Lattices and Automorphisms of Compact Complex Manifolds

by

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Contents

Statement

Chapter 1 collects definitions and basic known results in lattice and coding theory, sporadic finite simple groups, resolutions of singularities and toric geometry. See [**13**] [**21**] [**35**] [**2**] [**3**] [**27**] [**16**] [**33**] [**24**] [**52**].

The material in chapter 2 is largely known [**59**] [**50**] [**34**]. The description in terms of holomorphic surgery of line bundles is new (§1), as are the specific examples (§3). Again in chapter 3, the construction of generalised Kummer manifolds is given in terms of line bundles, though such resolutions are well documented. The examples of §3,4 & 5 are all new, as is the proof they are simply connected in §4 (though based on a proof of Spanier [**59**]).

Paragraph §1 of chapter 4 summarises known material on abelian varieties [**40**] [**28**]. §2 is original work, and §3,4 apply known theory and formulae to these constructions.

In chapter 5, paragraph §1 cites a theorem by Demazure [**20**]. In §2, 3 the construction of Δ_W is not new [**62**], but the calculations of the symmetries are. §5 is all original.

Abstract

This work makes use of well-known integral lattices to construct complex algebraic varieties reflecting properties of the lattices. In particular the automorphism groups of the lattices are closely related to the symmetries of varieties.

The constructions are of two types: generalised Kummer manifolds and toric varieties. In both cases the examples are of the most interest.

A generalised Kummer manifold is the resolution of the quotient of a complex torus by some finite group G. A description of the construction for certain cyclic groups G is given in terms of holomorphic surgery of disc bundles. The action of the automorphism groups is given explicitly. The most important example is a compact complex 12-dimensional manifold associated to the Leech lattice admitting an action of the finite simple Suzuki group. All these generalised Kummer manifolds are shown to be simply connected.

Toric varieties are associated to certain decompositions of $\mathbb{R}^n$ into convex cones. The automorphism groups of those associated to Weyl group decompositions of $\mathbb{R}^n$ are calculated. These are used to construct 24-dimensional singular varieties from some Niemeier lattices. Their symmetries are extensions of Mathieu groups and their singularities closely related to the Golay codes.

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Introduction

The main goal set out for this thesis was to construct compact complex manifolds admitting actions of finite simple groups. With hindsight another objective might have been complex analytic spaces reflecting properties of a lattice in various ways (the automorphisms being one of them). The motivations for such endeavours are several, and we shall return to them.

Throughout the text lattices are positive definite quadratic forms, with root lattices, the Niemeier lattices and the Leech lattice as main examples. The constructions of many of the lattices make use of codes and their automorphisms are often closely related to finite simple groups. All the examples and these connections are discussed in chapter 1.

We examine essentially two different constructions of complex analytic spaces making use of lattices, namely generalised Kummer manifolds and toric varieties. The first are by definition resolutions of quotients of complex tori by finite groups. The second are determined by a set of convex cones in $\mathbb{R}^n$ generated by vectors in a lattice. We study these constructions from the viewpoint of the lattice and see how properties of particular lattices are reflected in the geometries of the resulting varieties. In particular, what can be said about the group of automorphisms of the varieties (that is the group of biholomorphic tranformations or biregular maps).

Generalised Kummer manifolds (g.K.m.'s) are discussed in chapters 2,3 and 4. The classical Kummer surfaces are minimal resolutions of the quotient of a torus T of complex dimension 2 by the involution $\langle \pm 1 \rangle$ sending a point to its inverse in the group structure. These were extensively studied by Hudson [**34**] in 1905 and are examples of $K3$-surfaces. In §1 of chapter 2 we give details of the construction of an n-dimensional Kummer manifold K (involving attaching disc bundles by surgery) and how automorphisms of K can be described in this model. As throughout this is done from the perspective of the lattice involved, in this case in the construction of the torus $T = \mathbb{C}^n/L$ (the involution sends $l \in L$ to $-l$). In the 2 dimensional case this involves looking at lattices of rank 4 and their automorphisms. We give details of cases giving rise to interesting group actions. Finite symplectic automorphism groups of $K3$-surfaces were classified by Mukai [**50**] and we examine how the groups acting on the Kummer surfaces fit into this picture. Due to this connection we pay more attention to symplectic actions in the 2 dimensional case. The root lattice D_4 gives rise to the most interesting case with a symplectic action of $2^4 \rtimes A_4$ on the associated Kummer surface K_{D_4}.

Of course the more interesting simple groups appear in higher dimensions: we make use of the construction of ch.2 §1 to obtain Kummer manifolds admitting actions of these larger finite simple groups (ch.2 §3). Not surprisingly the Leech lattice provides the most interesting example: a 12-dimensional complex Kummer manifold $K_{\Lambda_{24}}$ admitting an action of the Suzuki group.

The generalised Kummer manifolds in chapter 3 are resolutions of quotients of complex tori by cyclic groups of the form $\langle\Theta\rangle$ where Θ acts on the universal covering $\mathbb{C}^n$ of T as multiplication by $\exp(2\pi i/d)$. These cyclic groups have a fixed-point-free action (apart from the origin) on the lattice L involved and correspond to some complex structure on L. The general construction is similar to that of the Kummer manifolds but involves slightly more complicated resolutions. We again view the resolution as attaching disc bundles by surgery. This description also allows us to determine the fundamental group of the generalised Kummer manifolds constructed (ch.3 §7). S.S.Roan's results tell us that we essentially cover all possible quotients of this type (ch.3 §8): only cyclic groups $\langle\Theta\rangle$ of order at most 6 can occur. We give details for the most interesting example, again arising from the Leech lattice: $X_{\Lambda_{24}}$ is a 12-dimensional generalised Kummer manifold admitting an action of the Suzuki group. The advantage of taking these larger quotients (as opposed to the $\langle\pm 1\rangle$ for Kummer manifolds) is that it eliminates "uninteresting" symmetries of the resulting manifold. In other words we obtain a symmetry group as close to the finite simple group as possible. These quotients also give rise to intricate combinatorial identifications among the attached bundles. Unfortunately we are unable to determine if the groups obtained are the entire group of (Kähler) transformations, although they are certainly all isometries coming from the torus.

In chapter 4 we show that most the examples of Kummer and generalised Kummer manifolds constructed are in fact algebraic vareties. This boils down to checking certain conditions for the lattices (the Riemann conditions). A complex structure on the lattice turns out to be sufficient for these conditions to hold and most the lattices considered have such a structure.

Toric varieties are the second lattice related construction we consider. As already mentioned these are determined by a collection of cones (called a fan) in euclidean space where each cone is generated by vectors in some lattice N (for details ch.1 §5). These are characterised as normal varieties containing a dense open algebraic torus whose action on itself extends to one on the entire variety.

The basic example of toric varieties we consider are those corresponding to the Weyl chamber decomposition associated to some root lattice R. The cones of the decomposition are in fact in the dual lattice R^*. These varieties are non-singular and we determine the exact automorphism group using a theorem of Demazure (the Weyl group is naturally involved). However these are essentially too symmetrical for our purposes of identifying interesting finite simple group actions. A natural next step is to apply these root lattice decompositions to (some of) the Niemeier lattices, each determined by its root sublattice of rank 24. The resulting 24-dimensional singular complex varieties inherit actions from symmetries of the lattices and reflect other properties of the lattices in their singularities. The main examples have symmetries involving the Mathieu groups and singularities related to the Golay codes. For these reasons and also the simplicity of the construction the varieties appear as natural geometric realisations of the Niemeier lattices.

Toric varieties are entirely different from the spaces considered so far. The trivial toric variety corresponding to the fan consisting only of the origin in $\mathbb{R}^n$ is the complex algebraic torus T_N of dimension n. A compact complex torus is the quotient of T_N by some free subgroup of rank n. In this sense generalised Kummer manifolds are (resolutions of) quotients of a (trivial) toric variety. This seems to be the limit of any direct link between the two different constructions, though indirectly they will be seen to be connected by their symmetry groups. Notice also that in the toric setting a $2n$-dimensional lattice L produces a $2n$-dimensional complex variety while the associated torus $\mathbb{C}^n/L$ is of dimension n. From the point of view of the symmetries, one has in a sense doubled the dimension of L and added a complex structure when comparing some g.K.m. with a toric variety, both constructed from L.

Singularities and their resolutions are present throughout this work. The resolutions involved in the generalised Kummer manifolds are very simple (involving a simple blow-up of the original torus), though the description given is less conventional. The singularities in the toric material are closely related to codes and open up new lines of inquiry in this respect. Minimal resolutions are resolutions yielding a space with trivial canonical bundle and these are important in the construction of mathematical models of field theories for example (see last paragraph). The resolutions in this work are mostly not minimal, and indeed most the singularities considered admit no such resolution. These questions fit into the framework of the generalised McKay correspondence discussed in chapter 3, which attempts to link the representation theory of the group $G < \mathrm{SL}(n, \mathbb{C})$ with minimal resolutions of the singularity $\mathbb{C}^n/G$. This stems from John Mckay's observations in dimension 2 [**47**].

There are several motivations behind this research. The constructions fit into the particular aim of obtaining all sporadic finite simple groups as automorphism groups of compact complex manifolds, with ultimate challenge the Monster simple group. Whether this last goal is achievable is still unclear. A more general aim is to give some geometrical interpretation to the list of sporadics. Obtaining them as symmetry groups of some family of geometrical objects (such as compact complex manifolds, or a particular class or family of these) would achieve this for example. Currently there is no unified way of describing the 26 sporadics, some best displayed as symmetries of codes (eg. the Mathieu groups), others of lattices (eg. the Conway groups), others of vertex operator algebras (eg. the Monster), others neither of these three (eg. the Fisher groups).

A different motivation though perhaps related to the discussion above, is the construction of models of field theories in theoretical physics. The methods involved in this work are very close to the techniques used in constructing these so-called sigma models. This is to the extent that much of the literature is to be found in mathematical physics. If one requires the resolution to be minimal as defined in chapter 3, then generalised Kummer manifolds are natural generalisations of models of field theories in dimension 3 (see [**44**]). Note however that most our examples admit no such resolutions by results of Roan (ch.3 §8). Indeed minimal resolutions can only occur for quotients of tori of dimension 2 and 3 by the cyclic groups of order 2 and 3 respectively. Other models are described as subvarieties of toric varieties. The hope is that the constructions in this work may through their symmetries lead to important such examples.

Notation

$\mathbb{H}$	quaternions
$\mathbb{F}_q$	finite field of $p = p^r$ elements
X, M	complex manifolds
$\mathrm{Aut}(X)$	group of biholomorphic transformations of X
$b_i(X)$	Betti numbers of X
$\tau(X)$	signature of X
$\chi(X)$	Euler charateristic of X
$\mathcal{L}$	universal line bundle over $\mathbb{C}\mathrm{P}^{n-1}$
$\mathcal{K}_X$	canonical line bundle of X
L, Λ	lattices
mL	$L \oplus \cdots \oplus L$ (m times)
R	root lattice
Φ	root system
$\mathrm{W} = \mathrm{W}(R) = \mathrm{W}(\Phi)$	Weyl group of a root system or lattice
$\mathrm{Aut}(L)$	group of automorphisms of a lattice L
$\mathbb{Z}^n, \mathrm{A}_n, \mathrm{D}_n, \mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8$	root lattices
Λ_{24}	Leech lattice
K_{12}	Coxeter-Todd lattice
$N = N(R) = R^+$	Niemeier lattice with root sublattice R
$\mathcal{G} = \mathbb{Z}[i]$	ring of Gaussian integers
$\mathcal{E} = \mathbb{Z}[\omega]$	ring of Eisenstein integers
$\mathcal{H}$	ring of Hurwitz integers
$T = T_L$	compact complex torus $T = \mathbb{C}^n/L$
$\mathrm{Hom}(T)$	group homomorphisms of the torus T
Tr	group of translations of the torus
K_L	Kummer manifold associated to T_L

C	linear code
$\mathrm{Aut}(C)$	group of automorphisms of the code C
$\mathcal{C}_{24}$	binary Golay code
$\mathcal{C}_{12}$	ternary Golay code
$\mathcal{C}_6$	hexacode

For groups A, B:

$A.B$	arbitrary extension of A by B
$A \rtimes B$	split extension of A by B
$A \times B$	direct sum of A and B
A^m	direct sum $A \times \cdots \times A$ (m times)
$\mathbb{Z}/n$ or n	cyclic group of order n
A_n, S_n	alternating and symmetric groups
D_n	dihedral group of order $2n$

In toric material:

N	lattice of rank n
M	lattice dual to N
σ	cone in $N \otimes \mathbb{R} = \mathbb{R}^n$
$\sigma^\vee$	dual cone to σ in $M \otimes R$
Δ	fan=collection of cones
$X(\Delta, N)$	toric variety from Δ in N
T_N	complex algebraic torus $(\mathbb{C}^*)^n$

In chapter 4:

Λ	lattice
L	line bundle over T

The particular changes in notation for chapter 4 are made to remain consistent with the referenced literature.

CHAPTER 1

Lattices, finite simple groups and toric geometry

1. Lattices

We give the principal definitions and properties of lattices.

DEFINITION 1. *A* lattice *L is a finitely generated free $\mathbb{Z}$-module L together with a non-degenerate symmetric bilinear form $\langle\ ,\ \rangle : L \times L \to \mathbb{R}$.*

The *dimension* of the lattice is the rank of the module, while a *basis* of L is just a module basis. A *positive definite* or *real* lattice is then a subset $L \subset \mathbb{R}^n$ of all $\mathbb{Z}$-linear combinations over a real basis $\{v_1, \ldots, v_n\}$ of $\mathbb{R}^n$:

$$L = \mathbb{Z}\{v_1, \ldots, v_n\}.$$

The usual euclidean dot product is the symmetric form in this case.

A general lattice L is an *integral lattice* if $\langle x, y\rangle \in \mathbb{Z}$ for all $x, y \in L$. L is called *even* if $\langle x, x\rangle \equiv 0 \mod 2$ for all $x \in L$, and *odd* otherwise.

Fix a basis $\{l_1, \ldots, l_n\}$ of L. The $n \times m$ matrix M with rows l_1 to l_n is a *generator matrix* for L. The *Gram matrix* $A = MM^t$ is the square matrix with entries $a_{ij} = \langle l_i, l_j\rangle$.

DEFINITION 2. *The* dual lattice *L^* is the set $\mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$ of all $\mathbb{Z}$-linear maps from L to $\mathbb{Z}$. Equivalently, for a positive definite lattice $L \subset \mathbb{R}^n$,*

$$L^* = \{x \in \mathbb{R}^n : \langle x, y\rangle \in \mathbb{Z}\ \forall y \in L\}$$

A lattice is *self-dual* if $L = L^*$, and *unimodular* if positive definite and self-dual. If L is integral then $L \subset L^*$ and the quotient $G = L^*/L$ is a finite abelian group whose order is called the *discriminant* of L. Note $\mathrm{discr}(L) = \det A$.

DEFINITION 3. *An* automorphism *(or* symmetry*) of a lattice L is a $\mathbb{Z}$-linear map $L \to L$ preserving the form $\langle\ ,\ \rangle$.*

We denote the group of all automorphisms of L by $\mathrm{Aut}(L)$.

If L is positive definite then an automorphism of L is a map $A \in \mathrm{O}(n)$ such that $A(L) = L$.

Two lattices L_1 and L_2 are *isomorphic* (denoted $L_1 \cong L_2$) if there is a bijection $\varphi : L_1 \to L_2$ such that $\langle x, y\rangle = \langle \varphi x, \varphi y\rangle$ for all $x, y \in L_1$. For positive definite lattices, the isomorphism is an invertible orthogonal map $A \in \mathrm{O}(n)$ mapping L_1 to L_2. Two lattices are *similar* (denoted $L_1 \simeq L_2$) if one also allows a change of scale, that is $cL_1 \cong L_2$ for some constant $c \in \mathbb{R}$.

The *direct sum* $L_1 \oplus L_2$ is the sum of the modules with the induced bilinear form on the two summands and $\langle x, y\rangle = 0$ for all $x \in L_1,\ y \in L_2$. A lattice is *irreducible* if it cannot be split as a direct sum.

A *shell* of a lattice L is the set of vectors in L of a given norm.

1.1. Root lattices and root systems. The *reflection* in a hyperplane $r^{\perp}$ $(r \in L)$ is the linear map

$$s_r(\lambda) = \lambda - \frac{2\langle \lambda, r\rangle}{\langle r, r\rangle} r$$

A *root* is an element $r \in L$ of norm 2, $\langle r, r\rangle = 2$. The associated reflection s_r is a symmetry of the lattice, that is $s_r(L) = L$, and reflections in all the roots of L generate a finite *reflection group*. A *root lattice* R is a positive definite lattice generated by its roots. Finite reflection groups are classified [**35**], leading to the following list of irreducible root lattices:

$$\mathbb{Z}, \mathrm{A_n}\ (n \geq 1), \mathrm{D_n}\ (n \geq 4), \mathrm{E_6}, \mathrm{E_7}, \mathrm{E_8}$$

The general root lattices are the direct sums of these. We now give the definitions of the (irreducible) root lattices and their duals both intrinsically and by a generator matrix.

The n-dimensional lattices $\mathrm{A_n}$ and $\mathrm{A_n^*}$.

$$\mathrm{A_n} = \{(x_1, \ldots, x_{n+1}) \in \mathbb{Z}^{n+1} : x_1 + \cdots + x_{n+1} = 0\}$$

and has generator matrix

$$M = \begin{bmatrix} -1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 \end{bmatrix}.$$

Its dual $\mathrm{A}_{\mathrm{n}}^*$ has generator matrix

$$M = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & -1 & 0 \\ -n/(n+1) & 1/(n+1) & 1/(n+1) & \cdots & 1/(n+1) & 1/(n+1) \end{bmatrix}.$$

The n-dimensional lattices D_{n} and $\mathrm{D}_{\mathrm{n}}^*$.

$$\mathrm{D}_{\mathrm{n}} = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n : x_1 + \cdots + x_n \text{ is even}\}$$

and has generator matrix

$$M = \begin{bmatrix} -1 & -1 & 0 & \cdots & 0 & 0 \\ 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{bmatrix}.$$

Its dual $\mathrm{D}_{\mathrm{n}}^*$ has generator matrix

$$M = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 1/2 & 1/2 & \cdots & 1/2 & 1/2 \end{bmatrix}.$$

The 8-dimensional self-dual lattice E_8.

$$E_8 = \{(x_1, \ldots, x_8) : \text{all } x_i \in \mathbb{Z} \text{ or all } x_i \in \mathbb{Z} + 1/2. \sum x_i \equiv 0 \mod 2\}$$

and has generator matrix

$$M = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \end{bmatrix}.$$

The 7-dimensional lattices E_7 and E_7^*.

$$E_7 = \{(x_1, \ldots x_8) \in E_8 : x_1 + \cdots + x_8 = 0\}.$$

A generator matrix is

$$M = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 1/2 & 1/2 & 1/2 & 1/2 & -1/2 & -1/2 & -1/2 & -1/2 \end{bmatrix}.$$

The dual lattice E_7^* has generator matrix

$$M = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ -3/4 & -3/4 & 1/4 & 1/4 & 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix}.$$

The 6-dimensional lattices E_6 and E_6^*.

$$E_6 = \{(x_1, \dots, x_8) \in E_8 : x_1 + x_8 = x_2 + \dots + x_6 = 0\}$$

and has generator matrix

$$M = \begin{bmatrix} 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 1/2 & 1/2 & 1/2 & 1/2 & -1/2 & -1/2 & -1/2 & -1/2 \end{bmatrix}.$$

The dual lattice E_6^* has generator matrix

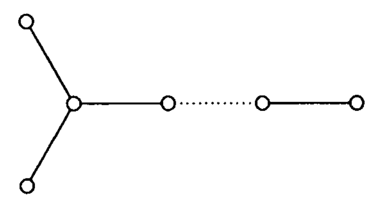
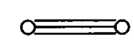
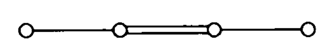
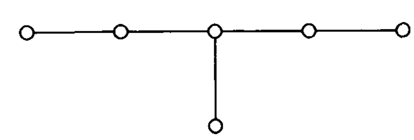
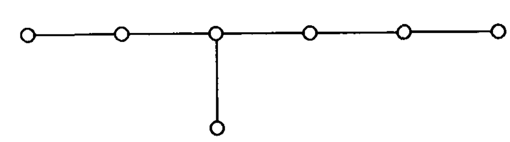
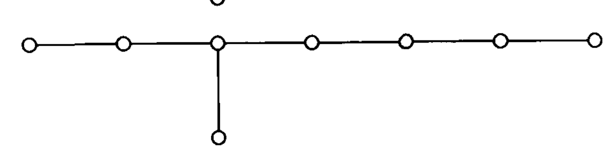
$$M = \begin{bmatrix} 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 2/3 & 2/3 & -1/3 & -1/3 & -1/3 & -1/3 & 0 \\ 1/2 & 1/2 & 1/2 & 1/2 & -1/2 & -1/2 & -1/2 & -1/2 \end{bmatrix}.$$

Both E_7 and E_6 are given as sublattices of E_8.

The groups R^*/R are as follows. Later we give explicit descriptions of the elements of some of these groups.

root lattice R	A_n	D_n (n even)	D_n (n odd)	E_6	E_7	E_8
group R^*/R	$\mathbb{Z}/(n+1)$	$\mathbb{Z}/2 \times \mathbb{Z}/2$	$\mathbb{Z}/4$	$\mathbb{Z}/3$	$\mathbb{Z}/2$	1

A *(crystallographic) root system* Φ is a set of vectors $\{v_i\} \subset V$ spanning a positive definite lattice for which the corresponding reflections s_{v_i} generate a finite group. The elements of Φ are called *roots* and the associated reflection group the *Weyl group* $W(\Phi)$. A *simple system* $S \subset \Phi$ is a basis of the root system in which a root is expressed with all non-positive or non-negative coefficients. The elements of a simple system are *simple roots.* The geometry of a root system (and lattice) can be encoded in a *Dynkin diagram* - take a node for each simple root and join two nodes v, w by $-\langle v, w\rangle$ edges. We refer to the standard literature for details **[35]** **[7]**. We give a table of all irreducible root systems together with the lattice they generate and the associated Dynkin diagram.

Root system	Lattice	Dynkin diagram
A_n	A_n	
B_n	$\mathbb{Z}^n$	
C_n	D_n	
D_n	D_n	
G_2	A_2	
F_4	D_4	
E_6	E_6	
E_7	E_7	
E_8	E_8	

The action of the Weyl group partitions the vector space $V = \Phi \otimes \mathbb{R}$ into $|W(\Phi)|$ simplices called the *Weyl chambers*. Each of these is a fundamental domain D of the Weyl group and is determined by a unique simple system $S \subset \Phi$. The n walls of D are the hyperplanes perpendicular to the roots of the associated simple system. By convention D consists of those vectors in V with inner product at least 0 with each simple root. The Weyl group acts simply transitively on both the Weyl chambers and the simple systems.

The table below gives the number of roots for each root system, the structure of the Weyl group and its order.

Root system Φ	$\lvert\Phi\rvert$	$W(\Phi)$	$\lvert W(\Phi)\rvert$
A_n	$n(n+1)$	S_{n+1}	$(n+1)!$
B_n and C_n	$2n^2$	$2^n \rtimes S_n$	$2^n n!$
D_n	$2n(n-1)$	$2^{n-1} \rtimes S_n$	$2^{n-1}n!$
F_4	48	$(2^3 \rtimes S_4) \rtimes S_3$	$2^7.3^2$
G_2	12	$S_3 \rtimes 2$	12
E_6	72	$U_4(2).2$	$2^7.3^4.5$
E_7	126	$2 \times S_6(2)$	$2^{10}.3^4.5.7$
E_8	240	$2.O_8^+(2).2$	$2^{14}.3^5.5^2.7$

Here $O_8^+(2)$, $S_6(2)$ and $U_4(2)$ are simple groups of Lie type - see the Atlas [**16**] and Humphreys [**35**] for further details.

We describe the automorphism group of a root lattice R. The *Weyl group* $W(R)$ is the normal subgroup of $\mathrm{Aut}(R)$ generated by reflections in the roots of R. The remaining symmetries are those of a fundamental domain of $W(R)$, or equivalently the graph symmetries of the Dynkin diagram. We denote this last group by $G(R)$. $\mathrm{Aut}(R)$ is then the split extension of $W(R)$ by $G(R)$, where the permutations act by conjugation on the Weyl group:

$$\mathrm{Aut}(R) = W(R) \rtimes G(R)$$

For reducible root lattices $G(R)$ includes all permutations of similar components. Here is a table of the graph automorphisms of the irreducible root lattices.

Root lattice	A_1	A_n $(n \geq 2)$	D_4	D_n $(n \neq 4)$	E_6	E_7	E_8
$G(R)$	1	$\mathbb{Z}/2$	S_3	$\mathbb{Z}/2$	$\mathbb{Z}/2$	1	1

We now make a few (hopefully clarifying) remarks on the connections between root systems and root lattices. The set of roots of a lattice form a root system. The root systems with all elements of equal length are exactly the minimal vectors of the root lattices; the remaining root systems are formed of the first two shells of some root lattices. We shall use *root* to mean both element of a root system and root of a lattice, though the latter can only have norm 2 according to our earlier definition. Also according to our definitions, the Weyl group of a root lattice is that generated by reflections in the vectors of norm 2, which ensures that the Weyl group of a root lattice R is the same as the Weyl group of the root system Φ of roots of R. Also note that $\mathrm{Aut}(\mathrm{D}_4) = \mathrm{W}(\mathrm{F}_4)$; F_4 is the first two shells of the lattice D_4. The same holds for A_2 and G_2, with $\mathrm{Aut}(\mathrm{A}_2) = \mathrm{W}(\mathrm{G}_2)$. Finally although E_8 and $\mathbb{Z}$ are the only self-dual irreducible root lattices, others are *self-similar* meaning $R \simeq R^*$. In particular, $\mathrm{D}_4 \simeq \mathrm{D}_4^*$ and $\mathrm{A}_2 \simeq \mathrm{A}_2^*$.

1.2. Even unimodular lattices. A now classical result states that a (positive definite) even unimodular lattice must have dimension divisible by 8 [48], p.24. These have been classified up to and including dimension 24, and the result is summarised in the table below

Dimension	Even unimodular lattices
8	E_8
16	$\mathrm{E}_8 \oplus \mathrm{E}_8$ and D_{16}^+
24	the Leech lattice Λ_{24} and the 23 Niemeier lattices

The Leech lattice Λ_{24} is the unique 24-dimensional even unimodular lattice with no roots. In other words $\mathrm{Aut}(\Lambda_{24})$ contains no reflections (see §1.5 below). To construct D_{16}^+ and the Niemeier lattices we make use of gluing theory.

1.3. Gluing theory. The idea is to build new integral lattices from known ones (e.g. root lattices). Gluing theory gives a convenient description of an arbitrary integral lattice L with a direct sum $L_1 \oplus \cdots \oplus L_m$ as sublattice. We suppose the L_i's are integral and the direct sum has the rank of L. A general vector in L then has the form $x = x^1 + \cdots + x^m$ where $x^j \in L_j^*$. L is generated by adding some vectors $\{y_1, \ldots, y_n\}$ of this type to the direct sum,

$$L = \mathbb{Z}\{L_1 \oplus \cdots \oplus L_m, y_1, \ldots, y_k\}$$

The y_i's are called *glue vectors* and the integral lattice L is obtained by gluing the *components* $L_1, \ldots, L_m$. The y_i^j's (where $y_i = y_i^1 + \cdots + y_i^m$) can be viewed as elements of L_j^*/L_j since all representatives of a coset generate the same lattice. The set of glue vectors form a group modulo $L_1 \oplus \cdots \oplus L_n$ called the *glue code*. In this context quotients Λ^*/Λ (for a lattice Λ) are called *glue groups*.

As an example take the integral lattice $\mathrm{D_n}$ and the glue vector

$$[1] = (1/2, \ldots, 1/2) \in \mathbb{R}^n$$

Then define $\mathrm{D}_n^+ = \langle \mathrm{D_n}, [1] \rangle$ (see Conway and Sloane [**13**] p.117).

When the automorphisms of L permute the components there is a convenient description of the symmetry group of L. This is for example the case if the sublattice is that generated by all roots of L. Let G_2 be the group of permutations of the L_j induced by automorphisms of L. G_2 is then the quotient of $\mathrm{Aut}(L)$ by the normal subgroup N fixing the components, $\mathrm{Aut}(L) = N.G_2$. Also $N = G_0.G_1$ where G_0 is the subgroup fixing the components $y_i^j \in L_j^*/L_j$ of the glue vectors in their cosets and G_1 is the permutations of the glue vectors y_i induced by N. Combining all these remarks we get

$$\mathrm{Aut}(L) = G_0.G_1.G_2 \ .$$

For more details on gluing theory see Conway and Sloane [**13**], ch.4.

