Nuclear and minimal atomic S-algebras

by

Helen S.E. Gilmour

A thesis submitted to the
Faculty of Information and Mathematical Sciences
at the University of Glasgow
for the degree of
Doctor of Philosophy

July 2006

Abstract

We begin in Chapter 1 by considering the original framework in which most work in stable homotopy theory has taken place, namely the stable homotopy category. We introduce the idea of structured ring and module spectra with the definition of ring spectra and their modules. We then proceed by considering the category of S-modules $\mathscr{M}_S$ constructed in [19]. The symmetric monoidal category structure of $\mathscr{M}_S$ allows us to discuss the notions of S-algebras and their modules, leading to modules over an S-algebra R. In Section 2.5 we use results of Strickland [43] to prove a result relating to the products on ko/w as a ko-module.

A survey of results on nuclear and minimal atomic complexes from [5] and [23] is given in the context of $\mathscr{M}_S$ in Chapter 3. We give an account of basic results for topological André–Quillen homology (HAQ) of commutative S-algebras in Chapter 4. In Section 4.2 we are able to set up a framework on HAQ for cell commutative S-algebras which allows us to extend results reported in Chapter 3 to the case of commutative S-algebras in Chapter 5. In particular, we consider the notion of a core of commutative S-algebras. We give examples of non-cores of MU, MSU, MO and MSO in Chapter 6. We construct commutative MU-algebra $MU//x_2$ in Chapter 7 and consider various calculations associated to this construction.

Acknowledgements

I would like to acknowledge my supervisor Dr. Andrew J. Baker for his guidance throughout the period of research. I am also extremely grateful to Professor Peter H. Kropholler for his invaluable help and support during the final stages of this project.

I would like to thank EPSRC and the University of Glasgow for their financial support during my studies.

Statement

This thesis is submitted in accordance with regulations for the degree of Doctor of Philosophy in the University of Glasgow. No part of this thesis has previously been submitted by me for a degree at any university.

The ideas for the main research themes of this thesis have arisen from discussion with my supervisor Dr Andrew J. Baker. The material in Chapter 4 is contained in [4]. Sections 2.5, 5.2 - 5.3, 6.3 and much of Chapter 7 consists of original work.

Contents

Introduction

Our research into nuclear, minimal and minimal atomic commutative S-algebras was motivated by results on cores of complexes (spaces and spectra) by Hu, Kriz and May in [23] and by a later paper on minimal atomic complexes by Baker and May [5]. Hu, Kriz and May [23] note that their work on complexes could equally well have taken place in the category of S-modules $\mathscr{M}_S$ that has been constructed by Elmendorf, Kriz, Mandell and May in [19]. It is in this context that we present, in Chapter 3, known results on minimal atomic and nuclear spectra contained in [23] and [5].

The category $\mathscr{M}_S$ has an associative, commutative and unital smash product and a derived category $\mathscr{D}_S$ that is equivalent to the stable homotopy category $\bar{h}\mathscr{S}$. This symmetric monoidal structure of $\mathscr{M}_S$ allows us in Section 2.3 to define an S-algebra and a commutative S-algebra which are versions of the earlier notions of A_∞ and E_∞ ring spectra. We discuss $\bar{h}\mathscr{S}$, also described as the derived homotopy category, in Chapter 1 where we draw a parallel between this and the derived category of an abelian category discussed in Section 1.3. The theory of model categories (Section 1.4) allows us to obtain $\bar{h}\mathscr{S}$ from the category of spectra $\mathscr{S}$.

Associated to an S-algebra R, we have the category $\mathscr{M}_R$ of R-modules and the derived category $\mathscr{D}_R$ of R-modules. In Section 2.5 we work in $\mathscr{D}_R$ and follow results by Strickland given in [43]. Strickland constructs R-modules R/x for $x \in \pi_n R$ and considers when these constructions inherit an associative and commutative product structure from R. We consider a particular example, namely $R = ko$. As $\pi_*(ko)$ is not concentrated in even degrees, this example does not satisfy the hypothesis used by Strickland. The main result of the section is given in Proposition 2.15.

It seems entirely appropriate to ask whether results of [23] and [5] that exist for S-modules also hold for S-algebras and commutative S-algebras. Hu, Kriz and May [23] begin work on this theory by constructing a core of a commutative S-algebra which in turn leads to the definition of a nuclear commutative S-algebra. We present this construction and the definition of a nuclear commutative

S-algebra in Chapter 5.

Hu, Kriz and May prove in [23, Proposition 1.5] (stated in Theorem 3.13) that a nuclear complex is atomic. The proof uses an argument based on a skeletal filtration of a nuclear complex and involves a commutative diagram of ordinary homology long exact sequences. This argument suggests that in order to prove that a nuclear commutative S-algebra is atomic we must employ a suitable homology theory for commutative S-algebras.

Such a homology theory was founded by Basterra [6] and is called topological André–Quillen homology. We give an account of some basic results on topological André–Quillen homology (HAQ) for CW commutative S-algebras in Chapter 4. This discussion includes the development of arguments based on skeletal filtrations and results in an isomorphism analogous to the classical Hurewicz theorem.

In Theorem 5.5 we are able to show that every nuclear commutative S-algebra is an atomic commutative S-algebra. This result was conjectured by Hu, Kriz and May in [23, Conjecture 2.9]. The proof mirrors that of [23, Proposition 1.5], but utilizes the topological André–Quillen homology machinery for cell S-algebras set up in Section 4.2 in place of ordinary homology. In particular we apply the associated HAQ long exact sequence for cell commutative S-algebras. The proof of [23, Proposition 1.5] makes use of the Hurewicz theorem to reduce the problem to a chase of a diagram of homotopy groups and application of the nuclear condition for S-modules. The proof of Theorem 5.5 also requires the application of the nuclear condition for commutative S-algebras. However, in place of the Hurewicz theorem we use the isomorphism $\pi_{n+1}\Sigma K_n \longrightarrow \mathrm{HAQ}_{n+1}(X_{n+1}/X_n)$ referred to above and given in diagram 4.16.

Proposition 5.6 characterizes minimal atomic commutative S-algebras in terms of nuclear commutative S-algebras. The result follows by considering a core $f : X \longrightarrow Y$ of minimal atomic commutative S-algebra Y.

Baker and May in [5, Proposition 2.5] show that every core of a nuclear complex is an equivalence. The proof of Theorem 5.5 strongly suggests that the same result will hold for commutative S-algebras (Conjecture 5.7). We are able to present a detailed account of how the proof of this result might work and the completion of the proof hinges upon proving that an epimorphism between the homotopy groups of two cofibres is in fact an isomorphism.

If we assume that Conjecture 5.7 holds we are able to show that a nuclear commutative S-

algebra is a minimal atomic commutative S-algebra (Conjecture 5.8). This is the commutative S-algebra analogue of [5, Theorem 2.6] which was originally conjectured by Hu, Kriz and May in [23, Conjecture 1.12] and is stated in terms of S-modules in Proposition 3.14. This result would give us a strengthening of Theorem 5.5 with Conjecture 5.7 playing a key role in the proof.

Supposing Conjecture 5.8, we can make the interesting observation given in Conjecture 5.10, which again compares the characterization of a commutative S-algebra with that of its underlying S-module.

The notion of minimality was introduced by Baker and May in [5], where they proved in [5, Theorem 3.3] that every complex is equivalent to a minimal one. In Section 5.3 we give a suitable definition of a minimal commutative S-algebra. We prove the analogous result to [5, Theorem 3.3] for commutative S-algebras in Theorem 5.12. We also show in Theorem 5.14 that a commutative S-algebra R is nuclear if and only if it is minimal and has no $\bmod p$ detectable homotopy. This result explains the relevance of minimality to the theory of nuclear and minimal atomic S-algebras and is in fact the analogous result to that for complexes given in [5, Theorem 3.4], which is stated for S-modules in Theorem 3.19. The proof requires a commutative diagram involving a $\bmod p$ HAQ long exact sequence. The diagram is obtained in Chapter 4.

We prove in Proposition 5.9 that for any core $g : Q \longrightarrow R$ of S-algebras, we have cores of S-modules $f : X \longrightarrow R$ and $\xi : X \longrightarrow Q$ such that $f = g \circ \xi$. This result leads us to consider examples of non-cores for commutative S-algebras MU, MSU, MO and MSO (cobordism Thom spectra) in Chapter 6. Our results rely on the action of the Dyer–Lashof algebra on the homology of the commutative S-algebras under consideration and are based on formulae of Kochman [24].

In Chapter 7 we concern ourselves with the Thom spectrum MU, considered as a commutative S-algebra. We begin by constructing a commutative MU-algebra, denoted $MU//x_2$ via a pushout construction involving homotopy element $x_2 \in \pi_4(MU)$. This construction can be thought of as the R-algebra analogue to the construction of R-module R/x in Section 2.5. In Section 7.1.1 we calculate $\pi_*(MU//x_2)$ via a Künneth spectral sequence which gives us an introduction to the techniques that could be employed to further investigate $MU//x_2$. It seems reasonable to expect that we could use the theory of topological André–Quillen homology (HAQ) for CW commutative S-algebras (Chapter 4) to calculate $\mathrm{HAQ}_k(MU//x_2/S)$ via HAQ long exact sequences. We consider this approach in Section 7.1.2 and note that this may require the calculation of $H_*(ku)$. To obtain

$H_*(ku)$ we need to consider the Hurewicz homomorphism $h : \pi_*(MU) \longrightarrow H_*(MU)$ and its image. In Section 7.2 we again consider the action of the Dyer–Lashof algebra on $H_*(MU)$. We carry out preliminary calculations with the aim of establishing a full description of $H_*(MU)$ in terms of the allowable Dyer–Lashof operations on homotopy element x_2.

Chapter 1

Spectra and the Stable Homotopy Category

In this chapter we explore the fundamental ideas on which the following chapters are based. We begin with a brief explanation of the term stable homotopy theory as a branch of algebraic topology. We discuss the framework in which, until recently, most work in stable homotopy theory has taken place, namely the stable homotopy category. This category was first introduced by Boardman [10] and was developed further by Adams [2].

For reasons we shall see later, topologists generally work with rings and modules in the stable homotopy category and with products and actions defined only up to homotopy. The resulting objects are known as ring spectra and are discussed in Section 2.1.

1.1 Stable Homotopy

Stable homotopy theory is a branch of algebraic topology which is concerned with invariants that are stable under suspension. Stable homotopy theory began around 1937 with the Freudenthal Suspension Theorem. In the simplest of terms this theorem states that homotopy groups are invariant under suspension (under some dimension limitations). In order to further explore stable homotopy theory we require several basic definitions accompanied by some explanation.

For based spaces (X, x_0) and (Y, y_0), the analogue of the cartesian product is the smash product

$X \wedge Y$, defined to be the quotient space

$$X \wedge Y = X \times Y / (X \times \{y_0\} \cup \{x_0\} \times Y).$$

The suspension of a based topological space (X, x_0), denoted ΣX, is defined to be the smash product $X \wedge S^1$. If we view S^1 as $I/\delta I$, then we can consider the suspension of X, ΣX as the double cone over X with the interval over the basepoint x_0 collapsed. Sometimes this construction is called the reduced suspension to distinguish it from the unreduced construction, where the line $\{x_0\} \times I$, through the basepoint of X is not collapsed to a point. The suspension of the n-sphere S^n for $n \geq 0$ is homeomorphic to the $(n+1)$-sphere S^{n+1}.

The most basic invariants in algebraic topology are the homotopy groups. The nth homotopy group $\pi_n(X)$ of a topological space X is given by the homotopy classes of based maps from the n-sphere S^n to X, denoted $[S^n, X]$, for $n \geq 0$. We think of two maps being homotopic if one can be continuously deformed into the other.

For a based topological space X, we can define the *suspension homomorphism* Σ as follows; we take $f : S^q \longrightarrow X$ as a representation of an element of $\pi_q(X)$ and let

$$\Sigma f = f \wedge id : S^{q+1} \cong S^q \wedge S^1 \longrightarrow X \wedge S^1 = \Sigma X.$$

We give the following theorem known as the Freudenthal Suspension Theorem, the proof of which can be found in [35].

Theorem 1.1. *Let X be a $(n-1)$-connected, based topological space, where $n \geq 1$. Then Σ : $\pi_q(X) \longrightarrow \pi_{q+1}(\Sigma X)$ is a bijection if $q < 2n-1$ and a surjection if $q = 2n-1$.*

By taking X in the above theorem to be the n-sphere S^n, we can consider the homotopy groups of spheres $\pi_{n+r}(S^n)$ and the suspension homomorphism

$$\Sigma : \pi_{n+r}(S^n) \longrightarrow \pi_{n+r+1}(\Sigma S^{n+1}).$$

We have that this homomorphism is an isomorphism for $n > r+1$ and so $\pi_{n+r}(S^n)$ stabilizes as n becomes large.

The homotopy groups $\pi_{n+r}(S^n)$, $n > r+1$ are called the stable homotopy groups of spheres, written as π_r^s.

Homology and cohomology groups are also invariant under suspension and provide a tool for solving stable problems. We can even go as far as to say that generalized homology and cohomology form part of stable homotopy theory.

1.2 Spectra

Before defining what we mean by the term spectrum, we should, as the name suggests, introduce the notion of a prespectrum. Once we have explored the concept of a prespectrum, the ideas of homotopy and spectra, particularly CW spectra, become much more transparent.

Definition 1.2. A *prespectrum* is a sequence of based topological spaces $\{E_n : n \geq 0\}$ and based continuous maps $\Sigma E_n \longrightarrow E_{n+1}$, or equivalently $E_n \longrightarrow \Omega E_{n+1}$.

For a space X with a basepoint x_0, the *loop space* of X, ΩX, is the space of all continuous basepoint preserving maps $(S^1, *) \longrightarrow (X, x_0)$, with the compact-open topology. As a basepoint for ΩX, we take the function ω_0, constant at x_0.

Definition 1.3. The *suspension prespectrum* of any based space X is denoted $\{\Sigma^n X\}$, the required maps $\Sigma(\Sigma^n X) \longrightarrow \Sigma^{n+1} X$ are the evident identifications.

Example 1.4. The *sphere prespectrum* (or 0-sphere prespectrum) is the suspension prespectrum of S^0. That is, the sphere prespectrum is a sequence of spaces with the nth term S^n for $n \geq 0$ (we take a point for negative dimension), and so, we get the sphere prespectrum by suspending the $(n-1)$-sphere at each step n.

Example 1.5. We denote the Eilenberg–Mac Lane prespectrum by $\{K(\pi, n), n \geq 1\}$. The spaces $K(\pi, n)$ are Eilenberg–Mac Lane spaces associated to abelian groups π. Eilenberg–Mac Lane spaces have the homotopy types of CW complexes and are constructed such that

$$\pi_q(K(\pi, n)) = \begin{cases} \pi & q = n \\ 0 & q \neq n. \end{cases}$$

Prespectra are stable objects that have associated homotopy, homology and cohomology groups. We aim to arrive at a good category of stable objects, analagous to the category of based spaces that has all the constructions we would expect from based spaces. These include suspensions, cofibre sequences and smash products. The objects of such a good category are known as *spectra*. There is a way of constructing a spectrum from a prespectrum without changing its homotopy, homology or cohomology groups. We use a *spectrification* functor which is the left adjoint $L : \mathscr{P} \longrightarrow \mathscr{S}$ to the forgetful functor $l : \mathscr{S} \longrightarrow \mathscr{P}$ from the category of spectra to the category of prespectra. We give the definition of a spectrum below and it is clear that if we drop the requirement that the maps are homeomorphisms than we get the notion of a prespectrum. The idea of a spectrum was first introduced by Lima [31] in 1958 and utilized by Adams [2]. The particular definition given below was presented by Lewis, May and Steinberger [30] in 1986 and replaced earlier definitions of spectra by a notion of greater generality.

Definition 1.6. A spectrum E is a prespectrum such that the adjoints $\tilde{\sigma} : E_n \longrightarrow \Omega E_{n+1}$ of the structure maps $\sigma : \Sigma E_n \longrightarrow E_{n+1}$ are homeomorphisms. A map $f : T \longrightarrow T'$ of prespectra is a sequence of maps $f_n : T_n \longrightarrow T'_n$ such that $\sigma'_n \circ \Sigma f_n = f_{n+1} \circ \sigma_n$ for all n. A map $f : E \longrightarrow E'$ of spectra is a map between E and E' regarded as prespectra.

So, a map of spectra is a sequence of based continuous maps $f_n : T_n \longrightarrow T'_n$ which are strictly compatible with the structure maps, we have the following commuting diagram for each n.

$$\begin{tikzcd}
\Sigma T_n \arrow[r, "\sigma_n"] \arrow[d, "\Sigma f_n"'] & T_{n+1} \arrow[d, "f_{n+1}"] \\
\Sigma T'_n \arrow[r, "\sigma'_n"] & T'_{n+1}
\end{tikzcd}$$

The *suspension* spectrum of based space X is given by

$$\Sigma^\infty X = L\{\Sigma^n X\}.$$

We define the sphere or 0-sphere spectrum to be $S = \Sigma^\infty S^0$ and we define sphere spectra for integers n by $S^n = \Sigma^\infty S^n$.

The *smash product* $T \wedge X$ of a prespectrum T with a space X is defined space-wise, that is, $(T \wedge X)_n = T_n \wedge X$. For a spectrum E, $E \wedge X$ is given by applying the functor L to the prespectrum level construction.

A *homotopy* in the category of spectra is a map $E \wedge I_+ \longrightarrow E'$ and we let $[E, E']$ denote the set of homotopy classes of maps between spectra E and E'.

Homotopy groups of spectra are given by

$$\pi_n(E) = [S^n, E], n \in \mathbb{Z}.$$

A *cofibration* of spectra is a map $i : E \longrightarrow E'$ that satisfies the homotopy extension property (HEP). This means that if $h \circ i_0 : E \longrightarrow F$ is a restriction of a map $f : E' \longrightarrow F$ then it extends to a homotopy $\tilde{h} : E' \wedge I_+ \longrightarrow F$ of f. We have $\tilde{h}$ making the following diagram commute.

$$\begin{tikzcd}
E \arrow[rr, "i_0"] \arrow[dd, "i"'] & & E \wedge I_+ \arrow[dl, "h"] \arrow[dd, "i \wedge \mathrm{id}"] \\
& F & \\
E' \arrow[ur, "f"] \arrow[rr, "i_0"'] & & E' \wedge I_+ \arrow[ul, "\tilde{h}"]
\end{tikzcd}$$

A *fibration* of spectra is a map $p : E \longrightarrow E'$ that satisfies the covering homotopy property (CHP). This means that a homotopy $h : F \wedge I_+ \longrightarrow E'$ of a projection $p \circ f$ where $f : F \longrightarrow E$, is covered by a homotopy $\tilde{h} : F \wedge I_+ \longrightarrow E$ of f. We have $\tilde{h}$ making the following diagram commute.

$$\begin{tikzcd}
F \arrow[r, "f"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\
F \wedge I_+ \arrow[ur, "\tilde{h}"] \arrow[r, "h"'] & E'
\end{tikzcd}$$

The notion of a spectrum is very natural if we begin with something called a cohomology theory. Essentially, a cohomology theory can be represented by a spectrum. This is a simple statement of Brown's representability theorem [13] and to explain this in more detail we need to introduce a large class of spaces called CW complexes.

A CW complex is built up from standard building blocks called cells. Each n-cell is homeomorphic to the open n-dimensional disc or ball e_n. A CW complex is a space which is the union of an expanding sequence of closed subspaces

$$X^0 \subset X^1 \subset X^2 \subset \cdots .$$

X^0 is a discrete set of points and X^n is called the n-skeleton of X and is obtained by attaching n-cells to X^{n-1} via attaching maps $j : S^{n-1} \longrightarrow X^{n-1}$, so from the boundary of a single n-disc to the $(n-1)$-skeleton.

As alluded to earlier, if we take a cohomology theory or more specifically, a reduced cohomology theory K^* on CW complexes then we have a sequence of contravariant functors $K^n : \mathscr{CW} \longrightarrow \mathscr{A}$ from the category of CW complexes to the category of abelian groups. As a cohomology theory this sequence satisfies certain properties. If, along with these properties the cohomology theory K^* satisfies something called the Wedge Axiom then we can apply the representability theorem of

Brown. Brown's Representability Theorem essentially means that there exists a sequence of CW complexes

$$K = \{K_n\}_{n\in\mathbb{Z}}$$

along with isomorphisms

$$K^n(X) \cong [X, K_n].$$

The sequence of spaces $K = \{K_n\}_{n\in\mathbb{Z}}$ form a spectrum and we say that this spectrum K represents the cohomology theory K^*.

A CW prespectrum has all spaces CW complexes D_n and all structure maps $\Sigma D_n \longrightarrow D_{n+1}$ are cellular. The theory of CW spectra is developed by taking the domains of the attaching maps to be sphere spectra S^n.

The fundamental invariants of spectra are their homotopy groups and we say that a map of spectra is a *weak equivalence* if it induces an isomorphism on homotopy groups. As with based spaces, the Whitehead theorem holds that is, a weak equivalence between CW spectra is a homotopy equivalence. As a consequence, we have for every spectrum E, a weak equivalence $\gamma : \Gamma E \longrightarrow E$ for some CW spectrum ΓE.

1.3 Derived Categories of Abelian Categories

We base this account of localization and derived categories on the texts of König and Zimmermann [25] and Weibel [46] and aim to provide a comprehensive overview of the processes involved. We begin with an abelian category $\mathscr{A}$, and aim to construct the derived category $D(\mathscr{A})$. We note that König chooses to write maps on the right and therefore denotes "g follows f" by fg. Throughout the rest of this thesis we shall be thinking in terms of maps on the left and will therefore give a close account of the material in [25] using this convention.

This construction involves performing a localization procedure on a homotopy category $K(\mathscr{A})$. This involves inverting morphisms which are quasi-isomorphisms, and so, in building the derived category $D(\mathscr{A})$ we are forcing certain complexes to become isomorphic. Essentially, if we have two complexes X and Y that are related by a morphism f that induces an isomorphism in cohomology,

$$H^n(f) : H^n(X) \xrightarrow{\cong} H^n(Y),$$

then these complexes will become isomorphic.

We begin with an abelian category $\mathscr{A}$ and we have the following definitions.

Definition 1.7. A *complex* X in $\mathscr{A}$ consists of two double infinite sequences $\{X^n\}_{n\in\mathbb{Z}}$ and $\{d_X^n\}_{n\in\mathbb{Z}}$ where X^n is an object in $\mathscr{A}$ and d_X^n is a *differential* or morphism in $\mathrm{Hom}_{\mathscr{A}}(X^n, X^{n+1})$. We have the condition that $d_X^{n+1}d_X^n = 0$.

Definition 1.8. A *homomorphism* between two complexes X and Y is a sequence $\{f^n\}_{n\in\mathbb{Z}}$ of morphisms $f^n \in \mathrm{Hom}_{\mathscr{A}}(X^n, Y^n)$ such that the diagram shown below commutes.

$$\begin{tikzcd}
\dots \arrow[r] & X^{n-1} \arrow[r, "d_X^{n-1}"] \arrow[d, "f^{n-1}"'] & X^n \arrow[r, "d_X^n"] \arrow[d, "f^n"'] & X^{n+1} \arrow[r] \arrow[d, "f^{n+1}"'] & \dots \\
\dots \arrow[r] & Y^{n-1} \arrow[r, "d_Y^{n-1}"] & Y^n \arrow[r, "d_Y^n"] & Y^{n+1} \arrow[r] & \dots
\end{tikzcd}$$

We consider the category $C(\mathscr{A})$ of complexes as defined above, which is also an abelian category. We also note that if $\mathscr{A}$ has enough projectives and enough injectives, then to any object M in our abelian category, we can associate a projective resolution and an injective resolution, all of these 'disguised' versions of M will become isomorphic in the derived category $D(\mathscr{A})$.

Definition 1.9. A *homotopy* between two morphisms f and g from X to Y is a sequence $\{s^n\}_{n\in\mathbb{Z}}$ of morphisms $s^n : X^n \longrightarrow Y^{n-1}$, such that $f^n - g^n = d_Y^{n-1}s^n + s^{n+1}d_X^n$ for each n.

$$\begin{tikzcd}
 & X^n \arrow[r, "d_X^n"] \arrow[dl, "s^n"'] \arrow[d, "f^n-g^n"] & X^{n+1} \arrow[dl, "s^{n+1}"] \\
Y^{n-1} \arrow[r, "d_Y^{n-1}"'] & Y^n &
\end{tikzcd}$$

We say that f and g, in the above definition, are *homotopic* and we have an equivalence relation. If a morphism is homotopic to a zero morphism then it is described as being null homotopic. If the identity morphism of a complex X is homotopic to a zero morphism then X itself is homotopy equivalent to the zero complex.

Example 1.10. Consider the example shown below of a complex in the category of abelian groups along with the identity morphism. The only possible homotopy map for $s^2 : \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}$ is the zero map. In fact we choose s^1 to be multiplication by m and all other homotopy morphisms are forced to be zero.

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "\times 2"] \arrow[d, "\times m"] \arrow[dl, "0"'] & \mathbb{Z} \arrow[r] \arrow[d] \arrow[dl] & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d] \arrow[dl, "0"'] & 0 \arrow[dl, "0"'] \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\times 2"'] & \mathbb{Z} \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

It is clear that this does not give a homotopy between the identity and the zero map and we can therefore say that the complex in question is not homotopy equivalent to the zero complex.

Example 1.11. Now consider an alternative example of a complex in the category of abelian groups along with the identity morphism. The maps in the complex are the obvious inclusion and projection maps. We put $s^1 : \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ as the inclusion into the second factor and $s^2 : \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}$ as the projection onto the first factor.

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d] \arrow[dl, "0"'] & \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d] \arrow[dl] & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d] \arrow[dl] & 0 \arrow[dl, "0"'] \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

These homotopy morphisms give a homotopy between the identity and the zero map and we can therefore say that the complex is homotopy equivalent to the zero complex.

We are now in the position to take the first step towards forming the derived category $D(\mathscr{A})$. That is, we now pass from $C(\mathscr{A})$ to the homotopy category, as given in the following definition.

Definition 1.12. For the category of complexes $C(\mathscr{A})$ of an abelian category $\mathscr{A}$, consider two complexes X and Y in $C(\mathscr{A})$ and let $\mathrm{Ht}(X, Y)$ be the set of morphisms from X to Y that are homotopic to zero. With X and Y fixed, note that $\mathrm{Hom}_{C(\mathscr{A})}(X, Y)$ is a group under addition, the new set $\mathrm{Ht}(X, Y)$ is a subgroup and the cosets of this subgroup are precisely the homotopy classes of maps, or in other words the equivalence classes of maps under the relation of homotopy. König [25, Definition 2.2.3] emphasizes this further by pointing out that the collection of all sets $\mathrm{Ht}(X, Y)$ as X and Y vary form in effect an ideal in the entire category. This is because the composite of two maps, one of which is null homotopic, will also be null homotopic. We form the homotopy category $K(\mathscr{A})$ whose objects are those of $C(\mathscr{A})$ but whose morphism sets are given by

$$\mathrm{Hom}_{K(\mathscr{A})}(X, Y) = \mathrm{Hom}_{C(\mathscr{A})}(X, Y) / \mathrm{Ht}(X, Y).$$

There is an obvious functor from $C(\mathscr{A})$ to $K(\mathscr{A})$, along with an associated universal property. Details can be found in Weibel's book [46, Proposition 10.1.2].

As stated in Chapter 2 of [25], the category $K(\mathscr{A})$ is a triangulated additive category, but in general not abelian. We introduce the concept of a *triangle* in $K(\mathscr{A})$ below and refer the reader to [46], Chapter 10 for further details on triangulated categories. We will however include a sketch of the definition of a triangulated category.

Definition 1.13. An additive category $\mathscr{K}$ is called a *triangulated category* if it has an automorphism $T : \mathscr{K} \longrightarrow \mathscr{K}$, a family of distinguished triangles and is subject to a number of axioms (six in König's account [25] and four in Weibel's account [46]). The distinguished triangles always consist of four objects and three morphisms like this:

$$A \longrightarrow B \longrightarrow C \longrightarrow T(A).$$

Notice that the fourth object is always the translate by T of the first.

(TR0) The collection of distinguished triangles is closed under isomorphism of triangles. This axiom is König's TR0 [25, Theorem 2.3.1], and Weibel includes this as a part of an axiom TR1 [46, Definition 10.2.1], hence the ambiguity on how many axioms are required.

(TR1) For each object X in $\mathscr{K}$ there is a distinguished triangle

$$X \xrightarrow{\mathrm{id}_X} X \longrightarrow 0 \longrightarrow T(X)$$

involving the identity map on X. This tells us that every object in the category appears in at least one distinguished triangle. This axiom is König's TR1 [25, Theorem 2.3.1], and Weibel includes this as a part of his TR1 [46, Definition 10.2.1].

(TR2) Given any homomorphism $f : X \longrightarrow Y$ in $\mathscr{K}$, there is a distinguished triangle

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow T(X)$$

involving f. This tells us that every morphism in the category appears in at least one distinguished triangle. This axiom is König's TR2 [25, Theorem 2.3.1], and is part of Weibel's TR1 [46, Definition 10.2.1].

(TR3) If the triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$$

is distinguished, then the triangles

$$Y \xrightarrow{g} Z \xrightarrow{h} T(X) \xrightarrow{-T(f)} T(Y)$$

and

$$T^{-1}[Z] \xrightarrow{-T^{-1}(h)} X \xrightarrow{f} Y \xrightarrow{g} Z$$

are distinguished. This is known as the rotation axiom and the two triangles above are *rotates* of the first. This axiom is König's TR3 [25, Theorem 2.3.1], and Weibel's TR2 [46, Definition 10.2.1].

(TR4) If

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow T(X)$$

and

$$X' \xrightarrow{f'} Y' \longrightarrow Z' \longrightarrow T(X')$$

are distinguished triangles, with commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ X' & \xrightarrow{f'} & Y' \end{array}$$

then there exists a morphism $w : Z \longrightarrow Z'$ giving a morphism of triangles, as shown below.

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & T(X) \\ {\scriptstyle u}\downarrow & & {\scriptstyle v}\downarrow & & {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle T(u)} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & T(X') \end{array}$$

This axiom is König's TR4 [25, Theorem 2.3.1], and Weibel's TR3 [46, Definition 10.2.1].

(TR5) Verdier's octahedral axiom holds. This axiom is often regarded as the most complicated and confusing of the axioms (see [46, Exegesis 10.2.3] for example). However there are good accounts in [46] and Chapter 2 of [25].

The translate $T(X)$ of an object X can also be written as $X[1]$. In this alternative notation we can write $X[n]$ to mean that we have applied the translate n times. Also, since we have the notion of translating X in the opposite direction given by $T^{-1}(X)$, it makes sense for n to be negative.

We are interested in a particular example of a triangulated category, namely $K(\mathscr{A})$. We consider the kth translate of a complex X in $C(\mathscr{A})$, described in [25] as applying the *shift functor.* This functor is defined for an integer k, and sends a complex X to another complex $X[k]$ defined by $X[k]^n := X^{n+k}$ and $d^n_{X[k]} := (-1)^k d^{n+k}_X$. It is clear that applying the shift functor for a positive k moves a complex to the left. The definition of the mapping cone below explains the reason for the differential sign convention.

