Cylindric Symmetric Functions and Quantum Integrable Systems

by

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Abstract

We employ the level-n action of the affine symmetric group to construct cylindric versions of elementary and complete symmetric functions. We identify their expansions in terms of the bases of ordinary elementary and complete symmetric functions with the structure constants of generalised Verlinde algebras. Then we describe statistical vertex models associated to the q-boson model, when evaluated at $q = 1$, and study the interplay between the partition functions of these models and the cylindric symmetric functions introduced above.

Contents

Abstract i

Acknowledgements v

Declaration vi

1 Introduction 1

1.1 Verlinde algebras . . . 3
1.2 Statistical vertex models . . . 4
1.3 Quantum integrable models . . . 6
1.4 Present work and outline of this thesis . . . 7

2 Symmetric functions 11

2.1 Partitions . . . 11
2.1.1 Plane partitions and tableaux . . . 12
2.1.2 The symmetric group . . . 14
2.2 The ring of symmetric functions . . . 15
2.2.1 Monomial symmetric functions . . . 16
2.2.2 Elementary and complete symmetric functions . . . 17
2.2.3 Schur functions . . . 19
2.2.4 Power sums and augmented monomial symmetric functions . . . 20
2.2.5 Hall inner product and an involution . . . 22
2.2.6 Hopf algebra structure on Λ . . . 23
2.3 Coproduct and skew symmetric functions . . . 24
2.3.1 Skew elementary and complete symmetric functions . . . 25
2.3.2 Weighted sums over reverse plane partitions . . . 28
2.3.3 Weighted sums over row strict tableaux . . . 31
2.3.4 Adjacent column tableaux . . . 34

3 Cylindric symmetric functions 38

3.1 Cylindric Reverse Plane Partitions . . . 38

3.1.1 Cylindric diagrams and cylindric reverse plane partitions 38
3.1.2 The extended affine symmetric group 43
3.2 Weighted sums over CRPPs 46
3.2.1 Generalisation of $\theta_{\lambda/\mu}$ to the cylinder 48
3.2.2 Generalisation of $\psi_{\lambda/\mu}$ to the cylinder 53
3.2.3 The main definitions 56
3.3 Symmetric functions at roots of unity 57
3.3.1 Product expansions at roots of unity 60
3.3.2 Expansion formulae for cylindric symmetric functions 66
3.3.3 Cylindric adjacent column tableaux 71
3.4 Properties of cylindric symmetric functions 76
3.4.1 An involution between cylindric reverse plane partitions 78

4 Quantum integrable systems 83

4.1 Vertex models in statistical physics 83
4.1.1 Vertex and lattice configurations 84
4.1.2 Solutions of the Yang-Baxter equation 88
4.1.3 Partition functions and cylindric symmetric functions 95
4.1.4 Vertex models for the adjoint operators 103
4.2 The conserved charges for the free boson model 111
4.2.1 Functional relations for the conserved charges 113
4.3 Conserved charges and symmetric functions 120
4.3.1 Expansions of cylindric symmetric functions 125
4.3.2 Computation of the partition functions 128
4.4 The action of the conserved charges 131
4.4.1 Power sums 131
4.4.2 Elementary and complete symmetric functions 133
4.4.3 Monomial symmetric functions 138
4.4.4 Schur functions 143
4.5 State-charge correspondence 146
4.5.1 Idempotents in the algebra $(\mathcal{F}_k(n), \circledast)$ 148

5 Generalised Verlinde algebras 154

5.1 The modular group 155
5.1.1 Properties of the $\mathcal{S}$-matrix 156
5.1.2 The charge conjugation matrix 157
5.1.3 A representation of the modular group 158
5.2 Frobenius algebras and 2D TQFT 159
5.2.1 The Verlinde formula 161

5.2.2 An infinite family of Frobenius Algebras 163
5.3 The generalised symmetric group 164
5.3.1 Irreducible representations 165
5.3.2 Representation theory and fusion coefficients 166

6 Conclusions and open problems 170

Bibliography 172

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Declaration

I declare that, except where explicit reference is made to the contribution of others, this thesis is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

Chapter 1

Introduction

The basis of the ring of symmetric functions Λ given by the Schur functions $\{s_\lambda\}_{\lambda\in\mathcal{P}^+}$, where λ runs over the set $\mathcal{P}^+$ of partitions, is one of the most important and well studied within the theory of symmetric functions [52, 60, 67]. The reason for this lies in its connection with several other branches in mathematics, such as representation theory, algebraic geometry and integrable systems. Consider for example the following product expansion,

$$s_\mu s_\nu = \sum_{\lambda\in\mathcal{P}^+} c_{\mu\nu}^\lambda s_\lambda \ , \tag{1.1}$$

where the non-negative integers $c_{\mu\nu}^\lambda$ are the celebrated Littlewood-Richardson coefficients (see *loc. cit.*). Recall [52] the inverse limit $\Lambda = \lim\limits_{\leftarrow} \Lambda_k$, where Λ_k is the ring of symmetric polynomials in k variables. The Schur polynomials $s_\lambda(x_1,\ldots,x_k)$ are the characters of the finite dimensional irreducible polynomial representations of the general linear group $\mathrm{GL}_k(\mathbb{C})$ [29], and the product expansion (1.1) then describes the tensor product multiplicities of the mentioned $\mathrm{GL}_k(\mathbb{C})$-representations. It is known (see e.g. [66] and references therein) that the cohomology ring $H^*(\mathrm{Gr}(k,n))$ of the Grassmannian $\mathrm{Gr}(k,n)$ is isomorphic to $\Lambda_k/\langle s_{(n+k-1)},\ldots,s_{(n)}\rangle$. A basis of the latter consists of the Schur polynomials $s_\lambda(x_1,\ldots,x_k)$ for which the Young diagram of λ fits inside a $k\times(n-k)$ rectangle. An isomorphism between these two rings is given as follows: the Schubert class $\sigma_\lambda \in H^*(\mathrm{Gr}(k,n))$, which is the cohomology class of the Schubert variety $\Omega_\lambda \subset \mathrm{Gr}(k,n)$, is mapped to $s_\lambda(x_1,\ldots,x_k)$. The coefficients $c_{\mu\nu}^\lambda$ appearing in (1.1) are the structure constants of $H^*(\mathrm{Gr}(k,n))$, and they have the geometric interpretation as the intersection number of the Schubert varieties $\Omega_\mu, \Omega_\nu, \Omega_\lambda$. Finally, there exists a purely combinatorial rule for the computation of the coefficients $c_{\mu\nu}^\lambda$. This is the celebrated Littlewood-Richardson rule [52, 60, 67].

The ring of symmetric functions Λ carries the structure of a Hopf algebra [73]. Employing the coproduct $\Delta : \Lambda \to \Lambda \otimes \Lambda$, one can define 'skew Schur functions' $s_{\lambda/\mu}$ via the equation (see e.g. [52, I.5])

$$\Delta(s_\lambda) = \sum_{\mu \in \mathcal{P}^+} s_{\lambda/\mu} \otimes s_\mu \ . \tag{1.2}$$

As explained in Chapter 2, the Hopf algebra structure on Λ can be used to obtain the Littlewood-Richardson coefficients via the expansion

$$s_{\lambda/\mu} = \sum_{\nu \in \mathcal{P}^+} c^\lambda_{\mu\nu} s_\nu \ . \tag{1.3}$$

This implies that the skew Schur functions are 'Schur positive', that is their expansion coefficients in terms of ordinary Schur functions are non-negative integers. An explicit construction of the irreducible representations of $\mathrm{GL}_k(\mathbb{C})$ is given by the Weyl modules [25], which can be generalised to skew shapes λ/μ. The corresponding characters are given by $s_{\lambda/\mu}(x_1, \dots, x_k)$, and the expansion (1.3) describes the decomposition of 'skew Weyl modules' into irreducible $\mathrm{GL}_k(\mathbb{C})$-representations (see *loc. cit.*). The adjective 'skew' stems from the fact that $s_{\lambda/\mu}$ has an alternative combinatorial description given by

$$s_{\lambda/\mu} = \sum_T x^T \ , \tag{1.4}$$

where the sum runs over all column strict tableaux T whose shape is given by the skew tableau λ/μ. We set $x^T = x_1^{\mathrm{wt}_1(T)} x_2^{\mathrm{wt}_2(T)} \cdots$, where $\mathrm{wt}_i(T)$ is the number of entries in T which is equal to i. We will provide further details in due course (see Figure 2.2 for an example of column strict tableau). A similar description holds for ordinary Schur functions thanks to the equality $s_\lambda = s_{\lambda/\emptyset}$, with $\emptyset$ the empty partition.

The combinatorial formula (1.4) can be generalised to cylindric skew shapes and cylindric row strict tableaux [5, 28, 53, 58]. In fact, skew Schur functions are particular cases of 'cylindric Schur functions', which are defined [58] as

$$s_{\lambda/d/\mu} = \sum_{\hat{T}} x^{\hat{T}} \ . \tag{1.5}$$

Here λ and μ are partitions fitting inside a $k \times (n-k)$ rectangle, and d is a non-negative integer. The sum runs over all cylindric column strict tableaux of shape $\lambda/d/\mu$, and $x^{\hat{T}} = x_1^{\mathrm{wt}_1(\hat{T})} x_2^{\mathrm{wt}_2(\hat{T})} \cdots$ (see Chapter 3 for further details). The representation theoretical interpretation of the expansion (1.3) generalises to toric Schur functions, which are the specialisation of $s_{\lambda/d/\mu}$ to k variables. It was conjectured in [58] that $s_{\lambda/d/\mu}(x_1, \dots, x_k)$ is the $\mathrm{GL}_k(\mathbb{C})$-character of a 'cylindric Schur module', and a potential proof of this statement with the help of positroid classes appears in [57]. It follows that the coefficients $C^{\lambda,d}_{\mu\nu}$ defined via the expansion [58]

$$s_{\lambda/d/\mu}(x_1,\ldots,x_k)=\sum_{\nu\in\mathcal{P}^+}C^{\lambda,d}_{\mu\nu}s_\nu(x_1,\ldots,x_k) \tag{1.6}$$

are non-negative integers. In particular, we have that $C^{\lambda,0}_{\mu\nu}=c^{\lambda}_{\mu\nu}$. While the cylindric Schur functions (1.5) are not Schur positive, they are instead 'cylindric Schur positive', as conjectured in [53]. In fact, we have the expansion [50]

$$s_{\lambda/d/\mu}=\sum_{d'=0}^{d}\sum_{\nu\in\mathcal{P}^+}C^{\lambda,d'}_{\mu\nu}s_{\nu/(d-d')/\emptyset}\ . \tag{1.7}$$

An alternative proof of (1.7) was presented in [45] by employing the analogues of the Schur polynomials as elements in the principal Heisenberg subalgebra. In the discussion of algebraic geometry, the coefficients $C^{\lambda,d}_{\mu\nu}$ are known as (3-point) Gromov-Witten invariants [71]. They count the number of rational curves of degree d in $\mathrm{Gr}(k,n)$ that meet fixed generic translates of the Schubert varieties $\Omega_\mu,\Omega_\nu,\Omega_\lambda$. The Gromov-Witten invariants are the structure constants of the (small) quantum cohomology $qH^*(\mathrm{Gr}(k,n))$ [26,32,69,71]. As a linear space, $qH^*(\mathrm{Gr}(k,n))$ is equal to the tensor product $H^*(\mathrm{Gr}(k,n))\otimes\mathbb{Z}[q]$, whereas the product of two Schubert classes in $qH^*(\mathrm{Gr}(k,n))$ is a q-deformation of the product in $H^*(\mathrm{Gr}(k,n))$. The map $\sigma_\lambda\mapsto s_\lambda(x_1,\ldots,x_k)$ introduced above is an isomorphism of rings [61]

$$qH^*(\mathrm{Gr}(k,n))\cong\Lambda_k\otimes\mathbb{Z}[q]/\langle s_{(n+k-1)},\ldots,s_{(n-1)},s_{(n)}+q(-1)^k\rangle\ . \tag{1.8}$$

and we have the following product expansion in the quotient ring (1.8),

$$s_\mu(x_1,\ldots,x_k)s_\nu(x_1,\ldots,x_k)=\sum_{d\geq0}q^d\sum_{\lambda\in\mathcal{P}^+}C^{\lambda,d}_{\mu\nu}s_\lambda(x_1,\ldots,x_k)\ . \tag{1.9}$$

A combinatorial proof for the non-negativity of $C^{\lambda,d}_{\mu\nu}$ is still missing. There exists a formula for $C^{\lambda,d}_{\mu\nu}$ as an alternating sum of Littlewood-Richardson coefficients [5], but this formula is not manifestly positive. Attempts have been made to solve this problem by means of Knutson-Tao puzzles [11,12,47].

1.1 Verlinde algebras

It is known [71] that the small quantum cohomology $qH^*(\mathrm{Gr}(k,n))$, when evaluated at $q=1$, is isomorphic to the $\hat{\mathfrak{gl}}_n$-Verlinde algebra at level k. The quotient of $qH^*(\mathrm{Gr}(k,n))$ obtained by imposing the further relations $q=s_{(1^k)}$ and $s_{(n)}=1$ is in turn isomorphic to the $\hat{\mathfrak{sl}}_n$-Verlinde algebra at level k [46]. The Verlinde algebra of an affine Lie algebra $\hat{\mathfrak{g}}$

is the fusion algebra of the integrable highest weight modules of level k (see i.e. [33] for details). The structure constants of the Verlinde algebra, the so-called fusion coefficients, are given by the celebrated Verlinde formula [70]

$$\mathcal{N}_{\mu\nu}^{\lambda} = \sum_{\sigma} \frac{\mathcal{S}_{\mu\sigma}\mathcal{S}_{\nu\sigma}\mathcal{S}_{\sigma\lambda}^{-1}}{\mathcal{S}_{\emptyset\sigma}} , \tag{1.10}$$

where λ, μ, ν are labels for the integrable dominant weights of level k. The characters of the integrable highest weight modules of level k yield a representation of the modular group $\mathrm{SL}_2(\mathbb{Z})$ [33]. The images of the generators of $\mathrm{SL}_2(\mathbb{Z})$ are known as the $\mathcal{S}$-matrix (which appears in the Verlinde formula above) and the $\mathcal{T}$-matrix of the Verlinde algebra. There is a geometrical interpretation of the fusion coefficients as the dimension of moduli spaces of generalised θ-functions, the so-called conformal blocks [4, 23].

The Verlinde algebra plays a central role in the discussion of conformal field theory (CFT). Wess-Zumino-Witten models, which are a subclass of rational CFTs, can be constructed from the integrable highest weight modules of affine Lie algebras, with the level k fixing the value of the central element, and the primary fields being in one-to-one correspondence with the highest weight vectors. The fusion of two primary fields is then described by the Verlinde formula. The representation of $\mathrm{SL}_2(\mathbb{Z})$ mentioned above is at the core of modular covariance in rational CFTs (see e.g. the textbook [17] for further details).

Modular tensor categories (MTCs) arise as representation categories of rational CFTs (see e.g. [55, 72] and references therein). The Verlinde algebra is the Grothendieck ring of a MTC [2], which is the ring generated by isomorphism classes of simple objects, and the Verlinde formula yields the structure constants of this ring. The $\mathcal{S}$-matrix and $\mathcal{T}$-matrix of a MTC, which represent the modular datum of the MTC itself, form a projective representation of the modular group $\mathrm{SL}_2(\mathbb{Z})$ (see e.g. [10]). The notion of modular data is important because MTCs are usually classified according to their modular data [21]. A MTC determines uniquely a three-dimensional topological quantum field theory (TQFT) [68], and the Verlinde algebra itself can be also seen as a TQFT, but a two-dimensional one. It is well known [1] that the category of two-dimensional TQFTs is canonically equivalent to the category of commutative Frobenius algebras.

1.2 Statistical vertex models

The study of statistical vertex models has attracted increasing attention over the last century, reaching its climax with Baxter's solution of the eight-vertex model [3]. The hallmark of an exactly solvable vertex model consists in the Yang-Baxter equation [3,9]

$$\mathcal{R}_{12}(x,y)\mathcal{L}_1(x)\mathcal{L}_2(y) = \mathcal{L}_2(y)\mathcal{L}_1(x)\mathcal{R}_{12}(x,y) \ . \tag{1.11}$$

Here $\mathcal{L}(x)$ is a (infinite dimensional) matrix whose entries belong to an appropriate algebra $\mathcal{H}$. In other words, we have that $\mathcal{L}(x) \in \mathrm{End}(V(x) \otimes \mathcal{H})$, where V is some complex vector space and $V(x) \equiv V \otimes \mathbb{C}[[x]]$. Moreover, $\mathcal{R}(x,y) \in \mathrm{End}(V(x) \otimes V(y))$. The Yang-Baxter equation is therefore an identity in $\mathrm{End}(V(x) \otimes V(y) \otimes \mathcal{H})$, and the subscripts indicate which copy of V the operators act on. The monodromy matrix $\mathcal{T}(x) = \mathcal{L}_n(x) \cdots \mathcal{L}_2(x)\mathcal{L}_1(x) \in \mathrm{End}(V(x) \otimes \mathcal{H}^{\otimes n})$, where the subscripts now indicate the copy of $\mathcal{H}$ in the tensor product $\mathcal{H}^{\otimes n}$, is also a solution of the Yang-Baxter equation. This can be employed to define Baxter's 'transfer matrix' as the operator $T(x) \in \mathcal{H}^{\otimes n} \otimes \mathbb{C}[[x]]$ given by the partial trace

$$T(x) = \mathrm{Tr}_V\, \mathcal{T}(x) \ . \tag{1.12}$$

As a direct consequence of the Yang-Baxter equation, we have the commutation relation

$$T(x)T(y) = T(y)T(x) \tag{1.13}$$

for arbitrary x, y. The partition function of a statistical vertex model encodes all the physical properties of the model itself. Imposing period boundary conditions in the horizontal direction (that is, defining the vertex model on a cylinder) the partition function can be identified with the matrix element

$$\langle \lambda | T(x_1)T(x_2) \cdots T(x_k) | \mu \rangle \ .$$

The algebra $\mathcal{H}^{\otimes n}$ acts on the vector space spanned by basis vectors $|\mu\rangle$ in bra-ket notation from physics, where μ are labels for the vertical boundary conditions of the lattice.

The commutation relation (1.13) implies that the partition function is symmetric in the variables $(x_1, \dots, x_k)$. A natural question is whether there exists an expansion in terms of known symmetric functions which exhibits interesting combinatorial features. It was shown [42] that the toric Schur functions (1.6) can be identified with the partition functions of two exactly solvable vertex models, namely the vicious and osculating models (see *loc. cit.* and references therein). A new class of cylindric symmetric functions $P'_{\lambda/d/\mu}(q)$, which can be interpreted as cylindric versions of q-Whittaker functions [27], originated from the computation of the partition function of the vertex model defined by Baxter's Q^+-operator associated to the q-boson model [41]. If $q = 0$ one recovers the cylindric Schur functions defined in (1.5), but with the cylindric loop associated to λ shifted in the vertical direction rather than in the diagonal one [41, 53, 58]. The expansion coefficients in terms of Schur functions are now the structure constants of the $\hat{\mathfrak{sl}}_n$-Verlinde algebra. In fact, with an appropriate labelling of the basis elements, the latter coincide with the Gromov-Witten invariants $C_{\mu\nu}^{\lambda,d}$ [38]. There is another Q-operator associated to the q-boson model, which has been identified with a quantum version of Baxter's Q^{-}-operator [43]. A combinatorial interpretation for the partition function of the vertex model defined by this operator is still missing for arbitrary q.

1.3 Quantum integrable models

The transfer matrix $T(x)$ introduced in Section 1.2 can be interpreted as the generator of quantum integrals of motion of a one-dimensional quantum system [3, 37]. That is, the coefficients $\{T_r\}_{r\geq 0}$ defined by the expansion

$$T(x) = \sum_{r\geq 0} x^r T_r \tag{1.14}$$

are the commuting Hamiltonians of a quantum integrable model. The basis vectors $|\mu\rangle$ described above span the Fock space of the algebra $\mathcal{H}^{\otimes n}$. This is slightly different from the XXX or XXZ models, where the quantum integrals of motion are obtained via logarithmic derivatives of $T(x)$ (see e.g. [18]). In the context of quantum integrability, the methodology described in Section 1.2 is part of what is known as the 'Quantum Inverse Scattering Method' (QISM), or 'Algebraic Bethe ansatz' [22, 62–64]. The main feature of the QISM is that one can employ the Yang-Baxter equation to diagonalise simultaneously the quantum integrable of motions (see *loc. cit.*).

There is a remarkable connection between quantum integrable models and Verlinde algebras. Employing non-commutative versions of Schur polynomials, one can endow the k-particle subspace of the phase model [9] with the structure of an algebra, which is isomorphic to the $\hat{\mathfrak{sl}}_n$-Verlinde algebra [46]. The phase model can be diagonalised using the QISM, and the transition matrix from the basis of eigenvectors to the particle basis coincides with the $\mathcal{S}$-matrix of the $\hat{\mathfrak{sl}}_n$-Verlinde algebra.

The preceding discussion was generalised to the q-boson model [41]. The k-particle subspace of this model is endowed with the structure of a Frobenius algebra. This algebra can be interpreted as a deformation of the $\hat{\mathfrak{sl}}_n$-Verlinde algebra, since for $q = 0$ it specialises to the latter. The q-boson model is also diagonalisable via the QISM for $q \neq 1$, and the transition matrix from the basis of eigenvectors to the particle basis defines a deformed $\mathcal{S}$-matrix, which can be employed to construct a deformed Verlinde formula. It is not known whether the deformed $\mathcal{S}$-matrix is the generator of a representation of the modular group $\mathrm{SL}_2(\mathbb{Z})$ for arbitrary q. The deformed Verlinde formula yields the structure constants $\mathcal{N}_{\mu\nu}^{\lambda,d}(q)$ of the deformed Verlinde algebra, and the latter are linked to the partition function of the vertex model defined by the Q^+ operator associated to the q-boson model. Namely, the coefficients $\mathcal{N}_{\mu\nu}^{\lambda,d}(q)$ appear in the expansion of the cylindric q-Whittaker function $P'_{\lambda/d/\mu}(q)$ introduced in Section 1.2 in terms of ordinary q-Whittaker functions [41]. A geometric interpretation or connection to MTCs is currently unknown, so it is of interest to investigate the simpler case $q = 1$ first.

1.4 Present work and outline of this thesis

We shall refer to the statistical vertex models defined by the Q^+ and Q^- operators associated to the q-boson model (see Section 1.2) as the Q^+ and Q^- vertex models. The partition functions of these models depend on a indeterminate z which keeps track of the winding number around the cylinder. Setting $z = 0$ and $q = 1$, the latter can be identified respectively with 'skew elementary symmetric functions' $e_{\lambda/\mu}$ and 'skew complete symmetric functions' $h_{\lambda/\mu}$ which are defined via the following co-product expansions,

$$\Delta e_\lambda = \sum_{\mu\in\mathcal{P}^+} e_\mu \otimes e_{\lambda/\mu}\ , \qquad \Delta h_\lambda = \sum_{\mu\in\mathcal{P}^+} h_\mu \otimes h_{\lambda/\mu}\ . \tag{1.15}$$

The basis $\{e_\lambda\}_{\lambda\in\mathcal{P}^+}$ and $\{h_\lambda\}_{\lambda\in\mathcal{P}^+}$ of elementary and complete symmetric functions will be described in Chapter 3. This is the main observation that motivated the present work from a combinatorial point of view.

We will show that the symmetric functions $e_{\lambda/\mu}$ and $h_{\lambda/\mu}$ have combinatorial expressions which resemble the one for skew Schur functions given by (1.3). These expressions are obtained by employing the cardinalities of sets involving the symmetric group. With the help of the affine symmetric group and cylindric reverse plane partitions [28], we will generalise $e_{\lambda/\mu}$ and $h_{\lambda/\mu}$ to cylindric analogues $e_{\lambda/d/\mu}$ and $h_{\lambda/d/\mu}$, the cylindric elementary and complete symmetric functions. We will prove the validity of the following expansions,

$$e_{\lambda/d/\mu} = \sum_{d'=0}^{d+n}\sum_{\nu\in\mathcal{P}^+} \mathcal{N}_{\mu\nu}^{\lambda,d'}(1) e_{\nu/(d-d')/\emptyset}\ , \qquad h_{\lambda/d/\mu} = \sum_{d'=0}^{d+n}\sum_{\nu\in\mathcal{P}^+} \mathcal{N}_{\mu\nu}^{\lambda,d'}(1) h_{\nu/(d-d')/\emptyset}\ , \tag{1.16}$$

where the coefficients $\mathcal{N}_{\mu\nu}^{\lambda,d}(1)$ coincide with the structure constants $\mathcal{N}_{\mu\nu}^{\lambda,d}(q)$ of the deformed Verlinde algebra discussed in Section 1.3, when evaluated at $q = 1$. We will show that the symmetric function $e_{\lambda/d/\mu}$ can be identified with the partition function of the Q^+ vertex model for $q = 1$ and generic z. In other words, $e_{\lambda/d/\mu}$ coincides with the cylindric

q-Whittaker function $P'_{\lambda'/d/\mu'}(q)$ discussed in Section 1.2, when evaluated at $q = 1$. On the other hand, the symmetric function $h_{\lambda/d/\mu}$ has not been introduced in the literature previously. We will describe the link between $h_{\lambda/d/\mu}$ and the partition function of the Q^- vertex model for $q = 1$ and generic z. Part of this thesis is a joint work with C. Korff that appears in [44, 45]. We will refer to these papers when necessary. In particular, the connection with quantum integrable systems is not in [44, 45].

We finish this introduction with an outline of this thesis. The reader may find it useful to refer to the flowchart presented in Figure 1.1.

Chapter 2 We recall the notions of partitions, plane partitions, and the symmetric group. Then we introduce the ring of symmetric functions Λ, and we describe some of its bases. We present some properties of the symmetric functions $e_{\lambda/\mu}$ and $h_{\lambda/\mu}$, and we provide their expansions in terms of the bases of Λ described previously. Finally, we give combinatorial expressions for $e_{\lambda/\mu}$ and $h_{\lambda/\mu}$ which resemble the one for skew Schur functions given by (1.3).

Chapter 3 We recall the notions of cylindric reverse plane partitions and the affine symmetric group. With the help of these mathematical objects, we generalise $e_{\lambda/\mu}$ and $h_{\lambda/\mu}$ to cylindric analogues $e_{\lambda/d/\mu}$ and $h_{\lambda/d/\mu}$. Then we provide the expansions of $e_{\lambda/d/\mu}$ and $h_{\lambda/d/\mu}$ in terms of the bases of Λ introduced in Chapter 2, and we prove the validity of the expansions (1.16). To this end, we first derive some product expansions which hold in a quotient of the ring $\Lambda_k \otimes \mathbb{Z}[z, z^{-1}]$. We finish this chapter by discussing some further properties of $e_{\lambda/d/\mu}$ and $h_{\lambda/d/\mu}$.

Chapter 4 We describe the Q^+ and Q^- vertex models for the case $q = 1$, and we present three solutions of the Yang-Baxter equation. Then we introduce two further vertex models, which are related to the previous ones by taking the adjoint of the transfer matrices. We evaluate the partition functions of all these vertex models in terms of $e_{\lambda/d/\mu}$ and $h_{\lambda/d/\mu}$. In the second part of this chapter we describe the conserved charges (i.e. the quantum integral of motions) for the free boson model, which is the $q = 1$ specialisation of the q-boson model. We employ the matrix elements of these conserved charges to present an alternative proof for the expansions of $e_{\lambda/d/\mu}$ and $h_{\lambda/d/\mu}$ described in Chapter 3. Then we use the same matrix elements to illustrate an alternative approach for computing the partition functions of the vertex models defined previously.

Chapter 5 We endow the k particle space of the free boson model with the structure of a Frobenius algebra. This coincides with the deformed Verlinde algebra discussed in Section 1.3, when evaluated at $q = 1$. We show that the transition matrix from the basis of un-normalised eigenvectors to the particle basis is a generator of a representation of the modular group $\mathrm{SL}_2(\mathbb{Z})$. We employ this matrix to construct a Verlinde-type formula, which yields the structure constants $\mathcal{N}_{\mu\nu}^{\lambda,d}(1)$ appearing in the expansion (1.16). Finally, we present a formula for $\mathcal{N}_{\mu\nu}^{\lambda,d}(1)$ in terms of tensor multiplicities for irreducible representations of the generalised symmetric group.

Figure 1.1: A flowchart connecting the various topics presented in this thesis. It is understood that $q = 1$.

Chapter 2

Symmetric functions

The first part of this chapter is devoted to the exposition of some basic concepts, which will be used throughout this thesis. We introduce the ring of symmetric functions Λ, we discuss its Hopf algebra structure, and we describe some of its bases. For this purpose, we first recall the notions of partitions, plane partitions, and the symmetric group. In the second part of this chapter we describe in detail the skew symmetric functions $e_{\lambda/\mu}$ and $h_{\lambda/\mu}$, which were introduced in (1.15). In particular, we provide their expansions in terms of the bases of Λ described earlier in this chapter, and we give combinatorial expressions by employing the cardinalities of sets involving the symmetric group. From here to the end of this thesis, we denote with $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \dots\}$ the non-negative integers and with $\mathbb{N} = \{1, 2, 3, \dots\}$ the positive integers. The main references for the first part of this chapter are [52, 60, 67].

2.1 Partitions

A partition $\lambda = (\lambda_1, \lambda_2, \dots)$ is a (finite or infinite) sequence of weakly decreasing non-negative integers, that is $\lambda_1 \geq \lambda_2 \geq \cdots$, with only finitely many non-zero terms [52]. More generally, a composition $\alpha = (\alpha_1, \alpha_2, \dots)$ is an analogous sequence which is not necessarily weakly decreasing. As a simple example we have that $(3, 2, 0)$ is a partition, whereas $(0, 1, 2)$ is a composition. We will not distinguish between two such sequences differing only by a string of zeroes at the end. So for example we regard $(0, 1, 2)$, $(0, 1, 2, 0)$ and $(0, 1, 2, 0, 0, \dots)$ as the same composition. We will refer to $\mathcal{P}^+$ as the set of all partitions, and to $\mathcal{P}$ as the set of all compositions. By definition it is clear that $\mathcal{P}^+ \subset \mathcal{P}$.

Let us introduce some notation. Call λ_i and α_i the parts (or the elements) of the partition λ and the composition α respectively. Denote with $\emptyset$ the empty partition, that is the partition whose parts are all equal to 0. The length of a partition λ is the number $\ell(\lambda)$ of non-zero elements in λ, whereas the weight $|\lambda|$ is the sum of these elements. The weight $|\alpha|$ of a composition α is defined in an analogous way. For $i \in \mathbb{N}$ denote with $m_i(\lambda)$

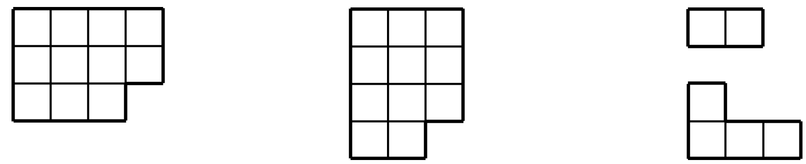


Figure 2.1: From left to right we have the Young diagrams of the partition $\lambda = (4, 4, 3)$, its conjugate partition $\lambda' = (3, 3, 3, 2)$, and the composition $\alpha = (2, 0, 1, 3)$.

the multiplicity of i in λ, that is the number of parts in λ equal to i. Then we have the equivalent notation for partitions

$$\lambda = (1^{m_1(\lambda)}, 2^{m_2(\lambda)}, \dots) \; . \tag{2.1}$$

The Young diagram of λ is a set of left-justified boxes with λ_i boxes in the i-th row. Formally one can think of this as a subset of points in the $\mathbb{Z} \times \mathbb{Z}$ plane, whose coordinates are increasing from left to right and downwards. Namely, one identifies the lower-right vertex of the box in the i-th row and j-th column of a Young diagram with the point $(i, j) \in \mathbb{Z} \times \mathbb{Z}$. Denote with λ' the conjugate partition of λ, that is the partition whose diagram is obtained by reflecting the boxes of the diagram of λ along the line $\{(i, i) \mid i \in \mathbb{Z}\}$. Then λ'_j is the number of boxes in the j-th column of λ, and one can show that

$$m_i(\lambda) = \lambda'_i - \lambda'_{i+1} \; . \tag{2.2}$$

It will be useful to extend the notion of Young diagram to compositions [30, 31]. Define the (Young) diagram of a composition α as a set of left-justified boxes with α_i boxes in the i-th row. See Figure 2.1 for an example.

Let $\lambda, \mu \in \mathcal{P}^+$, and write $\mu \subset \lambda$ if the diagram of μ is contained in the diagram of λ, that is if $\mu_i \leq \lambda_i$ for all $i \in \mathbb{N}$. Assuming that $\mu \subset \lambda$, we define the skew diagram $\lambda/\mu \subset \mathbb{Z} \times \mathbb{Z}$ as

$$\lambda/\mu = \{(i, j) \in \mathbb{Z} \times \mathbb{Z} \,|\, 1 \leq i \leq \ell(\lambda), \mu_i < j \leq \lambda_i\} \; . \tag{2.3}$$

Denote with $|\lambda/\mu| = |\lambda| - |\mu|$ its cardinality, that is the number of boxes in λ/μ. Notice that if $\mu = \emptyset$ this simply is the Young diagram of λ. If λ/μ has at most one box in each row (respectively column) we will call it a vertical (respectively horizontal) strip. By convention the skew diagram λ/λ, which has no boxes, is both a vertical and a horizontal strip. We will sometimes use the statement 'λ/μ is a skew diagram' to mean that $\mu \subset \lambda$.

2.1.1 Plane partitions and tableaux

Assume throughout this section that $\lambda, \mu \in \mathcal{P}^+$ with $\mu \subset \lambda$.

Definition 2.1.1. A plane partition π of shape λ/μ is a filling of the boxes of λ/μ with positive integers, called the entries of π, which are weakly decreasing from left to right in rows and down columns.

Since $\lambda/\mu \subset \mathbb{Z}\times\mathbb{Z}$ we can interpret a plane partition as a map $\pi : \lambda/\mu \to \mathbb{N}$, $(i,j) \mapsto \pi_{i,j}$ with the constraints

$$\begin{aligned} \pi_{i,j} &\geq \pi_{i+1,j}\ , &\quad &\text{if } (i+1,j) \in \lambda/\mu\ , \\ \pi_{i,j} &\geq \pi_{i,j+1}\ , &\quad &\text{if } (i,j+1) \in \lambda/\mu\ . \end{aligned}$$

We will call π a reverse plane partition (RPP) if the entries of π are weakly increasing from left to right and down columns instead. Denote with $\mathrm{wt}_i(\pi)$ the number of entries of π equal to i. The weight of a plane partition π is defined as the composition $\mathrm{wt}(\pi) = (\mathrm{wt}_1(\pi), \mathrm{wt}_2(\pi), \dots)$. A similar notation holds for RPPs. See Figure 2.2 for an example.

Lemma 2.1.2. *A plane partition π of shape λ/μ with highest entry $l \in \mathbb{N}$ is equivalent to a sequence $\{\lambda^{(r)}\}_{r=0}^{l}$ of partitions with*

$$\mu = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(l)} = \lambda\ . \tag{2.4}$$

An analogous statement is true for RPPs.

Proof. This is a well known result, nevertheless we present a proof for RPPs since in the next chapter we will generalise these objects to the 'cylinder'. The proof for plane partitions is similar and therefore we omit it. So suppose that π is a RPP with highest entry l, and for $r = 1, \dots, l$ let $\lambda^{(r)}$ be the partition whose Young diagram is obtained by joining the Young diagram of μ with the boxes of π containing the entries from 1 to r. In particular this gives $\lambda^{(l)} = \lambda$, and together with $\lambda^{(0)} = \mu$ we obtain a sequence of partitions $\{\lambda^{(r)}\}_{r=0}^{l}$ which by construction satisfies (2.4). Conversely, if for $r = 1, \dots, l$ we fill the boxes of the skew diagram $\lambda^{(r)}/\lambda^{(r-1)}$ with the integer r, we obtain a RPP π of shape λ/μ with highest entry l, and moreover $\mathrm{wt}_r(\pi) = |\lambda^{(r)}/\lambda^{(r-1)}|$. □

In the following we will focus our attention mostly on RPPs and their special cases of row strict and column strict tableaux. The latter are also known in the literature as semistandard tableaux.

Definition 2.1.3. A row (respectively column) strict tableau T of shape λ/μ is a RPP whose entries strictly increase along each row (respectively column).

Equivalently, a row (respectively column) strict tableau of shape λ/μ is a sequence of partitions $\{\lambda^{(r)}\}_{r=0}^{l}$ satisfying (2.4) such that $\lambda^{(r)}/\lambda^{(r-1)}$ is a vertical (respectively horizontal) strip for $r = 1, \dots, l$. See once again Figure 2.2 for an example.

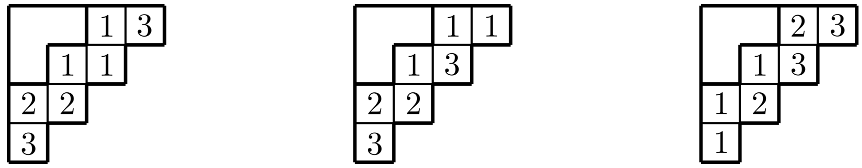


Figure 2.2: Let $\lambda = (4,3,2,1)$ and $\mu = (2,1)$. From left to right we have a RPP, a column strict tableau and a row strict tableau, all of which have weight $(3,2,2)$. The RPP on the left is equivalent to the sequence of partitions $(2,1)$, $(3,3)$, $(3,3,2)$, $(4,3,2,1)$.

Definition 2.1.4. A standard tableau T of shape λ/μ is a row strict tableau which contains each entry $1, 2, \ldots, |\lambda/\mu|$ exactly once.

It follows immediately from the definition above that a standard tableau has weight $\mathrm{wt}(T) = (1, 1, \ldots, 1)$. Furthermore, notice that a standard tableau is also a column strict tableau. Define $f_{\lambda/\mu}$ as the number of standard tableaux of shape λ/μ. For $\mu = \emptyset$ we have the formula (see e.g. [67, Ch. 3.10])

$$f_\lambda = \frac{|\lambda|!}{\prod_{(i,j)\in\lambda} h_\lambda(i,j)} \ , \qquad h_\lambda(i,j) = \lambda_i + \lambda'_j - i - j + 1 \ . \tag{2.5}$$

2.1.2 The symmetric group

For a review of the symmetric group, together with its representation theory, see for example [60]. Fix $k \in \mathbb{N}$ until the end of this section.

Definition 2.1.5. The symmetric group S_k in k letters is the Coxeter group A_{k-1}, that is the group generated by $\{\sigma_1, \ldots, \sigma_{k-1}\}$ subject to the relations

$$\sigma_i^2 = 1 \ , \qquad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \ , \qquad \sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{for } |i-j| > 1 \ . \tag{2.6}$$

This group can be realised as the group of bijections from $[k] \equiv \{1, 2, \ldots, k\}$ to itself, using composition as the group product. For this reason, the elements of S_k are called 'permutations'. With this realisation, the generators σ_i are given by the following maps,

$$\sigma_i(m) = \begin{cases} i+1 \ , & m = i \\ i \ , & m = i+1 \\ m \ , & \text{otherwise} \end{cases} \ .$$

Denote with $\mathfrak{gl}_k$ the Lie algebra of the general linear group $\mathrm{GL}_k(\mathbb{C})$. Moreover, let $\mathcal{P}_k = \bigoplus_{i=1}^k \mathbb{Z}\epsilon_i$ be the $\mathfrak{gl}_k$ weight lattice with standard basis $\epsilon_1, \ldots, \epsilon_k$ and inner product $(\epsilon_i, \epsilon_j) = \delta_{ij}$. We use the notation $\alpha = (\alpha_1, \ldots, \alpha_k)$ for $\alpha = \sum_{i=1}^k \alpha_i\epsilon_i \in \mathcal{P}_k$. Denote with $\mathcal{P}_k^{\geq 0} \subset \mathcal{P}_k$ the positive weights, that is

$$\mathcal{P}_k^{\geq 0} = \{\alpha \in \mathcal{P}_k \mid \alpha_i \geq 0 \text{ for } i = 1, \ldots, k\} \ , \tag{2.7}$$

and with $\mathcal{P}_k^+ \subset \mathcal{P}_k^{\geq 0} \subset \mathcal{P}_k$ the positive dominant weights, namely

$$\mathcal{P}_k^+ = \{\lambda \in \mathcal{P}_k \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0\} \ . \tag{2.8}$$

We will often identify the compositions α with $\alpha_i = 0$ for $i > k$ as weights in $\mathcal{P}_k^{\geq 0}$. In particular we identify the partitions λ with $\ell(\lambda) \leq k$ as weights in $\mathcal{P}_k^+$.

We shall use the right action $\mathcal{P}_k \times S_k \to \mathcal{P}_k$ given by $(\alpha, w) \mapsto \alpha.w = (\alpha_{w(1)}, \ldots, \alpha_{w(k)})$. In particular, $\alpha.\sigma_i$ is the weight obtained from α by permuting its entries at positions i and $i+1$. Let $\lambda \in \mathcal{P}_k^+$ and denote by $S_\lambda \subset S_k$ its stabilizer subgroup, that is the subgroup of permutations $w \in S_k$ such that $\lambda.w = \lambda$. The stabilizer subgroup S_λ of the weight $\lambda \in \mathcal{P}_k^+$ is a parabolic subgroup of S_k (see for example [8, Ch. 2.4] for further details). Its cardinality is given by $|S_\lambda| = \prod_{i \in \mathbb{Z}_{\geq 0}} m_i(\lambda)!$, where $m_i(\lambda)$ for $i \in \mathbb{Z}$ denotes the multiplicity of i in the weight λ. Denote with $S_\lambda \setminus S_k$ the set of right cosets $\{S_\lambda w \mid w \in S_k\}$ of S_λ in S_k. The following is a special case of a more general result involving Coxeter groups and parabolic subgroups, which can be found for instance in [8, Prop. 2.4.4 and Cor. 2.4.5]. For this purpose, define the length of $w \in S_k$ as

$$\ell(w) = \min\{r \in \mathbb{N} \mid w = \sigma_{i_1} \cdots \sigma_{i_r} \text{ for some } i_1, \ldots, i_r \in [k-1]\} \ . \tag{2.9}$$

Proposition 2.1.6. *(i) Each right coset $S_\lambda w$ has a unique representative of minimal length.*

(ii) Every element $w \in S_k$ has a unique decomposition $w = w_\lambda w^\lambda$, with $w_\lambda \in S_\lambda$ and w^λ a minimal length representative of one right coset in $S_\lambda \setminus S_k$.

Denote with S^λ the set of minimal length representatives of the right cosets $S_\lambda \setminus S_k$.

Example 2.1.7. Let $\lambda = (3, 3, 2) \in \mathcal{P}_3^+$. Then $S_\lambda = \{1, \sigma_1\}$ and $S^\lambda = \{1, \sigma_2, \sigma_2\sigma_1\}$.

2.2 The ring of symmetric functions

Let $x = \{x_1, x_2, \ldots\}$ be an infinite set of commuting indeterminates and consider the formal power series ring $\mathbb{Z}[[x_1, x_2, \ldots]]$. For each $k \in \mathbb{N}$ we have the left action $S_k \times \mathbb{Z}[[x_1, x_2, \ldots]] \to \mathbb{Z}[[x_1, x_2, \ldots]]$ given by

$$\big(w, f(x_1, x_2, \ldots)\big) \mapsto w.f(x_1, x_2, \ldots) = f(x_{w(1)}, \ldots, x_{w(k)}, x_{k+1}, \ldots) \ . \tag{2.10}$$

The next definition is equivalent to the one which is given for example in [67, p. 286] and in [60, p. 151].

Definition 2.2.1. The ring of symmetric functions Λ is the subring of $\mathbb{Z}[[x_1, x_2, \ldots]]$ whose elements $f(x_1, x_2, \ldots)$ satisfy the following conditions: (i) for every $k \in \mathbb{N}$ and $w \in S_k$ one has $w.f(x_1, x_2, \dots) = f(x_1, x_2, \dots)$, (ii) the degrees of the monomials in $f(x_1, x_2, \dots)$ are bounded.

We will most frequently drop the variable dependence, and we will write f rather than $f(x_1, x_2, \dots)$. We say that $f \in \Lambda$ has degree $n \in \mathbb{Z}_{\geq 0}$ if all the monomials appearing in f have degree n. The ring Λ then has the structure of a graded ring, where the grading is given by the degree. We will often be interested in dealing with a finite number of variables $(x_1, \dots, x_k)$ for some $k \in \mathbb{N}$. To this end, we will use the projection

$$\Lambda \to \Lambda_k = \mathbb{Z}[x_1, \dots, x_k]^{S_k} , \tag{2.11}$$

which is defined by setting $x_i = 0$ for $i > k$. The notation on the RHS of (2.11) stands for the set of S_k-invariants of $\mathbb{Z}[x_1, \dots, x_k]$. For $f \in \Lambda$ denote by $f(x_1, \dots, x_k)$ its projection onto Λ_k. Condition (ii) in Definition 2.2.1 ensures that this projection is well defined, that is $f(x_1, \dots, x_k)$ consists of a finite sum of monomials.

Remark 2.2.2. There is yet another alternative definition of the ring Λ as the inverse limit $\Lambda = \varprojlim \Lambda_k$ in the category of graded rings, see [52, I.2] for further details. For our purposes, it is enough to realise that each $f \in \Lambda$ can be equivalently defined as a sequence of functions $\{f_k(x_1, \dots, x_k)\}_{k \in \mathbb{Z}_{\geq 0}}$, with $f_k(x_1, \dots, x_k) \in \Lambda_k$, such that $f_{k'}(x_1, \dots, x_k, 0, \dots, 0) = f_k(x_1, \dots, x_k)$ whenever $k' \geq k$. In particular, we have that the projection $\Lambda \to \Lambda_k$ sends f to $f_k(x_1, \dots, x_k)$.

2.2.1 Monomial symmetric functions

We now proceed to describe various bases of Λ [52, 60, 67]. For λ a partition and α a composition, we write $\alpha \sim \lambda$ and say that α is a permutation of λ, if there exist distinct indices $\{i_1, i_2, \dots, i_{\ell(\lambda)}\}$ such that $\alpha_{i_j} = \lambda_j$ for $j = 1, \dots, \ell(\lambda)$ and if, furthermore, the other parts of α are 0. In particular $\lambda \sim \lambda$. The monomial symmetric functions are defined as

$$m_\lambda = \sum_{\alpha \sim \lambda} x^\alpha = \sum_{\alpha \sim \lambda} x_1^{\alpha_1} x_2^{\alpha_2} \cdots \ . \tag{2.12}$$

Notice that $m_\emptyset = 1$ as the only permutation of $\emptyset$ is given by itself.

Example 2.2.3. Some permutations of $\lambda = (3, 3, 2)$ are given by $(3, 3, 2)$, $(3, 2, 3)$, $(2, 3, 3)$, $(0, 3, 3, 2)$, $(2, 0, 3, 3)$. Thus $m_{(3,3,2)} = x_1^3 x_2^3 x_3^2 + x_1^3 x_2^2 x_3^3 + x_1^2 x_2^3 x_3^3 + x_2^3 x_3^3 x_4^2 + x_1^2 x_3^3 x_4^3 + \dots$.

The set $\{m_\lambda\}_{\lambda \in \mathcal{P}^+}$ is a basis of Λ, and thus the following expansion is well defined,

$$m_\mu m_\nu = \sum_{\lambda \in \mathcal{P}^+} f^\lambda_{\mu\nu} \, m_\lambda \ . \tag{2.13}$$

The coefficients $f^{\lambda}_{\mu\nu}$ are non-negative integers, and they equal the cardinality of the set [13]

$$\{(\alpha,\beta)\in\mathcal{P}\times\mathcal{P},\alpha\sim\mu,\beta\sim\nu \mid \alpha+\beta=\lambda\}\ , \tag{2.14}$$

where the sum of two compositions α and β is defined as the composition γ with parts $\gamma_i=\alpha_i+\beta_i$ for $i\in\mathbb{N}$. By definition we have that $f^{\lambda}_{\mu\nu}=f^{\lambda}_{\nu\mu}$, and taking advantage of (2.14) it follows that $f^{\lambda}_{\mu\nu}$ is non-zero only if $\mu,\nu\subset\lambda$.

Example 2.2.4. Let $\mu=(2,2,1)$, $\nu=(1,1)$ and $\lambda=$(3,2,2). Consider the compositions $\alpha_1,\alpha_2\sim\mu$ given by $\alpha_1=(2,2,1)$, $\alpha_2=(2,1,2)$ and the compositions $\beta_1,\beta_2\sim\nu$ given by $\beta_1=(1,0,1)$, $\beta_2=(1,1,0)$. Then $f^{\lambda}_{\mu\nu}=2$, since the pairs (α_1,β_1) and (α_2,β_2) are the only ones satisfying the constraint $\alpha_1+\beta_1=\alpha_2+\beta_2=\lambda$.

Lemma 2.2.5 ([52])**.** *Let $\lambda\in\mathcal{P}^+$, $k\in\mathbb{N}$ and project onto the ring Λ_k. We have that $m_\lambda(x_1,\ldots,x_k)=0$ if $\ell(\lambda)>k$, otherwise*

$$m_\lambda(x_1,\ldots,x_k)=\sum_{w\in S^\lambda} x_1^{\lambda_{w(1)}}\cdots x_k^{\lambda_{w(k)}}=\frac{1}{|S_\lambda|}\sum_{w\in S_k} x_1^{\lambda_{w(1)}}\cdots x_k^{\lambda_{w(k)}}\ . \tag{2.15}$$

For $\ell(\lambda)\le k$ it is understood that $\lambda\in\mathcal{P}_k^+$ and thus $S_\lambda,S^\lambda\subset S_k$.

Proof. Each monomial x^α appearing in (2.12) consists of a product containing $\ell(\lambda)$ variables x_i. If $\ell(\lambda)$ exceeds k then at least one of these terms equals 0 since $x_i=0$ for $i>k$. This means that $x^\alpha=0$ for all $\alpha\sim\lambda$ and thus $m_\lambda(x_1,\ldots,x_k)=0$. So suppose that $\ell(\lambda)\le k$. For $\alpha\sim\lambda$ we have that x^α is non-zero if and only if the non-zero parts of α are in the first k positions. Assume that x^α is non-zero, then both $\lambda,\alpha\in\mathcal{P}_k$ and we can take advantage of the right action $\mathcal{P}_k\times S_k\to\mathcal{P}_k$ described above. Since $\alpha\sim\lambda$ there exists a unique permutation $w\in S^\lambda\subset S_k$ such that $\alpha=\lambda.w$, that is $x^\alpha=x_1^{\lambda_{w(1)}}\cdots x_k^{\lambda_{w(k)}}$, and the first equality in (2.15) follows by applying the same argument to each non-zero monomial x^α in (2.12). To prove the second equality, decompose each $w\in S_k$ as in part (ii) of Proposition 2.1.6, and then use the fact that $\lambda.w_\lambda=\lambda$ for $w_\lambda\in S_\lambda$. □

Remark 2.2.6. For every $k\in\mathbb{N}$ we have that the set $\{m_\lambda(x_1,\ldots,x_k)\}_{\lambda\in\mathcal{P}_k^+}$ is a basis of Λ_k. See [52, I.2] for details.

2.2.2 Elementary and complete symmetric functions

The next two bases of Λ of interest are given by the elementary symmetric functions $\{e_\lambda\}_{\lambda\in\mathcal{P}^+}$ and the complete symmetric functions $\{h_\lambda\}_{\lambda\in\mathcal{P}^+}$ respectively. For $r\in\mathbb{N}$ these are defined as

$$e_r = \sum_{1<i_1<i_2<\cdots<i_r} x_{i_1}x_{i_2}\cdots x_{i_r} , \qquad e_\lambda = e_{\lambda_1}e_{\lambda_2}\cdots , \tag{2.16}$$

$$h_r = \sum_{1\le i_1\le i_2\le\cdots\le i_r} x_{i_1}x_{i_2}\cdots x_{i_r} , \qquad h_\lambda = h_{\lambda_1}h_{\lambda_2}\cdots , \tag{2.17}$$

where $e_0 = h_0 = 1$. In other words, e_r consists of the sum of all products of r distinct variables, whereas h_r is the sum of all monomials of degree r.

Example 2.2.7. We have $e_2 = x_1x_2 + x_1x_3 + \cdots + x_2x_3 + x_2x_4 + \ldots$ and moreover $h_2 = x_1^2 + x_2^2 + \cdots + x_1x_2 + x_1x_3 + \cdots + x_2x_3 + x_2x_4 + \ldots$.

For $r \in \mathbb{N}$ we have the equivalent expressions

$$e_r = \sum_{\alpha\in\mathcal{P}} x^\alpha , \qquad h_r = \sum_{\beta\in\mathcal{P}} x^\beta , \tag{2.18}$$

where the sums run over all compositions α and β with $|\alpha| = |\beta| = r$, and moreover $\alpha_i = 0, 1$ for $i \in \mathbb{N}$. A comparison with (2.12) shows that $e_r = m_{(1^r)}$ and $h_r = \sum_{\mu\in\mathcal{P}^+} m_\mu$, where the second sum is restricted to those $\mu \in \mathcal{P}^+$ with $|\mu| = r$. Let u be an indeterminate, then the generating functions for the elementary and complete symmetric functions are the elements in $\mathbb{Z}[[u]] \otimes_{\mathbb{Z}} \Lambda$ given by

$$E(u) = \sum_{r\ge0} u^r e_r = \prod_{j\ge1}(1 + ux_j) , \tag{2.19}$$

$$H(u) = \sum_{r\ge0} u^r h_r = \prod_{j\ge1} \frac{1}{1 - ux_j} . \tag{2.20}$$

From these equalities it can be readily seen that

$$E(-u)H(u) = 1 , \tag{2.21}$$

which is equivalent to the relations $\sum_{i=0}^r (-1)^i e_i h_{r-i} = 0$ for $r \in \mathbb{N}$.

We now want to express the symmetric functions $\{e_\lambda\}_{\lambda\in\mathcal{P}^+}$ and $\{h_\lambda\}_{\lambda\in\mathcal{P}^+}$ in terms of monomial symmetric functions. Given a matrix A, denote with r_i the sum of its elements in the i-th row and with c_j the sum of its elements in the j-th column. Define the compositions $\mathrm{row}(A) = (r_1, r_2, \ldots)$ and $\mathrm{col}(A) = (c_1, c_2, \ldots)$. Let $\lambda, \mu \in \mathcal{P}^+$ and denote with $M_{\lambda\mu}$ the number of $\ell(\lambda) \times \ell(\mu)$ matrices A with entries equal to 0 or 1 satisfying $\mathrm{row}(A) = \lambda$, $\mathrm{col}(A) = \mu$. Similarly let $L_{\lambda\mu}$ be the number of $\ell(\lambda) \times \ell(\mu)$ matrices A with entries in $\mathbb{Z}_{\ge0}$ satisfying $\mathrm{row}(A) = \lambda$, $\mathrm{col}(A) = \mu$. We then have the following expansions for the elementary and complete symmetric functions (see e.g. [67, Prop. 7.4.1 and Prop.

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&3\\\hline 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&2\\\hline 4\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&4\\\hline 2\\\cline{1-1}\end{array}$$

Figure 2.3: Let $\lambda = (2,1)$. Above there are displayed some of the column strict tableaux of shape λ. For example the first tableau has weight $(2,1)$ and then $x^T = x_1^2 x_2$. Thus $s_{(2,1)} = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + \cdots + 2x_1x_2x_3 + 2x_1x_2x_4 + \ldots$.

7.5.1])

$$e_\lambda = \sum_{\mu\in\mathcal{P}^+} M_{\lambda\mu} m_\mu \,, \qquad h_\lambda = \sum_{\mu\in\mathcal{P}^+} L_{\lambda\mu} m_\mu \,. \tag{2.22}$$

2.2.3 Schur functions

We now introduce one of the most important basis of Λ, which is given by the Schur functions $\{s_\lambda\}_{\lambda\in\mathcal{P}^+}$ (compare with the discussion presented in Chapter 1). For $\lambda \in \mathcal{P}^+$ the Schur function s_λ is defined combinatorially as

$$s_\lambda = \sum_T x^T \,, \tag{2.23}$$

where the sum runs over all column strict tableaux T of shape λ, and moreover we set $x^T = x_1^{\mathrm{wt}_1(T)} x_2^{\mathrm{wt}_2(T)} \cdots$. See Figure 2.3 for an example. For $\alpha \in \mathcal{P}$ with $|\alpha| = |\lambda|$, define the Kostka number $K_{\lambda\alpha}$ as the number of column strict tableaux of shape λ and weight α. If instead $|\alpha| \neq |\lambda|$, set $K_{\lambda\alpha} = 0$. We can then rearrange (2.23) as $s_\lambda = \sum_{\alpha\in\mathcal{P}} K_{\lambda\alpha} x^\alpha$ (see for example [67, Ch. 7.10]). Thanks to the relation $K_{\lambda\mu} = K_{\lambda\beta}$, which holds for all $\mu \in \mathcal{P}^+$ and $\beta \in \mathcal{P}$ such that $\beta \sim \mu$ (see *loc.cit.*), we deduce the following expansion of Schur functions in terms of monomial symmetric functions,

$$s_\lambda = \sum_{\mu\in\mathcal{P}^+} K_{\lambda\mu} m_\mu \,. \tag{2.24}$$

The Jacobi-Trudi determinants [60] provide expressions for Schur functions in terms of elementary and complete symmetric functions. These are given by

$$s_\lambda = \det\left(h_{\lambda_i - i + j}\right)_{1\le i,j\le \ell(\lambda)} = \det\left(e_{\lambda'_i - i + j}\right)_{1\le i,j\le \lambda_1} \,. \tag{2.25}$$

The Littlewood-Richardson coefficients $c_{\mu\nu}^\lambda \in \mathbb{Z}_{\ge 0}$ [52, 60, 67] are defined via the following product expansion,

$$s_\mu s_\nu = \sum_{\lambda\in\mathcal{P}^+} c_{\mu\nu}^\lambda s_\lambda \,. \tag{2.26}$$

These coefficients admit a combinatorial interpretation in terms of Littlewood-Richardson tableaux, which is given by the celebrated Littlewood-Richardson rule (see *loc. cit.*). The Schur functions $s_\lambda(x_1, \ldots, x_k)$ are the characters of the finite dimensional irreducible polynomials representations of the general linear group $\mathrm{GL}_k(\mathbb{C})$ (see i.e. [29]), and the product expansion (2.26) then describes the tensor product multiplicities of the mentioned $\mathrm{GL}_k(\mathbb{C})$-representations. The finite dimensional irreducible representations of the general linear and symmetric groups are related via Schur-Weyl duality [25]. The irreducible representations V_λ of the symmetric group S_k are labelled by partitions λ with $|\lambda| = k$. By means of the Frobenius characteristic map, one can show that the multiplicity of the irreducible V_λ in the representation $\mathrm{Ind}_{S_{|\mu|}\times S_{|\nu|}}^{S_{|\lambda|}} V_\mu \times V_\nu$ of $S_{|\lambda|}$ is equal to $c_{\mu\nu}^\lambda$ (see *loc. cit.*).

2.2.4 Power sums and augmented monomial symmetric functions

Consider now the ring $\Lambda_\mathbb{Q} = \Lambda \otimes_\mathbb{Z} \mathbb{Q}$ of symmetric functions with rational coefficients. There are two bases of $\Lambda_\mathbb{Q}$ of our interest, namely the power sums $\{p_\lambda\}_{\lambda\in\mathcal{P}^+}$ and the augmented monomial symmetric functions $\{m^\lambda\}_{\lambda\in\mathcal{P}^+}$ (see for example [52, p. 110]). For $r \in \mathbb{N}$ the former are defined as

$$p_r = \sum_{i\geq 1} x_i^r \,, \qquad p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots \,, \tag{2.27}$$

with the convention $p_0 = 1$, whereas the latter are given by

$$m^\lambda = u_\lambda m_\lambda \,, \qquad u_\lambda = \prod_{i\geq 1} m_i(\lambda)! \,. \tag{2.28}$$

Notice that $\{p_\lambda\}_{\lambda\in\mathcal{P}^+}$ and $\{m^\lambda\}_{\lambda\in\mathcal{P}^+}$ do not form bases of Λ. As a simple example, we have that $e_2 = \frac{1}{2}(p_1^2 - p_2)$ does not have integer coefficients when expressed in terms of power sums.

Lemma 2.2.8. *Let $\lambda \in \mathcal{P}^+$, $k \in \mathbb{N}$ and project onto the ring Λ_k. Then $m^\lambda(x_1, \ldots, x_k) = 0$ if $\ell(\lambda) > k$, otherwise*

$$m^\lambda(x_1, \ldots, x_k) = \frac{1}{m_0(\lambda)!} \sum_{w\in S_k} x_1^{\lambda_{w(1)}} \cdots x_k^{\lambda_{w(k)}} \,. \tag{2.29}$$

For $\ell(\lambda) \leq k$ it is understood that $m_0(\lambda)$ is the multiplicity of 0 in $\lambda \in \mathcal{P}_k^+$.

Proof. This follows by taking advantage of Lemma 2.2.5, and using the fact that $|S_\lambda| = \prod_{i\in\mathbb{Z}_{\geq 0}} m_i(\lambda)! = u_\lambda m_0(\lambda)!$ for $\ell(\lambda) \leq k$. □

Let once again u be an indeterminate, then the generating function for power sums is the element in $\mathbb{Q}[[u]] \otimes_\mathbb{Q} \Lambda_\mathbb{Q}$ given by

$$P(u) = \sum_{r\geq 1} u^{r-1} p_r = \sum_{i\geq 1} \frac{x_i}{1 - x_i u} = \frac{1}{E(-u)} \frac{d}{du} E(-u) = \frac{1}{H(u)} \frac{d}{du} H(u) \,. \tag{2.30}$$

We want to express the augmented monomial symmetric functions in terms of power sums. For this purpose, we first introduce some notation. Let $l \in \mathbb{N}$, and denote by P_l the set of all set partitions of $[l]$. In particular, call $\hat{0} \in P_l$ the set partition of $[l]$ into l singletons, that is $\hat{0} = \big\{\{1\}, \{2\}, \ldots, \{k\}\big\}$. Let $B_1, B_2, \ldots, B_s$ be the blocks of $\Pi \in P_l$, and $\operatorname{card} B_i$ the cardinality of the block B_i. Moreover, define the quantity $B(\Pi) = (-1)^{l-s} \prod_{i=1}^{s} (\operatorname{card} B_i - 1)!$. As an example, the set partition $\Pi = \big\{\{1,2\}, \{3\}\big\} \in P_3$ consists of the two blocks $B_1 = \{1,2\}$ and $B_2 = \{3\}$, and then $B(\Pi) = -1$. For $\Pi \in P_l$ and $\mu \in \mathcal{P}^+$ satisfying $\ell(\mu) = l$, denote with $\mu(\Pi)$ the partition whose parts are given by $\sum_{j \in B_i} \mu_j$, where $i = 1, \ldots, s$. Now, let $\lambda \in \mathcal{P}^+$ and set $l = \ell(\lambda)$. The augmented monomial symmetric function m^λ satisfies the recurrence relation [54, Theorem 1]

$$m^{(\lambda_1,\ldots,\lambda_l)} = p_{\lambda_l} m^{(\lambda_1,\ldots,\lambda_{l-1})} - \sum_{i=1}^{l-1} m^{(\lambda_1,\ldots,\lambda_i+\lambda_l,\ldots,\lambda_{l-1})}\ , \tag{2.31}$$

where it is understood that $m^\alpha = m^\lambda$ for $\alpha \sim \lambda$. The latter has the (unique) solution ([19] and [54, Theorem 2])

$$m^\lambda = \sum_{\Pi \in P_l} B(\Pi) p_{\lambda(\Pi)}\ . \tag{2.32}$$

Example 2.2.9. Let $\lambda = (3,2,2)$. The following quantities can be computed directly from the definitions,

Π	$B(\Pi)$	$\lambda(\Pi)$
$\big\{\{1\},\{2\},\{3\}\big\}$	1	$(3,2,2)$
$\big\{\{1,2\},\{3\}\big\}$	-1	$(5,2)$
$\big\{\{1,3\},\{2\}\big\}$	-1	$(5,2)$
$\big\{\{2,3\},\{1\}\big\}$	-1	$(4,3)$
$\big\{\{1,2,3\}\big\}$	2	(7)

Using (2.32) one then has that $m^{(3,2,2)} = p_{(3,2,2)} - 2p_{(5,2)} - p_{(4,3)} + 2p_{(7)}$.

We will also make use of the expansion of power sums in terms of monomial symmetric functions. Let $\lambda, \mu \in \mathcal{P}^+$ and define $R_{\lambda\mu}$ as the number of set partitions $\Pi \in P_{\ell(\lambda)}$ with $\mu = \lambda(\Pi)$. Then we have the expansion [67, Prop. 7.7.1]

$$p_\lambda = \sum_{\mu \in \mathcal{P}^+} R_{\lambda\mu} m_\mu\ . \tag{2.33}$$

Remark 2.2.10. The bases of symmetric functions labelled by $\lambda \in \mathcal{P}^+$ which we described so far are all of degree $|\lambda|$. As we explained at the beginning of Section 2.2, the ring Λ is a graded ring, where the grading is given by the degree. It follows that the expansions (2.22), (2.24) and (2.33) only involve partitions $\mu \in \mathcal{P}^+$ with $|\mu| = |\lambda|$. In particular, for every $r \in \mathbb{Z}_{\geq 0}$ the expansion coefficients form matrices labelled by $\lambda, \mu \in \mathcal{P}^+$, with $|\lambda| = |\mu| = r$, which are invertible. One can deduce further constraints on $\mu \in \mathcal{P}^+$ appearing in the expansions above by means of the natural ordering on $\mathcal{P}^+$. See [52, I.6] for details.

To conclude this section, we write down the equations relating the elementary and complete symmetric functions to the power sums. These are known as Newton's formulae [52], and read

$$re_r = \sum_{i=1}^{r}(-1)^{i-1}p_i e_{r-i} , \tag{2.34}$$

$$rh_r = \sum_{i=1}^{r} p_i h_{r-i} , \tag{2.35}$$

for $r \in \mathbb{N}$. The solutions of these recursive equations are given by

$$e_r = \sum_{\mu\in\mathcal{P}^+} \epsilon_\mu z_\mu^{-1} p_\mu , \tag{2.36}$$

$$h_r = \sum_{\mu\in\mathcal{P}^+} z_\mu^{-1} p_\mu , \tag{2.37}$$

where the sums are restricted to $\mu \in \mathcal{P}^+$ with $|\mu| = r$, and

$$\epsilon_\mu = (-1)^{|\mu|-\ell(\mu)} , \qquad z_\mu = \prod_{i\geq 1} i^{m_i(\mu)} m_i(\mu)! . \tag{2.38}$$

2.2.5 Hall inner product and an involution

We will often use the ring homomorphism $\omega : \Lambda \to \Lambda$ defined for $r \in \mathbb{Z}_{\geq 0}$ by

$$e_r \mapsto \omega(e_r) = h_r ,$$

and $\omega(e_\lambda) = h_\lambda$ for $\lambda \in \mathcal{P}^+$. The fact that $\{e_\lambda\}_{\lambda\in\mathcal{P}^+}$ is a basis of Λ implies that ω is well defined. We now review some properties of ω; see [52, I.2] for details. The symmetry of the relations $\sum_{i=0}^{r}(-1)^i e_i h_{r-i} = 0$ (which were discussed in section 2.2.2) as between the two bases $\{e_\lambda\}_{\lambda\in\mathcal{P}^+}$ and $\{h_\lambda\}_{\lambda\in\mathcal{P}^+}$ of Λ shows that w is an involution, that is ω^2 is the identity map. It follows that ω is an automorphism of Λ. From the Jacobi-Trudi determinants (2.25) we have the relation $\omega(s_\lambda) = s_{\lambda'}$. Furthermore, since ω interchanges the generating functions $E(u)$ and $H(u)$, it follows from (2.30) that $\omega(p_r) = (-1)^{r-1}p_r$, and in general $\omega(p_\lambda) = \epsilon_\lambda p_\lambda$ for $\lambda \in \mathcal{P}^+$. The fact that ω is an automorphism allows us to define another basis of Λ, the so called forgotten symmetric functions $\{f_\lambda\}_{\lambda\in\mathcal{P}^+}$, via the relation $f_\lambda = \omega(m_\lambda)$. These have no particularly simple direct description.

Remark 2.2.11. From here to the end of this thesis, if not stated otherwise, we shall always work with the ring $\Lambda \otimes_{\mathbb{Z}} \mathbb{C}$ of symmetric functions with complex coefficients. By abuse of notation we denote this with Λ. Similarly, we shall refer to $\Lambda_k \otimes_{\mathbb{Z}} \mathbb{C}$ simply as Λ_k.

We define the Hall inner product on Λ by requiring that the bases $\{h_\lambda\}_{\lambda\in\mathcal{P}^+}$ and $\{m_\lambda\}_{\lambda\in\mathcal{P}^+}$ are dual to each other, that is

$$\langle h_\lambda, m_\mu\rangle = \delta_{\lambda\mu} \ . \tag{2.39}$$

We recall some known facts about the Hall inner product; see [52, I.4] and [67, Ch. 7.9] for details. We have the following relations,

$$\langle s_\lambda, s_\mu\rangle = \delta_{\lambda\mu} \ , \qquad \langle p_\lambda, p_\mu\rangle = z_\lambda\delta_{\lambda\mu} \ , \tag{2.40}$$

which imply that $\{s_\lambda\}_{\lambda\in\mathcal{P}}$ is a orthonormal basis of Λ, and that $\{p_\lambda\}_{\lambda\in\mathcal{P}}$ is a orthogonal basis of Λ. The involution ω is an isometry for the Hall inner product, that is for $f, g \in \Lambda$ we have the relation $\langle f, g\rangle = \langle\omega(f), \omega(g)\rangle$. It follows that the elementary symmetric functions and the forgotten symmetric functions are duals of each other, that is

$$\langle e_\lambda, f_\mu\rangle = \delta_{\lambda\mu} \ . \tag{2.41}$$

2.2.6 Hopf algebra structure on Λ

We now introduce a coproduct $\Delta : \Lambda \to \Lambda\otimes\Lambda$ in the ring of symmetric functions. See [52, p. 91] and [67, p. 342] for further details. For this purpose, notice that $\Lambda\otimes\Lambda$ can be identified with the functions in the two sets of variables $x = \{x_1, x_2, \dots\}$ and $y = \{y_1, y_2, \dots\}$ which are symmetric in each set separately. As an example, for $f, g \in \Lambda$ the element $f \otimes g$ corresponds to $f(x)g(y)$. If $f \in \Lambda$ then $f(x, y) \in \Lambda\otimes\Lambda$, because if f is symmetric in $\{x, y\}$ then it will be symmetric in x and y separately. Thus we define the coproduct of f as

$$\Delta f \equiv f(x, y) \ . \tag{2.42}$$

For $r \in \mathbb{N}$ one has by direct inspection that

$$\Delta e_r = \sum_{i=0}^{r} e_i \otimes e_{r-i} \ , \qquad \Delta h_r = \sum_{i=0}^{r} h_i \otimes h_{r-i} \ , \qquad \Delta p_r = 1 \otimes p_r + p_r \otimes 1 \ ,$$

and furthermore $\Delta(1) = 1 \otimes 1$. It can be shown that Δ satisfies coassociativity and thus, together with the counit $\epsilon : \Lambda \to \mathbb{C}$ given by $f \mapsto f(0, 0, \dots)$, Λ becomes a coalgebra. Moreover, together with the usual multiplication of polynomials and the unit map $\eta : \mathbb{C} \to \Lambda$, it can be shown that Λ is endowed with the structure of a bialgebra. The Hall inner product is compatible with the bialgebra structure, namely for $f, g, h \in \Lambda$ one has that

$$\langle \Delta f, g \otimes h\rangle = \langle f, gh\rangle \ . \tag{2.43}$$

The scalar product on the LHS is defined as $\langle f_1 \otimes g_1, f_2 \otimes g_2\rangle = \langle f_1, f_2\rangle\langle g_1, g_2\rangle$ for $f_1, f_2, g_1, g_2 \in \Lambda$. An alternative interpretation of (2.43) is that the coproduct is the adjoint map of the multiplication map $m : \Lambda \otimes \Lambda \to \Lambda$. Finally, the antipode $\gamma : \Lambda \to \Lambda$ defined for $r \in \mathbb{Z}_{\geq 0}$ by

$$e_r \mapsto \gamma(e_r) = (-1)^r h_r$$

endows Λ with the structure of a cocommutative Hopf algebra over $\mathbb{C}$ (see [73] for further details). By definition the antipode of a Hopf algebra is the unique map satisfying the equalities (see e.g. [24, Ch. 4])

$$m \circ (1 \otimes \gamma) \circ \Delta = m \circ (\gamma \otimes 1) \circ \Delta = \eta \circ \epsilon \ . \tag{2.44}$$

Notice how the antipode is closely related to the involution ω defined above.

2.3 Coproduct and skew symmetric functions

From here to the end of this section, if not stated otherwise, we assume that $\lambda, \mu \in \mathcal{P}^+$. The coproduct $\Delta : \Lambda \to \Lambda \otimes \Lambda$ introduced earlier allows us to define new classes of symmetric functions. We shall start by considering the so called ‘skew Schur functions’, which are discussed for instance in [52, I.5]. Compare also with the discussion presented in Chapter 1.

Definition 2.3.1. Define the skew Schur function $s_{\lambda/\mu}$ via the equation

$$\Delta s_\lambda = \sum_{\mu \in \mathcal{P}^+} s_{\lambda/\mu} \otimes s_\mu \ . \tag{2.45}$$

Lemma 2.3.2. *The symmetric function $s_{\lambda/\mu}$ can be equivalently defined as the function satisfying the following relation, which is valid for all $\nu \in \mathcal{P}^+$,*

$$\langle s_{\lambda/\mu}, s_\nu\rangle = \langle s_\lambda, s_\mu s_\nu\rangle \ . \tag{2.46}$$

Notice that (2.46) fixes $s_{\lambda/\mu}$ entirely thanks to the expansion $s_{\lambda/\mu} = \sum_{\nu\in\mathcal{P}^+}\langle s_{\lambda/\mu}, s_\nu\rangle s_\nu$, which holds since $\{s_\lambda\}_{\lambda\in\mathcal{P}^+}$ is a basis of Λ. Furthermore, by linearity of the Hall inner product we have that $\langle s_{\lambda/\mu}, g\rangle = \langle s_\lambda, s_\mu g\rangle$ for all $g \in \Lambda$.

Proof (of Lemma 2.3.2). Starting from (2.45) one has that $\langle \Delta s_\lambda, s_\mu \otimes s_\nu\rangle$ equals $\langle s_\lambda, s_\mu s_\nu\rangle$ thanks to (2.43) and $\langle s_{\lambda/\mu}, s_\nu\rangle$ thanks to (2.40). This implies the validity of (2.46). Conversely, define $s_{\lambda/\mu}$ via (2.46), and consider the expansion $\Delta s_\lambda = \sum_{\mu,\nu\in\mathcal{P}^+}\langle \Delta s_\lambda, s_\nu \otimes s_\mu\rangle s_\nu \otimes s_\mu$. Taking advantage of (2.43), (2.46) and the expansion $s_{\lambda/\mu} = \sum_{\nu\in\mathcal{P}^+}\langle s_{\lambda/\mu}, s_\nu\rangle s_\nu$ one ends up with (2.45). □

The adjective 'skew' stems from the fact that $s_{\lambda/\mu}$ has an alternative combinatorial description given by

$$s_{\lambda/\mu} = \sum_T x^T \ , \tag{2.47}$$

where the sum runs over all column strict tableaux T of shape λ/μ, and once again $x^T = x_1^{\mathrm{wt}_1(T)} x_2^{\mathrm{wt}_2(T)} \cdots$. This is the generalisation to skew diagrams of the combinatorial definition (2.23) of Schur functions, and one has that $s_{\lambda/\emptyset} = s_\lambda$. Notice that (2.47) is well defined only if $\mu \subset \lambda$. In fact, one can show that if $\mu \not\subset \lambda$ then $s_{\lambda/\mu} = 0$ (see for example [52, I.5]). Define the skew Kostka number $K_{\lambda/\mu}(\alpha)$ as the number of column strict tableaux of shape λ/μ and weight $\alpha \in \mathcal{P}$. Notice that $K_{\lambda/\emptyset}(\alpha) = K_{\lambda\alpha}$, where the latter is the Kostka number defined in Section 2.2.3. The equalities $K_{\lambda/\mu}(\nu) = \langle s_{\lambda/\mu}, h_\nu \rangle = \langle s_\lambda, s_\mu h_\nu \rangle$, which follow by linearity of the Hall inner product, imply that the skew Kostka numbers also appear in the product expansion $s_\mu h_\nu = \sum_{\lambda \in \mathcal{P}^+} K_{\lambda/\mu}(\nu) s_\lambda$. The next result can be found for instance in [52, I.5].

Lemma 2.3.3. *The symmetric function $s_{\lambda/\mu}$ can be expanded as*

$$s_{\lambda/\mu} = \sum_{\nu \in \mathcal{P}^+} c^\lambda_{\mu\nu} s_\nu = \sum_{\nu \in \mathcal{P}^+} K_{\lambda/\mu}(\nu) m_\nu \ , \tag{2.48}$$

where the coefficient $c^\lambda_{\mu\nu}$ was introduced in (2.26).

Proof. The first equality follows from (2.26), the expansion $s_{\lambda/\mu} = \sum_{\nu \in \mathcal{P}^+} \langle s_{\lambda/\mu}, s_\nu \rangle s_\nu$ and the defining relation (2.46) of $s_{\lambda/\mu}$. One can show that $K_{\lambda/\mu}(\nu) = K_{\lambda/\mu}(\beta)$, which holds for all $\nu \in \mathcal{P}^+$ and $\beta \in \mathcal{P}$ such that $\beta \sim \nu$ (see [67, p. 311] for details). Rearranging (2.47) appropriately, one ends up with the second equality in (2.48). □

Remark 2.3.4. An explicit construction of the irreducible representations of $\mathrm{GL}_k(\mathbb{C})$ is given by the Weyl modules [25], which can be generalised to skew shapes λ/μ. The corresponding characters are given by $s_{\lambda/\mu}(x_1, \ldots, x_k)$, and the first equality in (2.48) therefore describes the decomposition of 'skew Weyl modules' into irreducible $\mathrm{GL}_k(\mathbb{C})$-representations (see *loc. cit.*).

2.3.1 Skew elementary and complete symmetric functions

We now discuss the symmetric functions which arise by taking the coproduct of elementary and complete symmetric functions. We were unable to find these symmetric functions anywhere in the literature.

Definition 2.3.5. Define the 'skew elementary symmetric function' $e_{\lambda/\mu}$ and the 'skew complete symmetric function' $h_{\lambda/\mu}$ via the equations

$$\Delta e_\lambda = \sum_{\mu\in\mathcal{P}^+} e_{\lambda/\mu}\otimes e_\mu\ , \qquad \Delta h_\lambda = \sum_{\mu\in\mathcal{P}^+} h_{\lambda/\mu}\otimes h_\mu\ . \tag{2.49}$$

Lemma 2.3.6. *The symmetric functions $e_{\lambda/\mu}$ and $h_{\lambda/\mu}$ can be equivalently defined as the functions satisfying the relations, which are valid for all $\nu\in\mathcal{P}^+$,*

$$\langle e_{\lambda/\mu}, f_\nu\rangle = \langle e_\lambda, f_\mu f_\nu\rangle\ , \qquad \langle h_{\lambda/\mu}, m_\nu\rangle = \langle h_\lambda, m_\mu m_\nu\rangle\ . \tag{2.50}$$

Recall that the basis $\{h_\lambda\}_{\lambda\in\mathcal{P}^+}$ and $\{m_\lambda\}_{\lambda\in\mathcal{P}^+}$ are dual to each other with respect to the Hall inner product (2.39), and so are the basis $\{e_\lambda\}_{\lambda\in\mathcal{P}^+}$ and $\{f_\lambda\}_{\lambda\in\mathcal{P}^+}$. Thus (2.50) fixes $e_{\lambda/\mu}$ and $h_{\lambda/\mu}$ entirely thanks to the expansions $e_{\lambda/\mu} = \sum_{\nu\in\mathcal{P}^+}\langle e_{\lambda/\mu}, f_\nu\rangle e_\nu$ and $h_{\lambda/\mu} = \sum_{\nu\in\mathcal{P}^+}\langle h_{\lambda/\mu}, m_\nu\rangle h_\nu$. Furthermore, by linearity of the Hall inner product we have that $\langle e_{\lambda/\mu}, g\rangle = \langle e_\lambda, f_\mu g\rangle$ and $\langle h_{\lambda/\mu}, g\rangle = \langle h_\lambda, m_\mu g\rangle$ for all $g\in\Lambda$. Since the involution ω described in Section 2.2.5 is an isometry for the Hall inner product, it follows from (2.50) that $\langle \omega(e_{\lambda/\mu}), m_\nu\rangle = \langle h_\lambda, m_\mu m_\nu\rangle$ for all $\nu\in\mathcal{P}^+$, and thus

$$\omega(e_{\lambda/\mu}) = h_{\lambda/\mu}\ . \tag{2.51}$$

Proof (of Lemma 2.3.6). Starting from (2.49) one has that $\langle\Delta e_\lambda, f_\mu\otimes f_\nu\rangle$ equals $\langle e_\lambda, f_\mu f_\nu\rangle$ thanks to (2.43) and $\langle e_{\lambda/\mu}, f_\nu\rangle$ thanks to (2.41). This implies the validity of the first equation in (2.50), whereas the second one follows in a similar manner. Conversely define $e_{\lambda/\mu}$ and $h_{\lambda/\mu}$ via (2.50), and consider the expansion $\Delta e_\lambda = \sum_{\mu,\nu\in\mathcal{P}^+}\langle \Delta e_\lambda, f_\nu\otimes f_\mu\rangle e_\nu\otimes e_\mu$. Taking advantage of (2.43), (2.50) and the expansion $e_{\lambda/\mu} = \sum_{\nu\in\mathcal{P}^+}\langle e_{\lambda/\mu}, f_\nu\rangle e_\nu$ one ends up with the first equation in (2.49). Starting from $\Delta h_\lambda = \sum_{\mu,\nu\in\mathcal{P}^+}\langle\Delta h_\lambda, m_\nu\otimes m_\mu\rangle h_\nu\otimes h_\mu$ instead one arrives at the second equation in (2.49). □

For $\nu\in\mathcal{P}^+$, define the coefficients $\psi_{\lambda/\mu}(\nu)$ and $\theta_{\lambda/\mu}(\nu)$ via the product expansions

$$m_\mu e_\nu = \sum_{\lambda\in\mathcal{P}^+}\psi_{\lambda/\mu}(\nu) m_\lambda\ , \qquad m_\mu h_\nu = \sum_{\lambda\in\mathcal{P}^+}\theta_{\lambda/\mu}(\nu) m_\lambda\ . \tag{2.52}$$

Plugging (2.22) into (2.52), and then employing (2.13), it follows at once that

$$\psi_{\lambda/\mu}(\nu) = \sum_{\sigma\in\mathcal{P}^+} f^\lambda_{\mu\sigma} M_{\nu\sigma}\ , \qquad \theta_{\lambda/\mu}(\nu) = \sum_{\sigma\in\mathcal{P}^+} f^\lambda_{\mu\sigma} L_{\nu\sigma}\ . \tag{2.53}$$

The coefficient $f^\lambda_{\mu\sigma}$ was introduced in (2.13), whereas $M_{\nu\sigma}$ and $L_{\nu\sigma}$ were described in Section 2.2.2. The relations (2.53) imply that $\psi_{\lambda/\mu}(\nu)$ and $\theta_{\lambda/\mu}(\nu)$ are non-zero only if $\mu\subset\lambda$, since in turn $f^\lambda_{\mu\sigma}$ is non-zero only if $\mu\subset\lambda$, as we showed in Section 2.2.1.

