APPLICATIONS OF PFAFFIANS TO SOLITON THEORY

by

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Summary

This thesis is concerned with solutions to nonlinear evolution equations. In particular we examine two soliton equations, namely the Novikov-Veselov-Nithzik (NVN) equations and the modified Novikov-Veselov-Nithzik (mNVN) equations. We are interested in the role that determinants and pfaffians play in determining new solutions to various soliton equations. The thesis is organised as follows.

In chapter 1 we give an introduction and historical background to the soliton theory and recall John Scott Russell's observation of a solitary wave, made in 1844. We explain the Lax method and Hirota method and discuss the relevant basic topics of soliton theory that are used throughout this thesis. We also discuss different types of solutions that are applicable to nonlinear evolution equations in soliton theory. These are wronskians, grammians and pfaffians.

In chapter 2 we give an introduction to pfaffians which are the main elements of this thesis. We give the definition of a pfaffian and a classical notation for the pfaffians is also introduced. We discuss the identities of pfaffians which correspond to the Jacobi identity of determinants. We also discuss the differentiation of pfaffians which is useful in pfaffian technique. By applying the pfaffian technique to the BKP equation, an example of soliton solutions to the BKP equation is also given.

In chapter 3 we study the asymptotic properties of dromion solutions written in terms of pfaffians. We apply the technique that is used in [35] for the Davey-Stewartson (DS) equations to the NVN equations. We study the asymptotic properties of the $(1,1)$-dromion solution and generalize them to the (M,N)-dromion solution. Summaries of these asymptotic properties are given. As an application, we apply the general results obtained for the (M,N)-dromion solution to the $(2,2)$-dromion solution and to the $(1,2)$-dromion solution and show the asymptotic calculations explicitly for each

dromion. In the last section we give a number of plots which show various kind of dromion scattering. These illustrate that dromion interaction properties are different than the usual soliton interactions.

In chapter 4 we exploit the algebraic structure of the soliton equations and find solutions in terms of fermion particles [54]. We show how determinants and pfaffians arise naturally in the fermionic approach to soliton equations. We write the τ-function for charged and neutral free fermions in terms of determinants and pfaffians respectively, and show that these two concepts are analogous to one another. Examples of how to get soliton and dromion solutions from τ-functions for the various soliton equations are given.

In chapter 5 we use some results from [61] and [62]. We study two nonlinear evolution equations, namely the Konopelchenko-Rogers (KR) equations and the modified Novikov-Veselov-Nithzik (mNVN) equations. We derive a new Lax pair for the mNVN equations which is gauge equivalent to a pair of operators. We apply the pfaffian technique to the KR and mNVN equations and show that these equations in the bilinear form reduce to a pfaffian identity.

In this thesis, chapter 1 is a general introduction to soliton theory and chapter 2 is an introduction to the main elements of this thesis. The contents of these chapters are taken from various references as indicated throughout the chapters. Chapters 3, 4, 5 are the author's own work with some results used from other references also indicated in the chapters.

Contents

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Chapter 1

Introduction

1.1 The First Observation of a Soliton

Over one hundred and fifty years ago, while conducting experiments to determine the most efficient design for canal boats, a young Scottish engineer named John Scott Russell (1808-1882) made a remarkable scientific discovery. As he described it in his "Report on Waves" [1] in 1844 :

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation".

Following this discovery, Scott Russell built a wave tank in his laboratory, and in order to study this phenomenon more carefully, he made further important observations of the properties of the solitary wave.

1.2 The Discovery of the Soliton and the KdV equation

Fifty years later in 1895, after extensive investigations, two Dutchmen, Korteweg and de Vries, developed a nonlinear partial differential equation governing long one dimensional, small amplitude, surface gravity waves propagating in a shallow channel of water. Their aim was to model the propagation of shallow water waves applicable to situation that Scott Russell saw. (The attempt, in 1982, to recreate the phenomena on the original site (Union canal) was a complete failure, but a more recent attempt, in 1995, to recreate the soliton wave was successful. See the URL: http://www.ma.hw.ac.uk/solitons.) This famous equation is known as the KdV equation [2] (named after Korteweg and de Vries)

$$U_t + 6UU_x + U_{xxx} = 0. \tag{1.1}$$

One of the interesting properties of the KdV equation is the existence of permanent wave solutions. To obtain a travelling wave solution of the KdV equation, we seek a solution in the following form

$$U(x,t) = W(x-ct) = W(z).$$

Substituting this into the KdV equation in (1.1) yields a third order ordinary differential equation

$$W''' + 6WW' - cW' = 0,$$

where $' \equiv \frac{d}{dz}$. Integrating this twice gives

$$\tfrac{1}{2}(W')^2 = -W^3 + \tfrac{1}{2}cW^2 + AW + B,$$

where A, B are constants. If we add the boundary conditions $W, W', W'' \to 0$ as $z \to \pm\infty$, the constants of integration are zero, and we have

$$\tfrac{1}{2}(W')^2 = -W^3 + \tfrac{1}{2}cW^2.$$

Solving this differential equation gives us the solitary wave solution

$$U(x,t) = W(z) = \tfrac{c}{2}\,\mathrm{sech}^2(-\tfrac{\sqrt{c}}{2}(x - ct)).$$

If we take $-\frac{\sqrt{c}}{2} = k$, then the solution can be written as

$$U(x,t) = 2k^2\,\mathrm{sech}^2(k(x - 4k^2 t)), \tag{1.2}$$

where k is the wave number. The speed is $c = 4k^2$ and the amplitude is $a = 2k^2$. A key feature of KdV is that the speed of solitary waves is proportional to their height. Therefore taller waves travel faster than shorter waves and there is a tall, fast, thin solitary wave or (it could be) a small, slow, fat solitary wave. The obvious question then arises: what happens when a taller solitary wave overtakes a shorter (and therefore slower) solitary wave; in particular, do the individual pulses survive the collision? The answer to this question was not known until 1965 when Norman Zabusky and Martin Kruskal [3] discovered numerically that KdV solitary waves maintained their identity following collisions, and reported that "here we have a nonlinear physical process in which interacting localized pulses do not scatter irreversibly." They considered the initial-value problem for the KdV equation

$$U_t + UU_x + \delta^2 U_{xxx} = 0$$
$$U(x,0) = \cos(\pi x) \qquad 0 \leqslant x \leqslant 2$$

and took $\delta = 0.022$. They discovered that after a short time the wave steepens and almost produces a shock, and later a train of at least eight (well-defined) solitary waves develop with the faster waves overtaking the slower waves. When two solitary waves given in the form of (1.2), with different speeds and are initially well separated with the faster one behind the slower one, the faster wave overlaps the slower wave and the waves interact nonlinearly. After the interaction, the waves separate with the larger one in front of the smaller one, and have their initial profiles. The only effect of the interaction is the phase shifts, that is the waves are at different positions than where they would have been. Zabusky and Kruskal coined the term 'soliton' to reflect the particle-like nature of these robust travelling solitary waves.

Definition 1.2.1 (Soliton) *A soliton is a solution of a nonlinear equation or system which represents a wave of permanent form, is localized and decaying at infinity and interacts with other solitons so that after the interaction it retains its form.*

1.3 Conservation Laws

An important stage in the development of the general method of solution for the KdV equation was the discovery that the KdV equation had an infinite number of independent conservation laws. A conservation law is an equation of the form

$$\frac{\partial T}{\partial t} + \frac{\partial X}{\partial x} = 0,$$

where T is density and X is the associated flux. If additionally $X \to 0$ as $|x| \to \infty$ then

$$\frac{d}{dt}\int_{-\infty}^{\infty} T(x,t)dx = -[X]_{-\infty}^{\infty} = 0.$$

Therefore

$$\int_{-\infty}^{\infty} T(x,t)dx = C$$

for all time, and so C is conserved. The first three conservation laws for the KdV equation are

$$\begin{aligned} U_t + (3U^2 + U_{xx})_x &= 0, \\ (U^2)_t + (4U^3 + 2UU_{xx} - U_x^2)_x &= 0, \\ (U^3 - \tfrac{1}{2}U_x^2)_t + (\tfrac{9}{2}U^4 + 3U^2U_{xx} - 6UU_x^2 - U_xU_{xxx} + \tfrac{1}{2}U_{xx}^2)_x &= 0. \end{aligned}$$

The first two of these conservation laws correspond to conservation of mass and momentum respectively. The third was discovered by Whitham [4] in 1965. The fourth and fifth conservation laws for the KdV equation were found by Kruskal and Zabusky [5] in 1963. Later four more conservation laws were found and subsequently Miura found the tenth conservation law for the KdV equation. The reason that they are called first, second etc. is due to the highest order of the density, for example, the conserved densities for the first three conservation laws are $T_1 = U$, $T_2 = U^2$ and $T_3 = U^3 - \frac{1}{2}U_x^2$.

After studying the conservation laws of the KdV equation, in 1968, Miura [7] discovered the following transformation, now known as *Miura's transformation*:

$$U = -(V^2 + V_x). \tag{1.3}$$

If V is a solution of the modified Korteweg-de Vries (mKdV) equation

$$V_t - 6V^2V_x + V_{xxx} = 0,$$

then U given by the Miura transformation (1.3) is a solution of the KdV equation (1.1). This can be seen from the relation

$$U_t + 6UU_x + U_{xxx} = -(2V + \frac{\partial}{\partial x})(V_t - 6V^2V_x + V_{xxx}). \tag{1.4}$$

Every solution of the KdV equation can be obtained from a solution of the mKdV equation via Miura's transformation, but the converse is not true. Miura's transformation leads to many other important results related to the KdV equation. Initially it formed the basis of a proof that the KdV and mKdV equations have an infinite number of conservation laws [8]. Let W be such that

$$U = W - \epsilon W_x - \epsilon^2 W^2, \tag{1.5}$$

which is called the *Gardner transformation* and may be thought of as generalization of Miura's transformation (1.3). Then the equivalent relation to (1.4) is

$$U_t + 6UU_x + U_{xxx} = (1 - \epsilon\frac{\partial}{\partial x} - 2\epsilon^2 W)(W_t + 6(W - \epsilon^2 W^2)W_x + W_{xxx}). \tag{1.6}$$

Hence, U given by equation (1.5), is a solution of the KdV equation if W is a solution of

$$W_t + 6(W - \epsilon^2 W^2)W_x + W_{xxx} = 0, \tag{1.7}$$

and again the converse is not true. It is clear that if we set $\epsilon = 0$ then (1.7) becomes the KdV equation and the Gardner transformation reduces to $U = W$. Since the KdV equation does not contain ϵ, then its solution U depends only on x and t; however W, a solution of equation (1.7), depends on x, t and ϵ. Then, in order to generate conservation laws for the KdV equation, we take a power series solution of (1.5) in the form

$$W(x,t;\epsilon) = \sum_{n=0}^{\infty} \epsilon^n W_n(x,t). \tag{1.8}$$

Since the equation (1.7) is in conservation form

$$\frac{\partial}{\partial t}(W) + \frac{\partial}{\partial x}(W_{xx} + 3W^2 - 2\epsilon^2 W^3) = 0,$$

then

$$\int_{-\infty}^{\infty} W(x,t;\epsilon)dx = \text{constant},$$

and so the power series in ϵ

$$\int_{-\infty}^{\infty} W_n(x,t)dx = \text{constant},$$

for each $n = 0, 1, 2, \ldots$. Substituting (1.8) into (1.5) and equating coefficients of powers of ϵ and solving recursively gives

$$\begin{aligned} W_0 &= U, \\ W_1 &= W_{0,x} = U_x, \\ W_2 &= W_{1,x} + W_0^2 = U_{xx} + U^2, \\ W_3 &= W_{2,x} + 2W_0W_1 = U_{xxx} + 4UU_x, \\ W_4 &= W_{3,x} + 2W_0W_2 + W_1^2 = U_{xxxx} + 6UU_{xx} + 5U_x^2 + 2U^3, \end{aligned} \tag{1.9}$$

etc.. Continuing to all powers of ϵ gives an infinite number of conserved densities. The corresponding conservation laws can be found by substituting (1.8) and (1.9) into equation (1.7) and equating coefficients of powers of ϵ. In particular we note that each odd power of ϵ gives an exact derivative and the corresponding integral repeats an earlier conservation law. However the even powers of ϵ give independent conservation laws for the KdV equation.

1.4 Lax Method

Shortly after the discovery of the "soliton", in 1967 Gardner, Greene, Kruskal and Miura [6] discovered a new method of solution for the KdV equation by making use of the ideas of direct and inverse scattering. They termed the procedure the inverse-scattering-transform (IST) method.

In 1968 Lax [9] put the inverse-scattering-transform method for solving the KdV equation into a more general framework which subsequently generalized as a method for solving other partial differential equations. Let us consider two operators L and M, where L is the operator of the spectral problem and M is the operator governing the associated time evolution of the eigenfunctions

$$Lv = \lambda v, \tag{1.10}$$

$$v_t = Mv. \tag{1.11}$$

Now we differentiate (1.10) with respect to t to get

$$L_t v + L v_t = \lambda_t v + \lambda v_t,$$

and hence, using (1.11), we get

$$\begin{aligned} L_t v + LMv &= \lambda_t v + \lambda M v, \\ &= \lambda_t v + M\lambda v, \\ &= \lambda_t v + MLv. \end{aligned}$$

Therefore we obtain

$$[L_t + (LM - ML)]v = \lambda_t v$$

and if we solve this equation for nontrivial eigenfunctions $v(x,t)$ we obtain

$$L_t + [L, M] = 0, \tag{1.12}$$

where

$$[L, M] := LM - ML, \tag{1.13}$$

if and only if $\lambda_t = 0$. Equation (1.12) is called *Lax's equation*, and (1.13) is the commutator. The Lax equation contains a nonlinear evolution equation for suitably chosen L and M. For example if we take

$$L = \frac{\partial^2}{\partial x^2} + U, \tag{1.14}$$

$$M = -4\frac{\partial^3}{\partial x^3} - 3U\frac{\partial}{\partial x} - 3\frac{\partial}{\partial x}U, \tag{1.15}$$

then L and M satisfy (1.12) provided that U satisfies the KdV equation (1.1). Therefore, the KdV equation can be thought of as the compatibility condition of the two linear operators given by (1.14) and (1.15). If a nonlinear partial differential equation arises as the compatibility condition of two such operators L and M, then (1.12) is called the *Lax representation* of the partial differential equation and L and M are the *Lax Pair.*

1.5 Hirota Bilinear Method

In 1971 Hirota [10] developed a direct method for finding N-soliton solutions of nonlinear evolution equations, which are obtained from a nonlinear evolution equation via a dependent variable transformation. This method is called the *Hirota bilinear method* and has application to a large class of nonlinear evolution equations [29].

Here we show how the KdV equation can be written in the Hirota bilinear form and hence we find soliton solutions of the KdV equation [11].

