Quasi-invariants of hyperplane arrangements

by

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Abstract

The ring of *quasi-invariants* Q_m can be associated with the root system R and multiplicity function m. It first appeared in the work of Chalykh and Veselov [CV90] in the context of quantum Calogero-Moser systems. One can define an analogue $Q_{\mathcal{A}}$ of this ring for a collection $\mathcal{A}$ of vectors with multiplicities. We study the algebraic properties of these rings. For the class of arrangements on the plane with at most one multiplicity greater than one we show that the Gorenstein property for $Q_{\mathcal{A}}$ is equivalent to the existence of the Baker-Akhiezer function, thus suggesting a new perspective on systems of Calogero-Moser type.

The rings of quasi-invariants Q_m have a well known interpretation as modules for the spherical subalgebra of the rational Cherednik algebra with integer valued multiplicity function. We explicitly construct the anti-invariant quasi-invariant polynomials corresponding to the root system A_n as certain representations of the spherical subalgebra of the Cherednik algebra $H_{1/m}(S_{mn})$. We also study the relation of the algebra $\Lambda_{n,1,k}$ introduced in [SV04] to the ring of quasi-invariants for the deformed root system $\mathcal{A}_n(k)$. We find the Poincaré series for a 'symmetric part' of $Q_{\mathcal{A}_n(k)}$ for positive integer values of k.

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Statement

This thesis is submitted in accordance with the regulations for the degree of Doctor of Philosophy at the University of Glasgow.

Chapter 1 introduces some of the main concepts used throughout the thesis.

Chapter 2 is the author's own work in collaboration with Misha Feigin, with the exception of those results explicitly referenced.

Chapter 3 gives an explanation of the Gorenstein property for a graded ring, following the approach of Benson [Ben93]. As such it contains no new results, except for Theorem 3.3.2 which had not thus far appeared in the literature.

Chapters 4 and 5 are the author's own work in collaboration with Misha Feigin, with the exception of those results explicitly referenced.

Chapter 6 consists of concluding remarks.

Contents

Chapter 1

Introduction

Recall the notation of the Coxeter root system R. This system consists of different pairs of vectors $\{\pm\alpha\}$ in some Euclidean space V. The corresponding reflections

$$s_\alpha x = x - 2\frac{(\alpha, x)}{(\alpha, \alpha)}\alpha, \ \ x \in V.$$

generate a group G (Coxeter group) which should be finite. It is assumed that the set $s_\alpha, \alpha \in R$ is the set of all hyperplane reflections in G and that the vectors $\alpha, \beta \in R$ are non-collinear unless $\alpha = \pm\beta$. We also consider a G-invariant function $m : R \to \mathbb{Z}_+$ which we call *multiplicity*. Say $\alpha \in R$ has multiplicity m_α under this function. Let $S = S(V)$ be the ring of all polynomials on V and let S^G be the subring of G-invariant polynomials. The ring of *quasi-invariants* Q_m can be associated with the root system R and multiplicity function m. It first appeared in the work of Chalykh and Veselov [CV90] in the context of quantum Calogero-Moser systems. This is a key object for our work.

Definition ([CV90], see also [FV02])**.** *We say a polynomial $p \in S$ is quasi-invariant if for any $\alpha \in R$*

$$p(s_\alpha(x)) = p(x) + O((\alpha, x)^{2m_\alpha+1})$$

near the hyperplane $(\alpha, x) = 0$. Equivalently, for any $\alpha \in R$ and $s = 1, 3, \ldots, 2m_\alpha - 1$

$$\partial_\alpha^s p = 0$$

on the hyperplane $(\alpha, x) = 0$, where $\partial_\alpha = (\alpha, \frac{\partial}{\partial x})$ is the normal derivative for this hyperplane.

These polynomials form a ring, the ring of *m-quasi-invariants* (or just quasi-invariants) Q_m. This ring contains the ring of invariant polynomials S^G.

The rings Q_m have an interesting algebraic structure. Their properties have been extensively studied. Feigin and Veselov [FV02, Theorem 8] showed that the ring Q_m is free over the invariants S^G for any dihedral system R (Cohen-Macaulay property). They also conjectured that this is true in the general Coxeter case, and that one can take so called *m-harmonic* polynomials H_m as a basis for the free module. They checked this for dihedral systems. Etingof and Ginzburg [EG02a, Theorem 1.1] established that Q_m is free over the invariants S^G for any Coxeter system R. However the choice of H_m as a free basis is impossible in general (counterexample in [EG02a, Section 7]).

Feigin and Veselov [FV02, Corollary 8] also calculated the Poincaré series $P(Q_m, t)$ for quasi-invariants of dihedral systems. Their formula makes it clear that the series is palindromic, and thus the ring is Gorenstein. Etingof and Ginzburg [EG02a, Theorem 1.2] established the Gorenstein property for Q_m in the general Coxeter case. They gave two proofs, one of which is based on earlier results of Felder and Veselov [FV03b]. In that paper the authors computed the Poincaré polynomial for the space H_m of m-harmonic polynomials. It is given by the formula

$$P(H_m, t) = \sum_{V_j} \dim(V_j) t^{\sum_a m_a d_a^-(V_j)} P_j(t).$$

Here the external sum is carried over all non-isomorphic irreducible representations of G and the internal sum is carried over the classes C_a of conjugate reflections in G. The polynomial P_j is the Poincare polynomial for the representation V_j in the space $H_0 \cong \mathbb{C}[x_1, \dots, x_n]/I_0$ where I_0 is the ideal in the ring of polynomials generated by the homogeneous G-invariant polynomials of positive degree. Also

$$d_a^-(V_j) = \frac{2N_a \dim V_{j,a}^-}{\dim V_j}$$

where N_a is the number of elements in the class C_a and

$$V_{j,a}^- = \{v \in V_j \mid s_\alpha v = -v \text{ for any } s_\alpha \in C_a\}.$$

Felder and Veselov observed that $P(H_m, t)$ satisfies the palindromicity relation $P(H_m, t) = t^M P(H_m, t^{-1})$ for some M. Etingof and Ginzburg then established the Gorenstein property for Q_m via this observation and the Cohen-Macaulay property. More exactly it is shown in [EG02a, Theorem 1.1] that

$$P(Q_m, t) = \frac{P(H_m, t)}{\prod_{i=1}^n (1 - t^{d_i})}$$

where d_i are the degrees of the basic invariants.

Berest, Etingof and Ginzburg [BEG03] presented a different approach to understanding the structure of the rings Q_m in a slightly later paper. Here it is shown that quasi-invariants can be seen as modules for certain subalgebras of the rational Cherednik algebra. The Cohen-Macaulay property is established and the previously calculated series $P(Q_m, t)$ is also recovered.

Now we review the context of quantum integrable systems, specifically systems of Calogero-Moser type, where the rings Q_m arose. This subject has roots in the work of Calogero in 1969-1971 [Cal69, Cal71]. He studied a quantum system which describes pairwise interaction of n particles on a line with quadratic and/or inversely quadratic potential. Moser [Mos75] later famously studied the classical system with the Hamiltonian

$$H = \frac{1}{2}p^2 + g\sum_{i<j}^{n} \frac{1}{(q_i - q_j)^2}. \tag{1.1}$$

and established its integrability.

Quantum analogs of the classical integrals of the Calogero-Moser system (1.1) first appeared in the work of Calogero, Marchioro and Ragnisco [CMR75]. Olshanetksy and Perelomov proved that these integrals commute [OP77].

Olshanetsky and Perelomov [OP77] later introduced integrable generalizations of the system (1.1) related to an arbitrary Coxeter group. The potential of the generalized system has singularities located on hyperplanes which are mirrors for the corresponding Coxeter group.

The quantum Hamiltonian corresponding to the Coxeter root system R has the form

$$L = \Delta - \sum_{\alpha \in R_+} \frac{m_\alpha(m_\alpha + 1)(\alpha, \alpha)}{(\alpha, x)^2} \tag{1.2}$$

where R_+ is the positive part of R. Heckman [Hec91] proved that the quantum problem (1.2) related to an arbitrary Coxeter system is integrable. This was achieved via an explicit construction for quantum integrals based on the Dunkl operators [Dun89]. He showed that if one replaces the variable k_i with the i-th Dunkl operator ∇_i in an invariant polynomial $p(k) \in S^G$ then the restriction of the resulting differential difference operator $p(\nabla)$ to the space of invariant functions is an integral for the quantum problem (1.2). The Hamiltonian corresponds to the polynomial $p(k) = k^2$. Explicitly, Heckman's theorem is given by the following. Consider differential operators D_R with coefficients from the algebra generated by $(\alpha, x)^{-1}, \alpha \in R$ and by constant functions. Let

$$D^G_{R,m} = \{A \in D_R : [L, A] = 0, g(A) = A \text{ for all } g \in G\}.$$

Theorem ([Hec91], Theorem 1.7)**.** *There exists an isomorphism γ_m between the ring S^G of G-invariant polynomials and the ring $D^G_{R,m}$ of quantum integrals for the Hamiltonian (1.2)*

$$\gamma_m : S^G \cong D^G_{R,m}.$$

Remarkably, the ring of quantum integrals for the generalized Calogero-Moser systems (1.2) becomes larger than S^G when the parameters m_α are integer. This crucial observation was made by Chalykh and Veselov in 1990 [CV90]. Their work can be explained by the following.

Theorem ([CV90], Theorem 1)**.** *There exists a homomorphism*

$$\chi_m : p(k) \mapsto L_p(x, \frac{\partial}{\partial x}) \tag{1.3}$$

mapping Q_m into the commutative ring of differential operators containing the Calogero-Moser operator (1.2).

Feigin and Veselov fully explained the relation between the rings of quasi-invariants and the rings of all quantum integrals of the Calogero-Moser systems.

Theorem ([FV02], Theorem 2)**.** *Let D_m be the maximal commutative ring containing $D^G_{R,m}$ as a subring. Then the map (1.3) is an isomorphism between the quasi-invariant ring Q_m and the ring D_m.*

The key motivation for the work in this thesis was the discovery of rings of quasi-invariants associated to non-symmetric, integrable generalizations of the quantum Calogero-Moser problem. In [VFC96, CFV98] Chalykh, Feigin and Veselov introduced the configurations $\mathcal{A}_N(m)$ and $\mathcal{C}_N(m, l)$ which can be thought of as deformations of the classical root systems $\mathcal{A}_N$ and $\mathcal{C}_N$. Two commuting families of differential operators corresponding to the systems $\mathcal{A}_N(m)$ and $\mathcal{C}_N(m, l)$ were found. A physical interpretation of the $\mathcal{A}_N(m)$ system is the description of Calogero-Moser-like pairwise interaction of particles where one particle has different mass from the others.

There are quasi-invariants associated to the configurations $\mathcal{A}_N(m)$ and $\mathcal{C}_N(m, l)$. One can consider polynomials satisfying a quasi-invariance property under the action of all hyperplane reflections associated to these configurations. In [FV03a] it was shown that these rings still have nice algebraic properties. Indeed, Feigin and Veselov were able to show that the rings $Q_{\mathcal{A}}$ corresponding to the configurations $\mathcal{A}_N(m)$ and $\mathcal{C}_N(m,l)$ are Cohen-Macaulay, following the scheme from [EG02a] where this property was established in the Coxeter case. They also calculated the Poincaré series for the quasi-invariants corresponding to the two-dimensional configurations $\mathcal{A}_2(m)$ and $\mathcal{C}_2(m,l)$. They point out that the series found in this way are palindromic and thus the corresponding rings are Gorenstein. The calculations are achieved via direct analysis of the quasi-invariance conditions. In a similar fashion to the Coxeter case they also show that the rings $Q_{\mathcal{A}}$ corresponding to the configurations $\mathcal{A}_N(m)$ and $\mathcal{C}_N(m,l)$ are maximal commutative rings containing all quantum integrals for the corresponding generalized Calogero-Moser operator.

Thus rings of quasi-invariants can be introduced for non-Coxeter arrangements of vectors with multiplicities. More exactly, let A be a finite set of non-collinear vectors $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n$, where $\alpha_i \in \mathbb{C}^N$. Denote by Π_i the hyperplane $(\alpha_i, x) = 0$. Let m be a function $m : A \to \mathbb{N}$, the multiplicity function. Say $\alpha_i \in A$ has multiplicity m_i under this function. The configuration $\mathcal{A} = (A, m)$ is then a set of vectors with multiplicities.

Definition. *We say a polynomial $p \in \mathbb{C}[x_1, x_2, \ldots, x_N]$ is* quasi-invariant *with respect to $\mathcal{A} = (A, m)$ if, for all $\alpha_i \in A$ and $s = 1, 3, \ldots, 2m_i - 1$:*

$$\partial_{\alpha_i}^s p = 0$$

on the hyperplane Π_i, where $\partial_{\alpha_i} = (\alpha_i, \frac{\partial}{\partial x})$.

These polynomials form a ring, the *quasi-invariant ring* $Q_{\mathcal{A}}$. These rings will not in general have the nice algebraic properties exhibited in the Coxeter case, although as we have seen there exist interesting examples ($\mathcal{A}_N(m)$ and $\mathcal{C}_N(m,l)$) where these properties are apparent.

The central theme of the work in this thesis is thus the study of algebraic properties for certain rings of quasi-invariants, in particular the Gorenstein property, and the ramifications of this property being satisfied. Before explaining what is shown in detail we would like to explain a further connection between rings of quasi-invariants and integrability of generalized Calogero-Moser operators.

Chalykh, Feigin and Veselov [CFV99] established the integrability of the operators corresponding to the deformed configurations $\mathcal{A}_N(m)$ and $\mathcal{C}_N(m,l)$ via existence of the so-called *Baker-Akhiezer function* (BA function). This notion was introduced by Krichever

[Kri77] in the theory of finite-gap solutions of nonlinear PDE's, integrable by the inverse scattering method. Chalykh, Styrkas and Veselov [CV90, VSC93] then introduced and studied multi-dimensional versions of such functions in the context of the quantum Calogero-Moser problem. The construction of [VSC93] relates a BA function ϕ to $\mathcal{A} = (A, m)$. More exactly, for $k = (k_1, k_2, \ldots, k_N)$ and $x = (x_1, x_2, \ldots, x_N)$ let $(k, x) = \sum_{i=1}^{N} k_i x_i$ be the standard inner product in $\mathbb{C}^N$. We assume that $(\alpha_i, \alpha_i) \neq 0$ for any $\alpha_i \in A$.

Definition. *A function $\phi(k, x), k, x \in \mathbb{C}^N$ will be called a Baker-Akhiezer function if the following two conditions are fulfilled:*

- *$\phi(k, x)$ has the form*

$$\phi(k, x) = P(k, x) e^{(k,x)}$$

  where $P(k, x)$ is a polynomial in k with highest term $A(k) = \prod_{\alpha_i \in A} (k, \alpha_i)^{m_i}$;

- *For all $\alpha_i \in A$*

$$\partial_{\alpha_i} \phi(k, x) = \partial^3_{\alpha_i} \phi(k, x) = \ldots = \partial^{2m_i - 1}_{\alpha_i} \phi(k, x) = 0$$

  on the hyperplane $(k, \alpha_i) = 0$, where $\partial_{\alpha_i} = (\alpha_i, \frac{\partial}{\partial k})$ is the normal derivative for this hyperplane.

The relation of the function ϕ to the integrability of the corresponding quantum Calogero-Moser problem is explained by the following results.

Theorem ([VSC93], Proposition 1)**.** *If the Baker-Akhiezer function ϕ exists then it is unique and satisfies the Schrödinger equation*

$$L\phi = -k^2 \phi$$

where

$$L = -\Delta + \sum_{\alpha_i \in A} \frac{m_i(m_i + 1)(\alpha_i, \alpha_i)}{(\alpha_i, x)^2}. \tag{1.4}$$

Moreover the operator (1.4) is part of a large commutative ring of partial differential operators described by the following theorem.

Theorem ([VSC93], Theorem 1)**.** *If the Baker-Akhiezer function ϕ exists then for any polynomial $f(k) \in Q_{\mathcal{A}}$ there exists some differential operator $L_f(x, \frac{\partial}{\partial x})$ such that*

$$L_f \phi(k, x) = f(k) \phi(k, x).$$

These operators form a commutative ring isomorphic to $Q_{\mathcal{A}}$. The operator (1.4) corresponds to $f(k) = -k^2$.

The function $\phi(k,x), k, x \in \mathbb{C}^N$ exists only for very special configurations. An elegant result of Chalykh, Styrkas and Veselov describes fully the situation where all multiplicities are taken to be one.

Theorem ([VSC93], Theorem 4)**.** *Suppose $m_i = 1 \,\forall\, i$. Then the Baker-Akhiezer function ϕ exists for $\mathcal{A} = (A, m)$ if and only if A is a set of normals to the reflection hyperplanes for some Coxeter group.*

The following remarkable result shows that there is an effective way to check whether the BA function ϕ exists for a given configuration.

Theorem ([VSC93], [CFV99])**.** *The Baker-Akhiezer function ϕ exists for a configuration $\mathcal{A}$ if and only if the following two sets of conditions hold $\forall\ \alpha_j \in A$*

$$\sum_{i=0,\, i\neq j}^{n} \frac{m_i(\alpha_j,\alpha_i)^{2k-1}}{(x,\alpha_i)^{2k-1}} = 0$$

on the hyperplane Π_j for $1 \le k \le m_j$,

$$\sum_{i=0,\, i\neq j}^{n} \frac{m_i(m_i+1)(\alpha_i,\alpha_i)(\alpha_j,\alpha_i)^{2k-1}}{(\alpha_i,x)^{2k+1}} = 0$$

on the hyperplane Π_j for $1 \le k \le m_j$.

It was these conditions that were used by Chalykh, Feigin and Veselov [CFV99] to show that the BA function ϕ exists for the deformed Coxeter configurations $\mathcal{A}_N(m)$ and $\mathcal{C}_{N+1}(m,l)$.

Thus far the only explicitly constructed configurations for which ϕ is known to exist were the Coxeter arrangements, $\mathcal{A}_N(m)$ and $\mathcal{C}_N(m,l)$.

We now turn to the results contained in this thesis. The notion underpinning the material is the importance of the Gorenstein property. We are interested in the ramifications of this property being satisfied. It appears that this leads, via the Baker-Akhiezer function, back to integrable systems. This provides another viewpoint on integrable systems of generalized Calogero-Moser type. We were able to demonstrate this within specific classes of arrangements.

We now turn to the original results of the thesis. The subsequent chapters are structured as follows. Chapter 2 does not in fact deal with Gorenstein rings of quasi-invariants.

Instead we introduce the configuration $\mathcal{A}_{(m,1^n)}$ (Chapter 2, Definition 2.4.7) which consists of $n+1$ non-collinear vectors in $\mathbb{C}^2$, n of which have multiplicity one and one of which has arbitrary multiplicity $m \in \mathbb{N}$. When $m = 1$ this configuration is dihedral and when $n = 2$ it coincides with $\mathcal{A}_2(m)$. We are interested in the importance of $\mathcal{A}_{(m,1^n)}$ within the class of arrangements in $\mathbb{C}^2$ with all but one multiplicity equal to one. We investigate in particular the possibility of existence of the BA function for $\mathcal{A}_{(m,1^n)}$

Theorem (Chapter 2, Theorems 2.5.1, 2.6.1 and 2.6.2)**.** *Let $N = 2$ and suppose $m_0 = m \in \mathbb{N}$ and $m_i = 1$ for $1 \leq i \leq n$. Then the Baker-Akhiezer function $\phi(k, x)$ exists for $\mathcal{A}$ if and only if $\mathcal{A} = \mathcal{A}_{(m,1^n)}$*

In Chapter 3 we provide an introduction to the Gorenstein property in the graded case, recalling some standard results from the literature. The material is presented in a fashion which allows us to review Stanley's famous criterion for when Cohen-Macaulay domains are Gorenstein.

Theorem ([Sta78, Theorem 4.4])**.** *Let R be a Noetherian graded commutative Cohen-Macaulay domain. Suppose also that R is a K-algebra for a field K. Then R is Gorenstein if and only if its Poincaré series $P_R(t)$ is palindromic, that is if:*

$$P_R(t^{-1}) = (-1)^d t^l P_R(t) \quad \text{where } d = dim(R) \text{ and } l \in \mathbb{Z}.$$

We also show that rings of quasi-invariants corresponding to configurations of vectors in $\mathbb{C}^2$ are always Cohen-Macaulay. Thus analysis of the Gorenstein property for such rings is reduced to analysis of the Poincaré series.

In Chapter 4 we investigate the structure of the ring of quasi-invariants $Q_\mathcal{A}$ for certain classes of arrangements $\mathcal{A}$ in $\mathbb{C}^2$, by employing the results explained in Chapter 3. Thus far the only known Gorenstein examples were the rings $Q_\mathcal{A}$ corresponding to the Coxeter arrangements, $\mathcal{A}_2(m)$ and $\mathcal{C}_2(m, l)$. In two dimensions we show that there is only one Gorenstein ring of quasi-invariants in the case where all multiplicities are one, namely that corresponding to the dihedral configuration.

Theorem (Chapter 4, Theorem 4.3.1)**.** *Let $N = 2$ and suppose $m_i = 1$ for $0 \leq i \leq n$. Denote by $\mathcal{A}$ any configuration with these properties. Then the quasi-invariant ring $Q_\mathcal{A}$ is Gorenstein if and only if A is a set of normals to reflection lines for the dihedral group.*

We also study the class of configurations with all but one multiplicity equal to one. In particular we calculate the Poincaré series for the quasi-invariant ring corresponding to the configuration $\mathcal{A}_{(m,1^n)}$ appearing in Chapter 2.

Theorem (Chapter 4, Theorem 4.10.1)**.** *The Poincaré series for the quasi-invariant ring $Q_{\mathcal{A}_{(m,1^n)}}$ is given by*

$$P(t) = \frac{1 - t^2 + t^{n+1} + t^{n+2} + t^{2m+n} + t^{2m+n+1} - t^{2m+2n} + t^{2m+2n+2}}{(t^2-1)^2}.$$

Thus $Q_{\mathcal{A}_{(m,1^n)}}$ is Gorenstein. Furthermore, it is proven that there is exactly one Gorenstein ring of quasi-invariants for this class of arrangements.

Theorem (Chapter 4, Theorems 4.9.1, 4.10.1)**.** *Let $N = 2$ and suppose $m_0 = m$ and $m_i = 1$ for $1 \leq i \leq n$. Denote by $\mathcal{A}$ any configuration with these properties. Then the quasi-invariant ring $Q_{\mathcal{A}}$ is Gorenstein if and only if $\mathcal{A} = \mathcal{A}_{(m,1^n)}$.*

Thus we see that for planar configurations with at most one multiplicity greater than one, the Gorenstein property for quasi-invariants is apparently equivalent to the existence of the Baker-Akhiezer function.

We would like to remark that Theorem 4.3.1 concerning planar configurations with all multiplicities equal to one actually follows from Theorem 4.9.1 which deals with the case in which at most one vector has multiplicity greater than or equal to one. We include Theorem 4.3.1 in the thesis for the following reasons: firstly, Theorem 4.3.1 is proved via slightly different techniques than those used in Theorem 4.9.1 and we feel the strategy is worth recording; secondly, and more importantly, the proof of Theorem 4.3.1 has structural and conceptual similarities to the considerably longer and more technical proof of Theorem 4.9.1. Thus the proof of Theorem 4.3.1 serves as a useful introduction to many of the key features of Theorem 4.9.1.

More exactly, Theorem 4.3.1 is established by calculating the Poincaré series for an arbitrary configuration of vectors of multiplicity one in the plane and showing that this series can only be palindromic when the arrangement is in fact a dihedral arrangement. We show that the Poincaré series for an arbitrary arrangement includes finitely many arbitrary coefficients. We then show that choosing the coefficient b_{2n} at degree $2n$ to be $n+1$ implies that the vectors of $\mathcal{A}$ form a dihedral arrangement (Proposition 4.5.1). To complete the proof we show that if the Poincaré series is palindromic then $b_{2n} = n+1$ (Lemma 4.5.4).

The proof of Theorem 4.9.1 follows a similar structure. In Section 4.8 we show in particular that choosing the coefficient of the Poincaré series $b_{2(n+m-1)}$ at degree $2(n+m-1)$ to be one implies that the vectors of $\mathcal{A}$ form an arrangement whose geometry is fully fixed (Proposition 4.8.4). We then show that this arrangement is in fact $\mathcal{A}_{(m,1^n)}$ (Theorem

4.8.5). In Section 4.9 we calculate the Poincaré series for an arbitrary arrangement of vectors and show that there are finitely many arbitrary coefficients. We consider two cases depending on the interaction of the parameters m and n (Lemmas 4.9.10, 4.9.15). We then show that if the series is palindromic then $b_{2(n+m-1)} = 1$ (Theorem 4.9.17).

In Chapter 5 we shift our attention to quasi-invariants in the context of Cherednik algebras. The fact that rings of quasi-invariants are modules for certain subalgebras of Cherdnik algebras is well known. This has been established in the paper [BEG03]. Let us explain this result. Let G be a finite Coxeter group acting in its complexified reflection representation $V = \mathbb{C}^N$. Let R be the corresponding Coxeter root system. To each G-invariant function $c : R \to \mathbb{C}$ one can attach an associative algebra $H_c(G)$ called the *rational Cherednik algebra*. For any $\zeta \in \mathbb{C}^N$ the *Dunkl operator* ∇_ζ is defined as

$$\nabla_\zeta = \partial_\zeta - \sum_{\alpha \in R_+} \frac{c_\alpha(\alpha, \zeta)}{(\alpha, x)}(1 - s_\alpha)$$

where $(\cdot, \cdot)$ is the standard scalar product in $\mathbb{C}^N$. We have the important vector space isomorphism

$$\mathbb{C}[X] \otimes_{\mathbb{C}} \mathbb{C}[\nabla] \otimes_{\mathbb{C}} \mathbb{C}G \to H_c(G)$$

where $\mathbb{C}[X] = \mathbb{C}[x_1, x_2, \ldots, x_N]$ and $\mathbb{C}[\nabla]$ is the commutative algebra generated by the Dunkl operators ∇_i corresponding to the basis directions e_i. The rational Cherednik algebra $H_c(G)$ can be defined as the algebra generated by the linear operators x_i, ∇_j, g $(i, j = 1, \ldots, N, g \in G)$ which act naturally in $\mathbb{C}^N$.

Let $e = \frac{1}{|G|} \sum_{g \in G} g$. The *spherical subalgebra* of $H_c(G)$ is defined as $eH_c(G)e$. The vector space

$$M_\tau = \mathbb{C}[X] \otimes \tau$$

where $\tau \in$Irrep(G) can be given the structure of an $H_c(G)$ module. This is the 'standard module'. Consider the ring of quasi-invariants Q_c corresponding to the Coxeter root system R and $\mathbb{Z}_+$ valued multiplicity function c. Let Q_c^b denote the image of the homomorphism (1.3). It is shown in [FV02, Theorem 3] (see also [Cha98] where a similar observation was first made) that for any $u \in Q_c^b$, $u(Q_c) \subset Q_c$. In particular $C_c(Q_c) \subset Q_c$ where C_c is the centralizer of the operator L (1.2). Thus there is a natural action on Q_c of the algebra $\mathcal{B}_c$ generated by C_c and $\mathbb{C}[X]^G$. This $\mathcal{B}_c$ action commutes with the G-action on Q_c. It can be shown that $\mathcal{B}_c$ is isomorphic to eH_ce and thus the space Q_c acquires an $(\mathbb{C}G \otimes eH_ce)$-module structure. This structure is completely described by the following.

For any irreducible representation τ of G let $\tau_c^{'}$ be the representation of G for which the monodromy representation of the Dunkl connection with values in $\tau_c^{'}$ is τ, see [BC11].

Theorem ([BEG03], Proposition 6.6)**.** *There is a $(\mathbb{C}G\otimes eH_ce)$-module isomorphism $Q_c = \bigoplus_{\tau\in Irrep(G)} \tau \otimes eM_{\tau_c'}$.*

Later, Feigin [Fei] constructed generalized Calogero-Moser systems via representations of Cherednik algebras. Feigin showed that the invariant polynomials corresponding to the Coxeter group S_n are a module for the spherical subalgebra of the Cherednik algebra $H_{1/m}(S_{mn}), m, n \in \mathbb{Z}_+$. This is achieved via consideration of some specific submodules in the polynomial representation of $H_{1/m}(S_{mn})$. In Chapter 5 we begin to investigate whether *all* quasi-invariant polynomials can be constructed in such a fashion by considering submodules in more complicated representations.

To explain fully what we mean by this, let us explain the construction of [Fei] more exactly. Let $N = mn$. Let us denote by $\pi_{m,n}$ the plane in $\mathbb{C}^N$ given by the equations

$$x_1 = x_2 = \ldots = x_m$$

$$x_{m+1} = x_{m+2} = \ldots = x_{2m}$$

$$x_{(n-1)m+1} = x_{(n-1)m+2} = \ldots = x_{mn}.$$

The associated parabolic stratum is defined as

$$D_{m,n} = \bigcup_{w\in S_N} w(\pi_{m,n})$$

while the corresponding parabolic ideal is

$$I = \{p \in \mathbb{C}[X] | p|_{D_{m,n}} = 0\}.$$

Feigin establishes the following.

Theorem ([Fei], Theorem 1)**.** *I and hence $\mathbb{C}[X]/I$ are representations for the rational Cherednik algebra $H_{1/m}(S_N)$.*

From this theorem it follows that $e(\mathbb{C}[X]/I)$ is a representation of the spherical subalgebra $e(H_{1/m}(S_N))e$. This representation can be identified with a subspace of polynomials in $\mathbb{C}[x_1, \ldots, x_n]$, namely the restrictions $p(x)|_{\pi_{m,n}}$ where $p(x) \in \mathbb{C}[X]^{S_N}$. Then we have

Theorem ([Fei])**.** *The representation $e(\mathbb{C}[X]/I)$ of $e(H_{1/m}(S_N))e$ is isomorphic to the invariant polynomials $\mathbb{C}[x_1, \ldots, x_n]^{S_n}$ in $\mathbb{C}^n$.*

We generalize the work of Feigin in the following way. We introduce submodules N_τ of the standard modules M_τ for some specific representations τ of S_{mn}. We then go on to establish

Theorem (Chapter 5, Theorem 5.5.5)**.** *The module $e(M_\tau/N_\tau)$ is isomorphic to the space of all anti-invariant quasi-invariants for S_n acting in $\mathbb{C}^n$.*

In the second half of Chapter 5 we are interested in other rings of quasi-invariants arising from the theory of Lie superalgebras and symmetric superspaces. In the paper [SV04] Sergeev and Veselov use the notion of generalized root systems to construct families of deformed quantum Calogero-Moser systems. In particular they introduce the commutative algebra $\Lambda_{R,B}$ related to the generalized root system (R, B). The generalized root systems are essentially the root systems R of the contragredient Lie superalgebras together with a bilinear form B. They include an infinite family $A(n, m)$ which depends on one parameter k. Sergeev and Veselov show that for these generalized root systems and generic values of k there is a monomorphism χ from $\Lambda_{R,B}$ into the algebra of differential operators on V such that $\chi(x^2)$ is the corresponding generalized Calogero-Moser operator.

The algebra of integrals $\Lambda_{R,B}$ is of independent interest. They can be seen as a version of algebras where quasi-invariant conditions on the hyperplanes with non-integer multiplicities are understood as symmetry of polynomials under reflection at these hyperplanes. In [SV04] it was shown that for the series $A(n, m)$ and generic values of k these algebras are finitely generated. Also the Poincaré series were calculated.

The algebras corresponding to the series $A(n, m)$ can be realized as the following. Let $\Lambda_{n,m,k} \subset \mathbb{C}[V^*] = \mathbb{C}[x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m]$ be the algebra consisting of polynomials symmetric in the variables $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_m$ separately and satisfying the conditions

$$(\frac{\partial}{\partial x_i} - k\frac{\partial}{\partial y_j})f \mid_{x_i=y_j} \equiv 0$$

for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$. It is easy to see that the deformed Newton sums

$$p_r(x, y, k) = \sum_{i=1}^{n} x_i^r + \frac{1}{k}\sum_{j=1}^{m} y^r$$

belong to $\Lambda_{n,m,k}$ for $r \geq 0$. Denote by $\mathcal{N}_{n,m,k}$ the algebra generated by the deformed Newton sums. We have the following

Theorem ([SV04], Theorem 2)**.** *If k is not a positive rational number then $\Lambda_{n,m,k} = \mathcal{N}_{n,m,k}$.*

Further, Sergeev and Veselov use this to establish

Theorem ([SV04], Theorem 3). *The Poincaré series of the algebra* $\Lambda_{n,m,k}$ *for generic* k *has the form*

$$P_{n,m}(t) = \frac{1}{(1-t)(1-t^2)\dots(1-t^n)}[1 + \sum_{i=1}^{m} \frac{t^{i(n+1)}}{(1-t)(1-t^2)\dots(1-t^i)}]. \tag{1.5}$$

The importance of these algebras in the context of our work is as follows. Recall the deformed root systems $\mathcal{A}_n(k)$ and corresponding rings of quasi-invariants. The algebra $\Lambda_{n,1,k}$ is isomorphic to the partially symmetric quasi-invariants for the deformed root system $\mathcal{A}_n(k)$. Thus we have an expression for the Poincaré series for a 'symmetric component' of $Q_{\mathcal{A}_n(k)}$ in the case of generic k. This is in agreement with [FV03a] where the Poincaré series for $Q_{\mathcal{A}_2(k)}$ was calculated.

More explicitly, we have $\Lambda_{n,1,k} \cong Q^{sym}_{\mathcal{A}_n(k)}$, where $Q^{sym}_{\mathcal{A}_n(k)}$ consists of polynomials $q(x_1,\dots,x_{n+1})$ which are invariant under the action of the group S_n which permutes the coordinates $x_1, x_2, \dots, x_n$, and are quasi-invariant at the hyperplanes $x_i = \sqrt{k}x_{n+1}$, $i = 1, 2, \dots, n$ which have multiplicity one. Thus the Poincaré series $P_{Q^{sym}_{\mathcal{A}_n(k)}}$ is given by the formula (1.5) for generic k. We begin to generalize this with the following.

Theorem (Chapter 5, Theorem 5.6.3).

$$P_{Q^{sym}_{\mathcal{A}_n(k)}}(t) = \frac{1 - t + t^{n+1}}{(1-t)^2(1-t^2)\dots(1-t^n)} \,\forall\, k \in \mathbb{Z}_+.$$

We also study the relation between $\Lambda_{n,1,k}$ and $\mathcal{N}_{n,1,k}$ for non-integer k.

Theorem (Chapter 5, Theorem 5.6.4). *Suppose* $k \geq n$. *Then* $\Lambda_{n,1,\frac{1}{k+1}} = \mathcal{N}_{n,1,\frac{1}{k+1}}$.

In Chapter 6 we review the results of the thesis and mention possible generalizations and open problems. We will discuss the conclusions presented in this thesis and highlight possible areas of interest for future research.

Chapter 2

Baker-Akhiezer Functions

2.1 Summary

In this chapter we focus on the Baker-Akhiezer function (BA function). We will investigate the existence of the BA function when considering the class of arrangements of non-collinear vectors in $\mathbb{C}^2$ with at most one multiplicity greater than one, the so-called 'type $(m, 1^n)$' configurations. We introduce the configuration $\mathcal{A}_{(m,1^n)}$ which has type $(m, 1^n)$. We will then show that the BA function exists for a configuration $\mathcal{A}$ of type $(m, 1^n)$ if and only if $\mathcal{A} = \mathcal{A}_{(m,1^n)}$.

The results in Chapter 2 are new except for the following. Section 2.2 consists of background information and as such has no new material. Theorem 2.4.3 is s standard result which is well known.

2.2 Background

First let us fix some notation. Let A be a finite set of non-collinear vectors $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n$ with $\alpha_i \in \mathbb{C}^2$. Let m be a function $m : A \to \mathbb{N}$, the multiplicity function. Say $\alpha_i \in A$ has multiplicity m_i under this function. Thus $\mathcal{A} = (A, m)$ is a set of vectors with multiplicities. For $k = (k_1, k_2)$ and $x = (x_1, x_2)$ let $(k, x) = k_1x_1 + k_2x_2$ be the standard inner product in $\mathbb{C}^2$. We will assume throughout this chapter that $(\alpha_i, \alpha_i) \neq 0, 0 \leq i \leq n$. We are going to study the following class of configurations in $\mathbb{C}^2$.

Definition 2.2.1. *Suppose a configuration $\mathcal{A} = (A, m)$, where $A \subset \mathbb{C}^2$, satisfies the following properties: up to relabelling of the vectors, $m_0 = m \in \mathbb{Z}_+$ and $m_i = 1$ for $i = 1, 2, \ldots, n$. Then we say $\mathcal{A}$ has type $(m, 1^n)$.*

At this point we would like to make an important remark about equivalence of configurations.

Remark 2.2.2. *We will consider two configurations to be equal if they are the same up to rotation and dilatation of the vectors. Thus throughout the rest of the thesis if we say that $\mathcal{A} = \mathcal{B}$, where $\mathcal{A}, \mathcal{B}$ are two configurations of vectors with multiplicities in $\mathbb{C}^2$, we are referring to equality up to this equivalence.*

We now introduce the Baker-Akhiezer function.

Definition 2.2.3. *A function $\phi(k,x), k, x \in \mathbb{C}^2$ will be called a Baker-Akhiezer function if the following two conditions are fulfilled:*

- *$\phi(k,x)$ has the form*

$$\phi(k,x) = P(k,x)e^{(k,x)}$$

  where $P(k,x)$ is a polynomial in k with highest term $A(k) = \prod_{\alpha_i \in A} (k,\alpha_i)^{m_i}$;

- *For all $\alpha_i \in A$*

$$\partial_{\alpha_i}\phi(k,x) = \partial^3_{\alpha_i}\phi(k,x) = \ldots = \partial^{2m_i-1}_{\alpha_i}\phi(k,x) = 0$$

  on the hyperplane $\Pi_i : (k,\alpha_i) = 0$, where $\partial_{\alpha_i} = (\alpha_i, \frac{\partial}{\partial k})$ is the normal derivative for this hyperplane.

Combining results from [VSC93] and [CFV99] we have conditions which are both necessary and sufficient for the existence of the Baker-Akhiezer function ϕ.

Theorem 2.2.4 ([VSC93, CFV99])**.** *The Baker-Akhiezer function ϕ exists for a configuration $\mathcal{A}$ if and only if the following two sets of conditions hold $\forall\ \alpha_j \in A$*

$$\sum_{i=0,\, i\neq j}^{n} \frac{m_i(\alpha_j,\alpha_i)^{2k-1}}{(\alpha_i,x)^{2k-1}} = 0 \qquad (\alpha_j(k))$$

on the hyperplane Π_j for $1 \leq k \leq m_j$,

$$\sum_{i=0,\, i\neq j}^{n} \frac{m_i(m_i+1)(\alpha_i,\alpha_i)(\alpha_j,\alpha_i)^{2k-1}}{(\alpha_i,x)^{2k+1}} = 0 \qquad (\widetilde{\alpha}_j(k))$$

on the hyperplane Π_j for $1 \leq k \leq m_j$.

Next we are going to mention a result which will be useful in ascertaining whether certain configurations in $\mathbb{C}^2$ are in fact real.

Theorem 2.2.5. *Let $\mathcal{A}$ be a configuration of vectors $\alpha_0, \alpha_1, \ldots, \alpha_n \in \mathbb{R}^2$ with corresponding multiplicities $m_i \in \mathbb{R}_+, 0 \le i \le n$. Let $\theta_j \in [0, \pi)$ be such that $\alpha_j = (\cos\theta_j, \sin\theta_j)$. Suppose that $0 = \theta_0 < \theta_1 < \ldots < \theta_n < \pi$. Then for any choice of the multiplicities $m_0, \ldots, m_n$ there exists a unique configuration $\mathcal{A}$ which satisfies the relations $(\alpha_j(1)) \,\forall\, j = 0, 1 \ldots, n$.*

This result is established in [FJ]. The proof is adapted from a similar result in [Mul10].

2.3 Coordinate systems

Although the above introductory discourse is phrased in terms of Cartesian coordinates it is helpful for us to make use of complex and polar coordinates. In particular we will often need to apply Theorem 2.2.4. To this end we show how to state Theorem 2.2.4 in polar coordinates with the following.

Lemma 2.3.1. *Consider the non-collinear vectors $(\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{C}^2$. Let us assume that $(\alpha_i, \alpha_i) = 1$ and let $\alpha_s = (\cos\phi_s, \sin\phi_s)$ for some ϕ_s. Put*

$$z_s = e^{2i\phi_s}.$$

Then for $1 \le k \le m_j$,

$$\sum_{i=0, i\neq j}^{n} \frac{m_i(\alpha_i, \alpha_j)^{2k-1}}{(x, \alpha_i)^{2k-1}} \,|_{\Pi_j} = 0 \iff \sum_{i=0, i\neq j}^{n} m_i \frac{(z_i + z_j)^{2k-1}}{(z_i - z_j)^{2k-1}} = 0$$

and

$$\sum_{i=0, i\neq j}^{n} \frac{m_i(m_i+1)(\alpha_i, \alpha_i)(\alpha_i, \alpha_j)^{2k-1}}{(\alpha_i, x)^{2k+1}} \,|_{\Pi_j} = 0 \iff \sum_{i=0, i\neq j} \frac{m_i(m_i+1) z_i z_j (z_i + z_j)^{2k-1}}{(z_i - z_j)^{2k+1}} = 0.$$

Proof. We prove the second identity, the first is very similar. Set $\widetilde{z}_j = e^{i\phi_j}$. Since $(\alpha_j, x) = 0$ we take $x = (-\sin\phi_j, \cos\phi_j)$. Then we have:

$$\sum_{i=0, i\neq j}^{n} \frac{m_i(m_i+1)(\cos\phi_i \cos\phi_j + \sin\phi_i \sin\phi_j)^{2k-1}}{(-\cos\phi_j \sin\phi_i + \cos\phi_i \sin\phi_j)^{2k+1}} = 0$$

$$\iff \sum_{i=0, i\neq j} m_i(m_i+1) \frac{\cos^{2k-1}(\phi_j - \phi_i)}{\sin^{2k+1}(\phi_j - \phi_i)} = 0$$

$$\iff \sum_{i=0, i\neq j} m_i(m_i+1) \frac{(e^{i(\phi_j-\phi_i)} + e^{-i(\phi_j-\phi_i)})^{2k-1}}{(e^{i(\phi_j-\phi_i)} - e^{-i(\phi_j-\phi_i)})^{2k+1}} = 0$$

$$\iff \sum_{i=0, i\neq j} m_i(m_i+1) \frac{(\frac{\widetilde{z}_j}{\widetilde{z}_i} + \frac{\widetilde{z}_i}{\widetilde{z}_j})^{2k-1}}{(\frac{\widetilde{z}_j}{\widetilde{z}_i} - \frac{\widetilde{z}_i}{\widetilde{z}_j})^{2k+1}} = 0$$

$$\iff \sum_{i=0,i\neq j} m_i(m_i+1)\widetilde{z}_j^2\widetilde{z}_i^2\frac{(\widetilde{z}_j^2+\widetilde{z}_i^2)^{2k-1}}{(\widetilde{z}_j^2-\widetilde{z}_i^2)^{2k+1}}=0$$

$$\iff \sum_{i=0,i\neq j}^{n} m_i(m_i+1)\widetilde{z}_j^2\widetilde{z}_i^2\frac{(z_j+z_i)^{2k-1}}{(z_j-z_i)^{2k+1}}=0$$

$$\iff \sum_{i=0,i\neq j}^{n} m_i(m_i+1)\frac{z_iz_j(z_i+z_j)^{2k-1}}{(z_j-z_i)^{2k+1}}=0.$$

□

2.4 The $\mathcal{A}_{(m,1^n)}$ configuration

In this section we are going to introduce a particular configuration of non-collinear vectors in $\mathbb{C}^2$ which we are going to denote by $\mathcal{A}_{(m,1^n)}$. Let z_i be the same as in Lemma 2.3.1 and denote by $\widehat{e}_i$ the i-th elementary symmetric polynomial in the variables z_i, $(1 \le i \le n)$. The configuration $\mathcal{A}_{(m,1^n)}$ will be defined by setting

$$\widehat{e}_i=(-1)^i\binom{n}{i}\binom{m+i-1}{i}\binom{m+n-1}{i}^{-1}$$

for $1 \le i \le n$. In order to see that specifying these values for the $\widehat{e}_i$ will actually lead to a collection of n pairwise non-collinear vectors we will establish the following.

Theorem 2.4.1. *The polynomial*

$$P=\prod_{i=1}^{n}(z-z_i)=\sum_{i=0}^{n}(-1)^i\widehat{e}_iz^{n-i}$$

has no repeated roots.

In order to prove this, let us recall some standard notation and results.

Definition 2.4.2. *Let*

$$f(x)=\sum_{i=1}^{n}a_ix^{n-i}$$

and

$$g(x)=\sum_{j=0}^{m}b_jx^{m-j}$$

be two polynomials in $\mathbb{C}[x]$. The resultant of f(x) and g(x) is the determinant

$$R = \begin{vmatrix} a_0 & a_1 & \dots & a_n & 0 & \dots & 0 & 0 & 0 \\ 0 & a_0 & a_1 & \dots & a_n & 0 & \dots & 0 & 0 \\ 0 & 0 & a_0 & \dots & \dots & a_n & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & a_{n-2} & a_{n-1} & a_n \\ b_0 & b_1 & \dots & \dots & b_m & \dots & 0 & 0 & 0 \\ 0 & b_0 & b_1 & \dots & \dots & b_m & \dots & 0 & 0 \\ 0 & 0 & b_0 & \dots & \dots & \dots & b_m & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & b_{m-2} & b_{m-1} & b_m \end{vmatrix}$$

of size $(m+n) \times (m+n)$.

The following result is well known.

Theorem 2.4.3. *Let*

$$f(x) = \sum_{i=1}^{n} a_i x^{n-i}$$

and

$$g(x) = \sum_{j=0}^{m} b_j x^{m-j}$$

be two polynomials in $\mathbb{C}[x]$, and let R be the resultant of f and g. Then f and g have a common root if and only if R=0. In particular f has a repeated root if and only if the resultant of f and f' is zero.

Thus to prove Theorem 2.4.1 it is sufficient to establish the following.

Lemma 2.4.4. *Let R be the resultant of P and P'. Then $R \neq 0$.*

Proof. We proceed by induction on n. We will write R_n for the resultant corresponding to P when P has degree n. First let $n = 2$ ($n = 1$ is trivial). Set $\widehat{e}_1 = \frac{2m}{m+1} = a$ for simplicity. Then

$$P = z_1^2 - az_1 + 1$$

and

$$P' = 2z_1 - a.$$

So

$$R_2 = \begin{vmatrix} 1 & -a & 1 \\ 2 & -a & 0 \\ 0 & 2 & -a \end{vmatrix} \sim \begin{vmatrix} 1 & -a & 1 \\ 0 & a & -2 \\ 0 & 2 & -a \end{vmatrix} = 4 - a^2.$$

However

$$a^2 = 4 \Rightarrow \frac{4m^2}{(m+1)^2} = 4 \Rightarrow m = -\frac{1}{2}$$

which is not possible. So $R_2 \neq 0$. Now assume $R_i \neq 0$ for $i < n$. We need to show $R_n \neq 0$. Note that the entries of R_n are the coefficients of P and P', which are given by the elementary symmetric polynomials $\widehat{e}_k$. Recall that the $\widehat{e}_k$ are given by

$$\widehat{e}_k = (-1)^k \binom{n}{k} \binom{m+k-1}{k} \binom{m+n-1}{k}^{-1}$$

for $1 \leq k \leq n$. Observe also that within the expression for each $\widehat{e}_k$ the denominator $\prod_{i=1}^{k}(m+n-i)$ is non-zero. These observations suggest we should work with the determinants

$$\widehat{R}_n = \prod_{i=1}^{n}(m+n-i)^{2n-1} R_n$$

for simplicity. So it is sufficient to establish the following:

Lemma 2.4.5.

$$\widehat{R}_n = \frac{(-1)}{(n-1)!} m^{n-3} n^{3n-5} (2m+n-1)^{3n-5} \prod_{i=1}^{n}(m+n-i) \prod_{i=1}^{n-1}(m+i-1)(n-i+1)\widehat{R}_{n-1}.$$

We claim it is simple to write down the entries of any row in the matrix corresponding to $\widehat{R}_n$. Denote by r_i the ith row of $\widehat{R}_n$, and denote by r_i^j the jth entry of r_i. Then, for instance

$$r_1^j = \begin{cases} \frac{1}{(j-1)!} \prod_{i=1}^{j-1}(m+i-1)(n-i+1) \prod_{i=j}^{n}(m+n-i) & (1 \leq j \leq n+1) \\ 0 & (n+2 \leq j \leq 2n-1). \end{cases} \tag{2.1}$$

This is obvious: the first $n+1$ entries are the coefficients of P and the remaining entries are 0, by definition. Similarly:

$$r_n^j = \begin{cases} \frac{(n-j+1)}{(j-1)!} \prod_{i=1}^{j-1}(m+i-1)(n-i+1) \prod_{i=j}^{n}(m+n-i) & (1 \leq j \leq n) \\ 0 & (n+1 \leq j \leq 2n-1). \end{cases} \tag{2.2}$$

Remark 2.4.6. *$r_i^j = r_{i-1}^{j-1}$, for $2 \le i < n$, $2 \le j \le 2n-1$ and $2n-1 \ge i > n$, $2 \le j \le 2n-1$.*

This is immediate from the structure of $\widehat{R}_n$. Note this makes it simple to write down the entries for any row of $\widehat{R}_n$. Now, for $0 \le k \le n-4$ we apply in turn the transformations:

$$\begin{pmatrix} {}^{k+1}r_{1+k} \\ {}^{k+1}r_{2+k} \\ {}^{k+1}r_{n+k} \\ {}^{k+1}r_{n+1+k} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \delta_{0k}n & 0 & -1 & 1 \\ \delta_{0k}mn & 0 & -m & -(m+n-1) \\ 0 & -n & 0 & 1 \end{pmatrix} \begin{pmatrix} {}^{k}r_{1+k} \\ {}^{k}r_{2+k} \\ {}^{k}r_{n+k} \\ {}^{k}r_{n+1+k} \end{pmatrix}$$

where by ${}^k r_i$ we mean the row r_i after the k-th transformation. Note that this transformation does not change any other rows: for any k, ${}^{k+1}r_i = {}^k r_i$ for $i \ne 1+k, 2+k, n+k$ or $n+k+1$. Also ${}^0 r_i^j = r_i^j$. Note that the $(k+1)$-st row changes at step k only, while the $(n+k)$-th row changes at the steps $k, k+1$ only.

We claim that after the k-th transformation we have the following:

$$ {}^{k+1}r_{k+1} = {}^k r_{k+1}, \tag{2.3}$$

$$ {}^{k+1}r_{2+k} = nr_{k+1} - r_{n+k} + r_{n+k+1}, \tag{2.4}$$

$$ {}^{k+1}r_{n+k} = mnr_{k+1} - mr_{n+k} - (m+n-1)r_{n+k+1}, \tag{2.5}$$

and

$$ {}^{k+1}r_{n+k+1} = -nr_{k+2} + r_{n+k+1}. \tag{2.6}$$

Let us prove this by induction. We shall deal with (2.4), the rest are very similar. The case $k = 0$ is clear. If $k = s-1$ we have in particular:

$$ {}^s r_{n+s} = -nr_{s+1} + r_{n+s}$$

by inductive assumption and

$$ {}^s r_{n+s+1} = r_{n+s+1}$$

since r_{n+s+1} will only be changed after applying $s+1$ transformations. Then applying the transformation with $k = s$ gives:

$$\begin{aligned} {}^{s+1}r_{2+s} &= -{}^{s}r_{n+s} + {}^{s}r_{n+s+1} \\ &= -(-nr_{s+1} + r_{n+s}) + {}^{s}r_{n+s+1} \\ &= nr_{s+1} - r_{n+s} + r_{n+s+1} \end{aligned}$$

as required. Let ${}^{k}A$ be the matrix corresponding to $\widehat{R}_n$ after the k-th transformation step. Then:

$$\det {}^{k+1}A = (-nm)(2m+n-1) \det {}^{k}A. \tag{2.7}$$

Denote by $r_i^j[n-1]$ the j-th entry of the i-th row of the matrix corresponding to $\widehat{R}_{n-1}$. We claim that:

$${}^{k+1}r_{2+k}^{j} = \begin{cases} 0, & j = 1, \\ \frac{1}{n(2m+n-1)} r_{1+k}^{j-1}[n-1], & 2 \le j \le 2n-2, \\ 0, & j = 2n-1, \end{cases} \tag{2.8}$$

and

$${}^{k+1}r_{n+k}^{j} = \begin{cases} 0, & j = 1 \\ \frac{-1}{n(2m+n-1)} r_{n+k-1}^{j-1}[n-1], & 2 \le j \le 2n-2 \\ 0, & j = 2n-1. \end{cases} \tag{2.9}$$

By Remark 2.4.6 it is sufficient to consider the case $k = 0$. Using (2.1) and (2.2) we have:

$${}^{1}r_{n}^{j} = \begin{cases} 0, & j = 1, \\ \frac{(2m+n-1)(j-n-1)(n-j+2)}{(j-2)!} \prod_{i=1}^{j-2} (m+i-1)(n-i+1) \prod_{i=j}^{n} (m+n-i), & 2 \le j \le n+1, \\ 0, & n+1 \le j \le 2n-1, \end{cases} \tag{2.10}$$

and

$${}^{1}r_{2}^{j} = \begin{cases} 0, & j = 1, \\ \frac{(2m+n-1)(n-j+2)}{(j-2)!} \prod_{i=1}^{j-2} (m+i-1)(n-i+1) \prod_{i=j}^{n} (m+n-i), & 2 \le j \le n+1, \\ 0, & n+2 \le j \le 2n-1. \end{cases} \tag{2.11}$$

However

$$r_1^{j-1}[n-1] = \frac{(n-j+2)}{n(j-2)!}\prod_{i=1}^{j-2}(m+i-1)(n-i+1)\prod_{i=j}^{n}(m+n-i), \ \ 2 \le j \le 2n-3,$$

and

$$r_{n-1}^{j-1}[n-1] = \frac{(n-j+1)(n-j+2)}{n(j-2)!}\prod_{i=1}^{j-2}(m+i-1)(n-i+1)\prod_{i=j}^{n}(m+n-i), \ \ 2 \le j \le 2n-3$$

making the equality clear. Now, note that this sequence of transformations:

- Does not affect r_1
- Does not affect r_{n-1}
- Does not affect r_{2n-2}
- Does not affect r_{2n-1}
- Changes r_{2n-3} only once at the last step $k = n-4$.

