Wave Propagation in Residually-stressed Materials

by

Moniba Shams

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Abstract

The research work included in this thesis concerns the study of wave propagation in elastic materials which are stressed in their initial state. This research is based on the non-linear theory of elasticity.

Using the theory of invariants, the general constitutive equation for an isotropic hyper-elastic material in the presence of initial stress is derived. These invariants depend on the finite deformation as well as the initial stress. In general, this derivation involves 10 invariants for a compressible material and 9 for an incompressible material. Making use of these invariants, the elasticity tensor is given in its most general form for both the deformed and the undeformed (i.e., the initially stressed reference) configurations. The equations governing infinitesimal motions superimposed on a finite deformations are then used to study the effects of initial stress and finite deformation on wave propagation. For each of the problems carried out in this thesis, the results are specialized for a prototype strain energy function which depends on the initial stress as well as the deformation. The basic theory in each of the problems is formed for the material in the deformed configuration and is later specialized for the undeformed reference configuration. Considering the special case when initial stress is zero, the results are compared with those from the linear theory of elasticity.

The problem of homogeneous plane waves in an initially stressed incompressible half-space is considered. The basic theory of the problem is later used to study the reflection of plane waves from the boundary of such a half-space. The reflection coefficients of waves are calculated and graphical representations are given to study the behaviour with reference to the magnitude of initial stress and finite deformation.

The study of Rayleigh and Love waves follows thereafter and the basic theory already developed in this thesis is used to study the effect of initial stress on the wave speed of these surface waves. In both cases, the secular equation is analysed in deformed and undeformed configurations and graphs are presented.

The problem of wave propagation in a residually stressed inhomogeneous thick-walled incompressible tube which is axially stretched and inflated due to internal pressure, is considered. On the basis of known experimental behaviour, a simple expression for the residual stress is chosen to calculate the internal pressure used to inflate the tube and the axial load to stretch it. The effect of initial stress and stretch on pressure and axial load is studied and graphs are presented. The general theory developed for the deformed configuration for the special model is specialized to the reference configuration and the dispersion relation is analysed numerically.

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Statement

This thesis is submitted in accordance with the regulations for the degree of Doctor of Philosophy in the University of Glasgow.

No part of this thesis has previously been submitted by me for a degree at this or any other university.

Only a part of the research work in the early sections of Chapter 3 are due to appear in [46].

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Chapter 1

Introduction

Many materials can be internally stressed in their unloaded reference configuration. In metals, for example, an initial stress is caused by thermal processing like welding and/or mechanical processing like bending or forging of the metals parts. In geophysics, internal stresses are imposed on a rock due to processes such as burial, heating, cooling, and past tectonic events, etc., and these stresses remain locked inside the material after the rock is freed of boundary loads. In soft biological tissues, processes of growth and development in the tissue cause such stresses whereas in arteries, high blood pressure might leave the internal walls of the arteries stressed even when the pressure recedes. In all such cases, the material is therefore considered to be stressed in its initial state.

The term *initial stress* is used in its widest sense irrespective of the process that causes this stress. In the case when there is an associated pre-strain from an unstressed configuration due to an applied load, the term *prestress* replaces the term initial stress. However, in the situations when an initial stress is present in the absence of applied loads (body forces and surface tractions) it is called *residual stress* according to the definition of Hoger [16]. The presence of initial stresses in a material has a substantial effect on the material elastic properties and wave propagation. For example, the presence of residual stresses in a metal body can cause distortion or splitting of metal parts and also can result in premature fracture. However, residual stresses can be beneficial in many cases. For example, the peak stress present in the tissues in vivo is minimized by the residual stress present in the arterial walls. The presence of residual stress in vessels and arteries had significant effect on response of the tissue. For detailed discussion on this, we refer to [12, 21]. More recent work in this regards has been carried out by Ogden and Singh [36] and Holzapfel and Ogden [22] and the references therein.

We follow the basic concepts on residual stress from the work by Hoger who studied in detail the possibility of existence of residual stress in an elastic body with material symmetry [16]. For further discussion on residual stress in various respects and the development of basic constitutive equations for residually stressed materials, we refer to [17, 18, 19, 26, 27]. By definition, residually stressed elastic bodies are in mechanical equilibrium in the absence of surface traction and body forces. Due to the zero-traction condition, a residual stress depends on the geometry of the material body [16]. Further, any non-zero residual stress field is necessarily inhomogeneous and anisotropic [20, 16, 26]. This follows from the Signorini's mean stress theorem [14], so some of the body must be in compression and some in tension. Since the material symmetry and elastic properties are dependent on the residual stress, so the mechanical properties of such a material are expected to be inhomogeneous and anisotropic.

In this thesis, we have studied the phenomenon of wave propagation in initially stressed materials. For this, we refer to the most basic work done by Biot in [4, 5]. Biot explored various cases of wave propagation in an initially stressed material and specialised the results in a geophysical context. We also refer to the work of Tang in [48] and the references therein. Tang [48] considered wave motion in an infinite and initially stressed material medium for various special cases and compared with already found results. Man and Lu [28] followed the work by Hoger and presented generalised results which relate to much earlier work of Biot.

The above mentioned phenomenon of waves in elastic materials, also referred to as *acoustoelasticity*, has major application in the area of biomechanics that interfaces with acoustic wave propagation and elastography in living soft biological tissues. The theoretical study of acoustic waves in soft tissues can play a major role and provide some impetus for ultrasonic assessment and for non-invasive, non-destructive medical diagnostics. In regard to acoustoelasticity, we present basic theoretical formulation which is generally applicable to any hyperelastic material which is initially stressed. As a specialization to understand the general results clearly, a prototype strain energy function is used which depends on the initial stress and, in turn, the components of the elasticity tensor include the effects of initial stress.

Irrespective of the cause which develops the initial stress and whether or not it has a finite deformation associated with it, it is interesting to note the effect of the initial stress on small (static or time dependent) deformations. We refer to these as the *incremental deformations* when these are *linearized* relative to the initially stressed undeformed reference state. If there is some associated finite deformation then we refer to the theory as the theory of small deformations superimposed on large deformations. We adopt this approach as it is more generalized and simple. In particular, we consider a general constitutive law for a finitely deformed material which is initially stressed and then apply the increment. In this case, the components of the elasticity tensor are functions of the residual stress as well as the deformation and we make use of the most generalized form of the elasticity tensor. This approach is different from that followed by Biot [4, 5, 48] who used a linearized theory by taking first order terms in the stress and strain components and a very specialized assumed form of the constitutive equation. Also, in contrast to our approach, Man and Lu [28] assume the initial stress to be small so that the terms are linear in the initial stress and make use of a different form of elasticity tensor in their calculations.

We consider the formulation of a problem in elasticity for an initially stressed hyperelastic material. The basic concepts for this are collected in Chapter 2 of this thesis. We present the governing equilibrium equations for finite elasticity together with the concept of the elasticity tensor when it depends on the initial stress. Various useful identities of this elasticity tensor are given which follow from [17]. As mentioned above, we use a linearized theory [17] for constitutive equations and superimpose small deformations on the finite deformations to obtain the incremental equations for an initially stressed deformed hyperelastic material, which are being used throughout this thesis. Also, in every problem within this thesis, we suppose that the response of the considered material relative to the undeformed configuration would be isotropic in the absence of initial stress.

In Chapter 3, as a constitutive law, we consider a general strain energy function (defined per unit reference volume) which depends on the combined invariants of the right Cauchy-Green deformation tensor and the initial stress tensor. The expressions for these invariants are motivated from [19]. In general, there are 10 independent invariants for a compressible material which reduce to 9 for an incompressible material. The general expressions for the Cauchy stress and nominal stress tensors are given both for compressible and incompressible material in the deformed and undeformed initially stressed configurations. From these expressions, we find that the expression for an initial stress has to follow a few restrictions, namely given by Eqs. (3.10) and (3.13) for compressible and incompressible materials, respectively.

We consider a general form of the elasticity tensor which depends on the initial stress tensor ($\boldsymbol{\tau}$) as well as the right Cauchy-Green deformation tensor ($\mathbf{C}$). The detailed expressions for this tensor in case of a compressible material and an incompressible material are given which are further specialized for the undeformed initially stressed reference configuration. In the absence of initial stress, these expressions reduce to the classical elasticity tensor for isotropic materials. For simplicity of calculations only, we consider the dependence of the strain energy function on a limited set of invariants while ensuring that adequate effect of initial stress is included.

The general form of the acoustic tensor [44] for an initially stressed material is given both for the compressible and incompressible materials in the deformed and undeformed configurations. Various forms of the initial stress are considered, for instance to observe the dependence of the wave speed on the initial stress in the material. For a general initial stress, we consider materials which follow a specific constitutive model. This simple prototype model (Eq. (3.111)) is selected such that the restrictions on the initial stress are followed and the effects of initial stress and deformation both are included properly. Also, we consider that there is no stored elastic energy associated with a initially stressed material. Therefore, we take the energy function to be zero in the initially stressed undeformed reference configuration.

The ongoing theory in Chapter 3 then is applied to a problem of plane incremental motions in an initially-stressed incompressible homogeneous elastic half space. The general formulation of the problem is presented first and then specialized using the same prototype strain energy function. Homogeneous plane waves are considered and the analysis is carried out for incompressible materials in both the deformed and the undeformed reference configuration. In addition to this, respective problems for wave reflection from the plane boundary of an initially stressed half space are also considered and graphical results are included which show the effect of initial stress on reflection. It is noted that the reflection coefficients in this case behave in a similar fashion to those recorded by Ogden and Sotiropoulos [39], who analysed the effect of prestress on the propagation and reflection of plane waves in incompressible elastic solids.

In Chapter 4, we consider two types of surface waves, namely Rayleigh and Love waves, in an initially stressed homogenous incompressible material. A secular equation is found which is analysed for the strain energy function mentioned before. The dependence of surface wave speed on the initial stress and the strain is considered separately in both cases. It is noted that the presence of a compressional stress results in an increase in the surface wave speed whereas the wave speed decreases for tensile stress in the material. Tang [48] found a matching behaviour of the wave speed for the case of an infinite initially stressed medium under hydrostatic pressure.

In the last Chapter, i.e. Chapter 5, we consider the problem of wave propagation in a residually stressed cylindrical tube with axial extension and radial inflation. We propose various forms of the inhomogeneous residual stress based on the experimental behaviour of stresses present in arteries. The effect of residual stress on pressure used to inflate the tube is considered. It is noted that pressure stability depends on the ratio (λ_a given by Eq. (5.6)) of the inner wall thickness in the deformed and the undeformed configurations. Higher values of the parameters related to the initial stress lead to a higher value of the lower limit of λ_a and much increased values of pressure. Various graphs are presented to show this fact for numerous choices of parameters and varying wall thicknesses. It is interesting to note that for negative values of certain parameters, the pressure follows an upper and lower bound with respect to the ratio λ_a. The values obtained for the pressure are then used to calculate the axial load and the effect of initial stress is observed. Various choices of the parameters and wall thicknesses lead to a mixed behaviour of the axial load, which is recorded graphically.

The problem of wave propagation in an inflated and stretched residually stressed tube is then considered. Equations governing this problem are given both in the deformed and the undeformed configuration. It is found that the expressions for the deformed configuration are extremely cumbersome. We intend to solve the more complicated problem elsewhere. Due to time constraints, we solve a less complicated two-point boundary value problem in the reference configuration where we suppose the material is undeformed and the stretches are therefore equal to 1. We formulate a two-point boundary value problem in the reference configuration and solve it using the built-in function '`BVP4C`' in MATLAB. The dispersion relation for this problem is obtained numerically and various modes are plotted generally and in particular the behaviour of first mode for various values of the parameters and wall thicknesses. In each of the graphs, we plot the dispersion modes so as to observe the change in the magnitude of the dimensionless wave speed in the presence of residual stress from the state when no residual stress in present. The effect of residual stress is included in these expressions through the respective parameters. It may be noted that higher and positive values of these parameters result in a decrease of the wave speed due to the presence of residual stress. Similarly, lower or negative values of parameters give higher wave speeds in a residually stressed tube.

Chapter 2

Basics of the Theory of Finite Deformations

A body B is a set whose elements can be put into one-to-one correspondence with points of a region $\mathcal{B}$ in three-dimensional Euclidean space. The elements of B are called the *material points* and $\mathcal{B}$ is called a configuration of B. Let $\mathcal{B}_r$ be an arbitrarily chosen fixed reference configuration of $\mathcal{B}$. We assume that in this reference configuration the body is at rest and no external forces are present. Let $\mathcal{B}_t$ be the current configuration of $\mathcal{B}$ at time t. For the basic material covered in this chapter, we generally refer to [14] and [31].

2.1 Analysis of Deformation in an Elastic Material

A deformation $\boldsymbol{\chi}$ from $\mathcal{B}_r$ is a smooth one-to-one mapping that carries point $\mathbf{X} \in \mathcal{B}_r$ into the point $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ in $\mathcal{B}_t$, with components

$$x_i = \chi_i(X_\alpha, t), \quad i, \alpha = \{1, 2, 3\}. \tag{2.1}$$

Note that $\mathbf{x} = x_i \mathbf{e}_i$ with respect to the basis $\{\mathbf{e}_i\}$ and $\mathbf{X} = X_\alpha \mathbf{E}_\alpha$ with respect to the basis $\{\mathbf{E}_\alpha\}$. We will use Greek and Roman letters throughout for indices associated with the reference configuration and the deformed configuration, respectively.

The deformation gradient tensor is

$$\mathbf{F}(\mathbf{X}, t) = \operatorname{Grad} \mathbf{x} = \operatorname{Grad} \boldsymbol{\chi}(\mathbf{X}, t), \tag{2.2}$$

with components $F_{i\alpha} = \partial x_i / \partial X_\alpha$. Here Grad is the gradient operator in the reference configuration.

We adopt the usual convention that $J = \det \mathbf{F} > 0$, hence defining the notation J, which is a local measure of change in the material volume and appears in the equation of mass conservation $\rho_r = J\rho$, where ρ_r and ρ are the mass densities of the material in the reference and the deformed configurations, respectively.

For an isochoric (volume preserving) deformation,

$$J = \det \mathbf{F} = 1. \tag{2.3}$$

For an incompressible material all deformations are isochoric and Eq. (2.3) hence forms the incompressibility constraint.

The polar decompositions of $\mathbf{F}$ are given by

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR}, \tag{2.4}$$

where $\mathbf{R}$ is a proper orthogonal tensor, while $\mathbf{U}$ and $\mathbf{V}$ are positive definite, symmetric tensors, called the *right* and the *left stretch tensors*, respectively.

The spectral forms of $\mathbf{U}$ and $\mathbf{V}$ are given as

$$\mathbf{U} = \sum_{i=1}^{3} \lambda_i \mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)}, \quad \mathbf{V} = \sum_{i=1}^{3} \lambda_i \mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)}, \tag{2.5}$$

where $\lambda_i > 0$, $i \in \{1, 2, 3\}$, are the eigenvalues called the *principal stretches*, and $\mathbf{u}^{(i)}$ and $\mathbf{v}^{(i)}$ are the eigenvectors of $\mathbf{U}$ and $\mathbf{V}$, respectively. $\mathbf{u}^{(i)}$ and $\mathbf{v}^{(i)}$ are called the *Lagrangian* and *Eulerian* principal axes. Here, $\otimes$ denotes the tensor product defined for any two vectors, say $\mathbf{a}$ and $\mathbf{b}$, by

$$(\mathbf{a} \otimes \mathbf{b})\mathbf{v} = (\mathbf{b} \cdot \mathbf{v})\mathbf{a}, \tag{2.6}$$

i.e. $\mathbf{a} \otimes \mathbf{b}$ is a tensor that assigns to each vector $\mathbf{v}$ the vector $(\mathbf{b} \cdot \mathbf{v})\mathbf{a}$. Also note that $J = \det \mathbf{F}$ can be expressed in terms of the principal stretches λ_i through

$$J = \det \mathbf{U} = \det \mathbf{V} = \lambda_1 \lambda_2 \lambda_3. \tag{2.7}$$

The vectors $\mathbf{u}^{(i)}$ and $\mathbf{v}^{(i)}$ are connected through

$$\mathbf{v}^{(i)} = \mathbf{R}\mathbf{u}^{(i)}, \quad i = \{1, 2, 3\}. \tag{2.8}$$

We define the *right* and *left Cauchy-Green deformation tensors*, $\mathbf{C}$ and $\mathbf{B}$, respectively, as

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2, \quad \mathbf{B} = \mathbf{F}\mathbf{F}^T = \mathbf{V}^2. \tag{2.9}$$

The tensors $\mathbf{C}$ and $\mathbf{B}$ play important roles in the formation of constitutive laws, in particular through their principal invariants defined (for either $\mathbf{C}$ or $\mathbf{B}$) as

$$I_1 = \mathrm{tr}(\mathbf{C}), \quad I_2 = \frac{1}{2}[{I_1}^2 - \mathrm{tr}(\mathbf{C}^2)], \quad I_3 = \det(\mathbf{C}). \tag{2.10}$$

In terms of principal stretches, these are

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad I_2 = \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 + \lambda_1^2\lambda_2^2, \quad I_3 = \lambda_1^2\lambda_2^2\lambda_3^2. \tag{2.11}$$

Alternatively, the principal invariants of $\mathbf{U}$ can be used. Thus, we define

$$i_1 = \mathrm{tr}(\mathbf{U}), \quad i_2 = \frac{1}{2}[{i_1}^2 - \mathrm{tr}(\mathbf{U}^2)],, \quad i_3 = \det(\mathbf{U}), \tag{2.12}$$

or, equivalently, in terms of the stretches

$$i_1 = \lambda_1 + \lambda_2 + \lambda_3, \quad i_2 = \lambda_2\lambda_3 + \lambda_3\lambda_1 + \lambda_1\lambda_2, \quad i_3 = \lambda_1\lambda_2\lambda_3. \tag{2.13}$$

The connections between I_j and $i_j, j = \{1, 2, 3\}$ follow from the above expressions and are given by

$$I_1 = i_1^2 - 2i_2, \quad I_2 = i_2^2 - 2i_1 i_3, \quad I_3 = i_3^2. \tag{2.14}$$

2.2 Analysis of Motion in an Elastic Material

The velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ of a material particle $\mathbf{X}$ are defined by

$$\mathbf{v} \equiv \mathbf{x}_{,t} = \frac{\partial}{\partial t}\boldsymbol{\chi}(\mathbf{X}, t), \quad \mathbf{a} \equiv \mathbf{v}_{,t} \equiv \mathbf{x}_{,tt} = \frac{\partial^2}{\partial t^2}\boldsymbol{\chi}(\mathbf{X}, t), \tag{2.15}$$

respectively. Here $_{,t}$ in the subscript denotes the material time derivative.

It is sometimes useful to treat $\mathbf{v}$ as a function of $\mathbf{x}$ and t and we then define the velocity gradient tensor, denoted $\mathbf{L}$, as

$$\mathbf{L} = \mathrm{grad}\,\mathbf{v}, \tag{2.16}$$

with components (with respect to the basis $\{\mathbf{e}_i\}$)

$$L_{ij} = \frac{\partial v_i}{\partial x_j}. \tag{2.17}$$

Using the identity,

$$\mathrm{Grad}\,\mathbf{v} = (\mathrm{grad}\,\mathbf{v})\mathbf{F}, \tag{2.18}$$

we can write, using Eq. (2.16),

$$\mathrm{Grad}\,\mathbf{v} = \mathbf{LF}. \tag{2.19}$$

Since $\mathbf{v} \equiv \mathbf{x}_{,t}$, we also have

$$\mathrm{Grad}\,\mathbf{x}_{,t} = \frac{\partial}{\partial t}\mathrm{Grad}\,\mathbf{x} = \mathbf{F}_{,t}. \tag{2.20}$$

Hence from Eq. (2.19) and (2.20), we have the important connection

$$\mathbf{F}_{,t} = \mathbf{LF}. \tag{2.21}$$

Using the result for the derivative of the determinant of a tensor, i.e.

$$\frac{\partial}{\partial t}(\det\mathbf{F}) = (\det\mathbf{F})\mathrm{tr}\,(\mathbf{F}^{-1}\mathbf{F}_{,t}) = J\mathrm{tr}\,(\mathbf{F}^{-1}\mathbf{F}_{,t}), \tag{2.22}$$

together with Eq. (2.21), we have

$$\frac{\partial}{\partial t}(\det\mathbf{F}) = J\mathrm{tr}\,(\mathbf{L}), \tag{2.23}$$

or,

$$J_{,t} = J\mathrm{tr}\,(\mathbf{L}) = J\mathrm{div}\,\mathbf{v}, \tag{2.24}$$

where $\mathrm{tr}\,(\mathbf{L}) = L_{ii} = \partial v_i/\partial x_i = \mathrm{div}\,\mathbf{v}$. Here, div is the divergence operating in the current configuration, i.e. with respect to $\mathbf{x}$. Therefore, $\mathrm{div}\,\mathbf{v}$ measures the rate at which the volume changes during the motion. For an *isochoric* motion, $J = 1$, $J_{,t} = 0$ and hence

$$\mathrm{div}\,\mathbf{v} = 0, \tag{2.25}$$

which is another linearized form of the incompressibility constraint.

Also, $\mathbf{FF}^{-1} = \mathbf{I}$, and therefore

$$(\mathbf{F}^{-1})_{,t} = -\mathbf{F}^{-1}\mathbf{F}_{,t}\mathbf{F}^{-1} = -\mathbf{F}^{-1}\mathbf{L}. \tag{2.26}$$

2.3 Analysis of Stress and Equilibrium Equations for an Elastic Material

During a motion, mechanical interactions between parts of a body or between a body and its environment are described by using the concept of force. In this regard, one of the most important theorems is Cauchy's Theorem, stated as:

Let $(\mathbf{t}, \mathbf{b})$ be a system of surface and body forces for a body during a motion. Then a necessary and sufficient condition that the momentum balance laws be satisfied is that there exists a spatial tensor field $\mathbf{T}$, called the *Cauchy stress*, such that

- for each unit vector $\mathbf{n}$,

$$\mathbf{t}(\mathbf{n}) = \mathbf{T}\mathbf{n}; \tag{2.27}$$

- $\mathbf{T}$ is symmetric;

- $\mathbf{T}$ satisfies the *equation of motion*

$$\operatorname{div}\mathbf{T} + \rho\mathbf{b} = \rho\mathbf{v}_{,t}, \tag{2.28}$$

where $\mathbf{b}$ represents the body force.

Let $\mathbf{T} = \mathbf{T}(\mathbf{x}, t)$ be the stress at a particular place and time. If

$$\mathbf{T}\mathbf{n} = \sigma\mathbf{n}, \quad |\mathbf{n}| = 1, \tag{2.29}$$

then σ is a principal Cauchy stress and $\mathbf{n}$ is a principal direction, so that the principal Cauchy stress and principal directions are eigenvalues and eigenvectors of $\mathbf{T}$, respectively. Also, note that the symmetry of $\mathbf{T}$ ensures that three principal directions exist which are mutually perpendicular and the three corresponding principal stresses are real.

Consider an arbitrary oriented plane surface with positive unit normal $\mathbf{n}$ at $\mathbf{x}$. Then the surface force $\mathbf{T}\mathbf{n}$ can be decomposed into a sum of a *normal force* and a *shearing force*, respectively given by

$$(\mathbf{n} \cdot \mathbf{T}\mathbf{n})\mathbf{n} = (\mathbf{n} \otimes \mathbf{n})\mathbf{T}\mathbf{n}, \tag{2.30}$$

$$(\mathbf{I} - \mathbf{n} \otimes \mathbf{n})\mathbf{T}\mathbf{n}, \tag{2.31}$$

and it follows that $\mathbf{n}$ is a principal direction if and only if the corresponding shear stress vanishes. $\mathbf{I}$ is the identity tensor.

A fluid at rest is incapable of exerting shearing forces. In this case, $\mathbf{Tn}$ is parallel to $\mathbf{n}$ for each unit vector $\mathbf{n}$, and every such vector is an eigenvector of $\mathbf{T}$. We can write

$$\mathbf{Tn} = -\pi\mathbf{n}, \tag{2.32}$$

$$\mathbf{T} = -\pi\mathbf{I}, \tag{2.33}$$

where $\pi > 0$ is a scalar which represents the *pressure* of the fluid.

Two other important states of stress are:

- *Pure tension* (or compression) with tensile stress σ in the direction $\mathbf{e}$, where $|\mathbf{e}| = 1$:

$$\mathbf{T} = \sigma(\mathbf{e} \otimes \mathbf{e}). \tag{2.34}$$

- *Simple shear* with shear stress τ relative to the direction pair $(\mathbf{k}, \mathbf{n})$:

$$\mathbf{T} = \tau(\mathbf{k} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{k}), \tag{2.35}$$

  where $\mathbf{k}$ and $\mathbf{n}$ are orthogonal unit vectors

The Cauchy stress $\mathbf{T}$ measures the contact force per unit area in the *deformed configuration*. In many problems, it is not convenient to work with $\mathbf{T}$, since the deformed configuration is not known in advance. For this reason we may define a stress tensor $\mathbf{S}$ which gives the force measured per unit area in the *reference configuration*.

Through the *Nanson's formula*, we know the elements of surface area are related by

$$\mathbf{n}da = J\mathbf{F}^{-T}\mathbf{N}dA, \tag{2.36}$$

where $(\mathbf{n}, da)$ and $(\mathbf{N}, dA)$ are the unit normal and area elements in the deformed and the reference configuration, respectively. We can write the traction $\mathbf{t}(\mathbf{n})$ on an area element da in the deformed configuration as

$$\mathbf{t}da = \mathbf{Tn}da = J\mathbf{TF}^{-T}\mathbf{N}dA = \mathbf{S}^T\mathbf{N}dA, \tag{2.37}$$

where the *first Piola-Kirchhoff stress tensor* $\mathbf{S}^T$ is defined as

$$\mathbf{S}^T = J\mathbf{TF}^{-T}. \tag{2.38}$$

Therefore, the second order tensor field $\mathbf{S}$, called the *nominal stress tensor*, is given by

$$\mathbf{S} = J\mathbf{F}^{-1}\mathbf{T}. \tag{2.39}$$

This is also referred to as the *engineering stress*. $\mathbf{S}$ satisfies the equation of motion

$$\mathrm{Div}\,\mathbf{S} + \rho_r \mathbf{b} = \rho_r \boldsymbol{\chi}_{,tt}, \tag{2.40}$$

and the symmetry condition

$$\mathbf{F}\mathbf{S} = \mathbf{S}^T\mathbf{F}^T, \tag{2.41}$$

where Div is the divergence operator with respect to $\mathbf{X}$. We shall be considering the case without body forces which reduces Eq. (2.40) to the form

$$\mathrm{Div}\,\mathbf{S} = \rho_r \boldsymbol{\chi}_{,tt}. \tag{2.42}$$

The equivalent form of Eq. (2.42) in the deformed configuration is

$$\mathrm{div}\,\mathbf{T} = \rho \mathbf{v}_{,t}, \tag{2.43}$$

Eqs. (2.42) and (2.43) in component form are

$$\frac{\partial S_{\alpha i}}{\partial X_\alpha} = \rho_r \chi_{i,tt}, \tag{2.44}$$

and

$$\frac{\partial T_{ij}}{\partial x_j} = \rho v_{i,t}, \tag{2.45}$$

respectively.

Using Eq. (2.39), we define the *second Piola-Kirchhoff stress tensor*, denoted $\mathbf{S}^{(2)}$, as

$$\mathbf{S}^{(2)} \equiv \mathbf{S}\mathbf{F}^{-T} = J\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-T}. \tag{2.46}$$

Note that this tensor is symmetric.

2.4 Constitutive Equations in Finite Elasticity

Generally, the equations governing the motion of a continuous body are given by

- equation of mass conservation

$$\dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0; \tag{2.47}$$

- equation of motion

$$\operatorname{div} \mathbf{T} + \rho \mathbf{b} = \rho \mathbf{v}_{,\mathbf{t}}; \tag{2.48}$$

- equation of angular momentum balance

$$\mathbf{T}^T = \mathbf{T}. \tag{2.49}$$

Given Eq. (2.49), Eqs. (2.47) and (2.48) provide 4 equations for 10 scalar fields, i.e. ρ, $\mathbf{v}$ (3 components), $\mathbf{T}$ (6 components). The above mentioned laws are insufficient to fully characterise the behaviour of bodies because they do not distinguish between different types of materials. Physical experience has shown that two bodies of the same size and subject to the same motion will generally not have the same resulting stress distribution. We therefore introduce constitutive equations which make up for the missing equations and serve to distinguish different types of material behaviour.

The following are basic types of constitutive assumptions:

- Constraints on the possible deformations the body may undergo; e.g., rigid body motion, incompressibility or isochoric deformation, etc.,
- Assumptions on the form of the stress tensor; e.g., stress may be a pressure, etc...,
- constitutive equations relating the stress to the motion.

In classical mechanics, Hooke's law is the basic constitutive law which gives a relation between force and change in length without depending on the history of deformation or the rate of deformation. We define an *elastic material* as one for which the stress $\mathbf{T}(\mathbf{x}, t)$ at $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ depends only on the deformation gradient. Therefore, for such a material, the elastic constitutive equation for the Cauchy stress is

$$\mathbf{T}(\mathbf{x}, t) = \hat{\mathbf{T}}(\mathbf{F}(\mathbf{X}, t), \mathbf{X}). \tag{2.50}$$

Formally, an elastic body is a material body whose constitutive class is defined by a smooth response function $\hat{\mathbf{T}} : \mathrm{Lin}^+ \times \mathcal{B} \to \mathrm{Sym}$, where Lin^+ is a set of all second order tensors $\mathbf{F}$ with $\det \mathbf{F} > 0$ and Sym is the set of all symmetric second order tensors. It may be noted that $\mathbf{T}$

depends on $\mathbf{X}$ explicitly if the material is inhomogeneous. If the material is homogeneous $\mathbf{T}$ depends on $\mathbf{X}$ only through $\mathbf{F}$.

One of the main axioms of mechanics is the requirement that material response be independent of observer. In the case of an elastic body, a necessary and sufficient condition that the response of the elastic material is independent of the observer is that the response function $\hat{\mathbf{T}}$ satisfies

$$\mathbf{Q}\hat{\mathbf{T}}(\mathbf{F})\mathbf{Q}^T = \hat{\mathbf{T}}(\mathbf{QF}), \tag{2.51}$$

for every $\mathbf{F} \in \mathrm{Lin}^+$ and $\mathbf{Q} \in \mathrm{Orth}^+$, where Orth^+ is the set of all proper orthogonal tensors. The explicit dependence on $\mathbf{X}$ has been suppressed here. Also, we assume henceforth that the response is independent of the observer, so that Eq. (2.51) holds.

Alternative forms of constitutive equation (2.50) can be given using the deformation tensors $\mathbf{U}$ and $\mathbf{C}$. The equations may be referred to as reduced constitutive equations. The response function $\hat{\mathbf{T}}$ is completely determined by its restriction to PSym^+, where PSym^+ is the set of all symmetric, positive definite tensors). Consider the right polar decomposition of $\mathbf{F}$ i.e. $\mathbf{F} = \mathbf{RU}$, where $\mathbf{R} \in \mathrm{Orth}^+$ is the rotation tensor and $\mathbf{U} \in \mathrm{PSym}^+$, the right stretch tensor corresponding to $\mathbf{F}$. Eq. (2.51), with $\mathbf{Q}^T = \mathbf{R}$, gives

$$\hat{\mathbf{T}}(\mathbf{F}) = \mathbf{R}\hat{\mathbf{T}}(\mathbf{U})\mathbf{R}^T, \tag{2.52}$$

for every $\mathbf{F} \in \mathrm{Lin}^+$.

Further there exist smooth response functions $\breve{\mathbf{T}}, \tilde{\mathbf{T}}, \bar{\mathbf{T}}$ from $\mathrm{PSym}^+ \to \mathrm{Sym}$ such that

$$\hat{\mathbf{T}}(\mathbf{F}) = \mathbf{F}\breve{\mathbf{T}}(\mathbf{U})\mathbf{F}^T, \tag{2.53}$$

$$\hat{\mathbf{T}}(\mathbf{F}) = \mathbf{R}\tilde{\mathbf{T}}(\mathbf{C})\mathbf{R}^T, \tag{2.54}$$

$$\hat{\mathbf{T}}(\mathbf{F}) = \mathbf{F}\bar{\mathbf{T}}(\mathbf{C})\mathbf{F}^T, \tag{2.55}$$

where $\mathbf{C} = \mathbf{U}^2 = \mathbf{F}^T\mathbf{F}$ is the Cauchy-Green deformation tensor.

A *symmetry transformation* at $\mathbf{X}$ is a tensor $\mathbf{Q} \in \mathrm{Orth}^+$ such that

$$\hat{\mathbf{T}}(\mathbf{F}) = \hat{\mathbf{T}}(\mathbf{FQ}), \tag{2.56}$$

for every $\mathbf{F} \in \mathrm{Lin}^+$. This means that response of an elastic material is the same before and after the rotation $\mathbf{Q}$ in the reference configuration. Let $\mathcal{G}$ be the set of all symmetry transformations at $\mathbf{X}$. Note that $\mathcal{G}$ is a sub-group of Orth^+.

It may be noted that the response functions $\hat{\mathbf{T}}, \dot{\mathbf{T}}, \tilde{\mathbf{T}}$ and $\bar{\mathbf{T}}$ are invariant under $\mathcal{G}$. Formally,

$$\mathbf{Q}\hat{\mathbf{T}}(\mathbf{F})\mathbf{Q}^T = \hat{\mathbf{T}}(\mathbf{Q}\mathbf{F}\mathbf{Q}^T), \tag{2.57}$$

$$\mathbf{Q}\bar{\mathbf{T}}(\mathbf{C})\mathbf{Q}^T = \bar{\mathbf{T}}(\mathbf{Q}\mathbf{C}\mathbf{Q}^T), \tag{2.58}$$

for every $\mathbf{Q} \in \mathcal{G}, \mathbf{F} \in \mathrm{Lin}^+$ and $\mathbf{C} \in \mathrm{PSym}^+$. The equations for $\dot{\mathbf{T}}$ and $\tilde{\mathbf{T}}$ are written in the same manner.

For an elastic body, the same kind of equations can be written for the nominal stress tensor $\mathbf{S}$ as well. $\mathbf{S}$ is given by a constitutive equation of the form

$$\mathbf{S} = \hat{\mathbf{S}}(\mathbf{F}), \tag{2.59}$$

with

$$\hat{\mathbf{S}}(\mathbf{F}) = (\det \mathbf{F})\mathbf{F}^{-1}\hat{\mathbf{T}}(\mathbf{F}), \tag{2.60}$$

assuming the dependence of $\hat{\mathbf{S}}$ on $\mathbf{X}$ is understood. Choose $\mathbf{Q} \in \mathrm{Orth}^+$. Using Eq. (2.51), it can be shown that the objectivity condition for $\hat{\mathbf{S}}$ is

$$\hat{\mathbf{S}}(\mathbf{Q}\mathbf{F}) = \hat{\mathbf{S}}(\mathbf{F})\mathbf{Q}^T, \tag{2.61}$$

for every $\mathbf{F} \in \mathrm{Lin}^+$ and $\mathbf{Q} \in \mathrm{Orth}^+$.

Using Eq. (2.56), the symmetry condition for $\hat{\mathbf{S}}$ is

$$\hat{\mathbf{S}}(\mathbf{F}\mathbf{Q}^T) = \mathbf{Q}\hat{\mathbf{S}}(\mathbf{F}). \tag{2.62}$$

If $\mathbf{Q} \in \mathcal{G}$, using Eqs. (2.61) and (2.62), the material response function $\hat{\mathbf{S}}$ satisfies the invariance requirement under $\mathcal{G}$ as

$$\hat{\mathbf{S}}(\mathbf{Q}\mathbf{F}\mathbf{Q}^T) = \mathbf{Q}\hat{\mathbf{S}}(\mathbf{F})\mathbf{Q}^T, \tag{2.63}$$

for each $\mathbf{F} \in \mathrm{Lin}^+$ and for all $\mathbf{Q} \in \mathcal{G}$.

2.5 Initial Stress in a Material

It is generally assumed that the reference configuration $\mathcal{B}_r$ is stress free. This is not the case in many situations and there may be an *initial stress* present. This may, for example, be induced by some manufacturing process or, in the case of biological tissues, be generated by the process of growth, remodelling or adaptation.

Here, the term *initial stress* is used in its widest sense irrespective of the process that causes this stress. In the case when there is an associated pre-strain from an unstressed configuration due to an applied load, the term *prestress* replaces the term initial stress. However, in the situations when an initial stress is present in the absence of applied loads (body forces and surface tractions) it is called *residual stress* according to the definition of Hoger [16].

We suppose that the unloaded reference configuration $\mathcal{B}_r$ is not stress free and $\boldsymbol{\tau}$ is the Cauchy initial stress present in $\mathcal{B}_r$. We may take $\mathcal{B}_r$ to be the reference configuration with $\mathbf{F} = \mathbf{I}$. Since this is the reference configuration, there is no distinction between the Cauchy initial stress in $\mathcal{B}_r$ and the nominal initial stress $\mathbf{S}^{(r)}$ relative to $\mathcal{B}_r$. Formally, the stress

$$\boldsymbol{\tau} = \hat{\mathbf{T}}(\mathbf{I}, \mathbf{X}) = \hat{\mathbf{S}}(\mathbf{I}, \mathbf{X}), \tag{2.64}$$

is called the *initial stress* at $\mathbf{X}$. Hence, $\boldsymbol{\tau}$ is the stress present in the body when no deformation has occurred and there are no external forces acting on the body. Here $\hat{\mathbf{T}}$ and $\hat{\mathbf{S}}$ are the smooth response functions given by Eqs. (2.50) and (2.60), respectively. In the absence of body forces, this initial stress must satisfy the equilibrium equation

$$\operatorname{div} \boldsymbol{\tau} = \operatorname{Div} \mathbf{S}^{(r)} = \mathbf{0}, \quad \text{in } \mathcal{B}_r. \tag{2.65}$$

It may be noted that Div and div are the same in the reference configuration.

Using Eq. (2.64) in Eq. (2.57) at $\mathbf{F} = \mathbf{I}$, we have

$$\mathbf{Q}\boldsymbol{\tau}\mathbf{Q}^T = \boldsymbol{\tau}, \tag{2.66}$$

at a material point $\mathbf{X}$, for every $\mathbf{Q} \in \mathcal{G}$, i.e. the initial stress at a point commutes with every element of the symmetry group at that point. In terms of nominal stress tensor, the equivalent form of Eq. (2.66) is

$$\mathbf{Q}\mathbf{S}^{(r)} = \mathbf{S}^{(r)}\mathbf{Q}, \tag{2.67}$$

for every member $\mathbf{Q}$ of the symmetry group $\mathcal{G}$. Thus Eq. (2.67) imposes restrictions on the form of $\mathbf{S}^{(r)}$ and equivalently of $\boldsymbol{\tau}$.

Residual stress (also denoted $\boldsymbol{\tau}$ in this thesis) is defined to be the stress present in a body in an unloaded reference configuration, i.e. when traction is everywhere zero on the boundary. Thus, in the absence of body forces and surface traction equal to zero, the residual stress field is in equilibrium. The residual stress possible in a particular body depends, due to boundary conditions, on the shape and symmetry of the body. This fact has been studied in detail in a paper by Hoger [16]. An important consequence of this feature is that it distinguishes residual stress from conventional elastic properties and imposes the condition that a non-zero residual stress be non-uniform [17]. Also, Coleman and Noll [7] have found the forms for the residual stress fields for various specific symmetry groups.The constitutive equations appropriate for the description of materials that behave elastically in deformations from the residually stressed state have been derived under the assumption of small displacement gradients in [29] and for the case of small strains with arbitrary rotations in [18].

2.6 Hyperelastic Materials

An elastic body is a *hyperelastic* or a *Green elastic material* if the nominal stress $\hat{\mathbf{S}}(\mathbf{F},\mathbf{X})$ is the derivative of a scalar function $W(\mathbf{F},\mathbf{X})$ for fixed $\mathbf{X}$, i.e.

$$\mathbf{S} = \hat{\mathbf{S}}(\mathbf{F},\mathbf{X}) = \mathrm{D}W(\mathbf{F},\mathbf{X}) = \frac{\partial W(\mathbf{F},\mathbf{X})}{\partial \mathbf{F}}, \tag{2.68}$$

where D denotes the derivative with respect to $\mathbf{F}$. The scalar function $W : \mathrm{Lin}^{+} \times \mathcal{B} \to \mathbb{R}$ is called the strain-energy density function. $W(\mathbf{F},\mathbf{X})$ represents the work done (per unit volume at $\mathbf{X}$) by the stress in deforming the material from $\mathcal{B}_r$ to $\mathcal{B}_t$ (i.e from $\mathbf{I}$ to $\mathbf{F}$) and is independent of the path taken in deformation space: see [32]. $W(\mathbf{F},\mathbf{X})$ possesses the property of being indifferent to observer transformations. For a hyperelastic material which is isotropic relative to $\mathcal{B}_r$, it can be shown that W is an isotropic scalar function of $\mathbf{V}$. This means the following equation holds

$$W(\mathbf{Q}\mathbf{V}\mathbf{Q}^T) = W(\mathbf{V}), \tag{2.69}$$

for all orthogonal $\mathbf{Q}$. This enables us to regard W as a function of the principal invariants I_1, I_2, I_3 given by Eq. (2.10) or, equivalently, as a symmetric function of the principal stretches $\lambda_1, \lambda_2, \lambda_3$.

The meaning of the tensor $\partial W/\partial \mathbf{F}$ is given by

$$\mathrm{D}W(\mathbf{F},\mathbf{X})[\mathbf{A}] = \frac{\partial W(\mathbf{F},\mathbf{X})}{\partial \mathbf{F}} \cdot \mathbf{A}, \tag{2.70}$$

where the inner product on Lin is defined to be

$$\mathbf{A}\cdot\mathbf{B} = \mathrm{tr}(\mathbf{A}^T\mathbf{B}), \tag{2.71}$$

for any $\mathbf{A}, \mathbf{B} \in$ Lin. Lin is the set of all second order tensors.

Equation (2.68) gives the nominal stress in terms of the strain energy function W. Using Eq. (2.39), the Cauchy stress $\mathbf{T}$ is therefore given by

$$\mathbf{T} = J^{-1}\mathbf{F}\frac{\partial W}{\partial \mathbf{F}}. \tag{2.72}$$

In component form, the nominal and Cauchy stresses are

$$S_{\alpha i} = \frac{\partial W}{\partial F_{i\alpha}}, \quad T_{ij} = J^{-1}F_{i\beta}\frac{\partial W}{\partial F_{j\beta}}. \tag{2.73}$$

Incorporating the incompressibility constraint, the counterparts of the above equations are

$$S_{\alpha i} = \frac{\partial W}{\partial F_{i\alpha}} - p(F^{-1})_{\alpha i}, \quad T_{ij} = F_{i\beta}\frac{\partial W}{\partial F_{j\beta}} - p\delta_{ij}, \tag{2.74}$$

where p is the Lagrange multiplier.

It is convenient to assume that W is measured from the reference configuration, so that

$$W(\mathbf{I}) = 0. \tag{2.75}$$

If the reference configuration is stress free then

$$\frac{\partial W}{\partial \mathbf{F}}(\mathbf{I}) = \mathbf{0}, \tag{2.76}$$

for an unconstrained material, and

$$\frac{\partial W}{\partial \mathbf{F}}(\mathbf{I}) - p_0\mathbf{I} = \mathbf{0} \tag{2.77}$$

for an incompressible material. Here p_0 is the value of p when evaluated in the reference configuration.

A detailed discussion on hyperelastic material with residual stress is carried out in Section 3.1.

2.7 The Elasticity Tensor

The behaviour of the constitutive equation $\mathbf{S} = \hat{\mathbf{S}}(\mathbf{F}, \mathbf{X})$ for an unconstrained hyperelastic material is governed by the linear transformation $\boldsymbol{\mathcal{A}} : \mathrm{Lin} \to \mathrm{Lin}$, defined by

$$\boldsymbol{\mathcal{A}}(\mathbf{F}, \mathbf{X})[\mathbf{A}] = D\hat{\mathbf{S}}(\mathbf{F}, \mathbf{X})[\mathbf{A}] = \frac{\partial^2 W(\mathbf{F}, \mathbf{X})}{\partial \mathbf{F}^2} \cdot \mathbf{A}, \tag{2.78}$$

for each fixed material point $\mathbf{X}$ (see, for example, [31, §5.1]). Here, $\boldsymbol{\mathcal{A}}$ is called the *elasticity tensor* at the material point $\mathbf{X}$ and represents a (fourth-order) tensor of elastic moduli associated with the conjugate pair $(\mathbf{S}, \mathbf{F})$. A detailed general discussion on elastic moduli tensors can be found in [31, Chap. 6].

In component form

$$\mathcal{A}_{\alpha i \beta j} = \frac{\partial^2 W}{\partial F_{i\alpha} \partial F_{j\beta}}. \tag{2.79}$$

It may be noted hat $\boldsymbol{\mathcal{A}}$ is a fourth rank tensor and has 81 components but this number is reduced to 45 independent components using the major symmetries. The major symmetry of $\boldsymbol{\mathcal{A}}$ is given by

$$\mathcal{A}_{\alpha i \beta j} = \mathcal{A}_{\beta j \alpha i}. \tag{2.80}$$

In the reference configuration, i.e. when $\mathbf{F} = \mathbf{I}$, we write

$$\boldsymbol{\mathcal{C}}(\mathbf{X}) = \boldsymbol{\mathcal{A}}(\mathbf{I}, \mathbf{X}). \tag{2.81}$$

From this point, $\boldsymbol{\mathcal{A}}$ (or $\boldsymbol{\mathcal{C}}$) will be assumed to depend on $\mathbf{X}$ unless otherwise stated and the expression of $\mathbf{X}$ in the argument will be suppressed.

The elasticity tensor $\boldsymbol{\mathcal{C}}$ possesses the following useful properties:

1.
$$\boldsymbol{\mathcal{C}}[\mathbf{W}] = -\boldsymbol{\tau}\mathbf{W}, \tag{2.82}$$

   for every skew-symmetric $\mathbf{W}$.

   This can be proved by assuming $\mathbf{Q} = \mathbf{Q}(t)$ in Eq. (2.61). Differentiating Eq. (2.61) with respect to t, we get

$$D\hat{\mathbf{S}}(\mathbf{QF})[\mathbf{Q}_{,t}\mathbf{F}] = \hat{\mathbf{S}}(\mathbf{F})\mathbf{Q}_{,t}^T. \tag{2.83}$$

   The choice $\mathbf{Q}(t) = e^{\mathbf{W}t}$, for every skew-symmetric $\mathbf{W}$, and evaluation of Eq. (2.83) at $t = 0$ gives

$$D\hat{\mathbf{S}}(\mathbf{F})[\mathbf{WF}] = -\hat{\mathbf{S}}(\mathbf{F})\mathbf{W}. \tag{2.84}$$

Evaluation of Eq. (2.84) at $\mathbf{F} = \mathbf{I}$ establishes Eq. (2.82).

2.

$$\text{Skw}\ \boldsymbol{\mathcal{C}}[\mathbf{E}] = \frac{1}{2}(\boldsymbol{\tau}\mathbf{E} - \mathbf{E}\boldsymbol{\tau}), \tag{2.85}$$

for every symmetric $\mathbf{E}$ and Skw means the skew-symmetric part of $\boldsymbol{\mathcal{C}}[\mathbf{E}]$.

This property can be proved by using the symmetry of Cauchy stress. Differentiating Eq. (2.60) with respect to t and using Eqs. (2.22) and (2.26), we get after some simplification

$$D\hat{\mathbf{S}}(\mathbf{F})[\mathbf{A}]\mathbf{F} + \hat{\mathbf{S}}(\mathbf{F})\mathbf{A} = (\det \mathbf{F})\text{tr}\,(\mathbf{F}^{-1}\mathbf{A})\hat{\mathbf{T}}(\mathbf{F}) + (\det \mathbf{F})D\hat{\mathbf{T}}(\mathbf{F})[\mathbf{A}], \tag{2.86}$$

where $\mathbf{A} = \mathbf{F}_{,t}$. For $\mathbf{F} = \mathbf{I}$, the above equation becomes

$$\begin{aligned}\boldsymbol{\mathcal{C}}[\mathbf{A}] &= \text{tr}\,(\mathbf{A})\hat{\mathbf{T}}(\mathbf{I}) + D\hat{\mathbf{T}}(\mathbf{I})[\mathbf{A}] - \hat{\mathbf{S}}(\mathbf{I})\mathbf{A} \\ &= \text{tr}\,(\mathbf{A})\boldsymbol{\tau} + D\hat{\mathbf{T}}(\mathbf{I})[\mathbf{A}] - \boldsymbol{\tau}\mathbf{A}.\end{aligned} \tag{2.87}$$

Both $\boldsymbol{\tau}$ and $D\hat{\mathbf{T}}(\mathbf{I})[\mathbf{A}]$ are symmetric. So for symmetric $\mathbf{A}$ in the above equation, we establish Eq. (2.85).

3.

$$\mathbf{Q}\boldsymbol{\mathcal{C}}[\mathbf{H}]\mathbf{Q}^T = \boldsymbol{\mathcal{C}}[\mathbf{Q}\mathbf{H}\mathbf{Q}^T], \tag{2.88}$$

for every $\mathbf{H} \in$ Lin and $\mathbf{Q} \in \mathcal{G}$. This, in other words, means that $\boldsymbol{\mathcal{C}}$ fulfils the invariance requirement under the symmetry group $\mathcal{G}$ for the material at $\mathbf{X}$.

From Eq. (2.63), $\mathbf{S}$ satisfies the invariance requirement under the symmetry group $\mathcal{G}$ and so does its derivative $D\hat{\mathbf{S}}(\mathbf{F})$. This means

$$D\hat{\mathbf{S}}(\mathbf{Q}\mathbf{F}\mathbf{Q}^T)[\mathbf{Q}\mathbf{H}\mathbf{Q}^T] = \mathbf{Q}D\hat{\mathbf{S}}(\mathbf{F})[\mathbf{H}]\mathbf{Q}^T, \tag{2.89}$$

for every $\mathbf{H} \in$ Lin and $\mathbf{Q} \in \mathcal{G}$. For the proof of invariance of the derivative, see [14]. Using the above property of the derivative, we can prove Eq. (2.88). Taking the left hand side of Eq. (2.88)

$$\begin{aligned}\mathbf{Q}\boldsymbol{\mathcal{C}}[\mathbf{H}]\mathbf{Q}^T &= \mathbf{Q}D\hat{\mathbf{S}}(\mathbf{I})[\mathbf{H}]\mathbf{Q}^T \\ &= D\hat{\mathbf{S}}(\mathbf{Q}\mathbf{I}\mathbf{Q}^T)[\mathbf{Q}\mathbf{H}\mathbf{Q}^T] \\ &= D\hat{\mathbf{S}}(\mathbf{I})[\mathbf{Q}\mathbf{H}\mathbf{Q}^T] \\ &= \boldsymbol{\mathcal{C}}[\mathbf{Q}\mathbf{H}\mathbf{Q}^T]. \end{aligned} \tag{2.90}$$

2.8 The Linearized Constitutive Equations for Materials Under Residual Stress

The displacement of a material point is

$$\mathbf{u}(\mathbf{X}) = \boldsymbol{\chi}(\mathbf{X}) - \mathbf{X}. \tag{2.91}$$

The displacement gradient

$$\mathbf{H} = \mathrm{Grad}\,\mathbf{u}, \tag{2.92}$$

is related to the deformation gradient by

$$\mathbf{F} = \mathbf{I} + \mathbf{H}. \tag{2.93}$$

From the above equation, $\hat{\mathbf{S}}(\mathbf{F})$ can be considered a function of $\mathbf{H}$. We take the special case of small displacements i.e. $|\mathbf{H}| << 1$. Therefore, we can write

$$\mathbf{S} = \hat{\mathbf{S}}(\mathbf{F}) = \hat{\mathbf{S}}(\mathbf{I}+\mathbf{H}) = \hat{\mathbf{S}}(\mathbf{I}) + \mathrm{D}\hat{\mathbf{S}}(\mathbf{I})[\mathbf{H}] + o[\mathbf{H}], \tag{2.94}$$

assuming the differentiability of $\hat{\mathbf{S}}(\mathbf{F})$. Consider $\mathbf{H} = \mathbf{E} + \mathbf{W}$, where $\mathbf{E}$ and $\mathbf{W}$ are the symmetric and skew-symmetric parts of $\mathbf{H}$, respectively. Using Eqs. (2.82) and (2.85) in Eq. (2.94), we can write

$$\mathbf{S} = \boldsymbol{\tau} - \boldsymbol{\tau}\mathbf{W} + \frac{1}{2}(\boldsymbol{\tau}\mathbf{E} - \mathbf{E}\boldsymbol{\tau}) + \boldsymbol{\mathcal{L}}[\mathbf{E}] + o[\mathbf{H}], \tag{2.95}$$

where $\boldsymbol{\mathcal{L}}$ is the linear transformation from Sym to Sym defined by

$$\boldsymbol{\mathcal{L}}[\mathbf{E}] = \mathrm{Symmetric}\{\boldsymbol{\mathcal{C}}[\mathbf{E}]\}. \tag{2.96}$$

We refer to $\boldsymbol{\mathcal{L}}$ as the *incremental elasticity tensor* and it is the symmetric part of the elasticity tensor $\boldsymbol{\mathcal{C}}$.

When the initial stress vanishes and $o[\mathbf{H}]$ is neglected, Eq. (2.95) reduces to the stress strain law of classical linear elasticity. When the residual stress does not vanish, the stress $\mathbf{S}$ is generally not symmetric. However, calculation of $\mathbf{FS}$ using Eqs. (2.93), (2.95), (2.96) and symmetry of $\boldsymbol{\tau}$ yields

$$\mathbf{FS} = \mathbf{S}^T\mathbf{F}^T + o[\mathbf{H}]. \tag{2.97}$$

Hence, the balance of moments is automatically satisfied within the o$[\mathbf{H}]$ when the constitutive equation (2.95) is used.

For $\mathbf{Q} \in \mathcal{G}$, using Eqs. (2.88) and (2.97) it can be deduced that

$$\boldsymbol{\mathcal{L}}[\mathbf{QEQ}^T] = \mathbf{Q}\boldsymbol{\mathcal{L}}[\mathbf{E}]\mathbf{Q}^T, \tag{2.98}$$

where $\mathbf{E} \in \mathrm{Sym}$.

For the considered hyperelastic material, using the major symmetry of $\boldsymbol{\mathcal{C}}$, we have

$$\hat{\mathbf{E}} \cdot \boldsymbol{\mathcal{C}}[\mathbf{E}] = \mathbf{E} \cdot \boldsymbol{\mathcal{C}}[\hat{\mathbf{E}}], \tag{2.99}$$

where $\mathbf{E}, \hat{\mathbf{E}} \in \mathrm{Sym}$. Using Eq. (2.99) with

$$\boldsymbol{\mathcal{C}}[\mathbf{E}] = \frac{1}{2}(\boldsymbol{\tau}\mathbf{E} - \mathbf{E}\boldsymbol{\tau}) + \boldsymbol{\mathcal{L}}[\mathbf{E}], \tag{2.100}$$

and considering the symmetry of $\boldsymbol{\tau}$, we get

$$\hat{\mathbf{E}} \cdot \boldsymbol{\mathcal{L}}[\mathbf{E}] = \mathbf{E} \cdot \boldsymbol{\mathcal{L}}[\hat{\mathbf{E}}], \tag{2.101}$$

i.e., $\boldsymbol{\mathcal{L}}$ possesses the major symmetry.

2.9 Incremental Equations

Let $\boldsymbol{\chi}$ with $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$, be a known time-independent deformation. Let $\boldsymbol{\chi}^{'}$, with $\mathbf{x}^{'} = \boldsymbol{\chi}^{'}(\mathbf{X}, t)$ be a finite time-dependent deformation which is "close" to $\boldsymbol{\chi}$. The displacement, which can be thought of as a perturbation of $\boldsymbol{\chi}$, is written as

$$\dot{\mathbf{x}} = \mathbf{x}' - \mathbf{x} = \boldsymbol{\chi}^{'}(\mathbf{X}, t) - \boldsymbol{\chi}(\mathbf{X}, t) \equiv \dot{\boldsymbol{\chi}}(\mathbf{X}), \tag{2.102}$$

and its gradient is

$$\text{Grad}\,\dot{\boldsymbol{\chi}} = \text{Grad}\ \boldsymbol{\chi}^{'} - \text{Grad}\,\boldsymbol{\chi} \equiv \dot{\mathbf{F}}, \tag{2.103}$$

which is exact since it is without any approximation.

Consider the linear approximation of the important stress tensor $\mathbf{S}$. The incremental stress tensor $\dot{\mathbf{S}}$ for an unconstrained material, in its exact form, is

$$\dot{\mathbf{S}} = \mathbf{S}' - \mathbf{S} = \frac{\partial W}{\partial \mathbf{F}}(\mathbf{F}') - \frac{\partial W}{\partial \mathbf{F}}(\mathbf{F}), \tag{2.104}$$

which has the linear approximation

$$\dot{\mathbf{S}} = \boldsymbol{\mathcal{A}}\dot{\mathbf{F}}, \tag{2.105}$$

where $\boldsymbol{\mathcal{A}}$ is the elasticity tensor, the components of which are given by Eq. (2.79). Equation (2.105) in its component form, is

$$\dot{S}_{\alpha i} = \mathcal{A}_{\alpha i\beta j}\dot{F}_{j\beta}, \tag{2.106}$$

where $\dot{F}_{j\beta} = \dot{x}_{j,\beta}$. In the case of an incompressible material, the counterpart of Eq. (2.106), using Eq. (2.74), is

$$\dot{\mathbf{S}} = \boldsymbol{\mathcal{A}}\dot{\mathbf{F}} - \dot{p}\mathbf{F}^{-1} + p\mathbf{F}^{-1}\dot{\mathbf{F}}\mathbf{F}^{-1}, \tag{2.107}$$

coupled with the incompressibility constraint $\det \mathbf{F} = 1$ in its linearized incremental form, given by

$$\text{tr}\,(\dot{\mathbf{F}}\mathbf{F}^{-1}) = 0. \tag{2.108}$$

Here $\dot{p}$ is the linearized incremental form of p.

