Cohomological finiteness conditions for a class of metabelian groups

by

Joseph Mullaney

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Abstract

This thesis is concerned with the study of certain metabelian groups which can be viewed as split extensions of a free abelian group Q by a finitely generated $\mathbb{Z}Q$-module A. In particular we want to know which cohomological finiteness properties these groups possess in order to shed further light on the FP_m-conjecture of Bieri and Groves.

In Chapter 1 we provide a historical context for the work and our motivation for carrying it out. Chapter 2 consists of background material and already-known results needed later.

In Chapter 3 we describe the metabelian groups $G_n = A_n \rtimes Q_n$ $(n \geq 1)$ we wish to study. First we prove, using a method of Baumslag [3], that they are finitely presented. By generalizing a theorem of Groves and Kochloukova [23, Theorem 5], using their proof as a framework, we find that the groups G_n possess the higher cohomological finiteness property of being of type FP_{n+1}. This is implied by Theorem A which we extend to cover a wider class of metabelian groups.

In Chapter 4 we look at the spherical invariant Σ^c_A associated to each $\mathbb{Z}Q$-module A and prove in Theorem B that $\Sigma^c_{A_n}$ satisfies a 'finiteness' condition of its own: namely, $(n+1)$-tameness. This is already implied by Theorem A but proved independently Theorem B provides further evidence in favour of the FP_m-conjecture.

Finally, in Chapter 5 we study the second cohomology group $H^2(Q, A)$ and show as Theorem C that it is finite cyclic whenever Q and A satisfy the criteria given in Theorem A. This demonstrates that there are few if any non-split extensions of A by Q. In particular we find that the group $H^2(Q_n, A_n)$ is trivial and so every extension of A_n by Q_n is split.

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Declaration

This thesis is submitted in accordance with the regulations for the degree of Doctor of Philosophy at the University of Glasgow.

Chapter 2 and Sections 4.1, 4.2 and 5.1 cover background material and preliminary results. The results elsewhere are the original work of the author except where referenced.

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Chapter 1

Introduction

A group G is said to be metabelian if its commutator subgroup G' is abelian. In this thesis we study a class of finitely presented metabelian groups having their origins in the work of Baumslag and Stammbach ([3], [8]) and show that these groups satisfy certain higher cohomological finiteness conditions. To do this we adapt the methods of Kochloukova and Groves and prove a generalization of Theorem 5 in their paper [23].

The original motivation for this work comes from Baumslag's discovery of a finitely presented metabelian group with an abelian normal subgroup of infinite rank [3]. This is the group G_1 generated by the matrices

$$\alpha = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \ \sigma = \begin{pmatrix} 1+x & 0 \\ 0 & 1 \end{pmatrix}, \ \tau = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$$

over $\mathbb{Q}(x)$. The commutator (or derived) subgroup of G_1 is generated by all conjugates of α and is free abelian of infinite rank. Hence G_1 is an example of a finitely presented metabelian group that is neither polycyclic nor constructible; that is, it does not admit a subnormal series where all the factor groups are cyclic, and it cannot be built up from the trivial group by forming finite extensions, amalgamations or finite-rank HNN-extensions. Thus G_1 differs radically from any finitely presented soluble group seen up to that point.

Much work on finitely generated metabelian groups was carried out by Philip

Hall, whose papers [24] and [25] can be viewed as the beginning of the serious study of the subject. Hall proved in [24] that finitely generated metabelian groups satisfy max-n, the maximal condition for normal subgroups: that is, every normal subgroup is the normal closure of a finite number of elements. His main result in [25] was that finitely generated metabelian groups are *residually finite*: their subgroups of finite index have trivial intersection. Such properties are not enjoyed by generic finitely generated, or even finitely presented, soluble groups: for a good discussion of these and related points see Baumslag's survey [6].

Hall's results showed that finitely generated metabelian groups tend to be well-behaved. However, they say nothing specific about the nature of finitely *presented* metabelian groups, and it was Baumslag who carried out the first substantial work in this area. In particular, he proved in 1973 that every finitely generated metabelian group can be embedded in a finitely presented metabelian group [5]. This begged the following question: is there a way of distinguishing finitely presented metabelian groups from among all those that are merely finitely generated?

Perhaps the simplest example of a finitely generated metabelian group that is *not* finitely presented was first given by Baumslag in [6]: it is the split extension G of $A = \mathbb{Z}[1/6]$ by the infinite cyclic group $Q = \langle t \rangle$, where t acts on A by multiplication with $\frac{2}{3}$. The group G is generated by the elements $(t, 0)$, $(1, 1)$ but has no presentation with a finite number of defining relations. The following is an infinite presentation for G:

$$G = \langle a, t : (a^2)^t = a^3, [a, a^t] = 1, [a, a^{t^2}] = 1, \ldots, [a, a^{t^j}] = 1, \ldots \rangle$$

Hence G is a 1-relator metabelian group: it is the largest metabelian quotient Γ/Γ'' of the Baumslag-Solitar group

$$\Gamma = \langle a, t : (a^2)^t = a^3 \rangle.$$

If instead we define the action of t on $A = \mathbb{Z}[1/6]$ to be multiplication with 6, then the split extension $A \rtimes Q$ behaves very differently: it is a finitely presented group H with defining relation $a^t = a^6$. What is it that makes the groups G and H so different? The answer lies in the structure of a geometric invariant that can be associated to the finitely generated $\mathbb{Z}Q$-module A. Now any metabelian group G lies at the centre of a short exact sequence

$$A \hookrightarrow G \stackrel{\pi}{\twoheadrightarrow} Q, \tag{1.1}$$

where A and Q are abelian. Then A can be viewed as a subgroup of G and thus becomes a $\mathbb{Z}Q$-module if we define the action of Q on A by $a.q = g^{-1}ag$ whenever $\pi(g) = q$. Since A is abelian this action is well-defined. The metabelian group G is then finitely generated if and only if Q is finitely generated as a group and A is finitely generated as a $\mathbb{Z}Q$-module: a well-known fact which we prove explicitly in Proposition 2.4.5.

It may or may not be the case that a finitely generated $\mathbb{Z}Q$-module A is finitely generated as a $\mathbb{Z}Q_v$-module, where $v : Q \to \mathbb{R}$ is a non-zero group homomorphism (a *character* of Q) and Q_v is the monoid

$$Q_v = \{q \in Q : v(q) \geq 0\}.$$

This information is encoded in the invariant

$$\Sigma_A = \{[v] \in S(Q) : A \text{ is finitely generated as a } \mathbb{Z}Q_v\text{-module }\}$$

where $S(Q)$ is the set of equivalence classes $(\mathrm{Hom}(Q, \mathbb{R}) \setminus \{0\})/\sim$ under the relation $\sim$ defined by

$$v_1 \sim v_2 \text{ if and only if } v_1 = \rho v_2 \text{ for some real number } \rho > 0.$$

Since $\mathrm{Hom}(Q, \mathbb{R})$ is a real vector space of dimension n the set $S(Q)$ inherits the structure of an $(n-1)$-dimensional sphere, where n is the $\mathbb{Z}$-rank of the finitely generated abelian group Q. Hence Σ_A is a spherical subset. In [17] Bieri and Strebel proved a criterion for finitely presented metabelian groups in terms of the geometry of Σ_A: a finitely generated metabelian group G as in (1.1) is finitely related if and only if the complement Σ_A^c of Σ_A in $S(Q)$ does not contain any pair of antipodal points, or equivalently

$$S(Q) = \Sigma_A \cup (-\Sigma_A)$$

where

$$-\Sigma_A = \{[-v] : [v] \in \Sigma_A\}$$

is the set of all antipodes of points lying in Σ_A. If Σ_A^c has this property we say that the module A is *tame*.

The definition of tameness was extended by Bieri and Groves in [13]: the module A is said to be *m-tame* if every subset of Σ_A^c consisting of m points or fewer lies in an open hemisphere of $S(Q)$. Tameness is then equivalent to 2-tameness, and, given the Bieri-Strebel result and further evidence accrued in [13], Bieri and Groves were able to make the following conjecture:

The FP_m-Conjecture ([13]). *Suppose we have the short exact sequence* (1.1)*, where G is finitely generated, and let m be a positive integer. Then G is of type FP_m if and only if A is m-tame as a $\mathbb{Z}Q$-module.*

A group G is said to be of type FP_m if there is a $\mathbb{Z}G$-projective resolution of the trivial module $\mathbb{Z}$ where the modules are finitely generated in dimensions $\leq m$. In particular a group is of type FP_1 if and only if it is itself finitely generated, and so the FP_1-conjecture is trivial as any finitely generated module is obviously 1-tame.

While a finitely presented group is always of type FP_2 it is not known whether the opposite implication is true for all soluble groups. It is certainly not true for groups in general: for example, Bestvina and Brady have shown that the right-angled Artin groups in [10] are of type FP_2 but are not finitely presented. However, Bieri and Strebel showed that the properties of being of type FP_2 and being finitely presented are equivalent for metabelian groups and so the FP_m-conjecture holds for $m = 2$ [17]. Hence there is a hierarchy of finiteness conditions for infinite metabelian groups: 'finitely generated' is implied by 'finitely presented', which is implied by 'type FP_2', which is implied by 'type FP_3', and so on.

$$\text{finitely generated} \Leftarrow \text{finitely presented} \Leftarrow \text{type } FP_2 \Leftarrow \text{type } FP_3 \Leftarrow \dots$$

We define a *cohomological finiteness condition* to be any group-theoretic property which holds for all groups that admit finite Eilenberg-Mac Lane spaces. Finite generation, finite presentability and type FP_m are examples of such conditions.

Both directions of the FP_m conjecture remain open for $m > 2$ but a number of specific cases have been proved. In [1] Hans Åberg established the full FP_m-conjecture for the case where G has finite Prüfer rank: that is, whenever there exists an integer d such that any finitely generated subgroup of G can be generated by at most d elements.

Noskov generalized Åberg's proof to show that the 'only-if' part of the conjecture holds if A is torsion-free and G is the split extension of A by Q [31] . Kochloukova extended this to the case where either the additive group of A is torsion or the extension is split and so removed the torsion-free condition from the latter [26]. In [26] it is also shown that the full conjecture is true for the case where A is torsion and of Krull dimension 1 as a $\mathbb{Z}Q$-module. This includes the case proved by Bux [21] where A is a subring of a global function field, Q is contained in the group of units of A and the extension is split. More recently Bieri and Harlander proved the FP_3-conjecture for the split extension case [15].

A natural question to ask in the wake of the paper [17] of Bieri and Strebel was whether or not Baumslag's embedding result in [5] could be extended from groups of type FP_2 to groups of type FP_m for arbitrary $m > 2$. Groves and Kochloukova proved that finitely generated metabelian groups can be embedded in *quotients* of metabelian groups of type FP_m [23]. It is still unknown whether such quotients are themselves of type FP_m for $m \geq 4$, although this would follow from the FP_m-conjecture. However in [27] Kochloukova and da Silva proved that the analogue of Baumslag's embedding result does indeed hold for groups of type FP_m and so provided further evidence in favour of the FP_m-conjecture.

Our main results shall provide evidence in support of the conjecture in the split extension case. Let us describe these results. First we note that Baumslag's example of a finitely presented metabelian group with a free abelian normal subgroup of infinite rank can be generalized if we fix a positive integer $n \geq 1$ and define G_n to be the group generated by the 2×2 matrices

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} i+x & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}$$

over $\mathbb{Q}(x)$, for all $1 \leq i \leq n$. Using methods similar to Baumslag's it can be shown that G_n has a finite presentation: this will be Theorem 3.1.2. Now G_n is isomorphic to the split extension $A_n \rtimes Q_n$, where

$$A_n = \mathbb{Z}\left[x, x^{-1}, (1+x)^{-1}, \ldots, (n+x)^{-1}, \frac{1}{n!}\right]$$

and Q_n is the free abelian subgroup of the group of units of A_n generated by

$$n!,\ x,\ 1+x,\ \ldots,\ n+x.$$

The action of Q_n on A_n by multiplication turns A_n into a cyclic $\mathbb{Z}Q_n$-module.

By studying the action of G_n on a subcomplex of a product X of trees Γ_v indexed by a finite set of discrete characters v of G_n we are able to show that G_n is a group of type FP_{n+1}. Now observe that $x,\ 1+x,\ \ldots\ ,\ n+x$ are pairwise coprime when viewed as elements of the polynomial ring $\mathbb{Z}[x, 1/n!]$: the ideal generated by any two of them is always the whole ring. This key property is needed in order to prove that G_n is of type FP_{n+1}, and by isolating it we have turned it into one of the core conditions required in the following generalization (Theorem 3.1.3) of that result.

Theorem A. *Suppose $n \geq 1$ and let Q be a free abelian group with a finitely generated $\mathbb{Z}Q$-module*

$$A = \mathbb{Z}[q_0, q_0^{-1}, f_1^{-1}, \ldots, f_n^{-1}, 1/k],$$

where k is a positive integer. Write $q_0 = f_0$ and assume that for $0 \leq i \leq n$ the f_i are irreducible non-constant monic polynomials in $\mathbb{Z}[q_0]$ that are pairwise coprime as elements of $\mathbb{Z}[q_0, 1/k]$. For $k = 1$ we assume that Q has a basis $q_0, q_1, \ldots, q_n$ acting on A by multiplication with $q_0, f_1, \ldots, f_n$ respectively. If $k > 1$ then Q has an additional basis element q_{-1} acting by multiplication with k. In each case the split extension $G = A \rtimes Q$ is of type FP_{n+1}. In particular the group $G_n = A_n \rtimes Q_n$ is of type FP_{n+1}.

The method used in the proof of Theorem A originated in the work of Åberg [1] and closely follows that of Theorem 5 of Groves and Kochloukova in [23]. The $l = 1$ case of that result covers the case where $k = 1$ in Theorem A. In order to extend the Groves-Kochloukova result to cover the groups G_n for $n > 1$, we need to invert $n!$ since the polynomials $x, 1+x, \dots, n+x$ are not coprime in $\mathbb{Z}[x]$.

The basic outline of the proof is as follows. We choose a set V of characters of Q with the property that A is not finitely generated as a Q-module for any $v \in V$, and so each $[v] \in \Sigma_A^c$. For each v a tree Γ_v equipped with a natural action of G is constructed. The cartesian product X of these trees is a CW-complex, and it has a subcomplex Y possessing three important properties:

(i) G acts co-compactly on Y;

(ii) Y is n-connected;

(iii) the stabilizers in G of cells in Y are of type FP_{n+1}.

That G is of type FP_{n+1} is then implied by a criterion of Brown [20]. Hence the group G_n is of type FP_{n+1}. It follows from Kochloukova's 'only if' result [26, Theorem B] that A_n is $(n+1)$-tame as a $\mathbb{Z}Q_n$-module, but by our choice of characters we will see that it is not $(n+2)$-tame and so G_n is not of type FP_{n+2}.

The module A_n was first studied by Baumslag and Stammbach [8]. Since the underlying abelian group of A_n is torsion-free it follows from Proposition 5.1 in [13] that its i-th exterior power is isomorphic to its i-th integral homology:

$$\bigwedge\nolimits_{\mathbb{Z}}^{i} A_n \cong H_i(A_n, \mathbb{Z}).$$

The exterior power is a $\mathbb{Z}Q_n$-module via the diagonal action of Q_n. It was proved in [8] that the exterior powers (and hence the homology) display the following unusual behaviour: $\bigwedge_{\mathbb{Z}}^{i} A_n$ is cyclic but not free as a $\mathbb{Z}Q_n$-module for $1 \le i \le n$, $\bigwedge_{\mathbb{Z}}^{n+1} A_n$ is free on one generator and $\bigwedge_{\mathbb{Z}}^{i} A_n$ is free on infinitely many generators for $i \ge n+2$. A more elegant proof of these facts was given by Kropholler and Stammbach [28]. Baumslag had already proved, with Dyer in [7], that the homology groups $H_m(G_1, \mathbb{Z})$ are free abelian of infinite rank for $m > 2$. The homology groups $H_m(G_n, \mathbb{Z})$ were determined in [8]. Together with Baumslag's discovery of the properties of G_1, it was the interesting behaviour of the exterior powers of A_n and the homology groups of G_n that prompted our interest in these examples.

Now, Bieri and Groves have shown [13, Theorem C] that finite generation of $\bigotimes_K^i M$, finite generation of $\bigwedge_K^i M$ and m-tameness are all equivalent whenever $1 \leq i \leq m$, K is a field, Q is a finitely generated abelian group and M is a finitely generated KQ-module. This is not true in general when we replace K with $\mathbb{Z}$. However, once we have proved that A_n is $(n+1)$-tame it follows from [13, Remark (2), p. 377] that the tensor product $\bigotimes_{\mathbb{Z}}^{n+1} A_n$ is finitely generated as a diagonal Q_n-module, and we know from [8] that $\bigwedge_{\mathbb{Z}}^i A_n$ is finitely generated for $1 \leq i \leq n+1$. Hence we would like to be able to prove *directly* that A_n is $(n+1)$-tame but not $(n+2)$-tame. Having already proved that the group $G_n = A_n \rtimes Q_n$ is of type FP_{n+1} but not of type FP_{n+2}, this will give self-contained evidence which supports the FP_m-conjecture.

To do this, we use the characterization of a finitely generated $\mathbb{Z}Q_v$-module A in terms of the centralizer $C(A)$ in $\mathbb{Z}Q$, as given in Proposition 2.1 of [17], and establish the following result (Theorem 4.3.1):

Theorem B. *Suppose $n \geq 2$ and let $v_i : Q_n \to \mathbb{R}$ be defined by $v_i(q_i) = 1, v_i(q_j) = 0$ for $i \neq j$, $-1 \leq i \leq n$. The v_i form a basis for $\mathrm{Hom}(Q_n, \mathbb{R})$ and so an arbitrary character is of the form $\chi = k_{-1}v_{-1} + k_0v_0 + \ldots + k_nv_n$ for some $k_i \in \mathbb{R}$. Then $[\chi] \in \Sigma^c_{A_n}$ implies that either*

(i) $k_{-1} > 0$ or

(ii) $k_{-1} = 0$ and $(k_0, k_1, \ldots, k_n)$ is a positive real multiple of some row of the $(n+2) \times (n+1)$ matrix

$$\begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ & \ldots & & \\ 0 & 0 & \ldots & 1 \\ -1 & -1 & \ldots & -1 \end{pmatrix}.$$

Moreover all $[\chi]$ as in (ii) are elements of $\Sigma^c_{A_n}$. In particular A_n is $(n+1)$-tame, *but not $(n+2)$-tame.*

It is already well-known that A_1 is 2-tame but not 3-tame: we give the full calculation of $\Sigma^c_{A_1}$ in Section 4.2. When $n = 2$ we calculate $\Sigma^c_{A_2}$ explicitly for $k_{-1} > 0$ in Proposition 4.5.1. This is done using both the method of Theorem B and, in addition, the relationship between Σ^c_A and the sets $\Delta^v_A(Q)$ of characters induced by valuations, where A is a Q-module and v a real valuation on $\mathbb{Z}$ with $v(\mathbb{Z}) \geq 0$. This relationship was proved by Bieri and Groves [14, Theorem 8.1]. We will show as a corollary of [14, Theorem A] that the set $\Delta^v_{A_n}(Q_n)$ is a homogeneous polyhedron of dimension 1 for $n \geq 1$ (Corollary 4.4.3) and use the relationship with $\Sigma^c_{A_n}$ to describe the geometric structure of the latter set.

In general the status of the FP_m-conjecture for non-split extensions seems to be more delicate and more technical than for split extensions. Our third main result focuses on calculating the second cohomology group $H^2(Q, A)$ whenever Q and A are as defined in Theorem A, in order to classify all group extensions of A by Q. What we are interested in is whether or not any non-split extensions exist.

One consequence of the main results of [17] is that whether or not a metabelian group G is finitely presented depends only upon the $\mathbb{Z}Q$-module A, and not on the extension class in $H^2(Q, A)$. In particular, G is finitely presented if and only if the split extension $A \rtimes Q$ is finitely presented. Using a Lyndon-Hochschild-Serre spectral sequence argument we have been able to show that $H^2(Q_n, A_n) = 0$ and so *every* extension of A_n by Q_n is split. In general we have the following (Theorem 5.2.1):

Theorem C. *In the situation of Theorem A the second cohomology group $H^2(Q, A)$ is cyclic of order dividing $k-1$ whenever $k \geq 2$. For $k = 1$ it is cyclic of order dividing l whenever $f_1(1) \neq 1$, where $l = |f_1(1) - 1|$. In the special case of $A = A_n$ and $Q = Q_n$, the cohomology group vanishes for all $n \geq 1$.*

When $k = 1$ and $f_1(1) = 1$ it has not been possible to find $H^2(Q, A)$ in the absence of more information about the polynomials $f_1, \ldots, f_n$, such as that available when $A = A_n$ and $Q = Q_n$. However, there would appear to be few, if any, non-split extensions of A by Q whenever A and Q satisfy the conditions in Theorem A, and so it may well be that $H^2(Q, A)$ is trivial in *all* such cases.

Chapter 2

Preliminaries

In this chapter we shall describe the basic concepts of homology and cohomology and relate these to groups. We then describe the known connections between the geometry of the Bieri-Strebel invariant associated to each finitely generated metabelian group and the cohomological type of these groups.

2.1 Modules, complexes and resolutions

In the discussion that follows R is a (possibly non-commutative) ring with identity element. We shall assume, unless it is stated otherwise, that all modules are right-modules. Most of the definitions and results in this section and Section 2.2 can be found in Peter Kropholler's unpublished notes on 'Cohomology of Groups and its Connections with Geometric Group Theory', and the author would like to thank Professor Kropholler for his permission to include this material.

Definition 2.1.1. A *chain complex* $C_\bullet$ of R-modules consists of a family

$$(C_n : n \in \mathbb{Z})$$

of R-modules together with R-module homomorphisms $d_n : C_n \to C_{n-1}$ for each $n \in \mathbb{Z}$ with the property that any composite $d_{n-1}d_n$ of consecutive maps is the zero map.

We say that a chain complex is *exact* at C_n if $\ker d_n = \mathrm{im}\ d_{n+1}$. The nth *homology* group $H_n = \ker d_n/\mathrm{im}\ d_{n+1}$ thus measures the extent to which exactness in a complex fails. A chain complex is said to be *acyclic* if it is exact everywhere, that is if the homology H_n is trivial in all dimensions n.

In a *cochain complex* $C^\bullet$ the maps go in the opposite direction; the modules are $(C^n : n \in \mathbb{Z})$ and the maps are of the form $d^n : C^n \to C^{n+1}$. The group $H^n = \ker d^n/\mathrm{im}\ d^{n-1}$ is the nth *cohomology* group of the cochain complex.

Note that in a chain complex the elements of $\ker d_n$ are the *cycles* and the elements of $\mathrm{im}\ d_n$ are the *boundaries*, for all n. In a cochain complex we call the kernels *cocycles* and the images *coboundaries*.

We can speak more generally and say that a sequence of algebraic objects and the maps between them is exact if the kernel of each map is equal to the image of the map preceding it in the sequence. Thus we have exact sequences of groups, modules, rings, in fact of any objects in an abelian category. The most important type of exact sequence is the *short* exact sequence. For modules these look like

$$0 \to A \to B \to C \to 0,$$

where $(A \to B)$ is injective, $(B \to C)$ surjective and $\ker(B \to C) = \mathrm{im}(A \to B)$. For groups we write 1 for the trivial group and then a short exact sequence has the form

$$1 \to H \to G \to Q \to 1.$$

Since $(H \to G)$ is an injection we can think of H as a subgroup of G, and then $Q \cong G/H$.

Definition 2.1.2. Let

$$0 \to A \to B \to C \to 0$$

be a short exact sequence in an abelian category P. If $F : P \to Q$ is a covariant functor from P to another abelian category Q, then we say that F is *left-exact* if the sequence

$$0 \to F(A) \to F(B) \to F(C)$$

in Q is exact, *right-exact* if

$$F(A) \to F(B) \to F(C) \to 0$$

is exact and *exact* if

$$0 \to F(A) \to F(B) \to F(C) \to 0$$

is exact. If $G : P \to Q$ is a contravariant functor, that is, one that reverses the direction of maps in P, then it is said to be left-exact whenever the sequence

$$0 \to G(C) \to G(B) \to G(A)$$

is exact and right-exact whenever the sequence

$$G(C) \to G(B) \to G(A) \to 0$$

is exact.

A functor is exact if and only if it is both left- and right-exact. It is also equivalent to say that an exact functor preserves exact sequences of any length.

For example, consider the covariant functor $\mathrm{Hom}_R(M,\ )$ from the category of R-modules to the category of abelian groups. This is a left-exact functor taking an R-module N to the set of R-module homomorphisms $\mathrm{Hom}_R(M, N)$. If M is an R-module such that $\mathrm{Hom}_R(M,\ )$ is an exact functor then we say that M is *projective*. Dually, the contravariant functor $\mathrm{Hom}_R(\ , N)$ is left-exact for all R-modules N. If it is exact for some particular N we say that N is *injective*.

An alternative (and often more convenient) definition of a projective module is that it is a direct summand of a free module. In particular, any free module is projective.

Lemma 2.1.3. (Schanuel's Lemma) *Given any two short exact sequences*

$$K \hookrightarrow P \twoheadrightarrow M,$$

$$K' \hookrightarrow P' \twoheadrightarrow M$$

such that P and P' are projective modules, then $P' \oplus K$ is isomorphic to $P \oplus K'$.

Definition 2.1.4. Let R, S be rings. A (covariant) *cohomological functor* U^* from the category $\mathfrak{Mod}_R$ of right R-modules to the category $\mathfrak{Mod}_S$ of right S-modules consists of a family $(U^n : n \in \mathbb{Z})$ of functors $U^n : \mathfrak{Mod}_R \to \mathfrak{Mod}_S$, along with natural connecting maps

$$\delta : U^n(M') \to U^{n+1}(M'')$$

for each short exact sequence $M'' \overset{\iota}{\hookrightarrow} M \overset{\pi}{\twoheadrightarrow} M'$ and each $n \in \mathbb{Z}$, such that there is a long exact sequence

$$\ldots \xrightarrow{\delta} U^n(M'') \xrightarrow{\iota_*} U^n(M) \xrightarrow{\pi_*} U^n(M') \xrightarrow{\delta} U^{n+1}(M'') \xrightarrow{\iota_*} U^{n+1}(M) \xrightarrow{\pi_*} \ldots$$

Here $\pi_* = U^m(\pi)$ and $\iota_* = U^m(\iota)$, where m is any integer. In addition, U^* is said to be *coeffaceable* if U^n is zero for all $n < 0$ and if $U^n(J) = 0$ whenever J is an injective module and $n > 0$.

There is an analogous definition for a contravariant cohomological functor W^*. We say that such a functor W^* is *effaceable* if W^n is zero for all $n < 0$ and if $W^n(P) = 0$ whenever $n > 0$ and P is projective.

