THE APPLICATION OF PICTURES TO DECISION PROBLEMS AND RELATIVE PRESENTATIONS

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Contents

STATEMENT

Chapter 1 covers basic materials and motivation for thesis. Most of these are taken from [48].

Results in §2.1 and §2.2 may be known even though I could not find in any literature but §2.3.3, §2.3.4 and §2.4 are my own work.

Chapters 3, 5 and 6 are my own work.

All materials in Chapter 4 (except §4.3.6) are fairly standard and can be found in at least one of these—[3, 8, 22].

ACKNOWLEDGEMENT

I would like to express my gratitude to my supervisor Prof. S.J. Pride, FRSE who introduced and suggested this work. I am indebted to him for his guidance, assistance, encouragement and ideas that he has provided.

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Finally, I would like to thank all my friends for all assistance and support.

ABSTRACT

Regard the presentation $\mathcal{P} =< \boldsymbol{x}; \boldsymbol{r} >$ as a 2-complex. Then we have the second homotopy module $\pi_2(\mathcal{P})$. The elements of $\pi_2(\mathcal{P})$ can be represented by spherical pictures. This is the key idea for this thesis. We give this preliminary background in Chapter 1.

In Chapter 2, we study properties of groups concerning $\pi_2(\mathcal{P})$. We show that all these properties are recursively unsolvable, that is, there is no effective methods which can be applied to an arbitrary finite presentation $\mathcal{P}$ to determine whether or not groups have these properties. Our main results are Theorems 2.3.1 and 2.3.2, that is, that p-Cockcroft and efficiency are recursively unsolvable.

Let $\boldsymbol{P}$ be a collection of spherical pictures over $\mathcal{P}$. Then we may form a 3-complex $\mathcal{K} =< \boldsymbol{x}; \boldsymbol{r}; \boldsymbol{P} >$. In Chapter 3 we establish the picture problem for $\mathcal{K}$—the analogue of the word problem for $\mathcal{P}$, a dimension higher. We prove Theorem 3.1.1—the existence of $\mathcal{K}$ with unsolvable picture problem.

From now onwards, we deal with relative presentations. We are interested in investigating the asphericity of $\mathcal{P} =< H, t; th_1th_2th_3t^{-1}h_4 >$ $(h_i \in H)$. In Chapter 4, we survey the basic concepts, the important theorems for relative presentations and the tests for asphericity.

The first major case that we consider is $< H, t; t^3at^{-1}b >$ where a and b are non-trivial elements of H. We investigate asphericity of this form in Chapter 5. Excluding some exceptions that are not yet decided, we state our results in Theorems 5.1.1 and 5.2.1.

In Chapter 6, we consider the second major case—$< H, t; t^2atbt^{-1}c >$ where a, b and c are non-trivial elements of H. As in Chapter 5, we have some exceptions and we state our results in Theorems 6.1.1, 6.2.1, 6.3.1 and 6.4.1.

NOTATIONS

Let G and K be groups

$G \oplus K$	the direct sum
$G \times K$	the direct product
$G * K$	the free product
$G \cong K$	G is isomorphic to K
G/K	the quotient group of G by K
$\mathbb{Z}G$	the integral group ring
IG	the augmentation ideal
$rk_{\mathbb{Z}}(G)$	the rank of torsion free part (when G is abelian)
$d(G)$	the least number of generators
$H_k(G, A)$	the k-th homology group of G with coefficient in A
$H^k(G, B)$	the k-th co-homology group of G with coefficient in B
$\delta(G)$	$= 1 - rk_{\mathbb{Z}}(H_1(G)) + d(H_2(G))$
cd$G$	cohomological dimension
G'	derived group (commutator subgroup) of G
$G*_{K,\sigma}$	HNN-extension with base group G,
	associated subgroup K with specified isomorphism σ
$G \times_\rho K$	split extension of G by K with K-action ρ
K^G	normal closure of K in G

If $a, b \in G$

$[a, b]$	the commutator of a and b $(= aba^{-1}b^{-1})$
$o(a)$	the order of a

Also

$\mathbb{Z}^n$	the free abelian group of rank n
$\mathbb{Z}_n$	the finite cyclic group of order n

and

$\oplus$	the direct sum
$*$	the free product

If $\mathcal{P} =< \boldsymbol{x}; \boldsymbol{r} >$ is a presentation

$G(\mathcal{P})$	the group defined by $\mathcal{P}$
$\pi_1(\mathcal{P})$	the first fundamental group
$\pi_2(\mathcal{P})$	the second homotopy module
$I_1(\mathcal{P})$	the first Fox ideal
$I_2(\mathcal{P})$	the second Fox ideal
$\chi(\mathcal{P})$	the Euler characteristic of $\mathcal{P}$
$\mathcal{P}^{st}$	the star graph
$M(\mathcal{P})$	the relation module

Also if W is a word on $\boldsymbol{x}$

$[W]$	equivalence classes containing W
$\overline{W}$	the element of $G(\mathcal{P})$ that represents W

and if $R \in \boldsymbol{r}$

$p(R)$	the period of R
$\dot{R}$	the root of R

Let $\mathbb{P}$ be a picture over $\mathcal{P}$

$W(\mathbb{P})$	the label of $\mathbb{P}$
$\partial\mathbb{P}$	the boundary of $\mathbb{P}$
$-\mathbb{P}$	the mirror image
$< \mathbb{P} >$	equivalence classes containing $\mathbb{P}$
$W(\gamma)$	the label of path γ
$W(c)$	the label of corner c
$\theta(c)$	the angle of corner c
$exp_R\mathbb{P}$	exponent sum of R in $\mathbb{P}$

If $\mathcal{K}$ is a complex

$\mathcal{K}^{(n)}$	the n-th skeleton
$\chi(\mathcal{K})$	the Euler characteristic of $\mathcal{K}$
$\pi_n(\mathcal{K})$	the n-th homotopy groups

Also if ρ is a path in $\mathcal{K}$

$[\rho]_{\mathcal{K}}$	equivalence classes containing ρ

We adopt the usual notation in set theory

$A \cup B$	the unions of the sets A and B
$A - B$	the set difference
$A \subseteq B$	A is a subset of B
$a \in A$	a belongs to A
$\mid A \mid$	the cardinality of A

Chapter 1

Preliminaries

Let $\boldsymbol{x}$ be a set (alphabet). A *word* W on $\boldsymbol{x}$ is of the form

$$x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_n^{\epsilon_n}$$

where $n \geq 0, x_i \in \boldsymbol{x},\ \epsilon_i = \pm 1 (i = 1, \cdots, n)$. If $x_i^{\epsilon_i} \neq x_{i+1}^{-\epsilon_{i+1}} (i = 1, \cdots, n-1)$, then we say that it is *reduced.* Furthermore it is *cyclically reduced* if in addition $x_1^{\epsilon_1} \neq x_n^{-\epsilon_n}$. Then we have a presentation

$$\mathcal{P} = < \boldsymbol{x}; \boldsymbol{r} >$$

where $\boldsymbol{r}$ is a set of non-empty cyclically reduced words on $\boldsymbol{x}$. We say that $\mathcal{P}$ is finite if $\boldsymbol{x}$ and $\boldsymbol{r}$ are both finite.

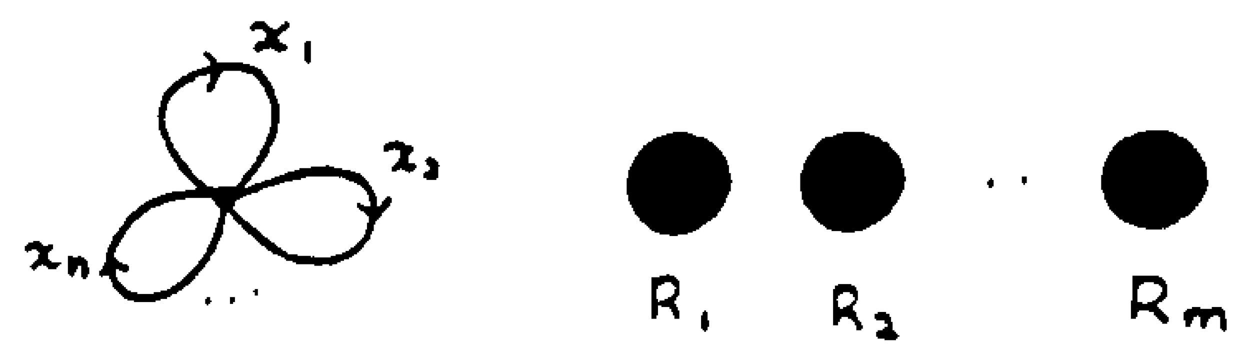
Throughout this thesis, all presentations will be assumed to be finite unless stated otherwise.

If $F(\boldsymbol{x})$ is the free group on $\boldsymbol{x}$ and $N = << \boldsymbol{r} >>$ is the normal closure of $\boldsymbol{r}$ in $F(\boldsymbol{x})$, then the quotient group

$$G(\mathcal{P}) = F(\boldsymbol{x})/N$$

is the *group defined by* $\mathcal{P}$. Denote a typical element of $G(\mathcal{P})$ by $\overline{W} = [W]N$ where W is a word on $\boldsymbol{x}$ and $[W]$ is the free equivalence class of W. A group G is said to be *finitely presented* if G can be defined by a finite presentation (that is $G = G(\mathcal{P})$ for some finite presentation $\mathcal{P}$).

We may regard $\mathcal{P}$ as a 2-complex



This complex has a single 0-cell, the 1-cells are in bijective correspondence with $\boldsymbol{x}$, and the 2-cells are in bijective correspondence with $\boldsymbol{r}$ and are attached by the boundary path determined by the spelling of the corresponding member of $\boldsymbol{r}$. Thus there are homotopy groups $\pi_1(\mathcal{P})$ and $\pi_2(\mathcal{P})$. As a 2-complex, a *path* in $\mathcal{P}$ is of the form

$$x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_n^{\epsilon_n}$$

where $n \geq 0, x_i \in \boldsymbol{x},\ \epsilon_i = \pm 1 (i = 1, \cdots, n)$. Clearly any path in $\mathcal{P}$ is closed since $\mathcal{P}$ has a single 0-cell. We define elementary operations on paths as follows:

(a) Insertion/deletion of an inverse pair $x^{\epsilon} x^{-\epsilon} (x \in \boldsymbol{x})$.

(b) Insertion/deletion of an element $\boldsymbol{r} \cup \boldsymbol{r}^{-1}$.

Two paths are said to be *homotopic (relative to $\mathcal{P}$)* if one can be obtained from the other by a finite number of the above operations. Denote the equivalence class containing path W by $[W]_{\mathcal{P}}$. For any paths U, V in $\mathcal{P}$, we may define a multiplication of equivalence classes $[U]_{\mathcal{P}}[V]_{\mathcal{P}} = [UV]_{\mathcal{P}}$. The first fundamental group $\pi_1(\mathcal{P})$ is simply the set of equivalence classes of paths with the above operations (see for example [17, 51]). The identity is the equivalence class containing the empty path and the inverse of $[W]_{\mathcal{P}}$ is $[W^{-1}]_{\mathcal{P}}$. Also there is a bijective homomorphism

$$\phi : \pi_1(\mathcal{P}) \longrightarrow G(\mathcal{P})$$

$$[W]_{\mathcal{P}} \mapsto \overline{W}.$$

The elements of the second homotopy module $\pi_2(\mathcal{P})$ can be represented by geometric configurations called spherical pictures, as described in §1.1.

1.1 Second homotopy modules

In this section we will introduce the basic concept which is needed in all chapters. There are many reference to this basic theory such as [9, 13, 25, 32, 33, 35, 48, 50].

1.1.1 Pictures

A picture $\mathbb{P}$ over $\mathcal{P}$ is a geometric configuration consisting of the following:

1. A disc D^2 with basepoint 0 on ∂D^2.

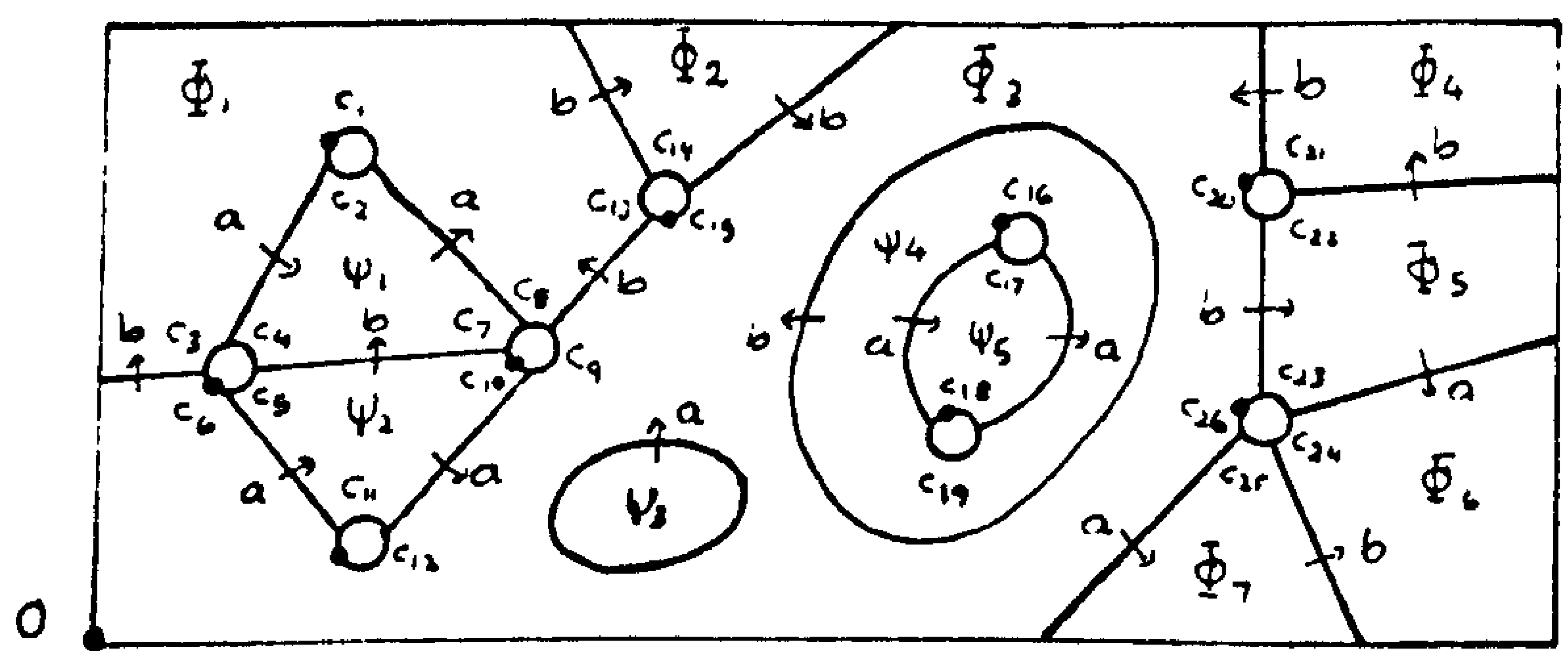
2. Disjoint discs $\Delta_1, \Delta_2, \cdots, \Delta_n$ in the interior of D^2. Each Δ_i has a basepoint 0_i on $\partial\Delta_i$.

3. A finite number of disjoint arcs $\alpha_1, \alpha_2, \cdots, \alpha_m$ where each arc lies in the closure of $D^2 - \bigcup_{i=1}^{i=n} \Delta_i$ and is either a simple closed curve having trivial intersection with $\partial D^2 \cup \partial\Delta_1 \cup \partial\Delta_2 \cup \cdots \cup \partial\Delta_n$, or is a simple non-closed curved which joins two points of $\partial D^2 \cup \partial\Delta_1 \cup \cdots \cup \partial\Delta_n$, neither point being a base point. Each arc has a normal orientation, indicated by a short arrow meeting with the arc transversely and is labelled by an element of $\boldsymbol{x} \cup \boldsymbol{x}^{-1}$.

4. If we travel around $\partial\Delta_i$ once in clockwise direction starting from 0_i and read off the labels on arcs encountered (if we cross an arc, labelled x say, in the direction of its normal orientation, then we read x, whereas if we cross the arc against the orientation, then we read x^{-1}), then we obtain a word which belongs to $\boldsymbol{r} \cup \boldsymbol{r}^{-1}$. We call this word the *label* of Δ_i.

For each disc Δ, a *corner* of Δ is the closure of a connected component of $\partial\Delta - \cup\{\beta_1, \beta_2, \cdots, \beta_k\}$ where $\beta_1, \cdots, \beta_k$ are the arcs of $\mathbb{P}$ meeting Δ. The *regions* of $\mathbb{P}$ are the closure of connected components of $D^2 - \{(\cup\{\Delta_1, \Delta_2, \cdots, \Delta_n\}) \bigcup(\cup\{\alpha_1, \alpha_2, \cdots, \alpha_m\})\}$. A region Φ of $\mathbb{P}$ is called an *outer* region if it meets ∂D^2 and an *interior* region otherwise. We say that $\mathbb{P}$ is *connected* if $(\cup\{\Delta_1, \Delta_2, \cdots, \Delta_n\}) \bigcup(\cup\{\alpha_1, \alpha_2, \cdots, \alpha_m\})$ is connected.

Example 1.1.1 Let $\mathcal{P} =< a, b; a^2, b^3, [a, b] >$. Then



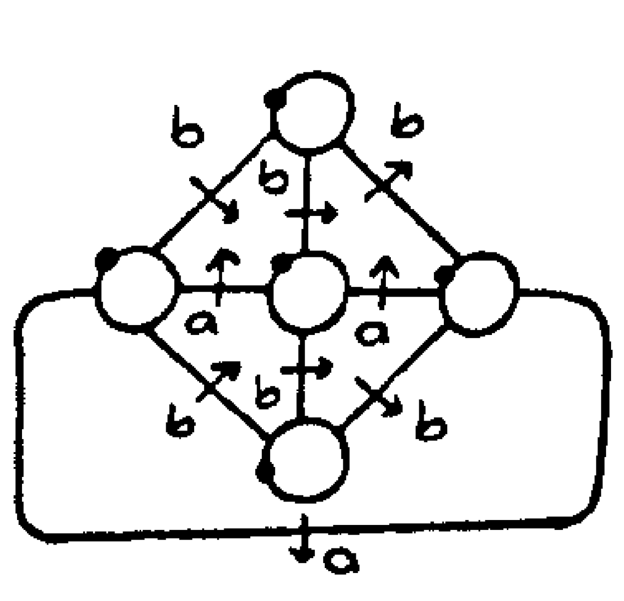
is a picture over $\mathcal{P}$. In this picture we have seven outer regions—$\Phi_1, \cdots, \Phi_7$ and five inner regions—$\Psi_1, \cdots, \Psi_5$. We also have twentysix corners— $c_1, \cdots, c_{26}$.

We define $\partial\mathbb{P}$ to be ∂D^2. The *label* on $\mathbb{P}$ (denoted by $W(\mathbb{P})$) is the word read off by travelling around $\partial\mathbb{P}$ once in the clockwise direction starting from 0. Thus in the above example, $W(\mathbb{P}) = bbbb^{-1}b^{-1}ab^{-1}a^{-1}$.

In some articles (see for example [13, 36, 43, 47]), the dual of pictures—diagrams—are considered. Thus there is a pictorial version of the 'van Kampen Lemma'

Lemma 1.1.2 *There exists a picture $\mathbb{P}$ over $\mathcal{P}$ with label W if and only if $\overline{W} = 1$ in $G(\mathcal{P})$.*

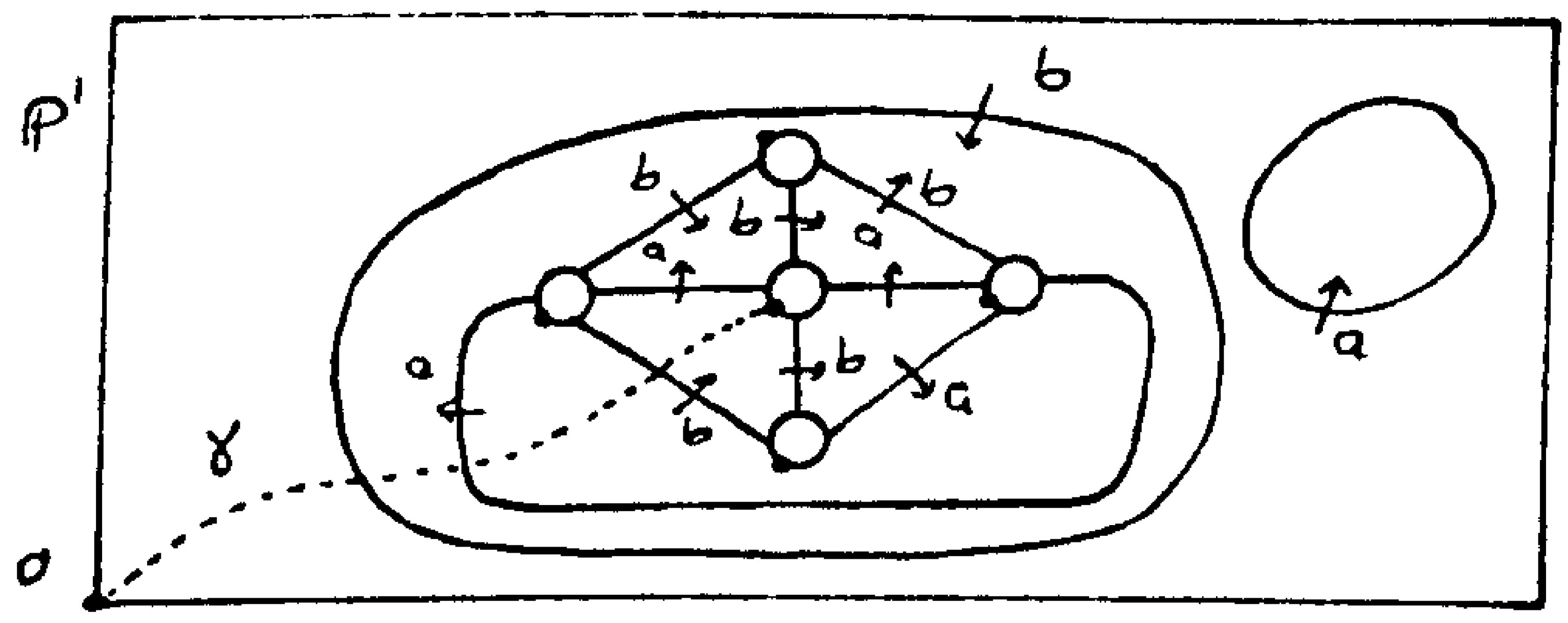
We say that $\mathbb{P}$ is *spherical* if no arcs meet $\partial\mathbb{P}$. If $\mathbb{P}$ is spherical we often omit $\partial\mathbb{P}$.
Example 1.1.1 (continued)



is a spherical picture over $\mathcal{P}$.

A *transverse path* in $\mathbb{P}$ is a path in the closure of $D^2 - \bigcup_{i=1}^{i=n} \Delta_i$ which intersects the arcs of $\mathbb{P}$ only finitely many times. If we travel along a path γ from its initial point to its terminal point we will cross various arcs. We can read off the labels of these arcs, giving a word $W(\gamma)$, the label on γ.

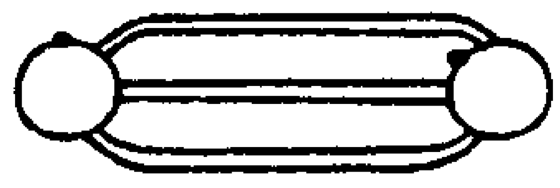
Example 1.1.1 (continued)



Then $W(\gamma) = ba^{-1}b$.

When we refer to the discs of $\mathbb{P}$ we mean the discs $\Delta_1, \Delta_2, \cdots, \Delta_n$, not the ambient disc D^2. A closed arc which encircles no disc or arc of $\mathbb{P}$ is called a *floating circle*. In the above picture $\mathbb{P}'$, the closed arc labelled by a is a floating circle but the closed arc labelled by b is not. A *cancelling pair* is a spherical picture with exactly two discs and where their basepoints lie in the same region. We only allow one basepoint on each disc. Thus if a relator $R \in \boldsymbol{r}$ is a proper power, we need to be cautious. This means that

are cancelling pairs, whereas



is not.

Now we introduce the basic operations on pictures.

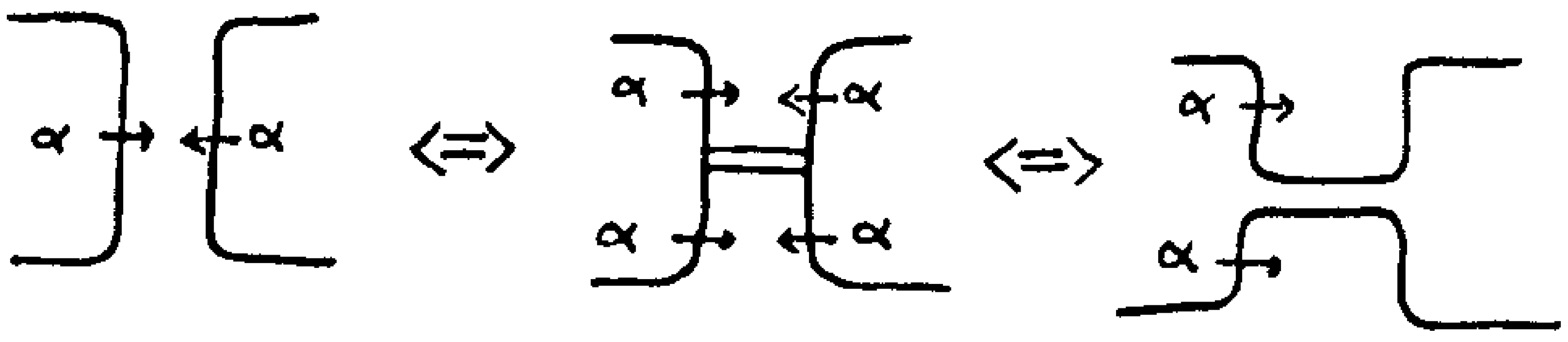
$A)$ Delete floating circle.

$A)^{-1}$ Insert floating circle.

$B)$ Delete cancelling pairs.

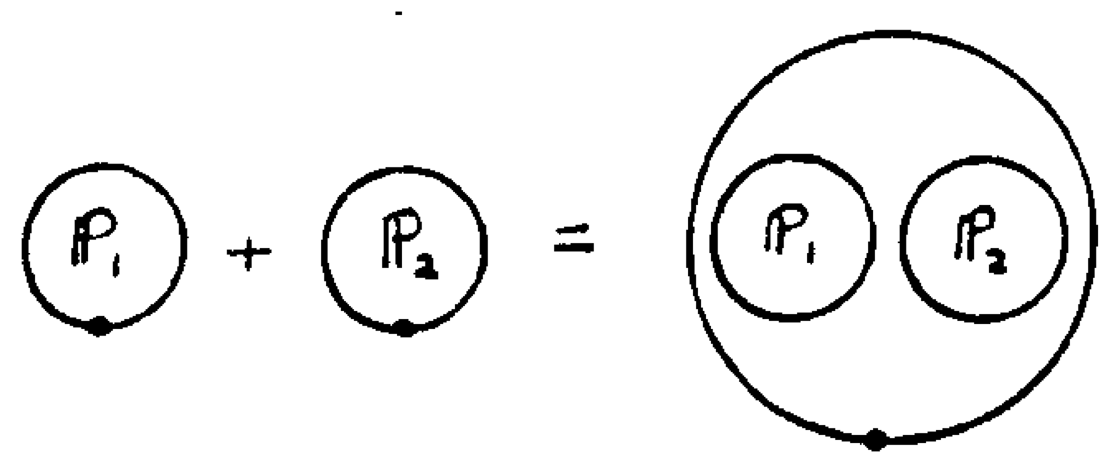
$B)^{-1}$ Insert cancelling pairs.

$C)$ Bridge moves



Two pictures will be said to be *equivalent* if one can be transformed to the other by a finite number of operations $A)^{\pm 1}, B)^{\pm 1}$ and $C)$. We let $<\mathbb{P}>$ denote the equivalence class containing $\mathbb{P}$.

Let $\mathbb{P}_1, \mathbb{P}_2$ be spherical pictures over $\mathcal{P}$. Then the mirror image of $\mathbb{P}_1$ will be denoted by $-\mathbb{P}_1$ and $\mathbb{P}_1 + \mathbb{P}_2$ is defined to be

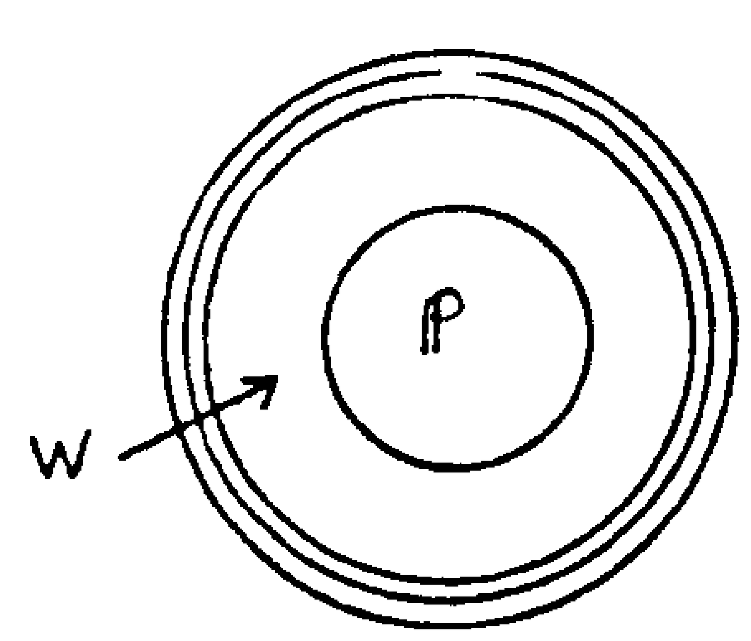


Note that the set of equivalence classes of all spherical pictures over $\mathbb{P}$ forms an abelian group, denoted by $\pi_2(\mathcal{P})$ under the following binary operation

$$<\mathbb{P}_1> + <\mathbb{P}_2> = <\mathbb{P}_1 + \mathbb{P}_2> .$$

The identity (zero) is the equivalence class containing the empty picture and the inverse (negative) of $<\mathbb{P}>$ is $<-\mathbb{P}>$.

Let $\mathbb{P}^W$ be the spherical picture obtained from spherical picture $\mathbb{P}$ by surrounding it by a collection of concentric closed arcs with total label W.



Then we can consider $\pi_2(\mathcal{P})$ as a left $\mathbb{Z}G(\mathcal{P})$-module where the (well-defined) $G(\mathcal{P})$-action is given by

$$\overline{W} \cdot <\mathbb{P}> = <\mathbb{P}^W> \quad (\overline{W} \in G).$$

We call $\pi_2(\mathcal{P})$ the *second homotopy module* of $\mathcal{P}$. There are known procedures for producing homotopy elements from pictures but we shall not pursue this here. See for example [9, 13, 25, 48, 50] for the connection with the topological definition of $\pi_2(\mathcal{P})$.

Consider a collection X of spherical pictures over $\mathcal{P}$. We introduce two further operations on spherical pictures.

$D)$ If there is a simple closed path β in a picture such that the part of the picture enclosed by β is a copy of some $\mathbb{P}$ or $-\mathbb{P}$ ($\mathbb{P} \in \boldsymbol{X}$) then delete that part of the picture enclosed by β.

$D)^{-1}$ The opposite of $E)$.

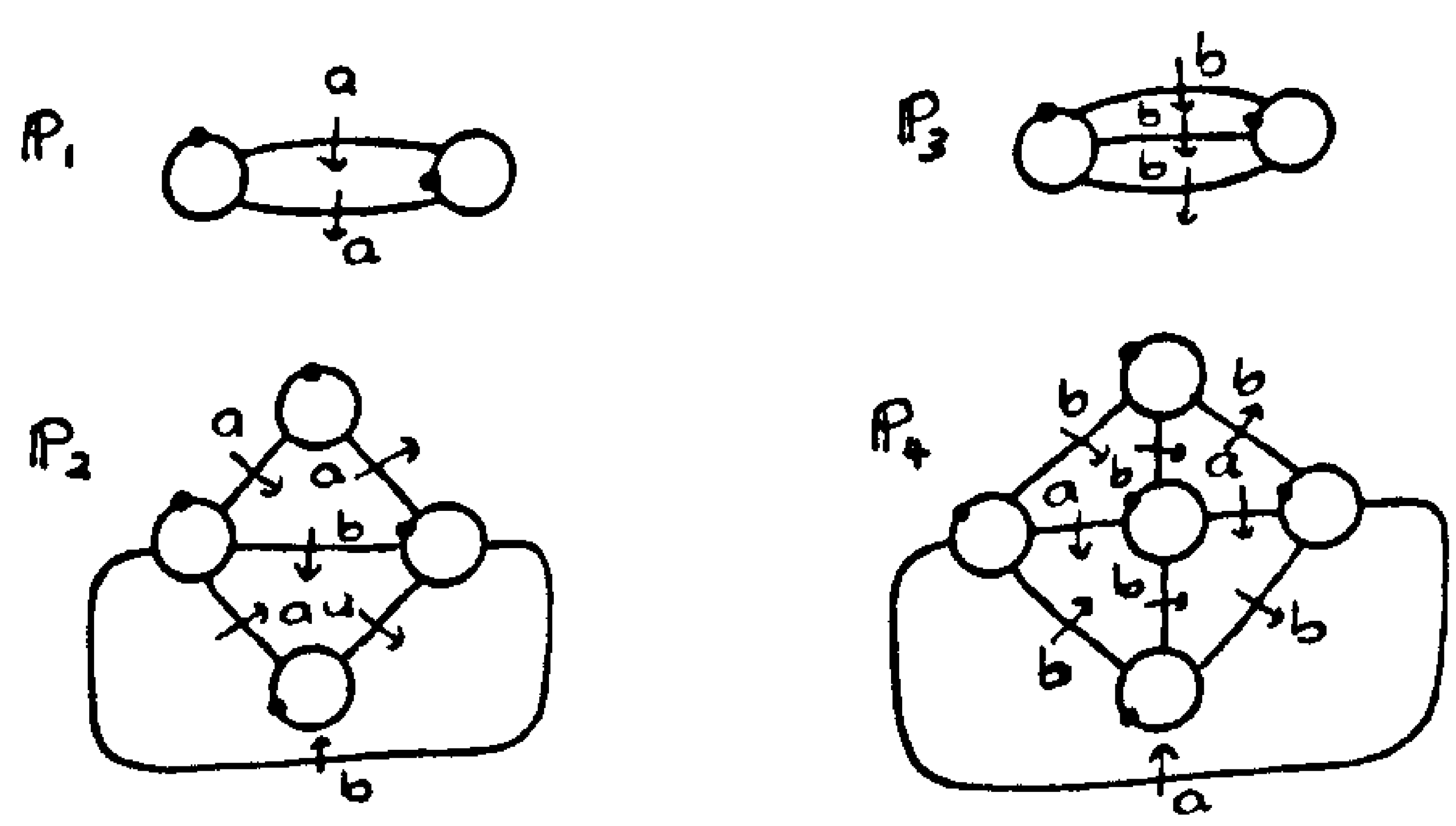
Two spherical pictures will be said to be *equivalent (relative to $\boldsymbol{X}$)* if one can be transformed to the other by a finite number of operations $A)^{\pm 1}, B)^{\pm 1}. C)$ and $D)^{\pm 1}$. Then by [48, Theorem 2.6*,Corollary 1] we have

Theorem 1.1.3 *The elements $< \mathbb{P} >$ ($\mathbb{P} \in \boldsymbol{X}$) generate $\pi_2(\mathcal{P})$ if and only if every spherical picture is equivalent to the empty picture (relative to $\boldsymbol{X}$).*

We say that $\boldsymbol{X}$ *generates* $\pi_2(\mathcal{P})$ if the elements $< \mathbb{P} >$ ($\mathbb{P} \in \boldsymbol{X}$) generate $\pi_2(\mathcal{P})$.

Example 1.1.1 (continued)

One may refer to [4] to show that $\pi_2(\mathcal{P})$ is generated by



1.1.2 Properties of $G(\mathcal{P})$ concerning $\pi_2(\mathcal{P})$

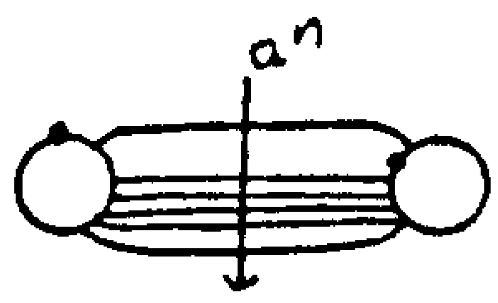
In this subsection we give some definitions which play a major role in Chapter 2. Throughout this subsection assume that $\mathcal{P} =< \boldsymbol{x}; \boldsymbol{r} >$.

Definition 1.1.4 *A presentation $\mathcal{P}$ is said to be* aspherical *if $\pi_2(\mathcal{P}) = 0$ and a group G is said to be* aspherical *if it is defined by an aspherical presentation.*

Note that all free groups are aspherical.

Definition 1.1.5 *A presentation $\mathcal{P}$ is said to be* combinatorially aspherical (CA) *if $\pi_2(\mathcal{P})$ is generated by a set of pictures containing exactly two discs, and a group G is said to be* combinatorially aspherical (CA) *if it can be defined by a CA presentation.*

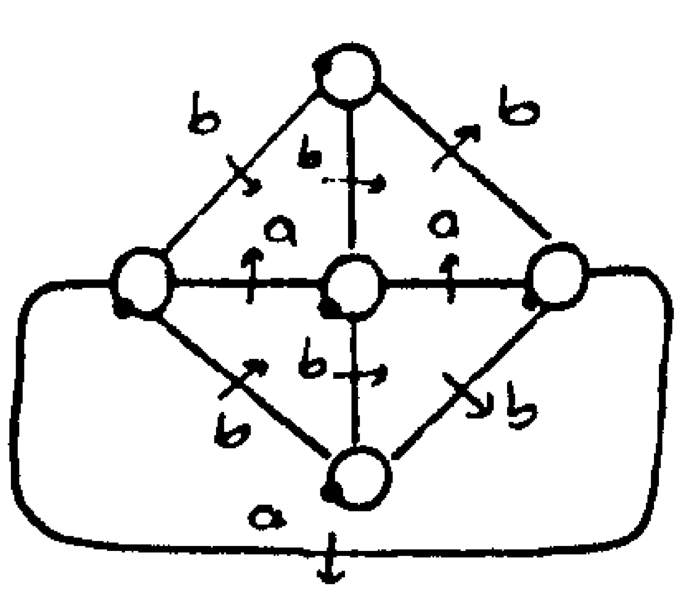
Example 1.1.6 Let $\mathcal{P} =< a; a^n >$ define a cyclic group of order n. It is known that $\pi_2(\mathcal{P})$ is generated by a single picture



Thus $\mathcal{P}$ is CA.

For any picture $\mathbb{P}$ over $\mathcal{P}$ and for any $R \in \boldsymbol{r}$, the *exponent sum of* R *in* $\mathbb{P}$, denoted by $exp_R(\mathbb{P})$ is the number of discs of $\mathbb{P}$ labelled by R minus the number labelled by R^{-1}. Note that if pictures $\mathbb{P}_1$ and $\mathbb{P}_2$ are equivalent, then $exp_R(\mathbb{P}_1) = exp_R(\mathbb{P}_2)$ for all $R \in \boldsymbol{r}$.

Example 1.1.7 Let $\mathcal{P} =< a, b; a^2, b^3, [a, b] >$ as in Example 1.1.1. Consider



Then $exp_{b^3}(\mathbb{P}) = 0$ and $exp_{[a,b]}(\mathbb{P}) = 3$.

Definition 1.1.8 *A presentation* $\mathcal{P}$ *is said to be* Cockcroft *if* $exp_R(\mathbb{P}) = 0$ *for all* $R \in \boldsymbol{r}$ *and for all spherical pictures* $\mathbb{P}$ *over* $\mathcal{P}$. *A group* G *is said to* Cockcroft *if* G *admits a Cockcroft presentation.*

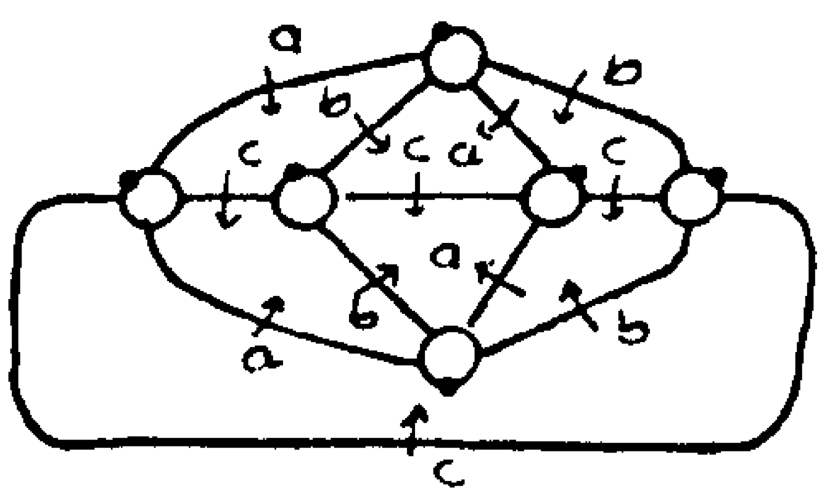
The Cockcroft property has recieved considerable attention in [21, 26, 27, 29, 38].

Definition 1.1.9 *A presentation* $\mathcal{P}$ *is said to be* p-Cockcroft *(p a prime) if* $exp_R(\mathbb{P}) \equiv 0(p)$ *for all* $R \in \boldsymbol{r}$ *and for all spherical pictures* $\mathbb{P}$ *over* $\mathcal{P}$. *A group* G *is said to be* p-Cockcroft *if* G *admits a p-Cockcroft presentation.*

The p-Cockcroft property is discussed for example in [38].

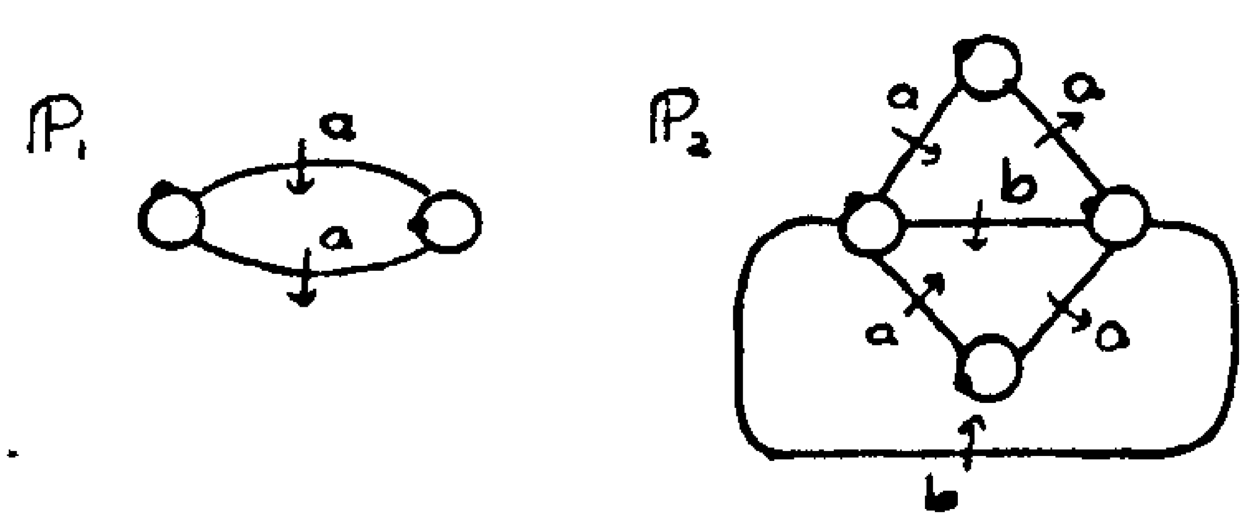
Remark 1.1.10 It should be noted that for Definitions 1.1.8 and 1.1.9, it is enough to check the exponent sum of R (for all $R \in \boldsymbol{r}$) in a set of generating pictures.

Example 1.1.11 Let $\mathcal{P} =< a, b, c; [a, b], [a, c], [b, c] >$. Then one may refer to [4] to show that $\pi_2(\mathcal{P})$ is generated by



Since $exp_{[a,b]}(\mathbb{P}) = exp_{[a,c]}(\mathbb{P}) = exp_{[b,c]}(\mathbb{P}) = 0$, then $\mathcal{P}$ is Cockcroft.

Example 1.1.12 Let $\mathcal{P} =< a, b; a^2, [a, b] >$. Then one may refer to [4] to show that $\pi_2(\mathcal{P})$ is generated by



Since $exp_{a^2}(\mathbb{P}_1) = exp_{[a,b]}(\mathbb{P}_1) = exp_{a^2}(\mathbb{P}_2) = 0$ and $exp_{[a,b]}(\mathbb{P}_2) = 2$, then $\mathcal{P}$ is 2-Cockcroft.

Example 1.1.13 Let $\mathcal{P} =< a, b; a^2, b^3, [a, b] >$ as in Example 1.1.1 and so $\pi_2(\mathcal{P})$ is generated by $\{\mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_3, \mathbb{P}_4\}$. Since $exp_{[a,b]}(\mathbb{P}_2) = 2$ and $exp_{[a,b]}(\mathbb{P}_4) = 3$ then clearly $\mathcal{P}$ is not p-Cockcroft for any prime p.

Note that

$$\text{Aspherical} \Rightarrow \text{CA} \Rightarrow \text{Cockcroft} \Rightarrow p\text{-Cockcroft}.$$

Since $\mathcal{P}$ can be regarded as a 2-complex we can consider the Euler characteristic of $\mathcal{P}$, $\chi(\mathcal{P}) = 1 - \mid \boldsymbol{x} \mid + \mid \boldsymbol{r} \mid$. It is known (see for example [24]) that $\chi(\mathcal{P})$ is bounded below by $\delta(G) = 1 - rk_{\mathbb{Z}} H_1(G) + d(H_2(G))$ (where $rk_{\mathbb{Z}}$ denotes the rank of torsion free part, $d()$ denotes the minimal number of generators and G is the group defined by $\mathcal{P}$).

Definition 1.1.14 *A presentation $\mathcal{P}$ is said to* minimal *if it has minimal Euler characteristic over all finite presentations defining the group G.*

Definition 1.1.15 *A presentation $\mathcal{P}$ is said to be* efficient *if $\chi(\mathcal{P}) = \delta(G(\mathcal{P}))$. A group G is said to be* efficient *if it admits an efficient presentation.*

Note that if a presentation is efficient then it is minimal.

Example 1.1.16 Note that the finite cyclic group $\mathbb{Z}_6$ can be defined by presentations $\mathcal{P}_1 =< a, b; a^2, b^3, [a, b] >$ and $\mathcal{P}_2 =< t; t^6 >$. Since $H_1(\mathbb{Z}_6) = \mathbb{Z}_6$ and $H_2(\mathbb{Z}_6) = 0$ then $\delta(\mathbb{Z}_6) = 1$. Thus $\mathcal{P}_2$ is an efficient presentation for $\mathbb{Z}_6$ while $\mathcal{P}_1$ is not.

Note that if we can find a minimal presentation $\mathcal{P}$ for a group G such that $\mathcal{P}$ is not efficient then $\chi(\mathcal{P}') \geq \chi(\mathcal{P}) > \delta(G)$ for all presentations $\mathcal{P}'$ defining the same group G. Thus there is no efficient presentation for G, that is G is not efficient. Examples of non-efficient groups were given in Swan [54] and their minimal presentations were given in [55]. New examples can be found in [4].

It is known (see [38] or §2.3.3) that a group G is efficient if and only if it is p-Cockcroft for some prime p.

There has been a lot of ad hoc work done on determining whether various types of groups are efficient (see for example [7, 14, 37, 56]). So one is led to ask whether these ad hoc calculations could be done algorithmically, that is

- *Is there any algorithm to decide for any given finite presentation whether the given group defined by the presentation is efficient?*

One of our main results (see Theorem 2.3.2) shows that no such algorithm exists.

Definition 1.1.17 *A group G is said to be* π_2-free *or* π_2-projective *if G admits a presentation $\mathcal{P}$ such that $\pi_2(\mathcal{P})$ is a free or projective left $\mathbb{Z}G$-module respectively.*

Note that all aspherical groups are π_2-free and π_2-projective.

1.1.3 An exact sequence concerning $\pi_2(\mathcal{P})$

A *projective resolution* of the trivial G-module $\mathbb{Z}$ is an exact positive chain complex

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

where each P_i is projective. This resolution will be known as a *free resolution* if each P_i is free. We say that this resolution is *n-finite* if $P_0, P_1, \cdots, P_n$ are finitely generated and G is *of type* FP_n $(0 \leq n \leq \infty)$ if G has an n-finite projective resolution. Clearly if G is of type FP_n then it is of type FP_{n-1}.

Let $\mathcal{P} =< \boldsymbol{x}; \boldsymbol{r} >$ define the group $G = F(\boldsymbol{x})/N$. Then the *relation module* $M(\mathcal{P})$ *of* $\mathcal{P}$ is the abelian group N/N', regarded as a left $\mathbb{Z}G$-module under the (well-defined) action

$$\overline{W} \cdot [U]N' = [WUW^{-1}]N' \ (\overline{W} \in G, [U] \in N).$$

Let P_1 and P_2 be the free left $\mathbb{Z}G$-modules with basis $\{e_x : x \in \boldsymbol{x}\}$ and $\{e_R : R \in \boldsymbol{r}\}$ respectively. Then there is a surjective module homomorphism (see for example [4, 48])

$$\rho_2 : P_2 \longrightarrow M(\mathcal{P})$$

$$e_R \mapsto [R]N' \ (R \in \boldsymbol{r})$$

and there is an injective module homomorphism (see for example [4, 48])

$$\mu_1 : M(\mathcal{P}) \longrightarrow P_1$$

$$[W]N' \mapsto \sum_{x \in \boldsymbol{x}} \theta(\frac{\partial [W]}{\partial x}) e_x$$

where $\frac{\partial}{\partial x}$ is Fox derivation (refer [43, §II.3]), θ is the ring homomorphism $\mathbb{Z}F \longrightarrow \mathbb{Z}G$ induced by the natural surjection $F \longrightarrow G$.

We have the following exact sequence (refer for example [4])

$$0 \longrightarrow \pi_2(\mathcal{P}) \xrightarrow{\mu_2} P_2 \xrightarrow{\mu_1 \rho_2} P_1 \xrightarrow{\rho_1} \mathbb{Z}G \xrightarrow{\rho_0} \mathbb{Z} \longrightarrow 0 \qquad (1.1)$$

where ρ_0 is the augmentation map which takes each elements of G to 1, and the module homomorphism

$$\rho_1 : P_1 \longrightarrow \mathbb{Z}G$$

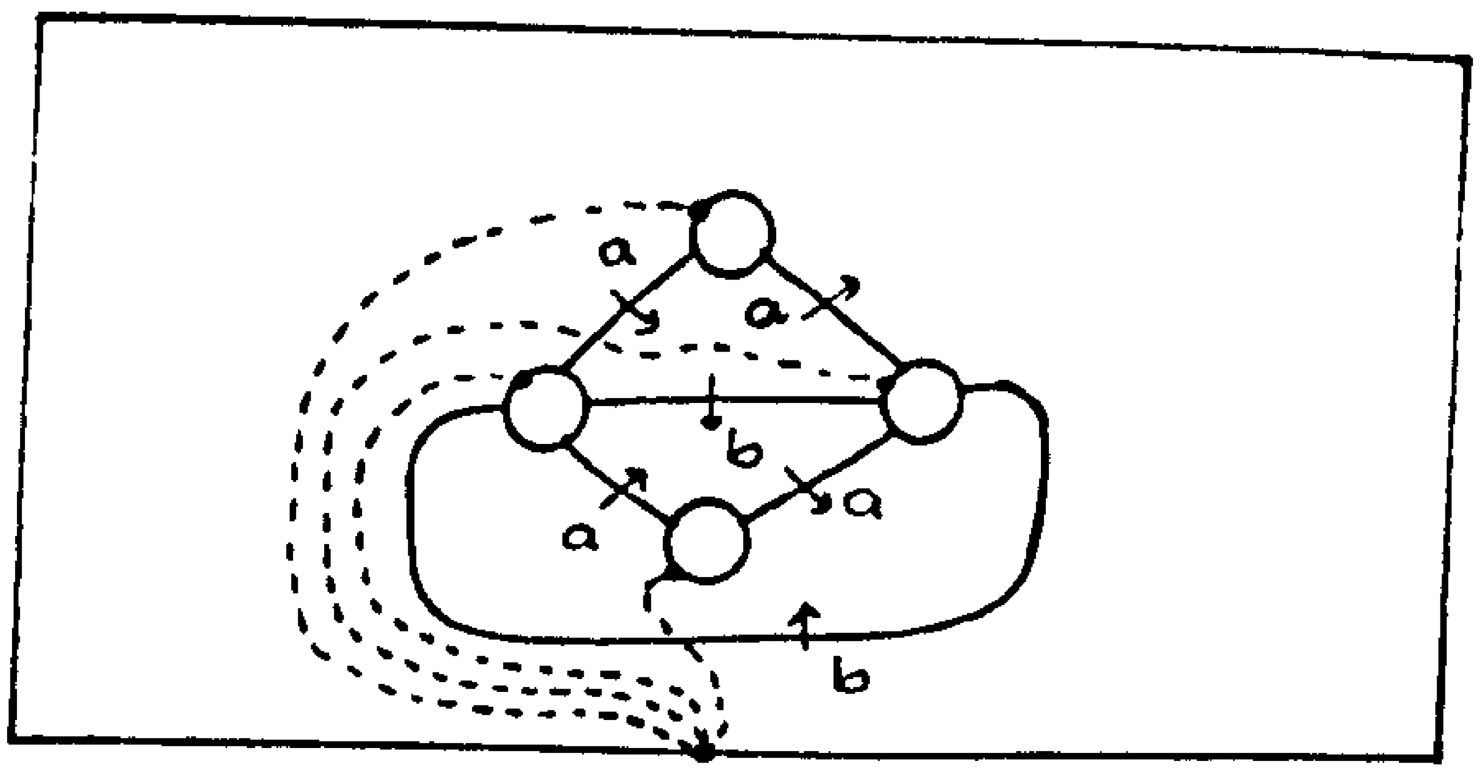
$$e_x \mapsto 1 - \overline{x}$$

is surjective. The embedding μ_2 is given as follows. Let $< \mathbb{P} > \in \pi_2(\mathcal{P})$ and suppose that $\mathbb{P}$ has discs $\Delta_1, \Delta_2, \cdots, \Delta_n$ with label $R_1^{\epsilon_1}, R_2^{\epsilon_2}, \cdots, R_n^{\epsilon_n}$ respectively ($R_i \in \boldsymbol{r}, \epsilon_i =$

$\pm 1, i = 1, 2, \cdots, n$). Let w_i be a transverse path from the basepoint of $\mathbb{P}$ to the basepoint of Δ_i for $1 \leq i \leq n$. Let W_i be the label on w_i. Note that if we choose another transverse path $\dot{w}_i$ from the basepoint of $\mathbb{P}$ to the basepoint of Δ_i, then $\overline{W_i} = \overline{\dot{W}_i}$ in G where $\dot{W}_i$ is the label on $\dot{w}_i$ (see for example [48]). Then

$$\mu_2(< \mathbb{P} >) = \sum_{i=1}^{n} \epsilon_i \overline{W_i} e_{R_i}.$$

Example 1.1.18 Let $G = \mathbb{Z}_2 \oplus \mathbb{Z}$ be defined by $\mathcal{P} =< a, b; a^2, [a, b] >$ and consider



Then $\mu_2(< \mathbb{P} >) = (\bar{b} - 1)e_{a^2} + (1 + \bar{a})e_{[a,b]}$.

Let $I_2(\mathcal{P})$ be the two sided ideal in $\mathbb{Z}G$ generated by the non-zero coefficient of elements of the image of μ_2. This ideal is called the *second Fox ideal*. Knowing a set of generators of $\pi_2(\mathcal{P})$ enables us to compute a set of generators of $I_2(\mathcal{P})$. In Example 1.1.18 (see also Example 1.1.12), $I_2(\mathcal{P})$ is generated by $\{\bar{b} - 1, 1 + \bar{a}, 1 - \bar{a}\}$. Note that there is also the first Fox ideal $I_1(\mathcal{P})$ corresponding to the image of μ_1. The zeroth Fox ideal is just the augmentation ideal IG. This concept of Fox ideals is discussed in [41, 42]. We need this concept for the test of minimality (see §2.3.3).

It follows from the above sequence (1.1) that every finitely presented aspherical group is of type FP_∞. In particular every finitely generated free group is of type FP_∞. Also from the above sequence, it follows (using Proposition 4.3 [12, page 193]) that a group G defined by a finite presentation $\mathcal{P}$ is of type FP_3 if and only if $\pi_2(\mathcal{P})$ is finitely generated.

Also note that a group G is of type FP_1 if and only if G is finitely generated (see for example [6, Proposition 2.1]).

The concept of FP_n will be discussed in §2.4.1.

1.2 Decision problems

This section introduces the main theme for Chapters 2 and 3.

Decision problems for finitely presented groups were formulated by Max Dehn [18] in 1911. The objective is to determine the existence and nature of algorithms which decide:

1. *Global properties*—whether or not groups as a whole possess certain properties or relationships.

2. *Local properties*—whether or not elements of a group have certain properties or relationships.

A rather comprehensive accounts of history, motivations and results on decision problems can be found in [45].

In Chapter 2 we deal with global properties concerning second homotopy modules. Our main results are Theorems 2.3.1 and 2.3.2.

- *For any given prime p, there exists a recursive class Ω_1 of finite presentations of groups such that the problem of determining whether an arbitrary member of Ω_1 defines a group which is p-Cockcroft is recursively unsolvable.*

- *There exists a recursive class Ω_2 of finite presentations of groups such that the problem of determining whether an arbitrary member of Ω_2 defines an efficient group is recursively unsolvable.*

A local fundamental property which has played a central role in decision problems is the *word problem.* Let G be a group given by a finite presentation $\mathcal{P} = <\boldsymbol{x}; \boldsymbol{r}>$. The word problem is the question of asking for the existence of an algorithm to determine of an arbitrary word W on $\boldsymbol{x}$ whether or not $\overline{W} = [W]N = 1$ in G. If we regard $\mathcal{P}$ as a 2-complex then the word problem is the problem of determining the existence of algorithm to determine for any arbitrary element of $\pi_1(<\boldsymbol{x}>)$ whether its image under the inclusion induced homomorphism

$$\pi_1(<\boldsymbol{x}>) \longrightarrow \pi_1(<\boldsymbol{x}; \boldsymbol{r}>)$$

is trivial.

The main unsolvabality result is the following Novikov-Boone Theorem [10, 46].

Theorem 1.2.1 *There exists a finitely presented group with unsolvable word problem.*

In Chapter 3, we will introduce the analogue of the word problem, a dimension higher called the *picture problem* and provide the analogue of the above theorem. Our main result is Theorem 3.1.1.

- *There exists a finite 3-presentation* $\mathcal{K} =< \boldsymbol{x}; \boldsymbol{r}; \boldsymbol{P} >$ *which has unsolvable picture problem. Moreover* $\mathcal{K}$ *can be chosen such that the word problem for the underlying presentation* $\mathcal{K}^{(2)} =< \boldsymbol{x}; \boldsymbol{r} >$ *has solvable word problem.*

Let H be a finitely generated subgroup of G. Then the *generalised word problem* for H in G is the problem of deciding for an arbitrary word W on $\boldsymbol{x}$, whether or not W defines an element of H (if H is trivial then the generalised word problem is just simply the word problem). This concept is useful in §3.1.

1.3 One relator relative presentation

This section introduces the main theme for Chapters 4, 5 and 6.

The concept of a *one relator relative presentation*

$$\mathcal{P} =< H, t; R >$$

will be introduced in Chapter 4. Here H is any arbitrary group, $< t >$ is an infinite cyclic group and R is an element of $H* < t >$ of the form

$$t^{\epsilon_1} h_1 t^{\epsilon_2} h_2 \cdots t^{\epsilon_n} h_n$$

where $\epsilon_i = \pm 1, h_i \in H$ for $i = 1, 2, \cdots, n$ (if $h_i = 1$ in H, then $\epsilon_i \neq -\epsilon_{i+1}$). Since t occurs n times, we say that this is a *case of t-length n*. The group G defined by $\mathcal{P}$, is the group

$$\frac{H* < t >}{<< R >>}$$

where $<< R >>$ is the normal closure of R. Relative presentations are considered for example in [3, 8, 22, 31].

There are two major issues that have been asked (see [9]):

1. When is the natural map $H \longrightarrow G$ is injective.

2. When is $\mathcal{P}$ aspherical in the sense of relative presentations (refer Definition 4.1.1)

A relative presentation $\mathcal{P}$ is *injective* if the natural map $H \longrightarrow G$ is injective. We are interested to see when $\mathcal{P}$ is aspherical. In Chapter 4 we survey the basic concepts, the important theorems for relative presentations and the tests for asphericity. The first case of interest is t-length 3. If R has the form $th_1th_2th_3$, Levin [40] shows the injectivity, and the asphericity is discussed in [8]. Howie [30] shows the injectivity for the case $th_1th_2t^{-1}h_3$ while Edjvet [22] considers asphericity for this form. Thus the t-length 4 is now considered. There are four main cases for length four (refer §4.1). The injectivity has been discussed in [23, 40]. For the case R has the form $th_1th_2th_3th_4$, the asphericity is considered in [3] . In Chapters 5 and 6 we cover the next case, that is all presentations of the form

$$< H, t; tatbtct^{-1}d >$$

where a, b, c and d are elements of H (c and d are not trivial). There are some exceptions that we can not decide at this moment. We list these exceptions in §6.5. Excluding these exceptions, our main results are:

- **Theorem 5.1.1**
  Let $\mathcal{P} =< H, t; t^3at^{-1}a >$ where a is a non-trivial element of H. Then $\mathcal{P}$ is aspherical if and only if a has infinite order in H.

- **Theorem 5.2.1**
  Suppose that $\mathcal{P} =< H, t; t^3at^{-1}b >$ is not an exceptional case and $2 \leq o(a) \leq o(b)$ where a and b are distinct elements in H. Then $\mathcal{P}$ is aspherical if and only if none of these holds:

1. $a^2 = 1, a = b^2$

2. $a^2 = 1, a = b^3$

3. $a^2 = b^3 = [a, b] = 1$

4. $o(a) = p, o(b) = q, o(ab^{-1}) = k$ *for* $\frac{1}{p} + \frac{1}{q} + \frac{1}{k} > 1$ *where* $\frac{1}{\infty} := 0$

5. $a = b^{-1}$ *of finite order*

- **Theorem 6.1.1**

  Let $\mathcal{P} =< H, t; t^2atat^{-1}c >$ *where* a *and* c *are non-trivial elements of* H *(*a *and* c *may be equal). Then* $\mathcal{P}$ *is aspherical if and only if* c *has infinite order.*

- **Theorem 6.2.1**

  Suppose that $\mathcal{P} =< H, t; t^2atbt^{-1}b >$ *is not an exceptional case where* a *and* b *are non-trivial elements of* H*. Then* $\mathcal{P}$ *is aspherical except when* $b = a^{-1}$ *and* b *has finite order.*

- **Theorem 6.3.1**

  Suppose that $\mathcal{P} =< H, t; t^2atbt^{-1}a >$ *is not an exceptional case where* a *and* b *are distinct non-trivial elements of* H*. Then* $\mathcal{P}$ *is aspherical if and only if none of these holds:*

  1. $o(a) = 2, o(b) < \infty$

  2. $b = a^{-2}$ *of finite order*

  3. $b = a^2, o(a) = 3$ *or* 4

- **Theorem 6.4.1**

  Suppose that $\mathcal{P} =< H, t; t^2atbt^{-1}c >$ *is not an exceptional case where a,b and c are distinct non-trivial elements of* H*. Then* $\mathcal{P}$ *is aspherical if and only if none of these holds:*

  1. $a = c^{-1}, o(b) < \infty$

  2. $b = ac$ *and* $\frac{1}{p} + \frac{1}{q} + \frac{1}{k} > 1$ *where o(a)=p, o(b)=q and o(c)=k (* $\frac{1}{\infty} := 0$*)*

Chapter 2

Global problems for second homotopy modules

Global problems for finitely presented groups are concerned with the question of determining the existence of algorithms to decide whether or not groups as a whole possess certain properties or relationships. We are interested in properties concerning second homotopy modules—aspherical, CA, Cockcroft, p-Cockcroft, efficient, π_2-free, π_2-projective and FP_3 (in fact we consider more general—FP_n for $n \geq 3$). All these properties will be shown to be *recursively unsolvable*, that is there are no effective method which can be applied to an arbitrary finite presentation $\mathcal{P}$ to determine whether or not $G(\mathcal{P})$ has these properties.

In §2.1 we show that properties aspherical, CA, π_2-free and π_2-projective are all Markov properties and hence unsolvable. The properties being of type FP_n and Cockcroft are homological Markov properties (see §2.2) and hence unsolvable. However for the properties p-Cockcroft and efficient it is not known whether or not they are Markov or homological Markov properties. We deal with these properties in §2.3. Originally we were not aware of the homological Markov property and we had our independent proofs for FP_n and Cockcroft. These proofs will be given in §2.4.

The main results for this chapter are Theorems 2.3.1 and 2.3.2.

2.1 Markov property

An abstract property P of finitely presented groups is said to be a *Markov property* if there are two finitely presented groups G_+ and G_- such that:

1. G_+ has the property P.

2. G_- can not be embedded into any finitely presented group that has the property P.

These groups G_+ and G_- will be known as *positive* and *negative witnesses* for the Markov property P respectively. It should be emphasized that if P is a Markov property then the negative witness does not have the property P, nor is it embedded in any finitely presented group with property P. A familiar example of a Markov property is being trivial. We may take G_- to be any finitely presented non-trivial group. Having rank 2 is not a Markov property since every finitely presented group can be embedded in a finitely presented group with rank 2 by the Higmann-Nuemann-Neumann embedding.