Lemma 2.3.7. *The symmetric functions $e_{\lambda/\mu}$ and $h_{\lambda/\mu}$ can be expanded as*

$$
\begin{aligned}
e_{\lambda/\mu} &= \sum_{\nu\in\mathcal{P}^+} f^{\lambda}_{\mu\nu} e_\nu = \sum_{\nu\in\mathcal{P}^+} \psi_{\lambda/\mu}(\nu) m_\nu \ , && (2.54)\\
h_{\lambda/\mu} &= \sum_{\nu\in\mathcal{P}^+} f^{\lambda}_{\mu\nu} h_\nu = \sum_{\nu\in\mathcal{P}^+} \theta_{\lambda/\mu}(\nu) m_\nu \ . && (2.55)
\end{aligned}
$$

Proof. Let us start from the equalities $h_{\lambda/\mu} = \sum_{\nu\in\mathcal{P}^+}\langle h_{\lambda/\mu}, m_\nu\rangle h_\nu = \sum_{\nu\in\mathcal{P}^+}\langle h_{\lambda/\mu}, h_\nu\rangle m_\nu$. Taking advantage of the second equation in (2.50), the product expansion $m_\mu m_\nu = \sum_{\lambda\in\mathcal{P}^+} f^{\lambda}_{\mu\nu} m_\lambda$ in (2.13) and the second equation in (2.52), one deduces the validity of (2.55). Similarly, consider the expansions $e_{\lambda/\mu} = \sum_{\nu\in\mathcal{P}^+}\langle e_{\lambda/\mu}, f_\nu\rangle e_\nu = \sum_{\nu\in\mathcal{P}^+}\langle e_{\lambda/\mu}, h_\nu\rangle m_\nu$. Applying the involution ω to both sides of (2.13) one ends up with $f_\mu f_\nu = \sum_{\lambda\in\mathcal{P}^+} f^{\lambda}_{\mu\nu} f_\lambda$. This equality, together with the first equation in (2.50), the fact that ω is an isometry for the Hall inner product, and the first equation in (2.52), implies the validity of (2.54). □

Remark 2.3.8. The first equalities in (2.54) and (2.55) imply that the symmetric functions $e_{\lambda/\mu}$ and $h_{\lambda/\mu}$ are non-zero only if $\mu\subset\lambda$, since in turn $f^{\lambda}_{\mu\nu}$ is non-zero only if $\mu\subset\lambda$.

Lemma 2.3.9. *We have the identities*

$$
\sum_{\nu\in\mathcal{P}^+} (-1)^{|\nu|-|\mu|} e_{\lambda/\nu} h_{\nu/\mu} = \sum_{\nu\in\mathcal{P}^+} (-1)^{|\lambda|-|\nu|} h_{\lambda/\nu} e_{\nu/\mu} = \delta_{\lambda\mu} \ . \qquad (2.56)
$$

Proof. The equality $(m_\lambda m_\mu) m_\nu = m_\lambda (m_\mu m_\nu)$, which simply reflects the associativity of the product in Λ, together with (2.13) implies the relation $\sum_{\sigma\in\mathcal{P}^+} f^{\sigma}_{\lambda\mu} f^{\rho}_{\sigma\nu} = \sum_{\sigma\in\mathcal{P}^+} f^{\sigma}_{\nu\mu} f^{\rho}_{\sigma\lambda}$ for $\rho\in\mathcal{P}^+$. Taking advantage of the latter and the first equalities in (2.54) and (2.55) one ends up with

$$
\Delta e_{\lambda/\mu} = \sum_{\nu\in\mathcal{P}^+} e_{\lambda/\nu}\otimes e_{\nu/\mu} \ , \qquad \Delta h_{\lambda/\mu} = \sum_{\nu\in\mathcal{P}^+} h_{\lambda/\nu}\otimes h_{\nu/\mu} \ . \qquad (2.57)
$$

The same equalities imply that $\gamma(e_{\lambda/\mu}) = h_{\lambda/\mu}(-1)^{|\lambda|-|\mu|}$ and $\gamma(h_{\lambda/\mu}) = e_{\lambda/\mu}(-1)^{|\lambda|-|\mu|}$, where γ is the antipode of Λ defined in Section 2.2.6. Employing the defining relations (2.44) of the antipode, one has that $(m\circ(1\otimes\gamma)\circ\Delta)(e_{\lambda/\mu}) = \sum_{\nu\in\mathcal{P}^+}(-1)^{|\nu|-|\mu|} e_{\lambda/\nu} h_{\nu/\mu}$, $(m\circ(\gamma\otimes 1)\circ\Delta)(e_{\lambda/\mu}) = \sum_{\nu\in\mathcal{P}^+}(-1)^{|\lambda|-|\nu|} h_{\lambda/\nu} e_{\nu/\mu}$ and $(\eta\circ\epsilon)(e_{\lambda/\mu}) = \delta_{\lambda\mu}$. Since these are all equal to each other the claim follows. Had we applied the same relations to $h_{\lambda/\mu}$, we would have still ended up with (2.56). □

Remark 2.3.10. The identities (2.56) represent the generalisation to skew functions of the equalities $\sum_{i=0}^{r}(-1)^i e_i h_{r-i} = 0$ for $r\in\mathbb{N}$, which are recovered by setting $\lambda = (r)$ and $\mu = \emptyset$.

To conclude this section, we provide the expansions for skew elementary and complete symmetric functions in terms of Schur functions. Define the coefficient $\chi^{\lambda}_{\mu\nu}$ as

$$\chi^{\lambda}_{\mu\nu} = \sum_{\sigma\in\mathcal{P}^+} f^{\lambda}_{\mu\sigma} K_{\nu\sigma} \ . \tag{2.58}$$

The coefficient $K_{\nu\sigma}$ is the Kostka number, which was introduced in Section 2.2.3. Notice that $\chi^{\lambda}_{\mu\nu}$ is a non-negative integer, since $f^{\lambda}_{\mu\sigma}$ and $K_{\nu\sigma}$ are non-negative integers as well.

Lemma 2.3.11. *The functions $e_{\lambda/\mu}$ and $h_{\lambda/\mu}$ can be expanded as*

$$e_{\lambda/\mu} = \sum_{\nu\in\mathcal{P}^+} \chi^{\lambda}_{\mu\nu} s_{\nu'} \ , \tag{2.59}$$

$$h_{\lambda/\mu} = \sum_{\nu\in\mathcal{P}^+} \chi^{\lambda}_{\mu\nu} s_{\nu} \ . \tag{2.60}$$

Proof. Plug the expansions $e_\nu = \sum_{\sigma\in\mathcal{P}^+} K_{\nu\sigma} s_{\sigma'}$ and $h_\nu = \sum_{\sigma\in\mathcal{P}^+} K_{\nu\sigma} s_\sigma$, which can be found for instance in [52, I.6], into the first equalities of (2.54) and (2.55) respectively. A comparison with (2.58) then proves the validity of the claim. □

Remark 2.3.12. Lemma 2.3.11 implies that the functions $e_{\lambda/\mu}$ and $h_{\lambda/\mu}$ are Schur-positive, that is the coefficients appearing in the expansions in terms of Schur functions are non-negative integers. As we discussed in Remark 2.3.4, the polynomial characters of the irreducible representations of $\mathrm{GL}_k(\mathbb{C})$ coincide with the Schur functions $s_\nu(x_1, \dots, x_k)$. It follows that there exist representations of $\mathrm{GL}_k(\mathbb{C})$ whose polynomial characters are given by $e_{\lambda/\mu}(x_1, \dots, x_k)$ and $h_{\lambda/\mu}(x_1, \dots, x_k)$ respectively. It would be interesting to present a more explicit construction of these representations.

2.3.2 Weighted sums over reverse plane partitions

We now wish to give combinatorial expressions for $e_{\lambda/\mu}$ and $h_{\lambda/\mu}$, and we want these to resemble the one for skew Schur functions given by (2.47). It turns out to be somewhat easier to start with $h_{\lambda/\mu}$, and for this purpose we generalise the notion of skew diagram described in (2.3) to compositions. Namely, for $\alpha, \beta \in \mathcal{P}$ we write $\alpha \subset \beta$ if $\alpha_i \leq \beta_i$ for all $i \in \mathbb{N}$, and we refer to the set $\beta/\alpha \subset \mathbb{Z}\times\mathbb{Z}$ as a 'skew diagram'. Recall that the notation $\alpha \sim \lambda$, which was introduced in Section 2.2.1, indicates that $\alpha \in \mathcal{P}$ is a permutation of $\lambda \in \mathcal{P}^+$.

Definition 2.3.13. For $\lambda, \mu \in \mathcal{P}^+$, denote with $\theta_{\lambda/\mu}$ the cardinality of the set

$$\{\alpha \in \mathcal{P} \mid \alpha \sim \mu, \alpha \subset \lambda\} \ . \tag{2.61}$$

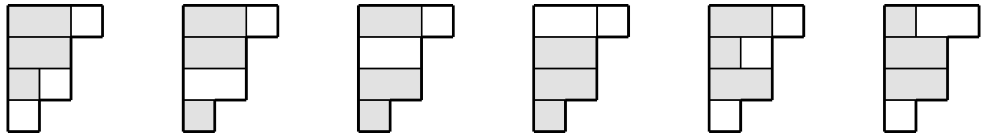


Figure 2.4: Let $\lambda = (3, 2, 2, 1)$ and $\mu = (2, 2, 1)$. Above there are represented (in grey) all the compositions $\alpha \sim \mu$ such that $\alpha \subset \lambda$, and thus $\theta_{\lambda/\mu} = 6$.

Lemma 2.3.14. *The set* (2.61) *is non-empty if and only if $\mu \subset \lambda$, that is if and only if λ/μ is a skew diagram.*

Proof. If $\mu \subset \lambda$ we have that $\alpha = \mu$ belongs to (2.61) which is then non-empty. Conversely assume that (2.61) is non-empty, that is there exists $\alpha \sim \mu$ such that $\alpha \subset \lambda$. Say that $\alpha_i = \mu_1$ for some $i \in \mathbb{N}$, then the composition $\beta \sim \mu$ obtained from α by permuting α_1 and α_i still satisfies $\beta \subset \lambda$, as $\beta_1 = \alpha_i \leq \lambda_i \leq \lambda_1$ and $\beta_i = \alpha_1 \leq \alpha_i \leq \lambda_i$. Notice that by construction $\beta_1 = \mu_1$. Say that $\beta_j = \mu_2$ for some $j \in \mathbb{N}$, then the composition $\gamma \sim \mu$ obtained from β by permuting β_j and β_2 still satisfies $\gamma \subset \lambda$, and furthermore $\gamma_1 = \mu_1$, $\gamma_2 = \mu_2$. Proceeding in a similar vein one eventually concludes that $\mu \subset \lambda$. $\square$

Lemma 2.3.15. *Suppose that $\mu \subset \lambda$, then the cardinality of the set* (2.61) *has the following explicit expression in terms of binomial coefficients,*

$$\theta_{\lambda/\mu} = \prod_{i \geq 1} \binom{\lambda'_i - \mu'_{i+1}}{\mu'_i - \mu'_{i+1}} . \tag{2.62}$$

Proof. Lemma 2.3.14 implies that for $\mu \subset \lambda$ the set (2.61) is non-empty. To prove the claim we will count the number of elements in (2.61), that is the number of distinct permutations α of μ satisfying $\alpha \subset \lambda$, recursively. For this purpose, set $l = \lambda_1$ and notice that all the parts in μ are smaller than l. To begin with, the $m_l(\mu) = \mu'_l$ parts of μ equal to l must be among the first $m_l(\lambda) = \lambda'_l$ parts of α, and there are $\binom{\lambda'_l}{\mu'_l}$ distinct ways to implement this constraint on α. Notice that we allow $m_l(\mu) = 0$, in which case there is only $\binom{\lambda'_l}{0} = 1$ 'way' to do this. Next, the $m_{l-1}(\mu) = \mu'_{l-1} - \mu'_l$ parts of μ equal to $l-1$ must be among the first $m_{l-1}(\lambda) + m_l(\lambda) = \lambda'_{l-1}$ parts of α, and since we have already fixed the μ'_l parts of α equal to l, there are $\binom{\lambda'_{l-1} - \mu'_l}{\mu'_{l-1} - \mu'_l}$ distinct ways to implement this further constraint on α.

The i-th step of this counting procedure would be that the $m_{l+1-i}(\mu) = \mu'_{l+1-i} - \mu'_{l+2-i}$ parts of μ equal to $l+1-i$ must be among the first $\sum_{j \geq l+1-i} m_j(\lambda) = \lambda'_{l+1-i}$ parts of α, and since we have already fixed the $\sum_{j \geq l+2-i} m_j(\mu) = \mu'_{l+2-i}$ parts of α greater or equal than $l+2-i$ there are $\binom{\lambda'_{l+1-i} - \mu'_{l+2-i}}{\mu'_{l+1-i} - \mu'_{l+2-i}}$ distinct ways to implement the constraint just mentioned on α. Finally, the cardinality of (2.61) is given by the product of all these possibilities, that is

$$\prod_{i=1}^{l} \binom{\lambda'_{l+1-i} - \mu'_{l+2-i}}{\mu'_{l+1-i} - \mu'_{l+2-i}} = \prod_{i\geq 1} \binom{\lambda'_i - \mu'_{i+1}}{\mu'_i - \mu'_{i+1}} .$$

□

Recall that a RPP π of shape λ/μ with highest entry l is equivalent to a sequence $\{\lambda^{(r)}\}_{r\in\mathbb{Z}_{\geq 0}}$ of partitions, with $\lambda^{(0)} = \mu$ and $\lambda^{(l)} = \lambda$, such that $\lambda^{(r-1)} \subset \lambda^{(r)}$ for $r \geq 1$. The only difference with Lemma 2.1.2 is that we set $\lambda^{(r)} = \lambda^{(l)}$ for $r \geq l$.

Lemma 2.3.16. *Suppose that $\mu \subset \lambda$, then the skew complete symmetric function $h_{\lambda/\mu}$ is the weighted sum*

$$h_{\lambda/\mu} = \sum_{\pi} \theta_\pi \, x^\pi \ , \qquad \theta_\pi = \prod_{r\geq 1} \theta_{\lambda^{(r)}/\lambda^{(r-1)}} \ , \tag{2.63}$$

over all RPPs of shape λ/μ. In particular, the coefficient $\theta_{\lambda/\mu}(\nu)$ defined in (2.53) *has the alternative expression*

$$\theta_{\lambda/\mu}(\nu) = \sum_{\pi} \theta_\pi \ , \tag{2.64}$$

where the sum runs over all RPPs of shape λ/μ and weight ν.

Proof. First we show the validity of the product expansion

$$m_\mu h_r = \sum_{\lambda\in\mathcal{P}^+} \theta_{\lambda/\mu} m_\lambda \ , \tag{2.65}$$

where the sum runs over all partitions λ such that $\mu \subset \lambda$ and $|\lambda/\mu| = r$. Notice that the coefficient of m_λ in $m_\mu h_r$ equals the coefficient of x^λ in the same product. From (2.12) and (2.18) it follows that each monomial appearing in the product $m_\mu h_r$ is of the form $x^\alpha x^\beta$ for some $\alpha, \beta \in \mathcal{P}$ with $\alpha \sim \mu$ and $|\beta| = r$. Fix $\alpha' \sim \mu$, then a composition β' with $|\beta'| = r$ and $x^{\alpha'} x^{\beta'} = x^\lambda$ exists if and only if $\alpha' \subset \lambda$ and $|\lambda| - |\mu| = r$, in which case $\beta' = \lambda - \alpha'$. Thus the coefficient of x^λ in $m_\mu h_r$ equals the cardinality of the set (2.61) provided that $|\lambda| - |\mu| = r$. This is by definition $\theta_{\lambda/\mu}$, which is non-zero if and only if $\mu \subset \lambda$ thanks to Lemma 2.3.14. In conclusion, the coefficient of m_λ in $m_\mu h_r$ is non-zero if and only if $\mu \subset \lambda$ and $|\lambda/\mu| = |\lambda| - |\mu| = r$, in which case it is equal to $\theta_{\lambda/\mu}$, thus proving the claim.

Applying the result just obtained repeatedly to the product $m_\mu h_\nu$, and comparing with the second equality in (2.52), one sees the validity of (2.64). Since $h_\beta = h_\nu$ for $\beta \sim \nu$ this also implies that $\theta_{\lambda/\mu}(\beta) = \theta_{\lambda/\mu}(\nu)$, where $\theta_{\lambda/\mu}(\beta)$ for β a composition is defined as in (2.64). Rearranging the equality $h_{\lambda/\mu} = \sum_{\nu\in\mathcal{P}^+} \theta_{\lambda/\mu}(\nu) m_\nu$ proved in Lemma 2.3.7 appropriately one then arrives at (2.63).

□

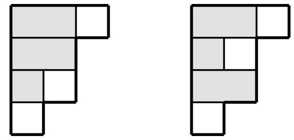


Figure 2.5: Let $\lambda = (3, 2, 2, 1)$ and $\mu = (2, 2, 1)$ as in Figure 2.4. Above there are represented (in grey) all the compositions $\alpha \sim \mu$ such that λ/α is a vertical strip, and thus $\psi_{\lambda/\mu} = 2$.

Corollary 2.3.17. *The coefficient $L_{\lambda\mu}$ introduced in* (2.22) *has the following alternative expression,*

$$L_{\lambda\mu} = \sum_{\pi} \theta_{\pi} \ , \tag{2.66}$$

where the sum runs over all RPPs of shape λ and weight μ.

Proof. First of all, notice that from (2.14) one has that $f^{\lambda}_{\sigma\emptyset} = \delta_{\lambda\sigma}$. Furthermore, from the definition of the coefficient $L_{\lambda\mu}$ it follows that $L_{\lambda\mu} = L_{\mu\lambda}$ (compare with [67, Cor. 7.5.2]). Finally, set $\mu = \emptyset$ in the second equation in (2.53) and take advantage of (2.64) to prove the claim. □

2.3.3 Weighted sums over row strict tableaux

In this section we shall provide a combinatorial expression for $e_{\lambda/\mu}$. For this purpose, we present a similar discussion to the one presented in the previous section for $h_{\lambda/\mu}$. We generalise the notion of vertical strips given in Section 2.1.1 to compositions, and we say that for $\alpha, \beta \in \mathcal{P}$ the skew diagram β/α is a vertical strip if $\beta_i - \alpha_i = 0, 1$ for all $i \in \mathbb{N}$.

Definition 2.3.18. For $\lambda, \mu \in \mathcal{P}^+$, denote with $\psi_{\lambda/\mu}$ the cardinality of the set

$$\{\alpha \in \mathcal{P} \mid \alpha \sim \mu, \lambda/\alpha \text{ is a vertical strip}\} \ . \tag{2.67}$$

Lemma 2.3.19. *The set* (2.67) *is non-empty if and only if λ/μ is a vertical strip.*

Proof. If λ/μ is a vertical strip we have that $\alpha = \mu$ belongs to (2.67), which is then non-empty. Conversely, assume that (2.67) is non-empty, that is there exists $\alpha \sim \mu$ such that λ/α is a vertical strip. Assume that $\alpha_i = \mu_1$ for some $i \in \mathbb{N}$, and consider the composition $\beta \sim \mu$ obtained from α by permuting α_1 and α_i. We show that λ/β is a vertical strip. From the hypothesis $\alpha_i \geq \alpha_1$ but we must also have $\alpha_i \leq \alpha_1 + 1$, for if $\alpha_i > \alpha_1 + 1$ we end up with $\lambda_1 - \alpha_1 > 1$, which is a contradiction since λ/α is a vertical strip. Thus we can only have $\alpha_1 = \alpha_i$, in which case $\alpha = \beta$ and thus λ/β is a vertical strip, or $\alpha_1 = \alpha_i + 1$. In the second case we must have $\lambda_1 - \alpha_1 = 1$, that is $\lambda_1 - \alpha_i = \lambda_1 - \beta_1 = 0$, otherwise

$\lambda_i - \alpha_i < 0$. We must also require $\lambda_i - \alpha_i = 0$, that is $\lambda_i - \alpha_1 = \lambda_i - \beta_i = 1$, otherwise $\lambda_1 - \alpha_1 > 1$. This implies once again that λ/β is a vertical strip.

Assume now that $\beta_j = \mu_2$ for some $j \in \mathbb{N}$, and denote with $\gamma \sim \mu$ the composition obtained from β by permuting β_j and β_2. In the same fashion as before one can show that λ/γ is a vertical strip, and furthermore $\gamma_1 = \mu_1$, $\gamma_2 = \mu_2$. Proceeding in a similar vein one eventually concludes that λ/μ must be a vertical strip. □

Lemma 2.3.20. *Suppose that λ/μ is a vertical strip, then the cardinality of the set* (2.67) *has the following explicit expression in terms of binomial coefficients,*

$$\psi_{\lambda/\mu} = \prod_{i\geq 1} \binom{\lambda'_i - \lambda'_{i+1}}{\lambda'_i - \mu'_i} . \tag{2.68}$$

Proof. Lemma 2.3.19 implies that if λ/μ is a vertical strip the cardinality of (2.67) is non-empty. In a similar fashion to Lemma 2.3.15, we will count the number of distinct permutations α of μ such that λ/α is a vertical strip recursively. For this purpose, we set again $l = \lambda_1$. The $m_l(\mu) = \mu'_l$ parts of μ equal to l must be among the first $m_l(\lambda) = \lambda'_l$ parts of α, and there are $\binom{\lambda'_l}{\mu'_l}$ distinct ways to implement this constraint on α. Notice that the other first λ'_l parts of α, in number $\lambda'_l - \mu'_l$, must be equal to $l-1$ since λ/α is a vertical strip. Next, we have that the remaining $m_{l-1}(\mu) - (\lambda'_l - \mu'_l) = \mu'_{l-1} - \lambda'_l$ parts of μ equal to $l-1$ must be among the parts α_j of α with $\lambda'_l + 1 \leq j \leq \lambda'_l + m_{l-1}(\lambda) = \lambda'_{l-1}$, and thus there are $\binom{\lambda'_{l-1} - \lambda'_l}{\mu'_{l-1} - \lambda'_l}$ distinct ways to implement this further constraint on α. The remaining parts of α in the positions just considered, in number $\lambda'_{l-1} - \lambda'_l - (\mu'_{l-1} - \lambda'_l) = \lambda'_{l-1} - \mu'_{l-1}$, must be equal to $l-2$.

This describes the first 2 steps of the counting procedure. In the i-th step one has that the remaining $m_{l+1-i}(\mu) - (\lambda'_{l+2-i} - \mu'_{l+2-i}) = \mu'_{l+1-i} - \lambda'_{l+2-i}$ parts of μ equal to $l+1-i$ must be among the parts α_j of α with $\lambda'_{l+2-i} + 1 \leq j \leq \lambda'_{l+2-i} + m_{l+1-i}(\lambda) = \lambda'_{l+1-i}$, and there are $\binom{\lambda'_{l+1-i} - \lambda'_{l+2-i}}{\mu'_{l+1-i} - \lambda'_{l+2-i}}$ distinct ways to implement this constraint on α. On the other hand, the remaining parts of α in the positions just considered, in number $\lambda'_{l+1-i} - \lambda'_{l+2-i} - (\mu'_{l+1-i} - \lambda'_{l+2-i}) = \lambda'_{l+1-i} - \mu'_{l+1-i}$, must be equal to $l-i$. Thus, we eventually get that the cardinality of (2.67) is given by

$$\prod_{i=1}^{l} \binom{\lambda'_{l+1-i} - \lambda'_{l+2-i}}{\mu'_{l+1-i} - \lambda'_{l+2-i}} = \prod_{i=1}^{l} \binom{\lambda'_{l+1-i} - \lambda'_{l+2-i}}{\lambda'_{l+1-i} - \mu'_{l+1-i}} = \prod_{i\geq 1} \binom{\lambda'_i - \lambda'_{i+1}}{\lambda'_i - \mu'_i} .$$

□

Lemma 2.3.21. *Suppose that $\mu \subset \lambda$, then the skew elementary symmetric function $e_{\lambda/\mu}$ is the weighted sum*

$$e_{\lambda/\mu} = \sum_T \psi_T \, x^T , \qquad \psi_T = \prod_{r\geq 1} \psi_{\lambda^{(r)}/\lambda^{(r-1)}} , \tag{2.69}$$

over all row strict tableaux of shape λ/μ. In particular, the coefficient $\psi_{\lambda/\mu}(\nu)$ defined in (2.52) *has the alternative expression*

$$\psi_{\lambda/\mu}(\nu) = \sum_T \psi_T \ , \tag{2.70}$$

where the sum runs over all row strict tableaux of shape λ/μ and weight ν.

Proof. This goes very similarly to the proof of Lemma 2.3.16. One first needs to show that

$$m_\mu e_r = \sum_{\lambda \in \mathcal{P}^+} \psi_{\lambda/\mu} m_\lambda \ , \tag{2.71}$$

where the sum runs over all partitions λ such that λ/μ is a vertical strip and $|\lambda/\mu| = r$. We will use once again the fact that the coefficient of m_λ in $m_\mu e_r$ equals the coefficient of x^λ in the same product. From (2.12) and (2.18) it follows that each monomial appearing in the product $m_\mu e_r$ is of the form $x^\alpha x^\beta$ for some $\alpha \sim \mu$ and $\beta \in \mathcal{P}$ with $|\beta| = r$ and $\beta_i = 0, 1$. Fix $\alpha' \sim \mu$, then a composition β' with $|\beta'| = r$, $\beta'_i = 0, 1$ and $x^{\alpha'} x^{\beta'} = x^\lambda$ exists if and only if λ/α' is a vertical strip and $|\lambda| - |\mu| = r$, in which case $\beta' = \lambda - \alpha'$. Thus the coefficient of x^λ in $m_\mu e_r$ equals the cardinality of the set (2.67) provided that $|\lambda| - |\mu| = r$. This is by definition $\psi_{\lambda/\mu}$, which is non-zero if and only if λ/μ is a vertical strip thanks to Lemma 2.3.19. In conclusion the coefficient of m_λ in $m_\mu e_r$ is non-zero if and only if λ/μ is a vertical strip with $|\lambda/\mu| = |\lambda| - |\mu| = r$ in which case it is equal to $\psi_{\lambda/\mu}$, thus proving the claim.

Applying the result just obtained repeatedly to the product $m_\mu e_\nu$ and comparing with the first equality in (2.52) one sees the validity of (2.70). Since $e_\beta = e_\nu$ for $\beta \sim \nu$ this also implies that $\psi_{\lambda/\mu}(\beta) = \psi_{\lambda/\mu}(\nu)$, where $\psi_{\lambda/\mu}(\beta)$ for β a composition is defined as in (2.70). Rearranging the equality $e_{\lambda/\mu} = \sum_{\nu \in \mathcal{P}^+} \psi_{\lambda/\mu}(\nu) m_\nu$, proved in Lemma 2.3.7, with the help of (2.70) one then arrives at (2.69).

□

Remark 2.3.22. The combinatorial interpretation of the coefficients $\theta_{\lambda/\mu}$ and $\psi_{\lambda/\mu}$ presented respectively in (2.65) and (2.71) is not new (see [34, Lemma 4.1] and [52, p. 215] in the limit $t = 1$). The novel aspect here is that these coefficients were determined via cardinalities of sets, compare with Definitions 2.3.13 and 2.3.18. As we will discuss in the next chapter, this provides a natural generalisation to cylindric partitions.

Corollary 2.3.23. *The coefficient $M_{\lambda\mu}$ introduced in* (2.22) *has the following alternative expression,*

$$M_{\lambda\mu} = \sum_T \psi_T \ , \tag{2.72}$$

where the sum runs over all row strict tableaux of shape λ and weight μ.

Proof. Notice that from the definition of the coefficient $M_{\lambda\mu}$ it follows that $M_{\lambda\mu} = M_{\mu\lambda}$ (compare with [67, Cor. 7.4.2]). Set $\mu = \emptyset$ in the first equation in (2.53), use the fact that $f^{\lambda}_{\sigma\emptyset} = \delta_{\lambda\sigma}$, and then take advantage of (2.70) to prove the claim. □

2.3.4 Adjacent column tableaux

We conclude this chapter by studying the expansions of the symmetric functions $e_{\lambda/\mu}$ and $h_{\lambda/\mu}$ in terms of power sums. For $\nu \in \mathcal{P}^+$ define the coefficient $\varphi_{\lambda/\mu}(\nu)$ via the product expansion

$$m_\mu p_\nu = \sum_{\lambda \in \mathcal{P}^+} \varphi_{\lambda/\mu}(\nu) m_\lambda \ . \tag{2.73}$$

Plugging the expansion (2.33) into (2.73), and taking advantage of (2.13), it follows at once that

$$\varphi_{\lambda/\mu}(\nu) = \sum_{\sigma \in \mathcal{P}^+} f^{\lambda}_{\mu\sigma} R_{\nu\sigma} \ , \tag{2.74}$$

where the coefficient $R_{\nu\sigma}$ was introduced in Section 2.2.4. The relation (2.74) implies that $\varphi_{\lambda/\mu}(\nu)$ is non-zero only if $\mu \subset \lambda$, since in turn $f^{\lambda}_{\mu\sigma}$ is non-zero only if $\mu \subset \lambda$, as we showed in Section 2.2.1.

Lemma 2.3.24. *The symmetric functions $h_{\lambda/\mu}$ and $e_{\lambda/\mu}$ can be expanded as*

$$e_{\lambda/\mu} = \sum_{\nu \in \mathcal{P}^+} \varphi_{\lambda/\mu}(\nu) z_\nu^{-1} \epsilon_\nu p_\nu \ , \tag{2.75}$$

$$h_{\lambda/\mu} = \sum_{\nu \in \mathcal{P}^+} \varphi_{\lambda/\mu}(\nu) z_\nu^{-1} p_\nu \ . \tag{2.76}$$

Proof. To prove (2.76) one starts from the expansion $h_{\lambda/\mu} = \sum_{\nu \in \mathcal{P}^+} \langle h_{\lambda/\mu}, p_\nu \rangle z_\nu^{-1} p_\nu$, which follows from $\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda$, and then proceeds in a similar fashion to the proof of Lemma 2.3.7. Applying the involution ω to both sides of (2.76), and using the fact that $\omega(h_{\lambda/\mu}) = e_{\lambda/\mu}$ and $\omega(p_\lambda) = \epsilon_\lambda p_\lambda$, one ends up with (2.75). □

The coefficient $\varphi_{\lambda/\mu}(\nu)$ has an alternative expression involving a new type of tableau, which we now describe.

Definition 2.3.25. Suppose that $\mu \subset \lambda$. We say that λ/μ is a 'adjacent column horizontal strip' (ACHS) if λ/μ is a horizontal strip and if furthermore the columns in $\mathbb{Z} \times \mathbb{Z}$ containing it are adjacent. A 'adjacent column tableau' (ACT) T of shape λ/μ is a sequence $\{\lambda^{(r)}\}_{r=0}^{l}$ of partitions, with $\lambda^{(0)} = \mu$ and $\lambda^{(l)} = \lambda$, such that $\lambda^{(r)}/\lambda^{(r-1)}$ is a ACHS for $r = 1, \dots, l$.

Definition 2.3.26. Let $a, r \in \mathbb{N}$ and suppose that $m_{a-1}(\mu) \neq 0$ (which is understood to be always true for $a = 1$). Define $\mu_{a,r}$ as the partition whose Young diagram is obtained

Figure 2.6: Let $\lambda = (4,4,3,1,1)$ and $\mu = (4,3,1,1)$. On the left we have in grey the ACHS λ/μ. Notice that $\lambda = \mu_{1,4}$, or equivalently that λ is obtained from μ by adding a part equal to 4. Since the part of λ intersecting with the rightmost column of the ACHS is equal to 4 we have that $\varphi_{\lambda/\mu} = m_4(\lambda) = 2$. On the right we have a ACT of shape $(5,4,2)/(2,1)$ and weight $(3,3,2)$.

by adding one box per column in the Young diagram of μ, starting at column a and ending at column $a+r-1$ for a total of r boxes.

Suppose that $\lambda = \mu_{a,r}$ for some $a, r \in \mathbb{N}$ with $m_{a-1}(\mu) \neq 0$. By definition we have that

$$\lambda'_i = \begin{cases} \mu'_i + 1 \,, & a \leq i \leq a+r-1 \\ \mu'_i \,, & \text{otherwise} \end{cases} . \tag{2.77}$$

Furthermore, using the relation $m_i(\lambda) = \lambda'_i - \lambda'_{i+1}$ it follows that

$$\lambda = \begin{cases} \left(\dots, (a-1)^{m_{a-1}(\mu)-1}, \dots, (a-1+r)^{m_{a-1+r}(\mu)+1}, \dots\right) \,, & a > 1 \\ \left(\dots, r^{m_r(\mu)+1}, \dots\right) \,, & a = 1 \end{cases} . \tag{2.78}$$

That is, λ is obtained from μ by removing a part equal to $a-1$ (or removing no parts if $a = 1$) and adding a part equal to $a-1+r$. See once again Figure 2.6 for an example.

Lemma 2.3.27. *The skew diagram λ/μ is a ACHS with $|\lambda/\mu| = r \in \mathbb{N}$ if and only if $\lambda = \mu_{a,r}$ for some $a \in \mathbb{N}$ with $m_{a-1}(\mu) \neq 0$.*

Proof. The claim follows from (2.77), since if $\lambda = \mu_{a,r}$ then one must add r boxes in adjacent columns of the Young diagram of μ to obtain the Young diagram of λ. □

Let $\lambda = \mu_{a,r}$ as before, and define

$$\varphi_{\lambda/\mu} = m_{a-1+r}(\lambda) \,. \tag{2.79}$$

In particular, set $\varphi_{\lambda/\lambda} = 1$. Stated otherwise, the coefficient $\varphi_{\lambda/\mu}$ is the multiplicity of the part in λ intersecting with the rightmost box of λ/μ. See Figure 2.6 for an example. Notice that $\varphi_{\lambda/\mu} = m_{a-1+r}(\mu) + 1$ thanks to (2.78).

Lemma 2.3.28. *Suppose that $\mu \subset \lambda$, then we have the equality*

$$\varphi_{\lambda/\mu}(\nu) = \sum_T \varphi_T \,, \qquad \varphi_T = \prod_{r \geq 1} \varphi_{\lambda^{(r)}/\lambda^{(r-1)}} \,, \tag{2.80}$$

where the sum runs over all ACT of shape λ/μ and weight ν.

Proof. We just need to show for $r \in \mathbb{N}$ the product expansion

$$m_\mu p_r = \sum_{\lambda \in \mathcal{P}^+} \varphi_{\lambda/\mu} m_\lambda \ , \tag{2.81}$$

where the sum runs over all partitions λ such that λ/μ is a ACHS with $|\lambda/\mu| = r$, that is $\lambda = \mu_{a,r}$ for some $a \in \mathbb{N}$ with $m_{a-1}(\mu) \neq 0$. The claim then follows after a repeated application of (2.81) to the product $m_\mu p_\nu$, which also implies that $\varphi_{\lambda/\mu}(\beta) = \varphi_{\lambda/\mu}(\nu)$ for $\beta \sim \nu$.

The coefficient of m_λ in $m_\mu p_r$ equals the coefficient of x^λ in the same product. We show that this is non-zero if and only if $\lambda = \mu_{a,r}$ for some $a \in \mathbb{N}$ with $m_{a-1}(\mu) \neq 0$. Using the definition $p_r = \sum_{i \geq 1} x_i^r$ of power sums, one has that each monomial appearing in $m_\mu p_r$ is of the form $x^\alpha x^\beta$ for some $\alpha \sim \mu$ and $\beta \in \mathcal{P}$ defined by $\beta_i = r\delta_{il}$ for some $l \in \mathbb{N}$. Assume that there exists two such compositions α and β with $x^\alpha x^\beta = x^\lambda$. Since $\alpha \sim \mu$ we see that λ is obtained from μ by removing one of its parts equal to α_l (or removing no parts if $\alpha_l = 0$) and replacing it with $\alpha_l + r$. Setting $a = \alpha_l + 1 \in \mathbb{N}$ we then have $\lambda = \mu_{a,r}$ thanks to (2.78). Conversely, suppose that $\lambda = \mu_{a,r}$ for some $a \in \mathbb{N}$ with $m_{a-1}(\mu) \neq 0$. Equation (2.78) implies that there exists $l \in \mathbb{N}$ and $\alpha \sim \mu$ with $\alpha_l = a - 1$ such that $\lambda = \alpha + \beta$, where again $\beta \in \mathcal{P}$ is defined by $\beta_i = r\delta_{il}$. Thus $x^\alpha x^\beta = x^\lambda$, and the coefficient of m_λ in $m_\mu p_r$ is non-zero.

So suppose that $\lambda = \mu_{a,r}$ for some $a \in \mathbb{N}$ with $m_{a-1}(\mu) \neq 0$, and let i and j be the smallest indices for which $\mu_i < a - 1 + r$ and $\mu_j = a - 1$ respectively. The monomials in $m_\mu p_r$ which equal x^λ are of the form

$$x_1^{\mu_1} \cdots x_{i-l-1}^{\mu_{i-l-1}} x_{i-l}^{\mu_j + r} x_{i-l+1}^{\mu_{i-l}} \cdots x_j^{\mu_{j-1}} x_{j+1}^{\mu_{j+1}} \cdots$$

for $l = 0, \ldots, m_{a-1+r}(\mu)$. This implies that $m_\mu p_r = \sum_a (m_{a-1+r}(\mu) + 1) m_{\mu_{a,r}}$, where the sum runs over all $a \in \mathbb{N}$ such that $m_{a-1}(\mu) \neq 0$, and applying the definition of $\varphi_{\lambda/\mu}$ the latter equals (2.81).

$\square$

Remark 2.3.29. The linear combination $\sum_{\nu \in \mathcal{P}^+} \varphi_{\lambda/\mu}(\nu) m_\nu$ is not equal to the symmetric function $p_{\lambda/\mu}$ defined via the relation $p_\lambda = \sum_{\mu \in \mathcal{P}^+} p_{\lambda/\mu} \otimes p_\mu$. There seems to be no natural definition of 'skew power sums' in terms of ACT.

Corollary 2.3.30. *The coefficient $R_{\lambda\mu}$ introduced in* (2.33) *has the following alternative expression,*

$$R_{\lambda\mu} = \sum_T \varphi_T \ , \tag{2.82}$$

where the sum runs over all ACT of shape μ and weight λ.

Proof. Set $\mu = \emptyset$ in (2.74), use the fact that $f^{\lambda}_{\sigma\emptyset} = \delta_{\lambda\sigma}$ and take advantage of (2.80). $\square$

Notice the difference between this result and Corollaries 2.3.17, 2.3.23. This is because in general $R_{\lambda\mu} \neq R_{\mu\lambda}$; see for example [52, Eq. (6.5)].

Chapter 3

Cylindric symmetric functions

The purpose of this chapter is to generalise the skew symmetric functions $e_{\lambda/\mu}$ and $h_{\lambda/\mu}$, which were discussed in Section 2.3, to the cylinder $\mathfrak{C}_{k,n}$. This is defined as the quotient

$$\mathfrak{C}_{k,n} = \mathbb{Z} \times \mathbb{Z}/(-k, n)\mathbb{Z} \ .$$

In words, $\mathfrak{C}_{k,n}$ is the quotient of the $\mathbb{Z} \times \mathbb{Z}$ plane modulo the shifting action which sends (i, j) to $(i - k, j + n)$. Equivalently, we will work with objects defined on $\mathbb{Z} \times \mathbb{Z}$ which admit a projection onto the cylinder $\mathfrak{C}_{k,n}$. Although strictly speaking ambiguous, we will call the latter 'cylindric'. Then we introduce a quotient of the ring $\Lambda_k \otimes \mathbb{C}[z, z^{-1}]$, and we describe some product expansions which hold in this quotient. We shall employ these product expansions to obtain the expansions of the 'cylindric' symmetric functions $e_{\lambda/d/\mu}$ and $h_{\lambda/d/\mu}$ (which are the generalisation of $e_{\lambda/\mu}$ and $h_{\lambda/\mu}$ to the cylinder $\mathfrak{C}_{k,n}$) in terms of the bases of Λ described in Section 2.2. From here to the end of this chapter, we assume that $k, n \in \mathbb{N}$.

3.1 Cylindric Reverse Plane Partitions

Our first task is to generalise Section 2.1 to the cylinder $\mathfrak{C}_{k,n}$. The objects defined on $\mathbb{Z} \times \mathbb{Z}$ in Chapter 2, such as skew diagrams or Young diagrams, do in general not admit a projection onto the cylinder, and for this reason they will be referred as 'non-cylindric'. Our purpose is to generalise the latter 'to the cylinder' or 'to the cylindric case', that is we will extend their definition such that their projection onto the cylinder exists.

3.1.1 Cylindric diagrams and cylindric reverse plane partitions

For the discussion in this section we take inspiration from [41, 53, 58], although we expose the material in a slightly different manner to accommodate for further developments.

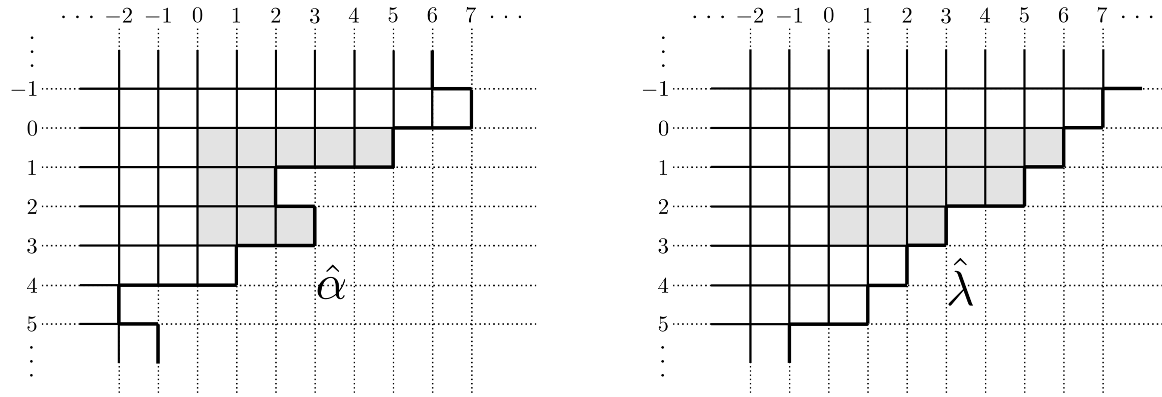


Figure 3.1: Let $k = 3$ and $n = 4$. Displayed above are represented the (cylindric) diagrams of the cylindric composition $\hat{\alpha} = (\dots, 5, 2, 3, \dots)$ and the cylindric partition $\hat{\lambda} = (\dots, 6, 5, 3, \dots)$ respectively. The boundary line of the diagram of $\hat{\lambda}$ is called a 'cylindric loop' in [58]. The shadowed boxes represent the (Young) diagrams of $\alpha = (5, 2, 3)$ and $\lambda = (6, 5, 3)$ respectively.

Definition 3.1.1. A cylindric composition $\hat{\alpha}$ of type (k, n) is defined as a doubly infinite sequence

$$(\dots, \hat{\alpha}_1, \hat{\alpha}_2, \dots, \hat{\alpha}_k, \dots) \tag{3.1}$$

in $\mathbb{Z}$, subject to the relation $\hat{\alpha}_{i+k} = \hat{\alpha}_i - n$ for all $i \in \mathbb{Z}$. A cylindric partition $\hat{\lambda}$ of type (k, n) is a cylindric composition satisfying the further constraint $\hat{\lambda}_i \geq \hat{\lambda}_{i+1}$ for all $i \in \mathbb{Z}$.

Denote with $\mathcal{P}_{k,n}$ the set of all cylindric compositions of type (k, n), and with $\mathcal{P}^+_{k,n} \subset \mathcal{P}_{k,n}$ the subset of all cylindric partitions of type (k, n). It will always be clear from the context what the type of a cylindric composition is. We now extend the notion of (Young) diagram to cylindric compositions.

Definition 3.1.2. Define the (cylindric) diagram of $\hat{\alpha} \in \mathcal{P}_{k,n}$ as the subset of $\mathbb{Z} \times \mathbb{Z}$ given by

$$\{(i, j) \in \mathbb{Z} \times \mathbb{Z} \mid j \leq \hat{\alpha}_i\} . \tag{3.2}$$

We identify each point in the diagram of $\hat{\alpha} \in \mathcal{P}_{k,n}$ with a box as we did for (non-cylindric) compositions. What is different is that now there is no lower boundary for the coordinate j. See Figure 3.1 for an example. For $\hat{\lambda} \in \mathcal{P}^+_{k,n}$ define the conjugate cylindric partition $\hat{\lambda}'$ as the cylindric partition whose diagram is obtained by reflecting the boxes of the diagram of $\hat{\lambda}$ along the diagonal $\{(i, i) \mid i \in \mathbb{Z}\}$. One has that $\hat{\lambda}' \in \mathcal{P}^+_{n,k}$ since $\hat{\lambda}'_{i+n} = \hat{\lambda}'_i - k$ for all $i \in \mathbb{Z}$. Denote with $m_i(\hat{\lambda})$ the multiplicity of i in $\hat{\lambda}$, then we have that $m_i(\hat{\lambda}) = \hat{\lambda}'_i - \hat{\lambda}'_{i+1}$ in analogy with the non-cylindric case.

Remark 3.1.3. We will often take advantage of the bijection $\mathcal{P}_{k,n} \to \mathcal{P}_k$ given by $\hat{\alpha} \mapsto \alpha = (\hat{\alpha}_1, \dots, \hat{\alpha}_k)$, which restricts to an injection $\mathcal{P}^+_{k,n} \to \mathcal{P}^+_k$. The inverse map sends $\alpha = (\alpha_1, \dots, \alpha_k)$ to the cylindric composition $\hat{\alpha}$ obtained by setting $\hat{\alpha}_i = \alpha_i$ for $i = 1, \dots, k$

and then $\hat{\alpha}_{i+k} = \hat{\alpha}_i - n$ for all $i \in \mathbb{Z}$. It is understood that whenever $\hat{\alpha} \in \mathcal{P}_{k,n}$ and $\alpha \in \mathcal{P}_k$ appear in the same context they are related via this bijection.

Remark 3.1.4. Let $\hat{\lambda} \in \mathcal{P}^+_{k,n}$ with $\hat{\lambda}_k > 0$, in which case we have that $\hat{\lambda}_1 > n$. Moreover, consider the partition λ which is the image of $\hat{\lambda}$ under the map introduced in the previous remark. View each box belonging to the diagram of λ as situated on the cylinder obtained in the following way: wrap the diagram of λ onto itself and glue together its first and last rows, so that for $j = 1, \ldots, \lambda_k$ the boxes associated respectively with the points (k, j) and $(1, j+n)$ are adjacent. On the cylinder constructed in this way, we have that for $i = 1, \ldots, k+1$ the box associated to (i, λ_i) is to the right of the box associated to $(i+1, \lambda_{i+1})$. This is in analogy with non-cylindric Young diagrams, compare with Figure 2.1. The observation presented above represents an alternative way to justify the epithet 'cylindric' for cylindric partitions (see [28] for further details).

Definition 3.1.5. Let $\hat{\lambda}, \hat{\mu} \in \mathcal{P}^+_{k,n}$, and write $\hat{\mu} \subset \hat{\lambda}$ if the diagram of $\hat{\mu}$ is contained in the diagram of $\hat{\lambda}$, or equivalently if $\hat{\mu}_i \leq \hat{\lambda}_i$ for all $i \in \mathbb{Z}$. Define the cylindric skew diagram $\hat{\lambda}/\hat{\mu} \subset \mathbb{Z} \times \mathbb{Z}$ as

$$\hat{\lambda}/\hat{\mu} = \{(i, j) \in \mathbb{Z} \times \mathbb{Z} \mid \hat{\mu}_i < j \leq \hat{\lambda}_i\} . \tag{3.3}$$

We will often use the expression '$\hat{\lambda}/\hat{\mu}$ is a cylindric skew diagram', and by this we mean that $\hat{\mu} \subset \hat{\lambda}$. In analogy with the non-cylindric case, we can think of $\hat{\lambda}/\hat{\mu}$ as the set of boxes which are placed between the boundaries of the diagrams of $\hat{\mu}$ and $\hat{\lambda}$. Denote with $|\hat{\lambda}/\hat{\mu}| = \sum_{i=1}^k (\hat{\lambda}_i - \hat{\mu}_i)$ the number of boxes in $\hat{\lambda}/\hat{\mu}$ which are located in lines 1 to k. If $\hat{\lambda}/\hat{\mu}$ has at most one box per row (respectively column) we will call it a cylindric vertical (respectively horizontal) strip. In particular, $\hat{\lambda}/\hat{\lambda}$ is both a cylindric vertical and horizontal strip. For $\hat{\lambda}, \hat{\mu} \in \mathcal{P}^+_{k,n}$ with $\hat{\lambda}_k, \hat{\mu}_k \geq 0$ we have that λ and μ are partitions, and then if we restrict $\hat{\lambda}/\hat{\mu}$ to the lines 1 to k we recover the skew diagram λ/μ. In particular let $\hat{\emptyset} \in \mathcal{P}^+_{k,n}$ be the cylindric partition with parts $\hat{\emptyset}_i = 0$ for $i = 1, \ldots, k$, then if we restrict $\hat{\lambda}/\hat{\emptyset}$ to the lines 1 to k we obtain the Young diagram of λ. In contrast with the non-cylindric case, a cylindric partition $\hat{\mu} \in \mathcal{P}^+_{k,n}$ such that the diagram of $\hat{\lambda}$ coincides with the diagram of $\hat{\lambda}/\hat{\mu}$ does not exists.

Definition 3.1.6 ([28])**.** Let $\hat{\lambda}, \hat{\mu} \in \mathcal{P}^+_{k,n}$. A cylindric reverse plane partition (CRPP) $\hat{\pi}$ of shape $\hat{\lambda}/\hat{\mu}$ is a map $\hat{\lambda}/\hat{\mu} \to \mathbb{N}$, $(i, j) \to \hat{\pi}_{i,j}$, subject to the constraints

$$\begin{aligned}
\hat{\pi}_{i,j} &= \hat{\pi}_{i+k,j-n} \ , && \\
\hat{\pi}_{i,j} &\leq \hat{\pi}_{i+1,j} \ , && \text{if } (i+1, j) \in \hat{\lambda}/\hat{\mu} \ , \\
\hat{\pi}_{i,j} &\leq \hat{\pi}_{i,j+1} \ , && \text{if } (i, j+1) \in \hat{\lambda}/\hat{\mu} \ .
\end{aligned}$$

In other words, $\hat{\pi}$ is a filling of the boxes of $\hat{\lambda}/\hat{\mu}$ with positive integers, called the entries of $\hat{\pi}$, which are weakly increasing from left to right in rows and down columns. Define the weight of $\hat{\pi}$ as the composition $\mathrm{wt}(\hat{\pi}) = (\mathrm{wt}_1(\hat{\pi}), \mathrm{wt}_2(\hat{\pi}), \dots)$, where $\mathrm{wt}_i(\hat{\pi})$ is the number of entries equal to i in lines 1 to k, or equivalently in columns 1 to n. See Figure 3.2 for an example.

Remark 3.1.7. If the entries of $\hat{\pi}$ are instead weakly decreasing from left to right in rows and down columns, we might refer to $\hat{\pi}$ as a cylindric plane partition, although in this thesis we will not make use of such object. In [28] this is what is called a 'cylindric partition'.

Remark 3.1.8. Let $\hat{\lambda}, \hat{\mu} \in \mathcal{P}^+_{k,n}$ with $\hat{\lambda}_k, \hat{\mu}_k > 0$. Moreover, let $\hat{\pi}$ be a CRPP of shape $\hat{\lambda}/\hat{\mu}$, and consider the RPP π of shape λ/μ which is obtained by restricting $\hat{\pi}$ to the lines 1 to k. View the entries of π as situated on the cylinder obtained in the following way: wrap π onto itself and glue together its first and last rows, so that for $j = 1, \dots, \lambda_k$ the entries $\pi_{k,j}$ and $\pi_{1,j+n}$ are adjacent (compare with the discussion presented in Remark 3.1.4). On the cylinder constructed in this way, we have that the entries of π are still weakly increasing from left to right in rows and down columns. This observation represents an alternative way to justify the epithet 'cylindric' for CRPPs (see [28] for further details).

Lemma 3.1.9. *A CRPP $\hat{\pi}$ of shape $\hat{\lambda}/\hat{\mu}$ with largest entry $l \in \mathbb{N}$ is equivalent to a sequence $\{\hat{\lambda}^{(r)}\}_{r=0}^{l}$ of cylindric partitions with*

$$\hat{\mu} = \hat{\lambda}^{(0)} \subset \hat{\lambda}^{(1)} \subset \cdots \subset \hat{\lambda}^{(l)} = \hat{\lambda}\ . \tag{3.4}$$

Proof. Suppose that $\hat{\pi}$ is a CRPP of shape $\hat{\lambda}/\hat{\mu}$. Set $\hat{\lambda}^{(0)} = \hat{\mu}$, and for $r = 1, \dots, l$ let $\hat{\lambda}^{(r)}$ be the cylindric partition whose diagram is obtained by joining the diagram of $\hat{\mu}$ with the boxes of $\hat{\pi}$ containing the entries from 1 to r. In particular $\hat{\lambda}^{(l)} = \hat{\lambda}$. It follows by construction that $\hat{\lambda}^{(r-1)} \subset \hat{\lambda}^{(r)}$ and thus the sequence $\{\hat{\lambda}^{(r)}\}_{r=0}^{l}$ of cylindric partitions satisfies the condition (3.4). Conversely, define a map $\hat{\pi} : \hat{\lambda}/\hat{\mu} \to \mathbb{N}$, $(i,j) \to \hat{\pi}_{i,j}$ as follows: for $r = 1, \dots, l$ set $\hat{\pi}_{i,j} = r$ if $(i,j) \in \hat{\lambda}^{(r)}/\hat{\lambda}^{(r-1)}$. Since the cylindric partitions $\{\hat{\lambda}^{(r)}\}_{r=0}^{l}$ satisfy the periodicity condition $\hat{\lambda}^{(r)}_{i+k} = \hat{\lambda}^{(r)}_{i} - n$, it follows that $\hat{\pi}_{i,j} = \hat{\pi}_{i+k,j-n}$. Let $(i,j) \in \hat{\lambda}^{(r)}/\hat{\lambda}^{(r-1)}$ for some $r = 1, \dots, l$. If $(i+1,j) \in \hat{\lambda}/\hat{\mu}$ it follows that $(i+1,j) \in \hat{\lambda}^{(r')}/\hat{\lambda}^{(r'-1)}$ for some $r' \geq r$, and then $\hat{\pi}_{i,j} \leq \hat{\pi}_{i+1,j}$. Similarly if $(i,j+1) \in \hat{\lambda}/\hat{\mu}$ we have that $\hat{\pi}_{i,j} \leq \hat{\pi}_{i,j+1}$, and thus $\hat{\pi}$ is a CRPP according to Definition 3.1.6. In particular it follows by construction that $\mathrm{wt}_r(\hat{\pi}) = |\hat{\lambda}^{(r)}/\hat{\lambda}^{(r-1)}|$ for $r = 1, \dots, l$. □

We shall make use of the following special case of CRPP. Compare with Definition 3.1.6.

Definition 3.1.10. Let $\hat{\lambda}, \hat{\mu} \in \mathcal{P}^+_{k,n}$. A *cylindric row strict tableau* (CRST) $\hat{T}$ of shape

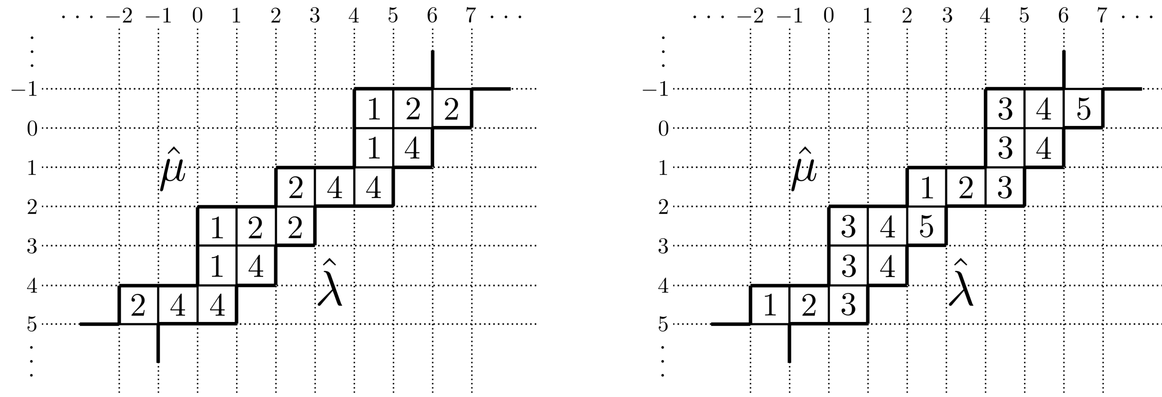


Figure 3.2: Let $k = 3$, $n = 4$, $\hat{\lambda} = (\ldots, 6, 5, 3, \ldots)$ and $\hat{\mu} = (\ldots, 4, 2, 0, \ldots)$. On the left we have a CRPP of shape $\hat{\lambda}/\hat{\mu}$ and weight $(2, 3, 0, 3)$. On the right we have a CRST of shape $\hat{\lambda}/\hat{\mu}$ and weight $(1, 1, 3, 2, 1)$. The CRPP on the left is equivalent to the sequence of cylindric partitions $(\ldots, 4, 2, 0, \ldots)$, $(\ldots, 5, 2, 1, \ldots)$, $(\ldots, 5, 3, 3, \ldots)$, $(\ldots, 5, 3, 3, \ldots)$, $(\ldots, 6, 5, 3, \ldots)$.

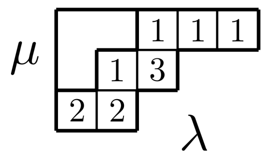
$\hat{\lambda}/\hat{\mu}$ is a map $\hat{T} : \hat{\lambda}/\hat{\mu} \to \mathbb{N}$, $(i, j) \to \hat{T}_{i,j}$, subject to the constraints

$$\begin{aligned} \hat{T}_{i,j} &= \hat{T}_{i+k,j-n} \ , \\ \hat{T}_{i,j} &\leq \hat{T}_{i+1,j} \ , \qquad \text{if } (i+1, j) \in \hat{\lambda}/\hat{\mu} \ , \\ \hat{T}_{i,j} &< \hat{T}_{i,j+1} \ , \qquad \text{if } (i, j+1) \in \hat{\lambda}/\hat{\mu} \ . \end{aligned}$$

Equivalently, $\hat{T}$ is a filling of the boxes of $\hat{\lambda}/\hat{\mu}$ with positive integers which are weakly increasing down columns but strictly increasing from left to right in rows. One can define cylindric column strict tableaux in a similar fashion [5, 53, 58], although we will not make use of such objects in our discussion.

Lemma 3.1.11. *A CRST of shape $\hat{\lambda}/\hat{\mu}$ with largest entry l is equivalent to a sequence $\{\hat{\lambda}^{(r)}\}_{r=0}^{l}$ of cylindric partitions satisfying* (3.4)*, such that $\hat{\lambda}^{(r)}/\hat{\lambda}^{(r-1)}$ is a cylindric vertical strip for $r = 1, \ldots, l$.*

Proof. The proof of this statement proceeds in a similar fashion to the proof of Lemma 3.1.9. Let $\hat{T}$ be a CRST of shape $\hat{\lambda}/\hat{\mu}$, set $\hat{\lambda}^{(0)} = \hat{\mu}$ and define $\hat{\lambda}^{(r)}$ for $r = 1, \ldots, l$ as therein. This implies that $\hat{\lambda}^{(r-1)} \subset \hat{\lambda}^{(r)}$, and furthermore $\hat{\lambda}^{(r)}/\hat{\lambda}^{(r-1)}$ is a cylindric vertical strip since in $\hat{T}$ there is at most one box per line containing the entry r. Conversely, define a map $\hat{T} : \hat{\lambda}/\hat{\mu} \to \mathbb{N}$, $(i, j) \mapsto \hat{T}_{i,j}$ as in the proof of Lemma 3.1.9, that is for $r = 1, \ldots, l$ set $\hat{T}_{i,j} = r$ if $(i, j) \in \hat{\lambda}^{(r)}/\hat{\lambda}^{(r-1)}$. Since $\hat{T}$ is a CRPP it follows that $\hat{T}_{i,j} = \hat{T}_{i+k,j-n}$, and morever $\hat{T}_{i,j} \leq \hat{T}_{i+1,j}$ provided that $(i + 1, j) \in \hat{\lambda}/\hat{\mu}$. Let $(i, j) \in \hat{\lambda}^{(r)}/\hat{\lambda}^{(r-1)}$ for some $r = 1, \ldots, l$, and suppose that $(i, j + 1) \in \hat{\lambda}/\hat{\mu}$. The fact that $\hat{\lambda}^{(r)}/\hat{\lambda}^{(r-1)}$ is a cylindric vertical strip implies that $(i, j + 1) \in \hat{\lambda}^{(r')}/\hat{\lambda}^{(r'-1)}$ for some $r' > r$, and then $\hat{T}_{i,j} < \hat{T}_{i,j+1}$. In conclusion, we have that $\hat{T}$ is a CRST according to Definition 3.1.10. $\square$



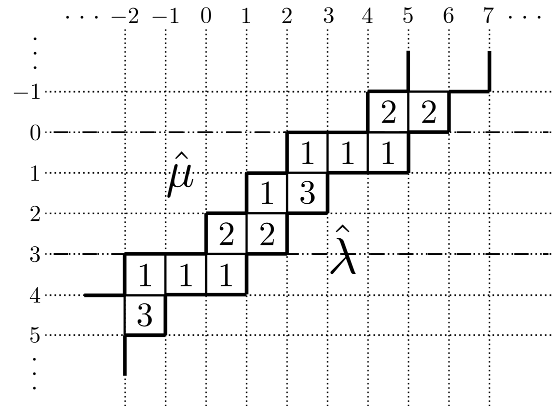


Figure 3.3: Let $k = 3$, $n = 4$, $\lambda = (5, 3, 2)$ and $\mu = (2, 1)$. The RPP of shape λ/μ on the left does not give rise to a CRPP of shape $\hat{\lambda}/\hat{\mu}$ when extended periodically to $\mathbb{Z} \times \mathbb{Z}$, as shown in the picture on the right.

Remark 3.1.12. Let $\hat{\lambda}, \hat{\mu} \in \mathcal{P}^+_{k,n}$ with $\hat{\lambda}_k, \hat{\mu}_k \geq 0$. Restricting a CRPP (respectively CRST) of shape $\hat{\lambda}/\hat{\mu}$ to the lines 1 to k one obtains a RPP (respectively row strict tableau) of shape λ/μ. The converse is not true in general, that is not all RPPs π of shape λ/μ extend to a map $\hat{\pi} : \hat{\lambda}/\hat{\mu} \to \mathbb{N}$ satisfying the requirements of Definition 3.1.6. See Figure 3.3 for a counterexample. Notice that this condition holds if the parts of $\lambda, \mu \in \mathcal{P}^+_k$ are smaller or equal than n, since in this case there are no boxes of λ/μ in columns greater than n.

3.1.2 The extended affine symmetric group

In analogy with the non-cylindric case, we want to introduce a notion of 'permutation' on the set of cylindric compositions. The mathematical object which is needed for this purpose is the following [49].

Definition 3.1.13. The extended affine symmetric group $\hat{S}_k$ is the group generated by $\{\sigma_0, \sigma_1, \ldots, \sigma_{k-1}, \tau\}$ subject to the relations

$$\sigma_i^2 = 1 \ , \qquad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \ , \qquad \sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{for } |i-j| > 1 \ , \tag{3.5}$$

together with

$$\tau\sigma_{i+1} = \sigma_i\tau \ ,$$

where the indices are understood modulo k.

We will also make extensive use of the affine symmetric group $\tilde{S}_k$, that is the affine Coxeter group $\tilde{A}_{k-1}$, which is the subgroup of $\hat{S}_k$ generated by $\{\sigma_0, \ldots, \sigma_{k-1}\}$. The affine symmetric group $\tilde{S}_k$ is isomorphic to the group of bijections $\tilde{w} : \mathbb{Z} \to \mathbb{Z}$, with composition as the group product, subject to the conditions

$$\tilde{w}(m+k) = \tilde{w}(m) + k\ , \quad \forall m \in \mathbb{Z} \qquad \text{and} \qquad \sum_{m=1}^{k} \tilde{w}(m) = \binom{k}{2}\ .$$

This statement first appeared in [51], and has been subsequently used by many authors [7, 20]. The following [45] is a generalisation of such statement to $\hat{S}_k$.

Proposition 3.1.14. *The extended affine symmetric group $\hat{S}_k$ is isomorphic to the group of bijections $\hat{w} : \mathbb{Z} \to \mathbb{Z}$, with composition as the group product, subject to the conditions*

$$\hat{w}(m+k) = \hat{w}(m) + k, \quad \forall m \in \mathbb{Z} \qquad \textit{and} \qquad \sum_{m=1}^{k} \hat{w}(m) = \binom{k}{2} \mod k\ . \tag{3.6}$$

The isomorphism between $\hat{S}_k$ and the group of bijections $\hat{w} : \mathbb{Z} \to \mathbb{Z}$ satisfying the constraints (3.6) is given as follows. The generator $\tau \in \hat{S}_k$ is mapped to the bijection $\tau : \mathbb{Z} \to \mathbb{Z}$ defined via $\tau(m) = m - 1$, and for this reason we will sometimes refer to τ as the 'shift operator'. Moreover, for $i = 0, \ldots, k-1$ the generator $\sigma_i \in \hat{S}_k$ is mapped to the bijection $\sigma_i : \mathbb{Z} \to \mathbb{Z}$ defined via

$$\sigma_i(m) = \begin{cases} m+1\ , & m = i \mod k \\ m-1\ , & m = (i+1) \mod k \\ m\ , & \text{otherwise} \end{cases}\ . \tag{3.7}$$

We now want to construct a right action of $\hat{S}_k$ on cylindric compositions. For this purpose, recall [49] the level-n right action $\mathcal{P}_k \times \hat{S}_k \to \mathcal{P}_k$, $(\alpha, \hat{w}) \mapsto \alpha.\hat{w}$, which is fixed by the following maps

$$(\alpha_1, \ldots, \alpha_i, \alpha_{i+1}, \ldots \alpha_k).\sigma_i = (\alpha_1, \ldots, \alpha_{i+1}, \alpha_i, \ldots, \alpha_k)\ , \tag{3.8}$$

$$(\alpha_1, \ldots, \alpha_k).\sigma_0 = (\alpha_k + n, \alpha_2, \alpha_3, \ldots, \alpha_{n-1}, \alpha_1 - n)\ , \tag{3.9}$$

$$(\alpha_1, \ldots, \alpha_k).\tau = (\alpha_k + n, \alpha_1, \alpha_2, \ldots, \alpha_{k-1})\ . \tag{3.10}$$

Notice that $i = 1, \ldots, k-1$ in the first equation. It can be shown that the 'alcove'

$$\mathcal{A}_k^+(n) = \{\lambda \in \mathcal{P}_k \mid n \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0\} \tag{3.11}$$

is a fundamental domain with respect to this action. That is, for any $\alpha \in \mathcal{P}_k$ the orbit $\alpha.\hat{S}_k$ intersects $\mathcal{A}_k^+(n)$ in a unique point. Denote with $\hat{\mathcal{A}}_k^+(n)$ the image of $\mathcal{A}_k^+(n)$ under the bijection $\lambda \mapsto \hat{\lambda}$.

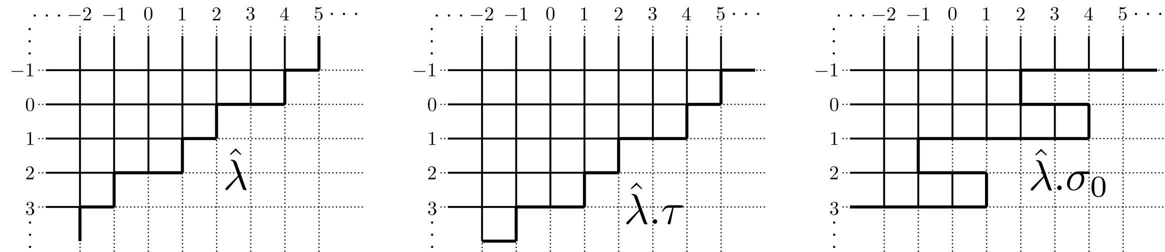


Figure 3.4: Let $k=2$, $n=3$ and $\hat\lambda=(\dots,2,1,\dots)$. From left to right we have the diagrams of $\hat\lambda$, $\hat\lambda.\tau$ and $\hat\lambda.\sigma_0$. Notice that the diagram of $\hat\lambda.\tau$ is obtained by translating each box in the diagram of $\hat\lambda$ by the vector $(0,1)$.

Lemma 3.1.15. *The map $\mathcal{P}_{k,n}\times\hat S_k\to\mathcal{P}_{k,n}$ defined as*

$$(\hat\alpha,\hat w)\mapsto\hat\alpha.\hat w=(\dots,\hat\alpha_{\hat w(1)},\hat\alpha_{\hat w(2)},\dots,\hat\alpha_{\hat w(k)},\dots)\ , \tag{3.12}$$

is a right group action of $\hat S_k$ on cylindric compositions. On the RHS of the equality in (3.12) *it is understood that $\hat w:\mathbb{Z}\to\mathbb{Z}$ (compare with Proposition 3.1.14).*

Proof. Let $\hat\alpha.\hat w$ be the image of $\alpha.\hat w\in\mathcal{P}_k$ under the bijection described in Remark 3.1.3. Then the map $\mathcal{P}_{k,n}\times\hat S_k\to\mathcal{P}_{k,n}$ given by $(\hat\alpha,\hat w)\mapsto\hat\alpha.\hat w$ is a group action, and moreover a straightforward computation shows that

$$(\dots,\hat\alpha_i,\hat\alpha_{i+1},\dots).\sigma_i \;=\; (\dots,\hat\alpha_{i+1},\hat\alpha_i,\dots)\ , \tag{3.13}$$
$$(\dots,\hat\alpha_m,\hat\alpha_{m+1},\dots).\tau \;=\; (\dots,\hat\alpha_{m-1},\hat\alpha_m,\dots)\ , \tag{3.14}$$

where $i=0,\dots,k-1$. That is, $\hat\alpha.\sigma_i$ is obtained from $\hat\alpha$ by permuting its parts at positions i and $i+1$ modulo k, whereas $\hat\alpha.\tau$ is obtained by shifting each part of $\hat\alpha$ by one position forward. It follows that the group action defined above corresponds to the map (3.12), since equations (3.13) and (3.14) imply that the action of these two maps coincide for the generators of $\hat S_k$, and thus they must coincide for every element in $\hat S_k$.

□

The extended affine symmetric group $\hat S_k$ has an alternative set of generators given by $\{\sigma_1,\dots,\sigma_{k-1}\}\cup\{y_1,\dots,y_k\}$. These satisfy the relations

$$y_iy_j=y_jy_i\ ,\qquad \sigma_iy_i=y_{i+1}\sigma_i\ ,\qquad y_i\sigma_j=\sigma_jy_i\quad\text{for } |i-j|>1\ , \tag{3.15}$$

where again the indices are understood modulo k. The link with the generators introduced in Definition 3.1.13 is given by $y_k=\tau\sigma_1\dots\sigma_{k-1}$ and $\sigma_0=\sigma_1\cdots\sigma_{k-1}\sigma_{k-2}\cdots\sigma_1y_1y_k^{-1}$. A straightforward computation shows that

$$(\dots,\hat\alpha_i,\dots).y_i=(\dots,\hat\alpha_i+n,\dots)\ . \tag{3.16}$$

We shall make use of the fact that every $\hat{w} \in \hat{S}_k$ can be expressed uniquely as $\hat{w} = wy^\alpha$ with $w \in S_k$ and $\alpha \in \mathcal{P}_k$, where we use the notation $y^\alpha \equiv y_1^{\alpha_1} \cdots y_k^{\alpha_k}$ [49]. In particular, if $\hat{w} \in \tilde{S}_k \subset \hat{S}_k$ we have that $|\alpha| = 0$.

For $\hat{\mu} \in \mathcal{P}_{k,n}^+$ denote with $\tilde{S}_{\hat{\mu}} \subset \tilde{S}_k$ its stabilizer subgroup, which is a parabolic subgroup of $\tilde{S}_k$. Moreover, denote with $\tilde{S}_{\hat{\mu}} \backslash \tilde{S}_k$ the set of right cosets $\{\tilde{S}_{\hat{\mu}}\tilde{w} \mid \tilde{w} \in \tilde{S}_k\}$ of $\tilde{S}_{\hat{\mu}}$ in $\tilde{S}_k$. We now state a similar result to Proposition 2.1.6 for the affine case, which can be found for instance in [8, Prop. 2.4.4 and Cor. 2.4.5]. For this purpose, define the length of $\tilde{w} \in \tilde{S}_k$ as

$$\ell(\tilde{w}) = \min\left\{r \in \mathbb{N} \mid \tilde{w} = \sigma_{i_1} \cdots \sigma_{i_r} \text{ for some } i_1, \ldots, i_r \in \{0, 1, \ldots, k-1\}\right\}. \tag{3.17}$$

Proposition 3.1.16. *(i) Each right coset $\tilde{S}_{\hat{\mu}}w$ has a unique representative of minimal length.*

(ii) Every element $\tilde{w} \in \tilde{S}_k$ has a unique decomposition $\tilde{w} = \tilde{w}_{\hat{\mu}}\tilde{w}^{\hat{\mu}}$, with $\tilde{w}_{\hat{\mu}} \in \tilde{S}_{\hat{\mu}}$ and $\tilde{w}^{\hat{\mu}}$ a minimal length representative of one right coset in $\tilde{S}_{\hat{\mu}} \setminus \tilde{S}_k$.

Denote with $\tilde{S}^{\hat{\mu}}$ the set of minimal length representatives of the right cosets $\tilde{S}_{\hat{\mu}} \setminus \tilde{S}_k$.

Remark 3.1.17. In the following we shall make extensive use of both the action $\mathcal{P}_{k,n} \times \hat{S}_k \to \mathcal{P}_{k,n}$ and the level-n right action $\mathcal{P}_k \times \hat{S}_k \to \mathcal{P}_k$. We adopt the notation $\tilde{S}_\mu \subset \tilde{S}_k$ for the stabiliser subgroup of $\mu \in \mathcal{P}_k^+$, which coincides with $\tilde{S}_{\hat{\mu}}$. Similarly, denote with $\tilde{S}_\mu \setminus \tilde{S}_k$ the set of right cosets $\{\tilde{S}_\mu\tilde{w} \mid \tilde{w} \in \tilde{S}_k\}$ of $\tilde{S}_\mu$ in $\tilde{S}_k$, which is the same as the set $\tilde{S}_{\hat{\mu}} \setminus \tilde{S}_k$ introduced above. Finally, the set $\tilde{S}^{\hat{\mu}}$ coincides with the set $\tilde{S}^\mu$ of minimal length representatives of the right cosets $\tilde{S}_\mu \setminus \tilde{S}_k$.

Lemma 3.1.18. *Suppose that $\mu \in \mathcal{A}_k^+(n)$. For every $\tilde{w} \in \tilde{S}^\mu$, there exists a unique element in the right coset $\tilde{S}_\mu\tilde{w}$ which can be written as wy^α for some $w \in S^\mu$ and $\alpha \in \mathcal{P}_k$ with $|\alpha| = 0$.*

Proof. Since $\mu \in \mathcal{A}_k^+(n)$ we have that $\mu_k + n > \mu_1$, and thus $\tilde{S}_\mu = S_\mu \subset S_k$. Let $\tilde{w} \in \tilde{S}^\mu$, and write $\tilde{w} = \bar{w}y^\alpha$ for some $\bar{w} \in S_k$ and $\alpha \in \mathcal{P}_k$ with $|\alpha| = 0$. Part (ii) of Proposition 2.1.6 implies that there exists a unique decomposition $\bar{w} = w_\mu w^\mu$, with $w_\mu \in S_\mu$ and $w^\mu \in S^\mu$. It follows that $(w_\mu)^{-1}\tilde{w}$ is the unique element in the right coset $\tilde{S}_\mu\tilde{w}$ which can be expressed as wy^α for some $w \in S^\mu$ and $\alpha \in \mathcal{P}_k$ with $|\alpha| = 0$. This proves the claim. □

3.2 Weighted sums over CRPPs

The goal of this section is to extend the skew elementary and complete symmetric functions defined in Chapter 2 to the cylinder. For this purpose, we generalise to the cylinder the expansions derived in Lemmas 2.3.16 and 2.3.21, namely

$$\begin{aligned} e_{\lambda/\mu} &= \sum_T \psi_T x^T\ , \qquad \psi_T = \prod_{i\geq 1} \psi_{\lambda^{(i)}/\lambda^{(i-1)}}\ , \\ h_{\lambda/\mu} &= \sum_\pi \theta_\pi x^\pi\ , \qquad \theta_\pi = \prod_{i\geq 1} \theta_{\lambda^{(i)}/\lambda^{(i-1)}}\ . \end{aligned}$$

Recall that these expansions are valid if $\mu \subset \lambda$, otherwise we have that $e_{\lambda/\mu} = h_{\lambda/\mu} = 0$. The weights $\psi_{\lambda/\mu}$ and $\theta_{\lambda/\mu}$ were defined in (2.67) and (2.61) respectively as the cardinalities of the sets

$$\{\alpha \in \mathcal{P} \mid \alpha \sim \mu, \lambda/\alpha \text{ is a vertical strip}\}\ , \qquad \{\alpha \in \mathcal{P} \mid \alpha \sim \mu, \alpha \subset \lambda\}\ . \tag{3.18}$$

Remark 3.2.1. Given $\alpha, \beta \in \mathcal{P}_k$, we write $\alpha \leq \beta$ if $\alpha_i \leq \beta_i$ for all $i = 1 \ldots, k$. We shall use the symbol $\subset$ for cylindric compositions instead. That is, for $\hat{\alpha}, \hat{\beta} \in \mathcal{P}_{k,n}$ we write $\hat{\alpha} \subset \hat{\beta}$ if the diagram of $\hat{\alpha}$ is contained in the diagram of $\hat{\beta}$. Taking advantage of the bijection introduced in Remark 3.1.3, we have that $\alpha \leq \beta$ if and only if $\hat{\alpha} \subset \hat{\beta}$.

Our first task is to generalise the sets (3.18), or equivalently the weights $\psi_{\lambda/\mu}$ and $\theta_{\lambda/\mu}$, to the cylinder. For this purpose, we reformulate them in terms of the symmetric group $S_k \subset \hat{S}_k$.

Lemma 3.2.2. *Suppose that $\mu \subset \lambda$ and $\ell(\lambda) \leq k$. The sets* (3.18) *are in bijection respectively with*

$$\{w \in S^\mu \mid \lambda_i - (\mu.w)_i = 0, 1\}\ , \qquad \{w \in S^\mu \mid \mu.w \leq \lambda\}\ . \tag{3.19}$$

Proof. If $\alpha \sim \mu$ belongs to one of the sets (3.18) then it must satisfy the constraint $\alpha \subset \lambda$. We can therefore identify both α and μ as weights in $\mathcal{P}_k$. Moreover, we can take advantage of the action $\mathcal{P}_k \times S_k \to \mathcal{P}_k$, which implies that there exists a unique element $w \in S^\mu \subset S_k$ such that $\alpha = \mu.w \leq \lambda$ in the notation of Remark 3.2.1. The permutations $\alpha \sim \mu$ belonging to the sets (3.18) are then labelled by elements in S^μ, and the claim follows. $\square$

Before proceeding with the generalisation of the sets (3.19) to the cylinder, we present some further preliminary results. The next lemma shows that, starting from the alcove (3.11), we can recover the set of all cylindric partitions by employing the action of the extended affine symmetric group.

Lemma 3.2.3. *Every element in $\mathcal{P}^+_{k,n}$ can be expressed uniquely as $\hat{\lambda}.\tau^d$ with $\lambda \in \mathcal{A}^+_k(n)$ and $d \in \mathbb{Z}$.*

Proof. Let $\hat{\nu} \in \mathcal{P}^+_{k,n}$. If $\hat{\nu} \in \hat{\mathcal{A}}^+_k(n)$ set $\hat{\lambda} = \hat{\nu}$ and $d = 0$, otherwise periodicity implies that either $\hat{\nu}_1 > n$ or $\hat{\nu}_k < 0$. Suppose that $\hat{\nu}_1 > n$, in which case $(\hat{\nu}.\tau^{-1})_k = \hat{\nu}_{k+1} = \hat{\nu}_1 - n > 0$.

