in Section 4.4.

4.1 Topological Landau-Ginzburg Models and the dKP Hierarchy

As stated in Section 2.4, the orbit spaces of finite Coxeter groups provide an important class of Frobenius manifolds, and the free energies in these cases are polynomials in the flat coordinates.

The Coxeter group A_N is isomorphic to the symmetric group σ_{N+1} and acts on $\mathbb{R}^{N+1} = \{(x_0, x_1, \ldots, x_N)\}$ via permutation of the coordinates x_i, which is to say via reflections in the diagonal hyperplanes $\pi_{ij} = \{(x_0, \ldots, x_N) : x_i + x_j = 0\}$. This action is reducible, since it leaves invariant the hyperplane defined by $x_0 + x_1 + \cdots + x_N = 0$. So we restrict to this hyperplane and thus obtain an action on $\mathbb{R}^N$, which we complexify to get the required action on $\mathbb{C}^N$.

We interpret the orbit space $\mathbb{C}^N/A_N$ as the space of polynomials

$$\lambda(p) = p^{N+1} + s_1 p^{N-1} + \ldots + s_N\,, \tag{4.3}$$

on which the Coxeter group acts by permuting the roots x_i. The s_i are thus polynomials in the x_i invariant under the action of A_N. Since the coefficient of p^N is zero, we have $x_0 + \cdots + x_N = 0$.

From (4.3), the Frobenius manifold structure may be derived via the Landau-Ginzburg formalism [25, 50, 51]. Namely, we use the formulas

$$\eta(\partial_{s_i}, \partial_{s_j}) = -\sum \operatorname*{res}_{d\lambda=0}\left\{\frac{\partial_{s_i}\lambda(p)\,\partial_{s_j}\lambda(p)}{\lambda'(p)}\,dp\right\} \tag{4.4}$$

and

$$c\left(\partial_{s_i},\partial_{s_j},\partial_{s_k}\right) = -\sum \operatorname*{res}_{d\lambda=0}\left\{\frac{\partial_{s_i}\lambda(p)\,\partial_{s_j}\lambda(p)\,\partial_{s_k}\lambda(p)}{\lambda'(p)}\,dp\right\} \tag{4.5}$$

with the function λ, which in this context is called the superpotential, given by (4.3), to recover the necessary metric and multiplication. The conformal structure of the Frobenius manifold is specified by the Euler vector field

$$E = \sum_{i=1}^{N} \frac{(1+i)}{N+1} s_i \frac{\partial}{\partial s_i}\,,$$

which can be shown to arise as a consequence of the invariance of the polynomial (4.3) under

$$\tilde{E} = \frac{1}{N+1} p \frac{\partial}{\partial p} + \sum_{i=1}^{N} \frac{(1+i)}{N+1} s^i \frac{\partial}{\partial s^i}\,.$$

The numbers $(1+i)$ arise as the degrees of the polynomials s_i when written in terms of the x_i.

Example 4.1.1. *With*

$$\lambda(p) = p^4 + s_1 p^2 + s_2 p + s_3$$

the formula (4.4) gives the metric

$$\eta = \frac{s_1}{8} ds_1^2 - \frac{1}{2} ds_1 ds_3 - \frac{1}{4} ds_2^2\,.$$

This result is calculated by using the fact that the sum of the residues of a meromorphic function on $\mathbb{C}P^1$ *is zero. Other than the zeroes of* $\lambda'(p)$*, the only possible pole of the argument in the right-hand side of (4.4) is at* $p=\infty$*. Thus we have, for instance,*

$$\begin{aligned}
\eta\left(\partial_{s_1},\partial_{s_3}\right) &= -\sum \operatorname*{res}_{d\lambda=0}\left\{\frac{p^2}{4p^3+2s_1p+s_2}dp\right\}\,,\\
&= \operatorname*{res}_{p=\infty}\left\{\frac{p^2}{4p^3+2s_1p+s_2}dp\right\}\,,\\
&= \operatorname*{res}_{\tilde{p}=0}\left\{\frac{\tilde{p}^{-2}}{4\tilde{p}^{-3}+2s_1\tilde{p}^{-1}+s_2}d\left(\frac{1}{\tilde{p}}\right)\right\}\,,\\
&= -\operatorname*{res}_{\tilde{p}=0}\left\{\frac{1}{\tilde{p}(4+2s_1\tilde{p}^2+s_2\tilde{p}^3)}dp\right\}\,,\\
&= -\frac{1}{4}\,.
\end{aligned}$$

While this metric is flat, the s^i *are not flat coordinates. With*

$$\begin{aligned}
s_3 &= t_1 + \frac{1}{8}t_3^2\,,\\
s_2 &= t_2\,,\\
s_1 &= t_3
\end{aligned}$$

one obtains a metric with constant coefficients. The tensor given by the formula (4.5) may then be used to construct the free energy

$$F = -\frac{1}{8}t_1^2 t_3 - \frac{1}{8}t_1 t_2^2 + \frac{1}{64}t_2^2 t_3^2 - \frac{1}{3840}t_3^5 \,,$$

which is homogeneous with respect to the Euler vector field

$$E = \frac{1}{4}\left(4t_1\frac{\partial}{\partial t_1} + 3t_2\frac{\partial}{\partial t_2} + 2t_3\frac{\partial}{\partial t_3}\right) .$$

We recognise in (4.3) the same polynomial used to specify the N^{th} dispersionless Gel'fand-Dikiĭ hierarchy in (1.54). We thus find the polynomial solutions obtained above arising from three approaches:

(i) as a basic example of an orbit space construction. Here the manifold is $\mathbb{C}^N/A_N$ where A_N is a Coxeter group;

(ii) as a topological Landau-Ginzburg field theory;

(iii) as a reduction of the dispersionless KP hierarchy.

We shall take (iii) as our starting point, constructing a solution to the equations of associativity from a specific reduction of the dispersionless KP hierarchy [50, 51]. In particular the so-called waterbag reductions [10, 46], which are specified by imposing the restriction

$$\lambda(p) = p + \sum_{i=1}^{N} k_i \log\left(\frac{p - p_i}{p - \tilde{p}_i}\right) \tag{4.6}$$

on the dispersionless Lax function (1.52), have been studied in this context by Chang [13, 14]. The original motivation for the work in this chapter was the construction of a two-dimensional solution of the WDVV equations arising from setting $N = 1$ in (4.6), which was presented in [13]. Here we generalise this setting and consider functions of the form

$$\lambda(p) = (\text{rational function})\,(p) + \sum_{i=1}^{M} k_i \log(p - b_i)\,.$$

Formally one may expand this function for large p as a series, but this will have terms of the form

$$\left(\sum_{i=1}^{M} k_i\right) \log p$$

and the constraint $\sum k_i = 0$ is often imposed. Here we show that one still gets a solution without such a constraint. The logarithmic terms mean that $\lambda(p)$ is multi-valued. However, since the constructions in Section 4.2 involve only the derivatives of λ, all physical quantities are well-defined. For simplicity we present proofs in the polynomial case, with

$$\lambda(p) = p^{N+1} + s_1 p^{N-1} + \ldots + s_N + \sum_{i=1}^{M} k_i \log(p - b_i) \tag{4.7}$$

and state the result for the rational case and its reductions to even superpotentials, since no essential new features will be present in the rational case that are not already present in the polynomial case. Note that without this constraint the function is not technically a reduction of the dKP hierarchy, but one may associate a 'regularised' function

$$\lambda_{\rm reg}(p) = \lambda(p) - \left(\sum_{i=1}^{M} k_i\right) \log p \tag{4.8}$$

which is considered in [63]. For this reason we call the form (4.7) a generalised waterbag reduction. We denote the space of such superpotentials $\mathcal{M}^{(M,N)}$ or just $\mathcal{M}$.

4.2 The Generalised Waterbag Reduction of the Dispersionless KP Hierarchy

We begin by proving that the formulas (4.4) and (4.5) with the function (4.7) define a commutative, associative, semi-simple multiplication on the tangent space to the manifold of parameters. This will be done using canonical coordinates - the critial values of λ (i.e. λ evaluated at its critical points). Since $\lambda(p)$ only involves logarithms, its derivative is a rational function which may be written in the form

$$\lambda'(p) = \frac{(N+1)\prod_{i=1}^{M+N}(p - \xi_i)}{\prod_{j=1}^{M}(p - b_j)}$$

(we assume that we are considering the generic case, where the poles and zeros are all distinct). The canonical coordinates are then

$$u^i = \lambda(\xi_i)\,, \qquad i = 1, \ldots, N+M \tag{4.9}$$

(for such a formula to be single-valued, various cuts have to be made in the complex plane).

Lemma 4.2.1. *The formulae (4.4) and (4.5) with λ given by (4.7) define, at a generic point, a semi-simple, commutative, associative multiplication*

$$\frac{\partial}{\partial u^i} \circ \frac{\partial}{\partial u^j} = \delta_{ij} \frac{\partial}{\partial u_i}\,, \tag{4.10}$$

compatible with the metric

$$\eta = -\sum_{r=1}^{M+N} \frac{du_i^2}{\lambda''(\xi_i)}\,, \tag{4.11}$$

where the coordinates u^i are given by (4.9).

Proof. The proof follows [25], Lemma 4.5. From the formulae

$$\frac{\partial}{\partial u^i}\lambda(p)\Big|_{p=\xi_j} = \delta_{ij}\,, \qquad\qquad i = 1\,\ldots\,, N+M$$

and

$$\frac{\partial}{\partial u^i}\lambda(p) = \left\{\prod_{r=1}^{M}(p-b_r)\right\}^{-1} B_i(p)$$

(where B_i is a polynomial of degree $N+M-1$) one obtains

$$B_i(\xi_j) = \begin{cases} 0\,, & i \neq j \\ \prod_{r=1}^{M}(\xi_i - b_r)\,, & i = j\,. \end{cases}$$

The Lagrange interpolation formula then gives

$$B_i(p) = \frac{\prod_{j\neq i}(p-\xi_j)\prod_{r=1}^{M}(\xi_i-b_r)}{\prod_{j\neq i}(\xi_i-\xi_j)}$$

and hence

$$\begin{aligned} \frac{\partial\lambda(p)}{\partial u^i} &= \frac{\prod_{j\neq i}(p-\xi_j)\prod_{r=1}^{M}(\xi_i-b_r)}{\prod_{j\neq i}(\xi_i-\xi_j)\prod_{r=1}^{M}(p-b_r)}\,, \\ &= \frac{1}{(p-\xi_i)}\lambda'(p)\left\{\frac{\prod_{r=1}^{M}(\xi_i-b_r)}{(N+1)\prod_{j\neq i}(\xi_i-\xi_j)}\right\}\,, \\ &= \frac{1}{(p-\xi_i)}\frac{\lambda'(p)}{\lambda''(\xi_i)}\,. \end{aligned} \tag{4.12}$$

Note that this is the same functional form as in the polynomial case (equation (4.52) in [25]). With this

$$\begin{aligned} \eta(\partial_{u_i},\partial_{u_j}) &= -\sum \operatorname*{res}_{d\lambda=0}\left\{\frac{1}{(p-\xi_i)(p-\xi_j)}\frac{\lambda'(p)}{\lambda''(\xi_i)\lambda''(\xi_j)}dp\right\}\,, \\ &= -\frac{1}{\lambda''(\xi_i)}\delta_{ij}\,. \end{aligned}$$

Similarly

$$c(\partial_{u_i},\partial_{u_j},\partial_{u_k}) = \begin{cases} -\dfrac{1}{\lambda''(\xi_i)}\,, & i=j=k\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

□

This multiplication has an identity. Since $e(\lambda) = 1$, where the vector field e is defined to be

$$e = \frac{\partial}{\partial s^N}\,,$$

it is immediate from equations (4.4) and (4.5) that

$$c(\partial, \partial', e) = \eta(\partial, \partial')\,.$$

From this it follows that e is the identity for the multiplication. In semi-simple coordinates it follows from the multiplication (4.10) that

$$e = \sum_{r=1}^{M+N} \frac{\partial}{\partial u^i}\,.$$

We prove next that the metric is flat and Egorov. In the pure-polynomial case (or A_N-case) the flat coordinates are defined by an inverse series, using the so-called thermodynamical identity (or Maxwell relation). The presence of the logarithms makes such an inversion problematic. However, it turns out that part of the flat-coordinates of the metric are exactly the same as in the polynomial case.

Lemma 4.2.2. *The formula (4.4) with λ given by (4.7) gives the following:*

$$\eta(\partial_{s_i}, \partial_{s_j}) = -\sum \operatorname*{res}_{d\lambda_+=0} \left\{ \frac{\partial_{s_i}\lambda_+(p)\partial_{s_j}\lambda_+(p)}{\lambda'_+(p)} dp \right\}\,, \qquad i\,, j = 1\,, \ldots\,, N\,,$$

where $\lambda_+(p) = p^{N+1} + s_1 p^{N-1} + \ldots + s_N$ is a truncation of λ, and

$$\begin{aligned} \eta(\partial_{b_r}, \partial_{s_j}) &= 0\,, & r = 1\,, \ldots\,, M\,, j = 1\,, \ldots\,, N\,,\\ \eta(\partial_{b_i}, \partial_{b_j}) &= k_i \delta_{ij}\,, & i, j = 1\,, \ldots\,, M\,. \end{aligned}$$

It follows from these formulae that the metric is flat.

Proof. These formulae just involve the use of basic ideas from complex variable theory, particulalrly that the sum of the residues of a meromorphic function on $\mathbb{C}P^1$ is zero.

$$\begin{aligned} \eta(\partial_{s_i}, \partial_{s_j}) &= -\sum \operatorname*{res}_{d\lambda=0} \left\{ \frac{p^{2N-i-j}}{\lambda'(p)} dp \right\}\,,\\ &= \operatorname*{res}_{p=\infty} \left\{ \frac{p^{2N-i-j}}{\lambda'(p)} dp \right\}\,. \end{aligned}$$

Now

$$\begin{aligned} \lambda'(p) &= \lambda'_+(p) + \sum_{r=1}^{M} \frac{k_i}{(p-b_i)}\,,\\ &= \lambda'_+(p) \left\{ 1 + \frac{1}{\lambda'_+(p)} \sum_{r=1}^{M} \frac{k_i}{(p-b_i)} \right\}\,. \end{aligned}$$

Hence

$$\begin{aligned}\eta(\partial_{s_i},\partial_{s_j}) &= \operatorname*{res}_{p=\infty}\left\{\frac{p^{2N-i-j}}{\lambda'_+(p)}\left[1+\frac{1}{\lambda'_+(p)}\sum_{r=1}^{M}\frac{k_i}{(p-b_i)}\right]^{-1}dp\right\},\\ &= -\operatorname*{res}_{\tilde p=0}\left\{\frac{\tilde p^{i+j-N-2}}{\mu(\tilde p)}\left[1+\frac{\tilde p^{N+1}}{\mu(\tilde p)}\sum_{r=1}^{M}\frac{k_i}{1-\tilde p b_i}\right]^{-1}d\tilde p\right\},\\ &= -\operatorname*{res}_{\tilde p=0}\left\{\frac{\tilde p^{i+j-N-2}}{\mu(\tilde p)}d\tilde p\right\},\end{aligned}$$

where $\tilde p = p^{-1}$ and $\lambda'_+(p)=\tilde p^{-N}\mu(\tilde p)$. Reversing the argument yields the result.

Similarly,

$$\begin{aligned}\eta(\partial_{s_i},\partial_{b_r}) &= -\sum\operatorname*{res}_{d\lambda=0}\left\{\frac{p^{N-i}}{\lambda'(p)}\frac{-k_r}{(p-b_r)}dp\right\},\\ &= -\frac{1}{N+1}\operatorname*{res}_{p=\infty}\left\{\frac{k_r p^{N-i}\prod_{r\neq i}(p-b_r)}{\prod_{j=1}^{M+N}(p-\xi_j)}dp\right\},\\ &= \frac{1}{N+1}\operatorname*{res}_{\tilde p=0}\left\{k_r\tilde p^{i-1}\frac{\prod_{r\neq i}(1-b_r\tilde p)}{\prod_{j=1}^{M+N}(1-\xi_j\tilde p)}d\tilde p\right\},\\ &= 0\,.\end{aligned}$$

Finally,

$$\eta(\partial_{b_i},\partial_{b_j}) = -\frac{1}{N+1}\sum\operatorname*{res}_{d\lambda=0}\left\{\frac{k_i}{(p-b_i)}\frac{k_j}{(p-b_j)}\frac{\prod_{r=1}^{M}(p-b_r)}{\prod_{k=1}^{M+N}(p-\xi_k)}dp\right\}.$$

For $i\neq j$ this, on deforming the contour around the Riemann sphere, gives zero: there is no pole at infinity, and the simple poles cancel. For $i=j$,

$$\begin{aligned}\eta(\partial_{b_i},\partial_{b_i}) &= -k_i^2\sum\operatorname*{res}_{d\lambda=0}\left\{\frac{1}{(p-b_i)^2}\frac{1}{\lambda'(p)}dp\right\},\\ &= k_i^2\frac{1}{N+1}\frac{\prod_{k\neq i}(b_i-b_k)}{\prod_k(b_i-\xi_k)}\,.\end{aligned}$$

On evaluating the residue at the poles using the two different formulae for $\lambda'(p)$,

$$(N+1)p^N+(N-1)s_1p^{N-2}+\ldots s_1+\sum_{r=1}^{M}\frac{k_i}{(p-b_r)} = (N+1)\frac{\prod_{i=1}^{M+N}(p-\xi_i)}{\prod_{j=1}^{M}(p-b_j)}$$

one obtains

$$k_i = (N+1)\frac{\prod_k(b_i-\xi_k)}{\prod_{k\neq i}(b_i-b_k)}$$

from which the final formula follows. □

Alternative proof via thermodynamical identity. Following the polynomial case in [25], invert $\lambda_+(p)$ as

$$p_+(k) = k+\frac{1}{N+1}\left(\frac{t^N}{k}+\frac{t^{N-1}}{k^2}+\ldots+\frac{t^1}{k^N}\right)+O\left(\frac{1}{k^{N+1}}\right),$$

where $\lambda_+ = k^{N+1}$. Then

$$\begin{aligned}\lambda(p_+(k,t),t,b) &= \lambda_+(p_+(k,t),t) + \sum_{i=1}^{M} k_i \log(p_+ - b_i)\,,\\ &= k^{N+1} + \sum_{i=1}^{M} k_i \log(p_+ - b_i)\,.\end{aligned}$$

Differentiating with respect to t^α gives

$$\begin{aligned}\left.\frac{d\lambda}{dp}\right|_{p=p_+(k)} \frac{\partial p_+}{\partial t^\alpha} + \frac{\partial \lambda}{\partial t^\alpha} &= \sum_{i=1}^{M} \frac{k_i}{p_+ - b_i}\frac{\partial p_+}{\partial t^\alpha}\,.\\ &= O\left(\frac{1}{k^{N+2-\alpha}}\right)\,.\end{aligned}$$

So we have as our thermodynamical identity in this case

$$\frac{\partial}{\partial t^\alpha}(\lambda dp) + \frac{\partial}{\partial t^\alpha}(p_+ d\lambda) = O\left(\frac{1}{k^{N+1-\alpha}}\right) dk\,. \tag{4.13}$$

Although the right hand side is not zero as it is for polynomial λ, this identity is sufficient to give

$$\frac{\partial}{\partial t^\alpha}(\lambda dp) = -k^{\alpha-1} dk + O\left(\frac{1}{k}\right) dk$$

(eqn. (4.68) in [25]), from which it follows, using

$$d\lambda = d\lambda_+ + O\left(\frac{1}{k}\right) dk\,, \tag{4.14}$$

that

$$\eta(\partial_{t^\alpha}, \partial_{t^\beta}) = -\frac{\delta_{\alpha+\beta,N+1}}{N+1}\,.$$

□

The flat coordinates are therefore

$$\{t^i\,, i = 1\,,\ldots\,, N\,; b_j\,, j = 1\,,\ldots\,, M\}$$

where the t^i are defined by the inverse series for the truncated function $\lambda_+ = \lambda_+(p)$, expanded as a Puiseaux series as $\lambda \to \infty$,

$$p(k) = k + \frac{1}{N+1}\left(\frac{t^N}{k} + \frac{t^{N-1}}{k^2} + \ldots + \frac{t^1}{k^N}\right) + O\left(\frac{1}{k^{N+1}}\right) \tag{4.15}$$

where $k = (\lambda_+)^{\frac{1}{N+1}}$, in the standard way [25]. Note that each t^i is a polynomial in the s_i and vice versa.

