A stably finite analogue of the Cuntz algebra $\mathcal{O}_2$

by

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Abstract

The Elliott Programme seeks classification of simple, separable, nuclear C^*-algebras via a functor based on K-theory. There are a handful of C^*-algebras, including the Cuntz algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$, that play particularly important roles in the programme. It is principally in this context that the Jiang-Su algebra $\mathcal{Z}$ is regarded as an analogue of $\mathcal{O}_\infty$, and this thesis proposes an analogue of $\mathcal{O}_2$ in a similar fashion.

More specifically, we construct a simple, nuclear, stably projectionless C^*-algebra W which has trivial K-theory and a unique tracial state, and we prove that W shares some of the properties of the C^*-algebras named above. In particular, we show that every trace-preserving endomorphism of W is approximately inner, and that W admits a trace-preserving embedding into the central sequences algebra $M(W) \cap W'$. While we do not quite prove that $W \otimes W \cong W$, we show how this can be deduced from a conjectured generalization of an existing classification theorem. Assuming this conjecture, we also show that W is absorbed tensorially by a large class of C^*-algebras with trivial K-theory. Finally, we provide presentations of both $\mathcal{Z}$ and W as universal C^*-algebras, leading us to suggest that, in addition to its position as a stably finite analogue of $\mathcal{O}_2$, W may be also thought of, both intrinsically and extrinsically, as a stably projectionless analogue of $\mathcal{Z}$.

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Statement

This thesis is submitted in accordance with the regulations for the degree of Doctor of Philosophy at the University of Glasgow. No part of this thesis has previously been submitted by me for a degree at this or any other institution.

The suggested proof of Conjecture 4.5.2 is due to Luis Santiago. (The argument I offered in an earlier version of this thesis was incorrect.) Chapter 5 contains joint work with Wilhelm Winter.

Contents

Chapter 1

Introduction

1.1 The Elliott Programme

In the study of operator algebras, classification has always been a central theme. The classification of nuclear C^*-algebras was initiated by Elliott, who used ordered K-theory to classify approximately finite dimensional (AF) C^*-algebras in [Ell76b] (building on earlier work of Glimm [Gli60] and Bratteli [Bra72]) and approximately circle (A$\mathbb{T}$) algebras of real rank zero in [Ell93]. Around 1990, Elliott conjectured that the class of all separable, nuclear C^*-algebras could be classified by an invariant $\mathrm{Ell}(\cdot)$ based on K-theory.

Conjecture 1.1.1 (Elliott)**.** *Let A and B be simple, separable, nuclear C^*-algebras. Then $A \cong B$ if and only if $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$.*

Augmenting the invariant to include tracial data (and the natural pairing between traces and K-theory), this has come to be known as the Elliott Conjecture, and the project to establish its veracity, the Elliott Programme. The survey article [ET08] gives an excellent overview of the history and recent developments of the Programme, and a more detailed exposition can be found in Rørdam's monograph [Rør02].

1.2 The invariant

For the sake of completeness, we will explain those terms used in the previous section (perhaps apart from 'simple', 'separable' and 'C^*-algebra') that appear later on.

Definition 1.2.1 (Nuclearity)**.** A C^*-algebra A is *nuclear* if it has the completely positive approximation property: for every finite subset $F \subset A$ and every $\epsilon > 0$, there exists a finite dimensional C^*-algebra B and completely positive contractive (c.p.c.) maps $\psi : A \to B$ and $\varphi : B \to A$ such that $\|\varphi \circ \psi(a) - a\| < \epsilon$ for every $a \in F$. Equivalently (see [CE78]), A is nuclear if, for every C^*-algebra B, there is a unique C^*-norm on the algebraic tensor product $A \odot B$.

It is well known that every nuclear C^*-algebra B is *exact*: if $0 \to J \to A \to A/J \to 0$ is a short exact sequence of C^*-algebras, then the natural sequence

$$0 \longrightarrow J \otimes B \longrightarrow A \otimes B \longrightarrow (A/J) \otimes B \longrightarrow 0$$

is also exact. Here (and throughout the thesis), $\otimes$ denotes the minimal tensor product. Where there is a need for emphasis we will write $\otimes_{min}$ for the minimal tensor product and $A \otimes_\alpha B$ for the completion of $A \odot B$ with respect to a C^*-norm $\|\cdot\|_\alpha$. We will occasionally make use of a theorem of Takesaki (see [Tak79, Corollary IV.4.21]): the tensor product $A \otimes_\alpha B$ of C^*-algebras A and B is simple if and only if both A and B are simple and $\|\cdot\|_\alpha$ is the minimal norm.

Definition 1.2.2 (Classification)**.** A *classification* of a category $\mathcal{C}$ consists of a category $\mathcal{C}'$ and a functor $F : \mathcal{C} \to \mathcal{C}'$ such that every isomorphism $\varphi : F(A) \to F(B)$ lifts to an isomorphism $\Phi : A \to B$ with $F(\Phi) = \varphi$.

One can make stronger demands on the functor F (which we will call an *invariant*), for example requiring the existence of lifts of arbitrary morphisms, or perhaps that lifts should be essentially unique. Moreover, one clearly wants the category $\mathcal{C}'$ to be 'less complicated' than the category $\mathcal{C}$ and the functor F to be well-behaved and computable. These criteria are all met by ordered K-theory (but considerably less so by tracial data).

Definition 1.2.3 (K-theory)**.** Let A be a unital C^*-algebra, and denote by $\mathcal{K}$ the algebra of compact operators on a separable Hilbert space. Recall that projections $p, q \in A \otimes \mathcal{K}$ are *Murray-von Neumann equivalent*, $p \sim q$, if $p = v^*v$ and $q = vv^*$ for some $v \in A \otimes \mathcal{K}$. Define $V(A) := \{\text{projections in } A \otimes \mathcal{K}\}/\sim$ and denote the equivalence class of p by $[p]$. Then $V(A)$ is a semigroup under the addition $[p] + [q] = \left[\left(\begin{smallmatrix} p & 0 \\ 0 & q \end{smallmatrix}\right)\right]$. The group $K_0(A)$ is defined to be the Grothendieck group of $V(A)$, while $K_0(A)_+$ is the image of $V(A)$ under this construction. Then $\big(K_0(A), K_0(A)_+, [1_A]\big)$ is a pre-ordered, pointed abelian group.

The group $K_1(A)$ is defined similarly, but with unitaries instead of projections. Let $U(A)$ denote the unitary group of A, and $U_0(A)$ the connected component of 1_A. Then

$K_1(A)$ is defined to be the abelian group

$$K_1(A) = \varinjlim (U(M_n(A))/U_0(M_n(A)), \varphi_n),$$

where $\varphi_n : U(M_n(A))/U_0(M_n(A)) \to U(M_{n+1}(A))/U_0(M_{n+1}(A))$ comes from the inclusion of $U(M_n(A))$ into $U(M_{n+1}(A))$ via the map $a \mapsto \left(\begin{smallmatrix} a & 0 \\ 0 & 1_A \end{smallmatrix}\right)$.

The invariant so far is

$$\mathrm{Ell}(A) = \left((K_0(A), K_0(A)_+, [1_A]), K_1(A) \right). \tag{1.1}$$

It becomes more or less complicated depending on whether or not traces need to be considered.

Definition 1.2.4 (Traces)**.** A *trace* on a unital C^*-algebra A is a unital linear map $\tau : A \to \mathbb{C}$ that is positive ($\tau(x^*x) \geq 0$ for $x \in A$) and tracial ($\tau(xy) = \tau(yx)$ for $x, y \in A$). We will denote by $T(A)$ the simplex of traces on A. Nonunital C^*-algebras may have unbounded traces, and we will denote the collection of these by T^+A; see Chapter 3 for a more detailed discussion.

Every trace $\tau \in T(A)$ gives rise to a well-defined *state* on $K_0(A)$ via $\tau_*([p]) = \tau(p)$. (That is, τ_* is a group homomorphism $K_0(A) \to \mathbb{R}$ with $\tau_*(K_0(A)_+) \subset \mathbb{R}_+$ and $\tau_*([1_A]) = 1$.) Hence there is a continuous affine map $r_A : T(A) \to S(K_0(A))$, which by theorems of Blackadar-Rørdam and Haagerup (see [Rør02, Theorem 1.1.11]) is surjective whenever A is unital and nuclear. In full generality, the invariant becomes

$$\mathrm{Ell}(A) = \Big((K_0(A), K_0(A)_+, [1_A]), K_1(A), T(A), r_A : T(A) \to S(K_0(A)) \Big). \tag{1.2}$$

In many important cases, the tracial part of the invariant disappears. For example, if projections (in $A \otimes \mathcal{K}$) separate traces (on A), (e.g. if A has real rank zero or if A is monotracial), then r_A is trivially injective, and hence is an affine homeomorphism. Thus all the tracial data is already contained in ordered K-theory. However, it follows respectively from work of Goodearl [Goo92] and Elliott [Ell96] that both $T(A)$ and r_A are necessary in general.

The invariant simplifies even further if A is an 'infinite' C^*-algebra.

Definition 1.2.5. A projection p in a C^*-algebra is said to be *infinite* it it is equivalent to a proper sub-projection of itself. Otherwise, p is called *finite*. A C^*-algebra A is *infinite* if A contains an infinite projection, is *finite* if all of its projections are finite, is *stably finite* if $A \otimes \mathcal{K}$ is finite, and is *stably projectionless* if $A \otimes \mathcal{K}$ contains no nonzero projections (equivalently, if $M_n(A)$ contains no nonzero projections for every n). Following [Rør02], let us say that a simple C^*-algebra is of type

(F0) if A is stably projectionless;

(F1) if A is stably finite and not stably projectionless;

(Inf) if $A \otimes \mathcal{K}$ is infinite.

Proposition 1.2.6 ([Rør02] Proposition 2.2.2)**.** *There are three disjoint and exhaustive possibilities for a simple, nuclear C^*-algebra A.*

(i) If A is of type (F0) then $K_0(A)_+ = 0$ and $T^+A \neq \{0\}$.

(ii) If A is of type (F1) then $K_0(A)_+ \cap -K_0(A)_+ = \{0\}$, $K_0(A)_+ - K_0(A)_+ = K_0(A) \neq \{0\}$ (i.e. $(K_0(A), K_0(A)_+)$ is an ordered *abelian group) and $T^+A \neq \{0\}$.*

(iii) If A is of type (Inf) then $K_0(A)_+ = K_0(A)$ (i.e. the order structure disappears) and $T^+A = \{0\}$.

Remark 1.2.7. (i) Of particular interest among (simple, separable, nuclear) infinite C^*-algebras are the *purely infinite* ones, i.e. those C^*-algebras for which every nonzero hereditary subalgebra contains an infinite projection. The classification (modulo the UCT) of these algebras by Kirchberg and Phillips is one of the crowning achievements of the classification programme (see [Kir], [Rør02, Theorem 8.4.1] and [Rør95]).

(ii) Particularly noteworthy examples of classification within (F1) are (in increasing order of generality): Elliott's classification [Ell93] of $A\mathbb{T}$-algebras of real rank zero; Dadarlat-Elliott-Gong classification [DG97] of simple, unital AH-algebras of slow dimension growth and real rank zero; and Elliott-Gong-Li classification [EGL07] of simple, unital AH-algebras of bounded dimension.

(iii) The invariant for simple, nonunital C^*-algebras is slightly more subtle, but one often does not lose much generality by only considering the unital case. This is because, by Brown's Theorem [Bro77], if $A \otimes \mathcal{K}$ contains a nonzero projection p, then A is stably isomorphic to the unital C^*-algebra $p(A \otimes \mathcal{K})p$. It is really only the stably projectionless case that is genuinely different in character.

Finally, we should probably comment on the UCT, which almost always appears (even if implicitly) in the preamble of classification theorems.

Definition 1.2.8 (UCT)**.** For separable C^*-algebras A and B, there is a sequence

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B)) \xrightarrow{\delta} KK_*(A,B) \xrightarrow{\gamma} \mathrm{Hom}(K_*(A), K_*(B)) \longrightarrow 0$$

in the category of ordered abelian groups, where $KK(\cdot,\cdot)$ is Kasparov's bivariant K-theory, γ has degree 0 and δ has degree 1. One says that A satisfies the *Universal Coefficient Theorem* (UCT) if this is a short exact sequence for every B.

It is shown in [RS87] that the UCT holds for a large class $\mathcal{N}$ of C^*-algebras, called the *bootstrap class*, which consists of all C^*-algebras 'KK-equivalent' to an abelian C^*-algebra, and which is closed under extensions, countable inductive limits, tensor products and crossed products by $\mathbb{Z}$ and $\mathbb{R}$. The reason for the omnipresence of the UCT as a background assumption is that for such C^*-algebras, KK-equivalence is the same as isomorphism of K-groups. A major open question is whether every nuclear C^*-algebra satisfies the UCT, but every C^*-algebra that we consider in this thesis certainly does.

1.3 Strongly self-absorbing C^*-algebras

Today, it is known that the Elliott Conjecture does not hold in full generality. The ideas of [Vil98] were used by Rørdam [Rør03] and Toms [Tom08] to construct simple, separable, nuclear non-isomorphic C^*-algebras with the same Elliott invariant. These counterexamples to the conjecture can be dealt with in two ways:

(i) enlarge the invariant to include the Cuntz semigroup (which is sensitive enough to be able to distinguish between the counterexamples of Rørdam and Toms, and, for well-behaved C^*-algebras, can be recovered functorially from K-theory and traces — see [BPT08]), or

(ii) impose further regularity conditions on the C^*-algebras to be classified.

This thesis is in the spirit of option (ii), and the relevant regularity property is $\mathcal{Z}$-stability.

The Jiang-Su algebra $\mathcal{Z}$ is a simple, separable, nuclear, infinite dimensional, projectionless C^*-algebra which has the same Elliott invariant as $\mathbb{C}$ (see [JS99] and also [RW10] for some alternative descriptions of $\mathcal{Z}$). A C^*-algebra A is '$\mathcal{Z}$-stable' if $A \otimes \mathcal{Z} \cong A$; the counterexamples of Rørdam and Toms each involve a pair of non-isomorphic C^*-algebras which have the same Elliott invariant, but one of which is not $\mathcal{Z}$-stable. Thus the class of $\mathcal{Z}$-stable C^*-algebras is the largest class in which the Elliott Conjecture can be expected to hold.

The term 'strongly self-absorbing' was coined by Toms and Winter in [TW07] to describe a handful of algebras which, like $\mathcal{Z}$, have played pivotal roles in the classification programme.

Definition 1.3.1 (Toms-Winter [TW07])**.** A separable unital C^*-algebra $\mathcal{D} \neq \mathbb{C}$ is *strongly self-absorbing* if there is an isomorphism $\varphi : \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$ such that φ is approximately unitarily equivalent to the first factor embedding $\mathrm{id}_{\mathcal{D}} \otimes 1_{\mathcal{D}}$ (written $\varphi \sim_{a.u.} \mathrm{id}_{\mathcal{D}} \otimes 1_{\mathcal{D}}$), which means that there is a sequence of unitaries $(u_n)_{n=1}^{\infty}$ in $\mathcal{D} \otimes \mathcal{D}$ such that $\varphi(a) = \lim_{n\to\infty} u_n(a \otimes 1)u_n^*$ for every $a \in \mathcal{D}$. As for $\mathcal{Z}$, a C^*-algebra A is '$\mathcal{D}$-stable' (or 'absorbs $\mathcal{D}$') if $A \otimes \mathcal{D} \cong A$.

If A is strongly self-absorbing then A is simple and nuclear, and is either purely infinite or stably finite with a unique trace (see [TW07, §1]). Moreover, the list of known strongly self-absorbing algebras is short (and closed under $\otimes$): the Cuntz algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$, UHF algebras of infinite type M_{p^∞} (such as the CAR algebra M_{2^∞}), $\mathcal{O}_\infty \otimes M_{p^\infty}$, and the Jiang-Su algebra $\mathcal{Z}$. One of the reasons that these algebras are important is that they provide localized versions of the Elliott Conjecture in the sense of [Win07], i.e. theorems of the form $\mathrm{Ell}(A) \cong \mathrm{Ell}(B) \implies A \otimes \mathcal{D} \cong B \otimes \mathcal{D}$. For example, the classification of Kirchberg algebras ([Kir], [Phi00]) can be interpreted as the classification up to $\mathcal{O}_\infty$-stability of all simple, separable, unital, nuclear C^*-algebras that satisfy the UCT. At the other end of the spectrum, Winter proves in [Win07] and [Win10] that the Elliott Conjecture holds for a large class of stably finite $\mathcal{Z}$-stable C^*-algebras (which have finite decomposition rank, as defined in [KW04]).

The known strongly self-absorbing algebras form a hierarchy, with $\mathcal{Z}$ at the bottom (every strongly self-absorbing algebra is $\mathcal{Z}$-stable — see [Win09]) and $\mathcal{O}_2$ at the top ($\mathcal{O}_2$ absorbs every strongly self-absorbing algebra — see [KP00]).

Stably finite	**Purely infinite**
?	$\mathcal{O}_2$
M_{p^∞}	$\mathcal{O}_\infty \otimes M_{p^\infty}$
$\mathcal{Z}$	$\mathcal{O}_\infty$

Every purely infinite algebra in this hierarchy has a stably finite analogue (most notably, $\mathcal{Z}$ corresponds to $\mathcal{O}_\infty$), except for $\mathcal{O}_2$. That is, there is no stably finite strongly self-absorbing C^*-algebra A with $K_*(A) = 0$. In fact, it is not hard to see that if A is separable and stably finite with $K_0(A) = 0$ then $A \otimes \mathcal{K}$ cannot have any full projections; in particular, A must be nonunital (in fact, stably projectionless if A is also simple). This explains the gap since the definition of 'strongly self-absorbing' conspicuously involves a unit. What is therefore needed to fill this gap is a well-behaved notion of 'strongly self-absorbing' that makes sense for nonunital C^*-algebras and agrees with the existing definition for unital algebras.

There are a few equivalent characterizations of strongly self-absorbing C^*-algebras, some more amenable than others to a sensible interpretation in the nonunital case. The above definition is obviously troublesome and highlights a general problem: If A is a nonunital C^*-algebra without projections then there are no obvious *-homomorphisms from A to $A \otimes A$; in particular, there is no obvious way of making sense of an infinite tensor product $A^{\otimes\infty}$. On the other hand, if $A \neq \mathbb{C}$ is separable and unital, then A is strongly self-absorbing if and only if (with some redundancy)

(i) $A \cong A \otimes A$;

(ii) every unital endomorphism of A is approximately inner;

(iii) A admits an asymptotically central sequence of unital endomorphisms (see section 4.6 of Chapter 4).

These conditions also make sense if A is nonunital (provided that we replace 'unital' by 'nondegenerate' where appropriate) and we could think of taking some subset of these as a general definition of strongly self-absorbing. It is not yet clear how well-behaved such a definition would be, but the goal of this thesis is to construct a stably projectionless C^*-algebra W with trivial K-theory and a unique tracial state, and at least see how far these properties hold for W. This is the content of Chapter 4, with Chapter 3 serving as a collection of auxiliary technical results. The suggested proof of Conjecture 4.5.2 ($W \otimes W \cong W$) depends on some nontrivial properties of noncommutative CW complexes, which are outlined in Chapter 2. (The suggested proof is due to Luis Santiago.) Chapter 4 also contains a proof that among the C^*-algebras classified in [Rob10], those that are simple and have trivial K_0-group are all of the form $A \otimes W$ for some AF algebra A. (We accomplish this by first extending a theorem of Blackadar [Bla80] and Goodearl [Goo78] about traces on AF algebras.) It would be unsurprising if in fact W-stability characterizes those stably projectionless algebras for which the Elliott Conjecture holds.

Finally, in Chapter 5 we use 'order zero' maps to show that W and $\mathcal{Z}$ can be presented as universal C^*-algebras in strikingly similar ways. In light of this and our earlier results, it seems reasonable to think of W, both structurally and functionally, as a stably projectionless analogue of $\mathcal{Z}$.

Chapter 2

Noncommutative CW complexes

In this chapter, we recall the definition of a noncommutative CW (NCCW) complex, focusing on those of dimension at most two.

2.1 Definition and notation

Definition 2.1.1 ([ELP98], [Ped99])**.** As in topology, the definition of an NCCW complex is by induction on the dimension of the underlying space. A *zero dimensional NCCW complex* is just a finite dimensional C^*-algebra. For $k \geq 1$, a *k-dimensional NCCW complex* A_k is a C^*-algebra which can be written as a pullback of the form

$$\begin{array}{ccc} A_k & \longrightarrow & A_{k-1} \\ \downarrow & & \downarrow \sigma_k \\ C([0,1]^k, F_k) & \xrightarrow[\partial]{} & C(S^{k-1}, F_k) \end{array} \tag{2.1}$$

where F_k is a finite dimensional C^*-algebra, S^{k-1} denotes the boundary of $[0,1]^k$ (with ∂ the natural restriction map), A_{k-1} is a $(k-1)$-dimensional NCCW complex and σ_k is any *-homomorphism. Given such a decomposition (which need not be unique), we will write A_k as the restricted direct sum $A_k = C([0,1]^k, F_k) \oplus_{C(S^{k-1},F_k)} A_{k-1}$.

Remark 2.1.2 ([ELP99a])**.** A NCCW complex A_k as in (2.1) is unital if and only if A_{k-1} is unital and so is the connecting *-homomorphism σ_k. In this case, A_k is an example of a *recursive subhomogeneous C^*-algebra* ; these are analogously constructed C^*-algebras that allow for more general topological spaces than CW complexes — see [Phi07b]. On the other hand, if A_k is nonunital then, by appropriately unitizing at each stage, one can write the unitization $\widetilde{A_k}$ as an NCCW complex, written as an iterated pullback consisting entirely of unital maps and C^*-algebras. In this way, one can deduce that the results of [ELP98] hold for NCCW complexes in general, not just unital ones (as assumed therein).

2.2 Dimension one

One-dimensional NCCW complexes may perhaps be thought of as noncommutative analogues of topological graphs (this is made precise in [ELP98, Lemma 3.1.1]), and they form an important source of building blocks for constructing more complicated C^*-algebras. Examples include the dimension drop algebras of [JS99] (described in Chapter 5) and the C^*-algebras considered in [Raz02] (which will be used to construct the stably projectionless C^*-algebra W in Chapter 4). These latter building blocks are pullbacks of the form

$$\begin{array}{ccc} A & \longrightarrow & M_n \\ \downarrow & & \downarrow{\scriptstyle \varphi_0 \oplus \varphi_1} \\ C([0,1], M_{n'}) & \xrightarrow[\mathrm{ev}_0 \oplus \mathrm{ev}_1]{} & M_{n'} \oplus M_{n'} \end{array} \tag{2.2}$$

(i.e. the zero-dimensional NCCW complex involved is a single matrix algebra). One can show that a C^*-algebra of this form has trivial K_0 when φ_0 and φ_1 have different multiplicities, and has trivial K_1 precisely when this difference is 1 (see for example [Rob10, Lemma 4]). Inductive limits of one-dimensional NCCW complexes with trivial K_1-groups have been classified by Leonel Robert in [Rob10] (see Theorem 4.1.2 below for a statement of this result in the simple case).

While every NCCW complex is finitely generated ([ELP98, Lemma 2.4.3]), one-dimensional NCCW complexes are particularly tractable because they are semiprojective, hence can be finitely presented by stable relations (see Theorem 6.2.2 and Proposition 2.2.11 of [ELP98]). For the purposes of this thesis (in particular, in the proof of Lemma 4.4.1 and in the suggested proof of Conjecture 4.5.2), it is enough for us to know that this implies the following property (see [Lor93, Lemma 3.7]).

Proposition 2.2.1 (Eilers-Loring-Pedersen)**.** *Let B be a one-dimensional NCCW complex, $C = \overline{\bigcup_{i=1}^{\infty} C_i}$ an increasing union of C^*-algebras and $\theta : B \to C$ a *-homomorphism. Fix a finite subset $F \subset B$ and a tolerance $\epsilon > 0$. Then there exist $k \in \mathbb{N}$ and a *-homomorphism $\psi_k : B \to C_k$ such that $\|\theta(f) - \psi_k(f)\| < \epsilon$ for every $f \in F$.*

2.3 Dimension two

Let us show that if A is a one-dimensional NCCW complex of the form (2.2) then $A \otimes A$ is an NCCW complex of dimension two. (In fact, it can be shown that if A and B are NCCW complexes of dimensions m and n then $A \otimes B$ is an NCCW complex of dimension $m + n$ — see [Ped99, Theorem 11.14].) The following Lemma will be useful for this; it is straightforward to prove and can be phrased more generally — see [Ped99, Proposition 3.1].

Lemma 2.3.1. *A commutative diagram of C^*-algebras*

$$\begin{array}{ccc} X & \xrightarrow{\gamma} & B \\ \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\beta} \\ A & \xrightarrow{\alpha} & C \end{array} \tag{2.3}$$

with surjective horizontal maps and injective vertical maps is a pullback if and only if $\delta(\ker \gamma) = \ker \alpha$.

For convenience, and since this is our primary case of interest, let us assume that φ_1 is unital. Since $M_n(B)$ is an NCCW complex (of the same dimension) whenever B is, we may further assume that $n = 1$. Then, for some natural numbers $m \leq k$, A is isomorphic to the C^*-algebra

$$\{f \in C([0,1], M_k) : f(0) = z1_m,\ f(1) = z1_k,\ z \in \mathbb{C}\}.$$

Let χ be the character of A corresponding to evaluation at the endpoints. Identify $C([0,1], M_k) \otimes C([0,1], M_k)$ with $C([0,1]^2, M_{k^2})$ (via $\iota(f \otimes g)(s,t) = f(s) \otimes g(t) \in M_k \otimes M_k \cong M_{k^2}$), and regard $A \otimes A$ as a subalgebra of $C([0,1]^2, M_{k^2})$, specified by boundary conditions on S^1. Since there is an obvious commutative diagram

$$\begin{array}{ccc} A \otimes A & \twoheadrightarrow & (A \otimes A)|_{S^1} \\ \downarrow & & \downarrow \\ C([0,1]^2, M_{k^2}) & \twoheadrightarrow & C(S^1, M_{k^2}) \end{array}$$

which by Lemma 2.3.1 is a pullback, it suffices to show that the restriction $(A \otimes A)|_{S^1}$ of $A \otimes A$ to the boundary S^1 is a one-dimensional NCCW complex. But this fits into the commutative diagram

$$\begin{array}{ccc} (A \otimes A)|_{S^1} & \xrightarrow{\gamma = \chi \otimes \chi} & \mathbb{C} \\ \downarrow{\scriptstyle\delta = (\mathrm{id} \otimes \chi) \oplus (\chi \otimes \mathrm{id})} & & \downarrow{\scriptstyle\beta} \\ C([0,1], M_k) \oplus C([0,1], M_k) & \xrightarrow{\alpha} & M_k \oplus M_k \oplus M_k \oplus M_k \end{array}$$

where α is evaluation at the endpoints and β has multiplicity (m, k, m, k). Moreover, if $(f, g) \in \ker\alpha = C_0((0,1), M_k) \oplus C_0((0,1), M_k)$ and we let $h \in A$ be the element $h(t) = 1_m \oplus t1_{k-m}$ then $F := f \otimes h + h \otimes g$ is in $\ker\gamma$ and $\delta(F) = (f, g)$. Thus $\ker\alpha = \delta(\ker\gamma)$, so this diagram is a pullback, and hence $(A \otimes A)|_{S^1}$ is a one-dimensional NCCW complex, as required.

This means that, once we have constructed W as an appropriate inductive limit of one-dimensional NCCW complexes, we may regard $W \otimes W$ as an inductive limit of NCCW complexes of dimension two. The key ingredient in the suggested proof of Conjecture 4.5.2 ($W \otimes W \cong W$) is then the following (which is a special case of [ELP98, Corollary 8.1.5]).