The main unsolvability result concerning the recognition of properties of finitely presented groups is the following:

Theorem 2.1.1 (Adian [1, 2], Rabin [49]) *If P is a Markov property of finitely presented groups, then P is not recursively solvable.*

The proof of this theorem (which can also be found in [43, Theorem 4.4.1] or [45, Theorem 3.3]) shows that many properties other than Markov properties are not recursively solvable. A property P is said to be *incompatible with free product* if:

1. There is a finitely presented group G_+ with the property P.

2. If A is any non-trivial finitely presented group, then $A * G_+$ does not have the property P.

Thus to show that a property is recursively unsolvable, one may check whether or not the property (or its negation) satisfies one of these properties. We will show that the properties of being aspherical, CA, π_2-free and π_2-projective (refer §1.1.2 for definitions) are all Markov properties and hence are undecidable.

We will need the following idea. A group K has *cohomological dimension* k (we write $\text{cd}K = k$) if $H^q(K, A) = 0$ for all integers $q > k$ and all left $\mathbb{Z}K$-modules A, but there exists a left $\mathbb{Z}K$-module B such that $H^k(K, B) \neq 0$.

Proposition 2.1.2 *If S is a subgroup of G, then $cdS \leq cdG$.*

The proof can be found in [6, Proposition 4.9] or [12, Proposition 8.2.4].

The following result is also well-known (see for example [4]).

Proposition 2.1.3 *Let G be a group defined by a presentation $\mathcal{P}$. Then*

i. $cdG \leq 2$ if $\pi_2(\mathcal{P}) = 0$.

ii. $cdG \leq 3$ if and only if $\pi_2(\mathcal{P})$ is projective.

Remark 2.1.4 Note that $\text{cd}\mathbb{Z}^n = n$ $(H^n(\mathbb{Z}^n, \mathbb{Z}) \simeq \mathbb{Z})$.

(Refer [12, page 185].)

We obtain the first result:

Theorem 2.1.5 *The properties aspherical, CA, π_2-free and π_2-projective are all Markov properties.*

Proof Note that $\mathbb{Z}$ has a presentation $< x >$ such that $\pi_2(< x >) = 0$. Thus $\mathbb{Z}$ has all of these properties. Hence we only need to choose four negative witness groups G_1, G_2, G_3 and G_4 of property aspherical, CA, π_2-free and π_2-projective respectively.

Consider $G_1 = \mathbb{Z}^3$, so $\text{cd}G_1 = 3$. Suppose that $\mathbb{Z}^3$ can be embedded into group G having a presentation $\mathcal{P}$ such that $\pi_2(\mathcal{P}) = 0$. Then by Proposition 2.1.3(i), $\text{cd}G \leq 2$ which contradicts Proposition 2.1.2.

Consider $G_2 = \mathbb{Z}^3$. We also claim that $\mathbb{Z}^3$ can not be embedded into any group G which is CA. Suppose it could be. Then by [15, Proposition 2.5], $\mathbb{Z}^3$ has a (not necessarily finite) presentation $\mathcal{P}_{\mathbb{Z}^3} =< \boldsymbol{x}; \boldsymbol{r} >$ which is CA. For all $R \in \boldsymbol{r}$, write

$R = \dot{R}^{p(R)}$ where $\dot{R}$ is not a proper power and $p(R)$ is a positive integer. Since $\mathcal{P}_{\mathbb{Z}^3}$ is CA, it follows that (refer [48, Theorem 1.7(i)]) $\dot{R}$ defines an element of order exactly $p(R)$ in $G(\mathcal{P}_{\mathbb{Z}^3}) = \mathbb{Z}^3$. Since $\mathbb{Z}^3$ is torsion free then $p(R) = 1$. This means that no element of $\boldsymbol{r}$ is a proper power and hence by [48, Theorem 1.8], $\mathcal{P}_{\mathbb{Z}^3}$ is aspherical. Then by Proposition 2.1.3(i), $\text{cd}\mathbb{Z}^3 \leq 2$ which contradicts Remark 2.1.4.

We may choose G_3 and G_4 to be $\mathbb{Z}^4$. Now $\mathbb{Z}^4$ can not be embedded into any group G which has property π_2-free or π_2-projective because otherwise G would have $\text{cd}G \leq 3$ which contradicts Proposition 2.1.2. •

2.2 Homological Markov property

Miller [45, Definition 8] (see also Gordon [28]) introduced a new property—concerned with decision-theoretic aspects of the homological invariants of finitely presented groups. An abstract property P of finitely presented groups is said to be a *homological Markov property* if there are two finitely presented groups G_+ and G_- such that:

1. G_+ has the property P.

2. If Y is a finitely presented group such that $H_n(G_-) \subseteq H_n(Y)$ for $n > 1$, then Y does not have the property P.

Generalising the argument of Adian-Rabin Theorem (Theorem 2.1.1), Miller [45, Theorem 8.6] proves

Theorem 2.2.1 *If P is a homological Markov property of finitely presented groups, then P is not recursively unsolvable.*

We will show that the properties of being of type $FP_n (n \geq 3)$ and of being Cockcroft are homological Markov properties.

Fix $n \geq 3$. Note that $\mathbb{Z}$ is of type FP_∞ and hence it will be the positive witness for type FP_n. For the negative witness, we may take the Stalling group A_2 (refer §2.4.1). Stallings [53] shows that $H_3(A_2)$ is not finitely generated. Now let Y be any finitely presented group such that $H_n(A_2) \subseteq H_n(Y)$ for $n > 1$. Suppose that Y is of type FP_n, then by [6, Proposition 2.15], $H_k(Y)$ is finitely generated for $0 \leq k \leq n$. In particular $H_3(Y)$ is finitely generated and this will contradicts the fact that $H_3(A_2) \subseteq H_3(Y)$.

For the Cockcroft property, we may again take $\mathbb{Z}$ to be the positive witness. We may take G_- to be $\mathbb{Z} \oplus \mathbb{Z}_4$ and hence $H_2(G_-) = \mathbb{Z}_4$ which is clearly not free abelian. Let Y be any finitely presented group such that $H_n(G_-) \subseteq H_n(Y)$ for $n > 1$. Suppose that Y is Cockcroft. Then by a theorem of Cockcroft (quoted in [11, Introduction]) $H_2(Y)$ is free abelian. This contradicts the fact that $H_2(G_-) \subseteq H_2(Y)$.

2.3 Decision for p-Cockcroft and efficiency

It is not clear whether or not these properties (or their negations) are Markov or homological Markov properties. Having been unable to settle these problems we prove directly

Theorem 2.3.1 *For any given prime p, there exists a recursive class Ω_1 of finite presentations of groups such that the problem of determining whether an arbitrary member of Ω_1 is p-Cockcroft is recursively unsolvable.*

As a generalisation of Theorem 2.3.1, we prove

Theorem 2.3.2 *There exists a recursive class Ω_2 of finite presentation of groups such that the problem of determining whether an arbitrary member of Ω_2 is efficient is recursively unsolvable.*

We will prove Theorem 2.3.1 by first showing the existence of a finitely presented aspherical group U with unsolvable word problem (see §2.3.2). Having established the group U , we will construct a family $\Omega_1 = \{L_W : W$ is a word in the generators of $U\}$ of finitely presented groups such that L_W is p-Cockcroft if and only if $\overline{W} \neq 1$ in U. Since U is finitely presented, Ω_1 is a recursive class. Any algorithm which would determine whether or not an arbitrary member of Ω_1 is p-Cockcroft would yield a solution of the word problem for U. Hence there is no algorithm which determines wether or not an arbitrary member of Ω_1 is p-Cockcroft.

It will turn out that the family Ω_2 can be chosen to be Ω_1 and hence efficiency is also unsolvable.

Note that Gordon [28] has shown the (easier) result that *deficiency* is not decidable. It should be noted that for each group in our class Ω_1, one can determine the deficiency. In fact, the presentations used to define the groups L_W (see §2.3.3) realise the deficiency.

2.3.1 Generalised HNN-extensions

We need the idea of *generalised HNN-extension* which was introduced by Klyachko [39, Theorem 2]. Suppose that we have the following:

1. A group G_0 defined by $\mathcal{P}_0 =< \boldsymbol{x}; \boldsymbol{r} >$.

2. A group V defined by $\mathcal{V} =< \boldsymbol{v}; \boldsymbol{t} >$.

3. Groups S and $\hat{S}$, both are subgroups of G_0 with specified isomorphism

$$\sigma : S \longrightarrow \hat{S}.$$

4. A non-trivial element $\xi \in G_0 * V$ written in normal form. We require that the first and the last term of ξ belong to V. We also require that every term of ξ has infinite order.

Then the generalised HNN-extension with base group G_0 and associated subgroups S and $\hat{S}$ is the group

$$G =< G_0, V; s\xi\sigma(s)^{-1}\xi^{-1}(s \in S) > . \tag{2.1}$$

Note that if we take the group V to be the infinite cyclic group $< t >$ and $\xi = t$, then G is simply the *ordinary HNN-extension.*

We will define a presentation for G. Let $a_i, \hat{a}_i (i \in I)$ be non-empty freely reduced words on $\boldsymbol{x}$ such that:

1. The subgroup S is generated by the elements represented by the a_i's.

2. The subgroup $\hat{S}$ is generated by the elements represented by the $\hat{a}_i$'s.

3. a_i and $\hat{a}_i$ correspond under the isomorphism σ.

Choose a reduced word T on $\boldsymbol{x} \cup \boldsymbol{v}$ which represents the element ξ and let $T_i = a_i T \hat{a}_i^{-1} T^{-1}$ and let $\boldsymbol{a} = \{T_i : i \in I\}$. Then

$$\mathcal{P} = < \boldsymbol{x}, \boldsymbol{v}; \boldsymbol{r}, \boldsymbol{t}, \boldsymbol{a} > \tag{2.2}$$

is a presentation for G. Thus if G is an ordinary HNN-extension, then G can be defined by

$$\mathcal{P} = < \boldsymbol{x}, t; \boldsymbol{r}, a_i t \hat{a}_i^{-1} t^{-1} (i \in I) > . \tag{2.3}$$

The following is a consequence of [4, Theorem 2].

Proposition 2.3.3 *Let G be a generalised HNN-extension as defined in (2.1). If $\mathcal{P}_0$ and $\mathcal{V}$ are aspherical and S is free on the a_i's then the presentation $\mathcal{P}$ as define in (2.2) is aspherical.*

Note that if G is an ordinary HNN-extension, then $\mathcal{V} = < t; \; >$ is always aspherical.

2.3.2 The existence of group U

The following result can be found in [45, Theorem 4.12].

Theorem 2.3.4 *There exists a finitely presented group U having unsolvable word problem. Indeed U can be obtained from a free group by applying three successive HNN-extensions where the associated subgroups are finitely generated free groups.*

We will choose this group U. Since U can be obtained from a free group then for each step of extensions we may choose the associated subgroups to be free on the given generators. Then by Proposition 2.2.3 we may obtain an aspherical presentation $\mathcal{U}$ for U.

2.3.3 The proof of Theorem 2.3.1

We let F_2 to be the free group $< a, b >$ and choose H and $\hat{H}$ to be the free subgroups of rank 2 generated by $\{a^2, ab\}$ and $\{a^{-3}, ba^3\}$ respectively. If $\mathcal{U} = < \boldsymbol{u}; \boldsymbol{s} >$

is the presentation for U as in §2.3.2, then for any word W on $\boldsymbol{u}$, let $\mathcal{L}_W$ denote the presentation

$$< \boldsymbol{u}, a, b; \boldsymbol{s}, a^2Wa^3W^{-1}, abWa^{-3}b^{-1}W^{-1} >$$

and let L_W be the group define by $\mathcal{L}_W$. We need the following result which is due to Lustig [41, Corollary 2] (see also [38, Theorem 1.4] and [42, §2.7]).

Proposition 2.3.5 (Test for minimality) *Let G be the group defined by $\mathcal{P}$. If there is a ring homomorphism ϕ from $\mathbb{Z}G$ into the matrix ring of all $k \times k$-matrices $(k \geq 1)$ over some commutative ring A with 1, such that $\phi(1) = 1$, and if ϕ maps the second Fox ideal $I_2(\mathcal{P})$ to 0, then $\mathcal{P}$ is minimal.*

The following result which is essentially due to Epstein [24] can be found in [38, Theorem 1.3].

Proposition 2.3.6 *A finite presentation $\mathcal{P}$ is efficient if and only if $\mathcal{P}$ is p-Cockcroft for some prime p.*

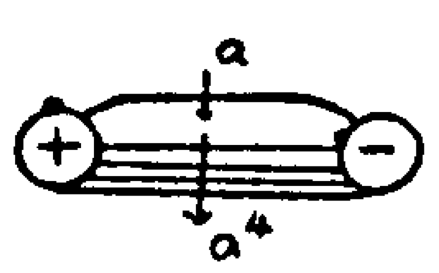
We use these results to prove

Lemma 2.3.7 *If $\overline{W} = 1$ in U then L_W is not p-Cockcroft (for any prime p).*

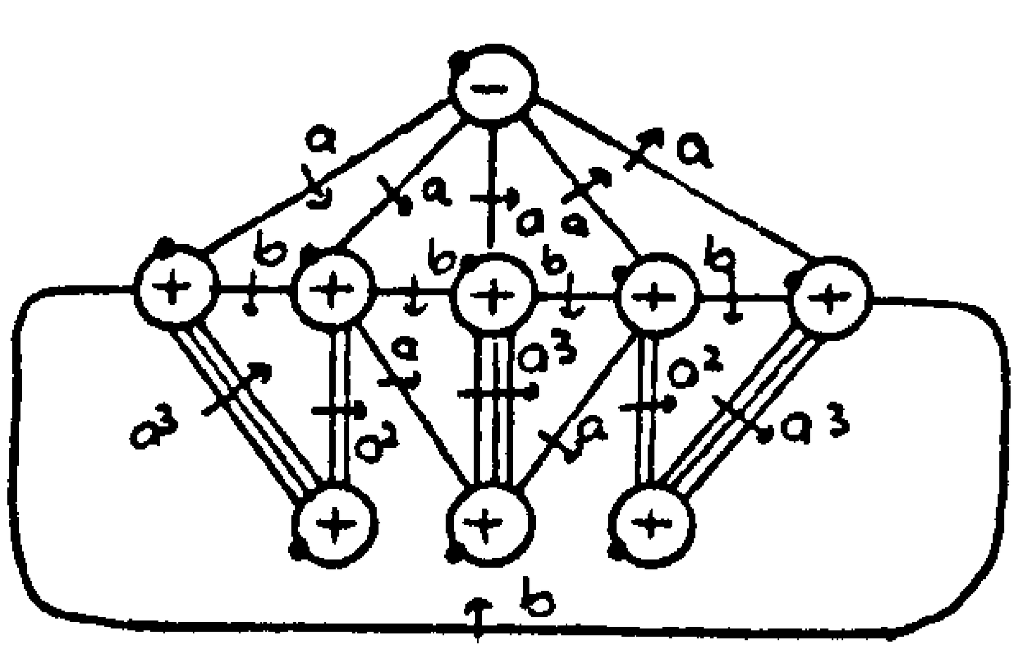
Proof If $\overline{W} = 1$ in U, then clearly

$$\mathcal{L}'_W = < \boldsymbol{u}, a, b; \boldsymbol{s}, a^5, aba^{-3}b^{-1} >$$

is a presentation of L_W. By [4, Theorem 2] and the fact that $\mathcal{U}$ is aspherical, then $\pi_2(\mathcal{L}'_W)$ is generated by



and



Thus $\mathcal{L}'_W$ is clearly not p-Cockcroft for any prime p and hence not efficient by Proposition 2.3.6. We will show that $\mathcal{L}'_W$ is minimal and hence there could not be an efficient presentation which defines the group L_W. Thus we can conclude that L_W is not p-Cockcroft for any prime p.

From the above pictures, $I_2(\mathcal{L}'_W)$ is generated (as a 2-sided ideal) by

$$I = \{1 - \bar{a}, 1 + \bar{a} + \bar{a^2} + \bar{a^3} + \bar{a^4}, 3\bar{b} - 1\}.$$

Let $< x >$ be an infinite cyclic group and consider the ring homomorphism

$$\mathbb{Z}L_W \longrightarrow \mathbb{Z} < x >$$

arising from the group homomorphism defined by

$$u \mapsto 1(u \in \boldsymbol{u}), a \mapsto 1,\ b \mapsto x.$$

If we consider

$$\mathbb{Z} < x > \longrightarrow \mathbb{Z}_5$$

by sending all integer coefficients to their congruence modulo 5 and sending x to the congruence class of 2, then the mapping

$$\mathbb{Z}L_W \longrightarrow \mathbb{Z} < x > \longrightarrow \mathbb{Z}_5$$

sends I to 0 and 1 to 1. Hence $\mathcal{L}'_W$ is minimal. •

Lemma 2.3.8 *If $\overline{W} \neq 1$ in U then L_W is p-Cockcroft (true for any prime p).*

Proof Let $\overline{W} \neq 1$ in U. Then L_W is a generalised HNN-extension define by $\mathcal{L}_W$ as in §2.3.1. Since $< a, b >$ and $\mathcal{U}$ are aspherical and H is free on the given generators then by Proposition 2.3.3, $\mathcal{L}_W$ is aspherical and hence is p-Cockcroft (for any prime p). •

2.3.4 The proof of Theorem 2.3.2

Let $\Omega_2 = \{L_W : W \text{ is a word in the generators of } U\} = \Omega_1$ as we defined in §2.3.3. Then Lemma 2.3.7, 2.3.8 and also Proposition 2.3.6 guarantee that L_W is efficient if and only if $\overline{W} \neq 1$ in U. Thus we obtain the desired result.

2.4 Alternative proofs for FP_n and Cockcroft

We obtained these independent proofs before we become aware of the existence of homological Markov property.

2.4.1 Decision problem for type FP_n

We prove

Theorem 2.4.1 *For a given $n \geq 3$, there exists a recursive class Ω_3 of finitely presented groups such that the problem of determining whether an arbitrary member of Ω_3 defines a group of type FP_n is recursively unsolvable.*

Let U be the group defined in §2.3.2. Then we may construct, for a fixed $n \geq 3$ a family $\Omega_3 = \{G_W : W$ is a word in the generators of $U\}$ of finitely presented groups such that G_W is of type FP_n if and only if $\overline{W} \neq 1$ in U. Since U is finitely presented, Ω_3 is a recursive class. Any algorithm which would determine whether or not an arbitrary member of Ω_3 is of type FP_n would yield a solution of the word problem for U. Hence there is no algorithm which determines whether or not an arbitrary member of Ω_3 is of type FP_n.

We will need the following result [6, Proposition 2.13(b)]:

Proposition 2.4.2 *Let $G = G_0 *_{S,\sigma}$ be the HNN-extension with base group G_0. Then*

i) if G_0 is of type FP_n, S is of type FP_{n-1} then G is of type FP_n.
ii) if G, S are of type FP_n then so is G_0.
iii) if G_0 is of type FP_{n-1}, G is of type FP_n then S is of type FP_{n-1}.

The following group appears in Beiri [6, page 37]. Let

$$D_m = <x_1, y_1> \times <x_2, y_2> \times \cdots \times <x_m, y_m>$$

be the direct product of m free groups of rank 2. Let F_∞ be the free group on the generators $\{a_k\}$ $(k \in \mathbb{Z})$. Define D_m-action (ρ) on F_∞ by

$$x_i \cdot a_j = y_i \cdot a_j = a_{j+1}.$$

Then define

$$A_m = F_\infty \times_\rho D_m$$

to be the split extension of F_∞ by D_m. Note that A_m can be considered as a two-step HNN-extension with base group A_{m-1}, first with stable letter x_m then with stable letter y_m. Thus A_{m-1} is a subgroup of A_m. Beiri shows that A_m is of type FP_m but not of type FP_{m+1} ([6, Proposition 2.14]). Note also that $A_m (m \geq 2)$ is finitely presented.

(For the case $m = 2$, this group is due to Stalling [53] and hence A_2 will be known as the Stalling group.)

Now we may construct the group G_W. Fix $n \geq 3$, let $B = A_n$, $C = A_{n-2}$ and let U be as in §2.3.2. Define

$$G^* = U * B * < x > .$$

Note that G^* is of type FP_n since U, B and $< x >$ are of type FP_n. It is also finitely presented. For any word W on the generator of U, define H_W to be the subgroup of G^* generated by the elements $(c_i x)W(c_i x)^{-1}W^{-1}c_i (i = 1, 2, \cdots, k)$ where $\{c_i : i = 1, 2, \cdots, k\}$ is a set of the generators of C. Note that since C is at least of type FP_1, C is finitely generated. Define

$$G_W = G^* *_{H_W, id}$$

to be the HNN-extension of G^* with the associated subgroup H_W. Then G_W is finitely presented since G^* is finitely presented and H_W is finitely generated.

Lemma 2.4.3 *If $\overline{W} = 1$ in U then G_W is not of type FP_n.*

Proof Let $\overline{W} = 1$ in U. Then $H_W = C$ is not of type FP_{n-1}. Since G^* is of type FP_n (in particular it is of type FP_{n-1}), then by Proposition 2.2.4(iii), G_W is not of type FP_n. •

Lemma 2.4.4 *If $\overline{W} \neq 1$ in U then G_W is of type FP_n.*

Proof We will show that H_W is free on the generators $(c_i x)W(c_i x)^{-1}W^{-1}c_i$ for $(i = 1, 2, \cdots, k)$ and hence is of type FP_∞ (in particular is of type FP_{n-1}). Then by Proposition 2.2.4(i), G_W is of type FP_n since G^* is of type FP_n.

Let $\alpha_i = (c_i x)W(c_i x)^{-1}W^{-1}c_i (i = 1, 2, \cdots, k)$. In order to show that H_W is free, we need to show that no non-empty reduced product of α_i's is equal to 1 in G^*. We will show by induction that for any reduced product

$$\alpha_{i_1}^{\beta_1}\alpha_{i_2}^{\beta_2}\cdots\alpha_{i_n}^{\beta_n} \text{ where } \alpha_{i_j}^{\beta_j} \neq \alpha_{i_{j+1}}^{-\beta_{j+1}} \text{ and } \beta_j = \pm 1 (j = 1, 2, \cdots, n-1)$$

when expressed in normal form in G^* will end with

1. $W^{-1}c_{i_n}$ if $\beta_n = 1$ or
2. $W^{-1}x^{-1}c_{i_n}^{-1}$ if $\beta_n = -1$.

The case for $n = 1$ is obviously true.

Assume $n > 1$. We can assume by inductive hypothesis that

$$\alpha_{i_1}^{\beta_1}\cdots\alpha_{i_{n-1}}^{\beta_{n-1}} = \begin{cases} W_1 W^{-1} c_{i_{n-1}} & \text{for } \beta_{n-1} = 1 \\ W_1 W^{-1} x^{-1} c_{i_{n-1}}^{-1} & \text{for } \beta_{n-1} = -1 \end{cases}$$

for some word W_1. Then

$$\alpha_{i_1}^{\beta_1}\cdots\alpha_{i_{n-1}}^{\beta_{n-1}}\alpha_{i_n}^{\beta_n} = \begin{cases} W_1 W^{-1} c_{i_{n-1}} c_{i_n} x W(c_{i_n}x)^{-1} W^{-1} c_{i_n} & \text{if } \beta_{n-1} = 1, \beta_n = 1 \\ W_1 W^{-1} c_{i_{n-1}} c_{i_n}^{-1} W(c_{i_n}x) W^{-1} x^{-1} c_{i_n}^{-1} & \text{if } \beta_{n-1} = 1, \beta_n = -1 \\ W_1 W^{-1} x^{-1} c_{i_{n-1}}^{-1} c_{i_n} x W(c_{i_n}x)^{-1} W^{-1} c_{i_n} & \text{if } \beta_{n-1} = -1, \beta_n = 1 \\ W_1 W^{-1} x^{-1} c_{i_{n-1}}^{-1} c_{i_n}^{-1} W(c_{i_n}x) W^{-1} x^{-1} c_{i_n}^{-1} & \text{if } \beta_{n-1} = -1, \beta_n = -1. \end{cases}$$

The word in the first case is clearly equivalent to $W_1 W^{-1} c x W(c_{i_n}x)^{-1} W^{-1} c_{i_n}$ where $c = c_{i_{n-1}} c_{i_n}$. This word does not collapse even if c is the identity.

The second and the third cases do not collapse since $c_{i_{n-1}} \neq c_{i_n}$ (otherwise our word would not be reduced).

The word in the fourth case is equivalent to $W_1 W^{-1} x^{-1} c W c_{i_n} x W^{-1} x^{-1} c_{i_n}^{-1}$ where $c = c_{i_{n-1}}^{-1} c_{i_n}^{-1}$ and so does not collapse.

In all cases, the word ends with the required form. Thus we can conclude that no non-empty reduced product is equal to 1 in G^* and hence H_W is free. •

2.4.2 Decision problem for Cockcroft

To show that Cockcroft is undecidable, a similar technique as in §2.3 will be used. We will prove

Theorem 2.4.5 *There exists a recursive class Ω_4 of finite presentations of groups such that the problem of determining whether an arbitrary member of Ω_4 is Cockcroft is recursively unsolvable.*

To prove this theorem, we will show the existence of a family $\Omega_4 = \{K_W : W$ is a word in the generators of $U\}$ of finitely presented group such that K_W is Cockcroft if and only if $\overline{W} \neq 1$ in U. The group U is the same group that we define in §2.3.2.

Let F_2 be the free group $< a, b >$ and H and $\hat{H}$ be free subgroups of rank two generated by $\{a^2, ab\}$ and $\{a^{-2}, ba\}$ respectively. Let $\mathcal{U} =< \boldsymbol{u}; \boldsymbol{s} >$ be the presentation for U as given in §2.3.2. For any word W on $\boldsymbol{u}$, define

$$\mathcal{K}_W =< \boldsymbol{u}, a, b; \boldsymbol{s}, a^2Wa^2W^{-1}, abWa^{-1}b^{-1}W^{-1} > .$$

Then let K_W be the group defined by $\mathcal{K}_W$.

Lemma 2.4.6 *If $\overline{W} = 1$ in U then K_W is not Cockcroft.*

Proof If $\overline{W} = 1$ in U then clearly

$$K_W = U * (\mathbb{Z}_4 \oplus \mathbb{Z})$$

and hence

$$\begin{aligned} H_2(K_W, \mathbb{Z}) &= H_2(U, \mathbb{Z}) \oplus H_2(\mathbb{Z}_4 \oplus \mathbb{Z}, \mathbb{Z}) \\ &= H_2(U, \mathbb{Z}) \oplus \mathbb{Z}_4. \end{aligned}$$

Then by a theorem of Cockcroft (quoted in [11, Introduction]), K_W is not Cockcroft since its second homology is not free abelian. •

Lemma 2.4.7 *If $\overline{W} \neq 1$ in U then K_W is Cockcroft.*

Proof Let $\overline{W} \neq 1$ in U. Then K_W is a generalised HNN-extension defined by presentation $\mathcal{K}_W$. Since $< a, b >$ and $\mathcal{U}$ are aspherical and H is free on the given generators, then by Proposition 2.3.3, $\mathcal{K}_W$ is aspherical and hence Cockcroft. •

Chapter 3

Picture problem

Recall that the word problem for a finite connected 2-complex $\mathcal{L}$ is the problem of determining the existence of algorithm to decide whether for any arbitrary element of $\pi_1(\mathcal{L}^{(1)})$, its image under the inclusion induced homomorphism

$$\pi_1(\mathcal{L}^{(1)}) \longrightarrow \pi_1(\mathcal{L})$$

is trivial. In this chapter we introduce the picture problem which is the analogue of the word problem, one dimension higher. This means that for any finite connected 3-complex $\mathcal{K}$, we may ask whether for any arbitrary element of $\pi_2(\mathcal{K}^{(2)})$, its image under the inclusion induced homomorphism

$$\pi_2(\mathcal{K}^{(2)}) \longrightarrow \pi_2(\mathcal{K})$$

is trivial.

Just as a presentation can be regarded as 2-complex with a single 0-cell, so we will consider a 3-complex with a single 0-cell, known as a 3-presentation. A *3-presentation* $\mathcal{K}$ is a triple

$$< \boldsymbol{x}; \boldsymbol{r}; \boldsymbol{P} >$$

where $\mathcal{K}^{(2)} =< \boldsymbol{x}; \boldsymbol{r} >$ is an arbitrary presentation and $\boldsymbol{P}$ is a set of spherical pictures over $\mathcal{K}^{(2)}$. We say that $\mathcal{K}$ is finite if $\boldsymbol{x}$, $\boldsymbol{r}$ and $\boldsymbol{P}$ are all finite. Note that this definition is actually the same thing as an *extended group presentation* $< \boldsymbol{x}; \boldsymbol{r}; \boldsymbol{I} >$ defined by Fenn [25, Definition 2.7.1]. The reason is because any collection of relation identities

$\boldsymbol{I}$ (as in Fenn) can be represented geometrically by a collection of spherical pictures $\boldsymbol{P}$ over the same presentation (see for example [48, Theorem 2.1]). Now using the equivalence relations on pictures (refer §1.1.1), we formulate the *picture problem*:

- *For a given finite 3-presentation $\mathcal{K} =< \boldsymbol{x}; \boldsymbol{r}; \boldsymbol{P} >$ is there an algorithm to decide for any spherical picture $\mathbb{P}$ over $\mathcal{K}^{(2)} =< \boldsymbol{x}; \boldsymbol{r} >$, whether $\mathbb{P}$ is equivalent (relative to $\boldsymbol{P}$) to the empty picture?*

We are interested in the analogue of Novikov-Boone Theorem [10, 46]—the existence of a finite presentation with unsolvable word problem. In §3.1, we give an example of a finite 3-presentation $\mathcal{K}$ with unsolvable picture problem but the word problem for $\mathcal{K}^{(2)}$ is solvable. We modified our construction in §3.2 and give a simpler example of a finite 3-presentation such that both word and picture problems are unsolvable.

3.1 Example 1

The main unsolvability result for 3-presentations is the following:

Theorem 3.1.1 *There exists a finite 3-presentation $\mathcal{K}$ which has unsolvable picture problem. Moreover $\mathcal{K}$ can be chosen such that the word problem for the underlying presentation $\mathcal{K}^{(2)}$ is solvable.*

To prove this theorem we need a group V (see §3.1.1) defined by a finite presentation $\mathcal{V} =< \boldsymbol{v}; \boldsymbol{t} >$ such that:

1. The word problem for V is solvable.

2. $\pi_2(\mathcal{V})$ is finitely generated.

3. There exists a finitely generated normal subgroup H of V such that the generalised word problem (refer §1.2) for H in V is unsolvable.

We will establish the existence of such a group V in §3.1.1. Having established the group V, then we will use it to construct a finite 3-presentation $\mathcal{K} =< \boldsymbol{x}; \boldsymbol{r}; \boldsymbol{P} >$ (see §3.1.2). Let W be any arbitrary word on $\boldsymbol{v}$. We will consider the set $\{\mathbb{P}_W : \mathbb{P}_W$ is a spherical picture over $\mathcal{K}^{(2)}\}$ and show that $\mathbb{P}_W$ is equivalent to the empty picture (relative to $\boldsymbol{P}$) if and only if W defines an element of H. Hence we can conclude that $\mathcal{K}$ has unsolvable picture problem since the generalised word problem for H in V is unsolvable.

3.1.1 The existence of group V

We will present the group of Miller [44, Example 4] (see also [5, Corollary 1]). Let $\mathcal{U} =< u_1, u_2, \cdots, u_n; S_1, S_2, \cdots, S_m >$ define the group U as in §2.3.2. Let F_{n+1} be the free group $< q, u_1, u_2, \cdots, u_n >$. Finally let $\mathcal{V}$ be the presentation with generators

$$q, u_1, \cdots, u_n, t_1, \cdots, t_m, d_1, \cdots, d_n$$

and defining relations

$$t_i^{-1}qt_i = qS_i, \ t_i^{-1}u_jt_i = u_j$$

$$d_j^{-1}qd_j = u_j^{-1}qu_j, \ d_j^{-1}u_kd_j = u_k$$

where $1 \leq i \leq m$, $1 \leq j \leq n$, and $1 \leq k \leq n$. Then let $\mathcal{V}$ define the group V.

Miller [44, Lemma 2] shows that V has solvable word problem. We may choose the normal subgroup H to be the subgroup generated by $\{q, t_1, \cdots, t_m, d_1, \cdots, d_n\}$ and it is mentioned in [44] that the generalised word problem for H in V is unsolvable. Thus we only need to show that condition 2 is satisfied. It is not hard to see (or refer to [44, Lemma 1]) that V is an $(m+n)$-step HNN-extension with base group F_{n+1} (defined by an aspherical presentation). In the first m steps, the stable letter for the ith step is t_i while d_j is the stable letter for the jth step in the last n steps. For each step, the associated subgroup is F_{n+1} itself and is free on the given generators. Thus by Proposition 2.3.3 (or [4, Theorem 2]) $\mathcal{V}$ is aspherical and hence condition 2 is trivially satisfied.

3.1.2 The construction of $\mathcal{K}$

Let V be any group satisfying conditions 1–3 above, and let V have presentation $\mathcal{V} =< \boldsymbol{v}; \boldsymbol{t} >$. Then let

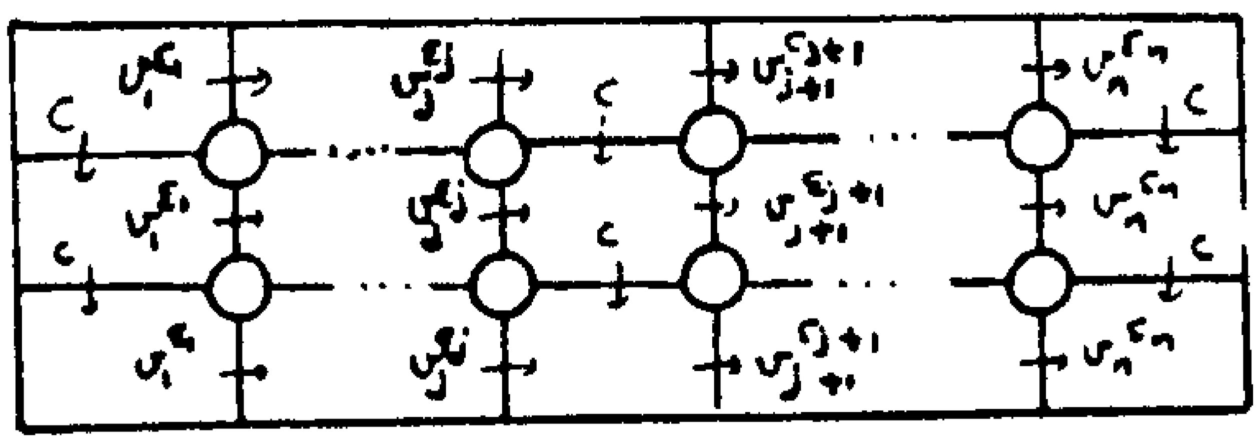
$$K = V \times \mathbb{Z}_2$$

defined by the presentation

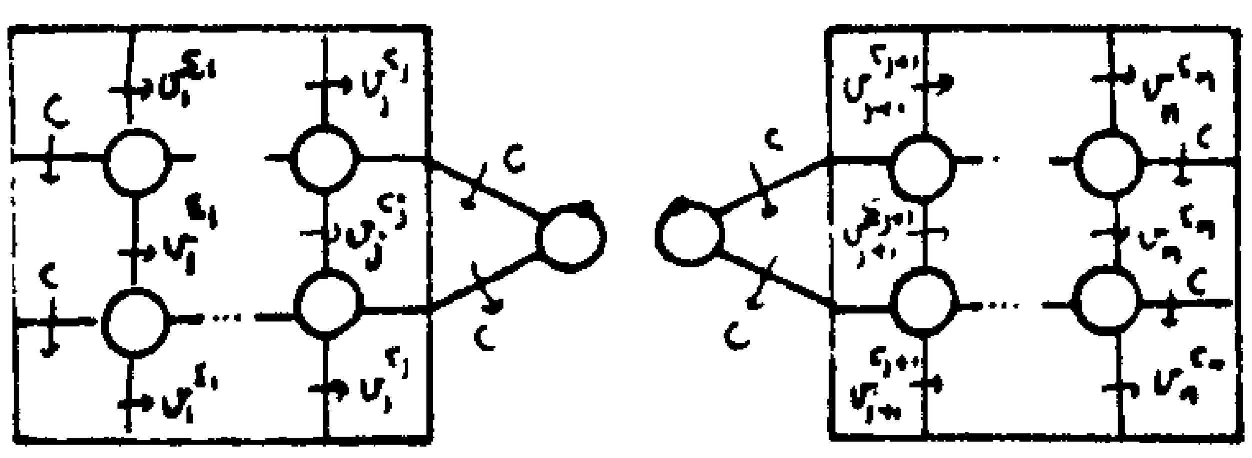
$$\mathcal{K}^{(2)} = < \boldsymbol{v}, c; \boldsymbol{t}, c^2, [v, c] (v \in \boldsymbol{v}) > .$$

Since V and $\mathbb{Z}_2$ have solvable word problem then so does K. Before we proceed, we need the following definition.

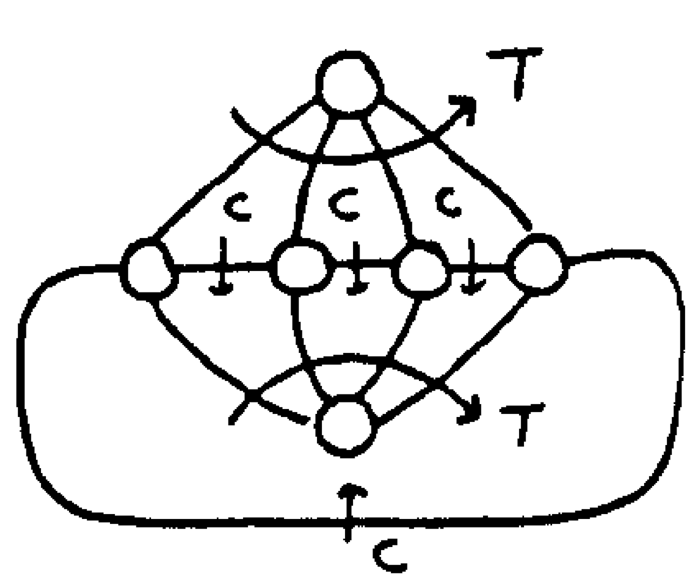
Definition 3.1.2 *Let $v_1^{\epsilon_1} \cdots v_j^{\epsilon_j} v_{j+1}^{\epsilon_{j+1}} \cdots v_n^{\epsilon_n}$ be a word on $\boldsymbol{v}$. Then a* commutator picture $\mathbb{C}_{v_1^{\epsilon_1} \cdots v_j^{\epsilon_j} v_{j+1}^{\epsilon_{j+1}} \cdots v_n^{\epsilon_n}}$ *is a picture over $< \boldsymbol{v}, c; [v, c] (v \in \boldsymbol{v}) >$ of the form*



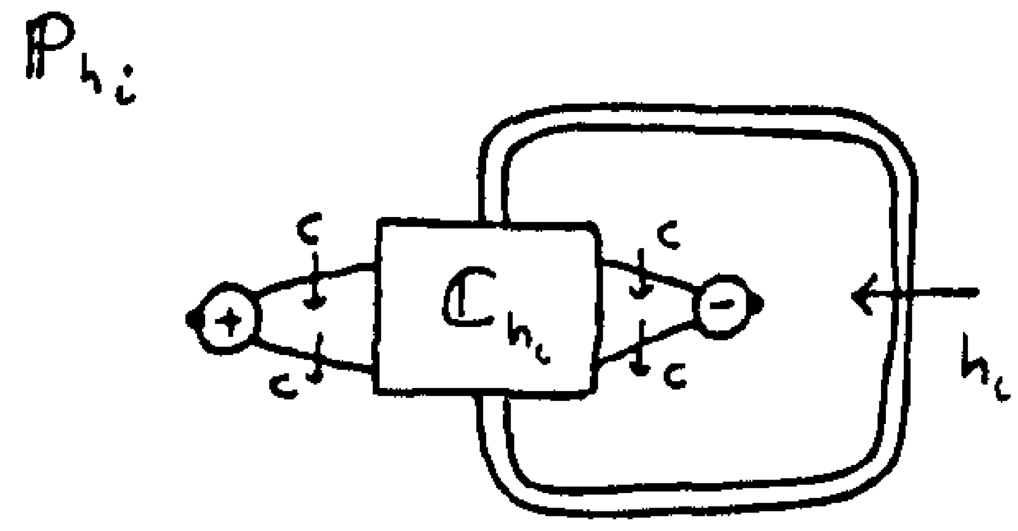
Remark 3.1.3 If we regard $\mathbb{C}_{v_1^{\epsilon_1} \cdots v_j^{\epsilon_j} v_{j+1}^{\epsilon_{j+1}} \cdots v_n^{\epsilon_n}}$ as a picture over $< \boldsymbol{v}, c; c^2, [v, c] (v \in \boldsymbol{v}) >$, then it is equivalent to



Let $\boldsymbol{h}$ be a set of words on $\boldsymbol{v}$ which represents a finite set of generators of H. Let $\boldsymbol{P}_1$ be a finite set of spherical pictures that generate $\pi_2(\mathcal{V})$. Let $\boldsymbol{P}_2$ be the finite set of spherical pictures over $\mathcal{K}^{(2)}$ of the form



for each $T \in \boldsymbol{t}$. Also let $\boldsymbol{P}_3$ be the finite set of spherical pictures over $\mathcal{K}^{(2)}$ of the form

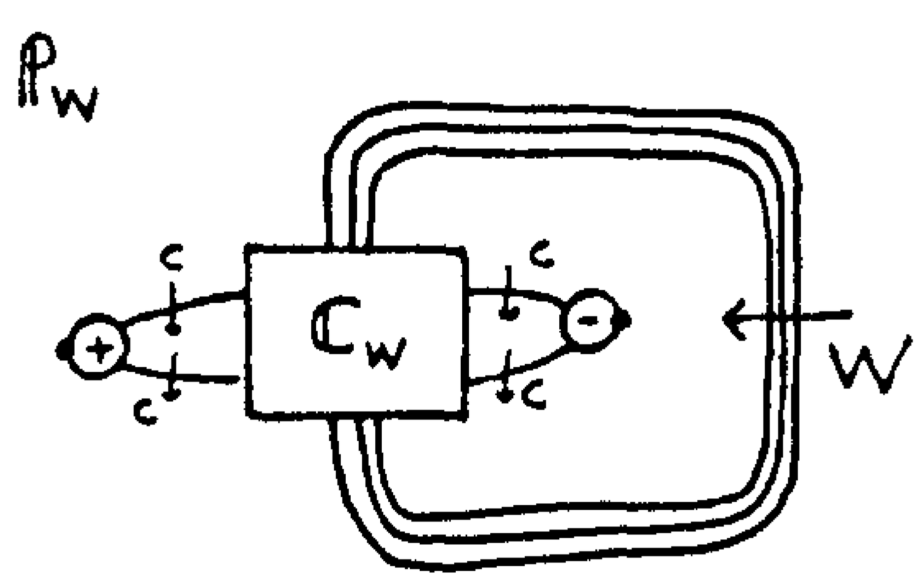


for each $h_i \in \boldsymbol{h}$ where $\mathbb{C}_{h_i}$ is a commutator picture. Let $\boldsymbol{P} = \boldsymbol{P}_1 \cup \boldsymbol{P}_2 \cup \boldsymbol{P}_3$. Note that $\boldsymbol{P}$ is finite. Thus we have a finite 3-presentation

$$\mathcal{K} = < \boldsymbol{v}, c; \boldsymbol{t}, c^2, [v, c](v \in \boldsymbol{v}); \boldsymbol{P} >$$

such that the underlying presentation $\mathcal{K}^{(2)}$ has solvable word problem.

For any word W on $\boldsymbol{v}$, let $\mathbb{P}_W$ be a spherical picture of the form

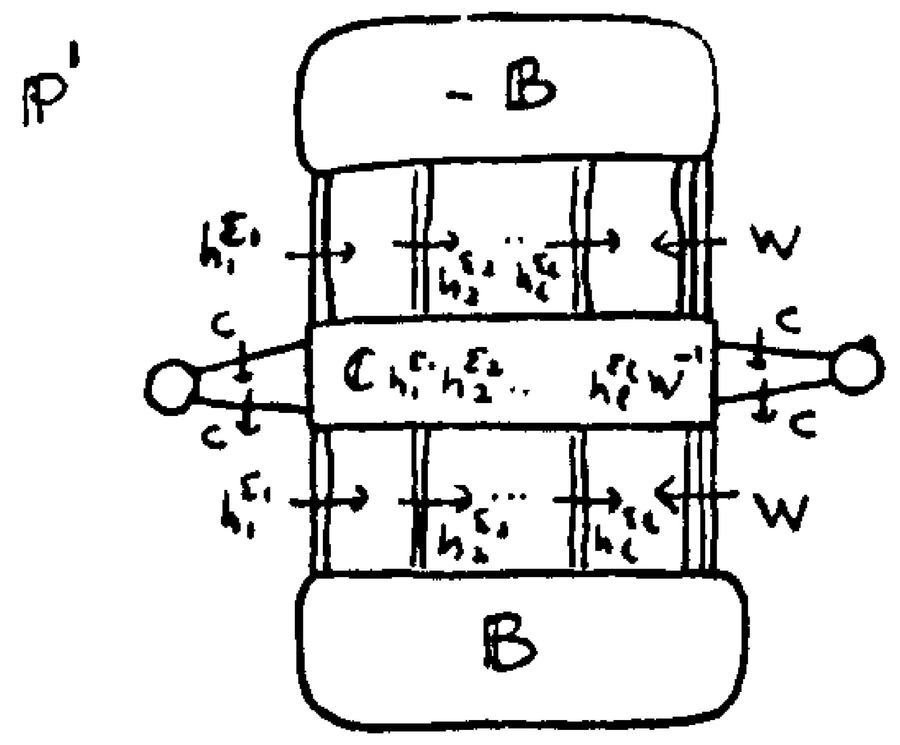


where $\mathbb{C}_W$ is a commutator picture. We will show that $\mathbb{P}_W$ is equivalent to the empty picture (relative to $\boldsymbol{P}$) if and only if W defines an element of H, and hence $\mathcal{K}$ has unsolvable picture problem since the generalised word problem for H in V is unsolvable.

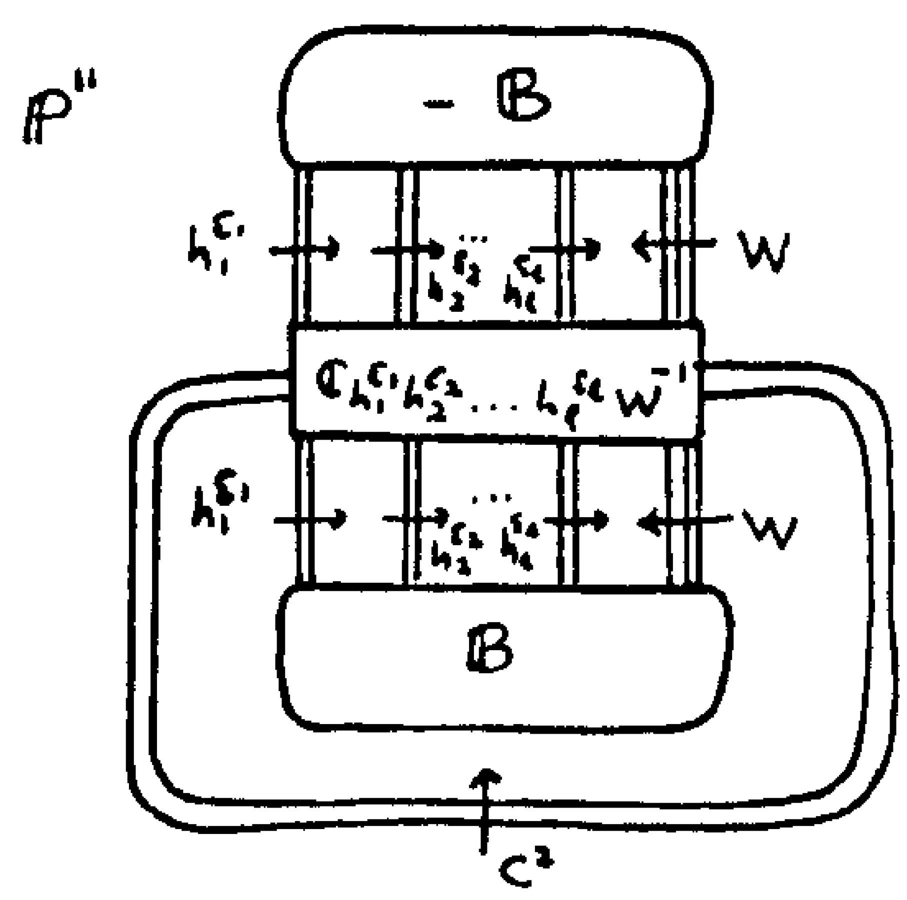
Lemma 3.1.4 *If W defines an element of H, then $\mathbb{P}_W$ is equivalent to the empty picture (relative to $\boldsymbol{P}$) over $\mathcal{K}^{(2)}$.*

Proof Let W define an element of H, then $\overline{W} = \overline{h_1^{\epsilon_1} h_2^{\epsilon_2} \cdots h_l^{\epsilon_l}}$ in V for some h_i's which belong to $\boldsymbol{h}$ and $\epsilon_i = \pm 1$. Thus by Lemma 1.1.2, there is a picture $\mathbb{B}$ over $< \boldsymbol{v}; \boldsymbol{t} >$ with boundary label $h_1^{\epsilon_1} h_2^{\epsilon_2} \cdots h_l^{\epsilon_l} W^{-1}$.

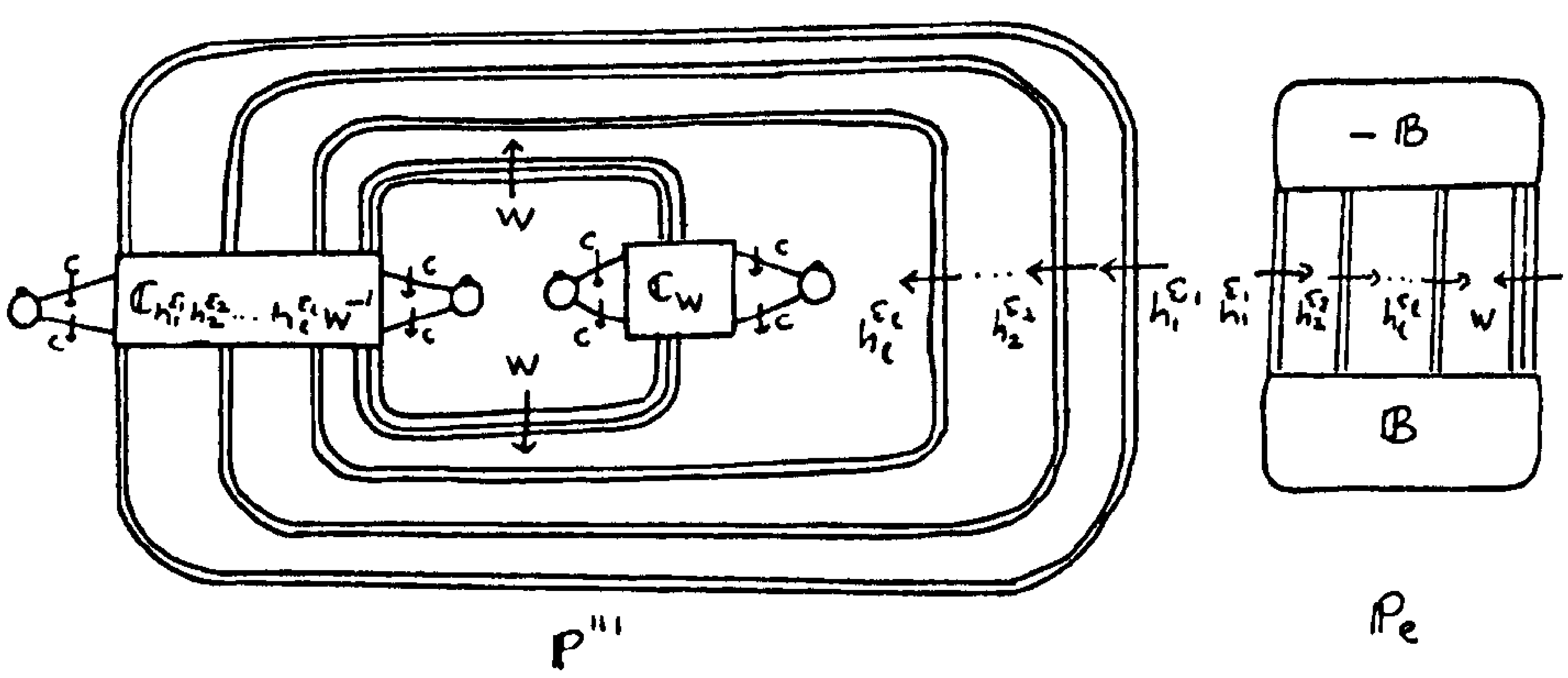
Now consider



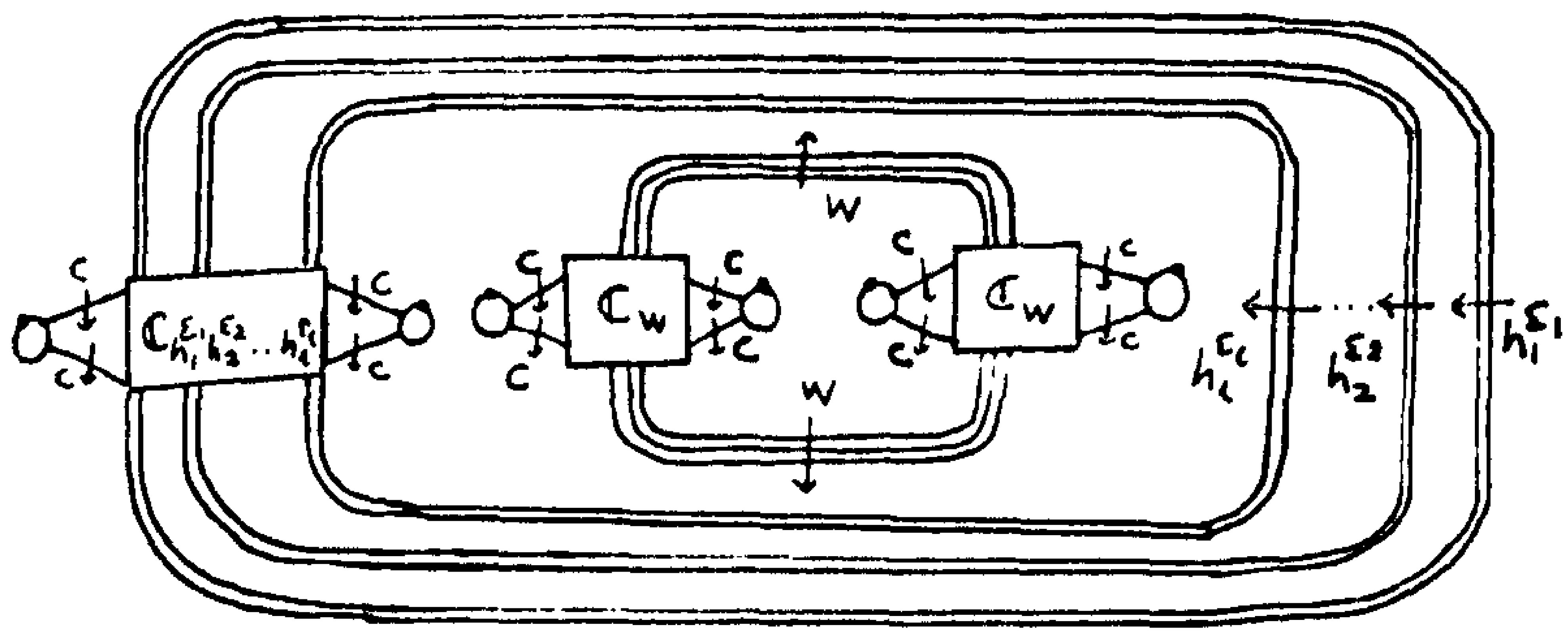
where $\mathbb{C}_{h_1^{\epsilon_1} h_2^{\epsilon_2} \cdots h_l^{\epsilon_l} W^{-1}}$ is a commutator picture. Note that $\mathbb{P}'$ is equivalent to



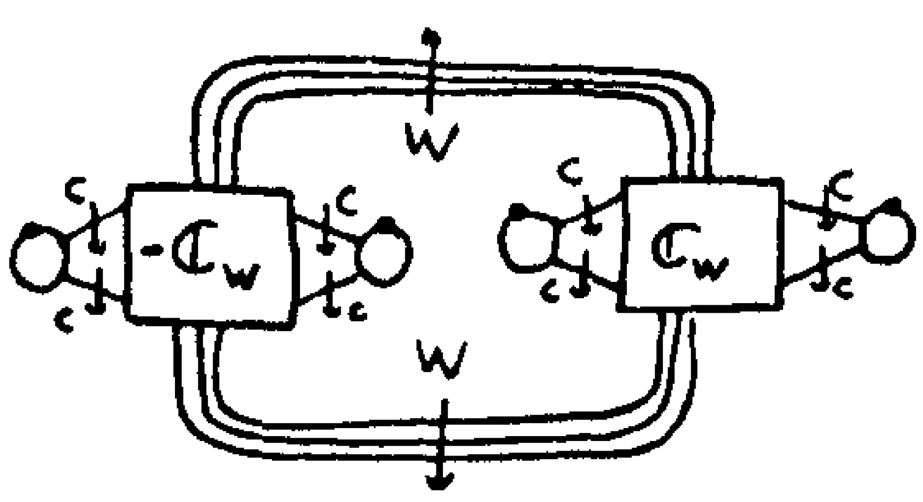
One may refer to [4] that $\boldsymbol{P}_1 \cup \boldsymbol{P}_2$ generates $\pi_2(<\boldsymbol{v}, c; \boldsymbol{t}, [v, c](v \in \boldsymbol{v}>)$. Thus by Theorem 1.1.3, $\mathbb{P}''$ is equivalent (relative to $\boldsymbol{P}_1 \cup \boldsymbol{P}_2$) to the empty picture and so is $\mathbb{P}'$. If we insert $\mathbb{P}'$ to the left side of $\mathbb{P}_W$ and perform some bridge moves, we obtain



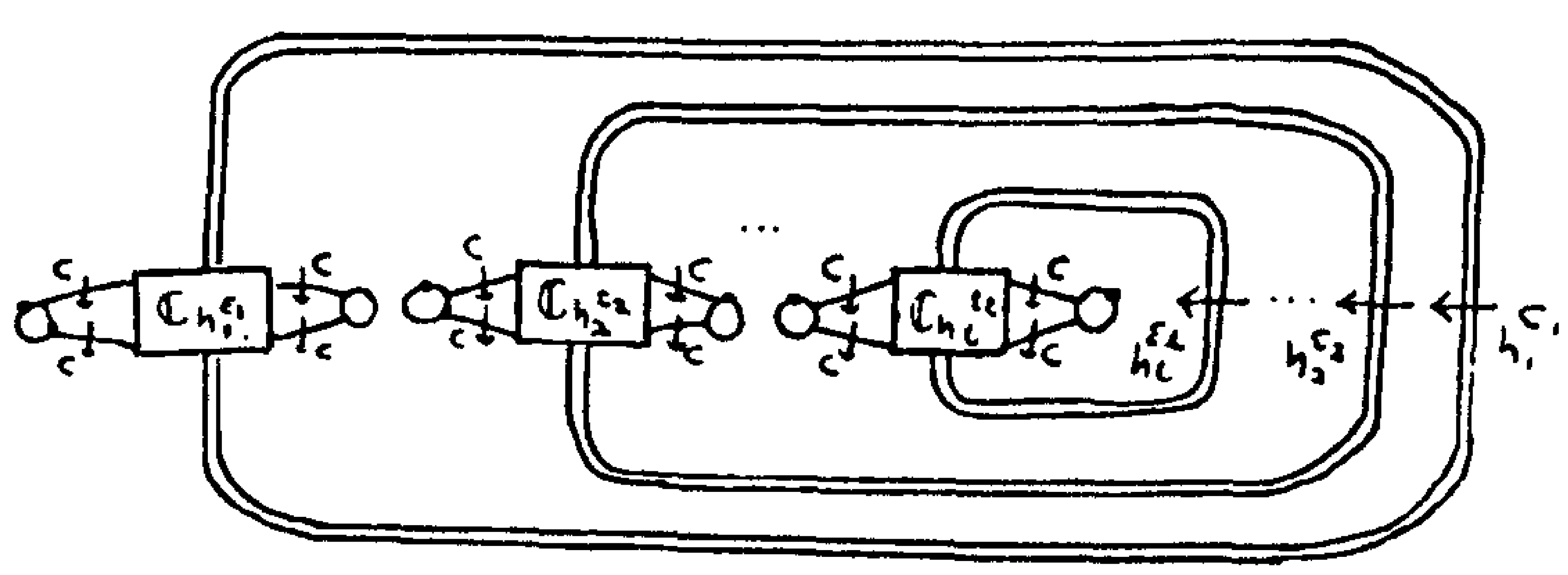
and so we may delete $\mathbb{P}_e$ since it is equivalent to the empty picture. Then by Remark 3.1.3, $\mathbb{P}'''$ is equivalent to



where $\mathbb{C}_{h_1^{\epsilon_1} h_2^{\epsilon_2} \cdots h_l^{\epsilon_l}}$ is a commutator picture. Then we may delete



since it is equivalent to the empty picture. Again by Remark 3.1.3 we may obtain



Clearly this last picture is equivalent (relative to $\boldsymbol{P}_3$) to the empty picture. Thus $\mathbb{P}_W$ is equivalent (relative to $\boldsymbol{P}$) to the empty picture as required. •

Lemma 3.1.5 *If W does not define any element of H, then $\mathbb{P}_W$ is not equivalent to the empty picture (relative to $\boldsymbol{P}$) over $\mathcal{K}^{(2)}$.*

Proof Suppose that W does not define an element of H and suppose that $\mathbb{P}_W$ can be obtained from $\boldsymbol{P}$. Let P_2 be the free $\mathbb{Z}K$-module with basis $\{e_T : T \in \boldsymbol{t}\} \cup \{e_{c^2}\} \cup \{e_{[v,c]} : v \in \boldsymbol{v}\}$. We will consider the image of $\mathbb{P}_W$ in P_2 (refer (1.1) in §1.1.3). Let $C = c^2$ and write

$$P_2 = \mathbb{Z}K e_C \oplus P_2'$$

where P_2' is the free $\mathbb{Z}K$-module with the above basis excluding e_C. Then the image of $\mathbb{P}_W$ in P_2 is

$$\gamma_W = (\overline{W} - 1)e_C + \lambda_W \text{ for some } \lambda_W \in P_2'.$$

and the image of $\mathbb{P}_{h_i}$ ($\mathbb{P}_{h_i} \in \boldsymbol{P}_3$) is

$$\gamma_i = (\overline{h_i} - 1)e_C + \lambda_i \text{ for some } \lambda_i \in P_2'.$$

Also let the image of each $\mathbb{P}_T$ ($T \in \boldsymbol{t}$) be γ_T and the image of $\mathbb{Q}$ ($\mathbb{Q} \in \boldsymbol{P}_1$) be $\gamma_{\mathbb{Q}}$. Note that γ_T and $\gamma_{\mathbb{Q}}$ lie entirely in P_2'.

Since $\mathbb{P}_W$ is obtainable from $\boldsymbol{P}$, we have

$$\gamma_W = \beta_1\gamma_1 + \beta_2\gamma_2 + \cdots + \beta_k\gamma_k + \sum_{T \in \boldsymbol{t}} \alpha_T\gamma_T + \sum_{\mathbb{Q} \in \boldsymbol{P}_1} \alpha_{\mathbb{Q}}\gamma_{\mathbb{Q}}$$

for some α's and β's which belong to $\mathbb{Z}K$. Equating the coefficients of e_C, we have

$$\overline{W} - 1 = \beta_1(\overline{h_1} - 1) + \beta_2(\overline{h_2} - 1) + \cdots + \beta_k(\overline{h_k} - 1).$$

Then consider the ring homomorphism

$$\mathbb{Z}K \longrightarrow \mathbb{Z}V \longrightarrow \mathbb{Z}(V/H)$$

arising from the group homomorphism specified by

$$\overline{v} \mapsto \overline{v}H \ (v \in \boldsymbol{v}), \ \overline{c} \mapsto 1H$$

Then we have $\overline{W}H - 1H = 0$, that is W defines the element $\overline{W}$ of H which contradicts our assumption. •

3.2 Example 2

If we just want a finite 3-presentation $\mathcal{L}$ with both the word and picture problems unsolvable, we need the following construction:

Let U be a finitely presented group defined by $\mathcal{U} =< \boldsymbol{u}; \boldsymbol{s} >$ such that:

1. The word problem for U is unsolvable.

2. $\pi_2(\mathcal{U})$ is finitely generated.

(To make a definite choice, we may take the aspherical group U given in §2.3.2. so condition 2 is trivially satisfied.)

Let

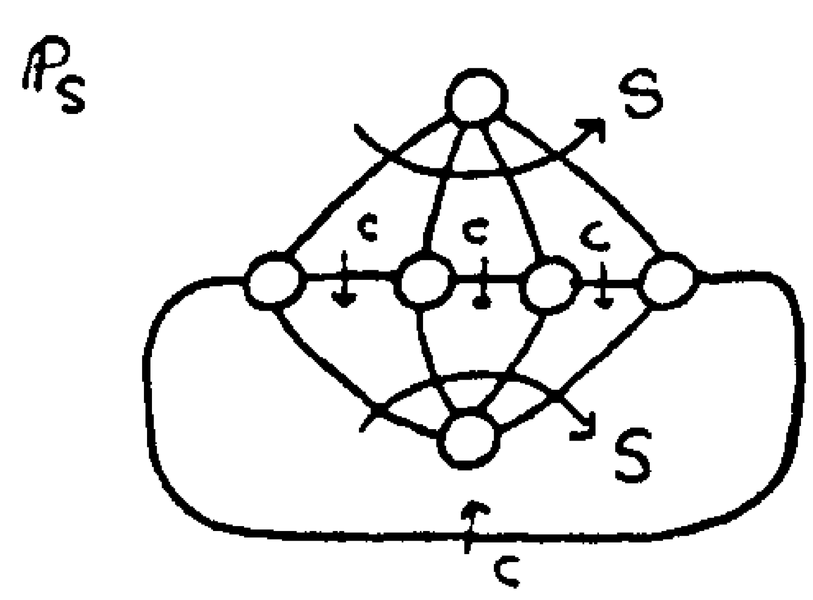
$$L = U \times \mathbb{Z}_2$$

defined by the presentation

$$\mathcal{L}^{(2)} =< \boldsymbol{u}, c; \boldsymbol{s}, c^2, [u, c](u \in \boldsymbol{u}) > .$$

Clearly L has unsolvable word problem since it contains the subgroup U with unsolvable word problem.

