Homological Methods for Graded $\mathbb{k}$-Algebras

by

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In memory of my granny, Christina Cameron Laird and my nana, Jeanie Taylor Watt.

Abstract

In this thesis we develop results concerning strongly group-graded $\Bbbk$-algebras. Chapter 1 is mainly expository: we set up a careful treatment of well-known facts and definitions regarding graded algebras so that later results run smoothly. A secondary reason for including the treatment is to give the reader a solid grounding in the basics: much use will be made of these initial observations throughout the thesis.

In Chapter 2 we establish generalisations of known work for group algebras. Here the paper *Complexity and Varieties for infinite groups, I* by D. J. Benson is key, with results of J. Cornick and P. H. Kropholler discussed and generalised as needed. The main theorems of this chapter characterise – albeit under specific conditions – modules of finite projective dimension over strongly group-graded $\Bbbk$-algebras for G an $\mathbf{LH}\mathfrak{F}$-group.

Chapter 3 sees us take a different tack with complete cohomology where we define the zeroeth cohomology group to be the set of morphisms in certain module categories. We show that these categories can be realised as quotients of the derived category of suitable subcategories. This work also generalises results due to Benson.

We introduce some vanishing theorems for modules of type FP_∞ over skew polynomial rings, with suitable finiteness conditions on the base ring in Chapter 4. Iterated skew polynomial rings are also investigated, as are iterated skew Laurent polynomial rings.

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Statement

This thesis is submitted in accordance with the regulations for the degree of Doctor of Philosophy at the University of Glasgow. No part of this thesis has previously been submitted by me at this or any other university.

Chapter 1 covers definitions and basic results. Chapters 2,3 and 4 comprise (unless otherwise stated) the author's original work.

Contents

List Of Notation

Introduction

Hilbert's Theorem on Syzygies states that if $\Bbbk$ is a field then the global dimension of the polynomial ring $\Bbbk[x_1, x_2, \ldots, x_n]$ is equal to n. In the 1960s, Fields [14] showed that if R is a ring of global dimension n and σ is an automorphism of R, then the polynomial ring $R[x; \sigma]$ (with multiplication defined by $rx = x\sigma(r)$ for all $r \in R$) has global dimension $n+1$. In 1975, Goodearl [17] showed that if R is a ring of finite global dimension and δ is an additive map, then the polynomial ring $R[x; \delta]$ (with multiplication defined by $rx = xr + \delta(r)$) has global dimension either equal to that of R, or more one than it and that both possibilities can occur. It is now known that if a ring R has global dimension at most n, then the skew polynomial ring (or polynomial ring of 'mixed type'), $R[x; \sigma, \delta]$ has global dimension at either n or $n+1$, [26], (where σ and δ are as before and multiplication is defined by $rx = x\sigma(r) + \delta(r)$ for all $r \in R$). Woodward [30] discusses the situations under which both possibilities occur. These identities imply that modules over polynomial rings have projective dimension at most one more than their projective dimension over the base ring (which is necessarily finite).

In Chapter 4 of this thesis, we investigate finiteness conditions on the base ring of skew polynomial rings and skew Laurent polynomial rings so that the projective dimension of modules of type FP_∞ over the polynomial ring is finite. The types of base ring we will consider are:

- Strongly G-graded $\Bbbk$-algebras R with $\Bbbk$ a commutative ring. This is a $\Bbbk$-algebra which admits a $\Bbbk$-module decomposition,

$$R = \bigoplus_{g \in G} R_g$$

in such a way that $R_g R_h = R_{gh}$. In particular, we will be interested in the case G belongs to the class of $\mathbf{H}\mathfrak{F}$-groups which is the smallest class of groups that contains all finite groups and which contains a group G whenever there is an admissible action of G on a finite dimensional contractible cell complex for which all isotropy groups already belong to $\mathbf{H}\mathfrak{F}$. A more conceptually accessible definition of this class of groups appears in Section 2.3.

- Rings R which are filtered colimits of rings possessing finite global dimension, with the further property that R is flat over each of the rings in the limit.

It turns out that if R is a strongly G-graded $\Bbbk$-algebra (with G an $\mathbf{H}\mathfrak{F}$-group), $S = R[x, \sigma, \delta]$ or $R[x^{-1}, x; \sigma]$ and M is a left S-module of type FP_∞ such that M has finite projective dimension as an R_H-module for all finite subgroups H of G, then M has finite projective dimension as an S-module. Also, if R is a filtered colimit of rings of finite global dimension and $S = R[x, \sigma, \delta]$ or $R[x^{-1}, x; \sigma]$, then every left S-module of type FP_∞ has finite projective dimension.

If T is an iterated skew polynomial ring or iterated skew Laurent polynomial ring and the base ring R is of either type described above, then more is true: for every intermediate polynomial ring S_j in the construction, every left T-module M of type FP_∞ (with the property that M has finite projective dimension as an R_H-module for each finite subgroup H of G, when R is the graded algebra case), all left T-modules N and all integers i,

$$\widehat{\mathrm{Ext}}^i_T(M, T \otimes_{S_j} N) = 0$$

from which the previous results follow as corollaries upon taking $N = M$, $i = 0$ and $S_j = T$.

We conclude Chapter 4 by defining the concepts of bricks, walls and foundations and we pose questions for further research.

In Chapter 2 we prove a number of results relating to complete cohomology and strongly group-graded $\Bbbk$-algebras. Notable in this chapter is the Vanishing Theorem (Theorem 2.4.13) which extends a result of Cornick and Kropholler (Vanishing Theorem, Section 8, page 50 [10]) to include the class of $\mathbf{LH}\mathfrak{F}$-groups which comprises those groups whose finitely generated subgroups are in $\mathbf{H}\mathfrak{F}$. Their result follows as a corollary.

Much work has been done in the area of finiteness conditions for modules over group algebras. Given that group algebras are the simplest examples of strongly group-graded algebras, it seems natural to ask to what extent one can recover results for group algebras in this greater generality. Some headway has been made in this respect: Aljadeff and Ginosar [1] in 1996 proved that if G is a finite group and R a strongly G-graded $\Bbbk$-algebra, then an R-module is projective if and only if it is projective as an R_E-module for each elementary abelian subgroup E of G. This generalises Chouinard's Theorem which was proved twenty years previously [8]. The theorem states that if $\Bbbk$ is a commutative ring and G is a finite group, then a $\Bbbk G$-module is projective if and only if it is projective as a $\Bbbk E$-module for each elementary abelian subgroup E of G.

It is in this spirit that we generalise results obtained by Benson for group algebras in [3]. Many of the results there concern group algebras $\Bbbk G$ with G finite and some condition (e.g. Noetherian or finite global dimension) on the base ring $\Bbbk$. In order to generalise these results to the case when R is a G-graded $\Bbbk$-algebra (G any group), we need similar conditions to work with. The key observation is:

- Let H be a subgroup of finite index in G. We then choose R_H to be an analogue of a base ring: we assume the conditions on $\Bbbk$ from the group algebra case on R_H in the strongly group-graded problem under consideration. Also, R is a projective R_H-module (just as $\Bbbk G$ is a projective (free) $\Bbbk$-module).

We successfully use this observation to state and prove a number of generalised results in Chapter 2.

If M is an R-module of type FP_∞ (that is, a module admitting a resolution by finitely generated projective modules), then the functors $\widehat{\mathrm{Ext}}^i_R(M,-)$ are continuous for all $i \in \mathbb{Z}$; that is if $\varinjlim_j N_j$ is a filtered colimit of R-modules, then $\widehat{\mathrm{Ext}}^i_R(M, \varinjlim_j N_j) = \varinjlim_j \widehat{\mathrm{Ext}}^i_R(M, N_j)$. It is because of this result coupled with the fact that $\widehat{\mathrm{Ext}}^0_R(M,M) = 0$ if and only if M has finite projective dimension as an R-module (Lemma 2.1.8), that modules of type FP_∞ are so often used when dealing with questions about finiteness of projective dimension. Indeed, many of our results concern these modules.

Modules of type FP_∞ play a rôle in Chapter 3. If R is a strongly G-graded $\Bbbk$-algebra, then we can define four categories: ${}_R\mathsf{mod}$, whose objects are left R-modules of type FP_∞ which have finite projective dimension as left R_1-modules and whose maps are the set of left R-module homomorphisms; ${}_R\mathsf{stmod}$, with the same objects as ${}_R\mathsf{mod}$ and maps $\widehat{\mathrm{Ext}}^0_R(M,N)$ from M to N; ${}_R\mathsf{Mod}$ has objects left R-modules M such that $B \otimes M$ has finite projective dimension (where B is the $\Bbbk G$-module of maps from G to $\Bbbk$ which take only finitely many different values in $\Bbbk$) and has maps the set of left R-module homomorphisms; and ${}_R\mathsf{StMod}$ which has the same objects as ${}_R\mathsf{Mod}$ and maps given by $\widehat{\mathrm{Ext}}^0_R(M,N)$ from M to N. The work done here also generalises results due to Benson who originally defined the objects of ${}_{\Bbbk G}\mathsf{Mod}$ and ${}_{\Bbbk G}\mathsf{StMod}$ (for $\Bbbk G$ a group algebra with $\Bbbk$ a Noetherian ring) to be countably presented left R-modules such that $B \otimes M$ has finite projective dimension. That the countability hypothesis can be dropped in the group algebra case was something first noted by him in [3]. Working with strongly group-graded algebras, we see that we can also drop this hypothesis, as well as the Noetherian condition.

In Section 3.2 we show that ${}_R\mathsf{stmod}$ and ${}_R\mathsf{StMod}$ are triangulated categories. Then, defining ${}_R\mathsf{Proj}$ to be the full subcategory of ${}_R\mathsf{Mod}$ whose objects are the projective modules, we show that the category ${}_R\mathsf{StMod}$ is equivalent to the quotient category $D^b({}_R\mathsf{Mod})/D^b({}_R\mathsf{Proj})$. We also show that when R is Noetherian and G is an $\mathbf{LH}\mathfrak{F}$-group, we obtain the same result for ${}_R\mathsf{mod}$; that is, it is equivalent to $D^b({}_R\mathsf{mod})/D^b({}_R\mathsf{proj})$.

Chapter 1

Preliminaries

1.1 Some basics

Throughout this chapter, $\Bbbk$ will denote a commutative ring and G a monoid unless otherwise stated. Unembellished tensors mean $\otimes_{\Bbbk}$.

Definition 1.1.1. Let X be a set. An ***X-graded $\Bbbk$-module*** is a $\Bbbk$-module M together with a $\Bbbk$-module decomposition $M = \bigoplus_{x\in X} M_x$. Elements of M_x are said to be ***homogeneous of degree*** x.

Example 1.1.2. $\mathbb{Z}^n$, $n \in \mathbb{N}$ is a $\{1, 2, \ldots, n\}$-graded $\mathbb{Z}$-module via

$$\mathbb{Z}^n = \bigoplus_{i=1}^{n} (\mathbb{Z}^n)_i$$

where each $\overline{\alpha} \in (\mathbb{Z}^n)_i$ has the form

$$(0, \ldots, 0, \alpha, 0, \ldots, 0)$$

for some $\alpha \in \mathbb{Z}$ appearing in the i^{th} position.

Example 1.1.3. $\mathbb{Z}^n$, $n \in \mathbb{N}$ is a $\mathbb{Z}$-graded $\mathbb{Z}$-module via

$$\mathbb{Z}^n = \bigoplus_{i=-\infty}^{\infty} (\mathbb{Z}^n)_i$$

where $(\mathbb{Z}^n)_i = \{(\alpha_1, \alpha_2, \ldots, \alpha_n) : \sum_{j=1}^{n} \alpha_j = i\}$.

Of course the above definition ignores any structure that the set X may possess. Much of what follows will depend on the next few definitions.

Definition 1.1.4. A $\Bbbk$-***algebra*** is a $\Bbbk$-module R with two $\Bbbk$-module maps: the ***unit***

$$\varepsilon : \Bbbk \to R$$

and ***multiplication***

$$\mu : R \otimes R \to R$$

such that the diagrams

$$\begin{tikzcd} R \otimes R & R \otimes \Bbbk \arrow[l, "1\otimes\varepsilon"'] \arrow[d] \\ \Bbbk \otimes R \arrow[u, "\varepsilon\otimes 1"] \arrow[r] & R \arrow[from=1-1, "\mu"] \end{tikzcd}$$

$$\begin{tikzcd} R \otimes R \otimes R \arrow[r, "\mu\otimes 1"] \arrow[d, "1\otimes\mu"] & R \otimes R \arrow[d, "\mu"] \\ R \otimes R \arrow[r, "\mu"] & R \end{tikzcd}$$

commute. As usual, $\mu(r \otimes r')$ will be written as rr'.

Definition 1.1.5. A ***G-graded*** $\Bbbk$-***algebra*** is a $\Bbbk$-algebra R which admits a $\Bbbk$-module decomposition $R = \bigoplus_{g\in G} R_g$ with the property that for all $g, h \in G, R_g R_h \subseteq R_{gh}$. For a subset $H \subseteq G$, $R_H = \bigoplus_{g\in H} R_g$ is a $\Bbbk$-submodule of R. When H is a submonoid, R_H is a subalgebra of R called the ***subalgebra of R supported on H***.

Example 1.1.6. Consider $\mathbb{N}_0$ as a monoid under addition. The polynomial ring in n commuting variables

$$R = \Bbbk[x_1, x_2, \ldots, x_n]$$

is an $\mathbb{N}_0$-graded $\Bbbk$-algebra if for each $n \in \mathbb{N}_0$ we define R_n to be the $\Bbbk$-module generated by all monomials of total degree n; that is, all $x_1^{e_1} x_2^{e_2} \ldots x_n^{e_n}$ with $\sum_i e_i = n$.

Definition 1.1.7. Let G be a group and R a G-graded $\Bbbk$-algebra. R is said to be ***strongly G-graded*** if for each $g \in G$, there exist $n_g \in \mathbb{N}$ and sequences of elements $x_1, x_2, \ldots, x_{n_g} \in R_g$ and $y_1, y_2, \ldots, y_{n_g} \in R_{g^{-1}}$ such that $\sum_i^{n_g} x_i y_i = 1 \in R_1$. An equivalent definition is $R_g R_h = R_{gh}$ for each pair $g, h \in G$.

Remark 1.1.8. Occasionally it will be necessary to indicate the degree of a homogeneous element. Where needed, this will be demonstrated through the use of a superscript: $x_n^{(g)}$ means "the n^{th} element x from R_g". It is hoped that this notation, while cumbersome, will facilitate understanding in large calculations.

Example 1.1.9. The simplest example of a strongly G-graded $\Bbbk$-algebra for a given group G is the group algebra $R = \Bbbk G = \bigoplus_{g \in G} \Bbbk g$. Note that $R_1 = \Bbbk$ here.

Example 1.1.10. The splitting field of the the polynomial $X^2 - 2 = 0$, $\mathbb{Q}(\sqrt{2})$, is a $\mathbb{Q}$-algebra which is strongly group-graded by its Galois group. To see this, observe that

$$\mathbb{Q}(\sqrt{2}) = \mathbb{Q} \oplus \sqrt{2}\mathbb{Q}.$$

The first summand is a multiplicatively closed field, however, the second summand is not multiplicatively closed: the product of two second summand elements results in an element of the first. Also, the second summand is closed under multiplication by elements of the first summand. There are two elements in $\mathrm{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$, namely the identity and a non-trivial element of order two, that is, the Galois group is $\mathbb{Z}/2\mathbb{Z}$. If we label the summands

$$(\mathbb{Q}(\sqrt{2}))_{1_{\mathbb{Z}/2\mathbb{Z}}} = \mathbb{Q}$$

and

$$(\mathbb{Q}(\sqrt{2}))_{-1_{\mathbb{Z}/2\mathbb{Z}}} = \sqrt{2}\mathbb{Q}$$

then $\mathbb{Q}(\sqrt{2})$ is a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{Q}$-algebra. It is strongly $\mathbb{Z}/2\mathbb{Z}$-graded because

$$\frac{1}{2}\sqrt{2}.\sqrt{2} = 1.$$

Example 1.1.11. Let n be a natural number greater than 1 and $\Bbbk$ a field. The $n \times n$ matrix ring over $\Bbbk$, $M_n(\Bbbk)$, is a $\Bbbk$-algebra if we map $k \in \Bbbk$ to kI_n where I_n is the $n \times n$ identity matrix. Decompose $M_n(\Bbbk)$ into two direct summands as follows: in the first summand, allow non-zero entries in positions

$$(i, j) : i, j \leq m \text{ for some } 1 \leq m < n$$

and

$$(k, l) : k, l > m.$$

The second summand is the complement of the first in the sense that non-zero entries appear in those positions which in the first summand were allocated zero, and zeroes appear everywhere else. It is obvious that the decomposition so described is not unique for $n > 2$: there are precisely $n - 1$ decompositions. For example, when $n = 5$, the four decompositions are

$$\text{(i)}\quad \begin{pmatrix} \Bbbk & 0 & 0 & 0 & 0 \\ 0 & \Bbbk & \Bbbk & \Bbbk & \Bbbk \\ 0 & \Bbbk & \Bbbk & \Bbbk & \Bbbk \\ 0 & \Bbbk & \Bbbk & \Bbbk & \Bbbk \\ 0 & \Bbbk & \Bbbk & \Bbbk & \Bbbk \end{pmatrix} \bigoplus \begin{pmatrix} 0 & \Bbbk & \Bbbk & \Bbbk & \Bbbk \\ \Bbbk & 0 & 0 & 0 & 0 \\ \Bbbk & 0 & 0 & 0 & 0 \\ \Bbbk & 0 & 0 & 0 & 0 \\ \Bbbk & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{(ii)}\quad \begin{pmatrix} \Bbbk & \Bbbk & 0 & 0 & 0 \\ \Bbbk & \Bbbk & 0 & 0 & 0 \\ 0 & 0 & \Bbbk & \Bbbk & \Bbbk \\ 0 & 0 & \Bbbk & \Bbbk & \Bbbk \\ 0 & 0 & \Bbbk & \Bbbk & \Bbbk \end{pmatrix} \bigoplus \begin{pmatrix} 0 & 0 & \Bbbk & \Bbbk & \Bbbk \\ 0 & 0 & \Bbbk & \Bbbk & \Bbbk \\ \Bbbk & \Bbbk & 0 & 0 & 0 \\ \Bbbk & \Bbbk & 0 & 0 & 0 \\ \Bbbk & \Bbbk & 0 & 0 & 0 \end{pmatrix}$$

$$\text{(iii)}\quad \begin{pmatrix} \Bbbk & \Bbbk & \Bbbk & 0 & 0 \\ \Bbbk & \Bbbk & \Bbbk & 0 & 0 \\ \Bbbk & \Bbbk & \Bbbk & 0 & 0 \\ 0 & 0 & 0 & \Bbbk & \Bbbk \\ 0 & 0 & 0 & \Bbbk & \Bbbk \end{pmatrix} \bigoplus \begin{pmatrix} 0 & 0 & 0 & \Bbbk & \Bbbk \\ 0 & 0 & 0 & \Bbbk & \Bbbk \\ 0 & 0 & 0 & \Bbbk & \Bbbk \\ \Bbbk & \Bbbk & \Bbbk & 0 & 0 \\ \Bbbk & \Bbbk & \Bbbk & 0 & 0 \end{pmatrix}$$

$$\text{(iv)}\quad \begin{pmatrix} \Bbbk & \Bbbk & \Bbbk & \Bbbk & 0 \\ \Bbbk & \Bbbk & \Bbbk & \Bbbk & 0 \\ \Bbbk & \Bbbk & \Bbbk & \Bbbk & 0 \\ \Bbbk & \Bbbk & \Bbbk & \Bbbk & 0 \\ 0 & 0 & 0 & 0 & \Bbbk \end{pmatrix} \bigoplus \begin{pmatrix} 0 & 0 & 0 & 0 & \Bbbk \\ 0 & 0 & 0 & 0 & \Bbbk \\ 0 & 0 & 0 & 0 & \Bbbk \\ 0 & 0 & 0 & 0 & \Bbbk \\ \Bbbk & \Bbbk & \Bbbk & \Bbbk & 0 \end{pmatrix}$$

As in the last example, these algebras (with any choice of decomposition) are strongly $\mathbb{Z}/2\mathbb{Z}$-graded if we allow $(M_n(\Bbbk))_{1_{\mathbb{Z}/2\mathbb{Z}}}$ to be the first summand and $(M_n(\Bbbk))_{-1_{\mathbb{Z}/2\mathbb{Z}}}$ the second.

In examples 1.1.9 and 1.1.10, R_1 coincided with the coefficient ring $\Bbbk$, but in example 1.1.11, $\Bbbk \subset (M_n(\Bbbk))_{1_{\mathbb{Z}/2\mathbb{Z}}} \cong M_m(\Bbbk) \oplus M_{n-m}(\Bbbk)$, with m depending on the choice of decomposition. Note that the axioms for algebras only require the existence of a homomorphism from $\Bbbk$ to R_1 (the unit).

Definition 1.1.12. Let R be a $\Bbbk$-algebra. A ***left R-module*** is a $\Bbbk$-module M together with a map $\theta : R \otimes M \to M$ such that the diagrams

$$\begin{array}{ccc} R\otimes R\otimes M & \xrightarrow{1\otimes\theta} & R\otimes M \\ \downarrow{\scriptstyle \mu\otimes 1} & & \downarrow{\scriptstyle \theta} \\ R\otimes M & \xrightarrow{\theta} & M \end{array}$$

$$\begin{array}{ccc} \Bbbk\otimes M & \xrightarrow{\varepsilon\otimes 1} & R\otimes M \\ & \searrow & \downarrow{\scriptstyle \theta} \\ & & M \end{array}$$

commute. As usual, $\theta(r\otimes m)$ will be written as rm. The first diagram tells us that there is an action of R on M, and the second that this action is unital.

Lemma 1.1.13. *Let G be a group, H a subgroup of G and R a strongly G-graded $\Bbbk$-algebra. Then R is projective as a left R_H-module and as a right R_H-module. If $[G:H]<\infty$, then R is finitely generated as a left R_H-module and as a right R_H-module.*

Proof. The first part of this lemma is Lemma 6.2 of [10].

Assume $[G:H]<\infty$. We prove that R is finitely generated as a right R_H-module, the proof of finite generation as a left R_H-module being similar. First note that $R=\bigoplus_{t\in T}R_{tH}$ where T is a transversal to H in G. For each $t\in T$ we can find a number n_t and sequences of elements $x_1,x_2,\dots,x_{n_t}\in R_t$ and $y_1,y_2,\dots,y_{n_t}\in R_{t^{-1}}$ such that $\sum_{i=1}^{n_t}x_iy_i=1$. For $r\in R_{tH}$,

$$\begin{aligned} r &= 1.r \\ &= (\sum_{i=1}^{n_t}x_iy_i)r \\ &= \sum_{i=1}^{n_t}(x_iy_i)r \\ &= \sum_{i=1}^{n_t}x_i(y_ir) \\ &= \sum_{i=1}^{n_t}x_ir_i \end{aligned}$$

where $r_i=y_ir\in R_H$. It follows that R_{tH} is finitely generated as a right R_H-module. Since $|T|=[G:H]<\infty$, it follows that R is a finitely generated right R_H-module. □

Lemma 1.1.14. *Let G be a group and R a strongly G-graded $\Bbbk$-algebra. If H is a subgroup of G with $[G:H]<\infty$, then R_H left (respectively right) Noetherian implies R left (respectively right) Noetherian.*

Proof. Let R_H be left Noetherian. Since $R = \bigoplus_{t\in T} R_{Ht}$, T is finite and each R_{Ht} is finitely generated over R_H by Lemma 1.1.13, it follows that R is left Noetherian. The right Noetherian case is proven similarly. □

Definition 1.1.15. Let R and S be $\Bbbk$-algebras. A ***$\Bbbk$-algebra map*** $f : R \to S$ is a $\Bbbk$-module map such that the following diagrams commute:

$$\begin{tikzcd} R\otimes R \arrow[r, "f\otimes f"] \arrow[d, "\mu_R"] & S\otimes S \arrow[d, "\mu_S"] \\ R \arrow[r, "f"] & S \end{tikzcd}$$

$$\begin{tikzcd} \Bbbk \arrow[r, "\varepsilon_S"] \arrow[dr, "\varepsilon_R"'] & S \\ & R \arrow[u, "f"'] \end{tikzcd}$$

where $\mu_R, \varepsilon_R, \mu_S, \varepsilon_S$ are the multiplication and unit of R and S respectively.

We will not need the following definition, but we include it for the sake of completeness.

Definition 1.1.16. Let G be a group and R a G-graded $\Bbbk$-algebra. A ***graded left R-module*** M is a left R-module admitting a decomposition

$$M = \bigoplus_{g\in G} M_g$$

into R_1-modules (where 1 is the identity element of G) such that

$$R_g M_h \subseteq M_{gh}$$

for all $g, h \in G$.

A ***strongly graded left R-module*** M is a graded left R-module with

$$R_g M_h = M_{gh}$$

for all $g, h \in G$.

Remark 1.1.17. The interested reader can find more details on graded modules over graded algebras in Dade's paper, [12].

1.2 Some dual constructions

There is the notion of a 'coalgebra', dual to that of 'algebra':

Definition 1.2.1. A $\Bbbk$-***coalgebra*** is a $\Bbbk$-module C together with two maps: the ***counit***

$$v : C \to \Bbbk$$

and ***comultiplication***

$$\gamma : C \to C \otimes C$$

such that the diagrams

$$\begin{tikzcd} C \otimes C \arrow[r, "1\otimes v"] \arrow[d, "v\otimes 1"] & C \otimes \Bbbk \\ \Bbbk \otimes C & C \arrow[l] \arrow[u] \arrow[ul, "\gamma"'] \end{tikzcd}$$

$$\begin{tikzcd} C \otimes C \otimes C & C \otimes C \arrow[l, "\gamma\otimes 1"'] \\ C \otimes C \arrow[u, "1\otimes\gamma"] & C \arrow[l, "\gamma"'] \arrow[u, "\gamma"'] \end{tikzcd}$$

commute.

Example 1.2.2. Let X be a set. Then the free $\Bbbk$-module on X, $\Bbbk X$, is a $\Bbbk$-coalgebra if we define the maps v and γ by

$$v : x \mapsto 1,$$

$$\gamma : x \mapsto x \otimes x.$$

Definition 1.2.3. Let C be a $\Bbbk$-coalgebra. A ***left*** C***-comodule*** is a $\Bbbk$-module V together with a map $\phi : V \to C \otimes V$ such that the diagrams

$$\begin{tikzcd} C \otimes C \otimes V & C \otimes V \arrow[l, "1\otimes\phi"'] \\ C \otimes V \arrow[u, "\gamma\otimes 1"] & V \arrow[l, "\phi"'] \arrow[u, "\phi"'] \end{tikzcd}$$

$$\begin{tikzcd} \Bbbk \otimes V & C \otimes V \arrow[l, "v\otimes 1"'] \\ & V \arrow[u, "\phi"'] \arrow[ul] \end{tikzcd}$$

commute. The first diagram says that we have a *coaction* and the second says that this coaction is *counital.*

Example 1.2.4. $\Bbbk G$ has a left $\Bbbk G$-comodule structure, the map $\phi : \Bbbk G \to \Bbbk G \otimes \Bbbk G$ defined by $g \mapsto g \otimes g$.

Definition 1.2.5. Let C and D be coalgebras. A $\Bbbk$-***coalgebra map*** $f : C \to D$ is a $\Bbbk$-module map such that the following diagrams commute:

$$\begin{tikzcd} C \otimes C \arrow[r, "f\otimes f"] & D \otimes D \\ C \arrow[u, "\gamma_C"] \arrow[r, "f"] & D \arrow[u, "\gamma_D"'] \end{tikzcd}$$

$$\begin{tikzcd} \Bbbk & D \arrow[l, "v_D"'] \\ & C \arrow[u, "f"'] \arrow[ul, "v_C"] \end{tikzcd}$$

where $\gamma_C, v_C, \gamma_D, v_D$ are the comultiplication and counit of C and D respectively.

The usefulness of the above definitions becomes evident with the following result.

Lemma 1.2.6. *Let X be a set. Then there is an equivalence between the category of (left) $\Bbbk X$-comodules and the category of X-graded $\Bbbk$-modules.*

Remark 1.2.7. This lemma appears as Example 1.6.7 in [27] and a terse proof is sketched. Since the lemma is so important for later work in this thesis, we shall include a detailed proof.

Proof. We show that every X-graded $\Bbbk$-module structure determines a $\Bbbk X$-comodule structure, and *vice versa*.

An X-graded $\Bbbk$-module is a $\Bbbk$-module V which decomposes as a direct sum of $\Bbbk$-submodules, indexed by the set X:

$$V = \bigoplus_{x \in X} V_x.$$

For each homogeneous element of degree x, define a map

$$V \to \Bbbk X \otimes V; \; v \mapsto x \otimes v.$$

This determines a $\Bbbk X$-comodule structure on V.

Conversely, if V is a $\Bbbk X$-comodule with map $\phi : V \to \Bbbk X \otimes V$, then define subsets of V as follows: for each $x \in X$, let $V_x = \{v \in V : \phi(v) = x \otimes v\}$. The natural map

$$\hat{\iota} : \bigoplus_{x \in X} V_x \to V$$

induced from the inclusion $\iota : V_x \hookrightarrow V$, is an isomorphism. To see this, choose any element $v \in V$. Then there are uniquely determined elements v_x for $x \in X$ such that $\phi(v) = \sum_x x \otimes v_x$ and since we have a coaction, it follows that

$$\sum_x x \otimes \phi(v_x) = \sum_x x \otimes x \otimes v_x$$

which implies that $\phi(v_x) = x \otimes v_x$, so $v_x \in V_x$ for each x. In this step we have used the commutativity of the square in Definition 1.2.3, just as Montgomery does with her reference to 1.6.2 b) in the proof of ([27], 1.6.7). Since the coaction is counital (the commutativity of the triangle in Definition 1.2.3), we have $v = \sum_x v_x$ and hence $\widehat{\iota}$ is surjective. Suppose that $\sum_x v_x = 0$ for some finite sum of elements $v_x \in V_x$. Applying ϕ, we see that $0 = \sum_x \phi(v_x) = \sum_x x \otimes v_x$ and so all the v_x are zero and $\widehat{\iota}$ is injective. $\square$

It is now clear that we can redefine G-graded $\Bbbk$-algebras as follows:

Definition 1.2.8. A G-graded $\Bbbk$-algebra is a $\Bbbk$-algebra R together with a $\Bbbk$-algebra map $\phi : R \to \Bbbk G \otimes R$ which makes R into a (left) $\Bbbk G$-comodule.

Example 1.2.9. Let G be a group and R a G-graded $\Bbbk$-algebra. Denote by $B = B(G, \Bbbk)$ the set of functions from G to $\Bbbk$ which take only finitely many different values in $\Bbbk$. This can be made into a commutative $\Bbbk$-algebra by pointwise multiplication and realised as a left $\Bbbk G$-module via $({}^g\sigma)(h) = \sigma(hg)$. Given a left R-module M, the tensor product $B \otimes M$ can be made into an R-module via the structure map $\phi : R \to \Bbbk G \otimes R$: for $r = \sum_{g \in G} r_g$,

$$r \cdot (\sigma \otimes m) = \sum_{g \in G} {}^g\sigma \otimes r_g m.$$

We have homomorphisms of $\Bbbk G$-modules: $i : \Bbbk \to B$ (the inclusion of the constant functions) and $\mu : B \otimes B \to B$ (multiplication). The cokernel of i is denoted $\overline{B}$.

Remark 1.2.10. This works more generally: if R is a G-graded $\Bbbk$-algebra, V a $(\Bbbk G, \Bbbk)$-bimodule and M a left R-module, then the tensor product $V \otimes M$ is naturally a left R-module via $r.(v \otimes m) = g.v \otimes rm$ for each homogeneous element r of degree g. This is called the ***semi-diagonal*** action of R.

Remark 1.2.11. Let R, V and M be as in the previous remark. We can make $\mathrm{Hom}_{\Bbbk}(V, M)$ into a left R-module via a semi-diagonal action: for each homogeneous element $r \in R$, of degree g, $(r\phi)(v) := r(\phi(g^{-1}.v))$.

Lemma 1.2.12. *Let R be a G-graded $\Bbbk$-algebra (G any group), V a $(\Bbbk G, \Bbbk)$-bimodule, and M and N left R-modules. Then there is a natural isomorphism*

$$\mathrm{Hom}_R(V \otimes M, N) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_{\Bbbk}(V, N)).$$

Proof. This is Lemma 3.2 of [10]. □

These ideas sometimes allow for more elegant arguments:

Lemma 1.2.13. *Let R be a $\Bbbk$-algebra and $\pi : G \to Q$ a monoid homomorphism. If R is G-graded then it is Q-graded.*

Proof. Define a map $\widehat{\pi} : \Bbbk G \to \Bbbk Q$ by

$$\widehat{\pi}(\sum_g k_g g) = \sum_g k_g \pi(g).$$

This is a ring homomorphism. We have the map $\widehat{\pi} \otimes 1 : \Bbbk G \otimes R \to \Bbbk Q \otimes R$, but we also have the structure map $\phi : R \to \Bbbk G \otimes R$ since R is a $\Bbbk G$-comodule. We can therefore define a map from R to $\Bbbk Q \otimes R$ as the composite $(\widehat{\pi} \otimes 1) \circ \phi$:

$$\begin{tikzcd} R \arrow[r, "\phi"] \arrow[dr, dotted] & \Bbbk G \otimes R \arrow[d, "\widehat{\pi}\otimes 1"] \\ & \Bbbk Q \otimes R \end{tikzcd}$$

This map makes R into a $\Bbbk Q$-comodule, which by Lemma 1.2.6 is equivalent to being a Q-graded $\Bbbk$-algebra. □

There is a similar result for strongly group-graded $\Bbbk$-algebras:

Lemma 1.2.14. *Let G and Q be groups and $\pi : G \to Q$ a surjective group homomorphism. If R is a strongly G-graded $\Bbbk$-algebra, then it is a strongly Q-graded $\Bbbk$-algebra.*

Proof. By the previous lemma, we know R is a Q-graded $\Bbbk$-algebra. For each $q \in Q$, there exists a $g \in G$ such that $\pi(g) = q$ because π is a surjective homomorphism. For each such g there exists a number n_g and lists of elements $x_1, x_2, \ldots, x_{n_g} \in R_g$, $y_1, y_2, \ldots, y_{n_g} \in R_{g^{-1}}$ such that $\sum_{i=1}^{n_g} x_i y_i = 1$ because R is a strongly G-graded $\Bbbk$-algebra. Under the structure map ϕ,

$$x_i \mapsto g \otimes x_i \text{ and } y_i \mapsto g^{-1} \otimes y_i$$

for each $1 \leq i \leq n_g$. Under the map $\widehat{\pi} \circ 1$ (defined in the proof of the previous lemma),

$$g \otimes x_i \mapsto \pi(g) \otimes x_i = q \otimes x_i$$

and

$$g^{-1} \otimes y_i \mapsto \pi(g^{-1}) \otimes y_i = \pi(g)^{-1} \otimes y_i = q^{-1} \otimes y_i$$

for each i. It follows that for each $q \in Q$, we can find a number n_q $(= n_g)$ and sequences of elements $x_1, x_2, \ldots, x_{n_q} \in R_q$ and $y_1, y_2, \ldots, y_{n_q} \in R_{q^{-1}}$ such that $\sum_{i=1}^{n_q} x_i y_i = 1$; that is, R is strongly Q-graded $\Bbbk$-algebra. □

1.3 Induction and coinduction

Definition 1.3.1. Let H be a submonoid of G and let R be a G-graded $\Bbbk$-algebra. If M is a left R_H-module, then the ***induced module*** is the left R-module

$$\mathrm{Ind}_H^G(M) = R \otimes_{R_H} M$$

with the action of R defined by

$$s(r \otimes m) = sr \otimes m$$

for each $s \in R$, $r \otimes m \in R \otimes_{R_H} M$.

Definition 1.3.2. Let H be a submonoid of G and let R be a G-graded $\Bbbk$-algebra. If M is a left R_H-module, then the ***coinduced module*** is the left R-module

$$\mathrm{Coind}_H^G(M) = \mathrm{Hom}_{R_H}(R, M)$$

with the action of R defined by

$$s\phi : r \mapsto \phi(rs)$$

for each $s \in R$, $\phi \in \mathrm{Hom}_{R_H}(R, M)$.

Remark 1.3.3. We can define another action of R on $\mathrm{Hom}_{R_H}(R, M)$ by $s\phi : r \mapsto \phi(sr)$, but this is not well-defined for non-commutative rings. For $r, s, t \in R$, $\phi \in \mathrm{Hom}_{R_H}(R, M)$,

$$(st)\phi : r \mapsto \phi(str)$$

but

$$s(t\phi) : r \mapsto (t\phi)(sr) = \phi(tsr)$$

which are not in general equal unless R is commutative. With our definition, however, we see that

$$(st)\phi : r \mapsto \phi(rst)$$

and

$$s(t\phi) : r \mapsto (t\phi)(rs) = \phi(rst)$$

which are equal for any ring R.