Proposition 2.3.2 (Eilers-Loring-Pedersen)**.** *Let A be a two-dimensional NCCW complex with trivial K-theory, C a C^*-algebra of stable rank one and $\varphi : A \to C$ a *-homomorphism. Fix a finite subset $F \subset A$ and a tolerance $\epsilon > 0$. Then there is a one-dimensional NCCW complex B, and *-homomorphisms $\alpha : A \to B$, $\beta : B \to C$ such that $\|\varphi(a) - \beta \circ \alpha(a)\| < \epsilon$ for every $a \in F$.*

(Recall that a unital C^*-algebra has *stable rank one* if it is the closure of its group of invertible elements, and a nonunital C^*-algebra is said to have stable rank one if its unitization does. Examples include $C(X)$ for X a compact metric space of covering dimension one and (inductive limits of) one-dimensional NCCW complexes.)

Remark 2.3.3. Unfortunately for us, the one-dimensional NCCW complex B given by Proposition 2.3.2 in general has nontrivial K_1-group. It can be seen from the proof of [ELP98, Corollary 8.1.5] (see also [ELP99b, §6.3]) that if $A \subset C([0,1]^2, M_n)$ is a two-dimensional NCCW complex, then B is the restriction of A to the grid

$$\Gamma_k = [0,1]^2 \cap \left\{ (x, y) \in \mathbb{R}^2 : x \in \frac{1}{k}\mathbb{Z} \text{ or } y \in \frac{1}{k}\mathbb{Z} \right\}$$

for some sufficiently large $k \in \mathbb{N}$. Equivalently, B is the quotient of A by the ideal of functions vanishing on Γ_k, and then one easily sees from the six-term exact sequence of K-theory that, if $K_*(A) = 0$, then $K_0(B) = 0$, but $K_1(B) = \mathbb{Z}^{k^2}$. This means that we cannot quite appeal to Theorem 4.1.2 to prove Conjecture 4.5.2. We will comment further on this in Chapter 4.

Chapter 3

Traces and convexity

One of the major technical hurdles associated to nonunital C^*-algebras is that they can have unbounded traces. In this chapter, we collect various facts about such traces that will be used for the remainder of the thesis. The right category in which to work is that of topological convex cones, or, dually, that of order unit spaces, so we start with rapid introductions to these categories.

3.1 Ordered vector spaces

An *ordered vector space* is a real vector space E together with a partial ordering $\leq$ such that

(i) $x \leq y \implies x + z \leq y + z$ for every $x, y, z \in E$; and

(ii) $x \leq y \implies \alpha x \leq \alpha y$ for every $x, y \in E$ and $\alpha \geq 0$.

Let E be an ordered vector space. Then:

(i) E is called *Archimedean* if for every $a \in E$, if the set $\{\alpha a : \alpha \geq 0\}$ has an upper bound, then $a \leq 0$; and

(ii) an element $e \in E$ is an *order unit* if e generates E as an order ideal, i.e. for every $a \in E$, there exists some $n \in \mathbb{N}$ such that $-ne \leq a \leq ne$.

It is not difficult to show that an Archimedean ordered vector space E with an order unit e admits a norm

$$\|a\| := \inf\{\lambda > 0 : -\lambda e \leq a \leq \lambda e\} \tag{3.1}$$

which satisifes

$$-\|a\|e \leq a \leq \|a\|e \qquad \forall a \in E. \tag{3.2}$$

Such a space (E, e) is called an *order unit space*, and if the corresponding *order unit norm* given by (3.1) is complete, then it is a *complete order unit space.*

Example 3.1.1. If A is a unital C^*-algebra and A_{sa} denotes the set of self-adjoint elements of A, then $(A_{sa}, 1)$ is a complete order unit space. The corresponding order unit norm is of course the restriction to A_{sa} of the C^*-norm on A. In particular, if X is compact, then $(C_{\mathbb{R}}(X), 1)$ is a complete order unit space, and the corresponding order unit norm is just the sup norm. If X is also convex, then the space $\mathrm{Aff}(X) \subset C_{\mathbb{R}}(X)$ of continuous affine functionals on X (i.e. continuous maps $f : X \to \mathbb{R}$ with $f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda)f(y)$ for $x, y \in X$ and $0 \leq \lambda \leq 1$) is also a complete order unit space.

Just as for C^*-algebras, the order and norm of an order unit space mutually determine each other, and lend certain rigidity to the corresponding morphisms. Specifically, it is not hard to show the following (see [Alf71, Proposition II.1.3]).

Lemma 3.1.2. *Let $\psi : (E, e) \to (E', e')$ be a linear map between order unit spaces with $\psi(e) = e'$. Then ψ is positive (i.e. order-preserving) if and only if ψ is bounded with $\|\psi\| = 1$. In particular, a nonzero linear functional q on an order unit space (E, e) is positive if and only if it is bounded with $\|q\| = q(e)$.*

A linear functional q on an order unit space (E, e) is called a *state* if $q \geq 0$ and $q(e) = 1$; by Lemma 3.1.2, this is equivalent to $\|q\| = q(e) = 1$. The states of (E, e) form a w*-compact convex subset of the unit ball of E^*, called the *state space* $S = S(E, e)$ of (E, e). As with C^*-algebras, order unit spaces possess 'sufficiently many' states, and it follows from a theorem of Kadison (see [Alf71, Theorem II.1.8]) that if (E, e) is a *complete* order unit space and S is its state space, then E and $\mathrm{Aff}(S)$ are isomorphic as order unit spaces.

A *convex cone* is a subset $C \neq \{0\}$ of some vector space E that satisfies $\lambda x + \mu y \in C$ whenever $x, y \in C$ and $\lambda, \mu \geq 0$. We will call C a *topological convex cone* if the ambient vector space E is locally convex (Hausdorff). By mapping a convex cone C to the ordered vector space $C - C$, and an ordered vector space E to its positive cone E^+, there is a bijective correspondence between convex cones and (positively generated) ordered vector spaces.

In fact, this correspondence is an equivalence of categories. The morphisms between (topological) cones C_1 and C_2 are the (continuous) linear maps $f : C_1 \to C_2$, i.e. those that satisfy $f(\lambda x + \mu y) = \lambda f(x) + \mu f(y)$ for $\lambda, \mu \geq 0$ and $x, y \in C_1$. Each such f extends to a genuine positive linear map from the ordered vector space $C_1 - C_1$ to the ordered vector space $C_2 - C_2$.

Notation 3.1.3. When $C_2 = \mathbb{R}^+$, we will denote the ordered vector space of such maps by $\mathrm{Aff}_0(C_1)$.

A (nonempty) convex subset X of C is said to be a *base* of C if for every $y \in C$ there exist unique $x \in X$ and $\alpha \geq 0$ such that $y = \alpha x$. (This implies that $0 \notin X$.) If C has a base then C is proper: $C \cap -C = \{0\}$. Moreover, a cone C is locally compact if and only if it has a compact base X (this is a theorem of Klee — see [Alf71, Theorem II.2.6]) and in this case the restriction map $(\mathrm{Aff}_0(C), e_X) \to (\mathrm{Aff}(X), 1)$ is an isomorphism of complete order unit spaces. Here, the order unit of $\mathrm{Aff}_0(C)$ is the map $e_X(\lambda x) = \lambda$ for $x \in X$, $\lambda \geq 0$, and the corresponding order-unit norm is $\|f\|_X = \|f|_X\|_\infty$.

Unsurprisingly, there is a Banach-Stone theorem for locally compact convex cones, i.e. such a cone C is determined up to affine isomorphism by $\mathrm{Aff}_0 C$. This is claimed in [Pul07] to be true for *all* topological convex cones, but there seems to be an error in the proof. However, when C is locally compact the argument is much simpler. We include it here for completeness.

Lemma 3.1.4. *Let X be a compact convex subset of a locally convex (Hausdorff) space. Then the evaluation morphism $\delta : X \to S(\mathrm{Aff}(X), 1)$, $\delta_x(a) = a(x)$ is an affine homeomorphism.*

Proof. The map δ is affine (because each $a \in \mathrm{Aff}(X)$ is), is injective (since $\mathrm{Aff}(X)$ separates X) and is continuous (since each $a \in \mathrm{Aff}(X)$ is continuous and S has the w*-topology). It follows that $\delta(X)$ is convex and compact, and hence closed (since S is Hausdorff). Now suppose that there is some $x_0 \in S \backslash \delta(X)$. By Hahn-Banach, there exists a continuous linear functional φ on $(\mathrm{Aff}(X))^*$ such that $\varphi(x) \leq 1$ for every $x \in \delta(X)$ and $\varphi(x_0) > 1$. Since $(\mathrm{Aff}(X))^*$ has the w*-topology, there exists $a \in \mathrm{Aff}(X)$ such that $\varphi(\omega) = \omega(a)$ for every $\omega \in (\mathrm{Aff}(X))^*$, and so $a(x) = \delta_x(a) = \varphi(\delta_x) \leq 1$ for every $x \in X$. Thus $a \leq 1$, so $x_0(a) \leq 1$ (since x_0 is a state) and this contradicts $x_0(a) = \varphi(x_0) > 1$. □

Theorem 3.1.5 (Banach-Stone)**.** *Let X_1 and X_2 be compact convex subsets of some locally convex (Hausdorff) spaces. Then X_1 and X_2 are affinely homeomorphic if and only if $\mathrm{Aff}(X_1)$ and $\mathrm{Aff}(X_2)$ are isomorphic as order unit spaces. Moreover, every isomorphism*

$\theta : \mathrm{Aff}(X_1) \to \mathrm{Aff}(X_2)$ is of the form $\theta(a) = a \circ h$ for some affine homeomorphism $h : X_2 \to X_1$.

Proof. This is easy, given Lemma 3.1.4. An isomorphism $\theta : \mathrm{Aff}(X_1) \to \mathrm{Aff}(X_2)$ descends to an affine homeomorphism $\theta^* : S(\mathrm{Aff}(X_2)) \to S(\mathrm{Aff}(X_1))$ and hence to an affine homeomorphism $h = (\delta^{(1)})^{-1} \circ \theta^* \circ \delta^{(2)} : X_2 \to X_1$. □

The following is now immediate.

Corollary 3.1.6. *Let C_1 and C_2 be locally compact convex cones. Then C_1 and C_2 are isomorphic as cones if and only if there are compact bases X_1 of C_1 and X_2 of C_2 such that $(\mathrm{Aff}_0(C_1), e_{X_1})$ and $(\mathrm{Aff}_0(C_2), e_{X_2})$ are isomorphic as order unit spaces.* □

3.2 The Pedersen ideal

Theorem 5.6.1 of [Ped79] says that each C^*-algebra A has a dense hereditary ideal, called the *Pedersen ideal* $\mathrm{Ped}(A)$ of A, which is minimal among all dense ideals. The Pedersen ideal is the linear span of the set

$$\left\{ x \in A^+ : x \leq \sum_{k=1}^{n} f_k(x_k) \text{ for some } x_k \in A^+ \text{ and } f_k \in C_c(0,\infty)^+ \right\}, \tag{3.3}$$

and can also be described as follows.

Lemma 3.2.1. $\mathrm{Ped}(A)$ *is generated as a two-sided ideal by the set*

$$\{a \in A^+ : ea = a \text{ for some } e \in A^+\}.$$

Proof. The given set, B say, is dense in A^+ by functional calculus. By minimality, it therefore suffices to show that $B \subset \mathrm{Ped}(A)$. Let $a \in A^+$ such that $ea = a$ for some $e \in A^+$. Note that e and a commute; under the Gelfand isomorphism of $C^*(e, a)$ with $C_0(X)$ (for some appropriate X), let us identify e with the function g and a with the function h. Let $f \in C(0,\infty)$ be such that f is positive, $f(1) = \|a\|$ and f has compact support. Then $f(e)$ corresponds to the function $f \circ g$, and $g = 1$ on the support of h. Since $f \circ g = \|a\|$ on the support of h, we have $a \leq f(e)$. Thus $a \in \mathrm{Ped}(A)$, as required. □

Note that this shows immediately that $\mathrm{Ped}(A)$ contains all the projections of A, and that $\mathrm{Ped}(A) = A$ whenever A is unital.

3.3 The tracial cone

For a C^*-algebra A, let us say that a map $\tau : A^+ \to [0, \infty]$ is a *trace* if it is linear (i.e. $\tau(\lambda x + \mu y) = \lambda\tau(x) + \mu\tau(y)$ for $x, y \in A^+$ and $\lambda, \mu \geq 0$) and satisfies the trace identity $\tau(x^*x) = \tau(xx^*)$ for $x \in A$. Recall that τ is *lower semicontinuous* if $\tau^{-1}[0, r]$ is closed in A^+ for every $r \in [0, \infty]$, and τ is *densely finite* (or *densely defined*) if $\tau^{-1}[0, \infty)$ is dense in A^+. We will write T^+A for the set of lower semicontinuous densely finite traces on a C^*-algebra A.

Lemma 3.3.1. *For every C^*-algebra A, there is a bijective correspondence between T^+A and the cone of positive linear functionals on* $\mathrm{Ped}(A)$ *that satisfy the trace identity.*

Proof. Let us write $T(\mathrm{Ped}(A))$ for the set of positive, tracial linear functionals on $\mathrm{Ped}(A)$. For $a \in A^+$ and $\epsilon > 0$, we use the standard notation $(a - \epsilon)_+$ to denote the element of $C^*(a)$ corresponding under functional calculus to the function $f(t) = \max\{0, t - \epsilon\}$. Note that, by (3.3), we have $(a - \epsilon)_+ \in \mathrm{Ped}(A)$ for every $a \in A^+$.

Every $\tau \in T^+A$ extends to a positive linear functional on a dense hereditary *-ideal A^τ of A, and satisfies $\tau(xy) = \tau(yx)$ for every $x \in A$ and $y \in A^\tau$ (see Chapter 5 of [Ped79]). Since A^τ contains $\mathrm{Ped}(A)$, we see that τ restricts to an element of $T(\mathrm{Ped}(A))$. Moreover, by [ERS09, Lemma 3.1], lower semicontinuity of τ is equivalent to

$$\tau(a) = \sup_{\epsilon > 0} \tau((a - \epsilon)_+) \qquad \text{for every } a \in A^+. \tag{3.4}$$

Conversely, every $\tau \in T(\mathrm{Ped}(A))$ also satisfies (3.4), for $a \in \mathrm{Ped}(A)^+$; this is because, by [Ped79, Proposition 5.6.2], $C^*(a)$ is contained in $\mathrm{Ped}(A)$ whenever a is, and positive linear functionals on a C^*-algebra are automatically continuous (see [Dix77, 2.1.8]). Thus, (3.4) defines an extension of τ to a map $A^+ \to [0, \infty]$; it follows from [ERS09, Proposition 2.3] that this extension is linear and satisfies the trace identity, and from [ERS09, Lemma 3.1] that it is lower semicontinuous. Therefore, restriction and extension give inverse bijections between T^+A and $T(\mathrm{Ped}(A))$. □

We can therefore regard T^+A as a subset of the (algebraic) dual of $\mathrm{Ped}(A)$; equipped with the weak*-topology, T^+A is then a topological convex cone. If A is unital, then the set $T_1^+A = T(A) := \{\tau \in T^+A : \|\tau\| = 1\}$ of tracial states on A is a compact base for this cone. We now prove (with thanks to Leonel Robert) that T^+A has a compact base whenever A is simple.

Lemma 3.3.2. *If A is a simple C^*-algebra then every nonzero trace $\tau \in T^+A$ is faithful.*

Proof. Let $I = \{x \in A : \tau(x^*x) = \tau(xx^*) = 0\}$. Suppose that $x, z \in I$ and $a \in A$. Write $y = (xx^*)^{1/2}$. Since τ is linear and order-preserving, we see that τ is zero on polynomials in xx^* and hence, by lower semicontinuity, is zero on $C^*(xx^*)$. Therefore, $\tau(y)$ is also zero (in particular, finite), so we can freely use the trace identity to deduce that

$$\tau((ax)(ax)^*) = \tau(xx^*a^*a) = \tau(ya^*ay) \leq \tau(y\|a\|^2y) = \|a\|^2\tau(xx^*) = 0,$$

and hence that $ax \in I$. Also, $x + z \in I$ by the Cauchy-Schwarz inequality (see [Ped79, Lemma 5.1.2]), and clearly $x^* \in I$. Hence I is an ideal of A, so $I = 0$. □

Proposition 3.3.3. *Let A be a simple C^*-algebra. Then T^+A has a compact base.*

Proof. Fix some nonzero $e \in \mathrm{Ped}(A)^+$, and let $K := \{\tau \in T^+A : \tau(e) = 1\}$. By Lemma 3.3.2, every $0 \neq \tau \in T^+A$ is faithful, and so K is a convex base of T^+A. It remains to show that K is compact (in the weak*-topology induced by $\mathrm{Ped}(A)$). Let $a \in \mathrm{Ped}(A)^+$. I would like to think that since A is simple, we can find $x_1, \ldots, x_n \in A$ such that $a \leq \sum_{i=1}^n x_iex_i^*$ — Cuntz proves in [Cun77, Proposition 1.10] that when A is *unital* you can even get equality — but we don't actually need this.

First note that $\mathrm{Ped}(A)$ is algebraically simple: For $0 \neq b \in \mathrm{Ped}(A)$, the set $I := \{\sum_{i=1}^n x_iby_i : x_i, y_i \in \mathrm{Ped}(A)\}$ is a nonzero two-sided ideal of A. Since A is simple, we have $\overline{I} = A$, and since $\mathrm{Ped}(A)$ is the *minimal* dense ideal of A and $I \subset \mathrm{Ped}(A)$ we thus have $I = \mathrm{Ped}(A)$. Hence $\mathrm{Ped}(A)$ is simple.

It follows that we can find elements $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathrm{Ped}(A)$ such that $a = \sum_{i=1}^n x_iey_i^*$. By passing to $M_n(A)$ we may assume that $n = 1$, i.e if $e' := e \otimes 1_n$, $a' := a \otimes e_{11}$, $x := \sum x_i \otimes e_{1i}$ and $y := \sum y_i \otimes e_{1i}$ (all in $\mathrm{Ped}(M_n(A))$) then $a' = xe'y^*$. So let us assume that $a = xey^*$ — we do this at the cost of scaling $\tau(a)$ but this doesn't matter since n is fixed. (Indeed, this entire assumption is unnecessary but it's a nice trick and spares us some indices.) The polarization identity [Ped79, Lemma 5.1.2] gives

$$y^*x = \frac{1}{4}\sum_{k=0}^{3} i^k(x + i^ky)^*(x + i^ky) =: \sum_{k=0}^{3} i^kz_k^*z_k. \tag{3.5}$$

Then for $\tau \in T^+A$ we have

$$\tau(a) = \tau(xey^*) = \tau(e^{1/2}y^*xe^{1/2}) = \sum_{k=0}^{3} i^k\tau(e^{1/2}z_k^*z_ke^{1/2}) \leq \left(\sum_{k=0}^{3} \|z_k^*z_k\|\right)\tau(e)$$

(the second equality is valid since $e^{1/2} \in \mathrm{Ped}(A)$). Hence the set $\{\tau(a) : \tau \in K\}$ is bounded for every $a \in \mathrm{Ped}(A)^+$. This gives a mapping from K into a product of discs, which by Tychonov's Theorem is a compact set. Note that the mapping is a homeomorphism onto its image (by Lemma 3.3.1 and by definition of the topology on T^+A). Moreover, the image of K is closed: if (τ_α) is a net in K converging pointwise to some τ, then it is immediate that $\tau(e) = 1$ and that τ is a positive linear functional on $\mathrm{Ped}(A)$ that satisfies the trace identity, so by Lemma 3.3.1, is an element of K. Therefore, K is compact. □

Proposition 3.2 of [ERS09] shows that the ordering induced on T^+A as a cone (i.e. $\tau_1 \leq \tau_2$ if and only if $\tau_1 + \tau = \tau_2$ for some $\tau \in T^+A$) coincides with the pointwise ordering induced by A (i.e. $\tau_1 \leq \tau_2$ if and only if $\tau_1(x) \leq \tau_2(x)$ for every $x \in A^+$). Pedersen proves in [Ped69, Theorem 3.1] that T^+A is a lattice under this order — see also [ERS09, Theorem 3.3], where the ('non-cancellative') cone of not necessarily densely finite traces is shown to be a complete lattice. Therefore, whenever T^+A has a compact base (e.g. if A is unital or simple), this base is a Choquet simplex. (Here, we consider a nonempty compact convex subset of a locally convex space to be a *Choquet simplex* if it is the base of a cone which is a lattice. See Chapter 10 of [Phe01] for an equivalent formulation involving uniqueness of representing measures.)

3.4 The functors $T^+(\cdot)$ and $\mathrm{Aff}_0 T^+(\cdot)$

It is easy to see that $T^+(\cdot)$ is a contravariant functor (from the category of C^*-algebras and *-homomorphisms to the category of topological convex cones and continuous linear maps). Note in particular that, by (3.3), if $\varphi : A \to B$ is a *-homomorphism then $\varphi(\mathrm{Ped}(A)) \subset \mathrm{Ped}(B)$. Moreover, $T^+(\cdot)$ is *continuous* in the sense that $T^+A \cong \varprojlim(T^+A_i, \varphi_i^*)_{i\in I}$ for every C^*-algebra $A = \varinjlim(A_i, \varphi_i)_{i\in I}$. To prove this, write $C = \varprojlim(T^+A_i, \varphi_i^*)_{i\in I} \subset \prod_{i\in I} T^+A_i$. Define $\alpha : T^+A \to C$ by $\alpha(\tau) := (\tau \circ \varphi_{i\infty})_{i\in I}$, and $\beta : C \to T^+A$ by $\beta((\tau_i)_{i\in I})(\varphi_{i\infty}(a)) := \tau_i(a)$ for $a \in A_i$. One checks (see for example [ERS09, Theorem 3.11]) that $\beta((\tau_i)_{i\in I})$ is well-defined on the set $\bigcup_{i\in I} \varphi_{i\infty}(A_i^+)$ and extends to a lower semicontinuous trace on A, and that α and β are continuous, linear and mutually inverse.

It is also clear that $\mathrm{Aff}_0 T^+(\cdot)$ is a covariant functor, with

$$\varphi_*(f)(\tau) = f(\varphi^*\tau) = f(\tau \circ \varphi) \tag{3.6}$$

for a *-homomorphism $\varphi : A \to B$, $f \in \mathrm{Aff}_0 T^+A$ and $\tau \in T^+B$. In the category of unital C^*-algebras and unital *-homomorphisms, $\mathrm{Aff}\ T_1^+(\cdot)$ is also functorial, and $\mathrm{Aff}_0 T^+A \cong$

Aff $T_1^+ A$ for every unital C^*-algebra A. Moreover, Aff $T_1^+(\cdot)$ is a *continuous* functor from this category to the category of complete order unit spaces. More generally, the following is true.

Proposition 3.4.1. *Let $(A_i, \varphi_i)_{i\in\mathbb{N}}$ be an inductive system of C^*-algebras, with $A = \varinjlim(A_i, \varphi_i)$. Suppose that there exists an order unit η_1 of $\mathrm{Aff}_0 T^+ A_1$ such that, with $\eta_i := (\varphi_{1i})_*(\eta_1)$, the set $\Sigma_i := \{\tau \in T^+ A_i : \eta_i(\tau) = 1\}$ is a compact base of $T^+ A_i$ (so that η_i is an order unit of $\mathrm{Aff}_0 T^+ A_i$) for every $i \geq 1$. Then*

$$\mathrm{Aff}_0 T^+(\varinjlim A_i) \cong \mathrm{Aff}_0(\varprojlim T^+ A_i) \cong \varinjlim(\mathrm{Aff}_0 T^+ A_i) \tag{3.7}$$

as complete order unit spaces. (Here, the order unit of $\mathrm{Aff}_0 T^+ A$ is the affine extension of the constant map 1 on $\Sigma := \varprojlim \Sigma_i$ to all of $T^+ A$, and the order unit of $\varinjlim \mathrm{Aff}_0 T^+ A_i$ is the image of η_1 in the inductive limit.)

Proof. This follows from [Tho94, Lemma 3.2]. If the A_i and φ_i are *unital*, then the order units in question are of course just the norm maps $T^+ A_i \to [0, \infty)$. □

3.5 Traces on tensor products

If A and B are C^*-algebras with $\sigma \in T^+ A$ and $\tau \in T^+ B$ then there is a well-defined product trace $\sigma \otimes \tau$ on the minimal tensor product $A \otimes B$ (see for example [ER78, Proposition 2.10]). Indeed, after combining [BC82], [BH82], [BK04] and an unpublished note of Haagerup, this is how one proves that the minimal tensor product $A \otimes B$ of simple, stably finite, exact C^*-algebras A and B is also (simple, exact and) stably finite. (The last two references can be replaced by [BW10] if one is willing to pass from the class of exact C^*-algebras to those of locally finite nuclear dimension. This class covers all simple C^*-algebras that have been classified by their Elliott invariants so far, and there is as yet no known example of a nuclear C^*-algebra which does not have locally finite nuclear dimension.) We show below that if A has a unique trace and so does B, then this product trace is the *only* trace on $A \otimes B$. The proof is again thanks to Leonel Robert.

Lemma 3.5.1. *Let A and B be C^*-algebras and let $\|\cdot\|_\alpha$ be a C^*-norm on $A \odot B$. Then* $\mathrm{Ped}(A) \odot \mathrm{Ped}(B) \subset \mathrm{Ped}(A \otimes_\alpha B)$.

Proof. Let $a \in \mathrm{Ped}(A)^+$ and $b \in \mathrm{Ped}(B)^+$. Choose positive elements $x_k \in A$, $y_l \in B$ and functions $f_k, g_l \in C_c(0, \infty)^+$ such that $a \leq \sum f_k(x_k)$ and $b \leq \sum g_l(y_l)$. Using the fact that if $x, y \geq 0$ then $x \otimes y = (x^{1/2} \otimes y^{1/2})^2 \geq 0$, we then have

$$a \otimes b \leq a \otimes \sum g_l(y_l) \leq \sum f_k(x_k) \otimes g_l(y_l).$$

By Lemma 3.2.1, the right hand side is in $\mathrm{Ped}(A \otimes_\alpha B)$: for every k and l let $\tilde{f}_k, \tilde{g}_l \in C_0(0, \infty)$ be such that $\tilde{f}_k = 1$ on the support of f_k and $\tilde{g}_l = 1$ on the support of g_l; then $\tilde{f}_k(x_k) f_k(x_k) = f_k(x_k)$ and $\tilde{g}_l(y_l) g_l(y_l) = g_l(y_l)$. Since $\mathrm{Ped}(A \otimes_\alpha B)$ is hereditary, it therefore also contains $a \otimes b$. □

Note that, while $\mathrm{Ped}(A) \odot \mathrm{Ped}(B)$ is certainly a dense subalgebra of $A \otimes_\alpha B$, it might not be an ideal. This is why we cannot deduce equality in Lemma 3.5.1.

Lemma 3.5.2. *Let A and B be C^*-algebras, let $\|\cdot\|_\alpha$ be a C^*-norm on $A \odot B$ and let $\omega \in T^+(A \otimes_\alpha B)$. Suppose that, up to scalar multiples, B has a unique trace τ. Then there exists $\sigma \in T^+ A$ such that $\omega(a \otimes b) = \sigma(a)\tau(b)$ for every $a \in \mathrm{Ped}(A)^+$ and $b \in \mathrm{Ped}(B)^+$.*

Proof. Fix some $a \in \mathrm{Ped}(A)^+$. Consider the function $\omega_a : B^+ \to [0, \infty]$ given by $b \mapsto \omega(a \otimes b)$. Then ω_a is positive linear on B^+, is lower semicontinuous (because ω is: for every $\lambda \in \mathbb{R}_+$ we have $\{x \in B^+ : \omega_a(x) \leq \lambda\} = \{x \in B^+ : \omega(a \otimes x) \leq \lambda\}$, which is closed since ω is lower semicontinuous) and moreover is a trace. Finally, Lemma 3.5.1 shows that ω_a is densely defined. Thus $\omega(a \otimes b) = \omega_a(b) = \sigma(a)\tau(b)$ for some non-negative real number $\sigma(a)$ and every $b \in B^+$. Now let $b_0 \in \mathrm{Ped}(B)^+$ such that $\tau(b_0) = 1$. Extend σ to A^+ by defining $\sigma(c) = \omega(c \otimes b_0)$ for $c \in A_+$. Then $\omega \in T^+ A$, by a similar argument, and it has the required property. □

Remark 3.5.3. If, in the above Lemma, ω is *bounded*, then of course $\omega = \sigma \otimes \tau$ on all of $A \otimes_\alpha B$, and $\|\omega\| = \|\sigma\| \|\tau\|$, but in general this is not so obvious. On the other hand, if A is an AF algebra, this remains true for *all* ω; this follows for example from [ERS09, Remark 3.5] upon noting that, for positive integers n_i, $\bigoplus_{i=1}^k M_{n_i}(\mathrm{Ped}(B)) = \mathrm{Ped}(\bigoplus_{i=1}^k M_{n_i}(B))$ is closed under under the functional calculus operation $a \mapsto (a - \epsilon)_+$ for positive elements a and $\epsilon > 0$. (Alternatively, one could appeal to the continuity of $T^+(\cdot)$, which itself depends on this fact; see also Lemma 3.5.5 below.) In particular, this will allow us to deduce that for every AF algebra A, there is an isomorphism between $T^+(A \otimes W)$ and $T^+ A$ that maps the tracial states of $A \otimes W$ onto the tracial states of A. See Corollary 4.7.2.