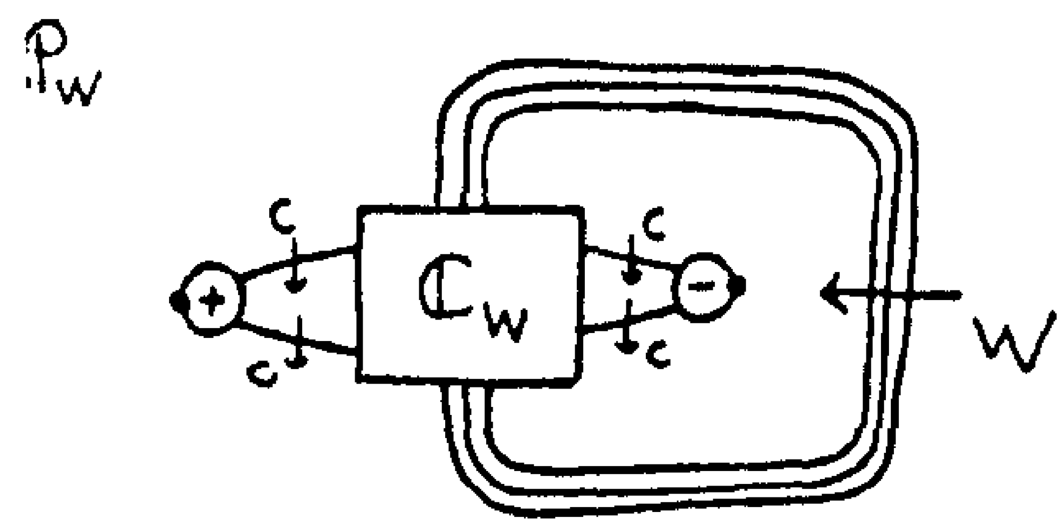
Let $\boldsymbol{P}_1$ be a finite set of spherical pictures that generate $\pi_2(\mathcal{U})$ and $\boldsymbol{P}_2$ be the finite set of spherical pictures over $\mathcal{L}^{(2)}$ of the form



for each $S \in \boldsymbol{s}$. Then let $\boldsymbol{P} = \boldsymbol{P}_1 \cup \boldsymbol{P}_2$, which is clearly finite. Thus we have a finite 3-presentation

$$\mathcal{L} =< \boldsymbol{u}, c; \boldsymbol{s}, c^2, [u, c](u \in \boldsymbol{u}); \boldsymbol{P} >$$

such that the word problem for $\mathcal{L}^{(2)}$ is unsolvable. Now for any word W on $\boldsymbol{u}$, we may consider spherical picture $\mathbb{P}_W$ (as in §3.1)



and show

Lemma 3.2.1 *If* $\overline{W} = 1$ *in* U, *then* $\mathbb{P}_W$ *is equivalent to the empty picture (relative to* $\boldsymbol{P}$*) over* $\mathcal{L}^{(2)}$.

Lemma 3.2.2 *If* $\overline{W} \neq 1$ *in* U, *then* $\mathbb{P}_W$ *is not equivalent to the empty picture (relative to* $\boldsymbol{P}$*) over* $\mathcal{L}^{(2)}$.

The proof of these lemmas are similar to the proof of Lemma 3.1.4 and 3.1.5 where we assume H to be the trivial subgroup. Hence $\mathcal{L}$ has unsolvable picture problem since the word problem for U is unsolvable.

Chapter 4

Introduction to relative presentations

From now onwards, we will deal with relative presentations. These presentations are considered for example in [3, 8, 22, 31]. As a matter of fact we only consider a special case—one relator relative presentations.

Our own work starts in Chapter 5. This chapter is rather an introduction to establish the methods in proving results in Chapters 5 and 6. In §4.1 we give some preliminary backgrounds. One may compare spherical pictures over relative presentations to the ordinary ones in §4.2. The asphericity test for relative presentations will be given in §4.3. Most material in this chapter is fairly standard (except §4.3.6) and can be found in at least one of these—[3], [8] and [22]. The observation in §4.4.1 plays an important role for the distribution test in the last two chapters while in §4.4.2 we identify that our work can be divided into two main classes—which will appear in Chapters 5 and 6 respectively.

4.1 Preliminaries

Let H be any arbitrary group and $< t >$ be an infinite cyclic group. We will construct a new group G as follows. Let R be an element of $H * < t >$ of the form

$$t^{\epsilon_1} h_1 t^{\epsilon_2} h_2 \cdots t^{\epsilon_n} h_n \tag{4.1}$$

where $\epsilon_i = \pm 1, h_i \in H$ for $i = 1, 2, \cdots, n$. Then we have the *one relator relative presentation*

$$\mathcal{P} = < H, t; R > .$$

The *one relator relative group* G, defined by $\mathcal{P}$, is the group

$$\frac{H * < t >}{<< R >>}$$

where $<< R >>$ is the normal closure of R. If H happens to be the trivial group, then $\mathcal{P}$ is simply an ordinary one relator presentation for G in the usual sense.

From now on when we refer to a relative presentation, we assume it is a one relator relative presentation.

To obtain an ordinary presentation $\tilde{\mathcal{P}}$ defining the group $\tilde{G}$, we have to select a presentation $\mathcal{Q} = < \boldsymbol{a}; \boldsymbol{s} >$ defining the group H. Thus there is an epimorphism

$$\phi : F(\boldsymbol{a}) \longrightarrow H$$

with kernel $<< \boldsymbol{s} >>$. For each h_i, choose a word w_i on $\boldsymbol{a}$ representing h_i and let

$$\tilde{R} = t^{\epsilon_1} w_1 t^{\epsilon_2} w_2 \cdots t^{\epsilon_n} w_n.$$

Then the *ordinary lifted presentation* is the presentation

$$\tilde{\mathcal{P}} = < \boldsymbol{a}, t; \boldsymbol{s}, \tilde{R} > .$$

The group $\tilde{G} = G(\tilde{\mathcal{P}})$ defined by $\tilde{\mathcal{P}}$ is the group

$$\frac{F(\boldsymbol{a}, t)}{<< \boldsymbol{s}, \tilde{R} >>}$$

where $<< \boldsymbol{s}, \tilde{R} >>$ is the normal closure of $\{\boldsymbol{s}, \tilde{R}\}$. We have an isomorphism

$$\nu : \tilde{G} \longrightarrow G$$

induced by the epimorphism

$$\phi * id : F(\boldsymbol{a}) * < t > \longrightarrow H * < t > .$$

Definition 4.1.1 *A relative presentation $\mathcal{P} =< H, t; R >$ is said to be* aspherical *if for some ordinary presentation $\mathcal{Q} =< \boldsymbol{a}; \boldsymbol{s} >$ for H and for some lifted presentation $\tilde{\mathcal{P}} =< \boldsymbol{a}, t; \boldsymbol{s}, \tilde{R} >$, the second homotopy module $\pi_2(\tilde{\mathcal{P}})$ is generated by $\pi_2(\mathcal{Q})$ as a left $\mathbb{Z}G$-module. In term of pictures, $\mathcal{P}$ is aspherical if every picture over $\tilde{\mathcal{P}}$ is equivalent (relative to $\mathcal{Q}$) to the empty picture.*

The above concept of asphericity is more general than that given in [8]. The restricted notion according to the theory developed in [8] will be given in §4.2.

There are two major issues that have always been asked:

1. When is the natural map $H \longrightarrow G$ injective?

2. When is $\mathcal{P}$ aspherical?

It is of interest to discuss these two issues. We say that $\mathcal{P}$ is *injective* if the natural map $H \longrightarrow G$ is injective. The main consequence is the following (see [3, Theorem 1])

Theorem 4.1.2 *Suppose that $\mathcal{P} =< H, t; R >$ is an injective aspherical relative presentation for a group G. Then*

i) $H_n(G, A) \cong H_n(H, A)$ for all $n \geq 3$ and for all right $\mathbb{Z}G$-modules A.

ii) $H^n(G, B) \cong H^n(H, B)$ for all $n \geq 3$ and for all left $\mathbb{Z}G$-modules B.

iii) Each finite subgroup of G is contained in a G-conjugate of H.

The first case of interest is t-length 3. If R has the form $th_1th_2th_3$, Levin [40] shows that $H \longrightarrow G$ is injective and the asphericity is decided in [8]. Howie [30] shows the injectivity for the case $R = th_1th_2t^{-1}h_3$ while Edjvet [22] considers asphericity of this form apart from eight exceptional cases. The t-length 4 is now considered. For the case $R = th_1th_2th_3th_4$, the injectivity again follows from Levin [40] while the asphericity is decided in [3] with five exceptional cases. When the powers of t are not all positive, the injectivity of $H \longrightarrow G$ has been discussed in [23]. Our results in Chapters 5 and 6 deal with asphericity when R has the form $th_1th_2th_3t^{-1}h_4$. There are two more cases for t-length 4 (up to equivalence as defined below) that need to be considered in future—the form $th_1th_2t^{-1}h_3t^{-1}h_4$ and the form $th_1t^{-1}h_2th_3t^{-1}h_4$. We found that S. Wreth is considering the last case for her thesis.

4.1.1 Equivalent presentations

Define some operations on words R as in (4.1) as follows:

I Replace R by $\bar{R} = t^{-\epsilon_n} h_{n-1}^{-1} \cdots t^{-\epsilon_2} h_1^{-1} t^{-\epsilon_1} h_n^{-1}$.

II Replace R by $S = t^{\epsilon_{i+1}} h_{i+1} \cdots t^{\epsilon_n} h_n t^{\epsilon_1} h_1 \cdots t^{\epsilon_i} h_i$.

III Let ϕ be an automorphism of $H* < t >$ fixing H. Replace R by $\phi(R)$ provided that $\phi(R)$ has the form (4.1).

Two words of the form (4.1) are said to be equivalent if one can be obtained from the other by a finite number of the above operations. We say that two relative presentations $\mathcal{P} =< H, t; R >$ and $\mathcal{P}' =< H, t; R' >$ are equivalent if R and R' are equivalent. Note that all results in Chapters 5 and 6 are up to equivalence.

4.2 Pictures over relative presentations

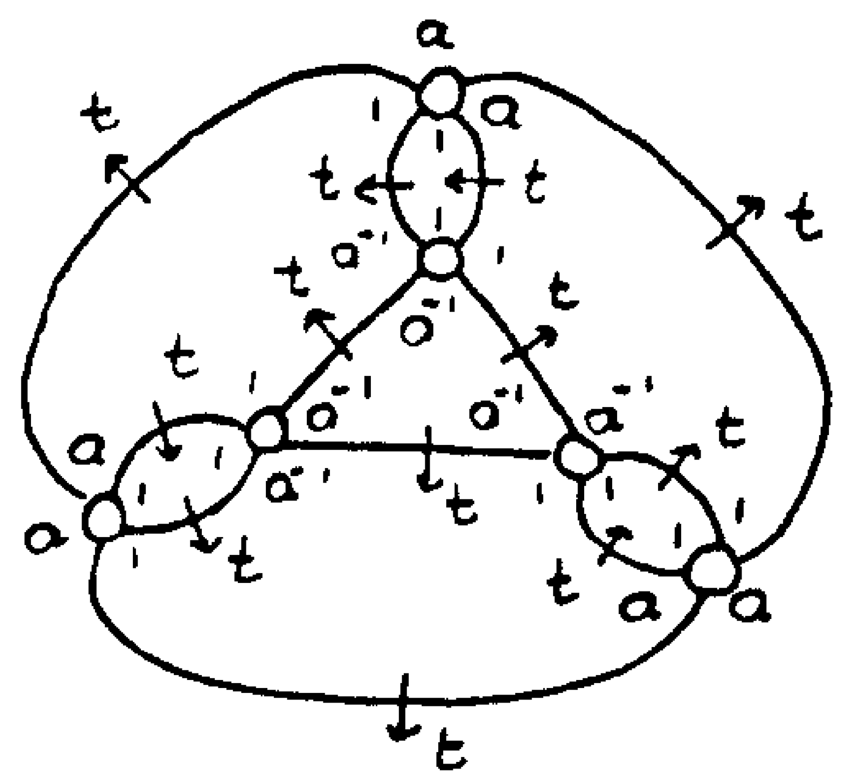
A picture $\mathbb{P}$ over a relative presentation $\mathcal{P}$ has the same geometric shape as an ordinary pictures as in §1.1.1, but the labelling is different and additional conditions are needed.

Fix a relative presentation $\mathcal{P} =< H, t; R >$. A picture $\mathbb{P}$ over $\mathcal{P}$ is labelled as follows. Each arc is labelled by t and each corner is to be oriented clockwise (with respect to the ambient disc of $\mathbb{P}$) and labelled by an element of H. If c is a corner of disc Δ, then denote by $W(c)$ the word obtained by reading in clockwise order around $\partial\Delta$ the labels on the arcs and corners meeting $\partial\Delta$ beginning with the label on the arc at the head of the clockwise oriented corner c. The following two conditions must be satisfied:

1. For each corner c of $\mathbb{P}$, $W(c)$ is a cyclic permutation of R or R^{-1}.

2. If $k_1, k_2, \cdots, k_m$ is the sequence of corner labels encountered in an anticlockwise traversal of the boundary of an inner region of $\mathbb{P}$, then $k_1 k_2 \cdots k_m = 1$ in H.

Example 4.2.1 Let $\mathcal{P} = < \mathbb{Z}_3, t; t^3at^{-1}a >$ where a generates $\mathbb{Z}_3$. Then



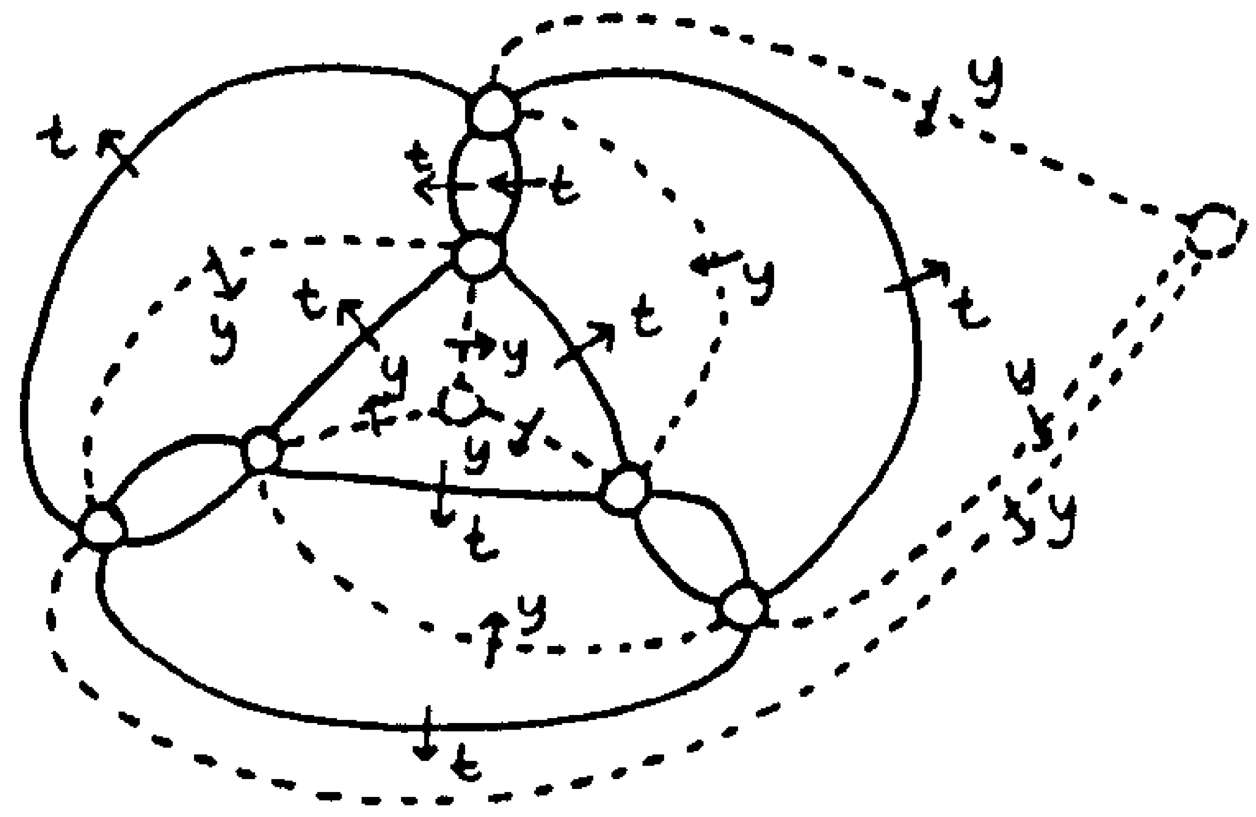
is a spherical picture over $\mathcal{P}$.

A connected spherical picture $\mathbb{P}$ over $\mathcal{P}$ is said to be *strictly spherical* if the product of the corner labels in the outer annular region defines the identity in H. Note that the above example is strictly spherical.

Recall (refer §4.1) that for any arbitrary relative presentation $\mathcal{P}$, we may obtain an ordinary lifted presentation $\tilde{\mathcal{P}}$. In a similar manner (refer [8] for more details), one may lift (though not uniquely) a spherical picture $\mathbb{P}$ over $\mathcal{P}$ to a picture $\tilde{\mathbb{P}}$ over $\tilde{\mathcal{P}}$. For each inner region Σ of $\mathbb{P}$, there is a picture $\tilde{\Sigma}$ over $\mathcal{Q}$ with boundary label equal to the product of the words in $\boldsymbol{a}$ which represent the corner labels for Σ. For the outer annular region, replace each corner label by a succession of $\boldsymbol{a}$-arcs reading the representative word. Thus we may obtain the lifted picture $\tilde{\mathbb{P}}$.

Example 4.2.1 (continued)

Let $\mathcal{Q} = < y; y^3 >$ be a presentation for $\mathbb{Z}_3$. Then

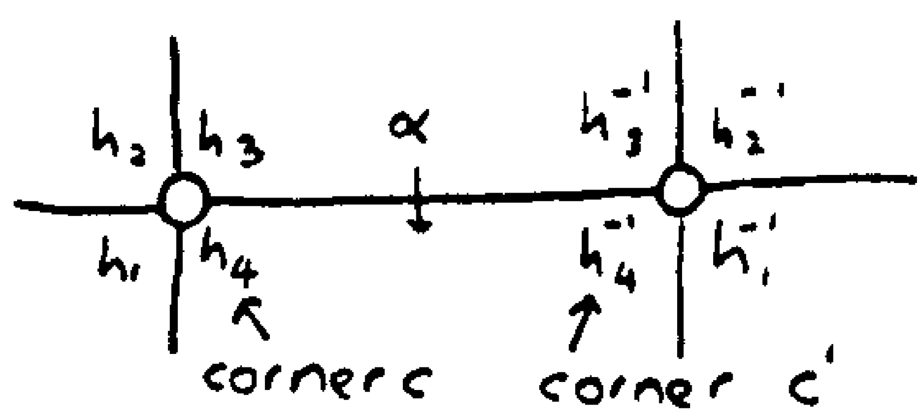


is a lifted picture for the above picture.

We will also discuss the restricted notion of asphericity according to the theory developed in [8]. A *dipole* in a picture $\mathbb{P}$ consists of a pair of corners c, c' of the picture together with an arc α joining the head of one corner with the tail of the other such that:

1. c and c' lie in the same region of $\mathbb{P}$.

2. $W(c) = W(c')$.



A picture $\mathbb{P}$ is said to be *reduced* if it does not contain a dipole. In [8] a relative presentation $\mathcal{P}$ was said to be aspherical if it is injective and every connected strictly spherical picture over $\mathcal{P}$ contains a dipole. It is quite clear that if a picture $\mathbb{P}$ is not reduced then for any lifted picture $\tilde{\mathbb{P}}$, it is not reduced in the sense of ordinary picture. Thus $\tilde{\mathbb{P}}$ is equivalent to the empty picture (relative to $\mathcal{Q}$). Hence the restricted notion of asphericity implies asphericity defined in Definition 4.1.1. Thus in order to show that an injective relative presentation $\mathcal{P}$ is aspherical, it is suffices to show that there is no non-empty reduced strictly spherical picture $\mathbb{P}$ over $\mathcal{P}$.

4.3 Tests for asphericity

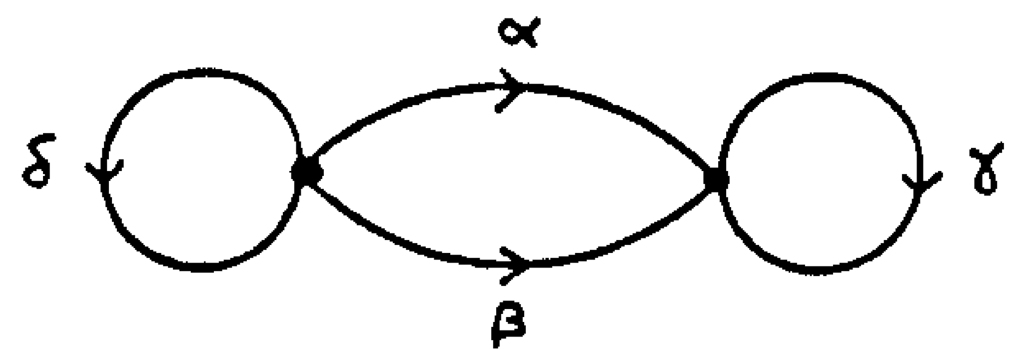
Here we give techniques and tests for asphericity. Some of these will be used in Chapters 5 and 6.

4.3.1 Small cancellation theory

The *star-complex* $\mathcal{P}^{st}$ of $\mathcal{P}$ is a certain graph with edges labelled by elements of H. There are two vertices, labelled t and t^{-1}. For each cyclic permutation that starts with t of $R^{\pm 1}$ say R^c, write $R^c = Sh$ where $h \in H$ and S begins and ends with t

or t^{-1}. Each R^c forms an edge such that the initial vertex is the first symbol of S and the terminal vertex is the inverse of the last symbol of S. The inverse $(R^c)^{-1}$, is defined to be $S^{-1}h^{-1}$. We label the edge R^c by $\lambda(R^c) = h^{-1}$ and extend it to paths in the obvious way. A non-empty cyclically reduced closed path in $\mathcal{P}^{st}$ will be called *admissible* if it has a trivial label in H.

Example 4.3.1 Let $\mathcal{P} =< H, t; t^2atbt^{-1}c >$. Then $\mathcal{P}^{st}$ is as follows:



where $\alpha \leftrightarrow 1, \beta \leftrightarrow a^{-1}, \gamma \leftrightarrow b^{-1}$ and $\delta \leftrightarrow c^{-1}$. If $ac = b$ in H then $\beta^{-1}\delta^{-1}\alpha\gamma$ is admissible.

Let q be a positive integer. Then we say that $\mathcal{P}$ *satisfies T(q)* if there are no admissible paths in $\mathcal{P}^{st}$ of length m for $3 \leq m < q$.

Example 4.3.1 (continued)

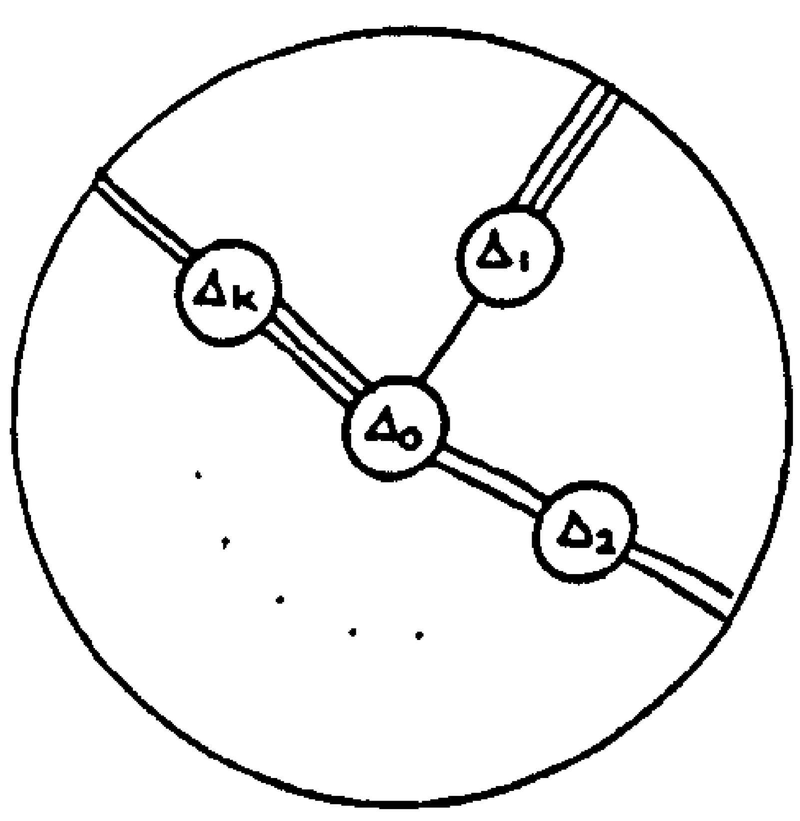
If a, b and c are distinct such that $b^{-1} \neq a \neq c^{-1}$ and $o(b), o(c) > 3$ then there is no admissible cycle of length 3. Thus $\mathcal{P}$ satisfies $T(4)$ in this case.

A (non-trivial) connected picture W over $\mathcal{P}$ is said to be a *k-wheel* over $\mathcal{P}$ (k is any positive integer) if W has discs $\{\Delta_0, \Delta_1, \cdots, \Delta_k\}$ such that:

1. Each arc of W meets a disc Δ_j for some $j \in \{1, 2, \cdots, k\}$.

2. Each arc of W either meets Δ_0 or ∂W.

3. Each disc of W has a corner which lies in a region of W that meets ∂W.



We say that $\mathcal{P}$ *satisfies C(p)* if there are no reduced k-wheels over $\mathcal{P}$ for $k < p$.

Example 4.3.1 (continued)

Suppose that $a \neq 1$ in H and $o(b), o(c) > 2$. Then if two discs Δ_1 and Δ_2 of a picture $\mathbb{W}$ share at least two consecutive t-arcs, they constitute a dipole. Thus there is no reduced k-wheel over $\mathcal{P}$ for $k < 4$ and hence $\mathcal{P}$ satisfies $C(4)$ in this case.

The following theorem can be found for example in [8, Theorem 2.2].

Theorem 4.3.2 *If $\mathcal{P}$ satisfies $C(p)$ and $T(q)$ where $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$ then $\mathcal{P}$ is aspherical.*

Example 4.3.1 (continued)

If a, b and c are distinct non-trivial element of H such that $b^{-1} \neq a \neq c^{-1}$ and $o(b), o(c) > 3$ then $\mathcal{P}$ satisfies $C(4)$ and $T(4)$ and so $\mathcal{P}$ is aspherical.

4.3.2 Weight test

A *weight function θ* on $\mathcal{P}^{st}$ is a real valued function θ on the set of edges of $\mathcal{P}^{st}$ such that $\theta(Sh) = \theta(S^{-1}h^{-1})$ for each edge Sh. The weight of a path is the sum of the weight of the constituent edges. We say that a weight function θ is *aspherical* if the following conditions hold.

1. If $R = t^{\epsilon_1}h_1 \cdots t^{\epsilon_n}h_n$ then

$$\sum_{i=1}^{n}(1 - \theta(t^{\epsilon_i}h_i \cdots t^{\epsilon_n}h_n t^{\epsilon_1}h_1 \cdots t^{\epsilon_{i-1}}h_{i-1})) \geq 2.$$

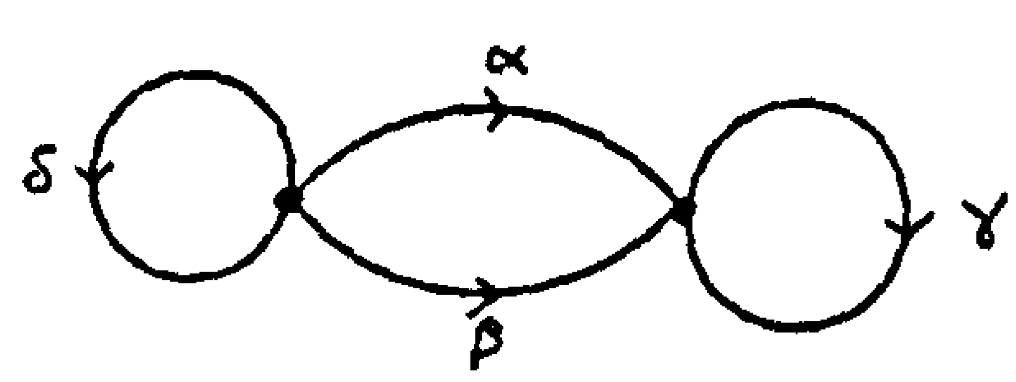
2. Each admissible cycle in $\mathcal{P}^{st}$ has weight at least 2.

3. Each edge of $\mathcal{P}^{st}$ has non-negative weight.

The significance of the weight test is (see for example in [8])

Theorem 4.3.3 *If $\mathcal{P}^{st}$ admits an aspherical weight function then $\mathcal{P}$ is aspherical.*

Example 4.3.4 Let $\mathcal{P} = < H, t : t^3at^{-1}a >$ such that a has infinite order in H. The star graph $\mathcal{P}^{st}$ is



where $\alpha \leftrightarrow 1, \beta \leftrightarrow 1, \gamma \leftrightarrow a^{-1}$ and $\delta \rightarrow a^{-1}$. We may assign the following weights

$$\theta(\alpha) = \theta(\beta) = 1, \theta(\gamma) = \theta(\delta) = 0.$$

Clearly the first and the third conditions hold. Since a has infinite order then for any integer $p \neq 0$, γ^p and δ^p are not admissible. Thus any admissible cycle must involve α and/or β at least twice and hence has a weight of at least two. Thus the second condition holds and so $\mathcal{P}$ is aspherical.

4.3.3 Curvature test

Let $\mathbb{P}$ be any strictly spherical picture over $\mathcal{P}$. Then we can assign a real valued angle function θ on the set of corners of $\mathbb{P}$. Associated to θ is a *curvature function* γ defined on a typical disc Δ of $\mathbb{P}$ by

$$\gamma(\Delta) = 2\pi - \sum_{c \subseteq \partial\Delta} \theta(c)$$

and on a typical region Φ of $\mathbb{P}$ by

$$\gamma(\Phi) = 2\pi - \sum_{c \subseteq \partial\Phi} (\pi - \theta(c))$$

where c denotes a corner in the boundary of a disc Δ or region Φ. Noting that $\mathbb{P}$ has twice as many corners as arcs, an Euler characteristic count reveals that

$$\sum_{\Delta} \gamma(\Delta) + \sum_{\Phi} \gamma(\Phi) = 2\pi\chi(S^2) = 4\pi$$

where the sum is taken over all discs and regions of $\mathbb{P}$ including the outer annular region. As a consequence we have

Lemma 4.3.5 *For any angle function on any connected spherical pictures, some disc or region has positive curvature.*

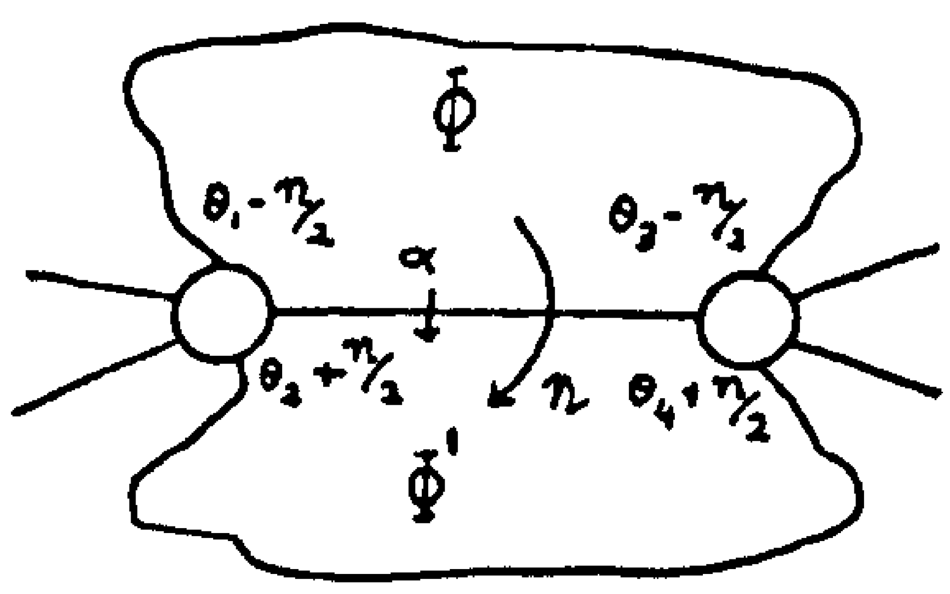
We will use the curvature test as follows. A strictly spherical picture $\mathbb{P}$ is *flat* at a disc Δ if the sum of the angles of the corners of Δ is exactly 2π, that is $\gamma(\Delta) = 0$. Thus Lemma 4.3.5 says that there exists a region Φ with positive curvature assuming that $\mathbb{P}$ is flat at every disc. This means that if Φ has d corners then the sum of the angles measures of the corners of Φ is greater than $(d-2)\pi$. We will call such a region Φ

an *exceptional* region. Since Φ is a region in picture $\mathbb{P}$, we can find the possibilities of the labels of the corners of Φ and since the product of these must be 1 in H. we obtain some restriction on $h_1, h_2, \cdots, h_n$.

4.3.4 Distribution test

This test was introduced by Edjvet [22]—applicable after the curvature test. Assign an angle function θ on a strictly spherical picture $\mathbb{P}$ such that every disc in $\mathbb{P}$ is flat as above. So we have an exceptional region Φ with positive curvature. It is possible to *flatten* such a region by distributing the curvature to its neighbours.

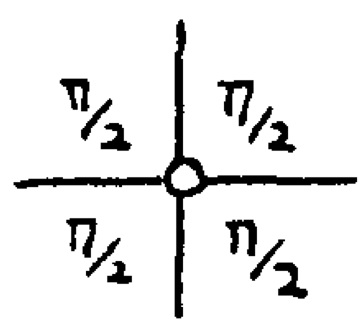
Suppose that Φ shares an arc with a region Φ'. Let η be any scalar. Subtract $\eta/2$ from each of the two corners in Φ that touch α and add $\eta/2$ to the two opposite corners in Φ'.



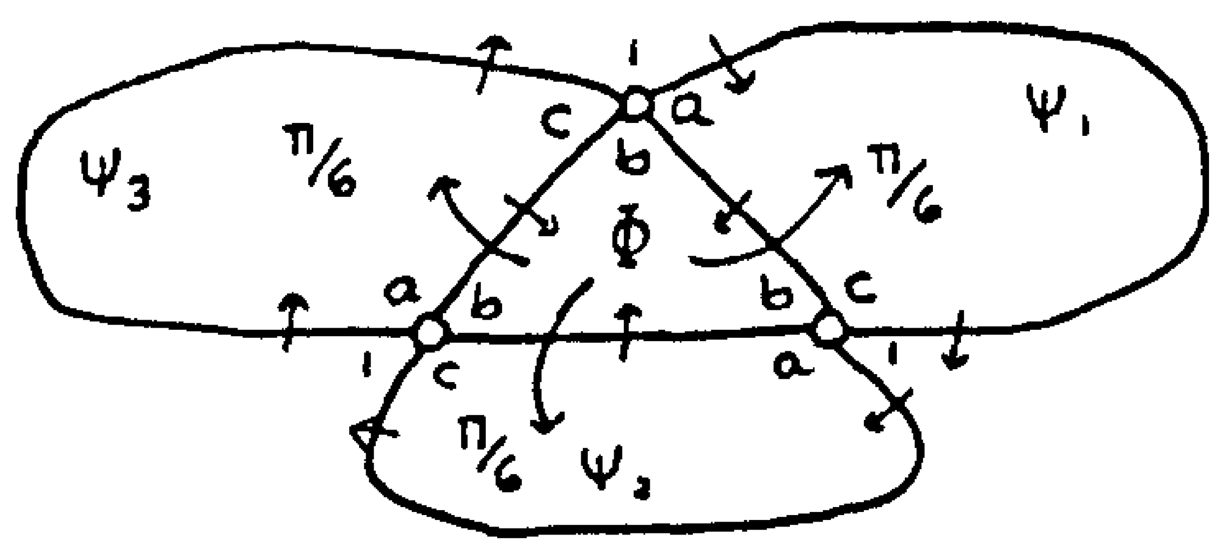
This means that we assign a new angle function θ^* on $\mathbb{P}$ and so we have a new curvature function γ^*. It is quite clear that $\gamma^*(\Delta) = \gamma(\Delta)$ for all discs Δ in $\mathbb{P}$, $\gamma^*(\Phi) = \gamma(\Phi) - \eta$ and $\gamma^*(\Phi') = \gamma(\Phi) + \eta$. Other regions are unaffected. We will say that we have *distributed* the curvature η from Φ to Φ'.

Given any region Φ with positive curvature, we can distribute curvature from Φ to some of the neighbouring regions of Φ, so that the new curvature $\gamma^*(\Phi)$ of Φ is non-positive. Of course, this reduction in the curvature of Φ will lead to an increase in the curvature of other regions in $\mathbb{P}$. However, if we can carry out this distribution in such a way that the new curvature of all regions is non-positive, then we obtain a contradiction to the fact that the curvature of the sphere is 4π. Then $\mathbb{P}$ could not exist.

Example 4.3.6 Let $\mathcal{P} =< H, t; t^2atbt^{-1}c >$ where a, b and c are distinct non-trivial element of H such that $a \neq b^{-1}, a \neq c^{-1}$ and $o(b) = 3, o(c) > 3$. Suppose that there is a reduced strictly spherical picture $\mathbb{P}$ over $\mathcal{P}$. Then we may assign a uniform angle



for each disc in $\mathbb{P}$. Then by Lemma 4.3.5, there is an exceptional region Φ of valence m with positive curvature. This means that $m \cdot \pi/2 > (m-2)\pi$ and so $m < 4$. Since $\mathbb{P}$ is reduced then one may check that there is only one possibility for Φ namely a region of valence three with label bbb. Since total sum of angle in Φ is $3\pi/2$, then the curvature $\gamma(\Phi) = \pi/2$. We may distribute



$\pi/6$ each to Ψ_1, Ψ_2 and Ψ_3. Clearly this is possible since they are not regions with label bbb. We need to make sure that the new curvature $\gamma^*(\Psi_i)(i = 1, 2, 3)$ remains non-positive. Note that in every three edges of Ψ_i, there is at most one such Φ shares an edge with Ψ_i (refer §4.4.1). Thus if Ψ has valence n , then we do not want $n \cdot \pi/2 + n/3 \cdot \pi/6 > (n-2)\pi$ that is $n < 4\frac{1}{2}$. Hence we just need to consider regions of valence four with label acW for some W of length two. Since $\mathbb{P}$ is reduced, the possibilities are $acc1, ac1b, ac1b^{-1}, aca^{-1}b$ and $aca^{-1}b^{-1}$. Clearly the last two are not possible since $o(b) \neq o(c)$. Thus if $a \neq c^{-2}$ and $b^{\pm 1} \neq ac$ then $\mathcal{P}$ must be aspherical.

4.3.5 Finiteness of element t

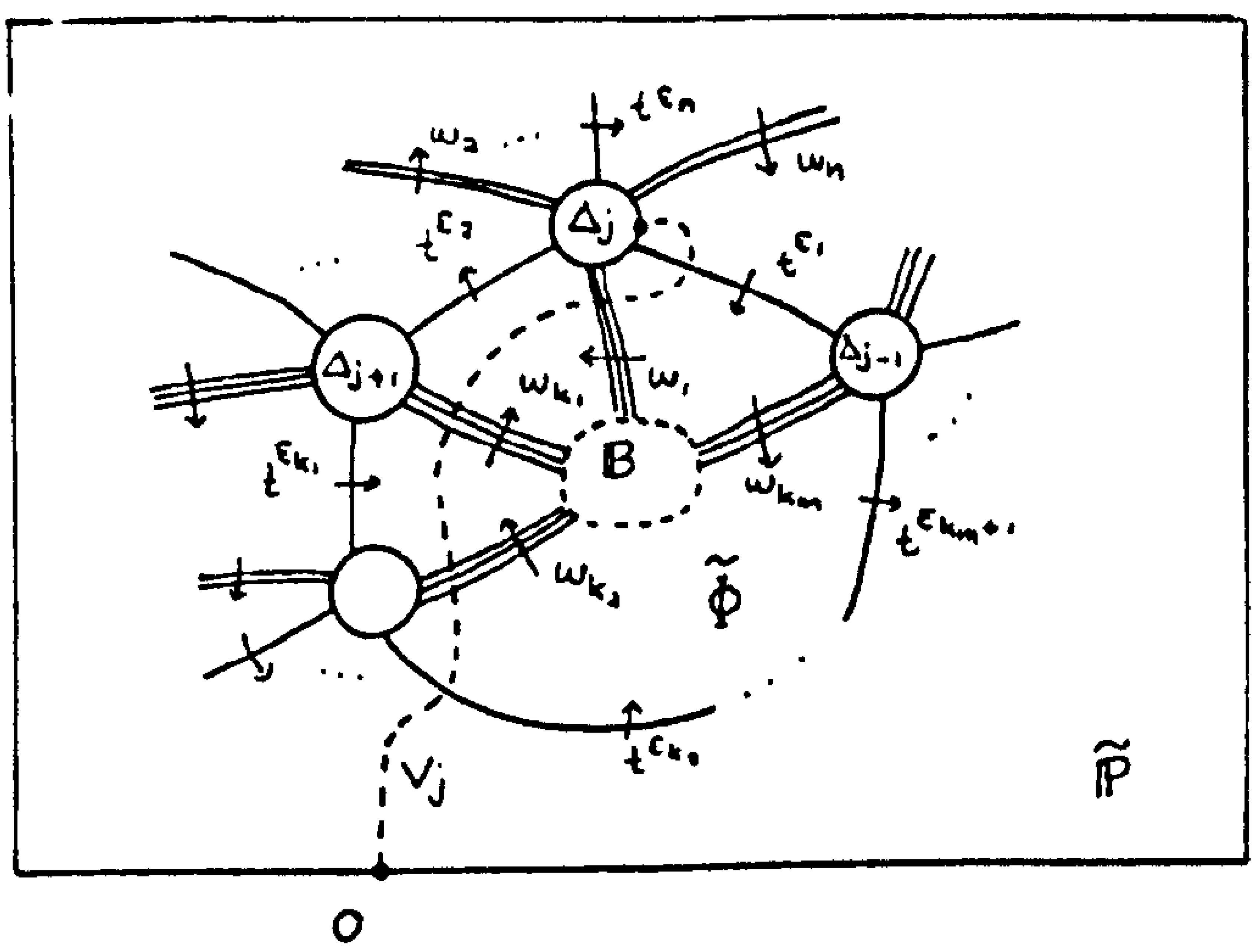
Recall that (see Theorem 4.1.1(iii)) if $\mathcal{P} =< H, t; t^{\epsilon_1}h_1t^{\epsilon_2}h_2 \cdots t^{\epsilon_n}h_n >$ is injective aspherical then every finite subgroup of G is contained in a G-conjugate of H. Also note that the factor group G/H^G is cyclic generated by tH^G of order $k = \epsilon_1 + \cdots + \epsilon_n$ (if $k = 0$ then it has infinite order). Now suppose that $k \neq \pm 1$, then clearly $t \notin H^G$. Hence if we can show that t has finite order then we know that $\mathcal{P}$ is not aspherical since the finite subgroup $\{t\}$ is not contained in any G-conjugate of H.

4.3.6 Degeneracy

Let $\mathbb{P}$ be a strictly spherical picture over $\mathcal{P}$. Recall that $\mathbb{P}$ can be lifted (though not uniquely) to a spherical picture $\tilde{\mathbb{P}}$ over $\tilde{\mathcal{P}}$ for some appropriate choice of $\mathcal{Q} =< \boldsymbol{a}; \boldsymbol{s} >$ (refer §4.2). We may now consider the image of $\tilde{\mathbb{P}}$ under the embedding (refer (1.1) in §1.1.3)

$$\mu_2 : \pi_2(\tilde{\mathcal{P}}) \longrightarrow (\oplus_{S\in \boldsymbol{s}} \mathbb{Z}\tilde{G}e_S) \oplus \mathbb{Z}\tilde{G}e_{\tilde{R}}. \tag{4.2}$$

In particular, we can consider the coefficient of $e_{\tilde{R}}$ say $\lambda_{\tilde{\mathbb{P}}}$ which belongs to the group ring $\mathbb{Z}\tilde{G}$. Since $\lambda_{\tilde{\mathbb{P}}}$ is the coefficient of $e_{\tilde{R}}$, in picture $\tilde{\mathbb{P}}$ we just need to consider all discs Δ_j ($j = 1, 2, \cdots, l$) with labels $\tilde{R}^{\pm 1}$. Let Φ be any region in $\mathbb{P}$ and $\mathbb{B}$ be a picture over $\mathcal{Q}$ with boundary label equal to the product of the words in $\boldsymbol{a}$ which represent the corner labels for Φ chosen for $\tilde{\mathbb{P}}$. Corresponding to Φ , we have a region $\tilde{\Phi}$ in $\tilde{\mathbb{P}}$ enclosed by t-arcs.



Let v_j be a transverse path from the basepoint of $\tilde{\mathbb{P}}$ to the basepoint of Δ_j such that v_j is always 'close' to t-arcs and $\tilde{R}$-discs (that is for any region $\tilde{\Phi}$ of $\tilde{\mathbb{P}}$, v_j does not cut through $\mathbb{B}$). Since the product of corners in any region $\tilde{\Phi}$ is the identity in H, it does not matter for v_j to be on the left or right of $\mathbb{B}$. Let V_j be the label on v_j. Note that V_j must be a word on $\boldsymbol{a} \cup \{t\}$. Then

$$\lambda_{\tilde{\mathbb{P}}} = \sum_{j=1}^{l} \delta_j \tilde{g}_j$$

where $\tilde{g}_j$ is the element of $\tilde{G}$ represented by V_j and $\delta_j = \pm 1$. Since the group isomorphism $\nu : \tilde{G} \longrightarrow G$ (refer §4.2) induces the ring isomorphism

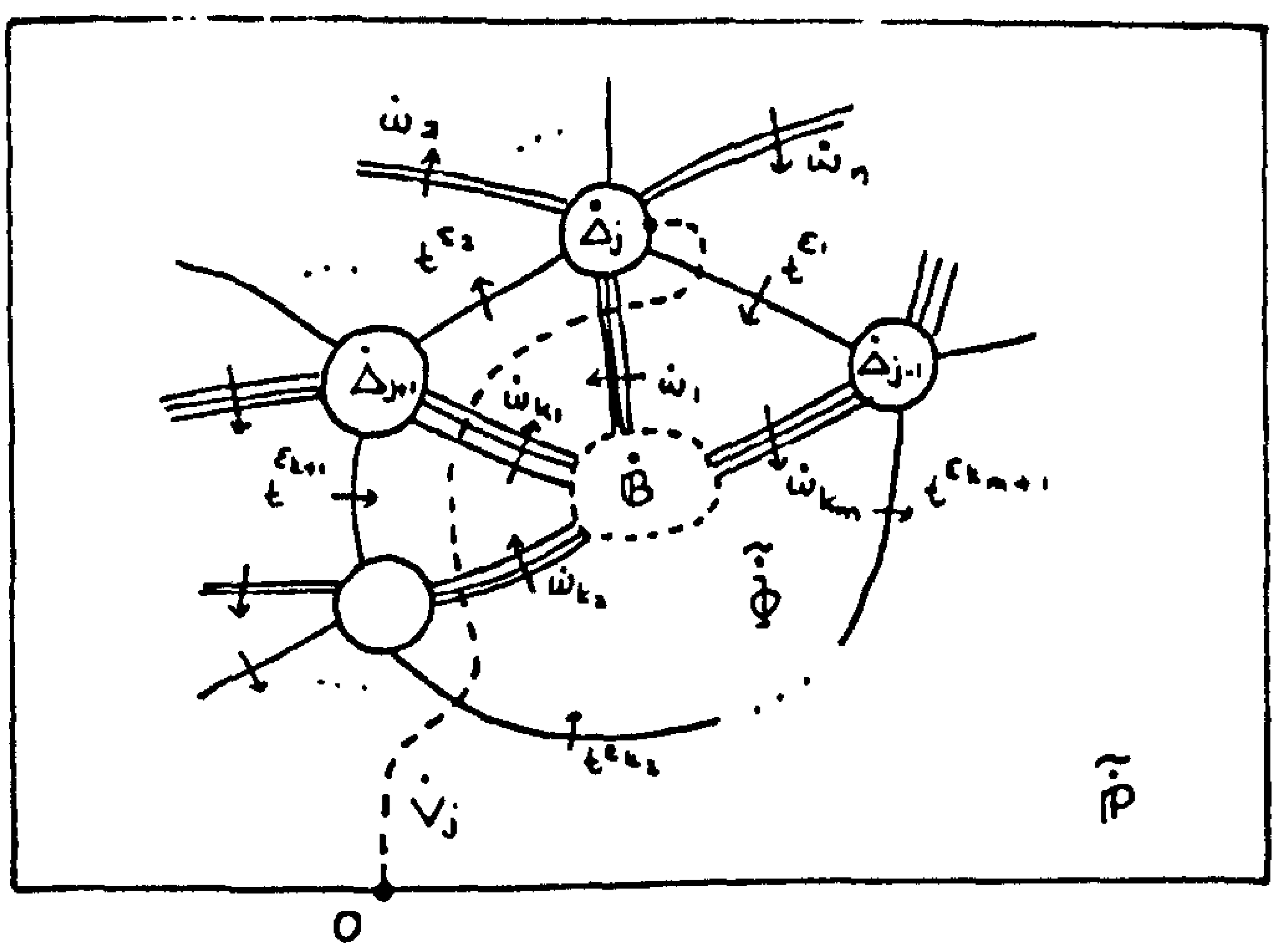
$$\nu_* : \mathbb{Z}\tilde{G} \longrightarrow \mathbb{Z}G,$$

let $\nu_*(\lambda_{\tilde{\mathbb{P}}}) = \lambda_{\mathbb{P}}$.

Suppose now we choose another presentation $\dot{\mathcal{Q}} =< \dot{a}; \dot{s} >$ for H, and so we have a lifted presentation

$$\tilde{\dot{\mathcal{P}}} =< \dot{a}, t; \dot{s}, \tilde{\dot{R}} >$$

and the group $\tilde{\dot{G}} = G(\tilde{\dot{\mathcal{P}}})$ where $\tilde{\dot{R}} = t_1^{\epsilon_1}\dot{w}_1 t_2^{\epsilon_2}\dot{w}_2 \cdots t_n^{\epsilon_n}\dot{w}_n$ such that $\dot{w}_i$ is a word on $\dot{a}$ representing h_i and $i = 1, 2, \cdots, n$. Thus we have a lifted picture $\tilde{\dot{\mathbb{P}}}$. Then by the similar set up and notation, if Φ is a region in $\mathbb{P}$ then let $\dot{\mathbb{B}}$ be a picture over $\dot{\mathcal{Q}}$ with boundary label equal to the product of the words in $\dot{a}$ which represent the corner labels for Φ chosen for $\tilde{\dot{\mathbb{P}}}$. Similarly let $\tilde{\dot{\Phi}}$ be the region in $\tilde{\dot{\mathbb{P}}}$ enclosed by t-arcs corresponds to Φ in $\mathbb{P}$.



Thus we may regard $\tilde{\dot{\mathbb{P}}}$ as a copy of $\tilde{\mathbb{P}}$ by replacing

1. $\tilde{R}$-discs by $\tilde{\dot{R}}$-discs,

2. successions of a-arcs with total label w_i by successions of $\dot{a}$-arcs with total label $\dot{w}_i$ and

3. picture $\mathbb{B}_k$ in region $\tilde{\Phi}_k$ by picture $\dot{\mathbb{B}}_k$

For each transverse path v_j in $\tilde{\mathbb{P}}$, the copy of v_j in $\dot{\tilde{\mathbb{P}}}$ is a transverse path from the basepoint of $\dot{\tilde{\mathbb{P}}}$ to the basepoint of $\dot{\Delta}_j$. Denote this path by $\dot{v}_j$ and let $\dot{V}_j$ be the label on $\dot{v}_j$. Note that $\dot{V}_j$ must be a word on $\dot{\boldsymbol{a}} \cup \{t\}$. Since there is an isomorphism

$$\frac{F(\dot{\boldsymbol{a}})}{<< \dot{\boldsymbol{s}} >>} \xrightarrow{\dot{\phi}_*} H \xrightarrow{\phi_*^{-1}} \frac{F(\boldsymbol{a})}{<< \boldsymbol{s} >>},$$

where $\dot{w}_j \leftrightarrow w_j$ under this isomorphism and $\dot{v}_j$ is a copy of v_j, then $\dot{V}_j \leftrightarrow V_j$. Thus if $\lambda_{\dot{\tilde{\mathbb{P}}}}$ is the image of the coefficient of $e_{\dot{\tilde{R}}}$ in picture $\dot{\tilde{\mathbb{P}}}$ under the embedding

$$\dot{\mu}_2 : \pi_2(\dot{\tilde{\mathcal{P}}}) \longrightarrow (\oplus_{\dot{S} \in \dot{\boldsymbol{s}}} \mathbb{Z}\check{G} e_{\dot{S}}) \oplus \mathbb{Z}\check{G} e_{\dot{\tilde{R}}},$$

then $\lambda_{\dot{\tilde{\mathbb{P}}}} \leftrightarrow \lambda_{\tilde{\mathbb{P}}}$.

This means that $\lambda_{\mathbb{P}}$ is independent of the choice of $\mathcal{Q}$, the choice of $\tilde{R}$ and the choice of lift. We say that $\mathbb{P}$ is *degenerate* if $\lambda_{\mathbb{P}} = 0$.

Recall that in order to show that $\mathcal{P}$ is aspherical, one must show that there is a lifted presentation $\tilde{\mathcal{P}}$ such that any picture over $\tilde{\mathcal{P}}$ is equivalent to the empty picture (relative to $\mathcal{Q}$). Since $\lambda_{\mathbb{P}}$ is independent of the lifting, then we have

Lemma 4.3.7 *If there exists a reduced strictly spherical picture $\mathbb{P}$ over $\mathcal{P}$ such that $\mathbb{P}$ is not degenerate (that is $\lambda_{\mathbb{P}} \neq 0$) then $\mathcal{P}$ is not aspherical.*

Proof Let $\mathbb{P}$ be a reduced strictly spherical picture over $\mathcal{P}$ such that $\lambda_{\mathbb{P}} \neq 0$ and suppose that $\mathcal{P}$ is aspherical. Then there is a presentation $\mathcal{Q} =< \boldsymbol{a}; \boldsymbol{s} >$ of H such that $\tilde{\mathbb{P}}$ is equivalent to the empty picture (relative to $\mathcal{Q}$). Thus the image of $\tilde{\mathbb{P}}$ under the embedding μ_2 as in (4.2) lies entirely in $\oplus_{S \in \boldsymbol{s}} \mathbb{Z}\tilde{G} e_S$. Hence the coeficient of $e_{\tilde{R}}$ must be zero that is $\lambda_{\mathbb{P}} = 0$. •

In order to show that $\mathbb{P}$ is not degenerate, we can consider the ring homomorphism

$$\psi : \mathbb{Z}G \longrightarrow \mathbb{Z} < x; x^k >$$

arising from the group homomorphism defined by

$$H \mapsto 1, t \mapsto x.$$

Here we assume that R has the form (4.1) and so $k = \epsilon_1 + \cdots + \epsilon_n$. Then we need to show that $\psi(\lambda_{\mathbb{P}}) \neq 0$.

Unless we manage to show that t has finite order (refer §4.3.5), we will show that a presentation $\mathcal{P}$ is not aspherical in this way.

Example 4.3.8 Let $\mathcal{P} =< \mathbb{Z}_3, t; t^3at^{-1}a >$ as in Example 4.2.1. We will show that $\mathcal{P}$ is not aspherical. Consider the reduced strictly spherical picture $\mathbb{P}$ and its lifted picture $\tilde{\mathbb{P}}$ given in Example 4.2.1. Then

$$\lambda_{\mathbb{P}} = (1 + a + a^2)t^{-1}a - (1 + a + a^2)t^{-1}at^2at^{-1}a.$$

Thus

$$\psi(\lambda_{\mathbb{P}}) = 3x - 3 \neq 0.$$

So $\mathbb{P}$ is not degenerate.

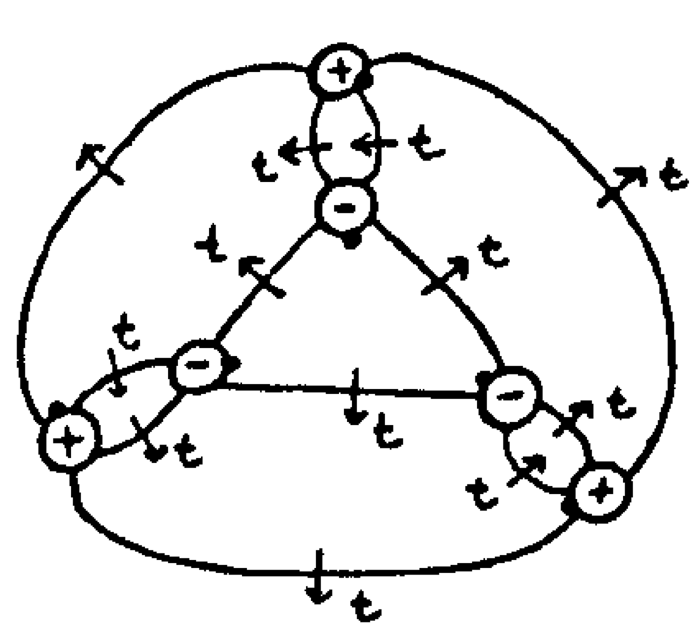
Let $\mathcal{P} =< H, t; t^{\epsilon_1}h_1t^{\epsilon_2}h_2 \cdots t^{\epsilon_n}h_n >$ and $\mathbb{P}$ be a spherical picture over $\mathcal{P}$. Also let $\mathbb{P}^{(o)}$ be the picture over $\mathcal{P}^{(o)} =< t; R^{(o)} >$ (where $R^{(o)} = t^{\epsilon_1}t^{\epsilon_2} \cdots t^{\epsilon_n}$) obtained from $\mathbb{P}$ by eliminating all corner labels. Note that $\mathcal{P}^{(o)}$ is a presentation for the cyclic group of order k. Then $\psi(\lambda_{\mathbb{P}})$ is just the coefficient of $e_{R^{(o)}}$ in the embedding of

$$\mu : \pi_2(\mathcal{P}^{(o)}) \longrightarrow \mathbb{Z}(\mathbb{Z}_k)e_{R^{(o)}}.$$

Thus we can work out $\psi(\lambda_{\mathbb{P}})$ without having to work out $\lambda_{\mathbb{P}}$ itself. This is useful in particular when $\mathbb{P}$ has too many discs.

Example 4.3.8 (continued)

We obtained a picture $\mathbb{P}^{(o)}$



The embedding μ in the usual way gives

$$\psi(\lambda_{\mathbb{P}}) = 3x - 3.$$

4.4 Preliminary classification

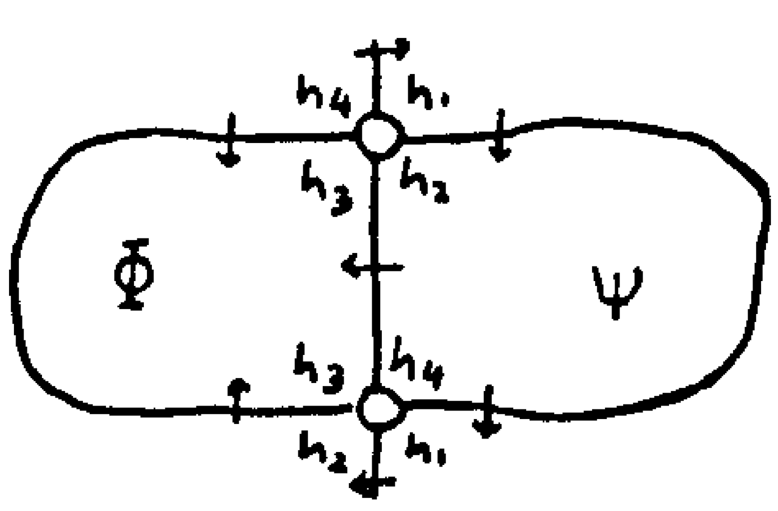
We are interested in the asphericity of relative presentation $\mathcal{P}$ where R has the form

$$th_1th_2th_3t^{-1}h_4 \ (h_3, h_4 \neq 1).$$

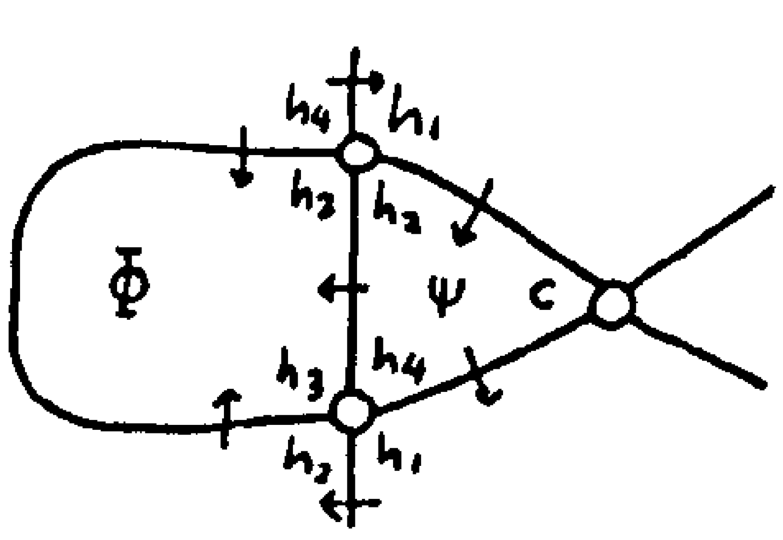
We give here (§4.4.1) a crucial observation concerning pictures over $\mathcal{P}$. Also we list (§4.4.2) the different subcases which must be considered in discussing the asphericity of $\mathcal{P}$.

4.4.1 Observation

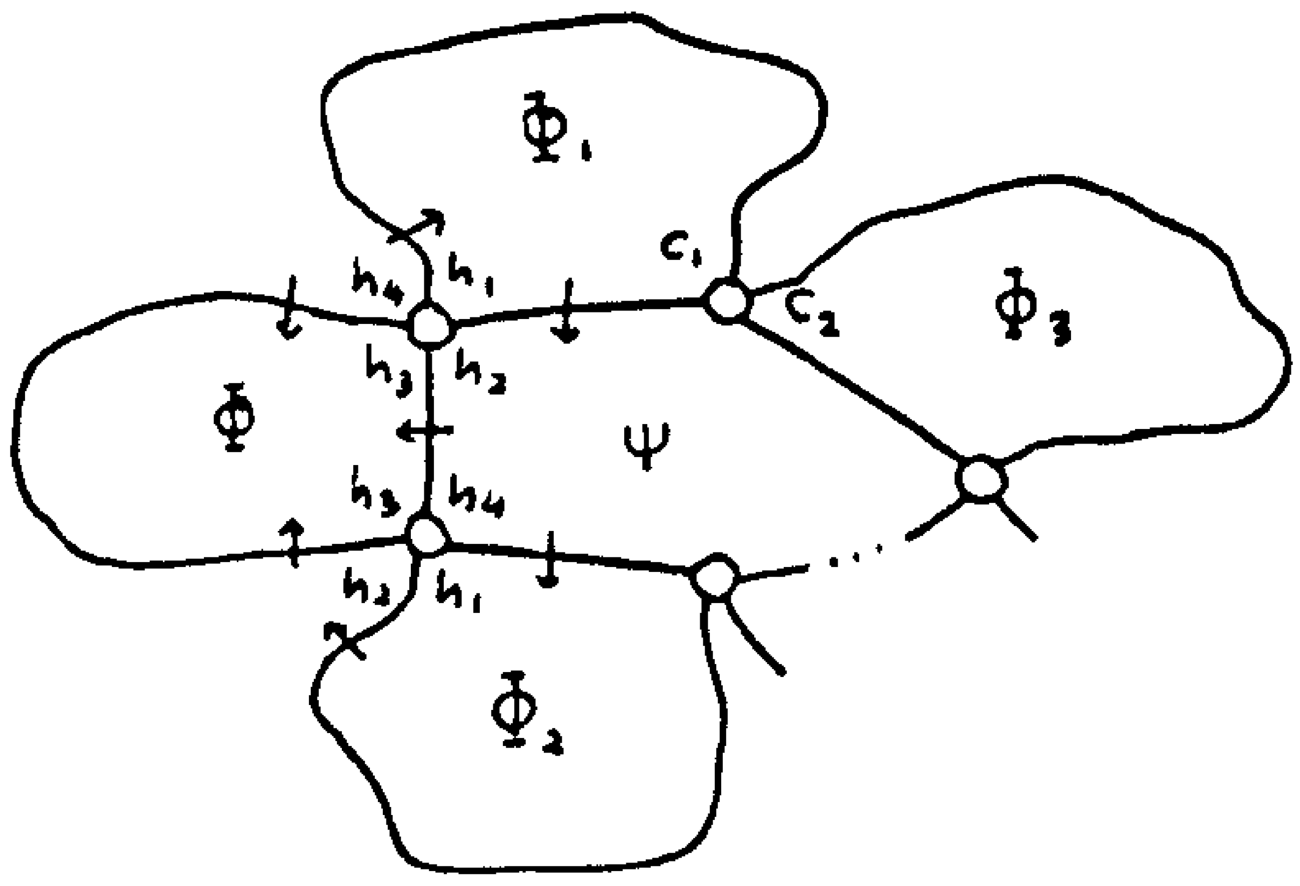
Fix a relative presentation $\mathcal{P} =< H, t; th_1th_2th_3t^{-1}h_4 >$. Let Φ be any region in a reduced strictly spherical picture $\mathbb{P}$ over $\mathcal{P}$. We say that Φ is a *3-region* or a *4-region* if every corner in Φ is labelled by h_3 or h_4 respectively. Let Ψ be any region in $\mathbb{P}$ that shares an edge with a 3-region Φ.