Consider the diagonal metric (4.11). Its rotation coefficients β_{ij} are defined by the formula

$$\beta_{ij} = \frac{\partial_{u_i} H_j}{H_i}\,, \qquad H_i^2 = \frac{1}{\lambda''(\xi_i)}\,.$$

Such a metric is said to be Egorov if the rotation coefficients are symmetric. This then implies that the metric may be written in terms of a single potential function $V(u)$,

$$\eta = \sum_{i=1}^{M+N} \frac{\partial V}{\partial u^i} \left(du^i\right)^2\,.$$

According to Gibbons et al. [45], there is an Egorov metric associated with any reduction of the dKP hierarchy. In the following we demonstarte that this is the metric we are considering.

Lemma 4.2.3. *The metric (4.11) is Egorov.*

Proof. This proof closely follows that of Hitchin [48] for the A_N-reduction. In canonical coordinates η is diagonal with i^{th} entry

$$-\frac{1}{\lambda''(\xi_i)}\,.$$

From (4.12)

$$\begin{aligned}\frac{\partial\lambda}{\partial u^i} &= \frac{1}{p-\xi_i}\frac{\lambda'(p)}{\lambda''(\xi_i)}\,,\\ &= \frac{N+1}{\lambda''(\xi_i)}\frac{\prod_{r\neq i}(p-\xi_r)}{\prod_{s=1}^{M}(p-b_s)}\,,\end{aligned}$$

so we have

$$\frac{\partial\lambda}{\partial u^i}\prod_{s=1}^{m}(p-b_s) = \frac{N+1}{\lambda''(\xi_i)}\prod_{r\neq i}(p-\xi_i)$$

where each side is a polynomial of degree $N+M-1$.

Also

$$\frac{\partial\lambda}{\partial u^i} = \frac{\partial s_1}{\partial u^i}p^{N-1} + \frac{\partial s_2}{\partial u^i}p^{N-2} + \cdots + \frac{\partial s_N}{\partial u^i} - \sum_{r=1}^{M}\frac{k_r}{p-b_r}\frac{\partial b_r}{\partial u^i}\,,$$

so

$$\frac{\partial\lambda}{\partial u^i}\prod_{s=1}^{m}(p-b_s) = \left(\frac{\partial s_1}{\partial u^i}p^{N-1} + \cdots + \frac{\partial s_N}{\partial u^i}\right)\prod_{s=1}^{M}(p-b_s) - \sum_{r=1}^{M}k_r\frac{\partial b_r}{\partial u^i}\prod_{s\neq r}(p-b_s)\,.$$

Comparing coefficients of p^{N+M-1} in $\frac{\partial\lambda}{\partial u^i}\prod_{s=1}^{M}(p-b_s)$ in these two expressions gives

$$\frac{N+1}{\lambda''(\xi_i)} = \frac{\partial s_1}{\partial u^i}\,.$$

Hence

$$\eta_{ii} = \eta(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^i}) = -\frac{1}{\lambda''(\xi_i)} = \frac{\partial}{\partial u^i}\left(-\frac{1}{N+1}s_1\right). \tag{4.16}$$

□

This Egorov property is equivalent to a potentiality condition on the $\left[\begin{smallmatrix}0\\3\end{smallmatrix}\right]$-tensor c, namely that the tensor ∇c is totally symmetric. Since the metric is flat one may, in flat-coordinates, integrate by the Poincaré lemma and express everything in terms of a free energy F which satisfies the WDVV equations. Collecting these results together we obtain:

Proposition 4.2.4. *The flat metric (4.4) and totally symmetric $\left[\begin{smallmatrix}0\\3\end{smallmatrix}\right]$-tensor (4.5), with λ given by*

$$\lambda = p^{N+1} + s_1 p^{N-1} + \ldots + s_N + \sum_{i=1}^{M} k_i \log(p - b_i), \quad k_i \text{ constant}$$

define, on the space of such functions, a solution to the WDVV equations. Geometrically they define a semi-simple, associative, commutative algebra with unity on the tangent space $T\mathcal{M}$ compatible with the flat metric.

Example 4.2.5. $\mathrm{N} = 0, \mathrm{M} = 2$. *In the above proofs it has been assumed that $N \neq 0$. However one may adapt these proofs to deal with this case. In particular, the identity field, normally associated to the variable s_N, has to be carefully defined. With*

$$\lambda(p) = p + k_1 \log\left[p - (t_1 + t_2)\right] + k_2 \log\left[p - (t_1 - t_2)\right]$$

one obtains the free energy

$$F = \frac{1}{6}\left\{k_1(t_1+t_2)^3 + k_2(t_1-t_2)^3\right\} + 2k_1k_2\, t_2^2 \,\log t_2\,.$$

Note that if the condition $k_1 + k_2 = 0$ is imposed, one obtains, after some rescalings, the solution obtained by Chang [13]. This example was the original motivation of this work.

Before giving some more examples, it must be remarked that we do not have a Frobenius manifold, just a free energy solving the equations of associativity, with a flat metric and a covariantly constant identity vector field. As was remarked in one of the earliest papers on waterbag reductions, such reductions do not have a scaling symmetry and this fact manifests itself in the non-existence of an Euler vector field, the consequence being that we have only the structure of a non-conformal Frobenius manifold.

However, as was remarked in the introduction, these solutions do possess a form of scaling invariance in that the free energy decomposes as

$$F = F^{(0)}(s) + \sum_{r=1}^{M} k_r F^{(1)} + \sum_{\substack{r,s=1 \\ r\neq s}}^{M} k_r k_s F^{(2)}$$

in which each of the functions $F^{(0)}$, $F^{(1)}$ and $F^{(2)}$ is separately quasihomogeneous with respect to the vector field

$$E = \sum_{i=1}^{N} \frac{(1+i)}{N+1} s_i \frac{\partial}{\partial s_i} + \sum_{j=1}^{M} \frac{b_j}{N+1} \frac{\partial}{\partial b_j} \,. \tag{4.17}$$

We proceed to prove this, and that the functions $F^{(0)}$ and $F^{(1)}$ are polynomials.

Lemma 4.2.6.

$$\begin{aligned}
c\left(\frac{\partial}{\partial b_\alpha}, \frac{\partial}{\partial b_\beta}, \frac{\partial}{\partial b_\gamma}\right) &= 0\,, \qquad \alpha,\beta,\gamma \textit{ distinct}\,,\\
c\left(\frac{\partial}{\partial b_\alpha}, \frac{\partial}{\partial b_\alpha}, \frac{\partial}{\partial b_\beta}\right) &= \frac{k_\alpha k_\beta}{b_\beta - b_\alpha}\,, \qquad \alpha\neq\beta\,,\\
c\left(\frac{\partial}{\partial b_\alpha}, \frac{\partial}{\partial b_\alpha}, \frac{\partial}{\partial b_\alpha}\right) &= k_\alpha \lambda'_+(b_\alpha) + \sum_{r\neq\alpha} \frac{k_\alpha k_r}{b_\alpha - b_r}\,,\\
\\
c\left(\frac{\partial}{\partial b_\alpha}, \frac{\partial}{\partial b_\beta}, \frac{\partial}{\partial s_\gamma}\right) &= 0\,, \qquad \alpha\neq\beta\,,\\
c\left(\frac{\partial}{\partial b_\alpha}, \frac{\partial}{\partial b_\alpha}, \frac{\partial}{\partial s_\gamma}\right) &= k_\alpha (b_\alpha)^{N-\gamma}\,,\\
\\
c\left(\frac{\partial}{\partial b_\alpha}, \frac{\partial}{\partial s_\beta}, \frac{\partial}{\partial s_\gamma}\right) &= k_\alpha S_{\beta+\gamma}(s_1,\ldots,s_N,b_\alpha)\,,
\end{aligned}$$

$$c\left(\frac{\partial}{\partial s_\alpha}, \frac{\partial}{\partial s_\beta}, \frac{\partial}{\partial s_\gamma}\right) = R^{(0)}_{\alpha+\beta+\gamma}(s_1,\ldots,s_N) + \sum_{j=1}^{M} k_j R^{(1)}_{\alpha+\beta+\gamma}(s_1,\ldots,s_N,b_j)$$

where S_σ, $R^{(0)}_\sigma$ and $R^{(1)}_\sigma$ are polynomial functions of their respective variables, and independent of all k_i's.

In particular, the term independent of k_j, $R^{(0)}_{\alpha+\beta+\gamma}(s_1,\ldots,s_N)$, is precisely the value of $c(\partial_{s_\alpha},\partial_{s_\beta},\partial_{s_\gamma})$ found from (4.5) using the polynomial $\lambda_+(p)$ as the Landau-Ginzburg potential (4.3).

Proof. Here we write

$$\lambda'(p) = \frac{\nu(p)}{\prod_{j=1}^{M}(p-b_j)}$$

where

$$\begin{aligned}\nu(p) &= \lambda_+'(p)\prod_{j=1}^{M}(p-b_j)+\sum_{j=1}^{M}k_j\prod_{k\neq j}(p-b_k)\,,\\ &= (N+1)\prod_{j=1}^{M+N}(p-\xi_j)\,.\end{aligned}$$

After the substitution $p \to 1/\tilde{p}$ we will have cause to refer to the polynomial

$$\mu(\tilde{p}) = \tilde{p}^N\lambda_+'\left(\frac{1}{\tilde{p}}\right) = (N+1)+(N-1)s_1\tilde{p}^2+(N-2)s_2\tilde{p}^3+\cdots+s_{N-1}\tilde{p}^N\,.$$

(**bbb**) From the definition (4.5),

$$c\left(\frac{\partial}{\partial b_\alpha},\frac{\partial}{\partial b_\beta},\frac{\partial}{\partial b_\gamma}\right) = \sum \operatorname*{res}_{\nu=0}\frac{k_\alpha k_\beta k_\gamma}{(p-b_\alpha)(p-b_\beta)(p-b_\gamma)}\frac{\prod_{j=1}^M(p-b_j)}{\nu(p)}dp\,.$$

This is evaluated by deforming the contour to encompass the poles at $p=\infty$ and possibly at $p=b_\alpha$ if there is repetition in the b's. The residue at infinity is zero, and so in particular $c(\partial_{b_\alpha},\partial_{b_\beta},\partial_{b_\gamma})=0$ for α,β,γ distinct.

For the case (α,α,β), the pole at $p=b_\alpha$ is simple, and the result follows immediately, noting that $\nu(b_\alpha)=k_\alpha\prod_{k\neq\alpha}(b_\alpha-b_k)$.

For the case $\alpha=\beta=\gamma$, the pole is second order, and is evaluated directly as

$$\begin{aligned}c\left(\frac{\partial}{\partial b_\alpha},\frac{\partial}{\partial b_\alpha},\frac{\partial}{\partial b_\alpha}\right) &= -\operatorname*{res}_{p=b_\alpha}\frac{k_\alpha^3}{(p-b_\alpha)^2}\frac{\prod_{k\neq\alpha}(p-b_k)}{\nu(p)}dp\,,\\ &= -k_\alpha^3\left.\frac{d}{dp}\right|_{p=b_\alpha}\frac{\prod_{k\neq\alpha}(p-b_k)}{\nu(p)}\,.\end{aligned}$$

(bbs)

$$\begin{aligned}c\left(\frac{\partial}{\partial b_\alpha},\frac{\partial}{\partial b_\beta},\frac{\partial}{\partial s_\gamma}\right) &= -\sum\operatorname*{res}_{\nu=0}\frac{k_\alpha k_\beta}{(p-b_\alpha)(p-b_\beta)}\frac{p^{N-\gamma}\prod_{j=1}^M(p-b_j)}{\nu(p)}dp\,,\\ &= \left(\operatorname*{res}_{p=\infty}+\operatorname*{res}_{p=b_\alpha}+\operatorname*{res}_{p=b_\beta}\right)\frac{k_\alpha k_\beta}{(p-b_\alpha)(p-b_\beta)}\frac{p^{N-\gamma}\prod_{j=1}^M(p-b_j)}{\nu(p)}dp\,.\end{aligned}$$

Once again there is no pole at infinity, and there exists a (simple) pole at $p=b_\alpha$ only if $\alpha=\beta$. The result again follows from $\nu(b_\alpha)=k_\alpha\prod_{j\neq\alpha}(b_\alpha-b_j)$.

(sss)

$$\begin{aligned}c\left(\frac{\partial}{\partial s_\alpha},\frac{\partial}{\partial s_\beta},\frac{\partial}{\partial s_\gamma}\right) &= \operatorname*{res}_{p=\infty}\frac{p^{3N-\alpha-\beta-\gamma}\prod_{j=1}^M(p-b_j)}{\lambda_+'(p)\prod_{j=1}^M(p-b_j)+\sum_{j=1}^M k_j\prod_{k\neq j}(p-b_k)}dp\,,\\ &= \operatorname*{res}_{p=\infty}\frac{p^{3N-\alpha-\beta-\gamma}}{\lambda_+'(p)}\left[1+\sum_{j=1}^M\frac{k_j}{\lambda_+'(p)(p-b_j)}\right]^{-1}dp\,.\end{aligned}$$

This is expanded as a Taylor series in $x = \sum k_j/\lambda'_+(p)(p-b_j)$ to give a series of terms

$$c\left(\frac{\partial}{\partial s_\alpha}, \frac{\partial}{\partial s_\beta}, \frac{\partial}{\partial s_\gamma}\right) = \sum_{i=0}^{\infty} \tilde{R}^{(i)}_{\alpha+\beta+\gamma}$$

where

$$\tilde{R}^{(i)}_\sigma = (-1)^i \operatorname*{res}_{p=\infty} \frac{p^{3N-\sigma}}{\lambda'_+(p)} \left[\frac{1}{\lambda'_+(p)} \sum_{j=1}^{M} \frac{k_j}{p-b_j}\right]^i dp\,.$$

So, in particular, $R^{(0)}_{\alpha+\beta+\gamma} := \tilde{R}^{(0)}_{\alpha+\beta+\gamma} = \operatorname*{res}_{p=\infty} \frac{\partial_{s_\alpha}\lambda_+ \partial_{s_\beta}\lambda_+ \partial_{s_\gamma}\lambda_+}{\lambda'_+} dp$ is $c_{\alpha\beta\gamma}$ from the A_N orbit space corresponding to λ_+.

$\tilde{R}^{(1)}_\sigma(s_1,\ldots,s_N,b_1,\ldots,b_M)$ can be decomposed as $\sum_{i=1}^M k_i R^{(1)}_\sigma(s_1,\ldots,s_N,b_i)$ where

$$\begin{aligned} R^{(1)}_\sigma(s_1,\ldots,s_N,b) &= -\operatorname*{res}_{p=\infty} \frac{p^{3N-\sigma}}{(p-b)(\lambda'_+(p))^2} dp\,, \\ &= \operatorname*{res}_{\tilde{p}=0} \frac{1}{(1-b\tilde{p})(\mu(\tilde{p}))^2} \tilde{p}^{\sigma-N-1} d\tilde{p}\,. \end{aligned}$$

This is seen to be zero for $\sigma \geq N+1$, and $1/(N+1)^2$ for $\sigma = N$. For $\sigma < N$ it is a pole of order $N+1-\sigma$ and can be evaluated as

$$\frac{1}{(N-\sigma)!} \left(\frac{d}{d\tilde{p}}\right)^{N-\sigma} \Bigg|_{\tilde{p}=0} \frac{1}{(1-b\tilde{p})(\mu(\tilde{p}))^2}\,. \tag{4.18}$$

Clearly this evaluates to a polynomial in $\{s_1,\ldots,s_N,b\}$. Finally, by making the substitution $p = 1/\tilde{p}$ it can be seen that $\tilde{R}^{(i)}_\sigma = 0$ for $i \geq 2$.