Definition 2.1.5. Fix an R-module M. A *projective resolution* $P_* \twoheadrightarrow M$ of M is an exact sequence

$$\ldots \to P_j \to \ldots \to P_2 \to P_1 \to P_0 \twoheadrightarrow M$$

in which the modules P_j are projective for all j. An *injective resolution* $M \hookrightarrow I_*$ of M is defined in a dual manner and comprises an exact sequence of injective modules.

A projective resolution can be found for any R-module M. For instance, we can always construct a *free* resolution of M using the following step-by-step procedure: choose a surjection $\varepsilon : F_0 \twoheadrightarrow M$ where F_0 is a free R-module, then choose a surjection $F_1 \twoheadrightarrow \ker \varepsilon$ with F_1 free, and so on.

The projective resolution in Definition 2.1.5 gives us a chain complex $P_\bullet$ where $P_n = 0$ for $n < 0$ and which satisfies $H_0(P_\bullet) \cong M$ and $H_n(P_\bullet) = 0$ for all other n. From $P_\bullet$ we obtain the cochain complex $\mathrm{Hom}_R(P_\bullet, N)$ below where N is a choice of any R-module N and the maps are given by composition with the appropriate maps in $P_\bullet$.

$$0 \to \mathrm{Hom}_R(P_0, N) \to \mathrm{Hom}_R(P_1, N) \to \mathrm{Hom}_R(P_2, N) \to \dots$$

Taking the cohomology groups of this cochain complex gives rise to the covariant cohomological functor

$$H^*(\mathrm{Hom}_R(P_\bullet,\ )) := \mathrm{Ext}^*_R(M,\ ).$$

The reason this definition makes sense is that $\mathrm{Ext}^*_R(M,\ )$ doesn't depend on the choice of projective resolution of M. We explain this as follows. Since the functor $\mathrm{Hom}_R(\ , N)$ is right-exact we get an exact sequence

$$0 \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(P_0, N) \to \mathrm{Hom}_R(P_1, N).$$

Hence $\mathrm{Ext}^0_R(M, N) = H^0(\mathrm{Hom}_R(P_\bullet, N))$ is naturally isomorphic to $\mathrm{Hom}_R(M, N)$, which clearly doesn't depend on the chain complex $P_\bullet$. Since $\mathrm{Hom}_R(\ , J)$ is an exact functor if J an injective module it follows that $\mathrm{Ext}^n_R(M, J)$ vanishes for all injective modules J and $n > 0$. Hence $\mathrm{Ext}^*_R(M,\ )$ is coeffaceable. We then apply the following result.

Theorem 2.1.6. *Suppose we have two cohomological functors U^*, V^* between categories of modules and that U^* is coeffaceable. Then any natural map $\nu^0 : U^0 \to V^0$ extends uniquely to a natural map $\nu^* : U^* \to V^*$.*

In the case above, if $U^* = H^*(\mathrm{Hom}_R(P_\bullet,\ ))$ and $V^* = H^*(\mathrm{Hom}_R(Q_\bullet,\ ))$ are cohomological functors arising from two different projective resolutions for M then $U^0 = V^0 = \mathrm{Ext}^0_R(M, N)$ and so the natural map ν^0 is just the identity. Hence $U^* = V^*$.

In an analogous way we can define the contravariant effaceable cohomological functor $\mathrm{Ext}^*_R(\ , M) := H^*(\mathrm{Hom}_R(\ , I_\bullet))$, where $I_\bullet$ is an injective resolution of M.

We establish the following terminology for an R-module M depending on what kind of projective resolution it possesses.

Definition 2.1.7. An R-module M is said to be of type

• FP_n ($n \in \mathbb{N}$) if there exists a projective resolution $P_* \twoheadrightarrow M$ where the P_i are finitely generated for $i \leq n$;

• FP_∞ if there exists a projective resolution $P_* \twoheadrightarrow M$ of finite type, that is, one where the P_i are finitely generated in *all* dimensions;

• FP if there exists a finite projective resolution, that is, one of both finite length and finite type;

• FL if there exists a finite free resolution $F_* \twoheadrightarrow M$.

In the next section we shall establish analogous versions of the properties in Definition 2.1.7 for groups.

2.2 The cohomology and homology of a group

We now introduce group cohomology. Let G be a group. Then the group ring $\mathbb{Z}G$ is formally defined to be the set of all functions from G to $\mathbb{Z}$ with finite support, with addition and scalar multiplication defined in the obvious way. The product of two such functions f and g is defined by the function

$$x \mapsto \sum_{uv=x} f(u)g(v)$$

for $x, u, v \in G$. However, it is often useful to regard the elements of $\mathbb{Z}G$ as $\mathbb{Z}$-linear combinations of the elements in G, with multiplication given by extending the multiplication in G linearly.

We can apply the theory of cohomological functors of module categories to the category of $\mathbb{Z}G$-modules. It is useful to think of a $\mathbb{Z}G$-module as an additive abelian group with a G-action respecting that structure. One important case is the abelian group $\mathbb{Z}$: we can let any group G act trivially on $\mathbb{Z}$, and so we will call $\mathbb{Z}$ the *trivial* $\mathbb{Z}G$-module. The cohomology of G can then be defined by looking at projective resolutions of $\mathbb{Z}$ consisting of $\mathbb{Z}G$-modules.

Definition 2.2.1. Define

$$H^*(G,\ ) := \mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z},\ ).$$

Then we say that $H^n(G, M)$ is the n-th *cohomology group of* G *with coefficients in* M, where M is some $\mathbb{Z}G$-module.

The group G is said to be of type FP_n (resp. FP_∞, FP) if the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ is of type FP_n (resp. FP_∞, FP). These are examples of cohomological finiteness conditions for groups, and we see below that in some sense they generalize the classical finiteness conditions of being finite, finitely generated and finitely presented.

Proposition 2.2.2 ([12, Proposition 2.1, p. 19])**.** *A group* G *is of type* FP_1 *if and only if it is finitely generated.*

Proof. There is a ring homomorphism $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$ defined by $\varepsilon(g) = 1$ for all $g \in G$ and extended linearly to $\mathbb{Z}G$. This is the *augmentation map*; its kernel is the two-sided ideal $\mathfrak{g}$ of $\mathbb{Z}G$ generated by all elements $g - 1$. We call $\mathfrak{g}$ the *augmentation ideal* and get the short exact sequence

$$\mathfrak{g} \hookrightarrow \mathbb{Z}G \overset{\varepsilon}{\twoheadrightarrow} \mathbb{Z}. \tag{2.1}$$

Now suppose that G is finitely generated, with generators $g_1, \ldots, g_n$. Then $\mathfrak{g}$ is generated as an ideal by the elements $g_i - 1$, and hence one can construct a free resolution

$$\ldots \to \bigoplus^n \mathbb{Z}G \twoheadrightarrow \mathfrak{g} \hookrightarrow \mathbb{Z}G \overset{\varepsilon}{\twoheadrightarrow} \mathbb{Z}.$$

So G is of type FP_1.

Conversely, assume that G is of type FP_1. Then we can choose a projective resolution

$$\ldots \to P_1 \to P_0 \overset{\pi}{\twoheadrightarrow} \mathbb{Z}$$

where P_0, P_1 are finitely generated, and so get a short exact sequence

$$\ker \pi \hookrightarrow P_0 \overset{\pi}{\twoheadrightarrow} \mathbb{Z} \tag{2.2}$$

where $\ker \pi$ is finitely generated since P_1 maps surjectively onto it. We claim that the augmentation ideal $\mathfrak{g}$ is finitely generated over $\mathbb{Z}G$. To see this, we apply Lemma 2.1.3 to the short exact sequences (2.1) and (2.2) to get $P_0 \oplus \mathfrak{g} \cong \mathbb{Z}G \oplus \ker \pi$. It follows that $\mathfrak{g}$ can be generated as a $\mathbb{Z}G$-module by a finite number n elements of the form $g_i - 1$, where $g_i \in G$. Let S be the subgroup generated by $g_1, \ldots, g_n$ and $\mathfrak{s}$ its augmentation ideal. Then $\mathfrak{s}\mathbb{Z}G = \mathfrak{g}$. Now consider the short exact sequence

$$\mathfrak{s} \hookrightarrow \mathbb{Z}S \stackrel{\varepsilon}{\twoheadrightarrow} \mathbb{Z}.$$

Tensoring with $\mathbb{Z}G$ via the right-exact functor $- \otimes_{\mathbb{Z}S} \mathbb{Z}G$ (see [2, Proposition 2.18]) yields

$$\mathfrak{s} \otimes_{\mathbb{Z}S} \mathbb{Z}G \hookrightarrow \mathbb{Z}G \twoheadrightarrow \mathbb{Z} \otimes_{\mathbb{Z}S} \mathbb{Z}G.$$

But $\mathbb{Z} \otimes_{\mathbb{Z}S} \mathbb{Z}G \cong \mathbb{Z}(G/S)$ via the map

$$n \otimes x \mapsto n.Sx,$$

while $\mathfrak{s} \otimes_{\mathbb{Z}S} \mathbb{Z}G \cong \mathfrak{s}\mathbb{Z}G$ via

$$(s-1) \otimes x \mapsto (s-1)x.$$

It follows that $\mathfrak{g} \hookrightarrow \mathbb{Z}G \twoheadrightarrow \mathbb{Z}(G/S)$ is short exact, and so $G = S$. □

All finitely presented groups are of type FP_2, however the converse is not true in general [10]. It is, however, true for all metabelian groups as we shall discover in Section 2.4.

We can interpret the low-dimensional cohomology groups in relatively simple ways. First consider $H^0(G, M)$. View $\mathbb{Z}$ as a trivial $\mathbb{Z}G$-module and observe that if $\phi : \mathbb{Z} \to M$ is a $\mathbb{Z}G$-module homomorphism then

$$\phi(n) = \phi(n.g) = \phi(n).g$$

for all $n \in \mathbb{Z}, g \in G$, so that $\phi(n)$ is a fixed point of M under the action of G. Conversely any fixed point m gives rise to a $\mathbb{Z}G$-module homomorphism θ defined by $\theta(1) = m$. Hence

$$H^0(G, M) = \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, M) \cong M^G$$

where $M^G = \{m \in M : m.g = m \text{ for all } g \in G\}$ is the group of invariants in M under the action of G. Thus M^G is the largest submodule of M on which G acts trivially.

What about the first cohomology group? We shall interpret this in terms of *derivations*.

Definition 2.2.3. A *derivation* from a group G to a $\mathbb{Z}G$-module M is a function δ with the property that, for all $g, h \in G$,

$$\delta(gh) = \delta(g)h + \delta(h).$$

If m is any fixed element of M then the function

$$g \mapsto m.g - m$$

is a derivation. Derivations of this type are called *inner*. We write $\mathrm{Der}(G, M)$ for the set of derivations from G to M. This can be turned into an additive group via addition of functions and the set $\mathrm{Ider}(G, M)$ of inner derivations is then a subgroup of $\mathrm{Der}(G, M)$.

Lemma 2.2.4. *There is a natural exact sequence*

$$0 \to M^G \to M \to \mathrm{Der}(G, M) \to H^1(G, M) \to 0$$

for $\mathbb{Z}G$-modules M. The image of the map $M \to \mathrm{Der}(G, M)$ is $\mathrm{Ider}(G, M)$ and so

$$H^1(G, M) \cong \mathrm{Der}(G, M)/\mathrm{Ider}(G, M).$$

Proof. Take the short exact sequence

$$0 \to \mathfrak{g} \hookrightarrow \mathbb{Z}G \stackrel{\varepsilon}{\twoheadrightarrow} \mathbb{Z} \to 0$$

introduced in Proposition 2.2.2 and substitute into the long exact sequence for the cohomological functor $\mathrm{Ext}^*_{\mathbb{Z}G}(\ , M)$ given in Definition 2.1.4. This yields the exact sequence

$$0 \to M^G \to M \to \mathrm{Hom}_{\mathbb{Z}G}(\mathfrak{g}, M) \to H^1(G, M) \to 0.$$

Note that $\mathrm{Ext}^1_{\mathbb{Z}G}(\mathbb{Z}G, M) = 0$ since $\mathrm{Ext}^1_{\mathbb{Z}G}(\ , M)$ is effaceable.

We need to show that $\mathrm{Hom}_{\mathbb{Z}G}(\mathfrak{g}, M)$ is isomorphic to $\mathrm{Der}(G, M)$. To see this, we need to prove that a function $d : G \to M$ is a derivation if and only if the map $\theta : \mathfrak{g} \to M$ defined by $\theta(g - 1) = d(g)$ is a $\mathbb{Z}G$-module homomorphism. The latter is the case if and only if

$$\begin{aligned} d(g).h = \theta(g-1).h = \theta((g-1).h) &= \theta(gh-h) \\ &= \theta(gh-1-(h-1)) \\ &= \theta(gh-1)-\theta(h-1) \\ &= d(gh)-d(h) \end{aligned}$$

for all $g, h \in G$, that is if and only if d is a derivation.

Finally, to see that the image of the map $M \to \mathrm{Der}(G, M)$ is $\mathrm{Ider}(G, M)$, note that $M \cong \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, M)$ and that the image of the map $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, M) \to \mathrm{Hom}_{\mathbb{Z}G}(\mathfrak{g}, M)$ consists of all G-maps ($\mathbb{Z}G$-module homomorphisms) that are restrictions to $\mathfrak{g}$ of G-maps from $\mathbb{Z}G$ to M. These amount to elements of M acted on by $\mathfrak{g}$, in other words they are of the form

$$m.(g-1) = m.g - m$$

for $g \in G$ and so correspond to inner derivations from G to M.

□

Definition 2.2.5. Let G be a group and let M be a $\mathbb{Z}G$-module. The *split extension* of M by G is the group $M \rtimes G$ whose elements consist of the set $G \times M$ and with group multiplication given by

$$(g, m)(g', m') = (gg', m.g' + m').$$

The identity element is $(1, 0)$ and inverses are given by $(g, m)^{-1} = (g^{-1}, -m.g^{-1})$. We get the short exact sequence

$$M \overset{\iota}{\hookrightarrow} M \rtimes G \overset{\pi}{\twoheadrightarrow} G$$

where $\iota(m) = (1, m)$ and $\pi((g, m)) = g$.

The following lemma describes the relationship between derivations and split extensions.

Lemma 2.2.6. *For a function $\delta : G \to M$ the following are equivalent:*

(i) δ is a derivation;

(ii) the function $\theta : G \to M \rtimes G$ defined by $\theta(g) = (g, \delta(g))$ is a group homomorphism. Hence there is a bijection between Der(G, M) and the set of homomorphisms $\theta : G \to M \rtimes G$ such that $\pi\theta = id_G$.

Proof. If $\theta : G \to M \rtimes G$ is a function with $\pi\theta = \mathrm{id}_G$ then $\theta(g) = (g, \delta(g))$ where $\delta : G \to M$ is some uniquely determined function. So we need only show that θ is a homomorphism if and only if δ is a derivation. This follows from the fact that

$$\theta(gg') = (gg', \delta(gg'))$$

and

$$\theta(g)\theta(g') = (gg', \delta(g)g' + \delta(g')).$$

□

We shall classify *all* extensions of a group G by a $\mathbb{Z}G$-module M in terms of the second cohomology group $H^2(G, M)$ and seek to determine when such an extension is split. This material will be covered in Chapter 5.

Dual to cohomology we have the notion of the *homology* of a group G.

Definition 2.2.7. Let $P_* \twoheadrightarrow \mathbb{Z}$ be a right projective resolution of the right $\mathbb{Z}G$-module $\mathbb{Z}$ and let M be a left $\mathbb{Z}G$-module. Then the *homology of G with coefficients in M* is defined to be

$$H_*(G, M) := H_*(P_\bullet \otimes_{\mathbb{Z}G} M).$$

Here $P_\bullet \otimes_{\mathbb{Z}G} M$ can be thought of as the chain complex obtained from $P_\bullet$ by applying the tensor product functor $- \otimes_{\mathbb{Z}G} M$. If M is a right $\mathbb{Z}G$-module then we can define the homology group as $H_*(G, M) = H_*(M \otimes_{\mathbb{Z}G} P_\bullet)$, where $P_\bullet$ is a left resolution of $\mathbb{Z}$.

In the cases that interest us G will always be an abelian group, hence $\mathbb{Z}G$ is commutative and so the distinction between left and right modules disappears.

Just as with cohomology groups, the homology groups $H_n(G, M)$ are independent of the choice of projective resolution of $\mathbb{Z}$. Let's consider these groups for $n = 0, 1$.

There is a simple interpretation of $H_0(G, M)$. Dual to the group of invariants M^G of a module M under the action of a group G we have the group of *coinvariants*. This is the quotient M_G of M given by factoring out the action of G:

$$M_G = M/\langle m.g - m : g \in G,\ m \in M\rangle.$$

Hence M_G is the largest quotient of M on which G acts trivially. There is an isomorphism

$$M_G \cong \mathbb{Z} \otimes_{\mathbb{Z}G} M;$$

see [19, §2.2] for more details. Now we have already observed that $H^0(G, M) = M^G$. A similar fact is true in homology: $H_0(G, M) = M_G$. This can be observed by applying the functor $- \otimes_{\mathbb{Z}G} M$ to a projective resolution for $\mathbb{Z}$ and using the fact that $- \otimes_{\mathbb{Z}G} M$ is right-exact [2, Proposition 2.18].

Now suppose that G is an infinite cyclic group with generator t. Then we have a resolution

$$0 \to \mathbb{Z}G \stackrel{t-1}{\to} \mathbb{Z}G \stackrel{\varepsilon}{\to} \mathbb{Z} \to 0 \tag{2.3}$$

where $t - 1$ denotes multiplication by this element and ε denotes the augmentation map $\varepsilon(g) = 1$. Tensoring with M via $- \otimes_{\mathbb{Z}G} M$ gives the exact sequence

$$M \stackrel{t-1}{\to} M \twoheadrightarrow M_G \to 0$$

and from this we observe that $H_1(G, M) = \ker(t - 1) = M^G$. If instead we apply the left-exact functor $\mathrm{Hom}_{\mathbb{Z}G}(\ , M)$ we obtain the exact sequence

$$M^G \hookrightarrow M \stackrel{t-1}{\to} M \to 0$$

and from this we see that $H^1(G, M) = M/(t - 1)M = M_G$. Hence

$$H_1(G, M) = H^0(G, M)$$

and

$$H^1(G, M) = H_0(G, M)$$

whenever G is an infinite cyclic group.

It is also clear from the choice of resolution (2.3) that $H_i(G,M) = H^i(G,M) = 0$ for $i > 1$, where G is an infinite cyclic group. This calculation of the first homology and cohomology groups for infinite cyclic groups will be useful to us when computing the cohomology of certain metabelian groups in Chapter 5.

2.3 Valuations and characters

In this section we introduce the concept of valuations and show that this leads to a way of using spherical geometry to study modules over finitely generated abelian groups.

Definition 2.3.1. Let R be a non-trivial commutative ring and let $\mathbb{R}_\infty$ denote the set of real numbers together with an additional point ∞, known as the 'point at infinity'. Then a *valuation* on R is a map $v : R \to \mathbb{R}_\infty$ satisfying $v(0) = \infty$, $v(1) = 0$, $v(ab) = v(a) + v(b)$ and $v(a + b) \geq \min\{v(a), v(b)\}$ for all $a, b \in R$.

For example, take $R = \mathbb{Q}(x)$, the field of rational functions in one variable. Let f be an irreducible polynomial in $\mathbb{Z}[x]$ and let $r(x)$ be a non-zero rational function. Since $\mathbb{Z}[x]$ is a unique factorization domain we can express r uniquely as

$$r(x) = (f(x))^\rho \frac{g(x)}{h(x)},$$

where ρ is some integer and g, h are a pair of coprime polynomials in $\mathbb{Z}[x]$, neither of which is divisible by f. We then define the *f-adic valuation* v by $v(r(x)) = \rho$. Note that in particular $v(f) = 1$ and that if $f = p$, where p is a prime number, and r is a rational number then v is the p-adic valuation on $\mathbb{Q}$.

As an example of an f-adic valuation take $f(x) = x+2$, $r(x) = g(x)/h(x)$ where $g(x) = x^3 + 5x + 6$ and $h(x) = 3x^3 + 24$. Observe that

$$g(x) = (x + 2)(x^2 - 2x + 9) - 12,$$

$$h(x) = 3(x + 2)(x^2 - 2x + 4)$$

and that $x^2 - 2x + 4$ is not divisible by $x + 2$. Hence $v(h) = -1$ and so $v(r(x)) = v(g) - v(h) = -1$.

We must check that an f-adic valuation v is indeed a valuation on $\mathbb{Q}(x)$ in accordance with Definition 2.3.1. Write $p = g/h$, $r = g'/h'$. It is easy to see that the first condition is satisfied. To see that the second condition holds first note that it certainly does in the special case where $h = h' = 1$ for then both p and r are polynomials in $\mathbb{Z}[x]$. In fact we get equality unless $v(p) = v(r) = \rho$ and the coefficient of $(f(x))^\rho$ in p is the negative of its coefficient in r. Note also that in this case $v(p), v(r) \geq 0$. Now suppose h is an arbitrary polynomial in $\mathbb{Z}[x]$. We have

$$p + r = (gh' + g'h)/hh'$$

and

$$\begin{aligned} v(gh' + g'h) &\geq \min\{v(gh'), v(g'h)\} \\ &= \min\{v(g) + v(h'), v(g') + v(h)\} \\ &= v(g) + v(h'), \end{aligned}$$

say. Then $v(g) - v(h) \leq v(g') - v(h')$ and so

$$\begin{aligned} v(p + r) &\geq v(g) + v(h') - v(hh') \\ &= v(g) - v(h) \\ &= \min\{v(p), v(r)\} \end{aligned}$$

as required.

Let us now relate valuations on rings to finitely generated abelian groups.

Definition 2.3.2. Let Q be a finitely generated abelian group. A real *character* of Q is a non-zero group homomorphism $v : Q \to \mathbb{R}$ into the additive group of real numbers.

If Q is a finitely generated abelian group then it is isomorphic to $\mathbb{Z}^n \oplus F$ where F is some finite group. The positive integer n is the *torsion-free rank* of Q, and the set $\mathrm{Hom}(Q, \mathbb{R})$ of real characters of Q is a real vector space of dimension n. One way to see this is that, since the torsion subgroup of $\mathbb{R}$ is trivial, any group homomorphism from Q to $\mathbb{R}$ is trivial on the torsion subgroup of Q and so we may assume without loss that Q is free abelian. Let $q_1, \ldots, q_n$ be a basis of Q and define v_i to be the real character of Q defined by

$$v_i(q_i) = 1,\ v_i(q_j) = 0 \quad (1 \leq i \neq j \leq n).$$

Then it is easily seen that the v_i form a vector space basis of $\mathrm{Hom}(Q, \mathbb{R})$.

The reason we will often use the letter v to denote a character of Q is that such a character can be extended to the group ring $\mathbb{Z}Q$ by defining $v(0) = \infty$ and

$$v(\lambda) = \min\{v(q) : q \in \mathrm{supp}(\lambda)\},$$

where $0 \neq \lambda \in \mathbb{Z}Q$ and the support $\mathrm{supp}(\lambda)$ of λ in Q is the (finite) set of all elements q with $\lambda(q) \neq 0$. Then v is a valuation on $\mathbb{Z}Q$, abusing notation. Clearly v satisfies

$$v(\lambda + \mu) \geq \min\{v(\lambda), v(\mu)\}$$

for all $\lambda, \mu \in \mathbb{Z}Q$.

To show that v is logarithmic, i.e. $v(\lambda\mu) = v(\lambda) + v(\mu)$, we choose a total ordering $\succeq$ on Q so that $q \succeq q'$ implies that $v(q) \geq v(q')$ and $qq'' \succeq q'q''$ for any $q'' \in Q$. For each non-zero $\lambda \in \mathbb{Z}Q$ let q_λ be the least element in $\mathrm{supp}(\lambda)$ under $\succeq$. It follows that $q_{\lambda\mu} = q_\lambda q_\mu$ for any $\mu \in \mathbb{Z}Q \setminus \{0\}$. We now have $v(\lambda\mu) = v(q_{\lambda\mu}) = v(q_\lambda) + v(q_\mu) = v(\lambda) + v(\mu)$.

We define an equivalence relation $\sim$ on the set $\mathrm{Hom}(Q, \mathbb{R}) \setminus \{0\}$ of all non-trivial real characters of Q as follows:

$$v_1 \sim v_2 \text{ if and only if } v_1 = \rho v_2 \text{ for some real number } \rho > 0.$$

Write $S(Q) = (\mathrm{Hom}(Q, \mathbb{R}) \setminus \{0\}) / \sim$ for the set of these equivalence classes $[v]$. We want to show that $S(Q)$ is homeomorphic to an $(n-1)$-dimensional sphere. Let $\theta : Q \to \mathbb{Z}^n \hookrightarrow \mathbb{R}^n$ be the isomorphism mapping the basis element q_i to $e_i = (0, 0, \ldots, 1, 0, \ldots, 0)$. The e_i form the usual basis for $\mathbb{Z}^n$ as a free abelian group. However they also form a basis for $\mathbb{R}^n$ as a real vector space, so given a character v of Q we can define $\alpha : \mathbb{Z}^n \to \mathbb{R}$ by $\alpha(e_i) = v(q_i)$ and extend linearly to $\mathbb{R}^n$ to get the unique $\mathbb{R}$-linear map $\bar{v} : \mathbb{R}^n \to \mathbb{R}$ with $\bar{v} \circ \theta = v$. Let $(y_1, \ldots, y_n) \in \mathbb{R}^n$. Then

$$\begin{aligned}\bar{v}(y) &= y_1\alpha(e_1) + \ldots + y_n\alpha(e_n)\\ &= y_1 v(q_1) + \ldots + y_n v(q_n)\\ &= \langle x_v, y\rangle\end{aligned}$$

where $x_v = (v(q_1), \ldots, v(q_n))$ and $\langle\ ,\ \rangle$ denotes the standard scalar product on $\mathbb{R}^n$. So $v(q) = \bar{v} \circ \theta(q) = \langle x_v, \theta(q)\rangle$. Clearly x_v is dependent only on the character v and so we have a one-to-one correspondence between non-zero characters of Q and points of $\mathbb{R}^n \setminus \{0\}$ given by

$$v \longleftrightarrow x_v.$$

Taking $(1/\|x_v\|)x_v$ yields a point on the unit sphere $S^{n-1} \subset \mathbb{R}^n \cong \mathrm{Hom}(Q, \mathbb{R})$. Since all positive multiples of v are identified in $S(Q)$ we have a bijection between $S(Q)$ and S^{n-1} given by

$$[v] \longleftrightarrow \frac{x_v}{\|x_v\|}.$$

Now let $Q_v = \{q \in Q : v(q) \geq 0\}$. This is a submonoid of Q and is clearly dependent only on the equivalence class $[v]$ of v, that is, $Q_v = Q_{v'}$ whenever $v \sim v'$. The definition of a monoid ring is analogous to the definition of a group ring. Given this we can form the monoid ring $\mathbb{Z}Q_v$ and then every $\mathbb{Z}Q$-module is also a $\mathbb{Z}Q_v$-module. Let A be a finitely generated $\mathbb{Z}Q$-module. One can associate to A the *Bieri-Strebel invariant*

$$\Sigma_A = \{[v] \in S(Q) : A \text{ is finitely generated as a } \mathbb{Z}Q_v\text{-module }\}.$$

So Σ_A is a subset of the $(n-1)$-sphere $S(Q)$. We write $\Sigma_A^c = S(Q) \setminus \Sigma_A$ for the complement of Σ_A in $S(Q)$. Define an *m-point set* to be any set consisting of $\leq m$ points. The $\mathbb{Z}Q$-module A, or alternatively Σ_A^c, is then said to be *m-tame* if every m-point subset of Σ_A^c lies in an open hemisphere of $S(Q)$. Equivalently, $v_1 + \ldots + v_m \neq 0$ for any $[v_1], \ldots, [v_m] \in \Sigma_A^c$.