Remark 1.3.4. Ind_H^G and Coind_H^G are functors from the category of left R_H-modules, ${}_{R_H}\mathfrak{Mod}$, to the category of left R-modules, ${}_R\mathfrak{Mod}$.

Lemma 1.3.5. *Let H be a submonoid of G and R a G-graded $\Bbbk$-algebra.*

(i) For each left R_H-module M, there exists a natural homomorphism

$$\nu_M : \mathrm{Ind}_H^G(M) \to \mathrm{Coind}_H^G(M)$$

of left R-modules.

(ii) If G is a group, H a subgroup of G and R a strongly G-graded $\Bbbk$-algebra, then ν_M is a monomorphism.

Proof. (i) Choose a left R_H-module M and define a map

$$\psi : R \times M \to \mathrm{Hom}_{R_H}(R, M)$$

by

$$\psi(r, m) = r \odot m$$

where $r \odot m$ is the R_H-map defined for each $s \in R$ by

$$r \odot m(s) = \sum_{gg' \in H} (s^{(g)} r^{(g')})m.$$

There are several things to check:

- $r \odot m$ is an R_H-module map. For $s_1, s_2 \in R$,

$$\begin{aligned}
r \odot m(s_1 + s_2) &= \sum_{gg' \in H} ((s_1 + s_2)^{(g)} r^{(g')})m \\
&= \sum_{gg' \in H} ((s_1^{(g)} + s_2^{(g)}) r^{(g')})m \\
&= \sum_{gg' \in H} (s_1^{(g)} r^{(g')} + s_2^{(g)} r^{(g')})m \\
&= \sum_{gg' \in H} ((s_1^{(g)} r^{(g')})m + (s_2^{(g)} r^{(g')})m) \\
&= \sum_{gg' \in H} (s_1^{(g)} r^{(g')})m + \sum_{gg' \in H} (s_2^{(g)} r^{(g')})m \\
&= r \odot m(s_1) + r \odot m(s_2).
\end{aligned}$$

For $t \in R_H$,

$$\begin{aligned}
t(r \odot m(s)) &= t(\sum_{gg' \in H} (s^{(g)} r^{(g')}) m) \\
&= \sum_{gg' \in H} t((s^{(g)} r^{(g')}) m) \\
&= \sum_{gg' \in H} (t(s^{(g)} r^{(g')})) m \\
&= \sum_{gg' \in H} ((t s^{(g)}) r^{(g')}) m \\
&= \sum_{gg' \in H} ((ts)^{(g)} r^{(g')}) m \\
&= r \odot m(ts).
\end{aligned}$$

- ψ is an R_H-biadditive map. For $s \in R$,

$$\begin{aligned}
(r_1 + r_2) \odot m(s) &= \sum_{gg' \in H} (s^{(g)} (r_1 + r_2)^{(g')}) m \\
&= \sum_{gg' \in H} (s^{(g)} (r_1^{(g')} + r_2^{(g')})) m \\
&= \sum_{gg' \in H} (s^{(g)} r_1^{(g')} + s^{(g)} r_2^{(g')}) m \\
&= \sum_{gg' \in H} ((s^{(g)} r_1^{(g')}) m + (s^{(g)} r_2^{(g')}) m) \\
&= \sum_{gg' \in H} (s^{(g)} r_1^{(g')}) m + \sum_{gg' \in H} (s^{(g)} r_2^{(g')}) m \\
&= r_1 \odot m(s) + r_2 \odot m(s).
\end{aligned}$$

That is, $\psi(r_1 + r_2, m) = \psi(r_1, m) + \psi(r_2, m)$.

Again, for $s \in R$,

$$\begin{aligned}
r \odot (m_1 + m_2)(s) &= \sum_{gg' \in H} (s^{(g)} r^{(g')})(m_1 + m_2) \\
&= \sum_{gg' \in H} ((s^{(g)} r^{(g')}) m_1 + (s^{(g)} r^{(g')}) m_2) \\
&= \sum_{gg' \in H} (s^{(g)} r^{(g')}) m_1 + \sum_{gg' \in H} (s^{(g)} r^{(g')}) m_2 \\
&= r \odot m_1(s) + r \odot m_2(s).
\end{aligned}$$

That is, $\psi(r, m_1 + m_2) = \psi(r, m_1) + \psi(r, m_2)$.

For $t \in R_H$, $s \in R$,

$$\begin{aligned}
r \odot tm(s) &= \sum_{gg' \in H} (s^{(g)} r^{(g')}) tm \\
&= \sum_{gg' \in H} s^{(g)} (r^{(g')}(tm)) \\
&= \sum_{gg' \in H} s^{(g)} ((r^{(g')} t) m) \\
&= \sum_{gg' \in H} (s^{(g)} (rt)^{(g')}) m \\
&= rt \odot m(s).
\end{aligned}$$

That is, $\psi(rt, m) = \psi(r, tm)$.

It follows by universality that we have a well-defined map $\nu_M : R \otimes_{R_H} M \to \mathrm{Hom}_{R_H}(R, M)$ of abelian groups. It remains to show that ν_M is a map of R-modules; that is, for $t \in R$, $t.\nu_M(r \otimes m) = \nu_M(tr \otimes m)$. Let $s \in R$. Then

$$\begin{aligned}
t.r \odot m(s) &= r \odot m(st) \\
&= \sum_{gg' \in H} ((st)^{(g)} r^{(g')}) m \\
&= \sum_{g''g'''g' \in H} ((s^{(g'')} t^{(g''')}) r^{(g')}) m \\
&= \sum_{g''g'''g' \in H} (s^{(g'')} (t^{(g''')} r^{(g')})) m \\
&= tr \odot m(s).
\end{aligned}$$

For naturality, choose left R_H-modules M and N and an R_H-map $\alpha : M \to N$, to get the following diagram:

$$\begin{tikzcd}
R \otimes_{R_H} M \arrow[r, "\nu_M"] \arrow[d, "\alpha_*"] & \mathrm{Hom}_{R_H}(R, M) \arrow[d, "\alpha_*"] \\
R \otimes_{R_H} N \arrow[r, "\nu_N"] & \mathrm{Hom}_{R_H}(R, N)
\end{tikzcd}$$

where the α_* are the maps induced from α by

$$\alpha_* : R \otimes_{R_H} M \to R \otimes_{R_H} N; \quad r \otimes m \mapsto r \otimes \alpha(m)$$

$$\alpha_* : \mathrm{Hom}_{R_H}(R, M) \to \mathrm{Hom}_{R_H}(R, N); \quad \psi \mapsto \alpha\psi.$$

We have:

$$\begin{aligned}\alpha_*(r \odot m(s)) &= \alpha(\sum_{gg' \in H} (s^{(g)} r^{(g')}) m) \\ &= \sum_{gg' \in H} (s^{(g)} r^{(g')}) \alpha(m) \\ &= r \odot \alpha(m)(s).\end{aligned}$$

That is, $\alpha_* \circ \nu_M = \nu_N \circ \alpha_*$ and the square commutes.

(ii) Note that

$$R \otimes_{R_H} M \cong \bigoplus_{t \in T} R_{tH} \otimes_{R_H} M$$

and

$$\mathrm{Hom}_{R_H}(R, M) \cong \mathrm{Hom}_{R_H}(\bigoplus_{t \in T} R_{Ht^{-1}}, M) \cong \prod_{t \in T} \mathrm{Hom}_{R_H}(R_{Ht^{-1}}, M)$$

where T is a transversal to H in G. The map ν_M carries $R_{tH} \otimes_{R_H} M$ into $\mathrm{Hom}_{R_H}(R_{Ht^{-1}}, M)$, so it suffices to show that for each $t \in T$, ν_M is injective. Fix t and choose elements $x_1, x_2, \ldots, x_{n_t} \in R_t, y_1, y_2, \ldots, y_{n_t} \in R_{t^{-1}}$ such that $\sum_{i=1}^{n_t} x_i y_i = 1$. Suppose $\sum_j r_j \otimes m_j \in \mathrm{Ker}\, \nu_M$, with the $r_j \in R_{tH}$, $m_j \in M$. Then $\sum_j r_j m_j = 0$ and

$$\begin{aligned}\sum_j r_j \otimes m_j &= \sum_j \sum_{i=1}^{n_t} x_i y_i r_j \otimes m_j \\ &= \sum_j \sum_{i=1}^{n_t} x_i \otimes y_i r_j m_j \\ &= \sum_{i=1}^{n_t} x_i \otimes y_i (\sum_j r_j m_j) \\ &= 0.\end{aligned}$$

□

Remark 1.3.6. It is required that R be a strongly G-graded $\Bbbk$-algebra for part (ii) of Lemma 1.3.5 to be true. For, let $R = \Bbbk[X]$, the polynomial ring in one indeterminate over a field $\Bbbk$. This can be made into a $\mathbb{Z}$-graded $\Bbbk$-algebra with the following grading:

- $R_0 = \Bbbk$
- $R_i = \{\text{all monomials of degree } i\}$ for $i > 0$
- $R_i = 0$ for $i < 0$.

R is not strongly $\mathbb{Z}$-graded because $R_i R_{-i} \neq R_0$. We know that ν_M carries the summand $R_{tH} \otimes_{R_H} M$ into $\mathrm{Hom}_{R_H}(R_{Ht^{-1}}, M)$. Let $H = 0$, the trivial subgroup of $\mathbb{Z}$. Then, for each $t > 0$, we have $R_t \otimes_{\Bbbk} M \neq 0$, but $\mathrm{Hom}_{\Bbbk}(R_{t^{-1}}, M) = 0$ since $R_{t^{-1}} = 0$. It follows that ν_M cannot be injective.

Lemma 1.3.7. *Let G be a group, H a subgroup of G and R a strongly G-graded $\Bbbk$-algebra. Suppose that $[G : H] < \infty$. Then the natural monomorphism*

$$\nu_M : \mathrm{Ind}_H^G(M) \to \mathrm{Coind}_H^G(M)$$

is an isomorphism for each left R_H-module M.

Proof. First note that

$$\begin{aligned}
\mathrm{Hom}_{R_H}(R, M) &\cong \mathrm{Hom}_{R_H}(\bigoplus_{t\in T} R_{Ht}, M) \\
&\cong \prod_{t\in T} \mathrm{Hom}_{R_H}(R_{Ht}, M) \\
&= \bigoplus_{t\in T} \mathrm{Hom}_{R_H}(R_{Ht}, M)
\end{aligned}$$

since T, a transversal to H in G, is finite with $[G : H]$ elements.

For a fixed $t \in T$, define

$$\mu_M^t : \mathrm{Hom}_{R_H}(R_{Ht^{-1}}, M) \to R_{tH} \otimes_{R_H} M$$

by

$$\phi \mapsto \sum_{i=1}^{n_t} x_i \otimes \phi(y_i)$$

where the $x_i \in R_t$, $y_i \in R_{t^{-1}}$ and $\sum_{i=1}^{n_t} x_i y_i = 1$.

Look at $\nu_M \circ \mu_M^t$:

$$\phi \overset{\mu_M^t}{\longmapsto} \sum_{i=1}^{n_t} x_i \otimes \phi(y_i) \overset{\nu_M}{\longmapsto} (s \mapsto \sum_{i=1}^{n_t} (s x_i)\phi(y_i))$$

but

$$\begin{aligned}
\sum_{i=1}^{n_t} (s x_i)\phi(y_i) &= \sum_{i=1}^{n_t} \phi((s x_i) y_i) \\
&= \sum_{i=1}^{n_t} \phi(s(x_i y_i)) \\
&= \phi(\sum_{i=1}^{n_t} s(x_i y_i)) \\
&= \phi(s(\sum_{i=1}^{n_t} x_i y_i)) \\
&= \phi(s).
\end{aligned}$$

Look at $\mu^t_M \circ \nu_M$:

$$r \otimes m \overset{\nu_M}{\longmapsto} (s \mapsto (sr)m) \overset{\mu^t_M}{\longmapsto} \sum_{i=1}^{n_t} x_i \otimes (y_i r)m$$

but

$$\begin{aligned}
\sum_{i=1}^{n_t} x_i \otimes (y_i r)m &= \sum_{i=1}^{n_t} x_i(y_i r) \otimes m \\
&= \sum_{i=1}^{n_t} (x_i y_i) r \otimes m \\
&= (\sum_{i=1}^{n_t} x_i y_i) r \otimes m \\
&= r \otimes m.
\end{aligned}$$

The result follows. □

1.4 H-injective and H-projective modules

Definition 1.4.1. Let G a group, H a subgroup of G and R a strongly G-graded $\Bbbk$-algebra. An R_H***-split monomorphism*** is an R-monomorphism

$$0 \to L \xrightarrow{\gamma} M$$

for which there exists an R_H-map $\eta : M \to L$ such that $\eta \circ \gamma = \mathrm{id}_L$. It is ***split*** if the map η is an R-map.

An R_H***-split epimorphism*** is an R-epimorphism

$$M \xrightarrow{\gamma} N \to 0$$

for which there exists an R_H-map $\eta : N \to M$ such that $\gamma \circ \eta = \mathrm{id}_N$. It is ***split*** if the map η is an R-map.

A short exact sequence of R-modules

$$0 \to L \xrightarrow{i} M \xrightarrow{p} N \to 0$$

is R_H-split if there exists a map $j \in \mathrm{Hom}_{R_H}(N, M)$ with $p \circ j = \mathrm{id}_N$. Equivalently, the short exact sequence is R_H-split if there exists a map $q \in \mathrm{Hom}_{R_H}(M, L)$ with $q \circ i = \mathrm{id}_L$. The short exact sequence is said to be split if the maps j and q are R-maps.

A long exact sequence

$$0 \to M_n \xrightarrow{d_n} M_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} M_0 \to 0$$

is R_H-split if for $1 \leq k \leq n-1$, each short exact sequence

$$0 \to \operatorname{Ker} d_k \to M_k \to \operatorname{Im} d_k \to 0$$

is R_H-split. The long exact sequence is split if each short exact sequence is.

We are now ready to introduce the notions of H- and R_H-projective modules and H- and R_H-injective modules.

Definition 1.4.2. Let G be a group, H a subgroup of G and R a strongly G-graded $\Bbbk$-algebra. A left R-module M is said to be ***H-projective*** if given an R_H-split epimorphism of left R-modules $\alpha : A \to B$ and an R-map $\gamma : M \to B$, there is at least one R-map $\beta : M \to A$ making the following diagram commute:

$$\begin{tikzcd} & M \arrow[dl, dotted, "\exists\beta"'] \arrow[d, "\gamma"] & \\ A \arrow[r, "\alpha"] & B \arrow[r] & 0. \end{tikzcd}$$

Definition 1.4.3. Let G be a group, H a subgroup of G and R a strongly G-graded $\Bbbk$-algebra. A left R-module M is ***R_H-projective*** if it is projective as a left R_H-module.

Lemma 1.4.4. *Let G be a group, H a subgroup of G with $[G : H] < \infty$ and R a strongly G-graded $\Bbbk$-algebra. Then a left R-module M is projective if and only if it is both H-projective and R_H-projective.*

Proof. ($\Rightarrow$) Projectivity of M allows us to lift γ in the following diagram of left R-modules

$$\begin{tikzcd} & M \arrow[d, "\gamma"] & \\ A \arrow[r, "\alpha"] & B \arrow[r] & 0 \end{tikzcd}$$

to a map $\beta : M \to A$. In particular, we can do this when α is R_H-split, so M is H-projective. Since all R-maps are R_H-maps, M is also R_H-projective.

($\Leftarrow$) Let $\alpha : A \to B$ be an R-module epimorphism and suppose we have an R-module map $\gamma : M \to B$. We have the following diagram:

$$\begin{tikzcd}
R \otimes_{R_H} A \arrow[rr, "\alpha^*"] \arrow[dd, "\phi_2"] & & R \otimes_{R_H} B \arrow[r] \arrow[dd, "\phi_1"] & 0 \\
& M \arrow[dr, "\gamma"] & & \\
A \arrow[rr, "\alpha"] \arrow[d] & & B \arrow[r] \arrow[d] & 0 \\
0 & & 0 &
\end{tikzcd}$$

where the R_H-split maps ϕ_1 and ϕ_2 are the natural maps sending a tensor $r \otimes m$ to rm (the splitting being given by $m \mapsto 1 \otimes m$) and α^* is the (surjective) map induced from α. Since M is H-projective, we can find an R-map $\psi : M \to R \otimes_{R_H} B$ such that $\gamma = \phi_1 \circ \psi$. Since M is R_H-projective, we can find an R_H-map ω from M to $R \otimes_{R_H} A$ such that $\psi = \alpha^* \circ \omega$, which we may take to be an R-map since the functor $\mathrm{Hom}_R(M, R \otimes_{R_H} -)$ is an exact functor on R_H-modules. To see this, note that

$$\begin{aligned}
\mathrm{Hom}_R(M, R \otimes_{R_H} -) &\cong \mathrm{Hom}_R(M, \mathrm{Hom}_{R_H}(R, -)) \\
&\cong \mathrm{Hom}_{R_H}(R \otimes_R M, -) \\
&\cong \mathrm{Hom}_{R_H}(M, -)
\end{aligned}$$

which is exact since M is projective over R_H. The first isomorphism comes from Lemma 1.3.7 and the second from the standard adjunction

$$\mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C)) \cong \mathrm{Hom}_S(B \otimes_R A, C)$$

for rings R and S and modules ${}_RA$, ${}_SB_R$ and ${}_SC$.

The composite $\phi_2 \circ \omega$ gives us an R-map from M to A. Filling in these maps:

$$\begin{tikzcd}
R \otimes_{R_H} A \arrow[rr, "\alpha^*"] \arrow[dd, "\phi_2"] & & R \otimes_{R_H} B \arrow[r] \arrow[dd, "\phi_1"] & 0 \\
& M \arrow[ul, "\omega"] \arrow[ur, "\psi"'] \arrow[dl, "\phi_2 \circ \omega"'] \arrow[dr, "\gamma"] & & \\
A \arrow[rr, "\alpha"] \arrow[d] & & B \arrow[r] \arrow[d] & 0 \\
0 & & 0 &
\end{tikzcd}$$

but

$$\begin{aligned}\alpha \circ \phi_2 \circ \omega &= \phi_1 \circ \alpha^* \circ \omega \\ &= \phi_1 \circ \psi \\ &= \gamma\end{aligned}$$

that is, M is projective. □

Remark 1.4.5. Only the second part of the proof required the hypothesis that $[G:H] < \infty$; that is, a projective left R-module M is both H-projective and R_H-projective.

Definition 1.4.6. Let G be a group, H a subgroup of G and R a strongly G-graded $\Bbbk$-algebra. A left R-module M is said to be ***H-injective*** if given an R_H-split monomorphism of left R-modules $\alpha : A \to B$ and an R-map $\gamma : A \to M$, there is at least one R-map $\beta : B \to M$ making the following diagram commute:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "\gamma"] & B \arrow[dl, dotted, "\exists\beta"] \\ & M & \end{tikzcd}$$

Remark 1.4.7. Modules which are H-injective for $H = 1_G$ are called ***weakly injective***.

Definition 1.4.8. Let G be a group, H a subgroup of G and R a strongly G-graded $\Bbbk$-algebra. A left R-module M is ***R_H-injective*** if it is injective as a left R_H-module.

Lemma 1.4.9. *Let G be a group, H a subgroup of G with $[G:H] < \infty$ and R a strongly G-graded $\Bbbk$-algebra. Then a left R-module M is injective if and only if it is both H-injective and R_H-injective.*

Proof. ($\Rightarrow$) Injectivity of M guarantees that we can find a map $\beta : B \to M$ which makes the following diagram commute:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "\gamma"] & B \\ & M & \end{tikzcd}$$

In particular, we can do this when α is R_H-split, so M is H-injective. Since all R-maps are R_H-maps, M is R_H-injective.

($\Leftarrow$) Let $\alpha : A \to B$ be an R-module monomorphism and suppose we have an R-module map $\gamma : A \to M$. We have the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] \\
0 \arrow[r] & A \arrow[rr, "\alpha"] \arrow[dr, "\gamma"] \arrow[dd, "\phi_1"] & & B \arrow[dd, "\phi_2"] \\
 & & M & \\
0 \arrow[r] & \mathrm{Hom}_{R_H}(R, A) \arrow[rr, "\alpha^*"] & & \mathrm{Hom}_{R_H}(R, B)
\end{tikzcd}$$

where the R_H-split maps ϕ_1 and ϕ_2 are the embeddings of A and B into their coinduced modules and α^* is the (injective) map induced from α. Since M is H-injective, we can find an R-map $\psi : \mathrm{Hom}_{R_H}(R, A) \to M$ such that $\gamma = \psi \circ \phi_1$. Since M is R_H-injective, we can find an R_H-map ω such that $\psi = \omega \circ \alpha^*$ which we may take to be an R-map since the functor $\mathrm{Hom}_R(\mathrm{Hom}_{R_H}(R, -), M)$ is an exact functor on R_H-modules. To see this, note that

$$\begin{aligned}
\mathrm{Hom}_R(\mathrm{Hom}_{R_H}(R, -), M) &\cong \mathrm{Hom}_R(R \otimes_{R_H} -, M) \\
&\cong \mathrm{Hom}_{R_H}(-, \mathrm{Hom}_R(R, M)) \\
&\cong \mathrm{Hom}_{R_H}(-, M)
\end{aligned}$$

which is exact since M is injective over R_H. The first isomorphism comes from Lemma 1.3.7 and the second from the standard adjunction

$$\mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C)) \cong \mathrm{Hom}_S(B \otimes_R A, C)$$

for rings R and S and modules ${}_RA$, ${}_SB_R$ and ${}_SC$.

The composite $\omega \circ \phi_2$ gives us an R-map from B to M. Filling in these maps:

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] \\
0 \arrow[r] & A \arrow[rr, "\alpha"] \arrow[dr, "\gamma"] \arrow[dd, "\phi_1"] & & B \arrow[dd, "\phi_2"] \arrow[dl, "\omega\circ\phi_2"'] \\
 & & M & \\
0 \arrow[r] & \mathrm{Hom}_{R_H}(R, A) \arrow[rr, "\alpha^*"] \arrow[ur, "\psi"] & & \mathrm{Hom}_{R_H}(R, B) \arrow[ul, "\omega"']
\end{tikzcd}$$

but

$$\begin{aligned}
\omega \circ \phi_2 \circ \alpha &= \omega \circ \alpha^* \circ \phi_1 \\
&= \psi \circ \phi_1 \\
&= \gamma
\end{aligned}$$

that is, M is injective over R. □

Lemma 1.4.10. *Let G be a group, H a subgroup of G, R a strongly G-graded $\Bbbk$-algebra and M a left R-module. Then the following are equivalent:*

(i) M is H-projective.

(ii) M is a direct summand of $R \otimes_{R_H} V$ for some left R_H-module V.

(iii) M is a direct summand of $R \otimes_{R_H} M$.

Proof. (i)⇒(iii) We have an R_H-split epimorphism of left R-modules $p : R \otimes_{R_H} M \to M$ given by $r \otimes m \mapsto rm$ (the splitting is defined by $m \mapsto 1 \otimes m$). Since M is H-projective, we can find an R-map β making the following diagram commute:

$$\begin{tikzcd}
& M \arrow[dl, "\beta"'] \arrow[d, equal] & \\
R \otimes_{R_H} M \arrow[r, "p"] & M \arrow[r] & 0
\end{tikzcd}$$

which gives us a split-exact sequence of R-modules

$$0 \to \operatorname{Ker} p \to R \otimes_{R_H} M \xrightarrow{p} M \to 0$$

and $R \otimes_{R_H} M \cong M \oplus \operatorname{Ker} p$.

(iii)⇒(ii) Trivial.

(ii)⇒(i) Suppose we have an R_H-split epimorphism of R-modules $\pi : B \to M$ and an R-map $\psi : R \otimes_{R_H} V \to M$, giving us the following diagram

$$\begin{tikzcd}
& R \otimes_{R_H} V \arrow[dl, dotted, "?"'] \arrow[d, "\psi"] & \\
B \arrow[r, bend left, "\pi"] & M \arrow[l, bend left, "\tau"] \arrow[r] & 0.
\end{tikzcd}$$

We want to find an R-map $\phi : R \otimes_{R_H} V \to B$, making the diagram commute. By the adjunction mentioned earlier, $\operatorname{Hom}_R(R \otimes_{R_H} V, M) \cong \operatorname{Hom}_{R_H}(V, M)$. Similarly for B. We have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_R(R \otimes_{R_H} V, M) \arrow[d, leftrightarrow, "\iota_M"] & & \mathrm{Hom}_R(R \otimes_{R_H} V, B) \arrow[ll, "\pi_*"'] \arrow[d, leftrightarrow, "\iota_B"] \\
\mathrm{Hom}_{R_H}(V, M) \arrow[rr, bend right, "\tau_*"'] & & \mathrm{Hom}_{R_H}(V, B) \arrow[ll, bend right, "\pi_*"']
\end{tikzcd}$$

where ι_M and ι_B are isomorphisms. We have

$$\begin{aligned}
\pi_* \circ \iota_B \circ \tau_* \circ \iota_M \circ \psi &= \iota_M \circ \pi_* \circ \tau_* \circ \iota_M \circ \psi \\
&= \iota_M \circ (\pi \circ \tau)_* \circ \iota_M \circ \psi \\
&= \iota_M \circ \iota_M \circ \psi \\
&= \psi
\end{aligned}$$

that is, π_* is surjective so $\psi = \pi_*(\phi) = \pi \circ \phi$ for some R-map $\phi : R \otimes_{R_H} V \to B$. $\square$

Lemma 1.4.11. *Let G be a group, H a subgroup of G, R a strongly G-graded $\Bbbk$-algebra and M a left R-module. Then the following are equivalent:*

(i) M is H-injective.

(ii) M is a direct summand of $\mathrm{Hom}_{R_H}(R, V)$ for some left R_H-module V.

(iii) M is a direct summand of $\mathrm{Hom}_{R_H}(R, M)$.

Proof. (i)$\Rightarrow$(iii) We have an R_H-split monomorphism of left R-modules $i : M \to \mathrm{Hom}_{R_H}(R, M)$ given by $m \mapsto \phi_m$ where ϕ_m is the R_H-map sending an element r of R to rm (the splitting is defined by $\phi \mapsto \phi(1)$). Since M is H-injective, we can find an R-map α making the following diagram commute:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "i"] \arrow[d, equal] & \mathrm{Hom}_{R_H}(R, M) \arrow[dl, "\alpha"] \\
& M &
\end{tikzcd}$$

which gives us a split-exact sequence of R-modules

$$0 \to M \xrightarrow{i} \mathrm{Hom}_{R_H}(R, M) \to \operatorname{Coker} i \to 0$$

and $\mathrm{Hom}_{R_H}(R, M) \cong M \oplus \operatorname{Coker} i$.

(iii)$\Rightarrow$(ii) Trivial.

(ii)⇒(i) Suppose we have an R_H-split monomorphism of R-modules $i : M \to B$ and an R-map $\psi : M \to \mathrm{Hom}_{R_H}(R, V)$, giving us the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, bend left, "i"] \arrow[d, "\psi"'] & B \arrow[l, bend left, "\rho"] \arrow[dl, dotted, "?"] \\
& \mathrm{Hom}_{R_H}(R, V). &
\end{tikzcd}$$

We want to find an R-map $\phi : B \to \mathrm{Hom}_{R_H}(R, V)$, making the diagram commute. By the adjunction mentioned earlier, $\mathrm{Hom}_R(B, \mathrm{Hom}_{R_H}(R, V)) \cong \mathrm{Hom}_{R_H}(B, V)$. Similarly for M. We have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_R(B, \mathrm{Hom}_{R_H}(R, V)) \arrow[r, "i^*"] \arrow[d, leftrightarrow, "\iota_B"'] & \mathrm{Hom}_R(M, \mathrm{Hom}_{R_H}(R, V)) \arrow[d, leftrightarrow, "\iota_M"] \\
\mathrm{Hom}_{R_H}(B, V) \arrow[r, bend left, "i^*"] & \mathrm{Hom}_{R_H}(M, V) \arrow[l, bend left, "\rho^*"]
\end{tikzcd}$$

where ι_B and ι_M are isomorphisms. We have

$$\begin{aligned}
i^* \circ \iota_B \circ \rho^* \circ \iota_M \circ \psi &= \iota_M \circ i^* \circ \rho^* \circ \iota_M \circ \psi \\
&= \iota_M \circ (\rho \circ i)^* \circ \iota_M \circ \psi \\
&= \iota_M \circ \iota_M \circ \psi \\
&= \psi
\end{aligned}$$

that is, i^* is surjective so $\psi = i^*(\phi) = \phi \circ i$ for some R-map $\phi : B \to \mathrm{Hom}_{R_H}(R, V)$. □

Lemma 1.4.12. *Let G be a group, H a subgroup of G with $[G : H] < \infty$ and R a strongly group-graded $\Bbbk$-algebra. Then a left R-module M is H-injective if and only if it is H-projective.*

Proof. In view of Lemma 1.4.10, we shall prove the equivalent statement "M is H-injective if and only if M is a direct summand of $R \otimes_{R_H} M$".

(⇒) We have an R_H-split monomorphism of left R-modules $i : M \to \mathrm{Hom}_{R_H}(R, M)$. Since $[G : H] < \infty$, we have that $\mathrm{Hom}_{R_H}(R, M) \cong R \otimes_{R_H} M$. Since M is H-injective, we can find an R-map α making the following diagram commute:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "i"] \arrow[d, equal] & R \otimes_{R_H} M \arrow[dl, dotted, "\alpha"] \\
& M &
\end{tikzcd}$$

which gives us a split-exact sequence of left R-modules

$$0 \to M \xrightarrow{i} R \otimes_{R_H} M \to \operatorname{Coker} i \to 0$$

and so $R \otimes_{R_H} M \cong M \oplus \operatorname{Coker} i$.

($\Leftarrow$) Suppose we have an R_H-split monomorphism of R-modules $\varepsilon : A \to B$ and an R-map $\psi : A \to R \otimes_{R_H} M$, giving us the following diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, bend left, "\varepsilon"] \arrow[d, "\psi"] & B \arrow[l, bend left, "\sigma"] \arrow[dl, dotted, "?"] \\ & R \otimes_{R_H} M & \end{tikzcd}$$

We want to find an R-map $\phi : B \to R \otimes_{R_H} M$ to make the diagram commute. Given that $[G : H] < \infty, R \otimes_{R_H} M \cong \operatorname{Hom}_{R_H}(R, M)$ and by the adjunction mentioned earlier, $\operatorname{Hom}_R(A, R \otimes_{R_H} M) \cong \operatorname{Hom}_{R_H}(A, M)$. Similarly for B. We have a commutative diagram

$$\begin{tikzcd} \operatorname{Hom}_R(B, R \otimes_{R_H} M) \arrow[r, "\varepsilon^*"] \arrow[d, leftrightarrow, "\iota_B"] & \operatorname{Hom}_R(A, R \otimes_{R_H} M) \arrow[d, leftrightarrow, "\iota_A"] \\ \operatorname{Hom}_{R_H}(B, M) \arrow[r, bend left, "\varepsilon^*"] & \operatorname{Hom}_{R_H}(A, M) \arrow[l, bend left, "\sigma^*"] \end{tikzcd}$$

where ι_B and ι_A are isomorphisms. We have

$$\begin{aligned} \varepsilon^* \circ \iota_B \circ \sigma^* \circ \iota_A \circ \psi &= \iota_A \circ \varepsilon^* \circ \sigma^* \circ \iota_A \circ \psi \\ &= \iota_A \circ (\varepsilon \circ \sigma)^* \circ \iota_A \circ \psi \\ &= \iota_A \circ \iota_A \circ \psi \\ &= \psi \end{aligned}$$

that is, ε^* is surjective so $\psi = \varepsilon^*(\phi) = \phi \circ \varepsilon$ for some R-map $\phi : B \to R \otimes_{R_H} M$. $\square$

1.5 Dimensions

Definition 1.5.1. Let R be a ring. A ***projective resolution*** of a left R-module M is an exact sequence,

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

in which each P_i $(i \in \mathbb{N}_0)$ is a projective left R-module.

Definition 1.5.2. Let R be a ring. A left R-module M has ***finite projective dimension*** if, in a projective resolution of M, there exists $n \in \mathbb{N}_0$ such that for all $i > n$, $P_i = 0$. If $M \neq 0$ has a finite projective resolution, then its ***projective dimension***, $pd_R(M)$, is the least number $n \in \mathbb{N}_0$ such that there exists a projective resolution of length n:

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0.$$

In case no finite projective resolution exists, $pd_R(M)$ is defined to be equal to ∞.

The zero module is defined to have projective dimension $-\infty$.

Example 1.5.3. Let R be a ring. A non-zero left R-module M is projective if and only if $pd_R(M) = 0$. In this case, $P_0 = M$.

Example 1.5.4. Let p be a prime number. As a left $\mathbb{Z}$-module, $\mathbb{Z}/p\mathbb{Z}$ has projective dimension 1:

$$0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0.$$

Definition 1.5.5. Let R be a ring. A left R-module M is said to be ***of type*** FP_∞ ***over*** R if it admits a resolution

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

by finitely generated projective left R-modules.

Example 1.5.6. If R is a Noetherian ring and M is a finitely generated left R-module, then M is of type FP_∞ over R because the kernel of every surjective R-map from a finitely generated free module onto M is finitely generated.

Definition 1.5.7. Let R be a ring and let

$$\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\pi} M \to 0$$

be a projective resolution of a left R-module M. For $n \geq 0$, the n^{th} ***syzygy*** is

$$\Omega_n(M) = \begin{cases} \operatorname{Ker} \pi & \text{if } n = 0 \\ \operatorname{Ker} d_n & \text{if } n \geq 1. \end{cases}$$

Syzygies help us compute projective dimension:

Lemma 1.5.8. *Let R be a ring. The following are equivalent for a left R-module M:*

(i) $pd_R(M) \leq n$.

(ii) $\mathrm{Ext}_R^k(M,N) = 0$ *for all left R-modules N and all* $k \geq n+1$.

(iii) For every projective resolution of M, $\Omega_{n-1}(M)$ *is projective.*

Proof. This is Lemma 11.123 of [28]. □

The projective dimensions of modules in an exact sequence are intimately related:

Lemma 1.5.9. *Let R be a ring and*

$$0 \to A \to B \to C \to 0$$

a short exact sequence of left R-modules. If it is the case that two modules have finite projective dimension then so does the third. In fact $pd_R(B) \leq max\{pd_R(A), pd_R(C)\}$ *with equality unless* $pd_R(B) < pd_R(C) = 1 + pd_R(A)$.

Proof. See, for example, page 247 of [26]. □

Lemma 1.5.10. *Let*

$$0 \to M_r \to \cdots \to M_1 \to M_0 \to M \to 0$$

be an exact sequence of R-modules. If M_i *has projective dimension at most n for all i then M has projective dimension at most* $n+r$.

Proof. This is Lemma 10.1 of [10]. □

The following definitions will be used later.

Definition 1.5.11. Let R be a ring. A ***flat resolution*** of a left R-module M is an exact sequence,

$$\cdots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$$

in which each F_i $(i \in \mathbb{N}_0)$ is a flat left R-module.

Definition 1.5.12. Let R be a ring. A left R-module M has ***finite weak dimension*** if there exists a flat resolution of M of finite length. If $M \neq 0$ has a finite flat resolution, then its ***weak dimension***, $wd_R(M)$, is the least number $n \in \mathbb{N}_0$ such that there exists a flat resolution of length n:

$$0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0.$$

In case no finite flat resolution exists, $wd_R(M)$ is defined to be equal to ∞.

The zero module is defined to have weak dimension $-\infty$.

The term "***flat dimension***" is sometimes used in place of "weak dimension".

Example 1.5.13. Let R be a ring. A left R-module M is flat if and only if $wd_R(M) = 0$.

Example 1.5.14. Let R be a ring. Since projective modules are flat, every projective resolution of a left R-module M is a flat resolution. It follows that if R is any ring, then $wd_R(M) \leq pd_R(M)$ for every left R-module M.

Definition 1.5.15. Let R be a ring and let

$$\cdots \to F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\pi} M \to 0$$

be a flat resolution of a left R-module M. For $n \geq 0$, the n^{th} ***yoke*** is

$$Y_n(M) = \begin{cases} \operatorname{Ker} \pi & \text{if } n = 0 \\ \operatorname{Ker} d_n & \text{if } n \geq 1. \end{cases}$$

Lemma 1.5.16. *Let R be a ring. The following are equivalent for a left R-module M:*

(i) $wd_R(M) \leq n$.

(ii) $\operatorname{Tor}_k^R(N, M) = 0$ *for all right R-modules N and all $k \geq n+1$.*

(iii) For every flat resolution of M, $Y_{n-1}(M)$ is flat.

Proof. This is Lemma 11.138 of [28]. □

Definition 1.5.17. The ***left global dimension*** of a ring R is defined as

$$lD(R) = \sup\{pd_R(M) : M \in \operatorname{obj}({}_R\mathfrak{Mod})\}.$$

There is a similar notion of right global dimension.

Definition 1.5.18. The ***left finitistic dimension*** of a ring R is defined as

$$lFPD = \sup\{pd_R(M)\}$$

where M runs through all left R-modules for which $pd_R(M)$ is finite.