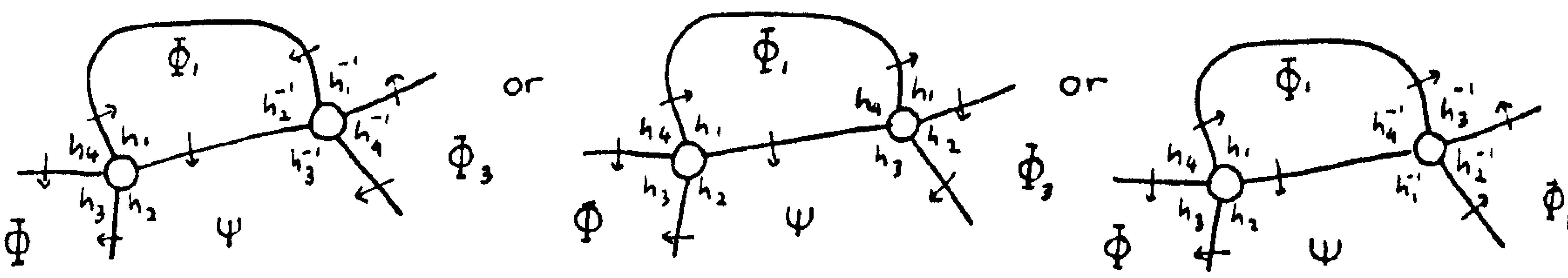
From the above diagram, we know that Ψ can not be a region of valence two. Now suppose that Ψ has valence three



Since $\mathbb{P}$ is reduced, the only possible label for c is h_1^{-1} and hence we have $h_1 = h_2h_4$. In most of our cases, we assume $h_1 \neq h_2h_4$ and so Ψ must have valence of at least four.



Clearly Φ_1 and Φ_2 are not 3-regions or 4-regions. Note that since $\mathbb{P}$ is reduced, c_1 must have label h_2^{-1}, h_4 or h_4^{-1} and so c_2 must have label h_4^{-1}, h_4 or h_2^{-1} respectively.



Thus Φ_3 can not be a 3-region but it may be a 4-region. Hence in every three edges of Ψ, there is at most one 3-region as its neighbour. By similar argument for a 4-region, we may conclude that if Ψ has valence m, then Ψ has at most $m/3$ 3-regions and $m/3$ 4-regions as its neighbours.

This observation is important in assigning any distribution scheme. This fact is needed in §6.3.3 and §6.3.4.

4.4.2 The forms of R

We are looking at the asphericity of $\mathcal{P}$ where R has the form $th_1th_2th_3t^{-1}h_4$ and $h_3, h_4 \neq 1$. By applying the automorphism (refer **III** of §4.1.1)

$$t \longrightarrow th_1^{-1}, H \xrightarrow{id} H$$

of $H* < t >$ if necessary, we may assume without loss of generality that $h_1 = 1$, so we have

$$R = t^2 h_2 t h_3 t^{-1} h_4 \ (h_3, h_4 \neq 1).$$

There are two main cases:

case 1: $h_1 = 1$

Then after changing notation, we have

$$R = t^3 a t^{-1} b \ (a, b \neq 1)$$

and so we have to consider:

1.1 $a = b$

1.2 $a \neq b$

case 2: $h_2 \neq 1$

Then after changing notation we have that R has one of the following form:

2.1 $t^2 atat^{-1} a$

2.2 $t^2 atat^{-1} c$

2.3 $t^2 atbt^{-1} b$

2.4 $t^2 atbt^{-1} a$

2.5 $t^2 atbt^{-1} c$

where a, b and c are all distinct non-trivial elements of H.

We will consider the first case in Chapter 5 and the second case in Chapter 6.

Chapter 5

The form $t^3at^{-1}b$

As we classified in §4.4.2, we have two main subcases:

1.1 $a = b$

1.2 a and b are distinct non-trivial elements of H

For the first subcase we will show that $\mathcal{P}$ is aspherical if and only if a has infinite order (Theorem 5.1.1). For the second subcase we can decide the asphericity of $\mathcal{P}$ apart from two exceptional families

FE1 $o(a) = 2, o(b) \geq 4$ and $ab = ba$

FE2 $3 \leq o(a) \leq \infty$ and $a = b^2$

The proof of Theorem 5.1.1 is not difficult (see §5.1). However Theorem 5.2.1 requires a fair amount of work (see §5.2).

Throughout this chapter ψ will denote the ring homomorphism (refer §4.3.6)

$$\mathbb{Z}G \longrightarrow \mathbb{Z} < x; x^2 >$$

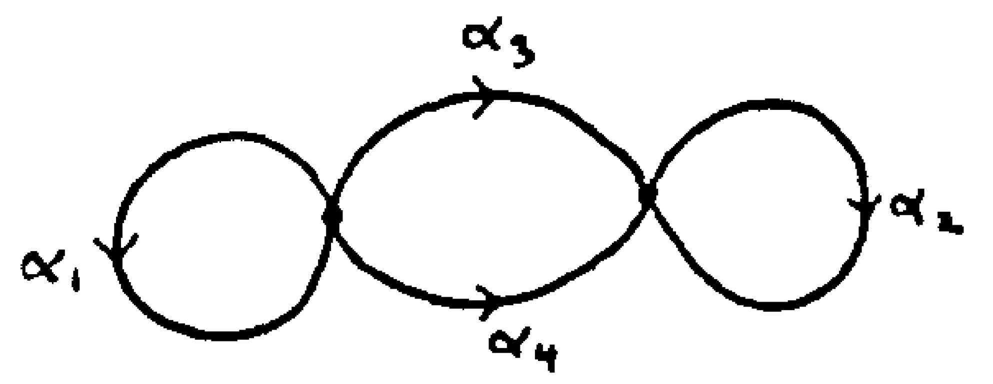
$$H \mapsto 1, t \mapsto x.$$

5.1 The case $a = b$

We obtain the following result.

Theorem 5.1.1 *Let $\mathcal{P} = < H, t; t^3at^{-1}a >$ where a is a non-trivial element in H. Then $\mathcal{P}$ is aspherical if and only if a has infinite order.*

Proof Let a have infinite order. Consider the star graph $\mathcal{P}^{st}$

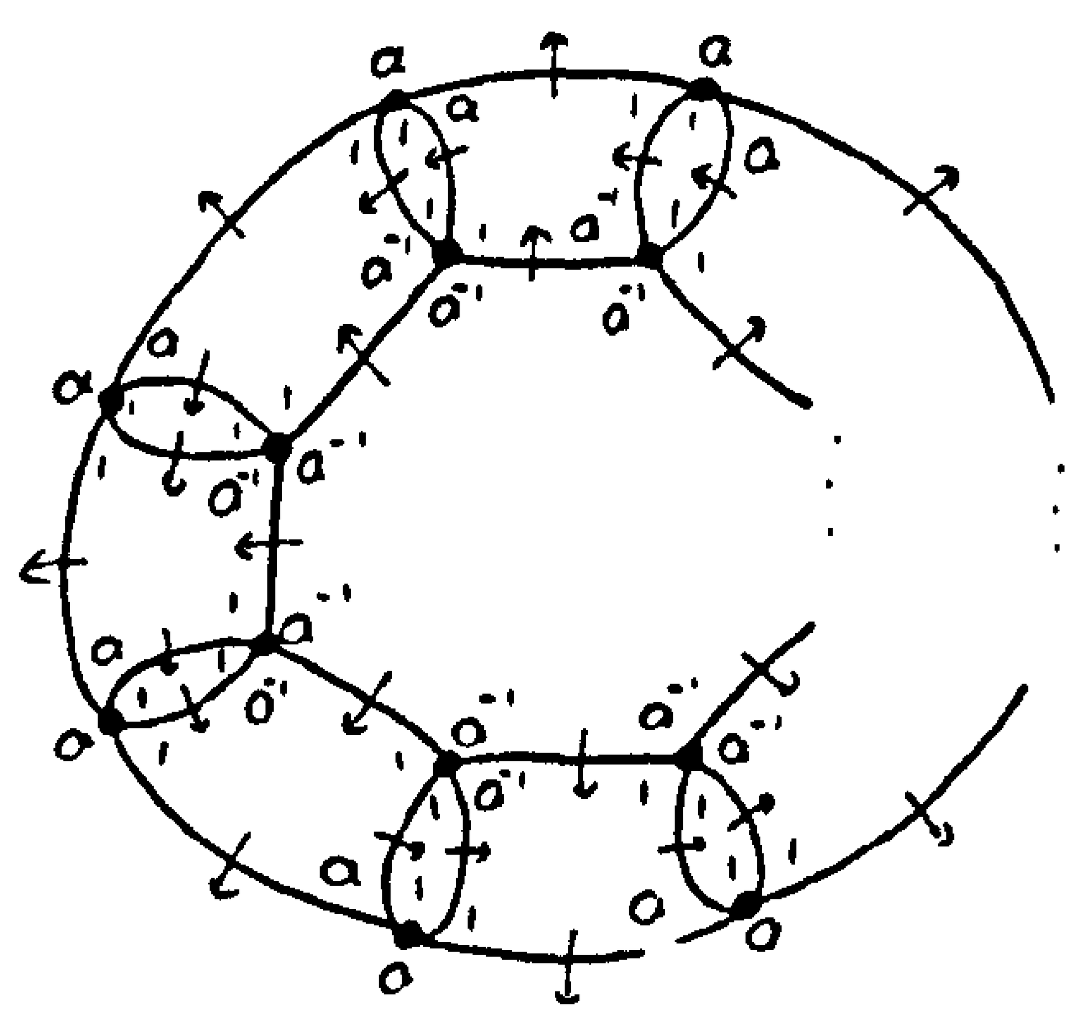


where $\alpha_1 \leftrightarrow a^{-1}, \alpha_2 \leftrightarrow a^{-1}, \alpha_3 \leftrightarrow 1$ and $\alpha_4 \leftrightarrow 1$. Assign the following weights

$$\theta(\alpha_1) = \theta(\alpha_2) = 0, \theta(\alpha_3) = \theta(\alpha_4) = 1.$$

Since a has infinite order, any admissible cycle must involve α_3 and/or α_4 at least twice, and so it has a weight of at least two. One may check that all three conditions given in §4.3.2 are satisfied. Hence by Theorem 4.3.3, $\mathcal{P}$ is aspherical.

Now let a have order $2 \leq n < \infty$. Since $a^n = 1$ in H, we may obtain a reduced strictly spherical picture $\mathbb{P}$ over $\mathcal{P}$ as below



and so (refer §4.3.6) $\psi(\lambda_{\mathbb{P}}) = n(x-1) \neq 0$. Thus $\mathbb{P}$ is not degenerate and so by Lemma 4.3.7 $\mathcal{P}$ is not aspherical. •

5.2 The case a and b are distinct

Consider presentation $\mathcal{P} =< H, t; t^3at^{-1}b >$ where $a \neq b$ $(a, b \neq 1)$ in H. Note that we can apply some operations (refer §4.1.1)

$$\begin{aligned} t^3at^{-1}b &\rightarrow ta^{-1}t^{-3}b^{-1} \text{ (operation I)} \\ &\rightarrow t^{-1}a^{-1}t^3b^{-1} \text{ (operation III)} \\ &\rightarrow t^3b^{-1}t^{-1}a^{-1} \text{(operation II)}. \end{aligned}$$

So up to equivalent presentations, we can identify a with b^{-1} and vice versa. Thus without loss of generality, we may assume that $o(a) \leq o(b)$. The following are two exceptional cases that we still can not decide:

FE1 $o(a) = 2, o(b) \geq 4$ and $ab = ba$

FE2 $3 \leq o(a) < \infty$ and $a = b^2$

Excluding these exceptions, we have

Theorem 5.2.1 *Suppose that $\mathcal{P} =< H, t; t^3at^{-1}b >$ is not an exceptional case and $2 \leq o(a) \leq o(b)$ where a and b are distinct non-trivial elements of H. Then $\mathcal{P}$ is aspherical if and only if none of these holds:*

1. $a^2 = 1, a = b^2$
2. $a^2 = 1, a = b^3$
3. $a^2 = b^3 = 1$ *and* $group\{a, b\} \simeq \mathbb{Z}_2 \times \mathbb{Z}_3$
4. $o(a) = p, o(b) = q$ *and* $o(ab^{-1}) = k$ *for* $\frac{1}{p} + \frac{1}{q} + \frac{1}{k} > 1$ *where* $\frac{1}{\infty} := 0$
5. $a = b^{-1}$ *and a has a finite order*

To prove this theorem, we will consider the following subcases separately.

i. a and b have infinite order

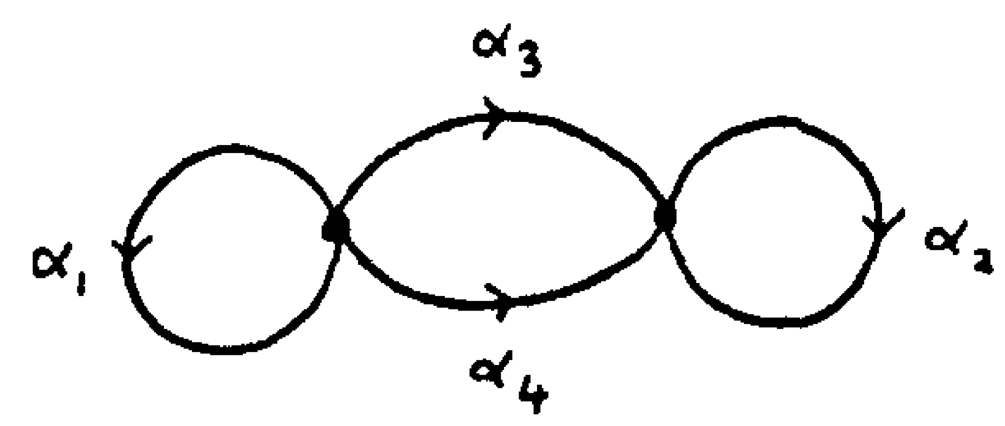
ii. $o(a) = o(b) = 2$

iii. $o(a) = 2, o(b) \geq 3$

iv. $3 \leq o(a) < \infty$

5.2.1 The subcase a and b have infinite order

None of the hypotheses is satisfied, so we will show that $\mathcal{P}$ is aspherical. Consider the star graph $\mathcal{P}^{st}$



where $\alpha_1 \leftrightarrow b^{-1}, \alpha_2 \leftrightarrow a^{-1}, \alpha_3 \leftrightarrow 1$ and $\alpha_4 \leftrightarrow 1$. Assign the following weights

$$\theta(\alpha_1) = \theta(\alpha_2) = 0, \theta(\alpha_3) = \theta(\alpha_4) = 1.$$

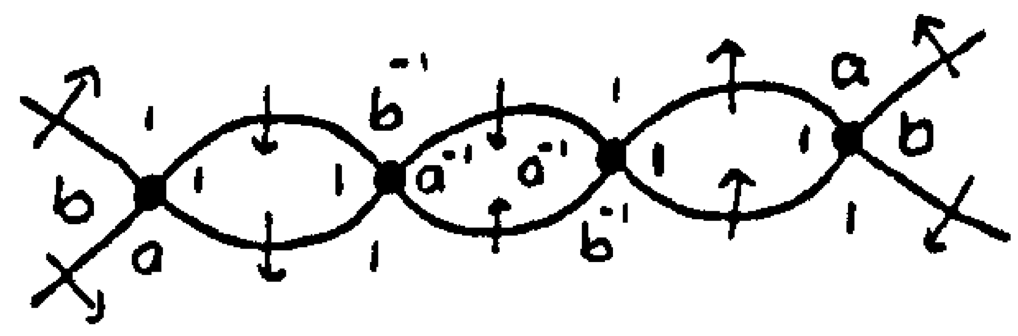
Since a and b have infinite order, any admissible cycle must involve α_3 and/or α_4 at least twice, and so it has a weight of at least two. One may check that all three conditions given in §4.3.2 are satisfied. Hence by Theorem 4.3.3, $\mathcal{P}$ is aspherical.

5.2.2 The subcase $o(a) = o(b) = 2$

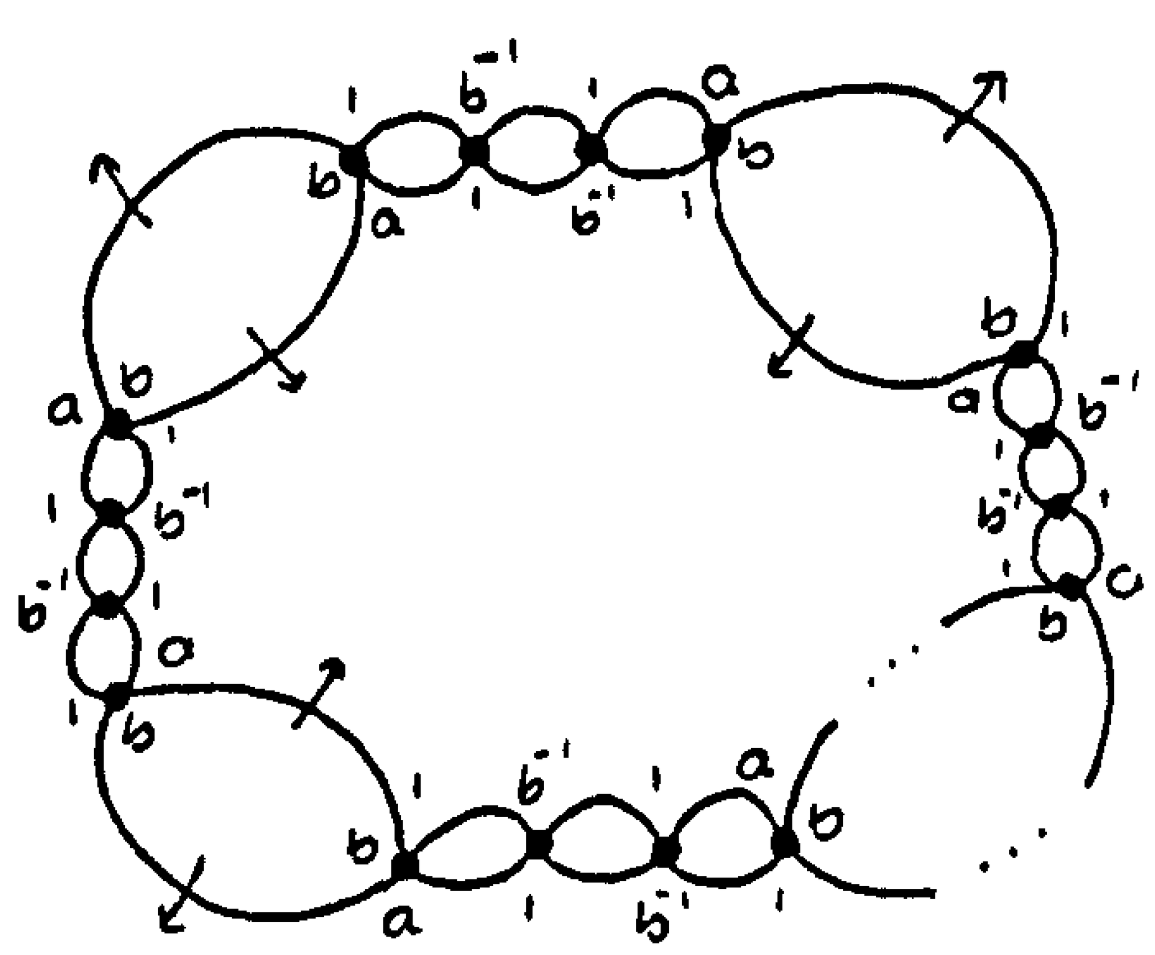
In this section, we have to show that $\mathcal{P}$ is aspherical if and only if ab^{-1} has infinite order in H.

1) ab^{-1} has finite order

Let ab^{-1} have order $2 \leq n < \infty$. Consider the chain



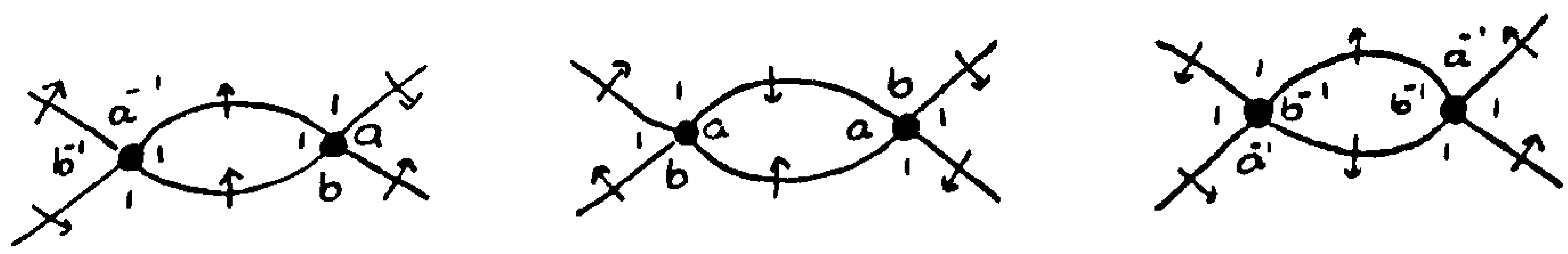
Since $(ab^{-1})^n = 1$, we can join n chain like this to obtain a reduced strictly spherical picture



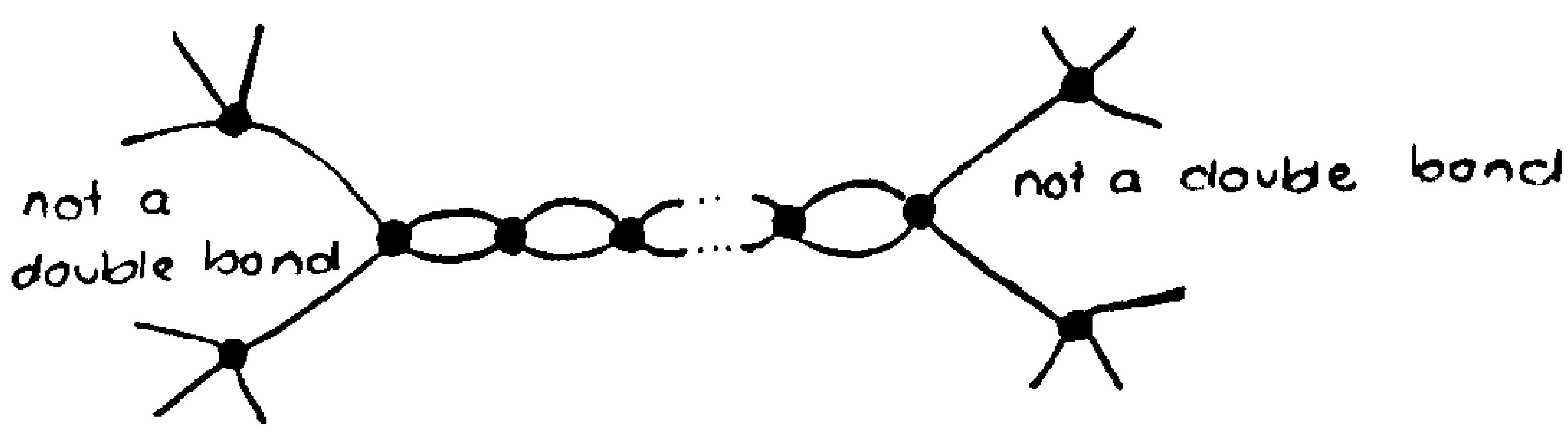
and so (refer §4.3.6) $\psi(\lambda_{\mathbb{P}}) = 2n(x-1) \neq 0$. Since $\mathbb{P}$ is not degenerate then by Lemma 4.3.7, $\mathcal{P}$ is not aspherical.

2) ab^{-1} has infinite order

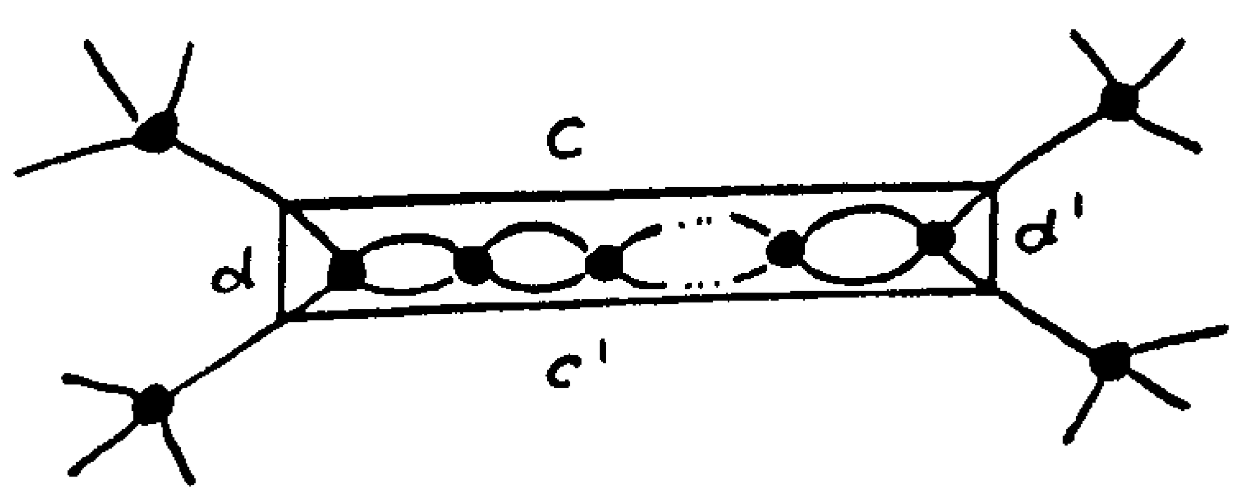
Now let ab^{-1} have infinite order. We will show that $\mathcal{P}$ is aspherical. It suffices to show that there is no reduced strictly spherical picture $\mathbb{P}$ over $\mathcal{P}$. Suppose there were. Since $\mathbb{P}$ is reduced, we only have the following double bonds



Some of them may form a chain



Regard each chain as a single disc

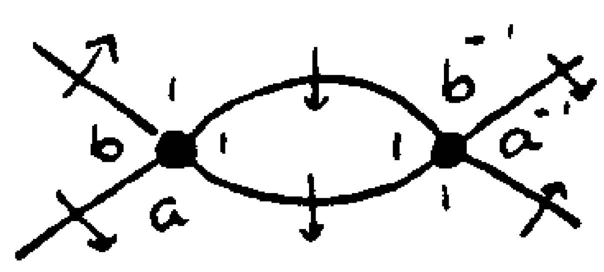
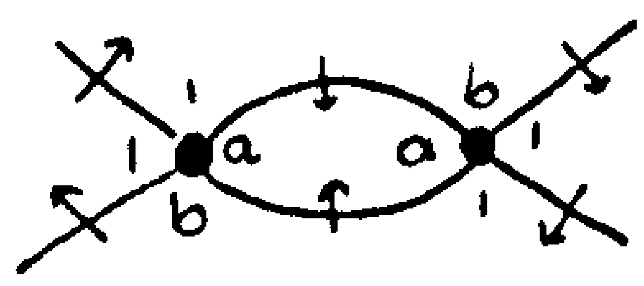


where labels c and c' must have the form $(ab^{-1}ab^{-1}\cdots ab^{-1})^{\pm 1}, (ab^{-1}ab^{-1}\cdots ab^{-1}a)^{\pm 1}$ or $(b^{-1}ab^{-1}a\cdots b^{-1}a)^{\pm 1}$. So we obtain a new *derived picture* $\mathbb{P}'$ such that each disc has valence four. Assign the angle function $\pi/2$ to each corner so that every disc in $\mathbb{P}'$ is flat. Then by Lemma 4.3.5, there exists an exceptional region Φ' such that the curvature $\gamma(\Phi') > 0$. If Φ' has m sides, this means that $\gamma(\Phi') > (m-2)\pi$. Since each corner has angle $\pi/2$, we have $m\pi/2 > (m-2)\pi$ which means $m < 4$. We will show that there is no such region by examining the label on Φ' (refer Appendix A.1.1) and so we conclude that $\mathcal{P}$ must be aspherical.

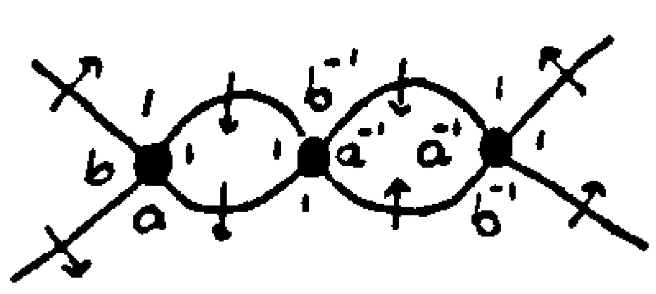
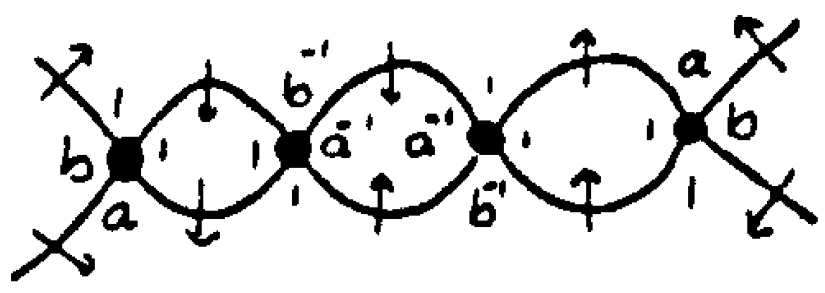
5.2.3 The subcase $o(a) = 2, o(b) \geq 3$

When is $\mathcal{P}$ aspherical?

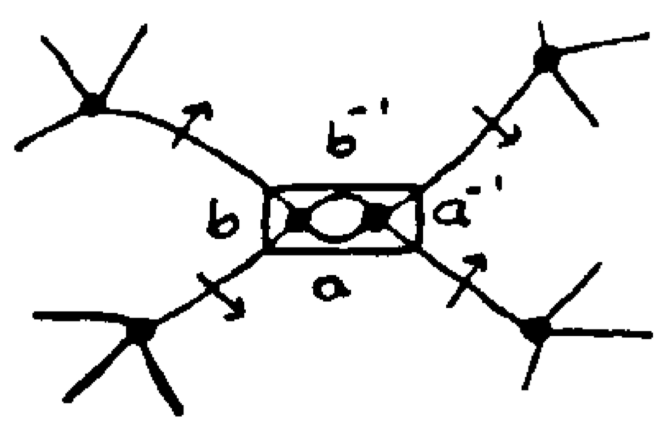
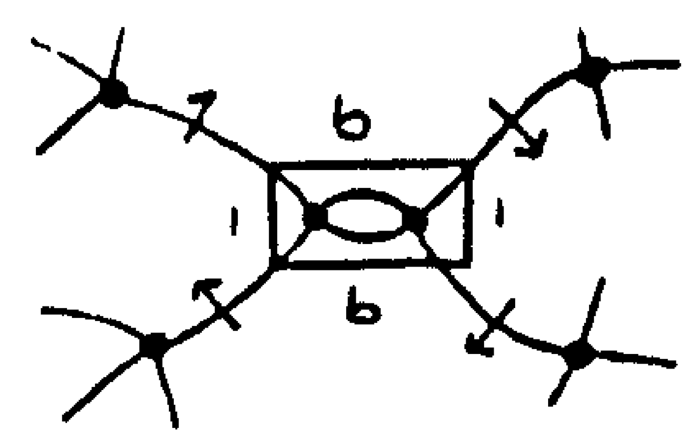
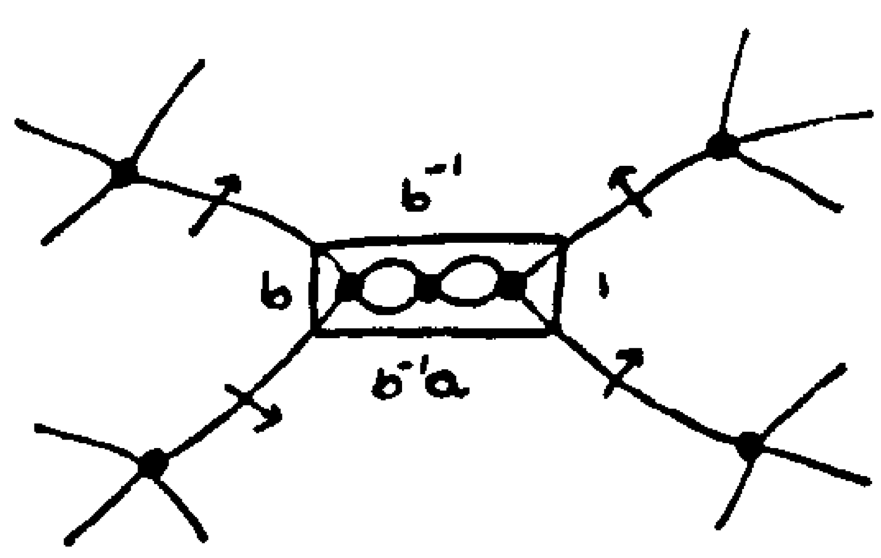
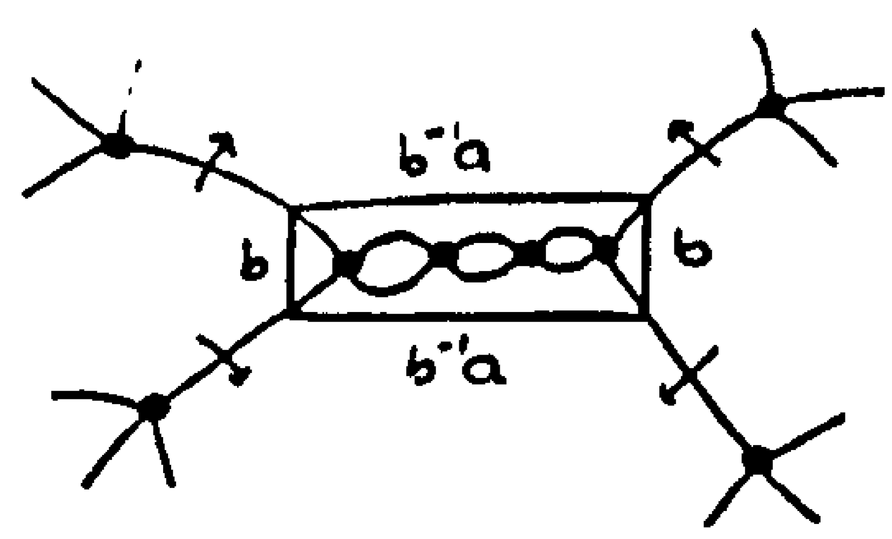
Assume that none of the relations in the hypothesis holds. We will show that $\mathcal{P}$ is aspherical. It is suffices to show that there is no reduced strictly spherical picture $\mathbb{P}$ over $\mathcal{P}$. Suppose there were. Since $\mathbb{P}$ is reduced and $o(b) \geq 3$, we have the following double bonds

Some of them may form chains of length two or three

Regard all bonds and chains

as single discs. Then we have a new derived picture $\mathbb{P}'$ such that each disc has valence four. Assign the angle function $\pi/2$ to each corner of the disc so that every disc in $\mathbb{P}'$ is flat. Then by Lemma 4.3.5 there is an exceptional region Φ' of valence m such that $\gamma(\Phi') > 0$. This means that $m\pi/2 > (m-2)\pi$ and so $m < 4$. Thus Φ' must have either valence two or three. Finding all possibilities for Φ' (refer Appendix A.2.1 for more details), we may conclude that $\mathcal{P}$ is aspherical except possibly if one of these holds:

1. $o(ab^{-1}) = 2$

2. $o(ab^{-1}) = 3$

3. $b = (b^{-1}a)^2$ (5.1)

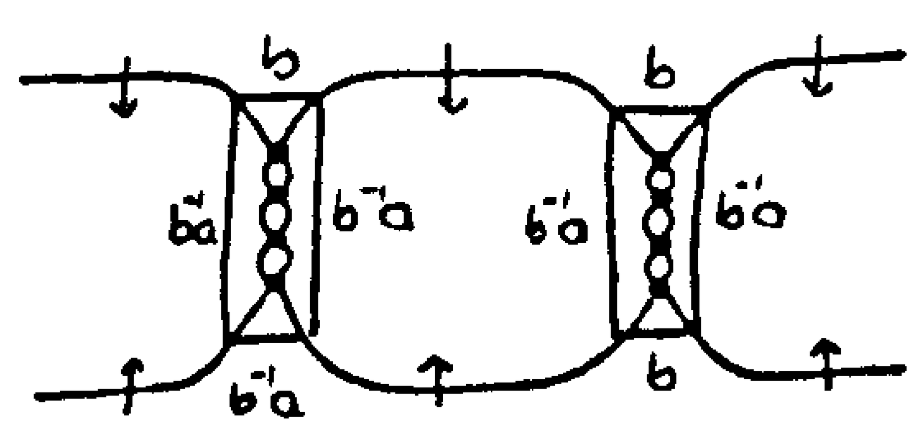
4. $o(b) = 3$

5. $ab^{-1}ab = 1$

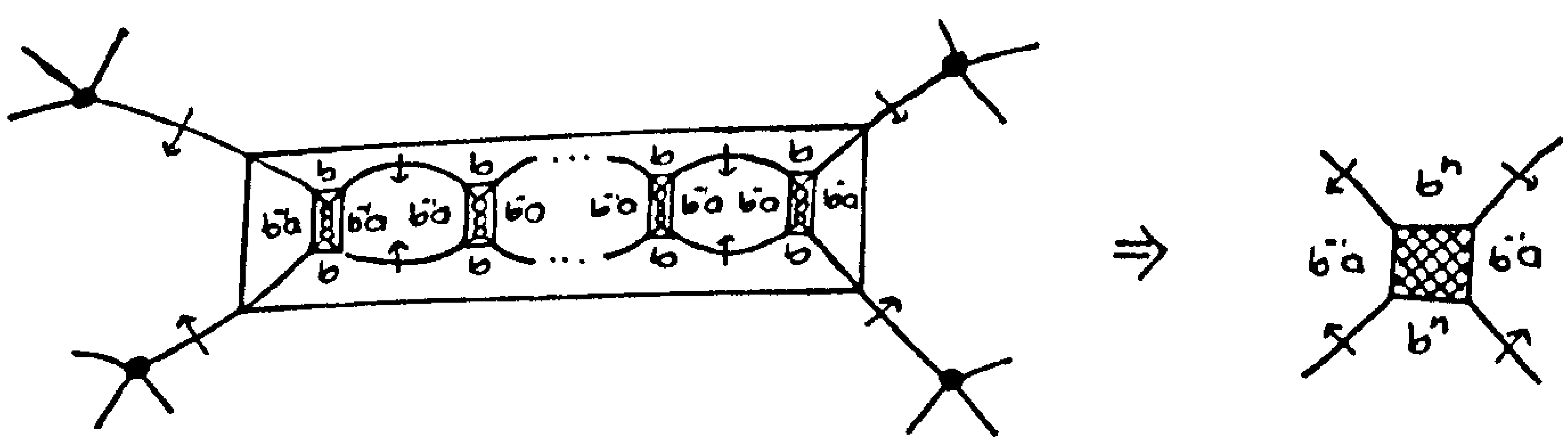
We will consider each of these separately.

1) $o(ab^{-1}) = 2$

Since we assume that (refer hyphothesis 4) $1/2 + 1/q + 1/2 \leq 1$, b must have infinite order. Note that we have a double bond



Some of them may form chains and so regard them as single discs



Thus we have the second derived spherical picture $\mathbb{P}''$ such that each disc has valence four. As before, we can assign the angle function $\pi/2$ to each corner so that each disc in $\mathbb{P}''$ is flat. Thus by Lemma 4.3.4, there is an exceptional region Φ'' with positive curvature. So Φ'' must have either two or three sides. Since we have already eliminated a lot of possibilities (refer Appendix A.2.2), we found that no more possible labels may hold and this leads to a contradiction. Thus $\mathcal{P}$ must be aspherical.

2) $o(ab^{-1}) = 3$

Note that since we assume that (refer hypothesis 4) $1/2+1/q+1/3 \leq 1$, then we must have $o(b) \geq 6$. We will use the distribution test and so we go back to the exceptional region Φ' in $\mathbb{P}'$. Now we will show that the only possible label for Φ' is $b^{-1}ab^{-1}ab^{-1}a$ since other relations in (5.1) do not hold.

Clearly $ab^{-1}ab^{-1} \neq 1$. We can not have relation $b = (b^{-1}a)^2$ since $(b^{-1}a)^3 = 1$ would give $a = 1$. Clearly $b^3 \neq 1$ since we assume that $o(b) \geq 6$. Also if relation $ab^{-1}ab = 1$ holds then we would have $b^{-1}ab^{-1} = b$ and hence $a = b^3$ which satisfies hypothesis 2.

Thus Φ' must have label $b^{-1}ab^{-1}ab^{-1}a$ of valence 3, and so the curvature $\gamma(\Phi') = \pi/2$. We will distribute $\pi/6$ to each region Ψ that shares an edge with Φ' since Ψ can not have label $b^{-1}ab^{-1}ab^{-1}a$ (refer Appendix A.2.3). Assume that Ψ has valence m, then total sum of angles in Ψ is less than or equal to $m\pi/2+m\pi/6$. We will make sure that the new curvature $\gamma^*(\Psi)$ remains non-positive, that is $(m-2)\pi \geq m\pi/2+m\pi/6$ which means that $m \geq 6$. So consider the case when Ψ has valence less than six. There is no possible labels for Ψ of valence two, three or four (refer Appendix A.2.3 for more details). There are some possible labels for Ψ of valence five. This gives rise to the following relations:

1. $a = b^4$

2. $b^2aba = 1$

3. $b^2ab^{-1}a = 1$

4. $b^3aba = 1$

We will show that they are not possible too.

If $a = b^4$ then $b^8 = 1$. Also we have $ab^{-1} = b^3$ and so $b^9 = 1$ since $o(ab^{-1}) = 3$. This leads to $b = 1$.

Suppose that $b^2aba = 1$. Then $ab^{-1}ab^{-2} = 1$ and so $b^{-1} = ab^{-1}$ which contradicts the fact that a is not trivial.

If the relation $b^2ab^{-1}a = 1$ holds then

$$b^3b^{-1}ab^{-1}a = 1 \;\Rightarrow\; b^3 = b^{-1}a \text{ since } (b^{-1}a)^3 = 1$$
$$\Rightarrow\; b^4 = a$$
$$\Rightarrow\; b^8 = 1.$$

Also $b^3 = b^{-1}a \Rightarrow b^9 = 1$ since $o(ab^{-1}) = 3$. This will imply that $b = 1$ a contradiction.

Now suppose that $b^3aba = 1$ and then $b^2 = a^{-1}b^{-1}a^{-1}b^{-1} = ab^{-1}ab^{-1}$ since $a^2 = 1$. Thus we have $b^2 = (ab^{-1})^{-1}$ since $(ab^{-1})^3 = 1$ which implies that $b = a$.

Thus none of the above labels is possible and so the above distribution guarantees that there is no exceptional region and hence we can conclude that $\mathcal{P}$ is aspherical.

3) $b = (b^{-1}a)^2$

We will show that this relation does not hold. If $b = (b^{-1}a)^2$ then we have $babab = 1$ since $a^2 = 1$.

$$\Rightarrow\; bababa^2 = 1$$
$$\Rightarrow\; ababab a = 1$$
$$\Rightarrow\; abababab = b$$
$$\Rightarrow\; (ab)^4 = b$$
$$\Rightarrow\; (ab)^4 = (b^{-1}a)^2$$
$$\Rightarrow\; (b^{-1}a)^{-4} = (b^{-1}a)^2$$
$$\Rightarrow\; (b^{-1}a)^6 = 1.$$

So we have $b^3 = 1$ since $b = (b^{-1}a)^2$. Also $b^2aba = 1$ will imply that $b^{-1}aba = 1$ in particular a and b commute. This will satisfy hypothesis 3 and contradict our assumption. So the relation $b = (b^{-1}a)^2$ does not hold.

4) $o(b) = 3$

We will use the distribution test on $\mathbb{P}'$ and so we go back to the exceptional region Φ' (one may also refer back to Appendix A.2.1). We will show that there is no possible label for Φ' except bbb since other relations in (5.1) do not hold.

Clearly $ab^{-1}ab^{-1} \neq 1$ and $ab^{-1}ab^{-1}ab^{-1} \neq 1$ since we assume that $1/2 + 1/3 + 1/k \leq 1$ where $o(ab^{-1}) = k$ (that is $k \geq 6$). We can not have relation $b = (b^{-1}a)^2$

since otherwise we would have $b^{-1}a^{-1}ba = 1$ (the last relation listed in (5.1)). Then hypothesis 3 is satisfied.

Since the only possible label for Φ' is bbb of valence three, the curvature $\gamma(\Phi') = \pi/2$. We will distribute $\pi/6$ to each region Ψ that shares an edge with Φ'. This is possible because Ψ can not be a region with label bbb (refer Appendix A.2.4). Since total sum of angle for region Ψ is less or equal to $m\pi/2 + m\pi/6$ (assume Ψ has valence m) then we must have $(m-2)\pi \geq m\pi/2 + m\pi/6$ which means that $m \geq 6$ in order to make sure that the new curvature $\gamma^*(\Psi)$ remains non-positive. So we will consider when Ψ has valence two, three, four or five. Finding the possibilities of the label of Ψ (refer Appendix A.2.4 for more details), we conclude that $\mathcal{P}$ is aspherical except possibily if one of these holds:

1. $b^{-1}ababa = 1$ (or $b^{-1}ab^{-1}aba = 1$)

2. $b^{-1}abababa = 1$ (or $b^{-1}ab^{-1}ab^{-1}aba = 1$)

3. $b^{-1}ab^{-1}ababa = 1$

If $b^{-1}ababa = 1$ then we have $a^{-1}b^{-1}aba = b^{-1}$ since $a^2 = 1$. This means that a is conjugate to b^{-1} which is not possible since they have different orders.

The relation $b^{-1}abababa = 1$ does not hold since this will imply that $a^{-1}bababa = b$ and so $babab$ has order 3 as b. Thus we have

$$(b^2aba)^3 = 1 \Rightarrow (b^{-1}aba)^3 = 1.$$

Also

$$b^{-1}abababa = 1 \Rightarrow b^{-1}aba = ab^{-1}ab^{-1} = (ab^{-1})^2.$$

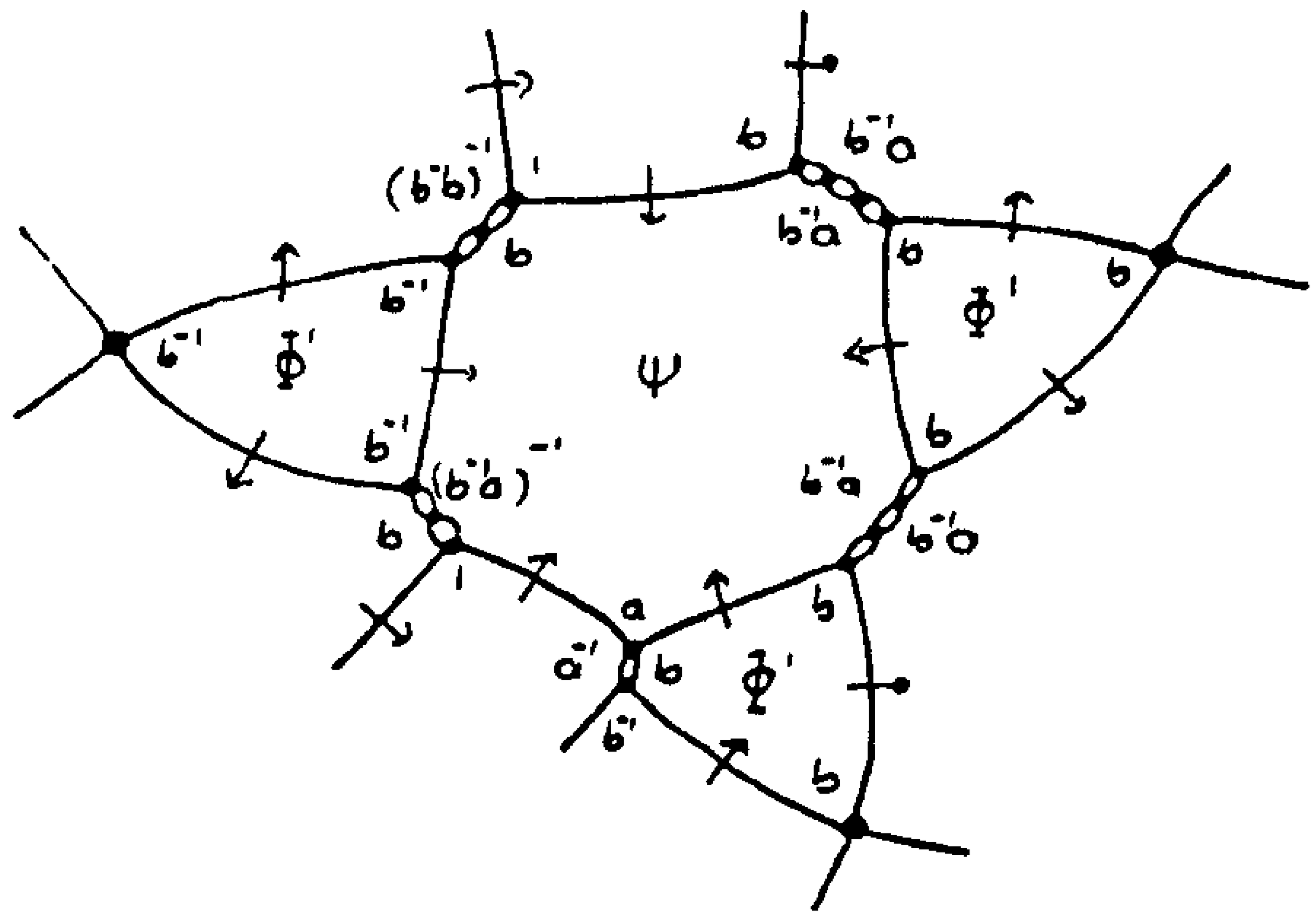
Thus we have $(ab^{-1})^6 = 1$ which implies $(ba)^6 = 1$ and so

$$b^{-1}abababa = 1 \Rightarrow b^{-1}a = (ba)^{-3} = (ba)^3.$$

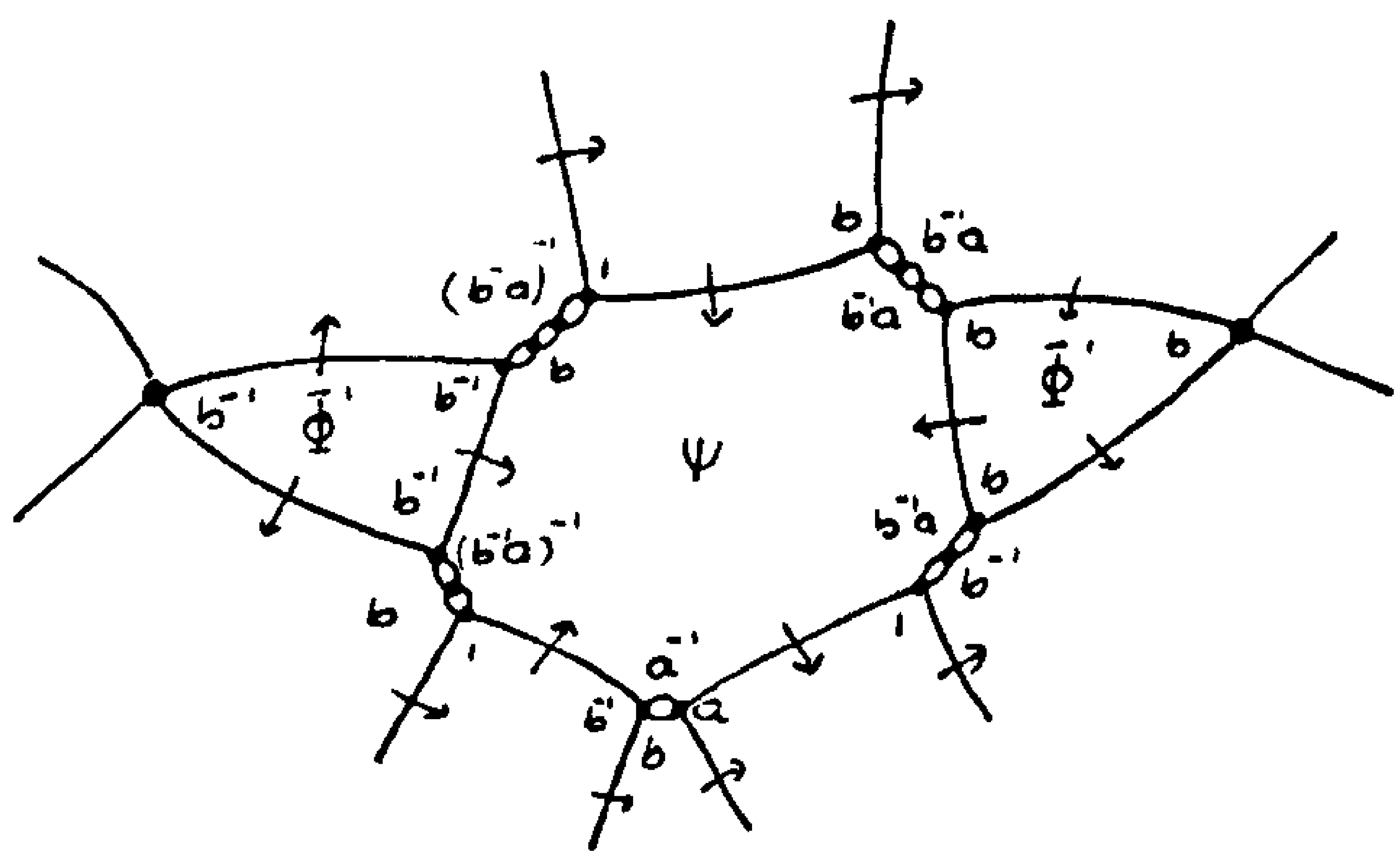
This means $b^{-1}a = bababa$ and again leads to $b = 1$.

For the last possibility is not known whether or not it holds. Thus we will draw all possibilities for Ψ and show that Ψ is not an exceptional region since the new curvature remains non-positive. There are two possibilities:

i)



ii)



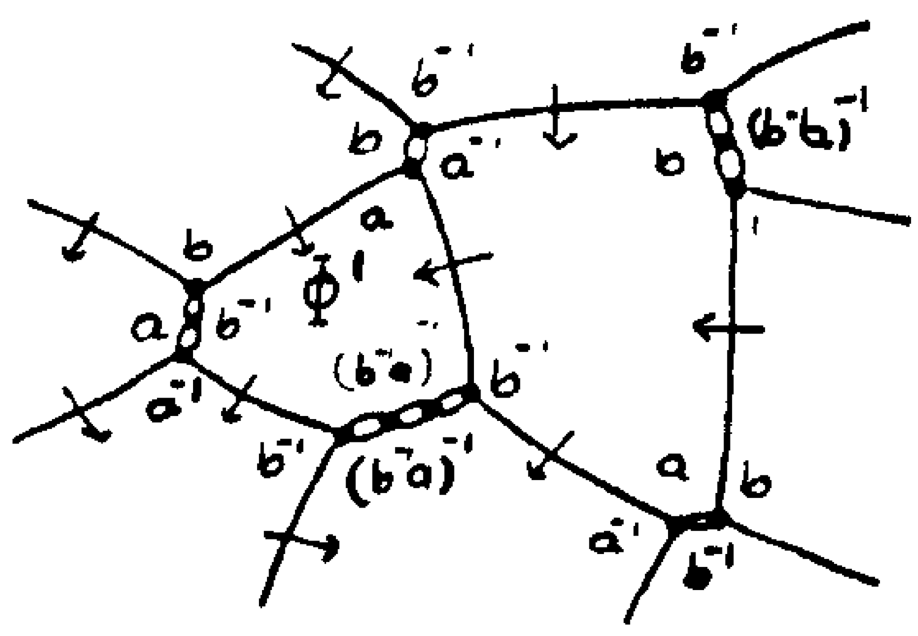
In the first case, Ψ only accepts the distribution angles from at most three out of five of its neighbours. Thus the curvature $\gamma^*(\Psi) \leq (5 \cdot \pi/2 + 3 \cdot \pi/6) - 3\pi$ which remains non-positive. Clearly the curvature in the second case is also non-positive since Ψ may accept the distribution angles from at most two out of five of its neighbours.

Thus we conclude that $\mathcal{P}$ is aspherical.

5) $ab^{-1}ab = 1$

Again there is no possible label for Φ' except $ab^{-1}ab$ since no other relations in (5.1) hold. Clearly $ab^{-1}ab^{-1} \neq 1$ since otherwise we would have $b^2 = 1$. If $(ab^{-1})^3 = 1$ then we would have $a = b^3$ and hence hypothesis 2 is satisfied. Now suppose that $b = (b^{-1}a)^2$. Then we obtained $b^3 = 1$ and so hypothesis 3 is satisfied.

We have a problem for this case because Φ' may have neighbours that have the same label $ab^{-1}ab$ such as following:



This is the first family exception that we can not decide (refer **FE1**).

When is $\mathcal{P}$ not aspherical?

We have to show that if one of these holds then $\mathcal{P}$ is not aspherical.

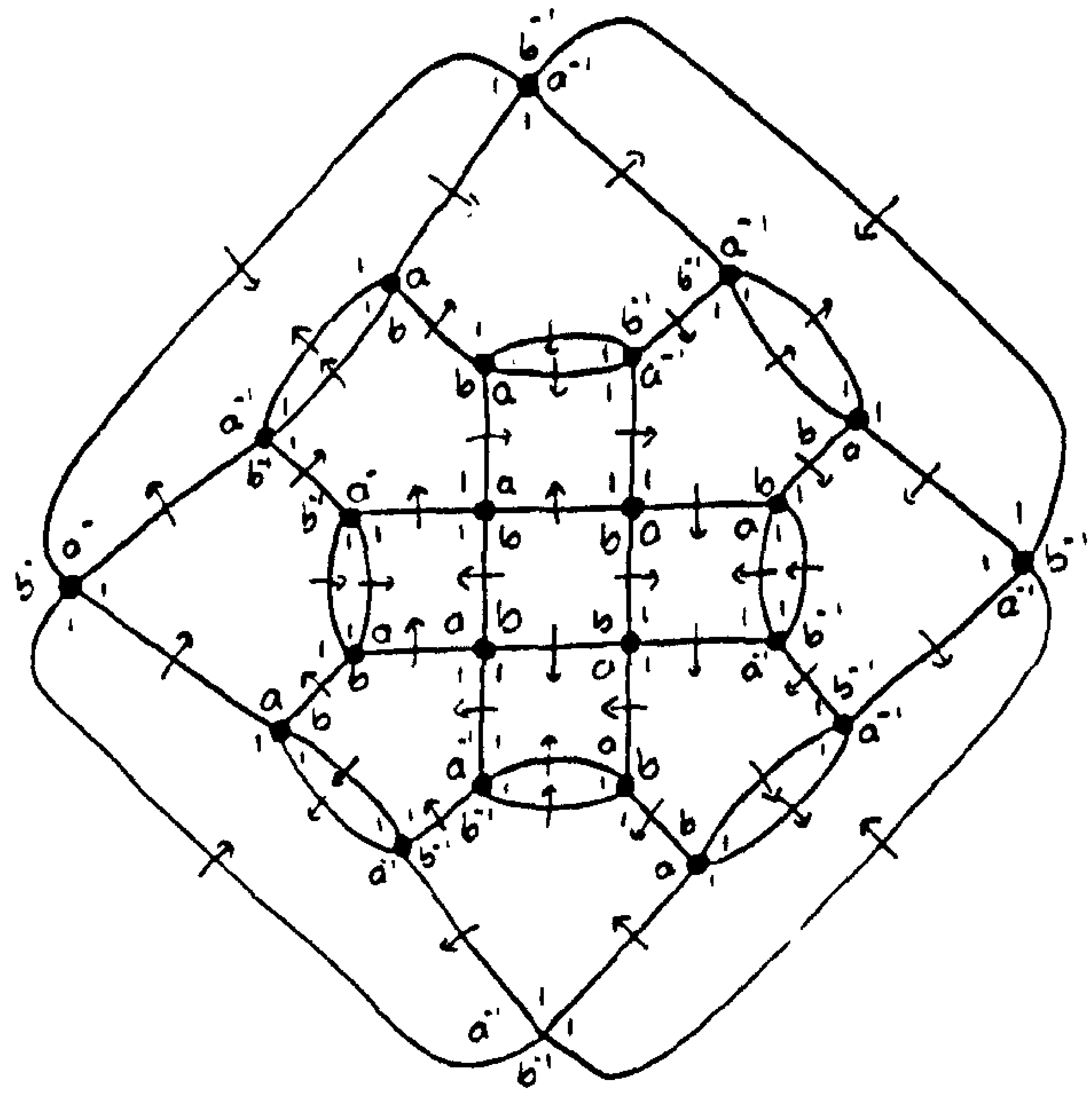
1. $a = b^2$
2. $a = b^3$
3. $o(b) = 3$ and group$\{a, b\} \simeq \mathbb{Z}_2 \times \mathbb{Z}_3$
4. $\frac{1}{2} + \frac{1}{q} + \frac{1}{k} > 1$ where $o(b) = q$ and $o(ab^{-1}) = k$

(Clearly the relation $a = b^{-1}$ (hypothesis 5) does not hold since $o(a) \neq o(b)$.)

For each case, we will draw a reduced strictly spherical picture (except in case 2) and show that it is not degenerate (refer §4.3.6).

1) $a = b^2$

$\Rightarrow b^4 = 1$



Thus (refer §4.3.6) $\psi(\lambda_{\mathbb{P}}) = 12(x-1) \neq 0$. Since $\mathbb{P}$ is not degenerate then by Lemma 4.3.7, $\mathcal{P}$ is not aspherical.

2) $a = b^3$

Clearly we have $b^6 = 1$ and so

$$\begin{aligned} t^3at^{-1}b = 1 \;&\Rightarrow\; t^3b^3t^{-1}b = 1 \\ &\Rightarrow\; t^3b^{-3}t^{-1}b = 1 \\ &\Rightarrow\; bt^3 = tb^3. \end{aligned}$$

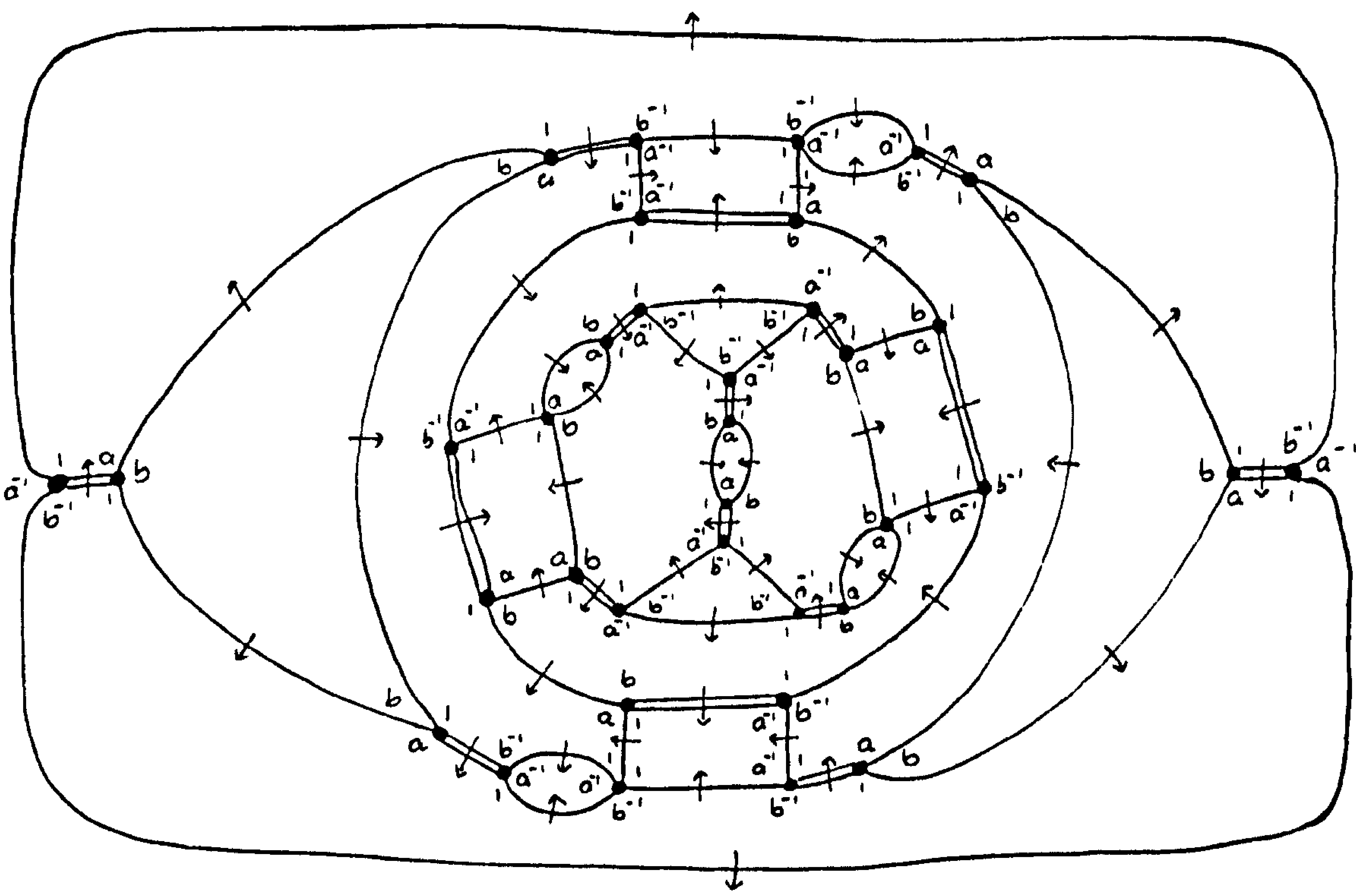
One may refer to [36, page 183] to see that the group defined by

$$< b, t; bt^3 = tb^3, b^6 = 1 >$$

is finite. Thus we conclude that t has finite order and hence (refer §4.3.5) $\mathcal{P}$ is not aspherical.