Now define the centralizer $C(A)$ of the Q-module A to be the set

$$C(A) = \{\lambda \in \mathbb{Z}Q : \lambda a = a \text{ for all } a \in A\}.$$

We then have the following criterion of Bieri and Strebel [17] for whether or not A is finitely generated as a module over the valuation monoid Q_v for a given v.

Proposition 2.3.3 ([17, Proposition 2.1])**.** *Let A be a finitely generated $\mathbb{Z}Q$-module and $v : Q \to \mathbb{R}$ a non-trivial character. Then A is finitely generated over Q_v if and only if there exists $\lambda \in C(A)$ with $v(\lambda) > 0$. Moreover if A is finitely generated over Q_v then any set generating A as a $\mathbb{Z}Q$-module generates A as a $\mathbb{Z}Q_v$-module.*

Thus the criterion determines which elements $[v]$ of the sphere $S(Q)$ are *not* in Σ_A^c, or equivalently which elements are in Σ_A. As a consequence of this result we have the following alternative description of Σ_A, which is more useful to work with in practice:

$$\Sigma_A = \bigcup_{\lambda \in C(A)} \{[v] \in S(Q) : v(\lambda) > 0\}. \tag{2.4}$$

We will give the proof of the criterion in Chapter 4.

2.4 Metabelian groups

In this final section of Chapter 2 we prove a necessary and sufficient condition for a metabelian group to be finitely generated and describe how the Bieri-Strebel invariant Σ_A defined in Section 2.3 relates to the study of finitely generated metabelian groups.

Definition 2.4.1. A group G is *metabelian* if there exists a short exact sequence

$$A \hookrightarrow G \overset{\pi}{\twoheadrightarrow} Q \tag{2.5}$$

of groups with A and Q abelian.

Equivalently the derived group G' is abelian. Every metabelian group is thus a soluble group of derived length 2. The abelian group A in the exact sequence (2.5) is a $\mathbb{Z}Q$-module via conjugation: we define

$$a \circ q = g^{-1} a g$$

whenever $\pi(g) = q$. Since A is abelian, this is well-defined. We want to give an explicit proof of the fact that the metabelian group G is finitely generated if and only if Q is finitely generated as a group and A is finitely generated as a $\mathbb{Z}Q$-module. This well-known fact is a consequence of the following lemma of Neumann.

Lemma 2.4.2 ([30, Lemma 8]). *Let a group G be generated by a system of m generators $g_1, \ldots, g_m$ with r defining relations $R_1(g_1, \ldots, g_m) = 1, \ldots, R_r(g_1, \ldots, g_m) = 1$; let $h_1, \ldots, h_n$ be another system of generators of G. Then G can be defined by s relations in $h_1, \ldots, h_n$ with $s \leq r + n$.*

Proof. Since both systems $g_1, \ldots, g_m$ and $h_1, \ldots, h_n$ generate G, the h_j can be expressed in terms of the g_i and vice versa: we have $h_j = H_j(g)$ and $g_i = G_i(h)$ where $g = (g_1, \ldots, g_m)$ and $h = (h_1, \ldots, h_n)$. Now G can be generated by the g_i together with the h_j and defined by the relations

$$R_1(g) = 1, \ldots, R_r(g) = 1, \tag{2.6}$$

$$h_1 = H_1(g), \ldots, h_n = H_n(g). \tag{2.7}$$

To these we add the relations

$$g_1 = G_1(h), \ldots, g_m = G_m(h). \tag{2.8}$$

We can then express (2.6) and (2.7) in terms of (2.8):

$$R_1(G(h)) = 1, \ldots, R_r(G(h)) = 1, \tag{2.9}$$

$$h_1 = H_1(G(h)), \ldots, h_n = H_n(G(h)) \tag{2.10}$$

where $G(h) = (G_1(h), \ldots, G_m(h))$. Now (2.8), (2.9) and (2.10) define G. But the generators g_i can be dropped, together with the relations (2.8) which determine them in terms of the h_j: the g_i no longer enter any other relations. Hence (2.9) and (2.10) are sufficient to define G. □

Corollary 2.4.3. *If G is a finitely presented group and $\theta : F \twoheadrightarrow G$ a homomorphism from a finitely generated free group onto G then $\ker\theta$ is the normal closure of a finite set.*

Proof. By Lemma 2.4.2 any presentation for G can be given in terms of a finite number of relations. $\square$

Lemma 2.4.4. *Let $K \hookrightarrow G \overset{\pi}{\twoheadrightarrow} Q$ be a group extension. If G is finitely generated and Q finitely presented then K is finitely generated as a normal subgroup of G.*

Proof. Choose a finite presentation $\theta : F \twoheadrightarrow Q$ for Q, where F is free on a set $X = \{f_1, \ldots, f_d\}$. Let $R = \ker \theta$. By Corollary 2.4.3 R is finitely generated as a normal subgroup of F. We have a mapping

$$t\gamma : X \to G,$$

where γ takes X into Q and $t : Q \to G$ is a choice of *transversal* of π: that is, a function such that πt is the identity on Q. Since F is free this mapping can be extended to a homomorphism $\alpha : F \to G$: we have $\pi\alpha = \theta$.

The aim is to get a surjection onto G from some free group. Suppose that G can be generated by the elements

$$g_1, \ldots, g_{d'}$$

and take E to be free on a finite set of d' elements $e_1, \ldots, e_{d'}$. Define a homomorphism $\beta : E \to G$ by $\beta(e_i) = g_i$. We then form the free product $E * F$ and define a homomorphism $\widehat{\alpha} : E * F \to G$ to coincide with α on F and β on E. This provides a surjection onto G.

If we also define $\widehat{\theta} : E * F \to Q$ to coincide with θ on F and $\pi\beta$ on E, then we get a commutative diagram

$$\begin{array}{ccccc} \widehat{R} & \hookrightarrow & E * F & \overset{\widehat{\theta}}{\twoheadrightarrow} & Q \\ \downarrow{\scriptstyle \widehat{\alpha}_{|\widehat{R}}} & & \downarrow{\scriptstyle \widehat{\alpha}} & & \| \\ K & \hookrightarrow & G & \overset{\pi}{\twoheadrightarrow} & Q \end{array}$$

where $\widehat{R} = \ker \widehat{\theta}$. By Corollary 2.4.3 $\widehat{R}$ is finitely generated as a normal subgroup of $E * F$. We want to show that $\widehat{\alpha}$ maps $\widehat{R}$ surjectively onto K. Let $k \in K$. Then $k = \widehat{\alpha}(x)$ for some $x \in E * F$, and so

$$\pi(\widehat{\alpha}(x)) = \pi(k) = 1.$$

But $\pi\widehat{\alpha} = \widehat{\theta}$, so $\widehat{\theta}(x) = 1$ implying that $x \in \widehat{R}$. Hence $\widehat{R}$ maps surjectively onto K. Now $\widehat{R}$ is the normal closure of a finite set of elements $r_1, \ldots, r_k$ in $E * F$, and so it follows that K is the normal closure in G of the $\widehat{\alpha}(r_i)$, $1 \leq i \leq k$. This completes the proof. □

Proposition 2.4.5. *The metabelian group G in* (2.5) *is finitely generated if and only if Q is finitely generated as a group and A is finitely generated as a $\mathbb{Z}Q$-module.*

Proof. Suppose that G is finitely generated. Then Q is a finitely generated abelian group, hence it is finitely presented. It follows from Lemma 2.4.4 that A is finitely generated as a normal subgroup of G and so it is finitely generated as a $\mathbb{Z}Q$-module via conjugation.

Conversely let $a_1, \ldots, a_n$ be generators of A as a $\mathbb{Z}Q$-module. Let $g_1, \ldots, g_k$ be elements of G whose images $\pi(g_1), \ldots, \pi(g_k)$ in Q generate Q, and write H for the subgroup of G generated by $g_1, \ldots, g_k$. Thus, for any $g \in G$, $\pi(g) = \pi(h)$ for some $h \in H$, and so $gh^{-1} \in \ker \pi = A$, implying that $g \in AH$. Hence $G = AH$.

It follows that any conjugate $a^g = a^{a'h} = a^h$ for some $a' \in A$, $h \in H$, since A is abelian and so A is generated as a subgroup of G by all conjugates of $a_1, \ldots, a_n$ by elements of H. Thus, writing

$$S = \langle a_1, \ldots, a_n, g_1, \ldots, g_k \rangle,$$

we have $A \subset S$ and since $H \subset S$ it follows that $G = S$. Hence G is finitely generated. □

Henceforth we shall always assume that a metabelian group G is finitely generated. Viewing A as a $\mathbb{Z}Q$-module opens up the study of finitely generated metabelian groups to commutative algebra. In particular, if A is finitely generated over Q then it is noetherian. Baumslag made heavy use of commutative ring theory, in particular localization, in order to prove the following result: that every finitely generated metabelian group embeds in one that is finitely presented.

Theorem 2.4.6 ([5])**.** *Every finitely generated metabelian group can be embedded in a finitely presented metabelian group.*

Proof. We outline a sketch of the proof, as given in the note [4]. Let G be a finitely generated metabelian group. First G is embedded in a factor group W/N of the wreath product W of two finitely generated abelian groups. This embedding is constructed in such a way that N is contained in the base group of W.

The crucial step in the proof is to show that W can then be embedded in a finitely presented metabelian group $\widehat{W}$ in a way that allows the normal closure $\widehat{N}$ of N in $\widehat{W}$ to meet W in N i.e.

$$\widehat{N} \cap W = N.$$

Formally, if we have an explicit injection $\phi : W \to \widehat{W}$ such that $\widehat{N} \cap (\phi(W)) = \phi(N)$, then we get an injection $\bar{\phi} : W/N \to \widehat{W}/\widehat{N}$ defined by

$$\bar{\phi}(wN) := \phi(w)\widehat{N}$$

and so G is embedded in $\widehat{W}/\widehat{N}$. By [24, Theorem 3, Corollary 1] the fact that finitely generated metabelian groups satisfy the max-n condition implies that every factor group of a finitely presented metabelian group is finitely presented, and so we have embedded G in a finitely presented metabelian group.

□

From Theorem 2.4.6 we can see that the subgroup structure of finitely presented metabelian groups is in general as complicated as that of finitely generated metabelian groups. It also leads to the obvious question: is there a way of distinguishing finitely presented metabelian groups from amongst all those that are finitely generated? Here we bring the concepts introduced in Section 2.3 into play. Write $-\Sigma_A = \{[-v] : [v] \in \Sigma_A\}$. Bieri and Strebel [17] proved that a metabelian group G as in (2.5) is finitely presented if and only if $\Sigma_A \cup (-\Sigma_A) = S(Q)$, or equivalently A is 2-tame. Hence whether or not such a metabelian group is finitely presented depends only on the Q-module A and not on the extension class of G in $H^2(Q, A)$: in particular, G is finitely presented if and only if the split extension $A \rtimes Q$ is finitely presented.

In addition, Bieri and Strebel proved in [17] that metabelian groups of type FP_2 are always finitely presented. Their results, together with those in [13], led to the

FP_m-conjecture of Bieri and Groves.

Conjecture 2.4.7 ([13])**.** Suppose we have the short exact sequence (2.5) and let m be a positive integer. Then G is of type FP_m if and only if A is m-tame as a $\mathbb{Z}Q$-module.

Our main results in Chapters 3 and 4 will provide evidence in favour of the FP_m-conjecture being true.

Chapter 3

A class of torsion-free finitely presented metabelian groups

We now introduce the torsion-free metabelian groups we wish to study. To do so we generalize an example of Baumslag [3], and prove that these groups are finitely presented. We then prove a generalized version of a theorem of Groves and Kochloukova in order to give more precise cohomological finiteness conditions for these, and related, groups.

3.1 Generalization of Baumslag's group

Let $n \geq 0$ be an integer. Define G_n to be the group generated by the following 2×2 matrices over $\mathbb{Q}(x)$:

$$\alpha = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \ \sigma_i = \begin{pmatrix} i+x & 0 \\ 0 & 1 \end{pmatrix}, \ \beta = \begin{pmatrix} n! & 0 \\ 0 & 1 \end{pmatrix}$$

for $0 \leq i \leq n$.

Consider now the split extension $A_n \rtimes Q_n$, where

$$A_n = \mathbb{Z}\left[x, x^{-1}, (1+x)^{-1}, \dots, (n+x)^{-1}, \frac{1}{n!}\right]$$

and Q_n is the subgroup of the group of units of A_n generated by

$$n!, \ x, \ 1+x, \ \dots, \ n+x.$$

The group Q_n is thus free abelian of rank $n+2$ for $n \geq 2$. For $n = 0, 1$, $n! = 1$ and so the rank of Q_n is $n+1$. The split extension can be formed because A_n is a cyclic $\mathbb{Z}Q_n$-module via multiplication, with generator 1: the action can be thought of as conjugation in the split extension since

$$(1, a \circ q) = (q, 0)^{-1}(1, a)(q, 0)$$

for any $a \in A_n$, $q \in Q_n$. Note that A_n is not free over $\mathbb{Z}Q_n$: writing $q_0 = x$ and $q_1 = 1 + x$ we see that the non-identity element $q_1 - q_0 - 1$ in $\mathbb{Z}Q_n$ acts trivially on A_n.

It can be readily seen that the matrix group G_n is isomorphic to the split extension $A_n \rtimes Q_n$ and so is metabelian. Indeed, any element of G_n is of the form

$$\begin{pmatrix} q & 0 \\ a & 1 \end{pmatrix},$$

where q is non-zero but a is possibly zero, and the isomorphism is given by

$$(q, a) \longleftrightarrow \begin{pmatrix} q & 0 \\ a & 1 \end{pmatrix}$$

where $q \in Q_n$ and $a \in A_n$. We have an exact sequence

$$A_n \overset{\iota}{\hookrightarrow} A_n \rtimes Q_n \overset{\pi}{\twoheadrightarrow} Q_n,$$

where $\iota(a) = (1, a)$ and $\pi((q, a)) = q$. Given this isomorphism we can write $G_n = A_n \rtimes Q_n$. It is then clear that the subgroup Q_n of G_n is generated by the matrices τ, β, σ_i ($1 \leq i \leq n$). Hence Q_n is an abelian quotient of G_n and so A_n contains the derived subgroup of G_n; that is, the subgroup generated by all commutators of the form $[x, y] = x^{-1}y^{-1}xy$ for $x, y \in G_n$. Since A_n is a Laurent polynomial ring over the subring $\mathbb{Z}[1/n!]$ of $\mathbb{Q}$, it is clearly a torsion-free abelian group of infinite rank when viewed additively.

Suppose now that we define $\widehat{G}_n = A_n \rtimes \widehat{Q}_n$, where $\widehat{Q}_n$ is the free abelian group of rank $n+2$ with basis elements

$$q_{-1}, q_0, q_1, \ldots, q_n$$

acting on A_n by multiplication with $n!$, x, $1+x$, $\dots$, $n+x$ respectively. Then $G_n \cong \widehat{G}_n$ for $n \geq 2$. For $n \leq 1$ however the the rank of Q_n is $n+1$, since $n! = 1$, and so q_{-1} acts trivially on A_n. Hence

$$G_n \cong \widehat{G}_n \times \mathbb{Z}$$

for $n = 0, 1$. By applying the following result to the short exact sequence

$$\mathbb{Z} \hookrightarrow \widehat{G}_n \twoheadrightarrow G_n,$$

and using the fact that $\mathbb{Z}$ is of type FP_∞, we see that $\widehat{G}_n$ is of type FP_{n+1} if and only if G_n is.

Proposition 3.1.1 ([12, Proposition 2.7])**.** *Let $N \hookrightarrow G \twoheadrightarrow Q$ be a short exact sequence of groups and assume that N is of type FP_∞. Then G is of type FP_n if and only if Q is.*

As our goal is to prove that G_n is of type FP_{n+1} it would thus be fine to work with the definition of $\widehat{G}_n$, without loss. In practice we shall use the definition of G_n but adopt the notation of $\widehat{G}_n$ and write the elements of Q_n as $q_{-1}, q_0, q_1, \dots, q_n$ for $n \geq 2$, q_0, q_1 for $n = 1$ and q_0 for $n = 0$.

Note that we use the notation $x^y = y^{-1}xy$ for the conjugate of x by y, where x and y are two group elements. Now, in the $n = 0$ case, G_0 is the split extension of an infinite cyclic group $\langle x \rangle$ by the Laurent polynomial ring $\mathbb{Z}[x, x^{-1}]$. This is a 2-generator, 1 relator-group: it has the presentation

$$G_0 = \langle a, t : [a^t, a] = 1 \rangle.$$

Baumslag [3] proved that G_1 is a 3-generator 3-relator group, with presentation

$$G_1 = \langle a, s, t : [s, t] = [a^t, a] = 1, a^s = aa^t \rangle,$$

and that it is an example of a finitely presented metabelian group whose derived group A_1 is free abelian of infinite rank. We generalize his proof here to encompass the groups G_n for all $n \geq 1$ and show that these are all finitely presented. First we note that, taking $a = \alpha$, $s_i = \sigma_i$ for $1 \le i \le n$, $t = \sigma_0$ and $b = \beta$, the following relations hold in G_n:

$$a^{s_i} = a^i a^t,\ a^b = a^{n!},\ [a^t, a] = [s_i, s_j] = [s_i, t] = [s_i, b] = [b, t] = 1. \tag{3.1}$$

Theorem 3.1.2. *Let G be the finitely presented group defined by the relations in* (3.1)*. Then G is metabelian. It follows that the group G_n is finitely presented.*

Proof. From the defining relations for G we can see that the derived group G' is generated by all conjugates

$$a^{{s_1}^{j_1} \ldots {s_n}^{j_n} t^k b^l}$$

of the generator a, where $j_1, \ldots, j_n, k, l$ are any integers. Hence it will be enough to prove that these conjugates commute.

The key part of the proof lies in showing that commutativity of the conjugates a^{t^i} follows from the relation $[a^t, a] = 1$. We do this by induction. Suppose we have proved that the group generated by $a, a^t, \ldots, a^{t^n}$ is abelian for some $n \ge 1$. Then the group generated by $a^t, a^{t^2}, \ldots, a^{t^{n+1}}$ is also abelian since, for $1 \le k \le n$,

$$a^{t^k} a^{t^{n+1}} = (a^{t^{k-1}} a^{t^n})^t = (a^{t^n} a^{t^{k-1}})^t = a^{t^{n+1}} a^{t^k}.$$

Therefore, using the defining relations,

$$\begin{aligned} 1 &= [a, a^{t^n}]^{s_1} = [a^{s_1}, a^{t^n s_1}] = [a^{s_1}, a^{s_1 t^n}] \\ &= [aa^t, (aa^t)^{t^n}] = [aa^t, a^{t^n} a^{t^{n+1}}] \end{aligned}$$

and, since a^t, a^{t^n}, $a^{t^{n+1}}$ all commute, we see that $[aa^t, a^{t^n} a^{t^{n+1}}] = [a, a^{t^{n+1}}]$. Hence the group generated by $a, a^t, \ldots, a^{t^{n+1}}$ is also abelian and as a result

$$[a, a^{t^i}] = 1 \tag{3.2}$$

for all positive integers i. It follows easily that $[a, a^{t^i}] = 1$ for all negative i also.

For $1 \le i \le n$, $a^{{s_i}^j}$ is a product of the elements $a, a^t, \ldots, a^{t^j}$ for a fixed $j \ge 0$ and so, by (3.2),

$$[a, a^{t^k {s_i}^j}] = 1.$$

Since $a^b = a^{n!}$, it follows that any conjugate of the form

$$a^{{s_1}^{j_1}{s_2}^{j_2}\ldots{s_n}^{j_n}t^k b^l}, \tag{3.3}$$

where the j_i are all nonnegative and k, l are any integers, commutes with a. If we have some $j_{i_1}, \ldots, j_{i_t} < 0$ then observe that

$$\begin{aligned} 1 &= [a^{{s_{i_1}}^{-j_{i_1}}\ldots{s_{i_t}}^{-j_{i_t}}}, a^{{s_1}^{j_1}\ldots\widehat{s_{i_1}}^{-j_{i_1}}\ldots\widehat{s_{i_t}}^{-j_{i_t}}\ldots{s_n}^{j_n}t^k b^l}] \\ &= [a^{{s_{i_1}}^{-j_{i_1}}\ldots{s_{i_t}}^{-j_{i_t}}}, a^{{s_1}^{j_1}\ldots\widehat{s_{i_1}}^{-j_{i_1}}\ldots\widehat{s_{i_t}}^{-j_{i_t}}\ldots{s_n}^{j_n}t^k b^l}]^{{s_{i_1}}^{-j_{i_1}}\ldots{s_{i_t}}^{-j_{i_t}}} \\ &= [a, a^{{s_1}^{j_1}\ldots{s_n}^{j_n}t^k b^l}], \end{aligned}$$

again using (3.2) and where $\widehat{s}$ denotes the omission of s from the given product. So a commutes with conjugates of the form (3.3) where $j_1, \ldots, j_n$ can be *any* integers.

Now, from the relations (3.1) it is clear that the subgroup Q of G generated by $s_1, \ldots, s_n, t, b$ is free abelian. Then the derived group G' is generated by all conjugates a^q, where $q \in Q$, and we have shown that

$$[a, a^q] = 1$$

whenever $q \in Q$. Hence, given $\theta, \phi \in Q$,

$$1 = [a, a^{\phi\theta^{-1}}] = [a^\theta, a^\phi]^{\theta^{-1}} = [a^\theta, a^\phi].$$

Thus G' is abelian and so G is metabelian. Since G_n is a homomorphic image of G it follows that G_n is finitely presented, because factor groups of finitely presented metabelian groups are finitely presented. □

The fact that G_n is finitely presented implies that it is also of cohomological type FP_2. Our aim is to show that it is in fact of type FP_{n+1} and that this result is sharp: in other words, G_n is not of type FP_{n+2} for $n \geq 1$. We shall show the former by proving the following result which applies to a generalized version of the groups G_n. First of all note that two elements of a ring are said to be *coprime* if together they do not lie in any proper ideal of the ring. Then it is clear that the polynomials x, $1 + x$, …, $n + x$ are pairwise coprime in $\mathbb{Z}[x, 1/n!]$, since the difference of any of these pairs is invertible in this ring.

Theorem 3.1.3. *Suppose $n \geq 1$ and let Q be a free abelian group with a finitely generated $\mathbb{Z}Q$-module*

$$A = \mathbb{Z}[q_0, q_0^{-1}, f_1^{-1}, \ldots, f_n^{-1}, k^{-1}],$$

where k is a positive integer. Write $q_0 = f_0$ and assume that for $0 \leq i \leq n$ the f_i are irreducible non-constant monic polynomials in $\mathbb{Z}[q_0]$ that are pairwise coprime as elements of $\mathbb{Z}[q_0, 1/k]$. For $k = 1$ we assume that Q has a basis $q_0, q_1, \ldots, q_n$ acting on A by multiplication with $q_0, f_1, \ldots, f_n$ respectively. If $k > 1$ then Q has an additional basis element q_{-1} acting by multiplication with k. In each case the split extension $G = A \rtimes Q$ is of type FP_{n+1}.

Corollary 3.1.4. *The metabelian group $G_n = A_n \rtimes Q_n$ is of type FP_{n+1}.*

Proof. Apply Theorem 3.1.3 with $k = n!$ and $f_i = i + q_0$ for $0 \leq i \leq n$. □

Notice that for convenience we have changed the notation used for the indeterminant in the polynomial ring A_n from x to q_0. To prove Theorem 3.1.3 we follow the method used by Groves and Kochloukova to prove Theorem 5 in their paper [23] and construct a CW-complex Y endowed with an action of the metabelian group G. We then prove three important properties possessed by Y: two concerning the action of G on Y and the third involving the structure of Y as a topological space. Given these three properties, we then invoke a result of Brown [20, Proposition 1.1] and it will follow that G is of type FP_{n+1}. The technicalities of this construction are discussed in the next section.

3.2 Character trees and the Åberg complex

In this section we shall describe how the G-CW-complex required to prove Theorem 3.1.3 is constructed. The construction is the same as that carried out by Groves and Kochloukova ([23], [26]) and is based on the work of Hans Åberg [1]. A similar method was also used by Bux [21].

Let G be the split extension of a finitely generated free abelian group Q by a finitely generated $\mathbb{Z}Q$-module A. For each non-zero group homomorphism (or character) $v : G \to \mathbb{R}$ we can construct a *tree* associated to v. A tree is a non-empty connected graph with no cycles. In order to carry out this construction we shall make some assumptions concerning the character v. First, that it is *discrete*; in other words its image is $\mathbb{Z}$. Furthermore, and viewing A and Q as subgroups of G via the isomorphisms $A \cong \{(1, a) : a \in A\}$ and $Q \cong \{(q, 0) : q \in Q\}$, we assume that $v(A) = 0$ and that there is a basis of Q containing an element q_v such that $v(q_v) = 1$ and $v(q) = 0$ for all other q in this basis. Hence v factors through the projection $\pi : G \twoheadrightarrow Q$ and $v(g) = v(q)$ whenever $\pi(g) = q$. So v restricts to a character of Q and we need only consider its values on Q.

Now choose a set of elements $a_1, \ldots, a_d$ generating A as a right $\mathbb{Z}Q$-module. If Q_v is the submonoid $\{q \in Q : v(q) \geq 0\}$ of Q then we define A_v to be the $\mathbb{Z}Q_v$-submodule of A generated by $a_1, \ldots, a_d$. Fix a negative integer β, which we shall define explicitly in Section 3.3, and let $G(v)$ be the subgroup of G generated by $A_v \circ {q_v}^\beta$ and $\ker(v|_Q)$, where $\circ$ denotes the action of Q on A by conjugation. Note that $G(v) \cap A = A_v \circ q_v^\beta$ and that since $q_v \in Q_v$ we have ${q_v}^{-1}G(v)q_v \subseteq G(v)$. Hence $G(v)q_v \subseteq q_vG(v)$. Define $G(v)^+$ to be the submonoid of G generated by q_v and the elements of $G(v)$. Then we have

$$G(v)^+ = \bigcup_{z \geq 0} {q_v}^z G(v). \tag{3.4}$$

It is easy to see that $G(v)^+ \supset \bigcup_{z\geq 0} {q_v}^z G(v)$. To show that $G(v)^+ \subseteq \bigcup_{z\geq 0} {q_v}^z G(v)$ observe that if $y = {q_v}^{k_1} g_1 {q_v}^{k_2} g_2 \ldots {q_v}^{k_m} g_m$ is an arbitrary element of $G(v)^+$, with the k_i nonnegative integers and the g_i elements of $G(v)$, then

$$y \in {q_v}^{k_1+k_2+\ldots+k_m} G(v),$$

because $G(v)q_v \subseteq q_vG(v)$.