Definition 1.14. For complexes X and Y in $C(\mathscr{A})$, we consider the morphism $f : X \longrightarrow Y$. The *mapping cone* of f is the complex $M(f)$ whose degree n part is $X^{n+1} \oplus Y^n$ and the differential is given by the matrix $d^n_{M(f)} := \begin{pmatrix} d^n_{X[1]} & 0 \\ f^{n+1} & d^n_Y \end{pmatrix}$.

We now define what is meant by a *distinguished triangle* in the category $K(\mathscr{A})$.

Definition 1.15. A *distinguished triangle* in the category $K(\mathscr{A})$ is a triangle which is isomorphic to a *strict triangle* which is of the form

$$X \xrightarrow{f} Y \longrightarrow M(f) \longrightarrow X[1].$$

By this, we mean that there is a commutative diagram of chain complexes in $K(\mathscr{A})$ as follows

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[r, "g"] \arrow[d, "v"'] & Z \arrow[r, "h"] \arrow[d, "w"'] & X[1] \arrow[d, "T(u)"'] \\
X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & M(f) \arrow[r, "h'"] & X'[1]
\end{tikzcd}$$

such that the maps u, v and w are isomorphisms in $K(\mathscr{A})$.

A distinguished triangle is sometimes referred to as an *exact triangle* (see for example [46, Definition 10.1.3]).

Moving from the category $C(\mathscr{A})$ to the homotopy category $K(\mathscr{A})$ forces complexes that are homotopy equivalent to zero to be isomorphic to zero. Before constructing the derived category $D(\mathscr{A})$, in which two complexes related by a morphism which induces an isomorphism in cohomology, will become isomorphic, we must define the *nth cohomology* of X in $C(\mathscr{A})$. For an integer n, we write $Z^n(X)$ for the kernel of the differential d_X^n and $B^n(X)$ for the image of d_X^{n-1}. The nth cohomology of X is written as $H^n(X)$ and is the quotient $Z^n(X)/B^n(X)$. Complexes with all cohomology zero are called *exact*. $H^n(-) : C(\mathscr{A}) \longrightarrow \mathscr{A}$ is an additive functor. We have the following result.

Lemma 1.16. *If $f : C \longrightarrow D$ is null homotopic then every map $f_n : H^n(C) \longrightarrow H^n(D)$ is zero. A complex which is homotopy equivalent to zero is exact. But notice, as we have seen, that exact complexes need not be homotopy equivalent to zero.*

As already mentioned, the derived category $D(\mathscr{A})$ is constructed via a localization procedure. $D(\mathscr{A})$ is defined to be the localization $Q^{-1}K(\mathscr{A})$ of the homotopy category $K(\mathscr{A})$ at a collection of quasi-isomorphisms Q. We explain what is meant by the term *localization* by including the definition [46, Definition 10.3.1] below.

Definition 1.17. Let S be a collection of morphisms in a category C. A *localization* of C with respect to S is a category $S^{-1}C$, together with a functor $q : C \longrightarrow S^{-1}C$ such that

i. $q(s)$ is an isomorphism in $S^{-1}C$ for every $s \in S$;

ii. Any functor $F : C \longrightarrow D$ such that $F(s)$ is an isomorphism for all $s \in S$ factors in a unique way through q.

The existence of localizations becomes a set theoretic problem when the class S is not a set (see [46, Set-Theoretic Remark 10.3.3]). The quasi-isomorphisms we are going to invert in our localization procedure are defined as follows.

Definition 1.18. Consider a morphism $f : X \longrightarrow Y$ in $K(\mathscr{A})$. If $H^n(f) : H^n(X) \longrightarrow H^n(Y)$ is an isomorphism in $\mathscr{A}$ for any integer n, then we call f a *quasi-isomorphism* and X and Y *quasi-isomorphic.*

König [25] mentions an equivalent condition for f to be a quasi-isomorphism, that is, the mapping cone $M(f)$ has zero cohomology. We can see this by considering the long exact cohomology sequence

$$\ldots \longrightarrow H^{n-1}(M(f)) \longrightarrow H^n(X) \xrightarrow{\cong} H^n(Y) \longrightarrow H^n(M(f)) \longrightarrow H^{n+1}(X) \longrightarrow \ldots$$

We can see that if f is a quasi-isomorphism then the induced map $H^n(f) : H^n(X) \longrightarrow H^n(Y)$ is an isomorphism and so by exactness $H^n(M(f)) = 0$ and vice-versa.

We localize a category $\mathscr{C}$ with respect to a certain class of morphisms. This class is called a multiplicative system and it satisfies certain conditions.

(S1) For each object X in $\mathscr{C}$, the identity morphism on X is in S.

(S2) If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are in S, then gf is in S.

(S3) If there is a diagram

$$\begin{tikzcd} & Z \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with g in S, then there exists some object W and $h : W \longrightarrow X$ in S and $k : W \longrightarrow Z$ such that the following diagram is commutative.

$$\begin{tikzcd} W \arrow[r, "k"] \arrow[d, "h"'] & Z \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

We can write this condition as $g^{-1}f = kh^{-1}$. This is sometimes called the Ore condition and allows us to define composition. It is named after the Norwegian mathematician Øystein Ore (1899–1968).

(S4) If $f, g : X \longrightarrow Y$ are parallel morphisms in $\mathscr{C}$ then the following two conditions are equivalent:

(a) $tf = tg$ for some $t \in S$, $t : Y \longrightarrow Y'$.

(b) $fs = gs$ for some $s \in S$, $s : X' \longrightarrow X$.

This condition is used when proving that compositions are well defined.

If we have a multiplicative system S, then we can form the category $\mathscr{C}_S$ called the localization of $\mathscr{C}$ at S. This category has the same objects as the original category $\mathscr{C}$. For any two objects X and Y, the morphisms from X to Y in $\mathscr{C}_S$ can be thought of as a triple (X', s, f) where X' is any object, $s : X' \longrightarrow X$ is in S and $f : X' \longrightarrow Y$ is any morphism. We can think of the morphisms from X to Y as fractions.

$$fs^{-1} : X \xleftarrow{s} X' \xrightarrow{f} Y$$

We define an equivalence relation by calling $X \xleftarrow{s} X' \xrightarrow{f} Y$ equivalent to $X \xleftarrow{t} X'' \xrightarrow{f} Y$ if and only if there is a fraction $X \longleftarrow X''' \longrightarrow Y$ fitting into a commutative diagram in $\mathscr{C}$.

$$\begin{tikzcd}
& X' \arrow[dl, "s"'] \arrow[dr, "f"] & \\
X & X''' \arrow[l] \arrow[r] \arrow[u] \arrow[d] & Y \\
& X'' \arrow[ul, "t"] \arrow[ur, "g"'] &
\end{tikzcd}$$

It can be shown that this is indeed an equivalence relation.

It is also important to consider the composition of our new morphisms. If we want to compose $X \xleftarrow{r} X' \xrightarrow{g} Y$ with $Y \xleftarrow{t} Y' \xrightarrow{h} Z$, we use the Ore condition to find a commutative diagram

$$\begin{tikzcd}
& W \arrow[r, "f"] \arrow[d, "s"] & Y' \arrow[r, "h"] \arrow[d, "t"] & Z \\
X & X' \arrow[l, "r"] \arrow[r, "g"'] & Y &
\end{tikzcd}$$

And so the composition given as

$$ht^{-1}gr^{-1}$$

and rewritten by the Ore condition, and expressed as $hf(rs)^{-1}$. Hence the composite is the class of the fraction $X \longleftarrow Y \longrightarrow Z$. We can prove that this composition is well defined.

It is clear that we have just defined localization with respect to a class of morphisms, it is also possible to localize with respect to a class of objects. This class of objects is known as a *null system* η and is defined by König [25, Definition 2.5.3]. The definition utilizes the triangulated category structure of $\mathscr{C}$ by working with distinguished triangles. To this null system or class of objects in a category $\mathscr{C}$, König associates a class of morphisms, denoted by $S(\eta)$. König defines $S(\eta)$ as follows.

Definition 1.19. For null system η, we define

$$S(\eta) = \{f : X \longrightarrow Y\}$$

such that X and Y are objects in $\mathscr{C}$ and there is a distinguished triangle

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow X[1]$$

with Z in η.

$S(\eta)$ is a multiplicative system. We form the category $\mathscr{C}_{S(\eta)}$ by localising with respect to $S(\eta)$ and write $\mathscr{C}_{S(\eta)}$ as $\mathscr{C}/\eta$, since the canonical functor $Q : \mathscr{C} \longrightarrow \mathscr{C}/\eta$ sends any object $Z \in \eta$ to zero as follows.

By TR1 for triangulated categories we have a distinguished triangle

$$Z \xrightarrow{\mathrm{id}_Z} Z \longrightarrow 0 \longrightarrow Z[1]$$

with $\mathrm{id}_Z \in S(\eta)$. Hence (by TR3 or rotation) we have distinguished triangle

$$Z \longrightarrow 0 \longrightarrow Z[1] \longrightarrow Z[1]$$

where $Z[1] \in \eta$ and hence $Z \longrightarrow 0 \in S(\eta)$ and so is inverted in $\mathscr{C}/\eta$ and becomes an isomorphism, and therefore Z is sent to 0.

To summarize, we have defined a class of objects in a triangulated category $\mathscr{C}$ known as the null system η and we have shown that we can associate to this class of objects a class of morphisms $S(\eta)$ giving us a multiplicative system which we can use to localize.

We now apply this same construction to the triangulated category $K(\mathscr{C})$. The null system is defined by

$$\eta := \{X \in \mathscr{C} : H^n(X) = 0, \forall n \in \mathbb{Z}\}.$$

Then $S(\eta)$ consists of those morphisms $f : X \longrightarrow Y$ such that there is a distinguished triangle

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow X[1]$$

with $Z \in \eta$, that is, $H^n(Z) = 0$. We pass to cohomology giving the following long exact sequence

$$0 = H^{n-1}(Z) \longrightarrow H^n(X) \longrightarrow H^n(Y) \longrightarrow H^n(Z) = 0$$

in which the cohomology of Z vanishes since Z is in the null system. Thus we have the isomorphisms

$$H^n(f) : H^n(X) \xrightarrow{\cong} H^n(Y)$$

and f is therefore a quasi-isomorphism. The derived category $D(\mathscr{C})$ is the localization of the homotopy category with respect to $S(\eta)$ as defined.

The following proposition is given by König [25, Proposition 2.5.2].

Proposition 1.20. *The functor which is a composition of the embedding and the localization functor*

$$\begin{tikzcd} C \arrow[rr] \arrow[dr] & & D(\mathscr{C}) \\ & K(\mathscr{C}) \arrow[ur] & \end{tikzcd}$$

defines an equivalence between the category $\mathscr{C}$ and the full subcategory of $D(\mathscr{C})$ having objects with cohomology vanishing in non-zero degrees.

The embedding sends any object M in our abelian category $\mathscr{C}$ to an associated complex with $X^0 = M$ and all other X^n as well as all differentials being zero. This gives a full and faithful embedding of $\mathscr{C}$ into $K(\mathscr{C})$. Complexes of this form have cohomology vanishing in non-zero degrees.

The motivation for introducing localization is that if we have a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow X[1]$ in our homotopy category $K(\mathscr{C})$ then Z is quasi isomorphic to the mapping cone $M(f)$. So by localizing, the quasi isomorphism $M(f) \longrightarrow Z$ becomes an isomorphism. This allows us to form the distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} M(f) \longrightarrow X[1]$$

which is sometimes called the strict triangle on f. The important point is that there may be no similar distinguished triangle in our homotopy category.

1.4 Model Categories

We include the definition of a model category, based on the account given in [17]. The notion of a model category allows us to obtain the *stable homotopy category* of the following section. We begin with the following important preliminary notions.

Definition 1.21. A map f in $\mathscr{C}$ is a *retract* of a map $g \in \mathscr{C}$ if and only if there is a commutative diagram of the following form, where the horizontal composites are identities.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & C \arrow[r] \arrow[d, "g"'] & A \arrow[d, "f"'] \\ B \arrow[r] & D \arrow[r] & B \end{tikzcd}$$

Definition 1.22. A *functorial factorization* is an ordered pair (α, β) of functors $Map\mathscr{C} \longrightarrow Map\mathscr{C}$ such that $f = \beta(f) \circ \alpha(f)$ for all $f \in Map\mathscr{C}$.

Definition 1.23. Suppose $i : A \longrightarrow B$ and $p : X \longrightarrow Y$ are maps in a category $\mathscr{C}$. Then i has the *left lifting property* with respect to p and p has the *right lifting property* with respect to i if, for every commutative diagram of the following form, there exists a lift $h : B \longrightarrow X$, such that $hi = f$ and $ph = g$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

Definition 1.24. ([17, Definition 3.3]) A model category is a category $\mathscr{C}$ with three distinguished classes of maps; weak equivalences, fibrations and cofibrations. Each class of maps is closed under composition and contains all identity maps. A fibration or cofibration is said to be acyclic if it is also a weak equivalence. We have the following axioms.

i. Finite limits and colimits exist in $\mathscr{C}$.

ii. If f and g are maps in $\mathscr{C}$ such that gf is defined and two out of three maps f, g and gf are weak equivalences, then so is the third.

iii. If f and g are maps in $\mathscr{C}$ such that f is a retract of g and g is a fibration, cofibration or weak equivalence, then so is f.

iv. The acyclic cofibrations have the left lifting property with respect to the fibrations and the cofibrations have the left lifting property with respect to the acyclic fibrations.

v. Any map f in $\mathscr{C}$ can be factored in two ways: $f = pi$ where i is a cofibration and p is an acyclic fibration or, $f = pi$ where i is an acyclic cofibration and p is a fibration.

We follow the definition above with a preliminary observation about model categories [17, Proposition 3.13].

Proposition 1.25. *Let $\mathscr{C}$ be a model category.*

i. The cofibrations in $\mathscr{C}$ are the maps which have the left lifting property with respect to acyclic fibrations.

ii. The acyclic cofibrations in $\mathscr{C}$ are the maps which have the left lifting property with respect to fibrations.

iii. The fibrations in $\mathscr{C}$ are the maps which have the have the right lifting property with respect to acyclic cofibrations.

iv. The acyclic fibrations in $\mathscr{C}$ are the maps which have the have the right lifting property with respect to cofibrations.

Proposition 1.25 implies that the axioms for a model category (Definition 1.24) are over prescribed (as remarked upon in [17], Remark on Proposition 3.13). In describing a model structure on a category, it is sufficient to choose the fibrations and weak equivalences or cofibrations and weak equivalences.

1.5 The Derived Homotopy Category

In this section we aim to describe the process by which we obtain the stable homotopy category from the category of spectra $\mathscr{S}$ described in Section 1.2.

There is a strong motivation for constructing this stable homotopy category to work in. It was in the 1960's that algebraic topologists realized that a good 'stable homotopy category' was needed in order to carry out calculations. The objects of said category should be the stabilized analogue of spaces, each of which represents a cohomology theory. The category in question was first constructed by Boardman [10] in 1964 and later reviewed by Adams [2].

A primary need in constructing a stable homotopy category was for multiplicative structures given by rings and modules. Hence, a smash product $E \wedge E$, was defined which is associative, commutative and unital (with unit the sphere spectrum S). A category with such a product is described as *symmetric monoidal* and this product structure does indeed allow us to use algebraic notions such as ring and module in stable homotopy theory. Note that we have a smash product $\wedge$ which is associative commutative and unital in $\bar{h}\mathscr{S}$, but we do not have these relations on the point-set level.

The stable homotopy category can be thought of as the topological derived category, that is, the analogue of the derived category $D(\mathscr{A})$ of an abelian category described in 1.3.

In what follows, we use the theory of model categories (see Section 1.4 and homotopy categories to obtain $\bar{h}\mathscr{S}$.

The terms cofibration and fibration will continue to be used to describe maps which satisfy the homotopy extension property and covering homotopy property respectively. For this reason we adopt the same nomenclature as [19] and use q-cofibrations and q-fibrations for the model category terms.

It is important that the category of spectra $\mathscr{S}$ is a model category and so we state the following theorem, which is contained within [19, VII Theorem 4.4].

Theorem 1.26. *The category $\mathscr{S}$ of spectra is a model category with respect to the weak equivalences and Serre fibrations.*

A Serre fibration of spectra is a map that satisfies the CHP with respect to the set of "cone spectra" (see [19, VII.]).

Using the basic theory of model categories set up in Section 1.4 and Definition 1.17 we are able to use [22, I.2] and [17, Theorem 6.2], to define the homotopy category $h\mathscr{C}$ associated to a model category $\mathscr{C}$. As we shall see, $h\mathscr{C}$ is in fact a localization of $\mathscr{C}$ obtained by inverting the subcategory of weak equivalences. This interpretation of a homotopy category depends only on the weak equivalences and suggests that these carry the fundamental homotopy theoretic information.

Theorem 1.27. *Let $\mathscr{C}$ be a model category and let W be the class of weak equivalences in $\mathscr{C}$. Then the functor $\gamma : \mathscr{C} \longrightarrow h\mathscr{C}$ is a localization of $\mathscr{C}$ with respect to W.*

Hence, if $\mathscr{C}$ is a model category and W is the class of weak equivalences in $\mathscr{C}$, then $W^{-1}\mathscr{C}$ exists and is isomorphic to $h\mathscr{C}$.

It is now clear that the stable homotopy category or topological derived category $\bar{h}\mathscr{S}$ is isomorphic to the localization $W^{-1}\mathscr{S}$, where W is the class of weak equivalences. We should note that in algebraic topology $h\mathscr{S}$ denotes the *homotopy category* where homotopic maps are identified and $\bar{h}\mathscr{S}$ denoted the *stable homotopy category* where we adjoin formal inverses to the weak equivalences via a localization procedure as discussed above.

We are now able to utilize the notion of CW spectra. We use a weak equivalence $\gamma : \Gamma E \longrightarrow E$ for spectrum E and CW spectrum ΓE.

We also have Whitehead's theorem which states that every weak homotopy equivalence of CW spectra is a homotopy equivalence, that is, an isomorphism in $h\mathscr{S}$. These results allow us to show the following.

Corollary 1.28. *The stable homotopy category* $\bar{h}\mathscr{S}$ *is equivalent to the homotopy category of CW spectra*

$$\bar{h}\mathscr{S} \cong h\mathscr{S}_{CW}.$$

We use a similar argument in 2.4 to show that the derived category of R-modules $\mathscr{D}_R$ is equivalent to $h\mathscr{C}_R$.

Chapter 2

Structured ring and module spectra

2.1 Ring spectra and their modules

In the previous chapter we formed homotopy category $h\mathscr{S}$, in which the homotopic maps are identified. The desired stable homotopy category $\bar{h}\mathscr{S}$ is obtained from $h\mathscr{S}$ by adjoining formal inverses to the weak equivalences. Every spectrum is weakly equivalent to a CW spectrum and $\bar{h}\mathscr{S}$ is equivalent to the homotopy category of CW spectra.

We have that the smash product on the stable homotopy category is associative, commutative and unital, with unit the sphere spectrum and therefore $\bar{h}\mathscr{S}$ is symmetric monoidal. The structure allows us to carry over algebraic notions such as *ring* and *module* to stable homotopy theory. Hence we are able to make the definition of a ring spectrum E and E-module M in terms of unit and product maps.

Definition 2.1. A *ring spectrum* R, is a spectrum R, together with a product $\phi\colon R \wedge R \longrightarrow R$, and a unit $\eta\colon S \longrightarrow R$, such that the unit and associativity diagrams commute in $\bar{h}\mathscr{S}$.

$$\begin{tikzcd}
S \wedge R \arrow[r, "\eta\wedge 1"] \arrow[dr, "\simeq"'] & R \wedge R \arrow[d, "\phi"] & R \wedge S \arrow[l, "1\wedge\eta"'] \arrow[dl, "\simeq"] \\
 & R &
\end{tikzcd}$$

$$\begin{tikzcd}
R \wedge R \wedge R \arrow[r, "1\wedge\phi"] \arrow[d, "\phi\wedge 1"'] & R \wedge R \arrow[d, "\phi"] \\
R \wedge R \arrow[r, "\phi"] & R
\end{tikzcd}$$

Further, R is said to be a *commutative* ring spectrum if the diagram

$$\begin{tikzcd} R \wedge R \arrow[rr, "\tau"] \arrow[dr, "\phi"'] & & R \wedge R \arrow[dl, "\phi"] \\ & R & \end{tikzcd}$$

commutes in $\bar{h}\mathscr{S}$.

Example 2.2. The Eilenberg–Mac Lane spectrum HR for R a ring is a *ring spectrum.* The Eilenberg–Mac Lane spectrum is obtained from the Eilenberg–Mac Lane prespectrum (Example 1.5) in the standard way.

We also have the following notion of a *module spectrum* over a ring spectrum R or an R-module.

Definition 2.3. For a ring spectrum R, an *R-module* is a spectrum M together with a map $\mu : R \wedge M \longrightarrow M$ such that the diagrams below commute in $\bar{h}\mathscr{S}$.

$$\begin{tikzcd} S \wedge M \arrow[rr, "\eta\wedge 1"] \arrow[dr] & & R \wedge M \arrow[dl, "\mu"] \\ & M & \end{tikzcd}$$

$$\begin{tikzcd} R \wedge R \wedge M \arrow[r, "\phi\wedge 1"] \arrow[d, "1\wedge\mu"'] & R \wedge M \arrow[d, "\mu"] \\ R \wedge M \arrow[r, "\mu"] & M \end{tikzcd}$$

Example 2.4. For R a ring and M a left R module, the Eilenberg–Mac Lane spectrum HM is an HR-module.

2.2 The category of S-modules

We begin by describing the basic object on which all subsequent work is based: the S-module. The following material is based on [19, II.1]. In [19] the construction of a symmetric monoidal category of S-modules $\mathscr{M}_S$ is preceded by a category which possesses nearly all of the desired properties. This is the category of $\mathbb{L}$-spectra which has an associative and commutative product denoted by $\wedge_{\mathscr{L}}$, which is not however unital. In fact, the natural map $\lambda : S \wedge_{\mathscr{L}} M \longrightarrow M$ is often an isomorphism, but always a weak equivalence. $\mathbb{L}$ is in fact a monad in the category $\mathscr{S}$ and is specified in [19, Notations 4.1]. An $\mathbb{L}$-spectrum is a spectrum that is an $\mathbb{L}$-algebra and we denote the category of $\mathbb{L}$-spectra by $\mathscr{S}[\mathbb{L}]$. We define an S-module as follows.

Definition 2.5. An S-module is an $\mathbb{L}$-spectrum M, such that $\lambda : S \wedge_{\mathscr{L}} M \longrightarrow M$ is an isomorphism.

For our purposes it is adequate to think of an S-module as a spectrum with additional structure. A map of S-modules $f : M \longrightarrow N$ is a map as $\mathbb{L}$-spectra.

We denote the category of S-modules by $\mathscr{M}_S$ and restricting $\wedge_{\mathscr{L}}$ to S-modules we rename it $\wedge_S$ and write $M \wedge_S N$ for the smash product of S-modules M and N in this category. We have that $S \wedge_S M \cong M$ for every object in $\mathscr{M}_S$ from Definition 2.5. We accept the existence of symmetric monoidal $\mathscr{M}_S$ and proceed by examining its properties.

The reason for the name S-module is illustrated by the following commutative diagrams.

$$\begin{tikzcd}
S \wedge_S S \wedge_S M \arrow[r, "\lambda \wedge_S 1"] \arrow[d, "1 \wedge_S \lambda"'] & S \wedge_S M \arrow[d, "\lambda"] \\
S \wedge_S M \arrow[r, "\lambda"] & M
\end{tikzcd}$$

$$\begin{tikzcd}
M \arrow[r, "\lambda^{-1}"] \arrow[dr, "1"'] & S \wedge_S M \arrow[d, "\lambda"] \\
& M
\end{tikzcd}$$

As well as the smash product $\wedge_S$ we have function S-module functor F_S. The following theorem is taken from [19, II.1.6] and displays an adjunction between $\wedge_S$ and F_S.

Theorem 2.6. *The category $\mathscr{M}_S$ is symmetric monoidal under $\wedge_S$ and for S-modules M, N and P there is an adjunction*

$$\mathscr{M}_S(M \wedge_S N, P) \cong \mathscr{M}_S(M, F_S(N, P)).$$

A *homotopy* in the category of S-modules is, as in the category of spectra (Section 1.2), a map $M \wedge I_+ \longrightarrow N$. We say that a map of S-modules is a *weak equivalence* if it is a weak equivalence of spectra.

Recall the process by which we formed the stable homotopy category $\bar{h}\mathscr{S}$ from the category of spectra $\mathscr{S}$, detailed in Section 1.5. In a similar manner, we arrive at the derived category $\mathscr{D}_S$ of the category of S-modules. We first construct the homotopy category $h\mathscr{M}_S$ and then localize with respect to the weak equivalences. This localization essentially adjoins inverses to the weak equivalences. This process is made rigorous by CW approximation, as for spectra. We use sphere S-modules S_S^n as the domains of the attaching maps in the definitions of cell and CW S-modules. We define sphere S-modules as in [19, Chapter II, (1.7)] by

$$S_S^n \equiv S \wedge_{\mathscr{L}} \mathbb{L}S^n \tag{2.1}$$

where $\mathbb{L}$ is the free functor taking CW spectra to CW $\mathbb{L}$-spectra. We also have the following, from [19, II.1.8]

$$\pi_n(M) \equiv h\mathscr{S}(S^n, M) \cong h\mathscr{M}_S(S_S^n, M) \tag{2.2}$$

where for S-module M, $h\mathscr{S}(S^n, M) \cong h\mathscr{M}_S(S_S^n, M)$ follows from results on the category of $\mathbb{L}$-spectra.

We can develop a theory of cell and CW S-modules as in the theory of cell and CW spectra. This theory is a specialization of that presented in Section 2.4. We can summarize by the following statements.

Theorem 2.7. *i. A weak equivalence of cell S-modules is a homotopy equivalence.*

ii. Any S-module is weakly equivalent to a CW S-module.

iii. $\mathscr{D}_S$ is equivalent to the homotopy category of CW S-modules.

Elmendorff, Kriz, Mandell and May [19] establish an equivalence between $\mathscr{D}_S$ and $\bar{h}\mathscr{S}$, via the equivalence between stable homotopy categories $\bar{h}\mathscr{S}$ and $\bar{h}\mathscr{S}[\mathbb{L}]$ as given in [19, I.4.6].

We can conclude that homotopy theory can be done in $\bar{h}\mathscr{S}$ or $\mathscr{D}_S$, since these categories are equivalent. When working on the point-set level we have constructed a category of S-modules $\mathscr{M}_S$ with an associative, commutative and unital smash product $\wedge_S$.

2.3 S-algebras and their modules

We work in the symmetric monoidal category of S-modules $\mathscr{M}_S$ and begin by defining the concepts of an S-algebra and a commutative S-algebra.

Definition 2.8. An *S-algebra* is an S-module R with unit $\eta : S \longrightarrow R$ and product $\phi : R \wedge_S R \longrightarrow R$

such that the following unity and associativity diagrams commute.

$$\begin{tikzcd} S \wedge_S R \arrow[r, "\eta\wedge\mathrm{id}"] \arrow[dr, "\cong"'] & R \wedge_S R \arrow[d, "\phi"] & R \wedge_S S \arrow[l, "\mathrm{id}\wedge\eta"'] \arrow[dl, "\cong"] \\ & R & \end{tikzcd}$$

$$\begin{tikzcd} R \wedge_S R \wedge_S R \arrow[r, "\mathrm{id}\wedge\phi"] \arrow[d, "\phi\wedge\mathrm{id}"'] & R \wedge_S R \arrow[d, "\phi"] \\ R \wedge_S R \arrow[r, "\phi"'] & R \end{tikzcd}$$

R is a *commutative S-algebra* if the following commutativity diagram also commutes.

$$\begin{tikzcd} R \wedge_S R \arrow[rr, "\tau"] \arrow[dr, "\phi"'] & & R \wedge_S R \arrow[dl, "\phi"] \\ & R & \end{tikzcd}$$

We progress by considering left modules over a (commutative) S-algebra R.

Definition 2.9. For an S-algebra or commutative S-algebra R, a *left R-module* is an S-module M with a map $\mu : R \wedge_S M \longrightarrow M$ such that the following diagrams commute.

$$\begin{tikzcd} S \wedge_S M \arrow[r, "\eta_R\wedge\mathrm{id}"] \arrow[dr, "\cong"'] & R \wedge_S M \arrow[d, "\mu"] \\ & M \end{tikzcd}$$

$$\begin{tikzcd} R \wedge_S R \wedge_S M \arrow[r, "\mathrm{id}\wedge\mu"] \arrow[d, "\phi\wedge\mathrm{id}"'] & R \wedge_S M \arrow[d, "\mu"] \\ R \wedge_S M \arrow[r, "\mu"'] & M \end{tikzcd}$$

Modules will mean left R-modules and we let $\mathscr{M}_R$ denote the category of left R-modules. It is worthwhile noting that the definitions of S-algebras and commutative S-algebras given above are *brave new* versions of A_∞ and E_∞ ring spectra. In [19] they are described as 'unital sharpenings' of A_∞ and E_∞ ring spectra, first introduced in [34]. We quote the following lemma given in [19, II Lemma 3.4].

Lemma 2.10. *An S-algebra or commutative S-algebra is an A_∞ or E_∞ ring spectrum which is also an S-module.*

We observe that S is a commutative S-algebra with unit id and product λ.

2.4 R-modules

In this section we explore the theory associated to modules over an S-algebra R. The theory of cell spectra can be generalized to give the theory of cell R-modules. We discuss the construction of the derived category $\mathscr{D}_R$ of R-modules which is equivalent to the homotopy category of cell R-modules. Again, the following material is based upon [19], with particular reference to Chapter III.

We work in the category of S-modules $\mathscr{M}_S$ and fix an S-algebra R. We begin by considering cell and CW theories for R-modules. We define sphere R-modules S^n_R via a free R-module functor on spectra and define a cell R-module as below (see [19, Definitions 2.1]). The following definition mirrors that for cell spectra and we quote it for completeness. We let CX denote the cone functor $CX = X \wedge I$.

Definition 2.11. A cell R-module M is the union of an expanding sequence of sub R-modules M_n, such that $M_0 = *$ and M_{n+1} is the cofiber of a map $\phi_n : F_n \longrightarrow M_n$ where F_n is a (possibly empty) wedge of sphere R-modules S^q_R (of varying dimensions). The restriction of ϕ_n to a wedge summand S^q_R is called an attaching map. The induced map $CS^q_R \longrightarrow M_{n+1} \subset M$ is called a cell.

We also have the following definition of CW R-modules [19, Definition 2.5].

Definition 2.12. A cell R-module M is said to be a CW R-module if each cell is attached only to cells of lower dimension. The n-skeleton M^n of a CW R-module is the union of its cells of dimension at most n. A map $f : M \longrightarrow N$ between CW R-modules is cellular if $f(M^n) \subset N^n$ for all n.

The following two results allow us to form the derived category of R-modules $\mathscr{D}_R$. The first is the R-module version of the Whitehead Theorem and is a formal consequence of the homotopy extension and lifting property for R-modules [19, III 2.7].