If $(\hat{\nu}.\tau^{-1})_1 = \hat{\nu}_2 \leq n$ set $\hat{\lambda} = \hat{\nu}.\tau^{-1} \in \hat{\mathcal{A}}_k^+(n)$ and $d = 1$, otherwise consider the cylindric partition $\hat{\nu}.\tau^{-2}$. If $(\hat{\nu}.\tau^{-2})_1 = \hat{\nu}_3 \leq n$ set $\hat{\lambda} = \hat{\nu}.\tau^{-2} \in \hat{\mathcal{A}}_k^+(n)$ and $d = 2$, otherwise consider the cylindric partition $\hat{\nu}.\tau^{-3}$. Proceeding in a similar vein, one concludes that there exists $d' \in \mathbb{N}$ such that $(\hat{\nu}.\tau^{-d'}) \leq n$, and the claim follows by setting $\hat{\lambda} = \hat{\nu}.\tau^{-d'} \in \hat{\mathcal{A}}_k^+(n)$ and $d = d'$. The proof for the case $\hat{\nu}_k < 0$ is similar. □

Suppose that $\hat{\nu}_1, \hat{\nu}_2 \in \mathcal{P}_{k,n}^+$. Thanks to Lemma 3.2.3, we can express $\hat{\nu}_1$ and $\hat{\nu}_2$ uniquely as $\hat{\lambda}_1.\tau^{d_1}$ and $\hat{\lambda}_2.\tau^{d_2}$, with $\lambda_1, \lambda_2 \in \mathcal{A}_k^+(n)$ and $d_1, d_2 \in \mathbb{Z}_{\geq 0}$. Notice that, if we translate the vertical axis by d_2 units in the positive direction, the cylindric skew diagram $\hat{\nu}_1/\hat{\nu}_2$ is mapped to $\hat{\lambda}_1.\tau^{d_1-d_2}/\hat{\lambda}_2$. After an appropriate translation of the vertical axis, every cylindric skew diagram can then be expressed as $\hat{\lambda}.\tau^d/\hat{\mu}$ for some $\lambda, \mu \in \mathcal{A}_k^+(n)$ and $d \in \mathbb{Z}_{\geq 0}$. We denote the latter by $\lambda/d/\mu$ in agreement with [58]. That is,

$$\lambda/d/\mu = \{(i,j) \in \mathbb{Z} \times \mathbb{Z} \mid \hat{\mu}_i < j \leq (\hat{\lambda}.\tau^d)_i\} \ . \tag{3.20}$$

To show that d is non-negative, use the constraint $\hat{\mu}'_1 \leq (\hat{\lambda}.\tau^d)'_1$ together with $(\hat{\lambda}.\tau^d)'_i = \hat{\lambda}'_i + d$ and $\hat{\mu}'_1 = \hat{\lambda}'_1 = k$. We will sometimes use the fact that $|\lambda/d/\mu| = \sum_{i=1}^k((\hat{\lambda}.\tau^d)_i - \hat{\mu}_i) = |\lambda| + dn - |\mu|$. To obtain the diagram of $\hat{\lambda}.\tau^d$ one needs to translate each box in the diagram of $\hat{\lambda}$ by the vector $(0, d)$, compare with Figure 3.4. Instead in [58] a different convention is used, namely one has to translate each box in the diagram of $\hat{\lambda}$ by the vector (d, d) to obtain the cylindric skew diagram $\lambda/d/\mu$. Thanks to Lemma 3.1.9, a CRPP of shape $\lambda/d/\mu$ with largest entry l is equivalent to a sequence

$$\hat{\mu} = \hat{\lambda}^{(0)}.\tau^{d_0} \subset \hat{\lambda}^{(1)}.\tau^{d_1} \subset \cdots \subset \hat{\lambda}^{(l)}.\tau^{d_l} = \hat{\lambda}.\tau^d \tag{3.21}$$

of cylindric partitions with $\hat{\lambda}^{(r)} \in \hat{\mathcal{A}}_k^+(n)$ and $d_r - d_{r-1} \geq 0$ for $r = 1, \ldots, l$. Similarly, a CRST of shape $\lambda/d/\mu$ is equivalent to a sequence (3.21) of cylindric partitions where $\hat{\lambda}^{(r)}.\tau^{d_r}/\hat{\lambda}^{(r-1)}.\tau^{d_{r-1}}$ is a cylindric vertical strip for $r = 1, \ldots, l$.

3.2.1 Generalisation of $\theta_{\lambda/\mu}$ to the cylinder

We already have a mathematical object which we can use to construct 'permutations' of cylindric compositions. This is the extended affine symmetric group $\hat{S}_k$, compare with (3.13) and (3.14). Let us generalise the notion of cylindric skew diagram described in Definition 3.1.5 to cylindric compositions. Suppose that $\hat{\alpha}, \hat{\beta} \in \mathcal{P}_{k,n}$ with $\hat{\alpha} \subset \hat{\beta}$ in the notation of Remark 3.2.1, then we refer to the set $\hat{\beta}/\hat{\alpha} \subset \mathbb{Z} \times \mathbb{Z}$ as a cylindric skew diagram.

Definition 3.2.4. For $\lambda, \mu \in \mathcal{A}_k^+(n)$ and $d \in \mathbb{Z}_{\geq 0}$ define $\theta_{\lambda/d/\mu}$ as the cardinality of the set

$$\{\tilde{w} \in \tilde{S}^{\hat{\mu}} \mid \hat{\mu}.\tilde{w} \subset \hat{\lambda}.\tau^d\} \ . \tag{3.22}$$

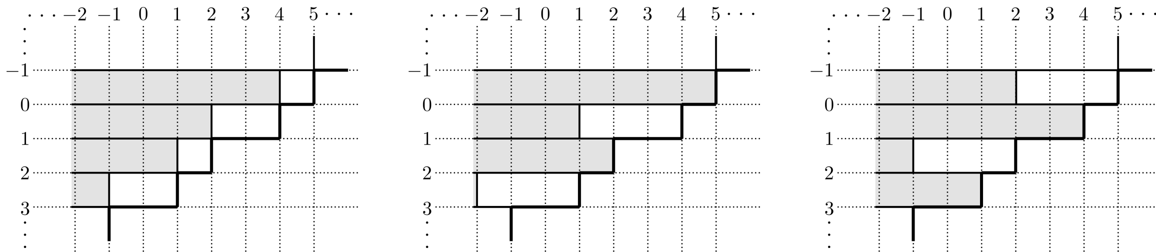


Figure 3.5: Let $k = 2$, $n = 3$, $d = 1$ and $\hat{\lambda} = \hat{\mu} = (\dots, 2, 1, \dots)$. From left to right we have, in grey, the diagrams of $\hat{\mu}$, $\hat{\mu}.\sigma_1$ and $\hat{\mu}.\sigma_0$. These are the only diagrams of the form $\hat{\mu}.\tilde{w}$, for $w \in \tilde{S}^{\hat{\mu}}$, that are contained in the diagram of $\hat{\lambda}.\tau$, whose boundary is indicated by the solid black line. It follows that $\theta_{\lambda/d/\mu} = 3$.

In Figure 3.5 there is an example which emphasises the combinatorial nature of the set (3.22). Compare with Figure 2.4 in Chapter 2. If instead we use the action of $\hat{S}_k$ on $\mathcal{P}_k$ (compare with Remark 3.1.17), this set can be expressed equivalently as

$$\{\tilde{w} \in \tilde{S}^{\mu} \mid \mu.\tilde{w} \leq \lambda.\tau^d\} \,. \tag{3.23}$$

Notice that in (3.23) we employed the notation introduced in Remark 3.2.1 for weights in $\mathcal{P}_k$. For practical reasons we will mostly work with the set (3.23) in proofs.

Remark 3.2.5. It is straightforward to show that (3.23) reduces to the second set in (3.19) for $d = 0$. In fact, let $\tilde{w} \in \tilde{S}^{\mu}$ belong to (3.23), and write $\tilde{w} = wy^{\alpha}$ for some $w \in S_k$ and $\alpha \in \mathcal{P}_k$ with $|\alpha| = 0$. Since $\lambda, \mu \in \mathcal{A}_k^+(n)$ we have for $d = 0$ that $\alpha = (0, \dots, 0)$, and thus $\tilde{w} \in S^{\mu} \subset S_k$ as $\tilde{w}$ is a minimal length coset representative. In particular, this implies that $\theta_{\lambda/0/\mu} = \theta_{\lambda/\mu}$.

Lemma 3.2.6. *The set* (3.23) *has the following alternative form,*

$$\left\{(w, \alpha) \in S^{\mu} \times \mathcal{P}_k^{\geq 0} \mid |\alpha| = d, \mu.wy^{-\alpha} \leq \lambda\right\} \,. \tag{3.24}$$

Proof. Notice that, for every $\tilde{w} \in \tilde{S}^{\mu}$, all the elements in the right coset $\tilde{S}_{\mu}\tilde{w}$ have the same action on the weight μ. Thanks to Lemma 3.1.18 it follows that (3.23) can be expressed as the set

$$\mathbb{S}_1 = \left\{(w, \alpha) \in S^{\mu} \times \mathcal{P}_k \mid |\alpha| = 0, \mu.wy^{\alpha}\tau^{-d} \leq \lambda\right\} \,.$$

Write the shift operator as $\tau = y_k\sigma_{k-1} \cdots \sigma_1$. Using the commutation relation $\sigma_i y_i = y_{i+1}\sigma_i$ one has that $\tau^{-d} = (\sigma_{k-1} \cdots \sigma_1)^{-d} y^{\beta}$ for some $\beta \in \mathcal{P}_k$ with $|\beta| = -d$. Let $(w, \alpha) \in \mathbb{S}_1$ and consider the weight $\gamma \in \mathcal{P}_k$ given by $\gamma = \alpha.(\sigma_{k-1} \cdots \sigma_1)^{-d} + \beta$, which satisfies the constraint $|\gamma| = -d$. Using the fact that $y^{\alpha}\, w' = w'\, y^{\alpha.w'}$ for $w' \in S_k$, which can be proved with the help of (3.15), one ends up with the equality $wy^{\alpha}\tau^{-d} = w(\sigma_{k-1} \cdots \sigma_1)^{-d}y^{\gamma}$. Thus

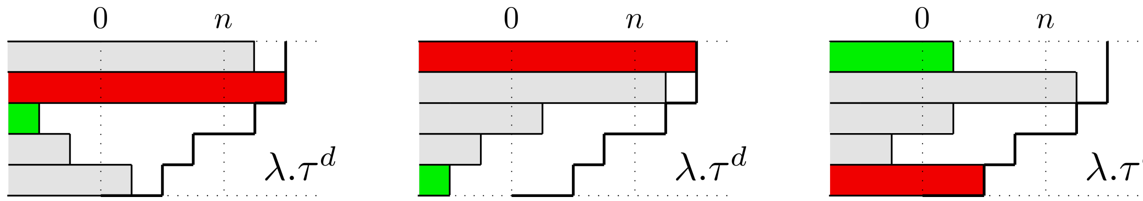


Figure 3.6: A pictorial representation of the weights μ, $\bar{\mu}$ and $\nu^{(1)}$, which were introduced in the proof of Proposition 3.2.7, from left to right. We have highlighted the parts $(\mu.\tilde{w})_i$ (red) and $(\mu.\tilde{w})_j$ (green), where $i < j$.

we can express the set $\mathbb{S}_1$ as

$$\mathbb{S}_2 = \left\{(w,\gamma) \in S^\mu \times \mathcal{P}_k \mid |\gamma| = -d, \mu.w(\sigma_{k-1}\cdots\sigma_1)^d y^\gamma \le \lambda\right\} .$$

If $(w,\gamma) \in \mathbb{S}_2$ then $\gamma_i \le 0$, because none of the parts of $\mu.w(\sigma_{k-1}\cdots\sigma_1)^d y^\gamma$ can exceed n since $\lambda \in \mathcal{A}_k^+(n)$. In other words we have that the weight $-\gamma$ belongs to $\mathcal{P}_k^{\ge 0}$. Part (ii) of Proposition 2.1.6 implies that $w(\sigma_{k-1}\cdots\sigma_1)^{-d}$ has a unique decomposition $w_\mu w^\mu$, with $w_\mu \in S_\mu$ and $w^\mu \in S^\mu$. Notice that for different elements $w \in S^\mu$ we end up with different elements $w^\mu \in S^\mu$, and moreover we have by definition that $\mu.w_\mu = \mu$. This finally implies that $\mathbb{S}_2$ is the same as the set (3.24).

□

Proposition 3.2.7. *The set* (3.22) *is non-empty if and only if* $\hat{\mu} \subset \hat{\lambda}.\tau^d$*, that is if and only if* $\lambda/d/\mu$ *is a cylindric skew diagram as defined in* (3.20)*.*

Proof. To simplify the proof we will work with the set (3.23) instead. In other words we will show that the latter is non-empty if and only if $\mu \le \lambda.\tau^d$ in the notation of Remark 3.2.1. Assume that $\mu \le \lambda.\tau^d$, then $\tilde{w} = 1$ belongs to the set (3.23), which is then non-empty. Conversely suppose that (3.23) is non-empty, that is there exists $\tilde{w} \in \tilde{S}^\mu$ such that $\mu.\tilde{w} \le \lambda.\tau^d$. If $d = 0$ then (3.23) reduces to the second set in (3.19) and we are done thanks to Lemma 2.3.14. So let $d > 0$ and write $\tilde{w} = wy^\alpha$ for some $w \in S_k$ and $\alpha \in \mathcal{P}_k$ with $|\alpha| = 0$. If $\alpha = (0,0,\ldots,0)$ one can prove that $\mu \le \lambda.\tau^d$ in a similar vein to the proof of Lemma 2.3.14, so assume that $\alpha \ne (0,0,\ldots,0)$ and set $l = (|\alpha_1| + \cdots + |\alpha_k|)/2$. We now construct recursively a sequence

$$\nu^{(1)}, \nu^{(2)}, \ldots, \nu^{(l)}$$

of weights in $\mathcal{P}_k$ satisfying the constraint $\nu^{(r)} \le \lambda.\tau^d$ for $r = 1,\ldots,l$. Refer to Figure 3.6 for a graphical depiction of this construction.

For $1 \le i,j \le n$ let $(\mu.\tilde{w})_i$ and $(\mu.\tilde{w})_j$ be respectively the greatest and smallest part of $\mu.\tilde{w}$. Call $\bar{\mu}$ the weight obtained from $\mu.\tilde{w}$ by swapping $(\mu.\tilde{w})_1$ with $(\mu.\tilde{w})_i$ and $(\mu.\tilde{w})_k$ with $(\mu.\tilde{w})_j$, compare with Figure 3.6. By construction we have that $\bar{\mu} \le \lambda.\tau^d$. Taking advantage of the relation $y^\alpha w' = w' y^{\alpha.w'}$ for $w' \in S_k$ and part (ii) of Proposition 2.1.6 we can write $\bar\mu = \mu.\bar w\, y^{\bar\alpha}$ for some $\bar w \in S^\mu$ and $\bar\alpha \in \mathcal{P}_k$. We have that $\bar\alpha_1 > 0$ and $\bar\alpha_k < 0$, otherwise $\bar\alpha = \alpha = (0,0,\dots,0)$ which contradicts the hypothesis, and thus $\bar\mu_k + n < \bar\mu_1$ since $\mu \in \mathcal{A}_k^+(n)$. Set $\nu^{(1)} = \bar\mu.\sigma_0$ and notice that $\nu^{(1)} \le \lambda.\tau^d$ as $\nu_1^{(1)} = \bar\mu_k + n < \bar\mu_1 \le (\lambda.\tau^d)_1$ and $\nu_k^{(1)} = \bar\mu_1 - n \le (\lambda.\tau^d)_1 - n \le (\lambda.\tau^d)_k$. Let $\alpha^{(1)} \in \mathcal{P}_k$ with parts $\alpha_1^{(1)} = \bar\alpha_k + 1$, $\alpha_k^{(1)} = \bar\alpha_1 - 1$ and $\alpha_i^{(1)} = \bar\alpha_i$ for $i \neq 1,k$. Taking advantage of the equality $\sigma_0 = \sigma_1 \cdots \sigma_{k-1}\sigma_{k-2}\cdots\sigma_1 y_1 y_k^{-1}$ and part (ii) of Proposition 2.1.6 it then follows that $\nu^{(1)} = \mu.w^{(1)} y^{\alpha^{(1)}}$ for some $w^{(1)} \in S^\mu$. The crucial point here is that $|\alpha_1^{(1)}| + \cdots + |\alpha_k^{(1)}| = |\bar\alpha_1| + \cdots + |\bar\alpha_k| - 2 = |\alpha_1| + \cdots + |\alpha_k| - 2$.

Starting from $\nu^{(1)}$ and repeating the procedure just described one can construct a second weight $\nu^{(2)}$ such that $\nu^{(2)} \le \lambda.\tau^d$. Writing $\nu^{(2)} = \mu.w^{(2)} y^{\alpha^{(2)}}$ for some $w^{(2)} \in S^\mu$ and $\alpha^{(2)} \in \mathcal{P}_k$ it follows by construction that $|\alpha_1^{(2)}| + \cdots + |\alpha_k^{(2)}| = |\alpha_1^{(1)}| + \cdots + |\alpha_k^{(1)}| - 2 = |\alpha_1| + \cdots + |\alpha_k| - 4$. Proceeding along similar lines we end up with weights $\{\nu^{(r)}\}_{r=1}^l$ such that $\nu^{(r)} \le \lambda.\tau^d$. Writing $\nu^{(l)} = \mu.w^{(l)} y^{\alpha^{(l)}}$ for some $w^{(l)} \in S^\mu$ and $\alpha^{(l)} \in \mathcal{P}_k$, we have by construction the constraint $|\alpha_1^{(l)}| + \cdots + |\alpha_k^{(l)}| = |\alpha_1| + \cdots + |\alpha_k| - 2l = 0$, which implies that $\alpha^{(l)} = (0,0,\dots,0)$. It follows that $\nu^{(l)} = \mu.w^{(l)}$ for some $w^{(l)} \in S^\mu$, and since $\nu^{(l)} \le \lambda.\tau^d$ this completes the proof.

□

We shall use the convention

$$\binom{a}{b} = \binom{a}{a-b} = 0$$

for integers a, b satisfying $a < b$ or $b < 0$.

Lemma 3.2.8. *Let $\lambda, \mu \in \mathcal{A}_k^+(n)$, $d \in \mathbb{Z}_{\ge 0}$ and suppose that $\lambda/d/\mu$ is a cylindric skew diagram. The cardinality of the set* (3.22) *has the following explicit expression in terms of cylindric partitions,*

$$\theta_{\lambda/d/\mu} = \prod_{i=1}^n \binom{(\hat\lambda.\tau^d)'_i - \hat\mu'_{i+1}}{\hat\mu'_i - \hat\mu'_{i+1}} - \prod_{i=1}^n \binom{(\hat\lambda.\tau^{d-1})'_i - \hat\mu'_{i+1}}{\hat\mu'_i - \hat\mu'_{i+1}} . \tag{3.25}$$

Proof. For $d = 0$ this expression reduces to $\theta_{\lambda/\mu}$, as the equality $(\hat\lambda.\tau^{-1})'_n - \hat\mu'_n = -1$ implies that the second term on the RHS is 0. So suppose that $d > 0$, and consider the two weights in $\mathcal{P}_{k+d}^+$ given by $\Lambda^{(d)} = (n,\dots,n,\lambda_1,\dots,\lambda_k)$ and $\mu^{(d)} = (\mu_1,\dots,\mu_k,0,\dots,0)$. To prove the claim, we construct a bijection between the set (3.24), that is

$$\mathbb{A} = \left\{(w,\alpha) \in S^\mu \times \mathcal{P}_k^{\ge 0} \mid |\alpha| = d, \mu.wy^{-\alpha} \le \lambda\right\} ,$$

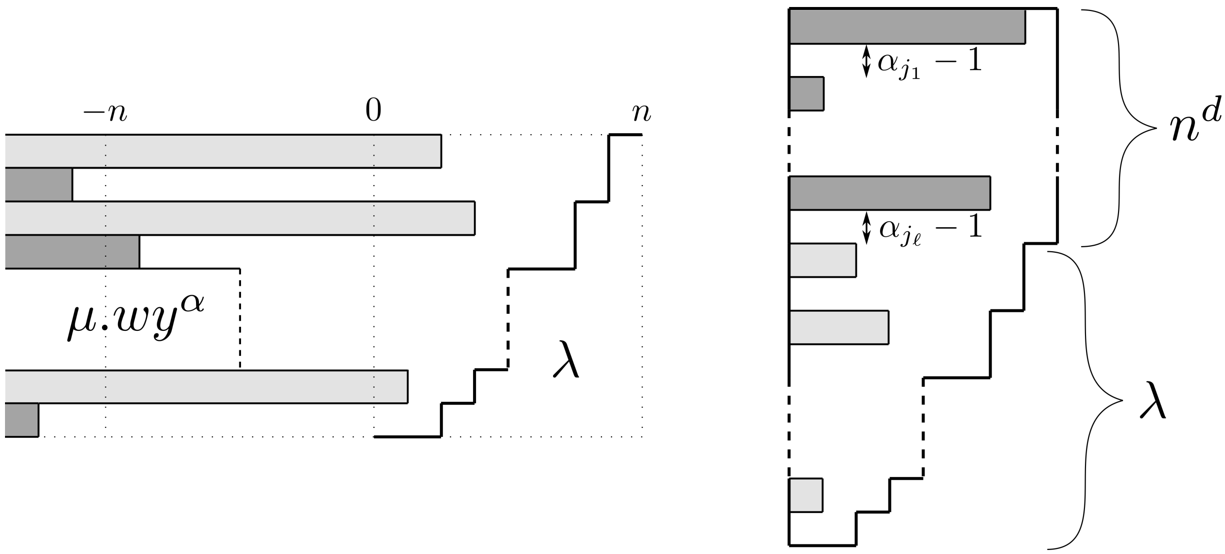


Figure 3.7: A graphical depiction for the proof of Lemma 3.2.8. On the left we have the weight $\mu.wy^{-\alpha}$ for $(w,\alpha)\in\mathbb{A}$. The parts $(\mu.wy^{-\alpha})_{j_i}$ for $i=1,\dots,l$ are represented in dark grey. On the right we have the weight γ.

and the following set involving weights in $\mathcal{P}_{k+d}$ and permutations in S_{k+d},

$$\mathbb{B}=\{\bar w\in S^{\mu^{(d)}}\mid \mu^{(d)}.\bar w\le\Lambda^{(d)},(\mu^{(d)}.\bar w)_1\neq 0\}\ .$$

We then show that the cardinality of the latter is given by (3.25). Let $(w,\alpha)\in\mathbb{A}$, denote with $J=\{j_1,\dots,j_l\}\subset[k]$ the set for which $\alpha_{j_i}\neq 0$, and with $\bar J=[k]\setminus J$ its complement. Define a weight $\gamma\in\mathcal{P}_{k+d}$ (compare with Figure 3.7) whose parts γ_j for $1\le j\le d$ are fixed by the vector

$$\big((\mu.w)_{j_1},\underbrace{0,\dots,0}_{\alpha_{j_1}-1},(\mu.w)_{j_2},\underbrace{0,\dots,0}_{\alpha_{j_2}-1},\dots,(\mu.w)_{j_l},\underbrace{0,\dots,0}_{\alpha_{j_l}-1}\big)\ ,$$

whereas for $1\le j\le k$ they are given by

$$\gamma_{d+j}=\begin{cases}(\mu.w)_j\ , & j\in\bar J\\ 0\ , & \text{otherwise}\end{cases}\ .$$

By definition we have that $\gamma\le\Lambda^{(d)}$, since all the parts of μ are smaller or equal than n and furthermore $(\mu.w)_j=(\mu.wy^{-\alpha})_j\le\lambda_j$ for $j\in\bar J$. Moreover, since by construction $m_i(\gamma)=m_i(\mu^{(d)})$ for $i=0,\dots,n$ it follows that there exists a unique permutation $\bar w\in S^{\mu^{(d)}}$ such that $\mu^{(d)}.\bar w=\gamma$ and $(\mu^{(d)}.\bar w)_1=\gamma_1=(\mu.w)_{j_1}\neq 0$. In conclusion, each $(w,\alpha)\in\mathbb{A}$ determines a unique element $\bar w\in\mathbb{B}$, that is $(w,\alpha)\mapsto\bar w$ defines a map $\mathbb{A}\to\mathbb{B}$.

To show that $\mathbb{A}$ and $\mathbb{B}$ are in bijection we need to create the inverse map $\mathbb{B}\to\mathbb{A}$.

The simplest approach is to merely reverse the preceding construction. For this purpose, suppose that $\bar{w} \in \mathbb{B}$. Define $\bar{J} \subset [k]$ as the set of indices $j \in \bar{J}$ satisfying $(\mu^{(d)}.\bar{w})_{d+j} \neq 0$, and denote with $J = [k] \setminus \bar{J} = \{j_1, \ldots, j_l\}$ its complement. Let $(\mu^{(d)}.\bar{w})_{p_1}, \ldots, (\mu^{(d)}.\bar{w})_{p_l}$ be the non-zero parts of $\mu^{(d)}.\bar{w}$ for indices $p_1, \ldots, p_l \leq d$, and consider the weight $\beta \in \mathcal{P}_k$ with parts

$$\beta_j = \begin{cases} (\mu^{(d)}.\bar{w})_{p_j} \ , & j \in J \\ (\mu^{(d)}.\bar{w})_{d+j} \ , & j \in \bar{J} \end{cases} .$$

Define the map $\bar{w} \mapsto (w, \alpha)$ as follows: $w \in S^{\mu}$ is the unique permutation such that $\mu.w = \beta$, and $\alpha \in \mathcal{P}_k^{\geq 0}$ is the weight with parts $\alpha_{j_i} = p_{i+1} - p_i$ for $i = 1, \ldots, l-1$, $\alpha_{j_l} = d - p_l$ and $\alpha_j = 0$ for $j \in \bar{J}$. The map just described is by construction the inverse map $\mathbb{B} \to \mathbb{A}$. That is, the composition of the two maps $\mathbb{A} \to \mathbb{B}$ and $\mathbb{B} \to \mathbb{A}$ gives the identity map on $\mathbb{A}$ and $\mathbb{B}$ respectively.

The cardinality of $\mathbb{B}$ equals the cardinality of the set $\{\bar{w} \in S^{\mu^{(d)}} \mid \mu^{(d)}.\bar{w} \leq \Lambda^{(d)}\}$, which is given by $\theta_{\Lambda^{(d)}/\mu^{(d)}}$ thanks to Lemma 3.2.2, minus the cardinality of the set $\{\bar{w} \in S^{\mu^{(d)}} \mid \mu^{(d)}.\bar{w} \leq \Lambda^{(d)}, (\mu^{(d)}.\bar{w})_1 = 0\}$. The latter is in bijection with the set $\{\bar{w}' \in S^{\mu^{(d-1)}} \mid \mu^{(d-1)}.\bar{w}' \leq \Lambda^{(d-1)}\}$, which has cardinality $\theta_{\Lambda^{(d-1)}/\mu^{(d-1)}}$. In particular, notice that $\Lambda^{(d-1)}, \mu^{(d-1)} \in \mathcal{P}^+_{d+k-1}$. A bijection $\bar{w} \mapsto \bar{w}'$ between these two sets can be constructed via the relation $\big((\mu^{(d)}.\bar{w})_2, \ldots, (\mu^{(d)}.\bar{w})_d\big) = \mu^{(d-1)}.\bar{w}'$. Thus,

$$\theta_{\lambda/d/\mu} = \theta_{\Lambda^{(d)}/\mu^{(d)}} - \theta_{\Lambda^{(d-1)}/\mu^{(d-1)}} \ ,$$

and equation (3.25) follows by taking advantage of Lemma 2.3.15 and the equality $(\Lambda^{(d)})'_i = \lambda'_i + d = (\hat{\lambda}.\tau^d)'_i$.

□

3.2.2 Generalisation of $\psi_{\lambda/\mu}$ to the cylinder

We proceed in close analogy to the previous section. For $\hat{\alpha}, \hat{\beta} \in \mathcal{P}_{k,n}$ with $\hat{\alpha} \subset \hat{\beta}$ we say that $\hat{\beta}/\hat{\alpha}$ is a cylindric vertical strip if $\hat{\beta}_i - \hat{\alpha}_i = 0, 1$ for all $i \in \mathbb{Z}$.

Definition 3.2.9. For $\lambda, \mu \in \hat{\mathcal{A}}_k^+(n)$ and $d \in \mathbb{Z}_{\geq 0}$ define $\psi_{\lambda/d/\mu}$ as the cardinality of the set

$$\{\tilde{w} \in \tilde{S}^{\hat{\mu}} \mid \hat{\lambda}.\tau^d/\hat{\mu}.\tilde{w} \text{ is a cylindric vertical strip}\} \ . \tag{3.26}$$

In Figure 3.8 there is an example which emphasises the combinatorial nature of the set (3.22). Compare with Figure 2.5 in Chapter 2. Notice that the set (3.26) can be equivalently expressed as

$$\{\tilde{w} \in \tilde{S}^{\mu} \mid (\lambda.\tau^d)_i - (\mu.\tilde{w})_i = 0, 1\} \ . \tag{3.27}$$

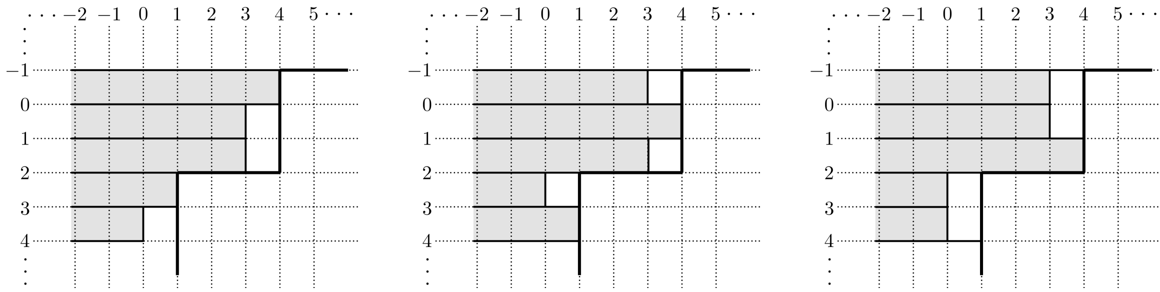


Figure 3.8: Let $k = n = 3$, $d = 2$, $\hat{\lambda} = (\ldots, 1, 1, 1, \ldots)$ and $\hat{\mu} = (\ldots, 3, 3, 1, \ldots)$. From left to right we have, in grey, the cylindric diagrams of $\hat{\mu}$, $\hat{\mu}.\sigma_0$ and $\hat{\mu}.\sigma_0\sigma_1$. These are the only diagrams of the form $\hat{\mu}.\tilde{w}$, for $w \in \tilde{S}^{\hat{\mu}}$, for which $\hat{\lambda}.\tau^2/\hat{\mu}.\tilde{w}$ is a cylindric vertical strip. It follows that $\psi_{\lambda/d/\mu} = 3$.

Following similar steps as described in Remark 3.2.5, one can show that (3.27) reduces to the first set in (3.19) for $d = 0$. In particular this implies that $\psi_{\lambda/0/\mu} = \psi_{\lambda/\mu}$.

Lemma 3.2.10. *The set* (3.27) *has the following alternative form,*

$$\left\{(w, \alpha) \in S^\mu \times \mathcal{P}_k^{\geq 0} \mid |\alpha| = d, \lambda_i - (\mu.wy^{-\alpha})_i = 0, 1\right\} . \tag{3.28}$$

Proof. For every $\tilde{w} \in \tilde{S}^\mu$, all the elements in the right coset $\tilde{S}_\mu \tilde{w}$ have the same action on the weight μ. Thanks to Lemma 3.1.18 we have that (3.27) can be expressed as the set

$$\left\{(w, \alpha) \in S^\mu \times \mathcal{P}_k \mid |\alpha| = 0, \lambda_i - (\mu.wy^\alpha \tau^{-d})_i = 0, 1\right\} .$$

The claim then follows by employing similar steps to the ones described in the proof of Lemma 3.2.6. □

Proposition 3.2.11. *The set* (3.26) *is non-empty if and only if $\lambda/d/\mu$ is a cylindric vertical strip.*

Proof. To make things easier we will prove the claim for the set (3.27) instead. In other words, we will show that the latter is non-empty if and only if $(\lambda.\tau^d)_i - \mu_i = 0, 1$ for all $i \in \mathbb{Z}$. We start by considering the case $d > m_n(\mu)$. Then there exists an index $j > m_n(\mu)$ such that $(\lambda.\tau^d)_j - \mu_j > 1$, because $\lambda.\tau^d$ has d parts strictly greater than n whereas μ has only $m_n(\mu) < d$ parts equal to n. This implies that $\lambda/d/\mu$ is not a cylindric vertical strip. Similarly the set (3.28), and thus the set (3.27), must be empty. In fact, suppose that the pair $(w, \alpha) \in S^\mu \times \mathcal{P}_k^{\geq 0}$ belongs to (3.28). Then $\mu.wy^{-\alpha}$ must have at least one part strictly smaller than 0, since the parts of α are non-negative, and moreover $|\alpha| = d > m_n(\mu)$. This implies that there exists an index j such that $\lambda_j - (\mu.wy^{-\alpha})_j > 1$, and since this is a contradiction, the claim follows for $d > m_n(\mu)$. One can prove in a similar fashion that for $d > m_1(\lambda)$ the cylindric skew diagram $\lambda/d/\mu$ is not a cylindric vertical strip, and that the set (3.27) is empty.

Suppose now that $d \leq \min(m_1(\lambda), m_n(\mu))$. If $\lambda/d/\mu$ is a vertical strip the element $\tilde{w} = 1$ belongs to (3.27), which is therefore non-empty. Conversely, assume that (3.27) is non-empty, that is there exists $\tilde{w} \in \tilde{S}^\mu$ such that $(\lambda.\tau^d)_i - (\mu.\tilde{w})_i = 0, 1$. For $d = 0$ the claim follows from Lemma 2.3.19 since (3.27) reduces to the first set in (3.19). So let $d > 0$, and write $\tilde{w} = wy^\alpha$ for some $w \in S_k$ and $\alpha \in \mathcal{P}_k$ with $|\alpha| = 0$. If $\alpha = (0, 0, \dots, 0)$ then $\lambda/d/\mu$ is a cylindric vertical strip by a similar argument as then one used in the proof of Lemma 2.3.19, so assume that $\alpha \neq (0, 0, \dots, 0)$ and set $l = (|\alpha_1| + \cdots + |\alpha_k|)/2$. We adopt the same strategy used in the proof of Proposition 3.2.7, namely we construct recursively a sequence

$$\nu^{(1)}, \nu^{(2)}, \dots, \nu^{(l)}$$

of weights in $\mathcal{P}_k$ such that $(\lambda.\tau^d)_i - \nu_i^{(r)} = 0, 1$ for $r = 1, \dots, l$.

For $1 \leq p, q \leq n$ let $(\mu.\tilde{w})_p$ and $(\mu.\tilde{w})_q$ be respectively the greatest and smallest part of $\mu.\tilde{w}$. Notice that $\alpha_p > 0$ and $\alpha_q < 0$, otherwise $\alpha = (0, \dots, 0)$ which contradicts the hypothesis. Since $d \leq m_1(\lambda)$ we have that $(\lambda.\tau^d)_1 = n + 1$ and $(\lambda.\tau^d)_k = 1$. In particular $(\lambda.\tau^d)_p \leq n+1$ which implies, together with $(\mu.\tilde{w})_p > n$, the constraint $(\mu.\tilde{w})_p = (\lambda.\tau^d)_p = n + 1$. Similarly, we have that $(\mu.\tilde{w})_q = 0$ and $(\lambda.\tau^d)_q = (\lambda.\tau^d)_k = 1$. Thus, calling $\bar{\mu}$ the weight obtained from $\mu.\tilde{w}$ by swapping $(\mu.\tilde{w})_1$ with $(\mu.\tilde{w})_p$ and $(\mu.\tilde{w})_k$ with $(\mu.\tilde{w})_q$, it follows that $(\lambda.\tau^d)_i - \bar{\mu}_i = 0, 1$. Finally, the weight $\nu^{(1)} = \bar{\mu}.\sigma_0$ satisfies the constraints $(\lambda.\tau^d)_i - \nu_i^{(1)} = 0, 1$ as well, thanks to the relations $(\lambda.\tau^d)_1 - (\bar{\mu}_k + n) = (\lambda.\tau^d)_k - \bar{\mu}_k = 0, 1$ and $(\lambda.\tau^d)_k - (\bar{\mu}_1 - n) = (\lambda.\tau^d)_1 - \bar{\mu}_1 = 0, 1$. In particular, writing $\nu^{(1)} = \mu.w^{(1)}y^{\alpha^{(1)}}$ for some $w^{(1)} \in S^\mu$ and $\alpha^{(1)} \in \mathcal{P}_k$, it follows that $|\alpha_1^{(1)}| + \cdots + |\alpha_k^{(1)}| = |\alpha_1| + \cdots + |\alpha_k| - 2$.

One can then construct, starting from $\nu^{(1)}$ and repeating the procedure just described, a second weight $\nu^{(2)}$ such that $(\lambda.\tau^d)_i - \nu_i^{(2)} = 0, 1$. Writing $\nu^{(2)} = \mu.w^{(2)}y^{\alpha^{(2)}}$ for some $w^{(2)} \in S^\mu$ and $\alpha^{(2)} \in \mathcal{P}_k$ it follows by construction that $|\alpha_1^{(2)}| + \cdots + |\alpha_k^{(2)}| = |\alpha_1| + \cdots + |\alpha_k| - 4$. Proceeding along the same line we end up with weights $\{\nu^{(r)}\}_{r=1}^l$ such that $(\lambda.\tau^d)_i - \nu_i^{(r)} = 0, 1$, and moreover $\nu^{(l)} = \mu.w^{(l)}y^{(0,0,\dots,0)}$ for some $w^{(l)} \in S^\mu$. Since $(\lambda.\tau^d)_i - \nu_i^{(l)} = 0, 1$ this completes the proof.

□

Lemma 3.2.12. *Let $\lambda, \mu \in \hat{\mathcal{A}}_k^+(n)$, $d \in \mathbb{Z}_{\geq 0}$ and suppose that $\lambda/d/\mu$ is a cylindric vertical strip. The cardinality of the set* (3.26) *has the following explicit expression in terms of cylindric diagrams,*

$$\psi_{\lambda/d/\mu} = \prod_{i=1}^{n} \binom{(\hat{\lambda}.\tau^d)'_i - (\hat{\lambda}.\tau^d)'_{i+1}}{(\hat{\lambda}.\tau^d)'_i - \hat{\mu}'_i} . \tag{3.29}$$

Proof. Since $\lambda/d/\mu$ is a cylindric vertical strip we must have that $d \leq \min(m_1(\lambda), m_n(\mu))$, as explained in the proof of Proposition 3.2.11. We now construct a bijection between the set (3.27), that is

$$\mathbb{A} = \left\{ (w, \alpha) \in S^\mu \times \mathcal{P}_k^{\geq 0} \mid |\alpha| = d, \lambda_i - (\mu.wy^{-\alpha})_i = 0, 1 \right\} ,$$

and the set

$$\mathbb{B} = \{\bar{w} \in S^{\mu.\tau^{-d}} \mid \lambda_i - (\mu.\tau^{-d}\bar{w})_i = 0, 1\} \ .$$

The cardinality of the latter is given by (3.29), as we will show below. Let $(w, \alpha) \in \mathbb{A}$, then $\alpha_j \neq 0$ only if $(\mu.w)_j = n$, in which case $\alpha_j = 1$. Since $d \leq m_n(\mu)$ we have that $\mu.\tau^{-d} = (\mu_{d+1}, \ldots \mu_k, 0, \ldots, 0)$. This implies that $m_i(\mu.\tau^{-d}) = m_i(\mu.wy^{-\alpha})$ for $i = 0, \ldots, n$, and thus there exists a unique permutation $\bar{w} \in S^{\mu.\tau^{-d}}$ such that $(\mu.\tau^{-d}).\bar{w} = \mu.\tau^{-d}\bar{w} = \mu.wy^{-\alpha}$. Since by construction $\lambda_i - (\mu.\tau^{-d}\bar{w})_i = 0, 1$ it follows that $\bar{w} \in \mathbb{B}$. In conclusion, each $(w, \alpha) \in \mathbb{A}$ determines a unique element $\bar{w} \in \mathbb{B}$, that is $(w, \alpha) \mapsto \bar{w}$ defines a map $\mathbb{A} \to \mathbb{B}$.

We now create the inverse map $\mathbb{B} \to \mathbb{A}$ by reversing the construction above. Let $\bar{w} \in \mathbb{B}$, and consider the weight $\alpha \in \mathcal{P}_k$ with parts $\alpha_j = 1$ if $(\mu.\tau^{-d}\bar{w})_j = 0$ and $\alpha_j = 0$ otherwise. Since the weight $\mu.\tau^{-d}\bar{w}$ has d parts equal to 0 it follows that $|\alpha| = d$. By construction we have that $m_i(\mu) = m_i(\mu.\tau^{-d}\bar{w}y^{\alpha})$ for $i = 1, \ldots, n$ and thus there exists a unique permutation $w \in S^{\mu}$ such that $\mu.w = \mu.\tau^{-d}\bar{w}y^{\alpha}$. This condition is equivalent to $\mu.wy^{-\alpha} = \mu.\tau^{-d}\bar{w}$, which implies that $\lambda_i - (\mu.wy^{-\alpha})_i = 0, 1$ and thus the pair (w, α) belongs to $\mathbb{A}$. The map $\bar{w} \mapsto (w, \alpha)$ just described is by construction the inverse map $\mathbb{B} \to \mathbb{A}$. That is, the composition of the two maps $\mathbb{A} \to \mathbb{B}$ and $\mathbb{B} \to \mathbb{A}$ gives the identity map on $\mathbb{A}$ and $\mathbb{B}$ respectively.

Now that we have established the bijection between the sets $\mathbb{A}$ and $\mathbb{B}$, notice that the latter has cardinality $\psi_{\lambda/\mu.\tau^{-d}}$ thanks to Lemma 2.3.20. Thus, taking advantage of the relations $(\mu.\tau^{-d})'_i = (\hat{\mu}.\tau^{-d})'_i = \hat{\mu}'_i - d$ for $i = 1, \ldots, n$ the claim follows from the equalities

$$\psi_{\lambda/d/\mu} = \psi_{\lambda/\mu.\tau^{-d}} = \prod_{i=1}^{n} \binom{\lambda'_i - \lambda'_{i+1}}{\lambda'_i - (\mu.\tau^{-d})'_i} = \prod_{i=1}^{n} \binom{(\hat{\lambda}.\tau^d)'_i - (\hat{\lambda}.\tau^d)'_{i+1}}{(\hat{\lambda}.\tau^d)'_i - \hat{\mu}'_i} \ .$$

□

3.2.3 The main definitions

We now have all the tools to extend the skew elementary and complete symmetric functions to the cylindric case. For $\lambda, \mu \in \mathcal{A}_k^+(n)$ and $d \in \mathbb{Z}_{\geq 0}$ let $\hat{\pi}$ be a CRPP of shape $\lambda/d/\mu$, that is a sequence $\{\hat{\lambda}^{(r)}.\tau^{d_r}\}_{r \in \mathbb{Z}_{\geq 0}}$ of cylindric partitions as described in (3.21). If $l \in \mathbb{N}$ is the largest entry of $\hat{\pi}$ we set $\hat{\lambda}^{(r)}.\tau^{d_r} = \hat{\lambda}.\tau^d$ for $r \geq l$. Notice that applying $\tau^{-d_{r-1}}$ to both cylindric partitions appearing in $\hat{\lambda}^{(r)}.\tau^{d_r}/\hat{\lambda}^{(r-1)}.\tau^{d_{r-1}}$ one obtains the cylindric skew diagram $\lambda^{(r)}/(d_r - d_{r-1})/\lambda^{(r-1)}$. Equivalently, the latter is recovered after translating the vertical axis by the vector $(0, d_{r-1})$. Set

$$\theta_{\hat{\pi}} = \prod_{r \geq 1} \theta_{\lambda^{(r)}/(d_r - d_{r-1})/\lambda^{(r-1)}} \ ,$$

and denote by $x^{\hat{\pi}}$ the monomial $x_1^{\mathrm{wt}_1(\hat{\pi})} x_2^{\mathrm{wt}_2(\hat{\pi})} \cdots$ in the indeterminates $\{x_1, x_2, \dots\}$. Similarly, consider a CRST $\hat{T}$ of shape $\lambda/d/\mu$, that is a sequence $\{\hat{\lambda}^{(r)}.\tau^{d_r}\}_{r \in \mathbb{Z}_{\geq 0}}$ of cylindric partitions where $\hat{\lambda}^{(r)}.\tau^{d_r}/\hat{\lambda}^{(r-1)}.\tau^{d_{r-1}}$ is a cylindric vertical strip for $r \geq 1$. Set $x^{\hat{T}} = x_1^{\mathrm{wt}_1(\hat{T})} x_2^{\mathrm{wt}_2(\hat{T})} \cdots$, and

$$\psi_{\hat{T}} = \prod_{r \geq 1} \psi_{\lambda^{(r)}/(d_r - d_{r-1})/\lambda^{(r-1)}} \ .$$

Definition 3.2.13. Let $\lambda, \mu \in \mathcal{A}_k^+(n)$, $d \in \mathbb{Z}_{\geq 0}$ and suppose that $\hat{\mu} \subset \hat{\lambda}.\tau^d$. Introduce the cylindric elementary symmetric function $e_{\lambda/d/\mu}$ and the cylindric complete symmetric function $h_{\lambda/d/\mu}$ as the weighted sums

$$e_{\lambda/d/\mu} = \sum_{\hat{T}} \psi_{\hat{T}} x^{\hat{T}} \ , \tag{3.30}$$

$$h_{\lambda/d/\mu} = \sum_{\hat{\pi}} \theta_{\hat{\pi}} x^{\hat{\pi}} \ , \tag{3.31}$$

over all CRSTs and CRPPs of shape $\lambda/d/\mu$ respectively. If $\lambda/d/\mu$ is not a cylindric skew diagram set $e_{\lambda/d/\mu} = h_{\lambda/d/\mu} = 0$.

Suppose that $\lambda, \mu \in \mathcal{A}_k^+(n)$ with $\mu \subset \lambda$. Since the parts of λ and μ are by definition smaller or equal than n, it follows from Remark 3.1.12 that there exists a bijection between CRPPs of shape $\lambda/0/\mu$ and RPPs of shape λ/μ, whose action consists in restricting a CRPP to the lines 1 to k. Calling $\hat{\pi} \mapsto \pi$ such bijection we have that $\mathrm{wt}(\hat{\pi}) = \mathrm{wt}(\pi)$, and thus $x^{\hat{\pi}} = x^{\pi}$. Moreover, we have that $\theta_{\hat{\pi}} = \theta_{\pi}$ thanks to the discussion in Remark 3.2.5. This implies that $h_{\lambda/0/\mu} = h_{\lambda/\mu}$, that is for $d = 0$ we recover the (non-cylindric) skew complete symmetric functions. In particular, if $\mu \not\subset \lambda$ then $h_{\lambda/0/\mu} = h_{\lambda/\mu} = 0$. Similarly one has that $e_{\lambda/0/\mu} = e_{\lambda/\mu}$.

3.3 Symmetric functions at roots of unity

The main goal of this section is to prove that the cylindric symmetric functions $e_{\lambda/d/\mu}$ and $h_{\lambda/d/\mu}$ introduced above are actually symmetric, that is they belong to the ring of symmetric functions Λ. For this purpose it is enough to show that they can be expanded in terms of the basis $\{m_\nu\}_{\nu \in \mathcal{P}^+}$ of Λ. Of greater interest is their expansions in terms of $\{e_\nu\}_{\nu \in \mathcal{P}^+}$ and $\{h_\nu\}_{\nu \in \mathcal{P}^+}$ respectively, since the expansion coefficients are related to the fusion coefficients of a 2D TQFT, as we will see in Chapter 5 (compare also with Corollaries 3.3.13, 3.4.3 and Remark 3.4.4 below). The proof of these expansions requires the generalisation of the product formulae

$$
\begin{aligned}
m_\mu m_\nu &= \sum_{\lambda\in\mathcal{P}^+} f^\lambda_{\mu\nu} m_\lambda \ , && (3.32)\\
m_\mu e_\nu &= \sum_{\lambda\in\mathcal{P}^+} \psi_{\lambda/\mu}(\nu) m_\lambda \ , && (3.33)\\
m_\mu h_\nu &= \sum_{\lambda\in\mathcal{P}^+} \theta_{\lambda/\mu}(\nu) m_\lambda \ , && (3.34)
\end{aligned}
$$

which were introduced in (2.13), (2.65) and (2.71) respectively, to an appropriate ring. The latter is given by the following quotient

$$\mathcal{V}_k(n) = \Lambda_k[z, z^{-1}]/\mathcal{I}_{k,n} \ , \tag{3.35}$$

where we set $\Lambda_k[z, z^{-1}] = \Lambda_k \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}]$, and moreover we define the two-sided ideal

$$\mathcal{I}_{k,n} = \langle p_n(x_1, \dots, x_k) - zk,\ p_{n+r}(x_1, \dots, x_k) - zp_r(x_1, \dots, x_k) \text{ for } r = 1, \dots, k-1\rangle \ .$$

It is important to keep in mind that the product formulae in $\mathcal{V}_k(n)$ involve symmetric functions in k variables, whereas the expansions of $e_{\lambda/d/\mu}$ and $h_{\lambda/d/\mu}$ mentioned above hold in Λ. The next result shows that for $z = 0$ one can think of $\mathcal{V}_k(n)$ as the ring of symmetric functions in k variables evaluated at the n-th roots of unity.

Lemma 3.3.1. *The two set of equations*

$$p_n(x_1, \dots, x_k) - zk = 0 \ , \quad p_{n+r}(x_1, \dots, x_k) - zp_r(x_1, \dots, x_k) = 0 \ , \quad 1 \le r \le k-1 \ , \tag{3.36}$$

and

$$x_r^n = z \ , \qquad 1 \le r \le k \ , \tag{3.37}$$

are equivalent.

Proof. The relations (3.36) are clearly satisfied if (3.37) hold (compare with the definition (2.27) of power sums). To show the converse, we shall take advantage of Newton's formula (2.34), which can be rearranged as $p_{r-1}e_1 = (-1)^r r e_r + (-1)^r \sum_{i=1}^{r-2} (-1)^i p_i e_{r-i} + p_r$. So assume that the relations (3.36) hold. A straightforward computation shows that

$$
\begin{aligned}
&\big(p_{n+k-1}(x_1, \dots, x_k) - zp_{k-1}(x_1, \dots, x_k)\big) e_1(x_1, \dots, x_k) \\
&\qquad\qquad\qquad = p_{n+k}(x_1, \dots, x_k) - zp_k(x_1, \dots, x_k) \ .
\end{aligned}
$$

Since the relations (3.36) are satisfied, it follows that the LHS of the equality just obtained is equal to 0, and so is the RHS. Similarly, one can show by induction that the relation

$p_{n+r}(x_1,\ldots,x_k) - zp_r(x_1,\ldots,x_k) = 0$ is valid for all $r \in \mathbb{N}$. Assume that u is an invertible indeterminate, then the generating function (2.30) for power sums obeys the constraint

$$\begin{aligned} P(u) &= \sum_{i=1}^{n} p_i(x_1,\ldots,x_k)u^{i-1} + \sum_{i>n} p_i(x_1,\ldots,x_k)u^{i-1} \\ &= \sum_{i=1}^{n} p_i(x_1,\ldots,x_k)u^{i-1} + zu^n \sum_{i\geq 1} p_i(x_1,\ldots,x_k)u^{i-1} \\ &= \sum_{i=1}^{n} p_i(x_1,\ldots,x_k)u^{i-1} + zu^n P(u) , \end{aligned}$$

where in the first line we used (3.36). This last identity can be rearranged as

$$\sum_{i=1}^{n} p_i(x_1,\ldots,x_k)u^{i-1} = (1-zu^n)P(u) = (1-zu^n)\sum_{i=1}^{k} \frac{x_i}{1-ux_i} .$$

It follows that the formal series expansion of $(1-zu^n)P(u)$ terminates after finitely many terms. Thus for $i = 1,\ldots,k$ the residue of $(1-zu^n)P(u)$ at $u^{-1} = x_i$ must vanish, and these conditions provide a set of equations which corresponds to (3.37).

□

Lemma 3.3.2. *The set* $\{m_\lambda(x_1,\ldots,x_k)\}_{\lambda\in\mathcal{A}_k^+(n)}$ *is a basis of* $\mathcal{V}_k(n)$.

Proof. First of all, notice that the elements $\{m_\lambda(x_1,\ldots,x_k)\}_{\lambda\in\mathcal{A}_k^+(n)}$ are linearly independent in $\mathcal{V}_k(n)$. In fact, suppose that $\sum_{\lambda\in\mathcal{A}_k^+(n)} a_\lambda m_\lambda(x_1,\ldots,x_k) = 0$ for some $a_\lambda \in \mathbb{C}$. For each $\lambda \in \mathcal{A}_k^+(n)$ we have that the monomial $x_1^{\lambda_1}\cdots x_k^{\lambda_k}$ appears only once in this linear combination thanks to Lemma 3.3.1 and the expansion (2.15), that is

$$m_\lambda(x_1,\ldots,x_k) = \sum_{w\in S^\lambda} x_1^{\lambda_{w(1)}}\cdots x_k^{\lambda_{w(k)}} = \frac{1}{|S_\lambda|}\sum_{w\in S_k} x_1^{\lambda_{w(1)}}\cdots x_k^{\lambda_{w(k)}} .$$

It follows that $a_\lambda = 0$ for all $\lambda \in \mathcal{A}_k^+(n)$. To prove the claim, it is then enough to show that in $\mathcal{V}_k(n)$ each element of the set $\{m_\lambda(x_1,\ldots,x_k)\}_{\lambda\in\mathcal{P}_k^+}$, which is a basis of Λ_k as we mentioned in Remark 2.2.6, can be expanded in terms of $\{m_\lambda(x_1,\ldots,x_k)\}_{\lambda\in\mathcal{A}_k^+(n)}$. Let $\lambda \in \mathcal{P}_k^+$, and denote with $\check{\lambda} \in \mathcal{A}_k^+(n)$ the unique intersection point of the orbit $\lambda.\hat{S}_k$ with $\mathcal{A}_k^+(n)$. Taking advantage once again of equation (2.15) and Lemma 3.3.1, one ends up with the following equality in $\mathcal{V}_k(n)$,

$$m_\lambda(x_1,\ldots,x_k) = \frac{|S_{\check{\lambda}}|}{|S_\lambda|} z^{\frac{|\lambda|-|\check{\lambda}|}{n}} m_{\check{\lambda}}(x_1,\ldots,x_k) . \tag{3.38}$$

By definition we have that $\check{\lambda} = \lambda.wy^\alpha$ for some $w \in S^\lambda$ and $\alpha \in \mathcal{P}_k$. It follows that $|\check{\lambda}| - |\lambda| = n|\alpha|$, and thus $\frac{|\lambda|-|\check{\lambda}|}{n} \in \mathbb{Z}$. This finally proves the claim. □

3.3.1 Product expansions at roots of unity

We now wish to generalise the product formulae (3.32), (3.33) and (3.34) to the quotient $\mathcal{V}_k(n)$. Let us start from the product formula (3.32). Namely, we want to expand the product $m_\mu(x_1,\ldots,x_k)m_\nu(x_1,\ldots,x_k)$ in terms of the basis of $\mathcal{V}_k(n)$ introduced in Lemma 3.3.2, and find a combinatorial interpretation for the expansion coefficients. The latter is given by the following definition, as we will show in Lemma 3.3.5 below.

Definition 3.3.3. For $\lambda \in \mathcal{A}_k^+(n)$, $\mu,\nu \in \mathcal{P}_k^+$ and $d \in \mathbb{Z}$ define $N_{\mu\nu}^{\lambda,d}$ as the cardinality of the set

$$\left\{(w,w') \in S^\mu \times S^\nu \mid \mu.w + \nu.w' = \lambda.y^\alpha \text{ for some } \alpha \in \mathcal{P}_k \text{ with } |\alpha| = d\right\}. \tag{3.39}$$

If we restrict the weights λ,μ,ν to the alcove $\mathcal{A}_k^+(n)$, then the coefficients $N_{\mu\nu}^{\lambda,d}$ become the fusion coefficients of a 2D TQFT, as we will see in Chapter 5. Notice furthermore that these coefficients are by definition non-negative integers. In Chapter 5 we provide a representation theoretical interpretation of this statement. Namely, we will present a formula for the coefficients $N_{\mu\nu}^{\lambda,d}$ in terms of tensor multiplicities for irreducible representations of the generalised symmetric group.

Lemma 3.3.4. *Let $\lambda \in \mathcal{A}_k^+(n)$, $\mu,\nu \in \mathcal{P}_k^+$ and $d \in \mathbb{Z}$. Then $N_{\mu\nu}^{\lambda,d}$ is non-zero only if the following conditions are satisfied.*

1. $|\mu| + |\nu| - |\lambda| = dn$.
2. $d \geq -k$.
3. $d \geq 0$, *provided that at least one of μ,ν belongs to $\mathcal{A}_k^+(n)$.*
4. $\hat{\mu} \subset \hat{\lambda}.\tau^d$, *that is $\lambda/d/\mu$ is a cylindric skew diagram, provided that $\mu \in \mathcal{A}_k^+(n)$.*

Proof. Suppose that $N_{\mu\nu}^{\lambda,d}$ is non-zero, that is the set (3.39) is non-empty. The relation $\mu.w + \nu.w' = \lambda.y^\alpha$ implies the constraint $|\mu| + |\nu| = |\lambda| + dn$, which is equivalent to Condition 1. Assume that $(w,w') \in S^\mu \times S^\nu$ belongs to (3.39), that is there exists $\alpha \in \mathcal{P}_k$ with $|\alpha| = d$ such that $\mu.w + \nu.w' = \lambda.y^\alpha$. Since $1 \leq \lambda_i \leq n$ we have that $\alpha_i \geq -1$ for $i = 1,\ldots,k$, and thus $d = |\alpha| \geq -k$. In particular, if at least one of μ,ν belongs to $\mathcal{A}_k^+(n)$ it follows that $\alpha \in \mathcal{P}_k^{\geq 0}$, since all the parts of $\mu.w + \nu.w'$ are positive, and thus we must have that $d \geq 0$. This proves Conditions 2 and 3.

We now show the validity of Condition 4, and for this purpose assume that $\mu \in \mathcal{A}_k^+(n)$. If $d \geq k$ all the parts of $\lambda.\tau^d$ are greater than n, and thus the relation $\hat{\mu} \subset \hat{\lambda}.\tau^d$ follows immediately. So let $d < k$ and suppose that $N_{\mu\nu}^{\lambda,d}$ is non-zero, that is there exists a pair $(w,w') \in S^\mu \times S^\nu$ and $\alpha \in \mathcal{P}_k$ with $|\alpha| = d$ such that $\mu.w + \nu.w' = \lambda.y^\alpha$. This last constraint implies that $\mu.w \leq \lambda.y^\alpha$, which can be rearranged as $(\mu.y^{-\gamma}).w \leq \lambda$ after setting $\gamma = \alpha.w^{-1}$. Following similar steps as described in the proof of Lemma 2.3.14, one has that $\mu.y^{-\gamma} \leq \lambda$, and thus $\mu \leq \lambda.y^{\gamma}$. Notice that the parts of γ must be non-negative since the parts of μ are all positive. Let $i \in \mathbb{N}$ and suppose that $d < i \leq k$. Since $|\gamma| = d$ there are at most d non-zero parts in γ, and then there exists $j \in \mathbb{N}$ with $i - d \leq j \leq i$ such that $\gamma_j = 0$. We then have the chain of inequalities $\lambda_{i-d} \geq \lambda_j \geq \mu_j \geq \mu_i$ which implies that $(\lambda.\tau^d)_i \geq \mu_i$. If instead $1 \leq i \leq d$ the inequality $(\lambda.\tau^d)_i \geq \mu_i$ follows from the fact that $(\lambda.\tau^d)_i > n$. In conclusion we have that $(\lambda.\tau^d)_i \geq \mu_i$ for $i = 1, \ldots, k$, and thus $\hat{\mu} \subset \hat{\lambda}.\tau^d$. □

Lemma 3.3.5. *For $\mu, \nu \in \mathcal{P}_k^+$ we have the following product expansion in $\mathcal{V}_k(n)$,*

$$m_\mu(x_1, \ldots, x_k) m_\nu(x_1, \ldots, x_k) = \sum_{d \in \mathbb{Z}} z^d \sum_{\lambda \in \mathcal{A}_k^+(n)} N_{\mu\nu}^{\lambda,d} m_\lambda(x_1, \ldots, x_k) \ , \tag{3.40}$$

where the first sum is restricted to $d \geq 0$ if at least one of μ, ν belongs to $\mathcal{A}_k^+(n)$, and to $d \geq -k$ otherwise.

The sum over d on the RHS involves only finitely many non-zero terms, for if d is large enough the constraint $|\mu| + |\nu| - |\lambda| = dn$ can no longer be fulfilled, and thus $N_{\mu\nu}^{\lambda,d}$ must equal 0 thanks to Property 1 of Lemma 3.3.4. The same constraint implies that for each $\lambda \in \mathcal{A}_k^+(n)$ there is at most one non-zero term $z^d N_{\mu\nu}^{\lambda,d}$ which contributes to the coefficient of $m_\lambda(x_1, \ldots, x_k)$ in $m_\mu(x_1, \ldots, x_k) m_\nu(x_1, \ldots, x_k)$.

Proof. For $\lambda \in \mathcal{A}_k^+(n)$ and $d \in \mathbb{Z}$ we have that the coefficient of $z^d m_\lambda(x_1, \ldots, x_k)$ in $m_\mu(x_1, \ldots, x_k) m_\nu(x_1, \ldots, x_k)$ equals the coefficient of $z^d x_1^{\lambda_1} \cdots x_k^{\lambda_k}$ in the same product. Each monomial appearing in the product $m_\mu(x_1, \ldots, x_k) m_\nu(x_1, \ldots, x_k)$ is of the form $x_1^{\mu_{w(1)}+\nu_{w'(1)}} \cdots x_k^{\mu_{w(k)}+\nu_{w'(k)}}$, for some $w \in S^\mu$ and $w' \in S^\nu$. Taking advantage of the set of equations (3.37), which was introduced in Lemma 3.3.1, one sees that this monomial equals $z^d x_1^{\lambda_1} \cdots x_k^{\lambda_k}$ if and only if there exists $\alpha \in \mathcal{P}_k$ with $|\alpha| = d$ satisfying the relation $\mu.w + \nu.w' = \lambda.y^\alpha$, that is if and only if the pair (w, w') belongs to the set (3.39). This shows that for $\lambda \in \mathcal{A}_k^+(n)$ and $d \in \mathbb{Z}$ the coefficient of $z^d m_\lambda(x_1, \ldots, x_k)$ in $m_\mu(x_1, \ldots, x_k) m_\nu(x_1, \ldots, x_k)$ equals $N_{\mu\nu}^{\lambda,d}$, thus proving equation (3.40). The restriction on d follows from Lemma 3.3.4. □

Lemma 3.3.6. *Let $\lambda, \rho \in \mathcal{A}_k^+(n)$, $\eta, \mu, \nu \in \mathcal{P}_k^+$ and $d \in \mathbb{Z}$. The coefficients $N_{\mu\nu}^{\lambda,d}$ satisfy the following properties.*

1. *Commutativity:*

$$N_{\mu\nu}^{\lambda,d} = N_{\nu\mu}^{\lambda,d} \ . \tag{3.41}$$

2. Associativity: if $d \geq -k$, then

$$\sum_{\substack{d_1+d_2=d \\ d_1 \geq -k, d_2 \geq 0}} \sum_{\sigma \in \mathcal{A}_k^+(n)} N_{\eta\mu}^{\sigma,d_1} N_{\sigma\nu}^{\rho,d_2} = \sum_{\substack{d_1+d_2=d \\ d_1 \geq -k, d_2 \geq 0}} \sum_{\sigma \in \mathcal{A}_k^+(n)} N_{\nu\mu}^{\sigma,d_1} N_{\sigma\eta}^{\rho,d_2} . \tag{3.42}$$

3. If $d \geq -k$, employing the coefficient $f_{\eta\mu}^{\sigma}$ introduced in (2.13), then

$$\sum_{\sigma \in \mathcal{P}_k^+} f_{\eta\mu}^{\sigma} N_{\sigma\nu}^{\rho,d} = \sum_{\substack{d_1+d_2=d \\ d_1 \geq -k, d_2 \geq 0}} \sum_{\sigma \in \mathcal{A}_k^+(n)} N_{\eta\mu}^{\sigma,d_1} N_{\sigma\nu}^{\rho,d_2} . \tag{3.43}$$

4. If $d = 0$ and at least one of μ, ν belongs to $\mathcal{A}_k^+(n)$, then

$$N_{\mu\nu}^{\lambda,0} = f_{\mu\nu}^{\lambda} . \tag{3.44}$$

5. If $\mu \in \mathcal{A}_k^+(n)$, setting $n^k = (n, \ldots, n) \in \mathcal{A}_k^+(n)$, then

$$N_{\mu n^k}^{\lambda,d} = \delta_{d,k} \delta_{\lambda\mu} . \tag{3.45}$$

6. If $\check{\nu}$ is the unique intersection point of the orbit $\nu.\hat{S}_k$ with $\mathcal{A}_k^+(n)$, then

$$N_{\mu\nu}^{\lambda,d} = N_{\mu\check{\nu}}^{\lambda,d+\frac{|\check{\nu}|-|\nu|}{n}} \frac{|S_{\check{\nu}}|}{|S_{\nu}|} . \tag{3.46}$$

Proof. Property 1 follows immediately from the definition of $N_{\mu\nu}^{\lambda,d}$. Let $\pi_{k,n} : \Lambda_k[z, z^{-1}] \to \mathcal{V}_k(n)$ be the quotient map, and consider the equality $(m_\eta m_\mu) m_\nu = m_\eta (m_\mu m_\nu)$ in $\Lambda[z, z^{-1}]$, which simply reflects the associativity of the product. Apply the projection $\Lambda[z, z^{-1}] \to \Lambda_k[z, z^{-1}]$, which was introduced in (2.11), to both sides of this equality first, and then apply the quotient map $\pi_{k,n}$. Using the product expansion (3.40), and comparing the terms with the same power of z, we end up with Property 2. Property 3 follows after similar steps, where one has to start from the equality $(m_\eta m_\mu) m_\nu = \sum_{\sigma \in \mathcal{P}^+} f_{\eta\mu}^{\sigma} m_\sigma m_\nu$ in $\Lambda[z, z^{-1}]$ instead. We now prove Property 4, and for this purpose notice that the coefficient $N_{\mu\nu}^{\lambda,0}$ can be expressed as the cardinality of the set

$$\{(w, w') \in S^\mu \times S^\nu \mid \mu.w + \nu.w' = \lambda\} .$$

In fact, if $(w, w') \in S^\mu \times S^\nu$ belongs to (3.39) then we must have $\alpha = (0, \ldots, 0)$, since $|\alpha| = 0$ and moreover $\alpha \in \mathcal{P}_k^{\geq 0}$ as mentioned in the proof of Lemma 3.3.4. But this is just an equivalent rewriting of the set (2.14), and the claim follows. To prove Property 5 notice that $N^{\lambda,d}_{\mu n^k}$ equals the cardinality of the set

$$\left\{w \in S^\mu \mid \mu.w + n^k = \lambda.y^\alpha \text{ for some } \alpha \in \mathcal{P}_k \text{ with } |\alpha| = d\right\} ,$$

since the stabilizer subgroup of n^k coincides with S_k, and thus S^{n^k} is just the identity element in S_k. The constraint $\mu.w + n^k = \lambda.y^\alpha$ is equivalent to $\mu.w = \lambda.y^{\alpha-(1,1,\ldots,1)}$, which can be fulfilled if and only if $\alpha = (1, 1, \ldots, 1)$ and $\lambda = \mu$, since we assumed that $\mu \in \mathcal{A}^+_k(n)$. This shows the validity of (3.45). Finally, Property 6 follows by applying (3.38) to the product expansion (3.40) in $\mathcal{V}_k(n)$. □

We shall now proceed to generalise the remaining product expansions (3.33) and (3.34) to the quotient $\mathcal{V}_k(n)$.

Definition 3.3.7. Let $\lambda, \mu \in \mathcal{A}^+_k(n)$, $d \in \mathbb{Z}_{\geq 0}$, $\nu \in \mathcal{P}$ and suppose that $\hat{\mu} \subset \hat{\lambda}.\tau^d$. Define the coefficients

$$\psi_{\lambda/d/\mu}(\nu) = \sum_{\hat{T}} \psi_{\hat{T}} , \tag{3.47}$$

$$\theta_{\lambda/d/\mu}(\nu) = \sum_{\hat{\pi}} \theta_{\hat{\pi}} , \tag{3.48}$$

where the sums run over all CRSTs and CRPPs of shape $\lambda/d/\mu$ and weight ν respectively. If $\lambda/d/\mu$ is not a cylindric skew diagram set $\psi_{\lambda/d/\mu}(\nu) = \theta_{\lambda/d/\mu}(\nu) = 0$.

If $\hat{\pi}$ is a CRPP of shape $\lambda/d/\mu$ and weight ν then we have the constraint $|\lambda/d/\mu| = |\lambda| + dn - |\mu| = |\nu|$. It follows that $\theta_{\lambda/d/\mu}(\nu)$ is non-zero only if $|\mu| + |\nu| - |\lambda| = dn$. Let $\mu \subset \lambda$, then we have that $\theta_{\lambda/0/\mu}(\nu) = \theta_{\lambda/\mu}(\nu)$, where the latter was described in Lemma 2.3.16 as a weighted sum over RPPs. This is because the bijection $\hat{\pi} \mapsto \pi$ between CRPPs of shape $\lambda/0/\mu$ and RPPs of shape λ/μ is such that $\mathrm{wt}(\hat{\pi}) = \mathrm{wt}(\pi)$ and $\theta_{\hat{\pi}} = \theta_\pi$. In particular, if $\mu \not\subset \lambda$ then $\theta_{\lambda/0/\mu}(\nu) = \theta_{\lambda/\mu}(\nu) = 0$. Similarly, one has that $\psi_{\lambda/d/\mu}(\nu)$ is non-zero only if $|\mu| + |\nu| - |\lambda| = dn$, and moreover $\psi_{\lambda/0/\mu}(\nu) = \psi_{\lambda/\mu}(\nu)$.

Lemma 3.3.8. *Let $\mu \in \mathcal{A}^+_k(n)$ and $\nu \in \mathcal{P}$. The following product rule holds in $\mathcal{V}_k(n)$,*

$$m_\mu(x_1, \ldots, x_k)h_\nu(x_1, \ldots, x_k) = \sum_{d\in\mathbb{Z}_{\geq 0}} z^d \sum_{\lambda\in\mathcal{A}^+_k(n)} \theta_{\lambda/d/\mu}(\nu)m_\lambda(x_1, \ldots, x_k) , \tag{3.49}$$

where the second sum is restricted to those $\lambda \in \mathcal{A}^+_k(n)$ for which $\lambda/d/\mu$ is a cylindric skew diagram with $|\mu| + |\nu| - |\lambda| = dn$.

The fact that $\theta_{\lambda/d/\mu}(\nu)$ is non-zero only if $|\mu| + |\nu| - |\lambda| = dn$ implies that the sum over $d \in \mathbb{Z}_{\geq 0}$ on the RHS of (3.49) terminates after finitely many terms, and that for each $\lambda \in \mathcal{A}^+_k(n)$ there is at most one non-zero term $z^d\theta_{\lambda/d/\mu}(\nu)$ which contributes to the coefficient of $m_\lambda(x_1, \ldots, x_k)$ in $m_\mu(x_1, \ldots, x_k)h_\nu(x_1, \ldots, x_k)$.

Proof. We show that the following product expansion holds in $\mathcal{V}_k(n)$

$$m_\mu(x_1,\ldots,x_k)h_r(x_1,\ldots,x_k) = \sum_{d\in\mathbb{Z}_{\geq 0}} z^d \sum_{\lambda\in\mathcal{A}_k^+(n)} \theta_{\lambda/d/\mu} m_\lambda(x_1,\ldots,x_k) \;, \tag{3.50}$$

where the second sum runs over all $\lambda \in \mathcal{A}_k^+(n)$ for which $\lambda/d/\mu$ is a cylindric skew diagram with $|\lambda/d/\mu| = r$, that is $|\mu| + r - |\lambda| = dn$. The claim then follows by applying (3.50) repeatedly to the product $m_\mu(x_1,\ldots,x_k)h_\nu(x_1,\ldots,x_k)$. This also implies that $\theta_{\lambda/d/\mu}(\beta) = \theta_{\lambda/d/\mu}(\nu)$ for $\beta \sim \nu$, where $\theta_{\lambda/d/\mu}(\beta)$ for β a composition is defined in an analogous way to (3.48).

For $\lambda \in \mathcal{A}_k^+(n)$ and $d \in \mathbb{Z}$ we have that the coefficient of $z^d m_\lambda(x_1,\ldots,x_k)$ in the product $m_\mu(x_1,\ldots,x_k)h_r(x_1,\ldots,x_k)$ equals the coefficient of $z^d x_1^{\lambda_1}\cdots x_k^{\lambda_k}$ in the same product. Projecting the expansion (2.18) of h_r onto Λ_k one has that $h_r(x_1,\ldots,x_k) = \sum_{\gamma\in\mathcal{P}_k^{\geq 0}} x_1^{\gamma_1}\cdots x_k^{\gamma_k}$, where the sum runs over all $\gamma \in \mathcal{P}_k^{\geq 0}$ such that $|\gamma| = r$. It then follows that each monomial appearing in the product $m_\mu(x_1,\ldots,x_k)h_r(x_1,\ldots,x_k)$ is of the form $x_1^{\mu_{w(1)}+\gamma_1}\cdots x_k^{\mu_{w(k)}+\gamma_k}$ for some $w \in S^\mu$ and $\gamma \in \mathcal{P}_k^{\geq 0}$ with $|\gamma| = r$. The latter equals $z^d x_1^{\lambda_1}\cdots x_k^{\lambda_k}$ if and only if there exists $\alpha \in \mathcal{P}_k$ with $|\alpha| = d$ satisfying the relation $\mu.w + \gamma = \lambda.y^\alpha$. If such a weight $\alpha \in \mathcal{P}_k$ exists then it must belong to $\mathcal{P}_k^{\geq 0}$ since $\mu \in \mathcal{A}_k^+(n)$, and this in turn implies that $d \in \mathbb{Z}_{\geq 0}$. Thus the coefficient of $z^d x_1^{\lambda_1}\cdots x_k^{\lambda_k}$ in $m_\mu(x_1,\ldots,x_k)h_r(x_1,\ldots,x_k)$ equals the cardinality of the set

$$\mathbb{A} = \left\{(w,\gamma,\alpha) \in S^\mu \times \mathcal{P}_k^{\geq 0} \times \mathcal{P}_k^{\geq 0} \mid |\gamma| = r, |\alpha| = d, \mu.w + \gamma = \lambda.y^\alpha\right\} ,$$

provided that $d \in \mathbb{Z}_{\geq 0}$. Notice that $\mathbb{A}$ is non-empty only if $|\mu| + r - |\lambda| = dn$ due to the constraint $\mu.w+\gamma = \lambda.y^\alpha$. In particular this implies that $\mathbb{A}$ is non-empty only if d satisfies the inequality $kn + r - k > dn$, since kn and k are respectively the largest and smallest possible values for $|\mu|$ and $|\lambda|$. So suppose that the relation $|\mu| + r - |\lambda| = dn$ is satisfied. With this assumption, the set $\mathbb{A}$ can be put in bijection with

$$\mathbb{B} = \left\{(w,\alpha) \in S^\mu \times \mathcal{P}_k^{\geq 0} \mid |\alpha| = d, \mu.w \leq \lambda.y^\alpha\right\} .$$

In fact, if $(w,\alpha,\gamma) \in \mathbb{A}$ then $\mu.w \leq \lambda.y^\alpha$, and thus $(w,\alpha) \in \mathbb{B}$. The assignment $(w,\alpha,\gamma) \mapsto (w,\alpha)$ therefore defines a map $\mathbb{A} \to \mathbb{B}$. To construct the inverse map $\mathbb{B} \to \mathbb{A}$, notice that if $(w,\alpha) \in \mathbb{B}$ then the weight $\gamma = \lambda.y^\alpha - \mu.w$ belongs to $\mathcal{P}_k^{\geq 0}$, and since $|\gamma| = |\lambda| + nd - |\mu| = r$ it follows that $(w,\alpha,\gamma) \in \mathbb{A}$. Thus, the assignment $(w,\alpha) \mapsto (w,\alpha,\gamma)$ is by construction the inverse map $\mathbb{B} \to \mathbb{A}$. Lemma 3.2.6 implies that the set $\mathbb{B}$ has cardinality $\theta_{\lambda/d/\mu}$, which is non-zero if and only if $\lambda/d/\mu$ is a cylindric skew diagram thanks to Proposition 3.2.7. In conclusion, for $\lambda \in \mathcal{A}_k^+(n)$ and $d \in \mathbb{Z}_{\geq 0}$ the coefficient of $z^d m_\lambda(x_1,\ldots,x_k)$ in $m_\mu(x_1,\ldots,x_k)h_r(x_1,\ldots,x_k)$ is non-zero if and only if $\lambda/d/\mu$ is a cylindric skew diagram with $|\mu| + r - |\lambda| = dn$, in which case it equals $\theta_{\lambda/d/\mu}$. This completes the proof of (3.50).

□

Lemma 3.3.9. *Let $\mu \in \mathcal{A}_k^+(n)$ and $\nu \in \mathcal{P}$. The following product rule holds in $\mathcal{V}_k(n)$,*

$$m_\mu(x_1,\ldots,x_k)e_\nu(x_1,\ldots,x_k) = \sum_{d\in\mathbb{Z}_{\geq 0}} z^d \sum_{\lambda\in\mathcal{A}_k^+(n)} \psi_{\lambda/d/\mu}(\nu) m_\lambda(x_1,\ldots,x_k) \,. \tag{3.51}$$

where the second sum is restricted to those $\lambda \in \mathcal{A}_k^+(n)$ for which $\lambda/d/\mu$ is a cylindric skew diagram with $|\mu|+|\nu|-|\lambda| = dn$.

As mentioned above, the coefficient $\psi_{\lambda/d/\mu}(\nu)$ is non-zero only if $|\mu|+|\nu|-|\lambda| = dn$. Thus, the sum over $d \in \mathbb{Z}_{\geq 0}$ on the RHS of (3.51) is finite, and for each $\lambda \in \mathcal{A}_k^+(n)$ there is at most one non-zero term $z^d\psi_{\lambda/d/\mu}(\nu)$ contributing to the coefficient of $m_\lambda(x_1,\ldots,x_k)$ in $m_\mu(x_1,\ldots,x_k)e_\nu(x_1,\ldots,x_k)$.

Proof. We proceed in close analogy to the proof of Lemma 3.3.8. That is, we first show the validity of the following product expansion in $\mathcal{V}_k(n)$,

$$m_\mu(x_1,\ldots,x_k)e_r(x_1,\ldots,x_k) = \sum_{d\in\mathbb{Z}_{\geq 0}} z^d \sum_{\lambda\in\mathcal{A}_k^+(n)} \psi_{\lambda/d/\mu} m_\lambda(x_1,\ldots,x_k) \,, \tag{3.52}$$

where the second sum runs over all $\lambda \in \mathcal{A}_k^+(n)$ for which $\lambda/d/\mu$ is a cylindric vertical strip with $|\mu|+r-|\lambda| = dn$. The claim then follows by applying repeatedly the equality above to the product $m_\mu(x_1,\ldots,x_k)e_\nu(x_1,\ldots,x_k)$. This also implies that $\psi_{\lambda/d/\mu}(\beta) = \psi_{\lambda/d/\mu}(\nu)$ for $\beta \sim \nu$, where $\psi_{\lambda/d/\mu}(\beta)$ for β a composition is defined in an analogous way to (3.47).

As usual, for $\lambda \in \mathcal{A}_k^+(n)$ and $d \in \mathbb{Z}$ we have that the coefficient of $z^d m_\lambda(x_1,\ldots,x_k)$ in $m_\mu(x_1,\ldots,x_k)e_r(x_1,\ldots,x_k)$ equals the coefficient of $z^d x_1^{\lambda_1}\cdots x_k^{\lambda_k}$ in the same product. Projecting the expansion (2.18) of e_r onto Λ_k one has that $e_r(x_1,\ldots,x_k) = \sum_{\gamma\in\mathcal{P}_k^{\geq 0}} x_1^{\gamma_1}\cdots x_k^{\gamma_k}$, where the sum runs over all $\gamma \in \mathcal{P}_k^{\geq 0}$ such that $\gamma_i = 0,1$ and $|\gamma| = r$. This implies that each monomial appearing in the product $m_\mu(x_1,\ldots,x_k)e_r(x_1,\ldots,x_k)$ is of the form $x_1^{\mu_{w(1)}+\gamma_1}\cdots x_k^{\mu_{w(k)}+\gamma_k}$ for some $w \in S^\mu$ and $\gamma \in \mathcal{P}_k^{\geq 0}$ with $\gamma_i = 0,1$ and $|\gamma| = r$. The latter equals $z^d x_1^{\lambda_1}\cdots x_k^{\lambda_k}$ if and only if there exists $\alpha \in \mathcal{P}_k^{\geq 0}$ with $|\alpha| = d$ satisfying the relation $\mu.w + \gamma = \lambda.y^\alpha$. If such a weight $\alpha \in \mathcal{P}_k$ exists then it must belong to $\mathcal{P}_k^{\geq 0}$, and this implies that $d \in \mathbb{Z}_{\geq 0}$. Thus the coefficient of $z^d x_1^{\lambda_1}\cdots x_k^{\lambda_k}$ in $m_\mu(x_1,\ldots,x_k)e_r(x_1,\ldots,x_k)$ equals the cardinality of the set

$$\mathbb{A} = \left\{(w,\gamma,\alpha) \in S^\mu \times \mathcal{P}_k^{\geq 0} \times \mathcal{P}_k^{\geq 0} \mid \gamma_i = 0,1, |\gamma| = r, |\alpha| = d, \mu.w + \gamma = \lambda.y^\alpha\right\} ,$$

which is non-empty only if $|\mu|+r-|\lambda| = dn$. Following similar steps as in the proof of Lemma 3.3.8, one can show that $\mathbb{A}$ is in bijection with the set

$$\mathbb{B} = \left\{(w,\alpha) \in S^\mu \times \mathcal{P}_k^{\geq 0} \mid |\alpha| = d, (\lambda.y^\alpha)_i - (\mu.w)_i = 0,1\right\} ,$$

provided that $|\mu| + r - |\lambda| = dn$. But the latter corresponds to the set (3.28), whose cardinality is by definition $\psi_{\lambda/d/\mu}$. Moreover, $\mathbb{B}$ is non-empty if and only if $\lambda/d/\mu$ is a cylindric vertical strip according to Proposition 3.2.11. In conclusion, for $\lambda \in \mathcal{A}_k^+(n)$ and $d \in \mathbb{Z}_{\geq 0}$ the coefficient of $z^d m_\lambda(x_1, \dots, x_k)$ in $m_\mu(x_1, \dots, x_k)e_r(x_1, \dots, x_k)$ is non-zero if and only if $\lambda/d/\mu$ is a cylindric vertical strip with $|\mu| + r - |\lambda| = dn$, in which case it is equal to $\psi_{\lambda/d/\mu}$. This shows the validity of (3.52).

□

The coefficients described in Definition 3.3.7 have alternative combinatorial expressions which are the generalisation of (2.53) to the cylinder. For $\lambda \in \mathcal{P}^+$ recall the expansions $e_\lambda = \sum_{\mu \in \mathcal{P}^+} M_{\lambda\mu} m_\mu$ and $h_\lambda = \sum_{\mu \in \mathcal{P}^+} L_{\lambda\mu} m_\mu$ in Λ, which were introduced in (2.22).