To rectify this consider the further transformation

$$\begin{pmatrix} \widehat{r}_{n-1} \\ \widehat{r}_{2n-3} \\ \widehat{r}_{2n-2} \\ \widehat{r}_{2n-1} \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 & 0 \\ 0 & -m & -(m+n-1) & 0 \\ mn & 0 & -m & -(m+n-1) \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r_{n-1} \\ {}^{n-3}r_{2n-3} \\ r_{2n-2} \\ r_{2n-1} \end{pmatrix}.$$

Let B be the matrix corresponding to $\widehat{R}_n$ after applying the above transformation. Then:

$$\det B = mn(2m+n-1)\det{}^{n-4}A. \tag{2.12}$$

We also claim that under this transformation

$$\widehat{r}_{2n-3}^{j} = \begin{cases} 0, \ \ 1 \le j \le n-3, \\ \frac{(2n-j-1)(2m+n-1)(j+2n-2)}{(j-n+1)!}\prod_{i=1}^{j-n+1}(m+i-1)(n-i+1)\prod_{i=j-n+3}^{n}(m+n-i), \ \ n-2 \le j \le 2n-1, \end{cases}$$

$$\widehat{r}_{n-1}^{j} = \begin{cases} 0, \ \ 1 \le j \le n-3, \\ \frac{(2m+n-1)(2n-j-1)}{(j-n+1)!}\prod_{i=1}^{j-n+1}(m+i-1)(n-i+1)\prod_{i=j-n+3}^{n}(m+n-i), \ \ n-2 \le j \le 2n-1, \end{cases}$$

and

$$\widehat{r}_{2n-2}^{j} = \begin{cases} 0, \ \ 1 \le j \le n-1, \\ \frac{(2n-j)(1+j-2n)(2m+n-1)}{(j-n)!}\prod_{i=1}^{j-n}(m+i-1)(n-i+1)\prod_{i=j-n+2}^{n}(m+n-i) \ \ n-2 \le j \le 2n-1. \end{cases}$$

This is clear since for $n-2 \le j \le 2n-1$:

$$\widehat{r}^{j}_{2n-2} = mnr^{j}_{n-1} - mr^{j}_{2n-2} - (m+n-1)r^{j}_{2n-1}$$
$$= \frac{mn}{(j-n+1)!}\prod_{i=1}^{j-n+1}(m+i-1)(n-i+1)\prod_{i=j-n+2}^{n}(m+n-i)$$
$$- \frac{m(2n-j-1)}{(j-n+1)!}\prod_{i=1}^{j-n+1}(m+i-1)(n-i+1)\prod_{i=j-n+2}^{n}(m+n-i)$$
$$- \frac{(m+n-1)(2n-j)}{(j-n)!}\prod_{i=1}^{j-n}(m+i-1)(n-i+1)\prod_{i=j-n+1}^{n}(m+n-i)$$
$$= \prod_{i=1}^{j-n}(m+i-1)(n-i+1)\prod_{i=j-n+2}^{n}(m+n-i)[\frac{mn(m+j-n)(2n-j)}{(j-n+1)!} - \frac{m(2n-j-1)(m+j-n)(2n-j)}{(j-n+1)!}$$
$$- \frac{(m+n-1)(2n-j)(2n+m-j-1)}{(j-n)!}]$$
$$= \prod_{i=1}^{j-n+1}(m+i-1)(n-i+1)\prod_{i=j-n+2}^{n}(m+n-i)\frac{(2n-j)}{(j-n+1)!}[mn(m+j-n) - m(2n-j-1)(m+j-n)$$
$$-(m+n-1)(2n+m-j-1)(j-n+1)]$$
$$= \prod_{i=1}^{j-n+1}(m+i-1)(n-i+1)\prod_{i=j-n+2}^{n}(m+n-i)\frac{(2n-j)(1+j-2n)(j-n+1)(2m+n-1)}{(j-n+1)!}.$$

Similarly

$$\widehat{r}^{j}_{2n-3} = mnr^{j}_{n-2} - mr^{j}_{2n-3} - (m+n-1)r^{j}_{2n-2}$$
$$= \frac{mn}{(j-n+2)!}\prod_{i=1}^{j-n+2}(m+i-1)(n-i+1)\prod_{i=j-n+3}^{n}(m+n-i)$$
$$- \frac{m(2n-j-2)}{(j-n+2)!}\prod_{i=1}^{j-n+2}(m+i-1)(n-i+1)\prod_{i=j-n+3}^{n}(m+n-i)$$
$$- \frac{(m+n-1)(2n-j-1)}{(j-n+1)!}\prod_{i=1}^{j-n+1}(m+i-1)(n-i+1)\prod_{i=j-n+2}^{n}(m+n-i)$$
$$= \prod_{i=1}^{j-n+1}(m+i-1)(n-i+1)\prod_{i=j-n+3}^{n}(m+n-i)[\frac{mn(m+j-n+1)(2n-j-1)}{(j-n+2)!}$$
$$- \frac{m(2n-j-2)(m+j-n+1)(2n-j-1)}{(j-n+2)!}$$
$$- \frac{(m+n-1)(2n-j-1)(m+2n-j-2)}{(j-n+1)!}]$$
$$= \prod_{i=1}^{j-n+1}(m+i-1)(n-i+1)\prod_{i=j-n+3}^{n}(m+n-i)\frac{(m+j-n+1)(2n-j-1)}{(j-n+2)!}[mn - m(2n-j-2)]$$
$$- \prod_{i=1}^{j-n+1}(m+i-1)(n-i+1)\prod_{i=j-n+3}^{n}(m+n-i)\frac{(2n-j-1)(m+n-1)(m+2n-j-2)}{(j-n+1)!}$$
$$= \prod_{i=1}^{j-n+1}(m+i-1)(n-i+1)\prod_{i=j-n+3}^{n}(m+n-i)\frac{(2n-j-1)[m(m+j-n+1) - (m+n-1)(m+2n-j-2)]}{(j-n+1)!}$$
$$= \prod_{i=1}^{j-n+1}(m+i-1)(n-i+1)\prod_{i=j-n+3}^{n}(m+n-i)\frac{(2n-j-1)(2m+n-1)(j+2-2n)}{(j-n+1)!}.$$

Finally:

$$
\begin{aligned}
&\widehat{r}^j_{n-1} = nr^j_{n-2} - r^j_{2n-3} + r^j_{2n-2} \\
&= \frac{n}{(j-n+2)!} \prod_{i=1}^{j-n+2} (m+i-1)(n-i+1) \prod_{i=j-n+3}^{n} (m+n-i) \\
&- \frac{(2n-j-2)}{(j-n+2)!} \prod_{i=1}^{j-n+2} (m+i-1)(n-i+1) \prod_{i=j-n+3}^{n} (m+n-i) \\
&+ \frac{(2n-j-1)}{(j-n+1)!} \prod_{i=1}^{j-n+1} (m+i-1)(n-i+1) \prod_{i=j-n+2}^{n} (m+n-i) \\
&= \prod_{i=1}^{j-n+1} (m+i-1)(n-i+1) \prod_{i=j-n+3}^{n} (m+n-i)[\frac{n(m+j-n+1)(2n-j-1)}{(j-n+2)!} \\
&- \frac{(2n-j-2)(m+j-n+1)(2n-j-1)}{(j-n+2)!} \\
&+ \frac{(2n-j-1)(m+2n-j-2)}{(j-n)+1!}] \\
&= \prod_{i=1}^{j-n+1} (m+i-1)(n-i+1) \prod_{i=j-n+3}^{n} (m+n-i) \frac{(m+j-n+1)(2n-j-1)}{(j-n+2)!}[n-(2n-j-2)] \\
&+ \prod_{i=1}^{j-n+1} (m+i-1)(n-i+1) \prod_{i=j-n+3}^{n} (m+n-i) \frac{(2n-j-1)(m+2n-j-2)}{(j-n+1)!} \\
&= \prod_{i=1}^{j-n+1} (m+i-1)(n-i+1) \prod_{i=j-n+3}^{n} (m+n-i) \frac{(2n-j-1)[(m+j-n+1)+(m+2n-j-2)]}{(j-n+1)!} \\
&= \prod_{i=1}^{j-n+1} (m+i-1)(n-i+1) \prod_{i=j-n+3}^{n} (m+n-i) \frac{(2n-j-1)(2m+n-1)}{(j-n+1)!}.
\end{aligned}
$$

We claim also that for $2 \leq j \leq 2n-2$:

$$
\begin{aligned}
\frac{-1}{n(2m+n-1)} \widehat{r}^j_{2n-2} &= r^{j-1}_{2n-3}[n-1], \\
\frac{-1}{n(2m+n-1)} \widehat{r}^j_{2n-3} &= r^{j-1}_{2n-4}[n-1], \\
\frac{1}{n(2m+n-1)} \widehat{r}^j_{n-1} &= r^{j-1}_{n-2}[n-1].
\end{aligned}
$$

This is clear since:

$$
\begin{aligned}
r^{j-1}_{2n-3}[n-1] &= \frac{(2n-j-1)}{(j-n)!} \prod_{i=1}^{j-n} (m+i-1)(n-i) \prod_{i=j-n+1}^{n-1} (m+n-i-1) \\
&= \frac{(2n-j)(2n-j-1)}{n(j-n)!} \prod_{i=1}^{j-n} (m+i-1)(n-i+1) \prod_{i=j-n+2}^{n} (m+n-i) \\
&= \frac{-1}{n(2m+n-1)} \widehat{r}^j_{2n-2}.
\end{aligned}
$$

Also,

$$\begin{aligned} r^{j-1}_{2n-4}[n-1] &= \frac{(2n-j-1)}{(j-n+1)!}\prod_{i=1}^{j-n+1}(m+i-1)(n-i)\prod_{i=j-n+2}^{n-1}(m+n-i-1)\\ &= \frac{(2n-j-1)(2n-j-2)}{n(j-n+1)!}\prod_{i=1}^{j-n}(m+i-1)(n-i+1)\prod_{i=j-n+2}^{n}(m+n-i)\\ &= \frac{-1}{n(2m+n-1)}\widehat{r}^{j}_{2n-3}, \end{aligned}$$

and

$$\begin{aligned} r^{j-1}_{n-2}[n-1] &= \frac{1}{(j-n+1)!}\prod_{i=1}^{j-n+1}(m+i-1)(n-i)\prod_{i=j-n+2}^{n-1}(m+n-i-1)\\ &= \frac{(2n-j-1)}{n(j-n+1)!}\prod_{i=1}^{j-n+1}(m+i-1)(n-i+1)\prod_{i=j-n+3}^{n}(m+n-i)\\ &= \frac{1}{n(2m+n-1)}\widehat{r}^{j}_{n-1}. \end{aligned}$$

In total the above transformations give rise to the following structure:

$$\widehat{R}_n = \begin{vmatrix} \prod_{i=1}^{n}(m+n-i) & - & 0\\ 0 & \widehat{A}_{n-1} & 0\\ 0 & - & \frac{m}{(n-1)!}\prod_{i=1}^{n-1}(m+i-1)(n-i+1) \end{vmatrix}$$

where $\widehat{A}_{n-1}$ is the matrix corresponding to $\widehat{R}_{n-1}$. So we can apply the $n-2$ transformations outlined above to obtain:

$$\widehat{R}_n = \frac{(-1)}{(n-1)!}m^{n-3}n^{3n-5}(2m+n-1)^{3n-5}\prod_{i=1}^{n}(m+n-i)\prod_{i=1}^{n-1}(m+i-1)(n-i+1)\widehat{R}_{n-1}.$$

□

We are now ready to introduce the configuration $\mathcal{A}_{(m,1^n)}$.

Definition 2.4.7. *Let z_i be the same as in Lemma 2.3.1. Let $\mathcal{A} = (A, m)$ be a finite collection of vectors $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}^2$ with multiplicities such that $m_0 = m$ and $m_i = 1$ for $1 \le i \le n$. Denote by $\widehat{e}_i$ the i-th elementary symmetric polynomial in the variables z_i, $(1 \le i \le n)$. We say that $\mathcal{A}$ is the $\mathcal{A}_{(m,1^n)}$ configuration if*

$$\widehat{e}_i = (-1)^i\binom{n}{i}\binom{m+i-1}{i}\binom{m+n-1}{i}^{-1}$$

for $1 \le i \le n$.

By Theorem 2.4.1 $\mathcal{A}_{(m,1^n)}$ consists of $n+1$ pairwise non-collinear vectors. In fact $\mathcal{A}_{(m,1^n)}$ is always a real arrangement. This follows from Theorems 2.2.5 and 2.5.1. We will also see in Theorem 2.5.6 that $\mathcal{A}_{(m,1^n)}$ is symmetric with respect to the vector α_0. This property can be rephrased as the following: $z_0 = 1$ and $z_i z_{n-i+1} = 1$ for $1 \leq i \leq n$, for appropriate ordering of the vectors with multiplicity 1. When $m = 1$ the configuration $\mathcal{A}_{(m,1^n)}$ is dihedral (that is, the vectors are a set of normals to reflection lines for some dihedral group). When $n = 2$, $\mathcal{A}_{(m,1^n)}$ coincides with the configuration $\mathcal{A}_2(m)$ introduced in [VFC96]. See [CFV99, FV03a] for more information regarding this configuration.

2.5 Type $(m, 1^n)$ configurations admitting Baker-Akhiezer functions

Let $\mathcal{A}$ have type $(m, 1^n)$. We are going to establish that the BA function $\phi(k, x), k, x \in \mathbb{C}^2$ exists for $\mathcal{A}$ if and only if $\mathcal{A} = \mathcal{A}_{(m,1^n)}$. We will first show that if the BA function exists for some type $(m, 1^n)$ configuration $\mathcal{A}$ then $\mathcal{A} = \mathcal{A}_{(m,1^n)}$ (Theorem 2.5.1). We will then show that the BA function exists for $\mathcal{A}_{(m,1^n)}$ (Theorems 2.6.1, 2.6.2).

2.5.1 $\mathcal{A}_{(m,1^n)}$ is necessary for BA existence

Theorem 2.5.1. *Let $\mathcal{A}$ have type $(m, 1^n)$. Suppose the BA function exists for $\mathcal{A}$. Then the identities $(\alpha_j(k))$ imply that $\mathcal{A} = \mathcal{A}_{(m,1^n)}$.*

The proof will consist of a series of lemmas. Suppose the BA function exists for $\mathcal{A}$. Then the conditions specified in Theorem 2.2.4 are satisfied. Let

$$S_i = z_0 \widehat{e}_{i-1} + a_i \widehat{e}_i \tag{2.13}$$

for some $a_i \in \mathbb{C}$.

Lemma 2.5.2. *The identity $(\alpha_0(1))$ is equivalent to the relation*

$$\sum_{i=1}^{n} C_i z_0^{n-i} S_i = 0$$

for some $C_i \in \mathbb{C}, 0 \leq i \leq n+1$. Moreover the C_i satisfy the following system of equations:

$$C_i a_i + C_{i+1} = (-1)^i (n - 2i), \;\; 0 \leq i \leq n \tag{2.14}$$

and $C_0 = C_{n+1} = 0$.

Proof. Consider the identity $(\alpha_0(1))$. This is the identity corresponding to the vector α_0 when $k = 1$. It takes the form:

$$\sum_{i=1}^{n} \frac{z_0 + z_i}{z_0 - z_i} = 0. \tag{2.15}$$

We can rearrange this to

$$\sum_{i=1}^{n} (z_0 + z_i) \prod_{j=1, j \neq i}^{n} (z_0 - z_i) = 0. \tag{2.16}$$

We claim that we can write (2.16) in the following form:

$$\sum_{i=0}^{n} (-1)^i (n - 2i) z_0^{n-i} \widehat{e}_i = 0. \tag{2.17}$$

The proof is essentially a simple observation, see Lemma 2.5.8. Let $C_1, C_2, \ldots, C_n \in \mathbb{C}$. We want to write the above relation in the form:

$$\sum_{i=1}^{n} C_i z_0^{n-i} S_i = 0 \tag{2.18}$$

for some appropriate choice of the a_i corresponding to the S_i, (see (2.13)) which we will determine later. Utilizing the S_i we write (2.18) as:

$$\sum_{i=0}^{n} (C_i a_i + C_{i+1}) \widehat{e}_i z_0^{n-i} = 0 \tag{2.19}$$

where $C_0 = C_{n+1} = 0$. If we now equate coefficients in (2.17) and (2.19) we derive the following system of equations:

$$C_i a_i + C_{i+1} = (-1)^i (n - 2i)$$

□

We will return to and solve the system of equations (2.14) using information derived from the identity $(\alpha_1(1))$, which we turn to now. Denote by $\widetilde{e}_i$ the i-th elementary symmetric polynomials in the variables $z_2, z_3, \ldots, z_n$.

Lemma 2.5.3. *The identity* $(\alpha_1(1))$ *is equivalent to the relation*

$$\sum_{i=1}^{n} D_i z_1^{n-i} S_i = 0$$

where $D_i, a_i \in \mathbb{C}, 0 \leq i \leq n + 1$. *Moreover the* D_i, a_i *satisfy the following two systems of equations:*

$$D_i + D_{i+1} = (-1)^{i+1} (m - n + 2i - 1), \ \ 1 \leq i \leq n, \tag{2.20}$$

$$D_i a_i + D_{i+1} a_{i+1} = (-1)^i (m+n-2i-1), \;\; 0 \le i \le n-1, \tag{2.21}$$

and $D_0 = D_{n+1} = 0$*.*

Proof. The identity ($\alpha_1(1)$) has the form:

$$m\frac{z_1+z_0}{z_1-z_0} + \sum_{i=2}^{n} \frac{z_1+z_i}{z_1-z_i} = 0. \tag{2.22}$$

We can write (2.22) in the form

$$\sum_{i=0}^{n} (-1)^{i+1}((m-n+2i-1)z_0\widetilde{e}_{i-1} - (m+n-2i-1)\widetilde{e}_i)z_1^{n-i} = 0 \tag{2.23}$$

where $\widetilde{e}_{-1} = \widetilde{e}_n = 0$. The proof is again easy, see Lemma 2.5.9. To proceed we attempt to write the above expression in a form analogous to equation (2.18). So we require

$$\sum_{i=1}^{n} D_i z_1^{n-i} S_i = 0 \tag{2.24}$$

using our previously defined S_i. Upon substitution of the S_i this gives:

$$\sum_{i=0}^{n} ((D_i + D_{i+1})z_0\widetilde{e}_{i-1} + (a_{i+1}D_{i+1} + a_i D_i)\widetilde{e}_i)z_1^{n-i} = 0 \tag{2.25}$$

with $\widetilde{e}_{-1} = \widetilde{e}_n = 0$. We now equate coefficients in (2.23) and (2.25) to deduce that:

$$(D_i+D_{i+1})z_0\widetilde{e}_{i-1}+(D_i a_i+D_{i+1}a_{i+1})\widetilde{e}_i = (-1)^{i+1}(m-n+2i-1)z_0\widetilde{e}_{i-1}+(-1)^i(m+n-2i-1)\widetilde{e}_i. \tag{2.26}$$

Equating coefficients in (2.26) we derive two systems of equations

$$D_i + D_{i+1} = (-1)^{i+1}(m-n+2i-1), \;\; 1 \le i \le n$$

where $D_{n+1} = 0$ and

$$D_i a_i + D_{i+1} a_{i+1} = (-1)^i (m+n-2i-1), \;\; 0 \le i \le n-1$$

where $D_0 = 0$. □

We can now solve the various systems of equations introduced. We have

Lemma 2.5.4. *The systems of equations* (2.14)*,* (2.20) *and* (2.21) *have the unique solution:*

$$\begin{aligned} a_i &= \frac{i(m+n-i)}{(n-i+1)(m+i-1)}, \\ C_i &= \frac{(-1)^{i+1}(n-i+1)(m+i-1)}{m}, \\ D_i &= (-1)^{i+1}(n-i+1)(m+i-1), \end{aligned}$$

where $1 \le i \le n$*.*

Proof. We will solve (2.20) first. Recall that we have

$$\begin{aligned} &D_n = (-1)^{n+1}(m+n-1), \\ &D_i + D_{i+1} = (-1)^{i+1}(m-n+2i-1) \text{ for } 1 \le i \le n-1. \end{aligned}$$

By substituting D_n and continuing recursively it is easy to see that

$$D_{n-r} = (-1)^{n-r+1}(r+1)(m+n-r-1), \;\; 0 \le r \le n-1$$

and that on setting $n-r=i$

$$D_i = (-1)^{i+1}(n-i+1)(m+i-1), \;\; 1 \le i \le n.$$

Moving to (2.21),

$$D_1 a_1 = m+n-1$$

and

$$D_i a_i + D_{i+1}a_{i+1} = (-1)^i(m+n-2i-1), \;\; 1 \le i \le n-1.$$

Substituting $D_1 a_1$ and continuing recursively we have

$$D_i a_i = (-1)^{i+1}(n-i+1)(m+i-1)$$

so that

$$\begin{aligned} a_i &= \frac{(-1)^{i+1} i(m+n-i)}{(-1)^{i+1}(n-i+1)(m+i-1)} \\ &= \frac{i(m+n-i)}{(n-i+1)(m+i-1)}. \end{aligned}$$

Using this we can solve the system of equations (2.14). By inspection

$$C_i a_i + C_{i+1} = (-1)^i(n-2i), \;\;\; 1 \le i \le n.$$

We claim that

$$C_i = \frac{(-1)^{i+1}(n-i+1)(m+i-1)}{m}, \quad 1 \leq i \leq n \tag{2.27}$$

so that for example $C_1 = n$ which is clear. We have

$$\begin{aligned} C_{i+1} &= -C_i a_i + (-1)^i(n-2i) \\ &= (-1)^{i+2}(n-2i) - \frac{(-1)^{i+1}(n-i+1)(m+i-1)i(m+n-i)}{m(n-i+1)(m+i-1)} \\ &= \frac{(-1)^{i+2}(n-2i)m + (-1)^{i+2}i(m+n-i)}{m} \\ &= \frac{(-1)^{i+2}(im+in-i^2-2im+nm)}{m} \\ &= \frac{(-1)^{i+2}(n-i)(m+i)}{m} \end{aligned}$$

which completes the induction. □

We can now prove the following:

Lemma 2.5.5. *Suppose a configuration $\mathcal{A}$ of type $(m, 1^n)$ satisfies the identities $(\alpha_j(k))$, $1 \leq j \leq n$, $1 \leq k \leq m_j$. Then we have $S_i = 0$ for $1 \leq i \leq n$.*

Proof. First, note that by Lemma 2.5.4

$$D_i = mC_i, \quad 1 \leq i \leq n. \tag{2.28}$$

It remains to observe that the identities $(\alpha_s(1))$, $2 \leq s \leq n$ are identical to the identity $(\alpha_1(1))$ up to a permutation of the variables. So we can write, using (2.28):

$$C_1 z_0^{n-1} S_1 + C_2 z_0^{n-2} S_2 + C_3 z_0^{n-3} S_3 + \ldots + C_{n-2} z_0^2 S_{n-2} + C_{n-1} z_0 S_{n-1} + C_n S_n = 0$$
$$C_1 z_1^{n-1} S_1 + C_2 z_1^{n-2} S_2 + C_3 z_1^{n-3} S_3 + \ldots + C_{n-2} z_1^2 S_{n-2} + C_{n-1} z_1 S_{n-1} + C_n S_n = 0$$

$$C_1 z_{n-1}^{n-1} S_1 + C_2 z_{n-1}^{n-2} S_2 + C_3 z_{n-1}^{n-3} S_3 + \ldots + C_{n-2} z_{n-1}^2 S_{n-2} + C_{n-1} z_{n-1} S_{n-1} + C_n S_n = 0.$$

Writing this in matrix form we have

$$\begin{pmatrix} z_0^{n-1} & z_0^{n-2} & \ldots & z_0^2 & z_0 & 1 \\ z_1^{n-1} & z_1^{n-2} & \ldots & z_1^2 & z_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ z_{n-1}^{n-1} & z_{n-1}^{n-2} & \ldots & z_{n-1}^2 & z_{n-1} & 1 \end{pmatrix} \begin{pmatrix} \bar{S}_1 \\ \bar{S}_2 \\ \vdots \\ \bar{S}_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where

$$\bar{S}_i = C_i S_i.$$

Since $z_i \neq z_j$ for $i \neq j$ the left hand matrix has non-zero determinant, and we conclude that $S_i = 0$ for $1 \leq i \leq n$. □

Proposition 2.5.6. *Suppose a configuration $\mathcal{A}$ of type $(m, 1^n)$ satisfies the identities $(\alpha_j(k))$, $0 \leq j \leq n$, $1 \leq k \leq m_j$. Then $\mathcal{A}$ is symmetric with respect to α_0.*

Proof. By Lemma 2.5.5, $S_i = 0$ for $1 \leq i \leq n$. It follows from (2.13) that

$$\widehat{e}_i = (-1)^i \frac{1}{\prod_{j=1}^{i} a_j}. \tag{2.29}$$

We note that

$$a_i a_{n-i+1} = 1, \quad 1 \leq i \leq n,$$

since we can use the values for the a_i obtained in Lemma 2.5.4 to see that:

$$a_i a_{n-i+1} = \frac{i(m+n-i)}{(n-i+1)(m+i-1)} \frac{(n-i+1)(m+n-(n-i+1))}{(n-(n-i+1)+1)(m+n-i+1-1)} = 1.$$

Thus

$$a_i = \frac{1}{a_{n-i+1}}.$$

Using this we conclude that

$$\widehat{e}_i = (-1)^n \widehat{e}_{n-i}, \;\; 1 \leq i \leq n. \tag{2.30}$$

Let

$$P = \prod_{i=1}^{n} (z - z_i) = \sum_{i=0}^{n} (-1)^i \widehat{e}_i z^{n-i} = \sum_{i=0}^{n} (-1)^i \widehat{e}_i z^i. \tag{2.31}$$

Thus, since z_i $(1 \leq i \leq n)$ is a root of P so too is $\frac{1}{z_i}$ due to the palindromicity of P given by (2.30). Thus up to relabelling $z_i z_{n-i+1} = 1$, $1 \leq i \leq n$. If we set $z_0 = 1$ the proposition follows. □

To complete the proof of Theorem 2.5.1 we need the following.

Lemma 2.5.7. *Let $\mathcal{A}$ have type $(m, 1^n)$. Suppose the BA function exists for $\mathcal{A}$. Then*

$$\widehat{e}_k = (-1)^k \frac{\prod_{j=1}^{k} (m+j-1)(n-j+1)}{k! \prod_{j=1}^{k} (m+n-j)} = (-1)^k \binom{n}{k} \binom{m+k-1}{k} \binom{m+n-1}{k}^{-1}. \tag{2.32}$$

for $1 \leq k \leq n$.

Proof. If the BA function exists for a configuration $\mathcal{A}$ of type $(m, 1^n)$ then in particular the identities $(\alpha_j(k))$, $0 \le j \le n$, $1 \le k \le m_j$ are satisfied. Recall from (2.29) that in this case

$$\widehat{e}_k = (-1)^k \frac{1}{\prod_{j=1}^k a_j}, \quad 1 \le k \le n.$$

Also, recall from Lemma 2.5.4 that

$$a_k = \frac{k(m+n-k)}{(n-k+1)(m+k-1)}, \quad 1 \le k \le n.$$

Combining these it is clear that

$$\widehat{e}_k = (-1)^k \frac{\prod_{j=1}^{k}(m+j-1)(n-j+1)}{k! \prod_{j=1}^{k}(m+n-j)} = (-1)^k \binom{n}{k} \binom{m+k-1}{k} \binom{m+n-1}{k}^{-1}.$$

□

Theorem 2.5.1 is now proven.

2.5.2 Equivalent forms for the identities $(\alpha_0(1))$ and $(\alpha_1(1))$

Lemma 2.5.8. *We can write the identity $(\alpha_0(1))$ in the form*

$$\sum_{i=0}^{n}(-1)^i(n-2i)z_0^{n-i}\widehat{e}_i = 0. \tag{2.33}$$

Proof. We proceed by induction on n. The first nontrivial case is $n = 3$. In this case the identity $\alpha_0(1)$ has the form

$$3z_0^3 - z_0^2\widehat{e_1} - z_0\widehat{e_2} + 3\widehat{e_3} = 0.$$

Consider the identity $(\alpha_0(1))$ in the $(m, 1^{n+1})$ case (that is, we have $n+1$ lines with multiplicity one.) Denote by $f(n)$ the left hand side of the expression (2.33) for fixed n. Then we can write the identity $(\alpha_0(1))$ in the $(m, 1^{n+1})$ case as

$$f(n)(z_0 - z_{n+1}) + (z_0 + z_{n+1}) \sum_{i=0}^{n}(-1)^i\widehat{e}_i = 0.$$

So we have

$$z_0(f(n) + \sum_{i=0}^{n}(-1)^i\widehat{e}_i) + z_{n+1}(\sum_{i=0}^{n}(-1)^i\widehat{e}_i - f(n)) = 0. \tag{2.34}$$

Let's consider the coefficient of the term at z_0^{n-i} in the expression (2.34). Denote this coefficient by b_{n-i}. For the induction to hold this should agree with replacing i by $i+1$ in $f(n+1)$. So we expect

$$b_{n-i} = (n-2i-1)\widehat{e}_{i+1} z_0^{n-i}.$$

Denote by $\widehat{e}_i(n)$ the ith elementary symmetric polynomial in n variables. Note that $\widehat{e}_{i+1}(n) + z_{n+1}\widehat{e}_i(n) = \widehat{e}_{i+1}(n+1)$. Moving to (2.34), we observe that

$$\begin{aligned} b_{n-i} &= (n-2i-1)(-1)^{i+1}\widehat{e}_{i+1}(n) + z_{n+1}(n-2i-1)(-1)^{i+1}\widehat{e}_i(n) \\ &= (n-2i-1)(-1)^{i+1}(\widehat{e}_{i+1}(n) + z_{n+1}\widehat{e}_i(n)) \\ &= (n-2i-1)\widehat{e}_{i+1}(n+1) \end{aligned}$$

as required. □

Lemma 2.5.9. *We can write the identity* $(\alpha_1(1))$ *in the form*

$$\sum_{i=0}^{n}(-1)^{i+1}((m-n+2i-1)z_0\widetilde{e}_{i-1} - (m+n-2i-1)\widetilde{e}_i)z_1^{n-i} = 0. \tag{2.35}$$

Proof. The first nontrivial case is $n = 3$. It is very easy in this case to check that the identity $(\alpha_1(1))$ has the form

$$(m+2)z_1^3 + ((m-2)z_0 - m\widetilde{e}_1)z_1^2 - (z_0 m\widetilde{e}_1 - (m-2)\widetilde{e}_2)z_1 + (m+2)z_0\widetilde{e}_2 = 0.$$

We proceed by induction on n. Consider the identity $(\alpha_1(1))$ in the $(m, 1^{n+1})$ case (that is, we have $n+1$ lines with multiplicity one.) Let us denote the left hand side of the expression (2.35) by $g(n)$. We can then write the identity $(\alpha_1(1))$ in the $(m, 1^{n+1})$ case as

$$g(n)(z_1 - z_{n+1}) + (z_1 + z_{n+1})(z_1 - z_0)(z_1 - z_2)(z_1 - z_3)\dots(z_1 - z_n) = 0.$$

In other words

$$g(n)(z_1 - z_{n+1}) + (z_1 + z_{n+1})(z_1 - z_0)\sum_{i=0}^{n-1}(-1)^i\widetilde{e}_i z_1^{n-1-i} = 0. \tag{2.36}$$

We consider now the coefficient of the term z_1^{n-i} in this expression (2.36). This has the form

$$\begin{aligned}
&(-1)^{i+2}((m-n+2i+1)z_0\widetilde{e}_i-(m+n-2i-3)\widetilde{e}_{i+1})\\
&-z_{n+1}(-1)^{i+1}((m-n+2i-1)z_0\widetilde{e}_{i-1}-(m+n-2i-1)\widetilde{e}_i)\\
&+(-1)^{i+1}\widetilde{e}_{i+1}-z_0(-1)^i\widetilde{e}_i+z_{n+1}(-1)^i\widetilde{e}_i-z_0z_{n+1}(-1)^{i-1}\widetilde{e}_{i-1}\\
&=(-1)^i((m-n+2i)z_0\widetilde{e}_i-(m+n-2i-2)\widetilde{e}_{i+1})\\
&-z_{n+1}(-1)^{i+1}((m-n+2i)z_0\widetilde{e}_{i-1}-(m+n-2i-2)\widetilde{e}_i)\\
&=(-1)^i((m-n+2i)z_0(\widetilde{e}_i+z_{n+1}\widetilde{e}_{i-1})-(m+n-2i-2)(\widetilde{e}_{i+1}+z_{n+1}\widetilde{e}_i)).
\end{aligned}$$

We can write this as

$$(m-n+2i)z_0\bar{e}_i-(m+n-2i-2)\bar{e}_{i+1}$$

where $\bar{e}_i$ is the ith elementary symmetric polynomial in the variables $z_2, z_3, \ldots, z_{n+1}$, using the fact that $\bar{e}_i = z_{n+1}\widetilde{e}_{i-1}+\widetilde{e}_i$. Now we consider the coefficient of the term z_1^{n-i} in $g(n+1)$. This can easily be seen from (2.35); it is equal to

$$(m-(n-2i))z_0\bar{e}_i-(m+n-2i-2)\bar{e}_{i+1}$$

which agrees with what we calculated. □

2.6 Existence of BA function

We will now show that the BA function function exists for $\mathcal{A}_{(m,1^n)}$. To begin with we have:

Theorem 2.6.1. *If $\mathcal{A}=\mathcal{A}_{(m,1^n)}$, the identities $(\alpha_j(k))$, $0\le j\le n$, $1\le k\le m_j$ and $(\widetilde{\alpha}_j(k))$, $j=0$, $1\le k\le m$ are satisfied.*

Proof. This follows immediately from the fact that $\mathcal{A}_{(m,1^n)}$ is symmetric with respect to α_0 (Proposition 2.5.6) and the fact that the identities $(\alpha_j(k))$, $1\le j\le n$, $k=1$ hold (Lemmas 2.5.2, 2.5.3, 2.5.5). □

The rest of this chapter is then devoted to establishing the following.

Theorem 2.6.2. *If $\mathcal{A}=\mathcal{A}_{(m,1^n)}$, the identities $(\widetilde{\alpha}_s(1))$, $1\le s\le n$ are satisfied.*

This is because we can combine Theorems 2.6.1 and 2.6.2 to show that the BA function exists for $\mathcal{A}_{(m,1^n)}$.

2.6.1 The form of the identity $(\widetilde{\alpha}_1(1))$

To prove Theorem 2.6.2 it suffices to check the identity $(\widetilde{\alpha}_1(1))$; the others are identical up to a permutation of variables. The proof of Theorem 2.6.2 will consist of a series of lemmas. Since some rather unwieldy calculations are involved we provide a brief outline for convenience. Using Lemmas 2.6.3, 2.6.4, and 2.6.5 we show that the identity $(\widetilde{\alpha}_1(1))$ holds if a certain polynomial Q is identically zero at z_1. Recalling from (2.31) that z_1 is a root of P, we go on to show that Q is divisible by P, completing the proof. We set

$$f(z) = \sum_{i=2}^{n} \frac{1}{z - z_i}. \tag{2.37}$$

Then we have

Lemma 2.6.3. *The identity* $(\widetilde{\alpha}_1(1))$ *is equivalent to*

$$\frac{m(m+1)(z_1+z_0)z_0}{2(z_1-z_0)^3} + f(z_1) + 3z_1 f^{'}(z_1) + z_1^2 f^{''}(z_1) = 0.$$

Proof. We need to check if the identity $(\widetilde{\alpha}_1(1))$ is satisfied, that is, we need to establish the relation:

$$\frac{m(m+1)(z_1+z_0)z_0}{2(z_1-z_0)^3} + \sum_{i=2}^{n} \frac{z_i(z_1+z_i)}{(z_1-z_i)^3} = 0. \tag{2.38}$$

We now write (2.38) in a more compact manner. By using (2.37) we can deduce the following:

$$\begin{aligned}
\sum_{i=2}^{n} \frac{(z_1+z_i)z_i}{(z_1-z_i)^3} &= z_1 \sum_{i=2}^{n} \frac{(z_i-z_1)+z_1}{(z_1-z_i)^3} + \sum_{i=2}^{n} \frac{(z_i^2-z_1^2)+z_1^2}{(z_1-z_i)^3} \\
&= -z_1 \sum_{i=2}^{n} \frac{1}{(z_1-z_i)^2} + z_1^2 \sum_{i=2}^{n} \frac{1}{(z_1-z_i)^3} \\
&+ \sum_{i=2}^{n} \frac{(z_i-z_1)(z_i+z_1)}{(z_1-z_i)^3} + z_1^2 \sum_{i=2}^{n} \frac{1}{(z_1-z_i)^3} \\
&= -3z_1 \sum_{i=2}^{n} \frac{1}{(z_1-z_i)^2} + 2z_1^2 \sum_{i=2}^{n} \frac{1}{(z_1-z_i)^3} + \sum_{i=2}^{n} \frac{1}{z_1-z_i} \\
&= f(z_1) + 3z_1 f^{'}(z_1) + z_1^2 f^{''}(z_1).
\end{aligned}$$

So we can write (2.38) as

$$\frac{m(m+1)(z_1+z_0)z_0}{2(z_1-z_0)^3} + f(z_1) + 3z_1 f^{'}(z_1) + z_1^2 f^{''}(z_1) = 0. \tag{2.39}$$

□

Next we will show that we can write $f(z_1), f^{'}(z_1)$ and $f^{''}(z_1)$ in terms of P and its derivatives.

Lemma 2.6.4. *We have the identities*

$$f(z_1) = \frac{P^{''}(z_1)}{2P^{'}(z_1)} = \frac{1}{2z_1}(n-1-\frac{m(z_1+z_0)}{z_1-z_0}), \tag{2.40}$$

$$f^{'}(z_1) = \frac{1}{P^{'}(z_1)}(\frac{P^{(3)}(z_1)}{3} - \frac{P^{(2)}(z_1)^2}{4P^{'}(z_1)}), \tag{2.41}$$

$$f^{''}(z_1) = \frac{1}{P^{'}(z_1)}(\frac{P^{(4)}(z_1)}{4} - \frac{P^{(3)}(z_1)P^{(2)}(z_1)}{2P^{'}(z_1)} + \frac{P^{(2)}(z_1)^3}{4P^{'}(z_1)^2}). \tag{2.42}$$

Proof. First recall from (2.37) that

$$f(z) = \sum_{j=2}^{n} \frac{1}{z-z_j}.$$

Since

$$P(z) = \prod_{i=1}^{n}(z-z_i)$$

it is easy to see that

$$f(z_1) = \frac{P^{''}(z_1)}{2P^{'}(z_1)}. \tag{2.43}$$

Recall that the identity $(\alpha_1(1))$ is satisfied since $\mathcal{A} = \mathcal{A}_{(m,1^n)}$, see Theorem 2.6.1. We can write the identity $(\alpha_1(1))$ in the form (2.22):

$$m\frac{z_1+z_0}{z_1-z_0} + \sum_{i=2}^{n} \frac{z_1+z_i}{z_1-z_i} = 0.$$

We note that

$$\begin{aligned}\sum_{i=2}^{n} \frac{z_1+z_i}{z_1-z_i} &= \sum_{i=2}^{n} \frac{2z_1+z_i-z_1}{z_1-z_i} \\ &= 2z_1\sum_{i=2}^{n} \frac{1}{z_1-z_i} - \sum_{i=2}^{n} 1 \\ &= 2z_1 f(z_1) - (n-1).\end{aligned}$$

Using (2.22) and this we conclude that

$$f(z_1) = \frac{1}{2z_1}(n-1-\frac{m(z_1+z_0)}{z_1-z_0}). \tag{2.44}$$

Now, observe that:

$$\frac{P^{'}(z)}{P(z)} = \sum_{i=1}^{n} \frac{1}{z-z_i} = f(z) + \frac{1}{z-z_1}.$$

So we have the identity

$$f(z) = \frac{P'(z)(z - z_1) - P(z)}{P(z)(z - z_1)}. \tag{2.45}$$

We want to derive identities for $f'(z_1)$, $f''(z_1)$ in terms of P. We do this by using (2.45) and considering the Taylor expansion at z_1. For brevity set $\varepsilon = z - z_1$. We have:

$$\begin{aligned}
f(z) &= \frac{P'(z)(z - z_1) - P(z)}{P(z)(z - z_1)} \\
&= \frac{(P'(z_1)\varepsilon + P^{(2)}(z_1)\varepsilon^2 + \frac{1}{2!}P^{(3)}(z_1)\varepsilon^3 + \frac{1}{3!}P^{(4)}(z_1)\varepsilon^4 + \ldots) - (P'(z_1)\varepsilon + \frac{1}{2!}P^{(2)}(z_1)\varepsilon^2 + \frac{1}{3!}P^{(3)}(z_1)\varepsilon^3 + \ldots)}{P'(z_1)\varepsilon^2 + \frac{1}{2!}P^{(2)}(z_1)\varepsilon^3 + \frac{1}{3!}P^{(3)}(z_1)\varepsilon^4 + \ldots} \\
&= \frac{\frac{1}{2}P^{(2)}(z_1) + \frac{1}{3}P^{(3)}(z_1)\varepsilon + \frac{1}{8}P^{(4)}(z_1)\varepsilon^2 + \ldots}{P'(z_1) + \frac{1}{2}P^{(2)}(z_1)\varepsilon + \frac{1}{6}P^{(3)}(z_1)\varepsilon^2 + \ldots} \\
&= \frac{\frac{1}{2}P^{(2)}(z_1) + \frac{1}{3}P^{(3)}(z_1)\varepsilon + \frac{1}{8}P^{(4)}(z_1)\varepsilon^2 + \ldots}{P'(z_1)(1 + \frac{P^{(2)}(z_1)}{2P'(z_1)}\varepsilon + \frac{P^{(3)}(z_1)}{6P'(z_1)}\varepsilon^2 + \frac{P^{(4)}(z_1)}{24P'(z_1)}\varepsilon^3 + \ldots)} \\
&= \frac{1}{P'(z_1)}(\frac{1}{2}P^{(2)}(z_1) + \frac{1}{3}P^{(3)}(z_1)\varepsilon + \frac{1}{8}P^{(4)}(z_1)\varepsilon^2 + ...)(1 - \frac{P^{(2)}(z_1)}{2P'(z_1)}\varepsilon - \frac{P^{(3)}(z_1)}{6P'(z_1)}\varepsilon^2 + ... + (\frac{P^{(2)}(z_1)}{2P'(z_1)})^2\varepsilon^2 + ...) \\
&= \frac{1}{P'(z_1)}(\frac{1}{2}P^{(2)}(z_1) + \frac{1}{3}P^{(3)}(z_1)\varepsilon + \frac{1}{8}P^{(4)}(z_1)\varepsilon^2 + \ldots)(1 - \frac{P^{(2)}(z_1)}{2P'(z_1)}\varepsilon + (\frac{P^{(2)}(z_1)^2}{4P'(z_1)^2} - \frac{P^{(3)}(z_1)}{6P'(z_1)})\varepsilon^2 + \ldots) \\
&= \frac{P^{(2)}(z_1)}{2P'(z_1)} + \frac{1}{P'(z_1)}\left(\frac{P^{(3)}(z_1)}{3} - \frac{P^{(2)}(z_1)^2}{4P'(z_1)}\right)\varepsilon + \frac{1}{P'(z_1)}\left(\frac{1}{8}P^{(4)}(z_1) - \frac{P^{(3)}(z_1)P^{(2)}(z_1)}{6P'(z_1)}\right. \\
&\left. + \frac{1}{4}P^{(2)}(z_1)(\frac{P^{(2)}(z_1)^2}{2P'(z_1)^2} - \frac{P^{(3)}(z_1)}{3P'(z_1)})\right)\varepsilon^2 \\
&+ O(\varepsilon^3).
\end{aligned}$$

So we have the identities:

$$\begin{aligned}
f'(z_1) &= \frac{1}{P'(z_1)}(\frac{P^{(3)}(z_1)}{3} - \frac{P^{(2)}(z_1)^2}{4P'(z_1)}), \\
f''(z_1) &= \frac{1}{P'(z_1)}(\frac{P^{(4)}(z_1)}{4} - \frac{P^{(3)}(z_1)P^{(2)}(z_1)}{2P'(z_1)} + \frac{P^{(2)}(z_1)^3}{4P'(z_1)^2}).
\end{aligned}$$

□

Lemma 2.6.5. *The identity* $(\widetilde{\alpha}_1(1))$ *is equivalent to the relation:*

$$\begin{aligned}
Q := &P'(z_1)\left(2m(m+1)z_0z_1(z_0 + z_1) + 2(z_1 - z_0)^2((n-1)(z_1 - z_0) - m(z_1 + z_0))\right. \\
&-3(z_1 - z_0)((n-1)^2(z_1 - z_0)^2 - 2m(n-1)(z_1 + z_0)(z_1 - z_0) + m^2(z_1 + z_0)^2) \\
&\left.+((n-1)(z_1 - z_0) - m(z_1 + z_0))^3\right) \\
&+ 2P^{(3)}(z_1)z_1^2(z_1 - z_0)^2\left((z_1 - z_0)(3 - n) + m(z_1 + z_0)\right) \\
&+ P^{(4)}(z_1)\left(z_1^3(z_1 - z_0)^3\right) = 0.
\end{aligned} \tag{2.46}$$

Proof. By Lemmas 2.6.3, 2.6.4, the identity $(\widetilde{\alpha}_1(1))$ takes the form:

$$\frac{m(m+1)(z_1+z_0)z_0}{2(z_1-z_0)^3} + f(z_1) + 3z_1 f^{'}(z_1) + z_1^2 f^{''}(z_1)$$
$$= \frac{m(m+1)(z_1+z_0)z_0}{2(z_1-z_0)^3}$$
$$+ \frac{1}{2z_1}(n-1-\frac{m(z_1+z_0)}{z_1-z_0})$$
$$+ 3z_1\frac{1}{P'(z_1)}(\frac{P^{(3)}(z_1)}{3} - \frac{P^{(2)}(z_1)^2}{4P'(z_1)})$$
$$+ z_1^2\frac{1}{P'(z_1)}(\frac{P^{(4)}(z_1)}{4} - \frac{P^{(3)}(z_1)P^{(2)}(z_1)}{2P'(z_1)} + \frac{P^{(2)}(z_1)^3}{4P'(z_1)^2}) = 0. \tag{2.47}$$

Recall from (2.43) that

$$f(z_1) = \frac{P^{''}(z_1)}{2P'(z_1)}$$

This means we can simplify (2.47) further to:

$$m(m+1)z_0z_1(z_1+z_0)$$
$$+ (z_1-z_0)^3(n-1-\frac{m(z_1+z_0)}{z_1-z_0})$$
$$+ 2z_1^2(z_1-z_0)^3(\frac{P^{(3)}(z_1)}{P'(z_1)} - 3f(z_1)^2)$$
$$+ z_1^3(z_1-z_0)^3(\frac{P^{(4)}(z_1)}{2P'(z_1)} - \frac{P^{(3)}(z_1)P^{(2)}(z_1)}{P'(z_1)^2} + 4f(z_1)^3) = 0.$$

In other words

$$P^{'}(z_1)(m(m+1)z_0z_1(z_1+z_0) + (z_1-z_0)^3(n-1-\frac{m(z_1+z_0)}{z_1-z_0}) - 6z_1^2(z_1-z_0)^3f(z_1)^2 + 4z_1^3(z_1-z_0)^3f(z_1)^3)$$
$$+ 2z_1^2(z_1-z_0)^3P^{(3)}(z_1) + \frac{1}{2}z_1^3(z_1-z_0)^3P^{(4)}(z_1) - 2z_1^3(z_1-z_0)^3f(z_1)P^{(3)}(z_1) = 0.$$

Now we recall from (2.44) that

$$f(z_1) = \frac{1}{2z_1}(n-1-\frac{m(z_1+z_0)}{z_1-z_0}).$$

So we have

$$P^{'}(z_1)\Bigg(m(m+1)(z_1)(z_1+z_0)z_0 + (z_1-z_0)^3(n-1-\frac{m(z_1+z_0)}{z_1-z_0})$$
$$-6z_1^2(z_1-z_0)^3(\frac{1}{2z_1}(n-1-\frac{m(z_1+z_0)}{z_1-z_0}))^2 + 4z_1^3(z_1-z_0)^3(\frac{1}{2z_1}(n-1-\frac{m(z_1+z_0)}{z_1-z_0}))^3\Bigg)$$
$$+ 2P^{(3)}(z_1)\left(z_1^2(z_1-z_0)^3 - z_1^3(z_1-z_0)^3(\frac{1}{2z_1}(n-1-\frac{m(z_1+z_0)}{z_1-z_0}))\right)$$
$$+ P^{(4)}(z_1)\left(\frac{1}{2}z_1^3(z_1-z_0)^3\right) = 0.$$

Cancelling the denominators we obtain:

$$
\begin{aligned}
&P^{'}(z_1)\big(2m(m+1)z_0z_1(z_0+z_1)+2(z_1-z_0)^2((n-1)(z_1-z_0)-m(z_1+z_0))\\
&-3(z_1-z_0)((n-1)^2(z_1-z_0)^2-2m(n-1)(z_1+z_0)(z_1-z_0)+m^2(z_1+z_0)^2)\\
&+((n-1)(z_1-z_0)-m(z_1+z_0))^3\big)\\
&+2P^{(3)}(z_1)z_1^2(z_1-z_0)^2\left(2(z_1-z_0)-(n-1)(z_1-z_0)+m(z_1+z_0)\right)\\
&+P^{(4)}(z_1)z_1^3(z_1-z_0)^3=0,
\end{aligned}
$$

giving

$$
\begin{aligned}
&P^{'}(z_1)\big(2m(m+1)z_0z_1(z_0+z_1)+2(z_1-z_0)^2((n-1)(z_1-z_0)-m(z_1+z_0))\\
&-3(z_1-z_0)((n-1)^2(z_1-z_0)^2-2m(n-1)(z_1+z_0)(z_1-z_0)+m^2(z_1+z_0)^2)\\
&+((n-1)(z_1-z_0)-m(z_1+z_0))^3\big)\\
&+2P^{(3)}(z_1)z_1^2(z_1-z_0)^2\left((z_1-z_0)(3-n)+m(z_1+z_0)\right)\\
&+P^{(4)}(z_1)z_1^3(z_1-z_0)^3=0.
\end{aligned}
$$

□

2.6.2 The identity $(\widetilde{\alpha}_1(1))$ holds

By Lemma 2.6.5 we now know that in order to show that the identity $(\widetilde{\alpha}_1(1))$ holds we must show that a polynomial of degree $n+2$ is zero at z_1. To do this we prove that the polynomial $P(z_1)$ is a factor of the polynomial Q, recalling from (2.31) that z_1 is a root of P.