3) $b^3 = 1$ and $ab = ba$

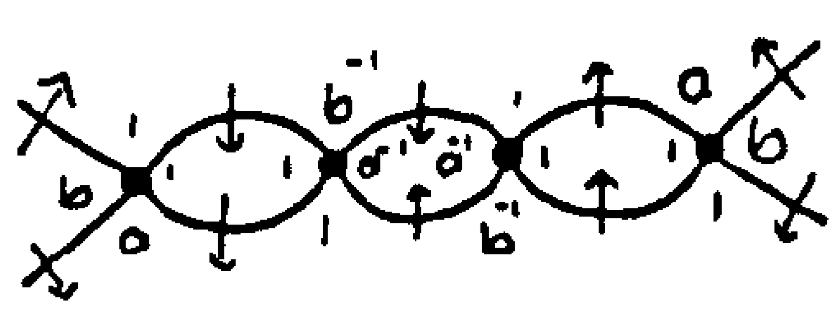
Since $a^2 = b^3 = aba^{-1}b^{-1} = 1$ then we have $abab^2 = 1$. We draw a reduced strictly spherical picture



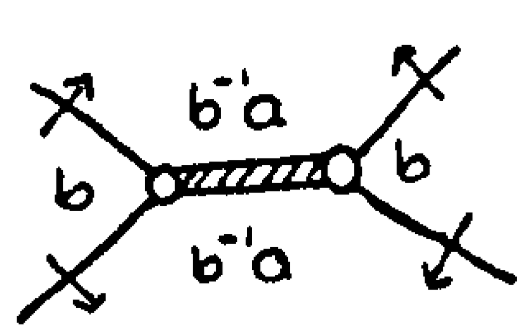
Thus (refer §4.3.6) $\psi(\lambda_{\mathbb{P}}) = 18(x-1) \neq 0$. Since $\mathbb{P}$ is not degenerate then by Lemma 4.3.7, $\mathcal{P}$ is not aspherical.

4) $\frac{1}{2} + \frac{1}{q} + \frac{1}{k} > 1$ where $o(b) = q$ and $o(ab^{-1}) = k$

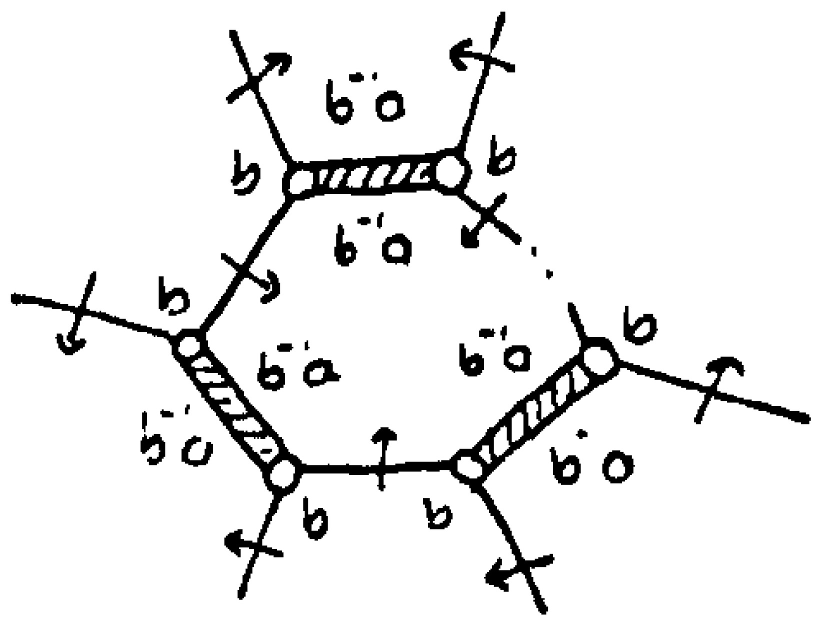
We will draw a family of group of strictly reduced spherical pictures over $\mathcal{P}$. To make drawing simpler, we draw the chain



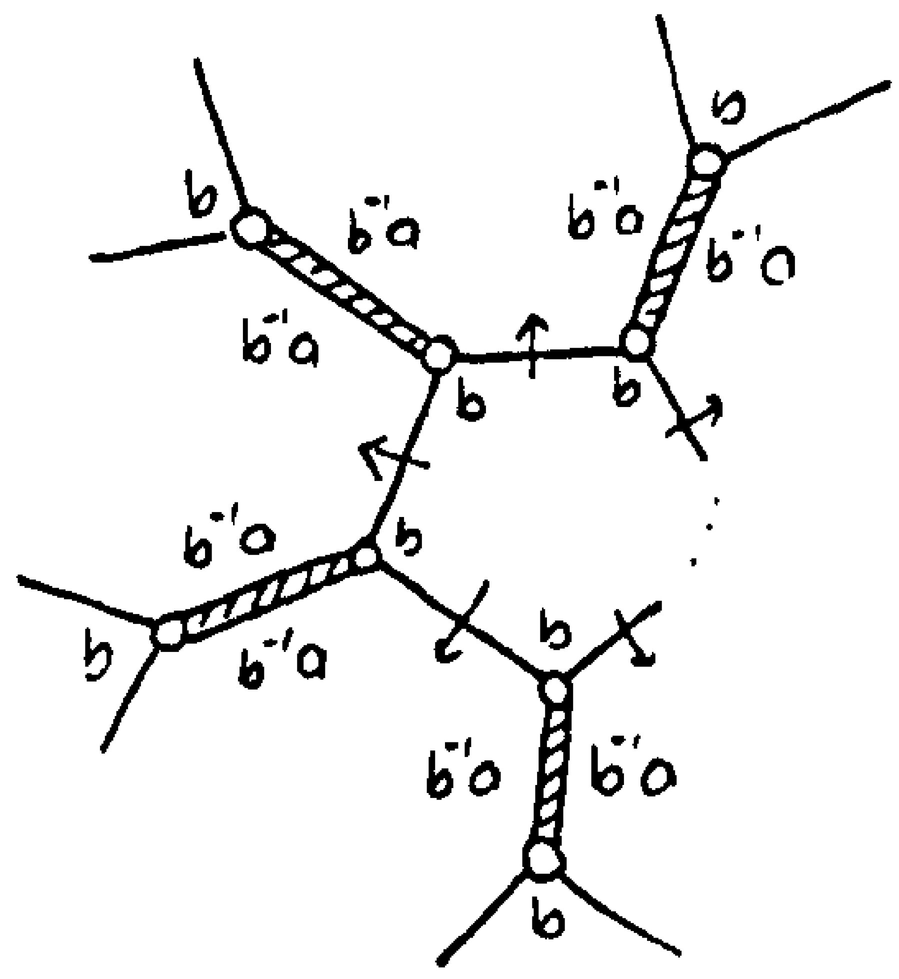
as



so that in a region with k-chain,

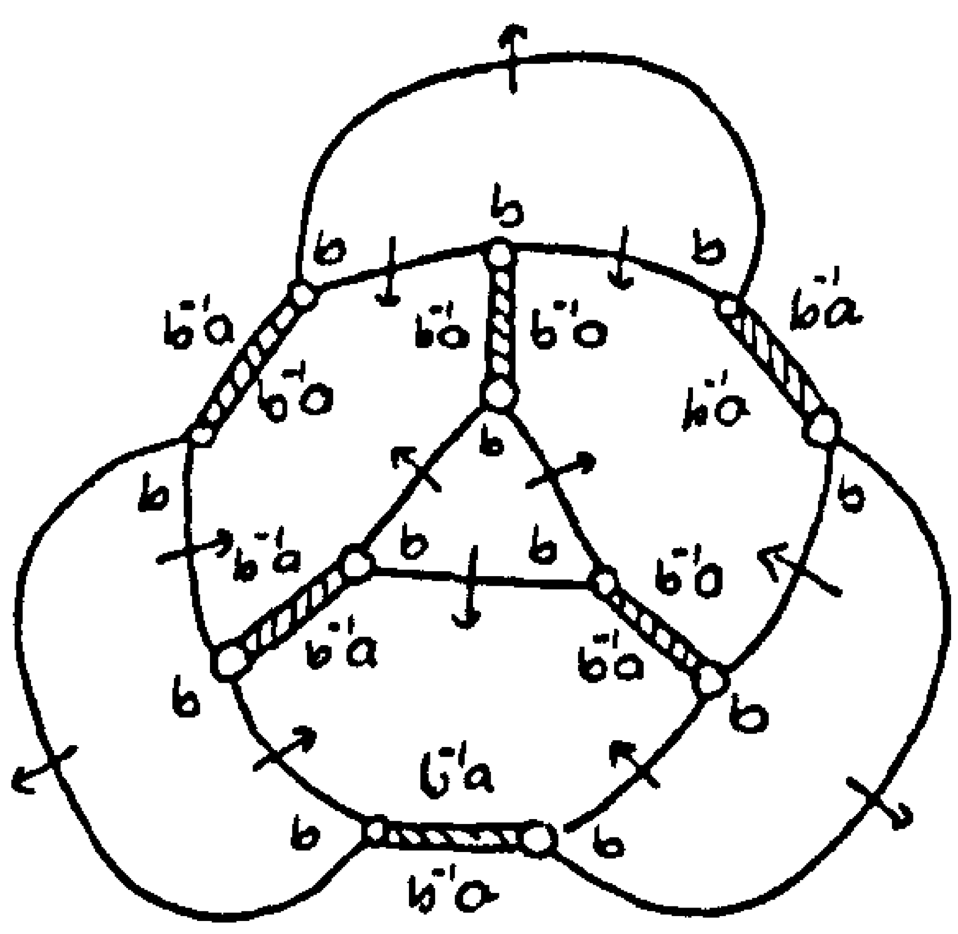


has label $(ab^{-1})^{\pm k}$. Also one should note that the label for any region where q-chain is attached,



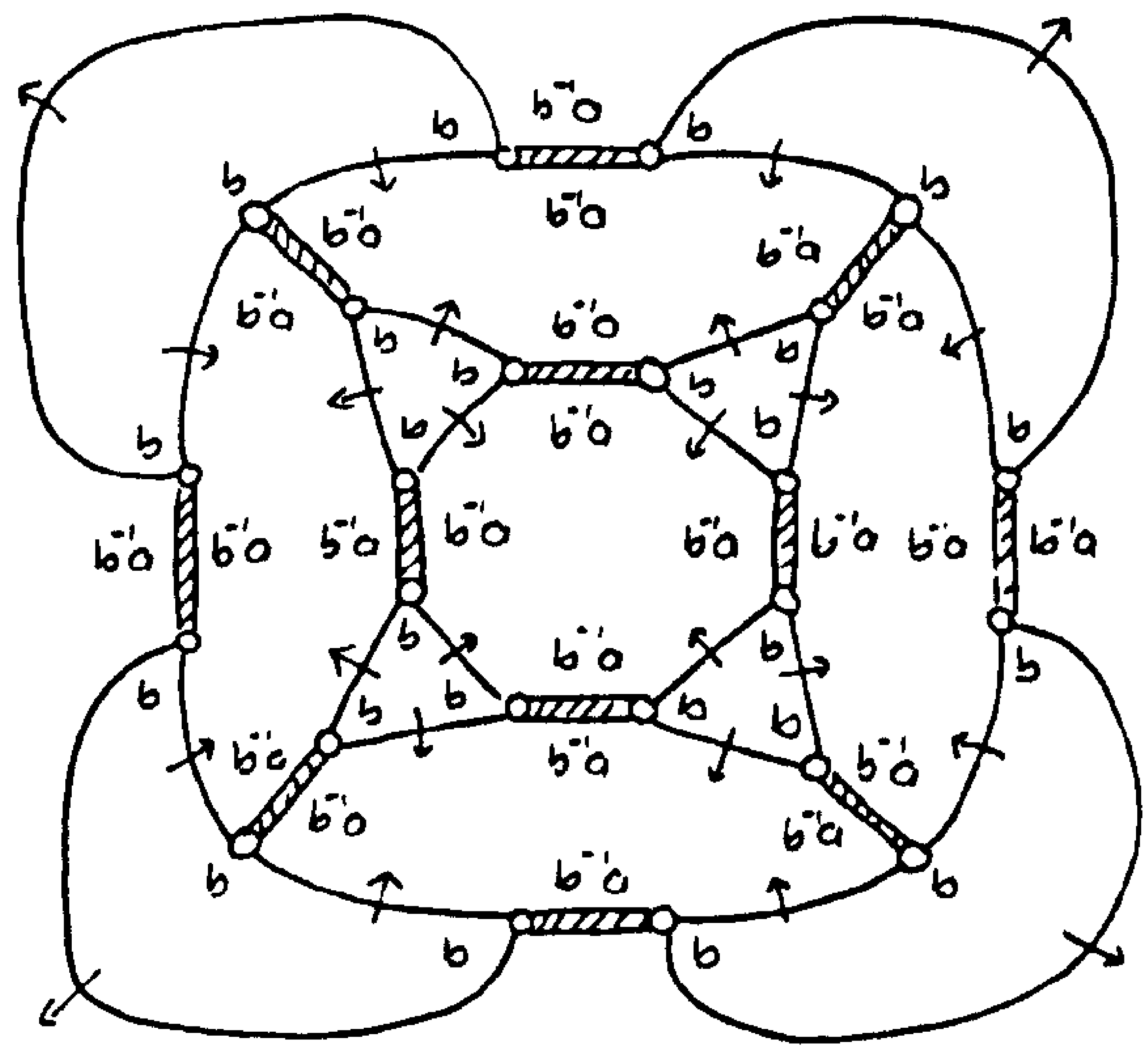
is $b^{\pm q}$.

i) $o(b) = 3, o(ab^{-1}) = 3$



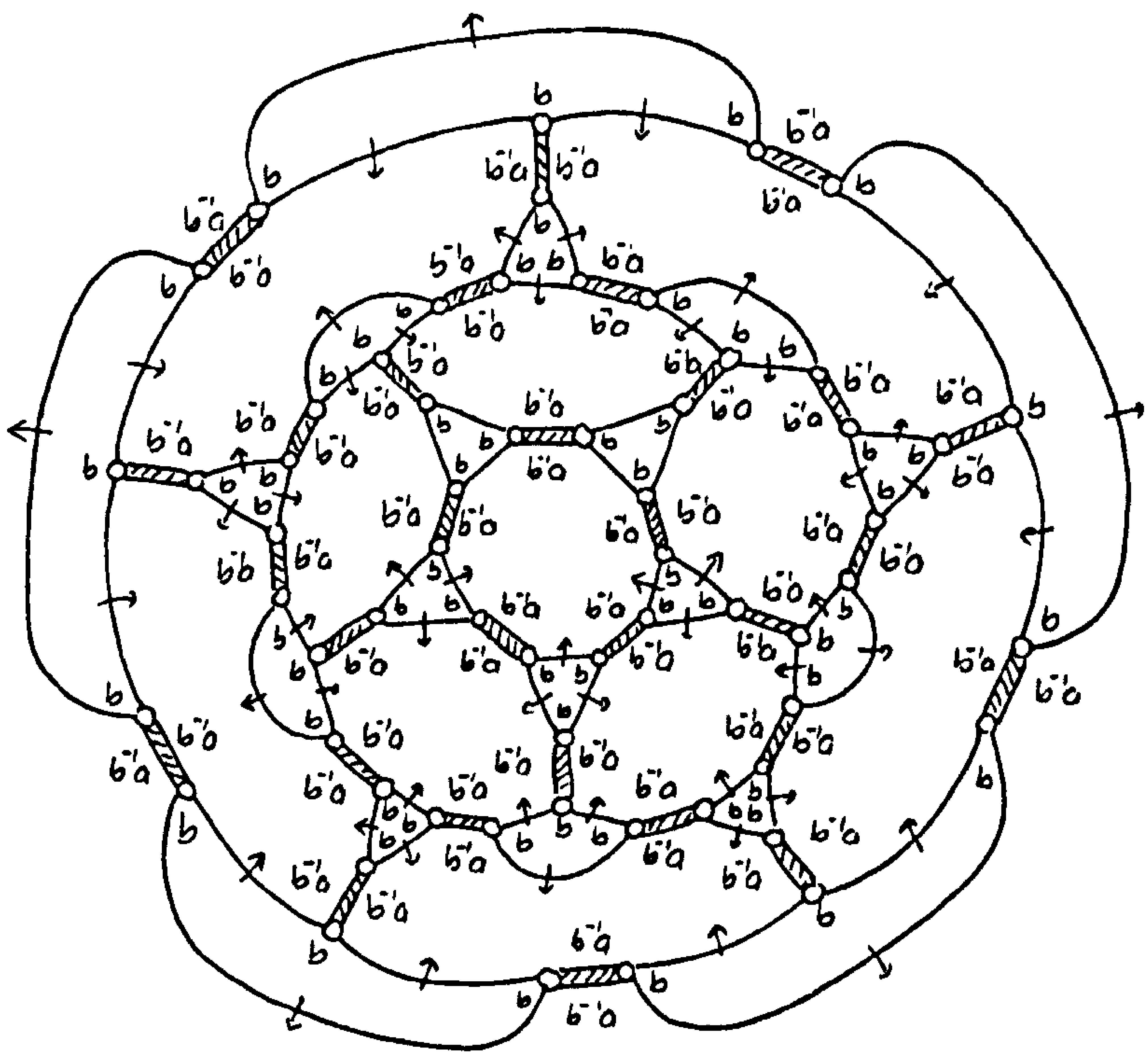
Since $\psi(\lambda_{\mathbb{P}}) = 12(x-1) \neq 0$ then $\mathbb{P}$ is not degenerate. Then by Lemma 4.3.7, $\mathcal{P}$ is not aspherical.

ii) $o(b) = 3, o(ab^{-1}) = 4$



Here we have $\psi(\lambda_{\mathbb{P}}) = 24(x-1) \neq 0$. Thus $\mathbb{P}$ is not degenerate and by Lemma 4.3.7, $\mathcal{P}$ is not aspherical.

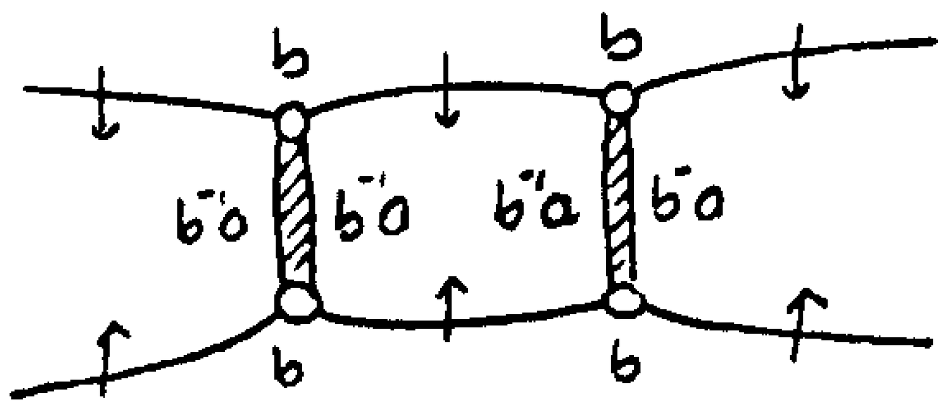
iii) $o(b) = 3, o(ab^{-1}) = 5$



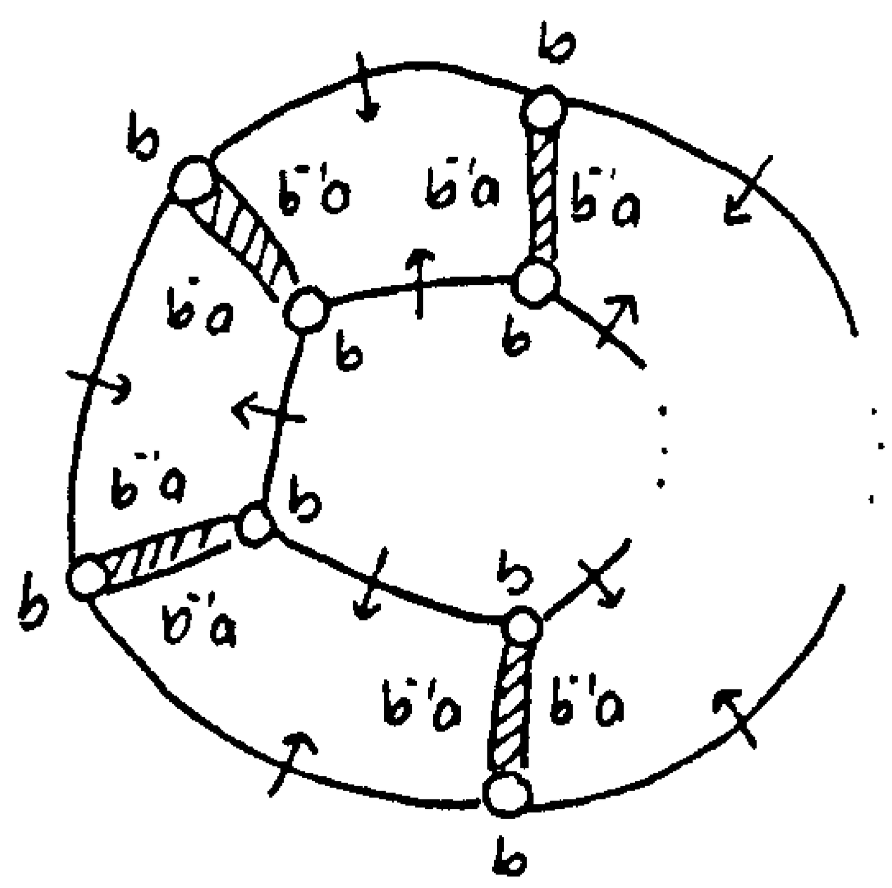
Since $\psi(\lambda_{\mathbb{P}}) = 60(x-1) \neq 0$ then $\mathbb{P}$ is not degenerate. Then by Lemma 4.3.7, $\mathcal{P}$ is not aspherical.

iv) $3 \leq o(b) < \infty, o(ab^{-1}) = 2$

Consider

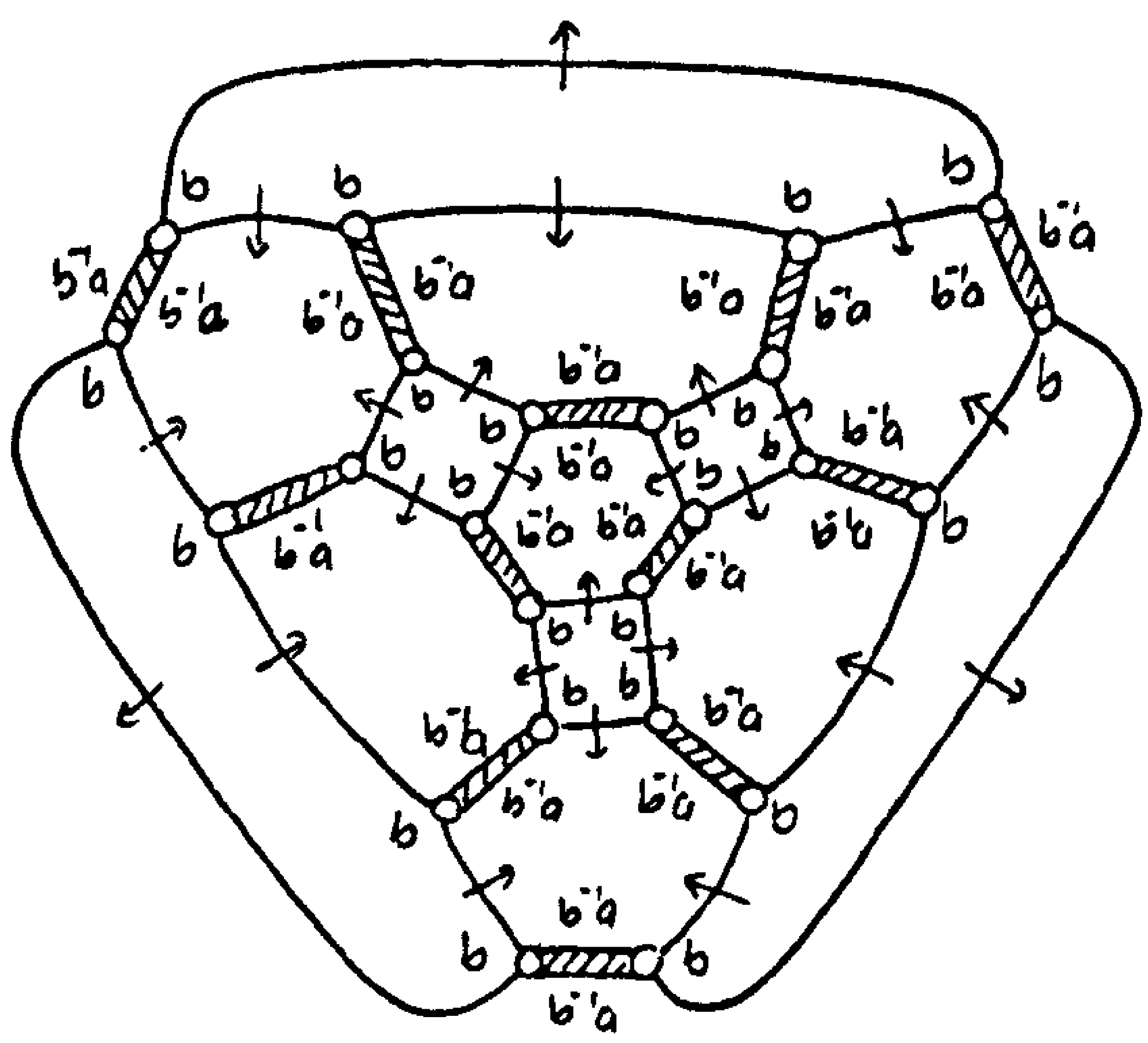


and if $o(b) = q$, then join q of these to form a strictly reduced spherical picture



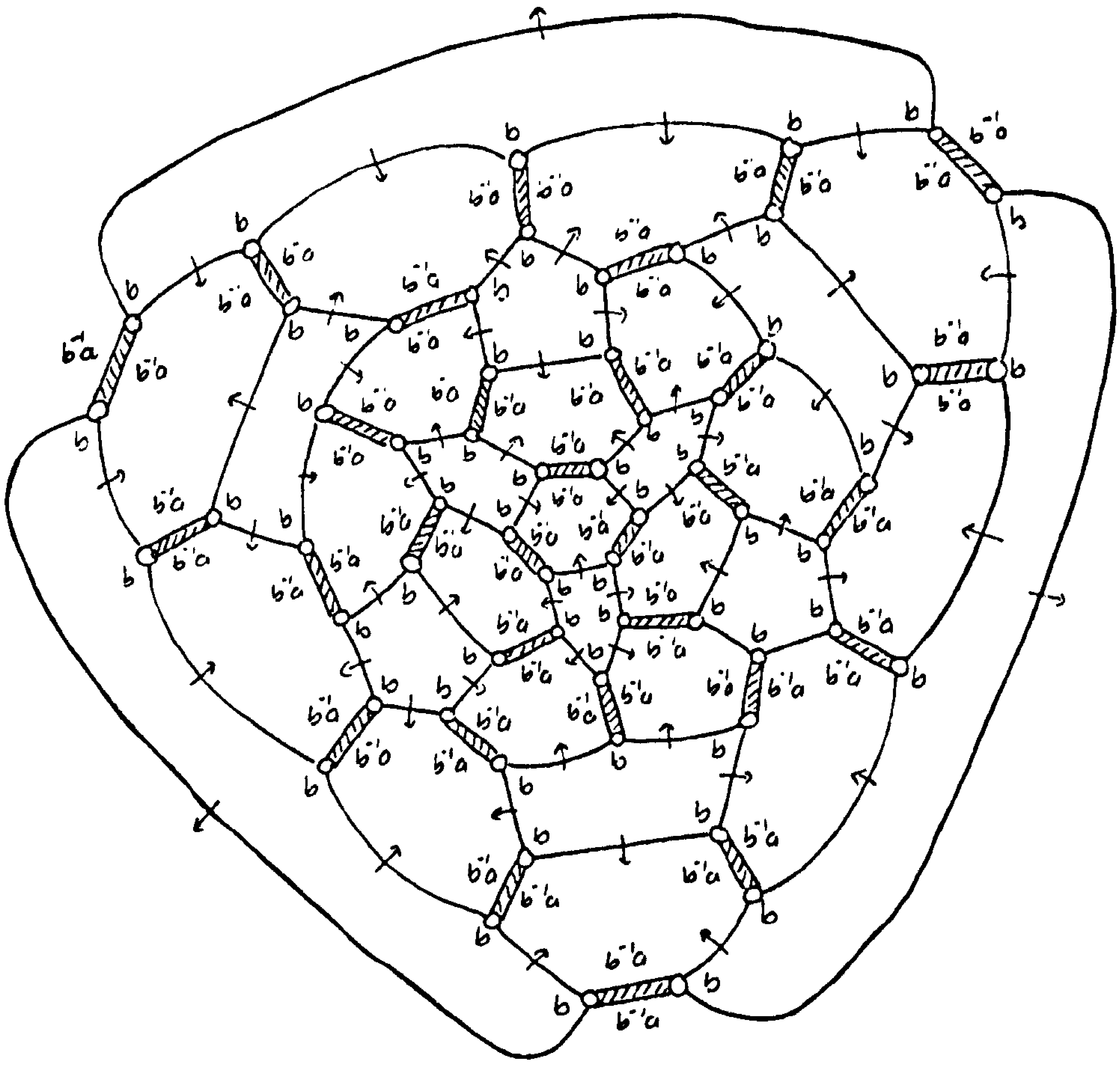
Then $\psi(\lambda_{\mathbb{P}}) = 2q(x-1) \neq 0$ and so $\mathbb{P}$ is not degenerate. Hence by Lemma 4.3.7, $\mathcal{P}$ is not aspherical.

vi) $o(ab^{-1}) = 3, o(b) = 4$



Since $\psi(\lambda_{\mathbb{P}}) = 24(x-1) \neq 0$ then $\mathbb{P}$ is not degenerate. Then by Lemma 4.3.7, $\mathcal{P}$ is not aspherical.

vii) $o(ab^{-1}) = 3, o(b) = 5$

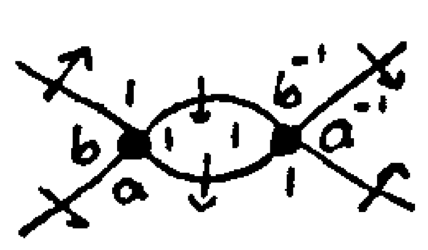


Then $\psi(\lambda_{\mathbb{P}}) = 60(x-1) \neq 0$ and hence $\mathbb{P}$ is not degenerate. Thus $\mathcal{P}$ is not aspherical.

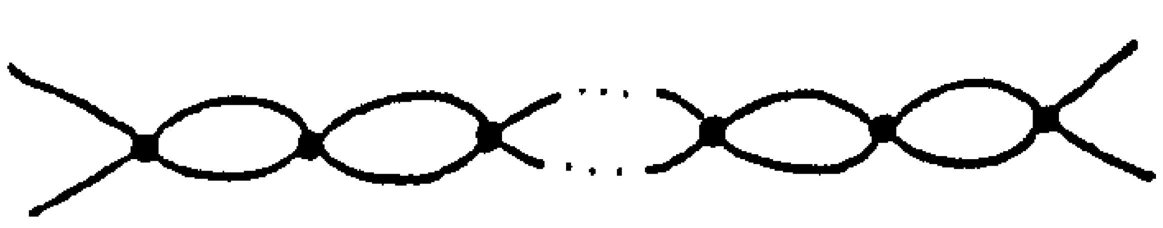
5.2.4 The subcase $3 \leq o(a)$

When is $\mathcal{P}$ aspherical?

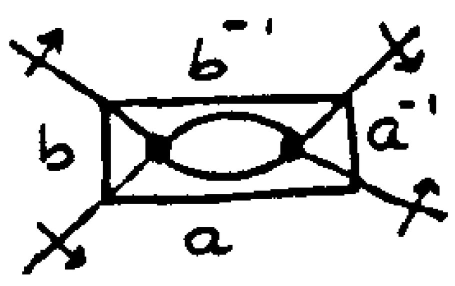
Assume that none of the relations in hypothesis holds. We will show that $\mathcal{P}$ is aspherical. It is suffices to show that there is no reduced strictly spherical picture $\mathbb{P}$ over $\mathcal{P}$. Suppose there were. Note that there is only one possibility for double bonds in $\mathbb{P}$ (since $\mathbb{P}$ is reduced) as follows.



Also note that there is no two or more double bonds forming a chain



Regard all double bonds as single discs



and so we obtain a new derived picture $\mathbb{P}'$ such that every disc has valence four. Assign the angle function $\pi/2$ to each corner, so that every disc in $\mathbb{P}'$ is flat. Then by Lemma 4.3.4, there exists an exceptional region Φ' with positive curvature. If Φ' has valence m, this means that $\gamma(\Phi') > (m-2)\pi$. Since each corner has angle $\pi/2$. Φ' must have valence three or four. We may now find all possible labels for Φ' (refer Appendix A.3.1). Thus we conclude that $\mathcal{P}$ is aspherical except possibly if one of these holds:

1. $o(a) = 3$
2. $o(b) = 3$
3. $a = b^2$
4. $b = a^2$

Up to equivalence, we just need to consider the first and the third case. At the moment we can not decide for the third case (refer **FE2**). Now assume that $o(a) = 3$ and $a \neq b^2$. Clearly $b \neq a^2$ since we also assume that $a \neq b^{-1}$. We will proceed by considering the order of b.

1) $o(b) = 3$

Note that Φ' may be a region with label aaa or with label bbb of valence three. The curvature is $\pi/2$ and so we may distribute $\pi/6$ to each of their three neighbours. This is possible since none of their neighbours have label aaa or bbb (refer Appendix A.3.2).

Let Ψ be a neighbour of Φ'. We have to make sure that the new curvature for Ψ remains non-positive. If Ψ has valence m, then we need

$$m\pi/2 + m\pi/6 \leq (m-2)\pi.$$

Since $m \geq 6$, we need to consider the case when Ψ has valence less than six. Finding all possibilities (refer Appendix A.3.2), we conclude that $\mathcal{P}$ is aspherical except possibly if one of these holds:

1. $ba^2 = ab$ (or $a^2b = ba$)

2. $b^2a = ab$ (or $ab^2 = ba$)

3. $ba = ab$

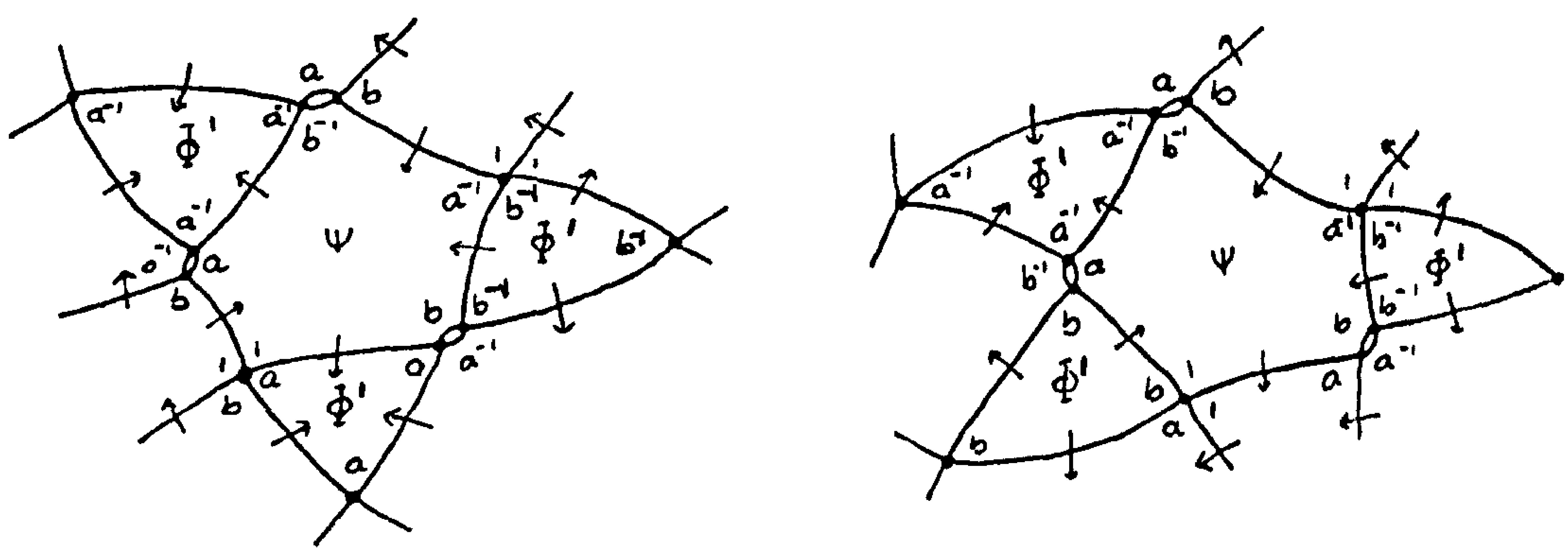
We will show that the first and second relations are not possible. Suppose that $ba^2 = ab$. Then

$$
\begin{aligned}
& b^{-1}ab = a^2 \\
\Rightarrow\ & b^{-2}ab^2 = b^{-1}(b^{-1}ab)b = b^{-1}a^2b = (b^{-1}ab)^2 = a^4 \\
\Rightarrow\ & b^{-3}ab^3 = b^{-1}(b^{-2}ab^2)b = b^{-1}a^4b = (b^{-1}ab)^4 = a^8 \\
\Rightarrow\ & a = a^8 \text{ since } b^3 = 1 \\
\Rightarrow\ & a^7 = 1
\end{aligned}
$$

Since $a^3 = 1$, then a would be trivial.

The second case can be shown similarly.

Now suppose that $ba = ab$. Then there are two possible regions Ψ in $\mathbb{P}'$ of valence five (refer appendix A.3.2). In each case Ψ may only accept distribution angles from at most three out of five of its neighbours.



$$\text{new curvature } \leq (5\pi/2 + 3\pi/6) - 3\pi = 0$$

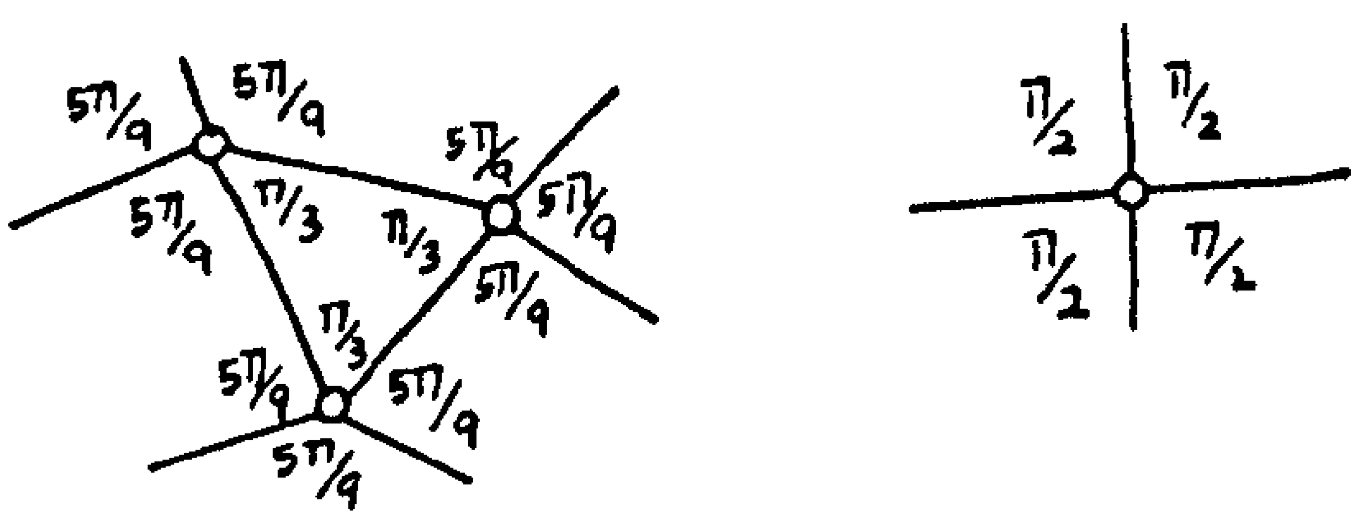
This means that Ψ is not exceptional.

2) $o(b) = 4, 5$

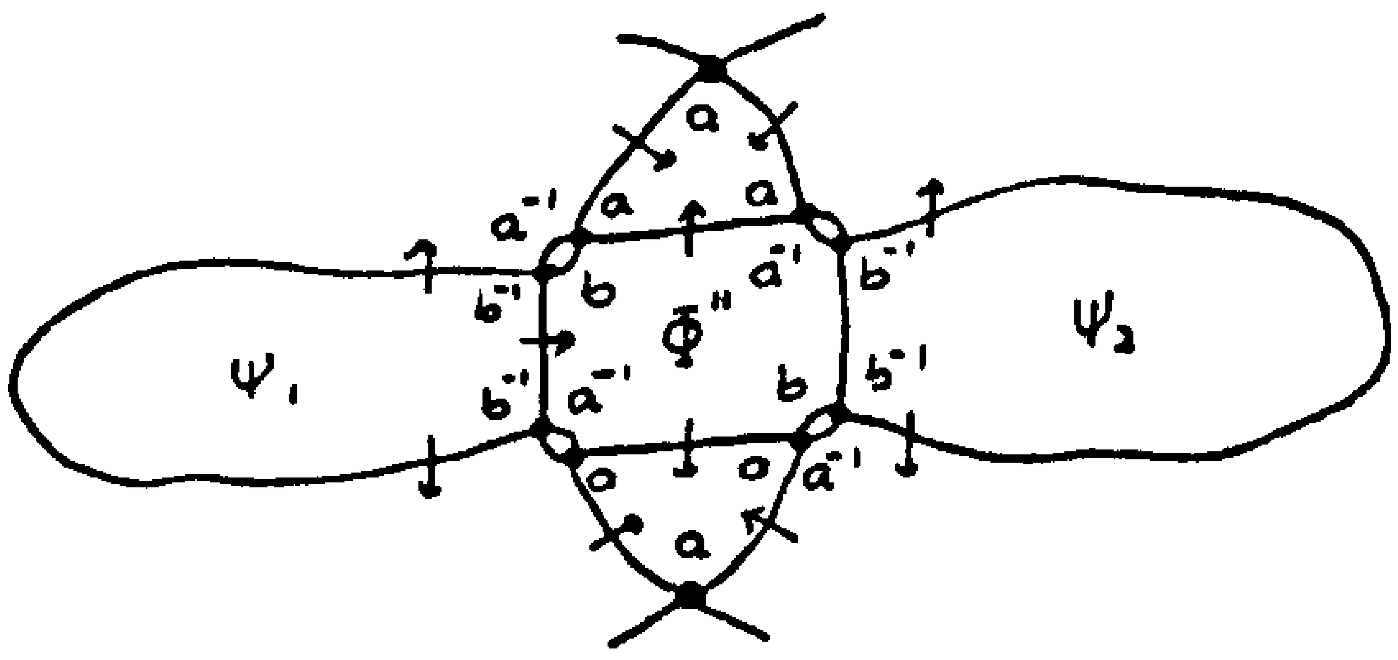
This case is similar to the first case except the only possible label for Φ' is just aaa. We will use the similar distribution scheme. If Ψ shares an edge with Φ', then Ψ will receive $\pi/6$ from Φ'. Since in every two edges of Ψ, there is at most one edge shared with Φ' and so we have $m\pi/2 + m/2 \cdot \pi/6 \leq (m-2)\pi$ where m is the valence of Ψ. This means that if Ψ has valence more than four than the new curvature for Ψ remains non-positive. There is no possible label for Ψ of valence two, three or four (refer Appendix A.3.2), and hence we may conclude that $\mathcal{P}$ is aspherical.

3) $o(b) \geq 6$

Note that there is no possible region of valence two in $\mathbb{P}'$ and the only possible region of valence three is Φ with label aaa (refer Appendix A.3.1). In order to make sure that Φ will not be the exceptional, we will assign the following angle



Since every disc is flat then by Lemma 4.3.4, there exists an exceptional region Φ'' with positive curvature. If Φ'' has valence m then $m \cdot 5\pi/9 > (m-2)\pi$ and so $m < 4\frac{1}{2}$. Thus Φ'' must be a region of valence four. There is only one such possible region (refer Appendix A.3.3) as follows



The curvature for Φ'' is at most $2\pi/9$ and so we distribute $\pi/9$ each to Ψ_1 and Ψ_2. Suppose that $\Psi_i (i = 1, 2)$ has valence m. We need to make sure that the new curvature for both of Ψ_i's remain non-positive that is $m \cdot 5\pi/9 + m \cdot \pi/9 \leq (m-2)\pi$. Since $m \geq 6$, we need to consider when Ψ_i has valence four or five. There is no possible region for Ψ_i of valence less than six (refer Appendix A.3.4) and so we may conclude that $\mathcal{P}$ is aspherical.

When is $\mathcal{P}$ not aspherical?

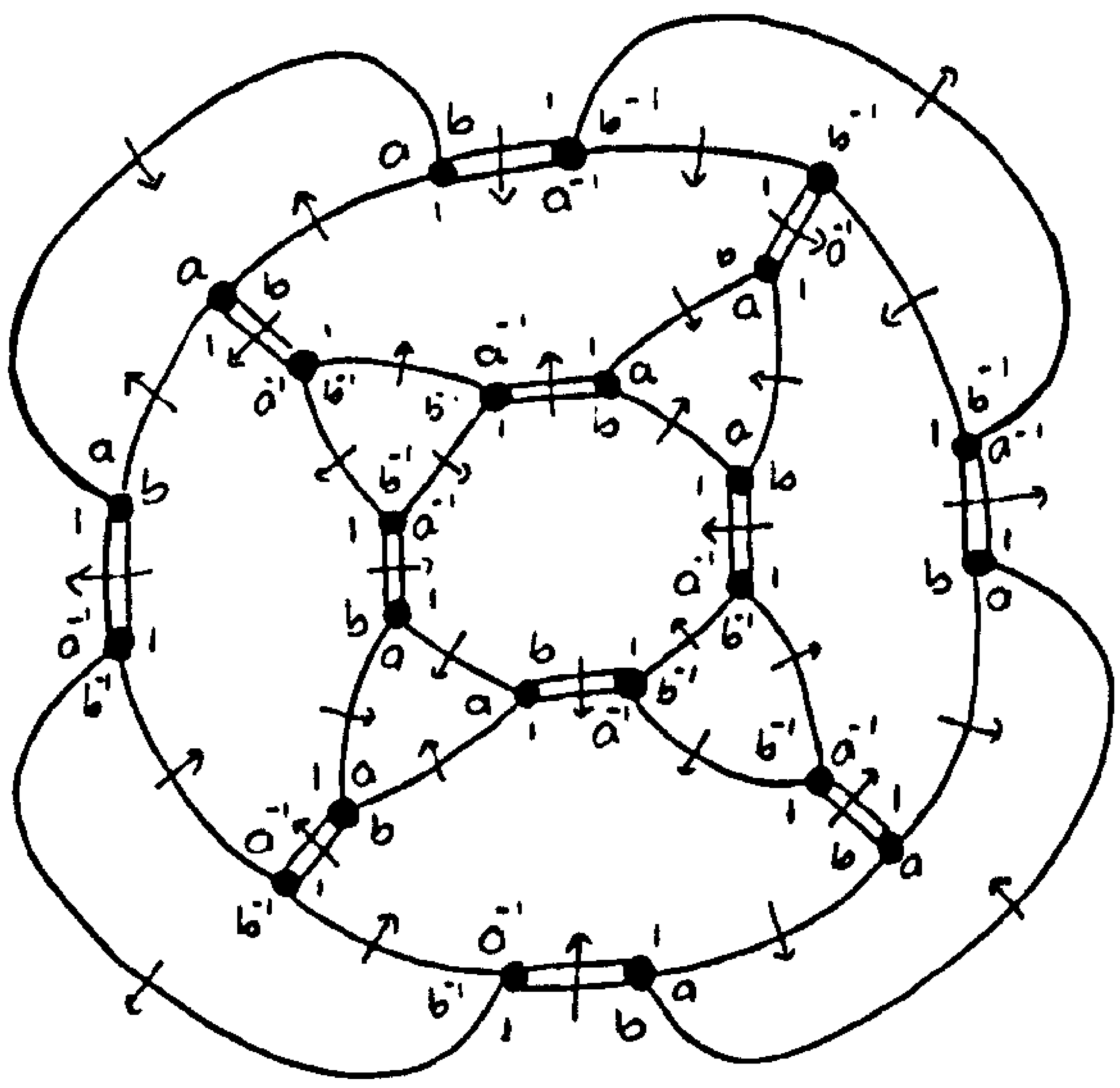
We have to show that if one of these is satisfied, then $\mathcal{P}$ is not aspherical.

1. $\frac{1}{p} + \frac{1}{q} + \frac{1}{k} > 1$ where $o(a) = p, o(b) = q$ and $o(ab^{-1}) = k$

2. $a = b^{-1}$ of finite order

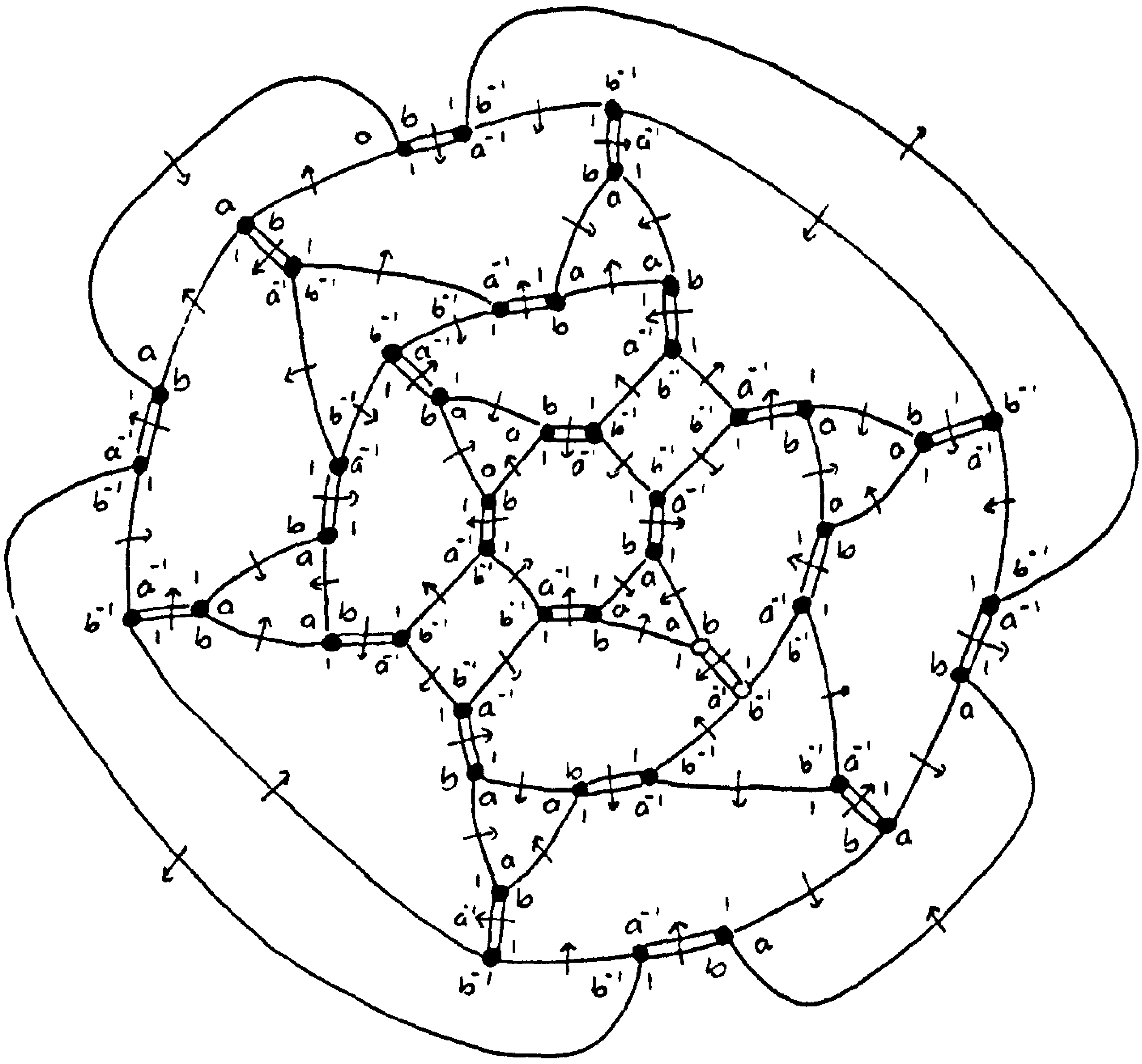
For the first case, we draw reduced strictly spherical pictures. Then we will show that they do not degenerate (refer §4.3.6). Then by Lemma 4.3.7 $\mathcal{P}$ is not aspherical.

i) $o(a) = 3, o(b) = 3, o(ab^{-1}) = 2$



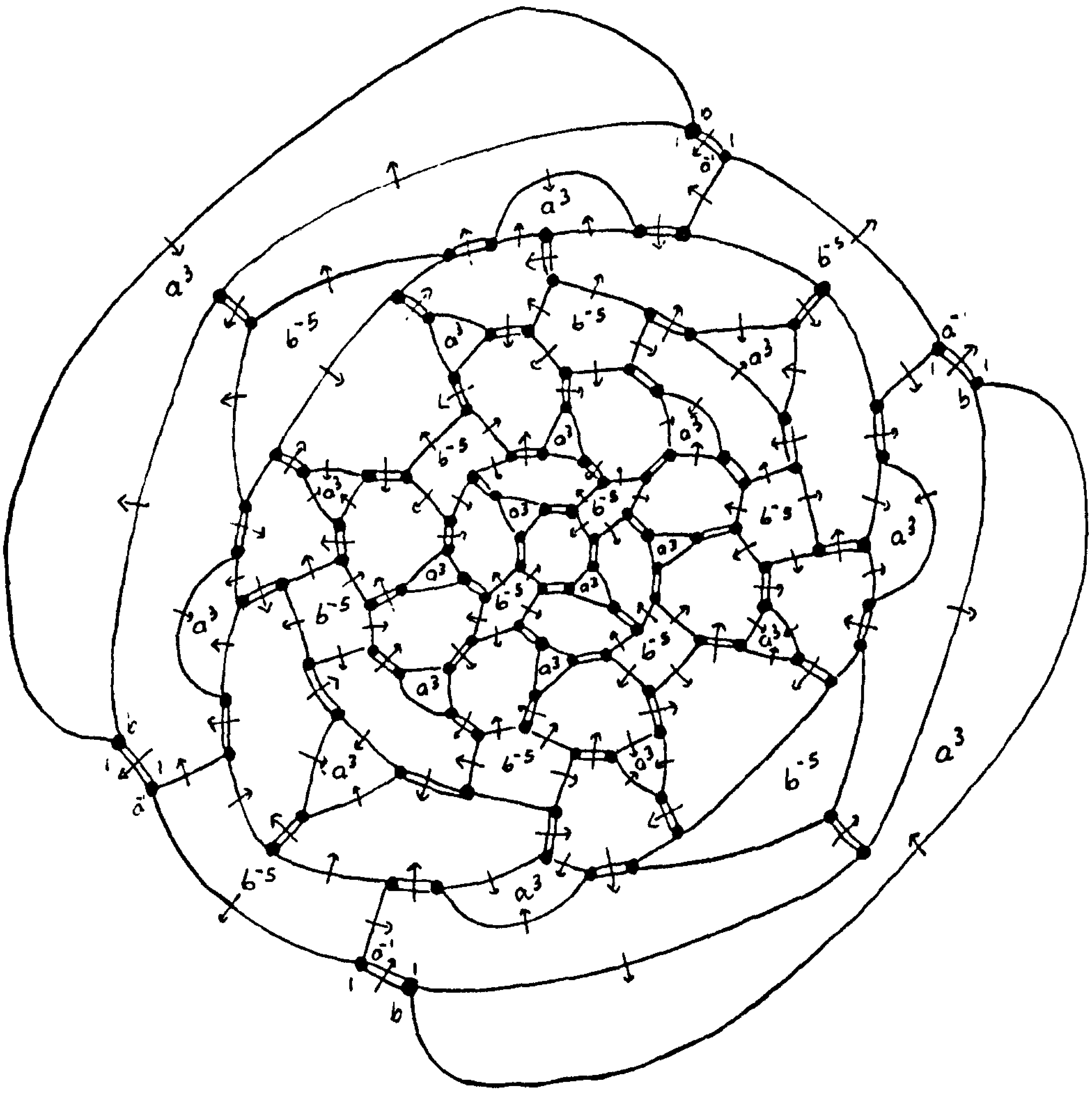
Then $\psi(\lambda_{\mathbb{P}}) = 12(x-1) \neq 0$ and hence $\mathbb{P}$ is not degenerate.

ii) $o(a) = 3, o(b) = 4, o(ab^{-1}) = 2$



Since $\psi(\lambda_{\mathbb{P}}) = 24(x-1) \neq 0$ then $\mathbb{P}$ is not degenerate.

iii) $o(a) = 3, o(b) = 5, o(ab^{-1}) \doteq 2$



Then we have $\psi(\lambda_{\mathbb{P}}) = 60(x-1) \neq 0$ and hence $\mathbb{P}$ is not degenerate.

We will not draw pictures for the second case. Note that if $a = b^{-1}$ then

$$\begin{aligned} 1 = t^3at^{-1}b \;&\Rightarrow\; 1 = t^3at^{-1}a^{-1} \\ &\Rightarrow\; ata^{-1} = t^3 \\ &\Rightarrow\; a^2ta^{-2} = a(ata^{-1})a^{-1} = at^3a^{-1} = t^{3^2} \end{aligned}$$

By induction, one may prove that for any integer n, $a^nta^{-n} = t^{3^n}$. Since $o(a) = p$, we have

$$t^{3^p} = a^pta^{-p} = t$$

which implies $t^{3^p-1} = 1$ and hence t has a finite order. Thus (refer §4.3.5) $\mathcal{P}$ is not aspherical.

Chapter 6

The form $t^2atbt^{-1}c$

This chapter covers all five subcases when R has the following form:

2.1 $t^2atat^{-1}a$

2.2 $t^2atat^{-1}c$

2.3 $t^2atbt^{-1}b$

2.4 $t^2atbt^{-1}a$

2.5 $t^2atbt^{-1}c$

where a, b and c are distinct non-trivial elements of H.

For the first subcase we will show that $\mathcal{P}$ is aspherical if and only if a has infinite order while for the second subcase we will show that $\mathcal{P}$ is aspherical if and only if c has infinite order (Theorem 6.1.1). We can decide the asphericity of $\mathcal{P}$ for the third subcase apart from a special exception and a family of exceptions (refer §6.2):

SE1 $o(a) = 2, o(b) = 3$

FE3 $o(a) < \infty, o(b) = 2$

There is a special exception and three families of exceptions for the fourth subcase (see §6.3) that are not yet decided.

SE2 $o(b) = 3, o(a) = 4$

FE4 $o(b) = 2$

FE5 $o(a) = 3, o(b) < \infty$

FE6 b has finite order and $b \neq a^{-1}, a^2, a^{\pm 3}, a^{\pm 4}$

or $b^2 \neq a^{\pm 2}$ or $a \neq b^{\pm 2}, b^{\pm 3}$

For the last subcase we can decide the asphericity of $\mathcal{P}$ apart from (see §6.4):

SE3 $a^2 = b^3 = 1$ and $a = c^{-2}$

SE4 $o(c) = 3, o(b) = 2, b = ac, c = a^2$

SE5 $o(c) = 6, o(b) = 2, b = ac, a = c^2$

FE7 $b^2 = 1, a = c^{-2}$ where $4 \leq o(c) < \infty$

Then we list again all exceptions for both forms in §6.6.

As in Chapter 5, ψ will denote the ring homomorphism (refer §4.3.6)

$$\mathbb{Z}G \longrightarrow \mathbb{Z} < x; x^2 >$$

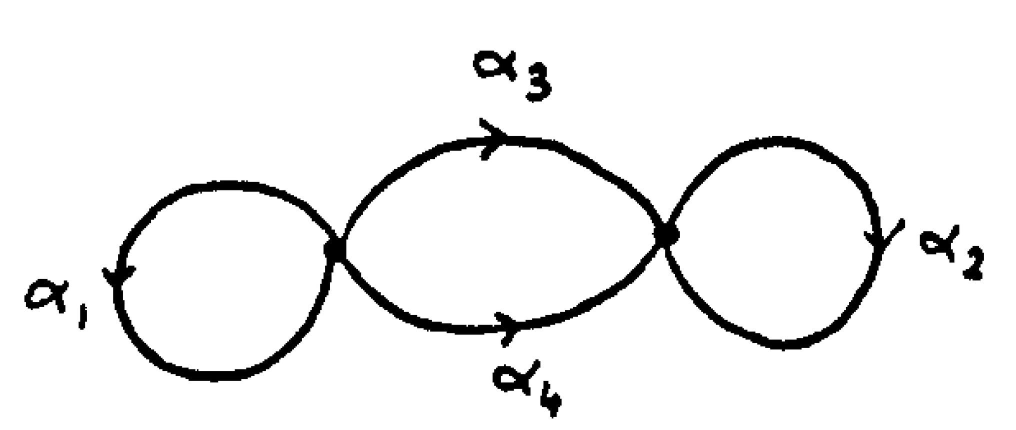
$$H \mapsto 1, t \mapsto x.$$

6.1 The subcase $a = b$

The main result is

Theorem 6.1.1 *Let $\mathcal{P} =< H, t; t^2atat^{-1}c >$ where a and c are non-trivial elements of H (a and c may be equal). Then $\mathcal{P}$ is aspherical if and only if c has infinite order.*

Proof Let c have infinite order and consider the star graph $\mathcal{P}^{st}$



where $\alpha_1 \leftrightarrow c^{-1}, \alpha_2 \leftrightarrow a^{-1}, \alpha_3 \leftrightarrow a^{-1}$ and $\alpha_4 \leftrightarrow 1$. We will assign the weights depending on the order of a.

If a has infinite order, assign

$$\theta(\alpha_1) = \theta(\alpha_2) = 0, \theta(\alpha_3) = \theta(\alpha_4) = 1.$$

Since a and c have infinite order, then any admissible cycle must involve α_3 and/or α_4 at least twice, and so it has a weight of at least two. One may check that all three conditions given in §4.3.2 are satified and hence by Theorem 4.3.3, $\mathcal{P}$ must be aspherical.

We assign the following weights

$$\theta(\alpha_1) = 0, \theta(\alpha_2) = 1, \theta(\alpha_3) = \theta(\alpha_4) = \frac{1}{2}$$

if a has finite order. Let γ be any cycle in $\mathcal{P}^{st}$. If γ

(a) involves α_2 at least twice or

(b) involves α_2 at least once and involves α_3 and/or α_4 at least twice or

(c) involves α_3 and/or α_4 at least four times

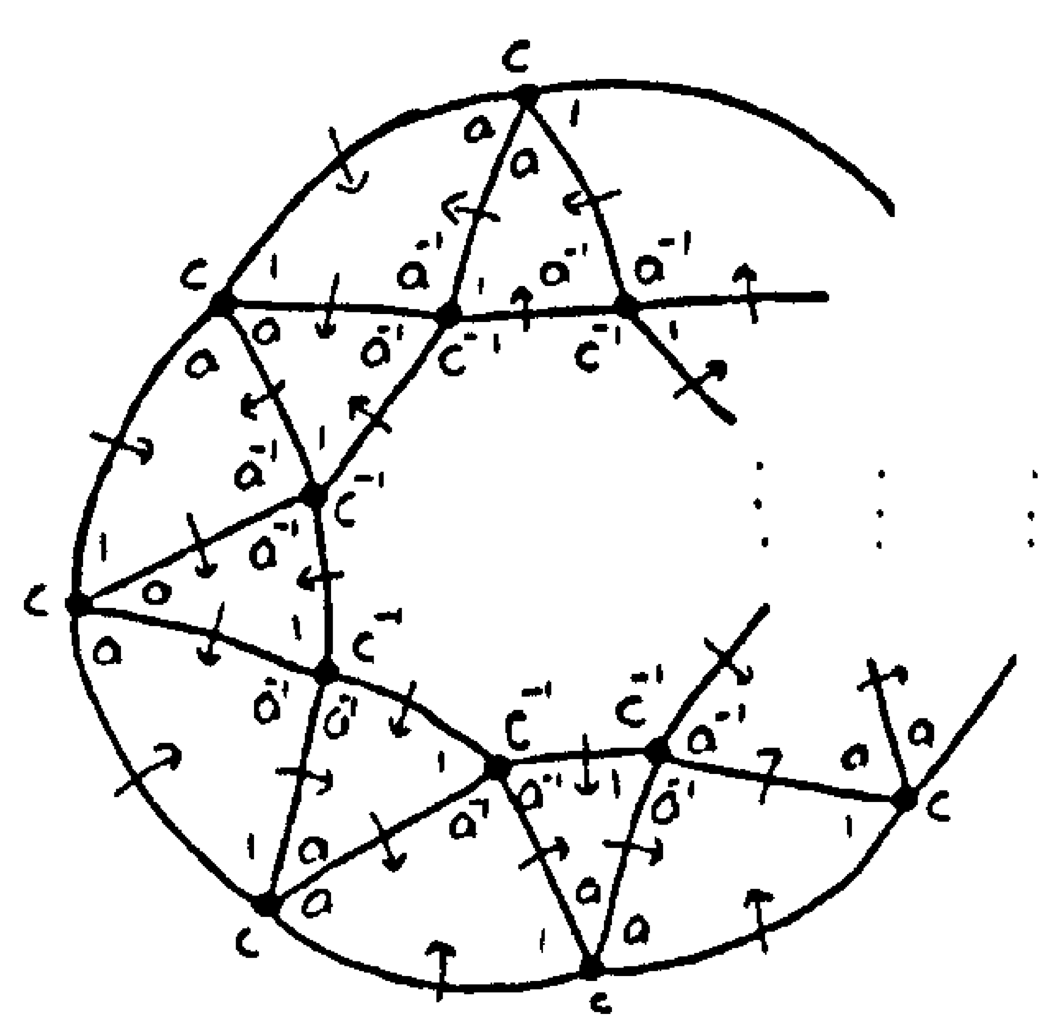
then γ has a weight of at least two. Thus we only need to check

(a) $\alpha_1^{\pm p}, p > 0$

(b) $\alpha_1^{\pm p}(\alpha_3\alpha_4^{-1}), p \geq 0$

Since c has infinite order, then clearly the first form is not admissible. Note that the second form is not admissible since $1 \neq a \neq c^{\pm p}$. With the above weights, all three conditions in §4.3.2 are satisfied and hence by Theorem 4.3.3, $\mathcal{P}$ is aspherical.

Now let c have order $2 \leq q < \infty$ and so we will show that $\mathcal{P}$ is not aspherical. Since $c^q = 1$, we may obtain a reduced strictly spherical picture $\mathbb{P}$ over $\mathcal{P}$ as below



and so (refer §4.3.6) $\psi(\lambda_{\mathbb{P}}) = q(1 - x) \neq 0$. Since $\mathbb{P}$ is not degenerate then by Lemma 4.3.7, $\mathcal{P}$ is not aspherical. •

6.2 The subcase $a \neq b = c$

If we have $\mathcal{P} =< H, t; t^2atbt^{-1}b >$ where a and b are distinct non-trivial elements of H, then we have two cases that we still can not decide.