We start by introducing the dependent variable transformation in the following form:

$$U = 2\frac{\partial^2}{\partial x^2}\log f, \tag{1.16}$$

where $f(x,t)$ is a new dependent variable. Substituting (1.16) into (1.1), integrating twice with respect to x and setting integration constants to zero, we obtain

$$f_{xt}f - f_x f_t + f_{xxxx}f - 4f_{xxx}f_x + 3f_{xx}^2 = 0, \tag{1.17}$$

which is the bilinear form of the KdV equation. This equation can be expressed in terms of Hirota bilinear derivatives in the following way

$$(D_x D_t + D_x^4)f \cdot f = 0, \tag{1.18}$$

where D_x and D_t are Hirota operators. These are defined in more general form by

$$D_x^l D_y^m D_t^n g \cdot f =$$
$$\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^l \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n g(x,y,t)f(x',y',t') \,\Big|_{\substack{x'=x\\ y'=y\\ t'=t}}$$

for nonnegative integers l, m, n. To find soliton solutions of the KdV equation from the Hirota form (1.18) we look for solutions in the form

$$f = 1 + \sum_{n=1}^{N} \epsilon^n f^{(n)}. \tag{1.19}$$

Substituting (1.19) into (1.18) and equating coefficients of powers of ϵ gives the following recursion relations:

$$\begin{aligned}
\epsilon: &\quad f_{xxxx}^{(1)} + f_{xt}^{(1)} &&= 0, \\
\epsilon^2: &\quad f_{xxxx}^{(2)} + f_{xt}^{(2)} &&= -\tfrac{1}{2}(D_x D_t + D_x^4)f^{(1)} \cdot f^{(1)}, \\
\epsilon^3: &\quad f_{xxxx}^{(3)} + f_{xt}^{(3)} &&= -(D_x D_t + D_x^4)f^{(1)} \cdot f^{(2)},
\end{aligned} \tag{1.20}$$

and so on. The N-soliton solution for the KdV equation is found by assuming that $f^{(1)}$ has the form

$$f^{(1)} = \sum_{i=1}^{N} \exp(\eta_i),$$

where $\eta_i = 2k_i x - w_i t + x_{i0}$ and k_i, $w_i = 8k_i^3$ and x_{i0} are constants. Then the N-soliton solution can be written in the bilinear form in the following form:

$$f_N = \sum_{\mu=0,1} \exp\left[\sum_{i=1}^{N} \mu_i \eta_i + \sum_{i>j}^{N} \mu_i \mu_j A_{ij}\right],$$

where

$$\exp(A_{ij}) = \left(\frac{k_i - k_j}{k_i + k_j}\right)^2.$$

For $N = 1$, we take

$$f^{(1)} = \exp(\eta_1),$$

and by solving (1.20) we find that

$$f^{(n)} = 0, \qquad \text{for} \qquad n \geqslant 2.$$

Therefore we have

$$f_1 = 1 + \exp(\eta_1)$$

and substituting this into (1.16), we get the same solution as in (1.2) apart from a phase constant $x_{1,0}$

$$U(x,t) = 2k_1^2 \operatorname{sech}^2(k_1 x - k_1^3 t + x_{1,0}),$$

the one-soliton solution for the KdV equation.

For $N = 2$, we take

$$f^{(1)} = \exp(\eta_1) + \exp(\eta_2),$$

and by solving (1.20) we find that

$$f^{(2)} = \exp(\eta_1 + \eta_2 + A_{12}),$$

where

$$\exp(A_{12}) = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2,$$

and

$$f^{(n)} = 0, \qquad \text{for} \qquad n \geqslant 3.$$

Therefore the two-soliton solution for the KdV equation is obtained from

$$U(x,t) = 2\frac{\partial^2}{\partial x^2}(\log f_2),$$

where

$$f_2 = 1 + \exp(\eta_1) + \exp(\eta_2) + \exp(\eta_1 + \eta_2 + A_{12}).$$

1.6 Wronskian, Grammian and Pfaffian Solutions

More recently the series ansatz (1.19) has been replaced in the Hirota method by expressions in terms of wronskians [12], grammians [15] or pfaffians [44]. Wronskians and grammians are special types of determinants and will be explained below. Pfaffians are explained in the next chapter. There are two main advantages of these types of solution over the series ansatz; they allow much easier methods for verifying solutions and are a convenient form in which to study the asymptotic properties of solutions. The method of solution in terms of wronskians, grammians or pfaffians can be applied to many nonlinear partial differential equations, and give rise to different type solutions, for instance soliton solutions, lump solutions and dromion solutions.

1.6.1 Wronskian Solutions

Writing the solution of evolution equations in terms of wronskians has the advantage of avoiding long complicated calculations, especially when verifying the N-soliton solution. The wronskians have nice properties when differentiated since each row of a wronskian is the derivative of the previous one. Therefore the derivative of a wronskian is a single determinant. Higher derivatives lead to sums of determinants that depend on the number of differentiations and not the number N of solitons. For example we consider the Kadomtsev-Petviashvili (KP) [37] equation

$$(U_t + 6UU_x + U_{xxx})_x + 3U_{yy} = 0 \tag{1.21}$$

which has the solution

$$U = 2\frac{\partial^2}{\partial x^2}\log f. \tag{1.22}$$

Substitution of (1.22) in (1.21) gives

$$ff_{xt} - f_x f_t + ff_{xxxx} + 3f_{xx}^2 - 4f_x f_{xxx} + 3ff_{yy} - 3f_y^2 = 0, \tag{1.23}$$

and this can be written in Hirota form as

$$(D_x D_t + D_x^4 + 3D_y^2)f \cdot f = 0. \tag{1.24}$$

Next we introduce the wronskian determinant, namely

$$f = W(\phi_1, \phi_2, \ldots, \phi_N),$$

where the wronskian W of the elements $\phi_i \quad (i = 1, \dots, N)$ is defined as

$$W(\phi_1, \phi_2, \dots, \phi_N) = \begin{vmatrix} \phi_1 & \cdots\cdots & \phi_N \\ \phi_1^{(1)} & \cdots\cdots & \phi_N^{(1)} \\ \vdots & & \vdots \\ \phi_1^{(N-1)} & \cdots\cdots & \phi_N^{(N-1)} \end{vmatrix}, \quad \text{with} \quad \phi_i^{(j)} = \frac{\partial^j \phi_i}{\partial x^j}$$

and ϕ_i satisfies the partial differential equations

$$\begin{aligned} \frac{\partial \phi_i}{\partial y} - \frac{\partial^2 \phi_i}{\partial x^2} &= 0, \\ \frac{\partial \phi_i}{\partial t} + 4\frac{\partial^3 \phi_i}{\partial x^3} &= 0. \end{aligned}$$

In order to make the differentiation of wronskians easier, it is convenient to use the following notation for f

$$\begin{aligned} f &= \begin{vmatrix} \phi_1 & \cdots\cdots & \phi_1^{(N-1)} \\ \phi_2 & \cdots\cdots & \phi_2^{(N-1)} \\ \vdots & & \vdots \\ \phi_N & \cdots\cdots & \phi_N^{(N-1)} \end{vmatrix} \\ &= \left|\phi \cdots\cdots \phi^{(N-1)}\right|. \end{aligned} \tag{1.25}$$

(See [12] for details). To obtain the soliton solutions we take

$$\phi_i = e^{-k_i x + k_i^2 y + 4k_i^3 t} + \alpha_i e^{k_i x + k_i^2 y - 4k_i^3 t}, \quad (i = 1, \dots, N) \tag{1.26}$$

where α_i are constants. Differentiating f in (1.25) and substituting the derivatives into the left hand side of (1.23) gives a $2N \times 2N$ determinant, which vanishes by virtue of Laplace expansion. Thus, using the relation in (1.26), (1.22) gives the N-soliton solutions for the KP equation (1.21).

1.6.2 Grammian Solutions

Grammian solutions are the determinants of a matrix whose elements are in an integral form. The grammian method is much more practical than the wronskian method, since the Nth order wronskian solution requires by definition $(N-1)$ differentiations, whereas the Nth order grammian needs only one integration. In this method the solution is expressed in terms of grammian determinants and is verified by using the

Jacobi identity. For example for the KP equation (1.21), the solution f is given in the following determinantal form

$$f = |f_{ij}| \tag{1.27}$$

with the entries

$$f_{ij} = c_{ij} + \int_{-\infty}^{x} \phi_i \psi_j dx,$$

where c_{ij} are constants and the ϕ_i and ψ_j are functions of x, y, t and satisfy the following linear partial differential equations

$$\begin{aligned}
(4\partial_x^3 - \partial_t)\phi_i &= 0, \\
(4\partial_x^3 - \partial_t)\psi_j &= 0, \\
(3\partial_x^2 + \partial_y)\psi_j &= 0, \\
(3\partial_x^2 - \partial_y)\phi_i &= 0.
\end{aligned}$$

(See [15] for details and that f given in (1.27) satisfies the bilinear equation (1.23) by virtue of a Jacobi identity.)

There is a similarity between grammian solutions and pfaffian solutions. The grammian solution of a soliton equation reduces to a Jacobi identity, whereas the pfaffian solution reduces to a pfaffian identity. These identities are described in the next chapter. The derivative of grammians and pfaffians also have similar structures. The derivative of grammians can be expressed in terms of bordered determinants. These expressions arise because, in general, for an $n \times n$ matrix A whose entries a_{ij} are such that $a'_{ij} = \alpha_i \beta_j$, the derivative of its determinant can be written as

$$\begin{aligned}
|A|' &= \sum_{i,j=1}^{n} (-1)^{i+j} \alpha_i \beta_j A_{ij} \\
&= - \begin{vmatrix} 0 & \beta_1 & \cdots & \beta_n \\ \alpha_1 & & & \\ \vdots & & A & \\ \alpha_n & & & \end{vmatrix},
\end{aligned}$$

where A_{ij} is the (i,j)th minor of A. The derivatives of pfaffians are explained in the next chapter.

Chapter 2

Introduction to Pfaffians

Roughly speaking, a pfaffian is the square root of the determinant of a skew-symmetric matrix. Let

$$A = (a_{ij})$$

be a $n \times n$ skew-symmetric matrix (i.e. $a_{ij} = -a_{ji}$ and consequently $a_{ii} = 0$ for $i, j = 1, 2, \cdots, n$). It is known that if n is odd, then $\det(A)$ is zero, but if n is even $\det(A)$ is a perfect square of a polynomial in the entries a_{ij}, called the *pfaffian* of A and denoted by $\mathrm{Pf}(A)$. To be precise, for even n

$$\mathrm{Pf}(A) = \sum_{\sigma} \epsilon(\sigma)\, a_{\sigma(1),\sigma(2)} \cdots a_{\sigma(n-1),\sigma(n)},$$

where σ runs over the permutations of $\{1, \cdots, n\}$ such that

$$\sigma(1) < \sigma(2), \sigma(3) < \sigma(4), \cdots, \sigma(n-1) < \sigma(n),$$

$$\sigma(1) < \sigma(3) < \cdots < \sigma(n-1),$$

and $\epsilon(\sigma)(= \pm 1)$ is the parity of this permutation. For example, if we take the dimension of matrix A to be 4 we have,

$$\det(A) = \begin{vmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{vmatrix} = (a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})^2$$

and we can write the pfaffian of A as a triangular array and expand it as

$$\mathrm{Pf}(A) = \left| \begin{array}{ccc} a_{12} & a_{13} & a_{14} \\ & a_{23} & a_{24} \\ & & a_{34} \end{array} \right| = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}.$$

A classical notation for the pfaffian of A [53] is

$$\mathrm{Pf}(A) = (1, 2, \cdots, n),$$

where $(i, j) = a_{ij}$. One expansion rule for pfaffians is given by

$$(1, 2, \cdots, n) = \sum_{i=2}^{n} (-1)^i (1, i)(2, 3, \cdots, \hat{i}, \cdots, n),$$

where $\hat{}$ indicates that the index underneath should be deleted. We can write the example above with pfaffian representation as

$$(1, 2, 3, 4) = (1, 2)(3, 4) - (1, 3)(2, 4) + (1, 4)(2, 3).$$

2.1 Identities of Pfaffians

Identities of pfaffians correspond to the Jacobi identity of determinants. The Jacobi identity is given as follows: for an $N \times N$ matrix A, we write $A^{i,\ldots,j}_{k,\ldots,l}$ for the minor obtained by omitting the ith,..., jth rows and the kth,..., lth columns, in this notation the Jacobi identity is

$$|A|\, A^{i,j}_{k,l} = \begin{vmatrix} A^i_k & A^j_k \\ A^i_l & A^j_l \end{vmatrix}.$$

If, for example, we take

$$A = \begin{pmatrix} P & a & b \\ c^T & 0 & 0 \\ d^T & 0 & 0 \end{pmatrix}$$

with $\{i, j\} = \{k, l\} = \{2, 3\}$, then we get the following identity

$$\begin{vmatrix} P & a & b \\ c^T & 0 & 0 \\ d^T & 0 & 0 \end{vmatrix} |P| = \begin{vmatrix} P & a \\ c^T & 0 \end{vmatrix} \begin{vmatrix} P & b \\ d^T & 0 \end{vmatrix} - \begin{vmatrix} P & a \\ d^T & 0 \end{vmatrix} \begin{vmatrix} P & b \\ c^T & 0 \end{vmatrix},$$

where P is a square matrix, and a, b, c, d are vectors.

Let m and n be positive integers. For the even case (even number of a_i) we have the following pfaffian identity

$$\begin{aligned}&(a_1, a_2, \cdots, a_{2m}, 1, 2, \cdots, 2n)(1, 2, \cdots, 2n)\\ &= \sum_{s=2}^{2m}(-1)^s(a_1, a_s, 1, 2, \cdots, 2n)(a_2, a_3, \cdots, \hat{a}_s, \cdots, a_{2m}, 1, 2, \cdots, 2n),\end{aligned} \tag{2.1}$$

and for the odd case (odd number of a_i)

$$\begin{aligned}&(a_1, a_2, \cdots, a_{2m-1}, 1, 2, \cdots, 2n-1)(1, 2, \cdots, 2n)\\ &= \sum_{s=1}^{2m-1}(-1)^{s-1}(a_s, 1, 2, \cdots, 2n-1)(a_1, a_2, \cdots, \hat{a}_s, \cdots, a_{2m-1}, 1, 2, \cdots, 2n),\end{aligned} \tag{2.2}$$

where a_i are just extra indices in the same way that the a, b, c^T, d^T are extra columns and rows. (See [60] for the proof of the identities (2.1), (2.2).)

For example from (2.1) and (2.2), for $m = 2$, we have the following pfaffian identities

$$\begin{aligned}&(a_1, a_2, a_3, a_4, 1, 2, \cdots, 2n)(1, 2, \cdots, 2n)\\ &= (a_1, a_2, 1, 2, \cdots, 2n)(a_3, a_4, 1, 2, \cdots, 2n)\\ &-(a_1, a_3, 1, 2, \cdots, 2n)(a_2, a_4, 1, 2, \cdots, 2n)\\ &+(a_1, a_4, 1, 2, \cdots, 2n)(a_2, a_3, 1, 2, \cdots, 2n)\end{aligned}$$

and

$$\begin{aligned}&(a_1, a_2, a_3, 1, 2, \cdots, 2n-1)(1, 2, \cdots, 2n)\\ &= (a_1, 1, 2, \cdots, 2n-1)(a_2, a_3, 1, 2, \cdots, 2n)\\ &-(a_2, 1, 2, \cdots, 2n-1)(a_1, a_3, 1, 2, \cdots, 2n)\\ &+(a_3, 1, 2, \cdots, 2n-1)(a_1, a_2, 1, 2, \cdots, 2n).\end{aligned}$$

2.2 Differentiation of Pfaffians

In this section we will show how the derivatives of pfaffians may be represented by the sum of pfaffians. Suppose that

$$(i, j)_x = g(\theta_i)f(\theta_j) - f(\theta_i)g(\theta_j),$$

where f, g are differential operators, then by defining indices f and g such that $(f,i) = f(\theta_i)$, $(g,i) = g(\theta_i)$ and $(f,g) = 0$, we have

$$\begin{aligned}
(i,j)_x = (f,g,i,j) &= \begin{vmatrix} (f,g) & (f,i) & (f,j) \\ & (g,i) & (g,j) \\ & & (i,j) \end{vmatrix} \\
&= \begin{vmatrix} 0 & f(\theta_i) & f(\theta_j) \\ & g(\theta_i) & g(\theta_j) \\ & & (i,j) \end{vmatrix} \\
&= g(\theta_i)f(\theta_j) - f(\theta_i)g(\theta_j).
\end{aligned}$$

In general, it can be shown that [60]

$$\frac{\partial}{\partial x}(1,2,\cdots,2n) = (f,g,1,2,\cdots,2n).$$

Higher order derivatives of pfaffians can be calculated in a similiar way. Let us assume that

$$\frac{\partial}{\partial y}(i,j) = l(\theta_i)k(\theta_j) - k(\theta_i)l(\theta_j),$$

where k, l are differential operators and $(k,i) = k(\theta_i)$, $(l,i) = l(\theta_i)$ and $(k,l) = 0$, and we also assume that $f \circ k = k \circ f$, $f \circ l = l \circ f$, $g \circ k = k \circ g$, $g \circ l = l \circ g$, where '$\circ$' is explained later. We then have

$$\frac{\partial}{\partial y}(1,2,\cdots,2n) = (k,l,1,2,\cdots,2n)$$

and

$$\begin{aligned}
\frac{\partial^2}{\partial y \partial x}(1,2,\cdots,2n) &= \frac{\partial}{\partial y}(f,g,1,2,\cdots,2n) \\
&= (f \circ k, g \circ l, 1,2,\cdots,2n) + (f \circ l, g \circ k, 1,2,\cdots,2n) \\
&\quad + (f,g,k,l,1,2,\cdots,2n),
\end{aligned}$$

where $(f \circ k, i) = f(k(\theta_i))$, $(g \circ l, i) = g(l(\theta_i))$, $(f \circ l, i) = f(l(\theta_i))$, $(g \circ k, i) = g(k(\theta_i))$ and $(f,k) = (f,l) = (g,k) = (g,l) = 0$.