Proposition 3.5.4. *Suppose that A and B are σ-unital C^*-algebras, each with a unique trace (up to scalar multiples). Then $A \otimes_\alpha B$ also has at most one nonzero trace, for any*

C^*-norm $\|\cdot\|_\alpha$ on $A \odot B$. In particular, if A and B are also simple, stably finite and exact then $A \otimes_{min} B$ has a unique trace.

Proof. Since A and B are σ-unital, we can find approximate units $(e_n)_{n\in\mathbb{N}}$ of A and $(f_n)_{n\in\mathbb{N}}$ of B with $e_{n+1}e_n = e_n$ and $f_{n+1}f_n = f_n$ for every n. (Note in particular that, by Lemma 3.2.1, $e_n \in \mathrm{Ped}(A)$ and $f_n \in \mathrm{Ped}(B)$, so they have finite trace.) Then $g_n := e_n \otimes f_n$ gives an approximate unit of $C := A \otimes_\alpha B$ which also satisfies $g_{n+1}g_n = g_n$. (Once again, each g_n lies in $\mathrm{Ped}(C)$, so has finite trace as well.)

For each n, let $C_n = \mathrm{Her}(g_n) = \overline{g_n C g_n}$ and let D_n be the set

$$\left\{ g_n \left(\sum a_i \otimes b_i \right) g_n = \sum e_n a_i e_n \otimes f_n b_i f_n : a_i \in A, b_i \in B \right\},$$

which is dense in C_n and contained in $\mathrm{Ped}(A) \odot \mathrm{Ped}(B)$. Let $x \in C_n^+$ and find a sequence (x_k) of self-adjoint elements in D_n with $x_k \to x$. Then for any $\epsilon > 0$ and every sufficiently large k we have $\|x_k - x\| < \epsilon$. Since g_{n+1} dominates g_n, hence acts as a multiplicative unit on C_n, by functional calculus we get $-\epsilon g_{n+1} \leq x_k - x \leq \epsilon g_{n+1}$. Hence $|\tau(x_k) - \tau(x)| \leq \epsilon\tau(g_{n+1})$ for large k and every $\tau \in T^+C$, i.e. $\tau(x_k) \to \tau(x)$ for every $\tau \in T^+C$.

Now suppose that $\tau_1, \tau_2 \in T^+(A \otimes_\alpha B)$ with (without loss of generality) $\tau_1 \neq 0$. By Lemma 3.5.2, since A and B both have a unique trace, there is some $\lambda \geq 0$ such that $\tau_2 = \lambda\tau_1$ on $\mathrm{Ped}(A) \odot \mathrm{Ped}(B)$. Hence $\tau_2 = \lambda\tau_1$ on each D_n, and hence on each C_n^+ by the argument just given. But the C_n are increasing sub-C^*-algebras of C with $C = \overline{\bigcup_{n=1}^\infty C_n}$ and T^+ is a continuous functor, so $T^+C \cong \varprojlim T^+C_n$. Therefore $\tau_2 = \lambda\tau_1$ on all of $C = A \otimes_\alpha B$, and so $A \otimes_\alpha B$ has at most one trace. □

Finally, we make the easy remark that tensoring with a matroid C^*-algebra preserves the tracial cone.

Lemma 3.5.5. *Let A be a C^*-algebra and let B be a matroid C^*-algebra (i.e. $B \cong \varinjlim(M_{n_i}, \varphi_i)$ for some increasing sequence of natural numbers n_i). Then $T^+(A \otimes B) \cong T^+A$. In particular, $T^+(A \otimes \mathcal{K}) \cong T^+A$ for every C^*-algebra A.*

Proof. The proof consists of the following two observations.

(i) $T^+M_n(A) \cong T^+A$ for every $n \in \mathbb{N}$.

To see this, note that $M_n(\mathrm{Ped}(A)) = M_n \odot \mathrm{Ped}(A)$ is a dense ideal of $M_n(A) = M_n \odot A$. By Lemma 3.5.1, we therefore have $\mathrm{Ped}(M_n(A)) = M_n(\mathrm{Ped}(A))$. Lemma 3.5.2, together with Lemma 3.3.1, then shows that every trace on $M_n(A)$ is of the form $\tau \otimes \mathrm{tr}_n$ for some $\tau \in T^+A$, where tr_n is the unique tracial state on M_n.

(ii) If $\varphi : M_n \to M_k$ is a nonzero *-homomorphism, then the induced map $(\mathrm{id} \otimes \varphi)^* : T^+M_k(A) \to T^+M_n(A)$ is an isomorphism.

We may assume that φ is of the form $a \mapsto \mathrm{diag}(\overbrace{a, \ldots, a}^{m}, 0)$ for some m. Suppose that $\sigma \in T^+M_k(A)$. Then $\sigma = \tau \otimes \mathrm{tr}_k$ for some $\tau \in T^+A$. For $a \in A$ and the matrix unit $e_{11} \in M_n$, we have $(\mathrm{id} \otimes \varphi)^*(\sigma)(a \otimes e_{11}) = m\tau(a)/k$, so $(\mathrm{id} \otimes \varphi)^*$ is injective. Given (i), $(\mathrm{id} \otimes \varphi)^*$ is also surjective, so is an isomorphism.

By continuity of T^+, i.e. since $T^+(A \otimes B) \cong \varprojlim(T^+M_{n_i}(A), (\mathrm{id} \otimes \varphi_i)^*)$, it is now immediate that $T^+(A \otimes B) \cong T^+A$. $\square$

Chapter 4

A stably finite analogue of $\mathcal{O}_2$

4.1 Background

In this chapter, we construct a simple, nuclear, stably projectionless C^*-algebra W which has trivial K-theory and a unique tracial state, and we prove that W shares some of the important properties of the Cuntz algebra $\mathcal{O}_2$. In particular, we show that every nondegenerate endomorphism of W is approximately inner, and we construct a trace-preserving embedding of W into the central sequences algebra $M(W)_\infty \cap W'$. We also show how one can deduce that $W \otimes W \cong W$ from a conjectured generalization (Conjecture 4.5.3) of an existing classification theorem (Theorem 4.1.2).

As elaborated in the introduction, the hope is that W will play a role in the classification of stably projectionless C^*-algebras, similar to the roles played by $\mathcal{O}_\infty$ and $\mathcal{Z}$ in the classification of purely infinite and stably finite algebras respectively. There is already an indication of this in [Rob11], where (assuming Conjecture 4.5.2) it is proved that the Cuntz semigroup of a W-absorbing C^*-algebra is determined by the cone of its lower semicontinuous (not necessarily densely finite) 2-quasitraces. Moreover, W itself arises from a class of stably projectionless algebras which have been completely classified (see [Raz02, Theorem 1.1], [Tsa05, Theorem 5.1] and also the more general classification result of [Rob10]). These algebras are inductive limits of building blocks of the form

$$A(n, n') := \{f \in C([0,1], M_{n'}) : f(0) = \mathrm{diag}(\overbrace{c, \dots, c}^{a}, 0_n),\ f(1) = \mathrm{diag}(\overbrace{c, \dots, c}^{a+1}),\ c \in M_n\},$$

where n and n' are natural numbers with $n|n'$ and $a := \frac{n'}{n} - 1 > 0$. Each building block is stably projectionless and has trivial K-theory, so the Elliott invariant is purely tracial.

Theorem 4.1.1 (Razak)**.** *Let A and B be simple inductive limits of (countably many) building blocks. If (T^+A, Σ_A) is isomorphic to (T^+B, Σ_B) then A is isomorphic to B.*

Here, T^+A is the cone of densely defined lower semicontinuous traces on A and Σ_A is the compact convex subset of T^+A consisting of those (bounded) traces of norm ≤ 1 (see Chapter 3). We will call an inductive limit of countably many building blocks a *Razak algebra*. (We may always assume that the connecting maps are injective — see section 4.2 below).

Theorem 4.1.1 has recently been superseded by the main result of [Rob10]. There, a functor based on the Cuntz semigroup is used to classify (not necessarily simple) inductive limits of one-dimensional NCCW-complexes with trivial K_1-groups (see Chapter 2). In the case of simple C^*-algebras with trivial K-theory, this reduces to the following (see [Rob10, Corollary 4]).

Theorem 4.1.2 (Robert)**.** *Let A and B be inductive limits of one-dimensional NCCW-complexes which have trivial K_1-groups, such that A and B are simple, nonunital and have trivial K-theory. If (T^+A, Σ_A) is isomorphic to (T^+B, Σ_B) then A is isomorphic to B.*

We can if we like appeal to Theorem 4.1.2 instead of Theorem 4.1.1 whenever necessary, but it turns out that something slightly stronger (Conjecture 4.5.3) may be needed to prove Conjecture 4.5.2: we want Theorem 4.1.2 to hold under the milder assumption that the *limits*, rather than all of the building blocks individually, have trivial K_1. While this more general result is almost certainly true, actually proving it is beyond the scope of this thesis.

For the range of the invariant, we still refer to Tsang (although an alternative proof will be provided in section 4.7).

Theorem 4.1.3 (Tsang)**.** *Let C be a topological convex cone that has a metrizable Choquet simplex as a base and let ω be a lower semicontinuous linear map $C \to [0, \infty]$. Then there exists a simple Razak algebra A such that $(T^+A, \Sigma_A) = (C, \omega^{-1}([0, 1]))$.*

Suppose that $C = \mathbb{R}_+$ and let $\omega(x) = x$. By Theorems 4.1.3 and 4.1.1, there exists a unique simple Razak algebra W which has a tracial state τ such that $T^+W = \mathbb{R}_+\tau$. That is, W has a unique tracial state and every trace on W is bounded. Note that $W \otimes \mathcal{K}$ is also a simple Razak algebra whose trace is unique up to scalar multiples, except that in this case the trace is unbounded. (This corresponds to taking $\omega(x) = 0$ if $x = 0$ and $\omega(x) = \infty$ otherwise.) Again, Theorem 4.1.1 says that $W \otimes \mathcal{K}$ is the unique simple Razak algebra with this property.

It is easy to find a simple, nuclear, nonunital C^*-algebra with a (unique) tracial state (for example, any proper hereditary subalgebra A of $\mathcal{Z}$ has a tracial state, which is unique since A is stably isomorphic to $\mathcal{Z}$), but to the author's knowledge, W may be the first example of such an algebra which is stably projectionless. In particular, W lives outside of the realm of C^*-algebras for which projections separate traces.

Note that Theorem 4.1.1 also shows that W absorbs the universal UHF algebra $\mathcal{Q}$, which implies that W absorbs $\mathcal{Z}$ (see also section 4.6), and the Kirchberg-Phillips classification theorem ([Kir], [Phi00]) shows that $W \otimes A \cong \mathcal{O}_2 \otimes \mathcal{K}$ for any simple, separable, nuclear, stably infinite C^*-algebra A that satisfies the UCT (see Theorem 8.4.1 and also Theorems 4.1.10 and 4.1.3 of [Rør02]). In particular, $W \otimes \mathcal{O}_2 \cong W \otimes \mathcal{O}_\infty \cong \mathcal{O}_2 \otimes \mathcal{K}$.

The chapter is organized as follows. We first recall some basic facts about building blocks and establish notation in section 4.2. In section 4.3 we will explicitly exhibit W in a manner analogous to the construction of the Jiang-Su algebra $\mathcal{Z}$. Next, in section 4.4 we characterize W as the unique terminal object in a certain category, from which it will follow that every nondegenerate endomorphism of W is approximately inner. We then conjecture in section 4.5 that $W \otimes W \cong W$, and suggest a proof of this using an argument provided by Luis Santiago. Section 4.6 contains a complete proof of [TW05, Proposition 4.1], which says that every simple Razak algebra is approximately divisible, and we show how the proof can be adapted to construct a trace-preserving embedding of W into the central sequences algebra $M(W)_\infty \cap W'$ (where $M(W)$ is the multiplier algebra of W). In section 4.7 we generalize a theorem of Blackadar and Goodearl (namely, we prove a version of Theorem 4.1.3 for AF algebras), and we combine this with Theorem 4.1.2 to show that, among the C^*-algebras classified therein, those that have trivial K_0 are all of the form $A \otimes W$ for some AF algebra A. Finally, we describe in section 4.8 the relationship between W and certain crossed products of $\mathcal{O}_2$ by $\mathbb{R}$.

4.2 The building blocks

Throughout this section, let A be the building block

$$A(n,n') = \{f \in C([0,1], M_{n'}) : f(0) = \mathrm{diag}(\overbrace{c,\ldots,c}^{a}, 0_n),\ f(1) = \mathrm{diag}(\overbrace{c,\ldots,c}^{a+1}),\ c \in M_n\},$$

where n and n' are natural numbers with $n|n'$ and $a := \frac{n'}{n} - 1 > 0$. Each building block is a one-dimensional NCCW-complex given by a pullback

$$\begin{array}{ccc} A & \xrightarrow{\pi_1} & M_n \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \varphi=\varphi_0\oplus\varphi_1} \\ C[0,1]\otimes M_{n'} & \xrightarrow{\partial} & M_{n'}\oplus M_{n'} \end{array}$$

(see section 2.2 and Lemma 2.3.1 of Chapter 2), is stably projectionless and has trivial K-theory. That A is projectionless is easily seen from the endpoint conditions, and since $M_k \otimes A(n,n') \cong A(kn,kn')$, the same argument shows that A is *stably* projectionless. As for the K-theory, since A is a pullback with ∂ surjective, we can appeal to the Mayer-Vietoris sequence of [Sch84, Theorem 4.5] (see also [Sch84, Theorem 4.1] and [Bla98, Theorem 21.5.1]). That is, there is an exact sequence

$$\begin{array}{ccccc} K_1(A) & \xrightarrow{(\pi_1)_*\oplus(\pi_2)_*} & K_1(M_n)\oplus K_1(C[0,1]\otimes M_{n'}) & \xrightarrow{(\varphi_*,-\partial_*)} & K_1(M_{n'}\oplus M_{n'}) \\ \uparrow & & & & \downarrow \\ K_0(M_{n'}\oplus M_{n'}) & \xleftarrow[(\varphi_*,-\partial_*)]{} & K_0(M_n)\oplus K_0(C[0,1]\otimes M_{n'}) & \xleftarrow[(\pi_1)_*\oplus(\pi_2)_*]{} & K_0(A) \end{array}$$

which reduces to the sequence

$$\begin{array}{ccccc} K_1(A) & \longrightarrow & 0 & \longrightarrow & 0 \\ \uparrow & & & & \downarrow \\ \mathbb{Z}\oplus\mathbb{Z} & \xleftarrow[\alpha]{} & \mathbb{Z}\oplus\mathbb{Z} & \longleftarrow & K_0(A) \end{array} .$$

Here, the map α is given by the matrix $\left(\begin{smallmatrix} a & -1 \\ a+1 & -1 \end{smallmatrix}\right)$, and it is then easily checked that $K_0(A) \cong \ker\alpha = 0$ and $K_1(A) \cong \operatorname{coker}\alpha = 0$.

We will call an inductive limit of countably many building blocks a *Razak algebra*. Razak algebras are separable, nuclear, satisfy the UCT and are completely classified by tracial data (Theorems 4.1.1 and 4.1.3). We now establish some basic notation that will be used throughout this chapter.

Irreducible representations.

It follows from [Dix77, Proposition 2.10.2] that every irreducible representation of A is a sub-representation of an irreducible representation of $C([0,1], M_{n'})$. Thus, the irreducible representations of A are (up to unitary equivalence) the evaluation maps $\mathrm{ev}_s : A \to M_{n'}$ for $s \in (0,1)$ together with 'evaluation at the irreducible fibre at infinity', i.e. $\mathrm{ev}_\infty : A \to M_n$

is such that $\mathrm{ev}_0 = \bigoplus_{i=1}^{a} \mathrm{ev}_\infty$. In particular, the primitive ideals of A are precisely those of the form $I_x = \{f \in A : f(x) = 0\}$ for $x \in [0,1]/\{0,1\}$, and hence every ideal I of A is of the form $I = \{f \in A : f|_{\Gamma(I)} = 0\}$ for some unique closed subset $\Gamma(I) \subseteq \mathbb{T} = [0,1]/\{0,1\}$.

Connecting maps.

Given the above characterization of ideals of A, it is easy to see that any proper quotient of A has nontrivial projections. Therefore, if B is another projectionless C^*-algebra then every nonzero *-homomorphism $\varphi : A \to B$ is injective. (This means that we may always assume that Razak algebras have injective connecting maps.) If B is also a building block then for each $x \in [0,1]$ we will denote by $\varphi^x : A \to B_x$ the morphism $\varphi^x(f) = \varphi(f)(x)$.

Traces.

We write T^+A for the cone of densely defined lower semicontinuous traces on A (as in Chapter 3). Every trace on the building block A is bounded and has the form $\tau = \mathrm{tr} \otimes \mu$, where tr is the normalized trace on $M_{n'}$ and μ is some positive Borel measure on $(0,1]$.

To see this, we may assume that $n = 1$ (since $A(n,kn) \cong M_n \otimes A(1,k)$; see also Lemma 3.5.5). Let $h \in A$ be the canonical strictly positive element $h(t) = \overbrace{1 \oplus \cdots \oplus 1}^{a} \oplus t$, and notice that A decomposes as the sum $A = \mathbb{C}h + I$, where $I := C_0((0,1), M_k)$ is an ideal of A. (This is because $f - \mathrm{ev}_\infty(f)h \in I$ for every $f \in A$.) Note also that $\mathrm{Ped}(A)$ contains $\mathrm{Ped}(C^*(h))$ and $\mathrm{Ped}(I)$, which are easily described. Every element of $C^*(h)$ is of the form $f(h) = \overbrace{f(1) \oplus \cdots \oplus f(1)}^{a} \oplus f$ for some $f \in C_0((0,1])$; we have $f(h) \in \mathrm{Ped}(C^*(h))$ if and only if f is an element of $C_c((0,1])$, the compactly supported continuous functions on $(0,1]$. On the other hand, we have $\mathrm{Ped}(I) = C_c((0,1), M_k)$.

Let $\tau \in T^+A$. Choose a partition of unity $\{\rho_i\}_{i\in\mathbb{N}}$ of $(0,1)$ such that each ρ_i is continuous with compact support. Write U_i for the support of ρ_i, so U_i is an open set whose closure is a compact subset of $(0,1)$. Note that, for every $f \in C([0,1], M_k)$ and $i \in \mathbb{N}$, we have $f \cdot \rho_i 1_k \in C_0(U_i, M_k) \subset \mathrm{Ped}(I) \subset \mathrm{Ped}(A)$, so $\tau_i(f) := \tau(f \cdot \rho_i 1_k)$ defines a finite trace on $C([0,1], M_k)$. Since these correspond to traces on $C([0,1])$ (as in Lemma 3.5.5), there is a positive Borel measure μ_i, supported on U_i, such that $\tau_i(f) = \int_0^1 \mathrm{tr}(f(t)) d\mu_i(t)$ for every $f \in C([0,1], M_k)$.

Define $\tau' : A^+ \to [0,\infty]$ by $\tau'(f) = \sum_{i=1}^{\infty} \tau_i(f)$. Then τ' is linear and satisfies the trace identity, and for $f \in A^+$ and every $N \in \mathbb{N}$, we have

$$\tau(f) \geq \tau(f \cdot (\rho_1 + \cdots + \rho_N)1_k) = \sum_{i=1}^{N} \tau_i(f),$$

so τ' is dominated by τ. It follows that τ' is finite on $\mathrm{Ped}(A)$ and so, by Lemma 3.3.1, is also lower semicontinuous. That is, $\tau' \in T^+A$ with $\tau' \leq \tau$. Therefore, there exists $\sigma \in T^+A$ such that $\tau = \tau' + \sigma$ (see [ERS09, Proposition 3.2]).

Let $f \in \mathrm{Ped}(I)^+$. Then there exists N such that $\rho_i = 0$ on the support of f for every $i > N$, hence $\tau'(f) = \sum_{i=1}^{N} \tau_i(f)$ and $\tau(f) = \tau(f \cdot \sum_{i=1}^{N} \rho_i 1_k) = \sum_{i=1}^{N} \tau_i(f) = \tau'(f)$. Thus, τ and τ' agree on $\mathrm{Ped}(I)$ and hence on all of I. In particular, we have $\sigma|_I = 0$, and so $\sigma = \lambda \mathrm{ev}_\infty$ for some $\lambda \geq 0$ (namely, $\lambda = \sigma(h)$). Finally, let $f \in C_c((0,1])$ with $f(1) = 1$. Then $\mathrm{tr}(f(h)(t)) \geq (k-1)/k$ for every $t \in [0,1]$, and for every $N \in \mathbb{N}$ we have

$$\tau(f(h)) \geq \sum_{i=1}^{N} \int_0^1 \mathrm{tr}(f(h)(t)) d\mu_i(t) \geq \frac{k-1}{k} \sum_{i=1}^{N} \mu_i([0,1]) = \frac{k-1}{k} \sum_{i=1}^{N} \|\tau_i\|.$$

Since $f(h) \in \mathrm{Ped}(A)$, so $\tau(f(h)) < \infty$, it follows that τ' is finite and moreover is of the form $\tau' = \mathrm{tr} \otimes \mu'$, with $\mu' = \sum_{i=1}^{\infty} \mu_i$. Therefore, if μ denotes the finite measure $\mu' + \lambda\delta_1$, then $\tau = \mathrm{tr} \otimes \mu$.

We will write T_1^+A for the simplex of tracial states on A (corresponding to Borel probability measures on $(0,1]$), Σ_A for those traces of norm at most one, and we let $\mathrm{Aff}_0 T^+A$ denote the ordered vector space of continuous real-valued linear functionals on the cone T^+A.

Following [Raz02], we equip $\mathrm{Aff}_0 T^+A$ with two different norms. First, we define $\|\cdot\|_A$ by $\|f\|_A := \sup\{|f(\tau)| : \tau \in \Sigma_A\}$. Second, we use the order unit norm given by the following: fix $\eta \in \mathrm{Aff}_0 T^+A$ with $\inf\{\eta(\tau) : \|\tau\| = 1\} > 0$, so that η is an order unit of $\mathrm{Aff}_0 T^+A$, and denote by Σ_η the closed convex set $\{\tau \in T^+A : \eta(\tau) = 1\}$; we then get a corresponding order unit norm $\|\cdot\|_\eta$ given by $\|f\|_\eta := \sup\{|f(\tau)| : \tau \in \Sigma_\eta\}$. It is not hard to show that the norms $\|\cdot\|_A$ and $\|\cdot\|_\eta$ are equivalent for the building block A, so in particular Σ_η is a *compact* base of T^+A. In fact, we can say exactly what the spaces $(\mathrm{Aff}_0 T^+A, \|\cdot\|_A)$ and $(\mathrm{Aff}_0 T^+A, \|\cdot\|_\eta)$ look like.

(i) Define $C[0,1]_a := \{f \in C[0,1] : f(0) = \frac{a}{a+1} f(1)\}$. Then there is an isometric isomorphism of ordered Banach spaces $\iota : (\mathrm{Aff}_0 T^+A, \|\cdot\|_A) \to C[0,1]_a$, $\iota(f)(t) = f(\mathrm{tr} \otimes \delta_t)$ (where δ_t denotes the point mass at t). A Krein-Milman argument shows that ι preserves infima:

$$\inf\{f(\tau) : \tau \in T_1^+A\} = \inf \iota(f) \quad \text{for every} \quad f \in \mathrm{Aff}_0 T^+A.$$

(ii) There is an isomorphism of order unit spaces $\iota'_\eta : (\mathrm{Aff}_0 T^+A, \eta) \to (C_{\mathbb{R}}(\mathbb{T}), 1)$ given by $\iota'_\eta(f) = \frac{\iota(f)}{\iota(\eta)}$, which again preserves infima: $\inf\{f(\tau) : \tau \in \Sigma_\eta\} = \inf \iota'_\eta(f)$ for $f \in \mathrm{Aff}_0 T^+A$.

Finally, there is an embedding ψ_A of $\mathrm{Aff}_0 T^+A$ into the set A_{sa} of self-adjoint elements of A given by

$$\psi_A(f)(t) = \frac{a+1}{a+t}\left(\overbrace{f(\mathrm{tr}\otimes\delta_t)1_n \oplus \cdots \oplus f(\mathrm{tr}\otimes\delta_t)1_n}^{a} \oplus t f(\mathrm{tr}\otimes\delta_t)1_n\right),$$

which (by another Krein-Milman argument) is right-inverse to the usual map $\rho_A : A_{sa} \to \mathrm{Aff}_0 T^+A$. That is, ψ_A satisfies $\tau(\psi_A(f)) = f(\tau)$ for every $\tau \in T^+A$. The embedding is natural in the sense that if B is another building block and $\varphi : A \to B$ is a *-homomorphism then

$$\tau(\varphi \circ \psi_A(f)) = \varphi^*\tau(\psi_A(f)) = f(\varphi^*\tau) = \varphi_*(f)(\tau) = \tau(\psi_B \circ \varphi_*(f))$$

for every $\tau \in T^+B$ and $f \in \mathrm{Aff}_0 T^+A$. This justifies suppressing the notation ψ_A and ρ_A, and in the sequel we will do so without comment (particularly in section 4.4).

4.3 The construction of W

In this section we construct explicit connecting maps for W by adapting the procedure of [JS99, Proposition 2.5].

Proposition 4.3.1. *There exists an inductive sequence (A_i, φ_i) of building blocks $A_i = A(n_i, (a_i+1)n_i)$ such that each connecting map $\varphi_{ij} : A_i \to A_j$ is a *-homomorphism of the form*

$$\varphi_{ij}(f) = u \begin{pmatrix} f\circ\xi_1 & & & \\ & f\circ\xi_2 & & \\ & & \ddots & \\ & & & f\circ\xi_m \end{pmatrix} u^* \tag{4.1}$$

for some unitary $u \in C([0,1], M_{n'_j})$ and continuous maps $\xi_k : [0,1] \to [0,1]$ that satisfy

$$|\xi_k(x) - \xi_k(y)| \le (1/2)^{j-i} \quad \textit{for every} \quad x, y \in [0,1] \quad \textit{and} \quad 1 \le k \le m = \frac{n'_j}{n'_i}; \tag{4.2}$$

$$\bigcup_{k=1}^{m} \xi_k([0,1]) = [0,1]. \tag{4.3}$$

Proof. Let A_1 be some building block $A(n_1,(a+1)n_1)$. We will find a building block $A_2 = A(n_2,(b+1)n_2)$ and an injective *-homomorphism $\varphi : A_1 \to A_2$ of the form (4.1) where each ξ_k is one of the maps

$$\xi(x) = x/2, \qquad \xi(x) \equiv 1/2 \quad \text{or} \quad \xi(x) = (x+1)/2. \tag{4.4}$$

Repeating this process then gives an inductive sequence where each $\varphi_i : A_i \to A_{i+1}$ has the right form. Note also that as defined, $\varphi_i(f)$ makes sense for any $f \in C([0,1], M_{n'_i})$, so φ_i extends to a unital *-homomorphism $C([0,1], M_{n'_i}) \to C([0,1], M_{n'_{i+1}})$ and in particular $\varphi_i(u)$ is unitary whenever u is. Therefore, each connecting map $\varphi_{ij} : A_i \to A_j$ will be of the form (4.1) with each ξ_k a composition of $j-i$ functions from the list (4.4), so will be one of the maps

$$\xi(x) = \frac{l}{2^{j-i}} \quad \text{or} \quad \xi(x) = \frac{x+l}{2^{j-i}},$$

for some integer l with $0 < l < 2^{j-i}$ in the former case and $0 \le l < 2^{j-i}$ in the latter. Hence (4.2) is satisfied.