1.4. Niemeier lattices. We define a *Niemeier lattice* to be an even unimodular lattice of rank 24 containing some roots. These were classified by Niemeier [**51**] in 1974. The root sublattice of a Niemeier lattice N is the sublattice $R \subset N$ generated by the roots of N. R has rank 24 and its irreducible components all have the same Coxeter number. In fact all such root lattices appear as sublattice of some Niemeier lattice. We list all possible such root lattices R, each corresponding to a unique Niemeier lattice $(R)^+$ (also denoted $N(R)$). nL stands for $L \oplus \cdots \oplus L$ (n times).

$$24\mathrm{A}_1, 12\mathrm{A}_2, 8\mathrm{A}_3, 6\mathrm{A}_4, 4\mathrm{A}_6, 3\mathrm{A}_8, 2\mathrm{A}_{12}, \mathrm{A}_{24}$$

$$6\mathrm{D}_4, 4\mathrm{D}_6, 3\mathrm{D}_8, 2\mathrm{D}_{12}, \mathrm{D}_{24}$$

$$4\mathrm{E}_6, 3\mathrm{E}_8$$

$$4\mathrm{A}_5 \oplus \mathrm{D}_4, 2\mathrm{A}_7 \oplus 2\mathrm{D}_5, 2\mathrm{A}_9 \oplus \mathrm{D}_6, \mathrm{A}_{15} \oplus \mathrm{D}_9$$

$$\mathrm{E}_8 \oplus \mathrm{D}_{16}, 2\mathrm{E}_7 \oplus \mathrm{D}_{10}, \mathrm{E}_7 \oplus \mathrm{A}_{11}$$

Using the gluing theory outlined above one can generate a Niemeier lattice by adding a set of glue vectors (glue code) to the appropriate root system. The glue code is denoted by G_∞ in accordance with [**13**]. We describe some interesting examples and refer to §2 for definitions of the codes encountered below. For more details see Conway and Sloane [**13**], ch.16.

D_{24}^+ was defined in §1.3. $3E_8$ is of course already even unimodular so no glue vectors are added. For the next cases we need more details on the glue groups of the irreducible root lattices. The lattice A_1 has two glue vectors,

$$[0] = (0,0),\ [1] = (1/2, -1/2).$$

Identifying these with the two elements of the finite field $\mathbb{F}_2$, the glue code for $(24A_1)^+$ is the binary Golay code $\mathcal{C}_{24} \subset \mathbb{F}_2^{24}$. Similarly for $(12A_2)^+$; A_2 has 3 glue vectors

$$[0],\ [1] = (1/3, 1/3, -2/3),\ [2] = (2/3, -1/3, -1/3).$$

The glue code is then the ternary Golay code $\mathcal{C}_{12} \subset \mathbb{F}_3^{12}$. The glue vectors of D_4 are

$$[0],\ [1] = (1/2, 1/2, 1/2, 1/2),\ [2] = (0,0,0,1),\ [3] = (1/2, 1/2, 1/2, -1/2)$$

and the glue code for the Niemeier lattice $(6D_4)^+$ is the hexacode $\mathcal{C}_6 \subset \mathbb{F}_4^6$. The glue vectors of E_6 are

$$[0],\ [1] = (0, -2/3, -2/3, 1/3, 1/3, 1/3, 1/3, 0),\ [2] = -[1].$$

The glue code of the Niemeier lattice $(4E_6)^+$ is the tetracode $\mathcal{C}_4 \subset \mathbb{F}_3^4$. This approach to Niemeier lattices by examining the associated codes is due to Venkov [**61**].

As explained in §1.3, the automorphism group of a Niemeier lattice N splits as

$$\mathrm{Aut}(N) = G_0(N).G_1(N).G_2(N).$$

$G_0(N)$ is always the Weyl group of the root lattice $R = R_1 \oplus \cdots \oplus R_m \subset N$ where R_i are the irreducible components:

$$G_0(N) = \mathrm{W}(R) = \mathrm{W}(R_1) \times \cdots \times \mathrm{W}(R_m).$$

G_1 is cyclic of order 1,2 or 3 and G_2 is the permutations of the irreducible components preserving the glue code. Of course only similar irreducible components can be permuted.

We work through the above examples. Since no glue (or only the trivial glue vector) is added to $3E_8$ all permutations of the 3 components arise.

$$\mathrm{Aut}(3E_8) = G_0.G_2 \simeq \mathrm{W}(E_8)^3.S_3$$

Also $\mathrm{Aut}(\mathrm{D}_{24}^{+}) = \mathrm{W}(\mathrm{D}_{24})$ since the 2 glue vectors cannot be permuted. At the other extreme G_2 is closely related to the automorphism group of the code formed by the glue vectors. For $12\mathrm{A}_2$ and $24\mathrm{A}_1$, G_2 are the finite simple Mathieu groups M_{12} and M_{24} respectively (see §2 and §3).

$$\mathrm{Aut}(12\mathrm{A}_2^{+}) = G_0.G_1.G_2 \simeq \mathrm{W}(\mathrm{A}_2)^{12}.\mathrm{Aut}(\mathcal{C}_{12}) \simeq 3^{12}.2.\mathrm{M}_{12}$$

$$\mathrm{Aut}(24\mathrm{A}_1^{+}) = G_0.G_2 \simeq \mathrm{W}(\mathrm{A}_1)^{24}.\mathrm{Aut}(\mathcal{C}_{24}) \simeq 2^{24}.\mathrm{M}_{24}$$

The following table (taken from Conway and Sloane [**13**] p.407) gives the order of the groups G_∞, G_1 and G_2 for all Niemeier lattices. The generators are given using standard notation for the glue vectors of the root lattices (as in the examples above). In each case the set of generators includes all those obtained by a cyclic permutation of the elements in round brackets. For example [1(012)] includes [1012], [1201] and [1120].

Components	Generators for glue code	$\lvert G_\infty\rvert$	$\lvert G_1\rvert$	$\lvert G_2\rvert$
D_{24}	[1]	2	1	1
$D_{16}E_8$	[10]	2	1	1
$3E_8$	[000]	1	1	6
A_{24}	[5]	5	2	1
$2D_{12}$	[(12)]	4	1	2
$A_{17}E_7$	[31]	6	2	1
$D_{10}2E_7$	[110], [301]	4	1	2
$A_{15}D_9$	[21]	8	2	1
$3D_8$	[(122)]	8	1	6
$2A_{12}$	[15]	13	2	2
$A_{11}D_7E_6$	[111]	12	2	1
$4E_6$	[1(012)]	9	2	24
$2A_9D_6$	[240], [501], [053]	20	2	2
$4D_6$	even perms of [0123]	16	1	24
$3A_8$	[(114)]	27	2	6
$2A_72D_5$	[1112], [1721]	32	2	4
$4A_6$	[1(216)]	49	2	12
$4A_5D_4$	[2(024)0], [33001], [30302], [30033]	72	2	24
$6D_4$	[111111], [0(02332)]	64	3	720
$6A_4$	[1(01441)]	125	2	120
$8A_3$	[3(2001011)]	256	2	1344
$12A_2$	[2(11211122212)]	729	2	$\lvert M_{12}\rvert$
$24A_1$	[1(00000101001100110101111)]	4096	1	$\lvert M_{24}\rvert$

1.5. The Leech lattice Λ_{24}. The Leech lattice was discovered by John Leech in 1965 [**41**]. It is the unique even unimodular lattice of rank 24 with no roots.

The most common definitions of Λ_{24} make use of the binary Golay code $\mathcal{C}_{24} \subset \mathbb{F}_2^{24}$ (see §2 on codes). For example the modulo 2 reduction map $\mathbb{Z} \to \mathbb{F}_2$ induces a map

$\rho : \mathbb{Z}^{24} \to \mathbb{F}_2^{24}$. The Leech lattice can then be defined as the preimage

$$\Lambda_{24} := \rho^{-1}(\mathcal{C}_{24}).$$

Also in the 1960s John Conway determined the automorphism group of this lattice [**12**]:

$$\mathrm{Aut}(\Lambda_{24}) = 2.\mathrm{Co}_1$$

where Co_1 is the largest Conway sporadic finite simple group. $2.\mathrm{Co}_1$ is also sometimes denoted Co_0 or $\cdot 0$. Of course $\mathrm{Aut}(\Lambda_{24})$ contains no reflections since Λ_{24} has no roots.

More details on the structure of the lattice are given in §1.9.

1.6. Complex lattices. Let $\mathcal{G}$ and $\mathcal{E}$ denote the rings $\mathbb{Z}[i]$ and $\mathbb{Z}[\omega]$ respectively, where ω is the third primitive root of unity. $\mathcal{G}$ is the ring of *Gaussian integers* and $\mathcal{E}$ the ring of *Eisenstein integers.* Let V be a complex vector space of dimension n.

DEFINITION 4. *A* complex lattice *(or J-*lattice*) is a finitely generated free J-module*

$$L = J\{v_1, \ldots, v_n\} \subset V$$

where $J = \mathcal{G}$ or $\mathcal{E}$ and $\{v_1, \ldots, v_n\}$ is a complex basis of V.

The form $\langle\ ,\ \rangle_h$ is now the usual hermitian form on $\mathbb{C}^n$:

$$\langle x, y\rangle_h = x \cdot \overline{y} = x_1\overline{y_1} + \cdots + x_n\overline{y_n} \quad \text{for } x, y \in \mathbb{C}^n.$$

The *generator matrix* and *Gram matrix* of a J-lattice are defined as in the real case. Also define the *dual lattice*

$$L^* = \{x \in \mathbb{C}^n : \langle x, y\rangle_h \in J \quad \forall y \in L\}$$

DEFINITION 5. *An* automorphism *A of a complex lattice $L \subset \mathbb{C}^n$ is a unitary complex linear map preserving the lattice: $A \in \mathrm{U}(n)$ s.t. $A(L) = L$.*

Equivalently, for a J-lattice L an automorphism is a J-linear module map $A : L \to L$ preserving the hermitian form $\langle\ ,\ \rangle_h$. The group of all complex automorphisms of a J-lattice L is denoted $\mathrm{Aut}_J(L)$.

Two complex lattices L_1, L_2 are *isomorphic* if there is a complex linear map $A \in \mathrm{U}(n)$ such that $A(L_1) = L_2$.

A complex lattice $L = J\{v_1, \ldots, v_n\}$ has an *underlying real lattice* $L_{\mathbb{R}}$:

$$L_{\mathbb{R}} = \mathbb{Z}\{\mathrm{Re}(v_1), \mathrm{Im}(v_1), \ldots, \mathrm{Re}(v_n), \mathrm{Im}(v_n)\}.$$

The converse is not always true. A real lattice L admits the structure of a complex $\mathcal{G}$-lattice (resp. $\mathcal{E}$-lattice) if and only if it has a fixed-point-free symmetry of order 4 (order 3 resp.). The automorphism is the i (resp. ω) of the corresponding complex structure. We tend to use the same symbol to denote a complex lattice and its underlying real lattice.

A real lattice may admit several distinct complex or J-structures. The Leech lattice for example has both a $\mathcal{G}$ and $\mathcal{E}$-structure. Of course for a complex lattice L

$$\mathrm{Aut}_J(L) < \mathrm{Aut}(L_{\mathbb{R}})$$

since the complex linear maps are also real linear and the distances are the same in both cases. $\mathrm{Aut}_J(L)$ consists of those real automorphisms commuting with the corresponding fixed point symmetry i or ω mentioned above. However different complex structures on the same real lattice may give rise to different automorphism groups. In the same vein, two non-isomorphic complex lattices may have isomorphic underlying real lattices.

A few examples. The rings $J = \mathcal{G}$, $\mathcal{E}$ are complex J-lattices themselves, naturally embedded in $\mathbb{C}$ (with trivial generator matrix 1).

The root lattices D_4 and E_6 have structures of $\mathcal{E}$-lattices with respective generator matrices

$$\begin{bmatrix} 2 & 0 \\ 1 & \theta \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \theta & 0 & 0 \\ 0 & \theta & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

where $\theta = \omega - \overline{\omega} = \sqrt{-3}$.

1.7. The Coxeter-Todd lattice K_{12}. The Coxeter-Todd lattice is a 12-dimensional integral lattice. It has a simple description as 6-dimensional complex lattice over the Eisenstein integers, with generators

$$\frac{1}{\sqrt{2}}(\pm\theta, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1)$$

where $\theta = \omega - \overline{\omega} = \sqrt{-3}$ may be in any position and there must be an even number of minus signs.

Its complex automorphism group $\mathrm{Aut}_{\mathcal{E}}(K_{12})$ is the Mitchell group, isomorphic to $6.\mathrm{U}_4(3).2$, where $\mathrm{U}_4(3) \simeq \mathrm{PSU}_4(3)$ is a finite simple group of Lie type of order $2^7 \cdot 3^6 \cdot 5 \cdot 7 = 3,265,920$. For more details see [**14**].

1.8. Quaternionic lattices. Let $\mathcal{H} \subset \mathbb{H}$ denote the ring of *Hurwitz integers*, generated as $\mathbb{Z}$-module over its 24 units in $\mathbb{H}$:

$$\pm 1,\ \pm i,\ \pm j,\ \pm k,\ 1/2(\pm 1 \pm i \pm j \pm k)$$

Its group of units $\mathcal{H}_u$ is isomorphic to the binary tetrahedral group $2.A_4$ and the ring $\mathcal{H}$ coincides with the root lattice D_4 under the usual identification $\mathbb{R}^4 \equiv \mathbb{H}$.

DEFINITION 6. *A* quaternionic lattice *(or $\mathcal{H}$-*lattice*) is an $\mathcal{H}$-module*

$$L = \mathcal{H}\{v_1, \ldots, v_n\} \subset \mathbb{H}^n$$

where $\{v_1, \ldots, v_n\}$ is a quaternionic basis of $\mathbb{H}$.

The concepts of *generator matrix* and *underlying real lattice* $L_{\mathbb{R}}$ go through as for complex lattices. An $\mathcal{H}$-lattice also has several complex structures both over $\mathcal{G}$ and $\mathcal{E}$, since for example multiplication by i and $1/2(-1+i+j+k)$ are fixed-point-free symmetries of order 4 and 3 respectively.

DEFINITION 7. *An* automorphism *of a quaternionic lattice L is those symmetries of $L_{\mathbb{R}}$ commuting with the action of $\mathcal{H}_u$.*

Denote the group of quaternionic automorphisms by $\mathrm{Aut}_{\mathcal{H}}(L)$.

As an example, E_8 admits the structure of an $\mathcal{H}$-lattice in $\mathbb{H}^2$ with generator matrix

$$\begin{bmatrix} 1+i & 0 \\ 1 & 1 \end{bmatrix}.$$

For more on lattices over general rings see the discussion in Conway and Sloane [**13**] ch.2, p.52.

1.9. Structure of the Leech lattice. The Leech lattice has structures of $\mathcal{G}$, $\mathcal{E}$ and $\mathcal{H}$-lattices. Similarly as for the real Leech lattice in §1.5 one can construct Λ_{24} as an $\mathcal{E}$-lattice by pulling back the ternary Golay code $\mathcal{C}_{12} \subset \mathbb{F}_3^{12}$. Take the quotient map

$$\rho : \mathbb{Z}[\omega]^{12} \to (\mathbb{Z}[\omega]/\beta)^{12} = \mathbb{F}_3^{12}$$

where β is the ideal generated by $1-\omega$. The pullback $\rho^{-1}(\mathcal{C}_{12})$ is then the complex Leech lattice. More on obtaining lattices from codes over finite fields can be found in Ebeling [**21**].

Other sporadic simple groups appear as automorphism groups of the $\mathcal{E}$ and $\mathcal{H}$-Leech lattices. These automorphism groups are of course the subgroups of $\mathrm{Aut}(\Lambda_{24}) = 2.\mathrm{Co}_1$ commuting with the units of the two rings:

$$\mathrm{Aut}_{\mathcal{E}}(\Lambda_{24}) \simeq 6.\mathrm{Suz}, \quad \mathrm{Aut}_{\mathcal{H}}(\Lambda_{24}) \simeq 2.\mathrm{J}_2.$$

Suz is the Suzuki group and J_2 is the Hall-Janko group. See §3 on finite simple groups. Robert Wilson [**63**], [**64**] identifies the maximal subgroups of these simple groups.

In [**60**] J.Tits provides a nice unified description of the situation, viewing Λ_{24} as a module over the endomorphism rings R_i of subgroups $U_i \subset 2.\mathrm{Co}_1$ where

$$U_9 \supset U_8 \supset \cdots \supset U_2$$

and U_i is isomorphic to the double cover $2.A_i$ of the alternating group A_i. Then for $i = 2, 3, 4$ we have $R_i = \mathbb{Z}, \mathcal{E}, \mathcal{H}$ and recover our previous constructions.

2. Codes

We give some of the basic definitions and describe the codes encountered in the previous section.

Definition 8. *A* (linear) code *is a linear subspace $C \subset \mathbb{F}_q^n$ where $\mathbb{F}_q$ is a finite field of order q.*

The *weight* $w(c)$ of a *codeword* $c = (c_1, \ldots, c_n) \in C$ is the number of non-zero coordinates c_i. A code is called *doubly even* if $w(c) \equiv 0 \bmod 4$ for all codewords $c \in C$. A code $C \in \mathbb{F}_q^n$ of dimension k and minimum weight d is called an $[n, k, d]$-code.

The *weight distribution* of a code records how many codewords are of each weight and is displayed as a sequence of terms n^m indicating that there are m codewords of weight n.

A *generator matrix* for an $[n,k,d]$-code C is a $k \times n$ matrix whose rows form a basis for C as a linear space over $\mathbb{F}_q$.

For $x, y \in \mathbb{F}_q^n$ define the *inner product* $x \cdot y$

$$x \cdot y = \sum_{i=1}^{n} x_i y_i$$

The *dual code* $C^\perp$ is then

$$C^\perp = \{x \in \mathbb{F}_q^n : x \cdot y = 0 \quad \forall y \in C\}.$$

A code is *self-dual* if $C = C^\perp$.

DEFINITION 9. *An* automorphism *of a code $C \subset \mathbb{F}_q^n$ is an $\mathbb{F}_q$-linear isomorphism $f : \mathbb{F}_q^n \to \mathbb{F}_q^n$ preserving the code, $f(C) = C$.*

The group of automorphisms of C is denoted $\mathrm{Aut}(C)$.

We give some important examples of codes and their properties.

2.1. The tetracode $\mathcal{C}_4$. The $[4,2,3]$ tetracode $\mathcal{C}_4 \subset \mathbb{F}_3^4$ is a ternary code with generator matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix}.$$

$\mathcal{C}_4$ has 9 codewords and weight distribution $0^1 3^8$. Its automorphism group is a non-split extension of the group of units of the field, $\mathrm{Aut}(\mathcal{C}_4) = 2.S_4$.

2.2. The hexacode $\mathcal{C}_6$. The $[6,3,4]$ hexacode $\mathcal{C}_6 \subset \mathbb{F}_4^6$ is a code over $\mathbb{F}_4 = \{0, 1, \omega, \overline{\omega}\}$ with generator matrix

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & \omega & \overline{\omega} \\ 1 & 0 & 0 & 1 & \overline{\omega} & \omega \end{bmatrix}.$$

$\mathcal{C}_6$ has 64 codewords and weight distribution

$$0^1 4^{45} 6^{18}$$

The group of automorphisms of $\mathcal{C}_6$ is an extension of the group of units of the field by the alternating group on 6 letters, namely $\mathrm{Aut}(\mathcal{C}_6) = 3.A_6$.

2.3. The binary Golay code $\mathcal{C}_{24}$. The $[24, 12, 8]$ code $\mathcal{C}_{24} \subset \mathbb{F}_2^{24}$ is an extended quadratic residue code. We will not be needing details of the construction of $\mathcal{C}_{24}$ or the ternary Golay code $\mathcal{C}_{12}$ so refer to Conway and Sloane [**13**], ch.3 and Ebeling [**21**] for precise definitions. $\mathcal{C}_{24}$ is the unique 24-dimensional self-dual doubly even code and can be used to define the Leech lattice (see §1.5). The code has 4096 elements with weight distribution

$$0^1 8^{759} 12^{2576} 16^{759} 24^1.$$

Its automorphisms form the largest Mathieu group:

$$\mathrm{Aut}(\mathcal{C}_{24}) = \mathrm{M}_{24}.$$

2.4. The ternary Golay code $\mathcal{C}_{12}$. The $[12, 6, 6]$ code $\mathcal{C}_{12} \subset \mathbb{F}_3^{12}$ is also an extended quadratic residue code. In §1.9 $\mathcal{C}_{12}$ is used to construct the complex Leech lattice over the Eisenstein integers $\mathbb{Z}[\omega]$. The code has 729 words and weight distribution

$$0^1 6^{264} 9^{440} 12^{24}.$$

Its automorphisms also essentially form a Mathieu group:

$$\mathrm{Aut}(\mathcal{C}_{12}) = 2 \rtimes \mathrm{M}_{12}.$$

In the context of the Golay codes, codewords are sometimes refered to as *$\mathcal{C}$-sets*. See the discussion on the Mathieu groups in §3.

3. Sporadic finite simple groups

The finite simple groups were completely classified by the mid-1980s and split into 4 categories

- groups of prime order
- the alternating groups A_n $(n \geq 5)$
- groups of Lie type
- the 26 sporadic groups

The groups of prime order are the abelian finite simple groups. A_5 is the smallest non-abelian finite simple group. In this work we will mainly encounter sporadic groups and alternating groups, though some groups of Lie type will arise in chapter 3. See Aschbacher [**2**] for a concise description of general finite simple groups. We list the sporadics, and for details refer to another of Aschbacher's books [**3**] and a recent book by R.Griess [**27**] on the sporadics associated to the Leech lattice. The Mathieu groups appear in several different contexts in coming chapters so their construction is outlined in some detail.

Notation	Name	Order
M_{11}	Mathieu	$2^4.3^2.5.11$
M_{12}		$2^6.3^3.5.11$
M_{22}		$2^7.3^2.5.7.11$
M_{23}		$2^7.3^2.5.7.11.23$
M_{24}		$2^{10}.3^3.5.7.11.23$
J_1	Janko	$2^3.3.5.7.11.19$
J_2		$2^7.3^3.5^2.7$
J_3		$2^7.3^5.5.17.19$
J_4		$2^{21}.3^3.5.7.11^3.23.29.31.37.43$
HS	Higman-Sims	$2^9.3^2.5^3.7.11$
Mc	McLaughlin	$2^7.3^6.5^3.7.11$
Suz	Suzuki	$2^{13}.3^7.5^2.7.11.13$
Ly	Lyons	$2^8.3^7.5^6.7.11.31.37.67$
He	Held	$2^{10}.3^3.5^2.7^3.17$
Ru	Rudvalis	$2^{14}.3^3.5^3.7.13.29$
O'N	O'Nan	$2^9.3^4.5.7^3.11.19.31$
Co_3	Conway	$2^{10}.3^7.5^3.7.11.23$
Co_2		$2^{18}.3^6.5^3.7.11.23$
Co_1		$2^{21}.3^9.5^4.7^2.11.13.23$
M(22)	Fischer	$2^{17}.3^9.5^2.7.11.13$
M(23)		$2^{18}.3^{13}.5^2.7.11.13.17.23$
M(24)		$2^{21}.3^{16}.5^2.7^3.11.13.17.23.29$
F_3	Thompson	$2^{15}.3^{10}.5^3.7^2.13.19.31$
F_5	Harada	$2^{14}.3^6.5^6.7.11.19$
F_2	Baby Monster	$2^{41}.3^{13}.5^6.7^2.11.13.17.19.23.31.47$
F_1	Monster	$2^{26}.3^{20}.5^9.7^6.11^2.13^3.17.19.23.29.31.41.47.59.71$

There is yet no unifying way of presenting the sporadic groups. The first finite simple groups to be discovered were the Mathieu groups back in the nineteenth century as multiply transitive permutation groups. The others have all been constructed in the second half of the twentieth century. Janko was the first to add to the list in 1965 with J_1. An important step was the discovery by Conway of Co_1, Co_2 and Co_3 as automorphisms of the Leech lattice. The previously known HS, Mc and Suz were then displayed as stabilisers of sublattices of Λ_{24}. In the early 1970s B.Fischer discovered the three Fischer groups as 3-transposition groups. The Monster is the largest sporadic group and was first constructed by Griess **[26]** as automorphism group of a 196,883-dimensional real algebra. Most other sporadics can be found in the monster but it is still unknown if this is true in general.

3.1. The Mathieu groups. We define M_{24}. Let $\mathbb{F}_q$ be the finite field of $q = p^m$ elements (p prime) and $Q = \{x^2 : x \in \mathbb{F}_q\}$ be the squares in $\mathbb{F}_q$. $GL_n(q)$ is the linear isomorphisms of an n-dimensional vector space over $\mathbb{F}_q$ and $SL_n(q)$ is the subgroup of elements of determinant 1. The centre of $GL_n(q)$ consists of multiplication by non-zero elements of the field. The quotient by the centre yields $PGL_n(q)$ and $PSL_n(q)$. $PSL_n(q)$ is denoted by $L_n(q)$ and is a simple group of Lie type for $n \geq 2$ except for the cases $n = 2$ and $q = 2, 3$.

From now on $n = 2$. $A = (a_{ij}) \in SL_2(q)$ acts on $\mathbb{F}_q^2$ as $(x, y) \mapsto (a_{11}x + a_{12}y, a_{21}x + a_{22}y)$ where $a_{11}a_{22} - a_{12}a_{21} = 1$. The projective line $PL(q)$ consists of the $q+1$ ratios x/y for $x, y \in \mathbb{F}_q$ and can be identified with the q elements of the field and ∞. We denote $PL(q)$ by Ω. $L_2(q)$ acts on Ω as

$$x \longmapsto \frac{a_{11}x + a_{12}}{a_{21}x + a_{22}}$$

So we have displayed a permutation representation of $L_2(q)$ on a set of order $q+1$.

Define the *Mathieu group* M_{24} to be the group generated by $L_2(23)$ and the permutation $x \mapsto x^3/9$ for $x \in Q \subset \Omega$.

The action of M_{24} is quintuply transitive on Ω (that is transitive on 5 element subsets of Ω - see Aschbacher **[2]** for basics on multiply transitive groups).

Let $P(\Omega)$ be the set of subsets of Ω. One can view $P(\Omega)$ as a 24-dimensional vector space over $\mathbb{F}_2$ where addition of sets is symmetric difference: for $A, B \in P(\Omega)$, $A + B = (A \backslash B) \cup (B \backslash A)$. $P(\Omega)$ inherits an action of M_{24}. The binary Golay code $\mathcal{C}_{24}$ is a 12-dimensional subspace of $\mathbb{F}_2^{24}$. M_{24} consists precisely of those permutations of Ω preserving $\mathcal{C}_{24}$.

From §2 there are $\mathcal{C}$-sets of order 8, 12, 16. Every 5-element subset of Ω is contained in precisely one $\mathcal{C}$-set of order 8 (octads). In other words the octads form a Steiner system $S(5,8,24)$, whose automorphism group is also the Mathieu group M_{24} (see Biggs [**5**] for more on this approach).

Define the *Mathieu groups* M_{24-k} ($k \leq 5$) to be the pointwise stabiliser of a k-element subset of Ω in M_{24}.

Also define the *Mathieu group* M_{12} to be the (setwise) stabiliser of a 12-element $\mathcal{C}$-set (dodecad). M_{12} can also be defined in a similar way to M_{24} above and is characterised as the automorphism group of the ternary Golay code $\mathcal{C}_{12}$ and the Steiner system $S(5,6,12)$.

4. Singularities, resolutions and blow-ups

A point x of a complex manifold M is *singular* if x is a singularity of the underlying differentiable manifold. The singularities arising in this work will always be isolated quotient singularities. In other words a neighbourhood U of $x \in M$ will be isomorphic (as germs of holomorphic functions) to

$$\mathbb{C}^n/G \text{ where } G < \mathrm{GL}(n,\mathbb{C}), \text{ G finite,} \tag{1}$$

with origin the only fixed point. A *resolution* of the singularity is a holomorphic map ρ from a non-singular complex space $\widetilde{\mathbb{C}^n/G}$,

$$\rho : \widetilde{\mathbb{C}^n/G} \to \mathbb{C}^n/G,$$

such that the restriction $\rho : \widetilde{\mathbb{C}^n/G} - \rho^{-1}(0) \to \mathbb{C}^n/G - 0$ is a biholomorphism.

By a (complex) *orbifold* we shall mean a complex space with only isolated quotient singularities (a concept first introduced by Satake [**57**]).

In resolving quotient singularities the key construction is that of a blow-up. Let n be the complex dimension of M. The *blow-up* of M at $x \in M$ yields a complex manifold $\widehat{M_x}$ and map

$$\sigma : \widehat{M_x} \to M$$

such that the restriction $\sigma : \widehat{M_x} - \sigma^{-1}(x) \to M - x$ is a biholomorphism and the inverse image $\sigma^{-1}(x)$ of x is isomorphic to $\mathbb{C}\mathrm{P}^{n-1}$. $\sigma^{-1}(x)$ is called the *exceptional divisor*. We define the *blow-up* of $\mathbb{C}^n$ at the origin 0; this is sufficient to obtain the general construction for a manifold M. Let $(w_1 : \cdots : w_n)$ be homogeneous coordinates for $\mathbb{C}P^{n-1}$ and $(z_1, \ldots, z_n)$ coordinates for $\mathbb{C}^n$. The blow-up $\widehat{\mathbb{C}^n}$ is the submanifold of $\mathbb{C}^n \times \mathbb{C}P^{n-1}$ satisfying the equations $w_i z_j = z_i w_j$. The projection $\widehat{\mathbb{C}^n} \hookrightarrow \mathbb{C}^n \times \mathbb{C}P^{n-1} \to \mathbb{C}^n$ is the map σ above.