For example if

$$\frac{\partial}{\partial x}(i,j) = \theta_i\theta_{j,x} - \theta_{i,x}\theta_j$$

$$\frac{\partial}{\partial y}(i,j) = \theta_{i,y}\theta_j - \theta_i\theta_{j,y}$$

then

$$
\begin{aligned}
\frac{\partial}{\partial x}(1,2,\cdots,2n) &= (\partial_x, I, 1, 2, \cdots, 2n), \\
\frac{\partial}{\partial y}(1,2,\cdots,2n) &= (I, \partial_y, 1, 2, \cdots, 2n), \\
\frac{\partial^2}{\partial y \partial x}(1,2,\cdots,2n) &= (\partial_{xy}, I, 1, 2, \cdots, 2n) + (\partial_x, \partial_y, 1, 2, \cdots, 2n),
\end{aligned}
$$

where $(\partial_z, i) = \theta_{i,z}$, $z = x, y, xy$, and $(I, i) = \theta_i$.

2.3 Soliton Solutions to the BKP Equation

The BKP equation

$$
(u_t + 15uu_{3x} + 15u_x^3 - 15u_x u_y + u_{5x})_x + 5u_{3x,y} - 5u_{yy} = 0
$$

may be written in the bilinear form as

$$
(D_x^6 - 5D_y^2 - 5D_x^3 D_y + 9D_x D_t)\tau \cdot \tau = 0. \tag{2.3}
$$

In this section we will show that the n-soliton solution τ_n satisfies the BKP equation by virtue of pfaffian identities. This was first proved by Hirota [60] and is called the pfaffian technique. The τ function is expressed in terms of pfaffian as

$$
\tau_n = (1, 2, \cdots, 2n) \tag{2.4}
$$

whose (i,j)th element is given by

$$
(i,j) = c_{ij} + \int_{-\infty}^{x} \left[\frac{\partial f_i}{\partial x} f_j - f_i \frac{\partial f_j}{\partial x} \right] dx, \qquad i,j = 1, 2, \cdots, 2n \tag{2.5}
$$

where f_i satisfies the linear differential equations

$$
\frac{\partial f_i}{\partial y} = \frac{\partial^3 f_i}{\partial x^3}, \qquad \frac{\partial f_i}{\partial t} = \frac{\partial^5 f_i}{\partial x^5}.
$$

For example, if we choose

$$
f_i = e^{\xi_i} \qquad \text{with} \qquad \xi_i = k_i x + k_i^3 y + k_i^5 t,
$$

where k_i, $\quad i = 1, 2, \cdots, n$ are constants, then the (i,j)th term is

$$
(i,j) = c_{ij} + \frac{k_i - k_j}{k_i + k_j} e^{\xi_i + \xi_j}.
$$

The two-soliton solution τ_2 can be obtained by choosing the constants $c_{12} = c_{34} = 1$ and $c_{13} = c_{14} = c_{23} = c_{24} = 0$. Hence, from (2.4)

$$\begin{aligned}
\tau_2 &= (1,2,3,4) = (1,2)(3,4) - (1,3)(2,4) + (1,4)(2,3) \\
&= \left(1 + \frac{k_1 - k_2}{k_1 + k_2} e^{\xi_1+\xi_2}\right)\left(1 + \frac{k_3 - k_4}{k_3 + k_4} e^{\xi_3+\xi_4}\right) \\
&\quad + \left(\frac{k_1 - k_4}{k_1 + k_4}\frac{k_2 - k_3}{k_2 + k_3} - \frac{k_1 - k_3}{k_1 + k_3}\frac{k_2 - k_4}{k_2 + k_4}\right) e^{\xi_1+\xi_2+\xi_3+\xi_4} \\
&= 1 + e^{\xi_1+\xi_2+\alpha_{12}} + e^{\xi_3+\xi_4+\alpha_{34}} + A_{12} e^{\xi_1+\xi_2+\xi_3+\xi_4+\alpha_{12}+\alpha_{34}},
\end{aligned}$$

where

$$\alpha_{12} = \log\left(\frac{k_1 - k_2}{k_1 + k_2}\right), \qquad \alpha_{34} = \log\left(\frac{k_3 - k_4}{k_3 + k_4}\right)$$

and

$$A_{12} = \frac{(k_1 - k_3)}{(k_1 + k_3)}\frac{(k_1 - k_4)}{(k_1 + k_4)}\frac{(k_2 - k_3)}{(k_2 + k_3)}\frac{(k_2 - k_4)}{(k_2 + k_4)},$$

gives the two-soliton solution for the BKP equation.

In order to show that the τ_n in (2.4) satisfy the bilinear form (2.3), we need to differentiate the pfaffian given by (2.4). We begin with the differentiation of the element (i, j) with respect to x, from (2.5)

$$\begin{aligned}
\frac{\partial}{\partial x}(i,j) &= \frac{\partial f_i}{\partial x} f_j - f_i \frac{\partial f_j}{\partial x}, \\
&= (d_0, d_1, i, j)
\end{aligned}$$

where we have used the pfaffians representing derivatives of functions $f_i(x)$, hence $(d_0, i) = f_i$, $(d_1, i) = \frac{\partial}{\partial x} f_i$ and $(d_0, d_1) = 0$. The derivative of the pfaffian (2.4) is given in the following form

$$\frac{\partial}{\partial x}(1, 2, \cdots, 2n) = (d_0, d_1, 1, 2, \cdots, 2n).$$

Higher order derivatives of the pfaffian $(1, 2, \cdots, 2n)$ can be calculated by using the following relation

$$\begin{aligned}
\frac{\partial}{\partial x}(d_m, d_n, 1, 2, \cdots, 2n) &= (d_{m+1}, d_n, 1, 2, \cdots, 2n) + (d_m, d_{n+1}, 1, 2, \cdots, 2n) \\
&\quad + (d_0, d_1, d_m, d_n, 1, 2, \cdots, 2n).
\end{aligned}$$

From this relation the higher order derivatives can be written as follows;

$$\begin{aligned}
\frac{\partial^2}{\partial x^2}(1,2,\cdots,2n) &= (d_0,d_2,1,2,\cdots,2n) \\
\frac{\partial^3}{\partial x^3}(1,2,\cdots,2n) &= (d_1,d_2,1,2,\cdots,2n)+(d_0,d_3,1,2,\cdots,2n) \\
\frac{\partial^4}{\partial x^4}(1,2,\cdots,2n) &= 2(d_1,d_3,1,2,\cdots,2n)+(d_0,d_4,1,2,\cdots,2n) \\
\frac{\partial^5}{\partial x^5}(1,2,\cdots,2n) &= 2(d_2,d_3,1,2,\cdots,2n)+3(d_1,d_4,1,2,\cdots,2n) \\
&\quad +(d_0,d_5,1,2,\cdots,2n) \\
\frac{\partial^6}{\partial x^6}(1,2,\cdots,2n) &= 5(d_2,d_4,1,2,\cdots,2n)+4(d_1,d_5,1,2,\cdots,2n) \\
&\quad +(d_0,d_6,1,2,\cdots,2n)+2(d_0,d_1,d_2,d_3,1,2,\cdots,2n)
\end{aligned}$$

Next we calculate the y-derivative of (i,j), from (2.5)

$$\frac{\partial}{\partial y}(i,j)=\int_{-\infty}^{x}\left[\frac{\partial^4 f_i}{\partial x^4}f_j+\frac{\partial f_i}{\partial x}\frac{\partial^3 f_j}{\partial x^3}-\frac{\partial^3 f_i}{\partial x^3}\frac{\partial f_j}{\partial x}-f_i\frac{\partial^4 f_j}{\partial x^4}\right]dx$$

where we have used $\frac{\partial f_i}{\partial y}=\frac{\partial^3 f_i}{\partial x^3}$. Integrating this integral by parts, we get

$$\begin{aligned}
\frac{\partial}{\partial y}(i,j) &= \frac{\partial^3 f_i}{\partial x^3}f_j-f_i\frac{\partial^3 f_j}{\partial x^3}-2\left(\frac{\partial^2 f_i}{\partial x^2}\frac{\partial f_j}{\partial x}-\frac{\partial f_i}{\partial x}\frac{\partial^2 f_j}{\partial x^2}\right) \\
&= (d_0,d_3,i,j)-2(d_1,d_2,i,j).
\end{aligned}$$

Higher order derivatives of the pfaffian $(1,2,\cdots,2n)$ are obtained by using the following relation

$$\begin{aligned}
\frac{\partial}{\partial y}(d_m,d_n,1,2,\cdots,2n) &= (d_{m+3},d_n,1,2,\cdots,2n)+(d_m,d_{n+3},1,2,\cdots,2n) \\
&\quad +(d_0,d_3,d_m,d_n,1,2,\cdots,2n)-2(d_1,d_2,d_m,d_n,1,2,\cdots,2n).
\end{aligned}$$

The t-derivative of the element (i,j) is calculated as follows

$$\frac{\partial}{\partial t}(i,j)=(d_0,d_5,i,j)-2(d_1,d_4,i,j)+2(d_2,d_3,i,j).$$

Hence, following the same procedure, we find the necessary derivatives for the proof as follows:

$$
\begin{aligned}
\frac{\partial}{\partial y}(1,2,\cdots,2n) &= (d_0,d_3,1,2,\cdots,2n) - 2(d_1,d_2,1,2,\cdots,2n),\\
\frac{\partial^2}{\partial x\partial y}(1,2,\cdots,2n) &= -(d_1,d_3,1,2,\cdots,2n) + (d_0,d_4,1,2,\cdots,2n),\\
\frac{\partial^3}{\partial x^2\partial y}(1,2,\cdots,2n) &= -(d_2,d_3,1,2,\cdots,2n) + (d_0,d_5,1,2,\cdots,2n),\\
\frac{\partial^4}{\partial x^3\partial y}(1,2,\cdots,2n) &= -(d_2,d_4,1,2,\cdots,2n) + (d_1,d_5,1,2,\cdots,2n)\\
&\quad + (d_0,d_6,1,2,\cdots,2n) - (d_0,d_1,d_2,d_3,1,2,\cdots,2n),\\
\frac{\partial^2}{\partial y^2}(1,2,\cdots,2n) &= 2(d_2,d_4,1,2,\cdots,2n) - 2(d_1,d_5,1,2,\cdots,2n)\\
&\quad + (d_0,d_6,1,2,\cdots,2n) - 4(d_0,d_1,d_2,d_3,1,2,\cdots,2n),\\
\frac{\partial}{\partial t}(1,2,\cdots,2n) &= (d_0,d_5,1,2,\cdots,2n) - 2(d_1,d_4,1,2,\cdots,2n)\\
&\quad + 2(d_2,d_3,1,2,\cdots,2n),\\
\frac{\partial^2}{\partial x\partial t}(1,2,\cdots,2n) &= -(d_1,d_5,1,2,\cdots,2n) + (d_0,d_6,1,2,\cdots,2n)\\
&\quad + 2(d_0,d_1,d_2,d_3,1,2,\cdots,2n).
\end{aligned}
$$

Substituting these results into the bilinear BKP equation (2.3), we get

$$
\begin{aligned}
&(d_0,d_1,d_2,d_3,1,2,\cdots,2n)(1,2,\cdots,2n)\\
&-(d_0,d_1,1,2,\cdots,2n)(d_2,d_3,1,2,\cdots,2n)\\
&+(d_0,d_2,1,2,\cdots,2n)(d_1,d_3,1,2,\cdots,2n)\\
&-(d_0,d_3,1,2,\cdots,2n)(d_1,d_2,1,2,\cdots,2n)
\end{aligned}
$$

which vanishes by virtue of the pfaffian identity (2.1).

In the next chapter we will study the asymptotic properties of dromion solutions written in terms of pfaffians. In chapter 4 we will see how pfaffians arise naturally in the fermionic approach to soliton equations and in chapter 5 the pfaffian technique will be applied to two other integrable systems.

Chapter 3

Dromion solutions of the Nizhnik-Veselov-Novikov equations and their asymptotic properties

3.1 Introduction

In recent years the generalizations of integrable (1+1)-dimensional equations to (2+1) dimensions have been widely studied. The integrable generalization of the nonlinear Schrödinger (NLS) equation are the Davey-Stewartson (DS) equations [34]. Gilson and Nimmo [35] studied the dromion solutions of the DS equations [25] and their asymptotic properties. The generalization of the Korteweg-de-Vries (KdV) equation has two possibilities which are the Kadometsev-Petviashvili (KP) equations [37] and the Nizhnik-Veselov-Novikov (NVN) equations [38]. These generalizations, the DS and NVN equations, have two dimensional localized hump solutions that decay exponentially in all directions, which are called two dimensional solitons or dromions. The KP equation does not have such solutions. The word dromion comes from the Greek word dromos, which means track and has been given [31] to these objects, because they are located at the intersection of plane waves, which can be thought to form tracks.

An alternative approach has been through direct methods using the bilinear form of the DS and NVN equations. Hietarinta and Hirota [40] and Jaulent et al. [41]

obtained a broader class of dromion solutions of the DS equations in terms of wronskian determinants and as polynomials in exponentials respectively. Athorne and Nimmo [43] and Ohta [44] obtained dromion solutions of the NVN equations in terms of pfaffians.

These (2+1)-dimensional generalizations also possess the usual features of (1+1)-dimension integrable equations, namely solvability by the inverse scattering transform, existence of Bäcklund transformations and Hamiltonian formulation [30], [31], [32].

3.2 A class of solutions of the NVN equations

In this section we recall some results obtained in [43]. The NVN equations are

$$U_t = U_{xxx} + U_{yyy} + 3(\Phi_{xx}U)_x + 3(\Phi_{yy}U)_y \tag{3.1}$$

$$U = \Phi_{xy}. \tag{3.2}$$

A class of solutions of (3.1) and (3.2) is given by

$$U = 2\left(\log(P(\theta_1, \cdots, \theta_n))\right)_{xy}, \tag{3.3}$$

where θ_i are solutions of the linear equations

$$\phi_{xy} + \Phi^{(0)}_{xy}\,\phi = 0 \tag{3.4}$$

$$\phi_t = \phi_{xxx} + \phi_{yyy} + 3\Phi^{(0)}_{xx}\phi_x + 3\Phi^{(0)}_{yy}\phi_y, \tag{3.5}$$

and

$$P(\theta_1, \cdots, \theta_n) = \begin{cases} (1, 2, \cdots, n) & \text{n even} \\ (1, 2, \cdots, n, I) & \text{n odd} \end{cases}$$

where $(i, j) = P(\theta_i, \theta_j)$, $(i, I) = \theta_i$ and

$$P(\theta_i, \theta_j) = \int W_x[\theta_i, \theta_j]\, dx - W_y[\theta_i, \theta_j]\, dy. \tag{3.6}$$

In (3.6)

$$W_X[\theta_i, \theta_j] = \theta_i\theta_{jX} - \theta_{iX}\theta_j$$

denotes the wronskian of θ_i and θ_j with respect to variable $X = x$ or $X = y$ and so $P(\theta_i, \theta_j)$ is skew-symmetric.