(bss) Proceeding as in the (sss) case, we are led to

$$c\left(\frac{\partial}{\partial b_\alpha}, \frac{\partial}{\partial s_\beta}, \frac{\partial}{\partial s_\gamma}\right) = k_\alpha \sum_{i=0}^{\infty} S^{(i)}_{\beta+\gamma}$$

where

$$\begin{aligned} S^{(i)}_\sigma &= (-1)^{i+1} \operatorname*{res}_{p=\infty} \frac{p^{2N-\sigma}}{p-b_\alpha} \frac{1}{(\lambda'_+(p))^{i+1}} \left[\sum_{j=1}^{M} \frac{k_j}{p-b_j}\right]^i dp\,, \\ &= (-1)^{i+1} \operatorname*{res}_{\tilde{p}=0} \frac{\tilde{p}^{\sigma-2N}}{\tilde{p}^{-1}-b_\alpha} \frac{1}{(\lambda'_+(\tilde{p}^{-1}))^{i+1}} \left[\sum_{j=1}^{M} \frac{k_j}{\tilde{p}^{-1}-b_j}\right]^i \left(-\frac{d\tilde{p}}{\tilde{p}^2}\right)\,, \\ &= (-1)^i \operatorname*{res}_{\tilde{p}=0} \frac{\tilde{p}^{\sigma-N-1+i(N+1)}}{(1-b_\alpha\tilde{p})(\mu(\tilde{p}))^{i+1}} \left[\frac{k_j}{1-b_j\tilde{p}}\right]^i d\tilde{p}\,. \end{aligned}$$

From this we can see that $S^{(i)}_\sigma = 0$ for $i \geq 1$. This leaves only

$$S_\sigma := S^{(0)}_\sigma = \operatorname*{res}_{\tilde{p}=0} \frac{\tilde{p}^{\sigma-N-1}}{(1-b_\alpha\tilde{p})\mu(\tilde{p})} d\tilde{p}\,,$$

which is zero for $\sigma \geq N+1$, and $1/(N+1)$ for $\sigma = N$, whilst for $\sigma \leq N-1$ it may be evaluated as

$$S_\sigma = \frac{1}{(N-\sigma)!}\left(\frac{d}{d\tilde{p}}\right)^{N-\sigma}\Bigg|_{\tilde{p}=0} \frac{1}{(1-b_\alpha\tilde{p})\mu(\tilde{p})}\,. \tag{4.19}$$

□

For the Frobenius structure on the space of polynomials

$$\lambda(p) = p^{N+1} + s_1 p^{N-1} + \ldots + s_N,$$

the variables s_i inherit a scaling symmetry from the scaling of the polynomial. Namely if $p \to \epsilon p$ and we ask $\lambda \to \epsilon^{N+1}\lambda$, then we require $s_i \to \epsilon^{i+1}s_i$. Thus we conclude s_i has degree $i+1$.

For the waterbag reduction

$$\lambda(p) = p^{N+1} + s_1 p^{N-1} + \cdots + s_N + \sum_{i=1}^{M} k_i \log(p-b_i),$$

the same degrees may be attached to the coefficients $\{s_i\}$, and to preserve homogeneity of the arguments of the logarithms, each b_i is assigned degree 1. If, in addition, a non-geometrically justified degree of $N+1$ is assigned to each k_i, then the regularised function $\lambda_{\text{reg}} = \lambda(p) - \sum k_i \log p$, introduced in (4.8), is homogeneous of degree $N+1$.

Lemma 4.2.7. *Under the rescalings*

$$\begin{aligned} s_i &\to \epsilon^{i+1}s_i \quad i = 1\ldots N\,,\\ b_i &\to \epsilon b_i \quad i = 1\ldots M\,,\\ k_i &\to \epsilon^{N+1}k_i \quad i = 1\ldots M \end{aligned}$$

the free energy F associated to the waterbag reduction(4.7) is homogeneous of degree $2N+4$.

Proof. This may be verified from the explicit expressions for the components of the tensor $c(\partial,\partial',\partial'')$ obtained in Lemma 4.2.6, remembering to add the degrees lost from differentiating along $\partial,\partial',\partial''$.

In particular, for $c(\partial_{b_\alpha},\partial_{b_\alpha},\partial_{b_\alpha}) = k_\alpha\lambda'_+(b_\alpha) + \sum_{r\neq\alpha}\frac{k_\alpha k_r}{b_\alpha - b_r}$ we note that $\lambda'_+(b_\alpha) = (N+1)(b_\alpha)^N + (N-1)s_1(b_\alpha)^{N-1} + \cdots + s_{N-1}$ has degree N.

The degrees of the polynomials $R^{(0)}_\sigma$, $R^{(1)}_\sigma$ and S_σ, when they are not zero or constant, can be determined from the differential expressions (4.18), (4.19) and the corresponding expression for $R^{(0)}_\sigma$, which is

$$R^{(0)}_\sigma = \begin{cases} 0 & \sigma \geq 2N+2 \\ -1/(N+1) & \sigma = 2N+1 \\ \dfrac{1}{(2N+1-\sigma)!}\left(\dfrac{d}{d\tilde{p}}\right)^{2N+1-\sigma}\Bigg|_{\tilde{p}=0} \dfrac{1}{\mu(\tilde{p})} & \sigma \leq 2N \end{cases} .$$

In this the degree of zero is undetermined, whilst for the middle case, the degree of a constant is 0. Integrating with respect to s_α,s_β and s_γ adds to this degree $(\alpha+1)+(\beta+1)+(\gamma+1) = \sigma+3 = 2N+4$. In the final case, if $\tilde{p} = 1/p$ is considered to have degree -1, then $\mu(\tilde{p})$ has degree zero. Thus on differentiation we obtain the quotient of two homogeneous polynomials with relative degrees $2N+1-\sigma$. Evaluation at $\tilde{p}=0$ merely makes this the ratio of constant terms, so that $R^{(0)}_\sigma$ has degree $2N+1-\sigma$. Integrating will add $\sigma+3$ to this, making $2N+4$ as required. S_σ and $R^{(1)}_\sigma$ proceed similarly. □

The degrees of the flat coordinates $\{t^i, i = 1\ldots N\}$ are inherited from the polynomial transformations rules relating them to the s_i. They can also be deduced from the Puiseaux series (4.15), in which we require both p and k to scale with degree 1.

Lemma 4.2.8. *The degree of t^i is $N+2-i$.*

Proof. Denoting by E_+ the part of the pseudo-Euler vector field (4.17) relating to the polynomial part $\lambda_+(p)$ of the superpotential $\lambda(p)$, i.e.

$$E_+ = \sum_{i=1}^{N} \frac{(1+i)}{N+1} s_i \frac{\partial}{\partial s_i} ,$$

then, from $\lambda_+(p_+(k)) = k^{N+1}$, we get both

$$\lambda'_+(p_+(k))\frac{dp_+}{dk} = (N+1)k^N$$

which gives

$$\frac{1}{N+1} k \frac{dp_+}{dk} = \frac{k^N}{\lambda'_+(p_+(k))} , \tag{4.20}$$

and

$$E_+(\lambda_+(p_+(k))) = 0 ,$$

which is equivalent to

$$E_+(\lambda_+(p))\big|_{p=p_+(k)} + \lambda'_+(p_+(k))E_+(p_+(k)) = 0 . \tag{4.21}$$

From

$$E_{+}(\lambda_{+}(p)) + \frac{1}{N+1} p \lambda'_{+}(p) = \lambda_{+}(p) \tag{4.22}$$

we have

$$-\lambda'_{+}(p_{+}(k)) E_{+}(p_{+}(k)) + \frac{1}{N+1} p_{+}(k) \lambda'_{+}(p_{+}(k)) = k^{N+1} \,.$$

Dividing by $\lambda'_{+}(p_{+}(k))$ and using (4.20) we get

$$E_{+}(p_{+}(k)) + \frac{1}{N+1} k \frac{dp_{+}}{dk} = \frac{1}{N+1} p_{+}(k) \,,$$

which allows us to read off the degrees of the t^i variables from the Puiseaux series. □

We now draw together some simple observations, which follow immediately from lemmas 4.2.2, 4.2.6 and 4.2.7.

Proposition 4.2.9. *The free energy is at most quadratic in the parameters k_i, that is, up to quadratic terms in the flat coordinates:*

$$\begin{aligned} F(t^1, \ldots, t^N, b^1, \ldots, b^M) &= F^{(0)}(t^1, \ldots, t^N) \\ &\quad + \sum_i k_i F^{(1)}(t^1, \ldots, t^N, b^i) \\ &\quad + \sum_{i \neq j} k_i k_j F^{(2)}(b^i, b^j) \end{aligned}$$

where $F^{(0)}, F^{(1)}, F^{(2)}$ are independent of the parameters k_i. $F^{(0)}$ is the free energy for the $\mathbb{C}^N/A_N$ orbit space with λ_{+} as the Landau-Ginzburg potential, and as such is a polynomial in the flat coordinates $\{t^1, \ldots, t^N\}$. $F^{(1)}$ is also a polynomial, and

$$F^{(2)}(b^i, b^j) = \frac{1}{8} (b^i - b^j)^2 \log (b^i - b^j)^2 \,.$$

In place of quasi-homogeneity we have

$$\begin{aligned} \deg\left(F^{(0)}\right) &= 2N + 4 \,, \\ \deg\left(F^{(1)}\right) &= N + 3 \,, \\ \deg\left(F^{(2)}\right) &= 2 \,, \qquad \textit{(modulo quadratic terms)} \,. \end{aligned}$$

The structure functions for the Frobenius algebra are always at most linear in the parameters k_i, that is:

$$c_{\alpha\beta}{}^{\gamma} = c^{(0)}_{\alpha\beta}{}^{\gamma} + \sum_i k_i \, c^{(i)}_{\alpha\beta}{}^{\gamma} \,.$$

where the $c^{(0)}_{\alpha\beta}{}^{\gamma}$ and $c^{(i)}_{\alpha\beta}{}^{\gamma}$ are independent of the parameters.

In fact, we can write out the multiplication in terms of the objects appearing in Lemma 4.2.6:

$$\begin{aligned}
\frac{\partial}{\partial b_i}\circ\frac{\partial}{\partial b_j} &= \frac{k_j}{b_j-b_i}\frac{\partial}{\partial b_i}+\frac{k_i}{b_i-b_j}\frac{\partial}{\partial b_j} \quad \text{for } i\neq j\,,\\
\frac{\partial}{\partial b_i}\circ\frac{\partial}{\partial b_i} &= \left\{\lambda_+'(b_i)+\sum_{j\neq i}\frac{k_j}{b_i-b_j}\right\}\frac{\partial}{\partial b_i}+\sum_{j\neq i}\frac{k_i}{b_j-b_i}\frac{\partial}{\partial b_j}\\
&\quad +\sum_{\alpha,\beta=1}^{N}\eta^{s_\alpha s_\beta}k_i b_i^{N-\beta}\frac{\partial}{\partial s_\alpha}\,,\\
\frac{\partial}{\partial b_i}\circ\frac{\partial}{\partial s_\alpha} &= b_i^{N-\alpha}\frac{\partial}{\partial b_i}+\sum_{\beta,\gamma=1}^{N}k_i\eta^{s_\beta s_\gamma}S_{\alpha+\gamma}(s,b_i)\frac{\partial}{\partial s_\beta}\,,\\
\frac{\partial}{\partial s_\alpha}\circ\frac{\partial}{\partial s_\beta} &= \sum_{i=1}^{M}S_{\alpha+\beta}(s,b_i)\frac{\partial}{\partial b_i}\\
&\quad +\sum_{\gamma,\delta=1}^{N}\eta^{s_\gamma s_\delta}\left(R^{(0)}_{\alpha+\beta+\delta}(s)+\sum_{j=1}^{M}k_jR^{(1)}_{\alpha+\beta+\delta}(s,b_j)\right)\frac{\partial}{\partial s_\gamma}\,.
\end{aligned}$$

This is not terribly illuminating since the (polynomial) quantities $\eta^{s_\gamma s_\beta}$, $R^{(0)}$, $R^{(1)}$ and S are hard to determine explicitly; however, it does highlight the dependence of the multiplication on the deformation parameters. In particular, we see from the final equation that the product of two vectors tangent to the submanifold obtained by setting each b_α to a constant is not tangent to that submanifold, even in the limit $k_1,\ldots,k_M\to 0$. ($S_N(s,b)$ is never zero for any value of b.) So the Frobenius structure on $\mathbb{C}^N/A_N$ does not appear as a natural submanifold of $\mathcal{M}^{(M,N)}$.

We are now able to demonstrate the existence of the special class of polynomial solutions mentioned in (4.1) which deform the A_N-polynomials.

Corollary 4.2.10. *For $M=1$, the free energy on the space of functions*

$$\lambda(p)=p^{N+1}+s_1p^{N-1}+\cdots+s_N+k\log(p-b)$$

is polynomial in the flat coordinates $\{t^i,b\}$ Conversely, if the free energy is polynomial in the flat coordinates then $M=1$ (or $M=0$).

Proof. This is an immediate consequence of the decomposition of F given in Proposition 4.2.9: the component $F^{(2)}$ contains all non-polynomial terms appearing in F, and is present if and only if $M\geq 2$. □

We finish this main section with two simple examples.

Example 4.2.11. $\mathrm{N}=2\,,\mathrm{M}=1\,.$ *With*

$$\lambda(p) = p^3 + t_2 p + t_1 + k\log(p-t_3)$$

one obtains the free energy

$$F = \frac{1}{6}t_1^2 t_2 - \frac{1}{216}t_2^4 - \frac{1}{2}k\,t_1 t_3^2 - \frac{1}{6}k\,(t_2^2 t_3 + t_2 t_3^3) - \frac{1}{20}k\,t_3^5\,.$$

Example 4.2.12. $\mathrm{N}=1\,,\mathrm{M}$ arbitrary. *In this case one has*

$$\lambda(p) = p^2 + t_1 + \sum_{i=1}^{M} k_i \log(p-b_i)\,.$$

With this, lemmas 4.2.2 and 4.2.6 give, on integrating, the following free energy:

$$F = -\frac{1}{12}t_1^3 + \sum_{i=1}^{M} k_i \left\{ \frac{t_1 b_i^2}{2} + \frac{b_i^4}{12} \right\} + \frac{1}{8}\sum_{i\neq j} k_i k_j (b_i - b_j)^2 \log(b_i - b_j)^2\,.$$

We note that the $z^2\log z$-type terms have appeared in the WDVV-literature before (see, for example, [37, 56]) but one normally considers these as being derived as examples of dual Frobenius manifolds [27]. Their functional form suggests the type of term that may be present in a construction of deformed solutions for other Coxeter group orbit spaces. It is also worth noting that the function $F^{(1)}(t^1,\ldots,t^N,b)$ does not depend upon the value of M, so one need only calculate the free energy in the case $M=1$ and the extension to arbitrary M is straightforward. Example 4.2.11 is given only for $M=1$ to provide an explicit example of a deformed polynomial solution.

4.3 Geometric and Algebraic Properties

In this section we study certain geometric and algebraic properties of the manifold.

4.3.1 Geometric Properties

The association of a flat pencil of metrics with a Frobenius manifold explained in Section 2.3 depended crucially on the scaling properties of the metric under the Euler vector field, since without this the intersection form defined by

$$g^{ij} = E^r c_r^{ij}$$

or

$$g(E\circ X, Y) = \eta(X,Y)$$

fails to be flat.

To understand the scaling properties of this metric we introduce an extended Lie derivative $\mathcal{L}_X^{ext}$,

$$\mathcal{L}_X^{ext} = \mathcal{L}_X + \sum_{r=1}^{M} k_r \frac{\partial}{\partial k_r}\,,$$

so, for an arbitrary tensor $\omega_{a\ldots b}^{i\ldots j}$,

$$\left(\mathcal{L}_X^{ext}\omega\right)_{a\ldots b}^{i\ldots j} = \left(\mathcal{L}_X\omega\right)_{a\ldots b}^{i\ldots j} + \sum_{r=1}^{M} k_r \frac{\partial}{\partial k_r}\omega_{a\ldots b}^{i\ldots j}\,.$$

This may be used to clarify the pseudo-quasi-homogeneity properties of the various structures with respect to the vector field (4.17), for example

$$\mathcal{L}_E^{ext}\,F = (3-d)F\,, \qquad\qquad d = \frac{N-1}{N+1}\,.$$

Similarly the metrics g and η have various pseudo-quasi-homogeneity properties:

Lemma 4.3.1. *The following equations hold:*

$$\begin{aligned} [e,E] &= e\,, \\ \mathcal{L}_E^{ext} g^{-1} &= (d-1)g^{-1}\,, \qquad \mathcal{L}_E^{ext}\eta^{-1} = (d-2)\eta^{-1}\,, \\ \mathcal{L}_e^{ext} g^{-1} &= \eta^{-1}\,, \qquad\qquad\quad \mathcal{L}_e^{ext}\eta^{-1} = 0\,. \end{aligned}$$

However, the metric g is not flat, and moreover, despite being linear in t^1 the pencil $g_\Lambda^{-1} = g^{-1} + \Lambda\eta^{-1}$ does not define an almost compatible pencil (the tensor $E\circ : T\mathcal{M} \to T\mathcal{M}$ fails to satisfy the Nijenhuis condition [17]), let alone a compatible pencil. The role of this second metric is therefore unclear. Given the origin of these structures in reductions of the dKP hierarchy one would expect bi-Hamiltonian structures of differential-geometric type. Non-local bi-Hamiltonian structures have been found recently [14]; however, these do not become the standard structures in the limit as $k_i \to 0$. Another possibility for the metric g is

$$g = \sum \frac{\eta_{ii}}{u_i} du_i^2\,,$$

where η_{ii} are the components of the metric η in canonical coordinates, given in (4.16). This choice for g coincides with the previous case when one is dealing with a true Frobenius manifold. It does define a non-local bi-Hamiltonian structure [67] but finding its form in the flat-coordinate system for the metric η is problematic. A related problem is to relate the Euler vector field (4.2) with the vector field

$$\mathcal{E} = \sum_{i=1}^{M+N} u^i \frac{\partial}{\partial u^i} \,,$$

the two being equal in the undeformed case. The properties of this vector field in the pure A_n case follow from establishing that it satisfies (4.22); this proof depends crucially on that fact that λ_+ is polynomial (although it extends easily to the rational case), not just that its derivatives are. For the waterbag models, $\mathcal{E}$ does not satisfy conditions 4(a) and 4(b) of a Frobenius manifold, namely $\nabla\nabla\mathcal{E} \neq 0$ and $\mathcal{L}_{\mathcal{E}}\eta \neq D\eta$.