The blow-up of $\mathbb{C}^n$ at the origin is equivariant with respect to an action of $G < GL(n, \mathbb{C})$. One attempts to resolve the quotient singularity (1) above by successive blow-ups, first at the origin of $\mathbb{C}^n$ then in the exceptional divisors, to obtain a new space P whose quotient by the G-action is now non-singular. This is always possible since in the 1960s Hironaka [**30**] proved that every singularity of an algebraic space can be resolved by successive blow-ups of the ambient space, though giving explicit resolutions can be very difficult.

5. Toric geometry

A *toric variety* is a variety containing an algebraic torus as dense open subset whose action on itself extends to an action on the entire variety. A toric variety can be specified by the following data:

1. a lattice $N \subset V = N_{\mathbb{R}} \cong \mathbb{R}^n$
2. a collection $\Delta = (\Delta, N)$ of strongly convex rational polyhedral cones σ called a fan

A *strongly convex rational cone* is a set

$$\sigma = \{\sum_i r_i x_i : r_i \geq 0\}$$

where $x_i \in N$ and containing no affine subspace through the origin. We say that $\sigma = [x_1, \ldots, x_m]$ is generated or spanned by the x_i. The cones in the fan must fit together nicely, intersecting in faces. Also all faces of a given cone in Δ are cones in Δ.

To each cone one can associate an affine variety as follows. Let $M \subset V^*$ be the lattice dual to N and $\sigma^\vee := \{w \in V^* \text{ s.t. } w(x) \geq 0 \ \forall x \in \sigma\}$ be the dual cone. Also let S_σ be the finitely generated semigroup $M \cap \sigma^\vee$. Now taking the $\mathbb{C}$-algebra $A_\sigma = \mathbb{C}[S_\sigma]$ generated by the semigroup, we define

$$U_\sigma := \mathrm{Spec}(A_\sigma) = \mathrm{Spec}(\mathbb{C}[S_\sigma]).$$

A general toric variety $X(\Delta, N)$ is constructed by gluing the affine parts along their intersections. Note the origin of V is part of any fan and corresponds to the algebraic torus T_N (hence toric variety) whose action on itself extends to an action on $X(\Delta, N)$.

If P is a rational convex polytope in V containing the origin then the cones of the associated fan Δ_P are those spanned by the proper faces of P. Denote $X(\Delta_P, N)$ by X_P.

The *support* $|\Delta| \subset V$ of a toric variety is the union of all its cones. A toric variety is always a normal variety, and there is a nice condition for compactness:

LEMMA 1. *A toric variety $X(\Delta, N)$ is complete if and only if it has support $|\Delta| = V$.*

Such a fan is called *complete.* The varieties X_P described above are always complete, though not all complete toric varieties arise from convex polytopes.

A map of fans $\varphi : (\Delta_1, N_1) \to (\Delta_2, N_2)$ is a $\mathbb{Z}$-linear map $\varphi : N_1 \to N_2$ such that for all cones $\sigma \in \Delta_1$, $\varphi(\sigma)$ is contained is some cone of Δ_2. A map of fans naturally gives rise to a holomorphic map $\varphi_* : X(\Delta_1, N_1) \to X(\Delta_2, N_2)$ equivariant with respect to the action of the tori T_{N_1}, T_{N_2} and the induced map

$$\varphi \otimes 1 : T_{N_1} = N_1 \otimes_{\mathbb{Z}} \mathbb{C}^* \to T_{N_2} = N_2 \otimes_{\mathbb{Z}} \mathbb{C}^*.$$

The following result tells us that these are all the maps in this category [**52**], p.19:

THEOREM 2. *Suppose $f : X(\Delta_1, N_1) \to X(\Delta_2, N_2)$ is a holomorphic map, equivariant w.r.t. a homomorphism $f' : T_{N_1} \to T_{N_2}$. Then there is a unique map of fans $\varphi : (N_1, \Delta_1) \to (N_2, \Delta_2)$ such that $\varphi_* = f$.*

Note an isomorphism of fans $\varphi : N \to N$ need not be an automorphism of the lattice N. As a counterexample, let $N = \mathbb{Z}\{1/2e_1, e_2\}$ be a lattice in $\mathbb{R}^2$ with 2-dimensional cones the 4 quadrants of the plane. The linear map $\varphi : N \to N$ swapping the two generators is clearly a map of fans but does not preserve distances. Similarly an automorphism of the lattice need not be a map of fans as can easily be seen by taking an asymmetrical decomposition of the plane (with for eg $N = \mathbb{Z}^2$). We will return to these aspects at a later stage when examining automorphisms of compact toric varieties. We quote a well-known result on singular affine toric varieties (see Fulton [**24**], p.29).

THEOREM 3. *U_σ is non-singular if and only if some set of generators $\{x_i\}$ of σ can be completed to a basis of the lattice N.*

We now have an ideal framework in which to describe quotient singularities $\mathbb{C}^n/G$ for finite abelian groups G. Let $N' \subset N$ be a finite index sublattice of N. Then the quotient $G = N/N'$ of abelian groups acts naturally on $\mathbb{C}[M']$ (i.e. on the algebraic torus and hence any other affine part). Let $X_{u'} \in \mathbb{C}[M']$ correspond to $u' \in S_\sigma$. Then $v \in N/N'$ acts as

$$v \cdot X_{u'} = \exp(2\pi i \langle v, u' \rangle) X_{u'}$$

where

$$\langle\ ,\ \rangle : N/N' \times M'/M \to \mathbb{Q}/\mathbb{Z}$$

is induced by the usual pairing. Now taking a cone σ generated by a basis of N' we get $X(\sigma, N') = \mathbb{C}^n$. G acts on this affine variety as above, and under this identification

$$\mathbb{C}^n = X(\sigma, N') \to X(\sigma, N) = \mathbb{C}^n/G$$

is the quotient map. In general the sublattice N' is generated by the shortest elements of N along the edges of the cone - this rules out simply obtaining affine space again as the quotient.

In fact given a finite abelian group $G < \mathrm{GL}(n, \mathbb{C})$ of diagonal matrices one can construct a lattice $N \supset \mathbb{Z}^n = \mathbb{Z}\{e_1, \ldots, e_n\}$ such that $N/\mathbb{Z}^n \simeq G$ [**56**] and $X(\sigma, N)$ is isomorphic to $\mathbb{C}^n/G$, where σ is the cone generated by the standard basis $\{e_1, \ldots, e_n\}$. Take the exponential map

$$\exp : \mathbb{R}^n \to \mathbb{C}^n, \qquad \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \mapsto \begin{bmatrix} e^{2\pi i x_1} \\ \vdots \\ e^{2\pi i x_n} \end{bmatrix}.$$

$G < (\mathbb{C}^*)^n$ and the lattice $N = \exp^{-1}(G)$ is the one required. Note $\mathbb{Z}^n \subset N$ is just the inverse image of the identity matrix.

5.1. Resolutions of toric varieties. Clearly resolutions are going to be obtained by subdivisions of the singular fan Δ (in the case above consisting of one n-dimensional cone and all its faces) where each new n-dimensional cone now generates the lattice. In fact this can always be done (see [**24**], p.48):

THEOREM 4. *Any singular fan Δ admits a non-singular subdivision Δ'.*

Brylinski [**9**] gives an equivariant form of this result.

THEOREM 5. *Let G be a finite group of automorphisms of a fan Δ. Then Δ admits a non-singular subdivision also invariant under G.*

5.2. Calculating topological invariants. The fan Δ allows for easy combinatorial calculations of many topological invariants of the associated variety $X(\Delta, N)$. Let d_i be the number of cones of Δ of dimension i. The odd Betti numbers of $X(\Delta, N)$ are 0 and

$$b_{2k} = \sum_{i=k}^{n} (-1)^{i-k} \binom{i}{k} d_{n-i}.$$

From this one determines the Euler characteristic χ

$$\chi(X(\Delta, N)) = d_n = \text{the number of top dimensional cones.}$$

The signature of the intersection form on the middle cohomology is

$$\tau(X(\Delta, N)) = \sum_{i=0}^{n} (-2)^i d_{n-i}.$$

Indefinite integral forms are determined by their signature and dimension, so in certain cases these calculations suffice in establishing the intersection form of the variety. As for the fundamental group of $X(\Delta, N)$ (see Oda [**52**], p.14),

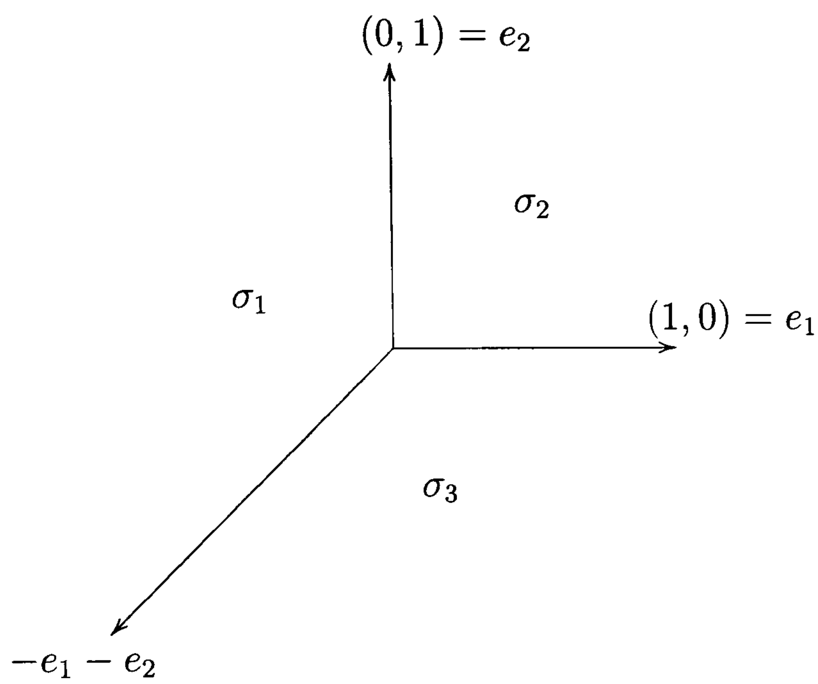
PROPOSITION 6. *The fundamental group $\pi_1(X(\Delta, N))$ is isomorphic to the abelian group N/N', where N' is the sublattice $\bigcup_{\sigma\in\Delta}(\sigma \cap N)$.*

In particular if Δ contains at least one n-dimensional cone then $N' = N$ and $X(\Delta, N)$ is simply-connected.

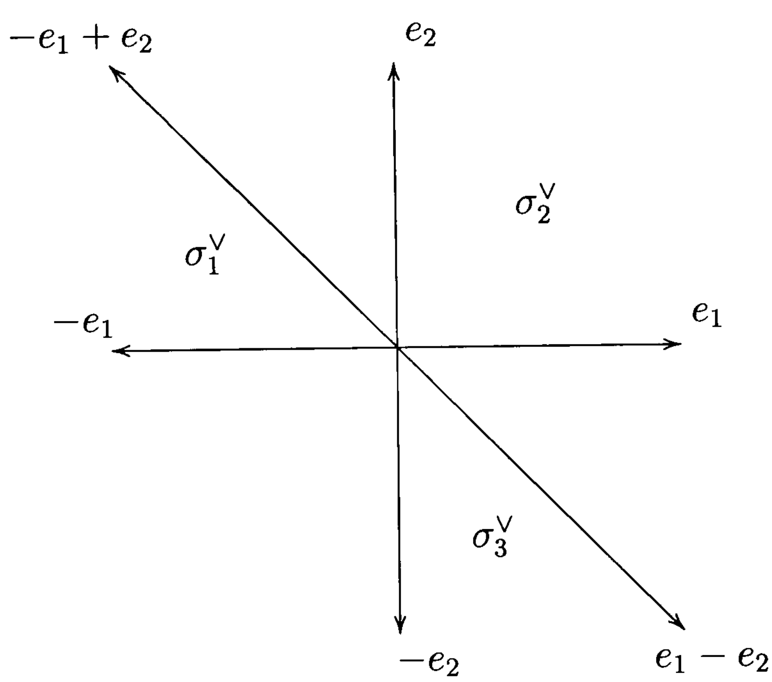
We give some examples of toric varieties.

5.3. Example. The origin in $\mathbb{R}^n$ is a fan by itself with associated variety $X(\Delta, N)$ the n-dimensional algebraic torus T_N.

5.4. Example. Let $N = \mathbb{Z}^2 = \mathbb{Z}\{e_1, e_2\}$ and Δ be the complete 2-dimensional fan below, with the 3 top-dimensional cones σ_1, σ_2 and σ_3.



The dual fan then takes the form:



All three σ_i's are clearly non-singular and yield the affine varieties

$$A_{\sigma_1} = \mathbb{C}[X^{-1}, X^{-1}Y]$$

$$A_{\sigma_3} = \mathbb{C}[Y^{-1}, XY^{-1}]$$

$$A_{\sigma_2} = \mathbb{C}[X, Y]$$

where X, Y correspond to the generators e_1, e_2 respectively. The gluing maps $(A, B) \mapsto (A^{-1}, A^{-1}B)$ between the affine components tells us these form a projective space $X(\Delta, \mathbb{Z}^2) = \mathbb{CP}^2$.

CHAPTER 2

Automorphisms of Kummer manifolds

A Kummer manifold is obtained by taking the quotient of a complex torus by an involution and resolving the singular points. They also appear in the literature before the resolution as singular algebraic varieties (or Kummer orbifolds). The study of the 2-dimensional Kummer surfaces is classical, the first substantial account being that of Hudson in 1905 [**34**]. This case has since attracted much attention. Both Hudson and more recently Gonzalez-Dorrego [**25**] described the striking link with combinatorial $(16, 6)$-configurations, leading to a classification of all 2-dimensional Kummer varieties.

In this chapter we shall be examining automorphism groups of both Kummer surfaces and higher dimensional Kummer manifolds. We start in §1 by a detailed construction of the manifolds leading to a nice description of their automorphisms. In §2 we concentrate on the two dimensional surfaces and how the results obtained fit in with previous work by Mukai on K3-surfaces [**50**]. §3 examines the situation in higher dimensions.

1. Kummer manifolds

Let us first describe the (complex) 2-dimensional construction disregarding the complex structure. Take a real 4-torus

$$T = S^1 \times S^1 \times S^1 \times S^1$$

carrying the group structure inherited from S^1. This topological group then clearly has 16 points of order two, namely the quadruples of the form $(\pm 1, \pm 1, \pm 1, \pm 1)$. We now remove a small open 4-disc B_i from around each of these double points, creating a manifold with boundary 16 distinct 3-spheres. The next step is to identify the remaining points of the orginal torus T with their respective inverses. This clearly identifies antipodal points on each boundary sphere, and we obtain a new manifold

X with boundary 16 3-dimensional real projective spaces $\mathbb{RP}^3$,

$$X = T - \{B_i\}/\sim \quad \text{where} \quad x \sim x^{-1}.$$

Let M_f be the mapping cylinder arising from

$$f : \mathbb{RP}^3 \to \mathbb{CP}^1, \quad (x_1 : x_2 : x_3 : x_4) \mapsto [(x_1 + ix_2) : (x_3 + ix_4)]$$

in homogenous coordinates. M_f has boundary $\mathbb{RP}^3$, so we can glue a copy of M_f onto each of the 16 $\mathbb{RP}^3$'s by identifying the boundaries. The resulting manifold (without boundary) is called the real 4-dimensional Kummer manifold.

Starting with a complex torus

$$T = \mathbb{C}^2/L$$

(where L is a lattice in $\mathbb{C}^2$ acting by addition) and repeating the above process we construct a complex *Kummer surface* K_L. In the additive notation now, the double points are $(\pm 1/2, \pm 1/2, \pm 1/2, \pm 1/2)$ in terms of a basis of L.

From now on we denote by $\langle \pm 1 \rangle$ the group of order 2 acting on a torus by sending an element to its inverse.

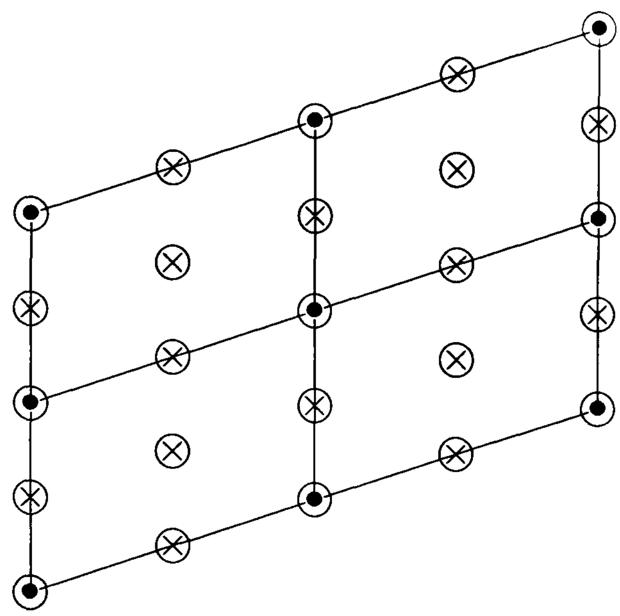
The construction above goes through in higher dimensions. Start with a complex n-torus ($n \geq 2$)

$$T = \mathbb{C}^n/L$$

where L is now a lattice in $\mathbb{C}^n$. The torus has 2^{2n} fixed points under $\langle \pm 1 \rangle$. Extract discs around each of these and take the quotient by $\langle \pm 1 \rangle$. The resulting manifold X_L has boundary 2^{2n} real projective planes $\mathbb{RP}^{2n-1}$. Attach a mapping cyclinder M_f to each of these, this time built from

$$f : \mathbb{RP}^{2n-1} \to \mathbb{CP}^{n-1}, \quad (x_1 : \cdots : x_{2n}) \mapsto [(x_1 + ix_2) : \cdots : (x_{2n-1} + ix_{2n})]$$

We obtain an n-dimensional complex *Kummer manifold* K_L. Below is a picture of the situation in a subspace generated by two elements of a basis of L. The fundamental domain of the torus is outlined; the dots are the lattice points and the crosses are the points of order two other than the lattice points; the circles outline the discs removed.



It is clear from this construction that all Kummer manifolds of the same dimension are homeomorphic. The complex structure however depends on the lattice.

By an *automorphism* of the Kummer maniold K_L we shall mean a biholomorphic transformation of K_L.

We now explain how to construct automorphisms of K_L starting from the lattice L. The following result (cf [**23**]) links biholomorphic homomorphisms of tori to the lattices.

THEOREM 7. *Two tori $T_1 = \mathbb{C}^n/L_1$ and $T_2 = \mathbb{C}^n/L_2$ are biholomorphic if and only if there exits an $A \in \mathrm{GL}(n, \mathbb{C})$ such that $A(L_1) = L_2$.*

In particular invertible $n \times n$ complex matrices preserving the lattice are biholomorphic homomorphisms of T. We shall denote the group of all such automorphisms by $\mathrm{Hom}(T)$ since these are also group homomorphisms. In fact the group of biholomorphic transformations of the torus $\mathrm{Aut}(T)$ splits as the semi-direct product of the group of translations Tr of the torus by $\mathrm{Hom}(T)$ (see [**1**], p.42).

THEOREM 8. $\mathrm{Aut}(T) = \mathrm{Hom}(T) \ltimes \mathrm{Tr}$

The homomorphisms act by conjugation on the translations. As an additive group $\mathrm{Hom}(T) \simeq \mathbb{Z}^m$ where $m \leq 4n^2$ (see Lange and Birkenhake [**40**] ch.1), so in particular is infinite. The torus has a natural hermitian metric inherited from $\mathbb{C}^n$. If we restrict to isometries:

LEMMA 9. *Let* $\mathrm{Hom}_u(T) = \mathrm{Hom}(T) \cap \mathrm{U}(n)$. *Then* $\mathrm{Hom}_u(T)$ *is finite.*

PROOF. Let L_d denote the elements of the lattice L of length d - these are finite sets. Then it is clear that the lattice is generated by a finite number of L_d's, say $J = L_1 \cup ... \cup L_n$. Hence any such homomorphism of T is determined by its action on the elements of J. The automorphisms must preserve lengths, so each L_d is mapped into itself, and J is mapped into J. However there are only a finite number of distinct permutations of the elements of J, and it follows that there is only a finite number of such automorphisms of T. □

There are of course infinitely many translations $x \mapsto t + x$ of the torus, one for each element $t \in T$.

We shall now restrict our attention to the homomorphisms Hom(T) and return to the translations at the end of the section. We describe how homomorphisms of the torus behave under the construction of K_L outlined above.

Let G be a subgroup of Hom(T). Suppose $G < \mathrm{U}(n)$ and that $\langle \pm 1 \rangle < G$; this last condition is always satisfied when G is the automorphism group of a lattice and will almost always be true in coming chapters. G acts on the set of double points of T and preserves distances, so acts on the truncated torus and its quotient by $\langle \pm 1 \rangle$. $\langle \pm 1 \rangle$ identifies the cosets

$$\begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \sim \begin{bmatrix} -z_1 \\ \vdots \\ -z_n \end{bmatrix} \quad \text{in } T, \text{ where } \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \in \mathbb{C}^n$$

So the action of G is no longer faithful since A and $-A$ in G act in the same way. $\langle \pm 1 \rangle$ is normal in G so we do get a faithful action of the group $G/\langle \pm 1 \rangle$.

It remains to determine automorphisms of the complex projective planes glued in and how the two constructions match up on the boundaries. We start by giving another description of the mapping cylinder: we can view the process as attaching a disc bundle. Recall the map $f : \mathbb{RP}^{2n+1} \to \mathbb{CP}^n$ defined above. It can easily be seen that the preimage of a complex line $[(x_1 + ix_2) : \cdots : (x_{2n-1} + ix_{2n})]$ consists of all lines in $\mathbb{R}^{2n}$ of the form

$$s\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{2n-1} \\ x_{2n} \end{bmatrix} + t\begin{bmatrix} -x_2 \\ x_1 \\ \vdots \\ -x_{2n} \\ x_{2n-1} \end{bmatrix} \quad \text{where } s,t \in \mathbb{R}$$

These lines clearly form a projective line $\mathbb{RP}^1 \cong S^1$ and so $\mathbb{RP}^{2n+1} \to \mathbb{CP}^n$ is a sphere bundle (the Hopf bundle for $n = 1$). On gluing the mapping cylinder each sphere is shrunk to a point, so the process could equally well be seen as attaching the corresponding disc bundle (i.e. with the sphere filled in).

LEMMA 10. *$\mathbb{RP}^{2n-1} \to \mathbb{CP}^{n-1}$ is the sphere bundle of the vector bundle $\mathcal{L}^2 = \mathcal{L} \otimes \mathcal{L}$ where $\mathcal{L} \to \mathbb{CP}^{n-1}$ is the universal line bundle over $\mathbb{CP}^{n-1}$.*

For completeness we reprove this known fact. Recall that $\mathcal{L}$ is the dual of the hyperplane bundle.

PROOF. We use the Gysin sequence derived from the sphere bundle $S^d \to E \to B$:

$$\ldots \to H^m(E;R) \to H^{m-d}(B;R) \xrightarrow{\cup e} H^{m+1}(B;R) \to H^{m+1}(E;R) \to \ldots$$

where R is a ring and $e = e(\lambda) \in H^{d+1}(B; H^0(S^d;R))$ is the Euler class of the associated vector bundle λ. Taking $R = \mathbb{Z}$ and $m = 1$ for the S^1-bundle $\mathbb{RP}^{2n-1} \to \mathbb{CP}^{n-1}$ this gives

$$\ldots \to H^0(\mathbb{CP}^{n-1}, \mathbb{Z}) \xrightarrow{\cup e} H^2(\mathbb{CP}^{n-1}, \mathbb{Z}) \xrightarrow{\phi} H^2(\mathbb{RP}^{2n-1}, \mathbb{Z}) \to 0 \to \ldots$$

Now the first two terms are isomorphic to $\mathbb{Z}$, the second group with generator $c_1(\mathcal{L})$, the first Chern class of the universal bundle. The last term is isomorphic to $\mathbb{Z}/2$. Since ϕ must be surjective the cup product by e map must be multiplication by 2 in $\mathbb{Z}$, that is $e(\lambda) = 2c_1(\mathcal{L})$. So we have shown that the Euler class of the vector bundle λ, which is also the first Chern class, is $2c_1(\mathcal{L})$. Finally since $c_1(\mathcal{L} \otimes \mathcal{L}) = 2c_1(\mathcal{L})$ and holomorphic line bundles are classified (up to isomorphism) by their first chern class we obtain $\lambda = \mathcal{L}^2$ and the lemma is proved. □

From the above lemma we can now view the surgery as attaching the disc bundle $D(\mathcal{L}^2)$ over $\mathbb{CP}^{n-1}$ where the fibers are restricted to discs of the appropriate radius. It remains to describe the group action on these bundles.

$A \in G < \mathrm{U}(n)$ acts on the total space $\mathcal{L} \subset \mathbb{CP}^{n-1} \times \mathbb{C}^n$ by sending

$$(x, tx) \longmapsto (Ax, A(tx))$$

where $t \in \mathbb{C}$ and (x, tx) is an element of $\mathbb{CP}^{n-1} \times \mathbb{C}^n$. This is of course a bundle map. Similarly A has a natural action on the tensor $\mathcal{L} \otimes \mathcal{L}$:

$$(x, t_1 x \otimes t_2 x) \longmapsto (Ax, A(t_1 X) \otimes A(t_2 x))$$

But again A and $-A$ have the exact same action on $\mathcal{L}^2$, so for $G < \mathrm{U}(n)$, $G/\langle \pm 1 \rangle$ has a faithful action on the bundle (still assuming $\langle \pm 1 \rangle$ is contained in G). Since $G < \mathrm{U}(n)$ preserves distances, the action restricts to the disc bundle $D(\mathcal{L}^2)$.

Finally the groups acting on the bundles and quotient space must match up on the boundaries along which they are glued. This amounts to choosing the same group in both cases. In other words, taking a linear transformation acting on the quotient of the truncated torus, the same transformation must be chosen on each of the 2^{2n} bundles to obtain an automorphism of the entire manifold. Another way of viewing this process is that if we choose an appropriate automorphism of a particular bundle it extends uniquely and identically out of it and down into the remaining $2^{2n} - 1$ bundles replacing the double points.

Summarising, we have shown that for a group $G < \mathrm{U}(n)$ preserving a lattice $L \subset \mathbb{C}^n$, $G/\langle \pm 1 \rangle$ acts faithfully on the Kummer manifold K_L. In particular let $\mathrm{Aut}_{\mathbb{C}}(L)$ denote the subgroup of complex linear maps of $\mathrm{Aut}(L)$. Then we have

LEMMA 11. *$\mathrm{Aut}_{\mathbb{C}}(L)/\langle \pm 1 \rangle$ has a faithful holomorphic action on K_L.*

Note that $\langle \pm 1 \rangle$ is always in $\mathrm{Aut}(L)$ as all lattices have the -1 symmetry. Of course if L has a structure of J-lattice for $J = \mathcal{E}$ or $\mathcal{G}$ then $\mathrm{Aut}_{\mathbb{C}}(L)$ is just $\mathrm{Aut}_J(L)$. See ch.1 §1 for more on complex lattices and their automorphisms.

In the two dimensional case $G/\langle \pm 1 \rangle$ is a subgroup of $\mathrm{PSU}(2)$. In higher dimensions however this is no longer the case in general as the centre of $\mathrm{SU}(n)$ is not $\langle \pm 1 \rangle$.

We now examine another source of biholomorphic maps of a Kummer manifold, this time independent of the lattice used in the construction, namely translations. First recall that the automorphism group of a torus T is the semidirect product

$$\mathrm{Aut}(T) = \mathrm{Hom}(T) \ltimes \mathrm{Tr}$$

of the group of translations of the torus Tr by the group $\mathrm{Hom}(T)$ of biholomorphic homomorphisms of T. Up to now we have dealt exclusively with the latter group. Most translations of the torus are not preserved under the idenfication $x \sim -x$ when forming the Kummer manifold. Indeed in n dimensions only the 2^{2n} translations by the points of order 2 remain well-defined. Let p denote a point of order 2 and $x \in T$. Then since $p \sim -p$,

$$x + p \sim -x - p \sim -x + p.$$

In other words adding a point of order 2 is well-defined on the quotient space $T/\langle \pm 1 \rangle$. It is easily seen that these are the only translations remaining. On the Kummer manifolds these induce translations of the attached disc bundles.

LEMMA 12. *The translations of the torus $T = \mathbb{C}^n/L$ induce a free action of $(\mathbb{Z}/2)^{2n}$ on the n-dimensional Kummer manifold K_L.*

Together with lemma 11 this implies:

THEOREM 13. $2^{2n} \rtimes \mathrm{Aut}_{\mathbb{C}}(L)/\langle \pm 1 \rangle$ *acts faithfully on the Kummer manifold K_L. Moreover these are all the isometries of K_L coming from the torus.*

PROOF. The action follows from lemmas 11 and 12 above. K_L inherits a metric from the torus and by definition $\mathrm{Aut}_{\mathbb{C}}(L) = \mathrm{Hom}_u(T)$ are the metric preserving homomorphisms of the torus and all translations preserve the form. The product is still semi-direct as in theorem 8. □

2. Kummer surfaces and $K3$-surfaces

The 2-dimensional Kummer manifolds or Kummer surfaces are of particular interest for several reasons. First of all non-singular Kummer surfaces are $K3$-surfaces. The oldest account of the connections with combinatorial configurations is Hudson's book [**34**]. A more modern and thorough account including work on $K3$-surfaces and jacobians can be found in Gonzalez-Dorrego [**25**]. Another good reference is Lange and Birkenhake [**40**]. More relevant to our work are the papers of Mukai [**50**] and more recently Kondō [**39**] and Xiao [**65**] on symplectic automorphisms of $K3$-surfaces. Mukai classifies all finite groups of such actions. See also Mason's account [**45**]. Finally Kummer surfaces over other fields arise in number theory as quotients of jacobians (eg. in [**11**]).