Following the equation of motion (2.40) for $\boldsymbol{\chi}'$ and subtracting its counterpart equation for $\boldsymbol{\chi}$, we obtain

$$\text{Div}\,\dot{\mathbf{S}} + \rho_r\dot{\mathbf{b}} = \rho_r\dot{\boldsymbol{\chi}}_{,tt}, \tag{2.109}$$

where $\dot{\mathbf{b}}$ and $\dot{\boldsymbol{\chi}}$, defined earlier in Eq. (2.102), are the incremental forms of the body force $\mathbf{b}$ and $\boldsymbol{\chi}$, respectively. Equation (2.109) is exact but can be linearly approximated (to the first order in $\dot{\mathbf{F}}$) using either Eq. (2.105) for an unconstrained material, or Eq. (2.107) along with Eq. (2.108) for an incompressible material. In respect of Eq. (2.105), Eq. (2.109) has the component form

$$\frac{\partial}{\partial X_\alpha}(\mathcal{A}_{\alpha i\beta j}\frac{\partial \dot{\chi}_j}{\partial X_\beta}) + \rho_r\dot{b}_i = \rho_r\dot{\chi}_{i,tt}, \tag{2.110}$$

or, equivalently

$$\mathcal{A}_{\alpha i\beta j}\frac{\partial^2 \dot{\chi}_j}{\partial X_\alpha \partial X_\beta} + \mathcal{A}_{\alpha i\beta j,\alpha}\frac{\partial \dot{\chi}_j}{\partial X_\beta} + \rho_r\dot{b}_i = \rho_r\dot{\chi}_{i,tt}. \tag{2.111}$$

Here

$$\mathcal{A}_{\alpha i\beta j,\alpha} = \frac{\partial \mathcal{A}_{\alpha i\beta j}}{\partial X_\alpha} = \mathcal{A}_{\alpha i\beta j\gamma k}\frac{\partial \chi_k}{\partial X_\gamma \partial X_\alpha}, \tag{2.112}$$

where

$$\mathcal{A}_{\alpha i\beta j\gamma k} = \frac{\partial^3 W}{\partial F_{i\alpha}\partial F_{j\beta}\partial F_{k\gamma}}. \tag{2.113}$$

If the deformation $\boldsymbol{\chi}$ is homogeneous, the expression in Eq. (2.112) vanishes as $\boldsymbol{\mathcal{A}}$ is independent of $\mathbf{X}$ and Eq. (2.111) reduces to

$$\mathcal{A}_{\alpha i\beta j}\frac{\partial^2 \dot{\chi}_j}{\partial X_\alpha \partial X_\beta} + \rho_r \dot{b}_i = \rho_r \dot{\chi}_{i,tt}. \tag{2.114}$$

For an incompressible material, using Eq. (2.107) along with Eq. (2.108) in Eq. (2.109) gives

$$\mathrm{Div}\,(\boldsymbol{\mathcal{A}}\dot{\mathbf{F}}) - \mathbf{F}^{-T}\mathrm{Grad}\,\dot{p} + \mathbf{F}^{-T}\dot{\mathbf{F}}^T\mathbf{F}^{-T}\mathrm{Grad}\,p + \rho_r\dot{\mathbf{b}} = \rho_r\dot{\boldsymbol{\chi}}_{,tt}. \tag{2.115}$$

For a homogeneous deformation, Eq. (2.115), in its component form, reduces to

$$\mathcal{A}_{\alpha i\beta j}\frac{\partial^2 \dot{\chi}_j}{\partial X_\alpha \partial X_\beta} - \frac{\partial \dot{p}}{\partial x_i} + \rho_r \dot{b}_i = \rho_r \dot{\chi}_{i,tt}. \tag{2.116}$$

Taking the increment of Eq. (2.41), we get

$$\mathbf{F}\dot{\mathbf{S}} + \dot{\mathbf{F}}\mathbf{S} = \dot{\mathbf{S}}^T\mathbf{F}^T + \mathbf{S}^T\dot{\mathbf{F}}^T. \tag{2.117}$$

In dealing with incremental deformations it is often convenient to choose the reference configuration to coincide with the current configuration. In that case all the quantities are updated accordingly and treated as functions of $\mathbf{x}$ instead of $\mathbf{X}$. For this purpose, we define the notations

$$\mathbf{u}(\mathbf{x},t) = \dot{\boldsymbol{\chi}}(\boldsymbol{\chi}^{-1}(\mathbf{x},t)), \quad \boldsymbol{\Gamma} = \dot{\mathbf{F}}\mathbf{F}^{-1}, \quad \dot{\mathbf{S}}_0 = J^{-1}\mathbf{F}\dot{\mathbf{S}}, \tag{2.118}$$

the latter being the push-forward of $\dot{\mathbf{S}}$ motivated by the connection $\mathbf{T} = J^{-1}\mathbf{F}\mathbf{S}$.

Let $\boldsymbol{\mathcal{A}}_0$ be the updated form of $\boldsymbol{\mathcal{A}}$. The updated elasticity tensor, in terms of $\boldsymbol{\mathcal{A}}$, is

$$\mathcal{A}_{0ijkl} = J^{-1}F_{i\alpha}F_{k\beta}\mathcal{A}_{\alpha j\beta l}. \tag{2.119}$$

For the derivation of this relation see [31]. The updated nominal stress for an unconstrained material is given by

$$\dot{\mathbf{S}}_0 = \boldsymbol{\mathcal{A}}_0\dot{\mathbf{F}}_0 = \boldsymbol{\mathcal{A}}_0\boldsymbol{\Gamma}, \tag{2.120}$$

where $\dot{\mathbf{F}}_0 = \mathbf{\Gamma}$ is the updated incremental form of the deformation gradient in the reference configuration.

The updated counterparts of Eqs. (2.109) and (2.117) are

$$\mathrm{Div}\,\dot{\mathbf{S}}_0 + \rho\dot{\mathbf{b}} = \rho\mathbf{u}_{,tt}, \tag{2.121}$$

$$\dot{\mathbf{F}}_0\mathbf{T} + \dot{\mathbf{S}}_0 = \dot{\mathbf{S}}_0^T + \mathbf{T}\dot{\mathbf{F}}_0^T, \tag{2.122}$$

where ρ is the current density and $\mathbf{T}$ is the Cauchy stress.

Using Eq. (2.120), Eq. (2.122) can be further reduced to the form

$$\mathbf{\Gamma}\mathbf{T} + \boldsymbol{\mathcal{A}}_0\mathbf{\Gamma} = (\boldsymbol{\mathcal{A}}_0\mathbf{\Gamma})^T + \mathbf{T}\mathbf{\Gamma}^T. \tag{2.123}$$

In the case of an incompressible material, the counterpart of Eq. (2.120) is

$$\dot{\mathbf{S}}_0 = \boldsymbol{\mathcal{A}}_0\mathbf{\Gamma} - \dot{p}\mathbf{I} + p\mathbf{\Gamma}, \tag{2.124}$$

along with the updated incompressibility condition, (following Eq. (2.108)),

$$\mathrm{tr}\,(\mathbf{\Gamma}) \equiv \mathrm{div}\,\mathbf{u} = 0. \tag{2.125}$$

In component form, Eqs. (2.120) and (2.124) are

$$\dot{S}_{0pi} = \mathcal{A}_{0piqj}u_{j,q}, \tag{2.126}$$

$$\dot{S}_{0pi} = \mathcal{A}_{0piqj}u_{j,q} - \dot{p}\delta_{pi} + p\Gamma_{pi}, \tag{2.127}$$

respectively. Hence, replacing $\dot{\mathbf{S}}_0$ with either of the above expressions, the equation of motion (2.121) can be linearized. In the above formulation, it is assumed that second and higher order terms in $\dot{\boldsymbol{\chi}}$ and its derivatives may be neglected. We have, therefore, derived the linearized theory of *incremental deformations superimposed on a finite deformation.*

Chapter 3

Plane Waves in Initially-stressed Materials

The effect of initial stress on the propagation of waves in elastic materials was initially studied by Biot in [4, 5]. Here, we use the concept of the strain energy function to develop the basic equations required to carry out the analysis of plane wave propagation when the materials is initially stressed, irrespective of the cause that develops this initial stress. We study plane wave propagation in both the cases of a homogeneous incompressible material in its deformed as well as undeformed states. The effect of initial stress is included generally through the invariants of the right Cauchy-Green deformation tensor. Later, the theory is applied to the problem of reflection of a plane wave in an initially stressed incompressible material. The effect of pre-stress on wave propagation was examined by Ogden and Sotiropoulos for compressible materials [37] and for incompressible materials [39]. We also refer to the work of Hussain and Ogden on reflection and transmission of plane waves [23, 24, 25].

3.1 Initial Stress in Hyperelastic Materials

We consider an initially stressed homogeneous hyperelastic material for which the strain energy function W per unit reference volume depends on the deformation gradient $\mathbf{F}$ and the initial stress $\boldsymbol{\tau}$. By objectivity we can regard W as a function of $\mathbf{F}$ through the right Cauchy-Green deformation tensor given by Eq. $(2.9)_1$. Thus $W = W(\mathbf{C}, \boldsymbol{\tau})$.

When subjected to a rotation $\mathbf{Q}$ in the reference configuration, $\mathbf{C}$ and $\boldsymbol{\tau}$ change to $\mathbf{QCQ}^T$ and $\mathbf{Q}\boldsymbol{\tau}\mathbf{Q}^T$, respectively. The strain energy is invariant under this change if it depends on the 10 invariants $I_1, ..., I_{10}$ of the two tensors $\mathbf{C}$ and $\boldsymbol{\tau}$ defined by

$$\begin{aligned} I_1 &= \mathrm{tr}(\mathbf{C}), \quad I_2 = \frac{1}{2}[I_1^2 - \mathrm{tr}(\mathbf{C}^2)], \quad I_3 = \det(\mathbf{C}), \quad I_4 = \mathrm{tr}(\boldsymbol{\tau}), \\ I_5 &= \frac{1}{2}[I_4^2 - \mathrm{tr}(\boldsymbol{\tau}^2)], \quad I_6 = \det(\boldsymbol{\tau}), \quad I_7 = \mathrm{tr}(\mathbf{C}\boldsymbol{\tau}), \\ I_8 &= \mathrm{tr}(\mathbf{C}^2\boldsymbol{\tau}), \quad I_9 = \mathrm{tr}(\mathbf{C}\boldsymbol{\tau}^2), \quad I_{10} = \mathrm{tr}(\mathbf{C}^2\boldsymbol{\tau}^2). \end{aligned} \tag{3.1}$$

These are the only independent principal invariants for the two tensors $\mathbf{C}$ and $\boldsymbol{\tau}$. Note, in particular, that $\mathrm{tr}(\mathbf{C}\boldsymbol{\tau}\mathbf{C}\boldsymbol{\tau})$ depends on $I_1, I_2, ..., I_{10}$. Similarly for $\mathrm{tr}(\mathbf{C}\boldsymbol{\tau}\mathbf{C}^2\boldsymbol{\tau})$, etc. Evaluating the expressions in Eqs. (3.1) for $\mathbf{F} = \mathbf{C} = \mathbf{I}$, we get the invariants in the reference configuration as

$$\begin{aligned} &I_1 = I_2 = 3, \quad I_3 = 1, \quad I_4 = I_7 = I_8 = \mathrm{tr}(\boldsymbol{\tau}), \\ &I_5 = \frac{1}{2}[I_4^2 - \mathrm{tr}(\boldsymbol{\tau}^2)], \quad I_6 = \det(\boldsymbol{\tau}), \quad I_9 = I_{10} = I_4^2 - 2I_5. \end{aligned} \tag{3.2}$$

Using the expressions in Eqs. (3.1) and considering the initial stress to be independent of the deformation, the first derivatives of these invariants are given by

$$\begin{aligned} &\frac{\partial I_1}{\partial F_{i\alpha}} = 2F_{i\alpha}, \quad \frac{\partial I_2}{\partial F_{i\alpha}} = 2(C_{\gamma\gamma}F_{i\alpha} - C_{\alpha\gamma}F_{i\gamma}), \quad \frac{\partial I_3}{\partial F_{i\alpha}} = 2I_3(F^{-1})_{\alpha i}, \\ &\frac{\partial I_4}{\partial F_{i\alpha}} = 0, \quad \frac{\partial I_5}{\partial F_{i\alpha}} = 0, \quad \frac{\partial I_6}{\partial F_{i\alpha}} = 0, \\ &\frac{\partial I_7}{\partial F_{i\alpha}} = 2\tau_{\alpha\gamma}F_{i\gamma}, \quad \frac{\partial I_8}{\partial F_{i\alpha}} = 2\tau_{\alpha\delta}C_{\delta\gamma}F_{i\gamma} + 2C_{\alpha\delta}\tau_{\delta\gamma}F_{i\gamma}, \\ &\frac{\partial I_9}{\partial F_{i\alpha}} = 2\tau^2_{\alpha\gamma}F_{i\gamma}, \quad \frac{\partial I_{10}}{\partial F_{i\alpha}} = 2\tau^2_{\alpha\delta}C_{\delta\gamma}F_{i\gamma} + 2C_{\alpha\delta}\tau^2_{\delta\gamma}F_{i\gamma}. \end{aligned} \tag{3.3}$$

Evaluating the above expressions in the reference configuration, we have the non-zero derivatives as

$$\begin{aligned} &\frac{\partial I_1}{\partial F_{i\alpha}} = 2\delta_{i\alpha}, \quad \frac{\partial I_2}{\partial F_{i\alpha}} = 4\delta_{i\alpha}, \quad \frac{\partial I_3}{\partial F_{i\alpha}} = 2\delta_{i\alpha}, \quad \frac{\partial I_7}{\partial F_{i\alpha}} = 2\tau_{\alpha i}, \\ &\frac{\partial I_8}{\partial F_{i\alpha}} = 4\tau_{\alpha i}, \quad \frac{\partial I_9}{\partial F_{i\alpha}} = 2\tau^2_{\alpha i}, \quad \frac{\partial I_{10}}{\partial F_{i\alpha}} = 4\tau^2_{\alpha i}. \end{aligned} \tag{3.4}$$

Now consider that the strain energy function W is a function of $I_1, I_2, ..., I_{10}$. Then

$$\frac{\partial W}{\partial \mathbf{F}} = \sum_{r=1}^{10} W_r \frac{\partial I_r}{\partial \mathbf{F}}, \tag{3.5}$$

where $W_r = \partial W/\partial I_r$.

By definition, the nominal stress tensor for an initially stressed (unconstrained) material is given by

$$\begin{aligned}\mathbf{S} &= 2W_1\mathbf{F}^T + 2W_2(I_1\mathbf{F}^T - \mathbf{F}^T\mathbf{B}) + 2W_3I_3\mathbf{F}^{-1} + 2W_7\boldsymbol{\tau}\mathbf{F}^T \\ &+ 2W_8(\boldsymbol{\tau}\mathbf{C}\mathbf{F}^T + \mathbf{C}\boldsymbol{\tau}\mathbf{F}^T) + 2W_9\boldsymbol{\tau}^{\mathbf{2}}\mathbf{F}^T + 2W_{10}(\boldsymbol{\tau}^2\mathbf{C}\mathbf{F}^T + \mathbf{C}\boldsymbol{\tau}^{\mathbf{2}}\mathbf{F}^T). \end{aligned} \tag{3.6}$$

For an incompressible material, $I_3 = 1$ and Eq. (3.6) is replaced by

$$\begin{aligned}\mathbf{S} &= 2W_1\mathbf{F}^T + 2W_2(I_1\mathbf{F}^T - \mathbf{F}^T\mathbf{B}) + 2W_7\boldsymbol{\tau}\mathbf{F}^T + 2W_8(\boldsymbol{\tau}\mathbf{C}\mathbf{F}^T + \mathbf{C}\boldsymbol{\tau}\mathbf{F}^T) \\ &+ 2W_9\boldsymbol{\tau}^{\mathbf{2}}\mathbf{F}^T + 2W_{10}(\boldsymbol{\tau}^{\mathbf{2}}\mathbf{C}\mathbf{F}^T + \mathbf{C}\boldsymbol{\tau}^{\mathbf{2}}\mathbf{F}^T) - p\mathbf{F}^{-1}. \end{aligned} \tag{3.7}$$

The Cauchy stress tensor for an initially stressed (unconstrained) material is given by

$$\begin{aligned}J\mathbf{T} = \mathbf{F}\mathbf{S} &= 2W_1\mathbf{B} + 2W_2\mathbf{B}^* + 2W_3I_3\mathbf{I} + 2W_7\boldsymbol{\Sigma} + 2W_8(\boldsymbol{\Sigma}\mathbf{B} + \mathbf{B}\boldsymbol{\Sigma}) \\ &+ 2W_9\boldsymbol{\Sigma}\mathbf{B}^{-1}\boldsymbol{\Sigma} + 2W_{10}(\boldsymbol{\Sigma}\mathbf{B}^{-1}\boldsymbol{\Sigma}\mathbf{B} + \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}^{-1}\boldsymbol{\Sigma}), \end{aligned} \tag{3.8}$$

where $\boldsymbol{\Sigma} = \mathbf{F}\boldsymbol{\tau}\mathbf{F}^T$ and $\mathbf{B}^* = I_1\mathbf{B} - \mathbf{B}^2$. If the above expression is evaluated in the reference configuration, we get the expression for $\boldsymbol{\tau}$ as

$$\boldsymbol{\tau} = 2(W_1 + 2W_2 + W_3)\mathbf{I} + 2(W_7 + 2W_8)\boldsymbol{\tau} + 2(W_9 + 2W_{10})\boldsymbol{\tau}^{\mathbf{2}}, \tag{3.9}$$

which suggests to set

$$W_1 + 2W_2 + W_3 = 0, \quad 2(W_7 + 2W_8) = 1, \quad W_9 + 2W_{10} = 0, \tag{3.10}$$

in the reference configuration. Similarly, the Cauchy stress for an incompressible initially stressed material is

$$\begin{aligned}\mathbf{T} &= 2W_1\mathbf{B} + 2W_2\mathbf{B}^* + 2W_7\boldsymbol{\Sigma} + 2W_8(\boldsymbol{\Sigma}\mathbf{B} + \mathbf{B}\boldsymbol{\Sigma}) + 2W_9\boldsymbol{\Sigma}\mathbf{B}^{-1}\boldsymbol{\Sigma} \\ &+ 2W_{10}(\boldsymbol{\Sigma}\mathbf{B}^{-1}\boldsymbol{\Sigma}\mathbf{B} + \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}^{-1}\boldsymbol{\Sigma}) - p\mathbf{I}. \end{aligned} \tag{3.11}$$

In the reference configuration, Eq. (3.11) reduces to

$$\boldsymbol{\tau} = (2W_1 + 4W_2 - p_0)\mathbf{I} + (2W_7 + 4W_8)\boldsymbol{\tau} + (2W_9 + 4W_{10})\boldsymbol{\tau}^{\mathbf{2}}, \tag{3.12}$$

where p_0 is the value of p when evaluated in the reference configuration. This suggests that in the reference configuration, we should set

$$2W_1 + 4W_2 - p_0 = 0, \quad 2(W_7 + 2W_8) = 1, \quad W_9 + 2W_{10} = 0. \tag{3.13}$$

3.2 The Elasticity Tensor for an Initially Stressed Hyperelastic Material

Using Eqs. (3.3), the non-zero second derivatives of the invariants are

$$\begin{aligned}
\frac{\partial^2 I_1}{\partial F_{i\alpha} F_{j\beta}} &= 2\delta_{\alpha\beta}\delta_{ij}, \\
\frac{\partial^2 I_2}{\partial F_{i\alpha} F_{j\beta}} &= 2I_1\delta_{\alpha\beta}\delta_{ij} + 4F_{i\alpha}F_{j\beta} - 2F_{i\beta}F_{j\alpha} - 2C_{\alpha\beta}\delta_{ij} - 2\delta_{\alpha\beta}B_{ij}, \\
\frac{\partial^2 I_3}{\partial F_{i\alpha} F_{j\beta}} &= 4I_3(F^{-1})_{\alpha i}(F^{-1})_{\beta j} - 2I_3(F^{-1})_{\alpha j}(F^{-1})_{\beta i}, \quad \frac{\partial^2 I_7}{\partial F_{i\alpha} F_{j\beta}} = 2\tau_{\alpha\beta}\delta_{ij}, \\
\frac{\partial^2 I_8}{\partial F_{i\alpha} F_{j\beta}} &= 2\tau_{\alpha\beta}B_{ij} + 2(\boldsymbol{\tau}\mathbf{C})_{\alpha\beta}\delta_{ij} + 2(\mathbf{C}\boldsymbol{\tau})_{\alpha\beta}\delta_{ij} + 2\delta_{\alpha\beta}\Sigma_{ij} + 2(\boldsymbol{\tau}\mathbf{F}^T)_{\alpha j}F_{i\beta} \\
&+ 2(\boldsymbol{\tau}\mathbf{F}^T)_{\beta i}F_{j\alpha}, \\
\frac{\partial^2 I_9}{\partial F_{i\alpha} F_{j\beta}} &= 2(\boldsymbol{\tau}^{\mathbf{2}})_{\alpha\beta}\delta_{ij}, \\
\frac{\partial^2 I_{10}}{\partial F_{i\alpha} F_{j\beta}} &= 2(\boldsymbol{\tau}^{\mathbf{2}})_{\alpha\beta}B_{ij} + 2(\boldsymbol{\tau}^{\mathbf{2}}\mathbf{C})_{\alpha\beta}\delta_{ij} + 2(\mathbf{C}\boldsymbol{\tau}^{\mathbf{2}})_{\alpha\beta}\delta_{ij} + 2\delta_{\alpha\beta}(\mathbf{F}\boldsymbol{\tau}^{\mathbf{2}}\mathbf{T})_{ij} \\
&+ 2(\boldsymbol{\tau}^{\mathbf{2}}\mathbf{F}^T)_{\alpha j}F_{i\beta} + 2(\boldsymbol{\tau}^{\mathbf{2}}\mathbf{F}^T)_{\beta i}F_{j\alpha}.
\end{aligned} \tag{3.14}$$

In the reference configuration, the above expressions reduce to

$$\begin{aligned}
\frac{\partial^2 I_1}{\partial F_{i\alpha} F_{j\beta}} &= 2\delta_{\alpha\beta}\delta_{ij}, \quad \frac{\partial^2 I_2}{\partial F_{i\alpha} F_{j\beta}} = 2\delta_{\alpha\beta}\delta_{ij} + 4\delta_{i\alpha}\delta_{j\beta} - 2\delta_{i\beta}\delta_{j\alpha}, \\
\frac{\partial^2 I_3}{\partial F_{i\alpha} F_{j\beta}} &= 4\delta_{\alpha i}\delta_{\beta j} - 2\delta_{\alpha j}\delta_{\beta i}, \quad \frac{\partial^2 I_7}{\partial F_{i\alpha} F_{j\beta}} = 2\tau_{\alpha\beta}\delta_{ij}, \\
\frac{\partial^2 I_8}{\partial F_{i\alpha} F_{j\beta}} &= 6\tau_{\alpha\beta}\delta_{ij} + 2\tau_{ij}\delta_{\alpha\beta} + 2\tau_{\alpha j}\delta_{i\beta} + 2\tau_{\beta i}\delta_{j\alpha}, \quad \frac{\partial^2 I_9}{\partial F_{i\alpha} F_{j\beta}} = 2(\boldsymbol{\tau}^{\mathbf{2}})_{\alpha\beta}\delta_{ij}, \\
\frac{\partial^2 I_{10}}{\partial F_{i\alpha} F_{j\beta}} &= 6(\boldsymbol{\tau}^{\mathbf{2}})_{\alpha\beta}\delta_{ij} + 2(\boldsymbol{\tau}^{\mathbf{2}})_{ij}\delta_{\alpha\beta} + 2(\boldsymbol{\tau}^{\mathbf{2}})_{\alpha j}\delta_{i\beta} + 2(\boldsymbol{\tau}^{\mathbf{2}})_{\beta i}\delta_{j\alpha},
\end{aligned} \tag{3.15}$$

where $\boldsymbol{\Sigma} = \boldsymbol{\tau} = \mathbf{S}$ in the reference configuration. From Eqs. (2.79) and (3.5)

$$\boldsymbol{\mathcal{A}} = \frac{\partial^2 W}{\partial \mathbf{F}\partial \mathbf{F}} = \sum_{r=1}^{10} W_r \frac{\partial^2 I_r}{\partial \mathbf{F}\partial \mathbf{F}} + \sum_{r,s=1}^{10} W_{rs}\frac{\partial I_r}{\partial \mathbf{F}} \otimes \frac{\partial I_s}{\partial \mathbf{F}}, \tag{3.16}$$

where $W_{rs} = \partial^2 W/\partial I_r \partial I_s$.

Also, using Eq. (2.119), the updated elasticity tensor in its component form is given by

$$\mathcal{A}_{0piqj} = J^{-1}\left(\sum_{r=1}^{10} W_r F_{p\alpha} F_{q\beta} \frac{\partial^2 I_r}{\partial F_{i\alpha}\partial F_{j\beta}} + \sum_{r,s=1}^{10} W_{rs} F_{p\alpha} F_{q\beta} \frac{\partial I_r}{\partial F_{i\alpha}} \frac{\partial I_s}{\partial F_{j\beta}}\right). \tag{3.17}$$

Taking $N = 10$ in Eq. (3.17), we then have for a compressible material

$$\begin{aligned}
J\mathcal{A}_{0piqj} &= 2W_1 B_{pq}\delta_{ij} + 2W_2[I_1 B_{pq}\delta_{ij} - B_{iq}B_{jp} + 2B_{pi}B_{qj} - \delta_{ij}(\mathbf{B}^2)_{pq} - B_{pq}B_{ij}] \\
&+ 2W_3 I_3(2\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}) + 2W_7\Sigma_{pq}\delta_{ij} + 2W_8[\Sigma_{pq}B_{ij} + (\mathbf{\Sigma B})_{pq}\delta_{ij} \\
&+ (\mathbf{B\Sigma})_{pq}\delta_{ij} + \Sigma_{ij}B_{pq} + \Sigma_{pj}B_{iq} + \Sigma_{qi}B_{jp}] + 2W_9(\mathbf{\Sigma}^2)_{pq}\delta_{ij} \\
&+ 2W_{10}[(\mathbf{\Sigma}^2)_{pq}B_{ij} + (\mathbf{\Sigma}^2\mathbf{B})_{pq}\delta_{ij} + (\mathbf{B\Sigma}^2)_{pq}\delta_{ij} + (\mathbf{\Sigma}^2)_{ij}B_{pq} + (\mathbf{\Sigma}^2)_{pj}B_{iq} \\
&+ (\mathbf{\Sigma}^2)_{qi}B_{jp}] + 4W_{11}B_{ip}B_{jq} + 4W_{22}B^*_{ip}B^*_{jq} + 4W_{33}I_3^2\delta_{ip}\delta_{jq} \\
&+ 4W_{12}[2I_1 B_{ip}B_{jq} - B_{ip}(\mathbf{B}^2)_{jq} - B_{jq}(\mathbf{B}^2)_{ip}] + 4W_{13}I_3(B_{ip}\delta_{jq} \\
&+ B_{jq}\delta_{ip}) + 4W_{17}(B_{ip}\Sigma_{jq} + B_{jq}\Sigma_{ip}) + 4W_{18}[B_{ip}(\mathbf{\Sigma B} + \mathbf{B\Sigma})_{jq} \\
&+ (\mathbf{\Sigma B} + \mathbf{B\Sigma})_{ip}B_{jq}] + 4W_{19}[B_{ip}(\mathbf{\Sigma}^2)_{jq} + B_{jq}(\mathbf{\Sigma}^2)_{ip}] \\
&+ 4W_{1(10)}[B_{ip}(\mathbf{\Sigma}^2\mathbf{B} + \mathbf{B\Sigma}^2)_{jq} + (\mathbf{\Sigma}^2\mathbf{B} + \mathbf{B\Sigma}^2)_{ip}B_{jq}] \\
&+ 4W_{23}I_3[I_1(B_{ip}\delta_{jq} + B_{jq}\delta_{ip}) - \delta_{ip}(\mathbf{B}^2)_{jq} - (\mathbf{B}^2)_{ip}\delta_{jq}] \\
&+ 4W_{27}(B^*_{ip}\Sigma_{jq} + \Sigma_{ip}B^*_{jq}) + 4W_{28}[B^*_{ip}(\mathbf{\Sigma B} + \mathbf{B\Sigma})_{jq} \\
&+ (\mathbf{\Sigma B} + \mathbf{B\Sigma})_{ip}B^*_{jq}] + 4W_{29}[B^*_{ip}(\mathbf{\Sigma}^2)_{jq} + (\mathbf{\Sigma}^2)_{ip}B^*_{jq}] \\
&+ 4W_{2(10)}[B^*_{ip}(\mathbf{\Sigma}^2\mathbf{B} + \mathbf{B\Sigma}^2)_{jq} + (\mathbf{\Sigma}^2\mathbf{B} + \mathbf{B\Sigma}^2)_{ip}B^*_{jq}] \\
&+ 4W_{37}I_3\delta_{ip}\Sigma_{jq} + 4W_{38}I_3[\delta_{ip}(\mathbf{\Sigma B} + \mathbf{B\Sigma})_{jq} + (\mathbf{\Sigma B} + \mathbf{B\Sigma})_{ip}\delta_{jq}] \\
&+ 4W_{39}I_3[\delta_{ip}(\mathbf{\Sigma}^2)_{jq} + (\mathbf{\Sigma}^2)_{ip}\delta_{jq}] + 4W_{3(10)}I_3[\delta_{ip}(\mathbf{\Sigma}^2\mathbf{B} + \mathbf{B\Sigma}^2)_{jq} \\
&+ (\mathbf{\Sigma}^2\mathbf{B} + \mathbf{B\Sigma}^2)_{ip}\delta_{jq}] + 4W_{77}\Sigma_{ip}\Sigma_{jq} + 4W_{78}[\Sigma_{ip}(\mathbf{\Sigma B} + \mathbf{B\Sigma})_{jq} \\
&+ (\mathbf{\Sigma B} + \mathbf{B\Sigma})_{ip}\Sigma_{jq}] + 4W_{79}[\Sigma_{ip}(\mathbf{\Sigma}^2)_{jq} + (\mathbf{\Sigma}^2)_{ip}\Sigma_{jq}] \\
&+ 4W_{7(10)}[\Sigma_{ip}(\mathbf{\Sigma}^2\mathbf{B} + \mathbf{B\Sigma}^2)_{jq} + (\mathbf{\Sigma}^2\mathbf{B} + \mathbf{B\Sigma}^2)_{ip}\Sigma_{jq}] \\
&+ 4W_{88}(\mathbf{\Sigma B} + \mathbf{B\Sigma})_{ip}(\mathbf{\Sigma B} + \mathbf{B\Sigma})_{jq} + 4W_{89}[(\mathbf{\Sigma}^2)_{ip}(\mathbf{\Sigma B} + \mathbf{B\Sigma})_{jq} \\
&+ (\mathbf{\Sigma B} + \mathbf{B\Sigma})_{ip})(\mathbf{\Sigma}^2)_{jq}] + 4W_{8(10)}[(\mathbf{\Sigma B} + \mathbf{B\Sigma})_{ip}(\mathbf{\Sigma}^2\mathbf{B} + \mathbf{B\Sigma}^2)_{jq} \\
&+ (\mathbf{\Sigma}^2\mathbf{B} + \mathbf{B\Sigma}^2)_{ip}(\mathbf{\Sigma B} + \mathbf{B\Sigma})_{jq})] + 4W_{99}(\mathbf{\Sigma}^2)_{ip}(\mathbf{\Sigma}^2)_{jq} \\
&+ 4W_{9(10)}[(\mathbf{\Sigma}^2)_{ip}(\mathbf{\Sigma}^2\mathbf{B} + \mathbf{B\Sigma}^2)_{jq} + (\mathbf{\Sigma}^2\mathbf{B} + \mathbf{B\Sigma}^2)_{ip}(\mathbf{\Sigma}^2)_{jq}] \\
&+ 4W_{(10)(10)}(\mathbf{\Sigma}^2\mathbf{B} + \mathbf{B\Sigma}^2)_{ip}(\mathbf{\Sigma}^2\mathbf{B} + \mathbf{B\Sigma}^2)_{jq}.
\end{aligned} \tag{3.18}$$

For simplicity of calculations only, we omit the dependence of W on I_5, I_6, I_9 and I_{10}, the invariants which are nonlinear in $\boldsymbol{\tau}$. Therefore, Eqs. $(3.10)_3$ and $(3.13)_3$ are automatically satisfied. The specialized expressions of the Cauchy stress for unconstrained material in this case follow from Eq. (3.8) and from Eq. (3.11) for an incompressible material. The specialized initial Cauchy stress for an unconstrained material follows from Eq. (3.9) and from Eq. (3.12) for an incompressible material.

Using Eq. (3.10) in Eq. (3.18), with $\mathcal{C}_{piqj}$ as the components of the updated elasticity tensor $\boldsymbol{\mathcal{C}}$, we have for an unconstrained material

$$\begin{aligned}\mathcal{C}_{piqj} = \mathcal{A}_{0piqj} &= \alpha_1(\delta_{ij}\delta_{pq} + \delta_{iq}\delta_{jp} - \delta_{ip}\delta_{jq}) + \alpha_2\delta_{ip}\delta_{jq} + \delta_{ij}\tau_{pq} + \alpha_3(\delta_{ij}\tau_{pq} \\ &+ \delta_{pq}\tau_{ij} + \delta_{iq}\tau_{jp} + \delta_{jp}\tau_{iq}) + \alpha_4(\delta_{ip}\tau_{jq} + \delta_{jq}\tau_{ip}) + \alpha_5\tau_{ip}\tau_{jq}, \qquad (3.19)\end{aligned}$$

in the reference configuration. Here, we have defined

$$\begin{aligned}&\alpha_1 = 2(W_1 + W_2), \quad \alpha_2 = 2(W_2 + W_3) + 4(W_{11} + 4W_{12} + 2W_{13} + 4W_{22} + 4W_{23} + W_{33}), \\ &\alpha_3 = 2W_8, \quad \alpha_4 = 4(W_{17} + 2W_{18} + 2W_{27} + 4W_{28} + W_{37} + 2W_{38}), \\ &\alpha_5 = 4(W_{77} + 4W_{78} + 4W_{88}), \qquad (3.20)\end{aligned}$$

evaluated in the reference configuration. When $\boldsymbol{\tau} = \mathbf{0}$, Eq. (3.19) gives

$$\mathcal{C}_{piqj} = \alpha_1(\delta_{pq}\delta_{ij} + \delta_{iq}\delta_{jp} - \delta_{ip}\delta_{jq}) + \alpha_2\delta_{ip}\delta_{jq}. \qquad (3.21)$$

In terms of invariants, the classical isotropic strain energy function is given by

$$W(I_1, I_2, I_3) = \frac{\mu}{4}[I_1^2 - 2(I_1 + I_2) + 3] + \frac{\lambda}{8}[I_1 - 3]^2, \qquad (3.22)$$

where λ and μ are the Lamé moduli. Using Eq. (3.22) in Eq. (3.21), we get $\mathcal{A}_{0piqj}$ in its classical form

$$\mathcal{C}_{piqj} = \mu(\delta_{pq}\delta_{ij} + \delta_{iq}\delta_{jp}) + \lambda\delta_{ip}\delta_{jq}, \qquad (3.23)$$

where we have used $\alpha_1 = \mu$ and $\alpha_2 - \alpha_1 = \lambda$.

Using Eq. (3.19), we can write, for $i \neq j \neq k \neq i$,

$$
\begin{aligned}
\mathcal{C}_{iiii} &= \alpha_1 + \alpha_2 + (1 + 4\alpha_3 + 2\alpha_4)\tau_{ii} + \alpha_5\tau_{ii}^2 && (3.24)\\
\mathcal{C}_{iijj} &= -\alpha_1 + \alpha_2 + \alpha_4(\tau_{ii} + \tau_{jj}) + \alpha_5\tau_{ii}\tau_{jj}, && (3.25)\\
\mathcal{C}_{ijij} &= \alpha_1 + (1 + \alpha_3)\tau_{ii} + \alpha_3\tau_{jj} + \alpha_5\tau_{ij}^2, && (3.26)\\
\mathcal{C}_{ijji} &= \alpha_1 + \alpha_3(\tau_{ii} + \tau_{jj}) + \alpha_5\tau_{ij}^2, && (3.27)\\
\mathcal{C}_{iiij} &= \mathcal{C}_{ijii} = (2\alpha_3 + \alpha_4)\tau_{ij} + \alpha_5\tau_{ii}\tau_{ij}, && (3.28)\\
\mathcal{C}_{iiji} &= \mathcal{C}_{jiii} = (1 + 2\alpha_3 + \alpha_4)\tau_{ij} + \alpha_5\tau_{ii}\tau_{ij}, && (3.29)\\
\mathcal{C}_{iikj} &= \mathcal{C}_{iijk} = \mathcal{A}_{0jkii} = \mathcal{A}_{0kjii} = \alpha_4\tau_{jk} + \alpha_5\tau_{ii}\tau_{jk}, && (3.30)\\
\mathcal{C}_{ijki} &= \mathcal{C}_{ijik} = \mathcal{A}_{0ikji} = \mathcal{A}_{0kiij} = \alpha_3\tau_{jk} + \alpha_5\tau_{ij}\tau_{ik}, && (3.31)\\
\mathcal{C}_{ikjk} &= \mathcal{C}_{jkik} = (1 + \alpha_3)\tau_{ij} + \alpha_5\tau_{ik}\tau_{jk}. && (3.32)
\end{aligned}
$$

For an incompressible material, the terms including the subscript '3' in Eq. (3.18) are omitted. We therefore have

$$
\begin{aligned}
\mathcal{A}_{0piqj} &= 2W_1B_{pq}\delta_{ij} + 2W_2(I_1B_{pq}\delta_{ij} - B_{iq}B_{jp} + 2B_{pi}B_{qj} - \delta_{ij}(\mathbf{B}^2)_{pq} - B_{pq}B_{ij})\\
&+ 2W_7\Sigma_{pq}\delta_{ij} + 2W_8(\Sigma_{pq}B_{ij} + (\mathbf{\Sigma B})_{pq}\delta_{ij} + (\mathbf{B\Sigma})_{pq}\delta_{ij} + \Sigma_{ij}B_{pq} + \Sigma_{pj}B_{iq}\\
&+ \Sigma_{qi}B_{jp}) + 4W_{11}B_{ip}B_{jq} + 4W_{22}B^*_{ip}B^*_{jq} + 4W_{12}(2I_1B_{ip}B_{jq} - B_{ip}(\mathbf{B}^2)_{jq}\\
&- B_{jq}(\mathbf{B}^2)_{ip}) + 4W_{17}(B_{ip}\Sigma_{jq} + B_{jq}\Sigma_{ip}) + 4W_{18}(B_{ip}(\mathbf{\Sigma B} + \mathbf{B\Sigma})_{jq}\\
&+ (\mathbf{\Sigma B} + \mathbf{B\Sigma})_{ip}B_{jq}) + 4W_{19}[B_{ip}(\mathbf{\Sigma}^2)_{jq} + B_{jq}(\mathbf{\Sigma}^2)_{ip}]\\
&+ 4W_{1(10)}[B_{ip}(\mathbf{\Sigma}^2\mathbf{B} + \mathbf{B\Sigma}^2)_{jq} + (\mathbf{\Sigma}^2\mathbf{B} + \mathbf{B\Sigma}^2)_{ip}B_{jq}] + 4W_{27}(B^*_{ip}\Sigma_{jq}\\
&+ \Sigma_{ip}B^*_{jq}) + 4W_{28}(B^*_{ip}(\mathbf{\Sigma B} + \mathbf{B\Sigma})_{jq}(\mathbf{\Sigma B} + \mathbf{B\Sigma})_{ip}B^*_{jq})\\
&+ 4W_{29}[B^*_{ip}(\mathbf{\Sigma}^2)_{jq} + (\mathbf{\Sigma}^2)_{ip}B^*_{jq}] + 4W_{2(10)}[B^*_{ip}(\mathbf{\Sigma}^2\mathbf{B} + \mathbf{B\Sigma}^2)_{jq}\\
&+ (\mathbf{\Sigma}^2\mathbf{B} + \mathbf{B\Sigma}^2)_{ip}B^*_{jq}] + 4W_{77}\Sigma_{ip}\Sigma_{jq} + 4W_{78}(\Sigma_{ip}(\mathbf{\Sigma B} + \mathbf{B\Sigma})_{jq}\\
&+ (\mathbf{\Sigma B} + \mathbf{B\Sigma})_{ip}\Sigma_{jq}) + 4W_{79}[\Sigma_{ip}(\mathbf{\Sigma}^2)_{jq} + (\mathbf{\Sigma}^2)_{ip}\Sigma_{jq}]\\
&+ 4W_{7(10)}[\Sigma_{ip}(\mathbf{\Sigma}^2\mathbf{B} + \mathbf{B\Sigma}^2)_{jq} + (\mathbf{\Sigma}^2\mathbf{B} + \mathbf{B\Sigma}^2)_{ip}\Sigma_{jq}]\\
&+ 4W_{88}(\mathbf{\Sigma B} + \mathbf{B\Sigma})_{ip}(\mathbf{\Sigma B} + \mathbf{B\Sigma})_{jq} + 4W_{89}[(\mathbf{\Sigma}^2)_{ip}(\mathbf{\Sigma B} + \mathbf{B\Sigma})_{jq}\\
&+ (\mathbf{\Sigma B} + \mathbf{B\Sigma})_{ip})(\mathbf{\Sigma}^2)_{jq}] + 4W_{8(10)}[(\mathbf{\Sigma B} + \mathbf{B\Sigma})_{ip}(\mathbf{\Sigma}^2\mathbf{B} + \mathbf{B\Sigma}^2)_{jq}\\
&+ (\mathbf{\Sigma}^2\mathbf{B} + \mathbf{B\Sigma}^2)_{ip}(\mathbf{\Sigma B} + \mathbf{B\Sigma})_{jq})] + 4W_{99}(\mathbf{\Sigma}^2)_{ip}(\mathbf{\Sigma}^2)_{jq}\\
&+ 4W_{9(10)}[(\mathbf{\Sigma}^2)_{ip}(\mathbf{\Sigma}^2\mathbf{B} + \mathbf{B\Sigma}^2)_{jq}(\mathbf{\Sigma}^2\mathbf{B} + \mathbf{B\Sigma}^2)_{ip}(\mathbf{\Sigma}^2)_{jq}]\\
&+ 4W_{(10)(10)}(\mathbf{\Sigma}^2\mathbf{B} + \mathbf{B\Sigma}^2)_{ip}(\mathbf{\Sigma}^2\mathbf{B} + \mathbf{B\Sigma}^2)_{jq}. && (3.33)
\end{aligned}
$$

We consider an incompressible material the elastic response of which is described by a general strain energy function $W(\mathbf{C}, \boldsymbol{\tau})$. Let this material be subject to a general pure homogeneous pre-strain such that $\lambda_1, \lambda_2, \lambda_3$ are the principal stretches corresponding the principal axes x_1, x_2 and x_3 respectively. Here, for simplicity of calculations only, we omit the dependence of W on I_5, I_6, I_9 and I_{10}. Following from Eq. (3.33), various expressions for elastic modulli in this case are given by, for $i \neq j$,

$$
\begin{aligned}
\mathcal{A}_{0iiii} &= 2W_1\lambda_i^2 + 2W_2\lambda_i^2(I_1 - \lambda_i^2) + 2W_7\Sigma_{ii} + 12W_8\lambda_i^2\Sigma_{ii} + 4W_{11}\lambda_i^4 + 4W_{22}\lambda_i^4(I_1 - \lambda_i^2)^2 \\
&+ 8W_{12}\lambda_i^4(I_1 - \lambda_i^2) + 8W_{17}\lambda_i^2\Sigma_{ii} + 16W_{18}\lambda_i^4\Sigma_{ii} + 8W_{27}\lambda_i^2(I_1 - \lambda_i^2)\Sigma_{ii} \\
&+ 16W_{28}\lambda_i^4(I_1 - \lambda_i^2)\Sigma_{ii} + 4W_{77}\Sigma_{ii}^2 + 16W_{78}\lambda_i^2\Sigma_{ii}^2 + 16W_{88}\lambda_i^4\Sigma_{ii}^2, \qquad (3.34) \\
\mathcal{A}_{0iijj} &= 4W_2\lambda_i^2\lambda_j^2 + 4W_{11}\lambda_i^2\lambda_j^2 + 4W_{22}\lambda_i^2\lambda_j^2(I_1 - \lambda_i^2)(I_1 - \lambda_j^2) + 4W_{12}\lambda_i^2\lambda_j^2(2I_1 - \lambda_i^2 - \lambda_j^2) \\
&+ 4W_{17}(\lambda_i^2\Sigma_{ii} + \lambda_j^2\Sigma_{jj}) + 8W_{18}\lambda_i^2\lambda_j^2(\Sigma_{ii} + \Sigma_{jj} + 4W_{27}[\lambda_i^2(I_1 - \lambda_i^2)\Sigma_{jj} \\
&+ \lambda_j^2(I_1 - \lambda_j^2)\Sigma_{ii}] + 8W_{28}\lambda_i^2\lambda_j^2[(I_1 - \lambda_i^2)\Sigma_{jj} + (I_1 - \lambda_j^2)\Sigma_{ii}] + 4W_{77}\Sigma_{ii}\Sigma_{jj} \\
&+ 8W_{78}(\lambda_i^2 + \lambda_j^2)\Sigma_{ii}\Sigma_{jj} + 16W_{88}\lambda_i^2\lambda_j^2\Sigma_{ii}\Sigma_{jj}, \qquad (3.35) \\
\mathcal{A}_{0ijij} &= 2W_1\lambda_i^2 + 2W_2(I_1\lambda_i^2 - \lambda_i^2\lambda_j^2 - \lambda_i^4) + 2W_7\Sigma_{ii} + 2W_8(\lambda_j^2\Sigma_{ii} + 2\lambda_i^2\Sigma_{ii} + \lambda_i^2\Sigma_{jj}) \\
&+ 4W_{77}\Sigma_{ij}^2 + 16W_{78}\lambda_i^2\Sigma_{ij}^2 + 16W_{88}\lambda_i^4\Sigma_{ij}^2, \qquad (3.36) \\
\mathcal{A}_{0ijji} &= -2W_2\lambda_i^2\lambda_j^2 + 2W_8(\lambda_j^2\Sigma_{ii} + \lambda_i^2\Sigma_{jj}) + 4W_{77}\Sigma_{ij}^2 + 16W_{78}\lambda_j^2\Sigma_{ij}^2 \\
&+ 16W_{88}\lambda_i^2\lambda_j^2\Sigma_{ij}^2, \qquad (3.37) \\
\mathcal{A}_{0ijjj} &= 2[W_7 + W_8(3\lambda_j^2 + \lambda_i^2) + 2W_{17}\lambda_j^2 + 2W_{18}\lambda_j^2(\lambda_i^2 + \lambda_j^2) + 2W_{27}\lambda_j^2(I_1 - \lambda_j^2) + 2W_{28}\lambda_j^2 \\
&\times (I_1 - \lambda_j^2)(\lambda_i^2 + \lambda_j^2)]\Sigma_{ij} + 4[W_{77} + W_{78}(3\lambda_j^2 + \lambda_i^2) + 2W_{88}\lambda_j^2(\lambda_i^2 + \lambda_j^2)]\Sigma_{ij}\Sigma_{jj}, \qquad (3.38) \\
\mathcal{A}_{0iiij} &= 2[2W_8\lambda_i^2 + 2W_{17}\lambda_i^2 + 2W_{18}\lambda_i^2(\lambda_i^2 + \lambda_j^2) + 2W_{27}\lambda_i^2(I_1 - \lambda_i^2) + 2W_{28}\lambda_i^2(I_1 - \lambda_i^2) \\
&\times (\lambda_i^2 + \lambda_j^2)]\Sigma_{ij} + 4[W_{77} + W_{78}(3\lambda_i^2 + \lambda_j^2) + 2W_{88}\lambda_i^2(\lambda_i^2 + \lambda_j^2)]\Sigma_{ij}\Sigma_{ii}, \qquad (3.39) \\
\mathcal{A}_{0iikj} &= \mathcal{A}_{0iijk} = \mathcal{A}_{0jkii} = \mathcal{A}_{0kjii} = 4[W_{17}\lambda_i^2 + W_{18}\lambda_i^2(\lambda_j^2 + \lambda_k^2) + W_{27}\lambda_i^2(I_1 - \lambda_i^2) \\
&+ W_{28}\lambda_i^2(I_1 - \lambda_i^2)(\lambda_j^2 + \lambda_k^2)]\Sigma_{jk} + 4[W_{77} + W_{78}(2\lambda_i^2 + \lambda_j^2 + \lambda_k^2) \\
&+ 2W_{88}\lambda_i^2(\lambda_j^2 + \lambda_k^2)]\Sigma_{jk}\Sigma_{ii}, \qquad (3.40) \\
\mathcal{A}_{0ijki} &= \mathcal{A}_{0ijik} = \mathcal{A}_{0ikji} = \mathcal{A}_{0kiij} = 2W_8\lambda_i^2\Sigma_{jk} + 4[W_{77} + W_{78}(2\lambda_i^2 + \lambda_j^2 + \lambda_k^2) \\
&+ W_{88}(\lambda_i^2 + \lambda_j^2)(\lambda_i^2 + \lambda_k^2)]\Sigma_{ij}\Sigma_{ik}, \qquad (3.41) \\
\mathcal{A}_{0ikjk} &= \mathcal{A}_{0jkik} = 2[W_7 + W_8(\lambda_i^2 + \lambda_j^2 + \lambda_k^2)]\Sigma_{ij} + 4[W_{77} + W_{78}(\lambda_i^2 + \lambda_j^2 + 2\lambda_k^2) \\
&+ W_{88}(\lambda_i^2 + \lambda_k^2)(\lambda_j^2 + \lambda_k^2)]\Sigma_{ik}\Sigma_{jk}. \qquad (3.42)
\end{aligned}
$$

For an incompressible material in the initially stressed reference configuration, Eq. (3.33) reduces to

$$\begin{aligned} \mathcal{C}_{piqj} &= \alpha_1\delta_{pq}\delta_{ij} + \tau_{pq}\delta_{ij} + \alpha_3(\tau_{pq}\delta_{ij} + \tau_{ij}\delta_{pq} + \tau_{pj}\delta_{iq} + \tau_{qi}\delta_{jp}) \\ &+ \alpha_5\tau_{ip}\tau_{jq} + \alpha_6(\delta_{ip}\tau_{jq} + \delta_{jq}\tau_{ip}), \end{aligned} \tag{3.43}$$

while the conditions in Eq. (3.13) hold. Here, we have defined

$$\alpha_6 = 4(W_{17} + 2W_{18} + 2W_{27} + 4W_{28}), \tag{3.44}$$

and α_1, α_3 and α_5 are given by Eq. (3.20). The expression for $\mathcal{A}_{0piqj}$ in the absence of initial stress, for an incompressible material, follows from Eq. (3.43) as

$$\mathcal{C}_{piqj} = \alpha_1\delta_{pq}\delta_{ij}. \tag{3.45}$$

In the case of an incompressible material there is an element of non-uniqueness in the components of $\boldsymbol{\mathcal{C}}$ since they depend on the point at which the incompressibility condition is applied during the differentiations. The counterpart of Eq. (3.23) in this case follows from Eq. (3.45) and Eq. (3.22) and is given by

$$\mathcal{C}_{iijj} = \mathcal{C}_{ijij} = \alpha_1 = \mu \quad i \neq j, \tag{3.46}$$

where μ is the shear modulus in $\mathcal{B}_r$. The differences between the expressions in Eq. (3.46) and any alternative expressions are absorbed by the incremental Lagrange multiplier $\dot{p}$ in Eq. (2.127).

From Eq. (3.43), we have, for $i \neq j \neq k \neq i$,

$$\mathcal{C}_{iiii} = \alpha_1 + (1 + 4\alpha_3 + 2\alpha_6)\tau_{ii} + \alpha_5\tau_{ii}^2, \tag{3.47}$$

$$\mathcal{C}_{iijj} = \alpha_6\tau_{jj} + \alpha_5\tau_{ii}\tau_{jj}, \tag{3.48}$$

$$\mathcal{C}_{ijij} = \alpha_1 + (1 + \alpha_3)\tau_{ii} + \alpha_3\tau_{jj} + \alpha_5\tau_{ij}\tau_{ij}, \tag{3.49}$$

$$\mathcal{C}_{ijji} = \alpha_3(\tau_{ii} + \tau_{jj}) + \alpha_5\tau_{ij}\tau_{ij}, \tag{3.50}$$

$$\mathcal{C}_{iiij} = \mathcal{C}_{ijii} = (2\alpha_3 + \alpha_6)\tau_{ij} + \alpha_5\tau_{ii}\tau_{ij}, \tag{3.51}$$

$$\mathcal{C}_{iiji} = \mathcal{C}_{jiii} = (1 + 2\alpha_3 + \alpha_6)\tau_{ij} + \alpha_5\tau_{ii}\tau_{ij}, \tag{3.52}$$

$$\mathcal{C}_{iikj} = \mathcal{C}_{iijk} = \mathcal{A}_{0jkii} = \mathcal{A}_{0kjii} = \alpha_6\tau_{jk} + \alpha_5\tau_{ii}\tau_{jk}, \tag{3.53}$$

$$\mathcal{C}_{ijki} = \mathcal{C}_{ijik} = \mathcal{A}_{0ikji} = \mathcal{A}_{0kiij} = \alpha_3\tau_{jk} + \alpha_5\tau_{ij}\tau_{ik}, \tag{3.54}$$

$$\mathcal{C}_{ikjk} = \mathcal{C}_{jkik} = (1 + \alpha_3)\tau_{ij} + \alpha_5\tau_{ik}\tau_{jk}. \tag{3.55}$$

The above expressions also follow from Eqs. (3.34)–(3.42) with $\lambda_i = 1$, $i = \{1, 2, 3\}$ and $\Sigma_{ij} = \tau_{ij}$, etc., in the reference configuration.

3.3 The Effect of Initial Stress on the Propagation of Homogenous Plane Waves in a Homogeneously Deformed Infinite Medium

Consider an initially stressed medium with initial stress $\boldsymbol{\tau}$ whose elastic response is characterised by the strain energy function $W(\mathbf{C}, \boldsymbol{\tau})$. We consider incremental motions in an infinite medium subject to homogeneous deformation and homogeneous initial stress. Using Eq. (2.126) in Eq. (2.121), the equation of motion for a compressible material is given by

$$\mathcal{A}_{0piqj}u_{j,pq} = \rho u_{i,tt}, \tag{3.56}$$

and from Eq. (2.124) in Eq. (2.121), the equation of motion for an incompressible material is given by

$$\mathcal{A}_{0piqj}u_{j,pq} - \dot{p}_{,i} = \rho u_{i,tt}, \tag{3.57}$$

along with the incompressibility condition (2.125), i.e.

$$u_{p,p} = 0. \tag{3.58}$$

Consider an incremental plane wave of the form

$$\mathbf{u} = \mathbf{m} f(\mathbf{n} \cdot \mathbf{x} - ct), \tag{3.59}$$

where $\mathbf{m}$ is a unit vector referred to as the *polarization vector*, c is the *wave speed* and f is a twice continuously differentiable function. For homogeneous plane waves the unit vector $\mathbf{n}$ is real and defines the *direction of propagation* of the wave.

Using the incremental displacement given by Eq. (3.59) in Eq. (3.56), we have for a compressible material

$$\mathbf{Q}(\mathbf{n})\mathbf{m} = \rho c^2 \mathbf{m}, \tag{3.60}$$

where $\mathbf{Q}(\mathbf{n})$ is the so-called *acoustic tensor* (see, for example, [34, 44]). It depends on $\mathbf{n}$ and is defined in its component form as

$$Q_{ij}(\mathbf{n}) = \mathcal{A}_{0piqj} n_p n_q. \tag{3.61}$$

Eq. (3.60) is called the *propagation equation*. For a particular choice of $\mathbf{n}$ it determines possible wave speeds and polarizations corresponding to plane waves propagating in that direction. The wave speeds are determined by the *characteristic equation*

$$\det[\mathbf{Q}(\mathbf{n}) - \rho c^2 \mathbf{I}] = 0, \tag{3.62}$$

where $\mathbf{I}$ is again the identity tensor in three dimensions.

For an incompressible material we also assume $\dot{p} = g(\mathbf{n} \cdot \mathbf{x} - ct)$, where g is another function. Substitution of $\dot{p}$ in Eq. (3.56) leads to

$$\mathcal{A}_{0piqj} n_p n_q m_j f'' - g' n_i = \rho c^2 m_i f'', \tag{3.63}$$

together with

$$m_i n_i = 0, \tag{3.64}$$

which comes out as a result of the incompressibility condition. From Eqs. (3.63) and (3.64) we obtain $g' = \mathcal{A}_{0piqj} n_p n_q m_j n_i f''$, which when substituted back in Eq. (3.63), yields

$$(\mathbf{I} - \mathbf{n} \otimes \mathbf{n})\mathbf{Q}(\mathbf{n})\mathbf{m} = \rho c^2 \mathbf{m}, \quad \mathbf{m} \cdot \mathbf{n} = 0. \tag{3.65}$$

Since $(\mathbf{I} - \mathbf{n} \otimes \mathbf{n})\mathbf{m} = \mathbf{m}$, we can define a symmetric tensor $\bar{\mathbf{Q}}(\mathbf{n})$ for an incompressible material such that

$$\bar{\mathbf{Q}}(\mathbf{n})\mathbf{m} = \rho c^2 \mathbf{m}, \quad \mathbf{m} \cdot \mathbf{n} = 0, \tag{3.66}$$

where

$$\bar{\mathbf{Q}}(\mathbf{n}) = (\mathbf{I} - \mathbf{n} \otimes \mathbf{n})\mathbf{Q}(\mathbf{n})(\mathbf{I} - \mathbf{n} \otimes \mathbf{n}), \tag{3.67}$$

which is the projection of $\mathbf{Q}(\mathbf{n})$ onto the plane normal to $\mathbf{n}$.