Definition 1.5.19. The ***weak dimension*** of a ring R is defined as

$$wD(R) = \sup\{wd_R(M) : M \in \operatorname{obj}({}_R\mathfrak{Mod})\}.$$

It will be noted that we do not speak of the left weak dimension of a ring R. This is because the weak dimension can be computed using Tor groups which involve left and right modules simultaneously.

Lemma 1.5.20. *Let G be a group, H a subgroup of G, R a strongly G-graded $\Bbbk$-algebra and M an R-module.*

(i) If M is a projective R-module then M is a projective R_H-module.

(ii) $pd_R(R \otimes_{R_H} M) = pd_{R_H}(M) \leq pd_R(M)$.

Proof. This is Lemma 6.3 parts (i) and (ii) of [10]. □

Lemma 1.5.21. *Let G be a group, H a subgroup of G with $[G:H] < \infty$ and R a strongly G-graded $\Bbbk$-algebra. Then for every left R-module M with $pd_R(M) < \infty$,*

$$pd_R(M) = pd_{R_H}(M).$$

Proof. This is Lemma 6.6 of [10]. □

1.6 Some useful results

In this section we introduce some results we shall need for later work but whose inclusion elsewhere would interrupt the flow of the thesis.

Recall the following:

Proposition 1.6.1. *Let R be a ring. A left R-module P is projective if and only if $\mathrm{Hom}_R(P, -)$ is an exact functor.*

Proof. This is Proposition 7.53 of [28] □

Lemma 1.6.2. *Let G be a group, H a subgroup of G with $[G:H] < \infty$ and R a strongly G-graded $\Bbbk$-algebra. Then, given a left R-module N which is projective as a left R_H-module, we can construct an R_H-split exact sequence of left R-modules*

$$0 \to N \xrightarrow{\iota_0} P_0 \xrightarrow{d_0} P_1 \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} P_n \to 0$$

of any desired length in which the P_i $(0 \leq i \leq n-1)$ are projective and P_n is projective as a left R_H-module. If N is finitely generated, then each of the P_i can be taken to be finitely generated as left R-modules.

Proof. We have an injection

$$N \xrightarrow{\iota_0} \mathrm{Hom}_{R_H}(R, N)$$

given by

$$n \mapsto (\phi_n : r \mapsto rn).$$

As we have already seen, this map is R_H-split via

$$\phi \mapsto \phi(1).$$

Thus we have an R_H-split short exact sequence

$$0 \to N \xrightarrow{\iota_0} \mathrm{Hom}_{R_H}(R, N) \xrightarrow{\pi_0} \mathrm{Coker}\, \iota_0 \to 0.$$

Note that

$$\begin{aligned} \mathrm{Hom}_R(\mathrm{Hom}_{R_H}(R, N), -) &\cong \mathrm{Hom}_R(R \otimes_{R_H} N, -) \\ &\cong \mathrm{Hom}_{R_H}(N, \mathrm{Hom}_R(R, -)) \\ &\cong \mathrm{Hom}_{R_H}(N, -) \end{aligned}$$

which is an exact functor because N is projective as an R_H-module (by Lemma 1.5.20, part 1). It follows by Proposition 1.6.1 that $\mathrm{Hom}_{R_H}(R, N)$ is a projective left R-module. Because the sequence is R_H-split, $\mathrm{Coker}\, \iota_0$ is projective as an R_H-module.

It is clear that we can repeat this process, obtaining another R_H-split short exact sequence

$$0 \to \mathrm{Coker}\, \iota_0 \xrightarrow{\iota_1} \mathrm{Hom}_{R_H}(R, \mathrm{Coker}\, \iota_0) \xrightarrow{\pi_0} \mathrm{Coker}\, \iota_1 \to 0$$

in which $\mathrm{Hom}_{R_H}(R, \mathrm{Coker}\, \iota_0)$ is a projective left R-module and $\mathrm{Coker}\, \iota_1$ is a projective left R_H-module.

Set

- $P_0 = \mathrm{Hom}_{R_H}(R, N)$
- $d_0 = \iota_1 \circ \pi_0$

and inductively (for $i \geq 1$),

- $P_i = \mathrm{Hom}_{R_H}(R, \mathrm{Coker}\, \iota_{i-1})$
- $d_i = \iota_{i+1} \circ \pi_i$

The construction is illustrated below:

$$\begin{tikzcd}
 & & & & 0 \arrow[dr] & & 0 & & \\
 & & & & & \mathrm{Coker}\,\iota_1 \arrow[ur] \arrow[dr, "\iota_2"] & & & \\
0 \arrow[r] & N \arrow[r, "\iota_0"] & P_0 \arrow[rr, "d_0"] \arrow[dr, "\pi_0"'] & & P_1'' \arrow[rr, "d_1"] \arrow[ur, "\pi_1"] & & P_2 \arrow[rr, "d_2"] \arrow[dr, "\pi_2"'] & & \cdots \\
 & & & \mathrm{Coker}\,\iota_0 \arrow[ur, "\iota_1"'] \arrow[dr] & & & & \mathrm{Coker}\,\iota_2 \arrow[ur, "\iota_3"'] \arrow[dr] & \\
 & & 0 \arrow[ur] & & 0 & & 0 \arrow[ur] & & 0
\end{tikzcd}$$

We still need to check that the sequence is exact:

- $\operatorname{Im} d_{i-1} \subseteq \operatorname{Ker} d_i$.

$$d_i \circ d_{i-1} = \iota_{i+1} \circ \pi_i \circ \iota_i \circ \pi_{i-1} = 0.$$

- $\operatorname{Ker} d_i \subseteq \operatorname{Im} d_{i-1}$. Suppose $d_i(p) = \iota_{i+1}(\pi_i(p)) = 0$ for some $p \in P_i$. ι_{i+1} is injective, so $\pi_i(p) \in \operatorname{Ker} \pi_i = \operatorname{Im} \iota_i$, by exactness. Thus there exists a $p' \in \operatorname{Coker} \iota_{i-1}$ such that $\iota_i(p') = p$. π_{i-1} is surjective, so there exists a $p'' \in P_{i-1}$ such that $\pi_{i-1}(p'') = p'$. It follows that $p \in \operatorname{Im} d_{i-1}$.

The sequence is easily seen to be R_H-split: the splitting of d_i is the map which is the composite of the the maps which split ι_{i+1} and π_i respectively.

Suppose N is finitely generated with generating set $\{n_1, n_2, \ldots, n_s\}$. Because $[G : H] < \infty$, we have by Lemma 1.1.13 that R is finitely generated as a (right or left) R_H-module with generating set $\{r_1, r_2, \ldots, r_t\}$. A simple check shows that $P_0 = \operatorname{Hom}_{R_H}(R, N)$ is finitely generated as a left R-module by a generating set containing at most st elements. It follows by induction that the P_i $(0 \leq i \leq n)$ are finitely generated. □

Theorem 1.6.3. *Let $R = \varinjlim_\alpha R_\alpha$ be a filtered colimit of rings, A a right R-module and B a left R-module. Then*

$$A \otimes_R B = \varinjlim_{\alpha} A \otimes_{R_\alpha} B.$$

Proof. For each R_α in the limit we have a ring homomorphism $\phi_\alpha : R_\alpha \to R$. It follows that $A \otimes_{R_\alpha} B = A \otimes_{\operatorname{Im} \phi_\alpha} B$ for each α and so $\varinjlim_\alpha A \otimes_{R_\alpha} B = \varinjlim_\alpha A \otimes_{\operatorname{Im} \phi_\alpha} B$. Thus it remains to show $A \otimes_R B = \varinjlim_\alpha A \otimes_{\operatorname{Im} \phi_\alpha} B$.

Elements of $\varinjlim_\alpha A \otimes_{\operatorname{Im} \phi_\alpha} B$ are equivalence classes $[a \otimes b, \alpha]$ where $a \otimes b \in A \otimes_{\operatorname{Im} \phi_\alpha} B$, and $[a \otimes b, \alpha] = [a' \otimes b', \beta]$ if and only if there exist $\gamma > \alpha, \beta$ and maps $\phi_\gamma^\alpha : A \otimes_{R_\alpha} B \to$

$A \otimes_{R_\gamma} B$ and $\phi_\gamma^\beta : A \otimes_{R_\beta} B \to A \otimes_{R_\gamma} B$ such that $\phi_\gamma^\alpha(a \otimes b) = \phi_\gamma^\beta(a' \otimes b')$; that is, an element of $A \otimes_{\operatorname{Im}\phi_\alpha} B$ and an element of $A \otimes_{\operatorname{Im}\phi_\beta} B$ become identified in the limit if they are equal as elements of $A \otimes_R B$.

We have a natural map $\varinjlim_\alpha A \otimes_{\operatorname{Im}\phi_\alpha} B \xrightarrow{\psi} A \otimes_R B$ defined by $[a \otimes b, \alpha] \mapsto a \otimes b$, which is well-defined: for $r \in \operatorname{Im}\phi_\alpha$,

$$\begin{tikzcd}
{[ar \otimes b, \alpha]} \arrow[r, mapsto] \arrow[d, equal] & ar \otimes b \arrow[d, equal] \\
{[a \otimes rb, \alpha]} \arrow[r, mapsto] & a \otimes rb.
\end{tikzcd}$$

It is easily checked ψ is an R-module homomorphism.

Let $[a \otimes b, \alpha] \in \operatorname{Ker}\psi$ (every element of the limit is of this form because the index set is directed). Then $a \otimes b = 0$ so $[a \otimes b, \alpha]$ is the equivalence class of the zero element and ψ is injective.

Any $a \otimes b \in A \otimes_R B$ is the image under ψ of $[a \otimes b, \alpha]$ for any label α. Thus ψ is surjective. $\square$

Chapter 2

Generalisations

In this chapter we will generalise known results for group rings to the strongly group-graded case. Throughout, $\Bbbk$ will denote a commutative ring.

2.1 Complete cohomology groups

The projective dimension of a left R-module M is the least $n \in \mathbb{N}_0$ such that $\operatorname{Ext}^i_R(M, N) = 0$ for all $i > n$ and all left R-modules N. As a method for finding projective dimension, the use of Ext groups is virtually useless; however, very often all we are interested in knowing is whether the projective dimension is finite. A more appropriate formulation for this scenario are the complete cohomology groups.

Lemma 2.1.1 (Schanuel). *Let R be a ring. Given exact sequences*

$$0 \to K \to P \to M \to 0$$

and

$$0 \to K' \to P' \to M \to 0,$$

of left R-modules with P and P' projective, then there is an isomorphism

$$K \oplus P' \cong K' \oplus P.$$

Proof. The result is well known. A proof can be found on page 10 of [2] □

Let R be a ring and M a left R-module. Because the category ${}_R\mathfrak{Mod}$ has enough projectives, we are always able to find a projective left R-module P and an epimorphism $P \to M$.

Let R be a ring and A a set. We can construct a free left R-module $F'A$, where each element has a unique expression of the form $\sum_a r_a a$, $a \in A$, and all but finitely many of the r_a are zero. Thus, we can define a functor $F' : \mathfrak{Sets} \to {}_R\mathfrak{Mod}$. This functor is left adjoint to the forgetful functor ${}_R\mathfrak{Mod} \to \mathfrak{Sets}$ which sends a module to its underlying set and regards homomorphisms as functions. If we define $F : {}_R\mathfrak{Mod} \to \mathfrak{Sets} \to {}_R\mathfrak{Mod}$ as the composite of these two functors, then the identity function from the set M to the module M induces an epimorphism $FM \to M$.

Definition 2.1.2. Given a left R-module M, ΩM is defined to be the kernel of an epimorphism $P \to M$ with P a projective left R-module. (c.f. Definition 1.5.7.) Ω is not a functor on ${}_R\mathfrak{Mod}$.

Remark 2.1.3. To avoid ambiguity, we will occasionally use brackets on this notation to distinguish $\Omega(M) \oplus N$ from $\Omega(M \oplus N)$, for example.

Remark 2.1.4. Schanuel's Lemma shows that the definition of Ω is well-defined up to adding and removing projective summands: given two partial projective resolutions of a module M,

$$0 \to \Omega M \to P \to M \to 0$$

and

$$0 \to \Omega M \to Q \to M \to 0,$$

we have $\Omega(M) \oplus Q \cong \Omega(M) \oplus P$.

If ϕ is an R-map between two left R-modules M, N, then we may lift as in the following diagram

$$\begin{tikzcd} 0 \arrow[r] & \Omega M \arrow[r] \arrow[d] & P_M \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \Omega N \arrow[r] & P_N \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

to obtain a map $\Omega(\phi) : \Omega M \to \Omega N$ which is unique up to the addition of maps factoring through a projective module.

The free module FM obtained from a module M provides an easy way to obtain a surjective map from a projective module to M.

Suppose we have an R-map $\phi : M \to N$, then we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \Omega M \arrow[r] \arrow[d, "\Omega(\phi)"] & FM \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "\phi"] & 0 \\ 0 \arrow[r] & \Omega N \arrow[r] & FN \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

We can iterate this process, giving us a commutative diagram (and free resolutions of M and N):

$$\begin{tikzcd}
 & & \Omega^2 M \arrow[dr] \arrow[dd, "\Omega^2(\phi)"] & & \Omega M \arrow[dr] \arrow[dd, "\Omega(\phi)"] & & & \\
\cdots \arrow[r] & F^3M \arrow[ur] \arrow[rr] \arrow[dd] & & F^2M \arrow[ur] \arrow[rr] \arrow[dd] & & FM \arrow[r] \arrow[dd] & M \arrow[r] \arrow[dd, "\phi"] & 0 \\
 & & \Omega^2 N \arrow[dr] & & \Omega N \arrow[dr] & & & \\
\cdots \arrow[r] & F^3N \arrow[ur] \arrow[rr] & & F^2N \arrow[ur] \arrow[rr] & & FN \arrow[r] & N \arrow[r] & 0.
\end{tikzcd}$$

Where $\Omega^i M = \mathrm{Ker}(F^i M \to \Omega^{i-1} M)$ and $F^i M = F(\Omega^{i-1} M)$.

Definition 2.1.5. Let R be a ring. Given any two left R-modules M and N, define $\mathrm{Phom}_R(M, N)$ to be the additive subgroup of $\mathrm{Hom}_R(M, N)$ comprising those homomorphisms which factor through a projective module. $\underline{\mathrm{Hom}}_R(M, N)$ is then defined as the factor group

$$\underline{\mathrm{Hom}}_R(M, N) := \mathrm{Hom}_R(M, N)/\mathrm{Phom}_R(M, N)$$

and there is a natural map $\mathrm{Hom}_R(M, N) \twoheadrightarrow \underline{\mathrm{Hom}}_R(M, N)$.

Remark 2.1.6. While Ω is not a functor on ${}_R\mathfrak{Mod}$, it is a functor on the category which has the same objects as ${}_R\mathfrak{Mod}$ but whose morphisms are given by $\underline{\mathrm{Hom}}_R(M, N)$ for each pair of modules M, N.

For any two left R-modules M and N, Ω induces a function

$$\Omega : \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(\Omega M, \Omega N)$$

which is not a homomorphism. However, if P is projective then ΩP is either projective or the zero module, therefore we have an induced map

$$\Omega : \underline{\mathrm{Hom}}_R(M, N) \to \underline{\mathrm{Hom}}_R(\Omega M, \Omega N)$$

which is an additive homomorphism and indeed we get a commutative square

$$\begin{tikzcd}
\mathrm{Hom}_R(M, N) \arrow[r, "\Omega"] \arrow[d, two heads] & \mathrm{Hom}_R(\Omega M, \Omega N) \arrow[d, two heads] \\
\underline{\mathrm{Hom}}_R(M, N) \arrow[r, "\Omega"] & \underline{\mathrm{Hom}}_R(\Omega M, \Omega N)
\end{tikzcd}$$

for any pair of left R-modules M, N. A proof of this appears in [25].

Successive applications of Ω gives us a direct system of R-modules:

$$\underline{\mathrm{Hom}}_R(M,N) \xrightarrow{\Omega} \underline{\mathrm{Hom}}_R(\Omega M,\Omega N) \xrightarrow{\Omega} \underline{\mathrm{Hom}}_R(\Omega^2 M,\Omega^2 N) \xrightarrow{\Omega} \dots$$

Definition 2.1.7. Let R be a ring and let M and N be R-modules. The ***zeroeth complete cohomology group*** $\widehat{\mathrm{Ext}}^0_R(M,N)$ is the colimit of the above direct system:

$$\widehat{\mathrm{Ext}}^0_R(M,N) := \varinjlim_i \underline{\mathrm{Hom}}_R(\Omega^i M,\Omega^i N).$$

For each $j \in \mathbb{Z}$, we can define $\widehat{\mathrm{Ext}}^j_R(M,N)$: for $j \geq 0$ as

$$\widehat{\mathrm{Ext}}^j_R(M,N) := \widehat{\mathrm{Ext}}^0_R(\Omega^j M,N) = \varinjlim_i \underline{\mathrm{Hom}}_R(\Omega^{i+j} M,\Omega^i N)$$

and for $j < 0$ as

$$\widehat{\mathrm{Ext}}^j_R(M,N) := \widehat{\mathrm{Ext}}^0_R(M,\Omega^{-j} N) = \varinjlim_i \underline{\mathrm{Hom}}_R(\Omega^i M,\Omega^{i-j} N).$$

It is easy to check that $\widehat{\mathrm{Ext}}^*_R(M,N)$ is a functor which is covariant in N and contravariant in M. There are a number of different (equivalent) definitions of these $\widehat{\mathrm{Ext}}^*_R$ groups. Cornick and Kropholler follow Mislin in defining the $\widehat{\mathrm{Ext}}^*_R$ in terms of satellites of functors [9]. Goichot [16] takes cohomology of a double complex of Homs in which he has factored out the 'bounded functions'. The above definition is from Benson and Carlson [4].

The following lemma was first proved by Kropholler in [22]. It illustrates the strength of the above definition so we reproduce the proof.

Lemma 2.1.8. *Let R be a ring and M an R-module. Then M has finite projective dimension if and only if $\widehat{\mathrm{Ext}}^0_R(M,M) = 0$*

Proof. If M has projective dimension r, then for all $i > r$, $\Omega^i M = 0$ and so

$$\widehat{\mathrm{Ext}}^0_R(M,M) = \varinjlim_i \underline{\mathrm{Hom}}_R(\Omega^i M,\Omega^i M) = 0.$$

On the other hand, if $\widehat{\mathrm{Ext}}^0_R(M,M) = 0$ then the identity map of M goes to zero in the limit and so there exists an $i \geq 0$ such that the identity map on M becomes zero under the natural map $\mathrm{Hom}_R(M,M) \twoheadrightarrow \underline{\mathrm{Hom}}_R(M,M) \xrightarrow{\Omega^i} \underline{\mathrm{Hom}}_R(\Omega^i M,\Omega^i M)$. But that means that the identity map on $\Omega^i M$ factors through a projective module. Therefore $\Omega^i M$ is projective and so by Lemma 1.5.8, M has projective dimension at most i. □

We now state a number of lemmata which will prove useful later, but first a definition.

Definition 2.1.9. A functor F between two module categories is said to be ***continuous*** (or ***finitary***) if and only if the natural map $\varinjlim_\lambda F(M_\lambda) \to F(\varinjlim_\lambda M_\lambda)$ is an isomorphism for all direct limit systems (M_λ) of modules.

Example 2.1.10. Let R be a ring. For any right R-module A, the functors $\mathrm{Tor}_*^R(A, -)$ are continuous. (For a proof of this, see Proposition 10.99 of [28].)

Lemma 2.1.11. *Let R be a ring and M a left R-module of type* FP_∞. *Then the functors* $\widehat{\mathrm{Ext}}^*_R(M, -)$ *are continuous.*

Proof. This is Lemma 5.2.7 of [25]. □

Remark 2.1.12. It follows that if (N_λ) is any family of left R-modules and M is a left R-module of type FP_∞ then the map $\bigoplus_\lambda \widehat{\mathrm{Ext}}^n_R(M, N_\lambda) \to \widehat{\mathrm{Ext}}^n_R(M, \bigoplus_\lambda N_\lambda)$ is an isomorphism because the direct sum is the colimit of all its finite subsums.

Lemma 2.1.13. *Let R be a ring,*

$$0 \to M_r \to M_{r-1} \to \cdots \to M_1 \to M_0 \to L \to 0$$

an exact sequence of left R-modules and N a left R-module. Then if i is an integer such that $\widehat{\mathrm{Ext}}^i_R(N, L)$ is non-zero then there exists a j with $0 \le j \le r$ such that $\widehat{\mathrm{Ext}}^{i+j}_R(N, M_j)$ is non-zero.

Proof. This is Lemma 3.1 of [22]. □

The next result is crucial. Results of this kind are well known for group algebras, but it has not been possible to find an explicit reference for the general form that is needed. However, the discussion in Chapter 1 is intended to facilitate proving general results over group-graded rings. We shall give a proof of this lemma and a careful discussion of two corollaries (Lemmata 2.1.15 and 2.1.16).

Lemma 2.1.14. *Let G be a group, H a subgroup with $[G : H] < \infty$, R a strongly G-graded $\Bbbk$-algebra and M a left R-module. If N is a left R-module which is projective as an R_H-module, then the natural map*

$$\Omega : \underline{\mathrm{Hom}}_R(N, M) \to \underline{\mathrm{Hom}}_R(\Omega N, \Omega M)$$

is an isomorphism.

Proof. We show that Ω is surjective. Let $\phi \in \underline{\mathrm{Hom}}_R(\Omega N, \Omega M)$. We have a commutative diagram in which the top row is R_H-split exact and the bottom row exact:

$$\begin{tikzcd}
0 \arrow[r] & \Omega N \arrow[r, "\iota_N"] \arrow[d, "\phi"'] \arrow[dr, "\iota_M \circ \phi"] & R \otimes_{R_H} N \arrow[r, "\pi_N"] \arrow[d, "\phi'"] & N \arrow[r] \arrow[d, "\phi''"] & 0 \\
0 \arrow[r] & \Omega M \arrow[r, "\iota_M"] & FM \arrow[r, "\pi_M"] & M \arrow[r] & 0
\end{tikzcd}$$

where FM is the free left R-module on the underlying set of M. Since FM is a projective R-module, it is H-projective by Lemma 1.4.4 and because $[G : H] < \infty$, it is H-injective by Lemma 1.4.12. It follows that we can find an R-map $\phi' : R \otimes_{R_H} N \to FM$ making the left-hand square commute.

If $n \in N$, then there is an $r_1 \otimes n_1 \in R \otimes_{R_H} N$ with $\pi_N(r_1 \otimes n_1) = n$ because π_N is surjective. Define $\phi'' : N \to M$ by $\phi''(n) = \pi_M \circ \phi'(r_1 \otimes n_1)$. ϕ'' is well-defined because if $r_2 \otimes n_2 \in R \otimes_{R_H} N$ satisfies $\pi_N(r_2 \otimes n_2) = n$, then $\pi_N(r_1 \otimes n_2 - r_2 \otimes n_2) = 0$. That is, $r_1 \otimes n_2 - r_2 \otimes n_2 \in \mathrm{Ker}\,\pi_N = \mathrm{Im}\,\iota_N$, by exactness. Hence $r_1 \otimes n_2 - r_2 \otimes n_2 = \iota_N(n')$, for some $n' \in \Omega N$. Thus

$$\begin{aligned}
\pi_M(\phi'(r_1 \otimes n_2 - r_2 \otimes n_2)) &= \pi_M(\phi'(\iota_N(n'))) \\
&= \pi_M(\iota_M(\phi(n'))) \\
&= 0
\end{aligned}$$

and the right-hand square commutes. It follows that the natural map Ω is surjective.

For injectivity, first recall that for a map $\phi : N \to M$, we obtain a map $\Omega(\phi) : \Omega N \to \Omega M$. If $\Omega(\phi) \in \underline{\mathrm{Hom}}_R(\Omega N, \Omega M)$ represents the zero map, then it factors through a projective module P so we have a commutative diagram with an R_H-split exact top row and exact bottom row:

$$\begin{tikzcd}
0 \arrow[r] & \Omega N \arrow[rr, "\iota_N"] \arrow[dd, "\Omega(\phi)"'] \arrow[dr, "\phi_1"] & & P_N \arrow[r, "\pi_N"] \arrow[dl, "\alpha"'] \arrow[dd, "F\phi"] & N \arrow[r] \arrow[dd, "\phi"] & 0 \\
& & P \arrow[dl, "\phi_2"] \arrow[dr, "\beta"'] & & & \\
0 \arrow[r] & \Omega M \arrow[rr, "\iota_M"] & & P_M \arrow[r, "\pi_M"] & M \arrow[r] & 0.
\end{tikzcd}$$

We already know that the left-hand triangle commutes. Since P is projective, it is H-injective (by the same reasoning above) so we can find an R-map $\alpha : P_N \to P$ which makes the top triangle commute. Defining β to be the composite $\iota_M \circ \phi_2$ ensures commutativity of the bottom triangle. It follows that

$$\begin{aligned}(F\phi - \beta \circ \alpha) \circ \iota_N &= F\phi \circ \iota_N - \beta \circ \alpha \circ \iota_N \\ &= \iota_M \circ \Omega(\phi) - \iota_M \circ \phi_2 \circ \alpha \circ \iota_N \\ &= \iota_M \circ \Omega(\phi) - \iota_M \circ \phi_2 \circ \phi_1 \\ &= \iota_M \circ \Omega(\phi) - \iota_M \circ \Omega(\phi) \\ &= 0,\end{aligned}$$

an observation that we will use shortly.

Define an R-map $s : N \to P_M$ by sending a pre-image $(n' \in P_N)$ under π_N of $n \in N$ to $(F\phi - \beta \circ \alpha)(n')$. s is well-defined because if $n'' \in P_N$ is another pre-image of n, then $\pi_N(n' - n'') = 0$, so that $n' - n'' \in \operatorname{Ker} \pi_N = \operatorname{Im} \iota_N$. Hence $n' - n'' = \iota_N(n''')$ for some $n''' \in \Omega N$. Thus, by the above observation,

$$\begin{aligned}0 &= (F\phi - \beta \circ \alpha) \circ \iota_N(n''') \\ &= (F\phi - \beta \circ \alpha)(n' - n'') \\ &= F\phi(n' - n'') - \beta \circ \alpha(n' - n'') \\ &= F\phi(n') - F\phi(n'') - \beta \circ \alpha(n') + \beta \circ \alpha(n'') \\ &= (F\phi - \beta \circ \alpha)(n') - (F\phi - \beta \circ \alpha)(n'') \\ &= s(n') - s(n'')\end{aligned}$$

as required.

By commutativity, $\phi \circ \pi_N = \pi_M \circ F\phi$ but

$$\begin{aligned}\pi_M \circ F\phi &= \pi_M \circ (s \circ \pi_N + \beta \circ \alpha) \\ &= \pi_M \circ s \circ \pi_N + \pi_M \circ \beta \circ \alpha \\ &= \pi_M \circ s \circ \pi_N + \pi_M \circ \iota_M \circ \phi_2 \circ \alpha \\ &= \pi_M \circ s \circ \pi_N\end{aligned}$$

and since π_N is surjective, we have that $\phi = \pi_M \circ s$; that is, ϕ factors through a projective R-module and so represents 0 in $\underline{\operatorname{Hom}}_R(N, M)$. It follows that the natural map Ω is injective. □

We immediately obtain

Lemma 2.1.15. *Let G be a group, H a subgroup with $[G : H] < \infty$, R a strongly G-graded $\Bbbk$-algebra and M a left R-module. If N is a left R-module which is projective as an R_H-module, then the natural map*

$$\underline{\mathrm{Hom}}_R(N, M) \to \widehat{\mathrm{Ext}}^0_R(N, M)$$

is an isomorphism.

Proof. By repeated use of Proposition 2.1.14, we see that the maps in the system

$$\underline{\mathrm{Hom}}_R(N, M) \xrightarrow{\Omega} \underline{\mathrm{Hom}}_R(\Omega N, \Omega M) \xrightarrow{\Omega} \underline{\mathrm{Hom}}_R(\Omega^2 N, \Omega^2 M) \xrightarrow{\Omega} \cdots$$

are all isomorphisms and so the colimit is equal to $\underline{\mathrm{Hom}}_R(N, M)$. □

Proposition 2.1.16. *Let G be a group, H a subgroup with $[G : H] < \infty$, R a strongly G-graded $\Bbbk$-algebra and M a left R-module. If the left global dimension of R_H is finite and equal to n, then for all $i \geq n$ and all left R-modules N, the natural map*

$$\underline{\mathrm{Hom}}_R(\Omega^i N, \Omega^i M) \to \widehat{\mathrm{Ext}}^0_R(N, M)$$

is an isomorphism.

Proof. Let $i \geq lD(R_H)$, then as an R_H-module, $\Omega^i N$ is projective. We have a commutative diagram

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_R(\Omega^i N, \Omega^i M) \arrow[r] \arrow[dr] & \widehat{\mathrm{Ext}}^0_R(\Omega^i N, \Omega^i M) \\
\underline{\mathrm{Hom}}_R(N, M) \arrow[r] \arrow[u] & \widehat{\mathrm{Ext}}^0_R(N, M) \arrow[u]
\end{tikzcd}$$

The top map is an isomorphism by Proposition 2.1.15 and the right-hand map is an isomorphism because the terms in the system

$$\underline{\mathrm{Hom}}_R(N, M) \xrightarrow{\Omega} \underline{\mathrm{Hom}}_R(\Omega N, \Omega M) \xrightarrow{\Omega} \cdots \xrightarrow{\Omega} \underline{\mathrm{Hom}}_R(\Omega^j N, \Omega^j M) \xrightarrow{\Omega} \cdots$$

agree with those of the limit system

$$\underline{\mathrm{Hom}}_R(\Omega^i N, \Omega^i M) \xrightarrow{\Omega} \underline{\mathrm{Hom}}_R(\Omega^{i+1} N, \Omega^{i+1} M) \xrightarrow{\Omega} \cdots$$

for $j \geq i$ and so their colimits are isomorphic. It follows that the diagonal map is an isomorphism also. □

2.2 Complete cohomology for group-graded $\Bbbk$-algebras

The first lemma of this section generalises Lemma 6.1 of Benson's paper [3]. An obvious approach when generalising any result is to investigate to what extent one can utilise the original argument. Benson's lemma concerns group algebras and his proof involves a somewhat complicated calculation with group elements, something not immediately transferable to the strongly group-graded case. As we shall see, the machinery set up in Chapter 1 allows for a more elegant argument; indeed, it was trying to generalise this result which motivated the careful treatment of H-injective and H-projective modules there.

Lemma 2.2.1. *Let G be a group, H a subgroup with $[G : H] < \infty$ and R a strongly G-graded $\Bbbk$-algebra. Let M and N be left R-modules and let $\gamma : N \to N'$ be an R_H-split monomorphism of left R-modules. If $\phi : N \to M$ factors through some projective module, then it factors through the map γ.*

Proof. Call the projective module that ϕ factors through P. We have the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[rr, "\gamma"] \arrow[dr, "\beta"] \arrow[dd, "\phi"'] & & N' \\
& & P \arrow[dl, "\alpha"] & \\
& M & &
\end{tikzcd}$$

Since P is projective, it is H-projective by Lemma 1.4.4 and H-injective by Lemma 1.4.12 so we can find an R-map $\eta : N' \to P$ with $\beta = \eta \circ \gamma$. It follows that

$$\begin{aligned} \alpha \circ \eta \circ \gamma &= \alpha \circ \beta \\ &= \phi. \end{aligned}$$

□

Remark 2.2.2. Compared to Benson's proof, ours is conceptually more accessible in that we can see exactly what is happening, and with fewer lines of argument. Also, we did not require the hypothesis that N be projective as an R_H-module: we may therefore conclude the hypothesis that N be projective as a $\Bbbk$-module in Lemma 6.1 of [3] is redundant.

We shall have need of the following lemma, which is a variation on well-known results.

Lemma 2.2.3. *Let R be a ring and let*

$$0 \to M_r \to \cdots \to M_1 \to M_0 \to M \to 0$$

be an exact sequence of left R-modules. If $pd_R(M_i) \leq r-i$ for each i then M has projective dimension at most r.

Proof. We proceed by induction on the length of the exact sequence. Let K denote the kernel of the map $M_0 \to M$. Then the exact sequence

$$0 \to M_r \to \cdots \to M_1 \to K \to 0$$

comprises modules M_i, each of whose projective dimension is at most $r - i$ and so by induction, $pd_R(K) \leq r - 1$. To the short exact sequence

$$0 \to K \to M_0 \to M \to 0$$

apply $\mathrm{Hom}_R(-, X)$, where X is any left R-module, to give us the long exact sequence

$$\cdots \to \mathrm{Ext}^r_R(K, X) \to \mathrm{Ext}^{r+1}_R(M, X) \to \mathrm{Ext}^{r+1}_R(M_0, X) \to \cdots .$$

The two outside terms are 0 since $pd_R(M_0) \leq r$ and $pd_R(K) \leq r - 1$, so exactness forces $\mathrm{Ext}^{r+1}_R(M, X) = 0$ for all X and

$$pd_R(M) \leq \max\{1 + pd_R(K), pd_R(M_0)\} \leq r. \qquad \square$$

The following generalises Benson's Proposition 6.3 of [3].

Proposition 2.2.4. *Let G be a group, H a subgroup of G with $[G : H] < \infty$ and R a strongly G-graded $\Bbbk$-algebra. If M and N are left R-modules such that N has projective dimension at most r as a left R_H-module and $\phi : N \to M$ is an R-module homomorphism which goes to zero in $\underline{\mathrm{Hom}}_R(\Omega^r N, \Omega^r M)$, then ϕ factors through a left R-module X of projective dimension at most r. If N and $\Omega^r N$ are finitely generated then X may also be taken to be finitely generated.*

Proof. Our proof closely mirrors that of Proposition 6.3 of [3]. We have occasionally included additional details and we have also used the methodology outlined in Chapter 1 to smooth certain arguments.

Choose resolutions and maps

$$\begin{tikzcd} 0 \arrow[r] & \Omega^r N \arrow[r, "d'_r"] \arrow[d, "\Omega^r(\phi)"] & P'_{r-1} \arrow[r, "d'_{r-1}"] \arrow[d, "\phi_{r-1}"] & \cdots \arrow[r, "d'_1"] & P'_0 \arrow[r, "d'_0"] \arrow[d, "\phi_0"] & N \arrow[r] \arrow[d, "\phi"] & 0 \\ 0 \arrow[r] & \Omega^r M \arrow[r, "d_r"] & P_{r-1} \arrow[r, "d_{r-1}"] & \cdots \arrow[r, "d_1"] & P_0 \arrow[r, "d_0"] & M \arrow[r] & 0 \end{tikzcd}$$

with $P_0, \dots, P_{r-1}, P_0', \dots, P_{r-1}'$ projective left R-modules. Since $\Omega^r N$ is left R_H-projective, we can, by Lemma 1.6.2 find a projective left R_H-module N' and projective left R-modules $P_0'', \dots, P_{r-1}''$ (which are finitely generated if $\Omega^r N$ is) so that there is an R_H-split sequence of R-modules

$$0 \longrightarrow \Omega^r N \xrightarrow{d_r''} P_{r-1}'' \xrightarrow{d_{r-1}''} \cdots \xrightarrow{d_1''} P_0'' \xrightarrow{d_0''} N' \longrightarrow 0.$$

We have the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \Omega^r N \arrow[r, "d_r''"] \arrow[d, equal] & P_{r-1}'' \arrow[r, "d_{r-1}''"] & \cdots \arrow[r, "d_1''"] & P_0'' \arrow[r, "d_0''"] & N' \arrow[r] & 0 \\
0 \arrow[r] & \Omega^r N \arrow[r, "d_r'"] & P_{r-1}' \arrow[r, "d_{r-1}'"] & \cdots \arrow[r, "d_1'"] & P_0' \arrow[r, "d_0'"] & N \arrow[r] & 0.
\end{tikzcd}$$

Because P_{r-1}', is a projective left R-module, we can appeal to Lemma 2.2.1 to find an R-map ψ_{r-1} making the triangle

$$\begin{tikzcd}
0 \arrow[r] & \Omega^r N \arrow[r, "d_r''"] \arrow[d, "d_r'"] & P_{r-1}'' \arrow[dl, dotted, "\exists \psi_{r-1}"] \\
& P_{r-1}' &
\end{tikzcd}$$

commute. In fact it is almost unnecessary to appeal to Lemma 1.6.2 because P_{r-1}' is projective, therefore H-projective and so also H-injective by the results of Chapter 1.