Due to the connections with $K3$-surfaces we shall be primarily interested in symplectic automorphisms of the Kummer surfaces. Quaternions provide a nice framework in this context.

The non-commutative field of quaternions is denoted by $\mathbb{H}$, with standard real basis $1, i, j, k$ and the usual (real) vector space isomorphism

$$\mathbb{R}^4 \to \mathbb{H} \text{ given by } (x_1, x_2, x_3, x_4) \to x_1 + x_2 i + x_3 j + x_4 k.$$

We also identify $\mathbb{H}$ with $\mathbb{C}^2$ in the standard way

$$\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j.$$

$\mathbb{H}^n$ $(n \geq 1)$ will be viewed as a right $\mathbb{H}$-vector space.

We are interested in finite multiplicative groups of quaternions. Since

$$|x^2| = |x||x|$$

(where the operation is quaternion multiplication), any quaternion with modulus not equal to 1 would generate an infinite number of elements. Hence

LEMMA 14. *Any finite multiplicative group of quaternions consists entirely of unit quaternions.*

In fact these groups are conveniently classified (see Coxeter [**17**])

THEOREM 15. *A finite multiplicative group of quaternions is one of the following*

(1) a cyclic group $\mathbb{Z}/n$ *of order* n

(2) a dicyclic group $\langle p, 2, 2\rangle$ *of order* $4p$

(3) the binary tetrahedral group $\langle 3, 3, 2\rangle$ *of order* 24

(4) the binary octahedral group $\langle 4, 3, 2\rangle$ *of order* 48

(5) the binary icosahedral group $\langle 5, 3, 2\rangle$ *of order* 120

The set of all unit quaternions form an infinite multiplicative group denoted Sp(1) which is also the group of distance preserving quaternionic linear maps of $\mathbb{H}$. One can view Sp(1) as acting by quaternionic multiplication on the left of the right $\mathbb{H}$-vector space $\mathbb{H}$.

There is a particular nice description of the above finite groups in terms of generators and relations. Apart from the cyclic groups with their obvious presentation the group $\langle p,q,2\rangle$ has presentation

$$\langle A,B,C : A^p = B^q = C^2 = ABC = -1\rangle$$

Note that strictly speaking C is not necessary since $AB = C$ so that A and B generate the group. Taking $C = k$ the following lemma gives us quaternions A and B that generate the group (see Coxeter [**17**]).

LEMMA 16. *The group* $\langle p,q,2\rangle$ *is generated by the quaternions*

$$A = \cos\frac{\pi}{p} + k\cos\frac{\pi}{q} + i\sin\frac{\pi}{h}, \quad B = \cos\frac{\pi}{q} + k\cos\frac{\pi}{p} + j\sin\frac{\pi}{h}, \quad \textit{and } C = k$$

$$\textit{where } \cos^2\frac{\pi}{h} = \cos^2\frac{\pi}{p} + \cos^2\frac{\pi}{q}$$

For example in the case of the dicyclic group $\langle 2,2,2\rangle$ the lemma produces the generators i, j, k satisfying of course

$$i^2 = j^2 = k^2 = ijk = -1$$

This group is sometimes denoted Q_8.

Finally bear in mind that the above lemma gives a possible set of generators of $\langle p,q,2\rangle$ and that there are other isomorphic subgroups in Sp(1).

LEMMA 17. *The group of all unit quaternions* Sp(1) *is isomorphic to* SU(2).

So in particular theorem 15 above also holds for SU(2) and its finite subgroups. Under the isomorphism the following correspond

$$i \longleftrightarrow \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad j \longleftrightarrow \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad k \longleftrightarrow \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

These three matrices also generate an algebra in $GL(2,\mathbb{C})$ isomorphic to $\mathbb{H}$. By generate here we mean generate as an algebra with multiplication the usual matrix multiplication.

Recall that $\mathrm{PSU}(n)$ is the quotient of $\mathrm{SU}(n)$ by its centre. In general the centre consists of those matrices $\mu \cdot \mathrm{Id}$ where μ is an nth root of unity and Id the $n \times n$ identity matrix. For $n = 2$, $\mathrm{PSU}(2)$ is obtained by factoring out $\langle \pm \mathrm{Id} \rangle$:

$$\mathrm{PSU}(2) = \frac{\mathrm{SU}(2)}{\langle \pm \mathrm{Id} \rangle}.$$

The following important result ties in with the quaternion case (theorem 15)

THEOREM 18. *A finite subgroup of* $\mathrm{PSU}(2)$ *is one of the following*

(1) a cyclic group $\mathbb{Z}/n$ *of order* n

(2) a dihedral group D_p *of order* $2p$

(3) the alternating A_4 *of order* 12

(4) the symmetric group S_4 *of order* 24

(5) the alternating group A_5 *of order* 60

For a proof of the theorem and further discussion of these groups see Jones [**37**]. We now exhibit the obvious link between the groups in theorems 15 and 18.

Recall $\mathrm{PSU}(2)$ is obtained from $\mathrm{SU}(2)$ by identifying matrices with their additive inverses. Hence the exact sequence

$$0 \longrightarrow \langle \pm \mathrm{Id} \rangle \longrightarrow \mathrm{SU}(2) \xrightarrow{\ \rho\ } \mathrm{PSU}(2) \longrightarrow 0$$

It is clear that a quaternion subgroup of type (i) of theorem 15 of order $2n$ is mapped (via the map ρ above) to the subgroup of type (i) of theorem 18 of order n (the cyclic groups of odd order excepted, since they do not contain $-\mathrm{Id}$). Indeed under ρ the presentation above becomes

$$\langle A, B, C : A^p = B^q = C^2 = ABC = 1 \rangle$$

where -1 has been identified with 1. With this in mind we may also denote the groups

$$D_p, A_4, S_4, A_5$$

by

$$(p, 2, 2), (3, 3, 2), (4, 3, 2), (5, 3, 2)$$

respectively. Note $(p, q, 2)$ has elements of order p, q and of course 2. We now have an accurate description of the different finite groups of quaternions and the resulting groups after the quotient by the centre $\langle \pm \mathrm{Id} \rangle$.

DEFINITION 10. *A $K3$-surface is a compact complex surface S with trivial canonical bundle $\mathcal{K}_S = 0$ and first Betti number $b_1(S) = 0$.*

Recall that the first Betti number is the dimension of the cohomology group $H^1(S, \mathcal{O}_S)$, where $\mathcal{O}_S$ denotes the structure sheaf of S.

An important consequence of the definition is that every $K3$-surface admits a nowhere vanishing holomorphic symplectic 2-form ω corresponding to a section of $\mathcal{K}_S$. A *symplectic automorphism* of a $K3$-surface S is a biholomorphic transformation of S preserving the form ω. On the tangent space at $x \in S$ the holomorphic symplectic form is

$$\omega_x(z, v) = z_2 v_1 - z_1 v_2$$

where $z = (z_1, z_2)$ and $v = (v_1, v_2) \in \mathbb{C}^2$. SU(2) consists of those unitary matrices preserving ω_x. Hence a subgroup $G < \mathrm{PSU}(2)$ acting on a Kummer manifold K_L as described in §1 is a group of symplectic automorphisms of the $K3$-surface K_L. For a proof that Kummer surfaces are indeed $K3$-surfaces see [**25**].

In [**50**] Mukai classifies the finite symplectic automorphism groups of $K3$-surfaces. First recall that the Mathieu group $\mathrm{M}_{23} < \mathrm{M}_{24}$ acts by permutations on the set Ω of order 24 (see ch.1 §3). M_{23} is the stabiliser of an element of Ω.

THEOREM 19 (Mukai). *A finite group G has a (faithful) symplectic action on a $K3$-surface if and only if G has an embedding in M_{23} splitting Ω into at least 5 orbits.*

This is shown to be equivalent to the next more explicit statement.

THEOREM 20 (Mukai). *A finite group G has a (faithful) symplectic action on a $K3$-surface $\Longleftrightarrow$ G is isomorphic to a subgroup of one of the 11 groups below*

	Group	Order	$K3$-surface
(1)	$\mathrm{L}_2(7) = \mathrm{PSL}_2(7)$	168	$x^3y + y^3z + z^3x + t^4 = 0$ in $\mathbb{C}\mathrm{P}^3$
(2)	A_6	360	$\sum_1^6 x_i = \sum_1^6 x_i^2 = \sum_1^6 x_i^3 = 0$ in $\mathbb{C}\mathrm{P}^5$
(3)	S_5	120	$\sum_1^5 x_1 = \sum_1^6 x_i^2 = \sum_1^5 x_i^3 = 0$ in $\mathbb{C}\mathrm{P}^5$
(4)	$\mathrm{M}_{20} = 2^4 \rtimes A_5$	960	$x^4 + y^4 + z^4 + t^4 + 12xyzt = 0$ in $\mathbb{C}\mathrm{P}^3$
(5)	$F_{384} = 4^2 \rtimes S_4$	384	$x^4 + y^4 + z^4 + t^4 = 0$ in $\mathbb{C}\mathrm{P}^3$
(6)	$A_{4,4} = 2^4 \rtimes A_{3,3}$	288	$x^2 + \omega y^2 + \omega^2 z^2 = \sqrt{3}v^2$ in $\mathbb{C}\mathrm{P}^5$ $x^2 + \omega^2 y^2 + \omega z^2 = \sqrt{3}w^2$ $x^2 + y^2 + z^2 = \sqrt{3}u^2$
(7)	$T_{192} = (Q_8 * Q_8) \rtimes S_3$	192	$x^4 + y^4 + z^4 + t^4 - 2\sqrt{-3}(x^2y^2 + z^2t^2) = 0$
(8)	$H_{192} = 2^4 \rtimes D_6$	192	$x_1^2 + x_3^2 + x_5^2 = x_2^2 + x_4^2 + x_6^2$ in $\mathbb{C}\mathrm{P}^5$ $x_1^2 + x_4^2 = x_2^2 + x_5^2 = x_3^2 + x_6^2$
(9)	$N_{72} = 3^2 \rtimes D_4$	72	$x_1^3 + x_2^3 + x_3^3 + x_4^3 = x_1x_2 + x_3x_4 + x_5^2 = 0$ in $\mathbb{C}\mathrm{P}^5$
(10)	$\mathrm{M}_9 = 3^2 \rtimes Q_8$	72	Double cover of $\mathbb{C}\mathrm{P}^2$ with branch $x^6 + y^6 + z^6 - 10(x^3y^3 + y^3z^3 + z^3x^3) = 0$
(11)	$T_{48} = Q_8 \rtimes S_3$	48	Double cover of $\mathbb{C}\mathrm{P}^2$ with branch $xy(x^4 + y^4) + z^6 = 0$

The proof of the ($\Rightarrow$) statement is group theoretical and rests on the observation that the number of fixed points of a symplectic automorphism of order m coincides with the number of fixed points of a permutation in M_{23} of order m. The ($\Leftarrow$) direction is proved by displaying an action of all 11 groups on a particular $K3$-surface given in the 3rd column of the above table. Mukai describes the surfaces as loci of homogeneous polynomials, that is as projective varieties.

Note that a K3-surface need not be algebraic, although Mukai only makes use of such surfaces in the theorem above. We will return to this question at a later stage when discussing algebraic embeddings of Kummer manifolds into projective space (chapter 4).

We now describe some explicit constructions of Kummer surfaces and groups acting on them. As mentioned at the beginning we shall make use of the quaternions by constructing a lattice L with automorphisms acting by quaternion left multiplication (ie in SU(2)). The corresponding subgroup of PSU(2) acts symplectically on K_L. In more detail now.

Let L be a lattice in $\mathbb{R}^4 = \mathbb{H}$ generated by unit quaternions forming a finite multiplicative group. Then the group acts on the lattice:

(1) on those elements of unit length in L by quaternion multiplication on the left
(2) on the rest of the lattice by linear expansion

This action on L is of course free. The image of $G < \mathrm{SU}(2)$ in PSU(2) is a finite subgroup H acting faithfully on K_L.

It remains to construct such a lattice. In general let G be a finite quaternion group of the form $\langle p, q, 2\rangle$ for some p, q (we are only omitting the cyclic groups). Lemma 16 gives us three generators $A, B, C \in \mathbb{H}$ for this group, where $C = k$. Now consider the lattice

$$L = \mathbb{Z}\{1, A, B, C\}.$$

In the examples to follow, the minimal vectors of the lattice form the group G.

So to recap: if $G = \langle p, q, 2\rangle$ consists of the unit elements of $L = \mathbb{Z}\{1, A, B, C\}$ then G acts on the torus and truncated torus. After identifying inverse points the order of the group is halved and we obtain $H = (p, q, 2) < \mathrm{PSU}(2)$ acting faithfully on K_L after extending the action into the attached bundles.

Let us start with an easy case.

2.1. The dihedral group D_2. Take the most obvious lattice in 4 dimensions, namely the root lattice $\mathbb{Z}^4$. Viewed in the quaternions,

$$\mathbb{Z}^4 = \mathbb{Z}\{1, i, j, k\}$$

The 8 elements of $\mathbb{Z}^4$ of unit length are of course

$$\{\pm 1, \pm i, \pm j, \pm k\}$$

These 8 elements form the multiplicative group of quaternions Q_8, and we are in the situation described above. Of course Q_8 must be one of the groups of theorem 15. Writing Q_8 as

$$Q_8 = \langle i, j, k : i^2 = j^2 = k^2 = ijk = -1 \rangle$$

we see from the presentation of $\langle p, q, 2 \rangle$ that

$$Q_8 = \langle 2, 2, 2 \rangle$$

where $\langle 2, 2, 2 \rangle$ is the dicyclic group of order 8. Hence we have obtained the dihedral group $D_2 = (2, 2, 2)$ of order 4 as a group of symplectic automorphisms of $K_{\mathbb{Z}^4}$.

2.2. The dihedral group D_3. We construct the lattice using the method described above. Using lemma 16 we obtain the generators A, B of the dicyclic group $\langle 3, 2, 2 \rangle$:

$$A = \frac{1}{2} + \frac{\sqrt{3}}{2} i, \quad B = \frac{1}{2} + \frac{\sqrt{3}}{2} j$$

$\langle 3, 2, 2 \rangle$ has order 12. Consider the lattice

$$L = \mathbb{Z}\{1, k, A, B\}$$

It remains to verify that the dicyclic group is included in the lattice, i.e. that all quaternions in the group are a linear combination of the four generators above (coefficients in $\mathbb{Z}$ of course) - a simple calculation. Once this has been checked, the image of the group in PSU(2) is a dihedral group D_3 of order 6 acting on K_L.

2.3. The alternating group A_4. Again we use the same method. The relevant quaternion group is of course the binary tetrahedral group $\langle 3,3,2\rangle = 2.A_4$ of order 24. Using lemma 16 we obtain generators for $2.A_4$:

$$A = \frac{1}{2} + \frac{1}{2}k + \frac{1}{\sqrt{2}}i, \quad B = \frac{1}{2} + \frac{1}{2}k + \frac{1}{\sqrt{2}}j.$$

Taking these together with 1 and k consider the lattice

$$L = \mathbb{Z}\{1, k, A, B\}.$$

As previously it remains to check that the 24 elements of $\langle 3,3,2\rangle$ are in L - a tedious but easy calculation. Thus we have obtained a group $A_4 = (3,3,2)$ of symplectic automorphisms of the Kummer surface K_L.

2.4. Another construction. We obtain the same action and surface using a different embedding of the same lattice. Consider the 4-dimensional lattice D_4 with generator matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Equivalently D_4 is the lattice

$$\mathrm{D}_4 = \mathbb{Z}\{i, j, k, \mu\}$$

where μ is a primitive sixth root of unity:

$$\mu = \frac{1}{2}(1 + i + j + k).$$

The 24 elements of unit length in D_4 are

$$\pm 1, \ \pm i, \ \pm j, \ \pm k, \quad \frac{1}{2}(\pm 1 \pm i \pm j \pm k)$$

where there are $2^4 = 16$ elements of the last form. These elements form a multiplicative group of quaternions $2.A_4 = \langle 3,3,2\rangle$.

So we have displayed another construction of K_{D_4} with an A_4-action. (see ch.1 §1 and Conway and Sloane [**13**], ch.4 for more on D_4)

Both the lattices A_2 and $\mathbb{Z}^4$ are clearly embedded as sublattices in the copy of D_4 just described, $\mathbb{Z}^4$ as mentioned in the above paragraph on D_2 and A_2 as the powers of the root μ.

The 24 unit vectors of D_4 form the units $\mathcal{H}_u$ of the ring of Hurwitz integers $\mathcal{H}$ described in ch.1 §1. Indeed D_4 is a one-dimensional quaternionic lattice with generator 1.

Given the list of subgroups of $\mathrm{PSU}(2)$ in theorem 18 it is natural to ask which of these can be made to act on some Kummer surface K_L as described above. Further motivation is given by Mukai's results described earlier; A_5 and D_6 would be of particular interest. We now show that there is one major obstruction on the size of the groups one may obtain in this way. Recall that in order for $H < \mathrm{PSU}(2)$ to act on K we required the corresponding group $G \subset \mathrm{SU}(2)$ to act by quaternion multiplication on the lattice in $\mathbb{C}^2$. In the next propositions we omit the cyclic groups $\mathbb{Z}/n < \mathrm{SU}(2)$ of odd order since these are preserved under the quotient by $\langle \pm 1 \rangle$.

LEMMA 21. *Let $H < \mathrm{PSU}(2)$ be a finite group (odd cyclic groups excluded) acting on K_L as described. Then the lattice L must have at least $2|H|$ points of any given length.*

PROOF. Let G denote the preimage of H in $\mathrm{SU}(2)$ under the quotient map. Recall that the group G consists of unit quaternions and acts by quaternionic multiplication. Hence the action preserves lengths. Also only multiplication by 1 fixes some element so the action on L is free.

Suppose now that there were less than $|G| = 2|H|$ points of length l in some lattice $L \subset \mathbb{R}^4$. Let a be such a point. Then since distances are preserved and there is at most $|G| - 2$ other lattice points of the same length l, there must exist two *distinct* elements $\alpha, \beta \in G < \mathrm{SU}(2) \simeq \mathrm{Sp}(1)$ such that

$$\alpha a = \beta a.$$

But this is impossible since G acts freely. Contradiction. □

So to construct a limit on the order of the groups obtainable we need an upper bound on the minimum number of points in shells of a 4 dimensional lattice.

For general sphere packings (see Conway and Sloane [**13**] ch.1 for a definition) the *kissing number* is the number of spheres touching a given sphere. A *lattice sphere packing* has spheres centered at the lattice points. The kissing number of a lattice sphere packing is the same for all the spheres, and we can define:

DEFINITION 11. *The* kissing number τ *of a lattice* L *is the number of minimal vectors of* L

LEMMA 22. *The maximum kissing number of a 4 dimensional lattice is 24.*

Indeed in dimensions $n \leq 9$ the laminated lattices Λ_n attain the maximum kissing number for lattice sphere packings - see Conway and Sloane [**13**] ch.1.

PROPOSITION 23. *A finite subgroup* H *of* PSU(2) *acting on a Kummer surface* K *as explained is of order at most 12.*

PROOF. Let L denote the lattice used in constructing K. By lemma 21 L must have at least $2|H|$ points of any given length (except zero). However by lemma 22 L has at most 24 minimal vectors. Also $2|H| \leq 24$ and $|H| \leq 12$, as required. □

COROLLARY 24. *No Kummer surface has a symplectic action of* A_5 *or* S_4 *induced from automorphisms of the lattice.*

The main action constructed is:

THEOREM 25. *There is a faithful symplectic action of* $2^4 \rtimes A_4$ *on the Kummer surface* K_{D_4}.

PROOF. This follows from the earlier construction where we obtained an A_4-action on K_{D_4} and from lemma 12 for the translations. The product splits since the group is a subgroup of the quotient of the automorphism group $\mathrm{Hom}(T) \ltimes \mathrm{Tr}$ of the torus (see theorem 8). The translations also act symplectically. □

How does this fit in with Mukai's results ? By theorem 20 one should find $2^4 \rtimes A_4$ as a subgroup of one of the 11 groups listed there. The $K3$-surfaces on which the groups are made to act need not of course be Kummer surfaces. This leads an interesting open question as to which of these groups can be obtained acting on the more restricted class of Kummer surfaces. Returning to the above result $2^4 \rtimes A_4$ does indeed appear as a subgroup of the Mathieu group $M_{20} = 2^4 \rtimes A_5$ listed as number (4) in theorem 20. I have so far been unable to determine whether the surface used by Mukai in this case

$$x^4 + y^4 + z^4 + t^4 + 12xyzt = 0 \text{ in } \mathbb{CP}^3$$

is in fact a Kummer surface.

2.5. Fixed points of symplectic actions. The number of fixed points f_d of a symplectic transformation of a $K3$-surface depends only on its order d (Mukai [**50**]):

d	2	3	4	5	6	7	8
f_d	8	6	4	4	2	3	2

We examine how these appear in our construction of Kummer surfaces. Consider the root lattice $\mathbb{Z}^4$ as complex lattice over the Gaussian integers as in §2.1: $\mathbb{Z}^4 = \mathbb{Z}[i] \oplus \mathbb{Z}[i]$. The Kummer surface $K_{\mathbb{Z}^4}$ inherits a symplectic transformation φ_A of order 2 corresponding to the symmetry

$$A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

of the lattice (multiplication by i in the quaternions). A has $2^2 = 4$ fixed points on the torus with representatives

$$(0,0,0,0),\ (1/2,1/2,0,0),\ (0,0,1/2,1/2),\ (1/2,1/2,1/2,1/2)$$

in $\mathbb{R}^4$. These are also fixed by the Kummer involution $\langle \pm 1 \rangle$ and are blown-up to form the Kummer surface. A acts on the disc bundle $D(\mathcal{L})$ as described in §1. The zero section complex projective line of the disc bundle is preserved by A and has two fixed points $(1:0),\ (0:1) \in \mathbb{CP}^1$. And φ_A has a total of $2.4 = 8$ fixed points as expected.

Take the root lattice $2A_2$ with Eisenstein structure $\mathbb{Z}[\omega] \oplus \mathbb{Z}[\omega]$. The Kummer surface K_{2A_2} then has a symplectic transformation φ_B of order 3 corresponding to the symmetry

$$B = \begin{bmatrix} \omega & 0 \\ 0 & \overline{\omega} \end{bmatrix}$$

of the lattice $2A_2$. B has $3^2 = 9$ fixed points on the torus of which only the origin is fixed by $\langle \pm 1 \rangle$. $\langle \pm 1 \rangle$ identifies the other fixed points in pairs (see chapter 3 for an explicit description). Again B has 2 fixed points $(1:0)$, $(0:1)$ on the zero section of the disc bundle. Since the origin is the only one of these points blown-up in the Kummer construction, φ_B has a total of $(3^2-1)/2 + 2.1 = 6$ fixed points.

2.6. Non-symplectic actions. Non-symplectic actions are constructed as outlined at the end of §1. Take the standard complex root lattice $\mathbb{Z}^4 = \mathbb{Z}[i] \oplus \mathbb{Z}[i] \subset \mathbb{C}^2$ as above. $\mathbb{Z}^4$ then has complex automorphism group of order 32:

$$\mathrm{Aut}_{\mathcal{G}}(\mathbb{Z}^4) = 4^2 \rtimes 2$$

where $(\mathbb{Z}/4)^2$ is multiplication by i in each component and $\mathbb{Z}/2$ is the permutation of the two. $\langle \pm 1 \rangle$ is then multiplication by -1 in each component. The quotient is

$$\mathrm{Aut}_{\mathcal{G}}(\mathbb{Z}^4)/\langle \pm 1 \rangle = (2 \times 4) \rtimes 2$$

D_4 also has a complex structure over the Gaussian integers. To determine $\mathrm{Aut}_{\mathcal{G}}(D_4)$ we use the quaternionic description: $D_4 = \mathcal{H} \subset \mathbb{H}$ with minimal vectors

$$\pm 1,\ \pm i,\ \pm j,\ \pm k,\ 1/2(\pm 1 \pm i \pm j \pm k)$$

$\mathrm{Aut}_{\mathcal{H}}(D_4) = 2.A_4 < \mathrm{Sp}(1)$ acts by quaternionic multiplication on the left of $D_4 \subset \mathbb{H}$ commuting with the $\mathbb{H}$-linear action on the right. Take the complex structure to be multiplication by i on the right in the quaternionic setting, that is multiplication by

$$\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$$

on $\mathbb{C}^2$. $\mathrm{Aut}_{\mathcal{H}}(D_4) = 2.A_4$ acting on the left then trivially commutes with the complex structure. In addition to these transformations, sign changes of the two complex coordinates are also complex automorphisms of the lattice (that these preserve the lattice can easily be seen in the explicit description of D_4 given above). They form a $(\mathbb{Z}/2)^2$ preserving $2.A_4$ under conjugation (this follows for example from the transitivity of the $2.A_4$-action on D_4), so

$$\mathrm{Aut}_{\mathcal{G}}(D_4) = 2.A_4.2^2$$

$\langle\pm1\rangle$ is multiplication by -1 in $2.A_4$, so the quotient is

$$\mathrm{Aut}_{\mathcal{G}}(\mathrm{D}_4)/\langle\pm1\rangle = A_4.2^2$$

Together with the translations by fixed points these calculations yield holomorphic (non-symplectic) actions on Kummer surfaces:

$$2^4 \rtimes (2\times4)\rtimes 2 \text{ on } K_{\mathbb{Z}^4}$$

$$2^4 \rtimes A_4.2^2 \text{ on } K_{\mathrm{D}_4}$$

3. Higher dimensional Kummer manifolds

In comparison with Kummer surfaces in the previous section much less is known about a general higher dimensional Kummer manifold. There is no longer a nice combinatorial approach using configurations and of course the desingularised manifold is no longer a K3-surface. Accordingly we no longer have any particular interest in special linear actions. Using results on lattice automorphism groups it is an easy step to apply our earlier construction and obtain interesting groups acting on certain Kummer manifolds. We then show how tensoring lattices helps us obtain all lattice automorphism groups acting on some Kummer manifold. A good general reference for the lattice theory is again Conway and Sloane [**13**]. For more on finite simple groups see the Atlas [**16**] and Aschbacher [**3**]. Higher dimensional Kummer manifolds are discussed in Lange and Birkenhake [**40**] and from a topological point of view by Spanier [**59**]. Spanier shows that all Kummer manifolds K (dim $K\geq 2$) are simply connected and calculates the Betti numbers.

THEOREM 26. *There are faithful holomorphic actions of*
(1) $2^{12}\rtimes 3.\mathrm{U}_4(3).2$ *on* $K_{\mathrm{K}_{12}}$
(2) $2^{24}\rtimes 3.\mathrm{Suz}$ *on* $K_{\Lambda_{24}}$.

PROOF. The Coxeter-Todd lattice K_{12} has a complex structure over the Eisenstein integers $\mathcal{E}=\mathbb{Z}[\omega]$ and

$$\mathrm{Aut}_{\mathcal{E}}(\mathrm{K}_{12}) = 6.\mathrm{U}_4(3).2$$

(see ch.1 §1.7). The Leech lattice as $\mathcal{E}$-lattice has automorphism group

$$\mathrm{Aut}_{\mathcal{E}}(\Lambda_{24}) = 6.\mathrm{Suz}$$

(see ch.1 §1.9). The result now follows from theorem 13. □

3.1. Complex lattices and tensoring. If the dimension of the manifolds is not an issue then one can obtain simple subgroups of real automorphism groups of lattices acting holomorphically on a Kummer manifold. This is done simply by tensoring the given real lattice by the Gaussian or Eisenstein integers.

Let L be an n-dimensional real lattice with basis $\{l_1, \dots, l_n\}$ and define

$$L_i = \mathbb{Z}[i] \otimes_{\mathbb{Z}} L \subset \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^n = \mathbb{R}^{2n}$$

This is again a real lattice, now of dimension $2n$ with basis $\{l_1, \dots, l_n, il_1, \dots, il_n\}$. By definition L_i admits a complex structure over the Gaussian integers. Its real automorphism group is made up of the cyclic group $\langle \pm i \rangle$ of order 4 (the units of the Gaussian integers) and the automorphism group $\mathrm{Aut}(L)$:

$$g \in \mathrm{Aut}(L) \text{ acts as } g \cdot (x \otimes l) = x \otimes gl$$

$$z \in \langle \pm i \rangle \text{ acts as } z \cdot (x \otimes l) = zx \otimes l$$

Clearly these all commute with the complex structure i, so writing $\mathrm{Aut}(L) = 2.G$ for some group G (all lattices have the $\langle \pm 1 \rangle$ symmetry):

$$\mathrm{Aut}_{\mathcal{G}}(L_i) = \mathrm{Aut}(L_i) = 4.G$$

The sum is not direct as (-1) acts on both sides of the tensor product. The Kummer involution on the new lattice L_i corresponds to $\langle \pm 1 \rangle$ inside $\langle \pm i \rangle$ and the involution on L:

$$-(z \otimes l) = (-z) \otimes l = z \otimes (-l)$$

in $\mathbb{Z}[i] \otimes_{\mathbb{Z}} L$. By lemma 11 the resulting n-dimensional Kummer manifold K_{L_i} admits a holomorphic action of $2 \times \mathrm{Aut}(L)$. The sum is direct as 2 is now $\{1, i\}$.