For a given direction of propagation $\mathbf{n}$, Eqs. (3.60) and (3.66) are the symmetric eigenvalue problems for determining the wave speeds and polarizations in compressible and incompressible materials, respectively. Since, $\mathbf{Q}(\mathbf{n})$ and $\bar{\mathbf{Q}}(\mathbf{n})$ are symmetric tensors, there are three (two) mutually orthogonal eigenvectors $\mathbf{m}$ for compressible (incompressible) materials corresponding to the direction of propagation $\mathbf{n}$. In the case of incompressible materials, $\mathbf{m}$ and $\mathbf{n}$ are normal to each other.

The characteristic equation in the case of an incompressible material is given by

$$\det[\bar{\mathbf{Q}}(\mathbf{n}) - \rho c^2 \bar{\mathbf{I}}] = 0, \tag{3.68}$$

where $\bar{\mathbf{I}} = \mathbf{I} - \mathbf{n} \otimes \mathbf{n}$, is the two dimensional identity tensor in the plane normal to $\mathbf{n}$.

The strong ellipticity condition, for arbitrary choice of non-zero vectors $\mathbf{m}$ and $\mathbf{n}$ for compressible materials and subject to the restriction (3.64) for an incompressible material, is given by (see, for example, [34])

$$Q_{ij} m_i m_j = \mathcal{A}_{0piqj} n_p n_q m_i m_j > 0, \quad \text{for all non-zero } \mathbf{m}, \mathbf{n}. \tag{3.69}$$

Taking the scalar product of Eq. (3.60) or the first equation in (3.66), we find

$$\rho c^2 = [\mathbf{Q}(\mathbf{n})\mathbf{m}] \cdot \mathbf{m} = \mathcal{A}_{0piqj} n_p n_q m_i m_j. \tag{3.70}$$

The above equation holds for both the compressible and incompressible materials. The strong ellipticity condition (3.69) thus guarantees positive values for ρc^2. However, c can be either negative or positive.

Using Eq. (3.18) in Eq. (3.61), we get for a compressible material

$$
\begin{aligned}
\mathbf{Q}(\mathbf{n}) \;=\;& [2(W_1 + W_2 I_1)B^{(\mathbf{n})} - 2W_2 B^{2(\mathbf{n})} + 2W_7\Sigma^{(\mathbf{n})} + 4W_8(\Sigma B)^{(\mathbf{n})}]\mathbf{I} - 2(W_2 B^{(\mathbf{n})} \\
-\;& W_8\Sigma^{(\mathbf{n})})\mathbf{B} + 2W_8 B^{(\mathbf{n})}\boldsymbol{\Sigma} + 2(W_3 I_3 + 2W_{33}I_3^2)\mathbf{n}\otimes\mathbf{n} + 2(W_2 + 2W_{11} + 4W_{12}I_1) \\
\times\;& \mathbf{Bn}\otimes\mathbf{Bn} + 2(W_8 + 2W_{17})(\mathbf{Bn}\otimes\boldsymbol{\Sigma}\mathbf{n} + \boldsymbol{\Sigma}\mathbf{n}\otimes\mathbf{Bn}) + 4W_{22}\mathbf{B}^*\mathbf{n}\otimes\mathbf{B}^*\mathbf{n} \\
-\;& 4W_{12}(\mathbf{Bn}\otimes\mathbf{B}^2\mathbf{n} + \mathbf{B}^2\mathbf{n}\otimes\mathbf{Bn}) + 4I_3(W_{13} + W_{23}I_1)(\mathbf{Bn}\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{Bn}) \\
+\;& 4W_{18}[\mathbf{Bn}\otimes(\boldsymbol{\Sigma}\mathbf{B} + \mathbf{B}\boldsymbol{\Sigma})\mathbf{n} + (\boldsymbol{\Sigma}\mathbf{B} + \mathbf{B}\boldsymbol{\Sigma})\mathbf{n}\otimes\mathbf{Bn}] + 4W_{19}(\mathbf{Bn}\otimes\boldsymbol{\Sigma}^2\mathbf{n} \\
+\;& \boldsymbol{\Sigma}^2\mathbf{n}\otimes\mathbf{Bn}) + 4W_{1(10)}[\mathbf{Bn}\otimes(\boldsymbol{\Sigma}^2\mathbf{B} + \mathbf{B}\boldsymbol{\Sigma}^2)\mathbf{n} + (\boldsymbol{\Sigma}^2\mathbf{B} + \mathbf{B}\boldsymbol{\Sigma}^2)\mathbf{n}\otimes\mathbf{Bn}] \\
-\;& 4W_{23}I_3(\mathbf{n}\otimes\mathbf{B}^2\mathbf{n} + \mathbf{B}^2\mathbf{n}\otimes\mathbf{n}) + 4W_{27}(\mathbf{B}^*\mathbf{n}\otimes\boldsymbol{\Sigma}\mathbf{n} + \boldsymbol{\Sigma}\mathbf{n}\otimes\mathbf{B}^*\mathbf{n}) + 4W_{28}[\mathbf{B}^*\mathbf{n} \\
\otimes\;& (\boldsymbol{\Sigma}\mathbf{B} + \mathbf{B}\boldsymbol{\Sigma})\mathbf{n} + (\boldsymbol{\Sigma}\mathbf{B} + \mathbf{B}\boldsymbol{\Sigma})\mathbf{n}\otimes\mathbf{B}^*\mathbf{n}] + 4W_{29}(\mathbf{B}^*\mathbf{n}\otimes\boldsymbol{\Sigma}^2\mathbf{n} + \boldsymbol{\Sigma}^2\mathbf{n}\otimes\mathbf{B}^*\mathbf{n}) \\
+\;& 4W_{2(10)}[\mathbf{B}^*\mathbf{n}\otimes(\boldsymbol{\Sigma}^2\mathbf{B} + \mathbf{B}\boldsymbol{\Sigma}^2)\mathbf{n} + (\boldsymbol{\Sigma}^2\mathbf{B} + \mathbf{B}\boldsymbol{\Sigma}^2)\mathbf{n}\otimes\mathbf{B}^*\mathbf{n}] \\
+\;& 4W_{37}I_3(\mathbf{n}\otimes\boldsymbol{\Sigma}\mathbf{n} + \boldsymbol{\Sigma}\mathbf{n}\otimes\mathbf{n}) + 4W_{38}I_3[\mathbf{n}\otimes(\boldsymbol{\Sigma}\mathbf{B} + \mathbf{B}\boldsymbol{\Sigma})\mathbf{n} + (\boldsymbol{\Sigma}\mathbf{B} + \mathbf{B}\boldsymbol{\Sigma})\mathbf{n}\otimes\mathbf{n}] \\
+\;& 4W_{39}I_3(\mathbf{n}\otimes\boldsymbol{\Sigma}^2\mathbf{n} + \boldsymbol{\Sigma}^2\mathbf{n}\otimes\mathbf{n}) + 4W_{3(10)}I_3[\mathbf{n}\otimes(\boldsymbol{\Sigma}^2\mathbf{B} + \mathbf{B}\boldsymbol{\Sigma}^2)\mathbf{n} + (\boldsymbol{\Sigma}^2\mathbf{B} \\
+\;& \mathbf{B}\boldsymbol{\Sigma}^2)\mathbf{n}\otimes\mathbf{n}] + 4W_{77}\boldsymbol{\Sigma}\mathbf{n}\otimes\boldsymbol{\Sigma}\mathbf{n} + 4W_{78}[\boldsymbol{\Sigma}\mathbf{n}\otimes(\boldsymbol{\Sigma}\mathbf{B} + \mathbf{B}\boldsymbol{\Sigma})\mathbf{n} \\
+\;& (\boldsymbol{\Sigma}\mathbf{B} + \mathbf{B}\boldsymbol{\Sigma})\mathbf{n}\otimes\boldsymbol{\Sigma}\mathbf{n}] + 4W_{79}(\boldsymbol{\Sigma}\mathbf{n}\otimes\boldsymbol{\Sigma}^2\mathbf{n} + \boldsymbol{\Sigma}^2\mathbf{n}\otimes\boldsymbol{\Sigma}\mathbf{n}) \\
+\;& 4W_{7(10)}[\boldsymbol{\Sigma}\mathbf{n}\otimes(\boldsymbol{\Sigma}^2\mathbf{B} + \mathbf{B}\boldsymbol{\Sigma}^2)\mathbf{n} + (\boldsymbol{\Sigma}^2\mathbf{B} + \mathbf{B}\boldsymbol{\Sigma}^2)\mathbf{n}\otimes\boldsymbol{\Sigma}\mathbf{n}] + 4W_{88}(\boldsymbol{\Sigma}\mathbf{B} + \mathbf{B}\boldsymbol{\Sigma})\mathbf{n} \\
\otimes\;& (\boldsymbol{\Sigma}\mathbf{B} + \mathbf{B}\boldsymbol{\Sigma})\mathbf{n} + 4W_{89}[\boldsymbol{\Sigma}^2\mathbf{n}\otimes(\boldsymbol{\Sigma}\mathbf{B} + \mathbf{B}\boldsymbol{\Sigma})\mathbf{n} + (\boldsymbol{\Sigma}\mathbf{B} + \mathbf{B}\boldsymbol{\Sigma})\mathbf{n}\otimes\boldsymbol{\Sigma}^2\mathbf{n}] \\
+\;& 4W_{8(10)}[(\boldsymbol{\Sigma}\mathbf{B} + \mathbf{B}\boldsymbol{\Sigma})\mathbf{n}\otimes(\boldsymbol{\Sigma}^2\mathbf{B} + \mathbf{B}\boldsymbol{\Sigma}^2)\mathbf{n} + (\boldsymbol{\Sigma}^2\mathbf{B} + \mathbf{B}\boldsymbol{\Sigma}^2)\mathbf{n}\otimes(\boldsymbol{\Sigma}\mathbf{B} + \mathbf{B}\boldsymbol{\Sigma})\mathbf{n}] \\
+\;& 4W_{99}\boldsymbol{\Sigma}^2\mathbf{n}\otimes\boldsymbol{\Sigma}^2\mathbf{n} + 4W_{9(10)}[\boldsymbol{\Sigma}^2\mathbf{n}\otimes(\boldsymbol{\Sigma}^2\mathbf{B} + \mathbf{B}\boldsymbol{\Sigma}^2)\mathbf{n} + (\boldsymbol{\Sigma}^2\mathbf{B} + \mathbf{B}\boldsymbol{\Sigma}^2)\mathbf{n}\otimes\boldsymbol{\Sigma}^2\mathbf{n}] \\
+\;& 4W_{(10)(10)}(\boldsymbol{\Sigma}^2\mathbf{B} + \mathbf{B}\boldsymbol{\Sigma}^2)\mathbf{n}\otimes(\boldsymbol{\Sigma}^2\mathbf{B} + \mathbf{B}\boldsymbol{\Sigma}^2)\mathbf{n}, \qquad (3.71)
\end{aligned}
$$

where we have defined $B^{(\mathbf{n})} = \mathbf{n}\cdot\mathbf{Bn}$, $\Sigma^{(\mathbf{n})} = \mathbf{n}\cdot\boldsymbol{\Sigma}\mathbf{n}$, $B^{2(\mathbf{n})} = \mathbf{n}\cdot\mathbf{B}^2\mathbf{n}$ and $(\Sigma B)^{(\mathbf{n})} = \mathbf{n}\cdot\boldsymbol{\Sigma}\mathbf{Bn}$.

For simplicity only, we ignore the dependence of W on I_5, I_6, I_9 and $I_{(10)}$ in Eq. (3.71) and therefore, for a compressible material in the reference configuration, we have

$$
\begin{aligned}
\mathbf{Q}(\mathbf{n}) \;=\;& (\alpha_1 + (1+\alpha_3)\tau^{(\mathbf{n})})\mathbf{I} + \alpha_2\mathbf{n}\otimes\mathbf{n} + \alpha_3\boldsymbol{\tau} \\
+\;& (\alpha_3 + \alpha_4)(\mathbf{n}\otimes\mathbf{n}\boldsymbol{\tau} + \boldsymbol{\tau}\mathbf{n}\otimes\mathbf{n}) + \alpha_5\boldsymbol{\tau}\mathbf{n}\otimes\boldsymbol{\tau}\mathbf{n}, \qquad (3.72)
\end{aligned}
$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and α_5 are given by Eq. (3.20).

From Eqs. (3.60) and (3.72), it follows, for arbitrary $\mathbf{m}$ and $\mathbf{n}$,

$$\begin{aligned} \rho c^2 &= [\mathbf{Q}(\mathbf{n})\mathbf{m}] \cdot \mathbf{m} \\ &= \alpha_1 + (1+\alpha_3)\tau^{(\mathbf{n})} + \alpha_2(\mathbf{n}\cdot\mathbf{m})^2 + \alpha_3(\mathbf{m}\cdot\boldsymbol{\tau}\mathbf{m}) \\ &+ 2(\alpha_3+\alpha_4)(\mathbf{m}\cdot\boldsymbol{\tau}\mathbf{n})(\mathbf{n}\cdot\mathbf{m}) + \alpha_5(\mathbf{m}\cdot\boldsymbol{\tau}\mathbf{n})^2. \end{aligned} \tag{3.73}$$

Choose axes such that $\mathbf{n} = \mathbf{e}_1$, and let $\mathbf{e}_2, \mathbf{e}_3$ be basis vectors in the plane normal to $\mathbf{n}$. Therefore, using Eq. (3.72) for an unconstrained material in the reference configuration, $\mathbf{Q}(\mathbf{n})$ has components $Q_{ij}, \{i,j\} \in \{1,2,3\}$, which are given by

$$\begin{aligned} &Q_{11} = \alpha_1 + \alpha_2 + (1 + 4\alpha_3 + 2\alpha_4)\tau_{11} + \alpha_5\tau_{11}^2, \\ &Q_{12} = Q_{21} = (2\alpha_3 + \alpha_4)\tau_{12} + \alpha_5\tau_{11}\tau_{12}, \\ &Q_{13} = Q_{31} = (2\alpha_3 + \alpha_4)\tau_{13} + \alpha_5\tau_{11}\tau_{13}, \\ &Q_{22} = \alpha_1 + (1+\alpha_3)\tau_{11} + \alpha_3\tau_{22} + \alpha_5\tau_{12}^2, \\ &Q_{23} = Q_{32} = \alpha_3\tau_{23} + \alpha_5\tau_{12}\tau_{13}, \\ &Q_{33} = \alpha_1 + (1+\alpha_3)\tau_{11} + \alpha_3\tau_{33} + \alpha_5\tau_{13}^2. \end{aligned} \tag{3.74}$$

Using Eqs. (3.62) and (3.74), the characteristic equation for compressible materials gives a cubic equation in ρc^2 from which actual values of ρc^2 are found independently of $\mathbf{m}$. The cubic equation is given by

$$\begin{aligned} &(\rho c^2)^3 - (Q_{11} + Q_{22} + Q_{33})(\rho c^2)^2 - (Q_{12}^2 + Q_{13}^2 + Q_{23}^2 \\ - \; &Q_{11}Q_{22} - Q_{11}Q_{33} - Q_{22}Q_{33})(\rho c^2) - (2Q_{12}Q_{13}Q_{23} \\ - \; &Q_{12}^2Q_{33} - Q_{13}^2Q_{22} - Q_{23}^2Q_{11} + Q_{11}Q_{22}Q_{33}) = 0, \end{aligned} \tag{3.75}$$

where $Q_{ij}, \{i,j\} \in \{1,2,3\}$, are given by Eq. (3.74).

The expression for $\bar{\mathbf{Q}}$ for an incompressible material follows from Eq. (3.71) as

$$\begin{aligned}
\bar{\mathbf{Q}}(\mathbf{n}) &= [2(W_1 + W_2 I_1)B^{(\mathbf{n})} - 2W_2 B^{2(\mathbf{n})} + 2W_7\Sigma^{(\mathbf{n})} + 4W_8(\Sigma B)^{(\mathbf{n})}]\bar{\mathbf{I}} - 2(W_2 B^{(\mathbf{n})} \\
&- W_8\Sigma^{(\mathbf{n})})\bar{\mathbf{B}} + 2W_8 B^{(\mathbf{n})}\bar{\boldsymbol{\Sigma}} + 2(W_2 + 2W_{11} + 4W_{12}I_1)\bar{\mathbf{I}}\mathbf{B}\mathbf{n}\otimes\bar{\mathbf{I}}\mathbf{B}\mathbf{n} + 2(W_8 \\
&+ 2W_{17})(\bar{\mathbf{I}}\mathbf{B}\mathbf{n}\otimes\bar{\mathbf{I}}\boldsymbol{\Sigma}\mathbf{n} + \bar{\mathbf{I}}\boldsymbol{\Sigma}\mathbf{n}\otimes\bar{\mathbf{I}}\mathbf{B}\mathbf{n}) + 4W_{22}\bar{\mathbf{I}}\mathbf{B}^*\mathbf{n}\otimes\bar{\mathbf{I}}\mathbf{B}^*\mathbf{n} - 4W_{12}(\bar{\mathbf{I}}\mathbf{B}\mathbf{n}\otimes\bar{\mathbf{I}}\mathbf{B}^2\mathbf{n} \\
&+ \bar{\mathbf{I}}\mathbf{B}^2\mathbf{n}\otimes\bar{\mathbf{I}}\mathbf{B}\mathbf{n}) + 4W_{18}[\bar{\mathbf{I}}\mathbf{B}\mathbf{n}\otimes\bar{\mathbf{I}}(\boldsymbol{\Sigma}\mathbf{B}+\mathbf{B}\boldsymbol{\Sigma})\mathbf{n} + \bar{\mathbf{I}}(\boldsymbol{\Sigma}\mathbf{B}+\mathbf{B}\boldsymbol{\Sigma})\mathbf{n}\otimes\bar{\mathbf{I}}\mathbf{B}\mathbf{n}] \\
&+ 4W_{19}(\mathbf{B}\mathbf{n}\otimes\boldsymbol{\Sigma}^2\mathbf{n} + \boldsymbol{\Sigma}^2\mathbf{n}\otimes\mathbf{B}\mathbf{n}) + 4W_{1(10)}[\mathbf{B}\mathbf{n}\otimes(\boldsymbol{\Sigma}^2\mathbf{B}+\mathbf{B}\boldsymbol{\Sigma}^2)\mathbf{n} \\
&+ (\boldsymbol{\Sigma}^2\mathbf{B}+\mathbf{B}\boldsymbol{\Sigma}^2)\mathbf{n}\otimes\mathbf{B}\mathbf{n}] + 4W_{27}(\bar{\mathbf{I}}\mathbf{B}^*\mathbf{n}\otimes\bar{\mathbf{I}}\boldsymbol{\Sigma}\mathbf{n} + \bar{\mathbf{I}}\boldsymbol{\Sigma}\mathbf{n}\otimes\bar{\mathbf{I}}\mathbf{B}^*\mathbf{n}) + 4W_{28}(\bar{\mathbf{I}}\mathbf{B}^*\mathbf{n} \\
&\otimes \bar{\mathbf{I}}(\boldsymbol{\Sigma}\mathbf{B}+\mathbf{B}\boldsymbol{\Sigma})\mathbf{n} + \bar{\mathbf{I}}(\boldsymbol{\Sigma}\mathbf{B}+\mathbf{B}\boldsymbol{\Sigma})\mathbf{n}\otimes\bar{\mathbf{I}}\mathbf{B}^*\mathbf{n}) + 4W_{29}(\mathbf{B}^*\mathbf{n}\otimes\boldsymbol{\Sigma}^2\mathbf{n} + \boldsymbol{\Sigma}^2\mathbf{n}\otimes\mathbf{B}^*\mathbf{n}) \\
&+ 4W_{2(10)}[\mathbf{B}^*\mathbf{n}\otimes(\boldsymbol{\Sigma}^2\mathbf{B}+\mathbf{B}\boldsymbol{\Sigma}^2)\mathbf{n} + (\boldsymbol{\Sigma}^2\mathbf{B}+\mathbf{B}\boldsymbol{\Sigma}^2)\mathbf{n}\otimes\mathbf{B}^*\mathbf{n}] + 4W_{77}\bar{\mathbf{I}}\boldsymbol{\Sigma}\mathbf{n}\otimes\bar{\mathbf{I}}\boldsymbol{\Sigma}\mathbf{n} \\
&+ 4W_{78}[\bar{\mathbf{I}}\boldsymbol{\Sigma}\mathbf{n}\otimes\bar{\mathbf{I}}(\boldsymbol{\Sigma}\mathbf{B}+\mathbf{B}\boldsymbol{\Sigma})\mathbf{n} + \bar{\mathbf{I}}(\boldsymbol{\Sigma}\mathbf{B}+\mathbf{B}\boldsymbol{\Sigma})\otimes\bar{\mathbf{I}}\boldsymbol{\Sigma}\mathbf{n}] + 4W_{79}(\boldsymbol{\Sigma}\mathbf{n}\otimes\boldsymbol{\Sigma}^2\mathbf{n} \\
&+ \boldsymbol{\Sigma}^2\mathbf{n}\otimes\boldsymbol{\Sigma}\mathbf{n}) + 4W_{7(10)}[\boldsymbol{\Sigma}\mathbf{n}\otimes(\boldsymbol{\Sigma}^2\mathbf{B}+\mathbf{B}\boldsymbol{\Sigma}^2)\mathbf{n} + (\boldsymbol{\Sigma}^2\mathbf{B}+\mathbf{B}\boldsymbol{\Sigma}^2)\mathbf{n}\otimes\boldsymbol{\Sigma}\mathbf{n}] \\
&+ 4W_{88}\bar{\mathbf{I}}(\boldsymbol{\Sigma}\mathbf{B}+\mathbf{B}\boldsymbol{\Sigma})\mathbf{n}\otimes\bar{\mathbf{I}}(\boldsymbol{\Sigma}\mathbf{B}+\mathbf{B}\boldsymbol{\Sigma})\mathbf{n} + 4W_{89}[\boldsymbol{\Sigma}^2\mathbf{n}\otimes(\boldsymbol{\Sigma}\mathbf{B}+\mathbf{B}\boldsymbol{\Sigma})\mathbf{n} + (\boldsymbol{\Sigma}\mathbf{B} \\
&+ \mathbf{B}\boldsymbol{\Sigma})\mathbf{n}\otimes\boldsymbol{\Sigma}^2\mathbf{n}] + 4W_{8(10)}[(\boldsymbol{\Sigma}\mathbf{B}+\mathbf{B}\boldsymbol{\Sigma})\mathbf{n}\otimes(\boldsymbol{\Sigma}^2\mathbf{B}+\mathbf{B}\boldsymbol{\Sigma}^2)\mathbf{n} + (\boldsymbol{\Sigma}^2\mathbf{B}+\mathbf{B}\boldsymbol{\Sigma}^2)\mathbf{n} \\
&\otimes (\boldsymbol{\Sigma}\mathbf{B}+\mathbf{B}\boldsymbol{\Sigma})\mathbf{n}] + 4W_{99}\boldsymbol{\Sigma}^2\mathbf{n}\otimes\boldsymbol{\Sigma}^2\mathbf{n} + 4W_{9(10)}[\boldsymbol{\Sigma}^2\mathbf{n}\otimes(\boldsymbol{\Sigma}^2\mathbf{B}+\mathbf{B}\boldsymbol{\Sigma}^2)\mathbf{n} + (\boldsymbol{\Sigma}^2\mathbf{B} \\
&+ \mathbf{B}\boldsymbol{\Sigma}^2\mathbf{n})\otimes\boldsymbol{\Sigma}^2\mathbf{n}] + 4W_{(10)(10)}(\boldsymbol{\Sigma}^2\mathbf{B}+\mathbf{B}\boldsymbol{\Sigma}^2)\mathbf{n}\otimes(\boldsymbol{\Sigma}^2\mathbf{B}+\mathbf{B}\boldsymbol{\Sigma}^2)\mathbf{n}, \qquad (3.76)
\end{aligned}$$

where we have defined $B^{2(\mathbf{n})} = \mathbf{n}\cdot\mathbf{B}^2\mathbf{n}, \bar{\mathbf{B}} = \bar{\mathbf{I}}\mathbf{B}\bar{\mathbf{I}}$ and $\bar{\boldsymbol{\Sigma}} = \bar{\mathbf{I}}\boldsymbol{\Sigma}\bar{\mathbf{I}}$.

In the reference configuration, $B^{2(\mathbf{n})} = B^{(\mathbf{n})} = 1, B^{*(\mathbf{n})} = 2$ and $(\Sigma B)^{(\mathbf{n})} = \tau^{(\mathbf{n})}$. Also, for simplicity of calculations, we ignore the dependence of W on I_5, I_6, I_9 and I_{10}. Therefore, for an incompressible material in the reference configuration, we have

$$\bar{\mathbf{Q}}(\mathbf{n}) = [\alpha_1 + (1+\alpha_3)\tau^{(\mathbf{n})}]\bar{\mathbf{I}} + \alpha_3\bar{\boldsymbol{\tau}} + \alpha_5\bar{\mathbf{I}}\boldsymbol{\tau}\mathbf{n}\otimes\bar{\mathbf{I}}\boldsymbol{\tau}\mathbf{n}, \qquad (3.77)$$

where α_1, α_3 and α_5 are given by Eq. (3.20).

The counterpart of Eq. (3.73) for incompressible materials follows from Eqs. (3.66) and (3.77), for arbitrary choice of $\mathbf{m}$ and $\mathbf{n}$, subject to $\mathbf{m}\cdot\mathbf{n} = 0$, as

$$\begin{aligned}
\rho c^2 &= [\bar{\mathbf{Q}}(\mathbf{n})\mathbf{m}]\cdot\mathbf{m} \\
&= \alpha_1 + (1+\alpha_3)\tau^{(\mathbf{n})} + \alpha_3\mathbf{m}\cdot\boldsymbol{\tau}\mathbf{m} + \alpha_5(\mathbf{m}\cdot\boldsymbol{\tau}\mathbf{n})^2. \qquad (3.78)
\end{aligned}$$

For instance, if $\mathbf{n} = \mathbf{e}_1$, for an incompressible material in the reference configuration, the components $\bar{Q}_{ij}, \{i,j\} \in \{2,3\}$ of $\bar{\mathbf{Q}}(\mathbf{n})$ follow from Eq. (3.77) and are given by

$$\bar{Q}_{22} = \alpha_1 + (1+\alpha_3)\tau_{11} + \alpha_3\tau_{22} + \alpha_5\tau_{12}^2, \quad \bar{Q}_{23} = \bar{Q}_{32} = \alpha_3\tau_{23} + \alpha_5\tau_{12}\tau_{13},$$
$$\bar{Q}_{33} = \alpha_1 + (1+\alpha_3)\tau_{11} + \alpha_3\tau_{33} + \alpha_5\tau_{13}^2. \tag{3.79}$$

Therefore, for an incompressible material, the counterpart of Eq. (3.75) is

$$(\rho c^2)^2 - (\bar{Q}_{22} + \bar{Q}_{33})\rho c^2 + \bar{Q}_{22}\bar{Q}_{33} - \bar{Q}_{23}^2 = 0, \tag{3.80}$$

where $\bar{Q}_{ij}, \{i,j\} \in \{2,3\}$, are given by Eq. (3.79). The strong ellipticity condition in this case gives

$$\bar{Q}_{22} > 0, \quad \bar{Q}_{22}\bar{Q}_{33} - \bar{Q}_{23}^2 > 0, \tag{3.81}$$

which ensure positive roots for Eq. (3.80). Also, it follows from the above conditions that $Q_{33} > 0$.

3.3.1 Isotropy

When $\boldsymbol{\tau}$ vanishes, the material is isotropic and the expression for $\mathbf{Q}(\mathbf{n})$ follows from Eq. (3.71) for a compressible material

$$\begin{aligned}\mathbf{Q}(\mathbf{n}) &= [2(W_1 + W_2 I_1)B^{(\mathbf{n})} - 2W_2 B^{2(\mathbf{n})}]\mathbf{I} + (W_1 + W_3)B^{(\mathbf{n})}\mathbf{B} + 2(W_3 I_3 + 2W_{33} I_3^2)\mathbf{n}\otimes\mathbf{n} \\ &+ 2(W_2 + 2W_{11} + 4W_{12} I_1)\mathbf{Bn}\otimes\mathbf{Bn} + 4I_3(W_{13} + W_{23} I_1)(\mathbf{Bn}\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{Bn}) \\ &- 4W_{12}(\mathbf{Bn}\otimes\mathbf{B}^2\mathbf{n} + \mathbf{B}^2\mathbf{n}\otimes\mathbf{Bn}) + 4W_{22}\mathbf{B}^*\mathbf{n}\otimes\mathbf{B}^*\mathbf{n} \\ &- 4W_{23} I_3(\mathbf{n}\otimes\mathbf{B}^2\mathbf{n} + \mathbf{B}^2\mathbf{n}\otimes\mathbf{n}).\end{aligned} \tag{3.82}$$

If we consider W to be independent of I_2, the terms involving the derivatives with respect to I_2 are omitted from Eq. (3.82) and it reduces to

$$\begin{aligned}\mathbf{Q}(\mathbf{n}) &= 2W_1 B^{(\mathbf{n})}\mathbf{I} + (W_1 + W_3)B^{(\mathbf{n})}\mathbf{B} + 2(W_3 I_3 + 2W_{33} I_3^2)\mathbf{n}\otimes\mathbf{n} \\ &+ 4W_{11}\mathbf{Bn}\otimes\mathbf{Bn} + 4I_3 W_{13}(\mathbf{Bn}\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{Bn}).\end{aligned} \tag{3.83}$$

Equation (3.72) gives the expression for $\mathbf{Q}(\mathbf{n})$ in the undeformed configuration for a compressible material when $\boldsymbol{\tau} \neq \mathbf{0}$, whereas in Eq. (3.83) $\mathbf{Q}(\mathbf{n})$ is calculated when deformation has occurred in the absence of the initial stress but is associated with a pre-stress. On comparison of Eq. (3.72) and Eq. (3.83), it is obvious that the roles of $\boldsymbol{\tau}$ and $\mathbf{B}$ have been reversed.

Further, if we consider W to be dependent on I_9 and I_{10}, Eq. (3.72) becomes

$$\begin{aligned}\mathbf{Q}(\mathbf{n}) &= [\alpha_1 + (1+\alpha_3)\tau^{(\mathbf{n})} + 3\alpha_7\tau^{2(\mathbf{n})}]\mathbf{I} + \alpha_2\mathbf{n}\otimes\mathbf{n} + \alpha_3\boldsymbol{\tau} + \alpha_7\boldsymbol{\tau}^2 \\ &+ (\alpha_3+\alpha_4)[\mathbf{n}\otimes\mathbf{n}\boldsymbol{\tau} + \boldsymbol{\tau}\mathbf{n}\otimes\mathbf{n}] + \alpha_5\boldsymbol{\tau}\mathbf{n}\otimes\boldsymbol{\tau}\mathbf{n} + \alpha_8[\mathbf{n}\otimes\mathbf{n}\boldsymbol{\tau}^2 \\ &+ \boldsymbol{\tau}^2\mathbf{n}\otimes\mathbf{n}] + \alpha_9[\boldsymbol{\tau}\mathbf{n}\otimes\mathbf{n}\boldsymbol{\tau}^2 + \boldsymbol{\tau}^2\mathbf{n}\otimes\boldsymbol{\tau}\mathbf{n}] + \alpha_{10}\boldsymbol{\tau}^2\mathbf{n}\otimes\boldsymbol{\tau}^2\mathbf{n},\end{aligned} \tag{3.84}$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and α_5 are given by Eq. (3.20) and

$$\begin{aligned}&\alpha_7 = 2W_{10}, \quad \alpha_8 = 2(W_{10} + 2W_{19} + 4W_{1(10)} + 4W_{29} + 8W_{2(10)} + 2W_{39} + 4W_{3(10)}), \\ &\alpha_9 = 4(W_{99} + 2W_{9(10)} + 4W_{(10)(10)}),\end{aligned} \tag{3.85}$$

evaluated in the reference configuration. Also, we have defined $\tau^{2(\mathbf{n})} = \mathbf{n}\cdot\boldsymbol{\tau}^2\mathbf{n}$. Comparing Eqs. (3.82) and (3.84), we again find the roles of $\boldsymbol{\tau}$ and $\mathbf{B}$ reversed.

A similar kind of comparison can be done for incompressible materials. Therefore, considering Eq. (3.76) in the absence of $\boldsymbol{\tau}$, we have

$$\begin{aligned}\bar{\mathbf{Q}}(\mathbf{n}) &= [2(W_1 + W_2I_1)B^{(\mathbf{n})} - 2W_2B^{2(\mathbf{n})} + 2W_7\tau^{(\mathbf{n})} + 4W_8(\tau B)^{(\mathbf{n})}]\bar{\mathbf{I}} - 2W_2B^{(\mathbf{n})}\bar{\mathbf{B}} \\ &+ 2(W_2 + 2W_{11} + 4W_{12}I_1)\bar{\mathbf{I}}\mathbf{B}\mathbf{n}\otimes\bar{\mathbf{I}}\mathbf{B}\mathbf{n} - 4W_{12}(\bar{\mathbf{I}}\mathbf{B}\mathbf{n}\otimes\bar{\mathbf{I}}\mathbf{B}^2\mathbf{n} + \bar{\mathbf{I}}\mathbf{B}^2\mathbf{n}\otimes\bar{\mathbf{I}}\mathbf{B}\mathbf{n}) \\ &+ 4W_{22}\bar{\mathbf{I}}\mathbf{B}^*\mathbf{n}\otimes\bar{\mathbf{I}}\mathbf{B}^*\mathbf{n}.\end{aligned} \tag{3.86}$$

Omitting only the terms with second order derivatives with respect to I_2, Eq. (3.86) reduces to

$$\begin{aligned}\bar{\mathbf{Q}}(\mathbf{n}) &= [2(W_1 + W_2I_1)B^{(\mathbf{n})} - 2W_2B^{2(\mathbf{n})} + 2W_7\tau^{(\mathbf{n})} + 4W_8(\tau B)^{(\mathbf{n})}]\bar{\mathbf{I}} - 2W_2B^{(\mathbf{n})}\bar{\mathbf{B}} \\ &+ 2(W_2 + 2W_{11})\bar{\mathbf{I}}\mathbf{B}\mathbf{n}\otimes\bar{\mathbf{I}}\mathbf{B}\mathbf{n}.\end{aligned} \tag{3.87}$$

Equation (3.87) gives the expression for $\bar{\mathbf{Q}}(\mathbf{n})$ for an incompressible material in the deformed configuration in the absence of initial stress. Comparison of Eq. (3.87) with Eq. (3.77) again shows the reversed roles of $\boldsymbol{\tau}$ and $\mathbf{B}$. Now, in the case of undeformed incompressible material, if we consider W to be dependent on I_9 and I_{10}, we have from Eq. (3.76)

$$\begin{aligned} \bar{\mathbf{Q}}(\mathbf{n}) &= [\alpha_1 + (1+\alpha_3)\tau^{(\mathbf{n})} + 3\alpha_7\tau^{2(\mathbf{n})}]\bar{\mathbf{I}} + \alpha_3\bar{\boldsymbol{\tau}} + \alpha_5\bar{\mathbf{I}}\boldsymbol{\tau}\mathbf{n}\otimes\bar{\mathbf{I}}\boldsymbol{\tau}\mathbf{n} + \alpha_7\bar{\boldsymbol{\tau}^2} \\ &+ \alpha_8[\bar{\mathbf{I}}\boldsymbol{\tau}\mathbf{n}\otimes\bar{\mathbf{I}}\boldsymbol{\tau}^2\mathbf{n} + \bar{\mathbf{I}}\boldsymbol{\tau}^2\mathbf{n}\otimes\bar{\mathbf{I}}\boldsymbol{\tau}\mathbf{n}] + \alpha_9\bar{\mathbf{I}}\boldsymbol{\tau}^2\mathbf{n}\otimes\bar{\mathbf{I}}\boldsymbol{\tau}^2\mathbf{n}, \end{aligned} \tag{3.88}$$

where α_1, α_3 and α_5 are given by Eq. (3.20) and α_7, α_8 and α_9 by Eq. (3.85). Here, we have defined $\bar{\boldsymbol{\tau}^2} = \bar{\mathbf{I}}\boldsymbol{\tau}^2\bar{\mathbf{I}}$. On the comparison of Eqs. (3.86) and (3.88), we find that the roles of $\boldsymbol{\tau}$ and $\mathbf{B}$ are not exactly reversed. However, except for only a few terms, they occur quite similarly.

3.3.2 Examples of Initial Stress

In this section we take two different forms of initial stress $\boldsymbol{\tau}$ to see the effect on the wave speeds, both in the case of compressible and incompressible materials.

Case A: $\boldsymbol{\tau} = \tau\mathbf{n}\otimes\mathbf{n}$ for Compressible Materials

In this case, for arbitrary $\mathbf{m}$ and $\mathbf{n}$, we have from Eq. (3.73)

$$\rho c^2 = \alpha_1 + (1+\alpha_3)\tau + [\alpha_2 + 3\alpha_3\tau + 2\alpha_4\tau + \alpha_5\tau^2](\mathbf{m}\cdot\mathbf{n})^2. \tag{3.89}$$

We have $\rho c^2 > 0$ for all possible values of $\mathbf{m}$ and $\mathbf{n}$ if and only if

$$\alpha_1 + (1+\alpha_3)\tau > 0 \quad \text{and} \quad \alpha_1 + \alpha_2 + (1+4\alpha_3)\tau + 2\alpha_4\tau + \alpha_5\tau^2 > 0. \tag{3.90}$$

Choosing $\mathbf{m} = \mathbf{n} = \mathbf{e}_1$ in Eq. (3.89), we have

$$\rho c^2 = \alpha_1 + \alpha_2 + (1 + 4\alpha_3 + 2\alpha_4)\tau + \alpha_5\tau^2, \tag{3.91}$$

where $\tau_{11} = \tau$. The strong ellipticity condition in this case gives

$$\rho c^2 = Q_{11} = \alpha_1 + \alpha_2 + (1 + 4\alpha_3 + 2\alpha_4)\tau + \alpha_5\tau^2 > 0, \tag{3.92}$$

which ensures positive values of ρc^2 in Eq. (3.75). The above inequality is quadratic in τ and for inequality (3.92) to hold $\forall\tau$, we require

$$\begin{aligned} & \alpha_1+\alpha_2>0, \quad \alpha_5>0, \\ - \quad & 2\sqrt{\alpha_5(\alpha_1+\alpha_2)}<1+4\alpha_3+2\alpha_4<2\sqrt{\alpha_5(\alpha_1+\alpha_2)}. \end{aligned} \tag{3.93}$$

If conditions (3.93) do not hold, we must have

$$\text{either } \tau>\frac{1}{2}(-B+\sqrt{B^2-4C}) \text{ or } \tau<\frac{1}{2}(-B-\sqrt{B^2-4C}), \tag{3.94}$$

where we have defined

$$B=(1+4\alpha_3+2\alpha_4)/\alpha_5, \quad C=(\alpha_1+\alpha_2)/\alpha_5. \tag{3.95}$$

It is obvious that for this particular case in a compressible material, wave speeds are dependent on the form of the initial stress and particularly on the sign of τ through (3.92).

If we choose $\mathbf{m}\cdot\mathbf{n}=0$ and $\mathbf{n}=\mathbf{e}_1$, we have from Eq. (3.89)

$$\rho c^2=\alpha_1+(1+\alpha_3)\tau. \tag{3.96}$$

As we will see in *Case* C below, this result is equivalent to taking $\boldsymbol{\tau}=\tau\mathbf{n}\otimes\mathbf{n}$ in an incompressible material for arbitrary $\mathbf{m}$ and $\mathbf{n}$. For a positive ρc^2, we thus require from the strong ellipticity condition

$$\alpha_1+(1+\alpha_3)\tau>0. \tag{3.97}$$

Let $(1+\alpha_3)>0$, then inequality (3.97) holds, if

$$\tau>-\alpha_1/(1+\alpha_3). \tag{3.98}$$

The wave speed thus depends on the form of initial stress.

***Case* B: $\boldsymbol{\tau}\mathbf{n}=\mathbf{0}$ for Compressible Materials**

In this case $\tau^{(\mathbf{n})}=0$. Thus, Eq. (3.73) gives

$$\rho c^2 \quad = \quad \alpha_1+\alpha_2(\mathbf{n}\cdot\mathbf{m})^2+\alpha_3(\mathbf{m}\cdot\boldsymbol{\tau}\mathbf{m}). \tag{3.99}$$

We have $\rho c^2 > 0$ for all possible values of $\mathbf{m}$ and $\mathbf{n}$ if

$$\alpha_1 + \alpha_3(\mathbf{m} \cdot \boldsymbol{\tau}\mathbf{m}) > 0 \quad \text{and} \quad \alpha_1 + \alpha_2 + \alpha_3(\mathbf{m} \cdot \boldsymbol{\tau}\mathbf{m}) > 0. \tag{3.100}$$

For $\mathbf{m} = \mathbf{n} = \mathbf{e}_1$, we have from Eq. (3.99)

$$\rho c^2 = \alpha_1 + \alpha_2. \tag{3.101}$$

It may be noted that in this case, the wave speeds are independent of $\boldsymbol{\tau}$ and this has a structure similar to the case of pure elasticity, i.e. in the absence of initial stress for compressible materials. The strong ellipticity condition, in this case, gives

$$\alpha_1 > -\alpha_2. \tag{3.102}$$

For $\mathbf{m} \cdot \mathbf{n} = 0$, we have from Eq. (3.99) for arbitrary $\mathbf{m}$ and $\mathbf{n}$

$$\rho c^2 \ = \alpha_1 + \alpha_3(\mathbf{m} \cdot \boldsymbol{\tau}\mathbf{m}). \tag{3.103}$$

We have $\rho c^2 > 0$ in this case if

$$\alpha_1 + \alpha_3(\mathbf{m} \cdot \boldsymbol{\tau}\mathbf{m}) > 0. \tag{3.104}$$

Case C: $\boldsymbol{\tau} = \tau\mathbf{n} \otimes \mathbf{n}$ for Incompressible Materials

In this case, for arbitrary $\mathbf{m}$ and $\mathbf{n}$ subject to $\mathbf{m} \cdot \mathbf{n} = 0$, we have from Eq. (3.78) for an incompressible material

$$\rho c^2 = \alpha_1 + (1 + \alpha_3)\tau. \tag{3.105}$$

For a positive ρc^2, we thus require from the strong ellipticity condition

$$\alpha_1 + (1 + \alpha_3)\tau > 0. \tag{3.106}$$

Therefore, the wave speed in the case of an initially stressed incompressible material is dependent on the form of the initial stress. In this case, τ is subject to condition (3.97).

***Case* D: $\boldsymbol{\tau}\mathbf{n} = \mathbf{0}$ for Incompressible Materials**

In this case, Eq. (3.78) gives

$$\rho c^2 = \alpha_1 + \alpha_3 \mathbf{m} \cdot \boldsymbol{\tau}\mathbf{m}. \tag{3.107}$$

The strong ellipticity condition gives

$$\alpha_1 + \alpha_3 \mathbf{m} \cdot (\boldsymbol{\tau}\mathbf{m}) > 0, \tag{3.108}$$

for arbitrary $\mathbf{m}$. This is a case similar to taking $\boldsymbol{\tau}\mathbf{n} = \mathbf{0}$ along with $\mathbf{m} \cdot \mathbf{n} = 0$ in the compressible materials as shown in Eqs. (3.103) and (3.104).

If $\mathbf{m} = \mathbf{e_2}$, we have

$$\rho c^2 = \alpha_1 + \alpha_3 \tau_{22}, \tag{3.109}$$

and if $\mathbf{m} = \mathbf{e_3}$, we have

$$\rho c^2 = \alpha_1 + \alpha_3 \tau_{33}. \tag{3.110}$$

3.3.3 Specific Strain Energy Function W for an Initially Stressed Incompressible Material

We consider an incompressible material whose elastic response is characterized by the strain energy function $W(\mathbf{C}, \boldsymbol{\tau})$ given by

$$W = \frac{\mu}{2}(I_1 - 3) + \frac{\bar{\mu}}{2}(I_7 - I_4)^2 + \frac{1}{2}(I_7 - I_4), \tag{3.111}$$

where μ and $\bar{\mu}$ are material constants. The material constant μ has the same dimension as stress and $\bar{\mu}$ has dimensions of stress^{-1}. The invariants I_1, I_4 and I_7 are given by Eq. (3.1). This simple model is chosen to illustrate the combined effect of finite deformation and initial stress. We can rewrite Eq. (3.111) in terms of the principal stretches $\lambda_1, \lambda_2, \lambda_3$ and the principal initial stresses τ_1, τ_2, τ_3 as

$$\begin{aligned} W &= \frac{\mu}{2}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) + \frac{\bar{\mu}}{2}[(\lambda_1^2 - 1)\tau_1 + (\lambda_2^2 - 1)\tau_2 + (\lambda_3^2 - 1)\tau_3]^2 \\ &+ \frac{1}{2}[(\lambda_1^2 - 1)\tau_1 + (\lambda_2^2 - 1)\tau_2 + (\lambda_3^2 - 1)\tau_3]. \end{aligned} \tag{3.112}$$

For the incompressible material (3.111) in the deformed configuration, we have

$$W_1 = \frac{\mu}{2}, \quad W_4 = -\bar{\mu} I_7, \quad W_7 = \bar{\mu}(I_7 - I_4) + \frac{1}{2}, \quad W_{77} = \bar{\mu}, \tag{3.113}$$

which in the reference configuration reduce to

$$W_1 = \frac{\mu}{2}, \quad W_4 = -\bar{\mu} I_4, \quad W_7 = \frac{1}{2}, \quad W_{77} = \bar{\mu}. \tag{3.114}$$

All other $W_r, W_{rs} = 0$. Using Eq. (3.113) in Eq. (3.76), we have

$$\bar{\mathbf{Q}}(\mathbf{n}) = [\mu B^{(\mathbf{n})} + [2\bar{\mu}(I_7 - I_4) + 1]\Sigma^{(\mathbf{n})}]\bar{\mathbf{I}} + 4\bar{\mu}\bar{\mathbf{I}}\boldsymbol{\Sigma}\mathbf{n} \otimes \bar{\mathbf{I}}\boldsymbol{\Sigma}\mathbf{n}, \tag{3.115}$$

for an incompressible material in the deformed configuration. Also, using Eq. (3.114) in Eq. (3.77) gives

$$\bar{\mathbf{Q}}(\mathbf{n}) = (\mu + \tau^{(\mathbf{n})})\bar{\mathbf{I}} + 4\bar{\mu}\bar{\mathbf{I}}\boldsymbol{\tau}\mathbf{n} \otimes \bar{\mathbf{I}}\boldsymbol{\tau}\mathbf{n}. \tag{3.116}$$

for an incompressible material in the reference configuration.

From Eq. (3.70), it follows for an incompressible material in the deformed configuration that

$$\rho c^2 = \mu B^{(\mathbf{n})} + [2\bar{\mu}(I_7 - I_4) + 1]\Sigma^{(\mathbf{n})} + 4\bar{\mu}(\mathbf{m} \cdot \boldsymbol{\Sigma}\mathbf{n})^2, \tag{3.117}$$

which in the reference configuration reduces to

$$\rho c^2 = \mu + \tau^{(\mathbf{n})} + 4\bar{\mu}(\mathbf{m} \cdot \boldsymbol{\tau}\mathbf{n})^2. \tag{3.118}$$

For the particular choice $\boldsymbol{\tau} = \tau\mathbf{n} \otimes \mathbf{n}$, we have $\boldsymbol{\Sigma} = \tau\mathbf{F}\mathbf{n} \otimes \mathbf{F}\mathbf{n}$. Equation (3.117) for an incompressible material in the deformed configuration thus reduces to

$$\rho c^2 = \mu B^{(\mathbf{n})} + \tau[(2\bar{\mu}(I_7 - I_4) + 1) + 4\bar{\mu}\tau(\mathbf{m} \cdot \mathbf{F}\mathbf{n})^2](\mathbf{n} \cdot \mathbf{F}\mathbf{n})^2. \tag{3.119}$$

For arbitrary $\mathbf{m}$ and $\mathbf{n}$, a real speed exists if

$$\mu B^{(\mathbf{n})} + \tau[(2\bar{\mu}(I_7 - I_4) + 1) + 4\bar{\mu}\tau(\mathbf{m} \cdot \mathbf{F}\mathbf{n})^2](\mathbf{n} \cdot \mathbf{F}\mathbf{n})^2 > 0. \tag{3.120}$$

In the undeformed configuration, for an incompressible material, Eq. (3.119) reduces to

$$\rho c^2 = \mu + \tau, \tag{3.121}$$

and a real wave speed is ensured if $\mu + \tau > 0$.

If we choose, $\mathbf{n} = \mathbf{e}_1$, $\mathbf{m} = \mathbf{e}_2$ or $\mathbf{m} = \mathbf{e}_3$ and $\boldsymbol{\Sigma}\mathbf{n} = \mathbf{0}$, we find from Eq. (3.117)

$$\rho c^2 = \mu B^{(\mathbf{n})}, \tag{3.122}$$

in the deformed configuration. Similarly, in the reference configuration, when $\boldsymbol{\tau}\mathbf{n} = \mathbf{0}$, we find from Eq. (3.118)

$$\rho c^2 = \mu. \tag{3.123}$$

Using Eq. (3.115), various components of $\bar{\mathbf{Q}}(\mathbf{n})$ can be calculated for the special model given by Eq. (3.111). In the deformed configuration, these components are given by

$$\bar{Q}_{22} = \mu B^{(\mathbf{n})} + [2\bar{\mu}(I_7 - I_4) + 1]\Sigma_{11} + 4\bar{\mu}\Sigma_{12}^2, \tag{3.124}$$

$$\bar{Q}_{23} = \bar{Q}_{32} = 4\bar{\mu}\Sigma_{12}\Sigma_{13}, \tag{3.125}$$

$$\bar{Q}_{33} = \mu B^{(\mathbf{n})} + [2\bar{\mu}(I_7 - I_4) + 1]\Sigma_{11} + 4\bar{\mu}\Sigma_{13}^2, \tag{3.126}$$

which, in the reference configuration, reduce to

$$\bar{Q}_{22} = \mu + \tau_{11} + 4\bar{\mu}\tau_{12}^2, \tag{3.127}$$

$$\bar{Q}_{23} = \bar{Q}_{32} = 4\bar{\mu}\tau_{12}\tau_{13}, \tag{3.128}$$

$$\bar{Q}_{33} = \mu + \tau_{11} + 4\bar{\mu}\tau_{13}^2, \tag{3.129}$$

Using Eqs. (3.124)–(3.126) in the characteristic equation (3.80) gives

$$\begin{aligned} & (\rho c^2)^2 - [2\mu B^{(\mathbf{n})} + 2[2\bar{\mu}(I_7 - I_4) + 1]\Sigma_{11} + 4\bar{\mu}(\Sigma_{12}^2 + \Sigma_{13}^2)](\rho c^2) \\ + \;& 4\bar{\mu}[4\bar{\mu}\Sigma_{12}^2\Sigma_{13}^2 + [(\mu B^{(\mathbf{n})} + 2\bar{\mu}(I_7 - I_4) + 1]\Sigma_{11})(\Sigma_{12}^2 + \Sigma_{13}^2)] \\ + \;& [\mu B^{(\mathbf{n})} + [2\bar{\mu}(I_7 - I_4) + 1]\Sigma_{11}]^2 = 0. \end{aligned} \tag{3.130}$$

The actual values of ρc^2, independent of $\mathbf{m}$ can be calculated from Eq. (3.130) in the deformed configuration. For Eq. (3.130) to give $\rho c^2 > 0$, we require the strong ellipticity conditions (3.81) to hold when $\bar{Q}_{22}, \bar{Q}_{23}$, and $\bar{Q}_{33}$ are given by Eqs. (3.124)–(3.126). Therefore, we have

$$\begin{aligned} \bar{Q}_{22} > 0 \quad &\Longrightarrow \quad \mu B^{(\mathbf{n})} + [2\bar{\mu}(I_7 - I_4) + 1]\Sigma_{11} + 4\bar{\mu}\Sigma_{12}^2 > 0, && (3.131) \\ \bar{Q}_{22}\bar{Q}_{33} - \bar{Q}_{23}^2 \quad &\Longrightarrow \quad (\mu B^{(\mathbf{n})} + [2\bar{\mu}(I_7 - I_4) + 1]\Sigma_{11})^2 \\ &+ \quad 4\bar{\mu}(\mu B^{(\mathbf{n})} + [2\bar{\mu}(I_7 - I_4) + 1]\Sigma_{11})[\Sigma_{12}^2 + \Sigma_{13}^2] > 0, && (3.132) \end{aligned}$$

which together imply

$$\bar{Q}_{33} > 0 \quad \Longrightarrow \quad \mu B^{(\mathbf{n})} + [2\bar{\mu}(I_7 - I_4) + 1]\Sigma_{11} + 4\bar{\mu}\Sigma_{13}^2 > 0. \qquad (3.133)$$

Similarly, using Eqs. (3.127)–(3.129) in Eq. (3.80), we have for the reference configuration

$$(\rho c^2)^2 - [2(\mu + \tau) + 4\bar{\mu}(\tau_{12}^2 + \tau_{13}^2)](\rho c^2) + (\mu + \tau)^2 + 4\bar{\mu}[(\mu + \tau)(\tau_{12}^2 + \tau_{13}^2) + 4\bar{\mu}\tau_{12}^2\tau_{13}^2] = 0. \qquad (3.134)$$

For Eq. (3.134) to give $\rho c^2 > 0$, we require the strong ellipticity conditions (3.81) to hold when $\bar{Q}_{22}, \bar{Q}_{23}$, and $\bar{Q}_{33}$ are given by Eqs. (3.127)–(3.129). Therefore, we have

$$\begin{aligned} \bar{Q}_{22} > 0 \quad &\Longrightarrow \quad \mu + \tau_{11} + 4\bar{\mu}\tau_{12}^2 > 0, && (3.135) \\ \bar{Q}_{22}\bar{Q}_{33} - \bar{Q}_{23}^2 \quad &\Longrightarrow \quad (\mu + \tau_{11})^2 + 4\bar{\mu}(\mu + \tau_{11})[\tau_{12}^2 + \tau_{13}^2], && (3.136) \end{aligned}$$

which together imply

$$\bar{Q}_{33} > 0 \quad \Longrightarrow \quad \mu + \tau_{11} + 4\bar{\mu}\tau_{13}^2. \qquad (3.137)$$

3.4 Plane Incremental Motions in an Initially Stressed Incompressible Elastic Half Space

3.4.1 Basic Equations

We consider an initially stressed incompressible material whose elastic response is characterised by the strain energy function $W(\mathbf{C}, \boldsymbol{\tau})$. Let $\lambda_1, \lambda_2, \lambda_3$ be the principal stretches corresponding the principal axes x_1, x_2 and x_3 respectively. Let $\tau_{ii}, (i = 1, 2, 3)$ denote the normal initial stress components and $\tau_{ij}, i \neq j$, $(i, j \in \{1, 2, 3\})$ denote the shear components of the initial stress. We assume $\tau_{ij} = 0, i \neq j$. Using Eq. (3.11), the principal Cauchy stresses are given by

$$\begin{aligned} T_{ii} &= -p + 2\lambda_i^2 W_1 + 2\lambda_i^2(I_1 - \lambda_i^2)W_2 + 2\Sigma_{ii}^{(r)} W_7 + 4\lambda_i^2 \Sigma_{ii}^{(r)} W_8 + 2\lambda_i^{-2} \Sigma_{ii}^{(r)^2} W_9 \\ &+ 4\Sigma_{ii}^{(r)^2} W_{10}. \end{aligned} \tag{3.138}$$

Considering W to be independent of I_5, I_9 and I_{10}, we can write the principal Cauchy stress components as

$$T_{11} = -p + 2\lambda_1^2 W_1 + 2\lambda_1^2(\lambda_2^2 + \lambda_3^2)W_2 + 2\lambda_1^2 \tau_{11} W_7 + 4\lambda_1^4 \tau_{11} W_8, \tag{3.139}$$

$$T_{22} = -p + 2\lambda_2^2 W_1 + 2\lambda_2^2(\lambda_1^2 + \lambda_3^2)W_2 + 2\lambda_2^2 \tau_{22} W_7 + 4\lambda_2^4 \tau_{22} W_8, \tag{3.140}$$

$$T_{33} = -p + 2\lambda_3^2 W_1 + 2\lambda_3^2(\lambda_1^2 + \lambda_2^2)W_2 + 2\lambda_3^2 \tau_{33} W_7 + 4\lambda_3^4 \tau_{33} W_8. \tag{3.141}$$

Considering the relation

$$\frac{\partial W}{\partial \lambda_i} = \sum_{k=1,2,7,8} \frac{\partial W}{\partial I_k} \frac{\partial I_k}{\partial \lambda_i}, \quad \{i = 1, 2, 3\}, \tag{3.142}$$

it can be easily deduced that

$$T_{ii} \equiv t_i = \lambda_i \frac{\partial W}{\partial \lambda_i} - p, \quad i = \{1, 2, 3\}. \tag{3.143}$$

In this section we consider plane incremental motions in the (x_1, x_2) plane with incremental displacement $\mathbf{u}$ having components

$$u_1(x_1, x_2, t), \quad u_2(x_1, x_2, t), \quad u_3 = 0. \tag{3.144}$$

From Eq. (3.57), we have in this case

$$\begin{aligned} &\mathcal{A}_{01111} u_{1,11} + 2\mathcal{A}_{02111} u_{1,12} + \mathcal{A}_{02121} u_{1,22} + \mathcal{A}_{01112} u_{2,11} \\ &+(\mathcal{A}_{01122} + \mathcal{A}_{02112}) u_{2,12} + \mathcal{A}_{02221} u_{2,22} - \dot{p}_{,1} = \rho u_{1,tt}, \end{aligned} \tag{3.145}$$

$$\begin{aligned} &\mathcal{A}_{01112} u_{1,11} + (\mathcal{A}_{01122} + \mathcal{A}_{02112}) u_{1,12} + \mathcal{A}_{02221} u_{1,22} \\ &+\mathcal{A}_{01212} u_{2,11} + 2\mathcal{A}_{01222} u_{2,12} + \mathcal{A}_{02222} u_{2,22} - \dot{p}_{,2} = \rho u_{2,tt}, \end{aligned} \tag{3.146}$$

and the incompressibility condition (3.58) reduces to

$$u_{1,1} + u_{2,2} = 0. \tag{3.147}$$

From Eq. (3.147) we deduce the existence of a scalar stream-like function $\psi(x_1, x_2, t)$ such that

$$u_1 = \psi_{,2}, \quad u_2 = -\psi_{,1}. \tag{3.148}$$

We substitute Eq. (3.148) in Eqs. (3.145) and (3.146). From the resulting expressions, $\dot{p}$ can be eliminated by cross differentiation followed by subtraction. As a result of this, we obtain an equation for ψ, namely

$$\beta_1\psi_{,1111} + 2\beta_2\psi_{,2112} + 2\beta_3\psi_{,1222} + 2\beta_4\psi_{,2111} + \beta_5\psi_{,2222} = \rho(\psi_{,11tt} + \psi_{,22tt}), \tag{3.149}$$

where

$$\begin{aligned} &\beta_1 = \mathcal{A}_{01212}, \quad 2\beta_2 = \mathcal{A}_{01111} - 2\mathcal{A}_{01122} - 2\mathcal{A}_{02112} + \mathcal{A}_{02222}, \\ &\beta_3 = \mathcal{A}_{02111} - \mathcal{A}_{02221}, \quad \beta_4 = \mathcal{A}_{01222} - \mathcal{A}_{01112}, \quad \beta_5 = \mathcal{A}_{02121}. \end{aligned} \tag{3.150}$$

Let $\mathbf{m} = (m_1, m_2, 0)$ and $\mathbf{n} = (n_1, n_2, 0)$ be two unit vectors and let $m_1 = n_2$ and $m_2 = -n_1$ so that the incompressibility condition (3.64) is satisfied without loss of generality. The strong ellipticity condition (3.69) in two dimensions, for this special case of an incompressible material, is given by

$$\beta_1 n_1^4 + 2\beta_4 n_1^3 n_2 + 2\beta_2 n_1^2 n_2^2 + 2\beta_3 n_1 n_2^3 + \beta_5 n_2^4 > 0. \tag{3.151}$$

Without loss of generality, we can further assume $n_1 = \cos\theta$ and $n_2 = \sin\theta$. Therefore, inequality (3.151), after some calculations, gives

$$\beta_5 t^4 + 2\beta_3 t^3 + 2\beta_2 t^2 + 2\beta_4 t + \beta_1 > 0, \tag{3.152}$$

where $t = \tan\theta$.