Proposition 3.3.10. *Let $\lambda, \mu \in \mathcal{A}_k^+(n)$, $d \in \mathbb{Z}_{\geq 0}$ and $\nu \in \mathcal{P}$. The following equalities hold*

$$\psi_{\lambda/d/\mu}(\nu) = \sum_{\sigma \in \mathcal{P}_k^+} N_{\mu\sigma}^{\lambda,d} M_{\nu\sigma} \ , \tag{3.53}$$

$$\theta_{\lambda/d/\mu}(\nu) = \sum_{\sigma \in \mathcal{P}_k^+} N_{\mu\sigma}^{\lambda,d} L_{\nu\sigma} \ . \tag{3.54}$$

Proof. Projecting onto Λ_k one has that $e_\nu(x_1, \dots, x_k) = \sum_{\sigma \in \mathcal{P}_k^+} M_{\nu\sigma} m_\sigma(x_1, \dots, x_k)$. This, together with the quotient map $\pi_{k,n} : \Lambda_k[z, z^{-1}] \to \mathcal{V}_k(n)$ and the product expansion (3.40), can be used to obtain the following identity in $\mathcal{V}_k(n)$,

$$m_\mu(x_1, \dots, x_k)e_\nu(x_1, \dots, x_k) = \sum_{d \in \mathbb{Z}_{\geq 0}} z^d \sum_{\lambda \in \mathcal{A}_k^+(n)} \left(\sum_{\sigma \in \mathcal{P}_k^+} N_{\mu\sigma}^{\lambda,d} M_{\nu\sigma} \right) m_\lambda(x_1, \dots, x_k) \ .$$

Thanks to part 4 of Lemma 3.3.4, the second sum on the RHS is restricted to those $\lambda \in \mathcal{A}_k^+(n)$ for which $\lambda/d/\mu$ is a cylindric skew diagram, together with the constraint $|\mu| + |\nu| - |\lambda| = dn$ since $N_{\mu\sigma}^{\lambda,d}$ is non-zero only if $|\mu| + |\sigma| - |\lambda| = dn$ and $M_{\nu\sigma}$ is non-zero only if $|\nu| = |\sigma|$. A comparison of the latter with (3.51) then yields (3.53) provided that $\lambda/d/\mu$ is a cylindric skew diagram. If instead $\hat{\mu} \not\subset \hat{\lambda}.\tau^d$ then both $\psi_{\lambda/d/\mu}(\nu)$ and $N_{\mu\sigma}^{\lambda,d}$ are 0, and thus (3.53) still holds. Equation (3.54) follows in an analogous way, using the product expansion (3.49) instead. □

3.3.2 Expansion formulae for cylindric symmetric functions

We are now ready to prove the main expansion formulae for cylindric symmetric functions, which are the generalisation of (2.54) and (2.55) to the cylinder.

Proposition 3.3.11. *Let $\lambda, \mu \in \mathcal{A}_k^+(n)$ and $d \in \mathbb{Z}_{\geq 0}$. The functions $e_{\lambda/d/\mu}$ and $h_{\lambda/d/\mu}$ can* *be expanded as*

$$e_{\lambda/d/\mu} = \sum_{\nu\in\mathcal{P}^+} \psi_{\lambda/d/\mu}(\nu) m_\nu \,, \tag{3.55}$$

$$h_{\lambda/d/\mu} = \sum_{\nu\in\mathcal{P}^+} \theta_{\lambda/d/\mu}(\nu) m_\nu \,. \tag{3.56}$$

into the basis $\{m_\nu\}_{\nu\in\mathcal{P}^+}$ *of* Λ*, and thus are symmetric.*

Proof. Suppose that $\lambda/d/\mu$ is a cylindric skew diagram. A simple rewriting of (3.30) shows that

$$e_{\lambda/d/\mu} = \sum_{\nu\in\mathcal{P}^+} \sum_{\beta\sim\nu} x^\beta \sum_{\hat{T}} \psi_{\hat{\pi}} \,, \tag{3.57}$$

where the last sum runs over all CRSTs $\hat{T}$ of shape $\lambda/d/\mu$ and weight β, and thus $e_{\lambda/d/\mu} = \sum_{\nu\in\mathcal{P}^+}\sum_{\beta\sim\nu} x^\beta \psi_{\lambda/d/\mu}(\beta)$. The latter can then be rearranged to give (3.55) thanks to the relation $\psi_{\lambda/d/\mu}(\beta) = \psi_{\lambda/d/\mu}(\nu)$ for $\beta\sim\nu$, which was proved in Lemma 3.3.9, and the definition $m_\nu = \sum_{\beta\sim\nu} x^\beta$ of monomial symmetric functions. If instead $\hat{\mu} \not\subset \hat{\lambda}.\tau^d$ then both $e_{\lambda/d/\mu}$ and $\psi_{\lambda/d/\mu}(\nu)$ are 0, and (3.55) still holds. Equation (3.56) follows in a completely analogous way.

□

Theorem 3.3.12. *Let* $\lambda,\mu \in \mathcal{A}_k^+(n)$ *and* $d\in\mathbb{Z}_{\geq 0}$*. We have the expansions*

$$e_{\lambda/d/\mu} = \sum_{\nu\in\mathcal{P}_k^+} N_{\mu\nu}^{\lambda,d} e_\nu \,, \tag{3.58}$$

$$h_{\lambda/d/\mu} = \sum_{\nu\in\mathcal{P}_k^+} N_{\mu\nu}^{\lambda,d} h_\nu \,, \tag{3.59}$$

into the basis $\{h_\nu\}_{\nu\in\mathcal{P}^+}$ *and* $\{e_\nu\}_{\nu\in\mathcal{P}^+}$ *of* Λ *respectively.*

Proof. Using (3.53) one has that $\sum_{\sigma\in\mathcal{P}^+}\psi_{\lambda/d/\mu}(\sigma)M^{-1}_{\nu\sigma}$ equals $N_{\mu\nu}^{\lambda,d}$ if $\ell(\nu)\leq k$, that is if $\nu\in\mathcal{P}_k^+$, and 0 otherwise. Starting from (3.55), and using the fact that the basis $\{e_\lambda\}_{\lambda\in\mathcal{P}^+}$ and $\{f_\lambda\}_{\lambda\in\mathcal{P}^+}$ of Λ are dual to each other, we then have that

$$e_{\lambda/d/\mu} = \sum_{\nu,\sigma\in\mathcal{P}^+} \psi_{\lambda/d/\mu}(\sigma)\langle m_\sigma, f_\nu\rangle e_\nu = \sum_{\nu\in\mathcal{P}^+}\left(\sum_{\sigma\in\mathcal{P}^+}\psi_{\lambda/d/\mu}(\sigma)M^{-1}_{\nu\sigma}\right)e_\nu \,,$$

which implies the validity of (3.58). To justify the second equality notice first of all that $\langle m_\sigma, f_\nu\rangle = \langle w(m_\sigma), w(f_\nu)\rangle = \langle f_\sigma, m_\nu\rangle$, where w is the involution in Λ described in Section 2.2.5. Then use the the expansion $m_\nu = \sum_{\rho\in\mathcal{P}^+} M^{-1}_{\nu\rho} e_\rho$ together with the orthogonality relation $\langle f_\sigma, e_\rho\rangle = \delta_{\sigma\rho}$. The proof of (3.59) is completely analogous and therefore we omit it. □

Theorem 3.3.12 has a few consequences, some of which we now explore. First of all, applying the involution ω to both sides of (3.58) it follows that

$$\omega(e_{\lambda/d/\mu}) = h_{\lambda/d/\mu}\ , \tag{3.60}$$

which means that $e_{\lambda/d/\mu}$ and $h_{\lambda/d/\mu}$ are 'dual' to each other. A further consequence is given by the following corollary. Namely, by expanding $e_{\lambda/d/\mu}$ and $h_{\lambda/d/\mu}$ in a suitable way, we recover the coefficients $N_{\mu\nu}^{\lambda,d}$ involving only weights in $\mathcal{A}_k^+(n)$. These are the fusion coefficients of a 2D TQFT, as we will see in Chapter 5.

Corollary 3.3.13 ([45])**.** *Let $\lambda,\mu\in\mathcal{A}_k^+(n)$ and $d\in\mathbb{Z}_{\geq 0}$. We have the expansions*

$$e_{\lambda/d/\mu} = \sum_{d'=0}^{d+k}\sum_{\sigma\in\mathcal{A}_k^+(n)} N_{\mu\sigma}^{\lambda,d'} e_{\sigma/(d+k-d')/n^k}\ , \tag{3.61}$$

$$h_{\lambda/d/\mu} = \sum_{d'=0}^{d+k}\sum_{\sigma\in\mathcal{A}_k^+(n)} N_{\mu\sigma}^{\lambda,d'} h_{\sigma/(d+k-d')/n^k}\ , \tag{3.62}$$

where $n^k=(n,\dots,n)\in\mathcal{A}_k^+(n)$.

Proof. Setting $\mu=\emptyset$ in (3.43) and renaming the partitions appropriately one has that $N_{\mu\nu}^{\lambda,d}=\sum_{\substack{d_1+d_2=d\\ d_1\geq -k,d_2\geq 0}}\sum_{\sigma\in\mathcal{A}_k^+(n)} N_{\mu\emptyset}^{\sigma,d_1}N_{\sigma\nu}^{\lambda,d_2}$ thanks to the relation $f_{\eta\emptyset}^{\sigma}=\delta_{\sigma\eta}$. Furthermore, notice that the coefficient $N_{\emptyset\nu}^{\sigma,d}$ equals the cardinality of the set

$$\left\{w\in S^\nu \mid \nu.w=\sigma.y^\alpha \text{ for some } \alpha\in\mathcal{P}_k \text{ with } |\alpha|=d\right\}.$$

The constraint $\nu.w=\sigma.y^\alpha$ holds if and only if $\nu.w+n^k=\sigma.y^{\alpha'}$ holds, where we defined $\alpha'=\alpha+(1,1,\dots,1)$, and this implies that $N_{\emptyset\nu}^{\sigma,d}=N_{n^k\nu}^{\sigma,d+k}$. Starting from (3.58) we end up with the following chain of equalities,

$$\begin{aligned}
e_{\lambda/d/\mu} &= \sum_{\nu\in\mathcal{P}_k^+} N_{\mu\nu}^{\lambda,d} e_\nu \\
&= \sum_{\nu\in\mathcal{P}_k^+}\sum_{\substack{d_1+d_2=d\\ d_1\geq -k,d_2\geq 0}}\sum_{\sigma\in\mathcal{A}_k^+(n)} N_{\mu\emptyset}^{\sigma,d_1}N_{\sigma\nu}^{\lambda,d_2}e_\nu \\
&= \sum_{\nu\in\mathcal{P}_k^+}\sum_{\substack{d_1+d_2=d\\ d_1\geq -k,d_2\geq 0}}\sum_{\sigma\in\mathcal{A}_k^+(n)} N_{n^k\nu}^{\sigma,d_1+k}N_{\sigma\mu}^{\lambda,d_2}e_\nu \\
&= \sum_{d'=0}^{d+k}\sum_{\sigma\in\mathcal{A}_k^+(n)} N_{\mu\sigma}^{\lambda,d'}\sum_{\nu\in\mathcal{P}_k^+} N_{n^k\nu}^{\sigma,d+k-d'}e_\nu \\
&= \sum_{d'=0}^{d+k}\sum_{\sigma\in\mathcal{A}_k^+(n)} N_{\mu\sigma}^{\lambda,d'} e_{\sigma/(d+k-d')/n^k}\ .
\end{aligned}$$

In the third line we used Properties 1 and 2 of Lemma 3.3.6, whereas in the last line we applied (3.58) once again. This proves (3.61), and applying the involution ω to both sides of the latter one ends up with (3.62). □

For practical purposes we defined the cylindric symmetric functions $e_{\lambda/d/\mu}$ and $h_{\lambda/d/\mu}$ for cylindric skew diagrams of the form $\lambda/d/\mu$ (compare with Definition 3.2.13). Recall from Lemma 3.2.3 that every cylindric partition can be written as $\hat{\lambda}.\tau^d$ for some $\lambda \in \mathcal{A}_k^+(n)$ and $d \in \mathbb{Z}$. Thus, the most general cylindric skew diagram is given by $\hat{\lambda}.\tau^{d_1}/\hat{\mu}.\tau^{d_2}$ for some $\lambda, \mu \in \mathcal{A}_k^+(n)$ and $d_1, d_2 \in \mathbb{Z}$ such that $d_1 - d_2 \geq 0$. But every CRPP (respectively CRST) of shape $\hat{\lambda}.\tau^{d_1}/\hat{\mu}.\tau^{d_2}$ can be translated along the vertical direction to a CRPP (respectively CRST) of shape $\lambda/(d_1-d_2)/\mu$. Moreover, it is straightforward to extend the definitions of $\psi_{\lambda/d/\mu}$ and $\theta_{\lambda/d/\mu}$ to the most general cylindric skew diagram $\hat{\lambda}.\tau^{d_1}/\hat{\mu}.\tau^{d_2}$, and these definitions turn out to be invariant under translation of the vertical axis. It is then natural to define cylindric symmetric functions for cylindric skew diagrams $\hat{\lambda}.\tau^{d_1}/\hat{\mu}.\tau^{d_2}$ as

$$e_{\hat{\lambda}.\tau^{d_1}/\hat{\mu}.\tau^{d_2}} = e_{\lambda/(d_1-d_2)/\mu}\ , \tag{3.63}$$

$$h_{\hat{\lambda}.\tau^{d_1}/\hat{\mu}.\tau^{d_2}} = h_{\lambda/(d_1-d_2)/\mu}\ . \tag{3.64}$$

In particular one has that $e_{\hat{\lambda}.\tau^d/\hat{\emptyset}} = e_{\lambda/(d+k)/n^k}$, which are the symmetric functions used in the expansion (3.61). Moreover, we set $e_{\hat{\lambda}.\tau^{d_1}/\hat{\mu}.\tau^{d_2}} = h_{\hat{\lambda}.\tau^{d_1}/\hat{\mu}.\tau^{d_2}} = 0$ if $\hat{\lambda}.\tau^{d_1}/\hat{\mu}.\tau^{d_2}$ is not a cylindric skew diagram. Notice that if $d \leq m_0(\hat{\lambda})$ then there are no boxes of $\lambda.\tau^d$ in columns greater than n. It follows from Remark 3.1.12 that there exists a bijection $\hat{T} \mapsto T$ between CRSTs of shape $\hat{\lambda}.\tau^d/\hat{\emptyset}$ and row strict tableaux of shape $\lambda.\tau^d$, which in turn implies that $e_{\hat{\lambda}.\tau^d/\hat{\emptyset}} = e_{\lambda.\tau^d}$. A completely analogous discussion holds for (3.64).

Lemma 3.3.14. *Let $\lambda \in \mathcal{A}_k^+(n)$ and $d \in \mathbb{Z}$. We have the equalities*

$$e_{\hat{\lambda}.\tau^d/\hat{\emptyset}} = \sum_{\nu\in\mathcal{P}_k^+} \frac{|S_\lambda|}{|S_\nu|} e_\nu\ , \tag{3.65}$$

$$h_{\hat{\lambda}.\tau^d/\hat{\emptyset}} = \sum_{\nu\in\mathcal{P}_k^+} \frac{|S_\lambda|}{|S_\nu|} h_\nu\ , \tag{3.66}$$

where the sums are restricted to those $\nu \in \lambda.\hat{S}_k$ with $|\nu| - |\lambda| = dn$.

Proof. Properties 4 and 5 of Lemma 3.3.6 imply that

$$N_{n^k\nu}^{\lambda,d+k} = N_{n^k\check{\nu}}^{\lambda,d+k+\frac{|\check{\nu}|-|\nu|}{n}} \frac{|S_{\check{\nu}}|}{|S_\nu|} = \delta_{d+\frac{|\check{\nu}|-|\nu|}{n},0}\,\delta_{\lambda\check{\nu}} \frac{|S_{\check{\nu}}|}{|S_\nu|}\ ,$$

where $\check{\nu}$ is the unique intersection point of the orbit $\nu.\hat{S}_k$ with $\mathcal{A}_k^+(n)$. Let $\check{\nu} = \lambda$, then by definition we have that $\nu = \lambda.wy^\alpha$ for some $w \in S^\lambda$ and $\alpha \in \mathcal{P}_k$. This implies that

$|\alpha| = \frac{|\nu|-|\lambda|}{n} \geq -m_n(\lambda)$ since the parts of ν are all positive, and thus $N_{n^k\nu}^{\lambda,d+k}$ is non-zero only if $d \geq -m_n(\lambda)$. Similarly we have that $e_{\hat{\lambda}.\tau^d/\hat{\emptyset}}$ is non-zero only if $d \geq -m_n(\lambda)$, because if $d < -m_n(\lambda)$ then $\hat{\lambda}.\tau^d/\hat{\emptyset}$ is not a cylindric skew diagram. This proves the claim for $d < -m_n(\lambda)$, so suppose that $d \geq -m_n(\lambda)$. Plugging the relation for $N_{n^k\nu}^{\lambda,d+k}$ just obtained into the expansion $e_{\hat{\lambda}.\tau^d/\hat{\emptyset}} = e_{\lambda/(d+k)/n^k} = \sum_{\nu\in\mathcal{P}_k^+} N_{n^k\nu}^{\lambda,d+k} e_\nu$ we get the validity of (3.65). Equation (3.66) follows by applying the involution ω to both sides of (3.65). □

The next result shows that the expansions (3.61) and (3.62) are unique.

Corollary 3.3.15 ([45]). *The sets*

$$\{e_{\hat{\lambda}.\tau^d/\hat{\emptyset}} \mid \lambda \in \mathcal{A}_k^+(n),\, d \geq -m_n(\lambda)\}\ , \tag{3.67}$$

$$\{h_{\hat{\lambda}.\tau^d/\hat{\emptyset}} \mid \lambda \in \mathcal{A}_k^+(n),\, d \geq -m_n(\lambda)\}\ , \tag{3.68}$$

are linearly independent in Λ.

Proof. Suppose that $\sum_{\lambda\in\mathcal{A}_k^+(n)} \sum_{d\in\mathbb{Z}} a_{\hat{\lambda}.\tau^d} e_{\hat{\lambda}.\tau^d/\hat{\emptyset}} = 0$ for some $a_{\hat{\lambda}.\tau^d} \in \mathbb{C}$, where the second sum is restricted to those $d \in \mathbb{Z}$ for which $d \geq -m_n(\lambda)$. For $\nu \in \mathcal{P}^+$ the symmetric function e_ν only appears once in this linear combination. This holds thanks to (3.65), and the fact that the set $\mathcal{A}_k^+(n)$ is a fundamental domain for the action of $\hat{S}_k$ on $\mathcal{P}_k$. Since the set $\{e_\nu\}_{\nu\in\mathcal{P}^+}$ is linearly independent in Λ it follows that $a_{\hat{\lambda}.\tau^d} = 0$ for all $\lambda \in \mathcal{A}_k^+(n)$ and $d \in \mathbb{Z}$ such that $d \geq -m_n(\lambda)$, thus proving the claim for the set (3.67). The proof for (3.67) is completely analogous. □

Remark 3.3.16. The set (3.67) does not form a spanning set for Λ. In fact, the symmetric function e_ν appears in the expansion (3.65) only if $\nu \in \mathcal{P}^+$ satisfies the constraint $\nu_1 - n \leq \nu_k$. This implies that if ν does not satisfy such constraint then e_ν cannot be written as a linear combination of elements belonging to (3.67). On the other hand, Λ is spanned by $\{e_\nu\}_{\nu\in\mathcal{P}^+}$, and therefore we conclude that the set (3.67) does not form a spanning set for Λ. An analogous statement holds for (3.68).

To conclude this section, we write down the expansions for cylindric elementary and complete symmetric functions in terms of Schur functions. These expansions are the generalisation of (2.59) (2.60) to the cylinder.

Definition 3.3.17. Let $\lambda, \mu \in \mathcal{A}_k^+(n)$, $d \in \mathbb{Z}_{\geq 0}$ and $\nu \in \mathcal{P}_k^+$. Define the weight

$$\chi_{\mu\nu}^{\lambda,d} = \sum_{\sigma\in\mathcal{P}_k^+} N_{\mu\sigma}^{\lambda,d} K_{\nu\sigma}\ . \tag{3.69}$$

The coefficient $K_{\nu\sigma}$ is the Kostka number, which was introduced in Section 2.2.3.

Recall that $N_{\mu\sigma}^{\lambda,d}$ is non-zero only if $|\mu|+|\sigma|-|\lambda|=dn$, as we showed in Lemma 3.3.4. Moreover, we have by definition that $K_{\nu\sigma}$ is non-zero only if $|\nu|=|\sigma|$. This implies that $\chi_{\mu\nu}^{\lambda,d}$ is non-zero only if $|\mu|+|\nu|-|\lambda|=dn$. Thanks to Property 4 of Lemma 3.3.6 we have the relation $\chi_{\mu\nu}^{\lambda,0}=\chi_{\mu\nu}^{\lambda}$, where the coefficient $\chi_{\mu\nu}^{\lambda}$ was introduced in (2.58). Finally, we have that $\chi_{\mu\nu}^{\lambda,d}$ is a non-negative integer, since $N_{\mu\sigma}^{\lambda,d}$ and $K_{\nu\sigma}$ are non-negative integers as well.

Proposition 3.3.18. *Let $\lambda,\mu\in\mathcal{A}_k^+(n)$ and $d\in\mathbb{Z}_{\geq 0}$. The functions $e_{\lambda/d/\mu}$ and $h_{\lambda/d/\mu}$ can be expanded as*

$$e_{\lambda/d/\mu} = \sum_{\nu\in\mathcal{P}_k^+}\chi_{\mu\nu}^{\lambda,d}s_{\nu'}\ , \tag{3.70}$$

$$h_{\lambda/d/\mu} = \sum_{\nu\in\mathcal{P}_k^+}\chi_{\mu\nu}^{\lambda,d}s_{\nu}\ . \tag{3.71}$$

Proof. Plug the expansions $e_\nu=\sum_{\sigma\in\mathcal{P}^+}K_{\nu\sigma}s_{\sigma'}$ and $h_\nu=\sum_{\sigma\in\mathcal{P}^+}K_{\nu\sigma}s_\sigma$, which can be found for instance in [52, I.6], into (3.58) and (3.59) respectively. A comparison with (3.69) then proves the validity of the claim. □

Remark 3.3.19. Lemma 2.3.11 implies that the functions $e_{\lambda/d/\mu}$ and $h_{\lambda/d/\mu}$ are Schur-positive. It follows that there exist representations of $\mathrm{GL}_r(\mathbb{C})$ whose polynomial characters are given by $e_{\lambda/d/\mu}(x_1,\ldots,x_r)$ and $h_{\lambda/d/\mu}(x_1,\ldots,x_r)$ respectively (compare with the discussion presented in Remark 2.3.12). It would be interesting to present a more explicit construction of these representations.

3.3.3 Cylindric adjacent column tableaux

We finally wish to expand $h_{\lambda/d/\mu}$ and $e_{\lambda/d/\mu}$ in terms of the basis $\{p_\nu\}_{\nu\in\mathcal{P}^+}$ of Λ. For this purpose we generalise the notion of ACHS defined in Section 2.3.4 to the cylinder, and to do this we take inspiration from Definition 2.3.26 and Lemma 2.3.27. Compare also with Figure 3.9.

Definition 3.3.20. Let $\hat{\mu}\in\mathcal{P}_{k,n}^+$, $r\in\mathbb{N}$ and $1\leq a\leq n$ with $m_{a-1}(\hat{\mu})\neq 0$. Define $\hat{\mu}_{a,r}$ as the cylindric partition whose diagram is obtained as follows: for every $p\in\mathbb{Z}$ add one box per column in the diagram of $\hat{\mu}$, starting at column $a+pn$ and ending at column $a+r-1+pn$, for a total of r boxes.

Lemma 3.3.21. *We have the equality*

$$\hat{\mu}_{a,r}=\hat{\mu}_{a,q}.\tau^s\ , \tag{3.72}$$

where we set $r=sn+q$ for some $s\in\mathbb{Z}_{\geq 0}$ and $1\leq q\leq n$.

Proof. For $r \leq n$, that is for $s = 0$, equation (3.72) reduces to the identity $\hat{\mu}_{a,q} = \hat{\mu}_{a,q}$. So suppose that $r > n$, that is $s > 0$. Notice that to obtain the diagram of $\hat{\mu}_{a,r}$ we need to add at least s boxes in each column of the diagram of $\hat{\mu}$. In other words, we have that $(\hat{\mu}_{a,r})'_i - \hat{\mu}'_i \geq s$ for all $i \in \mathbb{Z}$ and thus $\hat{\mu} \subset \hat{\mu}_{a,r}.\tau^{-s}$. This is because for each $p \in \mathbb{Z}$ we are adding, among others, a box per column in the diagram of $\hat{\mu}$ from columns $a + pn$ to $a + (p + s)n - 1$. Periodicity implies that the diagram of $\hat{\mu}_{a,r}.\tau^{-s}$ is obtained by adding a box per column in the diagram of $\hat{\mu}$ from columns $a + pn$ to $a + q - 1 + pn$, and repeating this procedure for all $p \in \mathbb{Z}$. But this is by definition $\hat{\mu}_{a,q}$ and the claim follows. $\square$

Definition 3.3.22. Let $\hat{\lambda}, \hat{\mu} \in \mathcal{P}^+_{k,n}$ with $\hat{\mu} \subset \hat{\lambda}$. We say that $\hat{\lambda}/\hat{\mu}$ is a 'cylindric adjacent column skew diagram' (CACSD) if either $\hat{\lambda} = \hat{\mu}$ or $\hat{\lambda} = \hat{\mu}_{a,r}$ for some $1 \leq a \leq n$ with $m_{a-1}(\hat{\mu}) \neq 0$ and $r \in \mathbb{N}$. A 'cylindric adjacent column reverse plane partition' (CACRPP) $\hat{\pi}$ of shape $\hat{\lambda}/\hat{\mu}$ is a sequence $\{\hat{\lambda}^{(r)}\}_{r=0}^{l}$ of cylindric partitions, with $\hat{\lambda}^{(0)} = \hat{\mu}$ and $\hat{\lambda}^{(l)} = \hat{\lambda}$, such that $\hat{\lambda}^{(r)}/\hat{\lambda}^{(r-1)}$ is a CACSD for $r = 1, \ldots, l$.

See Figure 3.9 for a depiction of two CACSDs. Notice that for $\lambda, \mu \in \mathcal{A}^+_k(n)$ the restriction of $\hat{\lambda}/\hat{\mu}$ to the lines 1 to k is a ACHS, and conversely the periodic continuation of λ/μ to the cylinder is a CACSD.

Remark 3.3.23. Whereas a ACHS is a special case of horizontal strip, from the proof of Lemma 3.3.21 it follows that a CACSD is not always a cylindric horizontal strip. Compare with Figure 3.9.

Let $\lambda, \mu \in \mathcal{A}^+_k(n)$ and $d \in \mathbb{Z}_{\geq 0}$ such that $\hat{\lambda}.\tau^d = \hat{\mu}_{a,r}$ for some $1 \leq a \leq n$ with $m_{a-1}(\hat{\mu}) \neq 0$. Suppose that $1 \leq r \leq n$, then by definition we have the equality

$$(\hat{\lambda}.\tau^d)'_i = \begin{cases} \hat{\mu}'_i + 1\,, & a + pn \leq i \leq a + r - 1 + pn \text{ for some } p \in \mathbb{Z} \\ \hat{\mu}'_i\,, & \text{otherwise} \end{cases}, \tag{3.73}$$

and since $(\hat{\lambda}.\tau^d)'_n - \hat{\mu}'_n = d$ it follows that $d = 0, 1$. For $j \in \mathbb{Z}$ define

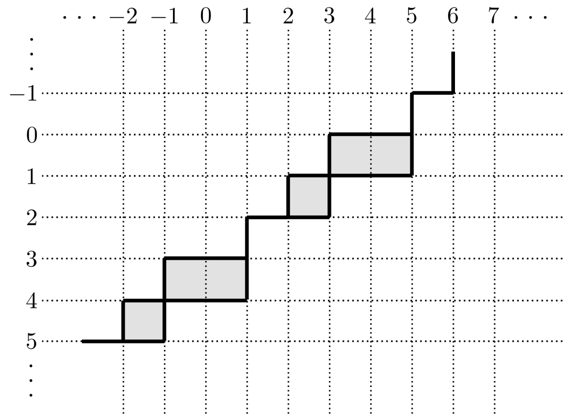
$$j \operatorname{Mod} n = j \bmod n + n\delta_{j \bmod n, 0}\,, \tag{3.74}$$

and notice that $1 \leq j \operatorname{Mod} n \leq n$. Using the relation $m_i(\hat{\lambda}.\tau^d) = (\hat{\lambda}.\tau^d)'_i - (\hat{\lambda}.\tau^d)'_{i+1} = m_i(\hat{\lambda})$, together with (3.73), and adopting the notation (2.1) for partitions we end up with

$$\lambda = \left(\ldots, \left((a-1) \operatorname{Mod} n\right)^{m_{a-1}(\hat{\mu})-1}, \ldots, \left((a-1+r) \operatorname{Mod} n\right)^{m_{a-1+r}(\hat{\mu})+1}, \ldots\right). \tag{3.75}$$

That is, λ is obtained from μ by removing a part equal to $(a-1) \operatorname{Mod} n$ and adding a part equal to $(a - 1 + r) \operatorname{Mod} n$. In particular, for $r = n$ it follows that $\lambda = \mu$ and $d = 1$.

Remark 3.3.24. Equation (3.73) implies that for $r < n$ the integer a is unique, and if in general $r \bmod n \neq 0$ we can just apply Lemma 3.3.21 to reduce to the previous case. On



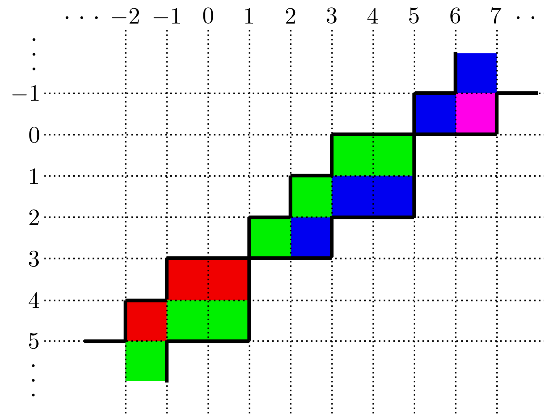


Figure 3.9: Let $\lambda, \mu \in \mathcal{A}_3^+(4)$ given by $\lambda = (3, 1, 1)$ and $\mu = (3, 2, 1)$. On the left we have the CACSD $\lambda/1/\mu$ with $\hat{\lambda}.\tau = \hat{\mu}_{3,3}$. In fact, if in the diagram of $\hat{\mu}$ we add one box in columns $-1, 0, 1$, and then in columns $3, 4, 5$, and so on we recover the cylindric diagram of $\hat{\lambda}.\tau$. Since $m_5(\hat{\lambda}.\tau) = m_5(\hat{\lambda}) = 2$ it follows that $\varphi_{\lambda/1/\mu} = 2$. On the right we have the CACSD $\lambda/2/\mu$ with $\hat{\lambda}.\tau^2 = \hat{\mu}_{3,7}$. In fact, if in the diagram of $\hat{\mu}$ we add one box in columns $-5, -4, -3, -2, -1, 0, 1$ (red) and then in columns $-1, 0, 1, 2, 3, 4, 5$ (green), in columns $3, 4, 5, 6, 7, 8, 9$ (blue), in columns $7, 8, 9, 10, 11, 12, 13$ (fuchsia) and so on we recover the cylindric diagram of $\hat{\lambda}.\tau^2$. Since $m_9(\hat{\lambda}.\tau^2) = m_9(\hat{\lambda}) = m_5(\hat{\lambda}) = 2$ it follows that $\varphi_{\lambda/2/\mu} = 2$. Notice that $\hat{\mu}_{3,7} = \hat{\mu}_{3,3}.\tau$ and that $\varphi_{\lambda/2/\mu} = \varphi_{\lambda/1/\mu}$.

the other hand, for $r = n$ the same equation implies that $\hat{\lambda}.\tau = \hat{\mu}_{a,n}$ for every $1 \leq a \leq n$ such that $m_{a-1}(\hat{\mu}) \neq 0$. A similar argument applies if $r \bmod n = 0$.

Let $r \in \mathbb{N}$ and suppose that $\hat{\lambda}.\tau^d = \hat{\mu}_{a,r}$ as above. Define

$$\varphi_{\lambda/d/\mu} = \begin{cases} m_{a-1+r}(\hat{\lambda}.\tau^d) \ , & r \bmod n \neq 0 \\ k \ , & \text{otherwise} \end{cases} \ , \tag{3.76}$$

and in particular set $\varphi_{\lambda/0/\lambda} = 1$. Notice that $\varphi_{\lambda/d/\mu} = m_{a-1+r}(\hat{\mu}) + 1$ for $r \bmod n \neq 0$ thanks to (3.75). The weight $\varphi_{\lambda/d/\mu}$ is the generalisation of the coefficient $\varphi_{\lambda/\mu}$ defined in (2.79) to the cylinder. In fact, the restriction of the CACSD $\lambda/0/\mu$ to the lines 1 to k generates the ACHS λ/μ. If $d = 0$ equation (3.73) implies that $a - 1 + r \leq n$, and thus it follows by definition that $\varphi_{\lambda/0/\mu} = \varphi_{\lambda/\mu}$.

Definition 3.3.25. Let $\lambda, \mu \in \mathcal{A}_k^+(n)$, $d \in \mathbb{Z}_{\geq 0}$, $\nu \in \mathcal{P}$ and suppose that $\hat{\mu} \subset \hat{\lambda}$. Define the coefficient

$$\varphi_{\lambda/d/\mu}(\nu) = \sum_{\hat{\pi}} \varphi_{\hat{\pi}} \ , \qquad \varphi_{\hat{\pi}} = \prod_{r \geq 1} \varphi_{\lambda^{(r)}/(d_r - d_{r-1})/\lambda^{(r-1)}} \ , \tag{3.77}$$

where the sum runs over all CACRPPs of shape $\lambda/d/\mu$ and weight ν. If $\lambda/d/\mu$ is not a cylindric skew diagram set $\varphi_{\lambda/d/\mu}(\nu) = 0$.

In particular, $\varphi_{\lambda/d/\mu}(\nu)$ is non-zero only if $|\mu|+|\nu|-|\lambda| = dn$. Moreover we have that $\varphi_{\lambda/0/\mu}(\nu) = \varphi_{\lambda/\mu}(\nu)$ where the latter is the coefficient appearing in the product expansion (2.73), which was described in Lemma 2.3.28 as a weighted sum over ACT.

Lemma 3.3.26. *Let $\mu \in \mathcal{A}_k^+(n)$ and $\nu \in \mathcal{P}^+$. The following product rule holds in $\mathcal{V}_k(n)$*

$$m_\mu(x_1,\ldots,x_k)p_\nu(x_1,\ldots,x_k) = \sum_{d\in\mathbb{Z}_{\geq 0}} z^d \sum_{\lambda\in\mathcal{A}_k^+(n)} \varphi_{\lambda/d/\mu}(\nu) m_\lambda(x_1,\ldots,x_k)\,, \tag{3.78}$$

where the second sum is restricted to those $\lambda \in \mathcal{A}_k^+(n)$ for which $\lambda/d/\mu$ is a cylindric skew diagram with $|\mu|+|\nu|-|\lambda| = dn$.

Since $\varphi_{\lambda/d/\mu}(\nu)$ is non-zero only if $|\mu|+|\nu|-|\lambda| = dn$ it follows that the sum over $d \in \mathbb{Z}_{\geq 0}$ on the RHS of (3.78) is finite, and that for each $\lambda \in \mathcal{A}_k^+(n)$ there is at most one non-zero term $z^d\varphi_{\lambda/d/\mu}(\nu)$ contributing to the coefficient of $m_\lambda(x_1,\ldots,x_k)$ in $m_\mu(x_1,\ldots,x_k)p_\nu(x_1,\ldots,x_k)$.

Proof. In analogy with the proofs of Lemmas 3.3.8 and 3.3.9 we show for $r \in \mathbb{N}$ the following identity in $\mathcal{V}_k(n)$

$$m_\mu(x_1,\ldots,x_k)p_r(x_1,\ldots,x_k) = \sum_{d\in\mathbb{Z}_{\geq 0}} z^d \sum_{\lambda\in\mathcal{A}_k^+(n)} \varphi_{\lambda/d/\mu} m_\lambda(x_1,\ldots,x_k)\,, \tag{3.79}$$

where the second sum runs over all $\lambda \in \mathcal{A}_k^+(n)$ for which $\lambda/d/\mu$ is a CACSD with $|\mu| + r - |\lambda| = dn$. The claim then follows by applying repeatedly the equality above to the product $m_\mu(x_1,\ldots,x_k)p_\nu(x_1,\ldots,x_k)$. This also implies that $\varphi_{\lambda/d/\mu}(\beta) = \varphi_{\lambda/d/\mu}(\nu)$ for $\beta \sim \nu$, where $\varphi_{\lambda/d/\mu}(\beta)$ for β a composition is defined in an analogous way to (3.77).

We first show that it is enough to prove (3.79) for $1 \leq r \leq n$. For this purpose suppose that $r > n$ and write $r = sn + q$ for some $s \geq 1$ and $1 \leq q \leq n$. Let $\lambda \in \mathcal{A}_k^+(n)$ and $d \in \mathbb{Z}_{\geq 0}$ such that $\hat\lambda.\tau^d = \hat\mu_{a,r}$ for some $1 \leq a \leq k$ with $m_{a-1}(\hat\mu) \neq 0$. From the discussion of Lemma 3.3.21 it follows that $(\hat\lambda.\tau^d)'_1 = k + d = (\hat\mu_{a,r})'_1 \geq k + s$, and thus we must have $d \geq s$. The same lemma implies that $\hat\lambda.\tau^d = \hat\mu_{a,r}$ if and only if $\hat\lambda.\tau^{d-s} = \hat\mu_{a,q}$. In other words, $\lambda/d/\mu$ is a CACSD if and only if $\lambda/(d-s)/\mu$ is, and moreover the equality $m_{a-1+r}(\hat\lambda.\tau^d) = m_{a-1+r}(\hat\lambda.\tau^{d-s})$ implies that $\varphi_{\lambda/d/\mu} = \varphi_{\lambda/(d-s)/\mu}$. Thus the RHS of (3.79) equals

$$\begin{aligned}\sum_{d\geq s} z^d \sum_{\lambda\in\mathcal{A}_k^+(n)} \varphi_{\lambda/d/\mu} m_\lambda(x_1,\ldots,x_k) &= z^s \sum_{d\geq s} z^{d-s} \sum_{\lambda\in\mathcal{A}_k^+(n)} \varphi_{\lambda/(d-s)/\mu} m_\lambda(x_1,\ldots,x_k) \\ &= z^s \sum_{d'\geq 0} z^{d'} \sum_{\lambda\in\mathcal{A}_k^+(n)} \varphi_{\lambda/d'/\mu} m_\lambda(x_1,\ldots,x_k)\,, \end{aligned}\tag{3.80}$$

where in the last equality the second sum runs over all $\lambda \in \mathcal{A}_k^+(n)$ for which $\lambda/d'/\mu$ is a CACSD with $|\mu| + q - |\lambda| = d'n$. On the other hand we have that $p_r(x_1,\ldots,x_k) - z^s p_q(x_1,\ldots,x_k)$ is in the ideal $\mathcal{I}_{k,n}$ introduced in (3.35). This shows that the product $m_\mu(x_1,\ldots,x_k)p_r(x_1,\ldots,x_k)$ equals (3.80) provided that (3.79) has been proved for $1 \leq r \leq n$. Suppose now that $r = n$. Then (3.79) follows since $m_\mu(x_1,\ldots,x_k)p_n(x_1,\ldots,x_k) = zkm_\mu(x_1,\ldots,x_k)$, and moreover the only weight $\lambda \in \mathcal{A}_k^+(n)$ such that $\hat{\lambda}.\tau^d = \hat{\mu}_{a,n}$ for some $d \in \mathbb{Z}_{\geq 0}$ and $1 \leq a \leq n$ with $m_{a-1}(\hat{\mu}) \neq 0$ is $\lambda = \mu$, in which case we have that $d = 1$ and $\varphi_{\lambda/1/\mu} = k$.

So suppose that $1 \leq r \leq n-1$. We now prove that for $\lambda \in \mathcal{A}_k^+(n)$ the coefficient of $x_1^{\lambda_1}\cdots x_k^{\lambda_k}$ in $m_\mu(x_1,\ldots,x_k)p_r(x_1,\ldots,x_k)$ is non-zero if and only if $\hat{\lambda}.\tau^d = \hat{\mu}_{a,r}$ for some $1 \leq a \leq n$ with $m_{a-1}(\hat{\mu}) \neq 0$ and $d = 0, 1$. Each monomial appearing in $m_\mu(x_1,\ldots,x_k)p_r(x_1,\ldots,x_k)$ is of the form $x_1^{\mu_{w(1)}}\cdots x_l^{\mu_{w(l)}+r}\cdots x_k^{\mu_{w(k)}}$ for some $w \in S^\mu$ and $1 \leq l \leq k$. Assume that the latter equals $z^d x_1^{\lambda_1}\cdots x_k^{\lambda_k}$ for some $d \in \mathbb{Z}_{\geq 0}$. Then λ is obtained from μ by removing a part equal to $\mu_{w(l)}$ and adding a part equal to $(\mu_{w(l)}+r) \operatorname{Mod} n$, which also implies that $d = 0$ for $\mu_{w(l)}+r \leq n$ and $d = 1$ otherwise. Setting $a = (\mu_{w(l)}+1) \operatorname{Mod} n$ we have that $\hat{\lambda}.\tau^d = \hat{\mu}_{a,r}$ thanks to (3.75), and this proves the 'if' part of the statement. For the 'only if' part, one can proceed in a similar fashion to the proof of Lemma 2.3.28.

Let $\lambda \in \mathcal{A}_k^+(n)$ and suppose that $\hat{\lambda}.\tau^d = \hat{\mu}_{a,r}$ for some $1 \leq a \leq n$ with $m_{a-1}(\hat{\mu}) \neq 0$ and $d = 0, 1$. Assume that $a-1+r \leq n$, and let i and j be the smallest indices for which $\mu_i < a-1+r$ and $\mu_j = a-1$ respectively. Then the monomials in $m_\mu(x_1,\ldots,x_k)p_r(x_1,\ldots,x_k)$ which equal $x_1^{\lambda_1}\cdots x_k^{\lambda_k}$ are of the form

$$x_1^{\mu_1}\cdots x_{i-l-1}^{\mu_{i-l-1}} x_{i-l}^{\mu_j+r} x_{i-l+1}^{\mu_{i-l}}\cdots x_j^{\mu_{j-1}} x_{j+1}^{\mu_{j+1}}\cdots x_k^{\mu_k}$$

for $l = 0,\ldots,m_{a-1+r}(\mu)$. Assume now that $a-1+r > n$, let i be the smallest index for which $\mu_i < a-1+r-n$ and let j be the greatest index for which $\mu_j = a-1$. Then the monomials in $m_\mu(x_1,\ldots,x_k)p_r(x_1,\ldots,x_k)$ which equal $zx_1^{\lambda_1}\cdots x_k^{\lambda_k}$ are of the form

$$x_1^{\mu_1}\cdots x_{j-1}^{\mu_{j-1}} x_j^{\mu_{j+1}}\cdots x_{i-l-2}^{\mu_{i-l-1}} x_{i-l-1}^{\mu_j+r} x_{i-l}^{\mu_{i-l}}\cdots x_k^{\mu_k}$$

for $l = 0,\ldots,m_{a-1+r-n}(\mu)$. In conclusion we have that

$$\begin{aligned} m_\mu(x_1,\ldots,x_k)p_r(x_1,\ldots,x_k) &= \sum_a (m_{a-1+r}(\mu)+1)m_\lambda(x_1,\ldots,x_k) \\ &\qquad +z\sum_a (m_{a-1+r-n}(\mu)+1)m_\lambda(x_1,\ldots,x_k)\,, \end{aligned}$$

where $\lambda \in \mathcal{A}_k^+(n)$ is defined via $\hat{\lambda} = \hat{\mu}_{a,r}$ in the first sum, and via $\hat{\lambda}.\tau = \hat{\mu}_{a,r}$ in the second sum. Both sums run over all $1 \leq a \leq n$ such that $m_{a-1}(\hat{\mu}) \neq 0$, together with the constraint $a-1+r \leq n$ for the first one, and the constraint $a-1+r > n$ for the second one. This finally implies the validity of (3.79) after applying the definition (3.76) of $\varphi_{\lambda/d/\mu}$. $\square$

We now generalise (2.74) to the cylinder, namely we give an alternative combinatorial description of the coefficient $\varphi_{\lambda/d/\mu}(\nu)$. For this purpose recall the expansion $p_\lambda = \sum_{\mu\in\mathcal{P}^+} R_{\lambda\mu} m_\mu$ in Λ, which was introduced in (2.33).

Lemma 3.3.27. *Let $\lambda, \mu \in \mathcal{A}_k^+(n)$, $d \in \mathbb{Z}_{\geq 0}$ and $\nu \in \mathcal{P}$. The following equality holds*

$$\varphi_{\lambda/d/\mu}(\nu) = \sum_{\sigma\in\mathcal{P}_k^+} N_{\mu\sigma}^{\lambda,d} R_{\nu\sigma}\ . \tag{3.81}$$

Proof. This statement follows by employing similar steps as the ones described in the proof of Proposition 3.3.10. Plug the expansion $p_\nu(x_1,\dots,x_k) = \sum_{\sigma\in\mathcal{P}_k^+} R_{\nu\sigma} m_\sigma(x_1,\dots,x_k)$ into $m_\mu(x_1,\dots,x_k)p_\nu(x_1,\dots,x_k)$, take the quotient map $\pi_{k,n} : \Lambda_k[z,z^{-1}] \to \mathcal{V}_k(n)$ and use the product expansion (3.40). A comparison with (3.78) then yields the desired equality. $\square$

Proposition 3.3.28 ([45]). *Let $\lambda, \mu \in \mathcal{A}_k^+(n)$ and $d \in \mathbb{Z}_{\geq 0}$. We have the expansions*

$$e_{\lambda/d/\mu} = \sum_{\nu\in\mathcal{P}^+} \varphi_{\lambda/d/\mu}(\nu) z_\nu^{-1} \epsilon_\nu p_\nu\ , \tag{3.82}$$

$$h_{\lambda/d/\mu} = \sum_{\nu\in\mathcal{P}^+} \varphi_{\lambda/d/\mu}(\nu) z_\nu^{-1} p_\nu\ , \tag{3.83}$$

into the basis $\{p_\nu\}_{\nu\in\mathcal{P}^+}$ of $\Lambda_{\mathbb{Q}}$.

Proof. Using (3.81) we have that $N_{\mu\nu}^{\lambda,d} = \sum_{\sigma\in\mathcal{P}^+} \varphi_{\lambda/d/\mu}(\sigma) R_{\nu\sigma}^{-1}$ for $\nu \in \mathcal{P}_k^+$. Furthermore we have the relation $\sum_{\nu\in\mathcal{P}_k^+} R_{\nu\sigma}^{-1} e_\nu = z_\sigma e_\sigma p_\sigma$, see for example [52, page 104]. Plugging these equalities into the expansion $e_{\lambda/d/\mu} = \sum_{\nu\in\mathcal{P}_k^+} N_{\mu\nu}^{\lambda,d} e_\nu$ we end up with (3.82). Equation (3.83) follows by applying the involution ω to both sides of (3.82). $\square$

3.4 Properties of cylindric symmetric functions

We start this section by evaluating the coproduct of the cylindric symmetric functions $e_{\lambda/d/\mu}$ and $h_{\lambda/d/\mu}$, and then by exploring some consequences of this computation. In particular, we shall identify certain subspaces as subcoalgebras of Λ, whose structure constants are given by the coefficients $N_{\mu\nu}^{\lambda,d}$ (see Definition 3.3.3) for $\lambda, \mu, \nu \in \mathcal{A}_k^+(n)$. In Chapter 5 we will identify these coefficients as the structure constants of a Frobenius algebra, i.e. a 2D TQFT.

Proposition 3.4.1 ([45]). *Let $\lambda, \mu \in \mathcal{A}_k^+(n)$ and $d \in \mathbb{Z}_{\geq 0}$. We have the following coprod-*

ucts of cylindric symmetric functions

$$\Delta(e_{\lambda/d/\mu}) = \sum_{\substack{d_1+d_2=d\\d_1,d_2\geq 0}} \sum_{\nu\in\mathcal{A}_k^+(n)} e_{\lambda/d_1/\nu}\otimes e_{\nu/d_2/\mu}\ , \tag{3.84}$$

$$\Delta(h_{\lambda/d/\mu}) = \sum_{\substack{d_1+d_2=d\\d_1,d_2\geq 0}} \sum_{\nu\in\mathcal{A}_k^+(n)} h_{\lambda/d_1/\nu}\otimes h_{\nu/d_2/\mu}\ . \tag{3.85}$$

Proof. Consider the chain of equalities

$$\begin{aligned}\Delta(e_{\lambda/d/\mu}) &= \sum_{\nu\in\mathcal{P}_k^+} N_{\mu\nu}^{\lambda,d}\Delta(e_\nu) = \sum_{\nu\in\mathcal{P}_k^+} N_{\mu\nu}^{\lambda,d}\sum_{\rho,\sigma\in\mathcal{P}_k^+} f_{\rho\sigma}^{\nu} e_\rho\otimes e_\sigma\\ &= \sum_{\rho,\sigma\in\mathcal{P}_k^+}\left(\sum_{\nu\in\mathcal{P}_k^+} f_{\rho\sigma}^{\nu}N_{\mu\nu}^{\lambda,d}\right)e_\rho\otimes e_\sigma\\ &= \sum_{\substack{d_1+d_2=d\\d_1,d_2\geq 0}} \sum_{\nu\in\mathcal{A}_k^+(n)}\left(\sum_{\rho\in\mathcal{P}_k^+} N_{\nu\rho}^{\lambda,d_1}e_\rho\right)\otimes\left(\sum_{\sigma\in\mathcal{P}_k^+} N_{\mu\sigma}^{\nu,d_2}e_\sigma\right).\end{aligned}$$

In the first line we used (3.58) and then we took advantage of the identities $\Delta(e_\nu) = \sum_{\sigma\in\mathcal{P}_k^+} e_{\nu/\sigma}\otimes e_\sigma$ and $e_{\nu/\sigma} = \sum_{\rho\in\mathcal{P}_k^+} f_{\sigma\rho}^{\nu}e_\rho$, which were introduced in (2.54) and (2.57) respectively. Notice that we restricted these sums to $\sigma,\rho\in\mathcal{P}_k^+$ since the coefficient $f_{\sigma\rho}^{\nu}$ is non-zero only if $\sigma,\rho\subset\nu\in\mathcal{P}_k^+$. In the third line we used Properties 2 and 3 in Lemma 5.2.5, and then we rearranged terms appropriately. Exploiting once again (3.58) in the last line, equation (3.84) follows. One can prove (3.85) in an analogous way.

□

Corollary 3.4.2. *Let $\lambda,\mu\in\mathcal{A}_k^+(n)$ and $d\in\mathbb{Z}_{\geq 0}$. We have the identities*

$$\sum_{\substack{d_1+d_2=d\\d_1,d_2\geq 0}} \sum_{\nu\in\mathcal{A}_k^+(n)} (-1)^{|\nu|+nd_2-|\mu|}e_{\lambda/d_1/\nu}\,h_{\nu/d_2/\mu} = \delta_{\lambda\mu}\delta_{d,0}\ , \tag{3.86}$$

$$\sum_{\substack{d_1+d_2=d\\d_1,d_2\geq 0}} \sum_{\nu\in\mathcal{A}_k^+(n)} (-1)^{|\lambda|+nd_1-|\nu|}h_{\lambda/d_1/\nu}\,e_{\nu/d_2/\mu} = \delta_{\lambda\mu}\delta_{d,0}\ . \tag{3.87}$$

Proof. Taking advantage of Theorem 3.3.12 we end up with $\gamma(e_{\lambda/d/\mu}) = (-1)^{|\lambda|+dn-|\mu|}h_{\lambda/d/\mu}$ and $\gamma(h_{\lambda/d/\mu}) = (-1)^{|\lambda|+dn-|\mu|}e_{\lambda/d/\mu}$, where γ is the antipode of Λ defined in Section 2.2.6. The claim then follows after applying the defining relations (2.44) of the antipode to $e_{\lambda/d/\mu}$. Had we started with $h_{\lambda/d/\mu}$ we would have reached the same equalities.

□

Corollary 3.4.3 ([45]). *The respective subspaces spanned by* (3.67) *and* (3.68) *each form a positive subcoalgebra of Λ with structure constants $N_{\mu\nu}^{\lambda,d}$ for $\lambda,\mu,\nu\in\mathcal{A}_k^+(n)$. That is, for*

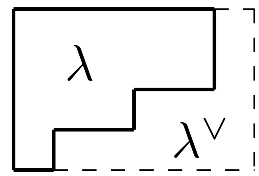


Figure 3.10: Let $\lambda = (5,5,3,1) \in \mathcal{A}_4^+(5)$. Then $\lambda^\vee = (5,3,1,1)$ is the complementary partition of λ in the bounding box of height 4 and width 6 shown in the Figure.

$\lambda \in \mathcal{A}_k^+(n)$ and $d \in \mathbb{Z}$ with $d \geq -m_n(\lambda)$ we have the coproduct expansion

$$\Delta(e_{\hat{\lambda}.\tau^d/\hat{\emptyset}}) = \sum_{\substack{d_1+d_2=d \\ d_1,d_2\geq -k}} \sum_{\mu\in\mathcal{A}_k^+(n)} e_{\lambda/d_1/\mu} \otimes e_{\hat{\mu}.\tau^{d_2}/\hat{\emptyset}} \,, \tag{3.88}$$

$$e_{\lambda/d_1/\mu} = \sum_{d_1'=0}^{d_1+n} \sum_{\nu\in\mathcal{A}_k^+(n)} N_{\mu\nu}^{\lambda,d_1'} e_{\hat{\nu}.\tau^{d_1-d_1'}/\hat{\emptyset}} \,, \tag{3.89}$$

and the analogous coproduct expansion holds for $h_{\hat{\lambda}.\tau^d/\hat{\emptyset}}$.

Proof. The claim is a direct consequence of Corollary 3.3.13 and Proposition 3.4.1, together with the equalities (3.63) and (3.64). □

Remark 3.4.4. Setting $e_{\hat{\nu}.\tau^{d_1-d_1'}/\hat{\emptyset}} = e_{\nu/(d_1-d_1')/\emptyset}$, we have that (3.89) corresponds to the first expansion in (1.16). In other words, the coefficients $N_{\mu\nu}^{\lambda,d}$ for $\lambda,\mu,\nu \in \mathcal{A}_k^+(n)$ coincide with the structure constants $\mathcal{N}_{\mu\nu}^{\lambda,d}(q)$ of the deformed Verlinde algebra discussed in Section 1.3, when evaluated at $q = 1$. Compare with Remark 5.2.4 in Chapter 5.

3.4.1 An involution between cylindric reverse plane partitions

Define now the map $\vee : \mathcal{A}_k^+(n) \to \mathcal{A}_k^+(n)$ by

$$\lambda \mapsto \lambda^\vee = (n+1-\lambda_k, n+1-\lambda_{k-1}, \ldots, n+1-\lambda_1) \,. \tag{3.90}$$

That is, $\lambda^\vee$ is the complementary partition of λ in a bounding box of height k and width $n+1$ (see Figure 3.10), and thus $\vee : \mathcal{A}_k^+(n) \to \mathcal{A}_k^+(n)$ is an involution. This map will play an important role in the next chapter, so we spend the rest of this section to describe some of its properties. The parts of the cylindric partition $\hat{\lambda}^\vee$ are given by $\hat{\lambda}_i^\vee = n+1-\hat{\lambda}_{k+1-i}$. In fact, for $1 \leq i \leq k$ this matches with the definition of $\lambda^\vee$ above, and moreover $\hat{\lambda}_{i+k}^\vee = n+1-(\hat{\lambda}_{k+1-i-k}) = n+1-\hat{\lambda}_{k+1-i}-n = \hat{\lambda}_i^\vee - n$ for $i \in \mathbb{Z}$. Similarly the parts of the conjugate cylindric partition $\hat{\lambda}^{\vee\prime}$ are given by $\hat{\lambda}_i^{\vee\prime} = k - \hat{\lambda}_{n+2-i}'$. This holds for $1 \leq i \leq n$ since λ and $\lambda^\vee$ fit in a bounding box of height k and width $n+1$, and furthermore $\hat{\lambda}_{i+n}^{\vee\prime} = k - (\hat{\lambda}_{n+2-i-n}') = k - \hat{\lambda}_{n+2-i}' - k = \hat{\lambda}_i^{\vee\prime} - k$ for $i \in \mathbb{Z}$. We will make use of the identities

$$(\hat{\lambda}.\tau^d)_i - \hat{\mu}_i = (\hat{\mu}^\vee.\tau^d)_{k+1-i+d} - \hat{\lambda}^\vee_{k+1-i+d} , \tag{3.91}$$
$$(\hat{\lambda}.\tau^d)'_i - \hat{\mu}'_i = (\hat{\mu}^\vee.\tau^d)'_{n+2-i} - \hat{\lambda}^{\vee\prime}_{n+2-i} , \tag{3.92}$$

which follow after a straightforward computation using $(\hat{\lambda}.\tau^d)_i = \hat{\lambda}_{i-d}$ and $(\hat{\lambda}.\tau^d)'_i = \hat{\lambda}'_i + d$ respectively.

Lemma 3.4.5. *Let $\lambda, \mu \in \mathcal{A}^+_k(n)$ and $d \in \mathbb{Z}_{\geq 0}$. Then $\hat{\mu} \subset \hat{\lambda}.\tau^d$ if and only if $\hat{\lambda}^\vee \subset \hat{\mu}^\vee.\tau^d$, and in particular $\lambda/d/\mu$ is a cylindric vertical strip if and only if $\mu^\vee/d/\lambda^\vee$ is. Moreover $\lambda/d/\mu$ is a CACSD if and only if $\mu^\vee/d/\lambda^\vee$ is. We have the identities*

$$\theta_{\lambda/d/\mu} = \frac{|S_\lambda|}{|S_\mu|}\theta_{\mu^\vee/d/\lambda^\vee} , \qquad \psi_{\lambda/d/\mu} = \frac{|S_\lambda|}{|S_\mu|}\psi_{\mu^\vee/d/\lambda^\vee} , \qquad \varphi_{\lambda/d/\mu} = \frac{|S_\lambda|}{|S_\mu|}\varphi_{\mu^\vee/d/\lambda^\vee} , \tag{3.93}$$

where in the last one it is understood that $\varphi_{\lambda/d/\mu} = \varphi_{\mu^\vee/d/\lambda^\vee} = 0$ if $\lambda/d/\mu$ is not a CACSD.

Proof. Equation (3.91) implies the first statement of the lemma. For the part involving the CACSDs, suppose that $\hat{\lambda}.\tau^d = \hat{\mu}_{a,r}$ for some $1 \leq a \leq n$ with $m_{a-1}(\hat{\mu}) \neq 0$ and $r \leq n$. Taking advantage of (3.73) and (3.92) we end up with

$$(\hat{\mu}^\vee.\tau^d)'_i = \begin{cases} \hat{\lambda}^{\vee\prime}_i + 1 , & n+2-(a+r-1)+pn \leq i \leq n+2-a+pn,\, p \in \mathbb{Z} \\ \hat{\lambda}^{\vee\prime}_i , & \text{otherwise} \end{cases} .$$

That is, $\hat{\mu}^\vee.\tau^d = \hat{\lambda}^\vee_{n+2-(a+r-1)\,\mathrm{Mod}\,n,r}$ and thus $\mu^\vee/d/\lambda^\vee$ is a CACSD (see Figure 3.11 for an example). Now suppose that $r > n$ and write $r = sn + q$ for some $s \geq 1$ and $1 \leq q \leq n$. Lemma 3.3.21 implies that $\hat{\lambda}.\tau^d = \hat{\mu}_{a,r}$ if and only if $\hat{\lambda}.\tau^{d-s} = \hat{\mu}_{a,q}$. Since $q \leq n$ we have that $\hat{\mu}^\vee.\tau^{d-s} = \hat{\lambda}^\vee_{n+2-(a+q-1)\,\mathrm{Mod}\,n,q}$, that is $\mu^\vee/(d-s)/\lambda^\vee$ is a CACSD, and applying Lemma 3.3.21 once again we conclude that $\mu^\vee/d/\lambda^\vee$ is a CACSD as well. This finally proves the second statement of the lemma thanks to the fact that $\vee : \mathcal{A}^+_k(n) \to \mathcal{A}^+_k(n)$ is an involution. The first identity in (3.93) is identically 0 if $\lambda/d/\mu$ is not a cylindric skew diagram, as Proposition 3.2.7 then implies that $\theta_{\lambda/d/\mu} = \theta_{\mu^\vee/d/\lambda^\vee} = 0$. Otherwise this follows after a manipulation of (3.25), with the help of $(\hat{\lambda}.\tau^d)'_i = \hat{\lambda}'_i + d$ and $\hat{\lambda}'_i = k - \hat{\lambda}^{\vee\prime}_{n+2-i}$. One can prove the second identity in (3.93) in a similar fashion. Assuming that $\lambda/d/\mu$ is a CACSD, that is $\hat{\lambda}.\tau^d = \hat{\mu}_{a,r}$ for some $1 \leq a \leq n$ and $r \in \mathbb{N}$ with $m_{a-1}(\hat{\mu}) \neq 0$, the third identity follows by noticing that

$$\frac{|S_\lambda|}{|S_\mu|} = \frac{m_{a+r-1}(\hat{\lambda})}{m_{a-1}(\hat{\mu})} = \frac{m_{a+r-1}(\hat{\lambda}.\tau^d)}{m_{n+2-a}(\hat{\mu}^\vee.\tau^d)} = \frac{\varphi_{\lambda/d/\mu}}{\varphi_{\mu^\vee/d/\lambda^\vee}} ,$$

where we used the equality $m_{a-1}(\hat{\mu}) = \hat{\mu}'_{a-1} - \hat{\mu}'_a = \hat{\mu}^{\vee\prime}_{n+2-a} - \hat{\mu}^{\vee\prime}_{n+3-a} = m_{n+2-a}(\hat{\mu}^\vee)$. □

Let $\Pi_{k,n}$ be the set of all CRPPs, and consider the map $\vee : \Pi_{k,n} \to \Pi_{k,n}$ which sends

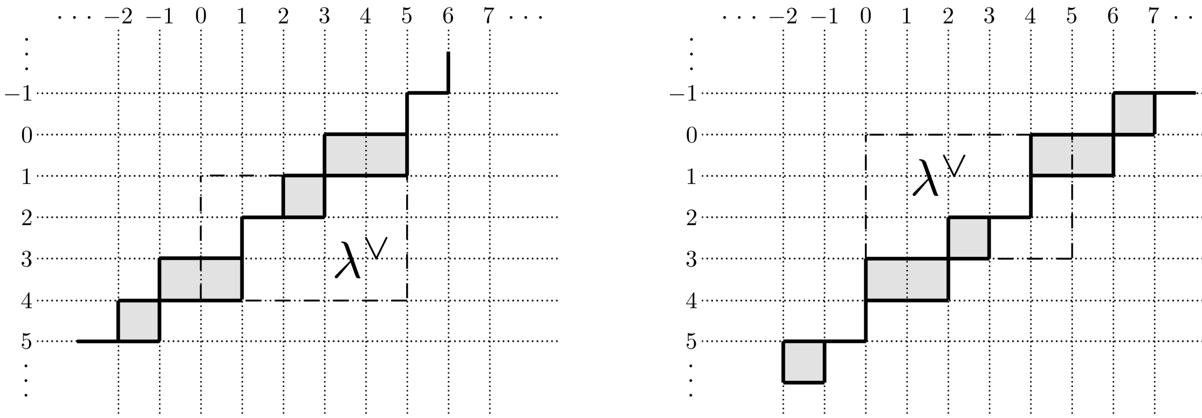


Figure 3.11: Let $\lambda, \mu \in \mathcal{A}_3^+(4)$ given by $\lambda = (3,1,1)$, $\mu = (3,2,1)$ and let $d = 1$. On the left we have the CACSD $\lambda/d/\mu$, whereas on the right we have the CACSD $\mu^\vee/d/\lambda^\vee$, where $\mu^\vee = (4,3,2)$ and $\lambda^\vee = (4,4,2)$. In the $\mathbb{Z} \times \mathbb{Z}$ plane on the left there is highlighted a bounding box of height 3 and width 5 containing λ and $\lambda^\vee$, whose top-left corner is on the point $(0,d)$. The same bounding box is reproduced on the right, rotated by 180° and with its top-left corner on the point $(0,0)$. Notice that the cylindric skew diagram on the right is obtained from the one on the left by rotation of 180°. Furthermore, whereas $\hat{\lambda}.\tau = \hat{\mu}_{3,3}$ we have that $\hat{\mu}^\vee.\tau = \hat{\lambda}^\vee_{4+2-(3+3-1)\,\mathrm{Mod}\,4,3} = \hat{\lambda}^\vee_{1,3}$.

the CRPP $\hat{\pi}$ of shape $\lambda/d/\mu$ given by

$$\hat{\mu} = \hat{\lambda}^{(0)}.\tau^{d_0} \subset \hat{\lambda}^{(1)}.\tau^{d_1} \subset \cdots \subset \hat{\lambda}^{(l)}.\tau^{d_l} = \hat{\lambda}.\tau^d$$

to the CRPP $\hat{\pi}^\vee$ of shape $\mu^\vee/d/\lambda^\vee$ given by

$$\hat{\lambda}^{(l)^\vee} = \hat{\lambda}^\vee \subset \hat{\lambda}^{(l-1)^\vee}.\tau^{d-d_{l-1}} \subset \cdots \subset \hat{\lambda}^{(1)^\vee}.\tau^{d-d_1} \subset \hat{\lambda}^{(0)^\vee}.\tau^{d-d_0} = \hat{\mu}^\vee.\tau^d\ .$$

This map is well defined thanks to Lemma 3.4.5, and it is an involution since $\vee : \mathcal{A}_k^+(n) \to \mathcal{A}_k^+(n)$ is. See Figure 3.12 for an example. In particular, Lemma 3.4.5 implies that $\hat{T}$ and $\hat{\pi}$ are respectively a CRST and a CACRPP if and only if $\hat{T}^\vee$ and $\hat{\pi}^\vee$ are. Notice that if $\hat{\pi}$ has weight $\big(\mathrm{wt}(\hat{\pi})_1, \ldots, \mathrm{wt}(\hat{\pi})_l\big)$ then $\hat{\pi}^\vee$ has weight $\big(\mathrm{wt}(\hat{\pi})_l, \ldots, \mathrm{wt}(\hat{\pi})_1\big)$. In fact, thanks to Lemma 3.1.9 and equation (3.91), we have the chain of equalities, for $r = 1, \ldots, l$,

$$\begin{aligned}\mathrm{wt}_r(\hat{\pi}^\vee) &= |\hat{\lambda}^{(l-r)^\vee}.\tau^{d-d_{l-r}}/\hat{\lambda}^{(l-r+1)^\vee}.\tau^{d-d_{l-r+1}}| \\ &= |\hat{\lambda}^{(l-r+1)}.\tau^{d_{l-r+1}}/\hat{\lambda}^{(l-r)}.\tau^{d_{l-r}}| = \mathrm{wt}_{l+1-r}(\hat{\pi})\ .\end{aligned}$$

Corollary 3.4.6. *Let $\lambda, \mu \in \mathcal{A}_k^+(n)$ and $d \in \mathbb{Z}_{\geq 0}$. Suppose that $\hat{\pi}_1$, $\hat{T}$ and $\hat{\pi}_2$ are respec-*

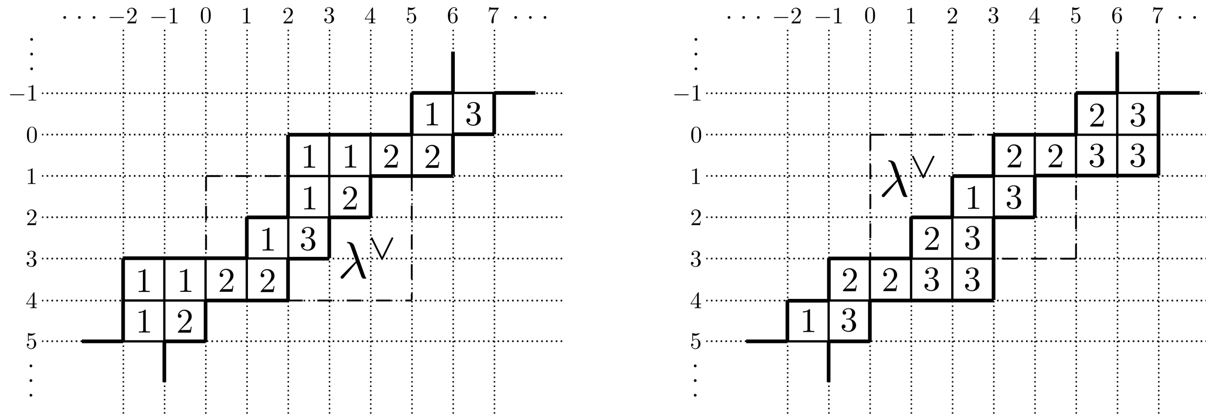


Figure 3.12: Let $\lambda, \mu \in \mathcal{A}_3^+(4)$ given by $\lambda = (4,3,2)$, $\mu = (2,2,1)$ and let $d = 1$. On the left we have a CRPP $\hat{\pi}$ of shape $\lambda/d/\mu$ and weight $(4,3,1)$, whereas on the right we have its image $\hat{\pi}^\vee$ under the map $\vee : \Pi_{k,n} \to \Pi_{k,n}$. This is a CRPP of shape $\mu^\vee/d/\lambda^\vee$ and weight $(1,3,4)$, where $\mu^\vee = (4,3,3)$ and $\lambda^\vee = (3,2,1)$. In the $\mathbb{Z}\times\mathbb{Z}$ plane on the left there is highlighted a bounding box of height 3 and width 5 containing λ and $\lambda^\vee$, whose top-left corner is on the point $(0,d)$. The same bounding box is reproduced on the right, rotated by 180° and with its top-left corner on the point $(0,0)$. Notice that $\hat{\pi}^\vee$ is obtained by first rotating $\hat{\pi}$ of 180°, and then applying the substitutions $1 \leftrightarrow 3$, $2 \leftrightarrow 2$ to the entries of the latter.

tively a CRPP, a CRST and a CACRPP of shape $\lambda/d/\mu$. We have the identities

$$\theta_{\hat{\pi}_1} = \frac{|S_\lambda|}{|S_\mu|}\theta_{\hat{\pi}_1^\vee}\ , \qquad \psi_{\hat{T}} = \frac{|S_\lambda|}{|S_\mu|}\psi_{\hat{T}^\vee}\ , \qquad \varphi_{\hat{\pi}_2} = \frac{|S_\lambda|}{|S_\mu|}\varphi_{\hat{\pi}_2^\vee}\ . \tag{3.94}$$

Proof. Using the first equality in (3.93) we have that

$$\begin{aligned}
\theta_{\hat{\pi}_1} &= \prod_{i=1}^{l} \theta_{\lambda^{(i)}/(d_i-d_{i-1})/\lambda^{(i-1)}} \\
&= \prod_{i=1}^{l} \frac{|S_{\lambda^{(i)}}|}{|S_{\lambda^{(i-1)}}|}\theta_{\lambda^{(i-1)\vee}/(d-d_{i-1})-(d-d_i)/\lambda^{(i)\vee}} \\
&= \frac{|S_\lambda|}{|S_\mu|}\prod_{i=1}^{l}\theta_{\lambda^{(i-1)\vee}/(d-d_{i-1})-(d-d_i)/\lambda^{(i)\vee}} = \frac{|S_\lambda|}{|S_\mu|}\theta_{\hat{\pi}_1^\vee}\ .
\end{aligned}$$

This prove the first identity in (3.94), whereas the other ones follow in a similar way. □

Proposition 3.4.7. *Let $\lambda, \mu \in \mathcal{A}_k^+(n)$ and $d \in \mathbb{Z}_{\geq 0}$. We have the identities*

$$e_{\lambda/d/\mu} = \frac{|S_\lambda|}{|S_\mu|}\, e_{\mu^\vee/d/\lambda^\vee}\ , \tag{3.95}$$

$$h_{\lambda/d/\mu} = \frac{|S_\lambda|}{|S_\mu|}\, h_{\mu^\vee/d/\lambda^\vee}\ . \tag{3.96}$$

Proof. As we noticed before, if the CRPP $\hat{\pi}$ of shape $\lambda/d/\mu$ has weight $\nu = (\nu_1, \dots, \nu_l)$ then the CRPP $\hat{\pi}^\vee$ of shape $\mu^\vee/d/\lambda^\vee$ has weight $(\nu_l, \dots, \nu_1)$. The first identity in (3.94) implies that

$$\theta_{\lambda/d/\mu}(\nu) = \sum_{\hat{\pi}} \theta_{\hat{\pi}} = \sum_{\hat{\pi}^\vee} \frac{|S_\lambda|}{|S_\mu|} \theta_{\hat{\pi}^\vee} = \frac{|S_\lambda|}{|S_\mu|} \theta_{\mu^\vee/d/\lambda^\vee}\big((\nu_l, \dots, \nu_1)\big) = \frac{|S_\lambda|}{|S_\mu|} \theta_{\mu^\vee/d/\lambda^\vee}(\nu) \,.$$

In the last equality we used the fact that $\theta_{\mu^\vee/d/\lambda^\vee}(\beta) = \theta_{\mu^\vee/d/\lambda^\vee}(\nu)$ for $\beta \sim \nu$. The expansion $h_{\lambda/d/\mu} = \sum_{\nu \in \mathcal{P}^+} \theta_{\lambda/d/\mu}(\nu) m_\nu$ then implies the validity of (3.96). Alternatively we could have proved the same identity by first showing that $\varphi_{\lambda/d/\mu}(\nu) = \frac{|S_\lambda|}{|S_\mu|} \varphi_{\mu^\vee/d/\lambda^\vee}(\nu)$ and then exploiting the expansion $h_{\lambda/d/\mu} = \sum_{\nu \in \mathcal{P}^+} \varphi_{\lambda/d/\mu}(\nu) z_\nu^{-1} p_\nu$. Equation (3.95) can be proved in a similar way. Namely one can show that $\psi_{\lambda/d/\mu}(\nu) = \frac{|S_\lambda|}{|S_\mu|} \psi_{\mu^\vee/d/\lambda^\vee}$, and the claim follows by applying the latter to the expansion $e_{\lambda/d/\mu} = \sum_{\nu \in \mathcal{P}^+} \psi_{\lambda/d/\mu}(\nu) m_\nu$.

□

Chapter 4

Quantum integrable systems

This chapter aims to make a connection between the cylindric symmetric functions $e_{\lambda/d/\mu}$ and $h_{\lambda/d/\mu}$ defined in Chapter 3, and the study of quantum integrable systems. In particular, we are interested in the vertex models defined by the Q^+ and Q^- operators associated to the q-boson model (compare with the discussion presented in Section 1.2), when evaluated at $q = 1$. We refer to the latter as the Q^+ and Q^- vertex models. We shall also consider two additional vertex models, which are related to the previous ones by taking the adjoint of the transfer matrices. We show that $e_{\lambda/d/\mu}$ and $h_{\lambda/d/\mu}$ can be identified with the partition functions of these vertex models with periodic boundary conditions in the horizontal direction. To this end, we provide a bijection between lattice configurations and CRPPs, as defined in Section 3.1.

The transfer matrices of the vertex models defined above commute with the Hamiltonian of the free boson model, which is the $q = 1$ specialisation of the q-boson model. We identify the matrix elements of the quantum integral of motions of the free boson model with the coefficients appearing in certain product expansions in $\mathcal{V}_k(n)$, the quotient of $\Lambda_k[z, z^{-1}]$ defined in Section 3.3. We exploit this observation to illustrate an alternative method for evaluating the expansions of $e_{\lambda/d/\mu}$ and $h_{\lambda/d/\mu}$ in terms of the bases of Λ introduced in Section 2.2. Then we present an alternative approach for computing the partition functions of the vertex models defined above. Finally, we employ the quantum integral of motions of the free boson model to endow the k-particle space with the structure of a unital, commutative and associative algebra.

4.1 Vertex models in statistical physics

We start with the formulation of the Q^+ and Q^- vertex models, and for the discussion presented in Sections 4.1.1, 4.1.2 and 4.1.3 we take inspiration from [40–43]. From here to the end of this thesis we assume, if not stated otherwise, that $l, n \in \mathbb{N}$ and that $k \in \mathbb{Z}_{\geq 0}$.

4.1.1 Vertex and lattice configurations

The Q^+ and Q^- vertex models are defined over a two dimensional lattice $\Gamma \subset \mathbb{Z} \times \mathbb{Z}$ with l rows and n columns, as the one depicted for example in Figure 4.2 (see [41, Ch. 4.1] for a formal definition). Denote with $\mathbb{E}$ the set of horizontal and vertical lattice edges, each edge consisting of two points in Γ.

Definition 4.1.1. A lattice configuration $\mathcal{C} : \mathbb{E} \to \mathbb{Z}_{\geq 0}$ is an assignment of non-negative integers to the lattice edges.

In particular, a vertex configuration is defined as a set of four non-negative integers $\{a, b, c, d\}$ attached to a point in Γ, called a 'vertex', and oriented as in Figure 4.1. To each vertex configuration we associate a (vertex) weight, and a vertex configuration is said to be 'allowed' if the related weight is non-zero. In Figure 4.1 we introduce the allowed vertex configurations for the $Q^{\pm}$ vertex models, together with the associated weights. The vertex weights for these models depend by definition on an indeterminate x. When evaluated in the complex numbers, this indeterminate is called a 'spectral parameter'. We assume that the weights of the vertices in the i-th row depend on the same indeterminate x_i. The weight of a lattice configuration $\mathcal{C}$ is defined as

$$\mathrm{wt}(\mathcal{C}) = \prod_{i=1}^{l} \prod_{j=1}^{n} \mathrm{wt}(v_{i,j}) , \tag{4.1}$$

where $v_{i,j}$ is the vertex obtained by intersecting the i-th row with the j-th column, and $\mathrm{wt}(v_{i,j})$ is the associated weight. For the sake of clarity, we will be using the symbols wt^+ and wt^- whenever we associate to a vertex configuration, or a lattice configuration, the weight defined for the Q^+ and Q^- vertex model respectively.

In the following we impose periodic boundary conditions in the horizontal direction, that is we identify the leftmost and rightmost edges in the same lattice row. In other words, we define the vertex models on a cylinder. We identify the boundary of this cylinder with a vertical line between columns n and 1.

Remark 4.1.2. The allowed lattice configurations for the $Q^{\pm}$ vertex models, that is the ones with non-zero weight, can be interpreted in terms of non-intersecting lattice paths travelling from North-West to South-East. See Figures 4.1 and 4.2. The constraint $a + b = c + d$ at each vertex of the lattice implies that on the cylinder the number of paths is conserved throughout the lattice. In other words, the number of vertical paths between rows i and $i + 1$ of the lattice is the same for every $i = 1, \ldots, l - 1$. This number coincides with the number of vertical paths above row 1, and the number of vertical paths below row l.

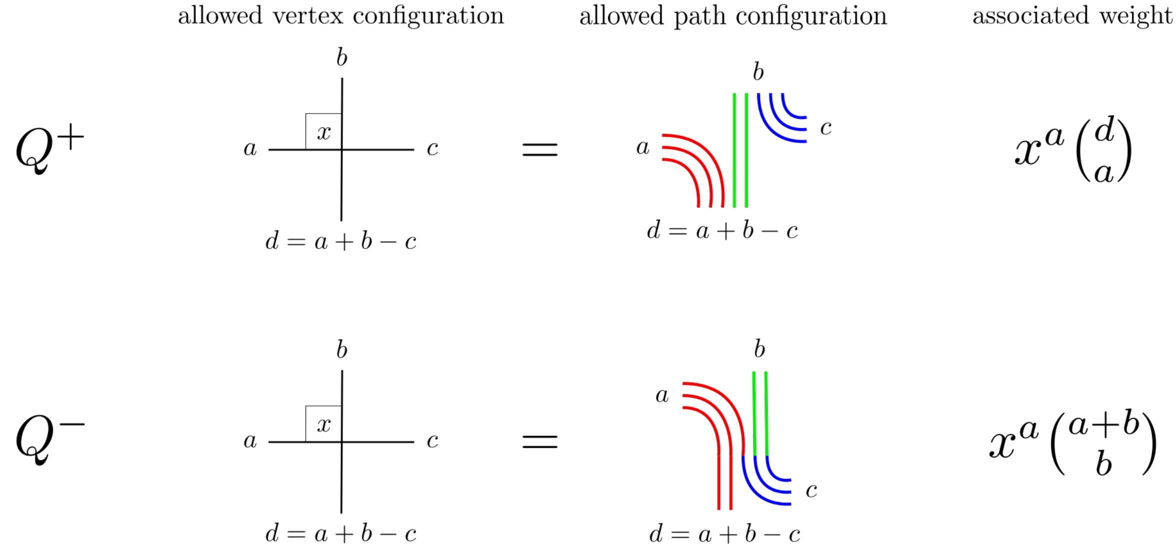


Figure 4.1: The allowed vertex configurations for the Q^+ and Q^- vertex models, together with their associated weights. For the Q^+ vertex model, we only allow for the vertex configurations such that $a + b = c + d$ and $b \geq c$, or equivalenty $d \geq a$. For the Q^- vertex model, a vertex configuration is 'allowed' if the only condition $a + b = c + d$ is satisfied. The allowed vertex configurations for these models can be interpreted in terms of non-intersecting paths travelling from North-West to South-East.

Given an allowed lattice configuration, let $k \in \mathbb{Z}_{\geq 0}$ be the number of non-intersecting paths. We fix the values of the outer vertical edges on top and bottom respectively with two partitions μ and λ belonging to the set $\mathcal{A}_k^+(n)$ (see Figure 4.2). The latter was introduced in (3.11), that is

$$\mathcal{A}_k^+(n) = \{\lambda \in \mathcal{P}_k \mid n \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0\} \; .$$

For $j = 1, \dots, n$, we have that $m_j(\mu)$ represents the number of paths starting from the j-th column of the lattice, or equivalently the value attached to the upmost edge of the same column. Similarly, $m_j(\lambda)$ represents the number of paths ending at the j-th column of the lattice, or equivalently the value attached the lowest edge of the same column.