Let T_0 be the right G-set $G(v) \setminus G = \{G(v)g : g \in G\}$ with action given by right multiplication. We define a relation $\leq$ on T_0 by $G(v)g \leq G(v)h$ if and only if $h \in G(v)^+g$.

Lemma 3.2.1. *The relation $\leq$ is a partial order on T_0.*

Proof. It is straightforward to check that $\leq$ is well-defined and transitive so we can move straight on to proving that, given $G(v)g \leq G(v)h$ and $G(v)h \leq G(v)g$, we have $G(v)g = G(v)h$.

Write $(G(v)^+)^{-1}$ for the set of inverses in G of the elements of $G(v)^+$. Since $gh^{-1} = (hg^{-1})^{-1} \in (G(v)^+)^{-1}$ we have $gh^{-1} \in G(v)^+ \cap (G(v)^+)^{-1}$. However, $G(v)^+ \cap (G(v)^+)^{-1} = G(v)$. To see this, let $x \in G(v)^+ \cap (G(v)^+)^{-1}$. Then $x = (q_v)^r g$ for some $g \in G(v)$ and $r \geq 0$. Suppose $r \geq 1$. Now $G(v)^+ \cap (G(v)^+)^{-1}$ is multiplicatively closed and $g \in G(v) \subset G(v)^+$, so $(q_v)^r \in G(v)^+ \cap (G(v)^+)^{-1}$. Hence $(q_v)^{-r} \in G(v)^+$, contradicting the definition of $G(v)^+$. Hence $r = 0$ and so $x \in G(v)$. The opposite inclusion is clear.

It follows that $gh^{-1} \in G(v)$ and so $G(v)g \subseteq G(v)h$. Similarly we can show that $G(v)h \subseteq G(v)g$ and so $G(v)g = G(v)h$. □

As $G(v) \subseteq \ker v$, the character $v : G \to \mathbb{Z}$ factors through a map $h_v : T_0 \to \mathbb{Z}$; that is, $v(g) = h_v(G(v)g)$ for all $g \in G$. We call two elements $\alpha \leq \gamma$ of T_0 *neighbours* if $h_v(\gamma) - h_v(\alpha) = 1$ and link every pair $\alpha \leq \gamma$ of neighbours with a closed unit interval. We obtain a connected graph Γ_v with vertex set T_0 and in which two vertices are joined by an edge if and only if the corresponding elements of T_0 are neighbours. Each edge can then be labelled with a unit interval and $h_v : T_0 \to \mathbb{Z}$ can be extended to an $\mathbb{R}$-linear map $h_v : \Gamma_v \to \mathbb{R}$ by setting

$$h_v((1-t)\alpha + t\gamma) = (1-t)h_v(\alpha) + th_v(\gamma) = h_v(\alpha) + t$$

whenever $\alpha \leq \gamma$ are neighbours, for $t \in [0, 1]$. If $\alpha \leq \gamma$ are neighbours then αg and γg are also neighbours. We make G act on Γ_v by setting

$$((1-t)\alpha + t\gamma) * g = (1-t)(\alpha g) + t(\gamma g)$$

whenever α and γ are neighbours.

For any fixed $g \in G$ we define $L_{g,v}$ to be the subgraph of Γ_v spanned by the set of vertices $\{G(v){q_v}^z g\}_{z \in \mathbb{Z}}$. Any two of these vertices are distinct, since assuming otherwise forces us to conclude that $q_v \in G(v)$, a contradiction. Note that $G(v){q_v}^z g$ and $G(v){q_v}^{z+1} g$ are neighbours and so $L_{g,v}$ is a line. Indeed two vertices $G(v){q_v}^m g$ and $G(v){q_v}^n g$ in $L_{g,v}$ are neighbours if and only if $|m - n| = 1$.

The image under h_v of a point

$$(1-t)G(v){q_v}^z g + tG(v){q_v}^{z+1}g$$

on the edge joining $G(v){q_v}^z g$ and $G(v){q_v}^{z+1}g$ in $L_{g,v}$ is $n+z+t$, where $n = v(g)$ and $t \in [0,1]$. Hence the restriction $h_v : L_{g,v} \to \mathbb{R}$ is bijective: for a fixed integer n, every non-integral real number r can be written in precisely one way as

$$r = n + z + t$$

where z is some integer and $t \in (0,1)$. If $m \in \mathbb{Z}$ then $m = n + (m-n-1) + 1 = n + (m-n) + 0$ but the pairs $(m-n-1, 1)$ and $(m-n, 0)$ for (z,t), $t \in [0,1]$, give the same point on $L_{g,v}$.

The following lemma shows that the graph Γ_v is the union of the subgraphs $L_{a,v}$, where $a \in A$.

Lemma 3.2.2. *The group* $G = \bigcup_{z\in\mathbb{Z}} G(v){q_v}^z A$.

Proof. Let $K = \ker(v|_Q)$. Then since $K{q_v}^z = \{q \in Q : v(q) = z\}$ for any integer z we have

$$Q = \bigcup_{z\in\mathbb{Z}} K{q_v}^z.$$

Hence every element $g \in G = QA$ can be expressed in the form $g = q{q_v}^z a$ where $q \in K$, $a \in A$. But $K \subseteq G(v)$ and so we conclude that $G = \bigcup_{z\in\mathbb{Z}} G(v){q_v}^z A$. □

It follows immediately that

$$\Gamma_v = \bigcup_{a\in A} L_{a,v}.$$

Let $a \in A$, $r \in \mathbb{R}$. We write $[(a,r)]$ for the unique element of $L_{a,v}$ whose image under h_v is r. By the definition of the G-action on Γ_v we have, for any two elements $a, b \in A$, and viewing A additively as an abelian group,

$$[(a,r)] * b = [(a+b, r)].$$

We can now prove that the graph Γ_v is a tree.

Lemma 3.2.3 ([26, Lemma 2.1]). *For every $a, b \in A$ we have*

$$L_{a,v} \bigcap L_{b,v} = \{[(a,r)] = [(b,r)] : r \leq z_0\},$$

where $z_0 = \sup\ \{z : a - b \in A_v \circ {q_v}^{\beta+z}\}$.

In other words, the intersection of any two lines labelled by different elements of A is either a line (extending infinitely in both directions) or a ray (extending infinitely in only one direction). From this result we can conclude that Γ_v cannot contain any cycles and so it is a tree.

Proof. Let r be a non-integral real number. Then r can be written uniquely as $z+t$ for some integer z and $t \in (0,1)$. If a, b are elements of A then $[(a,r)] = [(b,r)]$ if and only if $G(v){q_v}^z a = G(v){q_v}^z b$, and this will be the case if and only if $(a-b) \circ {q_v}^{-z} \in G(v) \cap A = A_v \circ {q_v}^{\beta}$. Then the result follows from the fact that if $z \leq z_0$ then $a - b \in A_v \circ {q_v}^{\beta+z_0} \subseteq A_v \circ {q_v}^{\beta+z}$, since $A_v \circ q_v \subseteq A_v$. □

We must choose some set V of $n+2$ discrete characters of Q satisfying the assumptions made at the start of this section, where $n \in \mathbb{N}$. Define the product $X = \prod_{v \in V} \Gamma_v$ and a map $h : X \to \mathbb{R}^{n+2}$ given by

$$h\left(\prod_{v \in V} [(a_v, r_v)]\right) := \prod_{v \in V} h_v([(a_v, r_v)]) = \prod_{v \in V} r_v.$$

Let $f : Q \otimes_{\mathbb{R}} \mathbb{R} \to \mathbb{R}^{n+2}$ be the $\mathbb{R}$-linear map extending the map

$$q \mapsto \prod_{v \in V} v(q)$$

and let W be the image of f. We then define

$$Y = \{x \in X : h(x) \in W\} = h^{-1}(W).$$

We can put CW-complex structures on X and its subspace Y as follows: X is a product of trees and can be viewed as built by gluing together $(n+2)$-dimensional cubes of unit length obtained by taking one edge of every tree Γ_v. Then the intersection of Y with any such $(n+2)$-cube or smaller dimensional cube of its boundary is either empty or some simplex. Such an intersection is homeomorphic to its image via the map h. These simplices are the cells of Y. Each cell is homeomorphic to $\mathbb{R}^m$ for some natural number m and its geometric dimension is equal to its real vector space dimension.

We define the diagonal action of G on X by

$$\left(\prod_{v\in V}[(a_v,r_v)]\right) * g = \prod_{v\in V}[(a_v,r_v)] * g.$$

Lemma 3.2.4 ([26, Lemma 2.2]). *The subspace Y of X is invariant under this G-action.*

Proof. Let $\prod_{v\in V}[(a_v,r_v)]$ be a point of Y and let g be an element of G. Then $\left(\prod_{v\in V}[a_v,r_v]\right) * g = \prod_{v\in V}[(a'_v, r_v+v(g))]$ where $G(v){q_v}^{\lceil r_v\rceil}a_v g = G(v){q_v}^{\lceil r_v\rceil+v(g)}a'_v$ for some element a'_v of A and where $\lceil r_v\rceil$ is the least integer greater than or equal to r_v. If $q=\pi(g)$ then $v(g)=v(q)$ and so $\prod_{v\in V} v(g)\in W$. Hence $\prod_{v\in V}(r_v+v(g))\in W$ and so

$$\prod_{v\in V}[(a'_v, r_v+v(g))]\in Y.$$

$\square$

The CW-complex Y is said to be of *Åberg type*. Next we shall begin the proof of Theorem 3.1.3 by choosing the set V of characters of the group Q appropriately.

3.3 The construction of the set of characters V

Our next step is to define an explicit set V of characters so that we can tackle the specific case of Theorem 3.1.3; namely, where G is a type of split extension encompassing the case $G=G_n$. The theorem we will prove is the following generalization of Theorem 5 in [23].

Theorem 3.3.1. *Let l be a positive integer and suppose that $Q=Q_0\times Q_1\times\ldots\times Q_l$ is a finitely generated free abelian group, where $Q_0=\langle q_{-1}\rangle$ and Q_i is a free abelian group of rank z_i+1 with basis $\{q_{i,j}\}_{0\le j\le z_i}$, for $1\le i\le l$. Let A be a finitely generated $\mathbb{Z}Q$-module with annihilator ideal I and let $I_i=I\cap\mathbb{Z}Q_i$. Write $M=\mathbb{Z}Q/I$ and*

$M_i = \mathbb{Z}Q_i/I_i$. We assume that the natural map

$$M \to M_0 \otimes M_1 \otimes \ldots \otimes M_l$$

is an isomorphism and that A is free as an M-module. Further we assume that $I_0 = \langle q_{-1} - k\rangle$ where k is some positive integer. If $k = 1$ we suppose that q_{-1} is trivial so that Q_0 is the trivial group: otherwise Q_0 is infinite cyclic. For $1 \leq i \leq l$, I_i is generated as an ideal by

$$\{q_{i,j} - f_{i,j}\}_{0 \leq j \leq z_i}$$

where for fixed i the $f_{i,j}$ are irreducible non-constant monic polynomials in $\mathbb{Z}[q_{i,0}]$ that are pairwise coprime in $\mathbb{Z}[q_{i,0}, 1/k]$. Assume also that $f_{i,0} = q_{i,0}$. Then the split extension $G = A \rtimes Q$ is of type FP_m, where $m = \min\{rk(Q_i) : 1 \leq i \leq l\}$.

Theorem 3.1.3 is then a special case of Theorem 3.3.1: take $l = 1$ and $A = M$. Theorem 5 in [23] deals with the case where $k = 1$ in Theorem 3.3.1. Thus Groves and Kochloukova did not allow for any positive integer $k > 1$ to be inverted in their theorem, but we will need this property to allow $k = n!$ to be inverted in the particular case of $G = G_n$.

From now on we are in the situation described in Theorem 3.3.1. We shall concentrate on the group $\widetilde{Q} = Q_0 \times Q_1$ as this will suffice to prove what we need. Write q_j for $q_{1,j}$, f_j for $f_{1,j}$ and $z_1 = n$ so that Q_1 has rank $n+1$. Hence $n+1 \geq m$. Let $\widetilde{M} = M_0 \otimes M_1$; then

$$\widetilde{M} \cong \mathbb{Z}[q_0, q_0^{-1}, f_1^{-1}, \ldots, f_n^{-1}, k^{-1}].$$

The q_j act on $\widetilde{M}$ by multiplication with f_j for $j \geq 1$ while q_{-1} acts by multiplication with the positive integer k. Since the case where $k = 1$ is proved in [23] we can assume that $k \geq 2$. Let $\phi : \mathbb{Z}\widetilde{Q} \to M$ be the ring homomorphism sending q_{-1} to k and q_i to f_i for $0 \leq i \leq n$. The restriction of ϕ to $\widetilde{Q}$ is a group homomorphism τ from the free abelian group $\widetilde{Q}$ to the group of units $(\widetilde{M})^\times$.

Lemma 3.3.2. *The group homomorphism $\tau : \widetilde{Q} \to (\widetilde{M})^\times$ is injective.*

Proof. Suppose $q = {q_{-1}}^{l_{-1}} {q_0}^{l_0} \dots {q_n}^{l_n} \in \ker \tau$. We then have

$$k^{l-1} q_0^{l_0} f_1^{l_1} \dots f_z^{l_n} = 1$$

in $(\widetilde{M})^\times$. Suppose that $l_{i_1}, \dots, l_{i_s}$ are precisely the positive powers occurring among the l_j $(0 \le j \le n)$ and $l_{j_1}, \dots, l_{j_t}$ the negative powers. Then, assuming without loss that l_{-1} is positive, we get

$$k^{l-1} f_{i_1}^{l_{i_1}} \dots f_{i_s}^{l_{i_s}} = f_{j_1}^{l_{j_1}} \dots f_{j_t}^{l_{j_t}}.$$

Since $\mathbb{Z}[q_0]$ is a unique factorization domain, and $q_0, f_1, \dots, f_n$ are all irreducible, then by writing k as a product of its prime factors we see that the powers on both sides must all be zero. Hence $q = 1$.

□

By Lemma 3.3.2 we may identify $\widetilde{Q}$ with its isomorphic copy in $(\widetilde{M})^\times$ via τ. We shall think of $\widetilde{M}$ as a subring of the field of rational functions $\mathbb{Q}(q_0)$, so $\widetilde{Q}$ embeds in $\mathbb{Q}(q_0)$. In particular Q_1 embeds in $\mathbb{Q}(q_0)$.

Now define v_j to be the f_j-adic valuation on $\mathbb{Q}(q_0)$ defined in Section 2.3, so that v_j satisfies $v_j(f_j) = 1$. In addition let w be the unique valuation on $\mathbb{Q}(q_0)$ that is zero on $\mathbb{Q} \setminus \{0\}$ and satisfies $w(g) = -\deg(g)$ for any polynomial $g \in \mathbb{Q}[q_0]$. Let χ_i be the character of Q that is defined to be zero on Q_0 and on each Q_j with $j > 1$ and on Q_1 to be the composite

$$Q_1 \overset{\tau|_{Q_1}}{\to} \mathbb{Q}(q_0) \overset{v_i}{\to} \mathbb{Z}$$

where $\tau(q_j) = f_j$. In addition define $\chi_w = w\tau|_{Q_1}$ and extend χ_w to Q by taking it to be trivial on $Q_0, Q_2, \dots, Q_l$. Thus for $0 \le i, j \le n$ we have $\chi_i(q_j) = \delta_{i,j}$ and $\chi_w(q_j) = -\deg(f_j)$. We identify Q_1 as a subgroup of $\mathbb{Q}(q_0)$ via the embedding τ and by abuse of notation write $v(q)$ for $v\tau(q)$ for $q \in Q_1$, where $v \in \{w, v_0, v_1, \dots, v_n\}$. Hence each v is a discrete character of Q.

It can then be readily seen that $w(q) = -(d_0 v_0 + d_1 v_1 + \dots + d_n v_n)(q)$ for any $q \in Q_1$, where $d_i = \deg(f_i)$. The set $V = \{w, v_0, v_1, \dots, v_n\}$ of $n + 2$ discrete characters of Q clearly satisfies the assumptions of Section 3.2 if we define $q_{v_i} = q_i$ for $0 \leq i \leq n$ and $q_w = {q_0}^{-1}$, for then $v(q_v) = 1$ for all $v \in V$. In addition we set

$$\beta = -\left(2 + \sum_{i=0}^{n} d_i\right).$$

For $v \in V$ let $(Q_1)_v$ be the submonoid of Q_1 of all elements q with $v(q) \geq 0$ and let $(M_1)_v$ be the $\mathbb{Z}(Q_1)_v$-submodule of M_1 generated by the image of 1_{Q_1} in M_1. Note that here we are thinking of $M_1 \cong \mathbb{Z}[q_0, 1/q_0, {f_1}^{-1}, \ldots, {f_n}^{-1}]$ as a subring of $\mathbb{Q}(q_0)$.

Lemma 3.3.3. *For $v \in V$ we have $(M_1)_v = \{m \in M_1 : v(m) \geq 0\}$.*

Proof. Observe that any element $m \in M_1$ can be written as

$$m = h(q_0, q_0^{-1}) f_1^{\alpha_1} \ldots f_n^{\alpha_n} = 1.h(q_0, q_0^{-1}) q_1^{\alpha_1} \ldots q_n^{\alpha_n} = 1.h(q_0, q_0^{-1}) q,$$

where $q = q_1^{\alpha_1} \ldots q_n^{\alpha_n}$, the α_i are some integers, $h \in \mathbb{Z}[q_0, q_0^{-1}]$ and none of the f_i are factors of h. Then, since v is a valuation on $\mathbb{Z}Q_1$, we have

$$v(m) = v(h(q_0, q_0^{-1})q) = \min\{v(q_0^i q)\}$$

where i runs through the powers of q_0 appearing in the monomials of h. But each $q_0^i q \in (Q_1)_v$ if and only if $m \in (M_1)_v$ and so $v(m) \geq 0$ if and only if $m \in (M_1)_v$. $\square$

Constructing the tree Γ_v described in Section 3.3 also requires us to fix a set of generators of A as a $\mathbb{Z}Q$-module. We choose a set of generators $a_1, \ldots, a_d$ that is a basis of A as a free M-module. With this, we can build the $(n+2)$ trees $\Gamma_w, \Gamma_{v_0}, \ldots, \Gamma_{v_n}$, their Cartesian product X and the Åberg complex Y defined in Section 3.2. By proving that Y possesses three important properties, which will be stated in Section 3.4, we shall prove that the group $G = A \rtimes Q$ in Theorem 3.3.1 is of type FP_m, where $m = \min\{rk(Q_i) : 1 \leq i \leq l\}$.

3.4 G acts cocompactly on Y

We now have a metabelian group $G = A \rtimes Q$, a set V of discrete real characters of Q, a tree Γ_v associated to each $v \in V$ and a CW-complex Y constructed by taking a particular subcomplex of the product X of these trees. We will shortly give the three important properties of Y that we need in order to prove Theorem 3.3.1. To do this we need some definitions. A topological space Y is said to be *n-connected* if it is path connected and the homotopy groups $\pi_1(Y), \ldots, \pi_n(Y)$ are trivial. An action of a group G on any topological space Y is said to be *cocompact* (resp. *cofinite*) if the quotient space Y/G is compact (resp. finite).

Proposition 3.4.1. *Let $G = A \rtimes Q = A \rtimes (Q_0 \times Q_1 \times \ldots \times Q_l)$ be the metabelian group with the properties stated in Theorem 3.3.1, let $v \in V = \{w, v_0, v_1, \ldots, v_n\}$ be a character of Q and let Y be the subcomplex of the product $X = \prod_v \Gamma_v$ given by $Y = h^{-1}(W)$, where $h : X \to \mathbb{R}^{n+2}$ is the map*

$$h\left(\prod_{v \in V} [(a_v, r_v)]\right) = \prod_{v \in V} r_v$$

and W is the subspace of $\mathbb{R}^{n+2}$ spanned by $\{\prod_{v \in V} v(q)\}_{q \in Q}$. Moreover define $m = \min\{rk(Q_i) : 1 \leq i \leq l\}$. Then Y has the following properties:
(i) G acts cocompactly on Y;
(ii) the stabilizers in G of cells in Y are of type FP_m; and
(iii) Y is $(m-1)$-connected.

After proving Proposition 3.4.1 we can then apply Brown's criterion below to see that G is of type FP_m.

Proposition 3.4.2 ([20, Proposition 1.1])**.** *Let G be a group acting on a CW-complex Y via permutation of the cells and such that the stabilizer of every cell fixes its vertices pointwise. Assume further that Y is $(m-1)$-acyclic, the stabilizers in G of cells of dimension $i \leq m$ are always of type FP_{m-i} and G acts cocompactly on Y. Then G is of homological type FP_m.*

By $(m-1)$-*acyclic* we mean that the homology groups $H_k(Y, \mathbb{Z})$ with integral coefficients are all trivial for $k < m$. We can then apply the Hurewicz theorem to show that it is enough to prove that Y is $(m-1)$-connected.

Theorem 3.4.3 (Hurewicz Theorem)**.** *Let X be a topological space and suppose that*

$\pi_k(X) = 0$ *for* $k < m$. *Then* $H_k(X, \mathbb{Z}) = 0$ *for* $k < m$ *and* $\pi_m(X) \cong H_m(X, \mathbb{Z})$ *for* $m \geq 1$.

First we prove part (i) of Proposition 3.4.1. The elements of $\widetilde{Q} = Q_0 \times Q_1$ are of the form $q_{-1}^{m_{-1}} {q_0}^{m_0} \ldots {q_n}^{m_n}$ and so, as a real vector space, W is spanned by elements in $\mathbb{Z}^{n+2}$ of the form

$$(-(d_0 m_0 + d_1 m_1 + \ldots d_n m_n), m_0, m_1, \ldots, m_n)$$

where $d_i = \deg(f_i)$ and the m_i are arbitrary integers for $0 \leq i \leq n$. Hence

$$W = \{(y_{-1}, y_0, y_1, \ldots, y_n) \in \mathbb{R}^{n+2} : y_{-1} + \sum_{i=0}^{n} d_i y_i = 0\}.$$

For a subset B of $\prod_{v \in V} \mathbb{R}$ we define

$$[[B]] = \left\{ \prod_{v \in V} \lceil b_v \rceil : \prod_{v \in V} b_v \in B \right\}$$

where $\lceil b_v \rceil$ denotes the least integer greater than or equal to b_v. In order to prove that the action of G on Y is cocompact we first show that the action of Q on $[[W]] \in \mathbb{Z}^{n+2}$ is cofinite.

Lemma 3.4.4 ([23, Lemma 7]). *If* $\prod_{v \in V} s_v \in [[W]]$, *then*

$$0 \leq s_w + \sum_{i=0}^{n} d_i s_{v_i} < 1 + \sum_{i=0}^{n} d_i.$$

Hence Q *acts cofinitely on* $[[W]]$.

Proof. Let $(s_w, s_{v_0}, \ldots, s_{v_n}) \in [[W]]$. Then there exists an element

$$(r_w, r_{v_0}, \ldots, r_{v_n}) \in W$$

such that $r_v \leq s_v < r_v + 1$ for all $v \in V$. Hence

$$0 = r_w + \sum_{i=0}^{n} d_i r_{v_i} \leq s_w + \sum_{i=0}^{n} d_i s_{v_i} < (r_w + 1) + \sum_{i=0}^{n} d_i (r_{v_i} + 1) = 1 + \sum_{i=0}^{n} d_i.$$

Now observe that $q \in Q$ acts on $\mathbb{R}^{n+2}$ by translation with $\prod_{v \in V} v(q)$. Since $v(Q_i) = 0$ for $i > 1$ we can suppose $q = q_{-1}^{k_{-1}} {q_0}^{k_0} \ldots {q_n}^{k_n}$, then

$$\begin{aligned}(s_w, s_{v_0}, \ldots, s_{v_z}) \circ q &= (s_w + w(q), s_{v_0} + v_0(q), \ldots, s_{v_n} + v_n(q)) \\ &= (s_w - (d_0 k_0 + \ldots + d_n k_n), s_{v_0} + k_0, \ldots, s_{v_n} + k_n).\end{aligned}$$

We see from this that Q acts transitively on the last $n+1$ coordinates of points in $[[W]]$: we can get any integers in these coordinates by appropriate choice of $k_0, k_1, \ldots, k_n$. Hence each Q-orbit in $[[W]]$ contains some point with the last $n+1$ coordinates equal to zero, so we can take $s_{v_0} = s_{v_1} = \ldots = s_{v_n} = 0$. But then the above inequalities bound s_w as an integer in the interval $[0, 1 + \sum_{i=0}^{n} d_i)$ and so there can be only finitely many Q-orbits.

□

As in [23] the following theorem is a key step in the proof that G acts cocompactly on Y. Here we make use of the fact that the polynomials f_i are pairwise coprime over $\mathbb{Z}[q_0, k^{-1}]$ as opposed to $\mathbb{Z}[q_0]$. To do so we need the Chinese Remainder Theorem for commutative rings.

Proposition 3.4.5. (Chinese Remainder Theorem) *Let R be a commutative ring and $I_1, \ldots, I_k$ be ideals of R which are pairwise coprime i.e. $I_i + I_j = R$ for any $i \neq j$. Then the product I of $I_1, \ldots, I_k$ is equal to their intersection, and the quotient ring R/I is isomorphic to the product ring $R/I_1 \times \ldots \times R/I_k$ via the isomorphism*

$$f : R/I \to R/I_1 \times \ldots \times R/I_k$$

given by

$$f(x + I) = (x + I_1, \ldots, x + I_k)$$

for all $x \in R$.

Theorem 3.4.6 (c.f. [23, Theorem 6])**.** *For every $\prod_{v \in V} s_v$ in $[[W]]$ and every set $\{a_v \in A : v \in V\}$, there exists an element $a \in A$ such that $\prod_{v \in V}[(a_v, s_v)] = \prod_{v \in V}[(a, s_v)]$. Consequently for every $\prod_{v \in V} r_v \in W$ such that $[[\prod_{v \in V} r_v]] = \prod_{v \in V} s_v$ one has $\prod_{v \in V}[(a_v, r_v)] = \prod_{v \in V}[(a, r_v)]$.*

Proof. Since A is free as an M-module, it is sufficient to prove that the result holds when A is cyclic as an M-module and so we suppose $A = M$. Then A is a cyclic $\mathbb{Z}Q$-module (since $M = \mathbb{Z}Q/I$ for some ideal I) and so is generated by the image of 1_Q. Hence by the definition given in Section 3.2 A_v is the $\mathbb{Z}Q_v$-submodule of A

generated by the image of 1_Q in $M = A$, and so by Lemma 3.3.3

$$A_v = M_0 \otimes (M_1)_v \otimes M_2 \otimes \ldots \otimes M_l$$

since v is trivial on $M_0, M_2, \ldots, M_l$.