In particular, setting $\Phi^{(0)} = 0$ in (3.4) and (3.5) yields separable θ_i

$$\theta_i = \theta_i^{(x)}(x,t) + \theta_i^{(y)}(y,t),$$

where

$$\theta_{i_t}^{(X)} = \theta_{i_{XXX}}^{(X)}$$

for $X = x$ or $X = y$.

To obtain plane wave soliton solutions, we choose

$$\theta_i^{(x)} = \alpha_i \exp(k_i x + k_i^3 t) \quad \text{and} \quad \theta_i^{(y)} = \beta_i \exp(l_i y + l_i^3 t)$$

and then (3.3) gives, in the case $n = 1$,

$$U = -\frac{kl}{2} \operatorname{sech}^2 \left(\frac{1}{2} \left[kx - ly + (k^3 - l^3)t + \log\left(\frac{\alpha}{\beta}\right) \right] \right).$$

If k or l tends to zero then U tends to the trivial solution. Then the individual solitons are a kind of "ghost" solitons [49], parallel to the x and y axes. These are given by

$$\theta = \alpha \exp(kx + k^3 t) + \beta \quad \text{and} \quad \theta = \alpha + \beta \exp(ly + l^3 t)$$

respectively.

A single dromion solution may be thought of as a two-soliton solution made out of two intersecting ghost solitons. Outside the interaction region, the solution is approximated by individual ghost solitons and the physical field U vanishes.

If we take $n = 2$ with

$$\theta_1 = \theta_1^{(x)} = \alpha \exp(kx + k^3 t) + 1 \quad \text{and} \quad \theta_2 = \theta_2^{(y)} = 1 + \beta \exp(ly + l^3 t)$$

and using equation (3.6), the pfaffian becomes

$$P(\theta_1, \theta_2) = \int W_x[\theta_1, \theta_2]\, dx - W_y[\theta_1, \theta_2]\, dy \tag{3.7}$$

$$= -(\theta_1 \theta_2 + c - 1), \tag{3.8}$$

where c is some constant. From equation (3.3)

$$U = 2\left(\log(P(\theta_1, \theta_2))\right)_{xy}$$

and thus

$$U = \frac{2kl(c-1)\alpha\beta}{(ce^{-\eta/2}e^{-\rho/2} + \alpha e^{\eta/2}e^{-\rho/2} + \beta e^{-\eta/2}e^{\rho/2} + \alpha\beta e^{\eta/2}e^{\rho/2})^2},$$

where

$$\eta = kx + k^3t \qquad \rho = ly + l^3t.$$

Here, we see that as $(x, y) \to \infty$ at least one of the terms in the denominator tends to infinity, and hence U is a localized solution which we call the $(1,1)-$dromion solution of the NVN equations. This is illustrated in figure 3.1.

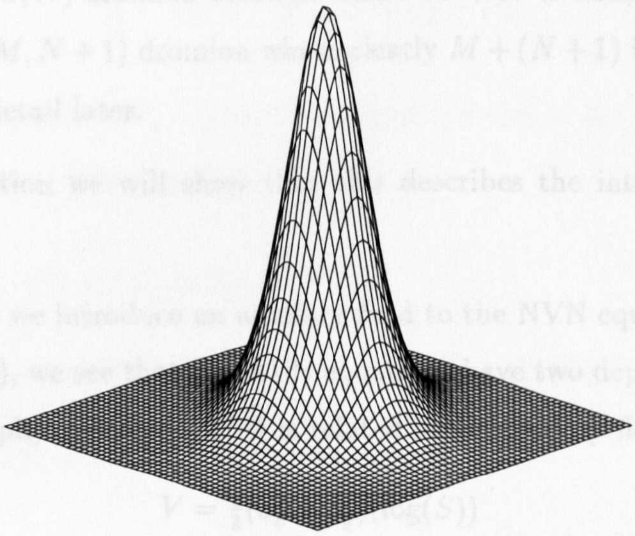


Figure 3.1: (1,1)-dromion plot for the parameters given by $k = l = \frac{1}{2}$, $c = \frac{3}{2}$ and $\alpha = \beta = 1$.

An (M, N)-dromion solution, in which $M + N$ is even, is obtained by choosing

$$\theta_i = \alpha_i \exp(k_i x + k_i^3 t) + 1 \quad \text{for} \quad i = 1, \cdots, M$$

and

$$\theta_{j+M} = 1 + \beta_j \exp(l_j y + l_j^3 t) \quad \text{for} \quad j = 1, \cdots, N.$$

To express the solution in a compact form, we take S as the square of the pfaffian $P(\theta_1, \cdots, \theta_{M+N})$, hence

$$S = |P(\theta_i, \theta_j)|.$$

Then S has the following block structure:

$$S = \begin{vmatrix} \int^x W_x[\theta_i, \theta_p]\, dx & -(\theta_i\, \theta_j + c_{ij}) \\ \\ (\theta_p\, \theta_q + c_{pq}) & -\int^y W_y[\theta_j, \theta_q]\, dy \end{vmatrix}, \tag{3.9}$$

where $i, p = 1, \cdots, M$ and $j, q = M+1, \cdots, M+N$. Then the solution is given by

$$U = (\log(S))_{xy}.$$

To obtain an (M, N)-dromion solution where $M + N$ is odd, we may consider a special case of an $(M, N+1)$ dromion where clearly $M + (N+1)$ is even. This will be discussed in more detail later.

In the next section we will show that this describes the interaction of $M \times N$ dromions.

For convenience we introduce an auxilary field to the NVN equations. From equations (3.1) and (3.2), we see that the NVN equations have two dependent variables; U corresponds to the physical field and Φ is related to an auxilary field

$$V = \tfrac{1}{2}(\partial_x^2 + \partial_y^2)(\log(S)) \tag{3.10}$$

which is introduced to help visualize the nature of the soliton. While U is localized, V is not in general. For the solution we discuss, the point of intersection of the plane wave solitons in the V-plane and the localized solitons in the U-plane coincide.

3.3 Asymptotic analysis of the $(1, 1)$-dromion solution

To understand the meaning of the parameters that appear in the (M, N)-dromion solution (3.9) we consider the simplest case in which $M = N = 1$. The form of S is then

$$S(\theta_1, \theta_2) = (P(\theta_1, \theta_2))^2 \tag{3.11}$$

and from (3.7) and (3.8) we get

$$P(\theta_1, \theta_2) = -\theta_1\theta_2 + \text{constant}. \tag{3.12}$$

To simplify the presentation, we use P instead of $P(\theta_1, \theta_2)$. Taking

$$\theta_1 = 1 + e^{\eta} \quad \text{and} \quad \theta_2 = 1 + e^{\rho}$$

P becomes

$$P = -c - e^{\eta} - e^{\rho} - e^{\eta + \rho}, \tag{3.13}$$

where c is constant and

$$\eta = k(x + k^2 t), \quad \rho = l(y + l^2 t).$$

The effect of the parameters c, k and l on the properties of the solution to the NVN equations is explained in the following theorem.

Theorem 3.3.1 *For $c > 0$ and $k, l \neq 0$,*

$$U = 2\, \partial_x \partial_y (\log(P)) \quad \textit{and} \quad V = (\partial_x^2 + \partial_y^2)(\log(P)) \tag{3.14}$$

with P given by (3.13) have the following properties:

1) V is the interaction a pair of plane wave solitons, one parallel to the y-axis, $V^{(x)}$, parametrized in terms of k and the other parallel to the x-axis, $V^{(y)}$, parametrized in terms of l. These waves have speeds $-k^2$ and $-l^2$, and amplitudes $\frac{k^2}{4}$ and $\frac{l^2}{4}$ respectively. The relative phase shifts of the plane wave solitons at the interaction may be expressed in terms of

$$F_{\perp} = \log(c),$$

the 'perpendicular phase shift'; for $V^{(x)}$ the relative phase shift is $\operatorname{sgn}(l) F_{\perp}$ and for $V^{(y)}$ it is $\operatorname{sgn}(k) F_{\perp}$.

2) U decays to zero exponentially as $(x, y) \to \infty$ in any direction and the amplitude is

$$\begin{aligned} U_0 &= \frac{kl}{2} \frac{\sqrt{c} - 1}{\sqrt{c} + 1} \\ &= \frac{kl}{2} \frac{e^{\frac{1}{2} F_{\perp}} - 1}{e^{\frac{1}{2} F_{\perp}} + 1}. \end{aligned}$$

At time t the maximum or minimum on the dromion is located at

$$\begin{aligned}(x,y) &= \left(\frac{\log(c)-2k^3t}{2k}, \frac{\log(c)-2l^3t}{2l}\right)\\ &= \left(\frac{1}{2}\left(\psi_x^+ - \psi_x^-\right) - k^2t, \frac{1}{2}\left(\psi_y^+ - \psi_y^-\right) - l^2t\right),\end{aligned}$$

where $\psi_x^\pm$ and $\psi_y^\pm$ are the phase constants in the plane waves in the V-plane at $y = \pm\infty$ and $x = \pm\infty$ respectively. The trajectory of the dromion is the line

$$k^2y - l^2x = \frac{1}{2kl}\,\log(c)\,(k^3 - l^3).$$

Proof. 1) To find the speeds and the amplitudes, we fix y and hence ρ in P

$$\begin{aligned}P &= -c - e^\eta - e^\rho - e^{\eta+\rho}\\ &= -c - e^\rho - (1+e^\rho)\,e^\eta\\ &= -(c+e^\rho)\left(1 + \frac{1+e^\rho}{c+e^\rho}\,e^\eta\right).\end{aligned}$$

Since ρ and c are constants, this expression for P gives the same U and V as

$$P = 1 + \alpha e^\eta,$$

where $\alpha = (1+e^\rho)/(c+e^\rho)$.

Hence

$$\begin{aligned}V &= (\partial_x^2 + \partial_y^2)(\log(P))\\ &= \partial_x^2(\log(1+\alpha\,e^\eta))\\ &= \frac{\alpha k^2 e^\eta}{(1+\alpha e^\eta)^2}\\ &= \frac{1}{4}k^2\,\mathrm{sech}^2\left(\frac{1}{2}k\left[(x+k^2t) + \frac{1}{k}\log(\alpha)\right]\right)\end{aligned}$$

which is a one-dimensional plane wave soliton propagating in the x direction. Hence the speed is $-k^2$ and the amplitude is $\frac{k^2}{4}$. If we fix x and hence η in P, the other one-dimensional plane wave soliton perpendicular to this one, propagating in the y direction would be of the form

$$\begin{aligned}V &= \partial_y^2(\log(1+\beta e^\rho))\\ &= \frac{1}{4}l^2\,\mathrm{sech}^2\left(\frac{1}{2}l\left[(y+l^2t) + \frac{1}{k}\log(\beta)\right]\right).\end{aligned}$$

Hence the speed is $-l^2$ and the amplitude is $\frac{l^2}{4}$.

To determine the phase shift we consider the change in P as x and y change from $-\infty$ to $+\infty$. To get the phase shift in the x direction we fix x and t in the limits as $y \to \infty$ and as $y \to -\infty$.

In what follows, the symbol $\simeq$ is used to denote functions that are 'asymptotically equivalent'. We say that two expressions $P = B$ and $P = C$ are equivalent under the changes of variable (3.14) from P to U and V if $C = \alpha e^{\beta x + \gamma y} B$ for some constants α, β, γ so that both B and C give the same U and V. So, we write $A \simeq B$ as $X \to a$ to mean that $A \sim C$ (A is asymptotic to C) as $X \to a$ and B and C are equivalent in the sense given above.

For $l > 0$, as $y \to \infty$

$$\begin{aligned} P &= -e^{\rho}(ce^{-\rho} + e^{\eta-\rho} + 1 + e^{\eta}) \\ &\simeq 1 + e^{\eta} \end{aligned}$$

hence

$$V = \frac{k^2}{4}\operatorname{sech}^2\left(\frac{1}{2}k(x + k^2 t)\right)$$

and as $y \to -\infty$

$$P \sim -c - e^{\eta}$$

hence

$$V = \frac{k^2}{4}\operatorname{sech}^2\left(\frac{1}{2}k\left[(x + k^2 t) - \frac{1}{k}\log(c)\right]\right).$$

Thus the phase constants at $y = \pm\infty$ are

$$\psi_x^+ = 0 \quad \text{and} \quad \psi_x^- = -\frac{1}{k}\log(c)$$

and then the phase shift in the plane wave parallel to the y-axis is

$$\psi_x^+ - \psi_x^- = \frac{1}{k}\log(c). \tag{3.15}$$

By a similar calculation, the phase shift when $l < 0$ is

$$\psi_x^+ - \psi_x^- = -\frac{1}{k}\log(c). \tag{3.16}$$

Hence the *relative* phase shift (= (wave number)×(absolute phase shift)) is $\operatorname{sgn}(l)\log(c)$ as required.

Similarly to get the phase shift in the y direction we fix y and t in the limits as $x \to \infty$ and as $x \to -\infty$. For $k > 0$, as $x \to \infty$

$$\begin{aligned} P &= -e^{\eta}(c\, e^{-\eta} + 1 + e^{\rho-\eta} + e^{\rho}) \\ &\simeq 1 + e^{\rho} \end{aligned}$$

hence

$$V = \frac{l^2}{4}\,\mathrm{sech}^2\left(\frac{1}{2}l(y + l^2 t)\right)$$

and as $x \to -\infty$

$$P \simeq -c - e^{\rho}$$

hence

$$V = \frac{l^2}{4}\,\mathrm{sech}^2\left(\frac{1}{2}l\left[(y + l^2 t) - \frac{1}{l}\log(c)\right]\right).$$

Thus the phase constants at $x = \pm\infty$ are

$$\psi_y^+ = 0 \quad \text{and} \quad \psi_y^- = -\frac{1}{l}\log(c)$$

and then the phase shift in the plane wave parallel to the x-axis is

$$\psi_y^+ - \psi_y^- = \frac{1}{l}\log(c) \tag{3.17}$$

and the phase shift when $k < 0$ is

$$\psi_y^+ - \psi_y^- = -\frac{1}{l}\log(c). \tag{3.18}$$

2) For P given by (3.13)

$$\begin{aligned} U &= \frac{2\,kl(c-1)e^{\eta+\rho}}{(c + e^{\eta} + e^{\rho} + e^{\eta+\rho})^2} \\ &= \frac{2\,kl(c-1)}{(c\, e^{-(\eta+\rho)/2} + e^{(\eta-\rho)/2} + e^{-(\eta-\rho)/2} + e^{(\eta+\rho)/2})^2} \end{aligned} \tag{3.19}$$

from which we see that U is exponentially localized since at least one of the exponential terms in the denominator tends to infinity as $(x, y) \to \infty$. To show this we consider the exponentials in the denominator in (3.19) and a ray in any direction $y = \alpha x$, where $\alpha \in \mathbb{R}$. With $y = \alpha x$ and the appropriate expressions for η and ρ,

$$c\, e^{-(\eta+\rho)/2} + e^{(\eta-\rho)/2} + e^{-(\eta-\rho)/2} + e^{(\eta+\rho)/2}$$

becomes

$$c\, e^{-((k+\alpha l)x+(k^3+l^3)t)/2} + e^{((k-\alpha l)x+(k^3-l^3)t)/2} + e^{-((k-\alpha l)x+(k^3-l^3)t)/2} + e^{((k+\alpha l)x+(k^3+l^3)t)/2}.$$

Here, if $x \to \infty$ or $x \to -\infty$ the expression tends to infinity, whatever the signs of k, l and α are.