A central object in the discussion of vertex models, which encodes all the physical properties of the models themselves, is the so called 'partition function'. Given $\lambda, \mu \in \mathcal{A}_k^+(n)$, denote with $\Gamma^+_{\lambda,\mu}(l)$ (respectively $\Gamma^-_{\lambda,\mu}(l)$) the set of allowed lattice configurations for the Q^+ (respectively Q^-) vertex model, where the values attached to the outer vertical edges on top and bottom are fixed respectively by μ and λ. The partition functions of the $Q^\pm$ vertex models, for the lattice with periodic boundary conditions in the horizontal direction, are defined as the weighted sums

$$Z^{\pm}_{\lambda,\mu}(x_1, \dots, x_l) = \sum_{\gamma^\pm \in \Gamma^\pm_{\lambda,\mu}(l)} z^d \, \mathrm{wt}^\pm(\gamma^\pm) \; . \tag{4.2}$$

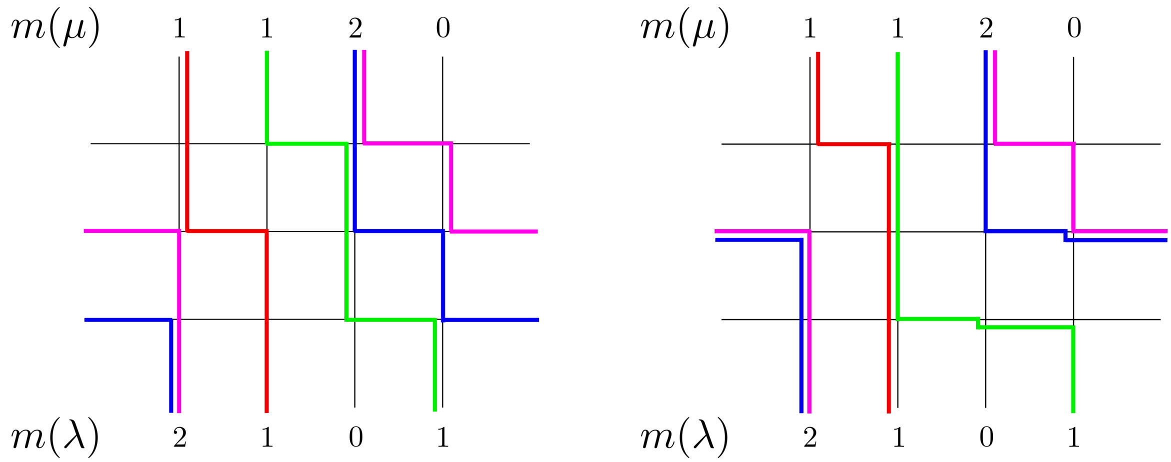


Figure 4.2: Allowed lattice configurations for the Q^+ vertex model (on the left) and for the Q^- vertex model (on the right), with periodic boundary conditions in the horizontal direction. The lattices have $l = 3$ rows and $n = 4$ columns. The lattice configurations consists of $k = 4$ non-intersecting paths. Two paths are crossing the boundary of the cylinder, and therefore $d = 2$ in the notation of equation (4.2). The upper and lower boundary conditions, that is the values attached to the outer vertical edges on top and bottom, are fixed respectively by the partitions $\mu = (3, 3, 2, 1)$ and $\lambda = (4, 2, 1, 1)$, which belong to $\mathcal{A}_4^+(4)$. We set $m(\mu) = (m_1(\mu), \ldots, m_k(\mu))$ and $m(\lambda) = (m_1(\lambda), \ldots, m_k(\lambda))$.

Here z is a formal variable, and for each lattice configuration $\gamma^\pm \in \Gamma^\pm_{\lambda,\mu}(l)$ we set $d = d(\gamma^\pm)$ to be the number of paths crossing the boundary of the cylinder. Compare with the notation introduced in [41, Eq. (5.1)].

Remark 4.1.3. The partition functions $Z^\pm_{\lambda,\mu}(x_1, \ldots, x_l)$ acquire a physical meaning when the indeterminates z and $(x_1, \ldots, x_l)$ are evaluated in the interval $(0, 1) \subset \mathbb{R}$. In this thesis we shall not discuss any physical properties related to the $Q^\pm$ vertex models. On the other hand, in Section 4.1.3 we provide a combinatorial interpretation for the partition functions (4.2), which relies on the cylindric symmetric functions defined in Chapter 3. For this reason, we will always regard $Z^\pm_{\lambda,\mu}(x_1, \ldots, x_l)$ as a formal power series in the indeterminates z and $(x_1, \ldots, x_l)$.

For $\lambda \in \mathcal{P}^+$ and $s \in \mathbb{Z}$ define the partition [45]

$$\operatorname{rot}^s \lambda = (1^{m_{1-s}(\lambda)}, 2^{m_{2-s}(\lambda)}, \ldots, n^{m_{n-s}(\lambda)}) \,, \tag{4.3}$$

where the indices are understood modulo n.

Lemma 4.1.4. *Let $\lambda, \mu \in \mathcal{A}_k^+(n)$ and $1 \le s \le n$. We have the identities*

$$Z^\pm_{\operatorname{rot}^{-s}\lambda, \operatorname{rot}^{-s}\mu}(x_1, \ldots, x_l) = z^{\sum_{i=1}^s (m_i(\mu) - m_i(\lambda))} Z^\pm_{\lambda,\mu}(x_1, \ldots, x_l) \,. \tag{4.4}$$

Proof. We shall only prove the claim for the Q^+ vertex model, since for the Q^- vertex model the claim follows in a completely analogous way. For this purpose, we first construct a bijection between the sets $\Gamma^+_{\lambda,\mu}(l)$ and $\Gamma^+_{\mathrm{rot}^{-s}\lambda,\mathrm{rot}^{-s}\mu}(l)$. Let $\gamma \in \Gamma^+_{\lambda,\mu}(l)$, and shift the paths in the horizontal direction by s units clockwise (compare with Figure 4.3). Denote by $\tilde{\gamma}$ the lattice configuration obtained via this shift, and notice that the upper and lower boundary conditions of $\tilde{\gamma}$ are fixed respectively by the partitions $\mathrm{rot}^{-s}\mu$ and $\mathrm{rot}^{-s}\lambda$. Since the Q^+ vertex model is defined on a cylinder, it follows that at each vertex of $\tilde{\gamma}$ we recover the path configuration depicted in Figure 4.1. This implies that $\tilde{\gamma}$ is an allowed lattice configuration for the Q^+ vertex model, and therefore $\tilde{\gamma} \in \Gamma^+_{\mathrm{rot}^{-s}\lambda,\mathrm{rot}^{-s}\mu}(l)$. In conclusion, each $\gamma \in \Gamma^+_{\lambda,\mu}(l)$ determines a unique element $\tilde{\gamma} \in \Gamma^+_{\mathrm{rot}^{-s}\lambda,\mathrm{rot}^{-s}\mu}(l)$, that is $\gamma \mapsto \tilde{\gamma}$ defines a map $\Gamma^+_{\lambda,\mu}(l) \to \Gamma^+_{\mathrm{rot}^{-s}\lambda,\mathrm{rot}^{-s}\mu}(l)$.

To show that $\Gamma^+_{\lambda,\mu}(l)$ and $\Gamma^+_{\mathrm{rot}^{-s}\lambda,\mathrm{rot}^{-s}\mu}(l)$ are in bijection, we need to create the inverse map $\Gamma^+_{\mathrm{rot}^{-s}\lambda,\mathrm{rot}^{-s}\mu}(l) \to \Gamma^+_{\lambda,\mu}(l)$. Let $\tilde{\gamma} \in \Gamma^+_{\mathrm{rot}^{-s}\lambda,\mathrm{rot}^{-s}\mu}(l)$, and shift the paths in the horizontal direction by s units counterclockwise. Denote by γ the lattice configuration obtained in this way. Following similar steps as the ones described above, one can show that $\gamma \in \Gamma^+_{\lambda,\mu}(l)$. The assignment $\tilde{\gamma} \mapsto \gamma$ therefore defines a map $\Gamma^+_{\mathrm{rot}^{-s}\lambda,\mathrm{rot}^{-s}\mu}(l) \to \Gamma^+_{\lambda,\mu}(l)$, which is clearly the inverse of the map $\Gamma^+_{\lambda,\mu}(l) \to \Gamma^+_{\mathrm{rot}^{-s}\lambda,\mathrm{rot}^{-s}\mu}(l)$ introduced above. We conclude that there exists a bijection between the sets $\Gamma^+_{\lambda,\mu}(l)$ and $\Gamma^+_{\mathrm{rot}^{-s}\lambda,\mathrm{rot}^{-s}\mu}(l)$.

Let $\gamma \in \Gamma^+_{\lambda,\mu}(l)$, and consider the lattice configuration $\tilde{\gamma} \in \Gamma^+_{\mathrm{rot}^{-s}\lambda,\mathrm{rot}^{-s}\mu}(l)$ which is the image under the map $\Gamma^+_{\lambda,\mu}(l) \to \Gamma^+_{\mathrm{rot}^{-s}\lambda,\mathrm{rot}^{-s}\mu}(l)$ described previously. Denote by $v_{i,j}$ the vertex obtained by intersecting the i-th row of the lattice with the j-th column. Employing the map $\Gamma^+_{\lambda,\mu}(l) \to \Gamma^+_{\mathrm{rot}^{-s}\lambda,\mathrm{rot}^{-s}\mu}(l)$ we have that the vertex configuration attached to $v_{i,j}$ becomes the vertex configuration attached to $v_{i,j-s}$, where the indices are understood modulo n. We can therefore take advantage of (4.1) to deduce the identity $\mathrm{wt}^+(\gamma) = \mathrm{wt}^+(\tilde{\gamma})$. By construction, the number of paths d associated to γ that are crossing the boundary of the cylinder coincides with the number of paths associated to $\tilde{\gamma}$ that are crossing the vertical line between columns $n-s$ and $n-s+1$. Let $\tilde{d}$ be the number of paths associated to $\tilde{\gamma}$ that are crossing the boundary of the cylinder. Notice that the constraint $a+b=c+d$ at each vertex of the lattice (compare with Figure 4.1) implies that

$$d + \sum_{i=n-s+1}^{n} m_i(\mathrm{rot}^{-s}\mu) = \tilde{d} + \sum_{i=n-s+1}^{n} m_i(\mathrm{rot}^{-s}\lambda)\ .$$

Starting from (4.2), and then employing this last identity, together with the relation $m_i(\mathrm{rot}^{-s}\lambda) = m_{i+s}(\lambda)$, one ends up with (4.4). $\square$

Remark 4.1.5. A natural question is how the partition functions (4.2) change if we identify the boundary of the cylinder with a vertical line between columns s and $s+1$, and then we redefine the number $d = d(\gamma^{\pm})$ as the number of paths crossing this new boundary. Employing the map $\tilde{\gamma} \mapsto \gamma$ defined in Lemma 4.1.4, we deduce that these

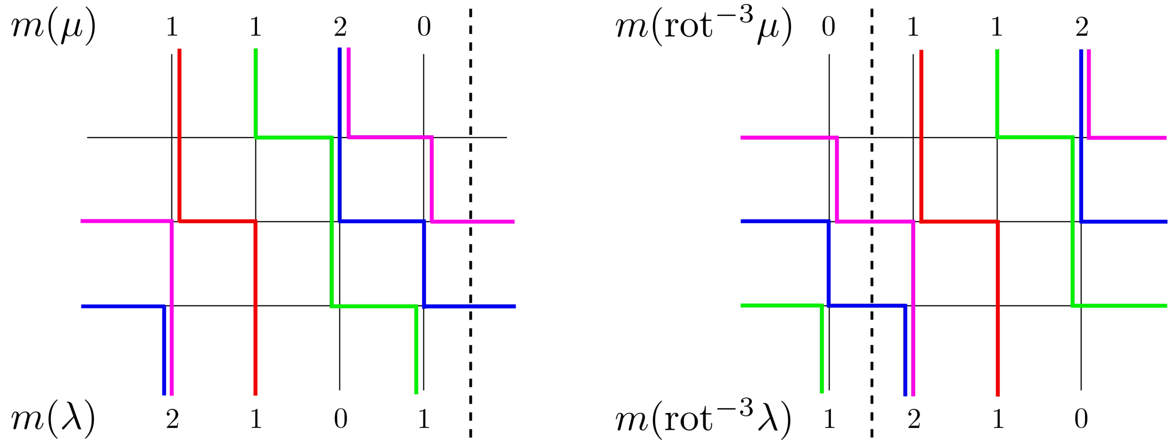


Figure 4.3: Let $l = 3$, $n = 4$, $k = 4$, $\mu = (3, 3, 2, 1)$ and $\lambda = (4, 2, 1, 1)$. On the left we have an allowed lattice configuration $\gamma \in \Gamma^{+}_{\lambda,\mu}(l)$ for the Q^{+} vertex model. On the right we have the allowed lattice configuration $\tilde{\gamma}$ for the Q^{+} vertex model which is obtained from γ by shifting the paths in the horizontal direction by 3 units clockwise. This is the image under the map $\Gamma^{+}_{\lambda,\mu}(l) \to \Gamma^{+}_{\mathrm{rot}^{-3}\lambda,\mathrm{rot}^{-3}\mu}(l)$ defined in the proof of Lemma 4.1.4. To ease the comparison, on the left we drew a vertical dashed line between columns n and 1, which corresponds to the boundary of the cylinder, and on the right we shifted this line by 3 units clockwise (that is, from right to left).

partition functions are given by $Z^{\pm}_{\mathrm{rot}^s\lambda,\mathrm{rot}^s\mu}(x_1, \dots, x_l)$, which equal

$$z^{\sum_{i=n-s+1}^{n}(m_i(\lambda)-m_i(\mu))} Z^{\pm}_{\lambda,\mu}(x_1, \dots, x_l)$$

thanks to (4.4) and the relation $m_i(\mathrm{rot}^s\lambda) = m_{i-s}(\lambda)$.

4.1.2 Solutions of the Yang-Baxter equation

As it is customary in the discussion of vertex models, we now wish to identify the vertex weights for the $Q^{\pm}$ vertex models as matrix elements of a vector space. For this purpose, we now interpret the values attached to the edges of the lattice as labels of basis vectors in the vector space $\mathcal{F} = \bigoplus_{m\in\mathbb{Z}_{\geq 0}} \mathbb{C}v_m$. We shall be using the the bra-ket notation from physics. Namely, we denote the vector $v_m \in \mathcal{F}$ with the 'ket' symbol $|m\rangle$, and the dual vector v^m belonging to the dual space $\tilde{\mathcal{F}} = \bigoplus_{m\in\mathbb{Z}_{\geq 0}} \mathbb{C}v^m$ with the 'bra' symbol $\langle m|$. The dual vectors are defined via the relation $v^m(v_{m'}) = \delta_{mm'}$, which corresponds to the 'braket' $\langle m|m'\rangle = \delta_{mm'}$ in the bra-ket notation.

Definition 4.1.6. The Heisenberg algebra $\mathcal{H}$ is the unital, associative $\mathbb{C}$-algebra generated by the two elements $\{b, b^*\}$ subject to the relation $[b, b^*] \equiv bb^* - b^*b = 1$.

Lemma 4.1.7. *The set $\{(b^*)^p b^q \mid p, q \in \mathbb{Z}_{\geq 0}\}$ forms a basis of $\mathcal{H}$.*

Proof. This is a well known result, see for example [65]. $\square$

Equip the vector space $\mathcal{F}$ with the map $\mathcal{H} \times \mathcal{F} \to \mathcal{F}$ defined as

$$\begin{aligned} b^* \left|m\right\rangle &= (m+1)\left|m+1\right\rangle \ , \\ b\left|m\right\rangle &= \left|m-1\right\rangle \ , \end{aligned}$$

together with $b\left|0\right\rangle = 0$. This map turns $\mathcal{F}$ into an infinite dimensional left module of the Heisenberg algebra $\mathcal{H}$ with highest vector $\left|0\right\rangle$. See for example [35, 5.1.1, Proposition 3] in the limit $q = 0$. Similarly, equip $\tilde{\mathcal{F}}$ with the map $\tilde{\mathcal{F}} \times \mathcal{H} \to \tilde{\mathcal{F}}$ defined as

$$\begin{aligned} \left\langle m\right| b^* &= \left\langle m-1\right| m \ , \\ \left\langle m\right| b &= \left\langle m+1\right| \ , \end{aligned}$$

together with $\left\langle 0\right| b^* = 0$. In this way, $\tilde{\mathcal{F}}$ becomes an infinite dimensional right module of the Heisenberg algebra with highest vector $\left\langle 0\right|$. As a consequence of the above actions, we have the relations

$$\left|m\right\rangle = \frac{(b^*)^m}{m!}\left|0\right\rangle \ , \qquad \text{and} \qquad \left\langle m\right| = \left\langle 0\right| b^m \ . \tag{4.5}$$

Let x be an indeterminate and set $\mathcal{F}((x)) = \mathbb{C}((x)) \otimes \mathcal{F}$. The L^+ and L^- operators are the operators $L^\pm(x) \in \mathrm{End}(\mathcal{F}((x))) \otimes \mathcal{H}$ defined via the relations

$$L^+(x)\left|m\right\rangle \otimes 1 = \sum_{m' \in \mathbb{Z}_{\geq 0}} \left|m'\right\rangle \otimes \frac{x^m}{m!}(b^*)^m b^{m'} \ , \tag{4.6}$$

$$L^-(x)\left|m\right\rangle \otimes 1 = \sum_{m' \in \mathbb{Z}_{\geq 0}} \left|m'\right\rangle \otimes \frac{x^m}{m!} b^{m'} (b^*)^m \ . \tag{4.7}$$

For $a, b \in \mathbb{Z}_{\geq 0}$, we shall use the notation $|a, b\rangle = |a\rangle \otimes |b\rangle$. Moreover, we set $\binom{a}{b} = 0$ whenever $a < b$.

Lemma 4.1.8. *The matrix elements of the $L^\pm$ operators are given by*

$$\langle c, d|L^+(x)|a, b\rangle = x^a \binom{d}{a} \delta_{a+b,c+d} \ , \tag{4.8}$$

$$\langle c, d|L^-(x)|a, b\rangle = x^a \binom{a+b}{b} \delta_{a+b,c+d} \ . \tag{4.9}$$

Proof. The claim follows after a straightforward computation. □

A comparison with Figure 4.1 shows that the matrix element (4.8) is non-zero if and only if the integers $\{a, b, c, d\}$ represent an allowed vertex configuration for the Q^+ vertex model. Similarly, the matrix element (4.9) is non-zero if and only if the vertex configuration defined by the integers $\{a, b, c, d\}$ is 'allowed' for the Q^- vertex model. If these matrix elements are non-zero, then they coincide with the vertex weights introduced in Figure 4.1. We can therefore identify the vertex weights for the $Q^{\pm}$ vertex models with the matrix elements of the $L^{\pm}$ operators. Let $P \in \mathrm{End}(\mathcal{F}\otimes\mathcal{F})$ be the flip operator $P\,|m_1,m_2\rangle = |m_2,m_1\rangle$. Define operators $R^{\pm} \in \mathrm{End}[\mathbb{C}((x,y))\otimes\mathcal{F}\otimes\mathcal{F}]$ via the relations

$$\langle m_1',m_2'|R^+(x/y)|m_1,m_2\rangle = \binom{m_2'}{m_1}\left(\frac{x}{y}\right)^{m_1}\left(1-\frac{x}{y}\right)^{m_2'-m_1}\delta_{m_1+m_2,m_1'+m_2'} \qquad (4.10)$$

and

$$R^-(x/y) = PR^+(y/x)P\;. \qquad (4.11)$$

Moreover, define $R \in \mathrm{End}[\mathbb{C}((x,y))\otimes\mathcal{F}\otimes\mathcal{F}]$ via

$$\langle m_1',m_2'|R(x/y)|m_1,m_2\rangle = \binom{m_1+m_2}{m_1}\left(\frac{x}{y}\right)^{m_1}\left(1+\frac{x}{y}\right)^{-m_1-m_2}\delta_{m_1+m_2,m_1'+m_2'}\;. \qquad (4.12)$$

We employ these operators to construct three solutions of the Yang-Baxter equation (YBE) in terms of the Heisenberg algebra.

Proposition 4.1.9. *The $L^{\pm}$ operators are solutions of the YBE*

$$\begin{aligned} R_{12}^+(x/y)L_1^+(x)L_2^+(y) &= L_2^+(y)L_1^+(x)R_{12}^+(x/y)\;, & (4.13)\\ R_{12}^-(x/y)L_1^-(x)L_2^-(y) &= L_2^-(y)L_1^-(x)R_{12}^-(x/y)\;, & (4.14)\\ R_{12}(x/y)L_1^-(x)L_2^+(y) &= L_2^+(y)L_1^-(x)R_{12}(x/y)\;. & (4.15)\end{aligned}$$

The three solutions of the YBE presented above are identities in $\mathrm{End}[\mathbb{C}((x,y))\otimes\mathcal{F}\otimes\mathcal{F}]\otimes\mathcal{H}$. The subscripts attached to the operators specify which copies of the Fock space these operators are acting on. For example, we have that

$$\begin{aligned} L_1^+(x)\,|m_1,m_2\rangle\otimes 1 &= \sum_{m'\in\mathbb{Z}_{\geq 0}} |m',m_2\rangle\otimes\frac{x^{m_1}}{m_1!}(b^*)^{m_1}b^{m'}\;,\\ L_2^+(y)\,|m_1,m_2\rangle\otimes 1 &= \sum_{m'\in\mathbb{Z}_{\geq 0}} |m_1,m'\rangle\otimes\frac{y^{m_2}}{m_2!}(b^*)^{m_2}b^{m'}\;.\end{aligned}$$

Notice that (4.13) is the limit $q=1$ of a similar identity proved in [41, Proposition 3.7]. Equations (4.14) and (4.15) are new results.

Proof. Denote by $L^{\pm}_{m',m}(x)\in\mathcal{H}\otimes\mathbb{C}((x))$ the elements defined via the relation $L^{\pm}(u)\,|m\rangle\otimes 1=\sum_{m'\in\mathbb{Z}_{\geq 0}}|m'\rangle\otimes L^{\pm}_{m',m}(u)$. Equations (4.6) and (4.7) imply that $L^+_{m',m}(x)=\frac{x^m}{m!}(b^*)^m b^{m'}$ and $L^-_{m',m}(x)=\frac{x^m}{m!}b^{m'}(b^*)^m$. Set $R^{\pm}_{m_1',m_2';m_1,m_2}(x/y)=\langle m_1',m_2'|R^{\pm}(x/y)|m_1,m_2\rangle$ and $R_{m_1',m_2';m_1,m_2}(x/y)=\langle m_1',m_2'|R(x/y)|m_1,m_2\rangle$. Applying both sides of (4.13) to the element $|m_1,m_2\rangle\otimes 1$, and then doing the same for (4.14) and (4.15), one arrives at the following set of constraints for the operators $L^{\pm}_{m',m}(x)$,

$$\sum_{m_1',m_2'\geq 0} R^{\pm}_{n_1,n_2;m_1',m_2'} L^{\pm}_{m_1',m_1}(x) L^{\pm}_{m_2',m_2}(y) = \sum_{m_1',m_2'\geq 0} L^{\pm}_{n_2,m_2'}(y) L^{\pm}_{n_1,m_1'}(x) R^{\pm}_{m_1',m_2';m_1,m_2}(x/y) \ ,$$

together with

$$\sum_{m_1',m_2'\geq 0} R_{n_1,n_2;m_1',m_2'} L^{-}_{m_1',m_1}(x) L^{+}_{m_2',m_2}(y) = \sum_{m_1',m_2'\geq 0} L^{+}_{n_2,m_2'}(y) L^{-}_{n_1,m_1'}(x) R_{m_1',m_2';m_1,m_2}(x/y) \ ,$$

for $m_1, m_2, n_1, n_2 \in \mathbb{Z}_{\geq 0}$. The constraints involving solely the operators $L^{+}_{m',m}(x)$ can be deduced from [41, Proposition 3.7] by taking the limit $q = 1$. The others follow after a straightforward but tedious computation, whose details we omit. For this purpose, one has to take advantage of the following commutation relation,

$$(b^*)^r b^s = \sum_{l=0}^{\min(r,s)} \frac{(-1)^l r! s!}{l!(r-l)!(s-l)!} b^{s-l} (b^*)^{r-l} \ ,$$

which can be proved by induction. □

Remark 4.1.10. The universal R-matrix $R \in \mathcal{U} \otimes \mathcal{U}$ of a quantum group $\mathcal{U}$ satisfies the relation

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12} \ . \tag{4.16}$$

In the context of quantum groups, the latter is what is known as the Yang-Baxter equation. Suppose now that X, Y and Z are $\mathcal{U}$-modules. Denote by R_{XY} the universal R-matrix that is acting on the first two spaces of the tensor product $X \otimes Y \otimes Z$. Define R_{XZ} and R_{YZ} in an analogous way. We then deduce from (4.16) the relation $R_{XY} R_{XZ} R_{YZ} = R_{YZ} R_{XZ} R_{XY}$. The spaces X and Y are usually referred to as the 'auxiliary spaces', whereas Z is the 'quantum space'. Moreover, it is customary to call $R_{XZ} = L_X$, and similarly $R_{YZ} = L_Y$, the 'Lax matrix'. See for example [15] for further details. In our discussion, the matrix R^+ defined in (4.10) is a degenerate limit of the universal R-matrix of $\mathcal{U}_q(\hat{\mathfrak{sl}}_2)$ [39]. Even though the L-operators defined in (4.6), (4.7) and the R-matrices defined in (4.10), (4.11), (4.12) are not related, it is customary in quantum integrable systems to call the equations of the form (4.13), (4.14) and (4.15) as the Yang-Baxter equation. See for example [41, Proposition 3.6].

We now generalise the discussion presented so far to include the partition functions (4.2). Namely, we wish to express the latter as the matrix elements of some suitable operators. For this purpose, consider the n-fold tensor product $\mathcal{H}_n = \mathcal{H}^{\otimes n}$, and denote with $\{b_i, b_i^*\}$ the generators belonging to the i-th copy of $\mathcal{H}$. It follows that $\mathcal{H}_n$ is generated by the elements $\{b_i, b_i^*\}_{i=1}^n$ subject to the relations

$$[b_i, b_j] = [b_i^*, b_j^*] = 0 \ , \qquad\qquad [b_i, b_j^*] = \delta_{ij} \ . \tag{4.17}$$

The n-fold tensor product $\mathcal{F}^{\otimes n}$ admits the decomposition $\mathcal{F}^{\otimes n} = \bigoplus_{k\in\mathbb{Z}_{\geq 0}} \mathcal{F}_k^{\otimes n}$. For $k \in \mathbb{Z}_{\geq 0}$, the subspace $\mathcal{F}_k^{\otimes n} \subset \mathcal{F}^{\otimes n}$ is spanned by the vectors $|m_1, m_2, \ldots, m_n\rangle = |m_1\rangle \otimes |m_2\rangle \otimes \cdots \otimes |m_n\rangle$ satisfying the constraint $\sum_{i=1}^n m_i = k$. We label these vectors with partitions $\lambda \in \mathcal{A}_k^+(n)$ as follows,

$$|\lambda\rangle = |m_1(\lambda), m_2(\lambda), \ldots, m_n(\lambda)\rangle = \frac{1}{u_\lambda} b_{\lambda_1}^* b_{\lambda_2}^* \cdots b_{\lambda_k}^* |0\rangle \ , \tag{4.18}$$

where the symbol $u_\lambda = \prod_{i\geq 1} m_i(\lambda)!$ was introduced in (2.28). Every partition $\mu \in \mathcal{P}^+$ with $\mu_1 \leq n$ can then be identified with a vector in $\mathcal{F}^{\otimes n}$ via the relation (4.18). Similarly, we have the decomposition $\tilde{\mathcal{F}}^{\otimes n} = \bigoplus_{k\in\mathbb{Z}_{\geq 0}} \tilde{\mathcal{F}}_k^{\otimes n}$. For $k \in \mathbb{Z}_{\geq 0}$, the subspace $\tilde{\mathcal{F}}_k^{\otimes n}$ is spanned by the vectors

$$\langle\lambda| = \langle m_1(\lambda), m_2(\lambda), \ldots, m_n(\lambda)| = \langle 0| b_{\lambda_k} \cdots b_{\lambda_1} \ , \tag{4.19}$$

where λ ranges over all partitions in $\mathcal{A}_k^+(n)$. By construction we have that $\langle\mu|\nu\rangle = \delta_{\mu\nu}$ for all $\mu, \nu \in \mathcal{P}^+$ with $\mu_1, \nu_1 \leq n$.

For $i = 1\ldots, n$, let $L_i^\pm(x) \in \mathcal{F}((x)) \otimes \mathcal{H}_n$ coincide with the $L^\pm$ operators defined in (4.6) and (4.7), where the elements $\{b, b^*\}$ of $\mathcal{H}$ are replaced by the elements $\{b_i, b_i^*\}$ of $\mathcal{H}_n$. Define the monodromy matrices $\mathbf{Q}^\pm(x) \in \mathcal{F}((x)) \otimes \mathcal{H}_n$ as

$$\mathbf{Q}^\pm(x) = L_n^\pm(x) \cdots L_2^\pm(x) L_1^\pm(x) \ . \tag{4.20}$$

Notice that the subscripts attached to the $L^\pm$ operators in (4.20) have a different meaning from the subscripts introduced in Proposition 4.1.9. Denote with $Q_{m',m}^\pm(x) \in \mathcal{H}_n \otimes \mathbb{C}((x))$ the elements defined via the relation $\mathbf{Q}^\pm(x)\,|m\rangle \otimes 1 = \sum_{m'\in\mathbb{Z}_{\geq 0}} |m'\rangle \otimes Q_{m',m}^\pm(x)$.

Corollary 4.1.11. *The monodromy matrices* (4.20) *are solutions of the YBE*

$$R_{12}^+(x/y)\mathbf{Q}_1^+(x)\mathbf{Q}_2^+(y) = \mathbf{Q}_2^+(y)\mathbf{Q}_1^+(x)R_{12}^+(x/y) \ , \tag{4.21}$$

$$R_{12}^-(x/y)\mathbf{Q}_1^-(x)\mathbf{Q}_2^-(y) = \mathbf{Q}_2^-(y)\mathbf{Q}_1^-(x)R_{12}^-(x/y) \ , \tag{4.22}$$

$$R_{12}(x/y)\mathbf{Q}_1^-(x)\mathbf{Q}_2^+(y) = \mathbf{Q}_2^+(y)\mathbf{Q}_1^-(x)R_{12}(x/y) \ , \tag{4.23}$$

where the operators R^+, R^- and R were defined respectively in (4.10), (4.11) *and* (4.12).

The three solutions of the YBE presented above are identities in $\mathrm{End}[\mathbb{C}((x,y)) \otimes \mathcal{F} \otimes \mathcal{F}] \otimes \mathcal{H}_n$. The subscripts attached to the operators have the same meaning of the subscripts introduced in Proposition 4.1.9. That is, they specify which copies of the Fock space these operators are acting on.

Proof. Taking advantage of Proposition 4.1.9, one can prove the claim via induction on n. This is a standard computation, which can be found for instance in [37, VI.1]. □

Lemma 4.1.12. *We have the equalities*

$$Q^{+}_{m',m}(x) = \frac{x^m}{m!} \sum_{\alpha\in\mathcal{P}^{\geq 0}_{n-1}} \frac{x^{|\alpha|}}{\alpha_1!\cdots\alpha_{n-1}!}(b_1^*)^m(b_1b_2^*)^{\alpha_1}\cdots(b_{n-1}b_n^*)^{\alpha_{n-1}}b_n^{m'} \ , \tag{4.24}$$

$$Q^{-}_{m',m}(x) = \frac{x^m}{m!} \sum_{\alpha\in\mathcal{P}^{\geq 0}_{n-1}} \frac{x^{|\alpha|}}{\alpha_1!\cdots\alpha_{n-1}!}b_n^{m'}(b_n^*b_{n-1})^{\alpha_{n-1}}\cdots(b_2^*b_1)^{\alpha_1}(b_1^*)^m \ . \tag{4.25}$$

where the set $\mathcal{P}^{\geq 0}_{n-1} = \{\alpha\in\mathcal{P}_{n-1} \mid \alpha_i \geq 0$ *for* $i = 1,\ldots,n-1\}$ *was introduced in* (2.7).

Proof. These equalities are the limit $q = 1$ of similar identities proved in [43, Lemma 4.1]. □

Introduce Q^+ and Q^- operators as the operators $Q^{\pm}(x) \in \mathcal{H}_n \otimes \mathbb{C}((z,x))$ defined via the following partial trace,

$$Q^{\pm}(x) = \mathrm{Tr}_{\mathcal{F}}\, z^N \mathbf{Q}^{\pm}(x) = \sum_{m\in\mathbb{Z}_{\geq 0}} z^m Q^{\pm}_{m,m}(x) \ , \tag{4.26}$$

where the operator $z^N \in \mathrm{End}(\mathcal{F})$ is defined via the relation $\langle m'|z^N|m\rangle = z^m\delta_{m'm}$. The partial trace of the monodromy matrix associated to some vertex model, which in our case coincides with the $Q^{\pm}$ operators defined above, is also known in the literature as Baxter's 'transfer matrix'. The following result shows that we can interpret the matrix elements of the $Q^{\pm}$ operators as the partition functions (4.2) of the $Q^{\pm}$ vertex models.

Lemma 4.1.13. *Let* $\lambda,\mu \in \mathcal{A}^+_k(n)$. *We have the identities*

$$Z^{\pm}_{\lambda,\mu}(x_1,x_2,\ldots,x_l) = \langle\lambda|Q^{\pm}(x_1)Q^{\pm}(x_2)\cdots Q^{\pm}(x_l)|\mu\rangle \ . \tag{4.27}$$

Proof. Suppose that γ is a lattice configuration belonging to either $\Gamma^+_{\lambda,\mu}(l)$ or $\Gamma^-_{\lambda,\mu}(l)$. We first label the values attached to the edges of the lattice in a suitable manner; compare with Figure 4.4. For $i = 1,\ldots,l$ and $j = 1,\ldots,n$ let $a^{(i)}_j$ be the value attached to the edge in the i-th row between columns $(j-1)\,\mathrm{Mod}\,n$ and j, where Mod was defined in (3.74). Similarly, for $i = 1,\ldots,l-1$ let $m^{(i)}_j$ be the value attached to the edge in the j-th column between rows i and $i+1$. Moreover, set $m^{(0)}_j = m_j(\mu)$ and $m^{(l)}_j = m_j(\lambda)$.

Notice that the vertex configuration associated to $v_{i,j}$, which is the vertex obtained by intersecting the i-th row with the j-th column, consists of the four non-negative integers

$\{a_j^{(i)}, m_j^{(i-1)}, a_{j+1}^{(i)}, m_j^{(i)}\}$ oriented as in Figure 4.1. By definition we have that γ is an allowed lattice configuration, which means that its associated weight is non-zero. Equation (4.1) implies that the vertex configuration associated to $v_{i,j}$ is also 'allowed'. A comparison with Figure 4.1 shows that for $i = 1, \ldots, l$ and $j = 1, \ldots, n$ the integers $\{a_j^{(i)}, m_j^{(i-1)}, a_{j+1}^{(i)}, m_j^{(i)}\}$ satisfy the constraint $a_j^{(i)} + m_j^{(i-1)} = a_{j+1}^{(i)} + m_j^{(i)}$. Moreover, thanks to (4.8) and (4.9) we have the identity

$$\mathrm{wt}^{\pm}(v_{i,j}) = \langle a_{j+1}^{(i)}, m_j^{(i)} | L_j^{\pm}(x_i) | a_j^{(i)}, m_j^{(i-1)} \rangle \ . \tag{4.28}$$

For $i = 0, \ldots, l$, define the partition $\lambda^{(i)}$ via the relation $m_j(\lambda^{(i)}) = m_j^{(i)}$, where $j = 1, \ldots, l$. In particular, notice that $\lambda^{(0)} = \mu$ and $\lambda^{(l)} = \lambda$. The lattice configuration γ consists of $k \in \mathbb{Z}_{\geq 0}$ non-intersecting paths, and since these paths are conserved throughout the lattice, we have for $i = 0, \ldots, l$ the relation $\sum_{j=1}^n m_j^{(i)} = k$. The latter can be also deduced from the constraints $a_j^{(i)} + m_j^{(i-1)} = a_{j+1}^{(i)} + m_j^{(i)}$ mentioned above. It follows that $\lambda^{(i)} \in \mathcal{A}_k^+(n)$ for $i = 0, \ldots, l$, and the partition functions (4.2) can then be expressed as the following weighted sum,

$$Z_{\lambda,\mu}^{\pm}(x_1, x_2, \ldots, x_l) = \sum_{\lambda^{(1)} \in \mathcal{A}_k^+(n)} \cdots \sum_{\lambda^{(l-1)} \in \mathcal{A}_k^+(n)} \sum_{a_1^{(i)} \in \mathbb{Z}_{\geq 0}} \cdots \sum_{a_n^{(i)} \in \mathbb{Z}_{\geq 0}} z^{\sum_{i=1}^l a_1^{(i)}} \prod_{i=1}^{l} \prod_{j=1}^{n} \mathrm{wt}^{\pm}(v_{i,j}) \ .$$

Notice that the number of paths crossing the boundary of the cylinder is equal to $\sum_{i=1}^l a_1^{(i)}$.

Taking advantage of (4.20) and (4.26), and identifying the vertex weight associated to $v_{i,j}$ with the matrix element (4.28), one can show after a straightforward computation that the matrix element $\langle \lambda | Q^{\pm}(x_1) Q^{\pm}(x_2) \cdots Q^{\pm}(x_l) | \mu \rangle$ coincides with the expansion for $Z_{\lambda,\mu}^{\pm}(x_1, x_2, \ldots, x_l)$ obtained above, thus proving the claim.

□

With the help of Corollary 4.1.11, we can deduce that the transfer matrices of the $Q^{\pm}$ vertex models, which are just the $Q^{\pm}$ operators, commute with themselves for arbitrary values of the spectral parameter. This is the main feature of exactly solvable vertex models.

Corollary 4.1.14. *We have the commutation relations*

$$[Q^{\pm}(x), Q^{\pm}(y)] = [Q^{+}(x), Q^{-}(y)] = 0 \ . \tag{4.29}$$

Proof. A straightforward computation shows that the operators R^+ and R^-, which were introduced in (4.10) and (4.11) respectively, satisfy $R^+(x/y)R^-(x/y) = R^-(x/y)R^+(x/y) = 1$. In other words, these operators are invertible, and they are each other's inverse. We then have the following chain of equalities,

$$\begin{aligned}
Q^{\pm}(x)Q^{\pm}(y) &= \mathrm{Tr}_{\mathcal{F}\otimes\mathcal{F}}\, z^{N_1}\mathbf{Q}_1^{\pm}(x)z^{N_2}\mathbf{Q}_2^{\pm}(y) \\
&= \mathrm{Tr}_{\mathcal{F}\otimes\mathcal{F}}\, R_{12}^{\mp}(x/y)R_{12}^{\pm}(x/y)z^{N_1}z^{N_2}\mathbf{Q}_1^{\pm}(x)\mathbf{Q}_2^{\pm}(y) \\
&= \mathrm{Tr}_{\mathcal{F}\otimes\mathcal{F}}\, R_{12}^{\pm}(x/y)z^{N_1}z^{N_2}\mathbf{Q}_1^{\pm}(x)\mathbf{Q}_2^{\pm}(y)R_{12}^{\mp}(x/y) \\
&= \mathrm{Tr}_{\mathcal{F}\otimes\mathcal{F}}\, R_{12}^{\pm}(x/y)z^{N_1}z^{N_2}R_{12}^{\mp}(x/y)\mathbf{Q}_2^{\pm}(y)\mathbf{Q}_1^{\pm}(x) \\
&= \mathrm{Tr}_{\mathcal{F}\otimes\mathcal{F}}\, z^{N_2}\mathbf{Q}_2^{\pm}(y)z^{N_1}\mathbf{Q}_1^{\pm}(x) \\
&= Q^{\pm}(y)Q^{\pm}(x)\ .
\end{aligned}$$

The subscripts attached to the operators indicate which copy of the Fock space in the tensor product $\mathcal{F}\otimes\mathcal{F}$ these operators are acting on. In the third line we used the cyclicity of the trace for the operators $R_{12}^{\mp}(x/y)$, whereas in the fourth line we took advantage of (4.21) and (4.22). In the fifth line we used the commutation relation $[R_{12}^{\pm}(x/y), z^{N_1}z^{N_2}] = 0$, which can be deduced from (4.10) and (4.11). This is a standard computation in the discussion of vertex models (see for example [37, VI.1]), and it shows that $[Q^{\pm}(x), Q^{\pm}(y)] = 0$. The same reasoning cannot be applied to derive the commutation relation $[Q^{+}(x), Q^{-}(y)] = 0$, since the operator R, which was introduced in (4.12), is not invertible. Acting with both sides of equation (4.23) on the element $|m_1, m_2\rangle \otimes 1$, we end up with the following constraint,

$$\begin{aligned}
&\sum_{m=0}^{m_1+m_2}\left(\frac{x}{y}\right)^{m}\left(1+\frac{x}{y}\right)^{-m_1-m_2}\binom{m_1+m_2}{m}Q^{-}_{m,n_1}(x)Q^{+}_{m_1+m_2-m,n_2}(y) \\
&\qquad = \left(\frac{x}{y}\right)^{n_1}\left(1+\frac{x}{y}\right)^{-n_1-n_2}\binom{n_1+n_2}{n_1}\sum_{m=0}^{n_1+n_2}Q^{+}_{m_2,n_1+n_2-m}(y)Q^{-}_{m_1,m}(x)\ ,
\end{aligned}$$

for $m_1, m_2, n_1, n_2 \in \mathbb{Z}_{\geq 0}$. Set $n_1 = m_2 = 0$ and $n_2 = m_1 = s$, multiply both sides of this last equality by z^s and sum over $s \in \mathbb{Z}_{\geq 0}$. Then the LHS equals $Q^{-}(x)Q^{+}(y)$, whereas the RHS equals $Q^{+}(x)Q^{-}(y)$, thus proving that $[Q^{+}(x), Q^{-}(y)] = 0$. $\square$

4.1.3 Partition functions and cylindric symmetric functions

Lemma 4.1.13 and Corollary 4.1.14 imply that the partition functions $Z^{\pm}_{\lambda,\mu}(x_1,\dots,x_l)$ defined in (4.2) are symmetric in the variables $\{x_1,\dots,x_l\}$. The goal of this section is to show that $Z^{\pm}_{\lambda,\mu}(x_1,\dots,x_l)$ can be expanded in terms of the cylindric symmetric functions $e_{\lambda/d/\mu}$ and $h_{\lambda/d/\mu}$ introduced in Section 3.2.3. To this end, we proceed as follows. First, we show that the allowed lattice configurations of the Q^{+} and Q^{-} vertex models are in bijection respectively with CRSTs and CRPPs, which were defined in Section 3.1.1. Then we provide the relation between the weight of these lattice configurations and the weights $\psi_{\lambda/d/\mu}$ and $\theta_{\lambda/d/\mu}$ introduced in Definitions 3.2.9 and 3.2.4 respectively. In Section 4.3.2 we shall evaluate the partition functions $Z^{\pm}_{\lambda,\mu}(x_1,\dots,x_l)$ via a different method, which

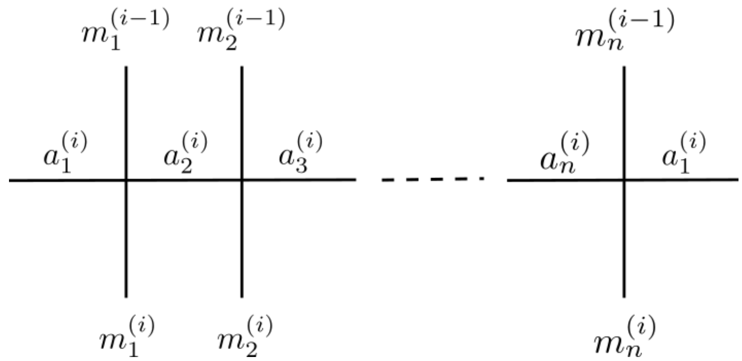


Figure 4.4: The i-th row of a lattice configuration γ belonging to either $\Gamma^+_{\lambda,\mu}(l)$ or $\Gamma^+_{\lambda,\mu}(l)$, together with the labels introduced in Lemma 4.1.13 for its lattice edges.

consists in computing the action of the $Q^\pm$ operators defined in (4.26) on the vectors (4.18) belonging to $\mathcal{F}^{\otimes n}$.

Let $\lambda, \mu \in \mathcal{A}^+_k(n)$ and $d \in \mathbb{Z}_{\geq 0}$. Define $\Gamma^\pm_{\lambda,\mu}(l,d) \subset \Gamma^\pm_{\lambda,\mu}(l)$ as the set of allowed lattice configurations for the $Q^\pm$ vertex models, where the values attached to the outer vertical edges on top and bottom are fixed respectively by μ and λ, and the number of paths crossing the boundary of the cylinder is equal to d. In particular, notice that $\Gamma^\pm_{\lambda,\mu}(l) = \bigsqcup_{d\in\mathbb{Z}_{\geq 0}} \Gamma^\pm_{\lambda,\mu}(l,d)$. If $\lambda/d/\mu$ is a cylindric skew diagram as defined in (3.3), denote with $\Pi_{\lambda/d/\mu}(l)$ (respectively $\mathrm{T}_{\lambda/d/\mu}(l)$) the set of CRPPs (respectively CRSTs) of shape $\lambda/d/\mu$, whose largest entry is smaller or equal than l. On the other hand, if $\lambda/d/\mu$ is not a cylindric skew diagram, that is if $\hat{\mu} \not\subset \hat{\lambda}.\tau^d$ in the notation of Remark 3.2.1, set $\Pi_{\lambda/d/\mu}(l) = \mathrm{T}_{\lambda/d/\mu}(l) = \emptyset$. Let $l' \in \mathbb{N}$ such that $l' \leq l$, and suppose that $\lambda/d/\mu$ is a cylindric skew diagram. Recall that a CRPP of shape $\lambda/d/\mu$ with largest entry l' is equivalent to a sequence

$$\hat{\mu} = \hat{\lambda}^{(0)}.\tau^{d_0} \subset \hat{\lambda}^{(1)}.\tau^{d_1} \subset \cdots \subset \hat{\lambda}^{(l')}.\tau^{d_{l'}} = \hat{\lambda}.\tau^d$$

of cylindric partitions with $\hat{\lambda}^{(r)} \in \hat{\mathcal{A}}^+_k(n)$ and $d_r - d_{r-1} \geq 0$ for $r = 1, \ldots, l'$. Compare with equation (3.21). Setting $\hat{\lambda}^{(r)}.\tau^{d_r} = \hat{\lambda}.\tau^d$ for $l' < r \leq l$, it follows that every CRPP belonging to $\Pi_{\lambda/d/\mu}(l)$ is equivalent to a sequence

$$\hat{\mu} = \hat{\lambda}^{(0)}.\tau^{d_0} \subset \hat{\lambda}^{(1)}.\tau^{d_1} \subset \cdots \subset \hat{\lambda}^{(l)}.\tau^{d_l} = \hat{\lambda}.\tau^d \tag{4.30}$$

of cylindric partitions with $\hat{\lambda}^{(r)} \in \hat{\mathcal{A}}^+_k(n)$ and $d_r - d_{r-1} \geq 0$ for $r = 1, \ldots, l$. Similarly, one has that every CRST belonging to $\mathrm{T}_{\lambda/d/\mu}(l)$ is equivalent to a sequence (4.30) of cylindric partitions, where $\hat{\lambda}^{(r)}.\tau^{d_r}/\hat{\lambda}^{(r-1)}.\tau^{d_{r-1}}$ is a cylindric vertical strip for $r = 1, \ldots, l$.

Notice that $\Pi_{\lambda/d/\mu}(l)$ is non-empty if and only if $\hat{\mu} \subset \hat{\lambda}.\tau^d$. In fact, if $\lambda/d/\mu$ is a cylindric skew diagram, then one can obtain a CRPP belonging to $\Pi_{\lambda/d/\mu}(l)$ by filling all the boxes of $\lambda/d/\mu$ with the entry 1. The same statement is not true for $\mathrm{T}_{\lambda/d/\mu}(l)$. For example, there are no CRSTs of shape $\lambda/d/\mu$ with $\lambda=(3,2,2)$, $\mu=(3,3,2)$, $d=1$ and largest entry $l=2$.

Consider the expansion (3.30) for the cylindric symmetric function $e_{\lambda/d/\mu}$. Projecting both sides of this expansion onto Λ_l, that is setting $x_i=0$ for $i>l$, one has that only the terms involving CRSTs with largest entry smaller or equal than l are non-zero. Similarly, projecting onto Λ_l both sides of the expansion (3.31) for the complete symmetric function $h_{\lambda/d/\mu}$, it follows that only the terms involving CRPPs with largest entry smaller or equal than l are non-zero. In other words, we have the identities

$$e_{\lambda/d/\mu}(x_1,\ldots,x_l) = \sum_{\hat{T}\in \mathrm{T}_{\lambda/d/\mu}(l)} \psi_{\hat{T}} x^{\hat{T}} \,, \tag{4.31}$$

$$h_{\lambda/d/\mu}(x_1,\ldots,x_l) = \sum_{\hat{\pi}\in \Pi_{\lambda/d/\mu}(l)} \theta_{\hat{\pi}} x^{\hat{\pi}} \,. \tag{4.32}$$

Proposition 4.1.15. *For every $\lambda,\mu\in\mathcal{A}_k^+(n)$ and $d\in\mathbb{Z}_{\geq 0}$, there exists a bijection between the sets $\Gamma^-_{\lambda,\mu}(l,d)$ and $\Pi_{\lambda/d/\mu}(l)$.*

Proof. Let $\lambda,\mu\in\mathcal{A}_k^+(n)$ and $d\in\mathbb{Z}_{\geq 0}$. Assume that $\Gamma^-_{\lambda,\mu}(l,d)$ is non-empty, and let $\gamma^-\in\Gamma^-_{\lambda,\mu}(l,d)$. We label the values attached to the lattice edges in the same way as described in the proof of Lemma 4.1.13 (see also Figure 4.4). In particular, for $i=0,\ldots,l$ define the partition $\lambda^{(i)}\in\mathcal{A}_k^+(n)$ as therein, that is via the relation $m_j(\lambda^{(i)})=m_j^{(i)}$, where $j=1,\ldots,n$. For $i=1,\ldots,l$ set $d_i=\sum_{p=1}^i a_1^{(p)}$, and notice that the number of paths crossing the boundary of the cylinder is given by $d=\sum_{p=1}^l a_1^{(p)}=d_l$.

By definition, we have that γ^- is an allowed lattice configuration for the Q^- vertex model. That is, for $i=1,\ldots,l$ and $j=1,\ldots,n$ the integers $\{a_j^{(i)},m_j^{(i-1)},a_{j+1}^{(i)},m_j^{(i)}\}$, which are the values attached to the vertex $v_{i,j}$, represent an allowed vertex configuration for the Q^- vertex model. A comparison with Figure 4.1 shows that for $i=1,\ldots,l$ and $j=1,\ldots,n$ we have the constraint $a_j^{(i)}+m_j^{(i-1)}=a_{j+1}^{(i)}+m_j^{(i)}$. Employing the latter, together with the identity $(\hat{\lambda}.\tau^d)'_j=\hat{\lambda}'_j+d$, it follows that $(\hat{\lambda}^{(i)}.\tau^{d_i})'_j-(\hat{\lambda}^{(i-1)}.\tau^{d_{i-1}})'_j=a_j^{(i)}\geq 0$, which in turn implies that $\hat{\lambda}^{(i-1)}.\tau^{d_{i-1}}\subset\hat{\lambda}^{(i)}.\tau^{d_i}$ for $i=1,\ldots,l$. We deduce that the sequence

$$\hat{\mu}=\hat{\lambda}^{(0)}\subset\hat{\lambda}^{(1)}.\tau^{d_1}\subset\cdots\subset\hat{\lambda}^{(l)}.\tau^{d_l}=\hat{\lambda}.\tau^d$$

of cylindric partitions is equivalent to a CRPP $\hat{\pi}$ belonging to $\Pi_{\lambda/d/\mu}(l)$, that is a CRPP of shape $\lambda/d/\mu$ whose largest entry is smaller or equal than l. In conclusion, the set $\Pi_{\lambda/d/\mu}(l)$ is non-empty, and each $\gamma^-\in\Gamma^-_{\lambda,\mu}(l,d)$ determines a unique element $\hat{\pi}\in\Pi_{\lambda/d/\mu}(l)$, that is $\gamma^-\mapsto\hat{\pi}$ defines a map $\Gamma^-_{\lambda,\mu}(l,d)\to\Pi_{\lambda/d/\mu}(l)$.

Let once again $\lambda,\mu\in\mathcal{A}_k^+(n)$ and $d\in\mathbb{Z}_{\geq 0}$. Assume that $\Pi_{\lambda/d/\mu}(l)$ is non-empty, and let $\hat{\pi}\in\Pi_{\lambda/d/\mu}(l)$. Moreover, define $\lambda^{(0)}=\mu$, $\lambda^{(l)}=\lambda$, $d_0=0$ and $d_l=d$. As we explained in (4.30), the CRPP $\hat{\pi}$ is equivalent to a sequence $\{\hat{\lambda}^{(i)}.\tau^{d_i}\}_{i=0}^l$ of cylindric

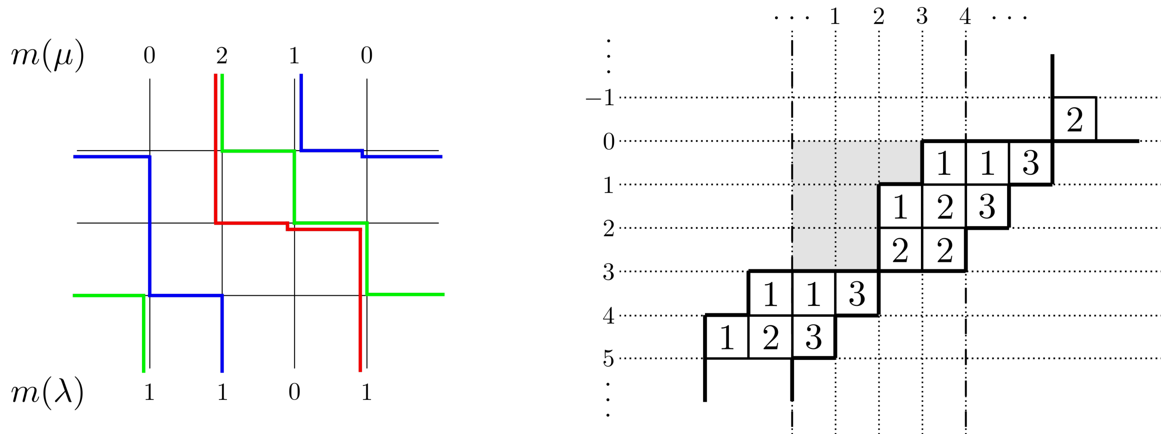


Figure 4.5: Let $n = 4$, $k = 4$, $l = 3$, $\lambda = (4, 2, 1)$, $\mu = (3, 2, 2)$ and $d = 2$. On the left we have a lattice configuration $\gamma^- \in \Gamma^-_{\lambda,\mu}(l, d)$. On the right we have the CRPP $\hat{\pi}$ which is the image of γ^- under the map $\Gamma^-_{\lambda,\mu}(l, d) \to \Pi_{\lambda/d/\mu}(l)$ defined in Proposition 4.1.15. We can construct $\hat{\pi}$ as follows. Let $p \in \mathbb{Z}$. In the notation of Figure 4.4, we have that $a^{(1)}_1 = 1$, $a^{(2)}_1 = 0$ and $a^{(3)}_1 = 1$. Thus, in column $1 + pn$ of the diagram of $\hat{\mu}$, add one box with entry 1, zero boxes with entry 2, and one box with entry 3. Moreover, we have that $a^{(1)}_2 = a^{(2)}_2 = 0$ and $a^{(3)}_2 = 1$. Thus, in column $2 + pn$ of the diagram of $\hat{\mu}$, add zero boxes with entries 1 and 2, and one box with entry 3. Proceed in an analogous way for columns $3 + pn$ and $4 + pn$ of the diagram of $\hat{\mu}$, and then repeat the algorithm just described for all $p \in \mathbb{Z}$.

partitions, with $\hat{\lambda}^{(i)} \in \hat{\mathcal{A}}^+_k(n)$ and $d_i - d_{i-1} \geq 0$, such that $\hat{\lambda}^{(i-1)}.\tau^{d_{i-1}} \subset \hat{\lambda}^{(i)}.\tau^{d_i}$ for $i = 1, \dots, l$. Define a lattice configuration γ^- as follows. For $i = 1, \dots, l$ and $j = 1, \dots, n$ set $a^{(i)}_j = (\hat{\lambda}^{(i)}.\tau^{d_i})'_j - (\hat{\lambda}^{(i-1)}.\tau^{d_{i-1}})'_j$. Furthermore, for $i = 1, \dots, l-1$ and $j = 1, \dots, n$ set $m^{(i)}_j = m_j(\lambda^{(i)})$. Finally, fix the upper and lower outer horizontal edges of γ^- with the two partitions $\lambda^{(0)} = \mu$ and $\lambda^{(l)} = \lambda$ respectively. By definition we have that $m^{(i)}_j \geq 0$, and since $\hat{\lambda}^{(i-1)}.\tau^{d_{i-1}} \subset \hat{\lambda}^{(i)}.\tau^{d_i}$ it follows that $a^{(i)}_j \geq 0$. Notice that $m^{(i)}_j = \hat{\lambda}^{(i)\prime}_j - \hat{\lambda}^{(i)\prime}_{j+1} = (\hat{\lambda}^{(i)}.\tau^{d_i})'_j - (\hat{\lambda}^{(i)}.\tau^{d_i})'_{j+1}$, where in the second equality we took advantage of the fact that $(\hat{\lambda}.\tau^d)'_i = \hat{\lambda}'_i + d$. It follows that $a^{(i)}_j + m^{(i-1)}_j = a^{(i)}_{j+1} + m^{(i)}_j = (\hat{\lambda}^{(i)}.\tau^{d_i})'_j - (\hat{\lambda}^{(i-1)}.\tau^{d_{i-1}})'_{j+1}$, which in turn implies that for $i = 1, \dots, l$ and $j = 1, \dots, n$ the vertex configuration associated to the vertex $v_{i,j}$ is 'allowed', and then γ^- is an allowed lattice configuration for the Q^- vertex model. Moreover, since the number of paths crossing the boundary of the cylinder is given by $\sum_{p=1}^{l} a^{(p)}_1 = d_l = d$, we deduce that $\gamma^- \in \Gamma^-_{\lambda,\mu}(l, d)$. In conclusion, the set $\Gamma^-_{\lambda,\mu}(l, d)$ is non-empty, and each $\hat{\pi} \in \Pi_{\lambda/d/\mu}(l)$ determines a unique element $\gamma^- \in \Gamma^-_{\lambda,\mu}(l, d)$, that is $\hat{\pi} \mapsto \gamma^-$ defines a map $\Pi_{\lambda/d/\mu}(l) \to \Gamma^-_{\lambda,\mu}(l, d)$.

From the discussion presented so far we deduce that the sets $\Gamma^-_{\lambda,\mu}(l, d)$ and $\Pi_{\lambda/d/\mu}(l)$ are either both empty or non-empty. If these sets are both non-empty, the map $\Pi_{\lambda/d/\mu}(l) \to \Gamma^-_{\lambda,\mu}(l, d)$ is by construction the inverse of the map $\Gamma^-_{\lambda,\mu}(l, d) \to \Pi_{\lambda/d/\mu}(l)$ defined above. That is, the composition of these two maps gives the identity map on $\Gamma^-_{\lambda,\mu}(l, d)$ and on $\Pi_{\lambda/d/\mu}(l)$ respectively. This proves the claim.

$\square$

Let $\gamma^- \in \Gamma^-_{\lambda,\mu}(l,d)$, and consider the CRPP $\hat{\pi}$ which is the image of γ^- under the map $\Gamma^-_{\lambda,\mu}(l,d) \to \Pi_{\lambda/d/\mu}(l)$ defined in the proof of Proposition 4.1.15. Notice that $(\hat{\lambda}^{(i)}.\tau^{d_i})'_j - (\hat{\lambda}^{(i-1)}.\tau^{d_{i-1}})'_j$ is the number of boxes which are placed in column j of the cylindric skew diagram $\hat{\lambda}^{(i)}.\tau^{d_i}/\hat{\lambda}^{(i-1)}.\tau^{d_{i-1}}$, and that this number is by construction equal to $a_j^{(i)}$. We can then construct $\hat{\pi}$ as follows (compare with Figure 4.5). Let $p \in \mathbb{Z}$. In column $1+pn$ of the diagram of $\hat{\mu}$, add $a_1^{(1)}$ boxes with entry 1, $a_1^{(2)}$ boxes with entry 2, and so on, up to $a_1^{(l)}$ boxes with entry l. In column $2+pn$ of the diagram of $\hat{\mu}$, add $a_2^{(1)}$ boxes with entry 1, $a_2^{(2)}$ boxes with entry 2, and so on, up to $a_2^{(l)}$ boxes with entry l. Proceed in a similar fashion for columns $3+pn$ to $k+pn$ of the diagram of $\hat{\mu}$. Finally, repeat the algorithm just described for all $p \in \mathbb{Z}$.

Proposition 4.1.16. *For every $\lambda, \mu \in \mathcal{A}_k^+(n)$ and $d \in \mathbb{Z}_{\geq 0}$, there exists a bijection between the sets $\Gamma^+_{\lambda,\mu}(l,d)$ and $\mathrm{T}_{\lambda/d/\mu}(l)$.*

Proof. The proof of this statement, which has been already presented in [41, Theorem 6.4], is closely related to the proof of Proposition 4.1.15. Let $\lambda, \mu \in \mathcal{A}_k^+(n)$ and $d \in \mathbb{Z}_{\geq 0}$. Assume that $\Gamma^+_{\lambda,\mu}(l,d)$ is non-empty, and let $\gamma^+ \in \Gamma^+_{\lambda,\mu}(l)$. Label the values attached to the lattice edges in the same way as described in the proof of Lemma 4.1.13. In particular, define for $i = 0, \ldots, l$ the partition $\lambda^{(i)} \in \mathcal{A}_k^+(n)$ as therein, that is via the relation $m_j(\lambda^{(i)}) = m_j^{(i)}$, where $j = 1, \ldots, n$. For $r = i, \ldots, l$ set $d_i = \sum_{p=1}^i a_1^{(p)}$, and notice that d_l is equal to the number of paths d crossing the boundary of the cylinder.

By definition, we have that γ^+ is an allowed lattice configuration for the Q^+ vertex model. That is, for $i = 1, \ldots, l$ and $j = 1, \ldots, n$ the integers $\{a_j^{(i)}, m_j^{(i-1)}, a_{j+1}^{(i)}, m_j^{(i)}\}$, which are the values attached to the vertex $v_{i,j}$, represent an allowed vertex configuration for the Q^+ vertex model. A comparison with Figure 4.1 shows that for $i = 1, \ldots, l$ and $j = 1, \ldots, n$ we have the constraint $a_j^{(i)} + m_j^{(i-1)} = a_{j+1}^{(i)} + m_j^{(i)}$, together with the inequality $m_j^{(i-1)} - a_{j+1}^{(i)} \geq 0$. It follows that $(\hat{\lambda}^{(i)}.\tau^{d_i})'_j - (\hat{\lambda}^{(i-1)}.\tau^{d_{i-1}})'_j = a_j^{(i)} \geq 0$, and moreover $(\hat{\lambda}^{(i-1)}.\tau^{d_{i-1}})'_j - (\hat{\lambda}^{(i)}.\tau^{d_i})'_{j+1} \geq 0$. These properties imply that for $i = 1, \ldots, l$ the cylindric skew diagram $\hat{\lambda}^{(i)}.\tau^{d_i}/\hat{\lambda}^{(i-1)}.\tau^{d_{i-1}}$ is a cylindric vertical strip. See [41, Theorem 6.4] for details. Thus, we have that the sequence

$$\hat{\mu} = \hat{\lambda}^{(0)} \subset \hat{\lambda}^{(1)}.\tau^{d_1} \subset \cdots \subset \hat{\lambda}^{(l)}.\tau^{d_l} = \hat{\lambda}.\tau^d$$

of cylindric partitions is equivalent to a CRST $\hat{T}$ belonging to $\mathrm{T}_{\lambda/d/\mu}(l)$, that is a CRST of shape $\lambda/d/\mu$ whose largest entry is smaller or equal than l. In conclusion, the set $\mathrm{T}_{\lambda/d/\mu}(l)$ is non-empty, and each $\gamma^+ \in \Gamma^+_{\lambda,\mu}(l,d)$ determines a unique element $\hat{T} \in \mathrm{T}_{\lambda/d/\mu}(l)$, that is $\gamma^+ \mapsto \hat{T}$ defines a map $\Gamma^+_{\lambda,\mu}(l,d) \to \mathrm{T}_{\lambda/d/\mu}(l)$.

Let once again $\lambda, \mu \in \mathcal{A}_k^+(n)$ and $d \in \mathbb{Z}_{\geq 0}$, and assume that $\mathrm{T}_{\lambda/d/\mu}(l)$ is non-empty. Following similar steps as the ones described in the proof of Proposition 4.1.15, one can show that $\Gamma^+_{\lambda,\mu}(l,d)$ is non-empty as well. Moreover, one can then construct a map

$\mathrm{T}_{\lambda/d/\mu}(l) \to \Gamma^+_{\lambda,\mu}(l,d)$, which turns out to be the inverse of the map $\Gamma^+_{\lambda,\mu}(l,d) \to \mathrm{T}_{\lambda/d/\mu}(l)$ defined above. See once again [41, Theorem 6.4] for details.

□

Remark 4.1.17. Proposition 4.1.15 implies that the set $\Gamma^-_{\lambda,\mu}(l,d)$ is non-empty if and only if $\lambda/d/\mu$ is a cylindric skew diagram, since we showed at the beginning of this section that an analogous statement holds for the set $\Pi_{\lambda/d/\mu}(l)$. On the other hand, the set $\Gamma^+_{\lambda,\mu}(l,d)$ is non-empty only if $\lambda/d/\mu$ is a cylindric skew diagram. This is because, as we discussed before, the set $T_{\lambda/d/\mu}(l)$ might be empty even if $\lambda/d/\mu$ is a cylindric skew diagram.

Theorem 4.1.18. *Let $\lambda,\mu \in \mathcal{A}^+_k(n)$. The partition functions of the $Q^\pm$ vertex models admit the following expansions,*

$$Z^+_{\lambda,\mu}(x_1,x_2,\ldots,x_l) = \sum_{d\in\mathbb{Z}_{\geq 0}} z^d e_{\lambda/d/\mu}(x_1,x_2,\ldots,x_l)\ , \tag{4.33}$$

$$Z^-_{\lambda,\mu}(x_1,x_2,\ldots,x_l) = \prod_{i=1}^{l} \frac{1}{1-zx_i^n} \sum_{d\in\mathbb{Z}_{\geq 0}} z^d h_{\lambda/d/\mu}(x_1,x_2,\ldots,x_l)\ . \tag{4.34}$$

Proof. Equation (4.33) is the limit $q=1$ of a similar identity proved in [41, Theorem 6.4]. Nevertheless, we present a proof of (4.33) for the sake of completeness. Let $d \in \mathbb{Z}_{\geq 0}$, and recall from Lemma 3.2.12 that the weight $\psi_{\lambda/d/\mu}$ has the following expression in terms of binomial coefficients,

$$\psi_{\lambda/d/\mu} = \prod_{j=1}^{n} \binom{(\hat{\lambda}.\tau^d)'_j - (\hat{\lambda}.\tau^d)'_{j+1}}{(\hat{\lambda}.\tau^d)'_j - \hat{\mu}'_j}\ .$$

Moreover, we shall employ the fact that the number $|\lambda/d/\mu|$ of boxes in $\lambda/d/\mu$ which are located in lines 1 to k, or equivalently in columns 1 to n, is equal to $\sum_{j=1}^n((\hat{\lambda}.\tau^d)'_j - \hat{\mu}'_j)$. Suppose that $\gamma^+ \in \Gamma^+_{\lambda,\mu}(l,d)$, and label the values attached to the lattice edges as described in Lemma 4.1.13 (see also Figure 4.4). For $i=1,\ldots,l$ and $j=1,\ldots,n$ we have that the vertex configuration associated to the vertex $v_{i,j}$ consists of the four integers $\{a^{(i)}_j, m^{(i-1)}_j, a^{(i)}_{j+1}, m^{(i)}_j\}$ oriented as in Figure 4.1, and then we have by definition the identity

$$\mathrm{wt}^+(v_{i,j}) = x_i^{a^{(i)}_j} \binom{m^{(i)}_j}{a^{(i)}_j}\ .$$

Consider the CRST $\hat{T}$ of shape $\lambda/d/\mu$ which is the image of γ^+ under the map $\Gamma^+_{\lambda,\mu}(l,d) \to \mathrm{T}_{\lambda/d/\mu}(l)$ defined in Proposition 4.1.16. By definition, $\hat{T}$ is equivalent to the sequence $\{\hat{\lambda}^{(i)}.\tau^{d_i}\}_{i=0}^l$ of cylindric partitions which are defined via the relations $m_j(\lambda^{(i)}) = m^{(i)}_j$ and $(\hat{\lambda}^{(i)}.\tau^{d_i})'_j - (\hat{\lambda}^{(i-1)}.\tau^{d_{i-1}})'_j = a^{(i)}_j$. In the proof of Lemma 3.1.9 we showed that the multiplicity of the entry $r \in \mathbb{N}$ in $\hat{T}$, between lines 1 to k, satisfies the equality $\mathrm{wt}_r(\hat{T}) = |\lambda^{(r)}/(d_r - d_{r-1})/\lambda^{(r-1)}|$. Taking advantage of the identity $(\hat{\lambda}.\tau^d)'_i = \hat{\lambda}'_i + d$ we have that

$a_j^{(i)} = (\hat{\lambda}^{(i)}.\tau^{d_i-d_{i-1}})'_j - (\hat{\lambda}^{(i-1)})'_j$, which can be used to show the relation $\mathrm{wt}_i(\hat{T}) = \sum_{j=1}^n a_j^{(i)}$ for $i = 1, \ldots, l$. We then have the following chain of equalities,

$$\begin{aligned}
\mathrm{wt}^+(\gamma^+) &= \prod_{i=1}^{l}\prod_{j=1}^{n} \mathrm{wt}^+(v_{i,j}) = \prod_{i=1}^{l}\prod_{j=1}^{n} x_i^{a_j^{(i)}} \binom{m_j^{(i)}}{a_j^{(i)}} \\
&= \prod_{i=1}^{l} x_i^{\sum_{j=1}^n a_j^{(i)}} \prod_{j=1}^{n} \binom{(\hat{\lambda}^{(i)}.\tau^{d_i-d_{i-1}})'_j - (\hat{\lambda}^{(i)}.\tau^{d_i-d_{i-1}})'_{j+1}}{(\hat{\lambda}^{(i)}.\tau^{d_i-d_{i-1}})'_j - (\hat{\lambda}^{(i-1)})'_j} \\
&= \prod_{i=1}^{l} x_i^{\mathrm{wt}_i(\hat{T})} \psi_{\lambda^{(i)}/(d_i-d_{i-1})/\lambda^{(i-1)}} = \psi_{\hat{T}} x^{\hat{T}} .
\end{aligned}$$

In the second line we employed the equalities $m_j^{(i)} = \lambda_j^{(i)\prime} - \lambda_{j+1}^{(i)\prime} = (\hat{\lambda}^{(i)}.\tau^{d_i-d_{i-1}})'_j - (\hat{\lambda}^{(i)}.\tau^{d_i-d_{i-1}})'_{j+1}$. Taking advantage of (4.2) and the identity $\Gamma^+_{\lambda,\mu}(l) = \bigsqcup_{d\in\mathbb{Z}_{\geq 0}} \Gamma^+_{\lambda,\mu}(l,d)$, we end up with

$$Z^+_{\lambda,\mu}(x_1, x_2, \ldots, x_l) = \sum_{d\in\mathbb{Z}_{\geq 0}} z^d \sum_{\hat{T}\in \mathrm{T}_{\lambda/d/\mu}(l)} \psi_{\hat{T}} x^{\hat{T}} .$$

A comparison of the latter with (4.31) then shows the validity of (4.33).

We now prove (4.34). For this purpose, let $d \in \mathbb{Z}_{\geq 0}$ and define the weight

$$\tilde{\theta}_{\lambda/d/\mu} = \sum_{\bar{d}=0}^{d} \theta_{\lambda/\bar{d}/\mu} = \prod_{j=1}^{n} \binom{(\hat{\lambda}.\tau^d)_j - \hat{\mu}'_{j+1}}{\hat{\mu}'_j - \hat{\mu}'_{j+1}} . \tag{4.35}$$

In the second equality we took advantage of the expression for $\theta_{\lambda/d/\mu}$ in terms of binomial coefficients, which was proved in Lemma 3.2.8. Suppose that $\gamma^- \in \Gamma^+_{\lambda,\mu}(l,d)$, and label the values attached to the lattice edges as described in Lemma 4.1.13. The weight of each vertex $v_{i,j}$ can then be expressed as

$$\mathrm{wt}^-(v_{i,j}) = x^{a_j^{(i)}} \binom{a_j^{(i)} + m_j^{(i-1)}}{m_j^{(i-1)}} .$$

Consider the CRPP $\hat{\pi}$ of shape $\lambda/d/\mu$ which is the image of γ^- under the map $\Gamma^-_{\lambda,\mu}(l,d) \to \Pi_{\lambda/d/\mu}(l)$ defined in Proposition 4.1.15. By definition, $\hat{\pi}$ is equivalent to the sequence $\{\hat{\lambda}^{(i)}.\tau^{d_i}\}_{i=0}^l$ of cylindric partitions which are defined via the relations $m_j(\lambda^{(i)}) = m_j^{(i)}$ and $(\hat{\lambda}^{(i)}.\tau^{d_i})'_j - (\hat{\lambda}^{(i-1)}.\tau^{d_{i-1}})'_j = a_j^{(i)}$. Setting $\tilde{\theta}_{\hat{\pi}} = \prod_{r=1}^l \tilde{\theta}_{\lambda^{(r)}/(d_r-d_{r-1})/\lambda^{(r-1)}}$, we end up with the following chain of equalities,

$$\begin{aligned}
\mathrm{wt}^-(\gamma^-) &= \prod_{i=1}^{l}\prod_{j=1}^{n}\mathrm{wt}^-(v_{i,j}) = \prod_{i=1}^{l}\prod_{j=1}^{n} x^{a_j^{(i)}}\binom{a_j^{(i)}+m_j^{(i-1)}}{m_j^{(i-1)}} \\
&= \prod_{i=1}^{l} x_i^{\sum_{j=1}^n a_j^{(i)}} \prod_{j=1}^{n}\binom{(\hat\lambda^{(i)}.\tau^{d_i-d_{i-1}})'_j-(\hat\lambda^{(i-1)})'_{j+1}}{(\hat\lambda^{(i-1)})'_j-(\hat\lambda^{(i-1)})'_{j+1}} \\
&= \prod_{i=1}^{l} x_i^{\mathrm{wt}_i(\hat\pi)}\tilde\theta_{\lambda^{(i)}/(d_i-d_{i-1})/\lambda^{(i-1)}} = \tilde\theta_{\hat\pi}x^{\hat\pi}\ ,
\end{aligned}$$

and employing (4.2) once again, we have that

$$Z^-_{\lambda,\mu}(x_1,x_2,\ldots,x_l) = \sum_{d\in\mathbb{Z}_{\geq 0}} z^d \sum_{\hat\pi\in\Pi_{\lambda/d/\mu}(l)} \tilde\theta_{\hat\pi}x^{\hat\pi}\ . \tag{4.36}$$

Thanks to Proposition 3.2.7, it follows that $\tilde\theta_{\lambda/d/\mu}$ is non-zero if and only if $\lambda/d/\mu$ is a cylindric skew diagram, that is if and only if $\hat\mu\subset\hat\lambda.\tau^d$. This implies that in the RHS of (4.36) we can replace the sum over $\hat\pi\in\Pi_{\lambda/d/\mu}(l)$ with the sum over all sequences $\{\hat\lambda^{(r)}.\tau^{d_r}\}_{r=0}^{l}$ of cylindric partitions, where $\lambda^{(0)}=\mu$, $\lambda^{(l)}=\lambda$, $d_0=0$ and $d_l=d$. The latter is equivalent to the joint sum over all sequences $\{\lambda^{(r)}\}_{r=1}^{l-1}$ of weights in $\mathcal{A}_k^+(n)$, and over all sequences $\{d_r\}_{r=1}^{l-1}$ of integers. The sum does not change if we use the restriction $d_r-d_{r-1}\geq 0$. This is because $\hat\lambda^{(r-1)}.\tau^{d_{r-1}}\subset\hat\lambda^{(r)}.\tau^{d_r}$ only if $d_r-d_{r-1}\geq 0$, as we showed in the discussion of equation 3.21. Setting $d'_r=d_r-d_{r-1}$ for $r=1,\ldots,l$, we have that $d'_r\in\mathbb{Z}_{\geq 0}$ and $d'_1+\cdots+d'_l=d$. The partition function $Z^-_{\lambda,\mu}(x_1,x_2,\ldots,x_l)$ is then equal to

$$\sum_{d'_1\in\mathbb{Z}_{\geq 0}}\cdots\sum_{d'_l\in\mathbb{Z}_{\geq 0}} z^{d'_1+\cdots+d'_l}\sum_{\lambda^{(1)}\in\mathcal{A}_k^+(n)}\cdots\sum_{\lambda^{(l-1)}\in\mathcal{A}_k^+(n)}\prod_{r=1}^{l}\tilde\theta_{\lambda^{(r)}/d'_r/\lambda^{(r-1)}}\prod_{i=1}^{l}x_i^{|\lambda^{(i)}/d'_i/\lambda^{(i-1)}|}\ . \tag{4.37}$$

Applying the definition (4.35) of the weight $\tilde\theta_{\lambda/d/\mu}$, it follows that

$$\prod_{r=1}^{l}\tilde\theta_{\lambda^{(r)}/d'_r/\lambda^{(r-1)}} = \sum_{d''_1=0}^{d'_1}\cdots\sum_{d''_l=0}^{d'_l}\prod_{r=1}^{l}\theta_{\lambda^{(r)}/d''_r/\lambda^{(r-1)}}\ .$$

Moreover, we have the identity $|\lambda^{(i)}/d'_i/\lambda^{(i-1)}|=|\lambda^{(i)}/d''_i/\lambda^{(i-1)}|+n(d'_i-d''_i)$, provided that $\lambda^{(i)}/d''_i/\lambda^{(i-1)}$ is a cylindric skew diagram. This identity can be deduced from the equality $|\lambda/d/\mu|=\sum_{j=1}^n((\hat\lambda.\tau^d)'_j-\hat\mu'_j)$, together with the fact that $(\hat\lambda.\tau^d)'_j=\hat\lambda'_j+d$. Applying the identities just obtained to (4.37), swapping the sums in d'_i and d''_i, and then employing the formula $\sum_{i\geq 0}x^i=(1-x)^{-1}$ for x a formal variable, we deduce that $Z^-_{\lambda,\mu}(x_1,x_2,\ldots,x_l)$ is

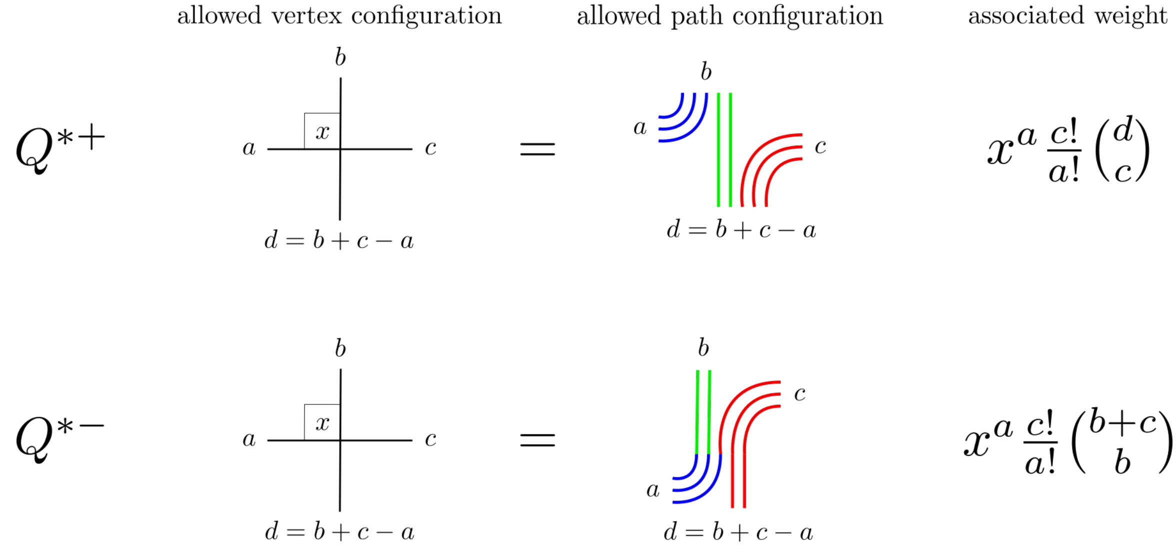


Figure 4.6: The allowed vertex configurations for the $Q^{*\pm}$ vertex models, together with their associated weights. For the Q^{*+} vertex model, we only allow for the vertex configurations such that $b + c = a + d$ and $b \geq a$, or equivalently $d \geq c$. For the Q^{*-} vertex model, a vertex configuration is 'allowed' if the only condition $b + c = a + d$ is satisfied. The allowed vertex configurations for these models can be interpreted in terms of non-intersecting paths travelling from North-East to South-West. Notice that the allowed path configurations depicted above are obtained by reflecting the allowed path configurations in Figure 4.1 about the vertical line passing through the edges labelled by b and d.

equal to $\prod_{i=1}^{l}(1 - zx_i^n)^{-1}$ times the following weighted sum,

$$\sum_{d_1'' \in \mathbb{Z}_{\geq 0}} \cdots \sum_{d_l'' \in \mathbb{Z}_{\geq 0}} z^{d_1'' + \cdots + d_l''} \sum_{\lambda^{(1)} \in \mathcal{A}_k^+(n)} \cdots \sum_{\lambda^{(l-1)} \in \mathcal{A}_k^+(n)} \prod_{r=1}^{l} \theta_{\lambda^{(r)}/d_r''/\lambda^{(r-1)}} \prod_{i=1}^{l} x_i^{|\lambda^{(i)}/d_i''/\lambda^{(i-1)}|} \,. \quad (4.38)$$

Notice that (4.38) is the same equation as (4.37), where the weights $\tilde{\theta}_{\lambda/d/\mu}$ are replaced by $\theta_{\lambda/d/\mu}$. Following similar steps as the ones leading from (4.36) to (4.37), but in reverse order, we conclude that (4.38) is equal to $\sum_{d \in \mathbb{Z}_{\geq 0}} z^d \sum_{\hat{\pi} \in \Pi_{\lambda/d/\mu}(l)} \theta_{\hat{\pi}} x^{\hat{\pi}}$. This finally completes the proof of (4.34), thanks to (4.32).

□

4.1.4 Vertex models for the adjoint operators

We shall now consider two additional vertex models, which we call the $Q^{*\pm}$ vertex models. The allowed vertex configurations for such models, together with the associated weights, are defined in Figure 4.6. We will be using the symbols wt^{*+} and wt^{*-} whenever we associate to a vertex configuration, or a lattice configuration, the weight defined for the Q^{*+} and Q^{*-} vertex model respectively.