Lemma 2.6.6. *Let the polynomial Q defined in (2.46) be expressed in the form:*

$$Q(z_1)=\sum_{j=0}^{n+2} t_j z_1^j$$

Let $z_0=1$. Then

$$t_j=\widehat{e}_j q_{j,0}+\widehat{e}_{j-1}q_{j,1}+\widehat{e}_{j-2}q_{j,2}+\widehat{e}_{j+1}q_{j,3}.$$

with

$$
\begin{aligned}
q_{j,0} =&(-1)^j j(2m(m+1)+2(m+3n-3)+3(n-1)((n-1)^2+(n-1)(m-3)-m(m+2))+3m^2(1-m)\\
&+(j-1)(j-2)(3j-6n-2m+9)),\\
q_{j,1} =&(-1)^{j-1}(j-1)(2m(m+1)+2(3-3n+m)+3(n-1)(-(n-1)^2+(n-1)(3+m)+m(m-2))\\
&-3m^2(m+1)+(j-2)(j-3)(6n-2m-3j-6)),\\
q_{j,2} =&(-1)^j(j-2)(2(n-1-m)+(n-1)((n-1)^2-3(n-1)(m+1)+3m(m+2))-m^2(3+m)\\
&+(j-3)(j-4)(1-2n+2m+j)),\\
q_{j,3} =&(-1)^{j+1}(j+1)(2(1-n-m)+(n-1)(-(n-1)^2+3(n-1)(1-m)+3m(2-m))+m^2(3-m)\\
&+j(j-1)(2n+2m-j-4)).
\end{aligned}
$$

Proof. By inspection of P, the terms which contribute to t_j are those proportional to $\widehat{e}_{j+1}$, $\widehat{e}_j$, $\widehat{e}_{j-1}$, $\widehat{e}_{j-2}$. We exhibit how these are calculated. First consider $q_{j,0}$, the term corresponding to $\widehat{e}_j$. Observe the appropriate term in P':

$$P'(z_1) = \ldots + (-1)^j j\widehat{e}_j z_1^{j-1} + \ldots$$

Then referring to Lemma 2.6.5 we must calculate:

$$
\begin{aligned}
q_{j,0} =&(-1)^j j[2m(m+1)z_0z_1(z_0+z_1)+2(z_1-z_0)^2((n-1)(z_1-z_0)-m(z_1+z_0))\\
&-3(z_1-z_0)((n-1)^2(z_1-z_0)^2-2m(n-1)(z_1+z_0)(z_1-z_0)+m^2(z_1+z_0)^2)\\
&+((n-1)(z_1-z_0)-m(z_1+z_0))^3]_1\\
&+(-1)^j j(j-1)(j-2)[2z_1^2(z_1-z_0)^2\,((z_1-z_0)(3-n)+m(z_1+z_0))]_3\\
&+(-1)^j j(j-1)(j-2)(j-3)[z_1^3(z_1-z_0)^3]_4
\end{aligned}
$$

where $[f]_k$ is the coefficient of z_1^k in the expression f. This gives:

$$
\begin{aligned}
q_{j,0} =&(-1)^j j(2m(m+1)+2(m+3n-3)+3(n-1)((n-1)^2+(n-1)(m-3)\\
&-m(m+2))+3m^2(1-m))+(j-1)(j-2)(3j-6n-2m+9)).
\end{aligned}
$$

For $q_{j,1}$ we must calculate:

$$
\begin{aligned}
q_{j,1} =&(-1)^{j-1}(j-1)[2m(m+1)z_0z_1(z_0+z_1)+2(z_1-z_0)^2((n-1)(z_1-z_0)-m(z_1+z_0))\\
&-3(z_1-z_0)((n-1)^2(z_1-z_0)^2-2m(n-1)(z_1+z_0)(z_1-z_0)+m^2(z_1+z_0)^2)\\
&+((n-1)(z_1-z_0)-m(z_1+z_0))^3]_2\\
&+(-1)^{j-1}(j-1)(j-2)(j-3)[2z_1^2(z_1-z_0)^2\,((z_1-z_0)(3-n)+m(z_1+z_0))]_4\\
&+(-1)^{j-1}(j-1)(j-2)(j-3)(j-4)[z_1^3(z_1-z_0)^3]_5
\end{aligned}
$$

giving

$$\begin{aligned}q_{j,1} =&(-1)^{j-1}(j-1)(2m(m+1)+2(3-3n+m)+3(n-1)(-(n-1)^2\\&+(n-1)(3+m)+m(m-2))\\&-3m^2(m+1)+(j-2)(j-3)(6n-2m-6-3j)).\end{aligned}$$

For $q_{j,2}$ we must calculate:

$$\begin{aligned}q_{j,2} =&(-1)^{j-2}(j-2)[2m(m+1)z_0z_1(z_0+z_1)+2(z_1-z_0)^2((n-1)(z_1-z_0)-m(z_1+z_0))\\&-3(z_1-z_0)((n-1)^2(z_1-z_0)^2-2m(n-1)(z_1+z_0)(z_1-z_0)+m^2(z_1+z_0)^2)\\&+((n-1)(z_1-z_0)-m(z_1+z_0))^3]_3\\&+(-1)^{j-2}(j-2)(j-3)(j-4)[2z_1^2(z_1-z_0)^2\,((z_1-z_0)(3-n)+m(z_1+z_0))]_5\\&+(-1)^{j-2}(j-2)(j-3)(j-4)(j-5)[z_1^3(z_1-z_0)^3]_6\end{aligned}$$

giving

$$\begin{aligned}q_{j,2} =&(-1)^{j-2}(j-2)(2(n-1-m)+(n-1)((n-1)^2-3(n-1)(m+1)+3m(m+2))\\&-m^2(3+m)+(j-3)(j-4)(1-2n+2m+j)).\end{aligned}$$

Finally for $q_{j,3}$ we must calculate:

$$\begin{aligned}q_{j,3} =&(-1)^{j+1}(j+1)[2m(m+1)z_0z_1(z_0+z_1)+2(z_1-z_0)^2((n-1)(z_1-z_0)-m(z_1+z_0))\\&-3(z_1-z_0)((n-1)^2(z_1-z_0)^2-2m(n-1)(z_1+z_0)(z_1-z_0)+m^2(z_1+z_0)^2)\\&+((n-1)(z_1-z_0)-m(z_1+z_0))^3]_0\\&+(-1)^{j+1}(j+1)(j)(j-1)[2z_1^2(z_1-z_0)^2\,((z_1-z_0)(3-n)+m(z_1+z_0))]_2\\&+(-1)^{j+1}(j+1)(j)(j-1)(j-2)[z_1^3(z_1-z_0)^3]_3\end{aligned}$$

giving

$$\begin{aligned}q_{j,3} =&(-1)^{j+1}(j+1)(2(1-n-m)+(n-1)(-(n-1)^2+3(n-1)(1-m)+3m(2-m))\\&+m^2(3-m)+j(j-1)(2n+2m-j-4)).\end{aligned}$$

□

Lemma 2.6.7.

$$Q = Pq + r \tag{2.48}$$

where $q,\, r \in \mathbb{C}[z_1]$*, deg* $r <$ *deg* $P = n$ *and* $q = az_1^2 + bz_1 + c$ *with:*

$$\begin{aligned} a =& mn(-m^2 + 3mn - n^2 - 6m + 6n - 7), \\ b =& mn(-2m^2 - mn + 2n^2 + 3m - 11n + 15), \\ c =& -mn(m+n-2)(m+n-3). \end{aligned}$$

Proof. We divide Q by P:

$$\begin{array}{r|llll} & t_{n+2}z_1^2 & +(t_{n+1}+t_{n+2}\widehat{e}_1)z_1 & +(t_n - t_{n+2}\widehat{e}_2 + t_{n+1}\widehat{e}_1 + t_{n+2}\widehat{e}_1^2) & \\ \hline & t_{n+2}z_1^{n+2} & +t_{n+1}z_1^{n+1} & +t_n z_1^n & +\cdots \\ z_1^n - \widehat{e}_1 z_1^{n-1} + \cdots & t_{n+2}z_1^{n+2} & -t_{n+2}\widehat{e}_1 z_1^{n+1} & +t_{n+2}\widehat{e}_2 z_1^n & -\cdots \\ & & (t_{n+1}+t_{n+2}\widehat{e}_1)z_1^{n+1} & +(t_n - t_{n+2}\widehat{e}_2)z_1^n & -\cdots \\ & & (t_{n+1}+t_{n+2}\widehat{e}_1)z_1^{n+1} & -\widehat{e}_1(t_{n+1}+t_{n+2}\widehat{e}_1)z_1^n & -\cdots \\ & & & (t_n - t_{n+2}\widehat{e}_2 + t_{n+1}\widehat{e}_1 + t_{n+2}\widehat{e}_1^2)z_1^n & -\cdots \end{array}$$

So the required quotient is

$$q = t_{n+2}z_1^2 + (t_{n+1} + t_{n+2}\widehat{e}_1)z_1 + (t_n - t_{n+2}\widehat{e}_2 + t_{n+1}\widehat{e}_1 + t_{n+2}\widehat{e}_1^2) \tag{2.49}$$

Thus to find a, b, c it will only be necessary to ascertain the coefficients t_n, t_{n+1}, t_{n+2}. We need to return to the polynomial Q introduced in Lemma 2.6.5. Recall from Lemma 2.6.6 that

$$t_j = \widehat{e}_j q_{j,0} + \widehat{e}_{j-1} q_{j,1} + \widehat{e}_{j-2} q_{j,2} + \widehat{e}_{j+1} q_{j,3}.$$

Recall also that $\widehat{e}_i = 0$ for $i \geq n+1$. So using Lemma 2.6.6 we have

$$\begin{aligned} t_{n+2} &= \widehat{e}_n q_{n+2,2} = (-1)^n q_{n+2,2} \\ &= n(2(n-1-m) + (n-1)((n-1)^2 - 3(n-1)(m+1) + 3m(m+2)) \\ &\quad - m^2(3+m) + (n-1)(n-2)(3-n+2m)) \\ &= -mn(m^2 - 3mn + n^2 + 6m - 6n + 7). \end{aligned}$$

Also

$$\begin{aligned} t_{n+1} &= \widehat{e}_n q_{n+1,1} + \widehat{e}_{n-1} q_{n+1,2} = (-1)^n q_{n+1,1} + (-1)^n \widehat{e}_1 q_{n+1,2} \\ &= n(2m(m+1) + 2(3-3n+m) + 3(n-1)(-(n-1)^2 + (n-1)(3+m) + m(m-2)) \\ &\quad - 3m^2(m+1) \\ &\quad + (n-1)(n-2)(3n-2m-9)) \\ &\quad - \widehat{e}_1(n-1)(2(n-1-m) + (n-1)((n-1)^2 - 3(n-1)(m+1) + 3m(m+2)) \\ &\quad - m^2(3+m) \end{aligned}$$

$$+ (n-2)(n-3)(2-n+2m))$$

$$= mn(9-4m-3m^2-6n+3mn+n^2)$$

$$- \widehat{e}_1(-m^3n+3m^2n^2-mn^3+m^3-9m^2n+3mn^2+n^3+6m^2-mn-6n^2-m+11n-6).$$

Finally

$$t_n = \widehat{e}_n q_{n,0} + \widehat{e}_{n-1} q_{n,1} + \widehat{e}_{n-2} q_{n,2} = (-1)^n q_{n,0} + (-1)^n \widehat{e}_1 q_{n,1} + (-1)^n \widehat{e}_2 q_{n,2}$$

$$= n(2m(m+1)+2(m+3n-3)+3(n-1)((n-1)^2+(n-1)(m-3)-m(m+2))$$

$$+ 3m^2(1-m)$$

$$+ (n-1)(n-2)(-3n-2m+9))$$

$$- \widehat{e}_1(n-1)(2m(m+1)+2(3-3n+m)$$

$$+ 3(n-1)(-(n-1)^2+(n-1)(3+m)+m(m-2))$$

$$- 3m^2(m+1)$$

$$+ (n-2)(n-3)(3n-2m-6))$$

$$+ \widehat{e}_2(n-2)(2(n-1-m)+(n-1)((n-1)^2-3(n-1)(m+1)+3m(m+2))$$

$$- m^2(3+m)$$

$$+ (n-3)(n-4)(1-n+2m))$$

$$= mn(-3m^2-3mn+n^2+8m-6n+9)$$

$$- \widehat{e}_1(-3m^3n+3m^2n^2+mn^3+3m^3-7m^2n$$

$$- 3mn^2-3n^3+4m^2+3mn+18n^2-m-33n+18)$$

$$+ \widehat{e}_2(-m^3n+3m^2n^2-mn^3+2m^3-12m^2n$$

$$+ 2n^3+12m^2+17mn-12n^2-26m+22n-12).$$

Recall now from (2.32) that:

$$\widehat{e}_1 = \frac{-mn}{m+n-1}$$

$$\widehat{e}_2 = \frac{mn(m+1)(n-1)}{2(m+n-1)(m+n-2)}.$$

So the required coefficients are:

$$
\begin{aligned}
a =& -mn(m^2+n^2-3mn+6(m-n)+7),\\
b =& mn(-3m^2+3mn+n^2-4m-6n+9)\\
&-\frac{mn}{m+n-1}(6+m-6m^2-m^3-11n-6mn+3m^2n+3mn^2+6n^2-n^3),\\
c =& mn(-3m^2-3mn+n^2+8m-6n+9)\\
&-\frac{mn}{m+n-1}(-3m^3+3m^2n-3mn^2+3n^3-4m^2+6mn-18n^2+m+33n-18)\\
&+\frac{mn(m+1)(n-1)}{2(m+n-1)(m+n-2)}(2m^3-6m^2n-6mn^2+2n^3+12m^2+24mn\\
&-12n^2-26m+22n-12)\\
&+\frac{m^2n^2}{(m+n-1)^2}(-m^3+3m^2n+3mn^2-n^3-6m^2-6mn+6n^2+m-11n+6).
\end{aligned}
$$

We now use the following trivial factorizations:

$$
\begin{aligned}
&-12-26m+12m^2+2m^3+22n+24mn-6m^2n-12n^2-6mn^2+2n^3\\
&=2(m+n-2)(3+8m+m^2-4n-4mn+n^2) \qquad (2.50)\\
&6+m-6m^2-m^3-11n-6mn+3m^2n+3mn^2+6n^2-n^3\\
&=-(m+n-1)(6+7m+m^2-5n-4mn+n^2) \qquad (2.51)\\
&-(-18+m-4m^2-3m^3+33n+6mn+3m^2n-18n^2-3mn^2+3n^3)\\
&+(m+1)(n-1)(3+8m+m^2-4n-4mn+n^2)-mn(6+7m+m^2-5n-4mn+n^2)\\
&=(-1+m+n)(-15-3m+2m^2+11n+mn-2n^2) \qquad (2.52)
\end{aligned}
$$

to obtain

$$
\begin{aligned}
a =& mn(-m^2+3mn-n^2-6m+6n-7),\\
b =& mn(-2m^2-mn+2n^2+3m-11n+15),\\
c =& -mn(m+n-2)(m+n-3).
\end{aligned}
$$

□

2.6.3 Final calculations

Referring now to (2.48) we should multiply P by q and determine the coefficient of the degree j term. Let

$$
S = Pq = \sum_{j=0}^{n+2} s_j z_1^{n-j}. \qquad (2.53)
$$

Then clearly

$$s_j = (-1)^j a\widehat{e}_{j-2} + (-1)^{j-1} b\widehat{e}_{j-1} + (-1)^j c\widehat{e}_j.$$

Thus the proof of Theorem 2.6.2 has been reduced to establishing the following lemma.

Lemma 2.6.8. *Q=S.*

Proof. To check that $Q = S$ we will show that $t_j = s_j$ for $1 \leq j \leq n+2$. We collect terms and subtract to get

$$\begin{aligned}
&t_j - s_j \\
&= \widehat{e}_{j+1}(-1)^{j+1}(j+1)(-11n + 6 - 11m + 6n^2 + 12mn - 3m^2n - 3n^2m + 6m^2 - n^3 - m^3 - 3j^2 - j^3 \\
&+ 2nj^2 + 2mj^2 + 4j - 2nj - 2mj) \\
&+ \widehat{e}_j(-1)^j[j(8m^2 - 3m^3 + 9m + 21n - 18n^2 - 12mn + 3n^3 + 3n^2m - 3m^2n + 3j^3 - 6nj^2 - 2mj^2 - 21j \\
&+ 18nj + 6mj) - mn(-m^2 - 2mn - n^2 + 5m + 5n - 6)] \\
&+ \widehat{e}_{j-1}[(-1)^{j-1}[(j-1)(9j^2 - 3j^3 + 6nj^2 - 2mj^2 - 30nj + 10mj + 12j - 18 + 3n + m - 4m^2 + 18n^2 \\
&- 3n^3 - 3m^3 - 12mn + 3mn^2 + 3nm^2) \\
&- mn(-2m^2 - mn + 2n^2 + 3m - 11n + 15)] \\
&+ \widehat{e}_{j-2}[(-1)^{j-2}[(j-2)(j^3 - 6j^2 + 5j - 2nj^2 + 2mj^2 + 14nj - 14mj - 13n + 13m + 6 - 6n^2 + 12mn \\
&- 3n^2m + 3m^2n - 6m^2 + n^3 - m^3) - mn(-m^2 + 3mn - n^2 - 6m + 6n - 7)].
\end{aligned}$$

Recall from Definition 2.4.7 that

$$\widehat{e}_j = (-1)^j \binom{n}{j}\binom{m+j-1}{j}\binom{m+n-1}{j}^{-1}$$

for $1 \leq i \leq n$. Substituting and multiplying through by $\dfrac{j!\prod_{r=1}^{j}(m+n-r)}{\prod_{r=1}^{j-1}(m+r-1)(n-r+1)}$ we obtain a final form for $t_j - s_j$, using the following easily checkable relations:

$$\begin{aligned}
&(-11n + 6 - 11m + 6n^2 + 12mn - 3m^2n - 3n^2m + 6m^2 - n^3 - m^3 - 3j^2 - j^3 + 2nj^2 + 2mj^2 + 4j \\
&- 2nj - 2j) \\
&= -(1 + j - m - n)(-6 + 2j + j^2 + 5m - jm - m^2 + 5n - jn - 2mn - n^2)
\end{aligned}$$

$$\begin{aligned}
&(j-2)(j^3 - 6j^2 + 5j - 2nj^2 + 2mj^2 + 14nj - 14mj - 13n + 13m + 6 - 6n^2 + 12mn - 3n^2m \\
&+ 3m^2n - 6m^2 + n^3 - m^3) - (-7mn - m^3n + 6mn^2 - mn^3 + 3n^2m^2 - 6m^2n) \\
&= (-2 + j + m)(-2 + j - n)(-3 - 4j + j^2 - 8m + jm - m^2 + 8n - jn + 3mn - n^2).
\end{aligned}$$

So we have

$$
\begin{aligned}
&F_j := t_j - s_j\\
&= (n-j)(m+j)(n-j+1)(m+j-1)(-6+2j+j^2+5m-mj-m^2+5n-nj-2mn-n^2)\\
&+ (n-j+1)(m+j-1)(j(8m^2-3m^3+9m+21n-18n^2-12mn+3n^3+3n^2m-3m^2n+3j^3\\
&- 6nj^2-2mj^2-21j+18nj+6mj)\\
&- mn(-6+5m-m^2+5n-2mn-n^2))\\
&+ (m+n-j)j(j-1)(9j^2-3j^3+6nj^2-2mj^2-30nj+10mj+12j-18+3n+m-4m^2+18n^2\\
&- 3n^3-3m^3-12mn+3mn^2+3nm^2\\
&- mn(15+3m-2m^2-11n-mn+2n^2))\\
&- j(j-1)(m+n-j)(m+n-j+1)(-3-4j+j^2-8m+mj-m^2+8n-nj+3mn-n^2).
\end{aligned}
$$

□

Thus to show that $Q = S$ we need the following.

Lemma 2.6.9. *The polynomial F_j is identically zero.*

Proof. We will split F_j into four summands. First we have

$$
\begin{aligned}
&(n-j)(m+j)(n-j+1)(m+j-1)(-6+2j+j^2+5m-mj-m^2+5n-nj-2mn-n^2)\\
&= j^6\\
&+ j^5(m-3n)\\
&+ j^4(-2m^2-6mn+2n^2+8m+6n-9)\\
&+ j^3(-3m^3-m^2n+7mn^2+n^3+16m^2-10n^2-28m+6n+14)\\
&+ j^2(-m^4+4m^3n+7m^2n^2-n^4+9m^3-18m^2n-18mn^2+3n^3-25m^2+29mn+7n^2\\
&+ 25m-15n-6)\\
&+ j(2m^4n+m^3n^2-3m^2n^3-2mn^4+m^4-13m^3n-4m^2n^2+n^4+9mn^3-6m^3+29m^2n\\
&+ 3mn^2-4n^3+11m^2-26mn+n^2-6m+6n)\\
&- m^4n^2-2m^3n^3-m^2n^4-m^4n+4m^3n^2+6m^2n^3+mn^4+6m^3n-4m^2n^2-4mn^3\\
&- 11m^2n+mn^2+6mn.
\end{aligned}
$$

For the second summand we have

$$(n-j+1)(m+j-1)(j(8m^2-3m^3+9m+21n-18n^2-12mn+3n^3+3n^2m-3m^2n+3j^3$$
$$-6nj^2-2mj^2-21j+18nj+6mj)$$
$$-mn(-6+5m-m^2+5n-2mn-n^2))$$
$$=-3j^6$$
$$+j^5(-m+9n+6)$$
$$+j^4(2m^2+7mn-6n^2-7m-33n+18)$$
$$+j^3(3m^3+m^2n-9mn^2-3n^3-16m^2-4mn+42n^2+26m-42)$$
$$+j^2(3m^4-m^3n-8m^2n^2-mn^3+3n^4-14m^3+25m^2n+35mn^2-12n^3+13m^2-51mn$$
$$-33n^2-9m+45n+21)$$
$$+j(-4m^4n-4m^3n^2+4m^2n^3+4mn^4-3m^4+15m^3n-2m^2n^2-21mn^3-3n^4+11m^3$$
$$-24m^2n+8mn^2+15n^3+m^2+36mn-3n^2-9m-21n)$$
$$+m^4n^2+2m^3n^3+m^2n^4+m^4n-4m^3n^2-6m^2n^3-mn^4-6m^3n+4m^2n^2+4mn^3+11m^2n$$
$$-mn^2-6mn.$$

For the third summand we have

$$(m+n-j)j(j-1)(9j^2-3j^3+6nj^2-2mj^2-30nj+10mj+12j-18+3n+m-4m^2+18n^2$$
$$-3n^3-3m^3-12mn+3mn^2+3nm^2$$
$$-mn(15+3m-2m^2-11n-mn+2n^2))$$
$$=3j^6$$
$$+j^5(-m-9n-12)$$
$$+j^4(-2m^2+4mn+6n^2+48n-3)$$
$$+j^3(3m^3-3m^2n-3mn^2+3n^3+16m^2-12mn-54n^2+12m-30n+30)$$
$$+j^2(-3m^4-2m^3n+5m^2n^2+2mn^3-3n^4-7m^3-10m^2n-2mn^2+15n^3-13m^2+27mn$$
$$+51n^2-29m-27n-18)$$
$$+j(2m^4n+3m^3n^2-m^2n^3-2mn^4+3m^4-3m^3n+2m^2n^2+11mn^3+3n^4+4m^3$$
$$+m^2n-21mn^2-18n^3-m^2-4mn-3n^2+18m+18n).$$

Finally the fourth summand is

$$- j(j-1)(m+n-j)(m+n-j+1)(-3-4j+j^2-8m+mj$$
$$- m^2+8n-nj+3mn-n^2)$$
$$= -j^6$$
$$+ j^5(m+3n+6)$$
$$+ j^4(2m^2-5mn-2n^2-m-21n-6)$$
$$+ j^3(-3m^3+3m^2n+5mn^2-n^3-16m^2+16mn+22n^2-10m+24n-2)$$
$$+ j^2(m^4-m^3n-4m^2n^2-mn^3+n^4+12m^3+3m^2n-15mn^2-6n^3+25m^2-5mn-25n^2$$
$$+ 13m-3n+3)$$
$$+ j(-m^4+m^3n+4m^2n^2+mn^3-n^4-9m^3-6m^2n+10mn^2+7n^3$$
$$- 11m^2-6mn+5n^2-3m-3n).$$

Now when we add all four summands it is easy to see that the result is zero. □

Chapter 3

Cohen-Macaulay and Gorenstein property

3.1 Summary

This Chapter comprises two sections. In Section 3.2 we describe the Gorenstein property for a graded module over a polynomial ring. We follow the approach of Benson [Ben93]. As such, Section 3.2 contains no original material. In Section 3.3 we will show that analysis of the Gorenstein property for rings of quasi-invariants in the plane reduces to analysis of the Poincaré series. The result in this section has been known for some time but no explicit reference exists.

3.2 Cohen-Macaulay and Gorenstein rings

We will be interested in establishing when the Gorenstein property holds for certain rings of quasi-invariants in the plane, and investigating the possible implications this may have. The importance of the Gorenstein property was recognized and explained by Bass [Bas63] in a famous paper. The subsequent development of the notion is a long and interesting story. An excellent reference is the paper [Hun99] which contains an introduction to key concepts as well as providing some historical background and motivation.

There are various definitions of the Gorenstein property available. Our case involves graded modules over polynomial rings. In Section 3.2 we will describe the setting and our notion of Gorenstein in more detail, with an aim to proving Stanley's remarkable criterion (Theorem 3.2.20, see [Sta78]) which will be a vital component of the work in Chapter

4. In Section 3.3 we will also show that analysis of the Gorenstein property for rings of quasi-invariants in the plane reduces to analysis of the Poincaré series.

Let us first review the most familiar definition of the Gorenstein property. Let R be a ring and let M be an R-module. Let inj $\dim_R M$ denote the injective dimension of M.

Definition 3.2.1. *A Noetherian local ring R is a Gorenstein ring if inj $\dim_R R < \infty$. A Noetherian ring is a Gorenstein ring if its localization at every maximal ideal is a Gorenstein local ring.*

Now we turn to the graded case. We present a condensed introduction to the Gorenstein property. All results in Section 3.2 are taken from Benson [Ben93] unless stated otherwise.

Let K be an algebraically closed field of characteristic zero and let $A = K[x_1, x_2, \dots, x_s]$ be a polynomial ring in s indeterminates, graded by degree. Let M be a graded, finitely generated A-module. Then we have the following, which is essentially Hilbert's Syzygy Theorem:

Theorem 3.2.2. *Any graded A-module M is projective if and only if it is free. Moreover M has a unique (up to isomorphism of chain complexes) free resolution of the form*

$$0 \to F_s \to F_{s-1} \to \dots \to F_0 \to M \to 0.$$

We now define the *homological dimension* $hdim_A(M)$ of a graded A-module M to be the minimal length of a projective resolution of M. The above theorem states that every graded A module has homological dimension at most equal to s. We also have

Theorem 3.2.3 ([BH93], Theorem 1.3.1)**.** *Let*

$$\mathfrak{F} : 0 \to F_s \to F_{s-1} \to \dots \to F_0 \to M \to 0$$

be a free A resolution of M. Then TFAE

- *The resolution $\mathfrak{F}$ is minimal.*
- *rank $F_i = dim_K Tor_i^A(M, K) \forall i \geq 0$*
- *rank $F_i = dim_K Ext_A^i(M, K) \forall i \geq 0$.*

In other words, $hdim_A(M)$ is equal to the largest value of t for which $Tor_t^A(M, K)$ or $Ext_A^t(M, K)$ is non-zero.

Example 3.2.4. *Let* $A = \mathbb{C}[x,y]$ *and let* $M = \mathbb{C}[x,y]/\langle x^2, y^2\rangle$. *We claim that* $hdim_A(M) = 2$. *To see this, recall that we can calculate* $Tor_i^R(M,\mathbb{C})$ *by resolving in either variable. Using the Koszul complex we have the following minimal free resolution of* $\mathbb{C} = \mathbb{C}[x,y]/\langle x,y\rangle$*:*

$$0 \to A \xrightarrow{\begin{pmatrix} y \\ x \end{pmatrix}} A \oplus A \xrightarrow{\begin{pmatrix} -x & y \end{pmatrix}} A \to \mathbb{C} \to 0.$$

Applying the functor $-\otimes_A M$ *to this resolution we obtain*

$$0 \to A \otimes_A M \xrightarrow{\begin{pmatrix} y \\ x \end{pmatrix}} (A \oplus A) \otimes_A M \xrightarrow{\begin{pmatrix} -x & y \end{pmatrix}} A \otimes_A M \to 0$$

giving

$$0 \to M \xrightarrow{\begin{pmatrix} y \\ x \end{pmatrix}} M \oplus M \xrightarrow{\begin{pmatrix} -x & y \end{pmatrix}} M \to 0. \tag{3.1}$$

Now we can calculate $Tor_2^A(M,\mathbb{C})$*, which is the homology of the complex (3.1) at the left* M *entry. It is easy to see this is given by* $\{\lambda xy + \langle x^2, y^2\rangle | \lambda \in \mathbb{C}\}$*, which is in fact the socle of* M*. So* $dim_{\mathbb{C}} Tor_2^A(M,\mathbb{C}) = 1 = rank_A A$ *and thus* $hdim_A(M) = 2$

We now turn to a brief discussion of depth and Cohen-Macaulay rings.

Definition 3.2.5. *If* A *is a commutative Noetherian ring and* M *is a finitely generated* A*-module, an element* $a \in A$ *is* regular *for* M *if* $0 \neq M \neq aM$ *and* a *is not a zero divisor on* M*. A sequence* $\theta_1, \theta_2, \ldots, \theta_r$ *is a* regular sequence *for* M *if each* θ_i *is regular for* $M/(\theta_1 M + \ldots + \theta_{i-1} M)$*. The* depth *of* M *is the length of the longest regular sequence for* M*. The depth of* A *is its depth as an* A *module. The ring* A *or module* M *is* Cohen-Macaulay *if its depth is equal to its Krull dimension.*

Example 3.2.6. *Let* $A = \mathbb{C}[x_1, x_2, \ldots, x_s]$*. Then* $x_1, x_2, \ldots, x_s$ *is a regular sequence for* A*. In fact, this is the longest possible regular sequence, so* A *has depth* s*. Moreover,* A *has Krull dimension* s*, and so* A *is Cohen-Macaulay.*

We now mention another characterization of Cohen-Macaulay in the case where A and M are graded and A is finitely generated over a field K by elements of positive degree. Let $Ann_A(M) = \{a \in A \mid am = 0 \,\forall\, m \in M\}$.

Theorem 3.2.7. *TFAE*

(i) M is Cohen-Macaulay.

(ii) There exist homogeneous elements $x_1, \ldots, x_n$ (here $n = \dim M$) generating a polynomial subring $K[x_1, \ldots, x_n] \subset A/Ann_A(M)$, such that M is a finitely generated free module over $K[x_1, \ldots, x_n]$.

(iii) Whenever $x_1, \ldots, x_n$ are homogeneous elements (here $n = \dim M$) generating a polynomial subring $K[x_1, \ldots, x_n] \subset A/Ann_A(M)$ over which M is a finitely generated, M is a free $K[x_1, \ldots, x_n]$ module.

Example 3.2.8. *In light of the above it is evident polynomial rings (with the variables in positive degree) are Cohen-Macaulay.*

We now move to a homological characterization of depth.

Theorem 3.2.9. *The depth of M is equal to the* homological codimension $hcodim_A(M)$ *which is defined to be the smallest $n \geq 0$ for which $Ext^n_A(K, M) \neq 0$, where $K = A/\oplus_{i>0} A_i$.*

Example 3.2.10. *Let $A = \mathbb{C}[x, y]$ and let $M = \mathbb{C}[x, y]/\langle x^2, y^2\rangle$. It is clear the depth(M)=0. We confirm that M has $hcodim_A(M) = 0$. Recall that to calculate $Ext^i_A(\mathbb{C}, M)$ we should first take an injective resolution of M. However we can also start with a free resolution of $\mathbb{C}$. So, consider the minimal free resolution of $\mathbb{C}$ given by the Koszul complex:*

$$0 \to A \xrightarrow{\begin{pmatrix} y \\ x \end{pmatrix}} A \oplus A \xrightarrow{\begin{pmatrix} -x & y \end{pmatrix}} A \to \mathbb{C} \to 0.$$

We apply $Hom_A(-, M)$ to this resolution to obtain

$$0 \leftarrow Hom_A(A, M) \xleftarrow{\begin{pmatrix} y & x \end{pmatrix}} Hom_A(A \oplus A, M) \xleftarrow{\begin{pmatrix} -x \\ y \end{pmatrix}} Hom_A(A, M) \leftarrow 0$$

giving

$$0 \leftarrow M \xleftarrow{\begin{pmatrix} y & x \end{pmatrix}} M \oplus M \xleftarrow{\begin{pmatrix} -x \\ y \end{pmatrix}} M \leftarrow 0. \tag{3.2}$$

Thus, $Ext^0_A(\mathbb{C}, M)$, which is the homology of (3.2) at the right hand M entry is given by $\{\lambda xy + \langle x^2, y^2\rangle | \lambda \in \mathbb{C}\}$. So $hcodim_A(M) = 0$. Notice that $hdim_A(M) + hcodim_A(M) = 2$, the number of generators of A. This is no accident, as we will see in the following theorem.

Theorem 3.2.11. *If M is a finitely generated graded module over a graded polynomial ring $A = K[x_1, x_2, \ldots, x_s]$, with each x_i is positive degree, then*

$$hdim_A(M) + hcodim_A(M) = s.$$

This theorem has the following useful corollary.

Corollary 3.2.12. *Let M be a Cohen-Macaulay graded module over $A = K[x_1, x_2, \ldots, x_s]$ and denote by dim(M) the Krull dimension of M. Then*

$$hdim_A(M) = s - dim(M).$$

In other words, the minimal free resolution of M stops after $s - dim(M)$ terms.

Example 3.2.13. *Let $A = \mathbb{C}[x, y]$ and let $M = \mathbb{C}[x, y]/\langle x^2, y^2\rangle$. We have the following minimal free resolution of M of length $2 - 0 = 2$ terms:*

$$0 \to A \xrightarrow{\begin{pmatrix} y^2 \\ x^2 \end{pmatrix}} A \oplus A \xrightarrow{\begin{pmatrix} -x^2 & y^2 \end{pmatrix}} A \to M \to 0.$$

Theorem 3.2.11 is the key step in establishing the following.

Theorem 3.2.14. *If M is a Cohen-Macaulay graded module of Krull dimension n for a graded polynomial ring $A = K[x_1, x_2, \ldots, x_s]$ with the x_i in positive degree then $Ext^i_A(M, A) = 0$ for $i \neq s - n$.*

Example 3.2.15. *We saw already that if $A = \mathbb{C}[x, y]$, $M = \mathbb{C}[x, y]/\langle x^2, y^2\rangle$ then*

$$0 \to A \xrightarrow{\begin{pmatrix} y^2 \\ x^2 \end{pmatrix}} A \oplus A \xrightarrow{\begin{pmatrix} -x^2 & y^2 \end{pmatrix}} A \to M \to 0$$

is a minimal free resolution of M. Applying $Hom_A(-, A)$ yields

$$0 \leftarrow Hom_A(A, A) \xleftarrow{\begin{pmatrix} y^2 & x^2 \end{pmatrix}} Hom_A(A \oplus A, A) \xleftarrow{\begin{pmatrix} -x^2 \\ y^2 \end{pmatrix}} Hom_A(A, A) \leftarrow 0$$

giving

$$0 \leftarrow A \xleftarrow{\begin{pmatrix} y^2 & x^2 \end{pmatrix}} A \oplus A \xleftarrow{\begin{pmatrix} -x^2 \\ y^2 \end{pmatrix}} A \leftarrow 0.$$

We can now calculate $Ext^i_A(M, A)$. *Firstly* $Ext^0_A(M, A) = 0$ *since the kernel of the right hand map is zero. Now, we claim that* $Ext^1_A(M, A) = 0$. *Denote by* $(y^2 : x^2)$ *the annihilator of* x^2 *modulo* y^2*, that is the set* $\{b \in A | \exists a \in A$ *such that* $ay^2 + bx^2 = 0\}$. *Note that an element* $(a, b) \in A \oplus A$ *is in the kernel of the middle map if and only if* $y^2a + x^2b = 0$. *This requires* $b \in (y^2 : x^2)$. *If* $b \in (y^2 : x^2)$ *there is an element* a *with* $x^2b + y^2a = 0$ *so that* (a, b) *will be in the kernel. Since* x^2 *is a nonzerodivisor* a *is uniquely determined by* b *and the association* $b \mapsto a$ *is a module homomorphism. So, the kernel is isomorphic to* $(y^2 : x^2)$. *Now an element is in the image of the right hand map if it is of the form* $(-cx^2, cy^2)$. *So the elements of* $(y^2 : x^2)$ *which correspond to elements of the image are the elements of* $\langle y^2 \rangle$. *So* $Ext^1_A(M, A) = (y^2 : x^2)/\langle y^2 \rangle = 0$*, since* x^2 *is not a zerodivisor modulo* y^2*, since* x^2, y^2 *is a regular sequence. Finally* $Ext^2_A(M, A) = A/\langle x^2, y^2 \rangle = M$. *(What we have described is an intrinsic property of the Koszul complex:* $H^1(M, A)$ *vanishes since* x^2, y^2 *is a regular sequence. Thus the Koszul complex can be used to determine the regularity of a sequence.)*

Now, suppose R is a commutative Cohen-Macaulay graded ring of dimension n which is finitely generated as an algebra over a field K by homogeneous elements $x_1, x_2, \ldots, x_s$ of positive degree. It is not hard to show that R is also a graded Cohen-Macaulay module for $A = K[x_1, x_2, \ldots, x_s]$. In light of this and Theorem 3.2.14 we introduce the following.

Definition 3.2.16. *The* canonical module *of* R *is the* A*-module*

$$\omega_A(R) = Ext^{s-n}_A(R, A).$$

In fact, since the kernel of $A \to R$ acts as zero on the left hand variable of $Ext^{s-n}_A(R, A)$, it acts as zero on the A-module $\omega_A(R)$ so we may regard $\omega_A(R)$ as an R-module. Furthermore it is not hard to show that the canonical module is actually independent of the choice of A, so we simply write $\omega(R)$.

Definition 3.2.17. *If* R *is a commutative Cohen-Macaulay graded ring which is finitely generated over a field* K *by homogeneous elements of positive degree, we say that* R *is* Gorenstein *if*

$$\omega(R) \cong R.$$

Example 3.2.18. *Let* $A = \mathbb{C}[x, y]$ *and let* $M = \mathbb{C}[x, y]/\langle x^2, y^2 \rangle$. *We saw already (Example 3.2.15) that*

$$\omega(M) = Ext^{2-0}_A(M, A) = M$$

and thus M is Gorenstein.

Definition 3.2.19. *Let $R = \oplus_{j\geq 0} R_j$ be a graded ring, so R_j is its graded component of degree j. The Poincare series of R is given by:*

$$P_R(t) = \sum_{j\geq 0} dim R_j t^j$$

where R_j denotes the graded component of R of degree j.

Next we present a remarkable criterion, due to Stanley, which gives a condition for when certain Cohen-Macaulay rings are Gorenstein.

Theorem 3.2.20 ([Sta78, Theorem 4.4])**.** *Let R be a Noetherian graded commutative Cohen-Macaulay domain. Suppose also that R is a K-algebra for a field K. Then R is Gorenstein if and only if its Poincaré series is palindromic, that is if:*

$$P_R(t^{-1}) = (-1)^d t^l P_R(t) \quad \text{where } d = dim(R) \text{ and } l \in \mathbb{Z}.$$

Proof. We reproduce the proof for the convenience of the reader. First we fix some notation. Let $y_1, y_2, \ldots y_s$ be a set of homogeneous generators for R. Let $\deg y_i = e_i \geq 1$. Let $A = K[Y_1, \ldots Y_s]$ be a polynomial ring over K in s independent variables $Y_i, 1 \leq i \leq s$. If we define $\deg Y_i = e_i$ then A has the structure of a graded K-algebra and there is a canonical degree preserving surjection $p : A \to R$ defined by $p(Y_i) = y_i$.

Suppose first that R is Gorenstein. By Theorem 3.2.2 and Corollary 3.2.12 we see that R has a minimal free resolution as an A-module which looks like

$$0 \to M_h \to \ldots \to M_1 \to M_0 \to R \to 0 \tag{3.3}$$

with $h = s - d$, where d is the Krull dimension of R. We can choose (3.3) so that each M_i is graded and all the homomorphisms preserve degree. Suppose M_i has free homogeneous generators $X_{i1}, \ldots X_{i\beta_i}$ with $\deg X_{ij} = g_{ij}$. Since R is Gorenstein, $M_h \cong A$. This essentially follows from Theorem 3.2.3, see also [Hun99, Proposition 3.2]. Moreover a result of Buchsbaum and Eisenbud [BE77, Theorem 1.5] shows that there is a degree preserving pairing $M_i \otimes M_{h-i} \to M_h \cong A$ which induces an isomorphism $M_i \to M_{h-i}^* = \mathrm{Hom}_A(M_{h-i}, A)$. Let $g_{h1} = g$. It follows that we can label the generators X_{ij} and $X_{h-i,j}$ so that $g_{ij} + g_{h-i,j} = g$ for $1 \leq j \leq \beta_i = \beta_{h-i}$. Then if we denote by $P_{M_i}(t)$ the Poincaré series for M_i we have

$$P_{M_i}(t) = \frac{\sum_{j=1}^{\beta_i} t^{g_{ij}}}{\prod_{j=1}^{s}(1 - t^{e_j})}. \tag{3.4}$$

Using a well-known property of exact sequences we obtain from (3.3) that

$$P_R(t) = P_{M_0}(t) - P_{M_1}(t) + \ldots + (-1)^h P_{M_h}(t). \tag{3.5}$$

Combining (3.4) and (3.5) we get an explicit expression for $P_R(t)$. Using the fact that $g_{ij} + g_{h-i,j} = g$ we see that $P_R(1/t) = (-1)^{s-h} t^\rho P_R(t) = (-1)^d t^\rho P_R(t)$ where $\rho = \sum e_i - g$.

Now suppose that $P_R(t^{-1}) = (-1)^d t^l P_R(t)$. We are going to show that $\omega(R)$ can be graded as an R-module so that for some $q \in \mathbb{Z}$

$$P_{\omega(R)}(t) = (-1)^d t^q P_R(1/t). \tag{3.6}$$

Let $*$ denote the functor $\mathrm{Hom}_A(-, A)$. Applying $*$ to (3.3) and using Theorem 3.2.14 we obtain

$$0 \to M_0^* \to \ldots \to M_{h-1}^* \to M_h^* \to \omega(R) \to 0 \tag{3.7}$$

as a minimal free resolution of $\omega(R)$. As A-modules we have $M_i^* \cong M_i$. Let $X_{i1}^*, \ldots, X_{i\beta_i}^*$ be the basis of M_i^* dual to the basis $X_{i1}, \ldots, X_{i\beta_i}$ of M_i. Setting $\deg X_{ij}^* = -g_{ij}$, the homomorphisms in (3.7) will be degree preserving. Thus $\omega(R)$ obtains the structure of a graded A-module and retains this grading as an R-module. From (3.7) we have

$$P_{\omega(R)}(t) = P_{M_h^*}(t) - P_{M_{h-1}^*}(t) + \ldots + (-1)^h P_{M_0^*}(t). \tag{3.8}$$

Since $\deg X_{ij}^* = -g_{ij}$ we have

$$P_{M_i^*}(t) = \frac{\sum_{j=1}^{\beta_i} t^{-g_{ij}}}{\prod_{j=1}^{s}(1 - t^{e_j})}. \tag{3.9}$$

We can now combine (3.8), (3.9), (3.4) and (3.5) to get (3.6). For simplicity we shift the grading of $\omega(R)$ so that the least degree of a non-zero element is zero. It follows from (3.6) and our shift of $\omega(R)$ that the elements of $\omega(R)$ of degree 0 form a vector space over K of dimension one. Let x be a non-zero element of $\omega(R)$ of degree 0. We use the result [BH93, Corollary 3.3.19] that since R is an integral domain, $\omega(R)$ is isomorphic to an ideal of R. We identify $\omega(R)$ with this ideal. Denote by R_n the n-th graded part of R and by ω_n the n-th graded part of $\omega(R)$. Since R is a domain we have $\dim_k xR_n = \dim_k R_n$. Also $\dim_k R_n = \dim_k \omega_n$, since by (3.6) $P_{\omega(R)}(t) = P_R(t)$. However by the definition of a graded R-module, $xR_n \subset \omega_n$. So we have $xR_n = \omega_n$. Thus $R \cong xR = \omega(R)$ and so R is Gorenstein. □

3.3 The Cohen-Macaulay property for rings of quasi-invariants in the plane

In this section we will show that analysis of the Gorenstein property for rings of quasi-invariants in the plane reduces to analysis of the Poincaré series. First let us introduce the ring of quasi-invariants.

Let A be a finite set of non-collinear vectors $(\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{C}^2$ such that $(\alpha_i, \alpha_i) \neq 0 \,\forall\, \alpha_i \in A$. Let m be a function $m : A \to \mathbb{N}$, the multiplicity function. Say $\alpha_i \in A$ has multiplicity m_i under this function. We refer to the pair (A, m) as the configuration $\mathcal{A}$; the configuration is then a collection of vectors with multiplicities.

Definition 3.3.1. *We say a polynomial $P \in \mathbb{C}[x_1, x_2]$ is quasi-invariant with respect to $\mathcal{A}$ if, for all $\alpha_i \in A$ and $s = 1, 3, \ldots, 2m_i - 1$:*

$$\partial_{\alpha_i}^s P = 0$$

on the hyperplane $\Pi_i : (\alpha_i, x) = 0$, where $\partial_{\alpha_i} = (\alpha_i, \partial_x)$.

These polynomials form a ring, the *quasi-invariant ring* $Q_{\mathcal{A}}$. We now turn to the key result of this section.

Theorem 3.3.2. *Let $\mathcal{A} = (A, m)$ be any configuration of vectors in $\mathbb{C}^2$ with some multiplicities. Let $Q_{\mathcal{A}}$ be the corresponding ring of quasi-invariants. Then $Q_{\mathcal{A}}$ is Cohen-Macaulay.*

Proof. Let us first introduce the polynomials

$$P_1 = \prod_{\alpha_i \in A} (\alpha_i, x)^{2m_i}$$

and

$$P_2 = x_1^2 + x_2^2 = (x_1 - ix_2)(x_1 + ix_2)$$

It is very easy to check that $P_1, P_2 \in Q_{\mathcal{A}}$. Since we are assuming that $(\alpha_j, \alpha_j) \neq 0 \,\forall\, \alpha_j \in A$ we have $(1, i), (1, -i) \notin A$, and so P_2 and P_1 have only $x_1 = x_2 = 0$ as a common zero.

Let I be the ideal generated by P_1, P_2. The dimension of the quotient $\mathbb{C}[x_1, x_2]/I$ is finite. This follows from standard results about isolated zeros of analytic maps [AVGZ85]. Thus the ring $\mathbb{C}[x_1, x_2]$ is finitely generated as a module over $\mathbb{C}[P_1, P_2]$. Since $\mathbb{C}[P_1, P_2]$ is Noetherian $Q_{\mathcal{A}}$ is finitely generated as a module over $\mathbb{C}[P_1, P_2]$. We also know that $Q_{\mathcal{A}}$ is a finitely generated algebra. This follows from the Hilbert-Noether Lemma [AM69, Proposition 7.8].

We are going to prove that $Q_{\mathcal{A}}$ is in fact free as a module over $\mathbb{C}[P_1, P_2]$. Let $q_1, \dots, q_N$ be a generating set for $Q_{\mathcal{A}}$ as a $\mathbb{C}[P_1, P_2]$ module. We claim that we can choose $q_1, \dots, q_N$ to be homogeneous with the following property: $q_i \neq \sum_{j \neq i} \widehat{\sigma}_j(P) q_j$, where $\widehat{\sigma}_j(P) \in \mathbb{C}[P_1, P_2]$. Indeed, we can construct $q_1, \dots, q_N$ inductively. Let $Q_{\mathcal{A}} = \sum_s Q_{\mathcal{A}_s}$ where $Q_{\mathcal{A}_s}$ is the graded component of $Q_{\mathcal{A}}$ of degree s. Let us take $q_1 = 1$ and suppose we have constructed $q_1, \dots, q_k$. Consider $Q^{(k)} = \{\sum_{i=1}^k \sigma_i(P) q_i | \sigma_i(P) \in \mathbb{C}[P_1, P_2]\}$. We take the lowest graded component $Q_{\mathcal{A}_{l_k}}$ such that $Q_{\mathcal{A}_{l_k}} \supsetneq Q^{(k)}_{\mathcal{A}_{l_k}}$ where $Q^{(k)}_{\mathcal{A}_{l_k}}$ is the graded component of degree l_k inside $Q^{(k)}$. We then extend $q_1, \dots, q_k$ by adding a basis $q_{k+1}, \dots, q_{k+s_k}$ in the complement of $Q^{(k)}_{\mathcal{A}_{l_k}}$ in $Q_{\mathcal{A}_{l_k}}$, where $s_k = \dim Q_{\mathcal{A}_{l_k}} / Q^{(k)}_{\mathcal{A}_{l_k}}$. Then we continue the process.

It is clear from the construction that q_i obtained in this way are homogeneous and satisfy the property $q_i \neq \sum_{j \neq i} \widehat{\sigma}_j(P) q_j$, where $\widehat{\sigma}_j(P) \in \mathbb{C}[P_1, P_2]$. To see that this generating set is indeed finite, recall that $Q_{\mathcal{A}}$ is finitely generated as a module over $\mathbb{C}[P_1, P_2]$ and is thus a noetherian $R = \mathbb{C}[P_1, P_2]$-module. By construction the $Q^{(k)}$ are $\mathbb{C}[P_1, P_2]$-submodules, and $q_{k+1} \notin Q^{(k)}$, so the chain

$$R = Q^{(1)} \subset Q^{(2)} \subset Q^{(3)} \subset \dots \tag{3.10}$$

is strictly ascending. By noetherianity it must be finite.

Now we are ready to show that the generating set $q_1 \dots, q_N$ is a basis. Suppose we have

$$\sum_{j=1}^N \sigma_j(P) q_j = 0 \tag{3.11}$$

where $\sigma_j(P) \in \mathbb{C}[P_1, P_2]$ for $j = 1, 2, \dots, N$ and $\sigma_k(P) \neq 0$ for some k. Since the q_j, $1 \leq j \leq N$ are homogeneous we may consider the relation expressed by (3.11) of least degree, that is, we choose σ_j, q_j such that the overall degree of the polynomial in x expressed by the left hand side is minimal, giving another expression

$$\sum_{j=1}^N \tau_j(P) q_j = 0 \tag{3.12}$$

where $\tau_j(P) \in \mathbb{C}[P_1, P_2]$ and not all of the $\tau_j(P)$ are 0. Notice that since $\{q_1, q_2, \dots, q_N\}$ are by construction linearly independent over $\mathbb{C}$, we can moreover say that the degree of the homogeneous polynomials $\tau_j(P)$ is strictly positive. Let us represent the polynomials $\tau_j(P)$ in (3.12) as linear combinations of monomials in P_1, P_2 and move terms containing P_2 to the right hand side. So the right hand side is divisible by P_2. Since we are assuming that the set of generators $q_1, \dots, q_N$ is minimal in the sense described above we have

$$P_1(\sum_i \widetilde{\sigma}_i q_i) = P_2(\sum_i \widetilde{\widetilde{\sigma}}_i q_i) \tag{3.13}$$

where $\widetilde{\sigma}_i$, $\widetilde{\widetilde{\sigma}}_i \in \mathbb{C}[P_1, P_2]$. We know that $P_2 \mid P_1(\sum_i \widetilde{\sigma}_i q_i)$. Thus $(x_1 \pm ix_2) \mid P_1(\sum_i \widetilde{\sigma}_i q_i)$. However $(x_1 \pm ix_2) \nmid P_1$ so $(x_1 \pm ix_2) \mid \sum_i \widetilde{\sigma}_i q_i$ and thus $P_2 \mid \sum_i \widetilde{\sigma}_i q_i$ and thus $\sum_i \widetilde{\sigma}_i q_i = P_2 \widehat{q}$. It is easy to see that $\widehat{q} \in Q_{\mathcal{A}}$. Thus $\widehat{q} = \sum_i \widehat{\sigma}_i q_i$ so $\sum_i \widetilde{\sigma}_i q_i = P_2 \sum_i \widehat{\sigma}_i q_i$, which contradicts the minimality of (3.12).

□

Combining Theorems 3.3.2 and 3.2.20, we now see that if $\mathcal{A} = (A, m)$ is a configuration of vectors in $\mathbb{C}^2$ then $Q_{\mathcal{A}}$ is Gorenstein if and only if its Poincaré series is palindromic. Thus we can approach the Gorenstein property via analysis of the Poincaré seres.

Chapter 4

Gorenstein rings of quasi-invariants in the plane

4.1 Summary

In this chapter we will investigate the structure of the ring of quasi-invariants $Q_{\mathcal{A}}$ for certain classes of arrangements $\mathcal{A}$ in $\mathbb{C}^2$. We will show that there is only one Gorenstein ring of quasi-invariants in the case where all multiplicities are one, namely that corresponding to the dihedral configuration.

We also study the class of configurations of type $(m, 1^n)$. In particular we calculate the Poincaré series for $Q_{\mathcal{A}_{(m,1^n)}}$ and use this to show that $Q_{\mathcal{A}_{(m,1^n)}}$ is Gorenstein. Furthermore, we will show that $Q_{\mathcal{A}_{(m,1^n)}}$ is the only Gorenstein ring of quasi-invariants for this class of arrangements.

Chapter 4 consists entirely of new results with the exception of the following. Section 4.2 consists of background information and as such contains no new results. Theorem 4.3.2 is taken from the literature.

4.2 Background

Let A be a finite set of non-collinear vectors $(\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{C}^N$, such that $(\alpha_i, \alpha_i) \neq 0 \, \forall \, \alpha_i \in A$. Let m be a function $m : A \to \mathbb{N}$, the multiplicity function. Say $\alpha_i \in A$ has multiplicity m_i under this function. We refer to the pair (A, m) as the configuration $\mathcal{A}$; the configuration is then a collection of vectors with multiplicities.