We now observe that if the theorem is true for any G-translate of $\prod_{v \in V}[(a_v, s_v)]$, then it is true for $\prod_{v \in V}[(a_v, s_v)]$ itself. To see this, note that

$$\left(\prod_{v \in V}[(a_v, s_v)]\right) * g = \left(\prod_{v \in V} G(v) q_v^{s_v} a_v\right) * g = \prod_{v \in V}[(a'_v, s_v + v(g))],$$

where $G(v)q_v^{s_v}a_v g = G(v)q_v^{s_v+v(g)}a'_v$ for some $a'_v \in A$. Suppose that there exists $a' \in A$ with

$$\prod_{v \in V}[(a', s_v + v(g))] = \prod_{v \in V}[(a'_v, s_v + v(g))].$$

Then $(\prod_{v \in V}[(a'_v, s_v + v(g))]) * g^{-1} = (\prod_{v \in V}[(a', s_v + v(g))]) * g^{-1}$, and so

$$\prod_{v \in V}[(a_v, s_v)] = \prod_{v \in V}[(a'', s_v)]$$

where $G(v){q_v}^{s_v}a'' = G(v){q_v}^{s_v+v(g)}a'g^{-1}$ for some $a'' \in A$. Now, since Q acts transitively on the last $n+1$ coordinates of $[[W]]$, we can find a Q-translate of $\prod_{v \in V} s_v$ in which each of the last $n+1$ coordinates is zero. By Lemma 3.4.4 we have

$$0 \leq s_w < -\beta.$$

Without affecting this latter property we can also find an A-translate of $\prod_{v \in V}[(a_v, s_v)]$ in which $a_w = 0$, namely $(\prod_{v \in V}[(a_v, s_v)]) * (-a_w)$. Thus we may assume that $s_v + \beta < 0$ for each $v \in V$ and that $a_w = 0$.

Let $a \in A$. Then

$$\prod_{v \in V}[(a_v, s_v)] = \prod_{v \in V}[(a, s_v)]$$

if and only if $a - a_v \in A_v \circ {q_v}^{s_v+\beta}$ for each $v \in V$: see the proof of Lemma 3.2.3. As the q_v act trivially on M_0 and on each M_i with $i > 1$ it will suffice to consider the case in which each $a_v \in M_1$, and we seek $a \in M_1$. Here we are identifying M_1 with the subring

$$\mathbb{Z} \otimes M_1 \otimes \mathbb{Z} \otimes \mathbb{Z} \otimes \ldots \otimes \mathbb{Z}$$

of $M = M_0 \otimes M_1 \otimes \ldots \otimes M_l$. Now $s_v + \beta < 0$ for each v and so, since $A_v \circ q_v \subseteq A_v$, we have $A_v \subseteq A_v \circ {q_v}^{s_v+\beta}$. Therefore by Lemma 3.3.3 it will suffice to find an $a \in M_1$ such that $v(a - a_v) \geq 0$ for each v, for then $a - a_v \in (M_1)_v \subseteq A_v$.

Let $F(q_0) = \prod_{i=0}^{n} f_i(q_0)$. Then, for some t, we have $F^t a_v \in \mathbb{Z}[q_0]$ for every $v \in \{v_0, v_1, \ldots, v_n\}$. Since the f_i are all pairwise coprime as elements of $\mathbb{Z}[q_0, 1/k]$ we can apply Proposition 3.4.5 to get a unique solution mod $F^t\mathbb{Z}[q_0, 1/k]$ to the congruences

$$a' \equiv a_{v_i} F^t \mod f_i^t \mathbb{Z}[q_0, 1/k]$$

for $0 \leq i \leq n$. The unique solution $a' \in A$ must have degree less than that of F^t. Set $a = a'F^{-t}$. Then

$$v_i(a - a_{v_i}) = v_i(a'F^{-t} - a_{v_i}) = v_i(F^{-t}(a' - a_{v_i}F^t)) = -t + v_i(a' - a_{v_i}F^t)$$

and since ${f_i}^t$ divides $a' - a_{v_i}F^t$ in $\mathbb{Z}[q_0, 1/k]$ we have $v_i(a - a_{v_i}) \geq 0$. Recall that $a_w = 0$. Thus

$$w(a - a_w) = w(a) = w(a'F^{-t}) = \deg(F^t) - \deg(a') > 0.$$

This completes the proof of Theorem 3.4.6. □

We can now prove that the action of G on Y is cocompact.

Theorem 3.4.7 ([23, Theorem 7]). *The group G acts cocompactly on Y.*

Proof. Let $\prod_{v \in V}[(a_v, r_v)]$ be a point of Y. Then by Theorem 3.4.6 there exists $a \in A$ such that $\prod_{v \in V}[(a_v, s_v)] = \prod_{v \in V}[(a, s_v)]$, where $\lceil r_v \rceil = s_v$. Then

$$\begin{aligned}\prod_{v \in V}[(a_v, s_v)] &= \prod_{v \in V}[(a, s_v)] \\ &= \prod_{v \in V} G(v){q_v}^{s_v} a \\ &= \left(\prod_{v \in V} G(v){q_v}^{s_v}\right) * a \\ &= \prod_{v \in V}[(0_A, s_v)] * a,\end{aligned}$$

where $*$ denotes the diagonal action of G on X, and so

$$\prod_{v\in V}[(a_v, r_v)] = \prod_{v\in V}[(0_A, r_v)] * a.$$

Hence $Y/A \cong \{\prod_{v\in V}[(0_A, r_v)] : \prod_{v\in V} r_v \in W\}$, where Y/A is the set of orbits of Y under the action of A. But under the map $h = \prod_{v\in V} h_v$ we have an isomorphism

$$\left\{\prod_{v\in V}[(0_A, r_v)] : \prod_{v\in V} r_v \in W\right\} \cong W$$

and $q \in Q$ acts on W via translation with $\prod_{v\in V} v(q)$. The quotient W/Q is compact as a topological space, since, under the map $f : Q \otimes \mathbb{R} \to \mathbb{R}^{n+2}$ defined in Section 3.2, W/Q is the continuous image of the quotient map

$$\bar{f} : Q \otimes \mathbb{R}/Q \to \mathbb{R}^{n+2}/\mathbb{Z}^{n+2}$$

induced by f. But

$$Q \otimes \mathbb{R}/Q \cong \mathbb{R}^{n+2}/\mathbb{Z}^{n+2}$$

and the latter is homeomorphic to the $(n+2)$-sphere S^{n+2}. Hence W/Q is compact.

Since $G = AQ = \{aq : a \in A, q \in Q\}$ we have

$$Y/G \cong (Y/A)/Q \cong W/Q$$

and so Y/G is compact.

□

3.5 Stabilizers in G of cells in Y and the proof that G is of type FP_m

The next step in the proof of Proposition 3.4.1 is to show that if P is the stabilizer in G of a cell in Y then P is of type FP_m. The following lemma implies that it is enough to prove that the stabilizer of a vertex $\prod_{v\in V} G(v)g_v$ of X lying in $h^{-1}[[W]]$ is of type FP_m.

Lemma 3.5.1 ([26, Lemma 2.9])**.** *If Γ is a cell of Y then the stabilizer of Γ in G coincides with the stabilizer in G of a vertex of X lying in $h^{-1}([[W]])$.*

Proof. Let Γ be the intersection of an i-subcell of the $(n+2)$-cell

$$J = \left\{ \prod_{v \in V} [(a_v, r_v)] : s_v \leq r_v \leq s_v + 1 \right\}$$

in X with Y, where the s_v are integers, let $\gamma_1, \ldots, \gamma_t$ be the vertices (0-subcells) of Γ, and let $g \in G$ stabilize Γ. Then g permutes the vertices, say $\gamma_i * g = \gamma_{\rho(i)}$ for some permutation ρ in the symmetric group on t letters. If $\gamma_i = \prod_{v \in V} [(a_v, s_{v,i})]$ we have $h_v(\gamma_i * g) = s_{v,i} + v(g)$ and so $s_{v,\rho(i)} = s_{v,i} + v(g)$. Summing over all i gives

$$\sum_{i=1}^{t} (s_{v,i} + v(g)) = \sum_{i=1}^{t} s_{v,\rho(i)} = \sum_{i=1}^{t} s_{v,i}.$$

Hence $v(g) = 0$ for all $v \in V$ and so $s_{v,i} = s_{v,\rho(i)}$ giving $\gamma_i = \gamma_{\rho(i)}$. Thus g stabilizes the vertices $\gamma_1, \ldots, \gamma_t$ and hence stabilizes Γ pointwise.

Note that g stabilizes a point $\prod_{v \in V} [(a_v, r_v)]$ of X if and only if g stabilizes the vertex $\prod_{v \in V} [(a_v, s_v)]$, where $\prod_v s_v = [[\prod_v r_v]]$, i.e. $s_v \in \mathbb{Z}$ and $s_v - r_v \in [0, 1)$ for all $v \in V$. Then

$$\{g \in G : g \text{ stabilizes } \Gamma \text{ pointwise}\} = \{g \in G : g \text{ stabilizes all points from the set } \Lambda\},$$

where

$$\Lambda = \left\{ \prod_{v \in V} [(a_v, s_v)] : \text{ there is a point } \prod_{v \in V} [(a_v, r_v)] \text{ in } \Gamma \text{ with } \prod_v s_v = \left[\left[\prod_v r_v \right]\right] \right\}.$$

So each element of Λ is a vertex of the $(n+2)$-cell J. List the elements of Λ as $T_1, \ldots, T_r$, where $T_j = \prod_{v \in V} [(a_v, s_{v,j})]$, and suppose that $h(T_j) = [[h(S_j)]]$ for some points $S_1, \ldots, S_r$ of Γ. We claim that the point $T = \prod_{v \in V} [(a_v, s_v + 1)]$ belongs to Λ. Indeed $(1/r)(S_1 + \ldots + S_r)$ is a point of Γ and $[[h(1/r)(S_1 + \ldots + S_r)]] = h(T)$. Thus $g \in G$ stabilizes Γ pointwise if and only if it stabilizes the vertex T, and T lies in $h^{-1}[[W]]$.

□

In fact we will be able to prove something stronger in the case where $l = 1$ in Theorem 3.3.1: that such a stabilizer is in fact of type FP_∞.

Let $\prod_{v\in V} G(v)g_v$ be a vertex of the CW-complex X and suppose that

$$\prod_{v\in V} G(v)g_v \in h^{-1}([[W]]).$$

We want to show that the stabilizer group $P \leq G$ of such a vertex satisfies the conditions laid out in Theorem 3.3.1 for some l.

Proposition 3.5.2 (c.f. [23, Section 4.5]). *Let P be the stabilizer subgroup in G of a vertex $\prod_{v\in V} G(v)g_v$ of X lying in $h^{-1}([[W]])$. Then P is the split extension of a finitely generated free $M' = \mathbb{Z}[1/k] \otimes M_2 \otimes \ldots \otimes M_l$-module by the free abelian group $Q_0 \times Q_2 \times \ldots \times Q_l$.*

Proof. First observe that $P = \bigcap_{v\in V} {g_v}^{-1}G(v)g_v$, and that the stabilizer of a G-translate $\prod_{v\in V} G(v)g_v g_1$ of $\prod_{v\in V} G(v)g_v$ is

$$\bigcap_{v\in V} (g_v g_1)^{-1}G(v)g_v g_1 = {g_1}^{-1}\left(\bigcap_{v\in V} {g_v}^{-1}G(v)g_v\right) g_1$$

where $g_1 \in G$. Since the two stabilizers are conjugate, they are isomorphic, and so it will suffice to prove the result for any vertex within each G-orbit.

By Theorem 3.4.6 and Lemma 3.2.2 we get

$$\prod_{v\in V} G(v)g_v = \prod_{v\in V} G(v){q_v}^{s_v}a = \left(\prod_{v\in V} G(v){q_v}^{s_v}\right) * a,$$

where the s_v are integers and $a \in A$, and so $\prod_{v\in V} G(v){q_v}^{s_v}$ and $\prod_{v\in V} G(v)g_v$ lie in the same G-orbit. Thus we can assume that $a_v = 0$ for all $v \in V$ and, given that Q acts transitively on the last $n+1$ coordinates of $[[W]]$, we can also assume that $s_v = 0$ for $v \neq w$. That is, we can suppose that $g_v = 1$ if $v \neq w$ and that $g_w = {q_w}^{s_w}$, with $0 \leq s_w < -\beta$ by Lemma 3.4.4.

Then, an element aq of G with $a \in A$ and $q \in Q$ lies in the stabilizer P of a vertex $(G(w){q_w}^{s_w}, G(v_0), G(v_1), \ldots, G(v_n))$ if and only if

$$a \in A \cap ({q_w}^{-s_w}G(w){q_w}^{s_w}) \text{ and}$$

$$a \in G(v_i) \cap A = A_{v_i} \circ {q_{v_i}}^{\beta} \text{ for } 0 \leq i \leq n, \quad q \in G(v) \text{ for all } v \in V.$$

Now $A \cap (q_w{}^{-s_w} G(w) q_w{}^{s_w}) = (A \cap G(w)) \circ q_w{}^{s_w}$. To see this, let $a = q_w{}^{-s_w} g q_w{}^{s_w}$ for some $g \in G(w)$. Since $q_w{}^{-s_w} G(w) q_w{}^{s_w} \subseteq G(w)$ we have $a \in G(w) \cap A = A_w \circ q_w{}^{\beta}$ and so $a = q_w{}^{-\beta} a_w q_w{}^{\beta}$ for some $a_w \in A_w$. Hence

$$q_w{}^{-s_w} g q_w{}^{s_w} = q_w{}^{-\beta} a_w q_w{}^{\beta}$$

and so

$$g = q_w{}^{s_w - \beta} a_w q_w{}^{-s_w + \beta}.$$

Thus $g \in A$. We get the reverse inclusion by noting that if $a \in A \cap G(w)$ then $a \circ q_w{}^{s_w} = q_w{}^{-s_w} a q_w{}^{s_w} \in A \cap (q_w{}^{-s_w} G(w) q_w{}^{s_w})$.

Given this, and the fact that $(A \cap G(w)) \circ q_w{}^{s_w} = A_w \circ q_w{}^{s_w + \beta}$, we can restate our previous conditions for an element of G to lie in P: $aq \in P$ if and only if

$$a \in (A_w \circ q_w{}^{s_w+\beta}) \cap (A_{v_0} \circ q_{v_0}{}^{\beta}) \cap \ldots \cap (A_{v_n} \circ q_{v_n}{}^{\beta}) \tag{3.5}$$

and

$$q \in G(v) \text{ for all } v \in V. \tag{3.6}$$

The condition (3.6) is equivalent to $q \in \ker v$ for all $v \in V$, that is $q \in Q_0 \times Q_2 \times \ldots \times Q_l$. This is essentially the same argument as in Section 4.5 of [23].

Now, since A is free as an M-module it suffices to consider the case when A is cyclic as an M-module, and so we can take $A = M$. Then $b \in A_v$ for all $v \neq w$ exactly if

$$b \in \mathbb{Z}[q_0, 1/k] \otimes M_2 \otimes \ldots \otimes M_l,$$

since $A_v = M_0 \otimes (M_1)_v \otimes M_2 \ldots \otimes M_l$. Recall that $q_{v_i} = q_i$, and let $\tilde{q} = q_0 q_1 \ldots q_n \in Q_1$. Then since $a \in \bigcap_{0 \leq i \leq n} A_{v_i} \circ q_i^{\beta}$ we have $a \circ (\tilde{q})^{-\beta} \in A_{v_i}$ for $0 \leq i \leq n$ and so $a \circ (\tilde{q})^{-\beta} \in \mathbb{Z}[q_0, 1/k] \otimes M_2 \otimes \ldots \otimes M_l$. Write the first component of $a \circ (\tilde{q})^{-\beta}$ as $g(q_0)$: a polynomial in q_0 with coefficients in $\mathbb{Z}[1/k]$. Now $q_w = q_0{}^{-1}$ and so

$$a \in A_w \circ q_0{}^{-(s_w+\beta)} \subseteq \{b \in A : w(b) \geq s_w + \beta\}.$$

It follows that

$$s_w + \beta \left(1 + \sum_{i=0}^{n} d_i\right) \leq w(a \circ (\tilde{q})^{-\beta}) < 0,$$

and so the degree of $g(q_0)$ is bounded above by

$$-\left(s_w + \beta\left(1 + \sum_{i=0}^{n} d_i\right)\right) := d.$$

Hence, since $q_1, \ldots, q_n$ act by multiplication with $f_1, \ldots, f_n$ respectively, the component $g(q_0) \circ \tilde{q}^\beta$ of a that is a polynomial in $\mathbb{Z}[q_0, 1/k]$ is a $\mathbb{Z}$-linear combination of the elements

$$k^{l_j} {q_0}^{j+\beta} {f_1}^\beta \ldots {f_n}^\beta$$

where $0 \leq j \leq d$ and the l_j are the powers of k appearing in the monomials of g. Thus $P \cap (M_0 \otimes M_1)$ is a free $\mathbb{Z}[1/k]$-module of finite rank $d+1$ and so $P \cap A = P \cap (\mathbb{Z}[q_0, 1/k] \otimes M_2 \otimes \ldots \otimes M_l)$ is a finitely generated free $M' = \mathbb{Z}[1/k] \otimes M_2 \otimes \ldots \otimes M_l$-module.

Condition (3.6) guarantees that P is the split extension of $P \cap A$ and $P \cap Q$. Hence P is the split extension of a finitely generated free $\mathbb{Z}[1/k] \otimes M_2 \otimes \ldots \otimes M_l$-module by $Q_0 \times Q_2 \times \ldots \times Q_l$. □

We now perform induction on l to show that P is always of type FP_m, where $m = \min\{rk(Q_i) : 1 \leq i \leq l\}$. First of all recall the concept of an *HNN-extension*. If G is a group with presentation $\langle S \mid R \rangle$ and we have an isomorphism $\alpha : H \to K$ between two subgroups of G then the *HNN-extension of G relative to α* is the group $G*_\alpha$ defined by

$$G*_\alpha = \langle S, t \mid R, t^{-1}ht = \alpha(h) \text{ for all } h \in H \rangle$$

where t is a new symbol not in S. We call the element t the *stable letter* of the extension. It turns out that when $l = 1$ the stabilizer P is such an HNN-extension, and we will use the following result of Bieri to show that P has type FP_∞.

Lemma 3.5.3 ([12, Proposition 2.13 (b)]). *Let P be the HNN-extension of a group G relative to an isomorphism $\alpha : H \to K$, where H, K are subgroups of G, and let t be the stable letter. If G is of type FP_n and H is of type FP_{n-1} then P is of type FP_n.*

Proposition 3.5.4. *Every stabilizer P in G of a cell in the Åberg complex Y is of type FP_m.*

Proof. First suppose $l = 1$. Then P is the split extension of a free $\mathbb{Z}[1/k]$-module of finite rank $d+1$ by the infinite cyclic group $Q_0 = \langle q_{-1} \rangle$. Since q_{-1} acts by multiplication with k the group $\mathbb{Z}[1/k] \rtimes Q_0$ has the presentation

$$\langle x, t : t^{-1}xt = x^k \rangle$$

and so we deduce that $P = (P \cap A) \rtimes Q_0$ has the presentation

$$\langle x_0, x_1, \ldots, x_d, t : t^{-1}x_i t = {x_i}^k \text{ for } 0 \leq i \leq d \rangle.$$

Hence P is an HNN-extension of $H = \langle x_0, x_1, \ldots, x_d \rangle$ relative to the isomorphism $\alpha : H \to K$ given by $\alpha(x_i) = {x_i}^k$, where $K = \langle {x_0}^k, \ldots, {x_d}^k \rangle$. It follows from Hilbert's Basis Theorem that since H and K are free abelian of finite rank their group rings are noetherian. It is always possible to choose a projective resolution of finite type for any module over a noetherian ring, and so free abelian groups of finite rank are of type FP_∞. Applying Lemma 3.5.3 we see that P must also be of type FP_∞. Hence P is certainly of type FP_m.

We now assume that Theorem 3.3.1 holds when M' is a tensor product of $\mathbb{Z}[1/k]$ and $l-1$ other components. By Proposition 3.5.2 P is the split extension of a free M'-module by the free abelian group $Q_0 \times Q_2 \times \ldots \times Q_l$ and so satisfies the conditions stated in Theorem 3.3.1 for $l-1$. Hence P is of type $FP_{m'}$, where $m' = \min\{rk(Q_i) : 2 \leq i \leq l\}$, and so P is of type FP_m, since $m \leq m'$.

□

We now just have to prove that the CW-complex Y is $(m-1)$-connected and we will have proved Proposition 3.4.1. Hence by Brown's criterion (Proposition 3.4.2) this will place the final stone in the proof that G is of type FP_m. We will make use of the following result of Åberg. First note that a *halfspace* of the affine space $\mathbb{R}^z$, where z is some positive integer, is either of the two parts into which $\mathbb{R}^z$ is divided by any hyperplane.

Lemma 3.5.5 ([1, Proposition III.3.3])**.** *Let V be a set of characters of Q satisfying the conditions of Section 3.2. If every m elements of V lie in an open halfspace of* $\operatorname{Hom}(Q, \mathbb{R}) \cong \mathbb{R}^z$ *then Y is $(m-1)$-connected.*

We prove below that the assumptions of Lemma 3.5.5 are satisfied for any $n+1$ elements of $V = \{w, v_0, v_1, \ldots, v_n\}$ and so Y is n-connected. By definition $n+1 \geq m = \min\{rk(Q_i) : 1 \leq i \leq l\}$, hence $n \geq m-1$ and so Y is $(m-1)$-connected.

Lemma 3.5.6 ([23, Lemma 8])**.** *Every set V' of $n+1$ elements of the set V lies in an open halfspace of* $\mathrm{Hom}(Q, \mathbb{R})$.

Proof. If $q \in Q$ with $q \neq 1$ then

$$U_q = \{\chi \in \mathrm{Hom}(Q, \mathbb{R}) : \chi(q) > 0\}$$

is an open halfspace of $\mathrm{Hom}(Q, \mathbb{R})$. First suppose that $w \notin V'$; then $V' = \{v_0, v_1, \ldots, v_n\}$. Set $q = q_0 q_1 \ldots q_n$. Then $V' \subseteq U_q$.

Now suppose that V' omits v_i for some $i \geq 0$. Set $\alpha > (\sum_{0 \leq j \neq i \leq n} d_j)/d_i$ and $q = q_0 q_1 \ldots q_{i-1} {q_i}^{-\alpha} q_{i+1} \ldots q_n$. Then $v_j(q) = 1$ for $0 \leq j \neq i \leq n$ and $w(q) = -\sum_{0 \leq j \neq i \leq n} d_j + \alpha d_i > 0$. Hence $V' \subseteq U_q$ and the proof is complete. □

This completes the proof of Theorem 3.3.1. This result, or more succinctly its corollary Theorem 3.1.3, implies that the group $G_n = A_n \rtimes Q_n$ is of type FP_{n+1}. To see that this statement is sharp, we appeal to the fact proved by Kochloukova that the 'only if' direction of the Bieri-Groves conjecture holds in the split extension case [26, Theorem B] . This implies that A_n is $(n+1)$-tame. However it is not $(n+2)$-tame, since

$$w + d_0 v_0 + d_1 v_1 + \ldots + d_n v_n = 0$$

and $[w], [v_0] = [d_0 v_0], \ldots, [v_n] = [d_n v_n] \in \Sigma^c_{A_n}$: by Proposition 2.3.3 $A_v = A$ if and only if $[v] \in \Sigma_A$ and we can see from Lemma 3.3.3 that $A_n \neq (A_n)_v$ for all $v \in V$.

In the next chapter we shall study the invariant Σ^c_A directly, and look to prove tameness properties without recourse to the FP_m-conjecture.

Chapter 4

The Bieri-Strebel invariant Σ_A^c

In Chapter 3 we proved that the finitely presented metabelian group $G_n = A_n \rtimes Q_n$ has cohomological type FP_{n+1}, where $A_n = \mathbb{Z}[x, x^{-1}, (1+x)^{-1}, \ldots, (n+x)^{-1}, 1/n!]$ and Q_n is the free abelian subgroup $\langle x, 1+x, \ldots, n+x, n! \rangle$ of the group of units of A_n acting by multiplication. By a result of Kochloukova [26, Theorem B], this implies that A_n is $(n+1)$-tame. In this chapter we give a direct proof of that fact by examining the structure of the spherical complement $\Sigma_{A_n}^c$ of the Bieri-Strebel invariant Σ_{A_n}, and so provide further independent evidence to support the FP_m-conjecture.

4.1 Recap of Σ_A^c and the FP_m-conjecture

First we shall recall the links between finitely generated metabelian groups and spherical geometry that were discussed in Section 2.3.

The *Bieri-Strebel invariant* is defined to be the subset Σ_A of the $(n-1)$-sphere $S(Q)$ given by

$$\Sigma_A = \{[v] \in S(Q) : A \text{ is finitely generated as a } \mathbb{Z}Q_v\text{-module } \}$$

whenever Q is a finitely generated abelian group of $\mathbb{Z}$-rank n, A a finitely generated $\mathbb{Z}Q$-module and $Q_v = \{q \in Q : v(q) \geq 0\}$, where $v : Q \to \mathbb{R}$ is a non-zero group homomorphism. This invariant can be used to determine which metabelian groups are finitely presented. We say that the module A is *m-tame* if $v_1 + v_2 + \ldots + v_m \neq 0$ for any m points $[v_i] \in \Sigma_A^c = S(Q) \setminus \Sigma_A$, $1 \leq i \leq m$, or equivalently any m-point subset of Σ_A^c lies in an open hemisphere of $S(Q)$.

A finitely generated $\mathbb{Z}Q$-module is always 1-tame, since a 1-point subset of $S(Q)$ plainly must be contained in an open hemisphere. Notice that A is 2-tame if and only if $\Sigma_A \cup -\Sigma_A = S(Q)$, where $-\Sigma_A = \{[-v] : [v] \in \Sigma_A\}$; or equivalently, if and only if Σ_A^c does not contain a pair of antipodal points. The following result characterizes finitely presented metabelian groups in terms of this property:

Theorem 4.1.1 ([17, Theorem 5.1])**.** *Suppose we have a group extension $A \hookrightarrow G \twoheadrightarrow Q$, with A and Q abelian and G finitely generated. Then G is finitely presented if and only if A is 2-tame as a $\mathbb{Z}Q$-module.*

Since it was proved by Bieri and Strebel that being finitely presented and being of type FP_2 are equivalent for metabelian groups [17, Theorem 5.4], we get the following corollary:

Corollary 4.1.2. *G is of cohomological type FP_2 if and only if A is 2-tame as a $\mathbb{Z}Q$-module.*

It is conjectured that the integer 2 in Corollary 4.1.2 can be replaced by any positive integer m; that is, Σ_A^c contains complete information about the higher finiteness properties FP_m of metabelian groups G, and not just finite presentability.

Conjecture 4.1.3 ([13])**.** If G is a finitely generated group which is an extension of abelian groups A and Q then G is of type FP_m if and only if A is m-tame as a $\mathbb{Z}Q$-module.