Theorem 2.13. *If M is a cell R-module and $e : N \longrightarrow P$ is a weak equivalence of R-modules, then $e_* : h\mathscr{M}_R(M, N) \longrightarrow h\mathscr{M}_R(M, P)$ is an isomorphism. Therefore a weak equivalence between cell R-modules is a homotopy equivalence.*

We also have the following theorem giving us an approximation by cell modules.

Theorem 2.14. *For any R-module M, there is a cell R-module ΓM and a weak equivalence $\gamma : \Gamma M \longrightarrow M$. If R is connective or (-1)-connected, ΓM can be chosen to be a CW R-module.*

For each R-module M, choose a cell R-module ΓM and a weak equivalence $\gamma : \Gamma M \longrightarrow M$. Considering an R-module N and weak equivalence $\gamma : \Gamma N \longrightarrow N$, we use the Whitehead theorem given above to get the following isomorphism.

$$e_* : h\mathscr{M}_R(\Gamma M, \Gamma N) \xrightarrow{\cong} h\mathscr{M}_R(\Gamma M, N) \tag{2.3}$$

For a map $f : M \longrightarrow N$, there is a map $\Gamma f : \Gamma M \longrightarrow \Gamma N$, unique up to homotopy, such that the following diagram is homotopy commutative

$$\begin{tikzcd} \Gamma M \arrow[r, "\Gamma f"] \arrow[d, "\gamma"'] & \Gamma N \arrow[d, "\gamma"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

and so Γ is a functor $h\mathscr{M}_R \longrightarrow h\mathscr{M}_R$ and γ a natural transformation of functors on the homotopy category.

The derived category $\mathscr{D}_R$ of R-modules is the category whose objects are the R-modules and whose morphisms are given by

$$\mathscr{D}_R(M, N) = h\mathscr{M}_R(\Gamma M, \Gamma N).$$

When M is a cell R-module, we use the isomorphism in 2.3, to see that

$$\mathscr{D}_R(M, N) \cong h\mathscr{M}_R(M, N).$$

We obtain a functor $i : h\mathscr{M}_R \longrightarrow \mathscr{D}_R$ by taking the identity on objects and Γ on morphisms. This functor sends weak equivalences to isomorphisms by Theorems 2.13 and 2.14. If we let $\mathscr{C}_R$ be the full subcategory with objects the cell R-modules, the functor Γ induces an equivalence of categories $\mathscr{D}_R \longrightarrow h\mathscr{C}_R$.

There is a smash product $M \wedge_R N$ of right R-module M and left R-module N, which is an S-module. There is a function S-module $F_R(M, N)$ for left R-modules M and N. Each $F_R(M, N)$ is an S-module. If R is commutative, then $M \wedge_R N$ and $F_R(M, N)$ are R-modules. In this case $\mathscr{M}_R$ and $\mathscr{D}_R$ have all of the properties of $\mathscr{M}_S$ and $\mathscr{D}_S$. This means that $\mathscr{D}_R$ is a symmetric monoidal category under $\wedge_R$ with unit R and we have the notion of a monoid or commutative monoid in $\mathscr{D}_R$. These are the analogues of ring spectra in classical stable homotopy theory and are referred to as R-ring spectra in [19, V.2].

2.5 An example: Products on ko/w

In this section we shall work 2-locally. In general for any prime p we let $\mathbb{Z}_{(p)}$ denote the integers localized at a fixed prime p. That is

$$\mathbb{Z}_{(p)} = \left\{ \frac{a}{b} \in \mathbb{Q} : p \nmid b; a, b \in \mathbb{Z} \right\}.$$

We first consider the algebraic construction of the localization of a group at prime p. An exposition of such a construction is given in [44].

We wish to consider the fixed prime p and allow division by all other primes. Of course the appropriate ground ring for this situation is $\mathbb{Z}_{(p)}$ and is used to localize Abelian groups by tensoring. We have, for an Abelian group G,

$$G_{(p)} = G \otimes \mathbb{Z}_{(p)}$$

and call $G_{(p)}$ the localization of G with respect to the prime p.

We can think of $G_{(p)}$ as constructed as follows. We let $p_1, p_2, \cdots, p_n, \cdots$ be an enumeration of the primes excluding our fixed prime p. We construct the following direct system indexed over directed set $\mathbb{N}$, where the map p_i is multiplication by p_i.

$$G_1 \xrightarrow{p_1} G_2 \xrightarrow{p_1 p_2} G_3 \xrightarrow{p_1 p_2 p_3} \cdots \xrightarrow{p_1 \cdots p_{n-1}} G_n \xrightarrow{p_1 \cdots p_n} G_{n+1} \xrightarrow{p_1 \cdots p_{n+1}} \cdots$$

We take the colimit of the above system:

$$\varinjlim G_i = \bigoplus_{i \in \mathbb{N}} G_i / \sim .$$

This construction ensures that multiplication by integers prime to p is an isomorphism and hence invertible. G is isomorphic to its localization at p if and only if G is a $\mathbb{Z}_{(p)}$-module. In this case G is described as p-local.

The example in this section is based on work by Strickland [43]. As in [43], we work in the derived category of R-modules $\mathscr{D}_R$ and think of this category as the brave-new analogue of the stable homotopy category. We construct R-modules R/x and consider when these constructions inherit an R-ring spectrum structure.

Let us begin by considering a commutative S-algebra R and an element $x \in \pi_n R$, which we write as $x \in R_n$. We think of x as a map $S^n_R \longrightarrow R$ and make use of the map

$$S^n_R \wedge_R R \xrightarrow{x \wedge \mathrm{id}} R \wedge_R R \cong R \tag{2.4}$$

We write $\Sigma^n R$ for $S^n_R \wedge_R R$ and $x : \Sigma^n R \longrightarrow R$ for the map 2.4. We define R/x to be the cofibre of the map 2.4.

In considering products on modules R/x, over commutative S-algebra R, such that $R_* = \pi_* R$ is concentrated in even degrees; Strickland [43] shows that there are never any obstructions to associativity and that the obstructions to commutativity are given by a certain naturally defined element $\bar{c}(x) \in \pi_{2d+2}(R)/(2,x)$ and ultimately by a certain power operation. The example we consider is $R = ko$, the spectrum representing real connective K-theory, and we use these results to show that there exists a unique associative and commutative product on ko/w, where $w \in \pi_8 ko$, noting that $ko_* = \pi_* ko$ is not concentrated in even degrees and does not therefore fulfil the hypothesis used by Strickland [43].

It is known that ko can be taken to be a q-cofibrant commutative S-algebra, and after 2-localization the homotopy ring of ko is

$$\mathbb{Z}_{(2)}[\eta, y, w]/(2\eta = \eta^3 = 0,\ y^2 = 4w,\ \eta y = 0)$$

where

$$\eta \in ko_1,\ y \in ko_4 \text{ and } w \in ko_8.$$

In order to compare any discussion with Strickland [43], we should note that for our example $R = ko$, $x = w \in ko_8$ and so $d = 8$. We have the cofibre sequence

$$\Sigma^8 ko \xrightarrow{w} ko \xrightarrow{\rho} ko/w \xrightarrow{\beta} \Sigma^9 ko.$$

Since w is not a zero divisor, we have $\pi_*(ko/w) = ko_*/w$. In particular, we consider $\pi_9(ko/w) = ko_9/w$. Since ko_9 is generated by ηw, we have $ko_9/w = 0$ (this corresponds to $R_{d+1}/x = 0$, in [43], which is due to $d+1$ being odd). Taking the cofibre sequence above and applying the contravariant functor $[-, ko/w]$, we get

$$[\Sigma^8 ko, ko/w] \xleftarrow{w^*} [ko, ko/w] \xleftarrow{\rho^*} [ko/w, ko/w] \xleftarrow{\beta^*} [\Sigma^9 ko, ko/w],$$

which gives

$$0 = ko_8/w \xleftarrow{w^*} [ko, ko/w] \xleftarrow{\rho^*} [ko/w, ko/w] \xleftarrow{\beta^*} ko_9/w = 0.$$

By exactness we find $\rho^* : [ko/w, ko/w] \cong [ko, ko/w]$. It follows, as in [43], that ko/w is unique up to unique isomorphism as an object under ko.

The following results are those in Strickland [43], modified for our particular example. The proofs in Strickland work for our modified results, and any potential problems due to ko_* not being even are given further clarification below.

In reconsidering the cofibre sequence above as follows

$$S^8_{ko} \xrightarrow{w} S^0_{ko} \xrightarrow{\rho} ko/w$$

and referring to the definition of cell R-modules (2.11), we can see that w is an attaching map. ko/w is therefore a cell ko-module with one 0-cell and one 9-cell. As in [43], we can also consider $(ko/w)^2 = ko/w \wedge_{ko} ko/w$ as a cell ko-module with one 0-cell, two 9-cells and one 18-cell and we say that $\phi : (ko/w)^2 \longrightarrow ko/w$ is a product if it agrees with ρ on the bottom cell, in other words $\phi \circ (\rho \circ \rho) = \rho : ko \longrightarrow ko/w$.

The main result is stated as follows.

Proposition 2.15. *In $\mathscr{D}_{ko}$, ko/w has the following properties:*

i. There exist products on ko/w.

ii. All products on ko/w are associative, and have ρ as a two-sided unit.

iii. ko/w admits a unique commutative product.

Lemma 2.16. *The map $w : \Sigma^8 ko/w \longrightarrow ko/w$ is zero.*

Proof. This proof works in exactly the same way as the proof for $x : \Sigma^d R/x \longrightarrow R/x$ is zero in [43]. We use the cofibration

$$\Sigma^8 ko \xrightarrow{\rho} \Sigma^8 ko/w \xrightarrow{\beta} \Sigma^{17} ko,$$

we also note that $ko_{17}/w = 0$ (in [43], $\pi_{2d+1}(R/x) = R_{2d+1}/x = 0$ since $2d+1$ is odd). We apply the contravariant functor $[-, ko/w]$ to the cofibration above and use $ko_{17}/w = 0$ along with exactness to find that $\rho^* : [\Sigma^8 ko/w, ko/w] \longrightarrow [\Sigma^8 ko, ko/w] = ko_8/w$ is injective. We have zero on the right hand side, since $ko_8/w = 0$ and injectivity implies zero on the left hand side, so $w : \Sigma^8 ko/w \longrightarrow ko/w$ is zero as claimed. □

Corollary 2.17. *There exist products on ko/w.*

Proof. As in [43], having proved Lemma 2.16, we consider the original cofibre sequence

$$\Sigma^8 ko \xrightarrow{w} ko \xrightarrow{\rho} ko/w \xrightarrow{\beta} \Sigma^9 ko.$$

Smashing with ko/w over ko we get the following cofibration

$$\Sigma^8 ko/w \xrightarrow{w} ko/w \xrightarrow{1\wedge\rho} ko/w \wedge_{ko} ko/w = (ko/w)^2,$$

where $1 : ko/w \longrightarrow ko/w$ is the identity map. From 2.16 above, the first map is zero. Hence by exactness, $1 \wedge \rho$ is a split monomorphism with splitting $\phi : (ko/w)^2 \longrightarrow ko/w$. This splitting is a product. □

Lemma 2.18. *If $\phi : (ko/w)^2 \longrightarrow ko/w$ is a product then ρ is a two-sided unit for ϕ in the sense that*

$$\phi \circ (\rho \wedge 1) = \phi \circ (1 \wedge \rho) = 1 : ko/w \longrightarrow ko/w.$$

Proof. As in [43]. □

Proposition 2.19. *Any product on ko/w is associative.*

Proof. As in [43], we use unital properties of products on ko/w along with [43, Lemma 3.6]. Strickland also uses the fact that $\pi_{3d+3}(R)/x = 0$ (since $3d+3$ is odd); likewise we have $\pi_{3d+3}ko/w = ko_{27}/w = 0$. □

Corollary 2.20. *ko/w admits a unique commutative product.*

Proof. By Strickland [43, Corollary 3.12], there is a naturally defined element $\bar{c}(x) \in \pi_{2d+2}(ko)/(2,w)$ such that ko/w admits a commutative product if and only if $\bar{c}(x) = 0$. In order to show that $\bar{c}(x) = 0$, it is sufficient to show that the group $\pi_{2d+2}(ko)/(2,w) = 0$. We have $\pi_{2d+2}(ko)/w = ko_{18}/w = (\mathbb{Z}/2)\eta^2 w^2/w = 0$ and hence $\pi_{2d+2}(ko)/(2,w) = 0$, therefore we have a commutative product on ko/w. Again by [43, Corollary 3.12], the set of commutative products has a free transitive action of the group $\mathrm{ann}(2, \pi_{2d+2}(ko)/w) := \{y \in \pi_{2d+2}ko/w : 2y = 0\}$. In particular, if $\pi_{2d+2}ko/w$ has no 2-torsion then there is a unique commutative product. Since $d = 8$, we have $\pi_{2d+2}ko/w = ko_{18}/w = 0$, hence $\mathrm{ann}(2, \pi_{2d+2}(ko)/w) = 0$ and so there exists a unique commutative product on ko/w. □

2.6 Localization for R-modules and R-algebras

In this section we introduce Bousfield localizations of R-modules and R-algebras and consider localization at a prime p as a particular example.

An overview of the concept of localization, both as an algebraic construction and for spaces within homotopy theory is provided in [44].

The classical Bousfield localization is a homological localization and commutes with suspension. For further details on localizations commuting with suspension see [14].

We begin by letting R be an S-algebra and taking a cell R-module E. We note that R could be S itself. We construct Bousfield localizations of R-modules at E. The following material is based on [19, VIII] and founded upon Bousfield's papers ([11], [12]). The model category structure on $\mathscr{M}_R$ (given in [19, VII, Corollary 4.8]) facilitates the construction of E-localizations of R-modules.

A map of R-modules $f : M \longrightarrow N$ is an *E-equivalence* or *E-acyclic map* if the induced map

$$\mathrm{id} \wedge_R f : E \wedge_R M \longrightarrow E \wedge_R N$$

is a weak equivalence. That is $f : M \longrightarrow N$ induces the following isomorphism in E-homology, for all $k \in \mathbb{Z}$.

$$f_* : E_k(M) \longrightarrow E_k(N)$$

Hence, homologically, we should call such maps *E_*^R-equivalences.*

An R-module W is *E-acyclic* if $E \wedge_R W \cong *$, or equivalently $E_k(W) = 0$ for all $k \in \mathbb{Z}$. It is evident that a map f is E-acyclic if and only if its cofibre is E-acyclic.

An R-module L is *E-local* if each E-equivalence $f : M \longrightarrow N$ induces an isomorphism

$$f^* : \mathscr{D}_R(N, L) \xrightarrow{\cong} \mathscr{D}_R(M, L),$$

or equivalently if $\mathscr{D}_R(W, L) = 0$ for any E-acyclic R-module W.

As Dwyer described in [16], L is local if E-equivalent R-modules cannot be distinguished by mapping them into L.

A *localization* of R-module M at E is a map $\lambda : M \longrightarrow M_E$ such that λ is an E-equivalence and M_E is E-local. The model structure on $\mathscr{M}_R$ in Theorem 2.21 below is constructed in [19, VII.4] and implies the existence of E-localizations of R-modules.

Theorem 2.21. *The category $\mathscr{M}_R$ is a topological model category where the weak equivalences are the E-equivalences and the cofibrations are the q-cofibrations of the original model category structure in* [19, VII, Corollary 4.8]. *The fibrations satisfy the right lifting property with respect to the E-acyclic q-cofibrations.*

We follow [19] by defining E-fibrations and E-cofibrations as follows.

Definition 2.22. A map $f : M \longrightarrow N$ is an *E-fibration* if it has the right lifting property with respect to the E-acyclic inclusions of subcomplexes in cell R-modules. A map $f : M \longrightarrow N$ is an *E-cofibration* if it satisfies the left lifting property with respect to the E-acyclic E-fibrations.

The motivation for using inclusions of subcomplexes is made transparent in [19, VIII, Lemma 1.9]. We have the following two results which consolidate the comparisons referred to in Theorem 2.21 and are given in [19, VIII, Lemma 1.3 and 1.4].

Lemma 2.23. *A map is an E-cofibration if and only if it is a q-cofibration.*

Lemma 2.24. *A map is an E-fibration if and only if it satisfies the right lifting property with respect to the E-acyclic q-cofibrations.*

An R-module L is said to be fibrant if the unique map $L \longrightarrow *$ is an E-fibration. We include the following proposition [19, VIII Proposition 1.5].

Proposition 2.25. *An R-module is E-fibrant if and only if it is E-local.*

Lemma 2.26. *Every R-module M has a localization $\lambda : M \longrightarrow M_E$.*

Proof. Consider the trivial map $M \longrightarrow *$. By Definition 1.24, any map can be factored as an E-acyclic E-cofibration and a fibration. This allows us to consider $M \longrightarrow *$ as the composite of $\lambda : M \longrightarrow M_E$ and E-fibration $M_E \longrightarrow *$. □

We proceed by stating further lemmas, as given in [19], that are key to the existence of the model category structure given in Theorem 2.21.

Lemma 2.27. *Any map $f : M \longrightarrow N$ factors as a composite*

$$M \xrightarrow{i} M' \xrightarrow{p} N$$

where p is an E-fibration and i is an E-acyclic q-cofibration that satisfies the left lifting property with respect to the E-fibrations.

Lemma 2.28. *The following conditions on a map $f : M \longrightarrow N$ are equivalent.*

i. f is an E-acyclic E-fibration.

ii. f is an E-acyclic map that satisfies the right lifting property with respect to all q-cofibrations.

iii. f is an acyclic q-fibration.

We note that Lemma 2.28 gives Lemma 2.23.

In [19] the authors progress the discussion of Bousfield localization by restricting R to a q-cofibrant commutative S-algebra and letting E once again be a cell R-module. In [19, VIII Theorems 2.1 and 2.2] it is proven that the localization at E of a cell R-algebra A can be constructed as a cell R-algebra and similarly for commutative cell R-algebras. Of course, any R-algebra is weakly equivalent to a cell R-algebra and we can therefore surmise that Bousfield localizations of R-algebras and commutative R-algebras are again R-algebras and commutative R-algebras respectively.

We now consider a particular example of Bousfield localization, namely localization at a prime p. A p-localization of S-module M may be defined with respect to the $\mathbb{Z}_{(p)}$ Moore spectrum $M\mathbb{Z}_{(p)}$. We have that the map $\lambda : M \longrightarrow M_{(p)}$ is an $M\mathbb{Z}_{(p)}$ equivalence and $M_{(p)}$ is $M\mathbb{Z}_{(p)}$-local.

We include some further details which are taken from [14, Section 3]. In considering M as an S-module, we take the map

$$\xi = \eta \wedge \mathrm{id} : M \cong S \wedge_S M \longrightarrow M\mathbb{Z}_{(p)} \wedge_S M$$

where η is given by the unit of $M\mathbb{Z}_{(p)}$ and is a localization of S at p. It can be shown that ξ is the localization of M at $M\mathbb{Z}_{(p)}$ and so p-localization is smashing;

$$M_{(p)} \cong M \wedge_S S_{(p)}.$$

For each $k \in \mathbb{Z}$ we have

$$\pi_k(M_{(p)}) \cong \pi_k(M) \otimes \pi_0(M\mathbb{Z}_{(p)}) \cong \pi_k(M) \otimes \mathbb{Z}_{(p)}.$$

Chapter 3

Nuclear and Minimal Atomic S-modules

The notion of a *minimal atomic* space or spectrum was first introduced by Hu, Kriz and May [23], inspired by work of Priddy [40] and relating to the more established idea of *atomic* spaces and spectra. Atomic spaces and spectra are tightly bound together so that a self map inducing an isomorphism on homotopy in the Hurewicz (bottom) dimension is necessarily an equivalence. We can associate various algebraic analogues to the concepts of atomic and minimal atomic spaces and spectra which help to illustrate the ideas involved. As a consequence of this algebraic link, we have the concept of an *irreducible* spectrum. We present the definition of an irreducible S-module (different to that in [23] where a spectrum was defined to be irreducible if it has no non-trivial retracts which is equivalent to wedge indecomposability), given in [5]. We also include the proof that the irreducible S-modules are precisely the minimal atomic S-modules. This is all included in Section 3.1, where we also provide a homological characterization of irreducible S-modules.

It is in Section 3.2 that we first introduce the term *nuclear*. The notion of a nuclear space or spectrum emerges by constructing something called a *core* of a space or spectrum. This construction of a core is in fact a generalization of a construction in [40] where the Brown–Peterson spectrum BP was constructed from $S_{(p)}$. In Section 3.2 we give a proof that nuclear S-modules are minimal atomic, as conjectured in [23] and shown in [5]. We also include the result that every minimal atomic S-module is equivalent to a nuclear S-module.

In [5] a notion of minimality was introduced that allows us to show that any S-module is equivalent to a minimal one. We examine the characterization of minimal S-modules in terms of nuclear S-modules in Section 3.3.

The ideas introduced in [23] including that of nuclear spaces and spectra and cores of spaces and spectra were developed further by Baker and May [5] providing a characterization of minimal atomic spaces and spectra. Both [23] and [5] give results on complexes, by which we mean CW spaces or spectra. We should note however, that these results can be adapted to other frameworks in which there is a notion of CW objects. In this chapter we present the known results on minimal atomic and nuclear spectra contained in [23] and [5] in terms of S-modules, that is, in the context of [19]. Working in the category $\mathscr{M}_S$ of S-modules allows us to extend the results contained in this chapter to the case of commutative S-algebras; details of which are contained in Chapter 5.

3.1 Definitions, basic constructions and characterization results

3.1.1 Notation and Terminology

In this section we provide the definitions of several of the concepts mentioned in the introduction, which allow us to characterize minimal atomic and irreducible S-modules. We work in the category $\mathscr{M}_S$ of S-modules and agree that all S-modules are localized at a fixed prime p. We also decide that all S-modules are to be p-local *CW* S-modules with the domains of the attaching maps being wedges of p-local sphere S-modules.

We take all S-modules to have Hurewicz dimension 0, which is equivalent to being (-1)-connected with π_0 as the first non-zero homotopy group. We say that X is a *Hurewicz S-module* if it has a single cell in dimension zero. If, $\pi_0(X)$ is cyclic over $\mathbb{Z}_{(p)}$ or, equivalently, $H_0(X;\mathbb{F}_p)=\mathbb{F}_p$, then X is weakly equivalent to a Hurewicz S-module. We may assume that X has no cells of negative dimension (except the base vertex) and we assume further that there are only finitely many cells in each dimension. We write X_n for the n-skeleton of X. We take $X_{-1}=*$ and, if X is a Hurewicz S-module, $X_0=S^0$. For $n\geq 0$, X_{n+1} is the cofiber of a map $j_n\colon J_n\longrightarrow X_n$, where J_n is a finite wedge of p-local n-sphere S-modules S^n.

We take $H_n(X)$ to be (reduced) homology with p-local coefficients and any S-module X has each $H_n(X)$ a finitely generated $\mathbb{Z}_{(p)}$-module.

3.1.2 Definitions and Remarks

We begin by stating the several basic definitions and observations required to give the characterizations of irreducible and minimal atomic S-modules found in Section 3.1.3.

The following are definitions of concepts which are invariant under equivalence.

Definition 3.1. A map $f\colon X \longrightarrow Y$ is a *monomorphism* if $f_*\colon \pi_0(X) \otimes \mathbb{F}_p \longrightarrow \pi_0(Y) \otimes \mathbb{F}_p$ and all $f_*\colon \pi_n(X) \longrightarrow \pi_n(Y)$ are monomorphisms.

If $f\colon X \longrightarrow Y$ is a *monomorphism* of *Hurewicz* S-modules then f induces an isomorphism on π_0 as follows. As Hurewicz S-modules X and Y have a single cell in dimension zero and so $\pi_0(X)$ and $\pi_0(Y)$ are cyclic over $\mathbb{Z}_{(p)}$. We have that $f_0 : \pi_0(X) \longrightarrow \pi_0(Y)$ is a monomorphism and is surjective, hence we have an isomorphism on π_0.

Definition 3.2. Y is *irreducible* if any monomorphism $f\colon X \longrightarrow Y$ is an equivalence.

The definition given above is different to that found in [23, Definition 1.1], where Y was defined to be irreducible if it does not admit any trivial retracts.

Definition 3.3. X is *atomic* if it is a Hurewicz S-module and a self-map $f\colon X \longrightarrow X$ that induces an isomorphism on π_0 is an equivalence.

An atomic S-module can be shrunk to S-modules with smaller homotopy groups; namely *minimal atomic* S-modules, which can be shrunk no further. Let us consider the definition of such an S-module.

Definition 3.4. Y is *minimal atomic* if it is atomic and any monomorphism $f\colon X \longrightarrow Y$ from an atomic complex X to Y is an equivalence.

We can think of minimal atomic S-modules as analogues of the algebraic notion of irreducible modules; whose only submodules are itself and the zero module. Obviously the definition of irreducible S-modules given above makes this analogy more transparent. In the next section (3.1.3) we give the result from [5] proving that the irreducible S-modules are precisely the minimal atomic S-modules. The implication that irreducible means minimal atomic is the analogue of Schur's lemma.

Definition 3.5. *Y has no homotopy detected by* $\operatorname{mod} p$ *homology* if Y is a Hurewicz S-module and and the $\operatorname{mod} p$ Hurewicz homomorphism $h : \pi_n(Y) \longrightarrow H_n(Y; \mathbb{F}_p)$ is zero for all $n > 0$.

3.1.3 Characterization Results

We will now begin to explore the relationships between the concepts defined above. Let us start by characterizing irreducible S-modules with the following theorem extracted from [5, Theorem 1.3].

Theorem 3.6. *The following two conditions on a Hurewicz S-module Y are equivalent.*

i. Y is irreducible.

ii. Y has no homotopy detected by $\bmod p$ *homology.*

Proof. We use proof by contradiction. Firstly, let us assume condition (i) and assume that the Hurewicz homomorphism $h : \pi_n(Y) \longrightarrow H_n(Y;\mathbb{F}_p)$ is non-zero, where $n > 0$. So there is a map $S^n \longrightarrow Y$ that is non-zero on $\bmod p$ homology, hence there is a map $g : Y \longrightarrow H\mathbb{F}_p$ that is non-zero on homotopy. Let $f : X \longrightarrow Y$ be the homotopy fibre of g. So we obtain a homotopy long exact sequence as follows

$$\cdots \longrightarrow \pi_n(X) \longrightarrow \pi_n(Y) \longrightarrow \pi_n(H\mathbb{F}_p) \longrightarrow \cdots \longrightarrow \pi_0(X) \longrightarrow \pi_0(Y) \longrightarrow \pi_0(H\mathbb{F}_p).$$

Clearly $f : X \longrightarrow Y$ induces an isomorphism on π_0 and at least a monomorphism on all $f_* : \pi_*(X) \longrightarrow \pi_*(Y)$. Therefore f is a monomorphism but not an equivalence, which contradicts (i).

Now let us assume condition (ii) and let $f : X \longrightarrow Y$ be a monomorphism. We will show that f is an equivalence. Let $g : Y \longrightarrow Z$ be the homotopy cofibre of f. The induced homotopy long exact sequence along with the assumptions we have made means that f is an equivalence if and only if $\pi_*(Z) = 0$. Let us suppose that $\pi_*(Z) \neq 0$ and let $\pi_n(Z)$ be the first non-zero homotopy group. Then $h : \pi_n(Z) \longrightarrow H_n(Z;\mathbb{F}_p)$ is non-zero, by the definition of the Hurewicz homomorphism. Let us consider the following portion of the homotopy long exact sequence

$$\cdots \longrightarrow \pi_n(X) \longrightarrow \pi_n(Y) \xrightarrow{g_n} \pi_n(Z) \xrightarrow{\delta_n} \pi_{n-1}(X) \xrightarrow{f_{n-1}} \pi_{n-1}(Y) \longrightarrow \pi_{n-1}(Z) \longrightarrow \cdots$$

in which $\pi_{n-1}(Z) = 0$. We have the assumption that f is a monomorphism and so f_{n-1} is an isomorphism. This implies that $\operatorname{im}\delta_n = 0$ and so g_n is an epimorphism. Considering the commutative diagram given below, we can see that the left arrow h is non-zero, which contradicts (ii).

$$\begin{array}{ccc} \pi_n(Y) & \xrightarrow{g_n} & \pi_n(Z) \\ {\scriptstyle h}\downarrow & & {\scriptstyle h}\downarrow \\ H_n(Y;\mathbb{F}_p) & \xrightarrow{g_*} & H_n(Z;\mathbb{F}_p) \end{array}$$

$\square$

We now give the following result which is proven using results from the following section (3.2) on nuclear S-modules.

Theorem 3.7. *For any S-module Y, there is a monomorphism $f\colon X \longrightarrow Y$ such that X is atomic.*

We now give the following key characterization theorem.

Theorem 3.8. *A Hurewicz S-module Y is irreducible if and only if it is minimal atomic.*

Proof. First, let us suppose that Hurewicz complex Y is irreducible. From Theorem 3.7, there exists a monomorphism $f : X \longrightarrow Y$ such that X is atomic and from the definition of irreducible f is an equivalence. This gives us that Y is minimal atomic, again by the definition.

Now, let Y be minimal atomic and $f : X \longrightarrow Y$ be a monomorphism. Suppose $g : W \longrightarrow X$ is a monomorphism such that W is atomic. The composite $f \circ g$ is an equivalence by the definition of minimal atomic and so we have that f is also an equivalence. Therefore we have that Y is irreducible. $\square$

3.2 Nuclear and Minimal Atomic S-modules

3.2.1 Motivation

As in [5], we are interested in describing minimal atomic S-modules in terms of *nuclear* S-modules. This description will allow us to prove Theorem 3.7. We will begin by introducing the concept of a *nuclear* S-module. The definition of a nuclear S-module emerges from the construction of a *core* of an S-module.

3.2.2 Definition and basic construction

In this section we shall attempt to explain what is meant by the concept of a nuclear S-module.

Hu, Kriz and May construct the core of a preassigned complex Y by building a *nuclear* complex X; essentially as an atomic space built up in an economical way [23, 1.6]. We especially do not want any cells attached trivially. It is illustrative to consider the following simple example. We can take the preassigned S-module Y to be $S^0 \vee S^n$, $n > 0$ and we can aim to construct the core of Y; namely a nuclear S-module X, along with a monomorphism $f\colon X \longrightarrow Y$. Firstly we note that sphere S-modules are nuclear. We begin our construction by taking the 0-skeleton X_0 of X to be the 0-sphere S^0. We aim to build the complex X by attaching cells, ensuring at each stage, that a map, $f_k\colon X_k \longrightarrow Y$ from the k-skeleton of X to our S-module Y induces a monomorphism on the homotopy groups in dimension k, namely, π_k. We know that the homotopy groups of the product $S^0 \vee S^n$ is equal to the direct sum of the homotopy groups of each sphere S-module, so we have,

$$\pi_*(S^0 \vee S^n) \cong \pi_*(S^0) \oplus \pi_*(S^n).$$

We also know that for an n-sphere the first non-trivial homotopy group is $\pi_n(S^n)$. So we have a monomorphism from the homotopy of S^0 to the homotopy of the product $S^0 \vee S^n$ and composing this with the projection onto the first factor gives the identity

$$\begin{tikzcd}
\pi_*(S^0) \arrow[r] \arrow[dr] & \pi_*(S^0) \oplus \pi_*(S^n) \arrow[d] \\
& \pi_*(S^0).
\end{tikzcd}$$

If we try to attach k-cells to X_0 to form X_k via attaching maps $S^{k-1} \longrightarrow X_0$, we find that the attaching maps must be null homotopic. Since we do not want any trivial attaching maps, we cannot attach any cells. And so in constructing the core of $S^0 \vee S^n$, we get S^0 only. We can state this more generally, by saying that the core of a wedge product $X \vee Y$, where X is nuclear, is just X itself.