To find any critical points we need to solve the equations $U_x = 0$ and $U_y = 0$. From

$$U = \frac{2\,kl e^{kx+k^3t+ly+l^3t}\,(c-1)}{\left(c+e^{k(x+k^2t)}+e^{l(y+l^2t)}+e^{kx+k^3t+ly+l^3t}\right)^2}$$

differentiating with respect to x gives

$$U_x = \frac{2\,k^2 l e^{kx+k^3t+ly+l^3t}\,(c-1)\left(c-e^{k\left(x+k^2t\right)}+e^{l\left(y+l^2t\right)}-e^{kx+k^3t+ly+l^3t}\right)}{\left(c+e^{k(x+k^2t)}+e^{l(y+l^2t)}+e^{kx+k^3t+ly+l^3t}\right)^3}$$

and differentiating with respect to y gives

$$U_y = \frac{2\,k l^2 e^{kx+k^3t+ly+l^3t}\,(c-1)\left(c+e^{k\left(x+k^2t\right)}-e^{l\left(y+l^2t\right)}-e^{kx+k^3t+ly+l^3t}\right)}{\left(c+e^{k(x+k^2t)}+e^{l(y+l^2t)}+e^{kx+k^3t+ly+l^3t}\right)^3}.$$

One obvious solution of this pair of equations is $c = 1$, but this corresponds to the trivial solution $U = 0$ and it is therefore excluded. Hence, solving the equations $U_x = 0$ and $U_y = 0$ for x and y is the same as solving the equations

$$c - e^{k(x+k^2t)} + e^{l(y+l^2t)} - e^{kx+k^3t+ly+l^3t} = 0$$

$$c + e^{k(x+k^2t)} - e^{l(y+l^2t)} - e^{kx+k^3t+ly+l^3t} = 0$$

which imply that

$$e^{kx+k^3t+ly+l^3t} = c \quad \text{and} \quad e^{k(x+k^2t)} = e^{l(y+l^2t)}.$$

This pair has a unique real solution

$$x = \frac{\log(\sqrt{c}) - k^3t}{k},\ y = \frac{\log(\sqrt{c}) - l^3t}{l}.$$

Since $U \to 0$ as $(x, y) \to \infty$ and there is a unique critical point, this clearly must be a local maximum or minimum located at the point

$$(x, y) = \left(\frac{\log(c) - 2k^3t}{2k}, \frac{\log(c) - 2l^3t}{2l}\right).$$

By eliminating t we get the trajectory of the dromion

$$k^2y - l^2x = \frac{1}{2kl}\log(c)\,(k^3 - l^3).$$

If we substitute the critical values of x and y into U, we obtain the amplitude of the dromion, namely

$$U_0 = \frac{kl}{2}\,\frac{\sqrt{c}-1}{\sqrt{c}+1}$$

which can be written in terms of the phase shift $F_\perp$ as follows:

$$U_0 = \frac{kl}{2}\,\frac{e^{\frac{1}{2}F_\perp}-1}{e^{\frac{1}{2}F_\perp}+1}.$$

□

Summary of results for $(1,1)$-dromion:

The main result is that there is a dromion in the U-plane and a pair of perpendicular plane waves in the V-plane. We observe further from Theorem 3.3.1 that

- The dromion may have arbitrary amplitude, positive, negative or zero. The amplitude is

  1. positive if $kl > 0$ and $c > 1$ or $kl < 0$ and $0 \le c < 1$,
  2. negative if $kl < 0$ and $c > 1$ or $kl > 0$ and $0 \le c < 1$,
  3. zero if $kl = 0$ or $c = 1$

- The plane waves always have positive amplitude and exert a phase-shift on one another. In particular, the directions of these phase shifts (forward or backward) depend on the signs of k, l and $\log(c)$. The phase shift is zero if and only if the dromion amplitude is zero.

- At any fixed time, the dromion (in the U-plane) is symmetrically located between the plane waves (in the V-plane). This is illustrated schematically for the case $k > 0$, $l > 0$, $\log(c) > 0$ in figure 3.2.

We next consider the general case in which V consists of $M + N$ plane waves, M plane waves parallel to the y-axis and N plane waves parallel to the x-axis, each set of plane waves interacting like one-dimensional multisolitons and U consists of $M \times N$ dromions situated symmetrically at the interaction of the plane waves in the V-plane.

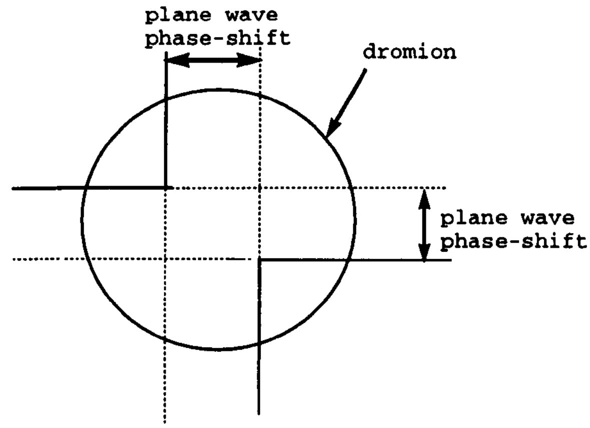


Figure 3.2: Phase shifts in the plane waves and the location of the dromion.

3.4 Asymptotic analysis of the (M, N)-dromion solution

In this section we consider the nature of the (M, N)-dromion solution as $t \to \pm\infty$. In order to get succinct expressions for these asymptotic forms of the solution we order the parameters k_i and l_j in this way:

$$k_1 < k_2 < \cdots < k_M \quad \text{and} \quad l_1 < l_2 < \cdots < l_N. \tag{3.20}$$

Also, to have non-singular solutions we make the following choices for the arbitrary constants appearing in the solution

$$\alpha_i = \beta_j = 1 \quad \text{for} \quad i, j \quad \text{odd}$$

$$\alpha_i = \beta_j = -1 \quad \text{for} \quad i, j \quad \text{even.}$$

Next we write the (M, N)-dromion solution given in (3.9), in the case when $M + N$ is even, in the following form

$$S = \begin{vmatrix} \Theta_1 & \Theta_2 \\ -\Theta_2^T & \Theta_3 \end{vmatrix} \tag{3.21}$$

where Θ_1 and Θ_3 are skew-symmetric matrices with entries

$$\Theta_{i,p} = c_{ip} - \alpha_i e^{\eta_i} + \alpha_p e^{\eta_p} + \alpha_i \alpha_p \tfrac{k_p - k_i}{k_p + k_i} e^{\eta_i + \eta_p} \qquad 1 \leq i < p \leq M$$

$$\Theta_{M+j,M+q} = c_{M+jM+q} + \beta_j e^{\rho_j} - \beta_q e^{\rho_q} + \beta_j \beta_q \tfrac{l_j - l_q}{l_j + l_q} e^{\rho_j + \rho_q} \qquad 1 \le j < q \le N$$

respectively, and Θ_2 has the entries

$$\Theta_{i,M+j} = c_{iM+j} - \alpha_i e^{\eta_i} - \beta_j e^{\rho_j} - \alpha_i \beta_j e^{\eta_i + \rho_j} \qquad i = 1, \ldots, M, j = 1, \ldots, N$$

where c_{ip}, c_{M+jM+q} and c_{iM+j} are arbitrary constants. We will only be interested in c_{ij} such that the solution has no singularities. The conditions on c_{ij} which give this property will be found by considering the asymptotic form of all of the $M \times N$ dromions.

To study the (M, N)-dromion solution in the case $M + N$ is odd, we may consider an $(M+1, N)$ or an $(M, N+1)$ dromion in which we set a k_i or an l_j equal to zero respectively. In making this choice it is important that the ordering (3.20) is preserved. An example of how a $(2, 1)$ dromion is obtained from a $(2, 2)$ dromion will be given later.

Further, it is convenient to express the determinant S in (3.21) in terms of other matrices, so that it can have simpler structure, namely

$$S = | \, C + D\,A - A^T D + D\,B\,D \, |. \tag{3.22}$$

The matrices in (3.22) have the following structure: A is a constant matrix with the (ij)-th entry

$$A_{ij} = \begin{cases} (-1)^i \, (1 - \delta_{ij}) & \text{for} \quad i = 1, \ldots, M \\ & \qquad j = 1, \ldots, M+N \\ (-1)^{i-(M+1)} \, (1 - \delta_{ij}) & \text{for} \quad i = M+1, \ldots, M+N \\ & \qquad j = 1, \ldots, M+N \end{cases};$$

where δ_{ij} is the Kronecker δ symbol; B is a skew-symmetric matrix with the block structure

$$B = \begin{pmatrix} K & R \\ -R^T & L \end{pmatrix} \tag{3.23}$$

in which K and L are constant skew-symmetric matrices with entries

$$K_{ij} = (-1)^{i+j} \frac{k_j - k_i}{k_j + k_i} \qquad 1 \le i < j \le M \tag{3.24}$$

$$L_{ij} = (-1)^{i+j+1} \frac{l_j - l_i}{l_j + l_i} \qquad 1 \le i < j \le N; \tag{3.25}$$

R is a rank-1 matrix with entries

$$R_{ij} = (-1)^{i+j+1} \qquad \text{for} \quad i = 1, \ldots, M, \quad j = 1, \ldots, N;$$

C is the general constant skew-symmetric matrix, which has the entries

$$c_{ij} \qquad \text{for} \qquad 1 \leqslant i < j \leqslant M + N$$

and

$$D = \operatorname{diag}(e^{\eta_1}, e^{\eta_2}, \ldots, e^{\eta_M}; e^{\rho_1}, e^{\rho_2}, \ldots, e^{\rho_N}) \tag{3.26}$$

is a diagonal matrix, where

$$\eta_i = k_i(x + k_i^2 t) \tag{3.27}$$

$$\rho_j = l_j(y + l_j^2 t) \tag{3.28}$$

for $i = 1, \ldots, M$ and $j = 1, \ldots, N$.

To determine the asymptotic form of the solution we fix the mth (x, t)-dependent plane wave and the nth (y, t)-dependent plane wave and we call the corresponding dromion the (m, n)th. We write S given by (3.22) in terms of

$$\hat{x} = x + {k_m}^2 t \quad \text{and} \quad \hat{y} = y + {l_n}^2 t$$

so that the (m, n)th dromion is independent of t (i.e. is stationary) when $\hat{x}$ and $\hat{y}$ are fixed. Also, we write the expressions (3.27) and (3.28) in terms of $\hat{x}$ and $\hat{y}$ we have

$$\begin{aligned} \eta_i &= k_i(\hat{x} + ({k_i}^2 - {k_m}^2)t) \\ \rho_j &= l_j(\hat{y} + ({l_j}^2 - {l_n}^2)t) \end{aligned}$$

for $i = 1, \ldots, M$ and $j = 1, \ldots, N$.

We will show that the asymptotic form of the solution as $t \to \pm\infty$ is a dromion. This will show that the (M, N)-dromion solution consists of $M \times N$ dromions separate asymptotically as $t \to \pm\infty$. Unlike solitons however amplitudes are not necessarily preserved. The study of these limits is rather technical but in the end we will obtain compact expressions for the change in amplitude of the dromions due to interaction.

Considering the limits of S with $\hat{x}$ and $\hat{y}$ fixed we have

as $t \to -\infty$

$$\eta_i \to \begin{cases} +\infty & \text{for } i < m \\ -\infty & \text{for } i > m \end{cases} \qquad \text{and} \qquad \rho_j \to \begin{cases} +\infty & \text{for } j < n \\ -\infty & \text{for } j > n \end{cases}$$

and as $t \to \infty$

$$\eta_i \to \begin{cases} -\infty & \text{for } i < m \\ +\infty & \text{for } i > m \end{cases} \qquad \text{and} \qquad \rho_j \to \begin{cases} -\infty & \text{for } j < n \\ +\infty & \text{for } j > n \end{cases}$$

while η_m and ρ_n are t-independent, and the limits of the exponentials are

as $t \to -\infty$

$$\begin{array}{llll} e^{-\eta_i} \to 0 & (i < m) & e^{-\rho_j} \to 0 & (j < n) \\ e^{\eta_i} \to 0 & (i > m) & e^{\rho_j} \to 0 & (j > n) \end{array}$$

and as $t \to \infty$

$$\begin{array}{llll} e^{\eta_i} \to 0 & (i < m) & e^{\rho_j} \to 0 & (j < n) \\ e^{-\eta_i} \to 0 & (i > m) & e^{-\rho_j} \to 0 & (j > n). \end{array}$$

To exploit these limits we must use appropriate equivalent forms for S given by (3.22). We factorize the diagonal matrix D given by (3.26) so that the factors or their inverses have finite limits as $t \to -\infty$ and as $t \to \infty$, as

$$D = D_- \, D_o \, D_+,$$

where

$$\begin{aligned} D_- &= \operatorname{diag}(1, \ldots, 1, 1, e^{\eta_{m+1}}, \ldots, e^{\eta_M}; 1, \ldots, 1, 1, e^{\rho_{n+1}}, \ldots, e^{\rho_N}), \\ D_o &= \operatorname{diag}(1, \ldots, 1, e^{\eta_m}, 1, \ldots, 1; 1, \ldots, 1, e^{\rho_n}, 1, \ldots, 1), \\ D_+ &= \operatorname{diag}(e^{\eta_1}, \ldots, e^{\eta_{m-1}}, 1, 1, \ldots, 1; e^{\rho_1}, \ldots, e^{\rho_{n-1}}, 1, 1, \ldots, 1). \end{aligned}$$

Hence we get, as $t \to -\infty$

$$\begin{aligned} D_- &\to \operatorname{diag}(1, \ldots, 1, 1, 0, \ldots, 0; 1, \ldots, 1, 1, 0, \ldots, 0) \\ D_+^{-1} &\to \operatorname{diag}(0, \ldots, 0, 1, 1, \ldots, 1; 0, \ldots, 0, 1, 1, \ldots, 1) \end{aligned}$$

and as $t \to \infty$

$$\begin{aligned} D_+ &\to \operatorname{diag}(0, \ldots, 0, 1, 1, \ldots, 1; 0, \ldots, 0, 1, 1, \ldots, 1) \\ D_-^{-1} &\to \operatorname{diag}(1, \ldots, 1, 1, 0, \ldots, 0; 1, \ldots, 1, 1, 0, \ldots, 0). \end{aligned}$$

To find the asymptotic forms of S as $t \to -\infty$ and $t \to \infty$ we take out factors of D_+ or D_- so that it is expressed solely in terms of matrices having finite limits.
As $t \to -\infty$

$$\begin{aligned} S &= | \, C + DA - A^T D + DBD \, | \\ &= | \, C + D_+ D_- D_o A - A^T D_- D_o D_+ + D_+ D_o D_- B D_- D_o D_+ \, | \\ &= | \, D_+ \, | \, | \, D_+^{-1} C D_+^{-1} \; + D_- D_o A D_+^{-1} - D_+^{-1} A^T D_- D_o + D_o D_- B D_- D_o \, | \, | \, D_+ \, | \end{aligned}$$

so the limit is

$$S_- \simeq | \, D_+^{-1} C D_+^{-1} \; + D_- D_o A D_+^{-1} - D_+^{-1} A^T D_o D_- + D_- D_o B D_o D_- \, |. \tag{3.29}$$

As $t \to \infty$

$$\begin{aligned} S &= | \, C + DA - A^T D + DBD \, | \\ &= | \, D_- \, | \, | \, D_-^{-1} C D_-^{-1} \; + D_+ D_o A D_-^{-1} - D_-^{-1} A^T D_+ D_o + D_o D_+ B D_+ D_o \, | \, | \, D_- \, | \end{aligned}$$

and the limit is

$$S_+ \simeq | \, D_-^{-1} C D_-^{-1} \; + D_+ D_o A D_-^{-1} - D_-^{-1} A^T D_o D_+ + D_+ D_o B D_o D_+ \, |. \tag{3.30}$$

We see from (3.29) and (3.30) that S_- and S_+ are the determinants of skew-symmetric matrices and are hence the squares of the pfaffians P_- and P_+ respectively. By expanding P_- and P_+ by their mth and $(M+n)$th lines one finds that

$$\begin{aligned} P_- &= P_1 + P_2 e^{\eta_m} + P_3 e^{\rho_n} + P_4 e^{\eta_m + \rho_n} \\ P_+ &= P_5 + P_6 e^{\eta_m} + P_7 e^{\rho_n} + P_8 e^{\eta_m + \rho_n}. \end{aligned}$$