Starting with the Eisenstein integers one proceeds in a similar fashion. Define

$$L_\omega = \mathbb{Z}[\omega] \otimes_{\mathbb{Z}} L.$$

Then

$$\mathrm{Aut}_{\mathcal{E}}(L_\omega) = 3 \times \mathrm{Aut}(L).$$

The only difference is that the involution is now only apparent in $\mathrm{Aut}(L)$. So for $\mathrm{Aut}(L) = 2.G$ we have

$$\mathrm{Aut}_{\mathcal{E}}(L_\omega) = 3 \times 2.G = 6.G$$

and

$$3 \times G < \mathrm{Aut}(K_{L_\omega}).$$

K_{L_ω} is an n-dimensional Kummer manifold.

3.2. Root lattices E_6, E_7, **and** E_8**.** The automorphism groups of root lattices consist essentially of their Weyl groups as explained in ch.1 §1. We construct Kummer manifolds admitting actions of the finite simple $\mathrm{U}_4(2)$, $\mathrm{S}_6(2)$ and $\mathrm{O}_8^+(2)$.

Define

$$\mathrm{E}_{6,\mathrm{i}} = \mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathrm{E}_6, \quad \mathrm{E}_{7,\mathrm{i}} = \mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathrm{E}_7, \quad \mathrm{E}_{8,\mathrm{i}} = \mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathrm{E}_8.$$

These are complex lattices in $\mathbb{C}^6$, $\mathbb{C}^7$ and $\mathbb{C}^8$ respectively with automorphism groups

$$\mathrm{Aut}_{\mathcal{G}}(\mathrm{E}_{6,\mathrm{i}}) = 4.\mathrm{W}(\mathrm{E}_{6,\mathrm{i}}) = 4.\mathrm{U}_4(2).2$$
$$\mathrm{Aut}_{\mathcal{G}}(\mathrm{E}_{7,\mathrm{i}}) = 2.\mathrm{W}(\mathrm{E}_7) = 4 \times \mathrm{S}_6(2)$$
$$\mathrm{Aut}_{\mathcal{G}}(\mathrm{E}_{8,\mathrm{i}}) = 2.\mathrm{W}(\mathrm{E}_8) = 4.\mathrm{O}_8^+(2).2$$

By theorem 13 the Kummer manifolds $K_{\mathrm{E}_{6,\mathrm{i}}}$, $K_{\mathrm{E}_{7,\mathrm{i}}}$ and $K_{\mathrm{E}_{8,\mathrm{i}}}$ of dimension 6, 7 and 8 respectively admit holomorphic actions of

$$2^{12} \rtimes (2 \times \mathrm{U}_4(2).2) < \mathrm{Aut}(K_{\mathrm{E}_{6,\mathrm{i}}})$$
$$2^{14} \rtimes (2 \times \mathrm{S}_6(2)) < \mathrm{Aut}(K_{\mathrm{E}_{7,\mathrm{i}}})$$
$$2^{16} \rtimes (2 \times \mathrm{O}_8^+(2).2) < \mathrm{Aut}(K_{\mathrm{E}_{8,\mathrm{i}}})$$

$\mathrm{E}_{8,\omega}$ appears in **[22]** as the unique indecomposable unimodular 8-dimensional lattice over the Eisenstein integers.

3.3. The Conway group Co_1. The automorphism group of the real Leech lattice Λ_{24} is $\mathrm{Aut}(\Lambda_{24}) = 2.\mathrm{Co}_1$ where Co_1 is the finite simple Conway group (see ch.1 §1). Proceeding as above, define

$$\Lambda_{24,i} = \mathbb{Z}[i] \otimes_{\mathbb{Z}} \Lambda_{24} \subset \mathbb{C}^{24}$$

The complex automorphisms are then

$$\mathrm{Aut}_{\mathcal{G}}(\Lambda_{24,i}) = 4.\mathrm{Co}_1.$$

The associated 24-dimensional Kummer manifold admits a holomorphic action of

$$2^{48} \rtimes (2 \times \mathrm{Co}_1)$$

3.4. Summary of constructions. We collect in a table the main Kummer manifolds K_L constructed in this chapter together with the groups acting on them. The dimensions are over $\mathbb{C}$.

$G < \mathrm{Aut}(K_L)$	Lattice L	Dimension of K_L
$2^4 \rtimes A_4$ (symplectic)	D_4	2
$2^4 \rtimes A_4.2^2$	D_4	2
$2^{12} \rtimes (2 \times \mathrm{U}_4(2).2)$	$\mathrm{E}_{6,\mathrm{i}}$	6
$2^{12} \rtimes 3.\mathrm{U}_4(3).2$	K_{12}	6
$2^{14} \rtimes (2 \times \mathrm{S}_6(2))$	$\mathrm{E}_{7,\mathrm{i}}$	7
$2^{16} \rtimes (2 \times \mathrm{O}_8^+(2).2)$	$\mathrm{E}_{8,\mathrm{i}}$	8
$2^{24} \rtimes 3.\mathrm{Suz}$	Λ_{24}	12
$2^{48} \rtimes (2 \times \mathrm{Co}_1)$	$\Lambda_{24,i}$	24

CHAPTER 3

Generalised Kummer manifolds

Chapter 2 examined Kummer manifolds K and their automorphisms, where K is the resolution of the quotient of a complex torus by the usual involution -1 sending a point to its inverse. A *generalised Kummer manifold* is obtained by resolving the quotient of an n-dimensional complex torus by a more general group of biholomorphic transformations. In certain contexts one requires the resolution to be *minimal* [**56**], a term we will define shortly. This greatly restricts the possibilities. In mathematical physics, generalised Kummer manifolds provide fundamental examples of field theory models. Much of the literature is to be found in this area; a good general reference from this point of view is the book by Hübsch [**33**].

In this chapter we shall be constructing generalised Kummer manifolds though the resolutions will not be minimal. The automorphisms will be induced as for the classical Kummer manifolds (chapter 2). In §7 we show that the fundamental group of a class of these is trivial. Finally we summarise some results of S.S.Roan on general quotients of complex tori by finite groups and their minimal resolutions (§8). We start by giving explicit resolutions for certain quotient singularities.

1. Resolutions of cyclic quotient singularities

Let $\theta \in \mathrm{GL}(n, \mathbb{C})$ be the order d linear transformation $\mu \cdot \mathrm{Id}$, where $\mu = \exp(2\pi i/d)$ and Id is the identity matrix. Consider the isolated singularity $\mathbb{C}^n/\langle\theta\rangle$ at the origin. This is a generalisation of the Kummer singularity $\mathbb{C}^n/\langle\pm 1\rangle$ encountered in chapter 2, and can be resolved in the same way. Let $\sigma : \widehat{\mathbb{C}^n} \to \mathbb{C}^n$ be the equivariant blow-up of $\mathbb{C}^n$ at the origin. The quotient $\widehat{\mathbb{C}^n}/\langle\theta\rangle$ is now non-singular and

$$\rho : \widehat{\mathbb{C}^n}/\langle\theta\rangle \to \mathbb{C}^n/\langle\theta\rangle$$

is a resolution of $\mathbb{C}^n/\langle\theta\rangle$ (ρ naturally induced by σ). So one blow-up of the underlying complex space is enough to desingularise in this case.

In line with chapter 2 we view this resolution as a surgery process by attaching a disc bundle. Remove a $2n$-disc D around the origin $0 \in \mathbb{C}^n$. θ still acts on $\mathbb{C}^n - D$, and the quotient $\mathbb{C}^n - D/\langle\theta\rangle$ is non-singular with boundary $S^{2n-1}/\langle\theta\rangle$. Let $\mathcal{L} \to \mathbb{C}\mathrm{P}^{n-1}$ be the universal line bundle, and $\mathcal{L}^d \to \mathbb{C}\mathrm{P}^{n-1}$ the dth tensor of $\mathcal{L}$. The disc bundle $D(\mathcal{L}^d) \to \mathbb{C}\mathrm{P}^{n-1}$ has boundary $S^{2n-1}/\langle\theta\rangle$. The resolution above is then obtained by gluing $D(\mathcal{L}^d)$ to $\mathbb{C}^n - D/\langle\theta\rangle$. Informally the line bundle "looks like" $\mathbb{C}^n/\langle\theta\rangle$ around but away from the origin. This construction is of course also equivariant, and we shall give details of the action in a future construction (§3).

2. Minimal resolutions

Section 4 in ch.1 discussed quotient singularities and their resolutions by successive blow-ups, §1 above providing a simple example. Let M be a complex orbifold. A resolution $\rho : \widetilde{M} \to M$ is *minimal* (or *canonical*) if $\widetilde{M}$ has a trivial canonical bundle, $\mathcal{K}_{\widetilde{M}} = 0$. An equivalent condition is for the first Chern class to be zero, $c_1(M) = 0$. Recall from ch.1 §4 that all singularities can be resolved. However not all have a minimal resolution. Locally now, a necessary condition for a quotient singularity to admit a minimal resolution is that the *canonical sheaf* of the singular space be trivial. Equivalently the finite group must be in the special linear group:

$$\mathbb{C}^n/G \text{ where } G < \mathrm{SL}(n, \mathbb{C}), \text{ G finite.}$$

This condition is far from sufficient however and much work has recently gone in to determining which quotient singularities admit a minimal resolution. In dimension $n = 2$ all admit minimal resolutions. These are the *Kleinian singularities* corresponding to the finite subgroups of $\mathrm{SL}(2, \mathbb{C})$ (or $\mathrm{SU}(2)$), see ch.2 §2. The minimal resolutions have exceptional divisors whose intersection form is either A_n, D_n, E_6, E_7 or E_8. The programme of finding minimal resolutions for all finite subgroups of $\mathrm{SL}(3, \mathbb{C})$ was recently completed by Roan [**54**]; see also [**4**],[**55**]. For $n \geq 4$ minimal resolutions no longer necessarily exist. For example the Kummer singularity $\mathbb{C}^n/\langle\pm 1\rangle$ has no minimal resolution for $n \geq 4$. This can be seen using toric geometry, as explained later in this section.

2.1. A generalised McKay correspondence. In 1981 John McKay found a connection between the classical Kleinian singularities and the representation theory of the corresponding finite subgroup of $\mathrm{SL}(2,\mathbb{C})$ [**47**],[**46**]. This has become known as the *McKay correspondence.* More recent work has attempted to link the minimal resolutions in higher dimensions n with the representation theory of the finite group $G < \mathrm{SL}(n,\mathbb{C})$. Several conjectures have been put forward in this vein, all versions of a *generalised McKay correspondence*, see for example [**36**],[**43**],[**53**]. However the first basic question of charaterising the groups $G < \mathrm{SL}(n,\mathbb{C})$ admitting a minimal resolution remains unanswered.

2.2. Minimal toric resolutions. In §5.1 of chapter 1 we quoted some results on resolutions of general toric varieties. In particular every quotient singularity $\mathbb{C}^n/G$, G abelian, could be expressed and resolved using toric methods. In these cases the minimality of a resolution can be readily seen in the geometry of the fan.

Recall that a resolution is given by adding new lattice vectors to the singular cone and subdividing into simplices. Let $\sigma = [x_1,\dots,x_m]$ be a singular cone in $\mathbb{R}^n$ and $\widetilde{\sigma} \to \sigma$ a resolution obtained by adding the vectors $\{v_1,\dots,v_k\}$ (the set could be empty). Define a trace map tr by

$$tr:\mathbb{R}^n \to \mathbb{R}, \quad x = a_1x_1+\cdots+a_mx_m \mapsto \sum_{i=1}^{m} a_i$$

We can now state (see [**56**])

THEOREM 27. *The resolution $\widetilde{\sigma}$ is minimal if and only $tr(v_i) = 1$ for all $1 \le i \le k$.*

3. The Suzuki group Suz and manifold $X_{\Lambda_{24}}$

We now construct a generalised Kummer manifold from the Leech lattice. As explained in ch.1 §1.9 one can view the Leech lattice Λ_{24} as a complex lattice over the Eisenstein integers $\mathcal{E} = \mathbb{Z}[\omega]$ such that

$$\mathrm{Aut}_{\mathcal{E}}(\Lambda_{24}) = 6.\mathrm{Suz}$$

where Suz is the Suzuki sporadic simple group of order $2^{13}.3^7.5^2.7.11.13$ (see chapter 1 and the Atlas [**16**]). Complex lattices and their automorphisms are discussed in ch.1 §1.6. The choice of complex structure induces an embedding $\Lambda_{24} \subset \mathbb{C}^{12}$ and we can construct the associated complex torus

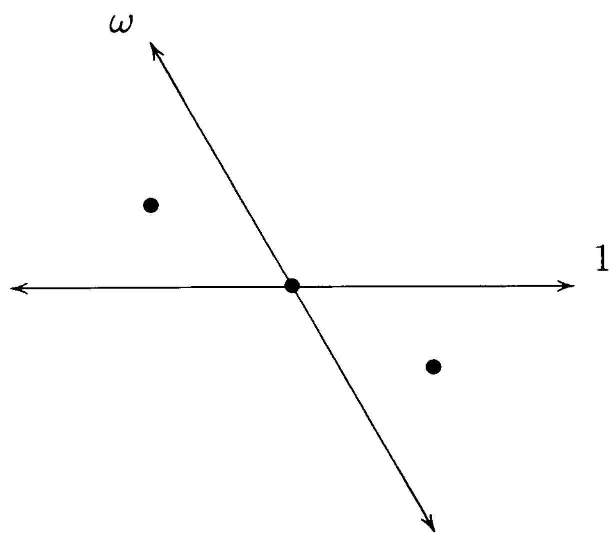
$$T = \mathbb{C}^{12}/\Lambda_{24}$$

The complex lattice Λ_{24} has a $\mathbb{Z}/3$-symmetry corresponding to multiplication by ω and a $\mathbb{Z}/2$-symmetry corresponding to the usual -1 involution of all integral lattices. We denote these groups of symmetries by $\langle\omega\rangle$ and $\langle\pm 1\rangle$ respectively. The product $\langle\pm 1, \omega\rangle$ of the two clearly form the normal $\mathbb{Z}/6$ in $\mathrm{Aut}_{\mathcal{E}}(\Lambda_{24})$ above.

As explained in ch.2 §1, $\mathrm{Aut}_{\mathcal{E}}(\Lambda_{24})$ is also the group of distance preserving biholomorphic homomorphisms of the torus $T = \mathbb{C}^{12}/\Lambda_{24}$. In this section we examine the orbifolds obtained as quotients of the torus by the cyclic $\langle\pm 1\rangle$ and $\langle\omega\rangle$ actions.

The case of the involution was extensively studied in chapter 2. The resulting orbifold has 2^{24} singular points of Kummer type and can be resolved by attaching a copy of the disc bundle of the squared universal bundle $\mathcal{L}^2$ over $\mathbb{CP}^{11}$ at each of these. The resolved orbifold is a 12-dimensional complex Kummer manifold $K_{\Lambda_{24}}$ admitting a holomorphic action of the group 3.Suz.

We approach the $\langle\omega\rangle$ case in a similar fashion. The root ω acts on $\mathbb{C}^{12}$ as $\omega\mathrm{Id}$ where Id is the 12×12 identity matrix. We now determine the fixed points on the torus, working in one of the twelve one-dimensional coordinate subspaces.



Let $x\omega + y \in \mathbb{C}$ be a coset representative of some point of the torus T. ω then acts by multiplication.

$$w \cdot (x\omega + y) = x\omega^2 + y\omega = x(-\omega - 1) + y\omega = (y - x)\omega - x.$$

The point is fixed in the torus if

$$(y - x)\omega - x = x\omega + y \mod \mathbb{Z}[\omega];$$

i.e. if

$$x = 1/3(-m - n) \text{ and } y = n + 2/3(-m - n).$$

Modulo $\mathbb{Z}$ there are only three solutions, namely

$$\{x=0,y=0\},\{x=1/3,y=2/3\} \text{ and } \{x=2/3,y=1/3\}.$$

So the fixed points on the torus in this coordinate space are the three marked by a dot on the above diagram, that is

$$\{0,\ -1/3+\omega/3,\ 1/3-\omega/3\}.$$

The torus T then clearly has 3^{12} fixed points under the $\langle\omega\rangle$ action, namely those with coset representatives $(s_1,\dots,s_{12})\in\mathbb{C}^{12}$ where each s_i is one of the three points determined above. The resulting orbifold can be resolved as follows. Each singular point is locally isomorphic to $\mathbb{C}^{12}/\langle\omega\rangle$. Removing a small neighbourhood of each singular point we are left with a 24-dimensional real manifold with boundary 3^{12} copies of $S^{23}/\langle\omega\rangle$. Let $\mathcal{L}\to\mathbb{CP}^{11}$ be the universal bundle over $\mathbb{CP}^{11}$. Take the third tensor $\mathcal{L}^3$ of the line bundle. Then the associated disc bundle $D(\mathcal{L}^3)$ has boundary $S^{23}/\langle\omega\rangle$ and we can glue a copy to each boundary component. The endproduct is a 12-dimensional complex manifold, which we shall denote X_ω.

This resolution allows for a nice description of the automorphisms of X_ω. The quotient $\mathrm{Aut}_{\mathcal{E}}(\Lambda_{24})/\langle\omega\rangle = 2.\mathrm{Suz}$ clearly acts faithfully on the truncated orbifold as it preserves distances. The symmetries extend down into the resolved points: let $A\in 6.\mathrm{Suz}$, then A acts on $\mathcal{L}^3$ as

$$(x,t_1x\otimes t_2x\otimes t_3x)\longmapsto (Ax,A(t_1x)\otimes A(t_2x)\otimes A(t_3x)).$$

This is a well defined holomorphic bundle automorphism. However $\omega A\in 6.\mathrm{Suz}$ and A now act in the same way, so we have a faithful action of 2.Suz, as required.

So far we have constructed two compact complex 12-dimensional manifolds $K_{\Lambda_{24}}$ and X_ω admitting holomorphic actions of 3.Suz and 2.Suz respectively. We now turn to the natural merger of the two, namely the quotient of the torus by the cyclic group $\langle\pm1,\omega\rangle$ of order 6.

The $\langle\pm1,\omega\rangle$ action fixes $2^{24}+3^{12}-1$ points on the torus and has three isomorphism classes of point stabilisers, namely $\langle\pm1\rangle$, $\langle\omega\rangle$ and $\langle\pm1,\omega\rangle$. The singular points on the quotient split into three categories:

1. $(2^{24}-1)/3$ singularities locally isomorphic to $\mathbb{C}^{12}/\langle\pm1\rangle$
2. $(3^{12}-1)/2$ singularities locally isomorphic to $\mathbb{C}^{12}/\langle\omega\rangle$

3. one singularity corresponding to the origin of type $\mathbb{C}^{12}/\langle\pm1,\omega\rangle$

Category 1 consists of Kummer type singularities, category 2 of the cyclic singularities discussed above, both for which appropriate resolutions have been given. The remaining singularity coming from the origin is resolved similarly: remove a neighbourhood of the singularity of the orbifold and attach the disc bundle $D(\mathcal{L}^6)$ of the sixth power of $\mathcal{L}$. Again the automorphisms descend. The subgroups $\langle\pm1\rangle$ and $\langle\omega\rangle$ commute and so preserve each others fixed point set. In particular the singularities of $T/\langle\pm1\rangle$ (resp. $T/\langle\omega\rangle$) are identified in 3's (resp. 2's) by the action of $\langle\omega\rangle$ (resp. $\langle\pm1\rangle$), except for the origin of course. This explains the numbers of type 1 and 2 singularities above.

So resolving the orbifold $T/\langle\pm1,\omega\rangle$ consists in attaching $(2^{24}-1)/3$ disc bundles $D(\mathcal{L}^2)$, $(3^{12}-1)/2$ bundles $D(\mathcal{L}^3)$ and one bundle $D(\mathcal{L}^6)$ - a total of $(2^{25}+3^{13}+1)/6$ bundles. The resulting 12-dimensional compact complex manifold $X_{\Lambda_{24}}$ admits a faithful holomorphic action of the Suzuki group $\mathrm{Suz} = \mathrm{Aut}_{\mathcal{E}}(\Lambda_{24})/\langle\pm1,\omega\rangle$.

Note that $X_{\Lambda_{24}}$ can also be described as the quotient of a manifold by a group action. Indeed if one attaches a disc bundle $D(\mathcal{L})$ at each fixed point of the torus to obtain a manifold $Y_{\Lambda_{24}}$, then $X_{\Lambda_{24}} = Y_{\Lambda_{24}}/\langle\pm1,\omega\rangle$. This facilitates the calculation of a number of topological invariants. Of course the surgery here is just blowing-up at smooth points - see ch.1 §4.

Unlike in the case of the classical Kummer manifolds in chapter 2, no translations of the torus are preserved after the quotient by $\langle\pm1,\omega\rangle$. Indeed those preserving the double points (fixed by $\langle\pm1\rangle$) do not preserve the points with stabiliser $\langle\omega\rangle$ and vice-versa.

Summarising we have constructed a compact complex manifold $X_{\Lambda_{24}}$ such that $\mathrm{Suz} < \mathrm{Aut}(X_{\Lambda_{24}})$.

4. The Conway group Co_1 and manifolds $X_{\Lambda_{24,\omega}}$ and $X_{\Lambda_{24,i}}$

Multiplication by ω in $\mathbb{Z}[\omega]$ induces a complex structure on $\Lambda_{24,\omega} = \mathbb{Z}[\omega] \otimes_{\mathbb{Z}} \Lambda_{24}$ (already discussed in ch.2 §3.1). Its complex automorphism group (those commuting with ω) is

$$\mathrm{Aut}_{\mathcal{E}}(\Lambda_{24,\omega}) = 6.\mathrm{Co}_1$$

where Co_1 is the finite simple Conway group (see ch.1 §3) and $\mathbb{Z}/6 = \langle \pm 1, \omega \rangle$ is the group encountered above when constructing $X_{\Lambda_{24}}$. We proceed here in a similar fashion.

Let T be the 24-dimensional compact complex torus $\mathbb{C}^{24}/\Lambda_{24,\omega}$. The $\langle \pm 1, \omega \rangle$-action on T has $2^{48} + 3^{24} - 1$ fixed points (points with nontrivial stabilisers) and the quotient $T/\langle \pm 1, \omega \rangle$ has 3 types of singularities:

1. $(2^{48} - 1)/3$ singularities locally isomorphic to $\mathbb{C}^{24}/\langle \pm 1 \rangle$
2. $(3^{24} - 1)/2$ singularities locally isomorphic to $\mathbb{C}^{24}/\langle \omega \rangle$
3. one singularity corresponding to the origin of type $\mathbb{C}^{24}/\langle \pm 1, \omega \rangle$

The quotient $T/\langle \pm 1, \omega \rangle$ can be resolved as in §3, this time with powers of the universal line bundle $\mathcal{L} \to \mathbb{CP}^{23}$. In total $(2^{49} + 3^{25} + 1)/6$ bundles are attached. The resulting compact complex manifold is denoted $X_{\Lambda_{24,\omega}}$ and admits a natural holomorphic action of the Conway group Co_1. But as in the case of $X_{\Lambda_{24}}$ none of the translations of the torus are preserved under the identification of inverse points of the torus:

$$\mathrm{Co}_1 < \mathrm{Aut}(X_{\Lambda_{24,\omega}})$$

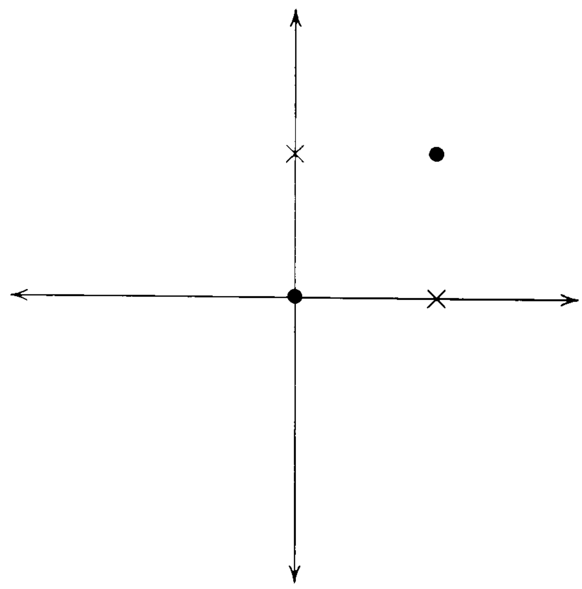
The construction of $X_{\Lambda_{24,i}}$ is similar. Take $T = \mathbb{C}^{24}/\Lambda_{24,i}$ where $\Lambda_{24,i} = \mathbb{Z}[i] \otimes_{\mathbb{Z}} \Lambda_{24}$. As explained in ch.2 §3 the complex automorphism group of the lattice is

$$\mathrm{Aut}_{\mathbb{C}}(\Lambda_{24,i}) = 4.\mathrm{Co}_1$$

where $\mathbb{Z}/4 = \langle \pm i \rangle$. 2^{48} points of the torus have nontrivial stabiliser under the $\langle \pm i \rangle$-action. The singularities of the quotient $T/\langle \pm i \rangle$ are of two types

1. 2^{24} singularities locally isomorphic to $\mathbb{C}^{24}/\langle \pm i \rangle$
2. $(2^{48} - 2^{24})/2 = 2^{23}(2^{24} - 1)$ singularities locally isomorphic to $\mathbb{C}^{24}/\langle \pm 1 \rangle$

We display a one-dimensional coordinate space $\mathbb{Z}[i]$ in the figure below.



The singularities of type 1 are all those made up of coordinates marked by a dot (including the origin), those of type 2 by coordinates taken from the dots and crosses but containing at least one cross coordinate. $\langle \pm i \rangle$ preserves the fixed point set of $\langle \pm 1 \rangle$ and identifies the two crosses in the above diagram. Hence there are half as many singularities of type 2 than fixed points of the torus with stabiliser $\langle \pm 1 \rangle$. Resolve using powers of the universal line bundle $\mathcal{L} \to \mathbb{C}\mathrm{P}^{23}$ (this time to the powers 2 and 4). The resulting 24-dimensional compact complex manifold denoted $X_{\Lambda_{24,i}}$ again admits a holomorphic action of the Conway group Co_1.

Unlike the quotients by $\langle \pm 1, \omega \rangle$ some translations are preserved by the construction: a subgroup of those acting on the Kummer manifolds in chapter 2. On the diagram above the diagonal translation by $1/2(1+i)$ preserves the two sets of fixed points and induces a map on $X_{\Lambda_{24,i}}$. These form a group $(\mathbb{Z}/2)^{24}$:

$$2^{24} \rtimes \mathrm{Co}_1 < \mathrm{Aut}(X_{\Lambda_{24,i}})$$

5. Miscellaneous manifolds X_L

Changing dimensions apart, sections 3 and 4 covered all types of quotients and singularities arising in such constructions. This will follow from Roan's results which we describe in §8. In this section we briefly describe the manifolds obtained for other lattices. The methods are identical to those of sections 3 and 4 so we omit most details. The only difference is in the dimensions of the tori and hence the number of singularities of different type. We start with tensors of the root lattices $\mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8$ as in the constructions of Kummer manifolds in ch.2 §3.2 and obtain a pair of manifolds for each group. This section is included more for sake of completeness than any obvious interest in the groups obtained.

5.1. 12-dimensional $X_{\mathrm{E}_{6,\omega}}$. Recall from ch.2 §3.2 that $\mathrm{E}_{6,\omega} = \mathcal{E} \otimes_{\mathbb{Z}} \mathrm{E}_6$ and $\mathrm{Aut}_{\mathcal{E}}(\mathrm{E}_{6,\omega}) = 6.\mathrm{U}_4(2).2$, where $6 = \langle \pm 1, \omega \rangle$.

Let $T_{\mathrm{E}_{6,\omega}} = \mathbb{C}^6/\mathrm{E}_{6,\omega}$, and consider the space $T_{\mathrm{E}_{6,\omega}}/\langle \pm 1, \omega \rangle$. The singularities of the quotient split as

1. $(2^{12}-1)/3$ of type $\mathbb{C}^6/\langle \pm 1 \rangle$
2. $(3^6-1)/2$ of type $\mathbb{C}^6/\langle \omega \rangle$
3. one singularity of type $\mathbb{C}^6/\langle \pm 1, \omega \rangle$

Resolve by attaching disc bundles $D(\mathcal{L}^2)$, $D(\mathcal{L}^3)$ and $D(\mathcal{L}^6)$ over $\mathbb{C}\mathrm{P}^5$. The resolution

$$X_{\mathrm{E}_{6,\omega}} := \widetilde{T_{\mathrm{E}_{6,\omega}}/\langle \pm 1, \omega \rangle}$$

inherits a holomorphic action of $\mathrm{U}_4(2).2$.