SE1 $o(a) = 2, o(b) = 3$

FE3 $o(a) < \infty, o(b) = 2$

Excluding these two exceptions, we have

Theorem 6.2.1 *Suppose that $\mathcal{P} =< H, t; t^2atbt^{-1}b >$ is not an exceptional case. Then $\mathcal{P}$ is aspherical except when $b = a^{-1}$ and b has finite order.*

To prove this theorem, we consider the order of b as follows:

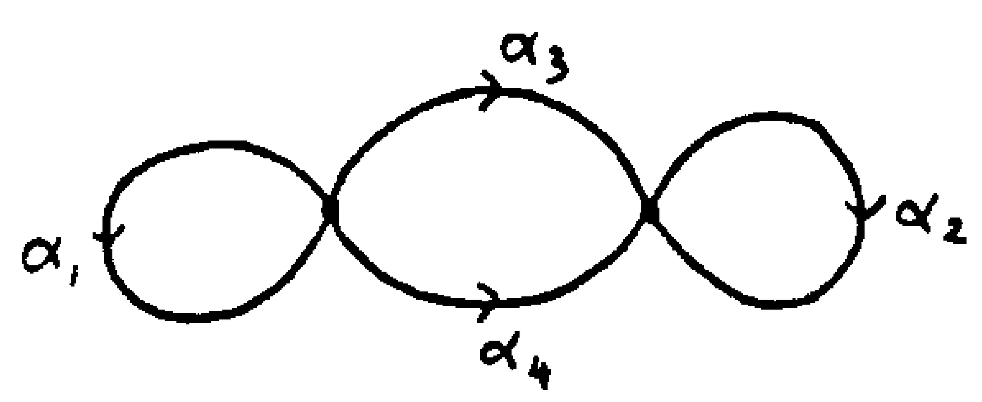
i. b has infinite order

ii. $4 \leq o(b) < \infty$

iii. $o(b) = 3$

iv. $o(b) = 2$

The star graph $\mathcal{P}^{st}$ for this case is



where $\alpha_1 \leftrightarrow b^{-1}, \alpha_2 \leftrightarrow b^{-1}, \alpha_3 \leftrightarrow a^{-1}$ and $\alpha_4 \leftrightarrow 1$.

6.2.1 The subsubcase $o(b) = \infty$

Since b has infinite order, we have to show that $\mathcal{P}$ is aspherical. Any admissible cycle in $\mathcal{P}^{st}$ must involve α_3 and/or α_4 at least twice. Thus we may assign the following weights

$$\theta(\alpha_1) = \theta(\alpha_2) = 0, \theta(\alpha_3) = \theta(\alpha_4) = 1.$$

Since all three conditions in §4.3.2 are satisfied, then by Theorem 4.3.3, $\mathcal{P}$ is aspherical.

6.2.2 The subsubcase $4 \leq o(b) < \infty$

Assume that $b \neq a^{-1}$ and so we will show that $\mathcal{P}$ is aspherical. Consider the weights

$$\theta(\alpha_1) = \theta(\alpha_2) = \theta(\alpha_3) = \theta(\alpha_4) = \frac{1}{2}.$$

Since all edges in $\mathcal{P}^{st}$ have weight $\frac{1}{2}$ then all admissible cycles of length four or more must have weight of at least two. Thus we only need to check all cycles of length two and three. There is no admissible cycle of length two because $b^2 \neq 1$ and $a \neq 1$. Any admissible cycle of length three will produce relations $b^3 = 1, ab = 1$ or $ab^{-1} = 1$. Since none of these relations holds, then there is no admissible cycle of length three and so all three conditions in §4.3.2 are satisfied. Thus by Theorem 4.3.3, $\mathcal{P}$ is aspherical.

Now let $b = a^{-1}$ of finite order. So we have to show that $\mathcal{P}$ is not aspherical. Clearly

$$\begin{aligned} t^2atbt^{-1}b = 1 \quad &\Rightarrow \quad t^2b^{-1}tbt^{-1}b = 1 \\ &\Rightarrow \quad t^2 = b^{-1}tb^{-1}t^{-1}b. \end{aligned}$$

Since t^2 is conjugate to b^{-1} of finite order, then t must have finite order and hence (refer §4.3.5) $\mathcal{P}$ is not aspherical.

6.2.3 The subsubcase $o(b) = 3$

Assign the folowing weights

$$\theta(\alpha_1) = \theta(\alpha_2) = \frac{2}{3}, \theta(\alpha_3) = \theta(\alpha_4) = \frac{1}{3}.$$

Since the minimum weight is 1/3 we need to check all cycles up to length five. Since any cycle of length five must involve α_1 or α_2 at least once then the minimum weight for a cycle of length five is $4 \cdot 1/3 + 2/3 = 2$. Thus we just need to check up to length four. The possibilities (up to cyclic permutation) are:

length 2	**length 3**	**length 4**
$\alpha_1^{\pm 2}$	$\alpha_1^{\pm 3}$	$\alpha_1^{\pm 4}$
$\alpha_2^{\pm 2}$	$\alpha_2^{\pm 3}$	$\alpha_2^{\pm 4}$
$(\alpha_3\alpha_4^{-1})^{\pm 1}$	$\alpha_1(\alpha_3\alpha_4^{-1})^{\pm 1}$	$\alpha_1^2(\alpha_3\alpha_4^{-1})^{\pm 1}$
	$\alpha_2(\alpha_3^{-1}\alpha_4)^{\pm 1}$	$\alpha_2^2(\alpha_3^{-1}\alpha_4)^{\pm 1}$
		$(\alpha_3\alpha_4^{-1})^{\pm 2}$

If we assume that $a^2 \neq 1$ (**SE1**) and $b \neq a^{-1}$, then we have to show that $\mathcal{P}$ is aspherical. By the above assumptions, the possible admissible cycles of length less than five are $\alpha_1^{\pm 3}, \alpha_2^{\pm 3}, \alpha_1^2(\alpha_3\alpha_4^{-1})^{\pm 1}$ and $\alpha_2^2(\alpha_3^{-1}\alpha_4)^{\pm 1}$. They have weight two, and so all three conditions in §4.3.2 are satisfied. Thus by Theorem 4.3.3, $\mathcal{P}$ is aspherical.

Now suppose that $b = a^{-1}$. Then

$$t^2atbt^{-1}b = 1 \Rightarrow t^2b^{-1}tbt^{-1}b = 1$$
$$\Rightarrow t^2 = b^{-1}tb^{-1}t^{-1}b.$$

Since $b^3 = 1$ then $t^6 = 1$ and hence (refer §4.3.5) $\mathcal{P}$ is not aspherical.

We still can not decide asphericity when $a^2 = 1$ (refer **SE1**).

6.2.4 The subsubcase $o(b) = 2$

Suppose that a have infinite order, then clearly $b \neq a^{-1}$. Thus we have to show that $\mathcal{P}$ is aspherical. Assign the following weights

$$\theta(\alpha_1) = \theta(\alpha_2) = 1, \theta(\alpha_3) = \theta(\alpha_4) = 0.$$

Since a has infinite order, any admissible cycle must involve α_1 and/or α_2 at least twice and hence has a weight of at least two. Thus by Theorem 4.3.3, $\mathcal{P}$ is aspherical.

For the case when a has finite order, then we still can not decide (refer **FE3**).

6.3 The subcase $a = c \neq b$

We encounter quite a number of exceptions in this case.

SE2 $o(b) = 3, o(a) = 4$

FE4 $o(b) = 2$

FE5 $o(a) = 3, o(b) < \infty$

FE6 b has finite order and $b \neq a^{-1}, a^2, a^{\pm 3}, a^{\pm 4}$
or $b^2 \neq a^{\pm 2}$ or $a \neq b^{\pm 2}, b^{\pm 3}$

Excluding these exceptions, we have

Theorem 6.3.1 *Suppose that $\mathcal{P} =< H, t; t^2atbt^{-1}a >$ is not an exceptional case where a and b are distinct non-trivial elements of H. Then $\mathcal{P}$ is aspherical if and only if none of these holds:*

1. $o(a) = 2, o(b) < \infty$
2. $b = a^{-2}$ *of finite order*
3. $b = a^2, o(a) = 3$ *or* 4

To prove this theorem, we consider

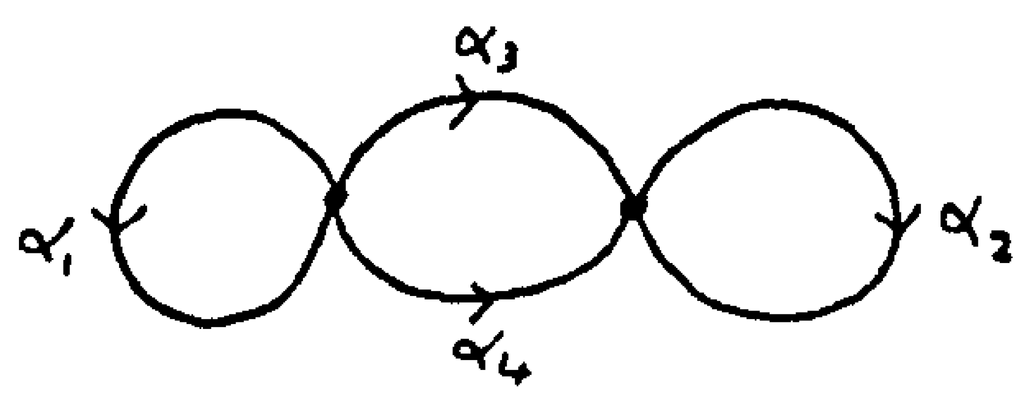
i. $o(a) = o(b) = \infty$

ii. $o(a) < o(b) = \infty$

iii. $o(b) < o(a) = \infty$

iv. $o(a), o(b) < \infty$

The star graph for this case is



where $\alpha_1 \leftrightarrow a^{-1}, \alpha_2 \leftrightarrow b^{-1}, \alpha_3 \leftrightarrow a^{-1}$ and $\alpha_4 \leftrightarrow 1$.

6.3.1 The subsubcase $o(a) = o(b) = \infty$

Note that none of the conditions 1,2,3 in the statement of the theorem holds, so we have to show that $\mathcal{P}$ is aspherical. Assign the following weights

$$\theta(\alpha_1) = \theta(\alpha_2) = 0, \theta(\alpha_3) = \theta(\alpha_4) = 1.$$

Since a and b have infinite order, then any admissible cycle must involve α_3 and/or α_4 at least twice. Thus all three conditions in §4.3.2 are satisfied and by Theorem 4.3.3, $\mathcal{P}$ is aspherical.

6.3.2 The subsubcase $o(a) < o(b) = \infty$

Clearly $b \notin < a >$ and so none of the conditions 1,2,3 in the statement is satisfied. Thus we have to show that $\mathcal{P}$ is aspherical. Consider the weights

$$\theta(\alpha_1) = 1, \theta(\alpha_2) = 0, \theta(\alpha_3) = \theta(\alpha_4) = \frac{1}{2}.$$

Let γ be any cycle in $\mathcal{P}^{st}$. If γ

(a) involves α_1 at least twice or

(b) involves α_1 at least once and involves α_3 and/or α_4 at least twice or

(c) involves α_3 and/or α_4 at least four times

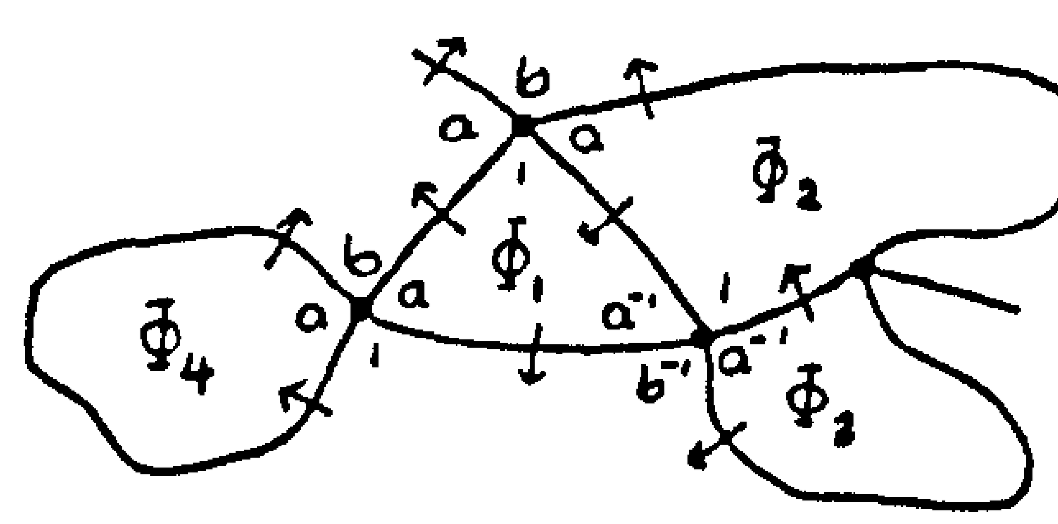
then γ has a weight of at least two. Also note that since b has infinite order then clearly $\alpha_2^{\pm k}$ is not admissible. Thus we only need to check (up to cyclic permutation) cycles of the form

$$\alpha_2^p(\alpha_3^{-1}\alpha_4)^{\pm 1}, p \geq 0.$$

They are not admissible since $1 \neq a \neq b^{\pm p}$. Since all three conditions in §4.3.2 are satisfied, then by Theorem 4.3.3, $\mathcal{P}$ is aspherical.

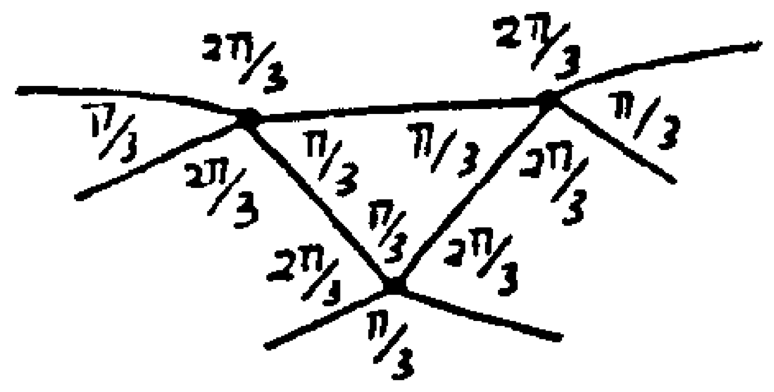
6.3.3 The subsubcase $o(b) < o(a) = \infty$

We can not use the weight test anymore. Again we have to show that $\mathcal{P}$ is aspherical since none of the relations in hypothesis is satisfied. It suffices to show that there is no reduced strictly spherical picture $\mathbb{P}$ over $\mathcal{P}$. Suppose there were. In picture $\mathbb{P}$ we may have a region Φ_1 of the form

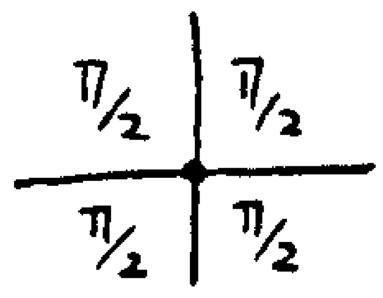


We call Φ_1 a *bad region*. Note that Φ_2 can not be a bad region but Φ_3 and Φ_4 are possible. Also note that a 3- or a 4-region (refer §4.4.1) can not share an edge with a bad region.

We may assign the angle function



to bad regions and for the rest, assign



Thus by Lemma 4.3.4, there exists an exceptional region Φ of valence m with positive curvature. Since the maximum angle is $2\pi/3$, then we have $m \cdot 2\pi/3 > (m-2)\pi$ which means that $m < 6$. From Appendix B.1.1 we conclude that $\mathcal{P}$ is aspherical except possibly when $o(b) \leq 5$.

Since a 3-region can not share an edge with a bad region then the maximum angle for any corner in a 3-region is $\pi/2$. Thus if $o(b) = 4$ or 5 then for any 3-region of valency four or five, the curvature is not positive. Hence we just need to check when $o(b) = 2$ or 3.

1) $o(b) = 3$

Clearly the only exceptional region is a 3-region Φ of valence three with label bbb. Since the maximum angle of any corner in Φ is $\pi/2$, then the curvature $\gamma(\Phi) \leq \pi/2$. We may distribute $\pi/6$ to each region Ψ that shares an edge with Φ. First note that Ψ is not a 3-region or a bad region. We have to make sure that the new curvature for Ψ remains non-positive. Note also that by the observation in §4.4.1, in every three edges of Ψ there is at most one 3-region. Thus we do not want (assume that Ψ has valence m) $(m-2)\pi < m \cdot 2\pi/3 + m/3 \cdot \pi/6$ which means that $m < 7\frac{1}{6}$. If Ψ has valence seven then Ψ may share an edge with at most two 3-regions. Then the curvature

$$\gamma^*(\Psi) \leq 7 \cdot 2\pi/3 + 2 \cdot \pi/6 - 5\pi = 0$$

and so it is not exceptional. From Appendix B.1.2, we know that there is no possible label if Ψ has valence six or less. Thus $\mathcal{P}$ must be aspherical.

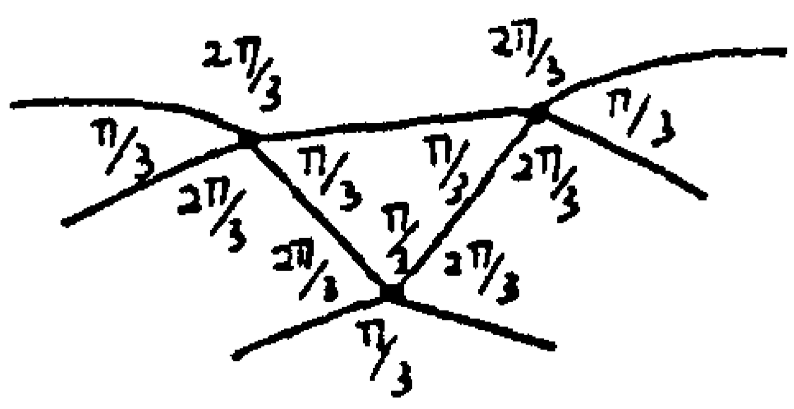
2) $o(b) = 2$

At this moment we can not decide asphericity for this case (refer **FE4**).

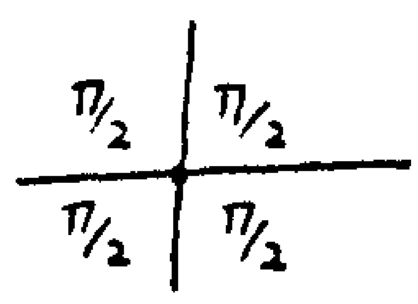
6.3.4 The subsubcase $o(a), o(b) < \infty$

When is $\mathcal{P}$ aspherical?

Excluding **FE6**, note that $a^2 \neq 1$ since otherwise hypothesis 1 is satisfied. It suffices to show that there is no reduced strictly spherical picture $\mathbb{P}$ over $\mathcal{P}$. Suppose there were. Then as in §6.3.3, assign the angle function

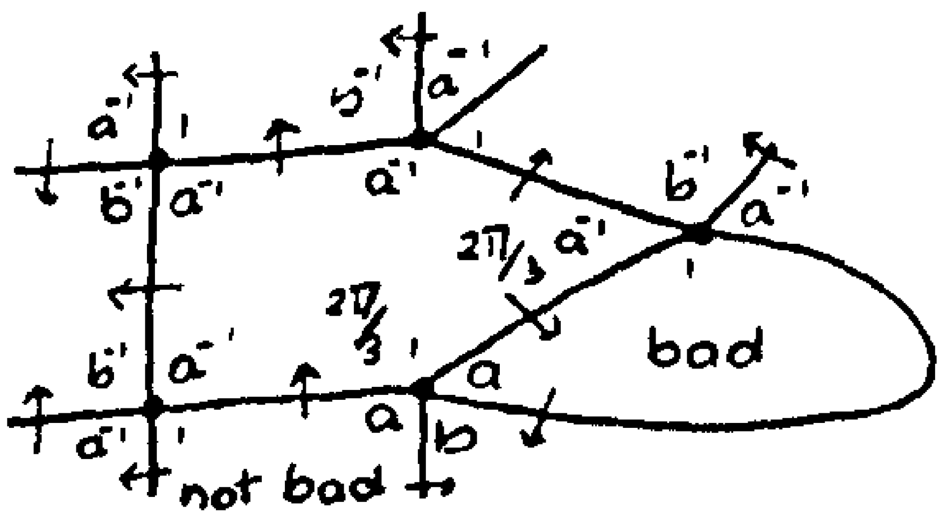


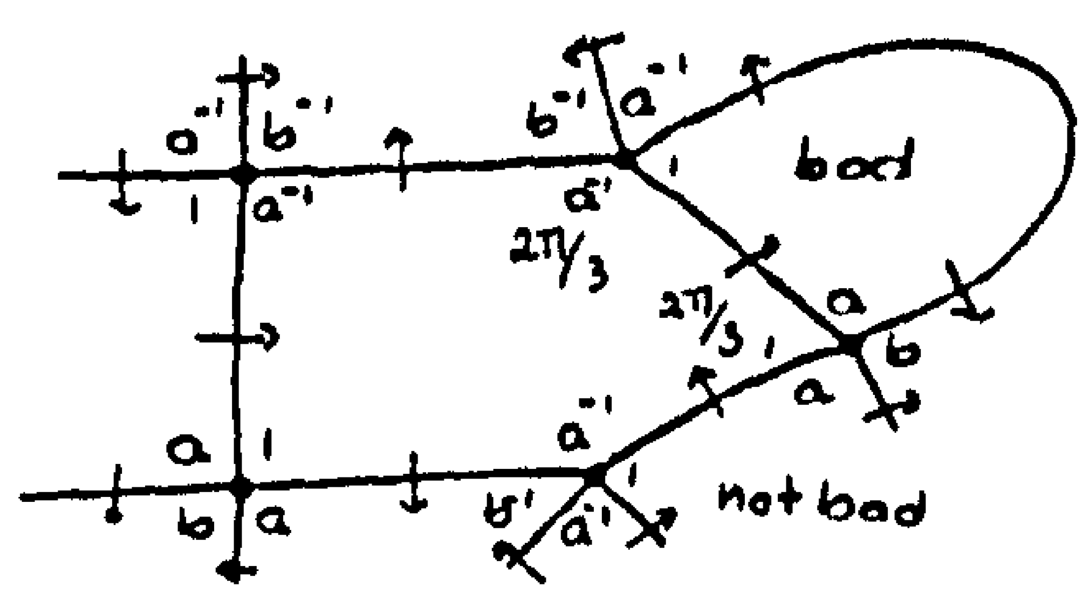
for bad regions and for the rest, assign the angle function



Since all discs in $\mathbb{P}$ are flat, then by Lemma 4.3.4, there is an exceptional region Φ of valence less than six (refer §6.3.3). From Appendix B.2.1, we may conclude that $\mathcal{P}$ is aspherical if $o(a), o(b) \geq 6$.

As in §6.3.3, there is no 3- or 4-region sharing an edge with a bad region. Thus the maximum angle for any corner in a 3- or a 4-region is $\pi/2$. Thus if they have valence four or more then they are not exceptional regions. Note that there are regions of valence five with label $1aaaa$ and $1a1aa$



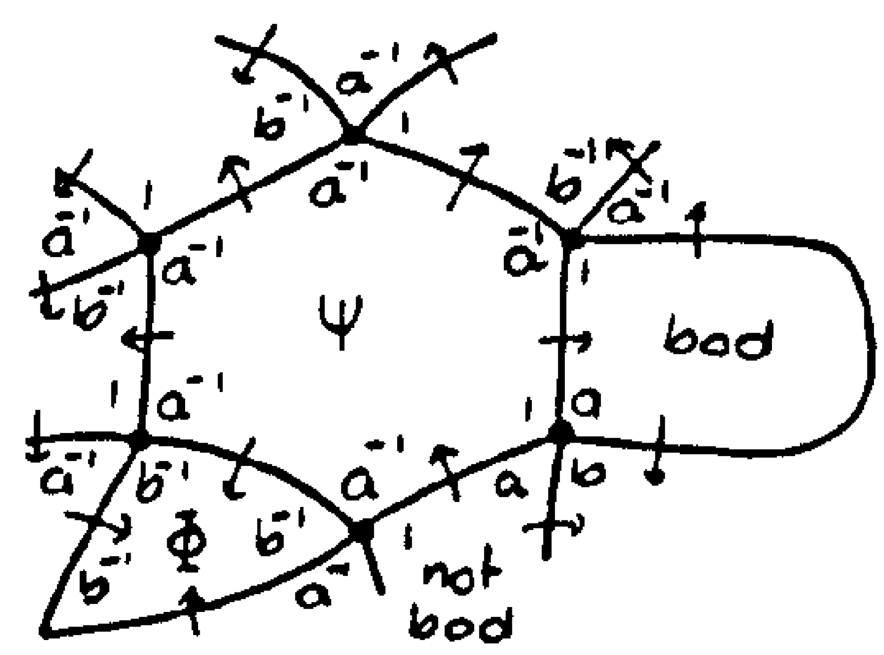


Since they share an edge with at most one bad region, the total sum of angles is less than or equal to $2 \cdot 2\pi/3 + 3 \cdot \pi/2 < 3\pi$. Thus they are not exceptional regions either.

However there is a possible exceptional region of valence four with label $1aaa$. The complexity of this region together with a bad region does not enable us to decide asphericity when $o(a) = 3$ (refer **FE5**). Thus if $o(a), o(b) \geq 4$ then $\mathcal{P}$ is aspherical and so we just need to consider when $o(b) = 2$ or 3.

1) $o(b) = 3$

Assume that $o(a) \neq 4$ (refer **SE2**). As in §6.3.3, the only exceptional region is a 3-region of valence three with label bbb. Using the same argument, we need to check a region Ψ with label aaW of valence six or less. From Appendix B.2.2 there is a possible label namely $1aaaaa$.



Since Ψ may share at most an edge with a bad region then there are most two corners having angle $2\pi/3$. Thus the curvature

$$\gamma^*(\Psi) \leq (2 \cdot 2\pi/3 + 4 \cdot \pi/2) + \pi/6 - 4\pi < 0.$$

Thus Ψ is not exceptional and hence $\mathcal{P}$ must be aspherical.

2) $o(b) = 2$

We can not decide asphericity for this case (refer **FE4**).

When is $\mathcal{P}$ not aspherical?

If one of these holds:

1. $a^2 = 1, o(b) < \infty$
2. $b = a^{-2}$ of finite order
3. $b = a^2, o(a) = 3$

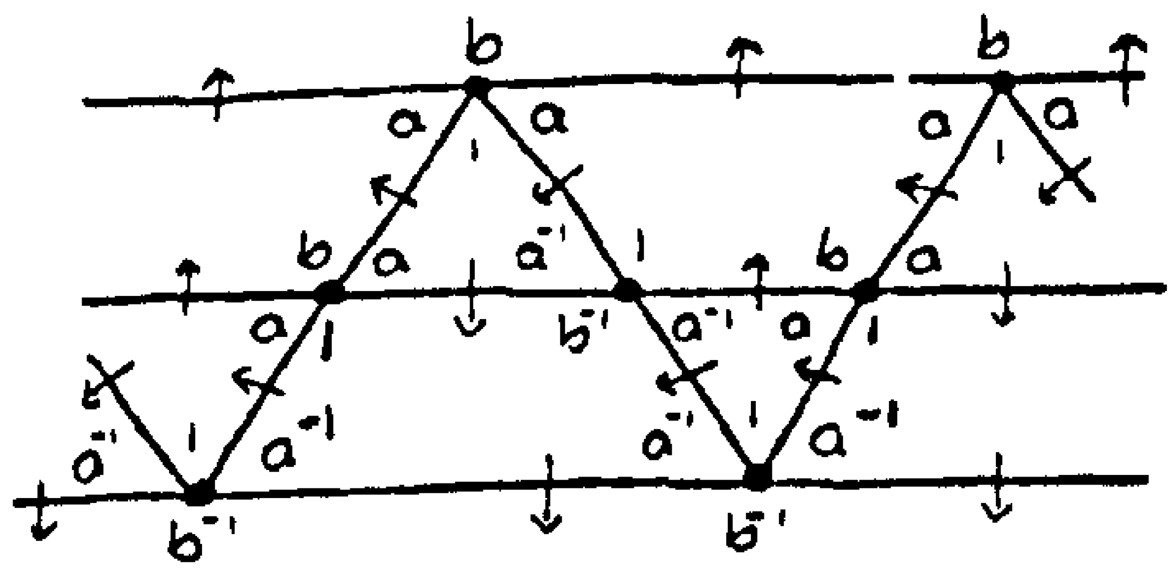
then $\mathcal{P}$ is not aspherical.

Suppose that $a^2 = 1$, then

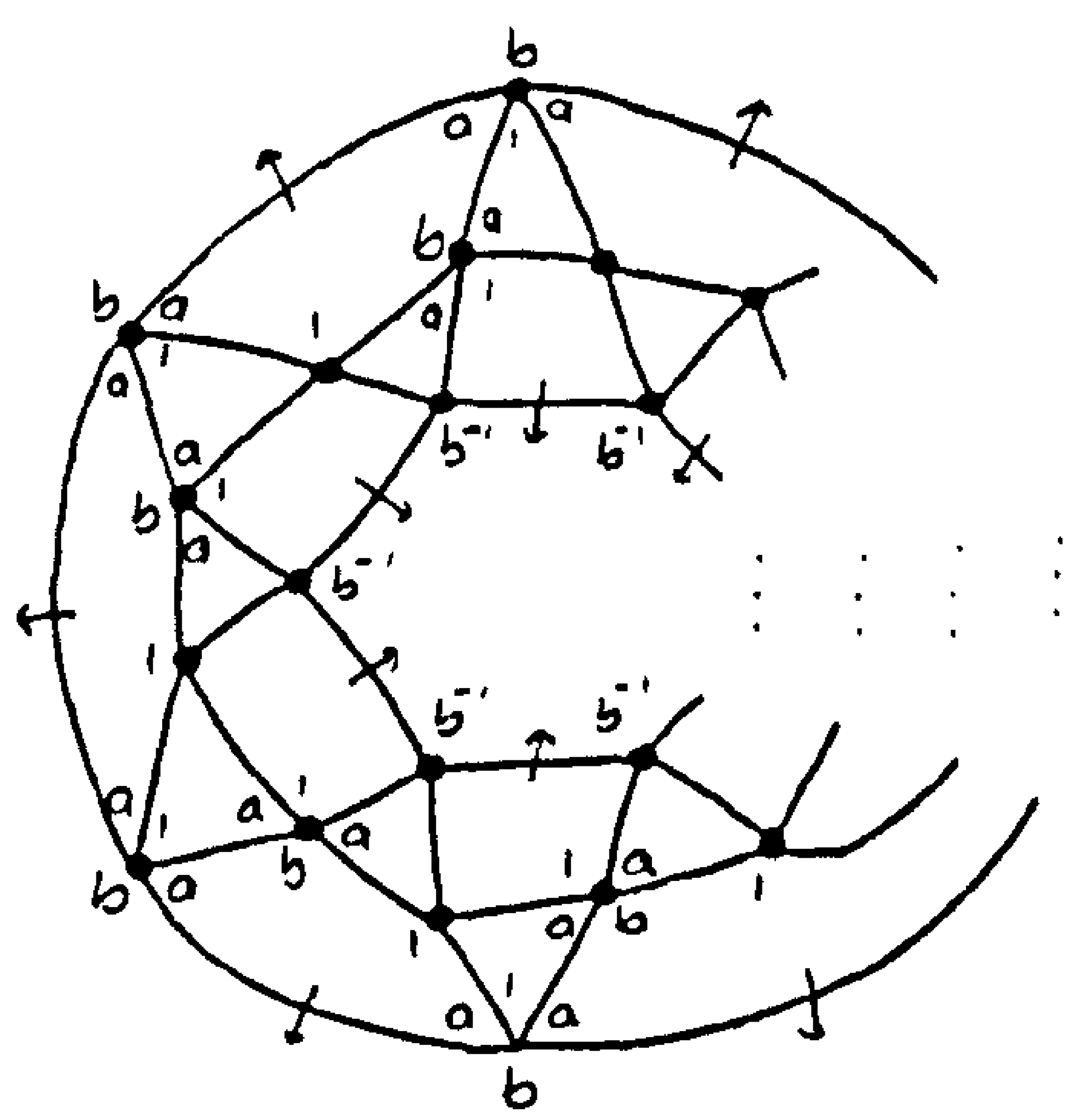
$$t^2atbt^{-1}a = 1 \Rightarrow t^2a^{-1}tbt^{-1}a = 1$$
$$\Rightarrow t^2 = atb^{-1}t^{-1}a^{-1}.$$

So we know that t has finite order since it is conjugate to b^{-1} of finite order and hence (refer §4.3.5) $\mathcal{P}$ is not aspherical.

If $b = a^{-2}$, then consider



We can join an appropriate number of these (according to the order of b) to form a reduced strictly spherical picture



and so (refer §4.3.6)

$$\lambda_{\mathbb{P}} = (1 + b + b^2 + \cdots + b^{q-1})(t^{-1}a + t^{-1}ata - t^{-1}atat - t^{-1}at^{-1}).$$

We can not use the mapping ψ as in §4.3.6 for this case and so we will consider another ring homomorphism. Consider

$$\phi : \mathbb{Z}G \longrightarrow \mathbb{Z}(H \times < x; x^2 >),$$

the ring homomorphism arising from the group homomorphism defined by

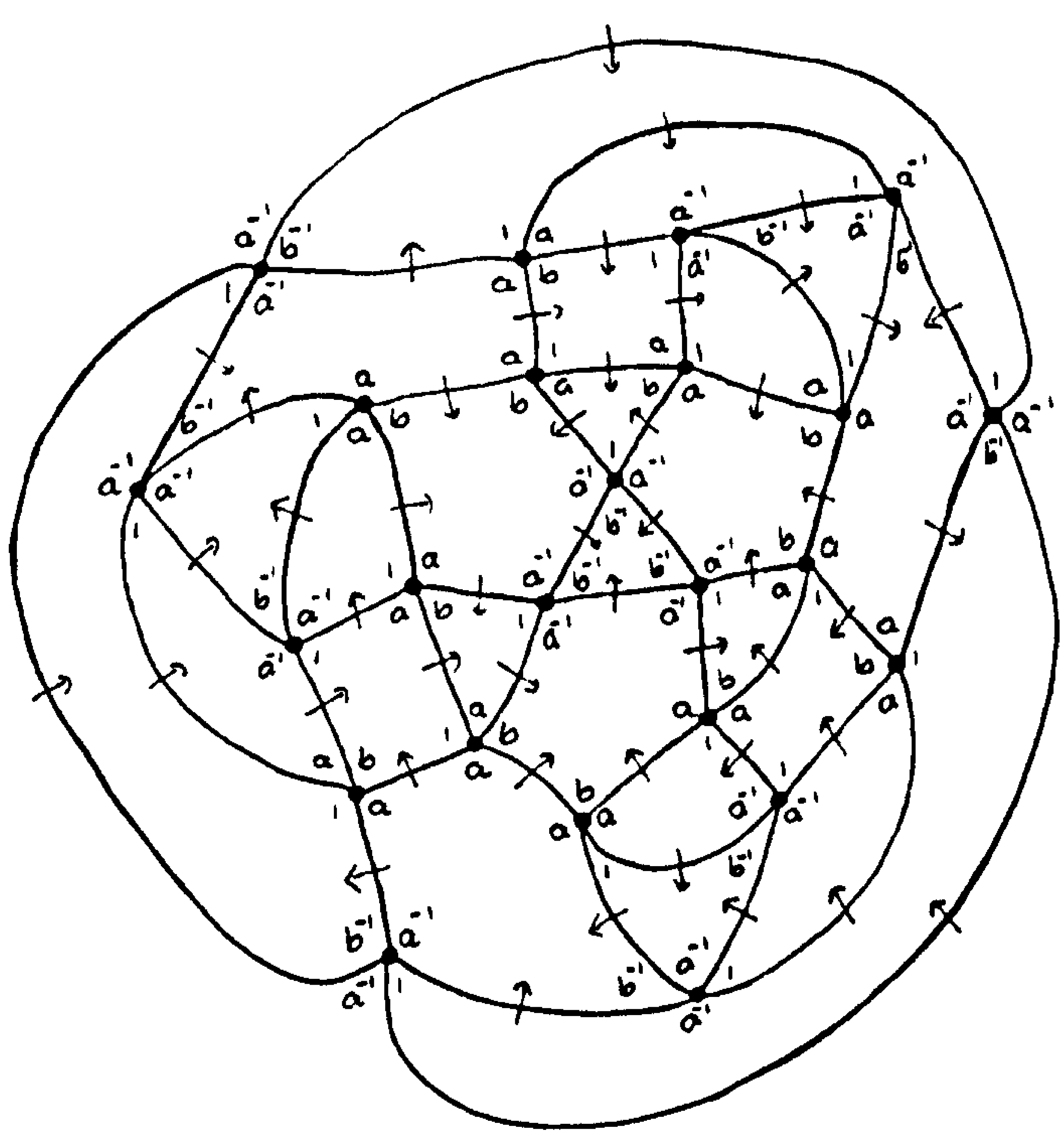
$$H \mapsto H, t \mapsto x.$$

Then

$$\begin{aligned} \phi(\lambda_{\mathbb{P}}) &= (1+b+\cdots+b^{q-1})(ax+a^2-a^2x-a) \\ &= (1+a+\cdots+a^{2q-2})(a^2-a)(1-x) \\ &\neq 0. \end{aligned}$$

Thus $\mathbb{P}$ is not degenerate and hence $\mathcal{P}$ is not aspherical.

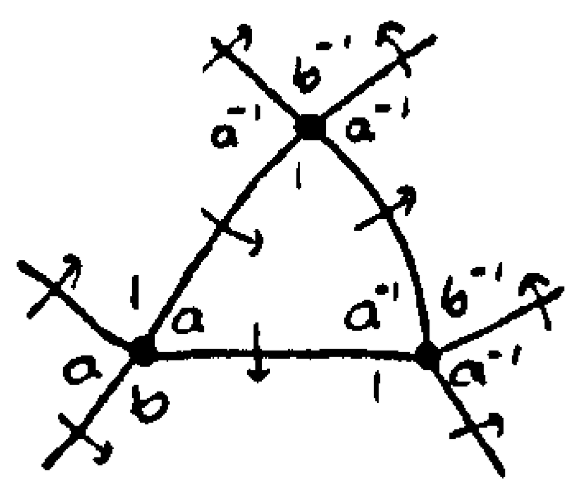
For the third case, we draw



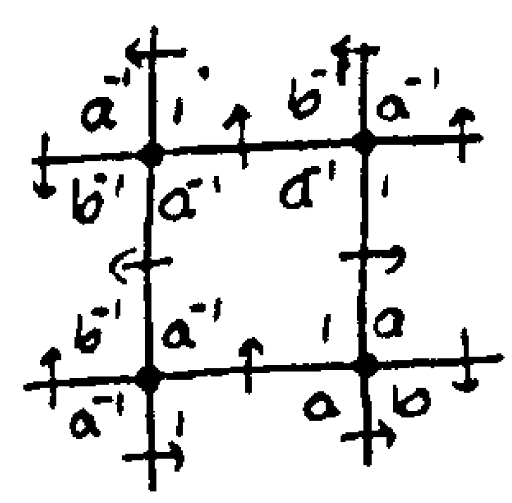
and so (refer §4.3.6) $\psi(\lambda_{\mathbb{P}}) = 3(1-x) \neq 0$. Then $\mathbb{P}$ is not degenerate and hence by Lemma 4.3.7, $\mathcal{P}$ is not aspherical.

Remark

This case is the most difficult to handle because we have a bad region



and if we have $a^3 = 1$, we have to consider



It is quite difficult to keep track in argument while handling these regions. Therefore we can not decide for the case when $o(a) = 3$ (refer **FE5**). The complexity of these regions also makes us unable to decide for the other exceptions.

6.4 The subcase a, b and c are distinct

Throughout this section we assume that $a \neq b^{-1}$ since (refer operations in §4.1.1)

$$\begin{aligned} t^2b^{-1}tbt^{-1}c &\stackrel{t\to tb}{\longrightarrow} tbtbb^{-1}tbbb^{-1}t^{-1}c = tbt^2bt^{-1}c \\ &\stackrel{\mathbf{I}}{\longrightarrow} tb^{-1}t^{-2}b^{-1}t^{-1}c^{-1} \\ &\stackrel{t\to t^{-1}}{\longrightarrow} t^{-1}b^{-1}t^2b^{-1}tc^{-1} \\ &\stackrel{\mathbf{II}}{\longrightarrow} t^2b^{-1}tc^{-1}t^{-1}b^{-1} \end{aligned}$$

which is subcase 6.3. Thus one may refer §6.3 when $a = b^{-1}$.

If we have $\mathcal{P} =< H, t; t^2atbt^{-1}c >$ where a, b and c are all distinct non-trivial elements of H, then we have three special exceptions and a family of exceptions that we still can not decide.

SE3 $a^2 = b^3 = 1$ and $a = c^{-2}$

SE4 $o(c) = 3, o(b) = 2, b = ac, c = a^2$

SE5 $o(c) = 6, o(b) = 2, b = ac, a = c^2$

FE7 $b^2 = 1$ and $a = c^{-2}$ where $4 \leq o(c) < \infty$

Excluding these exceptions, we have

Theorem 6.4.1 *Suppose that $\mathcal{P} = < H, t; t^2atbt^{-1}c >$ is not an exceptional case where a, b and c are distinct non-trivial elements of H. Then $\mathcal{P}$ is aspherical if and only if none of these holds:*

1. $a = c^{-1}, o(b) < \infty$

2. $b = ac$ *and* $\frac{1}{p} + \frac{1}{q} + \frac{1}{k} > 1$ *where* $o(a) = p, o(b) = q$ *and* $o(c) = k$ $(\frac{1}{\infty} := 0)$

To prove this theorem, we will consider the following subsubcases separately.

i. b and c have infinite order

ii. $o(b) < o(c) = \infty$

And if both b and c have finite orders

iii. $o(b) = o(c) = 2$

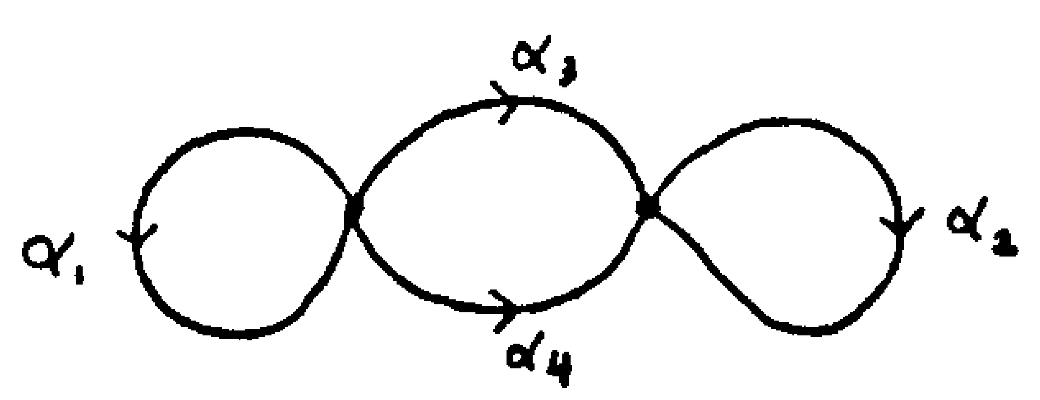
iv. $o(b) = 2, o(c) \geq 3$

v. $o(b) = 3, o(c) \geq 3$

vi. $3 < o(b), o(c) < \infty$

Note that up to equivalent presentations, we may identify $b \leftrightarrow c^{-1}$, so we do not consider the subsubcase $o(c) < o(b) = \infty$, the subsubcase $o(c) = 2, 3 \leq o(b) < \infty$ and the subsubcase $o(c) = 3, 3 \leq o(b) < \infty$ since the result and proof are similar to subsubcases 6.4.2, 6.4.4 and 6.4.5 respectively.

The star graph $\mathcal{P}^{st}$ for this case is



where $\alpha_1 \leftrightarrow c^{-1}, \alpha_2 \leftrightarrow b^{-1}, \alpha_3 \leftrightarrow a^{-1}$ and $\alpha_4 \leftrightarrow 1$.

6.4.1 The subsubcase $o(b) = o(c) = \infty$

Since b and c have infinite order, we have to show that $\mathcal{P}$ is aspherical. Any admissible cycle in $\mathcal{P}^{st}$ must involve α_3 and/or α_4 at least twice. Thus we may assign the following weights

$$\theta(\alpha_1) = \theta(\alpha_2) = 0, \theta(\alpha_3) = \theta(\alpha_4) = 1.$$

Since all three conditions in §4.3.2 are satisfied, then by Theorem 4.3.3, $\mathcal{P}$ is aspherical.

6.4.2 The subsubcase $o(b) < o(c) = \infty$

We have to show that $\mathcal{P}$ is aspherical if and only if $a \neq c^{-1}$.

Suppose first that $a \neq c^{-1}$.

If $a \neq c^{\pm p}, p > 1$ then we may assign the weights

$$\theta(\alpha_1) = 0, \theta(\alpha_2) = 1, \theta(\alpha_3) = \theta(\alpha_4) = \frac{1}{2}.$$

Let γ be any cycle in $\mathcal{P}^{st}$. If γ

(a) involves α_2 at least twice or

(b) involves α_2 at least once and involve α_3 and/or α_4 at least twice or

(c) involves α_3 and/or α_4 at least four times

then γ has a weight of at least two. Clearly $\alpha_1^{\pm k}, k > 0$ is not admissible since c has infinite order. Thus we just need to check cycles of the form

$$\alpha_1^q(\alpha_3\alpha_4^{-1})^{\pm 1}, q \geq 0.$$

They are not admissible since $1 \neq a \neq c^{\pm q}$. Then all three conditions in §4.3.2 are satisfied and hence by Theorem 4.3.3, $\mathcal{P}$ is aspherical.

If $a = c^{\pm p}, p \geq 2$ then a must have infinite order since $o(c) = \infty$. We may assign

$$\theta(\alpha_1) = \theta(\alpha_3) = \theta(\alpha_4) = \frac{1}{3}, \theta(\alpha_2) = 1.$$

Let γ be any cycle in $\mathcal{P}^{st}$. If γ

(a) involves α_2 more than once or

(b) involves α_1 and α_2 at least once each and involves α_3 and/or α_4 at least twice or

(c) involves α_3 and/or α_4 at least six times or

(d) involves α_1 at least twice and involve α_3 and/or α_4 at least twice or

(e) involves α_1 at least once and involves α_3 and/or α_4 at least four times

then γ has a weight of at least two. Thus we just need to check

(a) $\alpha_1^{\pm k}, k < 6$

(b) $(\alpha_3\alpha_4^{-1})^{\pm l}, l = 1, 2$

(c) $\alpha_1(\alpha_3\alpha_4^{-1})^{\pm 1}$

Since c and a have infinite order then the first two forms are not admissible. The last form is not admissible since $a \neq c^{\pm 1}$. Thus $\mathcal{P}$ must be aspherical.

Now suppose that $a = c^{-1}$. Then

$$t^2atbt^{-1}c = 1 \Rightarrow t^2 = c^{-1}tb^{-1}t^{-1}c.$$

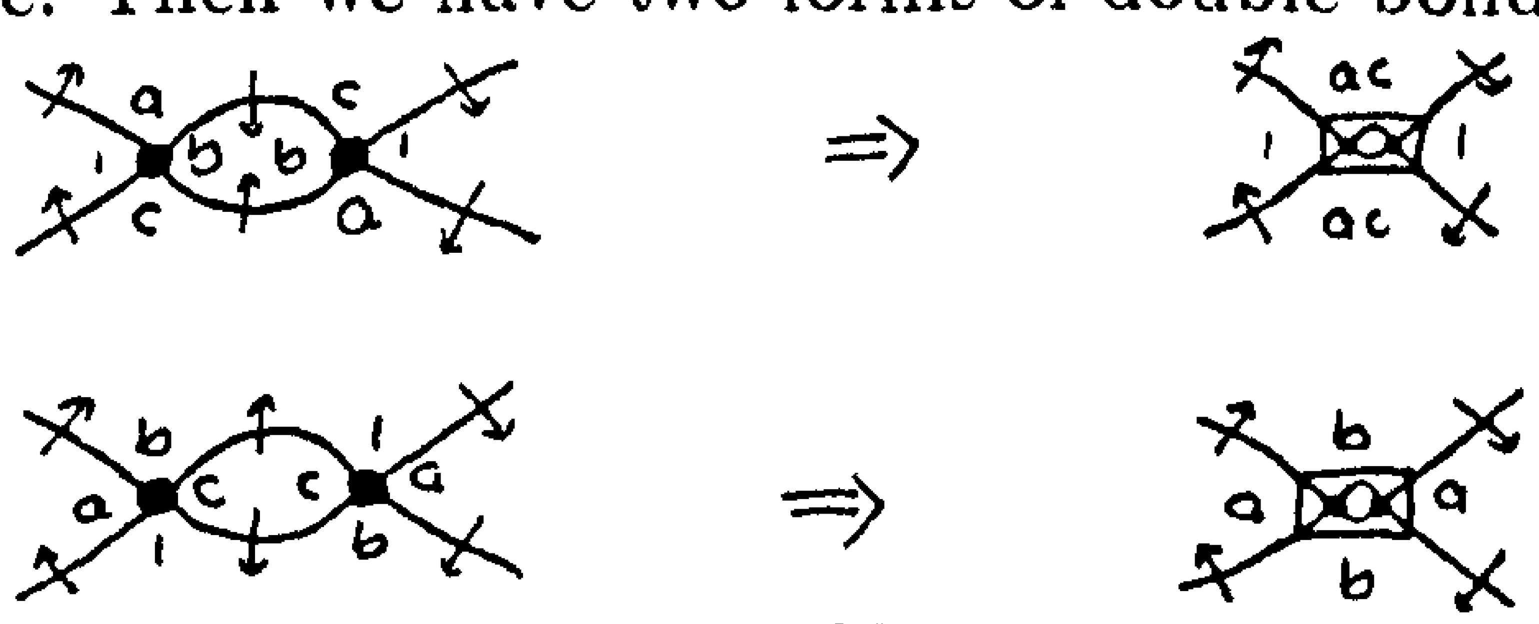
Since b has finite order, then so does t and hence (refer §4.3.5) $\mathcal{P}$ is not aspherical.

6.4.3 The subsubcase $o(b) = o(c) = 2$

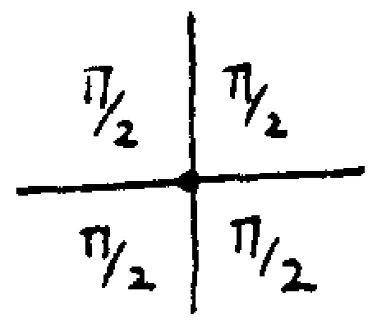
In this subsubcase, we have to show that $\mathcal{P}$ is aspherical if and only if either a has infinite order or $b \neq ac$.

When is $\mathcal{P}$ aspherical?

It is suffices to show that there is no reduced strictly spherical picture $\mathbb{P}$ over $\mathcal{P}$. Suppose there were. Then we have two forms of double bonds



and so we have a new derived picture $\mathbb{P}'$ such that every disc has valence four. We may assign the angle function



and so there is an exceptional region Φ' of valence less than four. Finding all possibilities (refer Appendix B.3.1), we conclude that $\mathcal{P}$ is aspherical except possibly if

1. $b = ac$

2. $ba = ac$

1) $b = ac$

Since $b = ac$ then a must have infinite order since otherwise hypothesis 4 would be satisfied. We may assign weights

$$\theta(\alpha_1) = \theta(\alpha_2) = 1, \theta(\alpha_3) = \theta(\alpha_4) = 0.$$

Since a has infinite order and $o(b) = o(c) = 2$ then clearly $(\alpha_3\alpha_4^{-1})^{\pm k}, \alpha_1(\alpha_3\alpha_4^{-1})^{\pm p}$ and $\alpha_2(\alpha_4^{-1}\alpha_3)^{\pm q}$ are not admissible. Thus any admissible cycle must involve α_1 and/or α_2 at least twice and has weight of at least two. Thus by Theorem 4.3.3, $\mathcal{P}$ is aspherical.

2) $ba = ac$

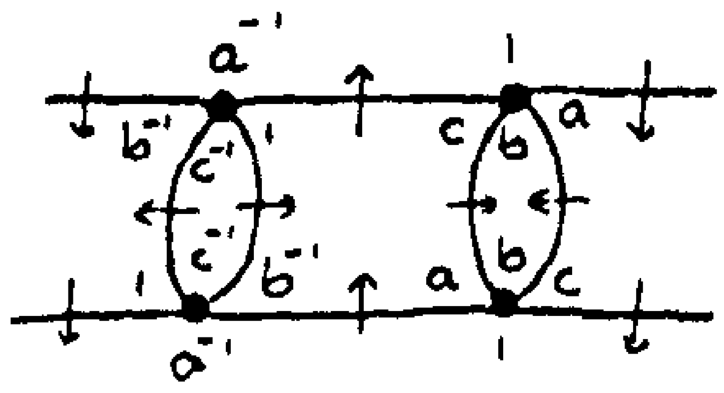
Clearly $b \neq ac$ since otherwise a would be trivial. Also $ac \neq ca$ since otherwise we would have $b = c$. Since (refer operations in §4.1.1)

$$\begin{aligned} t^2atbt^{-1}c &\xrightarrow{t \to ta^{-1}} ta^{-1}ta^{-1}ata^{-1}bat^{-1}c = ta^{-1}t^2ct^{-1}c \\ &\xrightarrow{\mathbf{I}} tc^{-1}t^{-2}at^{-1}c^{-1} \\ &\xrightarrow{t \to t^{-1}} t^{-1}c^{-1}t^2atc^{-1} \\ &\xrightarrow{\mathbf{II}} t^2atc^{-1}t^{-1}c^{-1} \end{aligned}$$

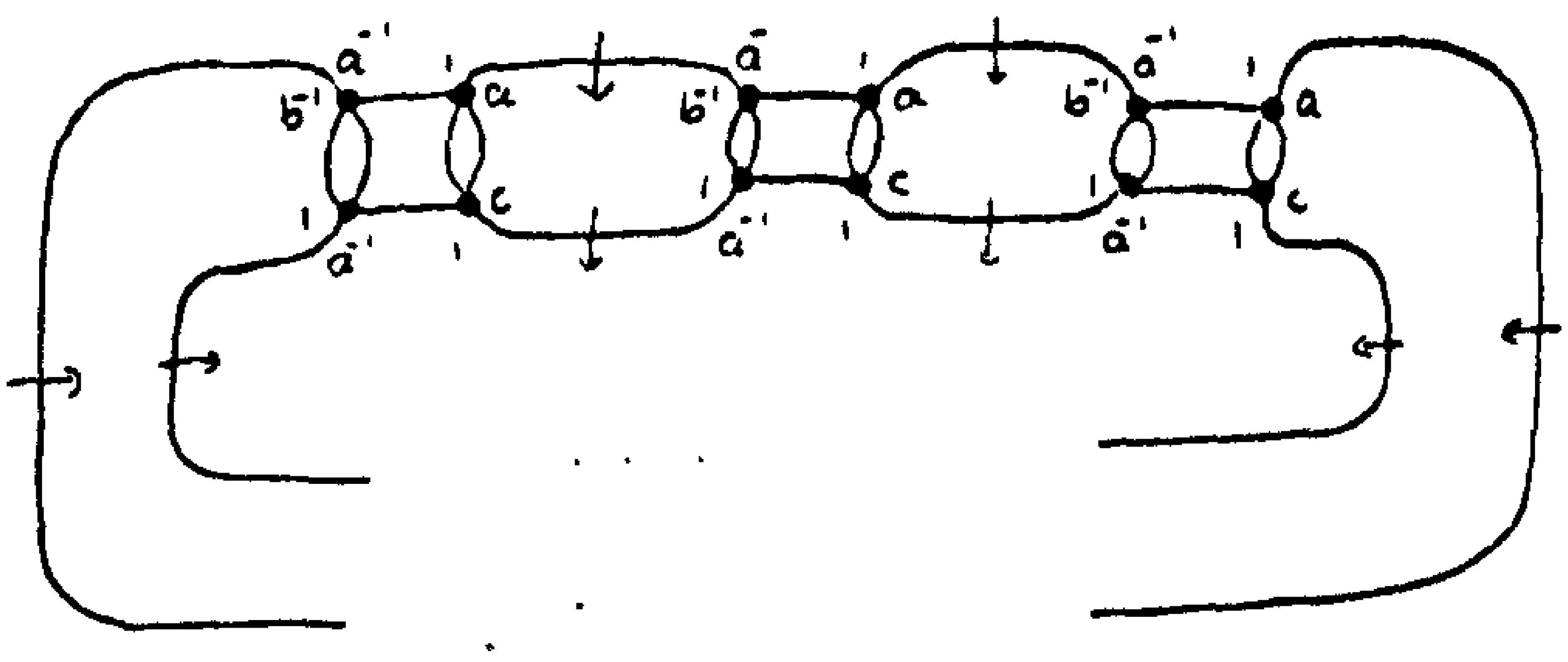
which is subcase 6.2, one may refer §6.2.4 for this case.

When is $\mathcal{P}$ not aspherical?

If $b = ac$ and a has finite order then $\mathcal{P}$ is not aspherical. Consider



If $a^p = 1$ then join p of this to form



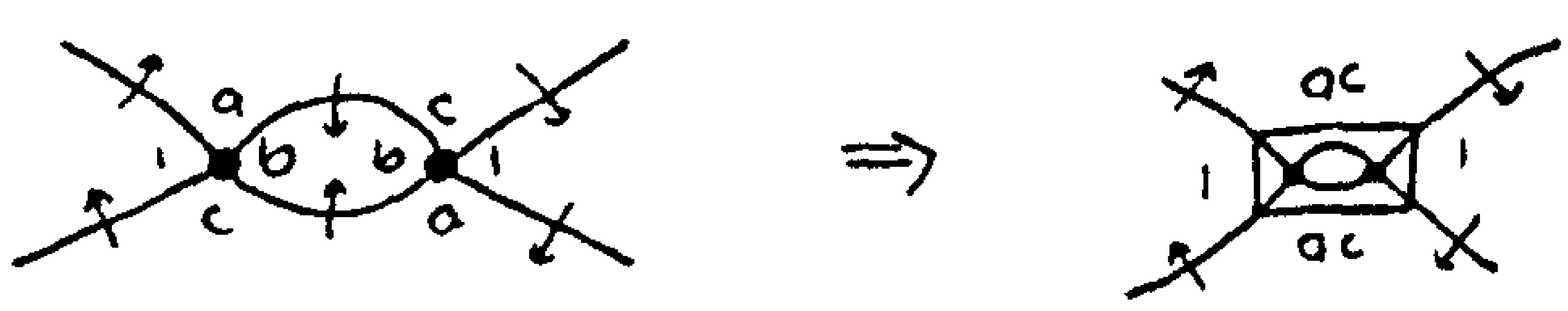
and hence (refer §4.3.6) $\psi(\lambda_{\mathbb{P}}) = 2p(x-1) \neq 0$. Since $\mathbb{P}$ is not degenerate then by Lemma 4.3.7 $\mathcal{P}$ is not aspherical.

6.4.4 The subsubcase $o(b) = 2, 3 \leq o(c) < \infty$

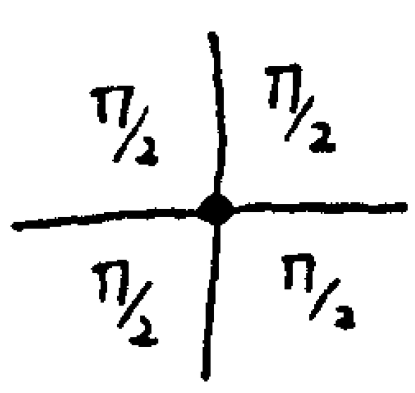
When is $\mathcal{P}$ aspherical?

Assume that none of the relations in hypothesis is satisfied. The technique used here is a combination of curvature and weight tests.

Suppose that there were a reduced strictly spherical picture $\mathbb{P}$ over $\mathcal{P}$ and so we may have a double bond



We may regard it as a single disc and so we have a new derived spherical picture $\mathbb{P}'$ such that every disc has valence four. Assign the angle function



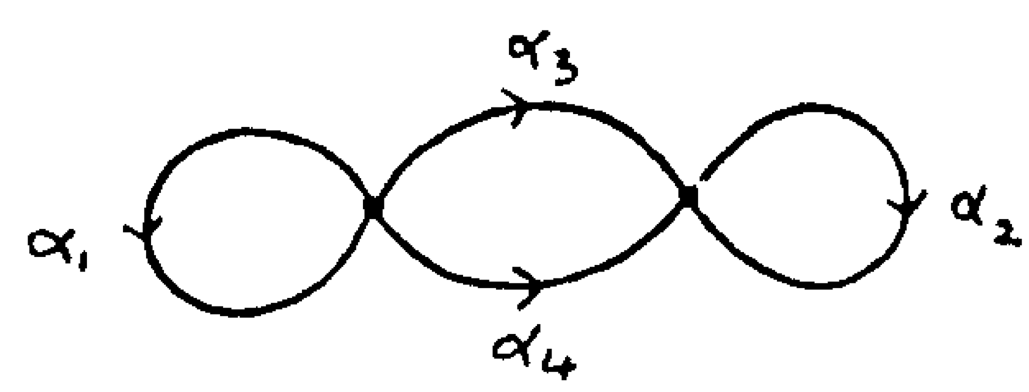
and then by Lemma 4.3.4, there exists an exceptional region Φ' with positive curvature. If Φ' has valence m, then $(m-2)\pi > m \cdot \pi/2$ which means that $m < 4$. Thus Φ' must have valency two or three. Then we may conclude that $\mathcal{P}$ must be aspherical (refer Appendix B.4.1) except possibily if $b = ac$ or $ac^2 = 1$.

1) $ac^2 = 1$

It is quite clear that if $ac^2 = 1$ then $o(c) \geq 4$ since otherwise a is trivial or equal to c. Also $b \neq ac$ since otherwise $b = c$. At the moment we can not decide for this case (refer **FE7**).

2) $b = ac$

Now assume that $b = ac$ and so $ac^2 \neq 1$. We will use the weight test to get more information. Recall that the star graph $\mathcal{P}^{st}$ for this case is



where $\alpha_1 \leftrightarrow c^{-1}, \alpha_2 \leftrightarrow b^{-1}, \alpha_3 \leftrightarrow a^{-1}$ and $\alpha_4 \leftrightarrow 1$.

a) $o(c) \geq 6, a \neq c^3$

By hypothesis 4, $o(a) > 2$. Assign weights

$$\theta(\alpha_1) = \theta(\alpha_3) = \theta(\alpha_4) = \frac{1}{3}, \theta(\alpha_2) = 1.$$

Since the minimum weight is $\frac{1}{3}$, then we need to check cycles up to length five. Note that any closed path involving α_2 twice or α_1 and α_2 at least once each has a weight of at least two. Since $o(a) > 2$ then clearly $(\alpha_3\alpha_4^{-1})^{\pm 1}$ and $(\alpha_3\alpha_4^{-1})^{\pm 2}$ are not admissible. Clearly $\alpha_1^{\pm p} (p < 6)$ is not admissible since $o(c) \geq 6$. Thus we only need to consider

$$\alpha_2^{\pm 1}(\alpha_3^{-1}\alpha_4)^2, \alpha_2^{\pm 2}(\alpha_3\alpha_4^{-1}), \alpha_1^{-3}(\alpha_3\alpha_4^{-1}) \text{ and } \alpha_1^{\pm 1}(\alpha_3\alpha_4^{-1})^2$$

that is $b = a^{\pm 2}, c^2 = a^{\pm 1}, c^3 = a^{-1}$ and $c = a^{\pm 2}$.

Clearly $b \neq a^2$ since $b = ac \Rightarrow a = c$. Similarly $b \neq a^{-2}$ since $b^2 = 1$.

If $c^2 = a$ then $b = ac \Rightarrow c^3 \Rightarrow c^6 = 1$ then refer to **SE5**.

Clearly $c^2 \neq a^{-1}$ since $b = ac \Rightarrow b = c$.

Also $c^3 \neq a^{-1}$ since $b = ac \Rightarrow b = c^{-2} \Rightarrow c^4 = 1$. This would contradict the fact that $o(c) \geq 6$.

Similarly if $c = a^2$ then $b = ac \Rightarrow b = a^3$. Then we would have $a^6 = 1$ and so $c^3 = 1$.

Clearly $c \neq a^{-2}$ since $b = ac \Rightarrow b = a$.

Thus none of them is admissible. Hence we may conclude that $\mathcal{P}$ is aspherical.

b) $o(c) \geq 3 (a = c^3$ if $o(c) \geq 6)$

Start off by assigning weights

$$\theta(\alpha_1) = 1 - 2\eta, \theta(\alpha_2) = 1, \theta(\alpha_3) = \theta(\alpha_4) = \eta$$

such that $o(c)(1 - 2\eta) = 2$. This guarantees that any admissible cycle involving just α_1 has a weight of at least two.. Note that any closed path involving α_2 twice or α_1 and α_2 at least once each has a weight of at least two.

Now we claim that any admissible cycle γ just involving α_3 and α_4 has a weight of at least two. Suppose that γ has weight less than two . Since γ has weight of at least $2\eta o(a)$ then

$$\begin{aligned} 2\eta o(a) < 2 \quad &\Rightarrow \quad (1 - \frac{2}{o(c)}) \cdot o(a) < 2 \\ &\Rightarrow \quad 1 < \frac{2}{o(a)} + \frac{2}{o(c)}. \end{aligned}$$

Also since we assume that $\frac{1}{o(a)} + \frac{1}{2} + \frac{1}{o(c)} \leq 1$ (refer hypothesis 4), then we would have

$$1 < \frac{2}{o(a)} + \frac{2}{o(c)} \leq 1$$

which is not possible.

Let γ be any admissible cycle of the form $\alpha_2(\alpha_3^{-1}\alpha_4)^{\pm q}$. For such γ to have weight less than two we require $1 + 2\eta q < 2$ and so we need $(1 - \frac{2}{o(c)})q < 1$ that is

$$q < \frac{o(c)}{o(c) - 2} = 1 + \frac{2}{o(c) - 2}.$$

Since $o(c) \geq 3$ then $q < 3$ and if $o(c) \geq 4$ then $q < 2$. Clearly $\alpha_2(\alpha_3^{-1}\alpha_4)^{\pm 1}$ is not admissible since $a \neq b^{\pm 1}$. If $o(c) = 3$ then we need to consider $q = 2$ that is $b = a^{\pm 2}$. Since $o(b) = 2$ then $b = ac \Rightarrow a = c$ if $b = a^2$. Thus $\alpha_2(\alpha_3^{-1}\alpha_4)^{\pm 2}$ is not admissible.