Definition 4.2.1. *We say a polynomial* $P \in \mathbb{C}[x_1, x_2, \dots, x_N]$ *is quasi-invariant with respect to* $\mathcal{A}$ *if, for all* $\alpha_i \in A$ *and* $s = 1, 3, \dots, 2m_i - 1$*:*

$$\partial^s_{\alpha_i} P = 0$$

on the hyperplane $\Pi_i : (\alpha_i, x) = 0$*, where* $\partial_{\alpha_i} = (\alpha_i, \partial_x)$*.*

Let $Q_\mathcal{A}$ be the ring of all quasi-invariant polynomials. Then $Q_\mathcal{A}$ is a graded ring, $Q_\mathcal{A} = \oplus_{k\geq 0} Q^{(k)}_\mathcal{A}$ where $Q^{(k)}_\mathcal{A}$ consists of homogeneous polynomials P such that $\deg P = k$. Let $P_\mathcal{A}(t)$ be the Poincaré series of the ring $Q_\mathcal{A}$,

$$P_\mathcal{A}(t) = \sum_{k=0}^{\infty} b_k t^k. \tag{4.1}$$

where $b_k = \dim Q^{(k)}_\mathcal{A}$.

4.2.1 Schur polynomials

A key ingredient in some of the proofs we present is that of elementary symmetric polynomials and Schur polynomials. For a detailed examination see [Mac98].

Definition 4.2.2. *Denote by* e_r *the* r*th elementary symmetric polynomial in the variables* $x_1, x_2, \dots, x_N$*, the sum of all products of the* r *distinct variables* x_i *so that* $e_0 = 1$ *and*

$$e_r = \sum_{1\leq i_1 < i_2 < \dots < i_r \leq N} x_{i_1} x_{i_2} \dots x_{i_r},$$

for $1 \leq r \leq N$. Let $\Lambda_N = \mathbb{Z}[x_1, x_2, \dots, x_N]^{S_N}$, the ring of symmetric polynomials in N variables. We have $\Lambda_N = \mathbb{Z}[e_1, e_2, \dots, e_N]$ and the e_i are algebraically independent. Consider the partition $\delta = (N-1, N-2, \dots, 1, 0)$ and we let λ be any partition of length at most N. Put $a_{\lambda+\delta} = \det(x_i^{\lambda_j + N - j})_{1\leq i,j\leq N}$. This determinant is divisible by the Vandermonde determinant:

$$\prod_{1\leq i<j\leq N} (x_i - x_j) = \det(x_i^{N-j}) = a_\delta.$$

Definition 4.2.3. *The quotient*

$$S_\lambda = a_{\lambda+\delta}/a_\delta$$

is called the Schur function in the variables $x_1, x_2, \dots, x_N$ *corresponding to the partition* λ*.*

It is clear that each Schur function $S_\lambda \in \Lambda_N$. Thus we can express each S_λ as a polynomial in the elementary symmetric functions e_r. We have

$$S_\lambda = \det(e_{\lambda_i'-i+j})_{1\le i,j\le N} \tag{4.2}$$

where $N \geq l(\lambda)$ and λ' denotes the conjugate partition of λ.

4.2.2 Hypergeometric series and alternating binomial sums

We will need to use some of the properties of the generalized hypergeometric series, as well as a famous identity of Gauss. The generalized hypergeometric series is given by the following, see [OLB10].

Definition 4.2.4.

$$_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;z) = \sum_{n=0}^{\infty} \frac{(a_1)_n\dots(a_p)_n}{(b_1)_n\dots(b_q)_n}\frac{z^n}{n!}$$

where $(a)_n = a(a+1)\dots(a+n-1)$.

We have the following well known identity, see [AS65].

Theorem 4.2.5 (Gauss)**.**

$$_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

for $c \neq 0,-1,-2,\dots,$ *Re*$(c-a-b) > 0$.

We will also need to use the following result.

Theorem 4.2.6 (Saalschütz, [GS85])**.**

$$_3F_2(a,b,-n;c,1+a+b-c-n;1) = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}.$$

Finally, the following simple result is well known.

Lemma 4.2.7. *Let* $n \in \mathbb{N}$. *Let* $P(x)$ *be a polynomial of* $\deg P < n$. *Then*

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}P(i) = 0. \tag{4.3}$$

Proof. This can be seen by evaluation of the identity

$$(1+x)^n = \sum_{i=0}^{n}\binom{n}{i}x^i$$

and its first $n-1$ derivatives at $x=-1$. The left hand side of the corresponding relations vanishes and (4.3) follows by taking appropriate linear combinations. □

4.3 Planar arrangements with multiplicities one

Assume for the rest of this chapter that $N = 2$. We are going to prove the following.

Theorem 4.3.1. *Suppose $m_i = 1$ for $0 \leq i \leq n$. Denote by $\mathcal{A}$ any configuration with these properties. Suppose the quasi-invariant ring $Q_{\mathcal{A}}$ is Gorenstein. Then A is a set of normals to reflection lines for the dihedral group.*

The Poincaré series for the quasi-invariant rings corresponding to dihedral arrangements in the case of constant multiplicity were found by Feigin and Veselov in [FV02].

Theorem 4.3.2 ([FV02], Theorem 8)**.** *Let $\mathcal{A}$ be the dihedral arrangement consisting of $n+1$ vectors with multiplicity one. Then we have*

$$P_{\mathcal{A}}(t) = \frac{1 + 2t^{n+2} + \ldots + 2t^{2n+1} + t^{3n+3}}{(1-t^2)(1-t^{n+1})}$$

These series are palindromic, so by establishing Theorem 4.3.1 we will have shown the following.

Theorem 4.3.3. *Suppose $m_i = 1$ for $0 \leq i \leq n$. Denote by $\mathcal{A}$ any configuration with these properties. Then the quasi-invariant ring $Q_{\mathcal{A}}$ is Gorenstein if and only if A is a set of normals to reflection lines for the dihedral group.*

4.3.1 Overview of proof

We give a rough outline of the proof of Theorem 4.3.1 for reference. The key idea is the following. We calculate the Poincaré series for a generic configuration of $n+1$ lines in the plane and show that this series can only be palindromic when the arrangement is in fact a dihedral arrangement (that is, a set of normals to reflection lines for the dihedral group.) This is tackled in a number of parts. We prove some general results about the form of the Poincaré series for any $n+1$ line configuration, in order to show that certain 'parts' of the series (this will be made explicit) do not depend on the geometry of our configuration. We then show that there are strong restrictions on the possibilities for the remaining coefficients in the Poincare series, and that choosing $b_{2n} = n+1$ (where b_{2n} is the coefficient of the term with degree $2n$ in the Poincaré series) implies that the vectors of A form a dihedral arrangement. To complete the proof we show that if $P_{\mathcal{A}}$ is palindromic then $b_{2n} = n+1$.

4.3.2 Coordinate systems

Although the key definition of quasi-invariance given above (4.2.1) is introduced through Cartesian coordinates, it is convenient to shift into the language of polar coordinates to tackle the proof of our theorem.

Lemma 4.3.4. *Let* $\alpha_i = (\cos\theta_i, \sin\theta_i) \in A$. *Let* $P(x,y) = \sum_{s=0}^{k} a_s z^s \overline{z}^{k-s}$ *be a homogeneous polynomial of degree* k. *Then* P *is quasi-invariant with respect to* $\mathcal{A}$ *if, for all* $\alpha_i \in A$ *and* $j = 1, 3, \ldots, 2m_i - 1$,

$$\sum_{s=0}^{k} a_s (2s-k)^j e^{i\theta_i(2s-k)} = 0. \tag{4.4}$$

Proof. Let $P(x,y)$ be a homogeneous polynomial of degree k. We transform (x,y) into the complex coordinates $(z,\overline{z})$ by putting

$$x = r\cos\theta,$$

$$y = r\sin\theta$$

and

$$z = re^{i\theta}$$

Then

$$P = \sum_{s=0}^{k} a_s z^s \overline{z}^{k-s} = r^k \sum_{s=0}^{k} a_s e^{is\theta} e^{-i\theta(k-s)} = r^k \sum_{s=0}^{k} a_s e^{i\theta(2s-k)}.$$

Note that

$$\partial_r = \partial_x \cos\theta + \partial_y \sin\theta,$$
$$\partial_\theta = \partial_x(-r\sin\theta) + \partial_y r\cos\theta$$

The first level quasi-invariant condition requires

$$(a\partial_x + b\partial_y)P = 0 \quad \text{on} \quad ax + by = 0$$

where $(a,b) = (-\sin\theta_i, \cos\theta_i)$. It is easily seen that:

$$\partial_x = \cos\theta\partial_r - \frac{1}{r}\sin\theta\partial_\theta,$$
$$\partial_y = \sin\theta\partial_r + \frac{1}{r}\cos\theta\partial_\theta$$

so

$$\begin{aligned} a\partial_x + b\partial_y &= -\sin\theta_i(\cos\theta\partial_r - \frac{1}{r}\sin\theta\partial_\theta) + \cos\theta_i(\sin\theta\partial_r + \frac{1}{r}\cos\theta\partial_\theta) \\ &= (-\sin\theta_i\cos\theta + \cos\theta_i\sin\theta)\partial_r + \frac{1}{r}(\sin\theta_i\sin\theta + \cos\theta_i\cos\theta)\partial_\theta \\ &= \sin(\theta-\theta_i)\partial_r + \frac{1}{r}\cos(\theta-\theta_i)\partial_\theta. \end{aligned} \tag{4.5}$$

We want to consider what happens on the line $\theta = \theta_i$. In this case the right hand side of (4.5) reduces to $\frac{1}{r}\partial_\theta$. Then we can see that

$$(a\partial_x + b\partial_y)P \mid_{ax+by=0} = \frac{1}{r}\partial_\theta P \mid_{\theta=\theta_i} .$$

Finally then, the first level quasi-invariant condition takes the form

$$\partial_\theta \sum_{s=0}^{k} a_s e^{i\theta(2s-k)} \mid_{\theta=\theta_i} = 0 \tag{4.6}$$

giving

$$\sum_{s=0}^{k} a_s(2s-k)e^{i\theta_i(2s-k)} = 0$$

and it is clear that we apply (4.6) repeatedly to obtain the higher odd derivatives, giving

$$\sum_{s=0}^{k} a_s(2s-k)^j e^{i\theta_i(2s-k)} = 0 \tag{4.7}$$

for $j = 1, 3, \ldots, 2m_i - 1$. □

4.4 Structure of the Poincaré series

We are going to study the Poincaré series (4.1) for a configuration of lines with all multiplicities equal to one. First we prove a useful result which can be established in a more general situation. Let $\mathcal{A}$ be a collection of $n+1$ vectors on the plane with arbitrary multiplicities. Then we have the following.

Lemma 4.4.1. *Let $k < n+1$. Then in the Poincaré series (4.1) the following hold:*

- *If k is even, $b_k = 1$.*
- *If k is odd, $b_k = 0$.*

Proof. Suppose first that $m_i = 1\ \forall i = 0, 1\ldots, n$. Let P be a homogeneous polynomial of odd degree k, with $k < n+1$. Our quasi-invariant condition for P takes the form

$$\sum_{s=0}^{k} a_s(2s-k)e^{i\theta_i(2s-k)} = 0 \tag{4.8}$$

for all θ_i with $0 \leq i \leq n$. Expressing these conditions in matrix form we have the equation

$$\begin{pmatrix} -ke^{-ki\theta_0} & (2-k)e^{(2-k)i\theta_0} & (4-k)e^{(4-k)i\theta_0} & \dots & ke^{ki\theta_0} \\ -ke^{-ki\theta_1} & (2-k)e^{(2-k)i\theta_1} & (4-k)e^{(4-k)i\theta_1} & \dots & ke^{ki\theta_1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -ke^{-ki\theta_n} & (2-k)e^{(2-k)i\theta_n} & (4-k)e^{(4-k)i\theta_n} & \dots & ke^{ki\theta_n} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{4.9}$$

Now let

$$e^{i\theta_j} = c_j \tag{4.10}$$

for $0 \leq j \leq n$. Put

$$A = \begin{pmatrix} -ke^{-ki\theta_0} & (2-k)e^{(2-k)i\theta_0} & (4-k)e^{(4-k)i\theta_0} & \dots & ke^{ki\theta_0} \\ -ke^{-ki\theta_1} & (2-k)e^{(2-k)i\theta_1} & (4-k)e^{(4-k)i\theta_1} & \dots & ke^{ki\theta_1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -ke^{-ki\theta_n} & (2-k)e^{(2-k)i\theta_n} & (4-k)e^{(4-k)i\theta_n} & \dots & ke^{ki\theta_n} \end{pmatrix}. \tag{4.11}$$

Then using (4.10) we have

$$A = \begin{pmatrix} -kc_0^{-k} & (2-k)c_0^{(2-k)} & (4-k)c_0^{(4-k)} & \dots & kc_0^k \\ -kc_1^{-k} & (2-k)c_1^{(2-k)} & (4-k)c_1^{(4-k)} & \dots & kc_1^k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -kc_n^{-k} & (2-k)c_n^{(2-k)} & (4-k)c_n^{(4-k)} & \dots & kc_n^k \end{pmatrix}.$$

Let us multiply the ith row of A by c_i^k. Under these elementary row operations A is transformed to the following matrix

$$B = \begin{pmatrix} -k & (2-k)c_0^2 & (4-k)c_0^4 & \dots & kc_0^{2k} \\ -k & (2-k)c_1^2 & (4-k)c_1^4 & \dots & kc_1^{2k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -k & (2-k)c_n^2 & (4-k)c_n^4 & \dots & kc_n^{2k} \end{pmatrix}.$$

Put $c_i^2 = d_i$. Then we can rewrite (4.9) as

$$B \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where

$$B = \begin{pmatrix} -k & (2-k)d_0 & (4-k)d_0^2 & \dots & kd_0^k \\ -k & (2-k)d_1 & (4-k)d_1^2 & \dots & kd_1^k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -k & (2-k)d_n & (4-k)d_n^2 & \dots & kd_n^k \end{pmatrix}. \tag{4.12}$$

Now, suppose the columns of B are linearly dependent. Then we have $r_i B_i = 0$ for some $r_i \in \mathbb{C}$, not all zero, where the B_i denote the columns of B. Let $p(x)$ be the polynomial

$$p(x) = r_0(-k) + r_1(2-k)x + r_2(4-k)x^2 + \ldots + r_k k x^k.$$

By inspection in (4.12) we can see that p has the $n+1$ distinct solutions $d_0, d_1, \ldots, d_n$. However the degree of the polynomial p is strictly less than $n+1$ so this is possible only if p is zero. We conclude that the columns of B are linearly independent and thus A has full rank, so $\mathrm{rk}A = k+1$. Thus for odd k, $b_k = (k+1) - (k+1) = 0$.

If k is even the situation is slightly different. In this case non-trivial dependence of the columns of B is possible but we still have to have $p(x) = 0$, so $r_i = 0$ for $i \neq k/2$. Hence $\mathrm{rk}A = k$. In order to establish the statement for any m_i it is sufficient to present a non-trivial quasi-invariant in the even degrees k. Such a quasi-invariant is given explicitly as $P = (z\bar{z})^{k/2}$. □

Assume that $m_i = 1$ $\forall i = 1, 2, \ldots, n+1$.

Lemma 4.4.2. *Let $k \geq n+1$. Then in (4.1) b_k can take only two possible values. They are:*

- *$b_k = k - n$ or*
- *$b_k = k + 1 - n$.*

Further, $b_k = k - n$ if k is odd or k is even such that $k \geq 2(n+1)$.

Proof. Consider first the case k is odd. Let P be a homogeneous polynomial of odd degree k, with $k \geq n+1$. Our quasi-invariant condition for P takes the form

$$\sum_{s=0}^{k} a_s(2s-k)e^{i\theta_i(2s-k)} = 0$$

for all θ_i with $0 \leq i \leq n$. For θ_0, this gives the expression

$$a_0(-k)e^{-ki\theta_0} + a_1(2-k)e^{(2-k)i\theta_0} + a_2(4-k)e^{(4-k)i\theta_0} + \ldots + a_k(k)e^{ki\theta_0} = 0$$

and a similar expression holds for the other values of θ_i. Expressing these conditions in matrix form we have the equation

$$M \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where

$$M = \begin{pmatrix} -ka_0^{-k} & (2-k)a_0^{(2-k)} & (4-k)a_0^{(4-k)} & \ldots & ka_0^k \\ -ka_1^{-k} & (2-k)a_1^{(2-k)} & (4-k)a_1^{(4-k)} & \ldots & ka_1^k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -ka_n^{-k} & (2-k)a_n^{(2-k)} & (4-k)a_n^{(4-k)} & \ldots & ka_n^k \end{pmatrix} \tag{4.13}$$

with $a_j = e^{i\theta_j}$, $0 \leq j \leq n$. Multiplying the ith row of M by a_i^k we see that M is row equivalent to

$$\widehat{M} = \begin{pmatrix} -k & (2-k)c_0 & (4-k)c_0^2 & \ldots & kc_0^k \\ -k & (2-k)c_1 & (4-k)c_1^2 & \ldots & kc_1^k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -k & (2-k)c_n & (4-k)c_n^2 & \ldots & kc_n^k \end{pmatrix}$$

where $c_j = a_j^2$, $0 \leq j \leq n$. Consider the following submatrix of $\widehat{M}$

$$N = \begin{pmatrix} -k & (2-k)c_0 & (4-k)c_0^2 & \ldots & (2n-k)c_0^n \\ -k & (2-k)c_1 & (4-k)c_1^2 & \ldots & (2n-k)c_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -k & (2-k)c_n & (4-k)c_n^2 & \ldots & (2n-k)c_n^n \end{pmatrix}.$$

Considering possible linear dependence in the columns of this matrix N produces a polynomial of degree at most n with $n+1$ solutions which is not possible unless the polynomial is zero, so that N has rank $n+1$ if k is odd. Thus the rank of the matrix M is $n+1$ in this case. We also note that the same arguments apply if $k/2 \geq n+1$, so in this case we also have $\text{rk}M = n+1$. However if k is even and $k/2 \leq n$ then the terms with coefficient $a_{k/2}$ in the quasi-invariant condition for each vector appear alongside zero and thus when k is even the matrix for the system of linear equations corresponding to the quasi-invariance conditions can be transformed to

$$\widetilde{M} = \begin{pmatrix} -k & (2-k)c_0 & (4-k)c_0^2 & \dots & -2c_0^{k/2-1} & 0 & 2c_0^{k/2+1} & \dots & kc_0^k \\ -k & (2-k)c_1 & (4-k)c_1^4 & \dots & -2c_1^{k/2-1} & 0 & 2c_1^{k/2+1} & \dots & kc_1^k \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -k & (2-k)c_n & (4-k)c_n^2 & \dots & -2c_n^{k/2-1} & 0 & 2c_n^{k/2+1} & \dots & kc_n^k \end{pmatrix}.$$

In this case we should consider the following submatrix of $\widetilde{M}$:

$$\widetilde{N} = \begin{pmatrix} -k & (2-k)c_0 & (4-k)c_0^2 & \dots & -2c_0^{k/2-1} & 0 & 2c_0^{k/2+1} & \dots & (2n-k)c_0^n \\ -k & (2-k)c_1 & (4-k)c_1^2 & \dots & -2c_1^{k/2-1} & 0 & 2c_1^{k/2+1} & \dots & (2n-k)c_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -k & (2-k)c_n & (4-k)c_n^2 & \dots & -2c_n^{k/2-1} & 0 & 2c_n^{k/2+1} & \dots & (2n-k)c_n^n \end{pmatrix}$$

and proceed as above to see that $\widetilde{N}$ has rank n and thus $\widetilde{M}$ has rank at least n. Thus the rank of the matrix for the system of linear equations coming from the quasi-invariance conditions is either n or $n+1$, and we are done. □

In light of Lemma 4.4.2 we will say that a configuration of $n+1$ vectors is generic if $b_k = k - n$ for even k with $n+1 \leq k \leq 2n$.

4.5 How the terms of the Poincaré series affect the configuration

Assume $m_i = 1 \,\forall i$ in Section 4.5. Lemmas 4.4.1 and 4.4.2 fix all the coefficients in the Poincaré series (4.1) for an arbitrary configuration with all multiplicities one except the coefficients b_k with k even, $n+1 \leq k \leq 2n$.

Proposition 4.5.1. *Suppose that in (4.1) $b_{2n} = n+1$. Then the configuration $\mathcal{A}$ is dihedral.*

Proof. Let P be a homogeneous polynomial of degree $2n$. Our quasi-invariant condition for P takes the form

$$\sum_{s=0}^{2n} a_s(2s-2n)e^{i\theta_i(2s-2n)} = 0$$

for all θ_i with $0 \le i \le n$. For θ_0 this gives the expression

$$a_0(-2n)e^{-2ni\theta_0} + a_1(2-2n)e^{(2-2n)i\theta_0} + a_2(4-2n)e^{(4-2n)i\theta_0} + \ldots + a_{2n}2ne^{2ni\theta_0} = 0$$

and a similar expression holds for the other values of θ_i. Expressing these conditions in matrix form we have the equation

$$\begin{pmatrix} -2ne^{(-2n)i\theta_0} & (2-2n)e^{(2-2n)i\theta_0} & (4-2n)e^{(4-2n)i\theta_0} & \ldots & 2ne^{(2n)i\theta_0} \\ -2ne^{(-2n)i\theta_1} & (2-2n)e^{(2-2n)i\theta_1} & (4-2n)e^{(4-2n)i\theta_1} & \ldots & 2ne^{(2n)i\theta_1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -2ne^{(-2n)i\theta_n} & (2-2n)e^{(2-2n)i\theta_n} & (4-2n)e^{(4-2n)i\theta_n} & \ldots & 2ne^{(2n)i\theta_n} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{2n} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Let

$$J = \begin{pmatrix} -2ne^{(-2n)i\theta_0} & (2-2n)e^{(2-2n)i\theta_0} & (4-2n)e^{(4-2n)i\theta_0} & \ldots & 2ne^{(2n)i\theta_0} \\ -2ne^{(-2n)i\theta_1} & (2-2n)e^{(2-2n)i\theta_1} & (4-2n)e^{(4-2n)i\theta_1} & \ldots & 2ne^{(2n)i\theta_1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -2ne^{(-2n)i\theta_n} & (2-2n)e^{(2-2n)i\theta_n} & (4-2n)e^{(4-2n)i\theta_n} & \ldots & 2ne^{(2n)i\theta_n} \end{pmatrix}$$

and for $0 \le j \le n$ let

$$e^{2i\theta_j} = a_j.$$

Multiplying the ith row of J by a_i^n we get

$$\widetilde{J} = \begin{pmatrix} -2n & (2-2n)a_0 & (4-2n)a_0^2 & \ldots & -2a_0^{n-1} & 0 & 2a_0^{n+1} & \ldots & 2na_0^{2n} \\ -2n & (2-2n)a_1 & (4-2n)a_1^2 & \ldots & -2a_1^{n-1} & 0 & 2a_1^{n+1} & \ldots & 2na_1^{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -2n & (2-2n)a_n & (4-2n)a_n^2 & \ldots & -2a_n^{n-1} & 0 & 2a_n^{n+1} & \ldots & 2na_n^{2n} \end{pmatrix}.$$

Now, suppose that in (4.1) $b_{2n} = n+1$. Thus the rank of $\widetilde{J}$ must be n. So any $(n+1)\times(n+1)$ minor of $\widetilde{J}$ must be zero (otherwise the rank of J is equal to $n+1$). Let us consider the minor

$$A_1 = \begin{vmatrix} -2n & (2-2n)a_0 & (4-2n)a_0^2 & \dots & -2a_0^{n-1} & 2a_0^{n+1} \\ -2n & (2-2n)a_1 & (4-2n)a_1^2 & \dots & -2a_1^{n-1} & 2a_1^{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -2n & (2-2n)a_n & (4-2n)a_n^2 & \dots & -2a_n^{n-1} & 2a_n^{n+1} \end{vmatrix}$$
$$= 2\prod_{j=0}^{n-1}(2j-2n) \begin{vmatrix} 1 & a_0 & a_0^2 & \dots & a_0^{n-1} & a_0^{n+1} \\ 1 & a_1 & a_1^2 & \dots & a_1^{n-1} & a_1^{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & a_n & a_n^2 & \dots & a_n^{n-1} & a_n^{n+1} \end{vmatrix}.$$

We know that $A_1 = 0$. We are going to find another expression for A_1. If we divide A_1 by the Vandermonde determinant

$$B_1 = \begin{vmatrix} 1 & a_0 & a_0^2 & \dots & a_0^{n-1} & a_0^{n} \\ 1 & a_1 & a_1^2 & \dots & a_1^{n-1} & a_1^{n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & a_n & a_n^2 & \dots & a_n^{n-1} & a_n^{n} \end{vmatrix}$$

we are left with the Schur polynomial corresponding to the partition $\lambda_1 = (1, 0, 0, \dots, 0)$ which we denote by S_{λ_1}. So we have

$$(1/2)\prod_{j=0}^{n-1}(2j-2n)^{-1}\frac{A_1}{B_1} = S_{\lambda_1}. \tag{4.14}$$

However by (4.2) we can express S_{λ_1} in terms of elementary symmetric polynomials, using the formula

$$S_{\lambda_1} = \det(e_{\lambda_i'-i+j}) = \begin{vmatrix} e_1 & e_2 & e_3 & \dots & e_{n-1} & e_{n+1} \\ 0 & 1 & e_1 & \dots & e_{n-2} & e_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \end{vmatrix} = e_1. \tag{4.15}$$

Then using (4.14) and (4.15) we see that

$$A_1 = 2\prod_{j=0}^{n-1}(2j-2n)\prod_{k<l}(a_k - a_l)e_1.$$

Recall that we require $A_1 = 0$. So, we conclude that $e_1 = 0$ since $a_i \neq a_j$ if $i \neq j$. Bearing this in mind we go on to consider

$$A_2 = \begin{vmatrix} -2n & (2-2n)a_0 & (4-2n)a_0^2 & \dots & -2a_0^{n-1} & 4a_0^{n+2} \\ -2n & (2-2n)a_1 & (4-2n)a_1^2 & \dots & -2a_1^{n-1} & 4a_1^{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -2n & (2-2n)a_n & (4-2n)a_n^2 & \dots & -2a_n^{n-1} & 4a_n^{n+2} \end{vmatrix}$$

$$= 4\prod_{j=0}^{n-1}(2j-2n) \begin{vmatrix} 1 & a_0 & a_0^2 & \dots & a_0^{n-1} & a_0^{n+2} \\ 1 & a_1 & a_1^2 & \dots & a_1^{n-1} & a_1^{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & a_n & a_n^2 & \dots & a_n^{n-1} & a_n^{n+2} \end{vmatrix}.$$

Following the same strategy as above we use (4.2) again to see the Schur polynomial derived is given by the equation

$$S_{\lambda_2} = \begin{vmatrix} e_1 & e_2 & e_3 & \dots & e_n & e_{n+1} \\ 1 & e_1 & e_2 & \dots & e_{n-1} & e_n \\ 0 & 0 & 1 & \dots & e_{n-3} & e_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \end{vmatrix} = e_1^2 - e_2$$

where $\lambda_2 = (2, 0, \dots, 0)$. So we conclude that $e_2 = 0$ since we know $e_1 = 0$. Continuing, we consider the minor

$$A_3 = \begin{vmatrix} -2n & (2-2n)a_0 & (4-2n)a_0^2 & \dots & -2a_0^{n-1} & 6a_0^{n+3} \\ -2n & (2-2n)a_1 & (4-2n)a_1^2 & \dots & -2a_1^{n-1} & 6a_1^{n+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -2n & (2-2n)a_n & (4-2n)a_n^2 & \dots & -2a_n^{n-1} & 6a_n^{n+3} \end{vmatrix}$$

$$= 6\prod_{j=0}^{n-1}(2j-2n) \begin{vmatrix} 1 & a_0 & a_0^2 & \dots & a_0^{n-1} & a_0^{n+3} \\ 1 & a_1 & a_1^2 & \dots & a_1^{n-1} & a_1^{n+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & a_n & a_n^2 & \dots & a_n^{n-1} & a_n^{n+3} \end{vmatrix}$$

and we arrive at

$$S_{\lambda_3} = \begin{vmatrix} 0 & 0 & e_3 & e_4 & \dots & e_{n+1} \\ 1 & 0 & 0 & e_3 & \dots & e_n \\ 0 & 1 & 0 & 0 & \dots & e_{n-1} \\ 0 & 0 & 0 & 1 & \dots & e_{n-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{vmatrix} = e_3$$

where $\lambda_3 = (3, 0, \dots, 0)$. Thus $e_3 = 0$. It is easy to see that we can continue on in this way to deduce that

$$e_1 = e_2 = e_3 = \dots = e_n = 0.$$

Indeed, at the i-th step we deal with the Schur polynomial for $\lambda_i = (i, 0, \dots, 0)$ and the left top corner submatrix of size $i \times i$ has determinant e_i if $e_j = 0$ for $1 \leq j \leq i-1$. Now, consider the polynomial

$$q = (x - a_0)(x - a_1)(x - a_2)\dots(x - a_n).$$

We can write this as

$$q = x^{n+1} - e_1 x^n + e_2 x^{n-1} + \dots + e_n x - a_0 a_1 a_2 \dots a_n$$

and since we know that $e_1 = e_2 = e_3 = \dots = e_n = 0$ we can rearrange q as

$$q = x^{n+1} - a_0 a_1 a_2 \dots a_n.$$

Put $a_0 a_1 a_2 \dots a_n = p$. Since each $a_i, 0 \leq i \leq n$ is a solution of the equation $q = 0$ we can see that for $0 \leq i \leq n$

$$a_i^{n+1} = p \tag{4.16}$$

which means the configuration $\mathcal{A}$ is dihedral. □

Suppose $\mathcal{A}$ is generic, so that $b_k = k - n$ for even k with $n+1 \leq k \leq 2n$. Denote by $P^g_{\mathcal{A}}$ the Poincaré series for a generic configuration. In order to complete the proof of Theorem 4.3.1 we need to calculate $P^g_{\mathcal{A}}$.

Lemma 4.5.2. *One has*

$$P^g_{\mathcal{A}}(t) = \frac{1 + 2\sum_{j=1}^{n+1}(-1)^j t^j}{(1-t)^2}$$

when $n+1$ is even and

$$P^g_{\mathcal{A}}(t) = \frac{1 + 2\sum_{j=1}^{n}(-1)^j t^j - t^{n+1} + t^{n+2}}{(1-t)^2}$$

when $n+1$ is odd.

Proof. Suppose $n+1$ is even. Lemmas 4.4.1 and 4.4.2 fix all the coefficients b_i except those b_i with i even, $n+1 \leq i \leq 2n$. For such b_i we have $b_i = i - n$ since $\mathcal{A}$ is generic. So we have

$$\begin{aligned} P^g_{\mathcal{A}}(t) &= \sum_{j=0}^{(n-1)/2} t^{2j} + \sum_{j=n+1}^{\infty} (j-n)t^j = \sum_{j=0}^{(n-1)/2} t^{2j} + t^{n+1}\sum_{j=0}^{\infty}(j+1)t^j \\ &= \sum_{j=0}^{(n-1)/2} t^{2j} + \frac{t^{n+1}}{(1-t)^2} = \frac{(\sum_{j=0}^{(n-1)/2} t^{2j})(1-2t+t^2) + t^{n+1}}{(1-t)^2} \\ &= \frac{1 + 2\sum_{j=1}^{n+1}(-1)^j t^j}{(1-t)^2}. \end{aligned}$$

Similarly, when $n+1$ is odd we have:

$$\begin{aligned} P^g_{\mathcal{A}}(t) &= \sum_{j=0}^{n/2} t^{2j} + \sum_{j=n+1}^{\infty} (j-n)t^j = \sum_{j=0}^{n/2} t^{2j} + t^{n+1}\sum_{j=0}^{\infty}(j+1)t^j \\ &= \sum_{j=0}^{n/2} t^{2j} + \frac{t^{n+1}}{(1-t)^2} = \frac{(\sum_{j=0}^{n/2} t^{2j})(1-2t+t^2) + t^{n+1}}{(1-t)^2} \\ &= \frac{1 + 2\sum_{j=1}^{n}(-1)^j t^j - t^{n+1} + t^{n+2}}{(1-t)^2}. \end{aligned}$$

□

Note that in both cases the generic series is not palindromic. Recall that Lemmas 4.4.1 and 4.4.2 show that, for any configuration $\mathcal{A}$, $P_{\mathcal{A}}$ can differ from $P^g_{\mathcal{A}}$ only in even degrees between $n+1$ and $2n$. Thus we can establish the following.

Lemma 4.5.3. *The only possible Poincaré series $P_{\mathcal{A}}(t)$ are given by the formulas*

$$P_{\mathcal{A}}(t) = \frac{1 + 2\sum_{j=1}^{n+1}(-1)^j t^j + S_{\mathcal{A}}(1-t)^2}{(1-t)^2}$$

when $n+1$ is even and

$$P_{\mathcal{A}}(t) = \frac{1 + 2\sum_{j=1}^{n}(-1)^j t^j - t^{n+1} + t^{n+2} + S_{\mathcal{A}}(1-t)^2}{(1-t)^2}$$

when $n+1$ is odd, where

$$S_{\mathcal{A}} = \sum_{i=0}^{(n-1)/2} c_i t^{n+2i+1} \tag{4.17}$$

when $n+1$ is even and

$$S_{\mathcal{A}} = \sum_{i=0}^{(n-2)/2} c_i t^{n+2i+2} \tag{4.18}$$

when $n+1$ is odd. Furthermore the coefficients c_i can take only the values 0 or 1 $\forall i$.

Proof. Lemmas 4.4.1, 4.4.2 and 4.5.2. □

Finally we need to check whether, given a palindromic $P_{\mathcal{A}}(t)$ it follows that $b_{2n} = n+1$ and thus the arrangement is dihedral.

Lemma 4.5.4. *Suppose $P_{\mathcal{A}}(t)$ is palindromic. Then in (4.1) $b_{2n} = n+1$.*

Proof. Recall that generically $b_{2n} = n$. So we only have to check that if $P_{\mathcal{A}}(t)$ is palindromic then $c_{\frac{n-1}{2}} = 1$ in (4.17) in the even case and $c_{\frac{n-2}{2}} = 1$ in (4.18) in the odd case. Consider the even case first. If $c_0 = 1$ then the numerator has the form

$$\begin{aligned} &1 + 2\sum_{j=1}^{n+1}(-1)^j t^j + t^{n+1} + O(t^{n+2}) \\ =&1 + 2\sum_{j=1}^{n}(-1)^j t^j + 3t^{n+1} + O(t^{n+2}) \end{aligned}$$

where $O(t^{n+2})$ denotes the terms in the numerator divisible by t^{n+2}. It is clear that no term other than t^{n+1} can have coefficient 3. The palindromicity of $P_{\mathcal{A}}(t)$ implies we have exactly $n+1$ non-zero terms to the right of t^{n+1}, so that $c_i = 1$ $\forall i = 0, 1, \dots, \frac{n-1}{2}$. In particular $c_{\frac{n-1}{2}} = 1$ and thus we are in the dihedral case. Suppose then that $c_0 = 0$. Then since the series is palindromic we must have $c_i = 1$ for some i with $1 \le i \le \frac{n-1}{2}$. Choose the first non-zero c_i. Then the numerator has the form

$$1 + 2\sum_{j=1}^{n+1}(-1)^j t^j + t^{n+2i+1} - 2t^{n+2i+2} + t^{n+2i+3} + O(t^{n+2i+3}).$$

Consider the term t^{n+2i+1}. By inspection this is not the 'mirror' term. (That is, the coefficients of the numerator are not symmetric around the term t^{n+2i+1}). Since we have less than $n+1$ terms to the right of t^{n+2i+1}, the mirror term must lie to the left of t^{n+2i+1}. However, to the left of t^{n+2i+1} only the term with degree 0 has coefficient 1. This means that if the mirror term does lie to the left of t^{n+2i+1} then the total degree of the numerator must be $n+2i+1$, a contradiction. Thus $c_0 = 1$ and we are done.

In the case when $n+1$ is odd the numerator has the form:

$$1 + 2\sum_{j=1}^{n}(-1)^j t^j - t^{n+1} + t^{n+2} + O(t^{n+2}).$$

Note that t^{n+1} has coefficient -1 and no other term can have coefficient -1. Thus t^{n+1} must be the 'mirror' term which immediately implies that $c_i = 1$ for all $i = 0, 1, \ldots, \frac{n-2}{2}$. This is easy to see, since if t^{n+1} is to be the mirror term then the numerator must have total degree $2n+2$. In particular this means $c_{\frac{n-2}{2}} = 1$ and we are done. $\square$

Theorem 4.3.1 now follows by combining Lemma 4.5.4 and Proposition 4.5.1.

4.6 Arrangements of type $(m, 1^n)$

Let $\mathcal{A}$ be a finite set of non-collinear vectors $\beta_0, \beta_1, \ldots, \beta_n$. Let $\beta_i = (1, \alpha_i)$, $\alpha_i \in \mathbb{C}$. We fix $\beta_0 = (0, 1)$. Recall the arrangements of type $(m, 1^n)$ from Definition 2.2.1. Recall also the arrangement $\mathcal{A}_{(m,1^n)}$ introduced in Definition 2.4.7. The remainder of this chapter will be devoted to proving the following:

Theorem 4.6.1. *Let $\mathcal{A}$ have type $(m, 1^n)$. Then the quasi-invariant ring $Q_{\mathcal{A}}$ is Gorenstein if and only if $\mathcal{A} = \mathcal{A}_{(m,1^n)}$.*

4.7 Preliminary lemmas

In this section we will prove some technical lemmas which will be useful when studying the matrix of the system of quasi-invariance conditions as equations for the coefficients of polynomials of certain degrees. This type of analysis will be the focus of Section 4.8.

Let $\Delta = \prod_{1 \le i < j \le [n/2]} (\alpha_i^2 - \alpha_j^2)$. Refer to the elementary symmetric polynomials in the variables $\alpha_1^2, \alpha_2^2, \ldots, \alpha_{[n/2]}^2$ as $e_i, 0 \le i \le [n/2]$ $(e_0 = 1)$. Throughout Section 4.7 let B be the $[n/2] \times (n-s)$ matrix, where $0 \le s \le n-1$, with the columns:

$$C_i(X) = \begin{pmatrix} (2i-1)\alpha_1^{2i-2} - (2X+2n-2i+1)\alpha_1^{2i} \\ (2i-1)\alpha_2^{2i-2} - (2X+2n-2i+1)\alpha_2^{2i} \\ (2i-1)\alpha_3^{2i-2} - (2X+2n-2i+1)\alpha_3^{2i} \\ \vdots \\ (2i-1)\alpha_{[n/2]}^{2i-2} - (2X+2n-2i+1)\alpha_{[n/2]}^{2i} \end{pmatrix}$$

where $1 \le i \le n-s$.

Lemma 4.7.1. *Let $s \le [n/2]$. Let B_L be the minor of B formed by taking the determinant of the square submatrix with columns $C_L(X), C_{L+1}(X), \dots, C_{L+[n/2]-1}(X)$ where $1 \le L \le n-s-[n/2]+1$. Then*

$$B_L = \Delta \prod_{i=1}^{[n/2]} \alpha_i^{2L-2} \sum_{i=0}^{[n/2]} (-1)^i \prod_{r=i}^{[n/2]-1} (2[n/2]-2r+2L-3) \prod_{r=0}^{i-1} (2X+2r+2n-2[n/2]-2L+3)e_i. \tag{4.19}$$

Proof. For $1 \le i \le n-s$ let us introduce the column vectors

$$x_i = (i+1)\begin{pmatrix} \alpha_1^i \\ \alpha_2^i \\ \alpha_3^i \\ \vdots \\ \alpha_{[n/2]}^i \end{pmatrix}, \quad y_i = (2X+2n-i+1)\begin{pmatrix} \alpha_1^i \\ \alpha_2^i \\ \alpha_3^i \\ \vdots \\ \alpha_{[n/2]}^i \end{pmatrix}.$$

Thus $C_i(X) = x_{2i-2} - y_{2i}$. Consider B_1 and suppose n is even, odd case is extremely similar. Let B_1^t be the determinant of the matrix with the columns $y_n, y_{n-2}, \dots, y_{n-2t+2}, x_{n-2t-2}, \dots, x_0$ where $0 \le t \le n/2$. Note that

$$B_1^t = \prod_{r=t}^{[n/2]-1} (n-2r-1) \prod_{r=0}^{t-1} (2X+2r+n+1) \begin{vmatrix} \alpha_1^n & \dots & \alpha_1^{n-2t+2} & \alpha_1^{n-2t-2} & \dots & 1 \\ \alpha_2^n & \dots & \alpha_2^{n-2t+2} & \alpha_2^{n-2t-2} & \dots & 1 \\ \vdots & & \vdots & \vdots & & \vdots \\ \alpha_{[n/2]}^n & \dots & \alpha_{[n/2]}^{n-2t+2} & \alpha_{[n/2]}^{n-2t-2} & \dots & 1 \end{vmatrix}$$

$$= \prod_{r=t}^{[n/2]-1} (n-2r-1) \prod_{r=0}^{t-1} (2X+2r+n+1)\Delta e_t$$

since we can express the elementary symmetric polynomials e_t via determinants, see Definition 4.2.3 and (4.2). So

$$\begin{aligned} B_1 &= \sum_{t=0}^{n/2} (-1)^t B_1^t \\ &= \Delta \sum_{t=0}^{[n/2]} (-1)^t \prod_{r=t}^{[n/2]-1} (n-2r-1) \prod_{r=0}^{t-1} (2X+2r+n+1) e_t. \end{aligned}$$

It is not hard to see that we can adopt the same strategy used to expand B_1 to deal with each B_L, $1 \le L \le n-s-[n/2]+1$. □

Lemma 4.7.2. *Let*

$$e_r = \binom{[n/2]}{r} \frac{\prod_{i=1}^{r} (2[n/2]-2i+1)}{\prod_{i=1}^{r} (2m+2i-1)}$$

where $0 \le r \le [n/2]$. *For* $1 \le s \le [n/2]$ *let* B_L *be defined by (4.19) with* $X = m-s$. *Then* $B_L = 0$.

Proof. Suppose first that n is even. Then for $1 \le L \le n/2 - s + 1$ we have

$$\begin{aligned} B_L &= \Delta \prod_{i=1}^{n/2} \alpha_i^{2L-2} \sum_{i=0}^{n/2} (-1)^i \prod_{r=i}^{n/2-1} (n-2r+2L-3) \prod_{r=0}^{i-1} (2m-2s+2r+n-2L+3) e_i \\ &= \Delta \prod_{i=1}^{n/2} \alpha_i^{2L-2} \sum_{i=0}^{n/2} (-1)^i \prod_{r=i}^{n/2-1} (n-2r+2L-3) \prod_{r=0}^{i-1} (2m-2s+2r+n-2L+3) \binom{n/2}{i} \frac{\prod_{r=1}^{i} (n-2r+1)}{\prod_{r=1}^{i} (2m+2r-1)} \\ &= y \sum_{i=0}^{n/2} (-1)^i \binom{n/2}{i} \prod_{r=i}^{n/2-1} (n-2r+2L-3) \prod_{r=1}^{i} (n-2r+1) \prod_{r=0}^{i-1} (2m-2s+2r+n-2L+3) \\ &\times \prod_{r=0}^{n/2-i-1} (2m+2i+2r+1) \\ &= yz \sum_{i=0}^{n/2} (-1)^i \binom{n/2}{i} \prod_{r=0}^{n/2-s-L} (2m+2i+2r+1) \prod_{r=0}^{L-2} (n-2i+2r+1). \end{aligned}$$

where

$$y = \prod_{r=1}^{n/2} (2m+2r-1)^{-1} \Delta \prod_{i=1}^{n/2} \alpha_i^{2L-2}$$

and

$$z = \prod_{r=n/2-s-L+1}^{n/2-1} (2m+2r+1) \prod_{r=2L-n/2-1}^{L-1} (n-2L+2r+1).$$

Now set

$$U(x) = \prod_{r=0}^{n/2-s-L} (2m+2x+2r+1) \prod_{r=0}^{L-2} (n-2x+2r+1).$$

Note that $U(x)$ has degree $n/2 - s$. Then we have

$$B_L = yz \sum_{i=0}^{n/2} (-1)^i \binom{n/2}{i} U(i) = 0$$

by Lemma 4.2.7. If n is odd we can proceed in exactly the same way to see that

$$B_L = yz \sum_{i=0}^{\frac{n-1}{2}} (-1)^i \binom{\frac{n-1}{2}}{i} \prod_{r=0}^{\frac{n+1}{2}-s-L} (2m+2i+2r+1) \prod_{r=0}^{L-3} (n-2i+2r)$$

where

$$y = \prod_{r=1}^{\frac{n-1}{2}} (2m+2r-1)^{-1} \Delta \prod_{i=1}^{\frac{n-1}{2}} \alpha_i^{2L-2}$$

and

$$z = \prod_{r=\frac{n+1}{2}-s-L+1}^{\frac{n-3}{2}} (2m+2r+1) \prod_{r=2L-\frac{n+1}{2}-1}^{L-2} (n-2L+2r+2).$$

Now set

$$V(x) = \prod_{r=0}^{\frac{n+1}{2}-s-L} (2m+2x+2r+1) \prod_{r=0}^{L-3} (n-2x+2r)$$

and continue as in the even case to complete the proof. □

Lemma 4.7.3. *Let $s = 0$ and let B_L be defined by (4.19) with $X = m + t$. Then for even n $\exists L, 1 \le L \le n/2 + 1$ s.t. $B_L \neq 0$. For odd n $\exists L, 2 \le L \le \frac{n+3}{2}$ s.t. $B_L \neq 0$.*

Proof. By Lemma 4.7.1 we have

$$B_L = \Delta \prod_{i=1}^{[n/2]} \alpha_i^{2L-2} \sum_{i=0}^{[n/2]} (-1)^i \prod_{r=i}^{[n/2]-1} (2[n/2]-2r+2L-3) \prod_{r=0}^{i-1} (2m+2t+2r+2n-2[n/2]-2L+3) e_i. \tag{4.20}$$

Suppose $B_L = 0$ for $1 \le L \le n/2 + 1$ if n is even and suppose $B_L = 0$ for $2 \le L \le \frac{n+3}{2}$ if n is odd. Let us cancel $\Delta \prod_{i=1}^{[n/2]} \alpha_i^{2L-2}$ and consider the resulting conditions as a system of linear equations for the unknowns $e_0, e_1, \ldots, e_{[n/2]}$. Refer to the corresponding matrix as Q. We will show that the determinant $|Q| \neq 0$, which would be a contradiction as there are non-zero e_i for some i. We consider $|Q|$ as a polynomial in m. First we show $|Q|$ is not identically zero in m. Set $2m + 2t + n + 1 = 0$. Let us enumerate the rows of Q by $L = 1, 2, \ldots, n/2 + 1$ when n is even and $L = 2, \ldots, \frac{n+3}{2}$ when n is odd, and the columns of Q by $i = 0, 1, \ldots, [n/2]$. Then it follows from (4.20) that the only non-zero entries in the i-th column are the entries of the last $[n/2] - i + 1$ rows. So $|Q|$ is just the product of the diagonal entries and this is clearly non-zero.

Next we will show that there are no positive values of m for which $|Q| = 0$. Note that as a polynomial in m $|Q|$ has degree $\sum_{i=0}^{[n/2]} i = \frac{[n/2]([n/2]+1)}{2}$. Let us subtract the $(i+1)$st column from the ith column of Q, $i = 0, 1 \ldots, [n/2]-1$. Then the Lth entry of the ith column is given by

$$(-1)^i \prod_{r=i}^{[n/2]-1} (2[n/2] - 2r + 2L - 3) \prod_{r=0}^{i-1} (2m + 2t + 2n - 2[n/2] - 2L + 3 + 2r)$$
$$+ (-1)^i \prod_{r=i+1}^{[n/2]-1} (2[n/2] - 2r + 2L - 3) \prod_{r=0}^{i} (2m + 2t + 2r + 2n - 2[n/2] - 2L + 3)$$
$$= (-1)^i \prod_{r=i+1}^{[n/2]-1} (2[n/2] - 2r + 2L - 3) \prod_{r=0}^{i-1} (2m + 2t + 2n - 2[n/2] - 2L + 3 + 2r)$$
$$\times ((2[n/2] - 2i + 2L - 3) + (2m + 2t + 2i + 2n - 2[n/2] - 2L + 3))$$
$$= (-1)^i \prod_{r=i+1}^{[n/2]-1} (2[n/2] - 2r + 2L - 3) \prod_{r=0}^{i-1} (2m + 2t + 2n - 2[n/2] - 2L + 3 + 2r)(2m + 2n + 2t).$$

Let us repeat this process and subtract the $(i+1)$st column from the ith column for $i = 0, 1, \ldots, [n/2]-2$. Then the Lth entry of the ith column becomes

$$(-1)^i \prod_{r=i+1}^{[n/2]-1} (2[n/2] - 2r + 2L - 3) \prod_{r=0}^{i-1} (2m + 2t + 2n - 2[n/2] - 2L + 3 + 2r)$$
$$+ (-1)^i \prod_{r=i+2}^{[n/2]-1} (2[n/2] - 2r + 2L - 3) \prod_{r=0}^{i} (2m + 2t + 2n - 2[n/2] - 2L + 3 + 2r)$$
$$= (-1)^i \prod_{r=i+2}^{[n/2]-1} (2[n/2] - 2r + 2L - 3) \prod_{r=0}^{i-1} (2m + 2t + 2n - 2[n/2] - 2L + 3 + 2r)$$
$$\times ((2[n/2] - 2i - 2 + 2L - 3) + (2m + 2t + 2n - 2[n/2] - 2L + 3 + 2i))$$
$$= (-1)^i \prod_{r=i+2}^{[n/2]-1} (2[n/2] - 2r + 2L - 3) \prod_{r=0}^{i-1} (2m + 2t + 2n - 2[n/2] - 2L + 3 + 2r)(2m + 2n + 2t - 2).$$

We can continue to see that the expression

$$\prod_{i=1}^{[n/2]} (2m + 2t + 2n + 2 - 2i)^{[n/2]-i+1}$$

is a factor of $|Q|$. This expression has the same total degree in m as $|Q|$ and is non-zero for positive values of m. So $|Q| \neq 0$ and we are done. □

Lemma 4.7.4. *Let $s = 1$ and let B_L be defined by (4.19) with $X = m - 1$. The system of equations $B_L = 0$ where $1 \leq L \leq n/2$ if n is even and $2 \leq L \leq \frac{n+1}{2}$ if n is odd as the system of linear equations for the unknowns $e_1, \ldots e_{[n/2]}$ has a unique solution.*

Proof. By Lemma 4.7.1 we have

$$B_L = \Delta \prod_{i=1}^{[n/2]} \alpha_i^{2L-2} \sum_{i=0}^{[n/2]} (-1)^i \prod_{r=i}^{[n/2]-1} (2[n/2] - 2r + 2L - 3) \prod_{r=0}^{i-1} (2m + 2r + 2n - 2[n/2] - 2L + 1) e_i. \tag{4.21}$$

Suppose $B_L = 0$ for $1 \leq L \leq n/2$ if n is even and suppose $B_L = 0$ for $2 \leq L \leq \frac{n+1}{2}$ if n is odd. Let us cancel $\Delta \prod_{i=1}^{[n/2]} \alpha_i^{2L-2}$ and consider the resulting conditions as a system of linear equations for the unknowns $e_1, \dots, e_{[n/2]}$. Refer to the corresponding matrix as R. We can follow the proof of Lemma 4.7.3 closely to show that $|R| \neq 0$. Setting $2m+n-1 = 0$ shows that $|R|$ is not identically zero as a polynomial in m. We can then proceed as in Lemma 4.7.3 to see that $|R|$ has no positive roots as a polynomial in m. □

Lemma 4.7.5. *Let D_L^k be the minor of B formed by taking the determinant of the square submatrix with columns $C_L(X), C_{L+1}(X), \dots, C_{L+[n/2]-2}(X)$ where $1 \leq L \leq n-s-[n/2]+2$ and we include all except the k-th row of B. Then*

$$D_L^k = \Delta \prod_{\substack{i=1 \\ i \neq k}}^{[n/2]} \alpha_i^{2L-2} \sum_{i=0}^{[n/2]-1} (-1)^i \prod_{r=i}^{[n/2]-2} (2[n/2] - 2r + 2L - 5) \prod_{r=0}^{i-1} (2X + 2r + 2n - 2[n/2] - 2L + 5) e_i^k \tag{4.22}$$

where we denote by e_i^k the elementary symmetric polynomials in the variables

$$\alpha_1^2, \alpha_2^2, \dots, \alpha_{k-1}^2, \alpha_{k+1}^2, \dots, \alpha_{[n/2]}^2.$$

Proof. It is easy to follow the method of Lemma 4.7.1 to see that

$$D_L^k = \Delta \prod_{\substack{i=1 \\ i \neq k}}^{[n/2]} \alpha_i^{2L-2} \sum_{i=0}^{[n/2]-1} (-1)^i \prod_{r=i}^{[n/2]-2} (2[n/2] - 2r + 2L - 5) \prod_{r=0}^{i-1} (2X + 2r + 2n - 2[n/2] - 2L + 5) e_i^k.$$

□

Lemma 4.7.6. *For $1 \leq s \leq [n/2]$ let D_L^k be defined by (4.22) with $X = m - s$. Then for any L, $1 \leq L \leq n - s - [n/2] + 2$, $\exists k$ such that $D_L^k \neq 0$.*

Proof. Using Lemma 4.7.5 we have

$$D_L^k = \Delta \prod_{\substack{i=1 \\ i \neq k}}^{[n/2]} \alpha_i^{2L-2} \sum_{i=0}^{[n/2]-1} (-1)^i \prod_{r=i}^{[n/2]-2} (2[n/2] - 2r + 2L - 5) \prod_{r=0}^{i-1} (2m + 2r + 2n - 2[n/2] - 2L - 2s + 5) e_i^k.$$

Let us fix L and set $D_L^k = 0$ for all $k = 1, 2, \dots, [n/2]$. We cancel the terms $\Delta \prod_{\substack{i=1 \\ i \neq k}}^{[n/2]} \alpha_i^{2L-2}$ and consider the resulting expressions as a system of linear equations for the unknowns

$$x_i = (-1)^i \prod_{r=i}^{[n/2]-2} (2[n/2] - 2r + 2L - 5) \prod_{r=0}^{i-1} (2m + 2r + 2n - 2[n/2] - 2L - 2s + 5),$$

where $j = 0, 1, \ldots, [n/2]-1$. Note $x_i \neq 0 \,\forall\, i$. The system takes the matrix form

$$AX = 0$$

where $A = (a_{kl})$, $1 \leq k \leq [n/2]$, $0 \leq l \leq [n/2]-1$, $a_{kl} = e_l^k$ and $X = (x_0, x_1, \ldots, x_{[n/2]-1})$. Note that the determinant of A has degree

$$1 + 2 + \ldots + ([n/2]-1) = \frac{([n/2]-1)([n/2])}{2} = \binom{[n/2]}{2}$$

as a polynomial in $\alpha_1^2, \ldots, \alpha_{[n/2]}^2$. We claim that $\det(A) \neq 0$. This can be seen by setting $\alpha_l^2 = \alpha_m^2$ for $l < m$. In this situation it is clear that $e_i^l = e_i^m$ for all $0 \leq i \leq [n/2]-1$ and thus $\det(A)$=0. Since $\det A$ has degree $\binom{[n/2]}{2}$ we are done. □

Lemma 4.7.7. *Let $[n/2]+1 \leq s \leq n-1$. Let $\vec{k} = (k_1, k_2, \ldots, k_{s+[n/2]-n})$ where $1 \leq k_i \leq [n/2]$. Let $E^{\vec{k}}$ be the minor of B formed by taking the determinant of the square submatrix with columns $C_1(m-s), C_2(m-s), \ldots, C_{n-s}(m-s)$ and we include all rows of B except rows $k_1, k_2, \ldots k_{s+[n/2]-n}$. Then*

$$E^{\vec{k}} = \Delta \prod_{\substack{i=1 \\ i \notin \vec{k}}}^{[n/2]} \alpha_i^{2L-2} \sum_{i=0}^{n-s} (-1)^i \prod_{r=i}^{n-s-1} (2n-2r-2s-1) \prod_{r=0}^{i-1} (2m+1+2r) e_i^{\vec{k}} \tag{4.23}$$

where we denote by $e_i^{\vec{k}}$ the elementary symmetric polynomials in the variables $\alpha_i^2, 1 \leq i \leq [n/2]$ but not including $\alpha_{k_1}^2, \alpha_{k_2}^2, \ldots, \alpha_{k_{s+[n/2]-n}}^2$.