Now follows the definition of a nuclear S-module.

Definition 3.9. Let X be a Hurewicz S-module whose CW structure is given by cofibrations $J_n \longrightarrow X_n \longrightarrow X_{n+1}$. We say that X is *nuclear* if the following condition is satisfied for each n;

$$\operatorname{Ker}(j_{n*}\colon \pi_n(J_n) \longrightarrow \pi_n(X_n)) \subset p.\pi_n(J_n). \tag{3.1}$$

Definition 3.10. A *core* of an S-module Y is a nuclear S-module X together with a monomorphism $f : X \longrightarrow Y$.

The definition of a core given above is more general than that in [23], where it was restricted to Hurewicz complexes.

We state and prove the following lemma, which is based on an observation of Priddy [40] and is given in [5, Lemma 3.6]. We will use this lemma to prove a result on minimal and nuclear S-modules in Section 3.3, as it gives us a way of redefining the notion of a nuclear S-module in terms of the Hurewicz homomorphisms of its skeleta.

Lemma 3.11. *A Hurewicz S-module is nuclear if and only if the* $\bmod p$ *Hurewicz homomorphism* $h : \pi_n(X_n) \longrightarrow H_n(X_n; \mathbb{F}_p)$ *is zero for all* $n > 0$.

Proof. Considering the cofibre sequence

$$J_n \longrightarrow X_n \longrightarrow X_{n+1},$$

we get the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
\pi_{n+1}(X_n) & \longrightarrow & \pi_{n+1}(X_{n+1}) & \longrightarrow & \pi_n(J_n) & \xrightarrow{j_*} & \pi_n(X_n) \\
 & & \Big\downarrow{\scriptstyle h} & & \Big\downarrow{\scriptstyle h} & & \Big\downarrow{\scriptstyle h} \\
0 & \longrightarrow & H_{n+1}(X_{n+1}; \mathbb{F}_p) & \longrightarrow & H_n(J_n; \mathbb{F}_p) & \xrightarrow{j_*} & H_n(X_n; \mathbb{F}_p)
\end{array}$$

where the vertical maps are Hurewicz homomorphisms. By considering the composition

$$\pi_{n+1}(X_{n+1}) \longrightarrow \pi_n(J_n) \longrightarrow H_n(J_n; \mathbb{F}_p)$$

we can see that condition (3.1) holds if and only if the left most arrow h is 0. $\square$

We include the following result, given in [23, Lemma 1.13] which is immediate from the definitions and is utilised in Chapter 5 to prove an important result for S-algebras (Proposition 5.9).

Lemma 3.12. *If $g : Y \longrightarrow Z$ is a map of S-modules that induces an isomorphism on π_0 and a monomorphism on all homotopy groups and if $f : X \longrightarrow Y$ is a core of Y, then $g \circ f : X \longrightarrow Z$ is a core of Z.*

3.2.3 Results

As in [5], we now want to begin to explore the relationship between nuclear S-modules and the various concepts that have been discussed previously in this chapter. This discussion leads to a theorem which describes minimal atomic S-modules. First we have the following result, which is proven in [23, Proposition 1.5] and is used to prove Theorem 3.7.

Theorem 3.13. *A nuclear S-module is atomic.*

The proof of this theorem begins with a nuclear S-module X along with a self map $f : X \longrightarrow X$ that induces an isomorphism on π_0. The authors show inductively that, for all n, the self maps f_n

of the skeleta X_n are equivalences. This gives us that f is a homotopy equivalence. We adapt the proof in Chapter 5 to give the analogous result for S-algebras in Theorem 5.5.

Using a similar construction to that given in [23, 1.6], of a core of a preassigned S-module Y along with the result above 3.13, we have Theorem 3.7.

The proof of 3.13 above, also adapts to give the following proposition which is a restatement of [5, Proposition 2.5] in the language of S-modules. This in turn implies the stronger result that a nuclear S-module is a minimal atomic S-module 3.15, conjectured in [23, 1.12].

Proposition 3.14. *Let X and Y be nuclear S-modules and let $f\colon X \longrightarrow Y$ be a core of Y. Then f is an equivalence.*

Proof. Let the given nuclear X and Y have CW structures given by the cofibrations

$$J_n \longrightarrow X_n \longrightarrow X_{n+1}$$

$$K_n \longrightarrow Y_n \longrightarrow Y_{n+1}.$$

By hypothesis X is a nuclear S-algebra and the map $f\colon X \longrightarrow Y$ is a map of S-modules that induces a monomorphism on all homotopy groups. We may assume that f is cellular, and so, we prove that $f\colon X_n \longrightarrow Y_n$ is an equivalence for all n. As $f : X \longrightarrow Y$ is a monomorphism between Hurewicz S-modules, $f : X_0 \longrightarrow Y_0$ is an equivalence. We assume inductively that $f\colon X_n \longrightarrow Y_n$ is an equivalence and deduce that $f\colon X_{n+1} \longrightarrow Y_{n+1}$ is an equivalence. We use the attaching maps of X and Y to give the following diagram of cofibre sequences

$$\begin{tikzcd} J_n \arrow[r, "j_n"] \arrow[d, "f"'] & X_n \arrow[r] \arrow[d, "f"'] & X_{n+1} \arrow[d, "f"'] \\ K_n \arrow[r, "k_n"] & Y_n \arrow[r] & Y_{n+1} \end{tikzcd}$$

When we pass to homology, we get the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H_{n+1}(X_{n+1}) \arrow[r] \arrow[d, "f_*"'] & H_n(J_n) \arrow[r, "(j_n)_*"] \arrow[d, "f_*"'] & H_n(X_n) \arrow[r] \arrow[d, "f_*"', "\cong"] & H_n(X_{n+1}) \arrow[r] \arrow[d, "f_*"'] & 0 \\ 0 \arrow[r] & H_{n+1}(Y_{n+1}) \arrow[r] & H_n(K_n) \arrow[r, "(k_n)_*"] & H_n(Y_n) \arrow[r] & H_n(Y_{n+1}) \arrow[r] & 0 \end{tikzcd} \tag{3.2}$$

in which the rows come from the long exact homology sequences for X and Y induced by the cofibre sequences. It is necessary to show that the left and right vertical arrows are isomorphisms. By the Five Lemma, this will hold if the induced map $f_* : H_n(J_n) \longrightarrow H_n(K_n)$ is an isomorphism. In view of the Hurewicz isomorphisms in dimension n it suffices to show that $\pi_n(J_n) \longrightarrow \pi_n(K_n)$ is an isomorphism. We have the following commutative diagram with exact rows

$$\begin{tikzcd} \cdots \arrow[r] & \pi_n(J_n) \arrow[r, "(j_n)_*"] \arrow[d, "f_*"'] & \pi_n(X_n) \arrow[r] \arrow[d, "f_*"', "\cong"] & \pi_n(X_{n+1}) \arrow[r] \arrow[d, "f_*"'] & 0 \\ \cdots \arrow[r] & \pi_n(K_n) \arrow[r, "(k_n)_*"] & \pi_n(Y_n) \arrow[r] & \pi_n(Y_{n+1}) \arrow[r] & 0 \end{tikzcd} \tag{3.3}$$

arising from the long exact sequences of homotopy groups. Using this diagram, we can see that the right vertical arrow is an epimorphism. Let us consider the following diagram

$$\begin{tikzcd} \pi_n(X_{n+1}) \arrow[r, "\cong"] \arrow[d, "f_*"'] & \pi_n(X) \arrow[d, "f_*"'] \\ \pi_n(Y_{n+1}) \arrow[r, "\cong"] & \pi_n(Y) \end{tikzcd} \tag{3.4}$$

In the above diagram, the right vertical arrow is a monomorphism because X is a core of Y and hence the left arrow is a monomorphism. We have then, that $\pi_n(X_{n+1}) \longrightarrow \pi_n(Y_{n+1})$ is an isomorphism. This implies that the right vertical arrow is an isomorphism in the following diagram

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Ker} j_{n*} \arrow[r, "i"] \arrow[d] & \pi_n(J_n) \arrow[r] \arrow[d, "f_*"'] & \operatorname{Im} j_{n*} \arrow[r] \arrow[d, "\cong"'] & 0 \\ 0 \arrow[r] & \operatorname{Ker} k_{n*} \arrow[r, "i"'] & \pi_n(K_n) \arrow[r] & \operatorname{Im} k_{n*} \arrow[r] & 0 \end{tikzcd}$$

Using the nuclear condition (3.1), both maps i become 0 after tensoring with $\mathbb{F}_p$. This implies that $f_* \otimes \mathbb{F}_p$ is an isomorphism. We can deduce that $f_* : \pi_n(J_n) \longrightarrow \pi_n(K_n)$ is an epimorphism by Nakayama's lemma [3, Corollary 2.7], and then, observing that $\pi_n(K_n)$ is a free $\mathbb{Z}_{(p)}$-module, it further follows that f_* is a monomorphism. □

We now give the following result, which is a stronger result than Theorem 3.13. The result was conjectured in [23, 1.12] and proven in [5].

Theorem 3.15. *A nuclear S-module is a minimal atomic S-module.*

Proof. Let Y be a nuclear S-module and let $f : X \longrightarrow Y$ be a monomorphism from an atomic S-module X. As Y is nuclear, it is atomic by 3.13 and we need to show that f is an equivalence in order to show that Y is minimal atomic. Let us consider the composite of f and a core $g : W \longrightarrow X$. By Proposition 3.14 $f \circ g : W \longrightarrow Y$ is an equivalence and so we have that f is also an equivalence. □

Now we have the following description of minimal atomic S modules.

Theorem 3.16. *The following conditions on an S-module Y are equivalent.*

i. Y is minimal atomic.

ii. Any core of Y is an equivalence.

iii. Y is equivalent to a nuclear S-module.

Proof. Theorem 3.15 gives $(iii) \Longrightarrow (i)$. If we assume that condition (i) holds, then by the definition of minimal atomic, any monomorphism from an atomic S-module X to Y is an equivalence and hence by Proposition 5.5 any core of Y is an equivalence i.e. condition (ii) holds. Finally, we can see that condition (ii) implies (iii) easily by considering the definition of a core. □

3.3 Minimal S-modules and Nuclear S-modules

As mentioned in the introduction to this chapter, Baker and May [5] introduced a notion of minimality that allows us to show that any S-module is equivalent to a minimal one. This notion of minimality is due to Cooke [15]. In this section we examine the characterization of minimal S-modules in terms of nuclear S-modules giving us Theorem 3.19, the key result of this section.

We begin by defining the term *minimal* in this context. Firstly we should note that a *CW* S-module X has a p-local chain complex

$$\cdots \longrightarrow C_n(X) \xrightarrow{d_n} C_{n-1}(X) \longrightarrow \cdots \tag{3.5}$$

with $C_n(X) = H_n(X_n, X_{n-1})$.

This *cellular* chain complex may be used to calculate the ordinary homology groups of an S-module X.

Definition 3.17. An S-module X is *minimal* if the differential on its mod p chain complex $C_*(X; \mathbb{F}_p)$ is zero.

In terms of the chain complex (3.5), we have that S-module X is minimal if $\operatorname{im} d_n \subseteq p.C_{n-1}(X)$. Tensoring with $\mathbb{F}_p$, (3.5) becomes

$$\cdots \longrightarrow C_n(X) \otimes \mathbb{F}_p \xrightarrow{d_n \otimes \mathrm{id}} C_{n-1}(X) \otimes \mathbb{F}_p \longrightarrow \cdots \tag{3.6}$$

with $d_n \otimes \mathrm{id} = 0$ for all n, as in Definition 3.17. We also have that

$$\begin{aligned} H_n(X;\mathbb{F}_p) &= H_n(C(X)\otimes \mathbb{F}_p) \\ &= \ker(d_n \otimes \mathrm{id})/\operatorname{im}(d_{n+1}\otimes \mathrm{id}) \\ &= C_n(X)\otimes \mathbb{F}_p. \end{aligned}$$

This is also true for X_n, namely $H_n(X_n;\mathbb{F}_p) = C_n(X_n)\otimes \mathbb{F}_p$.

From the cofiber sequence $X_n \longrightarrow X_{n+1} \longrightarrow X_{n+1}/X_n$, we consider the following long exact sequence.

$$0 \longrightarrow H_{n+1}(X_{n+1};\mathbb{F}_p) \longrightarrow H_{n+1}(X_{n+1}/X_n;\mathbb{F}_p) \longrightarrow H_n(X_n;\mathbb{F}_p) \longrightarrow H_n(X_{n+1};\mathbb{F}_p) \longrightarrow 0$$

Evidently we have an induced epimorphism

$$H_n(X_n;\mathbb{F}_p) \longrightarrow H_n(X_{n+1};\mathbb{F}_p). \tag{3.7}$$

For X minimal we have $H_{n+1}(X_{n+1};\mathbb{F}_p) = C_{n+1}(X_{n+1})\otimes \mathbb{F}_p = H_{n+1}(X_{n+1}/X_n;\mathbb{F}_p)$, giving an isomorphism

$$H_{n+1}(X_{n+1};\mathbb{F}_p) \xrightarrow{\cong} H_{n+1}(X_{n+1}/X_n;\mathbb{F}_p).$$

This implies that our induced epimorphism 3.7 is actually an isomorphism.

As a consequence we have an alternative formulation of the notion of minimality, namely there is an isomorphism $H_n(X_n;\mathbb{F}_p) \longrightarrow H_n(X_{n+1};\mathbb{F}_p)$ for each n, so that there isomorphisms

$$H_n(X_n;\mathbb{F}_p) \xrightarrow{\cong} H_n(X_{n+1};\mathbb{F}_p) \xrightarrow{\cong} H_n(X;\mathbb{F}_p). \tag{3.8}$$

We say that an S-module is *minimal Hurewicz* if it is minimal and Hurewicz. We should note that X is minimal if and only if each X_n is minimal.

3.3.1 Results

The first result given below is stated and proven for complexes in [5]. It originates from Cooke's paper [15] and has made a recent reappearance in [21]. We state the theorem below and give an outline of the proof from [5].

Theorem 3.18. *For any S-module Y, there is a minimal S-module X and an equivalence $f : X \longrightarrow Y$.*

The theorem above is proven in the following way. We begin with an S-module Y and build a minimal S-module X along with an equivalence $f : X \longrightarrow Y$. In order to prove that we have $f : X \longrightarrow Y$ with the desired properties it is sufficient to show that f induces an isomorphism on H_*.

From Section 3.1.1 we have assumed that each $H_n(Y)$ is a finitely generated $\mathbb{Z}_{(p)}$-module. Let us describe $H_n(Y)$ as a direct sum of finitely many $\mathbb{Z}_{(p)}$-modules. We construct the minimal S-module X by taking it to have an n-cell for each $\mathbb{Z}_{(p)}$-module.

The proof is inductive as it assumes the construction of the n-skeleton X_n together with a based map $f_n : X_n \longrightarrow Y$ which induces an isomorphism H_i for $i < n$ and on H_n a map from n-cells of X_n to chosen generators of the $\mathbb{Z}_{(p)}$-modules of $H_n(Y)$. Using the cofibre Cf_n of f_n, attaching maps for the construction of X_{n+1} from X_n are found. An extension $f_{n+1} : X_{n+1} \longrightarrow Y$ of f_n is obtained which induces an isomorphism on H_n and completes the inductive step in the construction of $f : X \longrightarrow Y$.

The following theorem brings together the main notions of this chapter and emphasizes the relevance of minimality to the current theory.

Theorem 3.19. *A minimal S-module is nuclear if and only if it has no homotopy detected by* $\bmod p$ *homology.*

Proof. We assume that X is a minimal S-module. Let us consider the following diagram relating the Hurewicz homomorphisms of X_n and X, where $n > n_0$.

$$\begin{array}{ccc} \pi_n(X_n) & \longrightarrow & \pi_n(X) \\ {\scriptstyle h}\big\downarrow & & {\scriptstyle h}\big\downarrow \\ H_n(X_n;\mathbb{F}_p) & \longrightarrow & H_n(X;\mathbb{F}_p) \end{array}$$

Since X is minimal, by 3.8 we have that the bottom arrow is an isomorphism. We also have that the top arrow is an epimorphism. The left Hurewicz arrow is zero if and only if the right Hurewicz arrow is zero and so we have the conclusion from 3.11. $\square$

The following theorem gives a description of minimal atomic S-modules and completes this discussion of nuclear and minimal atomic S-modules.

Theorem 3.20. *The following conditions on an S-module Y are equivalent.*

i. Y is minimal atomic.

ii. Any equivalence $f : X \longrightarrow Y$ from a minimal S-module X to Y is a core of Y.

iii. A minimal S-module equivalent to Y is nuclear.

Chapter 4

Topological André–Quillen Homology for Cellular Commutative S-algebras

In this chapter we give an account of some basic results on *topological André–Quillen homology and cohomology* for CW commutative S-algebras. This theory is required in order to extend the existing results for S-modules, contained in the previous chapter, to the case of CW commutative S-algebras.

In Chapter 5, we will define atomic and minimal atomic S-algebras and, in particular, consider the construction of a core of a commutative S-algebra, which leads to the definition of a nuclear commutative S-algebra. Essentially, we aim to characterize nuclear commutative S-algebras in terms of atomic and minimal atomic commutative S-algebras. In order to produce these results, which contribute to a theory of nuclear and minimal atomic S-algebras, we require arguments based on skeletal filtrations relying on a suitable homology theory, namely *topological André–Quillen homology.* As the name suggests, this theory is the topological analogue of André–Quillen homology for commutative algebras, discussed in [46], among others.

Our sources on topological André–Quillen (co)homology include [6, 8, 9, 33]. We also benefited from helpful comments by Birgit Richter and Maria Basterra.

4.1 Recollections on Topological André–Quillen (co)homology

We take the definition of topological André–Quillen homology and cohomology from [6] and [29]. In [6], Basterra begins with a discussion of the topological model categories in which the work takes place. We begin similarly, introducing the model categories on which the definition of topological André–Quillen (co)homology is founded.

For definiteness, we work in the category of commutative S-algebras described in [19]. For a commutative S-algebra A, we let $\mathscr{M}_A$ denote the category of A-modules and $\mathscr{C}_{A/B}$ denote the category of commutative A-algebras with A-algebra maps to B. We also have $\mathscr{N}_A$, the category of non-unital commutative A-algebras. Given a model category $\mathscr{C}$, we let $\bar{h}\mathscr{C}$ denote its derived homotopy category.

We then present a series of adjunctions, as in [6], between the model categories in question. These adjunctions lead to an equivalence of homotopy categories $\bar{h}\mathscr{C}_{A/A}$ and $\bar{h}\mathscr{N}_A$ which allow the description of an *abelianization* functor for commutative A-algebras over B.

The topological André–Quillen homology and cohomology of commutative S-algebras is defined in terms of the *abelianization* functor. This discussion is given in 4.1.3, along with the topological analogue of Quillen's transitivity exact sequence and a version of the classical Hurewicz isomorphism theorem. We end this section with a discussion of *ordinary topological André–Quillen homology* on sphere objects. The account of topological André–Quillen (co)homology found in this section is based on [6] and [29]. The theory is also included in an overview paper by Basterra and Richter [9] on existing homology theories for commutative S-algebras.

4.1.1 Categories of A-algebras

We let A be a commutative S-algebra and $\mathscr{M}_A$ denote the category of A-modules.

First let us consider the category of commutative A-algebras, $\mathscr{C}_A$. From [19] we have the monad $\mathbb{P} : \mathscr{M}_A \longrightarrow \mathscr{M}_A$ given by

$$\mathbb{P}M = \bigvee_{j \geq 0} M^j/\Sigma_j$$

where M is an A-module, M^j denotes the j-fold smash power over A and $M^0 = A$.

We also have that $\mathscr{C}_A$, the category of commutative A-algebras, is isomorphic to the category of $\mathbb{P}$-algebras in $\mathscr{M}_A$, denoted $\mathscr{M}_A[\mathbb{P}]$. That is, we have the monad $\mathbb{P}$ in $\mathscr{M}_A$ whose algebras are the commutative monoids in $\mathscr{M}_A$ (see [19, II, Proposition 4.5] for further details).

By [19, VII, Corollary 4.10] the category $\mathscr{C}_A$ is a *topological model category* in the sense of Quillen [41]. By this we mean that together with the usual model structure, the category is topologically enriched, that is, the Hom sets are topological spaces and the composition of morphisms is continuous.

Now let B be a commutative A-algebra and consider the category of commutative A-algebras over B, denoted $\mathscr{C}_{A/B}$. An object C of $\mathscr{C}_{A/B}$ is a commutative A-algebra equipped with an A-algebra map $\varepsilon : C \longrightarrow B$. A morphism between two such objects C and D is the following commutative diagram in $\mathscr{C}_A$.

$$\begin{tikzcd} C \arrow[rr] \arrow[dr] & & D \arrow[dl] \\ & B & \end{tikzcd} \tag{4.1}$$

$\mathscr{C}_{A/B}$ inherits a topological model category structure from $\mathscr{C}_A$. When $B = A$, we have the category $\mathscr{C}_{A/A}$ which is the category $\mathscr{C}_{A/B}$ over the terminal object A and so, as discussed in [22], is a pointed model category.

Let us denote by $\mathscr{N}_A$, the category of non-unital commutative A-algebras. An object in this category is an A-module M, together with a strictly associative multiplication map $M \wedge_A M \longrightarrow M$. Following [6] and [29] we call an object in $\mathscr{N}_A$ an A-NUCA. Basterra considers $\mathscr{N}_A$ as a category of algebras over a certain monad $\mathbb{A} : \mathscr{M}_A \longrightarrow \mathscr{M}_A$ defined as

$$\mathbb{A}M = \bigvee_{j>0} M^j/\Sigma_j$$

We have that $\mathscr{N}_A$ is the category of $\mathbb{A}$-algebras $\mathscr{M}_A[\mathbb{A}]$.

4.1.2 Equivalence of homotopy categories $\bar{h}\mathscr{C}_{A/A}$ and $\bar{h}\mathscr{N}_A$

In this section we describe an adjunction which gives us an equivalence of homotopy categories $\bar{h}\mathscr{C}_{A/A}$ and $\bar{h}\mathscr{N}_A$. The first functor from the adjoint equivalence is $K : \mathscr{N}_A \longrightarrow \mathscr{C}_A$ which assigns to an A-NUCA M, the commutative A-algebra $A \vee M$. The A-algebra $A \vee M$ can be considered as an object in $\mathscr{C}_{A/A}$ via the canonical projection map $\pi : A \vee M \longrightarrow A$. Therefore we can view the functor K as $K : \mathscr{N}_A \longrightarrow \mathscr{C}_{A/A}$.

Now we must introduce the second functor I. Let B be an A-algebra over A. So B is a commutative A-algebra with augmentation map $\varepsilon : B \longrightarrow A$. We denote the *augmentation ideal* of B by $I(B)$ and by this we mean the fibre of the augmentation map ε. $I(B)$ is then naturally an A-NUCA and we consider I as a functor from $\mathscr{C}_{A/A}$ to $\mathscr{N}_A$. Basterra [6, Proposition 2.1 and 2.2] states and proves the following result.

Proposition 4.1. *The functor K is left adjoint to I and this adjunction gives us an equivalence of homotopy categories $\bar{h}\mathscr{C}_{A/A}$ and $\bar{h}\mathscr{N}_A$.*

To see that the functors K and I are adjoint, that is, we have

$$\mathscr{C}_{A/A}(K(X), B) \cong \mathscr{N}_A(X, I(B)),$$

we use the monad $\mathbb{A} : \mathscr{M}_A \longrightarrow \mathscr{M}_A$ of Section 4.1.1. Recall that $\mathscr{N}_A$ is the category of $\mathbb{A}$-algebras $\mathscr{M}_A[\mathbb{A}]$ in A-modules and note that $K(\mathbb{A}M)$ is a free commutative A-algebra on M. Therefore for an A-module M we have;

$$\mathscr{C}_{A/A}(K(\mathbb{A}M), B) \cong \mathscr{M}_A(M, I(B)) \cong \mathscr{N}_A(\mathbb{A}M, I(B)).$$

To get the adjointness result for any A-NUCA X, we observe that there is a split coequalizer:

$$\mathbb{A}\mathbb{A}X \rightrightarrows \mathbb{A}X \rightarrow X.$$

We can illustrate the adjointness isomorphism explicitly as follows. If $g : X \longrightarrow I(B)$ is a morphism in $\mathscr{N}_A$ then its adjoint $\tilde{g} : A \vee X \longrightarrow B$ is given by the composite

$$\tilde{g} : A \vee X \xrightarrow{id \vee g} A \vee I(B) \xrightarrow{1 \vee i} B$$

where i is the canonical map $I(B) \longrightarrow B$.

In order to show that the adjunction $\mathscr{C}_{A/A}(K(X), B) \cong \mathscr{N}_A(X, I(B))$ passes to the homotopy categories, we use [17, 9.7]. This result shows that we are required to show that the functor K preserves cofibrations and acyclic cofibrations. We can then show, using [17, 9.7(ii)] that the adjunction gives the proposed equivalence of homotopy categories. Dwyer and Spalinski [17, 9.7] give a refinement of Quillen's total derived functor theorem [41, 1.4], which Quillen used heavily in order to show that two model categories have equivalent homotopy categories.

4.1.3 Abelianization functor

Let $\mathbf{R}I$ denote the total derived functor of I. Let us consider the functor $Q : \mathscr{N}_A \longrightarrow \mathscr{M}_A$ called the *indecomposables* functor which is given by the cofibre sequence:

$$N \wedge_A N \xrightarrow{m} N \longrightarrow Q(N),$$

where m denotes the multiplication map. The right adjoint functor of Q is the functor $Z : \mathscr{M}_A \longrightarrow \mathscr{N}_A$ which is given by considering A-modules as objects in $\mathscr{N}_A$ with zero multiplication. Let $\mathbf{L}Q$ denote the total derived functor of Q. Now we can define the *abelianization* functor:

$$Ab_A^B : \bar{h}\mathscr{C}_{A/B} \longrightarrow \bar{h}\mathscr{M}_B,$$

which is left adjoint to the trivial B-algebra extension functor

$$B \vee - : \bar{h}\mathscr{M}_B \longrightarrow \bar{h}\mathscr{C}_{A/B}.$$

We have the functor $- \wedge_A B : \mathscr{C}_{A/B} \longrightarrow \mathscr{C}_{A/A}$ and so $- \wedge_A^L B : \bar{h}\mathscr{C}_{A/B} \longrightarrow \bar{h}\mathscr{C}_{A/A}$.

Definition 4.2. Let C be a commutative A-algebra over B. Then

$$Ab_A^B(C) := \mathbf{L}Q\mathbf{R}I(C \wedge_A^{\mathbf{L}} B).$$

The adjunction described above gives us the following.

Theorem 4.3. *Let C be a commutative A-algebra over B and M a B-module. Then:*

$$\bar{h}\mathscr{C}_{A/B}(C, B \vee M) \cong \bar{h}\mathscr{M}_B(Ab_A^B(C), M).$$

We should note, as mentioned in [29] and [6], that the isomorphism of Theorem 4.3 holds on the level of homotopy categories, but does reflect an adjunction between strict categories.

4.1.4 Topological André–Quillen homology

For B a commutative A-algebra, we let $\Omega_{B/A}$ denote the B-module obtained by applying the *abelianization* functor to B, so that

$$\bar{h}\mathscr{C}_{A/B}(B, B \vee M) \cong \bar{h}\mathscr{M}_B(\Omega_{B/A}, M).$$

Then the *topological André–Quillen homology* and *cohomology* of B over A with coefficients in a B-module M are defined by

$$\mathrm{TAQ}_*(B/A; M) = \pi_*(\Omega_{B/A} \wedge_B M), \tag{4.2}$$

$$\mathrm{TAQ}^*(B/A; M) = \pi_{-*}(F_B(\Omega_{B/A}, M)) \tag{4.3}$$

where F_B denotes the internal function object in $\mathscr{M}_B$.

The definition of topological André–Quillen homology and cohomology leads us to a basic structural result, which is essentially the topological analogue of Quillen's transitivity exact sequence [46, 8.8.6]. The following result is given in [6, Proposition 4.2].

Proposition 4.4. *Let $B \longrightarrow C$ be an A-algebra map. Then*

$$\Omega_{B/A} \wedge_B C \longrightarrow \Omega_{C/A} \longrightarrow \Omega_{C/B}$$

is a homotopy cofibre sequence of C-modules.

And so, associated to an A-algebra map $B \longrightarrow C$ with M a C-module, there are natural long exact sequences,

$$\begin{aligned}\cdots \longrightarrow \mathrm{TAQ}_k(B/A;M) \longrightarrow \mathrm{TAQ}_k(C/A;M) \longrightarrow \mathrm{TAQ}_k(C/B;M)& \\ \longrightarrow \mathrm{TAQ}_{k-1}(B/A;M) \longrightarrow \cdots&\end{aligned} \tag{4.4a}$$

and

$$\begin{aligned}\cdots \longrightarrow \mathrm{TAQ}^k(C/B;M) \longrightarrow \mathrm{TAQ}^k(C/A;M) \longrightarrow \mathrm{TAQ}^k(B/A;M)& \\ \longrightarrow \mathrm{TAQ}^{k+1}(C/B;M) \longrightarrow \cdots&\end{aligned} \tag{4.4b}$$

We will be especially interested in the situation where A and B are connective and the map $\phi\colon A \longrightarrow B$ induces an isomorphism $A_0 \xrightarrow{\cong} B_0$; we will write $\Bbbk = A_0 = B_0$. Then there is an Eilenberg–Mac Lane object $H\Bbbk$, which can be taken to be a CW commutative A-algebra or B-algebra, which allows us to define the *ordinary topological André–Quillen homology* and *cohomology* of B over A:

$$\mathrm{HAQ}_*(B/A) = \mathrm{TAQ}_*(B/A;H\Bbbk) = \pi_*(\Omega_{B/A} \wedge_B H\Bbbk), \tag{4.5a}$$

$$\mathrm{HAQ}^*(B/A) = \mathrm{TAQ}^*(B/A;H\Bbbk) = \pi_{-*}(F_B(\Omega_{B/A}, H\Bbbk)). \tag{4.5b}$$

When $C_0 = \Bbbk$, the long exact sequences of (4.4) become long exact sequences as follows.

$$\cdots \longrightarrow \mathrm{HAQ}_k(B/A) \longrightarrow \mathrm{HAQ}_k(C/A) \longrightarrow \mathrm{HAQ}_k(C/B) \longrightarrow \mathrm{HAQ}_{k-1}(B/A) \longrightarrow \cdots \tag{4.6a}$$

$$\cdots \longrightarrow \mathrm{HAQ}^k(C/B) \longrightarrow \mathrm{HAQ}^k(C/A) \longrightarrow \mathrm{HAQ}^k(B/A) \longrightarrow \mathrm{HAQ}^{k+1}(C/B) \longrightarrow \cdots \tag{4.6b}$$

We can also introduce coefficients in a $\Bbbk$-module M by setting

$$\mathrm{HAQ}_*(B/A;M) = \mathrm{TAQ}_*(B/A;HM) = \pi_*(\Omega_{B/A} \wedge_B HM), \tag{4.7a}$$

$$\mathrm{HAQ}^*(B/A;M) = \mathrm{TAQ}^*(B/A;HM) = \pi_{-*}(F_B(\Omega_{B/A}, HM)). \tag{4.7b}$$

The following lemma is an important result on ordinary topological André–Quillen homology and is given in [6, lemma 8.2]. The result appears to be incorrectly stated and should read as below; we note that the proof appears to be correct. For a map θ of A-modules we let C_θ denote the mapping cone of θ in $\mathscr{M}_A$.