Let $b = 2a+1$, $n_2 = bn_1$ and $m = 2b$. Let $f \in A_1$, so that $f(0) = \mathrm{diag}(\overbrace{c, \cdots, c}^{a}, 0_{n_1})$ and $f(1) = \mathrm{diag}(\overbrace{c, \cdots, c}^{a+1})$ (in $M_{(a+1)n_1}$) for some $c \in M_{n_1}$. Write $d_f := \mathrm{diag}(f(1/2), \overbrace{c, \cdots, c}^{a}) \in M_{n_2}$. Then, in $M_{(b+1)n_2}$, the matrix

$$d_f \otimes 1_b = \begin{pmatrix} d_f & & & \\ & \ddots & & \\ & & d_f & \\ & & & 0_{n_2} \end{pmatrix}$$

consists (up to permutation) of ab copies of c, b copies of $f(1/2)$, and a zero matrix of size n_2. On the other hand, the matrix

$$\mathrm{diag}(\overbrace{f(0), \ldots, f(0)}^{b}, \overbrace{f(1/2), \ldots, f(1/2)}^{b}) \in M_{m(a+1)n_1} = M_{(b+1)n_2}$$

also consists of ab copies of c, b copies of $f(1/2)$, and a zero matrix of size $bn_1 = n_2$.

Therefore, there is a permutation unitary $u_0 \in M_{(b+1)n_2}$ such that

$$\begin{pmatrix} d_f & & & \\ & \ddots & & \\ & & d_f & \\ & & & 0_{n_2} \end{pmatrix} = u_0 \begin{pmatrix} f(0) & & & & & \\ & \ddots & & & & \\ & & f(0) & & & \\ & & & f(1/2) & & \\ & & & & \ddots & \\ & & & & & f(1/2) \end{pmatrix} u_0^*.$$

Similarly, again in $M_{(b+1)n_2}$, both of the matrices $\mathrm{diag}(\overbrace{f(1/2), \ldots, f(1/2)}^{b+1}, \overbrace{f(1), \ldots, f(1)}^{b-1})$ and $d_f \otimes 1_{b+1}$ consist up to permutation of $a(b+1)$ copies of c and $b+1$ copies of $f(1/2)$. Therefore, there is a permutation unitary $u_1 \in M_{(b+1)n_2}$ such that

$$\begin{pmatrix} d_f & & & \\ & d_f & & \\ & & \ddots & \\ & & & d_f \end{pmatrix} = u_1 \begin{pmatrix} f(1/2) & & & & & \\ & \ddots & & & & \\ & & f(1/2) & & & \\ & & & f(1) & & \\ & & & & \ddots & \\ & & & & & f(1) \end{pmatrix} u_1^*.$$

Now we just connect the endpoints: take u to be any continuous path of unitaries in $M_{(b+1)n_2}$ from u_0 to u_1, and define functions $\xi_1, \ldots, \xi_m : [0,1] \to [0,1]$ by

$$\xi_k(x) = \begin{cases} x/2 & \text{if} \quad 1 \le k \le b, \\ 1/2 & \text{if} \quad k = b+1, \\ (x+1)/2 & \text{if} \quad b+1 < k \le m. \end{cases}$$

Then the map φ as defined in (4.1) is by construction a *-homomorphism from A_1 to A_2.

Finally, note that φ_{ij} is injective if and only if condition (4.3) holds (since $\varphi_{ij}(f) = 0$ if and only if $f \in A_i$ is supported on the open set $[0,1] \setminus \bigcup_{k=1}^m \xi_k[0,1]$). By construction, (4.3) holds for the ξ_k used to define φ above, and it therefore follows that for every i and j, φ_{ij} is injective and satisfies (4.3). □

Remark 4.3.2. For the above inductive sequence we have $a_i \to \infty$ (and also $n_i/a_i = n_{i-1} \to \infty$) as $i \to \infty$; we will make use of this in section 4.4.

We want to show that the inductive limit A of any sequence (A_i, φ_i) as in Lemma 4.3.1 is simple and has a unique tracial state. First we observe that every trace on A is bounded.

Lemma 4.3.3. *If (A_i, φ_i) is any inductive sequence as in Proposition 4.3.1 then the connecting maps φ_{ij} are nondegenerate. In particular, $\varphi_{ij}^*(T_1^+(A_j)) \subseteq T_1^+(A_i)$ for every $j \geq i$. Moreover, if $A = \varinjlim(A_i, \varphi_i)$ then every $\tau \in T^+A$ is bounded.*

Proof. Here, 'nondegenerate' means that ϕ_{ij} maps an approximate unit of A_i to an approximate unit for A_j. It suffices to show that, for fixed $i > 1$, if $h \in A_{i-1}$ is the canonical strictly positive element $h(t) = (\overbrace{1 \oplus \cdots \oplus 1}^{a} \oplus t) \otimes 1_{n_{i-1}}$, then $h_i := \varphi_{i-1}(h)$ is strictly positive in A_i (which is equivalent to saying that $(h_i^{1/n})_{n\in\mathbb{N}}$ is an approximate unit for A_i).

Let $f \in A_i$ and let $\epsilon > 0$, and for convenience write $p = 1_{n_i'-n_i}$ and $q = 1_{n_i'}$. Certainly we have $\|h_i\| = 1$, and we may assume that $\|f\| = 1$ as well. Choose $\delta > 0$ such that if $0 \leq t < \delta$ then $\|f(t) - f(0)\| < \epsilon/5$ and $\|u_t - u_0\| < \epsilon/5$ (where u is as in (4.1) of Proposition 4.3.1). It is easy to see that as $n \to \infty$, $h_i^{1/n}(0)$ converges to p and $h_i^{1/n}$ converges to q uniformly on $[\delta, 1]$. Hence we can find some N such that $\|h_i^{1/n}(0) - p\| < \epsilon/5$ and $\|h_i^{1/n}(t) - q\| < \epsilon$ for every $\delta \leq t \leq 1$ and $n \geq N$. For $\delta \leq t \leq 1$ we therefore have

$$\|h_i^{1/n}(t)f(t) - f(t)\| = \|(h_i^{1/n}(t) - q)f(t)\| < \epsilon \qquad \forall n \geq N.$$

Now let $0 \leq t < \delta$ and write $g_n(t) = u_0 u_t^* h_i^{1/n}(t) u_t u_0^*$. Then $g_n(t)$ commutes with p (for example because $u_0^* g_n(t) u_0$ and $u_0^* p u_0$ are diagonal), and since the ξ_k are increasing, we have $\|p g_n(t) p - p\| \leq \|h_i^{1/n}(0) - p\| < \epsilon/5$ for $n \geq N$. Thus

$$\begin{aligned}
\|h_i^{1/n}(t)f(t) - f(t)\| &\leq \|h_i^{1/n}(t)f(t) - h_i^{1/n}(t)f(0)\| + \|h_i^{1/n}(t)f(0) - g_n(t)f(0)\| \\
&\quad + \|g_n(t)f(0) - p g_n(t) p f(0)\| + \|(p g_n(t) p - p)f(0)\| + \|f(0) - f(t)\| \\
&< (\epsilon + 2\epsilon + 0 + \epsilon + \epsilon)/5 \\
&= \epsilon
\end{aligned}$$

for $0 \leq t < \delta$ and $n \geq N$. Therefore, $\|h_i^{1/n} f - f\| < \epsilon$ for every $n \geq N$ and hence $(h_i^{1/n})_{n\in\mathbb{N}}$ is an approximate unit for A_i. It follows that the connecting maps φ_{ij} are nondegenerate.

Now suppose that $j \geq i$ and let ρ be a state on A_j. Then $\rho \circ \varphi_{ij}$ is a positive linear functional on A_i and moreover

$$\|\rho \circ \varphi_{ij}\| = \lim_{n\to\infty} \rho \circ \varphi_{ij}(h_i^{1/n}) = \lim_{n\to\infty} \rho(\varphi_{ij}(h_i)^{1/n}) = \lim_{n\to\infty} \rho(h_j^{1/n}) = \|\rho\| = 1,$$

so $\varphi_{ij}^*\rho = \rho \circ \varphi_{ij} \in S(A_i)$.

Finally, let $\tau \in T^+A$, which we identify with $\varprojlim(T^+A_i, \varphi_i^*)$. Then for every i, τ restricts to a bounded trace $\varphi_{i\infty}^*\tau$ on A_i, and we have

$$\|\varphi_{i\infty}^*\tau\| = \lim_{n\to\infty}(\varphi_{i\infty}^*\tau)(h_i^{1/n}) = \lim_{n\to\infty}\tau(\varphi_{i\infty}(h_i)^{1/n}) = \lim_{n\to\infty}\tau(\varphi_{1\infty}(h)^{1/n}) = \|\varphi_{1\infty}^*\tau\|.$$

Hence τ is bounded. □

To show that A is simple, we use the following well-known lemma (see [EGJS97, §4] for a unital version).

Lemma 4.3.4. *Let $A = \varinjlim(A_i, \varphi_{ij})$ be an inductive limit of building blocks. Then A is simple if and only if for every $i \in \mathbb{N}$ and every nonzero element a of A_i, the image $\varphi_{ij}(a)$ generates A_j as a closed two-sided ideal for all but finitely many $j \geq i$.*

We require a preliminary lemma (which is implicit but absent in [Raz02]).

Lemma 4.3.5. *Let A and B be building blocks, and let $\varphi : A \to B$ be an injective *-homomorphism. Let $x \in A$ with $\|x(t)\| \geq c$ for some $c > 0$ and every $t \in [0,1]$. Then $\|\varphi(x)(t)\| \geq c$ for every $t \in [0,1]$. In particular, if $f \in \mathrm{Aff}_0 T^+A$ with $\inf \iota(f) > 0$, then $\inf \iota(\varphi_*(f)) > 0$.*

Proof. By assumption, we have $\|\pi_A(x)\| \geq c$ for every $[\pi_A] \in \hat{A}$. Hence $\|\sigma(x)\| \geq c$ for every non zero representation σ of A, and so if $[\pi_B] \in \hat{B}$ then either $\pi_B \circ \varphi = 0$ on A or $\|\pi_B(\varphi(x))\| \geq c$. Since φ is injective, we have $\|\varphi(x)\| > 0$, so there must be some irreducible representation π_B of B such that $\|\pi_B(\varphi(x))\| > 0$. Therefore, by connectedness of $\hat{B}$ and continuity of the map $[\pi] \mapsto \|\pi(\varphi(x))\|$, i.e. $t \mapsto \|\varphi(x)(t)\|$, we must have $\|\pi(\varphi(x))\| \geq c$ for every irreducible representation π of B. Hence $\|\varphi(x)(t)\| \geq c$ for every $t \in [0,1]$, as claimed.

For the second statement, let $f \in \mathrm{Aff}_0 T^+A$ with $\inf \iota(f) > 0$. In particular, we have $f(\mathrm{tr} \otimes \delta_t) > 0$ for every $t \in [0,1]$. Then, by definition of ψ_A, we can find $c' > 0$ such that $\|\psi_A(f)(t)\| \geq c'$ for every $t \in [0,1]$. By the first part of the lemma, we then have $\|\varphi \circ \psi_A(f)(t)\| \geq c'$ for every $t \in [0,1]$. Thus, $\{\varphi \circ \psi_A(f)(t) : t \in [0,1]\}$ is contained in the compact set $\{y \in (M_{n'})_+ : c' \leq \|y\| \leq \|\psi_A(f)\|\} \subset M_{n'}$. Since tr is faithful and continuous, we can therefore find $c > 0$ such that $\mathrm{tr}(\varphi \circ \psi_A(f)(t)) \geq c$ for $t \in [0,1]$. But then (using p.35 for the first two steps)

$$\begin{aligned}
\inf \iota(\varphi_*(f)) &= \inf_{\tau \in T_1^+ B} \varphi_*(f)(\tau) \\
&= \inf_{\tau \in T_1^+ B} \tau(\varphi \circ \psi_A(f)) \\
&= \inf_{\mu \in M_1^+(0,1]} \int \operatorname{tr}(\varphi \circ \psi_A(f)(t)) d\mu(t) \\
&\geq c \\
&> 0.
\end{aligned}$$

□

Proof of Lemma 4.3.4. First suppose that the 'only if' condition holds, and that J is an ideal of $A = \overline{\bigcup_{i=1}^{\infty} A_i}$. If J is nonzero then there exists some n such that $J \cap A_n \neq 0$. (In fact, $J = \overline{\bigcup_{i=1}^{\infty} J \cap A_i}$ — see e.g. [Dav96, Lemma III.4.1].) So let a be a nonzero element of $J \cap A_n$. By assumption, a generates A as an ideal, and so $J = A$. Thus A is simple.

Conversely, suppose that A is simple. Let $h \in A_i$ be an element with $\|h(t)\| \geq 1$ for every $t \in [0,1]$. For every $j \geq i$, define $h_j := \varphi_{ij}(h)$ — by Lemma 4.3.5, we have $\|h_j(t)\| \geq 1$ for every $t \in [0,1]$. Now let a be a nonzero element of A_i. For every $j \geq i$, let I_j be the closed two-sided ideal generated by $a_j := \varphi_{ij}(a)$ in A_j and let $F_j = \Sigma(I_j)$ (i.e. I_j consists of those elements of A_j which vanish on the closed set F_j).

Note that $\varphi_j(I_j) \subset I_{j+1}$ for every j, so we let $I := \overline{\bigcup_{j=i}^{\infty} \varphi_{j\infty}(I_j)} \lhd A$. Since $\|h_j(t)\| \geq 1$ for every $t \in [0,1]$, we have $d(h_j, I_j) \geq 1$ whenever F_j is nonempty. Thus there must exist some j such that F_j *is* empty (otherwise, $h_\infty := \varphi_{i\infty}(h)$ would not be contained in I, so I would be a proper ideal of A). Therefore at some stage k, we must have $I_k = A_k$. In particular, I_k contains h_k and so for every $j \geq k$, I_j contains $h_j = \varphi_{kj}(h_k)$, so F_j is empty, so $I_j = A_j$. □

Proposition 4.3.6. *Let $A = \varinjlim(A_i, \varphi_i)$ for any inductive sequence (A_i, φ_i) as in Proposition 4.3.1. Then A is simple and has a unique tracial state.*

Proof. If f is a nonzero element of A_i then there is an interval $U \subset [0,1]$ on which f is nonzero. By (4.2) and (4.3) of Proposition 4.3.1, if $j \geq i$ is large enough then there is some $1 \leq k \leq m$ such that $\xi_k([0,1]) \subset U$. Then $f \circ \xi_k$, and hence $\varphi_{ij}(f)$, is nonzero on all of $[0,1]$, and so $\varphi_{ij}(f)$ generates A_j as a closed two-sided ideal. Lemma 4.3.4 therefore implies that A is simple.

Next, note that A has a nonzero trace. Since A is stably projectionless, simple and exact, we know this for abstract reasons (i.e. [Rør02, Theorem 1.1.5]). Slightly less abstractly, we can argue as follows. As in section 3.4 of Chapter 3, we have $T^+A \cong \varprojlim(T^+A_i, \varphi_i^*)$ in the category of topological convex cones. Fix an order unit $\eta_1 \in \mathrm{Aff}_0 T^+A_1$ with $\|\eta_1\|_{A_1} \le 1$ and write $\eta_i := (\varphi_{1i})_*(\eta_1)$ for each $i \in \mathbb{N}$; then Lemma 4.3.5 implies that for every i, η_i is an order unit of $\mathrm{Aff}_0 T^+A_i$, with Σ_{η_i} a compact base of T^+A_i. Thus, T^+A has a compact base $\Sigma = \varprojlim \Sigma_i$. Since the Σ_i are nonempty compact Hausdorff spaces it follows that Σ is nonempty (and does not contain zero since it is a base).

By Lemma 4.3.3, this trace is bounded and we now show that it is unique. Let $f \in A_i$ and $\epsilon > 0$ be given and choose $\delta > 0$ such that $\|f(y) - f(z)\| \le \epsilon/2$ whenever $|y - z| \le \delta$. Then provided $2^{j-i} > 1/\delta$, it follows from (4.2) that $\|f(\xi_k(x)) - f(\xi_k(y))\| \le \epsilon/2$ for every $x, y \in [0,1]$ and $1 \le k \le m$. It follows that for all sufficiently large j, every $\tau = \mathrm{tr} \otimes \mu \in T_1^+A_j$ and for fixed $y \in (0,1]$, we have

$$|\tau(\varphi_{ij}(f)) - \mathrm{tr} \otimes \delta_y(\varphi_{ij}(f))| = \left| \int \mathrm{tr}(\varphi_{ij}(f)(x) - \varphi_{ij}(f)(y)) d\mu(x) \right| \le \epsilon/2 \qquad (4.5)$$

and so

$$|\tau_{j,1}(\varphi_{ij}(f)) - \tau_{j,2}(\varphi_{ij}(f))| \le \epsilon \quad \text{for every} \quad \tau_{j,1}, \tau_{j,2} \in T_1^+A_j.$$

Hence A has at most one tracial state. □

As in the introduction to this chapter, we will denote the unique such inductive limit by W and its unique tracial state by τ.

4.4 A categorical description of W

In this section we characterize (W, τ) as a terminal object (Theorem 4.4.5), and use this description to prove that every trace-preserving endomorphism of W is approximately inner (Corollary 4.4.6), and that every simple Razak algebra embeds into $W \otimes \mathcal{K}$ (Corollary 4.4.7). We accomplish this via the following adaptation of [Rør04, Theorem 2.1].

Lemma 4.4.1. *Let B be a building block and let τ be the unique tracial state on W.*

*(i) For every faithful trace τ_0 on B with $\|\tau_0\| \le 1$ there exists a *-homomorphism $\psi : B \to W$ such that $\tau \circ \psi = \tau_0$.*

*(ii) Two *-homomorphisms $\psi_1, \psi_2 : B \to W$ are approximately unitarily equivalent if and only if $\tau \circ \psi_1 = \tau \circ \psi_2$.*

To prove this, we need to use Razak's local existence and local uniqueness theorems, which appear as [Raz02, Theorem 3.1] and [Raz02, Theorem 4.1] respectively. We restate them here for convenience, referring to section 4.2 for notation.

Proposition 4.4.2 (Local existence)**.** *Let B be a building block, and fix some finite subset $F \subset \mathrm{Aff}_0 T^+ B$ and some $\epsilon > 0$. Then there is a natural number N and some $\eta \in \mathrm{Aff}_0 T^+ B$ with $\|\eta\|_B \leq 1$ and $\inf \iota(\eta) \geq 1/2$ such that the following property holds. For any building block $A = A(n,(a+1)n)$ and contractive positive linear map $\xi : (\mathrm{Aff}_0 T^+ B, \|\cdot\|_B) \to (\mathrm{Aff}_0 T^+ A, \|\cdot\|_A)$, if $n \geq Na/\inf \iota(\xi(\eta))$ then there is a *-homomorphism $\psi : B \to A$ with $\|\xi(f) - \psi_*(f)\|_{\xi(\eta)} < \epsilon$ for every $f \in F$.*

Proposition 4.4.3 (Local uniqueness)**.** *Let B be a building block and let h be the canonical self-adjoint element of B (as in Lemma 4.3.3). Fix a finite subset $F \subset B$ and a tolerance $\epsilon > 0$. Then there exists a natural number M and two families of positive functions $\{\zeta_j\}_{j=1}^M, \{\sigma_j\}_{j=1}^M$ in the unit ball of $(\mathrm{Aff}_0 T^+ B, \|\cdot\|_B)$ such that for any building block A and any two *-homomorphisms $\varphi, \psi : B \to A$ that satisfy*

(i) $\varphi_*(\zeta_j)(\tau) > m$, $\psi_*(\zeta_j)(\tau) > m$, and $|\varphi_*(\sigma_j)(\tau) - \psi_*(\sigma_j)(\tau)| < m$ for some $m > 0$ and every $\tau \in T_1^+ A$ and $1 \leq j \leq M$;

(ii) $\varphi^t(h)$ and $\psi^t(h)$ have at least three distinct eigenvalues for every $t \in [0,1]$;

there exists a unitary $u \in \widetilde{A}$ such that $\|\varphi(f) - u\psi(f)u^*\| < \epsilon$ for every $f \in F$.

Typically, the eigenvalue condition (ii) of Proposition 4.4.3 is handled by the following standard consequence of Lemma 4.3.4.

Lemma 4.4.4 (δ-density)**.** *Let $A = \varinjlim(A_i, \varphi_{ij})$ be a simple inductive limit of building blocks $A_i = A(n_i,(a_i+1)n_i)$, and let $h \in A_1$ be the canonical self-adjoint element $h(t) = (\overbrace{1 \oplus \cdots \oplus 1}^{a} \oplus t) \otimes 1_{n_1}$. Then for every $\delta > 0$, there exists an integer N such that for every $j \geq N$ and every $x \in [0,1]$, the eigenvalues of $\varphi_{1j}^x(h)$ are δ-dense in $[0,1]$.*

Proof. Suppose not; then we may assume by passing to a subsequence that there exists $\delta > 0$ such that for every $i > 1$ there exist some $x_i, z_i \in [0,1]$ such that the spectrum of $\varphi_{1i}^{x_i}(h)$ does not intersect the ball $B_\delta(z_i) =: U_i$. (Note that $\varphi_{1i}^{x_i}(h)$ is a positive contraction, so its spectrum — i.e. the set of its eigenvalues — is contained in $[0,1]$.) By compactness of $[0,1]$, we may pass to a subsequence and assume that $z_i \to z \in [0,1]$. Then $U := B_{\delta/2}(z)$ is contained in each U_i for i sufficiently large. Let $f \in C_0(0,1]$ be a nonzero function supported on U. Under the Gelfand isomorphism of $C_0(\sigma(h))$ with $C^*(h)$, f corresponds to the map $f(h) : t \to (\overbrace{f(1) \oplus \cdots \oplus f(1)}^{a} \oplus f(t)) \otimes 1_{n_1}$ and we have

$$\varphi_{1i}^{x_i}(f(h)) = f(\varphi_{1i}^{x_i}(h)) = 0 \qquad \text{for all large } i.$$

This contradicts Lemma 4.3.4. $\square$

Note that Lemma 4.4.4 implies that $n_i \to \infty$ as $i \to \infty$ for any simple inductive limit of building blocks $A_i = A(n_i, (a_i + 1)n_i)$. Of course, we already know this for W by construction; either way, this will allow us to deal with the hypothesis of Proposition 4.4.2 (see also Remark 4.3.2).

We also need to use the fact that, since the building blocks are one-dimensional NCCW complexes, we can appeal to Proposition 2.2.1. We will use this in the proof of Lemma 4.4.1 (ii).

Proof of Lemma 4.4.1. As usual, write $W = \overline{\bigcup_{i=1}^{\infty} A_i}$ with $A_i = A(n_i, (a_i + 1)n_i)$.
(i) Let τ_0 be a faithful trace in Σ_B and fix an increasing sequence $F_1 \subset F_2 \subset \cdots$ of finite sets of self-adjoint elements of the unit ball B_1 of B such that $\bigcup_{k=1}^{\infty} F_k$ is dense in the self-adjoint part of B_1. We will find a sequence $(i_k)_{k=1}^{\infty}$ in $\mathbb{N}$, together with *-homomorphisms $\psi_k : B \to A_{i_k}$ and unitaries $u_k \in \widetilde{A_{i_k}}$ (with $u_1 = 1$) such that

$$\|\psi_k(f) - u_{k+1}\psi_{k+1}(f)u_{k+1}^*\| < 2^{-k} \quad \text{and} \quad |\tau(\psi_k(f)) - \tau_0(f)| < 1/k$$

for every $f \in F_k$. We will then have an approximately commutative diagram

$$\begin{array}{ccccccccc}
B & \xrightarrow{\text{id}} & B & \xrightarrow{\text{id}} & B & \xrightarrow{\text{id}} & \cdots & \longrightarrow & B \\
\downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \mathrm{Ad}_{u_2}\circ\psi_2} & & \downarrow{\scriptstyle \mathrm{Ad}_{u_2 u_3}\circ\psi_3} & & & & \vdots{\scriptstyle \psi} \\
A_{i_1} & \longrightarrow & A_{i_2} & \longrightarrow & A_{i_3} & \longrightarrow & \cdots & \longrightarrow & W
\end{array}$$

such that $|\tau(\mathrm{Ad}_{u_1 \cdots u_k} \circ \psi_k(f)) - \tau_0(f)| < 1/k$ for every $f \in F_k$, and we get a *-homomorphism $\psi : B \to W$ that satisfies $\psi(f) = \lim_{k \to \infty} u_1 \cdots u_k \psi_k(f) u_k^* \cdots u_1^*$ for every $f \in B$. This will imply that $\tau \circ \psi = \tau_0$.

The idea is to use local existence (Proposition 4.4.2) to find the ψ_k and local uniqueness (Proposition 4.4.3) to find the u_k. Working inductively, fix $k \geq 1$, and let $\{\zeta_j\}_{j=1}^{M}, \{\sigma_j\}_{j=1}^{M} \subset \mathrm{Aff}_0 T^+ B$ be the test functions in Proposition 4.4.3 corresponding to the finite set F_k and the tolerance $\epsilon_k = 2^{-k}$. Define G_k to be the finite set $F_k \cup \{\zeta_j\}_{j=1}^{M} \cup \{\sigma_j\}_{j=1}^{M} \cup G_{k-1} \subset B_{sa}$

(with $G_0 := \emptyset$). Set $c_k := 2/3 \min\{\tau_0(\zeta_j) : 1 \leq j \leq M\}$ and $\delta_k := \min\{1/k, c_k/2, c_{k-1}/2\}$. Since τ_0 is faithful, we have $c_k > 0$ and $\delta_k > 0$.

Let $\eta_k \in \mathrm{Aff}_0 T^+ B$ and $N_k \in \mathbb{N}$ be as in Proposition 4.4.2, corresponding to the finite set G_k and the tolerance $\delta_k/2$. For each $i \in \mathbb{N}$, fix $\nu_i \in \mathrm{Aff}_0 T^+ A_i \cong C[0,1]_{a_i}$ with, say, $\|\nu_i\|_{A_i} = 1$ and $\inf \iota(\nu_i) = a_i/(a_i+1)$. By construction of W (Proposition 4.3.1, see also Remark 4.3.2) we have $a_i, n_i/a_i \to \infty$ as $i \to \infty$, so we may choose $i_k > i_{k-1}$ (where $i_0 := 1$) such that $n_{i_k}/a_{i_k} > 4N_k/\|\tau_0\|$ and $a_{i_k}/(a_{i_k}+1) > 1 - \delta_k/2$.