A necessary and sufficient condition that P_- and P_+ have no zeros, and hence $U = 2(\log P_\pm)_{xy}$ has no singularities, is that $P_1, \ldots, P_4$ and $P_5, \ldots, P_8$ have the same sign. Furthermore, an overall change of sign in $P_\pm$ does not change U, and so, since U is supposed to be non-singular, without loss of generality we may write

$$\begin{aligned} P_- &= |P_1| + |P_2| e^{\eta_m} + |P_3| e^{\rho_n} + |P_4| e^{\eta_m + \rho_n} \\ P_+ &= |P_5| + |P_6| e^{\eta_m} + |P_7| e^{\rho_n} + |P_8| e^{\eta_m + \rho_n} \end{aligned} \tag{3.31}$$

where the pfaffians P_i ($i = 1..8$) satisfy the relation $P_i^2 = S_i$ and are defined in terms of minors of A, B and C, and S_i are the skew-symmetric determinants

$$S_1 = \left| \begin{array}{ccccc} 0 & (B)_{\substack{1\leqslant i<m \\ i<j<m}} & (A)_{\substack{1\leqslant i<m \\ m\leqslant j\leqslant M}} & (B)_{\substack{1\leqslant i<m \\ M+1\leqslant j<M+n}} & (A)_{\substack{1\leqslant i<m \\ M+n\leqslant j\leqslant M+N}} \\ & 0 & (C)_{\substack{m\leqslant i\leqslant M \\ i<j\leqslant M}} & (-A^T)_{\substack{m\leqslant i\leqslant M \\ M+1\leqslant j<M+n}} & (C)_{\substack{m\leqslant i\leqslant M \\ M+n\leqslant j\leqslant M+N}} \\ & & 0 & (B)_{\substack{M+1\leqslant i<M+n \\ i<j<M+n}} & (A)_{\substack{M+1\leqslant i<M+n \\ M+n\leqslant j\leqslant M+N}} \\ & & & 0 & (C)_{\substack{M+n\leqslant i\leqslant M+N \\ i<j\leqslant M+N}} \\ & & & & 0 \end{array} \right|$$

$$S_2 = \left| \begin{array}{ccccc} 0 & (B)_{\substack{1\leqslant i\leqslant m \\ i<j\leqslant m}} & (A)_{\substack{1\leqslant i\leqslant m \\ m<j\leqslant M}} & (B)_{\substack{1\leqslant i\leqslant m \\ M+1\leqslant j<M+n}} & (A)_{\substack{1\leqslant i\leqslant m \\ M+n\leqslant j\leqslant M+N}} \\ & 0 & (C)_{\substack{m<i\leqslant M \\ i<j\leqslant M}} & (-A^T)_{\substack{m<i\leqslant M \\ M+1\leqslant j<M+n}} & (C)_{\substack{m<i\leqslant M \\ M+n\leqslant j\leqslant M+N}} \\ & & 0 & (B)_{\substack{M+1\leqslant i<M+n \\ i<j<M+n}} & (A)_{\substack{M+1\leqslant i<M+n \\ M+n\leqslant j\leqslant M+N}} \\ & & & 0 & (C)_{\substack{M+n\leqslant i\leqslant M+N \\ i<j\leqslant M+N}} \\ & & & & 0 \end{array} \right|$$

$$S_3 = \left| \begin{array}{ccccc} 0 & (B)_{\substack{1\leqslant i<m \\ i<j<m}} & (A)_{\substack{1\leqslant i<m \\ m\leqslant j\leqslant M}} & (B)_{\substack{1\leqslant i<m \\ M+1\leqslant j\leqslant M+n}} & (A)_{\substack{1\leqslant i<m \\ M+n<j\leqslant M+N}} \\ & 0 & (C)_{\substack{m\leqslant i\leqslant M \\ i<j\leqslant M}} & (-A^T)_{\substack{m\leqslant i\leqslant M \\ M+1\leqslant j\leqslant M+n}} & (C)_{\substack{m\leqslant i\leqslant M \\ M+n<j\leqslant M+N}} \\ & & 0 & (B)_{\substack{M+1\leqslant i\leqslant M+n \\ i<j\leqslant M+n}} & (A)_{\substack{M+1\leqslant i\leqslant M+n \\ M+n<j\leqslant M+N}} \\ & & & 0 & (C)_{\substack{M+n<i\leqslant M+N \\ i<j\leqslant M+N}} \\ & & & & 0 \end{array} \right|$$

$$S_4 = \left| \begin{array}{ccccc} 0 & (B)_{\substack{1\leqslant i\leqslant m \\ i<j\leqslant m}} & (A)_{\substack{1\leqslant i\leqslant m \\ m<j\leqslant M}} & (B)_{\substack{1\leqslant i\leqslant m \\ M+1\leqslant j\leqslant M+n}} & (A)_{\substack{1\leqslant i\leqslant m \\ M+n<j\leqslant M+N}} \\ & 0 & (C)_{\substack{m<i\leqslant M \\ i<j\leqslant M}} & (-A^T)_{\substack{m<i\leqslant M \\ M+1\leqslant j\leqslant M+n}} & (C)_{\substack{m<i\leqslant M \\ M+n<j\leqslant M+N}} \\ & & 0 & (B)_{\substack{M+1\leqslant i\leqslant M+n \\ i<j\leqslant M+n}} & (A)_{\substack{M+1\leqslant i\leqslant M+n \\ M+n<j\leqslant M+N}} \\ & & & 0 & (C)_{\substack{M+n<i\leqslant M+N \\ i<j\leqslant M+N}} \\ & & & & 0 \end{array} \right|$$

$$S_5 = \left| \begin{array}{ccccc} 0 & (C)_{\substack{1\leqslant i\leqslant m\\ i<j\leqslant m}} & (-A^T)_{\substack{1\leqslant i\leqslant m\\ m<j\leqslant M}} & (C)_{\substack{1\leqslant i\leqslant m\\ M+1\leqslant j\leqslant M+n}} & (-A^T)_{\substack{1\leqslant i\leqslant m\\ M+n<j\leqslant M+N}} \\ & 0 & (B)_{\substack{m<i\leqslant M\\ i<j\leqslant M}} & (A)_{\substack{m<i\leqslant M\\ M+1\leqslant j\leqslant M+n}} & (B)_{\substack{m<i\leqslant M\\ M+n<j\leqslant M+N}} \\ & & 0 & (C)_{\substack{M+1\leqslant i\leqslant M+n\\ i<j\leqslant M+n}} & (-A^T)_{\substack{M+1\leqslant i\leqslant M+n\\ M+n<j\leqslant M+N}} \\ & & & 0 & (B)_{\substack{M+n<i\leqslant M+N\\ i<j\leqslant M+N}} \\ & & & & 0 \end{array} \right|$$

$$S_6 = \left| \begin{array}{ccccc} 0 & (C)_{\substack{1\leqslant i<m\\ i<j<m}} & (-A^T)_{\substack{1\leqslant i<m\\ m\leqslant j\leqslant M}} & (C)_{\substack{1\leqslant i<m\\ M+1\leqslant j\leqslant M+n}} & (-A^T)_{\substack{1\leqslant i<m\\ M+n<j\leqslant M+N}} \\ & 0 & (B)_{\substack{m\leqslant i\leqslant M\\ i<j\leqslant M}} & (A)_{\substack{m\leqslant i\leqslant M\\ M+1\leqslant j\leqslant M+n}} & (B)_{\substack{m\leqslant i\leqslant M\\ M+n<j\leqslant M+N}} \\ & & 0 & (C)_{\substack{M+1\leqslant i\leqslant M+n\\ i<j\leqslant M+n}} & (-A^T)_{\substack{M+1\leqslant i\leqslant M+n\\ M+n<j\leqslant M+N}} \\ & & & 0 & (B)_{\substack{M+n<i\leqslant M+N\\ i<j\leqslant M+N}} \\ & & & & 0 \end{array} \right|$$

$$S_7 = \left| \begin{array}{ccccc} 0 & (C)_{\substack{1\leqslant i\leqslant m\\ i<j\leqslant m}} & (-A^T)_{\substack{1\leqslant i\leqslant m\\ m<j\leqslant M}} & (C)_{\substack{1\leqslant i\leqslant m\\ M+1\leqslant j<M+n}} & (-A^T)_{\substack{1\leqslant i\leqslant m\\ M+n\leqslant j\leqslant M+N}} \\ & 0 & (B)_{\substack{m<i\leqslant M\\ i<j\leqslant M}} & (A)_{\substack{m<i\leqslant M\\ M+1\leqslant j<M+n}} & (B)_{\substack{m<i\leqslant M\\ M+n\leqslant j\leqslant M+N}} \\ & & 0 & (C)_{\substack{M+1\leqslant i<M+n\\ i<j<M+n}} & (-A^T)_{\substack{M+1\leqslant i<M+n\\ M+n\leqslant j\leqslant M+N}} \\ & & & 0 & (B)_{\substack{M+n\leqslant i\leqslant M+N\\ i<j\leqslant M+N}} \\ & & & & 0 \end{array} \right|$$

$$S_8 = \left| \begin{array}{ccccc} 0 & (C)_{\substack{1\leqslant i<m\\ i<j<m}} & (-A^T)_{\substack{1\leqslant i<m\\ m\leqslant j\leqslant M}} & (C)_{\substack{1\leqslant i<m\\ M+1\leqslant j<M+n}} & (-A^T)_{\substack{1\leqslant i<m\\ M+n\leqslant j\leqslant M+N}} \\ & 0 & (B)_{\substack{m\leqslant i\leqslant M\\ i<j\leqslant M}} & (A)_{\substack{m\leqslant i\leqslant M\\ M+1\leqslant j<M+n}} & (B)_{\substack{m\leqslant i\leqslant M\\ M+n\leqslant j\leqslant M+N}} \\ & & 0 & (C)_{\substack{M+1\leqslant i<M+n\\ i<j<M+n}} & (-A^T)_{\substack{M+1\leqslant i<M+n\\ M+n\leqslant j\leqslant M+N}} \\ & & & 0 & (B)_{\substack{M+n\leqslant i\leqslant M+N\\ i<j\leqslant M+N}} \\ & & & & 0 \end{array} \right|$$

It may be shown (see Appendix) that each of the S_i $(i = 1...8)$ may be factorized into skew-symmetric determinants defined in terms of minors of B and C; to be precise,

$$
\begin{aligned}
S_1 &= C_{\geqslant;\geqslant}B_{<;<}, & S_2 &= C_{>;\geqslant}B_{\leqslant;<}, \\
S_3 &= C_{\geqslant;>}B_{<;\leqslant}, & S_4 &= C_{>;>}B_{\leqslant;\leqslant}, \\
S_5 &= C_{\leqslant;\leqslant}B_{>;>}, & S_6 &= C_{<;\leqslant}B_{\geqslant;>}, \\
S_7 &= C_{\leqslant;<}B_{>;\geqslant}, & S_8 &= C_{<;<}B_{\geqslant;\geqslant}.
\end{aligned} \tag{3.32}
$$

The subscript notation in (3.32) is used to denote certain principal minors of an $(M+N)\times(M+N)$-dimensional matrix for fixed m and n. For instance, $X_{<;\leqslant}$ means the minor formed from rows and columns $1,\cdots,m-1,M+1,\cdots,M+n$ of the corresponding matrix X. In general, let W be a skew-symmetric $(M+N)\times(M+N)$-dimensional matrix with the entries

$$
W = \begin{pmatrix} X & Y \\ -Y^T & Z \end{pmatrix} \tag{3.33}
$$

where X is a $M\times M$, Y is a $M\times N$ and Z is a $N\times N$ dimensional block matrix. For a given m, n ($1\le m\le M$ and $1\le n\le N$) and inequalities $\prec_1$, $\prec_2$ (where $\prec_1$ and $\prec_2$ can be $<$, $\leqslant$, $>$, and $\geqslant$), we define $\tilde{W}$ to be a particular sub-matrix of W:

$$
\tilde{W} = \begin{pmatrix} (X)_{\substack{i\prec_1 m\\ j\prec_1 m}} & (Y)_{\substack{i\prec_1 m\\ j\prec_2 M+n}} \\ (-Y^T)_{\substack{i\prec_2 M+n\\ j\prec_1 m}} & (Z)_{\substack{i\prec_2 M+n\\ j\prec_2 M+n}} \end{pmatrix}
$$

then $W_{\prec_1;\prec_2} = \det(\tilde{W})$. We also define the pfaffian

$$
w_{\prec_1;\prec_2} = \begin{cases} Pf(\tilde{W}) & \text{for even-dimension} \\ Pf\begin{pmatrix} \tilde{W} & \epsilon \\ -\epsilon^T & 0 \end{pmatrix} & \text{for odd-dimension} \end{cases}
$$

where ϵ is the column matrix with all entries equal to 1.

We give an example on this for a $(2,2)$-dromion solution and a $(1,2)$-dromion solution in the next section.

Further, minors of B in (3.32) can also be factorized (see Appendix) into other minor skew-symmetric determinants, defined in terms of parameters k_i and l_j, which are given in (3.24) and (3.25) respectively. Hence, we have the fully factorized form of the skew-symmetric determinants S_i:

$$
\begin{aligned}
S_1 &= C_{\geqslant;\geqslant}K_{<}L_{<}, & S_2 &= C_{>;\geqslant}K_{\leqslant}L_{<}, \\
S_3 &= C_{\geqslant;>}K_{<}L_{\leqslant}, & S_4 &= C_{>;>}K_{\leqslant}L_{\leqslant}, \\
S_5 &= C_{\leqslant;\leqslant}K_{>}L_{>}, & S_6 &= C_{<;\leqslant}K_{\geqslant}L_{>}, \\
S_7 &= C_{\leqslant;<}K_{>}L_{\geqslant}, & S_8 &= C_{<;<}K_{\geqslant}L_{\geqslant},
\end{aligned}
\tag{3.34}
$$

where the single subscript notations are defined in a similar fashion. In general, for a given m and n ($1 \leq m \leq M$ and $1 \leq n \leq N$) and inequality $\prec$, (where $\prec$ can be $<$, $\leqslant$, $>$, and $\geqslant$), we define $\tilde{K}$ to be the sub-matrix of the ($M \times M$) block-matrix K in (3.23)

$$\tilde{K} = (K)_{\substack{i \prec m \\ j \prec m}}.$$

Then $K_{\prec} = \det(\tilde{K})$ and we also define the pfaffian

$$
k_{\prec} = \begin{cases} Pf(\tilde{K}) & \text{for even-dimension} \\ Pf\begin{pmatrix} \tilde{K} & \epsilon \\ -\epsilon^T & 0 \end{pmatrix} & \text{for odd-dimension} \end{cases}
$$

where $\epsilon = (-1, 1, \ldots, \pm 1)$. Similiarly, we define $\tilde{L}$ to be the sub-matrix of the ($N \times N$) block-matrix L in (3.23)

$$\tilde{L} = (L)_{\substack{i \prec n \\ j \prec n}},$$

$L_{\prec} = \det(\tilde{L})$ and

$$
l_{\prec} = \begin{cases} Pf(\tilde{L}) & \text{for even-dimension} \\ Pf\begin{pmatrix} \tilde{L} & \epsilon \\ -\epsilon^T & 0 \end{pmatrix} & \text{for odd-dimension.} \end{cases}
$$

Now we write the pfaffians P_i (i=1 ... 8) (3.31) of the skew-symmetric determinants S_i (i=1 ... 8) in terms of minor pfaffians of the corresponding minor determinants formed from C, B in (3.32) and C, K, L in (3.34). We will denote the pfaffians with the lowercase letters of the corresponding uppercase letters that have been used for denoting skew-symmetric determinants. For instance, $c_{\leqslant;<}$ is the pfaffian corresponding to the skew-symmetric determinant $C_{\leqslant;<}$. From (3.32) we have the pfaffians $|P_i| = (S_i)^{\frac{1}{2}}$:

$$\begin{aligned}
|P_1| &= |c_{\geqslant;\geqslant}b_{<;<}|, & |P_2| &= |c_{>;\geqslant}b_{\leqslant;<}|,\\
|P_3| &= |c_{\geqslant;>}b_{<;\leqslant}|, & |P_4| &= |c_{>;>}b_{\leqslant;\leqslant}|,\\
|P_5| &= |c_{\leqslant;\leqslant}b_{>;>}|, & |P_6| &= |c_{<;\leqslant}b_{\geqslant;>}|,\\
|P_7| &= |c_{\leqslant;<}b_{>;\geqslant}|, & |P_8| &= |c_{<;<}b_{\geqslant;\geqslant}|,
\end{aligned}$$

and the pfaffians in the fully factorized form, from (3.34):

$$\begin{aligned}
|P_1| &= |c_{\geqslant;\geqslant}k_{<}l_{<}|, & |P_2| &= |c_{>;\geqslant}k_{\leqslant}l_{<}|,\\
|P_3| &= |c_{\geqslant;>}k_{<}l_{\leqslant}|, & |P_4| &= |c_{>;>}k_{\leqslant}l_{\leqslant}|,\\
|P_5| &= |c_{\leqslant;\leqslant}k_{>}l_{>}|, & |P_6| &= |c_{<;\leqslant}k_{\geqslant}l_{>}|,\\
|P_7| &= |c_{\leqslant;<}k_{>}l_{\geqslant}|, & |P_8| &= |c_{<;<}k_{\geqslant}l_{\geqslant}|.
\end{aligned}$$

Here the pfaffian c determines the phase shifts $F_\perp$ between the interacting perpendicular plane waves and determines the amplitude of the dromions. The pfaffian b determines the phase shifts between the two sets of parallel plane waves in the V-plane. These interpretations may be made, because, if we choose the entries of pfaffian c so that the perpendicular phase shift and hence the amplitudes of the dromions vanish then the only phase shifts we get, determined by the pfaffian b, are the parallel phase shifts experienced by the parallel plane waves in the V-plane. This is achieved by setting all arbitrary constants in c to be 1, so that all minor pfaffians $c_{\prec_1;\prec_2}$ are equal to 1.