Lemma 4.5 (Basterra [6, lemma 8.2]). *Let $\phi\colon A \longrightarrow B$ be an n-equivalence, where $n \geq 1$. Then $\Omega_{B/A}$ is n-connected and there is a map of A-modules $\tau\colon \mathrm{C}_\phi \longrightarrow \Omega_{B/A}$ for which*

$$\tau_*\colon \pi_{n+1}(\mathrm{C}_\phi) \xrightarrow{\cong} \pi_{n+1}(\Omega_{B/A}).$$

Lemma 4.5 above allows, as an immediate consequence, the following version of the classical Hurewicz isomorphism theorem.

Corollary 4.6. *The map τ induces the following isomorphism*

$$\tau_*\colon \pi_{n+1}(\mathrm{C}_\phi) \xrightarrow{\cong} \mathrm{HAQ}_{n+1}(B/A)).$$

Proof. From [19] there is a Künneth spectral sequence for which

$$\mathrm{E}^2_{p,q} = \mathrm{Tor}^{B_*}_{p,q}(\pi_*(\Omega_{B/A}),\Bbbk) \Longrightarrow \pi_{p+q}(\Omega_{B/A} \wedge_B H\Bbbk) = \mathrm{HAQ}_{p+q}(B/A).$$

For dimensional reasons we have $\mathrm{E}^\infty_{0,n+1} = \mathrm{E}^2_{0,n+1}$ and so, on recalling that $A_0 = B_0 = \Bbbk$,

$$\mathrm{HAQ}_{n+1}(B/A) = [\pi_*(\Omega_{B/A}) \otimes_{B_*} \Bbbk]_{n+1} = \pi_{n+1}(\Omega_{B/A}) \otimes_{B_0} \Bbbk = \pi_{n+1}(\Omega_{B/A}). \qquad \square$$

Recall that for any A-module X, there is a free commutative A-algebra on X, $\mathbb{P}_A X = \bigvee_{i\geq 0} X^i/\Sigma_i$. The A-algebra map $\mathbb{P}_A X \longrightarrow \mathbb{P}_A * = A$ induced by collapsing X to a point makes A into an $\mathbb{P}_A X$-algebra.

Let X be a cell A-module. We can write

$$\Omega_{\mathbb{P}_A X/A} \simeq taq_{\mathbb{P}_A X}(\mathbb{P}_A X \wedge_A \mathbb{P}_A X),$$

where taq is the construction of Kuhn [27] generalised from S-modules to A-modules. From [27, Lemma 3.6] and [27, Example 3.8], both generalised from S-modules to A-modules, we get

$$taq_{\mathbb{P}_A X}(\mathbb{P}_A X \wedge_A \mathbb{P}_A X) \simeq \mathbb{P}_A X \wedge_A taq_A(\mathbb{P}_A X)$$

and

$$taq_A(\mathbb{P}_A X) \simeq X,$$

and it follows that

$$\Omega_{\mathbb{P}_A X/A} \simeq \mathbb{P}_A X \wedge_A X.$$

On taking $B = \mathbb{P}_A X$ and $C = A$, the cofibration sequence of Proposition 4.4 yields the following cofibration sequence of A-modules

$$\Omega_{\mathbb{P}_A X/A} \wedge_{\mathbb{P}_A X} A \longrightarrow \Omega_{A/A} \longrightarrow \Omega_{A/\mathbb{P}_A X}$$

in which $\Omega_{A/A} \simeq *$. Hence as A-modules,

$$\Omega_{A/\mathbb{P}_A X} \simeq \Sigma\Omega_{\mathbb{P}_A X/A} \wedge_{\mathbb{P}_A X} A. \tag{4.8}$$

For the A-sphere $S^n = S^n_A$ with $n > 0$ we obtain the commutative A-algebra $\mathbb{P}_A S^n$ and augmentation $\mathbb{P}_A S^n \longrightarrow A$; this allows us to view an A-module as a $\mathbb{P}_A S^n$-module.

Proposition 4.7. *For any $\mathbb{P}_A S^n$-module M we have*

$$\mathrm{TAQ}_*(\mathbb{P}_A S^n/A; M) \cong M_{*-n},$$

$$\mathrm{TAQ}^*(\mathbb{P}_A S^n/A; M) \cong M^{*-n}.$$

In particular,

$$\mathrm{HAQ}_k(\mathbb{P}_A S^n/A) = \mathrm{HAQ}^k(\mathbb{P}_A S^n/A) = \begin{cases} \Bbbk & \text{if } k = n, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Taking $X = S^n$ in (4.1.4) we obtain

$$\mathrm{TAQ}_*(\mathbb{P}_A S^n/A; M) = \pi_*\Omega_{\mathbb{P}_A S^n/A} \wedge_{\mathbb{P}_A S^n} M = \pi_* S^n \wedge M \cong M_{*-n},$$

$$\mathrm{TAQ}^*(\mathbb{P}_A S^n/A; M) = \pi_{-*}F_{\mathbb{P}_A S^n}(\Omega_{\mathbb{P}_A S^n/A}, M) = \pi_{-*}F(S^n, M) \cong M^{*-n}.$$

When $M = H\Bbbk$ with $\Bbbk = A_0$ this gives

$$\mathrm{HAQ}_k(\mathbb{P}_A S^n/A) = \mathrm{HAQ}^k(\mathbb{P}_A S^n/A) = \begin{cases} \Bbbk & \text{if } k = n, \\ 0 & \text{otherwise,} \end{cases}$$

as claimed. □

Proposition 4.8. *We have*

$$\mathrm{HAQ}_k(A/\mathbb{P}_A S^n) = \mathrm{HAQ}^k(A/\mathbb{P}_A S^n) = \begin{cases} \Bbbk & \text{if } k = n+1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Taking $X = S^n$ in (4.8) we have

$$\Omega_{A/\mathbb{P}_A X S^n} \wedge_A H\Bbbk \simeq \Sigma\Omega_{\mathbb{P}_A S^n/A} \wedge_{\mathbb{P}_A S^n} A \wedge_A H\Bbbk \simeq \Sigma\Omega_{\mathbb{P}_A S^n/A} \wedge_{\mathbb{P}_A S^n} H\Bbbk,$$

$$F_A(\Omega_{A/\mathbb{P}_A S^n}, H\Bbbk) \simeq F_A(\Sigma\Omega_{\mathbb{P}_A S^n/A} \wedge_{\mathbb{P}_A S^n} A, H\Bbbk) \simeq F_{\mathbb{P}_A S^n}(\Sigma\Omega_{\mathbb{P}_A S^n/A}, H\Bbbk).$$

Using Proposition 4.7 the result is now immediate. □

4.2 Topological André–Quillen homology for cell S-algebras

We will apply the results of Section 4.1 to the case of a CW commutative S-algebra R for which $R_{[0]} = S$ and the $(n+1)$-skeleton $R_{[n+1]}$ is obtained by attaching a wedge of $(n+1)$-cells to $R_{[n]}$ using a map $k_n : K_n \longrightarrow R_{[n]}$ from a wedge of n-spheres $K_n = \bigvee S^n$. Thus

$$R_{[n+1]} = R_{[n]} \wedge_{\mathbb{P}_S K_n} \mathbb{P}_S C K_n,$$

which is also the pushout of the following diagram.

$$\begin{tikzcd} & \mathbb{P}_S K_n \arrow[dl] \arrow[dr] & \\ R_{[n]} & & \mathbb{P}_S C K_n \end{tikzcd} \tag{4.9}$$

We will also assume that only cells of degree greater than 1 are attached, thus $R_{[1]} = R_{[0]} = S$ and $R_0 = \pi_0 S$. We will call such S-algebras *simply connected.*

Now by [6, proposition 4.6], for A-algebras $A \longrightarrow B$ and $A \longrightarrow C$ we have the following isomorphism

$$\Omega_{B \wedge_A C/C} \cong \Omega_{B/A} \wedge_A C. \tag{4.10}$$

For $n \geq 1$ this gives

$$\Omega_{R_{[n+1]}/R_{[n]}} = \Omega_{\mathbb{P}_S CK_n \wedge_{\mathbb{P}_S K_n} R_{[n]}/R_{[n]}} \cong \Omega_{\mathbb{P}_S CK_n/\mathbb{P}_S K_n} \wedge_{\mathbb{P}_S K_n} R_{[n]}.$$

We can consider the following long exact sequence derived from (4.6a),

$$\cdots \longrightarrow \mathrm{HAQ}_k(R_{[n]}/S) \longrightarrow \mathrm{HAQ}_k(R_{[n+1]}/S) \longrightarrow \mathrm{HAQ}_k(R_{[n+1]}/R_{[n]}) \\ \longrightarrow \mathrm{HAQ}_{k-1}(R_{[n]}/S) \longrightarrow \cdots$$

which by (4.10) becomes

$$\cdots \longrightarrow \mathrm{HAQ}_k(R_{[n]}/S) \longrightarrow \mathrm{HAQ}_k(R_{[n+1]}/S) \longrightarrow \mathrm{HAQ}_k(\mathbb{P}_S CK_n/\mathbb{P}_S K_n) \\ \longrightarrow \mathrm{HAQ}_{k-1}(R_{[n]}/S) \longrightarrow \cdots$$

in which there is an equivalence of $\mathbb{P}_S K_n$-algebras

$$\mathbb{P}_S CK_n \simeq \mathbb{P}_S * = S.$$

Hence we obtain the following long exact sequence

$$\cdots \longrightarrow \mathrm{HAQ}_k(R_{[n]}/S) \longrightarrow \mathrm{HAQ}_k(R_{[n+1]}/S) \longrightarrow \mathrm{HAQ}_k(S/\mathbb{P}_S K_n) \\ \longrightarrow \mathrm{HAQ}_{k-1}(R_{[n]}/S) \longrightarrow \cdots. \quad (4.11)$$

Using Proposition 4.8, we can now give an estimate on the size of $\mathrm{HAQ}_*(R/S)$ when R is a finite dimensional CW commutative S-algebra.

Proposition 4.9. *Let R be a CW commutative S-algebra with cells only in degrees at most n. Then $\mathrm{HAQ}_k(R/S) = 0$ when $n < k$.*

Corollary 4.10. *If R has only finitely many cells, then*

$$\sum_{k=0}^{n} \operatorname{rank} \mathrm{HAQ}_k(R/S) \leq \text{number of cells}.$$

We also have an analogue of a standard result on CW spectra.

Proposition 4.11. *Let $\phi : P \longrightarrow Q$ be a map of simply connected CW commutative S-algebras. Then ϕ is an equivalence if and only if $\phi_* : \mathrm{HAQ}_*(P/S) \longrightarrow \mathrm{HAQ}_*(Q/S)$ is an isomorphism.*

Proof. Let $n \geq 0$. Then ϕ is an n-equivalence if and only if the mapping cone C_ϕ is n-connected. But on combining Corollary 4.6 with the long exact sequence of (4.5a), we see that C_ϕ is n-connected if and only if $\phi_* : \mathrm{HAQ}_k(P/S) \longrightarrow \mathrm{HAQ}_k(Q/S)$ is an isomorphism for all $k \leq n$.

Since this holds for all n, the result follows. □

We will continue to work with a CW commutative S-algebra R as discussed at the beginning of this section. Let $i_n \colon R_{[n]} \longrightarrow R_{[n+1]}$ be the inclusion map (this is a map of S-algebras and therefore of S-modules). The following discussion provides us with the arguments that allow us to give results on nuclear and minimal atomic S-algebras in the following chapter. Let us consider the following two cofibration sequences in the category of S-modules;

$$K_n \xrightarrow{k_n} R_{[n]} \longrightarrow \mathrm{C}_{k_n},$$
$$R_{[n]} \xrightarrow{i_n} R_{[n+1]} \longrightarrow \mathrm{C}_{i_n}.$$

From the proof of [6, lemma 8.2], there is a homotopy commutative diagram

$$\begin{tikzcd}
R_{[n]} \arrow[r, "i_n"] \arrow[d] & R_{[n+1]} \arrow[r] \arrow[d] & \mathrm{C}_{i_n} \arrow[d, "\tau_n"] \\
R_{[n]} \arrow[r, "i_n"] & R_{[n+1]} \arrow[r, "u_n"] & \Omega_{R_{[n+1]}/R_{[n]}}
\end{tikzcd}$$

which we shall show to extend to a homotopy commutative diagram of the following form.

$$\begin{tikzcd}
R_{[n]} \arrow[r] \arrow[d, "="'] & \mathrm{C}_{k_n} \arrow[r] \arrow[d] & \Sigma K_n \arrow[d, "h_n"] \\
R_{[n]} \arrow[r, "i_n"] \arrow[d, "="'] & R_{[n+1]} \arrow[r] \arrow[d, "="'] & \mathrm{C}_{i_n} \arrow[d, "\tau_n"] \\
R_{[n]} \arrow[r, "i_n"] & R_{[n+1]} \arrow[r, "u_n"] & \Omega_{R_{[n+1]}/R_{[n]}}
\end{tikzcd} \tag{4.12}$$

First let us prove that the map $\mathrm{C}_{k_n} \longrightarrow R_{[n+1]}$ exists. Recall that $R_{[n+1]}$ is a pushout for the diagram (4.9), of commutative S-algebras, that is, we have the following commutative diagram.

$$\begin{tikzcd}
 & \mathbb{P}_S K_n \arrow[dl] \arrow[dr] & \\
R_{[n]} \arrow[dr] & & \mathbb{P}_S \, \mathrm{C}\, K_n \arrow[dl] \\
 & R_{[n+1]} &
\end{tikzcd}$$

We also have that C_{k_n} is defined as a pushout, and so we also have this commutative diagram.

$$\begin{tikzcd}
 & K_n \arrow[dl] \arrow[dr] & \\
R_{[n]} \arrow[dr] & & \mathrm{C}\,K_n \arrow[dl] \\
 & \mathrm{C}_{k_n} &
\end{tikzcd}$$

By considering the natural map $\mathrm{C}K_n \longrightarrow \mathbb{P}_S\mathrm{C}K_n \longrightarrow R_{[n+1]}$ as well, we get a commutative diagram of S-modules as follows

$$\begin{tikzcd}
 & K_n \arrow[dl, "k_n"'] \arrow[dr] & \\
R_{[n]} \arrow[dr, "i_n"'] & & \mathrm{C}K_n \arrow[dl] \\
 & R_{[n+1]} &
\end{tikzcd}$$

and so by the definition of C_{k_n} as a pushout we have a map $\mathrm{C}_{k_n} \longrightarrow R_{[n+1]}$.

As before, let us consider $\Omega_{R_{[n+1]}/R_{[n]}}$ and use (4.10) to get

$$\Omega_{R_{[n+1]}/R_{[n]}} \cong \Omega_{\mathbb{P}_S\,\mathrm{C}\,K_n/\mathbb{P}_S K_n} \wedge_{\mathbb{P}_S K_n} R_{[n]}$$

We now use the equivalence of $\mathbb{P}_S K_n$-algebras

$$\mathbb{P}_S\mathrm{C}K_n \simeq \mathbb{P}_S * = S.$$

to get

$$\Omega_{R_{[n+1]}/R_{[n]}} \cong \Omega_{\mathbb{P}_S\,\mathrm{C}\,K_n/\mathbb{P}_S K_n} \wedge_{\mathbb{P}_S K_n} R_{[n]} \simeq \Omega_{S/\mathbb{P}_S K_n} \wedge_{\mathbb{P}_S K_n} R_{[n]}$$

We now use (4.8) and (4.1.4) to get

$$\begin{aligned}
\Omega_{S/\mathbb{P}_S K_n} &\simeq \Sigma\Omega_{\mathbb{P}_S K_n/S} \wedge_{\mathbb{P}_S K_n} S \\
&\simeq \Sigma(\mathbb{P}_S K_n \wedge_S K_n) \wedge_{\mathbb{P}_S K_n} S \\
&= \Sigma K_n \wedge S.
\end{aligned}$$

And so,

$$\Omega_{R_{[n+1]}/R_{[n]}} \simeq R_{[n]} \wedge_{\mathbb{P}_S K_n} S \wedge \Sigma K_n$$

as modules over $R_{[n+1]} = R_{[n]} \wedge_{\mathbb{P}_S K_n} S$.

Let us now smash over $R_{[n+1]}$ with $H\mathbb{Z}$ to arrive at

$$H\mathbb{Z} \wedge_{R_{[n+1]}} \Omega_{R_{[n+1]}/R_{[n]}} \simeq H\mathbb{Z} \wedge \Sigma K_n.$$

Now consider the composite map $\tau_n \circ h_n$ from diagram (4.12) above, and let us smash with $H\mathbb{Z}$, giving the following map

$$H\mathbb{Z} \wedge \Sigma K_n \xrightarrow{\mathrm{id} \wedge \tau_n \circ h_n} H\mathbb{Z} \wedge \Omega_{R_{[n+1]}/R_{[n]}}.$$

We obtain a self map $f_n : H\mathbb{Z} \wedge \Sigma K_n \longrightarrow H\mathbb{Z} \wedge \Sigma K_n$ by following the map above with the natural map

$$H\mathbb{Z} \wedge \Omega_{R_{[n+1]}/R_{[n]}} \longrightarrow H\mathbb{Z} \wedge_{R_{[n+1]}} \Omega_{R_{[n+1]}/R_{[n]}} \simeq H\mathbb{Z} \wedge \Sigma K_n. \tag{4.13}$$

Since K_n is a wedge of n-spheres, $H\mathbb{Z} \wedge K_n$ is a wedge of copies of $H\mathbb{Z}$. In fact, the map of (4.13) induces an isomorphism on $\pi_{n+1}(\ \ )$.

Lemma 4.12. *$f_n : H\mathbb{Z} \wedge \Sigma K_n \longrightarrow H\mathbb{Z} \wedge \Sigma K_n$ is an equivalence. Equivalently, the following maps are isomorphisms:*

$$\pi_{n+1}\Sigma K_n \xrightarrow{(h_n)_*} \pi_{n+1}\,\mathrm{C}_{i_n}$$

$$\pi_{n+1}\Sigma K_n \xrightarrow{(h_n \circ \tau_n)_*} \pi_{n+1}\Omega_{R_{[n+1]}/R_{[n]}}.$$

Proof. The pairs $(\mathrm{C}_{k_n}, R_{[n]})$ and $(R_{[n+1]}, R_{[n]})$ occurring in (4.12) are relative cell complexes which have the same cells in degrees up to n. The cells in degree $n+1$ correspond to those on ΣK_n and therefore $(h_n)_* : \pi_{n+1}\Sigma K_n \longrightarrow \pi_{n+1}\,\mathrm{C}_{i_n}$ is an isomorphism. For a discussion of cellular structures in this context see [19, VII 3, X 2]. It now follows from the Hurewicz isomorphism theorem that f_n induces an isomorphism on $\pi_{n+1}(H\mathbb{Z} \wedge \Sigma K_n)$ which agrees with $H_{n+1}(\Sigma K_n)$. $\square$

Now applying homotopy to the diagram of (4.12), we obtain a commutative diagram of groups, a part of which is

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_{n+1}R_{[n]} \arrow[r] \arrow[d, "="'] & \pi_{n+1}\,\mathrm{C}_{k_n} \arrow[r] \arrow[d] & \pi_{n+1}\Sigma K_n \arrow[r] \arrow[d, "\cong"', "(h_n)_*"] & \pi_n R_{[n]} \arrow[r] \arrow[d, "="'] & \cdots \\
\cdots \arrow[r] & \pi_{n+1}R_{[n]} \arrow[r, "(i_n)_*"] \arrow[d, "="'] & \pi_{n+1}R_{[n+1]} \arrow[r] \arrow[d, "="'] & \pi_{n+1}\,\mathrm{C}_{i_n} \arrow[r] \arrow[d, "\cong"', "(\tau_n)_*"] & \pi_n R_{[n]} \arrow[r] \arrow[d, "="'] & \cdots \\
\cdots \arrow[r] & \pi_{n+1}R_{[n]} \arrow[r, "(i_n)_*"] & \pi_{n+1}R_{[n+1]} \arrow[r, "(u_n)_*"] & \pi_{n+1}\Omega_{R_{[n+1]}/R_{[n]}} \arrow[r] & \pi_n R_{[n]} \arrow[r] & \cdots
\end{tikzcd} \tag{4.14}$$

and in which the top two rows are exact. In the portion shown, the bottom row is also exact because of the isomorphism $(\tau_n)_*$ on $\pi_{n+1}(\quad)$.

Now we claim that the natural map

$$\pi_{n+1}\Omega_{R_{[n+1]}/R_{[n]}} \longrightarrow \mathrm{HAQ}_{n+1}(R_{[n+1]}/R_{[n]})$$

induced from the map

$$\Omega_{R_{[n+1]}/R_{[n]}} \cong S \wedge \Omega_{R_{[n+1]}/R_{[n]}} \longrightarrow H\mathbb{Z} \wedge \Omega_{R_{[n+1]}/R_{[n]}} \longrightarrow H\mathbb{Z} \wedge_{R_{[n+1]}} \Omega_{R_{[n+1]}/R_{[n]}},$$

extends to a diagram

$$\begin{array}{ccccc} \pi_{n+1}R_{[n+1]} & \longrightarrow & \pi_{n+1}\Omega_{R_{[n+1]}/R_{[n]}} & \longrightarrow & \pi_n R_{[n]} \\ \big\downarrow{\scriptstyle \theta_{n+1}} & & \cong\big\downarrow & & \big\downarrow{\scriptstyle \theta_n} \\ \mathrm{HAQ}_{n+1}(R_{[n+1]}/S) & \longrightarrow & \mathrm{HAQ}_{n+1}(R_{[n+1]}/R_{[n]}) & \longrightarrow & \mathrm{HAQ}_n(R_{[n]}/S) \end{array} \tag{4.15}$$

in which the bottom row is a portion of the usual long exact sequence (4.6) for $A = S$, $B = R_{[n]}$ and $C = R_{[n+1]}$. Furthermore, these diagrams are compatible for varying n.

We prove this by induction on n. The initial case $n = 0$ is trivial since $R_{[0]} = S$ and $\mathrm{HAQ}_0(R_{[0]}/S) = 0$. Now suppose we have the result for some $n \geq 0$. From Proposition 4.9, $\mathrm{HAQ}_{n+1}(R_{[n]}/S) = 0$, hence there is a commutative diagram

$$\begin{array}{ccccc} \pi_{n+1}R_{[n+1]} & \longrightarrow & \pi_{n+1}\Omega_{R_{[n+1]}/R_{[n]}} & \longrightarrow & \pi_n R_{[n]} \\ & & \cong\big\downarrow & & \big\downarrow{\scriptstyle \theta_n} \\ 0 \to \mathrm{HAQ}_{n+1}(R_{[n+1]}/S) & \longrightarrow & \mathrm{HAQ}_{n+1}(R_{[n+1]}/R_{[n]}) & \longrightarrow & \mathrm{HAQ}_n(R_{[n]}/S). \end{array}$$

Recalling that (4.14) has exact first and second rows, we see that both rows here are exact, we see that there is a map $\theta_{n+1} : \pi_{n+1}R_{[n+1]} \longrightarrow \mathrm{HAQ}_{n+1}(R_{[n+1]}/S)$ as desired.

Making use of the isomorphism $(\tau_n \circ h_n)_*$, we can replace the diagram (4.15) by

$$\begin{array}{ccccc} \pi_{n+1}R_{[n+1]} & \longrightarrow & \pi_{n+1}\Sigma K_n & \longrightarrow & \pi_n R_{[n]} \\ \big\downarrow{\scriptstyle \theta_{n+1}} & & \cong\big\downarrow & & \big\downarrow{\scriptstyle \theta_n} \\ 0 \to \mathrm{HAQ}_{n+1}(R_{[n+1]}/S) & \longrightarrow & \mathrm{HAQ}_{n+1}(R_{[n+1]}/R_{[n]}) & \longrightarrow & \mathrm{HAQ}_n(R_{[n]}/S). \end{array} \tag{4.16}$$

Using the evident natural transformation $\mathrm{HAQ}_n(\quad) \longrightarrow \mathrm{HAQ}_n(\quad;\mathbb{F}_p)$ we can map the bottom row of (4.16) into the exact sequence

$$0 \to \mathrm{HAQ}_{n+1}(R_{[n+1]}/S;\mathbb{F}_p) \longrightarrow \mathrm{HAQ}_{n+1}(R_{[n+1]}/R_{[n]};\mathbb{F}_p) \longrightarrow \mathrm{HAQ}_n(R_{[n]}/S;\mathbb{F}_p)$$

to obtain

$$\begin{tikzcd}
\pi_{n+1}R_{[n+1]} \arrow[r] \arrow[d, "\bar{\theta}_{n+1}"] & \pi_{n+1}\Sigma K_n \arrow[r] \arrow[d, "\mathrm{epi}"'] & \pi_n R_{[n]} \arrow[d, "\bar{\theta}_n"] \\
0 \to \mathrm{HAQ}_{n+1}(R_{[n+1]}/S;\mathbb{F}_p) \arrow[r] & \mathrm{HAQ}_{n+1}(R_{[n+1]}/R_{[n]};\mathbb{F}_p) \arrow[r] & \mathrm{HAQ}_n(R_{[n]}/S;\mathbb{F}_p)
\end{tikzcd} \tag{4.17}$$

in which the rows are exact and the middle vertical arrow is an epimorphism.

Chapter 5

Nuclear and Minimal Atomic S-algebras

Hu, Kriz and May [23] discussed the notions of *nuclear complexes* and *cores* of spaces, spectra and commutative S-algebras. In particular, they consider the construction of a *core of R as a commutative S-algebra* and this is the case of most interest to us. This construction leads to the definition of a *nuclear* commutative S-algebra. They remark that the constructions have non-commutative analogues. We explain in detail the analogous definition for *not necessarily commutative S-algebras*. We extend results of [5] and [23] that hold for S-modules to the case of commutative S-algebras. In particular, we characterize nuclear commutative S-algebras in terms of atomic and minimal atomic commutative S-algebras.

5.1 Definitions and Basic Constructions

We work in the context of [19]; specifically in the category of commutative S-algebras $\mathscr{C}\mathscr{A}_S$ and the category of S-algebras $\mathscr{A}_S$. We also work p-locally for a fixed prime p, where p-localizations of commutative S-algebras are commutative algebras over the p-local sphere S-module $S_{(p)}$ which we shall denote as S from this point forward. Such localizations are described in Section 2.6. We use results on *topological André–Quillen homology* given in Chapter 4. When we have a map of commutative connective S-algebras $\phi\colon A \longrightarrow B$, we set

$$\mathrm{HAQ}_*(B/A) = \mathrm{TAQ}_*(B/A; H\mathbb{Z}_{(p)})$$

where $\mathrm{TAQ}_*(B/A; H\mathbb{Z}_{(p)})$ is the topological André–Quillen homology of B over A with coefficients in the Eilenberg–Mac Lane object $H\mathbb{Z}_{(p)}$. We assume that all S-algebras are cellular and have only finitely many cells in each dimension with each homotopy group and $\mathrm{HAQ}_n(X/S)$ a finitely generated $\mathbb{Z}_{(p)}$-module.

We let $\mathscr{M}_S$ denote the category of S-modules. We have a forgetful functor $\mathscr{C}\mathscr{A}_S \longrightarrow \mathscr{M}_S$ with left adjoint free functor $\mathbb{P}\colon \mathscr{M}_S \longrightarrow \mathscr{C}\mathscr{A}_S$, producing from S-modules, free commutative S-algebras. The composite $\mathscr{M}_S \longrightarrow \mathscr{C}\mathscr{A}_S \longrightarrow \mathscr{M}_S$ of $\mathbb{P}$ and the forgetful functor is the monad associated with this adjunction. As before, we shall also denote this monad by $\mathbb{P}$. It is introduced in [19, II, Construction 4.4] and [19, II, Proposition 4.5] gives, as a consequence of the construction, that the category $\mathscr{C}\mathscr{A}_S$ is isomorphic to the category of $\mathbb{P}$-algebras in $\mathscr{M}_S$. There is an analogue for not necessarily commutative S-algebras using the monad $\mathbb{T}\colon \mathscr{M}_S \longrightarrow \mathscr{A}_S$ [19, II, Construction 4.4]. Let $\mathrm{C}\,X$ denote the cone on an S-module X and $\iota\colon X \longrightarrow \mathrm{C}\,X$ be the canonical inclusion. We let K_n be a wedge of finitely many copies of S^n and for an S-algebra Q_n we consider the map $k_n\colon K_n \longrightarrow Q_n$ as a map of S-modules. For commutative S-algebra Q we assume that the unit $\eta\colon S \longrightarrow Q$ induces an isomorphism on π_0, that is, $\pi_0(Q) = \mathbb{Z}_{(p)}$. We should note that this implies that any self map $Q \longrightarrow Q$ also induces an isomorphism on π_0.

Definition 5.1. A *nuclear* commutative S-algebra is a commutative S-algebra Q such that $Q = \operatorname{colim} Q_n$, where $Q_0 = S$ and inductively Q_{n+1} is the pushout of the following diagram.

$$\begin{tikzcd} \mathbb{P}K_n \arrow[r, "\widetilde{k_n}"] \arrow[d, "\mathbb{P}\iota"] & Q_n \\ \mathbb{P}\,\mathrm{C}\,K_n & \end{tikzcd}$$

Also, the map $k_n\colon K_n \longrightarrow Q_n$ satisfies the following condition.

$$\mathrm{Ker}(k_{n*}\colon \pi_n(K_n) \longrightarrow \pi_n(Q_n)) \subset p.\pi_n(K_n). \tag{5.1}$$

A *core* of a commutative S-algebra R is a nuclear commutative S-algebra Q together with a map $g\colon Q \longrightarrow R$ of S-algebras, such that, the induced map of homotopy groups

$$g_*\colon \pi_*(Q) \longrightarrow \pi_*(R)$$

is a monomorphism.

Below we include the explicit construction, given R, of the map $g\colon Q \longrightarrow R$, which is a *core of R as a commutative S-algebra*, as in [23, Construction 2.1]. The core is constructed by inductively killing homotopy groups.