As remarked in Section 2.2, the flatness of the deformed connection

$$\nabla^z_X Y = \nabla_X Y + zX \circ Y$$

does not depend on the presence of the Euler vector field, but only on the associativity and commutativity of $\circ$, and the potentiality condition on ∇c. Hence this structure is present on the manifold $\mathcal{M}$ (and on all subsequent examples of waterbag superpotentials).

This means that one can proceed to the solve the recurrence relations (2.15) to find the flat coordinates of ∇^z, and so find a family of commuting Hamiltonian densities for hydrodynamic type Poisson bracket defined by η. This reverses the approach taken by Chang in [13], in which free energy was reconstructed by analysing the recurrence relations satisfied by the conserved quantites of the two-dimensional reduction

$$\lambda(p) = p - c\frac{\log(p-p_1)}{\log(p-\tilde{p}_1)}$$

of the dispersionless KP hierarchy, and which was the original motivation for the work in this chapter.

4.3.2 Algebraic Deformation Theory

In this section we examine the linearity of the structure functions of the Frobenius algebra with respect to the parameters k^i from the point of view of deformation theory (we follow the notation of [12]). Let

$$M^k(V) = \{m : \underbrace{V \times \ldots \times V}_{k} \to V | m \text{ linear in each arguement}\}$$

Recall that a bilinear map $c \in M^2(V)$ defines an associative structure if and only if

$$[c,c]_{\mathcal{G}} = 0 \,,$$

where $[\cdot,\cdot]_{\mathcal{G}}$ is the Gerstenhaber bracket. Owing to the super-Jacobi identity one has $\delta_c^2 = 0$, where

$$\delta_c = [c,\cdot]_{\mathcal{G}} \quad : M^{\bullet}(V) \to M^{\bullet+1}(V)$$

and this gives rise to the Hochschild complex of (V,c).

From proposition 4.2.9 we have the following structure

$$c(k) = c^{(0)} + \sum_i k_i c^{(i)}\,,$$

that is, linearity of the structure functions of the associative algebra. Decomposing the condition $[c(k),c(k)]_{\mathcal{G}} = 0$ for all k one obtains the following conditions:

$$\begin{aligned} [c^{(0)},c^{(0)}]_{\mathcal{G}} &= 0\,, \\ [c^{(0)},c^{(i)}]_{\mathcal{G}} &= 0\,, \quad i = 1\,,\ldots\,,M\,, \\ [c^{(i)},c^{(j)}]_{\mathcal{G}} &= 0\,, \quad i,j = 1\,,\ldots\,,M\,. \end{aligned}$$

Thus each $c^{(i)}$, $i = 0\,,1\,,\ldots\,,M$ separately defines and associative structure on $T\mathcal{M}$. Each of these define a map $\delta_{c^{(i)}}$ and each $c^{(i)}$ is a cocycle with respect to each cohomology map $\delta_{c^{(j)}}$, that is:

$$\begin{aligned} [c^{(i)},c^{(i)}]_{\mathcal{G}} &= 0\,, \quad i = 0\,,1\,,\ldots\,,M\,, \\ \delta_{c^{(i)}} c^{(j)} &= 0\,, \quad i,j = 0\,,1\,,\ldots\,,M\,. \end{aligned}$$

It is also interesting to note that the pair $(\circ,E)$ satisfies the conditions

$$\mathcal{L}_{X\circ Y}(\circ) = X \circ \mathcal{L}_Y(\circ) + Y \circ \mathcal{L}_X(\circ)$$

and

$$\mathcal{L}_E^{ext}(\circ) = d\,\circ\,,$$

the former following from the semi-simplicity of the multiplication, and the latter from the pseudo-scaling property of the free energy. If one had $\mathcal{L}_E(\circ) = d\,\circ$ then one would have a F-manifold [47]. Here one has a modified version, where the scaling condition is replaced by the pseudo-scaling condition. One could also regard the multiplication as defining a deformation of the F-manifold based on the orbit space $\mathbb{C}^N/A_N$.

4.4 Further Results

In this section we consider generalisations of the results of Section 4.2 to waterbag deformations of other Frobenius manifolds which are specified in terms of a superpotential. In

Section 4.4.1 waterbag deformations of a rational superpotential are considered, and in Section 4.4.2 deformations of the superpotentials associated with the B_n Coxeter groups are consider. In Section 4.4.3 we consider deformations of rational superpotentials symmetric under $p \mapsto -p$, which include deformations of the D_n superpotentials.

4.4.1 Rational Water-bag Potentials

As remarked in Section 4.2, the calculations required to determine the properties of the Frobenius structure on the superpotential

$$\lambda(p) = p^{N+1} + s_1 p^{N-1} + \cdots + s_N + \sum_{i=1}^{M} k_i \log(p - b_i)$$

can be performed since they involve only the derivatives of this function, and thus one deals at worst with rational functions. As such, one may easily generalise the results of that section to include the more general case

$$\lambda(p) = (\text{rational function})(p) + \sum_{i=1}^{M} k_i \log(p - b_i)\,.$$

Proposition 4.4.1. *On the space of functions*

$$\begin{aligned}\lambda(p) \;&=\; p^{N+1} + s_1 p^{N-1} + \cdots + s_N \\ &\quad + \sum_{i=1}^{K} \left[\frac{v_{i,1}}{(p - v_{i,0})} + \cdots + \frac{v_{i,L_i}}{(p - v_{i,0})^{L_i}} \right] \\ &\quad + \sum_{i=1}^{M} k_i \log(p - b_i)\,, \end{aligned} \tag{4.23}$$

the formulas (4.4) and (4.5) define a solution of the WDVV equations.

Proof. Canonical coordinates are found as in Lemma 4.2.1. The Egorov potential is, as before, $-s_1/(N+1)$.

The flat coordinates are $\{b_1, \ldots, b_M\}$ together with those obtained for the purely rational case [2]; for, if one inverts $\lambda_+(p) = p^{N+1} + s_1 p^{N-1} + \cdots + s_N$ about $p = \infty$ using the Puiseaux series (4.15), and inverts

$$\lambda_{-i}(p) = \frac{v_{i,1}}{(p - v_{i,0})} + \cdots + \frac{v_{L_i}}{(p - v_{i,0})^{L_i}}$$

for $p \sim v_{i,0}$ as

$$p = \frac{1}{L_i} \left(x_{i,L_i+1} + \frac{x_{i,L_i}}{w} + \cdots + \frac{x_{i,1}}{w^{L_i}} \right) + O\left(\frac{1}{w^{L_i+1}} \right)\,,$$

where $\lambda_{-i} = w^{L_i}$, and $x_{i,L_i+1} = L_i v_{i,0}$, then statements analogous to (4.13) and (4.14) may be demonstrated. The flat coordinates are then $\{t^\alpha, x_{\beta,\gamma}, b_\delta\}$. In these coordinates the metric has only the following non-zero components:

$$\eta\left(\frac{\partial}{\partial t^\alpha}, \frac{\partial}{\partial t^\beta}\right) = -\frac{1}{N+1}\delta_{\alpha+\beta,N+1}\,,$$

$$\eta\left(\frac{\partial}{\partial x_{i,j}}, \frac{\partial}{\partial x_{i,k}}\right) = -\frac{1}{L_i}\delta_{j+k,L_i+2}\,,$$

$$\eta\left(\frac{\partial}{\partial b_\alpha}, \frac{\partial}{\partial b_\beta}\right) = k_\alpha \delta_{\alpha\beta}\,.$$

□

In the above proposition the location of the poles $\{s_i\}$ and the logarithmic poles $\{b^i\}$ were taken to be distinct. However, a modified version of the above proposition may be formulated which takes into account possible coincidences in these sets. Rather than state this we give an example.

Example 4.4.2. *The superpotential*

$$\lambda(p) = p^2 + t_1 + \frac{t_2}{(p-t_3)} + k\log(p-t_3)$$

leads to the following solution of the WDVV equation

$$F = \frac{1}{12}t_1^3 + t_1 t_2 t_3 - \frac{1}{2}k\, t_1 t_3^2 - \frac{3}{4}t_2^2 + \frac{1}{2}t_2^2\log t_2 + \frac{1}{3}t_2 t_3^3 - \frac{1}{12}k\, t_3^4\,. \tag{4.24}$$

This produces an interesting class of solutions, as no extra variables have had to be introduced, so in a sense they are true deformations of the original solution. The single pole case - generalisations of the above example - are isomorphic to deformations of the extended-affine-Weyl orbit space [30], since

$$H_{0,N+L+1}(N+1,L) \cong \mathbb{C}^{N+L+1}/\widetilde{W}^{(L)}(A_{N+L})\,.$$

Explicitly this is given by a Legendre transformation (which acts on solutions of the WDVV equations, not just to those solutions which define Frobenius manifolds).

Example 4.4.3. *Applying the Legendre transformation S_2 (using the notation of [25]) to the solution (4.24) yields the solution*

$$\hat{F} = \frac{1}{4}\hat{t}_1 + \frac{1}{2}\hat{t}_2^2\hat{t}_3 - \frac{1}{2}k\,\hat{t}_2\hat{t}_3^2 - \frac{1}{96}\hat{t}_1^4 + \hat{t}_1 e^{\hat{t}_3} - k\left(\frac{1}{4}\hat{t}_1^2\hat{t}_3 + \frac{1}{2}\hat{t}_2\hat{t}_3^2\right) + \frac{1}{6}k^2\hat{t}_3^3\,.$$

This defines a deformation of the extended-affine-Weyl space $\mathbb{C}^3/\widetilde{W}^1(A_2)$.

One would expect that the associated dispersionless integrable systems would be related to waterbag type-reductions of the dispersionless Toda equations and their generalisations [10, 15].

4.4.2 B_n-type Reductions

The results of Section 4.2 raise the question of whether or not the ideas may be applied to the Frobenius manifold structures on the space of orbits of Coxeter groups other than A_n. By this we mean is there a free energy schematically of the form

$$F(\mathbf{t}, \mathbf{b}) = F_W(\mathbf{t}) + kF^{(1)}(\mathbf{t}, \mathbf{b}) + k^2 F^{(2)}(\mathbf{t}, \mathbf{b})$$

based on the $\mathbb{C}^n/W$ free energy F_W which is pseudo-quasi-homogeneous with respect to some suitable Euler field?

For the group $W = B_n$ this is immediate, using the idea originally due to Zuber [74], of embedding the group B_n as a subgroup of A_{2n+1} , or geometrically, of regarding the B_n Frobenius manifold as the induced manifold on certain hyperplanes submanifolds in the A_{2n+1} Frobenius manifold.

Proposition 4.4.4. *On the space of functions*

$$\lambda(p) = p^{2N+2} + s_1 p^{2N} + s_3 p^{2N-2} + \cdots + s_{2N+1} + \sum_{i=1}^{M} k_i \log(p^2 - b_i^2) \tag{4.25}$$

the formulas (4.4) and (4.5) define a pseudo-quasi-homogeneous solution of the WDVV equations.

Proof. The function λ above is obtained from the following waterbag deformation of the A_{2N+1} superpotential:

$$\begin{aligned} \lambda_{A_N}(p) &= p^{2N+2} + s_1 p^{2N} + s_2 p^{2N-1} + s_3 p^{2N-2} + \cdots + s_{2N+1} \\ &\quad + \sum_{i=1}^{M} k_i \log(p - b_i) + \sum_{i=1}^{M} k_i \log(p - b_{i+M}) \,. \end{aligned}$$

We restrict this to the submanifold

$$\begin{aligned} s_r &= 0 \text{ for } r \text{ even}, \\ b_{i+M} &= -b_i \text{ for } 1 \leq i \leq M \,. \end{aligned}$$

The restriction of the s_r may be achieved in flat coordinates by setting all t^i of odd degree (i.e. even i) to zero. We introduce new flat coordinates $\tilde{b}_i = b_i$ and $\tilde{d}_i = b_i + b_{i+M}$

$(i = 1, \ldots, M)$, and restrict to $\tilde{d}_i = 0$. According to Zuber [74], this will provide us with a Frobenius manifold provided that the multiplication of any two vectors tangent to this submanifold is also tangent to it. That is, we must check that the following components of the multiplication tensor restrict to zero on the hyperplane $t^{2i+1} = 0$, $\tilde{d}^i = 0$:

$$\begin{array}{rr} c^{\tilde{d}_k}_{\tilde{b}_i\tilde{b}_j}\,, & c^{t^r}_{\tilde{b}_i\tilde{b}_j} \text{ for } r \text{ even,} \\ c^{\tilde{d}_k}_{\tilde{b}_i t^r} \text{ for } r \text{ odd,} & c^{t^s}_{\tilde{b}_i t^r} \text{ for } r \text{ odd, } s \text{ even,} \\ c^{\tilde{d}_k}_{t^r t^s} \text{ for } r, s \text{ odd,} & c^{t^u}_{t^r t^s} \text{ for } r, s \text{ odd, } u \text{ even.} \end{array}$$

Polynomial terms arising in these components can be seen to vanish from consideration of their degrees; all polynomials in $\{t^1, \ldots, t^{2N+1}\}$ of odd degree must vanish when all t^i of odd degree vanish, whereas polynomials in $\{t^i\}$ of even degree are always multiplied by (at least) a factor of $b_i + b_{i+M}$ for some i, and hence vanish on $d_i = 0$. Non-polynomial terms are given explicitly in Lemma 4.2.6. □

Remark: For the B_n waterbag models, we do not obtain any new polynomial solutions. This is because we must begin with an A_n waterbag superpotential with $M \geq 2$, and thus every B_n deformation will at least contain a term $k_1^2 b_1^2 \log b_1^2$ in the free energy.

4.4.3 Rational Waterbag Superpotentials with $\mathbb{Z}_2$ Symmetry

In this section we combine the results of Sections 4.4.1 and 4.4.2, and consider waterbag deformations of rational superpotentials which are symmetric under the involution $p \mapsto -p$. As a special case, we shall obtain deformations of the Frobenius structure on the orbit spaces of the D_n Coxeter groups.

The superpotential we consider here is

$$\lambda(p) = \lambda_+(p) + \lambda_-(p) + \sum_{r=1}^{L} \lambda_r(p) + \sum_{s=1}^{M} k_s \log(p^2 - b_s^2) \tag{4.26}$$

where

$$\begin{aligned} \lambda_+(p) &= p^{2N} + \sum_{i=1}^{N} s_i p^{2(N-i)}\,, \\ \lambda_-(p) &= \sum_{i=1}^{K_0} \frac{\nu_{0,i}}{p^{2i}}\,, \\ \lambda_r &= \sum_{i=1}^{K_r} \frac{\nu_{r,i}}{(p^2 - \nu_{r,0}^2)^i}\,. \end{aligned}$$

The fixed pole at $p = 0$ is included to make some calculations simpler (specifically it causes the roots of λ' to take the form $\pm q_i$ rather than have a fixed root at 0, which places all canonical coordinates on an equal footing), and because it is this pole which is present in D_n superpotentials.

Theorem 4.4.5. *On the space of functions (4.26), the formulas (4.4) and (4.5) define a solution of the WDVV equations.*

Proof. Coordinates on the space of superpotentials (4.26) are

$$\{s_1, \ldots, s_N, \nu_{0,1}, \ldots, \nu_{0,K_0}, b_1, \ldots, b_M\} \cup \bigcup_{r=1}^{L} \{\nu_{r,0}, \ldots, \nu_{r,K_r}\}.$$

For neatness of formulas, we set $A = N + L + M + \sum_{r=0}^{L} K_r$ to be the dimension of this space, and define $\mathcal{A} = \{1, \ldots, A\}$, and, for each $i \in \mathcal{A}$, $\mathcal{A}_i = \mathcal{A} - \{i\}$. We have

$$\lambda'(p) = \frac{2N \prod_{i\in\mathcal{A}}(p^2 - q_i^2)}{p^{2K_0+1} \prod_{r=1}^{L}(p^2 - \nu_{r,0}^2)^{K_r+1} \prod_{s=1}^{M}(p^2 - b_s^2)}$$

and

$$\lambda''(q_i) = \frac{4N \prod_{r\in\mathcal{A}_i}(q_i^2 - q_r^2)}{q^{2K_0+1} \prod_{r=1}^{L}(q_i^2 - \nu_{r,0}^2)^{K_r+1} \prod_{s=1}^{M}(q_i^2 - b_s^2)}.$$

Canonical coordinates are defined in the usual way as

$$u_i = \lambda(q_i) = \lambda(-q_i),$$

from which the formula

$$\frac{\partial \lambda}{\partial u_i} = \frac{2p}{p^2 - q_i^2} \frac{\lambda'(p)}{\lambda''(q_i)}$$

may derived, which is the same as the standard functional form for rational superpotentials invariant under $p \mapsto -p$. As such we have

$$\eta\left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_i}\right) = -\frac{2}{\lambda''(q_i)}$$

and

$$c\left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_i}\right) = -\frac{2}{\lambda''(q_i)},$$

with other components being zero. A similar argument to that used to prove Lemma 4.2.3 shows that

$$\eta\left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_i}\right) = \frac{\partial}{\partial u_i}\left(-\frac{s_1}{2N}\right),$$

and hence that the metric is Egorov.

As in the generic rational case, the metric splits into block diagonal form, with one block associated to each pole of λ according to the decomposition of coordinates:

$$\begin{aligned} \mathcal{S}_+ &= \{s_1, \ldots, s_N\}\,, \\ \mathcal{S}_- &= \{\nu_{0,1}, \ldots, \nu_{0,K_0}\}\,, \\ \mathcal{S}_i &= \{\nu_{i,0}, \ldots, \nu_{i,K_i}\} \quad \text{for } i = 1, \ldots, L\,, \\ \text{and} \quad \tilde{\mathcal{S}}_i &= \{b_i\} \quad \text{for } i = 1, \ldots, M\,. \end{aligned}$$

Each block is replaced by flat coordinates defined by a relevant inverse series close to the pole.

Flat coordinates replacing $\mathcal{S}_+$ are $\{t^0, \ldots, t^{N-1}\}$ obtained from inverting $\lambda_+ = k^{2N}$ for $p \sim \infty$ as

$$p = k - \frac{1}{2N} \sum_{r=0}^{\infty} \frac{t^{N-r}}{k^{2r+1}}\,.$$

Flat coordinates replacing $\mathcal{S}_-$ are $\{w_{0,1}, \ldots, w_{0,K_0}\}$ obtained from inverting $\lambda_- = m^{2K_0}$ for $p \sim 0$ as

$$p = \frac{1}{2K_0} \sum_{i=1}^{\infty} \frac{w_{0,i}}{m^{2i-1}}\,,$$

whilst those replacing $\mathcal{S}_i$ are $\{w_{i,0}, \ldots, w_{i,K_i}\}$ obtained from inverting $\lambda_i = m^{2K_i}$ for $p \sim \nu_{i,0}$ as

$$p = \frac{1}{2K_i} \sum_{j=0}^{\infty} \frac{w_{i,j}}{m^{2j}}\,, \tag{4.27}$$

where $w_{t,0} = 2K_t \nu_{i,0}$. The coordinates b_i are already flat coordinates.