We consider the half-space $x_2 < 0$ bounded by $x_2 = 0$. The incremental traction per unit area of the boundary is $\dot{\mathbf{S}}_0^T \boldsymbol{\nu}$, where $\boldsymbol{\nu}$ is the unit outward normal to the boundary. The component form of $\dot{\mathbf{S}}^T\boldsymbol{\nu}$ in this case follows from Eq. (2.127) as

$$\dot{S}_{0ji}\nu_j = (\mathcal{A}_{0jilk} + p\delta_{jk}\delta_{il})u_{k,l}\nu_j - \dot{p}\nu_i. \tag{3.153}$$

Since $\boldsymbol{\nu} = (0, 1, 0)$ in this case, the only non-vanishing components of $\dot{\mathbf{S}}_0^T\boldsymbol{\nu}$ are $\dot{S}_{021}$ and $\dot{S}_{022}$. These are given by

$$\dot{S}_{021} = \mathcal{A}_{02111}u_{1,1} + \mathcal{A}_{02121}u_{1,2} + (\mathcal{A}_{02112} + p)u_{2,1} + \mathcal{A}_{02122}u_{2,2}, \tag{3.154}$$

$$\dot{S}_{022} = \mathcal{A}_{01122}u_{1,1} + \mathcal{A}_{02221}u_{1,2} + \mathcal{A}_{02212}u_{2,1} + (\mathcal{A}_{02222} + p)u_{2,2} - \dot{p}. \tag{3.155}$$

Using Eq. (3.148) in Eqs. (3.154) and (3.155), respectively, we get

$$\dot{S}_{021} = -(\beta_5 - T_{22})\psi_{,11} + \beta_3\psi_{,12} + \beta_5\psi_{,22}, \tag{3.156}$$

$$\dot{S}_{022} = -\mathcal{A}_{02212}\psi_{,11} + (\mathcal{A}_{01122} - \mathcal{A}_{02222} - p)\psi_{,12} + \mathcal{A}_{02221}\psi_{,22} - \dot{p}. \tag{3.157}$$

As a result of taking the derivative of $\dot{S}_{022}$ in Eq. (3.157) with respect to x_1, the term $\dot{p}_{,1}$ appears which can be eliminated through Eq. (3.145). We therefore get

$$\dot{S}_{022,1} = \rho\psi_{,2tt} - \beta_4\psi_{,111} - (2\beta_2 + \beta_5 - T_{22})\psi_{,112} - 2\beta_3\psi_{,122} - \beta_5\psi_{,222}. \tag{3.158}$$

Here we have used the connection

$$\mathcal{A}_{0ijij} - \mathcal{A}_{0ijji} = T_{ii} + p, \quad i \neq j, \tag{3.159}$$

which follows from Eqs. (3.36), (3.37), (3.138) and (3.143).

Also, we have from Eq. (3.150)

$$\begin{aligned}
\beta_1 &= 2W_1\lambda_1^2 + 2W_2(I_1\lambda_1^2 - \lambda_1^2\lambda_2^2 - \lambda_1^4) + 2W_7\Sigma_{11} + 2W_8(\lambda_2^2\Sigma_{11} + 2\lambda_1^2\Sigma_{11} \\
&+ \lambda_1^2\Sigma_{22}) + 4[W_{77} + 4W_{78}\lambda_1^2 + 4W_{88}\lambda_1^4]\Sigma_{12}^2, \qquad (3.160) \\
2\beta_2 &= 2W_1(\lambda_1^2 + \lambda_2^2) + 2W_2[\lambda_1^2(I_1 - \lambda_1^2) + \lambda_2^2(I_1 - \lambda_2^2) - 2\lambda_1^2\lambda_2^2] \\
&+ 2W_7(\Sigma_{11} + \Sigma_{22}) + 4W_8[\lambda_1^2(3\Sigma_{11} - \Sigma_{22}) + \lambda_2^2(3\Sigma_{22} - \Sigma_{11})] \\
&+ 4W_{11}(\lambda_1^2 - \lambda_2^2)^2 + 4W_{22}[\lambda_1^2(I_1 - \lambda_1^2) - \lambda_2^2(I_1 - \lambda_2^2)]^2 \\
&+ 8W_{12}[\lambda_1^4(I_1 - \lambda_1^2) + \lambda_2^4(I_1 - \lambda_2^2) - \lambda_1^2\lambda_2^2(2I_1 - \lambda_1^2 - \lambda_2^2)] \\
&+ 16W_{18}[\lambda_1^4\Sigma_{11} + \lambda_2^4\Sigma_{22} - \lambda_1^2\lambda_2^2(\Sigma_{11} + \Sigma_{22})] \\
&+ 8W_{27}(\Sigma_{11} - \Sigma_{22})[\lambda_1^2(I_1 - \lambda_1^2) - \lambda_2^2(I_1 - \lambda_2^2)] + 16W_{28}[\lambda_1^4\Sigma_{11}(I_1 - \lambda_1^2) \\
&+ \lambda_2^4\Sigma_{22}(I_1 - \lambda_2^2) - \lambda_1^2\lambda_2^2(\Sigma_{22}(I_1 - \lambda_1^2) + \Sigma_{11}(I_1 - \lambda_1^2)] \\
&+ 4W_{77}[(\Sigma_{11} - \Sigma_{22})^2 - 2\Sigma_{12}^2] + 16W_{78}[(\Sigma_{11} - \Sigma_{22})^2 - 2\lambda_1^2\Sigma_{12}^2] \\
&+ 16W_{88}[(\Sigma_{11} - \Sigma_{22})^2 - 2\lambda_1^2\lambda_2^2\Sigma_{12}^2], \qquad (3.161) \\
\beta_3 &= 2[W_7 + W_8(3\lambda_1^2 - \lambda_2^2) + 2W_{17}(\lambda_1^2 - \lambda_2^2) + 2W_{18}(\lambda_1^2 + \lambda_2^2)(\lambda_1^2 - \lambda_2^2) \\
&+ 2W_{27}[\lambda_1^2(I_1 - \lambda_1^2) - \lambda_2^2(I_1 - \lambda_2^2)] + 2W_{28}[\lambda_1^2(I_1 - \lambda_1^2) \\
&- \lambda_2^2(I_1 - \lambda_2^2)](\lambda_1^2 + \lambda_2^2)]\Sigma_{12} + 4[W_{77}(\Sigma_{11} - \Sigma_{22}) + 4W_{78}[(3\lambda_1^2 + \lambda_2^2)\Sigma_{11} \\
&- (3\lambda_2^2 + \lambda_1^2)\Sigma_{22}] + 2W_{88}[\lambda_1^2\Sigma_{11} - \lambda_2^2\Sigma_{22}](\lambda_1^2 + \lambda_2^2)]\Sigma_{12}, \qquad (3.162) \\
\beta_4 &= 2[W_7 + W_8(3\lambda_2^2 - \lambda_1^2) + 2W_{17}(\lambda_2^2 - \lambda_1^2) + 2W_{18}(\lambda_2^2 + \lambda_1^2)(\lambda_2^2 - \lambda_1^2) \\
&+ 2W_{27}[\lambda_2^2(I_1 - \lambda_2^2) - \lambda_1^2(I_1 - \lambda_1^2)] + 2W_{28}[\lambda_2^2(I_1 - \lambda_2^2) \\
&- \lambda_1^2(I_1 - \lambda_1^2)](\lambda_1^2 + \lambda_2^2)]\Sigma_{12} + 4[W_{77}(\Sigma_{22} - \Sigma_{11}) + 4W_{78}[(3\lambda_2^2 + \lambda_1^2)\Sigma_{22} \\
&- (3\lambda_1^2 + \lambda_2^2)\Sigma_{11}] + 2W_{88}[\lambda_2^2\Sigma_{22} - \lambda_1^2\Sigma_{11}](\lambda_1^2 + \lambda_2^2)]\Sigma_{12}, \qquad (3.163) \\
\beta_5 &= 2W_1\lambda_2^2 + 2W_2(I_1\lambda_2^2 - \lambda_1^2\lambda_2^2 - \lambda_2^4) + 2W_7\Sigma_{22} + 2W_8(\lambda_1^2\Sigma_{22} + 2\lambda_2\Sigma_{22} \\
&+ \lambda_2^2\Sigma_{11}) + 4[W_{77} + 4W_{78}\lambda_2^2 + 4W_{88}\lambda_2^4]\Sigma_{12}^2. \qquad (3.164)
\end{aligned}$$

3.4.2 Analysis of Homogeneous Plane Waves in an Initially Stressed Incompressible Half Space in the Deformed Configuration

We now apply the foregoing theory to study wave propagation in an initially stressed incompressible homogenous half space ($x_2 < 0$) in the deformed configuration for the special model given by Eq. (3.111). For this specific strain energy function, we have $\alpha_1 = \mu, \alpha_3 =$

$0, \alpha_5 = 4\bar{\mu}$ and $\alpha_6 = 0$. We therefore have from Eqs. (3.160)–(3.164)

$$\begin{aligned} \beta_1 &= \mu\lambda_1^2 + [2\bar{\mu}(I_7 - I_4) + 1]\Sigma_{11} + 4\bar{\mu}\Sigma_{12}^2, && (3.165) \\ 2\beta_2 &= \mu(\lambda_1^2 + \lambda_2^2) + [2\bar{\mu}(I_7 - I_4) + 1](\Sigma_{11} + \Sigma_{22}) + 4\bar{\mu}[(\Sigma_{11} - \Sigma_{22})^2 - 2\Sigma_{12}^2], && (3.166) \\ \beta_3 &= [2\bar{\mu}(I_7 - I_4) + 1]\Sigma_{12} + 4\bar{\mu}[(\Sigma_{11} - \Sigma_{22})\Sigma_{12}, && (3.167) \\ \beta_4 &= [2\bar{\mu}(I_7 - I_4) + 1]\Sigma_{12} - 4\bar{\mu}[(\Sigma_{11} - \Sigma_{22})\Sigma_{12}, && (3.168) \\ \beta_5 &= \mu\lambda_2^2 + [2\bar{\mu}(I_7 - I_4) + 1]\Sigma_{22} + 4\bar{\mu}\Sigma_{12}^2. && (3.169) \end{aligned}$$

Noting that since $\mathbf{\Sigma}$ is symmetric, we can choose axes that correspond to the principal axes of $\mathbf{\Sigma}$, and therefore, $\Sigma_{12} = 0$ and Σ_{11} and Σ_{22} are the principal values. We therefore have from Eqs. (3.165–(3.169)

$$\begin{aligned} \beta_1 &= \mu\lambda_1^2 + [2\bar{\mu}(I_7 - I_4) + 1]\Sigma_{11}, && (3.170) \\ 2\beta_2 &= \mu(\lambda_1^2 + \lambda_2^2) + [2\bar{\mu}(I_7 - I_4) + 1](\Sigma_{11} + \Sigma_{22}) + 4\bar{\mu}(\Sigma_{11} - \Sigma_{22})^2, && (3.171) \\ \beta_3 &= 0, && (3.172) \\ \beta_4 &= 0, && (3.173) \\ \beta_5 &= \mu\lambda_2^2 + [2\bar{\mu}(I_7 - I_4) + 1]\Sigma_{22}. && (3.174) \end{aligned}$$

We consider an isochoric (homogenous plane strain) deformation such that $\lambda_3 = 1$. Here, we have the incompressibility condition (2.125) in the form

$$\lambda_1\lambda_2\lambda_3 = 1, \qquad (3.175)$$

and we define $\lambda_1 = \lambda$ and $\lambda_2 = \lambda^{-1}$.

In this case, using Eqs. (3.154) and (3.158), the non-zero traction components are given by

$$\begin{aligned} \dot{S}_{021} &= -(\beta_5 - T_{22})\psi_{,11} + \beta_5\psi_{,22}, && (3.176) \\ \dot{S}_{022,1} &= \rho\psi_{,2tt} - (2\beta_2 + \beta_5 - T_{22})\psi_{,112} - \beta_5\psi_{,222}. && (3.177) \end{aligned}$$

The strong ellipticity condition (3.152) in this case, is given by

$$\beta_5 t^4 + 2\beta_2 t^2 + \beta_1 > 0. \qquad (3.178)$$

For the above inequality to hold $\forall t$, the necessary and sufficient conditions are

$$\beta_1 > 0, \quad \beta_5 > 0, \quad \beta_2 > -\sqrt{\beta_1 \beta_5}. \tag{3.179}$$

Using Eqs. (3.170)–(3.174), Eq. (3.149) reduces to

$$\beta_1 \psi_{,1111} + 2\beta_2 \psi_{,2112} + \beta_5 \psi_{,2222} = \rho(\psi_{,11tt} + \psi_{,22tt}). \tag{3.180}$$

The stress-free incremental boundary conditions on $x_2 = 0$ follow from Eqs. (3.176) and (3.177) as

$$\psi_{,22} - \psi_{,11} = 0, \tag{3.181}$$

$$\rho\psi_{,2tt} - (2\beta_2 + \beta_5)\psi_{,112} - \beta_5 \psi_{,222} = 0, \tag{3.182}$$

respectively.

We now focus on plane waves in an initially stressed incompressible material in the reference configuration. Let us assume ψ is of the form

$$\psi = f[k(n_1 x_1 + n_2 x_2 - ct)], \tag{3.183}$$

where c is the wave speed, k is the wave number, t is time and f is a four times continuously differentiable function. Since we have already set $n_1 = \cos\theta$ and $n_2 = \sin\theta$, substitution of the above expression in Eq. (3.180) leads to

$$(\beta_1 + \beta_5 - 2\beta_2)\cos^4\theta + 2(\beta_2 - \beta_5)\cos^2\theta + \beta_5 = \rho c^2. \tag{3.184}$$

This determines the wave speed for any given direction of propagation in the (x_1, x_2) plane and it is easily shown that $\rho c^2 > 0$ follows from the strong ellipticity conditions (3.179). Alternatively, Eq. (3.184) determines possible directions in which waves may propagate for given wave speed, material properties and the principal initial stresses. In special cases, it is possible for Eq. (3.184) to yield two pairs of distinct directions of propagation.

In the classical theory of incompressible isotropic elasticity we have $\beta_1 = \beta_2 = \beta_5 = \mu$, where μ is the shear modulus given by Eq. (3.46). When $\boldsymbol{\Sigma}$ vanishes, the material is isotropic and Eq. (3.184) thus reduces to $\rho c^2 = \mu$ independently of the direction of propagation. This gives the speed of a classical shear wave.

We now consider two cases corresponding to different values of β_1, β_2 and β_5.

***Case* A:** $\beta_1 + \beta_5 = 2\beta_2$

This includes the special case of $\beta_1 = \beta_2 = \beta_5 = \mu$ when evaluated in the deformed configuration which entails $\lambda_1 = \lambda_2$ and $\Sigma_{11} = \Sigma_{22}$. This gives $\rho c^2 = \mu$ *independently* of the direction of propagation in the (x_1, x_2) plane. This also incorporates the classical theory with $\lambda_1 = \lambda_2 = 1$ and the stress assumed to be zero.

With $\beta_1 + \beta_5 = 2\beta_2$, Eq. (3.184) can be re-written as

$$\rho c^2 = \beta_1 \cos^2\theta + \beta_5 \sin^2\theta. \tag{3.185}$$

For either Eq. (3.184) or Eq. (3.185) a shear wave can propagate along the principal axis x_1, with $\rho c^2 = \beta_1$ or along the principal axis x_2, with $\rho c^2 = \beta_5$.

If $\beta_1 \neq \beta_5$, Eq. (3.185) gives

$$\cos^2\theta = \frac{\beta_5 - \rho c^2}{\beta_5 - \beta_1}. \tag{3.186}$$

For this to yield real values of $\cos\theta$ we must have

$$\text{either} \quad \beta_5 \leq \rho c^2 \leq \beta_1, \quad \text{or} \quad \beta_1 \leq \rho c^2 \leq \beta_5. \tag{3.187}$$

We can re-write Eq. (3.186) in the alternative form

$$\tan^2\theta = \frac{\rho c^2 - \beta_1}{\beta_5 - \rho c^2}. \tag{3.188}$$

For a given wave speed subject to conditions (3.187), Eq. (3.188) yields two (in general distinct) directions, symmetric with respect to the axes.

***Case* B:** $\beta_1 + \beta_5 \neq 2\beta_2$

For a given wave speed, the solutions of Eq. (3.184) may be written

$$\cos^2\theta = \frac{\beta_5 - \beta_2 \pm [\beta_2^2 - \beta_1\beta_5 + \rho c^2(\beta_1 + \beta_5 - 2\beta_2)]^{\frac{1}{2}}}{\beta_1 + \beta_5 - 2\beta_2}. \tag{3.189}$$

Considering $0 \le \cos^2\theta \le 1$, for real solutions from Eq. (3.189), we must have either

$$\beta_1 + \beta_5 - 2\beta_2 > 0, \quad \frac{\beta_1\beta_5 - \beta_2^2}{\beta_1 + \beta_5 - 2\beta_2} \le \rho c^2 \le \mathrm{Min}\{\beta_1, \beta_5\}, \tag{3.190}$$

or

$$\beta_1 + \beta_5 - 2\beta_2 < 0, \quad \mathrm{Max}\{\beta_1, \beta_5\} \le \rho c^2 \le \frac{\beta_1\beta_5 - \beta_2^2}{\beta_1 + \beta_5 - 2\beta_2}. \tag{3.191}$$

Equal roots arise when

$$\rho c^2 = \frac{\beta_1\beta_5 - \beta_2^2}{\beta_1 + \beta_5 - 2\beta_2}, \quad \cos^2\theta = \frac{\beta_1 - \beta_5}{\beta_1\beta_5 - \beta_2^2}. \tag{3.192}$$

We may write the directions of propagation in the form

$$\tan^2\theta = \frac{\rho c^2 - \beta_2 \pm [\beta_2^2 - \beta_1\beta_5 + \rho c^2(\beta_1 + \beta_5 - 2\beta_2)]^{\frac{1}{2}}}{\beta_5 - \rho c^2}. \tag{3.193}$$

Thus, for any given wave speed within the allowed range there are in general four possible distinct directions in which a plane shear wave may propagate. In a special case, these degenerate to two when conditions (3.192) hold.

For the special value $\rho c^2 = \beta_5$, the wave propagates *either* along the x_2 axis (as in case A) *or* in the direction given by

$$\cos^2\theta = \frac{2(\beta_5 - \beta_2)}{\beta_1 + \beta_5 - 2\beta_2}, \tag{3.194}$$

in which case either $\beta_1 < \beta_5 < \beta_2$ or $\beta_1 > \beta_5 > \beta_2$ must hold. Similarly, the special value $\rho c^2 = \beta_1$ means that the wave is *either* propagating along the x_1 direction or in the direction given by

$$\cos^2\theta = \frac{\beta_5 - \beta_1}{\beta_1 + \beta_5 - 2\beta_2}, \tag{3.195}$$

which requires that either $\beta_2 < \beta_1 < \beta_5$ or $\beta_2 > \beta_1 > \beta_5$ hold.

3.4.3 Wave Reflection from a Plane Boundary in the Deformed Configuration

We consider a plane wave of the form given by Eq. (3.183) incident on the boundary $x_2 = 0$ of the half space $x_2 < 0$. The boundary $x_2 = 0$ is taken to be free of incremental traction but subject to the normal initial stress in the x_1 direction. The incremental traction-free boundary conditions on $x_2 = 0$ are given by Eqs. (3.181) and (3.182).

Let the direction of propagation of this wave be $\mathbf{n} = (n_1, n_2) = (\cos\theta, \sin\theta)$ and c be its speed. As a result of this incidence, depending on the material properties and the state of deformation, one or two reflected waves and/or a surface wave are generated. We can write the general solution for ψ consisting of the incident and two reflected waves in the form

$$\psi = f[k(n_1x_1 + n_2x_2 - ct)] + Rf[k(n_1x_1 - n_2x_2 - ct)] + R^{'}f[k^{'}(n_1^{'}x_1 - n_2^{'}x_2 - c^{'}t)], \quad (3.196)$$

where R and $R^{'}$ are the reflection coefficients. Also, $k^{'}$ and $c^{'}$ are the wave number and wave speed of the second reflected wave. The first reflected wave has the same speed as the incident wave and is reflected at an angle θ to the boundary, while the angle of reflection of the second wave is $\theta^{'}$. We choose $n_1^{'} = \cos\theta^{'}, n_2^{'} = \sin\theta^{'}$. For the compatibility of the three waves, they should have the same frequency. For this we must set

$$kc = k^{'}c^{'}. \quad (3.197)$$

Using Eq. (3.196) in Eqs. (3.181) and (3.182), we find $kn_1 = k^{'}n_1^{'}$ and hence

$$c^{'}n_1 = cn_1^{'}, \quad (3.198)$$

which is a statement of Snell's law.

***Case* A:** $\beta_1 + \beta_5 = 2\beta_2$

In this case, from Eqs. (3.170)–(3.174), we require $\Sigma_{11} = \Sigma_{22} = 0$ which also implies that $\tau_{11} = \tau_{22} = 0$ and hence the case refers to vanishing of the initial stress. The wave speed in this case is given by

$$\rho c^2 = \beta_1 n_1^2 + \beta_5 n_2^2. \quad (3.199)$$

Writing Eq. (3.199) for the second reflected wave, we have

$$\rho c^{'2} = \beta_1 n_1^{'2} + \beta_5 n_2^{'2}. \tag{3.200}$$

Using Eq. (3.198) to eliminate c and $c^{'}$ between Eqs. (3.199) and (3.200), we have $n_1^{'2} = n_1^2$ for $\beta_1 + \beta_5 = 2\beta_2$. Thus, the two reflected waves coincide and and there is only one distinct reflected wave. Thus, without the loss of generality, we take $R^{'} = 0$. This behaviour is the same as is found for plane waves in homogeneous isotropic solid in the classical linearized theory [2].

Using Eqs. (3.196) and (3.199) in the boundary conditions given by Eqs. (3.181) and (3.182) we have

$$(1 + R)[1 - 2n_1^2] = 0, \tag{3.201}$$

$$\beta_5(1 - R)n_2 n_1^2 = 0, \tag{3.202}$$

respectively. Considering the possibility $R \neq \pm 1$, Eqs. (3.201) and (3.202) lead to

$$n_1^2 = \frac{1}{2} \quad \text{and} \quad n_2 n_1^2 = 0, \tag{3.203}$$

which are impossible to occur together. Therefore, R must take value either 1 or -1.

(i) $R = 1$. This case is possible only when a wave is incident at an angle of $\theta = \pi/4$ which results in a unique reflected wave at the same angle. The wave speed in this case is given by

$$\rho c^2 = (\beta_1 + \beta_5)/2. \tag{3.204}$$

In this case, the non-zero displacement component on the boundary $x_2 = 0$ is

$$u_2 = -\sqrt{2} k f[k(\frac{x_1}{\sqrt{2}} - ct)], \tag{3.205}$$

which means there is no displacement along the boundary in the x_1 direction. Since $\beta_1 = \beta_5 = \mu\lambda^2$ in this case, we recover the classical results for the speed of shear wave in a deformed isotropic material i. e., $\rho c^2/\mu = \lambda^2$.

(ii) $R = -1$. In this case, we have *either* the normal incidence $(n_1 = 0)$ *or* grazing incidence $(n_2 = 0)$. The grazing incidence in this case are not possible as for $n_2 = 0$, we have

$\psi \equiv 0$. The normal incidence results in a wave travelling along the vertical x_2 axis. From Eq. (3.199) when $n_1 = 0$, we have $\rho c^2/\mu = \lambda^2$.

***Case* B:** $\beta_5 + \beta_1 \neq 2\beta_2$

In the case of an incident wave Eq. (3.183) can be rewritten as

$$(\beta_1 + \beta_5 - 2\beta_2)n_1^4 + 2(\beta_2 - \beta_5)n_1^2 + \beta_5 = \rho c^2, \tag{3.206}$$

and when there is a reflected wave, we also have

$$(\beta_1 + \beta_5 - 2\beta_2){n_1^{'}}^4 + 2(\beta_2 - \beta_5){n_1^{'}}^2 + \beta_5 = \rho {c^{'}}^2, \tag{3.207}$$

together with the Snell's law given by Eq. (3.198).

From Eqs. (3.206) and (3.207), we find that either ${n_1^{'}}^2 = n_1^2$ or

$$n_1^2 {n_1^{'}}^2 = \beta_5/(\beta_1 + \beta_5 - 2\beta_2), \tag{3.208}$$

and, these two possibilities occur together when

$$n_1^4 = \beta_5/(\beta_1 + \beta_5 - 2\beta_2), \tag{3.209}$$

which defines the *transitional angle*, say θ_0. Thus, when Eq. (3.196) is applicable, a given incident wave generates two reflected waves in general. One of these waves is reflected at the same angle as the incident wave and the angle of reflection of the second wave is given by Eq. (3.208) with $n_1^{'} = \cos\theta^{'}$.

For a given angle of incidence θ, $\theta^{'}$ is calculated from Eq. (3.208), ρc^2 from Eq. (3.206) and $\rho {c^{'}}^2$ from Eq. (3.207). The reflection coefficients R and $R^{'}$ are calculated using the boundary conditions.

For a given θ, necessary and sufficient conditions for Eq. (3.208) to yield a real angle $\theta^{'}$ are

$$\beta_1 > 2\beta_2, \tag{3.210}$$

$$\tan^2\theta \leq \frac{(\beta_1 - 2\beta_2)}{\beta_5} \equiv \tan^2\theta_c, \tag{3.211}$$

where θ_c is defined as the critical angle given by the right-hand identity in Eq. (3.211).

Substitution of Eq. (3.196) in boundary conditions (3.181) and (3.182) after using the propagation condition (3.206) and Eq. (3.208), we get after some calculations

$$\beta_5(1+R+R^{'}) - (\beta_1+\beta_5-2\beta_2)[(1+R)n_1^{'2}n_2^2 + R^{'}n_1^2 n_2^{'2}] = 0, \quad (3.212)$$

$$2\beta_5[(1-R)n_1^{'}n_2 + R^{'}n_1 n_2^{'}] - (\beta_1+\beta_5-2\beta_2)[(1-R)n_1 n_2 + R^{'}n_1^{'}n_2^{'}]n_1 n_1^{'} = 0. \quad (3.213)$$

Explicit expressions for R and $R^{'}$ are given by

$$R = \frac{(1-2n_1^{'2})^2 n_1^3 n_2 - (1-2n_1^2)^2 n_1^{'3} n_2^{'}}{(1-2n_1^{'2})^2 n_1^3 n_2 + (1-2n_1^2)^2 n_1^{'3} n_2^{'}}, \quad (3.214)$$

$$R^{'} = \frac{-2(1-2n_1^2)(1-2n_1^{'2})n_1^{'2} n_1 n_2}{(1-2n_1^{'2})^2 n_1^3 n_2 + (1-2n_1^2)^2 n_1^{'3} n_2^{'}}. \quad (3.215)$$

From Eq. (3.215) it is obvious that $R^{'}$ vanishes for $n_1 = 0$ which means the angle of incidence $\theta = \pi/2$ (normal incidence).

We now consider three non-trivial cases where $R^{'}$ can possibly vanish.

(i) $R^{'} = 0, R \neq \pm 1$. In this case Eqs. (3.212) and (3.213) yield *either* $n_2 = 0$, i.e. grazing incidence, which is not possible since then $\psi = 0$, *or*

$$n_1^2 = \frac{1}{2} = \frac{2\beta_5}{\beta_1+\beta_5-2\beta_2} = n_1^{'2}. \quad (3.216)$$

For Eq. (3.216) to yield a real angle, inequality (3.210) along with the stability conditions (3.179) must hold. The wave speed in this case is given by

$$\rho c^2 = \beta_2 + \beta_5. \quad (3.217)$$

(ii) $R^{'} = 0, R = 1$. In this case, we have

$$n_1^2 = \frac{1}{2}. \quad (3.218)$$

This means an incident wave at an angle $\theta = \pi/4$ results in a unique wave reflected at the same angle. The wave speed in this case is given by

$$\rho c^2 = (\beta_1 + \beta_5 + 2\beta_2)/4. \quad (3.219)$$

In this case, the non-zero displacement component on the boundary $x_2 = 0$ is

$$u_2 = -\sqrt{2}kf[k(\frac{x_1}{\sqrt{2}} - ct)], \tag{3.220}$$

which means that there is no displacement along the boundary in the x_1 direction.

(iii) $R^{'} = 0, R = -1$. In this case, we have

$$n_1^2 = \frac{2\beta_5}{\beta_1 + \beta_5 - 2\beta_2} \quad \text{and} \quad n_1^{'2} = \frac{1}{2}. \tag{3.221}$$

For Eq. $(3.221)_1$ to yield a real angle, inequalities (3.210) and (3.179) must hold. The wave speed in this case is given by

$$\rho c^2 = \frac{\beta_5(\beta_1 + \beta_5 + 2\beta_2)}{\beta_1 + \beta_5 - 2\beta_2}. \tag{3.222}$$

In this case, the non-zero displacement component on the boundary $x_2 = 0$ is

$$u_1 = 2k(\frac{2\beta_5}{\beta_1 + \beta_5 - 2\beta_2})^{1/2} f[k(\frac{2\beta_5}{\beta_1 + \beta_5 - 2\beta_2})^{1/2} x_1 - ct)], \tag{3.223}$$

which means that there is no displacement normal to the boundary in the x_2 direction.

When Eqs. (3.210) and (3.211) are not satisfied a pair of reflected waves is not possible and an alternate expression for ψ should be used. Therefore, in Eq. (3.196) we have $n_1^{'} = 1$ and $n_2^{'} = -is$ where $s > 0$ so that the latter term in Eq. (3.196) decays as $x_2 \to \infty$. In this case, Eqs. (3.197) and (3.198) change such that

$$kn_1 = k^{'}, \quad c^{'}n_1 = c, \tag{3.224}$$

where $c^{'}$ now represents the speed of the surface wave whereas c is the speed of the incident wave and can be calculated using Eq. (3.206). The reflection coefficients in this case are given by explicit expressions for R and $R^{'}$ are given by

$$R = \frac{(1+s^2)^2 n_1^3 n_2 + is(1-2n_1^2)^2}{(1+s^2)^2 n_1^3 n_2 - is(1-2n_1^2)^2}, \tag{3.225}$$

$$R^{'} = \frac{-2(1+s^2)(2n_1^2 - 1)n_1 n_2}{(1+s^2)^2 n_1^3 n_2 - is(1-2n_1^2)^2}. \tag{3.226}$$

Using Eq. (3.180), we have the propagation conditions for the incident wave and the surface wave,

$$\beta_1 n_1^4 + 2\beta_2 n_1^2 n_2^2 + \beta_5 n_2^4 = \rho c^2, \tag{3.227}$$

$$\beta_1 + 2\beta_2 s^2 + \beta_5 s^4 = \rho c'^2 (1 - s^2), \tag{3.228}$$

respectively. Using Eq. $(3.224)_2$ and Eq. (3.227) to eliminate $\rho c'^2$ from Eq. (3.228) we have the counterpart of Eq. (3.208)

$$\beta_5 s^2 = \beta_5 - (\beta_1 + \beta_5 - 2\beta_2) n_1^2, \tag{3.229}$$

where $s > 0$. The right hand side of Eq. (3.229) is always positive for if $\beta_1 - 2\beta_2 \leq 0$. In the case when $\beta_1 - 2\beta_2 > 0$, the angle of incidence should be such that

$$\tan^2 \theta > (\beta_1 - 2\beta_2)/\beta_5. \tag{3.230}$$

It may be noted that in contrast to the upper bound (see [9]) or lower bound in certain circumstances (see [38]) on the surface wave speed in case of pre-stressed incompressible materials, there is not restriction observed here when the surface wave is generated by an incident wave. Numerical results illustrating the behaviour of the incident and two reflected waves or one reflected wave accompanied by a surface wave for angles of incidence greater than the critical angle are presented in the following section for a special model.

3.4.4 Analysis of Wave Reflection for a Special Model in the Deformed Configuration

We now apply the foregoing theory to the specific model given by Eq. (3.111) for an initially stressed homogeneous incompressible half space subject to homogeneous plane strain deformation.

We consider the boundary $x_2 = 0$ of the half space to be stress free. Therefore, $T_{22} = 0 = \Sigma_{22}$. After a few calculations using the definitions of I_4 and I_7 from Eq. (3.1), Eqs. (3.170)–(3.174) reduce to

$$\beta_1 = (\mu + \tau_{11})\lambda^2 + 2\bar{\mu}\lambda^2(\lambda^2 - 1)\tau_{11}^2, \tag{3.231}$$

$$2\beta_2 = (\mu + \tau_{11})\lambda^2 + \mu\lambda^{-2} + 2\bar{\mu}\lambda^2(3\lambda^2 - 1)\tau_{11}^2, \tag{3.232}$$

$$\beta_5 = \mu\lambda^{-2}. \tag{3.233}$$

The non-zero principal Cauchy stresses in this case are given by

$$T_{11} = [\mu + [2\bar{\mu}(I_7 - I_4) + 1]\Sigma_{11}]\lambda^2 - p, \tag{3.234}$$

$$T_{22} = \mu\lambda^{-2} - p. \tag{3.235}$$

Therefore, the stress-free boundary conditions lead to

$$p = \beta_5 = \mu\lambda^{-2}, \tag{3.236}$$

which holds everywhere in $x_2 < 0$ as the underlying state of the material is considered to be uniform.

Recasting Eqs. (3.231)–(3.233) in the dimensionless form, we have

$$\beta_1/\mu = \lambda^2\epsilon, \quad 2\beta_2/\mu = \lambda^2\epsilon + \lambda^{-2} + 4b_0\lambda^4\bar{\tau}^2, \quad \beta_5/\mu = \lambda^{-2}, \tag{3.237}$$

where

$$b_0 = \mu\bar{\mu}, \quad \bar{\tau} = \tau_{11}/\mu, \quad \epsilon = 2b_0(\lambda^2 - 1)\bar{\tau}^2 + \bar{\tau} + 1. \tag{3.238}$$

From the above expressions, we have

$$(\beta_1 + \beta_5 - 2\beta_2)/\mu = -4b_0\lambda^4\bar{\tau}^2, \quad (\beta_1 - 2\beta_2)/\mu = -\lambda^{-2} - 4b_0\lambda^4\bar{\tau}^2, \tag{3.239}$$

$$2(\beta_2 - \beta_5)/\mu = \lambda^2\epsilon - \lambda^{-2} + 4b_0\lambda^4\bar{\tau}^2. \tag{3.240}$$

We therefore have from Eqs. (3.210) and (3.211)

$$-4b_0\bar{\tau}^2\lambda^6 > 1, \tag{3.241}$$

and

$$\sec^2\theta \leq -4b_0\bar{\tau}^2\lambda^6 \equiv \sec^2\theta_c. \tag{3.242}$$

Since $0 < \sec^2\theta$, we require $\bar{\mu} < 0$ and hence $b_0 < 0$.

Provided that Eqs. (3.241) and (3.242) hold, we have from Eq. (3.208)

$$n_1 n_1' = 1/\sqrt{\bar{A}_1}, \tag{3.243}$$

where we have defined

$$\bar{A}_1 = -4b_0\bar{\tau}^2\lambda^6. \tag{3.244}$$

From Eqs. (3.306) and (3.307), we have

$$R = \frac{(\bar{A}_1 - 2\sec^2\theta)^2\cos^2\theta\sin\theta - (\sec^2\theta - 2)(\bar{A}_1 - \sec^2\theta)^{1/2}}{(\bar{A}_1 - 2\sec^2\theta)^2\cos^2\theta\sin\theta + (\sec^2\theta - 2)(\bar{A}_1 - \sec^2\theta)^{1/2}}, \tag{3.245}$$

$$R^{'} = \frac{-2(\sec^2\theta - 2)(\bar{A}_1 - 2\sec^2\theta)\sin\theta}{(\bar{A}_1 - 2\sec^2\theta)^2\cos^2\theta\sin\theta + (\sec^2\theta - 2)(\bar{A}_1 - \sec^2\theta)^{1/2}}, \tag{3.246}$$

respectively. It might be noted that for angle of incidence equal to the critical angle then $\theta^{'} = 0$ and there is a grazing reflection.

From the above expressions, if $\sec^2\theta > \bar{A}_1$ (i.e. angles of incidence greater than the critical angle), we have

$$R = \frac{(\bar{A}_1 - 2\sec^2\theta)^2\cos^2\theta\sin\theta - i(\sec^2\theta - 2)(\sec^2\theta - \bar{A}_1)^{1/2}}{(\bar{A}_1 - 2\sec^2\theta)^2\cos^2\theta\sin\theta + i(\sec^2\theta - 2)(\sec^2\theta - \bar{A}_1)^{1/2}}, \tag{3.247}$$

$$R^{'} = \frac{2(\sec^2\theta - 2)(2\sec^2\theta - \bar{A}_1)\sin\theta}{(\bar{A}_1 - 2\sec^2\theta)^2\cos^2\theta\sin\theta + i(\sec^2\theta - 2)(\sec^2\theta - \bar{A}_1)^{1/2}}, \tag{3.248}$$

respectively. The above expressions also follow from Eqs. (3.225) and (3.226), respectively. The value of s^2 in this case follows from Eq. (3.229) and is given by

$$s^2 = \cos^2\theta(\sec^2\theta - \bar{A}_1), \tag{3.249}$$

which has positive right hand side if either $\bar{A}_1 \leq 1$ or if $\bar{A}_1 > 1$ and $\sec^2\theta > \bar{A}_1$.

We can rewrite Eq. (3.242) in the form

$$\cos^2\theta \geq 1/\bar{A}_1. \tag{3.250}$$

This means, for real $\theta^{'}$, the inequality (3.250) should hold. For the values of θ out of this range, a surface wave exists whose reflection coefficient is given by Eq. (3.248). Figure 3.1 refers to the values of θ and $\bar{A}_1$ where inequality (3.250) holds.

The strong ellipticity conditions (3.179) in this case gives the sufficient conditions as

$$\lambda^2\epsilon > 0, \quad 4b_0\bar{\tau}^2\lambda^6 + \epsilon\lambda^4 + 2\epsilon^{\frac{1}{2}}\lambda^2 + 1 > 0, \tag{3.251}$$

which further imply that $\epsilon > 0$ where ϵ is given by Eq. (3.238). This holds $\forall\bar{\tau}$ when

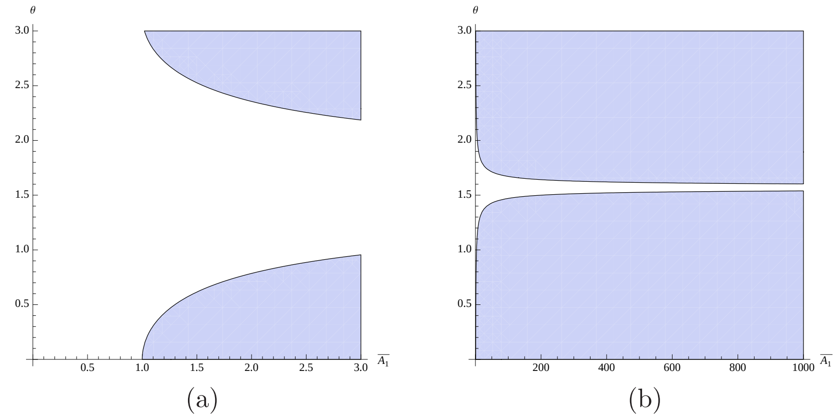


Figure 3.1: Plots of the stability region $(\theta, \bar{A}_1)$ (shaded area) for the reflected wave from inequality (3.250). (a) Stability region for the reflected wave for smaller values of $\bar{A}_1$, (b) Stability region for the reflected wave for very high values of $\bar{A}_1$

$8b_0(\lambda^2 - 1) > 1$. For $8b_0(\lambda^2 - 1) < 1$, we have

$$\frac{-1 + \sqrt{1 - 8b_0(\lambda^2 - 1)}}{4b_0(\lambda^2 - 1)} < \bar{\tau} < \frac{-1 - \sqrt{1 - 8b_0(\lambda^2 - 1)}}{4b_0(\lambda^2 - 1)}. \tag{3.252}$$

Using Eqs. (3.206) and (3.239) the dimensionless wave speed of the incident wave in this case is given by

$$\rho c^2/\beta_5 = (-4b_0\lambda^6\bar{\tau}^2)\cos^4\theta + (4b_0\lambda^6\bar{\tau}^2 + \lambda^4\epsilon - 1)\cos^2\theta + 1, \tag{3.253}$$

as a function of θ.

Similarly, for the reflected wave, the speed in its dimensionless form is given by

$$\rho c'^2/\beta_5 = (-4b_0\lambda^6\bar{\tau}^2)^{-1}[\sec^4\theta + (4b_0\lambda^6\bar{\tau}^2 + \lambda^4\epsilon - 1)\sec^2\theta - 4b_0\lambda^6\bar{\tau}^2], \tag{3.254}$$

as a function of θ. Since, for a reflected wave to exist, θ' in Eq. (3.254) must be real and should fall in the range to satisfy the inequality (3.250). For the angles out of this range, c' is the speed of a surface wave which increases indefinitely (and its amplitude vanishes) as the incident wave approaches normal incidence. The behaviours of $\rho c^2/\beta_5$ (dashed graph) and $\rho c'^2/\beta_5$ are shown in Fig. 3.2 for $0 \leq \theta \leq \theta_c$ where $\theta_c = \sec^{-1}(\sqrt{\bar{A}_1})$. For angles of incidence greater than the critical angle, c' represents the surface wave and is given by $c' = c/n_1$ where c is the speed of the incident wave given by Eq. (3.253). The speed of surface wave for $\theta > \theta_c$ is given by

$$\rho c'^2/\beta_5 = \sec^2\theta + \lambda^4 - 1. \tag{3.255}$$

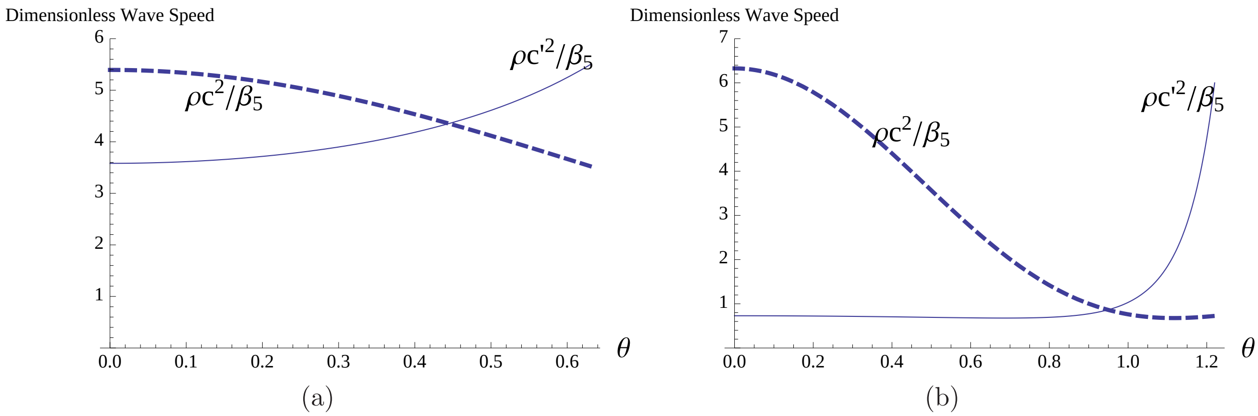


Figure 3.2: Plot of dimensionless wave speeds $\rho c^2/\beta_5$ (dashed) and $\rho c'^2/\beta_5$ for (a) $b_0 = -0.2, \lambda = 1.4, \bar{\tau} = 0.5, \bar{A}_1 = 1.50591$, (b) (a) $b_0 = -0.2, \lambda = 1.4, \bar{\tau} = 1.2, \bar{A}_1 = 8.67403$. The plots for $\rho c'^2/\beta_5$ refer to the dimensionless speed of a surface wave when the inequality (3.250) does not hold.

For the special value of $\rho c^2 = \beta_5$, Eq. (3.253) gives

$$\cos^2\theta = 0, \quad or \quad \cos^2\theta = (4b_0\lambda^6\bar{\tau}^2 + \lambda^4\epsilon - 1)/4b_0\lambda^6\bar{\tau}^2, \tag{3.256}$$

which means either the wave travels along the x_2 axis or in the direction given by Eq. $(3.256)_1$. For $(3.256)_1$ to give real angles, either of the following two should hold

$$\lambda^4\epsilon < 1 < 4b_0\lambda^6\bar{\tau}^2 + \lambda^4\epsilon - 1, \quad \lambda^4\epsilon > 1 > 4b_0\lambda^6\bar{\tau}^2 + \lambda^4\epsilon - 1. \tag{3.257}$$

For $\rho c^2 = \beta_1$, Eq. (3.253) gives

$$\cos^2\theta = 1, \quad or \quad \cos^2\theta = (\lambda^4\epsilon - 1)/4b_0\lambda^6\bar{\tau}^2, \tag{3.258}$$

where for real angles either of the following conditions should hold

$$4b_0\lambda^6\bar{\tau}^2 + \lambda^4\epsilon - 1 < \lambda^4\epsilon < 1, \quad 4b_0\lambda^6\bar{\tau}^2 + \lambda^4\epsilon - 1 > \lambda^4\epsilon > 1. \tag{3.259}$$

When $\bar{\tau} = 0$ in Eq. (3.237), that is the case when the initial stress vanishes, the case is equivalent to $\beta_1 + \beta_5 = 2\beta_2$ (see Section 3.4.3: *Case* A) and there is only one reflected wave at the same angle as the incident wave, with the dimensionless wave speed given by

$$\rho c^2/\beta_5 = (\lambda^4 - 1)\cos^2\theta + 1. \tag{3.260}$$

This result reduces to $\rho c^2/\mu = 1$ when specialized for reference configuration (i.e., $\lambda = 1$).

This is the same as that for classical linear theory. The result (3.253) is also deducible from Eq. (3.186) and from Eq. (3.187), either of the following two inequalities hold:

$$1 \leq \rho c^2/\beta_5 \leq \lambda^4, \quad \lambda^4 \leq \rho c^2/\beta_5 \leq 1. \tag{3.261}$$

From Eq. (3.244), we note that as $\bar{A}_1 \to 0$, the case corresponds to infinitesimally small initial stress. In this case, Eq. (3.237) give the expressions independent of the initial stress and the results are therefore comparable to the classical linearized theory for isotropic materials in the deformed configuration. We refer to Fig. 3.3 which shows the behaviour of $|R'|$ and $|R|(=1)$ when the initial stress has infinitesimally small magnitude. We see that $|R'|$ is bounded and vanishes at various angles of incidence. It is important to note that in the case of infinitely small values of $\bar{A}_1$, the inequality (3.250) does not hold and the plot of $|R^{'}|$ (for example, 3.3a) refers to a surface wave for all angles of incidence θ. The results are plotted using Eqs. (3.214) and (3.215). Similar behaviour is shown in the case of pre-stressed materials in [39] for pure homogeneous strain and in [23] for simple shear.

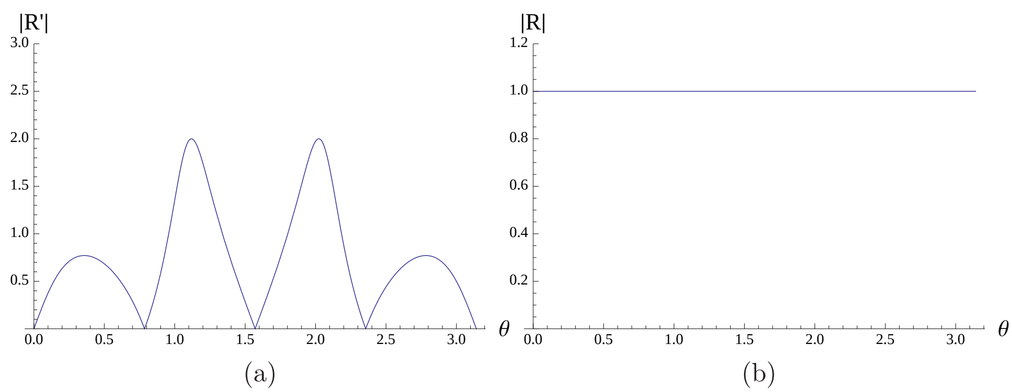
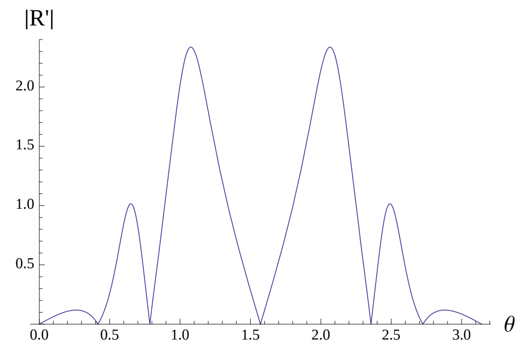
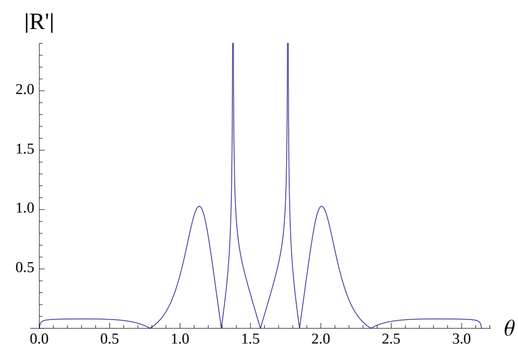


Figure 3.3: $|R^{'}|$ for (a) $\bar{\tau} = 10^{-5}, \lambda = 2, b_0 = -1, \bar{A}_1 = 2.56 \times 10^{-8}$ and $|R|(=1)$ for (b) $\bar{\tau} = 10^{-5}, \lambda = 2, b_0 = -1$, (b) $\bar{\tau} = 10^{-5}, \lambda = 3, b_0 = -4, \bar{A}_1 = 1.1664 \times 10^{-}6$, (Plotted as a function of the angle of incidence θ and $0 \leq \theta \leq \pi$)

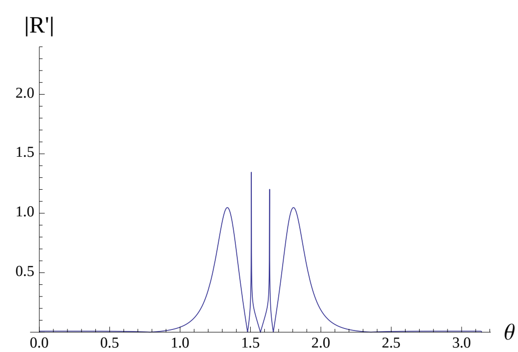
It is obvious that for $\bar{A}_1 \to \infty$ (very large magnitude of initial stress), $|R^{'}|$ tends to vanish and becomes more confined in the band along the normal angle of incidence (see Fig. 3.4d), where as $|R| \to 1$ (see Fig. 3.5d). For intermediate values of $\bar{A}_1$, we refer to Figs. 3.4 (for $|R'|$) and 3.5 (for $|R|$). It may be observed that since the stretches and initial stress occur in a product, a variation in any of the two leads to similar values of $\bar{A}_1$. Also, the plots in Fig. 3.4 refer to the amplitude of a surface wave for the range of values of θ where the inequality (3.250) does not hold. For instance, in Fig. 3.4a where $\bar{A}_1 = 2.391$, the inequality (3.250) does not hold for the range $0.867532 < \theta < 2.27406$ and hence in this range the plot refers to a surface wave.



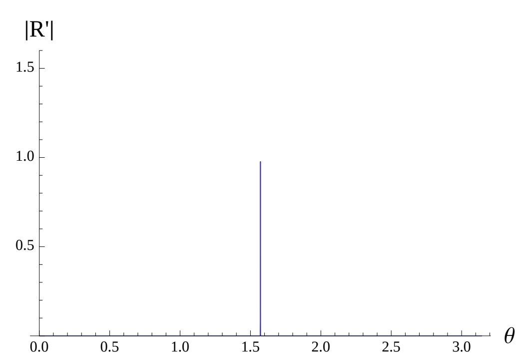
(a)



(b)

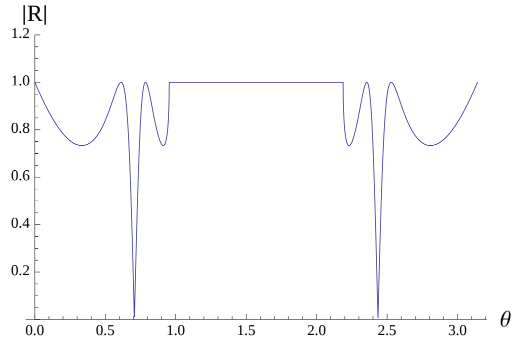


(c)

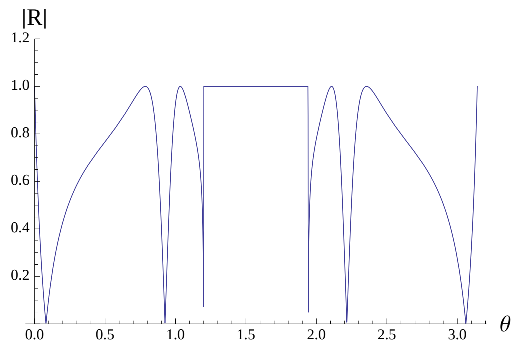


(d)

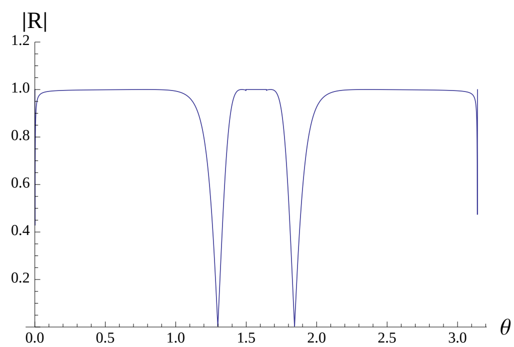
Figure 3.4: $|R'|$ for $\lambda = 0.9, b_0 = -0.5$ and (a) $\bar{\tau} = 1.5, \bar{A}_1 = 2.391$, (b) $\bar{\tau} = 5, \bar{A}_1 = 26.5721$, (c) $\bar{\tau} = 15, \bar{A}_1 = 106.288$, (d) $\bar{\tau} = 5^{10}, \bar{A}_1 = 1.01364 \times 10^{14}$, Plotted as a function of the angle of incidence θ and $0 \leq \theta \leq \pi$. The plots refer to the amplitude of a surface wave when the inequality (3.250) does not hold. For example, the plot refer to the amplitude of a surface wave for $0.867532 < \theta < 2.27406$ in (a) and for $1.37557 < \theta < 1.76603$ in (b).



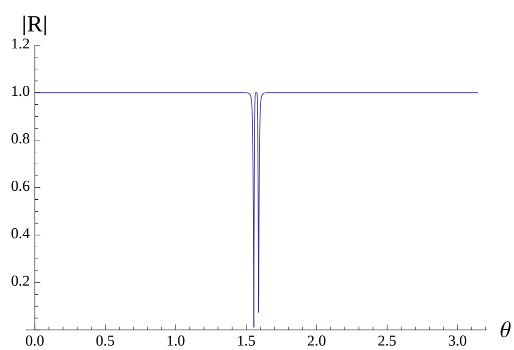
(a)



(b)



(c)



(d)

Figure 3.5: $|R|$ for $\lambda = 1.2, b_0 = -1$ and (a) $\bar{\tau} = 0.5, \bar{A}_1 = 2.98598$ (b) $\bar{\tau} = 0.8, \bar{A}_1 = 7.64412$, (c) $\bar{\tau} = 4, \bar{A}_1 = 191.103$, (d) $\bar{\tau} = 10^3, \bar{A}_1 = 1.19439 \times 10^7$, Plotted as a function of the angle of incidence θ and $0 \leq \theta \leq \pi$)

Another aspect is the symmetry of the amplitudes in Figs. 3.4 and 3.5 about the angle of incidence for all values of stretch. This aspect is also observed in [39] where the authors discuss a special class of materials with pre-stress and it is found that increasing or decreasing magnitude of the pre-stress (initial stress in our case) doesn't hinder the symmetrical behaviour. This is in contrast to the results for simple shear in [23] where the symmetrical behaviour is lost with the increase in stretch.