Then we just need to consider all cycles of the form $\alpha_1^{\pm p}(\alpha_3\alpha_4^{-1})^q$ for some p and q such that their total weights is less than two. We will show that they are not admissible. Note that $\alpha_1(\alpha_3\alpha_4^{-1})$ is not admissible since a and c are distinct. Also since we assume that $a \neq c^{-1}$ then $\alpha_1^{-1}(\alpha_3\alpha_4^{-1})$ is not admissible.

i) $o(c) = 3$

Since $\eta = \frac{1}{6}$ then we have to consider

$$\alpha_1^{\pm 2}(\alpha_3\alpha_4^{-1}), \alpha_1^{\pm 1}(\alpha_3\alpha_4^{-1})^2 \text{ and } \alpha_1^{\pm 1}(\alpha_3\alpha_4^{-1})^3$$

that is $c^2 = a^{\pm 1}, c = a^{\pm 2}$ and $c = a^{\pm 3}$.

Since $c^3 = 1$ then clearly $c^2 \neq a$. Otherwise hypothesis 1 would be satified.

Also $c^2 \neq a^{-1}$ since both a and c are distinct.

If $c = a^2$ then refer to **SE4**.

Now suppose that $c = a^{-2}$. Then $b = ac \Rightarrow b = a$ which is not possible.

If $c = a^3$ then $a^9 = 1$. Also $b = ac \Rightarrow b = a^4 \Rightarrow a^8 = 1$. Thus $a = 1$.

Now suppose that $c = a^{-3}$. Then $b = ac \Rightarrow b = a^2 \Rightarrow a^4 = 1$ Then we would have $a = 1$.

Thus none of them is admissible.

ii) $o(c) = 4$

Since $\eta = \frac{1}{4}$ then we have to consider

$$\alpha_1^{\pm 2}(\alpha_3\alpha_4^{-1}) \text{ and } \alpha_1^{\pm 1}(\alpha_3\alpha_4^{-1})^2$$

that is $c^2 = a^{\pm 1}$ and $c = a^{\pm 2}$.

Clearly $c^2 \neq a^{\pm 1}$ for otherwise since $b = ac$ we would have $b = c^3 = c^{-1} \Rightarrow b = c$.

If $c = a^2$ then $a^8 = 1$. Also $b = ac \Rightarrow b = a^3 \Rightarrow a^6 = 1$. Then we have $a^2 = 1$ which contradicts the fact that $o(a) \geq 4$.

We can not have relation $c = a^{-2}$ since $b = ac$ would imply $b = a$.

Thus none of them is admissible.

iii) $o(c) = 5$

Since $\eta = \frac{3}{10}$ then we have to consider

$$\alpha_1^{\pm 3}(\alpha_3\alpha_4^{-1}), \alpha_1^{\pm 2}(\alpha_3\alpha_4^{-1}) \text{ and } \alpha^{\pm 1}(\alpha_3\alpha^{-1})^2$$

that is $c^3 = a^{\pm 1}, c^2 = a^{\pm 1}$ and $c = a^{\pm 2}$.

If $c^3 = a$ then $b = ac \Rightarrow b = c^4 \Rightarrow c^8 = 1$. Since $o(c) = 5$, this is not possible.

Suppose that $c^3 = a^{-1}$. Then $b = ac \Rightarrow b = c^{-2} \Rightarrow c^4 = 1$. Again this is not possible.

If $c^2 = a$ then $b = ac \Rightarrow b = c^3 \Rightarrow c^6 = 1 \Rightarrow c = 1$.

Also $c^2 \neq a^{-1}$ since $b = ac \Rightarrow b = c$.

Suppose that $c = a^2$, then $a^{10} = 1$. Also $b = ac \Rightarrow b = a^3 \Rightarrow a^6 = 1$ and hence $a^2 = 1$, contradicts the fact that $o(a) \geq 4$.

If $c = a^{-2}$ then $b = ac \Rightarrow b = a$.

Thus none of them is admissible.

iv) $o(c) \geq 6, a = c^3$

Note that $a = c^3, b = ac \Rightarrow b = c^4$ and hence $c^8 = 1$. Since $o(c) \geq 6$ then clearly $o(c) = 8$. Thus $\eta = \frac{1}{4}$ and so we need to consider

$$\alpha_1^{\pm 2}(\alpha_3\alpha_4^{-1}) \text{ and } \alpha^{\pm 1}(\alpha_3\alpha_4^{-1})^2$$

that is $c^2 = a^{\pm 1}$ and $c = a^{\pm 2}$.

Since $a = c^3$ and $o(c) = 8$ then clearly $c^2 \neq a^{\pm 1}$.

Suppose that $c = a^2$ and so $a = c^3 \Rightarrow a^5 = 1$. Also $b = ac \Rightarrow b = a^3 \Rightarrow a^6 = 1$. Then $a = 1$.

Clearly $c \neq a^{-2}$ since $b = ac \Rightarrow b = a$.

Thus none of them is admissible.

When is $\mathcal{P}$ not aspherical?

We have to show that if one of these holds:

1. $a = c^{-1}$

2. $b = ac$ and $\frac{1}{p} + \frac{1}{2} + \frac{1}{k} > 1$ where $o(a) = p$ and $o(c) = k$

then $\mathcal{P}$ is not aspherical.

If $a = c^{-1}$ then we have

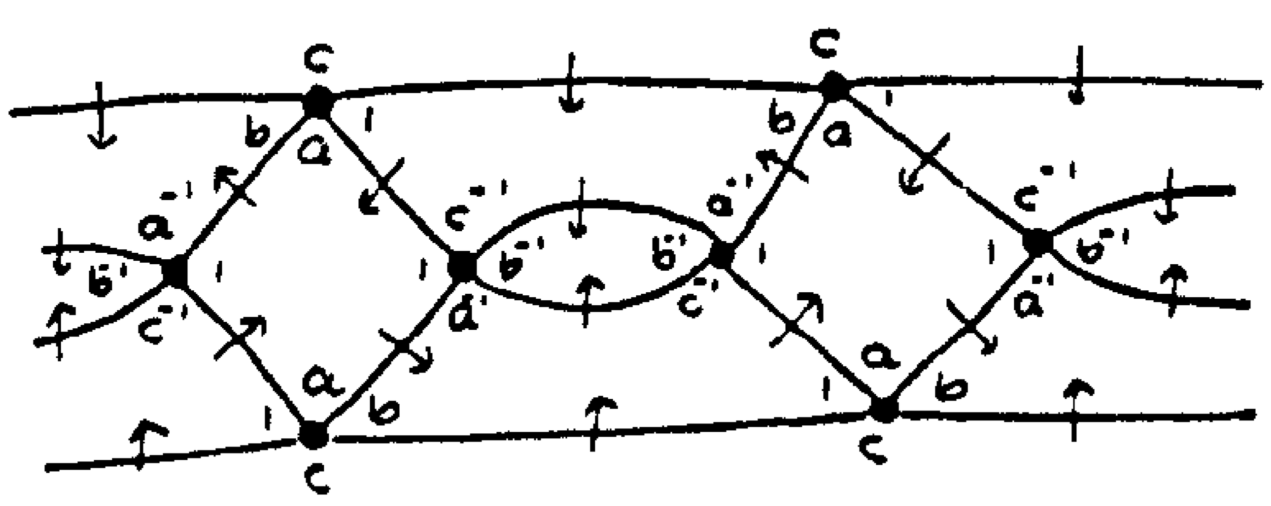
$$t^2atbt^{-1}c = 1 \Rightarrow t^2 = atb^{-1}t^{-1}a^{-1}.$$

Since $b^2 = 1$ then we have $t^4 = 1$ and hence (refer §4.3.5) $\mathcal{P}$ is not aspherical.

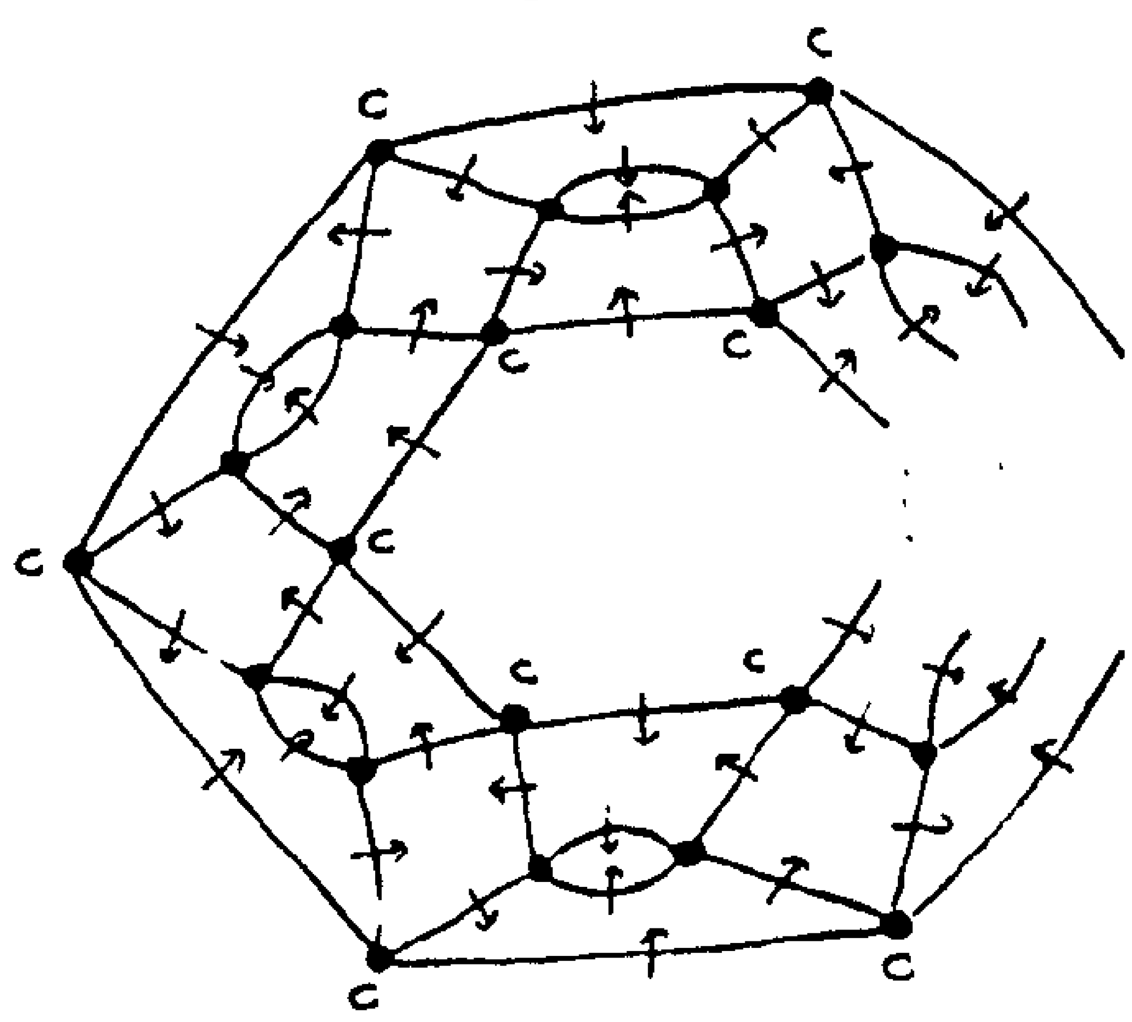
For the second case, consider

i) $o(a) = 2, o(c) < \infty$

Consider

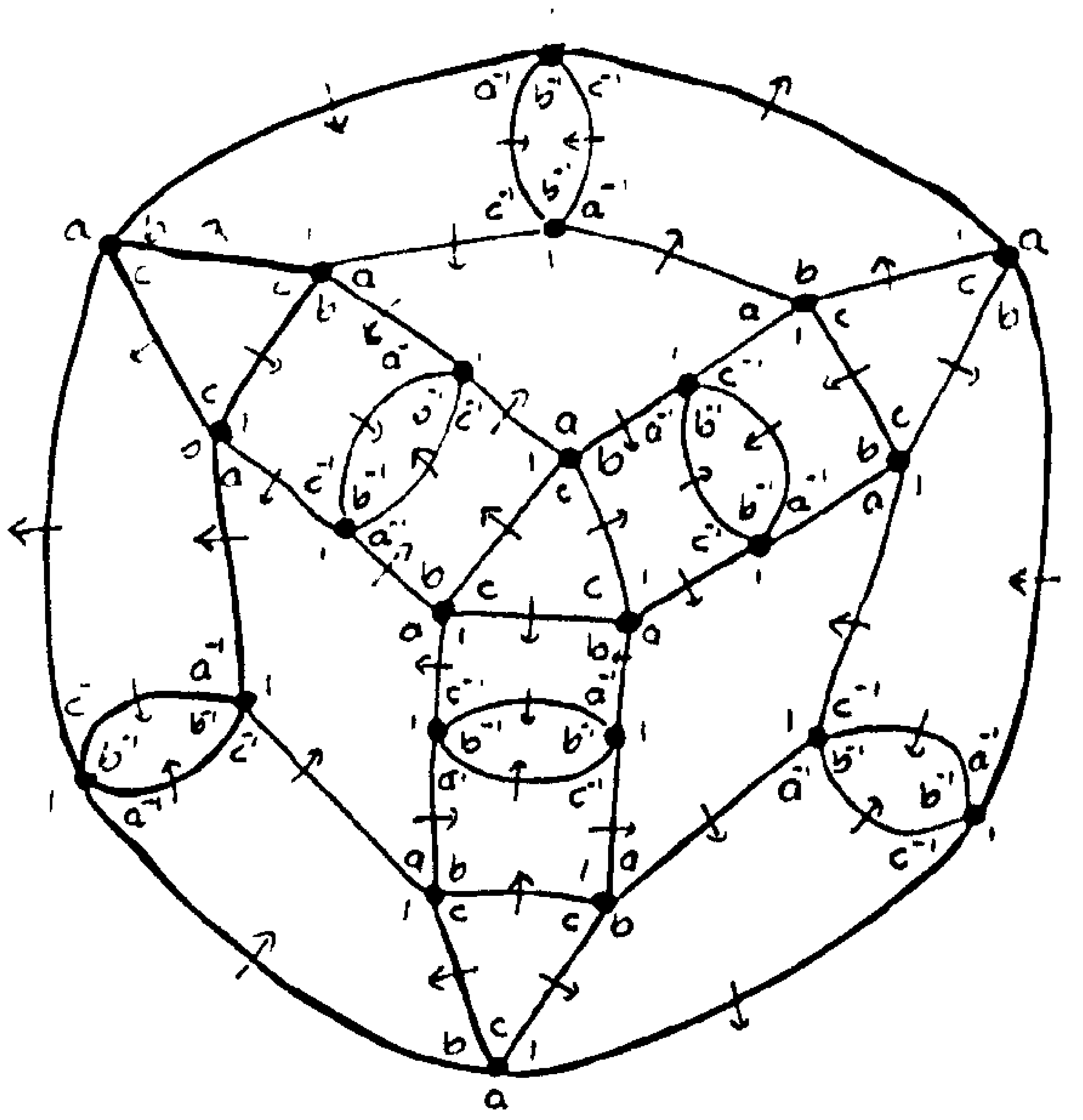


and then join this according to the order of c.



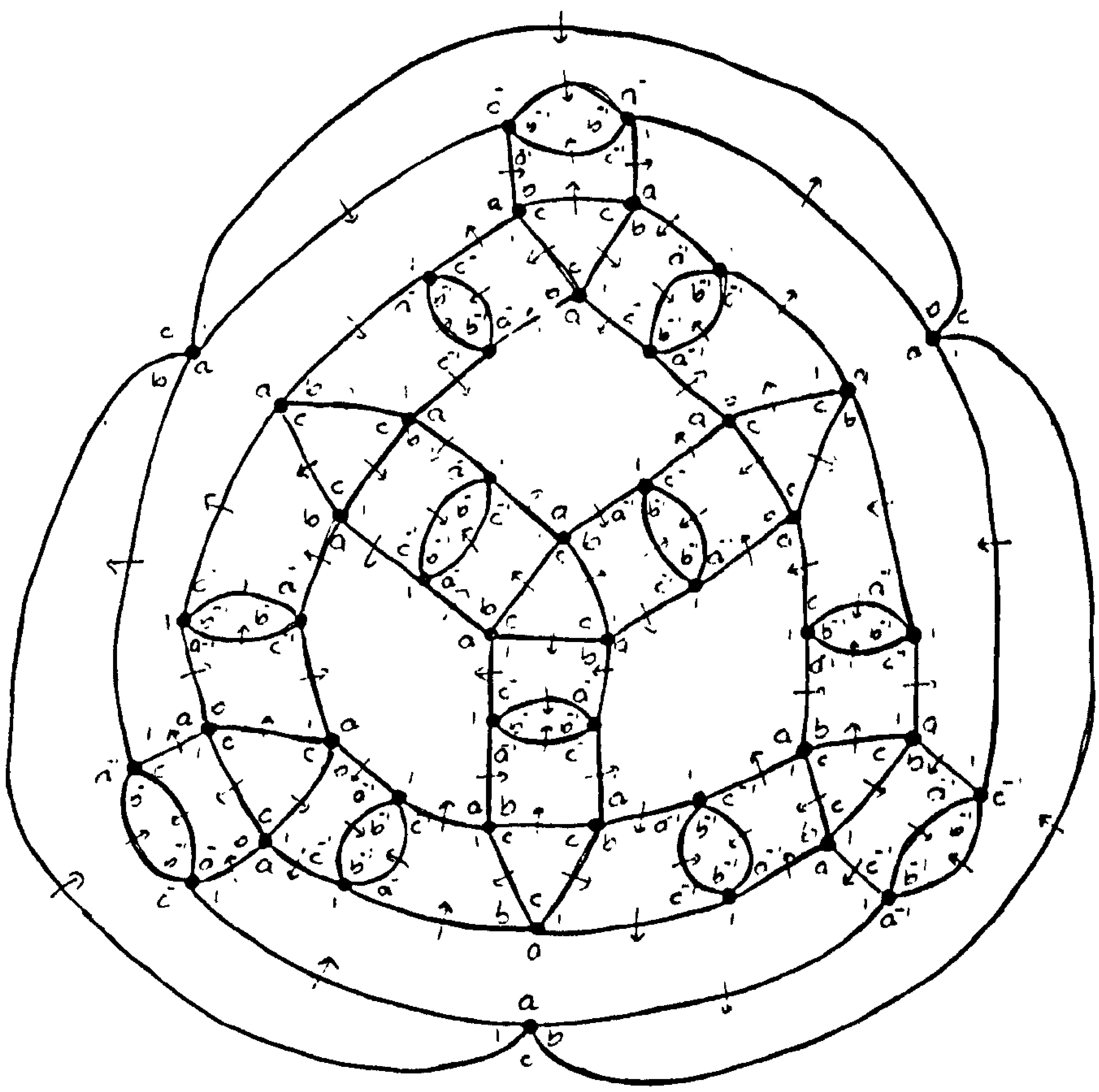
If $o(c) = q$ then (refer §4.3.6) $\chi(\lambda_{\mathbb{P}}) = 2q(1-x) \neq 0$. Thus $\mathbb{P}$ is not degenerate and hence $\mathcal{P}$ is not aspherical.

ii) $o(a) = 3, o(c) = 3$



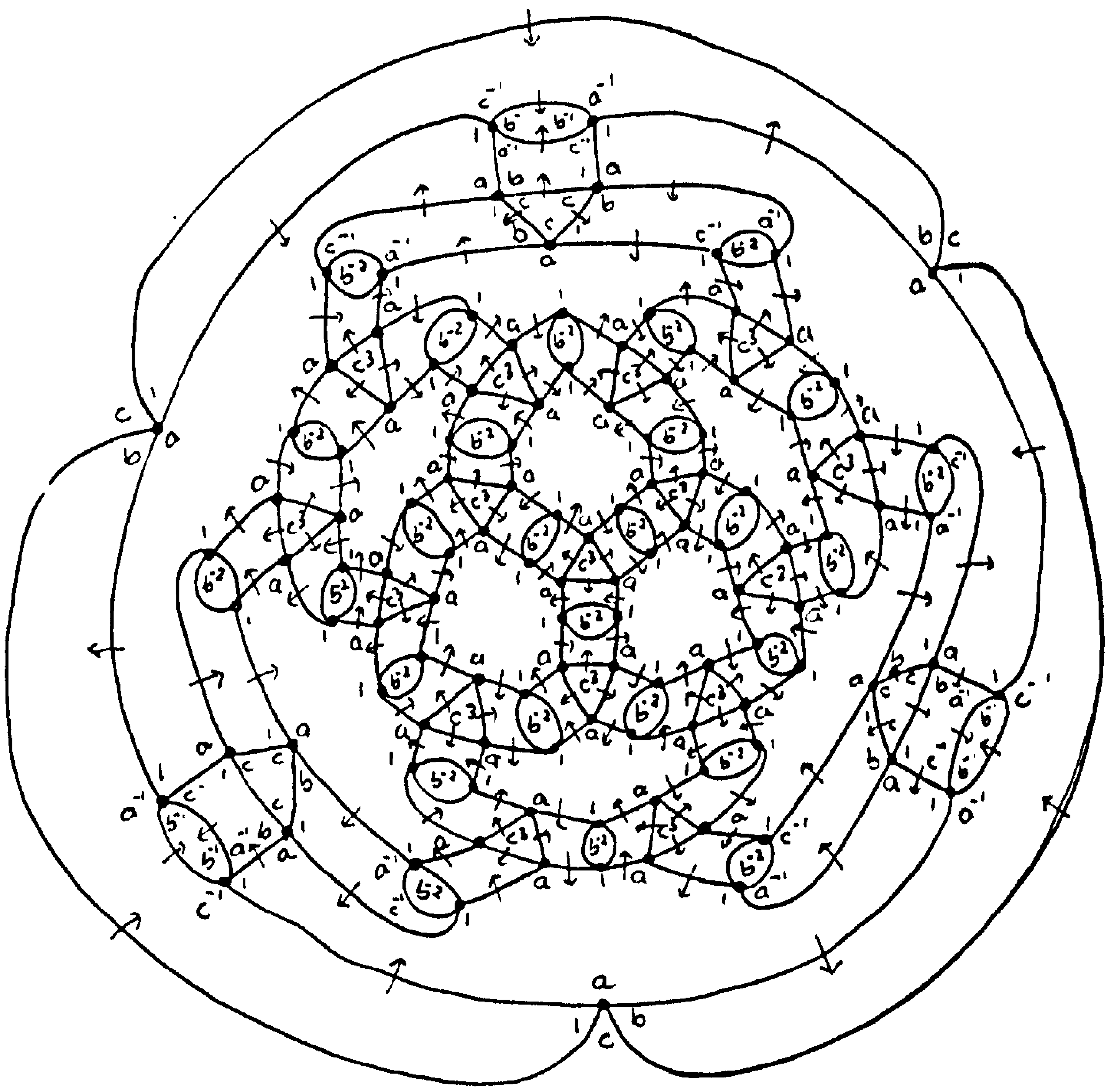
Note that (refer §4.3.6) $\psi(\lambda_{\mathbb{P}}) = 12(1 - x) \neq 0$. Thus $\mathcal{P}$ is not aspherical.

iii) $o(a) = 4, o(c) = 3$



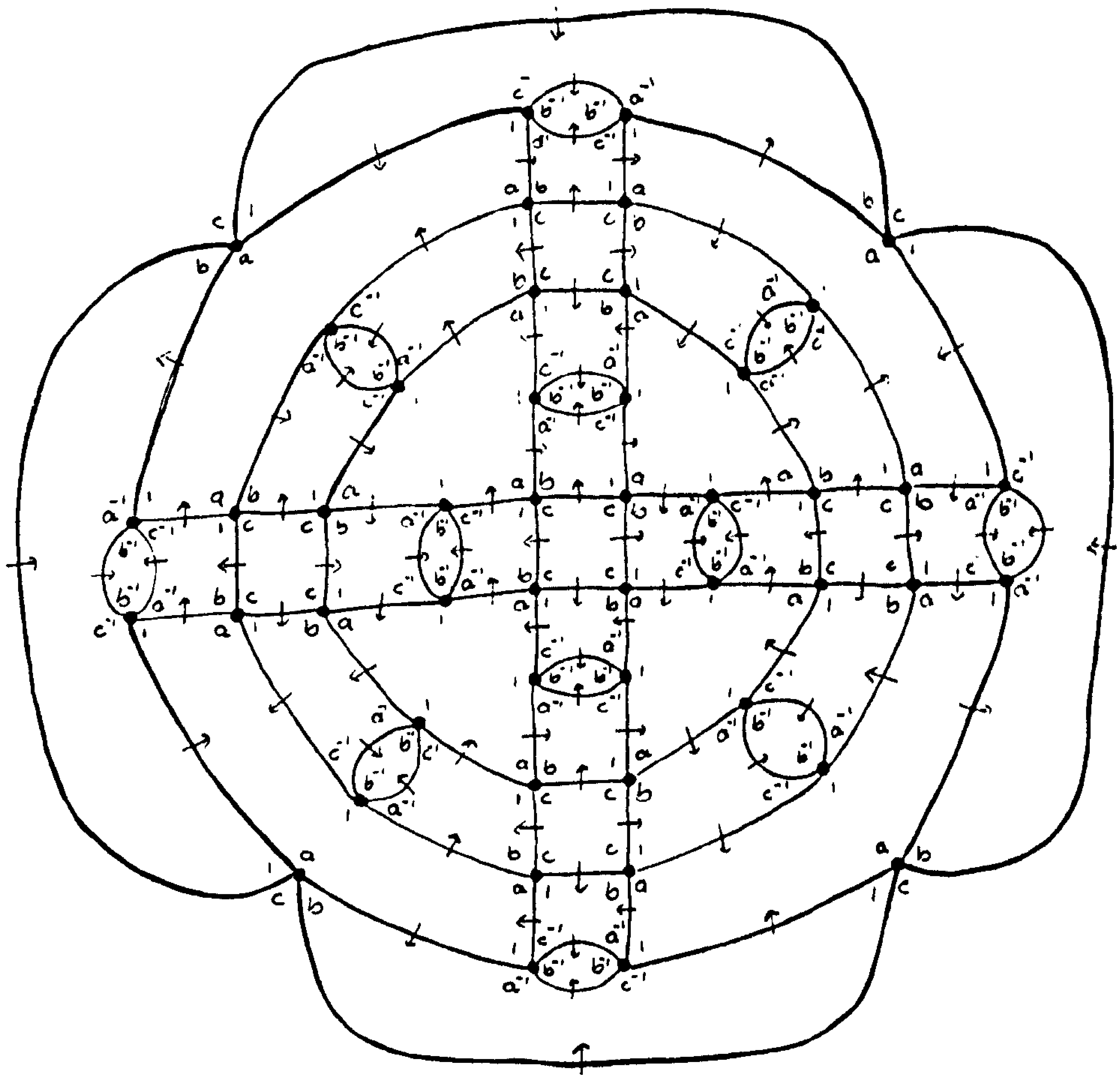
Here we have (refer §4.3) $\psi(\lambda_{\mathbb{P}}) = 24(1 - x) \neq 0$. Thus $\mathcal{P}$ is not aspherical.

iv) $o(a) = 5, o(c) = 3$



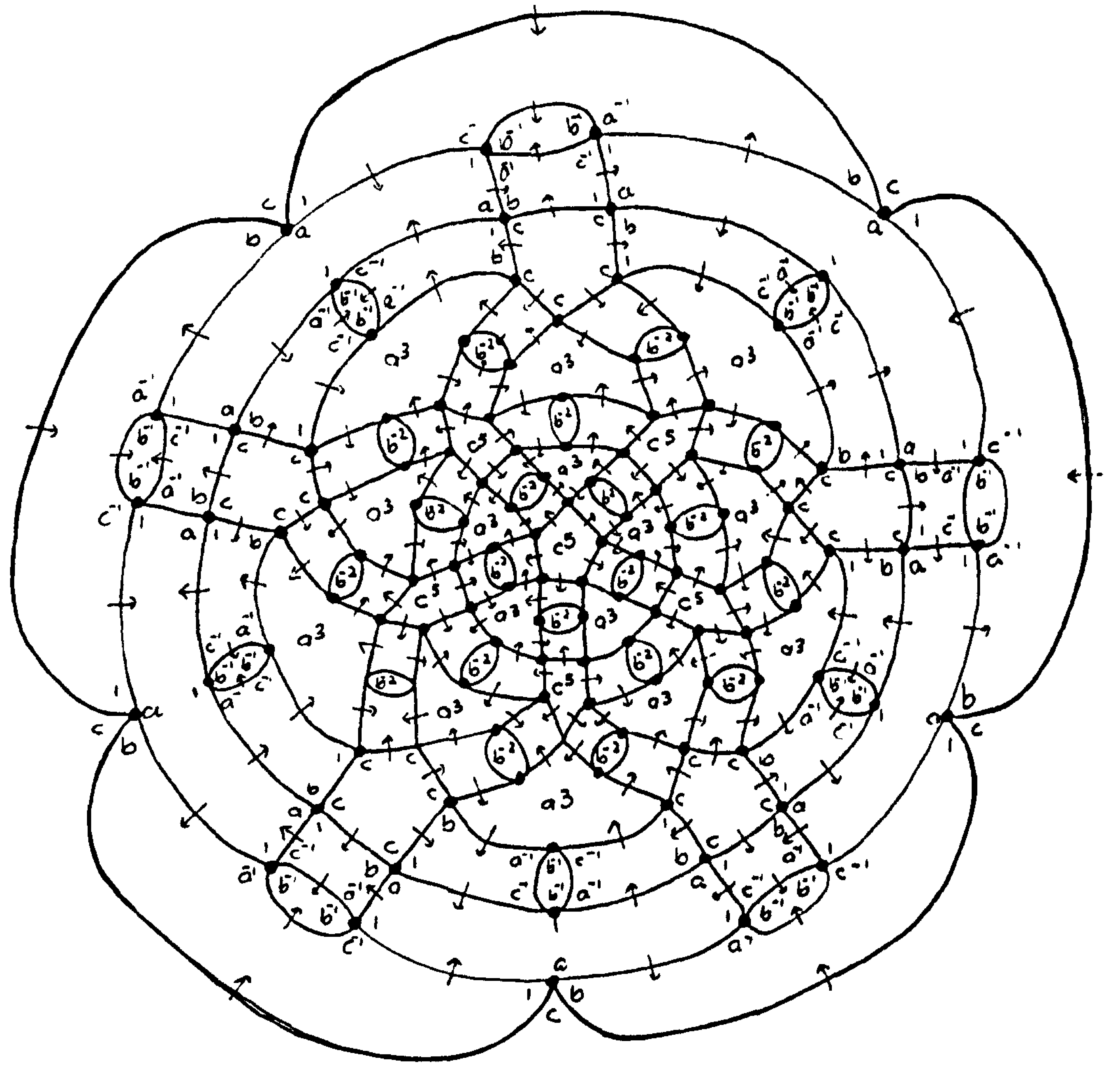
Thus (refer §4.3.6) $\psi(\lambda_{\mathbb{P}}) = 60(1 - x) \neq 0$ and hence $\mathcal{P}$ is not aspherical.

v) $o(a) = 3, o(c) = 4$



Thus (refer §4.3.6) $\psi(\lambda_{\mathbb{P}}) = 24(1 - x) \neq 0$. Since $\mathbb{P}$ is not degenerate then $\mathcal{P}$ is not aspherical.

vi) $o(a) = 3, o(c) = 5$



Thus (refer §4.3.6) $\psi(\lambda_{\mathbb{P}}) = 60(1-x) \neq 0$. Since $\mathbb{P}$ is not degenerate then $\mathcal{P}$ is not aspherical.

6.4.5 The subsubcase $o(b) = 3, 3 \leq o(c) < \infty$

When is $\mathcal{P}$ aspherical?

Assume that none of the relations 1-4 in the statement of Theorem 6.4.1 holds. Assign the weights

$$\theta(\alpha_1) = \theta(\alpha_2) = \frac{2}{3}, \theta(\alpha_3) = \theta(\alpha_4) = \frac{1}{3}.$$

Then the only possible admissible cycle with weight less than two is $(\alpha_3\alpha_4^{-1})^{\pm 2}$. So if $o(a) \neq 2$ then $\mathcal{P}$ must be aspherical.

If $o(c) \geq 6$, then assign the weights

$$\theta(\alpha_1) = \frac{1}{3}, \theta(\alpha_2) = \frac{2}{3}, \theta(\alpha_3) = \theta(\alpha_4) = \frac{1}{2}.$$

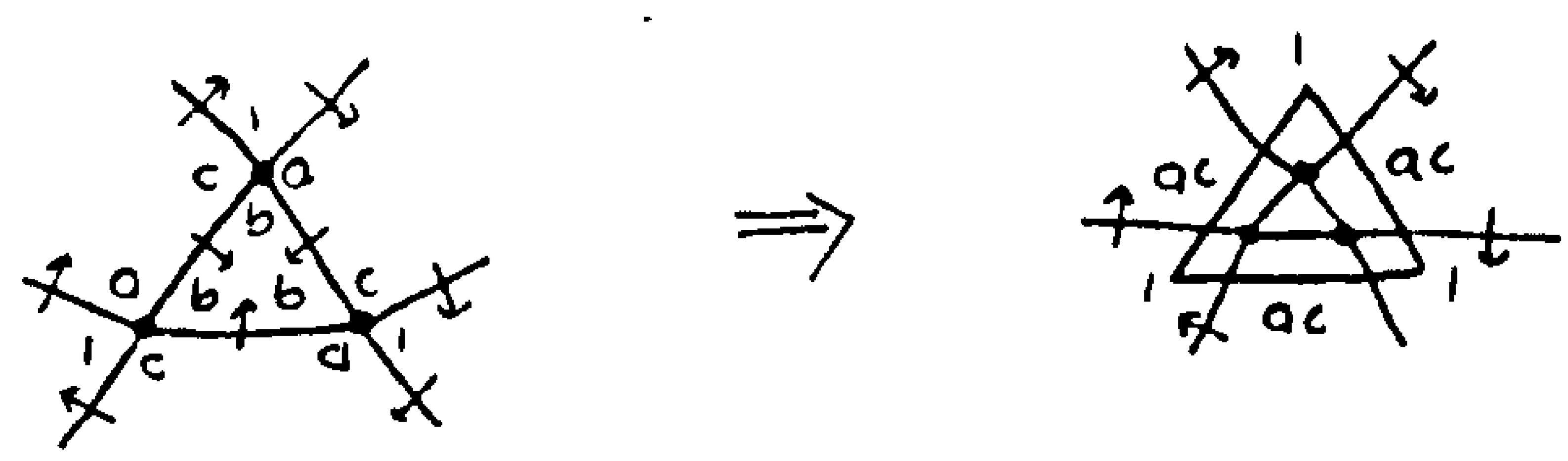
Since $a \neq b^{-1}$ or c^{-1} then one may check that all admissible cycles have weights of at least two. Thus by Theorem 4.3.3, $\mathcal{P}$ must be aspherical.

From the above two arguments, we know that we only have to consider:

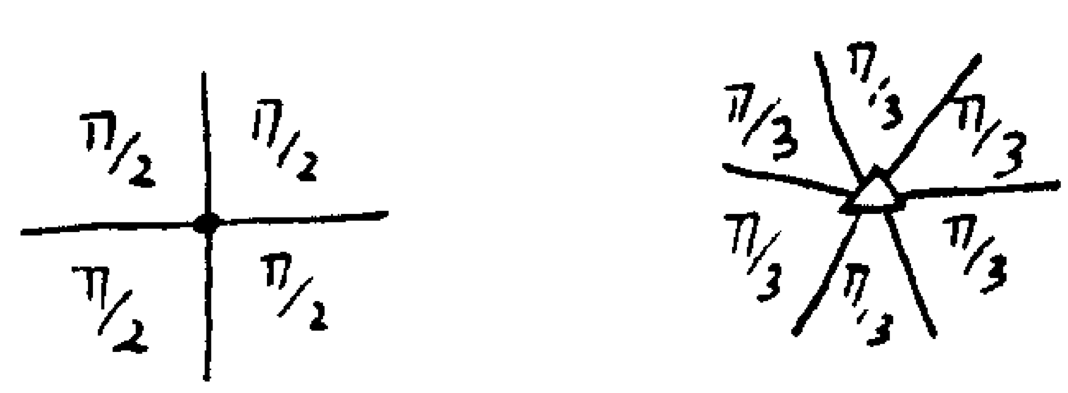
1. $o(c) = 5, o(a) = 2$
2. $o(c) = 4, o(a) = 2$
3. $o(c) = 3, o(a) = 2$

We also assume that $b \neq ac$ in any of these three cases because otherwise hypothesis 4 is satisfied.

Now suppose that there were a reduced strictly spherical picture $\mathbb{P}$ over $\mathcal{P}$. Consider any 3-region



So we have a new derived spherical picture $\mathbb{P}'$. Then assign the angle function



to $\mathbb{P}'$. Since every disc is flat then by Lemma 4.3.4, there is an exceptional region Φ' with positive curvature. If Φ' has valence m, then $(m-2)\pi < m\pi/2$ which means that $m < 4$. Finding the possibilities for Φ (refer Appendix B.5.1), we may conclude that $\mathcal{P}$ is aspherical except possibly $a = c^{-2}$.

From the above weight test argument, we know that $\mathcal{P}$ is aspherical except possibly if $3 \leq o(c) \leq 5$ and $a^2 = 1$. Combining with the curvature test ($a = c^{-2}$) immediately above we may conclude that $\mathcal{P}$ is aspherical except possibly $a^2 = b^3 = c^4 = 1$ and $a = c^{-2}$. This is an exceptional case that we can not decide (refer **SE3**).

When is $\mathcal{P}$ not aspherical?

We have to show that if one of these holds:

1. $a = c^{-1}, o(b) = 3$

2. $o(a) = 2, b = ac, o(c) = 3, 4$ or 5

then $\mathcal{P}$ is not aspherical.

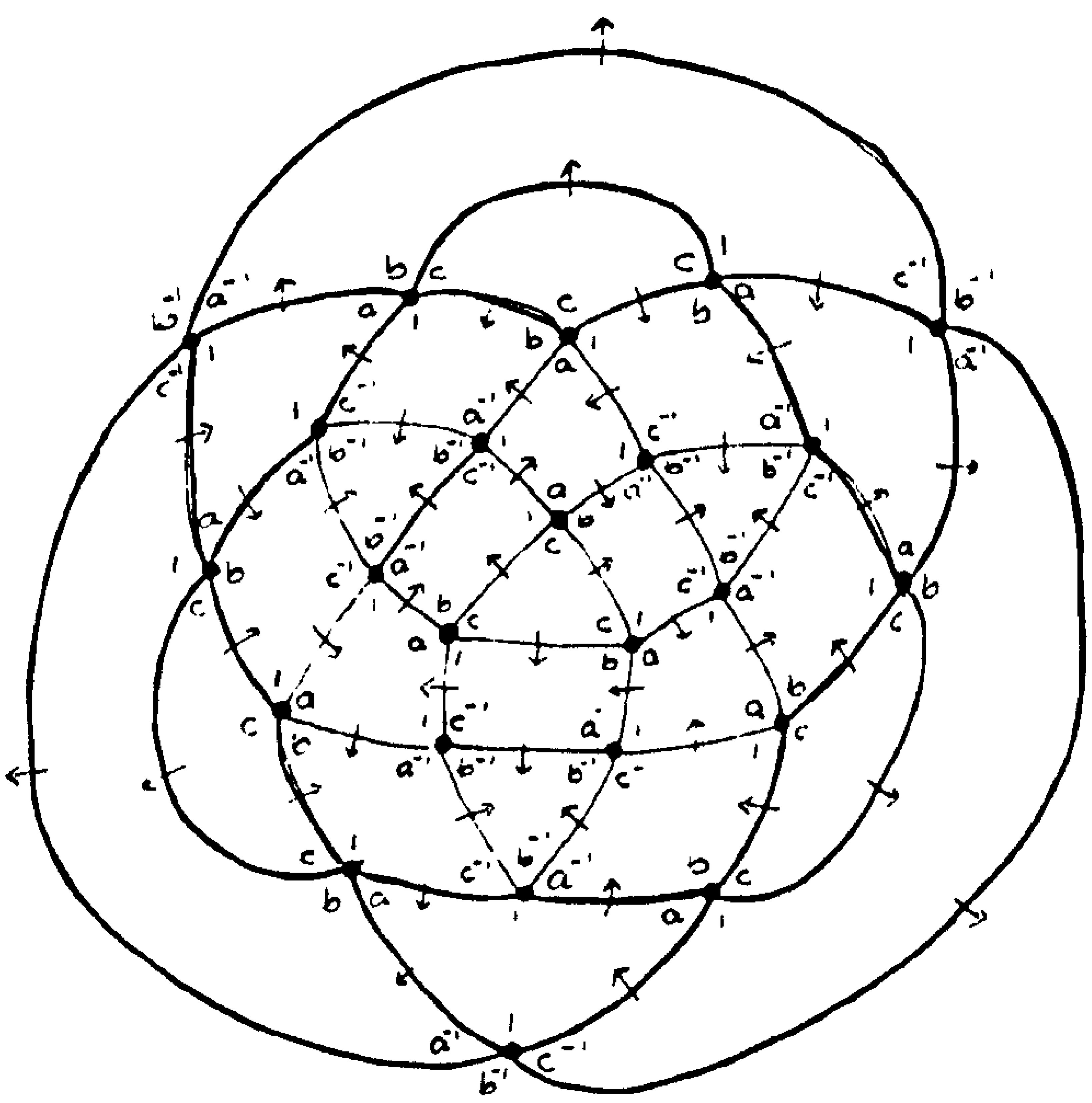
If $a = c^{-1}$ then

$$t^2atbt^{-1}c = 1 \Rightarrow t^2 = atb^{-1}t^{-1}a^{-1}$$

and so t has finite order. Hence (refer §4.3.5) $\mathcal{P}$ is not aspherical.

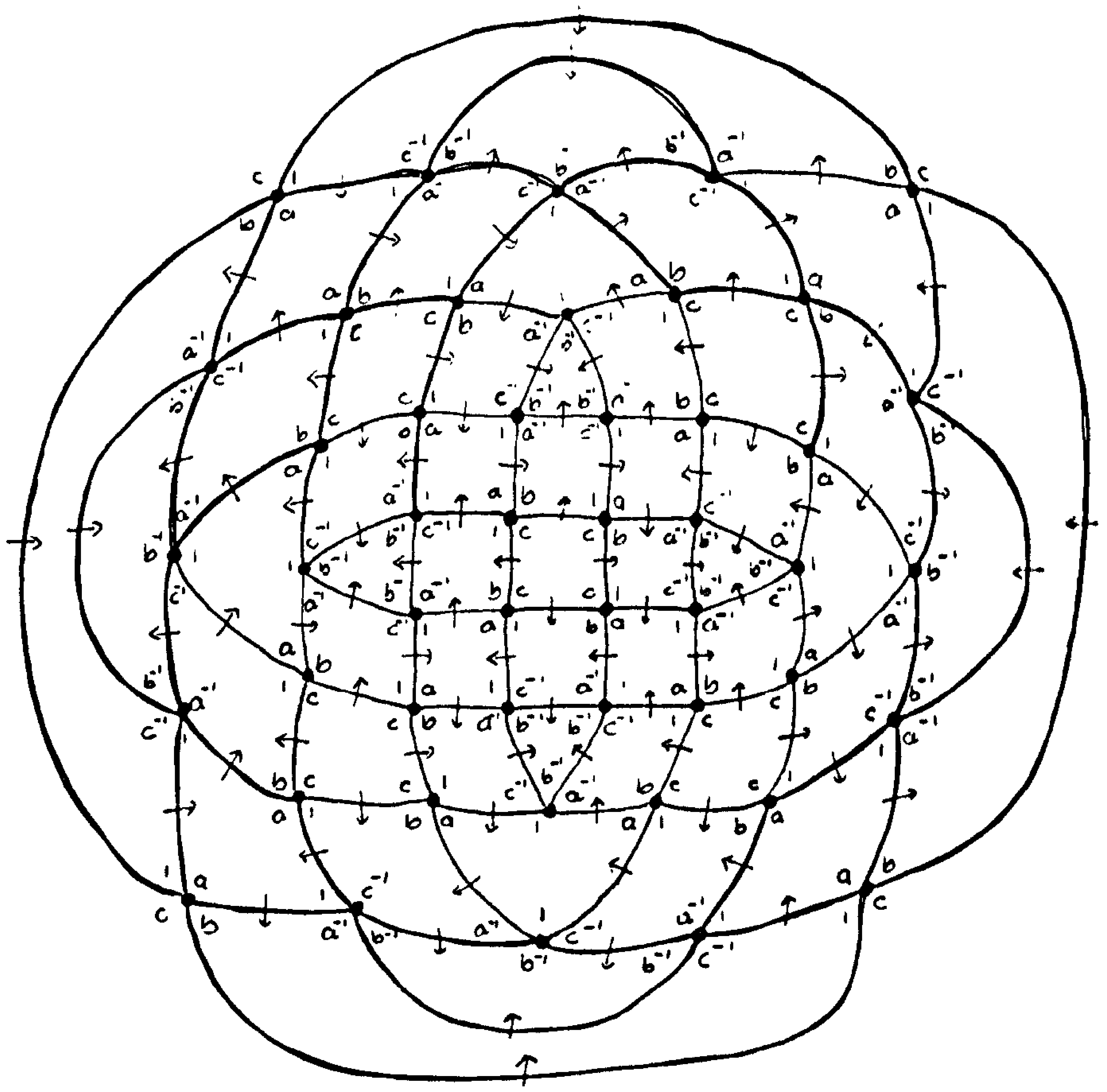
For the second case, we draw reduced strictly spherical pictures

i) $o(c) = 3$



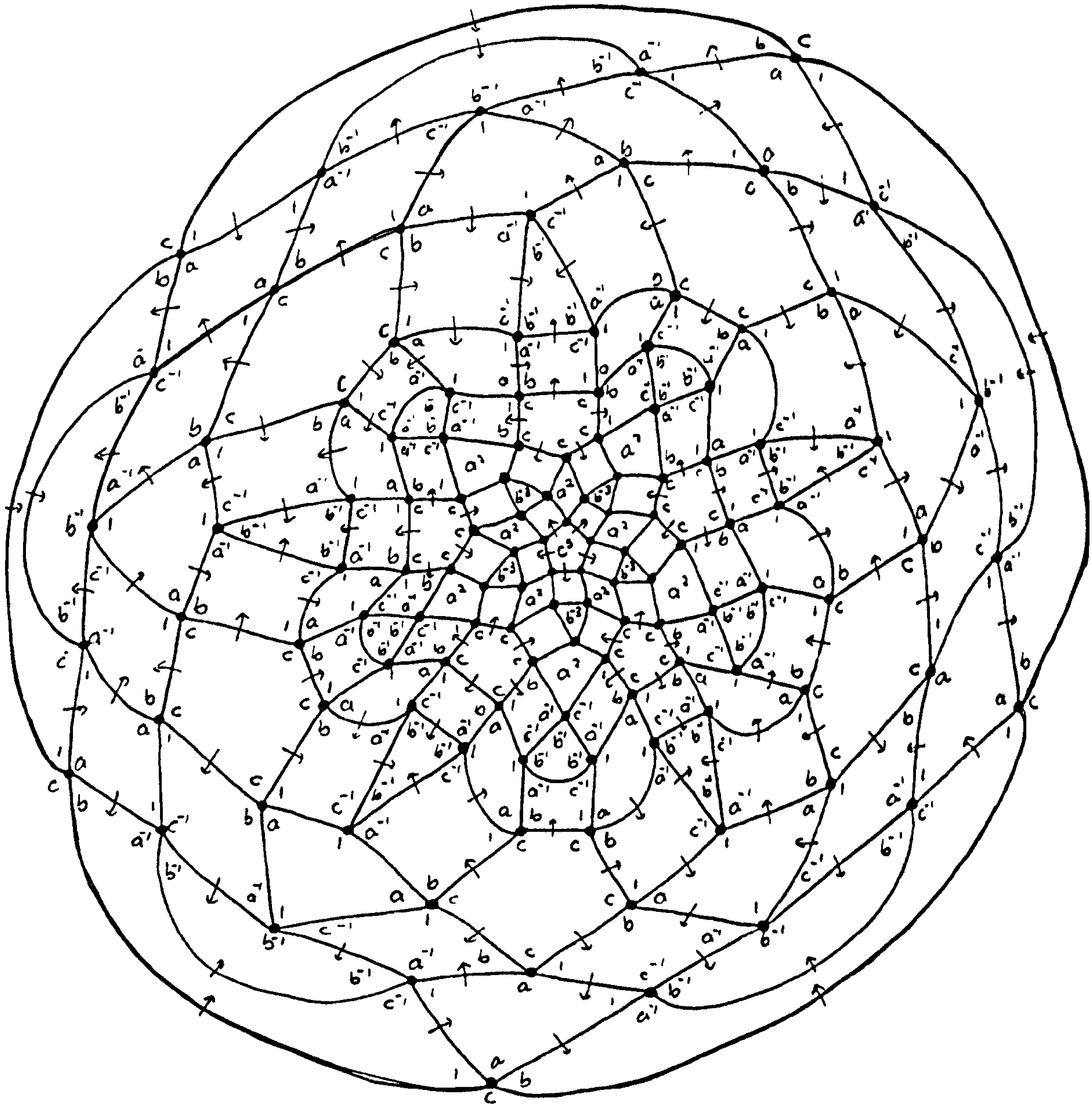
and so (refer §4.3.6) $v(\lambda_{\mathbb{P}}) = 12(1 - x) \neq 0$. Then $\mathcal{P}$ is not aspherical.

ii) $o(c) = 4$



and so (refer §4.3.6) $\psi(\lambda_{\mathbb{P}}) = 24(1 - x) \neq 0$. Thus $\mathcal{P}$ is not aspherical.

iii) $o(c) = 5$



Then $\mathcal{P}$ is not aspherical since $\psi(\lambda_{\mathbb{P}}) = 60(1 - x) \neq 0$.

6.4.6 The subsubcase $3 < o(b), o(c) < \infty$

We have to show that $\mathcal{P}$ is aspherical if and only if $a \neq c^{-1}$.

Suppose that $a \neq c^{-1}$. Then assign the weights

$$\theta(\alpha_1) = \theta(\alpha_2) = \theta(\alpha_3) = \theta(\alpha_4) = \frac{1}{2}.$$

Since $a \neq b^{-1}$ or c^{-1} then all admissible cycles have weights of at least two. Hence by Theorem 4.3.3, $\mathcal{P}$ is aspherical.

If $a = c^{-1}$ then

$$t^2atbt^{-1}c = 1 \Rightarrow t^2 = atb^{-1}t^{-1}a^{-1}.$$

Since t^2 is conjugate to b^{-1} of finite order, then t also has finite order and hence (refer §4.3.5) $\mathcal{P}$ is not aspherical.

6.5 Exceptions

The following are all special exceptions that we still can not decide.

SE1	$t^2atbt^{-1}b:$	$o(a) = 2, o(b) = 3$
SE2	$t^2atbt^{-1}a:$	$o(b) = 3, o(a) = 4$
SE3	$t^2atbt^{-1}c:$	$a^2 = b^3 = 1$ and $a = c^{-2}$
SE4	$t^2atbt^{-1}c:$	$o(c) = 3, o(b) = 2, b = ac, c = a^2$
SE5	$t^2atbt^{-1}c:$	$o(c) = 6, o(b) = 2, b = ac, a = c^2$

We also can not decide for the following family of exceptions:

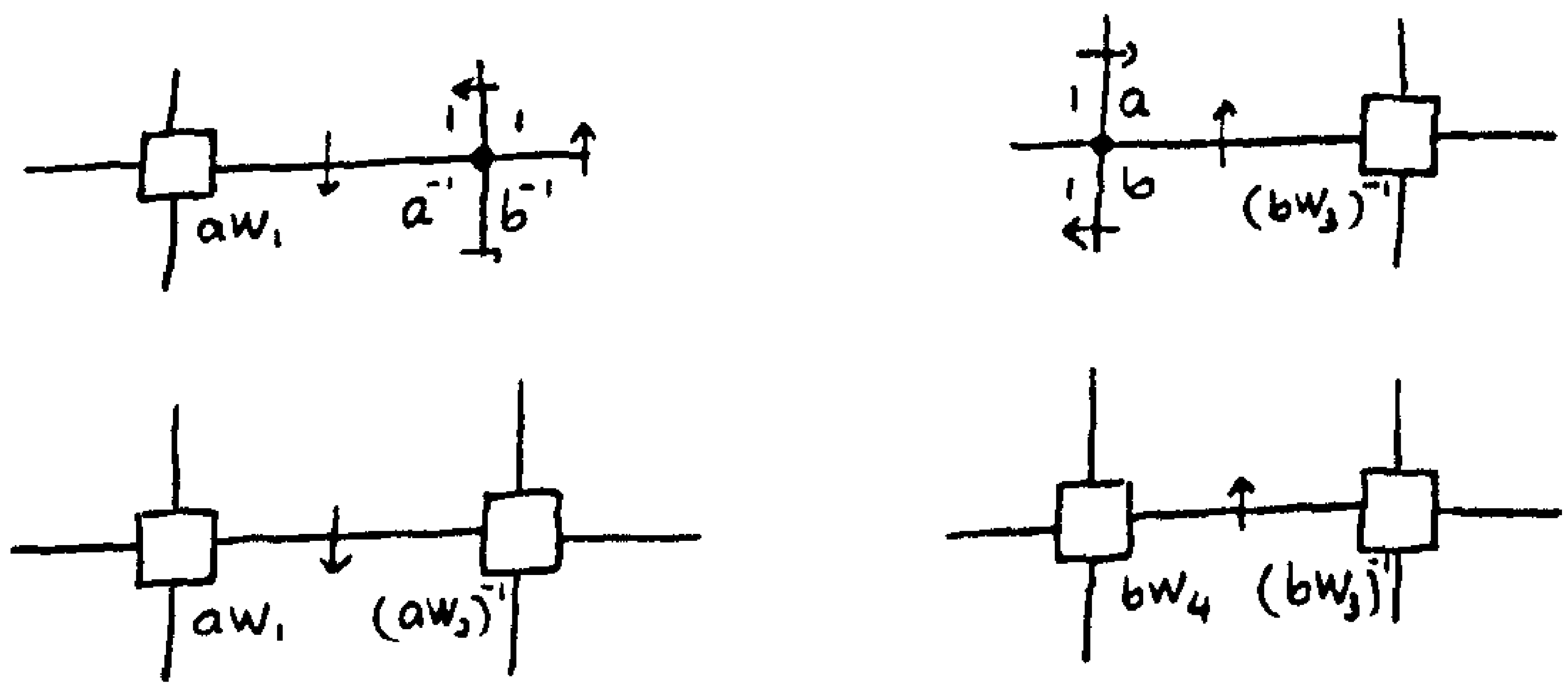
FE1	$t^3at^{-1}b:$	$o(a) = 2, o(b) \geq 4$ and $ab = ba$
FE2	$t^3at^{-1}b:$	$3 \leq o(a) < \infty$ and $a = b^2$
FE3	$t^2atbt^{-1}b:$	$o(a) < \infty, o(b) = 2$
FE4	$t^2atbt^{-1}a:$	$o(b) = 2$
FE5	$t^2atbt^{-1}a:$	$o(a) = 3, o(b) < \infty$
FE6	$t^2atbt^{-1}a:$	b has finite order and
		$b \neq a^{-1}, a^2, a^{\pm 3}, a^{\pm 4}$ or
		$b^2 \neq a^{\pm 2}$ or $a \neq b^{\pm 2}, b^{\pm 3}$
FE7	$t^2atbt^{-1}c:$	$a = c^{-2}$ of finite order, $b^2 = 1$ and $o(c) \geq 4$

Appendix A

Reference for Chapter 5

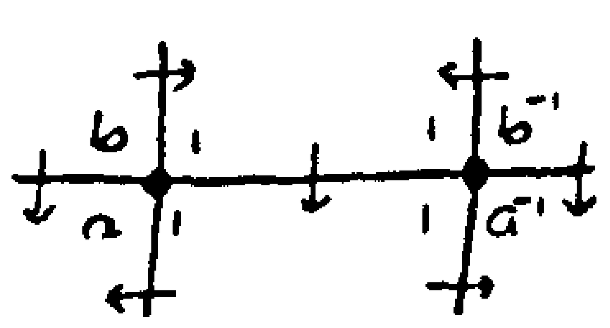
Let $\mathbb{P}$ be any reduced strictly spherical picture over $\mathcal{P}$. Thus we do not have the following

Hence in any first derived picture $\mathbb{P}'$, we do not have the following

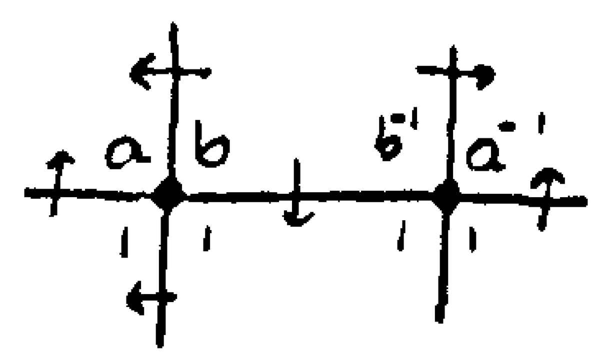


A^*

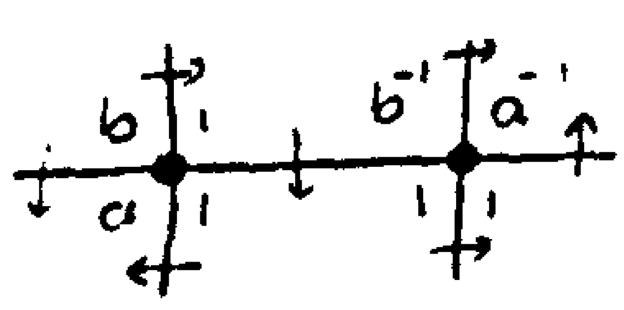
for some words W_1, W_2, W_3 and W_4 (possibly empty). Also note that



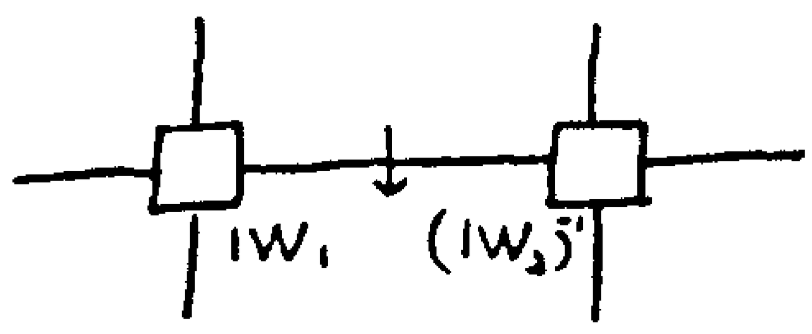
or



is not possible but

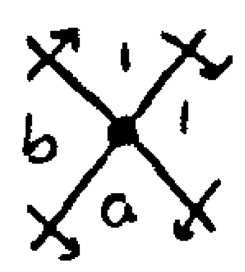


is possible and so we may have

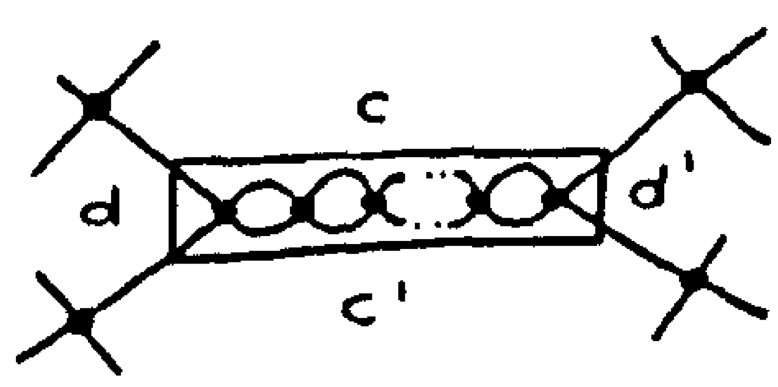


A.1 Reference for §5.2.2

We have the following



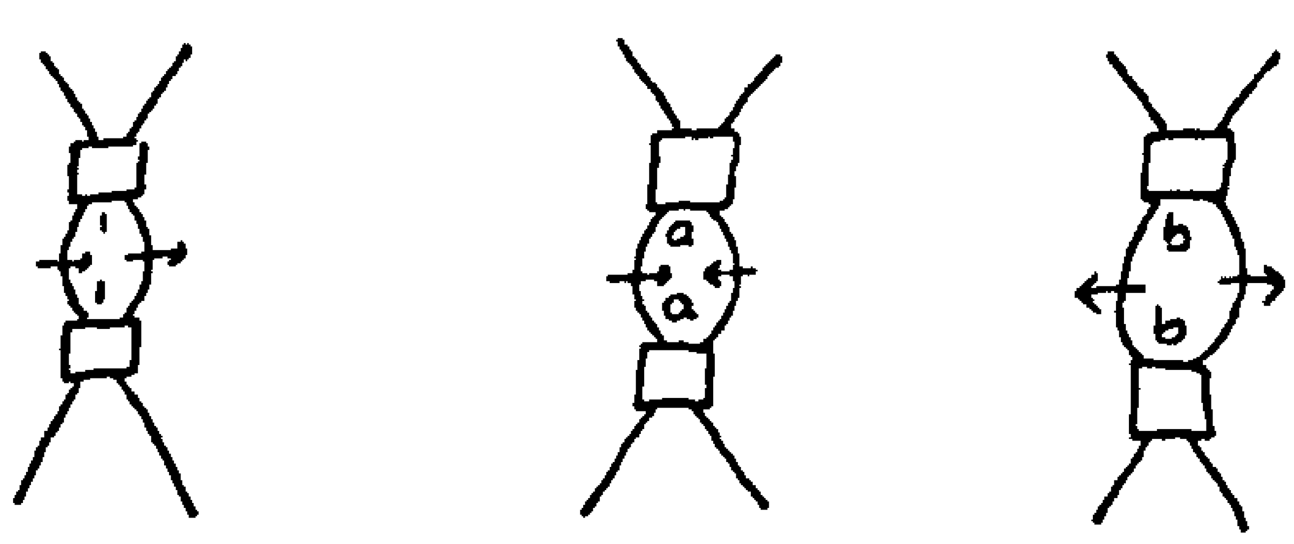
and 'square' discs of the form



where c and c' must have labels of the form

$$(ab^{-1}ab^{-1}\cdots ab^{-1})^{\pm1}, (ab^{-1}ab^{-1}\cdots ab^{-1}a)^{\pm1} \text{ or } (b^{-1}ab^{-1}a\cdots b^{-1}ab^{-1})^{\pm1}$$

while d and d' must have labels $1, a, a^{-1}, b$ or b^{-1}. Note that we have to find possible labels for Φ' of valence two and three. Note that there is no region in $\mathbb{P}'$ of valence two as follows:



since they would contradict our choice of 'square' discs. Thus aa and bb are not possible for Φ'. Clearly we can not have a region with label aaa or bbb since $o(a)$ =

$o(b) = 2$. Hence any possible label must be a mixture of c, c', d and d'. Since $o(a) = o(b) = 2$ and any product of labels in any region is 1 in H, then we obtain

$$ab^{-1}ab^{-1}\cdots ab^{-1} = 1, ab^{-1}ab^{-1}\cdots ab^{-1}a = 1 \text{ or } b^{-1}ab^{-1}a\cdots b^{-1}ab^{-1} = 1.$$

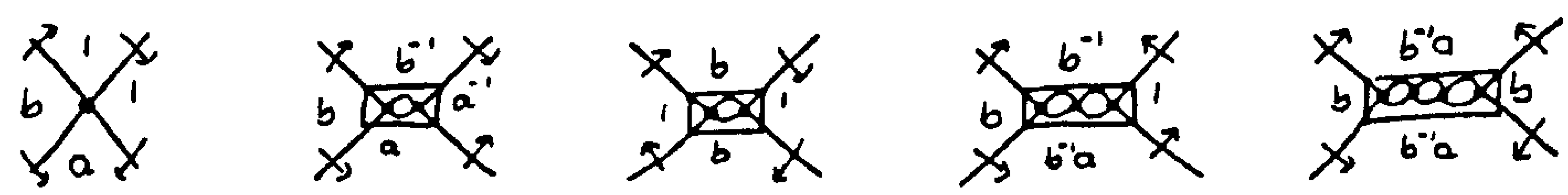
The first form is not possible since ab^{-1} has infinite order while the last two forms are not possible since a and b are not trivial. Thus there is no possible label for Φ'.

A.2 Reference for §5.2.3

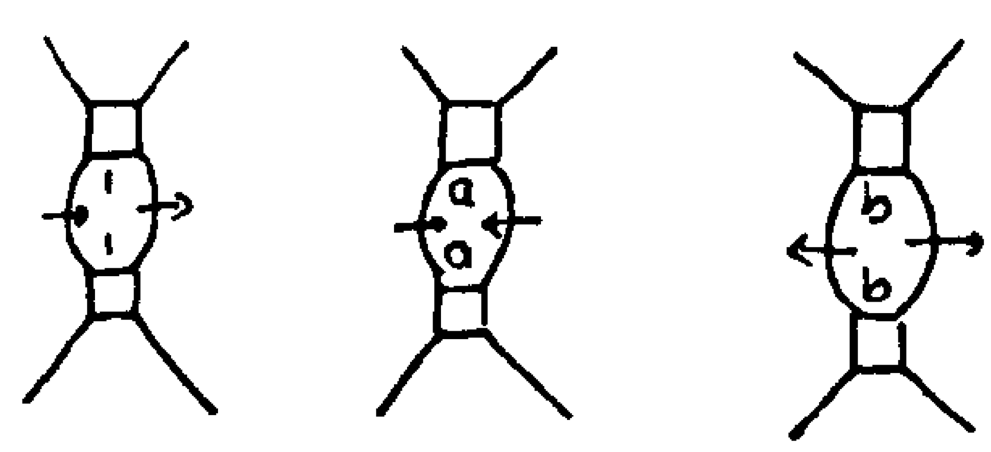
Throughout this section, all restrictions in A* is applicable.