Proof. It is easy to follow the method of Lemma 4.7.1 to see that

$$E^{\vec{k}} = \Delta \prod_{\substack{i=1 \\ i \notin \vec{k}}}^{[n/2]} \alpha_i^{2L-2} \sum_{i=0}^{n-s} (-1)^i \prod_{r=i}^{n-s-1} (2n-2r-2s-1) \prod_{r=0}^{i-1} (2m+1+2r) e_i^{\vec{k}}.$$

□

Lemma 4.7.8. *For $[n/2]+1 \leq s \leq n-1$ let $E^{\vec{k}}$ be defined by (4.23). Then $\exists \vec{k}$ such that $E^{\vec{k}} \neq 0$.*

Proof. Using Lemma 4.7.7 we have

$$E^{\vec{k}} = \Delta \prod_{\substack{i=1 \\ i \notin \vec{k}}}^{[n/2]} \alpha_i^{2L-2} \sum_{i=0}^{n-s} (-1)^i \prod_{r=i}^{n-s-1} (2n-2r-2s-1) \prod_{r=0}^{i-1} (2m+1+2r) e_i^{\vec{k}}.$$

Let us set $E^{\vec{k}} = 0$ for the following collection of $\vec{k}$: $\vec{k}_j = (1, 2, \ldots, s + [n/2] - n - 1, j)$ where $j = s + [n/2] - n, s + [n/2] - n + 1, \ldots, [n/2]$. We cancel the terms $\Delta \prod_{\substack{i=1 \\ i \notin \vec{k}}}^{[n/2]} \alpha_i^{2L-2}$ and consider the resulting expressions as a system of linear equations for the unknowns

$$x_i = (-1)^i \prod_{r=i}^{n-s-1} (2n - 2r - 2s - 1) \prod_{r=0}^{i-1} (2m + 1 + 2r),$$

where $i = 0, 1, \ldots, n - s$. Note $x_i \neq 0 \, \forall \, i$. The system takes the matrix form

$$AX = 0$$

where $A = (a_{jl})$, $1 \leq j \leq n - s + 1$, $0 \leq l \leq n - s$, $a_{jl} = e_l^{\vec{k}_j}$ and

$$X = (x_0, x_1, \ldots, x_{n-s}).$$

Note that the determinant of A has degree

$$1 + 2 + \ldots + (n - s) = \frac{(n - s)(n - s + 1)}{2} = \binom{n - s + 1}{2}$$

as a polynomial in the appropriate α_i^2. We claim that $\det A \neq 0$. This can be seen by setting $\alpha_l^2 = \alpha_m^2$ for $l < m$ where $s + [n/2] - n \leq l < m \leq [n/2]$. In this situation it is clear that $e_i^{\vec{k}_l} = e_i^{\vec{k}_m}$ for all $0 \leq i \leq n - s$ and thus $\det(A) = 0$. Since $\det A$ has degree $\binom{n-s+1}{2}$ we are done. □

Lemma 4.7.9. *For $s = 1$ let D_L^k be defined by (4.22) with $X = m - 1$. Let k be fixed. Then $\exists L, 2 \leq L \leq n - [n/2] + 1$ such that $D_L^k \neq 0$.*

Proof. Using Lemma 4.7.5 we have

$$D_L^k = \Delta \prod_{\substack{i=1 \\ i \neq k}}^{[n/2]} \alpha_i^{2L-2} \sum_{i=0}^{[n/2]-1} (-1)^i \prod_{r=i}^{[n/2]-2} (2[n/2] - 2r + 2L - 5) \prod_{r=0}^{i-1} (2m + 2r + 2n - 2[n/2] - 2L - 2s + 5) e_i^k.$$

Let us fix k and set $D_L^k = 0$ for $2 \leq L \leq n - [n/2] + 1$. We cancel the terms $\Delta \prod_{\substack{i=1 \\ i \neq k}}^{[n/2]} \alpha_i^{2L-2}$ and consider the resulting conditions as a system of linear equations for the unknowns $e_0^k, e_1^k, \ldots, e_{[n/2]-1}^k$. Refer to the corresponding matrix as Q. We can set $2m + n - 1 = 0$ to see that $|Q|$ is not identically zero as a polynomial in m. We can then proceed as in Lemma 4.7.3 to see that $|Q|$ has no positive roots as a polynomial in m. □

4.8 Statements about quasi-invariants of specific degree

We are now ready to begin the proof of Theorem 4.6.1. We will need various components to do this. Firstly, in Section 4.8 we are going to analyze the dimension of the space of quasi-invariants of certain degree. Recall the Poincaré series (4.1)

$$P_{\mathcal{A}}(t) = \sum_{k=0}^{\infty} b_k t^k. \tag{4.24}$$

In Proposition 4.8.2 we will show that for an arbitrary arrangement $\mathcal{A}$ of type $(m, 1^n)$, b_i is fixed for odd i with $i \geq 2m+n-1$. In Proposition 4.8.3 we will show that b_i is fixed for even i with $i \geq 2m+2n$. Finally in Proposition 4.8.4 we will show that $b_{2m+2n-2}$ is the same for any configuration $\mathcal{A}$, except one configuration whose geometry is fully fixed. From now on let $\mathcal{A}$ be of type $(m, 1^n)$.

Lemma 4.8.1. *If n is even then in (4.24) $b_{2m+n-1} = m$. If n is odd then in (4.24), $b_{2m+n-2} = m-1$.*

Proof. Let q be a homogeneous polynomial of degree $2m+n-1$;

$$q = \sum_{i=0}^{2m+n-1} a_i x^{2m+n-1-i} y^i.$$

for some $a_i \in \mathbb{C}$. Recall the definition of quasi-invariance (Definition 4.2.1). It is easy to see that if the quasi-invariance condition for the vector $\beta_0 = (0,1)$ is satisfied (that is, if $\partial_y^s q = 0$ when $y = 0$ for $s = 1, 3, \ldots 2m-1$) then $a_i = 0$ for $i = 1, 3, \ldots, 2m-1$. Before we consider the quasi-invariance conditions for the other vectors β_i we rewrite q as

$$q = \sum_{i=0}^{m} a_{2i} x^{2m+n-1-2i} y^{2i} + \sum_{i=2m+1}^{2m+n-1} a_i x^{2m+n-1-i} y^i. \tag{4.25}$$

The quasi-invariance conditions for the vectors β_j for $1 \leq j \leq n$ state that

$$(\partial_x + \alpha_j \partial_y) q = 0 \text{ on the line } x + \alpha_j y = 0.$$

That is

$$(\partial_x + \alpha_j \partial_y) q = \sum_{i=0}^{m} a_{2i}(2m+n-2i-1) x^{2m+n-2i-2} y^{2i} + \sum_{i=2m+1}^{2m+n-1} a_i (2m+n-i-1) x^{2m+n-i-2} y^i$$
$$+ \alpha_j \Big(\sum_{i=0}^{m} 2i a_{2i} x^{2m+n-1-2i} y^{2i-1} + \sum_{i=2m+1}^{2m+n-1} i a_i x^{2m+n-1-i} y^{i-1} \Big) = 0 \tag{4.26}$$

if $x = -\alpha_j y$ for $1 \leq j \leq n$. We can express the equations (4.26) in matrix form. We have

$$AC = 0,$$

where

$$C^T = \begin{pmatrix} a_0 & a_2 & . & . & a_{2m} & a_{2m+1} & a_{2m+2} & . & . & a_{2m+n-1} \end{pmatrix}$$

and the matrix A consists of the columns $A_1, A_2, \ldots, A_{m+n}$ given by the following. For $0 \leq i \leq m$:

$$A_{2i} = \begin{pmatrix} (2m+n-1-2i)\alpha_1^{2m+n-2i-2} - 2i\alpha_1^{2m+n-2i} \\ (2m+n-1-2i)\alpha_2^{2m+n-2i-2} - 2i\alpha_2^{2m+n-2i} \\ (2m+n-1-2i)\alpha_3^{2m+n-2i-2} - 2i\alpha_3^{2m+n-2i} \\ \vdots \\ (2m+n-1-2i)\alpha_n^{2m+n-2i-2} - 2i\alpha_n^{2m+n-2i} \end{pmatrix},$$

while for $2m+1 \leq i \leq 2m+n-1$:

$$A_i = (-1)^i \begin{pmatrix} (2m+n-1-i)\alpha_1^{2m+n-i-2} - i\alpha_1^{2m+n-i} \\ (2m+n-1-i)\alpha_2^{2m+n-i-2} - i\alpha_2^{2m+n-i} \\ (2m+n-1-i)\alpha_3^{2m+n-i-2} - i\alpha_3^{2m+n-i} \\ \vdots \\ (2m+n-1-i)\alpha_n^{2m+n-i-2} - i\alpha_n^{2m+n-i} \end{pmatrix}.$$

We need to find the rank of A. Recall that application of elementary row and column operations does not change the rank of the matrix. Note that

$$A_0 = (2m+n-1) \begin{pmatrix} \alpha_1^{2m+n-2} \\ \alpha_2^{2m+n-2} \\ \alpha_3^{2m+n-2} \\ \vdots \\ \alpha_n^{2m+n-2} \end{pmatrix}$$

and

$$A_{2m+n-1} = (-1)^n(2m+n-1) \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_n \end{pmatrix}.$$

Suppose now that n is even. We can apply appropriate elementary column transformations to reduce the matrix A to the form

$$\begin{pmatrix} \alpha_1^{2m+n-2} & \alpha_1^{2m+n-4} & \ldots & \alpha_1^2 & 1 & \alpha_1^{n-1} & \alpha_1^{n-3} \ldots & \alpha_1^3 & \alpha_1 \\ \alpha_2^{2m+n-2} & \alpha_2^{2m+n-4} & \ldots & \alpha_2^2 & 1 & \alpha_2^{n-1} & \alpha_2^{n-3} \ldots & \alpha_2^3 & \alpha_2 \\ \alpha_3^{2m+n-2} & \alpha_3^{2m+n-4} & \ldots & \alpha_3^2 & 1 & \alpha_3^{n-1} & \alpha_3^{n-3} \ldots & \alpha_3^3 & \alpha_3 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ \alpha_n^{2m+n-2} & \alpha_n^{2m+n-4} & \ldots & \alpha_n^2 & 1 & \alpha_n^{n-1} & \alpha_n^{n-3} \ldots & \alpha_n^3 & \alpha_n \end{pmatrix}.$$

Note that we have the $n \times n$ minor

$$Q = \begin{vmatrix} \alpha_1^{n-1} & \alpha_1^{n-2} & \alpha_1^{n-3} & \dots & \alpha_1^3 & \alpha_1^2 & \alpha_1 & 1 \\ \alpha_2^{n-1} & \alpha_2^{n-2} & \alpha_2^{n-3} & \dots & \alpha_2^3 & \alpha_2^2 & \alpha_2 & 1 \\ \alpha_3^{n-1} & \alpha_3^{n-2} & \alpha_3^{n-3} & \dots & \alpha_3^3 & \alpha_3^2 & \alpha_3 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ \alpha_n^{n-1} & \alpha_n^{n-2} & \alpha_n^{n-3} & \dots & \alpha_n^3 & \alpha_n^2 & \alpha_n & 1 \end{vmatrix} \tag{4.27}$$

Since $Q \neq 0$ we get rk$A$=n. Hence $b_{2m+n-1} = m + n - \text{rk}A = m$ as stated.

If n is odd the same type of analysis lets us transform the matrix of the system of linear equations for the coefficients of a homogeneous polynomial p of degree $2m + n - 2$ into the form

$$\begin{pmatrix} \alpha_1^{2m+n-3} & \alpha_1^{2m+n-5} & \dots & \alpha_1^2 & 1 & \alpha_1^{n-2} & \alpha_1^{n-4} \dots & \alpha_1^3 & \alpha_1 \\ \alpha_2^{2m+n-3} & \alpha_2^{2m+n-5} & \dots & \alpha_2^2 & 1 & \alpha_2^{n-2} & \alpha_2^{n-4} \dots & \alpha_2^3 & \alpha_2 \\ \alpha_3^{2m+n-3} & \alpha_3^{2m+n-5} & \dots & \alpha_3^2 & 1 & \alpha_3^{n-2} & \alpha_3^{n-4} \dots & \alpha_3^3 & \alpha_3 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ \alpha_n^{2m+n-3} & \alpha_n^{2m+n-5} & \dots & \alpha_n^2 & 1 & \alpha_n^{n-2} & \alpha_n^{n-4} \dots & \alpha_n^3 & \alpha_n \end{pmatrix}.$$

We still have the non-zero minor Q given by (4.27). Hence $b_{2m+n-2} = m+n-1-n = m-1$ as stated. □

It is easy to find all the odd coefficients of the Poincaré series for terms larger than a certain degree.

Proposition 4.8.2. *Let $i \geq 2m + n - 1$ with i odd. Then in (4.24) $b_i = i + 1 - m - n$.*

Proof. As in the proof of Lemma 4.8.1 we have $b_i = i + 1 - m - \text{rk}A$, where A is the matrix of the system of quasi-invariance conditions as equations for the coefficients of a polynomial of degree i. By the same reasons as in Lemma 4.8.1 we get rk$A = n$, so the Lemma follows. □

In the next Proposition we determine all the even coefficients of the Poincaré series starting with a certain degree.

Proposition 4.8.3. *Let $i = 2m+2n+2t$ where $t \in \mathbb{Z}_{\geq 0}$. Then in (4.24) $b_i = i+1-m-n$.*

Proof. Let q be a homogeneous polynomial of degree $2m + 2n + 2t$,

$$q = \sum_{i=0}^{2m+2n+2t} a_i x^{2m+2n+2t-i} y^i.$$

where $a_i \in \mathbb{C}$. The matrix of the system of linear equations for the coefficients of q has the following structure after elementary transformations

$$\left(\begin{array}{cccccc|c} \alpha_1^{2m+2n+2t-1} & \alpha_1^{2m+2n+2t-3} & \dots & \alpha_1^5 & \alpha_1^3 & \alpha_1 & \\ \alpha_2^{2m+2n+2t-1} & \alpha_2^{2m+2n+2t-3} & \dots & \alpha_2^5 & \alpha_2^3 & \alpha_2 & \\ \alpha_3^{2m+2n+2t-1} & \alpha_3^{2m+2n+2t-3} & \dots & \alpha_3^5 & \alpha_3^3 & \alpha_3 & A \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \\ \alpha_n^{2m+2n+2t-1} & \alpha_n^{2m+2n+2t-3} & \dots & \alpha_n^5 & \alpha_n^3 & \alpha_n & \end{array}\right) \tag{4.28}$$

where the block A consists of $n+t$ columns A_i

$$A_i = \begin{pmatrix} (2i-1)\alpha_1^{2i-2} - (2m+2n+2t-2i+1)\alpha_1^{2i} \\ (2i-1)\alpha_2^{2i-2} - (2m+2n+2t-2i+1)\alpha_2^{2i} \\ (2i-1)\alpha_3^{2i-2} - (2m+2n+2t-2i+1)\alpha_3^{2i} \\ \vdots \\ (2i-1)\alpha_n^{2i-2} - (2m+2n+2t-2i+1)\alpha_n^{2i} \end{pmatrix}$$

with $1 \le i \le n+t$. Suppose the rank of the matrix (4.28) is not full, so the rank is less than n. This means that any $n \times n$ minor must be zero. We are going to show that at least one $n \times n$ minor is non-zero, a contradiction which implies the rank of the original matrix is in fact n. First we assume that $\alpha_i = -\alpha_{[n/2]+i}$, $1 \le i \le [n/2]$, and $\alpha_n = 0$ if n is odd. This assumption will be justified later. Under this assumption the matrix (4.28) can be rearranged to the matrix

$$\left(\begin{array}{cccccc|c} \alpha_1^{2m+2n+2t-1} & \alpha_1^{2m+2n+2t-3} & \dots & \alpha_1^5 & \alpha_1^3 & \alpha_1 & \\ \alpha_2^{2m+2n+2t-1} & \alpha_2^{2m+2n+2t-3} & \dots & \alpha_2^5 & \alpha_2^3 & \alpha_2 & \\ \alpha_3^{2m+2n+2t-1} & \alpha_3^{2m+2n+2t-3} & \dots & \alpha_3^5 & \alpha_3^3 & \alpha_3 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \\ \alpha_{[n/2]}^{2m+2n+2t-1} & \alpha_{[n/2]}^{2m+2n+2t-3} & \dots & \alpha_{[n/2]}^5 & \alpha_{[n/2]}^3 & \alpha_{[n/2]} & \\ \hline & & & & & & \mathrm{B} \\ & & & 0 & & & \end{array}\right) \tag{4.29}$$

where block B consists of $n+t$ columns of the form

$$C_i = \begin{pmatrix} (2i-1)\alpha_1^{2i-2} - (2m+2n+2t-2i+1)\alpha_1^{2i} \\ (2i-1)\alpha_2^{2i-2} - (2m+2n+2t-2i+1)\alpha_2^{2i} \\ (2i-1)\alpha_3^{2i-2} - (2m+2n+2t-2i+1)\alpha_3^{2i} \\ \vdots \\ (2i-1)\alpha_{[n/2]}^{2i-2} - (2m+2n+2t-2i+1)\alpha_{[n/2]}^{2i} \\ (2i-1)0^{2i-2} \end{pmatrix} \tag{4.30}$$

where $1 \leq i \leq n+t$ and n is odd, so in particular $\alpha_n = 0$. When n is even C_i has the same form (4.30) with the final row removed. By Lemma 4.7.3 the block B contains at least one non-zero $n/2 \times n/2$ minor when n is even. When n is odd B contains a non-zero $\frac{n+1}{2} \times \frac{n+1}{2}$ minor which contains the last row and column of B. In order to show that the rank of the original matrix (4.28) is n we have to justify the assumption $\alpha_i^2 = \alpha_{[n/2]+i}^2$, $1 \leq i \leq [n/2]$, and $\alpha_n = 0$ if n is odd, which allowed us to write (4.28) in the form detailed in (4.29). We make the following elementary observation: since the block B contains at least one non-zero $[\frac{n+1}{2}] \times [\frac{n+1}{2}]$ minor, it contains at least one non-zero minor of each size $k \times k$, where $1 \leq k \leq [\frac{n+1}{2}]$. Armed with this observation, consider the matrix (4.28). If its rank is not n then any $n \times n$ minor must be zero. Consider

$$Q_1 = \begin{vmatrix} \alpha_1^{2n-1} & \alpha_1^{2n-3} & \alpha_1^{2n-5} & \dots & \alpha_1^3 & \alpha_1 \\ \alpha_2^{2n-1} & \alpha_2^{2n-3} & \alpha_2^{2n-5} & \dots & \alpha_2^3 & \alpha_2 \\ \alpha_3^{2n-1} & \alpha_3^{2n-3} & \alpha_3^{2n-5} & \dots & \alpha_3^3 & \alpha_3 \\ \vdots & & & & & \\ \alpha_n^{2n-1} & \alpha_n^{2n-3} & \alpha_n^{2n-5} & \dots & \alpha_n^3 & \alpha_n \end{vmatrix}$$

$$= \prod_{i=1}^{n} \alpha_i \begin{vmatrix} \alpha_1^{2n-2} & \alpha_1^{2n-4} & \alpha_1^{2n-6} & \dots & \alpha_1^2 & 1 \\ \alpha_2^{2n-2} & \alpha_2^{2n-4} & \alpha_2^{2n-6} & \dots & \alpha_2^2 & 1 \\ \alpha_3^{2n-2} & \alpha_3^{2n-4} & \alpha_3^{2n-6} & \dots & \alpha_3^2 & 1 \\ \vdots & & & & & \\ \alpha_n^{2n-2} & \alpha_n^{2n-4} & \alpha_n^{2n-6} & \dots & \alpha_n^2 & 1 \end{vmatrix}.$$

So $Q_1 = 0 \iff \alpha_i^2 = \alpha_j^2$ for some $1 \leq i < j \leq n$, or $\alpha_k = 0$ for some $k, 1 \leq k \leq n$. Suppose first that after relabelling $\alpha_1 = 0$. Then the original matrix has the following form

$$\left(\begin{array}{cccccc|c} 0 & 0 & \dots & 0 & 0 & 0 & \\ \alpha_2^{2m+2n+2t-1} & \alpha_2^{2m+2n+2t-3} & \dots & \alpha_2^5 & \alpha_2^3 & \alpha_2 & \\ \alpha_3^{2m+2n+2t-1} & \alpha_3^{2m+2n+2t-3} & \dots & \alpha_3^5 & \alpha_3^3 & \alpha_3 & \text{A} \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \\ \alpha_n^{2m+2n+2t-1} & \alpha_n^{2m+2n+2t-3} & \dots & \alpha_n^5 & \alpha_n^3 & \alpha_n & \end{array}\right),$$

where the block A consists of $n+t$ columns each with the following structure

$$D_i = \begin{pmatrix} (2i-1)0^{2i-2} \\ (2i-1)\alpha_2^{2i-2} - (2m+2n+2t-2i+1)\alpha_2^{2i} \\ (2i-1)\alpha_3^{2i-2} - (2m+2n+2t-2i+1)\alpha_3^{2i} \\ \vdots \\ (2i-1)\alpha_n^{2i-2} - (2m+2n+2t-2i+1)\alpha_n^{2i} \end{pmatrix}$$

with $1 \le i \le n+t$. In this situation the determinant

$$Q_1' = \left| \begin{array}{ccccc|c} 0 & 0 & 0 & \dots & 0 & \\ \alpha_2^{2n-3} & \alpha_2^{2n-5} & \alpha_2^{2n-7} & \dots & \alpha_2 & \\ \alpha_3^{2n-3} & \alpha_3^{2n-5} & \alpha_3^{2n-7} & \dots & \alpha_3 & D_1 \\ \vdots & \vdots & \vdots & & \vdots & \\ \alpha_n^{2n-3} & \alpha_n^{2n-5} & \alpha_n^{2n-7} & \dots & \alpha_n & \end{array} \right| \neq 0.$$

So we may assume that if the rank of the matrix (4.28) is less than n then $\alpha_1^2 = \alpha_2^2$ after relabelling. In this situation the matrix (4.28) is equivalent by row transformations to the matrix

$$\left(\begin{array}{cccccc|c} 0 & 0 & \dots & 0 & 0 & 0 & \\ \alpha_1^{2m+2n+2t-1} & \alpha_1^{2m+2n+2t-3} & \dots & \alpha_1^5 & \alpha_1^3 & \alpha_1 & \\ \alpha_3^{2m+2n+2t-1} & \alpha_3^{2m+2n+2t-3} & \dots & \alpha_3^5 & \alpha_3^3 & \alpha_3 & \widehat{A} \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \\ \alpha_n^{2m+2n+2t-1} & \alpha_n^{2m+2n+2t-3} & \dots & \alpha_n^5 & \alpha_n^3 & \alpha_n & \end{array} \right) \tag{4.31}$$

where the block $\widehat{A}$ consists of $n+t$ columns each with the following structure

$$\begin{pmatrix} (2i-1)\alpha_1^{2i-2} - (2m+2n+2t-2i+1)\alpha_1^{2i} \\ 0 \\ (2i-1)\alpha_3^{2i-2} - (2m+2n-2i+2t+1)\alpha_3^{2i} \\ \vdots \\ (2i-1)\alpha_n^{2i-2} - (2m+2n-2i+2t+1)\alpha_n^{2i} \end{pmatrix}$$

with $1 \le i \le n+t$. Note that the rows $1, 3 \dots, [n/2], [n/2]+1$ of $\widehat{A}$ coincide up to relabelling of the $\alpha_i s$ with the matrix $\widehat{B}$ formed by the first $[n/2]$ rows of the matrix B from (4.29), (4.30). As $\mathrm{rk} B = [\frac{n+1}{2}]$ the first row of the matrix $\widehat{A}$ contains some non-zero entry which we will denote by $B_{1\times 1}$. Now, consider the following minor of the matrix (4.31)

$$Q_2 = \begin{vmatrix} 0 & 0 & 0 & \dots & 0 & 0 & B_{1\times 1} \\ \alpha_1^{2n-3} & \alpha_1^{2n-5} & \alpha_1^{2n-7} & \dots & \alpha_1^3 & \alpha_1 & 0 \\ \alpha_3^{2n-3} & \alpha_3^{2n-5} & \alpha_3^{2n-7} & \dots & \alpha_3^3 & \alpha_3 & \\ \alpha_4^{2n-3} & \alpha_4^{2n-5} & \alpha_4^{2n-7} & \dots & \alpha_4^3 & \alpha_4 & * \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \\ \alpha_n^{2n-3} & \alpha_n^{2n-5} & \alpha_n^{2n-7} & \dots & \alpha_n^3 & \alpha_n & \end{vmatrix}.$$

Since $B_{1\times 1} \neq 0$, $Q_2 = 0 \iff \alpha_i^2 = \alpha_j^2$ for some $3 \leq i < j \leq n$ or $\alpha_k = 0$ for $k \neq 1, 2$. Suppose $\alpha_3 = 0$ up to relabelling. In this situation (4.31) is equivalent to a matrix which has minor

$$Q_2' = \begin{vmatrix} 0 & 0 & 0 & \dots & 0 & 0 & \\ 0 & 0 & 0 & \dots & 0 & 0 & B_{2\times 2} \\ \alpha_1^{2n-5} & \alpha_1^{2n-7} & \alpha_1^{2n-9} & \dots & \alpha_1^3 & \alpha_1 & \\ \alpha_4^{2n-5} & \alpha_4^{2n-7} & \alpha_4^{2n-9} & \dots & \alpha_4^3 & \alpha_4 & * \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \\ \alpha_n^{2n-5} & \alpha_n^{2n-7} & \alpha_n^{2n-9} & \dots & \alpha_n^3 & \alpha_n & \end{vmatrix}$$

where $B_{2\times 2}$ is a 2×2 submatrix of the first two rows of A such that the corresponding minor $|B_{2\times 2}| \neq 0$. Such a submatrix exists since $\mathrm{rk}\widehat{B} = [\frac{n+1}{2}]$. So we have $Q_2' \neq 0$. So we may assume that if the rank is less than n then $\alpha_1^2 = \alpha_2^2$ and $\alpha_3^2 = \alpha_4^2$ up to relabelling. It is not hard to see that we can continue in this way to deduce that if the rank is less than n then (up to relabelling) $\alpha_i^2 = \alpha_{[n/2]+i}^2$, $1 \leq i \leq [n/2]$. If n is even we stop at this point. Now, we need to record what happens when n is odd. Thus far we have seen that if the rank of the matrix (4.28) is less than n then we have the $\frac{n-1}{2}$ pairs $\alpha_i^2 = \alpha_{(n-1)/2+i}^2$, $1 \leq i \leq (n-1)/2$. In this situation (4.28) is equivalent to

$$\left(\begin{array}{cccccc|c} \alpha_1^{2m+2n+2t-1} & \alpha_1^{2m+2n+2t-3} & \dots & \alpha_1^5 & \alpha_1^3 & \alpha_1 & \\ \alpha_2^{2m+2n+2t-1} & \alpha_2^{2m+2n+2t-3} & \dots & \alpha_2^5 & \alpha_2^3 & \alpha_2 & \\ \alpha_3^{2m+2n+2t-1} & \alpha_3^{2m+2n+2t-3} & \dots & \alpha_3^5 & \alpha_3^3 & \alpha_3 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \\ \alpha_{\frac{n-1}{2}}^{2m+2n+2t-1} & \alpha_{\frac{n-1}{2}}^{2m+2n+2t-3} & \dots & \alpha_{\frac{n-1}{2}}^5 & \alpha_{\frac{n-1}{2}}^3 & \alpha_{\frac{n-1}{2}} & \\ \hline \multicolumn{6}{c|}{0} & \widehat{B} \\ \hline \alpha_n^{2m+2n+2t-1} & \alpha_n^{2m+2n+2t-3} & \dots & \alpha_n^5 & \alpha_n^3 & \alpha_n & * \end{array}\right) \tag{4.32}$$

where the block $\widehat{B}$ is up to relabelling the first $[n/2]$ rows of the block B introduced in (4.29), (4.30). Thus we can consider the following minor of (4.32)

$$Q'_{n-1} = \left| \begin{array}{cccccc|c} 0 & 0 & 0 & \dots & 0 & 0 & \\ 0 & 0 & 0 & \dots & 0 & 0 & B_{\frac{n-1}{2}\times\frac{n-1}{2}} \\ \alpha^n_{\frac{n+1}{2}} & \alpha^{n-2}_{\frac{n+1}{2}} & \alpha^{n-4}_{\frac{n+1}{2}} & \dots & \alpha^3_{\frac{n+1}{2}} & \alpha_{\frac{n+1}{2}} & \\ \alpha^n_{\frac{n+3}{2}} & \alpha^{n-2}_{\frac{n+3}{2}} & \alpha^{n-4}_{\frac{n+3}{2}} & \dots & \alpha^3_{\frac{n+3}{2}} & \alpha_{\frac{n+3}{2}} & * \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \\ \alpha^n_{n-1} & \alpha^{n-2}_{n-1} & \alpha^{n-4}_{n-1} & \dots & \alpha^3_{n-1} & \alpha_{n-1} & \\ \alpha^n_{n} & \alpha^{n-2}_{n} & \alpha^{n-4}_{n} & \dots & \alpha^3_{n} & \alpha_{n} & \end{array} \right|$$

where $B_{\frac{n-1}{2}\times\frac{n-1}{2}}$ corresponds to a non-zero minor of $\widehat{B}$. We have $Q'_{n-1} = 0 \iff \alpha_n = 0$. So if the rank of (4.28) is less than n and n is odd then $\alpha_i^2 = \alpha^2_{(n-1)/2+i}$, $1 \le i \le (n-1)/2$ and $\alpha_n = 0$. This is the case we started our analysis with. To conclude, we return to (4.29). If n is even consider the following $n \times n$ minor of (4.29)

$$\left| \begin{array}{cccccc|c} \alpha_1^{n-1} & \alpha_1^{n-3} & \dots & \alpha_1^5 & \alpha_1^3 & \alpha_1 & \\ \alpha_2^{n-1} & \alpha_2^{n-3} & \dots & \alpha_2^5 & \alpha_2^3 & \alpha_2 & \\ \alpha_3^{n-1} & \alpha_3^{n-3} & \dots & \alpha_3^5 & \alpha_3^3 & \alpha_3 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \\ \alpha_{n/2}^{n-1} & \alpha_{n/2}^{n-3} & \dots & \alpha_{n/2}^5 & \alpha_{n/2}^3 & \alpha_{n/2} & \\ \hline & & & 0 & & & B_{n/2\times n/2} \end{array} \right| .$$

It is non-zero. Similarly if n is odd consider the following $n \times n$ minor of (4.29)

$$\left| \begin{array}{cccccc|c} \alpha_1^{n-2} & \alpha_1^{n-4} & \dots & \alpha_1^5 & \alpha_1^3 & \alpha_1 & \\ \alpha_2^{n-2} & \alpha_2^{n-4} & \dots & \alpha_2^5 & \alpha_2^3 & \alpha_2 & \\ \alpha_3^{n-2} & \alpha_3^{n-4} & \dots & \alpha_3^5 & \alpha_3^3 & \alpha_3 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \\ \alpha_{\frac{n-1}{2}}^{n-2} & \alpha_{\frac{n-1}{2}}^{n-4} & \dots & \alpha_{\frac{n-1}{2}}^5 & \alpha_{\frac{n-1}{2}}^3 & \alpha_{\frac{n-1}{2}} & \\ \hline & & & 0 & & & B_{\frac{n+1}{2}\times\frac{n+1}{2}} \end{array} \right|$$

which is also non-zero, and we are done. □

Now we study the dimension of quasi-invariants of degree $2(m+n-1)$. It appears that this dimension is the same for any configuration $\mathcal{A}$ except one configuration whose geometry is fully fixed.

Proposition 4.8.4. *In (4.24),* $b_{2m+2n-2} = m+n-1$ *unless*

$$e_r = \binom{[n/2]}{r} \frac{\prod_{i=1}^{r}(2[n/2]-2i+1)}{\prod_{i=1}^{r}(2m+2i-1)}$$

where $0 \leq r \leq [n/2]$ *and*

$$\alpha_i^2 = \alpha_{[n/2]+i}^2,$$

$1 \leq i \leq [n/2]$*, and if* n *is odd,* $\alpha_n = 0$*, in which case* $b_{2m+2n-2} = m+n$*.*

Proof. Let q be a homogeneous quasi-invariant polynomial of degree $2(m+n-1)$, let

$$q = \sum_{i=0}^{2m+2n-2} a_i x^{2m+2n-2-i} y^i,$$

where $a_i \in \mathbb{C}$. The matrix of the system of linear equations for the coefficients of q has the structure.

$$M = \left(\begin{array}{cccccc|c} \alpha_1^{2m+2n-3} & \alpha_1^{2m+2n-5} & \dots & \alpha_1^5 & \alpha_1^3 & \alpha_1 & \\ \alpha_2^{2m+2n-3} & \alpha_2^{2m+2n-5} & \dots & \alpha_2^5 & \alpha_2^3 & \alpha_2 & \\ \alpha_3^{2m+2n-3} & \alpha_3^{2m+2n-5} & \dots & \alpha_3^5 & \alpha_3^3 & \alpha_3 & \mathrm{A} \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \\ \alpha_n^{2m+2n-3} & \alpha_n^{2m+2n-5} & \dots & \alpha_n^5 & \alpha_n^3 & \alpha_n & \end{array}\right), \tag{4.33}$$

where the block A consists of $n-1$ columns each with the following structure

$$A_i = \begin{pmatrix} (2i-1)\alpha_1^{2i-2} - (2m+2n-2i-1)\alpha_1^{2i} \\ (2i-1)\alpha_2^{2i-2} - (2m+2n-2i-1)\alpha_2^{2i} \\ (2i-1)\alpha_3^{2i-2} - (2m+2n-2i-1)\alpha_3^{2i} \\ \vdots \\ (2i-1)\alpha_n^{2i-2} - (2m+2n-2i-1)\alpha_n^{2i} \end{pmatrix}$$

with $1 \leq i \leq n-1$. We will show first that $\mathrm{rk}M \geq n-1$. Assume initially that $\alpha_i^2 = \alpha_{[n/2]+i}^2$, $1 \leq i \leq [n/2]$, and $\alpha_n = 0$ if n is odd. Under this assumption the matrix M is equivalent to

$$\left(\begin{array}{cccccc|c} \alpha_1^{2m+2n-3} & \alpha_1^{2m+2n-5} & \dots & \alpha_1^5 & \alpha_1^3 & \alpha_1 & \\ \alpha_2^{2m+2n-3} & \alpha_2^{2m+2n-5} & \dots & \alpha_2^5 & \alpha_2^3 & \alpha_2 & \\ \alpha_3^{2m+2n-3} & \alpha_3^{2m+2n-5} & \dots & \alpha_3^5 & \alpha_3^3 & \alpha_3 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \\ \alpha_{[n/2]}^{2m+2n-3} & \alpha_{[n/2]}^{2m+2n-5} & \dots & \alpha_{[n/2]}^5 & \alpha_{[n/2]}^3 & \alpha_{[n/2]} & \\ \hline & & 0 & & & & B \end{array}\right). \tag{4.34}$$

The block B consists of $n-1$ columns, each of which has the following structure

$$C_i = \begin{pmatrix} (2i-1)\alpha_1^{2i-2} - (2m+2n-2i-1)\alpha_1^{2i} \\ (2i-1)\alpha_2^{2i-2} - (2m+2n-2i-1)\alpha_2^{2i} \\ (2i-1)\alpha_3^{2i-2} - (2m+2n-2i-1)\alpha_3^{2i} \\ \vdots \\ (2i-1)\alpha_{[n/2]}^{2i-2} - (2m+2n-2i-1)\alpha_{[n/2]}^{2i} \\ (2i-1)0^{2i-2} \end{pmatrix} \tag{4.35}$$

where $1 \leq i \leq n-1$, and the last row should be removed if n is even. Now, by Lemma 4.7.9 there exists a non-zero $([n/2]-1) \times ([n/2]-1)$ minor of $\widehat{B}$ where $\widehat{B}$ is the submatrix of B formed by the first $[n/2]$ rows. The existence of this $([n/2]-1) \times ([n/2]-1)$ minor $B_{([n/2]-1)\times([n/2]-1)}$ allows us to reproduce the arguments from Proposition 4.8.3 which justify the assumption $\alpha_i^2 = \alpha_{[n/2]+i}^2$, $1 \leq i \leq [n/2]$, and $\alpha_n = 0$ if n is odd, if the rank of M is less than n. If n is even there is a non-zero $(n-1)\times(n-1)$ minor of (4.33) of the form

$$\left|\begin{array}{cccccc|c} \alpha_1^{n-1} & \alpha_1^{n-3} & \dots & \alpha_1^5 & \alpha_1^3 & \alpha_1 & \\ \alpha_2^{n-1} & \alpha_2^{n-3} & \dots & \alpha_2^5 & \alpha_2^3 & \alpha_2 & \\ \alpha_3^{n-1} & \alpha_3^{n-3} & \dots & \alpha_3^5 & \alpha_3^3 & \alpha_3 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \\ \alpha_{n/2}^{n-1} & \alpha_{n/2}^{n-3} & \dots & \alpha_{n/2}^5 & \alpha_{n/2}^3 & \alpha_{n/2} & \\ \hline & & & 0 & & & B_{(n/2-1)\times(n/2-1)} \end{array}\right|,$$

while if n is odd there is a non-zero $(n-1)\times(n-1)$ minor of (4.33) of the form

$$\left|\begin{array}{cccccc|c} \alpha_1^{n-2} & \alpha_1^{n-4} & \dots & \alpha_1^5 & \alpha_1^3 & \alpha_1 & \\ \alpha_2^{n-2} & \alpha_2^{n-4} & \dots & \alpha_2^5 & \alpha_2^3 & \alpha_2 & \\ \alpha_3^{n-2} & \alpha_3^{n-4} & \dots & \alpha_3^5 & \alpha_3^3 & \alpha_3 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \\ \alpha_{\frac{n-1}{2}}^{n-2} & \alpha_{\frac{n-1}{2}}^{n-4} & \dots & \alpha_{\frac{n-1}{2}}^5 & \alpha_{\frac{n-1}{2}}^3 & \alpha_{\frac{n-1}{2}} & \\ \hline & & & 0 & & & B_{\frac{n-1}{2}\times\frac{n-1}{2}} \end{array}\right|,$$

so the rank of M is at least $n-1$.

Now, suppose the rank of M is $n-1$. Then any $n \times n$ minor must vanish. Returning to the block B, by Lemma 4.7.1 in its notation we have

$$B_L = \Delta \prod_{i=1}^{[n/2]} \alpha_i^{2L-2} \sum_{i=0}^{[n/2]} (-1)^i \prod_{r=i}^{[n/2]-1} (2[n/2]-2r+2L-3) \prod_{r=0}^{i-1} (2m+2r+2n-2[n/2]-2L+1) e_i$$

where $1 \leq L \leq n/2$ if n is even and $2 \leq L \leq \frac{n+1}{2}$ if n is odd. Note that when n is odd we do not consider the minor B_1. In this case the minors B_L, $2 \leq L \leq \frac{n+1}{2}$, we consider have size $\frac{n-1}{2} \times \frac{n-1}{2}$. However we can then include the last row and column of B in each of these minors to create a minor of size $\frac{n+1}{2} \times \frac{n+1}{2}$. Adding the last row and column does not change the expression for each B_L, $2 \leq L \leq \frac{n+1}{2}$. Thus we can consider both the odd and even cases together. Suppose all the B_L vanish. Regard the resulting conditions as a system of linear equations for the unknowns $e_1, \ldots, e_{[n/2]}$. Then by Lemma 4.7.4 the corresponding matrix is non-degenerate. The system can be solved by setting

$$e_r = \binom{[n/2]}{r} \prod_{i=1}^{r} \frac{(2[n/2] - 2i + 1)}{(2m + 2i - 1)}$$

for $1 \leq r \leq [n/2]$ and $e_0 = 1$. This follows from Lemma 4.7.2. $\square$

Theorem 4.8.5. *Suppose the configuration $\mathcal{A}$ satisfies the following properties*

$$\alpha_i^2 = \alpha_{[n/2]+i}^2$$

where $1 \leq i \leq [n/2]$, $\alpha_n = 0$ if n is odd, and

$$e_r = \binom{[n/2]}{r} \prod_{i=1}^{r} \frac{(2[n/2] - 2i + 1)}{(2m + 2i - 1)} \tag{4.36}$$

where $0 \leq r \leq [n/2]$. Then $\mathcal{A} = \mathcal{A}_{(m,1^n)}$.

We will establish this theorem using the following strategy. Suppose that the elementary symmetric polynomials $\widehat{e}_i$ in the variables z_i (see Lemma 2.3.1) take the values specified in Definition 2.4.7, that $z_0 = 1$ and that $z_i z_{n-i+1} = 1$ for $1 \leq i \leq n$ which means that $\mathcal{A} = \mathcal{A}_{(m,1^n)}$. We will show that this happens if and only if the following two properties are satisfied after rotation by $\pi/2$ and relabelling: the elementary symmetric polynomials e_r, where $0 \leq r \leq [n/2]$, take the values specified in Theorem 4.8.5 and $\alpha_i^2 = \alpha_{[n/2]+i}^2$ where $1 \leq i \leq [n/2]$, and $\alpha_n = 0$ if n is odd. First we collect some useful observations and introduce some notation.

(i) Note that $\alpha_i^2 = \cot^2(\theta_i)$ where θ_i is such that the i-th vector of $\mathcal{A}_{(m,1^n)}$ has the form $(\cos\theta_i, \sin\theta_i)$. We have $\cot^2(\theta_i) = \frac{1}{\sin^2\theta_i} - 1$.

(ii) Let $u_i = \sin^2(\theta_i)$ for $1 \leq i \leq [n/2]$. Recall from Definition 2.4.7 that $z_j = \cos(2\theta_j) + i\sin(2\theta_j)$ and that (since $z_j z_{n-j+1} = 1$) $\theta_j = -\theta_{n-j+1}$, so that $z_j + z_{n-j+1} = 2\cos(2\theta_j) = 2(1 - 2\sin^2(\theta_j)) = 2(1 - 2u_j)$.

(iii) Finally note that, since $\theta_i = -\theta_{n-i+1}$, $\alpha_i^2 = \cot^2(\theta_i) = \cot^2(\theta_{n-i+1}) = \alpha_{n-i+1}^2$. So, up to relabelling the property $\alpha_i^2 = \alpha_{[n/2]+i}^2$ is satisfied. If n is odd note that (since

$z_j z_{n-j+1} = 1$), $z^2_{\frac{n+1}{2}} = 1$ so $\theta_{\frac{n+1}{2}} = \pi/2$ and thus $\alpha^2_{\frac{n+1}{2}} = \cot^2(\theta_{\frac{n+1}{2}}) = 0$. Thus up to relabelling $\alpha_n = 0$.

Denote by f_i, $0 \leq i \leq [n/2]$, the i-th elementary symmetric polynomials in the variables $u_j = \sin^2(\theta_j)$, $j = 1, 2, \ldots, [n/2]$. We are going to relate the configuration $\mathcal{A}$ with e_r given by (4.36) and the configuration $\mathcal{A}_{(m,1^n)}$ in two steps. First we find the values of the elementary symmetric polynomials f_i for the configuration $\mathcal{A}_{(m,1^n)}$ (Lemmas 4.8.6, 4.8.7 below). Then we check in the proof of Lemma 4.8.8 that these f_i lead to the relations (4.36).

Lemma 4.8.6. *When n is even we have*

$$\widehat{e_r} = \sum_{i=0}^{r} (-1)^i 2^{2i} \binom{n-2i}{r-i} f_i \tag{4.37}$$

for $0 \leq r \leq n/2$. When n is odd we have

$$\widehat{e_r} = \sum_{i=0}^{r} (-1)^i 2^{2i} \binom{n-2i-1}{r-i} \frac{2r-n}{n-i-r} f_i \tag{4.38}$$

for $0 \leq r \leq \frac{n-1}{2}$.

Proof. First suppose that n is even. Recall that the $\widehat{e_r}$ are the elementary symmetric polynomials in the variables z_i. Recall also that if $\mathcal{A} = \mathcal{A}_{(m,1^n)}$ then $z_i z_{n-i+1} = 1$ for $1 \leq i \leq n/2$. So for $0 \leq r \leq n/2$ we have

$$\begin{aligned}
\widehat{e_r} &= \sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq n} z_{i_1} z_{i_2} \ldots z_{i_r} \\
&= \sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq n/2} x_{i_1} x_{i_2} \ldots x_{i_r} + (n/2 - r + 2) \sum_{1 \leq i_1 < i_2 < \ldots < i_{r-2} \leq n/2} x_{i_1} x_{i_2} \ldots x_{i_{r-2}} \\
&+ (n/2 - r + 4) \sum_{1 \leq i_1 < i_2 < \ldots < i_{r-4} \leq n/2} x_{i_1} x_{i_2} \ldots x_{i_{r-4}} \\
&+ (n/2 - r + 6) \sum_{1 \leq i_1 < i_2 < \ldots < i_{r-6} \leq n/2} x_{i_1} x_{i_2} \ldots x_{i_{r-6}} \\
&+ \ldots
\end{aligned}$$

where $x_{i_t} = z_{i_t} + z_{n-i_t+1}$. Since $z_i + z_{n-i+1} = 2(1 - 2u_i)$ we have

$$\begin{aligned}
\widehat{e_r} = 2^r &\sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq n/2} (1 - 2u_{i_1})(1 - 2u_{i_2}) \ldots (1 - 2u_{i_r}) \\
&+ (n/2 - r + 2) 2^{r-2} \sum_{1 \leq i_1 < i_2 < \ldots < i_{r-2} \leq n/2} (1 - 2u_{i_1})(1 - 2u_{i_2}) \ldots (1 - 2u_{i_{r-2}}) \\
&+ (n/2 - r + 4) 2^{r-4} \sum_{1 \leq i_1 < i_2 < \ldots < i_{r-4} \leq n/2} (1 - 2u_{i_1})(1 - 2u_{i_2}) \ldots (1 - 2u_{i_{r-4}}) \\
&+ \ldots
\end{aligned}$$

In other words we have

$$\begin{aligned}\widehat{e}_r &= 2^r \sum_{j=0}^{r} (-1)^j \binom{n/2-j}{r-j} 2^j f_j \\ &+ 2^{r-2}(n/2-r+2) \sum_{j=0}^{r-2} (-1)^j \binom{n/2-j}{r-j-2} 2^j f_j \\ &+ 2^{r-4}(n/2-r+4) \sum_{j=0}^{r-4} (-1)^j \binom{n/2-j}{r-j-4} 2^j f_j \\ &+ \dots\end{aligned}$$

We can rearrange this to

$$\begin{aligned}\widehat{e}_r &= (-1)^r 2^{2r} \binom{n/2-r}{0} f_r \\ &+ (-1)^{r-1} 2^{2r-1} \binom{n/2-r+1}{1} f_{r-1} \\ &+ [(-1)^{r-2} 2^{2r-2} \binom{n/2-r+2}{2} + (-1)^{r-2} 2^{2r-4} \binom{n/2-r+2}{1} \binom{n/2-r+2}{0}] f_{r-2} \\ &+ [(-1)^{r-3} 2^{2r-3} \binom{n/2-r+3}{3} + (-1)^{r-3} 2^{2r-5} \binom{n/2-r+2}{1} \binom{n/2-r+3}{1}] f_{r-3} \\ &+ \dots \\ &= \sum_{i=0}^{r} ((-1)^{r-i} \sum_{s=0}^{[i/2]} 2^{2r-i-2s} \binom{n/2-r+2s}{s} \binom{n/2-r+i}{i-2s} f_{r-i}).\end{aligned}$$

Now, we are going to show that

$$\sum_{s=0}^{[i/2]} 2^{i-2s} \binom{n/2-r+2s}{s} \binom{n/2-r+i}{i-2s} = \binom{n-2r+2i}{i}. \tag{4.39}$$

We have

$$\begin{aligned}&\sum_{s=0}^{[i/2]} 2^{i-2s} \binom{n/2-r+2s}{s} \binom{n/2-r+i}{i-2s} \\ &= \frac{(n/2-r+i)!}{(n/2-r)!(i/2)!(i/2-1/2)(i/2-3/2)\dots 1/2} \sum_{s=0}^{[i/2]} \frac{(-i/2)_s (1/2-i/2)_s}{s!(n/2-r+1)_s} \\ &= \frac{(n/2-r+i)!}{(n/2-r)!(i/2)!(i/2-1/2)(i/2-3/2)\dots 1/2}\, {}_2F_1(-i/2, 1/2-i/2; n/2-r+1; 1) \\ &= \frac{(n/2-r+i)!}{(n/2-r)!(i/2)!(i/2-1/2)\dots 1/2} \frac{\Gamma(n/2-r+1)\Gamma(n/2-r+i/2+1/2)}{\Gamma(n/2-r+1+i/2)\Gamma(n/2-r+i/2+1/2)} \text{ (by Theorem 4.2.5)} \\ &= \frac{2^i (n/2-r+i)!}{i!} \frac{\Gamma(n/2-r+i+1/2)}{(n/2-r+i/2)!\Gamma(n/2-r+i/2+1/2)} \\ &= \frac{2^i (n/2-r+i)(n/2-r+i-1)\dots(n/2-r+i/2+1)(n/2-r+i-1/2)\dots(n/2-r+i/2+1/2)}{i!} \\ &= \frac{(n-2r+2i)(n-2r+2i-1)(n-2r+2i-2)\dots(n-2r+i+1)}{i!} \\ &= \binom{n-2r+2i}{i}.\end{aligned}$$

So if n is even we have

$$\widehat{e}_r = \sum_{i=0}^{r}(-1)^{r-i}2^{2r-2i}\binom{n-2r+2i}{i}f_{r-i} = \sum_{i=0}^{r}(-1)^{i}2^{2i}\binom{n-2i}{r-i}f_i$$

Now suppose that n is odd. Recall that in this case $z_{\frac{n+1}{2}} = -1$. Let us denote by $\widehat{e}_r(n-1)$ the elementary symmetric polynomials in the variables $z_1, z_2, \ldots, z_{n-1}$. Then up to relabelling of the z_i we have

$$\begin{aligned}
\widehat{e}_r &= \sum_{1\le i_1<i_2<\ldots<i_r\le n} z_{i_1}z_{i_2}\ldots z_{i_r} \\
&= \sum_{1\le i_1<i_2<\ldots<i_r\le n-1} z_{i_1}z_{i_2}\ldots z_{i_r} - \sum_{1\le i_1<i_2<\ldots<i_{r-1}\le n-1} z_{i_1}z_{i_2}\ldots z_{i_{r-1}} \\
&= \widehat{e}_r(n-1) - \widehat{e}_{r-1}(n-1) \\
&= \sum_{i=0}^{r}(-1)^i 2^{2i}\binom{n-2i-1}{r-i}f_i - \sum_{i=0}^{r-1}(-1)^i 2^{2i}\binom{n-2i-1}{r-i-1}f_i \\
&= \sum_{i=0}^{r}(-1)^i 2^{2i}(\binom{n-2i-1}{r-i} - \binom{n-2i-1}{r-i-1})f_i \\
&= \sum_{i=0}^{r}(-1)^i 2^{2i}\binom{n-2i-1}{r-i}\frac{n-2r}{n-i-r}f_i
\end{aligned}$$

as required. □

Lemma 4.8.7. *If the elementary symmetric polynomials f_i take the following values*

$$f_i = \binom{[n/2]}{i}\frac{\prod_{s=1}^{i}(2m+2[n/2]-2s+1)}{2^i\prod_{s=1}^{i}(m+n-s)}$$

for $0 \le i \le [n/2]$, then for n even the relation

$$\widehat{e}_r = (-1)^r\binom{n}{r}\binom{m+r-1}{r}\binom{m+n-1}{r}^{-1} = \sum_{i=0}^{r}(-1)^i 2^{2i}\binom{n-2i}{r-i}f_i$$

holds, while for n odd the relation

$$\widehat{e}_r = (-1)^r\binom{n}{r}\binom{m+r-1}{r}\binom{m+n-1}{r}^{-1} = \sum_{i=0}^{r}(-1)^i 2^{2i}\binom{n-2i-1}{r-i}\frac{n-2r}{n-i-r}f_i$$

holds.