Conjecture 4.1.3 is known as the 'FP_m-conjecture'. The FP_1-conjecture follows immediately from Propositions 2.2.2 and 2.4.5. According to the work of Kochloukova [26] and Noskov [31], the 'only if' half of Conjecture 4.1.3 holds whenever G is the split extension of A by Q. Hence we were able to show in Chapter 3 that for $n \geq 1$ the module $A_n = \mathbb{Z}[x, x^{-1}, (1+x)^{-1}, \ldots, (n+x)^{-1}, 1/n!]$ is $(n+1)$-tame by proving that the split extension $G_n = A_n \rtimes Q_n$ is of type FP_{n+1}. It would be useful to be able to prove this directly, by studying the invariant $\Sigma^c_{A_n}$, in order to provide further independent evidence in support of the FP_m-conjecture. In this chapter we show how this is done.

Let Q be a finitely generated abelian group and A a finitely generated RQ-module where R is an arbitrary integral domain. Recall that if $v : Q \to \mathbb{R}$ is a group homomorphism then we define $v(\lambda) = \min\{v(q) : q \in \operatorname{supp}(\lambda)\}$ for any $\lambda \in RQ$. In this way v extends to a valuation on the group ring RQ. The definition of Σ_A can then be extended to allow for A being such a module. The *centralizer* $C(A)$ of A is the set

$$C(A) = \{\lambda \in RQ : a\lambda = a \text{ for all } a \in A\}.$$

Hence $C(A) = 1 + \operatorname{Ann}_{RQ}(A)$, where

$$\operatorname{Ann}_{RQ}(A) = \{\mu \in RQ : a\mu = 0 \text{ for all } a \in A\}.$$

Our main tool for studying $\Sigma^c_{A_n}$ will be the following criterion of Bieri and Strebel.

Proposition 4.1.4 ([17, Proposition 2.1])**.** *Let A, Q and R be as above and let $v : Q \to \mathbb{R}$ be a non-trivial character. Then A is finitely generated over Q_v if and only if there is an element $\lambda \in C(A)$ with $v(\lambda) > 0$. Moreover if A is finitely generated over Q_v then any set generating A as an RQ-module generates A as an RQ_v-module.*

Proof. Let $\lambda \in C(A)$ with $v(\lambda) > 0$. If $0 \neq \mu$ is an arbitrary element of RQ we can choose an integer m with

$$mv(\lambda) \geq -v(\mu).$$

Then $v(\mu\lambda^m) \geq v(\mu) + mv(\lambda) \geq 0$, that is $\mu\lambda^m \in RQ_v$. It follows that $a\mu = a\mu\lambda^m \in aRQ_v$ for all $a \in A$, that is, $aRQ = aRQ_v$. Then since A is finitely generated as an RQ-module it is finitely generated as an RQ_v-module and any set generating A as an RQ-module also generates A as an RQ_v-module.

Now suppose that the elements $a_1, a_2, \ldots, a_k$ generate A as an RQ_v-module and choose $q \in Q$ such that $v(q) > 0$. We obtain a system of k linear equations

$$a_i q^{-1} = \sum_j a_j \lambda_{ij}$$

where $\lambda_{ij} \in RQ_v$. This can be expressed equivalently as

$$\sum_j a_j(\delta_{ij} - q\lambda_{ij}) = 0,$$

where $1 \leq i, j \leq k$, which gives us a matrix equation. Multiplying through on both sides by the adjoint of $(\delta_{ij} - q\lambda_{ij})$ we get

$$(a_1\ a_2\ \ldots\ a_k)\det(\delta_{ij} - q\lambda_{ij}) = 0$$

and so $\det(\delta_{ij} - q\lambda_{ij})$ annihilates A. Since $\det(\delta_{ij} - q\lambda_{ij}) = 1 - q\mu$ for some $\mu \in RQ_v$, and $v(q\mu) \geq v(q) + v(\mu) > 0$, we can choose $\lambda = q\mu$ to be the required element in $C(A)$. □

As a consequence of Proposition 4.1.4 we have the following alternative description of Σ_A:

$$\Sigma_A = \bigcup_{\lambda \in C(A)} \{[v] \in S(Q) : v(\lambda) > 0\}. \tag{4.1}$$

It follows that

$$\Sigma_A = \Sigma_{RQ/I} \tag{4.2}$$

where $I = \mathrm{Ann}_{RQ}(A)$.

4.2 Σ_A and finite presentability of metabelian groups

In the introduction to this thesis we mentioned two similar-looking metabelian groups, only one of which was finitely presented. This example was given by Lennox and Robinson [29, p. 264] and it provides a concrete example of the connection between the Bieri-Strebel invariant and finite presentability.

Let $G = A \rtimes Q$, where $A = \mathbb{Z}[1/6]$ and $Q = \langle t \rangle$ is an infinite cyclic group. It is clear that

$$S(Q) = \{[-1], [1]\}.$$

Thus if $0 \neq v \in \mathrm{Hom}(Q, \mathbb{R})$ we can assume that $v(t) = \pm 1$. First suppose that t acts on A by multiplication with 6. Let $r \in C(A)$ and assume $v(r) > 0$. Now

$\mathrm{Ann}_{\mathbb{Z}Q}(A)$ is a principal ideal of $\mathbb{Z}Q$, generated by the element $t-6$, and so since $C(A) = 1 + \mathrm{Ann}_{\mathbb{Z}Q}(A)$ we can write $r = 1 + (t-6)f(t)$ where $f(t) = \sum_n a_n t^n \in \mathbb{Z}[t]$.

First suppose that $v(t) = 1$. Then r can involve only positive powers of t, since $v(r) > 0$. If we equate the coefficients of all non-positive powers of t to 0, we obtain the equations

$$1 + a_{-1} - 6a_0 = 0 \quad \text{and} \quad a_{-j-1} - 6a_{-j} = 0, \quad j > 0.$$

Since only finitely many of the a_j can be non-zero, the second equation implies that in fact $a_{-j} = 0$ for all $j > 0$. Hence $a_{-1} = 0$ and so $1 - 2a_0 = 0$, a contradiction. Hence no such r can exist and so $[1] \in \Sigma_A^c$.

On the other hand if we take $v(t) = -1$ then $v(r) > 0$ implies that r can involve only negative powers of t. Equating coefficients again gives

$$1 + a_{-1} - 6a_0 = 0 \quad \text{and} \quad a_j - 6a_{j+1} = 0, \quad j \geq 0.$$

The second equation implies that $a_j = 0$ for $j \geq 0$ and so $a_{-1} = -1$. Hence $f(t) = -t^{-1}$ and so $r = 1 + (t-6)(-t^{-1}) = 6t^{-1} \in C(A)$ with $v(r) > 0$. Thus $[-1] \in \Sigma_A$, and so $S(Q) = \Sigma_A \cup (-\Sigma_A)$, in other words A is 2-tame as a $\mathbb{Z}Q$-module. The group $G = \mathbb{Z}[1/6] \rtimes \langle \times 6 \rangle$ is thus, as we already knew, finitely presented.

Now suppose that the action of t on A is given by multiplication with $\frac{2}{3}$. A similar calculation to the above shows that $[1] \notin \Sigma_A$. For $v = -1$, we again get $a_{-1} = -1$, an impossibility since all of the coefficients a_i must be divisible by 3 in order for all of the coefficients of the polynomial $f(t)$ to be integral. Hence $[-1] \notin \Sigma_A$ and so $\Sigma_A = \emptyset$, implying that A is not 2-tame. The group $G = \mathbb{Z}[1/6] \rtimes \langle \times \frac{2}{3} \rangle$ is not finitely presented.

Let us now consider the case of the module $A_1 = \mathbb{Z}[x, x^{-1}, (1+x)^{-1}]$. Then $G_1 = A_1 \rtimes Q_1$ is the Baumslag group defined in [3]. Here Q_1 has rank 2, with basis elements q_0 and q_1 acting on A_1 by multiplication with x and $1+x$ respectively, and so the only unit integers in A_1 are 1 and -1. That A_1 is 2-tame has already been shown explicitly by Bieri in [11] but it will be helpful for us to illustrate this example.

The crucial fact about the structure of A_1 is that its annihilator ideal $\mathrm{Ann}_{\mathbb{Z}Q_1}(A_1)$ is *principal*: it is generated by by the single element $q_1 - q_0 - 1$. Hence we can apply the following method of Bieri and Strebel [18, §5].

Let A be finitely generated over a group ring RQ, where R is some integral domain. Write I for the annihilator ideal $\mathrm{Ann}_{RQ}(A)$ and let μ be a non-zero element of RQ. We say that a group element $q \in \mathrm{supp}(\mu)$ is a *corner* of μ if there is a character $v : Q \to \mathbb{R}$ with the property that $v(q) < v(q')$ for all $q' \in \mathrm{supp}(\mu) \setminus \{q\}$. If Q is free abelian of rank n then we can identify Q with $\mathbb{Z}^n$ via a choice of isomorphism θ. We then see that the corners of μ correspond to the vertices of the convex hull in $\mathbb{R}^n$ of the set $\{\theta(q) : q \in \mathrm{supp}(\mu)\}$; that is, the smallest convex set in $\mathbb{R}^n$ containing the points $\{\theta(q) : q \in \mathrm{supp}(\mu)\}$. For an element $x \in Q$, we define $H_x = \{[v] : v(x) > 0\}$ to be the *open hemisphere* of $S(Q)$ defined by x. Each corner q then gives rise to a non-empty open convex subset

$$\begin{aligned} C_q &= \{[v] \in S(Q) : v(q) < v(p) \text{ for all } p \in \mathrm{supp}(\mu) \setminus \{q\}\} \\ &= \bigcap_p \{H_{pq^{-1}} : p \in \mathrm{supp}(\mu) \setminus \{q\}\}. \end{aligned}$$

We can now describe Σ_A explicitly as follows:

Theorem 4.2.1 ([18, Theorem 5.2])**.** *Let R, Q, A and I be as above with $I = RQ\mu$ for some $0 \neq \mu = \sum r_p.p \in RQ$, i.e. I is a principal ideal of RQ. Then Σ_A is the union of the sets C_q where q is any corner of μ with unit coefficient $\mu(q)$.*

Remark 4.2.2. If $\mu = 0$ then $I = 0$ and so $\Sigma_A = \Sigma_{RQ}$ by (4.2). But using (4.1) we see that Σ_{RQ} is empty, as the only element in the centralizer of RQ is the identity in RQ. Hence we can assume $\mu \neq 0$.

Remark 4.2.3. Theorem 4.2.1 can in fact be proved for the more general case where I is a product of a principal ideal with a fractional ideal of the quotient ring of R. As we only require I to be principal we give Strebel's proof [32, Lemma 23].

Proof. If $v : Q \to \mathbb{R}$ is any character of Q and λ is a non-zero element of RQ, we set

$$\lambda_v = \sum_{v(q)=v(\lambda)} \lambda(q)q.$$

Note that $(\lambda\lambda')_v = \lambda_v\lambda'_v$ for any pair λ, λ' of non-zero elements. Now assume that $[v] \in \Sigma_A$. By Proposition 4.1.4 the centralizer $C(A) = 1 + I$ contains some element $1 + \lambda\mu$, $\lambda \in RQ$, such that $v(1 + \lambda\mu) > 0$. Since $v(1 + \lambda\mu) = \min\{v(1), v(\lambda\mu)\}$ and $v(1) = 0$ it follows that 1 must be in the support of $\lambda\mu$, with coefficient -1. Hence $v(\lambda\mu) = 0$ and $-1 = (\lambda\mu)_v = \lambda_v\mu_v$. It follows that μ_v is a unit, and so it must be of the form $\mu(q_0)q_0$ for some $q_0 \in Q$ and unit $\mu(q_0)$ in R, as Q is free abelian and R is an integral domain. Then q_0 is a corner of μ with unit coefficient and $[v] \in C_{q_0}$.

Conversely, if $[v] \in C_{q_1}$ and $\mu(q_1)$ is a unit, then $\lambda = 1 + \mu(-\mu(q_1)q_1)^{-1}$ lies in $C(A) = 1 + RQ\mu$ and $v(\lambda) > 0$.

□

Now $A_1 = \mathbb{Z}[x, x^{-1}, (1 + x)^{-1}]$ is a cyclic $\mathbb{Z}Q_1$-module with generator 1, and is isomorphic to

$$\mathbb{Z}Q_1/\langle q_1 - q_0 - 1\rangle$$

where q_0 acts by multiplication with x and q_1 by multiplication with $1 + x$. Hence $\mathrm{Ann}_{\mathbb{Z}Q_1}(A_1) = \langle q_1 - q_0 - 1\rangle$ and so we can apply Theorem 4.2.1. Setting $\mu = q_1 - q_0 - 1$ we see that each element of $\mathrm{supp}(\mu) = \{1, q_0, q_1\}$ is a corner of μ: we have

$$-v_i(q_i) = -1 < -v_i(1) = -v_i(q_j) = 0$$

for $i, j \in \{0, 1\}$, $i \neq j$, where the v_i are as defined in Section 3.3, while taking $v = v_0 + v_1$ we have

$$v(1) = 0 < v(q_0) = v(q_1) = 1.$$

Let $\theta : Q \to \mathbb{Z}^2$ be the group isomorphism mapping q_0 to $(1, 0)$ and q_1 to $(0, 1)$. Then the convex hull of the set $\{\theta(q) : q \in \mathrm{supp}(\mu)\}$ is the triangle in $\mathbb{R}^2$ with vertices at $(0, 0)$, $(0, 1)$ and $(1, 0)$. By Theorem 4.2.1, Σ_{A_1} is the union of the sets $C_1 = \{[v] : v(q_0) > 0, v(q_1) > 0\}$, $C_{q_0} = \{[v] : v(q_1) > v(q_0), v(q_0) < 0\}$ and $C_{q_1} = \{[v] : v(q_0) > v(q_1), v(q_1) < 0\}$. By the identification of $S(Q)$ with the unit sphere S^{n-1} described in Section 2.3 we can interpret each pair $(v(q_0), v(q_1))$ as a point on the unit circle, and if two characters v and v' are equivalent then the points $(v(q_0), v(q_1))$ and $(v'(q_0), v'(q_1))$ will be the same.

Now as $(v(q_0), v(q_1))$ lies on the unit circle it must satisfy $(v(q_0))^2 + (v(q_1))^2 = 1$. Since $(v(q_0), v(q_1)) \in \Sigma_{A_1}$ whenever $v(q_0) > 0$ or $v(q_1) > 0$ we know that $v(q_0) \leq 0$ and $v(q_1) \leq 0$ for $(v(q_0), v(q_1)) \in \Sigma^c_{A_1}$. If $v(q_0) = 0$ then we get the point $(0, 1) \in \Sigma^c_{A_1}$: similarly, $(1, 0) \in \Sigma^c_{A_1}$.

Now suppose that $v(q_0) < 0$. Then we must have $0 > v(q_0) \geq v(q_1)$, so $v(q_0) = v(q_1)$. We draw the same conclusion by assuming that $v(q_1) < 0$, and so in both cases get the point $(-1/\sqrt{2}, -1/\sqrt{2})$. Thus we have proved the following.

Proposition 4.2.4. *The set $\Sigma^c_{A_1}$ consists of the three points*

$$(0, 1), (1, 0), (-1/\sqrt{2}, -1/\sqrt{2})$$

on the unit circle S^1.

Clearly no two of these points lie diametrically opposite and so A_1 is 2-tame. However, taken together the 3 points in $\Sigma^c_{A_1}$ are not contained in any open semicircle and so A_1 is not 3-tame.

Referring back to the identification of $S(Q)$ with the unit sphere described in Section 2.3 we see that, in the notation used there, $x_{v_0} = (1, 0)$ and $x_{v_1} = (0, 1)$, so that $(1, 0)$ and $(0, 1)$ correspond to $[v_0]$ and $[v_1]$ respectively. Given $w = -(v_0 + v_1)$ we have $x_w = (w(q_0), w(q_1)) = (-1, -1)$, and so $[w]$ corresponds to $(1/\|x_w\|)x_w = (-1/\sqrt{2}, -1/\sqrt{2})$. Hence we can also describe $\Sigma^c_{A_1}$ as

$$\Sigma^c_{A_1} = \{[v_0], [v_1], [w]\}.$$

When $n > 1$ the annihilator ideal of $A_n = \mathbb{Z}[x, x^{-1}, (1+x)^{-1}, \ldots, (n+x)^{-1}, 1/n!]$ is not principal, so we take a different approach to showing that A_n is $(n+1)$-tame.

4.3 Proving the $(n + 1)$-tameness of A_n

In the last section we showed that the module $A_1 = \mathbb{Z}[x, x^{-1}, (1 + x)^{-1}]$ is 2-tame by direct calculation of $\Sigma^c_{A_1}$. This was possible because of the fact that the annihilator ideal $\mathrm{Ann}_{\mathbb{Z}Q_1}(A_1)$ is principal. That isn't the case for $A_n = \mathbb{Z}[x, x^{-1}, (1 + x)^{-1}, \ldots, (n + x)^{-1}, 1/n!]$ when $n > 1$, so we take a different approach to proving that A_n is $(n+1)$-tame. In particular we shall make use of Proposition 4.1.4 but our method will also be applicable to the $n = 1$ case. What we shall prove is the following.

Theorem 4.3.1. *Suppose $n \geq 2$ is a positive integer. Let Q_n be the free abelian group with basis elements $q_{-1}, q_0, q_1, \ldots, q_n$ acting on the module A_n by multiplication with $n!$, x, $1+x$, …, $n+x$ respectively. Let $v_i : Q_n \to \mathbb{R}$ be defined by $v_i(q_i) = 1, v_i(q_j) = 0$ for $i \neq j$, $-1 \leq i \leq n$. The v_i form a basis for $\mathrm{Hom}(Q_n, \mathbb{R})$ and so an arbitrary character is of the form $v = k_{-1}v_{-1} + k_0 v_0 + \ldots + k_n v_n$ for some $k_i \in \mathbb{R}$. Then $[v] \in \Sigma^c_{A_n}$ implies that either*

(i) $k_{-1} > 0$ or

(ii) $k_{-1} = 0$ and $(k_0, k_1, \ldots, k_n)$ is a positive real multiple of any row of the $(n+2) \times (n+1)$ matrix

$$\begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ & \ldots & & \\ 0 & 0 & \ldots & 1 \\ -1 & -1 & \ldots & -1 \end{pmatrix}.$$

Moreover all $[v]$ as in (ii) are elements of $\Sigma^c_{A_n}$. In particular A_n is $(n+1)$-tame, but not $(n+2)$-tame.

That A_n is $(n+1)$-tame is already hinted at by Lemma 3.5.6 in the last chapter. There it was shown that any $(n+1)$-subset of the set $V = \{w, v_0, v_1, \ldots, v_n\}$ of characters of Q_n, where $w = -(v_0 + v_1 + \ldots + v_n)$, lies in an open halfspace of $\mathrm{Hom}(Q_n, \mathbb{R})$. Now $[w]$ and each of the $[v_i]$ are in $\Sigma^c_{A_n}$, and what Theorem 4.3.1 shows is that *any* $(n+1)$-subset of $\Sigma^c_{A_n}$ lies in an open hemisphere of $S(Q_n)$.

Proof. First observe that to prove A_n is $(n+1)$-tame it will be enough to show that, for any $[v] \in \Sigma^c_{A_n}$, either (i) or (ii) holds. Recall that $(n+1)$-tameness holds if there do not exist any $n+1$ elements in $\Sigma^c_{A_n}$ summing to zero. The $n+2$ rows of the matrix in (ii) sum to zero, and so A_n cannot be $(n+2)$-tame, but any sum taken over a subset of $n+1$ of these rows will give an $(n+1)$-tuple with at least one non-zero entry. This deals with $(n+1)$-tameness for characters of the form given in (ii).

Now choose $n+1$ characters with at least one having the form given in (i) and the rest taking the form in (ii). Summing these gives a character where the coefficient of v_{-1} is greater than zero, since $k_{-1} = 0$ for all characters of the form in (ii). Hence we get an $(n+1)$-tuple with at least one non-zero entry and so A_n is $(n+1)$-tame.

We now prove that, for all $v \in \mathrm{Hom}(Q_n, \mathbb{R})$, $[v] \in \Sigma^c_{A_n}$ implies either (i) or (ii). In Section 2.3 it was observed that the elements $v_{-1}, v_0, \ldots, v_n$ form a basis for $\mathrm{Hom}(Q_n, \mathbb{R})$, and so any element of $\mathrm{Hom}(Q_n, \mathbb{R})$ can be written in the form

$$v = k_{-1}v_{-1} + k_0v_0 + \ldots + k_nv_n,$$

where the k_i are arbitrary real numbers. Extend any non-zero v to a valuation on the group ring $\mathbb{Z}Q_n$ in the manner described in Section 2.3. We then apply Proposition 4.1.4 and look for $\lambda \in C(A_n)$ with $v(\lambda) > 0$. Whenever this is the case $[v] \in \Sigma_{A_n}$.

First of all suppose that $k_{-1} < 0$: then $v(\lambda) > 0$ if $\lambda = (n!)q_{-1}^{-1}$. Then suppose $k_{-1} = 0$ and let $0 \leq i, j \leq n$. Whenever $k_i, k_j > 0$ for some $i \neq j$ we have $v(\lambda) > 0$ when

$$\lambda = \frac{n!}{i-j}\left(q_jq_{-1}^{-1} - q_iq_{-1}^{-1}\right).$$

Suppose now that $k_{-1} = 0$ and $k_i < 0$ for some i. If we can choose (by relabelling if necessary) some j such that $k_j > k_i$ then $v(\lambda) > 0$ for $\lambda = q_j{q_i}^{-1} + (i-j)q_i^{-1}$. If instead $k_j = k_i$ for all i, j then $[v] = [w] \in \Sigma^c_{A_n}$.

Finally suppose that $k_{-1} = 0$, precisely one of the $k_i > 0$ and the rest of the k_i are zero. Then $[v] \in \{[v_0], [v_1], \ldots, [v_n]\} \subset \Sigma^c_{A_n}$ since, by Lemma 3.3.3, $A_n \neq (A_n)_{v_i}$ for any of the v_i, and then by applying Proposition 4.1.4. In all other cases we have $k_{-1} > 0$. □

Now suppose $n = 2$. In this case we have been able to provide the complete picture by examining each case of a character $v = k_{-1}v_{-1} + k_0v_0 + k_1v_1 + k_2v_2$ with $k_{-1} > 0$. Let v be a character of this form. For the cases where the coefficients in v are as in the table below we can choose the corresponding λ to give $v(\lambda) > 0$, and so have $[v] \in \Sigma_{A_2}$.

$k_{-1}, k_0 > 0,\ k_1 = k_2 = 0$	$\lambda = q_{-1}q_2^{-1} + q_0 q_2^{-1}$
$k_{-1}, k_2 > 0,\ k_0 = k_1 = 0$	$\lambda = q_2 q_0^{-1} - q_{-1}q_0^{-1}$
$k_{-1}, k_0 > k_2 > 0,\ k_1 = 0$	$\lambda = q_0 q_2^{-1} + q_{-1}q_2^{-1}$
$k_2, k_0 > k_{-1} > 0,\ k_1 = 0$	$\lambda = q_2 q_{-1}^{-1} - q_0 q_{-1}^{-1}$
$k_{-1}, k_2 > k_0 > 0,\ k_1 = 0$	$\lambda = q_2 q_0^{-1} - q_{-1}q_0^{-1}$

Now consider the v with $k_{-1} = 0$ that were shown in the proof of Theorem 4.3.1 to lie in Σ_{A_2}. For each of these cases suppose instead that $k_{-1} > 0$ but that the coefficients k_0, k_1, k_2 are the same. Then it is possible to choose the same λ as in the proof of the Theorem 4.3.1 and get $v(\lambda) > 0$, implying $[v] \in \Sigma_{A_2}$.

This leaves the cases where v is of one of the forms $k_{-1}v_{-1}$, $k_{-1}v_{-1} + k_1 v_1$, $k_{-1}v_{-1} + kw$ or $k_{-1}v_{-1} + k_0 v_0 + k_2 v_2$ where either $k_{-1} \geq k_0 = k_2$, $k_0 \geq k_1 = k_2$ or $k_2 \geq k_{-1} = k_0$. For each we are assuming that the coefficients k_{-1}, k_0, k_2, k are all strictly positive. In each of these cases it turns out that $[v] \in \Sigma_{A_2}^c$. To prove this, we must demonstrate positively that such characters do indeed give rise to elements of the Bieri-Strebel invariant $\Sigma_{A_2}^c$, and to do this we require the theory of *Bieri-Groves sets*. This relates the character theory of finitely generated abelian groups to real valuations on the ring $\mathbb{Z}$, or more generally to arbitrary commutative rings. We discuss this in the next section.

4.4 Characters induced by valuations and the geometric structure of the Bieri-Strebel invariant

We want to explicitly calculate the Bieri-Strebel invariant $\Sigma_{A_2}^c$. In addition we would like to know more about the general structure of the sets $\Sigma_{A_n}^c$. To do these things we first define the set of characters of Q *induced* by valuations on a non-trivial commutative ring R. Throughout this section we shall assume that Q is a finitely generated abelian group of $\mathbb{Z}$-rank n. The definitions and results that follow are taken from the Bieri-Groves paper [14].

Let $v : R \to \mathbb{R}_\infty$ be a valuation and A an algebra over the group ring RQ via a ring homomorphism $\phi : RQ \to A$. Then $\Delta_A^v(Q) \subseteq \mathrm{Hom}(Q, \mathbb{R})$ is defined to be the set of all real characters χ of Q with the property that there is a valuation $\bar{v} : A \to \mathbb{R}_\infty$ such that $\bar{v}\phi|_R = v$ and $\bar{v}\phi|_Q = \chi$. We shall say that the character χ is *induced* by the valuation $\bar{v}$, and refer to $\Delta_A^v(Q)$ as a *Bieri-Groves set.*

It can be shown that the sets $\Delta_A^v(Q)$ are *polyhedral.* Define a subset C of $\mathbb{R}^n$ to be a *convex polyhedron* if it can be written as

$$C = H_1 \cap H_2 \cap \ldots \cap H_s,$$

where the H_i are all closed halfspaces in $\mathbb{R}^n$. If these halfspaces can all be defined in terms of inequalities of the form

$$r_1x_1 + r_2x_2 + \ldots r_nx_n \geq a,$$

where the coefficients $r_i \in \mathbb{Q}$, $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and a is contained in some fixed subgroup S of $\mathbb{R}$, then C is said to be *rationally defined* over S.

The *dimension* of a convex polyhedron C is the vector space dimension of the affine subspace spanned by C. A subset Δ of $\mathbb{R}^n$ is said to be a *polyhedron (rationally defined over S)* if

$$\Delta = C_1 \cup \ldots \cup C_t,$$

where the C_i are convex polyhedrons (rationally defined over S). Such a polyhedron Δ is said to be *homogeneous of dimension* m if the decomposition can be chosen such that each C_i is of dimension m.

Now let $A = RQ/P$ be an integral domain containing both R and Q, where P is some prime ideal of RQ. Let $k \subseteq K$ denote the fields of fractions of $R \subseteq A$ and let $v : R \to \mathbb{R}_\infty$ be a valuation. We shall assume that $v^{-1}(\infty) = 0$ so that v can be extended to a valuation on the field k.