Because "cokernel is functorial," we can find a well-defined map $\psi_{r-1}' : \operatorname{Coker} d_r'' \to \operatorname{Coker} d_r'$. As we observed in the proof of Lemma 1.6.2, the map $\iota'' : \operatorname{Coker} d_r'' \to P_{r-2}''$ is an R_H-split monomorphism, so we can again appeal to Lemma 2.2.1 to find an R-map $\psi_{r-2} : P_{r-2}'' \to P_{r-2}'$.

$$\begin{tikzcd}
& & & \operatorname{Coker} d_r'' \arrow[dr, "\iota''"] \arrow[dd, "\psi_{r-1}'"] \arrow[dddr, "\iota' \circ \psi_{r-1}'"] & & \\
0 \arrow[r] & \Omega^r N \arrow[r, "d_r''"] \arrow[dd, equal] & P_{r-1}'' \arrow[ur, "\widehat{d_r''}"] \arrow[rr, "d_{r-1}''"] \arrow[dd, "\psi_{r-1}"] & & P_{r-2}'' \arrow[r, "d_{r-2}''"] \arrow[dd, dotted, "\psi_{r-2}"] & \cdots \\
& & & \operatorname{Coker} d_r' \arrow[dr, "\iota'"] & & \\
0 \arrow[r] & \Omega^r N \arrow[r, "d_r'"] & P_{r-1}' \arrow[ur, "\widehat{d_r'}"] \arrow[rr, "d_{r-1}'"] & & P_{r-2}' \arrow[r, "d_{r-2}'"] & \cdots
\end{tikzcd}$$

Also,

$$\begin{aligned} d'_{r-1}\circ\psi_{r-1} &= \iota'\circ\widehat{d'_r}\circ\psi_{r-1} \\ &= \iota'\circ\psi'_{r-1}\circ\widehat{d''_r} \\ &= \psi_{r-2}\circ\iota''\circ\widehat{d''_r} \\ &= \psi_{r-2}\circ d''_{r-1} \end{aligned}$$

so the square commutes. We may proceed in this way along the diagram, finding R-maps $\psi_i : P''_i \to P'_i$ for $0 \le i \le r-1$ and $\psi : N' \to N$.

Applying the (exact) functor $\mathrm{Hom}_{R_H}(R,-)$ to the exact sequence $P''_* \to N' \to 0$ gives us an exact sequence of projective R-modules

$$0 \to Q_r \xrightarrow{\widetilde{d''_r}} \cdots \xrightarrow{\widetilde{d''_1}} Q_0 \xrightarrow{\widetilde{d''_0}} Q \to 0$$

where

- $Q_i = \mathrm{Hom}_{R_H}(R, P_i)$ for $0 \le i \le r-1$,
- $Q_r = \mathrm{Hom}_{R_H}(R, \Omega^r N)$
- $Q = \mathrm{Hom}_{R_H}(R, N')$.

and $\widetilde{d''_i} : Q_i \to Q_{i-1}$ is the map induced from d''_i. For each P''_i, we have a natural map $\iota_i : P''_i \to Q_i$ (the inclusion of P''_i into Q_i) so that for each i we have a commutative square

$$\begin{tikzcd} P''_i \arrow[r, "d''_i"] \arrow[d, "\iota_i"] & P''_{i-1} \arrow[d, "\iota_{i-1}"] \\ Q_i \arrow[r, "\widetilde{d''}_i"] & Q_{i-1} \end{tikzcd}$$

Adding our new sequence to $P'_* \to N \to 0$ gives us an exact sequence $P'_* \oplus Q_* \to N \oplus Q \to 0$ if we define $\widetilde{d'_i} : P'_i \oplus Q_i \to P'_{i-1} \oplus Q_{i-1}$ by $(p,q) \mapsto (d'_i(p), \widetilde{d''_i}(q))$. Augmenting the ψ_i by defining $\widetilde{\psi_i} = (\psi_i, \iota_i)$ has the effect of making them injective. We have, for each i, a square

$$\begin{tikzcd} P''_i \arrow[r, "d''_i"] \arrow[d, "\widetilde{\psi}_i"] & P''_{i-1} \arrow[d, "\widetilde{\psi}_{i-1}"] \\ P'_i \oplus Q_i \arrow[r, "\widetilde{d'_i}"] & P'_{i-1} \oplus Q_{i-1} \end{tikzcd}$$

which is commutative, since for $p \in P_i''$,

$$\begin{aligned}
\widetilde{\psi}_{i-1}(d_i''(p)) &= (\psi_{i-1}(d_i''(p)), \iota_{i-1}(d_i''(p))) \\
&= (d_i'(\psi_i(p)), \iota_{i-1}(d_i''(p))) \\
&= (d_i'(\psi_i(p)), \widetilde{d_i''}(\iota_i(p))) \\
&= \widetilde{d_i'}(\psi_i(p), \iota_i(p)) \\
&= \widetilde{d_i'}(\widetilde{\psi}_i(p)).
\end{aligned}$$

If we extend ϕ by the zero map on the new projective summand, these changes do not affect the hypothesis or the conclusion of the theorem. (We shall drop the tilde on each of the $\widetilde{\psi}_i$ from this point.)

Turning our commutative diagram upside down

$$\begin{tikzcd}
0 \arrow[r] & \Omega^r(N) \oplus Q_r \arrow[r, "d_r'"] & P_{r-1}' \oplus Q_{r-1} \arrow[r, "d_{r-1}'"] & \cdots \arrow[r, "d_1'"] & P_0' \oplus Q_0 \arrow[r, "d_0'"] & N \oplus Q \arrow[r] & 0 \\
0 \arrow[r] & \Omega^r N \arrow[r, "d_r''"] \arrow[u, "\psi_r"] & P_{r-1}'' \arrow[r, "d_{r-1}''"] \arrow[u, "\psi_{r-1}"] & \cdots \arrow[r, "d_1''"] & P_0'' \arrow[r, "d_0''"] \arrow[u, "\psi_0"] & N' \arrow[r] \arrow[u, "\psi"] & 0
\end{tikzcd}$$

and thinking of it as lying in a sea of zeroes, we may regard it as a double complex and therefore as a first quadrant spectral sequence. The differentials on the E^0-page are the vertical maps. Remembering that the ψ_i are injective, on the E^1-page we get:

$$\operatorname{Coker}\psi_r \xrightarrow{d_r'} \operatorname{Coker}\psi_{r-1} \xrightarrow{d_{r-1}'} \cdots \xrightarrow{d_1'} \operatorname{Coker}\psi_0 \xrightarrow{d_0'} \operatorname{Coker}\psi.$$

On the E^2-page there are no relevant differentials, so we are looking at E^∞.

Reflecting the initial diagram in the diagonal gives rise to a second first quadrant spectral sequence, so that the $\widehat{E}^0$-page looks as follows:

$$\begin{tikzcd}
N' \arrow[r, "\psi"] & N \\
P_0'' \arrow[r, "\psi_0"] \arrow[u, "d_0''"] & P_0' \oplus Q \arrow[u, "d_0'"] \\
\vdots \arrow[u, "d_1''"] & \vdots \arrow[u, "d_1'"] \\
P_{r-1}'' \arrow[r, "\psi_{r-1}"] \arrow[u, "d_{r-1}''"] & P_{r-1}' \oplus Q_{r-1} \arrow[u, "d_{r-1}'"] \\
\Omega^r N \arrow[r, "\psi_r"] \arrow[u, "d_r''"] & \Omega^r(N) \oplus Q_r \arrow[u, "d_r'"]
\end{tikzcd}$$

The differentials on $\widehat{E}^0$ are the vertical maps, but the columns are exact sequences and so the spectral sequence collapses: the $\widehat{E}^1$-page consists entirely of zeroes. Since the two spectral sequences converge to the cohomology of the total complex, it follows that the original spectral sequence must have collapsed to zero as well and we conclude that we had an exact sequence

$$0 \to \operatorname{Coker}\psi_r \xrightarrow{d'_r} \operatorname{Coker}\psi_{r-1} \xrightarrow{d'_{r-1}} \cdots \xrightarrow{d'_1} \operatorname{Coker}\psi_0 \xrightarrow{d'_0} \operatorname{Coker}\psi \to 0;$$

that is, $\operatorname{Coker}\psi$ admits a resolution by modules of projective dimension at most 1 for $0 \leq i \leq r-1$, ending with a projective module at $i = r$. By Lemma 2.2.3, $\operatorname{Coker}\psi$ has projective dimension at most r.

By hypothesis, $\Omega^r(\phi)$ factors through a projective module and so by Lemma 2.2.1 there exists a map $h_{r-1} : P''_{r-1} \to \Omega^r M$ satisfying $\Omega^r(\phi) = h_{r-1} \circ d''_r$. We have the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \Omega^r N \arrow[r, "d''_r"] \arrow[d, "\Omega^r(\phi)"'] & P''_{r-1} \arrow[r, "d''_{r-1}"] \arrow[d, "\phi_{r-1}\circ\psi_{r-1}"] \arrow[dl, "h_{r-1}"] & \cdots \\
0 \arrow[r] & \Omega^r M \arrow[r, "d_r"'] & P_{r-1} \arrow[r, "d_{r-1}"'] & \cdots
\end{tikzcd}$$

By commutativity of the square,

$$\begin{aligned}
\phi_{r-1} \circ \psi_{r-1} \circ d''_r &= d_r \circ \Omega^r(\phi) \\
&= d_r \circ h_{r-1} \circ d''_r
\end{aligned}$$

and so if we define $\delta_{r-1} := \phi_{r-1} \circ \psi_{r-1} - d_r \circ h_{r-1}$, then $\operatorname{Im} d''_r \subseteq \operatorname{Ker}\delta_{r-1}$ because $\delta_{r-1} \circ d''_r = 0$. Thus we have the following:

$$\begin{tikzcd}
 & \operatorname{Im} d''_{r-1} \arrow[dr, hook, "\iota"] \arrow[ddl, dotted, "g_{r-1}"] & \\
P''_{r-1} \arrow[ur, two heads, "d''_{r-1}"] \arrow[rr, "d''_{r-1}"] \arrow[d, "\delta_{r-1}"'] & & P''_{r-2} \arrow[dll, dotted, "h_{r-2}"] \\
P_{r-1} & &
\end{tikzcd}$$

where ι is the inclusion of $\operatorname{Im} d''_{r-1}$ into P''_{r-2}. The map $g_{r-1} : \operatorname{Im} d''_{r-1} \to P_{r-1}$ is defined by $g_{r-1}(\alpha) = \delta_{r-1}(\beta)$ where β is a pre-image of α, which is well-defined: if γ is another pre-image of α, then $\beta - \gamma \in \operatorname{Ker} d''_{r-1} \subseteq \operatorname{Ker}\delta_{r-1}$ and so $\delta_{r-1}(\beta) = \delta_{r-1}(\gamma)$.

Because ι is a split monomorphism, Lemma 2.2.1 again provides a map $h_{r-2} : P''_{r-2} \to P_{r-1}$ satisfying $g_{r-1} = h_{r-2} \circ \iota$, and so

$$\begin{aligned} h_{r-2} \circ d''_{r-1} &= h_{r-2} \circ \iota \circ d''_{r-1} \\ &= g_{r-1} \circ d''_{r-1} \\ &= \delta_{r-1}. \end{aligned}$$

In particular, this means that $\phi_{r-1} \circ \psi_{r-1} = h_{r-2} \circ d''_{r-1} + d_r \circ h_{r-1}$, which by induction we can extend to

$$\phi_i \circ \psi_i = h_{i-1} \circ d''_i + d_{i+1} \circ h_i$$

for all $0 \leq i \leq r-1$.

We have the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega^r N \arrow[r, "d''_r"] \arrow[d, "\psi_r"] & P''_{r-1} \arrow[r, "d''_{r-1}"] \arrow[d, "\psi_{r-1}"] \arrow[ddl, dotted, "h_{r-1}"'] & \cdots \arrow[r, "d''_1"] & P''_0 \arrow[r, "d''_0"] \arrow[d, "\psi_0"] \arrow[ddl, dotted, "h_0"'] & N' \arrow[r] \arrow[d, "\psi"] \arrow[ddl, dotted, "h_{-1}"'] & 0 \\
0 \arrow[r] & \Omega^r(N) \oplus Q_r \arrow[r, "d'_r"] \arrow[d, "\Omega^r(\phi)"] & P'_{r-1} \oplus Q_{r-1} \arrow[r, "d'_{r-1}"] \arrow[d, "\phi_{r-1}"] & \cdots \arrow[r, "d'_1"] & P'_0 \oplus Q_0 \arrow[r, "d'_0"] \arrow[d, "\phi_0"] & N \oplus Q \arrow[r] \arrow[d, "\phi"] & 0 \\
0 \arrow[r] & \Omega^r M \arrow[r, "d_r"] & P_{r-1} \arrow[r, "d_{r-1}"] & \cdots \arrow[r, "d_1"] & P_0 \arrow[r, "d_0"] & M \arrow[r] & 0
\end{tikzcd}$$

in which

$$\begin{aligned} \phi \circ \psi \circ d''_0 &= d_0 \circ \phi_0 \circ \psi_0 \\ &= d_0 \circ (h_{-1} \circ d''_0 + d_1 \circ h_0) \\ &= d_0 \circ h_{-1} \circ d''_0. \end{aligned}$$

Since d''_0 is surjective, it follows that $\phi \circ \psi = d_0 \circ h_{-1}$.

Consider the pushout of the maps ψ and h_{-1}:

$$\begin{tikzcd}
N' \arrow[r, "h_{-1}"] \arrow[d, "\psi"] & P_0 \arrow[d, "\theta"] \arrow[ddr, "d_0"] & \\
N \oplus Q \arrow[r, "\eta"] \arrow[drr, "\phi"'] & X \arrow[dr, dotted, "\omega"] & \\
& & M
\end{tikzcd}$$

Since ψ is injective we have, by the properties of pushout, that θ is injective and $\phi = \omega \circ \eta$.

Look at the following diagram:

$$
\begin{tikzcd}
N' \arrow[r, tail, "h_{-1}"] \arrow[d, tail, "\psi"] & P_0 \arrow[d, tail, "\theta"] \\
N \oplus Q \arrow[r, "\eta"] \arrow[d, two heads] & X \arrow[d, two heads] \\
\operatorname{Coker} \psi \arrow[r, equal] & \operatorname{Coker} \theta
\end{tikzcd}
$$

Equality in the bottom row follows from exactness of the columns. Since $\operatorname{Coker} \psi$ has projective dimension at most r and P_0 is projective, we may use the Horseshoe Lemma to conclude X has projective dimension at most r.

If N and $\Omega^r N$ are finitely generated, then X may be taken to be finitely generated. For, let us suppose P_0 was chosen to be free at the beginning. Since N' is finitely generated, P_0 contains a finitely generated projective summand F containing the image of h_{-1}. If we replace P_0 by F and let X be the pushout of the maps $h_{-1} : N' \to F$ and ψ, then X is finitely generated because F, N and Q are. From here, we may proceed as before. $\square$

Remark 2.2.5. We could have avoided using a spectral sequence argument via the following reasoning:

Let

$$
\begin{tikzcd}
0 \arrow[r] & \Omega^r N \arrow[r, "d''_r"] \arrow[d, "\psi_r"] & P''_{r-1} \arrow[r, "d''_{r-1}"] \arrow[d, "\psi_{r-1}"] & \cdots \arrow[r, "d''_1"] & P''_0 \arrow[r, "d''_0"] \arrow[d, "\psi_0"] & N' \arrow[r] \arrow[d, "\psi"] & 0 \\
0 \arrow[r] & \Omega^r(N) \oplus Q_r \arrow[r, "d'_r"] & P'_{r-1} \oplus Q_{r-1} \arrow[r, "d'_{r-1}"] & \cdots \arrow[r, "d'_1"] & P'_0 \oplus Q_0 \arrow[r, "d'_0"] & N \oplus Q \arrow[r] & 0
\end{tikzcd}
$$

be our original diagram. If we add the cokernels:

$$
\begin{tikzcd}
0 \arrow[r] & \Omega^r N \arrow[r, "d''_r"] \arrow[d, "\psi_r"] & P''_{r-1} \arrow[r, "d''_{r-1}"] \arrow[d, "\psi_{r-1}"] & \cdots \arrow[r, "d''_1"] & P''_0 \arrow[r, "d''_0"] \arrow[d, "\psi_0"] & N' \arrow[r] \arrow[d, "\psi"] & 0 \\
0 \arrow[r] & \Omega^r(N) \oplus Q_r \arrow[r, "d'_r"] \arrow[d] & P'_{r-1} \oplus Q_{r-1} \arrow[r, "d'_{r-1}"] \arrow[d] & \cdots \arrow[r, "d'_1"] & P'_0 \oplus Q_0 \arrow[r, "d'_0"] \arrow[d] & N \oplus Q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{Coker} \psi_r \arrow[r, "\widetilde{d_r}"] & \operatorname{Coker} \psi_{r-1} \arrow[r, "\widetilde{d_{r-1}}"] & \cdots \arrow[r, "\widetilde{d_0}"] & \operatorname{Coker} \psi_0 \arrow[r, "\widetilde{d_0}"] & \operatorname{Coker} \psi \arrow[r] & 0
\end{tikzcd}
$$

bearing in mind that each of the ψ_i are injective, we see that we have a short exact sequence of chain complexes. If we call the top sequence $\mathbf{C}''_*$, the middle $\mathbf{C}_*$ and the bottom $\mathbf{C}'_*$, we get a long exact sequence

$$\begin{tikzcd}[column sep=small] \cdots \arrow[r] & H_{n+1}(\mathbf{C}'_*) \arrow[dr] & & & \\ & & H_n(\mathbf{C}''_*) \arrow[r] & H_n(\mathbf{C}_*) \arrow[r] & H_n(\mathbf{C}'_*) \arrow[dr] & \\ & & & & & H_{n-1}(\mathbf{C}''_*) \arrow[r] & \cdots \end{tikzcd}$$

in homology, which is actually equal to

$$\begin{tikzcd}[column sep=small] \cdots \arrow[r] & H_{n+1}(\mathbf{C}'_*) \arrow[dr] & & & \\ & & 0 \arrow[r] & 0 \arrow[r] & H_n(\mathbf{C}'_*) \arrow[dr] & \\ & & & & & 0 \arrow[r] & \cdots \end{tikzcd}$$

since $\mathbf{C}''_*$ and $\mathbf{C}_*$ are exact sequences. But this means that $H_n(\mathbf{C}'_*) = 0$ for all n and so $\mathbf{C}'_*$ is an exact sequence also.

Lemma 2.2.6. *Let R be a ring and M be a countably presented R-module. If M is flat then it has projective dimension at most one.*

Proof. See Lemma 4.4 of Bieri [6]. $\square$

Proposition 2.2.7. *Let G be a group, H a subgroup of G with $[G : H] < \infty$ and R a strongly G-graded $\Bbbk$-algebra. If M is a countably presented R-module which is projective as an R_H-module and flat as an R-module, then M is projective as an R-module.*

Proof. By Lemma 2.2.6, M has projective dimension at most one. Let

$$0 \to P_1 \xrightarrow{\alpha} P_0 \xrightarrow{\beta} M \to 0$$

be a projective resolution of M. Since M is projective as an R_H-module, we can find an R_H-map γ making the diagram

$$\begin{tikzcd} & M \arrow[dl, dotted, "\exists\gamma"'] \arrow[d, equal] & \\ P_0 \arrow[r, "\beta"] & M \arrow[r] & 0 \end{tikzcd}$$

commute and so the resolution splits as a sequence of R_H-modules. Since P_1 is projective, it is H-projective by Lemma 1.4.4 and H-injective by Lemma 1.4.12. Therefore we can find an R-map γ' making the diagram

$$\begin{tikzcd} 0 \arrow[r] & P_1 \arrow[r, "\alpha"] \arrow[d, equal] & P_0 \arrow[dl, dotted, "\exists\gamma'"] \\ & P_1 & \end{tikzcd}$$

commute; that is, the resolution splits as a sequence of R-modules, $P_0 \cong M \oplus P_1$ and so M is projective. $\square$

Proposition 2.2.8. *Let R be a ring for which the left finitistic dimension is finite. Then any flat left R-module has finite projective dimension.*

Proof. This is Proposition 6 of [19]. $\square$

Theorem 2.2.9. *Let G be a group, H a subgroup of G with $[G : H] < \infty$ and R a strongly G-graded $\Bbbk$-algebra. Suppose R_H is left Noetherian and has left global dimension at most r. Then the following are equivalent for a left R-module M:*

(i) M has finite projective dimension.

(ii) M has finite weak dimension.

(iii) M is a filtered colimit of finitely generated left R-modules of finite projective dimension.

(iv) For every finitely generated left R-module N we have $\widehat{\mathrm{Ext}}^0_R(N, M) = 0$.

Proof. (i) $\Rightarrow$ (iv): If M has projective dimension at most n, then $\Omega^n M$ is projective, so any homomorphism from $\Omega^n N$ to $\Omega^n M$ factors through a projective module, $\underline{\mathrm{Hom}}_R(\Omega^i N, \Omega^i M) = 0$ for all $i \geq n$ and so $\varinjlim_i \underline{\mathrm{Hom}}_R(\Omega^i N, \Omega^i M) = 0$.

(iv) $\Rightarrow$ (iii): Let I be a small skeleton of the category whose objects are the homomorphisms $N \xrightarrow{\alpha} M$ with N finitely generated and whose morphisms are the commutative triangles; i.e. for objects $N \xrightarrow{\alpha} M$, $N' \xrightarrow{\beta} M$, a morphism from α to β is an R-map γ making the diagram

$$\begin{tikzcd} N \arrow[dr, "\alpha"] \arrow[dd, "\gamma"'] & \\ & M \\ N' \arrow[ur, "\beta"'] & \end{tikzcd}$$

commute.

We show that I is filtered. Given two objects $N \xrightarrow{\alpha} M$ and $N' \xrightarrow{\beta} M$, we can build the commutative diagram

$$\begin{tikzcd} N \arrow[d, "\iota"'] \arrow[dr, "\alpha"] & \\ N \oplus N' \arrow[r, "(\alpha,\beta)"] & M \\ N' \arrow[u, "\iota'"] \arrow[ur, "\beta"'] & \end{tikzcd}$$

where ι, ι' are respectively the inclusions of N, N' into $N \oplus N'$. It follows that the first condition is satisfied.

For the second condition in the definition of a filtered category, suppose we have two morphisms from α to β:

$$\begin{tikzcd} N \arrow[d, bend right, "\phi"'] \arrow[d, bend left, "\psi"] \arrow[dr, "\alpha"] & \\ N' \arrow[r, "\beta"'] & M \end{tikzcd}$$

Taking the cokernel of $\phi - \psi$, we naturally obtain a commutative diagram

$$\begin{tikzcd} N \arrow[d, bend right, "\phi"'] \arrow[d, bend left, "\psi"] \arrow[dr, "\alpha"] & \\ N' \arrow[r, "\beta"] \arrow[d, "\pi"'] & M \\ \operatorname{Coker}(\phi - \psi) \arrow[ur, "\alpha_*"'] & \end{tikzcd}$$

Now, $0 = \pi \circ (\phi - \psi) = \pi \circ \phi - \pi \circ \psi$; that is, $\pi \circ \phi = \pi \circ \psi$ so the second condition is satisfied.

Let J be the full subcategory of I whose objects are the morphisms $N \xrightarrow{\alpha} M$ with N a finitely generated left R-module of projective dimension at most r. We show that J is cofinal in I; that is, every every object of I is the domain of a morphism whose codomain is an object of J. Let $N \xrightarrow{\alpha} M$ be an object of I. Because R_H is left Noetherian, we have by Lemma 1.1.14 that R is Notherian and so $\Omega^r N$ is finitely generated if N is. $\widehat{\operatorname{Ext}}^0_R(N, M) = 0$ for all finitely generated left R-modules N by hypothesis, so by Proposition 2.1.16, α goes to zero in $\underline{\operatorname{Hom}}_R(\Omega^r N, \Omega^r M)$ where r is the global dimension of R_H, and Proposition 2.2.4 tells us that α factors through a finitely generated left R-module

X of projective dimension at most r. We have a commutative diagram

$$\begin{tikzcd} N \arrow[dd, "\gamma"'] \arrow[dr, "\alpha"] & \\ & M \\ X \arrow[ur, "\beta"] & \end{tikzcd}$$

Viewing $X \xrightarrow{\beta} M$ as an object of J and γ as a morphism between α and β, we see that since the object $N \xrightarrow{\alpha} M$ of I was arbitrary, J is a cofinal subcategory. The module M is the filtered colimit of the functor from I to the module category which sends an object $N \xrightarrow{\alpha} M$ to N. It follows that M is also the filtered colimit of the same functor on the subcategory J; that is, M is a filtered colimit of finitely generated left R-modules of finite projective dimension.

(iii) $\Rightarrow$ (ii): Let $M = \varinjlim_{\lambda} M_{\lambda}$, a colimit of finitely generated left R-modules of finite projective dimension. It follows from Lemma 1.5.21 and the fact that R_H has finite global dimension that R has finite finitistic dimension since for any left R-module A of finite projective dimension,

$$pd_R(A) = pd_{R_H}(A) \leq r.$$

Each M_{λ} has finite projective dimension by hypothesis, so the weak dimension of each M_{λ} is finite (see Example 1.5.14). Let $n := \max\{wd_R(M_{\lambda})\}$. It follows by Lemma 1.5.16 and Example 2.1.10 that

$$\begin{aligned} \mathrm{Tor}_n^R(R, M) &= \mathrm{Tor}_n^R(R, \varinjlim_{\lambda} M_{\lambda}) \\ &= \varinjlim_{\lambda} \mathrm{Tor}_n^R(R, M_{\lambda}) \\ &= 0; \end{aligned}$$

that is, M has finite weak dimension.

(ii) $\Rightarrow$ (i): Since M has finite weak dimension, it admits a finite resolution

$$0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$$

by flat R-modules. As was noted above, R has finite finitistic dimension so that by Proposition 2.2.8, each F_i in the resolution has finite projective dimension. It follows by Lemma 1.5.10 that M has finite projective dimension also. $\square$

Corollary 2.2.10. *Let G be a group, H a subgroup of G with $[G : H] < \infty$ and R a strongly G-graded $\Bbbk$-algebra. Suppose R_H is left Noetherian and has finite left global dimension. If M is a left R-module which does not have finite projective dimension, then there exists a finitely generated left R-module N such that $\widehat{\operatorname{Ext}}^0_R(N, M) \neq 0$.*

2.3 $\mathbf{LH}\mathfrak{F}$ groups

In this short section we recall the definition of Kropholler's class of $\mathbf{LH}\mathfrak{F}$-groups.

Given the class $\mathfrak{F}$ of finite groups, define a class of groups $\mathbf{H}_\alpha\mathfrak{F}$ for each ordinal α as follows. For $\alpha = 0$ we define $\mathbf{H}_0\mathfrak{F} = \mathfrak{F}$. For $\alpha > 0$ we define a group G to be in $\mathbf{H}_\alpha\mathfrak{F}$ if G acts cellularly on a finite-dimensional contractible CW-complex X, in such a way that the setwise stabiliser of each cell is equal to the pointwise stabiliser and is in $\mathbf{H}_\beta\mathfrak{F}$ for some $\beta < \alpha$. The class $\mathbf{H}\mathfrak{F}$ is then defined to be the union of all the $\mathbf{H}_\alpha\mathfrak{F}$. A group G is said to be an $\mathbf{LH}\mathfrak{F}$-group if every finitely generated subgroup of G is an $\mathbf{H}\mathfrak{F}$-group.

The following useful observation appears in [23].

Lemma 2.3.1. *If G is a countable group whose finitely generated subgroups belong to $\mathbf{H}\mathfrak{F}$ then G is in $\mathbf{H}\mathfrak{F}$.*

The interested reader can find a detailed discussion of the class of $\mathbf{LH}\mathfrak{F}$ groups and indeed the closure operations $\mathbf{H}$ and $\mathbf{L}$ in Kropholler's paper [22]. We will use the definition of $\mathbf{LH}\mathfrak{F}$-groups throughout the remainder of this thesis without further comment.

2.4 Induced modules

Chouinard's theorem states that if $\Bbbk$ is a commutative ring of coefficients and G is a finite group, then a $\Bbbk G$-module is projective if and only if its restriction to every elementary abelian subgroup is projective. The result was subsequently generalised to the strongly group-graded case by Aljadeff and Ginosar in [1]. In this short section we will apply their theorem to generalise a result of Benson.

Here is a useful observation:

Lemma 2.4.1. *Let G be a group and R a strongly G-graded $\Bbbk$-algebra. Then*

(i) If V is projective over $\Bbbk$ then $V \otimes -$ carries projective R-modules to projective R-modules;

(ii) if V is free over $\Bbbk$, then $V \otimes -$ carries free R-modules to free R-modules.

Proof. This is Corollary 3.3 of [10]. □

The following lemmata were first proved in [24]. The first result is simply proved so we reproduce the details.

Lemma 2.4.2. *Let G be a group and R a strongly G-graded $\Bbbk$-algebra. If M is a finitely presented left R-module, then there is a finitely generated subgroup $H \leq G$ and a finitely presented R_H-module M' such that M is isomorphic to the induced module $R \otimes_{R_H} M'$.*

Proof. Let

$$(R)^r \to (R)^d \to M \to 0$$

be a finite presentation for M. Then the the first map in this sequence is represented by a $d \times r$ matrix with entries from R. Each entry of this matrix is equal to a (unique) finite sum of non-zero homogeneous elements, each coming from some $R_g, g \in G$. Since the number of such homogeneous elements is finite, the number of $g \in G$ is finite. Let H be the subgroup of G generated by these elements. We can use the same matrix over H to present a module $M' : (R_H)^r \to (R_H)^d \to M' \to 0$. Applying $R \otimes_{R_H} -$ to this sequence gives a presentation $(R)^r \to (R)^d \to R \otimes_{R_H} M' \to 0$ since induction is right exact. Exactness of the rows and equality of the first two columns of

$$\begin{tikzcd}
(R)^r \arrow[r] \arrow[d, equal] & (R)^d \arrow[r] \arrow[d, equal] & R \otimes_{R_H} M' \arrow[r] & 0 \\
(R)^r \arrow[r] & (R)^d \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

imply that M is isomorphic to $R \otimes_{R_H} M'$. □

Lemma 2.4.3. *Let G be a group, H a subgroup of G, R a strongly G-graded $\Bbbk$-algebra and M a left R_H-module. Then $R \otimes_{R_H} M$ is a left R-module of type FP_∞ if and only if M is a left R_H-module of type FP_∞.*

Proposition 2.4.4. *Let G be a group and R a strongly G-graded $\Bbbk$-algebra. If M is an R-module of type FP_∞, then there is a finitely generated subgroup $H \leq G$ and an R_H-module M' of type FP_∞ such that M is isomorphic to the induced module $R \otimes_{R_H} M'$.*

Proof. Every module of type FP_∞ is finitely presented. Now apply Lemma 2.4.2 then Lemma 2.4.3. □

Proposition 2.4.5. *Let G be a group, R a strongly G-graded $\Bbbk$-algebra and M a countably presented left R-module. Then there is a countable subgroup $H \leq G$ and a countably presented left R_H-module M' such that M is isomorphic to the induced module $R \otimes_{R_H} M'$.*

Proof. Let

$$A \xrightarrow{\alpha} B \to M \to 0$$

be a countable presentation for M and let $\{a_1, a_2, \ldots\}$ be a fixed basis for A and $\{b_1, b_2, \ldots\}$ a fixed basis of B. For each $n \in \mathbb{N}$, define $A_n = \oplus_{i=1}^{n} Ra_i$ and $B_n = \oplus_{i=1}^{n} Rb_i$. Then $A = \varinjlim_i A_i$ and $B = \varinjlim_i B_i$.

For each $i \in \mathbb{N}$, there exists $\alpha(i) \in \mathbb{N}$ such that $\alpha(A_i) \subseteq B_{\alpha(i)}$. Let α_i denote the restriction of α to A_i. Then for each $i \in \mathbb{N}$, we have maps

$$A_i \xrightarrow{\alpha_i} B_{\alpha(i)}$$

in which the map α_i can be expressed as an $\alpha(i) \times i$ matrix with entries from R. Fix i. As in the previous theorem, we can find a finitely generated subgroup H_i of G such that all the entries in the matrix associated to α_i come from R_{H_i}. Thus we have a chain

$$H_1 < H_2 < \ldots$$

of subgroups of G. Let $H := \varinjlim_i H_i$. Then H is a countably generated subgroup of G and the matrix associated to each α_i can be thought of as having entries from R_H. Let L_i be the left R_H-module presented by α_i, then we have an exact sequence

$$A_i \xrightarrow{\alpha_i} B_{\alpha(i)} \to L_i \to 0$$

and we see that L_i is a finitely presented left R_H-module. It follows that we have an exact sequence of colimit systems:

$$(A_i) \xrightarrow{\alpha_i} (B_{\alpha(i)}) \to (L_i) \to 0.$$

Taking the colimit of the columns, we get an exact sequence

$$A \xrightarrow{\alpha} B \to \varinjlim_i L_i \to 0,$$

and so $M' := \varinjlim_i L_i$ is a countably presented left R_H-module. Since $R \otimes_{R_H} -$ is a right exact functor, we see that $M \cong R \otimes_{R_H} M'$, as required. □

The following definition requires us to recall the module B, introduced in Example 1.2.9.

Definition 2.4.6. Let G be a group and R a strongly G-graded $\Bbbk$-algebra. Left R-modules M for which $B \otimes M$ is projective as a left R-module (with the semi-diagonal action of R) are said to be ***cofibrant***.

Proposition 2.4.7. *Let G be a group and R a strongly G-graded $\Bbbk$-algebra. Suppose that M is a cofibrant left R-module. If M has finite projective dimension then M is projective.*

Proof. We have a split short exact sequence of $\Bbbk$-modules:

$$0 \to \Bbbk \to B \to \overline{B} \to 0.$$

Tensoring this with M over $\Bbbk$ yields the short exact sequence

$$0 \to M \to B \otimes M \to \overline{B} \otimes M \to 0.$$

$\overline{B}$ is a free $\Bbbk$-module (see Lemma 5.1 of [4]) and so by Lemma 2.4.1, $\overline{B} \otimes P$ is a projective left R-module for any projective left R-module P. On tensoring a projective R-resolution of M with $\overline{B}$, we see that $pd_R(\overline{B} \otimes M) \leq pd_R(M)$. It follows, by Lemma 1.5.9, that $0 = pd_R(B \otimes M) = \max\{pd_R(M), pd_R(\overline{B} \otimes M)\}$. That is, M is a projective R-module. □

Lemma 2.4.8. *Suppose that Γ_α $(\alpha < \gamma)$ is an ascending chain of groups, for some ordinal γ, with union $\bigcup_{\alpha<\gamma} \Gamma_\alpha = \Gamma$ and R is a strongly Γ-graded $\Bbbk$-algebra. If M is a left R-module which is projective as a left R_{Γ_α}-module for each subgroup Γ_α, then M has projective dimension at most one.*

Proof. Define $\Gamma_\gamma := \Gamma$. We may assume γ is a limit ordinal for if it were a successor ordinal, then there would be a greatest ordinal β less than γ and so $\Gamma = \Gamma_{\beta+1}$. We may also assume that the union $\Gamma = \bigcup_{\alpha<\gamma} \Gamma_\alpha$ is continuous in the sense that if $\beta < \gamma$ and β is a limit ordinal, then $\Gamma_\beta = \bigcup_{\alpha<\beta} \Gamma_\alpha$.

Since M is projective as a left R_1-module, $P = R \otimes_{R_1} M$ is a projective R-module (by Lemma 1.5.20) and the map from P to M is a surjective homomorphism. For each of the subgroups Γ_α, let $Q_\alpha := R \otimes_{R_{\Gamma_\alpha}} M$ and let P_α be defined as the kernel of the natural surjective map from P to Q_α. Thus for each $\alpha < \gamma$, we have a short exact sequence

$$0 \to P_\alpha \to P \to Q_\alpha \to 0$$

in which Q_α is projective. Define $Q_\gamma := M$.

If $\alpha < \beta < \gamma$, then $Q_\beta \leq Q_\alpha$ and $P_\alpha \leq P_\beta$. It follows that the surjective map $Q_\alpha \to Q_\beta$ splits and its kernel K is projective. We have the following diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & K \arrow[d] & \\
0 \arrow[r] & P_\alpha \arrow[r] \arrow[d] & P \arrow[r] \arrow[d, equal] & Q_\alpha \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_\beta \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & Q_\beta \arrow[r] \arrow[d] & 0 \\
 & P_\beta/P_\alpha & 0 & 0 &
\end{tikzcd}$$

and so by The Snake Lemma, $K \cong P_\beta/P_\alpha$. Thus the short exact sequence $0 \to P_\alpha \to P_\beta \to P_\beta/P_\alpha \to 0$ splits and $P_\beta = P_\alpha \oplus (P_\beta/P_\alpha)$. Furthermore, if $\beta < \gamma$ and β is a limit ordinal, then $P_\beta = \bigcup_{\alpha<\beta} P_\alpha$. It follows that the kernel P_γ of $P \to M$ has a transfinite filtration by projective modules.

We have a colimit system

$$\varinjlim_{\alpha<\gamma}(0 \to P_\alpha \to P \to Q_\alpha)$$

of short exact sequences which is equal to

$$0 \to \varinjlim_{\alpha<\gamma} P_\alpha \to P \to M \to 0$$

because $\varinjlim$ is left exact and $\varinjlim_{\alpha<\gamma} Q_\alpha = M$, by the above discussion. Thus it remains to show $\varinjlim_{\alpha<\gamma} P_\alpha$ $(= \varinjlim P_\alpha)$ is a projective R-module. We will do this by showing $\mathrm{Hom}_R(\varinjlim P_\alpha, -)$ is an exact functor.