The plots in Fig. 3.6 and 3.7 show the real and imaginary plots for $R^{'}$ and R respectively. These plots use varying values of λ for fixed b_0 and $\bar{\tau}$ and the respective values of $\bar{A}_1$ are stated. The plots for Real($R^{'}$) show a sharp rise for a particular intermediate range of values of $\bar{A}_1$ (see Fig. 3.6c) and the amplitude drops as the values of λ increase (or in other words for very large values of $\bar{A}_1$). The changing range of the vertical axis may be noted in Figs. 3.4 and 3.6. This is a behaviour expected from Eq. (3.246) as $\bar{A}_1 \to \infty$. Figure 3.7 shows the symmetrical and bounded behaviour of R and it is worth noting that for even very small increase in the values of λ, the imaginary part of the amplitude vanishes and the real part is such that $|R| \leq 1$. This is obvious from Eq. (3.245) when $\bar{A}_1 \to \infty$.

In reference to the discussion carried out in Section 3.4.3 for vanishing of $R^{'}$, we note from Figs. 3.3a, 3.4 and 3.6 that $R^{'}$ vanishes as θ approaches $\pi/2$. Also, from Eqs. (3.218) and (3.219), we expect $R^{'}$ to vanish at either or both $\theta = \pi/4$ and

$$\cos^2\theta = 2/\bar{A}_1 = 2\cos^2\theta_c. \tag{3.262}$$

This fact is recoverable from Eq. (3.246) and is obvious in the above mentioned figures.

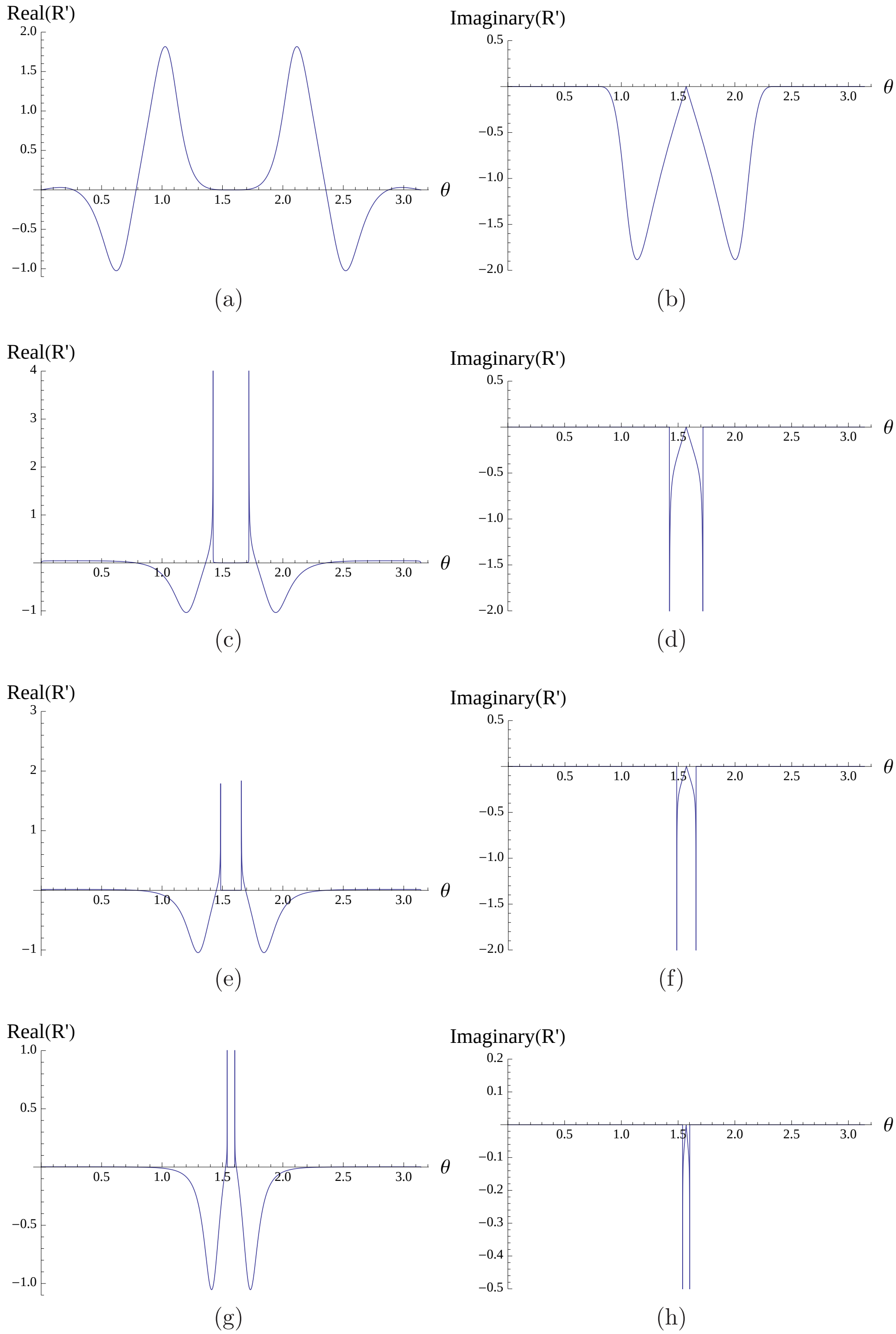


Figure 3.6: Real($R^{'}$) (left column) and Imaginary($R^{'}$) for $\bar{\tau} = 0.3, b_0 = -2$ and (a), (b) $\lambda = 1.2, \bar{A}_1 = 2.1499$, (c), (d) $\lambda = 2, \bar{A}_1 = 46.08$, (e), (f) $\lambda = 2.4, \bar{A}_1 = 137.594$, (g), (h) $\lambda = 3.35, \bar{A}_1 = 1017.66$, plotted as a function of the angle of incidence θ and $0 \le \theta \le \pi$. The plots refer to the reflection coefficient of a surface wave when the inequality (3.250) does not hold. The changing vertical scale may be noted.

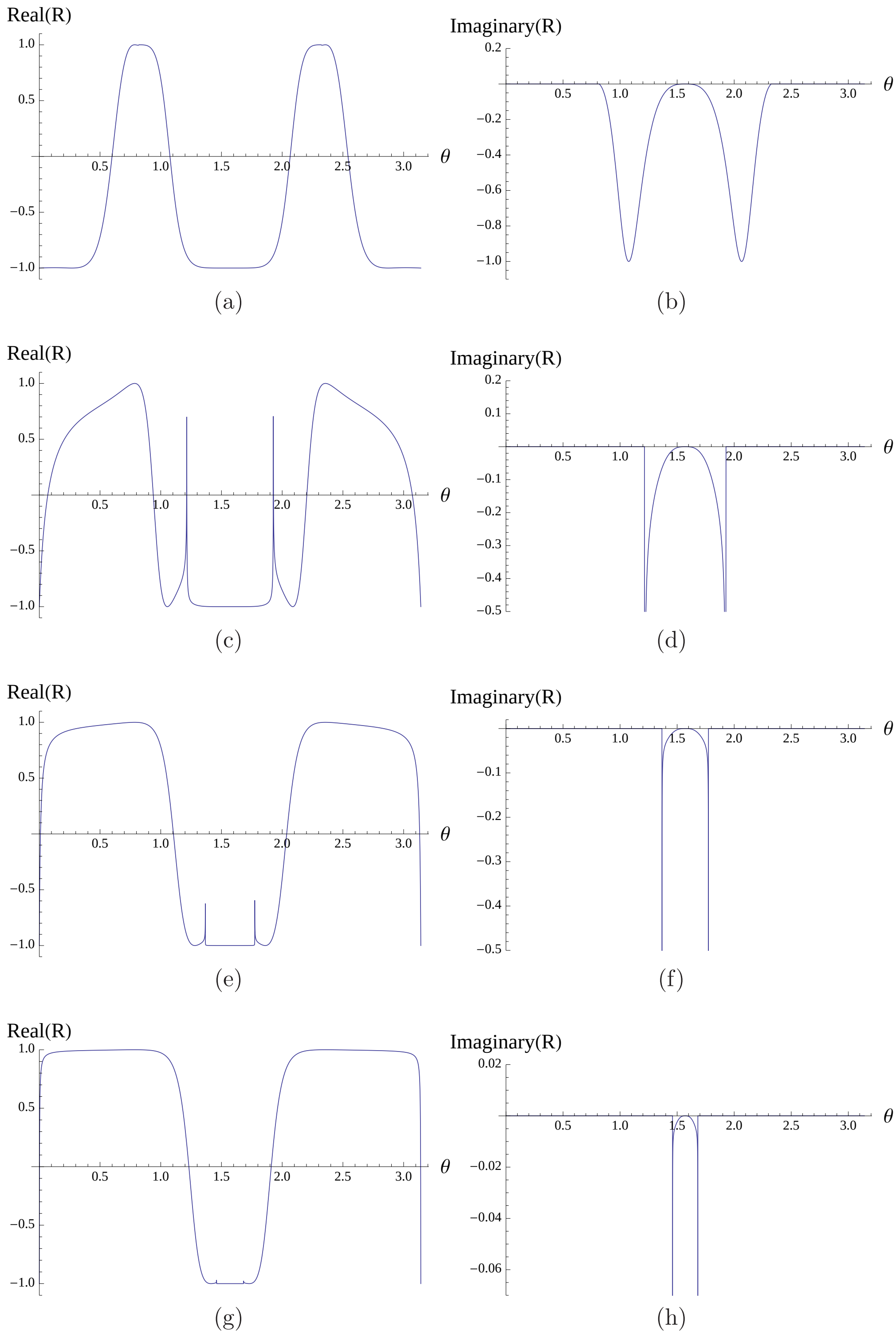


Figure 3.7: Real(R) (left column) and Imaginary(R) for $\bar{\tau} = 0.3, b_0 = -2$ and (a), (b) $\lambda = 1.2, \bar{A}_1 = 2.1499$, (c), (d) $\lambda = 1.5, \bar{A}_1 = 8.20125$, (e), (f) $\lambda = 1.8, \bar{A}_1 = 24.4888$, (g), (h) $\lambda = 2.2, \bar{A}_1 = 81.6335$, plotted as a function of the angle of incidence θ and $0 \leq \theta \leq \pi$.

3.4.5 Analysis of Homogeneous Plane Waves in an Initially Stressed Incompressible Half Space in the Reference Configuration

For an initially stressed incompressible material in the reference configuration, Eqs. (3.160)–(3.164) reduce to

$$\beta_1 = \alpha_1 + \tau_{11} + \alpha_3(\tau_{11} + \tau_{22}) + \alpha_5\tau_{12}^2, \quad (3.263)$$

$$2\beta_2 = 2\alpha_1 + \tau_{11} + \tau_{22} + 2\alpha_3(\tau_{11} + \tau_{22}) + \alpha_5[(\tau_{11} - \tau_{22})^2 - 2\tau_{12}^2)], \quad (3.264)$$

$$\beta_3 = \tau_{12} + \alpha_5\tau_{12}(\tau_{11} - \tau_{22}), \quad (3.265)$$

$$\beta_4 = \tau_{12} - \alpha_5\tau_{12}(\tau_{11} - \tau_{22}), \quad (3.266)$$

$$\beta_5 = \alpha_1 + \tau_{22} + \alpha_3(\tau_{11} + \tau_{22}) + \alpha_5\tau_{12}^2. \quad (3.267)$$

Considering the specific model given by Eq. (3.111) in the reference configuration, the counterparts of Eqs. (3.165)–(3.169) are

$$\beta_1 = \mu + \tau_{11} + 4\bar{\mu}\tau_{12}^2, \quad (3.268)$$

$$2\beta_2 = 2\mu + \tau_{11} + \tau_{22} + 4\bar{\mu}[(\tau_{11} - \tau_{22})^2 - 2\tau_{12}^2], \quad (3.269)$$

$$\beta_3 = 4\bar{\mu}[(\tau_{11} - \tau_{22})\tau_{12}, \quad (3.270)$$

$$\beta_4 = -4\bar{\mu}[(\tau_{11} - \tau_{22})\tau_{12}, \quad (3.271)$$

$$\beta_5 = \mu + \tau_{22} + 4\bar{\mu}\tau_{12}^2. \quad (3.272)$$

Considering $\tau_{12} = 0$, we have from the above equations for an incompressible material in the reference configuration

$$\beta_1 = \alpha = \mu + \tau_{11}, \quad (3.273)$$

$$2\beta_2 = 2\beta = 2\mu + \tau_{11} + \tau_{22} + 4\bar{\mu}(\tau_{11} - \tau_{22})^2, \quad (3.274)$$

$$\beta_5 = \gamma = \mu + \tau_{22}. \quad (3.275)$$

Using the strong ellipticity condition (3.152) for this special case, we have

$$\gamma t^4 + 2\beta t^2 + \alpha > 0. \quad (3.276)$$

For inequality (3.276) to hold generally, the necessary and sufficient conditions are simply

$$\alpha > 0, \quad \gamma > 0, \quad \beta > -\sqrt{\alpha\gamma}. \tag{3.277}$$

The above conditions thus ensure positive real values for ρc^2 through the strong ellipticity condition and Eq. (3.70).

For the special model given by Eq. (3.111), when $T_{22} = 0$ on $x_2 = 0$, we find

$$p_0 = \gamma, \tag{3.278}$$

which holds everywhere as the underlying state of the material is considered to be uniform. From Eq. $(3.13)_1$, we also have $p_0 = \mu$ and from Eq. (3.169), $\tau_{22} = 0$, for consistency.

Using Eqs. (3.273)–(3.275), Eq. (3.149) reduces to

$$\alpha\psi_{,1111} + 2\beta\psi_{,2112} + \gamma\psi_{,2222} = \rho(\psi_{,11tt} + \psi_{,22tt}). \tag{3.279}$$

The stress-free boundary conditions on $x_2 = 0$ in this case follow from Eqs. (3.176) and (3.177)

$$\psi_{,22} - \psi_{,11} = 0, \tag{3.280}$$

$$\rho\psi_{,2tt} - (2\beta + \gamma)\psi_{,112} - \gamma\psi_{,222} = 0. \tag{3.281}$$

We now focus on plane waves in an initially stressed incompressible material in the reference configuration. Let us assume ψ is of the form (3.175) which, when substituted in Eq. (3.279) leads to the propagation condition

$$(\alpha + \gamma - 2\beta)\cos^4\theta + 2(\beta - \gamma)\cos^2\theta + \gamma = \rho c^2. \tag{3.282}$$

This determines the wave speed for any given direction of propagation in the (x_1, x_2) plane and it is easily shown that $\rho c^2 > 0$ follows from the strong ellipticity conditions (3.277). Alternatively, Eq. (3.282) determines possible directions in which waves may propagate for given wave speed, material properties and the principal initial stresses. In special cases, it is possible for Eq. (3.282) to yield two pairs of distinct directions of propagation. It may be noted that Eq. (3.282) shows that a shear wave, polarized in the (x_1, x_2) plane, can propagate in any direction in the same plane provided the strong ellipticity conditions (3.277) hold.

In the classical theory of incompressible isotropic elasticity we have $\alpha = \beta = \gamma = \mu$, where μ is the shear modulus given by Eq. (3.46). Eq. (3.282) thus reduces to $\rho c^2 = \mu$ independently of the direction of propagation. This gives the speed of a classical shear wave.

We now consider two cases corresponding to different values of α, β and γ.

***Case* A:** $\alpha + \gamma = 2\beta$

This includes the special case of $\alpha = \beta = \gamma = \mu$ when evaluated in the undeformed configuration. This gives $\rho c^2 = \alpha$ *independently* of the direction of propagation in the (x_1, x_2) plane. This incorporates the classical theory with initial stress assumed to be zero in the undeformed configuration. Also, from Eqs. (3.273)–(3.275), for this special case, we require $\tau_{11} = \tau_{22} = 0$ since we have already assumed $\tau_{22} = 0$.

With $\alpha + \gamma = 2\beta$, Eq. (3.282) can be re-written as

$$\rho c^2 = \alpha \cos^2 \theta + \gamma \sin^2 \theta. \tag{3.283}$$

For either Eq. (3.282) or Eq. (3.283) a shear wave can propagate along the principal axis x_1, with $\rho c^2 = \alpha$ or along the principal axis x_2, with $\rho c^2 = \gamma$.

If $\alpha \neq \gamma$, Eq. (3.283) gives

$$\cos^2 \theta = \frac{\gamma - \rho c^2}{\gamma - \alpha}. \tag{3.284}$$

For this to yield real values of $\cos\theta$ we must have

$$\text{either} \quad \gamma \leq \rho c^2 \leq \alpha, \quad \text{or} \quad \alpha \leq \rho c^2 \leq \gamma. \tag{3.285}$$

We can re-write Eq. (3.284) in the alternative form

$$\tan^2 \theta = \frac{\rho c^2 - \alpha}{\gamma - \rho c^2}. \tag{3.286}$$

For a given wave speed subject to conditions (3.285), Eq. (3.286) yields two (in general distinct) directions, symmetric with respect to the axes.

***Case* B:** $\gamma + \alpha \neq 2\beta$

For a given wave speed, the solutions of Eq. (3.282) may be written

$$\cos^2\theta = \frac{\gamma - \beta \pm [\beta^2 - \alpha\gamma + \rho c^2(\alpha + \gamma - 2\beta)]^{\frac{1}{2}}}{\alpha + \gamma - 2\beta}. \tag{3.287}$$

Considering $0 \leq \cos^2\theta \leq 1$, for real solutions from Eq. (3.287), we must have either

$$\alpha + \gamma - 2\beta > 0, \quad \frac{\alpha\gamma - \beta^2}{\alpha + \gamma - 2\beta} \leq \rho c^2 \leq \mathrm{Min}\{\alpha, \gamma\}, \tag{3.288}$$

or

$$\alpha + \gamma - 2\beta < 0, \quad \mathrm{Max}\{\alpha, \gamma\} \leq \rho c^2 \leq \frac{\alpha\gamma - \beta^2}{\alpha + \gamma - 2\beta}. \tag{3.289}$$

Equal roots arise when

$$\rho c^2 = \frac{\alpha\gamma - \beta^2}{\alpha + \gamma - 2\beta}, \quad \cos^2\theta = \frac{\alpha - \beta}{\alpha + \gamma - 2\beta}. \tag{3.290}$$

We may write the directions of propagation in the form

$$\tan^2\theta = \frac{\rho c^2 - \beta \pm [\beta^2 - \alpha\gamma + \rho c^2(\alpha + \gamma - 2\beta)]^{\frac{1}{2}}}{\gamma - \rho c^2}. \tag{3.291}$$

Thus, for any given wave speed within the allowed range there are in general four possible distinct directions in which a plane shear wave may propagate. In a special case, these degenerate to two when conditions (3.290) hold.

For the special value $\rho c^2 = \gamma$, the wave propagates *either* along the x_2 axis (as in case A) *or* in the direction given by

$$\cos^2\theta = \frac{2(\gamma - \beta)}{\alpha + \gamma - 2\beta}, \tag{3.292}$$

in which case either $\alpha < \gamma < \beta$ or $\alpha > \gamma > \beta$ must hold. Similarly, the special value $\rho c^2 = \alpha$ means that the wave is *either* propagating along the x_1 direction or in the direction given by

$$\cos^2\theta = \frac{\gamma - \alpha}{\alpha + \gamma - 2\beta}, \tag{3.293}$$

which requires either $\beta < \alpha < \gamma$ or $\beta > \alpha > \gamma$ hold.

3.4.6 Wave Reflection from a Plane Boundary in the Reference Configuration

We consider a plane wave of the form given by Eq. (3.183) incident on the boundary $x_2 = 0$ in the half space $x_2 < 0$ in the reference configuration. The boundary $x_2 = 0$ is taken to be free of incremental traction but subject to the normal initial stresses τ_{11}, τ_{22}.

For the special model given by Eq. (3.111), p_0 is given by Eq. (3.278), $\tau_{22} = 0$ and the stress-free incremental boundary conditions in this case are given by Eqs. (3.280) and (3.281).

Let the direction of propagation of this wave be $\mathbf{n} = (n_1, n_2) = (\cos\theta, \sin\theta)$ and c be its speed. As a result of this incidence, depending on the material properties and the state of deformation, one or two reflected waves and/or a surface wave are generated. We can write the general solution for ψ consisting of the incident and two reflected waves in the form

$$\begin{aligned}\psi &= f[k(n_1x_1 + n_2x_2 - ct)] + Rf[k(n_1x_1 - n_2x_2 - ct)] \\ &+ R^{'} f[k^{'}(n_1^{'}x_1 - n_2^{'}x_2 - c^{'}t)],\end{aligned} \tag{3.294}$$

where R and $R^{'}$ are the reflection coefficients. Also, $k^{'}$ and $c^{'}$ are the wave number and wave speed of the second reflected wave. The first reflected wave has the same speed as the incident wave and is reflected at an angle θ to the boundary, while the angle of reflection of the second wave is $\theta^{'}$. We choose $n_1^{'} = \cos\theta^{'}, n_2^{'} = \sin\theta^{'}$. For the compatibility of three waves, they should have the same frequency. For this we must set

$$kc = k^{'}c^{'}. \tag{3.295}$$

Using Eq. (3.294) in Eqs. (3.280) and (3.281), we find $kn_1 = k^{'}n_1^{'}$ and hence

$$c^{'}n_1 = cn_1^{'}. \tag{3.296}$$

which is a statement of Snell's law.

We now consider separately the cases in which $\gamma + \alpha = 2\beta$ and $\gamma + \alpha \neq 2\beta$.

***Case* A:** $\gamma + \alpha = 2\beta$

In this case, from Eqs. (3.273)–(3.275), we require $\tau_{11} = \tau_{22} = 0$. This special case therefore corresponds to vanishing of the initial stress just as in classical mechanics. Also, it follows that $\alpha = \gamma = \beta$ in this case. We therefore have

$$\rho c^2 = \alpha. \tag{3.297}$$

This shows that for the initially stressed materials following $\gamma + \alpha = 2\beta$ in the reference configuration, a single wave travels *independently* of the direction of propagation with a fixed speed α or γ.

***Case* B:** $\gamma + \alpha \neq 2\beta$

We are now considering the case of two reflected SV waves. In the case of an incident wave Eq. (3.282) can be rewritten as

$$(\alpha + \gamma - 2\beta)n_1^4 + 2(\beta - \gamma)n_1^2 + \gamma = \rho c^2, \tag{3.298}$$

and when there is a reflected wave, we also have

$$(\alpha + \gamma - 2\beta){n_1^{'}}^4 + 2(\beta - \gamma){n_1^{'}}^2 + \gamma = \rho {c^{'}}^2, \tag{3.299}$$

together with the Snell's law given by Eq. (3.296).

From Eqs. (3.298) and (3.299), we find that either ${n_1^{'}}^2 = n_1^2$ or

$$n_1^2 {n_1^{'}}^2 = \gamma/(\alpha + \gamma - 2\beta), \tag{3.300}$$

and, these two possibilities occur in the case when

$$n_1^4 = \gamma/(\alpha + \gamma - 2\beta), \tag{3.301}$$

which defines the *transitional angle*, say, θ_0. Thus, when Eq. (3.294) is applicable, a given incident wave generates two reflected waves in general. One of these waves is reflected at the same angle as the incident wave and the angle of reflection of the second wave is given by Eq. (3.300).

For a given angle of incidence θ, $\theta^{'}$ is calculated from Eq. (3.300), ρc^2 from Eq. (3.298) and $\rho {c^{'}}^2$ from Eq. (3.299). The reflection coefficients R and $R^{'}$ are calculated using the boundary conditions.

For a given θ, necessary and sufficient conditions for Eq. (3.300) to yield a real angle $\theta^{'}$

are

$$\alpha > 2\beta, \tag{3.302}$$

$$\tan^2\theta \leq \frac{\alpha - 2\beta}{\gamma} \equiv \tan^2\theta_c, \tag{3.303}$$

where θ_c is defined as the critical angle given by the right-hand identity in Eq. (3.303).

Substitution of Eq. (3.294) in boundary conditions (3.280) and (3.281) after using the propagation condition (3.298) and Eq. (3.300), we get after some calculations

$$\gamma(1 + R + R') - (\alpha + \gamma - 2\beta)[(1 + R)n_1'^2 n_2^2 + R' n_1^2 n_2'^2] = 0, \tag{3.304}$$

$$2\gamma[(1 - R)n_1' n_2 + R' n_1 n_2'] - (\alpha + \gamma - 2\beta)[(1 - R)n_1 n_2 + R' n_1' n_2'] n_1 n_1' = 0. \tag{3.305}$$

Explicit expressions for R and R' are given by

$$R = \frac{(1 - 2n_1'^2)^2 n_1^3 n_2 - (1 - 2n_1^2)^2 n_1'^3 n_2'}{(1 - 2n_1'^2)^2 n_1^3 n_2 - (1 - 2n_1^2)^2 n_1'^3 n_2'}, \tag{3.306}$$

$$R' = \frac{-2(1 - 2n_1^2)(1 - 2n_1'^2) n_1'^2 n_1 n_2}{(1 - 2n_1'^2)^2 n_1^3 n_2 + (1 - 2n_1^2)^2 n_1'^3 n_2'}. \tag{3.307}$$

It is obvious from the above expressions that R' vanishes for the normal incidence.

We now consider three non-trivial cases where R' can possibly vanish.

(i) $R' = 0, R \neq \pm 1$. In this case, Eqs. (3.304) and (3.305), yield *either* $n_2 = 0$, i.e. grazing incidence, which is not possible since then $\psi = 0$, *or*

$$n_1^2 = \frac{1}{2} = \frac{2\gamma}{\alpha + \gamma - 2\beta} = n_1'^2. \tag{3.308}$$

For Eq. (3.308) to yield a real angle, inequality (3.302) must hold along with Eq. (3.277). The wave speed in this case is given by

$$\rho c^2 = \beta + \gamma. \tag{3.309}$$

(ii) $R' = 0, R = 1$. In this case, we have

$$n_1^2 = \frac{1}{2}. \tag{3.310}$$

The wave speed in this case is given by

$$\rho c^2 = (\alpha + \gamma + 2\beta)/4. \tag{3.311}$$

In this case, the non-zero displacement component on the boundary $x_2 = 0$ is

$$u_1 = 0, \quad u_2 = -\sqrt{2} k f[k(\frac{x_1}{\sqrt{2}} - ct)], \tag{3.312}$$

which means there is no displacement along the boundary in the x_1 direction.

(iii) $R^{'} = 0, R = -1$. In this case, we have

$$n_1^2 = \frac{2\gamma}{\alpha + \gamma - 2\beta} \quad \text{and} \quad n_1^{'\,2} = \frac{1}{2}. \tag{3.313}$$

For Eq. $(3.313)_1$ to yield a real angle, inequality (3.302) must hold along with Eq. (3.277). The wave speed in this case is given by

$$\rho c^2 = \frac{\gamma(\alpha + \gamma + 2\beta)}{\alpha + \gamma - 2\beta}. \tag{3.314}$$

In this case, the non-zero displacement component on the boundary $x_2 = 0$ is

$$u_1 = 2k(\frac{2\gamma}{\alpha + \gamma - 2\beta})^{1/2} f[k((\frac{2\gamma}{\alpha + \gamma - 2\beta})^{1/2} x_1 - ct)], \tag{3.315}$$

which means that there is no displacement normal to the boundary in the x_2 direction.

3.4.7 Analysis of Wave Reflection for A Special Model in the Reference Configuration

For the specific model given by Eq. (3.111), for an initially stressed incompressible material in the reference configuration, we have

$$\alpha = \mu + \tau_{11}, \quad 2\beta = 2\mu + \tau_{11} + 4\bar{\mu}\tau_{11}^2, \quad \gamma = \mu. \tag{3.316}$$

The case when $\alpha + \gamma = 2\beta$ refers to the case when the initial stress vanishes. The results are therefore equivalent to those for an isotropic linear elastic materials in the classical theory. For details, see the discussion in Section 3.4.6: *Case* A.

For the material model given by Eq. (3.111), $\alpha + \gamma \neq 2\beta$ and the general results in

Section 3.4.6: *Case* B, should apply. For this, we define the dimensionless quantity

$$\bar{A}_2 = (\alpha + \gamma - 2\beta)/\beta = -4b_0\bar{\tau}^2. \tag{3.317}$$

Since $\gamma > 0$ due to strong ellipticity, and $\alpha > 2\beta$ from Eq. (3.300), we must have $\bar{\mu} < 0$. Also we have from Eqs. (3.302) and (3.303) for real angles

$$\bar{A}_2 < 1, \tag{3.318}$$

and

$$0 < \sec^2\theta \quad \leq \quad \bar{A}_2 \equiv \sec^2\theta_c. \tag{3.319}$$

The above inequality may be written as

$$\cos^2\theta \geq 1/\bar{A}_2, \tag{3.320}$$

which gives the range of values for the angle of incidence θ for which a reflected wave exists. For $0 \leq \bar{A}_2 < 1$, the inequality (3.320) does not hold and hence for these values of $\bar{A}_2$, the second reflected wave is replaced by a surface wave. For the plot of inequality (3.320), we refer to Fig. 3.1 with $\bar{A}_1$ replaced by $\bar{A}_2$.

The strong ellipticity conditions (3.179), in this case gives the sufficient conditions as

$$\bar{\tau} + 1 > 0, \quad 4b_0\bar{\tau}^2 + \bar{\tau} + 2(\bar{\tau}+1)^{\frac{1}{2}} + 2 > 0. \tag{3.321}$$

Using Eqs. (3.206) and (3.239) the dimensionless wave speed of the incident wave in this case is given by

$$\rho c^2/\gamma = (-4b_0\bar{\tau}^2)\cos^4\theta + (4b_0\bar{\tau}^2 + \bar{\tau})\cos^2\theta + 1, \tag{3.322}$$

as a function of θ.

Similarly, for the reflected wave, the speed in its dimensionless form is given by

$$\rho c'^2/\gamma = (-4b_0\bar{\tau}^2)^{-1}[\sec^4\theta + (4b_0\bar{\tau}^2 + \bar{\tau})\sec^2\theta - 4b_0\bar{\tau}^2], \tag{3.323}$$

as a function of θ. Since, for a reflected wave to exist, θ' in Eq. (3.323) must be real and should fall in the range to satisfy the inequality (3.250). For the angles out of this range, $c^{'}$ is the speed of a surface wave which increases indefinitely (and its amplitude vanishes) as the incident wave approaches normal incidence. The behaviours of $\rho c^2/\gamma$ (dashed graph) and $\rho {c^{'}}^2/\gamma$ are shown in Fig. 3.8 for $0 \leq \theta \leq \theta_c$ where $\theta_c = \sec^{-1}(\sqrt{\bar{A}_2})$. For angles of incidence greater than the critical angle, $c^{'}$ represents the surface wave and is given by $c^{'} = c/n_1$ where c is the speed of the incident wave given by Eq. (3.253). The speed of surface wave for $\theta > \theta_c$ is given by

$$\rho {c^{'}}^2/\gamma = \sec^2\theta. \tag{3.324}$$

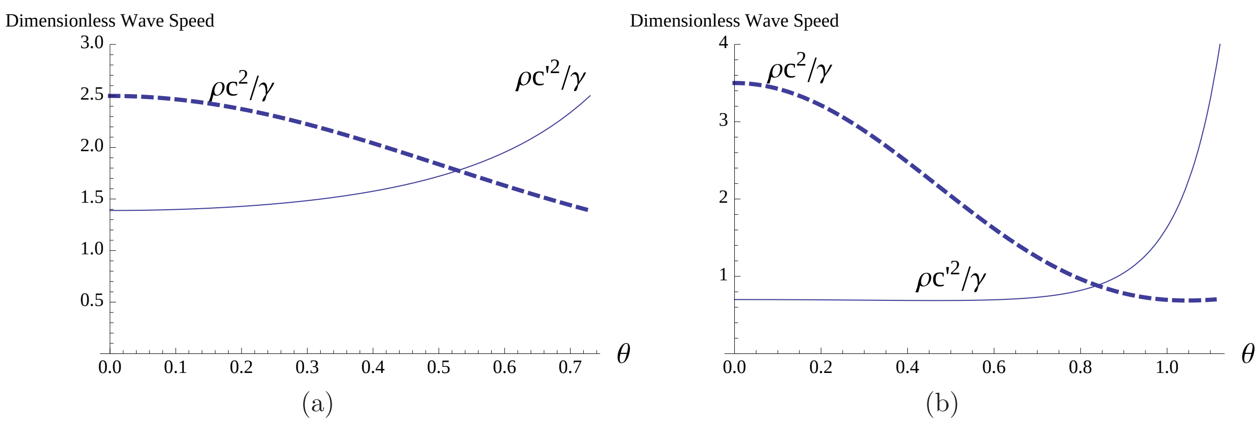


Figure 3.8: Plot of dimensionless wave speeds $\rho c^2/\gamma$ (dashed) and $\rho {c^{'}}^2/\gamma$ for (a) $b_0 = -0.2, \bar{\tau} = 0.5, \bar{A}_2 = 2$, (b) (a) $b_0 = -0.2, \bar{\tau} = 2, \bar{A}_2 = 32$.

We hence observe that the stability of waves in the reference configuration in this case depends on the magnitude of initial stress when b_0 is fixed. Figures 3.9 and 3.10 are counterpart plots (for $\lambda = 1$) of Figs. 3.4 and 3.5. Replacing $\bar{A}_1$ by $\bar{A}_2$, the expressions for R and $R^{'}$ follow from Eqs. (3.245) for a second reflected wave and from Eqs. (3.247) for a surface wave, respectively. A comparison shows that in the absence of stretches, $R^{'}$ allows a large value of $\bar{\tau}$ until it vanishes. The behaviour is similar for intermediate values of $\bar{A}_2$ that is we see a sharp increase in the magnitude of $R^{'}$ for particular choices of $\bar{\tau}$ for fixed b_0. The behaviour in the absence of stretches for very small and very large values is illustrated in Figs. 3.9(a, b) and 3.9(g, h) for $R^{'}$, respectively and in Figs. 3.10(a, b) and 3.10(g, h) for R, respectively. The real and imaginary parts of $R^{'}$ and R are shown in Figs. 3.11 and 3.12. In these figures, the symmetry of the curves about the angle of incidence is obvious. In the case of each plot for $R^{'}$, only those values of θ refer to a reflected wave for which the inequality (3.320) holds. For instance, in Fig. 3.9a, since the value of $\bar{A}_2 = 2$, which means a reflected wave does not exists for the range $\pi/4 < \theta < 3\pi/4$ (or equivalently $0.7853 < \theta < 2.3561$)

and rather a surface wave exists.

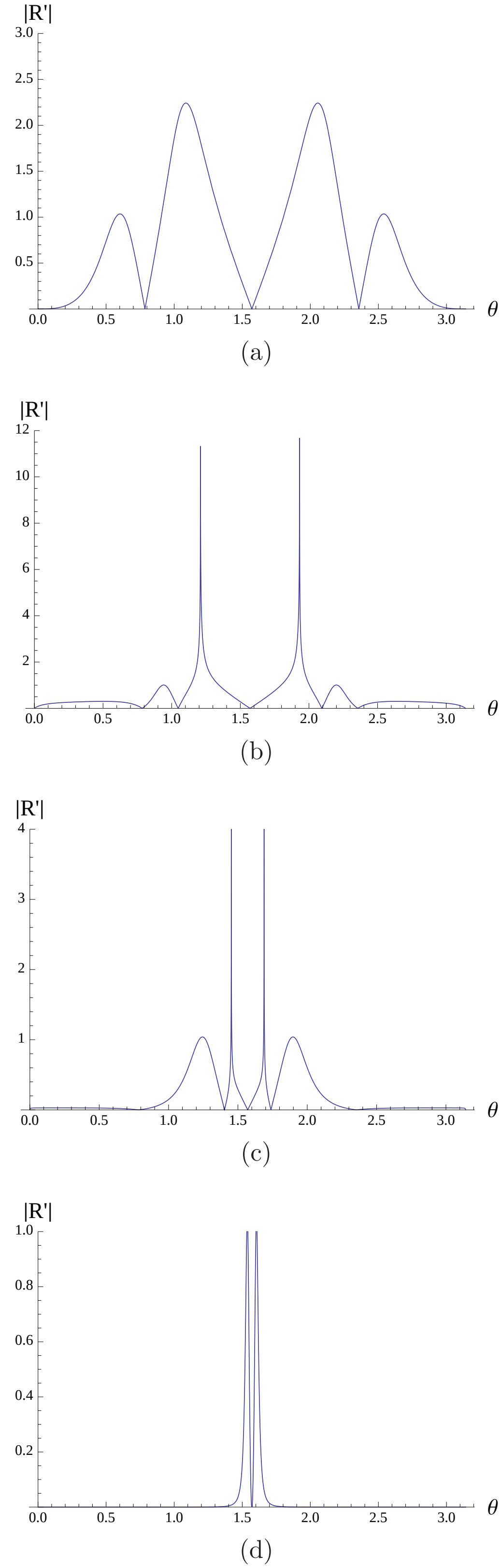
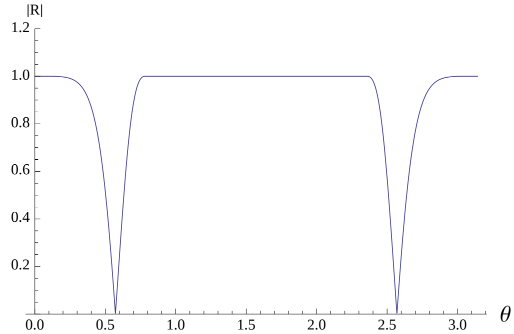
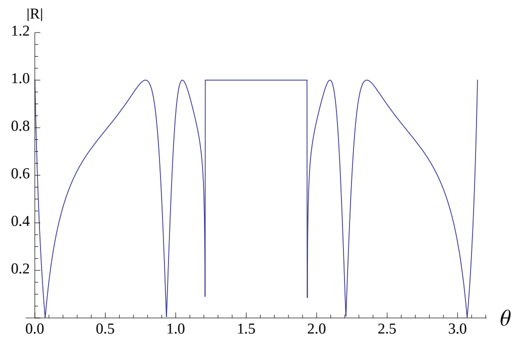


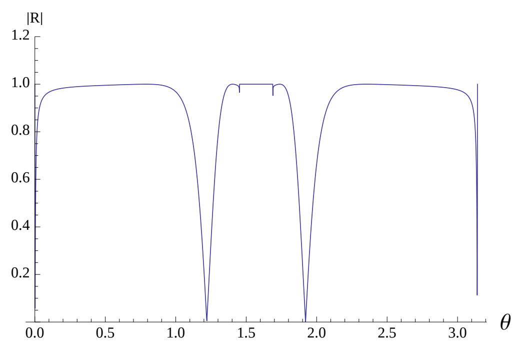
Figure 3.9: $|R^{'}|$ for $b_0 = -2$ and (a) $\bar{\tau} = 0.5, \bar{A}_2 = 2$, (b) $\bar{\tau} = 1, \bar{A}_2 = 8$, (c) $\bar{\tau} = 3, \bar{A}_2 = 72$, (d) $\bar{\tau} = 250, \bar{A}_2 = 5 \times 10^5$, Plotted as a function of the angle of incidence θ and $0 \leq \theta \leq \pi$. The plots refer to the amplitude of a surface wave when the inequality (3.320) does not hold. For example, a surface wave exists for $0.7853 < \theta < 2.3561$ in (a) and for $1.20943 < \theta < 1.93216$ in (b).



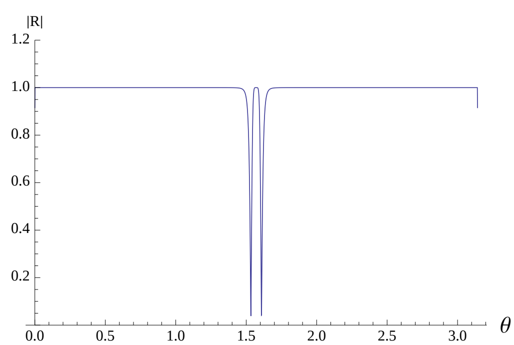
(a)



(b)



(c)



(d)

Figure 3.10: $|R^{'}|$ for $b_0 = -2$ and (a) $\bar{\tau} = 0.5, \bar{A}_2 = 2$, (b) $\bar{\tau} = 1, \bar{A}_2 = 8$, (c) $\bar{\tau} = 3, \bar{A}_2 = 73$, (d) $\bar{\tau} = 250, \bar{A}_2 = 5 \times 10^5$, Plotted as a function of the angle of incidence θ and $0 \leq \theta \leq \pi$

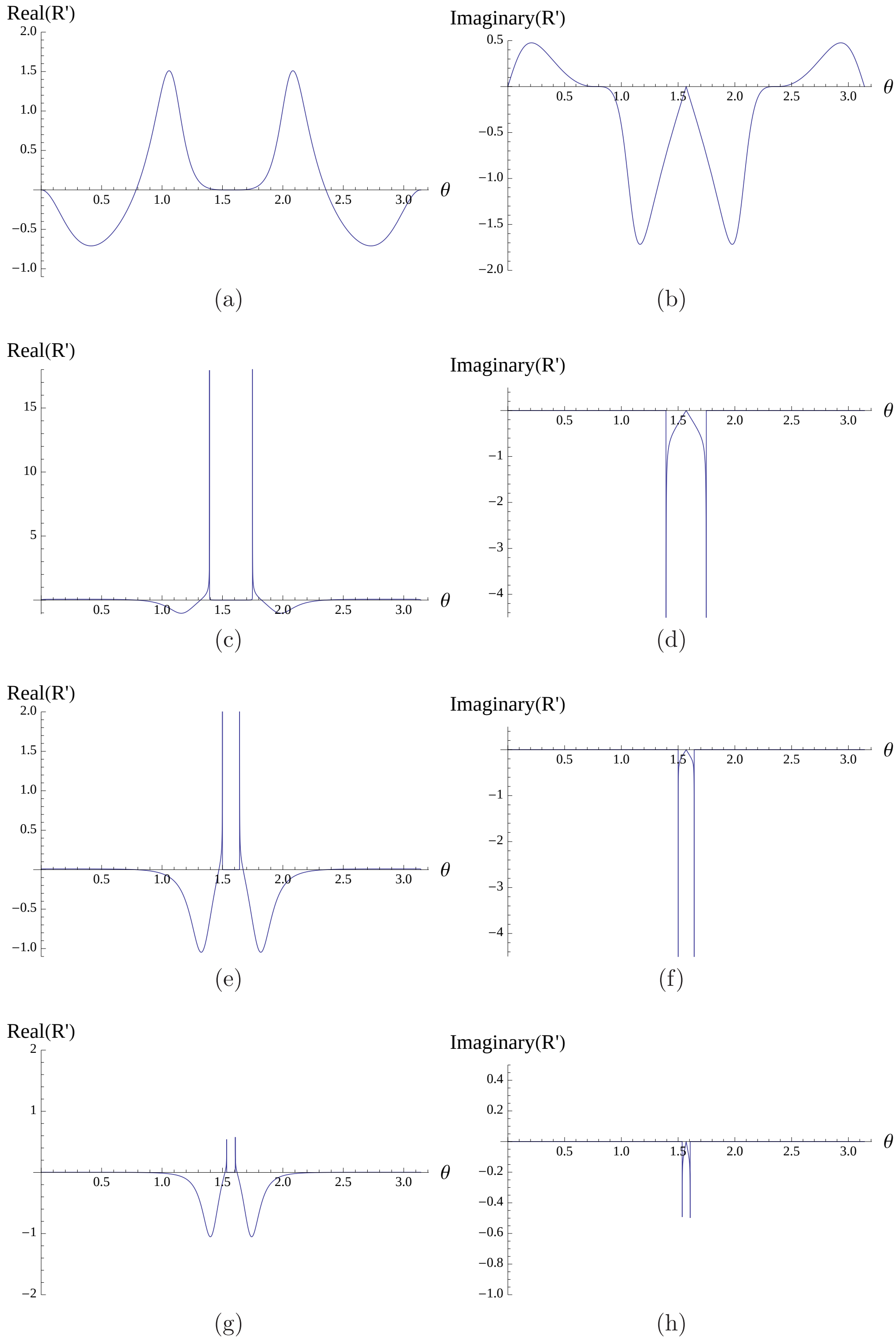


Figure 3.11: Real(R') (left column) and Imaginary(R') for $b_0 = -2$ and (a), (b) $\bar{\tau} = 0.00001$, $\bar{A}_2 = 8 \times 10^{-10}$, (c), (d) $\bar{\tau} = 2$, $\bar{A}_2 = 32$, (e), (f) $\bar{\tau} = 5$, $\bar{A}_2 = 200$, (g), (h) $\bar{\tau} = 10$, $\bar{A}_2 = 800$, plotted as a function of the angle of incidence θ and $0 \leq \theta \leq \pi$. The plots refer to the reflection coefficient of a surface wave when the inequality (3.320) does not hold. The changing vertical scale may be noted.

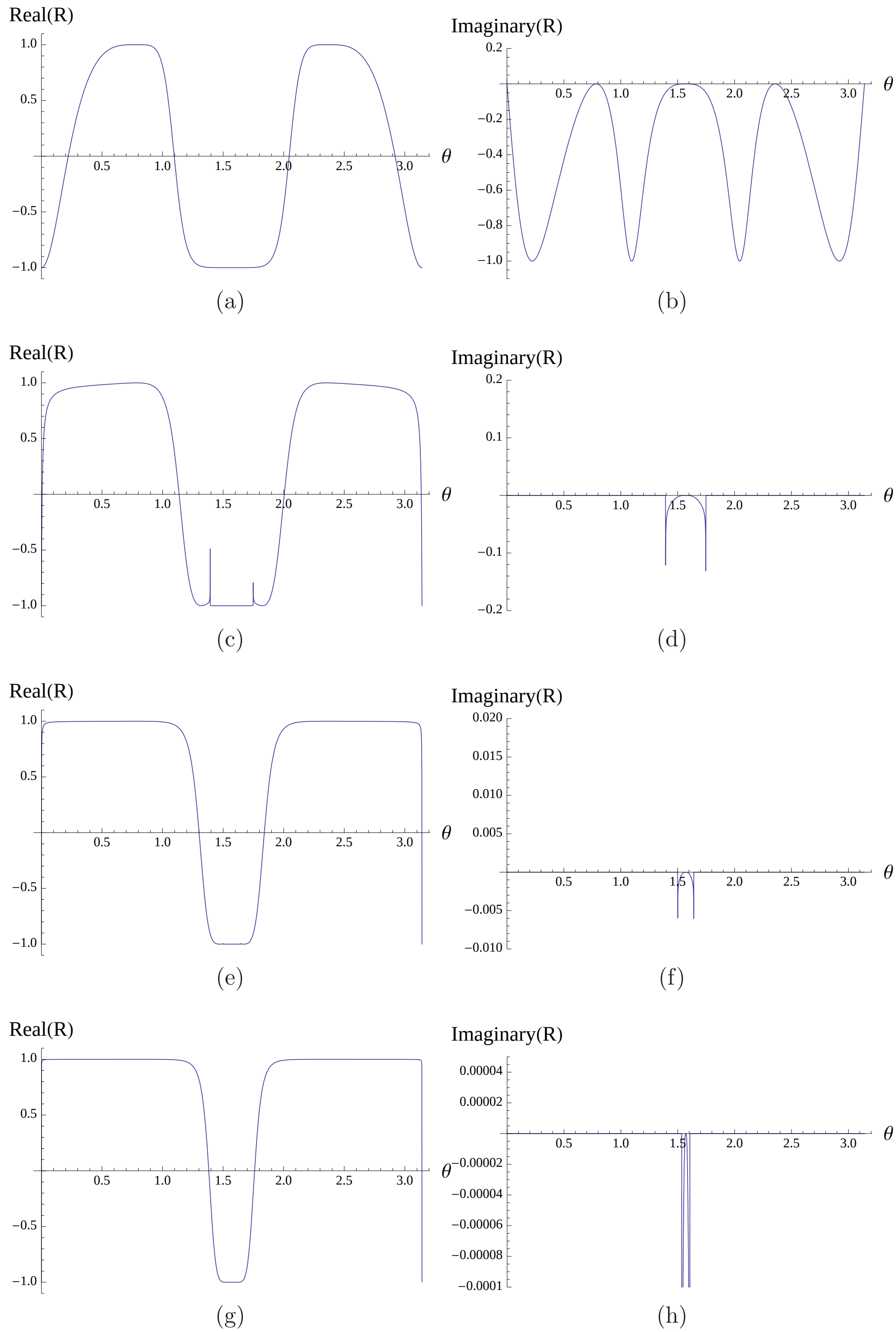


Figure 3.12: Real(R) (left column) and Imaginary(R) for $b_0 = -2$ and (a), (b) $\bar{\tau} = 0.00001, \bar{A}_2 = 8 \times 10^{-10}$, (c), (d) $\bar{\tau} = 2, \bar{A}_2 = 32$, (e), (f) $\bar{\tau} = 5, \bar{A}_2 = 200$, (g), (h) $\lambda = 10, \bar{A}_2 = 800$, plotted as a function of the angle of incidence θ and $0 \leq \theta \leq \pi$.

Chapter 4

Surface Waves in Initially-stressed Materials

In this chapter, we apply the theory of the superposition of infinitesimal deformations on finite deformations in a initially-stressed incompressible hyperelastic material to the study of the propagation of surface waves in a half-space which is subjected to a pure homogeneous deformation. The influence of initial stress on elastic waves was first studied by Biot [5]. Further work on the propagation of surface waves in a compressible and an incompressible elastic body which is not stress-free in its undisturbed state was carried out by Hayes and Rivlin [15]. Flavin [11] studied surface waves for Mooney-Rivlin and neo-Hookean materials. In [49, 50, 51], Willson investigated the properties of surface wave propagation for a variety of isotropic materials for different states of pre-stress and pre-strain. In [9], Dowaikh and Ogden analyzed in detail the propagation of infinitesimal Rayleigh surface waves along a principal direction in a pre-stressed incompressible elastic half-space whereas Dowaikh [8] carried out a similar analysis for Love waves. The effect of pre-stress on the speed of Love waves in an incompressible material subject to a high two-dimensional initial stress was studied by Ohnabe and Nowinski [40] and numerical results were presented for a neo-Hookean material.

4.1 Rayleigh Surface Waves in an Initially-stressed Incompressible Half-space

The possibility of a wave travelling along the free surface of an elastic half-space such that the disturbance is largely confined to the neighbourhood of the boundary was considered by Rayleigh [42]. The displacement of surface waves or Rayleigh waves decays exponentially with increasing distance from the boundary.

In this section, we consider the propagation of Rayleigh surface waves in an initially-stressed incompressible half-space subject to general pure homogeneous pre-strain. Expressed in terms of Cartesian coordinates (x_1, x_2, x_3), we suppose that the deformed half-space occupies the region $x_2 \leq 0$ and we consider waves propagating along the x_1 axis. For simplicity, we also take the x_1 axis to correspond to a principal axis of the underlying deformation. We assume that the incremental (or infinitesimal) displacement associated with the wave has no component normal to the (x_1, x_2) plane and that the x_1 and x_2 components are independent of x_3. As a result, the basic equations derived in the Section 3.4.1 apply. We suppose a surface wave of the form

$$\psi = A \exp[skx_2 - ik(x_1 - ct)], \tag{4.1}$$

where c is the speed of propagation of the wave and k is the wave number and A is a constant. We require s to have positive real part so that the wave decays as $x_2 \to -\infty$.

Using Eq. (4.1) in Eq. (3.149), we get

$$\beta_5 s^4 - 2i\beta_3 s^3 - (2\beta_2 - \rho c^2)s^2 + 2i\beta_4 s + (\beta_1 - \rho c^2) = 0. \tag{4.2}$$

For the particular strain-energy function given by Eq. (3.111) and choosing $\Sigma_{12} = 0$, $\beta_1, \ldots, \beta_5$ are given by Eqs. (3.170)–(3.174). Therefore, in this case, Eq. (4.2) yields the following quadratic for s^2:

$$\beta_5 s^4 - (2\beta_2 - \rho c^2)s^2 + (\beta_1 - \rho c^2) = 0. \tag{4.3}$$

If we assume that the boundary is traction free in the underlying configuration, then Σ_{22}, τ_{22} and T_{22} vanish identically. In this case, $\beta_1, 2\beta_2$ and β_5 are given by Eqs. (3.231)–(3.233). Also assuming the incremental traction to be zero on the boundary $x_2 = 0$, the incremental boundary conditions are given by Eqs. (3.181) and (3.182). The two-dimensional strong ellipticity conditions in this case are given by Eq. (3.179).

Let s_1^2 and s_2^2 be the roots of Eq. (4.3). Then

$$s_1^2 + s_2^2 = (2\beta_2 - \rho c^2)/\beta_5, \quad s_1^2 s_2^2 = (\beta_1 - \rho c^2)/\beta_5. \tag{4.4}$$

For decaying solutions for ψ, we require s_1 and s_2 to have positive real part or, exceptionally, if real, then at most one of s_1 and s_2 may vanish. In either case, we require $s_1^2 s_2^2 \geq 0$. Since $\beta_5 > 0$ by the strong ellipticity condition, we deduce from Eq. (4.4) that the wave speed lies within the bounds

$$0 \leq \rho c^2 \leq \beta_1. \tag{4.5}$$

If c_s denotes the speed of a plane (shear) wave propagating in the x_1-direction with displacement in the x_2-direction in an unbounded body subject to the same homogeneous pure strain, then

$$\xi_s = \beta_1/\mu, \tag{4.6}$$

where we have introduced the notation $\xi_s = \rho c_s^2/\mu$. This corresponds to the upper limit in Eq. (4.5).

Taking s_1 and s_2 to denote the solutions of Eq. (4.3) with positive real part, the solution for ψ may be written in the form

$$\psi = (A\mathrm{e}^{s_1 k x_2} + B\mathrm{e}^{s_2 k x_2}) \exp[ik(ct - x_1)], \tag{4.7}$$

where A and B are constants. Substituting Eq. (4.7) into the boundary conditions (3.181) and (3.182) yields

$$(s_1^2 + 1)A + (s_2^2 + 1)B = 0, \tag{4.8}$$

$$(2\beta_2 + \beta_5 - \rho c^2 - \beta_5 s_1^2)s_1 A + (2\beta_2 + \beta_5 - \rho c^2 - \beta_5 s_2^2)s_2 B = 0. \tag{4.9}$$

For a nontrivial solution of the system of Eqs. (4.8) and (4.9) for A and B, the determinant of the coefficients must vanish. Therefore,

$$\det \begin{bmatrix} 1 + s_1^2 & 1 + s_2^2 \\ (1 + s_2^2)s_1 & (1 + s_1^2)s_2 \end{bmatrix} = 0, \tag{4.10}$$

where we have made use of Eq. $(4.4)_1$. We therefore have

$$(s_1 - s_2)[s_1 s_2(s_1^2 + s_2^2 + s_1 s_2 + 2) - 1] = 0. \tag{4.11}$$

It may be noted that vanishing of the factor $(s_1 - s_2)$ in Eq. (4.11) yields a solution from

Eq. (4.7) which is recoverable from the second factor in Eq. (4.11) for $s_1 = s_2$. Therefore, after some calculations and noting that $s_1 s_2$ must be positive, removal of the factor $(s_1 - s_2)$ from Eq. (4.11) leads to

$$\beta_5(\beta_1 - \rho c^2) + (2\beta_2 + 2\beta_5 - \rho c^2)[\beta_5(\beta_1 - \rho c^2)]^{\frac{1}{2}} = \beta_5^2. \tag{4.12}$$

Equation (4.12) is the *secular equation* which determines the speed c of propagation of surface (Rayleigh) waves of the type considered.

The square root term in Eq. (4.12) can be removed by squaring after rearrangement to yield the cubic

$$(\rho c^2)^3 - p(\rho c^2)^2 + q(\rho c^2) - r = 0, \tag{4.13}$$

where

$$p = 4\beta_2 + 3\beta_5 + \beta_1, \tag{4.14}$$

$$q = (2\beta_2 + 2\beta_5)^2 + 2\beta_1(2\beta_2 + 2\beta_5) + 2\beta_5^2 - 2\beta_1\beta_5, \tag{4.15}$$

$$\begin{aligned} r = {} & [(\beta_1\beta_5)^{\frac{1}{2}}(2\beta_2 + 2\beta_5) + \beta_5^2 - \beta_1\beta_5] \\ \times {} & [(\beta_1\beta_5)^{\frac{1}{2}}(2\beta_2 + 2\beta_5) - \beta_5^2 + \beta_1\beta_5]/\beta_5. \end{aligned} \tag{4.16}$$

Equation (4.13) gives solutions of ρc^2 but the squaring process may introduce solutions of Eq. (4.13) which are not solutions of Eq. (4.12). For example, when $c = 0$, $r = 0$ and either of the factors in Eq. (4.16) may vanish, but only the second of these, given by

$$r' = (\beta_1\beta_5)^{\frac{1}{2}}(2\beta_2 + 2\beta_5) - \beta_5^2 + \beta_1\beta_5, \tag{4.17}$$

corresponds to a solution of Eq. (4.12). To prevent this problem, we introduce the notation

$$\eta = [(\beta_1 - \rho c^2)/\beta_5]^{\frac{1}{2}} \tag{4.18}$$

so that

$$\rho c^2 = \beta_1 - \beta_5\eta^2. \tag{4.19}$$

In view of Eq. (4.5), we must have

$$0 \leq \eta \leq (\beta_1/\beta_5)^{\frac{1}{2}}. \tag{4.20}$$

4.2 Analysis of the Secular Equation

4.2.1 Analysis of the Secular Equation in the Deformed Configuration

In terms of η, Eq. (4.12) reduces to a simple cubic equation

$$\eta^3 + \eta^2 + (2\beta_2 + 2\beta_5 - \beta_1)\eta/\beta_5 - 1 = 0. \tag{4.21}$$

A secular equation similar to Eq. (4.21) was obtained as a special case for a general strain-energy function by Dowaikh and Ogden [9] for an incompressible isotropic solid.