Theorem 4.4.1 ([14, Theorem A])**.** *The subset* $\Delta_K^v(Q) \subseteq \mathrm{Hom}(Q, \mathbb{R})$ *is a polyhedron rationally defined over the value group* $v(k^\times)$ *of* v *and homogeneous of dimension equal to the transcendence degree of* K *over* k*.*

The special case of Theorem 4.4.1 where v is the trivial valuation is enough to prove a conjecture of G. M. Bergman [9] concerning the logarithmic limit sets of algebraic varieties.

As a corollary to another of their theorems Bieri and Groves show that $\Delta_K^v(Q) = \Delta_{k[Q]}^v(Q)$ [14, Corollary 6.3], where $k[Q]$ is the smallest subring of K containing k and Q. The following result then shows that $\Delta_K^v(Q)$ is in fact equal to $\Delta_A^v(Q)$.

Proposition 4.4.2 ([16, §3, Proposition 1])**.** $\Delta_K^v(Q) = \Delta_A^v(Q)$.

Proof. Since $\Delta_K^v(Q) = \Delta_{k[Q]}^v(Q)$ take $\chi \in \Delta_{k[Q]}^v(Q)$. Then $\chi = w_{|Q}$, where $w : k[Q] \to \mathbb{R}_\infty$ is a valuation which restricts to v on k. Since A is generated as an R-algebra by Q, that is $A = R[Q] \subset k[Q]$, it follows that w can be restricted to A and so $\chi \in \Delta_A^v(Q)$.

Conversely suppose that $\chi : Q \to \mathbb{R}$ is induced by a valuation $w : A \to \mathbb{R}_\infty$ restricting to v on R. Then w can be extended to the field of fractions k of R and hence to $k[Q]$. □

Hence by Theorem 4.4.1 $\Delta_A^v(Q)$ is a polyhedron of dimension equal to the transcendence degree of $k(Q)$ over k. As a result we have the following:

Corollary 4.4.3. *The sets* $\Delta_{A_1}^v(Q_1) \subseteq \mathbb{R}^2$ *and* $\Delta_{A_n}^v(Q_n) \subseteq \mathbb{R}^{n+2}$ $(n \geq 2)$ *are homogeneous polyhedrons of dimension* 1 *for all valuations* $v : \mathbb{Q} \to \mathbb{R}_\infty$.

Proof. Here $R = \mathbb{Z}$, $k = \mathbb{Q}$ and $K = \mathbb{Q}(x)$ is the field of fractions of A. Since Q_n embeds in $\mathbb{Q}(x)$ for $n \geq 1$ the field $k(Q_n)$ is just $\mathbb{Q}(x)$, which has transcendence degree 1 over $\mathbb{Q}$. So it follows from Theorem 4.4.1 and Proposition 4.4.2 that $\Delta_{A_n}^v(Q_n)$ is a homogeneous polyhedron of dimension 1 for $n \geq 1$ and all valuations v on $\mathbb{Q}$; namely, either the trivial valuation or any constant multiple of the p-adic valuations for primes p. Moreover, $\Delta_{A_n}^v(Q_n)$ is rationally defined over $v(\mathbb{Q} \setminus \{0\})$.

□

Let us now relate the material so far in this section to the invariant $\Sigma_A^c \subseteq S(Q)$.

A point on the sphere $S(Q)$ is said to be *rational* if it is represented by a discrete character $\chi : Q \to \mathbb{R}$, that is one whose image is a discrete subgroup of $\mathbb{R}$. A closed (open) hemisphere $H \subset S(Q)$ is *rational* if its spherical centre (or rather, the point on the sphere equivalent to its centre) is a rational point or, equivalently, it is of the form $H = \{\chi : \chi(q) \geq 0\}$ (if closed) or $H = \{\chi : \chi(q) > 0\}$ (if open), where q is some element of Q with infinite order.

We can then define convex *spherical* polyhedrons. A (rational) *convex spherical polyhedron* $C \subseteq S(Q)$ is any subset of $S(Q)$ which can be written as the intersection of finitely many closed (rational) hemispheres. A (rational) *spherical polyhedron* in general is a subset of $S(Q)$ which can be written as the union of finitely many convex (rational) spherical polyhedrons.

Now consider the set Σ_A. Recall from (4.1) that

$$\Sigma_A = \bigcup_{\lambda \in C(A)} \{[v] \in S(Q) : v(\lambda) > 0\}.$$

From this formula we see that Σ_A is open in $S(Q)$ with respect to the topology inherited from $\mathbb{R}^n$. Indeed, the set

$$\{[v] \in S(Q) : v(\lambda) > 0\}$$

is equal to the intersection of the open sets

$$\{[v] \in S(Q) : v(q) > 0\}$$

taken over all $q \in Q$ in the support of λ. Clearly there can be only finitely many of these sets and so Σ_A is open. Hence the complement $\Sigma_A^c = S(Q) \setminus \Sigma_A$ is closed.

We now state the relationship between the Bieri-Strebel invariant and Bieri-Groves sets.

Theorem 4.4.4 ([14, Theorem 8.1])**.** *Let R be a non-trivial commutative ring and Q a finitely generated abelian group of rank n. Suppose that M is a finitely generated RQ-module and let $A = RQ/Ann_{RQ}(M)$ denote the quotient of the group ring RQ modulo the annihilator ideal of M. Then*

$$\Sigma_M^c = \bigcup_v [\Delta_A^v(Q)] = \bigcup_v \{[\chi] \in S(Q) : \chi \in \Delta_A^v(Q)\},$$

where v runs through all valuations $R \to \mathbb{R}_\infty$ with $v(R) \geq 0$.

Note that we are always assuming that $A = M$, since this is true when $M = A_n$ and $Q = Q_n$ in Theorem 4.4.4.

From the definition it is clear that a spherical polyhedron must always be closed. The projection $[\Delta]$ of a polyhedron Δ onto the sphere may not be a spherical polyhedron since it might not be closed. However if we suppose that $R = D$ is a Dedekind domain then the following result shows that the union in Theorem 4.4.4 is finite and so $\bigcup_v \Delta_A^v(Q)$ is a polyhedron.

Theorem 4.4.5 ([14, Theorem B]). *There is a finite set Π of primes in D with the property that*

$$\Delta_A^{v_{\mathfrak{p}}}(Q) = \Delta_A^0(Q)$$

for all primes $\mathfrak{p} \in \text{Spec}\,(D) \setminus \Pi$, where $v_{\mathfrak{p}}$ denotes the $\mathfrak{p}$-adic valuation on D and 0 the trivial valuation.

It remains to see that Σ_A^c is a rational spherical polyhedron whenever A is a finitely generated DQ-module. We already know that it is a closed subset of $S(Q)$, via (4.1). In fact, while the description of Σ_A in (4.1) is sufficient for our purposes, Bieri and Harlander improved on it in [16] by showing that there is always a *finite* set of centralizers $\Lambda \subseteq C(A)$ with

$$\Sigma_A = \bigcup_{\lambda \in \Lambda} \{[v] \in S(Q) : v(\lambda) > 0\}. \tag{4.3}$$

It follows immediately that:

Theorem 4.4.6 ([14, Theorem E]). *Σ_A^c is a rational spherical polyhedron.*

In particular this applies to the case we are interested in, where $D = \mathbb{Z}$.

There is a modern point of view in which the sets Σ_A^c can be expressed using tropical geometry. Such an approach is discussed in the work of Einsiedler, Kapranov and Lind [22].

4.5 Calculation of $\Sigma_{A_2}^c$

We now return to the module $A_2 = \mathbb{Z}[x, x^{-1}, (1+x)^{-1}, (2+x)^{-1}, 1/2]$. Our aim is to calculate the Bieri-Strebel invariant $\Sigma_{A_2}^c$ explicitly. We prove the following.

Proposition 4.5.1. *Let* $\chi = k_{-1}v_{-1} + k_0v_0 + k_1v_1 + k_2v_2$. *This is a real character of the group* Q_2 *with basis elements* q_{-1}, q_0, q_1, q_2 *acting on the module* A_2 *by multiplication with* 2, x, $1+x$, $2+x$ *respectively. Then* $[\chi] \in \Sigma^c_{A_2}$ *if and only if* $(k_{-1}, k_0, k_1, k_2) \in \mathbb{R}^4_{\geq 0} \setminus \{(0,0,0,0)\}$ *takes one of the following forms:*

(a) $(0, k_0, 0, 0), (0, 0, k_1, 0), (0, 0, 0, k_2), (0, -k_0, -k_0, -k_0)$;

(b) $(k_{-1}, 0, 0, 0)$, $(k_{-1}, 0, k_1, 0)$, $(k_{-1}, -k_0, -k_0, -k_0)$; *or*

(c) $(k_{-1}, k_{-1}, 0, k_2)$ $(k_2 \geq k_{-1})$, $(k_{-1}, k_0, 0, k_{-1})$ $(k_0 \geq k_{-1})$, $(k_{-1}, k_0, 0, k_0)$ $(k_{-1} \geq k_0)$.

Proof. It was proved in Theorem 4.3.1 that $[\chi] \in \Sigma^c_{A_2}$ whenever (k_{-1}, k_0, k_1, k_2) is of one of the forms given in (a). We also showed that all characters other than those of the form (a), (b) or (c) led to elements of Σ_{A_2}. Hence we can assume that $k_{-1} > 0$ and χ is of the form in either (b) or (c) . Our aim is to apply Theorem 4.4.4. By this result, we need to consider all valuations $v : \mathbb{Z} \to \mathbb{R}_\infty$ with $v(\mathbb{Z}) \geq 0$ and the property that $\chi \in \Delta^v_{A_2}(Q_2)$.

First note that we have the ring homomorphism $\phi : \mathbb{Z}Q_2 \to A_2$ which is the identity map on $\mathbb{Z}$ and is defined by

$$\phi(q_{-1}) = 2,\ \phi(q_i) = i + x \quad (0 \leq i \leq 2)$$

on Q_2. Observe that the characters v_0, v_1, v_2, when extended to valuations on the group ring $\mathbb{Z}Q_2$, and hence to A_2 via the embedding of the group of units of Q_2 in A_2, all restrict to the trivial valuation 0 on $\mathbb{Z}$. Hence $\{v_0, v_1, v_2\} \subset \Delta^0_{A_2}(Q_2)$ and so this provides an alternative way of seeing that $[v_i] \in \Sigma^c_{A_2}$ for $0 \leq i \leq 2$. Indeed, this argument could apply to $\Sigma^c_{A_n}$ for any $n \geq 1$.

We now show that each χ is induced by a valuation $\bar{v}$ on A_2 extending the 2-adic valuation $|\ |_2$ on $\mathbb{Z}$. First consider $\chi = k_{-1}v_{-1} + k_1v_1$, $k_{-1}, k_1 > 0$. Then, viewing Q_2 as a subgroup of the group of units of A_2, for the property $\chi \in \Delta^v_{A_2}(Q_2)$ to hold for some valuation $v : \mathbb{Z} \to \mathbb{R}_\infty$ we must have $\chi = \bar{v}\phi|_{Q_2}$ for some valuation $\bar{v} : A_2 \to \mathbb{R}_\infty$ on the ring A_2. Hence $\bar{v}$ must satisfy

$$\bar{v}(2) = k_{-1},\ \bar{v}(1 + x) = k_1,\ \bar{v}(x) = \bar{v}(2 + x) = 0. \tag{4.4}$$

We must also have $v = \bar{v}\phi|_{\mathbb{Z}}$. Since $\bar{v}(2) = k_{-1}$, this gives

$$v(2) = \bar{v}\phi(2) = \bar{v}(2) = k_{-1}.$$

Since $k_{-1} > 0$, and v must be a valuation on $\mathbb{Z}$ satisfying $v(\mathbb{Z}) \geq 0$, it follows that v is a positive multiple of the 2-adic valuation, that is

$$v = k_{-1}|\ |_2.$$

To confirm that $\bar{v}$ is indeed a valuation on A_2, we must check that it satisfies the property $\bar{v}(a+b) \geq \min\{\bar{v}(a), \bar{v}(b)\}$ for all $a, b \in A_2$. First observe that for any valuation v on a ring R we have $v(-1) = 0$, since

$$0 = v(1) = v((-1).(-1)) = v(-1) + v(-1).$$

It follows easily that $v(r) = v(-r)$ for any $r \in R$. Since we can take $\bar{v}$ to have arbitrary value on all monomials in A_2 other than those determined by the values given in (4.4), we only have to check the values $\bar{v}(a+b)$ for a, b, $a+b \in \{1, 2,\ x,\ 1+x,\ 2+x\}$. Moreover, since $\bar{v}(a) \geq 0$ for all a in this set, we need only check $\bar{v}(a+b) > 0$ whenever $\bar{v}(a), \bar{v}(b) > 0$. This leaves the following:

$$\bar{v}(3+x) = \bar{v}(2 + (1+x)) \geq \min\{k_{-1}, k_1\} > 0.$$

This forces a relationship between the values of $\bar{v}$ on 2, $1+x$ and $3+x$ but that need not concern us. What matters is that $\bar{v}$ can clearly be chosen to be a valuation, and so $\chi \in \Delta^v_{A_2}$. By Theorem 4.4.4 it follows that $[\chi] \in \Sigma^c_{A_2}$.

Now suppose that $\chi = k_{-1}v_{-1} + k_0 v_0 + k_2 v_2$. Then $\bar{v}$ must satisfy

$$\bar{v}(2) = k_{-1}, \quad \bar{v}(x) = k_0, \quad \bar{v}(1+x) = 0, \quad \bar{v}(2+x) = k_2.$$

As in the previous case let us check for $\bar{v}(a+b) \geq \min\{\bar{v}(a), \bar{v}(b)\}$. We get

$$k_2 = \bar{v}(2+x) \geq \min\{k_{-1}, k_0\}, \tag{4.5}$$

$$k_0 = \bar{v}(x) = \bar{v}((2+x) - 2) \geq \min\{k_{-1}, k_2\} \tag{4.6}$$

and

$$k_{-1} = \bar{v}(2) = \bar{v}((2+x) - x) \geq \min\{k_0, k_2\}. \tag{4.7}$$

Suppose that $k_{-1} \geq k_0$. Then from (4.5) we get $k_2 \geq k_0$ but (4.6) implies that either $k_0 \geq k_{-1}$ or $k_0 \geq k_2$. Hence either $k_0 = k_{-1} \leq k_2$ or $k_0 = k_2 \leq k_{-1}$.

Now suppose instead that $k_0 \geq k_{-1}$. From (4.5) we get $k_2 \geq k_{-1}$ but (4.7) implies that either $k_{-1} \geq k_0$, in which case $k_0 = k_{-1} \leq k_2$, or $k_{-1} \geq k_2$, in which case $k_2 = k_{-1} \leq k_0$. A similar check in the other cases confirms that in order for $\bar{v}$ to be a valuation on A_2 one of the following relationships between the positive integers k_{-1}, k_0, k_2, as given in (c) in the statement of this theorem, must hold:

$$k_2 \geq k_{-1} = k_0, \quad k_0 \geq k_{-1} = k_2, \quad k_{-1} \geq k_0 = k_1.$$

Hence in each case $\chi \in \Delta^v_{A_2}(Q_2)$, where v is a positive multiple $k_{-1}|\ |_2$ of the 2-adic valuation on $\mathbb{Z}$, and so $[\chi] \in \Sigma^c_{A_2}$.

We must complete the proof of (b). It is easy to see that $\chi = k_{-1}v_{-1} \in \Delta^v_{A_2}(Q_2)$, where $v = k_{-1}|\ |_2$. This leaves only the case $\chi = k_{-1}v_{-1} + kw$, where $w = -(v_0 + v_1 + v_2)$. It is straightforward to show that $\chi \in \Delta^v_{A_2}(Q_2)$ by using similar methods to those used in the other cases.

□

When $n = 3$ we find that the character $\chi = v_{-1} + v_0$ is not induced by any valuation on A_3, and yet we have not been able to find $\lambda \in C(A_3)$ such that $\chi(\lambda) > 0$. For any valuation $\bar{v}$ inducing χ must satisfy

$$\bar{v}(x) = 1,\ \bar{v}(1 + x) = \bar{v}(2 + x) = \bar{v}(3 + x) = 0$$

and, since q_{-1} acts on A_3 as multiplication by 6, it must also satisfy $\bar{v}(6) = 1$. Hence either $\bar{v}(2) = 1$ or $\bar{v}(3) = 1$, that is the valuation $v : \mathbb{Z} \to \mathbb{R}_\infty$ is either the 2-adic or the 3-adic valuation on $\mathbb{Z}$. However if $\bar{v}(2) = 1$ we get the contradiction

$$0 = \bar{v}(2 + x) \geq \min\{\bar{v}(2), \bar{v}(x)\} = 1,$$

and similarly $\bar{v}(3) = 1$ implies that

$$0 = \bar{v}(3 + x) \geq \min\{\bar{v}(3), \bar{v}(x)\} = 1.$$

This points towards $[\chi] \in \Sigma_{A_3}$. The difficulty with finding any $\lambda \in C(A_3)$ that would prove this assertion seems to lie in the fact that for $n > 2$ the element q_{-1} acts on A_n by multiplication with $n!$, which is not equal to n for $n > 2$.

For arbitrary n, we have the following necessary condition for $\chi = k_{-1}v_{-1} + k_0v_0 + \ldots + k_nv_n$ to be induced by a valuation $\bar{v}$ on A_n extending the p-adic valuation on $\mathbb{Z}$. Note that in such a case p is always a prime dividing $n!$, since we have $\bar{v}(n!) = \chi(q_{-1}) = k_{-1}$.

Proposition 4.5.2. *Let $k_{-1} > 0$. If $\chi = k_{-1}v_{-1} + k_0v_0 + \ldots + k_nv_n$ is such that $[\chi] \in \Sigma^c_{A_n}$, and if $k_i > 0$, $i > -1$, then*

$$k_{i+lp} > 0$$

for some fixed prime p and for any integer l such that $0 \leq i + lp \leq n$.

Proof. The fact that $[\chi] \in \Sigma^c_{A_n}$ implies that χ is induced by a valuation $\bar{v} : A_n \to \mathbb{R}_\infty$ extending a valuation v on $\mathbb{Z}$ with $v(\mathbb{Z}) \geq 0$, by Theorem 4.4.4. Such a valuation v must either be the trivial valuation or a positive multiple of a p-adic valuation for some prime p, and since $k_{-1} > 0$ it follows that it must be the latter. Then $\bar{v}(p) > 0$ and if $k_i > 0$ we have $\bar{v}(i + x) = k_i > 0$. Hence

$$k_{i+lp} = \bar{v}((i + lp) + x) \geq \min\{\bar{v}(i + x), \bar{v}(lp)\} > 0$$

since $v(l) \geq 0$ and so $\bar{v}(lp) = \bar{v}(l) + \bar{v}(p) > 0$. □

As we showed above in the $n = 2$ case, additional restrictions on $k_{-1}, k_0, k_1, \ldots, k_n$ may also apply.

Chapter 5

Group extensions

In this chapter we aim to calculate the second cohomology group $H^2(Q, A)$, where Q and A are the groups defined in Theorem 3.1.3. First of all we shall demonstrate how this will enable us to classify all possible extensions of A by Q. Specifically, $H^2(Q, A)$ corresponds bijectively to the set of equivalence classes of all such extensions, and the zero element in $H^2(Q, A)$ corresponds to the split extension. In the specific case where $Q = Q_n$ and $A = A_n$ we shall find that $H^2(Q, A) = 0$ and so *every* extension is in fact split. Hence the Baumslag group G_n is the unique extension of A_n by Q_n up to equivalence.

5.1 The second cohomology group and group extensions

Let G be a group and M a $\mathbb{Z}G$-module. We want to demonstrate how the elements of the second cohomology group $H^2(G, M)$ can be interpreted as the equivalence classes of group extensions of M by G. This material builds on that covered in Sections 2.1 and 2.2 and is based on §18 of Kropholler's unpublished notes 'Cohomology of Groups and its Connections with Geometric Group Theory'.

The second cohomology group is best understood by looking at a fixed resolution. Let X be a set. We can associate a chain complex $C_*(X)$ to X in the following way: for $n \geq 0$ define $C_n(X)$ to be the free abelian group on the set of ordered $(n+1)$-tuples of elements of X and for $n < 0$ set $C_n(X) = 0$. For $n \geq 1$ the differentials

$d : C_n(X) \to C_{n-1}(X)$ are the additive maps determined by

$$d(x_0, x_1, \ldots, x_n) = \sum_{i=0}^{n} (-1)^i (x_0, \ldots, \hat{x}_i, \ldots, x_n).$$

The notation $\hat{x}_i$ means that x_i is omitted. So the right hand side is an alternating sum of n-tuples and since $d^2 = 0$ we get a chain complex. The augmented chain complex $\widetilde{C}_*(X)$ is defined by setting $\widetilde{C}_{-1}(X) = \mathbb{Z}$ and $\widetilde{C}_i(X) = C_i(X)$ for $i \neq -1$, and by defining the map $\widetilde{C}_0(X) \to \widetilde{C}_{-1}(X)$ to be the augmentation map $x \mapsto 1$ for $x \in X$.

Lemma 5.1.1. *If X is non-empty then the augmented chain complex $\tilde{C}_*(X)$ is an exact sequence.*

Proof. It suffices to prove that every cycle is also a boundary. Fix a choice of $\dot{x} \in X$. Define maps $h : \tilde{C}_n(X) \to \tilde{C}_{n+1}(X)$ to be additive with

$$h(x_0, x_1 \ldots, x_n) = (\dot{x}, x_0, \ldots, x_n)$$

for $n \geq -1$. It can be easily shown that the map $hd + dh$ is the identity on each $\tilde{C}_n(X)$. Now if ξ is a cycle then $d\xi = 0$ and so $\xi = (hd + dh)\xi = d(h\xi)$ i.e. ξ is the boundary of $h\xi$. □

Now let G be a group. Then the chain complexes $C_*(G)$ and $\tilde{C}_*(G)$ both inherit an action of G from the right action

$$(g_0, \ldots, g_n)g = (g_0 g, \ldots, g_n g).$$

The action of G is trivial on $\tilde{C}_{-1}(G) = \mathbb{Z}$. Therefore we have an exact sequence

$$\ldots \to C_3(G) \to C_2(G) \to C_1(G) \to C_0(G) \to \mathbb{Z} \to 0$$

of $\mathbb{Z}G$-modules resolving the trivial module $\mathbb{Z}$. In fact this is a projective resolution, since:

Lemma 5.1.2. *For each $n \geq 0$, $C_n(G)$ is a free $\mathbb{Z}G$-module.*

Proof. Let $C'_n(G)$ denote the $\mathbb{Z}G$-module with the same underlying abelian group as $C_n(G)$ but where the action is given by multiplication in the rightmost coordinate only:

$$(g_0, \ldots, g_{n-1}, g_n)g = (g_0, \ldots, g_{n-1}, g_n g).$$

Then $C'_n(G)$ is a free module with basis consisting of all $(n+1)$-tuples of the form $(g_0, \ldots, g_{n-1}, 1)$. Moreover $C'_n(G)$ and $C_n(G)$ are isomorphic as $\mathbb{Z}G$-modules for each n, since we have the following pair of mutually inverse $\mathbb{Z}G$-maps:

$$\begin{aligned} C_n(G) &\to C'_n(G) \\ (g_0, g_1, \ldots, g_n) &\mapsto (g_0 g_1^{-1}, g_1 g_2^{-1}, \ldots, g_{n-1} g_n^{-1}, g_n) \\ C'_n(G) &\to C_n(G) \\ (h_0, h_1, \ldots, h_n) &\mapsto (h_0 h_1 h_2 \ldots h_n, h_1 h_2 \ldots h_n, h_2 \ldots h_n, \ldots, h_{n-1} h_n, h_n) \end{aligned}$$

□

We now want to define differentials $d : C'_n(G) \to C'_{n-1}(G)$ in the unique way that will give us the following commutative diagram describing an isomorphism of chain complexes:

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & C_3(G) & \longrightarrow & C_2(G) & \longrightarrow & C_1(G) & \longrightarrow & C_0(G) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \| & & \\ \cdots & \longrightarrow & C'_3(G) & \longrightarrow & C'_2(G) & \longrightarrow & C'_1(G) & \longrightarrow & C'_0(G) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \end{array}$$

This is done by setting

$$d((h_0, h_1, \ldots, h_n)) = (h_1, \ldots, h_n) + \sum_{i=0}^{n} (-1)^i (h_0, h_1, \ldots, h_{i-2}, h_{i-1} h_i, h_{i+1}, \ldots, h_n).$$

We shall use the resolution $C'_* \twoheadrightarrow \mathbb{Z}$ to compute the cohomology of G. For a $\mathbb{Z}G$-module M there is a bijective correspondence between $\mathrm{Hom}_{\mathbb{Z}G}(C'_n(G), M)$ and the set of all functions $G^n \to M$, where G^n denotes the cartesian product of n copies of G: this correspondence is defined by restricting maps $C'_n(G) \to M$ to the subset $G^n \times \{1\} \subset C'_n(G)$. Let $C^n(G, M)$ denote this set of functions. The cochain complex $\mathrm{Hom}_{\mathbb{Z}G}(C'_*(G), M)$ can now be identified with

$$\ldots \xrightarrow{\delta} C^0(G, M) \xrightarrow{\delta} C^1(G, M) \xrightarrow{\delta} C^2(G, M) \xrightarrow{\delta} C^3(G, M) \xrightarrow{\delta} \ldots \tag{5.1}$$

The differential $\delta : C^2(G, M) \to C^3(G, M)$ is given by $\delta(f) = f \circ d$, where d denotes the differential $d : C_3'(G) \to C_2'(G)$ and $f : C_2'(G) \to M$ is a $\mathbb{Z}G$-module homomorphism. Hence

$$\begin{aligned}\delta(f(g_1, g_2, g_3)) &= f(d(g_1, g_2, g_3, 1)) \\ &= f(g_2, g_3, 1) - f(g_1g_2, g_3, 1) + f(g_1, g_2g_3, 1) - f(g_1, g_2, g_3) \\ &= f(g_2, g_3, 1) - f(g_1g_2, g_3, 1) + f(g_1, g_2g_3, 1) - f(g_1, g_2, 1).g_3.\end{aligned}$$

Restricting f to $G \times G \times \{1\}$ means that it can be identified with a function from $G \times G$ to M and so

$$\delta(f(g_1, g_2, g_3)) = f(g_2, g_3) - f(g_1g_2, g_3) + f(g_1, g_2g_3) - f(g_1, g_2)g_3. \tag{5.2}$$

Similarly we find that $\delta : C^1(G, M) \to C^2(G, M)$ is given by

$$\delta(f(g_1, g_2)) = f(g_2) - f(g_1g_2) + f(g_1).g_2$$

and $\delta : C^0(G, M) \to C^1(G, M)$ by $\delta(f(g_1)) = f(1) - f(1).g_1$. We can see from these formulae that 1-cocycles (kernels of $C^1 \to C^2$) are derivations and 1-coboundaries (images of $C^0 \to C^1$) are inner derivations; see Definition 2.2.3.