In these coordinates the non-zero components of the metric are

$$\begin{aligned} \eta\left(\frac{\partial}{\partial t^\alpha}, \frac{\partial}{\partial t^\beta}\right) &= -\frac{1}{2N}\delta_{\alpha+\beta,N+1}\,, \\ \eta\left(\frac{\partial}{\partial w_{0,i}}, \frac{\partial}{\partial w_{0,j}}\right) &= -\frac{1}{2K_0}\delta_{i+j,K_0}\,, \\ \eta\left(\frac{\partial}{\partial w_{r,i}}, \frac{\partial}{\partial w_{r,j}}\right) &= -\frac{1}{2K_r}\delta_{i+j,K_r}\,, \\ \text{and} \quad \eta\left(\frac{\partial}{\partial b_i}, \frac{\partial}{\partial b_i}\right) &= 2k_i\,. \end{aligned}$$

□

The Frobenius structure given by the construction of Section 2.4 on the orbit space of the Coxeter group D_N can also be derived from the superpotential

$$\lambda_D(p) = p^{2(N-1)} + \sum_{r=0}^{N-2} a_r p^{2r} - \frac{1}{2}\frac{a_{-1}^2}{p^2}\,, \tag{4.28}$$

as is studied in [33].

Waterbag deformations of (4.28) are obtained in (4.26) by setting $L = 0$ and $K_0 = 1$.

Proposition 4.4.6. *On the space of functions*

$$\lambda(p) = p^{2(N-1)} + \sum_{r=0}^{N-2} s_r p^{2r} - \frac{1}{2}\frac{s_{-1}^2}{p^2} + \sum_{s=1}^{M} k_s \log(p^2 - b_s^2)\,. \tag{4.29}$$

the formulas (4.4) and (4.5) define a pseudo-quasi-homogeneous solution of the WDVV equations which deforms the free energy of the D_N model.

This result was also obtained independently in [75].

As with the B_n waterbag models, this deformation does not produce any new polynomial solutions to the equations of associativity, for in particular we have

$$c\left(\frac{\partial}{\partial b_\alpha}, \frac{\partial}{\partial b_\beta}, \frac{\partial}{\partial b_\gamma}\right) = \begin{cases} 0 & \alpha, \beta, \gamma \text{ distinct} \\ -4k_\alpha k_\beta \dfrac{b_\beta}{b_\alpha^2 - b_\beta^2} & \alpha = \gamma \neq \beta \\ 2k_\alpha \lambda_D'(b_\alpha) + 2\displaystyle\sum_{s\neq\alpha} k_\alpha k_s \frac{2b_\alpha}{b_\alpha^2 - b_s^2} + 4\frac{k_\alpha^2}{b_\alpha} & \alpha = \beta = \gamma \end{cases},$$

which shows that the free energy contains a logarithmic term which is quadratic in the deformation parameters even for $M = 1$, and also

$$c\left(\frac{\partial}{\partial a_{-1}}, \frac{\partial}{\partial a_{-1}}, \frac{\partial}{\partial b_\alpha}\right) = \frac{2k_\alpha}{b_\alpha}\,.$$

which shows that the term linear in the deformation parameters is not polynomial.

Thus the waterbag deformation can be extended to all classical Coxeter group orbit spaces. It would be of interest to see if these ideas can be applied to an arbitrary Coxeter group orbit space.

4.5 Conclusions

In this chapter we have constructed pseudo-quasi-homogeneous deformations of the WDVV solutions associated to the classical Coxeter groups A_N, B_N and D_N. This was done by introducing a corresponding deformation of the related superpotentials. It would be of interest to see if this deformation can be applied independently of the Landau-Ginzburg construction. That is: is it possible to see the origin of these deformations via a modified Saito-type construction. The absence of a flat ‘intersection form’ would seem problematical. A related question is whether one can formulate axiomatically a theory of pseudo-quasi-homogeneous solutions of the WDVV equations.

The space of rational functions which the rational waterbag superpotentials of Section 4.4.1 deformed may be viewed as the genus zero Hurwitz space $H_{0;N,L_1-1,L_2-1,\ldots,L_K-1}$, i.e. the space of branched coverings of the Riemann sphere, with poles of order $N+1, L_1, \ldots, L_K$. This suggests that one may be able to consider deformations of the WDVV solutions associated with an arbitrary Hurwitz space. The calculations in Section 4.4.1 were able to proceed since they only relied on that fact that the derivatives of the superpotential were meromorphic, not that the superpotential itself was. This suggests that one should look at generalisations where λ lies in some extension of the field of meromorphic functions.

All of the Frobenius structures found were semi-simple, which means there exist interesting submanifolds: discriminants and caustics [67]. What are the properties of such structures in the present case?

Although there is no obvious candidate for the intersection form, we still have the flatness of the Dubrovin connection ∇^{λ}, which means that the integrable systems one obtains as reductions of the dKP hierarchy can still be derived from the geometry of the Frobenius structure. However, the (non-local) bi-Hamiltonian structure [15], especially in the flat coordinates system for the metric η, is unknown in general, as is precisely how that passage from undeformed to deformed geometry is mirrored in the integrable systems [13]. Some attempt is made to address this point in Chapter 5.

Finally, does any of this have an interpretation in terms of topological quantum field theory or Gromov-Witten theory? There appears to be some connection with the integrable hierarchies found by Milanov and Tseng in their study of the orbifold cohomology of the projective line [57].

Chapter 5

Integrable Hierarchies from Waterbag Models

In this section we study examples of integrable systems associated to some low-dimensional waterbag models via the deformed flat connections and hydrodynamic type Poisson brackets as discussed in Sections 2.2 and 4.3.1.

The motivation for this work is to understand to what extent the construction of a bi-Hamiltonian structure on the loop space of a Frobenius manifold from section 2.3 goes through for these pseudo-quasi-homogeneous structures. It is reasonable to hope such a structure exists since the deformed flat connection demonstrates the integrability of the hierarchy; however, the quasi-homogeneity property is crucial to the flatness of the intersection form defined by (2.25) and it is not clear to what extent pseudo-quasi-homogeneity preserves this. This has already been alluded to in Section 4.3.1; here we investigate it in detail for two specific cases in the hope that this analysis may shed some light onto what happens in the general case.

In [14] Chang investigated the bi-Hamiltonian structure associated to the following reduction of the dispersionless KP hierarchy:

$$\lambda(p) = p - \sum_{i=1}^{M} k_i \log(p - b^i)\,. \tag{5.1}$$

For this superpotential, the free energy may be written explicitly as

$$F = -\sum_{i=1}^{M} k_i (b^i)^3 + \frac{1}{8}\sum_{i\neq j} k_i k_j (b^i - b^j)^2 \log(b^i - b^j)^2\,,$$

and thus one may write down the Frobenius structure. By introducing the objects

$$W^i_j = \eta^{ir}\frac{\partial^2 F}{\partial b^r \partial b^j}$$

where $\eta = \sum_{r=1}^{M} k_r db^r \otimes db^r$, Chang exhibited the Hamiltonian operators

$$\begin{aligned} P_1^{ij} &= \eta^{ij}\frac{d}{dx}\,, \\ P_2^{ij} &= \sum_{r,s=1}^{M} (W^i_r)_x \left(\frac{d}{dx}\right)^{-1} (W^j_s)_x\,, \end{aligned}$$

as a bi-Hamiltonian structure for the associated integrable hierarchy.

From the point of view taken in the previous chapter -of regarding waterbag models as deforming Frobenius manifolds- this is unsatisfactory. Firstly, P_2 is a purely non-local operator, and, as such, does not deform any hydrodynamic type Poisson bracket. Secondly, Chang's superpotential (5.1) does not deform the superpotential of an existing Frobenius manifold, as do (4.7), (4.25), (4.23), and (4.26), but rather deforms the zero-dimensional Frobenius manifold given by $\lambda(p) = p$. Thus it is impossible to tell from it what structures may be preserved in P_2 from the undeformed case.

We work with two examples of two-component systems, which deform the dispersionless Toda lattice and the dispersionless KdV equation respectively, and attempt to derive their Hamiltonian structures by finding their Riemann invariants and using a formula of Tsarev [68], which says that the metric generating a hydrodynamic type Poisson bracket for a diagonal system

$$u^i_t = v_i(u^1, \ldots, u^n)u^i_x \quad \text{(no summation)}$$

must be diagonal, so $g = \sum_{i=1}^{n} g_{ii} du^i \otimes du^i$, and satisfy

$$\frac{\partial_i v_k}{v_i - v_k} = \frac{1}{2}\partial_i \log g_{kk}\,.$$

5.1 Deformation of the Long Wave System

First we consider,similarly to Example 4.4.2, the superpotential

$$\lambda(p) = p + \frac{t_2}{p - t_1} + k\log(p - t_1)\,, \tag{5.2}$$

in which the poles of the rational function and the logarithmic term coincide. This deforms the Frobenius manifold studied in Examples 2.1.4, 2.2.3 and 2.3.3, without the addition of any new coordinates. (There is a change of sign in the Free energy in this section compared to these examples, which means there are also sign changes in η_{ij} and c_{ijk}, but not c^k_{ij}.)

The metric and multiplication are given by

$$\eta_{..} = \begin{pmatrix} k & -1 \\ -1 & 0 \end{pmatrix} \tag{5.3}$$

and

$$\begin{aligned} \frac{\partial}{\partial t_1} \circ \frac{\partial}{\partial t_1} &= \frac{\partial}{\partial t_1}\,, \\ \frac{\partial}{\partial t_1} \circ \frac{\partial}{\partial t_2} &= \frac{\partial}{\partial t_2}\,, \\ \frac{\partial}{\partial t_2} \circ \frac{\partial}{\partial t_2} &= \frac{1}{t_2}\frac{\partial}{\partial t_1} + \frac{k}{t_2}\frac{\partial}{\partial t_2}\,. \end{aligned}$$

This gives the free energy

$$F = -\frac{1}{2}t_1^2 t_2 - \frac{1}{4}t_2^2 \log t_2^2 + \frac{1}{6}k t_1^3\,,$$

which is pseudo-quasi-homogeneous with respect to the Euler vector field

$$E = t_1 \frac{\partial}{\partial t_1} + 2t_2 \frac{\partial}{\partial t_2}\,.$$

5.1.1 Associated Hierarchy

The function $t_2 - kt_1$ is a flat coordinate for (5.3). If we deform this to a flat coordinate $\phi^{t_2-kt_1}$ of the deformed flat connection (2.11) we obtain

$$\begin{aligned} \phi^{t_2-kt_1}(t_1,t_2,z) &= \sum_{r\geq 0} z^r \phi_r^{t_2-kt_1}(t_1,t_2) \\ &= t_2 - kt_1 \\ &\quad +z\left\{t_1 t_2 - \frac{1}{2}k t_1^2\right\} \\ &\quad +z^2\left\{\frac{1}{2}t_1 t_2^2 + \frac{1}{2}t_2^2 - \frac{1}{6}k t_1^3\right\} \\ &\quad +O(z^3)\,. \end{aligned}$$

It can be shown that all of the terms $\phi_r^{t_2-kt_1}$ in this expansion are polynomial in t_1, t_2 and k.

Using $\phi_1^{t_2-kt_1}$ as the Hamiltonian density we get the flow

$$\frac{d}{d\tau_1}\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = -\begin{pmatrix} t_{1,x} \\ t_{2,x} \end{pmatrix}$$

describing spatial translations, whilst for $\phi_2^{t_2-kt_1}$ we get

$$\frac{d}{d\tau_2}\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = -\begin{pmatrix} t_1 t_{1,x} + t_{2,x} \\ (t_1 t_2)_x + k t_{2,x} \end{pmatrix},$$

which deforms the one-dimensional long wave system (2.17).[1]

One also obtains a chain of conserved quantities from the flat coordinate ϕ^{t_2} of ∇^z which deforms t_2 as

$$\begin{aligned}\phi^{t_2} = \; & t_2 \\ & + z\left\{t_1 t_2 + k(t_2 \log t_2 - t_2)\right\} \\ & + z^2\left\{\frac{1}{2}t_2^2 + \frac{1}{2}t_1^2 t_2 + k(t_1 t_2 \log t_2 - t_1 t_2) + \frac{1}{2}k^2(t_2(\log t_2)^2 - 2t_2 \log t_2 + 2t_2)\right\} \\ & + O(z^3)\,.\end{aligned}$$

One may apply the Legendre transformation S_2 to obtain a deformation of the Frobenius manifold of Example 2.1.8. This can also be obtained from the formulas (4.4) and (4.5) by making the substitution $p \mapsto \hat{p}$ in the superpotential (5.2), where $\hat{p}$ is defined by

$$d\hat{p} = \frac{\partial}{\partial t_2}\left(\lambda(p)dp\right),$$

giving

$$\hat{\lambda}(\hat{p}) = e^{\hat{p}} + \nu + e^{-\hat{p}+b} + c(\hat{p} - b)\,,$$

where we write $b = \hat{t}^1$ and $\nu = \hat{t}^2$ for clarity in formulas.

The metric $\hat{\eta}$ is the same as (5.3), while the multiplication is given by

$$\begin{aligned}\frac{\partial}{\partial b} \circ \frac{\partial}{\partial b} &= e^b \frac{\partial}{\partial \nu} - k\frac{\partial}{\partial b}\,, \\ \frac{\partial}{\partial \nu} \circ \frac{\partial}{\partial b} &= \frac{\partial}{\partial b}\,, \\ \frac{\partial}{\partial \nu} \circ \frac{\partial}{\partial \nu} &= \frac{\partial}{\partial \nu}\,.\end{aligned}$$

[1] It is perhaps interesting to note that in [52] Kupershmidt studied a similar, but apparently distinct, deformation of the long wave system which was also Hamiltonian with respect to the hydrodynamic type Poisson bracket given by (5.3).

The flat coordinate for ∇^z corresponding to ν is

$$\begin{aligned}\phi^{(\nu)} &= \nu \\ &+z\left\{\frac{1}{2}\nu^2+e^b\right\} \\ &+z^2\left\{\frac{1}{6}\nu^3+e^b\nu-ke^b\right\} \\ &+z^3\left\{\frac{1}{24}\nu^4+\frac{1}{2}\nu^2e^b-k\nu e^b+\frac{1}{4}e^{2b}+k^2e^b\right\} \\ &+O(z^4)\,,\end{aligned}$$

from which we obtain the flow

$$\frac{d}{d\tau_{\nu,1}}\begin{pmatrix} b \\ \nu \end{pmatrix} = -\begin{pmatrix} \nu_x \\ b_xe^b+k\nu_x \end{pmatrix} \tag{5.4}$$

corresponding to (2.19). Writing this as a single second-order equation, we obtain the following deformation of the continuous Toda lattice ($t=\tau_{\nu,1}$):

$$b_{tt}+kb_{xt}=(e^b)_{xx}\,. \tag{5.5}$$

At this stage one may follow the pattern of [34] and introduce new variables u and v by

$$\begin{aligned} e^b &= uv\,, \\ \nu &= u+v\,, \end{aligned}$$

which allows us to write (5.4) as

$$\begin{aligned} u_t &= -uv_x-ku\frac{u_x+v_x}{u-v}\,, \\ v_t &= -vu_x-kv\frac{v_x+u_x}{v-u}\,, \end{aligned}$$

or

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} -k\dfrac{u}{u-v} & -u-k\dfrac{u}{u-v} \\ -v-k\dfrac{v}{v-u} & -k\dfrac{v}{v-u} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}_x . \tag{5.6}$$

5.1.2 Associated Metrics

In this section we consider the system (5.6) in terms of Riemann invariants. This will allow us to demonstrate that, while there is a flat metric associated to the system which deforms the first flat metric of the continuous Toda lattice, there is none which deforms the intersection form.

The eigenvalues of the matrix in the right-hand side of (5.6) are

$$\lambda_{\pm} = -\frac{1}{2}k \pm \frac{1}{2}\sqrt{k^2+4uv}$$

so the Riemann invariants must satisfy

$$\frac{\partial R^{\pm}}{\partial t} = \left(-\frac{1}{2}k \pm \frac{1}{2}\sqrt{k^2+4uv}\right)\frac{\partial R^{\pm}}{\partial x}\,.$$

Hence we find

$$R^{\pm} = \phi^{\pm}\left(u+v \mp \sqrt{k^2+4uv} - k\log(\sqrt{k^2+4uv} \mp k)\right)\,,$$

for some arbitrary functions $\phi^{\pm}$. We make the choice $\phi^{\pm} =$ identity, so

$$R^{\pm} = u+v \mp \sqrt{k^2+4uv} - k\log(\sqrt{k^2+4uv} \mp k)\,. \tag{5.7}$$

The Jacobian of this transformation is

$$\begin{aligned} J &= \begin{pmatrix} \dfrac{\partial R^+}{\partial u} & \dfrac{\partial R^-}{\partial u} \\ \dfrac{\partial R^+}{\partial v} & \dfrac{\partial R^-}{\partial v} \end{pmatrix}, \\ &= \begin{pmatrix} \dfrac{-\sqrt{k^2+4uv}+k+2v}{-\sqrt{k^2+4uv}+k} & \dfrac{\sqrt{k^2+4uv}+k+2v}{\sqrt{k^2+4uv}+k} \\ \dfrac{-\sqrt{k^2+4uv}+k+2u}{-\sqrt{k^2+4uv}+k} & \dfrac{\sqrt{k^2+4uv}+k+2u}{\sqrt{k^2+4uv}+k} \end{pmatrix}, \end{aligned}$$

and by inverting this one can obtain the derivatives of u and v along R^+ and R^-.