Then the asymptotic expressions in (3.31) can be written
as $t \to -\infty$

$$P_- = |b_{<;<}| + |b_{\leqslant;<}|e^{\eta_m} + |b_{<;\leqslant}|e^{\rho_n} + |b_{\leqslant;\leqslant}|e^{\eta_m+\rho_n}, \tag{3.35}$$

and as $t \to \infty$

$$P_+ = |b_{>;>}| + |b_{\geqslant;>}|e^{\eta_m} + |b_{>;\geqslant}|e^{\rho_n} + |b_{\geqslant;\geqslant}|e^{\eta_m+\rho_n}. \tag{3.36}$$

(3.31) can also be written in the following way as $t \to -\infty$

$$\begin{aligned}
P_- &= |k_< l_<| + |k_\lessgtr l_<| e^{\eta_m} + |k_< l_\lessgtr| e^{\rho_n} + |k_\lessgtr l_\lessgtr| e^{\eta_m+\rho_n} \\
&\sim 1 + \left|\frac{k_\lessgtr}{k_<}\right| e^{\eta_m} + \left|\frac{l_\lessgtr}{l_<}\right| e^{\rho_n} + \left|\frac{k_\lessgtr l_\lessgtr}{k_< l_<}\right| e^{\eta_m+\rho_n} \\
&= 1 + e^{\eta_m + F^-_{\|x}} + e^{\rho_n + F^-_{\|y}} + e^{\eta_m+\rho_n+F^-_{\|x}+F^-_{\|y}}
\end{aligned}$$

and as $t \to \infty$

$$\begin{aligned}
P_+ &= |k_> l_>| + |k_\gtrless l_>| e^{\eta_m} + |k_> l_\gtrless| e^{\rho_n} + |k_\gtrless l_\gtrless| e^{\eta_m+\rho_n} \\
&\sim 1 + \left|\frac{k_\gtrless}{k_>}\right| e^{\eta_m} + \left|\frac{l_\gtrless}{l_>}\right| e^{\rho_n} + \left|\frac{k_\gtrless l_\gtrless}{k_> l_>}\right| e^{\eta_m+\rho_n} \\
&= 1 + e^{\eta_m + F^+_{\|x}} + e^{\rho_n + F^+_{\|y}} + e^{\eta_m+\rho_n+F^+_{\|x}+F^+_{\|y}}
\end{aligned}$$

where $F^\pm_{\|x}$ and $F^\pm_{\|y}$ are the relative phase shifts experienced by the (x,t)-dependent and (y,t)-dependent parallel plane waves in the V-plane respectively. This is a case in which all of the dromions have zero amplitude and the solution for this case is $U \equiv 0$. As a consequence of these two pfaffians b and c being independent, these two kinds of phase shifts, determined by b and c, are independent from each other.

We have now shown that the (M,N)-dromion solution U decomposes into $M \times N$ dromions determined by (3.31) as $t \to -\infty$ and as $t \to \infty$. To identify the properties of the resulting dromions, we compare the asymptotic expressions in (3.31) with (3.13) generalizing Theorem 3.3.1, and give the theorem for the general case:

Theorem 3.4.1

$$U = \partial_x \partial_y (\log(S)) \quad \textit{and} \quad V = \tfrac{1}{2}(\partial_x^2 + \partial_y^2)(\log(S))$$

with S given by (3.22) have the following properties:

1) V is the interaction of M plane-wave solitons parallel to the y-axis and N plane-wave solitons parallel to the x-axis which decomposes asymptotically into $M \times N$ solutions as described in Theorem 3.3.1 part 1.

2) U decomposes asymptotically into $M \times N$ dromions as described in Theorem 3.3.1 part 2. The amplitude of the (m,n)th dromion is

$$U_0^- = \frac{1}{2}\, k_m \, l_n \frac{e^{\frac{1}{2}F_\perp^-} - 1}{e^{\frac{1}{2}F_\perp^-} + 1} \qquad \textit{and} \qquad U_0^+ = \frac{1}{2}\, k_m \, l_n \frac{e^{\frac{1}{2}F_\perp^+} - 1}{e^{\frac{1}{2}F_\perp^+} + 1} \tag{3.37}$$

as $t \to -\infty$ and $t \to \infty$ respectively, and the perpendicular phase shifts are

$$F_\perp^- = \log\left|\frac{c_{>;>}c_{\gtrless;\gtrless}}{c_{\gtrless;>}c_{>;\gtrless}}\right| \qquad \textit{and} \qquad F_\perp^+ = \log\left|\frac{c_{<;<}c_{\lessgtr;\lessgtr}}{c_{\lessgtr;<}c_{<;\lessgtr}}\right| \tag{3.38}$$

as $t \to -\infty$ and $t \to \infty$ respectively.

The location at time t of the (m,n)th dromion moves from

$$(x,y) = \left(\frac{1}{2k_m}\log\left|\frac{c_{\gtrless;\gtrless}c_{\gtrless;>}k_<^2}{c_{>;\gtrless}c_{>;>}k_\lessgtr^2}\right| - k_m^2 t, \frac{1}{2l_n}\log\left|\frac{c_{>;\gtrless}c_{>;\gtrless}l_<^2}{c_{\gtrless;>}c_{>;>}l_\lessgtr^2}\right| - l_n^2 t\right)$$

as $t \to -\infty$ to

$$(x,y) = \left(\frac{1}{2k_m}\log\left|\frac{c_{\lessgtr;\lessgtr}c_{\lessgtr;<}k_>^2}{c_{<;\lessgtr}c_{<;<}k_\gtrless^2}\right| - k_m^2 t, \frac{1}{2l_n}\log\left|\frac{c_{\lessgtr;\lessgtr}c_{<;\lessgtr}l_>^2}{c_{\lessgtr;<}c_{<;<}l_\gtrless^2}\right| - l_n^2 t\right)$$

as $t \to \infty$, giving the two-dimensional phase shift due to all interactions

$$\left(\frac{1}{2k_m}\log\left|\frac{c_{\lessgtr;\lessgtr}c_{\lessgtr;<}c_{>;\gtrless}c_{>;>}k_>^2 k_\lessgtr^2}{c_{<;\lessgtr}c_{<;<}c_{\gtrless;\gtrless}c_{\gtrless;>}k_<^2 k_\gtrless^2}\right|, \frac{1}{2l_n}\log\left|\frac{c_{\lessgtr;\lessgtr}c_{<;\lessgtr}c_{\gtrless;>}c_{>;>}l_>^2 l_\lessgtr^2}{c_{\lessgtr;<}c_{<;<}c_{\gtrless;\gtrless}c_{>;\gtrless}l_<^2 l_\gtrless^2}\right|\right).$$

Summary of results for (M,N)-dromion:

We observe from Theorem 3.4.1 that the summary of results for $(1,1)$-dromion can be generalized to (M,N)-dromion. The main result is that there are $M \times N$ dromions in the U-plane and $M+N$ perpendicular plane waves in the V-plane. We observe further that

- The (m,n)th dromion may have arbitrary amplitude, positive, negative or zero and varies as $t \to -\infty$ and $t \to \infty$. The amplitude is

  1. positive

     (a) as $t \to -\infty$: if $k_m l_n > 0$ and $|c_{>;>}c_{\gtrless;\gtrless}| > |c_{\gtrless;>}c_{>;\gtrless}|$

     or $k_m l_n < 0$ and $0 < \left|\frac{c_{>;>}c_{\gtrless;\gtrless}}{c_{\gtrless;>}c_{>;\gtrless}}\right| < 1$,

     (b) as $t \to \infty$: if $k_m l_n > 0$ and $|c_{<;<}c_{\lessgtr;\lessgtr}| > |c_{\lessgtr;<}c_{<;\lessgtr}|$

     or $k_m l_n < 0$ and $0 < \left|\frac{c_{<;<}c_{\lessgtr;\lessgtr}}{c_{\lessgtr;<}c_{<;\lessgtr}}\right| < 1$,

  2. negative

(a) as $t \to -\infty$: if $k_m l_n < 0$ and $|c_{>;>}c_{\geqslant;\geqslant}| > |c_{\geqslant;>}c_{>;\geqslant}|$

or $k_m l_n > 0$ and $0 < \left|\dfrac{c_{>;>}c_{\geqslant;\geqslant}}{c_{\geqslant;>}c_{>;\geqslant}}\right| < 1$,

(b) as $t \to \infty$: if $k_m l_n < 0$ and $|c_{<;<}c_{\leqslant;\leqslant}| > |c_{\leqslant;<}c_{<;\leqslant}|$

or $k_m l_n > 0$ and $0 < \left|\dfrac{c_{<;<}c_{\leqslant;\leqslant}}{c_{\leqslant;<}c_{<;\leqslant}}\right| < 1$,

3. zero

(a) as $t \to -\infty$: if $k_m l_n = 0$ or $|c_{>;>}c_{\geqslant;\geqslant}| = |c_{\geqslant;>}c_{>;\geqslant}|$,

(b) as $t \to \infty$: if $k_m l_n = 0$ or $|c_{<;<}c_{\leqslant;\leqslant}| = |c_{\leqslant;<}c_{<;\leqslant}|$

- The plane waves in the V-plane always have positive amplitude and exert a phase-shift on one another. The directions of these phase shifts (forward or backward) depend on the signs of k_m, l_n and

$$\log\left|\frac{c_{>;>}c_{\geqslant;\geqslant}}{c_{\geqslant;>}c_{>;\geqslant}}\right| \quad \text{or} \quad \log\left|\frac{c_{<;<}c_{\leqslant;\leqslant}}{c_{\leqslant;<}c_{<;\leqslant}}\right|.$$

The phase shifts in (3.38) are zero if and only if the corresponding dromion amplitude is zero.

- At any fixed time, the (m,n)th dromion (in the U-plane) is symmetrically located between the mth plane-wave soliton parallel to the y-axis and the nth plane-wave soliton parallel to the x-axis (in the V-plane).

3.5 A class of $(2,2)$-dromion solutions

In this section we apply the general results obtained in section 3.4 on the asymptotics of the (M,N)-dromion solution to the $(2,2)$-dromion solution. In order to clarify the results of the last section, we present the asymptotics explicitly for these solutions. From (3.21) the $(2,2)$-dromion solution is

$$S^{(2,2)} = \begin{vmatrix} 0 & \Theta_{12} & \Theta_{13} & \Theta_{14} \\ -\Theta_{12}^T & 0 & \Theta_{23} & \Theta_{24} \\ -\Theta_{13}^T & -\Theta_{23}^T & 0 & \Theta_{34} \\ -\Theta_{14}^T & -\Theta_{24}^T & -\Theta_{34}^T & 0 \end{vmatrix} \tag{3.39}$$

where

$$\begin{aligned}
\Theta_{12} &= c_{12} - e^{\eta_1} - e^{\eta_2} - \frac{k_2 - k_1}{k_2 + k_1} e^{\eta_1 + \eta_2} \\
\Theta_{13} &= c_{13} - e^{\eta_1} - e^{\rho_1} - e^{\eta_1 + \rho_1} \\
\Theta_{14} &= c_{14} - e^{\eta_1} + e^{\rho_2} + e^{\eta_1 + \rho_2} \\
\Theta_{23} &= c_{23} + e^{\eta_2} - e^{\rho_1} + e^{\eta_2 + \rho_1} \\
\Theta_{24} &= c_{24} + e^{\eta_2} + e^{\rho_2} - e^{\eta_2 + \rho_2} \\
\Theta_{34} &= c_{34} + e^{\rho_1} + e^{\rho_2} + \frac{l_2 - l_1}{l_2 + l_1} e^{\rho_1 + \rho_2}.
\end{aligned}$$

Here the arbitrary constants $c_{12}, c_{13}, c_{14}, c_{23}, c_{24}, c_{34}$ will be chosen so that the solution has no singularities. The conditions on the constants which give this property will be found by considering the asymptotic form of all of the $(2) \times (2)$ dromions. Also,

$$\eta_1 = k_1(x + k_1^2 t) \qquad \eta_2 = k_2(x + k_2^2 t)$$

$$\rho_1 = l_1(y + l_1^2 t) \qquad \rho_2 = l_2(y + l_2^2 t),$$

where the parameters k_1, k_2, l_1, l_2 have the ordering

$$0 < k_1 < k_2 \qquad \text{and} \qquad 0 < l_1 < l_2.$$

Next we expand the (2,2)-dromion solution (3.39), as in the form (3.22)

$$S^{(2,2)} = |\, C + D\,A - A^T\,D + D\,B\,D\,|, \tag{3.40}$$

where

$$A = \begin{pmatrix} 0 & -1 & -1 & -1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{pmatrix}, \tag{3.41}$$

$$B = \begin{pmatrix} 0 & -\frac{k_2-k_1}{k_2+k_1} & -1 & 1 \\ \frac{k_2-k_1}{k_2+k_1} & 0 & 1 & -1 \\ 1 & -1 & 0 & \frac{l_2-l_1}{l_2+l_1} \\ -1 & 1 & -\frac{l_2-l_1}{l_2+l_1} & 0 \end{pmatrix}, \tag{3.42}$$

$$C=\begin{pmatrix} 0 & c_{12} & c_{13} & c_{14} \\ -c_{12} & 0 & c_{23} & c_{24} \\ -c_{13} & -c_{23} & 0 & c_{34} \\ -c_{14} & -c_{24} & -c_{34} & 0 \end{pmatrix} \tag{3.43}$$

and

$$D = \mathrm{diag}(\ e^{\eta_1},\ e^{\eta_2};\ e^{\rho_1},\ e^{\rho_2}\ ).$$

To obtain the asymptotic solution for the $(2,2)$-dromion solution we focus on, for example, the 2nd (x,t)-dependent plane wave and the 1st (y,t)-dependent plane wave, hence η_2 and ρ_1 are fixed, and the corresponding dromion is the $(2,1)$th dromion. Then, from (3.31), one can write the asymptotic solution for the $(2,2)$-dromion solution

$$\begin{aligned} P_-^{(2,2)} &= |P_1^{(2,2)}| + |P_2^{(2,2)}|e^{\eta_2} + |P_3^{(2,2)}|e^{\rho_1} + |P_4^{(2,2)}|e^{\eta_2+\rho_1} \\ P_+^{(2,2)} &= |P_5^{(2,2)}| + |P_6^{(2,2)}|e^{\eta_2} + |P_7^{(2,2)}|e^{\rho_1} + |P_8^{(2,2)}|e^{\eta_2+\rho_1}, \end{aligned} \tag{3.44}$$

where the pfaffians $P_i^{(2,2)}$ $(i = 1...8)$ are defined in terms of minors of A, B, C in (3.41), (3.42) and (3.43) respectively.