5.2. 12-dimensional $X_{\mathrm{E}_{6,\mathrm{i}}}$. Recall that $\mathrm{E}_{6,\mathrm{i}} = \mathcal{G} \otimes_{\mathbb{Z}} \mathrm{E}_6$ and that $\mathrm{Aut}_{\mathcal{G}}(\mathrm{E}_{6,\mathrm{i}}) = 4.\mathrm{U}_4(2).2$, where $4 = \langle \pm i \rangle$.

Let $T_{\mathrm{E}_{6,\mathrm{i}}} = \mathbb{C}^6/\mathrm{E}_{6,\mathrm{i}}$. The quotient $T_{\mathrm{E}_{6,\mathrm{i}}}/\langle \pm i \rangle$ has singularities which split into

1. 2^6 of type $\mathbb{C}^6/\langle\langle \pm i \rangle\rangle$
2. $(2^{12}-2^6)/2 = 2^5(2^6-1)$ of type $\mathbb{C}^6/\langle \pm 1 \rangle$

Resolve by attaching disc bundles $D(\mathcal{L}^4)$ and $D(\mathcal{L}^2)$ over $\mathbb{C}\mathrm{P}^5$. The resulting

$$X_{\mathrm{E}_{6,\mathrm{i}}} := \widetilde{T_{\mathrm{E}_{6,\mathrm{i}}}/\langle \pm i \rangle}$$

admits a holomorphic action of $\mathrm{U}_4(2).2$.

5.3. 14-dimensional $X_{\mathrm{E}_{7,\omega}}$. Recall that $\mathrm{E}_{7,\omega} = \mathcal{E} \otimes_{\mathbb{Z}} \mathrm{E}_7$ and $\mathrm{Aut}_{\mathcal{E}}(\mathrm{E}_{7,\omega}) = 6.\mathrm{S}_6(2)$ with $6 = \langle \pm 1, \omega \rangle$. Let $T_{\mathrm{E}_{7,\omega}} = \mathbb{C}^7/\mathrm{E}_{7,\omega}$. Then $T_{\mathrm{E}_{7,\omega}}/\langle \pm 1, \omega \rangle$ has singularities:

1. $(2^{14}-1)/3$ of type $\mathbb{C}^7/\langle \pm 1 \rangle$
2. $(3^7-1)/2$ of type $\mathbb{C}^7/\langle \omega \rangle$
3. one singularity of type $\mathbb{C}^7/\langle \pm 1, \omega \rangle$

Resolve by attaching the disc bundles over $\mathbb{C}\mathrm{P}^6$. The resolution

$$X_{\mathrm{E}_{7,\omega}} := \widetilde{T_{\mathrm{E}_{7,\omega}}/\langle \pm 1, \omega \rangle}$$

admits a holomorphic action of $\mathrm{S}_6(2)$.

First a small well-known lemma we will be needing.

LEMMA 28. *Let Z be a simply-connected manifold with a free action of a finite group G. Then $\pi_1(Z/G) \simeq G$.*

Let $T = \mathbb{C}^n/L$ be a complex n-torus ($n \geq 2$) with $N = \langle \pm 1, \omega \rangle < \mathrm{Aut}(L)$, where ± 1 and ω act by complex multiplication on the universal covering $\mathbb{C}^n$. For convenience of notation we denote the involution (-1) by g. N is always normal in the group of automorphism $\mathrm{Aut}(T)$. The orbifold T/N can be resolved as in the construction of §3. Let X denote the resolved orbifold $\widetilde{T/N}$.

THEOREM 29. *$X = \widetilde{T/N}$ is simply connected.*

PROOF. We first fix some notation for the purposes of this proof. All dimensions will refer to real dimensions. T is a compact torus of dimension $2n$ (the complex structure is irrelevant to the problem). Let $\overline{Y}$ denote the manifold with boundary $2^{2n} + 3^n - 1$ spheres S^{2n-1} obtained by removing a small $2n$-disc around each of the fixed points of the action of N on T. Y is the manifold $\overline{Y}/N$ with boundary $(2^{2n} - 1)/3$ copies of $\mathbb{RP}^{2n-1}$, $(3^n - 1)/2$ copies of $S^{2n-1}/\langle\omega\rangle$ and one S^{2n-1}/N. For the three types of boundary component B there is a natural map $f : B \to \mathbb{CP}^{n-1}$. Let M be the mapping cylinder associated to f. Then $\partial M = B$ and the manifold X is obtained by attaching the appropriate mapping cylinders to all boundary components. We index the boundary spheres of $\overline{Y}$ as follows: S_0, S_i and S^j are the spheres around the origin, the fixed points of g and the fixed points of ω respectively (the origin excluded from the last two sets). Similarly denote the corresponding boundary components of Y by P_0, P_i and P^j. These are isomorphic to S^{2n-1}/N, $\mathbb{RP}^{2n-1}$ and $S^{2n-1}/\langle\omega\rangle$ respectively. Let M_0, M_i and M^j denote the corresponding mapping cylinders.

Since the discs removed from the torus have dimension $2n$ and $n \geq 2$, the fundamental group is not affected and $\pi_1(\overline{Y}) = \pi_1(T)$ is a free abelian group of rank $2n$. Choose some x_0 on S_0 as base point of $\overline{Y}$. Generators for $\pi_1(\overline{Y})$ can be taken to be the paths $\overline{y}_1, \ldots, \overline{y}_{2n}$ where $\overline{y}_i$ is a composition of the paths

$$\overline{y}_i = ab_ic_i(gb_i)^{-1}.$$

above. The resulting manifold is denoted $X_{K_{12}}$ and admits a holomorphic action of $U_4(3).2$.

6. Summary of constructions

We give a table of all the manifolds constructed in this chapter together with the groups acting on these. In previous sections we determined the translations preserved according to the cyclic group factored out: none remain for $\langle \pm 1, \omega \rangle$ while some of order 2 remain for $\langle \pm i \rangle$. The dimensions are over $\mathbb{C}$.

$G < \mathrm{Aut}(X_L)$	Lattice L	Dimension of X_L
$U_4(2).2$	$E_{6,\omega}$	6
$2^6 \rtimes U_4(2).2$	$E_{6,i}$	6
$S_6(2)$	$E_{7,\omega}$	7
$2^7 \rtimes S_6(2)$	$E_{7,i}$	7
$O_8^+(2).2$	$E_{8,\omega}$	8
$2^8 \rtimes O_8^+(2).2$	$E_{8,i}$	8
$U_4(3).2$	K_{12}	6
Suz	Λ_{24}	12
Co_1	$\Lambda_{24,\omega}$	24
$2^{24} \rtimes Co_1$	$\Lambda_{24,i}$	24

As explained at the end of §3 for $X_{\Lambda_{24}}$, all these complex manifolds X can be viewed as a quotient of another manifold by the cyclic group $N = \langle \pm i \rangle$ or $\langle \pm 1, \omega \rangle$ (as indeed can the Kummer manifolds of chapter 2 by $N = \langle \pm 1 \rangle$). Blow-up the torus at all the (smooth) fixed points of N to obtain a complex manifold Y. Blowing-up is equivariant and N still acts on Y. The quotient Y/N is X. For details on blowing-up see §1 and ch.1 §4.

7. The fundamental groups

In [**59**] Spanier shows that all Kummer manifolds are simply-connected. Using a similar method we show here that the same is true for the generalised Kummer manifolds constructed in this chapter. We go through the argument for manifolds whose construction resembles that of the manifold $X_{\Lambda_{24}}$ and the same argument can be applied to the other generalised Kummer manifolds.

First a small well-known lemma we will be needing.

LEMMA 28. *Let Z be a simply-connected manifold with a free action of a finite group G. Then $\pi_1(Z/G) \simeq G$.*

Let $T = \mathbb{C}^n/L$ be a complex n-torus ($n \geq 2$) with $N = \langle \pm 1, \omega \rangle < \mathrm{Aut}(L)$, where ± 1 and ω act by complex multiplication on the universal covering $\mathbb{C}^n$. For convenience of notation we denote the involution (-1) by g. N is always normal in the group of automorphism $\mathrm{Aut}(T)$. The orbifold T/N can be resolved as in the construction of §3. Let X denote the resolved orbifold $\widetilde{T/N}$.

THEOREM 29. $X = \widetilde{T/N}$ *is simply connected.*

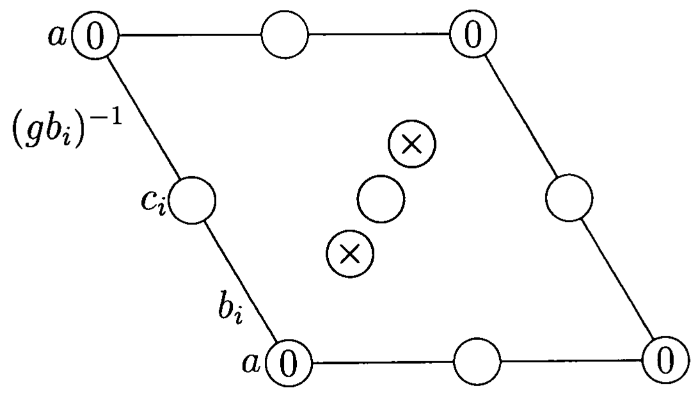
PROOF. We first fix some notation for the purposes of this proof. All dimensions will refer to real dimensions. T is a compact torus of dimension $2n$ (the complex structure is irrelevant to the problem). Let $\overline{Y}$ denote the manifold with boundary $2^{2n} + 3^n - 1$ spheres S^{2n-1} obtained by removing a small $2n$-disc around each of the fixed points of the action of N on T. Y is the manifold $\overline{Y}/N$ with boundary $(2^{2n} - 1)/3$ copies of $\mathbb{RP}^{2n-1}$, $(3^n - 1)/2$ copies of $S^{2n-1}/\langle\omega\rangle$ and one S^{2n-1}/N. For the three types of boundary component B there is a natural map $f : B \to \mathbb{CP}^{n-1}$. Let M be the mapping cylinder associated to f. Then $\partial M = B$ and the manifold X is obtained by attaching the appropriate mapping cylinders to all boundary components. We index the boundary spheres of $\overline{Y}$ as follows: S_0, S_i and S^j are the spheres around the origin, the fixed points of g and the fixed points of ω respectively (the origin excluded from the last two sets). Similarly denote the corresponding boundary components of Y by P_0, P_i and P^j. These are isomorphic to S^{2n-1}/N, $\mathbb{RP}^{2n-1}$ and $S^{2n-1}/\langle\omega\rangle$ respectively. Let M_0, M_i and M^j denote the corresponding mapping cylinders.

Since the discs removed from the torus have dimension $2n$ and $n \geq 2$, the fundamental group is not affected and $\pi_1(\overline{Y}) = \pi_1(T)$ is a free abelian group of rank $2n$. Choose some x_0 on S_0 as base point of $\overline{Y}$. Generators for $\pi_1(\overline{Y})$ can be taken to be the paths $\overline{y}_1, \dots, \overline{y}_{2n}$ where $\overline{y}_i$ is a composition of the paths

$$\overline{y}_i = ab_ic_i(gb_i)^{-1}.$$

The path a is on S_0 from x_0 to gx_0. The path b_i goes from gx_0 to a point $x_i \in S_i$. The path c_i on S_i goes from x_i to gx_i.

The diagram below is a one dimensional coordinate slice of a fundamental domain of the torus. The spheres bound the removed discs and surround the fixed points. The sphere marked 0 is S_0, those marked by a cross are the S^j's and the others the S_i's.



The quotient map $f : \overline{Y} \to Y$ is a 6-covering of Y with covering transformation $g\omega$. Let Q be a based set of order 6 and consider the exact sequence of the fibration $Q \to \overline{Y} \to Y$:

$$\cdots \to \pi_1(Q) \to \pi_1(\overline{Y}) \to \pi_1(Y) \to \pi_0(Q) \to \pi_0(\overline{Y}) \to \pi_0(Y)$$

Now $\pi_1(Q) = \pi_0(\overline{Y})$ are both trivial, so

$$1 \to \pi_1(\overline{Y}) \to \pi_1(Y) \to \mathbb{Z}/6 \to 1$$

And the fundamental group of Y is a split extension of $\pi_1(\overline{Y})$ by a cyclic group $C \simeq \mathbb{Z}/6$:

$$\pi_1(Y) = \pi_1(\overline{Y}) \rtimes C$$

where the action of the cyclic group $C = \langle g, \omega \rangle$ on $\pi_1(\overline{Y})$ is

$$g : \overline{y}_i \mapsto (\overline{y}_i)^{-1}$$
$$\omega : \overline{y}_{2i-1} \mapsto \overline{y}_{2i}, \quad \overline{y}_{2i} \mapsto (\overline{y}_{2i})^{-1}(\overline{y}_{2i-1})^{-1}, \quad (\overline{y}_{2i})^{-1}(\overline{y}_{2i-1})^{-1} \mapsto \overline{y}_{2i-1}$$

for $i \geq 1$.

By lemma 28 $\pi_1(S^{2n-1}/N)$ is cyclic of order 6; let $z, v \in \pi_1(Y)$ be the images of generators of order 2 and 3 respectively under the injection $\pi_1(P_0) \to \pi_1(Y)$ induced by $P_0 \hookrightarrow Y$. Let $y_1, \ldots, y_{2n}$ be the images of $\overline{y}_1, \ldots, \overline{y}_{2n}$ under f. $\pi_1(Y)$ is generated by $y_1, \ldots, y_n$, z and v. The generators z, v act by conjugation on the y_i as g, ω did on the $\overline{y}_i$ above.

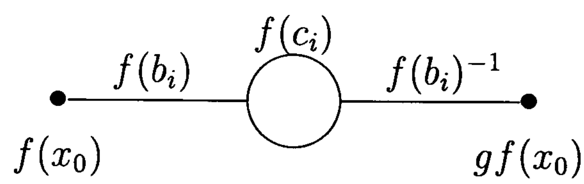
Now the path a represents z, that is $z = f(a)$.

$$\begin{aligned} y_i &= f(\overline{y}_i) \\ &= f(a)f(b_i)f(c_i)f[(gb_i)^{-1}] \\ &= zf(b_i)f(c_i)f(b_i)^{-1} \end{aligned}$$

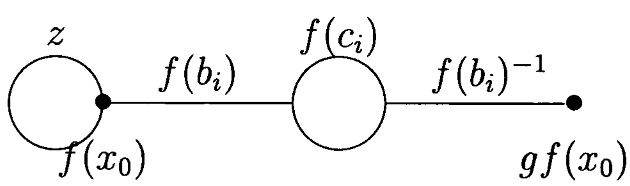
So since z is of order 2,

$$zy_i = f(b_i)f(c_i)f(b_i)^{-1}$$

But this is a closed path in Y with base point $f(x_0)$ identified with $gf(x_0)$ in Y:



In other words y_i is the composition of two closed paths z and zy_i with base point $f(x_0)$:



$\pi_1(Y)$ is generated by z, v and $zy_1, \ldots, zy_{2n}$. But zy_i is just the generator $f(c_i)$ of $\pi_1(P_i) = \pi_1(\mathbb{R}\mathrm{P}^{2n-1})$ moved to the base point $f(x_0)$. Hence $\pi_1(Y)$ is generated by the nontrivial elements of $\pi_1(P_0)$ and $\pi_1(P_i)$. In effect the fundamental group coming from the torus has disappeared.

Let (M, P) be one of the pairs (M_0, P_0), (M_i, P_i) or (M^j, P^j). We have the associated exact sequence:

$$\cdots \to \pi_2(M) \to \pi_2(M, P) \to \pi_1(P) \to \pi_1(M) \to \pi_1(M, P) \to 0$$

But $\pi_1(M) = \pi_1(\mathbb{C}\mathrm{P}^{n-1}) = 0$ since $n \geq 2$. So $\pi_2(M, P) \to \pi_1(P)$ is onto (and $\pi_1(M, P) = 0$). By attaching the mapping cylinders we therefore kill off all generators of $\pi_1(Y)$. The procedure cannot add to the fundamental group as $\pi_1(M, P) = 0$, so X is simply-connected. □

The proof goes through for the case $N = \langle \pm i \rangle$, the only difference being that all the fixed points are now fixed by $\langle \pm 1 \rangle$ and the relations for $\pi_1(\overline{Y})$ are slightly different.

COROLLARY 30. *All generalised Kummer manifolds of the table in §6 are simply-connected.*

PROOF. All are of the form $\widetilde{T/N}$ for $N = \langle \pm i \rangle$ or $\langle \pm 1, \omega \rangle$. □

8. Quotients of tori and S.S.Roan's results

In sections 3 and 4 we took quotients of complex tori by the cyclic groups $\langle \pm 1, \omega \rangle$ and $\langle \pm 1, \pm i \rangle$. This raises the question of which other cyclic groups arise as automorphisms of tori. The results we present here answer this question for cyclic groups $N = \langle \exp(2\pi i/d) \rangle$ acting on the universal cover $\mathbb{C}^n$ of the torus as diagonal matrices $\exp(2\pi i/d) \cdot \mathrm{Id}$, where Id is the identity matrix. The examples mentioned above are of this type. The second of Roan's theorems determines which of the T/N admit minimal resolutions (N as above). The material in this section can be found in Roan's paper of 1989 [**56**].

To state the result more conveniently, view $T = \mathbb{C}^n/L$ as a Lie group. Let $\Theta \in \mathrm{Hom}(T)$ be a Lie group homomorphism. The cyclic groups mentioned above are then those with Lie derivative at the origin having equal eigenvalues, that is $(d\Theta)_0 = \mu \cdot \mathrm{Id}$ where $\mu = \exp(2\pi i/d)$. We denote the fixed point set of Θ by T^Θ.

THEOREM 31 (S.S.Roan). *Let T be a complex torus and $\Theta \in \mathrm{Hom}(T)$ an order d homomorphism with $(d\Theta)_0 = \mu \cdot \mathrm{Id}$ and $\mu = \exp(2\pi i/d)$. Then $d = 2, 3, 4$ or 6 and*

$$d = 2, 3 \text{ or } 4 \quad \Leftrightarrow \quad |T^\Theta| \geq 2$$

$$d = 6 \quad \Leftrightarrow \quad |T^\Theta| = 1$$

So the cyclic groups acting on a torus with unique eigenvalue can have order at most 6. All such groups appeared as automorphisms of complex tori in our constructions of generalised Kummer manifolds in sections 3 and 4. Quotients of the tori by these cyclic groups yield isolated singularities of type $\mathbb{C}^n/\langle \pm 1 \rangle$ (Kummer singularity), $\mathbb{C}^n/\langle \omega \rangle$, $\mathbb{C}^n/\langle \pm i \rangle$ and $\mathbb{C}^n/\langle \pm 1, \omega \rangle$. By the theorem above these are all such singularities appearing in quotients of tori. A singularity of type $\mathbb{C}^n/\langle \pm 1, \omega \rangle$ will always be unique (and at the origin) as in the construction of $X_{\Lambda_{24}}$.

To prove theorem 31 Roan constructs a homomorphism from the torus to an elliptic curve. He then uses the fact that elliptic curves only have automorphisms of order $2, 3, 4$ and 6 and that those of order 6 have only the origin as fixed point.

THEOREM 32 (S.S.Roan). *Let* $T = \mathbb{C}^n/L$ *be a complex torus and* $\Theta \in \mathrm{Hom}(T)$ *an order* d *homomorphism with* $(d\Theta)_0 = \mu \cdot \mathrm{Id}$ *and* $\mu = \exp(2\pi i/d)$. *If there is a minimal resolution of* $T/\langle\Theta\rangle$ *then either* $n = d = 2$ *or* $n = d = 3$.

In other words the only quotients of complex tori by the cyclic groups of theorem 31 admitting minimal resolutions are 2 and 3 dimensional tori having singularities $\mathbb{C}^2/\langle \pm 1\rangle$ and $\mathbb{C}^2/\langle\omega\rangle$ respectively. The resolutions of these are the classical Kummer surfaces (ch.2 §2) and Calabi-Yau 3-folds [**44**]. In particular none of the quotients of tori (of dimension $n \geq 4$) of sections 3,4 and 5 have minimal desingularisations.

Of course Roan's results do not rule out finding minimal resolutions of quotients by cyclic groups $\langle\Theta\rangle$ where $(d\Theta)_0$ has different eigenvalues. Such an example is given in Roan's paper [**56**].

CHAPTER 4

The algebraic category

Here we address the question of whether the manifolds discussed up to now are complex algebraic varieties. Indeed much of the literature deals with them as algebraic rather than analytic objects. However as we shall see the interesting cases from our point of view are not ruled out by this restriction.

A complex structure on a lattice L turns out to be sufficient for the corresponding torus $T = \mathbb{C}^n/L$ to be algebraic (i.e. an abelian variety). Then by general results all corresponding g.K.m.'s will also be algebraic. Making use of the GAGA principles (Serre [**58**]) results in the previous chapters will then apply to the algebraic category. For this we shall be making use of the category of complex analytic spaces which includes singular spaces as well as the usual complex manifolds.

We end the chapter with a discussion of the canonical bundles of the manifolds and how these affect their automorphism groups. Let us start by introducing abelian varieties along with the machinery and results needed in this chapter. We follow the exposition of Lange and Birkenhake [**40**], an excellent reference for this topic.

1. Abelian and Kummer varieties

Notation. In this chapter only we use L to denote a line bundle and Λ a $2n$ dimensional lattice. This is to be consistent with references we make. V denotes a complex vector space of dimension n.

An *abelian variety* is a complex torus $T = V/\Lambda$ of complex dimension n together with a (holomorphic) positive definite line bundle $L \to T$, also called a polarisation. Such a line bundle is ample, i.e. L^m defines an analytic embedding φ_{L^m} of T into projective space for some $m \geq 1$, giving T the structure of an algebraic variety by Chow's theorem (see [**28**] and §3).

The *singular Kummer variety* associated to T is the quotient

$$K_\Lambda = T/\langle \pm 1 \rangle$$

where $\langle \pm 1 \rangle$ is the $\mathbb{Z}/2$-action on T, sending x to its inverse in the torus group. K_Λ is also an algebraic variety of dimension n, but with 2^{2n} singular points corresponding to the points of order 2 on T under the natural map

$$p : T \to K_\Lambda$$

Generalised Kummer varieties are defined in the obvious way as algebraic versions of those in chapter 3. If L is a particular kind of polarisation, namely an irreducible principle polarisation, then the embedding φ_{L^2} corresponding to the tensor of L with itself induces an embedding ψ of K_Λ into $\mathbb{CP}^{2^n-1}$. Indeed the following diagram commutes

$$\begin{array}{ccc} T & \xrightarrow{\varphi_{L^2}} & \mathbb{CP}^{2^n-1} \\ {\scriptstyle p}\downarrow & & \uparrow{\scriptstyle \psi} \\ K_\Lambda & = & K_\Lambda \end{array}$$

Before proceeding any further we must examine in more detail this idea of a polarisation as it will be essential in what follows.

First note that strictly speaking the map induced by the line bundle $L \to T$ above only depends on the first Chern class of the bundle. Also the first Chern class has a nice interpretation as an alternating form. Indeed there is a canonical isomorphism

$$H^2(T, \mathbb{Z}) \xrightarrow{\simeq} \mathrm{Alt}^2(\Lambda, \mathbb{Z}) := \bigwedge^2 \mathrm{Hom}(\Lambda, \mathbb{Z})$$

between the second cohomology group of the torus with integer coefficients and the alternating integer-valued 2-forms on the lattice Λ. By linear expansion we can also view the Chern class as an alternating form $E : V \times V \to \mathbb{R}$. We can give a precise condition for such an alternating form E to be the Chern class of some holomorphic line bundle. Indeed the following are equivalent

(1) there is a holomorphic line bundle such that $c_1(L) = E$ under the above isomorphism

(2) $E(\Lambda, \Lambda) \subseteq \mathbb{Z}$ and $E(iv, iw) = E(v, w)$ for all $v, w \in V$

Finally there is a $1 - 1$ correspondence between the alternating forms E on V satisfying the second part of condition (2) and the hermitian forms H on V. This correspondence is given by

$$H(v,w) = E(iv,w) + iE(v,w)$$

Denote by $NS(T)$ the Neron-Severi group of T, defined as the image of the first Chern class map

$$H^1(T, \mathcal{O}_T^*) \xrightarrow{c_1} H^2(T, \mathbb{Z})$$

where $\mathcal{O}_T^*$ is the sheaf of non-zero holomorphic functions on T. Recall that $H^1(X, \mathcal{O}_X^*)$ is isomorphic to the group of line bundles on a complex manifold X ([**28**], p.133). So we have

PROPOSITION 33. *We can view $NS(T)$ as either*
(A) the group of hermitian forms H on V satisfying $\mathrm{Im}H(\Lambda, \Lambda) \subseteq \mathbb{Z}$ *or*
(B) the group of alternating forms E on V satisfying both

$$E(\Lambda, \Lambda) \subseteq \mathbb{Z} \textit{ and } E(iv, iw) = E(v,w) \textit{ for all } v, w \in V.$$

Returning now to the idea of a polarisation on the torus T, let us recall how a holomorphic line bundle induces a map into projective space (this holds for any manifold X). Simply pick a basis $v_0 \ldots, v_n$ of $H^0(L) = H^0(X, \mathcal{O}(L))$, the space of holomorphic sections of L, and define

$$\varphi_L : X \to \mathbb{CP}^n$$

by

$$\varphi_L(x) = (v_0(x) : \cdots : v_n(x))$$

Of course this map is only defined if for any point $x \in X$ there is a section s of L not vanishing at x. This is equivalent to the linear system $|L| = \{(s) | s \in H^0(L)\}$ having no base points. We now ask the question of whether this map is an embedding of the manifold into projective space.

Returning to the tori, recall that a line bundle L on a torus T is said to be *ample* if φ_{L^m} is an embedding for some $m \geq 1$. As mentioned earlier a polarization of a torus T consists of a *positive definite line bundle* L on T, that is a line bundle whose first chern class $c_1(L) = H$ is a positive definite hermitian form.

PROPOSITION 34. *L is ample $\Longleftrightarrow$ L is positive definite*

Let us now examine the map φ_L in more detail. Choosing an appropriate basis for the lattice Λ the symplectic form E is given by a matrix

$$\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$$

where $D = \operatorname{diag}(d_1, \ldots, d_n)$ is a diagonal $n \times n$ matrix with integer entries. Such a basis is called a *symplectic basis* of Λ while $(d_1, \ldots, d_n)$ is called the *type* of the polarization. A *principal polarization* of some torus T is a polarization of type $(1, \ldots, 1)$.

Denote by $h^0(L)$ the dimension of $H^0(L)$, and by $Pf(E)$ the Pfaffian of the alternating form E. Recall that $Pf(E) = \det D = d_1 \cdots d_n$.

PROPOSITION 35. *For any positive definite line bundle L, $h^0(L) = Pf(E)$*

Hence given a positive definite line bundle L on some torus T and using the notation introduced above, φ_L is a map

$$\varphi_L : T \to \mathbb{C}\mathrm{P}^{Pf(E)-1}$$

into projective space of dimension $(d_1 \cdots d_n) - 1$. It remains to determine when φ_L is an embedding. The main result is that of Lefschetz:

THEOREM 36. *Let L be a polarisation of type $(d_1 \ldots d_n)$ with $d_1 \geq 3$. Then φ_L is an embedding.*

Noting that $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$ where addition is in the group of alternating forms this implies :

COROLLARY 37. *For any positive definite line bundle L on T, φ_{L^m} is an embedding for $m \geq 3$.*

1.1. Riemann Conditions. It remains to determine *when* a given complex torus T admits a positive definite line bundle, leading to an embedding of T and the associated Kummer manifold as algebraic varieties. The Riemann conditions are necessary and sufficient conditions for this to be the case.

First we define the period matrix Π of a complex torus $T = V/\Lambda$ of dimension n. Pick a basis $B = \{v_1, \ldots, v_n\}$ of V and a basis $\lambda_1, \ldots, \lambda_{2n}$ of Λ. Expressing the basis of Λ in terms of B, say $\lambda_j = (\lambda_{1,j}, \dots, \lambda_{n,j})$, and arranging them in a matrix we define the *period matrix* Π of T as

$$\Pi = \begin{pmatrix} \lambda_{1,1} & \cdots & \lambda_{1,2n} \\ \vdots & & \vdots \\ \lambda_{n,1} & \cdots & \lambda_{n,2n} \end{pmatrix}$$

The real lattice $\Lambda \subset \mathbb{R}^{2n}$ embeds in $V = \mathbb{C}^n$ via the usual identification $\mathbb{R}^2 \equiv \mathbb{C}$. Note that the period matrix depends on the basis picked for V and that only as a pair do they determine the lattice in $\mathbb{C}^n$. There are several formulations of the Riemann conditions. The most common is

THEOREM 38 (Riemann conditions I). *A complex torus T with period matrix Π is an abelian variety if and only if there is an integral skew-symmetric $n \times n$ matrix Q such that*

$$\text{(i)} \quad \Pi Q^{-1} \Pi^T = 0$$
$$\text{(ii)} \quad i\Pi Q^{-1} \overline{\Pi}^T \text{ is positive definite}$$

Here $\overline{\Pi}$ denotes the complex conjugate of Π.

For our computations the following restatement of the conditions will also prove helpful. Griffiths and Harris [**28**] give a detailed account of the various formulations and their equivalence.