Given a surjective map $U \xrightarrow{\pi} V \to 0$, we have the following diagram

$$\begin{tikzcd}
 & P_\alpha \arrow[d] \arrow[ddl, "\hat{\theta}_\alpha"'] \arrow[dd, bend left, "\theta_\alpha"] & \\
 & \varinjlim P_\alpha \arrow[d, "\theta"] \arrow[dl, dotted] & \\
U \arrow[r, "\pi"] & V \arrow[r] & 0
\end{tikzcd}$$

where $\hat{\theta}_\alpha$ is the map obtained from θ_α by projectivity of P_α. We want to show that we can always find a map $\varinjlim P_\alpha \to U$ making the bottom triangle commute in such a way that $\hat{\theta}_\beta|_{P_\alpha} = \hat{\theta}_\alpha$ for all $\alpha < \beta < \gamma$.

First choose $\hat{\theta}_0$. Then, for the choice of $\hat{\theta}_\alpha$, we have two options:

- α is a limit ordinal.

  Define

$$\hat{\theta}_\alpha := \bigcup_{\alpha' < \alpha} \hat{\theta}_{\alpha'}.$$

- α is a successor ordinal.

$$\begin{tikzcd}
 & & P_\alpha \arrow[d, tail, "\iota_\alpha"] \arrow[dddll, bend right=40] \arrow[dddl, "\hat{\theta}_\alpha"'] \arrow[ddd, bend left=70, "\theta_\alpha"] & \\
 & & P_{\alpha+1} \arrow[d] \arrow[ddl, "\phi"'] \arrow[dd, bend left=40, "\theta_{\alpha+1}"] & \\
 & & \varinjlim P_\alpha \arrow[dl] \arrow[d, "\theta"] & \\
W \arrow[r, tail] & U \arrow[r, "\pi"] & V \arrow[r] & 0.
\end{tikzcd}$$

First choose any map $\phi : P_{\alpha+1} \to U$ such that $\pi \circ \phi = \theta_{\alpha+1}$. On P_α, look at the following composite:

$$\begin{aligned}
\pi \circ (\hat{\theta}_\alpha - \phi \circ \iota_\alpha) &= \pi \circ \hat{\theta}_\alpha - \pi \circ \phi \circ \iota_\alpha \\
&= \theta_\alpha - \theta_{\alpha+1} \circ \iota_\alpha \\
&= \theta_\alpha - \theta_\alpha \\
&= 0.
\end{aligned}$$

Thus $\hat{\theta}_\alpha - \phi \circ \iota_\alpha$ factors through $W := \operatorname{Ker} \pi$.

We can extend $\hat{\theta}_\alpha - \phi \circ \iota_\alpha$ to a map $\psi : P_{\alpha+1} \to W$ (because $P_{\alpha+1} = P_\alpha \oplus (P_{\alpha+1}/P_\alpha)$, as we saw above). Consider $\phi + \psi : P_{\alpha+1} \to U$:

$$\begin{aligned}
(\phi + \psi)|_{P_\alpha} &= \phi \circ \iota_\alpha - \phi \circ \iota_\alpha + \hat{\theta}_\alpha \\
&= \hat{\theta}_\alpha.
\end{aligned}$$

Define $\hat{\theta}_{\alpha+1} := \phi + \psi$. The result follows. □

Here is a Lemma and a Corollary due to Cornick and Kropholler:

Lemma 2.4.9 (The tensor identity for strongly graded algebras). *If G is a group, H a subgroup of G and R a strongly G-graded $\Bbbk$-algebra, then for every left R-module M, there is an isomorphism*

$$R \otimes_{R_H} M \cong \Bbbk[G/H] \otimes M; r \otimes m \mapsto gH \otimes rm$$

for r of degree g. $R \otimes_{R_H} M$ has the action of R on the left and $\Bbbk[G/H] \otimes M$ has the semi-diagonal action of R.

Proof. This is Lemma 5.1 of [10]. □

Corollary 2.4.10. *If G is a group, H a subgroup of G and R a strongly G-graded $\Bbbk$-algebra, then for every left R-module M, there is an isomorphism*

$$\mathrm{Hom}_{\Bbbk}(\Bbbk[G/H], M) \cong \mathrm{Hom}_{R_H}(R, M); \phi \mapsto (r \mapsto r(\phi(g^{-1}H)))$$

for r of degree g. $\mathrm{Hom}_{R_H}(R, M)$ has the usual action of R and $\mathrm{Hom}_{\Bbbk}(\Bbbk[G/H], N)$ has the semi-diagonal action of R.

Proof. This is Corollary 5.2 of [10]. □

The following theorem generalises Theorem 5.7 of [3] although the argument is essentially the same.

Theorem 2.4.11. *Let G be an $\mathbf{LH}\mathfrak{F}$-group and R a strongly G-graded $\Bbbk$-algebra. If M is a left R-module such that $B \otimes M$ has projective dimension at most r and M has projective dimension at most r as a left R_H-module for all finite subgroups H of G, then M has projective dimension at most r.*

Proof. Observe that $\Omega^r M$ is a cofibrant R-module: if $P_* \to M \to 0$ is a projective resolution of M as an R-module, then the r^{th} kernel of $B \otimes P_* \to B \otimes M \to 0$ is $B \otimes \Omega^r M$ which is projective because $B \otimes M$ has projective dimension at most r. Also, $\Omega^r M$ is projective as an R_H-module for all finite subgroups H of G. It follows that we may assume $B \otimes M$ is projective and that M is projective as an R_H-module.

First suppose that G is an $\mathbf{H}\mathfrak{F}$-group. We use an inductive argument to prove that M is projective as an R_Γ-module for all subgroups Γ of G. The proof is on the least ordinal α such that Γ belongs to the subclass $\mathbf{H}\mathfrak{F}_\alpha$ of $\mathbf{H}\mathfrak{F}$. If $\alpha = 0$, then Γ belongs to $\mathbf{H}\mathfrak{F}_0$ which is the class of finite groups and so M is projective as an R_Γ-module by hypothesis.

If $\alpha > 0$ then there is an action of Γ on a finite dimensional contractible cell complex X with each isotropy group belonging to $\mathbf{H}\mathfrak{F}_\beta$ for some $\beta < \alpha$. The cellular chain complex of X is an exact sequence

$$0 \to C_r \to \cdots \to C_1 \to C_0 \to \Bbbk \to 0$$

of permutation modules over $\Bbbk\Gamma$. Tensoring with M and using the semi-diagonal action of R_Γ yields the exact sequence

$$0 \to C_r \otimes M \to \cdots \to C_1 \otimes M \to C_0 \otimes M \to M \to 0$$

of R_Γ-modules. Now each C_j is a direct sum of modules of the form $\Bbbk[\Gamma/K]$ where K is a cell stabiliser for the action of Γ on X and so $C_j \otimes M$ is a direct sum of modules of the form $\Bbbk[\Gamma/K] \otimes M$. Using the tensor identity, we have

$$\Bbbk[\Gamma/K] \otimes M \cong R_\Gamma \otimes_{R_K} M.$$

By induction, M is projective as an R_K-module for all isotropy groups K. By Lemma 1.5.20, each of the $R_\Gamma \otimes_{R_K} M$ are projective R_Γ-modules and so M has finite projective dimension as an R_Γ-module. Applying Proposition 2.4.7, we deduce that M is a projective R_Γ-module. The case $\Gamma = G$ proves the theorem for $\mathbf{H}\mathfrak{F}$-groups.

Now suppose Γ is an $\mathbf{LH}\mathfrak{F}$-group which is not an $\mathbf{H}\mathfrak{F}$-group. Then Γ is necessarily uncountable and so can be expressed as an ascending union $\Gamma = \bigcup_{\alpha<\gamma} \Gamma_\alpha$ (for some ordinal γ) in such a way that each Γ_α has strictly smaller cardinality than Γ. Thus $R = \bigcup_{\alpha<\gamma} R_{\Gamma_\alpha}$. By induction, M is projective as a left R_{Γ_α}-module for each subgroup Γ. By Lemma 2.4.8, M has projective dimension at most one. By Proposition 2.4.7, M is projective. □

Here is the statement of Aljadeff and Ginosar's theorem:

Theorem 2.4.12. *Let G be a finite group, R a strongly G-graded $\Bbbk$-algebra and N a left R-module. Then the following are equivalent:*

(i) N is projective over R.

(ii) N is projective over R_E for each elementary abelian subgroup E of G.

The following is a generalised and strengthened version of the Vanishing Theorem of Cornick and Kropholler ([10], Section 8). Their theorem covers the case when G is an $\mathbf{H}\mathfrak{F}$-group. We show that their result holds for any $\mathbf{LH}\mathfrak{F}$-group and in fact that it makes sense to state the result for a pair consisting of an arbitrary group together with an $\mathbf{LH}\mathfrak{F}$-subgroup. When this subgroup is an $\mathbf{H}\mathfrak{F}$-subgroup, our argument is similar to that of Cornick and Kropholler, but even this special case is not simply a corollary of the Cornick Kropholler result.

Theorem 2.4.13. *Let G be a group, R a strongly G-graded $\Bbbk$-algebra, M a left R-module of type* FP_∞ *and N a left R-module which has finite projective dimension over R_H for all finite subgroups H of G. Then $\widehat{\operatorname{Ext}}^i_R(M, R\otimes_{R_\Gamma} N)$ is zero for all i and all $\mathbf{LH}\mathfrak{F}$-subgroups Γ of G. If G is an $\mathbf{LH}\mathfrak{F}$-group, then $\widehat{\operatorname{Ext}}^i_R(M,N)$ is zero for all i.*

Proof. First assume that Γ is an $\mathbf{H}\mathfrak{F}$-group. We use an inductive argument on the least ordinal α such that Γ belongs to the subclass $\mathbf{H}_\alpha\mathfrak{F}$ of $\mathbf{H}\mathfrak{F}$. If $\alpha = 0$, then Γ belongs to $\mathbf{H}_0\mathfrak{F}$ which is just the class of finite groups and so $R\otimes_{R_\Gamma} N$ has finite projective dimension by Lemma 1.5.20. Thus the claim holds in this case because complete cohomology groups vanish on modules of finite projective dimension.

If $\alpha > 0$ then there is an action of Γ on a finite dimensional contractible cell complex X with with each isotropy group belonging to $\mathbf{H}_\beta\mathfrak{F}$ for some $\beta < \alpha$. The cellular chain complex of X is an exact sequence

$$0 \to C_r \to \cdots \to C_1 \to C_0 \to \Bbbk \to 0$$

of permutation modules over $\Bbbk\Gamma$. Tensoring with N and using the semi-diagonal action of R_Γ, yields the exact sequence

$$0 \to C_r\otimes N \to \cdots \to C_1\otimes N \to C_0\otimes N \to N \to 0$$

of R_Γ-modules. Applying the functor $R\otimes_{R_\Gamma} -$ preserves exactness because R is projective as a right R_Γ-module, by Lemma 1.1.13 and yields the exact sequence

$$0 \to R\otimes_{R_\Gamma}(C_r\otimes N) \to \cdots \to R\otimes_{R_\Gamma}(C_1\otimes N) \to R\otimes_{R_\Gamma}(C_0\otimes N) \to R\otimes_{R_\Gamma} N \to 0$$

of R-modules. Each C_j is a direct sum of modules of the form $\Bbbk[\Gamma/K]$ where K is a cell stabiliser for the action of Γ on X and so $R\otimes_{R_\Gamma}(C_j\otimes N)$ is a direct sum of modules of the form $R\otimes_{R_\Gamma}(\Bbbk[\Gamma/K]\otimes N)$. Using the tensor identity, we have

$$R\otimes_{R_\Gamma}(\Bbbk[\Gamma/K]\otimes N) \cong R\otimes_{R_\Gamma}(R_\Gamma\otimes_{R_K} N) = R\otimes_{R_K} N.$$

By induction, $\widehat{\operatorname{Ext}}^i_R(M, R\otimes_{R_K} N)$ vanishes for all isotropy groups K. Since M is of type FP_∞, $\widehat{\operatorname{Ext}}^i_R(M,-)$ commutes with arbitrary direct sums and so we deduce that

$$\widehat{\operatorname{Ext}}^i_R(M, R\otimes_{R_\Gamma}(C_j\otimes N)) = 0$$

for all i, j. Thus the functors $\widehat{\operatorname{Ext}}^*_R(M,-)$ vanish on all but the right-hand module in the exact sequence of R-modules, *vide supra*. By Lemma 2.1.13, it follows that these functors vanish on the right-hand module as well.

Assume now that Γ is an $\mathbf{LH}\mathfrak{F}$-group. Then, $\Gamma = \varinjlim \Gamma'$ where the limit is taken over all finitely generated subgroups Γ' of Γ, ordered by inclusion and so $R_\Gamma = \varinjlim R_{\Gamma'}$. Recalling that every finitely generated $\mathbf{LH}\mathfrak{F}$-group is an $\mathbf{H}\mathfrak{F}$-group, it follows that for all i.

$$\begin{aligned}\widehat{\operatorname{Ext}}^i_R(M, R \otimes_{R_\Gamma} N) &= \widehat{\operatorname{Ext}}^i_R(M, \varinjlim R \otimes_{R_{\Gamma'}} N)\\ &= \varinjlim \widehat{\operatorname{Ext}}^i_R(M, R \otimes_{R_{\Gamma'}} N)\\ &= 0.\end{aligned}$$

If G is an $\mathbf{LH}\mathfrak{F}$-group, then taking $\Gamma = G$ establishes the theorem. $\square$

Kropholler outlines a version of the Eckmann–Shapiro lemma for complete cohomology in [24]:

Lemma 2.4.14. *Let G be a group, H a subgroup of G, R a strongly G-graded $\Bbbk$-algebra, M a left R-module and N a left R_H-module. Then for all n, there is a natural isomorphism between $\widehat{\operatorname{Ext}}^n_R(R \otimes_{R_H} N, M)$ and $\widehat{\operatorname{Ext}}^n_{R_H}(N, M)$.*

The concluding result of this chapter was inspired by Theorem 7.6 of [3].

Theorem 2.4.15. *Let G be an $\mathbf{LH}\mathfrak{F}$-group and R a strongly G-graded $\Bbbk$-algebra such that R_1 is left Noetherian and has finite left global dimension. Let M be a left R-module such that for every finite elementary abelian subgroup E of G and every finitely generated left R_E-module N we have $\widehat{\operatorname{Ext}}^0_R(R \otimes_{R_E} N, M) = 0$. Then*

(i) M has finite projective dimension as a left R_H-module for every finite subgroup H of G.

(ii) If M is of type FP_∞, then M has finite projective dimension.

(iii) If $B \otimes M$ has finite projective dimension then M has finite projective dimension.

Proof. By Lemma 2.4.14 we have that

$$\widehat{\operatorname{Ext}}^0_{R_E}(N, M) \cong \widehat{\operatorname{Ext}}^0_R(R \otimes_{R_E} N, M) = 0,$$

by hypothesis. By Theorem 2.2.9, we deduce that M has finite projective dimension as a left R_E-module for every finite elementary abelian subgroup E of G.

Fix a finite subgroup H of G. Then H necessarily contains finitely many elementary abelian subgroups E. Define $n := \max\{pd_{R_E}(M)\}$. Then any partial projective resolution of M as a left R_H-module

$$0 \to K \to P_n \to \cdots \to P_0 \to M \to 0$$

(where K is the kernel of the map $P_n \to P_{n-1}$) is necessarily a projective resolution of M as a left R_E-module; that is, K is a projective R_E-module for each elementary abelian subgroup of H. By Theorem 2.4.12, K is a projective R_H-module and so M has finite projective dimension as a left R_H-module.

If M is of type FP_∞, we may apply Theorem 2.4.13 (taking $N = M$ and $i = 0$) and Lemma 2.1.8 to deduce M has finite projective dimension. If $B \otimes M$ has finite projective dimension, we may apply Theorem 2.4.11 to deduce M has finite projective dimension. $\square$

Chapter 3

Categorical Considerations

3.1 Some basics

We recall some basic notions. Much of what follows in this section can be found in [21] or [29].

Definition 3.1.1. Let $\mathcal{C}$ be an abelian category. A ***complex*** $(X_\bullet, d_\bullet)$ in $\mathcal{C}$ consists of a sequence of modules and maps

$$\cdots \to X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \to \cdots$$

where each X_i is an object of $\mathcal{C}$ and each of the d_i (called ***differentials***) are morphisms in $\mathrm{Hom}_{\mathcal{C}}(X_i, X_{i-1})$ subject to the condition $d_{i-1} \circ d_i = 0$ for all $i \in \mathbb{Z}$.

Definition 3.1.2. If $(X_\bullet, d_\bullet)$ and $(Y_\bullet, d'_\bullet)$ are complexes, then a ***chain map***

$$f = f_\bullet : (X_\bullet, d_\bullet) \to (Y_\bullet, d'_\bullet)$$

is a sequence of maps $f_i : X_i \to Y_i$ for all $i \in \mathbb{Z}$ making the following diagram commute:

$$\begin{tikzcd}
\cdots \arrow[r] & X_{i+1} \arrow[r, "d_{i+1}"] \arrow[d, "f_{i+1}"'] & X_i \arrow[r, "d_i"] \arrow[d, "f_i"'] & X_{i-1} \arrow[r] \arrow[d, "f_{i-1}"'] & \cdots \\
\cdots \arrow[r] & Y_{i+1} \arrow[r, "d'_{i+1}"] & Y_i \arrow[r, "d'_i"] & Y_{i-1} \arrow[r] & \cdots
\end{tikzcd}$$

Remark 3.1.3. The category of complexes is again abelian (see, for example [29]) and is denoted by $C(\mathcal{C})$.

Definition 3.1.4. A complex is ***bounded above*** if $X_i = 0$ for large enough n, ***bounded below*** if $X_i = 0$ for small enough n and ***bounded*** if bounded above and below.

Example 3.1.5. Every short exact sequence $0 \to A \to B \to C \to 0$ is a complex if we define $X_1 = A$, $X_0 = B$, $X_{-1} = C$ and $X_i = 0$ for all $i \notin \{-1, 0, 1\}$. The differentials agree with the maps of the short exact sequence and are the zero map elsewhere. This is an example of a bounded complex.

Similarly, any we can associate a complex to any module by defining $X_0 = M$ and $X_i = 0$ for all $i \neq 0$ and letting all differentials be the zero map. Alternatively, we could associate to M a projective resolution of M and define $X_{-1} = M$, $X_i = 0$ for $i \leq -2$ and $X_i = P_i$ for $i \geq 0$ where the P_i come from the projective resolution of M. The first complex associated to M is bounded and the second bounded below.

Definition 3.1.6. A chain map $f : (X_\bullet, d_\bullet) \to (Y_\bullet, d'_\bullet)$ is said to be ***nullhomotopic*** if, for all i, there are maps $s_n : X_i \to Y_{n+1}$ with

$$f_i = d'_{i+1} \circ s_i + s_{i-1} \circ d_i$$

$$\begin{tikzcd}
\cdots \arrow[r] & X_{i+1} \arrow[r, "d_{i+1}"] \arrow[d, "f_{i+1}"'] & X_i \arrow[r, "d_i"] \arrow[d, "f_i"] \arrow[dl, "s_i"] & X_{i-1} \arrow[r] \arrow[d, "f_{i-1}"] \arrow[dl, "s_{i-1}"] & \cdots \\
\cdots \arrow[r] & Y_{i+1} \arrow[r, "d'_{i+1}"'] & Y_i \arrow[r, "d'_i"'] & Y_{i-1} \arrow[r] & \cdots
\end{tikzcd}$$

If $f, g : (X_\bullet, d_\bullet) \to (Y_\bullet, d'_\bullet)$ are chain maps, then f is ***homotopic*** (or homotopically equivalent) to g, denoted $f \approx g$, if $f - g$ is nullhomotopic. $\approx$ is an equivalence relation on chain maps.

For a complex X in $C(\mathcal{C})$, the n^{th}***-homology*** of X is the quotient $\operatorname{Ker} d_n / \operatorname{Im} d_{n+1}$. Thus we have an additive functor $H_n(-) : C(\mathcal{C}) \to \mathcal{C}$. Complexes with zero homology are called ***exact*** or ***acyclic***. Chain maps which induce isomorphisms on the level of homology are called ***quasi-isomorphisms***.

Example 3.1.7. Let M be a module in an abelian category $\mathcal{C}$. If M has a projective resolution $\cdots \to P_1 \to P_0 \xrightarrow{\varepsilon} M \to 0$, then we can form the following map of complexes:

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d] & P_1 \arrow[r, "d_1"] \arrow[d] & P_0 \arrow[r] \arrow[d, "\varepsilon"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

with M appearing in degree 0. Upon taking homology, we obtain

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & P_0/\operatorname{Im} d_1 \arrow[r] \arrow[d, "\cong"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

and we see that the map of complexes defined before is a quasi-isomorphism.

Proposition 3.1.8. *Homotopic chain maps induce the same homomorphism between homology groups: if $f, g : (X_\bullet, d_\bullet) \to (Y_\bullet, d'_\bullet)$ are chain maps and $f \approx g$, then*

$$f_{*n} = g_{*n} : H_n(X_\bullet) \to H_n(Y_\bullet).$$

Definition 3.1.9. Let $(X_\bullet, d_\bullet) \to (Y_\bullet, d'_\bullet)$ be complexes and f a chain map between them. The ***mapping cone*** of f is the complex $M(f)$ where

$$M(f)_i := X_{i-1} \oplus Y_i$$

and the differentials are given by

$$d''_i = \begin{pmatrix} -d_{i-1} & 0 \\ -f_{i-1} & d'_i \end{pmatrix}$$

that is,

$$d''_i(x, y) = (-d_{i-1}(x), d'_i(y) - f_{i-1}(x))$$

for $x \in X_{i-1}, y \in Y_i$. $M(f)$ is related to X and Y via $\alpha(f) : Y \to M(f); y \mapsto (0, y)$ and $\beta(f) : M(f) \to X[1]; (x, y) \mapsto -x$.

Lemma 3.1.10. *A chain map $f : (X_\bullet, d_\bullet) \to (Y_\bullet, d'_\bullet)$ is a quasi-isomorphism if and only if the mapping cone $M(f)$ is exact.*

Proof. This is Corollary 1.5.4 of [29]. □

Given the category of complexes $C(\mathcal{C})$ for some abelian category $\mathcal{C}$, the ***homotopy category*** $K(\mathcal{C})$, is the category whose objects are the same as those of $C(\mathcal{C})$, but whose morphisms are

$$\mathrm{Hom}_{K(\mathcal{C})}(X, Y) = \mathrm{Hom}_{C(\mathcal{C})}(X, Y)/\sim$$

where $\sim$ denotes homotopic equivalence. $K(\mathcal{C})$ is an additive category, but may no longer be abelian.

Definition 3.1.11. Let n be an integer. The ***shift (by n) functor*** $[n]$ is an automorphism of the category $C(\mathcal{C})$ which sends a complex $(X_\bullet, d_\bullet)$ to another complex $(X[n]_\bullet, d_{\bullet[n]})$ defined by $X[n]_i := X_{i+n}$ and $d_{i[n]} := (-1)^n d_{i+n}$.

Remark 3.1.12. It is a feature of $K(\mathcal{C})$ that $M(\alpha(f)) = X[1]$.

Suppose we have a category $\mathbf{K}$ equipped with an automorphism τ. A ***triangle*** on an ordered triple (A, B, C) of objects of $\mathbf{K}$ is a triple (u, v, w) of morphisms, where $u : A \to B$, $v : B \to C$ and $w : C \to \tau A$. This is displayed figuratively thus:

$$\begin{tikzcd} & C \arrow[dl, "w"'] & \\ A \arrow[rr, "u"] & & B \arrow[ul, "u"'] \end{tikzcd}$$

A ***morphism of triangles*** is a triple (f, g, h) forming a commutative diagram in $\mathbf{K}$:

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"] & B \arrow[r, "v"] \arrow[d, "g"] & C \arrow[r, "w"] \arrow[d, "h"] & \tau A \arrow[d, "\tau f"] \\ A' \arrow[r, "u'"] & B' \arrow[r, "v'"] & C' \arrow[r, "w'"] & \tau A' \end{tikzcd}$$

We now proceed to the definition of a triangulated category, essentially as it appears in Weibel's book (page 374 of [29]). It will be noted that Weibel's definition is the same as that of Verdier (see page 266 of [13]). König's treatment (see page 11 of [21]) is different: he has six axioms to Verdier and Weibel's four. What König and Verdier call a 'distinguished triangle,' Weibel refers to as an 'exact triangle,' and Verdier has no name for the axiom the other two call 'the octahedral axiom'.

Definition 3.1.13. An additive category $\mathbf{K}$ is called a ***triangulated category*** if it is equipped with an automorphism $\tau : \mathbf{K} \to \mathbf{K}$ (called the ***translation functor***) and with a family of triangles (u, v, w) (called the ***distinguished triangles*** in $\mathbf{K}$) which are subject to the following four axioms:

(TR1) Every morphism $u : A \to B$ can be embedded in a distinguished triangle (u, v, w). If $A = B$ and $C = 0$, then the triangle $(id_A, 0, 0)$ is distinguished. If (u, v, w) is a triangle on (A, B, C), isomorphic to a distinguished triangle (u', v', w') on (A', B', C'), then (u, v, w) is also distinguished.

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "\cong"] & B \arrow[r, "v"] \arrow[d, "\cong"] & C \arrow[r, "w"] \arrow[d, "\cong"] & \tau A \arrow[d, "\cong"] \\ A' \arrow[r, "u'"] & B' \arrow[r, "v'"] & C' \arrow[r, "w'"] & \tau A' \end{tikzcd}$$

This is really three axioms; indeed, they are the first three axioms (TR0, TR1, TR2) in König's paper.

(TR2) (Rotation). If (u, v, w) is a distinguished triangle on (A, B, C), then both its 'rotates' $(v, w, -\tau u)$ and $(-\tau^{-1}w, u, v)$ are distinguished triangles on $(B, C, \tau A)$ and

$(\tau^{-1}C, A, B)$ respectively. König's rotation axiom states that (u, v, w) is distinguished if and only if $(v, w, -\tau u)$ is distinguished, which is seen to be equivalent to Weibel and Verdier's axiom.

(TR3) (Morphisms). Given two distinguished triangles

$$\begin{tikzcd} & C \arrow[dl, "w"'] & \\ A \arrow[rr, "u"] & & B \arrow[ul, "u"'] \end{tikzcd}$$

and

$$\begin{tikzcd} & C' \arrow[dl, "w'"'] & \\ A' \arrow[rr, "u'"] & & B' \arrow[ul, "u'"'] \end{tikzcd}$$

with morphisms $f : A \to A'$, $g : B \to B'$ such that $gu = u'f$, there exists a morphism $h : C \to C'$ so that (f, g, h) is a morphism of triangles.

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"] & B \arrow[r, "v"] \arrow[d, "g"] & C \arrow[r, "w"] \arrow[d, "\exists h"] & \tau A \arrow[d, "\tau f"] \\ A' \arrow[r, "u'"] & B' \arrow[r, "v'"] & C' \arrow[r, "w'"] & \tau A' \end{tikzcd}$$

(TR4) (The octahedral axiom). Given objects A, B, C, A', B', C' in $\mathbf{K}$, suppose there are three distinguished triangles: (u, j, ∂) on (A, B, C'); (v, x, i) on (B, C, A'); (vu, y, δ) on (A, C, B'). Then there is a fourth distinguished triangle $(f, g, (\tau j)i)$ on (C', B', A')

$$\begin{tikzcd} & A' \arrow[dl, "(\tau j)i"'] & \\ C' \arrow[rr, "f"] & & B' \arrow[ul, "g"'] \end{tikzcd}$$

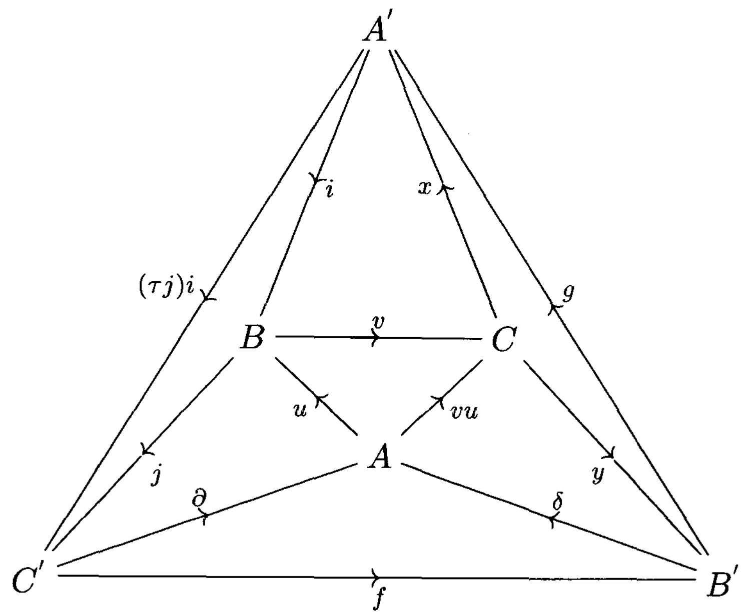
such that in the following octahedron we have

(1) the four distinguished triangles form four of the faces;

(2) the remaining four faces commute (that is, $\partial = \delta f : C' \to B' \to \tau A$ and $x = gy : C \to B' \to A'$);

(3) $yv = fj : B \to B'$;

(4) $u\delta = ig : B' \to B$.



Remark 3.1.14. The octahedral axiom is notoriously confusing. Verdier does not provide a pictorial representation of it, which is probably why he did not call it the 'octahedral axiom': it is not immediately obvious that one can associate a Platonic solid to the rather dry description of **TR4** in [13]. In contrast, it would appear that every treatment of this axiom since Verdier has included some diagram or another in an attempt to make the octahedron explicit. König's picture looks nothing like an octahedron, it being a 4×4 commutative diagram although the reader is assured that the octahedral axiom can be viewed as a "kind of first isomorphism theorem," which of course depends on how one numbers the isomorphism theorems. Weibel provides a very nice diagram in his book of an octahedron with appropriately labelled vertices and edges. One drawback is that it is difficult to discern a given face. The current author feels that a diagram retaining aspects of clarity from both authors would be one which is planar (König) and demonstrates the octahedron (Weibel), hence the use of the octahedral graph in **TR4**, above.

Example 3.1.15. Proposition 10.2.4 of [29] shows that the category $K(\mathcal{C})$ with automorphism $\tau = -[1]$ (shift by 1 composed with multiplication by -1) is a triangulated category. The triangles are diagrams of the form

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

and the distinguished triangles are triangles of the form

$$X \xrightarrow{f} Y \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} X[1].$$

Remark 3.1.16. The category $K(\mathcal{C})$ contains the full subcategories $K^+(\mathcal{C})$ of bounded below chain complexes, $K^-(\mathcal{C})$ of bounded above chain complexes and $K^b(\mathcal{C})$ of bounded chain complexes, all of which are triangulated.

3.2 Triangulated categories

The following lemma requires us to recall the module B, introduced in Example 1.2.9. Throughout this section, $\Bbbk$ will denote a commutative ring

Lemma 3.2.1. *Let G be a group and $\Bbbk$ a commutative ring. Let $B := B(G, \Bbbk)$. Then*

(i) B is free as a $\Bbbk$-module;

(ii) B is free as a $\Bbbk H$-module for each finite subgroup H of G.

Proof. This is Lemma 9.1 of [10]. □

The sequence

$$0 \to \Bbbk \to B \to \overline{B} \to 0$$

is split as a sequence of $\Bbbk$-modules because the module $\overline{B}$ is free as a $\Bbbk$-module (see Section 3 of [9]). If R is a strongly group-graded $\Bbbk$-algebra and P a projective left R-module, then Lemma 2.4.1 tells us that $B \otimes P$ and $\overline{B} \otimes P$ are projective left R-modules with the semi-diagonal action of R. Also, given any left R-module M, we have a short exact sequence

$$0 \to M \to B \otimes M \to \overline{B} \otimes M \to 0$$

of left R-modules.

The following is a strengthened version of Proposition 9.2 of [10] in that we extend the result to include $\mathbf{LH}\mathfrak{F}$-groups. The argument is essentially the same, the only difference being that we make use of Theorem 2.4.13.

Proposition 3.2.2. *Suppose that G is an $\mathbf{LH}\mathfrak{F}$-group and that R is a strongly G-graded $\Bbbk$-algebra. Let M be a left R-module of type FP_∞. Then the following are equivalent:*

(i) $B \otimes M$ has finite projective dimension over R.

(ii) M has finite projective dimension as an R_1-module.

Proof. (i) $\Rightarrow$ (ii) If $B \otimes M$ has finite projective dimension over R, then by Lemma 1.5.20 it has finite projective dimension over R_1. By Lemma 3.2.1, B is free as a $\Bbbk$-module and so M has finite projective dimension as an R_1-module because it is an R_1-module direct summand of $B \otimes M$.

(ii) $\Rightarrow$ (i) Suppose $pd_{R_1}(M) < \infty$. We show that $B \otimes M$ has finite projective dimension as an R_H-module for all finite subgroups H of G. Fix a finite subgroup H of G. Then B is free as a $\Bbbk H$-module by Lemma 3.2.1 and so $B \otimes M$ is a direct sum of copies of $\Bbbk H \otimes M$ as an R_H-module. By the tensor identity, Lemma 2.4.9, $\Bbbk H \otimes M$ is isomorphic to $R_H \otimes_{R_1} M$ which has finite projective dimension over R_H by Lemma 1.5.20. Since this is true for all choices of H, it follows from Theorem 2.4.13 that $\widehat{\operatorname{Ext}}^0_R(M, B \otimes M) = 0$. As was remarked in Section 9 of [10], this implies that $\widehat{\operatorname{Ext}}^0_R(B \otimes M, B \otimes M) = 0$. It follows by Lemma 2.1.8 that $B \otimes M$ has finite projective dimension as an R-module. $\square$

Let G be a group, H a subgroup of G, $\Bbbk$ a commutative ring and R a strongly G-graded $\Bbbk$-algebra. Write ${}_R\mathsf{mod}$ for the category in which the objects are finitely generated left R-modules of type FP_∞ which have finite projective dimension as left R_1-modules. The maps from M to N in ${}_R\mathsf{mod}$ are given by $\operatorname{Hom}_R(M, N)$. We write ${}_R\mathsf{stmod}$ for the category with the same objects as ${}_R\mathsf{mod}$, but whose maps are given instead by $\widehat{\operatorname{Ext}}^0_R(M, N)$.

We also define ${}_R\mathsf{Mod}$ to be the category in which the objects are left R-modules M such that $B \otimes M$ has finite projective dimension. The maps are given as before by $\operatorname{Hom}_R(M, N)$. We write ${}_R\mathsf{StMod}$ for the category with the same objects as ${}_R\mathsf{Mod}$, but whose maps are given instead by $\widehat{\operatorname{Ext}}^0_R(M, N)$. Note that since B is free as a $\Bbbk$-module, every module in ${}_R\mathsf{Mod}$ has finite projective dimension as an R_1-module because M is an R_1-module direct summand of $B \otimes M$.

By dint of Proposition 3.2.2, we see that when G is an $\mathbf{LH}\mathfrak{F}$-group and R is a strongly G-graded $\Bbbk$-algebra, ${}_R\mathsf{mod}$ can be realised as a full subcategory of ${}_R\mathsf{Mod}$, and ${}_R\mathsf{stmod}$ can be realised as a full subcategory of ${}_R\mathsf{StMod}$.

Proposition 3.2.3. *Let G be a group and R a strongly G-graded $\Bbbk$-algebra. For left modules M, N in ${}_R\mathsf{StMod}$, the following are equivalent:*

(i) M is isomorphic to N in ${}_R\mathsf{StMod}$.

(ii) There exist $r \in \mathbb{N}_0$ and projective left R-modules P and Q such that

$$\Omega^r(M) \oplus P \cong \Omega^r(N) \oplus Q.$$

Proof. We use $\simeq$ to denote an isomorphism in ${}_R\mathsf{StMod}$.

($\Rightarrow$) If $M \simeq N$, then there exist $f \in \widehat{\mathrm{Ext}}_R^0(M, N)$ and $g \in \widehat{\mathrm{Ext}}_R^0(N, M)$ such that $g \circ f$ represents the identity in $\widehat{\mathrm{Ext}}_R^0(M, M)$ and $f \circ g$ represents the identity in $\widehat{\mathrm{Ext}}_R^0(N, N)$.

If $g \circ f$ represents the identity in $\widehat{\mathrm{Ext}}_R^0(M, M)$, then there exist an integer $r \geq 0$ and $\rho \in \underline{\mathrm{Hom}}_R(\Omega^r M, \Omega^r N)$, $\sigma \in \underline{\mathrm{Hom}}_R(\Omega^r N, \Omega^r M)$ representing f and g, such that $\sigma \circ \rho = id_{\Omega^r M}$ in $\underline{\mathrm{Hom}}_R(\Omega^r M, \Omega^r M)$. Let ϕ and ψ be R-module representatives for ρ and σ respectively. Then there exist projective modules Q_1, Q_2 and R-maps $\alpha_1 : \Omega^r M \to Q_1$, $\beta_1 : Q_1 \to \Omega^r M$, $\alpha_2 : \Omega^r M \to Q_2$, $\beta_2 : Q_2 \to \Omega^r M$ with $\psi \circ \phi = id_{\Omega^r M} + \beta_1 \circ \alpha_1$ and $\phi \circ \psi = id_{\Omega^r N} + \beta_2 \circ \alpha_2$.