Hence one can solve Tsarev's equations, which in this case are

$$\begin{aligned} \frac{1}{2}\partial_- \log G_{++} &= \frac{\partial_-\lambda_+}{\lambda_- - \lambda_+}\,, \\ &= -\frac{uv}{(k^2+4uv)^{3/2}}\,, \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2}\partial_+ \log G_{--} &= \frac{\partial_+\lambda_-}{\lambda_+ - \lambda_-}\,, \\ &= \frac{uv}{(k^2+4uv)^{3/2}}\,. \end{aligned}$$

One obtains

$$
\begin{aligned}
G_{++} &= \frac{1}{\sqrt{k^2+4uv}}\frac{1}{f_+(R^+)}\,,\\
G_{--} &= -\frac{1}{\sqrt{k^2+4uv}}\frac{1}{f_-(R^-)}\,,
\end{aligned}
\tag{5.8}
$$

where f_+ and f_- are two arbitrary functions.

By making the choice $f_+(R^+) = f_-(R^-) = 1$ we recover the metric

$$
\begin{aligned}
\eta &= \frac{1}{\sqrt{k^2+4uv}}(dR^+\otimes dR^+ - dR^-\otimes dR^-)\,,\\
&= kdb\otimes db - (db\otimes d\nu + d\nu\otimes db)\,.
\end{aligned}
$$

If $k = 0$, then the choice $f^+(R^+) = R^+$, $f_-(R^-) = R^-$ gives the intersection form of the undeformed Frobenius manifold. In fact, if $k = 0$ the choice $f_+(R^+) = \mu + R^+$, $f_-(R^-) = \mu + R^-$ gives a metric which is flat for all μ, thus exhibiting the bi-Hamiltonian structure for (2.19). However, this is not flat for $k \neq 0$.

To find another metric which deforms the intersection form, we work in formal power series in k, and consider functions f_+ and f_- of the form

$$
\begin{aligned}
f_+(R^+) &= \sum_{s\geq 0} k^s f_+^s(R^+)\,,\\
f_-(R^-) &= \sum_{s\geq 0} k^s f_-^s(R^-)\,,
\end{aligned}
\tag{5.9}
$$

in which $f_+^0(R^+) = R^+$ and $f_-^0(R^-) = R^-$, and attempt to fix the functions $f_\pm^s$ in such a way that the curvature of G is zero.[2]

Expressing quantities in terms of Riemann invariants is problematic, since we cannot explicitly invert the change of coordinates (5.7). However, we may work with the necessary quantities by noting that

$$
R^+ - R^- = -2\sqrt{k^2+4uv} - k\log\left(\frac{\sqrt{k^2+4uv}-k}{\sqrt{k^2+4uv}+k}\right)\,, \tag{5.10}
$$

so $R^+ - R^-$ is a function of $\sqrt{k^2+4uv}$. By inverting this function we may write the factor in the metric G as

$$
\frac{1}{\sqrt{k^2+4uv}} = e^{\omega(R^+-R^-)}
$$

[2]The change of variables (5.7) depends upon the value of k. We work as though R^+ and R^- are independent of k, so that the values of u and v depend upon the parameter.

for some function ω.

Since we are working formally over the deformation parameter k it is possible to invert (5.10) and obtain an expression for $\sqrt{k^2+4uv}$ in terms of $R^+ - R^-$, and hence find an expression for the function ω. If we write

$$\begin{aligned} A &= -\frac{1}{2}\left(R^+ - R^-\right) , \\ \Delta &= \sqrt{k^2+4uv} , \end{aligned}$$

then (5.10) is equivalent to

$$\Delta = A + \left(-\frac{k}{2}\right) \log\left(\frac{\Delta - k}{\Delta + k}\right) .$$

The Lagrange reversion formula can then be used to express Δ in terms of A as

$$\Delta = A + \sum_{r=1}^{\infty} \frac{1}{r!} \left(-\frac{k}{2}\right)^r \left(\frac{\partial}{\partial A}\right)^{r-1} \left[\left(\log\left(\frac{A-k}{A+k}\right)\right)^r\right] , \tag{5.11}$$

and so $\omega(A) = -\log(\Delta)$ can be expressed in terms of its argument as a power series in k.

If we call

$$\Delta_r = \left(-\frac{k}{2}\right)^r \left(\frac{\partial}{\partial A}\right)^{r-1} \left[\left(\log\left(\frac{A-k}{A+k}\right)\right)^r\right]$$

then, because

$$\log\left(\frac{A-k}{A+k}\right) = -\sum_{s\geq 1} \frac{1}{s}\left(\frac{k}{A}\right)^{2s} ,$$

the lowest power of k appearing in Δ_r is k^{3r}. As will be seen, it is sufficient for us to work to $O(k^3)$, so only Δ_1 needs to be calculated.

Since the system (5.6) is two dimensional, the metric (5.8) is flat if and only if its scalar curvature is zero. The scalar curvature is

$$\begin{aligned} R &= \frac{1}{2} e^{-\omega(R^+-R^-)} \left\{ \left(f'_+(R^+) + f'_-(R^-)\right) \omega'(R^+-R^-) \right. \\ &\qquad \left. + 2\left(f_+(R^+) - f_-(R^-)\right)\omega''(R^+-R^-) \right\} \\ &= \frac{1}{2} e^{-\omega(R^+-R^-)} \left(\frac{\partial}{\partial R^+} - \frac{\partial}{\partial R^-}\right) \left\{ \left(f_+(R^+) - f_-(R^-)\right) \omega'(R^+-R^-) \right\} , \end{aligned}$$

and so $R=0$ if and only if the quantity

$$\left(f_+(R^+) - f_-(R^-)\right) \omega'(R^+-R^-) \tag{5.12}$$

is a function of $(R^+ + R^-)$.

Using (5.11) we may expand (5.12) as a power series in k. Choosing the functions f_+ and f_- such that G is flat is equivalent to choosing the functions f^i_+ and f^i_- in (5.9) such that every term in this power series is a function of $(R^+ + R^-)$.

Because of our choice $f^0_\pm(R^\pm) = R^\pm$, the k^0 term in (5.12) vanishes, and the k^1 and k^2 terms vanish if we choose $f^1_+(R^+) = f^2_+(R^+) = f^1_-(R^-) = f^2_-(R^-) = 0$. Regardless of our choices for $f^1_\pm$ and $f^2_\pm$, the order k^3 term is

$$-2\frac{f^3_+(R^+) - f^3_-(R^-)}{R^+ - R^-} - \frac{8}{3(R^+ - R^-)^3}\,,$$

which is a function of $(R^+ + R^-)$ if and only if

$$(R^+ - R^-)^3({f^3_+}'(R^+) + {f^3_-}'(R^-)) - 2(R^+ - R^-)^2(f^3_+(R^+) - f^3_-(R^-)) - 8 = 0\,.$$

However, one can show that this differential equation has no solutions: repeated differentiation shows that

$$\frac{d^5 f^3_+}{d{R^+}^5} = \frac{d^5 f^3_-}{d{R^-}^5} = 0\,,$$

so that if a solution exists, it must be specified by two (quartic) polynomial solutions; substitution then shows that no solution of this form is possible.

From this we conclude that there is no choice of the functions f_+ and f_-, at least in the ring of formal power series in k, such that the metric (5.8) is flat for all k and reduces to the intersection form of the dispersionless Toda system in the limit $k \to 0$.

5.1.3 Conformal Symmetry

As suggested by the discussion of alternatives to the Euler vector field in Section 4.3.1, we consider the vector field

$$\mathcal{E} = R^+\frac{\partial}{\partial R^+} + R^-\frac{\partial}{\partial R^-}\,.$$

By expressing the metric η as

$$\eta = e^{\omega(R^+ - R^-)}(dR^+ \otimes dR^+ - dR^- \otimes dR^-)\,,$$

we see that $\mathcal{E}$ satisfies

$$\mathcal{L}_{\mathcal{E}}\eta = \big(2 + (R^+ - R^-)\omega'(R^+ - R^-)\big)\,\eta\,,$$

so that it is a conformal symmetry of η, but not a special conformal symmetry as in a Frobenius manifold (that is, the conformal factor is not a constant).

In the conventional coordinates b, ν, this vector field is

$$\begin{aligned}\mathcal{E} \;=\; & \left\{2+\frac{c}{\sqrt{k^2+4e^b}}\log\left(\frac{\sqrt{k^2+4e^b}-c}{\sqrt{k^2+4e^b}+c}\right)\right\}\frac{\partial}{\partial b}\\ & +\left\{\nu+c\right.\\ & \quad -\frac{c}{2\sqrt{k^2+4e^b}}\Big((\sqrt{k^2+4e^b}-c)\log(\sqrt{k^2+4e^b}-c)\Big)\\ & \quad \left.-\frac{c}{2\sqrt{k^2+4e^b}}\Big((\sqrt{k^2+4e^b}+c)\log(\sqrt{k^2+4e^b}+c)\Big)\right\}\frac{\partial}{\partial \nu}\,, \qquad (5.13)\end{aligned}$$

which can be see to deform the Euler vector field

$$E = 2\frac{\partial}{\partial b}+\nu\frac{\partial}{\partial \nu}$$

of Example 2.1.8, corresponding to the undeformed superpotential

$$\hat{\lambda}(\hat{p}) = e^{\hat{p}}+\nu+e^{-\hat{p}+b}\,.$$

$\mathcal{E}$ also satisfies

$$\mathcal{L}_{\mathcal{E}}g = (1+(R^+-R^-)\omega'(R^+-R^-))g$$

where g is the (curved) metric obtained by choosing $f_+(R^+)=R^+$ and $f_-(R^-)=R^-$ in the solution of Tsarev's equations, (5.8).

Although the metric g is curved, η and g are compatible metrics as defined in [17]. The identity vector field in the Riemann invariants is

$$e = \frac{\partial}{\partial R^+}+\frac{\partial}{\partial R^-}\,,$$

then we have the relations

$$\begin{aligned}[e,\mathcal{E}] &= e\,,\\ \mathcal{L}_{\mathcal{E}}g^{ij} &= (f-1)g^{ij}\,,\\ \mathcal{L}_{e}g^{ij} &= \eta^{ij}\,,\\ \mathcal{L}_{e}\eta^{ij} &= 0\,,\end{aligned}$$

which are similar to those of Definition 2.3.1 defining the quasihomogeneity of a pencil of metrics, except that here one has that f is a function rather than a constant. One must also add the condition $\mathcal{L}_{\mathcal{E}}\eta^{ij}=(f-2)\eta^{ij}$ (or equivalently $e(f)=0$), and then a prescription similar to that of Section 2.3 allows one to reconstruct the multiplication. This is detailed in Appendix A.

5.2 Deformation of the Dispersionless KdV Equation

Here we consider the simplest example of a waterbag model of the form (4.7), namely

$$\lambda(p) = p^2 + a + k\log(p-b)\,. \tag{5.14}$$

Equations (4.4) and (4.5) give the metric as

$$\eta_{..} = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & k \end{pmatrix} \tag{5.15}$$

and the multiplication as

$$\begin{aligned} \frac{\partial}{\partial a}\circ\frac{\partial}{\partial a} &= \frac{\partial}{\partial a}\,, \\ \frac{\partial}{\partial a}\circ\frac{\partial}{\partial b} &= \frac{\partial}{\partial b}\,, \\ \frac{\partial}{\partial b}\circ\frac{\partial}{\partial b} &= 2b\frac{\partial}{\partial b} - 2k\frac{\partial}{\partial a}\,. \end{aligned}$$

So the free energy is

$$F = -\frac{1}{12}a^3 + \frac{1}{2}kab^2 + \frac{1}{12}kb^4\,,$$

which is pseudo-quasi-homogeneous with respect to the Euler vector field

$$E = a\frac{\partial}{\partial a} + \frac{1}{2}b\frac{\partial}{\partial b}\,.$$

5.2.1 Associated Hierarchy

The functions a and b are a flat coordinate system for η. The flat coordinate ϕ^a of the deformed flat connection ∇^z which deforms a can be expanded as

$$\begin{aligned} \phi^{(a)} &= a \\ &\quad + z\left\{\frac{1}{2}a^2 - kb^2\right\} \\ &\quad + z^2\left\{\frac{1}{6}a^3 - kab^2 - \frac{1}{3}kb^4\right\} \\ &\quad + O(z^3)\,. \end{aligned}$$

This gives the flows

$$\begin{aligned} \frac{d}{d\tau_{a,0}}\begin{pmatrix} a \\ b \end{pmatrix} &= 0\,, \\ \frac{d}{d\tau_{a,1}}\begin{pmatrix} a \\ b \end{pmatrix} &= -2\begin{pmatrix} a_x \\ b_x \end{pmatrix}, \\ \frac{d}{d\tau_{a,2}}\begin{pmatrix} a \\ b \end{pmatrix} &= -2\begin{pmatrix} aa_x - 2kbb_x \\ (ab)_x + 2b^2 b_x \end{pmatrix}. \end{aligned}$$

So it is the $\tau_{a,2}$ flow in which a satisfies the dispersionless KdV equation in the limit $k \to 0$.

The flat coordinate ϕ^b deforming b can be expanded as

$$\begin{aligned} \phi^{(b)} &= b \\ &\quad +z\left\{\frac{1}{3}b^3 + ab\right\} \\ &\quad +z^2\left\{\frac{1}{10}b^5 - \frac{1}{3}kb^3 + \frac{1}{3}ab^3 + \frac{1}{2}a^2 b\right\} \\ &\quad +O(z^3)\,, \end{aligned}$$

so the corresponding flows are

$$\begin{aligned} \frac{d}{d\tau_{b,0}}\begin{pmatrix} a \\ b \end{pmatrix} &= 0\,, \\ \frac{d}{d\tau_{b,1}}\begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} -2b_x \\ \frac{2}{k}bb_x + \frac{1}{k}a_x \end{pmatrix}, \\ \frac{d}{d\tau_{b,2}}\begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} -2b^2 b_x - 2(ab)_x \\ -6bb_x + \frac{1}{k}(2b^3 b_x + 2abb_x + b^2 a_x + aa_x) \end{pmatrix}. \end{aligned}$$

5.2.2 Canonical Coordinates

We focus on the $\tau_{a,2}$ flow

$$\frac{d}{dt}\begin{pmatrix} a \\ b \end{pmatrix} = -2\begin{pmatrix} aa_x - 2kbb_x \\ (ab)_x + 2b^2 b_x \end{pmatrix}. \tag{5.16}$$

Canonical coordinates for the Frobenius structure are obtained by evaluating (5.14) at its critical points. This gives

$$u_\pm = \lambda(q_\pm) = a + \frac{1}{2}b^2 - \frac{1}{2}k \pm \frac{1}{2}b\sqrt{b^2 - 2k} + k\log\left(-\frac{1}{2}b \pm \frac{1}{2}\sqrt{b^2 - 2k}\right). \tag{5.17}$$

The system (5.16) diagonalises in these coordinates with

$$\frac{du_\pm}{dt} = (-2a - 2b^2 \mp b\sqrt{b^2 - 2k})\frac{du_\pm}{dx}\,.$$

The Jacobian for this transformation is

$$\begin{aligned} J &= \begin{pmatrix} \dfrac{\partial u_+}{\partial a} & \dfrac{\partial u_-}{\partial a} \\ \dfrac{\partial u_+}{\partial b} & \dfrac{\partial u_-}{\partial b} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ b + \sqrt{b^2 - 2k} & b - \sqrt{b^2 - 2k} \end{pmatrix}, \end{aligned}$$

and its inverse is

$$\begin{aligned} J^{-1} &= \begin{pmatrix} \dfrac{\partial a}{\partial u_+} & \dfrac{\partial b}{\partial u_+} \\ \dfrac{\partial a}{\partial u_-} & \dfrac{\partial b}{\partial u_-} \end{pmatrix} \\ &= \begin{pmatrix} \dfrac{1}{2} - \dfrac{b}{2\sqrt{b^2 - 2k}} & \dfrac{1}{2\sqrt{b^2 - 2k}} \\ \dfrac{1}{2} + \dfrac{b}{2\sqrt{b^2 - 2k}} & -\dfrac{1}{2\sqrt{b^2 - 2k}} \end{pmatrix}. \end{aligned}$$

From Lemma 4.2.3, we know the metric η is Egorov in these coordinates with potential $-\frac{1}{2}a$, so we have

$$\eta_{\pm\pm} = -\frac{1}{4} \pm \frac{b}{4\sqrt{b^2 - 2k}}\,,$$

which provides us with a basic solution of Tsarev's equations for (5.16). The general solution is then

$$g_{\pm\pm} = \frac{\eta_{\pm\pm}}{f_\pm(u_\pm)}\,, \tag{5.18}$$

where f_+ and f_- are two arbitrary functions.

The scalar curvature of g can be shown to be

$$R = -\eta^{++}\left(\frac{\partial}{\partial u_+} - \frac{\partial}{\partial u_-}\right)\left[(f_+(u_+) - f_-(u_-))\Gamma^-_{++}\right]\,, \tag{5.19}$$

where Γ^-_{++} is the appropriate Christoffel symbol for the Levi-Civita connection of η, which is

$$\Gamma^-_{++} = \frac{1}{2}\frac{k}{(b^2 - 2k)^{\frac{3}{2}}}\frac{1}{b + \sqrt{b^2 - 2k}}\,.$$

Note that b, and therefore Γ^-_{++}, is a function of $(u_+ - u_-)$, so the choice $f_+(u_+) = u_+$, $f_-(u_-) = u_-$ does not result in a flat metric.

Because, unlike in Section 5.1, the superpotential here deforms that of the KdV equation by the addition of a second coordinate b, attempting to study the system or to solve (5.19) for $R = 0$ by regarding k as a small parameter and working formally over it proves problematic. Firstly, the metric (5.15) degenerates at $k = 0$. Secondly, although b is a function of $(u_+ - u_-)$, the dependence of this function on the parameter k is logarithmic, with the result that the scalar curvature cannot by expanded as a power series in k.

Further, although the vector field

$$\mathcal{E} = u_+ \frac{\partial}{\partial u_+} + u_- \frac{\partial}{\partial u_-} \tag{5.20}$$

certainly scales the multiplication and identity vector fields according to

$$\begin{aligned} \mathcal{L}_{\mathcal{E}} \circ &= \circ \,, \\ \mathcal{L} e &= -e \end{aligned}$$

(since we are in canonical coordinates), we have

$$\mathcal{L}_{\mathcal{E}} \eta_{++} = \left(2 - (u_+ - u_-) \frac{k}{(b^2 - 2k)^{\frac{3}{2}} (b - \sqrt{b^2 - 2k})} \right) \eta_{++} \,,$$

whilst

$$\mathcal{L}_{\mathcal{E}} \eta_{--} = \left(2 - (u_+ - u_-) \frac{k}{(b^2 - 2k)^{\frac{3}{2}} (b + \sqrt{b^2 - 2k})} \right) \eta_{--} \,,$$

so $\mathcal{E}$ is not a conformal symmetry of the metric in this case.