Proof. First suppose n is even. We need to check if the relation

$$\begin{aligned}
\widehat{e}_r &= (-1)^r\binom{n}{r}\binom{m+r-1}{r}\binom{m+n-1}{r}^{-1} \\
&= \sum_{i=0}^{r}(-1)^i 2^i\binom{n-2i}{r-i}\binom{n/2}{i}\frac{\prod_{s=1}^{i}(2m+n-2s+1)}{\prod_{s=1}^{i}(m+n-s)}
\end{aligned} \tag{4.40}$$

holds. We claim that the identity (4.40) can be seen as a special case of Saalschütz's Theorem, see Theorem 4.2.6. The identity (4.40) follows upon considering

$$ {}_3F_2(-r, r-n, -m-n/2+1/2; -n/2+1/2, -m-n+1; 1). $$

To see this, note first that

$$ \begin{aligned} &\frac{(-n/2+1/2+r)_{m+n/2-1/2}(n/2+1/2-r)_{m+n/2-1/2}}{(-n/2+1/2)_{m+n/2-1/2}(n/2+1/2)_{m+n/2-1/2}} \\ &= (-1)^r \frac{m(m+1)(m+2)\dots(m+i-1)}{(m+n-1)(m+n-2)\dots(m+n-i)}. \end{aligned} $$

Also, note that

$$ \begin{aligned} &{}_3F_2(-r, r-n, -m-n/2+1/2; -n/2+1/2, -m-n+1; 1) \\ &= \sum_{i=0}^{\infty} \frac{(-r)_i (r-n)_i (-m-n/2+1/2)_i}{(-n/2+1/2)_i (-m-n+1)_i i!} \\ &= \sum_{i=0}^{r} (-1)^i \frac{r(r-1)\dots(r-i+1)(n-r)(n-r-1)\dots(n-r-i+1)\prod_{s=1}^{i}(2m+n-2s+1)}{2^i(n/2-1/2)\dots(n/2-1/2-i+1)(m+n-1)\dots(m+n-i)i!} \\ &= \sum_{i=0}^{r} (-1)^i r! \frac{n(n-1)\dots(n-r+1)(n-r)(n-r-1)\dots(n-r-i+1)\prod_{s=1}^{i}(2m+n-2s+1)}{(r-i)!(n-1)\dots(n-2i+1)\prod_{s=1}^{i}(m+n-s)i!n(n-1)\dots(n-r+1)} \\ &= \sum_{i=0}^{r} (-1)^i r! 2^i \frac{\prod_{s=0}^{i-1}(n/2-s)\prod_{s=0}^{r-i-1}(n-2i-s)\prod_{s=1}^{i}(2m+n-2s+1)}{(r-i)!\prod_{s=1}^{i}(m+n-s)i!n(n-1)\dots(n-r+1)} \\ &= \sum_{i=0}^{r} (-1)^i 2^i \binom{n-2i}{r-i}\binom{n/2}{i} \frac{\prod_{s=1}^{i}(2m+n-2s+1)}{\prod_{s=1}^{i}(m+n-s)} \frac{r!}{n(n-1)\dots(n-r+1)}. \end{aligned} $$

So

$$ \begin{aligned} &{}_3F_2(-r, r-n, -m-n/2+1/2; -n/2+1/2, -m-n+1; 1) \\ &= \sum_{i=0}^{r} (-1)^i 2^i \binom{n-2i}{r-i}\binom{n/2}{i} \frac{\prod_{s=1}^{i}(2m+n-2s+1)}{\prod_{s=1}^{i}(m+n-s)} \frac{r!}{n(n-1)\dots(n-r+1)} \\ &= (-1)^r \frac{m(m+1)(m+2)\dots(m+i-1)}{(m+n-1)(m+n-2)\dots(m+n-i)}. \end{aligned} $$

Thus we have

$$\begin{aligned}
&= \sum_{i=0}^{r}(-1)^i 2^i \binom{n-2i}{r-i}\binom{n/2}{i}\frac{\prod_{s=1}^{i}(2m+n-2s+1)}{\prod_{s=1}^{i}(m+n-s)} \\
&= (-1)^r \frac{m(m+1)(m+2)\dots(m+i-1)n(n-1)\dots(n-r+1)}{r!(m+n-1)(m+n-2)\dots(m+n-i)} \\
&= (-1)^r \binom{n}{r}\binom{m+r-1}{r}\binom{m+n-1}{r}^{-1}.
\end{aligned}$$

Now suppose n is odd. We need to check if the relation

$$\begin{aligned}
\widehat{e}_r &= (-1)^r \binom{n}{r}\binom{m+r-1}{r}\binom{m+n-1}{r}^{-1} \\
&= \sum_{i=0}^{r}(-1)^i 2^i \binom{n-2i-1}{r-i}\binom{\frac{n-1}{2}}{i}\frac{n-2r}{n-i-r}\frac{\prod_{s=1}^{i}(2m+n-2s)}{\prod_{s=1}^{i}(m+n-s)}
\end{aligned} \tag{4.41}$$

holds. The identity (4.41) can again be seen as a special case of Saalschütz's Theorem. It follows upon considering

$$_3F_2(-r, r-n, -m-\frac{n-1}{2}+1/2; -\frac{n-1}{2}+1/2, -m-n+1; 1).$$

To see this note that

$$\begin{aligned}
&\frac{(-\frac{n-1}{2}+1/2+r)_{m+\frac{n-1}{2}-1/2}(-\frac{n-1}{2}+1/2-r+n)_{m+\frac{n-1}{2}-1/2}}{(-\frac{n-1}{2}+1/2)_{m+\frac{n-1}{2}-1/2}(-\frac{n-1}{2}+1/2+n)_{m+\frac{n-1}{2}-1/2}} \\
&= (-1)^r \frac{m(m+1)(m+2)\dots(m+i-1)}{(m+n-1)(m+n-2)\dots(m+n-i)}\frac{n}{n-2r}.
\end{aligned}$$

Also, note that

$$\begin{aligned}
&_3F_2(-r, r-n, -m-\frac{n-1}{2}+1/2; -\frac{n-1}{2}+1/2, -m-n+1; 1) \\
&= \sum_{i=0}^{r}(-1)^i 2^i \binom{n-2i-1}{r-i}\binom{\frac{n-1}{2}}{i}\frac{\prod_{s=1}^{i}(2m+n-2s)}{\prod_{s=1}^{i}(m+n-s)}\frac{r!}{(n-1)\dots(n-r+1)}\frac{1}{n-i-r}.
\end{aligned}$$

The identity (4.41) now follows. □

Lemma 4.8.8. *Suppose that for* $0 \le r \le [n/2]$

$$\widehat{e}_r = (-1)^r \binom{n}{r}\binom{m+r-1}{r}\binom{m+n-1}{r}^{-1}.$$

Then for $0 \le r \le [n/2]$ *we have*

$$e_r = \binom{[n/2]}{r}\frac{\prod_{s=1}^{r}(2[n/2]-2s+1)}{\prod_{s=1}^{r}(2m+2s-1)}. \tag{4.42}$$

Proof. First note that by Lemma 4.8.7 we know that for $0 \leq i \leq [n/2]$

$$f_i = \binom{[n/2]}{i} \frac{\prod_{s=1}^{i}(2m + 2[n/2] - 2s + 1)}{2^i \prod_{s=1}^{i}(m + n - s)}.$$

We have

$$\begin{aligned}
e_r &= \sum_{i_1 < i_2 < \ldots < i_r} (\frac{1}{u_{i_1}} - 1)(\frac{1}{u_{i_2}} - 1)\ldots(\frac{1}{u_{i_r}} - 1) \\
&= (-1)^r \binom{[n/2]}{r} - \binom{[n/2]-1}{r-1} \sum_i \frac{1}{u_i} + \binom{[n/2]-2}{r-2} \sum_{i<j} \frac{1}{u_i u_j} - \binom{[n/2]-3}{r-3} \sum_{i<j<k} \frac{1}{u_i u_j u_k} \\
&+ \ldots \\
&+ \sum_{i_1 < i_2 < \ldots < i_r} \frac{1}{u_{i_1} u_{i_2} \ldots u_{i_r}} \\
&= \sum_{i=0}^{r} (-1)^{r-i} \binom{[n/2]-i}{r-i} \widetilde{f}_i
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{f}_i = \frac{f_{[n/2]-i}}{f_{[n/2]}} &= \binom{[n/2]}{[n/2]-i} \frac{\prod_{s=1}^{[n/2]-i}(2m + 2[n/2] - 2s + 1) 2^{[n/2]} \prod_{s=1}^{[n/2]}(m + n - s)}{2^{[n/2]-i} \prod_{s=1}^{[n/2]-i}(m + n - s) \prod_{s=1}^{[n/2]}(2m + 2[n/2] - 2s + 1)} \\
&= 2^i \binom{[n/2]}{i} \frac{\prod_{s=0}^{i-1}(m + [n/2] + s)}{\prod_{s=0}^{i-1}(2m + 2s + 1)}.
\end{aligned}$$

So we have

$$e_r = \sum_{i=0}^{r} (-1)^i 2^i \binom{[n/2]-i}{r-i} \binom{[n/2]}{i} \frac{\prod_{s=0}^{i-1}(m + n - [n/2] + s)}{\prod_{s=0}^{i-1}(2m + 2s + 1)}.$$

So we just need to check if the identity

$$\sum_{i=0}^{r} (-1)^{r-i} 2^i \binom{[n/2]-i}{r-i} \binom{[n/2]}{i} \frac{\prod_{s=0}^{i-1}(m + n - [n/2] + s)}{\prod_{s=0}^{i-1}(2m + 2s + 1)} = \binom{[n/2]}{r} \frac{\prod_{s=1}^{r}(2[n/2] - 2s + 1)}{\prod_{s=1}^{r}(2m + 2s - 1)} \tag{4.43}$$

holds. Multiplying by the denominator of the right hand side the left hand side becomes

$$\begin{aligned}
&\sum_{i=0}^{r} (-1)^{r-i} 2^i \binom{[n/2]-i}{r-i} \binom{[n/2]}{i} \prod_{s=0}^{i-1}(m + n - [n/2] + s) \prod_{s=i}^{r-1}(2m + 2s + 1) \\
&= \binom{[n/2]}{r} \sum_{i=0}^{r} (-1)^{r-i} 2^i \binom{r}{i} \prod_{s=0}^{i-1}(m + n - [n/2] + s) \prod_{s=i}^{r-1}(2m + 2s + 1).
\end{aligned}$$

We are going to consider each side of (4.43) as a polynomial in n and show that these polynomials are equal. First, note that the coefficient of the highest term in n (namely n^r) is clearly the same on both sides. Both sides are zero when $n = 0$. Suppose now that

n is even. The non-zero roots in n of the polynomial in the right hand side of (4.43) are $n = 1, 3, 5, \ldots, 2r-1$. When $n = 1$ the left hand side of (4.43) becomes

$$\binom{n/2}{r} \sum_{i=0}^{r} (-1)^{r-i} 2^i \binom{r}{i} \prod_{s=0}^{i-1} (m + 1/2 + s) \prod_{s=i}^{r-1} (2m + 2s + 1)$$
$$= \prod_{s=0}^{r-1} (2m + 2s + 1) \sum_{i=0}^{r} (-1)^i \binom{r}{i} = 0$$

using Lemma 4.2.7. Similarly if $n = 3$ the left hand side becomes

$$\binom{n/2}{r} \sum_{i=0}^{r} (-1)^{r-i} 2^i \binom{r}{i} \prod_{s=0}^{i-1} (m + 3/2 + s) \prod_{s=i}^{r-1} (2m + 2s + 1)$$
$$= \prod_{s=1}^{r-1} (2m + 2s + 1) \sum_{i=0}^{r} (-1)^i \binom{r}{i} (2m + 2i + 1) = 0$$

again using Lemma 4.2.7. If we continue in this way, it is easy to see that for each $n = 1, 3, 5, 7, \ldots, 2r-1$ we can write the left hand side of (4.43) in the form

$$P(m) \sum_{i=0}^{r} (-1)^i \binom{r}{i} Q(i)$$

where Q has degree less than r and P is some polynomial in m. (When n is $2r+1$ or higher Q has degree r or higher). Thus for $n = 1, 3, 5, \ldots, 2r-1$ Lemma 4.2.7 shows that the left hand side of (4.43) is equal to zero. Suppose now that n is odd. The polynomial in the right hand side of (4.43) has roots $n = 2, 4, \ldots, 2r$. It is easy to see that the polynomial in the left hand side has the same roots, so both polynomials coincide. Hence (4.43) holds for odd n as well. □

Theorem 4.8.5 now follows from Lemmas 4.8.6, 4.8.7 and 4.8.8.

4.9 $Q_{\mathcal{A}}$ is Gorenstein for type $(m, 1^n) \Rightarrow \mathcal{A} = \mathcal{A}_{(m,1^n)}$

In this section we are going to prove the following theorem.

Theorem 4.9.1. *Suppose $Q_{\mathcal{A}}$ is Gorenstein where $\mathcal{A}$ has type $(m, 1^n)$. Then $\mathcal{A} = \mathcal{A}_{(m,1^n)}$.*

We will do this by completing the following steps. First we will calculate the Poincaré series for an arbitrary arrangement of type $(m, 1^n)$. We refer to this series throughout as $P(t)$. We show that any restrictions on the geometry of the arrangement lead to only finite freedom in the coefficients of $P(t)$. We will state these possible series explicitly. We will then show that if any such series is palindromic then the arrangement in question has to be $\mathcal{A}_{(m,1^n)}$. More specifically when calculating $P(t)$ we will make use of the important fact that the rank of the matrices at certain degrees depends only on two parameters: the number of different α_i^2 and the number of α_i that are equal to zero. The latter is 0 or 1. By this we mean that the rank at certain degrees can change by a natural number determined wholly by the number of pairs $\alpha_i^2 = \alpha_j^2$, $i \neq j$ and the number of α_i that are equal to zero. This will make calculating possible Poincaré series significantly easier. To this end we set

$$r = \text{number of different } \alpha_i^2, i = 1, 2, \ldots, n$$

and we introduce the parameter δ which is defined by

$$\delta = \begin{cases} 1 & \text{if } \alpha_j = 0 \text{ for some } j, \\ 0 & \text{if } \alpha_j \neq 0 \ \forall\ j. \end{cases}$$

The interaction of the parameters m and n is important in the considerations to follow. We will need to modify our analysis slightly when moving between the cases $m \geq n$ and $m < n$. With this in mind we first state and prove lemmas which do not require any assumptions regarding the interaction of m and n before analyzing specific cases.

Recall the definition of the Poincaré series (4.24). We intend to calculate $P(t)$ making use wherever possible of the parameters r, δ. To this end let

$$P_{r,\delta}^{k,l} = \sum_{i=k}^{l} b_i t^i$$

and

$$P_{r,\delta,odd}^{k,l} = \sum_{k \leq 2i+1 \leq l} b_{2i+1} t^{2i+1}$$

and

$$P_{r,\delta,even}^{k,l} = \sum_{k \leq 2i \leq l} b_{2i} t^{2i}$$

for any arrangement $\mathcal{A}$, where the series calculated depend on r, δ. If r or δ or both do not affect the answer we will omit them from this notation.

At this point we will prove two elementary lemmas which will be useful in the calculation of $P(t)$.

Lemma 4.9.2. *Let $a, b \in \mathbb{Z}$ with $b > a$ and suppose that $\frac{b-a}{2} \in \mathbb{Z}$. Then*

$$\sum_{s=0}^{\frac{b-a}{2}} t^{a+2s} = \frac{t^a - t^{a+2} - t^{b+2} + t^{b+4}}{(t^2-1)^2}.$$

Proof.

$$\begin{aligned}\sum_{s=0}^{\frac{b-a}{2}} t^{a+2s} &= t^a + t^{a+2} + \ldots + t^b \\ &= t^a(1 + t^2 + \ldots + t^{b-a}) \\ &= t^a(\frac{1 - t^{b-a+2}}{1 - t^2}) \\ &= t^a \frac{(1 - t^{b-a+2})(1 - t^2)}{(t^2 - 1)^2} \\ &= t^a \frac{(1 - t^2 - t^{b-a+2} + t^{b-a+4})}{(t^2 - 1)^2} \\ &= \frac{t^a - t^{a+2} - t^{b+2} + t^{b+4}}{(t^2 - 1)^2}\end{aligned}$$

□

Lemma 4.9.3. *Let $a, b \in \mathbb{Z}$ with $b > a$ and suppose that $\frac{b-a}{2} \in \mathbb{Z}$. Then*

$$\sum_{s=0}^{\frac{b-a}{2}} (a + 2s)t^{a+2s} = \frac{at^a + (2 - a)t^{a+2} - (b + 2)t^{b+2} + bt^{b+4}}{(t^2 - 1)^2}.$$

Proof.

$$\begin{aligned}\sum_{s=0}^{\frac{b-a}{2}} (a + 2s)t^{a+2s} &= at^a + (a + 2)t^{a+2} + \ldots + bt^b \\ &= at^a(1 + t^2 + \ldots + t^{b-a}) + 2t^{a+2}(1 + 2t^2 + \ldots + (\frac{b-a}{2})t^{b-a-2}) \\ &= at^a(\frac{1 - t^{b-a+2}}{1 - t^2}) + 2t^{a+2}(\frac{1 - t^{b-a+2}}{(1 - t^2)^2} - (\frac{b - a + 2}{2})\frac{t^{b-a}}{1 - t^2}) \\ &= at^a \frac{(1 - t^{b-a+2})(1 - t^2)}{(1 - t^2)^2} + 2t^{a+2} \frac{(1 - t^{b-a+2})}{(1 - t^2)^2} - (b - a + 2)t^{a+2} \frac{(1 - t^2)t^{b-a}}{(1 - t^2)^2} \\ &= \frac{at^a - at^{b+2} - at^{a+2} + t^{b+4} + 2t^{a+2} - 2t^{b+4} - (b - a + 2)(t^{b+2} - t^{b+4})}{(t^2 - 1)^2} \\ &= \frac{at^a + (2 - a)t^{a+2} - (b + 2)t^{b+2} + bt^{b+4}}{(t^2 - 1)^2}\end{aligned}$$

□

Now we are ready to begin calculating $P(t)$. We have the following.

Lemma 4.9.4.

$$P^{0,n} = \frac{1 - t^2 - t^{n+2} + t^{n+4}}{(t^2 - 1)^2}$$

when n is even and

$$P^{0,n} = \frac{1 - t^2 - t^{n+1} + t^{n+3}}{(t^2 - 1)^2}$$

when n is odd.

Proof. From Lemma 4.4.1 we know that for n even

$$\begin{aligned} P^{0,n} &= 1 + t^2 + t^4 + \ldots + t^{n-2} + t^n \\ &= \frac{1 - t^2 - t^{n+2} + t^{n+4}}{(t^2-1)^2}. \end{aligned}$$

A similar argument holds for odd n. □

Lemma 4.9.5. *Suppose n is even. Then*

$$P^{2m+n+1,2m+2n-3}_{odd} = \frac{(m+n-2)t^{2m+2n+1} - (m+n)t^{2m+2n-1} - mt^{2m+n+3} + (m+2)t^{2m+n+1}}{(t^2-1)^2}.$$

Similarly if n is odd

$$P^{2m+n,2m+2n-3}_{odd} = \frac{(m+n-2)t^{2m+2n+1} - (m+n)t^{2m+2n-1} + (m+1)t^{2m+n} - (m-1)t^{2m+n+2}}{(t^2-1)^2}.$$

Proof. If n is even from Lemma 4.8.2 we have

$$\begin{aligned} P^{2m+n+1,2m+2n-3}_{odd} &= (m+2)t^{2m+n+1} + (m+4)t^{2m+n+3} + (m+6)t^{2m+n+5} + \ldots + (m+n-2)t^{2m+2n-3} \\ &= \frac{(m+n-2)t^{2m+2n+1} - (m+n)t^{2m+2n-1} - mt^{2m+n+3} + (m+2)t^{2m+n+1}}{(t^2-1)^2}. \end{aligned}$$

Similarly if n is odd we have

$$\begin{aligned} P^{2m+n,2m+2n-3}_{odd} &= (m+1)t^{2m+n} + (m+3)t^{2m+n+2} + (m+5)t^{2m+n+4} + \ldots + (m+n-2)t^{2m+2n-3} \\ &= \frac{(m+n-2)t^{2m+2n+1} - (m+n)t^{2m+2n-1} + (m+1)t^{2m+n} - (m-1)t^{2m+n+2}}{(t^2-1)^2}. \end{aligned}$$

□

Lemma 4.9.6.

$$P^{2m+2n-1,\infty} = \frac{(m+n)t^{2m+2n-1} + (m+n+1)t^{2m+2n} + (-m-n+2)t^{2m+2n+1} - (m+n-1)t^{2m+2n+2}}{(t^2-1)^2}.$$

Proof. Using Proposition 4.8.2 and Proposition 4.8.3 we have

$$\begin{aligned} P^{2m+2n-1,\infty} &= \sum_{i \geq 2m+2n-1} (i+1-m-n)t^i = \sum_{i \geq 2m+2n-1} it^i - (m+n-1) \sum_{i \geq 2m+2n-1} t^i \\ &= \frac{(m+n)t^{2m+2n-1} + (m+n+1)t^{2m+2n} + (-m-n+2)t^{2m+2n+1} - (m+n-1)t^{2m+2n+2}}{(t^2-1)^2} \\ &= \frac{(m+n)t^{2m+2n-1} + (1-m-n)t^{2m+2n}}{(1-t)^2}. \end{aligned}$$

□

4.9.1 The case $m \geq n$

Lemma 4.9.7. *Suppose that $m \geq n$. Then*

$$\begin{aligned} P_{r,\delta}^{n+1,2m} &= \frac{t^{n+2} - t^{n+4} + t^{2r+1} + t^{2r-2\delta+2} - (m-r+1)t^{2m+1}}{(t^2-1)^2} \\ &+ \frac{(m-r)t^{2m+3} - (m-r+\delta+2)t^{2m+2} + (m-r+\delta+1)t^{2m+4}}{(t^2-1)^2} \end{aligned}$$

if n is even and

$$\begin{aligned} P_{r,\delta}^{n+1,2m} &= \frac{t^{n+1} - t^{n+3} + t^{2r+1} + t^{2r-2\delta+2} - (m-r+1)t^{2m+1}}{(t^2-1)^2} \\ &+ \frac{(m-r)t^{2m+3} - (m-r+\delta+2)t^{2m+2} + (m-r+\delta+1)t^{2m+4}}{(t^2-1)^2} \end{aligned}$$

if n is odd.

Proof. Let q be a homogeneous quasi-invariant of degree i. We suppose $n+1 \leq i \leq 2m$, so q has no odd powers of y. Suppose first that i is odd. The matrix of linear equations on the coefficients of q, which express quasi-invariance conditions, is equivalent to

$$A = \begin{pmatrix} \alpha_1^{i-1} & \dots & \alpha_1^2 & 1 \\ \alpha_2^{i-1} & \dots & \alpha_2^2 & 1 \\ \vdots & & \vdots & \vdots \\ \alpha_r^{i-1} & \dots & \alpha_r^2 & 1 \end{pmatrix},$$

where we removed zero rows. There are $\frac{i+1}{2}$ columns and r rows, so $\mathrm{rk}A = \min(\frac{i+1}{2}, r)$. Now, the dimension of homogeneous quasi-invariants of degree i is $b_i = i+1-\frac{i+1}{2} - \mathrm{rk}A = \frac{i+1}{2} - \mathrm{rk}A$, so $b_i = 0$ unless $\frac{i+1}{2} > r$, that is unless $i \geq 2r+1$. Further, for $i \geq 2r+1$ we have $b_i = \frac{i+1}{2} - r$. Notice that the parity of n does not affect these considerations. So the odd part of the segment of the series has the form

$$\begin{aligned} P_{r,odd}^{n+1,2m-1} &= t^{2r+1}(1 + 2t^2 + 3t^4 + \ldots + (m-r)t^{2m-2r-2}) \\ &= t^{2r+1}\left(\frac{(m-r)t^{2m-2r+2} - (m-r+1)t^{2m-2r} + 1}{(t^2-1)^2}\right). \end{aligned}$$

Now let i be even. The matrix expressing the quasi-invariance conditions is equivalent to

$$\widetilde{A} = \begin{pmatrix} \alpha_1^{i-1} & \dots & \alpha_1^3 & \alpha_1 \\ \alpha_2^{i-1} & \dots & \alpha_2^3 & \alpha_2 \\ \vdots & & \vdots & \vdots \\ \alpha_{r-\delta}^{i-1} & \dots & \alpha_{r-\delta}^3 & \alpha_{r-\delta} \end{pmatrix},$$

where we again omit zero rows. There are $\frac{i}{2}$ columns and r rows, so $\mathrm{rk}\widetilde{A} = \min(\frac{i}{2}, r-\delta)$. The dimension of homogeneous quasi-invariants of degree i is $b_i = \frac{i}{2} + 1 - \mathrm{rk}\widetilde{A}$, so $b_i = 1$ unless $r - \delta < i/2$, that is unless $2r + 2 - 2\delta \leq i$. In the latter case $b_i = i/2 + 1 - r + \delta$. So the even part of the segment of the series has the form

$$\begin{aligned} P^{n+2,2m}_{r,\delta,even} =& t^{n+2} + t^{n+4} + \ldots + t^{2r-2\delta-4} + t^{2r-2\delta-2} \\ &+ t^{2r-2\delta}(1 + 2t^2 + 3t^4 + \ldots + (m-r+1+\delta)t^{2m-2r+2\delta}) \\ &= \frac{t^{n+2} + t^{2r-2\delta+2} - t^{n+4} - t^{2r-2\delta}}{(t^2-1)^2} \\ &+ t^{2r-2\delta}(\frac{(m-r+\delta+1)t^{2m-2r+2\delta+4} - (m-r+\delta+2)t^{2m-2r+2\delta+2} + 1}{(t^2-1)^2}) \end{aligned}$$

if n is even and

$$\begin{aligned} P^{n+1,2m}_{r,\delta,even} =& t^{n+1} + t^{n+3} + \ldots + t^{2r-2\delta-4} + t^{2r-2\delta-2} \\ &+ t^{2r-2\delta}(1 + 2t^2 + 3t^4 + \ldots + (m-r+1+\delta)t^{2m-2r+2\delta}) \\ &= \frac{t^{n+1} + t^{2r-2\delta+2} - t^{n+3} - t^{2r-2\delta}}{(t^2-1)^2} \\ &+ t^{2r-2\delta}(\frac{(m-r+\delta+1)t^{2m-2r+2\delta+4} - (m-r+\delta+2)t^{2m-2r+2\delta+2} + 1}{(t^2-1)^2}) \end{aligned}$$

if n is odd. □

Lemma 4.9.8. *Suppose that $m \geq n$. If n is even then*

$$\begin{aligned} &P^{2m+1,2m+n-1}_{r,odd} \\ &= \frac{(r-m)t^{2m+3} + t^{2m+2n-2r+1} + (m+1-r)t^{2m+1} - (m+2)t^{2m+n+1} + mt^{2m+n+3}}{(t^2-1)^2} \end{aligned}$$

while if n is odd

$$\begin{aligned} &P^{2m+1,2m+n-2}_{r,odd} \\ &= \frac{(r-m)t^{2m+3} + t^{2m+2n-2r+1} + (m+1-r)t^{2m+1} - (m+1)t^{2m+n} + (m-1)t^{2m+n+2}}{(t^2-1)^2}. \end{aligned}$$

Proof. Let q be a homogeneous quasi-invariant of degree i with $2m+1 \leq i \leq 2m+n-1$, and i is odd. The matrix expressing quasi-invariance conditions as equations on the coefficients of q can be rearranged to the form

$$D = \begin{pmatrix} 0 & A \\ B & * \end{pmatrix}$$

where the block A consists of $\frac{i+1}{2} - m$ columns A_j

$$A_j = \begin{pmatrix} \alpha_1^{2j-1} \\ \alpha_2^{2j-1} \\ \alpha_3^{2j-1} \\ \vdots \\ \alpha_{n-r}^{2j-1} \end{pmatrix}$$

where $1 \le j \le \frac{i+1}{2} - m$, while the block B consists of $\frac{i+1}{2}$ columns B_j

$$B_j = \begin{pmatrix} \alpha_1^{2j-2} \\ \alpha_2^{2j-2} \\ \alpha_3^{2j-2} \\ \vdots \\ \alpha_r^{2j-2} \end{pmatrix}$$

where $1 \le j \le \frac{i+1}{2}$.

Thus $\text{rk}D \ge \min(r, \frac{i+1}{2}) + \min(\frac{i+1}{2} - m, n - r)$. However, $\frac{i+1}{2} \ge m + 1 > n \ge r$ so $\min(r, \frac{i+1}{2}) = r$. It is easy to see that in this case $\text{rk}D = r + \min(\frac{i+1}{2} - m, n - r)$, so we have

$$\text{rk}D = \begin{cases} r + \frac{i+1}{2} - m & \text{if } \frac{i+1}{2} - m \le n - r, \\ n & \text{if } \frac{i+1}{2} - m > n - r. \end{cases}$$

So the dimension of the space of homogeneous quasi-invariants of degree i

$$b_i = \begin{cases} \frac{i+1}{2} - r & \text{if } \frac{i+1}{2} - m \le n - r, \\ i - m - n + 1 & \text{if } \frac{i+1}{2} - m > n - r. \end{cases}$$

Thus when n is even

$$P_{r,odd}^{2m+1,2m+n-1} = \sum_{\substack{i=2m+1 \\ i odd}}^{2m+2n-2r-1} (\frac{i+1}{2} - r)t^i + \sum_{\substack{i=2m+2n-2r+1 \\ i odd}}^{2m+n-1} (i - m - n + 1)t^i.$$

Now,

$$\begin{aligned} \sum_{\substack{i=2m+1 \\ i odd}}^{2m+2n-2r-1} (\frac{i+1}{2} - r)t^i &= \frac{1}{2} \sum_{\substack{i=2m+1 \\ i odd}}^{2m+2n-2r-1} (i+1)t^i - r \sum_{\substack{i=2m+1 \\ i odd}}^{2m+2n-2r-1} t^i \\ &= t^{2m+1}(m+1-r)\frac{t^{2n+2-2r} - t^2 - t^{2n-2r} + 1}{(t^2-1)^2} \\ &+ t^{2m+3}\frac{(n-r-1)t^{2n-2r} - (n-r)t^{2n-2r-2} + 1}{(t^2-1)^2} \end{aligned}$$

and

$$\sum_{\substack{i=2m+2n-2r+1\\ i odd}}^{2m+n-1} (i-m-n+1)t^i = \sum_{\substack{i=2m+2n-2r+1\\ i odd}}^{2m+n-1} it^i - (m+n-1) \sum_{\substack{i=2m+2n-2r+1\\ i odd}}^{2m+n-1} t^i$$
$$= (m+n-2r+2)t^{2m+2n-2r+1}\frac{t^{2r-n+2}-t^2-t^{2r-n}+1}{(t^2-1)^2}$$
$$+ t^{2m+2n-2r+3}\frac{(2r-n-2)t^{2r-n}-(2r-n)t^{2r-n-2}+2}{(t^2-1)^2}.$$

Adding these expressions we get

$$P^{2m+1,2m+n-1}_{r,odd}$$
$$= \frac{(r-m)t^{2m+3}+t^{2m+2n-2r+1}+(m+1-r)t^{2m+1}-(m+2)t^{2m+n+1}+mt^{2m+n+3}}{(t^2-1)^2}.$$

When n is odd a similar argument shows that

$$P^{2m+1,2m+n-2}_{r,odd}$$
$$= \frac{(r-m)t^{2m+3}+t^{2m+2n-2r+1}+(m+1-r)t^{2m+1}-(m+1)t^{2m+n}+(m-1)t^{2m+n+2}}{(t^2-1)^2}.$$

□

Suppose now that the arrangement $\mathcal{A}$ is generic in the sense that the rank of the matrix corresponding to the quasi-invariance conditions for polynomials of degree i with i even, $2m+2 \le i \le 2m+2n-2$ is n. Then we have the following.

Lemma 4.9.9. *Suppose that $m \ge n$. Then*

$$P^{2m+2,2m+2n-2}_{even} = \frac{(m+3-n)t^{2m+2}+(n-m-1)t^{2m+4}+(m+n-1)t^{2m+2n+2}+(-m-n-1)t^{2m+2n}}{(t^2-1)^2}.$$

Proof. We assume the arrangement is generic so the rank of the matrix corresponding to the quasi-invariance conditions for the polynomials of degree i with $2m+2 \le i \le 2m+2n-2$ is n. Then

$$P^{2m+2,2m+2n-2}_{even} = (m+3-n)t^{2m+2}+(m+5-n)t^{2m+4}+(m+7-n)t^{2m+6}+\ldots+(m+n-1)t^{2m+2n-2}$$
$$= \frac{(m+3-n)t^{2m+2}+(n-m-1)t^{2m+4}+(m+n-1)t^{2m+2n+2}+(-m-n-1)t^{2m+2n}}{(t^2-1)^2}.$$

□

Lemma 4.9.10. *Suppose that $m \ge n$. Then for any arrangement of type $(m,1^n)$ the* *possible Poincaré series (4.24) are given by the following*

$$
\begin{aligned}
P(t) =& \frac{1 - t^2 + t^{2r+1} + t^{2r-2\delta+2} + t^{2m+2n-2r+1} + (1 + r - n - \delta + a_{2m+2})t^{2m+2}}{(t^2-1)^2} \\
&+ \frac{(a_{2m+4} - 2a_{2m+2} + n - r + \delta)t^{2m+4}}{(t^2-1)^2} \\
&+ \frac{\sum_{j=m+3}^{m+n-1} (a_{2j-4} - 2a_{2j-2} + a_{2j})t^{2j}}{(t^2-1)^2} \\
&+ \frac{(a_{2m+2n-4} - 2a_{2m+2n-2})t^{2m+2n} + a_{2m+2n-2}t^{2m+2n+2}}{(t^2-1)^2}
\end{aligned}
$$

where $a_{2j} \in \mathbb{N}$, $m + 1 \leq j \leq m + n - 1$.

Proof. Lemmas 4.9.4, 4.9.5, 4.9.6, 4.9.7, 4.9.8 and 4.9.9 make it clear that the only degrees for which the rank is not controlled by the choice of the parameters r and δ are even degrees i with $2m+2 \leq i \leq 2m+2n-2$. Assuming that the arrangement is generic within fixed r, δ we have that the Poincaré series is given by

$$
P_g(t) = \frac{1 - t^2 + t^{2r+1} + t^{2r+2-2\delta} + (1 + r - n - \delta)t^{2m+2} + (n - r + \delta)t^{2m+4} + t^{2m+2n-2r+1}}{(t^2-1)^2}
$$

where we denote by $P_g(t)$ the Poincaré series under this specific genericity assumption. The only possible modifications to $P_g(t)$ are given by adding the terms in the even degrees i with $2m + 2 \leq i \leq 2(m + n - 1)$ with arbitrary coefficient. So the possible forms for (4.24) are

$$
\begin{aligned}
P(t) &= P_g(t) + a_{2m+2}t^{2m+2} + a_{2m+4}t^{2m+4} + \ldots + a_{2m+2n-2}t^{2m+2n-2} \\
&= \frac{1 - t^2 + t^{2r+1} + t^{2r-2\delta+2} + t^{2m+2n-2r+1} + (1 + r - n - \delta + a_{2m+2})t^{2m+2}}{(t^2-1)^2} \\
&+ \frac{(a_{2m+4} - 2a_{2m+2} + n - r + \delta)t^{2m+4} + \sum_{j=m+3}^{m+n-1} (a_{2j-4} - 2a_{2j-2} + a_{2j})t^{2j}}{(t^2-1)^2} \\
&+ \frac{(a_{2m+2n-4} - 2a_{2m+2n-2})t^{2m+2n} + a_{2m+2n-2}t^{2m+2n+2}}{(t^2-1)^2}
\end{aligned}
$$

for some $a_{2j} \in \mathbb{N}$, $m + 1 \leq j \leq m + n - 1$, as required. □

Theorem 4.9.11. *Suppose* $m \geq n$ *and suppose* $Q_{\mathcal{A}}$ *is Gorenstein where* $\mathcal{A}$ *has type* $(m, 1^n)$. *Then* $\mathcal{A} = \mathcal{A}_{(m,1^n)}$.

Proof. We have $P(t)$ is palindromic. Consider the possibilities for $P(t)$ given by Lemma 4.9.10. Observe that the only terms of odd degree in the numerator are t^{2r+1} and $t^{2m+2n-2r+1}$. Now, suppose the degree of the numerator is not $2m + 2n + 2$. Then t^{2r+1} and $t^{2m+2n-2r+1}$ cannot 'match', so the total degree must be odd. However this means $-t^2$ must match with some term with odd power, which is not possible. So the total degree is

$2m+2n+2$. This means that $a_{2m+2n-2}$ is non-zero. Now, we see from Lemma 4.8.4 that $a_{2m+2n-2}$ can only take the values 0 or 1. So $a_{2m+2n-2}=1$ and from Lemma 4.8.4 and Theorem 4.8.5 it follows that $\mathcal{A}=\mathcal{A}_{(m,1^n)}$. □

4.9.2 The case $m<n$

Lemma 4.9.12. *Suppose $m<n$. Let $n+1\leq i\leq 2m+n-1$ with i odd. Let M be the matrix expressing the quasi-invariance conditions for a polynomial q of degree i. Suppose the arrangement $\mathcal{A}$ is generic in the sense that if*

$$\frac{i+1}{2}\leq r \text{ and } n-r<\frac{i+1}{2}-m$$

then M has maximal possible rank, that is $\mathrm{rk}M=\min(i+1-m,n)$. Then if n,m are even

$$P_{odd}^{n+1,2m+n-1}=\frac{2t^{m+n+1}-(m+2)t^{2m+n+1}+mt^{2m+n+3}}{(t^2-1)^2}.$$

If n,m are odd then

$$P_{odd}^{n+1,2m+n-1}=\frac{2t^{m+n+1}-(m+1)t^{2m+n}+(m-1)t^{2m+n+2}}{(t^2-1)^2}.$$

If n is even, m is odd then

$$P_{odd}^{n+1,2m+n-1}=\frac{t^{m+n}+t^{m+n+2}-(m+2)t^{2m+n+1}+mt^{2m+n+3}}{(t^2-1)^2}.$$

If n is odd, m is even then

$$P_{odd}^{n+1,2m+n-1}=\frac{t^{m+n}+t^{m+n+2}-(m+1)t^{2m+n}+(m-1)t^{2m+n+2}}{(t^2-1)^2}.$$

Further, if

$$2r<m+n$$

then

$$P_{r,odd}^{n+1,2m+n-1}=\frac{t^{2r+1}+t^{2n+2m-2r+1}-(m+2)t^{2m+n+1}+mt^{2m+n+3}}{(t^2-1)^2}$$

if n is even and

$$P_{r,odd}^{n+2,2m+n-2}=\frac{t^{2r+1}+t^{2n+2m-2r+1}-(m+1)t^{2m+n}+(m-1)t^{2m+n+2}}{(t^2-1)^2}$$

if n is odd.

Proof. First let q be a homogeneous quasi-invariant of degree i with $n+1 \leq i \leq 2m-1$, i odd. The matrix M expressing quasi-invariance conditions as equations on the coefficients of q consists of $\frac{i+1}{2}$ columns B_j

$$B_j = \begin{pmatrix} \alpha_1^{2j-2} \\ \alpha_2^{2j-2} \\ \alpha_3^{2j-2} \\ \vdots \\ \alpha_r^{2j-2} \end{pmatrix}$$

where $1 \leq j \leq \frac{i+1}{2}$. We have $\text{rk}M = \min(r, \frac{i+1}{2})$, so that

$$b_i = \begin{cases} 0 & \text{if } n+1 \leq i \leq 2r-1, \\ \frac{i+1}{2} - r & \text{if } 2r+1 \leq i \leq 2m-1. \end{cases}$$

Now let q be a homogeneous quasi-invariant of degree i with $\max(n+1, 2m+1) \leq i \leq 2m+n-1$, and i is odd. The matrix expressing quasi-invariance conditions as equations on the coefficients of q can be rearranged to the form

$$M = \begin{pmatrix} 0 & A \\ B & * \end{pmatrix}$$

where the block A consists of $\frac{i+1}{2} - m$ columns A_j

$$A_j = \begin{pmatrix} \alpha_1^{2j-1} \\ \alpha_2^{2j-1} \\ \alpha_3^{2j-1} \\ \vdots \\ \alpha_{n-r}^{2j-1} \end{pmatrix}$$

where $1 \leq j \leq \frac{i+1}{2} - m$, while the block B consists of $\frac{i+1}{2}$ columns B_j

$$B_j = \begin{pmatrix} \alpha_1^{2j-2} \\ \alpha_2^{2j-2} \\ \alpha_3^{2j-2} \\ \vdots \\ \alpha_r^{2j-2} \end{pmatrix}$$

where $1 \leq j \leq \frac{i+1}{2}$. Suppose first that $2r < m+n$ and n is even. Then there are three possibilities for the shapes of the blocks A, B:

(I) $\frac{i+1}{2} \leq r, \quad \frac{i+1}{2} - m \leq n - r,$

(II) $\frac{i+1}{2} > r, \quad \frac{i+1}{2} - m \leq n - r,$

(III) $\frac{i+1}{2} > r, \quad \frac{i+1}{2} - m > n - r.$

Note that rk$M = i+1-m$ in the case (I), rk$M = r + \frac{i+1}{2} - m$ in the case (II) and rk$M = n$ in the case (III). Hence the dimension b_i of homogeneous quasi-invariants of degree i where $n+1 \leq i \leq 2m+n-1$, with i odd, is given by the following.

$$b_i = \begin{cases} 0 & \text{if } i \leq 2r-1, \\ \frac{i+1}{2} - r & \text{if } 2r+1 \leq i \leq 2m+2n-2r-3, \\ i+1-m-n & \text{if } 2m+2n-2r-1 \leq i \leq 2m+n-1. \end{cases}$$

Thus we have

$$\begin{aligned} P^{n+1,2m+n-1}_{r,odd} &= \sum_{\substack{i=2r+1 \\ i odd}}^{2m+2n-2r-3} \left(\frac{i+1}{2} - r\right)t^i + \sum_{\substack{i=2n+2m-2r-1 \\ i odd}}^{2m+n-1} (i+1-m-n)t^i \\ &= \frac{t^{2r+1} + t^{2n+2m-2r+1} - (m+2)t^{2m+n+1} + mt^{2m+n+3}}{(t^2-1)^2}. \end{aligned}$$

When n is odd a very similar set of arguments can be applied to see that

$$P^{n+2,2m+n-2}_{r,odd} = \frac{t^{2r+1} + t^{2n+2m-2r+1} - (m+1)t^{2m+n} + (m-1)t^{2m+n+2}}{(t^2-1)^2}.$$

Suppose now that $2r \geq m+n$, n is even and m is odd. The other cases are very similar. Then there are three possibilities for the shapes of the blocks A, B:

(I) $\frac{i+1}{2} \leq r, \quad \frac{i+1}{2} - m \leq n - r,$

(II) $\frac{i+1}{2} \leq r, \quad \frac{i+1}{2} - m > n - r,$

(III) $\frac{i+1}{2} > r, \quad \frac{i+1}{2} - m > n - r.$

Note that rk$M = i+1-m$ in the case (I), rk$M = \min(i+1-m, n)$ in the case (II) and rk$M = n$ in the case (III). Hence the dimension b_i of homogeneous quasi-invariants of degree i where $n+1 \leq i \leq 2m+n-1$, with i odd, is given by the following.

$$b_i = \begin{cases} 0 & \text{if } i \leq m+n-2, \\ i+1-m-n & \text{if } m+n \leq i \leq 2m+n-1. \end{cases}$$

Thus we have

$$P_{r,odd}^{n+1,2m+n-1} = \sum_{\substack{i=m+n \\ i odd}}^{2m+n-1} (i+1-m-n)t^i = \frac{t^{m+n} + t^{m+n+2} - (m+2)t^{2m+n+1} + mt^{2m+n+3}}{(t^2-1)^2}.$$

□

Lemma 4.9.13. *Suppose $m < n$. Suppose that the arrangement $\mathcal{A}$ is generic in the sense that the rank of the matrix corresponding to the quasi-invariance conditions for polynomials of degree i with i even, $n+1 \leq i \leq 2m+2n-2$ is maximal. Then we have the following. Suppose n is even. Then*

$$P_{even}^{n+2,2m+2n-2} = \frac{t^{n+2} - t^{n+4} + t^{m+n+1} + t^{m+n+3} + (m+n-1)t^{2m+2n+2} - (m+n+1)t^{2m+2n}}{(t^2-1)^2}$$

when m is odd and

$$P_{even}^{n+2,2m+2n-2} = \frac{t^{n+2} - t^{n+4} + 2t^{m+n+2} + (m+n-1)t^{2m+2n+2} - (m+n+1)t^{2m+2n}}{(t^2-1)^2}$$

when m is even. Similarly if n is odd

$$P_{even}^{n+1,2m+2n-2} = \frac{t^{n+1} - t^{n+3} + t^{m+n+1} + t^{m+n+3} + (m+n-1)t^{2m+2n+2} - (m+n+1)t^{2m+2n}}{(t^2-1)^2}$$

when m is even and

$$P_{even}^{n+1,2m+2n-2} = \frac{t^{n+1} - t^{n+3} + 2t^{m+n+2} + (m+n-1)t^{2m+2n+2} - (m+n+1)t^{2m+2n}}{(t^2-1)^2}$$

when m is odd.

Proof. Suppose that n is even, m is odd. The other cases are very similar. Consider the matrix M corresponding to the quasi-invariance conditions for the polynomials of degree i with $n+2 \leq i \leq 2m+2n-2$, i even. M has n rows and $i-m$ columns. Thus if $i < m+n$, M has at least as many columns as rows. We assume the arrangement is generic so that $\mathrm{rk}M = \min(i-m, n)$.

$$P_{even}^{n+2,2m+2n-2} = \sum_{i=n+2}^{m+n-1} t^i + \sum_{i=m+n+1}^{2m+2n-2} (i+1-m-n)t^i = \frac{t^{n+2} - t^{n+4} + t^{m+n+1} + t^{m+n+3} + (m+n-1)t^{2m+2n+2} - (m+n+1)t^{2m+2n}}{(t^2-1)^2}$$

□

Suppose now that the arrangement is generic (in the sense of Lemma 4.9.13) within fixed r. Let us denote by $P_g(t)$ the Poincaré series (4.24) under this assumption. Then one has the following.

Lemma 4.9.14. *Suppose $m < n$. If $2r < m+n$ and $m+n$ is odd then*

$$P_g(t) = \frac{1 - t^2 + t^{2r+1} + t^{2n+2m-2r+1} + t^{m+n+1} + t^{m+n+3}}{(t^2-1)^2}.$$

If $2r < m+n$ and $m+n$ is even then

$$P_g(t) = \frac{1 - t^2 + t^{2r+1} + t^{2n+2m-2r+1} + 2t^{m+n+2}}{(t^2-1)^2}.$$

If $2r \geq m+n$ and $m+n$ is odd then

$$P_g(t) = \frac{1 - t^2 + t^{m+n} + t^{m+n+1} + t^{m+n+2} + t^{m+n+3} + (t^2-1)^2 \sum_{i=[\frac{n+1}{2}]}^{m+[n/2]-2} a_{2i+1} t^{2i+1}}{(t^2-1)^2}.$$

If $2r \geq m+n$ and $m+n$ is even then

$$P_g(t) = \frac{1 - t^2 + 2t^{m+n+1} + 2t^{m+n+2} + (t^2-1)^2 \sum_{i=[\frac{n+1}{2}]}^{m+[n/2]-2} a_{2i+1} t^{2i+1}}{(t^2-1)^2}.$$

where $a_{2i+1} \in \mathbb{N}$, $[\frac{n+1}{2}] \leq i \leq m + [n/2] - 2$.

Proof. Combine Lemmas 4.9.4, 4.9.5 and 4.9.6, 4.9.12 and 4.9.13

□

Lemma 4.9.15. *Suppose that $m < n$. Then for any arrangement of type $(m, 1^n)$ the possible Poincaré series (4.24) are given by the following. If $2r < m+n$ and $n+m$ is odd we have*

$$\begin{aligned} P(t) = {} & \frac{1 - t^2 + t^{2r+1} + t^{2n+2m-2r+1} + t^{m+n+1} + t^{m+n+3}}{(t^2-1)^2} \\ & + \frac{a_{2[\frac{n+2}{2}]} t^{2[\frac{n+2}{2}]} + (a_{2[\frac{n+4}{2}]} - 2a_{2[\frac{n+2}{2}]}) t^{2[\frac{n+4}{2}]}}{(t^2-1)^2} \\ & + \frac{\sum_{j=[\frac{n+6}{2}]}^{m+n-1} (a_{2j-4} - 2a_{2j-2} + a_{2j}) t^{2j}}{(t^2-1)^2} \\ & + \frac{(a_{2m+2n-4} - 2a_{2m+2n-2}) t^{2m+2n} + a_{2m+2n-2} t^{2m+2n+2}}{(t^2-1)^2}. \end{aligned}$$

If $2r < m+n$ and $n+m$ is even we have

$$P(t) = \frac{1 - t^2 + t^{2r+1} + t^{2n+2m-2r+1} + 2t^{m+n+2}}{(t^2-1)^2} + \frac{a_{2[\frac{n+2}{2}]}t^{2[\frac{n+2}{2}]} + (a_{2[\frac{n+4}{2}]} - 2a_{2[\frac{n+2}{2}]})t^{2[\frac{n+4}{2}]}}{(t^2-1)^2} + \frac{\sum_{j=[\frac{n+6}{2}]}^{m+n-1}(a_{2j-4} - 2a_{2j-2} + a_{2j})t^{2j}}{(t^2-1)^2} + \frac{(a_{2m+2n-4} - 2a_{2m+2n-2})t^{2m+2n} + a_{2m+2n-2}t^{2m+2n+2}}{(t^2-1)^2}.$$

Here $a_{2j} \in \mathbb{N}$, $[\frac{n+2}{2}] \le j \le m+n-1$. If $2r \ge m+n$ and $m+n$ is odd we have

$$P(t) = \frac{1 - t^2 + t^{m+n} + t^{m+n+1} + t^{m+n+2} + t^{m+n+3} + (t^2-1)^2\sum_{i=[\frac{n+1}{2}]}^{m+[n/2]-2} a_{2i+1}t^{2i+1}}{(t^2-1)^2} + \frac{a_{2[\frac{n+2}{2}]}t^{2[\frac{n+2}{2}]} + (a_{2[\frac{n+4}{2}]} - 2a_{2[\frac{n+2}{2}]})t^{2[\frac{n+4}{2}]}}{(t^2-1)^2} + \frac{\sum_{j=[\frac{n+6}{2}]}^{m+n-1}(a_{2j-4} - 2a_{2j-2} + a_{2j})t^{2j}}{(t^2-1)^2} + \frac{(a_{2m+2n-4} - 2a_{2m+2n-2})t^{2m+2n} + a_{2m+2n-2}t^{2m+2n+2}}{(t^2-1)^2}.$$

If $2r \ge m+n$ and $n+m$ is even we have

$$P(t) = \frac{1 - t^2 + 2t^{m+n+1} + 2t^{m+n+2} + (t^2-1)^2\sum_{i=[\frac{n+1}{2}]}^{m+[n/2]-2} a_{2i+1}t^{2i+1}}{(t^2-1)^2} + \frac{a_{2[\frac{n+2}{2}]}t^{2[\frac{n+2}{2}]} + (a_{2[\frac{n+4}{2}]} - 2a_{2[\frac{n+2}{2}]})t^{2[\frac{n+4}{2}]}}{(t^2-1)^2} + \frac{\sum_{j=[\frac{n+6}{2}]}^{m+n-1}(a_{2j-4} - 2a_{2j-2} + a_{2j})t^{2j}}{(t^2-1)^2} + \frac{(a_{2m+2n-4} - 2a_{2m+2n-2})t^{2m+2n} + a_{2m+2n-2}t^{2m+2n+2}}{(t^2-1)^2}.$$

Here $a_{2j} \in \mathbb{N}$, $[\frac{n+2}{2}] \le j \le m+n-1$ and $a_{2i+1} \in \mathbb{N}$, $[\frac{n+1}{2}] \le i \le m+[n/2]-1$.