We can turn the product $G \times M$ into a group by means of *factor sets*. A factor set $f : G \times G \to M$ is a 2-cocycle in a cochain complex of the form (5.1). Hence from (5.2) f satisfies

$$f(g_1, g_2g_3) + f(g_2, g_3) = f(g_1g_2, g_3) + f(g_1, g_2).g_3.$$

Lemma 5.1.3. *Let G be a group and let M be a $\mathbb{Z}G$-module. Let $f : G \times G \to M$ be any function. Define a product on the set $G \times M$ by*

$$(g, m)(g', m') = (gg', mg' + m' + f(g, g')).$$

Write $G \overset{f}{\times} M$ for the set $G \times M$ endowed with this product. Then $G \overset{f}{\times} M$ is a group if and only f is a factor set.

Proof. For the associative law to hold we see that f must be a 2-cocycle:

$$((g, m)(g', m'))(g'', m'') = (gg'g'', mg'g'' + m'g'' + m'' + f(g, g')g'' + f(gg', g'')),$$

$$(g, m)((g', m')(g'', m'')) = (gg'g'', mg'g'' + m'g'' + m'' + f(g', g'') + f(g, g'g'')).$$

Whenever this condition holds there is an identity element

$$(1, -f(1, 1))$$

and the inverse of an element (g, m) is

$$(g, m)^{-1} = (g^{-1}, -mg^{-1} - f(g, g^{-1} - f(1, 1)).$$

□

Now suppose we have an extension $M \hookrightarrow E \overset{\pi}{\twoheadrightarrow} G$ of groups, where M is abelian. Since M can be viewed as a subgroup of E it has a natural action of E by conjugation and so we can think of M as a $\mathbb{Z}G$-module in the way described in Section 2.4. Define a *transversal* of π to be any function $t : G \to E$ with the property that πt is the identity on G. We write $t(g) = t_g$ and define a function

$$\phi : E \to G \times M$$

by

$$e \mapsto (\pi(e), t_{\pi(e)}^{-1} e).$$

It can be readily checked that first of all $t_{\pi(e)}^{-1} e \in M$ and secondly ϕ is a bijection with inverse $G \times M \to E$ defined by

$$(g, m) \mapsto t_g m.$$

We now want to define a factor set $f : G \times G \to M$ which will give a product on $G \times M$ ensuring that $\phi : E \to G \overset{f}{\times} M$ is in fact an isomorphism of groups. Defining f by

$$f(g, g') = t_{gg'}^{-1} t_g t_{g'} \tag{5.3}$$

turns ϕ into a homomorphism and, since it is bijective, an isomorphism. Hence the groups E and $G \overset{f}{\times} M$ are isomorphic. What we have shown then is that every extension E of M by G is isomorphic to a group of the form $G \overset{f}{\times} M$ for some suitable choice of factor set f. Conversely, if we have a factor set f then we can obtain a group extension $M \hookrightarrow G \overset{f}{\times} M \twoheadrightarrow G$ where the injection from M is given by

$$m \mapsto (1, m - f(1,1))$$

and the epimorphism to G by $(g, m) \mapsto g$.

Now it makes sense to consider when two factor sets determine the same group extension. First we must define what we mean by same in this context. We say that two group extensions $M \hookrightarrow E \twoheadrightarrow G$ and $M \hookrightarrow E' \twoheadrightarrow G$ are *equivalent* whenever there is a commutative diagram

$$\begin{array}{ccccc} M & \overset{\iota_1}{\hookrightarrow} & E & \overset{\theta_1}{\twoheadrightarrow} & G \\ \| & & \downarrow \phi & & \| \\ M & \overset{\iota_2}{\hookrightarrow} & E' & \overset{\theta_2}{\twoheadrightarrow} & G \end{array}$$

with the extensions for its rows.

Lemma 5.1.4. *The map $\phi : E \to E'$ in the commutative diagram above is an isomorphism.*

Proof. Here we use multiplicative notation since, although M is abelian, G and E need not be. First we prove that ϕ is injective. Suppose $e \in E$ is such that $\phi(e) = 1$. Then by commutativity $\theta_1(e) = 1$ and so there exists $m \in M$ with $\iota_1(m) = e$. But, again by commutativity,

$$\iota_2(m) = \phi(\iota_1(m)) = \phi(e) = 1$$

and by injectivity of ι_2 we have $m = 1$, so $e = 1$.

Now suppose $e' \in E'$ and let $g = \theta_2(e')$. Since θ_1 is surjective there exists $e \in E$ with $\theta_1(e) = g = \theta_2(e')$. But $\theta_2 \phi = \theta_1$, so $\theta_2(\phi(e)) = \theta_2(e')$, implying that $\phi(e)(e')^{-1} \in \ker \theta_2$. Hence $\phi(e)(e')^{-1} = \iota_2(m)$ for some $m \in M$. But by commutativity $\phi(\iota_1(m)) = \iota_2(m)$, giving $e' = \phi(e\iota_1(m^{-1}))$. Hence ϕ is surjective. □

We can now state the connection between group extensions and the second cohomology group.

Theorem 5.1.5. *Let G be a group and let M be a $\mathbb{Z}G$-module. Then two factor sets $f, f' : G \times G \to M$ determine equivalent group extensions if and only if they are cohomologous: that is, if and only if $f - f' = \delta(s)$ for some function $s : G \to M$.*

Proof. A function $\phi : G \overset{f}{\times} M \to G \overset{f'}{\times} M$ which makes the diagram

$$\begin{array}{ccccc} M & \hookrightarrow & G \overset{f}{\times} M & \twoheadrightarrow & G \\ \| & & \downarrow \phi & & \| \\ M & \hookrightarrow & G \overset{f'}{\times} M & \twoheadrightarrow & G \end{array}$$

commute must be of the form

$$\phi(g, m) = (g, m + s(g))$$

for some function $s : G \to M$. Now ϕ is a homomorphism if and only if s satisfies the identity

$$f(g, g') + s(gg') = s(g)g' + s(g') + f'(g, g'),$$

and this says that $f - f' = \delta(s)$ where δ is the differential $\delta : C^1(G, M) \to C^2(G, M)$. □

Corollary 5.1.6. *The equivalence classes of group extensions of a $\mathbb{Z}G$-module M by a group G are in bijective correspondence with the second cohomology group $H^2(G, M)$.*

From the definition of $G \overset{f}{\times} M$ in Lemma 5.1.3 it is clear that the zero element of $H^2(G, M)$ - that is, the zero factor set - corresponds to the split extension of M by G. Hence from (5.3) it follows that the extension $M \hookrightarrow E \overset{\pi}{\twoheadrightarrow} G$ is split if and only if the transversal function $t : G \to E$ can be chosen to be a homomorphism.

5.2 Calculating cohomology via the Lyndon-Hochschild-Serre spectral sequence

The Lyndon-Hochschild-Serre spectral sequence will allow us to compute cohomology, and the second cohomology group in particular. Suppose that $K \hookrightarrow G \overset{\pi}{\twoheadrightarrow} Q$

is a group extension and let M be a $\mathbb{Z}G$-module. Note that K and Q need not be abelian. First of all we give some preliminary definitions. Since K is a subgroup of G, any element of M that is invariant under the induced action of G must be invariant under the action of K. Hence we have an inclusion $M^G \hookrightarrow M^K$ or equivalently

$$H^0(G, M) \hookrightarrow H^0(K, M).$$

By Theorem 2.1.6, this inclusion extends uniquely to a map

$$H^*(G, \ ) \to H^*(K, \ )$$

of cohomological functors on $\mathbb{Z}G$-modules, known as the *restriction map*. Since K is normal, the set $H^0(K, M) = M^K$ of K-invariants is a $\mathbb{Z}G$-submodule on which K acts trivially, and so $H^0(K, M)$ has the structure of a $\mathbb{Z}Q$-module since $Q \cong G/K$. In fact, this Q-action extends to a $\mathbb{Z}Q$-module structure on *all* of the cohomology groups $H^n(K, M)$. Theorem 2.1.6 can be used again to show this.

Now observe that via the surjection $\pi : G \twoheadrightarrow Q$ any $\mathbb{Z}Q$-module M becomes a $\mathbb{Z}G$-module, and an element of M is fixed under the action of Q if and only if it is fixed under the inherited action of G. Hence we get a natural isomorphism

$$H^0(Q, \ ) \to H^0(G, \ )$$

of cohomological functors on $\mathbb{Z}Q$-modules and, again by Theorem 2.1.6, this can be extended to a map of cohomological functors

$$\pi^* : H^*(Q, \ ) \to H^*(G, \ ).$$

The *inflation map* is then defined to be the composite

$$H^n(Q, M^K) \xrightarrow{\pi^*} H^n(G, M^K) \xrightarrow{\iota^*} H^n(G, M)$$

where $\iota : M^K \hookrightarrow M$ is the inclusion of M^K in M.

Given these definitions we can now describe the Lyndon-Hochschild-Serre (LHS) spectral sequence for the short exact sequence $K \hookrightarrow G \twoheadrightarrow Q$ and $\mathbb{Z}G$-module M. This consists of abelian groups $E_r^{p,q}$ for $p, q \in \mathbb{Z}$, $2 \leq r \leq \infty$, and homomorphisms (usually called *differentials*) $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,\ q-r+1}$ for $p, q \in \mathbb{Z}$, $2 \leq r < \infty$, such that the following six conditions hold:

(i) For all p, q there is an isomorphism $E_2^{p,q} \cong H^p(Q, H^q(K, M))$.

(ii) For all p, q and all $2 \leq r < \infty$, the composite

$$d_r^{p,q} \circ d_r^{p-r,\ q+r-1} : E_r^{p-r,\ q+r-1} \to E_r^{p+r,\ q-r+1}$$

is zero and there is an isomorphism

$$E_{r+1}^{p,q} \cong \ker(d_r^{p,q})/\mathrm{im}(d_r^{p-r,\ q+r-1}).$$

(iii) In the case where $r > \max\{p,\ q+1\}$ then $E_\infty^{p,q} \cong E_r^{p,q}$.

(iv) For each $n \geq 0$ there is a filtration

$$H^n(G, M) = F_n^n \geq \ldots \geq F_1^n \geq F_0^n \geq 0 = F_{-1}^n$$

such that, for $0 \leq i \leq n$, $F_i^n / F_{i-1}^n \cong E_\infty^{n-i,\ i}$.

(v) For each $n \geq 0$, the inflation map $H^n(Q, M^K) \to H^n(G, M)$ has image $F_0^n \cong E_\infty^{n,0}$.

(vi) For each $n \geq 0$, the restriction map $H^n(G, M) \to H^n(K, M)$ has kernel F_{n-1}^n and image $E_\infty^{0,n} \subseteq H^n(K, M)^Q$.

Now in Section 5.1 it was shown that the equivalence classes of extensions of a group G by a $\mathbb{Z}G$-module M correspond bijectively to the elements of the second cohomology group $H^2(G, M)$, and that in particular the split extension of M by G corresponds to the zero element in $H^2(G, M)$. Suppose that we are in the situation of Theorem 3.1.3. Then

$$A = \mathbb{Z}[q_0, {q_0}^{-1}, {f_1}^{-1}, \ldots, {f_n}^{-1}, k^{-1}],$$

where k is some positive integer and $f_1, \ldots, f_n$ are monic irreducible polynomials in $\mathbb{Z}[q_0]$ that are pairwise coprime as elements of $\mathbb{Z}[q_0, k^{-1}]$. We want to classify all possible extensions of A by Q, where Q is the free abelian subgroup of the group of units of A generated by $k, q_0, f_1, \ldots, f_n$. Suppose first that $k > 1$. Then we write $q_{-1}, q_0, q_1, \ldots, q_n$ for the basis elements of Q acting by multiplication with $k, q_0, f_1, \ldots, f_n$ respectively.

We shall derive the LHS spectral sequence for the short exact sequence

$$\langle q_{-1} \rangle = B \hookrightarrow Q \twoheadrightarrow C = \langle q_0, q_1, \ldots, q_n \rangle,$$

where B is infinite cyclic and C free abelian of rank $n+1$, and use it to calculate the second cohomology group $H^2(Q, A)$.

Now from property (iii) of the sequence we have a filtration

$$H^2(Q, A) = F_2^2 \geq F_1^2 \geq F_0^2 \geq F_{-1}^2 = 0, \tag{5.4}$$

where $F_2^2/F_1^2 \cong E_\infty^{0,2}$, $F_1^2/F_0^2 \cong E_\infty^{1,1}$ and $F_0^2/F_{-1}^2 \cong E_\infty^{2,0}$. Recall from Section 2.2 that

$$H^1(G, M) = H_0(G, M), \qquad H^i(G, M) = 0 \quad (i > 1)$$

whenever G is an infinite cyclic group. The first thing to observe is that both $E_\infty^{0,2}$ and $E_\infty^{2,0}$ are trivial for $k > 1$. Now $E_2^{2,0} \cong H^2(C, H^0(B, A))$. We know that $H^0(B, A)$ is the submodule of A consisting of all fixed points in A under the action of B, which is multiplication by k. If $k > 1$ there are no fixed points and so $H^0(B, A) = 0$. Hence $E_2^{2,0} = 0$: then $E_3^{2,0} = 0$ since it is a quotient of a subgroup of $E_2^{2,0}$. Property (iii) of the LHS spectral sequence then implies that $E_\infty^{2,0} = E_3^{2,0} = 0$.

To show that $E_\infty^{0,2} = 0$, note that $E_2^{0,2} \cong H^0(C, H^2(B, A)) = 0$ since B is infinite cyclic, and then use the same argument as for $E_\infty^{2,0}$.

From the filtration (5.4) we deduce that

$$H^2(Q, A) \cong E_\infty^{1,1}.$$

Now, from property (iii) of the LHS spectral sequence we see that $E_3^{1,1} \cong E_\infty^{1,1}$, but from (ii) it is evident that $E_3^{1,1} \cong E_2^{1,1}$ since cohomology vanishes in negative dimensions and because $E_2^{3,0} = H^3(C, H^0(B, A)) = 0$, as $H^0(B, A) = 0$. Hence

$$H^2(Q, A) \cong E_2^{1,1} \tag{5.5}$$

and so it is sufficient to calculate $E_2^{1,1} \cong H^1(C, H^1(B, A))$.

To do this we repeat the Lyndon-Hochschild-Serre argument and get a spectral sequence from the group extension

$$\langle q_0 \rangle = X \hookrightarrow C \twoheadrightarrow C' = \langle q_1, \ldots, q_n \rangle,$$

with abelian groups ${E'}_r^{p,q}$, and calculate

$${E'}_2^{0,1} \cong H^0(C', H^1(X, H^1(B, A)))$$

and

$${E'}_2^{1,0} \cong H^1(C', H^0(X, H^1(B, A))).$$

From the filtration

$$H^1(C, H^1(B, A)) = {F'}_1^1 \geq {F'}_0^1 \geq {F'}_{-1}^1 = 0$$

we get ${F'}_0^1 \cong {E'}_\infty^{1,0}$. Now

$$H^1(B, A) = H_0(B, A) = A/(k-1)A = (\mathbb{Z}/(k-1)\mathbb{Z})[q_0, q_0^{-1}, f_1^{-1}, \ldots, f_n^{-1}].$$

It follows that

$$H^0(X, H^1(B, A)) = \{ \text{ fixed points in } A/(k-1)A \text{ under the action of } q_0\} = 0,$$

so that ${E'}_2^{1,0} = 0$. Hence ${E'}_\infty^{1,0} = 0$, giving ${F'}_0^1 = 0$, and so

$$H^2(Q, A) = H^1(C, H^1(B, A)) = {E'}_\infty^{0,1} \cong {E'}_2^{0,1}.$$

We then see that

$$H^1(X, H^1(B, A)) = H_0(X, H^1(B, A)) = A/\langle k-1, q_0 - 1\rangle,$$

and so

$$H^2(Q, A) = H^0(C', A/\langle k-1, q_0 - 1\rangle). \tag{5.6}$$

If $k = 2$ this is equal to the trivial group and so $H^2(Q, A) = 0$. In general $H^2(Q, A)$ will be cyclic of order dividing $k-1$, for the following reason. We have

$$\begin{aligned} A/\langle k-1, q_0 - 1\rangle &= (\mathbb{Z}/(k-1)\mathbb{Z})[f_1(1)^{-1}, \ldots, f_n(1)^{-1}] \\ &\cong (\mathbb{Z}/(k-1)\mathbb{Z}) \otimes \mathbb{Z}[1/m], \end{aligned}$$

where m is the least common multiple of the integers $f_1(1), \ldots, f_n(1)$. But this is always cyclic: we have an exact sequence

$$\mathbb{Z} \hookrightarrow \mathbb{Z}[1/m] \twoheadrightarrow D$$

where D is a divisible abelian group, and tensoring with $\mathbb{Z}/(k-1)\mathbb{Z}$ yields an exact sequence

$$\ldots \to \mathbb{Z}/(k-1)\mathbb{Z} \to (\mathbb{Z}/(k-1)\mathbb{Z}) \otimes \mathbb{Z}[1/m] \to (\mathbb{Z}/(k-1)\mathbb{Z}) \otimes D \to 0.$$

Since $(\mathbb{Z}/(k-1)\mathbb{Z}) \otimes D = 0$ it follows that $(\mathbb{Z}/(k-1)\mathbb{Z}) \otimes \mathbb{Z}[1/m]$ is a quotient of $\mathbb{Z}/(k-1)\mathbb{Z}$. It then follows that $H^2(Q, A)$ is cyclic as claimed.

Now suppose $A = A_n$ and $Q = Q_n$. Then $k = n!$ and $f_i = i + q_0$ for $0 \leq i \leq n$. First of all suppose $n = 1$. Then $Q = \langle q_0, q_1 \rangle$ and we have a short exact sequence

$$\langle q_0 \rangle = B \hookrightarrow Q \twoheadrightarrow C = \langle q_1 \rangle. \tag{5.7}$$

Our approach is the same as for the general $k > 1$ case but we derive the LHS spectral sequence for the short exact sequence (5.7) instead. Since B is infinite cyclic we have $H^2(B, A) = 0$ and clearly $H^0(B, A) = 0$. So it suffices to calculate $H^1(B, A)$. But

$$H^1(B, A) = H_0(B, A) = A/\langle q_0 - 1 \rangle = \mathbb{Z}[1/2].$$

Hence $H^2(Q, A) = H^1(C, H^1(B, A))$ is the group of all fixed points in $\mathbb{Z}[1/2]$ under multiplication by 2, which is clearly trivial.

Now assume $n \geq 2$. From (5.6) we have

$$H^2(Q, A) = H^0(C', (\mathbb{Z}/(n! - 1)\mathbb{Z})[(n+1)^{-1}]),$$

since $2, 3, \ldots, n$ are coprime to $n! - 1$ and so are all units modulo $n! - 1$. Here C' acts via multiplication by the positive integers $2, 3, \ldots, n+1$. For $n = 2$ $H^2(Q, A)$ clearly vanishes. For $n > 2$ it is easy to see that there are no non-trivial fixed points under the action of C', since $2x = x$ implies that $x = 0$. Hence $H^2(Q, A) = 0$.

Now consider what happens when $k = 1$. Here we use the short exact sequence

$$\langle q_0 \rangle = B \hookrightarrow Q \twoheadrightarrow C = \langle q_1, \ldots, q_n \rangle.$$

In a similar way to the above we find that $H^2(Q, A) \cong E_2^{1,1} = H^1(C, H^1(B, A))$. To calculate $H^1(C, H^1(B, A))$ we derive the LHS spectral sequence from the short exact sequence

$$\langle q_1 \rangle = X \hookrightarrow C \twoheadrightarrow C' = \langle q_2, \ldots, q_n \rangle.$$

Analogously to the $k > 1$ case this involves calculating the groups $H^0(C', H^1(X, K))$ and $H^1(C', H^0(X, K))$ where $K = H^1(B, A)$. However it is not necessarily the case that $H^0(X, K)$, and so by implication $H^1(C', H^0(X, K))$, is trivial when $X = \langle q_1 \rangle$ acts by multiplication with f_1. We find that

$$K = H^1(B, A) = H_0(B, A) = A/\langle q_0 - 1 \rangle = \mathbb{Z}[f_1(1)^{-1}, \ldots, f_n(1)^{-1}].$$

Now X acts on K by multiplication with the integer $f_1(1)$ and so $H^0(X, K)$ is trivial unless $f_1(1) = 1$, in which case $H^0(X, K) = K$. Suppose $f_1(1) \neq 1$ and write $l = |f_1(1) - 1|$. Then

$$H^1(X, K) = K/\langle f_1(1) - 1 \rangle = (\mathbb{Z}/l\mathbb{Z})[f_2(1)^{-1}, \ldots, f_n(1)^{-1}],$$

which is cyclic of order dividing l by the same argument used for (5.6). It follows that $H^2(Q, A) = H^0(C', H^1(X, K))$ is also cyclic of order dividing l.

Suppose instead that $f_1(1) = 1$. Then $H^0(X, K) = K$ and since $H^1(X, K)$ is also non-trivial we are unable to determine $H^2(Q, A)$ in the absence of further information about the polynomials $f_1, \ldots, f_n$ in $\mathbb{Z}[q_0]$. For example, as shown in Theorem 5.2.1 we know that cohomology vanishes whenever $A = A_1$ and $Q = Q_1$.

We can summarize all we have proved in this theorem:

Theorem 5.2.1. *Let $A = \mathbb{Z}[q_0, {q_0}^{-1}, {f_1}^{-1}, \ldots, {f_n}^{-1}, k^{-1}]$, where k is some positive integer and $f_1, \ldots, f_n$ are monic irreducible polynomials in $\mathbb{Z}[q_0]$ that, together with q_0, are pairwise coprime as elements of $\mathbb{Z}[q_0, k^{-1}]$. Let Q be the free abelian subgroup of the group of units of A generated by $k, q_0, f_1, \ldots, f_n$. Then the second cohomology group $H^2(Q, A)$ is cyclic of order dividing $k - 1$ whenever $k \geq 2$. For $k = 1$ it is cyclic of order dividing l whenever $f_1(1) \neq 1$, where $l = |f_1(1) - 1|$. In the special case of $A = A_n$ and $Q = Q_n$, the second cohomology group vanishes for all $n \geq 1$.*

5.3 Further problems

In this final section we take a look at some examples of cohomology groups $H^2(Q, A)$ which are still to be determined, and also consider properties of the higher cohomology groups $H^m(Q, A)$ for $m > 2$.

Let us first give an example of an extension of groups A by Q satisfying the conditions of Theorem 3.1.3, and so also Theorem 5.2.1. Perhaps the simplest is when

$$A = \mathbb{Z}[x, x^{-1}, (k + x)^{-1}, k^{-1}]$$

and Q is the free abelian subgroup $\langle k, x, k+x \rangle$ of the group of units $A^{\times}$, where $k \geq 1$ is some positive integer. Clearly x and $k + x$ are coprime over $\mathbb{Z}[x, k^{-1}]$ since their difference is invertible, and so by Theorem 3.1.3 the metabelian group $G = A \rtimes Q$ is of type FP_2, hence finitely presented. Note that the case where $k = 1$ and the extension is split gives us the Baumslag group G_1.

From (5.6) we see that the second cohomology group is

$$H^2(Q, A) = H^0(\langle k + 1 \rangle, (\mathbb{Z}/(k - 1)\mathbb{Z})[(k + 1)^{-1}]),$$

in other words the set of fixed points in $(\mathbb{Z}/(k-1)\mathbb{Z})[(k+1)^{-1}]$ under multiplication by $k + 1$. Now $k + 1 \equiv 2 \mod (k - 1)$, so the action is given by multiplication by 2, which has no non-trivial fixed points. Hence the second cohomology group vanishes, and so every extension of A by Q is split.

In fact, there would appear to be no example of a non-split extension for the groups in Theorem 5.2.1. Hence it is our belief that Theorem 5.2.1 could be improved to state that $H^2(Q, A)$ is always trivial whenever A and Q satisfy the conditions of Theorem 3.1.3. If this is true it would ease the difficulty of the Bieri-Groves FP_m-conjecture considerably, as the non-split case of the conjecture has been by far the harder to make progress with.

To illustrate the problems in finding non-split extensions, let's generalize the example above, and indeed the case $A = A_n$, $Q = Q_n$. Let $b_1, \ldots, b_n, k$ be positive integers and take

$$A = \mathbb{Z}[x, x^{-1}, (b_1 + x)^{-1}, \ldots, (b_n + x)^{-1}, k^{-1}]$$

to be the $\mathbb{Z}Q$-module acted on by the free abelian subgroup

$$Q = \langle k, x, b_1 + x, \ldots, b_n + x\rangle$$

of the group of units of A. Here we define k to be the least common multiple of the differences $b_i - b_j$ for $i \neq j$, $b_i > b_j$, where $b_0 = 0$. The metabelian group $G = A \rtimes Q$ then clearly satisfies the conditions stated in Theorem 3.1.3, and so G is of type FP_{n+1}.

Now let's calculate $H^2(Q, A)$. Since $k = 1$ implies that $n = 1$ and $b_1 = 1$, giving $A = A_1$, we can assume that $k \geq 2$. First of all note that k must be even, since for $i \geq 1$ either at least one of the b_i is even or they are all odd, in which case $b_j - b_i$ is always even for $1 \leq i \neq j \leq n$. Suppose $n = 2$. Then $H^2(Q, A)$ is the set of fixed points in $(\mathbb{Z}/(k-1)\mathbb{Z})[(b_1+1)^{-1}, (b_2+1)^{-1}]$ under multiplication by b_1+1 and b_2+1. The question to consider is: are b_1+1 and b_2+1 always coprime to $k-1$? If so, they are always units modulo $k-1$ and so

$$H^2(Q, A) = H^0(C', \mathbb{Z}/(k-1)/\mathbb{Z}).$$

Then, if either b_1+1 or b_2+1 is even the cohomology will vanish, since $k-1$ is odd. The simplest case is where the least common multiple of b_1, b_2 and $b_2 - b_1$ is their product but we have been unable to answer the question even for this case.

Recall from (5.5) that $H^2(Q, A) = E_2^{1,1}$ when Q and A are as in Theorem 5.2.1. The higher cohomology groups $H^m(Q, A)$, for $m > 2$, can also be expressed in a simple fashion. Given the short exact sequence

$$\langle q_{-1}\rangle = B \hookrightarrow Q \twoheadrightarrow C = \langle q_0, q_1, \ldots, q_n\rangle$$

we have $H^i(B, A) = 0$ whenever $i \neq 1$ and so, by property (i) of the LHS spectral sequence, $E_2^{m-i,i} = 0$ for $i \neq 1$. It follows from property (ii) that $E_\infty^{m-i,i} = 0$ for $i \neq 1$ and so by using the filtration in property (iv) we get

$$H^m(Q, A) = E_2^{m-1,1}.$$

The group $E_2^{m-1,1}$ can then be studied using essentially the same arguments as those used to find $E_2^{1,1}$. When $m \geq n+3$ the groups $H^m(Q, A)$ vanish since the free abelian group Q has rank $n+2$.

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