Construction 5.2. We let $Q_0 = S$ and let $g_0 : Q_0 \longrightarrow R$ be the unit of commutative S-algebra R. We construct Q inductively and begin by assuming that we have constructed a commutative S-algebra Q_n and a map $g_n : Q_n \longrightarrow R$ of S-algebras that induces a monomorphism on homotopy groups in dimension less than n. Let $u_1, \ldots, u_m$ be a minimal set of generators of the kernel of $g_{n*} : \pi_n(Q_n) \longrightarrow \pi_n(R)$. We take a wedge K_n of copies of S^n; one for each u_i. We take $k_n : K_n \longrightarrow Q_n$ to be a map of S-modules that realizes each generator u_i of $\ker g_{n*}$ through the following diagram.

$$\begin{tikzcd}
K_n = \bigvee_i S^n \arrow[r, "\bigvee u_i"] & \bigvee_i Q_n \arrow[r, "\nabla"] & Q_n \\
S^n \arrow[u, "i\text{th factor}"] \arrow[urr, "u_i"'] & &
\end{tikzcd}$$

We have $k_{n*} : \pi_n(K_n) \twoheadrightarrow \ker g_{n*}$. The minimality of our chosen family of generators (u_i) for $\ker g_{n*}$ implies that

$$\ker(k_{n*} : \pi_n(K_n) \longrightarrow \pi_n(Q_n)) \subset p.\pi_n(K_n).$$

We have the induced map $\widetilde{k_n} : \mathbb{P}K_n \longrightarrow Q_n$ of S-algebras which gives Q_n as a $\mathbb{P}K_n$-algebra. We define $Q_{n+1} = \mathbb{P}CK_n \wedge_{\mathbb{P}K_n} Q_n$, that is Q_{n+1} is the pushout of the following diagram.

$$\begin{tikzcd}
\mathbb{P}K_n \arrow[r, "\widetilde{k_n}"] \arrow[d, "\mathbb{P}\iota"] & Q_n \\
\mathbb{P}CK_n &
\end{tikzcd}$$

Evidently, by our construction, $g_n \circ k_n$ is null homotopic. Let us now choose a null homotopy $h_n : CK_n \longrightarrow R$, with $\widetilde{h_n} : \mathbb{P}CK_n \longrightarrow R$ the induced map of S-algebras. By the universal property of pushouts there exists a map $g_{n+1} : Q_{n+1} \longrightarrow R$ that restricts to g_n on Q_n. Define $Q = \operatorname{colim} Q_n$ and let $g : Q \longrightarrow R$ be the map of S-algebras obtained by passage to colimits from the g_n. We have that, by construction, the induced map of homotopy groups $g_* : \pi_*(Q) \longrightarrow \pi_*(R)$ is a monomorphism.

We can also define a nuclear not necessarily commutative S-algebra and the core of a not necessarily commutative S-algebra in a similar way as follows.

Definition 5.3. A *nuclear* not necessarily commutative S-algebra is a not necessarily commutative S-algebra Q such that $Q = \operatorname{colim} Q_n$, where $Q_0 = S$ and inductively Q_{n+1} is the pushout of the following diagram in $\mathscr{A}_S$;

$$\begin{tikzcd} \mathbb{T}K_n \arrow[r, "\widetilde{k_n}"] \arrow[d, "\mathbb{T}\iota"] & Q_n \\ \mathbb{T}CK_n & \end{tikzcd}$$

Also, the map $k_n \colon K_n \longrightarrow Q_n$ satisfies the following condition;

$$\operatorname{Ker}(k_{n*} \colon \pi_n(K_n) \longrightarrow \pi_n(Q_n)) \subset p.\pi_n(K_n).$$

A core of a not necessarily commutative S-algebra R is a nuclear not necessarily commutative S-algebra Q together with a map $g \colon Q \longrightarrow R$ of S-algebras, such that, the induced map of homotopy groups

$$g_* \colon \pi_*(Q) \longrightarrow \pi_*(R)$$

is a monomorphism.

Definition 5.4. A commutative S-algebra Q whose unit induces an isomorphism on π_0 is

i. *atomic* if any map of S-algebras $f \colon Q \longrightarrow Q$ is an equivalence

ii. *minimal atomic* if it is atomic and if a map $g \colon P \longrightarrow Q$ of commutative S-algebras from an atomic S-algebra P to Q that induces a monomorphism on all homotopy groups is an equivalence.

5.2 Results on Nuclear and Minimal Atomic S-algebras

In this section we extend results of [5] and [23] that hold for S-modules to the case of commutative S-algebras. In particular, we characterize nuclear commutative S-algebras in terms of atomic and minimal atomic commutative S-algebras. The penultimate result of this section allows us to factorize a core of S-modules as a core of S-modules composed with a core of S-algebras, and it is this result that leads us to the examples contained in the next chapter.

We can now consider the analogue of [23, 1.5] which was conjectured in [23, 2.9]

Theorem 5.5. *Every nuclear commutative S-algebra is an atomic commutative S-algebra.*

Proof. Let X be a nuclear commutative S-algebra whose unit induces an isomorphism on π_0. Let $f\colon X \longrightarrow X$ be a map of S-algebras. We may assume that f is cellular, and so, we prove that $f : X_n \longrightarrow X_n$ is an equivalence for all n.

Since X is nuclear, we have $X_0 = S$ and the $(n+1)$-skeleton X_{n+1} is the pushout of the following diagram;

$$\begin{tikzcd} \mathbb{P}K_n \arrow[r, "\widetilde{k_n}"] \arrow[d, "\mathbb{P}\iota"] & X_n \\ \mathbb{P}\,C\,K_n \end{tikzcd}$$

where K_n is a wedge of finitely many copies of S^n. So, $X_{n+1} = X_n \wedge_{\mathbb{P}K_n} \mathbb{P}\,C\,K_n$. We assume inductively that $f\colon X_n \longrightarrow X_n$ is a homotopy equivalence and deduce that $f\colon X_{n+1} \longrightarrow X_{n+1}$ is a homotopy equivalence.

By Proposition 4.11, a map $\psi\colon P \longrightarrow Q$ of simply connected CW commutative S-algebras is an equivalence if and only if $\psi\colon \mathrm{HAQ}_*(P/S) \longrightarrow \mathrm{HAQ}_*(Q/S)$ is an isomorphism. For the nuclear commutative S-algebra X we can derive the following long exact sequence

$$\ldots \longrightarrow \mathrm{HAQ}_k(X_n/S) \longrightarrow \mathrm{HAQ}_k(X_{n+1}/S) \longrightarrow \mathrm{HAQ}_k(X_{n+1}/X_n) \longrightarrow \mathrm{HAQ}_{k-1}(X_n/S) \longrightarrow \ldots$$

which by 4.10 and the equivalence of $\mathbb{P}K_n$-algebras, $\mathbb{P}\,C\,K_n \simeq S$ becomes (4.11)

$$\ldots \longrightarrow \mathrm{HAQ}_k(X_n/S) \longrightarrow \mathrm{HAQ}_k(X_{n+1}/S) \longrightarrow \mathrm{HAQ}_k(S/\mathbb{P}K_n) \longrightarrow \mathrm{HAQ}_{k-1}(X_n/S) \longrightarrow \ldots$$

We now note that by the definition and (4.5) we have that

$$\mathrm{HAQ}_k(S/\mathbb{P}K_n) = \pi_k(\Omega_{S/\mathbb{P}K_n} \wedge_S H\mathbb{Z}_{(p)}).$$

We also use 4.8 to give

$$\Omega_{S/\mathbb{P}K_n} \simeq \Sigma\Omega_{\mathbb{P}K_n/S} \wedge_{\mathbb{P}K_n} S.$$

This allows us to see the the following

$$\begin{aligned} \mathrm{HAQ}_k(S/\mathbb{P}K_n) &= \pi_k(\Sigma\Omega_{\mathbb{P}K_n/S} \wedge_{\mathbb{P}K_n} H\mathbb{Z}_{(p)}) \\ &= \pi_{k-1}(\Omega_{\mathbb{P}K_n/S} \wedge_{\mathbb{P}K_n} H\mathbb{Z}_{(p)}) \\ &= \mathrm{HAQ}_{k-1}(\mathbb{P}K_n/S). \end{aligned}$$

We are therefore able to use the isomorphism below in the discussion that follows.

$$\mathrm{HAQ}_k(S/\mathbb{P}K_n) \cong \mathrm{HAQ}_{k-1}(\mathbb{P}K_n/S)$$

We have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 \to \mathrm{HAQ}_{n+1}(X_{n+1}/S) & \longrightarrow & \mathrm{HAQ}_n(\mathbb{P}K_n/S) & \longrightarrow & \mathrm{HAQ}_n(X_n/S) & \longrightarrow & \mathrm{HAQ}_n(X_{n+1}/S) \to 0 \\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*}\,\cong & & \downarrow{\scriptstyle f_*} \\
0 \to \mathrm{HAQ}_{n+1}(X_{n+1}/S) & \longrightarrow & \mathrm{HAQ}_n(\mathbb{P}K_n/S) & \longrightarrow & \mathrm{HAQ}_n(X_n/S) & \longrightarrow & \mathrm{HAQ}_n(X_{n+1}/S) \to 0
\end{array} \tag{5.2}$$

It is necessary to show that the left and right vertical arrows are isomorphisms. By the Five Lemma, this will hold if $\mathrm{HAQ}_n(\mathbb{P}K_n/S) \longrightarrow \mathrm{HAQ}_n(\mathbb{P}K_n/S)$ is an isomorphism. That is, if, $\mathrm{HAQ}_{n+1}(X_{n+1}/X_n) \longrightarrow \mathrm{HAQ}_{n+1}(X_{n+1}/X_n)$ is an isomorphism. By 4.16, we have an isomorphism $\pi_{n+1}\Sigma K_n \xrightarrow{\cong} \mathrm{HAQ}_{n+1}(X_{n+1}/X_n)$. And so it suffices to show that f induces an isomorphism $f_* : \pi_n(K_n) \longrightarrow \pi_n(K_n)$.

We have cofibre sequence

$$K_n \xrightarrow{k_n} X_n \longrightarrow \mathrm{C}_{k_n}$$

and this leads to a long exact sequence in homotopy:

$$\cdots \longrightarrow \pi_{j+1}(\mathrm{C}_{k_n}) \longrightarrow \pi_j(K_n) \longrightarrow \pi_j(X_n) \longrightarrow \pi_j(\mathrm{C}_{k_n}) \longrightarrow \pi_{j-1}(K_n) \longrightarrow \cdots .$$

Of course $\pi_j(K_n)$ vanishes for $j < n$ because K_n is a wedge of copies of S^n. Moreover, the self map f respects the cofibration because it is cellular and f induces a self map of this long exact sequence. Hence we obtain the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
\pi_n(K_n) & \longrightarrow & \pi_n(X_n) & \longrightarrow & \pi_n(\mathrm{C}_{k_n}) & \longrightarrow & 0 \\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \\
\pi_n(K_n) & \longrightarrow & \pi_n(X_n) & \longrightarrow & \pi_n(\mathrm{C}_{k_n}) & \longrightarrow & 0.
\end{array}$$

We have assumed that $f : X_n \longrightarrow X_n$ is a homotopy equivalence and hence $f_* : \pi_n(X_n) \longrightarrow \pi_n(X_n)$ is an isomorphism in the diagram above. By this and a diagram chase we deduce that the right vertical arrow is an epimorphism, but we cannot deduce that it is a monomorphism from the diagram because at this stage we do not know that the left vertical arrow is an epimorphism. However, we know that an epimorphic endomorphism of a Noetherian module is an isomorphism and this implies that the right vertical arrow is an isomorphism. It follows that in the diagram below the right hand map is an isomorphism.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ker}\, k_{n*} & \xrightarrow{i} & \pi_n(K_n) & \longrightarrow & \mathrm{Im}\, k_{n*} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle f_*} & & \cong\downarrow & & \\
0 & \longrightarrow & \mathrm{Ker}\, k_{n*} & \xrightarrow[i]{} & \pi_n(K_n) & \longrightarrow & \mathrm{Im}\, k_{n*} & \longrightarrow & 0
\end{array} \tag{5.3}$$

We use this diagram to show that

$$\pi_n(K_n) = \operatorname{Im} f_* + p.\pi_n(K_n). \tag{5.4}$$

Let us choose $x \in \pi_n(K_n)$ and find from the diagram $\hat{x} \in \pi_n(K_n)$ such that $f_*(\hat{x})$ and x have the same image under k_{n*}. Hence,

$$x - f_*(\hat{x}) \in \operatorname{Ker} k_{n*} \subset p.\pi_n(K_n),$$

using (5.1). Hence $x \in \operatorname{Im} f_* + p.\pi_n(K_n)$ and so we have shown that (5.4) holds. By Nakayama's Lemma, (see [3, Corollary 2.7]) $\operatorname{Im} f_* = \pi_n(K_n)$ and so we have that f_* is an epimorphism in the diagram 5.3. We can now apply the Noetherian argument used previously to get that f_* is an isomorphism. □

Proposition 5.6. *A minimal atomic commutative S-algebra is equivalent to a nuclear commutative S-algebra.*

Proof. Let Y be a minimal atomic commutative S-algebra and consider a core of Y, that is, a nuclear commutative S-algebra X along with a map $f\colon X \longrightarrow Y$ that induces a monomorphism on all homotopy groups. X is a nuclear commutative S-algebra and therefore by Theorem 5.5 it is an atomic commutative S-algebra. Since Y is minimal atomic, we use Definition 5.4(ii) to see that the map $f\colon X \longrightarrow Y$ is an equivalence. Hence we have minimal atomic commutative S-algebra Y being equivalent to nuclear commutative S-algebra X. □

The proof of Theorem 5.5 above, strongly suggests that the following may also hold.

Conjecture 5.7. Every core of a nuclear commutative S-algebra is an equivalence.

This conjecture is a natural generalization of Baker and May's [5, Proposition 2.5] about nuclear *complexes* and cores. In fact, we can already give a detailed picture of how the proof might work.

Let Y be a nuclear commutative S-algebra whose unit induces an isomorphism on π_0. And let $f\colon X \longrightarrow Y$ be a core of Y, that is, X is a nuclear commutative S-algebra and the map $f\colon X \longrightarrow Y$ is a map of commutative S-algebras that induces a monomorphism on all homotopy groups.

We may assume that f is cellular, and so, we prove that $f\colon X_n \longrightarrow Y_n$ is an equivalence for all n. As X and Y are both nuclear, we have $X_0 = Y_0 = S$ and the respective $(n+1)$-skeleta are given by $X_{n+1} = X_n \wedge_{\mathbb{P}J_n} \mathbb{P}\,\mathrm{C}\,J_n$ and $Y_{n+1} = Y_n \wedge_{\mathbb{P}K_n} \mathbb{P}\,\mathrm{C}\,K_n$, where J_n and K_n are wedges of copies of

S^n. We assume inductively that $f\colon X_n \longrightarrow Y_n$ is an equivalence and deduce that $f\colon X_{n+1} \longrightarrow Y_{n+1}$ is an equivalence.

We obtain the following commutative diagram with exact rows in a similar way to (5.2). We note again, that a map $\psi\colon P \longrightarrow Q$ of simply connected CW commutative S-algebras is an equivalence if and only if $\psi\colon \mathrm{HAQ}_*(P/S) \longrightarrow \mathrm{HAQ}_*(Q/S)$ is an isomorphism.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{HAQ}_{n+1}(X_{n+1}/S) \arrow[r] \arrow[d, "f_*"'] & \mathrm{HAQ}_n(\mathbb{P}J_n/S) \arrow[r] \arrow[d, "f_*"'] & \mathrm{HAQ}_n(X_n/S) \arrow[r] \arrow[d, "f_*"', "\cong"] & \mathrm{HAQ}_n(X_{n+1}/S) \arrow[r] \arrow[d, "f_*"'] & 0 \\
0 \arrow[r] & \mathrm{HAQ}_{n+1}(Y_{n+1}/S) \arrow[r] & \mathrm{HAQ}_n(\mathbb{P}K_n/S) \arrow[r] & \mathrm{HAQ}_n(Y_n/S) \arrow[r] & \mathrm{HAQ}_n(Y_{n+1}/S) \arrow[r] & 0
\end{tikzcd} \tag{5.5}$$

It is necessary to show that the left and right vertical arrows are isomorphisms. By the Five Lemma, this will hold if $\mathrm{HAQ}_n(\mathbb{P}J_n/S) \longrightarrow \mathrm{HAQ}_n(\mathbb{P}K_n/S)$ is an isomorphism. It suffices to show that $\pi_n(J_n) \longrightarrow \pi_n(K_n)$ is an isomorphism. We have the following map of cofibre sequences

$$\begin{tikzcd}
J_n \arrow[r, "j_n"] \arrow[d] & X_n \arrow[r] \arrow[d] & \mathrm{C}_{j_n} \arrow[d] \\
K_n \arrow[r, "k_n"] & Y_n \arrow[r] & \mathrm{C}_{k_n}
\end{tikzcd}$$

which gives us the following commutative diagram with exact rows

$$\begin{tikzcd}
\pi_n(J_n) \arrow[r, "j_{n*}"] \arrow[d, "f_*"'] & \pi_n(X_n) \arrow[r] \arrow[d, "f_*"'] & \pi_n(\mathrm{C}_{j_n}) \arrow[r] \arrow[d, "f_*"'] & 0 \\
\pi_n(K_n) \arrow[r, "k_{n*}"] & \pi_n(Y_n) \arrow[r] & \pi_n(\mathrm{C}_{k_n}) \arrow[r] & 0
\end{tikzcd}$$

By the diagram, the right vertical arrow is an epimorphism. At this point, if we could show that this was also a monomorphism then the proof could be completed as follows. We would have that the right vertical arrow in the diagram above as an isomorphism. This would imply that the right vertical arrow is an isomorphism in the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker} j_{n*} \arrow[r, "i"] \arrow[d] & \pi_n(J_n) \arrow[r] \arrow[d, "f_*"'] & \operatorname{Im} j_{n*} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{Ker} k_{n*} \arrow[r, "i"'] & \pi_n(K_n) \arrow[r] & \operatorname{Im} k_{n*} \arrow[r] & 0
\end{tikzcd}$$

Using the nuclear condition (5.1), both maps i become 0 after tensoring with $\mathbb{F}_p$. The argument can now be completed in the same way as the proof for Proposition 3.14.

It seems reasonable that the following S-algebra analogue of [5, Theorem 2.6], which was stated for S-modules in Theorem 3.15, should hold.

Conjecture 5.8. A nuclear commutative S-algebra is a minimal atomic commutative S-algebra

Proof that 5.7 $\Rightarrow$ 5.8.

The proof of [5, Theorem 2.6] adapts to S-algebras as follows. Let Y be a nuclear commutative S-algebra and let $f\colon X \longrightarrow Y$ be a map from an atomic S-algebra X to Y that induces a monomorphism on all homotopy groups. We aim to show that $f\colon X \longrightarrow Y$ is an equivalence. Consider the composite of f and a core $g\colon W \longrightarrow X$. This composite will induce a monomorphism on all homotopy groups and hence is a core. Since Y is nuclear, we use the result that a core of a nuclear commutative S-algebra is an equivalence (Conjecture 5.7) to show that the composite of f and g is an equivalence and we have that f is also an equivalence.

The following proposition is stated in [23, Proposition 2.10]. We have provided a shorter proof using a result on S-modules given in 3.12.

Proposition 5.9. *For any core $g\colon Q \longrightarrow R$ of commutative S-algebras, there exists a core $f\colon X \longrightarrow R$ of S-modules and a map $\xi\colon X \longrightarrow Q$ of S-modules such that $f = g \circ \xi$. In particular, ξ induces a monomorphism on all homotopy groups.*

Proof. We have a core $g\colon Q \longrightarrow R$ of commutative S-algebras. So Q is a nuclear commutative S-algebra and $g\colon Q \longrightarrow R$ is a map of S algebras that induces monomorphisms on all homotopy groups . Let $\xi\colon X \longrightarrow Q$ be a core of S-modules, then X is a nuclear S-module and the map ξ also induces monomorphisms on all homotopy groups. Hence by (Lemma 3.12) ([23, Lemma 1.13]), $g \circ \xi\colon X \longrightarrow R$ is a core of R as S-modules. So the proposition holds, since for any core $g\colon Q \longrightarrow R$, we have a core $f\colon X \longrightarrow R$ of S-modules such that $f = g \circ \xi$, by choosing the map $\xi\colon X \longrightarrow Q$ of S-modules to be a core. □

We should note that $\xi\colon X \longrightarrow Q$ is a core of S-modules. We have that Q is nuclear as a commutative S-algebra, but not nuclear as an S-module and so we cannot use Theorems 3.15 and 3.16 ([5, Theorems 2.6 and 2.7]) to get that ξ is an equivalence. For the examples of commutative S-algebras R in the following chapter, we show in fact that ξ cannot be an equivalence, by showing that there is no map of commutative S-algebras $X \longrightarrow R$, where X is a core of R as S-modules. These examples produce some interesting examples of non-cores. In one of our examples we look at the spectrum BoP introduced by [37] which is known to be a core of MSU when both spectra are considered as S-modules (see [5, Example 6.1]). We use the same techniques as Hu, Kriz and

May [23, Proposition 2.11] to show that there is no map $BoP \longrightarrow MSU$ of S-algebras. This work relies heavily upon the action of the Dyer–Lashof algebra on the homology of various commutative S-algebras.

The following relies upon Conjecture 5.8.

Conjecture 5.10. Any commutative S-algebra R, which is minimal atomic as an S-module is equivalent to a core of R as S-algebras and therefore equivalent to a minimal atomic S-algebra.

Proof that 5.8 $\Rightarrow$ 5.10.

Suppose we have a commutative S-algebra R which is minimal atomic as an S-module. If we let $g\colon Q \longrightarrow R$ be a core of commutative S-algebras, then by Proposition 5.9, there is a core $f\colon X \longrightarrow R$ of S-modules and a core $\xi\colon X \longrightarrow Q$ of S-modules such that $f = g \circ \xi$. Since R is minimal atomic as an S-module, the core $f\colon X \longrightarrow R$ is an equivalence by Definition 5.4(ii). Hence we have $g\colon Q \longrightarrow R$ as an equivalence, and therefore, R is equivalent to a nuclear commutative S-algebra. Since a nuclear commutative S-algebra is a minimal atomic commutative S-algebra (Conjecture 5.8), we have that R is equivalent to a minimal atomic S-algebra.

5.3 Minimal and Nuclear Commutative S-algebras

In Chapter 3, the notion of a *minimal* S-module X was introduced. Such an S-module was defined to have a zero differential on its cellular chain complex $C_*(X;\mathbb{F}_p)$. Theorem 3.18 states that every S-module Y is equivalent to a minimal one. We continue to take the view that HAQ_* is a good substitute for ordinary homology when considering commutative S-algebras and consider the notion of minimality in that situation.

Continuing to work with p-local CW commutative S-algebras, we give a suitable definition of a minimal commutative S-algebra.

Definition 5.11. Let R be a commutative S-algebra with n-skeleton $R_{[n]}$. Then R is *minimal* if, for each n, the induced epimorphism

$$\mathrm{HAQ}_n(R_{[n]}/S;\mathbb{F}_p) \longrightarrow \mathrm{HAQ}_n(R_{[n+1]}/S;\mathbb{F}_p) = \mathrm{HAQ}_n(R/S;\mathbb{F}_p)$$

is actually an isomorphism.

We present and prove the following theorem which is analogous to Theorem 3.18. The details of the proof are closely based upon the proof of Theorem 3.18 presented in [5].

Theorem 5.12. *For a commutative S-algebra Y, there is a minimal commutative S-algebra X and an equivalence of commutative S-algebras $f : X \longrightarrow Y$.*

Proof. We are given a p-local CW commutative S-algebra Y. We have assumed Y to have finitely many cells in each degree and so we take $\mathrm{HAQ}_n(Y/S)$ to be a direct sum of finitely many cyclic $\mathbb{Z}_{(p)}$-modules, which we shall denote as $A_{n,i}$. We must construct a minimal commutative S-algebra X along with an equivalence $f : X \longrightarrow Y$. Recall from Theorem 4.11 that a map $f : X \longrightarrow Y$ of S-algebras is an equivalence if and only if $f_* : \mathrm{HAQ}_*(X/S) \longrightarrow HAQ_*(Y/S)$ is an isomorphism.

The S-algebra X will have an n-cell $j_{n,i}$ for each summand $A_{n,i}$ and an $(n+1)$-cell with differential $q_i j_{n,i}$, for the HAQ cellular chain complex where $A_{n,i}$ is of order q_i. Since each q_i must be a power of p it is clear that the differential on the HAQ cellular chain complex is zero and so we induce the isomorphisms given in Definition 5.11.

We assume inductively that we have constructed minimal n-skeleton X_n together with a map $f_n : X_n \longrightarrow Y$ that induces isomorphisms on HAQ in dimensions less than n and an epimorphism on HAQ_n. That is, we assume that $\mathrm{HAQ}_n(X_n/S)$ is a free $\mathbb{Z}_{(p)}$-module on basis given by cells $j_{n,i}$ that map to the chosen generators of the $A_{n,i}$.

We have the following long exact sequence associated to S-algebra map $f_n : X_n \longrightarrow Y$.

$$\cdots \longrightarrow \mathrm{HAQ}_k(X_n/S) \longrightarrow \mathrm{HAQ}_k(Y/S) \longrightarrow \mathrm{HAQ}_k(Y/X_n) \longrightarrow \mathrm{HAQ}_{k-1}(X_n/S) \longrightarrow \cdots \tag{5.6}$$

We have $\mathrm{HAQ}_k(Y/X_n) = 0$ for $k \leq n$. Consider the kernel of $f_* : \mathrm{HAQ}_n(X_n/S) \longrightarrow \mathrm{HAQ}_n(Y/S)$ and note that it will be free on the basis $q_i j_{n,i}$ for those i such that $A_{n,i}$ has finite order. These elements will be the images of elements $k''_{n,i}$ in $\mathrm{HAQ}_{n+1}(Y/X_n)$.

We use a version of the Hurewicz isomorphism given in Corollary 4.6 to get $k''_{n,i} = \tau_*(k'_{n,i})$ for unique elements $k'_{n,i}$ in $\pi_{n+1}(Cf_n)$.

Similarly, the chosen generators for the $A_{n+1,i} \subset \mathrm{HAQ}_{n+1}(Y/S)$ map to elements $j''_{n+1,i}$ in $\mathrm{HAQ}_{n+1}(Y/X_n)$ with $j''_{n+1,i} = \tau_*(j'_{n+1,i})$ for unique elements $j'_{n+1,i}$ in $\pi_{n+1}(Cf_n)$.

For commutative S-algebras, the connecting homomorphism $\pi_{n+1}(Cf_n) \longrightarrow \pi_n(X_n)$ allows us to choose maps $S^n \longrightarrow X_n$ that represent $k'_{n,i}$ and $j'_{n+1,i}$ and consider them as attaching maps for the construction of X_{n+1} from X_n, by attaching cells $k_{n,i}$ and $j_{n+1,i}$. Since the sequence

$\pi_{n+1}(Cf_n) \longrightarrow \pi_n(X_n) \longrightarrow \pi_n(Y)$ is exact, the attaching maps are null homotopic in Y and so there is an extension $f_{n+1} : X_{n+1} \longrightarrow Y$.

We therefore have the following map of cofiber sequences.

$$\begin{tikzcd}
X_{[n]} \arrow[r] \arrow[d, equal] & X_{[n+1]} \arrow[r] \arrow[d, "f_{n+1}"] & \Omega_{X_{[n+1]}/X_{[n]}} \arrow[r] \arrow[d] & \Sigma X_n \arrow[d, equal] \\
X_{[n]} \arrow[r] & Y \arrow[r] & Cf_n \arrow[r] & \Sigma X_n
\end{tikzcd} \tag{5.7}$$

which gives rise to the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \to \mathrm{HAQ}_{n+1}(X_{n+1}/S) \arrow[r] \arrow[d, "(f_{n+1})_*"'] & \mathrm{HAQ}_{n+1}(X_{n+1}/X_n) \arrow[r] \arrow[d] & \mathrm{HAQ}_n(X_n/S) \arrow[r] \arrow[d, equal] & \mathrm{HAQ}_n(X_{n+1}/S) \to 0 \arrow[d, "(f_{n+1})_*"'] \\
0 \to \mathrm{HAQ}_{n+1}(Y/S) \arrow[r] & \mathrm{HAQ}_{n+1}(Cf_n) \arrow[r] & \mathrm{HAQ}_n(X_n/S) \arrow[r] & \mathrm{HAQ}_n(Y/S) \to 0
\end{tikzcd} \tag{5.8}$$

Again consider the HAQ cellular chain complex, where the differential $\mathrm{HAQ}_{n+1}(X_{n+1}/X_n) \longrightarrow \mathrm{HAQ}_n(X_n/X_{n-1})$ is the composite given below.

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{HAQ}_{n+1}(X_{n+1}/X_n) \arrow[rr] \arrow[dr] & & \mathrm{HAQ}_n(X_n/X_{n-1}) \arrow[r] & \cdots \\
& & \mathrm{HAQ}_n(X_n/S) \arrow[ur] & &
\end{tikzcd}$$

where the map $\mathrm{HAQ}_n(X_n/S) \longrightarrow \mathrm{HAQ}_n(X_n/X_{n-1})$ is a monomorphism.

The map $\mathrm{HAQ}_{n+1}(X_{n+1}/X_n) \longrightarrow \mathrm{HAQ}_n(X_n/S)$ will send basis elements $k_{n,i}$ to $q_i j_{n,i}$ and $j_{n+1,i}$ to zero. We therefore have that $\mathrm{HAQ}_{n+1}(X_{n+1}/S)$ is a free $\mathbb{Z}_{(p)}$-module on the basis elements $j_{n+1,i}$.

By a diagram chase we have that f_{n+1} induces an isomorphism on HAQ_n and we can complete the inductive step in constructing $f : X \longrightarrow Y$ by sending the basis elements $j_{n+1,i}$ to the generators of the $A_{n+1,i}$. □

Definition 5.13. A commutative S-algebra is said to have no $\bmod p$ detectable homotopy if, for $n > 0$, the map $\pi_n R_{[n]} \longrightarrow \mathrm{HAQ}_n(R/S; \mathbb{F}_p)$ is trivial.

Theorem 5.14. *A commutative S-algebra R is nuclear if and only if it is minimal and has no $\bmod p$ detectable homotopy.*

Proof. By a diagram chase we have that the nuclear condition 5.1 holds for R if and only if $\bar{\theta}_{n+1} = 0$ for $n \geq 0$ in the following commutative diagram with exact rows, obtained in 4.17.

$$\begin{tikzcd}
 & \pi_{n+1}R_{[n+1]} \arrow[r] \arrow[d, "\bar{\theta}_{n+1}"] & \pi_{n+1}\Sigma K_n \arrow[r] \arrow[d, "\text{epi}"'] & \pi_n R_{[n]} \arrow[d, "\bar{\theta}_n"] \\
0 \arrow[r] & \mathrm{HAQ}_{n+1}(R_{[n+1]}/S;\mathbb{F}_p) \arrow[r] & \mathrm{HAQ}_{n+1}(R_{[n+1]}/R_{[n]};\mathbb{F}_p) \arrow[r] & \mathrm{HAQ}_n(R_{[n]}/S;\mathbb{F}_p)
\end{tikzcd} \tag{5.9}$$

Let us now suppose that R is nuclear, that is $\bar{\theta}_{n+1} = 0$. Note that for $n = 0$, we have

$$\mathrm{HAQ}_0(R_{[0]}/S;\mathbb{F}_p) = \mathrm{HAQ}_0(S/S;\mathbb{F}_p) = 0.$$

And for $n > 0$, the image of the boundary map

$$\mathrm{HAQ}_{n+1}(R_{[n+1]}/R_{[n]};\mathbb{F}_p) \longrightarrow \mathrm{HAQ}_n(R_{[n]}/S;\mathbb{F}_p)$$

is trivial since $\operatorname{im}\bar{\theta}_n = 0$. We have therefore shown that the epimorphism

$$HAQ_n(R_{[n]}/S;\mathbb{F}_p) \longrightarrow \mathrm{HAQ}_n(R_{[n+1]}/S;\mathbb{F}_p)$$

is a monomorphism and so R is minimal. R can also be described as having no $\bmod p$ detectable homotopy in the sense that the map, $\pi_n R_{[n]} \longrightarrow \mathrm{HAQ}_n(R/S;\mathbb{F}_p)$ is trivial. This can be seen via the following diagram.

$$\begin{tikzcd}
\pi_n R_{[n]} \arrow[r, "\bar{\theta}_n"] \arrow[ddr] & \mathrm{HAQ}_n(R_{[n]}/S;\mathbb{F}_p) \arrow[d] \\
 & \mathrm{HAQ}_n(R_{[n+1]}/S;\mathbb{F}_p) \arrow[d, "\cong"] \\
 & \mathrm{HAQ}_n(R/S;\mathbb{F}_p)
\end{tikzcd} \tag{5.10}$$

Conversely, if R is minimal and has no $\bmod p$ detectable homotopy then we have isomorphism

$$\mathrm{HAQ}_n(R_{[n]}/S;\mathbb{F}_p) \xrightarrow{\cong} \mathrm{HAQ}_n(R_{[n+1]}/S;\mathbb{F}_p)$$

and using diagram 5.10 above, we also have that $\bar{\theta}_n : \pi_n R_{[n]} \longrightarrow \mathrm{HAQ}_n(R_{[n]}/S;\mathbb{F}_p)$ is trivial.