5.3 Conclusions

In this chapter we have attempted to study two of the waterbag superpotentials of Chapter 4 by considering the hierarchies which correspond to each via the hydrodynamic type Poisson bracket given by the metric η, and the Hamiltonians given by the flat coordinates of the deformed Levi-Civita connection $\nabla + \lambda \circ$.

Of these, the superpotential

$$\lambda(p) = p + \frac{t_2}{p - t_1} + k \log(p - t_1) \,, \tag{5.21}$$

or rather its Legendre transformed version

$$\lambda(p) = e^p + \nu + e^{-p+b} + c(p - b) \,,$$

proved the most tractable, and properties analogous to those of the original Frobenius manifold corresponding to $k = 0$ were present. In particular, the Euler vector field

$$\mathcal{E} = R^+ \frac{\partial}{\partial R^+} + R^- \frac{\partial}{\partial R^-}$$

which deformed the Euler vector field

$$E = \sum_{i=1}^{2} u_i \frac{\partial}{\partial u_i} = b\frac{\partial}{\partial b} + \frac{\partial}{\partial \nu}$$

of the underlying Frobenius manifold as described in Example 2.1.4, provides a scaling of the multiplication $\circ$, and is a conformal symmetry of the associated metric η. (Although, as is illustrated in 5.13, it is not linear in the flat coordinates of η.) However, this vector field $\mathcal{E}$ is too much of a generalisation of the Euler vector field to provide us with a flat intersection form. Further, it was shown that no flat deformation of the $k = 0$ intersection form exists.

This superpotential is a degenerate case of the generic rational waterbag model (4.23) for $N = 0\,, K = L_1 = M = 1$, obtained by having the pole of the logarithm coincide with the pole of the rational part. It is not clear that we can expect the scaling properties of the vector field $\mathcal{E}$ to persist in any of the generic waterbag models of Chapter 4.

The consideration of the superpotential

$$\lambda(p) = p^2 + a + k\log(p - b)$$

given in Section 5.2 suggests that (5.21) may be too special a degeneration from which to infer generic properties. Here we found that the analogous $\mathcal{E}$, (5.20), was not a conformal symmetry of the metric. However, because of the existence of canonical coordinates, one can pursue the construction of the multiplication given in Section 2.3, using the vector field $\mathcal{E}$ and using (5.18) with $f_+(R^+) = R^+\,, f_-(R^-) = R^-$ in place of the intersection form. However, this provides little insight as to why this construction gives a solution of the WDVV equations.

Chapter 6

Further Problems

In Chapter 3 we described the geometry associated to bi-Hamiltonian structures of degree 2, and how this can be applied to bi-Hamiltonian structures involving such an operator. An obvious open problem is how to extend this approach to higher degree Hamiltonian operators. Already at degree 2 we encounter an object, the coefficient c^{ij}_{kl}, which does not appear in standard differential geometry. At degree 3, which is considered in [59], more nonstandard objects appear. However, as at degree 2, if we wish the operator to be Hamiltonian all of these nonstandard terms are determined by the standard objects. Indeed, a degree 3 Hamiltonian operator is specified by a triple consisting of a rank 2 tensor, and two connections, which must satisfy some conditions. Although we consider a degree 1, i.e. hydrodynamic type, Hamiltonian operator as being defined by a metric tensor and a connection, since this connection is the Levi-Civita connection, it is determined entirely by the metric. Thus, at degree 1 we must specify a tensor only; at degree 2 we must specify a tensor and a connection; whilst at degree 3 we must specify a tensor and two connections. It would be interesting to know if a degree n Hamiltonian operator is specified by a tensor and $n-1$ connections satisfying some constraints, as this would mean that at all degrees one is only required to work with standard objects of differential geometry.

A disadvantage of differential-geometric type Poisson brackets is that as the degree increases, the number of coefficients increases rapidly, and even at degree 3 it is becoming unmanageable. ([59] works in special coordinates in which the multiplicative part of the operator vanishes. This removes three coefficients, but the remaining four must satisfy seven relations.) Clearly a more holistic/coordinate free approach is needed, particularly if one wishes to consider non-homogeneous operators. However, the concept of a differential-geometric type Poisson bracket was introduced partly because it could be studied in terms of its coefficients; attempting to study them in another fashion may make it an unnatural class. By this we mean that such operators are studied in terms of their properties which remain invariant under coordinates transformations on the target space; extending the group of allowed transforms, say to Miura or Quasi-Miura transformations as defined in [31], would require one to study a different set of properties, and possibly extend the class of allowed operators. For example, the class of $(0, n)$-brackets, or more generally the (p, q)-brackets of [31], is closed under Miura transformations. Whether the compatibility condition for such brackets can be formulated as a multiplicative structure is an interesting question, perhaps involving some kind of Frobenius algebra structure on the space of evolutionary vector fields.

We have already discussed some open problems arising from the construction of WDVV solutions from waterbag reductions of the dKP hierarchy at the end of Chapter 4. Specifically, we mentioned the possibility of constructing analogous solutions for higher genus Hurwitz spaces, and also raised the question of what one can say about the associated integrable systems. In Chapter 5, we attempted to study two specific integrable hierarchies arising from waterbag deformations of Frobenius manifolds. [45] provides formulas for the Hamiltonian structures of any reduction of the dKP hierarchy in terms of the function $\lambda(p)$ specifying the reduction (i.e. in terms of the superpotential). It may be that this can be extended to cover the superpotentials of Chapter 4. The main problem would be inverting $\lambda(p)$ to obtain $p(\lambda)$; perhaps the 'almost inversion' used in the second proof of Lemma 4.2.2 can be used here.

Another issue raised was whether one can formulate a set of axioms describing pseudo-quasi-homogeneous solutions of the WDVV equations. This was the purpose behind the extended Lie derivative $\mathcal{L}^{\text{ext}}$ of Section 4.3.1. A related issue would then be to associate such an object either to a pair of metrics satisfying the equations of Lemma 4.3.1, or to a pair of compatible metrics giving us a bi-Hamiltonian structure. An understanding of the relationship between the operations $\mathcal{L}_E^{\text{ext}}$ and $\mathcal{L}_{\mathcal{E}}$ would be useful in this context.

Appendix A

Compatible Metrics with Shared Conformal Killing Vector

In this appendix we present a slightly more general version of the construction of Section 2.3 of a commutative, associative algebra of vector fields from a pair of compatible metrics, which allows the Euler vector field to be a conformal symmetry of the metrics involved, but with a non-constant conformal factor. This includes, as a special case, the Frobenius structure of Section 5.1 for the waterbag deformation of the one-dimensional long wave system.

Throughout this appendix, η and g will be two metrics on a manifold, M, with Levi-Civita connections ∇ and $\bar{\nabla}$ given by Christoffel symbols Γ^k_{ij} and $\bar{\Gamma}^k_{ij}$, respectively. We will also use contravariant Christoffel symbols as defined by $\Gamma^{ij}_k = -\eta^{ir}\Gamma^j_{rk}$ and $\bar{\Gamma}^{ij}_k = -g^{ir}\bar{\Gamma}^j_{rk}$. We will also have cause to refer to the curvatures of these connections in the following form:

$$\begin{aligned} R^{ij}_{kl} &= \eta^{jr} R^i_{klr}\,, \\ \bar{R}^{ij}_{kl} &= g^{jr} \bar{R}^i_{klr}\,. \end{aligned}$$

Definition A.1. *[17, 60] The metrics η and g are said to be*

(i) Almost compatible *if and only if the contravariant Christoffel symbols of the metric $g^{ij}_\lambda = \eta^{ij} + \lambda g^{ij}$ are given by ${\Gamma_\lambda}^{ij}_k = \Gamma^{ij}_k + \lambda\bar{\Gamma}^{ij}_k$.*

(ii) Compatible *if and only if the curvature ${R_\lambda}^{ij}_{kl}$ of g_λ satisfies ${R_\lambda}^{ij}_{kl} = R^{ij}_{kl} + \lambda\bar{R}^{ij}_{kl}$.*

From now on we shall assume that in addition to the metrics η and g above, we have functions τ and f such that the following are satisfied:

(1) The metrics η and g are compatible.

(2) The vector field $E = g^{-1}d\tau$ satisfies

$$\begin{aligned} \mathcal{L}_E g^{-1} &= (f-1)g^{-1}\,, && \text{(A.1)} \\ \mathcal{L}_E \eta^{-1} &= (f-2)\eta^{-1}\,. && \text{(A.2)} \end{aligned}$$

(3) The vector field $e = \eta^{-1}d\tau$ satisfies

$$\begin{aligned} \mathcal{L}_e g^{-1} &= \eta^{-1}\,, \\ \mathcal{L}_E e &= -e\,. \end{aligned}$$

(4) The tensor defined by

$$T^*(X) = \frac{f-1}{2}X + \nabla_X E$$

is invertible as map from $\Gamma(TM)$ to $\Gamma(TM)$. We shall call the dual map from one-forms to one-forms T.

Note: given the rest of conditions (2) and (3), $\mathcal{L}_E\eta^{-1} = (f-2)\eta^{-1}$ is equivalent to $e(f) = 0$.

We shall demonstrate that one can define from the above data the structure of a smoothly varying commutative associative Frobenius algebra on each tangent space, and that the vector field E satisfies the scaling property $\mathcal{L}_E \circ = \circ$.

We first define a multiplication of covectors '$\diamond$' by, for two one-forms θ, ϕ,

$$\theta \diamond \phi = \bar{\nabla}_{g^{-1}\theta}\phi - \nabla_{g^{-1}\theta}\phi\,,$$

i.e. by $(\theta \diamond \phi)_k = \theta_i \phi_j \Delta^{ij}_k$ where

$$\Delta^{ij}_k = g^{ir}\left(\Gamma^j_{rk} - \bar{\Gamma}^j_{rk}\right)\,.$$

Result A.2. *[17]*

$$\begin{aligned} \eta^{-1}(\alpha \diamond \gamma, \beta) &= \eta^{-1}(\alpha, \beta \diamond \gamma)\,, && \text{(A.3)} \\ g^{-1}(\alpha \diamond \gamma, \beta) &= g^{-1}(\alpha, \beta \diamond \gamma)\,, && \text{(A.4)} \\ (\alpha \circ \beta) \diamond \gamma &= (\alpha \circ \gamma) \diamond \beta\,, && \text{(A.5)} \end{aligned}$$

for all one-forms α, β, γ.

Proof. As shown in [17], this follows immediately from the compatibility of η and g. □

Results A.3.

1. $\bar{\nabla}_X E = \frac{1-f}{2} X$.
2. *For all one-forms* ψ, $T(\psi) = d\tau \diamond \psi$, *i.e.* $T^i_j = E^r \left(\Gamma^i_{rj} - \bar{\Gamma}^i_{rj} \right)$.

Proof. Both parts of this proof proceed as in [17]; it affects nothing that f is a function rather than a constant.

1. Expressing the Lie derivative in (A.1) in terms of the Levi-Civita connection of g^{ij}, we get

$$\begin{aligned}(f-1)g^{ij} &= \mathcal{L}_E g^{ij} ,\\ &= E^r \bar{\nabla}_r g^{ij} - gir \bar{\nabla}_r E^j - g^{rj} \bar{\nabla}_r E^i ,\\ &= -g^{ir} g^{js} \bar{\nabla}_r \bar{\nabla}_s \tau - g^{rj} g^{is} \bar{\nabla}_r \bar{\nabla}_s \tau ,\\ &= -2 g^{ir} g^{js} \bar{\nabla}_r \bar{\nabla}_s \tau ,\end{aligned}$$

and so

$$\begin{aligned}(f-1)\delta^j_i &= -2 g^{js} \bar{\nabla}_i \bar{\nabla}_s \tau ,\\ &= -2 \bar{\nabla}_i E^j .\end{aligned}$$

2. For any vector field X and any 1-form ψ we have

$$\begin{aligned}\langle X | d\tau \diamond \psi \rangle &= \langle X | \bar{\nabla}_E \psi - \nabla_E \psi \rangle ,\\ &= E \langle X | \psi \rangle - \langle \bar{\nabla}_E X | \psi \rangle - E \langle X | \psi \rangle + \langle \nabla_E X | \psi \rangle ,\\ &= -\langle \bar{\nabla}_E X | \psi \rangle + \langle \nabla_E X | \psi \rangle ,\\ &= \langle -\bar{\nabla}_X E - [E, X] | \psi \rangle + \langle \nabla_X E + [E, X] | \psi \rangle ,\\ &= \langle \nabla_X E - \bar{\nabla}_X E | \psi \rangle ,\\ &= \left\langle \frac{f-1}{2} X + \nabla_X E \middle| \psi \right\rangle ,\\ &= \langle T^*(X) | \psi \rangle ,\\ &= \langle X | T(\psi) \rangle .\end{aligned}$$

□

Result A.4. *The multiplication '$\circ$' defined by*

$$\theta \circ \phi = \theta \diamond T^{-1}(\phi) \tag{A.6}$$

for all 1-forms θ, ϕ is commutative and associative, with identity $d\tau = \eta(e)$. Both of the contravariant metrics satisfy the Frobenius compatibility condition (2.4) with it. Consequently one may use the duality of T^*M and TM induced by the metric η to obtain a commutative, associative multiplication of vector fields with identity η, compatible with the covariant metric η.

Further, the metric g satisfies the usual definition of the intersection form on a Frobenius manifold, (A.6), i.e., for all one-forms θ, ϕ,

$$g^{-1}(\theta, \phi) = \langle E | \theta \circ \phi \rangle \,.$$

Proof. This proceeds from $T(\psi) = d\tau \diamond \psi$ and Result A.2 as follows. Let α, β, γ be one-forms. By equation (A.5)

$$\begin{aligned} T(\alpha) \circ T(\beta) &= T(\alpha) \diamond \beta \,, \\ &= (d\tau \diamond \alpha) \diamond \beta \,, \\ &= (d\tau \diamond \beta) \diamond \alpha \,, \\ &= T(\beta) \diamond \alpha \,, \\ &= T(\beta) \circ T(\alpha) \,, \end{aligned}$$

so $\circ$ is commutative. $\circ$ also satisfies $(\alpha \circ \beta) \circ \gamma = (\alpha \circ \gamma) \circ \beta$, and this is equivalent to associativity for a commutative multiplication.

Compatibility with the metrics follows from substituting $\gamma = T^{-1}(\gamma')$ into equations (A.3) and (A.4).

$d\tau = \eta(e)$ is the identity for $\circ$ since

$$d\tau \circ \alpha = d\tau \diamond T^{-1}(\alpha) = T(T^{-1}(\alpha)) = \alpha \,.$$

Finally

$$\begin{aligned} g^{-1}(\alpha, T(\beta)) &= g^{-1}(\alpha, d\tau \diamond \beta) \,, \\ &= g^{-1}(\alpha \diamond \beta, d\tau) \,, \\ &= \langle g^{-1}(d\tau) | \alpha \diamond \beta \rangle \,, \\ &= \langle E | \alpha \diamond \beta \rangle \,. \end{aligned}$$

□

Lemma A.5. *The Lie derivatives of the Christoffel symbols Γ^k_{ij} ad $\bar{\Gamma}^k_{ij}$ satisfy*

$$\mathcal{L}_E\bar{\Gamma}^a_{bc} = \frac{1}{2}\left(g_{bc}g^{ar}f_{,r} - \delta^a_c f_{,b} - \delta^a_b f_{,c}\right)$$

and

$$\mathcal{L}_E\Gamma^a_{bc} = \frac{1}{2}\left(\eta_{bc}\eta^{ar}f_{,r} - \delta^a_c f_{,b} - \delta^a_b f_{,c}\right) .$$

Proof. If we write $\mathcal{L}_E g_{ab} = (f-1)g_{ab}$ as $\bar{\nabla}_a E_b + \bar{\nabla}_b E_a = (f-1)g_{ab}$, where $E_a{=}g_{ar}E^r$ then

$$\begin{aligned}
\bar{\nabla}_a\bar{\nabla}_b E_c + \bar{\nabla}_a\bar{\nabla}_c E_b &= g_{bc}\bar{\nabla}_a f\,, \quad \text{(abc)}\\
\bar{\nabla}_b\bar{\nabla}_c E_a + \bar{\nabla}_b\bar{\nabla}_a E_c &= g_{ca}\bar{\nabla}_b f\,, \quad \text{(bca)}\\
\bar{\nabla}_c\bar{\nabla}_a E_b + \bar{\nabla}_c\bar{\nabla}_b E_a &= g_{ab}\bar{\nabla}_c f\,. \quad \text{(cab)}
\end{aligned}$$

Taking the sum (abc)-(bca)-(cab), we see

$$\begin{aligned}
&(\bar{\nabla}_a\bar{\nabla}_b - \bar{\nabla}_b\bar{\nabla}_a)E_c\\
&+(\bar{\nabla}_a\bar{\nabla}_c - \bar{\nabla}_c\bar{\nabla}_a)E_b\\
&-(\bar{\nabla}_b\bar{\nabla}_c + \bar{\nabla}_c\bar{\nabla}_b)E_a \quad = \quad g_{bc}\bar{\nabla}_a f - g_{ca}\bar{\nabla}_b f - g_{ab}\bar{\nabla}_c f\,.
\end{aligned}$$

The left hand side of this equation is

$$\begin{aligned}
\text{LHS} &= -E^r\bar{R}^r_{abc} - E^r\bar{R}^r_{acb} - (2\bar{\nabla}_b\bar{\nabla}_c + \bar{\nabla}_c\bar{\nabla}_b - \bar{\nabla}_b\bar{\nabla}_c)E_a\,,\\
&= -E_r\bar{R}^r_{abc} - E_r\bar{R}^r_{acb} - E_r\bar{R}^r_{bca} - 2\bar{\nabla}_b\bar{\nabla}_c E_a\,,\\
&= E_r\bar{R}^r_{cab} - E_r\bar{R}^r_{acb} - 2\bar{\nabla}_b\bar{\nabla}_c E_a\,,\\
&= 2E_r\bar{R}^r_{cab} - 2\bar{\nabla}_b\bar{\nabla}_c E_a\,,\\
&= 2E^r\bar{R}_{cabr} - 2\bar{\nabla}_b\bar{\nabla}_c E_a\,,\\
&= -2E^r\bar{R}_{rbca} - 2\bar{\nabla}_b\bar{\nabla}_c E_a\,.
\end{aligned}$$