Proof. Lemmas 4.9.4, 4.9.5, 4.9.6, 4.9.12 and 4.9.13 make it clear that the only degrees for which the rank is not controlled by the choice of the parameter r are even degrees i with $n+2 \le i \le 2m+2n-2$ when n is even and $n+1 \le i \le 2m+2n-2$ when n is odd. This means the only possible modifications to $P_g(t)$ are given by adding these degrees with arbitrary coefficient. □

Theorem 4.9.16. *Suppose $m < n$ and suppose $Q_{\mathcal{A}}$ is Gorenstein where $\mathcal{A}$ has type $(m, 1^n)$. Then $\mathcal{A} = \mathcal{A}_{(m,1^n)}$.*

Proof. We have $P(t)$ is palindromic. Suppose first that $2r < m+n$. Consider the possibilities for $P(t)$ given by Lemma 4.9.15. Observe that the only terms of odd degree in the numerator are t^{2r+1} and $t^{2m+2n-2r+1}$. Now, suppose the degree of the numerator is not $2m+2n+2$. Then t^{2r+1} and $t^{2m+2n-2r+1}$ cannot 'match', so the total degree must be odd. However this means $-t^2$ must match with some term with odd power, which is not possible. So the total degree is $2m+2n+2$. This means that $a_{2m+2n-2}$ is non-zero. Now, we see from Lemma 4.8.4 that $a_{2m+2n-2}$ can only take the values 0 or 1. So $a_{2m+2n-2} = 1$ and from Lemma 4.8.4 and Theorem 4.8.5 it follows that $\mathcal{A} = \mathcal{A}_{(m,1^n)}$.

Now suppose that $2r \geq m+n$. Note that in this case $r \neq [\frac{n+1}{2}]$ and so $\mathcal{A} \neq \mathcal{A}_{(m,1^n)}$. This means that $a_{2m+2n-2} = 0$. Suppose that n, m are even, the other cases are similar. First we will show that the total degree of $P(t)$ has to be even. For, consider the coefficient of the term t^{m+n+2} in the numerator of $P(t)$. Suppose this coefficient is zero. Then we have

$$a_{m+n+2} - 2a_{m+n} + a_{m+n-2} + 2 = 0$$

so that in particular $a_{m+n} > 0$. Continuing, consider the coefficient of the term t^{m+n+4} in the numerator of $P(t)$. Suppose this coefficient is zero. Then we have

$$a_{m+n+4} - 2a_{m+n+2} + a_{m+n} = 0$$

so that in particular $a_{m+n+2} > 0$. It is easy to see that we can continue this process to deduce that $a_{2m+2n-4} \neq 0$ so that the total degree is even. So we can assume that t^{m+n+2} or some higher even power appears in the numerator with non-zero coefficient. Since $P(t)$ is palindromic t^{m+n+2} must match with some degree $\geq n+1$, so that the total degree is $\geq m+2n+3 \geq 2m+n+3$ so that in particular the total degree is even. Suppose that the total degree is $2m+2n-2s$, $s = 0, 1, 2, \ldots, [n/2]-2$. It follows immediately that

$$a_{2m+n-2s-4} = a_{2m+n-2s-2} = \ldots = a_{2m+2n-2s-6} = a_{2m+2n-2s-4} = 1.$$

Since $P(t)$ is palindromic we know that the coefficients are equal at degrees $2m+n-2s-2$ and $n+2$. This means that

$$_{2m+n-2s-6} - a_{n+2} = 1$$

By comparing degrees $2m+n-2s-4$ and $n+4$ we see that

$$a_{2m+n-2s-8} - a_{n+4} = 1$$

We can continue like this to deduce that

$$a_{2m+n-2s-2t} - a_{n+2t-4} = 1$$

where $3 \leq t \leq \frac{m-2s}{2}$. At the next stage we compare the degrees $m+n+2$ and $m+n-2s-2$. This time we have

$$a_{m+n-2} - a_{m+n-2s-2} = -1.$$

We can continue to compare the degrees $m+n$ and $m+n-2s$. This time we have

$$a_{m+n-4} - a_{m+n-2s} = -3.$$

It is not hard to see that we can continue this process to deduce that

$$a_{2m+n-2s-2t} - a_{n+2t-4} = 2(a_{2m+n-2s-2t+2} - a_{n+2t-6}) - (a_{2m+n-2s-2t+4} - a_{n+2t-8}).$$

where $\frac{m-2s+2}{2} \leq t \leq [\frac{m-s+1}{2}]$. If $m-s$ is even it remains to check if the coefficients at degrees $m+n-s+2$ and $m+n-s-2$ are equal. This would require that

$$2(a_{m+n-s} - a_{m+n-s-4}) - (a_{m+n-s+2} - a_{m+n-s-6}) = 0$$

which is not possible. If $m-s$ is odd it remains to check if the coefficients at degrees $m+n-s+1$ and $m+n-s-1$ are equal. This would require that

$$-3(a_{m+n-s-1} - a_{m+n-s-3}) + (a_{m+n-s+1} - a_{m+n-s-5}) = 0$$

which is not possible. Thus we can conclude that if $2r \geq m+n$ then $P(t)$ cannot be palindromic, and we are done.

$\square$

Theorem 4.9.17. *Suppose $Q_{\mathcal{A}}$ is Gorenstein where $\mathcal{A}$ has type $(m, 1^n)$. Then $\mathcal{A} = \mathcal{A}_{(m,1^n)}$.*

Proof. Combine Theorems 4.9.11 and 4.9.16. $\square$

4.10 $Q_{\mathcal{A}_{(m,1^n)}}$ is Gorenstein

In this section we are going to prove the following theorem.

Theorem 4.10.1. *Let $\mathcal{A} = \mathcal{A}_{(m,1^n)}$. Then the Poincaré series for the quasi-invariant ring $Q_{\mathcal{A}}$ is given by*

$$P(t) = \frac{1 - t^2 + t^{n+1} + t^{n+2} + t^{2m+n} + t^{2m+n+1} - t^{2m+2n} + t^{2m+2n+2}}{(t^2-1)^2}$$

An immediate corollary is the fact that the quasi-invariant ring $Q_{\mathcal{A}_{(m,1^n)}}$ is Gorenstein.

We will approach the proof in the following manner. In the previous section we calculated all possible expressions for the Poincaré series $P(t)$. The expressions calculated (we considered three cases depending on the interaction of the parameters m and n) involved some arbitrary coefficients. We will show how to specify these coefficients in the case $\mathcal{A} = \mathcal{A}_{(m,1^n)}$. Also, we will show that these values give $P(t)$ palindromic, thus implying that $Q_{\mathcal{A}}$ is Gorenstein.

Lemma 4.10.2. *Suppose $\mathcal{A} = \mathcal{A}_{(m,1^n)}$. Let $i = 2(m+n-s)$, $s = 2, 3, \ldots, [n/2]$. Then in (4.24) $b_i = i - m - n + 2$.*

Proof. Let q be a homogeneous quasi-invariant of degree i, where $i = 2(m+n-s)$, $s = 2, 3, \ldots, [n/2]$. Recall that $\alpha_i^2 = \alpha_{[n/2]+i}^2$, $1 \leq i \leq [n/2]$, and $\alpha_n = 0$ if n is odd. Then the matrix M expressing the quasi-invariant conditions as linear equations for the coefficients of q is equivalent to

$$M = \left(\begin{array}{cccccc|c} \alpha_1^{2m+2n-2s-1} & \alpha_1^{2m+2n-2s-3} & \ldots & \alpha_1^5 & \alpha_1^3 & \alpha_1 & \\ \alpha_2^{2m+2n-2s-1} & \alpha_2^{2m+2n-2s-3} & \ldots & \alpha_2^5 & \alpha_2^3 & \alpha_2 & \\ \alpha_3^{2m+2n-2s-1} & \alpha_3^{2m+2n-2s-3} & \ldots & \alpha_3^5 & \alpha_3^3 & \alpha_3 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \\ \alpha_{[n/2]}^{2m+2n-2s-1} & \alpha_{[n/2]}^{2m+2n-2s-3} & \ldots & \alpha_{[n/2]}^5 & \alpha_{[n/2]}^3 & \alpha_{[n/2]} & \\ \hline & & & & & & B \\ & & & 0 & & & \end{array}\right) \tag{4.44}$$

The block B consists of $n-s$ columns $C_1, \ldots, C_{n-s}$ with the following structure

$$C_j = \begin{pmatrix} (2j-1)\alpha_1^{2j-2} - (2m+2n-2j+1-2s)\alpha_1^{2j} \\ (2j-1)\alpha_2^{2j-2} - (2m+2n-2j+1-2s)\alpha_2^{2j} \\ (2j-1)\alpha_3^{2j-2} - (2m+2n-2j+1-2s)\alpha_3^{2j} \\ \vdots \\ (2j-1)\alpha_{[n/2]}^{2j-2} - (2m+2n-2j+1-2s)\alpha_{[n/2]}^{2j} \\ (2j-1)0^{2j-2} \end{pmatrix}$$

where $1 \leq j \leq n-s$, and the last row should be removed if n is even. We are going to show that the rank of the matrix M cannot be n. For, consider any $n \times n$ minor. Such a minor is automatically zero unless we take exactly $[\frac{n+1}{2}]$ columns from the block B. Now, consider the block B. Let B_L be the minor formed by taking the determinant of the square submatrix with columns $C_L, C_{L+1}, \ldots, C_{L+[n/2]-1}$. Suppose n is even. By Lemma 4.7.2 $B_L = 0$ for $1 \leq L \leq n-s-[n/2]+1$. Consider B_1. Since $B_1 = 0$ we have $\sum_{i=1}^{[n/2]} \lambda_i C_i$

for some $\lambda_i \in \mathbb{Z}$. We claim $\lambda_1 \neq 0$. Indeed, by Lemma 4.7.6, we can find a non-zero $([n/2]-1) \times ([n/2]-1)$ minor in the block with columns $C_2, \ldots, C_{[n/2]}$. We can construct the linear dependence $\sum_{i=1}^{[n/2]} \lambda_i C_i = 0$ using appropriate $([n/2]-1) \times ([n/2]-1)$ minors as coefficients so we can take $\lambda_1 \neq 0$. Thus $C_1 \in < C_2, C_3, \ldots, C_{[n/2]} >$. By repeated application of Lemma 4.7.6 we can go on to deduce that for $1 \leq j \leq n-s-[n/2]$, $C_j \in < C_{n-s-[n/2]+1}, \ldots, C_{n-s} >$. This means that any $[n/2] \times [n/2]$ minor taken from the block B must be zero. So the rank of the matrix (4.44) is at most $n-1$. If n is odd by Lemma 4.7.2 $B_L = 0$ for $2 \leq L \leq n-s-[n/2]+1$. We can proceed as in the even case to see that the dimension of the space spanned by the columns of B is $\frac{n-1}{2}$.

We will now show that the rank of the matrix (4.44) is precisely $n-1$. To see this we will construct a non-zero $(n-1)\times(n-1)$ minor. For $1 \leq k \leq [n/2]$, $2 \leq L \leq n-s-[n/2]+2$ let D_L^k be the minors formed by taking the determinant of the square submatrix with columns $C_L, C_{L+1}, \ldots, C_{L+[n/2]-2}$ where we include all but the kth row of B. Then by Lemma 4.7.6 $\exists k$ such that $D_L^k \neq 0$. Thus we can find a non-zero minor of B of size $[\frac{n-1}{2}] \times [\frac{n-1}{2}]$. Denote this minor by $D_{[\frac{n-1}{2}]\times[\frac{n-1}{2}]}$. Consider the following $(n-1) \times (n-1)$ minor of (4.44)

$$\left|\begin{array}{cccccc|c}
\alpha_1^{n-1} & \alpha_1^{n-3} & \ldots & \alpha_1^5 & \alpha_1^3 & \alpha_1 & \\
\alpha_2^{n-1} & \alpha_2^{n-3} & \ldots & \alpha_2^5 & \alpha_2^3 & \alpha_2 & \\
\alpha_3^{n-1} & \alpha_3^{n-3} & \ldots & \alpha_3^5 & \alpha_3^3 & \alpha_3 & 0 \\
\vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \\
\alpha_{[n/2]}^{n-1} & \alpha_{[n/2]}^{n-3} & \ldots & \alpha_{[n/2]}^5 & \alpha_{[n/2]}^3 & \alpha_{[n/2]} & \\
\hline
 & & & 0 & & & D_{[\frac{n-1}{2}]\times[\frac{n-1}{2}]}
\end{array}\right| \neq 0$$

The result follows. □

Lemma 4.10.3. *Suppose $\mathcal{A} = \mathcal{A}_{(m,1^n)}$. Let $i = 2m+2n-2s$, $s = [n/2]+1, [n/2]+2, \ldots, \min(n-1, m+[\frac{n-1}{2}])$. Then in (4.24) $b_i = i/2 - [n/2] + 1$.*

Proof. Let $i = 2m+2n-2s$, $s = [n/2]+1, [n/2]+2, \ldots, n-1$. Then the matrix M of the system of linear equations for the coefficients of q at such degrees under the assumption $\mathcal{A} = \mathcal{A}_{(m,1^n)}$ is equivalent to the form (4.44) with the block B as detailed in the previous Lemma. It is not hard to see that the maximal possible rank for M is $[n/2]+n-s$. We will show the rank is exactly $[n/2]+n-s$. To do this we will construct an $([n/2]+n-s)\times([n/2]+n-s)$ minor which is non-zero, in a similar fashion to the previous Lemma. Using Lemma 4.7.8 we see that the block B contains a non-zero $(n-s)\times(n-s)$

minor $B_{(n-s)\times(n-s)}$. So we can construct the $([n/2]+n-s)\times([n/2]+n-s)$ minor

$$\left|\begin{array}{cccccc|c} \alpha_1^{n-1} & \alpha_1^{n-3} & \dots & \alpha_1^5 & \alpha_1^3 & \alpha_1 & \\ \alpha_2^{n-1} & \alpha_2^{n-3} & \dots & \alpha_2^5 & \alpha_2^3 & \alpha_2 & \\ \alpha_3^{n-1} & \alpha_3^{n-3} & \dots & \alpha_3^5 & \alpha_3^3 & \alpha_3 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \\ \alpha_{[n/2]}^{n-1} & \alpha_{[n/2]}^{n-3} & \dots & \alpha_{[n/2]}^5 & \alpha_{[n/2]}^3 & \alpha_{[n/2]} & \\ \hline & & & 0 & & & B_{(n-s)\times(n-s)} \end{array}\right|$$

which is non-zero. Thus the rank of M is $[n/2]+n-s$, and since $i=2m+2n-2s$ we have $\mathrm{rk}M=[n/2]+n-(m+n-i/2)=[n/2]-m+i/2$ as required. □

Theorem 4.10.4. *Let $\mathcal{A}=\mathcal{A}_{(m,1^n)}$. Suppose that $m\geq n$. Then the Poincaré series (4.24) is given by*

$$P(t)=\frac{1-t^2+t^{n+1}+t^{n+2}+t^{2m+n}+t^{2m+n+1}-t^{2m+2n}+t^{2m+2n+2}}{(t^2-1)^2}.$$

In particular it is palindromic.

Proof. Recall from Lemma 4.9.10 that the possible Poincaré series (4.24) in the situation $m\geq n$ are given by the following

$$\begin{aligned} P(t)=&\frac{1-t^2+t^{2r+1}+t^{2r-2\delta+2}+t^{2m+2n-2r+1}+(1+r-n-\delta+a_{2m+2})t^{2m+2}}{(t^2-1)^2}\\ &+\frac{(a_{2m+4}-2a_{2m+2}+n-r+\delta)t^{2m+4}+\sum_{j=m+3}^{m+n-1}(a_{2j-4}-2a_{2j-2}+a_{2j})t^{2j}}{(t^2-1)^2}\\ &+\frac{(a_{2m+2n-4}-2a_{2m+2n-2})t^{2m+2n}+a_{2m+2n-2}t^{2m+2n+2}}{(t^2-1)^2}\end{aligned}$$

where $a_{2j}\in\mathbb{N}$, $m+1\leq j\leq m+n-1$. Now using Lemmas 4.10.2 and 4.10.3 we can specify the values of each parameter a_{2j}, $m+1\leq j\leq m+n-1$. We have

$$a_{2j}=\begin{cases} n-([n/2]+j-m), & m+1\leq j\leq m+[\frac{n+1}{2}]-1,\\ 1, & m+[\frac{n+1}{2}]\leq j\leq m+n-1.\end{cases}$$

Suppose n is even. Then since $\mathcal{A}=\mathcal{A}_{(m,1^n)}$ we know that $r=n/2$, $\delta=0$. It is easy to check that $\sum_{j=m+3}^{m+n-1}(a_{2j-4}-2a_{2j-2}+a_{2j})t^{2j}=t^{2m+n}$. Thus we have

$$\begin{aligned} P(t)=&\frac{1-t^2+t^{n+1}+t^{n+2}+t^{2m+n+1}+(1+n/2-n+n/2-1)t^{2m+2}+(-n/2+n-n/2)t^{2m+4}}{(t^2-1)^2}\\ &+\frac{t^{2m+n}-t^{2m+2n}+t^{2m+2n+2}}{(t^2-1)^2}\\ =&\frac{1-t^2+t^{n+1}+t^{n+2}+t^{2m+n+1}+t^{2m+n}-t^{2m+2n}+t^{2m+2n+2}}{(t^2-1)^2}\end{aligned}$$

which is clearly palindromic. If n is odd, $r = \frac{n+1}{2}$, $\delta = 1$ and thus we have

$$\begin{aligned} P(t) =& \frac{1 - t^2 + t^{n+2} + t^{n+1} + t^{2m+n}}{(t^2-1)^2} \\ &+ \frac{t^{2m+n+1} - t^{2m+2n} + t^{2m+2n+2}}{(t^2-1)^2} \\ &= \frac{1 - t^2 + t^{n+2} + t^{n+1} + t^{2m+n} + t^{2m+n+1} - t^{2m+2n} + t^{2m+2n+2}}{(t^2-1)^2} \end{aligned}$$

which is again palindromic. □

Theorem 4.10.5. *Let $\mathcal{A} = \mathcal{A}_{(m,1^n)}$. Suppose that $m < n$. Then the Poincaré series (4.24) is given by*

$$P(t) = \frac{1 - t^2 + t^{n+1} + t^{n+2} + t^{2m+n} + t^{2m+n+1} - t^{2m+2n} + t^{2m+2n+2}}{(t^2-1)^2}.$$

In particular it is palindromic.

Proof. Note that $r = [\frac{n+1}{2}]$, so $2r < m + n$ and there are two possibilities for $P(t)$ given by Lemma 4.9.15. Note that $2m + n > m + n$ so we can directly apply Lemma 4.10.2 to find the parameters a_{2j} with $m + [n/2] \leq j \leq m + n - 1$ and deduce that these parameters are all equal to one. We now apply Lemma 4.10.3 to find the remaining a_{2j}, $[\frac{n+2}{2}] \leq j \leq m + [n/2] - 1$. Armed with these observations we have the following. When n is even and m is odd we have

$$a_{2j} = \begin{cases} j - n/2, & \frac{n+2}{2} \leq j \leq \frac{m+n-1}{2}, \\ n/2 - j + m, & \frac{m+n+1}{2} \leq j \leq m + n/2 - 1, \\ 1, & m + n/2 \leq j \leq m + n - 1. \end{cases}$$

When n is even and m is even

$$a_{2j} = \begin{cases} j - n/2, & \frac{n+2}{2} \leq j \leq \frac{m+n}{2}, \\ n/2 - j + m, & \frac{m+n+2}{2} \leq j \leq m + n/2 - 1, \\ 1, & m + n/2 \leq j \leq m + n - 1. \end{cases}$$

When n is odd and m is even

$$a_{2j} = \begin{cases} j - \frac{n-1}{2}, & \frac{n+1}{2} \leq j \leq \frac{m+n-1}{2}, \\ \frac{n+1}{2} - j + m, & \frac{m+n+1}{2} \leq j \leq m + \frac{n-1}{2} - 1, \\ 1, & m + \frac{n-1}{2} \leq j \leq m + n - 1. \end{cases}$$

Finally when n is odd and m is odd

$$a_{2j} = \begin{cases} j - \frac{n-1}{2}, & \frac{n+1}{2} \leq j \leq \frac{m+n}{2}, \\ \frac{n+1}{2} - j + m, & \frac{m+n+2}{2} \leq j \leq m + \frac{n-1}{2} - 1, \\ 1, & m + \frac{n-1}{2} \leq j \leq m+n-1. \end{cases}$$

Putting all of this together we have

$$\begin{aligned} P(t) &= \frac{1 - t^2 + t^{n+1} + t^{n+2m+1} + t^{m+n+1} + t^{m+n+3}}{(t^2-1)^2} \\ &+ \frac{t^{n+2} - t^{m+n+1} - t^{m+n+3} + t^{2m+n} - t^{2m+2n} + t^{2m+2n+2}}{(t^2-1)^2} \\ &= \frac{1 - t^2 + t^{n+1} + t^{n+2} + t^{2m+n} + t^{n+2m+1} - t^{2m+2n} + t^{2m+2n+2}}{(t^2-1)^2} \end{aligned}$$

when n is even and m is odd,

$$\begin{aligned} P(t) &= \frac{1 - t^2 + t^{n+1} + t^{n+2m+1} + 2t^{m+n+2}}{(t^2-1)^2} \\ &+ \frac{t^{n+2} - 2t^{m+n+2} + t^{2m+n} - t^{2m+2n} + t^{2m+2n+2}}{(t^2-1)^2} \\ &= \frac{1 - t^2 + t^{n+1} + t^{n+2} + t^{2m+n} + t^{n+2m+1} - t^{2m+2n} + t^{2m+2n+2}}{(t^2-1)^2} \end{aligned}$$

when n is even and m is even,

$$\begin{aligned} P(t) &= \frac{1 - t^2 + t^{n+2} + t^{2m+n} + t^{m+n+1} + t^{m+n+3}}{(t^2-1)^2} \\ &+ \frac{t^{n+1} - t^{m+n+1} - t^{m+n+3} + t^{2m+n+1} - t^{2m+2n} + t^{2m+2n+2}}{(t^2-1)^2} \\ &= \frac{1 - t^2 + t^{n+1} + t^{n+2} + t^{2m+n} + t^{n+2m+1} - t^{2m+2n} + t^{2m+2n+2}}{(t^2-1)^2} \end{aligned}$$

when n is odd, m even and

$$\begin{aligned} P(t) &= \frac{1 - t^2 + t^{n+2} + t^{2m+n} + 2t^{m+n+2}}{(t^2-1)^2} \\ &+ \frac{t^{n+1} - 2t^{m+n+2} + t^{2m+n+1} - t^{2m+2n} + t^{2m+2n+2}}{(t^2-1)^2} \\ &= \frac{1 - t^2 + t^{n+1} + t^{n+2} + t^{2m+n} + t^{n+2m+1} - t^{2m+2n} + t^{2m+2n+2}}{(t^2-1)^2} \end{aligned}$$

when n is odd, m is odd. So $P(t)$ is palindromic in all cases. □

Theorem 4.10.6. *Suppose $\mathcal{A} = \mathcal{A}_{(m,1^n)}$. Then the quasi-invariant ring $Q_{\mathcal{A}}$ is Gorenstein.*

Proof. Theorems 4.10.4 and 4.10.5. □

Chapter 5

Quasi-invariant modules for Cherednik algebras

5.1 Summary

This chapter is composed of two related parts. In the first part we show the following. We consider all anti-invariant quasi-invariant polynomials for the group S_n acting in $\mathbb{C}^n$ with multiplicity m. We realize this space as a module for the spherical subalgebra of the Cherednik algebra $H_{1/m}(S_{mn})$. This is achieved by considering certain submodules of the standard modules.

In the second part we study the algebra $\Lambda_{n,1,k}$ introduced by Sergeev and Veselov. This consists of the partially symmetric quasi-invariants for the deformed root system $\mathcal{A}_n(k)$. For generic values of the parameter k Sergeev and Veselov [SV04] show that this algebra is generated by deformed Newton sums and use this fact to calculate the Poincaré series of $\Lambda_{n,1,k}$. We find the Poincaré series for $\Lambda_{n,1,k}$ for any $k \in \mathbb{Z}_+$. We also study the relation between $\Lambda_{n,1,k}$ and the algebra generated by deformed Newton sums for non-integer k.

Chapter 5 consists entirely of new results with the exception of the following. Section 5.2 consists of background material and as such contains no new results. Theorems 5.6.1, 5.6.2 and 5.8.1 appear in the literature.

5.2 Background

We provide a short history of quasi-invariants in the context of Cherednik algebras. The fact that rings of quasi-invariants are modules for certain subalgebras of Cherednik algebras is well known. This has been established in the paper [BEG03]. In order to explain the results of this paper, let us first fix some notation. We follow [BEG03]. Let G be a finite Coxeter group acting in its complexified reflection representation $V = \mathbb{C}^N$. Let R be the corresponding Coxeter root system. To each G-invariant function $c : R \to \mathbb{C}$ one can attach an associative algebra $H_c(G)$ called the *rational Cherednik algebra*. This is the algebra generated by the vector spaces V, V^* and the set G with defining relations

$$wxw^{-1} = w(x), wyw^{-1} = w(y), \forall y \in V, x \in V^*, w \in G$$
$$x_1x_2 = x_2x_1, y_1y_2 = y_2y_1, \forall y_1, y_2 \in V, x_1, x_2 \in V^*$$
$$yx - xy = \langle y, x\rangle - \sum_{\alpha \in R/\{\pm 1\}} c_\alpha \langle y, \alpha\rangle \langle \alpha^\vee, x\rangle s_\alpha, \forall y \in V, x \in V^*.$$

For any $\zeta \in \mathbb{C}^N$ the *Dunkl operator* ∇_ζ is defined as

$$\nabla_\zeta = \partial_\zeta - \sum_{\alpha \in R_+} \frac{c_\alpha(\alpha, \zeta)}{(\alpha, x)}(1 - s_\alpha)$$

where $(\cdot, \cdot)$ is the standard scalar product in $\mathbb{C}^N$. We have the important vector space isomorphism

$$\mathbb{C}[X] \otimes_{\mathbb{C}} \mathbb{C}[\nabla] \otimes_{\mathbb{C}} \mathbb{C}G \to H_c(G)$$

where $\mathbb{C}[X] = \mathbb{C}[x_1, x_2, \ldots, x_N]$ and $\mathbb{C}[\nabla]$ is the commutative algebra generated by the Dunkl operators ∇_i corresponding to the basis directions e_i. The rational Cherednik algebra $H_c(G)$ can be defined as the algebra generated by the linear operators x_i, ∇_j, g $(i, j = 1, \ldots, N, g \in G)$ which act naturally in $\mathbb{C}^N$. We will always deal with the *rational* Cherednik algebra, and we will refer to this as the Cherednik algebra from now on.

Next we are going to describe certain subalgebras of $H_c(G)$. Let $e = \frac{1}{|G|}\sum_{g \in G} g$. The *spherical subalgebra* of $H_c(G)$ is defined as $eH_c(G)e$. The vector space

$$M_\tau = \mathbb{C}[X] \otimes \tau$$

where $\tau \in$Irrep(G) can be given the structure of an $H_c(G)$ module. This is the 'standard module'. The module M_τ has the natural action of the Cherednik algebra by polynomials and G. If we let $p \otimes v \in M_\tau$ then the Dunkl operator ∇_ζ acts as follows (see [EG02b])

$$\nabla_\zeta(p \otimes v) = \partial_\zeta p - \sum_{\alpha \in R_+} \frac{c_\alpha(\alpha, \zeta)}{(\alpha, x)}(1 - s_\alpha)p \otimes s_\alpha v.$$

We now explain how quasi-invariants can be seen as a module for the spherical subalgebra. Consider the ring of quasi-invariants Q_c corresponding to the Coxeter root system R and

$\mathbb{Z}_+$ valued multiplicity function c. Let Q_c^b denote the image of the homomorphism (1.3). It is shown in [FV02, Theorem 3] (see also [Cha98] where a similar observation was first made) that for any $u \in Q_c^b$, $u(Q_c) \subset Q_c$. In particular $C_c(Q_c) \subset Q_c$ where C_c is the centralizer of the operator L (1.2). Thus there is a natural action on Q_c of the algebra $\mathcal{B}_c$ generated by C_c and $\mathbb{C}[X]^G$. This $\mathcal{B}_c$ action commutes with the G-action on Q_c. It can be shown that $\mathcal{B}_c$ is isomorphic to eH_ce and thus the space Q_c acquires an $(\mathbb{C}G \otimes eH_ce)$-module structure. This structure is completely described by the following. For any irreducible representation τ of G let $\tau_c^{'}$ be the representation of G for which the monodromy representation of the Dunkl connection with values in $\tau_c^{'}$ is τ, see [BC11].

Theorem 5.2.1 ([BEG03], Proposition 6.6)**.** *There is a* $(\mathbb{C}G \otimes eH_ce)$*-module isomorphism* $Q_c = \bigoplus_{\tau \in Irrep(G)} \tau \otimes eM_{\tau_c^{'}}$.

Later, Feigin [Fei] constructed generalized Calogero-Moser systems via representations of Cherednik algebras. This paper provides the primary motivation for the work in the first half of this chapter. Feigin showed that the invariant polynomials corresponding to the Coxeter group S_n are a module for the spherical subalgebra of the Cherednik algebra $H_{1/m}(S_{mn}), m, n \in \mathbb{Z}_+$. This is achieved via consideration of some specific submodules in the polynomial representation of $H_{1/m}(S_{mn})$. Note that the multiplicity function takes the constant non-integer value $1/m$, in contrast with the theorem from [BEG03] explained above. In this chapter we begin to investigate whether *all* quasi-invariant polynomials can be constructed in such a fashion by considering submodules in more complicated representations.

To explain fully what we mean by this, let us explain the construction of [Fei] more exactly. Let $N = mn$. Let us denote by $\pi_{m,n}$ the plane in $\mathbb{C}^{mn}$ given by the equations

$$x_1 = x_2 = \ldots = x_m$$

$$x_{m+1} = x_{m+2} = \ldots = x_{2m}$$

$$x_{(n-1)m+1} = x_{(n-1)m+2} = \ldots = x_{mn}.$$

The associated parabolic stratum is defined as

$$D_{m,n} = \bigcup_{w \in S_N} w(\pi_{m,n})$$

while the corresponding parabolic ideal is

$$I = \{p \in \mathbb{C}[X] | p|_{D_{m,n}} = 0\}.$$

Feigin establishes the following.

Theorem 5.2.2 ([Fei], Theorem 1)**.** *I and hence $\mathbb{C}[X]/I$ are representations for the rational Cherednik algebra $H_{1/m}(S_{mn})$.*

From this theorem it follows that $e(\mathbb{C}[X]/I)$ is a representation of the spherical subalgebra $e(H_{1/m}(S_N))e$. This representation can be identified with a subspace of polynomials in $\mathbb{C}[x_1, \ldots, x_n]$, namely the restrictions $p(x)|_{\pi_{m,n}}$ where $p(x) \in \mathbb{C}[X]^{S_N}$. Then we have the following.

Theorem 5.2.3 ([Fei])**.** *The representation $e(\mathbb{C}[X]/I)$ of $e(H_{1/m}(S_N))e$ is isomorphic to the invariant polynomials $\mathbb{C}[x_1, \ldots, x_n]^{S_n}$ in $\mathbb{C}^n$.*

Indeed, this is clear since the basis $B_N = \{\sum_{i=1}^N x_i^k, k \geq 1\}$ of Newton sums for the invariant polynomials $\mathbb{C}[X]^{S_N}$ becomes a basis $B_n = \{\sum_{i=1}^n x_i^k, k \geq 1\}$ for $\mathbb{C}[x_1, \ldots, x_n]^{S_n}$ under the restriction $p(x) \to p(x)|_{\pi_{m,n}}$, $p(x) \in \mathbb{C}[X]^{S_N}$. We would also like to mention that Etingof, Enriquez and Calaque have obtained $\mathbb{C}[x_1, \ldots, x_n]^{S_n}$ as a representation of $eH_{1/m}(S_N)e$ via a different construction, see [CEE10, Section 8]

In the first half of this chapter we will generalize this construction in the following way. Let us first detail a general notion. Consider the anti-invariant quasi-invariants corresponding to some Coxeter group G and integer valued multiplicity function c: these are polynomials $p \in Q_c$ satisfying $s_\alpha p = -p$ for all $s_\alpha \in G$. These polynomials actually have the form $\prod_{\alpha \in R_+} (\alpha, x)^{2c_\alpha + 1} q(x)$, where R is the root system corresponding to G and $q(x) \in \mathbb{C}[X]^G$.

We are going to introduce submodules N_τ of the standard modules M_τ for some specific representations τ of S_{mn}. We will then show that $e(M_\tau/N_\tau)$ is isomorphic to the space of all anti-invariant quasi-invariant polynomials for S_n acting in $\mathbb{C}^n$.

5.3 Definitions and notation

In this section we will fix some notation and explain some concepts which will be of use throughout this chapter. This exposition is taken from the book [Ful97]. First we recall some information about representations of the symmetric group.

Let τ be a partition of $n \in \mathbb{N}$. A *Young diagram* is a collection of n boxes arranged in left-justified rows, with a weakly decreasing number of boxes in each row. Any way of putting the integers one until n in the boxes will be called a Young tableau. A tabloid is an equivalence class of Young tableaux in which tableaux are equivalent up to row permutations. S_n acts on the set of tabloids and thus on the module with the tabloids as a basis. Now, let T be a Young tableau and set

$$V_T = \sum_{q \in C(T)} \epsilon(q) q(T)$$

where $C(T)$ is the column group of T (that is, those $q \in S_n$ that permute the entries of each column amongst themselves), ϵ is the sign of a permutation and $q(T)$ is the tabloid corresponding to the Young tableau obtained from T by the action of q. The Specht module of τ is the module generated by the V_T as T runs through all tableaux with shape τ.

Let τ be an irreducible representation of S_{mn} with rectangular Young diagram consisting of n rows with m boxes in each row. Let dim(τ)=s and let $\{V_{T_\nu}\}$, $1 \leq \nu \leq s$, be the basis of the corresponding Specht module. One can take the standard Young tableaux to get a basis in the Specht module (that is, each T_ν has a numbering which is increasing along the rows and down the columns). Identify S_{mn} with A_{mn-1}; S_{mn} acts naturally on Specht modules, and acts on $\mathbb{C}[X] = \mathbb{C}[x_1, x_2, \ldots, x_{mn}]$ by permuting the subscripts. In $\mathbb{C}^{mn}$ consider the plane given by, for $1 \leq j \leq n$

$$x_{(j-1)m+1} = x_{(j-1)m+2} = \ldots = x_{jm}. \tag{5.1}$$

We are going to associate to this plane the $n \times m$ rectangular tabloid

$v_1 =$ (5.2)

1	2	3	$\ldots$	m
$m+1$	$m+2$	$m+3$	$\ldots$	$2m$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
$(n-1)m+1$	$(n-1)m+2$	$(n-1)m+3$	$\ldots$	mn

Define an equivalence relation $\sim$ on the set of tabloids as follows. We will say two tabloids are equivalent under $\sim$ if we can obtain one from the other via swapping rows. So for instance

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array} \sim \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 1 & 2 \\ \hline\end{array}.$$

We are going to introduce a correspondence between planes and tabloids with the following form

$$\text{Planes} \simeq \text{Tabloids}/\sim .$$

Consider the set of planes in $\mathbb{C}^{mn}$ obtained by action of S_{mn} on the plane with equations given by (5.1). Refer to the planes obtained in this way as Π_i, $1 \leq i \leq Z$. Let v_i be a tabloid corresponding to the plane Π_i for each i, $1 \leq i \leq Z$.

Let T_i be a Young tableau that is row increasing and corresponds to the tabloid v_i. Let V_{T_i} be the corresponding element of the Specht module. Throughout, let Π_1 be the plane given by (5.1) with associated tabloid v_1. Now, note that there are $n!Z$ tabloids of shape $n \times m$. Let us denote the set of all such tabloids as $\{u_{jk}, 1 \leq j \leq Z, 1 \leq k \leq n!\}$, so that any two tabloids with the same first index j are equivalent under $\sim$ and correspond to the plane Π_j. Further, suppose $u_{j1} = v_j \;\forall j$.

5.4 The Submodules N_τ

Let $F = \sum_\nu f_\nu \otimes V_{T_\nu} \in M_\tau$ and let $V_{T_\nu} = \sum_s a_s^\nu u_{sk}$. Here the V_{T_ν} are the basis elements of the Specht module corresponding to τ, $f_\nu \in \mathbb{C}[X]$, and we expand each basis element through column preserving permutations to obtain a sum of tabloids of shape $n \times m$ which we denoted by u_{sk}. Define $V_{T_\nu}^{u_{sk}} = a_s^\nu u_{sk}$. We also define

$$F^{v_j} = \sum_\nu a_j^\nu f_\nu$$

where $a_j^\nu = a_{j1}^\nu$. This is the sum of terms in F containing the tabloid v_j after expanding the basis of the Specht module through tabloids. Let N_τ be defined as

$$N_\tau = \{F \in M_\tau \mid F^{v_j}\,|_{\Pi_j} = 0 \,\forall\, j\}.$$

We are going to establish the following.

Theorem 5.4.1. *N_τ is an $H_{1/m}(S_{mn})$ module.*

The proof will consist of a series of lemmas. We will show that N_τ is invariant under the action of S_{mn} and the Dunkl operators ∇_i.

Lemma 5.4.2. *Let $g \in S_{mn}$ and let $F \in N_\tau$. Then*

$$[gF]^{v_j} \mid_{\Pi_j} = 0 \ \forall j.$$

Proof. For $g \in S_{mn}$ and $f(X) \in \mathbb{C}[X]$ let $gf(X) = f^g$. We have

$$\begin{aligned}
[gF]^{v_j} \mid_{\Pi_j} &= \sum_\nu f_\nu^g (\text{coefficient of } v_j \text{ in } gV_{T_\nu}) \mid_{\Pi_j} \\
&= \sum_\nu f_\nu (\text{coefficient of } g^{-1}v_j \text{ in } V_{T_\nu}) \mid_{g^{-1}\Pi_j} \\
&= 0 \ \ (\text{since } F \in N_\tau).
\end{aligned}$$

□

To show that N_τ is invariant under the Dunkl operators, it is sufficient to consider

$$\nabla_1 = \partial_1 \otimes 1 - \sum_{\alpha \in (A_{mn-1})_+} \frac{(1/m)(\alpha, e_1)(1 - s_\alpha)}{(\alpha, x)} \otimes s_\alpha. \tag{5.3}$$

Thus Theorem 5.4.1 is reduced to

Lemma 5.4.3. $[\nabla_1 F]^{v_j} \mid_{\Pi_j} = 0 \ \forall j.$

Note that

$$[\nabla_1 F]^{v_j} \mid_{\Pi_j} = [\partial_1 F - \frac{1}{m} \sum_{\alpha \in (A_{mn-1})_+} \frac{(\alpha, e_1)}{(\alpha, x)} \sum_\nu (1 - s_\alpha) f_\nu \otimes s_\alpha V_{T_\nu}]^{v_j} \mid_{\Pi_j} = (A + B + C) \mid_{\Pi_j}$$

where A, B, C are given by the following.

$$A = -\frac{1}{m} \left[\sum_{\alpha \in (A_{mn-1})_+} \frac{(\alpha, e_1)}{(\alpha, x)} \sum_\nu f_\nu^{s_\alpha} \otimes s_\alpha V_{T_\nu} \right]^{v_j} \quad \text{where } (\alpha, x) \mid_{\Pi_j} \not\equiv 0.$$

We call A the 'reflected' terms.

$$B = -\frac{1}{m} \sum_{\alpha \in (A_{mn-1})_+} \frac{(\alpha, e_1)}{(\alpha, x)} \left[\sum_\nu f_\nu \otimes s_\alpha V_{T_\nu} \right]^{v_j} \quad \text{where } (\alpha, x) \mid_{\Pi_j} \not\equiv 0.$$

We call B the 'non-reflected' terms.

$$C = \partial_1 F - \frac{1}{m} \sum_{\alpha \in (A_{mn-1})_+} \frac{(\alpha, e_1)}{(\alpha, x)} \left[\sum_\nu (f_\nu - s_\alpha f_\nu) \otimes s_\alpha V_{T_\nu} \right]^{v_j} \quad \text{where } (\alpha, x) \mid_{\Pi_j} \equiv 0$$

We call C the 'singular' terms.

5.4.1 'Reflected terms'

Lemma 5.4.4. *Let $\alpha \in A_{mn-1}$ with $(\alpha, x) |_{\Pi_j} \neq 0$. Then*

$$\frac{1}{(\alpha, x)} \left[\sum_{\nu} f_{\nu}^{s_\alpha} \otimes s_\alpha V_{T_\nu} \right]^{v_j} |_{\Pi_j} = 0.$$

Proof. For any such $\alpha \in A_{mn-1}$ apply Lemma 5.4.2. □

5.4.2 'Non-reflected terms'

Let $F = \sum_\nu f_\nu \otimes V_{T_\nu} \in N_\tau$. Let $b_1, b_2 \ldots, b_m$ be the entries of any row of v_j not containing 1. Suppose $\alpha_1, \ldots, \alpha_m$ are the vectors in A_{mn-1} of the form $\alpha_i = e_1 - e_{b_i}$. Consider the terms in $[\nabla_1 F]^{v_j} |_{\Pi_j}$ of the form:

$$[\sum_{k=1}^{m} \frac{1}{(\alpha_k, x)} \sum_{\nu} f_\nu \otimes s_{\alpha_k} V_{T_\nu}]^{v_j} |_{\Pi_j}$$

Lemma 5.4.5. *Let $b_1, b_2 \ldots, b_m$ be the entries of any row of a tabloid v_j and suppose that this row does not contain one. Then*

$$[\sum_{k=1}^{m} \frac{1}{x_1 - x_{b_k}} \sum_{\nu} f_\nu \otimes s_{1b_k} V_{T_\nu}]^{v_j} |_{\Pi_j} = 0$$

Proof. First we note that

$$\frac{1}{x_1 - x_{b_1}} |_{\Pi_j} = \frac{1}{x_1 - x_{b_2}} |_{\Pi_j} = \ldots = \frac{1}{x_1 - x_{b_m}} |_{\Pi_j} . \tag{5.4}$$

Recall that since $F \in N_\tau$

$$[\sum_{\nu} f_\nu \otimes V_{T_\nu}]^{v_j} |_{\Pi_j} = 0. \tag{5.5}$$

Using (5.4) and (5.5) we see we must show that

$$[\sum_{\nu} f_\nu \otimes V_{T_\nu} + \sum_{k=1}^{m} \sum_{\nu} f_\nu \otimes s_{1b_k} V_{T_\nu}]^{v_j} |_{\Pi_j} = 0.$$

It is sufficient to establish that

$$[V_{T_\nu} + \sum_{k=1}^{m} s_{1b_k} V_{T_\nu}]^{v_j} = 0 \,\forall\, \nu.$$

□

Thus Lemma 5.4.5 follows from the following useful result.

Lemma 5.4.6. *Let $b_1, b_2 \ldots, b_m$ be the entries of any row of a tabloid v_j and suppose that this row does not contain 1. Then we have*

$$[V_T + \sum_{k=1}^{m} s_{1b_k} V_T]^{v_j} = 0 \tag{5.6}$$

for all Young tableaux T.

Proof. Let

$$V_T = \sum_k a_{sk} u_{sk}.$$

Recall $u_{s1} = v_s$. Consider first the case $a_{j1} \neq 0$. Let

$$s_{1bt} V_T = \sum a^t_{sk} u_{sk}.$$

Let $v_r = s_{1b_t} v_j$. Then

$$a^t_{j1} \neq 0 \iff a_{r1} \neq 0.$$

Now, via column operations we can obtain $s_{1b_1} v_j$ from v_j but we cannot obtain $s_{1b_t} v_j$ for $t \neq 1$. So we conclude that $a_{r1} \neq 0 \iff t = 1$ and that in the case $t = 1$, $a_{r1} = -a_{j1}$. So we have shown that, if $a_{j1} \neq 0$ then

$$[V_T + \sum_{k=1}^{m} s_{1b_k} V_T]^{v_j} = 0.$$

Now suppose $a_{j1} = 0$ but $a^1_{j1} \neq 0$, say. We show that we can reduce this situation to the previous case. Let $2 \leq k \leq m$. We have

$$\begin{aligned}
[s_{1b_k} V_T]^{v_j} &= [s_{b_1 b_k} s_{1b_k} V_T]^{s_{b_1 b_k} v_j} \\
&= [s_{b_1 b_k} s_{1b_k} V_T]^{v_j} \text{ (since } s_{b_1 b_k} v_j = v_j) \\
&= [s_{1b_k} s_{1b_1} V_T]^{v_j} \text{ (since } s_{1b_k} s_{1b_1} = s_{b_1 b_k} s_{1b_k}) \\
&= [s_{1b_k} V_{s_{1b_1} T}]^{v_j}.
\end{aligned}$$

This shows that

$$[V_T + \sum_{k=1}^{m} s_{1b_k} V_T]^{v_j} = [V_{s_{1b_1} T} + \sum_{k=1}^{m} s_{1b_k} V_{s_{1b_1} T}]^{v_j}. \tag{5.7}$$

Now if we let $V_{\widetilde{T}} = V_{s_{1b_1} T}$ the right hand side of (5.7) becomes

$$[V_{\widetilde{T}} + \sum_{k=1}^{m} s_{1b_k} V_{\widetilde{T}}]^{v_j} \tag{5.8}$$

where $[V_{\widetilde{T}}]^{v_j} \neq 0$. This is the original case we considered. □

Lemma 5.4.5 is now proven.

5.4.3 'Singular terms'

Lemma 5.4.7. *Consider the row of v_j with entries $1, a_2, a_3, \ldots, a_m$. Let $\alpha_2, \alpha_3, \ldots, \alpha_m$ be the vectors from A_{mn-1} of the form $\alpha_i = e_1 - e_{a_i}$, $i = 2, 3, \ldots, m$. Then*

$$[\partial_1 \sum_\nu f_\nu \otimes V_{T_\nu} - \sum_{k=2}^m \frac{1/m}{(\alpha_k, x)} \sum_\nu (f_\nu - s_{\alpha_k} f_\nu) \otimes s_{\alpha_k} V_{T_\nu}]^{v_j} \mid_{\Pi_j} = 0$$

Proof. The relevant terms take the form

$$C = [\partial_1 \sum_\nu f_\nu \otimes V_{T_\nu} - \frac{1}{m} \sum_{k=2}^m \frac{1}{x_1 - x_{a_k}} \sum_\nu (f_\nu - f_\nu^{s_{1a_k}}) \otimes s_{1a_k} V_{T_\nu}]^{v_j}. \tag{5.9}$$

Let

$$V_{T_\nu} = \lambda_\nu v_j + \sum_{u_{rs} \neq v_j} \lambda_{rs} u_{rs}$$

where $\lambda_\nu, \lambda_{rs} \in \mathbb{Z}$. Since $F \in N_\tau$ we have

$$\sum_\nu \lambda_\nu f_\nu \mid_{\Pi_j} = 0$$

so we can write

$$\sum_\nu \lambda_\nu f_\nu = \sum_{k=2}^m (x_1 - x_{a_k}) g_k + h$$

where the g_k are some polynomials that do not vanish on Π_j and by h we mean combinations of the terms of the type $(x_{b_i} - x_{b_j}) h_{ij}$ where b_i, b_j are elements of any row of v_j not containing 1. Thus we have

$$[\partial_1 \sum_\nu f_\nu \otimes V_{T_\nu}]^{v_j} \mid_{\Pi_j} = \partial_1 \sum_\nu \lambda_\nu f_\nu \mid_{\Pi_j} = \sum_{k=2}^m g_k.$$

Now, note that

$$[V_{T_\nu}]^{v_j} = [s_{1a_t} V_{T_\nu}]^{v_j}$$

since $s_{1a_t} v_j = v_j$. Using this we can conclude that

$$[\sum_\nu (f_\nu - f_\nu^{s_{1a_t}}) \otimes s_{1a_t} V_{T_\nu}]^{v_j} = [\sum_\nu (f_\nu - f_\nu^{s_{1a_t}}) \otimes V_{T_\nu}]^{v_j} = \sum_\nu \lambda_\nu (f_\nu - f_\nu^{s_{1a_t}}).$$

Observe that

$$\sum_\nu \lambda_\nu (f_\nu - f_\nu^{s_{1a_t}}) = (x_1 - x_{a_t})(g_t + g_t^{s_{1a_t}}) + \sum_{\substack{k \neq t \\ k=2}}^m ((x_1 - x_{a_k}) g_k - (x_{a_t} - x_{a_k}) g_k^{s_{1a_t}}) + h - h^{s_{1a_t}}$$

So (5.9) gives, in total

$$C\mid_{\Pi_j}=\sum_{i=2}^{m} g_i\mid_{\Pi_j}-\frac{1}{m}\sum_{t=2}^{m}\frac{1}{x_1-x_{a_t}}((x_1-x_{a_t})(g_t+g_t^{s_{1a_t}})$$
$$+\sum_{\substack{k\neq t\\k=2}}^{m}((x_1-x_{a_k})g_k-(x_{a_t}-x_{a_k})g_k^{s_{1a_t}})+h-h^{s_{1a_t}}))\mid_{\Pi_j}$$
$$=\sum_{i=2}^{m} g_i\mid_{\Pi_j}-\frac{1}{m}\sum_{t=2}^{m}\frac{1}{x_1-x_{a_t}}(2(x_1-x_{a_t})g_t+(x_1-x_{a_t})\sum_{\substack{k\neq t\\k=2}}^{m} g_k)\mid_{\Pi_j}$$

$$=\sum_{i=2}^{m} g_i\mid_{\Pi_j}-\frac{1}{m}\sum_{t=2}^{m}(2g_t+\sum_{\substack{r=2\\r\neq t}}^{m} g_r)\mid_{\Pi_j}=\sum_{i=2}^{m} g_i\mid_{\Pi_j}-\frac{1}{m}\sum_{t=2}^{m} mg_t\mid_{\Pi_j}=0.$$

□

Lemma 5.4.3 now follows from Lemmas 5.4.4, 5.4.5, 5.4.7. Hence Theorem 5.4.1 is established.

5.5 The Operator $\sum_i \nabla_i^2$

Lemma 5.5.1. *There is an injection* θ

$$\theta : e(M_\tau/N_\tau)\to K$$

where $K=\{p\in\mathbb{C}[y_1,\dots y_n]\mid \sigma_{ij}p=(-1)^m p$ *for any simple transposition* $\sigma_{ij}\in S_n\}$.

Proof. A element $F\in M_\tau/N_\tau$ is uniquely determined by the collection of functions $\{F^{v_j}\mid_{\Pi_j}\}$. Further, for the invariants $F\in(M_\tau/N_\tau)^{S_{mn}}$ these collections are fully determined by a single function, e.g. $F^{v_1}\mid_{\Pi_1}$. This is because $F^{v_1}\mid_{\Pi_1}=F^{g^{-1}v_1}\mid_{g^{-1}\Pi_1}\ \forall g\in S_{mn}$. Moreover by considering $g\in S_{mn}$ s.t. $g(\Pi_1)=\Pi_1$ (i.e. g swaps two rows in v_1) we get the property $g(F^{v_1}\mid_{\Pi_1})=(-1)^m F^{v_1}\mid_{\Pi_1}$. □

Lemma 5.5.1 allows us to consider operators from $eH_{1/m}(S_{mn})e$ acting in $e(M_\tau/N_\tau)$ as operators acting on a subspace of anti-invariant polynomials. If $L\in eH_{1/m}(S_{mn})e$ then let us define $L\mid_{\Pi_1}$ as the action of L on $F^{v_1}\mid_{\Pi_1}\in e(M_\tau/N_\tau)$.