$\square$

Chapter 6

Examples of non-cores

In Chapter 5 we proved in Proposition 5.9 that for any core $g : Q \longrightarrow R$ of S-algebras, we have cores of S-modules $f : X \longrightarrow R$ and $\xi : X \longrightarrow Q$ such that $f = g \circ \xi$. In what follows, we show that the S-module core $\xi : X \longrightarrow Q$ cannot be an equivalence for particular examples of commutative S-algebras R. For each example we show that there is no map of commutative *S-algebras* $X \longrightarrow R$. We note that if ξ were an equivalence then we would have commutative S-algebra core Q of R being equivalent to X contradicting our results.

We give examples of non-cores for commutative S-algebras MU, MSU, MO and MSO. These results are motivated by Proposition 5.9 as well as [23, Proposition 2.11] where it was proven that BP is not a core of MU considered as commutative S-algebras. The proofs of the results which follow rely heavily upon the action of the Dyer–Lashof algebra on the mod p homology of the commutative S-algebras under consideration. For this reason we begin with an account of Dyer–Lashof operations including an introduction to infinite loop space theory, homology operations and leading to the definition of the Dyer–Lashof operations admitted by the homology of the infinite loop space X. We also give results based on formulae derived by Kochman [24] which we use directly in the final section containing the main results of this chapter; examples of non-cores.

6.1 Dyer–Lashof operations

The following material is based on the account of infinite loop spaces and their homology operations given in [32, Chapter 6]. Throughout this section $H_*(X)$,denotes the homology of X with $\mathbb{F}_2$-

coefficients.

For an infinite loop space X, we have Dyer–Lashof operations:

$$Q^k \colon H_i(X) \longrightarrow H_{i+k}(X)$$

which are natural with respect to infinite loop maps. The definition of the operation Q^k is analogous to the definition of the mod 2 Steenrod operations, or Steenrod squares:

$$Sq^k \colon H^i(X) \longrightarrow H^{i+k}(X)$$

which exist for any space X.

Recall that, passing to homotopy classes, we find a natural one to one correspondence

$$[Y, \Omega X] \longleftrightarrow [\Sigma Y, X] \tag{6.1}$$

We can iterate this one to one correspondence to give

$$[Y, \Omega^n X] \longleftrightarrow [\Sigma^n Y, X]$$

where $\Omega^n X = \Omega(\Omega^{n-1}X)$. In particular $\Omega^n X$ is the set of basepoint preserving maps $(S^n, *) \longrightarrow (X, x_0)$.

X is an *infinite loop space* if there is a sequence of spaces $X_0, X_1, X_2, \ldots$ with $X_0 = X$ and with weak equivalences

$$X_n \xrightarrow{\simeq} \Omega X_{n+1}.$$

We can associate to each weak equivalence above, the map

$$\Sigma X_n \longrightarrow X_{n+1}.$$

We can see, recalling details from Section 1.2, that the sequence X_n is a particular type of prespectrum.

Let us consider a prespectrum E, with structure maps

$$\varepsilon_n \colon \Sigma E_n \longrightarrow E_{n+1},$$

or equivalently

$$\varepsilon_n' \colon E_n \longrightarrow \Omega E_{n+1}.$$

If the structure maps ε_n' are weak equivalences, we call E an *Ω-spectrum.* So we can say that an infinite loop space X is the 0th term of an Ω-spectrum $X = \{X_n\}$. We note that in defining a spectrum we require the structure maps to be homeomorphisms.

Eilenberg–Mac Lane spaces (Example 1.5) are actually examples of infinite loop spaces. We see this as follows. Consider $\Omega K(\pi, n+1)$ and note that

$$\pi_q(\Omega K(\pi, n+1)) \cong \pi_{q+1}(K(\pi, n+1)) = \begin{cases} \pi & q = n, \\ 0 & q \neq n. \end{cases}$$

There is therefore a weak equivalence $K(\pi, n) \longrightarrow \Omega K(\pi, n+1)$.

Let X be an infinite loop space, that is, a single space in an Ω-spectrum. We have a natural embedding of $\Omega^i \Sigma^i X$ in $\Omega^{i+1}\Sigma^{i+1}X$. Let us take

$$Q(X) = \varinjlim(\Omega^n \Sigma^n(X))$$

that is, $Q(X)$ denotes the direct limit of the spaces $\Omega^n\Sigma^n X$ under this embedding and $\pi_i(Q(X)) = \pi_i^s(X)$, the ith stable homotopy group of X. Dyer and Lashof [18] were the first to study the structure of this infinite loop space and gave a geometric construction $C(X)$ which is taken as a model for $Q(X)$. This construction is reviewed in [32, Chapter 3B].

We obtain the following structure maps for infinite loop space X from the Dyer–Lashof model for $Q(X)$:

$$d_2 \colon E\Sigma_2 \times_{\Sigma_2} X \times X \longrightarrow X, \tag{6.2}$$

where Σ_2 is the symmetric group on two letters, $E\Sigma_2$ is the covering space for $B\Sigma_2$ and Σ_2 acts on $X \times X$ by permutation of coordinates. We have, by [32, Lemma 3.20], that $H_*(E\Sigma_2 \times_{\Sigma_2} X \times X)$ is generated by elements $e_i \otimes a \otimes a$ and $e_0 \otimes a \otimes b$ for $a, b \in H_*(X)$.

We now make the following definition for the Dyer–Lashof operations.

Definition 6.1. Let X be an infinite loop space. The jth *lower* Dyer–Lashof operation

$$Q_j \colon H_i(X) \longrightarrow H_{j+2i}(X)$$

is defined by

$$Q_j(a) = (d_2)_*(e_j \otimes a \otimes a)$$

where $a \in H_i(X)$ and $(d_2)_*$ is the map induced by d_2 in homology, that is,

$$(d_2)_* \colon H_*(E\Sigma_2 \times_{\Sigma_2} X \times X) \longrightarrow H_*(X).$$

Many of the properties of the Dyer–Lashof operations are usually and, in the case of the Adem relations, more conveniently described in the notation of the *upper* Dyer–Lashof relations, denoted Q^k. We have the following connection between the upper and lower operations.

$$Q^k(a) = Q_{k-|a|}(a) \quad |a| = \deg(a)$$

So, if we take $a \in H_i(X)$,we get

$$Q^k(a) = Q_{k-i}(a) \in H_{(k-i)+2i}(X) = H_{k+i}(X),$$

and so,

$$Q^k \colon H_i(X) \longrightarrow H_{k+i}(X)$$

that is, Q^k raises degree by k.

The Dyer–Lashof operations satisfy many properties, including the following.

For $x \in H_n(X)$,

$$Q^n(x) = x^2 \text{ if } \deg x = n$$

where the product is the Pontrjagin product. We also have

$$Q^0(\phi) = \phi \text{ and } Q^n(\phi) = 0 \text{ if } n > 0,$$

where $\phi \in H_0(X)$ is the identity element for the multiplication in $H_*(X)$. These operations also satisfy the multiplicative Cartan formula, given by

$$Q^r(xy) = \sum_{i=0}^{r} Q^i(x)Q^{r-i}(y),$$

as well as Adem relations, namely

$$Q^kQ^l(a) = \Sigma(-1)^{k+\nu}\binom{\nu-l-1}{2\nu-k}Q^{k+l-\nu}Q^\nu(a)$$

for $k > 2l$ and all $a \in H_*(X)$. We note at this stage that Kochman [24] chooses an alternative notation for the binomial coefficients throughout his paper, he writes

$$(i,j) = \begin{cases} (i+j)!/i!j! & i \geq 0 \text{ and } \mathrm{j} \geq 0, \\ 0 & i < 0 \text{ or } \mathrm{j} < 0. \end{cases}$$

6.2 Dyer–Lashof Operations for Cobordism Thom spectra

We consider non-cores for the cobordism Thom spectra in the following section. Kochman computes the Dyer–Lashof algebra on the homology of the infinite classical groups in [24]. We have, by a result of Lewis [30, IX, Proposition 7.4], that the Thom isomorphisms commute with Dyer–Lashof operations. We therefore use results in [24] to give the action of the Dyer–Lashof algebra R, generated by Dyer–Lashof operations Q^n, on $H_*(MU;\mathbb{F}_2)$, $H_*(MSU;\mathbb{F}_2)$, $H_*(MO;\mathbb{F}_2)$ and $H_*(MSO;\mathbb{F}_2)$.

Notation 6.2. We write $a_1, a_2, a_3, \ldots$ for the standard sequence of generators for $H_*(MU;\mathbb{Z})$. Thus

$$H_*(MU;\mathbb{Z}) = \mathbb{Z}[a_1, a_2, a_3, \ldots],$$

is a polynomial ring in which $\deg a_i = 2i$. We shall need to work mod p and by abuse of notation we shall again denote the mod p generator corresponding to a_i by a_i. The mod 2 homology ring $H_*(MSU;\mathbb{F}_2)$ can be identified with a subring of $H_*(MU;\mathbb{F}_2)$. This is also a polynomial ring

$$H_*(MSU;\mathbb{F}_2) = \mathbb{F}_2[b_2, b_3, b_4, \ldots],$$

in which the generators b_i have degree $2i$ and are related to the a_i by

$$b_i = a_i + \text{ terms involving } a_j\text{'s with } j < i.$$

If we write $\mathfrak{m}$ for the ideal of $H_*(MU,\mathbb{F}_p)$ generated by all the a_i then this says that (in case $p = 2$)

$$b_i \equiv a_i \mod \mathfrak{m}^2,$$

or equivalently in Kochman's terminology [24], $b_i \equiv a_i$ modulo decomposables.

From [24, Theorem 6 and Theorem 18] we have the following theorem.

Theorem 6.3. *In $H_*(MU;\mathbb{F}_p)$ and with $r \geq 0$ and $n \geq 1$, we have*

$$Q^r(a_n) = (-1)^{r+n+1}\binom{r-1}{n} a_{n+r(p-1)} \mod \mathfrak{m}^2, \tag{6.3}$$

for an odd prime p and

$$Q^{2r}(a_n) = \binom{r-1}{n} a_{n+r} \mod \mathfrak{m}^2 \tag{6.4}$$

for $p = 2$.

Then we have in $H_*(MSU;\mathbb{F}_2)$ *for* $r \geq 0$, $n \geq 2$ *and* $k \geq 1$*:*

$$Q^{2r}(a_n) = \binom{r-1}{n} a_{n+r} \mod \mathfrak{m}^2 \tag{6.5}$$

if n *is not a power of* 2 *and*

$$Q^r(a_{2^k}) \in \mathfrak{m}^2. \tag{6.6}$$

Notation 6.4. We write $z_1, z_2, z_3, \ldots$ for the standard sequence of generators for $H_*(MO;\mathbb{F}_2)$. Thus

$$H_*(MO;\mathbb{F}_2) = \mathbb{Z}[z_1, z_2, z_3, \ldots],$$

is a polynomial ring in which $\deg z_n = n$. We shall always work mod 2. The mod 2 homology ring $H_*(MSO;\mathbb{F}_2)$ can be identified with a subring of $H_*(MO;\mathbb{F}_2)$. This is also a polynomial ring

$$H_*(MSO;\mathbb{F}_2) = \mathbb{F}_2[y_2, y_3, y_4, \cdots],$$

in which the generators y_n have degree n and are related to the z_n by

$$y_n = z_n + \text{ terms involving } z_m\text{'s with } m < n.$$

If we write $\mathfrak{j}$ for the ideal of $H_*(MO,\mathbb{F}_2)$ generated by all the z_n then this says that

$$y_n \equiv z_n \mod \mathfrak{j}^2,$$

or equivalently in Kochman's terminology [24], $y_n \equiv z_n$ modulo decomposables.

From [24, Theorem 36 and Theorem 53] we have the following theorem.

Theorem 6.5. *In* $H_*(MO;\mathbb{F}_2)$ *and with* $r \geq 0$ *and* $n \geq 1$*, we have*

$$Q^r(z_n) = \binom{r-1}{n} z_{n+r} \mod \mathfrak{j}^2 \tag{6.7}$$

Then we have in $H_*(MSO;\mathbb{F}_2)$ *and with* $r \geq 0$ *and* $n \geq 2$ *:*

$$Q^r(y_n) = \binom{r-1}{n} y_{y+r} \mod \mathfrak{j}^2 \tag{6.8}$$

if n *is not a power of* 2 *and*

$$Q^r(y_{2^k}) \in \mathfrak{j}^2 \tag{6.9}$$

6.3 Examples of non-cores

6.3.1 MU

We can consider Proposition 5.9 in the previous chapter for commutative S-algebra MU: by taking $R = MU$ and any commutative S-algebra core $g\colon Q \longrightarrow MU$, we have a factorization of an S-module core $f\colon BP \longrightarrow MU$ as $g \circ \xi$ for a core $\xi\colon BP \longrightarrow Q$ of S-modules. If ξ were an equivalence then we would have any commutative S-algebra core Q of MU being equivalent to BP, which would contradict the following result, given in [23, Proposition 2.11].

Proposition 6.6. *(Hu, Kriz, May) There is no map $g\colon BP \longrightarrow MU$ of commutative S-algebras.*

The proof of Proposition 6.6 above will be mirrored for the examples which follow. It is assumed that there is such a map $g\colon BP \longrightarrow MU$ of commutative S-algebras. By hypothesis BP and MU are both commutative S-algebras and so the map $g_*\colon H_*(BP;\mathbb{F}_p) \longrightarrow H_*(MU;\mathbb{F}_p)$ on mod p homology would be a monomorphism that commutes with Dyer–Lashof operations. Using the Dyer–Lashof operations on $H_*(MU)$ computed by Kochman [24] and re-stated in 6.3, it is found that the commutativity condition is not satisfied and we conclude that there is no such map $g\colon BP \longrightarrow MU$ of commutative S-algebras.

6.3.2 MSU

We consider a further example; $R = MSU$, the Thom spectrum associated with special unitary cobordism, and ask the analogous question to [23, Proposition 2.11]. The 2-localization $MSU_{(2)}$ is equivalent to a wedge of suspensions of BP and an indecomposable 2-local spectrum BoP. In [38], Pengelley constructs BoP by applying the Sullivan–Bass construction to MSU to produce a spectrum whose localization is closely related to the indecomposable spectrum BoP.

Proposition 6.7. *There is no map $f\colon BoP \longrightarrow MSU$ of commutative S-algebras.*

Proof. Let us suppose that there is such a map f. Since BoP and MSU are commutative S-algebras, they have unit maps $\eta\colon S \longrightarrow BoP$ and $\varepsilon\colon S \longrightarrow MSU$. The map f will commute with the units, forming the following commutative diagram

$$\begin{tikzcd} S \arrow[r, "\eta"] \arrow[dr, "\varepsilon"'] & BoP \arrow[d, "f"] \\ & MSU \end{tikzcd}$$

which gives us

$$
\begin{tikzcd}
\pi_0(S) \arrow[r, "\eta_*"] \arrow[dr, "\varepsilon_*"'] & \pi_0(BoP) \arrow[d, "f_*"] \\
& \pi_0(MSU)
\end{tikzcd}
$$

and since BoP and MSU are both commutative S-algebras, whose units induce isomorphisms on π_0, we have that f_* is an isomorphism. In [5], Baker and May construct BoP as a nuclear spectrum and show in [5, Example 6.1] that BoP is a core of MSU as S-modules. By Theorem 3.13 [5, Proposition 2.3], a nuclear S-module is atomic, and so any self map $BoP \longrightarrow BoP$ is an equivalence. If we consider the composite of f and a splitting map $MSU \longrightarrow BoP$, we have a self equivalence of BoP. So we have that f is the inclusion of a retract and so the map

$$f_*\colon H_*(BoP;\mathbb{F}_2) \longrightarrow H_*(MSU;\mathbb{F}_2)$$

on mod 2 homology would be a monomorphism that commutes with Dyer–Lashof operations. The Thom isomorphism $\theta\colon H_*(MSU) \longrightarrow H_*(BSU)$ commutes with the Dyer–Lashof operations, by a result of Lewis. Kochman [24] has computed the Dyer–Lashof operations on $H_*(BSU;\mathbb{F}_p)$ and hence on $H_*(MSU;\mathbb{F}_p)$, for any prime p. We refer to Notation 6.2 and Theorem 6.3. Pengelley [37] makes the following identification

$$H_*(BoP;\mathbb{F}_2) = B \otimes \overline{Y}.$$

Here B is isomorphic to

$$\mathbb{F}_2[\zeta_1^4, \zeta_2^2, ..., \zeta_j^2, ...]$$

as a comodule over the dual Steenrod algebra A (where ζ_j is the conjugate of Milnor's generator ξ_j, and so $\deg \xi_j = 2^j - 1$) and $\overline{Y}$ is the exterior algebra

$$\overline{Y} = \mathbb{F}_2[x_{2^r}; r \geq 3, \deg x_{2^r} = 2^r].$$

Let us consider the generator of $H_*(BoP;\mathbb{F}_2)$ in degree 6, we can apply the monomorphism $f_*\colon H_*(BoP;\mathbb{F}_2) \longrightarrow H_*(MSU;\mathbb{F}_2)$ to this element to give an element of degree 6 that is equivalent to a_3 modulo $\mathfrak{m}^2$. We can now consider $Q^{16}(a_3)$, which by 6.5 gives the following

$$
\begin{aligned}
Q^{16}(a_3) &= \binom{7}{3} a_{11} \mod \mathfrak{m}^2 \\
&= 35\, a_{11} \mod \mathfrak{m}^2 \\
&= a_{11} \mod \mathfrak{m}^2
\end{aligned}
$$

Applying Q^{16} to our element of degree 6 in $H_*(BoP;\mathbb{F}_2)$, we get an element of degree 22, however $H_*(BoP;\mathbb{F}_2)$ has no indecomposable elements in degree 22, and so we have 0 mod decomposables, therefore, f_* does not commute with Dyer–Lashof operations. Hence there is no map $f\colon BoP \longrightarrow MSU$ of commutative S-algebras. $\square$

6.3.3 MO

Thom showed that MO, the spectrum representing unoriented bordism, is a wedge of mod 2 Eilenberg–Mac Lane spectra $H\mathbb{F}_2$.

Proposition 6.8. *There is no map $h\colon H\mathbb{F}_2 \longrightarrow MO$ of commutative S-algebras.*

Proof. Suppose there is such a map h of commutative S-algebras. By a similar argument to that in Proposition 6.7 and noting that $H\mathbb{F}_2$ is atomic as an S-module we have that $h_*\colon H_*(H\mathbb{F}_2;\mathbb{F}_2) \longrightarrow H_*(MO;\mathbb{F}_2)$ is a monomorphism that commutes with the Dyer–Lashof operations. We recall Notation 6.4 and note that we also have

$$
H_*(H\mathbb{F}_2;\mathbb{F}_2) = A = \mathbb{F}_2[\xi_i; i \geq 1, \deg \xi_i = 2^i - 1]
$$

where A is the dual Steenrod algebra. Consider $\xi_1 \in H_1(H\mathbb{F}_2;\mathbb{F}_2)$. We can apply the monomorphism $h_*\colon H_*(H\mathbb{F}_2;\mathbb{F}_2) \longrightarrow H_*(MO;\mathbb{F}_2)$ to ξ_1 to give $h_*(\xi_1) \in H_1(MO;\mathbb{F}_2)$, that is, an element of degree 1, so $h_*(\xi_1) = z_1$. We can now consider $Q^4(z_1)$, which by (6.7) gives the following

$$
\begin{aligned}
Q^4(z_1) &= \binom{3}{1} z_5 \mod \mathfrak{j}^2 \\
&= 3\, z_5 \mod \mathfrak{j}^2 \\
&= z_5 \mod \mathfrak{j}^2
\end{aligned}
$$

Applying Q^4 to $\xi_1 \in H_1(H\mathbb{F}_2;\mathbb{F}_2)$, we get an element of degree 5, however $H_*(H\mathbb{F}_2;\mathbb{F}_2))$ has no indecomposable elements in degree 5, and so $Q^4(\xi_1) \equiv 0 \mod \mathfrak{j}^2$, and so $h_*(Q^4(\xi_1)) \equiv 0 \mod \mathfrak{j}^2$. So we have that h_* does not commute with Dyer–Lashof operations. Hence there is no map $h\colon H\mathbb{F}_2 \longrightarrow MO$ of commutative S-algebras. $\square$

6.3.4 MSO

Thom showed that MSO, the spectrum representing oriented bordism, when localized at the prime 2 is a wedge of integral and mod 2 Eilenberg–Mac Lane spectra $H\mathbb{F}_2$.

Proposition 6.9. *There is no map $j\colon H\mathbb{Z} \longrightarrow MSO$ of commutative S-algebras.*

Proof. Suppose there is such a map j of commutative S-algebras. By a similar argument to that in Proposition 6.7 and noting that $H\mathbb{Z}$ is atomic as an S-module we have that $j_*\colon H_*(H\mathbb{Z};\mathbb{F}_2) \longrightarrow H_*(MSO;\mathbb{F}_2)$ is a monomorphism that commutes with the Dyer–Lashof operations. We recall Notation 6.4 and Theorem 6.5 and note that we also have

$$H_*(H\mathbb{Z};\mathbb{F}_2) = \mathbb{F}_2[\xi_1^2, \xi_2, \xi_3, \ldots; \deg \xi_i = 2^i - 1] \subseteq A$$

where A is the dual Steenrod algebra. Consider $\xi_2 \in H_3(H\mathbb{Z};\mathbb{F}_2)$. We can apply the monomorphism $j_*\colon H_*(H\mathbb{Z};\mathbb{F}_2) \longrightarrow H_*(MSO;\mathbb{F}_2)$ to ξ_2 to give $j_*(\xi_2) \in H_3(MSO;\mathbb{F}_2)$, that is, an element of degree 3, so $j_*(\xi_2) = y_3 \mod \mathfrak{j}^2$. We can now consider $Q^8(y_3)$, which by 6.8 gives the following

$$\begin{aligned} Q^8(y_3) &= \binom{7}{3} y_{11} \mod \mathfrak{j}^2 \\ &= 35\, y_{11} \mod \mathfrak{j}^2 \\ &= y_{11} \mod \mathfrak{j}^2 \end{aligned}$$

Applying Q^8 to $\xi_2 \in H_3(H\mathbb{Z};\mathbb{F}_2)$, we get an element of degree 11, however $H_*(H\mathbb{Z};\mathbb{F}_2))$ has no indecomposable elements in degree 11, and so $Q^8(\xi_2) \equiv_{(2)} 0 \mod \mathfrak{j}^2$, and so $j_*(Q^8(\xi_2)) \equiv_{(2)} 0 \mod \mathfrak{j}^2$. So we have that j_* does not commute with Dyer–Lashof operations. Hence there is no map $j\colon H\mathbb{Z} \longrightarrow MSO$ of commutative S-algebras. $\square$

Chapter 7

Further calculations

In this chapter we explore the theory of cellular commutative S-algebras set up in Chapters 4 and 5. In particular we consider MU, the Thom spectrum associated with unitary cobordism. We have already met MU as a commutative S-algebra in Chapter 5, and we note in Chapter 6 that Hu, Kriz and May [23] showed by contradiction that there is no map $g : BP \longrightarrow MU$ of commutative S-algebras. MU was originally constructed as an E_∞ ring spectrum, but, as in Chapter 5, can be thought of as a commutative S-algebra. The homotopy groups of MU are concentrated in even degrees. We focus on one homotopy element $x_2 \in \pi_4 MU$ and construct $MU \wedge_{\mathbb{P}_{MU} S^4} \mathbb{P}_{MU} \mathrm{C}\, S^4$, denoted by $MU//x_2$. The material in this chapter is designed to provide an insight into some of the techniques required to investigate $MU//x_2$ and other similar constructions. It would also be interesting to consider the relationship between MU and ku, the spectrum representing complex K-theory KU, by killing homotopy elements of MU via a similar pushout construction.

We begin in Section 7.1.1 by calculating the homotopy of commutative MU algebra $MU//x_2$ via a Künneth spectral sequence. In Section 7.1.2 we continue this examination of $MU//x_2$ by considering one possible approach to the calculation of $\mathrm{HAQ}_k(MU//x_2/S)$. We discuss the possibility of using results on long exact sequences from Chapter 4 and note that this may require the calculation of $H_*(ku)$, leading us to consider the Hurewicz homomorphism $h : \pi_*(MU) \longrightarrow H_*(MU)$ and its image.

In Section 7.2 we examine Dyer–Lashof operations on a_2, the generator of degree 4 in $H_*(MU; \mathbb{F}_2)$. The element a_2 is the image of $x_2 \in \pi_4(MU)$ under the Hurewicz homomorphism. The techniques used in this section could be used in establishing a full description of $H_*(MU)$ in terms of the allowable Dyer–Lashof operations.

7.1 Calculations on $MU//x_2$

7.1.1 Calculation of $\pi_*(MU//x_2)$

This section serves as an introduction to the techniques employed later in this chapter. We construct a commutative MU-algebra via a pushout construction involving a homotopy element. We then use a Künneth spectral sequence to arrive at the final answer. For efficiency we write X_* for $\pi_*(X)$.

Consider the commutative S-algebra MU and take homotopy element $x_2 \in \pi_4 MU$. We can think of x_2 as a map $S^4 \longrightarrow MU$ from the 4-sphere MU-module $S^4 = S^4_{MU}$ to MU itself. Recall from Chapter 4, that for any MU-module X, there is a free commutative MU-algebra on X, denoted $\mathbb{P}_{MU}X$. Taking the MU-sphere S^4_{MU} we obtain commutative MU-algebra $\mathbb{P}_{MU}S^4$. The MU-algebra map

$$\mathbb{P}_{MU}S^4 \xrightarrow{\tilde{x_2}} \mathbb{P}_{MU}* = MU$$

is induced by collapsing S^4 to a point. We therefore have MU as a $\mathbb{P}_{MU}S^4$-algebra, allowing us to view an MU-module as a $\mathbb{P}_{MU}S^4$-module. Again, we have $\mathrm{C}\,S^4$ as the cone on MU-module S^4 and ι to be the canonical inclusion.

We can construct the commutative MU-algebra $MU \wedge_{\mathbb{P}_{MU}S^4} \mathbb{P}_{MU}\,\mathrm{C}\,S^4$ as the pushout of the following diagram:

$$\mathbb{P}_{MU}\,\mathrm{C}\,S^4 \xleftarrow{\mathbb{P}\iota} \mathbb{P}_{MU}S^4 \xrightarrow{\tilde{x_2}} MU.$$

Let us denote $MU \wedge_{\mathbb{P}_{MU}S^4} \mathbb{P}_{MU}\,\mathrm{C}\,S^4$ as $MU//x_2$. The reason for this notation will become clear as we proceed.

In order to calculate $\pi_*(MU//x_2)$ we use the Künneth spectral sequence, given and constructed in [19, IV,(6.1)].

$$E^2_{s,t} = \mathrm{Tor}^{(\mathbb{P}_{MU}S^4)_*}_{s,t}(MU_*, (\mathbb{P}_{MU}\,\mathrm{C}\,S^4)_*) \Longrightarrow \pi_*(MU \wedge_{\mathbb{P}_{MU}S^4} \mathbb{P}_{MU}\,\mathrm{C}\,S^4) = \pi_*(MU//x_2). \qquad (7.1)$$

We have that $(\mathbb{P}_{MU}S^4)_* = MU_*[u]$ where u is a polynomial generator in degree 4. We also have the following equivalence of $\mathbb{P}_{MU}S^4$-algebras

$$\mathbb{P}_{MU}\,\mathrm{C}\,S^4 \simeq \mathbb{P}_{MU}* = MU.$$

We have the maps

$$\begin{aligned} (\mathbb{P}_{MU}S^4)_* &\longrightarrow (\mathbb{P}_{MU}\,\mathrm{C}\,S^4)_*; \\ u &\longmapsto 0 \end{aligned} \tag{7.2}$$

and

$$\begin{aligned} (\mathbb{P}_{MU}S^4)_* &\longrightarrow MU_*; \\ u &\longmapsto x_2. \end{aligned} \tag{7.3}$$

Hence we adopt the following notation; we write $(\mathbb{P}_{MU}\,\mathrm{C}\,S^4)_*$ as MU_* in (7.2) and MU_* as $(MU_*)_{x_2}$ in (7.3). We consider both as $(\mathbb{P}_{MU}S^4)_*$-modules, giving the following;

$$E^2_{s,t} = \mathrm{Tor}_{s,t}^{(\mathbb{P}_{MU}S^4)_*}((MU_*)_{x_2}, MU_*) \Rightarrow \pi_*(MU \wedge_{\mathbb{P}_{MU}S^4} \mathbb{P}_{MU}\,\mathrm{C}\,S^4) = \pi_*(MU//x_2). \tag{7.4}$$

In order to proceed with our spectral sequence calculation we need to form a free (Koszul) resolution for MU_*, recalling that it is a $(\mathbb{P}_{MU}S^4)_*$-module, that is, an $MU_*[u]$-module. We consider the following complex

$$0 \longrightarrow MU_*[u] \xrightarrow{d} MU_*[u] \longrightarrow 0.$$

It is useful to give separate names to the generators of the free modules, so we write;

$$0 \longrightarrow MU_*[u]e_1 \xrightarrow{d} MU_*[u]e_0 \longrightarrow 0 \tag{7.5}$$

for the Koszul resolution of MU_*. The differential is given by

$$d(e_1) = ue_0,$$

but we could also write $d(e_1) = u$.