We can re-write Eq. (4.21) in the form

$$f(\eta) = \eta^3 + \eta^2 + d\eta - 1 = 0, \tag{4.22}$$

where $d = (2\beta_2 + 2\beta_5 - \beta_1)/\beta_5$.

From Eq. (4.22), it may be noted that $f(0) < 0$ for any value of d. To ensure that at least one real root corresponding to a nonzero wave speed exists in the interval (4.20), we require $f(\beta_1/\beta_5)^{\frac{1}{2}} > 0$, which after some rearrangement yields

$$\beta_5[\beta_1 - \beta_5 + (\beta_1/\beta_5)^{\frac{1}{2}}(2\beta_2 + 2\beta_5)] > 0. \tag{4.23}$$

In the undeformed configuration, $\beta_5 = \mu$ and $\mu > 0$, and we therefore require from continuity

$$\beta_5 > 0, \tag{4.24}$$

$$\beta_1 - \beta_5 + (\beta_1/\beta_5)^{\frac{1}{2}}(2\beta_2 + 2\beta_5) > 0. \tag{4.25}$$

We see that the condition (4.24) entails the strong ellipticity condition $(3.179)_2$. Moreover, we have

$$2\beta_2 + 2(\beta_1\beta_5)^{\frac{1}{2}} - [2\beta_2 + 2\beta_5 + (\beta_5/\beta_1)^{\frac{1}{2}}(\beta_1 - \beta_5)] = (\beta_1/\beta_5)^{\frac{1}{2}}(\beta_1^{\frac{1}{2}} - \beta_5^{\frac{1}{2}})^2 \geq 0, \tag{4.26}$$

and it follows that (4.26) entails $(3.179)_3$.

It is easy to see that any possible turning points of $f(\eta)$ should occur at $\eta = (-1 \pm \sqrt{1-3d})/3$. If $d = 1/3$, the two points coincide. If $d > 1/3$, there are no real turning points, which means $f(\eta)$ is a monotonically non-decreasing function. For $0 < d < 1/3$, the maximum and minimum both occur in $\eta < 0$. For $d = 0$, the minimum value of $f(\eta)$ occurs at $\eta = 0$ and the maximum occurs in $\eta < 0$. For $d < 0$, the minimum value of $f(\eta)$ occurs in $\eta > 0$; however, the maximum remains in $\eta < 0$. In all these cases, there exists a unique positive solution for η which satisfies Eq. (4.22). Since the maximum occurs in $\eta < 0$, the maximum value of $f(\eta)$ may be positive or negative. Therefore, the two solutions other than the real positive solution can be both negative or complex conjugates, depending on the value of d. It is therefore ensured that a unique surface wave exists when the surface $x_2 = 0$ is free of traction. The exact solutions, say η_1, η_2, η_3, of Eq. (4.22) are given by

$$\begin{aligned}
\eta_1 &= -\frac{1}{3} - \frac{2^{1/3}(3d-1)}{3(25+9d+3\sqrt{3}\sqrt{23+18d-d^2+4d^3})^{1/3}} \\
&+ \frac{(25+9d+3\sqrt{3}\sqrt{23+18d-d^2+4d^3})^{1/3}}{3\times 2^{1/3}}, \qquad (4.27)\\
\eta_2 &= -\frac{1}{3} + \frac{(1+i\sqrt{3})(3d-1)}{3\times 2^{2/3}(25+9d+3\sqrt{3}\sqrt{23+18d-d^2+4d^3})^{1/3}} \\
&+ \frac{(1-i\sqrt{3})(25+9d+3\sqrt{3}\sqrt{23+18d-d^2+4d^3})^{1/3}}{6\times 2^{1/3}}, \qquad (4.28)\\
\eta_3 &= -\frac{1}{3} + \frac{(1-i\sqrt{3})(3d-1)}{3\times 2^{2/3}(25+9d+3\sqrt{3}\sqrt{23+18d-d^2+4d^3})^{1/3}} \\
&+ \frac{(1+i\sqrt{3})(25+9d+3\sqrt{3}\sqrt{23+18d-d^2+4d^3})^{1/3}}{6\times 2^{1/3}}. \qquad (4.29)
\end{aligned}$$

Here, η_1 gives the positive real solution and η_2 and η_3 are the complex conjugate solutions or the negative real solutions of Eq. (4.22), depending on the value of d.

For an initially-stressed incompressible half-space in its deformed configuration subject to a homogeneous deformation, let $\lambda_1, \lambda_2, \lambda_3$ be the principal stretches corresponding to the principal axes x_1, x_2 and x_3 respectively. In particular, if the underlying deformation of the half-space corresponds to plane strain then $\lambda_3 = 1$ and we write $\lambda_1 = \lambda$ and $\lambda_2 = \lambda^{-1}$. We assume $\tau_{12} = 0$ and τ_{11} and τ_{22} are the principal initial stresses.

For the special model given by Eq. (3.111) and assuming $\tau_{22} \equiv 0$, we can re-write the values of $\beta_1, 2\beta_2$ and β_5 from Eq. (3.237) as

$$\beta_1/\mu = \lambda^2\epsilon, \quad 2\beta_2/\mu = \lambda^2\epsilon + \lambda^{-2} + 4b_0\lambda^4\bar{\tau}^2, \quad \beta_5/\mu = \lambda^{-2}, \qquad (4.30)$$

where

$$b_0 = \mu\bar{\mu}, \quad \bar{\tau} = \tau_{11}/\mu, \quad \epsilon = 2b_0(\lambda^2 - 1)\bar{\tau}^2 + \bar{\tau} + 1. \tag{4.31}$$

Also, d in Eq. (4.22) is given by

$$d = 3 + 4b_0\bar{\tau}^2\lambda^6. \tag{4.32}$$

In this case, through Eq. (4.20), Eq. (4.32) is subject to

$$0 \leq \eta \leq \lambda^2\sqrt{\epsilon}. \tag{4.33}$$

For a real upper bound of η, we must have

$$\epsilon = 2b_0(\lambda^2 - 1)\bar{\tau}^2 + \bar{\tau} + 1 > 0, \tag{4.34}$$

which holds $\forall\bar{\tau}$ when $8b_0(\lambda^2 - 1) > 1$. For $8b_0(\lambda^2 - 1) < 1$ (Eq. (3.252) re-written), we have

$$\frac{-1 + \sqrt{1 - 8b_0(\lambda^2 - 1)}}{4b_0(\lambda^2 - 1)} < \bar{\tau} < \frac{-1 - \sqrt{1 - 8b_0(\lambda^2 - 1)}}{4b_0(\lambda^2 - 1)}. \tag{4.35}$$

It may be noted that inequality (4.34) entails the strong ellipticity condition $(3.179)_1$.

In terms of η, we can re-write Eq. (4.19) as

$$\xi = \lambda^2\epsilon - \lambda^{-2}\eta^2, \tag{4.36}$$

where we have defined $\xi = \rho c^2/\mu$. The underlying deformation is stable for $\lambda^4\epsilon > \eta^2$.

The relation of plane shear wave speed with the initial stress is given by

$$\xi_s = \lambda^2\epsilon. \tag{4.37}$$

Using the solution from Eq. (4.27) in Eq. (4.36), Fig. 4.1 shows the behaviour of ξ with respect to λ for various choices of b_0 and $\bar{\tau}$. The dashed graph here represents the shear wave velocity for zero, a positive and a negative value of the initial stress. From Fig. 4.1, when the initial stress vanishes in Eq. (4.36) (represented by the graph labelled as 'a'), the surface wave speed ξ approaches the plane shear wave ξ_s with increasing stretch. From Fig. 4.1, we can see that under the effect of the compressional initial stress the surface wave speed decreases, whereas for tensile initial stress, it increases with increasing stretch and also, it is obvious that a similar behaviour is observed for the shear wave velocity in a initially- stressed material. Also, when $\xi = 0$ or $\eta = \sqrt{\beta_1/\beta_5}$, from Eq. (4.36), the value of λ is

$$\lambda = \sqrt{\eta}/\epsilon^{1/4} \tag{4.38}$$

This value of λ gives the point of instability for particular choice of $\bar{\tau}$.

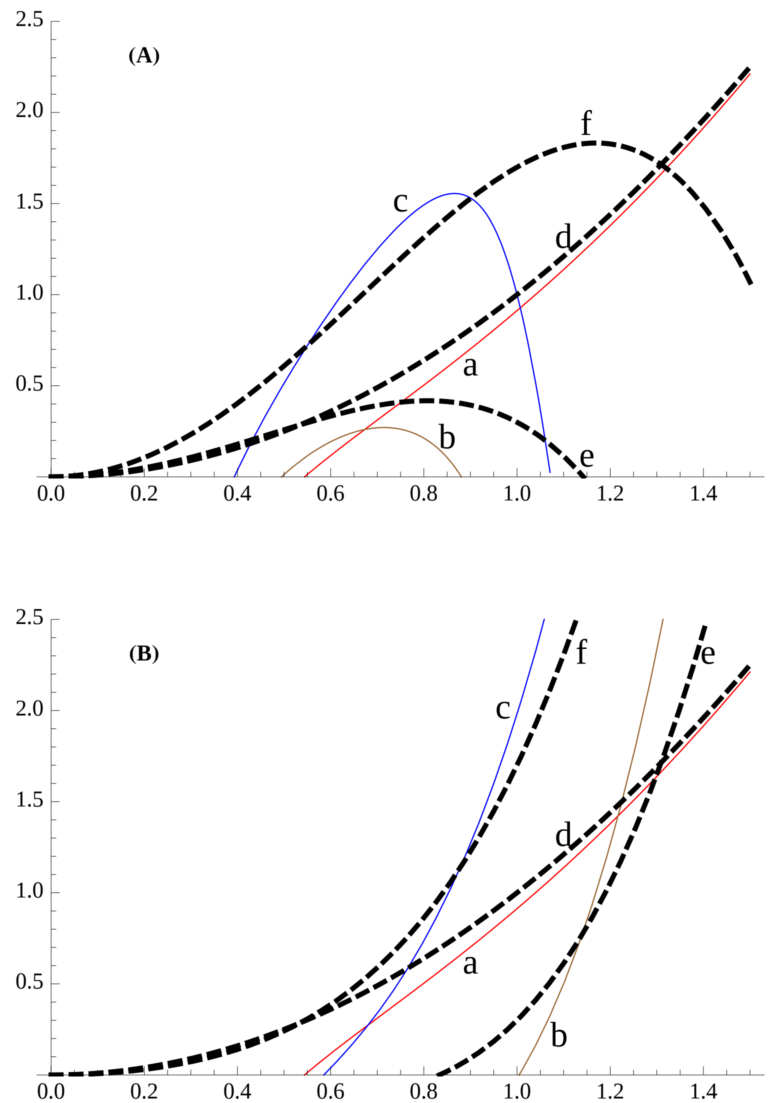
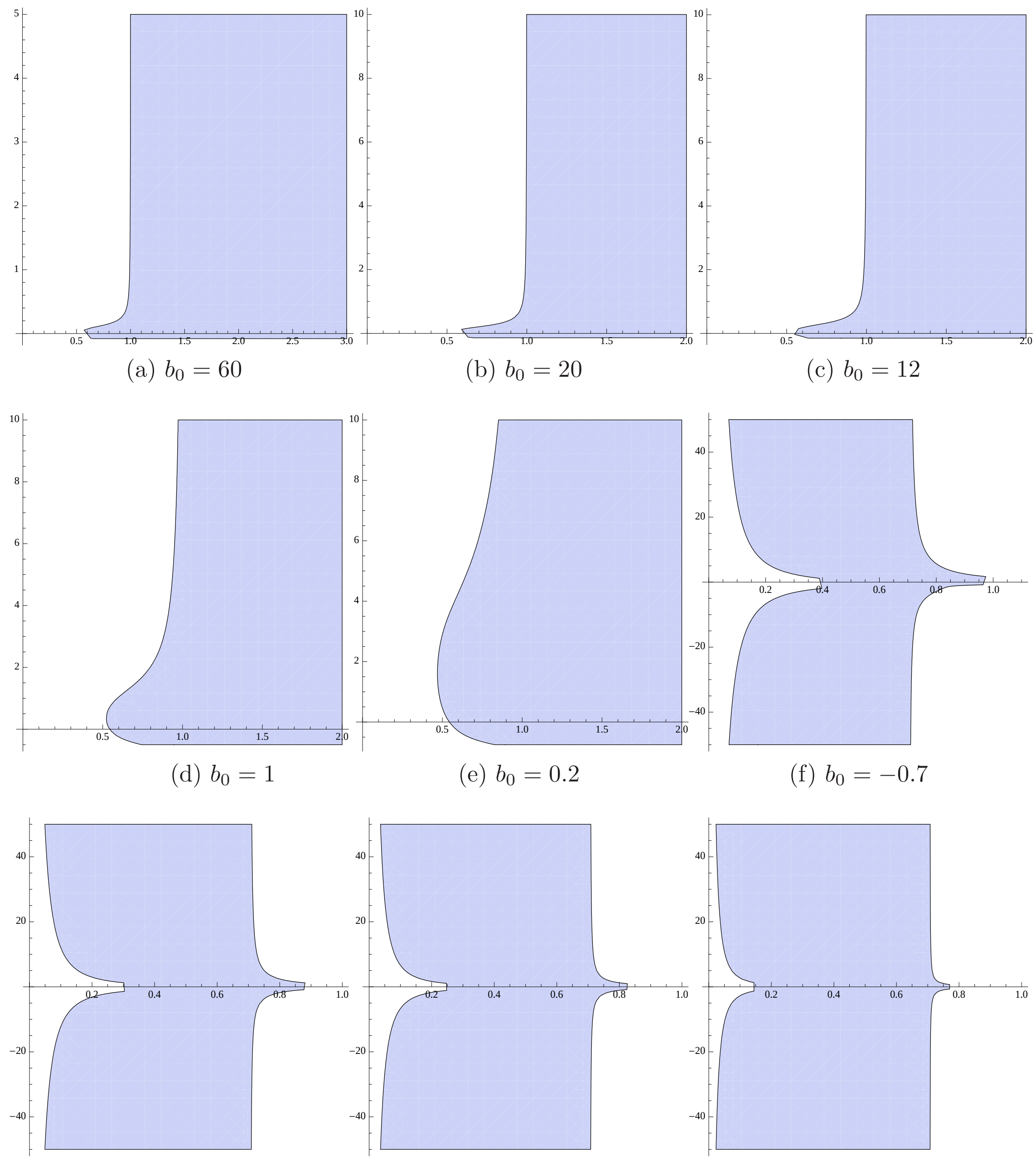


Figure 4.1: ξ (Vertical axis) with respect to λ. (A) For $b_0 = -1$, ξ (continuous graph) from Eq. (4.36), $\bar{\tau} =$ (a) 0, (b) -1, (c) 1, ξ_s (dashed graph) from Eq. (4.37) for $\bar{\tau} =$ (d) 0, (e) -0.7, (f) 0.7, (B) For $b_0 = 1$, ξ (continuous graph) from Eq. (4.36), $\bar{\tau} =$ (a) 0, (b) -1, (c) 1, ξ_s (dashed graph) from Eq. (4.37) for $\bar{\tau} =$ (d) 0, (e) -0.7, (f) 0.7

For the particular strain-energy function under consideration, the above inequalities



(a) $b_0 = 60$ (b) $b_0 = 20$ (c) $b_0 = 12$

(d) $b_0 = 1$ (e) $b_0 = 0.2$ (f) $b_0 = -0.7$

(g) $b_0 = -3$ (h) $b_0 = -10$ (i) $b_0 = -60$

Figure 4.2: Plot of $\bar{\tau}$ (Vertical axis) with respect to λ from the inequality (4.40). The shaded graph is the stability region for different values of b_0

(4.24) and (4.26) reduce to

$$\mu\lambda^{-2} > 0, \tag{4.39}$$

$$\lambda^2\epsilon - \lambda^{-2} + \lambda^2\epsilon^{\frac{1}{2}}[\lambda^2(\epsilon + 4b_0\lambda^2\bar{\tau}^2) + 3\lambda^{-2}] > 0. \tag{4.40}$$

The inequalities (4.39) and (4.40) along with $\epsilon > 0$ imply that the (two-dimensional) strong ellipticity condition holds. The stability regions for different values of b_0 are shown in Fig. 4.2 following the inequality (4.40).

Using Eq. (4.13), we can rewrite the secular equation in its non-dimensionalized form

$$g(\xi) = \xi^3 - p_1\xi^2 + q_1\xi - r_1 = 0, \tag{4.41}$$

and from Eqs. (4.14)–(4.16), we have p, q and r in the non-dimensionalized form

$$p_1 = p/\mu = 3\lambda^2\epsilon + 5\lambda^{-2} + 8b_0\lambda^4\bar{\tau}^2, \tag{4.42}$$

$$\begin{aligned} q_1 = q/\mu^2 &= [\lambda^2\epsilon + 3\lambda^{-2} + 4b_0\lambda^4\bar{\tau}^2]^2 + 2\lambda^2\epsilon[\lambda^2\epsilon + 3\lambda^{-2} + 4b_0\lambda^4\bar{\tau}^2] \\ &+ 2\lambda^{-4} - 2\epsilon, \end{aligned} \tag{4.43}$$

$$r_1 = r/\mu^3 = \lambda^2\epsilon[\lambda^2\epsilon + 3\lambda^{-2} + 4b_0\lambda^4\bar{\tau}^2]^2 - \lambda^{-2}(\lambda^{-2} - \lambda^2\epsilon)^2. \tag{4.44}$$

The behaviour of ξ as a function of $\bar{\tau}$, from Eq. (4.41), is illustrated in Fig. 4.3 for different choices of λ and b_0. Figure 4.3 shows that for various choices of λ and b_0, the squaring process may introduce solutions of Eq. (4.41) which are not solutions of Eq. (4.22). For example, the arrows in Figs. 4.3a and 4.3b refer to the extra values of $\bar{\tau}$ for which $\xi = 0$. The dashed graph represents the second factor in Eq. (4.17) which in this case is given by

$$r_1'(\bar{\tau}) = \epsilon[\lambda^2\epsilon + 3\lambda^{-2} + 4b_0\lambda^4\bar{\tau}^2] - \lambda^{-4} + \epsilon. \tag{4.45}$$

Also, it is obvious from Fig. 4.3 that the solutions are unaltered for $b_0 > 0$ and the solid graph ‘b’ and dashed graphs ‘c’ coincide.

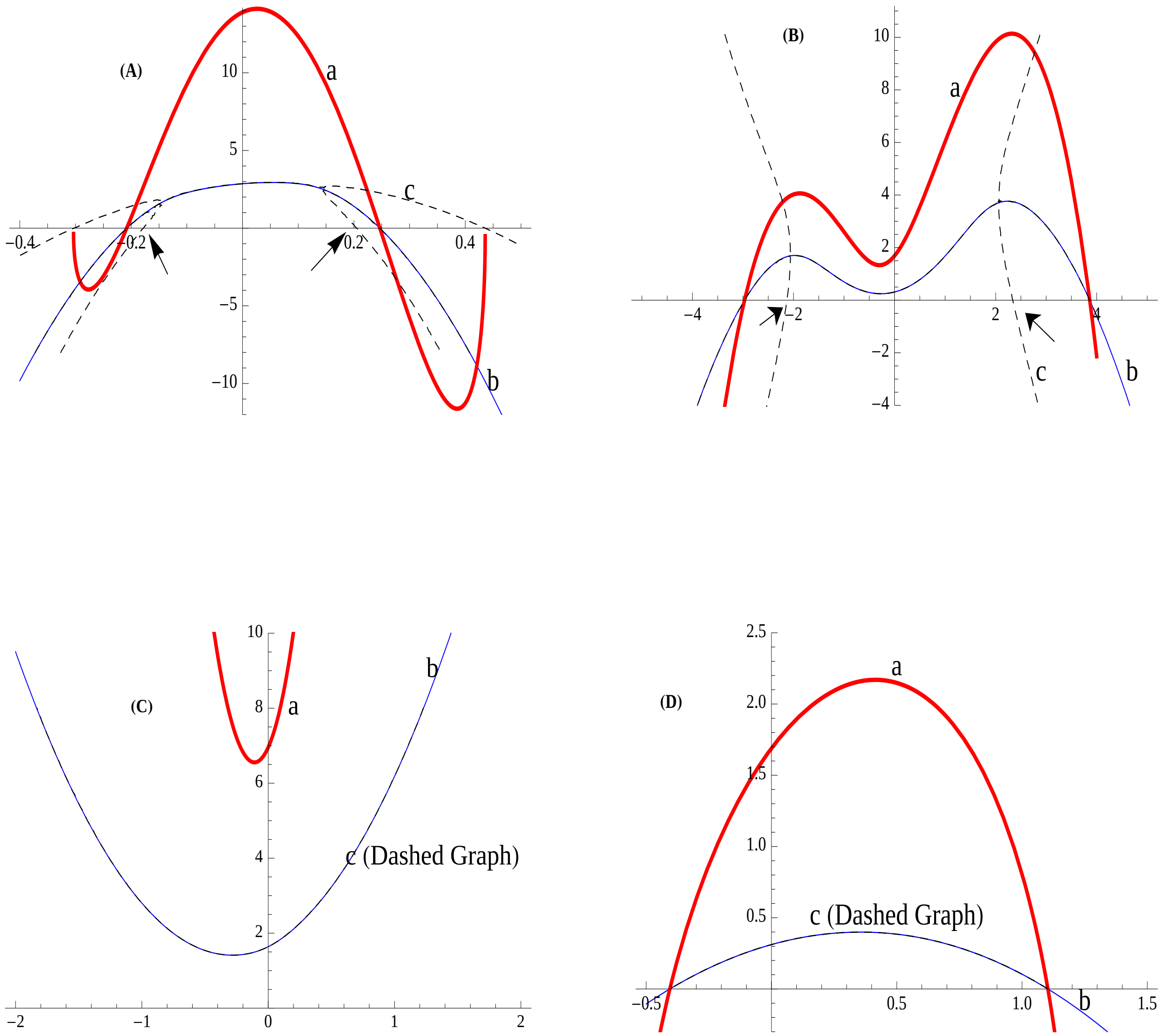


Figure 4.3: (a) r_1' (vertical) with respect to $\bar{\tau}$ (horizontal) from Eq. (4.45), (b) ξ (vertical) with respect to $\bar{\tau}$ (horizontal) from Eq. (4.36), (c) Dashed Graph: ξ (vertical) with respect to $\bar{\tau}$ (horizontal) from Eq. (4.41), for various choices of λ and b_0, (A) $\lambda = 1.7, b_0 = -2$, (B) $\lambda = 0.7, b_0 = -2$, (C) $\lambda = 1.3, b_0 = 1.2$, (D) $\lambda = 0.7, b_0 = 1.5$. For $b_0 > 0$ in (C) and (D), the solid graph 'b' and dashed graph 'c' coincide.

4.2.2 Analysis of the Secular Equation in the Reference Configuration

In the undeformed but initially-stressed reference configuration we have $\mathbf{F} = \mathbf{I}$ or $\lambda_1 = \lambda_2 = 1$. We assume $\tau_{12} = 0$ and τ_{11} and τ_{22} are the principal initial stresses. For the special model given by Eq. (3.111), the values of $\beta_1, 2\beta_2$ and β_5 are given by Eqs. (3.273)–(3.275). If we further assume $\tau_{22} \equiv 0$, d in Eq. (4.22) is given by

$$d = 3 + 4b_0\bar{\tau}^2. \tag{4.46}$$

In this case, through Eq. (4.20), Eq. (4.22) is subject to

$$0 \leq \eta \leq \sqrt{1 + \bar{\tau}}. \tag{4.47}$$

For a real upper bound of η, we must have

$$\bar{\tau} > -1. \tag{4.48}$$

This entails the strong ellipticity condition $(3.277)_1$ when evaluated in the reference configuration.

In the classical limit when the initial stress vanishes, i.e. when $\bar{\tau} = 0$, Eq. (4.22) reduces to

$$\eta^3 + \eta^2 + 3\eta - 1 = 0, \tag{4.49}$$

subject to

$$0 \leq \eta \leq 1. \tag{4.50}$$

The real root of Eq. (4.49) lies at $\eta \approx 0.2956$, which by Eq. (4.19) leads to the approximate value of 0.9162 for $\rho c^2/\mu$. This agrees with the classical case of Rayleigh surface wave in incompressible isotropic materials (see, e.g., [10]).

For the considered special model, we must have from inequalities (4.24) and (4.26)

$$\mu > 0, \tag{4.51}$$

$$\bar{\tau} + (1 + \bar{\tau})^{1/2}(4 + \bar{\tau} + 4b_0\bar{\tau}^2) > 0. \tag{4.52}$$

The inequalities (4.51) and (4.52) along with $\bar{\tau} > -1$ imply that the (two dimensional) strong ellipticity condition holds. Figure 4.4 shows the stability region $(b_0, \bar{\tau})$.

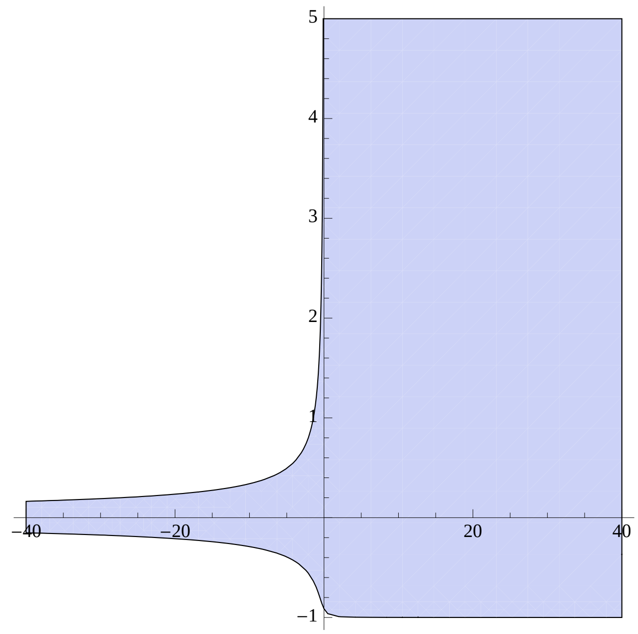


Figure 4.4: Plot of $\bar{\tau}$ (vertical axis) with respect to b_0 (horizontal axis) from Eq. (4.52). The shaded part shows the stability region for $(b_0, \bar{\tau})$

Specialising the discussion in Section 4.2.1 for $\lambda = 1$, for the special material under consideration, a unique real positive root, $\eta_0(\bar{\tau})$, for the secular equation (4.22) exists. The wave speed in this case relates to the initial stress through

$$\xi = \rho c^2/\mu = \bar{\tau} + (1 - \eta_0^2), \tag{4.53}$$

where for positive real values of ξ, we require $\bar{\tau} \geq -(1 - \eta_0^2)$. It is therefore ensured that a unique surface wave exists when the surface $x_2 = 0$ is free of traction. The exact expression for this solution is obtained by using the appropriate value of d, given by Eq. (4.46), in Eq. (4.27). The normalized plane shear wave speed ξ_s in this case is given by

$$\xi_s = \bar{\tau} + 1. \tag{4.54}$$

For Eqs. (4.53) and (4.54), we refer to Fig. 4.5. The point of instability i.e., when $\xi = 0$, is given by

$$\bar{\tau} = \eta_0^2 - 1, \tag{4.55}$$

whereas for shear wave ξ_s the corresponding value is $\bar{\tau} = -1$. From Fig. 4.5, we observe that for positive values of b_0, ξ behaves almost in the same manner as ξ_s. However, in the case when $b_0 < 0$, the behaviour is similar for some values of $\bar{\tau}$. Further, unlike the almost linear behaviour of ξ for $b_0 > 0$, the plot is non-linear and shows the two solutions where the wave speed vanishes.

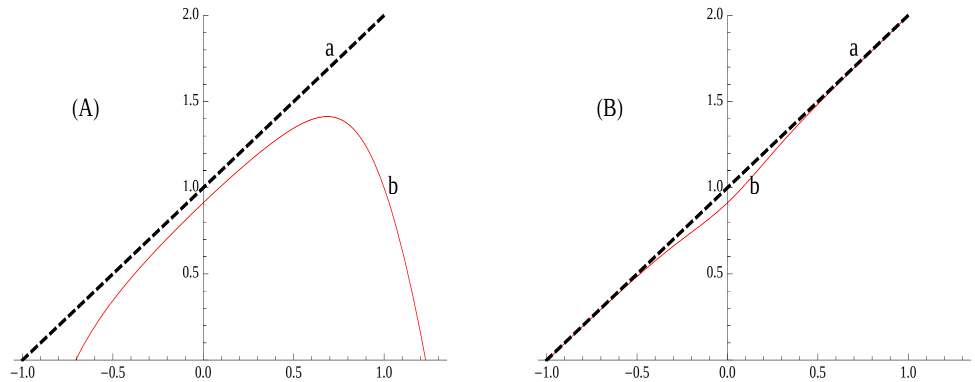


Figure 4.5: (A) for $b_0 = -1$: (a) Dashed Graph: ξ_s (vertical) from Eq. (4.54) with respect to $\bar{\tau}$, (b) Continuous Graph: ξ (vertical) with respect to $\bar{\tau}$ (horizontal) from Eq. (4.53), (B) for $b_0 = 5$: (a) Dashed Graph: ξ_s (vertical) from Eq. (4.54) with respect to $\bar{\tau}$, (b) Continuous Graph: ξ (vertical) with respect to $\bar{\tau}$ (horizontal) from Eq. (4.53)

For the same special model given by Eq. (3.111), using Eqs. (4.13)–(4.16), we can rewrite the secular equation in its non-dimensionalised form as

$$g(\xi) = \xi^3 - p_2\xi^2 + q_2\xi - r_2 = 0, \tag{4.56}$$

where

$$p_2 = p/\mu = 8b_0\bar{\tau}^2 + 3\bar{\tau} + 8, \tag{4.57}$$

$$q_2 = q/\mu^2 = [4b_0\bar{\tau}^2 + \bar{\tau} + 4]^2 + 2(\bar{\tau} + 1)[4b_0\bar{\tau}^2 + \bar{\tau} + 4] - 2\bar{\tau}, \tag{4.58}$$

$$r_2 = r/\mu^3 = (\bar{\tau} + 1)[4b_0\bar{\tau}^2 + \bar{\tau} + 4]^2 - \bar{\tau}^2. \tag{4.59}$$

The solutions of Eqs. (4.22) and (4.56) are such that $\xi = 1 + \bar{\tau} - \eta_0^2$. In the instance when the initial stress vanishes, i.e. $\bar{\tau} = 0$, Eq. (4.56) takes the form

$$\xi^3 - 8\xi^2 + 24\xi - 16 = 0, \tag{4.60}$$

and has the unique solution occurring at $\xi_0 = 0.9126$ mentioned above.

The behaviour of ξ as a function of $\bar{\tau}$ from Eq. (4.56) is illustrated in Fig. 4.6 for various values of b_0. From Fig. 4.6b, we can see various values of $\bar{\tau}$ which give $\xi = 0$. However, some solutions of Eq. (4.56) do not satisfy Eq. (4.22) as a result of the squaring process carried out to obtain Eq. (4.56). The arrows show the extra solutions in Fig. 4.6b for $b_0 = -3$. Also, from Eq. (4.17), in this case, we have

$$r_2'(\bar{\tau}) = \epsilon[4 + \epsilon + 4b_0\bar{\tau}^2] - 1, \tag{4.61}$$

where $\epsilon = 1 + \bar{\tau}$ in the reference configuration.

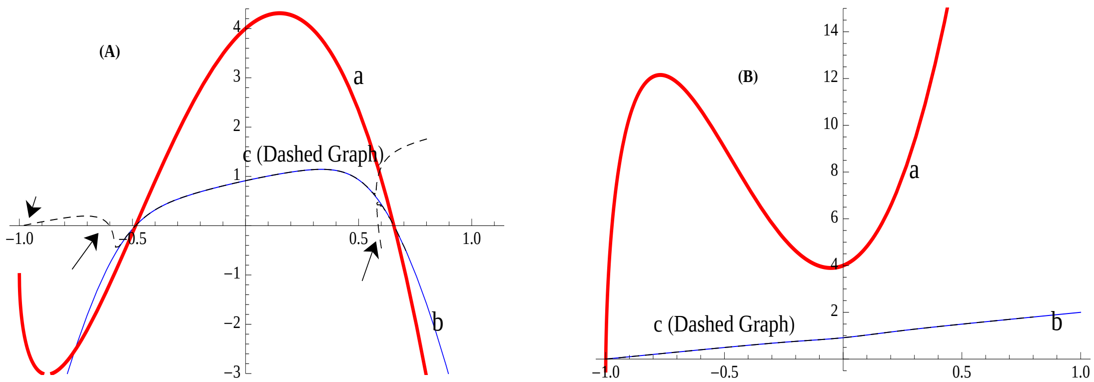


Figure 4.6: (a) r_2' (vertical) with respect to $\bar{\tau}$ (horizontal) from Eq. (4.61), (b) ξ (vertical) with respect to $\bar{\tau}$ (horizontal) from Eq. (4.56), (c) Dashed Graph: ξ (vertical) with respect to $\bar{\tau}$ (horizontal) from Eq. (4.53), for (A) $b_0 = -3$, (B) $b_0 = 10$. For $b_0 > 0$ in (B), the solid graph 'b' and dashed graph 'c' coincide.

4.3 Love Waves

It was first shown by Love, (see, for example, [2]) that surface SH waves are possible if the half-space is covered by a layer of a different material. He suspected that such waves were a consequence of formation of layers in the earth, and that SH waves were trapped in a superficial layer and propagated by multiple reflections within the layer.

In this section, we consider two different initially-stressed materials for the layer and the half-space. The half-space is defined by $x_2 < 0$ and the layer, of thickness h, has boundaries $x_2 = 0$ and $x_2 = h$. Let B denote the deformed half-space in the region $x_2 < 0$ and B^* the deformed layer (see Fig. 4.7). The deformation in both the half-space and layer is homogeneous plane strain with principal axes aligned with the co-ordinate axes. The quantities in the layer are specified by a superscript '$*$'. Let the principal stretches of the deformations in the half-space and layer be denoted $\lambda_1, \lambda_2, \lambda_3$ and $\lambda_1^*, \lambda_2^*, \lambda_3^*$, respectively. The initial stress tensors are denoted $\boldsymbol{\tau}$ and $\boldsymbol{\tau}^*$ in the half-space and the layer, respectively. The normal and shear initial stress components in the half-space (layer) are τ_{ii} and $\tau_{ij}(\tau_{ii}^*$

and τ^*_{ij}) for $i, j \in \{1, 2, 3\}$ and $i \neq j$, respectively. We assume $\tau_{ij} = 0 = \tau^*_{ij}, i \neq j$.

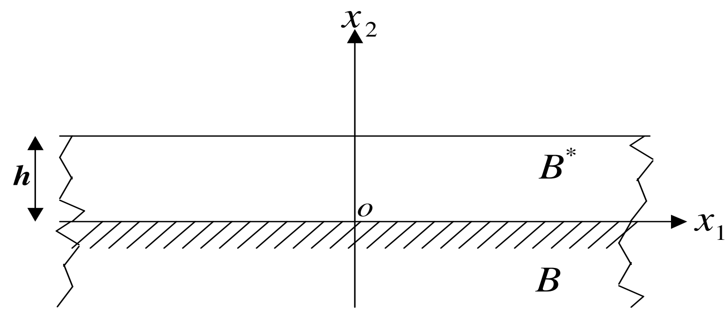


Figure 4.7: A initially-stressed deformed layer (B) of thickness h over a initially-stressed deformed half space (B^*).

Let the material response of B be specified by the strain-energy function per unit volume, $W(\lambda_i, \tau_{ii})$, and that of B^* by $W^*(\lambda^*_i, \tau^*_{ii})$. Following Eqs. (3.138)–(3.143), the principal Cauchy stresses t_i, t^*_i are given by

$$t_i = \lambda_i \frac{\partial W}{\partial \lambda_i} - p, \quad t^*_i = \lambda^*_i \frac{\partial W^*}{\partial \lambda^*_i} - p^*, \quad i \in \{1, 2, 3\}, \tag{4.62}$$

where p and p^* denoted the Lagrange multipliers associated with the incompressibility constraints given by

$$\lambda_1 \lambda_2 \lambda_3 = 1, \quad \lambda^*_1 \lambda^*_2 \lambda^*_3 = 1, \tag{4.63}$$

in the half-space and the layer respectively.

The plane strain deformation allows us to choose $\lambda_3 = 1$ in the half-space and $\lambda^*_3 = 1$ in the layer. From Eq. (4.63), we therefore have

$$\lambda_1 = \lambda_1, \quad \lambda_2 = \lambda_1^{-1} \quad \text{and} \quad \lambda^*_1 = \lambda^*_1, \quad \lambda^*_2 = (\lambda^*_1)^{-1}, \tag{4.64}$$

in the half-space and the layer, respectively.

The equations of incremental motions for $x_2 < 0$ and $0 < x_2 < h$ are, respectively, given by

$$\mathcal{A}_{0piqj} u_{j,pq} - \dot{p}_{,i} = \rho u_{i,tt}, \tag{4.65}$$

$$\mathcal{A}^*_{0piqj} u^*_{j,pq} - \dot{p}^*_{,i} = \rho^* u^*_{i,tt}, \tag{4.66}$$

where $\mathcal{A}_{0piqj}$ are the elastic moduli given by Eq. (3.33) with $\Sigma_{ij} = 0 = \Sigma^*_{ij}$ for $i \neq j$ and $i, j \in \{1, 2, 3\}$ which follows from the assumption that the shear components of the initial stress vanish for the considered homogenous strain. u_i are the components of the incremental displacement and $\dot{p}$, the corresponding increment in p and ρ (ρ^*) is the density of the material in the half-space (layer). Similarly, for the other material in $0 < x_2 < h$.

We now consider a plane harmonic SH wave, propagating along the x_1 principal direction with displacement in the x_3 direction. The incremental displacements $\mathbf{u}$ and $\mathbf{u}^*$ are such that $\mathbf{u} = (0, 0, u_3)$ and $\mathbf{u}^* = (0, 0, u_3^*)$. We suppose a wave of the form

$$u_3 = A \exp[skx_2] \exp[ik(x_1 - ct)], \tag{4.67}$$

$$u_3^* = [A_1 \cos s^*kx_2 + A_2 \sin s^*kx_2] \exp[ik(x_1 - ct)], \tag{4.68}$$

where $k = \omega/c$ is the wave number, ω is the angular frequency, c is the wave speed and s and s^* are to be determined by substitution of u_3 and u_3^* in the appropriate equations of motion. From Eqs. (4.65) and (4.66), we get $\dot{p}_{,1} = \dot{p}_{,2} = \dot{p}^*_{,1} = \dot{p}^*_{,2} = 0$ for $i = 1, 2$, and for $i = 3$

$$\begin{aligned} \mathcal{A}_{01313} u_{3,11} + \mathcal{A}_{02323} u_{3,22} &= \rho u_{3,tt}, && (4.69) \\ \mathcal{A}^*_{01313} u^*_{3,11} + \mathcal{A}^*_{02323} u_{3,22} &= \rho^* u^*_{3,tt}. && (4.70) \end{aligned}$$

On substitution of Eq. (4.67) in Eq. (4.69), we get

$$s^2 = \frac{\mathcal{A}_{01313} - \rho c^2}{\mathcal{A}_{02323}}. \tag{4.71}$$

Similarly, for the layer, Eqs. (4.68) and (4.70) lead to

$$s^{*2} = \frac{\rho^* c^2 - \mathcal{A}^*_{01313}}{\mathcal{A}^*_{02323}}. \tag{4.72}$$

Following the notation defined by [8], we can write

$$\rho c_{ij}^2 = \mathcal{A}_{0ijij}, \quad \rho^* c_{ij}^{*2} = \mathcal{A}^*_{0ijij}, \tag{4.73}$$

where c_{ij} is speed of a plane shear body wave propagating in the x_i direction with displacement in the x_j direction. Therefore, from Eqs. (4.71) and (4.72), we obtain

$$s^2 = \frac{c_{13}^2 - c^2}{c_{23}^2}, \quad s^{*2} = \frac{c^2 - c_{13}^{*2}}{c_{23}^{*2}}. \tag{4.74}$$

It follows from Eq. (2.127) that the incremental nominal stress tensor has components in the half-space and the layer

$$\dot{S}_{0pi} = \mathcal{A}_{0piqj}u_{j,q} - \dot{p}\delta_{pi} + pu_{p,i}, \tag{4.75}$$

$$\dot{S}^*_{0pi} = \mathcal{A}^*_{0piqj}u^*_{j,q} - \dot{p}^*\delta_{pi} + p^*u^*_{p,i}, \tag{4.76}$$

respectively.

The boundary conditions for the considered problem are given by the vanishing of the shear stress at the free surface $x_2 = h$ and continuity of the shear stress and the displacement at the interface $x_2 = 0$. Namely,

$$\dot{S}^*_{023} = 0 \quad \text{on} \quad x_2 = h, \tag{4.77}$$

$$u_3 = u^*_3, \quad \dot{S}_{023} = \dot{S}^*_{023}, \quad \text{on} \quad x_2 = 0. \tag{4.78}$$

Using Eq. (4.76) in Eq. (4.77), we get

$$A_1 \sin s^*kh - A_2 \cos s^*kh = 0, \tag{4.79}$$

and from Eqs. (4.67), (4.68), (4.75) and (4.76) in Eq. (4.78), we have

$$A = A_1, \quad s\rho c_{23}^2 A = s^*\rho^* c_{23}^{*2} A_2. \tag{4.80}$$

We require $s > 0$ for the wave to decay in the half-space as $x_2 \to -\infty$ and therefore, from Eq. (4.73), we have

$$c^2 < c_{13}^2. \tag{4.81}$$

However, s^{*2} may either be positive or negative. If $s^{*2} > 0$ then Eqs. (4.79) and (4.80) yield the secular equation in the form

$$\tan s^*kh = \frac{s\rho c_{23}^2}{s^*\rho^* c_{23}^{*2}}, \tag{4.82}$$

subject to

$$c_{13}^{*2} < c^2 < c_{13}^2. \tag{4.83}$$

On the other hand if $s^{*2} < 0$ then Eq. (4.82) is replaced by

$$\tanh|s^*|kh = \frac{s\rho c_{23}^2}{|s^*|\rho^* c_{23}^{*2}}, \tag{4.84}$$

subject to

$$c^2 < \min\{c_{13}^2, c_{13}^{*2}\}. \tag{4.85}$$

Using Eq. (4.74) in Eq. (4.82), we get

$$\tan(kh\sqrt{\frac{c^2/c_{13}^{*2}-1}{c_{23}^{*2}/c_{13}^{*2}}}) = \frac{\rho c_{23}c_{13}\sqrt{1-c^2/c_{13}^2}}{\rho^* c_{23}^* c_{13}^* \sqrt{c^2/c_{13}^{*2}-1}}, \tag{4.86}$$

subject to Eq. (4.83). If we take $\rho c_{23}^2 = \rho c_{13}^2 = \rho c_T^2 = \mu$ and $\rho^* c_{23}^{*2} = \rho^* c_{13}^{*2} = \rho^* c_T^{*2} = \mu^*$, the above equation reduces to the classical form given by [2]. Here c_T and c_T^* are the classical transverse wave velocities in the half-space and the layer, respectively.

Similarly, Eq. (4.84) in its explicit form is given by

$$\tanh(kh\sqrt{\frac{1-c^2/c_{13}^{*2}}{c_{23}^{*2}/c_{13}^{*2}}}) = \frac{\rho c_{23}c_{13}\sqrt{1-c^2/c_{13}^2}}{\rho^* c_{23}^* c_{13}^* \sqrt{1-c^2/c_{13}^{*2}}}, \tag{4.87}$$

subject to Eq. (4.85).

By using the relation $\omega = ck$, Eq. (4.86) (or Eq. (4.87)) may be expressed in terms of frequency and wave number. It is thus obvious that Love waves are dispersive. That is, for various successive values of k, the roots of Eq. (4.86) (or Eq. (4.87)) will result in $c = c(k)$ or in $\omega = \omega(k)$. The tangent function gives multiple branches and therefore suggests that multiple roots will exist for a given value of k. Thus, the dispersion curve and frequency spectrum should have multiple branches, corresponding to various modes of propagation. It may be noted that $c \to c_{13}$, as $kh\sqrt{\frac{1-c^2/c_{13}^{*2}}{c_{23}^{*2}/c_{13}^{*2}}} \to n\pi$, where n is any natural number. This means as the wavelength decreases, the wave corresponds to a plane shear wave propagating in the half-space. Also, $c \to c_{13}^*$ as $kh \to \infty$, i.e. as the wavelength becomes large compared to the thickness of the layer the Love wave takes on the speed of the upper medium.

4.3.1 Analysis of the Dispersion Relation in the Deformed Configuration

Following the special model given by Eq. (3.111), we can write for the half-space in the deformed configuration

$$\rho c_{ij}^2 = \mu\lambda_i^2 + \lambda_i^2[2\bar{\mu}(\lambda_i^2 - 1)\tau_{ii} + 1]\tau_{ii}, \tag{4.88}$$

which reduces to the classical expression $\rho c_s^2/\mu = \lambda^2$ in the absence of initial stress i.e., when $\tau_{ii} = 0$. Here, c_s is the shear wave speed in the deformed configuration. Similarly, for the layer, we have

$$\rho^* c_{ij}^{*2} = \mu^*\lambda_i^{*2} + \lambda_i^{*2}[2\bar{\mu}^*(\lambda_i^{*2} - 1)\tau_{ii}^* + 1]\tau_{ii}^*. \tag{4.89}$$

Since we are considering the displacement in the x_3 direction, we have $j = 3$. For $i = 1, 2$, the above expressions in Eqs. (4.88) and (4.89) can be re-written as

$$\frac{\rho c_{i3}^2}{\mu} = \epsilon_i\lambda_i^2, \quad \frac{\rho^* c_{i3}^{*2}}{\mu^*} = \epsilon_i^*\lambda_i^{*2}, \tag{4.90}$$

where

$$\epsilon_i = 2b_0(\lambda_i^2 - 1)\bar{\tau}_i^2 + \bar{\tau}_i + 1, \quad \epsilon_i^* = 2b_0^*(\lambda_i^{*2} - 1)\bar{\tau}_i^{*2} + \bar{\tau}_i^* + 1, \tag{4.91}$$

and

$$\bar{\tau}_i = \frac{\tau_{ii}}{\mu}, \quad \bar{\tau}_i^* = \frac{\tau_{ii}^*}{\mu^*}, \quad b_0 = \mu\bar{\mu}, \quad b_0^* = \mu^*\bar{\mu}^*. \tag{4.92}$$

We therefore have from Eqs. (4.86), (4.88), (4.89) and (4.90)

$$\tan[kh\sqrt{\frac{\rho'\mu'\xi - \epsilon_1^*\lambda_1^{*2}}{\epsilon_2^*\lambda_2^{*2}}}] = \mu'\frac{\epsilon_2\lambda_2}{\epsilon_2^*\lambda_2^*}\sqrt{\frac{\epsilon_1\lambda_1^2 - \xi}{\rho'\mu'\xi - \epsilon_1^*\lambda_1^{*2}}}, \tag{4.93}$$

subject to

$$\frac{\epsilon_1^*\lambda_1^{*2}}{\rho'\mu'} < \xi < \epsilon_1\lambda_1^2, \tag{4.94}$$

where we have introduced the notations $\rho' = \rho^*/\rho$, $\mu' = \mu/\mu^*$ and $\xi = \rho c^2/\mu$. Also,

$$\epsilon_1 = 2b_0(\lambda_1^2 - 1)\bar{\tau}_1^2 + \bar{\tau}_1 + 1, \quad \epsilon_1^* = 2b_0^*(\lambda_1^{*2} - 1)(\bar{\tau}_1^*)^2 + \bar{\tau}_1^* + 1, \tag{4.95}$$

$$\epsilon_2 = 2b_0(\lambda_2^2 - 1)\bar{\tau}_2^2 + \bar{\tau}_2 + 1, \quad \epsilon_2^* = 2b_0^*(\lambda_2^{*2} - 1)(\bar{\tau}_2^*)^2 + \bar{\tau}_2^* + 1. \tag{4.96}$$

Since we have assumed $\tau_{22} \equiv 0 \equiv \tau_{22}^*$, it implies that $\epsilon_2 = 1 = \epsilon_2^*$. Equation (4.93) becomes

$$\tan[kh\sqrt{\frac{\rho'\mu'\xi - \epsilon_1^*\lambda_1^{*2}}{\lambda_2^{*2}}}] = \mu'\frac{\lambda_2}{\lambda_2^*}\sqrt{\frac{\epsilon_1\lambda_1^2 - \xi}{\rho'\mu'\xi - \epsilon_1^*\lambda_1^{*2}}}, \tag{4.97}$$

subject to the inequality (4.94). Figure 4.8 shows the lowest modes for ξ with respect to kh in Eq. (4.97) in the classical limit ($\bar{\tau}_1 = 0$ and all stretches equal to unity) subject to (4.94). See, for example, [2]. For various other choices of parameters, see Figs. 4.9–4.11.

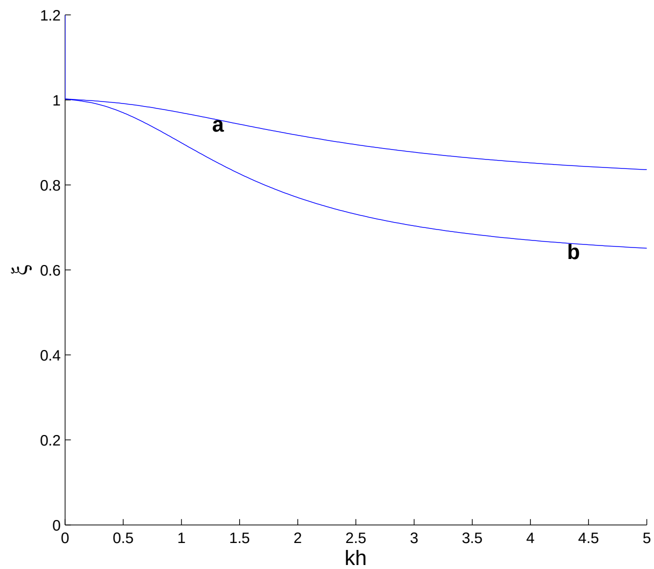


Figure 4.8: Plot of ξ for the lowest modes of Love waves from Eq. (4.97) for the classical limit $\lambda_1 = \lambda_2 = \lambda_1^* = \lambda_2^* = b_0 = b_0^* = 1$ and $\bar{\tau}_1 = 0 = \bar{\tau}_1^*$ (a) $\rho' = 0.9018, \mu' = 1.4$ (b) $\rho' = 0.9131, \mu' = 1.8$

From Figs. 4.9-4.11, it is obvious that wave speed increases in the presence of tensile initial stress ($\tau_1 > 0$) with increasing kh or equivalently decreasing wavelength. Whereas for compressional initial stress ($\tau_1 < 0$), the wave speed decreases.

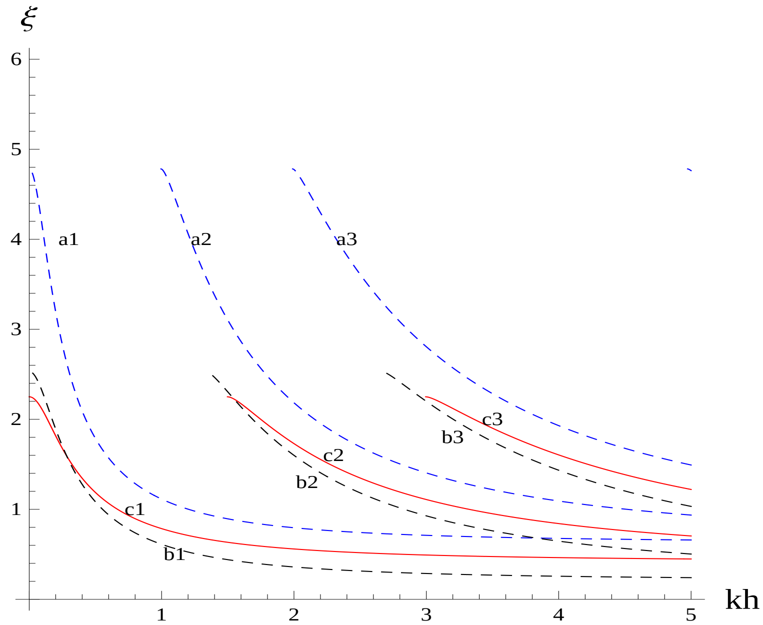


Figure 4.9: Plot of ξ for first three modes of Love waves from Eq. (4.97) for $\lambda_1 = 1.5, \lambda_2 = 1/1.5, \lambda_1^* = \lambda_2^* = b_0 = b_0^* = 1, \rho' = 1.5, \mu' = 1.6$. For a1, a2, a3 (representing first three modes, respectively), $\bar{\tau}_1 = 0.5 = \bar{\tau}_1^*$. For b1, b2, b3, $\bar{\tau}_1 = -0.5 = \bar{\tau}_1^*$. For c1, c2, c3, $\bar{\tau}_1 = 0 = \bar{\tau}_1^*$ (continuous graphs)

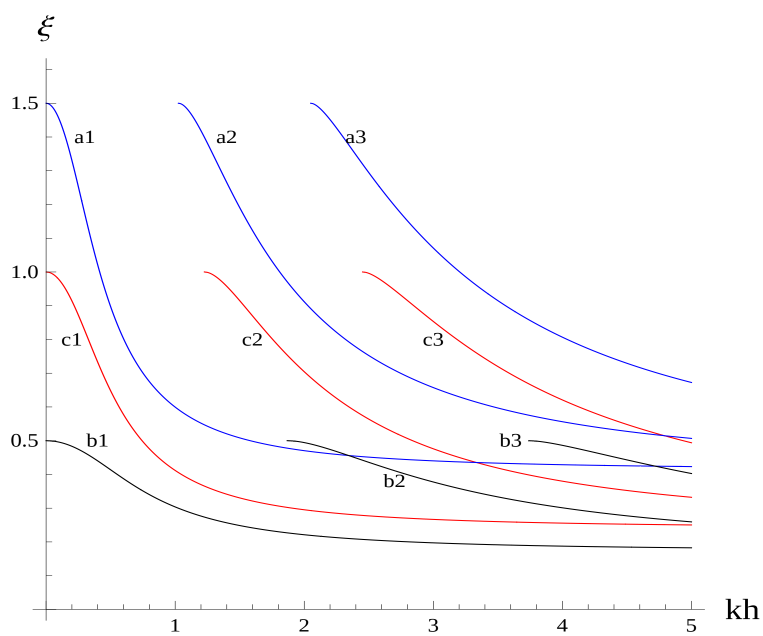


Figure 4.10: Plot of ξ for first three modes of Love waves from Eq. (4.97) for $\lambda_1 = 1 = \lambda_2, \lambda_1^* = 1.2, \lambda_2^* 1/1.2, b_0 = b_0^* = 1, \rho' = 2, \mu' = 3$. For a1, a2, a3 (representing first three modes, respectively), $\bar{\tau}_1 = 0.5 = \bar{\tau}_1^*$. For b1, b2, b3, $\bar{\tau}_1 = -0.5 = \bar{\tau}_1^*$. For c1, c2, c3, $\bar{\tau}_1 = 0 = \bar{\tau}_1^*$

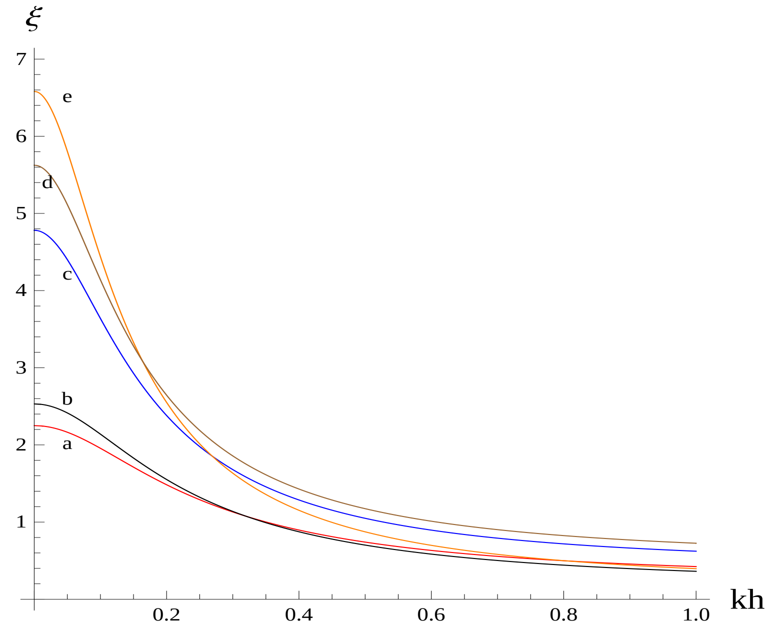


Figure 4.11: Plot of ξ for the lowest mode of Love waves from Eq. (4.97) for $\lambda_1 = 1.5, \lambda_2 = 1/1.5, \lambda_1^* = 1.2, \lambda_2^* = 1/1.2, b_0 = b_0^* = 1, \rho' = 2, \mu' = 3$ and (a) $\bar{\tau}_1 = 0 = \bar{\tau}_1^*$, (b) $\bar{\tau}_1 = -0.5 = \bar{\tau}_1^*$, (c) $\bar{\tau}_1 = 0.5 = \bar{\tau}_1^*$, (d) $\bar{\tau}_1 = -1, \bar{\tau}_1^* = 0.7$, (e) $\bar{\tau}_1 = 0.7, \bar{\tau}_1^* = -0.7$

If the initial stress vanishes, i.e. $\bar{\tau}_i = 0$ and $\lambda_1 = \lambda_2 = \lambda_3 = 1$ and $\lambda_1^* = \lambda_2^* = \lambda_3^* = 1$, Eq. (4.93) reduces to

$$\tan(kh\sqrt{\rho'\mu'\xi - 1}) - \mu'\frac{\sqrt{1-\xi}}{\sqrt{\rho'\mu'\xi - 1}} = 0, \tag{4.98}$$

which is the well-known dispersion relation for Love waves in the linear theory. See, for example, [2].