Remark 4.1.19. The allowed lattice configurations for the $Q^{*\pm}$ vertex models can be interpreted in terms of non-intersecting lattice paths travelling from North-West to South-East, in contrast to the ones for the $Q^{\pm}$ vertex models, where the paths are travelling from North-East to South-West instead. See Figures 4.1 and 4.6. The constraint $a+d=b+c$ at each vertex of the lattice implies that on the cylinder the number of paths is conserved throughout the lattice (compare with Remark 4.1.2).

Given $\lambda, \mu \in \mathcal{A}_k^+(n)$, denote with $\Gamma^{*\pm}_{\lambda,\mu}(l)$ the set of allowed lattice configurations for the $Q^{*\pm}$ vertex models, such that the values of the outer vertical edges on top and bottom are fixed respectively by μ and λ. Moreover, assume that the formal variable z introduced in (4.2) is invertible. The partition functions of the $Q^{*\pm}$ vertex models, for the lattice with periodic boundary conditions in the horizontal direction, are defined as the weighted sums

$$Z^{*\pm}_{\lambda,\mu}(x_1,\dots,x_l) = \sum_{\gamma^{*\pm}\in\Gamma^{\pm}_{\lambda,\mu}(l)} z^{-d}\,\mathrm{wt}^{*\pm}(\gamma^{*\pm}) \ . \tag{4.39}$$

For each lattice configuration $\gamma^{*\pm} \in \Gamma^{*\pm}_{\lambda,\mu}(l)$, we set $d = d(\gamma^{*\pm})$ to be the number of paths crossing the boundary of the cylinder.

We now present for the $Q^{*\pm}$ vertex models a similar discussion to the one described in Section 4.1.2. The $L^{*\pm}$ operators are the operators $L^{*\pm}(x) \in \mathrm{End}(\mathcal{F}((x))) \otimes \mathcal{H}$ defined via the relations

$$L^{*+}(x)\,|m\rangle \otimes 1 \;=\; \sum_{m'\in\mathbb{Z}_{\geq 0}} |m'\rangle \otimes \frac{x^m}{m!}(b^*)^{m'} b^m \ , \tag{4.40}$$

$$L^{*-}(x)\,|m\rangle \otimes 1 \;=\; \sum_{m'\in\mathbb{Z}_{\geq 0}} |m'\rangle \otimes \frac{x^m}{m!} b^m (b^*)^{m'} \ . \tag{4.41}$$

The next result shows that we can identify the vertex weights for the $Q^{*\pm}$ vertex models with the matrix elements of the $L^{*\pm}$ operators.

Lemma 4.1.20. *The matrix elements of the $L^{*\pm}$ operators are given by*

$$\langle c,d|L^{*+}(x)|a,b\rangle \;=\; x^a \binom{c}{a} \delta_{a+d,b+c} \ , \tag{4.42}$$

$$\langle c,d|L^{*-}(x)|a,b\rangle \;=\; x^a \binom{b+c}{b} \delta_{a+d,b+c} \ . \tag{4.43}$$

Proof. The claim follows after a straightforward computation. □

Consider the vector space isomorphism $\iota : \mathcal{F}^{\otimes n} \to \tilde{\mathcal{F}}^{\otimes n}$ defined as [41]

$$|\lambda\rangle \mapsto \frac{1}{u_\lambda}\,\langle\lambda| \ . \tag{4.44}$$

This induces a scalar product on $\mathcal{F}^{\otimes n}$ which we denote by $\langle\ |\ \rangle_\iota$, and which we assume to be anti-linear in the first factor. A straightforward computation shows that the generators b_i and b_i^* are the Hermitian adjoints of each other, with respect to the scalar product $\langle\,|\,\rangle_\iota$.

Remark 4.1.21. The Heisenberg algebra admits an involutive anti-automorphism * : $\mathcal{H}_n \to \mathcal{H}_n$, whose action is given by (see e.g. [6])

$$(b_j)^* = b_j^* \ , \qquad\qquad (b_j^*)^* = b_j \ . \tag{4.45}$$

The Hermitian adjoints with respect to the scalar product $\langle\,|\,\rangle_\iota$ of the elements in $\mathcal{H}_n$ are therefore the images under the map $^* : \mathcal{H}_n \to \mathcal{H}_n$, treated as operators in $\mathrm{End}(\mathcal{F}^{\otimes n})$. Although strictly speaking ambiguous, we refer to the images under $^* : \mathcal{H}_n \to \mathcal{H}_n$ as the 'adjoint' operators.

Let $L^{\pm}_{m',m}(x)$ and $L^{*\pm}_{m',m}(x)$ be the elements defined via the relations $L^{\pm}(x)\,|m\rangle \otimes 1 = \sum_{m'\in\mathbb{Z}_{\geq 0}} |m'\rangle \otimes L^{\pm}_{m',m}(x)$ and $L^{*\pm}(x)\,|m\rangle \otimes 1 = \sum_{m'\in\mathbb{Z}_{\geq 0}} |m'\rangle \otimes L^{*\pm}_{m',m}(x)$ respectively. Comparing equations (4.6) and (4.7) respectively with (4.40) and (4.41), it follows at once that

$$L^{*\pm}_{m',m}(x) = L^{\pm}_{m',m}(x)^* \ , \tag{4.46}$$

provided that $x^* = x$. In words, the operator $L^{*\pm}_{m',m}(x)$ is the adjoint of $L^{\pm}_{m',m}(x)$.

Proposition 4.1.22. *The $L^{*\pm}$ operators are solutions of the YBE*

$$\begin{aligned} R^{+}_{12}(x/y)L^{*+}_2(y)L^{*+}_1(x) &= L^{*+}_1(x)L^{*+}_2(y)R^{+}_{12}(x/y) \ , && (4.47)\\ R^{-}_{12}(x/y)L^{*-}_2(y)L^{*-}_1(x) &= L^{*-}_1(x)L^{*-}_2(y)R^{-}_{12}(x/y) \ , && (4.48)\\ R_{12}(x/y)L^{*+}_2(y)L^{*-}_1(x) &= L^{*-}_1(x)L^{*+}_2(y)R_{12}(x/y) \ , && (4.49) \end{aligned}$$

where the operators R^+, R^- and R were defined respectively in (4.10), (4.11) *and* (4.12).

Proof. Applying both sides of (4.47) to the element $|m_1, m_2\rangle \otimes 1$, and then doing the same for (4.48) and (4.49), one ends up with a set of constraints for the operators $L^{*\pm}_{m',m}(x)$ which are the adjoint of the constraints appearing in the proof of Proposition 4.1.9 for the operators $L^{\pm}_{m',m}(x)$. It follows that the YBE (4.13), (4.14) and (4.15) are equivalent to the YBE (4.47), (4.48) and (4.49). Proposition 4.1.9 then implies the validity of the claim. □

Define the monodromy matrices $\mathbf{Q}^{*\pm}(x) \in \mathcal{F}((x)) \otimes \mathcal{H}_n$ as the operators

$$\mathbf{Q}^{*\pm}(x) = L^{*\pm}_n(x)\cdots L^{*\pm}_1(x) \ . \tag{4.50}$$

Moreover, denote with $Q^{*\pm}_{m',m}(x) \in \mathcal{H}_n \otimes \mathbb{C}((x))$ the elements defined via the relation $\mathbf{Q}^{*\pm}(x)\,|m\rangle \otimes 1 = \sum_{m'\in\mathbb{Z}_{\geq 0}} |m'\rangle \otimes Q^{*\pm}_{m',m}(x)$.

Corollary 4.1.23. *The monodromy matrices* (4.50) *are solutions of the YBE*

$$
\begin{aligned}
R^{+}_{12}(x/y)\mathbf{Q}^{*+}_{2}(y)\mathbf{Q}^{*+}_{1}(x) &= \mathbf{Q}^{*+}_{1}(x)\mathbf{Q}^{*+}_{2}(y)R^{+}_{12}(x/y)\ , && (4.51)\\
R^{-}_{12}(x/y)\mathbf{Q}^{*-}_{2}(y)\mathbf{Q}^{*-}_{1}(x) &= \mathbf{Q}^{*-}_{1}(x)\mathbf{Q}^{*-}_{2}(y)R^{-}_{12}(x/y)\ , && (4.52)\\
R_{12}(x/y)\mathbf{Q}^{*+}_{2}(y)\mathbf{Q}^{*-}_{1}(x) &= \mathbf{Q}^{*-}_{1}(x)\mathbf{Q}^{*+}_{2}(y)R_{12}(x/y)\ . && (4.53)
\end{aligned}
$$

Proof. Taking advantage of Proposition 4.1.22, one can prove the claim via induction on n (see for example [37, VI.1]). □

Lemma 4.1.24. *We have the equalities*

$$Q^{*+}_{m',m}(x) = \frac{x^m}{m!}\sum_{\alpha\in\mathcal{P}^{\geq 0}_{n-1}} \frac{x^{|\alpha|}}{\alpha_1!\cdots\alpha_{n-1}!}(b^*_n)^{m'}(b_n b^*_{n-1})^{\alpha_{n-1}}\cdots(b_2 b^*_1)^{\alpha_1} b^m_1\ , \tag{4.54}$$

$$Q^{*-}_{m',m}(x) = \frac{x^m}{m!}\sum_{\alpha\in\mathcal{P}^{\geq 0}_{n-1}} \frac{x^{|\alpha|}}{\alpha_1!\cdots\alpha_{n-1}!}(b_1)^{m}(b^*_1 b_2)^{\alpha_1}\cdots(b^*_{n-1} b_n)^{\alpha_{n-1}}(b^*_n)^{m'}\ . \tag{4.55}$$

Proof. These equalities follow by induction on n. Compare with the proof of Lemma 4.1 in [43]. □

Lemma 4.1.25. *Suppose that $x^* = x$. We have the identities*

$$Q^{*\pm}_{m',m}(x) = Q^{\pm}_{m',m}(x)^*\ . \tag{4.56}$$

Proof. The claim follows immediately by comparing the expressions for $Q^{\pm}_{m',m}(u)$ and $Q^{*\pm}_{m',m}(u)$ derived in Lemmas 4.1.12 and 4.1.24 respectively. □

Introduce Q^{*+} and Q^{*-} operators as the operators in $\mathcal{H}_n\otimes\mathbb{C}((z,x))$ which are defined via the following partial traces,

$$Q^{*\pm}(x) = \mathrm{Tr}_{\mathcal{F}}\, z^{-N}\mathbf{Q}^{*\pm}(x) = \sum_{m\in\mathbb{Z}_{\geq 0}} z^{-m} Q^{*\pm}_{m,m}(x)\ . \tag{4.57}$$

Lemma 4.1.26. *We have the identities*

$$Z^{*\pm}_{\lambda,\mu}(x_1,x_2,\ldots,x_l) = \langle\lambda|Q^{*\pm}(x_1)Q^{*\pm}(x_2)\cdots Q^{*\pm}(x_l)|\mu\rangle\ . \tag{4.58}$$

Proof. The proof of this statement is completely analogous to the one of Lemma 4.1.13, and therefore we omit it. □

Corollary 4.1.27. *We have the commutation relations*

$$[Q^{*\pm}(x), Q^{*\pm}(y)] = [Q^{*+}(x), Q^{*-}(y)] = 0\ . \tag{4.59}$$

Proof. The claim follows by employing completely analogous steps as the ones described in the proof of Corollary 4.1.14. □

Corollary 4.1.28. *Suppose that $z^* = z^{-1}$ and $x^* = x$. We have the identities*

$$Q^{*\pm}(x) = Q^{\pm}(x)^* . \tag{4.60}$$

Proof. The claim is a direct consequence of Lemma 4.1.25, together with equations (4.26) and (4.57). □

We now show that the partition functions $Z^{*\pm}_{\lambda,\mu}(x_1,\ldots,x_l)$ can be expanded in terms of the cylindric symmetric functions $e_{\lambda/d/\mu}$ and $h_{\lambda/d/\mu}$. For this purpose, we shall take advantage of the fact that, at each vertex, the allowed path configurations for the $Q^{*\pm}$ vertex models are obtained by reflecting the ones for the $Q^{\pm}$ vertex models about the vertical line passing through the edges labelled by b and d. Compare with Figures 4.1 and 4.6. In Section 4.3.2 we will evaluate $Z^{*\pm}_{\lambda,\mu}(x_1,\ldots,x_l)$ by employing the $Q^{*\pm}$ operators.

Given $\lambda,\mu \in \mathcal{A}^+_k(n)$ and $d \in \mathbb{Z}_{\geq 0}$, define $\Gamma^{*\pm}_{\lambda,\mu}(l,d) \subset \Gamma^{*\pm}_{\lambda,\mu}(l)$ as the set of allowed lattice configurations for the $Q^{*\pm}$ vertex models, where the values attached to the outer vertical edges on top and bottom are fixed respectively by μ and λ, and the number of paths crossing the boundary of the cylinder is equal to d. We shall take advantage of the involution $\vee : \mathcal{A}^+_k(n) \to \mathcal{A}^+_k(n)$, which was defined in Section 3.4.1 as

$$\lambda \mapsto \lambda^\vee = (n+1-\lambda_k, n+1-\lambda_{k-1}, \ldots, n+1-\lambda_1) . \tag{4.61}$$

Proposition 4.1.29. *(i) For every $\lambda,\mu \in \mathcal{A}^+_k(n)$ and $d \in \mathbb{Z}_{\geq 0}$, there exists a bijection between the sets $\Gamma^{*+}_{\lambda,\mu}(l,d)$ and $\mathrm{T}_{\lambda^\vee/d/\mu^\vee}(l)$. (ii) For every $\lambda,\mu \in \mathcal{A}^+_k(n)$ and $d \in \mathbb{Z}_{\geq 0}$, there exists a bijection between the sets $\Gamma^{*-}_{\lambda,\mu}(l,d)$ and $\Pi_{\lambda^\vee/d/\mu^\vee}(l)$.*

Proof. We prove part (i) first, and for this purpose we construct, for every $\lambda,\mu \in \mathcal{A}^+_k(n)$ and $d \in \mathbb{Z}_{\geq 0}$, a bijection between the sets $\Gamma^{*+}_{\lambda,\mu}(l,d)$ and $\Gamma^{+}_{\lambda^\vee,\mu^\vee}(l,d)$.

Let $\lambda,\mu \in \mathcal{A}^+_k(n)$ and $d \in \mathbb{Z}_{\geq 0}$. Suppose that $\Gamma^{*+}_{\lambda,\mu}(l,d)$ is non-empty, and let $\gamma^{*+} \in \Gamma^{*+}_{\lambda,\mu}(l,d)$. Draw a vertical line between columns $n/2$ and $n/2+1$ of the lattice if n is even, or a vertical line overlapping column $(n+1)/2$ if n is odd. Reflect the lattice, together with the paths travelling along the lattice itself, about this line. Refer to Figure 4.7 for an example. The paths are now travelling from North-West to South-East, and at each vertex one recovers the path configuration depicted in Figure 4.1 for the Q^+ vertex model. Stated otherwise, by reflecting the lattice in the way just described we end up with a lattice configuration for the Q^+ vertex model, which we denote by γ^+. The values attached to the outer vertical edges of γ^+ on top and bottom are given by $(m_n(\mu),\ldots,m_1(\mu))$ and $(m_n(\lambda),\ldots,m_1(\lambda))$ respectively. In the notation of Figure 4.2, these values coincide with

$m(\mu^\vee)$ and $m(\lambda^\vee)$, and thus $\gamma^+ \in \Gamma^+_{\lambda^\vee,\mu^\vee}(l,d)$. In conclusion, the set $\Gamma^+_{\lambda,\mu}(l,d)$ is non-empty, and the assignment $\gamma^{*+} \mapsto \gamma^+$ defines a map $\Gamma^{*+}_{\lambda,\mu}(l,d) \to \Gamma^+_{\lambda^\vee,\mu^\vee}(l,d)$.

Let once again $\lambda,\mu \in \mathcal{A}^+_k(n)$ and $d \in \mathbb{Z}_{\geq 0}$. Suppose that $\Gamma^+_{\lambda^\vee,\mu^\vee}(l,d)$ is non-empty, and let $\gamma^+ \in \Gamma^+_{\lambda^\vee,\mu^\vee}(l,d)$. Reflecting the lattice about the same vertical line introduced above, and using the fact that the map $\vee : \mathcal{A}^+_k(n) \to \mathcal{A}^+_k(n)$ is an involution, we end up with a unique lattice configuration $\gamma^{*+} \in \Gamma^{*+}_{\lambda,\mu}(l,d)$. It follows that $\Gamma^{*+}_{\lambda,\mu}(l,d)$ is non-empty, and that $\gamma^+ \mapsto \gamma^*$ defines a map $\Gamma^+_{\lambda^\vee,\mu^\vee}(l,d) \to \Gamma^{*+}_{\lambda,\mu}(l,d)$, which is clearly the inverse of the map $\Gamma^{*+}_{\lambda,\mu}(l,d) \to \Gamma^+_{\lambda^\vee,\mu^\vee}(l,d)$ described above. We conclude that for every $\lambda,\mu \in \mathcal{A}^+_k(n)$ and $d \in \mathbb{Z}_{\geq 0}$ there exists a bijection between the sets $\Gamma^{*+}_{\lambda,\mu}(l,d)$ and $\Gamma^+_{\lambda^\vee,\mu^\vee}(l,d)$. Employing Proposition 4.1.16, we then deduce the validity of part (i) of the claim.

The proof of part (ii) is completely analogous. One can prove that the sets $\Gamma^{*-}_{\lambda,\mu}(l,d)$ and $\Gamma^-_{\lambda^\vee,\mu^\vee}(l,d)$ are either both empty or non-empty. In the second case, one can define a map $\Gamma^{*-}_{\lambda,\mu}(l,d) \to \Gamma^-_{\lambda^\vee,\mu^\vee}(l,d)$, together with the inverse $\Gamma^-_{\lambda^\vee,\mu^\vee}(l,d) \to \Gamma^{*-}_{\lambda,\mu}(l,d)$, by reflecting the lattice configurations about the same vertical line described in the proof of part (i). It follows that for every $\lambda,\mu \in \mathcal{A}^+_k(n)$ and $d \in \mathbb{Z}_{\geq 0}$ there exists a bijection between the sets $\Gamma^{*-}_{\lambda,\mu}(l,d)$ and $\Gamma^-_{\lambda^\vee,\mu^\vee}(l,d)$. Proposition 4.1.15 then implies the validity of part (ii) of the claim.

□

Theorem 4.1.30. *Let $\lambda,\mu \in \mathcal{A}^+_k(n)$. The partition functions of the $Q^{*\pm}$ vertex models have the expansion*

$$Z^{*+}_{\lambda,\mu}(x_1,x_2,\ldots,x_l) = \sum_{d\in\mathbb{Z}_{\geq 0}} z^{-d} e_{\lambda^\vee/d/\mu^\vee}(x_1,x_2,\ldots,x_l)\,, \tag{4.62}$$

$$Z^{*-}_{\lambda,\mu}(x_1,x_2,\ldots,x_l) = \prod_{i=1}^{l} \frac{1}{1-z^{-1}x_i^n} \sum_{d\in\mathbb{Z}_{\geq 0}} z^{-d} h_{\lambda^\vee/d/\mu^\vee}(x_1,x_2,\ldots,x_l)\,. \tag{4.63}$$

Proof. We shall prove (4.62) first. Let $d \in \mathbb{Z}_{\geq 0}$, $\gamma^{*+} \in \Gamma^{*+}_{\lambda,\mu}(l,d)$, and denote with γ^+ the lattice configuration of the Q^+ vertex model which is the image of γ^{*+} under the map $\Gamma^{*+}_{\lambda,\mu}(l,d) \to \Gamma^+_{\lambda^\vee,\mu^\vee}(l,d)$ introduced in the proof of Proposition 4.1.29. Denote with $v_{i,j}$ the vertex obtained by intersecting the i-th row of the lattice with the j-th column. If we reflect the lattice about the vertical line described in the proof of Proposition 4.1.29, it follows that $v_{i,j}$ is mapped to $v_{i,n+1-j}$. Label the edges associated to the lattice configuration γ^+ as in the proof of Lemma 4.1.13 (compare with Figure 4.4). In particular, the vertex configuration assigned to the 'reflected vertex' $v_{i,n+1-j}$ consists of the four integers $\{a^{(i)}_{n+1-j}, m^{(i-1)}_{n+1-j}, a^{(i)}_{n+2-j}, m^{(i)}_{n+1-j}\}$ oriented as in Figure 4.1. It follows that the vertex configuration associated to the 'non reflected vertex' $v_{i,j}$ is given by the integers

$\{a^{(i)}_{n+2-j}, m^{(i-1)}_{n+1-j}, a^{(i)}_{n+1-j}, m^{(i)}_{n+1-j}\}$, and then we have the identity

$$\mathrm{wt}^{*+}(v_{i,j}) = x^{a^{(i)}_{n+2-j}} \frac{a^{(i)}_{n+1-j}!}{a^{(i)}_{n+2-j}!} \binom{m^{(i)}_{n+1-j}}{a^{(i)}_{n+1-j}} . \tag{4.64}$$

Let $\hat{T}$ be the CRST of shape $\lambda^\vee/d/\mu^\vee$ and highest entry l which is the image of γ^+ under the map $\Gamma^+_{\lambda^\vee,\mu^\vee}(l,d) \to \mathrm{T}_{\lambda^\vee/d/\mu^\vee}(l)$ described in Proposition 4.1.15. By definition, $\hat{T}$ is equivalent to the sequence $\{\hat{\lambda}^{(i)}.\tau^{d_i}\}_{i=0}^l$ of cylindric partitions which are defined via the relations $m_j(\lambda^{(i)}) = m^{(i)}_j$ and $(\hat{\lambda}^{(i)}.\tau^{d_i})'_j - (\hat{\lambda}^{(i-1)}.\tau^{d_{i-1}})'_j = a^{(i)}_j$. Employing (4.64), we have that

$$\begin{aligned}
\mathrm{wt}^{*+}(\gamma^{*+}) &= \prod_{i=1}^{l}\prod_{j=1}^{n} \mathrm{wt}^{*+}(v_{i,j}) = \prod_{i=1}^{l}\prod_{j=1}^{n} \mathrm{wt}^{*+}(v_{i,n+1-j}) = \prod_{i=1}^{l}\prod_{j=1}^{n} x_i^{a^{(i)}_{j+1}} \frac{a^{(i)}_j!}{a^{(i)}_{j+1}!}\binom{m^{(i)}_j}{a^{(i)}_j} \\
&= \prod_{i=1}^{l} x^{\sum_{j=1}^n a^{(i)}_j} \prod_{j=1}^{n} \binom{(\hat{\lambda}^{(i)}.\tau^{d_i-d_{i-1}})'_j - (\hat{\lambda}^{(i)}.\tau^{d_i-d_{i-1}})'_{j+1}}{(\hat{\lambda}^{(i)}.\tau^{d_i-d_{i-1}})'_j - (\hat{\lambda}^{(i-1)})'_j} \\
&= \prod_{i=1}^{l} x_i^{\mathrm{wt}_i(\hat{T})} \psi_{\lambda^{(i)}/(d_i-d_{i-1})/\lambda^{(i-1)}} = \psi_{\hat{T}} x^{\hat{T}} .
\end{aligned}$$

Taking advantage of (4.39), the relation $\Gamma^{*+}_{\lambda,\mu}(l) = \bigsqcup_{d\in\mathbb{Z}_{\geq 0}} \Gamma^{*+}_{\lambda,\mu}(l,d)$, and Proposition 4.1.29, we end up with the identity

$$Z^{*+}_{\lambda,\mu}(x_1,x_2,\ldots,x_l) = \sum_{d\in\mathbb{Z}_{\geq 0}} z^{-d} \sum_{\hat{T}\in \mathrm{T}_{\lambda^\vee/d/\mu^\vee}(l)} \psi_{\hat{T}} x^{\hat{T}} .$$

The latter then implies the validity of (4.62), thanks to equation (4.31).

We now prove (4.63). Let $\gamma^{*-} \in \Gamma^{*-}_{\lambda,\mu}(l,d)$, and denote with γ^- the lattice configuration of the Q^- model which is the image of γ^{*-} under the map $\Gamma^{*-}_{\lambda,\mu}(l,d) \to \Gamma^-_{\lambda^\vee,\mu^\vee}(l,d)$ introduced in the proof of Proposition 4.1.29. Label the edges associated to the lattice configuration γ^- as in the proof of Lemma 4.1.13. We can then express $\mathrm{wt}^{*-}(v_{i,j})$ in terms of the vertex configuration of the 'reflected vertex' v_{n+1-j} via the following equality,

$$\mathrm{wt}^{*-}(v_{i,j}) = x^{a^{(i)}_{n+2-j}} \frac{a^{(i)}_{n+1-j}!}{a^{(i)}_{n+2-j}!} \binom{a^{(i)}_{n+1-j} + m^{(i-1)}_{n+1-j}}{m^{(i-1)}_{n+1-j}} . \tag{4.65}$$

Let $\hat{\pi}$ be the CRPP of shape $\lambda^\vee/d/\mu^\vee$ and highest entry l which is the image of γ^- under the map $\Gamma^-_{\lambda^\vee,\mu^\vee}(l,d) \to \Pi_{\lambda^\vee/d/\mu^\vee}(l)$ described in Proposition 4.1.15. By definition, $\hat{\pi}$ is equivalent to the sequence $\{\hat{\lambda}^{(i)}.\tau^{d_i}\}_{i=0}^l$ of cylindric partitions which are defined via the relations $m_j(\lambda^{(i)}) = m^{(i)}_j$ and $(\hat{\lambda}^{(i)}.\tau^{d_i})'_j - (\hat{\lambda}^{(i-1)}.\tau^{d_{i-1}})'_j = a^{(i)}_j$. Thanks to (4.65), we have

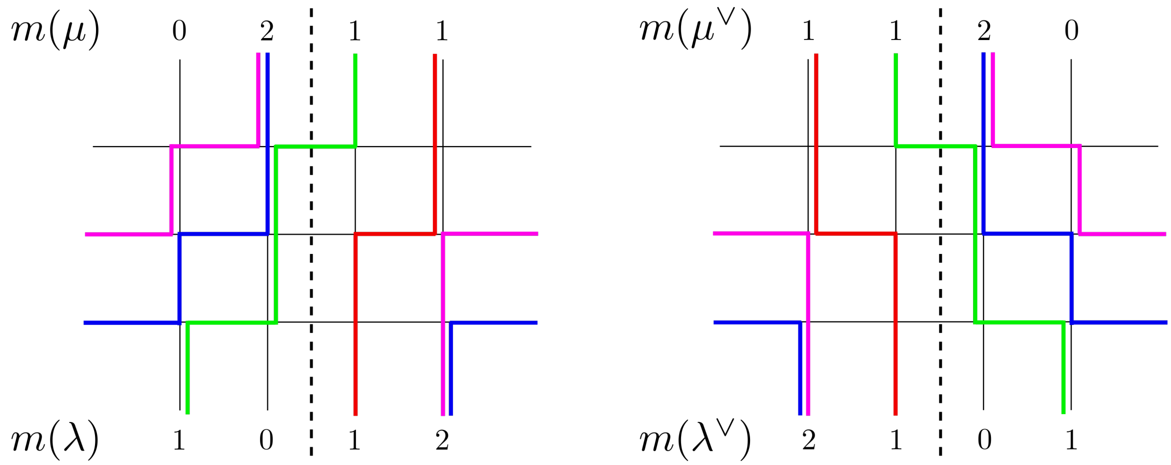


Figure 4.7: Let $l=3$, $n=4$, $k=4$, $\mu=(4,3,2,2)$ and $\lambda=(4,4,3,1)$. On the left we have an allowed lattice configuration $\gamma^{*+}\in\Gamma^{*+}_{\lambda,\mu}(l,d)$ for the Q^{*+} vertex model. On the right we have the allowed vertex configuration γ^{+} for the Q^{+} vertex model which is obtained from γ^{*+} by reflecting the lattice and the paths about the vertical dashed line passing between columns 2 and 3. This is the image of γ^{*+} under the map $\Gamma^{*+}_{\lambda,\mu}(l,d)\to\Gamma^{+}_{\lambda^\vee,\mu^\vee}(l,d)$ defined in the proof of Proposition 4.1.29.

that

$$\begin{aligned}
\mathrm{wt}^{*-}(\gamma^{*-}) &= \prod_{i=1}^{l}\prod_{j=1}^{n}\mathrm{wt}^{*-}(v_{i,j}) = \prod_{j=1}^{n}\mathrm{wt}^{*-}(v_{i,n+1-j}) = \prod_{i=1}^{l}\prod_{j=1}^{n} x_i^{a_{j+1}^{(i)}}\frac{a_{j+1}^{(i)}!}{a_j^{(i)}!}\binom{a_j^{(i)}+m_j^{(i-1)}}{m_j^{(i-1)}} \\
&= \prod_{i=1}^{l} x^{\sum_{j=1}^{n}a_j^{(i)}}\prod_{j=1}^{n}\binom{(\hat\lambda^{(i)}.\tau^{d_i-d_{i-1}})'_j-(\hat\lambda^{(i-1)})'_{j+1}}{(\hat\lambda^{(i-1)})'_j-(\hat\lambda^{(i-1)})'_{j+1}} \\
&= \prod_{i=1}^{l} x_i^{\mathrm{wt}_i(\hat\pi)}\tilde\theta_{\lambda^{(i)}/(d_i-d_{i-1})/\lambda^{(i-1)}} = \tilde\theta_{\hat\pi}x^{\hat\pi}\,.
\end{aligned}$$

In the last line we employed the relation (4.35) for the weight $\tilde\theta_{\lambda/d/\mu}$. Taking advantage of (4.39) once again, it follows that

$$Z^{*-}_{\lambda,\mu}(x_1,x_2,\ldots,x_l) = \sum_{d\in\mathbb{Z}_{\geq 0}} z^{-d}\sum_{\hat\pi\in\Pi_{\lambda^\vee/d/\mu^\vee}(l)}\tilde\theta_{\hat\pi}x^{\hat\pi}\,.$$

One can then prove the validity of (4.63) by employing similar steps as the ones described after equation (4.36).

□

Corollary 4.1.31. *Let $\lambda,\mu\in\mathcal{A}_k^+(n)$ and suppose that $z^*=z^{-1}$ and $x_i^*=x_i$ for $i=1,\ldots,l$. We have the following identities involving the partition functions of the $Q^\pm$ and*

$Q^{*\pm}$ vertex models,

$$Z^{*\pm}_{\lambda,\mu}(x_1, x_2, \ldots, x_l) = Z^{\pm}_{\lambda^\vee,\mu^\vee}(x_1, x_2, \ldots, x_l)^* \ , \tag{4.66}$$

and

$$Z^{*\pm}_{\lambda,\mu}(x_1, x_2, \ldots, x_l) = \frac{u_\lambda}{u_\mu} Z^{\pm}_{\mu,\lambda}(x_1, x_2, \ldots, x_l)^* \ . \tag{4.67}$$

Proof. Equation (4.66) follows immediately from Theorems 4.1.18 and 4.1.30, since the map $\vee : \mathcal{A}^+_k(n) \to \mathcal{A}^+_k(n)$ is an involution. Recall that the stabiliser subgroup of $\lambda \in \mathcal{P}^+_k$ has cardinality $|S_\lambda| = \prod_{i\geq 0} m_i(\lambda)!$ (see Section 2.1.2). Moreover, notice that we have the equalities $|S_\lambda| = u_\lambda$ and $|S_\mu| = u_\mu$, which follow from the fact that $m_0(\lambda) = m_0(\mu) = 0$ as $\lambda, \mu \in \mathcal{A}^+_k(n)$. Equation (4.67) is then a direct consequence of Proposition 3.4.7. □

4.2 The conserved charges for the free boson model

It is well known [3] that in many situations of interest the transfer matrix of a 2-dimensional classical vertex model commutes with the Hamiltonian of a 1-dimensional quantum model. As explained below, this is also true for the vertex models described in the previous section: the $Q^\pm$ and $Q^{*\pm}$ operators, which were introduced respectively in (4.26) and (4.57), commute with the Hamiltonian of the free boson model with periodic boundary conditions. This arises as the limit $q = 1$ for the Hamiltonian of the q-boson model [48], and it is defined as

$$H = -\sum_{i=1}^{n} (b^*_{i+1} b_i + b^*_i b_{i+1} - 2 b^*_i b_i) \ , \tag{4.68}$$

We set $b^*_{n+1} = z b_1$ and $b_{n+1} = z^{-1} b_1$, where the formal variable z was introduced in (4.2). We shall consider free bosons on a 1-dimensional lattice defined on a circle. Assuming that the lattice has n sites, we identify the boundary of the circle with a point placed between sites n and 1. The 'ket' vector $|\lambda\rangle$, which was introduced in (4.18), represents a quantum state with $m_i(\lambda)$ bosons sitting at site i, and the vector space $\mathcal{F}^{\otimes n}$ spanned by all the states $|\lambda\rangle$ is known as the 'Fock space'. The generator b^*_i creates a boson at site i, whereas b_i annihilates a boson at the same site. Thus, each term of the form $b^*_i b_j$ moves one boson from site j (if there are any) to site i. For this reason, the free boson model defined by the Hamiltonian (4.68) is sometimes referred as a 'hopping' model. Whenever a boson crosses the boundary of the circle clockwise via the action of the operator $b^*_{n+1} b_n = z b^*_1 b_n$, the quantum states acquire a factor equal to z. Similarly, if a boson crosses the boundary of the circle counterclockwise, then the quantum states acquire a factor equal to z^{-1}. This is because we have the relation $b^*_n b_{n+1} = z^{-1} b^*_n b_1$. While z is treated as a formal variable here, in the quantum mechanical setting it would be evaluated in the unit circle and, hence, we will call it a ‘phase factor’. The operator $b_i^* b_i$ counts the number of bosons at site i via the eigenvalue equation $b_i^* b_i \,|\lambda\rangle = m_i(\lambda)\,|\lambda\rangle$, which follows after a straightforward computation. In particular, the number operator $N = \sum_{i=1}^n b_i^* b_i$ counts how many bosons are sitting in the lattice.

In the context of quantum integrability, the methodology described in Section 4.1.2 is part of what is known as the ‘Quantum Inverse Scattering Method’ (QISM) [22, 62–64]. The starting point of the QISM is represented by the Yang-Baxter equation (see Propositions 4.1.9 and 4.1.22), which can be employed to construct a set of quantum commuting operators. We explain how to achieve this from the discussion presented in Sections 4.1.2 and 4.1.4. Set $\mathcal{H}_n[z, z^{-1}] = \mathcal{H}_n \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}]$, and define the operators $\{Q_r^\pm\}_{r\in\mathbb{Z}_{\geq 0}} \in \mathcal{H}_n[z, z^{-1}]$ via the following expansion of the $Q^\pm$ operators,

$$Q^\pm(x) = \sum_{r\in\mathbb{Z}_{\geq 0}} x^r Q_r^\pm \ . \tag{4.69}$$

Similarly, define the operators $\{Q_r^{*\pm}\}_{r\in\mathbb{Z}_{\geq 0}} \in \mathcal{H}_n[z, z^{-1}]$ via the expansion

$$Q^{*\pm}(x) = \sum_{r\in\mathbb{Z}_{\geq 0}} x^r Q_r^{*\pm} \ . \tag{4.70}$$

In the following, we assume that z satisfies the relations $z^* = \bar{z} = z^{-1}$.

Lemma 4.2.1. *For $r \in \mathbb{Z}_{\geq 0}$, we have the identity*

$$Q_r^{*\pm} = (Q_r^\pm)^* \ . \tag{4.71}$$

Proof. The claim follows by taking advantage of equations (4.69), (4.70) and Corollary 4.1.28. □

Corollaries 4.1.14 and 4.1.27 immediately imply that for every $r, s \in \mathbb{Z}_{\geq 0}$ we have the commutation relations

$$[Q_r^\pm, Q_s^\pm] = [Q_r^+, Q_s^-] = 0 \ , \qquad \text{and} \qquad [Q_r^{*\pm}, Q_s^{*\pm}] = [Q_r^{*+}, Q_s^{*-}] = 0 \ . \tag{4.72}$$

Notice that these commutation relations are not independent. Namely, thanks to Lemma 4.2.1, we can recover the ones on the right by taking the adjoint of the ones on the left, and vice versa.

Lemma 4.2.2. *For $r \in \mathbb{Z}_{\geq 0}$, we have the identities*

$$Q_r^+ = \sum_{\alpha \in \mathcal{P}_n^{\geq 0}} \frac{(zb_1^*)^{\alpha_n}(b_1 b_2^*)^{\alpha_1} \cdots (b_{n-1}b_n^*)^{\alpha_{n-1}} b_n^{\alpha_n}}{\alpha_1! \cdots \alpha_n!} , \tag{4.73}$$

$$Q_r^- = \sum_{\alpha \in \mathcal{P}_n^{\geq 0}} \frac{b_n^{\alpha_n}(b_{n-1}b_n^*)^{\alpha_{n-1}} \cdots (b_1 b_2^*)^{\alpha_1}(zb_1^*)^{\alpha_n}}{\alpha_1! \cdots \alpha_n!} , \tag{4.74}$$

where both sums are restricted to those $\alpha \in \mathcal{P}_n^{\geq 0}$ with $|\alpha| = r$. In particular, we have that $Q_0^+ = Q_0^- = 1$.

Proof. The claim follows from [43, Lemma 4.1] by taking the limit $q = 1$. □

The goal of this section is to prove that the free boson model is 'quantum integrable'. Stated otherwise, we will show that the operators $\{Q_r^\pm\}_{r \in \mathbb{Z}_{\geq 0}}$ are quantum integrals of motion, that is they commute with the Hamiltonian (4.68). This is one of the many characterisations for quantum integrability that are adopted in the literature; see [14] for details. Since the Hamiltonian (4.68) is self-adjoint, it follows from Lemma 4.2.1 that the operators $\{Q_r^{*\pm}\}_{r \in \mathbb{Z}_{\geq 0}}$ commute with (4.68) as well. Moreover, thanks to (4.69) and (4.70), we deduce that the $Q^\pm$ and $Q^{*\pm}$ operators also commute with the Hamiltonian (4.68). In agreement with the physics literature, we refer to the quantum integrals of motion as 'conserved charges'.

4.2.1 Functional relations for the conserved charges

The strategy we employ to prove quantum integrability of the free boson model is to show that the operators $\{Q_r^\pm\}_{r \in \mathbb{Z}_{\geq 0}}$ and $\{Q_r^{*\pm}\}_{r \in \mathbb{Z}_{\geq 0}}$ belong to the same commutative subalgebra of $\mathcal{H}_n[z, z^{-1}]$ (see Definition 4.2.6 and Proposition 4.2.10 below). To this end, we will make use of the following simpler operators,

$$T_r = \sum_{i=1}^{n} b_{i+r}^* b_i , \tag{4.75}$$

$$\tilde{T}_r = \sum_{i=1}^{n} b_i b_{i+r}^* , \tag{4.76}$$

for $r \in \mathbb{Z} \setminus \{0\}$, together with $T_0 = \tilde{T}_0 = 1$. The operators b_i and b_i^* for $i > n$ and $i < -1$ are defined via the relations

$$b_{i-n} = zb_i , \qquad \text{and} \qquad b_{i+n}^* = zb_i^* . \tag{4.77}$$

Since $z^* = z^{-1}$, it follows that the operators b_i^* and b_i are adjoint of each other for all $i \in \mathbb{Z}$. Notice that for $i, j \in \mathbb{Z}$ we have the commutation relation

$$[b_i, b_j^*] = \begin{cases} 0, & (j-i) \bmod n \neq 0 \\ z^{\frac{j-i}{n}}, & (j-i) \bmod n = 0 \end{cases} . \tag{4.78}$$

Setting $z = 1$, we have that the operators $\{T_1, \ldots, T_{n-1}\}$ are the limit $q = 1$ of similar operators appearing in the discussion of the q-boson model [9]. The latter arise from a solution of the YBE which holds only for $q \neq 1$. Notice that $z^{-1}T_n$ is equal to the number operator $N = \sum_{i=1}^n b_i^* b_i$ introduced above, thanks to the relations (4.77). Let us present some properties for the operators $\{T_r\}_{r\in\mathbb{Z}}$ and $\{\tilde{T}_r\}_{r\in\mathbb{Z}}$.

Lemma 4.2.3. *For $r \in \mathbb{Z}$ we have the relations*

$$T_r^* = T_{-r} , \tag{4.79}$$
$$\tilde{T}_r^* = \tilde{T}_{-r} . \tag{4.80}$$

Proof. These relations follow after a straightforward computation, with the help of (4.77). □

Lemma 4.2.4. *For $r \in \mathbb{Z} \setminus \{0\}$, we have the identity*

$$\tilde{T}_r = \begin{cases} T_r, & r \bmod n \neq 0 \\ T_r + n z^{\frac{r}{n}}, & r \bmod n = 0 \end{cases} . \tag{4.81}$$

Proof. The claim follows by taking advantage of (4.77) and (4.78). □

Lemma 4.2.5. *For $r \in \mathbb{Z}$ with $r \neq 0, n$ we have the relations*

$$T_r = zT_{r-n} , \qquad \text{and} \qquad \tilde{T}_r = z\tilde{T}_{r-n} . \tag{4.82}$$

Proof. The claim can be deduced by employing (4.77). □

Thanks to Lemmas 4.2.4 and 4.2.5, it follows that $\{T_r\}_{r\in\mathbb{Z}}$ and $\{\tilde{T}_r\}_{r\in\mathbb{Z}}$ can be expressed solely in terms of the operators $\{1\} \cup \{T_1, \ldots, T_n\}$. A similar statement is true for the Hamiltonian (4.68) of the free boson model, which can be expressed as

$$H = -(T_1 + T_{-1} - 2z^{-1}T_n) . \tag{4.83}$$

Moreover, the operators $\{T_1, \ldots, T_n\}$ are algebraically independent, as we will show in Lemma 4.2.11.

Definition 4.2.6. Let $\mathcal{T}_n[z, z^{-1}]$ be the unital subalgebra of $\mathcal{H}_n[z, z^{-1}]$ generated by the operators $\{T_1, \ldots, T_n\}$.

We will show in Proposition 4.2.10 that $\mathcal{T}_n[z, z^{-1}]$ is abelian, and that the operators $\{Q_r^{\pm}\}_{r\in\mathbb{Z}_{\geq 0}}$ and $\{Q_r^{*\pm}\}_{r\in\mathbb{Z}_{\geq 0}}$ belong to $\mathcal{T}_n[z, z^{-1}]$. This implies in particular that the conserved charges are normal operators, thanks to (4.79). Since the Hamiltonian (4.68) belongs to $\mathcal{T}_n[z, z^{-1}]$ as well, it follows that the free boson model is quantum integrable, according to the definition of quantum integrability that we discussed at the beginning of this section. Before proceeding with the proof of the above statement, we need to show the validity of some functional relations between the operators defined in this section (see Proposition 4.2.8 and Corollary 4.2.9). The simplest among these relations is given by $T_1 = Q_1^+ = Q_1^-$, which can be deduced from Lemma 4.2.2 and equation (4.75). Let us present a preliminary result.

Lemma 4.2.7. *For $r \in \mathbb{N}$ and $j \in \mathbb{Z}$ we have the commutation relations*

$$b_j Q_r^+ = Q_r^+ b_j + Q_{r-1}^+ b_{j-1} \,, \tag{4.84}$$

$$Q_r^+ b_j^* = b_j^* Q_r^+ + b_{j+1}^* Q_{r-1}^+ \,, \tag{4.85}$$

together with

$$b_j Q_r^- = Q_r^- b_j + b_{j-1} Q_{r-1}^- \,, \tag{4.86}$$

$$Q_r^- b_j^* = b_j^* Q_r^- + Q_{r-1}^- b_{j+1}^* \,. \tag{4.87}$$

Proof. These equations are the limit $q = 1$ of similar commutation relations appearing in [43, Theorem 5.2 and Lemma 5.3]. Nevertheless we present a proof of their validity, and at the same time we introduce some notation which will be used again in the proof of Proposition 4.2.8. Set

$$Q^+(\alpha) = \frac{(zb_1^*)^{\alpha_n}(b_1 b_2^*)^{\alpha_1} \cdots (b_{n-1} b_n^*)^{\alpha_{n-1}} b_n^{\alpha_n}}{\alpha_1! \cdots \alpha_n!}$$

for $\alpha \in \mathcal{P}_n^{\geq 0}$, and $Q^+(\alpha) = 0$ if at least one part of $\alpha \in \mathcal{P}_n$ is negative. Then we can write $Q_r^+ = \sum_{\alpha\in\mathcal{P}_n^{\geq 0}} Q^+(\alpha)$, where the sum is restricted to those $\alpha \in \mathcal{P}_n^{\geq 0}$ with $|\alpha| = r$. Denote with $\epsilon_1, \ldots, \epsilon_n$ the standard basis of the $\mathfrak{gl}_n$ weight lattice $\mathcal{P}_n$, and recall that the notation $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathcal{P}_n$, which was introduced in Section 2.1.2, stands for $\alpha = \sum_{i=1}^n \alpha_i \epsilon_i$. Taking advantage of the commutation relation $[(b_{j-1}b_j^*)^{\alpha_{j-1}}, b_j] = -\alpha_{j-1}(b_{j-1}b_j^*)^{\alpha_{j-1}-1} b_{j-1}$, we end up with

$$\begin{aligned} [Q^+(\alpha), b_j] &= \frac{(zb_1^*)^{\alpha_n}(b_1 b_2^*)^{\alpha_1} \cdots [(b_{j-1}b_j^*)^{\alpha_{j-1}}, b_j] \cdots (b_{n-1} b_n^*)^{\alpha_{n-1}} b_n^{\alpha_n}}{\alpha_1! \cdots \alpha_n!} \\ &= -Q^+(\alpha - \epsilon_{j-1}) b_{j-1} \end{aligned}$$

for $j = 2, \dots, n$, whereas the commutation relation $[(b_1^*)^{\alpha_n}, b_1] = -\alpha_n (b_1^*)^{\alpha_n - 1}$ implies that $[Q^+(\alpha), b_1] = -zQ^+(\alpha - \epsilon_n) b_n$. With the help of (4.77), it follows that

$$[Q^+(\alpha), b_j] = -Q^+(\alpha - \epsilon_{(j-1)\,\mathrm{Mod}\,n}) b_{j-1} \tag{4.88}$$

for all $j \in \mathbb{Z}$, where Mod was defined in (3.74). Summing both sides of this last equality over all $\alpha \in \mathcal{P}_n^{\geq 0}$ with $|\alpha| = r$, we have that

$$[Q_r^+, b_j] = -\Big(\sum_{\alpha \in \mathcal{P}_n^{\geq 0}} Q^+(\alpha - \epsilon_{(j-1)\,\mathrm{Mod}\,n}) \Big) b_{j-1} \ .$$

Notice that, on the RHS of this last equality, the weight $\gamma = \alpha - \epsilon_{(j-1)\,\mathrm{Mod}\,n}$ ranges over all $\mathcal{P}_n^{\geq 0}$ with $|\gamma| = r - 1$, and that $Q^+(\gamma) = 0$ if any of the parts of γ are negative. It follows that $\sum_{\alpha \in \mathcal{P}_n^{\geq 0}} Q^+(\alpha - \epsilon_{(j-1)\,\mathrm{Mod}\,n}) = Q_{r-1}^+$, and then we deduce the validity of (4.84) from the equality above. Equation (4.85) can be proved in a similar way. For this purpose, one has to take advantage of the commutation relation

$$[Q^+(\alpha), b_j^*] = b_{j+1}^* Q^+(\alpha - \epsilon_j) \ , \tag{4.89}$$

which follows after a straightforward computation.

The proof of (4.86) and (4.87) is analogous to the proof of (4.84) and (4.85). Set

$$Q^-(\alpha) = \frac{b_n^{\alpha_n} (b_{n-1} b_n^*)^{\alpha_{n-1}} \cdots (b_1 b_2^*)^{\alpha_1} (z b_1^*)^{\alpha_n}}{\alpha_1! \cdots \alpha_n!}$$

for $\alpha \in \mathcal{P}_n^{\geq 0}$, and $Q^-(\alpha) = 0$ if at least one of the parts of $\alpha \in \mathcal{P}_n$ is negative. This implies that $Q_r^- = \sum_{\alpha \in \mathcal{P}_n^{\geq 0}} Q^-(\alpha)$, where the sum is restricted to those $\alpha \in \mathcal{P}_n^{\geq 0}$ with $|\alpha| = r$. Following similar steps as the ones described above, one can show that

$$[Q^-(\alpha), b_j] = -b_{j-1} Q^-(\alpha - \epsilon_{(j-1)\,\mathrm{Mod}\,n}) \ , \tag{4.90}$$

$$[Q^-(\alpha), b_j^*] = Q^-(\alpha - \epsilon_j) b_{j+1}^* \ , \tag{4.91}$$

and summing both sides of these commutation relations over all weights $\alpha \in \mathcal{P}_n^{\geq 0}$ with $|\alpha| = r$, one can finally deduce the validity of (4.86) and (4.87) respectively.

□

Proposition 4.2.8. *The following functional equations are valid for all $r \in \mathbb{N}$,*

$$rQ_r^+ = \sum_{l=1}^{r} (-1)^{l-1} T_l Q_{r-l}^+ \ , \tag{4.92}$$

$$rQ_r^- = \sum_{l=1}^{r} \tilde{T}_l Q_{r-l}^- \ . \tag{4.93}$$

Proof. We adopt the same notation used in the proof of Lemma 4.2.7. Let us start by showing the validity of (4.92) first. If $r = 1$, the latter reduces to the identity $Q_1^+ = T_1$, so assume that $r > 1$. Using the relation $b_i(b_{i-1}b_i^*)^{\alpha_{i-1}} = (b_{i-1}b_i^*)^{\alpha_{i-1}}b_i + \alpha_{i-1}(b_{i-1}b_i^*)^{\alpha_{i-1}-1}b_{i-1}$, we end up for $i = 2, \dots, n-1$ with the chain of equalities

$$\begin{aligned} b_{i+1}^* b_i Q^+(\alpha) &= z^{\alpha_n}\frac{(b_1^*)^{\alpha_n}\cdots b_i(b_{i-1}b_i^*)^{\alpha_{i-1}}b_{i+1}^*(b_ib_{i+1}^*)^{\alpha_i}\cdots b_n^{\alpha_n}}{\alpha_1!\cdots\alpha_n!} \\ &= z^{\alpha_n}\frac{(b_1^*)^{\alpha_n}\cdots(b_{i-1}b_i^*)^{\alpha_{i-1}}(b_ib_{i+1}^*)^{\alpha_i+1}\cdots b_n^{\alpha_n}}{\alpha_1!\cdots\alpha_n!} \\ &\quad + z^{\alpha_n}\frac{(b_1^*)^{\alpha_n}\cdots(b_{i-1}b_i^*)^{\alpha_{i-1}-1}b_{i-1}b_{i+1}^*(b_ib_{i+1}^*)^{\alpha_i}\cdots b_n^{\alpha_n}}{\alpha_1!\cdots(\alpha_{i-1}-1)!\cdots\alpha_n!} \\ &= (\alpha_i+1)Q^+(\alpha+\epsilon_i) + b_{i+1}^*Q^+(\alpha-\epsilon_{i-1})b_{i-1}\,. \end{aligned}$$

A similar computation shows that $b_1^* b_n Q^+(\alpha) = (\alpha_n+1)Q^+(\alpha+\epsilon_n) + b_1^*Q^+(\alpha-\epsilon_{n-1})b_{n-1}$ and $b_2^*b_1Q^+(\alpha) = (\alpha_1+1)Q^+(\alpha+\epsilon_1) + b_2^*Q^+(\alpha-\epsilon_n)zb_n$. Taking advantage of (4.77), we deduce the identity

$$b_{i+1}^* b_i Q^+(\alpha) = (\alpha_i+1)Q^+(\alpha+\epsilon_i) + b_{i+1}^*Q^+(\alpha-\epsilon_{(i-1)\,\mathrm{Mod}\,n})b_{i-1}\,,$$

which is valid for all $i \in \mathbb{Z}$. This, together with a repeated application of (4.88), implies that

$$b_{i+1}^* b_i Q^+(\alpha) = (\alpha_i+1)Q^+(\alpha+\epsilon_i) + \sum_{l=1}^{|\alpha|} b_{i+1}^* b_{i-l}(-1)^{l-1}Q^+\Big(\alpha - \sum_{j=1}^{l}\epsilon_{(i-j)\,\mathrm{Mod}\,n}\Big)\,.$$

The first sum on the RHS is restricted to those $l \in \mathbb{N}$ with $l \leq |\alpha|$. In fact, for $l > |\alpha|$ the weight $\alpha - \sum_{j=1}^{l}\epsilon_{(i-j)\,\mathrm{Mod}\,n}$ has at least one part smaller than 0, and then we have that $Q^+\big(\alpha - \sum_{j=1}^{l}\epsilon_{(i-j)\,\mathrm{Mod}\,n}\big) = 0$. Summing both sides of the equality above over $i = 1, \dots, n$ and $\alpha \in \mathcal{P}_n^{\geq 0}$ with $|\alpha| = r-1$, we end up with

$$\begin{aligned} T_1 Q_{r-1}^+ &= \underbrace{\sum_{i=1}^{n}\sum_{\alpha\in\mathcal{P}_n^{\geq 0}}(\alpha_i+1)Q^+(\alpha+\epsilon_i)}_{1} \\ &\quad \underbrace{+\sum_{i=1}^{n}\sum_{l=1}^{r-1} b_{i+1}^* b_{i-l}(-1)^{l-1}\sum_{\alpha\in\mathcal{P}_n^{\geq 0}} Q^+\Big(\alpha - \sum_{j=1}^{l}\epsilon_{(i-j)\,\mathrm{Mod}\,n}\Big)}_{2}\,. \end{aligned}$$

For every $\beta \in \mathcal{P}_n^{\geq 0}$ with $|\beta| = r$, the term $Q^+(\beta)$ appears $\sum_{i=1}^{n}\beta_i = r$ times in 1, and thus $1 = rQ_r^+$. On the other hand, we have that

$$\sum_{\alpha\in\mathcal{P}_n^{\geq 0}} Q^+\left(\alpha - \sum_{j=1}^{l} \epsilon_{(i-j)\,\mathrm{Mod}\,n}\right) = Q^+_{r-1-l}\,.$$

This is because, on the LHS of this last equality, the weight $\gamma = \alpha - \sum_{j=1}^{l} \epsilon_{(i-j)\,\mathrm{Mod}\,n}$ ranges over all $\mathcal{P}_n^{\geq 0}$ with $|\gamma| = r-1-l$, and furthermore $Q^+(\gamma) = 0$ if any of the parts of γ are negative. With the help of (4.75), it then follows that $2 = \sum_{l=1}^{r-1}(-1)^{l-1}T_{l+1}Q^+_{r-1-l}$. In conclusion, we end up with the identity $T_1 Q^+_{r-1} = rQ_r^+ + \sum_{l=1}^{r-1}(-1)^{l-1}T_{l+1}Q^+_{r-1-l}$, which coincides with (4.92) after a simple rearrangement of terms.

Equation (4.93) can be proved by employing similar steps as the ones described for the proof of (4.92). If $r = 1$, this equation reduces to the identity $Q_1^- = \tilde{T}_1 = T_1$, so assume that $r > 1$. First of all, one can show after a straightforward computation the validity of the following relation,

$$b_i b^*_{i+1} Q^-(\alpha) = (\alpha_i + 1)Q^-(\alpha + \epsilon_i) - b_i Q^-(\alpha - \epsilon_{(i+1)\,\mathrm{Mod}\,n}) b^*_{i+2}\,.$$

This, together with a repeated application of (4.91), implies that

$$b_i b^*_{i+1} Q^-(\alpha) = (\alpha_i + 1)Q^-(\alpha + \epsilon_i) - \sum_{l=1}^{|\alpha|} b_i b^*_{i+1+l} Q^-\left(\alpha - \sum_{j=1}^{l} \epsilon_{(i+j)\,\mathrm{Mod}\,n}\right)\,.$$

Summing both sides of this last identity over $i = 1, \ldots, n$ and $\alpha \in \mathcal{P}_n^{\geq 0}$ with $|\alpha| = r-1$, one ends up with

$$\tilde{T}_1 Q^-_{r-1} = \sum_{i=1}^{n} \sum_{\alpha\in\mathcal{P}_n^{\geq 0}} (\alpha_i + 1)Q^-(\alpha + \epsilon_i) - \sum_{i=1}^{n}\sum_{l=1}^{r-1} b_i b^*_{i+1+l} \sum_{\alpha\in\mathcal{P}_n^{\geq 0}} Q^-\left(\alpha - \sum_{j=1}^{l} \epsilon_{(i+j)\,\mathrm{Mod}\,n}\right)\,.$$

Finally, proceeding in a similar fashion as described above, it follows that $\tilde{T}_1 Q^-_{r-1} = rQ_r^- - \sum_{l=1}^{r-1} Q^-_{r-l-1}\tilde{T}_{l+1}$, which is equivalent to (4.93). □

Corollary 4.2.9. *The following functional equations are valid for all $r \in \mathbb{N}$:*

$$rQ_r^{*+} = \sum_{l=1}^{r} (-1)^{l-1} T_{-l} Q^{*+}_{r-l}\,, \tag{4.94}$$

$$rQ_r^{*-} = \sum_{l=1}^{r} \tilde{T}_{-l} Q^{*-}_{r-l}\,. \tag{4.95}$$

Proof. These equations follow by taking the adjoint of both sides of (4.92) and (4.93) respectively, and then by employing Lemma 4.2.3. □

Proposition 4.2.10. *(i) $\mathcal{T}_n[z, z^{-1}]$ is a commutative subalgebra of $\mathcal{H}_n[z, z^{-1}]$.*
(ii) The operators $\{Q_r^{\pm}\}_{r\in\mathbb{Z}_{\geq 0}}$ and $\{Q_r^{*\pm}\}_{r\in\mathbb{Z}_{\geq 0}}$ belong to $\mathcal{T}_n[z, z^{-1}]$.

Proof. To prove part (i) of the claim, we just need to show the validity of the commutation relation $[T_r, T_s] = 0$ for all $r, s \in \mathbb{Z}$ such that $1 \leq r, s \leq n$. Taking advantage of (4.75), we have that

$$[T_r, T_s] = \sum_{i,j=1}^{n} b_{j+s}^* b_i [b_{i+r}^*, b_j] + \sum_{i,j=1}^{n} b_{i+r}^* b_j [b_i, b_{j+s}^*] \,.$$

Thanks to (4.78), together with the inequalities $1 \leq i + r \leq 2n$, it follows that $[b_{i+r}^*, b_j] = -\delta_{i+r,j} - z\delta_{i+r,j+n}$, and with the help of (4.77) we have the chain of equalities

$$\sum_{i,j=1}^{n} b_{j+s}^* b_i [b_{i+r}^*, b_j] = -\sum_{i=1}^{n-r} b_{i+r+s}^* b_i - z \sum_{i=n-r+1}^{n} b_{i+r+s-n}^* b_i = -\sum_{i=1}^{n} b_{i+r+s}^* b_i \,.$$

Similarly, we have that $\sum_{i,j=1}^{n} b_{i+r}^* b_j [b_i, b_{j+s}^*] = \sum_{i=1}^{n} b_{i+r+s}^* b_i$, and thus $[T_r, T_s] = 0$.

We now prove part (ii) of the claim. Employing (4.92) and (4.93), one can show by induction that $\{Q_r^+\}_{r\in\mathbb{Z}_{\geq 0}}$ and $\{Q_r^-\}_{r\in\mathbb{Z}_{\geq 0}}$ can be expressed in terms of $\{T_r\}_{r\in\mathbb{Z}_{\geq 0}}$ and $\{\tilde{T}_r\}_{r\in\mathbb{Z}_{\geq 0}}$ respectively. Similarly, with the help of (4.94) and (4.95), we deduce that $\{Q_r^{*+}\}_{r\in\mathbb{Z}_{\geq 0}}$ and $\{Q_r^{*-}\}_{r\in\mathbb{Z}_{\geq 0}}$ can be written in terms of $\{T_{-r}\}_{r\in\mathbb{Z}_{\geq 0}}$ and $\{\tilde{T}_{-r}\}_{r\in\mathbb{Z}_{\geq 0}}$ respectively. As we pointed out in the discussion preceding Definition 4.2.6, the operators $\{T_r\}_{r\in\mathbb{Z}}$ and $\{\tilde{T}_r\}_{r\in\mathbb{Z}}$ can be expressed solely in terms of $\{1\} \cup \{T_1, \ldots, T_n\}$. Since the latter generate $\mathcal{T}_n[z, z^{-1}]$, the claim follows.

□

Lemma 4.2.11. *The operators $\{T_1, \ldots, T_n\}$ are algebraically independent over $\mathbb{C}[z, z^{-1}]$.*

Proof. Taking advantage of Lemma 4.1.7, it follows that a basis of $\mathcal{H}_n[z, z^{-1}]$ is given by

$$\{(b_1^*)^{\alpha_1} \cdots (b_n^*)^{\alpha_n} b_1^{\gamma_1} \cdots b_n^{\gamma_n} \mid \alpha, \gamma \in \mathcal{P}_n^{\geq 0}\} \,. \tag{4.96}$$

Suppose that $\sum_\lambda a_\lambda T_\lambda = 0$ for some $a_\lambda \in \mathbb{C}[z, z^{-1}]$, where we set $T_\lambda = T_{\lambda_1} T_{\lambda_2} \cdots$, and the sum runs over all $\lambda \in \mathcal{P}^+$ for which $\lambda_1 \leq n$. Each operator T_λ belonging to this sum contains the term $b_{1+\lambda_1}^* b_{1+\lambda_2}^* \cdots b_{1+\lambda_{\ell(\lambda)}}^* b_1^{\ell(\lambda)}$, which equals

$$z^{m_n(\lambda)} (b_1^*)^{m_n(\lambda)} (b_2^*)^{m_1(\lambda)} \cdots (b_n^*)^{m_{n-1}(\lambda)} b_1^{\ell(\lambda)} \tag{4.97}$$

thanks to (4.77). We deduce that the element (4.97) does not appear in the operators T_μ for which $\mu \neq \lambda$. Apart from the factor $z^{m_n(\lambda)}$, the element (4.97) belongs to the basis (4.96) of $\mathcal{H}_n[z, z^{-1}]$. It follows that $a_\lambda = 0$ for all $\lambda \in \mathcal{P}^+$ with $\lambda_1 \leq n$, thus proving the claim. □

4.3 Conserved charges and symmetric functions

Proposition 4.2.8 suggests a connection between the conserved charges of the free boson model and the symmetric functions defined in Section 2.2. Namely, notice that the functional relation (4.92) has the same form of Newton's formula (2.34), that is

$$re_r = \sum_{l=1}^{r}(-1)^{l-1}p_l e_{r-l} \;, \tag{4.98}$$

where $\{p_\lambda\}_{\lambda\in\mathcal{P}^+}$ and $\{e_\lambda\}_{\lambda\in\mathcal{P}^+}$ are the power sums and the elementary symmetric functions. An analogous statement holds for the adjoint equation (4.94). We formalise this observation as follows.

Proposition 4.3.1. *The maps $\Xi_n^\pm : \Lambda \to \mathcal{T}_n[z,z^{-1}]$ defined via*

$$p_r \mapsto \Xi_n^\pm(p_r) = T_{\pm r} \tag{4.99}$$

for $r \in \mathbb{Z}_{\geq 0}$, and $\Xi_n^\pm(p_\lambda) = T_{\lambda_1}T_{\lambda_2}\cdots$ for $\lambda \in \mathcal{P}^+$, are algebra homomorphisms. The following relations hold for every $r \in \mathbb{Z}_{\geq 0}$,

$$\Xi_n^+(e_r) \;=\; Q_r^+ \;, \tag{4.100}$$

$$\Xi_n^-(e_r) \;=\; Q_r^{*+} \;. \tag{4.101}$$

Proof. The maps $\Xi_n^\pm : \Lambda \to \mathcal{T}_n[z,z^{-1}]$ are well defined since $\{p_\lambda\}_{\lambda\in\mathcal{P}^+}$ is a basis of Λ. Moreover, these maps are algebra homomorphisms, since the operators $\{T_r\}_{r\in\mathbb{Z}}$ commute with each other thanks to Proposition 4.2.10. This proves the first part of the claim. We shall now prove (4.100) via induction. For $r = 1$, equation (4.100) holds thanks to the identities $p_1 = e_1$ and $T_1 = Q_1^+$, so assume that $r > 1$. Using the induction hypothesis, together with (2.34) and (4.92), we end up with the following chain of equalities,

$$\Xi_n^+(e_r) = \Xi_n^+\left(\frac{1}{r}\sum_{l=1}^{r}(-1)^{l-1}p_l e_{r-l}\right) = \frac{1}{r}\sum_{l=1}^{r}(-1)^{l-1}T_l Q_{r-l}^+ = Q_r^+ \;,$$

which completes the induction proof. Equation (4.101) follows in a completely analogous way, with the help of (4.94). □

Remark 4.3.2. The identification (4.99) between symmetric functions and conserved charges is also based on the eigenvalues of the latter, when evaluating on the space $\mathcal{F}^{\otimes n}$ spanned by the states (4.18). Compare with Lemma 4.5.5.

Remark 4.3.3. Notice that both equation (4.93) and its adjoint (4.95) are of the same form as Newton's formula (2.35), that is

$$rh_r = \sum_{l=1}^{r} p_l h_{r-l} \ , \tag{4.102}$$

where $\{h_\lambda\}_{\lambda\in\mathcal{P}^+}$ are the complete symmetric functions (see Section 2.2). Lemmas 4.2.3 and 4.2.5, together with Proposition 4.2.10, imply that the operators $\{\tilde{T}_r\}_{r\in\mathbb{Z}}$ commute with each other. It follows that the maps $\tilde{\Xi}_n^\pm : \Lambda \to \mathcal{T}_n[z, z^{-1}]$ defined via

$$p_r \mapsto \tilde{\Xi}_n^\pm(p_r) = \tilde{T}_{\pm r} \tag{4.103}$$

for $r \in \mathbb{Z}_{\geq 0}$, and $\tilde{\Xi}_n^\pm(p_\lambda) = \tilde{T}_{\lambda_1}\tilde{T}_{\lambda_2}\cdots$ for $\lambda \in \mathcal{P}^+$, are algebra homomorphisms. Moreover, proceeding in close analogy to Proposition 4.99, we have for all $r \in \mathbb{Z}_{\geq 0}$ that

$$\tilde{\Xi}_n^+(h_r) \ = \ Q_r^- \ , \tag{4.104}$$

$$\tilde{\Xi}_n^-(h_r) \ = \ Q_r^{*-} \ . \tag{4.105}$$

The algebra homomorphisms $\tilde{\Xi}_n^\pm$ are of little importance in our discussion, so we shall only focus on the ones introduced in Proposition 4.3.1.

With the help of the map Ξ_n^+, we now introduce the analogoues of the symmetric functions defined in Section 2.2 as elements in $\mathcal{T}_n[z, z^{-1}]$, and denote these with capital letters if not defined previously (compare with the discussion presented in [45, Ch. 2.5]). Set $T_\lambda = T_{\lambda_1}T_{\lambda_2}\cdots$ and $Q_\lambda^+ = Q_{\lambda_1}^+ Q_{\lambda_2}^+ \cdots$ for $\lambda \in \mathcal{P}^+$. By definition of the map Ξ_n^+ we have that

$$T_\lambda \ = \ \Xi_n^+(p_\lambda) \ , \tag{4.106}$$

whereas from Proposition 4.3.1 it follows that

$$Q_\lambda^+ \ = \ \Xi_n^+(e_\lambda) \ . \tag{4.107}$$

Let us define for $r \in \mathbb{Z}_{\geq 0}$ the operator

$$H_r \ = \ \Xi_n^+(h_r) \ . \tag{4.108}$$

In Section 4.3.2 we show the relation between (4.108) and the conserved charges $\{Q_r^-\}_{r\in\mathbb{Z}_{\geq 0}}$. Setting $H_\lambda = H_{\lambda_1}H_{\lambda_2}\cdots$ for $\lambda \in \mathcal{P}^+$, it follows from Proposition 4.3.1 that

$$H_\lambda \ = \ \Xi_n^+(h_\lambda) \ . \tag{4.109}$$

Moreover, let us introduce the operators

$$M_\lambda = \Xi_n^+(m_\lambda) , \tag{4.110}$$

$$S_\lambda = \Xi_n^+(s_\lambda) , \tag{4.111}$$

which are the images under Ξ_n^+ of the monomial symmetric functions $\{m_\lambda\}_{\lambda\in\mathcal{P}^+}$ and the Schur functions $\{s_\lambda\}_{\lambda\in\mathcal{P}^+}$ respectively (see Section 2.2). The next result shows that we can recover the operators belonging to the image of Ξ_n^- by taking the adjoint of the operators just defined. Denote by $\Lambda_\mathbb{R} \subset \Lambda$ the ring of symmetric functions with real coefficients.

Lemma 4.3.4. *For every $g \in \Lambda_\mathbb{R}$ we have the equality*

$$\Xi_n^-(g) = \Xi_n^+(g)^* . \tag{4.112}$$

Proof. Since $\{p_\lambda\}_{\lambda\in\mathcal{P}^+}$ is a basis of $\Lambda_\mathbb{R}$, we can write that $g = \sum_{\lambda\in\mathcal{P}^+} g_\lambda p_\lambda$ for some coefficients $g_\lambda \in \mathbb{R}$. Using the fact that the operators $\{T_r\}_{r\in\mathbb{Z}}$ commute with each other, together with the relation $g_\lambda = g_\lambda^*$, and taking advantage of (4.79) we end up with the chain of equalities

$$\Xi_n^-(g) = \sum_{\lambda\in\mathcal{P}^+} g_\lambda T_{-\lambda_1} T_{-\lambda_2} \cdots = \sum_{\lambda\in\mathcal{P}^+} g_\lambda T^*_{\lambda_1} T^*_{\lambda_2} \cdots = \left(\sum_{\lambda\in\mathcal{P}^+} g_\lambda T_{\lambda_1} T_{\lambda_2} \cdots \right)^* = \Xi_n^+(g)^* .$$

□

Let $\lambda, \mu \in \mathcal{A}_k^+(n)$ and $\nu \in \mathcal{P}^+$. In Section 4.4 we will prove the following identities for the matrix elements of the operators introduced above,

$$\langle\lambda|T_\nu|\mu\rangle = \sum_{d\in\mathbb{Z}_{\geq 0}} z^d \varphi_{\lambda/d/\mu}(\nu) , \tag{4.113}$$

$$\langle\lambda|Q_\nu^+|\mu\rangle = \sum_{d\in\mathbb{Z}_{\geq 0}} z^d \psi_{\lambda/d/\mu}(\nu) , \tag{4.114}$$

$$\langle\lambda|H_\nu|\mu\rangle = \sum_{d\in\mathbb{Z}_{\geq 0}} z^d \theta_{\lambda/d/\mu}(\nu) , \tag{4.115}$$

$$\langle\lambda|M_\nu|\mu\rangle = \sum_{d\in\mathbb{Z}_{\geq 0}} z^d N_{\mu\nu}^{\lambda,d} , \tag{4.116}$$

$$\langle\lambda|S_\nu|\mu\rangle = \sum_{d\in\mathbb{Z}_{\geq 0}} z^d \chi_{\mu\nu}^{\lambda,d} , \tag{4.117}$$

where it is understood that $N_{\mu\nu}^{\lambda,d} = \chi_{\mu\nu}^{\lambda,d} = 0$ if $\ell(\nu) > k$. These identities are obtained by evaluating the action of such operators, treated as elements in $\mathrm{End}(\mathcal{F}^{\otimes n})$, on the vectors $|\mu\rangle$ introduced in (4.18), and then by applying the dual vector $\langle\lambda|$ defined in (4.19). The coefficients $\varphi_{\lambda/d/\mu}(\nu)$, $\psi_{\lambda/d/\mu}(\nu)$, $\theta_{\lambda/d/\mu}(\nu)$, and $\chi_{\mu\nu}^{\lambda,d}$ were introduced respectively in (3.77), (3.47), (3.48), and (3.69), whereas $N_{\mu\nu}^{\lambda,d}$ was defined as the cardinality of the set (3.39). Recall that such coefficients are non-zero only if $|\mu|+|\nu|-|\lambda|=dn$, and thus the sums on the RHS of the identities above involve at most one non-zero term.

We shall employ the identities (4.113) to (4.117) for the following purposes. First, we present an alternative method to the one presented in Section 3.3.2 for evaluating the expansions of the cylindric symmetric functions $e_{\lambda/d/\mu}$ and $h_{\lambda/d/\mu}$ in terms of the bases of Λ introduced in Section 2.2. Then we will illustrate an alternative approach to the one described in Sections 4.1.3 and 4.1.4 for computing the partition functions of the $Q^{\pm}$ and $Q^{*\pm}$ vertex models. Let us present some preliminary results.

Lemma 4.3.5. *For each $l\in\mathbb{Z}_{\geq 0}$ we have the chain of equalities*

$$\begin{aligned} Q^+(x_1)Q^+(x_2)\cdots Q^+(x_l) &= \sum_{\nu\in\mathcal{P}_l^+} Q_\nu^+ m_\nu(x_1,\dots,x_l) && (4.118)\\ &= \sum_{\nu\in\mathcal{P}^+} M_\nu e_\nu(x_1,\dots,x_l) && (4.119)\\ &= \sum_{\nu\in\mathcal{P}^+} \epsilon_\nu z_\nu^{-1} T_\nu p_\nu(x_1,\dots,x_l) && (4.120)\\ &= \sum_{\nu\in\mathcal{P}^+} S_\nu s_{\nu'}(x_1,\dots,x_l)\ . && (4.121)\end{aligned}$$

Proof. As pointed out in Section 4.1.3, the product $Q^+(x_1)Q^+(x_2)\cdots Q^+(x_l)$ is symmetric in the indeterminates $(x_1,\dots,x_l)$. This product can therefore be expanded in terms of the symmetric functions in l variables, with the expansion coefficients belonging to $\mathcal{H}_n[z,z^{-1}]$. To obtain the expansion in terms of monomial symmetric functions, notice that for $\nu\in\mathcal{P}_l^+$ the coefficient of $m_\nu(x_1,\dots,x_l)$ in $Q^+(x_1)Q^+(x_2)\cdots Q^+(x_l)$ equals the coefficient of $x_1^{\nu_1}\cdots x_l^{\nu_l}$ in the same product, which is just Q_ν^+. This proves (4.118). The other expansions follow by taking advantage of the relationships between the various bases of Λ, together with their images under Ξ_n^+. Consider for example the relation $e_\nu=\sum_{\sigma\in\mathcal{P}^+}M_{\nu\sigma}m_\sigma$, which was introduced in (2.22), and where the matrix $M_{\nu\sigma}$ was defined in Section 2.2.2. Applying the map Ξ_n^+ to both sides of the latter, it follows that $Q_\nu^+=\sum_{\sigma\in\mathcal{P}^+}M_{\nu\sigma}M_\sigma$. Moreover, projecting onto Λ_l we have that $e_\nu(x_1,\dots,x_l)=\sum_{\sigma\in\mathcal{P}_l^+}M_{\nu\sigma}m_\sigma(x_1,\dots,x_l)$. This is because $m_\sigma(x_1,\dots,x_l)=0$ if $\ell(\sigma)>l$, as shown in Lemma 2.2.5. The identities just described, together with the fact that $M_{\nu\sigma}=M_{\sigma\nu}$ (compare with [67, Cor. 7.5.2]), can then be used to prove (4.119). For the proof of (4.120) and (4.121), one can take advantage of the expansions $m_\nu=\sum_{\sigma\in\mathcal{P}^+}K_{\nu\sigma}^{-1}s_\sigma=\sum_{\sigma\in\mathcal{P}^+}R_{\nu\sigma}^{-1}p_\sigma$, $s_\nu=\sum_{\sigma\in\mathcal{P}^+}K_{\sigma\nu'}^{-1}e_\sigma$ and $p_\nu\epsilon_\nu z_\nu^{-1}=\sum_{\sigma\in\mathcal{P}^+}R_{\nu\sigma}^{-1}e_\sigma$. The matrices $K_{\nu\sigma}$ and $R_{\nu\sigma}$ were defined respectively in (2.22) and (2.24). The expansions of s_ν and p_ν in terms of the basis $\{e_\sigma\}_{\sigma\in\mathcal{P}^+}$ of Λ can be found for instance in [52, I.6]. □

We shall now make use of the following operator,

$$H(u) = \sum_{r\in\mathbb{Z}_{\geq 0}} u^r H_r \ , \tag{4.122}$$

where the operators $\{H_r\}_{r\in\mathbb{Z}_{\geq 0}}$ are defined in (4.108). We are adopting the same notation for the generating function of complete symmetric functions, which was introduced in (2.20). Nevertheless, it will be clear from the context which one of these two objects we are using.

Lemma 4.3.6. *For each $l \in \mathbb{Z}_{\geq 0}$ we have the chain of equalities*

$$\begin{aligned} H(x_1)H(x_2)\cdots H(x_l) &= \sum_{\nu\in\mathcal{P}_l^+} H_\nu m_\nu(x_1,\dots,x_l) && (4.123)\\ &= \sum_{\nu\in\mathcal{P}^+} M_\nu h_\nu(x_1,\dots,x_l) && (4.124)\\ &= \sum_{\nu\in\mathcal{P}^+} z_\nu^{-1} T_\nu p_\nu(x_1,\dots,x_l) && (4.125)\\ &= \sum_{\nu\in\mathcal{P}^+} S_\nu s_\nu(x_1,\dots,x_l) \ . && (4.126)\end{aligned}$$

Proof. The product $H(x_1)H(x_2)\cdots H(x_l)$ is symmetric in the indeterminates $(x_1,\dots,x_l)$. This is because Propositions 4.2.10 and 4.3.1 imply that the operators $\{H_r\}_{r\in\mathbb{Z}_{\geq 0}}$ commute with each other. To obtain the expansion in terms of monomial symmetric functions, notice that for $\nu \in \mathcal{P}_l^+$ the coefficient of $m_\nu(x_1,\dots,x_l)$ in $H(x_1)H(x_2)\cdots H(x_l)$ equals the coefficient of $x_1^{\nu_1}\cdots x_l^{\nu_l}$ in the same product, which is just H_ν. This proves (4.123). One can then proceed as in the proof of Lemma 4.3.5 to deduce the remaining equalities. For this purpose one can take advantage of the relation $h_\nu = \sum_{\sigma\in\mathcal{P}^+} L_{\nu\sigma} m_\sigma$, which was introduced in (2.22), and where the matrix $L_{\nu\sigma}$ was defined in Section 2.2.2, together with the expansions $s_\nu = \sum_{\sigma\in\mathcal{P}^+} K_{\sigma\nu}^{-1} h_\sigma$ and $p_\nu z_\nu^{-1} = \sum_{\sigma\in\mathcal{P}^+} R_{\nu\sigma}^{-1} h_\sigma$ (see for instance [52, I.6]). □

4.3.1 Expansions of cylindric symmetric functions

We now give an alternative proof for the following expansions of the cylindric symmetric functions $e_{\lambda/d/\mu}$ and $h_{\lambda/d/\mu}$,

$$\begin{aligned} e_{\lambda/d/\mu} &= \sum_{\nu\in\mathcal{P}^+} \psi_{\lambda/d/\mu}(\nu) m_\nu && (4.127)\\ &= \sum_{\nu\in\mathcal{P}_k^+} N_{\mu\nu}^{\lambda,d} e_\nu && (4.128)\\ &= \sum_{\nu\in\mathcal{P}^+} \varphi_{\lambda/d/\mu}(\nu)\epsilon_\nu z_\nu^{-1} p_\nu && (4.129)\\ &= \sum_{\nu\in\mathcal{P}_k^+} \chi_{\mu\nu}^{\lambda,d} s_{\nu'}\ , && (4.130)\end{aligned}$$

together with

$$\begin{aligned} h_{\lambda/d/\mu} &= \sum_{\nu\in\mathcal{P}^+} \theta_{\lambda/d/\mu}(\nu) m_\nu && (4.131)\\ &= \sum_{\nu\in\mathcal{P}_k^+} N_{\mu\nu}^{\lambda,d} h_\nu && (4.132)\\ &= \sum_{\nu\in\mathcal{P}^+} \varphi_{\lambda/d/\mu}(\nu) z_\nu^{-1} p_\nu && (4.133)\\ &= \sum_{\nu\in\mathcal{P}_k^+} \chi_{\mu\nu}^{\lambda,d} s_\nu\ . && (4.134)\end{aligned}$$

These expansions have been already proved in Propositions 3.3.11, 3.3.18, 3.3.28 and in Theorem 3.3.12. The new aspect here is that on the RHS of the expansions (4.127) to (4.134) the coefficients are the matrix elements (4.113) to (4.117). Let $\lambda, \mu \in \mathcal{A}_k^+(n)$ and $d \in \mathbb{Z}_{\geq 0}$. We start from the definitions

$$\begin{aligned} e_{\lambda/d/\mu}(x_1,\ldots,x_l) &= \sum_{\hat{T}\in\mathrm{T}_{\lambda/d/\mu}(l)} \psi_{\hat{T}} x^{\hat{T}}\ , && (4.135)\\ h_{\lambda/d/\mu}(x_1,\ldots,x_l) &= \sum_{\hat{\pi}\in\Pi_{\lambda/d/\mu}(l)} \theta_{\hat{\pi}} x^{\hat{\pi}}\ , && (4.136)\end{aligned}$$

which were discussed in (4.31) and (4.32) respectively. Let $\nu \in \mathcal{P}^+$, and for $\beta \in \mathcal{P}$ a composition set $Q_\beta^+ = Q_{\beta_1}^+ Q_{\beta_2}^+ \cdots$. Since the operators $\{Q_r^+\}_{r\in\mathbb{Z}_{\geq 0}}$ commute with each other, we have that $Q_\beta^+ = Q_\nu^+$ for every $\beta \sim \nu$, with the notation introduced in Section 2.2.1. We deduce from (4.114) that $\psi_{\lambda/d/\mu}(\beta) = \psi_{\lambda/d/\mu}(\nu)$ for every $\beta \sim \nu$, where $\psi_{\lambda/d/\mu}(\beta)$ is defined in an analogous way to (3.47). The validity of this last identity was shown by other means in the proof of Lemma 3.3.9. Similarly, setting $H_\beta = H_{\beta_1} H_{\beta_2} \cdots$ for $\beta \in \mathcal{P}$, we have that $H_\beta = H_\nu$ for every $\beta \sim \nu$, as the operators $\{H_r\}_{r\in\mathbb{Z}_{\geq 0}}$ commute with each other. It follows from (4.115) that $\theta_{\lambda/d/\mu}(\beta)=\theta_{\lambda/d/\mu}(\nu)$ for every $\beta\sim\nu$, where $\theta_{\lambda/d/\mu}(\beta)$ is defined in an analogous way to (3.48). Compare with the proof of Lemma 3.3.8. Following similar steps as the ones described in the proof of Proposition 3.3.11, and then taking advantage of the relation $m_\nu(x_1,\ldots,x_l)=0$ for $\ell(\nu)>l$, we can rearrange (4.135) and (4.136) to obtain the expansions

$$e_{\lambda/d/\mu}(x_1,\ldots,x_l) = \sum_{\nu\in\mathcal{P}^+}\psi_{\lambda/d/\mu}(\nu)m_\nu(x_1,\ldots,x_l)\,, \tag{4.137}$$

$$h_{\lambda/d/\mu}(x_1,\ldots,x_l) = \sum_{\nu\in\mathcal{P}^+}\theta_{\lambda/d/\mu}(\nu)m_\nu(x_1,\ldots,x_l)\,. \tag{4.138}$$

Now, act on the vector $|\mu\rangle$ with the RHS of (4.118), and then apply the dual vector $\langle\lambda|$ defined in (4.19). Do the same with the expressions appearing in (4.119), (4.120) and (4.121). Taking advantage of the identities (4.113) to (4.117), together with the expansion (4.137), and comparing the terms with the same power of z, we end up with the following chain of equalities,

$$\begin{aligned} e_{\lambda/d/\mu}(x_1,\ldots,x_l) &= \sum_{\nu\in\mathcal{P}_k^+}N_{\mu\nu}^{\lambda,d}e_\nu(x_1,\ldots,x_l) &(4.139)\\ &= \sum_{\nu\in\mathcal{P}^+}\varphi_{\lambda/d/\mu}(\nu)\epsilon_\nu z_\nu^{-1}p_\nu(x_1,\ldots,x_l) &(4.140)\\ &= \sum_{\nu\in\mathcal{P}_k^+}\chi_{\mu\nu}^{\lambda,d}s_{\nu'}(x_1,\ldots,x_l)\,. &(4.141)\end{aligned}$$

Similarly, starting from Lemma 4.3.6 and proceeding as above, we have that

$$\begin{aligned} h_{\lambda/d/\mu}(x_1,\ldots,x_l) &= \sum_{\nu\in\mathcal{P}_k^+}N_{\mu\nu}^{\lambda,d}h_\nu(x_1,\ldots,x_l) &(4.142)\\ &= \sum_{\nu\in\mathcal{P}^+}\varphi_{\lambda/d/\mu}(\nu)z_\nu^{-1}p_\nu(x_1,\ldots,x_l) &(4.143)\\ &= \sum_{\nu\in\mathcal{P}_k^+}\chi_{\mu\nu}^{\lambda,d}s_{\nu}(x_1,\ldots,x_l)\,. &(4.144)\end{aligned}$$

As we discussed in Remark 2.2.2, a symmetric function $f\in\Lambda$ is equivalent to a sequence of functions $\{f_l(x_1,\ldots,x_l)\}_{l\in\mathbb{Z}_{\geq 0}}$, with $f_l(x_1,\ldots,x_l)\in\Lambda_l$, satisfying for $l'\geq l$ the constraint $f_{l'}(x_1,\ldots,x_l,0,\ldots,0)=f_l(x_1,\ldots,x_l)$. For $f,g\in\Lambda$, it follows that $f=g$ if and only if the equality $f_l(x_1,\ldots,x_l)=g_l(x_1,\ldots,x_l)$ holds in Λ_l for all $l\in\mathbb{N}$. But the expansions obtained above for $e_{\lambda/d/\mu}(x_1,\ldots,x_l)$ and $h_{\lambda/d/\mu}(x_1,\ldots,x_l)$ are identities in Λ_l which are valid for all $l\in\mathbb{N}$, and thus they must hold on Λ as well. In this way we recover the expansions (4.127) to (4.134) for the cylindric symmetric functions $e_{\lambda/d/\mu}$ and $h_{\lambda/d/\mu}$.