THEOREM 39 (Riemann conditions II). *A complex torus $T = V/\Lambda$ is an abelian variety if and only if for some choice of basis for V and Λ the period matrix takes the form*

$$\Pi = \begin{pmatrix} \Delta & , & Z \end{pmatrix}$$

where

$$\Delta = \begin{pmatrix} \delta_1 & & \\ & \ddots & \\ & & \delta_n \end{pmatrix}$$

is a $n \times n$ diagonal matrix with integer entries and Z is a $n \times n$ symmetric matrix with imaginary part $\mathrm{Im}Z$ *positive definite.*

In fact this immediately produces the polarisation of T, since

$$E = \begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix}$$

is the matrix of the alternating form corresponding to the first Chern class. In particular $(\delta_1, \ldots, \delta_n)$ is the type of the polarisation.

Note that deducing the second form of the theorem from the first simply involves picking a symplectic basis of Λ and modifying the two conditions. In particular the skew-symmetric matrix Q becomes the same E above.

2. Tori from complex integral lattices

The vast majority of tori are not abelian varieties. We now turn to some of the tori come across so far and show that they satisfy the Riemann conditions.

2.1. The lattice of Hurwitz integral quaternions D_4**.** Recall from ch.2 §2 that D_4 is the densest 4-dimensional lattice and that the associated Kummer surface has (symplectic) automorphism group $2^4 \rtimes A_4$. A period matrix for D_4 is

$$P = \begin{bmatrix} 1 & i & 0 & 1/2+i/2 \\ 0 & 0 & 1 & 1/2+i/2 \end{bmatrix}$$

Now taking the alternating matrix

$$Q = \begin{bmatrix} 0 & 10 & 1 & 5 \\ -10 & 0 & 1 & -4 \\ -1 & -1 & 0 & 1 \\ -5 & 4 & -1 & 0 \end{bmatrix}$$

one can easily check that both

$$\text{(i) } PQ^{-1}P^T = 0 \text{ and (ii) } iPQ^{-1}\overline{P}^T \text{ is positive definite}$$

Hence Q satisfies the first statement of the Riemann conditions laid out above. It follows that the torus $\mathbb{C}^2/\mathrm{D}_4$ is an abelian variety (or equivalently that it is algebraic: see the next section on the analytic and algebraic categories). In particular this Kummer surface may be one of Mukai's $K3$-surfaces in theorem 20. The matrix Q was obtained by taking a general 4×4 skew-symmetric matrix and turning the two conditions (i) and (ii) above into conditions on the coefficients. These were then simplified and it became an easy task picking integers satisfying the equations.

2.2. The integral lattices $E_6, E_8, K_{12}, \Lambda_{24}$. One encounters obvious difficulties in applying the method used for D_4 to higher dimensional lattices due simply to the greater number of variables and bigger matrices involved. However the second form of the Riemann conditions stated above proves fruitful for complex lattices. The method employed here consists in finding an appropriate basis for $\mathbb{C}^n$ and the lattice (the basis of the lattice given in terms of the basis of $\mathbb{C}^n$) for the period matrix to take on the required form. For details on the lattices we refer to chapter 1.

THEOREM 40. *A complex torus $\mathbb{C}^n/\Lambda$ is an abelian variety if the lattice Λ admits a complex structure.*

First recall from ch.2 §1 that by a complex lattice we mean a lattice over the Gaussian or Eisenstein integers, that is the $\mathbb{Z}[i]$ or $\mathbb{Z}[\omega]$ span of a complex basis $\{v_1, \ldots, v_n\}$ of V. Λ is said to admit a complex structure if there is a complex lattice Λ_c with underlying real lattice Λ (identifying $\mathbb{C}^n$ and $\mathbb{R}^{2n}$ as usual).

PROOF. Since Λ admits a complex structure we can pick a basis $\{v_1, \ldots, v_n\}$ of Λ over J, where $J = \mathbb{Z}[i]$ or $\mathbb{Z}[\omega]$. The real basis for the underlying real lattice can be taken as $B = \{v_1, \ldots, v_n, gv_1, \ldots, gv_n\}$ where $g = i$ or ω ($gv_i \in \Lambda$ since by assumption Λ has a g symmetry, and the set is linearly independent over $\mathbb{R}$ since $\{v_1, \ldots, v_n\}$ are complex linearly independent). But now take $\{v_1, \ldots, v_n\}$ as complex basis of V. The period matrix w.r.t. the basis B of Λ above takes the form

$$(\mathrm{Id}, g \cdot \mathrm{Id})$$

where Id is the $n \times n$ identity matrix. But $g \cdot \mathrm{Id}$ is symmetric and with $\mathrm{Im}(g) = 1$ or $\sqrt{3}/2$ clearly also has imaginary part $\mathrm{Im}(g \cdot \mathrm{Id})$ positive definite. Hence by the second version of the Riemann relations the torus $\mathbb{C}^n/\Lambda$ is an abelian variety. □

The proof tells us more - all complex lattices admit a principal polarisation, that is one of type $(1, \ldots, 1)$.

COROLLARY 41. *$\mathbb{C}^3/E_6$, $\mathbb{C}^4/E_8$, $\mathbb{C}^6/K_{12}$ and $\mathbb{C}^{12}/\Lambda_{24}$ are all abelian varieties, as are all the tori used to construct generalised Kummer manifolds in chapter 3.*

PROOF. E_6 as Eisenstein lattice has generator matrix

$$\begin{bmatrix} \theta & 0 & 0 \\ 0 & \theta & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

where $\theta = \omega - \overline{\omega} = \sqrt{-3}$ (see Conway and Sloane [**13**], p.126). For E_8 also see [**13**] and for the Leech lattice Λ_{24} [**13**]. More elegantly, the above three lattices all appear in the chain of complex laminated $\mathbb{Z}[\omega]$-lattices, as shown in [**15**]. Finally K_{12} has generator matrix

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & \omega & \omega & 1 & 0 & 0 \\ \omega & 1 & \omega & 0 & 1 & 0 \\ \omega & \omega & 1 & 0 & 0 & 1 \end{bmatrix}$$

as Eisenstein matrix. (see Conway and Sloane [**13**], p.128 and [**14**] for more details).

Finally the tori in chapter 3 not mentioned so far are tensors of integral lattices with the rings $\mathcal{E}$ and $\mathcal{G}$, so are complex lattices by construction. □

For completeness D_4 is also a complex $\mathbb{Z}[\omega]$-lattice (see ch.1 §1.6) and so admits a principal polarisation.

So according to corollary 37 the tori $\mathbb{C}^2/D_4$, $\mathbb{C}^3/E_6$, $\mathbb{C}^4/E_8$, $\mathbb{C}^6/K_{12}$ and $\mathbb{C}^{12}/\Lambda_{24}$ all admit an analytic embedding into projective space. In fact since the polarisations are all principal these give embeddings into $\mathbb{CP}^8$, $\mathbb{CP}^{26}$, $\mathbb{CP}^{80}$, $\mathbb{CP}^{728}$ and $\mathbb{CP}^{3^{12}-1} = \mathbb{CP}^{531440}$ respectively !

3. Complex analytic and algebraic categories

Serre's paper [**58**] gives a clear account of this material. See also appendix A in Lange & Birkenhake [**40**] and Griffiths and Harris [**28**]. Now recall that one can associate a complex analytic space X^h to any algebraic variety X over $\mathbb{C}$. The map is functorial where the morphisms in the two categories are regular maps and holomorphic maps respectively. A non-singular complex analytic space corresponds to the usual idea of complex manifold. A complex analytic space Y is said to be algebraic if there is an algebraic variety X such that $X^h \simeq Y$.

THEOREM 42 (Chow). *An analytic subspace of projective space is algebraic.*

And indeed in this case the holomorphic maps in the analytic category are regular maps in the algebraic category.

So by Chow's theorem tori are abelian varieties if and only if they are algebraic varieties. Before examining Kummer and generalised Kummer varieties we need a result on group actions (see Cartan [**10**]). A group G is said to act *properly and discontinuously* on a topological space X if

(a) if x_1 and x_2 are not congruent modulo G there are neighbourhoods A_1, A_2 of x_1, x_2 respectively such that $gA_1 \cap gA_2 = \emptyset$ for all $g \in G$

(b) for all $x \in X$ the isotropy group G_x is finite and there is a neighbourhood A of x stable under G_x such that if $gs \in A$ for $s \in A, g \in G$ then $g \in G_x$.

THEOREM 43. *Let X be an analytic space and G a group of automorphisms of X acting properly and discontinuously. Then X/G is also an analytic space.*

Finite group actions always satisfy conditions (a) and (b), so in particular the Kummer involution $\langle \pm 1 \rangle$ and other cyclic groups $\langle \omega \rangle$, $\langle \pm i \rangle$, $\langle \pm 1, \omega \rangle$ act properly and discontinuously on tori. Then by the above theorem all generalised Kummer orbifolds T/N where N is one of the cyclic groups above, are complex analytic spaces. Their automorphism groups as complex analytic spaces are those constructed in previous sections.

Furthermore, if the torus is an abelian variety then the resulting generalised Kummer manifolds are also algebraic: the blow-up of the abelian variety remains algebraic and the quotient by the cyclic group also does (see Griffiths and Harris [**28**] p.192).

THEOREM 44. *All manifolds in the tables of ch.2 §3.4 and ch.3 §6 are algebraic varieties with algebraic actions of the corresponding groups (groups of biregular maps).*

4. The canonical bundles

Although the results of this section are inconclusive we include it to illustrate the methods involved. The motivation behind these calculations is this following theorem [38], p.82. Recall that Aut(M) denotes all holomorphic transformations of M.

THEOREM 45. *If M is a compact complex manifold with ample canonical bundle then* Aut(M) *is finite.*

Let $\sigma : \widehat{M_p} \to M$ be the blow-up of an n-dimensional manifold at a smooth point $p \in M$ and $E \cong \mathbb{CP}^{n-1}$ the exceptional divisor. Then

$$\mathcal{K}_{\widehat{M_p}} = \sigma^* \mathcal{K}_m \otimes [E]^{n-1}$$

If p is a singular point of type $\mathbb{C}^n/\langle\theta\rangle$ where $\theta = \mu \cdot \mathrm{Id}$, $\mu = \exp(2\pi i/d)$ and $\widehat{M_p}/\langle\theta\rangle$ is the resolution described in chapter 3 then the corresponding formula is

$$\mathcal{K}_{\widehat{M_p}/\langle\theta\rangle} = \sigma^* \mathcal{K}_M \otimes [E]^{n-d}$$

(for both these formulas see Hübsch [33], p.115 and Griffiths and Harris [28]).

$[E]|_E$ is the universal bundle $\mathcal{L} \to \mathbb{CP}^{n-1}$ ([28], p.185) so in particular is not ample (the dual hyperplane bundle $H \to \mathbb{CP}^{n-1}$ is ample). For a torus T, $\mathcal{K}_T = 0$ is the trivial bundle. Apart from the 2-dimensional case $n > d$. The canonical bundle of the resolved torus is obtained by applying the second formula for each singularity successively. This amounts to adding a power of $\mathcal{L}^{n-d}$ for each type of singularity. We illustrate this with two typical examples.

For a Kummer surface $K = K_T$,

$$\begin{aligned}\mathcal{K}_K &= \sigma^* \mathcal{K}_T \otimes \mathcal{L}^{16(2-2)} \\ &= 0 \otimes \mathcal{L}^0 \\ &= 0\end{aligned}$$

And K has trivial canonical bundle.

As in the construction of $X_{\Lambda_{24}}$ (ch.3 §3), let X be the resolution of $T/\langle \pm 1, \omega\rangle$ where T is an n-dimensional torus ($n > 6$). Denote by m_1, m_2 the number of singularities of type $\mathbb{C}^n/\langle\omega\rangle$, $\mathbb{C}^n/\langle\pm 1\rangle$. Then

$$\begin{aligned}\mathcal{K}_X &= \sigma^*\mathcal{K}_T \otimes \mathcal{L}^{n-6} \otimes \mathcal{L}^{m_1(n-3)} \otimes \mathcal{L}^{m_2(n-2)} \\ &= \mathcal{L}^{n-6} \otimes \mathcal{L}^{m_1(n-3)} \otimes \mathcal{L}^{m_2(n-2)}\end{aligned}$$

But $\mathcal{L}$ is not ample and these are all positive powers as $n > 6$, so $\mathcal{K}_X$ is not ample. In particular theorem 45 cannot be applied.

However we knew this would be the case. There is an infinite number of holomorphic transformations coming from the torus : all act on the quotient space and on the equivariant blow-up. Only when we restrict to isometries does the group become finite.

CHAPTER 5

Automorphisms of toric varieties

In this chapter we examine constructions of another type of variety involving lattices, namely toric varieties. A toric variety is determined by a lattice in $\mathbb{R}^n$ and a set of cones spanned by vectors of the lattice (as explained in ch.1 §5). Properties of the varieties appear in the combinatorics of the cones.

We examine the toric varieties obtained from standard lattices discussed so far with natural cone decompositions, and observe how properties of the lattices are reflected in the geometry and symmetries of these spaces.

The toric varieties associated to root lattices are dealt with in §3 and §4. These appear in the study of symmetric toric fano varieties [**62**][**19**]. We are interested in their symmetries and make use of Demazure's structure theorem (§1) to determine their automorphism groups. §5 shows how these naturally lead to toric "geometric realisations" of some Niemeier lattices. We use this term as the toric varieties constructed reflect many of the properties of these Niemeier lattices in a natural way.

We start by stating Demazure's theorem in the appropriate form for our purposes.

1. Demazure's structure theorem

Let $X = X(\Delta, N)$ be a complete nonsingular toric variety. Demazure gives a description of the automorphism group $\mathrm{Aut}(X)$ of algebraic morphisms of X. The finite group $\mathrm{Aut}(N, \Delta)$ of isomorphisms of the fan (together with the action of the torus T_N) are precisely those commuting with the torus action as explained in ch.1 §5. One starts by associating to the fan a root system $R(\Delta, N)$ in the following way. Denote by $\Delta(1)$ the set of cones in Δ of dimension 1, and by Δ_{min} the elements of N of minimal length along all $x \in \Delta(1)$. Then define $R(\Delta, N) \subset M$ as

$$R(\Delta, N) = \{\alpha \in M : \exists \rho_\alpha \in \Delta_{min} \text{ s.t. } \langle \alpha, \rho_\alpha \rangle = 1 \text{ and } \langle \alpha, \rho \rangle \leq 0 \ \forall \rho \in \Delta_{min}, \rho \neq \rho_\alpha\}.$$

We now state the principal theorem in this area. For more details see Oda's book [**52**] and the orginal paper [**20**]. Let $\mathrm{Aut}^0(X)$ be the connected component of $\mathrm{Aut}(X)$, and split $R(\Delta, N)$ as

$$R_s(\Delta, N) = R(\Delta, N) \cap (-R(\Delta, N)),\ R_u(\Delta, N) = R(\Delta, N) \backslash R_s(\Delta, N).$$

Finally $W(N, \Delta)$ is the subgroup of $\mathrm{Aut}(N, \Delta)$ generated by the elements $w_\alpha : N \to N$

$$w_\alpha(x) = x - \langle x, \alpha \rangle (\rho_\alpha - \rho_{-\alpha}).$$

for $\alpha \in R_s(\Delta, N)$.

THEOREM 46 (Demazure's Structure Theorem). *Let $X = X(\Delta, N)$ be a complete nonsingular toric variety. Then $\mathrm{Aut}(X)$ is a linear algebraic group with the following properties:*
(1) T_N is a maximal algebraic torus in $\mathrm{Aut}(X)$. $R(\Delta, N)$ is the root system of $\mathrm{Aut}^0(X)$ w.r.t. the maximal algebraic torus T_N
(2) the reductive Levi subgroup G_s of $\mathrm{Aut}^0(X) = G_u \rtimes G_s$ (where G_u is the unipotent radical) has root system $R_s(\Delta, N)$ and simple components of type A.
(3) $W(N, \Delta)$ coincides with the Weyl group of G_s and

$$\mathrm{Aut}(X)/\mathrm{Aut}^0(X) \simeq \mathrm{Aut}(N, \Delta)/W(N, \Delta)$$

2. A motivating example - the octahedron

Let P be the octahedron with vertices $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$ in $\mathbb{Z}^3$. Let Δ_P be the fan with cones spanned by the proper faces of P and $X_P = X(\Delta_P, \mathbb{Z}^3)$ the associated variety. The symmetry group of the octahedron is often denoted $[4, 3]$:

$$[4, 3] \simeq 2 \times S_4$$

This coincides with $\mathrm{Aut}(\mathbb{Z}^3, \Delta_P)$.

We give another description of the situation. View P in the root lattice $3\mathrm{A}_1$. The top dimensional cones are then fundamental domains of the Weyl group $W(3\mathrm{A}_1)$. The root lattice has automorphism group

$$\mathrm{Aut}(3\mathrm{A}_1) = 2^3 \rtimes S_3$$

This group coincides with the description of $[4,3]$ given above, where the 2 is the diagonal in 2^3.

We generalise this to other root lattices.

3. Toric varieties from root lattices

We apply Demazure's theorem to some particular fans for which the root system is empty, hence determining the automorphism groups of the associated complex manifolds. §3.1 and §3.2 examine examples we will be using in §5.

PROPOSITION 47. *Let (N, Δ) be a complete fan such that for any element $\alpha \in M$ there are at least two elements $x, y \in \Delta_{min}$ forming an angle of less than 90 degrees with α (i.e. with strictly positive inner product). Then $R(N, \Delta) = \emptyset$ and*

$$\mathrm{Aut}(X(\Delta, N)) = T_N \times \mathrm{Aut}(N, \Delta).$$

PROOF. The proposition follows easily from Demazure, since the empty root system corresponds to the maximal torus T_N, and clearly $W(N, \Delta)$ is trivial. The sum is a direct sum as $\mathrm{Aut}(N, \Delta)$ consists of those morphisms commuting with T_N and both are subgroups of automorphisms. □

Let R be an n-dimensional root lattice and Φ its set of roots.

DEFINITION 12. *Define $\Delta_{\mathrm{W}} = \Delta_{\mathrm{W}(R)}$ to be the fan whose top dimensional cones are fundamental domains of the Weyl group $\mathrm{W}(R)$.*

These top dimensional cones are n-simplices whose geometry is specified by the usual Dynkin diagrams : the nodes correspond to the walls of the simplex and the edges determine their angles of intersection. (From now on we omit $R = \mathbb{Z}^n$, similar to $n\mathrm{A}_1$ for the purposes of this chapter). We now show that the fan is defined in the dual root lattice R^* and is nonsingular.

LEMMA 48. *Let R be an n-dimensional root lattice. The 1-dimensional faces (edges) of the fundamental simplex of the Weyl group $W(R)$ are each spanned by a vector of R^*.*

PROOF. An edge E is the intersection of $n-1$ of the hyperplanes perpendicular to the simple generators $S = \{v_1, \ldots, v_n\}$ of R. Let D be the fundamental domain of the Weyl group corresponding to the simple system S. Let x be a vector in E. Then x has inner product $x \cdot v_i = 0$ with the given $n-1$ simple roots, and say $x \cdot v_k = t$ say, with the remaining simple root v_k. Under our convention $t > 0$ as $x \in D$ (and $x \cdot v_i = 0$ for all other simple roots v_i). But then $y = x/t \in E$ now has inner product 1 with v_k and still inner product 0 with the remaining generators. So $y \cdot R \in \mathbb{Z}$ by linear expansion and $y \in R^*$. □

Hence the fan defined is indeed in the lattice R^*. We now show that the resulting variety is nonsingular.

LEMMA 49. *The vectors $w_1, \dots, w_n \in R^*$ spanning a fundamental simplex of the Weyl group $W(R)$ generate R^*.*

PROOF. Again $R \subset V$ is a root lattice generated by a simple system $S = \{v_1, \dots, v_n\}$. Let $B = \{b_1, \dots, b_n\}$ be the vectors obtained by applying the above lemma to each edge of the fundamental simplex in V^*. Then by construction B is the basis of V^* dual to S. Let $y \in R^*$. Then $y = \alpha_1 b_1 + \cdots + \alpha_n b_n$ where $\alpha_i \in \mathbb{R}$. But

$$v_i \cdot y = v_i \cdot (\alpha_1 b_1 + \cdots + \alpha_n b_n) = \alpha_i$$

and by definition $v_i \cdot y \in \mathbb{Z}$ (since $v_i \in R$). Hence $\alpha_i \in \mathbb{Z}$ and any element $y \in R^*$ is a $\mathbb{Z}$-linear combination of B. □

The toric variety $X(\Delta_W, R^*)$ associated to the fan defined above is hence nonsingular and also clearly complete (see lemmas 3 and 1). Before proceeding any further we recall in a lemma some well-known facts about a lattice and its dual.

LEMMA 50. *For any lattice $L \subset \mathbb{R}^n$,*
(a) $(L^*)^* = L$
(b) $\mathrm{Aut}(L^*) = \mathrm{Aut}(L)$.

PROOF. Let Q be a generator matrix for L. Then $P = (Q^{-1})^T$ is a generator matrix for L^* (see Conway and Sloane [**13**],p.11). (a) clearly holds since $(P^{-1})^T = Q$. For (b) recall $B \in \mathrm{Aut}(L)$ if and only if there is an integral matrix U with $\det U = \pm 1$ such that $UQ = QB$ ([**13**] p.90).

$$(U^{-1})^T (Q^{-1})^T = (Q^{-1})^T (B^{-1})^T$$

So $\{(B^{-1})^T : B \in \mathrm{Aut}(L)\}$ is isomorphic to $\mathrm{Aut}(L)$ and $\mathrm{Aut}(L) < \mathrm{Aut}(L^*)$. The converse clearly holds also and $\mathrm{Aut}(L) = \mathrm{Aut}(L^*)$. □

We now show that for a root lattice R containing no irreducible component of type A_1, (Δ_W, R^*) satisfies the hypothesis of proposition 47 (we examine the case $R = \mathrm{A}_1$ separately). First by (a) of the above lemma M is the root lattice R, N being of course R^*. Vertices of a typical fundamental simplex for $R = \mathrm{A_n}$ $(n \geq 1), \mathrm{D_n}$ $(n \geq 3), \mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8$ (in $\Delta_{min} \subset R^*$) are given in the table pp.460-61 of Conway and Sloane [**13**] (omitting the origin vertex). One can easily check that all inner products among the edges are strictly positive. Now suppose $x \in M = R$. Then $x \in D$ for some fundamental domain D, and x has positive inner product with all the edges of D. If A_1 is not among the components then corollary 47 can be applied. Hence we have proved

PROPOSITION 51. *Let R be a root lattice containing no irreducible component of type A_1 and Δ_{W} the fan in R^* consisting of the Weyl chambers and their faces. Then*

$$\mathrm{Aut}(X(\Delta_{\mathrm{W}}, R^*)) = T_{R^*} \times \mathrm{Aut}(R^*, \Delta_{\mathrm{W}}).$$

By (b) of the above lemma $\mathrm{Aut}(R^*) = \mathrm{Aut}(R)$. As we shall now see this group coincides with the group $\mathrm{Aut}(R^*, \Delta_W)$ of fan maps and hence acts on the manifold $X(\Delta_{\mathrm{W}}, R^*)$. Recall the nice description of the automorphism group of a root lattice, namely $\mathrm{Aut}(R) = \mathrm{W} \rtimes G$ where $\mathrm{W} = \mathrm{W}(R)$ is the Weyl group of R and $G = G(R)$ the graph automorphisms of the associated Dynkin diagram.

PROPOSITION 52. *For any root lattice R,*

$$\mathrm{Aut}(R^*, \Delta_{\mathrm{W}}) = \mathrm{Aut}(R) = \mathrm{W}(R) \rtimes G(R).$$

PROOF. We must show that the group automorphisms $\varphi : R^* \to R^*$ preserving the fan (not a priori distance preserving) form the group $\mathrm{Aut}(R) = \mathrm{W} \rtimes G$ of lattice automorphisms.

Fix a simple system $S = \{v_1, \ldots, v_n\} \subset R$ of the root system Φ of R. The walls of the corresponding Weyl chamber D are then the hyperplanes $v_i^{\perp}$.

Since the top-dimensional cones in Δ_{W} are fundamental domains of the action of the Weyl group W, $w \in \mathrm{W}$ clearly acts on the fan, so $\mathrm{W}(R) < \mathrm{Aut}(R^*, \Delta_{\mathrm{W}})$.

Let $\varphi \in \operatorname{Aut}(R^*, \Delta_W)$. Since φ preserves the cones, $\varphi S = \{\varphi v_1, \ldots, \varphi v_n\}$ is also a simple system and determines a cone φD in Δ_W. The Weyl group W acts simply transitively on the simple systems (and chambers), so there exists a unique $w \in \mathrm{W}$ such that $\{wv_1, \ldots, wv_n\} = \{\varphi v_1, \ldots, \varphi v_n\}$ as sets. In other words the simple systems φS and wS are the same and the maps $\varphi, w : R \to R$ only differ by a permutation. To prove the proposition it remains to show that only those permutations preserving distances are automorphisms of the fan.

Let γ be a permutation of the simple system S. The action of γ extends to R by linear expansion. If γ is in $\mathrm{O}(n)$ then $\gamma \in G < \operatorname{Aut}(R)$ and acts on Δ_W (since it preserves the roots Φ and maps simple system to simple system). Hence $G < \operatorname{Aut}(R^*, \Delta_W)$.

Conversely now. In the Dynkin diagram of S, two roots v_i and v_j have either inner product -1 or 0. In the first case $v_i + v_j$ is also a root. Suppose γ did not preserve the angle between v_i and v_j. Then $v_i \cdot v_j \neq \gamma v_i \cdot \gamma v_j$ and $\gamma(v_i + v_j)$ is no longer a root. Similarly for $v_i \cdot v_j = 0$: $v_i + v_j$ is not a root, but if the inner product changed to -1 under γ, then $\gamma(v_i + v_j) \in \Phi$. In both cases the fan Δ_W is not preserved.

Hence only those permutations of S preserving the Dynkin graph are automorphisms of the fan and $\operatorname{Aut}(R^*, \Delta_W) = \operatorname{Aut}(R) = \mathrm{W} \rtimes G$. □

Combining propositions 51 and 52 we have proved:

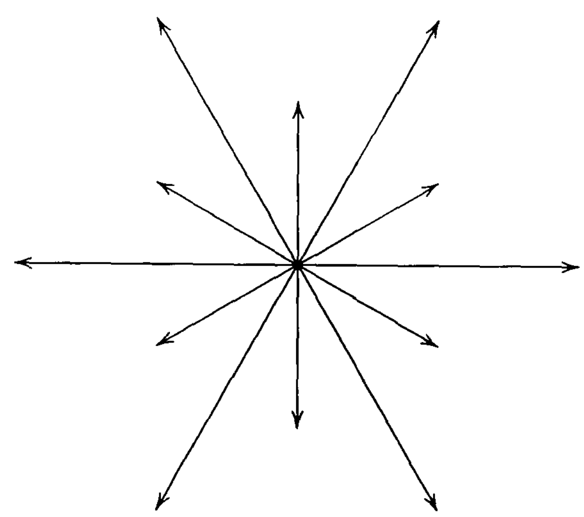
THEOREM 53. *For R and Δ_W as in proposition 51,*

$$\operatorname{Aut}(X(\Delta_W, R^*)) = T_{R^*} \times \operatorname{Aut}(R) = T_{R^*} \times (\mathrm{W}(R) \rtimes G(R))$$

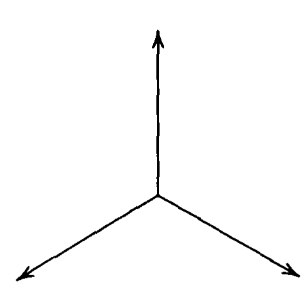
One can give a nice geometrical feeling to this result. Indeed one can view a toric variety as a thickened up version of the polygon determined by the fan, where each face is extended to an affine variety. The theorem then tells us that the automorphisms of this variety are the combinatorial symmetries of the polygon together with the action of the torus on each separate affine part.

Let us make a remark about automorphisms of a general fan Δ in a lattice L. These are the linear isomorphisms (bijections) $L \to L$ preserving the cones in Δ. In particular these need not be distance preserving as automorphisms of a lattice, although they coincide in theorem 53 above. Of course all the linear isomorphisms of L form the group $\mathrm{GL}(n,\mathbb{Z})$, and potentially any of these could preserve the fan. $\mathrm{Aut}(N,\Delta) < \mathrm{GL}(n,\mathbb{Z})$ is however finite as the number of edges in Δ is also finite.