A.2.1 Possibilities for Φ'

We have the following discs:



We will find the possible labels for Φ' of valence two and three. Any possible label will give a relation and so most of them can be eliminated immediately. We mark * if they seem possible and leave unmarked if they are obviously not possible. Here we also assume that a and b are distinct such that $a \neq b^{-1}, b^2$ or b^3 while $a^2 = 1$. As in Appendix A.1, we do not have the following



valence 2

$1b^{-1}$	ba	$b^{-1}ab$	ab	$a1$	ba	$b^{-1}aa$
$1b^{-1}a$	ba^{-1}	$b^{-1}a$	ab^{-1}	ab	$b1$	$b^{-1}ab^{-1}a*$
$1a$	$ba^{-1}b$	$b^{-1}a1$	ab^{-1}			

valence 3

$11a$	$1b1$	$bbb*$	$bbb*$	$b^{-1}a1b^{-1}$
$11a^{-1}$	$1b^{-1}1$	$b11$	bba	$b^{-1}ab^{-1}b^{-1}$
$11b$	$1ba$	$b1b$	$bba^{-1}b$	$b^{-1}aa1$
$11b^{-1}$	$1b^{-1}a$	$b1a^{-1}$	baa	$b^{-1}ab1$
$11b^{-1}a$	$1bb$	bab	$ba^{-1}a^{-1}$	$b^{-1}ab^{-1}b^{-1}$
$11b^{-1}aa$	$1b^{-1}b^{-1}a$	$ba1$	$ba^{-1}ba^{-1}b*$	$b^{-1}ab^{-1}a1*$
$1b^{-1}ab$	$1ab$		$b11$	$b^{-1}ab^{-1}aa$
$1b^{-1}ab^{-1}$	$1ab^{-1}$		bba	
	$1ab^{-1}a$		$b1b^{-1}a$	
	$1b^{-1}a$		$ba^{-1}b1$	
	$1b^{-1}b^{-1}$			
	$1b^{-1}b^{-1}a$			

aaa	abb	bba	$b^{-1}ab^{-1}ab^{-1}a*$
aab	$a1b$	$bb1$	$b^{-1}ab^{-1}aa$
aab^{-1}	$aa1$	baa	$b^{-1}ab^{-1}ab^{-1}*$
$aab^{-1}a$	aab	bab	$b^{-1}a11$
$a11$	$ab1$	$bab^{-1}a*$	$b^{-1}ab1$
$a1b^{-1}$	$aba^{-1}b*$	$b1a$	$b^{-1}aba*$
$a1b^{-1}a$	$ab^{-1}ab*$	$b1b$	$b^{-1}a1b^{-1}a*$
$ab1$	$ab^{-1}a1$	$b1b^{-1}a$	$b^{-1}a1a$
aba	$ab^{-1}1$		
	$ab^{-1}a^{-1}b*$		

Since any possible label gives a relation, we have the following possible relations:

1. $ab^{-1}ab^{-1} = 1$

2. $ab^{-1}ab^{-1}ab^{-1} = 1$

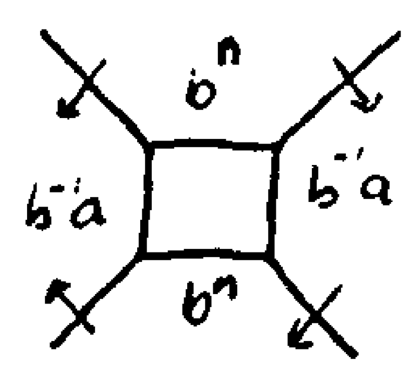
3. $b = (b^{-1}a)^2$

4. $b^3 = 1$

5. $bab^{-1}a = 1$

A.2.2 Possibilties for Φ''

We are dealing with the second derived picture $\mathbb{P}''$. Note that all restrictions A* is also applicable. In picture $\mathbb{P}''$, we have the same discs as in A.2.1 together with extra discs of the form



for some n. Since no other relations listed in (5.1) hold (except $(ab^{-1})^2 = 1$), then all possibilities listed in A.2.1 are no longer possible. Since we have extra discs, we also some extra possibilities for Φ'' as follows:

valence 2

$$b^n$$

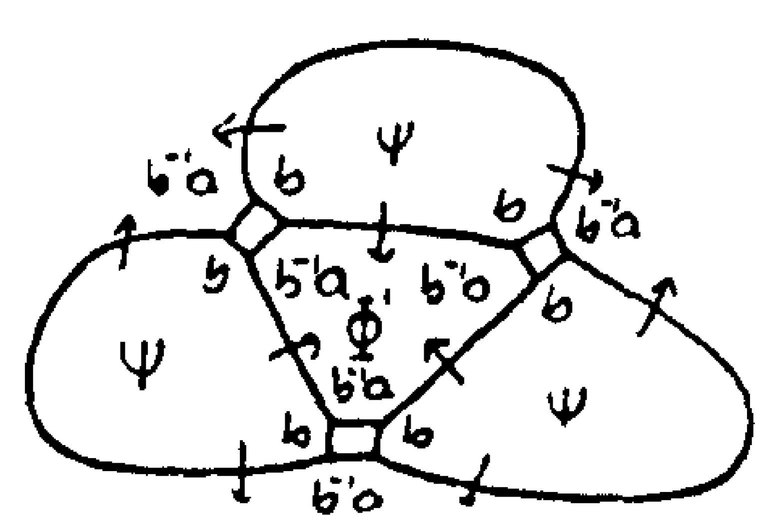
valence 3

$$b^n11 \quad b^n1b \quad b^n1a^{-1} \quad b^nb1 \quad b^nba^{-1} \quad b^na1$$

However all of these are not possible since we assume that b has infinite order.

A.2.3 Possibilities for Ψ ($o(ab^{-1}) = 3$)

Since there is only one possible region Φ' with label $b^{-1}ab^{-1}ab^{-1}a$ (refer A.2.1)



any possible label for Ψ must be in in the form bbW for some word W. We mark * if they seem possible and leave unmarked if they are obviously not possible. Here we assume that $o(b) \geq 6, b \neq (b^{-1}a)^2, ab^{-1}ab \neq 1$ and $a \neq b^{-1}, b^2$ or b^3 where $a^2 = 1$.

valence 2

$$b^2$$

valence 3

$$b^2bb$$

valence 4

$$b^2bb \quad b^21 1 \quad b^21a^{-1} \quad b^2a1 \quad b^2b1 \quad b^2ba^{-1}$$

valence 5

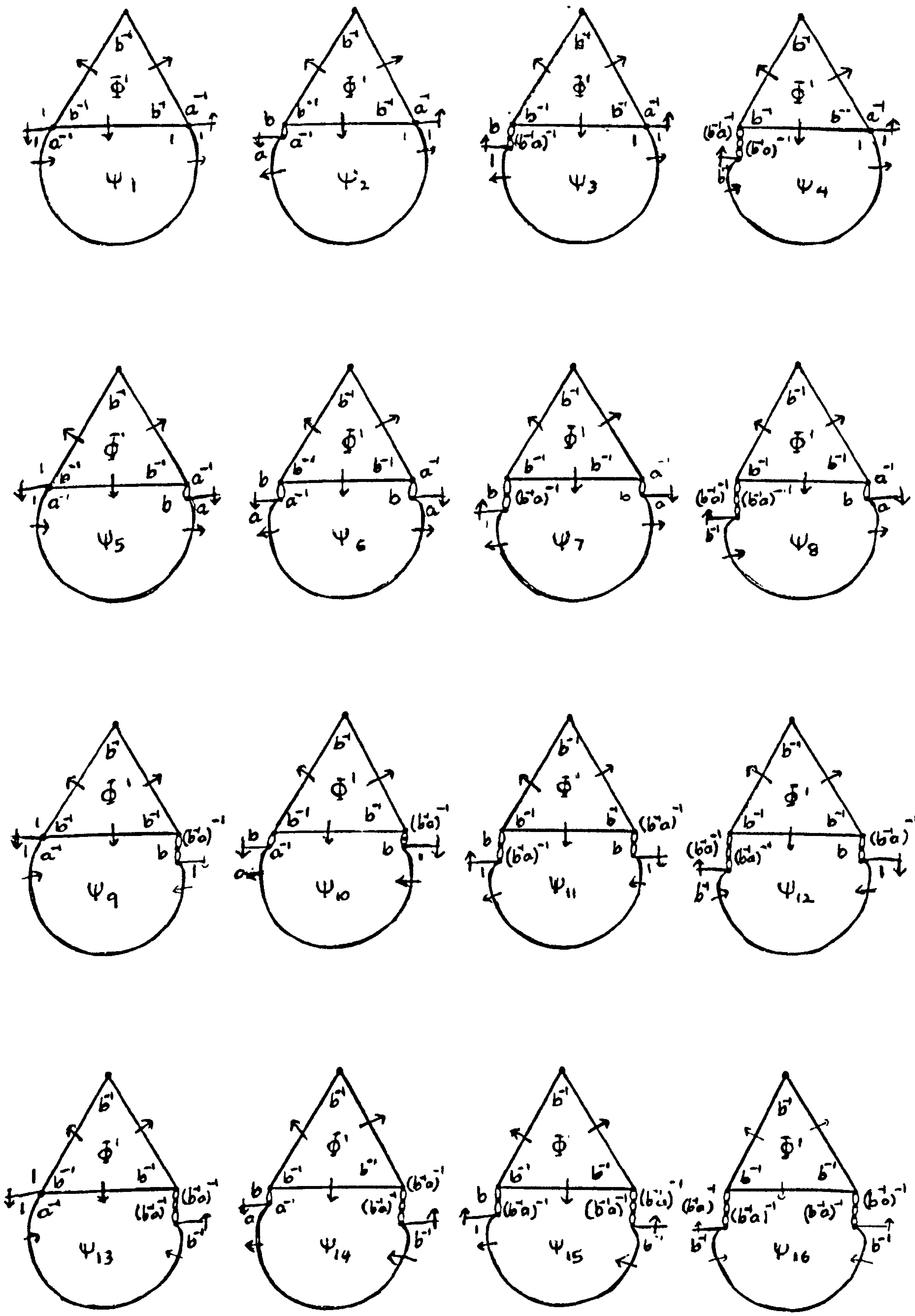
b^2bbb	b^21a1	b^2ba1	$b^2b(b^{-1}a)1$
b^2b11	$b^21a^{-1}1$	b^2ab1	$b^2b(b^{-1}a)a^{-1}$
b^2b1a^{-1}	b^21b1	$b^2a(b^{-1}a)1*$	$b^2b(b^{-1}a)^{-1}1*$
b^2ba1	b^21ba^{-1}	b^2ba1	$b^2b(b^{-1}a)^{-1}a^{-1}*$
b^2bb1	$b^21(b^{-1}a)1$	$b^2ba^{-1}1$	
$b^2bba^{-1}*$	$b^21(b^{-1}a)^{-1}1$	b^2bb1	
	$b^21(b^{-1}a)^{-1}a^{-1}*$	$b^2bba^{-1}*$	

Thus we have to consider the following relations:

1. $a = b^4$
2. $b^2aba = 1$
3. $b^2ab^{-1}a = 1$
4. $b^3aba = 1$

A.2.4 Possibilities for Ψ $(o(b) = 3)$

There are a lot of possible regions for Φ' with label bbb and none of their neighbours that sharing an edge with Φ' has the same label bbb. We identify sixteen different possible forms for Ψ.


Ψ1
Ψ2
Ψ3
Ψ4
Ψ5
Ψ6
Ψ7
Ψ8
Ψ9
Ψ10
Ψ11
Ψ12
Ψ13
Ψ14
Ψ15
Ψ16

Thus there are a lot of possibilities for Ψ but of course most of them are obviously not possible. We mark * if they seem possible and leave unmarked if they are obviously not possible. Here we assume that $o(ab^{-1}) \geq 6$, $a \neq b^2$, $abab^{-1} \neq 1$ and $b \neq (b^{-1}a)^2$.

valence 2

1 :	2 :	3 : $b^{-1}a1$	4 :
5 :	6 : ab^{-1}	7 : $(b^{-1}a)b^{-1}$	8 :
9 : ab^{-1}	10 :	11 :	12 : $(b^{-1}a)b^{-1}$
13 : $a(b^{-1}a)$	14 :	15 :	16 : $(b^{-1}a)^2$

valence 3

1 : $a1b$ $a1b^{-1}a$ $a11$	2 : $a1b^{-1}$ $a1b$	3 : $b^{-1}a1b$ $b^{-1}a1b^{-1}$	4 : $b^{-1}a1b^{-1}$ $b^{-1}aa1$ $b^{-1}a1b^{-1}a$ $b^{-1}a11$
5 : $ab^{-1}b^{-1}$ $ab^{-1}b^{-1}$ $ab^{-1}1$ $ab^{-1}b^{-1}a$	6 : $ab^{-1}b$ $ab^{-1}b^{-1}$	7 : $b^{-1}ab^{-1}b$ $b^{-1}ab^{-1}b^{-1}$	8 : $b^{-1}ab^{-1}b^{-1}$ $b^{-1}ab^{-1}a$ $b^{-1}ab^{-1}1$ $b^{-1}ab^{-1}b^{-1}a$
9 : $ab^{-1}a$ $ab^{-1}a^{-1}$ $ab^{-1}ab^{-1}a$ $ab^{-1}a(b^{-1}a)^{-1}$ $ab^{-1}b^{-1}$	10 : $ab^{-1}b$ $ab^{-1}a^{-1}$ $ab^{-1}1$ $ab^{-1}(b^{-1}a)^{-1}$	11 : $b^{-1}ab^{-1}b$ $b^{-1}ab^{-1}a$ $b^{-1}ab^{-1}1$ $b^{-1}ab^{-1}(b^{-1}a)^{-1}$	12 : $b^{-1}ab^{-1}a$ $b^{-1}ab^{-1}a^{-1}$ $b^{-1}ab^{-1}b^{-1}a$ $b^{-1}ab^{-1}(b^{-1}a)^{-1}$ $b^{-1}ab^{-1}b^{-1}$
13 : $ab^{-1}aa$ $ab^{-1}ab^{-1}a$ $ab^{-1}ab$ $ab^{-1}ab^{-1}$	14 : $ab^{-1}ab$ $ab^{-1}a1$	15 : $b^{-1}ab^{-1}ab$ $b^{-1}ab^{-1}a1$	16 : $b^{-1}ab^{-1}a$ $b^{-1}ab^{-1}ab^{-1}a$ $b^{-1}ab^{-1}ab^{-1}$

valence 4

1 :

$a1b^{-1}a$
$a1b^{-1}a^{-1}$
$a1b^{-1}b^{-1}a$
$a1b^{-1}a^{-1}b$
$a1a^2$
$a1ab^{-1}a$
$a1b^{-1}aa$
$a1b^{-1}ab^{-1}a$
$a1ba$
$a1b^{-1}b^{-1}$
$a1b^{-1}b^{-1}a$

2 :

$a1ab$
$a1a1$
$a1a(b^{-1}a)^{-1}$
$a1a^{-1}b$
$a1a^{-1}1$
$a1a^{-1}(b^{-1}a)^{-1}$
$a1b^{-1}ab$
$a1b^{-1}a1$
$a1b^{-1}a^{-1}b$
$a1(b^{-1}a)^{-1}b$
$a1a^{-1}ba^{-1}aa^{-1}b1$
$a1a^{-1}ba^{-1}b$
$a1bb$
$a1a^{-1}b$
$a1a^{-1}b^{-1}a1b$
$a1b^{-1}$
$a1a^{-1}bb$

3 :

$b^{-1}a1b^{-1}a$
$b^{-1}a1b^{-1}1$
$b^{-1}a1b^{-1}a^{-1}b$
$b^{-1}a1ab$
$b^{-1}a1a1$
$b^{-1}a1aa^{-1}b$
$b^{-1}a1b$
$b^{-1}a1a^{-1}$
$b^{-1}a1a^{-1}b$
$b^{-1}a1b^{-1}ab$
$b^{-1}a1b^{-1}aa^{-1}$
$b^{-1}a1b^{-1}a1$
$b^{-1}a1bb$
$b^{-1}a1b^{-1}b^{-1}$

4 :

$b^{-1}a1b^{-1}a$
$b^{-1}a1b^{-1}a^{-1}$
$b^{-1}a1aa^{-1}b$
$b^{-1}a1aa$
$b^{-1}a1aa^{-1}$
$b^{-1}a1ab^{-1}a$
$b^{-1}a1aa^{-1}b$
$b^{-1}a1b^{-1}aa$
$b^{-1}a1b^{-1}aa^{-1}$
$b^{-1}a1b^{-1}ab^{-1}a$
$b^{-1}a1b^{-1}aa^{-1}b$
$b^{-1}a1bb^{-1}$
$b^{-1}a1ba$
$b^{-1}a1b^{-1}a$
$b^{-1}a1bb^{-1}b^{-1}$
$b^{-1}a1b^{-1}a$
$b^{-1}a1b^{-1}b^{-1}a$
$b^{-1}a1b^{-1}ab^{-1}$

5 :

$ab^{-1}b^{-1}a$
$ab^{-1}b^{-2}a$
$ab^{-1}b^{-1}ab$
ab^{-1}
$ab^{-1}ab^{-1}a$
$ab^{-1}b$
$ab^{-1}a$
$ab^{-1}ab$
$ab^{-1}b^{-1}$
$ab^{-1}ba$
$ab^{-1}b^{-2}$

6 :

ab^{-1}
$ab^{-1}b^{-1}a$
$ab^{-1}b^{-1}$
$ab^{-1}b^{-1}ab$
$ab^{-1}ab$
$ab^{-1}a$
$ab^{-1}b$
$ab^{-1}ab^{-1}$
$ab^{-1}b^2$
$ab^{-1}b^{-2}$

7 :

$b^{-1}ab^{-1}$
$b^{-1}ab^{-1}b^{-1}a$
$b^{-1}ab^{-1}b^{-1}$
$b^{-1}ab^{-2}ab$
$b^{-1}ab^{-1}ab$
$b^{-1}ab^{-1}a$
$b^{-1}aab^{-1}$
$b^{-1}ab^{-1}b^2$
$b^{-1}ab^{-1}b^{-2}$

8 :

$b^{-1}ab^{-1}b^{-1}a$
$b^{-1}ab^{-1}b^{-2}a$
$b^{-1}ab^{-1}1$
$b^{-1}ab^{-1}ab^{-1}a$
$b^{-1}ab^{-1}ba$
$b^{-1}ab^{-1}a$
$b^{-1}ab^{-1}bab$
$b^{-1}ab^{-1}b^{-2}$
$b^{-1}ab^{-1}ab^{-1}$
$b^{-1}ab^{-2}aab^{-1}$

9 :

ab^{-1}

$ab^{-1}ab^{-1}a$

$ab^{-1}b$

$ab^{-1}b^{-1}$

$ab^{-1}ba$

$ab^{-1}a$

$ab^{-1}aba$

$ab^{-1}(b^{-1}a)^2$

$ab^{-1}a^{-1}bab^{-1}*$

$ab^{-1}ab^{-1}$

$ab^{-1}ab$

$ab^{-1}b^{-1}a$

$ab^{-1}ab^2$

$ab^{-1}b^{-1}ab$

$ab^{-1}b^{-1}ab^{-1}$

10 :

$ab^{-1}ab$

$ab^{-1}b^{-1}$

ab^{-1}

$ab^{-1}a$

$ab^{-1}b$

$ab^{-1}aba$

$ab^{-1}ab^{-1}$

$ab^{-1}b^{-1}ab$

$ab^{-1}b^{-1}a$

$ab^{-1}ab$

$ab^{-1}a^{-1}ba^{-1}b*$

$ab^{-1}ab^2$

11 :

$b^{-1}ab^{-1}ab$

$b^{-1}ab^{-1}$

$b^{-1}ab^{-1}a$

$b^{-1}ab^{-1}b$

$b^{-1}ab^{-1}aba$

$b^{-1}ab^{-1}ab^{-1}$

$b^{-1}ab^{-1}b^2$

$b^{-1}ab^{-1}babb$

$b^{-1}ab^{-1}ba$

$b^{-1}ab^{-}b^{-1}ab$

$b^{-1}ab^{-1}b^{-1}$

$b^{-1}ab^{-1}b^{-1}a$

$b^{-1}ab^{-1}ab^2$

$b^{-1}ab^{-1}a^{-1}ba^{-1}b$

12 :

$b^{-1}ab^{-1}$	$b^{-1}ab^{-1}ab^{-1}a$	$b^{-1}ab^{-1}b$
$b^{-1}ab^{-1}b^{-1}$	$b^{-1}ab^{-1}ba$	$b^{-1}ab^{-1}a$
$b^{-1}ab^{-1}aba*$	$b^{-1}ab^{-1}b^{-1}ab^{-1}a*$	$b^{-1}ab^{-1}abab$
$b^{-1}ab^{-1}ab^{-1}$	$b^{-1}ab^{-1}ab$	$b^{-1}ab^{-1}b^{-1}a$
$b^{-1}ab^{-1}a$	$b^{-1}ab^{-1}ab^{-1}$	$b^{-1}ab^{-1}b^{-1}a$
$b^{-1}ab^{-1}ab^2$	$b^{-1}ab^{-1}b^{-1}ab$	$b^{-1}ab^{-1}b^{-1}ab^{-1}*$

13 :

$ab^{-1}a$
$ab^{-1}aab^{-1}a$
$ab^{-1}ab$
$ab^{-1}ab^{-1}$
$ab^{-1}aba$
$ab^{-1}aa$
$ab^{-1}aaba$
$ab^{-1}ab^{-1}ab^{-1}a$
$ab^{-1}aa^{-1}ba^{-1}b$
$ab^{-1}aab^{-1}$
$ab^{-1}aab$
$ab^{-1}ab^{-1}a$
$ab^{-1}ab^2$
$ab^{-1}ab$
$ab^{-1}ab^{-1}ab*$
$ab^{-1}ab^{-1}ab^{-1}$

14 :

$ab^{-1}aab$
$ab^{-1}a$
$ab^{-1}ab*$
$ab^{-1}aaba$
$ab^{-1}aab^{-1}$
$ab^{-1}ab^{-1}ab*$
$ab^{-1}ab^{-1}$
$ab^{-1}ab^{-1}a$
$ab^{-1}aab^2$
$ab^{-1}aa^{-1}ba^{-1}b$
$ab^{-1}ab^2$
$ab^{-1}aba$
$ab^{-1}aba$

15 :

$b^{-1}ab^{-1}aab$
$b^{-1}ab^{-1}a*$
$b^{-1}ab^{-1}ab$
$b^{-1}ab^{-1}aaba$
$b^{-1}ab^{-1}aab^{-1}$
$b^{-1}ab^{-1}ab^{-1}ab$
$b^{-1}ab^{-1}ab^{-1}$
$b^{-1}ab^{-1}aab^2$
$b^{-1}ab^{-1}aa^{-1}baa^{-1}b$
$b^{-1}ab^{-1}ab^2$
$b^{-1}ab^{-1}abab$

16 :

$b^{-1}ab^{-1}a1$
$b^{-1}ab^{-1}ab^{-1}$
$b^{-1}ab^{-1}ab^{-1}ab^{-1}*$
$b^{-1}ab^{-1}aa^{-1}ba^{-1}b$
$b^{-1}ab^{-1}ab^{-1}a$
$b^{-1}ab^{-1}ab^{-1}a$

$b^{-1}ab^{-1}aab^{-1}a$
$b^{-1}ab^{-1}aba*$
$b^{-1}ab^{-1}aaba$
$b^{-1}ab^{-1}aab^{-1}$
$b^{-1}ab^{-1}aba*$

$b^{-1}ab^{-1}ab$
$b^{-1}ab^{-1}aa$
$b^{-1}ab^{-1}ab^{-1}ab^{-1}a$
$b^{-1}ab^{-1}aab$
$b^{-1}ab^{-1}ab^2$

valence 5

1 :

$a1b^{-1}b^{-2}$	$a1b^{-1}b^{-3}$
$a1ab$	$a1ab^{-1}$
$a1ab^{-2}$	$a1ba$
$a1b^{-1}a$	$a1b^2a$
$a1b^{-2}a$	$a1b^{-3}a$
$a1bab$	$a1b^{-1}ab$
$a1b^{-2}ab$	$a1b^{-1}abab$
$a1b^{-1}ab^2$	$a1b^{-2}ab^{-1}$
$a1b^{-1}ab^{-1}$	$a1b^{-1}ab^{-2}$

2 :

$a1bab$
$a1b^{-1}ab$
$a1b^{-1}ab^{-1}$
$a1b^{-1}ab^2$
$a1b^{-2}ab$
$a1b^{-1}ab^{-2}$
$a1b^{-1}abab*$

3 :

$b^{-1}a1b^{-1}a$
$b^{-1}a1b^{-2}a$
$b^{-1}a1ba$
$b^{-1}a1bab^{-1}a*$
$b^{-1}a1b^{-1}ab^{-2}$

4 :

$b^{-1}a1b^{-1}a$
$b^{-1}a1bab^{-1}a*$
$b^{-1}a1b^{-1}ab^{-1}a$
$b^{-1}a1b^{-1}ab^{-2}$
$b^{-1}a1b^{-1}ab^2$
$b^{-1}ab^{-2}ab^{-1}$
$b^{-1}abab^{-1}$
$b^{-1}aab^{-1}ab^{-1}$
$b^{-1}aab^{-1}ab$

5 :

$ab^{-1}ab^{-1}$
$ab^{-1}b^{-1}ab$
$ab^{-1}ab^2$
$ab^{-1}b^{-1}ab^{-2}$
$ab^{-1}b^{-1}ab^2$
$ab^{-1}b^{-2}ab$
$ab^{-1}ab^{-2}$
$ab^{-1}ab^{-1}ab^{-1}$
$ab^{-1}ab^{-1}ab*$

6 :

$ab^{-1}ab^{-1}$
$ab^{-1}b^{-1}ab^{-1}$
$ab^{-1}b^{-1}ab^{-2}$
$ab^{-1}abab^{-1}*$

7 :

$b^{-1}ab^{-1}a$
$b^{-1}ab^{-1}b^{-2}$
$b^{-1}ab^{-1}b^{-3}$
$b^{-1}ab^{-1}b^{-1}a$
$b^{-1}ab^{-1}b^{-2}a$
$b^{-1}ab^{-1}ab^{-1}a$
$b^{-1}ab^{-1}ab^{-2}$

8 :

$b^{-1}ab^{-1}a$
$b^{-1}ab^{-1}b^{-3}$
$b^{-1}ab^{-1}b^{-1}a$
$b^{-1}ab^{-1}b^{-2}a$
$b^{-1}ab^{-1}b^{-3}a$
$b^{-1}ab^{-1}aba*$
$b^{-1}ab^{-1}ab^{-1}a$
$b^{-1}ab^{-1}ab^{-2}a*$
$b^{-1}ab^{-1}b^{-1}aba*$
$b^{-1}ab^{-1}b^{-1}ab^{-1}a*$
$b^{-1}ababa*$
$b^{-1}ab^{-1}a(b^{-1}a)^2$

9 :

$ab^{-1}ab$
$ab^{-1}ab^{-1}$
$ab^{-1}ab^2$
$ab^{-1}ab^{-2}$
$ab^{-1}abab*$
$ab^{-1}abab^{-1}*$
$ab^{-1}ab^{-1}ab*$
$ab^{-1}ababab*$
$ab^{-1}b^{-3}$
$ab^{-1}(ab^{-1})^2$

10 :

$ab^{-1}ab$
$ab^{-1}ab^{-1}$
$ab^{-1}ab^2$
$ab^{-1}ab^{-2}$
$ab^{-1}ab^3$
$ab^{-1}b^{-1}ab$
$ab^{-1}b^{-1}ab^{-1}$
$ab^{-1}abab*$
$ab^{-1}abab^{-1}*$
$ab^{-1}abab^2*$
$ab^{-1}abab*$
$ab^{-1}b^{-1}abab$
$ab^{-1}b^{-1}ab^{-1}ab$

11 :

$b^{-1}ab^{-1}a$
$b^{-1}ab^{-1}b^{-2}$
$b^{-1}ab^{-1}aba$
$b^{-1}ab^{-1}ab^{-1}$
$b^{-1}ab^{-1}b^{-1}ab^{-1}a*$
$b^{-1}ab^{-1}a(ba)^2*$
$b^{-1}ab^{-1}b^2a$
$b^{-1}ab^{-1}b^{-1}abab$
$b^{-1}ab^{-1}b^{-1}ab^{-1}ab$

12 :

$b^{-1}ab^{-1}a$
$b^{-1}ab^{-1}b^{-2}$
$b^{-1}ab^{-1}b^{-1}a$
$b^{-1}ab^{-1}b^{-2}a$
$b^{-1}ab^{-1}aba*$
$b^{-1}ab^{-1}ab^{-1}a$
$b^{-1}ab^{-1}ab^2a$
$b^{-1}ab^{-1}ab^{-2}a*$
$b^{-1}ab^{-1}b^{-1}aba*$
$b^{-1}ab^{-1}ababa*$
$b^{-1}ab^{-1}abab^{-1}a*$
$b^{-1}ab^{-1}a(b^{-1}a)^2$

13 :

$ab^{-1}ab$
$ab^{-1}ab^{-1}$
$ab^{-1}ab^2$
$ab^{-1}ab^{-2}$
$ab^{-1}abab^{-1}*$
$ab^{-1}b^{-1}ab^{-1}$
$ab^{-1}abab*$
$ab^{-1}ab^{-1}ab*$
$ab^{-1}(ab^{-1})^2$
$ab^{-1}b^{-1}ab$
$ab^{-1}a(b^{-1}a)^2$

	$ab^{-1}ab$		$(b^{-1}a)^2$		$(b^{-1}a)^2 1$
	$ab^{-1}ab^2$		$(b^{-1}a)^2ba*$		$(b^{-1}a)^2b^{-1}a$
	$ab^{-1}ab^{-2}$		$(b^{-1}a)^2bab^{-1}*$		$(b^{-1}a)^2ba*$
	$ab^{-1}ab^3$		$(b^{-1}a)^2b^{-1}a$		$(b^{-1}a)^2b^{-2}a*$
	$ab^{-1}ab^{-1}$		$(b^{-1}a)^2(b^{-1}a)^2$		$(b^{-1}a)^2b^{-2}$
14 :	$ab^{-1}abab*$	15 :	$(b^{-1}a)^2b^2a$	16 :	$(b^{-1}a)^2b^{-1}ab^{-1}*$
	$ab^{-1}ab^{-1}ab*$		$(b^{-1}a)^2b^{-2}a*$		$(b^{-1}a)^2b^{-1}aba*$
	$ab^{-1}(ab^{-1})^2$		$(b^{-1}a)^2bab^2*$		$(b^{-1}a)^2(b^{-1}a)^2$
	$ab^{-1}abab^2*$		$(b^{-1}a)^2bab^{-1}*$		$(b^{-1}a)^2b^{-2}a*$
	$ab^{-1}abab^{-1}*$		$(b^{-1}a)^2b^{-1}abab$		$(b^{-1}a)^2(b^{-1}a)^3$
	$ab^{-1}ab^{-1}(ab)^2*$		$(b^{-1}a)^2(b^{-1}a)^2b$		$(b^{-1}a)^2bab^{-1}*$

Thus we obtain the following possible relations:

1. $b^{-1}ababa = 1$ (or $b^{-1}ab^{-1}aba = 1$)

2. $b^{-1}abababa = 1$ (or $b^{-1}ab^{-1}ab^{-1}aba = 1$)

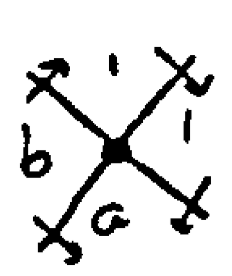
3. $b^{-1}ab^{-1}ababa = 1$

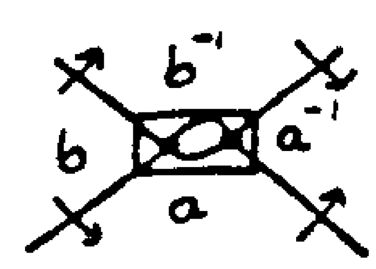
A.3 Reference for §5.2.4

Throughout this section, all restriction in A* is also applicable.

A.3.1 Possibilities for Φ'

We have two type of discs





Now we will find the possible label for Φ' of valence two and three. As usual, we mark * if they seem possible and leave unmarked if they are obviously not possible. Here we assume that a and b are distinct and $a \neq b^{-1}$ or b^2.

valence 2

$1a \quad 1b \quad a1 \quad aa \quad ab^{-1} \quad b1 \quad bb \quad ba^{-1}$

valence 3

$aaa*$	$a11$	$ab^{-1}a*$	$bb1$
$bbb*$	$a^{-1}11$	$ab^{-1}1$	$bba^{-1}*$
	$a1a$	$a^{-1}ba$	$b1a^{-1}$
	$a1b$	$a^{-1}b^{-1}1$	$b11$
	$a^{-1}1b$		bab^{-1}
			$ba1$

Thus we need to consider the following relations:

1. $a^3 = 1$

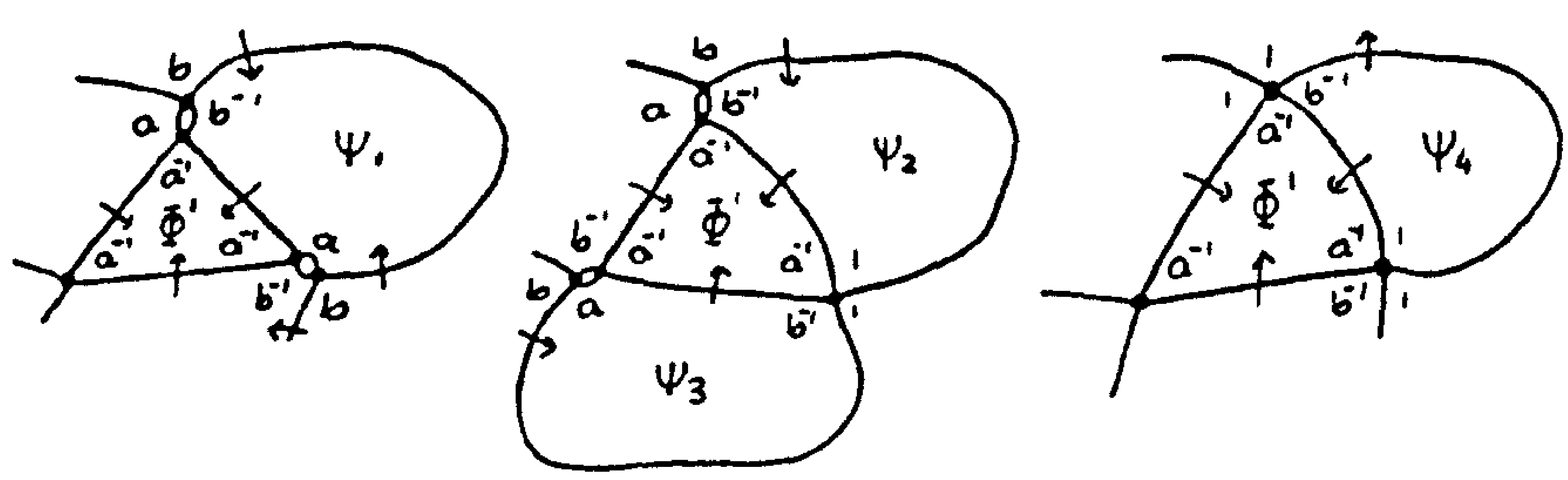
2. $b^3 = 1$

3. $a = b^2$

4. $b = a^2$

A.3.2 Possibilities for Ψ

There are two possibilities for Φ'—with label aaa or with label bbb. If Φ' is a 3-region with label aaa, then Ψ must have label of the form $a^{-1}bW_1$ or $1bV_1$ for some words W_1 and V_1. However if Φ' is a 4-region with label bbb, then Ψ must have label of the form $b^{-1}aW_2$ or $a1V_2$ for some words W_2 and V_2. Up to equivalent we just consider when Ψ has label $a^{-1}bW$ or $1bV$.



We need to consider up to valence five. From Appendix A.3.1, we know that there is no possible label for Ψ of valence two or three, and so we just consider when it has valence four or five. We mark * if they seem possible and leave unmarked if they are obviously not possible. Here we assume that a and b are distinct and $a \neq b^{-1}$ or b^2. Also if $o(b) = 3, 4$ or 5, then we assume that $(b^{-1}a)^2 \neq 1$ since otherwise hypothesis 4 in Theorem 5.2.1 would be satisfied.

valence 4

Ψ_1 :	Ψ_2 :	Ψ_3 :	Ψ_4 :
$a^{-1}b11$	$1b11$	$a^{-1}bb1$	$1b1b$
$a^{-1}b1a$	$1b1a$	$a^{-1}bba$	$1bba$
$a^{-1}b1b$	$1b1b$	$a^{-1}b1a$	$1b1a$
$a^{-1}ba^{-1}1$	$1ba^{-1}1$	$a^{-1}b1a^{-1}$	$1b1a^{-1}$
$a^{-1}ba^{-1}b$	$1ba^{-1}b$	$a^{-1}ba^{-1}a^{-1}$	$1ba^{-1}a^{-1}$
$a^{-1}baa$	$1baa$		
	$1ba^{-1}a^{-1}$		

valence 5

Ψ_1 :

$a^{-1}b1b1$	$a^{-1}b1a$	$a^{-1}b1b$	$a^{-1}ba1a^{-1}$
$a^{-1}ba1b^{-1}*$	$a^{-1}bab^{-1}1*$	$a^{-1}bab^{-1}a^{-1}*$	$a^{-1}bab^{-1}b^{-1}*$
$a^{-1}ba^{-1}11$	$a^{-1}ba^{-1}1a^{-1}$	$a^{-1}ba^{-1}1b^{-1}*$	$a^{-1}bab^{-1}1*$
$a^{-1}ba^{-1}b^{-1}a^{-1}*$	$a^{-1}ba^{-1}b^{-1}b$	$a^{-1}ba^{-1}a^{-1}1$	$a^{-1}ba^{-1}a^{-1}a^{-1}$
$a^{-1}ba^{-1}a^{-1}b*$	$a^{-1}b11a$	$a^{-1}b11a^{-1}$	$a^{-1}b1aa$
$a^{-1}b1ba$	$a^{-1}b1ba^{-1}$	$a^{-1}ba^{-1}1a$	$a^{-1}ba^{-1}1a^{-1}$
$a^{-1}ba^{-1}ba$	$a^{-1}ba^{-1}ba^{-1}*$	$a^{-1}bb^{-1}1a$	$a^{-1}bb^{-1}1a$
$a^{-1}bb^{-1}1a^{-1}$	$a^{-1}b^{-1}aa$	$a^{-1}baaa$	$a^{-1}ba^{-1}a^{-1}a^{-1}$

Ψ_2 :

$1b1b1$	$1b1ba^{-1}$	$1ba11$	$1ba1a^{-1}$
$1ba1b^{-1}$	$1bab^{-1}1$	$1bab^{-1}a^{-1}*$	$1bab^{-2}$
$1ba^{-1}11$	$1ba^{-1}1a^{-1}$	$1ba^{-1}1b^{-1}$	$1ba^{-1}b^{-1}1$
$1ba^{-1}b^{-1}a^{-1}*$	$1ba^{-1}b^{-2}$	$1ba^{-2}1$	$1ba^{-3}$
$1ba^{-2}b^{-1}$	$1b11a$	$1b11a^{-1}$	$1b1a^2$
$1b1ba$	$1b1ba^{-1}$	$1ba^{-1}1a$	$1ba^{-1}1a^{-1}$
$1ba^{-1}ba*$	$1ba^{-1}ba^{-1}$	$1ba^3$	$1ba^{-3}$

$$\Psi_3 : \begin{array}{llll} a^{-1}bb1a & a^{-1}bb1a^{-1} & a^{-1}bba^2 & a^{-1}b1a^2 \\ a^{-1}b1a^{-2} & a^{-1}ba^{-3} & a^{-1}bb^{-1}a^2 & a^{-1}bb^{-1}a^{-2} \\ a^{-1}bb^21 & a^{-1}bb^2a & a^{-1}bb^3 & a^{-1}bb^{-2}1 \\ a^{-1}bb^{-2}a & a^{-1}b111 & a^{-1}b11a & a^{-1}b1bb \\ a^{-1}b1a^{-1}1 & a^{-1}b1a^{-1}b & a^{-1}b1b^{-1}1 & a^{-1}b1b^{-1}a \\ a^{-1}ba11 & a^{-1}ba1a & a^{-1}ba1b* & a^{-1}bab^{-1}* \\ a^{-1}bab^{-1}a & a^{-1}bb11 & a^{-1}bb1a & a^{-1}bb1b \\ a^{-1}bba^{-1}1 & a^{-1}bba^{-1}b* & & \end{array}$$

$$\Psi_4 : \begin{array}{llll} 1bb1a & 1bb1a^{-1} & 1bba^2 & 1b1a^2 \\ 1b1a^{-2} & 1ba^{-3} & 1bb^{-1}a^2 & 1bb^{-1}a^{-2} \\ 1bb^21 & 1bb^2a & 1bb^3 & 1bb^{-2}1 \\ 1bb^{-1}a & 1b111 & 1b11a & 1b11b \\ 1b1a^{-1}1 & 1b1a^{-1}b & 1b1b^{-1}1 & 1b1b^{-1}a \\ 1ba11 & 1ba1a & 1ba1b & 1bab^{-1}1 \\ 1bab^{-1}a* & 1bb11 & 1bb1a & 1bb1b \\ 1bba^{-1}1 & 1bba^{-1}b & & \end{array}$$

If $o(b) = 3$ then we have to consider the following possible relations:

1. $ba^2 = ab$ (or $a^2b = ba$)

2. $b^2a = ab$ (or $ab^2 = ba$)

3. $ab = ba$

However if $o(b) = 4$ or 5 then we just have to check when Ψ has valence four. From the above list, there is no such possible label.

A.3.3 Possibilities for Φ''

The following are the list of label for regions of valence four:

$aaaa$	$a1b1$	$1a1a$
$aa11$	$a1b^{-1}1$	$1b1b$
$aa1a$	$a1ba$	$ab^{-1}ab^{-1}*$
$aa1b$	$a1b^{-1}a$	
$aab^{-1}1$	$a1bb$	
$aab^{-1}a$	$ab^{-1}b^{-1}1$	
	$ab^{-1}b^{-1}a$	

Since $a^3 = 1$ and $a \neq b^{-1}, b^2$ then the only possible label for Φ'' is $ab^{-1}ab^{-1}$.

A.3.4 Possibilities for Ψ_1 and Ψ_2

From the diagram we know that Ψ_i's must have label of the form bbW for some word W. Here we assume that $a \neq b^{-1}, b^2$ and $o(b) \geq 6$. The following are the list:

valence 4

$bbbb$	$bbb^{-1}b^{-1}$	$bb11$	$bb1a^{-1}$
$bb1b^{-1}$	$bba1$	$bbb1$	$bbba*$

valence 5

$bbbbb$	$bbb11$	$bb1a1$	$bba^{-2}1$
bbb^{-3}	$bbb1a^{-1}*$	$bb1ab^{-1}$	bba^{-3}
$bb11b$	$bbb1b^{-1}$	$bb1a^{-1}1$	$bba^{-2}b^{-1}$
$bb11b^{-1}$	$bbba1*$	$bb1a^{-2}$	$bbb^{-1}a1$
$bb1a^{-1}b*$	$bbbab^{-1}$	$bb1a^{-1}b^{-1}$	$bbb^{-1}ab^{-1}$
$bb1a^{-1}b^{-1}$			$bbb^{-1}a^{-1}1$
$bb1b^{-2}$			$bbb^{-1}a^{-2}$
			$bbb^{-1}a^{-1}b^{-1}$

Thus we need to consider the following:

1. $a = b^3$
2. $a = b^{-3}$

If $a = b^3$ then $ab^{-1}ab^{-1} = 1 \Rightarrow b^4 = 1$ which is impossible since $o(b) \geq 6$. Note that $a \neq b^{-3}$ since otherwise $a^3 = 1 \Rightarrow b^9 = 1$ and $ab^{-1}ab^{-1} = 1 \Rightarrow b^8 = 1$. Then b would be trivial.

Thus there is no possible label for Ψ_i of valence four or five.

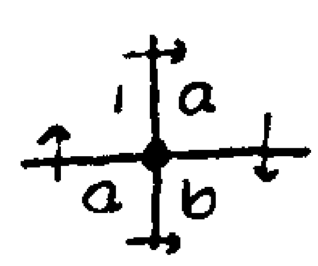
Appendix B

Reference for Chapter 6

B.1 Reference for §6.3.3

B.1.1 Possibilities for Φ

We only have one type of disc



and we need to list down all labels for Φ of valence up to five. As usual, we mark * if they seem possible and leave unmarked if they are obviously not possible. Since a has infinite order, then clearly $b \notin < a >$.

valence 2

$$a1 \quad aa \quad bb*$$

valence 3

$$aa1 \quad a1b \quad a1b^{-1} \quad aaa \quad bbb*$$

valence 4

$aaaa$	$a^2 1 a$	$a^{-2} 1 a$	$1 a 1 a$
$bbbb*$	$b^2 a 1$	$b^{-2} a 1$	
$a 1 b 1$	$a 1 b^{-1} 1$	$a 1 b a$	$a 1 b^{-1} a$
$a^{-1} 1 b 1$	$a^{-1} 1 b^{-1} 1$	$a^{-1} 1 b a$	$a^{-1} 1 b^{-1} a$
$a a^{-1} b a$	$a a^{-1} b^{-1} a$	$a a^{-1} b 1$	$a a^{-1} b^{-1} 1$
$a^{-1} a^{-1} b a$	$a^{-1} a^{-1} b^{-1} a$	$a^{-1} a^{-1} b 1$	$a^{-1} a^{-1} b^{-1} 1$

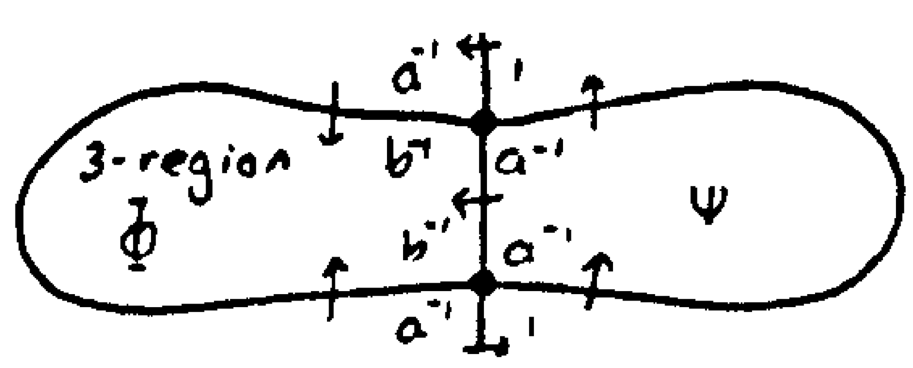
valence 5

$aaaaa$	$a^3 1 a$	$a^{-3} 1 a$	$a 1 a 1 a$	$a^{-1} 1 a 1 a$
$bbbbb*$	$b^3 1 a^{-1}$	$b^3 a 1$	$b 1 a 1 a$	$b^{-1} 1 a 1 a$
$b a a^2 a^{-1}$	$b 1 a^2 a^{-1}$	$b^2 a a a^{-1}$	$b^2 1 a 1$	
$b a a^{-2} a^{-1}$	$b 1 a^{-2} a^{-1}$	$b^2 a a^{-1} a^{-1}$	$b^2 1 a^{-1} 1$	
$b a a^2 1$	$b 1 a^2 1$	$b^{-2} a a a^{-1}$	$b^{-2} 1 a 1$	
$b a a^{-2} 1$	$b 1 a^{-2} 1$	$b^{-2} a a^{-1} a^{-1}$	$b^{-2} 1 a^{-1} 1$	
$b^{-1} a a^2 a^{-1}$	$b^{-1} 1 a^2 a^{-1}$	$b^2 a a 1$	$b^2 1 a a^{-1}$	
$b^{-1} a a^{-2} a^{-1}$	$b^{-1} 1 a^{-2} a^{-1}$	$b^2 a a^{-1} 1$	$b^2 1 a^{-1} a^{-1}$	
$b^{-1} a a^2 1$	$b^{-1} 1 a^2 1$	$b^{-2} a a a^{-1}$	$b^{-2} 1 a a^{-1}$	
$b^{-1} a a^{-2} 1$	$b^{-1} 1 a^{-2} 1$	$b^{-2} a a^{-1} 1$	$b^{-2} 1 a^{-1} a^{-1}$	

Thus we need to consider when $o(b) \leq 5$.

B.1.2 Possibilities for Ψ

Since Ψ shares an edge with Φ



then Ψ must have label of the form aaW for some word W. Below are the list for Ψ of valence up to six.

valence 2

valence 3

$$aa1$$

valence 4

$$aaa1 \quad aa1b \quad aa1b^{-1} \quad aaa^{-1}b \quad aaa^{-1}b^{-1}$$

valence 5

$$aaaa1 \quad aaa1b \quad aaa1b^{-1} \quad aaaa^{-1}b \quad aaaa^{-1}b^{-1}$$

$$aa1a1 \quad aa1b^2 \quad aa1b^{-2} \quad aaa^{-1}b^2 \quad aaa^{-1}b^{-2}$$

valence 6

$aaa^3 1$	$aaa1b^2$	$aa1b^3$	$aa1a1a$	$aa1a1b$
$aaa^2 1b$	$aa1b^{-2}$	$aa1b^{-3}$	$aa1ba1$	$aa1a1b^{-1}$
$aaa^2 1b^{-1}$	$aaaa^{-1}b^2$	$aaa^{-1}b^3$	$aa1b^{-1}a1$	$aaa^{-1}1a^{-1}b$
$aaa^2 a^{-1}b$	$aaaa^{-1}b^{-2}$	$aaa^{-1}b^{-3}$	$aaa^{-1}ba1$	$aaa^{-1}1a^{-1}b^{-1}$
$aaa^2 a^{-1}b^{-1}$			$aaa^{-1}b^{-1}a1$	

Since a has infinite order and $o(b) = 3$ then clearly none of these is possible.

B.2 Reference for §6.3.4

B.2.1 Possibilities for Φ

Assume that $a^2 \neq 1$. Then the list are

valence 2

$$a1 \quad aa \quad bb*$$

valence 3

$$aa1 \quad a1b \quad a1b^{-1} \quad aaa* \quad bbb*$$

valence 4

$aaaa*$	$a^2 1a*$	$a^{-2}1a$	$1a1a$
$bbbb*$	$b^2 a1$	$b^{-2}a1$	
$a1b1$	$a1b^{-1}1$	$a1ba$	$a1b^{-1}a$
$a^{-1}1b1$	$a^{-1}1b^{-1}1$	$a^{-1}1ba$	$a^{-1}1b^{-1}a$
$aa^{-1}ba$	$aa^{-1}b^{-1}a$	$aa^{-1}b1$	$aa^{-1}b^{-1}1$
$a^{-1}a^{-1}ba$	$a^{-1}a^{-1}b^{-1}a$	$a^{-1}a^{-1}b1$	$a^{-1}a^{-1}b^{-1}1$

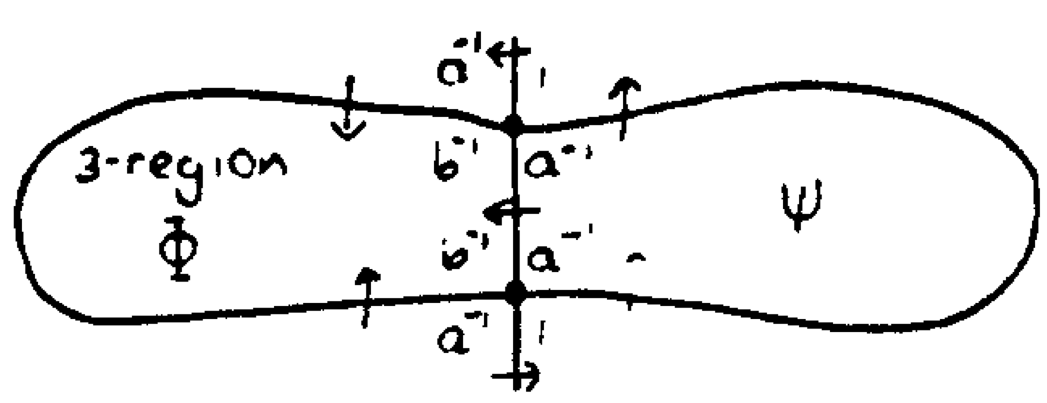
valence 5

$aaaaa*$	$a^3 1 a*$	$a^{-3} 1 a$	$a 1 a 1 a*$	$a^{-1} 1 a 1 a$
$bbbbb*$	$b^3 1 a^{-1}$	$b^3 a 1$	$b 1 a 1 a$	$b^{-1} 1 a 1 a$
baa^2a^{-1}	$b 1 a^2 a^{-1}$	$b^2 aaa^{-1}$	$b^2 1 a 1$	
$baa^{-2}a^{-1}$	$b 1 a^{-2} a^{-1}$	$b^2 aa^{-1}a^{-1}$	$b^2 1 a^{-1} 1$	
$baa^2 1$	$b 1 a^2 1$	$b^{-2} aaa^{-1}$	$b^{-2} 1 a 1$	
$baa^{-2} 1$	$b 1 a^{-2} 1$	$b^{-2} aa^{-1}a^{-1}$	$b^{-2} 1 a^{-1} 1$	
$b^{-1} aa^2 a^{-1}$	$b^{-1} 1 a^2 a^{-1}$	$b^2 aa 1$	$b^2 1 aa^{-1}$	
$b^{-1} aa^{-2} a^{-1}$	$b^{-1} 1 a^{-2} a^{-1}$	$b^2 aa^{-1} 1$	$b^2 1 a^{-1} a^{-1}$	
$b^{-1} aa^2 1$	$b^{-1} 1 a^2 1$	$b^{-2} aaa^{-1}$	$b^{-2} 1 aa^{-1}$	
$b^{-1} aa^{-2} 1$	$b^{-1} 1 a^{-2} 1$	$b^{-2} aa^{-1} 1$	$b^{-2} 1 a^{-1} a^{-1}$	

Excluding **FE6**, then we need to consider when $o(a), o(b) \leq 5$.

B.2.2 Possibilities for Ψ

Since Ψ shares an edge with Φ



then Ψ must have label of the form aaW for some word W. Below are the list for Ψ of valence up to six.

valence 2

valence 3

$aa1$

valence 4

$aaa1 \quad aa1b \quad aa1b^{-1} \quad aaa^{-1}b \quad aaa^{-1}b^{-1}$

valence 5

$aaaa1 \quad aaa1b \quad aaa1b^{-1} \quad aaaa^{-1}b \quad aaaa^{-1}b^{-1}$

$aa1a1 \quad aa1b^2 \quad aa1b^{-2} \quad aaa^{-1}b^2 \quad aaa^{-1}b^{-2}$

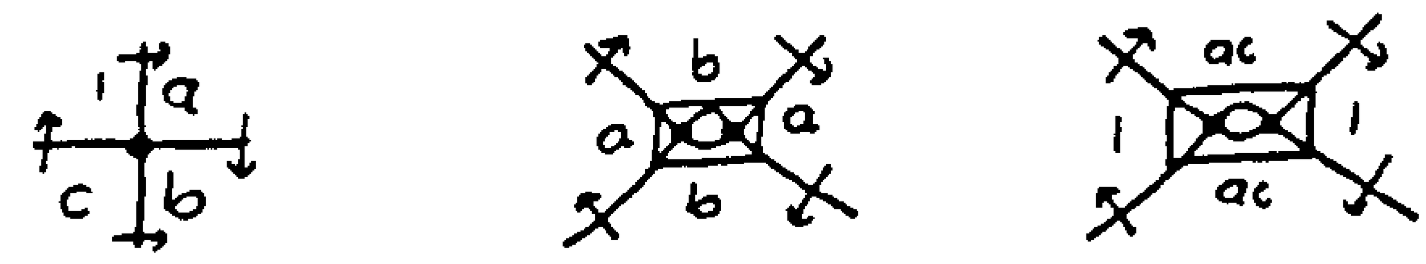
valence 6

$aaa^3 1$	$aaa1b^2$	$aa1b^3$	$aa1a1a$	$aa1a1b$
$aaa^2 1b$	$aa1b^{-2}$	$aa1b^{-3}$	$aa1ba1$	$aa1a1b^{-1}$
$aaa^2 1b^{-1}$	$aaaa^{-1}b^2$	$aaa^{-1}b^3$	$aa1b^{-1}a1$	$aaa^{-1}1a^{-1}b$
$aaa^2 a^{-1}b$	$aaaa^{-1}b^{-2}$	$aaa^{-1}b^{-3}$	$aaa^{-1}ba1$	$aaa^{-1}1a^{-1}b^{-1}$
$aaa^2 a^{-1}b^{-1}$			$aaa^{-1}b^{-1}a1$	

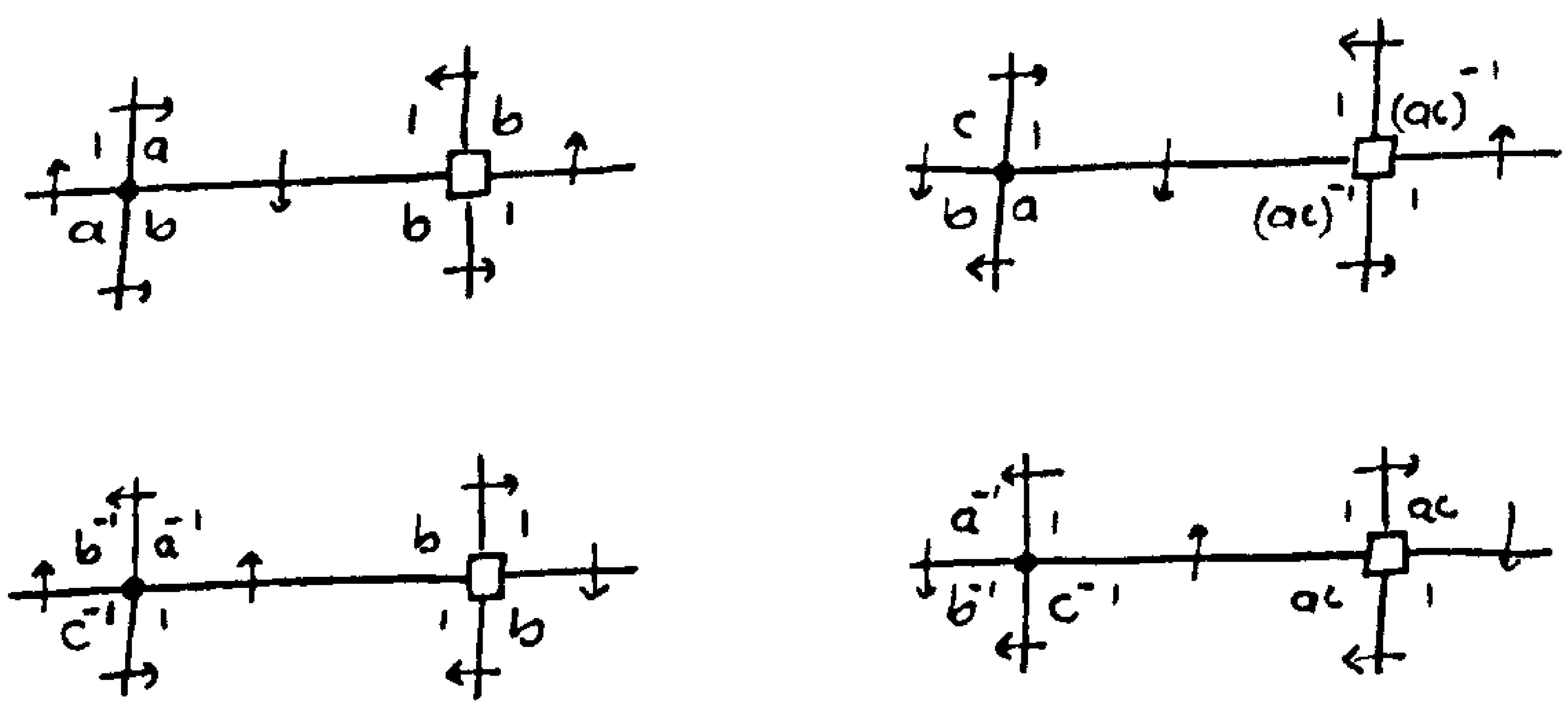
B.3 Reference for §6.4.3

B.3.1 Possibilities for Φ'

We have three type of discs



Note that in picture $\mathbb{P}'$ we do not have



since otherwise picture $\mathbb{P}$ is not reduced. So the possibilities are:

valence 2

$1(ac)^{-1} \quad 1a^{-1} \quad 1b^{-1}$

$a(ac)^{-1} \quad ab^{-1} \quad b(ac)^{-1}*$

valence 3

$c1a$	$c1ac$	$c1b$	$ca^{-1}1$	$ca^{-1}b*$
$cb^{-1}1$	$cb^{-1}a*$	$cb^{-1}ac$	$b1a^{-1}$	$b1(ac)^{-1}*$
$ba1$	$ba(ac)^{-1}*$	$bb(ac)^{-1}$	$bac1*$	$baca^{-1}*$
c^3	b^3			

Thus we have to consider:

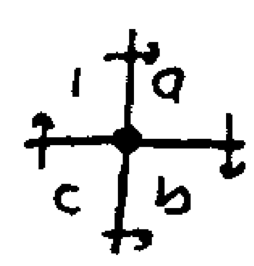
1. $b = ac$

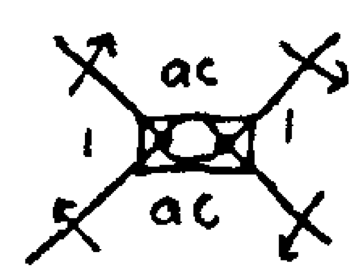
2. $ba = ac$

B.4 Reference for §6.4.4

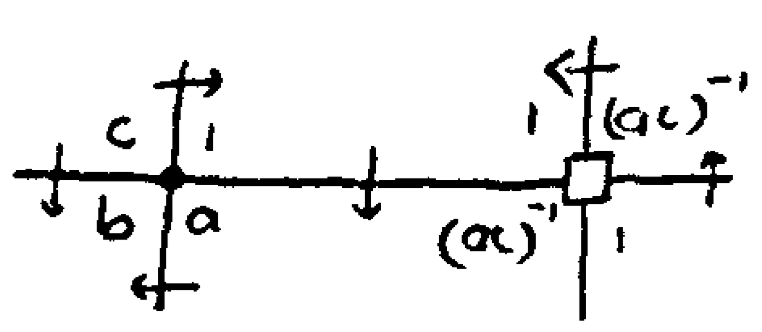
B.4.1 Possibilities for Φ'

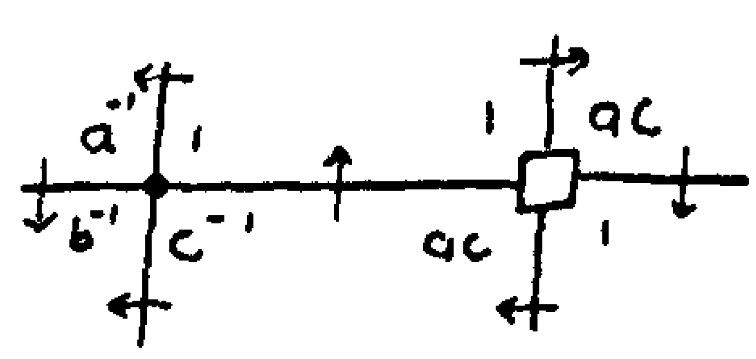
We have two type of discs





Note that in picture $\mathbb{P}'$ we do not have





since otherwise picture $\mathbb{P}$ is not reduced. So the possibilities are:

valence 2

$1a \quad 1ac \quad a(ac)^{-1} \quad cc$

valence 3

$b1a^{-1}$	$ba(ac)^{-1}$	$c1a$
$b1(ac)^{-1}*$	$baca^{-1}$	$c1ac*$
$ba1$		

Clearly $ba(ac)^{-1}$ and $baca^{-1}$ are not possible since $o(b) \neq o(c)$. Thus we just need to consider

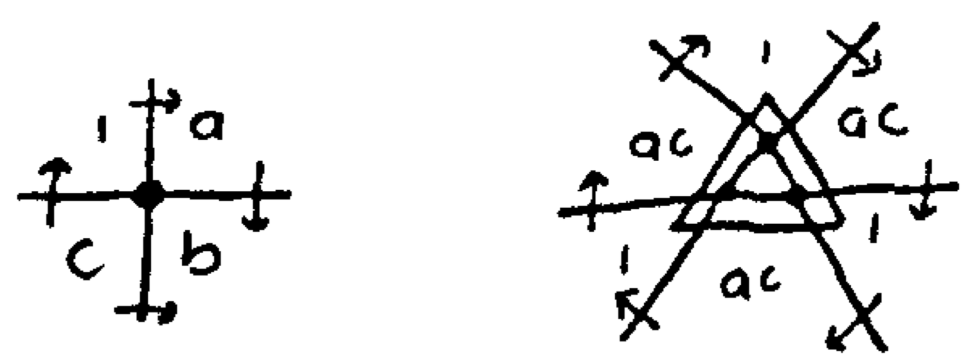
1. $a = c^2$

2. $b = ac$

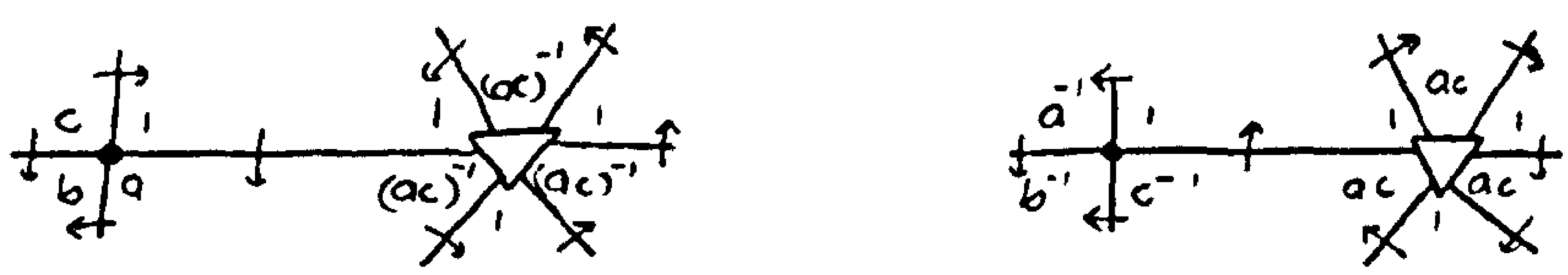
B.5 Reference for §6.4.5

B.5.1 Possibilities for Φ

We have two type of discs



and all restrictions similar to Appendix B.4.1 are applicable here.



Thus the list of possibilities are as in Appendix B.4.1. However there is only one possibilities namely $c1ac$.

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