Define $\xi_k : \mathrm{Aff}_0 T^+ B \to \mathrm{Aff}_0 T^+ A_{i_k}$ by $\xi_k(f)(\tau') := \nu_{i_k}(\tau') f(\tau_0)$. This ξ_k is positive and linear, and we have

$$\|\xi_k(f)\|_{A_{i_k}} = \sup\{|\xi_k(f)(\tau')| : \|\tau'\| = 1\} = \|\nu_{i_k}\|_{A_{i_k}} |f(\tau_0)| \leq \|f\|_B$$

(since $\|\tau_0\| \leq 1$), so ξ_k is a contraction from $(\mathrm{Aff}_0 T^+ B, \|\cdot\|_B)$ to $(\mathrm{Aff}_0 T^+ A_{i_k}, \|\cdot\|_{A_{i_k}})$. Since

$$\inf \iota(\xi_k(\eta_k)) = \inf_{\tau' \in T_1^+ A_{i_k}} \xi_k(\eta_k)(\tau') = \eta_k(\tau_0) \inf \iota(\nu_{i_k}) \geq \frac{\eta_k(\tau_0)}{2} \geq \frac{\|\tau_0\|}{4} \geq \frac{N_k a_{i_k}}{n_{i_k}},$$

Proposition 4.4.2 implies that there exists a *-homomorphism $\psi_k : B \to A_{i_k}$ such that $\|\xi_k(f) - (\psi_k)_*(f)\|_{\xi_k(\eta_k)} < \delta_k/2$ for every $f \in G_k$. Moreover, we have

$$|\nu_{i_k}(\tau') - 1| \leq 1 - a_{i_k}/(a_{i_k}+1) < \delta_k/2$$

for every $\tau' \in T_1^+ A_{i_k}$. Hence

$$\begin{aligned}
\delta_k/2 &> \sup_{\xi_k(\eta_k)(\tau')=1} |\nu_{i_k}(\tau') f(\tau_0) - (\psi_k)_*(f)(\tau')| \\
&= \sup_{\nu_{i_k}(\tau')=1/\eta_k(\tau_0)} |\nu_{i_k}(\tau') f(\tau_0) - (\psi_k)_*(f)(\tau')| \\
&\geq \sup_{\|\tau'\|=1} |\nu_{i_k}(\tau') f(\tau_0) - (\psi_k)_*(f)(\tau')| \\
&\geq \sup_{\|\tau'\|=1} |f(\tau_0) - (\psi_k)_*(f)(\tau')| - \delta_k/2
\end{aligned}$$

for every $f \in G_k$. (For the penultimate inequality we have used the fact that $\eta_k(\tau_0) \leq 1$ and $\|\nu_{i_k}\|_{A_{i_k}} = 1$.) Thus

$$|\tau'(\psi_k(f)) - \tau_0(f)| < \delta_k \quad \forall f \in G_k \quad \forall \tau' \in T_1^+ A_{i_k}. \tag{4.6}$$

In particular, noting Lemma 4.3.3 and its proof, $|\tau(\psi_k(f)) - \tau_0(f)| < 1/k$ for every $f \in F_k$, and for every $\tau' \in T_1^+ A_{i_{k+1}} \subseteq T_1^+ A_{i_k}$ and $1 \leq j \leq M$ we have

$$(\psi_k)_*(\zeta_j)(\tau') = \tau'(\psi_k(\zeta_j)) > \tau_0(\zeta_j) - \delta_k \geq \frac{3}{2} c_k - \frac{1}{2} c_k = c_k,$$

$$(\psi_{k+1})_*(\zeta_j)(\tau') > \tau_0(\zeta_j) - \delta_{k+1} \geq \frac{3}{2}c_k - \frac{1}{2}c_k = c_k,$$

and

$$\begin{aligned}|(\psi_{k+1})_*(\sigma_j)(\tau') - (\psi_k)_*(\sigma_j)(\tau')| &\leq |\tau'(\psi_{k+1}(\sigma_j)) - \tau_0(f)| + |\tau'(\psi_k(\sigma_j)) - \tau_0(f)| \\ &< \delta_{k+1} + \delta_k \\ &\leq c_k.\end{aligned}$$

By Lemma 4.4.4, we may also assume (by replacing each ψ_k by $\varphi_{i_k,l} \circ \psi_k$ as necessary) that for every k, $\psi_k^t(h)$ has at least three distinct eigenvalues for every $t \in [0,1]$. Hence, by Proposition 4.4.3, there exists a unitary $u_{k+1} \in \widetilde{A_{i_{k+1}}}$ such that $\|\psi_k(f) - u_{k+1}\psi_{k+1}(f)u_{k+1}^*\| < 2^{-k}$ for every $f \in F_k$, as required.

(ii) The 'only if' part is obvious. Suppose conversely that $\psi_1, \psi_2 : B \to W$ are *-homomorphisms with $\tau \circ \psi_1 = \tau \circ \psi_2$. If either map is zero, then the statement is trivial, so we may assume that both ψ_1 and ψ_2 are injective (see section 4.2). Fix a finite subset $F \subset B_{sa}$ and a tolerance $\epsilon > 0$, and let $\{\zeta_j\}_{j=1}^M, \{\sigma_j\}_{j=1}^M \subset \mathrm{Aff}_0 T^+B$ be the test functions in Proposition 4.4.3 corresponding to F and $\epsilon/3$. Set $F' := F \cup \{\zeta_j\}_{j=1}^M \cup \{\sigma_j\}_{j=1}^M$ and $\delta := \min\{\epsilon/3, \tau(\psi_1(\zeta_j))/6 : 1 \leq j \leq M\}$; since ψ_1 and ψ_2 are injective, we have $\delta > 0$. By Proposition 2.2.1, for all sufficiently large k there exist *-homomorphisms $\psi_1^{(k)}, \psi_2^{(k)} : B \to A_k$ such that

$$\|\psi_m^{(k)}(f) - \psi_m(f)\| < \delta \qquad \forall f \in F',\ m = 1, 2. \tag{4.7}$$

Note in particular that

$$|\tau \circ \psi_1^{(k)}(f) - \tau \circ \psi_2^{(k)}(f)| < 2\delta \qquad \forall f \in F'. \tag{4.8}$$

We may also assume that k is large enough so that

$$\sup\{|\tau(x) - \tau'(x)| : \tau' \in T_1^+ A_k\} < \delta \qquad \forall x \in \psi_1^{(k)}(F') \cup \psi_2^{(k)}(F') \tag{4.9}$$

(as in (4.5) of Proposition 4.3.6) and so that condition (ii) of Proposition 4.4.3 holds. By (4.8) and (4.9) we have

$$|(\psi_1^{(k)})_*(\sigma_j)(\tau') - (\psi_2^{(k)})_*(\sigma_j)(\tau')| < 4\delta$$

and by (4.9), (4.7) and our choice of δ we have

$$(\psi_m^{(k)})_*(\zeta_j)(\tau') > \tau(\psi_m^{(k)}(\zeta_j)) - \delta > \tau(\psi_m(\zeta_j)) - 2\delta \geq 4\delta$$

for $m = 1, 2$, $1 \leq j \leq M$ and $\tau' \in T_1^+ A_k$. Hence, by Proposition 4.4.3, there exists a unitary $u \in \widetilde{A_k}$ such that $\|\psi_1^{(k)}(f) - u\psi_2^{(k)}(f)u^*\| < \epsilon/3$ for every $f \in F$, which implies that

$$\|\psi_1(f) - u\psi_2(f)u^*\| < \epsilon \qquad \forall f \in F.$$

This proves that ψ_1 and ψ_2 are approximately unitarily equivalent. □

An object T in a category $\mathcal{C}$ is *terminal* if for every object X in $\mathcal{C}$ there exists a unique morphism from X to T; such objects are unique up to isomorphism. For example, the Cuntz algebra $\mathcal{O}_2$ is the unique terminal object in the category of strongly self-absorbing C^*-algebras, where the morphisms are approximate unitary equivalence classes of unital *-homomorphisms (see [KP00] and [TW07]). We now use Lemma 4.4.1 and an intertwining argument to characterize (W, τ) as a terminal object.

Theorem 4.4.5. *(W, τ) is the unique terminal object in the category whose objects are pairs (A, τ_A) with A a simple Razak algebra and $\tau_A \in \Sigma_A$, and where a morphism from (A, τ_A) to (B, τ_B) is (the approximate unitary equivalence class of) a *-homomorphism $\psi : A \to B$ with $\psi^*\tau_B = \tau_A$.*

Proof. Let $B = \varinjlim(B_i, \beta_i)$ be a simple Razak algebra and let $\tau_0 \in \Sigma_B$. We need to show that there is a *-homomorphism $\psi : B \to W$ with $\psi^*\tau = \tau_0$, and prove that, up to approximate unitary equivalence, ψ is the unique map $B \to W$ with this property. This is obvious if $\tau_0 = 0$ (since τ is faithful), so we may assume that τ_0 is nonzero, hence faithful (since B is simple). Write $\tau_0 = (\tau_i)_{i=1}^\infty$, where for $i \in \mathbb{N}$, τ_i is a faithful trace on B_i with $\|\tau_i\| \leq 1$ and $\tau_{i+1} \circ \beta_i = \tau_i$. By Lemma 4.4.1(i), for each i there exists a *-homomorphism $\psi_i : B_i \to W$ with $\tau \circ \psi_i = \tau_i$. But $\tau \circ \psi_{i+1} \circ \beta_i = \tau_{i+1} \circ \beta_i = \tau_i$, so by Lemma 4.4.1(ii), we have $\psi_{i+1} \circ \beta_i \sim_{a.u.} \psi_i$. It then follows from a (one-sided) approximate intertwining (as in the proof of Lemma 4.4.1(i)) that there is a *-homomorphism $\psi : B \to W$ which, by construction, satisfies $\psi^*\tau = \tau_0$. By restricting ψ to the building blocks B_i, Lemma 4.4.1(ii) says that, up to approximate unitary equivalence, ψ is the unique *-homomorphism $B \to W$ with this property. □

The next two results are immediate corollaries of Theorem 4.4.5.

Corollary 4.4.6. *Every trace-preserving endomorphism (hence every nondegenerate endomorphism) of W is approximately inner. That is, for any such endomorphism θ, there is a sequence of unitaries u_n in $\widetilde{W}$ such that $\theta(a) = \lim_{n\to\infty} u_n a u_n^*$ for every $a \in W$.*

Corollary 4.4.7. *Let B be a simple Razak algebra. Then B admits a tracial state if and only if B is isomorphic to a subalgebra of W. If B has no nonzero bounded traces, then B is stable and is isomorphic to a subalgebra of $W \otimes \mathcal{K}$.*

Proof. The first assertion follows from Theorem 4.4.5. For the second assertion, note that if B is a simple Razak algebra then so is $B \otimes \mathcal{K}$, and $(T^+(B \otimes \mathcal{K}), \Sigma_{B\otimes\mathcal{K}}) \cong (T^+B, 0)$ (by Lemma 3.5.5). Therefore, Theorem 4.1.1 implies that $B \cong B \otimes \mathcal{K}$ whenever B has no nonzero bounded traces, and Theorems 4.1.3 and 4.1.1 imply that *every* simple Razak algebra is stably isomorphic to a simple Razak algebra which has no unbounded traces. (To see the latter, given a simple Razak algebra A_1, apply Theorem 4.1.3 to the cone $C = T^+A_1$, with ω defined to be 1 on a base of C, to obtain a simple Razak algebra A_2 with the required properties.) The second statement therefore follows from the first. □

4.5 $W \otimes W \cong W$

Let us first make a remark on the proof that $\mathcal{Z}$ is strongly self-absorbing. Jiang and Su adopt the following strategy.

(i) Prove that the two maps $\mathrm{id} \otimes 1, 1 \otimes \mathrm{id} : \mathcal{Z} \to \mathcal{Z} \otimes \mathcal{Z}$ are approximately unitarily equivalent.

(ii) Show that there exists a unital *-homomorphism $\psi : \mathcal{Z} \otimes \mathcal{Z} \to \mathcal{Z}$.

(iii) Prove that $(\mathrm{id} \otimes 1) \circ \psi \sim_{a.u.} \mathrm{id}_{\mathcal{Z}\otimes\mathcal{Z}}$ and note that $\psi \circ (\mathrm{id} \otimes 1) \sim_{a.u.} \mathrm{id}_{\mathcal{Z}}$ since every unital endomorphism of $\mathcal{Z}$ is approximately inner. A standard intertwining argument [Rør94, Proposition A] then shows that there exists an isomorphism $\varphi : \mathcal{Z} \to \mathcal{Z} \otimes \mathcal{Z}$. Again, since every unital endomorphism of $\mathcal{Z}$ is approximately inner, it follows easily that $\varphi \sim_{a.u.} \mathrm{id} \otimes 1$.

Step (ii) in this procedure goes roughly as follows: Write the Jiang-Su algebra as $\mathcal{Z} = \varinjlim(A_n, \varphi_n)$, where each $A_n = I[p_n, d_n, q_n]$ is a prime dimension drop algebra (i.e. p_n and q_n are coprime, $d_n = p_n q_n$ and $A_n = \{f \in C([0,1], M_{d_n}) : f(0) \in M_{p_n} \otimes 1_{q_n}, f(1) \in 1_{p_n} \otimes M_{q_n}\}$). Define B_n to be the diagonal of $A_n \otimes A_n$, i.e. B_n consists of all continuous functions $f : [0,1] \to M_{d_n^2}$ such that $f(0) \in (M_{p_n} \otimes 1_{q_n})^{\otimes 2}$ and $f(1) \in (1_{p_n} \otimes M_{q_n})^{\otimes 2}$. Then $B_n \cong I[p_n^2, d_n^2, q_n^2]$ is a prime dimension drop algebra and we have a *-homomorphism $\rho_n : A_n \otimes A_n \to B_n$ given by restriction: $\rho_n(f)(x) = f(x,x)$ for $f \in A_n \otimes A_n$ and $x \in [0,1]$.

Jiang and Su construct connecting maps $\psi_n : B_n \to B_{n+1}$ such that the diagram

$$\begin{array}{ccccccccc}
A_1 \otimes A_1 & \xrightarrow{\varphi_1 \otimes \varphi_1} & A_2 \otimes A_2 & \xrightarrow{\varphi_2 \otimes \varphi_2} & A_3 \otimes A_3 & \longrightarrow & \cdots & \longrightarrow & \mathcal{Z} \otimes \mathcal{Z} \\
\downarrow{\scriptstyle \rho_1} & & \downarrow{\scriptstyle \rho_2} & & \downarrow{\scriptstyle \rho_3} & & & & \downarrow{\scriptstyle \rho} \\
B_1 & \xrightarrow{\psi_1} & B_2 & \xrightarrow{\psi_2} & B_3 & \longrightarrow & \cdots & \longrightarrow & B
\end{array}$$

commutes approximately (where $B := \varinjlim(B_n, \psi_n)$), so there is an induced morphism $\rho : \mathcal{Z} \otimes \mathcal{Z} \to B$. They also show that B is simple, and since it is an inductive limit of (prime) dimension drop algebras, their Theorem 6.2 then shows that there exists a morphism from B to $\mathcal{Z}$; the composition of this morphism with ρ is the required morphism $\psi : \mathcal{Z} \otimes \mathcal{Z} \to \mathcal{Z}$.

In fact, B is just $\mathcal{Z}$ in disguise. All of the restriction maps ρ_n are surjective, and hence so is the induced *-homomorphism ρ. Since $\mathcal{Z}$ is simple, so is $\mathcal{Z} \otimes \mathcal{Z}$ and hence ρ is also injective, so is an isomorphism. Thus B is simple and has the same invariant as $\mathcal{Z} \otimes \mathcal{Z}$, which has the same invariant as $\mathcal{Z}$. Since B is a simple unital inductive limit of dimension drop algebras, the classification theorem [JS99, Theorem 6.2] then shows that in fact $B \cong \mathcal{Z}$ and thus $\mathcal{Z} \otimes \mathcal{Z} \cong \mathcal{Z}$.

This means that that steps (i) and (iii) are unnecessary, so we obtain a shorter version of Jiang and Su's proof. Moreover, one could try adapting the argument to prove that $W \otimes W \cong W$, as indeed I claimed to be able to do in an earlier version of this thesis. However, because the building blocks of $W \otimes W$ have as spectrum the torus, rather than the square, this approach contains serious difficulties (specifically, in constructing connecting maps $\psi_n : B_n \to B_{n+1}$ in the manner of Jiang and Su).

The closest we get to a proof is via the following argument, due to Luis Santiago, which relies on Proposition 2.3.2. First we observe that $W \otimes W$ has stable rank one (i.e. the set of invertible elements of $(W \otimes W)^\sim$ is dense in $(W \otimes W)^\sim$).

Proposition 4.5.1. *$W \otimes W$ has stable rank one.*

Proof. Write $W = \varinjlim(A_i, \varphi_i)$ in the usual way, so that, with $\Phi_i := \varphi_i \otimes \varphi_i$, we have $W \otimes W = \varinjlim(A_i \otimes A_i, \Phi_i)$. We showed in Chapter 2 that $A_i \otimes A_i$ and its unitization $(A_i \otimes A_i)^\sim$ are two-dimensional NCCW complexes, and we regard them as subalgebras of $C([0,1]^2, M_{(n_i')^2})$. Then Φ_i extends to a unital *-homomorphism $\Phi_i : (A_i \otimes A_i)^\sim \to$

$(A_{i+1} \otimes A_{i+1})^{\sim}$ and has the form

$$\Phi_i(g)(x,y) = v(x,y)\mathrm{diag}(g(\xi_k(x), \xi_l(y)) : 1 \le k, l \le m)v(x,y)^* \tag{4.10}$$

for $g \in (A_i \otimes A_i)^{\sim}$ and $(x,y) \in [0,1]^2$, where v is an appropriate unitary in $C([0,1]^2, M_{(n_i')^2})$.

Now, $(W \otimes W)^{\sim} = \varinjlim((A_i \otimes A_i)^{\sim}, \Phi_i)$ is an inductive limit of two-dimensional unital NCCW complexes which, though not simple, does have the following properties. Let a be a nonzero element of $(A_i \otimes A_i)^{\sim}$. Then:

(i) for all large j, the image $\Phi_{ij}(a)$ of a in $(A_j \otimes A_j)^{\sim}$ is nonzero on all of $[0,1]^2$;

(ii) if a is positive then the hereditary subalgebra generated by $\Phi_{i\infty}(a)$ in $(W \otimes W)^{\sim}$ is stably isomorphic to a simple, nonelementary C^*-algebra or the unitization of such an algebra.

The first of these can be proved as in Lemma 4.3.6, using the form (4.10) of the connecting maps Φ_i, together with properties (4.2) and (4.3) of the functions ξ_k. For the second, the nonelementary C^*-algebra referred to is of course just $W \otimes W$: it is not isomorphic to $\mathcal{K}(H)$ for a Hilbert space H. The assertion is true because $W \otimes W$ is simple and (hence) is the unique proper ideal of $(W \otimes W)^{\sim}$; stable isomorphism then follows from Brown's Theorem [Bro77, Theorem 2.8].

It is proved in [Phi07a, Theorem 3.6], using a notion of approximate polar decomposition, that if A is a simple inductive limit of recursive subhomogeneous C^*-algebras of bounded dimension then A has stable rank one. But simplicity is only needed to prove [Phi07a, Lemma 1.8], for which properties (i) and (ii) suffice. (The Lemma in question guarantees that nonzero elements of the inductive system eventually have arbitrarily large rank in each fibre.) Since unital NCCW complexes are recursive subhomogeneous (see Remark 2.1.2), it follows that $W \otimes W$ has stable rank one. □

Conjecture 4.5.2. *$W \otimes W \cong W$.*

One way of proving this would be to first prove the following generalization of Theorem 4.1.2.

Conjecture 4.5.3. *Let A and B be inductive limits of one-dimensional NCCW-complexes, such that A and B are simple, nonunital and have trivial K-theory. If (T^+A, Σ_A) is isomorphic to (T^+B, Σ_B) then A is isomorphic to B.*

(This would follow from Theorem 4.1.2 if one could show, for example, that if $\varphi : A \to B$ is a *-homomorphism between one-dimensional NCCW complexes whose induced map $\varphi_* : K_1(A) \to K_1(B)$ is trivial, then φ factors through a one-dimensional NCCW complex C with $K_1(C) = 0$. However, it is not clear at present whether this is true, even for the very specific NCCW complexes that we actually consider.)

Proof of Conjecture 4.5.2 assuming Conjecture 4.5.3. Since $W \otimes W$ has stable rank one, and is an inductive limit of two-dimensional NCCW complexes $A_i \otimes A_i$ with trivial K-theory, Propositions 2.3.2 and 2.2.1 allow us to obtain one-dimensional NCCW complexes B_i, and a diagram

$$\begin{array}{ccccccccc}
A_{n_1}\otimes A_{n_1} & \longrightarrow & A_{n_2}\otimes A_{n_2} & \longrightarrow & A_{n_3}\otimes A_{n_3} & \longrightarrow & \cdots & \longrightarrow & W\otimes W \\
\downarrow \alpha_1 & \nearrow \beta_1 & \downarrow \alpha_2 & \nearrow \beta_2 & \downarrow \alpha_3 & \nearrow \beta_3 & & & \downarrow \alpha \\
B_1 & \xrightarrow{\alpha_2\circ\beta_1} & B_2 & \xrightarrow{\alpha_3\circ\beta_2} & B_3 & \longrightarrow & \cdots & \longrightarrow & B
\end{array} \tag{4.11}$$

whose upper triangles commute arbitrarily well on increasingly large finite sets. That is, (4.11) is an approximate intertwining of C^*-algebras and so induces an isomorphism $\alpha : W \otimes W \to B$. Thus B is is a simple inductive limit of one-dimensional NCCW complexes and has the same invariant as $W \otimes W$, which (by Proposition 3.5.4 and the Künneth Theorem for K-theory) has the same invariant as W (since W has a unique trace, which is bounded). Conjecture 4.5.3 then shows that in fact $B \cong W$ and thus $W \otimes W \cong W$. □

Corollary 4.5.4 (assuming Conjecture 4.5.2)**.** *W has approximately inner flip, i.e. there is a sequence $(u_n)_{n=1}^{\infty}$ of unitaries in the unitization of $W \otimes W$ such that $\lim_{n\to\infty} u_n(a \otimes b)u_n^* = b \otimes a$ for $a, b \in W$.*

Proof. This follows immediately from Corollary 4.4.6 and Conjecture 4.5.2. □

4.6 Asymptotically central sequences

If A and B are C^*-algebras and C is a sub-C^*-algebra of B then a sequence of *-homomorphisms $\varphi_n : A \to B$ is said to be *asymptotically central for C* if $\|[\varphi_n(a), c]\| \to 0$ as $n \to \infty$ for every $a \in A$ and $c \in C$. Such a sequence induces a *-homomorphism $\varphi : A \to B_\infty \cap C'$. Here, B_∞ is the limit algebra $l_\infty(B)/c_0(B)$, and there is a canonical inclusion of B into B_∞ as constant sequences; $B_\infty \cap C'$ denotes the relative commutant of C in B_∞.

Every strongly self-absorbing C^*-algebra $\mathcal{D}$ admits an asymptotically central sequence of unital endomorphisms [TW07, Proposition 1.10]. Conversely, exhibiting asymptotically central *-homomorphisms can be used to prove $\mathcal{D}$-stability — see [Rør94], [Rør02, Theorem 7.2.2] and [TW07, Theorem 2.3]. It is not clear whether there exists such a relationship between asymptotically central sequences and tensor products in the nonunital case, essentially because we lack the notion of an infinite tensor product. Nevertheless, asymptotically central sequences are still interesting and useful in their own right, and we show below that we can adapt the approximate divisibility of W to at least find an embedding of W into the limit algebra $M(W)_\infty \cap W'$.

Definition 4.6.1. A C^*-algebra A is said to be *approximately divisible* if for any $N \in \mathbb{N}$ there is a sequence of unital *-homomorphisms μ_n from $M_N \oplus M_{N+1}$ into the multiplier algebra $M(A)$ which is asymptotically central for A.

Approximate divisibility for unital C^*-algebras was studied in [BKR92], and the nonunital case appears for example in [Rør02, Definition 3.1.10]. Toms and Winter prove in [TW05, §2] that separable approximately divisible C^*-algebras are $\mathcal{Z}$-stable. Moreover, their Proposition 4.1 says that every simple Razak algebra is approximately divisible (hence $\mathcal{Z}$-stable). We show below that this is in fact an immediate consequence of classification (Theorem 4.1.1).

Let $\mathcal{Q}$ denote the universal UHF algebra (characterized by $K_0(\mathcal{Q}) = \mathbb{Q}$); $\mathcal{Q}$ is isomorphic to its infinite tensor product $\mathcal{Q}^{\otimes\infty}$ and (so) is strongly self-absorbing.

Proposition 4.6.2. *Every simple Razak algebra is $\mathcal{Q}$-stable (so absorbs every UHF algebra).*

Proof. Let $B = \varinjlim(B_i, \beta_i)$ be a simple Razak algebra and let $U = \varinjlim(M_{k_i}, \alpha_i)$ be a UHF algebra. Note that if A is a building block then so is $A \otimes M_k$ for every k, so $B \otimes U \cong \varinjlim(B_i \otimes M_{k_i}, \beta_i \otimes \alpha_i)$ is also a simple Razak algebra. Moreover, $(T^+(B \otimes U), \Sigma_{B\otimes U}) \cong (T^+B, \Sigma_B)$ by Lemma 3.5.5. Therefore, Theorem 4.1.1 implies that $B \otimes U \cong B$. □

Corollary 4.6.3. *Let A be a simple Razak algebra. Then A is approximately divisible and there is an isomorphism $\varphi : A \to A \otimes \mathcal{Z}$ such that $\varphi \sim_{a.u.} \mathrm{id}_A \otimes 1_{\mathcal{Z}}$.*

Proof. For each $k \in \mathbb{N}$, let ι_k be a unital embedding of M_k into $\mathcal{Q}$ and define a sequence of unital *-homomorphisms $\mu_{k,m} : M_k \to M(A \otimes \mathcal{Q}^{\otimes\infty})$ by

$$\mu_{k,m} := 1_{\widetilde{A}} \otimes 1_{\mathcal{Q}} \otimes \cdots \otimes 1_{\mathcal{Q}} \otimes \underbrace{\iota_k}_{m} \otimes 1_{\mathcal{Q}} \otimes \cdots \tag{4.12}$$

(where $\widetilde{A}$ is the minimal unitization of A). Then the sequence $(\mu_{k,m})_{m=1}^{\infty}$ is asymptotically central for $A \otimes \mathcal{Q}^{\otimes\infty}$. Thus $A \cong A \otimes \mathcal{Q}^{\otimes\infty}$ is approximately divisible. The second claim follows directly from Proposition 4.6.2 since UHF algebras are $\mathcal{Z}$-stable, or alternatively from [TW05, Theorem 2.3], which says that separable approximately divisible C^*-algebras are $\mathcal{Z}$-stable. That the isomorphism φ satisfies $\varphi \sim_{a.u.} \mathrm{id}_A \otimes 1_{\mathcal{Z}}$ is automatic (see [TW07, Theorem 2.2]). □

Next, we combine the proofs of Corollary 4.6.3 and [TW05, Proposition 2.2] to construct an embedding of W into the central sequences algebra $M(W)_\infty \cap W'$.

Theorem 4.6.4. *Let B be a separable $\mathcal{Q}$-stable C^*-algebra. Then there exists a *-homomorphism $\sigma = (\sigma_i)_{i=1}^{\infty} : W \to (\widetilde{B} \otimes \mathcal{Q}^{\otimes\infty})_\infty \cap (B \otimes \mathcal{Q}^{\otimes\infty})' \subset M(B \otimes \mathcal{Q}^{\otimes\infty})_\infty \cap (B \otimes \mathcal{Q}^{\otimes\infty})' \cong M(B)_\infty \cap B'$ which satisfies the nondegeneracy condition*

$$\|\sigma(a)b\| = \|a\|\|b\| \quad \text{for every } a \in W \text{ and } b \in B, \tag{4.13}$$

and which is trace-preserving in the sense that

$$\lim_{i\to\infty} \rho(\sigma_i(a)) = \tau(a) \quad \text{for every } a \in W \text{ and every } \rho \in T_1^+(B) \tag{4.14}$$

(where τ is the unique tracial state on W).

Remark 4.6.5. If B is a $\mathcal{Q}$-stable C^*-algebra then every tracial state ρ on $B \otimes \mathcal{Q}^{\otimes\infty}$

(i) is of the form $\tau \otimes \tau_{\mathcal{Q}}$ for some tracial state τ on B, where $\tau_{\mathcal{Q}}$ is the unique tracial state on $\mathcal{Q}$, and

(ii) extends uniquely to a tracial state on $\widetilde{B} \otimes \mathcal{Q}^{\otimes\infty} \subset M(B \otimes \mathcal{Q}^{\otimes\infty}) \cong M(B)$.

This is what is meant in (4.14). It may also be more natural to replace B_∞ with an ultrapower B_ω for some free ultrafilter ω, and the proof works equally well in this setting.