Proposition 5.5.2. *Let* $L=\sum_{i=1}^{mn}\nabla_i^2$. *Then*

$$L\mid_{\Pi_1}=fL_{CM,m}f^{-1}$$

where

$$L_{CM,m} = \Delta_y - \sum_{i<j}^{n} \frac{2m}{y_i - y_j}(\partial_{y_i} - \partial_{y_j}) \tag{5.10}$$

and

$$f = \prod_{i<j}(y_i - y_j)^{-(m+1)}$$

with $y = (y_1, \ldots, y_n)$ *orthonormal coordinates on the plane* Π_1 *and* Δ_y *the Laplacian in the* y_i *coordinates.*

Proof. We are going to show that

$$L\mid_{\Pi_1} = \Delta_y + \sum_{\substack{i,j=1\\ i<j}} \frac{2(\partial_{y_i} - \partial_{y_j})}{y_i - y_j} - \sum_{\substack{i,j=1\\ i<j}}^{n} \frac{2m(m+1)}{(y_i - y_j)^2}$$

where $y = (y_1, \ldots, y_n)$ are orthonormal coordinates on the plane Π_1 and Δ_y is the Laplacian in the y_i coordinates. (It is easy to see that $f^{-1}L\mid_{\Pi_1} f = L_{CM,m}$.) It is easily seen (see for instance [Fei] for a similar calculation) that the operator $L = \sum_i \nabla_i^2$ acts on $(M_\tau/N_\tau)^{S_{mn}}$ by

$$\Delta \otimes 1 - \sum_{\alpha \in A_{mn-1}} \frac{(2/m)\partial_\alpha}{(\alpha, x)} \otimes s_\alpha + \sum_{\alpha \in A_{mn-1}} \frac{(1/m)(\alpha,\alpha)}{(\alpha, x)^2} \otimes (s_\alpha - 1).$$

We want to trace how L acts on F^{v_1} after restriction to Π_1. For each block of colliding coordinates

$$x_{(k-1)m+1} = \ldots = x_{km},\ 1 \le k \le n$$

let us introduce the orthogonal coordinates

$$y_k := \frac{x_{(k-1)m+1} + \ldots + x_{km}}{\sqrt{m}},\ 1 \le k \le n$$

$$y_{n+(k-1)m+r} := \frac{x_{(k-1)m+r} - x_{(k-1)m+r+1}}{\sqrt{m}},\ 1 \le k \le n\ , 1 \le r \le m-1.$$

First consider

$$[(\sum_{\alpha \in A_{mn-1}} \frac{(2/m)\partial_\alpha}{(\alpha, x)} \otimes s_\alpha)F]^{v_1}.$$

Suppose s_α stabilizes Π_1. Consider constant extension of the functions f_ν from Π_1 in the normal direction to Π_1. Then

$$\partial_\alpha f_\nu \mid_{\Pi_1} = 0. \tag{5.11}$$

Now suppose $b_1, b_2, \ldots, b_m$ are the entries of any row of v_1 not containing 1. Let $\alpha_i = e_1 - e_{b_i}$ for $1 \le i \le m$. Now, suppose $1 \le g < h \le m$. Then for any ν

$$\frac{(2/m)\partial_{\alpha_g}}{(\alpha_g, x)} f_\nu \mid_{\Pi_1} = \frac{(2/m)\partial_{\alpha_h}}{(\alpha_h, x)} f_\nu \mid_{\Pi_1}.$$

So we have

$$\begin{aligned}
&[(\sum_{k=1}^{m}\frac{(2/m)\partial_{\alpha_k}}{(\alpha_k,x)}\otimes s_{\alpha_k})F]^{v_1}\mid_{\Pi_1}\\
&=[\sum_{k=1}^{m}(\frac{(2/m)\partial_{\alpha_k}}{(\alpha_k,x)}\sum_{\nu}(f_\nu\otimes s_{\alpha_k}V_{T_\nu}))]^{v_1}\mid_{\Pi_1}\\
&=[\sum_{k=1}^{m}(Q\sum_{\nu}(f_\nu\otimes s_{\alpha_k}V_{T_\nu}))]^{v_1}\mid_{\Pi_1}\\
&=Q\sum_{\nu}[f_\nu\otimes\sum_{k=1}^{m}s_{\alpha_k}V_{T_\nu}]^{v_1}\mid_{\Pi_1}
\end{aligned}$$

where $Q=Q_k=\frac{(2/m)\partial_{\alpha_k}}{(\alpha_k,x)}$. Then using Lemma 5.4.6 we have

$$Q\sum_{\nu}[f_\nu\otimes\sum_{k=1}^{m}s_{\alpha_k}V_{T_\nu}]^{v_1}\mid_{\Pi_1}=Q\sum_{\nu}[f_\nu\otimes-(V_{T_\nu})]^{v_1}\mid_{\Pi_1}=-QF^{v_1}\mid_{\Pi_1}$$

Thus we have

$$\begin{aligned}
&[(\sum_{k=1}^{m}\frac{(2/m)\partial_{\alpha_k}}{(\alpha_k,x)}\otimes s_{\alpha_k})F]^{v_1}\mid_{\Pi_1}\\
&=-QF^{v_1}\mid_{\Pi_1}\\
&=-\sum_{\substack{i,j=1\\ i<j}}\frac{2(\partial_{y_i}-\partial_{y_j})}{y_i-y_j}F^{v_1}.
\end{aligned}$$

Now consider

$$[(\sum_{\alpha\in(A_{mn-1})_+}\frac{(1/m)(\alpha,\alpha)}{(\alpha,x)^2}\otimes(s_\alpha-1))(\sum_{\nu}f_\nu\otimes V_{T_\nu})]^{v_1}. \tag{5.12}$$

Firstly, let s_α be a reflection with the property $s_\alpha v_1=v_1$. Then s_α stabilizes Π_1 and

$$[(s_\alpha-1)F]^{v_1}\mid_{\Pi_1}=0$$

Now, let $b_1,b_2,\ldots,b_m$ be the entries of any row of v_1 except the row containing 1. Let $\alpha_k=e_1-e_{b_k}$. If $1\le g<h\le m$ it is clear that for any ν

$$\frac{(1/m)(\alpha_g,\alpha_g)}{(\alpha_g,x)^2}f_\nu\mid_{\Pi_1}=\frac{(1/m)(\alpha_h,\alpha_h)}{(\alpha_h,x)^2}f_\nu\mid_{\Pi_1}.$$

So using Lemma 5.4.6 we can see that

$$
\begin{aligned}
&[\sum_{k=1}^{m}(\frac{(1/m)(\alpha_k,\alpha_k)}{(\alpha_k,x)^2}\otimes(s_{\alpha_k}-1)(\sum_{\nu} f_\nu\otimes V_{T_\nu}))]^{v_1}\mid_{\Pi_1}\\
&=[\sum_{k=1}^{m}(\frac{(1/m)(\alpha_k,\alpha_k)}{(\alpha_k,x)^2}\sum_{\nu} f_\nu\otimes(s_{\alpha_k}-1)V_{T_\nu})]^{v_1}\mid_{\Pi_1}\\
&=R[\sum_{k=1}^{m}\sum_{\nu} f_\nu\otimes s_{\alpha_k}V_{T_\nu}]^{v_1}\mid_{\Pi_1} -R[\sum_{k=1}^{m}\sum_{\nu} f_\nu\otimes V_{T_\nu}]^{v_1}\mid_{\Pi_1}\\
&-RF^{v_1}(1+m)\mid_{\Pi_1}
\end{aligned}
$$

where $R=R_k=\frac{(1/m)(\alpha_k,\alpha_k)}{(\alpha_k,x)^2}$. So we have

$$
\begin{aligned}
&[\sum_{k=1}^{m}(\frac{(1/m)(\alpha_k,\alpha_k)}{(\alpha_k,x)^2}\otimes(s_{\alpha_k}-1)(\sum_{\nu} f_\nu\otimes V_{T_\nu}))]^{v_1}\mid_{\Pi_1}\\
&=\sum_{i<j}^{n}\frac{-m(m+1).2/m.(\sqrt{m})^2}{(y_i-y_j)^2}F^{v_1}\\
&=\sum_{i<j}^{n}\frac{-2m(m+1)}{(y_i-y_j)^2}F^{v_1}.
\end{aligned}
$$

In total then

$$
L\mid_{\Pi_1}=\Delta_y+\sum_{\substack{i,j=1\\ i<j}}\frac{2(\partial_{y_i}-\partial_{y_j})}{y_i-y_j}-\sum_{\substack{i,j=1\\ i<j}}^{n}\frac{2m(m+1)}{(y_i-y_j)^2}
$$

upon noting that $\Delta_x=\Delta_y$, and so the Proposition is proved. □

Next we will show that we can identify $\widehat{K}=e(M_\tau/N_\tau)\subset K$ with the space of *all* anti-invariant quasi-invariants in $\mathbb{C}^n$. Let us denote by $L^{(j)}_{CM,m}$ the j-th quantum integral for the operator $L_{CM,m}$. This operator has the form $L^{(j)}_{CM,m}=\sum_{i=1}^{n}\partial^j_{y_i}$+lower order terms, and is homogeneous of degree $-j$. We define these operators as follows. Let $L_j:\widehat{K}\to\widehat{K}, P_j:\widehat{K}\to\widehat{K}$ be defined by

$$
\begin{aligned}
L_j&=\sum_{i=1}^{nm}\nabla_i^j,\\
P_j&=\sum_{i=1}^{nm}x_i^j.
\end{aligned}
$$

Then Proposition 5.5.2 above shows that

$$
f^{-1}L_2f=L_{CM,m} \tag{5.13}
$$

where $f=\prod_{i<j}(y_i-y_j)^{-(m+1)}$. It is also clear that

$$
f^{-1}P_jf=P_j.
$$

Using (5.13) and the commutativity of Dunkl operators we have

$$[f^{-1}L_j f, L_{CM,m}] = 0$$

and since $f^{-1}L_j f$ has homogeneous degree $-j$

$$f^{-1}L_j f = L^{(j)}_{CM,m}.$$

Let $\mathcal{L}$ be the algebra generated by $L^{(j)}_{CM,m}, P_j\ \forall j \geq 1$.

Lemma 5.5.3. *$\mathcal{L} : f^{-1}\widehat{K} \to f^{-1}\widehat{K}$.*

Proof. $f^{-1}L_2 f(f^{-1}\widehat{K}) = f^{-1}L_2(\widehat{K}) \subset f^{-1}\widehat{K}$ so that $L_{CM,m} : f^{-1}\widehat{K} \to f^{-1}\widehat{K}$. □

Finally then we have

Lemma 5.5.4. *$f^{-1}\widehat{K} = \{$all anti-invariant quasi-invariants in $\mathbb{C}^n\}$.*

Proof. Let $w \in f^{-1}\widehat{K}$ of minimal degree. Then $L^{(j)}_{CM,m}w = 0$. Recall the m-harmonic polynomials which are defined as the joint kernel of the operators $L^{(j)}_{CM,m}$, $1 \leq j \leq n$. Recall $\exists!$ anti-invariant m-harmonic polynomial w of minimal degree, where $w = \prod_{\alpha\in(A_{n-1})_+}(\alpha, y)^{2m+1}$, see [FV02]. So $f^{-1}\widehat{K} \supset \{\mathbb{C}[y_i]^{S_n}w\}$. Take $q = \prod_{\alpha\in(A_{n-1})_+}(\alpha, y)^{\widehat{m}}p$ where p is some invariant polynomial $\neq 0$ on $(\alpha, x) = 0$. Take $\widehat{m}$ minimal for $q \in f^{-1}\widehat{K}$. Assume $\widehat{m} < 2m + 1$. If $t = (\alpha, x)$ and $\Pi : (\alpha, x) = 0$ then $L^{(j)}_{CM,m}q = (\widehat{m}(\widehat{m} - 1) - 2m\widehat{m})t^{\widehat{m}-2}r+$(higher order zeros on Π), where r is a polynomial which does not vanish on Π. Thus, since $\widehat{m} - 1 < 2m$, $L^{(j)}_{CM,m}q$ has a lower order zero at $(\alpha, x) = 0$ than q. So $\widehat{m} = 2m + 1$ and we are done. □

Theorem 5.5.5. *The module $e(M_\tau/N_\tau)$ is isomorphic to the space of all anti-invariant quasi-invariants in $\mathbb{C}^n$.*

Proof. Combine Lemmas 5.5.4 and 5.5.1. □

5.6 The algebra $\Lambda_{R,B}$ and relation to quasi-invariants

In the paper [SV04] Sergeev and Veselov use the notion of generalized root systems to construct families of deformed quantum Calogero-Moser systems. In particular they introduce the commutative algebra $\Lambda_{R,B}$ related to the generalized root system (R, B). The generalized root systems are essentially the root systems of the contragredient Lie superalgebras together with a bilinear form B. They include an infinite family $A(n, m)$ which depends on one parameter k. Sergeev and Veselov showed that for these generalized root systems R and generic values of k there is a monomorphism χ from $\Lambda_{R,B}$ into the algebra of differential operators on V such that $\chi(x^2)$ is the corresponding Calogero-Moser operator related to R.

The algebra of integrals $\Lambda_{R,B}$ is of independent interest. They can be seen as a version of algebras where quasi-invariant conditions on the hyperplanes with non-integer multiplicities are understood as symmetry of polynomials under reflection at these hyperplanes. In [SV04] it was shown that for the series $A(n,m)$ and generic values of k these algebras are finitely generated. Also the Poincaré series were calculated.

The algebra corresponding to the series $A(n,m)$ can be realized as the following. Let $\Lambda_{n,m,k} \subset \mathbb{C}[V^*] = \mathbb{C}[x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m]$ be the algebra consisting of polynomials symmetric in the variables $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_m$ separately and satisfying the conditions

$$(\frac{\partial}{\partial x_i} - k\frac{\partial}{\partial y_j})f \mid_{x_i=y_j} \equiv 0 \tag{5.14}$$

for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$. It is easy to see that the deformed Newton sums

$$p_r(x, y, k) = \sum_{i=1}^{n} x_i^r + \frac{1}{k}\sum_{j=1}^{m} y_j^r \tag{5.15}$$

belong to $\Lambda_{n,m,k}$ for $r \geq 0$. Denote by $\mathcal{N}_{n,m,k}$ the algebra generated by the Newton sums (5.15).

We would like to remark here that it follows from [Fei] that $\mathcal{N}_{n,m,k}$ and $\Lambda_{n,m,k}$ are both representations of the spherical subalgebra of the Cherednik algebra $H_{1/k}(S_{nk+m})$. This is based on an analogue of Theorem 5.2.3 above. More exactly, consider the plane $\Pi \subset \mathbb{C}^{nk+m}$ with equations

$$x_1 = x_2 = \ldots = x_k$$

$$x_{k+1} = x_{k+2} = \ldots = x_{2k}$$

$$x_{(n-1)k+1} = x_{(n-1)k+2} = \ldots = x_{nk}.$$

Then let $D = \cup_{w \in S_{nk+m}} w(\Pi)$. It is proven in [Fei] that $I = \{p \in \mathbb{C}[x_1, \ldots, x_{nk+m}], p \mid_D = 0\}$ is a $H_{1/k}(S_{nk+m})$ module. Then $e(\mathbb{C}[x_1, \ldots, x_{nk+m}]/I)$ is a module for $e(H_{1/k}(S_{nk+m}))e$ generated by the deformed Newton sums (5.15) in appropriate variables, i.e. $\mathcal{N}_{n,m,k} \cong e(\mathbb{C}[x_1, \ldots, x_{nk+m}]/I)$.

We are going to study the embedding of the algebras $\mathcal{N}_{n,m,k} \subset \Lambda_{n,m,k}$. We have the following

Theorem 5.6.1 ([SV04], Theorem 2). *If k is not a positive rational number then $\Lambda_{n,m,k} = \mathcal{N}_{n,m,k}$.*

Further, Sergeev and Veselov use this to establish

Theorem 5.6.2 ([SV04], Theorem 3). *The Poincaré series of the algebra $\Lambda_{n,m,k}$ for generic k has the form*

$$P_{n,m}(t) = \frac{1}{(1-t)(1-t^2)\dots(1-t^n)}[1 + \sum_{i=1}^{m} \frac{t^{i(n+1)}}{(1-t)(1-t^2)\dots(1-t^i)}]. \tag{5.16}$$

The importance of these algebras in the context of our work is as follows. Recall the deformed root systems $\mathcal{A}_n(k)$ and corresponding rings of quasi-invariants. The configuration $\mathcal{A}_n(k)$ consists of the vectors $e_i - e_j$ with multiplicity k where $1 \le i < j \le n$ and the vectors $e_i - \sqrt{k}e_{n+1}$ with multiplicity 1.

The algebra $\Lambda_{n,1,k}$ is isomorphic to the partially symmetric quasi-invariants for the deformed root system $\mathcal{A}_n(k)$. Thus we have an expression for the Poincaré series for a 'symmetric component' of $Q_{\mathcal{A}_n(k)}$ in the case of generic k. This is in agreement with [FV03a] where the Poincaré series for $Q_{\mathcal{A}_2(k)}$ was calculated.

More explicitly, we have $\Lambda_{n,1,k} \cong Q^{sym}_{\mathcal{A}_n(k)}$, where $Q^{sym}_{\mathcal{A}_n(k)}$ consists of polynomials $q(x_1, \dots, x_{n+1})$ which are invariant under the action of the group S_n which permutes the coordinates $x_1, x_2, \dots, x_n$, and are quasi-invariant at the hyperplanes $x_i = \sqrt{k}x_{n+1}$, $i = 1, 2, \dots, n$ which have multiplicity one. Thus the Poincaré series $P_{Q^{sym}_{\mathcal{A}_n(k)}}$ is given by the formula (5.16) for generic k. We generalize the work of [SV04] with the following theorem.

Theorem 5.6.3. *One has*

$$P_{Q^{sym}_{\mathcal{A}_n(k)}}(t) = \frac{1 - t + t^{n+1}}{(1-t)^2(1-t^2)\dots(1-t^n)} \; \forall\, k \in \mathbb{Z}_+.$$

We also study the relation between $\Lambda_{n,1,k}$ and $\mathcal{N}_{n,1,k}$ for non-integer k.

Theorem 5.6.4. *Let $k \ge n$. Then $\Lambda_{n,1,\frac{1}{k+1}} = \mathcal{N}_{n,1,\frac{1}{k+1}}$.*

5.7 The algebra $\Lambda_{n,1,k}$ and deformed Newton sums

Denote by $\Lambda^N_{n,1,k}$ the homogeneous component of $\Lambda_{n,1,k}$ of degree N. Let λ be any partition of N with corresponding Young diagram and denote by $D_N(n)$ the number of partitions of N such that $\lambda_{n+1} \leq m$, which is the number of diagrams lying in the 'fat hook'. In [SV04] Sergeev and Veselov established in particular that $\dim \Lambda^N_{n,1,k} = D_N(n)$ for generic k. Thus to prove Theorem 5.6.3 it suffices to establish the following.

Proposition 5.7.1. *If $k \neq 0$ then $\dim \Lambda^N_{n,1,k} = D_N(n)$.*

We follow [SV04] in introducing the following notation. Let $I = (i_1, i_2, \dots i_n)$, $J = (j)$ be some (unordered) sequences of non-negative integers such that

$$\sum_{r=1}^{n} i_r + j = N.$$

Let $N(J)$ be the number of non-zero elements of J and let $M(I,J)$ be the number of elements of I that are greater than or equal to $N(J)$, so that $M(I,J) \leq n$. Following [SV04] let

$$E_{reg} = \{(I,J) \mid M(I,J) = n\} \tag{5.17}$$

and

$$E_{nreg} = \{(I,J) \mid M(I,J) < n\}. \tag{5.18}$$

Elements of E_{reg} are in 1:1 correspondence with the partitions inside the 'fat hook' of height of semi-infinite horizontal strip n and width of semi-infinite vertical vertical strip 1. Prescribe to each pair (I,J) a variable $C(I,J)$ such that $C(I,J) = C(\sigma(I),J)$ for any $\sigma \in S_n$. Let us write $C(I,J)_{reg}$ for any $C(I,J)$ with $(I,J) \in E_{reg}$ and $C(I,J)_{nreg}$ correspondingly. The number of different $C(I,J)_{reg}$ is equal to $D_N(n,1)$. This is due to the following. From a given sequence $I = (i_1, i_2, \dots, i_n)$ construct a Young diagram by reordering to weakly decreasing order and placing x_k boxes in the kth row where x_k is the k-th entry of I after reordering. Then attach the transposed diagram obtained in the same way from $J = (j)$, which is a single column, to the bottom. If $j = 0$ then any sequence I produces a diagram. Suppose $j \neq 0$. Then the condition $M(I,J) = n$ ensures we have a diagram since each element in I is ≥ 1 so there are no zero rows. Obviously such a diagram lies in the fat hook as there are n rows in I and 1 column in J^T.

Let $f = \sum C(I,J) x^I y^J$ be a homogeneous polynomial of degree N symmetric in the variables x. Choose $r, p \in \mathbb{N}$ with $1 \leq r \leq n$ and $1 \leq p \leq N$ and consider

$$\sum_{i+j=p} (i - kj) C(I,J) = 0 \tag{5.19}$$

where i occupies the rth place in I and the other elements of I are fixed. Then the system (5.19) is nothing more than the quasi-invariance condition (5.14) for polynomials

f from $\Lambda_{n,1,k}$, see [SV04]. The meaning of p is that it is the degree of some terms of f as polynomials in x_r and y. Thus to prove Proposition 5.7.1 it is sufficient to show that every element $C(I,J)_{nreg}$ can be uniquely expressed via the system (5.19) as a linear combination of elements of $C(I,J)_{reg}$.

Lemma 5.7.2. *Let $f = \sum C(I,J)x^I y^J \in \Lambda_{n,1,k}$ with $k \neq 0$. Then every element of $C(I,J)_{nreg}$ can be uniquely expressed via the system (5.19) as a linear combination of elements of $C(I,J)_{reg}$.*

Proof. We collect the coefficients $C(I,J)_{nreg}$ into subsets defined by the number of zero entries in I. Let

$$I = I_0 \sqcup I_1 \sqcup \ldots \sqcup I_n$$

where I_j is the set of those $I = \{i_1, i_2, \ldots, i_n\}$ where exactly j entries i_r, $1 \leq r \leq n$ are equal to zero. Then

$$E(I,J)_{nreg} = \bigsqcup_{i=1}^{n} E(I_i,J)_{nreg}. \tag{5.20}$$

Note that

$$E(I_i,0)_{nreg} = \emptyset \ \forall\ i = 0,1,2,\ldots,n \tag{5.21}$$

$$E(I_0,j)_{nreg} = \emptyset \text{ if } j \neq 0. \tag{5.22}$$

Before we can use this to our advantage, let's consider the equations in the system (5.19). As above, Let $f = \sum C(I,J)x^I y^J$ be a homogeneous polynomial of degree N symmetric in the variables x. Consider an arbitrary monomial term $g = x_1^{a_1}x_2^{a_2}\ldots x_n^{a_n}$, where $\sum_{i=1}^{n} a_i = N-1$, in the polynomial obtained by applying the quasi-invariance condition (5.14) to f. Each equation in (5.19) represents the vanishing of the sum of the coefficients of such a term. We consider which monomial terms in f give rise to g and what the coefficients of these terms are. It is clear that the operation

$$(\partial_{x_1} - k\partial_y)f\ |_{x_1=y}$$

applied to each of the monomials

$$x_1^{a_1+1-t}x_2^{a_2}\ldots x_n^{a_n}y^t \tag{5.23}$$

where $0 \leq t \leq a_1+1$, gives rise to $x_1^{a_1}x_2^{a_2}\ldots x_n^{a_n}$. The monomials (5.23) appear with the coefficients

$$C_{a_1+1-t,a_2,a_3,\ldots,a_n;t} = C(I,J) \text{ with } I = (a_1+1-t, a_2, a_3, \ldots, a_n) \text{ and } J = (t) \tag{5.24}$$

where $0 \leq t \leq a_1 + 1$. So each equation in (5.19) has the form

$$\sum_{t=0}^{a_1+1} (a_1 + 1 - t(k+1)) C_{a_1+1-t,a_2,a_3,\ldots,a_n;t} = 0. \tag{5.25}$$

Now, let $\{a_2, \ldots, a_n\}$ be non-zero elements in I_1. We are going to show that any element of $C(I_1, J)_{nreg}$ can be expressed through coefficients belonging to $C(I, J)_{reg}$ from the equation (5.25). To do this we show that the coefficient $C_{0,a_2,a_3,\ldots,a_n;p} \in C(I, J)_{nreg}$ can be expressed through coefficients belonging to $C(I, J)_{reg}$ from the equation (5.25). To see this, let $p = a_1 + 1$. Then (5.25) is simply

$$\sum_{t=0}^{p} (p - t(k+1)) C_{p-t,a_2,a_3,\ldots,a_n;t} = 0. \tag{5.26}$$

For any value of p, $1 \leq p \leq N$, it is clear that the equations (5.26) will include one and only one coefficient that lies in $C(I, J)_{nreg}$: the coefficient in (5.26) corresponding to $t = p$, which is the aforementioned $C_{0,a_2,a_3,\ldots,a_n;p}$. Clearly all elements of $C(I_1, J)_{nreg}$ have this form. We see that these coefficients appear alongside $-pk \neq 0$ and using (5.21) and (5.22) we see that all other coefficients correspond to elements of $C(I, J)_{reg}$. Moving on, let us now show that $C(I_2, J)_{nreg}$ can be expressed through $C(I_2, J)_{reg}$ and $C(I_1, J)$ using the equation (5.25). Let $\{a_3, \ldots, a_n\}$ be non-zero elements in I_2. This time (5.25) becomes

$$\sum_{t=0}^{p} (p - t(k+1)) C_{p-t,0,a_3,\ldots,a_n,t} = 0. \tag{5.27}$$

Now, if $t \neq 0$ or p the corresponding coefficient is an element of $C(I_1, J)_{nreg}$. These were expressed through $C(I, J)_{reg}$ at the previous stage. The only other entries in $C(I, J)_{nreg}$ that can appear lie in $C(I_2, J)_{nreg}$: these are the entries corresponding to $t = p$. These occur with coefficient $-pk$ which is non-zero. We can continue in this way and express all elements of the set $C(I, J)_{nreg}$ as a linear combination of elements of $C(I, J)_{reg}$. It is clear from this process that each element of $C(I, J)_{nreg}$ is uniquely expressed through elements of $C(I, J)_{reg}$ and the elements of $C(I, J)_{reg}$ can be chosen to be completely arbitrary. □

Proposition 5.7.1 now follows. Hence Theorem 5.6.3 is proven.

5.8 Poincaré series for deformed Newton sums

We now turn to the proof of Theorem 5.6.4. Recall the Poincaré series

$$p_R(t) = \sum_{j \geq 0} \dim R_j t^j$$

of an algebra R, where R_j denotes the graded component of R of degree j. Recall from Theorem 5.6.2 that for generic k the Poincaré series of the algebra $\Lambda_{n,1,k}$ has the form

$$P_{n,1}(t) = \frac{1 - t + t^{n+1}}{(1-t)^2(1-t^2)\dots(1-t^n)}. \tag{5.28}$$

We are going to investigate a way to realize the algebra generated by the Newton polynomials. For $M \in \mathbb{N}$ let $\mathbb{C}[x_1, x_2, x_3, \dots, x_M]^{S_M} = \mathbb{C}[X]^{S_M}$ be the algebra of symmetric polynomials in M variables. Define the ideal $I_k = \{f(x_1, x_2, \dots, x_M) \in \mathbb{C}[X]^{S_M} : f\mid_{x_1=x_2=\dots=x_{k+1}} = 0, 1 \le k \le M-1\}$.

For $k, M \ge 0$ let

$$A(k,M) = \sum_{\substack{N_1 \ge N_2 \ge \dots \ge N_k \\ \sum N_i = M}} \frac{t^{N_1^2+N_2^2+N_3^2+\dots N_k^2 - M}}{(t)_{N_1-N_2}(t)_{N_2-N_3}\dots(t)_{N_{k-1}-N_k}(t)_{N_k}} \tag{5.29}$$

where $(t)_N = \prod_{i=1}^{N}(1-t^i)$ and by convention $(t)_0 = 1$. We put $A(k, M) = 0$ if $M < 0$.

Theorem 5.8.1 ([FS94], Theorem 2.7.1)**.** *The Poincaré series of I_k is given by $P(I_k) = A(k, M)$.*

It is clear that $\mathcal{N}_{n,1,\frac{1}{k+1}} = \mathbb{C}[X]^{S_M}/I_k$. Then immediately we have the following

Corollary 5.8.2. *The Poincaré series for $\mathcal{N}_{n,1,\frac{1}{k+1}}$ is given by*

$$P(\mathcal{N}_{n,1,\frac{1}{k+1}}) = \prod_{i=1}^{M} \frac{1}{1-t^i} - A(k,M). \tag{5.30}$$

We are now ready to begin the proof of Theorem 5.6.4. We need some preliminary results.

Lemma 5.8.3.

(i) If $M \ge k$ then

$$A(k,M) = \sum_{r=0}^{[\frac{M}{k}]} \frac{t^{2rM - r(r+1)k}}{(t)_r} A(k-1, M-rk). \tag{5.31}$$

(ii) If $M < k$ then

$$A(k,M) = A(M,M). \tag{5.32}$$

Proof. (i) Suppose $M \geq k$. Consider N_k. This partition element can take any value between 0 and $[\frac{M}{k}]$. If we set $N_k = 0$ the corresponding term in the sum $A(k, M)$ is precisely $A(k-1, M)$. Motivated by this we consider $N_k = 0, 1, 2, \ldots, [\frac{M}{k}]$ to see that

$$A(k, M) = A(k-1, M) + \sum_{r=1}^{[\frac{M}{k}]} \sum_{\substack{N_1 \geq N_2 \ldots \geq N_{k-1} \\ \sum N_i = M - rk}} \frac{t^{(N_1+r)^2+(N_2+r)^2+(N_3+r)^2+\ldots+(N_{k-1}+r)^2+r^2-M}}{(t)_{N_1-N_2}(t)_{N_2-N_3}\ldots(t)_{N_{k-2}-N_{k-1}}(t)_{N_{k-1}}(t)_r}.$$

We know from the definition that

$$A(k-1, M-rk) = \sum_{\substack{N_1 \geq N_2 \ldots \geq N_{k-1} \\ \sum N_i = M - rk}} \frac{t^{N_1^2+N_2^2+N_3^2+\ldots N_{k-1}^2-(M-rk)}}{(t)_{N_1-N_2}(t)_{N_2-N_3}\ldots(t)_{N_{k-2}-N_{k-1}}(t)_{N_{k-1}}}.$$

So we can conclude that

$$\begin{aligned}
A(k, M) =& A(k-1, M) + \sum_{r=1}^{[\frac{M}{k}]} \sum_{\substack{N_1 \geq N_2 \ldots \geq N_{k-1} \\ \sum N_i = M - rk}} \frac{t^{\sum N_i^2 + 2r \sum N_i + kr^2 - M}}{(t)_{N_1-N_2}(t)_{N_2-N_3}\ldots(t)_{N_{k-2}-N_{k-1}}(t)_{N_{k-1}}(t)_r} \\
=& A(k-1, M) + \sum_{r=1}^{[\frac{M}{k}]} \frac{t^{2rM - r(r+1)k}}{(t)_r} A(k-1, M-rk) \\
=& \sum_{r=0}^{[\frac{M}{k}]} \frac{t^{2rM - r(r+1)k}}{(t)_r} A(k-1, M-rk).
\end{aligned}$$

(ii) If $M < k$ then the final $k - M$ integers N_i in the sum $A(k, M)$ must be zero. Thus $A(k, M) = A(M, M)$. □

Lemma 5.8.4.

$$A(k, k+1) = \prod_{i=1}^{k+1} \frac{1}{1-t^i} - \frac{1}{1-t}. \tag{5.33}$$

Proof. Let $M = k+1$ and consider $I_k = \{f(x_1, x_2, \ldots, x_{k+1}) \in \mathbb{C}[X]^{S_M} : f\mid_{x_1=x_2=\ldots=x_{k+1}} = 0\} \subset \mathbb{C}[X]^{S_M}$. If we consider any homogeneous polynomial f of degree N in I_k then it is clear that the ideal membership restriction imposed merely implies that the sum of the coefficients of every term of f must vanish. So, the dimension of the homogeneous component of degree N of I_k is just one less than that of the homogeneous component of degree N of $\mathbb{C}[X]^{S_M}$. □

Recall from Corollary 5.8.2 that $P(\mathcal{N}_{n,1,\frac{1}{k+1}}) = \prod_{i=1}^{M} \frac{1}{1-t^i} - A(k, M)$. With this in mind we establish the following.

Proposition 5.8.5. *Assume that $k \geq n$. Then $\prod_{i=1}^{k+n+1} \frac{1}{1-t^i} - A(k, n+k+1) = P_{n,1}(t)$ where $P_{n,1}(t)$ is given by (5.28).*

Proof. We proceed by induction on n. The base case $n = 0$ is immediate from (5.33). Let us assume that for $k \geq n$

$$\begin{aligned} B(k, n+k+1) := & \prod_{i=1}^{k+n+1} \frac{1}{1-t^i} - A(k, n+k+1) \\ = & P_{n,1}(t). \end{aligned} \tag{5.34}$$

Now, we can use (5.31) and the inductive hypothesis (5.34) to see that for $k \geq n+1$

$$\begin{aligned} & B(k, n+k+2) \\ = & \prod_{i=1}^{n+k+2} \frac{1}{1-t^i} - A(k, n+k+2) \\ = & \prod_{i=1}^{n+k+2} \frac{1}{1-t^i} - A(k+1, n+k+2) + \sum_{r=1}^{[\frac{n+k+2}{k+1}]} \frac{t^{2r(n+k+2)-r(r+1)(k+1)}}{(t)_r} A(k, n+k+2-r(k+1)) \\ = & \frac{1-t+t^{n+1}}{(1-t)^2(1-t^2)\dots(1-t^n)} + \sum_{r=1}^{[\frac{n+k+2}{k+1}]} \frac{t^{2r(n+k+2)-r(r+1)(k+1)}}{(t)_r} A(k, n+k+2-r(k+1)). \end{aligned}$$

Since $k \geq n+1$, $k > n$ so $A(k, n+k+2-r(k+1)) \neq 0 \iff r = 1$. If $r = 1$ then

$$B(k, n+k+2) = \frac{1-t+t^{n+1}}{(1-t)^2(1-t^2)\dots(1-t^n)} + \frac{t^{2n+2}}{(t)_1} A(k, n+1).$$

Again since $k > n$ we see from (5.32) that $A(k, n+1) = A(n+1, n+1)$. Using (5.31) we have

$$A(n+1, n+1) = A(n, n+1) + \frac{t^0}{(t)_1} A(n, 0). \tag{5.35}$$

Again using (5.32) note that $A(n, 0) = A(0, 0) = 1$. We can use the inductive hypothesis once again to see that

$$A(n, n+1) = \prod_{i=1}^{n+1} \frac{1}{1-t^i} - \frac{1}{1-t}. \tag{5.36}$$

Putting this together (5.35) becomes

$$A(n+1, n+1) = \prod_{i=1}^{n+1} \frac{1}{1-t^i} - \frac{1}{1-t} + \frac{1}{1-t} = \prod_{i=1}^{n+1} \frac{1}{1-t^i}. \tag{5.37}$$

So we can see that

$$\begin{aligned} B(k, n+k+2) &= \frac{1-t+t^{n+1}}{(1-t)^2(1-t^2)\dots(1-t^n)} + \frac{t^{2n+2}}{(t)_1}\prod_{i=1}^{n+1}\frac{1}{1-t^i} \\ &= \frac{1-t+t^{n+1}}{(1-t)^2(1-t^2)\dots(1-t^n)} + \frac{t^{2n+2}}{(t)_1(t)_{n+1}} \\ &= \frac{1}{(t)_1(t)_{n+1}}[(1-t+t^{n+1})(1-t^{n+1})+t^{2n+2}] \\ &= \frac{1}{(t)_1(t)_{n+1}}[1-t+t^{n+2}] \\ &= P_{n+1,1}(t) \end{aligned}$$

which completes the induction. □

Proposition 5.8.5 gives a simplification of the general formula (5.29) for the Poincaré series of the ideal $I_k \subset \mathbb{C}[x_1, \dots, x_{n+k+1}]$ in the case $k \geq n$. Further we have the following corollary.

Corollary 5.8.6. *Let $k \geq n$. Then $\mathcal{N}_{n,1,\frac{1}{k+1}} = \Lambda_{n,1,\frac{1}{k+1}}$.*

Proof. Let $k \geq n$. By Proposition 5.8.5 $\dim \mathcal{N}^N_{n,1,\frac{1}{k+1}} = \dim \Lambda^N_{n,1,\frac{1}{k+1}}$ for generic k. By the results of [SV04] this dimension equals $D_N(n)$ and we know from Proposition 5.7.1 that $\dim \Lambda^N_{n,1,\delta} = D_N(n)$ for $\delta \neq 0$ and in particular for $\delta = \frac{1}{k+1}$. □

Theorem 5.6.4 is now proven.

5.9 Further analysis of deformed Newton sums and Λ-algebra

To prove Theorem 5.6.4 we had to assume that $k \geq n$. We now investigate the case $k < n$. Firstly, suppose that $k+1 \leq n < 2(k+1)$. So, $0 \leq n-(k+1) \leq k$ and $k > n/2-1$. This is the case where we allow two 'colliding particles'. We are going to establish the following.

Proposition 5.9.1. *Suppose that $n-(k+1) = r$ for $0 \leq r \leq k$. Then*

$$P(\mathcal{N}_{n,1,\frac{1}{k+1}}) = P(\Lambda_{n,1,\frac{1}{k+1}}) - \frac{t^{2n+1-r}}{(t)_2(t)_r}(1-t^2+t^{r+2}). \tag{5.38}$$

Proof. We proceed by induction on n. For $n=1$ the condition $n-(k+1)=r$ cannot be satisfied as $k \geq 1$. Suppose $n=2$. Then the only possibility is $k=1$ and in this case $n-(k+1)=0$. The inequality $k > n/2-1$ is also satisfied. We can easily see that

$$P(\mathcal{N}_{2,1,1/2}) = P(\Lambda_{2,1,1/2}) - \frac{t^5}{(1-t)(1-t^2)}. \tag{5.39}$$

For,

$$P(\mathcal{N}_{2,1,1/2}) = \prod_{i=1}^{4} \frac{1}{1-t^i} - A(1,4) = \prod_{i=1}^{4} \frac{1}{1-t^i} - \frac{t^{12}}{(t)_4} = \frac{1-t^{12}}{(t)_4} = \frac{1+t^4+t^8}{(t)_3} = \frac{t^6-t^5+t^3-t+1}{(1-t)^2(1-t^2)}$$

and since

$$P(\Lambda_{2,1,1/2}) = \frac{1-t+t^3}{(1-t)^2(1-t^2)}$$

the statement is clear. Now we proceed inductively. We have

$$\begin{aligned} P(\mathcal{N}_{n+1,1,\frac{1}{k+1}}) &= \prod_{i=1}^{n+k+2} \frac{1}{1-t^i} - A(k, n+k+2) \\ &= \prod_{i=1}^{n+k+2} \frac{1}{1-t^i} - A(k+1, n+k+2) + \frac{t^{2n+2}}{(t)_1} A(k, n+k+2-(k+1)) \\ &+ \frac{t^{4n-2k+2}}{(t)_2} A(k, n+k+2-2(k+1)) \text{ using (5.31)} \\ &= P(\mathcal{N}_{n,1,\frac{1}{k+2}}) + \frac{t^{2n+2}}{(t)_1} A(k, n+1) + \frac{t^{4n-2k+2}}{(t)_2} A(k, n-k). \end{aligned}$$

Recall that we are assuming $n+1-(k+1) = r$ so that $n-k = r$. Also, $n-(k+1+1) = n-(k+2) = n-k-2 = r-2$. So, we can express $P(\mathcal{N}_{n,1,\frac{1}{k+2}})$ for fixed r through the case $r-2$. This means that we have

$$P(\mathcal{N}_{n+1,1,\frac{1}{k+1}}) = P(\Lambda_{n,1,\frac{1}{k+1}}) - \frac{t^{2n+3-r}}{(t)_2(t)_{r-2}}(1-t^2+t^r) + \frac{t^{2n+2}}{(t)_1} A(k, k+r+1) + \frac{t^{2n+2r+2}}{(t)_2} A(k,r) \tag{5.40}$$

upon noting that

$$n+1 = k+r+1$$

and

$$4n-2k+2 = 2(n-k)+2n+2 = 2r+2n+2$$

Since $r \leq k$, using (5.31) we have

$$A(r,r) = A(r-1,r) + \frac{t^0}{(t)_1} A(0,0) = \prod_{i=1}^{r} \frac{1}{1-t^i} - \frac{1}{1-t} + \frac{1}{1-t} = \frac{1}{(t)_r}. \tag{5.41}$$

Now consider the term $A(k, k+r+1)$. We use (5.31) again to deduce that

$$\begin{aligned} A(k, k+r+1) &= A(k+1, k+r+1) - \frac{t^{2(k+r+1)-2(k+1)}}{(t)_1} A(k,r) \\ &= A(k+1, k+r+1) - \frac{t^{2r}}{(t)_1} A(k,r). \end{aligned}$$

However we can again apply (5.31), this time to the term $A(k+1,k+r+1)$ in the above. We see that

$$\begin{aligned} A(k+1,k+r+1) &= A(k+2,k+r+1) - \frac{t^{2(k+r+1)-2(k+2)}}{(t)_1}A(k+1,r-1) \\ &= A(k+2,k+r+1) - \frac{t^{2r-2}}{(t)_1}A(r-1,r-1). \end{aligned}$$

We can continue in this way, applying (5.31) to $A(k+2,k+r+1)$, $A(k+3,k+r+1)$ and so on until we reach the term

$$\begin{aligned} A(k+r-1,k+r+1) &= A(k+r,k+r+1) - \frac{t^{2(k+r+1)-2(k+r)}}{(t)_1}A(k+r-1,1) \\ &= A(k+r,k+r+1) - \frac{t^2}{(t)_1}A(1,1). \end{aligned}$$

Recall from Lemma 5.33 that

$$A(k+r,k+r+1) = \prod_{i=1}^{k+r+1} \frac{1}{1-t^i} - \frac{1}{1-t}. \tag{5.42}$$

In total we have

$$\begin{aligned} A(k,k+r+1) = &\prod_{i=1}^{k+r+1} \frac{1}{1-t^i} - \frac{1}{1-t} - \frac{t^2}{(t)_1}A(1,1) - \frac{t^4}{(t)_1}A(2,2) - \frac{t^6}{(t)_1}A(3,3) - \dots \\ &\dots - \frac{t^{2r-2}}{(t)_1}A(r-1,r-1) - \frac{t^2r}{(t)_1}A(r,r) \\ = &\prod_{i=1}^{k+r+1} \frac{1}{1-t^i} - \frac{1}{1-t} - \sum_{j=1}^{r} \frac{t^{2j}}{(t)_1}A(j,j) \\ = &\frac{1}{(t)_{k+r+1}} - \frac{1}{(t)_1} - \frac{t^2}{(t)_1}\sum_{j=0}^{r-1}\frac{t^{2j}}{(t)_{j+1}} \end{aligned} \tag{5.43}$$

upon recalling that $A(j,j) = \frac{1}{(t)_j}$. Let us take stock of the situation. We can now write (5.40) as

$$\begin{aligned} P(\mathcal{N}_{n+1,1,\frac{1}{k+1}}) = P(\Lambda_{n,1,\frac{1}{k+1}}) &+ \frac{t^{2n+2}}{(t)_1(t)_{n+1}} - \frac{t^{2n+3-r}}{(t)_2(t)_{r-2}}(1-t^2+t^r) \\ &+ \frac{t^{2n+2r+2}}{(t)_2(t)_r} + \frac{t^{2n+2}}{(t)_1}\left(-\frac{1}{(t)_1} - \frac{t^2}{(t)_1}\sum_{j=1}^{r}\frac{t^{2j}}{(t)_j}\right). \end{aligned}$$

Recall from the final step in the proof of Proposition 5.8.5 that $P(\Lambda_{n,1,\frac{1}{k+1}}) + \frac{t^{2n+2}}{(t)_1(t)_{n+1}} = P(\Lambda_{n+1,1,\frac{1}{k+1}})$. Also note that

$$\sum_{j=0}^{s} \frac{t^{2j}}{(t)_j} = \frac{1-t+t^{s+1}}{(t)_s} \tag{5.44}$$

This is easy to see: first note that $1 + \frac{t^2}{(t)_1} = \frac{1-t+t^2}{(t)_1}$. Then proceed by induction and consider

$$\begin{aligned}\sum_{j=0}^{s+1} \frac{t^{2j}}{(t)_j} &= \sum_{j=0}^{s} \frac{t^{2j}}{(t)_j} + \frac{t^{2s+2}}{(t)_{s+1}} = \frac{1-t+t^{s+1}}{(t)_s} + \frac{t^{2s+2}}{(t)_{s+1}} \\ &= \frac{1}{(t)_s}(1-t+t^{s+1} + \frac{t^{2s+2}}{1-t^{s+1}}) = \frac{1}{(t)_{s+1}}(1-t+t^{s+2}).\end{aligned}$$

Putting all of this together we have

$$\begin{aligned}P(\mathcal{N}_{n+1,1,\frac{1}{k+1}}) &= P(\Lambda_{n+1,1,\frac{1}{k+1}}) - \frac{t^{2n+3-r}}{(t)_2(t)_{r-2}}(1-t^2+t^r) + \frac{t^{2n+2r+2}}{(t)_2(t)_r} - \frac{t^{2n+2}}{(t)_1(t)_1}(\frac{1}{(t)_r}(1-t+t^{r+1})) \\ &= P(\Lambda_{n+1,1,\frac{1}{k+1}}) - \frac{t^{2n+3-r}}{(t)_2(t)_{r-2}}(1-t^2+t^r) \\ &- \frac{t^{2n+2}}{(t)_1(t)_1(t)_r}(1-t+t^{r+1} - \frac{t^{2r}}{(t)_2}(1-t)(1-t)) \\ &= P(\Lambda_{n+1,1,\frac{1}{k+1}}) - \frac{t^{2n+3-r}}{(t)_2(t)_{r-2}}(1-t^2+t^r) - \frac{t^{2n+2}}{(t)_1(t)_1(t)_r}(1-t+t^{r+1} - \frac{t^{2r}}{1+t}) \\ &= P(\Lambda_{n+1,1,\frac{1}{k+1}}) - \frac{t^{2n+3-r}}{(t)_2(t)_{r-2}}(1-t^2+t^r) - \frac{t^{2n+2}}{(t)_2(t)_r}(1-t^2+t^{r+1}+t^{r+2}-t^{2r}) \\ &= P(\Lambda_{n+1,1,\frac{1}{k+1}}) \\ &- \frac{t^{2n+3-r}}{(t)_2(t)_r}((1-t^2+t^r)(1-t^r)(1-t^{r-1}) + t^{r-1}(1-t^2+t^{r+1}+t^{r+2}-t^{2r})) \\ &= P(\Lambda_{n+1,1,\frac{1}{k+1}}) - \frac{t^{2n+3-r}}{(t)_2(t)_r}(1-t^2+t^{r+2})\end{aligned}$$

as required. □

We now consider the situation $2(k+1) \leq n < 3(k+1)$. This corresponds to allowing three possible 'colliding particles'. To illustrate what happens consider the case $n-2(k+1) = 0$. We will look at $k=1, n=4$ first. Clearly this satisfies both inequalities. We have

$$\begin{aligned}P(\mathcal{N}_{4,1,1/2}) &= \prod_{i=1}^{6} \frac{1}{1-t^i} - A(1,6) \\ &= \frac{1-t^{30}}{(t)_6} \\ &= \frac{1+t^6+t^{12}+t^{18}+t^{24}}{(t)_5} \\ &= \frac{1-t+t^5-t^7+t^{10}-t^{13}+t^{15}-t^{19}+t^{20}}{(t)_1(t)_4}.\end{aligned}$$

Recall that $P(\Lambda_{4,1,\frac{1}{k+1}}) = \frac{1-t+t^5}{(t)_1(t)_4}$. So

$$\begin{aligned}P(\Lambda_{4,1,1/2}) - P(\mathcal{N}_{4,1,1/2}) &= \frac{t^7(1-t^3+t^6-t^8+t^{12}-t^{13})}{(t)_1(t)_4} \\ &= \frac{t^7(1-t)(1+t^2)(1+t-t^3+t^5+t^6-t^8+t^{10})}{(t)_1(t)_4}.\end{aligned}$$

Now consider $n = 6, k = 2$. Again this satisfies both inequalities. We have

$$\begin{aligned}
P(\mathcal{N}_{6,1,1/3}) &= \prod_{i=1}^{9} \frac{1}{1-t^i} - A(2,9) \\
&= \frac{1-t^{72}}{(t)_9} - \frac{t^{56}}{(t)_7(t)_1} - \frac{t^{64}}{(t)_5(t)_2} - \frac{t^{36}}{(t)_3(t)_3} - \frac{t^{32}}{(t)_1(t)_4} \\
&= \frac{1}{(t)_9}(1 - t^{72} - t^{56}(1+t+t^2+t^3+t^4+t^5+t^6+t^7)(1-t^9) \\
&- t^{64}(1+t+t^2+t^3+t^4+t^5)(1-t^7)(1+t^4)(1+t^2)(1-t^9) \\
&- t^{36}(1+t+t^2+t^3)(1-t^5)(1+t^3)(1-t^7)(1+t^4)(1+t^2)(1-t^9) \\
&- t^{32}(1+t+t^2+t^3+t^4)(1-t^6)(1-t^7)(1-t^8)(1-t^9)) \\
&= \frac{1-t+t^7-t^{10}+t^{14}+t^{16}-t^{17}-t^{19}+t^{21}+t^{23}-t^{26}+t^{30}-t^{33}-t^{36}}{(t)_1(t)_7} \\
&+ \frac{t^{37}+t^{44}+t^{46}+t^{48}-t^{64}-t^{66}-t^{68}}{(t)_1(t)_7}.
\end{aligned}$$

So we have

$$\begin{aligned}
&P(\Lambda_{6,1,1/3}) - P(\mathcal{N}_{6,1,1/3}) \\
&= \frac{1}{(t)_1(t)_7}(t^{10}(1-t)(1+t)(1+t^2)(1-t+t^2)(1+t+t^2)(1+t+t^2+t^3+t^4) \\
&\times (1-t-t^2+t^3+t^5-t^6-t^{11}+2t^{12}-2t^{14}+t^{15}-t^{34}+t^{35}-t^{38}+t^{40}-t^{42}+t^{45}-t^{46})).
\end{aligned}$$

This information suggests that, somewhat surprisingly, it is not possible to prove an analogous result to Proposition 5.9.1 in the situation $2(k+1) \le n < 3(k+1)$. This is because the two cases we considered, $k = 1, n = 4$ and $k = 2, n = 6$ both satisfy $n-2(k+1) = 0$ but the differences $P(\Lambda_{4,1,1/2}) - P(\mathcal{N}_{4,1,1/2})$ and $P(\Lambda_{6,1,1/3}) - P(\mathcal{N}_{6,1,1/3})$ have the following structure. We can rearrange to

$$P(\Lambda_{4,1,1/2}) - P(\mathcal{N}_{4,1,1/2}) = t^7 \frac{P(t)}{Q(t)}$$

and

$$P(\Lambda_{6,2,1/3}) - P(\mathcal{N}_{6,2,1/3}) = t^{10} \frac{P'(t)}{Q(t)}$$

where P, P', Q are polynomials in t and the degrees of P/Q, P'/Q are different. Thus it seems $P(\Lambda) - P(\mathcal{N})$ has the structure $t^{q(r,n)}R(t)/Q(t)$ where $q(r,n)$ is some function of r, n and $R(t)$ is a polynomial in t whose degree depends on n.

Chapter 6

Conclusions and open problems

The work contained in this thesis suggests a number of possible avenues for further exploration. We have answered a number of questions within the specific areas we considered. However, there is still much interesting work to be done. We highlight three areas of particular interest that we feel could reap rewards if investigated further.

The material contained in Chapters 2 and 4 can be summarized succinctly with the following statement. We have shown that within the class of arrangements of type $(m, 1^n)$ the Gorenstein property for the rings of quasi-invariants is equivalent to the existence of the Baker-Akhiezer function ϕ. We expect that this is true in the much more general setting of arbitrary multi-dimensional configurations of vectors with multiplicities. In principle moving from existence of the Baker-Akhiezer function to the Gorenstein property is easier due to the wealth of structures associated with the Baker-Akhiezer function. However even this direction is unclear at present. Establishing this conjecture would provide a very interesting new perspective on systems of Calogero-Moser type.

Another question, which is long-standing and predates the work in this thesis, is about the algebraic structure of the rings $Q_{\mathcal{A}_n(m)}$ and $Q_{\mathcal{C}_n(m,l)}$. Recall that these rings have been shown to be Cohen-Macaulay in arbitrary dimension, and Gorenstein in two-dimensions [FV03a]. It is known that the Baker-Akhiezer function exists for $\mathcal{A}_n(m)$ and $\mathcal{C}_n(m, l)$. Presumably the rings $Q_{\mathcal{A}_n(m)}$ and $Q_{\mathcal{C}_n(m,l)}$ are Gorenstein in arbitrary dimension. Thus calculation of the Poincaré series for $Q_{\mathcal{A}_n(m)}$ and $Q_{\mathcal{C}_n(m,l)}$ would be a useful result.

The final direction we would like to mention is the relation to representations of Cherednik algebras. We have seen in Chapter 5 that it is possible to construct the space of all anti-invariant polynomials in the quasi-invariant ring corresponding to the root system A_n as a module for the spherical subalgebra of the Cherednik algebra $H_{1/m}(S_{mn})$. A natural question is whether *all* quasi-invariants can be constructed in such a fashion as special representations of Cherednik algebras. In particular it would be interesting to ascertain whether all quasi-invariants corresponding to A_n can be realized in such a fashion, and not just the anti-invariants. We have attempted to complete such a construction via considering different representations of S_{mn} but this is not clear at present. A similar question would be whether the quasi-invariants $Q_{\mathcal{A}_n(m)}$ and $Q_{\mathcal{C}_n(m,l)}$ can be realized in this way. As a whole this direction is potentially exciting and interesting due to the rich representation theory of $H_{1/m}$.

In conclusion we have seen that questions about the algebraic structure of rings of quasi-invariants lead to interesting areas of exploration in integrable systems, algebra and representation theory. It seems there any many more questions to be answered and hopefully many exciting discoveries.

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