The statement of the lemma is that this quantity is $-2\mathcal{L}_E\bar{\Gamma}^a_{bc}$. To see this, we use the result [32] that

$$\left(\mathcal{L}_E\bar{\nabla}\right)_X Y = [E, \bar{\nabla}_X Y] - \bar{\nabla}_{[E,X]} - \bar{\nabla}_X[E,Y]$$

from which we see

$$\begin{aligned}
\left(\mathcal{L}_E\bar{\nabla}\right)_X Y &= \bar{\nabla}_E\bar{\nabla}_X Y - \bar{\nabla}_{\bar{\nabla}_X Y}E - \bar{\nabla}_{[E,X]}Y - \bar{\nabla}_X\bar{\nabla}_E Y - \bar{\nabla}_X\bar{\nabla}_Y E\,,\\
&= \left(\bar{\nabla}_E\bar{\nabla}_X Y - \bar{\nabla}_X\bar{\nabla}_E Y - \bar{\nabla}_{[E,X]}Y\right) + \left(\bar{\nabla}_X\bar{\nabla}_Y E - \bar{\nabla}_{\bar{\nabla}_X Y}E\right)\,,\\
&= R_{E,X}Y + \left(\bar{\nabla}_X\bar{\nabla}_Y E - \bar{\nabla}_{\bar{\nabla}_X Y}E\right)\,.
\end{aligned}$$

The result for $\mathcal{L}_E\Gamma^k_{ij}$ proceeds in the same fashion. □

As a consequence of this, we have also:

$$\begin{aligned}
\mathcal{L}_E \Delta^{ij}_k &= (f-1)\Delta^{ij}_k + \frac{1}{2} g^{ir}\left(\eta_{rk}\eta^{js} - g_{rk}g^{js}\right) f_{,s}\,, \\
&= (f-1)\Delta^{ij}_k + \frac{1}{2}\left(E^r c^i_{rk}\eta^{js} - \delta^i_k g^{js}\right) f_{,s}\,, \\
\mathcal{L}_E T^i_j &= \frac{1}{2}E^r\left(\eta_{rj}\eta^{is} - g_{rj}g^{is}\right) f_{,s}\,,
\end{aligned}$$

Result A.6.

$$\mathcal{L}_E c^k_{ij} = c^k_{ij}\,.$$

Proof. $\Delta^{ij}_k = c^{ir}_k T^j_r$, and so

$$\begin{aligned}
\left(\mathcal{L}_E c^{ir}_k\right) T^j_r &= \mathcal{L}_E \Delta^{ij}_k - c^{ir}_k \mathcal{L}_E T^j_r\,, \\
&= (f-1)\Delta^{ij}_k + \frac{1}{2}\left(E^r c^i_{rk}\eta^{js} - \delta^i_k g^{js}\right) f_{,s} - c^{ir}_k \frac{1}{2} E^t \left(\eta_{tr}\eta^{js} - g_{tr}g^{js}\right) f_{,s}\,, \\
&= (f-1)\Delta^{ij}_k + \frac{1}{2}\left(E^r c^i_{rk}\eta^{js} - \delta^i_k g^{js} - E^t c^i_{kt}\eta^{js} + g(E)_r c^{ir}_k g^{js}\right) f_{,s}\,, \\
&= (f-1)\Delta^{ij}_k\,, \\
&= (f-1)c^{ir}_k T^j_r\,,
\end{aligned}$$

where we have used the fact that $g(E)$ is the identity for c^{ij}_k. Hence, $\mathcal{L}_E c^{ij}_k = (f-1)c^{ij}_k$ and the result follows from $\mathcal{L}_E \eta_{ij} = (2-f)\eta_{ij}$. □

Although we have $\mathcal{L}_E c^i_{jk} = c^i_{jk}$, because the vector field E is not a special conformal symmetry of η, we get

$$\begin{aligned}
\mathcal{L}_E c_{ijk} &= \mathcal{L}_E \eta_{ir} c^r_{jk}\,, \\
&= \left(\mathcal{L}_E \eta_{ij}\right) c^r_{jk} + \eta_{ir}\left(\mathcal{L}_E c^r_{jk}\right)\,, \\
&= (2-f)\eta_{ir} c^r_{jk} + \eta_{ir} c^r_{jk}\,, \\
&= (3-f) c_{ijk}\,,
\end{aligned}$$

so E does not scale c_{ijk} by a constant factor. Because of this, and because $\nabla\nabla E \neq 0$, Result 5.1 does not lead to $E(F) = \text{constant} \times F$ if a prepotential exists for c_{ijk}.

The above construction, starting with the data g, η, e and E and the conditions imposed upon them, does not seem to provide us with the potentiality condition $\nabla_i c_{jkl} = \nabla_j c_{ikl}$, even in the case that η is flat, which is a property satisfied by the motivating example of Section 5.1.

References

[1] M. Adler. On a trace functional for formal pseudodifferential operators and the symplectic structure of the Korteweg-de Vries type equations. *Inv. Math.*, 50:219–248, 1979.

[2] S. Aoyama and Y. Kodama. Topological conformal field theory with a rational W potential and the dispersionless KP hierarchy. *Modern Phys. Lett. A*, 9(27):2481–2492, 1994.

[3] S. Aoyama and Y. Kodama. Topological Landau-Ginzburg theory with a rational potential and the dispersionless KP hierarchy. *Comm. Math. Phys.*, 182(1):185–219, 1996.

[4] C. Bai, D. Meng, and L. He. On Fermionic Novikov algebras. *J. Phys. A*, 35(47):10053–10063, 2002.

[5] A. A. Balinskiĭ and S. P. Novikov. Poisson brackets of hydrodynamic type, Frobenius algebras and Lie algebras. *Dokl. Akad. Nauk SSSR*, 283(5):1036–1039, 1985. English translation: *Soviet Math. Dokl.* 32 (1985), no. 1, 228–231.

[6] S. Benenti. Inertia tensors and Stäckel systems in the Euclidean spaces. *Rend. Sem. Mat. Univ. Politec. Torino*, 50(4):315–341, 1992.

[7] M. Bertola. Frobenius manifold structure on orbit space of Jacobi groups. I. *Differential Geom. Appl.*, 13(1):19–41, 2000.

[8] M. Bertola. Frobenius manifold structure on orbit space of Jacobi groups. II. *Differential Geom. Appl.*, 13(3):213–233, 2000.

[9] P. Bieliavsky, M. Cahen, S. Gutt, J. Rawnsley, and L. Schwachhöfer. Symplectic connections. *Int. J. Geom. Methods Mod. Phys.*, 3(3):375–420, 2006.

[10] L. V. Bogdanov and B. G. Konopelchenko. Symmetry constraints for dispersionless integrable equations and systems of hydrodynamic type. *Phys. Lett. A*, 330(6):448–459, 2004.

[11] A. Cannas da Silva. Symplectic geometry. In *Handbook of differential geometry. Vol. II*, pages 79–188. Elsevier/North-Holland, Amsterdam, 2006.

[12] A. Cannas da Silva and A. Weinstein. *Geometric models for noncommutative algebras*, volume 10 of *Berkeley Mathematics Lecture Notes*. American Mathematical Society, Providence, RI, 1999.

[13] J.-H. Chang. On the waterbag model of dispersionless KP hierarchy. *J. Phys. A*, 39(36):11217–11230, 2006.

[14] J.-H. Chang. On the waterbag model of dispersionless KP hierarchy (II). *J. Phys. A*, 40:12973–12985, 2007.

[15] J.-H. Chang. Remarks on waterbag model of dispersionless Toda hierarchy. *arXiv:0709.3859v1*, 2007.

[16] C. Chevalley. Invariants of finite groups generated by reflections. *Amer. J. Math.*, 77:778–782, 1955.

[17] L. David and I. A. B. Strachan. Compatible metrics on a manifold and nonlocal bi-Hamiltonian structures. *Int. Math. Res. Not.*, (66):3533–3557, 2004.

[18] R. Dijkgraaf. Intersection theory, integrable hierarchies and topological field theory. In *New symmetry principles in quantum field theory (Cargèse, 1991)*, volume 295 of *NATO Adv. Sci. Inst. Ser. B Phys.*, pages 95–158. Plenum, New York, 1992.

[19] R. Dijkgraaf, H. Verlinde, and E. Verlinde. Notes on topological string theory and 2D quantum gravity. In *String theory and quantum gravity (Trieste, 1990)*, pages 91–156. World Sci. Publ., River Edge, NJ, 1991.

[20] I. Dorfman. *Dirac structures and integrability of nonlinear evolution equations*. Nonlinear Science: Theory and Applications. John Wiley & Sons Ltd., Chichester, 1993.

[21] P. W. Doyle. Differential geometric Poisson bivectors in one space variable. *J. Math. Phys.*, 34(4):1314–1338, 1993.

[22] P. G. Drazin and R. S. Johnson. *Solitons: an introduction*. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 1989.

[23] B. Dubrovin. Differential geometry of the space of orbits of a Coxeter group. In *Surveys in differential geometry: integral systems [integrable systems]*, Surv. Differ. Geom., IV, pages 181–211. Int. Press, Boston, MA, 1998.

[24] B. Dubrovin. Painlevé transcendents in two-dimensional topological field theory. In *The Painlevé property*, CRM Ser. Math. Phys., pages 287–412. Springer, New York, 1999.

[25] B. A. Dubrovin. Geometry of 2D topological field theories. In *Integrable systems and quantum groups (Montecatini Terme, 1993)*, volume 1620 of *Lecture Notes in Math.*, pages 120–348. Springer, Berlin, 1996.

[26] B. A. Dubrovin. Flat pencils of metrics and Frobenius manifolds. In *Integrable systems and algebraic geometry (Kobe/Kyoto, 1997)*, pages 47–72. World Sci. Publ., River Edge, NJ, 1998.

[27] B. A. Dubrovin. On almost duality for Frobenius manifolds. In *Geometry, topology, and mathematical physics*, volume 212 of *Amer. Math. Soc. Transl. Ser. 2*, pages 75–132. Amer. Math. Soc., Providence, RI, 2004.

[28] B. A. Dubrovin and S. P. Novikov. Poisson brackets of hydrodynamic type. *Dokl. Akad. Nauk SSSR*, 279(2):294–297, 1984.

[29] B. A. Dubrovin and S. P. Novikov. Hydrodynamics of weakly deformed soliton lattices. Differential geometry and Hamiltonian theory. *Uspekhi Mat. Nauk*, 44(6(270)):29–98, 203, 1989.

[30] B. A. Dubrovin and Y. Zhang. Extended affine Weyl groups and Frobenius manifolds. *Compositio Math.*, 111(2):167–219, 1998.

[31] B. A. Dubrovin and Y. Zhang. Normal forms of hierarchies of integrable pde's, Frobenius manifolds and Gromov-Witten invariants. *arXiv.org:math.DG/0108160*, 2001.

[32] S. Dubrovskiy. Moduli space of Fedosov structures. *Ann. Global Anal. Geom.*, 27(3):273–297, 2005.

[33] T. Eguchi, Y. Yamada, and S.-K. Yang. Topological field theories and the period integrals. *Modern Phys. Lett. A*, 8(17):1627–1637, 1993.

[34] D. B. Fairlie and I. A. B. Strachan. The Hamiltonian structure of the dispersionless Toda hierarchy. *Phys. D*, 90(1-2):1–8, 1996.

[35] G. Falqui and M. Pedroni. Separation of variables for bi-Hamiltonian systems. *Math. Phys. Anal. Geom.*, 6(2):139–179, 2003.

[36] B. V. Fedosov. A simple geometrical construction of deformation quantization. *J. Differential Geom.*, 40(2):213–238, 1994.

[37] M. V. Feigin and A. P. Veselov. Logarithmic Frobenius structures and Coxeter discriminants. *Adv. Math.*, 212(1):143–162, 2007.

[38] E. V. Ferapontov. Compatible Poisson brackets of hydrodynamic type. *J. Phys. A*, 34:2377–2388, 2001.

[39] J. T. Ferguson. Flat pencils of symplectic connections and Hamiltonian operators of degree 2. *J. Geom. Phys.*, 58(4):468–486, 2008.

[40] J. T. Ferguson and I. A. B. Strachan. Logarithmic deformations of the rational superpotential/ Landau-Ginzburg construction of solutions of the WDVV equations. *arXiv:math-ph/0605078*, 2006. To appear in *Communications in Mathematical Physics.*

[41] E. Fermi, J. Pasta, and S. M. Ulam. Studies in nonlinear problems. In *Technical Report.* Los Alomos Science Laboratory, 1940.

[42] I. M. Gel'fand and L. A. Dikiĭ. The resolvent, and Hamiltonian systems. *Funkcional. Anal. i Priložen.*, 11(2):11–27, 95, 1977.

[43] I. M. Gel'fand, V. Retakh, and M. Shubin. Fedosov manifolds. *Adv. Math.*, 136(1):104–140, 1998.

[44] E. Getzler. A Darboux theorem for Hamiltonian operators in the formal calculus of variations. *Duke Math. J.*, 111(3):535–560, 2002.

[45] J. Gibbons, P. Lorenzoni, and A. Raimondo. Hamiltonian structure of reductions of the benney system. *arXiv.org:0802.1984*, 2008.

[46] J. Gibbons and S. P. Tsarev. Reductions of the Benney equations. *Phys. Lett. A*, 211(1):19–24, 1996.

[47] C. Hertling. *Frobenius manifolds and moduli spaces for singularities*, volume 151 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2002.

[48] N. Hitchin. Frobenius manifolds. In *Gauge theory and symplectic geometry*, pages 69–112. Kluwer Acad. Publ., Dordrecht, 1997.

[49] A. Ibort, F. Magri, and G. Marmo. Bihamiltonian structures and Stäckel separability. *J. Geom. Phys.*, 33(3-4):210–228, 2000.

[50] I. M. Krichever. The dispersionless Lax equations and topological minimal models. *Comm. Math. Phys.*, 143(2):415–429, 1992.

[51] I. M. Krichever. The τ-function of the universal Whitham hierarchy, matrix models and topological field theories. *Comm. Pure Appl. Math.*, 47(4):437–475, 1994.

[52] B. A. Kupershmidt. Equations of long waves with a free surface IV. the case of constant shear. *arXiv.nlin.SI/0610021v1*, 2006.

[53] B. A. Kupershmidt. Extended equations of long waves. *Stud. Appl. Math.*, 116(4):415–434, 2006.

[54] P. D. Lax. Integrals of nonlinear equations of evolution and solitary waves. *Comm. Pure Appl. Math.*, 21:467–490, 1968.

[55] F. Magri. A simple model of the integrable Hamiltonian equation. *J. Math. Phys.*, 19(5):1156–1162, 1978.

[56] R. Martini and L. K. Hoevenaars. Trigonometric solutions of the WDVV equations from root systems. *Lett. Math. Phys.*, 65(1):15–18, 2003.

[57] T. E. Milanov. Equivariant orbifold structures on the projective line and integrable hierarchies. *arXiv:0707.3172v1*, 2007.

[58] R. M. Miura, C. S. Gardner, and M. D. Kruskal. Korteweg-de Vries equation and generalizations. II. Existence of conservation laws and constants of motion. *J. Mathematical Phys.*, 9:1204–1209, 1968.

[59] O. I. Mokhov. Symplectic and Poisson structures on loops spaces of smooth manifolds, and integrable systems. *Russian Mathematical Surveys*, 53(3):515–622, 1998.

[60] O. I. Mokhov. Compatible and almost compatible metrics. *Uspekhi Mat. Nauk*, 55(4(334)):217–218, 2000.

[61] S. P. Novikov. The geometry of conservative systems of hydrodynamic type. the method of averaging for field-theoretical systems. *Russian Mathematical Surveys*, 40(4):85–98, 1985.

[62] W. Oevel. A note on the Poisson brackets associated with Lax operators. *Phys. Lett. A*, 186(1-2):79–86, 1994.

[63] M. V. Pavlov. The Hamiltonian approach in classification and integrability of hydrodynamic chains. *arXiv:nlin.SI/0603057*, 2006.

[64] G. V. Potëmin. Poisson brackets of differential-geometric type. *Dokl. Akad. Nauk SSSR*, 286(1):39–42, 1986.

[65] K. Saito. On a linear structure of the quotient variety by a finite reflexion group. *Publ. Res. Inst. Math. Sci.*, 29(4):535–579, 1993.

[66] A. Sergyeyev. A simple way of making a Hamiltonian system into a bi-Hamiltonian one. *Acta Appl. Math.*, 83(1-2):183–197, 2004.

[67] I. A. B. Strachan. Frobenius manifolds: natural submanifolds and induced bi-Hamiltonian structures. *Differential Geom. Appl.*, 20(1):67–99, 2004.

[68] S. P. Tsarëv. Classical differential geometry and integrability of systems of hydrodynamic type. In *Applications of analytic and geometric methods to nonlinear differential equations (Exeter, 1992)*, volume 413 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 241–249. Kluwer Acad. Publ., Dordrecht, 1993.

[69] I. Vaisman. *Lectures on the geometry of Poisson manifolds*, volume 118 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1994.

[70] N. M. J. Woodhouse. *Geometric quantization*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1992. Oxford Science Publications.

[71] X. Xu. Variational calculus of supervariables and related algebraic structures. *J. Algebra*, 223(2):396–437, 2000.

[72] K. Yano and S. Ishihara. *Tangent and cotangent bundles: differential geometry.* Marcel Dekker Inc., New York, 1973. Pure and Applied Mathematics, No. 16.

[73] N. J. Zabusky and M. D. Kruskal. Interaction of 'solitons' in a collisionless plasma and recurrence of initial states. *Phys. Rev. Lett.*, 15:240–243, 1965.

[74] J.-B. Zuber. On Dubrovin topological field theories. *Modern Phys. Lett. A*, 9(8):749–760, 1994.

[75] D. Zuo. Frobenius manifolds associated to B_l and D_l, revisited. *Int. Math. Res. Not. IMRN*, (8):Art. ID rnm020, 25, 2007.