To avoid notational overload, let us denote $id_{\Omega^r M}$ by 1. Choose a projective module P which maps onto $\Omega^r M$. Since $\psi \circ \phi - 1$ factors through Q_1, we have the following diagram

$$\begin{tikzcd}
& Q_1 \arrow[dl, dotted, "\gamma"'] \arrow[d, "\beta_1"] & \\
P \arrow[r, "\delta"] & \Omega^r M \arrow[r, "\psi\circ\phi-1"] & \Omega^r M \arrow[ul, "\alpha_1"']
\end{tikzcd}$$

where

$$\psi \circ \phi - 1 = \beta_1 \circ \alpha_1 = \delta \circ \gamma \circ \alpha_1$$

and we see that $\psi \circ \phi - 1$ maps though P.

Denote the composite $\gamma \circ \alpha_1$ by ε. Then $\psi \circ \phi - \delta \circ \varepsilon = 1$. We have the following sequence:

$$\Omega^r M \xrightarrow{(\varepsilon,\phi)} P \oplus \Omega^r N \xrightarrow{(-\delta,\psi)} \Omega^r M$$

and so $\Omega^r M$ is a direct summand of $P \oplus \Omega^r N$.

Composing (ε, ϕ) and $(-\delta, \psi)$ the other way, we obtain

$$e : P \oplus \Omega^r N \xrightarrow{(-\delta,\psi)} \Omega^r M \xrightarrow{(\varepsilon,\phi)} P \oplus \Omega^r N.$$

The composite e is an idempotent endomorphism of $P \oplus \Omega^r N$ because

$$\begin{aligned}
e^2 &= (\varepsilon(\psi - \delta), \phi(\psi - \delta))^2 \\
&= (\varepsilon(-\delta\varepsilon + \psi\phi)(\psi - \delta), \phi(-\delta\varepsilon + \psi\phi)(\psi - \delta)) \\
&= (\varepsilon(\psi - \delta), \phi(\psi - \delta)) \\
&= e.
\end{aligned}$$

Because e represents the identity map in $\underline{\mathrm{Hom}}_R(P \oplus \Omega^r N, P \oplus \Omega^r N)$, so $1' - e$ represents the zero map (where $1' = id_{P \oplus \Omega^r N}$) and so factors through a projective module Q', say.

$1' - e$ is also an idempotent endomorphism of $P \oplus \Omega^r N$. We have the following diagram:

$$\begin{tikzcd} \mathrm{Im}(1'-e) \arrow[r, tail] & P \oplus \Omega^r N \arrow[r] \arrow[rr, bend left, "1'-e"] & Q' \arrow[r] & P \oplus \Omega^r N \arrow[r, two heads] & \mathrm{Im}(1'-e) \end{tikzcd}$$

The composition of the maps is the identity on $\mathrm{Im}(1'-e)$. It follows that $\mathrm{Im}(1'-e) = \mathrm{Ker}\, e$ is a direct summand of Q' and so is projective. Thus $\mathrm{Ker}\, e \oplus \Omega^r M \cong P \oplus \Omega^r N$.

($\Leftarrow$) We have $\underline{\mathrm{Hom}}_R(\Omega^i(M) \oplus P, \Omega^i(N) \oplus Q) \cong \underline{\mathrm{Hom}}_R(\Omega^i M, \Omega^i N)$ and $\underline{\mathrm{Hom}}_R(\Omega^i(N) \oplus Q, \Omega^i(M) \oplus P) \cong \underline{\mathrm{Hom}}_R(\Omega^i N, \Omega^i M)$. It follows that there exist maps $f_i \in \underline{\mathrm{Hom}}_R(\Omega^i M, \Omega^i N)$ and $g_i \in \underline{\mathrm{Hom}}_R(\Omega^i N, \Omega^i M)$ such that $f_i \circ g_i = id_{\Omega^i M}$ in $\underline{\mathrm{Hom}}_R(\Omega^i M, \Omega^i M)$, and $g_i \circ f_i = id_{\Omega^i N}$ in $\underline{\mathrm{Hom}}_R(\Omega^i N, \Omega^i N)$. Because Ω is a functor on these $\underline{\mathrm{Hom}}_R$ sets, it follows, by the definition of $\widehat{\mathrm{Ext}}^0_R(M, N)$, that M and N are isomorphic in ${}_R\mathsf{StMod}$. □

Next, let ${}_R\mathsf{Mod}_c$ be the full subcategory of ${}_R\mathsf{Mod}$ consisting of the cofibrant R-modules and let ${}_R\mathsf{StMod}_c$ be the corresponding full subcategory of ${}_R\mathsf{StMod}$. Similar definitions exist for ${}_R\mathsf{mod}_c$ and ${}_R\mathsf{stmod}_c$.

Lemma 3.2.4. *Let G be a group, $\Bbbk$ a commutative ring of coefficients and R a strongly G-graded $\Bbbk$-algebra. The inclusion of the subcategory ${}_R\mathsf{StMod}_c$ into ${}_R\mathsf{StMod}$ is an equivalence of categories.*

Proof. Let M be a module in ${}_R\mathsf{StMod}$ and suppose that $B \otimes M$ is projective. Then M is a module in ${}_R\mathsf{StMod}_c$ and in particular, $\overline{B} \otimes M$ lies in ${}_R\mathsf{StMod}_c$ also.

Now suppose $B \otimes M$ has projective dimension r. Recall that there is a $\Bbbk$-split exact sequence

$$0 \to \Bbbk \to B \to \overline{B} \to 0$$

where B is the free $\Bbbk$-module described in the last section, and $\overline{B}$ is the free $\Bbbk$-module $\mathrm{Coker}(\Bbbk \to B)$. Then given a projective resolution $P_* \to M \to 0$ of M, we see that

$$0 \to B \otimes \Omega^r M \to B \otimes P_{r-1} \to \cdots \to B \otimes P_0 \to B \otimes M \to 0$$

is a projective resolution of $B \otimes M$ and so in particular, $\Omega^r M$ is a cofibrant R-module.

We claim that in ${}_R\mathsf{StMod}$, M is isomorphic to $\overline{B}^{\otimes r} \otimes \Omega^r M$ where $\overline{B}^{\otimes r} = \overline{B} \otimes \cdots \otimes \overline{B}$ (r copies). We have short exact sequences

$$0 \to \Omega^r M \to B \otimes \Omega^r M \to \overline{B} \otimes \Omega^r M \to 0$$

$$0 \to \overline{B} \otimes \Omega^r M \to B \otimes \overline{B} \otimes \Omega^r M \to \overline{B}^{\otimes 2} \otimes \Omega^r M \to 0$$

$$\vdots$$

$$0 \to \overline{B}^{\otimes i-1} \otimes \Omega^r M \to B \otimes \overline{B}^{\otimes i-1} \otimes \Omega^r M \to \overline{B}^{\otimes i} \otimes \Omega^r M \to 0$$

$$\vdots$$

$$0 \to \overline{B}^{\otimes r-1} \otimes \Omega^r M \to B \otimes \overline{B}^{\otimes r-1} \otimes \Omega^r M \to \overline{B}^{\otimes r} \otimes \Omega^r M \to 0$$

in which each middle term is projective: $B \otimes \Omega^r M$ is projective and $\overline{B}$ is free as a $\Bbbk$-module so by Lemma 2.4.1, $B \otimes \overline{B}^{\otimes i} \otimes \Omega^r M$ is projective for each $0 \leq i \leq r-1$ where $\overline{B}^{\otimes 0} := \Bbbk$. Splicing these sequences together, we obtain a long exact sequence

$$0 \to \Omega^r M \to B \otimes \Omega^r M \to B \otimes \overline{B} \otimes \Omega^r M \to \cdots \to B \otimes \overline{B}^{\otimes r-1} \otimes \Omega^r M \to \overline{B}^{\otimes r} \otimes \Omega^r M \to 0$$

which can be interpreted as a partial projective resolution of $\overline{B}^{\otimes r} \otimes \Omega^r M$ with r^{th} kernel $\Omega^r M$. It follows that the projective resolution of M, mentioned at the start, agrees with the projective resolution of $\overline{B}^{\otimes r} \otimes \Omega^r M$ just constructed from the r^{th} term onwards and so, by Proposition 3.2.3, M is isomorphic to $\overline{B}^{\otimes r} \otimes \Omega^r M$ which lies in ${}_R\mathsf{StMod}_c$. □

We need the following:

Lemma 3.2.5. *Let G be a group and R a strongly G-graded $\Bbbk$-algebra. Let M be a cofibrant left R-module. Then there are projective left R-modules P_1, P_2 and P_3 such that*

$$M \oplus P_1 \cong (\overline{B} \otimes \Omega M) \oplus P_2 \cong \Omega(\overline{B} \otimes M) \oplus P_3.$$

Proof. We have the short exact sequence $0 \to \Omega M \to FM \to M \to 0$. Since $\overline{B}$ is a free $\Bbbk$-module, we have the short exact sequence

$$0 \to \overline{B} \otimes \Omega M \to \overline{B} \otimes FM \to \overline{B} \otimes M \to 0.$$

We also have

$$0 \to M \to B \otimes M \to \overline{B} \otimes M \to 0$$

because $0 \to \Bbbk \to B \to \overline{B} \to 0$ is $\Bbbk$-split. By definition, we have the short exact sequence

$$0 \to \Omega(\overline{B} \otimes M) \to F(\overline{B} \otimes M) \to \overline{B} \otimes M \to 0.$$

Applying Schanuel's Lemma to the first two short exact sequences,

$$(\overline{B} \otimes \Omega M) \oplus (B \otimes M) \cong M \oplus (\overline{B} \otimes FM).$$

Applying the Lemma to the second and third sequence and then to the first and third, we see

$$M \oplus F(\overline{B} \otimes M) \cong \Omega(\overline{B} \otimes M) \oplus (B \otimes M)$$

and

$$(\overline{B} \otimes \Omega M) \oplus F(\overline{B} \otimes M) \cong \Omega(\overline{B} \otimes M) \oplus (\overline{B} \otimes FM).$$

Taking $P_1 = F(\overline{B} \otimes M) \oplus (\overline{B} \otimes FM)$, $P_2 = (B \otimes M) \oplus F(\overline{B} \otimes M)$ and $P_3 = (\overline{B} \otimes FM) \oplus (B \otimes M)$ establishes the lemma. □

Lemma 3.2.6. *Let R be a ring. Suppose we have a commutative diagram*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A'' \arrow[r, "\iota''"] \arrow[d, "\alpha"] & B'' \arrow[r, "\pi''"] \arrow[d, "\beta"] & C'' \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A \arrow[r, "\iota"] \arrow[d, "\alpha'"] & B \arrow[r, "\pi"] \arrow[d, "\beta'"] & C \arrow[r] \arrow[d, "\gamma'"] & 0 \\
0 \arrow[r] & A' \arrow[r, "\iota'"] \arrow[d] & B' \arrow[r, "\pi'"] \arrow[d] & C' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

of left R-modules with exact rows and columns. If the middle row is split, then we have a diagram

$$\begin{tikzcd}
0 \arrow[r] & A'' \arrow[r] \arrow[d, equal] & B'' \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A'' \arrow[r] & A \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

with exact rows in which the outside squares commute and the middle square anticommutes.

Proof. We have the following diagram

$$\begin{tikzcd}
0 \arrow[r] & A'' \arrow[r, "\iota''"] \arrow[dd, equal] & B'' \arrow[rr, "\gamma\circ\pi''"] \arrow[dd, "\kappa\circ\beta"'] \arrow[dr, "\beta"] & & C \arrow[r, "\gamma'"] \arrow[dd, "\beta'\circ\rho"] \arrow[dl, bend right, "\rho"'] & C' \arrow[r] \arrow[dd, equal] & 0 \\
 & & & B \arrow[ur, "\pi"] \arrow[dr, "\beta'"] \arrow[dl, bend right, "\kappa"'] & & & \\
0 \arrow[r] & A'' \arrow[r, "\alpha"'] & A \arrow[rr, "\iota'\circ\alpha'"'] \arrow[ur, "\iota"'] & & B' \arrow[r, "\pi'"'] & C' \arrow[r] & 0
\end{tikzcd}$$

with exact rows where ρ and κ are the splittings of the exact sequence $0 \to A \to B \to C \to 0$. Observe that the left-hand square commutes:

$$\begin{aligned}\kappa \circ \beta \circ \iota'' - \alpha &= \kappa \circ \beta \circ \iota'' - \kappa \circ \iota \circ \alpha \\ &= \kappa \circ (\beta \circ \iota'' - \iota \circ \alpha) \\ &= 0\end{aligned}$$

by commutativity of the top left square of our original 3×3 diagram and the fact that the composite $\kappa \circ \iota$ is the identity on A.

The right-hand square commutes because

$$\begin{aligned}\pi' \circ \beta' \circ \rho - \gamma' &= \pi' \circ \beta' \circ \rho - \gamma' \circ \pi \circ \rho \\ &= (\pi' \circ \beta' - \gamma' \circ \pi) \circ \rho \\ &= 0\end{aligned}$$

by commutativity of the bottom right square of the 3×3 diagram and the fact that the composite $\pi \circ \rho$ is the identity on C.

For the middle square, consider $\beta' \circ \rho \circ \gamma \circ \pi'' + \iota' \circ \alpha' \circ \kappa \circ \beta$. By commutativity of the top right square of the 3×3 diagram, $\gamma \circ \pi'' = \pi \circ \beta$ and by commutativity of the bottom left square of the 3×3 diagram, $\iota' \circ \alpha' = \beta' \circ \iota$. It follows that

$$\begin{aligned}\beta' \circ \rho \circ \gamma \circ \pi'' + \iota' \circ \alpha' \circ \kappa \circ \beta &= \beta' \circ \rho \circ \pi \circ \beta + \beta' \circ \iota \kappa \circ \beta \\ &= \beta' \circ (\rho \circ \pi + \iota \circ \kappa) \circ \beta \\ &= \beta' \circ \beta \\ &= 0.\end{aligned}$$

□

Lemma 3.2.7. *Let G be any group and R a G-graded $\Bbbk$-algebra. Let $0 \to K \to L \to M \to 0$ be a short exact sequence with M cofibrant. If Q is a projective R-module, then the sequence*

$$0 \to \mathrm{Hom}_R(M, Q) \to \mathrm{Hom}_R(L, Q) \to \mathrm{Hom}_R(K, Q) \to 0$$

is exact.

Proof. $\mathrm{Hom}_R(-, Q)$ is left exact, so it suffices to show that the map $\mathrm{Hom}_R(L, Q) \to \mathrm{Hom}_R(K, Q)$ is an epimorphism.

Applying the functor $\mathrm{Hom}_R(-, Q)$ to the split short exact sequence $0 \to \Bbbk \to B \to \overline{B} \to 0$ gives rise to the short exact sequence

$$0 \to \mathrm{Hom}_{\Bbbk}(\overline{B}, Q) \to \mathrm{Hom}_{\Bbbk}(B, Q) \to \mathrm{Hom}_{\Bbbk}(\Bbbk, Q) \to 0$$

which is split because (applying Corollary 2.4.10) $\mathrm{Hom}_{\Bbbk}(\Bbbk, Q) \cong \mathrm{Hom}_R(R. Q) \cong Q$ is a projective R-module, and so Q is a direct summand of $\mathrm{Hom}_{\Bbbk}(B, Q)$. We have the following diagram

$$\begin{tikzcd}
\mathrm{Hom}_R(L, Q) \arrow[r] \arrow[d, tail] & \mathrm{Hom}_R(K, Q) \arrow[d, tail] \\
\mathrm{Hom}_R(L, \mathrm{Hom}_{\Bbbk}(B, Q)) \arrow[r] & \mathrm{Hom}_R(K, \mathrm{Hom}_{\Bbbk}(B, Q))
\end{tikzcd}$$

and so it suffices to show that the bottom map is a surjection.

By Lemma 1.2.12, the bottom row of the above diagram is equivalent to

$$\mathrm{Hom}_R(B \otimes L, Q) \to \mathrm{Hom}_R(B \otimes K, Q).$$

Tensoring $0 \to K \to L \to M \to 0$ with B gives the short exact sequence $0 \to B \otimes K \to B \otimes L \to B \otimes M \to 0$ which is split because $B \otimes M$ is projective. It follows that the map $\mathrm{Hom}_R(B \otimes L, Q) \to \mathrm{Hom}_R(B \otimes K, Q)$ is surjective. □

Lemma 3.2.8. *Let G be a group, $\Bbbk$ a commutative ring of coefficients and R a strongly G-graded $\Bbbk$-algebra. For modules M and N in ${}_R\mathsf{StMod}_c$, the natural map $\underline{\mathrm{Hom}}_R(M, N) \to \widehat{\mathrm{Ext}}^0_R(M, N)$ is an isomorphism.*

Proof. We show the natural map $\Omega : \underline{\mathrm{Hom}}_R(M, N) \to \underline{\mathrm{Hom}}_R(\Omega M, \Omega N)$ is an isomorphism from which we deduce that $\underline{\mathrm{Hom}}_R(\Omega^i M, \Omega^i N) \to \underline{\mathrm{Hom}}_R(\Omega^{i+1} M, \Omega^{i+1} N)$ is an isomorphism for all i and so $\widehat{\mathrm{Ext}}^0_R(M, N) = \underline{\mathrm{Hom}}_R(M, N)$.

Surjectivity: Given a map $\phi_1 : \Omega M \to \Omega N$, we can find a map $\phi_2 : P_M \to P_N$ making the left-hand square commute in the diagram below. For, by Lemma 3.2.7, the map $\mathrm{Hom}_R(P_M, P_N) \xrightarrow{\iota_M^*} \mathrm{Hom}_R(\Omega M, P_N)$ is surjective, and so we can find $\phi_2 \in \mathrm{Hom}_R(P_M, P_N)$ satisfying $\iota_M^*(\phi_2) = \iota_N \circ \phi_1$; that is, $\phi_2 \circ \iota_M = \iota_N \circ \phi_1$. The proof of surjectivity is now the same as that of Lemma 2.1.14, giving us a map $\phi_3 : M \to N$ which makes the right-hand square commute.

$$\begin{tikzcd}
0 \arrow[r] & \Omega M \arrow[r, "\iota_M"] \arrow[d, "\phi_1"'] & P_M \arrow[r, "\pi_M"] \arrow[d, "\phi_2"'] & M \arrow[r] \arrow[d, "\phi_3"'] & 0 \\
0 \arrow[r] & \Omega N \arrow[r, "\iota_N"] & P_N \arrow[r, "\pi_N"] & N \arrow[r] & 0
\end{tikzcd}$$

Injectivity: Suppose we have a map $\phi : M \to N$ with the property that $\Omega(\phi)$ represents 0 in $\underline{\mathrm{Hom}}_R(\Omega M, \Omega N)$; that is, $\Omega(\phi)$ factors through a projective module P. Then, we have the following diagram in which $\phi' : P_M \to P_N$ is the map induced from ϕ:

$$\begin{tikzcd}
0 \arrow[r] & \Omega M \arrow[rr, "\iota_M"] \arrow[dr, "\alpha"] \arrow[dd, "\Omega(\phi)"'] & & P_M \arrow[r, "\pi_M"] \arrow[dl, "\beta"'] \arrow[dd, "\phi'"] & M \arrow[r] \arrow[dd, "\phi"] & 0 \\
 & & P \arrow[dl, "\gamma"'] \arrow[dr, "\delta"] & & & \\
0 \arrow[r] & \Omega N \arrow[rr, "\iota_N"] & & P_N \arrow[r, "\pi_N"] & N \arrow[r] & 0
\end{tikzcd}$$

The map $\beta : P_M \to P$ exists and makes the top triangle commute by Lemma 3.2.7. $\delta : P \to P_N$ is defined to be the composite $\iota_N \circ \gamma$. From here, the proof of injectivity is the same as that of Lemma 2.1.14: ϕ factors through P_N and so represents 0 in $\underline{\mathrm{Hom}}_R(M, N)$. $\square$

Let us now consider the structure of ${}_R\mathsf{StMod}$. To that end, replace an object M by an object M_0 in ${}_R\mathsf{StMod}_c$ isomorphic to it. We will show that ${}_R\mathsf{StMod}$ is a triangulated category. Define a translation functor on ${}_R\mathsf{StMod}$ by $\tau(M) = \overline{B} \otimes M_0$; this acts as an inverse to the functor Ω:

Lemma 3.2.9. *τ and Ω are mutually inverse functors.*

Proof. It is easily checked that τ is indeed a functor, so we prove that it acts as an inverse to Ω.

We have

$$M \xrightarrow{\tau} \overline{B} \otimes M_0 \xrightarrow{\Omega} \Omega(\overline{B} \otimes M_0).$$

By Lemma 3.2.5, $M_0 \oplus P \cong \Omega(\overline{B} \otimes M_0) \oplus Q$, so M_0 is isomorphic to $\Omega(\overline{B} \otimes M_0)$ in ${}_R\mathsf{StMod}_c$. Since $M \simeq M_0$, it follows that $M \simeq \Omega(\overline{B} \otimes M_0)$.

Composing the other way:

$$M \xrightarrow{\Omega} \Omega M \xrightarrow{\tau} \overline{B} \otimes (\Omega M)_0.$$

If $B \otimes M$ has projective dimension at most r, then $B \otimes \Omega M$ has projective dimension at most $r-1$. $(\Omega M)_0$ is the module $\overline{B}^{\otimes r-1} \otimes \Omega^{r-1}(\Omega M) = \overline{B}^{\otimes r-1} \otimes \Omega^r M$ (see the reasoning in the proof of Lemma 3.2.4). It follows that $\overline{B} \otimes (\Omega M)_0 = \overline{B} \otimes (\overline{B}^{\otimes r-1} \otimes \Omega^r M) = \overline{B}^{\otimes r} \otimes \Omega^r M$, which is isomorphic to M in ${}_R\mathsf{StMod}_c$. $\square$

Define the distinguished triangles in ${}_R\mathsf{StMod}$ to be the triangles which are isomorphic to triangles of the form

$$M \to M' \to M'' \to \tau(M)$$

where

$$0 \to M \xrightarrow{\phi} M' \xrightarrow{\psi} M'' \to 0$$

is a short exact sequence whose modules lie in ${}_R\mathsf{StMod}_c$ and τ is the functor defined above. Note that it makes sense to discuss injective and surjective maps in this category (up to the addition of maps which factor through a projective module) because of Lemma 3.2.8. Given such a short exact sequence, the map $M' \to M''$ can be made injective by replacing M'' by $M'' \oplus (B \otimes M')$ which is isomorphic to it in ${}_R\mathsf{StMod}_c$: given a projective resolution

$$\cdots \to P_r \to \cdots \to P_1 \to P_0 \to M'' \to 0,$$

the sequence

$$\cdots \to P_r \to \cdots \to P_1 \to P_0 \oplus (B \otimes M') \to M'' \oplus (B \otimes M') \to 0$$

is also exact and indeed a projective resolution of $M'' \oplus (B \otimes M')$ (because $B \otimes M'$ is projective). It follows by Proposition 3.2.3 that these two modules are isomorphic in ${}_R\mathsf{StMod}_c$ and we have a short exact sequence

$$0 \to M' \xrightarrow{\psi'} M'' \oplus (B \otimes M') \to \operatorname{Coker} \psi' \to 0.$$

We have the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & M \arrow[r, "\phi'"] \arrow[d, "\phi"] & B \otimes M' \arrow[d, "\iota"] & & \\
0 \arrow[r] & M' \arrow[r, "\psi'"] \arrow[d, "\psi"] & M'' \oplus (B \otimes M') \arrow[r, "\overline{\psi}'"] \arrow[d, "\pi"] & \operatorname{Coker} \psi' \arrow[r] & 0 \\
 & M'' \arrow[r, equal] \arrow[d] & M'' \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

where $\phi' : m \mapsto 1 \otimes \phi(m)$, $\psi' : m' \mapsto (\psi(m'), 1 \otimes m')$ and ι and π are the inclusion and projection respectively. Note that the top square commutes:

$$\begin{aligned}
\iota(\phi(m)) &= \iota(1 \otimes \phi(m)) \\
&= (0, 1 \otimes \phi(m)) \\
&= (\psi(\phi(m)), 1 \otimes \phi(m)) \\
&= \psi'(\phi(m))
\end{aligned}$$

as does the bottom square:

$$\begin{aligned}\pi(\psi'(m')) &= \pi(\psi(m'), 1\otimes m') \\ &= \psi(m').\end{aligned}$$

We have a sequence

$$0 \to M \xrightarrow{\phi'} B\otimes M' \xrightarrow{\overline{\psi}' \circ \iota} \operatorname{Coker}\psi' \to 0$$

in which $\operatorname{Im}\phi' \subseteq \operatorname{Ker}(\overline{\psi}' \circ \iota)$:

$$\overline{\psi}' \circ \iota \circ \phi' = \overline{\psi}' \circ \psi' \circ \phi = 0.$$

Let $\sum_i b_i \otimes m_i' \in \operatorname{Ker}(\overline{\psi}' \circ \iota)\backslash \operatorname{Im}\phi'$. Then $\iota(\sum_i b_i \otimes m_i') \in \operatorname{Ker}\overline{\psi}' = \operatorname{Im}\psi'$, so we can find $\sum_i n_i' \in M'$ such that $\psi'(\sum_i n_i') = \iota(\sum_i b_i\otimes m_i')$. In particular, $\sum_i(\psi(n_i'), 1\otimes n_i') = \sum_i(0, b_i\otimes m_i')$ and so $\sum_i \psi(n_i') = 0$. This means that there exist $n_i \in M$ such that $\phi(\sum_i n_i) = \sum_i n_i'$, but it also means that $(\iota\circ\phi')(\sum_i n_i) = (\psi'\circ\phi)(\sum_i n_i)$, which equals $\iota(\sum_i b_i\otimes m_i')$, implying $\phi'(\sum_i n_i) = \sum_i b_i \otimes m_i'$ (because ι is injective). Contradiction. Thus $\operatorname{Ker}(\overline{\psi}' \circ \iota) \subseteq \operatorname{Im}\phi'$ and the above sequence is exact.

We have the diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & B\otimes M \arrow[r] \arrow[d] & \overline{B}\otimes M \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M \arrow[r] & B\otimes M' \arrow[r] & \operatorname{Coker}\psi' \arrow[r] & 0 \end{tikzcd}$$

and because the left hand vertical map is an equality, we can appeal to Proposition 3.2.3 to deduce that $\tau(M) = \overline{B}\otimes M$ is isomorphic to $\operatorname{Coker}\psi'$ in ${}_R\mathsf{StMod}_c$. Thus it is clear what to take for the map for the third morphism in the triangle

$$M \to M' \to M'' \to \tau(M).$$

We now show that the family of distinguished triangles just defined satisfy the axioms for a triangulated category:

• **(TR 1)** Let $u : A \to B$ be a morphism in ${}_R\mathsf{StMod}$. Replace A and B by modules A_0 and B_0 isomorphic to them in ${}_R\mathsf{StMod}_c$. Then we have a short exact sequence in ${}_R\mathsf{StMod}_c$

$$0 \to A_0 \xrightarrow{u} B_0 \to \operatorname{Coker} u \to 0$$

and we deduce that

$$A \to B \to \operatorname{Coker} u \to \tau(A)$$

is a distinguished triangle. It follows that every morphism can be embedded in a distinguished triangle.

The triangle $(id_M, 0, 0)$ is distinguished. For, replace a module M by a module M_0 in ${}_R\mathsf{StMod}_c$ isomorphic to it. The sequence

$$0 \to M_0 \xrightarrow{\cong} M_0 \to 0 \to 0$$

is exact and the result follows.

Suppose (u, v, w) is a triangle on (A, B, C) and suppose it is isomorphic to a distinguished triangle (u', v', w') on (A', B', C'). Then we have a commutative diagram

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "\cong"] & B \arrow[r, "v"] \arrow[d, "\cong"] & C \arrow[r, "w"] \arrow[d, "\cong"] & \tau(A) \arrow[d, "\cong"] \\ A' \arrow[r, "u'"] & B' \arrow[r, "v'"] & C' \arrow[r, "w'"] & \tau(A'). \end{tikzcd}$$

We may assume that these modules lie in ${}_R\mathsf{StMod}_c$. Since the bottom triangle is distinguished, it remains to check that that the top row in the diagram

$$\begin{tikzcd} & A \arrow[r, "u"] \arrow[d, "\cong"] & B \arrow[r, "v"] \arrow[d, "\cong"] & C \arrow[d, "\cong"] & \\ 0 \arrow[r] & A' \arrow[r, "u'"] & B' \arrow[r, "v'"] & C' \arrow[r, "w'"] & 0 \end{tikzcd}$$

is exact, which is patently the case.

• **(TR 2)** Let (u, v, w) be an exact triangle on (L, M, N). We have to show that $M \xrightarrow{v} N \xrightarrow{w} \tau(L)$ is an exact sequence in ${}_R\mathsf{StMod}_c$ and also that the map $\tau(L) \to \tau(M)$ is equal to $-\tau u$.

As we saw above, $0 \to M \xrightarrow{v'} N \oplus (B \otimes M) \xrightarrow{w} \tau(L) \to 0$ is a short exact sequence which in ${}_R\mathsf{StMod}_c$ is isomorphic to $0 \to M \xrightarrow{v} N \xrightarrow{w} \tau(L) \to 0$. The second point is easily proved. It follows that $(v, w, -\tau u)$ is a distinguished triangle.

A similar argument can be had for the triangle $(-\Omega w, u, w)$ on the other 'rotate' $(\Omega N, L, M)$.

• **(TR 3)** Suppose we have two distinguished triangles $L \xrightarrow{u} M \xrightarrow{v} M \xrightarrow{w} \tau(L)$ and $L' \xrightarrow{u'} M' \xrightarrow{v'} M' \xrightarrow{w'} \tau(L')$ and maps $f : L \to L'$ and $g : M \to M'$ making the first square

$$\begin{tikzcd} L \arrow[r, "u"] \arrow[d, "f"'] & M \arrow[r, "v"] \arrow[d, "g"'] & N \arrow[r, "w"] & \tau(L) \\ L' \arrow[r, "u'"] & M' \arrow[r, "v'"] & N' \arrow[r] & \tau(L') \end{tikzcd}$$

commute. We can find a well-defined map $h : N \to N'$ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "u"] \arrow[d, "f"] & M \arrow[r, "v"] \arrow[d, "g"] & N \arrow[r] \arrow[d, "\exists h"] & 0 \\ 0 \arrow[r] & L' \arrow[r, "u'"] & M' \arrow[r, "v'"] & N' \arrow[r] & 0 \end{tikzcd}$$

commutes (by exactness of the rows). Similarly, we can find a map $i : \tau(L) \to \tau(L')$ which can be shown to be equal to τf and so (f, g, h) is a morphism of triangles

- (**TR 4**) The Third Isomorphism Theorem for modules can be employed here because we are dealing with short exact sequences.

3.3 Derived categories

In this section we will show that there is an equivalence of categories between ${}_R\mathsf{StMod}$ and the quotient category formed by the derived categories of two familiar examples. First we introduce the notion of localisation. The following is from [29]

Definition 3.3.1. Let S be a collection of morphisms in a category $\mathcal{C}$. A ***localisation of $\mathcal{C}$ with respect to S*** is a category $S^{-1}\mathcal{C}$, together with a functor $q : \mathcal{C} \to S^{-1}\mathcal{C}$ such that

(i) $q(s)$ is an isomorphism in $S^{-1}\mathcal{C}$ for every $s \in S$.

(ii) Any functor $F : \mathcal{C} \to \mathcal{D}$ such that $F(s)$ is an isomorphism for all $s \in S$ factors uniquely through q. (It follows that $S^{-1}\mathcal{C}$ is unique up to equivalence.)

Definition 3.3.2. The ***derived category*** $D(\mathcal{C})$ of an abelian category $\mathcal{C}$ is the category

$$D(\mathcal{C}) := Q^{-1}K(\mathcal{C})$$

where Q is the collection of quasi-isomorphisms. It is a triangulated category.

Remark 3.3.3. Because we will be working with categories of modules there are various set-theoretic difficulties which we can ignore. Further information on these considerations can be found in Set-Theoretic Remark 10.3.3 of [29].

We now give an explicit description of the morphisms in $Q^{-1}K(\mathcal{C})$.

Definition 3.3.4. A collection S of morphisms in a category $\mathcal{C}$ is called a ***multiplicative system*** in $\mathcal{C}$, if it satisfies the folowing three self-dual axioms:

(i) S is closed under composition (if $s, t \in S$ are composable, then $st \in S$) and contains all identity morphisms ($id_X \in S$ for all objects X of $\mathcal{C}$).

(ii) (Ore condition) If $t : Z \to Y$ is in S, then for every $g : X \to Y$ in $\mathcal{C}$ there is a commutative diagram "$gs = tf$" in $\mathcal{C}$ with $s \in S$.

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d, "s"'] & Z \arrow[d, "t"] \\ X \arrow[r, "g"] & Y \end{tikzcd}$$

(The slogan is "$t^{-1}g = fs^{-1}$ for some f and s".)

(iii) (Cancellation) If $f, g : X \to Y$ are parallel morphisms in $\mathcal{C}$, then the following two conditions are equivalent:

- $sf = sg$ for some $s \in S$ with source Y.
- $ft = gt$ for some $t \in S$ with target X.

Definition 3.3.5. A chain in $\mathcal{C}$ of the form

$$fs^{-1} : X \xleftarrow{s} X_1 \xrightarrow{f} Y$$

is called ***a (left) "fraction"*** if $s \in S$. fs^{-1} is ***equivalent*** to $X \xleftarrow{t} X_2 \xrightarrow{g} Y$ just in case there is a fraction $X \leftarrow X_3 \to Y$ fitting into a commutative diagram in $\mathcal{C}$:

$$\begin{tikzcd} & X_1 \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & X_3 \arrow[l] \arrow[r] \arrow[u] \arrow[d] & Y \\ & X_2 \arrow[ul, "t"] \arrow[ur, "g"'] & \end{tikzcd}$$

$\mathrm{Hom}_S(X, Y)$ is used to denote the family of equivalence classes of such fractions.

Composition of fractions is defined as follows: given a fraction $X \leftarrow X' \xrightarrow{g} Y$ and $Y \xleftarrow{t} Y' \to Z$, we use the Ore condition to find a diagram

$$\begin{tikzcd} & W \arrow[r, "f"] \arrow[d, "s"] & Y' \arrow[r] \arrow[d, "t"] & Z \\ X & X' \arrow[l] \arrow[r, "g"] & Y & \end{tikzcd}$$

with $s \in S$; the composite is the class of the fraction $X \leftarrow W \to Z$ in $\mathrm{Hom}_S(X, Z)$. The $\mathrm{Hom}_S(X, Y)$ form the morphisms of $S^{-1}\mathcal{C}$, which has the same objects as $\mathcal{C}$.

Remark 3.3.6. There is a similar notion of "right fraction", from which one can also construct $S^{-1}\mathcal{C}$ using the equivalence classes of $t^{-1}g : X \xrightarrow{g} Y' \xleftarrow{t} Y$.

It is noted in [29], that the set Q of quasi-isomorphisms form a multiplicative system in $K(\mathcal{C})$, so we can describe the morphisms of the category $Q^{-1}K(\mathcal{C})$ using the above formulation.

Definition 3.3.7. Let $\mathcal{B}$ be a full subcategory of $\mathcal{C}$ and let S be a multiplicative system in $\mathcal{C}$ whose restriction $S \cap \mathcal{B}$ to $\mathcal{B}$ is also a multiplicative system. For legibility, write $S^{-1}\mathcal{B}$ for $(S \cap \mathcal{B})^{-1}\mathcal{B}$. $\mathcal{B}$ is called a ***localising subcategory of*** $\mathcal{C}$ (for S) if the natural functor $S^{-1}\mathcal{B} \to S^{-1}\mathcal{C}$ identifies $S^{-1}\mathcal{B}$ with the full subcategory of $S^{-1}\mathcal{C}$ on the objects of $\mathcal{B}$.

Lemma 3.3.8. *A full subcategory $\mathcal{B}$ of $\mathcal{C}$ is localising for S if whenever $C \to B$ is a morphism in S with B an object of $\mathcal{B}$, there is a morphism $B' \to C$ in $\mathcal{C}$ with B' an object of $\mathcal{B}$ such that the composite $B' \to B$ is in S.*

Proof. This is Lemma 10.3.13 (2) of [29]. □

Corollary 3.3.9. *If $\mathcal{B}$ is a localising subcategory of $\mathcal{C}$, and for every object C of $\mathcal{C}$ there is a morphism $C \to B$ is S with B an object of $\mathcal{B}$, then $S^{-1}\mathcal{B}$ is equivalent to $S^{-1}\mathcal{C}$.*

Suppose in addition that $S \cap \mathcal{B}$ consists of isomorphisms. Then $\mathcal{B}$ is equivalent to $S^{-1}\mathcal{C}$.