Remark 4.3.7. The proof presented in Chapter 3 for the expansions (4.127) to (4.134) relies on $\mathcal{V}_k(n)$, which is the quotient of $\Lambda_k[z,z^{-1}]$ introduced in (3.35). On the other hand, the approach we use here to show the validity of the same expansions does not involve any quotient of $\Lambda_k[z,z^{-1}]$.

We now describe how the expansions (4.137) to (4.144) can be also obtained by means of the adjoint operators, that is the operators belonging to the image of the map Ξ_n^- introduced in (4.99), see also Lemma 4.3.4. For this purpose, define the operator $\mathcal{P} \in \mathrm{End}(\mathcal{F}^{\otimes n})$ via the relation

$$\mathcal{P}\,|\lambda\rangle = |\lambda^\vee\rangle \ , \tag{4.145}$$

where $\lambda \in \mathcal{A}_k^+(n)$, and the map $\vee : \mathcal{A}_k^+(n) \to \mathcal{A}_k^+(n)$ was introduced in (4.61). Since the latter is an involution in $\mathrm{End}(\mathcal{A}_k^+(n))$, it follows that $\mathcal{P}$ is an involution in $\mathrm{End}(\mathcal{F}^{\otimes n})$.

Lemma 4.3.8. *For every $g \in \Lambda_\mathbb{R} \subset \Lambda$ we have the following equality in* $\mathrm{End}(\mathcal{F}^{\otimes n})$,

$$\Xi_n^-(g) = \mathcal{P}\,\overline{\Xi_n^+(g)}\,\mathcal{P} \ , \tag{4.146}$$

where $\overline{\Xi_n^+(g)}$ is the complex conjugate of $\Xi_n^+(g)$.

Proof. Recall that $|\lambda\rangle = |m_1(\lambda),\dots,m_n(\lambda)\rangle$, as we described in (4.18). It follows that $|\lambda^\vee\rangle = |m_n(\lambda),\dots,m_1(\lambda)\rangle$, and then we can deduce the following identities in $\mathrm{End}(\mathcal{F}^{\otimes n})$,

$$b_j = \mathcal{P}\,b_{n+1-j}\,\mathcal{P} \ , \qquad \text{and} \qquad b_j^* = \mathcal{P}\,b_{n+1-j}^*\,\mathcal{P} \ . \tag{4.147}$$

Taking advantage of (4.147) and the relation $\bar{z} = z^{-1}$, together with Lemmas 4.2.2, 4.2.1 and Proposition 4.99, we end up with the identity

$$\Xi_n^-(e_r) = \mathcal{P}\,\overline{\Xi_n^+(e_r)}\,\mathcal{P} \ . \tag{4.148}$$

The claim then follows from the fact that $\{e_\lambda\}_{\lambda\in\mathcal{A}_k^+(n)}$ is a basis of $\Lambda_\mathbb{R}$, and that $\mathcal{P}$ is an involution.

□

Since the operators $\{Q_r^{*\pm}\}_{r\in\mathbb{Z}_{\geq 0}}$ and $\{H_r^*\}_{r\in\mathbb{Z}_{\geq 0}}$ commute with each other, we can employ Lemma 4.3.8, together with (4.114) and (4.115), to deduce once again the validity of the identities $\psi_{\lambda/d/\mu}(\beta) = \psi_{\lambda/d/\mu}(\nu)$ and $\theta_{\lambda/d/\mu}(\beta) = \theta_{\lambda/d/\mu}(\nu)$ for every $\beta \sim \nu$. This allows us to recover the expansions (4.137) and (4.138). Take the adjoint of the RHS of (4.118), act on the vector $|\mu^\vee\rangle$ and then apply the dual vector $\langle\lambda^\vee|$. Do the same with the RHS of (4.123), and with the expressions appearing in (4.119), (4.120), (4.121), (4.124), (4.125) and (4.126). Employing Lemma 4.3.8, and comparing the terms with the same power of $\bar{z} = z^{-1}$, we finally end up with the expansions (4.139) to (4.144).

4.3.2 Computation of the partition functions

In the previous section we employed the identities (4.113) to (4.117) to re-establish the validity of the expansions (4.127) to (4.134) of the cylindric symmetric functions $e_{\lambda/d/\mu}$ and $h_{\lambda/d/\mu}$. We now present an alternative proof of the identities stated in Theorems 4.1.18 and 4.1.30 for the partition functions of the $Q^{\pm}$ and $Q^{*\pm}$ vertex models respectively. For this purpose, we shall take advantage of the matrix elements (4.114), (4.115) and the expansions (4.137), (4.138). We start by considering the Q^+ vertex model.

Lemma 4.3.9. *Let $\lambda, \mu \in \mathcal{A}_k^+(n)$ and $l \in \mathbb{N}$. We have the equality*

$$\langle \lambda | Q^+(x_1) Q^+(x_2) \cdots Q^+(x_l) | \mu \rangle = \sum_{d \in \mathbb{Z}_{\geq 0}} z^d e_{\lambda/d/\mu}(x_1, \ldots, x_l) \, . \tag{4.149}$$

Proof. Taking matrix elements of both sides of (4.118), and then using (4.114), we obtain the identity

$$\langle \lambda | Q^+(x_1) Q^+(x_2) \cdots Q^+(x_l) | \mu \rangle = \sum_{d \in \mathbb{Z}_{\geq 0}} z^d \sum_{\nu \in \mathcal{P}_l^+} \psi_{\lambda/d/\mu}(\nu) m_\nu(x_1, \ldots, x_l) \, .$$

The claim then follows by employing (4.137). □

Thanks to Lemmas 4.1.13 and 4.3.9 we then recover (4.33), which is the expansion of the partition function $Z^+_{\lambda,\mu}(x_1, x_2, \ldots, x_l)$ in terms of cylindric elementary symmetric functions. We now present a similar discussion for the Q^- vertex model. To this end, we first need to understand the connection between $\{Q_r^-\}_{r \in \mathbb{Z}_{\geq 0}}$ and the operators belonging to the image of the map Ξ_n^+ introduced in (4.99). Applying Ξ_n^+ to both sides of Newton's formula $rh_r = \sum_{l=1}^r p_l h_{r-l}$ we end up with the equality

$$rH_r = \sum_{l=1}^r T_l H_{r-l} \, , \tag{4.150}$$

where the operators $\{H_r\}_{r \in \mathbb{Z}_{\geq 0}}$ are defined in (4.108). Notice the similarity between this last equality and (4.93). It is then natural to seek a relation between the operators $\{Q_r^-\}_{r \in \mathbb{Z}_{\geq 0}}$ and $\{H_r\}_{r \in \mathbb{Z}_{\geq 0}}$, which is given by the following result.

Lemma 4.3.10. *For $r \in \mathbb{Z}_{\geq 0}$ we have the identity*

$$H_r = \begin{cases} Q_r^- \, , & r < n \\ Q_r^- - zQ_{r-n}^- , & r \geq n \end{cases} \, . \tag{4.151}$$

Proof. We first show by induction that

$$Q^-_{sn+q} = \sum_{t=0}^{s} z^t H_{(s-t)n+q} \ , \tag{4.152}$$

where $s \in \mathbb{Z}_{\geq 0}$ and $0 \leq q \leq n-1$. The claim is true for $s = q = 0$, in which case $Q^-_0 = H_0 = 1$. So let $sn+q \neq 0$, and suppose that (4.152) holds for all $s' \in \mathbb{Z}_{\geq 0}$ and $0 \leq q' \leq n-1$ such that $s'n+q' < sn+q$. Thanks to (4.81) and (4.93), we have that

$$\begin{aligned} (sn+q)Q^-_{sn+q} &= \sum_{j=1}^{sn+q} T_j Q^-_{sn+q-j} + n\sum_{t'=1}^{s} z^{t'} Q^-_{(s-t')n+q} \\ &= \underbrace{\sum_{j=0}^{sn+q-1} T_{sn+q-j} Q^-_j}_{1} + \underbrace{n\sum_{t'=1}^{s} z^{t'} Q^-_{(s-t')n+q}}_{2} \end{aligned}$$

Taking advantage of the induction hypothesis, it follows that

$$\begin{aligned} 2 &= n\sum_{t'=1}^{s} z^{t'} \sum_{t''=0}^{s-t'} z^{t''} H_{(s-t'-t'')n+q} = n\sum_{t'=1}^{s}\sum_{t''=0}^{s-t'} z^{t'+t''} H_{(s-t'-t'')n+q} \\ &= n\sum_{t=1}^{s} t z^t H_{(s-t)n+q} \ . \end{aligned}$$

To express 1 in terms of the operators $\{H_r\}_{r\in\mathbb{Z}_{\geq 0}}$, we need to distinguish between the two cases $1 \leq q \leq n-1$ and $q = 0$. Setting $j = s''n + q''$, where $s'' \in \mathbb{Z}_{\geq 0}$ and $0 \leq q'' \leq n-1$, we have in the first case the chain of equalities

$$\begin{aligned} 1 &= \sum_{s''=0}^{s}\sum_{q''=0}^{q-1} T_{sn+q-(s''n+q'')} Q^-_{s''n+q''} \\ &= \sum_{s''=0}^{s}\sum_{q''=0}^{q-1} T_{sn+q-(s''n+q'')} \sum_{t=0}^{s''} z^t H_{(s''-t)n+q''} \\ &= \sum_{t=0}^{s} z^t \sum_{s''=t}^{s}\sum_{q''=0}^{q-1} T_{sn+q-(s''n+q'')} H_{(s''-t)n+q''} \\ &= \sum_{t=0}^{s} z^t \sum_{s''=0}^{s-t}\sum_{q''=0}^{q-1} T_{(s-t)n+q-(s''n+q'')} H_{s''n+q''} \\ &= \sum_{t=0}^{s} z^t \big((s-t)n+q\big) H_{(s-t)n+q} \ . \end{aligned}$$

In the second line we used the induction hypothesis, whereas in the third line we swapped the sums in t and in r. In the last line we took advantage of (4.150). If instead $q = 0$, we end up after similar steps with the identity $① = \sum_{t=0}^{s-1} z^t \big((s-t)n\big) H_{(s-t)n}$. Putting together all the results just described, we conclude that

$$① + ② = (sn + q) \sum_{t=0}^{s} z^t H_{(s-t)n+q} \ ,$$

which shows that Q^-_{sn+q} satisfies the relation (4.152) as well. This completes the induction proof.

We now use induction again to show the validity of (4.151). Notice that, thanks to (4.152), the latter holds already for $r \leq n$. So let $r = sn+q$ with $s \geq 1$ and $0 \leq q \leq n-1$, and assume that (4.151) holds for all $r' < r$. Starting from (4.152), and using the induction hypothesis, we then have that

$$\begin{aligned}
H_r &= Q^-_r - \sum_{t=1}^{s} z^t H_{(s-t)n+q} \\
&= Q^-_r - \sum_{t=1}^{s-1} z^t (Q^-_{(s-t)n+q} - z Q^-_{(s-t-1)n+q}) - Q^-_q \\
&= Q^-_r - \sum_{t=1}^{s} z^t Q^-_{(s-t)n+q} + \sum_{t=2}^{s} z^t Q^-_{(s-t)n+q} \\
&= Q^-_r - z Q^-_{(s-1)n+q} = Q^-_r - z Q^-_{r-n} \ .
\end{aligned}$$

□

Corollary 4.3.11. *We have the identity*

$$H(u) = Q^-(u)(1 - zu^n) \ , \tag{4.153}$$

where the operator $H(u)$ was introduced in (4.122).

Proof. The claim follows after a straightforward computation, with the help of (4.151). □

Lemma 4.3.12. *Let $\lambda, \mu \in \mathcal{A}^+_k(n)$ and $l \in \mathbb{N}$. We have the equality*

$$\langle \lambda | H(x_1) H(x_2) \cdots H(x_l) | \mu \rangle = \sum_{d \in \mathbb{Z}_{\geq 0}} z^d h_{\lambda/d/\mu}(x_1, \ldots, x_l) \ . \tag{4.154}$$

Proof. Take matrix elements of both sides of (4.123). Taking advantage of equation (4.115), we have that

$$\langle \lambda | H(x_1) H(x_2) \cdots H(x_l) | \mu \rangle = \sum_{d \in \mathbb{Z}_{\geq 0}} z^d \sum_{\nu \in \mathcal{P}^+_k} \theta_{\lambda/d/\mu}(\nu) m_\nu(x_1, \ldots, x_l) \ .$$

The claim then follows by taking advantage of (4.138). □

Employing Lemmas 4.1.13 and 4.3.12, together with the relation (4.153), we finally end up with the expansion (4.34) for the partition function $Z^{-}_{\lambda,\mu}(x_1, x_2, \ldots, x_l)$.

Assume that $\overline{x_i} = x_i$. The expansions (4.62) and (4.63) for the partition functions of the $Q^{*\pm}$ vertex models can be now deduced immediately by taking matrix elements of both sides of the following identity in $\mathrm{End}(\mathcal{F}^{\otimes n})$,

$$Q^{*\pm}(x_1)\cdots Q^{*\pm}(x_l) = \mathcal{P}Q^{\pm}(x_1)\cdots Q^{\pm}(x_l)\mathcal{P} \ , \tag{4.155}$$

which is a direct consequence of Proposition 4.99 and Lemma 4.3.8, and then by applying Lemma 4.1.26.

4.4 The action of the conserved charges

The goal of this section is to prove the identities (4.113) to (4.117). In other words, we will evaluate the action of the operator $\Xi^{+}_{n}(g)$, where $g \in \Lambda$ can be any of the functions $\{p_\nu\}_{\nu\in\mathcal{P}^+}$, $\{e_\nu\}_{\nu\in\mathcal{P}^+}$, $\{h_\nu\}_{\nu\in\mathcal{P}^+}$, $\{m_\nu\}_{\nu\in\mathcal{P}^+}$ and $\{s_\nu\}_{\nu\in\mathcal{P}^+}$, on the state (4.18), that is

$$|\mu\rangle = \frac{1}{u_\mu} b^{*}_{\mu_1} \cdots b^{*}_{\mu_k} \,|0\rangle \ . \tag{4.156}$$

We adopt the following strategy. We first compute the commutation relation $[\Xi^{+}_{n}(g), b^{*}_{i}]$ for all $i \in \mathbb{Z}$, and then we apply repeatedly such commutation relation to the state $\Xi^{+}_{n}(g)\,|\mu\rangle$. For the operator $\Xi^{+}_{n}(p_\nu)$ we instead evaluate its action on $|\mu\rangle$ directly, compare with Lemma 4.4.2 below. The action of $\Xi^{-}_{n}(g)$ on $|\mu\rangle$ can be computed by either using the identity $\Xi^{-}_{n}(g) = \mathcal{P}\,\overline{\Xi^{+}_{n}(g)}\,\mathcal{P}$ in $\mathrm{End}(\mathcal{F}^{\otimes n})$, which was proved in Lemma 4.3.8, or by employing a similar approach to the one described above. In the second case, one needs to evaluate the commutation relation $[\Xi^{-}_{n}(g), b^{*}_{i}]$ for all $i \in \mathbb{Z}$. Equivalently, thanks to the relation $\Xi^{-}_{n}(g) = \Xi^{+}_{n}(g)^{*}$, which was showed in Lemma 4.3.4, one can first evaluate the commutation $[b_i, \Xi^{+}_{n}(g)]$ for all $i \in \mathbb{Z}$, and then take the adjoint.

4.4.1 Power sums

We start with the operator introduced in (4.106) and in Proposition 4.3.1, that is

$$T_\lambda \;=\; \Xi^{+}_{n}(p_\lambda) \ . \tag{4.157}$$

We shall make use of CACSDs and CACRPPs, which were described in Section 3.3.3. In particular, we will take advantage of the weight $\varphi_{\lambda/d/\mu}$ defined in equation (3.76), and of the weight $\varphi_{\lambda/d/\mu}(\nu)$ introduced in Definition 3.3.25.

Lemma 4.4.1. *For $r, j \in \mathbb{Z}$ we have the commutation relations*

$$T_r b_j^* = b_j^* T_r + b_{j+r}^* , \tag{4.158}$$

$$b_j T_r = T_r b_j - b_{j-r} . \tag{4.159}$$

Proof. These identities follow by taking advantage of (4.77), together with the commutation relation (4.78). □

Lemma 4.4.2. *Let $\nu \in \mathcal{P}^+$ and $\mu \in \mathcal{A}_k^+(n)$. We have the equalities*

$$T_\nu |\mu\rangle = \sum_{d \in \mathbb{Z}_{\geq 0}} z^d \sum_{\lambda \in \mathcal{A}_k^+(n)} \varphi_{\lambda/d/\mu}(\nu) |\lambda\rangle , \tag{4.160}$$

$$T_\nu^* |\mu\rangle = \sum_{d \in \mathbb{Z}_{\geq 0}} z^{-d} \sum_{\lambda \in \mathcal{A}_k^+(n)} \varphi_{\lambda^\vee/d/\mu^\vee}(\nu) |\lambda\rangle , \tag{4.161}$$

The second sum in (4.160) runs over all $\lambda \in \mathcal{A}_k^+(n)$ for which $\lambda/d/\mu$ is a cylindric skew diagram with $|\mu| + |\nu| - |\lambda| = dn$. The second sum in (4.161) runs instead over all $\lambda \in \mathcal{A}_k^+(n)$ for which $\lambda^\vee/d/\mu^\vee$ is a cylindric skew diagram with $|\mu^\vee| + |\nu| - |\lambda^\vee| = dn$.

Proof. We prove (4.160) first. For this purpose, it is enough to show that

$$T_r |\mu\rangle = \sum_{d \in \mathbb{Z}_{\geq 0}} z^d \sum_{\lambda \in \mathcal{A}_k^+(n)} \varphi_{\lambda/d/\mu} |\lambda\rangle \tag{4.162}$$

for $r \geq 0$, where the second sum runs over all $\lambda \in \mathcal{A}_k^+(n)$ for which $\lambda/d/\mu$ is a CACSD with $|\mu| + r - |\lambda| = dn$. Then (4.160) follows after a repeated application of (4.162) to the state $T_\nu |\mu\rangle$. If $r = 0$, we have that (4.162) reduces to the identity $|\mu\rangle = |\mu\rangle$. This is because $T_0 = 1$, and moreover the only CACSD $\lambda/d/\mu$ such that $|\lambda/d/\mu| = |\lambda| + dn - |\mu| = 0$ is given by $\mu/0/\mu$, in which case $\varphi_{\mu/0/\mu} = 1$ by definition. Suppose now that $r = sn$ for some $s \in \mathbb{N}$, and let $\lambda \in \mathcal{A}_k^+(n)$ and $d \in \mathbb{Z}_{\geq 0}$. Notice that $\lambda/d/\mu$ is a CACSD satisfying the constraint $|\mu| + sn - |\lambda| = dn$, that is $\hat{\lambda}.\tau^d = \hat{\mu}_{a,sn}$ for some $1 \leq a \leq n$ with $m_{a-1}(\hat{\mu}) \neq 0$, if and only if $\lambda = \mu$ and $d = s$. This follows from (3.75) for $s = 1$, whereas for $s > 1$ this can be deduced from the fact that $\hat{\lambda}.\tau^s = \hat{\mu}_{a,sn}$ if and only if $\hat{\lambda}.\tau = \hat{\mu}_{a,n}$, which is a direct consequence of Lemma 3.3.21. Since $\psi_{\mu/s/\mu} = k$, we have that the RHS of (4.162) is equal to $z^s k |\mu\rangle$. Equation (4.162) then follows, as its LHS is also equal to $z^s k |\mu\rangle$ thanks to the identity $T_r = z^s N$, which can be derived from (4.75).

We now assume that $r > 1$ and $r \bmod n \neq 0$. We evaluate the action of the operator $T_r = \sum_{j=1}^n b_{j+r}^* b_j$ on the state $|\mu\rangle$ directly, and for this purpose let $i \in \mathbb{N}$ with $1 \leq i \leq n$. Notice that $b_{i+r}^* b_i |\mu\rangle = 0$ if $m_i(\mu) = 0$, so suppose that $m_i(\mu) \neq 0$. Let $\sigma \in \mathcal{A}_k^+(n)$ be the partition obtained from μ by removing a part equal to i and adding a part equal to $(i + r) \operatorname{Mod} n$, where Mod was defined in (3.74). Equation (3.75), together with Lemma

3.3.21, implies that $\hat{\sigma}.\tau^{d''} = \hat{\mu}_{(i+1)\,\mathrm{Mod}\,n,r}$, where $d'' \in \mathbb{Z}_{\geq 0}$ is defined via the relation $|\mu| + r - |\sigma| = d''n$. Moreover, taking advantage of (4.77), it follows that

$$b^*_{i+r} b_i \,|\mu\rangle = z^{d''} m_{(i+r)\,\mathrm{Mod}\,n}(\sigma)\,|\sigma\rangle = z^{d''} m_{i+r}(\hat{\sigma}.\tau^{d''})\,|\sigma\rangle \ .$$

Thus, we have the identity

$$T_r\,|\mu\rangle = \sum_a z^{d'} m_{a-1+r}(\hat{\rho}.\tau^{d'})\,|\rho\rangle \ ,$$

where the sum runs over all $1 \leq a \leq n$ such that $m_{a-1}(\hat{\mu}) \neq 0$, and $\rho \in \mathcal{A}^+_k(n)$, $d' \in \mathbb{Z}_{\geq 0}$ are defined via the relation $\hat{\rho}.\tau^{d'} = \hat{\mu}_{a,r}$. Let $\lambda \in \mathcal{A}^+_k(n)$, $d \in \mathbb{Z}_{\geq 0}$ and suppose that $\lambda/d/\mu$ is a cylindric skew diagram. If $\lambda/d/\mu$ is a CACSD satisfying the constraint $|\mu| + r - |\lambda| = dn$, that is if $\hat{\lambda}.\tau^d = \hat{\mu}_{a,r}$ for some $1 \leq a \leq n$ with $m_{a-1}(\hat{\mu}) \neq 0$, it follows from the equality above that $\langle\lambda|T_r|\mu\rangle = z^d\varphi_{\lambda/d/\mu}$, where we employed the relation (3.76) for the weight $\varphi_{\lambda/d/\mu}$. Otherwise, we have that $\langle\lambda|T_r|\mu\rangle = 0$. This completes the proof of (4.162), since by applying the dual vector $\langle\lambda|$ on (4.162) we recover the same identity.

Equation (4.161) follows immediately by employing the identity $T^*_\nu = \mathcal{P}\overline{T_\nu}\mathcal{P}$ in $\mathrm{End}(\mathcal{F}^{\otimes n})$, which is a consequence of Lemmas 4.3.4 and 4.3.8. The involution $\mathcal{P} \in \mathrm{End}(\mathcal{F}^{\otimes n})$ was introduced in (4.145).

□

Remark 4.4.3. The weight $\varphi_{\lambda/d/\mu}(\nu)$ is non-zero only if $\lambda/d/\mu$ is a cylindric skew diagram with $|\mu|+|\nu|-|\lambda| = dn$ (compare with Definition 3.3.25), and then (4.160) does not change if we allow λ to run over all the weights in $\mathcal{A}^+_k(n)$. It follows that

$$\langle\lambda|T_\nu|\mu\rangle = \sum_{d\in\mathbb{Z}_{\geq 0}} z^d\varphi_{\lambda/d/\mu}(\nu) \ , \tag{4.163}$$

which is the identity (4.113).

4.4.2 Elementary and complete symmetric functions

The next operators of our interest are the images under Ξ^+_n of the elementary and complete symmetric functions, which were defined in (4.107) and (4.109) as

$$Q^+_\lambda = \Xi^+_n(e_\lambda) \ , \tag{4.164}$$

$$H_\lambda = \Xi^+_n(h_\lambda) \ . \tag{4.165}$$

We shall take advantage of the weights $\psi_{\lambda/d/\mu}$ and $\theta_{\lambda/d/\mu}$, which were introduced in Definitions 3.2.9 and 3.2.4 respectively. We will also make use of the weights $\psi_{\lambda/d/\mu}(\nu)$ and $\theta_{\lambda/d/\mu}(\nu)$. See Definition 3.3.7.

Lemma 4.4.4. *For $r \in \mathbb{N}$ and $j \in \mathbb{Z}$ we have the commutation relations*

$$H_r b_j^* = b_j^* H_r + H_{r-1} b_{j+1}^* \ , \tag{4.166}$$

$$b_j H_r = H_r b_j + b_{j-1} H_{r-1} \ . \tag{4.167}$$

Proof. These relations can be obtained after a straigthforward computation, with the help of (4.86), (4.87) and (4.151). □

Lemma 4.4.5. *Let $\nu \in \mathcal{P}^+$ and $\mu \in \mathcal{A}_k^+(n)$. We have the equalities*

$$Q_\nu^+ |\mu\rangle = \sum_{d \in \mathbb{Z}_{\geq 0}} z^d \sum_{\lambda \in \mathcal{A}_k^+(n)} \psi_{\lambda/d/\mu}(\nu) |\lambda\rangle \ , \tag{4.168}$$

$$Q_\nu^{*+} |\mu\rangle = \sum_{d \in \mathbb{Z}_{\geq 0}} z^{-d} \sum_{\lambda \in \mathcal{A}_k^+(n)} \psi_{\lambda^\vee/d/\mu^\vee}(\nu) |\lambda\rangle \ . \tag{4.169}$$

The second sum in (4.168) runs over all $\lambda \in \mathcal{A}_k^+(n)$ for which $\lambda/d/\mu$ is a cylindric skew diagram with $|\mu| + |\nu| - |\lambda| = dn$. The second sum in (4.169) runs instead over all $\lambda \in \mathcal{A}_k^+(n)$ for which $\lambda^\vee/d/\mu^\vee$ is a cylindric skew diagram with $|\mu^\vee| + |\nu| - |\lambda^\vee| = dn$.

Proof. We prove the validity of (4.168) first. For this purpose, it is enough to show that

$$Q_r^+ |\mu\rangle = \sum_{d \in \mathbb{Z}_{\geq 0}} z^d \sum_{\lambda \in \mathcal{A}_k^+(n)} \psi_{\lambda/d/\mu} |\lambda\rangle \tag{4.170}$$

for $r \geq 0$, where the second sum runs over all $\lambda \in \mathcal{A}_k^+(n)$ for which $\lambda/d/\mu$ is a cylindric vertical strip with $|\mu| + r - |\lambda| = dn$. Equation (4.168) then follows after a repeated application of (4.170) to the state $Q_\nu^+ |\mu\rangle$. If $r = 0$, we have that (4.170) reduces to the identity $|\mu\rangle = |\mu\rangle$. This is because $Q_0^+ = 1$, and moreover the only cylindric vertical strip $\lambda/d/\mu$ such that $|\lambda/d/\mu| = |\lambda| + dn - |\mu| = 0$ is given by $\mu/0/\mu$, in which case $\psi_{\mu/0/\mu} = 1$. So assume that $r > 0$. Let $s \in \mathbb{Z}_{\geq 0}$, and notice that $Q_s^+ |0\rangle = 0$ unless $s = 0$, in which case $Q_0^+ |0\rangle = |0\rangle$. This can be deduced from the expansion (4.73) of Q_s^+ in terms of the generators of $\mathcal{H}_n[z, z^{-1}]$, together with the fact that $b_i |0\rangle = 0$. Set

$$b_\alpha^* = b_{\alpha_1}^* b_{\alpha_2}^* \cdots \tag{4.171}$$

for $\alpha \in \mathcal{P}$, and consider the expansion (4.156) for the state $|\mu\rangle$. After a repeated application of the commutation relation (4.84), we end up with the identity

$$Q_r^+ |\mu\rangle = \frac{1}{u_\mu} \sum_{\eta \in \mathcal{P}_k^{\geq 0}} b_{\mu+\eta}^* |0\rangle \ ,$$

where the sum is restricted to those weights $\eta \in \mathcal{P}_k^{\geq 0}$ with $\eta_i = 0, 1$ and $|\eta| = r$. Let

$\lambda \in \mathcal{A}_k^+(n)$. Thanks to (4.77), together with the fact that $\langle\lambda|\sigma\rangle = \delta_{\lambda\sigma}$ for all $\sigma \in \mathcal{A}_k^+(n)$, it follows that $\langle\lambda|b^*_{\mu+\eta}|0\rangle$ is non-zero if and only if there exists a pair $(w',\beta) \in S^\lambda \times \mathcal{P}_k^{\geq 0}$ satisfying the constraint $\mu+\eta = \lambda.w'y^\beta$. Employing part (ii) of Proposition 2.1.6, we have that if the pair (w',β) exists then it is unique, and moreover $\langle\lambda|b^*_{\mu+\eta}|0\rangle = z^{|\beta|}u_\lambda$. Thus, we have the identity

$$\langle\lambda|b^*_{\mu+\eta}|0\rangle = \sum_{d\in\mathbb{Z}_{\geq 0}} z^d u_\lambda \sum_{w'\in S^\lambda} \sum_{\beta\in\mathcal{P}_k^{\geq 0}} \delta_{\mu+\eta,\lambda.w'y^\beta} \ ,$$

where the third sum is restricted to those weights $\beta \in \mathcal{P}_k^{\geq 0}$ with $|\beta| = d$, and then

$$\langle\lambda|Q_r^+|\mu\rangle = \sum_{d\in\mathbb{Z}_{\geq 0}} z^d \frac{u_\lambda}{u_\mu} \sum_{w'\in S^\lambda} \sum_{\eta,\beta\in\mathcal{P}_k^{\geq 0}} \delta_{\mu+\eta,\lambda.w'y^\beta} \ .$$

Let $f : \mathcal{P}_k \to \mathbb{C}$, and notice that

$$\sum_{w\in S_k} f(\lambda.w) = u_\lambda \sum_{w\in S^\lambda} f(\lambda.w) \ . \tag{4.172}$$

This can be deduced by first employing part (ii) of Proposition 2.1.6, and then by taking advantage of the identity $|S_\lambda| = u_\lambda$, which follows from the fact that $m_0(\lambda) = 0$ as $\lambda \in \mathcal{A}_k^+(n)$. With the help of (4.172), we then have the equality

$$\langle\lambda|Q_r^+|\mu\rangle = \sum_{d\in\mathbb{Z}_{\geq 0}} z^d \frac{1}{u_\mu} \sum_{w'\in S_k} \sum_{\eta,\beta\in\mathcal{P}_k^{\geq 0}} \delta_{\mu+\eta,\lambda.w'y^\beta} \ .$$

Set $w = (w')^{-1}$, $\gamma = \eta.w$ and $\alpha = \beta.w$. Taking advantage of the relation $w'y^\beta = y^{\beta.(w')^{-1}}w'$, we can rewrite the constraint $\mu+\eta = \lambda.w'y^\beta$ as $\mu.w+\gamma = \lambda.y^\alpha$. Using (4.172) once more, we end up with

$$\langle\lambda|Q_r^+|\mu\rangle = \sum_{d\in\mathbb{Z}_{\geq 0}} z^d \sum_{w\in S^\mu} \sum_{\gamma,\alpha\in\mathcal{P}_k^{\geq 0}} \delta_{\mu.w+\gamma,\lambda.y^\alpha} \ , \tag{4.173}$$

where the second sum is restricted to those weights $\gamma,\alpha \in \mathcal{P}_k^{\geq 0}$ with $\gamma_i = 0,1$, $|\gamma| = r$ and $|\alpha| = d$. Let $d \in \mathbb{Z}_{\geq 0}$, and notice that the sum $\sum_{w\in S^\mu}\sum_{\gamma,\alpha\in\mathcal{P}_k^{\geq 0}} \delta_{\mu.w+\gamma,\lambda.y^\alpha}$ appearing in (4.173) is non-zero only if $|\mu|+r-|\lambda| = dn$. Assuming that the relation $|\mu|+r-|\lambda| = dn$ is satisfied, this sum equals the cardinality of the set

$$\{(w,\gamma,\alpha) \in S^\mu \times \mathcal{P}_k^{\geq 0} \times \mathcal{P}_k^{\geq 0} \mid \gamma_i = 0,1, |\gamma| = r, |\alpha| = d, \mu.w+\gamma = \lambda.y^\alpha\} \ .$$

But the latter coincides with the set $\mathbb{A}$ introduced in the proof of Lemma 3.3.9, the cardinality of which is given by $\psi_{\lambda/d/\mu}$. Lemma 3.2.11 implies that $\psi_{\lambda/d/\mu}$ is non-zero if and only if $\lambda/d/\mu$ is a cylindric vertical strip, and then we finally deduce that $\langle\lambda|Q_r^+|\mu\rangle =$

$\sum_{d\in\mathbb{Z}_{\geq 0}} z^d \psi_{\lambda/d/\mu}$, where the sum runs over all $d \in \mathbb{Z}_{\geq 0}$ for which $\lambda/d/\mu$ is a cylindric vertical strip with $|\mu|+r-|\lambda| = dn$. This proves the validity of (4.170), since by applying the dual vector $\langle\lambda|$ on (4.170) we recover the same equality.

The validity of (4.169) follows by employing the identity $Q^*_\nu = \mathcal{P}\,\overline{Q_\nu}\,\mathcal{P}$ in $\mathrm{End}(\mathcal{F}^{\otimes n})$, which can be deduced from Lemmas 4.3.4 and 4.3.8. For the sake of completeness, we present an alternative proof of (4.169) that relies on the commutation relation (4.84). For this purpose, it is enough to show the validity of

$$Q^{*+}_r\,|\mu\rangle = \sum_{d\in\mathbb{Z}_{\geq 0}} z^{-d} \sum_{\lambda\in\mathcal{A}^+_k(n)} \psi_{\lambda^\vee/d/\mu^\vee}\,|\lambda\rangle \tag{4.174}$$

for $r \geq 0$, where the second sum runs over all $\lambda \in \mathcal{A}^+_k(n)$ for which $\lambda^\vee/d/\mu^\vee$ is a cylindric vertical strip with $|\mu^\vee| + r - |\lambda^\vee| = dn$. Equation (4.169) then follows after a repeated application of (4.174) to the state $Q^{*+}_\nu\,|\mu\rangle$. Let $\lambda \in \mathcal{A}^+_k(n)$. Employing the adjoint of equation (4.84), which is given by $Q^{*+}_r b^*_j = b^*_j Q^{*+}_r + b^*_{j-1} Q^{*+}_{r-1}$, and following similar steps as the ones described above, we end up with the equality

$$\langle\lambda|Q^{*+}_r|\mu\rangle = \sum_{d\in\mathbb{Z}_{\geq 0}} z^{-d} \sum_{\bar{w}\in S^\mu} \sum_{\gamma,\bar{\alpha}\in\mathcal{P}^{\geq 0}_k} \delta_{\mu.\bar{w}-\gamma,\lambda.y^{\bar{\alpha}}}\ ,$$

where the second sum is restricted to those weights $\gamma, \bar{\alpha} \in \mathcal{P}^{\geq 0}_k$ with $\gamma_i = 0, 1$, $|\gamma| = r$ and $|\bar{\alpha}| = d$. Notice that the constraint $\mu.\bar{w} - \gamma = \lambda.y^{\bar{\alpha}}$ can be rewritten as $\mu^\vee.w + \gamma = \lambda^\vee.y^\alpha$, where $\alpha = (\bar{\alpha}_k, \ldots, \bar{\alpha}_1)$, and moreover $w \in S_k$ is obtained from $\bar{w} = \sigma_{i_1}\cdots\sigma_{i_{\ell(\bar{w})}}$ by replacing each generator σ_{i_j} with σ_{k-i_j}. Taking advantage of equation (4.172), together with the identity $u_\mu = u_{\mu^\vee}$, we have that

$$\langle\lambda|Q^{*+}_r|\mu\rangle = \sum_{d\in\mathbb{Z}_{\geq 0}} z^{-d} \sum_{w\in S^{\mu^\vee}} \sum_{\gamma,\alpha\in\mathcal{P}^{\geq 0}_k} \delta_{\mu^\vee.w+\gamma,\lambda^\vee.y^\alpha}\ , \tag{4.175}$$

where the second sum is restricted to those weights $\gamma, \alpha \in \mathcal{P}^{\geq 0}_k$ with $\gamma_i = 0, 1$, $|\gamma| = r$ and $|\alpha| = d$. Let $d \in \mathbb{Z}_{\geq 0}$, and notice that the sum $\sum_{w\in S^{\mu^\vee}} \sum_{\gamma,\alpha\in\mathcal{P}^{\geq 0}_k} \delta_{\mu^\vee.w+\gamma,\lambda^\vee.y^\alpha}$ appearing in (4.175) is non-zero only if $|\mu^\vee|+r-|\lambda^\vee| = dn$. Assuming that the relation $|\mu^\vee|+r-|\lambda^\vee| = dn$ is satisfied, this sum equals the cardinality of the set

$$\{(w, \gamma, \alpha) \in S^{\mu^\vee} \times \mathcal{P}^{\geq 0}_k \times \mathcal{P}^{\geq 0}_k \mid \gamma_i = 0, 1, |\gamma| = r, |\alpha| = d, \mu^\vee.w + \gamma = \lambda^\vee.y^\alpha\}\ ,$$

which is given by $\psi_{\lambda^\vee/d/\mu^\vee}$. It follows that $\langle\lambda|Q^{*+}_r|\mu\rangle = \sum_{d\in\mathbb{Z}_{\geq 0}} z^{-d}\psi_{\lambda^\vee/d/\mu^\vee}$, where the sum runs over all $d \in \mathbb{Z}_{\geq 0}$ for which $\lambda^\vee/d/\mu^\vee$ is a cylindric vertical strip with $|\mu^\vee|+r-|\lambda^\vee| = dn$. This proves the validity of (4.174), since by applying the dual vector $\langle\lambda|$ on (4.174) we recover the same equality. $\square$

Lemma 4.4.6. *Let $\nu \in \mathcal{P}^+$ and $\mu \in \mathcal{A}_k^+(n)$. We have the equalities*

$$H_\nu \left|\mu\right\rangle = \sum_{d\in\mathbb{Z}_{\geq 0}} z^d \sum_{\lambda\in\mathcal{A}_k^+(n)} \theta_{\lambda/d/\mu}(\nu) \left|\lambda\right\rangle \tag{4.176}$$

$$H_\nu^* \left|\mu\right\rangle = \sum_{d\in\mathbb{Z}_{\geq 0}} z^{-d} \sum_{\lambda\in\mathcal{A}_k^+(n)} \theta_{\lambda^\vee/d/\mu^\vee}(\nu) \left|\lambda\right\rangle \tag{4.177}$$

The second sum in (4.176) runs over all $\lambda \in \mathcal{A}_k^+(n)$ for which $\lambda/d/\mu$ is a cylindric skew diagram with $|\mu| + |\nu| - |\lambda| = dn$. The second sum in (4.177) runs instead over all $\lambda \in \mathcal{A}_k^+(n)$ for which $\lambda^\vee/d/\mu^\vee$ is a cylindric skew diagram with $|\mu^\vee| + |\nu| - |\lambda^\vee| = dn$.

Proof. The proof of (4.176) is similar to the one of (4.168), and for this purpose we just need to shown that

$$H_r \left|\mu\right\rangle = \sum_{d\in\mathbb{Z}_{\geq 0}} z^d \sum_{\lambda\in\mathcal{A}_k^+(n)} \theta_{\lambda/d/\mu} \left|\lambda\right\rangle \tag{4.178}$$

for $r \geq 0$, where the second sum runs over all $\lambda \in \mathcal{A}_k^+(n)$ for which $\lambda/d/\mu$ is a cylindric skew diagram with $|\mu| + r - |\lambda| = dn$. If $r = 0$, we have that (4.178) reduces to the identity $\left|\mu\right\rangle = \left|\mu\right\rangle$. This is because $H_0 = 1$, and moreover the only cylindric skew diagram $\lambda/d/\mu$ with $|\lambda/d/\mu| = |\lambda| + dn - |\mu| = 0$ is given by $\mu/0/\mu$, in which case $\theta_{\mu/0/\mu} = 1$. So assume that $r > 0$. Let $s \in \mathbb{Z}_{\geq 0}$, and notice that $H_s \left|0\right\rangle = 0$ unless $s = 0$, in which case $H_0 \left|0\right\rangle = \left|0\right\rangle$. This can be deduced from (4.74) and (4.151). A repeated application of (4.166) yields the equality $H_r b_j^* = \sum_{t=0}^r b_{j+t}^* H_{r-t}$, which can be used to show that

$$H_r \left|\mu\right\rangle = \frac{1}{u_\mu} \sum_{\gamma\in\mathcal{P}_k^{\geq 0}} b_{\mu+\gamma}^* \left|0\right\rangle \ ,$$

where the sum is restricted to those weights $\gamma \in \mathcal{P}_k^{\geq 0}$ with $|\gamma| = r$. Let $\lambda \in \mathcal{A}_k^+(n)$. Following similar steps as the ones described in the proof of (4.170), one ends up with

$$\left\langle\lambda\right|H_r\left|\mu\right\rangle = \sum_{d\in\mathbb{Z}_{\geq 0}} z^d \sum_{w\in S^\mu} \sum_{\gamma,\alpha\in\mathcal{P}_k^{\geq 0}} \delta_{\mu.w+\gamma,\lambda.y^\alpha} \ , \tag{4.179}$$

where the second sum is restricted to those weights $\gamma, \alpha \in \mathcal{P}_k^{\geq 0}$ with $|\gamma| = r$ and $|\alpha| = d$. Let $d \in \mathbb{Z}_{\geq 0}$, and notice that the sum $\sum_{w\in S^\mu} \sum_{\gamma,\alpha\in\mathcal{P}_k^{\geq 0}} \delta_{\mu.w+\gamma,\lambda.y^\alpha}$ appearing in (4.179) is non-zero only if $|\mu| + r - |\lambda| = dn$. Assuming that the relation $|\mu| + r - |\lambda| = dn$ is satisfied, this sum equals the cardinality of the set

$$\{(w, \gamma, \alpha) \in S^\mu \times \mathcal{P}_k^{\geq 0} \times \mathcal{P}_k^{\geq 0} \mid |\gamma| = r, |\alpha| = d, \mu.w + \gamma = \lambda.y^\alpha\} \ .$$

The latter coincides with the set $\mathbb{A}$ introduced in the proof of Lemma 3.3.8, the cardinality of which is given by $\theta_{\lambda/d/\mu}$. Lemma 3.2.7 implies that $\theta_{\lambda/d/\mu}$ is non-zero if and only if

$\lambda/d/\mu$ is a cylindric skew diagram, and then we end up with the equality $\langle\lambda|Q_r^+|\mu\rangle = \sum_{d\in\mathbb{Z}_{\geq 0}} z^d\psi_{\lambda/d/\mu}$, where the sum runs over all $d\in\mathbb{Z}_{\geq 0}$ for which $\lambda/d/\mu$ is a cylindric skew diagram with $|\mu|+r-|\lambda| = dn$. This proves the validity of (4.178), since by applying the dual vector $\langle\lambda|$ on (4.178) we recover the same equality.

Equation (4.169) follows from the identity $H_\nu^* = \mathcal{P}\,\overline{H_\nu}\,\mathcal{P}$ in $\mathrm{End}(\mathcal{F}^{\otimes n})$, which can be deduced from Lemmas 4.3.4 and 4.3.8. Alternatively, one can prove (4.169) by employing similar steps as the ones described above. For this purpose, one has to take advantage of the commutation relation $H_r^*b_j^* = \sum_{t=0}^r b_{j-t}^*H_{r-t}^*$, which follows after a repeated application of the adjoint equation of (4.167).

□

Remark 4.4.7. The weights $\psi_{\lambda/d/\mu}(\nu)$ and $\theta_{\lambda/d/\mu}(\nu)$ are non-zero only if $\lambda/d/\mu$ is a cylindric skew diagram with $|\mu|+|\nu|-|\lambda| = dn$ (compare with Definition 3.3.7). This means that (4.168) and (4.176) do not change if we allow λ to run over all the weights in $\mathcal{A}_k^+(n)$, and then we have the identities

$$\langle\lambda|Q_\nu^+|\mu\rangle = \sum_{d\in\mathbb{Z}_{\geq 0}} z^d\psi_{\lambda/d/\mu}(\nu)\ , \tag{4.180}$$

$$\langle\lambda|H_\nu|\mu\rangle = \sum_{d\in\mathbb{Z}_{\geq 0}} z^d\theta_{\lambda/d/\mu}(\nu)\ . \tag{4.181}$$

These are the identities (4.114) and (4.115) respectively.

4.4.3 Monomial symmetric functions

We shall now focus on the operator introduced in (4.110), that is

$$M_\lambda = \Xi_n^+(m_\lambda)\ . \tag{4.182}$$

In Section 4.5 we will employ the operators $\{M_\mu\}_{\mu\in\mathcal{A}_k^+(n)}$ to construct an algebra which is isomorphic to $\mathcal{V}_k(n)$, the quotient of $\Lambda_k[z,z^{-1}]$ defined in Section 3.3. Let

$$M^\lambda = \Xi_n^+(m^\lambda) \tag{4.183}$$

be the image under Ξ_n^+ of the augmented monomial symmetric function $m^\lambda = u_\lambda m_\lambda$ (see Section 2.2.4), and notice that by definition we have the identity $M^\lambda = u_\lambda M_\lambda$. Applying the map Ξ_n^+ to both sides of (2.32), it follows that

$$M^\lambda = \sum_{\Pi\in P_{\ell(\lambda)}} B(\Pi)T_{\lambda(\Pi)}\ . \tag{4.184}$$

Recall that we can express every partition $\lambda \in \mathcal{P}^+$ with the notation $(1^{m_1(\lambda)}, 2^{m_2(\lambda)}, \dots)$, which was introduced in (2.1). Set $I_0\lambda = \lambda$, and for $j > 0$ define the partition

$$I_j\lambda = (1^{m_1(\lambda)}, 2^{m_2(\lambda)}, \dots, (j-1)^{m_{j-1}(\lambda)}, j^{m_j(\lambda)+1}, (j+1)^{m_{j+1}(\lambda)}, \dots)\ . \tag{4.185}$$

Moreover, set $R_0\lambda = \lambda$, and for $j > 0$ with $m_j(\lambda) \neq 0$ define the partition

$$R_j\lambda = (1^{m_1(\lambda)}, 2^{m_2(\lambda)}, \dots, (j-1)^{m_{j-1}(\lambda)}, j^{m_j(\lambda)-1}, (j+1)^{m_{j+1}(\lambda)}, \dots)\ . \tag{4.186}$$

Stated otherwise, we have that $I_j\lambda$ is obtained from λ by adding a part equal to j, whereas $R_j\lambda$ is obtained from λ by removing a part equal to j, provided that $m_j(\lambda) \neq 0$. Applying the map Ξ_n^+ to both sides of (2.31), and taking advantage of (4.185) and (4.186), we have that for $\ell(\lambda) > 1$ the operator M^λ satisfies the recurrence formula

$$M^\lambda = T_{\lambda_l} M^{R_{\lambda_l}\lambda} - \sum_{i=1}^{\ell(\lambda)-1} M^{I_{\lambda_i+\lambda_l}R_{\lambda_i}R_{\lambda_l}\lambda}\ , \tag{4.187}$$

where we set $l = \ell(\lambda)$. If instead $\ell(\lambda) = 1$, that is if $\lambda = (r)$ for some $r \in \mathbb{N}$, it follows from (4.184) that $M^{(r)} = M_{(r)} = T_r$.

Proposition 4.4.8. *For $\lambda \in \mathcal{P}^+$ and $j \in \mathbb{Z}$ we have the commutation relations*

$$M^\lambda b_j^* = b_j^* M^\lambda + \sum_{i=1}^{\ell(\lambda)} b_{j+\lambda_i}^* M^{R_{\lambda_i}\lambda}\ , \tag{4.188}$$

$$b_j M^\lambda = M^\lambda b_j + \sum_{i=1}^{\ell(\lambda)} M^{R_{\lambda_i}\lambda} b_{j-\lambda_i}\ , \tag{4.189}$$

where we set $M^{R_i\lambda} = 0$ whenever $m_i(\lambda) = 0$.

Proof. We shall only prove (4.188), since (4.189) follows after employing completely analogous steps. For this purpose we use induction on the length of λ. If $\ell(\lambda) = 0$, that is if $\lambda = \emptyset$, then (4.188) reduces to the identity $b_j^* = b_j^*$, as $M_\emptyset = 1$. If instead $\ell(\lambda) = 1$, that is if $\lambda = (r)$ for some $r \in \mathbb{N}$, then (4.188) coincides with the relation $T_r b_j^* = b_j^* T_r + b_{j+r}^*$, which is just (4.158). So let $\lambda \in \mathcal{P}^+$ with $\ell(\lambda) > 1$, and assume that (4.188) holds for all partitions $\tilde{\lambda}$ such that $\ell(\tilde{\lambda}) < \ell(\lambda)$. Moreover, set $l = \ell(\lambda)$. Taking advantage of (4.187), together with the induction hypothesis, we have that

$$
\begin{aligned}
M^{\lambda}b_j^* &= \left(T_{\lambda_l}M^{R_{\lambda_l}\lambda} - \sum_{i=1}^{l-1} M^{I_{\lambda_i+\lambda_l}R_{\lambda_i}R_{\lambda_l}\lambda}\right)b_j^* \\
&= T_{\lambda_l}\left(b_j^* M^{R_{\lambda_l}\lambda} + \sum_{i=1}^{l-1} b_{j+\lambda_i}^* M^{R_{\lambda_i}R_{\lambda_l}\lambda}\right) \\
&\quad - \sum_{i=1}^{l-1}\left(b_j^* M^{I_{\lambda_i+\lambda_l}R_{\lambda_i}R_{\lambda_l}\lambda} + \sum_{\substack{s=1\\ s\neq i}}^{l-1} b_{j+\lambda_s}^* M^{R_{\lambda_s}I_{\lambda_i+\lambda_l}R_{\lambda_i}R_{\lambda_l}\lambda} + b_{j+\lambda_i+\lambda_l}^* M^{R_{\lambda_i}R_{\lambda_l}\lambda}\right) .
\end{aligned}
$$

Employing the commutation relation (4.158), it follows that

$$
\begin{aligned}
M^{\lambda}b_j^* &= b_j^* T_{\lambda_l} M^{R_{\lambda_l}\lambda} + b_{j+\lambda_l}^* M^{R_{\lambda_l}\lambda} + \sum_{i=1}^{l-1} b_{j+\lambda_i}^* T_{\lambda_l} M^{R_{\lambda_i}R_{\lambda_l}\lambda} + \sum_{i=1}^{l-1} b_{j+\lambda_i+\lambda_l}^* M^{R_{\lambda_i}R_{\lambda_l}\lambda} \\
&- b_j^* \sum_{i=1}^{l-1} M^{I_{\lambda_i+\lambda_l}R_{\lambda_i}R_{\lambda_l}\lambda} - \sum_{i=1}^{l-1}\sum_{\substack{s=1\\ s\neq i}}^{l-1} b_{j+\lambda_s}^* M^{I_{\lambda_i+\lambda_l}R_{\lambda_s}R_{\lambda_i}R_{\lambda_l}\lambda} - \sum_{i=1}^{l-1} b_{j+\lambda_i+\lambda_l}^* M^{R_{\lambda_i}R_{\lambda_l}\lambda} .
\end{aligned}
$$

In the double sum appearing in the second line, we took advantage of the equality $R_{\lambda_s}I_{\lambda_i+\lambda_l}R_{\lambda_i}R_{\lambda_l}\lambda = I_{\lambda_i+\lambda_l}R_{\lambda_s}R_{\lambda_i}R_{\lambda_l}\lambda$, which is valid for $s \neq i$ and $s \neq l-1$. In the same double sum, let us first swap the summations in i and in s, and then make the change of variables $i \leftrightarrow s$. The equality above then becomes

$$
\begin{aligned}
M^{\lambda}b_j^* &= b_j^*\left(T_{\lambda_l}M^{R_{\lambda_l}\lambda} - \sum_{i=1}^{l-1} M^{I_{\lambda_i+\lambda_l}R_{\lambda_i}R_{\lambda_l}\lambda}\right) + b_{j+\lambda_l}^* M^{R_{\lambda_l}\lambda} \\
&\quad + \sum_{i=1}^{l-1} b_{j+\lambda_i}^*\left(T_{\lambda_l}M^{R_{\lambda_l}R_{\lambda_i}\lambda} - \sum_{\substack{s=1\\ s\neq i}}^{l-1} M^{I_{\lambda_s+\lambda_l}R_{\lambda_s}R_{\lambda_l}R_{\lambda_i}\lambda}\right) .
\end{aligned}
$$

Using once again (4.187) for the terms in brackets, we then recover (4.188), and this completes the proof by induction. □

Corollary 4.4.9. *For $\lambda \in \mathcal{P}^+$ and $j \in \mathbb{Z}$ we have the commutation relations*

$$M_{\lambda}b_j^* = b_j^* M_{\lambda} + \sum_{i\geq 1} b_{j+i}^* M_{R_i\lambda} \, , \tag{4.190}$$

$$b_j M_{\lambda} = M_{\lambda} b_j + \sum_{i\geq 1} M_{R_i\lambda} b_{j-i} \, , \tag{4.191}$$

where we set $M_{R_i\lambda} = 0$ whenever $m_i(\lambda) = 0$.

Proof. Notice that (4.188) can be written in the following equivalent form,

$$M^\lambda b_j^* = b_j^* M^\lambda + \sum_{i\geq 1} b_{j+i}^* M^{R_i\lambda} m_i(\lambda)\ .$$

Dividing both sides of this equality by u_λ one then recovers (4.190), thanks to the fact that $M^\lambda = u_\lambda M_\lambda$, and that $u_\lambda = u_{R_i\lambda} m_i(\lambda)$ whenever $m_i(\lambda) \neq 0$. Equation (4.191) follows after a similar computation, starting from (4.189) instead.

□

Lemma 4.4.10. *Let $\nu \in \mathcal{P}^+$ and $\mu \in \mathcal{A}_k^+(n)$. We have that $M_\nu\,|\mu\rangle = M_\nu^*\,|\mu\rangle = 0$ if $\ell(\nu) > k$, otherwise*

$$M_\nu\,|\mu\rangle = \sum_{d\in\mathbb{Z}_{\geq 0}} z^d \sum_{\lambda\in\mathcal{A}_k^+(n)} N_{\mu\nu}^{\lambda,d}\,|\lambda\rangle\ , \tag{4.192}$$

$$M_\nu^*\,|\mu\rangle = \sum_{d\in\mathbb{Z}_{\geq 0}} z^{-d} \sum_{\lambda\in\mathcal{A}_k^+(n)} N_{\mu^\vee\nu}^{\lambda^\vee,d}\,|\lambda\rangle\ . \tag{4.193}$$

The coefficient $N_{\mu\nu}^{\lambda,d}$ was defined in Chapter 3 as the cardinality of the set (3.39).

Proof. We start by evaluating the action of M_ν on the state $|\mu\rangle$. For this purpose, we shall use the identity

$$M_\nu b_\mu^* = \sum_{\gamma\in\mathcal{P}_k^{\geq 0}} b_{\mu+\gamma}^* M_{R_\gamma\nu}\ , \tag{4.194}$$

where $R_\gamma\nu = R_{\gamma_1}R_{\gamma_2}\cdots\nu$ if $m_i(\gamma)\leq m_i(\nu)$ for all $i\in\mathbb{N}$, and otherwise we set $M_{R_\gamma\nu} = 0$. This identity follows after a repeated application of the commutation relation (4.190), which can be written as $M_\nu b_j^* = \sum_{i\in\mathbb{Z}_{\geq 0}} b_{j+i}^* M_{R_i\nu}$. Let $r\in\mathbb{Z}$, and notice that $T_r\,|0\rangle = 0$ unless $r = 0$, in which case $T_0\,|0\rangle = |0\rangle$. This can be deduced from (4.75), together with the fact that $b_i\,|0\rangle = 0$. Moreover, let $\sigma\in\mathcal{P}^+$. Taking advantage of (4.184) and the equality $M^\sigma = u_\sigma M_\sigma$, it follows that $M_\sigma\,|0\rangle = 0$ unless $\sigma = \emptyset$, in which case $M_\emptyset\,|0\rangle = |0\rangle$. Suppose that $\ell(\nu) > k$. Let $\gamma\in\mathcal{P}_k^{\geq 0}$ and notice that γ, by definition, has fewer non-zero parts than ν. We then have that $M_{R_\gamma\nu}\,|0\rangle = 0$, since $R_\gamma\nu\neq\emptyset$ if $m_i(\gamma)\leq m_i(\nu)$ for all $i\in\mathbb{N}$, and $M_{R_\gamma\nu} = 0$ otherwise. Exploiting the expansion (4.156) for the state $|\mu\rangle$, together with (4.194), we conclude that $M_\nu\,|\mu\rangle = \frac{1}{u_\mu} M_\nu b_\mu^*\,|0\rangle = 0$.

We now show the validity of (4.192), so suppose that $\ell(\nu)\leq k$. Let $\gamma\in\mathcal{P}_k^{\geq 0}$, and notice that $M_{R_\gamma\nu}\,|0\rangle = 0$ unless $\gamma = \nu.w''$ for some $w''\in S^\nu$, in which case $R_\gamma\nu = \emptyset$, and then $M_{R_\gamma\nu}\,|0\rangle = |0\rangle$. Taking advantage of (4.156) and (4.194), it follows that

$$M_\nu\,|\mu\rangle = \frac{1}{u_\mu}\sum_{w''\in S^\nu} b_{\mu+\nu.w''}^*\,|0\rangle\ .$$

Let $\lambda\in\mathcal{A}_k^+(n)$. Thanks to (4.77), we have that $\langle\lambda|b_{\mu+\nu.w''}^*|0\rangle$ is non-zero if and only if there exists a pair $(w''', \beta) \in S^\lambda \times \mathcal{P}_k^{\geq 0}$ satisfying the constraint $\mu + \nu.w'' = \lambda.w'''y^\beta$. If the pair (w''', β) exists, then it is unique thanks to part (ii) of Proposition 2.1.6, and moreover $\langle \lambda | b^*_{\mu+\nu.w''} | 0 \rangle = z^{|\beta|} u_\lambda$. With the help of (4.172), we then end up with the equalities

$$\begin{aligned} \langle \lambda | M_\nu | \mu \rangle &= \sum_{d \in \mathbb{Z}_{\geq 0}} z^d \frac{u_\lambda}{u_\mu} \sum_{w'' \in S^\nu} \sum_{w''' \in S^\lambda} \sum_{\beta \in \mathcal{P}_k^{\geq 0}} \delta_{\mu+\nu.w'', \lambda.w'''y^\beta} \\ &= \sum_{d \in \mathbb{Z}_{\geq 0}} z^d \frac{1}{u_\mu} \sum_{w'' \in S^\nu} \sum_{w''' \in S_k} \sum_{\beta \in \mathcal{P}_k^{\geq 0}} \delta_{\mu+\nu.w'', \lambda.w'''y^\beta} \ , \end{aligned}$$

where the sums are restricted to those $\beta \in \mathcal{P}_k^{\geq 0}$ with $|\beta| = d$. Let us rewrite the constraint $\mu + \nu.w'' = \lambda.w'''y^\beta$ as $\mu.(w''')^{-1} + \nu.w''(w''')^{-1} = \lambda.y^\alpha$, where we used the relation $w'''y^\beta = y^{\beta.(w''')^{-1}} w'''$, and we set $\alpha = \beta.(w''')^{-1}$. Part (ii) of Proposition 2.1.6 implies that $w''(w''')^{-1} = w_\nu w^\nu$, for some $w_\nu \in S_\nu$ and $w^\nu \in S^\nu$, and then $\nu.w''(w''')^{-1} = \nu.w^\nu$. Notice that for different elements $w'' \in S^\nu$ we end up with different elements $w^\nu \in S^\nu$. Set $w = (w''')^{-1}$ and $w' = w^\nu$. Using (4.172) once again, we have that

$$\langle \lambda | M_\nu | \mu \rangle = \sum_{d \in \mathbb{Z}_{\geq 0}} z^d \sum_{w \in S^\mu} \sum_{w' \in S^\nu} \sum_{\alpha \in \mathcal{P}_k^{\geq 0}} \delta_{\mu.w+\nu.w', \lambda.y^\alpha} \ .$$

Let $d \in \mathbb{Z}_{\geq 0}$, and notice that $\sum_{w \in S^\mu} \sum_{w' \in S^\nu} \sum_{\alpha \in \mathcal{P}_k^{\geq 0}} \delta_{\mu.w+\nu.w', \lambda.y^\alpha}$, where the third sum is restricted to those weights $\alpha \in \mathcal{P}_k^{\geq 0}$ with $|\alpha| = d$, equals the cardinality of the set

$$\{(w, w') \in S^\mu \times S^\nu \mid \mu.w + \nu.w' = \lambda.y^\alpha \text{ for some } \alpha \in \mathcal{P}_k^{\geq 0} \text{ with } |\alpha| = d\} \ ,$$

which is given by $N_{\mu\nu}^{\lambda,d}$ (compare with Definition 3.3.3). In conclusion, we have that

$$\langle \lambda | M_\nu | \mu \rangle = \sum_{d \in \mathbb{Z}_{\geq 0}} z^d N_{\mu\nu}^{\lambda,d} \ , \tag{4.195}$$

which is the identity (4.116), and this finally implies the validity of (4.192).

The claim for the operator M^*_λ is a consequence of the identity $M^*_\nu = \mathcal{P}\,\overline{M_\nu}\,\mathcal{P}$ in $\mathrm{End}(\mathcal{F}^{\otimes n})$, which can be deduced from Lemmas 4.3.4 and 4.3.8. Alternatively, one can prove the claim for M^*_λ by employing similar steps as the ones described above. For this purpose, one has to take advantage of the commutation relation $M^*_\nu b^*_j = \sum_{i \in \mathbb{Z}_{\geq 0}} b^*_{j-i} M^*_{R_i \nu}$, which follows after a repeated application of the adjoint equation of (4.191).

□

Remark 4.4.11. The coefficients $N_{\mu\nu}^{\lambda,d}$ for $\lambda, \mu, \nu \in \mathcal{A}_k^+(n)$ correspond to the fusion coefficients of a 2D TQFT, as we will see in Chapter 5. The interest aspect here is that these coefficients are obtained by taking the matrix elements of the operators (4.183), which are defined in terms of the conserved charges of a quantum integrable model. Recall that the same fusion coefficients were obtained in Chapter 3 by different means, that is via the expansions (3.61), (3.62) and via the coproducts (3.88), (3.89).

4.4.4 Schur functions

Consider now the operator

$$S_\lambda \quad = \quad \Xi_n^+(s_\lambda)\ , \tag{4.196}$$

which was defined in (4.111). From here to the end of this section, we set $Q_r^+ = H_r = 0$ for $r < 0$. With this convention, we have for example that the commutation relation $Q_r^+ b_j^* = b_j^* Q_r^+ + b_{j+1}^* Q_{r-1}^+$, which was shown in Lemma 4.2.7, holds for all $r \in \mathbb{Z}$. Moreover, applying the map Ξ_n^+ to the Jacobi-Trudi determinants (2.25), it follows that

$$S_\lambda = \det\left(H_{\lambda_i - i + j}\right)_{1\le i,j\le \ell(\lambda)} = \det\left(Q^+_{\lambda_i' - i + j}\right)_{1\le i,j\le\lambda_1}\ . \tag{4.197}$$

Suppose that λ/μ is a cylindric skew diagram with r boxes. Write $\lambda/\mu = (r)$ if λ/μ is a horizontal strip, and $\lambda/\mu = (1^r)$ if λ/μ is a vertical strip.

Lemma 4.4.12. *For $\lambda \in \mathcal{P}^+$ and $j \in \mathbb{Z}$ we have the commutation relations*

$$S_\lambda b_j^* \quad = \quad b_j^* S_\lambda + \sum_{r=1}^{\lambda_1} b_{j+r}^* \sum_{\substack{\mu\in\mathcal{P}^+ \\ \lambda/\mu=(r)}} S_\mu\ , \tag{4.198}$$

$$b_j S_\lambda \quad = \quad S_\lambda b_j + \sum_{r=1}^{\lambda_1}\Bigg(\sum_{\substack{\mu\in\mathcal{P}^+ \\ \lambda/\mu=(r)}} S_\mu\Bigg) b_{j-r}\ . \tag{4.199}$$

Proof. We proceed in close analogy to the proof of Proposition 11.4 in [46]. We shall only prove (4.198), since (4.199) follows after employing completely analogous steps. Let us first rewrite (4.198) in terms of the conjugate partitions, that is

$$S_{\lambda'} b_j^* = b_j^* S_{\lambda'} + \sum_{r=1}^{\ell(\lambda)} b_{j+r}^* \sum_{\substack{\mu\in\mathcal{P}^+ \\ \lambda/\mu=(1^r)}} S_{\mu'}\ . \tag{4.200}$$

We will prove (4.200) by induction on the length of λ. If $\ell(\lambda) = 0$, that is if $\lambda = \emptyset$, this reduced to the identity $b_j^* = b_j^*$, as $S_\emptyset = 1$. If instead $\ell(\lambda) = 1$, that is if $\lambda = (r)$ for some $r \in \mathbb{N}$, it follows from (4.197) that $S_{(r)'} = Q_r^+$, and then (4.200) reduces to the identity $Q_r^+ b_j^* = b_j^* Q_r^+ + b_{j+1}^* Q_{r-1}^+$. So let $\ell(\lambda) > 1$, and assume that (4.200) holds for all partitions $\tilde\lambda$ with $\ell(\tilde\lambda) < \ell(\lambda)$. Moreover, write $\ell(\lambda) = l + 1$ for some $l \in \mathbb{N}$. Set $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_l)$, and define the partitions

$$
\begin{aligned}
\lambda^{[0]} &= (\lambda_1, \ldots, \lambda_l) , \\
\lambda^{[i]} &= (\lambda_0 + 1, \ldots, \lambda_{i-1} + 1, \lambda_{i+1}, \ldots, \lambda_l) ,
\end{aligned}
$$

where $i = 1, \ldots, l$. Expanding the determinant in second equality of (4.197) along the first column, it follows that

$$
S_{\lambda'} = \sum_{i=0}^{l} (-1)^i Q^+_{\lambda_i - i} S_{\lambda^{[i]\prime}} . \tag{4.201}
$$

Employing this identity, together with the induction hypothesis and the commutation relation $Q^+_r b^*_j = b^*_j Q^+_r + b^*_{j+1} Q^+_{r-1}$, we have that

$$
\begin{aligned}
S_{\lambda'} b^*_j &= \sum_{i=0}^{l} (-1)^i Q^+_{\lambda_i - i} \Bigg(b^*_j S_{\lambda^{[i]\prime}} + \sum_{r=1}^{l} b^*_{j+r} \sum_{\substack{\mu \in \mathcal{P}^+ \\ \lambda^{[i]}/\mu = (1^r)}} S_{\mu'} \Bigg) \\
&= \sum_{i=0}^{l} (-1)^i (b^*_j Q^+_{\lambda_i - i} + b^*_{j+1} Q^+_{\lambda_i - i - 1}) S_{\lambda^{[i]\prime}} \\
&\qquad + \sum_{i=0}^{l} (-1)^i \sum_{r=1}^{l} (b^*_{j+r} Q^+_{\lambda_i - i} + b^*_{j+r+1} Q^+_{\lambda_i - i - 1}) \sum_{\substack{\mu \in \mathcal{P}^+ \\ \lambda^{[i]}/\mu = (1^r)}} S_{\mu'} .
\end{aligned}
$$

Let ρ_s be the partition of length $s \in \mathbb{Z}_{\geq 0}$ whose parts are all equal to 1. Since $\ell(\lambda^{[i]}) = l$, it follows that the only partition μ such that $\lambda^{[i]}/\mu = (1^l)$ is given by $\mu = \lambda^{[i]} - \rho_l$, and then $\sum_{\lambda^{[i]}/\mu = (1^l)} S_{\mu'} = S_{(\lambda^{[i]} - \rho_l)'}$. Moreover, since the only partition μ satisfying the constraint $\lambda^{[i]}/\mu = (1^0)$ is given by $\mu = \lambda^{[i]}$, we have that $S_{\lambda^{[i]\prime}} = \sum_{\lambda^{[i]}/\mu = (1^0)} S_{\mu'}$. Taking advantage of the results just described, and rearranging terms, we end up with

$$
\begin{aligned}
S_{\lambda'} b^*_j &= \underbrace{b^*_j S_{\lambda'}}_{1} + \underbrace{\sum_{r=1}^{l} b^*_{j+r} \sum_{i=0}^{l} (-1)^i \Bigg(Q^+_{\lambda_i - i} \sum_{\substack{\mu \in \mathcal{P}^+ \\ \lambda^{[i]}/\mu = (1^r)}} S_{\mu'} + Q^+_{\lambda_i - i - 1} \sum_{\substack{\mu \in \mathcal{P}^+ \\ \lambda^{[i]}/\mu = (1^{r-1})}} S_{\mu'} \Bigg)}_{2} \\
&\qquad \underbrace{+ b^*_{j+l+1} \sum_{i=0}^{l} (-1)^i Q^+_{\lambda_i - i - 1} S_{(\lambda^{[i]} - \rho_l)'}}_{3}
\end{aligned}
$$

Let $\nu = (\nu_0, \ldots, \nu_{l'})$ be a partition with $l' \leq l$, set $\nu^{[0]} = (\nu_1, \ldots, \nu_l)$ and moreover

$\nu^{[j]} = (\nu_0 + 1, \dots, \nu_{j-1} + 1, \nu_{j+1}, \dots, \nu_{l'})$ for $1 \le j \le l'$. Notice that if $\lambda/\nu = (1^r)$ for some $r \in \mathbb{Z}_{\ge 0}$, then for every $i \in \mathbb{N}$ with $1 \le i \le l'$ we have that either $\lambda_i = \nu_i$ or $\lambda_i = \nu_i + 1$. We can use this fact to prove that $\lambda/\nu = (1^r)$ and $\lambda_i = \nu_i$ if and only if $\lambda^{[i]}/\mu^{[i]} = (1^r)$. Similarly, we have that $\lambda/\nu = (1^r)$ and $\lambda_i = \nu_i + 1$ if and only if $\lambda^{[i]}/\mu^{[i]} = (1^{r-1})$. See [46, Proposition 11.4] for details. Taking advantage of (4.201), and employing the results which we just discussed, we have that

$$\begin{aligned}
\sum_{\substack{\mu \in \mathcal{P}^+ \\ \lambda/\mu = (1^r)}} S_{\mu'} &= \sum_{i=0}^{l} (-1)^i Q^+_{\mu_i - i} \sum_{\substack{\mu \in \mathcal{P}^+ \\ \lambda/\mu = (1^r)}} S_{\mu^{[i]\prime}} \\
&= \sum_{i=0}^{l} (-1)^i \Bigg(Q^+_{\lambda_i - i} \sum_{\substack{\mu \in \mathcal{P}^+ \\ \lambda^{[i]}/\mu = (1^r)}} S_{\mu'} + Q^+_{\lambda_i - i - 1} \sum_{\substack{\mu \in \mathcal{P}^+ \\ \lambda^{[i]}/\mu = (1^{r-1})}} S_{\mu'} \Bigg) .
\end{aligned}$$

Notice that the sum in the index i runs up to l, since for $i > l'$ we have that $Q^+_{\mu_i - i} = 0$. It then follows that 2 $= \sum_{r=1}^{l} b^*_{j+r} \sum_{\lambda/\mu = (1^r)} S_\mu$. Finally, taking advantage of (4.201), together with the identity $\lambda^{[i]} - \rho_l = (\lambda - \rho_{l+1})^{[i]}$, which can be deduced after a straightforward computation, we end up with

$$3 = b^*_{j+l+1} \sum_{i=0}^{l} (-1)^i Q^+_{\lambda_i - i - 1} S_{(\lambda - \rho_{l+1})^{[i]\prime}} = b^*_{j+l+1} S_{(\lambda - \rho_{l+1})'} = \sum_{\substack{\mu \in \mathcal{P}^+ \\ \lambda/\mu = (1^{l+1})}} S_{\mu'} .$$

In conclusion, the combination 1 + 2 + 3 coincides with the RHS of (4.200). This completes the induction proof. □

Lemma 4.4.13. *Let $\nu \in \mathcal{P}^+$ and $\mu \in \mathcal{A}^+_k(n)$. We have that $S_\nu |\mu\rangle = S^*_\nu |\mu\rangle = 0$ if $\ell(\nu) > k$, otherwise*

$$S_\nu |\mu\rangle = \sum_{d \in \mathbb{Z}_{\ge 0}} z^d \sum_{\lambda \in \mathcal{A}^+_k(n)} \chi^{\lambda,d}_{\mu\nu} |\lambda\rangle , \tag{4.202}$$

$$S^*_\nu |\mu\rangle = \sum_{d \in \mathbb{Z}_{\ge 0}} z^{-d} \sum_{\lambda \in \mathcal{A}^+_k(n)} \chi^{\lambda^\vee,d}_{\mu^\vee\nu} |\lambda\rangle . \tag{4.203}$$

The coefficient $\chi^{\lambda,d}_{\mu\nu}$ was introduced in Definition 3.3.17.

Proof. The claim follows after a straightforward but tedious computation, which resembles the one presented in the proof of Lemma 4.4.10. This computation consists in a repeated application of (4.198) and the adjoint equation of (4.199) to the states $S_\nu |\mu\rangle$ and $S^*_\nu |\mu\rangle$ respectively.

We now present an alternative proof of the claim. Applying the maps Ξ^+_n and Ξ^-_n to both sides of the expansion $s_\nu = \sum_{\sigma \in \mathcal{P}^+} K_{\nu\sigma} m_\sigma$, which was introduced in (2.24), we end up respectively with the identities $S_\nu = \sum_{\sigma\in\mathcal{P}^+} K_{\nu\sigma} M_\sigma$ and $S^*_\nu = \sum_{\sigma\in\mathcal{P}^+} K_{\nu\sigma} M^*_\sigma$. Moreover, we have that that $K_{\nu\sigma} = 0$ if $\ell(\sigma) < \ell(\nu)$; see for example [52, I.6]. Taking advantage of these results, together with Lemma 4.4.10 and Definition 3.3.17, we deduce that $S_\nu |\mu\rangle = S^*_\nu |\mu\rangle = 0$ if $\ell(\nu) > k$, and that equations (4.202) and (4.203) hold if $\ell(\nu) \le k$.

□

Remark 4.4.14. Applying the dual vector $\langle\lambda|$, where $\lambda \in \mathcal{A}^+_k(n)$, on the state (4.202) we end up with the equality

$$\langle\lambda|S_\nu|\mu\rangle = \sum_{d\in\mathbb{Z}_{\ge 0}} z^d \chi^{\lambda,d}_{\mu\nu} , \tag{4.204}$$

provided that $\ell(\nu) \le k$. Otherwise we have that $\langle\lambda|S_\nu|\mu\rangle = 0$. This completes the proof of the identities (4.113) to (4.117).

Remark 4.4.15. The result presented in Lemma 4.4.13 can be used to prove some combinatorial identities between the coefficients appearing in (4.160), (4.168), (4.176), (4.192) and (4.202). As an example, consider the expansion $e_\nu = \sum_{\sigma\in\mathcal{P}^+} M_{\nu\sigma} m_\sigma$, which first appeared in (2.22). Applying the map Ξ^+_n to both sides of this expansion, we have that $Q^+_\nu = \sum_{\sigma\in\mathcal{P}^+} M_{\nu\sigma} M_\sigma$, and taking advantage of this last relation, together with Lemma 4.4.10, we can then deduce that

$$Q^+_\nu |\mu\rangle = \sum_{d\in\mathbb{Z}_{\ge 0}} z^d \sum_{\lambda\in\mathcal{A}^+_k(n)} \left(\sum_{\sigma\in\mathcal{P}^+_k} N^{\lambda,d}_{\mu\sigma} M_{\nu\sigma} \right) |\lambda\rangle \ .$$

A comparison with (4.168) yields the following combinatorial identity,

$$\psi_{\lambda/d/\mu}(\nu) = \sum_{\sigma\in\mathcal{P}^+_k} N^{\lambda,d}_{\mu\sigma} M_{\nu\sigma} ,$$

which has been shown already in Proposition 3.3.10 by different means.

4.5 State-charge correspondence

In Section 4.4 we considered the images under the maps $\Xi^\pm_n$ of the bases of Λ introduced in Section 2.2, and we studied their action on the vectors $|\mu\rangle \in \mathcal{F}^{\otimes n}$. Notice that in Lemmas 4.4.2, 4.4.5, 4.4.6 and 4.4.10 we recover the coefficients appearing in the product expansions (3.40), (3.49), (3.51) and (3.78). These product expansions belong to $\mathcal{V}_k(n)$, which is the quotient of $\Lambda_k[z, z^{-1}]$ defined in Section 3.3. We shall formalise this fact in the next lemma. Denote by $\Lambda_\mathbb{R} \subset \Lambda$ the ring of symmetric functions with real coefficients. For

$\lambda, \mu \in \mathcal{A}_k^+(n)$ and $g \in \Lambda_{\mathbb{R}}$, define the coefficient $g_\mu^{\lambda,d}$ via the following product expansion in $\mathcal{V}_k(n)$,

$$g(x_1, \ldots, x_k) m_\mu(x_1, \ldots, x_k) = \sum_{d\in\mathbb{Z}} z^d \sum_{\lambda\in\mathcal{A}_k^+(n)} g_\mu^{\lambda,d} m_\lambda(x_1, \ldots, x_k) \ . \tag{4.205}$$

Lemma 4.5.1. *Let $\mu \in \mathcal{A}_k^+(n)$ and $g \in \Lambda_{\mathbb{R}}$. We have the equalities*

$$\Xi_n^+(g) \left|\mu\right\rangle \ = \ \sum_{d\in\mathbb{Z}} z^d \sum_{\lambda\in\mathcal{A}_k^+(n)} g_\mu^{\lambda,d} \left|\lambda\right\rangle \ , \tag{4.206}$$

$$\Xi_n^-(g) \left|\mu\right\rangle \ = \ \sum_{d\in\mathbb{Z}} z^{-d} \sum_{\lambda\in\mathcal{A}_k^+(n)} g_{\mu^\vee}^{\lambda^\vee,d} \left|\lambda\right\rangle \ . \tag{4.207}$$

Proof. Since $g \in \Lambda_{\mathbb{R}}$, we can write that $g = \sum_{\sigma\in\mathcal{P}^+} g_\sigma m_\sigma$ for some coefficients $g_\sigma \in \mathbb{R}$ satisfying the relation $g_\sigma = g_\sigma^*$. Projecting both sides of this equality onto Λ_k, we arrive at the identity $g(x_1, \ldots, x_k) = \sum_{\sigma\in\mathcal{P}_k^+} g_\sigma m_\sigma(x_1, \ldots, x_k)$. This is because $m_\sigma(x_1, \ldots, x_k) = 0$ if $\ell(\sigma) > k$, as shown in Lemma 2.2.5. Taking advantage of the quotient map $\pi_{k,n}$: $\Lambda[z, z^{-1}] \to \mathcal{V}_k(n)$, together with Lemma 3.3.5, we end up with the following identity in $\mathcal{V}_k(n)$,

$$g(x_1, \ldots, x_k) m_\mu(x_1, \ldots, x_k) = \sum_{d\in\mathbb{Z}} z^d \sum_{\lambda\in\mathcal{A}_k^+(n)} \Bigg(\sum_{\sigma\in\mathcal{P}_k^+} g_\sigma N_{\sigma\mu}^{\lambda,d} \Bigg) m_\lambda(x_1, \ldots, x_k) \ ,$$

and a comparison with (4.205) yields the equality $g_\mu^{\lambda,d} = \sum_{\sigma\in\mathcal{P}_k^+} g_\sigma N_{\sigma\mu}^{\lambda,d}$. Thanks to this last equality, together with Lemma 4.4.10, we can then employ the relations $\Xi_n^+(g) = \sum_{\sigma\in\mathcal{P}^+} g_\sigma M_\sigma$ and $\Xi_n^-(g) = \sum_{\sigma\in\mathcal{P}^+} g_\sigma M_\sigma^*$ to prove the claim.