Proof. Write $W = \overline{\bigcup_{i=1}^{\infty} A_i}$ with the building blocks $A_i = A(n_i, (a_i+1)n_i)$ and inclusion maps φ_{ij} given by Proposition 4.3.1. As in (4.12), for each $k \in \mathbb{N}$ let ι_k be a unital embedding of M_k into $\mathcal{Q}$, and define *-homomorphisms $\mu_{k,m} : M_k \to \widetilde{B} \otimes \mathcal{Q}^{\otimes\infty} \subset M(B \otimes \mathcal{Q}^{\otimes\infty})$ by

$$\mu_{k,m} := 1_{\widetilde{B}} \otimes 1_{\mathcal{Q}} \otimes \cdots \otimes 1_{\mathcal{Q}} \otimes \underbrace{\iota_k}_{m} \otimes 1_{\mathcal{Q}} \otimes \cdots .$$

Note that the sequence $(\mu_{k,m})_{m=1}^{\infty}$ is asymptotically central for $B \otimes \mathcal{Q}^{\otimes\infty}$ and also that $\|\mu_{k,m}(a)b\| \to \|a\|\|b\|$ as $m \to \infty$ for every $a \in M_k$ and $b \in B \otimes \mathcal{Q}^{\otimes\infty}$ (since this is true for finite tensors). Moreover, if ρ is a tracial state on $B \otimes \mathcal{Q}^{\otimes\infty}$ and tr_k denotes the unique tracial state on M_k, then in the sense of Remark 4.6.5 we have $\rho(\mu_{k,m}(x)) = \mathrm{tr}_k(x)$ for every $x \in M_k$ and $m \in \mathbb{N}$.

For every $i \in \mathbb{N}$, let $\pi_i : A_i \to M_{n_i}$ be the irreducible representation ev_∞ (actually, any point evaluation will do), and define $\sigma_{i,m} := \mu_{n_i,m} \circ \pi_i : A_i \to \widetilde{B} \otimes \mathcal{Q}^{\otimes\infty}$. Let $(b_i)_{i=1}^{\infty}$ be dense in $B \otimes \mathcal{Q}^{\otimes\infty}$ and fix finite subsets $F_i \subset A_i$ such that $F_i \subset F_{i+1}$ and $\overline{\bigcup_{i=1}^{\infty} F_i} = W$. For each i, we can use the properties of the *-homomorphisms $\mu_{n_i,m}$ to choose $m_i \geq m_{i-1}$ such that for $a \in F_i$ and $1 \leq j \leq i$ we have

$$\|[\sigma_{i,m_i}(a), b_j]\| \leq 1/i \tag{4.15}$$

and

$$\Big| \|\sigma_{i,m_i}(a)b_j\| - \|\pi_i(a)\|\|b_j\| \Big| \leq 1/i. \tag{4.16}$$

Note also that, for every $a \in A_i$ and as $j \to \infty$ we have

$$\|\pi_j \circ \varphi_{ij}(a)\| \to \|a\| \tag{4.17}$$

by (4.2) and (4.3) of Proposition 4.3.1 and

$$\mathrm{tr}_{n_j}(\pi_j \circ \varphi_{ij}(a)) \to \tau(a) \tag{4.18}$$

by (4.5) of Proposition 4.3.6. Now we just patch together the σ_{i,m_i} to get the desired map σ (as in the proof of [TW05, Proposition 2.2]). Since σ_{i,m_i} is finite rank, by Arveson's extension theorem we can extend it to a linear, contractive (in fact c.c.p.) map $\overline{\sigma}_{i,m_i} : W \to \widetilde{B} \otimes \mathcal{Q}^{\otimes\infty}$. Define σ to be the map $\sigma := (\overline{\sigma}_{i,m_i})_{i=1}^{\infty} : W \to (\widetilde{B} \otimes \mathcal{Q}^{\otimes\infty})_\infty$. Then σ is linear, contractive and (since the σ_{i,m_i} are *-homomorphisms) is multiplicative on $\bigcup_{i=1}^{\infty} A_i$, hence on all of W. That is, σ is a *-homomorphism. By (4.15), we have $\sigma(W) \subset (B \otimes \mathcal{Q}^{\otimes\infty})'$. For $a \in F_i$ and fixed b_j we have

$$\|\sigma(a)b_j\| = \limsup_k \|\sigma_{k,m_k}(\varphi_{ik}(a))b_j\| = \limsup_k \|\pi_k(\varphi_{ik}(a))\|\|b_j\| = \|a\|\|b_j\|$$

by (4.16) and (4.17). Finally, for $a \in A_i$ and $\rho \in T_1^+(B \otimes \mathcal{Q}^{\otimes\infty})$ we have

$$\lim_{k\to\infty} \rho(\sigma_{k,m_k}(a)) = \lim_{k\to\infty} \rho(\mu_{n_k,m_k} \circ \pi_k(\varphi_{ik}(a))) = \lim_{k\to\infty} \mathrm{tr}_{n_k}(\pi_k(\varphi_{ik}(a))) = \tau(a)$$

by (4.18). Therefore, σ is a *-homomorphism $W \to (\widetilde{B} \otimes \mathcal{Q}^{\otimes\infty})_\infty \cap (B \otimes \mathcal{Q}^{\otimes\infty})'$ that satisfies (4.13) and (4.14). □

It is easy to check that, by taking an approximate unit $(e_i)_{i=1}^\infty$ of B and cutting σ down by an appropriate subsequence of $(e_i \otimes 1_{\mathcal{Q}^{\otimes\infty}})_{i=1}^\infty$, we can get a trace-preserving completely positive contractive map from W into $B_\infty \cap B'$ which preserves orthogonality (i.e. is 'order zero' — see [WZ09] and also Chapter 5 below).

Corollary 4.6.6. *There exists a *-homomorphism $\sigma = (\sigma_i)_{i=1}^\infty : W \to M(W)_\infty \cap W'$ which satisfies $\|\sigma(a)b\| = \|a\|\|b\|$ for every $a, b \in W$ and $\lim_{i\to\infty} \tau(\sigma_i(a)) = \tau(a)$ for every $a \in W$. In particular, there is a τ-preserving c.p.c. order zero embedding $W \to W_\infty \cap W'$.*

4.7 W-stability

In the previous section we observed a dearth of available machinery for characterizing W-stability. However, we *can* appeal to classification. This, after all, is what makes classification useful: if you want to show that an object A has certain properties, you only need to find some object B which obviously has said properties, and which has the same invariant as A. In the present context, if we want to show that W is absorbed by a given class of C^*-algebras (which has already been classified), and assuming that $W \otimes W \cong W$, we just need to prove that the range of the classifying invariant is exhausted by algebras of the form $B \otimes W$. The class we consider includes all simple Razak algebras, and it turns out that we only need to tensor W with AF algebras.

The main result of this section is the following extension of a theorem of Blackadar [Bla80] and Goodearl [Goo78]. It is likely to be already known by experts.

Theorem 4.7.1. *Let Δ be a metrizable Choquet simplex, let C be the cone with Δ as its base, and let ω be a lower semicontinuous affine map $\Delta \to (0, \infty]$. Then there exists a simple AF algebra A and an isomorphism $T^+A \cong C$ under which ω corresponds to the norm map on T^+A.*

Proof. By [Bla80, Theorem 3.10], there exists a simple unital AF algebra B with $T_1^+B = \Delta$. Next, by [Alf71, Corollary I.1.4], we can find a strictly increasing sequence of continuous affine maps $\omega_n : \Delta \to \mathbb{R}$ that converges pointwise to ω; since ω is strictly positive, we may choose the ω_n to be strictly positive as well. It is well known that, since B is a simple unital AF algebra, it has the following properties.

(i) By tensoring with a UHF algebra if necessary, we may assume that $K_0(B) \ncong \mathbb{Z}$. Then the image of the natural map $K_0(B) \to \mathrm{Aff}(\Delta)$ is uniformly dense in $\mathrm{Aff}(\Delta)$ (see [Bla80, Proposition 3.1]).

(ii) $K_0(B)$ has the strict ordering from its states, which means that an element $z \in K_0(B)$ is in $K_0(B)_+$ if and only if $z = 0$ or $\tau_*(z) > 0$ for every $\tau \in \Delta$ (see [Bla98, Theorem 6.8.5]).

(iii) B has strict comparison, so that if p and q are projections in $B \otimes \mathcal{K}$ with $\tau(p) < \tau(q)$ for every $\tau \in T^+(B \otimes \mathcal{K}) \cong C$, then p is subequivalent to q (see [Bla98, Corollary 6.9.2]).

Using (i) and (ii), we can find projections $p_n \in B \otimes \mathcal{K}$ such that $\omega(\tau) = \lim_{n\to\infty} \tau(p_n)$ for every $\tau \in \Delta$. By (iii) and compactness of Δ, we may assume that $p_n \leq p_{n+1}$ for every n. Now we just set $A := \overline{\bigcup_{n=1}^{\infty} p_n(B \otimes \mathcal{K})p_n}$. Then A is a (full) hereditary subalgebra of $B \otimes \mathcal{K}$, so is a simple AF algebra which is stably isomorphic to B (for example by [Bro77, Theorem 2.8] and [Ell76a, Theorem 3.1]). Moreover, every trace on A extends uniquely to a trace on B (see [BK04, Remark 2.27 (viii)]), so we can identify T^+A with C; since $(p_n)_{n=1}^{\infty}$ is an approximate unit for A, we then have

$$\|\tau\| = \lim_{n\to\infty} \tau(p_n) = \omega(\tau)$$

for every $\tau \in \Delta$ (and hence, by linearity, for every $\tau \in T^+A$). □

Corollary 4.7.2. *Let A be a simple inductive limit of one-dimensional NCCW complexes A_i such that $K_1(A_i) = 0$ for every i and $K_0(A) = 0$. Then there exists a simple AF algebra B such that $A \cong B \otimes W$. In particular, if Conjecture 4.5.2 is true, then $A \otimes W \cong A$.*

Proof. Taking $C = T^+A$ and $\omega = \|\cdot\|$, Theorem 4.7.1 gives a simple AF algebra B and an isomorphism between T^+B and T^+A that maps the tracial states of B onto the tracial states of A. The C^*-algebra W is by construction a simple inductive limit of one-dimensional NCCW complexes each of which has trivial K-theory, so $B \otimes W$ is also of this form. Moreover, T^+W is generated by a tracial state, so there is an isomorphism between $T^+(B \otimes W)$ and T^+B that maps the tracial states of $B \otimes W$ onto the tracial states of B (see Remark 3.5.3). Theorem 4.1.2 then implies that $A \cong B \otimes W$, and the second statement follows from Conjecture 4.5.2. □

In the opposite direction, we note that a certain dichotomy holds for well-behaved, simple, W-stable C^*-algebras.

Proposition 4.7.3. *Suppose that A is a simple, separable, nuclear C^*-algebra that satisfies the UCT, such that $A \otimes W \cong A$. Then either $A \cong \mathcal{O}_2 \otimes \mathcal{K}$ or A is stably projectionless.*

Proof. Since W lies in the UCT class, we can apply the Künneth Theorem to deduce that $K_*(A) = K_*(A \otimes W) = 0$. Suppose that $A \otimes \mathcal{K}$ contains a nonzero projection p, and let $B := p(A \otimes \mathcal{K})p$. Then B is a unital C^*-algebra which, by Brown's Theorem [Bro77, Theorem 2.8], is stably isomorphic to A; in particular, $K_*(B) = 0$. Hence $[p] = [0]$ in $K_0(B)$, so $p \oplus q \sim 0 \oplus q$ for some projection $q \in M_n(B)$. Hence $p \oplus q$ is infinite, so B is 'stably infinite' and so is A. Then, since $A \cong A \otimes W$ is tensorially non-prime, A must in fact be purely infinite [Rør02, Theorem 4.1.10] and stable [Rør02, Theorem 4.1.3]. The Kirchberg-Phillips classification theorem [Rør02, Theorem 8.4.1] then shows that $A \cong \mathcal{O}_2 \otimes \mathcal{K}$. □

4.8 Quasi-free flows on $\mathcal{O}_2$

Write $\mathcal{O}_2 = C^*(s_1, s_2)$ (where of course s_1 and s_2 are isometries with $s_1 s_1^* + s_2 s_2^* = 1$). For $\lambda \in \mathbb{R}$, consider the action α of $\mathbb{R}$ on $\mathcal{O}_2 = C^*(s_1, s_2)$ given by

$$\alpha_t(s_1) = e^{it} s_1, \qquad \alpha_t(s_2) = e^{\lambda it} s_2.$$

These actions, and the associated crossed products $\mathcal{O}_2 \rtimes_\lambda \mathbb{R}$, have been studied by many authors including Evans, Kishimoto and Kumjian (see [Eva80], [KK96] and [KK97]). Kishimoto and Kumjian proved that $\mathcal{O}_2 \rtimes_\lambda \mathbb{R}$ is simple if and only if λ is irrational; in this case $\mathcal{O}_2 \rtimes_\lambda \mathbb{R}$ is stable, and is purely infinite if $\lambda < 0$ and projectionless with a unique (unbounded) trace if $\lambda > 0$. Moreover, Dean proves in [Dea01] that for generic positive irrational λ, $\mathcal{O}_2 \rtimes_\lambda \mathbb{R}$ can be written as a countable inductive limit of of one-dimensional NCCW complexes, which are shown in [Rob10] to have trivial K_1-groups. Since, by [Con81, Theorem IV.2], these crossed products have trivial K-theory, we can apply the Kirchberg-Phillips classification theorem in the former case, and Theorem 4.1.2

in the latter, to deduce that

$$\mathcal{O}_2 \rtimes_\lambda \mathbb{R} \cong \begin{cases} \mathcal{O}_2 \otimes \mathcal{K} & \text{for every} \quad \lambda \in \mathbb{R}_- \backslash \mathbb{Q} \\ W \otimes \mathcal{K} & \text{for generic} \quad \lambda \in \mathbb{R}_+ \backslash \mathbb{Q}. \end{cases}$$

(A consequence of this, together with Conjecture 4.5.2, would be that $\mathcal{O}_2 \rtimes_\lambda \mathbb{R}$ is self-absorbing for generic irrational λ.)

It is perhaps worth noting that one cannot hope to express the Jiang-Su algebra in an analogous manner. That is, $\mathcal{Z} \otimes \mathcal{K}$ is not isomorphic to the crossed product of a Kirchberg algebra by $\mathbb{R}$. This is because, by Propositions 3 and 4 of [KK97], any such crossed product which is simple is automatically either traceless or stable and projectionless.

Chapter 5

Universal C^*-algebras

In this chapter, we use order zero maps to express the Jiang-Su algebra $\mathcal{Z}$ and the stably projectionless algebra W described in the previous chapter as universal C^*-algebras on countably many generators and relations. The presentation of $\mathcal{Z}$ was discovered by Wilhelm Winter, but everything below has been written up by the present author.

5.1 Order zero maps

In this section, we collect some well-known facts about order zero maps that will be used throughout this chapter. Recall that a completely positive (c.p.) map $\varphi : A \to B$ has *order zero* if it preserves orthogonality. Any *-homomorphism has order zero and, more generally, if $\pi : A \to B$ is a *-homomorphism and $h \in M(B)$ is a positive element which commutes with $\pi(A)$, then $\varphi(\cdot) := \pi(\cdot)h$ defines a c.p. order zero map. (In fact, every c.p. order zero map is essentially of this form — see [WZ09].) The following Proposition comes from [Win03, Proposition 3.2(a)] and [Win05, 1.2]. We denote by e_{ij} (or $e_{ij}^{(n)}$) the canonical (i,j)-th matrix unit in $M_n = M_n(\mathbb{C})$.

Proposition 5.1.1 (Winter). *Let A be a C^*-algebra, let $n \in \mathbb{N}$ and let $\varphi : M_n \to A$ be a c.p.c. order zero map.*

*(i) There is a unique *-homomorphism $\rho_\varphi : C_0(0,1] \otimes M_n \to A$ such that $\rho_\varphi(\mathrm{id} \otimes x) = \varphi(x)$ for $x \in M_n$. (Conversely, any *-homomorphism $\rho : C_0(0,1] \otimes M_n \to A$ induces a c.p.c. order zero map $\varphi_\rho : M_n \to A$ with $\varphi_\rho(x) = \rho(\mathrm{id} \otimes x)$.)*

*(ii) There is a unique *-homomorphism $\pi_\varphi : M_n \to A^{**}$ (the 'canonical supporting *-homomorphism' of φ) such that $\pi_\varphi(e_{ij})$ is the partial isometry in the polar decompo-*

sition of $\varphi(e_{ij})$ in A^{**}. We have

$$\varphi(x) = \pi_\varphi(x)\varphi(1_n) = \varphi(1_n)\pi_\varphi(x) \tag{5.1}$$

for $x \in M_n$.

Remark 5.1.2. (i) Proposition 5.1.1(ii) provides a notion of positive functional calculus for c.p.c. order zero maps. It is clear from (5.1) that if $f \in C_0(0,1]$ is positive with $\|f\| \leq 1$ then the map $f(\varphi) : M_n \to A$ given by

$$f(\varphi)(x) := \pi_\varphi(x) f(\varphi(1_n)) \tag{5.2}$$

is a well-defined c.p. order zero map. On approximating f uniformly by polynomials, (5.2) and (5.1) yield

$$f(\varphi)(p) = f(\varphi(p)) \tag{5.3}$$

whenever $p \in M_n$ is a projection. If $\alpha : A \to B$ is a *-homomorphism then $\pi_{\alpha\circ\varphi} = \alpha^{**} \circ \pi_\varphi$ by uniqueness, and so

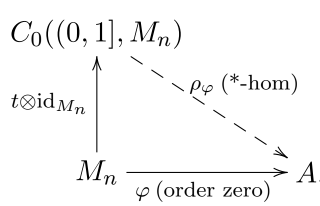
$$\alpha \circ f(\varphi) = f(\alpha \circ \varphi). \tag{5.4}$$

Note also that for positive contractions $f, g \in C_0(0,1]$ and $x, y \in M_n$ we have

$$f(\varphi)(x)g(\varphi)(y) = (fg)(\varphi)(xy). \tag{5.5}$$

(ii) It follows from (5.1) that if $\varphi : M_n \to A$ is c.p.c. order zero and $\varphi(1_n)$ is a projection, then φ is in fact a *-homomorphism.

One way of interpreting Proposition 5.1.1(i) is to say that the cone $C_0((0,1], M_n)$ has the universal property given by the commutative diagram



We say that $C_0((0,1], M_n)$ is *the universal C^*-algebra generated by a c.p.c. order zero map on M_n*. Alternatively, it is easy to check that $C_0((0,1], M_n)$ is the universal C^*-algebra on generators $x_1, \ldots, x_n$ subject to the relations $\mathcal{R}_n^{(0)}$ given by

$$\|x_i\| \leq 1, \quad x_1 \geq 0, \quad x_i x_i^* = x_1^2, \quad x_j^* x_j \perp x_i^* x_i \quad \text{for} \quad 1 \leq i \neq j \leq n. \tag{5.6}$$

(The isomorphism is given by sending x_j to $t^{1/2} \otimes e_{1j}$, so that $t \otimes e_{ij}$ corresponds to $x_i^* x_j$.) One can therefore view the statement

$$C_0((0,1], M_n) = C^*(\varphi \mid \varphi \text{ c.p.c. order zero on } M_n) \tag{5.7}$$

as an abbreviation for these relations.

Remark 5.1.3 ($n = 2$)**.** $C_0((0,1], M_2)$ is the universal C^*-algebra $C^*(x \mid \|x\| \leq 1, x^2 = 0)$. Therefore, if A is a C^*-algebra and $v \in A$ is a contraction with $v^2 = 0$, then there is a unique c.p.c. order zero map $\psi : M_2 \to A$ with $\psi(e_{12}) = v$ (so that $\psi(e_{11}) = |v^*|$ and $\psi(e_{22}) = |v|$).

Recall that a *prime dimension drop algebra* is a C^*-algebra of the form

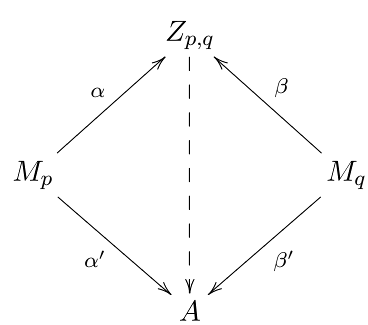
$$Z_{p,q} := I(p, pq, q) = \{f \in C([0,1], M_p \otimes M_q) : f(0) \in M_p \otimes 1_q, f(1) \in 1_p \otimes M_q\}, \tag{5.8}$$

where p and q are coprime natural numbers. The Jiang-Su algebra $\mathcal{Z}$ is the unique inductive limit of prime dimension drop algebras which is simple and has a unique tracial state (see [JS99]).

The order zero notation essentially appears in [RW10, Proposition 2.5], where the presentation of prime dimension drop algebras described in [JS99, Proposition 7.3] is reinterpreted in terms of order zero maps. Specifically, the prime dimension drop algebra $Z_{p,q}$ is the universal unital C^*-algebra

$$\begin{aligned} C^*(\alpha, \beta \mid \alpha \text{ c.p.c. order zero on } M_p, \beta \text{ c.p.c. order zero on } M_q,& \\ \alpha(1_p) + \beta(1_q) = 1,& \\ [\alpha(M_p), \beta(M_q)] = 0).& \end{aligned}$$

The universal property can be expressed by the diagram



(where α and β correspond to the obvious embeddings of $C_0([0,1), M_p)$ and $C_0((0,1], M_q)$ into $Z_{p,q}$, and α' and β' are c.p.c. order zero maps that satisfy the given relations).

When $q = p+1$, there is another presentation of $Z_{p,p+1}$ in terms of order zero maps that does not involve a commutation relation. The following is essentially contained in [RW10, Proposition 5.1], and we note that these relations have already proved highly useful, for example in [Win10].

Proposition 5.1.4 (Rørdam-Winter)**.** *Define $Z^{(n)}$ to be the universal unital C^*-algebra $C^*(\varphi, \psi|\mathcal{R}_n)$, where $\mathcal{R}_n$ denotes the set of relations:*

(i) φ and ψ are c.p.c. order zero maps on M_n and M_2 respectively;

(ii) $\psi(e_{11}) = 1 - \varphi(1_n)$;

(iii) $\psi(e_{22})\varphi(e_{11}) = \psi(e_{22})$.

Then $Z^{(n)} \cong Z_{n,n+1}$.

Following [Sat10, Proposition 2.1], we can take the map $\varphi : M_n \to Z_{n,n+1}$ to be

$$\varphi(a)(t) = u(t)(a \otimes 1_n)u(t)^* \oplus (1-t)(a \otimes e_{n+1,n+1}) \tag{5.9}$$

for $a \in M_n$ and $t \in [0,1]$. Here, u is a unitary in the algebra $C([0,1], M_n \otimes M_n)$ (included nonunitally in the top left corner of $C([0,1], M_n \otimes M_{n+1})$) whose purpose is to ensure that boundary conditions are satisfied. One can also explicitly write down the order zero map ψ, namely

$$\psi(e_{21})(t) = tu(t)\sum_{j=1}^{n} e_{1j} \otimes e_{j,n+1}. \tag{5.10}$$

However, it is the function of ψ rather than its precise form that is important: $\psi(e_{12})(t)$ should perhaps be thought of as something like a partial isometry that takes $1 - \varphi(1_n)(t)$ (which lives in the bottom corner of $M_n \otimes 1_{n+1}$) and spreads it out underneath the copies of e_{11} in the remaining corners.

Note also that, for $F \in C_0(0,1]$, we have

$$F(\varphi)(a)(t) = F(1)u(t)(a \otimes 1_n)u(t)^* \oplus F(1-t)(a \otimes e_{n+1,n+1}). \tag{5.11}$$

The C^*-algebra W was constructed in Chapter 4 as a simple, monotracial inductive limit of building blocks which resemble dimension drop algebras but are nonunital (in fact, stably projectionless). We shall see in section 5.3 that certain of these building blocks admit presentations very similar to that given in Proposition 5.1.4, and we use this to adapt the presentation of $\mathcal{Z}$ to obtain one for W.

5.2 Generators and relations for the Jiang-Su algebra

Suppose that $(A_n, \theta_n)_{n=1}^{\infty}$ is an inductive sequence of universal C^*-algebras

$$A_n = C^*\langle\{x_1^{(n)}, \ldots, x_{k_n}^{(n)}\}|R_n\rangle,$$

with connecting maps $\theta_n : A_n \to A_{n+1}$. One way of presenting the inductive limit $A = \varinjlim(A_n, \theta_n)$ as a universal C^*-algebra is to keep track of the images $\theta_n(x_i^{(n)})$ of the generators $x_i^{(n)}$ for every i and n. That is, if we can express $y_i^{(n)} := \theta_n(x_i^{(n)})$ in terms of the generators $x_j^{(n+1)}$, then A is isomorphic to the universal C^*-algebra on generators $\{x_i^{(n)} : n \in \mathbb{N}, 1 \leq i \leq k_n\}$, with relations $\bigcup_{n\in\mathbb{N}} R_n$ together with $x_i^{(n)} = y_i^{(n)}$ for $n \in \mathbb{N}$ and $1 \leq i \leq k_n$.

Given Proposition 5.1.4, this is exactly how we will exhibit the Jiang-Su algebra $\mathcal{Z}$ as a universal C^*-algebra. We will construct an inductive sequence $(Z^{(q(k))}, \alpha_k)$, where $q(k) = p^{3^k}$ for some fixed prime p and

$$Z^{(q(k))} = C^*(\varphi_k, \psi_k|\mathcal{R}_{q(k)}) \cong Z_{q(k),q(k)+1}$$

(as in Proposition 5.1.4), and we will check that the inductive limit is simple with a unique tracial state. It will then follow from the classification theorem of [JS99] that $\mathcal{Z} \cong \varinjlim(Z^{(q(k))}, \alpha_k)$.

In order to obtain an explicit presentation of $\mathcal{Z}$ in this way, we need to describe the connecting maps α_k in terms of generators and relations. (This is perhaps the key difference between our construction and the original construction of $\mathcal{Z}$ as an inductive limit in [JS99].) In other words, for every $k \in \mathbb{N}$ we will find c.p.c. order zero maps $\hat{\varphi}_k : M_{q(k)} \to Z^{(q(k+1))}$ and $\hat{\psi}_k : M_2 \to Z^{(q(k+1))}$ that satisfy the relations $\mathcal{R}_{q(k)}$ of Proposition 5.1.4. By universality, we will then have unital connecting maps $\alpha_k : Z^{(q(k))} \to Z^{(q(k+1))}$ with $\alpha_k \circ \varphi_k = \hat{\varphi}_k$ and $\alpha_k \circ \psi_k = \hat{\psi}_k$.

Before giving the connecting maps, it is instructive to note that there are obvious choices for $\hat{\varphi}_k$ and $\hat{\psi}_k$. Since $q(k+1) = q(k)^3$, we can identify $M_{q(k+1)}$ with $M_{q(k)} \otimes M_{q(k)} \otimes M_{q(k)}$ and then set $\hat{\varphi}_k = \varphi_{k+1} \circ (\mathrm{id}_{M_{q(k)}} \otimes 1_{q(k)} \otimes 1_{q(k)})$ and $\hat{\psi}_k = \psi_{k+1}$. It is easy to see that these maps satisfy the relations $\mathcal{R}_{q(k)}$, but the corresponding inductive limit certainly will not be simple. The idea is therefore to define $\hat{\varphi}_k$ in such a way as to ensure that $[0, 1]$ is chopped up into suitably small pieces under the induced *-homomorphism α_k; as usual, $\hat{\psi}_k(e_{12})$ will just be some partial-isometry-like element that facilitates the relations $\mathcal{R}_{q(k)}$.

One way of doing this is as follows. Define $\rho_k : M_{q(k)} \to M_{q(k+1)}$ by

$$\rho_k = (\mathrm{id}_{M_{q(k)}} \otimes 1_{q(k)-1} \otimes 1_{q(k)}) \oplus \left(\bigoplus_{i=1}^{q(k)} \frac{i}{q(k)} \left(\mathrm{id}_{M_{q(k)}} \otimes e_{q(k),q(k)} \otimes e_{ii} \right) \right). \tag{5.12}$$

Note that ρ_k is c.p.c. order zero, with canonical supporting *-homomorphism

$$\pi_{\rho_k} = \mathrm{id}_{M_{q(k)}} \otimes 1_{q(k)} \otimes 1_{q(k)}$$

(as in Proposition 5.1.1(ii)). We would essentially like to define $\hat{\varphi}_k := \varphi_{k+1} \circ \rho_k$. For this to work, we would need to be able to transport $1 - \varphi_{k+1}(\rho_k(1_{q(k)}))$ underneath $\varphi_{k+1}(\rho_k(e_{11}^{(q(k))}))$, and the basic idea is to do this in two steps.