Proof. This is Corollary 10.3.14 of [29]. □

It is also shown in Weibel's book that the subcategories $K^b(\mathcal{A}), K^+(\mathcal{A}), K^-(\mathcal{A})$ of $K(\mathcal{A})$ are localising for Q.

3.4 Equivalences

We will now study complexes comprising modules from categories we have already considered. Throughout, $\Bbbk$ will denote a commutative ring.

Following Chapter II of the Appendix to [13], we write $K^{+,b}({}_R\mathsf{Mod})$ for the category whose objects are the chain complexes of objects in ${}_R\mathsf{Mod}$ which are bounded below (this corresponds to the first index, $+$) and whose homology is bounded (second index, b). The maps are chain homotopy classes of maps. We write $K^{b,b}({}_R\mathsf{Mod})$ for the full subcategory whose objects are bounded chain complexes and $K^{b,\phi}({}_R\mathsf{Mod})$ for the full subcategory with the further restriction that the homology is zero; i.e. exact complexes. The derived category $D^b({}_R\mathsf{Mod})$ is the quotient category $K^{b,b}({}_R\mathsf{Mod})/K^{b,\phi}({}_R\mathsf{Mod})$, formed by formally inverting quasi-isomorphisms. Details on forming quotient categories can be found in Gabriel [15], page 365.

We can also consider ${}_R\mathsf{Proj}$, the full subcategory of ${}_R\mathsf{Mod}$ whose objects are projective modules and we can form categories $K^{+,b}({}_R\mathsf{Proj})$, $K^{b,b}({}_R\mathsf{Proj})$, $K^{b,\phi}({}_R\mathsf{Proj})$ and $D^b({}_R\mathsf{Proj})$ in the same way as before. It is easy to see that $K^{b,\phi}({}_R\mathsf{Proj})$ is the trivial category with one object and one morphism (any complex is homotopic to zero by the Comparison Theorem for exact sequences), so that $K^{b,b}({}_R\mathsf{Proj})$ and $D^b({}_R\mathsf{Proj})$ are equivalent. Since ${}_R\mathsf{Mod}$ has enough projectives we may replace objects by projective resolutions. Given any object X_* in $K^{b,b}({}_R\mathsf{Mod})$, there is an object P_* in $K^{+,b}({}_R\mathsf{Mod})$ and a map of chain complexes $P_* \to X_*$ inducing an isomorphism on homology. Such a map is called a projective resolution of X_*. The following shows this explicitly:

Definition 3.4.1. Let $\mathcal{A}$ be an abelian category that has enough projectives. A ***(left) Cartan-Eilenberg*** resolution $P_{*,*}$ of a chain complex A_* in $\mathcal{A}$ is an upper half-plane double-complex ($P_{p,q} = 0$ if $q < 0$), consisting of projective objects of $\mathcal{A}$, together with a chain map ("augmentation") $P_{*,0} \xrightarrow{\varepsilon} A_*$ such that for every p

(i) If $A_p = 0$, the column $P_{p,*}$ is zero,

(ii) The maps on boundaries and homology

$$B_p(\varepsilon) : B_p(P, d^h) \to B_p(A)$$

$$H_p(\varepsilon) : H_p(P, d^h) \to H_p(A)$$

are projective resolutions in $\mathcal{A}$, where $B_p(P, d^h)$ denotes the horizontal boundaries in the (p, q) spot, that is, the chain complex whose q^{th} term is $d^h(P_{p+1,q})$. The chain complexes $Z_p(p, d^h)$ and $H_p(P, d^h) = Z_p(p, d^h)/B_p(P, d^h)$ are defined similarly.

Remark 3.4.2. It is shown in [29] that the total complex $\mathrm{Tot}^{\oplus}(P) \to A$ is a quasi-isomorphism in $\mathcal{A}$. In case A is bounded above, the object P_* of $K^{+,b}({}_R\mathsf{Mod})$ described above is exactly this total complex.

Lemma 3.4.3. *Let Q an object of $K^{+,b}({}_R\mathsf{Mod})$ and P an object of $K^{+,b}({}_R\mathsf{Proj})$. Then every quasi-isomorphism $t : Q \to P$ is a split surjection in $K^{+,b}({}_R\mathsf{Mod})$.*

Proof. The mapping cone $M(t) = T(Q) \oplus P$ is exact by Lemma 3.1.10 and there is a natural map $\alpha(t) : P \to M(t); p \mapsto (0, p)$. The Comparison Theorem shows us that $\alpha(t)$

is null-homotopic by a chain homotopy $v = (s, k)$ from P to $Q \oplus T^{-1}(P)$. Thus

$$\begin{aligned}(0,p) &= \alpha(t)(p) \\ &= (d^{M(t)}v + vd^P)(p) \\ &= (-d^Q s(p), d^P k(p) - ts(p)) + (sd^P(p), kd^P(p)) \\ &= (-d^Q s(p) + sd^P(p), d^P k(p) - ts(p) + kd^P(p)).\end{aligned}$$

From the first co-ordinate, we see that s is a chain map, and from the second, that $-ts = id_P - d^P k(p) - kd^P(p)$ (i.e. k is a chain homotopy equivalence $-ts \approx id_P$. Hence, $-ts = id_P$ in $K^{+,b}({}_R\mathsf{Mod})$. □

Remark 3.4.4. Because the minus sign is annoying, we will suppress it on $-s$ and write $ts = id_P$ in $K^{+,b}({}_R\mathsf{Mod})$.

Theorem 3.4.5. *There is an equivalence of categories*

$$D^b({}_R\mathsf{Mod}) \to K^{+,b}({}_R\mathsf{Proj}).$$

Proof. We have already seen that to each object X of $K^{+,b}({}_R\mathsf{Mod})$ we can associate an object P of $K^{+,b}({}_R\mathsf{Proj})$ with $P \to X$ a quasi-isomorphism. If $X \to Y$ is a quasi-isomorphism, then so is $P \to Y$. It follows from Lemma 3.3.8 that $K^{+,b}({}_R\mathsf{Proj})$ is a localising subcategory of $K^{+,b}({}_R\mathsf{Mod})$. By Chapter II of the appendix to [13], $D^b({}_R\mathsf{Mod})$ is equivalent to $K^{+,b}({}_R\mathsf{Mod})/K^{+,\phi}({}_R\mathsf{Mod})$, so $D^b({}_R\mathsf{Mod})$ is equivalent to $K^{+,b}({}_R\mathsf{Proj})/K^{+,\phi}({}_R\mathsf{Proj})$. By Corollary 3.3.9, it suffices to show that every quasi-isomorphism in $K^{+,b}({}_R\mathsf{Proj})$ is an isomorphism to obtain the theorem.

Let P, Q be objects of $K^{+,b}({}_R\mathsf{Proj})$ and $t : P \to Q$ a quasi-isomorphism. By Lemma 3.4.3, there is a map $s : Q \to P$ such that $ts = id_Q$ in $K^{+,b}({}_R\mathsf{Proj})$. Interchanging the rôles of P and Q, s and t, we see that $su = id_P$ for some u. In $K^{+,b}({}_R\mathsf{Proj})$, we have $t = tsu = u$ and so t is an isomorphism in $K^{+,b}({}_R\mathsf{Proj})$ with $t^{-1} = s$. □

$D^b({}_R\mathsf{Proj})$ is a thick subcategory of $D^b({}_R\mathsf{Mod})$ and so we may form the quotient category

$$D^b({}_R\mathsf{Mod})/D^b({}_R\mathsf{Proj}).$$

The discussion of equivalences above shows that this quotient is equivalent to

$$K^{+,b}({}_R\mathsf{Proj})/K^{b,b}({}_R\mathsf{Proj}).$$

In the quotient, truncating a chain complex of projectives by replacing the finitely many modules below a given degree by zero results in an isomorphic complex. It follows that we can truncate above the homology so that every complex is isomorphic to a complex which is exact except in a single degree. Such a complex is a projective resolution of a module in ${}_R\mathsf{Mod}$, possibly shifted in degree. It follows that every object in the quotient is isomorphic to an object of this form.

There is a functor

$$ {}_R\mathsf{Mod} \to K^{+,b}({}_R\mathsf{Proj}) $$

which sends a module to a projective resolution. Such a resolution is well-defined up isomorphism in $K^{+,b}({}_R\mathsf{Proj})$ (by the Comparison Theorem for exact sequences).

The following digression includes material from [23] which serendipitously, is exactly the formulation we need to advance our discussion.

Definition 3.4.6. Let $\mathbf{C}$ and $\mathbf{D}$ be bounded below chain complexes of projectives. Then we can form three chain complexes called the ***total complex***, the ***hypercohomology complex*** and the ***Vogel complex***.

The hypercohomology complex is denoted $\mathbf{Hom}_R(\mathbf{C},\mathbf{D})$, the group of n-chains is defined by

$$ \mathbf{Hom}_R(\mathbf{C},\mathbf{D})_n := \prod_{p+q=n} \mathrm{Hom}_R(C_{-p}, D_q) $$

and the differential $\mathbf{d} : \mathbf{Hom}_R(\mathbf{C},\mathbf{D})_n \to \mathbf{Hom}_R(\mathbf{C},\mathbf{D})_{n-1}$ is defined by

$$ \mathbf{d}(\phi) := d' \circ \phi - (-1)^n \phi \circ d, $$

where d and d' denote the differentials in $\mathbf{C}$ and $\mathbf{D}$ respectively. For example, $\mathbf{Hom}_R(\mathbf{C},\mathbf{D})_1$ comprises all maps ϕ of the form

$$ \begin{tikzcd} \cdots \arrow[r] & C_{i+1} \arrow[r] \arrow[dl, "\phi_{i+1}"'] & C_i \arrow[r] \arrow[dl, "\phi_i"'] & C_{i-1} \arrow[r] \arrow[dl, "\phi_{i-1}"'] & \cdots \arrow[dl] \\ \cdots \arrow[r] & D_{i+1} \arrow[r] & D_i \arrow[r] & D_i \arrow[r] & \cdots \end{tikzcd} $$

and $\mathbf{Hom}_R(\mathbf{C},\mathbf{D})_0$ comprises all maps ϕ of the form

$$ \begin{tikzcd} \cdots \arrow[r] & C_{i+1} \arrow[r] \arrow[d, "\phi_{i+1}"] & C_i \arrow[r] \arrow[d, "\phi_i"] & C_{i-1} \arrow[r] \arrow[d, "\phi_{i-1}"] & \cdots \\ \cdots \arrow[r] & D_{i+1} \arrow[r] & D_i \arrow[r] & D_i \arrow[r] & \cdots \end{tikzcd} $$

In particular, chain maps are those maps $\phi \in \mathbf{Hom}_R(\mathbf{C},\mathbf{D})_0$ for which

$$d' \circ \phi - \phi \circ d = 0;$$

that is, the chain maps are the elements of $\operatorname{Ker}\mathbf{d}_0$. Also, the nullhomotopic maps are those maps $\phi \in \mathbf{Hom}_R(\mathbf{C},\mathbf{D})_0$ for which

$$\phi = d' \circ \psi + \psi \circ d$$

for some $\psi \in \mathbf{Hom}_R(\mathbf{C},\mathbf{D})_1$; that is, $\phi = \mathbf{d}_1(\psi) \in \operatorname{Im}\mathbf{d}_1$.

The total complex $\operatorname{Tot}(\mathbf{C},\mathbf{D})$ is the subcomplex of the hypercohomology complex in which the n^{th} chain group is given by

$$\operatorname{Tot}(\mathbf{C},\mathbf{D})_n := \bigoplus_{p+q=n} \operatorname{Hom}_R(C_{-p}, D_q).$$

The Vogel complex is defined to be the quotient of the hypercohomology complex by the total complex. Thus we have a short exact sequence of chain complexes:

$$0 \to \operatorname{Tot}(\mathbf{C},\mathbf{D}) \to \mathbf{Hom}_R(\mathbf{C},\mathbf{D}) \to \widehat{\mathbf{Hom}}_R(\mathbf{C},\mathbf{D}) \to 0,$$

with the Vogel complex at the right.

Theorem 3.4.7. *Let* $\mathbf{P}$ *and* $\mathbf{Q}$ *be non-negative chain complexes of projective left* R*-modules and suppose that both have homology concentrated in degree zero, with* $H_0(\mathbf{P}) = M$ *and* $H_0(\mathbf{Q}) = N$*. Then*

(i) the n^{th} *homology group of* $\mathbf{Hom}_R(\mathbf{P},\mathbf{Q})$ *is equal to* $\operatorname{Ext}_R^{-n}(M,N)$*; and*

(ii) the n^{th} *homology group of* $\widehat{\mathbf{Hom}}_R(\mathbf{P},\mathbf{Q})$ *is equal to* $\widehat{\operatorname{Ext}}_R^{-n}(M,N)$.

The morphisms of $K^{+,b}({}_R\mathsf{Proj})/K^{b,b}({}_R\mathsf{Proj})$ are just elements of the zeroeth homology of the Vogel complex: the maps in this category are chain maps and so elements of $\operatorname{Ker}\mathbf{d}_0$. We restrict ourselves to chain homotopy classes of maps, so we factor out the nullhomotopic maps; i.e. $\operatorname{Im}\mathbf{d}_1$. Thus we have a functor

$$\Phi : {}_R\mathsf{StMod} \to K^{+,b}({}_R\mathsf{Proj})/K^{b,b}({}_R\mathsf{Proj})$$

which induces an isomorphism on Hom sets by dint of Theorem 3.4.7. Every object in the quotient category can be truncated as described before to give a projective resolution of some module M sitting in degree $d \geq 0$ and hence is isomorphic to the image under Φ of the translate $\tau^d(M)$ in ${}_R\mathsf{StMod}$. We have:

Theorem 3.4.8. *Let G be a group and R a strongly G-graded $\Bbbk$-algebra. Then there is an equivalence of categories*

$$_R\mathsf{StMod} \to D^b(_R\mathsf{Mod})/D^b(_R\mathsf{Proj}).$$

We would like to arrive at a similar result for $_R\mathsf{stmod}$, $_R\mathsf{mod}$ and $_R\mathsf{proj}$, but this is not possible with the machinery used to obtain the above theorem: given a module M of type FP_∞, there is no reason why the translate $\tau(M) = \overline{B} \otimes M$ should be of type FP_∞; however, if we restrict the class of groups we work with and re-define the translation functor, we will see in the next section that we can achieve our aim.

3.5 The inclusion of stmod in StMod

Much of the work we need to do in this section was done in the previous section.

Let G be an $\mathbf{LH}\mathfrak{F}$-group and R a Noetherian strongly G-graded $\Bbbk$-algebra. If M is an object of $_R\mathsf{stmod}$, then by Lemma 3.2.2, $B \otimes M$ has finite projective dimension as an R-module and so $_R\mathsf{stmod}$ can be realised as a full subcategory of $_R\mathsf{StMod}$. A result of Kaplansky says that every projective module is a direct sum of countably generated projective modules. It follows that if $B \otimes M$ is projective then it has a finitely generated projective summand P which contains the image of $i \otimes id_M : M \to B \otimes M$ and so we have a diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & P \arrow[r] \arrow[d, tail] & P/M \arrow[r] \arrow[d, tail] & 0 \\
0 \arrow[r] & M \arrow[r] & B \otimes M \arrow[r] & \overline{B} \otimes M \arrow[r] & 0.
\end{tikzcd}$$

It follows that $P/M \cong \overline{B} \otimes M$ in $_R\mathsf{StMod}$. Note that P/M is a finitely generated R-module of type FP_∞.

If $B \otimes M$ has projective dimension $r \geq 0$, then in any projective resolution of M, $\Omega^r M$ is cofibrant and finitely generated (because R is Noetherian). By repeated use of Kaplansky's result, we obtain a series of commutative diagrams with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \Omega^r M \arrow[r] \arrow[d, equal] & P_{r-1} \arrow[r] \arrow[d, tail] & N_{r-1} \arrow[r] \arrow[d, tail] & 0 \\
0 \arrow[r] & \Omega^r M \arrow[r] & B \otimes \Omega^r M \arrow[r] & \overline{B} \otimes \Omega^r M \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & N_{r-1} \arrow[r] \arrow[d, tail] & P_{r-2} \arrow[r] \arrow[d, tail] & N_{r-2} \arrow[r] \arrow[d, tail] & 0 \\
0 \arrow[r] & \overline{B} \otimes \Omega^r M \arrow[r] & B \otimes \overline{B} \otimes \Omega^r M \arrow[r] & \overline{B}^{\otimes 2} \otimes \Omega^r M \arrow[r] & 0
\end{tikzcd}$$

$$\vdots$$

$$\begin{tikzcd}
0 \arrow[r] & N_{i+1} \arrow[r] \arrow[d, tail] & P_i \arrow[r] \arrow[d, tail] & N_i \arrow[r] \arrow[d, tail] & 0 \\
0 \arrow[r] & \overline{B}^{\otimes r-i-1} \otimes \Omega^r M \arrow[r] & B \otimes \overline{B}^{\otimes r-i-1} \otimes \Omega^r M \arrow[r] & \overline{B}^{\otimes r-i} \otimes \Omega^r M \arrow[r] & 0
\end{tikzcd}$$

$$\vdots$$

$$\begin{tikzcd}
0 \arrow[r] & N_0 \arrow[r] \arrow[d, tail] & P_{-1} \arrow[r] \arrow[d, tail] & N_{-1} \arrow[r] \arrow[d, tail] & 0 \\
0 \arrow[r] & \overline{B}^{\otimes r} \otimes \Omega^r M \arrow[r] & B \otimes \overline{B}^{\otimes r} \otimes \Omega^r M \arrow[r] & \overline{B}^{\otimes r+1} \otimes \Omega^r M \arrow[r] & 0
\end{tikzcd}$$

In which each of the P_i are projective and finitely generated, each of the $N_i := P_i/N_{i+1}$, and each of the middle terms in the bottom rows of the diagrams are projective. Splicing these diagrams together we obtain two exact sequences

$$0 \to \Omega^r M \to P_{r-1} \to P_{r-2} \to \cdots \to P_{-1} \to N_{-1} \to 0$$

$$0 \to \Omega^r M \to B \otimes \Omega^r M \to B \otimes \overline{B} \otimes \Omega^r M \to \cdots \to B \otimes \overline{B}^{\otimes r} \otimes \Omega^r M \to \overline{B}^{\otimes r+1} \otimes \Omega^r M \to 0.$$

In particular, $\overline{B}^{\otimes r+1} \otimes \Omega^r M \cong N_{-1}$ in ${}_R\mathsf{StMod}$ and N_{-1} is a finitely generated module of type FP_∞. Thus we can define a translation functor on ${}_R\mathsf{stmod}$ by $\tau(M) = \overline{B} \otimes M$, because

$$\tau(M) = \overline{B} \otimes M \cong \overline{B} \otimes \overline{B}^{\otimes r} \otimes \Omega^r M \cong N_{-1}$$

which, as we have observed, lies in ${}_R\mathsf{stmod}$. It follows that this functor is the same as the one defined before on ${}_R\mathsf{StMod}$, up to isomorphism and that ${}_R\mathsf{stmod}$ is a thick subcategory of ${}_R\mathsf{StMod}$ (that is, closed under taking direct summands (every direct summand of a module of type FP_∞ is again a module of type FP_∞) and triangulated with the same definition for distinguished triangles). We have the following:

Theorem 3.5.1. *If G is an arbitrary group and R a strongly G-graded $\Bbbk$-algebra, then there is an equivalence of categories*

$${}_R\mathsf{StMod} \to D^b({}_R\mathsf{Mod})/D^b({}_R\mathsf{Proj}).$$

If G is an $\mathbf{LH}\mathfrak{F}$*-group and R a Noetherian strongly G-graded* $\Bbbk$*-algebra, then in addition to above equivalence, we have that*

$$_R\mathsf{stmod} \to D^b({}_R\mathsf{mod})/D^b({}_R\mathsf{proj})$$

is an equivalence of categories. These equivalences are compatible with the inclusions induced by the inclusion of $_R\mathsf{mod}$ *as a subcategory of* $_R\mathsf{Mod}$.

Chapter 4

Finiteness conditions and polynomial rings

Let M be a left module over a ring R. At first glance the questions "does M admit a resolution by finitely generated projective modules?" and "does M have finite projective dimension?" would, as in the words of Cornick and Kropholler, "appear to address different qualitative properties of projective resolutions, and one would not expect either one to imply the other". However, it turns out that for certain rings one does indeed imply the other. In this chapter we will look at examples of rings where this is the case.

4.1 Some motivating examples

We begin with a lemma which we will not need immediately.

Lemma 4.1.1. *Let $\Bbbk$ be a field, G a group, H a subgroup of G and M a left $\Bbbk G$-module. Then as $\Bbbk G$-modules,*

$$\Bbbk G \otimes_{\Bbbk H} M \cong \Bbbk[G/H] \otimes M$$

where $\Bbbk G \otimes_{\Bbbk H} M$ has the action of $\Bbbk G$ on the left and $\Bbbk[G/H] \otimes M$ has the diagonal action.

Proof. Define maps

$$\phi : \Bbbk G \otimes_{\Bbbk H} M \to \Bbbk[G/H] \otimes M;\; g \otimes m \mapsto gH \otimes gm$$

and

$$\psi : \Bbbk[G/H] \otimes M \to \Bbbk G \otimes_{\Bbbk H} M;\; gH \otimes m \mapsto g \otimes g^{-1}m.$$

ϕ and ψ are well-defined since

$$\begin{array}{ccc} gh \otimes m & \xrightarrow{\phi} & ghH \otimes ghm \\ \| & & \| \\ g \otimes hm & \xrightarrow{\phi} & gH \otimes ghm \end{array}$$

and

$$\begin{array}{ccc} gH \otimes m & \xrightarrow{\psi} & g \otimes g^{-1}m \\ \| & & \| \\ ghH \otimes m & \xrightarrow{\psi} & gh \otimes h^{-1}g^{-1}m. \end{array}$$

ϕ and ψ are obviously additive maps. They are $\Bbbk G$-maps since

$$g^{'}.\phi(g \otimes m) = g^{'}.(gH \otimes gm) = g^{'}gH \otimes g^{'}gm = \phi(g^{'}.(g \otimes m))$$

and

$$g^{'}.\psi(gH \otimes m) = g^{'}.(g \otimes g^{-1}m) = g^{'}g \otimes g^{-1}m = \psi(g^{'}.(gH \otimes m)).$$

ϕ and ψ are mutually inverse since

$$\psi(\phi(g \otimes m)) = \psi(gH \otimes gm) = g \otimes m$$

and

$$\phi(\psi(gH \otimes m)) = \phi(g \otimes g^{-1}m) = gH \otimes m.$$ □

Proposition 4.1.2. *Let $\Bbbk$ be a field, G a free abelian group of finite rank and M a left $\Bbbk G$-module of type* FP_∞. *Then M has finite projective dimension. Moreover, the left global dimension of $\Bbbk G$ is finite and equal to the rank of G.*

Proof. Let the rank of G be n, then the trivial $\Bbbk G$-module $\Bbbk$ has projective dimension at most n:

$$0 \to P_n \to \ldots \to P_0 \to \Bbbk \to 0$$

(this is the Koszul resolution. For details on its construction see Corollary 4.5.5 on page 114 of [29]). Because this sequence is split as a sequence of $\Bbbk$-modules, applying $- \otimes M$ gives us an exact sequence of $\Bbbk G$-modules

$$0 \to P_n \otimes M \to \ldots \to P_0 \otimes M \to M \to 0$$

which is a projective resolution of M because

$$\mathrm{Hom}_{\Bbbk G}(P_i \otimes M, -) \cong \mathrm{Hom}_{\Bbbk}(M, \mathrm{Hom}_{\Bbbk G}(P_i, -))$$

which is a composite of exact functors and so, by Proposition 1.6.1, each $P_i \otimes M$ is a projective $\Bbbk G$-module. It follows that M has finite projective dimension.

The fact that M is of type FP_∞ was not used above, we may therefore conclude that the left global dimension of $\Bbbk G$ is finite and equal to n. □

Proposition 4.1.3. *Let $\Bbbk$ be a field, G a free abelian group of infinite rank and M a left $\Bbbk G$-module of type FP_∞. Then M has finite projective dimension.*

Proof. Since all modules of type FP_∞ are finitely presented, we have an exact sequence

$$(\Bbbk G)^r \xrightarrow{\alpha} (\Bbbk G)^d \to M \to 0$$

of $\Bbbk G$-modules. The map α can be represented as an $r \times d$ matrix L with entries from $\Bbbk G$. Let H be the subgroup of G generated by those $g \in G$ appearing in entries of L. Then we have a finite presentation of a $\Bbbk H$-module U:

$$(\Bbbk H)^r \xrightarrow{\alpha} (\Bbbk H)^d \to U \to 0.$$

Since H is free abelian of finite rank, we have, by Proposition 4.1.2, that U has finite projective dimension over $\Bbbk H$. Since $\Bbbk G = \oplus_{t \in T} t\Bbbk H$ where T is a transversal to H in G; that is, $\Bbbk G$ is free as a $\Bbbk H$-module and so $\Bbbk G \otimes_{\Bbbk H} -$ is an exact functor. Therefore

$$(\Bbbk G)^r \xrightarrow{\alpha} (\Bbbk G)^d \to \Bbbk G \otimes_{\Bbbk H} U \to 0$$

is an exact sequence, $M \cong \Bbbk G \otimes_{\Bbbk H} U$ and so M has finite projective dimension. □

Let $\Bbbk$ be a field and let $G = C_\infty \wr C_\infty = B \rtimes C_\infty$ where $C_\infty = \langle t \rangle$, $B = \langle x_i : i \in \mathbb{Z} \rangle$ and for all i, j,

$$t^{-1} x_i t = x_{i+1} \text{ and } [x_i, x_j] = 1.$$

Then we may view $\Bbbk G$ as a polynomial ring $\Bbbk B[t^{-1}, t]$, subject to the above relations.

Lemma 4.1.4. *Let $\Bbbk$ be a field and $G = B \rtimes C_\infty$, as defined above. Then we have a short exact sequence*

$$0 \to \Bbbk[G/B] \xrightarrow{\alpha} \Bbbk[G/B] \xrightarrow{\beta} \Bbbk \to 0$$

of left $\Bbbk G$-modules.

Proof. As $\Bbbk G$-modules, $\Bbbk[G/B] \cong \Bbbk[t^{-1}, t]$. For each $i \in \mathbb{Z}$, define maps

$$\alpha : \Bbbk[t^{-1}, t] \to \Bbbk[t^{-1}, t] \ ; \ k_i t^i \mapsto k_i t^{i+1} - k_i t^i$$

$$\beta : \Bbbk[t^{-1}, t] \to \Bbbk \; ; \; k_i t^i \mapsto k_i.$$

We must check the following:

- β is surjective. If $k \in \Bbbk$, then $\beta(k) = k$.
- $\operatorname{Im}\alpha \subseteq \operatorname{Ker}\beta$.

$$\beta(\alpha(k_i t^i)) = \beta(k_i t^{i+1} - k_i t^i) = k_i - k_i = 0.$$

- $\operatorname{Ker}\beta \subseteq \operatorname{Im}\alpha$. If $u = \sum_{i=-n}^{m} k_i t^i \in \operatorname{Ker}\beta$, then $\sum_{i=-n}^{m} k_i = 0$. Hence

$$\begin{aligned} u &= \sum_{i=-n}^{m} k_i t^i - \sum_{i=-n}^{m} k_i \\ &= \sum_{i=-n}^{m} (k_i t^i - k_i) \\ &= \sum_{i=-n}^{-1} (k_i t^i - k_i) + \sum_{i=1}^{m} (k_i t^i - k_i) \end{aligned}$$

because $k_0 t^0 - k_0 = k_0 - k_0 = 0$.

We now show that for each $i \geq 1$, $k_i t^i - k_i \in \operatorname{Im}\alpha$, the proof for each $i \leq -1$ being similar. These results, taken together, will prove that $u \in \operatorname{Im}\alpha$. It is apparent that $\sum_{j=1}^{i-1}(-k_i t^{i-j} + k_i t^{i-j}) = 0$. Upon re-associating the terms in the sum, we see that

$$\begin{aligned} k_i t^i - k_i &= k_i t^i + [\sum_{j=1}^{i-1}(-k_i t^{i-j} + k_i t^{i-j})] - k_i \\ &= (k_i t^i - k_i t^{i-1}) + (k_i t^{i-1} - k_i t^{i-2}) + \ldots + (k_i t - k_i) \\ &= \alpha(k_i t^{i-1}) + \alpha(k_i t^{i-2}) + \ldots + \alpha(k_i). \end{aligned}$$

- α is injective. If z is a non-zero element of $\operatorname{Ker}\alpha$ with $z = \sum k_i t^i$ and n is the largest index with $k_n \neq 0$, then the coefficient of $t^{n+1} = 0$ in $\alpha(z)$ is k_n, a contradiction since if $k_n t^n$ is non zero, then so is $k_n t^{n+1}$. Thus $\operatorname{Ker}\alpha = 0$. □

The following will be used in the next section.

Proposition 4.1.5. *Let $\Bbbk$ be a field and let $G = C_\infty \wr C_\infty$. Then every $\Bbbk G$-module of type FP_∞ has finite projective dimension.*

Remark 4.1.6. The proposition can be deduced from Theorem A on page 53 of [11] and more directly from Theorem B on page 44 of [10]. For, $G = C_\infty \wr C_\infty$ belongs to $\mathbf{H}\mathfrak{F}$ and is torsion free. It follows that if M is a $\Bbbk G$-module of type FP_∞ then M is trivially projective over $\Bbbk F$ for all finite subgroups F of G because $\Bbbk$ is a field. Taking $M = N$

and $i = 0$, these theorems imply $\widehat{\operatorname{Ext}}^0_{\Bbbk G}(M,M) = 0$; that is, $pd_{\Bbbk G}(M) < \infty$. We prove the proposition directly to make explicit why the method of attack employed for this example won't work for a related result that we will consider in the next section.

Proof. By Lemma 4.1.4 we have a short exact sequence

$$0 \to \Bbbk[G/B] \to \Bbbk[G/B] \to \Bbbk \to 0.$$

Let M be a $\Bbbk G$-module of type FP_∞. The functor $- \otimes M$ is exact (because M is free as a $\Bbbk$-module) and we get an exact sequence

$$0 \to \Bbbk[G/B] \otimes M \to \Bbbk[G/B] \otimes M \to M \to 0.$$

By Lemma 4.1.1, $\Bbbk[G/B] \otimes M \cong \Bbbk G \otimes_{\Bbbk B} M$ with the action of $\Bbbk G$ on the left, so our exact sequence becomes

$$0 \to \Bbbk G \otimes_{\Bbbk B} M \to \Bbbk G \otimes_{\Bbbk B} M \to M \to 0.$$

Consider now the long exact sequence

$$\ldots \to \widehat{\operatorname{Ext}}^i_{\Bbbk G}(M, \Bbbk G \otimes_{\Bbbk B} M) \to \widehat{\operatorname{Ext}}^i_{\Bbbk G}(M,M) \to \widehat{\operatorname{Ext}}^{i+1}_{\Bbbk G}(M, \Bbbk G \otimes_{\Bbbk B} M) \to \ldots$$

of complete cohomology groups arising from our short exact sequence. Assume (for a contradiction) that M doesn't have finite projective dimension. Then by Lemma 2.1.8, $\widehat{\operatorname{Ext}}^0_{\Bbbk G}(M,M) \neq 0$. Feeding this information into the long exact sequence tells us that $\widehat{\operatorname{Ext}}^i_{\Bbbk G}(M, \Bbbk G \otimes_{\Bbbk B} M)$ cannot be zero for both $i = 0, 1$.

Now, $\Bbbk G \otimes_{\Bbbk B} M = \varinjlim_H \Bbbk G \otimes_{\Bbbk H} M$ where the limit is taken over the direct system comprising the finitely generated subgroups H of B, ordered by inclusion. Since M is of type FP_∞, we have by Lemma 2.1.11 that $\widehat{\operatorname{Ext}}^i_{\Bbbk G}(M, -)$ commutes with filtered colimits; that is,

$$\widehat{\operatorname{Ext}}^i_{\Bbbk G}(M, \varinjlim_H \Bbbk G \otimes_{\Bbbk H} M) = \varinjlim_H \widehat{\operatorname{Ext}}^i_{\Bbbk G}(M, \Bbbk G \otimes_{\Bbbk H} M)$$

so if $\widehat{\operatorname{Ext}}^i_{\Bbbk G}(M, \Bbbk G \otimes_{\Bbbk H} M) \neq 0$ for one of $i = 0, 1$, then it is non-zero for at least one finitely generated subgroup H. Fix such an H. Then, since H is free abelian of finite rank, the left global dimension of $\Bbbk H$ is finite by Proposition 4.1.2 and M admits a finite resolution by projective $\Bbbk H$-modules:

$$0 \to P_n \to \ldots \to P_0 \to M \to 0.$$

The functor $\Bbbk G \otimes_{\Bbbk H} -$ is exact (because $\Bbbk G = \oplus_{t \in T} t\Bbbk H$ where T is a transversal to H in G) so applying this to the $\Bbbk H$-projective resolution of M gives us a $\Bbbk G$-projective resolution of $\Bbbk G \otimes_{\Bbbk H} M$:

$$0 \to \Bbbk G \otimes_{\Bbbk H} P_n \to \ldots \to \Bbbk G \otimes_{\Bbbk H} P_0 \to \Bbbk G \otimes_{\Bbbk H} M \to 0$$

but this means that

$$\widehat{\operatorname{Ext}}^0_{\Bbbk G}(M, \Bbbk G \otimes_H M) = \varinjlim_j \underline{\operatorname{Hom}}_{\Bbbk G}(\Omega^j M, \Omega^j(\Bbbk G \otimes_{\Bbbk H} M)) = 0$$

since $\Omega^j(\Bbbk G \otimes_{\Bbbk H} M) = 0$ for all $j \geq n$. Also,

$$\widehat{\operatorname{Ext}}^1_{\Bbbk G}(M, \Bbbk G \otimes_H M) = \varinjlim_j \underline{\operatorname{Hom}}_{\Bbbk G}(\Omega^{j+1} M, \Omega^j(\Bbbk G \otimes_{\Bbbk H} M)) = 0.$$

Contradiction. Therefore M has finite projective dimension. □

4.2 Skew polynomial rings

Let $\Bbbk$ be a field. We have already seen that the twisted polynomial algebra $\Bbbk G = \Bbbk B[t^{-1}, t]$ (where $G = C_\infty \wr C_\infty$) has the property that every module of type FP_∞ has finite projective dimension. Since B is a free abelian group of infinite rank, Proposition 4.1.3 tells us that every $\Bbbk B$-module of type FP_∞ has finite projective dimension. In the group G we have the relation $t^{-1}x_i t = x_{i+1}$ for all i. If we rewrite this as $x_i t = t x_{i+1}$, we can form the twisted polynomial algebra $\Bbbk B[t]$. Since $\Bbbk B \subset \Bbbk B[t] \subset \Bbbk B[t^{-1}, t]$, it raises the question whether every $\Bbbk B[t]$-module of type FP_∞ has finite projective dimension.

Immediately we are presented with the problem that we are no longer dealing with group algebras and so cannot appeal to group properties as we did in the proofs of the $\Bbbk B$ and $\Bbbk B[t^{-1}, t]$ cases. As we shall see, this difficulty can be obviated by considering the $\Bbbk B[t]$ case as one of skew polynomial rings.

Definition 4.2.1. Let R be a ring, M a left R-module and σ an automorphism of R. The ***conjugate module*** ${}^\sigma M$ is defined to be the left R-module whose underlying set is $\{{}^\sigma m : m \in M\}$, with the same abelian group structure as that of M and scalar multiplication given by $r.{}^\sigma m = {}^\sigma(\sigma(r)m)$ for all $m \in M$ and $r \in R$. Details of the construction of ${}^\sigma M$ can be found in [14].

Definition 4.2.2. Let R be a ring and σ an automorphism of R. A ***(right) σ-derviation*** is an additive map $\delta : R \to R$ such that for all $r, s \in R$,

$$\delta(rs) = r\delta(s) + \delta(r)\sigma(s).$$

Definition 4.2.3. Let R be a ring. A ***(right) skew derivation*** is a pair (σ, δ) where σ is a ring automorphism of R and δ is a right σ-derivation.

Definition 4.2.4. Let R be a ring and (σ, δ) a right skew derivation on R. The ***(right) skew polynomial ring over*** R, $R[x;\sigma,\delta]$, is the polynomial ring $R[x]$ with multiplication defined by

$$rx = x\sigma(r) + \delta(r)$$

for all $r \in R$.