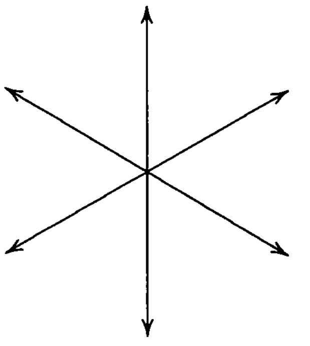
3.1. $R = \mathrm{A}_2$. . As an easy example, take the root system A_2, together with the minimal vectors of its dual A_2^*:



The longer vectors are the roots of A_2, the shorter ones the minimal vectors of A_2^*. The plane is tiled by the 6 fundamental simplices of the Weyl group spanned by the minimal vectors of the dual. In this case one could take the cones spanned by the roots of A_2 and obtain the same fan, but this does not work in general. The toric variety $X := X(\Delta_W, \mathrm{A}_2^*)$ is in fact projective space $\mathbb{C}\mathrm{P}^2$ blown up at three points, as can be seen by starting with the fan



The 3 top dimensional cones are the three parts of the affine covering of $\mathbb{C}\mathrm{P}^2$. Adding the negatives of the three vectors corresponds to a blow-up in each of these, yielding the fan Δ_W below:



By theorem 53,

$$\mathrm{Aut}(X) = T_{\mathrm{A}_2^*} \times (\mathrm{W}(\mathrm{A}_2) \rtimes \mathbb{Z}/2) \simeq T_{\mathrm{A}_2^*} \times (S_3 \rtimes \mathbb{Z}/2)$$

As expained in the introduction on toric geometry (ch.1 §5.2), several interesting invariants of the resulting manifold X can easily be computed from the fan. First the Betti numbers b_m. Denoting by d_i the number of cones of dimension i in the fan, we have for this example $d_0 = 1, d_1 = 6, d_2 = 6$. Then the odd Betti numbers are 0 and for $m = 2k$,

$$b_{2k} = \sum_{i=k}^{2} (-1)^{i-k} \binom{i}{k} d_{2-i}$$

yielding $b_0 = 1, b_2 = 4, b_4 = 1$. So the Euler charateristic is $\chi(X) = 6$ (also equals the number of top dimensional cones). Now turning to the intersection form on the middle cohomology we already know the lattice has rank 4 and is indefinite since the exceptional divisor of the 3 blow-ups has opposite orientation to that of the original $\mathbb{C}\mathrm{P}^2$. For an arbitrary nonsingular compact toric variety the signature $\tau(X)$ is

$$\tau(X) = \sum_{i=0}^{n} (-2)^i d_{n-i}.$$

So in this case we have $\tau(X) = -2$ and by the classification of indefinite quadratic forms the middle cohomology is the lattice

$$\langle 1 \rangle \oplus \langle -1 \rangle \oplus \langle -1 \rangle \oplus \langle -1 \rangle.$$

The fundamental group of X is trivial as for all these examples (see ch.1 §5.2).

For larger root lattices one is faced with the extra task of counting the number of cones in each dimension. We now work through the case $R = \mathrm{D}_4$.

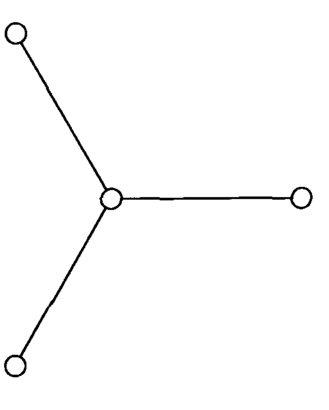
3.2. $R = D_4$. Let $X = X(\Delta_W, D_4^*)$ and $W = W(D_4)$ be the Weyl group. First since the 4-dimensional cones are fundamental domains of the Weyl group, clearly $d_4 = |W(D_4)| = 192$. Also $d_0 = 1$ as always. To determine the number of 1, 2 and 3-dimensional cones we use the action of the Weyl group and work out the appropriate stabilisers (parabolic subgroups). First recall that for a finite group G permuting a finite set of objects O, in our case the Weyl group acting on the faces of a given dimension, we have

$$|Gx| = |G|/|G_x|$$

where $x \in O$, Gx is the orbit of x under G, and G_x is the stabiliser of x in G. Let us start with the 1-dimensional cones of the fan. These are the edges of the Weyl chambers, and correspond to the intersection of 3 of the 4 hyperplanes determined by a simple system of D_4. The Weyl group is transitive on the simplices and 2 edges of a given simplex are not in the same orbit of W, so the number of edges is

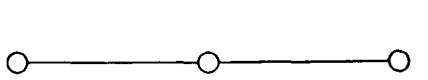
$$d_1 = |Wx_1| + |Wx_2| + |Wx_3| + |Wx_4|$$

where $x_1, \ldots, x_4$ span a Weyl chamber of W. The stabiliser of an edge x is generated by the reflections in the 3 walls whose intersection is x. The Dynkin diagram allows for a nice description of the situation (as already mentioned the nodes correspond to the walls of the simplex and the graph edges determine the angles of intersection). The 4 edges $\{x_1, \ldots, x_4\}$ correspond to the 4 choices of three nodes in the Dynkin diagram of D_4 (shown below), and reflections in the hyperplanes corresponding to the nodes generate the stabiliser of that edge.



Dynkin diagram D_4

Clearly 3 of the edges are of type A_3:



type (A_3)

and one is of type $3\mathrm{A}_1$:

$$\circ \qquad \circ \qquad \circ \qquad \text{type } (3\mathrm{A}_1)$$

The three edges of type (A_3) have stabiliser $\mathrm{W}(\mathrm{A}_3) \simeq S_4$ of order 24, and the one of type $(3\mathrm{A}_1)$ has stabiliser $\mathrm{W}(\mathrm{A}_1)^3 \simeq (\mathbb{Z}/2)^3$ of order 8, yielding

$$d_1 = 3 \cdot 192/24 + 192/8 = 48.$$

In a similar fashion we can compute the number of 2 and 3-dimensional faces d_2 and d_3: these are intersections of 2 of the walls of the simplex and the walls themselves respectively. In other words they correspond to subgraphs of the Dynkin diagram of one and two vertices respectively. Clearly the 4 walls are fixed only by reflection in themselves, while the 2-dimensional faces split into 3 of type (A_2) and 3 of type $(2\mathrm{A}_1)$ with stabilisers of order 6 and 4.

$$d_2 = 3 \cdot 192/6 + 3 \cdot 192/4 = 240, \quad d_3 = 4 \cdot 192/2 = 384$$

Summarising we have

$$d_0 = 1,\ d_1 = 48,\ d_2 = 240,\ d_3 = 384,\ d_4 = 192.$$

Using the formula for the Betti numbers (ch.1 §5.2) one calculates

$$b_i = 0 \ (i \text{ odd}),\ b_0 = 1,\ b_2 = 44,\ b_4 = 102,\ b_6 = 44,\ b_8 = 1.$$

The Euler characteristic is $\chi(X) = d_4 = 192$ and the middle cohomology of X is a lattice of rank 102 with signature

$$\tau(X) = d_4 - 2d_3 + 4d_2 - 8d_1 + 16 = 16.$$

(see ch.1 §5.2 for the formula). And again by the classification of indefinite quadratic forms the intersection form of the manifold X is

$$59\langle 1\rangle \oplus 43\langle -1\rangle.$$

As with all these constructions X is simply-connected as it contains at least one top-dimensional cone. Finally by theorem 53 X has automorphism group

$$\mathrm{Aut}(X) = T_{\mathrm{D}_4^*} \times (\mathrm{W}(\mathrm{D}_4) \rtimes S_3) \simeq T_{\mathrm{D}_4^*} \times ((2^3 \rtimes S_4) \rtimes S_3)$$

4. More fans in R^*

We construct a family of fans starting with a Δ_{W}. These new fans will also admit an action of the Weyl group. Let D be a top dimensional cone in Δ_{W} or equivalently a fundamental domain of the Weyl group W. The idea is to construct a new cone C by reflecting D in several of its walls and then generate the new fan by taking images of C under the Weyl group.

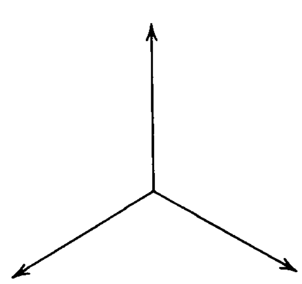
So we start by reflecting the fundamental domain D in several of its walls, say $s_o, \dots, s_k \in \mathrm{W}$. For W to preserve the new cone the image of D under all combinations of these must be included. So letting $\mathrm{W}_0 < \mathrm{W}$ be the parabolic subgroup generated by the reflections $s_0, \dots, s_k$, define

$$C = \{wF : w \in \mathrm{W}_0\}.$$

One must of course check C remains a strongly convex cone (clearly not the case for $\mathrm{W}_0 = \mathrm{W}$, for which $C = \mathbb{R}^n$). Define $\Delta_{\mathrm{W}}^{\mathrm{W}_0}$ to be the fan whose top dimensional cones are the images of C under the action of the Weyl group (W_0 fixing C of course):

$$\Delta_{\mathrm{W}}^{\mathrm{W}_0} = \{wC : w \in \mathrm{W}\} \cup \{\text{all faces}\}.$$

By construction these fans $\Delta_{\mathrm{W}}^{\mathrm{W}_0}$ admit an action of the Weyl group. As an easy example take $R = \mathrm{A}_2$ and $\mathrm{W}_0 = \mathrm{W}(\mathrm{A}_1) \simeq \mathbb{Z}/2$ the parabolic subgroup generated by one of the two simple roots of A_2. The fan $\Delta_{\mathrm{W}}^{\mathrm{W}_0}$ (already encountered in §3.1) is then:



The associated variety is $X(\Delta_{\mathrm{W}}^{\mathrm{W}_0}, \mathrm{A}_2^*) = \mathbb{C}\mathrm{P}^2$.

As this example illustrates proposition 47 no longer holds in general for fans $(\Delta_{\mathrm{W}}^{\mathrm{W}_0}, R^*)$. For details on which subgroups W_0 yield a fan see Voskresenskii [62], theorem 4.

5. Fans in the Niemeier lattices

Let R be an n-dimensional root lattice (not necessarily irreducible) and $\mathrm{W} = \mathrm{W}(R)$ its Weyl group. In §3 we assigned a fan Δ_{W} to R. Say R splits into the irreducible components $R = R_1 \oplus \cdots \oplus R_m$. Then $\mathrm{W}(R) = \mathrm{W}(R_1) \times \cdots \times \mathrm{W}(R_m)$ and Δ_{W} is the product of the fans of the irreducible components (definition given in ch.1 §5):

$$\Delta_{\mathrm{W}} = \Delta_{\mathrm{W}(R_1)} \times \cdots \times \Delta_{\mathrm{W}(R_m)}.$$

Also

$$X(\Delta_{\mathrm{W}}, R^*) = X(\Delta_{\mathrm{W}(R_1)}, R_1^*) \times \cdots \times X(\Delta_{\mathrm{W}(R_m)}, R_m^*).$$

Let N be a Niemeier lattice and R the sublattice generated by its roots. Then since N is self-dual

$$R < N = N^* < R^* \tag{2}$$

All four are equal only for $N = R = 3\mathrm{E}_8$. Up to now the fans defined in sections 3 and 4 and above have been fans in the dual root lattice R^* (we denote Δ_{W} by $(\Delta_{\mathrm{W}}, R^*)$ to emphasise this). These are clearly not in general in the Niemeier lattice by the relation (2) above. However in some of the most interesting cases $6\mathrm{D}_4$, $12\mathrm{A}_2$ and $24\mathrm{A}_1$ one can overcome this as the components are self-similar:

$$\mathrm{A}_1 \simeq \mathrm{A}_1^*,\ \mathrm{A}_2 \simeq \mathrm{A}_2^*,\ \mathrm{D}_4 \simeq \mathrm{D}_4^*$$

For each of these root lattices L there is a constant $c \in \mathrm{L}$ and isomorphism $\mu : cL \to L^*$,

$$\varphi := \mu \circ c : L \to L^*.$$

The fan $(\Delta_{\mathrm{W}}, L^*)$ can then be pulled back from L^* to L:

$$(\Delta_{\mathrm{W}}, L) = \{\varphi^{-1}(\sigma) : \text{for all cones } \sigma \in (\Delta_{\mathrm{W}}, L^*)\}.$$

In this way we obtain fans

$$(\Delta_{\mathrm{W}}, \mathrm{D}_4),\ (\Delta_{\mathrm{W}}, \mathrm{A}_2),\ (\Delta_{\mathrm{W}}, \mathrm{A}_1)$$

and taking products (ch.1 §5) one constructs

$$(\Delta_{\mathrm{W}}, 6\mathrm{D}_4),\ (\Delta_{\mathrm{W}}, 12\mathrm{A}_2),\ (\Delta_{\mathrm{W}}, 24\mathrm{A}_1).$$

The corresponding varieties are of course products of those defined in section 3:

$$X(\Delta_{\mathrm{W}}, 6\mathrm{D}_4) = X(\Delta_{\mathrm{W}}, \mathrm{D}_4^*) \times \cdots \times X(\Delta_{\mathrm{W}}, \mathrm{D}_4^*) \quad (6 \text{ times})$$
$$X(\Delta_{\mathrm{W}}, 12\mathrm{A}_2) = X(\Delta_{\mathrm{W}}, \mathrm{A}_2^*) \times \cdots \times X(\Delta_{\mathrm{W}}, \mathrm{A}_2^*) \quad (12 \text{ times})$$
$$X(\Delta_{\mathrm{W}}, 24\mathrm{A}_1) = X(\Delta_{\mathrm{W}}, \mathrm{A}_1^*) \times \cdots \times X(\Delta_{\mathrm{W}}, \mathrm{A}_1^*) \quad (24 \text{ times})$$

Restrict R to one of $6\mathrm{D}_4$, $12\mathrm{A}_2$ or $24\mathrm{A}_1$. Since $R < N$, Δ_{W} can be viewed as a fan in the associated Niemeier lattice N. The variety $X(\Delta_{\mathrm{W}}, N)$ is now singular as the vectors spanning a cone no longer generate the lattice but only the sublattice $R < N$. Recall from chapter 1 §5 that the singularities are quotient singularities $\mathbb{C}^{24}/G$ where G is the quotient of the lattice by the sublattice generated by the vectors spanning the cone. But N/R is precisely the gluecode so the singularities of $X(\Delta_{\mathrm{W}}, N)$ are of type $\mathbb{C}^{24}/G$, where $G < \mathrm{GL}(24, \mathbb{C})$ is isomorphic to G_∞.

We will determine the groups $\mathrm{Aut}(N, \Delta_{\mathrm{W}})$ for each case individually. Demazure's structure theorem can be generalised to non-compact toric varieties but not to non-singular ones. As a consequence we are unable to determine the group of automorphisms of the singular complex spaces $X(\Delta_{\mathrm{W}}, N)$.

We now make use of theorem 5 due to Brylinski. His result states that any singular fan admitting an action of a finite group H has a non-singular subdivision also invariant under H. In other words the corresponding singular toric variety has a resolution whose fan automorphisms still include H.

Definition 13. *Let* $N = N(24\mathrm{A}_1)$, $N(12\mathrm{A}_2)$ *or* $N(6\mathrm{D}_4)$. *Define* (Δ_N, N) *to be any non-singular fan obtained from* (Δ_W, N) *by Brylinski's algorithm* **[9]** *w.r.t. the finite group* $\mathrm{Aut}(N, \Delta_{\mathrm{W}})$.

We have constructed a manifold $X(\Delta_N, N) = X(\widetilde{\Delta_{\mathrm{W}}, N})$ for which $\mathrm{Aut}(N, \Delta_{\mathrm{W}}) < \mathrm{Aut}(N, \Delta_N)$.

We now examine the three cases $R = 24\mathrm{A}_1$, $12\mathrm{A}_2$ and $6\mathrm{D}_4$ in turn. To simplify notation we define $X_\mathrm{W} := X(\Delta_\mathrm{W}, R)$, $Y_N := X(\Delta_\mathrm{W}, N)$ and $X_N := X(\Delta_N, N)$.

5.1. The Niemeier lattice $(24\mathrm{A}_1)^+$. For the root lattice A_1, the fan Δ_W is

And $X(\Delta_\mathrm{W}, \mathrm{A}_1) = \mathbb{CP}^1$. So the product fan $(\Delta_\mathrm{W}, 24\mathrm{A}_1)$ has associated variety

$$X_\mathrm{W} := X(\Delta_\mathrm{W}, 24\mathrm{A}_1) = \mathbb{CP}^1 \times \cdots \times \mathbb{CP}^1 \quad \text{(24 times)}.$$

The connected component of the automorphism group is just the product of the connected components [**1**], p.31:

$$\begin{aligned} \mathrm{Aut}^0(X_\mathrm{W}) &= \mathrm{Aut}^0(\mathbb{CP}^1) \times \cdots \times \mathrm{Aut}^0(\mathbb{CP}^1) \quad \text{(24 times)} \\ &= \prod_{24} \mathrm{PGL}(2, \mathbb{C}) \end{aligned}$$

The structure of $\mathrm{Aut}(X_\mathrm{W})$ now follows from Demazure (theorem 46). First $W(24\mathrm{A}_1, \Delta_\mathrm{W})$ coincides with the Weyl group of $\prod \mathrm{PGL}(2, \mathbb{C})$, so

$$\mathrm{W} = \mathrm{W}(24\mathrm{A}_1, \Delta_\mathrm{W}) = \prod_{24} \mathrm{W}(A_1) \simeq 2^{24}$$

Also by proposition 52,

$$\mathrm{Aut}(24\mathrm{A}_1, \Delta_\mathrm{W}) = \mathrm{W}(24\mathrm{A}_1) \rtimes S_{24}$$

$\mathrm{Aut}(24\mathrm{A}_1)$ contains all permutations of the simple roots in this case. By Demazure

$$\begin{aligned} \mathrm{Aut}(X_\mathrm{W})/\mathrm{Aut}^0(X_\mathrm{W}) &= \mathrm{Aut}(24\mathrm{A}_1, \Delta_\mathrm{W})/\mathrm{W}(24\mathrm{A}_1, \Delta_\mathrm{W}) \\ &= (\mathrm{W}(24\mathrm{A}_1).S_{24})/\mathrm{W}(24\mathrm{A}_1) \\ &= S_{24} \end{aligned}$$

So finally

$$\mathrm{Aut}(X_\mathrm{W}) = (\prod_{24} \mathrm{PGL}(2, \mathbb{C})).S_{24}$$

Let $N = (24\mathrm{A}_1)^+$. As $24\mathrm{A}_1 \subset N$, Δ_W is also a fan in N. The toric variety $Y_N := X(\Delta_\mathrm{W}, N)$ is now an orbifold, that is Y_N has isolated quotient singularities. The fan has 2^{24} top-dimensional cones $[\pm\sqrt{2}e_1, \ldots, \pm\sqrt{2}e_{24}]$ where the e_i's are the standard generators of $\mathbb{R}^{24}$. As mentioned above each is an affine variety $\mathbb{C}^{24}/G$ where G is a group of holomorphic transformations isomorphic to the glue code G_∞.

Abstractly G is just the 2-group $(\mathbb{Z}/2)^{12}$. We identify $G < GL(24, \mathbb{C})$ more precisely. First recall from ch.1 §1.4 that the gluecode $G_\infty < 24A_1^*/24A_1$ is the binary Golay code $\mathcal{C}_{24} \subset \mathbb{F}_2^{24}$ where $0, 1 \in \mathbb{F}_2$ are identified with the glue vectors $[0], [1] \in A_1^*/A_1$. $[0]$ is the trivial glue vector while $[1] = (1/2, -1/2)$ is of index 2 in A_1. So following the procedure of ch.1 §5, $G < GL(24, \mathbb{C})$ is a group of 24×24 diagonal matrices and can be identified as a multiplicative version of the Golay code $\mathcal{C}_{24}$:

$$G = \{\text{diag}(\pm 1, \ldots, \pm 1) : \text{ the minus signs are indexed by } \mathcal{C}_{24}\}.$$

Summarising, Y_N is a compact complex orbifold with 2^{24} isolated singularities locally isomorphic to $\mathbb{C}^{24}/G$ where $G < GL(24, \mathbb{C})$ is the finite abelian 2-group described above. Y_N is the quotient of the product of complex projective lines by the natural induced action of G:

$$Y_N = \frac{X(\Delta_W, 24A_1)}{G} = \frac{\mathbb{C}P^1 \times \cdots \times \mathbb{C}P^1}{G}$$

A 24-dimensional cone in the fan $(\Delta_W, 24A_1)$ corresponds to a subset $U_1 \times \cdots \times U_{24} \subset \mathbb{C}P^1 \times \cdots \times \mathbb{C}P^1$ where the U_i's are one of the two affine parts of $\mathbb{C}P^1$.

As already mentioned we cannot determine the automorphism group $\text{Aut}(Y_N)$ of the complex orbifold Y_N as Demazure's theorem does not apply to singular spaces. However we know the fan automorphisms preserved in the new fan (now in the Niemeier lattice):

LEMMA 54. $\text{Aut}(N, \Delta_W)$ *is the subgroup of* $\text{Aut}(24A_1, \Delta_W)$ *sill acting on* (Δ_W, N) *and equals*

$$\text{Aut}(N, \Delta_W) = W(24A_1) \rtimes M_{24} < W(24A_1) \rtimes S_{24}$$

PROOF. The cones in (Δ_W, N) are the same as those in $(\Delta_W, 24A_1)$, so the only extra requirement for a map of $(\Delta_W, 24A_1)$ to preserve (Δ_W, N) is that the Niemeier lattice $(24A_1)^+$ be preserved. These are precisely the automorphisms of the Niemeier lattice. The lemma then follows from the properties of the Niemeier lattices in ch.1 §1.4. □

Applying Brylinski's algorithm we obtain a new fan (Δ_N, N) such that

$$W(24A_1) \rtimes M_{24} < \text{Aut}(N, \Delta_N)$$

where M_{24} is the Mathieu group. This group of course acts on the associated manifold $X(N, \Delta_N)$.

5.2. The Niemeier lattice $(12\mathrm{A}_2)^+$. The toric variety $X(\Delta_\mathrm{W}, \mathrm{A}_2)$ was discussed in detail in section 3.1 and is the complex projective plane $\mathbb{C}\mathrm{P}^2$ blown-up at three points which we denote $\overline{\mathbb{C}\mathrm{P}^2}$. The product fan $(\Delta_\mathrm{W}, 12\mathrm{A}_2) = (\Delta_\mathrm{W}, \mathrm{A}_2) \times \cdots \times (\Delta_\mathrm{W}, \mathrm{A}_2)$ then has associated variety

$$X_\mathrm{W} := X(\Delta_\mathrm{W}, 12\mathrm{A}_2) = \overline{\mathbb{C}\mathrm{P}^2} \times \cdots \times \overline{\mathbb{C}\mathrm{P}^2} \quad \text{(12 times)}$$

By proposition 52 and theorem 53,

$$\mathrm{Aut}(12\mathrm{A}_2, \Delta_\mathrm{W}) = \mathrm{W}(12\mathrm{A}_2) \rtimes 2^{12}.S_{12}$$

and

$$\begin{aligned}\mathrm{Aut}(X_\mathrm{W}) &= T_{12\mathrm{A}_2} \times (\mathrm{W}(12\mathrm{A}_2) \rtimes 2^{12}.S_{12}) \\ &= T_{12\mathrm{A}_2} \times (S_3^{12} \rtimes 2^{12}.S_{12})\end{aligned}$$

Here $2^{12}.S_{12}$ are the graph symmetries $G(12\mathrm{A}_2)$ of the Dynkin diagram of $12\mathrm{A}_2$.

Let $N = (12\mathrm{A}_2)^+$ be the corresponding Niemeier lattice. As $12\mathrm{A}_2 \subset N$, Δ_W is also a fan in N. The associated toric variety $Y_N = X(\Delta_\mathrm{W}, N)$ is an orbifold with 6^{12} singularities, one for each top-dimensional cone. The singularities are of type $\mathbb{C}^{24}/G$ where $G < \mathrm{GL}(24, \mathbb{C})$ is isomorphic to $N/12\mathrm{A}_2$. By construction $N/12\mathrm{A}_2$ is the ternary Golay code $\mathcal{C}_{12} \subset \mathbb{F}_3^{12}$ with additive group structure $(\mathbb{Z}/3)^6$. Let

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} \omega & 0 \\ 0 & \overline{\omega} \end{pmatrix}, \quad A_2 = \begin{pmatrix} \overline{\omega} & 0 \\ 0 & \omega \end{pmatrix}.$$

Then G is the group of block diagonal 24×24 matrices:

$$G = \{\mathrm{diag}(A_{i_1}, \ldots, A_{i_{12}}) \ : \ \text{subscripts in } \mathcal{C}_{12}\}$$

G has a global action on X_W restricting locally to that described above.

$$Y_N = \frac{X(\Delta_\mathrm{W}, 12\mathrm{A}_2)}{G} = \frac{\overline{\mathbb{C}\mathrm{P}^2} \times \cdots \times \overline{\mathbb{C}\mathrm{P}^2}}{G}$$

LEMMA 55. $\mathrm{Aut}(N, \Delta_\mathrm{W}) = W(12\mathrm{A}_2) \rtimes 2.\mathrm{M}_{12} = S_3^{12} \rtimes 2.\mathrm{M}_{12}$

PROOF. As for $24A_1$, a transformation of (Δ_W, N) is determined by its restriction to the roots (generating $12A_2$). So $\mathrm{Aut}(N, \Delta_W) < \mathrm{Aut}(12A_2, \Delta_W)$ is those linear maps also preserving N. The Weyl group preserves N. The subgroup of $2^{12}.S_{12}$ preserving the Niemeier lattice is precisely the group $G_1.G_2$ of automorphisms of N described in ch.1 §1.4 and isomorphic to $2.M_{12}$. The cyclic group consists in swapping the glue vectors [1] and [2] in each component and the Mathieu group is the induced permutations of the components. □

Again this variety can be resolved by Brylinski into a manifold X_N admitting a torus equivariant action of $S_3^{12} \rtimes 2.M_{12}$.

5.3. The Niemeier lattice $(6D_4)^+$**.** The toric variety $X = X(\Delta_W, D_4)$ was discussed in §3.2.

$$X_W := X(\Delta_W, 6D_4) = X \times \cdots \times X \quad \text{(6 times)}$$

By proposition 52 and theorem 53,

$$\mathrm{Aut}(6D_4, \Delta_W) = W(6D_4) \rtimes S_3^6.S_6$$

and

$$\begin{aligned} \mathrm{Aut}(X_W) &= T_{6D_4} \times (W(6D_4) \rtimes S_3^6.S_6) \\ &\simeq T_{6D_4} \times (2^3.S_4)^6 \rtimes S_3^6.S_6. \end{aligned}$$

Here $S_3^6.S_6$ is the symmetry group of the Dynkin diagram of $6D_4$.

Let $N = (6D_4)^+$ be the associated Niemeier lattice. Δ_W is a fan in N and $Y_N = X(\Delta_W, N)$ is an orbifold with 192^6 singularities of type $\mathbb{C}^{24}/G$ where $G < GL(24, \mathbb{C})$ is isomorphic to the group $C_6 \simeq 2^6$ $(= 2^3 \times 2^3)$. Let

$$A_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Then G is the group of 24×24 matrices formed of block components A_i down the diagonal:

$$G = \{\mathrm{diag}(A_{i_1}, \dots, A_{i_6}) : \text{subscripts in } \mathcal{C}_6\}$$

G acts on X_{W} and

$$Y_N = X_{\mathrm{W}}/G = (X \times \cdots \times X)/G$$

LEMMA 56. $\mathrm{Aut}(N, \Delta_{\mathrm{W}}) = \mathrm{W}(6\mathrm{D}_4) \rtimes 3.S_6$

PROOF. Similar to that of lemma 55. □

Resolving equivariantly we obtain a manifold X_N admitting a holomorphic torus-equivariant action of $\mathrm{Aut}(N, \Delta_{\mathrm{W}})$

Summing-up

Certainly the main achievement of this work is in bringing together several areas to study these manifolds and their properties. In other words examining generalised Kummer manifolds and toric varieties from the viewpoint of the lattice, and determining how well-known deep properties of the lattices are reflected in the associated spaces. The world of lattice theorists/finite simple group theorists seems to have remained surprisingly disjoint from that of the algebraic geometers working with toric or Kummer varieties (the exceptions appearing mainly to be physicists). As a consequence this thesis certainly poses as many questions as it answers and leaves open several promising lines of research, which we now discuss.

Recall that to a complex algebraic curve of genus g one can associate an abelian variety of (real) dimension $2g$ called the jacobian of the curve. It would be interesting to determine if for example the abelian variety $\mathbb{C}^{12}/\Lambda_{24}$ was a jacobian and if so of which curve. One could then ask questions about the curve itself, such as determining its automorphism group. Can lattice theory somehow be reflected in curves via this connection ? For curves over finite fields some linking constructions do exist [**49**].

The most significant generalised Kummer manifold constructed is undoubtedly $X_{\Lambda_{24}}$ (ch.3) admitting an action of the Suzuki group. The possibly difficult problem left open in chapters 2 and 3 is establishing the entire automorphism group of the manifolds involved. In the example above, does in fact $\mathrm{Aut}(X_{\Lambda_{24}}) = \mathrm{Suz}$? In general the groups obtained are certainly all the distance preserving transformations inherited from the torus, but the question remains open.

The toric constructions have proved more successful in this sense, where Demazure's theorem allows one to determine the exact automorphism group, at least of a non-singular variety. It is of no use for the singular spaces associated to the Niemeier lattices (Niemeier spaces) but would pin down the symmetries of any resolution of these. The Niemeier spaces are pleasing by the simplicity of the construction and the remark on bringing together two separate areas is no more true than in this context. An understanding of the lattices and toric geometry leads one naturally to consider them. As the next obvious step one asks if the Leech lattice gives rise to some toric object. It seems the right framework for the answer could be described as toric schemes. These arise from similar toric decompositions of Lorentzian space this time with cones in a Lorentzian lattice (not discussed in this work). The Leech lattice appears as simple system of the reflections in roots of the (even unimodular) Lorentzian lattice $II_{25,1}$ (see f.eg. Borcherds [**6**]).

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