1. Use $\psi_{k+1}(e_{12})$ to transport the 'bottom' corner of $1 - \varphi_{k+1}(\rho_k(1_{q(k)}))$ underneath the 'upper' corners of $\varphi_{k+1}(e_{11}^{(q(k+1))}) < \varphi_{k+1}(\rho_k(e_{11}^{(q(k))}))$ (in the sense of (5.9)).

2. Use a partial isometry $v_{k+1} \in M_{q(k+1)}$ to transport (a projection bigger than) $1_{q(k+1)} - \rho_k(1_{q(k)})$ underneath (a projection smaller than) $\rho_k(e_{11}^{(q(k))}) - e_{11}^{(q(k+1))}$, so that $\varphi_{k+1}(v_{k+1})$ transports each of the upper corners of $1 - \varphi_{k+1}(\rho_k(1_{q(k)}))$ underneath the corresponding corner of $\varphi_{k+1}(\rho_k(e_{11}^{(q(k))}))$.

Although this is essentially the right idea, it needs fine-tuning. For example, in step 1, we would really like to use not ψ_{k+1}, but its supporting *-homomorphism. Since this is not an element of $Z^{(q(k+1))}$, we instead use $d(\psi_{k+1})$ for an appropriate $d \in C_0(0,1]$ (i.e. apply functional calculus to the order zero map ψ_{k+1}). Functional calculus inevitably appears elsewhere in the relations that we describe below, using the following positive contractions in $C_0(0,1]$:

$$d(t) := \begin{cases} 16t/3, & 0 \le t \le 3/16 \\ 1, & 3/16 \le t \le 1 \end{cases} \tag{5.13}$$

$$f(t) := \begin{cases} 0, & 0 \le t \le 1/4 \\ 4t-1, & 1/4 \le t \le 1/2 \\ 1, & 1/2 \le t \le 1 \end{cases} \tag{5.14}$$

$$g(t) := \begin{cases} 0, & 0 \le t \le 1/4 \\ 4t-1, & 1/4 \le t \le 1/2 \\ 3-4t, & 1/2 \le t \le 3/4 \\ 0, & 3/4 \le t \le 1 \end{cases} \tag{5.15}$$

$$h(t) := \begin{cases} 0, & 0 \le t \le 1/2 \\ 4t-2, & 1/2 \le t \le 3/4 \\ 1, & 3/4 \le t \le 1. \end{cases} \tag{5.16}$$

These are chosen so that, writing $\bar{d}(t) = d(1-t)$, we have

$$g = f - h, \quad fh = h, \quad \bar{d}(1-f) = 1-f \quad \text{and} \quad \bar{d}g = g. \tag{5.17}$$

For use in section 5.3, we also note that if $\hat{d}$ is the function $\hat{d}(t) = d(t(1-t))$ then we have

$$\hat{d}(f - f^2) = f - f^2 \quad \text{and} \quad \hat{d}g = g. \tag{5.18}$$

(This is why d is chosen slightly oddly.)

Finally, to accomplish step 2, we choose a partial isometry $v_{k+1} \in M_{q(k+1)}$ such that

$$v_{k+1}v_{k+1}^* = 1_{q(k)} \otimes e_{q(k),q(k)} \otimes 1_{q(k)-1}$$

and

$$v_{k+1}^* v_{k+1} = (e_{11} \otimes 1_{q(k)-1} \otimes 1_{q(k)}) + (e_{11} \otimes e_{q(k),q(k)} \otimes e_{q(k),q(k)}) - (e_{11} \otimes e_{11} \otimes e_{11}).$$

This is possible since both of these projections have rank $q(k)^2 - q(k)$; since they are orthogonal, we moreover have $v_{k+1}^2 = 0$. This v_{k+1} then satisfies:

(i) $v_{k+1}^* v_{k+1} \perp e_{11} \otimes e_{11} \otimes e_{11}$;

(ii) $v_{k+1}^* v_{k+1}$ is dominated by $\rho_k(e_{11})$ (and hence by $\rho_k(e_{11}^{q(k)}) - e_{11}^{q(k+1)}$); and

(iii) $v_{k+1}v_{k+1}^*$ acts like a unit on

$$1_{q(k+1)} - \rho_k(1_{q(k)}) = \bigoplus_{i=1}^{q(k)} \left(1 - \frac{i}{q(k)}\right) (1_{q(k)} \otimes e_{q(k),q(k)} \otimes e_{ii}). \tag{5.19}$$

Theorem 5.2.1. *Let the functions $d, f, g, h \in C_0(0,1]$, the partial isometries $v_k \in M_{q(k)}$, and the c.p.c. order zero maps $\rho_k : M_{q(k)} \to M_{q(k+1)}$ be as above for each $k \in \mathbb{N}$. Define $\mathcal{Z}_U$ to be the universal unital C^*-algebra generated by c.p.c. order zero maps φ_k on $M_{q(k)}$ ($k \in \mathbb{N}$) and ψ_k on M_2 ($k \in \mathbb{N}$) such that for each k, these maps satisfy the relations $\mathcal{R}_{q(k)}$, i.e.*

$$\psi_k(e_{11}) = 1 - \varphi_k(1_{q(k)}) \tag{5.20}$$

and

$$\psi_k(e_{22})\varphi_k(e_{11}) = \psi_k(e_{22}), \tag{5.21}$$

together with the additional relations $\mathcal{S}_{q(k)}$ given by

$$\varphi_k = f(\varphi_{k+1}) \circ \rho_k, \tag{5.22}$$

$$\sqrt{\psi_k}(e_{12}) = \left(1 - f(\varphi_{k+1})\left(1_{q(k+1)}\right) + g(\varphi_{k+1})\left(1_{q(k+1)} - \rho_k(1_{q(k)})\right)\right)^{1/2} d(\psi_{k+1})(e_{12}) \tag{5.23}$$
$$+ h(\varphi_{k+1})\left(1_{q(k+1)} - \rho_k(1_{q(k)})\right)^{1/2} f(\varphi_{k+1})(v_{k+1}).$$

Then $\mathcal{Z}_U \cong \mathcal{Z}$.

Proof. First, (5.4) implies that the given set of relations is preserved under (unital) *-homomorphisms. Also, the only norm conditions are ones of the form $\|x\| \leq 1$ so, for example by [Lor97, Theorem 3.1.1], $\mathcal{Z}_U$ does indeed exist. For each k, define $\hat{\varphi}_k : M_{q(k)} \to Z^{(q(k+1))} = C^*(\varphi_{k+1}, \psi_{k+1} | \mathcal{R}_{q(k+1)})$ and $\hat{\psi}_k : M_2 \to Z^{(q(k+1))}$ by

$$\hat{\varphi}_k = f(\varphi_{k+1}) \circ \rho_k \tag{5.24}$$

and

$$\sqrt{\hat{\psi}_k}(e_{12}) = \gamma_k + \delta_k, \tag{5.25}$$

where

$$\gamma_k := (1 - f(\varphi_{k+1})(1_{q(k+1)}) + g(\varphi_{k+1})(1_{q(k+1)} - \rho_k(1_{q(k)})))^{1/2} d(\psi_{k+1})(e_{12}) \tag{5.26}$$

and

$$\delta_k := h(\varphi_{k+1})\left(1_{q(k+1)} - \rho_k(1_{q(k)})\right)^{1/2} f(\varphi_{k+1})(v_{k+1}). \tag{5.27}$$

That is, we insist that

$$\sqrt{\hat{\psi}_k}(e_{21}) = \left(\sqrt{\hat{\psi}_k}(e_{12})\right)^*, \tag{5.28}$$

$$\left(\sqrt{\hat{\psi}_k}(e_{11})\right)^2 = \sqrt{\hat{\psi}_k}(e_{12})\sqrt{\hat{\psi}_k}(e_{21}) = \hat{\psi}_k(e_{11}), \tag{5.29}$$

$$\left(\sqrt{\hat{\psi}_k}(e_{22})\right)^2 = \sqrt{\hat{\psi}_k}(e_{21})\sqrt{\hat{\psi}_k}(e_{12}) = \hat{\psi}_k(e_{22}), \tag{5.30}$$

$$\sqrt{\hat{\psi}_k}(e_{11})\sqrt{\hat{\psi}_k}(e_{12}) = \hat{\psi}_k(e_{12}) = \hat{\psi}_k(e_{21})^*. \tag{5.31}$$

We first need to check that $\hat{\varphi}_k$ and $\hat{\psi}_k$ satisfy the relations $\mathcal{R}_{q(k)}$. First, it is obvious that $\hat{\varphi}_k$ is c.p.c. order zero since φ_{k+1} and ρ_k are, and f is contractive. It follows from Remark 5.1.3 that to show that $\sqrt{\hat{\psi}_k}$ and, equally, $\hat{\psi}_k$ are c.p.c. order zero, it suffices to check that $\sqrt{\hat{\psi}_k}(e_{12})$ is a contraction that squares to zero. The following two lemmas will be useful for this.

Lemma 5.2.2. *Let $F, G \in C_0(0,1]$ be positive contractions and let $k \in \mathbb{N}$. Then in $Z^{(q(k))}$ we have*

$$F(\psi_k)(e_{12})G(\varphi_k)(1_{q(k)}) = G(1)F(\psi_k)(e_{12}).$$

In particular, if $G(1) = 0$ then $F(\psi_k)(e_{12})G(\varphi_k)(a) = 0$ for every $a \in M_{q(k)}$.

Proof. Write $w_k = \psi_k(e_{12})$, so that $\psi_k(e_{11}) = |w_k^*|$ and $\psi_k(e_{22}) = |w_k|$. Also let $\psi_k(\cdot) = \pi_{\psi_k}(\cdot)y_k$ be the canonical decomposition of ψ_k, and write $x_k = \varphi_k(1_{q(k)})$, so $G(\varphi_k)(1_k) = G(x_k)$. From (5.20) we have $(w_k w_k^*)^{1/2} = 1 - x_k$, which, on multiplying on the left by w_k (and using $w_k^2 = 0$), gives $w_k x_k = w_k$. In other words, we have

$$\pi_{\psi_k}(e_{12})y_k x_k = \pi_{\psi_k}(e_{12})y_k.$$

Approximating F uniformly by polynomials, and using the fact that y_k commutes with $\pi_{\psi_k}(e_{12})$, we then obtain

$$F(\psi_k)(e_{12})x_k = \pi_{\psi_k}(e_{12})F(y_k)x_k = \pi_{\psi_k}(e_{12})F(y_k) = F(\psi_k)(e_{12}).$$

Approximating G by polynomials then gives

$$F(\psi_k)(e_{12})G(\varphi_k)(1_k) = F(\psi_k)(e_{12})G(x_k) = G(1)F(\psi_k)(e_{12}). \qquad \square$$

Lemma 5.2.3. *Let $F, G \in C_0(0,1]$ be positive contractions, let $k \in \mathbb{N}$ and suppose that $a \in M_{q(k)}$ satisfies $e_{11}^{(q(k))}a = 0$. Then in $Z^{(q(k))}$ we have $F(\psi_k)(e_{12})G(\varphi_k)(a) = 0$.*

Proof. Using (5.21), (5.5) and the fact that $e_{11}a = 0$, we have

$$\psi_k(e_{22})G(\varphi_k)(a) = \psi_k(e_{22})\varphi_k(e_{11})G(\varphi_k)(a) = 0.$$

Since $e_{12} = e_{12}e_{22}$, it then follows from (5.1) that

$$\psi_k(e_{12})G(\varphi_k)(a) = 0,$$

and by approximating F uniformly by polynomials we see that

$$F(\psi_k)(e_{12})G(\varphi_k)(a) = 0. \qquad \square$$

Lemma 5.2.4. $\left(\sqrt{\hat{\psi}_k}(e_{12})\right)^2 = 0.$

Proof. We check that each of the four terms in the expansion of $\left(\sqrt{\hat{\psi}_k}(e_{12})\right)^2$ is zero. Since $f(1) = 1$ and $g(1) = 0$, it follows immediately from Lemma 5.2.2 that

$$\begin{aligned} d(\psi_{k+1})(e_{12})(1 - f(\varphi_{k+1})(1_{q(k+1)}) + g(\varphi_{k+1})(1_{q(k+1)} - \rho_k(1_{q(k)}))) \\ = d(\psi_{k+1})(e_{12})(1 - 1 + 0) \\ = 0. \end{aligned}$$

Hence (by functional calculus) $\gamma_k^2 = 0$. To show that $\gamma_k \delta_k = 0$, we use Lemma 5.2.3 and the fact that $e_{11}^{(q(k+1))} = (e_{11}^{(q(k))} \otimes e_{11}^{(q(k))} \otimes e_{11}^{(q(k))}) \perp (1_{q(k+1)} - \rho_k(1_{q(k)}))$ (for example from (5.19)) to see that

$$d(\psi_{k+1}(e_{12}))h(\varphi_{k+1})(1_{q(k+1)} - \rho_k(1_{q(k)})) = 0.$$

To show that $\delta_k^2 = 0$, we use (5.5), the fact that $fh = h$, and property (iii) of v_{k+1} to see that

$$f(\varphi_{k+1})(v_{k+1})h(\varphi_{k+1})(1_{q(k+1)} - \rho_k(1_{q(k)})) = h(\varphi_{k+1})(v_{k+1}(1_{q(k+1)} - \rho_k(1_{q(k)}))) = 0.$$

A similar argument shows that

$$\begin{aligned} h(\varphi_{k+1})((1_{q(k+1)} - \rho_k(1_{q(k)}))v_{k+1})(1 - f(\varphi_{k+1})(1_{q(k+1)}) + g(\varphi_{k+1})(1_{q(k+1)} - \rho_k(1_{q(k)}))) \\ = 0, \end{aligned}$$

and hence that $\delta_k \gamma_k = 0$. We have thus shown that $\left(\sqrt{\hat{\psi}_k}(e_{12})\right)^2 = 0$. ☐

That $\sqrt{\hat{\psi}_k}(e_{12})$ is a contraction will follow from $\hat{\psi}_k(e_{11}) = 1 - \hat{\varphi}_k(1_{q(k)})$, which we now check.

Lemma 5.2.5. $\hat{\psi}_k(e_{11}) = 1 - \hat{\varphi}_k(1_{q(k)})$.

Proof. It follows from Lemma 5.2.3 and property (i) of v_{k+1} that the cross terms $\gamma_k \delta_k^*$ and $\delta_k \gamma_k^*$ in the expansion of $\hat{\psi}_k(e_{11}) = \left(\sqrt{\hat{\psi}_k}(e_{12})\right)\left(\sqrt{\hat{\psi}_k}(e_{12})\right)^*$ vanish. Using (5.5), the fact that $fh = h$, and property (iii) of v_{k+1}, we have

$$\begin{aligned} h(\varphi_{k+1})(1_{q(k+1)} - \rho_k(1_{q(k)}))f(\varphi_{k+1})(v_{k+1})f(\varphi_{k+1})(v_{k+1}^*) \\ = h(\varphi_{k+1})((1_{q(k+1)} - \rho_k(1_{q(k)})v_{k+1}v_{k+1}^*) \\ = h(\varphi_{k+1})(1_{q(k+1)} - \rho_k(1_{q(k)})). \end{aligned}$$

Hence

$$\delta_k \delta_k^* = h(\varphi_{k+1})(1_{q(k+1)} - \rho_k(1_{q(k)})).$$

Using (5.3) and (5.20), we have

$$d(\psi_{k+1})(e_{11}) = d(\psi_{k+1}(e_{11})) = d(1 - \varphi_{k+1}(1_{q(k+1)})) = \bar{d}(\varphi_{k+1}(1_{q(k+1)})),$$

where $\bar{d}(t) = d(1-t)$ as in (5.17). Another application of (5.3), together with (5.17), gives

$$\begin{aligned}(1 - f(\varphi_{k+1})(1_{q(k+1)}))d(\psi_{k+1})(e_{11}) &= (1 - f)(\varphi_{k+1}(1_{q(k+1)}))\bar{d}(\varphi_{k+1}(1_{q(k+1)})) \\ &= (1 - f)(\varphi_{k+1}(1_{q(k+1)})) \\ &= 1 - f(\varphi_{k+1})(1_{q(k+1)}).\end{aligned}$$

Similarly, we have

$$g(\varphi_{k+1})(1_{q(k+1)})d(\psi_{k+1})(e_{11}) = g(\varphi_{k+1})(1_{q(k+1)}),$$

and hence

$$g(\varphi_{k+1})(1_{q(k+1)} - \rho_k(1_{q(k)}))d(\psi_{k+1})(e_{11}) = g(\varphi_{k+1})(1_{q(k+1)} - \rho_k(1_{q(k)}))$$

from (5.2). Using (5.5), we therefore have

$$\gamma_k \gamma_k^* = 1 - f(\varphi_{k+1})(1_{q(k+1)}) + g(\varphi_{k+1})(1_{q(k+1)} - \rho_k(1_{q(k)})).$$

Since $g + h = f$, it follows that

$$\begin{aligned}\hat{\psi}_k(e_{11}) &= \gamma_k \gamma_k^* + \delta_k \delta_k^* \\ &= 1 - f(\varphi_{k+1})(1_{q(k+1)}) + g(\varphi_{k+1})(1_{q(k+1)} - \rho_k(1_{q(k)})) + h(\varphi_{k+1})(1_{q(k+1)} - \rho_k(1_{q(k)})) \\ &= 1 - f(\varphi_{k+1})(\rho_k(1_{q(k)})) \\ &= 1 - \hat{\varphi}_k(1_{q(k)}).\end{aligned}$$

□

We have now verified that $\hat{\varphi}_k$ and $\hat{\psi}_k$ satisfy all but one of the relations in $\mathcal{R}_{q(k)}$. We now verify the remaining relation.

Lemma 5.2.6. $\hat{\psi}_k(e_{22})\hat{\varphi}_k(e_{11}) = \hat{\psi}_k(e_{22})$.

Proof. Using (5.5), the fact that $fh = h$, and the fact that v_{k+1} is a partial isometry with property (ii), we have

$$\begin{aligned}h(\varphi_{k+1})(1_{q(k+1)} - &\rho_k(1_{q(k)}))f(\varphi_{k+1})(v_{k+1})f(\varphi_{k+1})(\rho_k(e_{11})) \\ &= h(\varphi_{k+1})((1_{q(k+1)} - \rho_k(1_{q(k)}))v_{k+1}v_{k+1}^* v_{k+1}\rho_k(e_{11})) \\ &= h(\varphi_{k+1})((1_{q(k+1)} - \rho_k(1_{q(k)}))v_{k+1}) \\ &= h(\varphi_{k+1})(1_{q(k+1)} - \rho_k(1_{q(k)}))f(\varphi_{k+1})(v_{k+1}).\end{aligned}$$

Thus $\delta_k\hat{\varphi}_k(e_{11}) = \delta_k$. Next, it follows from (5.21), upon approximating d uniformly by polynomials, that

$$d(\psi_{k+1})(e_{22})\varphi_{k+1}(e_{11}) = d(\psi_{k+1})(e_{22}).$$

Approximating f by polynomials then gives

$$d(\psi_{k+1})(e_{22})f(\varphi_{k+1})(e_{11}) = f(1)d(\psi_{k+1})(e_{22}) = d(\psi_{k+1})(e_{22}).$$

Since $e_{11} \perp (\rho_k(e_{11}) - e_{11})$ and $f(\varphi_{k+1})$ is order zero, we therefore have

$$d(\psi_{k+1})(e_{22})f(\varphi_{k+1})(\rho_k(e_{11})) = d(\psi_{k+1})(e_{22}).$$

Since $e_{12} = e_{12}e_{22}$, it then follows from (5.1) that

$$d(\psi_{k+1})(e_{12})f(\varphi_{k+1})(\rho_k(e_{11})) = d(\psi_{k+1})(e_{12}).$$

Hence $\gamma_k\hat{\varphi}_k(e_{11}) = \gamma_k$, and so

$$\hat{\psi}_k(e_{22})\hat{\varphi}_k(e_{11}) = (\gamma_k^*\gamma_k + \delta_k^*\delta_k)\hat{\varphi}_k(e_{11}) = \hat{\psi}_k(e_{22}).$$ □

We have now shown that $\hat{\varphi}_k$ and $\hat{\psi}_k$ satisfy the relations $\mathcal{R}_{q(k)}$. This means that (5.22) and (5.23) do not introduce any new relations on φ_{k+1} and ψ_{k+1}, and so the C^*-algebra generated by φ_{k+1} and ψ_{k+1} within $\mathcal{Z}_U$ is (isomorphic to) the universal C^*-algebra on relations $\mathcal{R}_{q(k+1)}$. That is, $C^*(\varphi_{k+1}, \psi_{k+1}) \cong Z^{(q(k+1))}$. Moreover, since $C^*(\varphi_k, \psi_k) \subset C^*(\varphi_{k+1}, \psi_{k+1})$ for every k (by (5.22) and (5.23)), Proposition 5.1.4 implies that $\mathcal{Z}_U = \overline{\bigcup_{k=0}^{\infty} C^*(\varphi_k, \psi_k)}$ is isomorphic to an inductive limit of prime dimension drop algebras.

The strategy for the remainder of the proof is to pass from the abstract picture of $\mathcal{Z}_U$ as a universal C^*-algebra, to a concrete description as an inductive limit $\varinjlim(Z^{(q(k))}, \alpha_k)$, where we identify $Z^{(q(k))}$ with $Z_{q(k),q(k)+1}$, and where the (unital) connecting maps $\alpha_k : Z^{(q(k))} \to Z^{(q(k+1))}$ are determined by (5.22) and (5.23) (i.e. $\alpha_k \circ \varphi_k = \hat{\varphi}_k$ and $\alpha_k \circ \psi_k = \hat{\psi}_k$). We will obtain explicit descriptions of the maps α_k, and use these to show that $\mathcal{Z}_U$ is simple and has a unique tracial state.

Fix $k \in \mathbb{N}$. For each $t \in [0,1]$, let us write α_k^t for the map $\mathrm{ev}_t \circ \alpha_k : Z^{(q(k))} \to M_{q(k+1)} \otimes M_{q(k+1)+1}$, where ev_t denotes evaluation at t. Then α_k^t is a (finite-dimensional) representation of $Z^{(q(k))}$, so there are finitely many irreducible representations $\pi_1^t, \ldots, \pi_m^t$ of $Z^{(q(k))}$ (corresponding to point evaluations at some $t_1, \ldots, t_m \in [0,1]$) such that $\alpha_k^t \sim_u \bigoplus_{i=1}^m \pi_i^t$, where $\sim_u$ denotes unitary equivalence. Since $C^*(\varphi_k(M_{q(k)}))$ separates the points of $[0,1]$ (for example by (5.9)), it is easy to see that α_k is determined up to unitary equivalence by its values on $\varphi_k(M_{q(k)})$.

Moreover, we can calculate $\alpha_k|_{\varphi_k(M_{q(k)})}$ since $\alpha_k(\varphi_k(a)) = f(\varphi_{k+1})(\rho_k(a))$ and we have concrete descriptions of the generators $\varphi_k(\cdot)$ from (5.9) and (5.11), namely

$$\varphi_k(a)(t) = u_k(t)(a \otimes 1_{q(k)})u_k(t)^* \oplus (1-t)(a \otimes e_{q(k)+1,q(k)+1}) \tag{5.32}$$

and

$$f(\varphi_{k+1})(a)(t) = u_{k+1}(t)(a \otimes 1_{q(k+1)})u_{k+1}(t)^* \oplus f(1-t)(a \otimes e_{q(k+1)+1,q(k+1)+1}). \tag{5.33}$$

Also recall from (5.12) that

$$\rho_k = (\mathrm{id}_{M_{q(k)}} \otimes 1_{q(k)-1} \otimes 1_{q(k)}) \oplus \left(\bigoplus_{i=1}^{q(k)} \frac{i}{q(k)} \left(\mathrm{id}_{M_{q(k)}} \otimes e_{q(k),q(k)} \otimes e_{ii} \right) \right).$$

We have, for $a \in M_{q(k)}$,

$$\begin{aligned}
\alpha_k^t(\varphi_k(a)) &= f(\varphi_{k+1})(\rho_k(a))(t) \\
&= u_{k+1}(t)(a \otimes 1_{q(k)-1} \otimes 1_{q(k)} \otimes 1_{q(k+1)})u_{k+1}(t)^* \\
&\oplus u_{k+1}(t) \left(\bigoplus_{i=1}^{q(k)} \frac{i}{q(k)} \left(a \otimes e_{q(k),q(k)} \otimes e_{ii} \otimes 1_{q(k+1)} \right) \right) u_{k+1}(t)^* \\
&\oplus f(1-t)(a \otimes 1_{q(k)-1} \otimes 1_{q(k)} \otimes e_{q(k+1)+1,q(k+1)+1}) \\
&\oplus f(1-t) \left(\bigoplus_{i=1}^{q(k)} \frac{i}{q(k)} \left(a \otimes e_{q(k),q(k)} \otimes e_{ii} \otimes e_{q(k+1)+1,q(k+1)+1} \right) \right) \\
&\sim_u \left(\bigoplus_{m=1}^{q(k+1)} \bigoplus_{i=1}^{q(k)-1} \varphi_k(a) \left(1 - \frac{i}{q(k)}\right) \right) \oplus \left(\bigoplus_{m=1}^{q(k)(q(k)-1)} \varphi_k(a)(1-f(1-t)) \right) \\
&\oplus \left(\bigoplus_{i=1}^{q(k)} \varphi_k(a) \left(1 - \frac{if(1-t)}{q(k)}\right) \right).
\end{aligned}$$

Write $h_i(t) := 1 - \frac{if(1-t)}{q(k)}$ (so that, in fact, $h_{q(k)} = 1 - f(1-t) = h(t)$). We then have

$$\alpha_k^t = v_k(t) \left(\left(\bigoplus_{m=1}^{q(k+1)} \bigoplus_{i=1}^{q(k)-1} \mathrm{ev}_{\frac{i}{q(k)}} \right) \oplus \left(\bigoplus_{m=1}^{q(k)(q(k)-1)} \mathrm{ev}_{h(t)} \right) \oplus \left(\bigoplus_{i=1}^{q(k)} \mathrm{ev}_{h_i(t)} \right) \right) v_k(t)^* \tag{5.34}$$

for some unitary $v_k \in C([0,1], M_{q(k+1)} \otimes M_{q(k+1)+1})$ which is constructed from u_k, u_{k+1} and a unitary which is independent of t.

We can also give a description of the connecting map $\alpha_{k,k+n} = \alpha_{k+n-1} \circ \cdots \circ \alpha_k$. For each $j \in \mathbb{N}$, let Λ_j be the sequence of continuous functions given by listing each constant function $i/q(j)$ (for $1 \leq i \leq q(j)-1$) with multiplicity $q(j+1)$, then h with multiplicity

$q(j)(q(j)-1)$ and then each h_i for $1 \le i \le q(j)$. Then $\alpha_{k,k+n}$ is unitarily equivalent to the direct sum of all maps of the form $\mathrm{ev}_{F_1\circ\cdots\circ F_n}$ with $F_j \in \Lambda_{k+j-1}$ for $1 \le j \le n$.

Lemma 5.2.7. *$\mathcal{Z}_U$ is simple and has a unique tracial state.*

Proof. Let us write $T(A)$ for the space of tracial states on a C^*-algebra A. Recall that every tracial state on $Z^{(q(j))}$ is of the form $\mathrm{tr}\otimes\mu$ for some Borel probability measure μ on $[0,1]$, where tr is the unique tracial state on $M_{q(j)} \otimes M_{q(j)+1}$. (See for example [JS99, Lemma 2.4].) In particular, every such trace extends to a trace on $C([0,1], M_{q(j)} \otimes M_{q(j)+1})$.