□

We now show that the conserved charges of the free boson model generate an algebra which is isomorphic to $\mathcal{V}_k(n)$. Compare with [46, Theorem 6.12] and [41, Theorem 7.11].

Theorem 4.5.2. *Set $\mathcal{F}_k(n) = \mathcal{F}_k^{\otimes n} \otimes \mathbb{C}[z, z^{-1}]$, and define for $\mu, \nu \in \mathcal{A}_k^+(n)$ the product*

$$\left|\mu\right\rangle \circledast \left|\nu\right\rangle \equiv M_\mu \left|\nu\right\rangle \ . \tag{4.208}$$

Then $(\mathcal{F}_k(n), \circledast)$ is a commutative, associative and unital $\mathbb{C}[z, z^{-1}]$-algebra. The unit is given by $z^{-k} \left|n^k\right\rangle$, where $n^k = (n, \ldots, n) \in \mathcal{A}_k^+(n)$. So, in particular $\left|\lambda\right\rangle = z^{-k} M_\lambda \left|n^k\right\rangle$.

Proof. Thanks to Lemma 4.4.10, we have that

$$\left|\mu\right\rangle \circledast \left|\nu\right\rangle = \sum_{d\in\mathbb{Z}_{\geq 0}} z^d \sum_{\lambda\in\mathcal{A}_k^+(n)} N_{\mu\nu}^{\lambda,d} \left|\lambda\right\rangle \ . \tag{4.209}$$

The fact that $(\mathcal{F}_k(n), \circledast)$ is a commutative algebra follows from the equality $N_{\mu\nu}^{\lambda,d} = N_{\nu\mu}^{\lambda,d}$, which was shown in Lemma 3.3.6. Taking advantage of (4.209), together with Property 5 of Lemma 3.3.6, which is the identity $N_{n^k\nu}^{\lambda,d} = \delta_{d,k}\delta_{\lambda\nu}$, it follows that $|n^k\rangle \circledast |\nu\rangle = z^k |\nu\rangle$. This implies that the unit element of $(\mathcal{F}_k(n), \circledast)$ is given by $z^{-k}|n^k\rangle$. Finally, associativity is a consequence of the following chain of equalities,

$$
\begin{aligned}
(|\eta\rangle \circledast |\mu\rangle) \circledast |\nu\rangle &= \sum_{d_1 \in \mathbb{Z}_{\geq 0}} z^{d_1} \sum_{\sigma \in \mathcal{A}_k^+(n)} N_{\eta\mu}^{\sigma,d_1} |\sigma\rangle \circledast |\nu\rangle \\
&= \sum_{d_1 \in \mathbb{Z}_{\geq 0}} \sum_{d_2 \in \mathbb{Z}_{\geq 0}} z^{d_1+d_2} \sum_{\sigma,\rho \in \mathcal{A}_k^+(n)} N_{\eta\mu}^{\sigma,d_1} N_{\sigma\nu}^{\rho,d_2} |\rho\rangle \\
&= \sum_{d \in \mathbb{Z}_{\geq 0}} z^{d} \sum_{\rho \in \mathcal{A}_k^+(n)} \left(\sum_{\substack{d_1+d_2=d \\ d_1,d_2 \geq 0}} \sum_{\sigma \in \mathcal{A}_k^+(n)} N_{\eta\mu}^{\sigma,d_1} N_{\sigma\nu}^{\rho,d_2} \right) |\rho\rangle \\
&= \sum_{d \in \mathbb{Z}_{\geq 0}} z^{d} \sum_{\rho \in \mathcal{A}_k^+(n)} \left(\sum_{\substack{d_1+d_2=d \\ d_1,d_2 \geq 0}} \sum_{\sigma \in \mathcal{A}_k^+(n)} N_{\mu\nu}^{\sigma,d_1} N_{\sigma\eta}^{\rho,d_2} \right) |\rho\rangle \\
&= \sum_{d_1 \in \mathbb{Z}_{\geq 0}} \sum_{d_2 \in \mathbb{Z}_{\geq 0}} z^{d_1+d_2} \sum_{\sigma,\rho \in \mathcal{A}_k^+(n)} N_{\mu\nu}^{\sigma,d_1} N_{\sigma\eta}^{\rho,d_2} |\rho\rangle \\
&= |\eta\rangle \circledast \left(\sum_{d_1 \in \mathbb{Z}_{\geq 0}} z^{d_1} \sum_{\sigma \in \mathcal{A}_k^+(n)} N_{\mu\nu}^{\sigma,d_1} |\sigma\rangle \right) \\
&= |\eta\rangle \circledast (|\mu\rangle \circledast |\nu\rangle) ,
\end{aligned}
$$

where in the fourth line we used Property 2 of Lemma 3.3.6. □

Theorem 4.5.3. *The assignment $|\lambda\rangle \mapsto m_\lambda(x_1, \ldots, x_k) \in \mathcal{V}_k(n)$ for all $\lambda \in \mathcal{A}_k^+(n)$ defines an isomorphism of algebras*

$$(\mathcal{F}_k(n), \circledast) \cong \mathcal{V}_k(n) . \tag{4.210}$$

Proof. Let $\phi : (\mathcal{F}_k(n), \circledast) \to \mathcal{V}_k(n)$ be the map defined as $\phi(|\lambda\rangle) = m_\lambda(x_1, \ldots, x_k)$ for all $\lambda \in \mathcal{A}_k^+(n)$. Taking advantage of Lemma 3.3.5, together with (4.209), one has that $\phi(|\mu\rangle \circledast |\nu\rangle) = \phi(|\mu\rangle)\phi(|\nu\rangle)$. This implies that ϕ defines an algebra homomorphism. But since $\{|\lambda\rangle\}_{\lambda \in \mathcal{A}_k^+(n)}$ and $\{m_\lambda(x_1, \ldots, x_k)\}_{\lambda \in \mathcal{A}_k^+(n)}$ are bases of $(\mathcal{F}_k(n), \circledast)$ and $\mathcal{V}_k(n)$ respectively, it follows that ϕ is an algebra isomorphism, thus proving the claim. □

4.5.1 Idempotents in the algebra $(\mathcal{F}_k(n), \circledast)$

The goal of this section is to show that the eigenvectors of the operators belonging to the image of the maps $\Xi_n^{\pm}$ coincide with the idempotents of the algebra $(\mathcal{F}_k(n), \circledast)$ introduced in Theorem 4.5.2. This statement is formalised in Proposition 4.5.6 below. This is analogous to the case of the Verlinde algebra $\mathcal{V}_k(\hat{\mathfrak{sl}}_n)$ in relation to the phase model [46]. Assume that $z^{\pm 1/n}$ exists, and let χ be a primitive n-th root of unity. For $j = 1, \ldots, n$, set $\zeta_j = z^{1/n}\chi^j$ and consider the following Fourier transforms of the creation operators in the Heisenberg algebra $\mathcal{H}_n$,

$$B_j^* = \frac{1}{\sqrt{n}} \sum_{l=1}^{n} (\zeta_j)^{-l}\, b_l^* \ . \tag{4.211}$$

Moreover, for $\lambda \in \mathcal{A}_k^+(n)$ set $\zeta_\lambda = (\zeta_{\lambda_1}, \ldots, \zeta_{\lambda_k})$ and define

$$|\zeta_\lambda\rangle = B_{\lambda_1}^* B_{\lambda_2}^* \cdots B_{\lambda_k}^* \,|0\rangle = \frac{1}{n^{k/2}} \sum_{\sigma\in\mathcal{A}_k^+(n)} \overline{m^\sigma(\zeta_\lambda)}\,|\sigma\rangle \ . \tag{4.212}$$

In the last equality we used the identity $m^\sigma(\zeta_\lambda) = \sum_{w\in S_k} \zeta_{\lambda_1}^{\sigma_{w(1)}} \cdots \zeta_{\lambda_k}^{\sigma_{w(k)}}$, which follows from Lemma 2.2.8, and moreover we employed the relation $\bar{z} = z^{-1}$. We will show in Lemma 4.5.5 below that the states (4.212) are simultaneous eigenvectors of all the operators belonging to the image of the maps $\Xi_n^\pm$. We shall take advantage of the scalar product $\langle\,|\,\rangle_\iota$ introduced in Section 4.1.4. This is the scalar product induced by the vector space isomorphism $\iota : \mathcal{F}^{\otimes n} \to \tilde{\mathcal{F}}^{\otimes n}$, which was defined in (4.44) as

$$|\lambda\rangle \mapsto \frac{1}{u_\lambda}\,\langle\lambda| \ .$$

Recall that $\langle\,|\,\rangle_\iota$ is by definition antilinear in the first factor.

Lemma 4.5.4. *The states $\{|\zeta_\lambda\rangle\}_{\lambda\in\mathcal{A}_k^+(n)}$ form an orthogonal basis of $\mathcal{F}_k^{\otimes n}$ with respect to the scalar product $\langle\,|\,\rangle_\iota$. Their norm is given by $||\zeta_\lambda|| \equiv \langle\zeta_\lambda|\zeta_\lambda\rangle_\iota^{1/2} = u_\lambda^{1/2}$.*

Proof. Thanks to Lemma 2.2.5, we can deduce that

$$m_\lambda(\zeta_\mu) = \frac{z^{\frac{|\lambda|}{n}}}{u_\lambda} \sum_{w\in S_k} \prod_{i=1}^{k} \chi^{\lambda_i\,\mu_{w(i)}} \ . \tag{4.213}$$

Taking advantage of this last identity, together with the relation $m^\sigma = u_\sigma m_\sigma$, we have that the scalar product between the states $|\zeta_\lambda\rangle$ and $|\zeta_\mu\rangle$ is given by

$$\begin{aligned} \langle\zeta_\lambda\,|\,\zeta_\mu\rangle_\iota &= \frac{1}{n^k} \sum_{\sigma\in\mathcal{A}_k^+(n)} m_\sigma(\zeta_\lambda)\overline{m^\sigma(\zeta_\mu)} \\ &= \frac{1}{n^k} \sum_{\sigma\in\mathcal{A}_k^+(n)} \frac{1}{u_\sigma} \sum_{w',w''\in S_k} \prod_{i=1}^{k} \chi^{\sigma_i(\lambda_{w'(i)}-\mu_{w''(i)})} \ . \end{aligned}$$

Let $f : \mathcal{P}_k \to \mathbb{C}$, and consider the identity

$$\sum_{\sigma_1=1}^{n} \cdots \sum_{\sigma_k=1}^{n} f(\sigma) \quad = \quad \sum_{\sigma\in\mathcal{A}_k^+(n)} \sum_{w\in S^\sigma} f(\sigma.w) \ ,$$

where we set $\sigma = (\sigma_1, \ldots, \sigma_k) \in \mathcal{P}_k$. If the function f satisfies the constraint $f(\sigma.w) = f(\sigma)$ for all $w \in S^\sigma$, then the RHS of the identity above is equal to $\sum_{\sigma\in\mathcal{A}_k^+(n)} |S^\sigma| f(\sigma)$. Moreover, for $\sigma \in \mathcal{A}_k^+(n)$ we have the relation $|S_\sigma| = u_\sigma$, and then $|S^\sigma| = |S_k|/|S_\sigma| = k!/u_\sigma$. Notice that if we set $f(\sigma) = \sum_{w',w''\in S_k} \prod_{i=1}^k \chi^{\sigma_i(\lambda_{w'(i)} - \mu_{w''(i)})}$, we have that

$$
\begin{aligned}
f(\sigma.w) &= \sum_{w',w''\in S_k} \prod_{i=1}^{k} \chi^{\sigma_{w(i)}(\lambda_{w'(i)} - \mu_{w''(i)})} \\
&= \sum_{w',w''\in S_k} \prod_{i=1}^{k} \chi^{\sigma_i(\lambda_{w^{-1}w'(i)} - \mu_{w^{-1}w''(i)})} \\
&= \sum_{w',w''\in S_k} \prod_{i=1}^{k} \chi^{\sigma_i(\lambda_{w'(i)} - \mu_{w''(i)})} = f(\sigma) \; .
\end{aligned}
$$

In the last line we renamed $w^{-1}w'$ and $w^{-1}w''$ respectively as w' and w''. We then have the following chain of equalities,

$$
\begin{aligned}
\langle \zeta_\lambda \,|\, \zeta_\mu \rangle_\iota &= \frac{1}{n^k k!} \sum_{w',w''\in S_k} \sum_{\sigma_1=1}^{n} \cdots \sum_{\sigma_k=1}^{n} \prod_{i=1}^{n} \chi^{\sigma_i(\lambda_{w'(i)} - \mu_{w''(i)})} \\
&= \frac{1}{n^k k!} \sum_{w',w''\in S_k} \prod_{i=1}^{k} \sum_{\sigma_i=1}^{n} \chi^{\sigma_i(\lambda_{w'(i)} - \mu_{w''(i)})} \\
&= \frac{1}{k!} \sum_{w',w''\in S_k} \prod_{i=1}^{k} \delta_{\lambda_{w'(i)},\mu_{w''(i)}} \\
&= \frac{1}{k!} \sum_{w',w''\in S_k} \prod_{i=1}^{k} \delta_{\lambda_i,\mu_{(w')^{-1}w''(i)}} = u_\mu \sum_{w\in S^\mu} \prod_{i=1}^{k} \delta_{\lambda_i,\mu_{w(i)}} \; .
\end{aligned}
$$

In the third line we employed for $r \in \mathbb{Z}$ the relation $\sum_{j=1}^n \chi^{rj} = n\delta_{r \bmod n, 0}$, which follows from the fact that χ is a n-th root of unity. In the last equality, we first renamed $(w')^{-1}w''$ as w, and then we applied equation (4.172). It follows that

$$
\langle \zeta_\lambda \,|\, \zeta_\mu \rangle_\iota = u_\mu \delta_{\lambda\mu} \; , \tag{4.214}
$$

which shows that the vectors $\{|\zeta_\lambda\rangle\}_{\lambda\in\mathcal{A}_k^+(n)}$ are orthogonal, with norm $||\zeta_\lambda|| = u_\lambda^{1/2}$. This also implies the validity of the following identity,

$$
\frac{1}{n^k} \sum_{\sigma\in\mathcal{A}_k^+(n)} m_\sigma(\zeta_\lambda)\overline{m^\sigma(\zeta_\mu)} = u_\lambda \delta_{\lambda\mu} \; . \tag{4.215}
$$

Taking advantage of (4.212) and (4.215), together with the equality $z^{-|\sigma|/n} m^\sigma(\zeta_\lambda) =$

$z^{-|\lambda|/n}m^\lambda(\zeta_\sigma)$, we end up with

$$|\mu\rangle = \sum_{\lambda\in\mathcal{A}_k^+(n)} \frac{m_\mu(\zeta_\lambda)}{n^{k/2}u_\lambda}\,|\zeta_\lambda\rangle \ . \tag{4.216}$$

Since this last relation holds for all $\mu \in \mathcal{A}_k^+(n)$, and since $\{|\mu\rangle\}_{\mu\in\mathcal{A}_k^+(n)}$ is a basis of $\mathcal{F}_k^{\otimes n}$, we conclude that the states $\{|\zeta_\lambda\rangle\}_{\lambda\in\mathcal{A}_k^+(n)}$ form a basis of $\mathcal{F}_k^{\otimes n}$ as well, thus proving the claim. □

Lemma 4.5.5. *For every $g \in \Lambda$, the states $\{|\zeta_\lambda\rangle\}_{\lambda\in\mathcal{A}_k^+(n)}$ are eigenvectors of the operators $\Xi_n^\pm(g)$, with eigenvalues*

$$\begin{aligned}
\Xi_n^+(g)\,|\zeta_\lambda\rangle &= g(\zeta_\lambda)\,|\zeta_\lambda\rangle \ , &(4.217)\\
\Xi_n^-(g)\,|\zeta_\lambda\rangle &= \overline{g(\zeta_\lambda)}\,|\zeta_\lambda\rangle \ . &(4.218)
\end{aligned}$$

Proof. Multiplying both sides of the commutation relation $Q_r^+b_l^* = b_l^*Q_r^+ + b_{l+1}^*Q_{r-1}^+$ by $n^{-1/2}(\zeta_j)^{-l}$, and summing over all $l = 1,\dots,m$ and $r \in \mathbb{Z}_{\geq 0}$, we end up with the equality $Q^+(u)B_j^* = (1+u\zeta_j)B_j^*Q^+(u)$. This, together with (4.212), can be used to prove that the states $\{|\zeta_\lambda\rangle\}_{\lambda\in\mathcal{A}_k^+(n)}$ are eigenvectors of the Q^+ operator, with eigenvalue

$$Q^+(u)\,|\zeta_\lambda\rangle = \prod_{i=1}^{k}(1+u\zeta_{\lambda_i})\,|\zeta_\lambda\rangle \ . \tag{4.219}$$

A comparison of (4.219) with the generating function (2.19) of the elementary symmetric functions, which for k variables is given by $E(u) = \prod_{j=1}^k(1+ux_j)$, implies that $Q_r^+\,|\zeta_\lambda\rangle = e_r(\zeta_\lambda)\,|\zeta_\lambda\rangle$ for all $r \in \mathbb{Z}_{\geq 0}$ and $\lambda \in \mathcal{A}_k^+(n)$. Thanks to Proposition 4.99 we have that $Q_r^+ = \Xi_n^+(e_r)$, and since $\{e_\lambda\}_{\lambda\in\mathcal{P}^+}$ is a basis of Λ we deduce the validity of (4.217). Let $s \in \mathbb{Z}$ with $0 < s < n$, and consider the following chain of equalities,

$$\begin{aligned}
T_{-s}\,|\zeta_\lambda\rangle &= z^{-1}T_{n-s}\,|\zeta_\lambda\rangle = z^{-1}p_{n-s}(\zeta_\lambda)\,|\zeta_\lambda\rangle\\
&= z^{-1}\sum_{j=1}^{n}\zeta_{\lambda_j}^{n-s}\,|\zeta_\lambda\rangle = \sum_{i=1}^{n}\zeta_{\lambda_j}^{-s}\,|\zeta_\lambda\rangle = \overline{p_s(\zeta_\lambda)}\,|\zeta_\lambda\rangle \ .
\end{aligned}$$

In the first line we employed equations (4.82) and (4.217). In the second line we used the identity $\zeta_j^n = z$ for $j = 1,\dots,n$, together with the relation $z^{-1} = \bar{z}$. Moreover, since $T_{-n} = z^{-1}N$, it follows that $T_{-n}\,|\zeta_\lambda\rangle = \overline{p_n(\zeta_\lambda)}\,|\zeta_\lambda\rangle$. This is because, as we mentioned at the beginning of Section 4.2, the number operator $N = \sum_{i=1}^n b_i^*b_i$ satisfies the eigenvalue equation $N\,|\zeta_\lambda\rangle = k\,|\zeta_\lambda\rangle$. Finally, taking advantage of the relation $T_{-i} = z^{-1}T_{-i+n}$ for $i > n$, we have that $T_{-r}\,|\zeta_\lambda\rangle = \overline{p_r(\zeta_\lambda)}\,|\zeta_\lambda\rangle$ for all $r \in \mathbb{Z}_{\geq 0}$. This last equality implies the validity of (4.218), since $\Xi_n^-(p_r) = T_{-r}$ by definition, and moreover $\{p_\lambda\}_{\lambda\in\mathcal{P}^+}$ is a basis of

Λ.

$\square$

The next result implies that the states $\{|e_\lambda\rangle\}_{\lambda\in\mathcal{A}_k^+(n)}$ defined via the relation

$$|e_\lambda\rangle \equiv \frac{1}{n^{k/2}u_\lambda}\,|\zeta_\lambda\rangle \tag{4.220}$$

are a complete set of orthogonal idempotents of the algebra $(\mathcal{F}_k(n),\circledast)$. Compare with [41, Prop. 7.15].

Proposition 4.5.6. *For every $\lambda,\mu\in\mathcal{A}_k^+(n)$ we have the relation*

$$|e_\lambda\rangle\circledast|e_\mu\rangle=\delta_{\lambda\mu}\,|e_\lambda\rangle\ . \tag{4.221}$$

Moreover, the unit of $(\mathcal{F}_k(n),\circledast)$ admits the following decomposition,

$$z^{-k}\,|n^k\rangle=\sum_{\lambda\in\mathcal{A}_k^+(n)}|e_\lambda\rangle\ , \tag{4.222}$$

where $n^k=(n,\ldots,n)\in\mathcal{A}_k^+(n)$.

Proof. We have the chain of equalities

$$\begin{aligned}
|\zeta_\lambda\rangle\circledast|\zeta_\mu\rangle &= \frac{1}{n^{k/2}}\sum_{\sigma\in\mathcal{A}_k^+(n)}\overline{m^\sigma(\zeta_\lambda)}\,|\sigma\rangle\circledast|\zeta_\mu\rangle\\
&= \frac{1}{n^{k/2}}\sum_{\sigma\in\mathcal{A}_k^+(n)}\overline{m^\sigma(\zeta_\lambda)}M_\sigma\,|\zeta_\mu\rangle\\
&= \frac{1}{n^{k/2}}\sum_{\sigma\in\mathcal{A}_k^+(n)}\overline{m^\sigma(\zeta_\lambda)}m_\sigma(\zeta_\mu)\,|\zeta_\mu\rangle=\delta_{\lambda\mu}\,n^{k/2}u_\lambda\,|\zeta_\lambda\rangle\ .
\end{aligned}$$

In the first line we used the expansion (4.212), whereas in the second line we took advantage of the product (4.208). In the third line we employed Lemma 4.5.5 for the operator $M_\sigma=\Xi_n^+(m_\sigma)$, and then we made use of the relation (4.215). Equation (4.221) then follows by plugging the relation $|\zeta_\nu\rangle=n^{k/2}u_\nu\,|e_\nu\rangle$ for $\nu\in\mathcal{A}_k^+(n)$ into the equality $|\zeta_\lambda\rangle\circledast|\zeta_\mu\rangle=\delta_{\lambda\mu}\,n^{k/2}u_\lambda\,|\zeta_\lambda\rangle$. Starting from (4.215), and taking advantage of the relation $z^{|\mu|/n}u_\lambda m_\lambda(\zeta_\mu)=z^{|\lambda|/n}u_\mu m_\mu(\zeta_\lambda)$, which follows from (4.213), we end up with the identity

$$\frac{1}{n^k}\sum_{\sigma\in\mathcal{A}_k^+(n)}\frac{1}{u_\sigma}m_\lambda(\zeta_\sigma)\overline{m^\mu(\zeta_\sigma)}=\delta_{\lambda\mu}\ . \tag{4.223}$$

We can then employ (4.223) to obtain the equalities

$$
\begin{aligned}
\sum_{\lambda \in \mathcal{A}_k^+(n)} |e_\lambda\rangle &= \sum_{\sigma \in \mathcal{A}_k^+(n)} \left(\frac{1}{n^k} \sum_{\lambda \in \mathcal{A}_k^+(n)} \frac{1}{u_\lambda} \overline{m^\sigma(\zeta_\lambda)} \right) |\sigma\rangle \\
&= z^{-k} \sum_{\sigma \in \mathcal{A}_k^+(n)} \left(\frac{1}{n^k} \sum_{\lambda \in \mathcal{A}_k^+(n)} \frac{1}{u_\lambda} m_{n^k}(\zeta_\lambda) \overline{m^\sigma(\zeta_\lambda)} \right) |\sigma\rangle = z^{-k} \left| n^k \right\rangle \ ,
\end{aligned}
$$

thus proving the validity of (4.222). In the first line we took advantage of (4.212) and (4.220), whereas in the second line we used the relation $m_{n^k}(\zeta_\lambda) = z^k$, which can be deduced from (4.213). $\square$

Chapter 5

Generalised Verlinde algebras

The Verlinde algebra of an affine Lie algebra $\hat{\mathfrak{g}}$ is the fusion algebra of the integrable highest weight modules of level k [33]. The basis of the Verlinde algebra is indexed by the elements from the set of integral dominant weights of level k. For the special case of the $\hat{\mathfrak{sl}}_n$-Verlinde algebra, the latter is in bijection with the set $\mathcal{A}_k^+(n)$ introduced in (3.11) [41]. The $\hat{\mathfrak{sl}}_n$-Verlinde algebra is therefore defined as the $\mathbb{C}$-algebra with basis indexed by the elements from $\mathcal{A}_k^+(n)$ with the multiplication

$$\lambda * \mu = \sum_{\nu \in \mathcal{A}_k^+(n)} \mathcal{N}_{\mu\nu}^{\lambda} \, \nu \; , \tag{5.1}$$

where the structure constants $\mathcal{N}_{\mu\nu}^{\lambda}$, the so-called fusion coefficients, are given in terms of the Verlinde formula

$$\mathcal{N}_{\mu\nu}^{\lambda} = \sum_{\sigma \in \mathcal{A}_k^+(n)} \frac{\mathcal{S}_{\mu\sigma}\mathcal{S}_{\nu\sigma}\mathcal{S}^{-1}_{\sigma\lambda}}{\mathcal{S}_{n^k\sigma}} \; . \tag{5.2}$$

The characters of the integrable highest weight modules of level k yield a representation of the modular group $\mathrm{SL}_2(\mathbb{Z})$ [33]. The images of the generators of $\mathrm{SL}_2(\mathbb{Z})$ are known as the $\mathcal{S}$-matrix (which appears in the Verlinde formula) and the $\mathcal{T}$-matrix of the Verlinde algebra. The $\mathcal{S}$-matrix is given by the Kac-Peterson formula [33], but for the $\hat{\mathfrak{sl}}_n$-Verlinde algebra it can be alternatively computed via Schur functions (see Section 2.2) evaluated at roots of unity [46]. Verlinde algebras can be identified with the Grothendieck ring of modular tensor categories [2], and therefore they are endowed with the structure of a Frobenius algebra, that is a 2D TQFT (see Section 1.1).

As we discussed in Chapter 1, we have the isomorphism of rings

$$\mathcal{V}_k(\hat{\mathfrak{sl}}_n) \cong \Lambda_k / \langle s_{(n)} - 1, s_{(n+1)}, \ldots, s_{(n+k-1)}, s_{(n+k)} + (-1)^k s_{(1^k)} \rangle \; , \tag{5.3}$$

where $\mathcal{V}_k(\hat{\mathfrak{sl}}_n)$ is the $\hat{\mathfrak{sl}}_n$-Verlinde algebra, and $\{s_\lambda\}_{\lambda\in\mathcal{P}^+}$ are the Schur functions. If $k=1$, we have that $\mathcal{V}_k(\hat{\mathfrak{sl}}_n)$ is isomorphic to $\mathcal{V}_k(n)$, the quotient of $\Lambda[z,z^{-1}]$ introduced in Section 3.3, when specialised to $z=1$ [44]. We therefore refer to $\mathcal{V}_k(n)$ as a 'generalised Verlinde algebra'. We show that $\mathcal{V}_k(n)$ is a Frobenius algebra, and that its structure constants $N_{\mu\nu}^{\lambda,d}$ (see Definition 3.3.3) satisfy a Verlinde-type formula. We describe a representation of the modular group $\mathrm{SL}_2(\mathbb{Z})$, and we identify the image of a generator of $\mathrm{SL}_2(\mathbb{Z})$ as the transition matrix from the basis of eigenvectors for the free boson model to the particle basis (see Section 4.5.1). The latter is the matrix which enters into the Verlinde-type formula mentioned above. Finally, we present an alternative formula for $N_{\mu\nu}^{\lambda,d}$ in terms of tensor multiplicities for irreducible representations of the generalised symmetric group. We shall work with the algebra $(\mathcal{F}_k(n),\circledast)$ introduced in Theorem 4.5.2, which is isomorphic to $\mathcal{V}_k(n)$ as showed in Theorem 4.5.3.

5.1 The modular group

The modular group is by definition the special linear group $\mathrm{SL}_2(\mathbb{Z})$, which is the group of 2×2 matrices with integer entries and determinant 1 (see e.g. [21, Ch. 8.16]). The group $\mathrm{SL}_2(\mathbb{Z})$ is isomorphic to the group generated by two elements s and t, which satisfy the relations

$$(st)^3 = s^2\ , \qquad s^4 = 1\ . \tag{5.4}$$

Some authors define the modular group to be the projective special linear group $\mathrm{PSL}_2(\mathbb{Z})$ instead, which is the quotient of $\mathrm{SL}_2(\mathbb{Z})$ over the integers by its centre $\{I,-I\}$ (see e.g. [17, Ch. 10]).

Let χ be a n-th primitive root of unity, and recall from Section 4.5.1 the notation $\zeta_\lambda=(\zeta_{\lambda_1},\dots,\zeta_{\lambda_k})$ for $\lambda\in\mathcal{A}_k^+(n)$, where $\zeta_j=z^{1/n}\chi^j$ for $j=1,\dots,n$. Employing equation (4.216), we can introduce the transition matrix [45, Lemma 4.1]

$$\mathcal{S}_{\lambda\mu}=\frac{m_\lambda(\zeta_\mu)}{n^{k/2}} \tag{5.5}$$

from the basis $\{u_\mu^{-1}\,|\zeta_\mu\rangle\}_{\mu\in\mathcal{A}_k^+(n)}$ to the basis $\{|\lambda\rangle\}_{\lambda\in\mathcal{A}_k^+(n)}$ of $\mathcal{F}_k^{\otimes n}$, where the vector $|\zeta_\lambda\rangle$ was introduced in (4.212). Notice that both these bases are orthogonal with respect to the scalar product $\langle\ |\ \rangle_\iota$ introduced in Section 4.1.4. Namely, setting $|\tilde\zeta_\lambda\rangle=u_\lambda^{-1}\,|\zeta_\lambda\rangle$ we have that $\langle\mu|\lambda\rangle_\iota=\langle\tilde\zeta_\mu|\tilde\zeta_\lambda\rangle_\iota=u_\lambda^{-1}\delta_{\lambda\mu}$. We call (5.5) the $\mathcal{S}$-matrix, although at present such matrix is not related to any known Verlinde algebra. The goal of this section is to prove that the $\mathcal{S}$-matrix introduced above, together with the $\mathcal{T}$-matrix defined via the relation

$$\mathcal{T}_{\lambda\mu}=\delta_{\lambda\mu}\prod_{j=1}^k\theta_{\lambda_j}\ , \qquad \theta_{\lambda_j}=e^{-\frac{\pi i}{n}\left(\lambda_j^2+\frac{1}{12}n(n-1)\right)}\ , \tag{5.6}$$

yields a representation of the modular group. Notice that both these matrices are labelled in terms of partitions belonging to the set $\mathcal{A}_k^+(n)$.

5.1.1 Properties of the $\mathcal{S}$-matrix

Define an involution $^* : \mathcal{A}_k^+(n) \to \mathcal{A}_k^+(n)$ via

$$\lambda \mapsto \lambda^* = (1^{m_{n-1}(\lambda)}, 2^{m_{n-2}(\lambda)}, \ldots, (n-1)^{m_1(\lambda)}, n^{m_n(\lambda)}) \;, \tag{5.7}$$

where we use the notation introduced in (2.1) for partitions. Notice that we have the alternative expression

$$\lambda^* = (n, \ldots, n, n - \lambda_{\ell(\lambda)-m_n(\lambda)}, \ldots, n - \lambda_2, n - \lambda_1) \;. \tag{5.8}$$

Lemma 5.1.1. *The $\mathcal{S}$-matrix introduced in* (5.5) *satisfies the following properties.*

1. $\mathcal{S}_{\lambda\mu} = z^{\frac{|\lambda|-|\mu|}{n}} \dfrac{u_\mu}{u_\lambda} \mathcal{S}_{\mu\lambda}$.
2. $\mathcal{S}^{-1}_{\lambda\mu} = z^{\frac{|\lambda|-|\mu|}{n}} \overline{\mathcal{S}_{\lambda\mu}}$.
3. $\overline{\mathcal{S}_{\lambda\mu}} = z^{-\frac{|\lambda|+|\lambda^*|}{n}} \mathcal{S}_{\lambda^*\mu} = z^{-2\frac{|\lambda|}{n}} \mathcal{S}_{\lambda\mu^*}$.

Proof. Thanks to Lemma 2.2.5 we have the identity

$$m_\lambda(\zeta_\mu) = \frac{z^{\frac{|\lambda|}{n}}}{u_\lambda} \sum_{w \in S_k} \prod_{i=1}^{k} \chi^{\lambda_i \mu_{w(i)}} \;, \tag{5.9}$$

which was mentioned in the proof of Proposition 4.5.4. Taking advantage of the equality $z^{|\mu|/n} u_\lambda m_\lambda(\zeta_\mu) = z^{|\lambda|/n} u_\mu m_\mu(\zeta_\lambda)$, which follows from (5.9), we can then immediately deduce the validity of Property 1. A comparison of equation (4.223), that is

$$\frac{1}{n^k} \sum_{\sigma \in \mathcal{A}_k^+(n)} \frac{1}{u_\sigma} m_\lambda(\zeta_\sigma) \overline{m^\mu(\zeta_\sigma)} = \delta_{\lambda\mu} \;,$$

with (5.5) shows that

$$\mathcal{S}^{-1}_{\sigma\mu} = \frac{\overline{m^\mu(\zeta_\sigma)}}{n^{k/2} u_\sigma} = z^{\frac{|\sigma|-|\mu|}{n}} \frac{\overline{m_\sigma(\zeta_\mu)}}{n^{\frac{k}{2}}} = z^{\frac{|\sigma|-|\mu|}{n}} \overline{\mathcal{S}_{\sigma\mu}} \;, \tag{5.10}$$

which proves Property 2. Let $\bar{w} \in S^{\lambda^*}$ be the permutation defined via the relation $\lambda^*.\bar{w} =$

$(\lambda_k^*, \lambda_{k-1}^*, \ldots, \lambda_1^*)$, and consider the following chain of equalities,

$$\begin{aligned}
\overline{m_\lambda(\zeta_\mu)} &= \frac{z^{-\frac{|\lambda|}{n}}}{u_\lambda} \sum_{w\in S_k} \prod_{i=1}^{k} \chi^{-\lambda_i \mu_{w(i)}} \\
&= \frac{z^{-\frac{|\lambda|}{n}}}{u_\lambda} \sum_{w\in S_k} \prod_{i=1}^{k} \chi^{(n-\lambda_i)\mu_{w(i)}} \\
&= \frac{z^{-\frac{|\lambda|}{n}}}{u_\lambda} \sum_{w\in S_k} \prod_{i=1}^{k} \chi^{\lambda^*_{\bar{w}(i)}\mu_{w(i)}} \\
&= \frac{z^{-\frac{|\lambda|}{n}}}{u_\lambda} \sum_{w\in S_k} \prod_{i=1}^{k} \chi^{\lambda^*_i \mu_{\bar{w}^{-1}w(i)}} = z^{-\frac{|\lambda^*|+|\lambda|}{n}} m_{\lambda^*}(\zeta_\mu)\ .
\end{aligned}$$

In the first line we used the relation $\bar{z} = z^{-1}$, whereas in the third line we employed (5.8), together with the fact that $\chi^n = 1$. In the last line we first renamed $\bar{w}^{-1}w$ as w, and then we used (5.9). A similar computation shows that $\overline{m_\lambda(\zeta_\mu)} = z^{-2\frac{|\lambda|}{n}} m_\lambda(\zeta_{\mu^*})$. Thanks to these equalities, together with (5.5), we can finally deduce the validity of Property 3.

□

5.1.2 The charge conjugation matrix

The $\mathcal{S}$-matrix of the Verlinde algebra satisfies the identity $\mathcal{S}^2_{\lambda\mu} = \delta_{\lambda\mu^*}$, where λ, μ are labels for the integrable dominant weights of level k, and λ^*, μ^* are the weights corresponding to the conjugate representations. The matrix $\mathcal{C} = \mathcal{S}^2$ is known as the charge conjugation matrix (or $\mathcal{C}$-matrix) of the Verlinde algebra. See for example [17] for details. Let us define a $\mathcal{C}$-matrix via the relation

$$\mathcal{C}_{\lambda\mu} = \delta_{\lambda\mu^*}\ , \tag{5.11}$$

where $\lambda, \mu \in \mathcal{A}_k^+(n)$, and the map $^* : \mathcal{A}_k^+(n) \to \mathcal{A}_k^+(n)$ was introduced in (5.7). Since the map $^* : \mathcal{A}_k^+(n) \to \mathcal{A}_k^+(n)$ is an involution, it follows that

$$\mathcal{C}^2_{\lambda\mu} = \delta_{\lambda\mu}\ . \tag{5.12}$$

Stated otherwise, $\mathcal{C}^2$ corresponds to the identity matrix.

Lemma 5.1.2. *Set $z = 1$. The $\mathcal{S}$-matrix, $\mathcal{T}$-matrix and $\mathcal{C}$-matrix introduced respectively in* (5.5)*,* (5.6) *and* (5.11) *satisfy the identities*

$$\mathcal{S}^2 = \mathcal{C} \tag{5.13}$$

and

$$\mathcal{C}\mathcal{T} = \mathcal{T}\mathcal{C}\ . \tag{5.14}$$

Proof. Suppose first that $k = 1$. Equations (5.5), (5.6) and (5.11) then coincide respectively with the $\mathcal{S}$-matrix, $\mathcal{T}$-matrix and $\mathcal{C}$-matrix from the $\hat{\mathfrak{sl}}_n$-Verlinde algebra at level $k = 1$ (see e.g. [33] for details). The proof of (5.13) and (5.14) for $k = 1$ can be found in *loc. cit.* Assume now that $k > 1$, and notice that we have the identities

$$\mathcal{S}_{\lambda\mu} = \sum_{w\in S^\lambda}\prod_{i=1}^{k}\mathcal{S}_{w(\lambda_i),\mu_i}\ , \qquad \mathcal{T}_{\lambda\mu} = \prod_{i=1}^{k}\mathcal{T}_{\lambda_i,\mu_i}\ , \qquad \mathcal{C}_{\lambda\mu} = \prod_{i=1}^{k}\mathcal{C}_{\lambda_i,\mu_i}\ , \tag{5.15}$$

where $(\mathcal{S}_{rs})_{1\le r,s\le n}$, $(\mathcal{T}_{rs})_{1\le r,s\le n}$ and $(\mathcal{C}_{rs})_{1\le r,s\le n}$ are the $\mathcal{S}$-matrix, $\mathcal{T}$-matrix and $\mathcal{C}$-matrix for the case $k = 1$. In the proof of Proposition 4.5.4 we showed the validity of the following equality,

$$\sum_{\sigma_1=1}^{n}\cdots\sum_{\sigma_k=1}^{n} f(\sigma) = \sum_{\sigma\in\mathcal{A}_k^+(n)}\frac{k!}{u_\sigma}f(\sigma)\ , \tag{5.16}$$

where $\sigma = (\sigma_1,\dots,\sigma_k)\in\mathcal{P}_k$, and $f : \mathcal{P}_k\to\mathbb{C}$ satisfies the constraint $f(\sigma.w) = f(\sigma)$ for all $w\in S_k$. Starting from the expansion $\mathcal{S}^2_{\lambda\mu} = \sum_{\sigma\in\mathcal{A}_k^+(n)}\mathcal{S}_{\lambda\sigma}\mathcal{S}_{\sigma\mu}$ we deduce the following chain of equalities,

$$\mathcal{S}^2_{\lambda\mu} = \sum_{w\in S^\lambda}\prod_{i=1}^{k}\mathcal{S}^2_{\lambda_{w(i)},\mu_i} = \sum_{w\in S^\lambda}\prod_{i=1}^{k}\mathcal{C}_{\lambda_{w(i)},\mu_i} = \sum_{w\in\mathcal{S}^\lambda}\prod_{i=1}^{k}\delta_{(\lambda_{w(i)}-\mu_i)\,\mathrm{mod}\,n,0} = \mathcal{C}_{\lambda\mu}\ ,$$

which proves the validity of the relation $\mathcal{S}^2 = \mathcal{C}$ for $k > 1$. The first equality follows after a straightforward computation, with the help of (5.15) and (5.16). For the second one we employed the relation $\mathcal{S}^2 = \mathcal{C}$ for the case $k = 1$. In the third we used the identity $\mathcal{C}_{rs} = \delta_{(r-s)\,\mathrm{mod}\,n,0}$, which follows from (5.11) for $k = 1$. The relation (5.14) follows instead from the equalities

$$(\mathcal{C}\mathcal{T})_{\lambda\mu} = \mathcal{C}_{\lambda\mu}\prod_{i=1}^{k}\theta_{\mu_i} = \prod_{i=1}^{k}\mathcal{C}_{\lambda_i\mu_i}\theta_{\mu_i} = \prod_{i=1}^{k}\theta_{\lambda_i}\mathcal{C}_{\lambda_i\mu_i} = (\mathcal{T}\mathcal{C})_{\lambda\mu}\ ,$$

where in the third one we employed the relation $\mathcal{C}\mathcal{T} = \mathcal{T}\mathcal{C}$ for the case $k = 1$. □

5.1.3 A representation of the modular group

We are now ready to prove the main result of this section.

Proposition 5.1.3 ([45]). *Set $z = 1$. The $\mathcal{S}$-matrix and $\mathcal{T}$-matrix introduced respectively in* (5.5) *and* (5.6) *define a representation of the modular group $SL_2(\mathbb{Z})$. That is, we have the relations*

$$(\mathcal{S}\mathcal{T})^3 = \mathcal{S}^2\ , \qquad \mathcal{S}^4 = 1\ . \tag{5.17}$$

Proof. Notice that the relation $\mathcal{S}^4 = 1$ follows immediately from Lemma 5.1.2, since $\mathcal{C}^2$

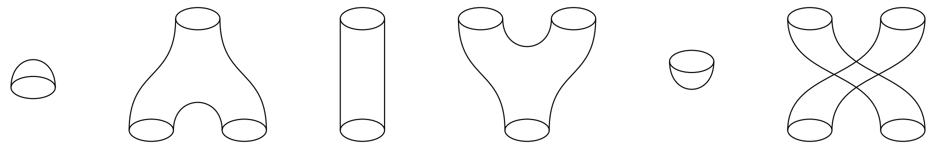


Figure 5.1: The cobordisms which represent a generating set of arrows for **2Cob**.

coincides with the identity matrix. Suppose that $k = 1$. As we pointed out in the proof of Lemma 5.1.2, equations (5.5), (5.6) and (5.11) coincide respectively with the $\mathcal{S}$-matrix, $\mathcal{T}$-matrix and $\mathcal{C}$-matrix from the $\hat{\mathfrak{sl}}_n$-Verlinde algebra at level $k = 1$ (see e.g. [33] for details). The proof of the relation $(\mathcal{ST})^3 = \mathcal{S}^2$ for $k = 1$ can then be found in *loc. cit.* Assume now that $k > 1$, and consider the expansion $(\mathcal{ST})^3_{\lambda\mu} = \sum_{\rho,\sigma\in\mathcal{A}^+_k(n)} \mathcal{S}_{\lambda\rho}\theta_\rho\mathcal{S}_{\rho\sigma}\theta_\sigma\mathcal{S}_{\sigma\mu}\theta_\mu$, where we set $\theta_\nu = \prod_{i=1}^k \theta_{\nu_i}$ for $\nu \in \mathcal{A}^+_k(n)$. With the help of (5.15) and (5.16), we end up with the following chain of equalities,

$$(\mathcal{ST})^3_{\lambda\mu} = \sum_{w\in\mathcal{S}^\mu}\prod_{i=1}^k(\mathcal{ST})^3_{\lambda_i,\mu_{w(i)}} = \sum_{w\in\mathcal{S}^\mu}\prod_{i=1}^k\mathcal{C}_{\lambda_i,\mu_{w(i)}} = \sum_{w\in\mathcal{S}^\mu}\prod_{i=1}^k\delta_{(\lambda_i-\mu_{w(i)})\bmod n,0} = \mathcal{C}_{\lambda\mu}\ .$$

which shows the validity of the relation $(\mathcal{ST})^3 = \mathcal{S}^2$ for $k > 1$ thanks to equation (5.13). In the second equality we employed the relations $(\mathcal{ST})^3 = \mathcal{S}^2 = \mathcal{C}$ for the case $k = 1$.

□

5.2 Frobenius algebras and 2D TQFT

In this section we will show that the algebra $(\mathcal{F}_k^{\otimes n}, \circledast)$, which was introduced in Theorem 4.5.2, can be endowed with the structure of a Frobenius algebra. We first recall some known facts about Frobenius algebras, and we refer the reader to [36, Ch.2] for further details.

Let A be an algebra over a field $\mathbb{K}$, and suppose that A is equipped with a bilinear form $\beta : A \otimes A \to \mathbb{K}$. The bilinear form β is non-denegerate if and only if the relation $\beta(a, b) = 0$ for all $a \in A$ implies that $b = 0$. Moreover, β is called invariant if for every $a, b, c \in A$ the following relation is satisfied,

$$\beta(a, bc) = \beta(ab, c)\ . \tag{5.18}$$

Definition 5.2.1. A Frobenius algebra is a finite dimensional, unital and associative algebra A over a field $\mathbb{K}$ equipped with a non-degenerate and invariant bilinear form.

Equivalently, one can define a Frobenius algebra as a finite dimensional, unital and associative algebra A over a field $\mathbb{K}$ equipped with a linear functional $\epsilon : A \to \mathbb{K}$ whose kernel contains no non-trivial left ideals. The functional $\epsilon \in A^*$ is called a Frobenius form, and we have the identity

$$\beta = \epsilon \circ \mu \ , \tag{5.19}$$

where $\mu : A \otimes A \to A$ is the multiplication map. Since the bilinear form β is non-degenerate, one can show that there exists a unique co-form $\gamma : \mathbb{K} \to A \otimes A$ such that the two compositions

$$A \xrightarrow{\mathrm{Id}_A \otimes \gamma} A \otimes A \otimes A \xrightarrow{\beta \otimes \mathrm{Id}_A} A \ ,$$
$$A \xrightarrow{\gamma \otimes \mathrm{Id}_A} A \otimes A \otimes A \xrightarrow{\mathrm{Id}_A \otimes \beta} A \ ,$$

are the identity map $\mathrm{Id}_A : A \to A$. The following compositions coincide,

$$A \xrightarrow{\mathrm{Id}_A \otimes \gamma} A \otimes A \otimes A \xrightarrow{\mu \otimes \mathrm{Id}_A} A \otimes A \ ,$$
$$A \xrightarrow{\gamma \otimes \mathrm{Id}_A} A \otimes A \otimes A \xrightarrow{\mathrm{Id}_A \otimes \mu} A \otimes A \ ,$$

and define a coproduct $\delta : A \to A \otimes A$. The latter satisfies the identity

$$\gamma = \delta \circ \eta \ , \tag{5.20}$$

where $\eta : \mathbb{K} \to A$ is the unit map. The coproduct δ is co-associative, and the Frobenius algebra A is therefore endowed with the structure of a co-algebra, where the co-unity map is given by the Frobenius form ϵ.

It is well known [1] that the category of commutative Frobenius algebras and the category of 2D TQFT are canonically equivalent. Let us briefly introduce the notion of 2D TQFT. See [36, Ch. 3] for further details.

Definition 5.2.2. A 2D TQFT is a monoidal functor $\boldsymbol{C} : \mathbf{2Cob} \to \mathbf{Vect}_{\mathbb{K}}$, from the category **2Cob** of two dimensional cobordisms (2-cobordisms) to the category $\mathbf{Vect}_{\mathbb{K}}$ of finite dimensional vector spaces over $\mathbb{K}$.

The objects of **2Cob** are given by $\{\mathbf{0}, \mathbf{1}, \mathbf{2}, \dots\}$, where $\boldsymbol{j}$ represents the disjoint union of j circles. Identifying the 2-cobordisms which are homeomorphic, one can show that the cobordisms depicted in Figure 5.1 represent a generating set of arrows for **2Cob**. That is, every 2-cobordism can be constructed via concatenation of the 2-cobordisms belonging to this set.

Suppose that a vector space A is the image of the object $\mathbf{1}$ under a 2D TQFT $\boldsymbol{C}$. Then A carries the structure of a commutative Frobenius algebra. The images of the 2-cobordisms represented in Figure 5.1 under the functor $\boldsymbol{C}$ are (from left to right) the Frobenius form $\epsilon : A \to \mathbb{K}$, the multiplication map $\mu : A \otimes A \to A$, the identity map $\mathrm{Id}_A : A \to A$, the co-multiplication map $\delta : A \to A \otimes A$, the unit map $\eta : \mathbb{K} \to A$, and the twist map $\sigma : A \otimes A \to A \otimes A$. The twist map ensures that the Frobenius algebra A is

Figure 5.2: An illustration of two cobordisms which are obtained via concatenation of cobordisms belonging to the generating set depicted in Figure 5.1.

commutative. Notice that the cobordism appearing on the left of Figure 5.2 is obtained via concatenation of the first and second cobordisms appearing in Figure 5.1. Its image under $\boldsymbol{C}$ is then given by $\epsilon \circ \mu$, which coincides with the bilinear form $\beta : A \otimes A \to \mathbb{K}$ thanks to (5.19). Similarly, the cobordism appearing on the right of Figure 5.2 is obtained via concatenation of the fourth and fifth cobordisms appearing in Figure 5.1. Its image under $\boldsymbol{C}$ is then given by $\delta \circ \eta$, which coincides with the co-form $\gamma : \mathbb{K} \to A \otimes A$ thanks to (5.20).

Conversely, given a Frobenius algebra A one can construct a unique 2D TQFT, that is a monoidal functor $\boldsymbol{C} : \mathbf{2Cob} \to \mathbf{Vect}_{\mathbb{K}}$. We refer the reader to [36] for details.

5.2.1 The Verlinde formula

We shall now employ the $\mathcal{S}$-matrix introduced in (5.5) to show that the structure constants of the algebra $(\mathcal{F}_k^{\otimes n}, \circledast)$, that is the coefficients $N_{\mu\nu}^{\lambda,d}$ defined in Chapter 3 as the cardinality of the set (3.39), satisfy a Verlinde-type formula. We will then take advantage of this formula to show some further properties for the coefficients $N_{\mu\nu}^{\lambda,d}$ (compare with Lemmas 3.3.4 and 3.3.6), which will be crucial for the proof of Theorem 5.2.6 below. Given $\lambda, \mu, \nu \in \mathcal{A}_k^+(n)$, define d via the constraint $|\mu| + |\nu| - |\lambda| = dn$, and set

$$N_{\mu\nu}^{\lambda} = \begin{cases} N_{\mu\nu}^{\lambda,d}, & d \in \mathbb{Z} \\ 0, & \text{otherwise} \end{cases} . \tag{5.21}$$

Proposition 5.2.3. *Let $\lambda, \mu, \nu \in \mathcal{A}_k^+(n)$, $d \in \mathbb{Z}$, and suppose that the relation $|\mu| + |\nu| - |\lambda| = dn$ holds. The coefficient $N_{\mu\nu}^{\lambda}$ satisfies the following Verlinde-type formula,*

$$N_{\mu\nu}^{\lambda} = z^{k-d} \sum_{\sigma \in \mathcal{A}_k^+(n)} \frac{\mathcal{S}_{\mu\sigma}\mathcal{S}_{\nu\sigma}\mathcal{S}_{\sigma\lambda}^{-1}}{\mathcal{S}_{n^k\sigma}} , \tag{5.22}$$

where $n^k = (n, \ldots, n) \in \mathcal{A}_k^+(n)$.

Proof. We shall make use of the operator M_ν, which was introduced in (4.182) as the image of m_ν under the map Ξ_n^+ defined in (4.99). Consider the equalities

$$\begin{aligned} M_\nu \left|\mu\right\rangle &= \sum_{\sigma\in\mathcal{A}_k^+(n)} M_\nu \mathcal{S}_{\mu\sigma}\frac{\left|\zeta_\sigma\right\rangle}{u_\sigma} \\ &= \sum_{\sigma\in\mathcal{A}_k^+(n)} m_\nu(\zeta_\sigma)\mathcal{S}_{\mu\sigma}\frac{\left|\zeta_\sigma\right\rangle}{u_\sigma} \\ &= \sum_{\lambda\in\mathcal{A}_k^+(n)} \left(\sum_{\sigma\in\mathcal{A}_k^+(n)} m_\nu(\zeta_\sigma)\mathcal{S}_{\mu\sigma}\mathcal{S}^{-1}_{\sigma\lambda}\right)\left|\lambda\right\rangle \\ &= \sum_{\lambda\in\mathcal{A}_k^+(n)} \left(z^k \sum_{\sigma\in\mathcal{A}_k^+(n)} \frac{\mathcal{S}_{\mu\sigma}\mathcal{S}_{\nu\sigma}\mathcal{S}^{-1}_{\sigma\lambda}}{\mathcal{S}_{n^k\sigma}}\right)\left|\lambda\right\rangle \ . \end{aligned} \tag{5.23}$$

In the first and third line we used the fact that the $\mathcal{S}$-matrix (5.5) is by definition the transition matrix from the basis of eigenvectors $\{u_\sigma^{-1}\left|\zeta_\sigma\right\rangle\}_{\sigma\in\mathcal{A}_k^+(n)}$ to the basis $\{\left|\mu\right\rangle\}_{\mu\in\mathcal{A}_k^+(n)}$. In the second line we employed the eigenvalue equation $M_\nu\left|\zeta_\sigma\right\rangle = m_\nu(\zeta_\sigma)\left|\zeta_\sigma\right\rangle$, which is a consequence of Lemma 4.5.5. The identity in the last line follows by taking advantage of equation (5.5), together with the fact that $\mathcal{S}_{n^k\sigma} = z^k n^{-1/2}$, which can be deduced after a straightforward computation. Thanks to Lemma 4.4.10 we have that

$$\langle\lambda|M_\nu|\mu\rangle = z^d N_{\mu\nu}^\lambda \ ,$$

which is non-zero only if $d\in\mathbb{Z}$. A comparison with the matrix elements of (5.23) implies the validity of the claim.

□

Remark 5.2.4. The structure constants $\mathcal{N}_{\mu\nu}^{\lambda,d}(q)$ of the deformed Verlinde algebra discussed in Section 1.3 satisfy a similar Verlinde-Type formula [41], which specialises to (5.22) when evaluated at $q=1$. In other words, we have the equality $N_{\mu\nu}^{\lambda,d} = \mathcal{N}_{\mu\nu}^{\lambda,d}(1)$ for $\lambda,\mu,\nu\in\mathcal{A}_k^+(n)$. Compare with Remark 3.4.4.

The next lemma is the special case $q=1$ of a similar result presented in [41, Cor. 7.13].

Lemma 5.2.5. *Let $\lambda,\mu,\nu\in\mathcal{A}_k^+(n)$. The coefficient $N_{\mu\nu}^\lambda$ satisfy the following properties.*

1. $N_{\mu\nu}^\lambda = N_{\mu^*\nu^*}^{\lambda^*}$.
2. $N_{\mu\nu}^{n^k} = \dfrac{k!}{u_\mu}\,\delta_{\mu\nu^*}$.
3. $\dfrac{N_{\mu\nu}^\lambda}{u_\lambda} = \dfrac{N_{\mu\lambda^*}^{\nu^*}}{u_\nu}$.

Proof. We shall set $z = 1$ for convenience. Starting from (5.22), and taking advantage of Property 3 of Lemma 5.1.1, we end up with

$$N_{\mu^*\nu^*}^{\lambda^*} = \sum_{\sigma \in \mathcal{A}_k^+(n)} \frac{\mathcal{S}_{\mu\sigma^*}\mathcal{S}_{\nu\sigma^*}\mathcal{S}^{-1}_{\sigma^*\lambda}}{\mathcal{S}_{n^k\sigma^*}} \ .$$

Since the map $* : \mathcal{A}_k^+(n) \to \mathcal{A}_k^+(n)$ is an involution, it follows that the RHS of this last identity is equal to $N_{\mu\nu}^{\lambda}$, and this proves Property 1. With the help of Lemma 5.1.1, one can show that

$$\mathcal{S}^{-1}_{\lambda\mu} = \frac{u_\mu}{u_\lambda}\mathcal{S}_{\mu^*\lambda} \ . \tag{5.24}$$

Thanks to this last identity, together with the relations $(n^k)^* = n^k$ and $u_{n^k} = k!$, we can employ (5.22) to deduce the equality

$$N_{\mu\nu}^{n^k} = \frac{k!}{u_\nu}\mathcal{S}^2_{\mu\nu} \ .$$

Property 2 then follows, since $\mathcal{S}^2_{\mu\nu} = \mathcal{C}_{\mu\nu} = \delta_{\mu\nu^*}$ thanks to Lemma 5.1.2. Finally, the validity of Property 3 follows from the chain of equalities

$$\begin{aligned}
N_{\mu\nu}^{\lambda} &= \sum_{\sigma \in \mathcal{A}_k^+(n)} \frac{\mathcal{S}_{\mu\sigma}\mathcal{S}_{\nu\sigma}\mathcal{S}^{-1}_{\sigma\lambda}}{\mathcal{S}_{n^k\sigma}} \\
&= \sum_{\sigma \in \mathcal{A}_k^+(n)} \frac{\frac{u_\sigma}{u_\mu}\mathcal{S}^{-1}_{\sigma\mu^*}\mathcal{S}_{\nu\sigma}\frac{u_\lambda}{u_\sigma}\mathcal{S}_{\lambda^*\sigma}}{\mathcal{S}_{n^k\sigma}} \\
&= \frac{u_\lambda}{u_\mu}\sum_{\sigma \in \mathcal{A}_k^+(n)} \frac{\mathcal{S}_{\lambda^*\sigma}\mathcal{S}_{\nu\sigma}\mathcal{S}^{-1}_{\sigma\mu^*}}{\mathcal{S}_{n^k\sigma}} = \frac{u_\lambda}{u_\mu}N_{\lambda^*\nu}^{\mu^*} \ ,
\end{aligned}$$

where in the second line we employed (5.24). □

5.2.2 An infinite family of Frobenius Algebras

We are now ready to show that the algebra $(\mathcal{F}_k^{\otimes n}, \circledast)$ can be endowed with the structure of a Frobenius algebra, i.e. a 2D TQFT. Notice that, with the help of (5.21), we can express the product (4.208) of the algebra $(\mathcal{F}_k^{\otimes n}, \circledast)$ as

$$|\mu\rangle \circledast |\nu\rangle = \sum_{\lambda \in \mathcal{A}_k^+(n)} N_{\mu\nu}^{\lambda}\, |\lambda\rangle \ . \tag{5.25}$$

Theorem 5.2.6 ([41]). *Set $z = 1$, and equip the algebra $(\mathcal{F}_k^{\otimes n}, \circledast)$ with the bilinear form*

$\beta : \mathcal{F}_k^{\otimes n} \otimes \mathcal{F}_k^{\otimes n} \to \mathbb{C}$ defined by

$$\beta\big(|\mu\rangle \otimes |\nu\rangle\big) \equiv N_{\mu\nu}^{n^k} \ , \tag{5.26}$$

where $n^k = (n, \ldots, n) \in \mathcal{A}_k^+(n)$. Then $(\mathcal{F}_k^{\otimes n}, \circledast, \beta)$ is a commutative Frobenius algebra over $\mathbb{C}$ with unit $|n^k\rangle$.

Proof. We proceed in close analogy to the proof of [41, Th. 7.11]. Thanks to Property 2 of Lemma 5.2.5 we can express the bilinear form β as

$$\beta\big(|\mu\rangle \otimes |\nu\rangle\big) = \frac{k!}{u_\mu}\, \delta_{\mu\nu^*} \ . \tag{5.27}$$

In view of Theorem 4.5.2, we just need to show that the bilinear form β is non-degenerate and invariant. Non-degeneracy is a consequence of the fact that the map $^* : \mathcal{A}_k^+(n) \to \mathcal{A}_k^+(n)$ is an involution. Let a be an element of the algebra $(\mathcal{F}_k^{\otimes n}, \circledast, \eta)$, and suppose that $\beta\big(|\mu\rangle \otimes a\big) = 0$ for all $\mu \in \mathcal{A}_k^+(n)$. It follows that $a = 0$, because if $a = |\nu\rangle$ for some $\nu \in \mathcal{A}_k^+(n)$ equation (5.27) then implies that $\beta(|\nu^*\rangle \otimes |\nu\rangle) \neq 0$, which is a contradiction. Taking advantage of (5.25), together with Property 3 of Proposition 5.2.5, we end up with the following chain of equalities

$$\begin{aligned}
\beta\big((|\lambda\rangle \circledast |\mu\rangle) \otimes |\nu\rangle\big) \ &= \sum_{\rho \in \mathcal{A}_k^+(n)} N_{\lambda\mu}^{\rho} \beta(|\rho\rangle \circledast |\nu\rangle) \\
&= \frac{k!}{u_\nu} N_{\lambda\mu}^{\nu^*} = \frac{k!}{u_\lambda} N_{\mu\nu}^{\lambda^*} \\
&= \sum_{\rho \in \mathcal{A}_k^+(n)} N_{\mu\nu}^{\rho} \beta(|\lambda\rangle \circledast |\rho\rangle) = \beta\big(\,|\lambda\rangle \otimes (|\mu\rangle \circledast |\nu\rangle)\big) \ ,
\end{aligned}$$

which prove that β is invariant. □

Remark 5.2.7. The commutative Frobenius algebra $(\mathcal{F}_k^{\otimes n}, \circledast, \beta)$ coincides with the deformed Verlinde algebra discussed in Section 1.3, when evaluated at $q = 1$.

5.3 The generalised symmetric group

The goal of this section is to derive a formula for the structure constants $N_{\mu\nu}^{\lambda}$ of the Frobenius algebra $(\mathcal{F}_k^{\otimes n}, \circledast, \beta)$ in terms of tensor multiplicities for irreducible representations of the generalised symmetric group. Let $\mathcal{C}_n$ be the cyclic group of order n, that is the group generated by an element of order n. Consider the k-fold group product $\mathcal{C}_n^{\times k}$, and denote with y_i the generator of the i-th copy of $\mathcal{C}^{\times k}$. By definition we have the relations $y_i^n = 1$ and $y_i y_j = y_j y_i$. Moreover, notice that every element in $\mathcal{C}^{\times k}$ can be expressed as $y^\alpha \equiv y_1^{\alpha_1} \cdots y_k^{\alpha_k}$ for some $\alpha \in \mathcal{P}_k$.

Definition 5.3.1. The generalised symmetric group $S(n,k)$ is the wreath product $\mathcal{C}_n \wr S_k$, that is

$$S(n,k) = \mathcal{C}_n^{\times k} \rtimes S_k \ . \tag{5.28}$$

Stated otherwise, the group $S(n,k)$ consists of the set of pairs (y^α, w), where $y^\alpha \in \mathcal{C}^{\times k}$ and $w \in S_k$, with multiplication rule

$$(y^{\alpha_1}, w_1).(y^{\alpha_2}, w_2) = (y^{\alpha_1+\alpha_2.w_1}, w_1 w_2) \ . \tag{5.29}$$

The group $S(n,k)$ admits the following presentation [16]: $S(k,n)$ is isomorphic to the group generated by the elements $\{\sigma_1,\dots,\sigma_{k-1}\} \cup \{y_1,\dots,y_n\}$ subject to the relations

$$\sigma_i^2 = 1 \ , \qquad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \ , \qquad \sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{for } |i-j| > 1 \ , \tag{5.30}$$

and

$$y_i^n = 1 \ , \qquad y_i y_j = y_j y_i \ , \tag{5.31}$$

together with

$$\sigma_i y_i = y_{i+1}\sigma_i \ , \qquad \sigma_i y_j = y_j \sigma_i \quad \text{for } j \neq i, i+1 \ . \tag{5.32}$$

It is clear that the subgroup of $S(n,k)$ generated by $\{\sigma_1,\dots,\sigma_{k-1}\}$ is isomorphic to S_k (see Section 2.1.2), whereas the normal subgroup $\mathcal{N}$ of $S(n,k)$ generated by $\{y_1,\dots,y_n\}$ is isomorphic to $\mathcal{C}_n^{\times k}$.

5.3.1 Irreducible representations

Let us briefly describe the irreducible representations of $S(n,k)$ (compare with the discussion presented in [45, Appendix B] and references therein). Define a n-multipartition $\boldsymbol{\lambda}$ as a sequence

$$\boldsymbol{\lambda} = (\lambda^{(1)},\dots,\lambda^{(n)}) \tag{5.33}$$

of partitions. It was shown in [56] that the finite dimensional irreducible representations of $S(n,k)$ are in bijection with n-multipartitions $\boldsymbol{\lambda}$ satisfying the constraint $|\boldsymbol{\lambda}| \equiv \sum_{i=1}^n |\lambda^{(i)}| = k$. We denote the resulting representations by $\mathcal{L}(\boldsymbol{\lambda})$. Define the type of a n-multipartition $\boldsymbol{\lambda}$ as the unique partition λ satisfying the relation $m_i(\lambda) = |\lambda^{(i)}|$ for $i = 1,\dots,n$, and $m_i(\lambda) = 0$ otherwise. It follows that $|\boldsymbol{\lambda}| = k$ if and only if $\lambda \in \mathcal{A}_k^+(n)$, where the latter is the set of partitions introduced in (3.11).

Definition 5.3.2. A standard n-tableau $\boldsymbol{T}$ of shape $\boldsymbol{\lambda}$ is a sequence

$$\boldsymbol{T} = (T^{(1)}, T^{(2)},\dots,T^{(n)}) \tag{5.34}$$

of row strict tableau, where $T^{(j)}$ has shape $\lambda^{(j)}$, and furthermore the entries $1,2,\dots,|\boldsymbol{\lambda}|$

appear exactly once in $\boldsymbol{T}$.

Let $\boldsymbol{\lambda}$ be a n-multipartition of type $\lambda \in \mathcal{A}_k^+(n)$, that is a n-multipartition which satisfies the constraint $|\boldsymbol{\lambda}| = k$. We now recall an explicit construction [59] for the irreducible representation $\mathcal{L}(\boldsymbol{\lambda})$, and for this purpose we employ the presentation of $S(n,k)$ introduced in (5.30), (5.31) and (5.32). The irreducible representation $\mathcal{L}(\boldsymbol{\lambda})$ is spanned by the set of all standard n-tableaux $\boldsymbol{T}$ of shape $\boldsymbol{\lambda}$. Let χ be a primitive n-th root of unity, and for $j = 1,\dots,k$ set $p_j(\boldsymbol{T}) = i$ if the entry i belongs to the tableau $T^{(j)}$ of $\boldsymbol{T}$. For $\alpha \in \mathcal{P}_k$ we have the action

$$y^\alpha.\boldsymbol{T} = \chi^{\alpha_1 p_1(\boldsymbol{T})+\cdots+\alpha_k p_k(\boldsymbol{T})}\,\boldsymbol{T}\,. \tag{5.35}$$

Define the content of the box $(a,b) \in T^{(j)}$ in $\boldsymbol{T}$ containing the entry i as $c_i(\boldsymbol{T}) = a - b$. Set

$$t_i = \frac{1}{c_{i+1}(\boldsymbol{T}) - c_i(\boldsymbol{T})} \tag{5.36}$$

if the entries i and $i+1$ of $\boldsymbol{T}$ belong to the same tableau $T^{(j)}$, and set $t_i = 0$ otherwise. Denote with $\boldsymbol{T}_{(i,i+1)}$ the standard n-tableau which is obtained from $\boldsymbol{T}$ by swapping the entries i and $i+1$ if the result is another standard n-tableau, otherwise set $\boldsymbol{T}_{(i,i+1)} = 0$. The action of the generators $\{\sigma_1,\dots,\sigma_{k-1}\}$ of $S(n,k)$ is given by

$$\sigma_i.\boldsymbol{T} = t_i\boldsymbol{T} + \sqrt{1-t_i^2}\,\boldsymbol{T}_{(i,i+1)}\,. \tag{5.37}$$

5.3.2 Representation theory and fusion coefficients

Let $\operatorname{Rep} S(n,k)$ be the representation ring of the finite dimensional modules of the generalised symmetric group, with structure constants

$$\mathcal{L}(\boldsymbol{\mu}) \otimes \mathcal{L}(\boldsymbol{\nu}) = \bigoplus_{\boldsymbol{\lambda}} c_{\boldsymbol{\mu}\boldsymbol{\nu}}^{\boldsymbol{\lambda}} \mathcal{L}(\boldsymbol{\lambda})\,. \tag{5.38}$$

In Proposition 5.3.5 we shall derive a formula which relates the coefficients $N_{\mu\nu}^{\lambda}$ introduced in (5.21) to the structure constants $c_{\boldsymbol{\mu}\boldsymbol{\nu}}^{\boldsymbol{\lambda}}$ appearing in (5.38), but first we present some preliminary results. Recall from Section 4.5.1 the notation $\zeta_\lambda = (\zeta_{\lambda_1},\dots,\zeta_{\lambda_k})$ for $\lambda \in \mathcal{A}_k^+(n)$, where $\zeta_j = z^{1/n}\chi^j$ for $j = 1,\dots,n$. Moreover, set $\chi_\alpha = (\chi^{\alpha_1},\dots,\chi^{\alpha_k})$ for $\alpha \in \mathcal{P}_k$.

Lemma 5.3.3 ([45]). *Let $\boldsymbol{\lambda} = (\lambda^{(1)},\dots,\lambda^{(n)})$ be a n-multipartition of type $\lambda \in \mathcal{A}_k^+(n)$. The characters of $\mathcal{L}(\boldsymbol{\lambda})$ restricted to the normal subgroup $\mathcal{N} \cong \mathcal{C}_n^{\times k}$ of $S(n,k)$ generated by $\{y_1,\dots,y_k\}$ are given by*

$$\operatorname{Tr}_{\mathcal{L}(\boldsymbol{\lambda})} y^\alpha = f_{\boldsymbol{\lambda}} m_\lambda(\chi_\alpha)\,, \qquad f_{\boldsymbol{\lambda}} = \prod_{i=1}^k f_{\lambda^{(i)}}\,, \tag{5.39}$$

where f_λ is defined in Section 2.1.1 as the number of standard tableaux of shape λ.

Proof. Set $\overline{\boldsymbol{\lambda}} = \big((|\lambda^{(1)}|),\dots,(|\lambda^{(n)}|)\big)$. That is, $\overline{\boldsymbol{\lambda}}$ is the n-partition whose parts are given by the partitions $(|\lambda^{(j)}|)$ with length 1. Notice that the action (5.35) of y^α on the standard n-tableaux of shape $\boldsymbol{\lambda}$ does not depend on the shape of each $\lambda^{(j)}$, but only on $|\lambda^{(j)}|$. It follows that

$$\mathrm{Tr}_{\mathcal{L}(\boldsymbol{\lambda})}\, y^\alpha = f_{\boldsymbol{\lambda}}\, \mathrm{Tr}_{\mathcal{L}(\overline{\boldsymbol{\lambda}})}\, y^\alpha\ . \tag{5.40}$$

We now show that there exists a bijection between the set of standard n-tableaux of shape $\overline{\boldsymbol{\lambda}}$ and the set of permutations in S^λ. Let $\boldsymbol{T}$ be a standard n-tableau of shape $\overline{\boldsymbol{\lambda}}$, and consider the weight $\beta \in \mathcal{P}_k$ with parts $\beta_j = p_j(\boldsymbol{T})$ for $j = 1,\dots,k$. Since $\overline{\boldsymbol{\lambda}}$ has type λ, it follows that $m_i(\beta) = m_i(\lambda)$ for all $i = 1,\dots,n$, and then there exists a unique element $w \in S^\lambda$ such that $\beta = \lambda.w$. This procedure therefore defines a map $\boldsymbol{T} \mapsto w$. Conversely, given $w \in S^\lambda$, define the standard n-tableau $\boldsymbol{T}$ of shape $\overline{\boldsymbol{\lambda}}$ via the relation $(\lambda.w)_j = p_j(\boldsymbol{T})$ for $j = 1,\dots,n$. This defines a map $w \mapsto \boldsymbol{T}$ which is by construction the inverse of the map $\boldsymbol{T} \mapsto w$ defined above. Employing the bijection just described it follows that $\mathrm{Tr}_{\mathcal{L}(\overline{\boldsymbol{\lambda}})}\, y^\alpha = \sum_{w\in S^\lambda} \chi^{\alpha_1(\lambda.w)_1+\cdots+\alpha_k(\lambda.w)_k} = m_\lambda(\chi_\alpha)$, where the last equality follows from Lemma 2.2.5. This implies the validity of the claim thanks to (5.40).

□

Lemma 5.3.4. *Let $f, g \in \Lambda$. The identity $f(x_1,\dots,x_k) = g(x_1,\dots,x_k)$ holds in $\mathcal{V}_k(n)$ if and only if the relation $f(\zeta_\sigma) = g(\zeta_\sigma)$ is valid for all $\sigma \in \mathcal{A}_k^+(n)$.*

Proof. We shall employ completely analogous steps as the ones described in [41, Lemma 7.4]. First of all, notice that $\Lambda_k = \mathbb{C}[p_1,\dots,p_k]$. In other words, the ring of symmetric functions Λ_k in k variables is freely generated by the power sums $\{p_1,\dots,p_k\}$ [52]. Set

$$\tilde{p}_r = \begin{cases} p_r - z p_{r-n}, & r = n+1,\dots,n+k-1 \\ p_n - zk, & r = n \\ p_r, & \text{otherwise} \end{cases}\ , \tag{5.41}$$

and without any loss of generality suppose that $z = 1$. Denote by

$$\mathcal{V}_{k,n} = \{(\pi_1,\dots,\pi_k) \in \mathbb{C}^k \ :\ \tilde{p}_n\big|_{p=\pi} = \cdots = \tilde{p}_{n+k-1}\big|_{p=\pi} = 0\} \tag{5.42}$$

the solutions of the equations (3.36) for $z = 1$ in the affine space $\mathbb{C}^k$, where it is understood that $\tilde{p}_r \in \Lambda_k$, and moreover $\tilde{p}_r\big|_{p=\pi}$ is obtained by replacing $p_r \to \pi_r$ in the expansion (5.41). The two set of equations (3.36) and (3.37) are equivalent, as proved in Lemma 3.3.1. The claim therefore follows from the equality

$$\mathcal{I}(\mathcal{V}_{k,n}) = \mathcal{I}_{k,n}\ , \tag{5.43}$$

where $\mathcal{I}(\mathcal{V}_{k,n})$ is the vanishing ideal of the affine variety (5.42), and $\mathcal{I}_{k,n}$ is the two-sided ideal generated by the set of equations (3.36), which was introduced in Section 3.3. We now prove that $\mathcal{I}_{k,n}$ is a radical ideal. Equation (5.43) then follows from Hilbert's Nullstellensatz.

The elements $\{p_r\}_{r\in\mathbb{Z}_{\geq 0}}$ are algebraically independent in Λ [52], and so are the elements $\{\tilde{p}_r\}_{r\in\mathbb{Z}_{\geq 0}}$, since p_{n+r} and p_r have different degree. We deduce that the set $\{\tilde{p}_\lambda\}_{\lambda\in\mathcal{P}^+}$, where $\tilde{p}_\lambda = \tilde{p}_{\lambda_1}\tilde{p}_{\lambda_2}\cdots$, is linearly independent in Λ. Let $\mathbf{I}_{k,n}\subset\Lambda$ be the ideal generated by $\{\tilde{p}_n,\tilde{p}_{n+1},\ldots,\tilde{p}_{n+k-1}\}$, and notice that projecting $\mathbf{I}_{k,n}$ onto Λ_k we obtain the ideal $\mathcal{I}_{k,n}\subset\Lambda_k$. Suppose that $f=\sum_{\lambda\in\mathcal{P}^+}c_\lambda\tilde{p}_\lambda\in\Lambda$ is not in $\mathbf{I}_{k,n}$. This means that there must exist at least one partition μ such that $\mu_j\notin\{n,n+1,\ldots,n+k-1\}$ for all $j\in\mathbb{N}$. Let $m\in\mathbb{N}$, and notice that the expansion of f^m contains $\tilde{p}_{\mu^n}$, where μ^n is the partition containing each part $\mu_j>0$ exactly m times. It follows that $f^m\notin\mathcal{I}_{k,n}$ for all $m\in\mathbb{N}$, which implies that $\mathbf{I}_{k,n}$ is radical. Projecting onto Λ_k we conclude that $\mathcal{I}_{k,n}$ is radical as well, thus proving the validity of the claim. □

Proposition 5.3.5 ([45]). *Let $\lambda,\mu,\nu\in\mathcal{A}_k^+(n)$. Moreover, suppose that $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ are two n-multipartitions of type μ and ν respectively. We have the identity*

$$N_{\mu\nu}^{\lambda}=\sum_{\boldsymbol{\lambda}}c_{\boldsymbol{\mu}\boldsymbol{\nu}}^{\boldsymbol{\lambda}}\frac{f_{\boldsymbol{\lambda}}}{f_{\boldsymbol{\mu}}f_{\boldsymbol{\nu}}}\ , \tag{5.44}$$

where the sum runs over all n-multipartitions $\boldsymbol{\lambda}$ of type λ.

Proof. Notice that we can rewrite the product expansion (3.40), which holds in $\mathcal{V}_k(n)$, as

$$m_\mu(x_1,\ldots,x_k)m_\nu(x_1,\ldots,x_k)=\sum_{\rho\in\mathcal{A}_k^+(n)}z^dN_{\mu\nu}^{\rho}m_\rho(x_1,\ldots,x_k)\ . \tag{5.45}$$

The integer d appearing on the RHS is defined via the relation $|\mu|+|\nu|-|\rho|=dn$. From (5.38) it follows that $\operatorname{Tr}_{\mathcal{L}(\boldsymbol{\mu})\otimes\mathcal{L}(\boldsymbol{\nu})}y^\sigma=\sum_{\boldsymbol{\rho}}c_{\boldsymbol{\mu}\boldsymbol{\nu}}^{\boldsymbol{\rho}}\operatorname{Tr}_{\mathcal{L}(\boldsymbol{\rho})}y^\alpha$, and taking advantage of Lemma 5.3.3 we end up with with the relation

$$m_\mu(\chi_\sigma)m_\nu(\chi_\sigma)=\sum_{\boldsymbol{\rho}}c_{\boldsymbol{\mu}\boldsymbol{\nu}}^{\boldsymbol{\rho}}\frac{f_{\boldsymbol{\rho}}}{f_{\boldsymbol{\mu}}f_{\boldsymbol{\nu}}}m_\rho(\chi_\sigma)\ ,$$

which is valid for all $\sigma\in\mathcal{A}_k^+(n)$. The partition $\rho\in\mathcal{A}_k^+(n)$ on the RHS of this relation corresponds to the type of $\boldsymbol{\rho}$. Setting $z=1$, in which case $\chi_\sigma=\zeta_\sigma$ for all $\sigma\in\mathcal{A}_k^+(n)$, we can employ Lemma 5.3.4 to deduce the following equalities in $\mathcal{V}_k(n)$,

$$\begin{aligned}m_\mu(x_1,\ldots,x_k)m_\nu(x_1,\ldots,x_k) &= \sum_{\boldsymbol{\rho}}c_{\boldsymbol{\mu}\boldsymbol{\nu}}^{\boldsymbol{\rho}}\frac{f_{\boldsymbol{\rho}}}{f_{\boldsymbol{\mu}}f_{\boldsymbol{\nu}}}m_\rho(x_1,\ldots,x_k)\\ &= \sum_{\rho\in\mathcal{A}_k^+(n)}\left(\sum_{\boldsymbol{\rho}}c_{\boldsymbol{\mu}\boldsymbol{\nu}}^{\boldsymbol{\rho}}\frac{f_{\boldsymbol{\rho}}}{f_{\boldsymbol{\mu}}f_{\boldsymbol{\nu}}}\right)m_\rho(x_1,\ldots,x_k)\ ,\end{aligned}$$

where the second sum in the second line runs over all n-multipartitions $\boldsymbol{\rho}$ of type ρ. Using the fact that $\{m_\rho(x_1,\ldots,x_k)\}_{\rho\in\mathcal{A}_k^+(n)}$ is a basis of $\mathcal{V}_k(n)$, which was proved in Lemma 3.3.2, the claim follows after a comparison with (5.45). $\square$

Chapter 6

Conclusions and open problems

We conclude this thesis with a summary of some open problems and potential avenues of research.

Representation theory of the general linear group $\mathbf{GL}_k(\mathbb{C})$

As pointed out in Remark 3.3.19, the cylindric symmetric functions $e_{\lambda/d/\mu}$ and $h_{\lambda/d/\mu}$ are Schur positive. It would be interesting to present an explicit construction of the $\mathrm{GL}_r(\mathbb{C})$-representations whose characters are given by $e_{\lambda/d/\mu}(x_1, \ldots, x_r)$ and $h_{\lambda/d/\mu}(x_1, \ldots, x_r)$ respectively. Perhaps a breakthrough could be achieved if we had a combinatorial interpretation for the coefficient $\chi^{\lambda;d}_{\mu\nu}$ (see Definition 3.3.17) which resembles the one for $\theta_{\lambda/d/\mu}(\nu)$ and $\psi_{\lambda/d/\mu}(\nu)$ (see Definition 3.3.7). Such a combinatorial interpretation is still missing.

The Q^+ and Q^- vertex models for arbitrary q

The symmetric function $e_{\lambda/d/\mu}$, which was defined from a purely combinatorially approach by employing the level-n action of the affine symmetric group, can be identified with the partition function of the Q^+ vertex model for $q = 1$. Similarly, the symmetric function $h_{\lambda/d/\mu}$ plays a central role in the computation of the partition function of the Q^- vertex model for $q = 1$. See Theorem 4.1.18. The next step would be to construct, from a purely combinatorially approach, a q-deformation of $e_{\lambda/d/\mu}$ that coincides with the cylindric q-Whittaker function $P'_{\lambda'/d/\mu'}(q)$ discussed in Section 1.2. The latter can be identified with the partition function of the Q^+ vertex model for arbitrary q [41]. One could then construct a q-deformation of $h_{\lambda/d/\mu}$ in a similar way, and see whether such q-deformation of $h_{\lambda/d/\mu}$ provides a combinatorial interpretation for the partition function of the Q^- vertex model for arbitrary q.

Modular tensor categories

The Verlinde formula and the existence of the modular group representation (see Chapter 5) are a 'fingerprint' of a richer structure: a modular tensor category (MTC). The $\mathcal{S}$-matrix of a MTC is symmetric, and the coefficients arising from the Verlinde formula are non-negative integers, as they coincide with the structure constants of the Grothendieck ring of the MTC (see e.g. [10]). Notice that the $\mathcal{S}$-matrix defined in (5.5) is not symmetric. On the other hand, let us introduce the transition matrix

$$\mathcal{S}_{\lambda\mu} = \frac{u_\lambda^{1/2}}{u_\mu^{1/2}} \frac{m_\lambda(\zeta_\mu)}{n^{k/2}} \tag{6.1}$$

from the basis $\{u_\mu^{-1/2}\,|\zeta_\mu\rangle\}_{\mu\in\mathcal{A}_k^+(n)}$ to the basis $\{u_\lambda^{1/2}\,|\lambda\rangle\}_{\lambda\in\mathcal{A}_k^+(n)}$ of $\mathcal{F}_k^{\otimes n}$. This $\mathcal{S}$-matrix is symmetric thanks to Lemma (2.2.5), and together with the $\mathcal{T}$-matrix defined in (5.6) we still have a representation of the modular group. But if we employ the Verlinde formula with this $\mathcal{S}$-matrix we end up with coefficients that are in general not integral. It would be interesting to investigate in more depth whether there exists a family of MTCs associated to the free boson model. Then one could ask himself how this discussion extends to the q-boson model for arbitrary q.

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