Similarly, Eq. (4.87) gives

$$\tanh[kh\sqrt{\frac{\epsilon_1^*\lambda_1^{*2} - \rho'\mu'\xi}{\epsilon_2^*\lambda_2^{*2}}}] = \mu'\frac{\epsilon_2\lambda_2}{\epsilon_2^*\lambda_2^*}\sqrt{\frac{\epsilon_1\lambda_1^2 - \xi}{\epsilon_1^*\lambda_1^{*2} - \rho'\mu'\xi}}, \tag{4.99}$$

subject to

$$\xi < \min\{\epsilon_1\lambda_1^2, \frac{\epsilon_1^*\lambda_1^{*2}}{\rho'\mu'}\}. \tag{4.100}$$

Since $\tau_{22} \equiv 0 \equiv \tau_{22}^*$, we have from Eq. (4.99)

$$\tanh[kh\sqrt{\frac{\epsilon_1^*\lambda_1^{*2} - \rho'\mu'\xi}{\lambda_2^{*2}}}] = \mu'\frac{\lambda_2}{\lambda_2^*}\sqrt{\frac{\epsilon_1\lambda_1^2 - \xi}{\epsilon_1^*\lambda_1^{*2} - \rho'\mu'\xi}}, \tag{4.101}$$

subject to the inequality (4.100). Figures 4.12–4.13 show plot of ξ with respect to kh for Eq.

(4.101) for various choices of pre-strain and initial stress subject to (4.100).

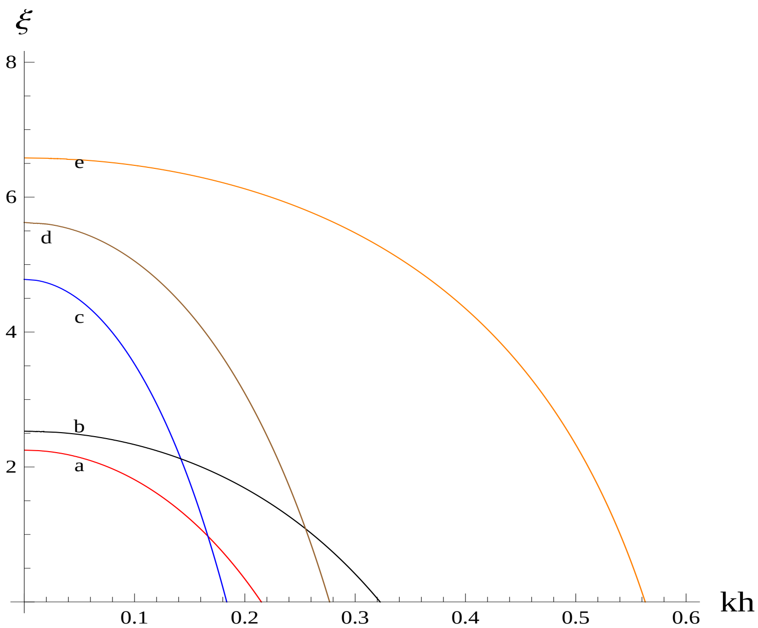


Figure 4.12: Plot of ξ with respect to kh from Eq. (4.101) for $\lambda_1 = 1.5, \lambda_2 = 1/1.5, \lambda_1^* = 1.2, \lambda_2^* = 1/1.2, b_0 = b_0^* = 1, \rho' = 2, \mu' = 3$ and (a) $\bar{\tau}_1 = 0 = \bar{\tau}_1^*$, (b) $\bar{\tau}_1 = -0.5 = \bar{\tau}_1^*$, (c) $\bar{\tau}_1 = 0.5 = \bar{\tau}_1^*$, (d) $\bar{\tau}_1 = -1, \bar{\tau}_1^* = 0.7$, (e) $\bar{\tau}_1 = 0.7, \bar{\tau}_1^* = -0.7$

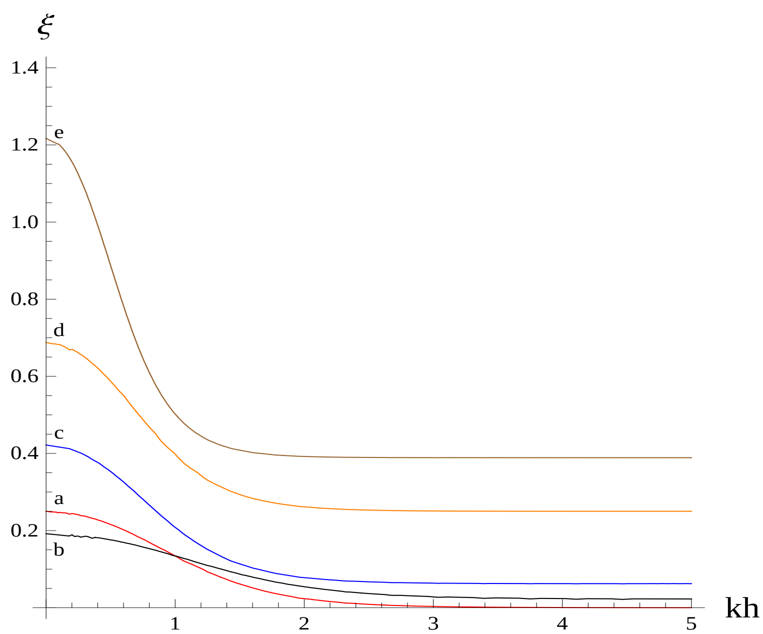


Figure 4.13: Plot of ξ with respect to kh from Eq. (4.101) for $\lambda_1 = 1/2, \lambda_2 = 2, \lambda_1^* = \lambda_2^* = \rho' = \mu' = 1, b_0 = -0.5, b_0^* = 2.5$ and (a) $\bar{\tau}_1 = 0 = \bar{\tau}_1^*$, (b) $\bar{\tau}_1 = -0.3 = \bar{\tau}_1^*$, (c) $\bar{\tau}_1 = 0.5 = \bar{\tau}_1^*$, (d) $\bar{\tau}_1 = 1 = \bar{\tau}_1^*$, (e) $\bar{\tau}_1 = 1.7, \bar{\tau}_1^* = 2.7$

4.3.2 Analysis of the Dispersion Relation in the Reference Configuration

For an initially- stressed reference or undeformed configuration, we have $\lambda_1 = \lambda_2 = \lambda_3 = 1$. As a result, Eq. (4.97) becomes

$$\tan[kh\sqrt{\rho'\mu'\xi - \epsilon_1^*}] \;=\; \mu'\sqrt{\frac{\epsilon_1 - \xi}{\rho'\mu'\xi - \epsilon_1^*}}, \tag{4.102}$$

subject to

$$\frac{\epsilon_1^*}{\rho'\mu'} < \xi < \epsilon_1. \tag{4.103}$$

Similarly, Eq. (4.101) gives

$$\tanh[kh\sqrt{\epsilon_1^* - \rho'\mu'\xi}] \;=\; \mu'\sqrt{\frac{\epsilon_1 - \xi}{\epsilon_1^* - \rho'\mu'\xi}}, \tag{4.104}$$

subject to

$$\xi \;<\; \min\{\epsilon_1, \frac{\epsilon_1^*}{\rho'\mu'}\}. \tag{4.105}$$

Figures 4.14–4.15 refer to Eqs. (4.102)–(4.105) for various choices of parameters whereas the stretches are all fixed as unity. Figure 4.14 shows that for positive values of $\bar{\tau}_1$, the wave speed increases for small kh (or large wavelength) whereas for negative values of $\bar{\tau}_1$, the wave speed decreases for small kh. A similar behaviour is shown in Fig. 4.15 for the negative and positive values of $\bar{\tau}_1$.

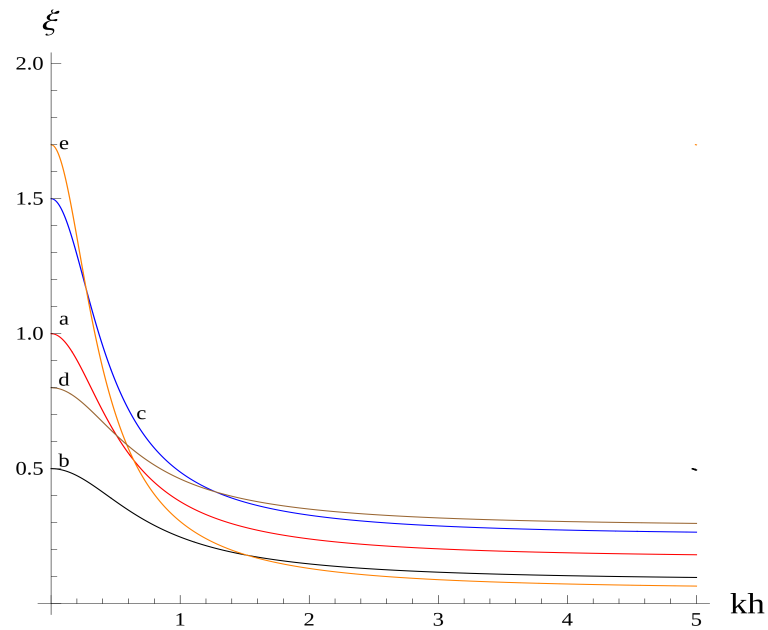


Figure 4.14: Plot of ξ with respect to kh from Eq. (4.102) for $\lambda_1 = \lambda_2 = \lambda_1^* = \lambda_2^* = b_0 = b_0^* = 1, \rho' = 2, \mu' = 3$ and (a) $\bar{\tau}_1 = 0 = \bar{\tau}_1^*$, (b) $\bar{\tau}_1 = -0.5 = \bar{\tau}_1^*$, (c) $\bar{\tau}_1 = 0.5 = \bar{\tau}_1^*$, (d) $\bar{\tau}_1 = -0.2, \bar{\tau}_1^* = 0.7$, (e) $\bar{\tau}_1 = 0.7, \bar{\tau}_1^* = -0.7$

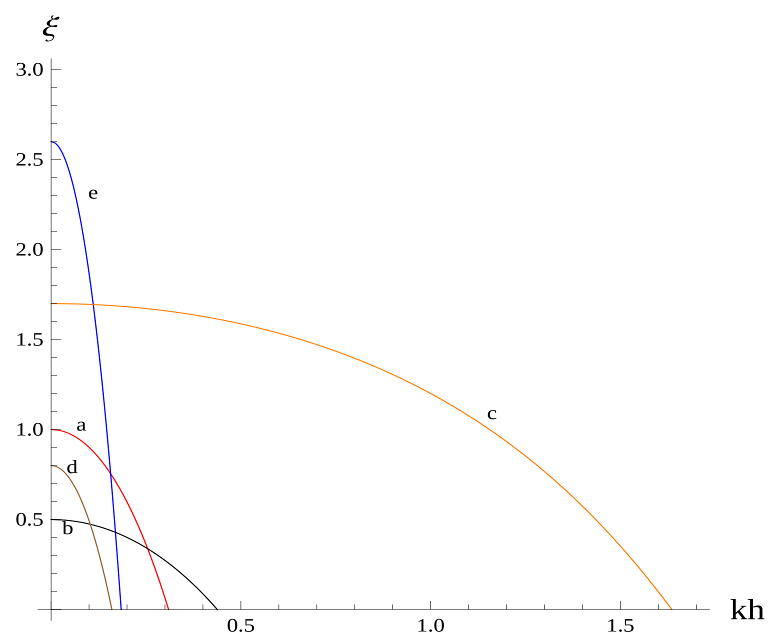


Figure 4.15: Plot of ξ with respect to kh from Eq. (4.104) for $\lambda_1 = \lambda_2 = \lambda_1^* = \lambda_2^* = b_0 = 2, b_0^* = 5, \rho' = 0.2, \mu' = 0.3$ and (a) $\bar{\tau}_1 = 0 = \bar{\tau}_1^*$, (b) $\bar{\tau}_1 = -0.5 = \bar{\tau}_1^*$, (c) $\bar{\tau}_1 = 0.7, \bar{\tau}_1^* = -0.7$, (d) $\bar{\tau}_1 = -0.2, \bar{\tau}_1^* = 0.7$, (e) $\bar{\tau}_1 = 1.6, \bar{\tau}_1^* = 1.7$

Chapter 5

Waves in a Residually-stressed Elastic Tube

5.1 Axial Extension and Radial Inflation of a Residually Stressed Thick-Walled Tube

We consider a thick-walled circular cylindrical tube which has a reference geometry defined by

$$A \leq R \leq B, \quad 0 \leq \Theta \leq 2\pi, \quad 0 \leq Z \leq L, \tag{5.1}$$

where A and B are the internal and external radii, respectively, and L is the length in the reference configuration. R, Θ, Z are cylindrical polar coordinates associated with basis vectors $\{\mathbf{E}_R, \mathbf{E}_\Theta, \mathbf{E}_Z\}$. The deformed geometry is given in terms of cylindrical polar coordinates r, θ, z, with basis vectors $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$, such that

$$a \leq r \leq b, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq z \leq l, \tag{5.2}$$

where a and b are the internal and external radii, respectively, and l is the length in the deformed configuration. The tube is deformed so that the circular cylindrical shape is maintained. See Fig. 5.1. Under the assumption that the material is incompressible, the deformation is described by

$$r^2 = a^2 + \lambda_z^{-1}(R^2 - A^2), \quad \theta = \Theta, \quad z = \lambda_z Z, \tag{5.3}$$

where λ_z is the uniform axial stretch.

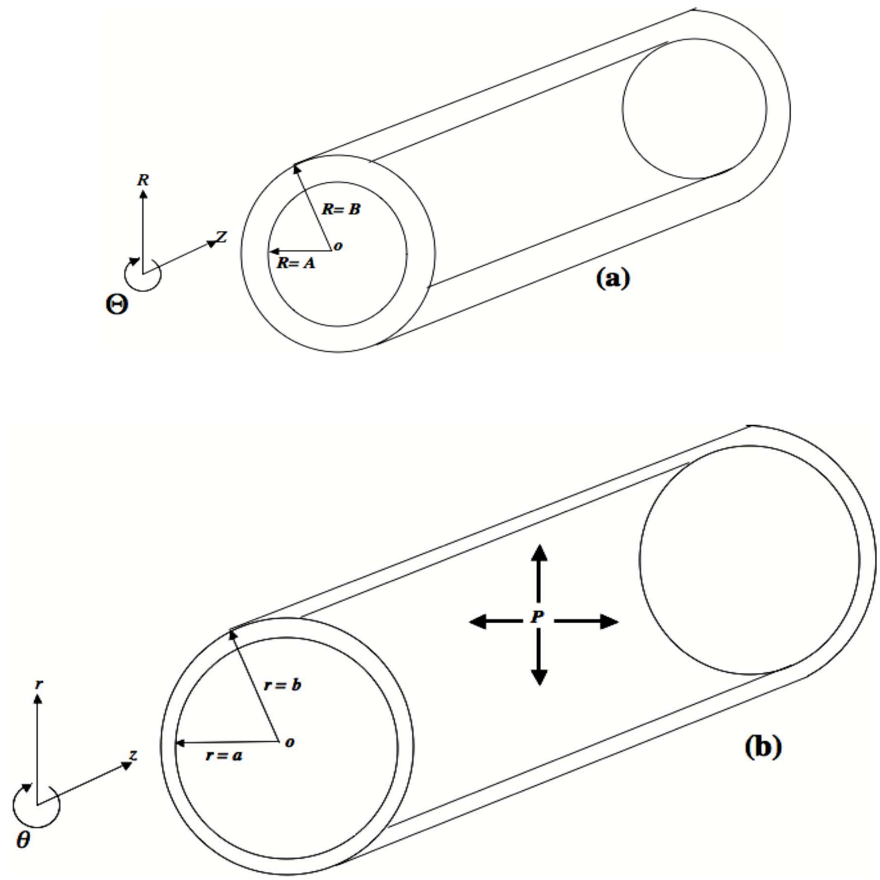


Figure 5.1: The residually-stressed circular cylindrical tube in (a) reference configuration and (b) deformed configuration when subject to axial load and internal pressure P.

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ denote the unit basis vectors corresponding to the coordinates θ, z, r respectively. Let $\lambda_1, \lambda_2, \lambda_3$ denote the corresponding principal stretches. Then, from the incompressibility condition (3.175) together with Eq. (5.3), we have

$$\lambda_1 = \frac{r}{R} \equiv \lambda, \quad \lambda_2 = \lambda_z, \quad \lambda_3 = \lambda^{-1}\lambda_z^{-1}, \tag{5.4}$$

where we have introduced the azimuthal stretch λ which is a function of r (or R) from Eq. (5.3).

It follows from Eqs. (5.3) and (5.4) that

$$\lambda_a^2\lambda_z - 1 = \frac{R^2}{A^2}(\lambda^2\lambda_z - 1) = \frac{B^2}{A^2}(\lambda_b^2\lambda_z - 1), \tag{5.5}$$

where the definitions

$$\lambda_a = a/A, \quad \lambda_b = b/B, \tag{5.6}$$

have been introduced. It follows from the above expressions for fixed λ_z that

$$\lambda_a \geq \lambda \geq \lambda_b, \tag{5.7}$$

holds during inflation of the tube. The equality holds if and only if $\lambda = \lambda_z^{-1/2}$ for $A \leq R \leq B$ and the deformation corresponds to simple tension.

Assuming that the residual stress is diagonal with respect to the cylindrical polar axes, let τ_1, τ_2, τ_3 denote the principal residual Cauchy stresses. Then, from Eq. (3.1), we can write the invariants $I_1, I_2, ..., I_{10}$ in terms of $\lambda, \lambda_z, \tau_1, \tau_2, \tau_3$ as follows

$$\begin{aligned}
I_1 &= \lambda^2 + \lambda_z^2 + \lambda^{-2}\lambda_z^{-2}, \quad I_2 = \lambda^2\lambda_z^2 + \lambda_z^{-2} + \lambda^{-2}, \quad I_3 = 1, \\
I_4 &= \tau_1 + \tau_2 + \tau_3, \quad I_5 = \tau_1\tau_2 + \tau_1\tau_3 + \tau_2\tau_3, \quad I_6 = \tau_1\tau_2\tau_3, \\
I_7 &= \lambda^2\tau_1 + \lambda_z^2\tau_2 + \lambda^{-2}\lambda_z^{-2}\tau_3, \quad I_8 = \lambda^4\tau_1 + \lambda_z^4\tau_2 + \lambda^{-4}\lambda_z^{-4}\tau_3, \\
I_9 &= \lambda^2\tau_1^2 + \lambda_z^2\tau_2^2 + \lambda^{-2}\lambda_z^{-2}\tau_3^2, \quad I_{10} = \lambda^4\tau_1^2 + \lambda_z^4\tau_2^2 + \lambda^{-4}\lambda_z^{-4}\tau_3^2.
\end{aligned} \tag{5.8}$$

The principal Cauchy stress components are given by Eq. (3.138). Considering W to be independent of I_5, I_6, I_9 and I_{10}, the principal Cauchy stress components are given by (3.139). Considering $\tau_{ij} = 0, i \neq j$, Eq. (3.143) holds.

We make use of Eq. (5.8) and recast the strain energy function W as a function of $\lambda_1 = \lambda$ and $\lambda_2 = \lambda_z$ and τ_1, τ_2, τ_3 and define

$$\hat{W}(\lambda, \lambda_z, \tau_1, \tau_2, \tau_3) = W(\lambda, \lambda_z, \lambda^{-1}\lambda_z^{-1}, \tau_1, \tau_2, \tau_3), \tag{5.9}$$

which is generally not symmetric in λ, λ_z. Using Eqs. (3.143), we obtain the stress differences

$$t_1 - t_3 = \lambda \frac{\partial \hat{W}}{\partial \lambda}, \quad t_2 - t_3 = \lambda_z \frac{\partial \hat{W}}{\partial \lambda_z}. \tag{5.10}$$

In its expanded form Eq. (5.10) is

$$\begin{aligned}
\lambda\frac{\partial \hat{W}}{\partial \lambda} = t_1 - t_3 &= 2(\lambda^2 - \lambda^{-2}\lambda_z^{-2})W_1 + 2(\lambda^2\lambda_z^2 - \lambda^{-2})W_2 + 2\lambda^2\tau_1(W_7 + 2\lambda^2 W_8) \\
&- 2\lambda^{-2}\lambda_z^{-2}\tau_3(W_7 + 2\lambda^{-2}\lambda_z^{-2}W_8), \qquad (5.11) \\
\lambda_z\frac{\partial \hat{W}}{\partial \lambda_z} = t_2 - t_3 &= 2(\lambda_z^2 - \lambda^{-2}\lambda_z^{-2})W_1 + 2(\lambda^2\lambda_z^2 - \lambda_z^{-2})W_2 + 2\lambda_z^2\tau_2(W_7 + 2\lambda_z^2 W_8) \\
&- 2\lambda^{-2}\lambda_z^{-2}\tau_3(W_7 + 2\lambda^{-2}\lambda_z^{-2}W_8). \qquad (5.12)
\end{aligned}$$

For our specific problem, we assume $\tau_2 = 0$. In the unloaded configuration the residual stresses must satisfy

$$\frac{d\tau_3}{dR} + \frac{1}{R}(\tau_3 - \tau_1) = 0, \qquad (5.13)$$

and this is coupled with the boundary conditions

$$\tau_3 = 0 \quad \text{on} \quad R = A \quad \text{and} \quad R = B. \qquad (5.14)$$

If there are no body forces in the deformed configuration, the equilibrium equation gives

$$\frac{dt_3}{dr} + \frac{1}{r}(t_3 - t_1) = 0, \qquad (5.15)$$

along with the boundary conditions

$$t_3 = \begin{cases} -P & \text{on} \quad r = a \\ 0 & \text{on} \quad r = b, \end{cases} \qquad (5.16)$$

where P is the internal pressure used to inflate the tube.

By making use of Eqs. (5.3), (5.4) and (5.5) we obtain (after some rearrangement)

$$r\frac{d\lambda}{dr} = -\lambda(\lambda^2\lambda_z - 1), \qquad (5.17)$$

and it is convenient to use the above expression to change the independent variable r to λ. Then, integrating Eq. (5.15) with application of the boundary conditions Eq. (5.16) we get

$$P = \int_a^b \frac{\lambda}{r}\frac{\partial \hat{W}}{\partial \lambda}dr = \int_{\lambda_b}^{\lambda_a} (\lambda^2\lambda_z - 1)^{-1}\frac{\partial \hat{W}}{\partial \lambda}d\lambda, \qquad (5.18)$$

where the variable of integration has been changed from r to λ by using Eq. (5.17).

In view of (5.5) we regard P as a function of λ_z and λ_a. Differentiating Eq. (5.18) with respect to λ_a, we get

$$\lambda_a^{-1}(\lambda_a^2\lambda_z - 1)\frac{\partial P}{\partial \lambda_a} = \frac{1}{\lambda_a}\frac{\partial \hat{W}(\lambda_a, \lambda_z)}{\partial \lambda} - \frac{1}{\lambda_b}\frac{\partial \hat{W}(\lambda_b, \lambda_z)}{\partial \lambda}. \tag{5.19}$$

Using Eq. (5.10) in Eq. (5.15) and integrating, we get

$$t_3(r) = \int_a^r \frac{\lambda}{r}\frac{\partial \hat{W}}{\partial \lambda} dr = \int_\lambda^{\lambda_a} (\lambda^2\lambda_z - 1)^{-1}\frac{\partial \hat{W}}{\partial \lambda} d\lambda, \tag{5.20}$$

where, as before, we have made use of Eq. (5.17) to transform the variable of integration from r to λ. The above expression will give the radial Cauchy stress component once a specific strain energy function is used.

Integration of Eq. (5.13) gives

$$R\tau_3(R) = \int_A^R \tau_1(R)dR. \tag{5.21}$$

Due to the boundary conditions (5.14), we must have

$$\int_A^B \tau_1(R)dR = 0. \tag{5.22}$$

This means, in particular, that $\tau_1(R)$ must be positive for some R and negative for other R. We cannot have $\tau_1(R) \equiv 0$ since this will render $\tau_3(R) \equiv 0$.

In [33] and [32], the author considered the consequences of using the assumption of uniform circumferential stress. As a result, it has been found the radial residual stress τ_3 is very small and negative except at the boundaries (where its value is zero due to Eq. (5.14)), whereas the circumferential stress τ_1 is compressive at the inner boundary ($R = A$) and tensile at the outer boundary ($R = B$). A similar discussion can be found in [6], [47] and [41]. However, Ogden and Schulze-Bauer [35] consider uniform circumferential strain distribution in addition to uniform circumferential stress which gives opposite signs of the residual radial and residual circumferential stresses compared to those in [32] and [33]. Further, for a particular choice of the circumferential growth stretch ratio, the behaviours of the residual stresses found in [32] and [33] match those presented in [43], i.e. when growth is considered in addition to the uniform circumferential stress. More recently, Guillou and Ogden [13] have considered growth which results in the development of residual stresses and they obtained very similar results to those in [33] for a different strain energy function.

In view of the above cited papers, we expect a particular behaviour of residual circumferential stress inside a thick-walled tube. For simplicity, we choose $\tau_1(R)$ to be linear in R and given by

$$\tau_1(R) = c_1(R - A) - c_2(B - A), \tag{5.23}$$

where c_1 and c_2 are constants. Since $B > A$, it is obvious that

$$\begin{aligned} \tau_1(A) &= -c_2(B - A) < 0 \quad \text{if} \quad c_2 > 0, \\ \tau_1(B) &= (c_1 - c_2)(B - A) > 0 \quad \text{if} \quad c_1 > c_2 > 0. \end{aligned} \tag{5.24}$$

This is in accordance with the typical behaviour of the residual circumferential stress which is negative on the inner boundary and positive on the outer boundary. Using Eq. (5.22), we have

$$\int_A^B \tau_1 dR = (c_1 - 2c_2)(B - A)^2 = 0. \tag{5.25}$$

Hence, $c_1 = 2c_2$. Integrating Eq. (5.21), we have

$$R\tau_3 = c_2(R - A)(R - B). \tag{5.26}$$

It is obvious that the above expression vanishes for $R = A$ and $R = B$.

We can re-write Eqns. (5.23) and (5.26) as

$$\tau_1 = c_2 A(2R/A - 1 - B/A), \tag{5.27}$$

$$\tau_3 = c_2 A(1 - A/R)(R/A - B/A). \tag{5.28}$$

Figure 5.2 shows the variation of these stresses, in non-dimensionalized form, for fixed tube thickness. These are very similar to the graphs obtained in ([33], [32], [47], [41], [43], [13]).

Based on the expected behaviour of the circumferential residual stress, τ_1, we can assume various other forms which may not be linearly dependent on R. For instance,

$$\tau_1 = c_1 \cos((B - R)/A) + c_2 \cos((R - A)/A), \tag{5.29}$$

and similar calculations as for Eqns. (5.24) and (5.25) give $c_1 = -c_2$ and $c_1 > 0$. Further

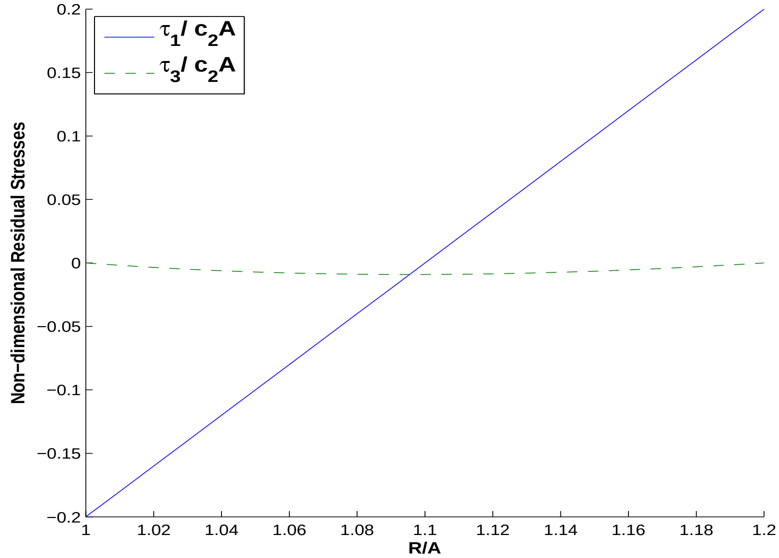


Figure 5.2: Plot of the principal residual stresses based on Eqs. (5.27) and (5.28) for $B/A = 1.2$

calculations lead to

$$\tau_1 \;=\; c_1(\cos((B-R)/A) - \cos((R-A)/A), \qquad (5.30)$$

$$\tau_3 \;=\; \frac{c_1 A}{R}(\sin((B-A)/A - \sin((R-A)/A) - \sin((B-R)/A)). \qquad (5.31)$$

Figure 5.3 shows the plot of the non-dimensional residual stresses for the above choice of τ_1 and τ_3.

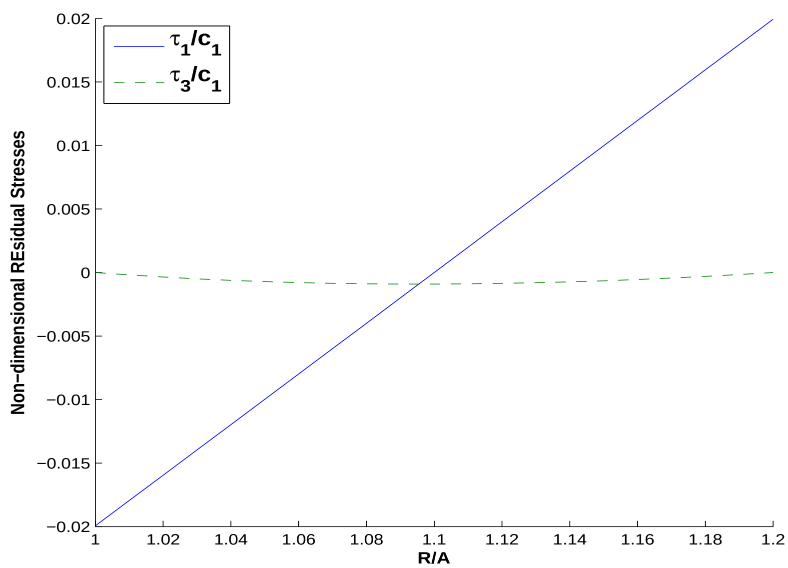


Figure 5.3: Plot of the principal residual stresses based on Eqs. (5.30) and (5.31) for $B/A = 1.2$

Similarly, consider the choice

$$\tau_1 = c_1(R-A)^2 - c_2(R-B)^2, \tag{5.32}$$

which, after similar calculations, gives $c_1 = c_2$ and $c_1 > 0$. Further calculations lead to

$$\tau_1(R) = c_1A^2((R/A-1)^2 - (R/A-B/A)^2), \tag{5.33}$$

$$\tau_3(R) = c_1A^3[(R/A-1)^3 - (R/A-B/A)^3 + (1-B/A)^3)]/R. \tag{5.34}$$

Figure 5.4 shows the plot of the residual stresses for the above choice of τ_1 and τ_3.

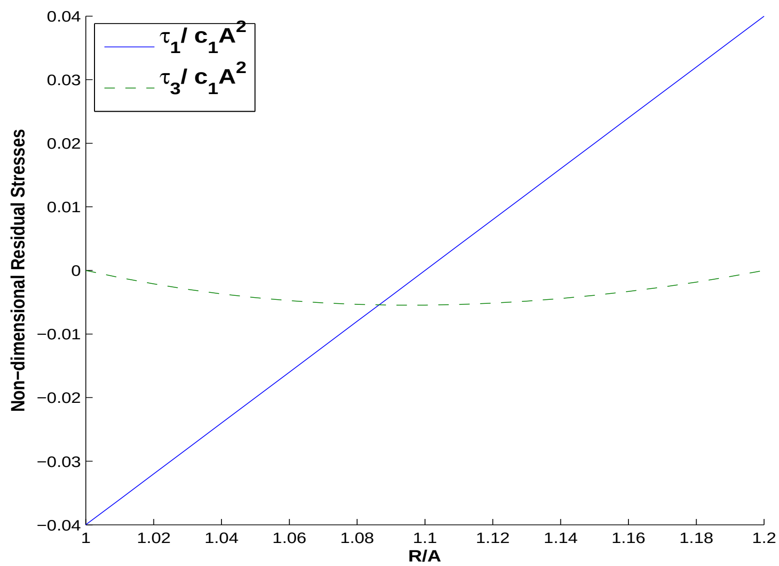


Figure 5.4: Plot of the principal residual stresses based on Eqs. (5.33) and (5.34) for $B/A = 1.2$

5.2 Internal Pressure in a Residually-stressed Thick-walled Tube

Using Eqs. (3.111) and (5.4), we can write

$$\begin{aligned} W &= \frac{\mu}{2}(\lambda^2 + \lambda_z^2 + \lambda^{-2}\lambda_z^{-2} - 3) + \frac{\bar{\mu}}{2}[(\lambda^2-1)\tau_1 + (\lambda_z^2-1)\tau_2 + (\lambda^{-2}\lambda_z^{-2}-1)\tau_3]^2 \\ &+ \frac{1}{2}[(\lambda^2-1)\tau_1 + (\lambda_z^2-1)\tau_2 + (\lambda^{-2}\lambda_z^{-2}-1)\tau_3]. \end{aligned} \tag{5.35}$$

From Eq. (5.35), we have

$$\begin{aligned}\lambda\frac{\partial\hat{W}}{\partial\lambda} &= \mu(\lambda^2-\lambda^{-2}\lambda_z^{-2})+2\bar{\mu}[(\lambda^4-\lambda^2)\tau_1^2+(\lambda^{-2}\lambda_z^{-2}-\lambda^2)\tau_1\tau_3+\lambda^2(\lambda_z^2-1)\tau_1\tau_2 \\ &- \lambda^{-2}\lambda_z^{-2}(\lambda_z^2-1)\tau_2\tau_3-(\lambda^{-4}\lambda_z^{-4}-\lambda^{-2}\lambda_z^{-2})\tau_3^2]+\lambda^2\tau_1-\lambda^{-2}\lambda_z^{-2}\tau_3, \quad (5.36)\\ \lambda_z\frac{\partial\hat{W}}{\partial\lambda_z} &= \mu(\lambda_z^2-\lambda^{-2}\lambda_z^{-2})+2\bar{\mu}[\lambda_z^2(\lambda_z^2-1)\tau_2^2-\lambda^{-2}\lambda_z^{-2}(\lambda^{-2}\lambda_z^{-2}-1)\tau_3^2+\lambda_z^2(\lambda^2-1)\tau_1\tau_2 \\ &- \lambda^{-2}\lambda_z^{-2}(\lambda^2-1)\tau_1\tau_3+(\lambda^{-2}\lambda_z^{-2}-\lambda_z^2)\tau_2\tau_3]+\lambda_z^2\tau_2-\lambda^{-2}\lambda_z^{-2}\tau_3, \quad (5.37)\end{aligned}$$

which, in the reference configuration, reduce to

$$\lambda\frac{\partial\hat{W}}{\partial\lambda}=\tau_1-\tau_3, \quad \lambda_z\frac{\partial\hat{W}}{\partial\lambda_z}=\tau_2-\tau_3. \quad (5.38)$$

Also, from Eq. (3.112), we have

$$\lambda_3\frac{\partial W}{\partial\lambda_3}=\lambda_3^2(\mu+\tau_3)+2\bar{\mu}\lambda_3^2\tau_3[(\lambda_1^2-1)\tau_1+(\lambda_2^2-1)\tau_2+(\lambda_3^2-1)\tau_3], \quad (5.39)$$

which, in the reference configuration, reduces to

$$\frac{\partial W}{\partial\lambda_3}=\mu+\tau_3. \quad (5.40)$$

Using Eq. (5.5) in Eqs. (5.27) and (5.28), we can write the dimensionless form of τ_1 and τ_3 in terms of $\lambda, \lambda_z, \lambda_a$, as

$$\begin{aligned}\hat{\tau}_1 &= b_2\Big(2\sqrt{\frac{\lambda_a^2\lambda_z-1}{\lambda^2\lambda_z-1}}-1-\frac{B}{A}\Big), \quad (5.41)\\ \hat{\tau}_3 &= b_2\Big(1-\sqrt{\frac{\lambda^2\lambda_z-1}{\lambda_a^2\lambda_z-1}}\Big)\sqrt{\Big(\frac{\lambda_a^2\lambda_z-1}{\lambda^2\lambda_z-1}}-\frac{B}{A}\Big), \quad (5.42)\end{aligned}$$

where we have introduced the dimensionless quantity

$$b_2=c_2A/\mu. \quad (5.43)$$

Although, λ appears in the expressions above but the residual stress doesn't explicitly depend on the stretches and the λ here serves as the dummy variable only for the purpose of integration.

It follows from Eq. (5.19)

$$\begin{aligned}\frac{dP^*}{d\lambda_a} &= \frac{\lambda_a}{\lambda_a^2\lambda_z - 1}[1 - \lambda_a^{-4}\lambda_z^{-2} + 2b_1(\lambda_a^2 - 1)(1 - B/A)^2 + b_2(1 - B/A) \\ &- (1 - \lambda_b^{-4}\lambda_z^{-2}) - 2b_1(\lambda_b^2 - 1)(B/A - 1)^2 - b_2(B/A - 1)]. \end{aligned} \tag{5.44}$$

Here, we have introduced the notation

$$b_1 = b_0 b_2^2 = \bar{\mu} c_2^2 A^2/\mu^2, \tag{5.45}$$

where b_0 is given by Eq. (4.31) and b_2 by Eq. (5.43).

Figure 5.5 shows the variation of $\frac{dP^*}{d\lambda_a}$ with respect to λ_a. From Eqs. (5.5) and (5.18), in the absence of residual stress, the pressure vanishes at $\lambda_a = \lambda_z^{-1/2}$ as is shown in Fig. 5.6 and the pressure tends to remain constant as the value of λ_a increases. This behaviour is similar for materials following neo-Hookean or Mooney-Rivlin models. This trend is illustrated in plots (a) and (b) in Fig. 5.5. The same figure shows plots (c) and (d) for the derivative of pressure in the presence of residual stress. In the presence of residual stress, the respective value of λ_a where the pressure vanishes (for fixed axial stretch) is shifted depending on the values of b_1 and b_2 and the thickness of the wall B/A. Also, the pressure is expected to increase or decrease depending on the values of respective parameters and this fact is obvious in the figures to follow.

Using Eqs. (5.35) in (5.18), we have the non-dimensionalized pressure

$$\begin{aligned} P^* = P/\mu &= \int_{\lambda_b}^{\lambda_a} \frac{\lambda - \lambda^{-3}\lambda_z^{-2}}{\lambda^2\lambda_z - 1} d\lambda + 2b_0[\int_{\lambda_b}^{\lambda_a} \frac{\lambda(\lambda^2 - 1)}{\lambda^2\lambda_z - 1}\hat{\tau}_1^2 d\lambda \\ &+ \int_{\lambda_b}^{\lambda_a} \frac{\lambda^{-3}\lambda_z^{-2} - \lambda}{\lambda^2\lambda_z - 1}\hat{\tau}_1\hat{\tau}_3 d\lambda - \int_{\lambda_b}^{\lambda_a} \frac{\lambda^{-3}\lambda_z^{-2}(\lambda^{-2}\lambda_z^{-2} - 1)}{\lambda^2\lambda_z - 1}\hat{\tau}_3^2 d\lambda] \\ &+ \int_{\lambda_b}^{\lambda_a} \frac{\lambda}{\lambda^2\lambda_z - 1}\hat{\tau}_1 d\lambda - \int_{\lambda_b}^{\lambda_a} \frac{\lambda^{-3}\lambda_z^{-2}}{\lambda^2\lambda_z - 1}\hat{\tau}_3 d\lambda, \end{aligned} \tag{5.46}$$

where b_0 is given by Eq. (4.31). The behaviour of pressure from Eq. (5.46) in the absence of residual stress is shown in Fig. 5.6 whereas graphs in the presence of the residual stress and for various choices of $\lambda_z, B/A, b_1$ and b_2 are shown in Figs. 5.7–5.9. We observe from these figures that for b_1, b_2 both positive, the pressure increases whereas for b_1, b_2 both negative, the pressure decreases with increasing λ_a. For fixed wall thickness and axial stretch, Fig. 5.9 shows the increasing or decreasing trend for various combinations of b_1 and b_2 which characterise the presence of residual stress.

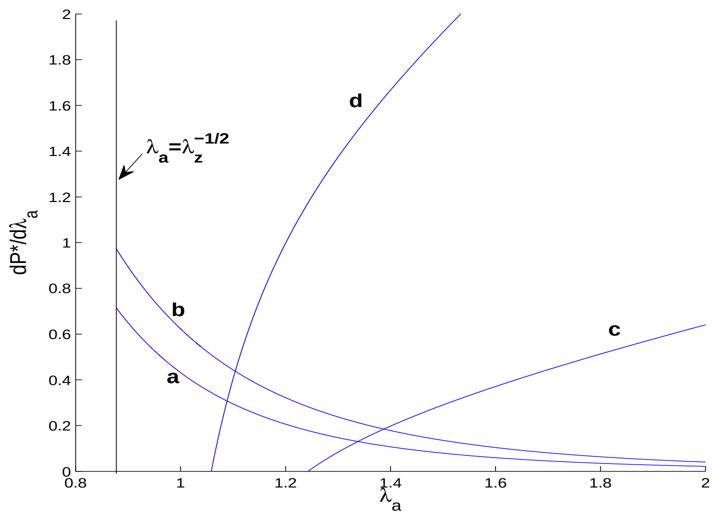


Figure 5.5: Plot of $\frac{1}{\mu}\frac{dP}{d\lambda_a}$ with respect to λ_a, (a) $b_1 = 0 = b_2, \lambda_z = 1.3 = B/A$, (b) $b_1 = 0 = b_2, \lambda_z = 1.3, B/A = 1.5$, (c) $b_1 = 8, b_2 = 1, \lambda_z = 1.3 = B/A$, (d) $b_1 = 8, b_2 = 1, \lambda_z = 1.3, B/A = 1.5$

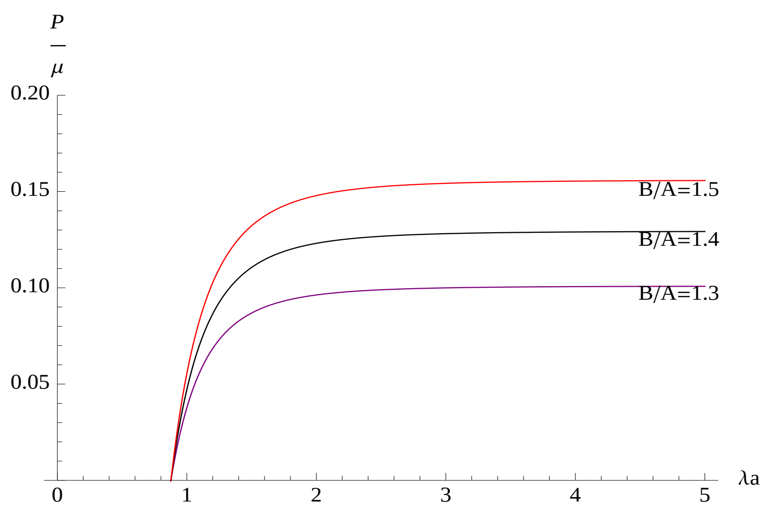


Figure 5.6: Plot of dimensionless pressure P^* with respect to λ_a in the absence of residual stress and for varying wall thickness B/A, $(\lambda_z = 1.3)$.

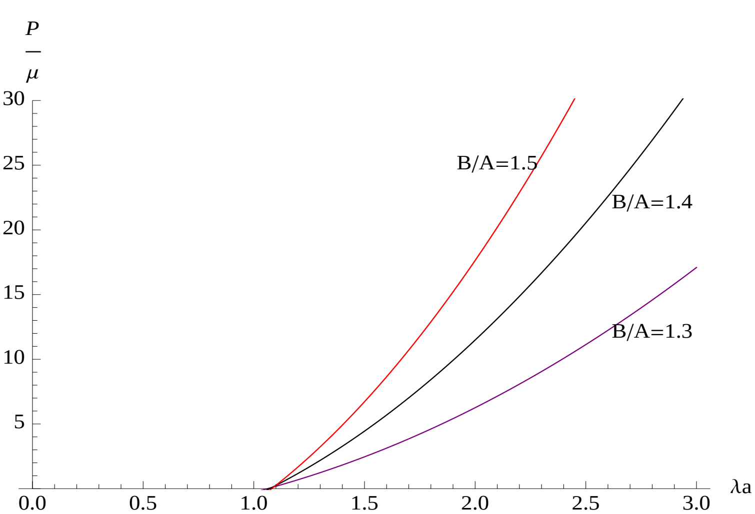


Figure 5.7: Plot of dimensionless pressure P^* with respect to λ_a for varying wall thickness B/A, $b_1 = 2 = b_2$ and $\lambda_z = 1.3$.

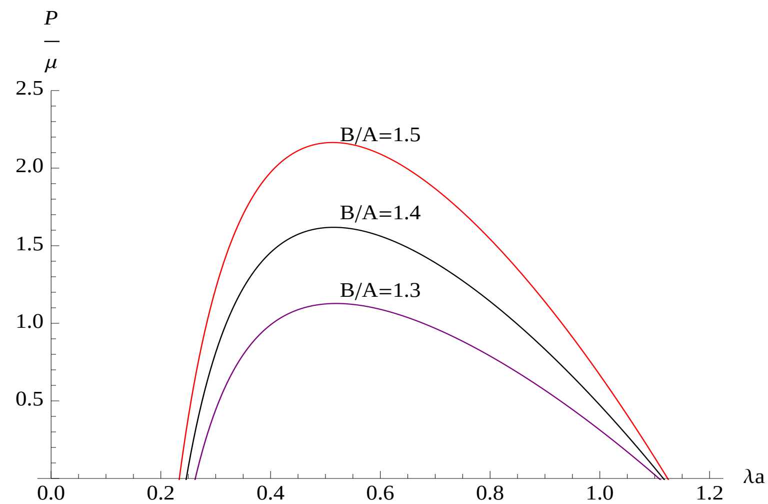


Figure 5.8: Plot of dimensionless pressure P^* with respect to λ_a for varying wall thickness B/A, $b_1 = -2 = b_2$ and $\lambda_z = 2$.

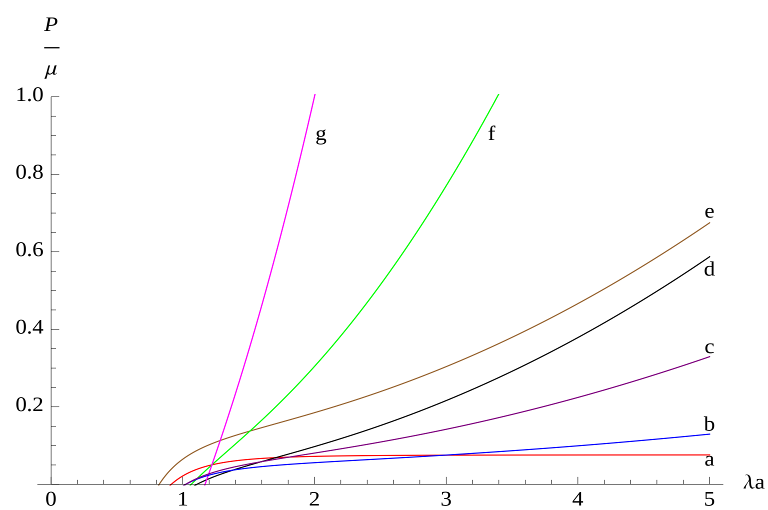


Figure 5.9: Plot of dimensionless pressure P^* with respect to λ_a for $B/A = 1.2$, $\lambda_z = 1.2$ and (a) $b_1 = 0 = b_2$, (b) $b_1 = 0.2, b_2 = 0.3$, (c) $b_1 = 0.7, b_2 = 0.3$, (d) $b_1 = 0.5, b_2 = 0.5$, (e) $b_1 = 0.5, b_2 = -0.5$, (f) $b_1 = 2, b_2 = 0.5$, (g) $b_1 = 0.5, b_2 = 2$.

5.3 Axial Load on a Residually-stressed Thick-walled Tube

In order to hold the axial stretch λ_z fixed an axial load N must be applied to the ends of the tube. This is given by

$$N = 2\pi \int_a^b t_2 r dr. \tag{5.47}$$

Using Eqs. (5.7) and (5.10)–(5.17), we have

$$N/\pi A^2 = (\lambda_a^2\lambda_z - 1)\int_{\lambda_b}^{\lambda_a} (\lambda^2\lambda_z - 1)^{-2}(2\lambda_z\frac{\partial \hat{W}}{\partial \lambda_z} - \lambda\frac{\partial \hat{W}}{\partial \lambda})\lambda d\lambda + P\lambda_a^2. \tag{5.48}$$

Using Eqs. (5.36) and (5.37) in Eq. (5.48), we have

$$\begin{aligned} N/A' &= (\lambda_a^2\lambda_z - 1)[\int_{\lambda_b}^{\lambda_a} \frac{2\lambda_z^2\lambda - \lambda^3 - \lambda^{-1}\lambda^{-2}}{(\lambda^2\lambda_z - 1)^2} d\lambda - 2b_0\int_{\lambda_b}^{\lambda_a}[\frac{\lambda^{-3}\lambda_z^{-4}(1-\lambda^2\lambda_z^2)}{1-\lambda^2\lambda_z}\hat{\tau}_3^2 \\ &+ \frac{3\lambda^{-1}\lambda_z^{-2}(\lambda^2-1)}{(1-\lambda^2\lambda_z)^2}\hat{\tau}_1\hat{\tau}_3 d\lambda + \frac{\lambda^3(\lambda^2-1)}{(1-\lambda^2\lambda_z)^2}\hat{\tau}_1^2]d\lambda - \int_{\lambda_b}^{\lambda_a}\frac{\lambda^3\hat{\tau}_1 + \lambda^{-1}\lambda_z^{-2}\hat{\tau}_3}{(1-\lambda^2\lambda_z)^2}d\lambda] \\ &+ P\lambda_a^2, \end{aligned} \tag{5.49}$$

where $A' = \pi\mu A^2$. The behaviour of axial load with respect to λ_a for various choices of parameters and wall-thicknesses is shown in Fig. 5.10. In the absence of residual stress, the load tends to be constant for increasing λ_a which is similar to the behaviour of neo-Hookean materials and Mooney-Rivlin materials. For a fixed wall-thickness, Fig. 5.11 shows the effect of residual stress on the axial load and it is observed that with increasing λ_a, the axial load either increases or decreases depending on the parameters that characterise the magnitude of residual stress.

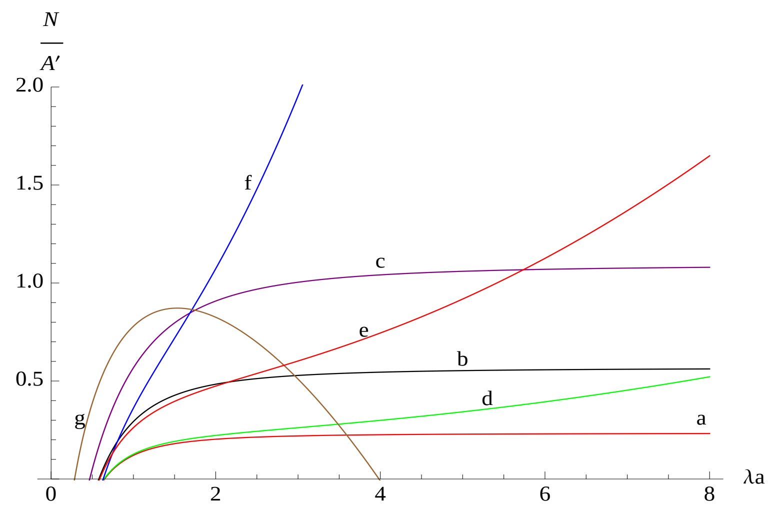


Figure 5.10: Plot of the non-dimensional axial load N/A' with respect to λ_a for $\lambda_z = 1.2$ and (a) $B/A = 1.2, b_1 = 0 = b_2$, (b) $B/A = 1.5, b_1 = 0 = b_2$, (c) $B/A = 2, b_1 = 0 = b_2$, (d) $B/A = 1.2, b_1 = -0.5, b_2 = 0.8$, (e) $B/A = 1.4, b_1 = -0.5, b_2 = 0.8$, (f) $B/A = 1.5, b_1 = -0.8, b_2 = 1.5$, (g) $B/A = 2, b_1 = 0.3, b_2 = 0.8$

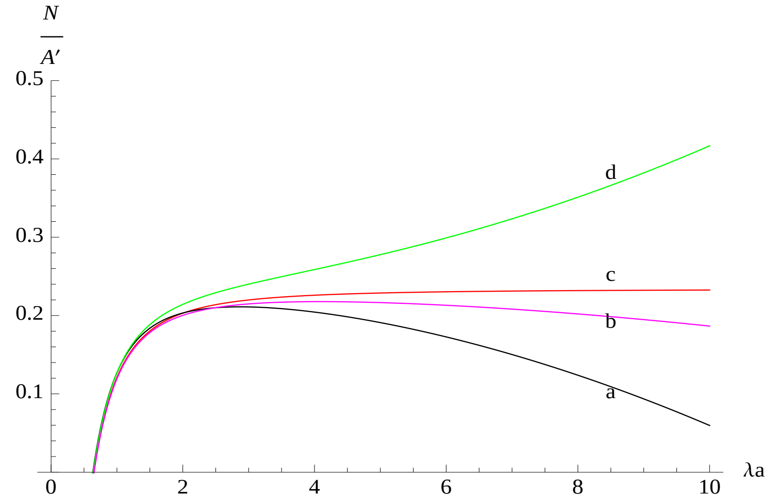


Figure 5.11: Plot of the non-dimensional axial load N/A' with respect to λ_a for $B/A = 1.2, \lambda_z = 1.2$ and (a) $b_1 = 0.2, b_2 = 0.8$, (b) $b_1 = 0.8, b_2 = -0.2$, (c) $b_1 = 0 = b_2$, (d) $b_1 = -0.2, b_2 = 0.8$

5.4 Analysis of Infinitesimal Wave Propagation in a Residually Stressed Thick-Walled Tube: Axisymmetric Case

We now consider the problem of propagation of an infinitesimal wave in a residually stressed thick-walled cylindrical tube subject to a finite axial extension and radial inflation. For this special case we choose $\mathbf{e}_1 = \mathbf{e}_\theta, \mathbf{e}_2 = \mathbf{e}_z, \mathbf{e}_3 = \mathbf{e}_r$ to represent the basis vectors. Also, the azimuthal, axial and radial residual stress components are denoted as τ_1, τ_2, τ_3, respectively.

Considering small time-dependent deformations imposed on a finite deformation, i.e. for small $\mathbf{H}$, we have from Eq. (2.93)

$$\dot{\mathbf{F}} \simeq \mathbf{H} \simeq \mathbf{\Gamma}. \tag{5.50}$$

In the absence of body forces and for small $\mathbf{H}$, the incremental updated form of equilibrium equation (2.121) is

$$\mathrm{div}\dot{\mathbf{S}}_0 \equiv \mathrm{div}[\boldsymbol{\mathcal{A}}_\mathbf{0}(\mathbf{H}) - \dot{p}\mathbf{I} + p\mathbf{H}] = \rho \mathbf{u}_{,tt}, \tag{5.51}$$

and in component form

$$\dot{S}_{0ji,j} + \dot{S}_{0ji}\mathbf{e}_k \cdot \mathbf{e}_{j,k} + \dot{S}_{0kj}\mathbf{e}_i \cdot \mathbf{e}_{j,k} = \rho u_{i,tt}, \quad i = 1, 2, 3, \tag{5.52}$$

with summation over indices j and k from 1 to 3, where the subscript j $(= 1, 2, 3)$ following a comma represents the derivatives $(\partial/r\partial\theta, \partial/\partial z, \partial/\partial r)$. The only non-zero components of $\mathbf{e}_i \cdot \mathbf{e}_{j,k}$ are

$$\mathbf{e}_1 \cdot \mathbf{e}_{3,1} = 1/r, \quad \mathbf{e}_3 \cdot \mathbf{e}_{1,1} = -1/r. \tag{5.53}$$

If $\mathbf{u} = v\mathbf{e}_\theta + w\mathbf{e}_z + u\mathbf{e}_r$, we have

$$[\mathbf{H}] = [\text{grad}\mathbf{u}] = \begin{pmatrix} (u + v_\theta)/r & v_z & v_r \\ w_\theta/r & w_z & w_r \\ (u_\theta - v)/r & u_z & u_r \end{pmatrix}, \tag{5.54}$$

where the square brackets indicate the matrix of components of the enclosed quantity and the subscripts (r, θ, z) denote the standard partial derivatives.

For the axisymmetric case, Eq. (5.54) becomes

$$\mathbf{H} = \begin{pmatrix} u/r & 0 & 0 \\ 0 & w_z & w_r \\ 0 & u_z & u_r \end{pmatrix}. \tag{5.55}$$

Here, the subscripts show the derivative with respect to the respective coordinate.