Since $\mathcal{Z}_U \cong \varinjlim Z^{(q(k))}$ with unital connecting maps α_k, we have $T(\mathcal{Z}_U) \cong \varprojlim T(Z^{(q(k))})$. Thus $T(\mathcal{Z}_U)$ is an inverse limit of nonempty compact Hausdorff spaces, so is nonempty. That is, $\mathcal{Z}_U$ has at least one tracial state. For uniqueness, we need to show that for every $k \in \mathbb{N}$, every $\epsilon > 0$, and every $b \in Z^{(q(k))}$ we have

$$|\tau_1(\alpha_{k,k+n}(b)) - \tau_2(\alpha_{k,k+n}(b))| < \epsilon \tag{5.35}$$

for all sufficiently large n and every $\tau_1, \tau_2 \in T(Z^{(q(k+n))})$. The key observation for this is that for each j, most of the elements in the sequence Λ_j defined above are constant functions. In fact, the proportion of functions in Λ_j that are *not* constant is

$$\frac{q(j)(q(j)-1)+q(j)}{q(j+1)(q(j)-1)+q(j)(q(j)-1)+q(j)} = \frac{q(j)^2}{q(j)^4-q(j)^3+q(j)^2} = \frac{1}{q(j)^2-q(j)+1}. \tag{5.36}$$

Since $F_1 \circ \cdots \circ F_n$ is constant if any of the F_i are constant, it follows that for fixed $b \in Z^{(q(k))}$, $\alpha_{k,k+n}(b)$ is unitarily equivalent to a direct sum of continuous $M_{q(k)} \otimes M_{q(k)+1}$-valued functions, most of which are constant except for a small corner. But any two tracial states on $Z^{(q(k+n))}$ agree on the constant pieces, and the small corner has trace at most $\|b\| \prod_{j=k}^{k+n-1} \frac{1}{q(j)^2-q(j)+1}$, which of course converges to 0 as $n \to \infty$. Thus (5.35) holds, and so $\mathcal{Z}_U$ has a unique tracial state.

For simplicity, it is enough to show that if b is a nonzero element of $Z^{(q(k))}$, then $\alpha_{k,r}(b)$ generates $Z^{(q(r))}$ as a (closed, two-sided) ideal for every sufficiently large r, which is the case if and only if $\alpha^t_{k,r}(b)$ is nonzero for every $t \in [0,1]$. Suppose that b is such an element, so that there is an interval in $(0,1)$ of width $\epsilon > 0$ on which b is nonzero. For each $n \in \mathbb{N}$ and $t \in [0,1]$, $\alpha^t_{k,k+n+1}(b)$ contains summands unitarily equivalent to $b\left(h^{(n)}\left(\frac{i}{q(k+n)}\right)\right)$ for

$1 \leq i \leq q(k+n)-1$, where $h^{(n)} := \overbrace{h \circ \cdots \circ h}^{n}$. Moreover, $h^{(n)}$ is of the form

$$h^{(n)}(t) = \begin{cases} 0, & 0 \leq t \leq l_n/4^n \\ 4^n t - l_n, & l_n/4^n \leq t \leq (1+l_n)/4^n \\ 1, & (1+l_n)/4^n \leq t \leq 1 \end{cases}$$

for some l_n, and so it suffices to show that for large n we have $\frac{1}{q(k+n)} < \frac{\epsilon}{4^n}$. But this is true for all large n since $\frac{4^n}{q(k+n)} = \frac{4^n}{p^{3^{k+n}}} \longrightarrow 0 \quad \text{as} \quad n \to \infty$. Thus $\mathcal{Z}_U$ is simple. □

It now follows from the classification theorem of [JS99] that $\mathcal{Z}_U \cong \mathcal{Z}$. □

5.3 W as a universal C^*-algebra

For each $n \in \mathbb{N}$, define W_n to be the C^*-algebra

$$W_n = \{f \in C([0,1], M_n \otimes M_{n+1}) : f(0) = a \otimes 1_{n+1}, f(1) = a \otimes 1_n,\ a \in M_n\}. \tag{5.37}$$

Then W_n is isomorphic to the building block $A(n,(n+1)n)$ as defined in Chapter 4. (To be consistent with the usual definition of dimension drop algebras, we have reversed the orientation of the interval. Obviously this does not matter.) Comparing with Proposition 5.1.4, the following Proposition indicates that W_n can be thought of as a nonunital version of the prime dimension drop algebra $Z_{n,n+1}$.

Proposition 5.3.1. *W_n is isomorphic to the universal C^*-algebra $C^*(\varphi, \psi | \hat{\mathcal{R}}_n)$, where $\hat{\mathcal{R}}_n$ denotes the set of relations:*

(i) φ and ψ are c.p.c. order zero maps on M_n and M_2 respectively;

(ii) $\psi(e_{11}) = \varphi(1_n)(1-\varphi(1_n))$;

(iii) $\psi(e_{22})\varphi(e_{11}) = \psi(e_{22})$.

Remark 5.3.2. In terms of generators and relations in the usual sense, this says that W_n is the universal C^*-algebra on generators, $v, x_1, \ldots, x_n$ such that the x_i satisfy the relations $\mathcal{R}_n^{(0)}$ of (5.6) and v is a contraction with $v^2 = 0$, together with the additional relations

$$vv^* = \left(\sum_{i=1}^n x_i^* x_i\right)\left(1 - \sum_{i=1}^n x_i^* x_i\right) \tag{5.38}$$

and

$$vx_1 = v. \tag{5.39}$$

One should first note that these relations are certainly bounded in the sense of [Lor97, Theorem 3.1.1], i.e. the corresponding universal C^*-algebra does indeed exist.

Proof. The proof is basically the same as that of Proposition 5.1.4. Define $\varphi : M_n \to W_n$ by

$$\varphi(a)(t) = (a \otimes 1_n) \oplus (1-t)(a \otimes e_{n+1,n+1}) \tag{5.40}$$

for $a \in M_n$ and $t \in [0,1]$. Then φ is clearly a c.p.c. order zero map. Equivalently, if we write

$$x_i(t) = (e_{1i} \otimes 1_n) \oplus (1-t)^{1/2}(e_{1i} \otimes e_{n+1,n+1}) = \sqrt{\varphi}(e_{1i})(t) \tag{5.41}$$

for $1 \leq i \leq n$, then the x_i satisfy the order zero relations $\mathcal{R}_n^{(0)}$ and $\varphi(e_{ij}) = x_i^* x_j$. Next, define

$$v(t) = t^{1/2}(1-t)^{1/2} \sum_{j=1}^{n} e_{j1} \otimes e_{n+1,j}. \tag{5.42}$$

Then $v^2 = 0$, and we have $vv^* = \varphi(1_n)(1-\varphi(1_n))$, so $\|v\| = 1/2$, hence v is a contraction. In particular, there is a unique c.p.c. order zero map $\psi : M_2 \to W_n$ with $\sqrt{\psi}(e_{12}) = v$, i.e.

$$\psi(e_{12})(t) = t(1-t) \sum_{j=1}^{n} e_{j1} \otimes e_{n+1,j}, \tag{5.43}$$

so that $\psi(e_{11}) = vv^*$ and $\psi(e_{22}) = v^*v$. Finally, it is clear that $vx_1 = v$, and so φ and ψ satisfy all of the relations $\hat{\mathcal{R}}_n$.

Next, we check that v and the x_i generate W_n as a C^*-algebra. Write $A := C^*(\{v, x_i : 1 \leq i \leq n\})$. We have

$$v^* x_i(t) = t^{1/2}(1-t)(e_{1i} \otimes e_{1,n+1})$$

and

$$v^* x_i v x_j(t) = t(1-t)^{3/2}(e_{1j} \otimes e_{1i})$$

for $1 \leq i,j \leq n$. Hence for $t \in (0,1)$, the elements $v^*x_i(t)$ and $v^*x_ivx_j(t)$ give all matrix units $\{e_{1k} \otimes e_{1l} : 1 \leq k \leq n, 1 \leq l \leq n+1\}$, so generate all of $M_n \otimes M_{n+1}$, and thus the irreducible representation $\mathrm{ev}_t : W_n \to M_n \otimes M_{n+1}$ restricts to an irreducible representation of A. For $t \in \{0,1\}$, the x_i generate all the matrix units of M_n in the endpoint irreducible representation $\mathrm{ev}_\infty : W_n \to M_n$. Thus every irreducible representation of W_n restricts to an irreducible representation of A. Also, since $x_1(s)$ is not unitarily equivalent to $x_1(t)$ for distinct $s,t \in (0,1)$, it follows that inequivalent irreducible representations of W_n restrict to inequivalent irreducible representations of A. Therefore, by Stone-Weierstrass (i.e. [Dix77, Proposition 11.1.6]), we do indeed have $C^*(\{v, x_i : 1 \leq i \leq n\}) = W_n$.

It remains to show that these generators of W_n enjoy the appropriate universal property: for every representation $\hat{v}, \hat{x}_1, \dots, \hat{x}_n$ of the given relations, we need to show that there is a *-homomorphism $W_n \to C^*(\hat{v}, \hat{x}_1, \dots \hat{x}_n)$ sending v to $\hat{v}$ and x_i to $\hat{x}_i$. By [Lor97, Lemma 3.2.2], it suffices to consider the case where $\{\hat{v}, \hat{x}_1, \dots, \hat{x}_n\} \subset \mathfrak{B}(H)$ is an *irreducible* representation, i.e. has trivial commutant. Suppose that this is the case, and let $\hat{\varphi}$ and $\hat{\psi}$ be the corresponding order zero maps. Define

$$b := \hat{v}\hat{v}^* + \sum_{i=1}^{n} \hat{x}_i^* \hat{v}^* \hat{v} \hat{x}_i \in C^*(\{\hat{v}, \hat{x}_i\}) =: B.$$

We claim that b is central in B. To see this, note that $\hat{v}\hat{v}^* = \hat{\varphi}(1_n)(1 - \hat{\varphi}(1_n))$ commutes with $\hat{x}_j$ for every $1 \leq j \leq n$ and that the relations $\mathcal{R}_n^{(0)}$ imply that $\hat{x}_i \hat{x}_j = 0$ for $i \neq 1$ and $\hat{x}_i \hat{x}_j^* = \delta_{ij} \hat{x}_1^2$. (These assertions all follow easily from the fact that $\sqrt{\hat{\varphi}}$ is order zero with $\hat{x}_i = \sqrt{\hat{\varphi}}(e_{1i})$.) Then

$$b\hat{x}_j = \hat{v}\hat{v}^*\hat{x}_j + \hat{x}_1\hat{v}^*\hat{v}\hat{x}_1\hat{x}_j = \hat{v}\hat{v}^*\hat{x}_j + \hat{v}^*\hat{v}\hat{x}_j;$$

$$\hat{x}_j b = \hat{x}_j \hat{v}\hat{v}^* + \hat{x}_1^2 \hat{v}^* \hat{v} \hat{x}_j = \hat{v}\hat{v}^*\hat{x}_j + \hat{v}^*\hat{v}\hat{x}_j = b\hat{x}_j;$$

$$b\hat{v}^* = b\hat{x}_1\hat{v}^* = \sum_{i=1}^{n} \hat{x}_i^* \hat{v}^* \hat{v} \hat{x}_i \hat{x}_1 \hat{v}^* = \hat{v}^*\hat{v}\hat{v}^*;$$

and

$$\begin{aligned}
\hat{v}\hat{v}^*\hat{x}_i^*\hat{v}^*\hat{v}\hat{x}_i &= \left(\sum_{k=1}^{n} \hat{x}_k^*\hat{x}_k\right)\left(\hat{x}_i^*\hat{v}^*\hat{v}\hat{x}_i - \sum_{k=1}^{n} \hat{x}_k^*\hat{x}_k\hat{x}_i^*\hat{v}^*\hat{v}\hat{x}_i\right) \\
&= \left(\sum_{k=1}^{n} \hat{x}_k^*\hat{x}_k\right)\left(\hat{x}_i^*\hat{v}^*\hat{v}\hat{x}_i - \hat{x}_i^*\hat{v}^*\hat{v}\hat{x}_i\right) \\
&= 0,
\end{aligned}$$

so that

$$\|\hat{v}^*\hat{x}_i^*\hat{v}^*\hat{v}\hat{x}_i\|^2 = \|\hat{x}_i^*\hat{v}^*\hat{v}\hat{x}_i\hat{v}\hat{v}^*\hat{x}_i^*\hat{v}^*\hat{v}\hat{x}_i\| = 0 \quad \text{for every } i,$$

which shows that

$$\hat{v}^* b = \hat{v}^*\hat{v}\hat{v}^* + \sum_{i=1}^{n} \hat{v}^*\hat{x}_i^*\hat{v}^*\hat{v}\hat{x}_i = \hat{v}^*\hat{v}\hat{v}^* = b\hat{v}^*.$$

Thus b is indeed central in B, and is therefore $\beta 1$ for some scalar β. Moreover, b is positive, and by functional calculus we have $\|\hat{v}\hat{v}^*\| = \|\hat{\varphi}(1_n)(1 - \hat{\varphi}(1_n))\| \leq 1/4$, so that $\|b\| \leq 1/2$. Hence $0 \leq \beta \leq 1/2$.

If $\beta = 0$ then $\hat{v} = 0$ and hence $\hat{\varphi}(1_n)$ is a projection. It follows that $\hat{\varphi}$ is a *-homomorphism giving an irreducible representation of M_n on H (see Remark 5.1.2). Thus

$H = \mathbb{C}^n$ and (up to unitary equivalence) the $\hat{x}_i$ are the matrix units e_{1i}, so are the images of $x_i \in W_n$ under the endpoint irreducible representation ev_∞.

Suppose that $\beta > 0$. Then

$$\beta\hat{v}\hat{v}^* = b\hat{v}\hat{v}^* = (\hat{v}\hat{v}^*)^2 + \sum_{i=1}^n \hat{x}_i^*\hat{v}^*\hat{v}\hat{x}_i\hat{v}\hat{v}^* = (\hat{v}\hat{v}^*)^2 + \sum_{i=1}^n \hat{x}_i^*\hat{v}^*\hat{v}\hat{v}\hat{v}^*\hat{x}_i = (\hat{v}\hat{v}^*)^2,$$

so $p := \beta^{-1}\hat{v}\hat{v}^*$ and $q := \beta^{-1}\hat{v}^*\hat{v}$ are orthogonal, nonzero projections (with $\beta^{-1/2}\hat{v}$ implementing a Murray-von Neumman equivalence between them). Since p commutes with every $\hat{x}_i$, the maps $\hat{\varphi}(\cdot)p$ and $\hat{\varphi}(\cdot)(1-p)$ are c.p.c. order zero. In fact,

$$\beta\hat{\varphi}(1_n)(1-p) = \hat{\varphi}(1_n)(b - \hat{v}\hat{v}^*) = \sum_{i,j} \hat{x}_j^*\hat{x}_j\hat{x}_i^*\hat{v}^*\hat{v}\hat{x}_i = \sum_i \hat{x}_i^*\hat{v}^*\hat{v}\hat{x}_i = b - \hat{v}\hat{v}^* = \beta(1-p),$$

i.e. $\hat{\varphi}(1_n)(1-p) = 1-p$. Thus $\hat{\varphi}(\cdot)(1-p)$ is a *unital* c.p.c. order zero map into the corner $(1-p)\mathfrak{B}(H)(1-p) \cong \mathfrak{B}((1-p)H)$, so is a *-homomorphism into this corner (see Remark 5.1.2(ii)). Also, $\hat{\varphi}(1_n)p$ commutes with (the WOT-closure of) the corner $pC^*(\{\hat{v}, \hat{x}_1, \ldots, \hat{x}_n\})p = pC^*(\{\hat{x}_1, \ldots, \hat{x}_n\})p$ (which, by irreducibility, is all of $p\mathfrak{B}(H)p \cong \mathfrak{B}(pH)$) so $\hat{\varphi}(1_n)p = tp$ for some $t \in [0,1]$. Hence $\hat{\varphi}(\cdot)t^{-1}p$ is also a *-homomorphism, and is in fact an *irreducible* representation of M_n on pH. Thus (again up to unitary equivalence) we have $pH = \mathbb{C}^n$ and $t^{-1}\hat{x}_i^*\hat{x}_jp = e_{ij}^{(n)}$ for $1 \le i, j \le n$.

Moreover, we have

$$\beta x_1^2 = bx_1^2 = \hat{v}\hat{v}^*\hat{x}_1^2 + \hat{v}^*\hat{v} = \beta(\hat{x}_1^2p + q),$$

and hence $\hat{x}_1^2(1-p) = q$. In particular, $\beta^{-1/2}\hat{v}$ implements an equivalence between p and $\hat{x}_1^2(1-p)$, which shows that the orthogonal projections $\hat{x}_i^*\hat{x}_i(1-p)$ in the representation $\hat{\varphi}(\cdot)(1-p)$ all have trace n $(= \mathrm{tr}(p))$. Thus (up to unitary equivalence) $(1-p)H = \mathbb{C}^{n^2}$ (so $H = \mathbb{C}^{n(n+1)}$) and $\hat{\varphi}(\cdot)(1-p) : M_n \to M_{n^2}$ is just $a \mapsto \mathrm{diag}(a, \ldots, a)$.

Finally, since

$$t(1-t)p = tp(p - tp) = \hat{\varphi}(1_n)p(p - \hat{\varphi}(1_n)p) = p\hat{v}\hat{v}^* = \beta p,$$

we have $t(1-t) = \beta$. Therefore, if $\mathrm{ev}_t : W_n \to \mathbb{C}^{n(n+1)}$ is the irreducible representation given by evaluation at $t \in (0,1)$, then

$$\mathrm{ev}_{1-t}(x_i) = \hat{x}_i \quad \text{and} \quad \mathrm{ev}_{1-t}(v) = \mathrm{ev}_t(v) = \hat{v}.$$

Hence W_n has the required universal property. □

Theorem 5.3.3. *Choose positive functions $d, f, g, h \in C_0(0,1]$, partial isometries $v_k \in M_{q(k)}$, and c.p.c. order zero maps $\rho_k : M_{q(k)} \to M_{q(k+1)}$ as in Theorem 5.2.1. Define W_U to be the universal C^*-algebra generated by c.p.c. order zero maps φ_k on $M_{q(k)}$ ($k \in \mathbb{N}$) and ψ_k on M_2 ($k \in \mathbb{N}$) such that for each k, these maps satisfy the relations $\hat{\mathcal{R}}_{q(k)}$, i.e.*

$$\psi_k(e_{11}) = \varphi_k(1_{q(k)})(1 - \varphi_k(1_{q(k)})) \tag{5.44}$$

and

$$\psi_k(e_{22})\varphi_k(e_{11}) = \psi_k(e_{22}), \tag{5.45}$$

together with the additional relations $\hat{\mathcal{S}}_{q(k)}$ given by

$$\varphi_k = f(\varphi_{k+1}) \circ \rho_k, \tag{5.46}$$

$$\begin{aligned}\sqrt{\psi_k}(e_{12}) = f(\varphi_{k+1})(\rho_k(1_{q(k)}))^{1/2}\Big(& h(\varphi_{k+1})(1_{q(k+1)} - \rho_k(1_{q(k)}))^{1/2} f(\varphi_{k+1})(v_{k+1}) \\ &+ (1 - f(\varphi_{k+1})(1_{q(k+1)}) + g(\varphi_{k+1})(1_{q(k+1)} - \rho_k(1_{q(k)})))^{1/2} d(\psi_{k+1})(e_{12})\Big).\end{aligned} \tag{5.47}$$

Then $W_U \cong W$.

Proof. The proof is essentially the same as that of Theorem 5.2.1, so we omit most of the details. As before, let us write

$$\hat{\varphi}_k = f(\varphi_{k+1}) \circ \rho_k \tag{5.48}$$

and

$$\sqrt{\hat{\psi}_k}(e_{12}) = \gamma_k + \delta_k, \tag{5.49}$$

where this time

$$\gamma_k := f(\varphi_{k+1})(\rho_k(1_{q(k)}))^{1/2}\lambda_k d(\psi_{k+1})(e_{12}) \tag{5.50}$$

and

$$\delta_k := f(\varphi_{k+1})(\rho_k(1_{q(k)}))^{1/2}\mu_k f(\varphi_{k+1})(v_{k+1}), \tag{5.51}$$

with

$$\lambda_k := (1 - f(\varphi_{k+1})(1_{q(k+1)}) + g(\varphi_{k+1})(1_{q(k+1)} - \rho_k(1_{q(k)})))^{1/2} \tag{5.52}$$

and

$$\mu_k := h(\varphi_{k+1})(1_{q(k+1)} - \rho_k(1_{q(k)}))^{1/2}. \tag{5.53}$$

Note that $f(\varphi_{k+1})(\rho_k(1_{q(k)}))^{1/2}$ commutes with λ_k and μ_k (for example using (5.5)), so we can show that $\left(\sqrt{\hat{\psi}_k}(e_{12})\right)^2 = 0$ in the same way as before. While Lemma 5.2.3 remains the same, the version of Lemma 5.2.2 in this context (with the same proof) is the following.

Lemma 5.3.4. *For every positive contractions $F, G \in C_0(0,1]$ and every $k \in \mathbb{N}$ we have*

$$F(\psi_k)(e_{12})G(\varphi_k)(1_{q(k)}) = G(1)F(\psi_k)(e_{12})\varphi_k(1_{q(k)}).$$

In particular, if $G(1) = 0$ then $F(\psi_k)(e_{12})G(\varphi_k)(\cdot) = 0$.

To show that $\hat{\psi}_k(e_{11}) = \hat{\varphi}_k(1_{q(k)})(1 - \hat{\varphi}_k(1_{q(k)}))$, we proceed exactly as in the proof of Lemma 5.2.5. The only difference is that we now have

$$d(\psi_{k+1})(e_{11}) = d(\psi_{k+1}(e_{11})) = d(\varphi_{k+1}(1_{q(k+1)})(1 - \varphi_{k+1}(1_{q(k+1)}))) = \hat{d}(\varphi_{k+1}(1_{q(k+1)})),$$

where $\hat{d}(t) = d(t(1-t))$ as in (5.18). Using (5.2), we also have

$$f(\varphi_{k+1})(\rho_k(1_{q(k)}))(1 - f(\varphi_{k+1})(1_{q(k+1)})) = \pi_{\varphi_{k+1}}(\rho_k(1_{q(k+1)}))(f - f^2)(\varphi_{k+1}(1_{q(k+1)})).$$

Since $\hat{d}(f - f^2) = f - f^2$, this therefore gives

$$\begin{aligned} f(\varphi_{k+1})(\rho_k(1_{q(k)}))(1 - f(\varphi_{k+1})(1_{q(k+1)}))d(\psi_{k+1})(e_{11}) \\ = f(\varphi_{k+1})(\rho_k(1_{q(k)}))(1 - f(\varphi_{k+1})(1_{q(k+1)})), \end{aligned}$$

and the rest of the argument carries over mutatis mutandis. The proof that $\hat{\psi}_k(e_{22})\hat{\varphi}_k(e_{11}) = \hat{\psi}_k(e_{22})$ is literally the same as the proof of Lemma 5.2.6.

We now know that W_U is isomorphic to an inductive limit $\varinjlim(W_{q(k)}, \beta_k)$. Moreover, arguing exactly as before, we see that these connecting maps β_k are unitarily equivalent to the connecting maps α_k obtained earlier. That is, we have

$$\beta_k^t = w_k(t)\left(\left(\bigoplus_{m=1}^{q(k+1)}\bigoplus_{i=1}^{q(k)-1} \mathrm{ev}_{\frac{i}{q(k)}}\right) \oplus \left(\bigoplus_{m=1}^{q(k)(q(k)-1)} \mathrm{ev}_{h(t)}\right) \oplus \left(\bigoplus_{i=1}^{q(k)} \mathrm{ev}_{h_i(t)}\right)\right) w_k(t)^* \tag{5.54}$$

for some unitary $w_k \in C([0,1], M_{q(k+1)} \otimes M_{q(k+1)+1})$.

The same arguments as with $\mathcal{Z}_U$ show that W_U is simple and has at most one tracial state. (One has to perhaps be slightly careful about arguing as in the proof of Lemma 5.2.7 to deduce the *existence* of a trace, since the space of tracial states of a nonunital C^*-algebra need not be compact. But this is not an issue – see the argument of Proposition 4.3.6.) The only minor technicality is that, since the building blocks $W_{q(k)}$ are nonunital and the connecting maps β_k are degenerate, W_U may have unbounded traces. However, one can easily show, using (5.36), that this is not the case.

Lemma 5.3.5. *Every trace on W_U is bounded.*

Proof. Suppose that $\tau = (\tau_k)_{k=1}^{\infty}$ is a trace on $\varinjlim(W_{q(k)}, \beta_k)$. That is, for every k we have $\tau_k \in T^+W_{q(k)}$ with $\tau_{k+1} \circ \beta_k = \tau_k$. Then for each k we have $\tau_k = \mathrm{tr}_k \otimes \mu_k$ for some positive Borel measure μ_k on $[0,1)$, where tr_k is the unique tracial state on $M_{q(k)} \otimes M_{q(k)+1}$.

For every $i \in \mathbb{N}$, let $c_i \in W_{q(i)}$ be the canonical strictly positive element, i.e. $c_i = \varphi_i(1_{q(i)})$. Then $(c_i^{1/n})_{n=1}^{\infty}$ is an approximate unit of $W_{q(i)}$, and $\mathrm{tr}_i(c_i^{1/n})$ converges pointwise as $n \to \infty$ to the function $y_i \in L^{\infty}([0,1], \mu_i)$ given by

$$y_i(t) = \begin{cases} 1, & 0 \le t < 1 \\ \frac{q(i)}{q(i)+1}, & t = 1. \end{cases}$$

By the monotone convergence theorem, we then have

$$\|\tau_i\| = \lim_{n\to\infty} \tau_i(c_i^{1/n}) = \lim_{n\to\infty} \int_{[0,1)} \mathrm{tr}_i(c_i^{1/n}) d\mu_i = \int_{[0,1)} y_i d\mu_i \qquad (= \mu_i([0,1))).$$

Now fix $k \in \mathbb{N}$. Note that $\mathrm{tr}_{k+1}(\beta_k(c_k^{1/n}))$ also converges pointwise, to some $z_{k+1} \in L^{\infty}([0,1], \mu_{k+1})$. From the definition (5.54) of β_k, and from (5.36), we see that

$$|z_{k+1}(t) - y_{k+1}(t)| \le \frac{1}{q(k)^2 - q(k) + 1} \le \frac{1}{(q(k)-1)^2}$$

for every $t \in [0,1)$. Thus

$$\int_{[0,1)} (y_{k+1} - z_{k+1}) d\mu_{k+1} \le \frac{1}{(q(k)-1)^2} \mu_{k+1}([0,1)) = \frac{\|\tau_{k+1}\|}{(q(k)-1)^2},$$

and so (applying the monotone convergence theorem once again)

$$\begin{aligned} \|\tau_{k+1}\| &= \int_{[0,1)} y_{k+1} d\mu_{k+1} \\ &\le \int_{[0,1)} z_{k+1} d\mu_{k+1} + \frac{\|\tau_{k+1}\|}{(q(k)-1)^2} \\ &= \lim_{n\to\infty} \int_{[0,1)} \mathrm{tr}_{k+1}(\beta_k(c_k^{1/n})) d\mu_{k+1} + \frac{\|\tau_{k+1}\|}{(q(k)-1)^2} \\ &= \lim_{n\to\infty} \tau_{k+1}(\beta_k(c_k^{1/n})) + \frac{\|\tau_{k+1}\|}{(q(k)-1)^2} \\ &= \lim_{n\to\infty} \tau_k(c_k^{1/n}) + \frac{\|\tau_{k+1}\|}{(q(k)-1)^2} \\ &= \|\tau_k\| + \frac{\|\tau_{k+1}\|}{(q(k)-1)^2}. \end{aligned}$$

Therefore

$$\|\tau_{k+1}\| \le \frac{(q(k)-1)^2}{(q(k)-1)^2 - 1} \|\tau_k\| \le \prod_{i=1}^{k} \left(1 + \frac{1}{(q(i)-1)^2 - 1}\right) \|\tau_1\|.$$

But $\sum_{i=1}^{\infty} \frac{1}{(q(i)-1)^2-1}$ converges, hence so does $\prod_{i=1}^{\infty} \left(1 + \frac{1}{(q(i)-1)^2-1}\right)$, and so there is some $M > 0$ such that $\|\tau_k\| \le M\|\tau_1\|$ for every $k \in \mathbb{N}$. Hence τ is bounded. $\square$

It therefore follows from Theorem 4.1.2 that $W_U \cong W$. □

Corollary 5.3.6. *There exists a trace-preserving embedding of W into $\mathcal{Z}$. Such an embedding is canonical at the level of the Cuntz semigroup, and is unique up to approximate unitary equivalence.*

Proof. This follows immediately from Theorem 5.3.3 and Theorem 5.2.1. The result can already be deduced from the main theorem of [Rob10], which also gives the uniqueness statement. □

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