Remark 4.2.5. We can write every element of $R[x;\sigma,\delta]$ in the form $\sum_{i=0}^{n} x^i r_i$ for some $n \in \mathbb{N}$ and $r_i \in R$. We shall not prove this suffice to say that the calculations required to express a given element in this form become somewhat involved as n increases:

$$rx = x\sigma(r) + \delta(r)$$

$$rx^2 = x^2\sigma^2(r) + x\{\delta(\sigma(r)) + \sigma(\delta(r))\} + \delta^2(r)$$

$$\begin{aligned} rx^3 &= x^3\sigma^3(r) + x^2\{\sigma^2(\delta(r)) + \sigma(\delta(\sigma(r))) + \delta(\sigma^2(r))\} \\ &\quad + x\{\sigma(\delta^2(r)) + \delta(\sigma(\delta(r))) + \delta^2(\sigma(r))\} + \delta^3(r). \end{aligned}$$

Remark 4.2.6. In light of Definition 4.2.4, we see that $\Bbbk B[t]$ is a right skew polynomial ring with $\delta = 0$.

Example 4.2.7. Let R be a ring and $T = R[x;\sigma,\delta]$. In general, ${}^{\sigma}N$ cannot be given the structure of a T-module without the following relationship between σ and δ:

Define an action $*$ of T on ${}^{\sigma}N$ by

$$r * {}^{\sigma}n = {}^{\sigma}(\sigma(r)n)$$

for $r \in R$ and

$$x * {}^{\sigma}n = {}^{\sigma}(xn).$$

We have that $rx = x\sigma(r) + \delta(r)$ in T, so we we need

$$(rx) * {}^{\sigma}n = (x\sigma(r)) * {}^{\sigma}n + \delta(r) * {}^{\sigma}n.$$

On one hand,

$$\begin{aligned} (rx) * {}^{\sigma}n &= r * (x * {}^{\sigma}n) \\ &= r * {}^{\sigma}(xn) \\ &= {}^{\sigma}(\sigma(r)xn) \\ &= {}^{\sigma}(x\sigma^2(r)n + \delta(\sigma(r))n). \end{aligned}$$

On the other,

$$\begin{aligned}(x\sigma(r)) * {}^{\sigma}n + \delta(r) * {}^{\sigma}n &= x * (\sigma(r) * {}^{\sigma}n) + {}^{\sigma}(\sigma(\delta(r))n) \\ &= x * {}^{\sigma}(\sigma^2(r)n) + {}^{\sigma}(\sigma(\delta(r))n) \\ &= {}^{\sigma}(x\sigma^2(r)n) + {}^{\sigma}(\sigma(\delta(r))n) \\ &= {}^{\sigma}(x\sigma^2(r)n + \sigma(\delta(r))n)\end{aligned}$$

so we have a well-defined action of T on ${}^{\sigma}N$ if σ and δ commute.

The following result will prove useful:

Lemma 4.2.8. *Let R be a ring and $R[x;\sigma,\delta]$ the right skew polynomial ring over R for some right skew derivation (σ,δ). Then $R[x;\sigma,\delta]$ is free as an R-module with basis $\{1,x,x^2,\ldots\}$.*

Proof. This is 2.4 on page 16 of [26]. □

Proposition 4.2.9. *Let R be a ring, $S = R[x;\sigma,\delta]$ the right skew polynomial ring over R for some right skew derivation (σ,δ) and M an S-module. Then there is a short exact sequence of S-modules*

$$0 \to S \otimes_R {}^{\sigma}M \xrightarrow{\lambda} S \otimes_R M \xrightarrow{\pi} M \to 0$$

where $\lambda(s \otimes {}^{\sigma}m) = sx \otimes m - s \otimes xm$ and $\pi(s \otimes m) = sm$.

Proof. This is the proposition on page 262 of [26]. □

We are now able to answer in the positive the question posed at the beginning of this section:

Proposition 4.2.10. *Every $\Bbbk B[t]$-module of type FP_∞ has finite projective dimension.*

Proof. Let M be a $\Bbbk B[t]$ module of type FP_∞. By Proposition 4.2.9 we have a short exact sequence

$$0 \to \Bbbk B[t] \otimes_{\Bbbk B} {}^{\sigma}M \to \Bbbk B[t] \otimes_{\Bbbk B} M \to M \to 0.$$

Applying the functor $\widehat{\mathrm{Ext}}^*_{\Bbbk B[t]}(M,-)$, we get the long exact sequence

$$\ldots \to \widehat{\mathrm{Ext}}^*_{\Bbbk B[t]}(M, \Bbbk B[t] \otimes_{\Bbbk B} M) \to \widehat{\mathrm{Ext}}^*_{\Bbbk B[t]}(M, M) \to \widehat{\mathrm{Ext}}^{*+1}_{\Bbbk B[t]}(M, \Bbbk B[t] \otimes_{\Bbbk B} {}^{\sigma}M) \to \ldots$$

Assume (for a contradiction) that M does not have finite projective dimension. Then $\widehat{\operatorname{Ext}}^0_{\Bbbk B[t]}(M, M) \neq 0$ by Lemma 2.1.8 and so exactness requires $\widehat{\operatorname{Ext}}^0_{\Bbbk B[t]}(M, \Bbbk B[t] \otimes_{\Bbbk B} M)$ and $\widehat{\operatorname{Ext}}^1_{\Bbbk B[t]}(M, \Bbbk B[t] \otimes_{\Bbbk B} {}^{\sigma}M)$ not both be zero.

Now, $\Bbbk B = \varinjlim_{\lambda} \Bbbk B_{\lambda}$ where the B_{λ} are the finitely generated subgroups of B, so $\Bbbk B[t] \otimes_{\Bbbk B} M = \varinjlim_{\lambda} \Bbbk B[t] \otimes_{\Bbbk B_{\lambda}} M$. Since M is of type FP_{∞}, $\widehat{\operatorname{Ext}}^*_{\Bbbk B[t]}(M, -)$ commutes with filtered colimits by Lemma 2.1.11, so

$$\begin{aligned}\widehat{\operatorname{Ext}}^0_{\Bbbk B[t]}(M, \Bbbk B[t] \otimes_{\Bbbk B} M) &= \widehat{\operatorname{Ext}}^0_{\Bbbk B[t]}(M, \varinjlim_{\lambda} \Bbbk B[t] \otimes_{\Bbbk B_{\lambda}} M) \\ &= \varinjlim_{\lambda} \widehat{\operatorname{Ext}}^0_{\Bbbk B[t]}(M, \Bbbk B[t] \otimes_{\Bbbk B_{\lambda}} M).\end{aligned}$$

Similarly,

$$\widehat{\operatorname{Ext}}^1_{\Bbbk B[t]}(M, \Bbbk B[t] \otimes_{\Bbbk B} {}^{\sigma}M) = \varinjlim_{\lambda} \widehat{\operatorname{Ext}}^1_{\Bbbk B[t]}(M, \Bbbk B[t] \otimes_{\Bbbk B_{\lambda}} {}^{\sigma}M).$$

If $\widehat{\operatorname{Ext}}^0_{\Bbbk B[t]}(M, \Bbbk B[t] \otimes_{\Bbbk B} M) \neq 0$ then $\widehat{\operatorname{Ext}}^0_{\Bbbk B[t]}(M, \Bbbk B[t] \otimes_{\Bbbk B_{\lambda}} M) \neq 0$ for some λ, but B_{λ} is free abelian of finite rank, so by Proposition 4.1.2, the left global dimension of $\Bbbk B_{\lambda}$ is finite and M admits a finite resolution

$$0 \to P_n \to \ldots \to P_0 \to M \to 0$$

by projective $\Bbbk B_{\lambda}$-modules. Now, $\Bbbk B$ is a free $\Bbbk B_{\lambda}$-module (because $\Bbbk B = \oplus_{s \in S} s \Bbbk B_{\lambda}$ where S is a transversal to B_{λ} in B) and $\Bbbk B[t]$ is a free $\Bbbk B$-module by Lemma 4.2.8, thus $\Bbbk B[t]$ is free as a $\Bbbk B_{\lambda}$-module and the functor $\Bbbk B[t] \otimes_{\Bbbk B_{\lambda}} -$ is exact. It follows that

$$0 \to \Bbbk B[t] \otimes_{\Bbbk B_{\lambda}} P_n \to \ldots \to \Bbbk B[t] \otimes_{\Bbbk B_{\lambda}} P_0 \to \Bbbk B[t] \otimes_{\Bbbk B_{\lambda}} M \to 0$$

is an exact sequence and so $\Bbbk B[t] \otimes_{\Bbbk B_{\lambda}} M$ has finite projective dimension as a $\Bbbk B[t]$-module. As a consequence of this,

$$\widehat{\operatorname{Ext}}^0_{\Bbbk B[t]}(M, \Bbbk B[t] \otimes_{\Bbbk B_{\lambda}} M) = \varinjlim_{i} \underline{\operatorname{Hom}}_{\Bbbk B[t]}(\Omega^i(M), \Omega^i(\Bbbk B[t] \otimes_{\Bbbk B_{\lambda}} M)) = 0$$

since $\Omega^i(\Bbbk B[t] \otimes_{\Bbbk B_{\lambda}} M) = 0$ for all i greater than some $n \geq 0$. An identical argument shows

$$\widehat{\operatorname{Ext}}^1_{\Bbbk B[t]}(M, \Bbbk B[t] \otimes_{\Bbbk B_{\lambda'}} {}^{\sigma}M) = \varinjlim_{i} \underline{\operatorname{Hom}}_{\Bbbk B[t]}(\Omega^{i+1}(M), \Omega^i(\Bbbk B[t] \otimes_{\Bbbk B_{\lambda'}} {}^{\sigma}M)) = 0$$

for all λ'. Therefore M has finite projective dimension. $\square$

The proof of Proposition 4.2.10 was essentially that of 4.1.5 – the skew polynomial methods were used only in setting up a short exact sequence. Also, the only property of $\Bbbk B$ we have used is that it is expressible as a filtered colimit of rings of finite left global dimension. This suggests we might be able to exploit the argument to arrive at a general theorem. Indeed, we have the following:

Theorem 4.2.11. *Let $R = \varinjlim_\lambda R_\lambda$ be a filtered colimit of rings of finite left global dimension with the property that R is flat as an R_λ-module for each λ and let $S = R[x;\sigma,\delta]$ be the right skew polynomial ring over R for some right skew derivation (σ,δ). Then every S-module of type* FP_∞ *has finite projective dimension.*

Proof. We exploit the method employed in the proof of 4.2.10.

- Let M be an S-module of type FP_∞.
- We have the short exact sequence $0 \to S \otimes_R {}^\sigma M \to S \otimes_R M \to M \to 0$ by Lemma 4.2.9
- Applying the functor $\widehat{\mathrm{Ext}}^*_S(M,-)$ yields the long exact sequence of complete cohomology groups in which $\widehat{\mathrm{Ext}}^0_S(M, S\otimes_R M)$ and $\widehat{\mathrm{Ext}}^1_S(M, S\otimes_R {}^\sigma M)$ cannot both be zero if M does not have finite projective dimension (by Lemma 2.1.8).
- $\widehat{\mathrm{Ext}}^*_S(M,-)$ commutes with filtered colimits since M is of type FP_∞, so $\widehat{\mathrm{Ext}}^0_S(M, S\otimes_{R_\lambda} M) \neq 0$ and $\widehat{\mathrm{Ext}}^1_S(M, S\otimes_{R_{\lambda'}} {}^\sigma M) \neq 0$ for some λ, λ'.
- As an R_λ-module, M has finite projective dimension since R_λ has finite global dimension by hypothesis. S is a free R-module by Lemma 4.2.8 and R is a flat R_λ-module, so the functor $S\otimes_{R_\lambda} -$ is exact, meaning $S\otimes_{R_\lambda} M$ has finite projective dimension as an S-module. Similarly, $S\otimes_{R_{\lambda'}} M^\sigma$ has finite projective dimension as an S-module.
- $\widehat{\mathrm{Ext}}^0_S(M, S\otimes_{R_\lambda} M) = 0$ and $\widehat{\mathrm{Ext}}^1_S(M, S\otimes_{R_{\lambda'}} M^\sigma) = 0$. Contradiction. Therefore M has finite projective dimension. □

4.3 Skew Laurent polynomial rings

In this short section we introduce skew Laurent polynomial rings and extend the main theorem of the previous section.

Definition 4.3.1. Let R be a ring and σ an automorphism. The *skew Laurent polynomial ring over R* is the the ring $R[x^{-1},x;\sigma]$ of polynomials over R subject to $rx = x\sigma(r)$ for all $r\in R$.

Remark 4.3.2. Each element of $R[x^{-1}, x; \sigma]$ has a unique representation of the form $\sum_{i \in \mathbb{Z}} x^i r_i$ with all but finitely many coefficients being zero.

We can improve Proposition 4.2.9:

Proposition 4.3.3. *Let R be a ring, $S = R[x; \sigma, \delta]$ or $S = R[x^{-1}, x; \sigma]$ and let M be a left S-module. Then there is a short exact sequence of S-modules*

$$0 \to S \otimes_R {}^{\sigma}M \to S \otimes_R M \to M \to 0$$

where $\lambda(s \otimes {}^{\sigma}m) = sx \otimes m - s \otimes xm$ and $\pi(s \otimes m) = sm$.

Proof. This is the full statement of the Proposition on page 262 of [26]. □

Another result we can improve upon is:

Lemma 4.3.4. *Let R be a ring and $S = R[x^{-1}, x; \sigma]$ or $S = R[x; \sigma, \delta]$. Then S is free as an R-module.*

Proof. For the proof that $S = R[x^{-1}, x; \sigma]$ is a free R-module, see Proposition 1.16 on page 16 of [18]. □

We now have for skew Laurent polynomial rings the same tools needed to make the skew polynomial ring problem in the previous section work: a short exact sequence of $R[x^{-1}, x; \sigma]$-modules for each $R[x^{-1}, x; \sigma]$-module M and the observation that $R[x^{-1}, x; \sigma]$ is free as an R-module. Thus, we can extend the statement of Theorem 4.2.11:

Theorem 4.3.5. *Let $R = \varinjlim_{\lambda} R_{\lambda}$ be a filtered colimit of rings of finite left global dimension with the property that R is flat as an R_{λ}-module for each λ and let $S = R[x; \sigma, \delta]$ or $S = R[x^{-1}, x; \sigma]$. Then every S-module of type* FP_{∞} *has finite projective dimension.*

Proof. The proof is the same as that of 4.2.11. □

It is now evident that to draw a distinction between the $\Bbbk B[t^{-1}, t]$ and $\Bbbk B[t]$ cases introduced in the previous section on the grounds that $\Bbbk B[t^{-1}, t]$ is a group algebra serves only to mask the problem.

4.4 When the base ring is strongly group-graded

In the last section we saw that FP_{∞} modules over a skew (Laurent) polynomial ring have finite projective dimension when the base ring has a suitable finiteness condition (in the case we considered, finite left global dimension of the rings in the limit was the condition). In this section we will see that a suitable finiteness condition when the base ring is a strongly group-graded $\Bbbk$-algebra is that the grading group belong to the class $\mathbf{H}\mathfrak{F}$. Throughout this section and the next, $\Bbbk$ will denote a commutative ring.

Recall the definition of the class $\mathbf{H}\mathfrak{F}$:

Definition 4.4.1. Let $\mathfrak{F}$ denote the class of finite groups. For each ordinal α we define operations $\mathbf{H}_\alpha$ inductively:

- $\mathbf{H}_0\mathfrak{F} = \mathfrak{F}$

and for ordinals $\alpha > 0$

- $\mathbf{H}_\alpha\mathfrak{F}$ is the class of groups G which admit a cellular action on a finite-dimensional contractible CW-complex in such a way that each isotropy group belongs to $\mathbf{H}_\beta\mathfrak{F}$ for some $\beta < \alpha$ (where β may depend on the isotropy group).

$\mathbf{H}\mathfrak{F}$ is then the union of all the $\mathbf{H}_\alpha\mathfrak{F}$.

Theorem 4.4.2. *Let G be an $\mathbf{H}\mathfrak{F}$-group and R a strongly G-graded $\Bbbk$-algebra. Let $S = R[x;\sigma,\delta]$ or $S = R[x^{-1},x;\sigma]$ and let M be an S-module of type* FP_∞ *such that M has finite projective dimension over R_H for all finite subgroups H of G. Then M has finite projective dimension as an S-module.*

Proof. We show that for all subgroups $H \le G$ and all i, $\widehat{\operatorname{Ext}}^i_S(M, S\otimes_{R_H} M)$ and $\widehat{\operatorname{Ext}}^i_S(M, S\otimes_{R_H} {}^\sigma M)$ are zero, from which the main result will follow. For, taking $G = H$, exactness of the sequence

$$\cdots \to \widehat{\operatorname{Ext}}^0_S(M, S\otimes_R M) \to \widehat{\operatorname{Ext}}^0_S(M,M) \to \widehat{\operatorname{Ext}}^1_S(M, S\otimes_R {}^\sigma M) \to \cdots$$

arising from the short exact sequence provided by Proposition 4.3.3, forces $\widehat{\operatorname{Ext}}^0_S(M,M) = 0$, which by Lemma 2.1.8 means M has finite projective dimension.

The proof is by induction on the least ordinal α such that H belongs to the subclass $\mathbf{H}_\alpha\mathfrak{F}$ of $\mathbf{H}\mathfrak{F}$. If $\alpha = 0$ then H belongs to $\mathbf{H}_0\mathfrak{F}$ which is the class of finite groups and so $R\otimes_{R_H} M$ and $R\otimes_{R_H} {}^\sigma M$ have finite projective dimension. As a consequence of Lemma 4.3.4, the functor $S\otimes_R -$ is exact and so $S\otimes_R R\otimes_{R_H} M = S\otimes_{R_H} M$ has finite projective dimension as an S-module. Similarly, $S\otimes_{R_H} {}^\sigma M$ has finite projective dimension as an S-module. Thus the claim holds because complete cohomology groups vanish on modules of finite projective dimension.

Assume now that $\alpha > 0$. Then H acts on a finite dimensional contractible cell complex X. Let

$$0 \to C_r \to \cdots \to C_1 \to C_0 \to \Bbbk \to 0$$

be the cellular chain complex of X. Each C_j is isomorphic to

$$\bigoplus_{\phi \in \Sigma_j} \Bbbk[H/H_\phi]$$

where Σ_j is a set of H-orbit representatives of j-dimensional cells in X and H_ϕ ($\in \mathbf{H}_\beta\mathfrak{F}$, $0 \leq \beta < \alpha$) is the isotropy group of the cell $\phi \in \Sigma_j$. Tensoring the sequence with M and using the semi-diagonal action of R_H, yields the exact sequence

$$0 \to C_r \otimes M \to \cdots \to C_1 \otimes M \to C_0 \otimes M \to M \to 0$$

of R_H-modules. As was seen above, the functor $S \otimes_{R_H} -$ is exact so applying it to the above sequence of R_H-modules yields the exact sequence

$$0 \to S \otimes_{R_H} (C_r \otimes M) \to \cdots \to S \otimes_{R_H} (C_1 \otimes M) \to S \otimes_{R_H} (C_0 \otimes M) \to S \otimes_{R_H} M \to 0$$

of S-modules. Applying the tensor identity we see that

$$\begin{aligned} S \otimes_{R_H} (C_j \otimes M) &= \bigoplus_{\phi \in \Sigma_j} S \otimes_{R_H} (\Bbbk[H/H_\phi] \otimes M) \\ &\cong \bigoplus_{\phi \in \Sigma_j} S \otimes_{R_H} (R_H \otimes_{R_{H_\phi}} M) \\ &= \bigoplus_{\phi \in \Sigma_j} S \otimes_{R_{H_\phi}} M. \end{aligned}$$

By induction, $\widehat{\operatorname{Ext}}^i_S(M, S \otimes_{R_{H_\phi}} M)$ vanishes for all isotropy groups H_ϕ. Since M is of type FP_∞ we have, by the comment following Lemma 2.1.11 that

$$\widehat{\operatorname{Ext}}^i_S(M, \bigoplus_{\phi \in \Sigma_j} S \otimes_{R_{H_\phi}} M) = \bigoplus_{\phi \in \Sigma_j} \widehat{\operatorname{Ext}}^i_S(M, S \otimes_{R_{H_\phi}} M)$$

and so we can deduce that $\widehat{\operatorname{Ext}}^i_S(M, S \otimes_{R_H} (C_j \otimes M)) = 0$ for all i, j. Thus the functors vanish on all modules in the above exact sequence of S-modules except $S \otimes_{R_H} M$. But this is incompatible with Lemma 2.1.13, so in fact these functors vanish on $S \otimes_{R_H} M$ also. An identical argument using ${}^\sigma M$ instead of M yields the same conclusion for $S \otimes_{R_H} {}^\sigma M$. Thus the claim holds for the subgroup H.

The strategy outlined at the beginning of the proof establishes the theorem. □

4.5 Some vanishing theorems

We now turn our attention to iterated skew polynomial rings over base the base rings we have already considered.

Definition 4.5.1. Let R be a ring. An ***n*-fold iterated skew polynomial ring over** R is defined as follows: let $S_0 = R$ and for each $n \geq 1$, $S_n := S_{n-1}[x_n; \sigma_{n-1}, \delta_{n-1}]$ for some right skew derivation $(\sigma_{n-1}, \delta_{n-1})$ on the ring S_{n-1}.

Definition 4.5.2. Let R be a ring. An ***n*-fold iterated skew Laurent polynomial ring over** R is defined as follows: let $S_0 = R$ and for each $n \geq 1$, $S_n := S_{n-1}[x_n^{-1}, x_n; \sigma_{n-1}]$ for some automorphism σ_{n-1} of the ring S_{n-1}.

Theorem 4.5.3. *Let G be an $\mathbf{H}\mathfrak{F}$-group and R a strongly G-graded $\Bbbk$-algebra with the property that every R-module has finite projective dimension as an R_H-module for each finite subgroup H of G. Let T be an n-fold iterated skew polynomial ring or an iterated n-fold skew Laurent polynomial ring over R. Then, for all left T-modules M of type* FP_∞*, all left T-modules N and all intermediate polynomial rings S_j, $0 \leq j \leq n$,*

$$\widehat{\operatorname{Ext}}^i_T(M, T \otimes_{S_j} N) = 0$$

for all i.

Proof. We proceed by induction on j.

• $j = 0$. The proof is similar to that of Lemma 4.4.2. We proceed by induction on the least ordinal α such that G belongs to the subclass $\mathbf{H}_\alpha\mathfrak{F}$ of $\mathbf{H}\mathfrak{F}$. If $\alpha = 0$ then G belongs to $\mathbf{H}_0\mathfrak{F}$ which is the class of finite groups and so as an R-module, N has finite projective dimension. The functor $T \otimes_R -$ is naturally equivalent to $T \otimes_{S_{n-1}} S_{n-1} \otimes_{S_{n-2}} \cdots \otimes_{S_1} S_1 \otimes_R -$. By Lemma 4.3.4, each S_{k+1} is free as an S_k-module $(0 \leq k \leq n-1)$ so $T \otimes_R -$ is an exact functor. It follows that $T \otimes_R N$ has finite projective dimension as a T-module and so $\widehat{\operatorname{Ext}}^i_T(M, T \otimes_R N) = 0$ for all i.

Assume now that $\alpha > 0$. Then G acts on a finite dimensional contractible cell complex X. Let

$$0 \to C_r \to \cdots \to C_1 \to C_0 \to \Bbbk \to 0$$

be the cellular chain complex of X. Each C_k is isomorphic to

$$\bigoplus_{\phi \in \Sigma_k} \Bbbk[G/G_\phi],$$

where $\sum_k$ is a set of G-orbit representatives of k-dimensional cells in X and G_ϕ ($\in \mathrm{H}_\beta\mathfrak{F}$ with $0 \le \beta < \alpha$) is the isotropy group of the cell $\phi \in \sum_k$. Tensoring the sequence with N and using the semi-diagonal action of R, yields the exact sequence

$$0 \to C_r \otimes N \to \cdots \to C_1 \otimes N \to C_0 \otimes N \to N \to 0$$

of R-modules. As we saw in the first part of the proof, the functor $T \otimes_R -$ is exact so applying it to the above sequence of R-modules yields the exact sequence

$$0 \to T \otimes_R (C_r \otimes N) \to \cdots \to T \otimes_R (C_1 \otimes N) \to T \otimes_R (C_0 \otimes N) \to T \otimes_R N \to 0.$$

Applying the tensor identity, we see that for $0 \le k \le r$,

$$\begin{aligned} T \otimes_R (C_k \otimes N) &= \bigoplus_{\phi \in \sum_k} T \otimes_R (\Bbbk[G/G_\phi] \otimes N) \\ &\cong \bigoplus_{\phi \in \sum_k} T \otimes_R (R \otimes_{RG_\phi} N) \\ &= \bigoplus_{\phi \in \sum_k} T \otimes_{RG_\phi} N. \end{aligned}$$

By induction, $\widehat{\operatorname{Ext}}^i_T(M, T \otimes_{RG_\phi} N)$ vanishes for all isotropy groups G_ϕ. M is a T-module of type FP_∞, so by the comment following Lemma 2.1.11,

$$\widehat{\operatorname{Ext}}^i_T(M, \bigoplus_{\phi \in \sum_k} T \otimes_{RG_\phi} N) = \bigoplus_{\phi \in \sum_k} \widehat{\operatorname{Ext}}^i_T(M, T \otimes_{RG_\phi} N);$$

that is, $\widehat{\operatorname{Ext}}^i_T(M, T \otimes_R (C_k \otimes N)) = 0$ for all i and all k. The long exact sequence for complete cohomology groups shows us that in fact $\widehat{\operatorname{Ext}}^i_T(M, T \otimes_R N) = 0$ for all i.

- $1 \le j \le n$. By Lemma 4.3.3, we have a short exact sequence

$$0 \to S_j \otimes_{S_{j-1}} {}^{\sigma_{j-1}}N \to S_j \otimes_{S_{j-1}} N \to N \to 0$$

of S_j-modules. As has been observed, the functor $T \otimes_{S_j} -$ is exact and so

$$0 \to T \otimes_{S_{j-1}} {}^{\sigma_{j-1}}N \to T \otimes_{S_{j-1}} N \to T \otimes_{S_j} N \to 0$$

is an exact sequence of T-modules. By induction,

$$\widehat{\operatorname{Ext}}^i_T(M, T \otimes_{S_{j-1}} {}^{\sigma_{j-1}}N) = \widehat{\operatorname{Ext}}^i_T(M, T \otimes_{S_{j-1}} N) = 0$$

for all i and so exactness of the sequence

$$\cdots \to \widehat{\operatorname{Ext}}^i_T(M, T \otimes_{S_{j-1}} N) \to \widehat{\operatorname{Ext}}^i_T(M, T \otimes_{S_j} N) \to \widehat{\operatorname{Ext}}^{i+1}_T(M, T \otimes_{S_{j-1}} {}^{\sigma_{j-1}}N) \to \cdots$$

forces $\widehat{\operatorname{Ext}}^i_T(M, T \otimes_{S_j} N) = 0$ for all i. □

Remark 4.5.4. Taking $S_j = T$, $N = M$ and $i = 0$ gives us Theorem 4.4.2.

Theorem 4.5.5. *Let $R = \varinjlim_\lambda R_\lambda$ be a filtered colimit of rings of finite left global dimension with the property that R is flat as an R_λ-module for each λ. Let T be an n-fold skew polynomial ring or n-fold iterated skew Laurent polynomial ring over R. Then, for all left T-modules of type FP_∞, all left N-modules and all intermediate polynomial rings S_j, $0 \le j \le n$,*

$$\widehat{\operatorname{Ext}}^i_T(M, T \otimes_{S_j} N) = 0$$

for all i.

Proof. We proceed by induction on j.

- $j = 0$. M is of type FP_∞ so by Lemma 2.1.11,

$$\widehat{\operatorname{Ext}}^i_T(M, T \otimes_R N) = \widehat{\operatorname{Ext}}^i_T(M, \varinjlim_\lambda T \otimes_{R_\lambda} N) = \varinjlim_\lambda \widehat{\operatorname{Ext}}^i_T(M, T \otimes_{R_\lambda} N)$$

for all i. Each R_λ has finite left global dimension, so we immediately see that $T \otimes_{R_\lambda} N$ has finite projective dimension as a T-module because the functor $T \otimes_{R_\lambda} -$ is exact by repeated application of Lemma 4.3.4 and the fact that R is flat as an R_λ-module. As a consequence of this, each $\widehat{\operatorname{Ext}}^i_T(M, T \otimes_{R_\lambda} N)$ is zero and so $\widehat{\operatorname{Ext}}^i_T(M, T \otimes_R N) = 0$ for each i.

- $1 \le j \le n$. By Lemma 4.3.3, we have a short exact sequence

$$0 \to S_j \otimes_{S_{j-1}} {}^{\sigma_{j-1}}N \to S_j \otimes_{S_{j-1}} N \to N \to 0$$

of S_j-modules. As has been observed, the functor $T \otimes_{S_j} -$ is exact and so

$$0 \to T \otimes_{S_{j-1}} {}^{\sigma_{j-1}}N \to T \otimes_{S_{j-1}} N \to T \otimes_{S_j} N \to 0$$

is an exact sequence of T-modules. By induction,

$$\widehat{\operatorname{Ext}}^i_T(M, T \otimes_{S_{j-1}} {}^{\sigma_{j-1}}N) = \widehat{\operatorname{Ext}}^i_T(M, T \otimes_{S_{j-1}} N) = 0$$

for all i and so exactness of the sequence

$$\cdots \to \widehat{\operatorname{Ext}}^i_T(M, T \otimes_{S_{j-1}} N) \to \widehat{\operatorname{Ext}}^i_T(M, T \otimes_{S_j} N) \to \widehat{\operatorname{Ext}}^{i+1}_T(M, T \otimes_{S_{j-1}} {}^{\sigma_{j-1}}N) \to \cdots$$

forces $\widehat{\operatorname{Ext}}^i_T(M, T \otimes_{S_j} N) = 0$ for all i. □

4.6 Bricks, Walls and Foundations

Definition 4.6.1. Let $\mathcal{B}$ be a poset of subrings of a ring T with the following properties

- $T \in \mathcal{B}$.
- $\mathcal{B}$ satisfies DCC.
- T is flat as an S-module for every $S \in \mathcal{B}$.
- For all $S \in \mathcal{B}$, one of the following is true
  (i) $S = S_0[x; \sigma, \delta]$ for some $S_0 \in \mathcal{B}$.
  (ii) $S = \varinjlim_\lambda S_\lambda$ for some $S_\lambda \in \mathcal{B}$.

then the $S \in \mathcal{B}$ are called ***bricks***, each minimal element of $\mathcal{B}$ is called a ***foundation brick*** and $\mathcal{B}$ is called a T-***wall***.

Remark 4.6.2. In order to perform various calculations, we have to stipulate that T be flat as an S-module above because, as the following example shows, we cannot assume it.

Example 4.6.3. Given rings R, S and T with $R \leq S \leq T$ such that S and T are flat as R-modules, it does not necessarily follow that T is flat as an S-module. For, take $R = \Bbbk$, a field, $S = \Bbbk[x]$ and $T = \Bbbk[x,y]/(xy, y^2)$. $\Bbbk[x]$ and $\Bbbk[x,y]/(xy,y^2)$ are free $\Bbbk$-modules since $\Bbbk$ is a field and so they are also flat. As a $\Bbbk[x]$-module, however, $\Bbbk[x,y]/(xy,y^2)$ is not free or even flat. To see this, first observe that as a $\Bbbk[x]$-module,

$$\Bbbk[x,y]/(xy,y^2) \cong \Bbbk \oplus \Bbbk[x]$$

via the map $\phi : (\alpha y + \beta_0 + \beta_1 x + \beta_2 x^2 + \cdots) \mapsto (\alpha, \beta_0 + \beta_1 x + \beta_2 x^2 + \cdots)$ where $\Bbbk$ is the $\Bbbk[x]$-module on which x acts as multiplication by zero; but this means that

$$\begin{aligned}
\mathrm{Tor}_n^{\Bbbk[x]}(\Bbbk, \Bbbk[x,y]/(xy,y^2)) &= \mathrm{Tor}_n^{\Bbbk[x]}(\Bbbk, \Bbbk \oplus \Bbbk[x]) \\
&\cong \mathrm{Tor}_n^{\Bbbk[x]}(\Bbbk, \Bbbk) \oplus \mathrm{Tor}_n^{\Bbbk[x]}(\Bbbk, \Bbbk[x]) \\
&= \mathrm{Tor}_n^{\Bbbk[x]}(\Bbbk, \Bbbk)
\end{aligned}$$

since Tor_n vanishes on free modules. Thus if $\mathrm{Tor}_1^{\Bbbk[x]}(\Bbbk, \Bbbk)$ is non-zero, then $\Bbbk[x,y]/(xy,y^2)$ cannot be flat as a $\Bbbk[x]$-module.

The Koszul complex provides us with a free resolution of the $\Bbbk[x]$-module $\Bbbk$:

$$0 \to \Bbbk[x] \xrightarrow{x} \Bbbk[x] \to \Bbbk \to 0.$$

Considering the complex

$$0 \to \Bbbk \otimes_{\Bbbk[x]} \Bbbk[x] \xrightarrow{1\otimes x} \Bbbk \otimes_{\Bbbk[x]} \Bbbk[x] \to 0,$$

we see that the map $1 \otimes x$ sends the element $1 \otimes 1$ to $1 \otimes x = x \otimes 1 = 0$; that is, $\mathrm{Ker}(1 \otimes x) = \Bbbk \otimes_{\Bbbk[x]} \Bbbk[x]$ so $\mathrm{Tor}_1^{\Bbbk[x]}(\Bbbk, \Bbbk) = \Bbbk \otimes_{\Bbbk[x]} \Bbbk[x] \neq 0$ and therefore $\Bbbk[x,y]/(xy, y^2)$ cannot be flat as a $\Bbbk[x]$-module.

Definition 4.6.4. Let T be a ring and $\mathcal{B}$ a T-wall. A T-module which is projective over each foundation brick is called a ***foundation module***.

Theorem 4.6.5. *Let T be a ring and $\mathcal{B}$ a T-wall. Then for all bricks S, all T-modules M of type* FP_∞ *and all foundation modules N,*

$$\widehat{\mathrm{Ext}}^*_T(M, T \otimes_S N) = 0.$$

Proof. In case S is a foundation brick, N is a projective S-module and so $T \otimes_S N$ is projective (since T is flat over every brick) and the claim holds.

Suppose now that the claim is not true for some brick which is not a foundation brick. Let S be the least such brick. Then $\widehat{\mathrm{Ext}}^j_T(M, T \otimes_S N) \neq 0$ for some j. Either $S = \varinjlim_\lambda S_\lambda$ or $S = S_0[x; \sigma, \delta]$.

- If $S = \varinjlim_\lambda S_\lambda$, then $T \otimes_S N = \varinjlim_\lambda T \otimes_{S_\lambda} N$. M is a T-module of type FP_∞ so

$$\widehat{\mathrm{Ext}}^j_T(M, T \otimes_S N) = \varinjlim_\lambda \widehat{\mathrm{Ext}}^j_T(M, T \otimes_{S_\lambda} N)$$

  so that if $\widehat{\mathrm{Ext}}^j_T(M, T \otimes_S N)$ is non-zero, then $\widehat{\mathrm{Ext}}^j_T(M, T \otimes_{S_\lambda} N)$ is non-zero for some λ. But $\widehat{\mathrm{Ext}}^j_T(M, T \otimes_{S_\lambda} N) = 0$ for all λ (since S was the least brick for which the claim is not true) and the result follows.

- If $S = S_0[x; \sigma, \delta]$, then $\widehat{\mathrm{Ext}}^*_T(M, T \otimes_{S_0} N) = 0$ for all T-modules of type FP_∞ and all foundation modules N.

  We have a short exact sequence of S-modules

$$0 \to S \otimes_{S_0} {}^{\sigma}N \to S \otimes_{S_0} N \to N \to 0$$

  (where σ is the automorphism of S_0 used in the construction of S). Applying the functor $T \otimes_S -$ to this sequence preserves exactness (because T is flat as an S-module) and yields the exact sequence

$$0 \to T \otimes_{S_0} {}^{\sigma}N \to T \otimes_{S_0} N \to T \otimes_S N \to 0$$

of T-modules. Applying $\widehat{\operatorname{Ext}}_T^*(M,-)$ to this sequence yields the long exact sequence

$$\cdots \to \widehat{\operatorname{Ext}}_T^i(M, T\otimes_{S_0} N) \to \widehat{\operatorname{Ext}}_T^i(M, T\otimes_S N) \to \widehat{\operatorname{Ext}}_T^{i+1}(M, T\otimes_{S_0} {}^{\sigma}N) \to \cdots$$

Each $\widehat{\operatorname{Ext}}_T^i(M, T\otimes_{S_0} N) = 0$ by hypothesis, so if $\widehat{\operatorname{Ext}}_T^i(M, T\otimes_S N)$ is non-zero for some i, then $\widehat{\operatorname{Ext}}_T^{i+1}(M, T\otimes_{S_0} {}^{\sigma}N)$ is non-zero.

Questions I don't know how to proceed from here. Does it follow that if N is a foundation module then ${}^{\sigma}N$ (where σ is an automorphism of some brick) is a foundation module also? If σ is an automorphism of the brick S_0, then we certainly have that $pd_T(T\otimes_{S_0} {}^{\sigma}N) \leq pd_{S_0}({}^{\sigma}N) = pd_{S_0}(N)$.

□

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