In this case, the incompressibility condition, $H_{pp} = 0$, gives

$$u/r + w_z + u_r = 0, \tag{5.56}$$

which can be rewritten as

$$(ru)_r + (rw)_z = 0. \tag{5.57}$$

From Eq. (5.57), we deduce the existence of a potential function $\phi = \phi(r, z)$ such that

$$ru = \phi_z \Rightarrow (ru)_r = \phi_{rz}, \quad rw = -\phi_r \Rightarrow (rw)_z = -\phi_{rz}. \tag{5.58}$$

From Eq. (5.52), we have for $i = 2$ and $i = 3$ respectively

$$\dot{S}_{0j2,j} + \frac{1}{r}\dot{S}_{032} = \rho\ddot{w}, \tag{5.59}$$

$$\dot{S}_{0j3,j} + \frac{1}{r}\dot{S}_{033} - \frac{1}{r}\dot{S}_{011} = \rho\ddot{u}. \tag{5.60}$$

In the expanded form, Eqs. (5.59) and (5.60) give

$$\begin{aligned}\dot{p}_z + \rho w_{tt} &= \mathcal{A}_{03232}w_{rr} + \mathcal{A}_{02222}w_{zz} + (r\mathcal{A}'_{03232} + \mathcal{A}_{03232})w_r/r \\ &+ (\mathcal{A}_{02233} + \mathcal{A}_{03223})u_{rz} + (r\mathcal{A}'_{03223} + \mathcal{A}_{03223} + \mathcal{A}_{01122} + rp')u_z/r, \qquad (5.61)\\ \dot{p}_r + \rho u_{tt} &= (r\mathcal{A}'_{01133} - \mathcal{A}_{01111})u/r^2 + (r\mathcal{A}'_{03333} + \mathcal{A}_{03333} + rp')u_r/r + \mathcal{A}_{03333}u_{rr} \\ &+ \mathcal{A}_{02323}u_{zz} + (r\mathcal{A}'_{02233} + \mathcal{A}_{02233} - \mathcal{A}_{01122})w_z/r + (\mathcal{A}_{02233} + \mathcal{A}_{03223})w_{rz}, \qquad (5.62)\end{aligned}$$

respectively. The primes ' $'$ ' in the superscript denote the derivative with respect to 'r'.

Taking r-derivative of Eq. (5.61), z-derivative of Eq. (5.62) and subtracting, we get

$$\begin{aligned}& r^4\mathcal{A}_{03232}w_{rrr} + r^3(2r\mathcal{A}'_{03232} + \mathcal{A}_{03232})w_{rr} + r^2(r^2\mathcal{A}''_{03232} + r\mathcal{A}'_{03232} - \mathcal{A}_{03232})w_r \\ +\; & r^3(r\mathcal{A}'_{02222} - r\mathcal{A}'_{02233} - \mathcal{A}_{02233} + \mathcal{A}_{01122})w_{zz} + r^4(\mathcal{A}_{02222} - \mathcal{A}_{02233} - \mathcal{A}_{03223})w_{rzz} \\ -\; & r^4\mathcal{A}_{02323}u_{zzz} + r^4(\mathcal{A}_{02233} + \mathcal{A}_{03223} - \mathcal{A}_{03333})u_{rrz} + r^3(r\mathcal{A}'_{02233} + 2r\mathcal{A}'_{03223} + \mathcal{A}_{03223} \\ +\; & \mathcal{A}_{01122} - r\mathcal{A}'_{03333} - \mathcal{A}_{03333})u_{rz} + r^2(r^2\mathcal{A}''_{03223} + r\mathcal{A}'_{03223} + r\mathcal{A}'_{01122} + r^2p'' \\ -\; & \mathcal{A}_{03223} - \mathcal{A}_{01122} - r\mathcal{A}'_{01133} + \mathcal{A}_{01111})u_z = \rho r^4(w_{rtt} - u_{ztt}). \qquad (5.63)\end{aligned}$$

From Eq. (5.58), we have

$$u = \frac{1}{r}\phi_z, \quad w = -\frac{1}{r}\phi_r, \tag{5.64}$$

which, in turn, gives

$$\begin{aligned}& u_z = \frac{1}{r}\phi_{zz}, \quad u_{zzz} = \frac{1}{r}\phi_{zzzz}, \quad u_{rz} = -\frac{1}{r^2}\phi_{zz} + \frac{1}{r}\phi_{rzz}, \\ & u_{rrz} = \frac{2}{r^3}\phi_{zz} - \frac{2}{r^2}\phi_{rzz} + \frac{1}{r}\phi_{rrzz}, \qquad (5.65)\end{aligned}$$

and

$$
\begin{aligned}
w_{zz} &= -\frac{1}{r}\phi_{rzz}, \quad w_{zzr} = \frac{1}{r^2}\phi_{rzz} - \frac{1}{r}\phi_{rrzz}, \\
w_r &= \frac{1}{r^2}\phi_r - \frac{1}{r}\phi_{rr}, \quad w_{rr} = -\frac{2}{r^3}\phi_r + \frac{2}{r^2}\phi_{rr} - \frac{1}{r}\phi_{rrr}, \\
w_{rrr} &= \frac{6}{r^4}\phi_r - \frac{6}{r^3}\phi_{rr} + \frac{3}{r^2}\phi_{rrr} - \frac{1}{r}\phi_{rrrr}. \qquad (5.66)
\end{aligned}
$$

Using the above expressions in Eq. (5.63), we get

$$
\begin{aligned}
& \mathcal{A}_{03232}(r^4\phi_{rrrr}) + 2[r\mathcal{A}^{'}_{03232} - \mathcal{A}_{03232}](r^3\phi_{rrr}) + [r^2\mathcal{A}^{''}_{03232} - 3r\mathcal{A}^{'}_{03232} + 3\mathcal{A}_{03232}](r^2\phi_{rr}) \\
- \; & [r^2\mathcal{A}^{''}_{03232} - 3r\mathcal{A}^{'}_{03232} + 3\mathcal{A}_{03232}](r\phi_r) + [\mathcal{A}_{02222} + \mathcal{A}_{03333} - 2\mathcal{A}_{02233} - 2\mathcal{A}_{03223}](r^4\phi_{rrzz}) \\
+ \; & \mathcal{A}_{02323}(r^4\phi_{zzzz}) + [r\mathcal{A}^{'}_{02222} + r\mathcal{A}^{'}_{03333} - 2r\mathcal{A}^{'}_{02233} - 2r\mathcal{A}^{'}_{03223} - \mathcal{A}_{02222} - \mathcal{A}_{03333} \\
+ \; & 2\mathcal{A}_{02233} + 2\mathcal{A}_{03223}](r^3\phi_{rzz}) - [r^2\mathcal{A}^{''}_{03223} + r^2p'' + r\mathcal{A}^{'}_{01122} + r\mathcal{A}^{'}_{03333} - r\mathcal{A}^{'}_{01133} \\
- \; & r\mathcal{A}^{'}_{03223} - r\mathcal{A}^{'}_{02233} + 2\mathcal{A}_{02233} - 2\mathcal{A}_{01122} - \mathcal{A}_{03333} + \mathcal{A}_{01111}](r^2\phi_{zz}) \\
= \; & \rho r^3[r\phi_{zztt} + r\phi_{rrtt} - \phi_{rtt}]. \qquad (5.67)
\end{aligned}
$$

Consider a solution for ϕ of the form

$$
\phi(r,z) = F(r)e^{i(kz-\omega t)}, \qquad (5.68)
$$

where k is the wave number and ω is the frequency. Using Eq. (5.68) in Eq. (5.67), after some rearrangement, we get

$$
\begin{aligned}
& \mathcal{A}_{03232}r^4F^{''''} + 2[r\mathcal{A}^{'}_{03232} - \mathcal{A}_{03232}]r^3F^{'''} + [r^2\mathcal{A}^{''}_{03232} - 3r\mathcal{A}^{'}_{03232} + 3\mathcal{A}_{03232} \\
& +\rho\omega^2r^2]r^2F^{''} - [r^2\mathcal{A}^{''}_{03232} - 3r\mathcal{A}^{'}_{03232} + 3\mathcal{A}_{03232} + \rho\omega^2r^2]rF^{'} + k^2r^2[(2\mathcal{A}_{02233} + 2\mathcal{A}_{03223} \\
& -\mathcal{A}_{02222} - \mathcal{A}_{03333})r^2F^{''} + (2r\mathcal{A}^{'}_{02233} + 2r\mathcal{A}^{'}_{03223} - r\mathcal{A}^{'}_{02222} - r\mathcal{A}^{'}_{03333} - 2\mathcal{A}_{02233} \\
& -2\mathcal{A}_{03223} + \mathcal{A}_{02222} + \mathcal{A}_{03333})rF^{'} + (r^2\mathcal{A}^{''}_{03223} + r^2p^{''} + r\mathcal{A}^{'}_{01122} + r\mathcal{A}^{'}_{03333} - r\mathcal{A}^{'}_{01133} \\
& -r\mathcal{A}^{'}_{03223} - r\mathcal{A}^{'}_{02233} + 2\mathcal{A}_{02233} - 2\mathcal{A}_{01122} - \mathcal{A}_{03333} + \mathcal{A}_{01111} - \rho\omega^2r^2)F] \\
& +k^4r^4\mathcal{A}_{02323}F = 0. \qquad (5.69)
\end{aligned}
$$

Let

$$
\begin{aligned}
&\gamma_1 = \mathcal{A}_{03232}, \quad \gamma_2 = 2\mathcal{A}_{02233} + 2\mathcal{A}_{03223} - \mathcal{A}_{02222} - \mathcal{A}_{03333},\\
&\gamma_3 = \mathcal{A}_{03223}, \quad \gamma_4 = \mathcal{A}_{01122} + \mathcal{A}_{03333} - \mathcal{A}_{01133} + \mathcal{A}_{03223} - \mathcal{A}_{02233},\\
&\gamma_5 = 2\mathcal{A}_{02233} - 2\mathcal{A}_{01122} - \mathcal{A}_{03333} + \mathcal{A}_{01111}, \quad \gamma_6 = \mathcal{A}_{02323}.
\end{aligned} \tag{5.70}
$$

Therefore, Eq. (5.69) becomes

$$
\begin{aligned}
&\gamma_1 r^4 F'''' + 2[r\gamma_1' - \gamma_1]r^3 F''' + [r^2\gamma_1'' - 3r\gamma_1' + 3\gamma_1 + \rho\omega^2 r^2]r^2 F''\\
&-[r^2\gamma_1'' - 3r\gamma_1' + 3\gamma_1 + \rho\omega^2 r^2]rF' + k^2r^2[\gamma_2 r^2 F'' + (r\gamma_2' - \gamma_2)rF'\\
&+(r^2\gamma_3'' + r^2 p'' + r\gamma_4' + \gamma_5 - \rho\omega^2 r^2)F] + k^4 r^4 \gamma_6 F = 0.
\end{aligned} \tag{5.71}
$$

The values of p' and p'' can be calculated using Eqs. (3.143) for $i = 3$ and (5.15). We get

$$
p'(r) = \frac{d}{dr}(\lambda_3 \frac{\partial W}{\partial \lambda_3}) - \frac{\lambda}{r}\frac{\partial W}{\partial \lambda}, \tag{5.72}
$$

$$
p''(r) = \frac{d^2}{dr^2}(\lambda_3 \frac{\partial W}{\partial \lambda_3}) - \frac{1}{r}(\frac{d\lambda}{dr}\frac{\partial \hat{W}}{\partial \lambda} + \lambda\frac{d}{dr}(\frac{\partial \hat{W}}{\partial \lambda})) + \frac{1}{r^2}\lambda\frac{\partial \hat{W}}{\partial \lambda}. \tag{5.73}
$$

Consider the boundary conditions of *pressure loading*, which, with respect to the reference configuration, may be written

$$
\mathbf{S}^T\mathbf{N} = -P\mathbf{F}^{-T}\mathbf{N}, \tag{5.74}
$$

where P is the pressure on the boundary per unit area of the deformed configuration. On taking the increment of Eq. (5.74) and updating to the deformed configuration we obtain

$$
\dot{\mathbf{S}}_{\mathbf{0}}^T\mathbf{n} = P\mathbf{H}^T\mathbf{n} - \dot{P}\mathbf{n}. \tag{5.75}
$$

Considering an infinite cylinder, we apply the specialisation of Eq. (5.75) on the cylindrical boundaries where we assume the outer boundary (i.e. $r = b$) free of incremental traction and the inner boundary (i.e. $r = a$) subject to pressure P. Taking $\dot{P} = 0$ in Eq. (5.75) we then have, for $i = 2, 3$,

$$
\dot{S}_{03i} = \begin{cases} PH_{3i} & \text{on} \quad r = a \\ 0 & \text{on} \quad r = b. \end{cases} \tag{5.76}
$$

For the considered axisymmetric case, the boundary conditions (5.76) along with Eq. (5.56), we get

$$(\mathcal{A}_{03333} - \mathcal{A}_{02233} + \lambda_3 \partial W/\partial \lambda_3) u_r + (\mathcal{A}_{01133} - \mathcal{A}_{02233}) u/r - \dot{p} = 0 \quad \text{on } r = a, b, \tag{5.77}$$

$$w_r + u_z = 0 \quad \text{on } r = a, b. \tag{5.78}$$

After a few algebraic manipulations and using Eqs. (3.34)–(3.42) and (5.68), the above expressions yield

$$\begin{aligned} &\mathcal{A}_{03232} r^3 F''' + [r\mathcal{A}'_{03232} - \mathcal{A}_{03232}] r^2 F'' - [r\mathcal{A}'_{03232} - \mathcal{A}_{03232} - \rho r^2 \omega^2] r F' \\ &- r^2 k^2 [(\mathcal{A}_{03333} + \mathcal{A}_{02222} - 2\mathcal{A}_{02233} - \mathcal{A}_{03223} + \lambda_3 \partial W/\partial \lambda_3] r F' \\ &- (r\mathcal{A}'_{03223} + \mathcal{A}_{01122} + \mathcal{A}_{03333} - \mathcal{A}_{01133} - \mathcal{A}_{02233} + r p' + \lambda_3 \partial W/\partial \lambda_3) F] = 0, \end{aligned} \tag{5.79}$$

and

$$r^2 F'' - r F' + r^2 k^2 F = 0, \tag{5.80}$$

respectively. Here, we have also made use of Eqs. (3.143) and (5.16).

5.4.1 Analysis of Wave Propagation for a Specific Model

We now apply the foregoing theory to a material following the special model given by Eq. (3.111). In the deformed configuration, using Eq. (5.3), the principal residual stresses from Eqs. (5.27) and (5.28) are given by

$$\hat{\tau}_1 = \tau_1/\mu = b_2(2\sqrt{1 + \lambda_z \lambda_a^2 (\hat{r}^2 - 1)} - 1 - \sqrt{1 + \lambda_z \lambda_a^2 (b^2/a^2 - 1)}), \tag{5.81}$$

$$\begin{aligned} \hat{\tau}_3 = \tau_3/\mu &= b_2(1 - 1/\sqrt{1 + \lambda_z \lambda_a^2 (\hat{r}^2 - 1)}) \\ &\times (\sqrt{1 + \lambda_z \lambda_a^2 (\hat{r}^2 - 1)} - \sqrt{1 + \lambda_z \lambda_a^2 (\hat{b}^2 - 1)}, \end{aligned} \tag{5.82}$$

where we have introduced the dimensionless quantities

$$\hat{r} = r/a, \quad \hat{b} = b/a. \tag{5.83}$$

and $b_2 = c_2 A/\mu$, already defined in Eq. (5.43).

We therefore have Eq. (5.71) in its specialized form

$$\begin{aligned}&\hat{\gamma}_1\hat{r}^4F'''' + 2[\hat{r}\hat{\gamma}_1' - \hat{\gamma}_1]\hat{r}^3F''' + [\hat{r}^2\hat{\gamma}_1'' - 3\hat{r}\hat{\gamma}_1' + 3\hat{\gamma}_1 + \hat{\omega}^2\hat{r}^2]\hat{r}^2F''\\ &-[\hat{r}^2\hat{\gamma}_1'' - 3\hat{r}\hat{\gamma}_1' + 3\hat{\gamma}_1 + \omega^2\hat{r}^2]\hat{r}F' + \hat{k}^2\hat{r}^2[\hat{\gamma}_2\hat{r}^2F'' + (\hat{r}\hat{\gamma}_2' - \hat{\gamma}_2)\hat{r}F'\\ &+(\hat{r}^2(p''/\mu) + \hat{r}\hat{\gamma}_4' + \hat{\gamma}_5 - \hat{\omega}^2\hat{r}^2)F] + \hat{k}^4\hat{r}^4\hat{\gamma}_6F = 0,\end{aligned} \tag{5.84}$$

where

$$\hat{\omega} = \sqrt{(\rho/\mu)}a\omega, \quad \hat{k} = ka, \quad \hat{\gamma}_i(\hat{r}) = \gamma_i(\hat{r})/\mu, \quad i \in \{1,2,4,5,6\}, \tag{5.85}$$

in the deformed configuration. We can re-write Eq. (5.84) as

$$\hat{f}_4(\hat{r})\hat{r}^4\frac{d^4F}{d\hat{r}^4} + \hat{f}_3(\hat{r})\hat{r}^3\frac{d^3F}{d\hat{r}^3} + \hat{f}_2(\hat{r})\hat{r}^2\frac{d^2F}{d\hat{r}^2} - \hat{f}_1(\hat{r})\hat{r}\frac{dF}{d\hat{r}} + \hat{f}(\hat{r})F = 0, \tag{5.86}$$

where

$$\begin{aligned}&\hat{f}_4(\hat{r}) = \hat{\gamma}_1, \quad \hat{f}_3(\hat{r}) = 2(\hat{r}\hat{\gamma}_1' - \hat{\gamma}_1),\\ &\hat{f}_2(\hat{r}) = \hat{r}^2\hat{\gamma}_1'' - 3\hat{r}\hat{\gamma}_1' + 3\hat{\gamma}_1 + \hat{\omega}^2\hat{r}^2 + \hat{k}^2\hat{r}^2\hat{\gamma}_2,\\ &\hat{f}_1(\hat{r}) = \hat{r}^2\hat{\gamma}_1'' - 3\hat{r}\hat{\gamma}_1' + 3\hat{\gamma}_1 + \hat{\omega}^2\hat{r}^2 - \hat{k}^2\hat{r}^2(\hat{r}\hat{\gamma}_2' - \hat{\gamma}_2),\\ &\hat{f}(\hat{r}) = \hat{k}^2\hat{r}^2(\hat{r}^2(p''/\mu) + \hat{r}\hat{\gamma}_4' + \hat{\gamma}_5 - \hat{\omega}^2\hat{r}^2 + \hat{k}^2\hat{r}^2\hat{\gamma}_6),\end{aligned} \tag{5.87}$$

where, for the special model under consideration, using Eqs. (3.47)–(3.55) with $\tau_{ij} = 0, i \neq j$, we have from Eq. (5.70)

$$\hat{\gamma}_1(\hat{r}) = \lambda^{-2}\lambda_z^{-2}[1 + \hat{\tau}_3 + 2b_0(I_7 - I_4)\hat{\tau}_3/\mu], \tag{5.88}$$

$$\begin{aligned}\hat{\gamma}_2(\hat{r}) &= -(1+\hat{\tau}_2)\lambda_z^2 - (1+\hat{\tau}_3)\lambda^{-2}\lambda_z^{-2} - 2b_0[(I_7 - I_4)(\lambda_z^2\hat{\tau}_2 + \lambda^{-2}\lambda_z^{-2}\hat{\tau}_3)/\mu\\ &+ 2(\lambda_z^2\hat{\tau}_2 - \lambda^{-2}\lambda_z^{-2}\hat{\tau}_3)^2],\end{aligned} \tag{5.89}$$

$$\gamma_3(\hat{r}) = 0, \tag{5.90}$$

$$\begin{aligned}\gamma_4(\hat{r}) &= (1+\hat{\tau}_3)\lambda^{-2}\lambda_z^{-2} + 2b_0\lambda^{-2}\lambda_z^{-2}\hat{\tau}_3[(I_7 - I_4)/\mu - 2\lambda^2\hat{\tau}_1 - 2\lambda_z^2\hat{\tau}_2 + 2\lambda^{-2}\lambda_z^{-2}\hat{\tau}_3]\\ &+ 4b_0\lambda^2\lambda_z^2\hat{\tau}_1\hat{\tau}_2,\end{aligned} \tag{5.91}$$

$$\begin{aligned}\gamma_5(\hat{r}) &= (1+\hat{\tau}_1)\lambda_1^2 - (1+\hat{\tau}_3)\lambda^{-2}\lambda_z^{-2} + 2b_0(\lambda^2\tau_1 - \lambda^{-2}\lambda_z^{-2}\hat{\tau}_3)[(I_7 - I_4)/\mu + 2\lambda^2\hat{\tau}_1\\ &+ 2\lambda^{-2}\lambda_z^{-2}\hat{\tau}_3 - 4\lambda_z^2\hat{\tau}_2],\end{aligned} \tag{5.92}$$

$$\gamma_6(\hat{r}) = \lambda_z^2 + 2b_0(I_7 - I_4)\lambda_z^2\tau_2/\mu + \lambda_z^2\tau_2, \tag{5.93}$$

where $b_0 = \bar{\mu}\mu$, already defined in Eq. (4.31). Also, in this case

$$(I_7 - I_4)/\mu = (\lambda^2 - 1)\hat{\tau}_1 + (\lambda_z^2 - 1)\hat{\tau}_2 + (\lambda^{-2}\lambda_z^{-2} - 1)\hat{\tau}_3, \tag{5.94}$$

which vanishes in the reference configuration.

The boundary conditions (5.79) and (5.80), at $\hat{r} = 1$ and $\hat{r} = \hat{b}$, give

$$\begin{aligned}&\hat{\gamma}_1\hat{r}^3 F''' + [\hat{r}a\gamma_1' - \gamma_1]\hat{r}^2 F'' - [\hat{r}a\gamma_1' - \gamma_1 - \hat{\omega}^2\hat{r}^2]\hat{r}F' \\ &-\hat{r}^2\hat{k}^2[(\hat{\gamma}_2 + \lambda_3\partial W/\partial\lambda_3(\hat{r})]\hat{r}F' - (\hat{\gamma}_4 + \hat{r}(p'/\mu) + \lambda_3\partial W/\partial\lambda_3(\hat{r}))F] = 0,\end{aligned} \tag{5.95}$$

and

$$\hat{r}^2 F'' - \hat{r}F' + \hat{r}^2\hat{k}^2 F = 0, \tag{5.96}$$

where

$$\lambda_3\partial W/\partial\lambda_3(\hat{r}) = \lambda^{-2}\lambda_z^{-2}, \tag{5.97}$$

since $\hat{\tau}_3$ vanishes at the boundaries. Both Eqs. (5.95) and (5.96) hold at $\hat{r} = 1$ and $\hat{r} = \hat{b}$.

The solution of Eq. (5.86) can be obtained numerically only due to its complexity. We introduce the following notation

$$F(\hat{r}) = z_1(\hat{r}), \quad F'(\hat{r}) = z_2(\hat{r}), \quad F''(\hat{r}) = z_3(\hat{r}), \quad F'''(\hat{r}) = z_4(\hat{r}), \tag{5.98}$$

which, from Eq. (5.86) gives

$$\begin{aligned}&\frac{dz_1(\hat{r})}{d\hat{r}} = z_2(\hat{r}), \quad \frac{dz_2(\hat{r})}{d\hat{r}} = z_3(\hat{r}), \quad \frac{dz_3(\hat{r})}{d\hat{r}} = z_4(\hat{r}), \\ &\frac{dz_4(\hat{r})}{d\hat{r}} = -\frac{\hat{f}_3}{\hat{f}_4}\hat{r}^{-1}z_4(\hat{r}) - \frac{\hat{f}_2}{\hat{f}_4}\hat{r}^{-2}z_3(\hat{r}) + \frac{\hat{f}_1}{\hat{f}_4}\hat{r}^{-3}z_2(\hat{r}) - \frac{\hat{f}}{\hat{f}_4}\hat{r}^{-4}z_1(\hat{r}),\end{aligned} \tag{5.99}$$

along with the four boundary conditions

$$\hat{\gamma}_1(1)z_4(1) + [\hat{\gamma}_1^{'}(1) - \hat{\gamma}_1(1)]z_3(1) - [\gamma_1^{'}(1) - \hat{\omega}^2]z_2(1) - \hat{k}^2[(\hat{\gamma}_2(1)$$
$$+\lambda^{-2}\lambda_z^{-2})z_2(1) - (\hat{\gamma}_4(1) + (p'/\mu) + \lambda^{-2}\lambda_z^{-2})z_1(1)] = 0, \tag{5.100}$$
$$z_3(1) - z_2(1) + \hat{k}^2 z_1(1) = 0, \tag{5.101}$$
$$\hat{\gamma}_1(\hat{b})\hat{b}^3 z_4(\hat{b}) + [\hat{b}\hat{\gamma}_1^{'}(\hat{b}) - \hat{\gamma}_1(\hat{b})]z_3(\hat{b}) - [\hat{b}\gamma_1^{'}(\hat{b}) - \hat{\omega}^2]z_2(\hat{b}) - \hat{b}^2\hat{k}^2[(\hat{\gamma}_2(\hat{b})$$
$$+\lambda^{-2}\lambda_z^{-2})\hat{b}z_2(\hat{b}) - (\hat{\gamma}_4(\hat{b}) + \hat{b}(p'/\mu) + \lambda^{-2}\lambda_z^{-2})z_1(\hat{b})] = 0, \tag{5.102}$$
$$\hat{b}^2 z_3(\hat{b}) - \hat{b}z_2(\hat{b}) + \hat{b}^2\hat{k}^2 z_1(\hat{b}) = 0. \tag{5.103}$$

5.4.2 Special Case: Boundary Value Problem for an Infinite Thick-Walled Cylindrical Tube in the Reference Configuration

Using the expressions from Eqs. (5.88)–(5.93) in the equation of motion (5.71) and the boundary conditions (5.79), we get the required equations to be solved in the deformed configuration for the materials specified by the special model (3.111). For simplicity of calculations, we assume $\tau_2 \equiv 0$ and solve the system in the reference configuration as a special case. In the reference configuration, the stretches are assumed as unity and their derivative with respect to r vanish. We therefore have from Eqs. (5.88)–(5.93)

$$\hat{\gamma}_1 = 1 + \hat{\tau}_3 \quad \hat{\gamma}_2 = -2 - \hat{\tau}_3 - 4b_0\hat{\tau}_3^2$$
$$\hat{\gamma}_3 = 0, \quad \hat{\gamma}_4 = 1 + \hat{\tau}_3 - 4b_0\hat{\tau}_3(\hat{\tau}_1 - \hat{\tau}_3),$$
$$\hat{\gamma}_5 = \hat{\tau}_1 - \hat{\tau}_3 + 4b_0(\hat{\tau}_1^2 - \hat{\tau}_3^2), \quad \hat{\gamma}_6 = 1, \tag{5.104}$$

where the principal residual stresses in the non-dimensionalized form are

$$\hat{\tau}_1 = b_2(2R/A - 1 - B/A), \tag{5.105}$$
$$\hat{\tau}_3 = b_2(1 - A/R)(R/A - B/A). \tag{5.106}$$

Also, in the reference configuration, using Eq. (5.73), the expression for $p_0''(R)$ reduces to

$$p_0''/\mu = \frac{d^2\hat{\tau}_3}{dR^2} - \frac{1}{R}(\frac{d\hat{\tau}_1}{dR} - \frac{d\hat{\tau}_3}{dR}) + \frac{1}{R^2}(\hat{\tau}_1 - \hat{\tau}_3), \tag{5.107}$$

which is evaluated using Eqs. (5.41) and (5.42) and we find that $p_0'' = 0$ for this special case.

For the purpose of non-dimensionalization, we introduce the notations

$$\hat{R} = R/A, \quad \hat{\beta} = B/A, \quad \hat{k} = kA,$$
$$\hat{\omega} = \sqrt{(\rho/\mu)}A\omega, \quad \hat{\gamma}_i(\hat{R}) = \gamma_i(\hat{R})/\mu, \quad i \in \{1, 2, 4, 5, 6\}. \tag{5.108}$$

Using above along with Eqs. (5.71) and (5.104)-(5.107), the specialized equation of motion in the reference configuration is given by

$$\hat{f}_4(\hat{R})\hat{R}^4 \frac{d^4 F}{d\hat{R}^4} + \hat{f}_3(\hat{R})\hat{R}^3 \frac{d^3 F}{d\hat{R}^3} + \hat{f}_2(\hat{R})\hat{R}^2 \frac{d^2 F}{d\hat{R}^2} - \hat{f}_1(\hat{R})\hat{R}\frac{dF}{d\hat{R}} + \hat{f}(\hat{R})F = 0, \tag{5.109}$$

where

$$\hat{f}_4(\hat{R}) = \hat{\gamma}_1, \quad \hat{f}_3(\hat{R}) = 2(\hat{R}\hat{\gamma}_1' - \hat{\gamma}_1),$$
$$\hat{f}_2(\hat{R}) = \hat{R}^2\hat{\gamma}_1'' - 3\hat{R}\hat{\gamma}_1' + 3\hat{\gamma}_1 + \hat{\omega}^2\hat{R}^2 + \hat{k}^2\hat{R}^2\hat{\gamma}_2,$$
$$\hat{f}_1(\hat{R}) = \hat{R}^2\hat{\gamma}_1'' - 3\hat{R}\hat{\gamma}_1' + 3\hat{\gamma}_1 + \hat{\omega}^2\hat{R}^2 - \hat{k}^2\hat{R}^2(\hat{R}\hat{\gamma}_2' - \hat{\gamma}_2),$$
$$\hat{f}(\hat{R}) = \hat{k}^2\hat{R}^2(\hat{R}\hat{\gamma}_4' + \hat{\gamma}_5 - \hat{\omega}^2\hat{R}^2 + \hat{k}^2\hat{R}^2\hat{\gamma}_6), \tag{5.110}$$

and for the special model under consideration, we have from Eqs. (5.104)

$$\hat{\gamma}_1(\hat{R}) = 1 + b_2(1 - 1/\hat{R})(\hat{R} - \hat{\beta}), \tag{5.111}$$
$$\hat{\gamma}_2(\hat{R}) = -2 - b_2(1 - 1/\hat{R})(\hat{R} - \hat{\beta}) - 4b_1[(1 - 1/\hat{R})(\hat{R} - \hat{\beta})]^2, \tag{5.112}$$
$$\hat{\gamma}_4(\hat{R}) = 1 + b_2(1 - 1/\hat{R})(\hat{R} - \hat{\beta}) - 4b_1(1 - 1/\hat{R})(\hat{R} - \hat{\beta})(\hat{R} - \hat{\beta}), \tag{5.113}$$
$$\hat{\gamma}_5(\hat{R}) = (\hat{R} - \hat{\beta}/\hat{R})[b_2 + 4b_1(3\hat{R} - 3 - 2\hat{\beta} + \hat{\beta}/\hat{R})], \quad \hat{\gamma}_6(\hat{R}) = 1, \tag{5.114}$$

and the respective derivatives

$$\hat{\gamma}_1'(\hat{R}) = b_2(1 - \hat{\beta}/\hat{R}^2), \tag{5.115}$$
$$\hat{\gamma}_1''(\hat{R}) = 2b_2\beta/\hat{R}^3, \tag{5.116}$$
$$\hat{\gamma}_2'(\hat{R}) = -(1 - \hat{\beta}/\hat{R}^2)[b_2 + 8b_1(1 - 1/\hat{R})(\hat{R} - \hat{\beta})], \tag{5.117}$$
$$\begin{aligned}\hat{\gamma}_4'(\hat{R}) &= b_2(1 - \hat{\beta}/\hat{R}^2) - 4b_1[(1 - \hat{\beta}/\hat{R}^2)(\hat{R} - \beta/\hat{R}) \\ &+ (1 - 1/\hat{R})(\hat{R} - \hat{\beta})(1 + \hat{\beta}/\hat{R}^2)],\end{aligned} \tag{5.118}$$

in the reference configuration.

The value of $p_0^{'}/\mu$ is given by

$$p_0^{'}/\mu = \hat{\gamma}_1^{'} - (\hat{\tau}_1 - \hat{\tau}_3)/R, \tag{5.119}$$

which vanishes in this special case. Also, the radial stress τ_3 vanishes at the boundary. Therefore, the boundary conditions from Eqs. (5.79) and (5.80), appropriately non-dimensionalized, respectively specialise to

$$\begin{aligned} \hat{R}^3 F^{'''} + (\hat{R}\hat{\gamma}_1^{'} - 1)\hat{R}^2 F^{''} - (\hat{R}\hat{\gamma}_1^{'} - 1 - \hat{\omega}^2\hat{R}^2 + 3\hat{k}^2\hat{R}^2)\hat{R}F^{'} + 2\hat{k}^2\hat{R}^2 F &= 0, \qquad (5.120) \\ \hat{R}^2 F^{''} - \hat{R}F^{'} + \hat{R}^2\hat{k}^2 F &= 0, \qquad (5.121) \end{aligned}$$

both of which hold at $\hat{R} = 1, \hat{\beta}$.

We seek a numerical solution of the ODE (5.109) and the problem may be converted into a system of first order linear ODEs. Let

$$F(\hat{R}) = y_1(\hat{R}), \quad F^{'}(\hat{R}) = y_2(\hat{R}), \quad F^{''}(\hat{R}) = y_3(\hat{R}), \quad F^{'''}(\hat{R}) = y_4(\hat{R}), \tag{5.122}$$

which, from Eq. (5.120) gives

$$\begin{aligned} \frac{dy_1}{d\hat{R}} &= y_2(\hat{R}), \quad \frac{dy_2}{d\hat{R}} = y_3(\hat{R}), \quad \frac{dy_3}{d\hat{R}} = y_4(\hat{R}), \\ \frac{dy_4}{d\hat{R}} &= -\frac{\hat{f}_3}{\hat{f}_4}\hat{R}^{-1}y_4 - \frac{\hat{f}_2}{\hat{f}_4}\hat{R}^{-2}y_3 + \frac{\hat{f}_1}{\hat{f}_4}\hat{R}^{-3}y_2 - \frac{\hat{f}}{\hat{f}_4}\hat{R}^{-4}y_1, \end{aligned} \tag{5.123}$$

along with the four boundary conditions (5.120) and (5.121) on $\hat{R} = 1$ and $\hat{R} = \hat{\beta}$ are given by

$$y_4(1) + (\hat{\gamma}_1^{'}(1) - 1)y_3(1) + (1 + \hat{\omega}^2 - 3\hat{k}^2)y_2(1) + 2\hat{k}^2 y_1(1) = 0, \tag{5.124}$$

$$y_3(1) - y_2(1) + \hat{k}^2 y_1(1) = 0, \tag{5.125}$$

$$\hat{\beta}^3 y_4(\hat{\beta}) + (\hat{\beta}\hat{\gamma}_1^{'}(\hat{\beta}) - 1)\hat{\beta}^2 y_3(\hat{\beta}) + (1 + \hat{\omega}^2\hat{\beta}^2 - 3\hat{k}^2\hat{\beta}^2)\hat{\beta}y_2(\hat{\beta}) + 2\hat{k}^2\hat{\beta}^2 y_1(\hat{\beta}) = 0, \tag{5.126}$$

$$\hat{\beta}^2 y_3(\hat{\beta}) - \hat{\beta}y_2(\hat{\beta}) + \hat{\beta}^2\hat{k}^2 y_1(\hat{\beta}) = 0. \tag{5.127}$$

The numerical solutions of this boundary value problem are presented in Section 5.4.4.

5.4.3 Isotropy

In the case when the residual stress vanishes, the material is isotropic. We therefore reduce the above boundary value problem to the classical case in linear elasticity and solve it analytically. For this special case, $b_1 = 0 = b_2$. Dropping the notation defined in Eq. (5.108), we have $\gamma_1 = 1, \gamma_2 = -2, \gamma_4 = 1, \gamma_6 = 1, \gamma_5 = 0 = \gamma_1' = \gamma_1'' = \gamma_2' = \gamma_4'$. The equation of motion (5.109) therefore becomes

$$\begin{aligned} & R^4 F'''' - 2R^3 F''' + (3 + \omega^2 R^2 - 2k^2 R^2) R^2 F'' - (3 + \omega^2 R^2 - 2k^2 R^2) R F' \\ + \; & k^2 R^4 (\omega^2 - k^2) F = 0, \end{aligned} \tag{5.128}$$

with boundary conditions from Eqs. (5.124)–(5.127)

$$F''' - F'' + (1 + \omega^2 - 3k^2) F' + 2k^2 F = 0, \quad \text{on } R = 1, \tag{5.129}$$

$$\beta^3 F''' - \beta^2 F'' + (1 + \omega^2 \beta^2 - 3k^2 \beta^2) \beta F' + 2k^2 \beta^2 F = 0, \quad \text{on } R = \beta, \tag{5.130}$$

$$F'' - F' + k^2 F = 0, \quad \text{on } R = 1, \tag{5.131}$$

$$\beta^2 F'' - \beta F' + \beta^2 k^2 F = 0, \quad \text{on } R = \beta. \tag{5.132}$$

Factoring the differential operator in Eq. (5.128), we can write

$$LM[F(R)] = 0, \tag{5.133}$$

where

$$L = r^4 \left(\frac{d^2}{dR^2} + \frac{1}{R} \frac{d}{dR} - k^2 \right), \quad M = \frac{d^2}{dR^2} - \frac{1}{R} \frac{d}{dR} - k^2 + \omega^2. \tag{5.134}$$

The solution of Eq. (5.133) is given by

$$\begin{aligned} F(R) &= C_1 R I_1(kR) + C_2 R K_1(kR) \\ &+ C_3 R J_1(\sqrt{\omega^2 - k^2} R) + C_4 R Y_1(\sqrt{\omega^2 - k^2} R), \end{aligned} \tag{5.135}$$

where C_1, C_2, C_3, C_4 are constants yet to be determined using the boundary conditions (5.129)–(5.130). Also, J_1 and Y_1 are the Bessel functions of first and the second kind of order one whereas the functions I_1 and K_1 are the modified Bessel functions of the first and the second kind of order one, respectively. A detailed discussion on the behaviour of these functions and their derivatives is given in [1].

Using Eq. (5.135) in (5.129)–(5.132), we get four algebraic equations in four unknowns C_1, C_2, C_3 and C_4. For a non-trivial solution the determinant of the matrix of coefficient, say $\mathbf{A}$ must vanish. Therefore, the dispersion relation relating k and ω, in this case is given by

$$\det \mathbf{A} = 0. \tag{5.136}$$

Here, we do not give the expression for $\det \mathbf{A}$ due its complexity. For a fixed β, we use a simple code in MAPLE to find the exact expression for $\det \mathbf{A}$ and solving Eq. (5.136). Further the use of command '`implicitplot`' gives the required curves for dispersion relation. The continuous graphs in Fig. 5.12 represent the dispersion relation for an isotropic material in linear elasticity.

5.4.4 Numerical Results

Unlike an initial value problem (IVP), a boundary value problem (BVP) may not have a solution at all, or may have a unique solution, or may have more than one solution. Because there might be more than one solution, BVP solvers require an initial guess for the solution of interest. Often there are parameters that need to be determined such that the BVP has a non-trivial solution. Associated with the solution, there might be one set of parameters, a finite number of possible sets, or an infinite number of possible sets.

For the purpose of solving Eq. (5.123) numerically with the boundary conditions (5.124)–(5.127), we have used a built-in MATLAB function '`BVP4C`'. A detailed discussion on the structure of this function can be found in [45]. This built-in function uses the system of first order ordinary differential equations and an initial guess for the solution as well as for the unknown parameters. For fixed $\hat{k}$, $\hat{\beta}$, b_1 and b_2, the solver gives a solution and hence obtains a dispersion relation between the dimensionless wave number $\hat{k}$ and the dimensionless frequency $\hat{\omega}$ (or equivalently, the dimensionless phase speed $\hat{c} = \hat{\omega}/\hat{k}$) and the related solution to the problem.

The special case when $b_1 = 0 = b_2$ refers to vanishing of the residual stress. For this special case, dispersion curves are shown in Fig. 5.12 which exactly match the results obtained analytically in Section 5.4.3. These results are also similar to those obtained for the case of axially symmetric waves in a hollow elastic rod in [30] for a complete range of wall thicknesses and frequencies. In [30], the theory was developed using expansions of the displacements in a series of orthogonal polynomials in the radial coordinate, retaining only the earliest terms.

To offset the error due to omission of terms various adjustment factors were introduced for the frequency spectrum to match the exact theory.

In a recent paper by Akbarov and Guz [3], the authors present results for an initially stressed and pre-stretched compound cylinder and specialized results are given for a hollow cylinder both in the presence and absence of initial stretch. The dispersion curves presented in Fig. 5.13 are quite similar to those presented in [3] for various thicknesses in the absence of initial stress. The same paper presents the plots of dispersion curves when the initial stress and pre-stretch are both included. Since we assume the pre-stretch as unity, our graphs show modes without any further branches in contrast to the results presented in [3]. Apart from this, the graphs are similar in their first few modes.

The effect of residual stress on different modes is presented in Figs. 5.14 –5.16. In the absence of residual stress, the strain energy function reduces to the neo-Hookean case. The dispersion curves for the first modes in this case are plotted in Fig. 5.13 and it is worth noting that with increasing wall thickness, the speed is the same for very small dimensionless wave number. This plots in Fig. 5.14 show a shift from this behaviour and with increasing wall thickness, the first modes have different phase speeds from small $\hat{k}$. From Fig. 5.15, the dispersion curves for the first modes for fixed wall thickness and varying parameters are shown. The graph labelled as $'a'$ is the behaviour in the case of neo-Hookean type material. We observe that as positive values of b_1 and b_2 increase, the phase speed decreases from that in neo-Hookean type materials. For decreasing values (see the negative values of b_1 or b_2 for plots (d) and (e)) of the parameters, the phase speed increases. Figure 5.16 shows the first four modes for the case when the residual stress is present (continuous plots) and when the residual stress vanishes (dashed plots).

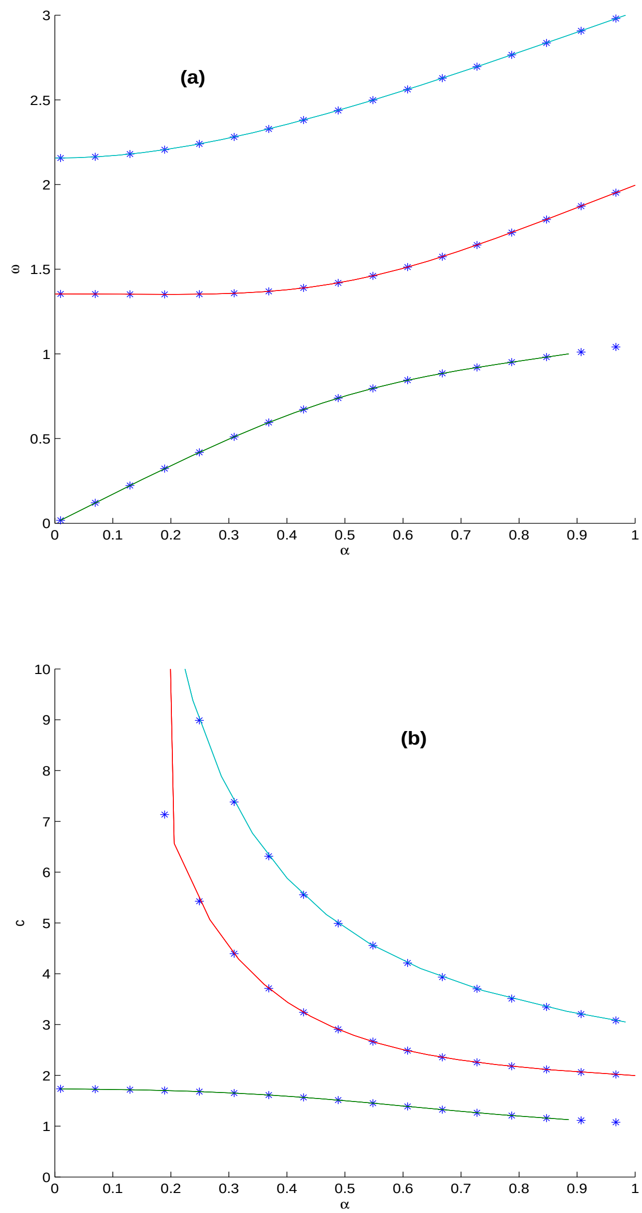


Figure 5.12: Comparison of the dispersion curves of the first three modes for the linear elasticity case (continuous curve) from Eq. (5.136) and the numerical results from Eq. (5.123)–(5.127) in the absence of residual stress, $b_1 = 0 = b_2, \beta = \hat{\beta} = 2.5$. (a) ω with respect k, (b) c with respect to k

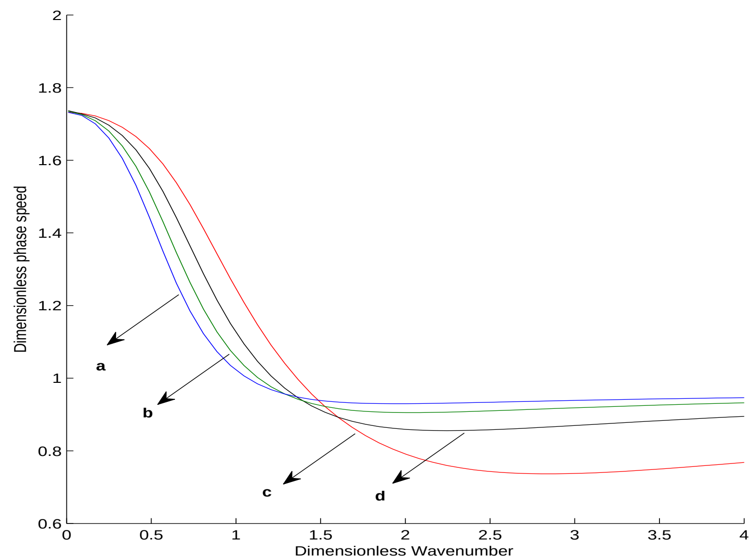


Figure 5.13: The dispersion curves of the first mode from Eqs. (5.123)–(5.127) in the absence of residual stress, $b_1 = 0 = b_2$, (a) $\hat{\beta} = 3$, (b) $\hat{\beta} = 2.5$, (c) $\hat{\beta} = 2$, (d) $\hat{\beta} = 1.5$.

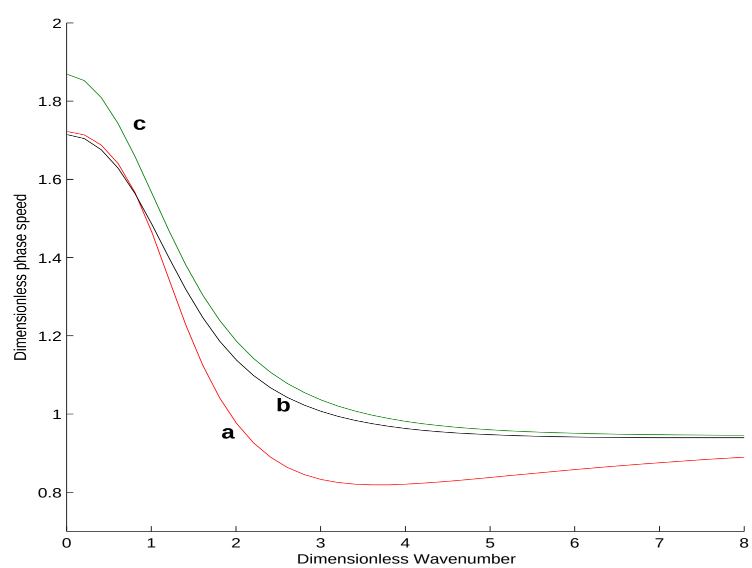


Figure 5.14: The dispersion curve for the first mode from Eqs. (5.123)–(5.127) for $b_1 = 7, b_2 = 2$ and (a) $\hat{\beta} = 1.5$, (b) $\hat{\beta} = 2$, (c) $\hat{\beta} = 2.5$.

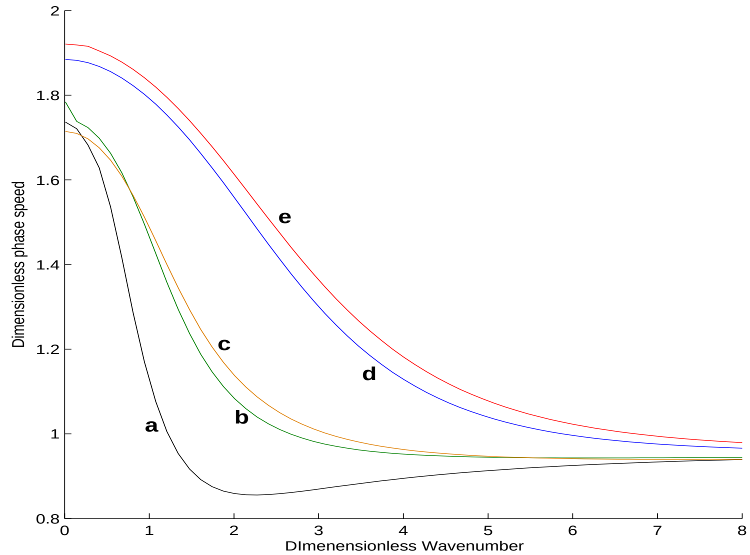


Figure 5.15: The dispersion curves of the first mode from Eqs. (5.123)–(5.127) for $\hat{\beta} = 2.5$ and (a) $b_1 = 0 = b_2$, (b) $b_1 = 4, b_2 = 1$, (c) $b_1 = 7, b_2 = 2$, (d) $b_1 = -4, b_2 = 1$, (e) $b_1 = -5, b_2 = -1$.

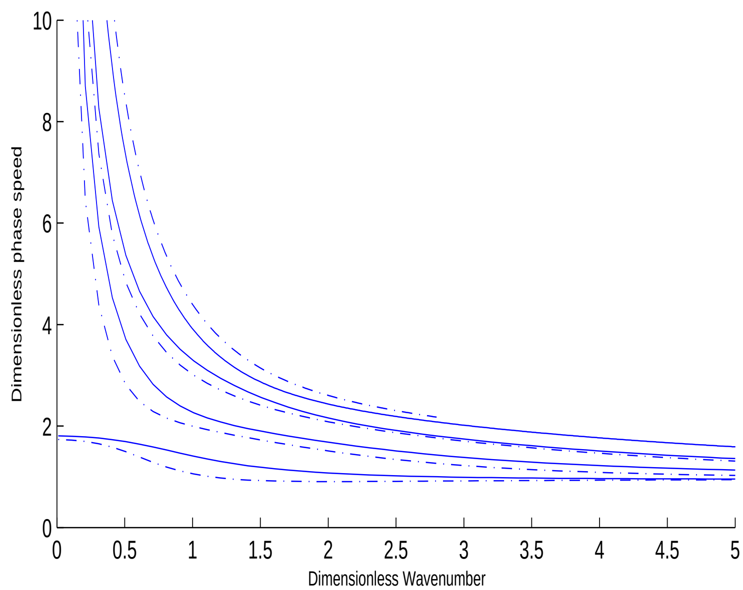


Figure 5.16: First four modes from Eqs. (5.123)–(5.127) for $\hat{\beta} = 2.5$, Continuous graph for $b_1 = 2, b_2 = 1$, dashed graph for $b_1 = 0 = b_2$.

Chapter 6

Conclusion and Future Work

6.1 Conclusions

The existence of initial/residual stress in materials has proven to be an interesting field of research and relatively less work has been done in this area. The work done by Biot [4, 5] in 1939 gave a foundation for this research. However, more substantial and recent development is due to Hoger [16, 17, 18, 19, 26, 27, 20] and Man and Lu [28]. We consider various problems to study the effect of initial/residual stress in elastic materials and develop the basic formulation by using the theory of infinitesimal deformations superimposed on finite deformations. The problems discussed include the plane wave reflection from the boundary of an initially-stressed half-space, the analysis of propagation of a surface wave in an initially-stressed hyperelastic material and wave propagation in an inflated and axially stretched elastic tube.

Chapter 2 gives the very basic concepts that are used in this thesis.

Using the general form of elasticity tensor in Chapter 3, we have developed the basic constitutive equations for a hyperelastic material. The theory for the propagation of small amplitude plane waves is developed and is then implemented to study the reflection of plane homogeneous waves from the traction free boundary of an initially-stressed incompressible half-space. It is found that a unique reflected wave exists for some angles of incidence explicitly mentioned in section 3.4.3. An additional reflected wave exists if the angle of incidence is either greater or less than a critical value. The amplitude of this wave vanishes however for some angles of incidence for a special class of materials. The reflection coefficients of the waves are calculated and it is found that these depend on the state of deformation and the magnitude of the initial stress. We carry out some specialized results for a special constitutive model which reduces the neo-Hookean case when the initial stress vanishes. For infinitesimally small initial stress, the behaviour of the reflection coefficients (see Fig. 3.6 for $|R'|$ and $|R|(= 1)$) is similar to that in the case of linear compressible theory. These results are also comparable to the results for neo-Hookean type materials in [39] for pure homogeneous strain in the absence of a pre-stress and to those in [23] for simple shear (for a special class of materials) in the absence of pre-stress. The presence of initial stress affects the wave speed and it is found that the wave speed for the first reflected wave decreases and that of the second wave increases infinitely (while its amplitude vanishes) with the increase in the magnitude of initial stress or for larger values of stretch and as the angle of incidence approaches normal. A similar behaviour of wave speed was found in [39] for the Varga strain energy function.

For materials when initial stress (τ) does not vanish and stretches (λ) are not unity, there is a reflected wave for every angle of incidence. Though, the amplitude of this reflected may vanish at various angles of incidence depending on the choice of τ and λ. It is obvious that for very high values of τ or λ, there is only one reflected wave whose amplitude approaches unity as the angle of incidence approaches normal.

In Chapter 3, the research is concerned with the behaviour of surface waves in an initially-stressed incompressible material. We use the work of Dowaikh and Ogden [9] for Rayleigh waves and that of Dowaikh [8] for Love wave.

In the case of Rayleigh waves, the secular equation is analysed and it is found that within the region of stability of $(\lambda, \bar{\tau})$ where $\bar{\tau}$ is the dimensionless initial stress, a unique surface wave exists in a specific interval dependent upon the initial stress and deformation within the stability region. Here, the strong ellipticity condition provides the region of stability. The general theory developed here is then applied to the special prototype model to study the effect of initial stress and deformation on the Rayleigh surface waves. The cubic secular equation in its dimensionless form is solved exactly and the solution is a function of initial stress and deformation. This secular equation reduces to its counterpart in the linear theory when initial stress vanishes. The solution from the secular equation is then used to calculate the dimensionless speed ξ of the Rayleigh surface wave for various choices of the initial stress which is plotted with respect to λ. The dimensionless speed for a plane shear wave ξ_s is also calculated in the presence of the initial stress. As a special case, it is inferred that for zero initial stress the dimensionless Rayleigh wave speed approaches the shear wave speed for increasing λ which is a result from the classical linear theory. The graphs show the plots in reference to the plots when residual stress vanishes. It is found that for positive values of initial stress, ξ decreases with increasing stretch and vice versa for negative values of the initial stress. It is also found that for a particular choice of $\bar{\tau}$ and other constants, the wave speed vanishes at a specific value or values of λ. The secular equation is also analysed in a more convenient form after a squaring process. However, this process introduces several solutions of the modified equation which are not the solutions of the original secular equation. This is illustrated graphically at the end of the problem. Also, a similar discussion is extended for the undeformed reference configuration and graphs are presented to study the effect initial stress on Rayleigh surface waves for $\lambda = 1$.

In the later part of Chapter 3, the discussion on Love waves in an initially-stressed layer of thickness h bounded below by an initially stressed infinite half-space is carried out. The theory developed generally is then applied to the special model and the dispersion relation is dependent on $\bar{\tau}, \lambda$ and the dimensionless wave number kh. It is shown through various plots that for increasing kh (or decreasing wave length) the speed of Love waves decreases (from that in the absence of initial stress) when the initial stress is compressional and vice versa for tensile initial stress.

The initial stress is the so-called *residual stress* when the boundary of the material is traction free. This traction-free boundary requires the material to be necessarily inhomogeneous for the residual stress to exist. In Chapter 4, the effect of a non-homogeneous initial stress i.e., the residual stress (also represented by τ) on small amplitude waves is observed in axisymmetric case of a thick-walled incompressible elastic cylinder. A simple expression of residual stress depending on the radius of the cylinder is chosen on the basis of experimental behaviour of such kind of stress in arteries and vessels. The pressure and axial load are calculated and the results are plotted and compared to those when the material is not residually stressed. It is found that the presence of residual stress does effect the pressure and the axial load. A sharp increase is observed for various choices of the respective parameters characterising the magnitude of residual stress. It is also observed that, for a fixed wall thickness, in contrast to the neo-Hookean materials (or Mooney-Rivlin type materials), the pressure either increases or decreases and may vanish at more that one values of the stretch ratio $\lambda_a = a/A$ where 'a' ('A') is the radius in the deformed (undeformed reference) configuration. A similar behaviour is observed for axial load which is illustrated in the graphs.

In the later part of Chapter 4, the analysis of small amplitude wave propagation is carried out in a residually stressed thick-walled incompressible elastic cylinder. The generalized equations for the axisymmetric case in the deformed configuration are presented and the problem is later specialized for the simple prototype model which depends on both the residual stress and the finite deformation. The more general problem in the deformed configuration is then specialized to the reference configuration by considering the stretches equal to unity. To avoid complexity, the problem is solved numerically to study the dispersion relation. The dispersion curves for the case of zero residual stress are obtained as reference in each graph. On comparison, it is found that the presence of residual stress either increases or decreases the phase speed depending on the parameters related to the magnitude of the residual stress. It is worth noting that for higher positive values of the parameters, the phase speed drops from the reference speed in a non-residually stressed material. For lower or negative values of these parameters, an increase in the phase speed is observed. The results are also compared to the case discussed in [3] and [30]. In the latter paper, similar results are obtained for pre-stressed materials using a different theory. The results also match those presented in the latter paper for the hollow cylinder when the pre-stretch is unity.

6.2 Future Work

A straightforward extension of this research is to carry out respective problems with compressibility included. Further, the case of reflection of homogeneous plane waves can be extended to transmission through an interface where the half-spaces can be initially stressed and the material may be compressible or incompressible. Also, the effect of viscoelasticity may be included in the problems where possible.

The problem related to the thick walled cylinder in this thesis is solved for material in the reference configuration only. The general problem is much more complicated. However, it is possible to solve it numerically when an efficient code is used in any mathematical software. In respect of this, the case of wave propagation in a cylinder with thin-walled tube or the case of solid cylinder can also be considered.

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