Now we take the Koszul resolution for MU_* (7.5) and tensor with $(MU_*)_{x_2}$ to yield

$$0 \longrightarrow (MU_*)_{x_2} \otimes_{MU_*[u]} MU_*[u]e_1 \xrightarrow{id\otimes d} (MU_*)_{x_2} \otimes_{MU_*[u]} MU_*[u]e_0 \longrightarrow 0.$$

This gives us the following complex, which we shall denote as T_*.

$$\begin{aligned} 0 \longrightarrow (MU_*)_{x_2}e_1 &\xrightarrow{\mathrm{id}\otimes d} (MU_*)_{x_2}e_0 \longrightarrow 0 \\ (a)e_1 &\longmapsto (a)ue_0 = (a)x_2e_0 \end{aligned} \tag{7.6}$$

Using this complex, we calculate the following.

$$\begin{aligned}
\mathrm{Tor}_{0,t}^{(\mathbb{P}_{MU}S^4)_*}((MU_*)_{x_2}, MU_*) &= H_0(T_*) \\
&= (MU_*)_{x_2}/(MU_*)_{x_2}x_2 && (7.7) \\
\mathrm{Tor}_{1,t}^{(\mathbb{P}_{MU}S^4)_*}((MU_*)_{x_2}, MU_*) &= H_1(T_*) \\
&= \{a \in (MU_*)_{x_2} : ax_2 = 0\} && (7.8) \\
&= 0
\end{aligned}$$

And hence, $\pi_*(MU//x_2) = (MU_*)_{x_2}/(MU_*)_{x_2}x_2$. This conclusion makes explicit the reason for the chosen notation $MU//x_2$.

7.1.2 Working towards $\mathrm{HAQ}_k(MU//x_2/S)$

This section provides an insight into one possible approach to calculating $\mathrm{HAQ}_k(MU//x_2/S)$. It was hoped the material that follows could be used, along with results on long exact sequences from Chapter 4, to perform the afore mentioned calculation.

To illustrate, we begin with one such long exact sequence, as in 4.6a. This sequence is associated to an A-algebra map $B \longrightarrow C$.

$$\cdots \longrightarrow \mathrm{HAQ}_k(B/A) \longrightarrow \mathrm{HAQ}_k(C/A) \longrightarrow \mathrm{HAQ}_k(C/B) \longrightarrow \mathrm{HAQ}_{k-1}(B/A) \longrightarrow \cdots \qquad (7.9)$$

In the previous section we constructed the commutative MU-algebra $MU \wedge_{\mathbb{P}_{MU}S^4} \mathbb{P}_{MU} \, \mathrm{C}\, S^4$ via a pushout construction and denoted it by $MU//x_2$. Now we consider the map $i : MU \longrightarrow MU//x_2$ as a map of S-algebras, allowing us to arrive at the following long exact sequence.

$$\begin{aligned}
\cdots \longrightarrow \mathrm{HAQ}_k(MU/S) \longrightarrow \mathrm{HAQ}_k(MU//x_2/S) \longrightarrow \mathrm{HAQ}_k(MU//x_2/MU) \\
\longrightarrow \mathrm{HAQ}_{k-1}(MU/S) \longrightarrow \cdots
\end{aligned}$$

We have that $\pi_k C_i = 0$ for $k \leq 4$ where C_i is the mapping cone of i in $\mathscr{M}_A$ and we use a version of the Hurewicz isomorphism theorem as given in Corollary 4.6 to find that $\pi_5 C_i \cong \mathrm{HAQ}_5(MU//x_2/MU)$. We can therefore consider the following portion of the long exact sequence.

$$\begin{aligned}
\cdots \longrightarrow \mathrm{HAQ}_5(MU/S) \longrightarrow \mathrm{HAQ}_5(MU//x_2/S) \longrightarrow \mathrm{HAQ}_5(MU//x_2/MU) \\
\longrightarrow \mathrm{HAQ}_4(MU/S) \longrightarrow \mathrm{HAQ}_4(MU//x_2/S) \longrightarrow \mathrm{HAQ}_4(MU//x_2/MU) = 0
\end{aligned}$$

We notice that we can apply a corollary from [7, page 5] stating that the cotangent complex $\Omega_{MU/S}$ is the MU-module $MU \wedge \Sigma^2 ku$. By equations 4.5 and this result, we have

$$\begin{aligned} \mathrm{HAQ}_k(MU/S) &= \pi_k(\Omega_{MU/S} \wedge_{MU} H\mathbb{Z}) \\ &= \pi_k(MU \wedge \Sigma^2 ku \wedge_{MU} H\mathbb{Z}) \\ &= \pi_k(\Sigma^2 ku \wedge H\mathbb{Z}) \\ &= H_{k-2}(ku). \end{aligned}$$

We progress this section by developing the calculation of $H_*(ku)$, where $H = H\mathbb{Z}$. We begin with the Künneth spectral sequence, involving the Hurewicz homomorphism. The material on the Hurewicz homomorphism will also be of use in the last section of this chapter.

We use the Künneth spectral sequence, as given below.

$$E^2_{s,t} = Tor^{MU_*}_{s,t}(H_*(MU), ku_*) \Rightarrow \pi_*((H \wedge MU) \wedge_{MU} ku) = \pi_*(H \wedge ku) = H_*(ku), \tag{7.10}$$

We recall that $H_*(MU) = \pi_*(H\mathbb{Z} \wedge MU)$.

We are now required to form a free resolution of ku_* over MU_*. We use the following.

$$\pi_*(ku) = \mathbb{Z}[t], \deg t = 2 \tag{7.11}$$

$$\pi_*(MU) = \mathbb{Z}[x_i], i \geq 1, \deg x_i = 2i \tag{7.12}$$

Let us choose generators as below.

$$\begin{aligned} MU_* &\longrightarrow ku_* \\ x_1 &\longmapsto t \qquad (7.13) \\ x_i &\longmapsto 0, i > 1. \qquad (7.14) \end{aligned}$$

We can attempt to follow the same recipe used in the calculation in Section 1. We begin by forming a Koszul resolution for ku_*, recalling that it is a MU_*-module. We should note however, that in this case we are looking for a free resolution of $ku_* = MU_*/\underline{x}MU_*$, where $\underline{x}$ is regular sequence $(x_2, x_3, \cdots)$.

We use the classical Koszul resolution $K(\underline{x})$ given in [46, Corollary 4.5.5]. The degree $(p-1)$ part of $K(\underline{x})$ is a free MU_* module generated by the symbols

$$e_2 \wedge \cdots \wedge e_p.$$

The differential

$$K_{p-1}(\underline{x}) \longrightarrow K_{p-2}(\underline{x})$$

sends

$$e_2 \wedge \cdots \wedge e_p$$

to

$$\Sigma(-1)^{k+1} x_k e_2 \wedge \cdots \wedge \widehat{e_k} \wedge \cdots \wedge e_p$$

for $k \geq 2$. So, for example the differential

$$K_3(\underline{x}) \longrightarrow K_2(\underline{x})$$

would map

$$e_2 \wedge e_3 \wedge e_4$$

to

$$-x_2 e_3 \wedge e_4 + x_3 e_2 \wedge e_4 - x_4 e_2 \wedge e_3.$$

We therefore have differential map $d : K_p(\underline{x}) \longrightarrow K_{p-1}(\underline{x})$ given by $d(e_i) = x_i$ and satisfying the Leibnitz rule.

We take the Koszul resolution $K(\underline{x})$ for ku_* and tensor over MU_* giving $H_*(MU) \otimes_{MU_*} K(\underline{x})$. On taking homology, this gives the following.

$$\begin{aligned} E^2_{**} &= H_*(H_*(MU) \otimes_{MU_*} \bigwedge_{MU_*} (e_i : i \geq 2)) \\ &= H_* \left(\bigwedge_{H_*(MU)} (e_i : i \geq 2) \right) \end{aligned}$$

For the complex in the spectral sequence above, the differential is given by $de_i = h(x_i)$ where $h : \pi_*(MU) \longrightarrow H_*(MU)$ is the Hurewicz homomorphism.

In order to calculate the E^2_{**} terms, we need to understand the differential $de_i = h(x_i)$, that is, we require the image of x_i under the Hurewicz homomorphism. Kozma [26] constructs explicit polynomial generators for $\pi_*(MU)$. Based on Notation 6.2 we have;

$$H_*(MU; \mathbb{Z}) = \mathbb{Z}[a_1, a_2, a_3, \ldots],$$

in which $\deg a_i = 2i$. Kozma defines elements $T_{p,i} \in H_{2(pi-1)}(MU)$ for every prime p by the induction formula

$$T_{p,i} = pa_{pi-1} - \sum_{sd=1, d<i} a_{s-1} T_{p,d}^s. \tag{7.15}$$

Kozma then proves the following theorem.

Theorem 7.1. [26] *$T_{p,i} \in \mathrm{Im}(h)$ for every i and p.*

We use a theorem due to Witt–Milnor, see [42], which states that a is a polynomial generator in dimension $2n$ if and only if $a \equiv \lambda a_n$ mod reducibles, where $\lambda = p$ if $n+1 = p^k$ for a prime p and $\lambda = 1$ otherwise. Using this theorem, we find the following; T_{p,p^k} is a polynomial generator for $\mathrm{Im}(h)$ in dimension $2(p^{k+1}-1)$. If $n+1$ is divisible by two primes $p \neq q$ (if $n+1 \neq p^k$, then $n+1 =$ a product of prime powers) write $n+1 = ps_1 = qs_2$ to give the elements $T_{p,s_1}, T_{q,s_2} \in H_{2n}(MU)$, and so a suitable linear combination of T_{p,s_1} and T_{q,s_2} will be a polynomial generator in dimension $2n$ (if $aT_{p,s_1} + bT_{q,s_2}$ is a polynomial generator in dimension $2n$, then $ap + bq = 1$ for $\lambda = 1$).

We can consider the following examples;

Example 7.2. Take $p = 2$ and $k = 1$, giving $n = 3$. And so we have $n + 1 = 4 = 2^2$.

$$\begin{aligned} T_{2,2} &= 2a_3 - \sum_{sd=2} a_{s-1} T_{2,d}^s \\ &= 2a_3 - a_1 T_{2,1}^2 \\ &= 2a_3 - 4a_1^3 \end{aligned}$$

since $T_{2,1} = 2a_1$.

$T_{2,2}$ is a polynomial generator in degree $2(2^2 - 1) = 6$.

Example 7.3. Take $p = 2$ and $k = 2$, giving $n = 7$. And so we have $n + 1 = 8 = 2^3$.

$$\begin{aligned} T_{2,4} &= 2a_7 - \sum_{sd=4} a_{s-1} T_{2,d}^s \\ &= 2a_7 - a_3 T_{2,1}^4 - a_1 T_{2,2}^2 \\ &= 2a_7 - a_3(2a_1)^4 - a_1(2a_3 - 4a_1^3)^2 \\ &= 2a_7 - 16a_1^4 a_3 - 4a_1 a_3^2 + 16a_1^4 a_3 - 16a_1^7 \end{aligned}$$

$T_{2,4}$ is a polynomial generator in degree $2(2^3 - 1) = 14$.

Example 7.4. Take $p = 3$ and $k = 1$, giving $n = 8$. And so we have $n + 1 = 9 = 3^2$.

$$\begin{aligned} T_{3,3} &= 3a_8 - \sum_{sd=3} a_{s-1} T_{3,d}^s \\ &= 3a_8 - a_2 T_{3,1}^3 \\ &= 3a_8 - a_2(3a_2)^3 \end{aligned}$$

since $T_{3,1} = 3a_2$. $T_{3,3}$ is a polynomial generator in degree $2(3^2 - 1) = 16$.

Now consider two examples where $n + 1 \neq p^k$;

Example 7.5. Take $n + 1 = 6 = 2.3$, and so a suitable linear combination of $T_{2,3}$ and $T_{3,2}$ will be a polynomial generator in dimension $2n = 10$.

$$\begin{aligned} T_{2,3} &= 2a_5 - \sum_{sd=3} a_{s-1} T_{2,d}^s \\ &= 2a_5 - a_2 T_{2,1}^3 \\ &= 2a_5 - a_2(2a_1)^3 \\ &= 2a_5 - 8(a_1)^3 a_2 \end{aligned}$$

$$\begin{aligned} T_{3,2} &= 3a_5 - \sum_{sd=2} a_{s-1} T_{3,d}^s \\ &= 3a_5 - a_1 T_{3,1}^2 \\ &= 3a_5 - a_1(3a_2)^2 \\ &= 3a_5 - 9a_1 a_2^2 \end{aligned}$$

Hence we have that

$$T_{3,2} - T_{2,3} = a_5 - 9a_1 a_2^2 + 8a_1^3 a_2$$

is a polynomial generator in degree 10.

Example 7.6. Take $n + 1 = 10 = 2.5$, and so a suitable linear combination of $T_{2,5}$ and $T_{5,2}$ will be a polynomial generator in dimension $2n = 18$.

$$\begin{aligned} T_{2,5} &= 2a_9 - \cdots \\ T_{5,2} &= 5a_9 - \cdots \end{aligned}$$

Hence we have that

$$3T_{2,5} - T_{5,2} = a_9 - \cdots$$

is a polynomial generator in degree 18.

7.2 Dyer–Lashof operations and $H_*(MU)$

In this section we examine the Dyer–Lashof operations on a homotopy element of MU, namely $x_2 \in \pi_4(MU)$. Our hope is that the techniques which follow could be used to establish a full description of $H_*(MU)$ in terms of the allowable Dyer–Lashof operations on x_2, denoted $Q^I x_2$.

Let us again consider generator x_2 of degree 4 in $\pi_*(MU) = \mathbb{Z}[x_i]$. In Chapter 6 we used Kochman's [24] results on the Dyer–Lashof operations on $H_*(BU;\mathbb{F}_p)$ to give the Dyer–Lashof operations on $H_*(MU;\mathbb{F}_p)$ in Theorem 6.3. We require the Hurewicz homomorphism $h : \pi_*(MU) \longrightarrow H_*(MU)$ along with these Dyer–Lashof operations on $H_*(MU;\mathbb{F}_p)$, in order to calculate the Dyer–Lashof operations on x_2.

We use Notation 6.2 and write

$$H_*(MU;\mathbb{Z}) = \mathbb{Z}[a_1, a_2, a_3, \ldots],$$

in which $\deg a_i = 2i$. We also have that the image of x_2 under the Hurewicz homomorphism is element $3a_2$. We calculate the Dyer–Lashof operations on a_2, working modulo 2. Using Theorem 6.3 we have in $H_*(MU;\mathbb{F}_2)$ and with $r \geq 0$ and $n \geq 1$,

$$Q^{2r}(a_n) = \binom{r-1}{n} a_{n+r} \mod \mathfrak{m}^2 \tag{7.16}$$

Since all the calculations that follow involve binomial coefficients modulo 2, for the binomial coefficient $\binom{a}{b}$, we can write

$$\begin{aligned} a &= a_0 + 2a_1 + 2^2 a_2 + \cdots + 2^r a_r \\ b &= b_0 + 2b_1 + 2^2 b_2 + \cdots + 2^r b_r \end{aligned}$$

where $a_i, b_i = 0, 1$. Then,

$$\binom{a}{b} \equiv_{(2)} \binom{a_0}{b_0}\binom{a_1}{b_1}\binom{a_2}{b_2}\cdots\binom{a_r}{b_r}$$

We also recall that

$$\binom{i}{j} = \begin{cases} 1 & \text{if } i = j = 0 \text{ or } i = 1, j = 0 \text{ or } i = j = 1, \\ 0 & \text{if } i = 0, j = 1. \end{cases}$$

By applying various Dyer–Lashof operations to a_2 we can attempt to reach the generators of $H_*(MU; \mathbb{F}_2)$ of higher degree. We begin by using single Dyer–Lashof operation and get the following lemma.

Lemma 7.7. *For $q \geq 1$, we have, working modulo decomposables;*

$$Q^{2(4q+1)}(a_2) \equiv_{(2)} 0 \tag{7.17}$$

$$Q^{2(4q+2)}(a_2) \equiv_{(2)} 0 \tag{7.18}$$

$$Q^{2(4q)}(a_2) \equiv_{(2)} a_{4q+2} \tag{7.19}$$

$$Q^{2(4q+3)}(a_2) \equiv_{(2)} a_{4q+5} \tag{7.20}$$

Proof. To show (7.17), we use equation (7.16) to get

$$Q^{2(4q+1)}(a_2) = \binom{4q}{2} a_{4q+3}.$$

Now we write

$$q = q_0 + 2q_1 + \cdots + 2^r q_r,$$

giving

$$4q = 0.1 + 0.2 + 2^2(q_0 + 2q_1 + \cdots + 2^r q_r).$$

Note that

$$2 = 0.1 + 1.2.$$

Hence

$$\begin{aligned} \binom{4q}{2} &\equiv_{(2)} \binom{0}{0}\binom{0}{1}\binom{q_0}{0}\binom{q_1}{0}\cdots \\ &= 0 \end{aligned}$$

and we have that

$$Q^{2(4q+1)}(a_2) \equiv_{(2)} 0.$$

Now to show (7.18), we use equation (7.16) to get

$$\begin{aligned} Q^{2(4q+2)}(a_2) &= \binom{4q+1}{2} a_{4q+4} \\ &= \binom{4q+1}{2} a_{4p} \text{ for } p = q+1 \end{aligned}$$

We have

$$4q+1 = 1.1 + 0.2 + 2^2(q_0 + 2q_1 + \cdots + 2^r q_r).$$

Giving

$$\begin{aligned} \binom{4q+1}{2} &\equiv_{(2)} \binom{1}{0}\binom{0}{1}\binom{q_0}{0}\binom{q_1}{0}\cdots \\ &= 0 \end{aligned}$$

Hence we have

$$Q^{2(4q+2)}(a_2) \equiv_{(2)} 0.$$

To show (7.19), we use

$$\begin{aligned} Q^{2(4q)}(a_2) &= \binom{4q-1}{2} a_{4q+2} \\ &= \binom{4r+3}{2} a_{4q+2} \text{ for } r = q-1 \end{aligned}$$

We have

$$4r+3 = 1.1 + 1.2 + 2^2(r_0 + 2r_1 + \cdots + 2^s r_s),$$

which gives us

$$\begin{aligned} \binom{4r+3}{2} &\equiv_{(2)} \binom{1}{0}\binom{1}{1}\binom{r_0}{0}\binom{r_1}{0}\cdots \\ &= 1 \end{aligned}$$

and hence we have

$$Q^{2(4q)}(a_2) \equiv_{(2)} a_{4q+2}.$$

Finally we consider (7.20)

$$\begin{aligned} Q^{2(4q+3)}(a_2) &= \binom{4q+2}{2} a_{4q+5} \\ &= \binom{4q+2}{2} a_{4p+1} \text{ for } p = q+1 \\ &= \binom{4q+2}{2} a_{4p+1}. \end{aligned}$$

We write

$$4q + 2 = 0.1 + 1.2 + 2^2(q_0 + 2q_1 + \cdots + 2^r q_r),$$

giving

$$\begin{aligned} \binom{4q+2}{2} &\equiv_{(2)} \binom{0}{0}\binom{1}{1}\binom{q_0}{0}\binom{q_1}{0}\cdots \\ &= 1. \end{aligned}$$

Hence we have

$$Q^{2(4q+3)}(a_2) \equiv_{(2)} a_{4p+1}.$$

□

Now we consider composing two Dyer–Lashof operations in an attempt to reach the missing generators a_{4q} and a_{4q+3}. We begin from the the generators we have reached already, namely a_{4q+1} and a_{4q+2}, we then apply an appropriate Dyer–Lashof operation to try to reach a_{4q} and a_{4q+3}. We work modulo decomposables and have the following four cases;

Case 1: $a_{4q+1} \longrightarrow a_{4r}$

Here we ask if

$$Q^{2[4(r-q)-1]}(a_{4q+1}) \equiv_{(2)} a_{4r}$$

is a possibility.

We have the following;

$$Q^{2[4(r-q)-1]}(a_{4q+1}) = \binom{4r-4q-2}{4q+1} a_{4r}.$$

We can write

$$q = q_0 + 2q_1 + \cdots + 2^s q_s$$

and

$$r = r_0 + 2r_1 + \cdots + 2^s r_s.$$

Now write $4r - 4q - 2$ as $4(r - q - 1) + 2$, giving

$$4(r-q-1) + 2 = 0.1 + 1.2 + (r_0 - q_0 - 1)2^2 + (r_1 - q_1)2^3 + \cdots.$$

We also have

$$4q+1 = 1.1 + 0.2 + (q_0 + 2q_1 + \cdots + 2^s q_s)2^2.$$

Hence we have that

$$\begin{aligned} \binom{4r-4q-2}{4q+1} &\equiv_{(2)} \binom{0}{1}\binom{1}{0}\cdots \\ &= 0 \end{aligned}$$

and we fail to reach m_{4r}.

Case 2: $a_{4q+2} \longrightarrow a_{4r}$

Here we ask if

$$Q^{2[4(r-q)-2]}(a_{4q+2}) \equiv_{(2)} a_{4r}$$

is a possibility.

We have the following;

$$Q^{2[4(r-q)-2]}(a_{4q+2}) = \binom{4r-4q-3}{4q+2} a_{4r}$$

We can write

$$q = q_0 + 2q_1 + \cdots + 2^s q_s$$

and

$$r = r_0 + 2r_1 + \cdots + 2^s r_s.$$

Rewriting $4r-4q-3$ as $4(r-q-1)+1$, we have

$$4(r-q-1)+1 = 1.1 + 0.2 + (r_0 - q_0 - 1)2^2 + (r_1 - q_1)2^3 + \cdots .$$

We also have

$$4q+2 = 0.1 + 1.2 + (q_0 + 2q_1 + \cdots + 2^s q_s)2^2.$$

Hence

$$\begin{aligned} \binom{4r-4q-3}{4q+2} &\equiv_{(2)} \binom{1}{0}\binom{0}{1}\cdots \\ &= 0 \end{aligned}$$

and again we fail to reach m_{4r}.

Case 3: $a_{4q+2} \longrightarrow a_{4r+3}$

Here we ask if

$$Q^{2[4(r-q)+1]}(a_{4q+2}) \equiv_{(2)} a_{4r+3}$$

is a possibility.

We have the following;

$$Q^{2[4(r-q)+1]}(a_{4q+2}) = \binom{4r-4q}{4q+2} a_{4r+3}$$

We write

$$q = q_0 + 2q_1 + \cdots + 2^s q_s$$

and

$$r = r_0 + 2r_1 + \cdots + 2^s r_s.$$

Rewriting $4r - 4q$ as $4(r-q)$, we have

$$4(r-q) = 0.1 + 0.2 + (r_0 - q_0)2^2 + (r_1 - q_1)2^3 + \cdots .$$

We also have

$$4q + 2 = 0.1 + 1.2 + (q_0 + 2q_1 + \cdots + 2^s q_s)2^2.$$

Hence

$$\begin{aligned} \binom{4r-4q}{4q+2} &\equiv_{(2)} \binom{0}{0}\binom{0}{1}\cdots \\ &= 0 \end{aligned}$$

and so we fail to reach m_{4r+3}.

Case 4: $a_{4q+1} \longrightarrow a_{4r+3}$

Here we ask if

$$Q^{2[4(r-q)+2]}(a_{4q+1}) \equiv_{(2)} a_{4r+3}$$

is a possibility.

We have the following;

$$Q^{2[4(r-q)+2]}(a_{4q+1}) = \binom{4r-4q+1}{4q+1} a_{4r+3}$$

We can write

$$q = q_0 + 2q_1 + \cdots + 2^s q_s$$

and

$$r = r_0 + 2r_1 + \cdots + 2^s r_s$$

with each $q_i, r_i = 0, 1$. If there is no i such that $r_i = 0$ and $q_i = 1$, then

$$4r - 4q + 1 = 1 + 2^2(r_0 - q_0) + \cdots + 2^{s+2}(r_s - q_s)$$

and

$$4q + 1 = 1 + 2^2 q_0 + \cdots + 2^{s+2} q_s,$$

so

$$\binom{4r - 4q + 1}{4q + 1} \equiv_{(2)} \binom{1}{1}\binom{0}{0}\binom{r_0 - q_0}{q_0} \cdots \binom{r_s - q_s}{q_s},$$

which is congruent to 0 modulo 2 if $r_i = q_i = 1$ for any i.

By similar reasoning we get the following.

i. If r and q are both odd we get $\binom{4r-4q+1}{4q+1} \equiv_{(2)} 0$ and therefore do not reach a missing generator.

ii. If r and q are both powers of 2 with $r \geq 2q$ then $\binom{4r-4q+1}{4q+1} \equiv_{(2)} 1$ and we do reach a missing generator.

This is the only case out of the four considered that gives us some of the missing generators. It is not straight forward to say for certain when generators will be reached.

References

[1] J. F. Adams, *Infinite loop spaces*, Annals of Mathematics Studies, vol. 90, Princeton University Press, Princeton, N.J., 1978.

[2] ———, *Stable homotopy and generalised homology*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1995. Reprint of the 1974 original.

[3] M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[4] A. J. Baker and H. Gilmour, *Notes on Topological André–Quillen homology for cellular commutative S-algebras*, Glasgow University Mathematics Department preprint, no. 04/13.

[5] A. J. Baker and J. P. May, *Minimal atomic complexes*, Topology **43** (2004), no. 3, 645–665.

[6] M. Basterra, *André–Quillen cohomology of commutative S-algebras*, J. Pure Appl. Algebra **144** (1999), no. 2, 111–143.

[7] M. Basterra and M. A. Mandell, *Homology and cohomology of E_∞ ring spectra*, Math. Z. **249** (2005), no. 4, 903–944.

[8] M. Basterra and R. McCarthy, *Γ-homology, topological André–Quillen homology and stabilization*, Topology Appl. **121** (2002), no. 3, 551–566.

[9] M. Basterra and B. Richter, *(Co-)homology theories for commutative S-algebras*, Structured Ring Spectra, 2004, pp. 115–131.

[10] J. M. Boardman, *Stable Homotopy theory*, Thesis, Warwick, 1964.

[11] A. K. Bousfield, *The localization of spaces with respect to homology*, Topology **14** (1975), 133–150.

[12] ———, *The localization of spectra with respect to homology*, Topology **18** (1979), no. 4, 257–281.

[13] E. H. Brown Jr., *Abstract homotopy theory*, Trans. Amer. Math. Soc. **119** (1965), 79–85.

[14] C. Casacuberta and J. J. Gutiérrez, *Homotopical localizations of module spectra*, Trans. Amer. Math. Soc. **357** (2005), no. 7, 2753–2770 (electronic).

[15] G. Cooke, *Embedding certain complexes up to homotopy type in euclidean space.*, Ann. of Math. (2) **90** (1969), 144–156.

[16] W. G. Dwyer, *Localizations*, Axiomatic, enriched and motivic homotopy theory, NATO Sci. Ser. II Math. Phys. Chem., vol. 131, Kluwer Acad. Publ., Dordrecht, 2004, pp. 3–28.

[17] W. G. Dwyer and J. Spaliński, *Homotopy theories and model categories*, Handbook of algebraic topology, 1995, pp. 73–126.

[18] E. Dyer and R. K. Lashof, *Homology of iterated loop spaces*, Amer. J. Math. **84** (1962), 35–88.

[19] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May, *Rings, modules, and algebras in stable homotopy theory*, Mathematical Surveys and Monographs, vol. 47, American Mathematical Society, Providence, RI, 1997. with an appendix by M. Cole.

[20] V. Giambalvo and D. J. Pengelley, *The homology of $MSpin$*, Math. Proc. Cambridge Philos. Soc. **95** (1984), no. 3, 427–436.

[21] A. Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002.

[22] M. Hovey, *Model categories*, Mathematical Surveys and Monographs, vol. 63, American Mathematical Society, Providence, RI, 1999.

[23] P. Hu, I. Kriz, and J. P. May, *Cores of spaces, spectra and E_∞ ring spectra*, Homology, Homotopy and Applications **3** (2001), no. 2, 341–54.

[24] S. O. Kochman, *Homology of the classical groups over the Dyer–Lashof algebra*, Trans. Amer. Math. Soc. **185** (1973), 83–136.

[25] S. König and A. Zimmermann, *Derived equivalences for group rings*, Lecture Notes in Mathematics, vol. 1685, Springer-Verlag, Berlin, 1998. With contributions by Bernhard Keller, Markus Linckelmann, Jeremy Rickard and Raphaël Rouquier.

[26] I. Kozma, *Generators for $\Pi_*(M\mathrm{U})$*, Math. Proc. Cambridge Philos. Soc. **78** (1975), no. part 2, 321–322.

[27] N. J. Kuhn, *Localization of André-Quillen-Goodwillie towers, and the periodic homology of infinite loopspaces*, Adv. Math. **201** (2006), no. 2, 318–378.

[28] T. Lance, *Steenrod and Dyer–Lashof operations on $B\mathbf{U}$*, Trans. Amer. Math. Soc. **276** (1983), no. 2, 497–510.

[29] A. Lazarev, *(Co-)homology theories for highly structured ring spectra*, Structured Ring Spectra, 2004, pp. 201–231.

[30] L. G. Lewis Jr., J. P. May, M. Steinberger, and J. E. McClure, *Equivariant stable homotopy theory*, Lecture Notes in Mathematics, vol. 1213, Springer-Verlag, Berlin, 1986. With contributions by J. E. McClure.

[31] E. L. Lima, *The Spanier-Whitehead duality in new homotopy categories*, Summa Brasil. Math. **4** (1959), 91–148 (1959).

[32] I. Madsen and R. J. Milgram, *The classifying spaces for surgery and cobordism of manifolds*, Annals of Mathematics Studies, vol. 92, Princeton University Press, Princeton, N.J., 1979.

[33] M. A. Mandell, *Topological André–Quillen cohomology and E_∞ André–Quillen cohomology*, Adv. Math. **177** (2003), no. 2, 227–279.

[34] J. P. May, *E_∞ ring spaces and E_∞ ring spectra*, Springer-Verlag, Berlin, 1977. With contributions by Frank Quinn, Nigel Ray, and Jørgen Tornehave; Lecture Notes in Mathematics, Vol. 577.

[35] ———, *A concise course in algebraic topology*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1999.

[36] R. McCarthy and V. Minasian, *HKR theorem for smooth S-algebras*, J. Pure Appl. Algebra **185** (2003), no. 1-3, 239–258.

[37] D. J. Pengelley, *The A-algebra structure of Thom spectra: MSO as an example*, Current trends in algebraic topology, Part 1 (London, Ont., 1981), 1982, pp. 511–513.

[38] ———, *The homotopy type of MSU*, Amer. J. Math. **104** (1982), no. 5, 1101–1123.

[39] ———, *The mod two homology of MSO and MSU as A comodule algebras, and the cobordism ring*, J. London Math. Soc. (2) **25** (1982), no. 3, 467–472.

[40] S. Priddy, *A cellular construction of BP and other irreducible spectra*, Math. Z. **173** (1980), no. 1, 29–34.

[41] D. G. Quillen, *Homotopical algebra*, Lecture Notes in Mathematics, No. 43, Springer-Verlag, Berlin, 1967.

[42] R. E. Stong, *Notes on cobordism theory*, Mathematical notes, Princeton University Press, Princeton, N.J., 1968.

[43] N. P. Strickland, *Products on* MU*-modules*, Trans. Amer. Math. Soc. **351** (1999), no. 7, 2569–2606.

[44] D. Sullivan, *Genetics of homotopy theory and the Adams conjecture*, Ann. of Math. (2) **100** (1974), 1–79.

[45] ———, *Geometric topology: localization, periodicity and Galois symmetry*, K-Monographs in Mathematics, vol. 8, Springer, Dordrecht, 2005. The 1970 MIT notes; Edited and with a preface by Andrew Ranicki.

[46] C. A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.