We now represent the pfaffians $P_i^{(2,2)}$ of the corresponding skew-symmetric determinants $S_i^{(2,2)}$ with the notations used for the general case in section 3.4. To show this, we rearrange the skew-symmetric determinants $S_i^{(2,2)}$ and its block matrices, formed from minors of the matrices (3.41), (3.42), (3.43), (for further explanation see section 3.4). It is done as follows: If the entries of any of the determinants $S_i^{(2,2)}$, formed from minors of the matrices (3.41), (3.42), (3.43), are not in the block structure, then we interchange the appropriate rows and columns of the determinants $S_i^{(2,2)}$. This is the case for $S_3^{(2,2)}$ and $S_6^{(2,2)}$ and in each case we need to interchange a pair of rows and a pair of columns to make them factorizable (see Appendix for the general case). If the block entries of any of the determinants $S_i^{(2,2)}$ are odd-dimensional, then we enlarge the determinants $S_i^{(2,2)}$ by adding an extra row and column to each block matrix, so that the block matrices are not singular, yet the determinant of the overall matrix is unchanged. The extra row and column to the block matrix formed from (3.41) are in the same structure as the block matrix itself, so that the enlarged sub-matrix is a rank-1 matrix. The extra column to the block matrix formed from (3.42) is in the form $\epsilon = (\ -1,\ 1,\ -1,\ 1, \ldots,\ \pm 1)$, and the corresponding row is $-\epsilon^T$, and the extra column to the block matrix formed from (3.43) is in the form $\epsilon = (1, 1, 1, \ldots, 1)$, and the corresponding row is $-\epsilon^T$.

In the following, first we show the skew-symmetric determinants $S_i^{(2,2)}$ in terms of minor skew-symmetric determinants, formed from (3.42) and (3.43), and then the pfaffians $P_i^{(2,2)}$ in terms of minor pfaffians of the minor skew-symmetric determinants.

The $S_i^{(2,2)} = \left(P_i^{(2,2)}\right)^2$ are as follows:

$$
S_1^{(2,2)} = \left|\begin{array}{c|ccc} 0 & -1 & -1 & -1 \\ \hline 1 & 0 & c_{23} & c_{24} \\ 1 & -c_{23} & 0 & c_{34} \\ 1 & -c_{24} & -c_{34} & 0 \end{array}\right| = \left|\begin{array}{cc|cccc} 0 & -1 & -1 & -1 & -1 & -1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ \hline 1 & -1 & 0 & c_{23} & c_{24} & 1 \\ 1 & -1 & -c_{23} & 0 & c_{34} & 1 \\ 1 & -1 & -c_{24} & -c_{34} & 0 & 1 \\ 1 & -1 & -1 & -1 & -1 & 0 \end{array}\right|
$$

$$
= \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} \begin{vmatrix} 0 & c_{23} & c_{24} & 1 \\ -c_{23} & 0 & c_{34} & 1 \\ -c_{24} & -c_{34} & 0 & 1 \\ -1 & -1 & -1 & 0 \end{vmatrix} = B_{<;<}\, C_{\geqslant;\geqslant} = (c_{23} - c_{24} + c_{34})^2
$$

$$
|P_1^{(2,2)}| = |b_{<;<}\, c_{\geqslant;\geqslant}| = |c_{23} - c_{24} + c_{34}|
$$

$$
S_2^{(2,2)} = \left|\begin{array}{cc|cc} 0 & -\frac{k_2-k_1}{k_2+k_1} & -1 & -1 \\ \frac{k_2-k_1}{k_2+k_1} & 0 & 1 & 1 \\ \hline 1 & -1 & 0 & c_{34} \\ 1 & -1 & -c_{34} & 0 \end{array}\right|
$$

$$
= \begin{vmatrix} 0 & -\frac{k_2-k_1}{k_2+k_1} \\ \frac{k_2-k_1}{k_2+k_1} & 0 \end{vmatrix} \begin{vmatrix} 0 & c_{34} \\ -c_{34} & 0 \end{vmatrix} = B_{\leqslant;<}\, C_{>;\geqslant} = \left(c_{34}\frac{k_2-k_1}{k_2+k_1}\right)^2
$$

$$
|P_2^{(2,2)}| = |b_{\leqslant;<}\, c_{>;\geqslant}| = \left|c_{34}\frac{k_2-k_1}{k_2+k_1}\right|
$$

$$S_3^{(2,2)} = \begin{vmatrix} 0 & -1 & -1 & -1 \\ 1 & 0 & -1 & c_{24} \\ 1 & 1 & 0 & 1 \\ 1 & -c_{24} & -1 & 0 \end{vmatrix} = \left|\begin{array}{cc|cc} 0 & -1 & -1 & -1 \\ 1 & 0 & 1 & 1 \\ \hline 1 & -1 & 0 & c_{24} \\ 1 & -1 & -c_{24} & 0 \end{array}\right|$$

$$= \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} \begin{vmatrix} 0 & c_{24} \\ -c_{24} & 0 \end{vmatrix} = B_{<;\leqslant}\, C_{\geqslant;>} = (c_{24})^2$$

$$|P_3^{(2,2)}| = |b_{<;\leqslant}\, c_{\geqslant;>}| = |c_{24}|$$

$$S_4^{(2,2)} = \left|\begin{array}{ccc|c} 0 & -\frac{k_2-k_1}{k_2+k_1} & -1 & -1 \\ \frac{k_2-k_1}{k_2+k_1} & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ \hline 1 & -1 & -1 & 0 \end{array}\right| = \left|\begin{array}{cccc|cc} 0 & -\frac{k_2-k_1}{k_2+k_1} & -1 & -1 & -1 & -1 \\ \frac{k_2-k_1}{k_2+k_1} & 0 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & -1 & 1 & 1 \\ 1 & -1 & 1 & 0 & -1 & -1 \\ \hline 1 & -1 & -1 & 1 & 0 & 1 \\ 1 & -1 & -1 & 1 & -1 & 0 \end{array}\right|$$

$$= \begin{vmatrix} 0 & -\frac{k_2-k_1}{k_2+k_1} & -1 & -1 \\ \frac{k_2-k_1}{k_2+k_1} & 0 & 1 & 1 \\ 1 & -1 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{vmatrix} \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} = B_{\leqslant;\leqslant}\, C_{>;>} = \left(\frac{k_2-k_1}{k_2+k_1}\right)^2$$

$$|P_4^{(2,2)}| = |b_{\leqslant;\leqslant}\, c_{>;>}| = \left|\frac{k_2-k_1}{k_2+k_1}\right|$$

$$S_5^{(2,2)} = \left|\begin{array}{ccc|c} 0 & c_{12} & c_{13} & 1 \\ -c_{12} & 0 & c_{23} & 1 \\ -c_{13} & -c_{23} & 0 & 1 \\ \hline -1 & -1 & -1 & 0 \end{array}\right| = \left|\begin{array}{cccc|cc} 0 & c_{12} & c_{13} & 1 & 1 & -1 \\ -c_{12} & 0 & c_{23} & 1 & 1 & -1 \\ -c_{13} & -c_{23} & 0 & 1 & 1 & -1 \\ -1 & -1 & -1 & 0 & 1 & -1 \\ \hline -1 & -1 & -1 & -1 & 0 & -1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{array}\right|$$

$$= \begin{vmatrix} 0 & c_{12} & c_{13} & 1 \\ -c_{12} & 0 & c_{23} & 1 \\ -c_{13} & -c_{23} & 0 & 1 \\ -1 & -1 & -1 & 0 \end{vmatrix} \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} = C_{\leqslant;\leqslant}\, B_{>;>} = (c_{12} - c_{13} + c_{23})^2$$

$$|P_5^{(2,2)}| = |c_{\leqslant;\leqslant}\, b_{>;>}| = |c_{12} - c_{13} + c_{23}|$$

$$
\begin{aligned}
S_6^{(2,2)} &= \left|\begin{array}{cccc} 0 & -1 & c_{13} & 1 \\ 1 & 0 & 1 & -1 \\ -c_{13} & -1 & 0 & 1 \\ -1 & 1 & -1 & 0 \end{array}\right| = \left|\begin{array}{cc|cc} 0 & c_{13} & -1 & 1 \\ -c_{13} & 0 & -1 & 1 \\ \hline 1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{array}\right| \\
&= \left|\begin{array}{cc} 0 & c_{13} \\ -c_{13} & 0 \end{array}\right| \left|\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right| = C_{<;\leqslant}\, B_{\geqslant;>} = (c_{13})^2 \\
|P_6^{(2,2)}| &= |c_{<;\leqslant}\, b_{\geqslant;>}| = |c_{13}| \\
S_7^{(2,2)} &= \left|\begin{array}{cc|cc} 0 & c_{12} & -1 & 1 \\ -c_{12} & 0 & -1 & 1 \\ \hline 1 & 1 & 0 & \frac{l_2-l_1}{l_2+l_1} \\ -1 & -1 & -\frac{l_2-l_1}{l_2+l_1} & 0 \end{array}\right| \\
&= \left|\begin{array}{cc} 0 & c_{12} \\ -c_{12} & 0 \end{array}\right| \left|\begin{array}{cc} 0 & \frac{l_2-l_1}{l_2+l_1} \\ -\frac{l_2-l_1}{l_2+l_1} & 0 \end{array}\right| = C_{\leqslant;<}\, B_{>;\geqslant} = \left(c_{12}\tfrac{l_2-l_1}{l_2+l_1}\right)^2 \\
|P_7^{(2,2)}| &= |c_{\leqslant;<}\, b_{>;\geqslant}| = \left|c_{12}\tfrac{l_2-l_1}{l_2+l_1}\right| \\
S_8^{(2,2)} &= \left|\begin{array}{c|ccc} 0 & -1 & -1 & 1 \\ \hline 1 & 0 & 1 & -1 \\ 1 & -1 & 0 & \frac{l_2-l_1}{l_2+l_1} \\ -1 & 1 & -\frac{l_2-l_1}{l_2+l_1} & 0 \end{array}\right| = \left|\begin{array}{cc|cccc} 0 & 1 & -1 & -1 & 1 & -1 \\ -1 & 0 & -1 & -1 & 1 & -1 \\ \hline 1 & 1 & 0 & 1 & -1 & -1 \\ 1 & 1 & -1 & 0 & \frac{l_2-l_1}{l_2+l_1} & 1 \\ -1 & -1 & 1 & -\frac{l_2-l_1}{l_2+l_1} & 0 & -1 \\ 1 & 1 & 1 & -1 & 1 & 0 \end{array}\right| \\
&= \left|\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right| \left|\begin{array}{cccc} 0 & 1 & -1 & -1 \\ -1 & 0 & \frac{l_2-l_1}{l_2+l_1} & 1 \\ 1 & -\frac{l_2-l_1}{l_2+l_1} & 0 & -1 \\ 1 & -1 & 1 & 0 \end{array}\right| = C_{<;<}\, B_{\geqslant;\geqslant} = \left(\tfrac{l_2-l_1}{l_2+l_1}\right)^2 \\
|P_8^{(2,2)}| &= |c_{<;<}\, b_{\geqslant;\geqslant}| = \left|\tfrac{l_2-l_1}{l_2+l_1}\right| .
\end{aligned}
$$

Hence, from (3.44), the asymptotic expressions, (when the $(2,1)$th dromion is fixed) for the $(2,2)$-dromion solution are

$$
\begin{aligned}
P_-^{(2,2)} &= |c_{23} - c_{24} + c_{34}| + \left|c_{34}\tfrac{k_2-k_1}{k_2+k_1}\right| e^{\eta_2} + |c_{24}| e^{\rho_1} + \left|\tfrac{k_2-k_1}{k_2+k_1}\right| e^{\eta_2+\rho_1} \\
P_+^{(2,2)} &= |c_{12} - c_{13} + c_{23}| + |c_{13}| e^{\eta_2} + \left|c_{12}\tfrac{l_2-l_1}{l_2+l_1}\right| e^{\rho_1} + \left|\tfrac{l_2-l_1}{l_2+l_1}\right| e^{\eta_2+\rho_1} .
\end{aligned}
$$

3.6 A class of $(1,2)$-dromion solutions

Next we show the $(1,2)$-dromion solution as a reduction from the $(2,2)$-dromion solution. As mentioned earlier, the $(2,2)$-dromion solution has two plane waves parallel to the y-axis and two plane waves parallel to the x-axis. If we cancel out one of these plane waves by making its amplitude zero, then we get the $(1,2)$-dromion solution. Let us take $k_1 = 0$. Hence we have one plane wave parallel to the y-axis and two plane waves parallel to the x-axis. From (3.39) we write the $(1,2)$-dromion solution with the arbitrary constants chosen as $c_{12} = c_{13} = c_{14} = 2$:

$$S^{(1,2)} = \begin{vmatrix} 0 & \Theta_{12} & \Theta_{13} & \Theta_{14} \\ -\Theta_{12}^T & 0 & \Theta_{23} & \Theta_{24} \\ -\Theta_{13}^T & -\Theta_{23}^T & 0 & \Theta_{34} \\ -\Theta_{14}^T & -\Theta_{24}^T & -\Theta_{34}^T & 0 \end{vmatrix}, \tag{3.45}$$

where

$$\begin{aligned} \Theta_{12} &= 1 - 2e^{\eta_2} \\ \Theta_{13} &= 1 - 2e^{\rho_1} \\ \Theta_{14} &= 1 + 2e^{\rho_2} \\ \Theta_{23} &= c_{23} + e^{\eta_2} - e^{\rho_1} + e^{\eta_2+\rho_1} \\ \Theta_{24} &= c_{24} + e^{\eta_2} + e^{\rho_2} - e^{\eta_2+\rho_2} \\ \Theta_{34} &= c_{34} + e^{\rho_1} + e^{\rho_2} + \frac{l_2 - l_1}{l_2 + l_1} e^{\rho_1+\rho_2}. \end{aligned}$$

Here the arbitrary constants c_{23}, c_{24}, c_{34} are chosen such that the solution has no singularities. The conditions, on the constants, which give this property will be found by considering the asymptotic form of all of the $(1) \times (2)$ dromions. The phases are

$$\eta_2 = k_2(x + k_2^2 t) \qquad \rho_1 = l_1(y + l_1^2 t) \qquad \rho_2 = l_2(y + l_2^2 t),$$

where the parameters k_2, l_1, l_2 have the ordering

$$0 < k_2 \qquad \text{and} \qquad 0 < l_1 < l_2.$$

Next we expand the $(1,2)$-dromion solution (3.45), as in the form (3.22), namely

$$S^{(1,2)} = |\, C \,+\, D\,A \,-\, A^T\,D \,+\, D\,B\,D\,|, \tag{3.46}$$

where

$$A = \begin{pmatrix} 0 & -1 & -1 & -1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{pmatrix}, \tag{3.47}$$

$$B = \begin{pmatrix} 0 & -1 & -1 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & -1 & 0 & \frac{l_2-l_1}{l_2+l_1} \\ -1 & 1 & -\frac{l_2-l_1}{l_2+l_1} & 0 \end{pmatrix}, \tag{3.48}$$

$$C = \begin{pmatrix} 0 & 2 & 2 & 2 \\ -2 & 0 & c_{23} & c_{24} \\ -2 & -c_{23} & 0 & c_{34} \\ -2 & -c_{24} & -c_{34} & 0 \end{pmatrix} \tag{3.49}$$

and

$$D = \mathrm{diag}(\ 1,\ e^{\eta_2};\ e^{\rho_1},\ e^{\rho_2}\ ).$$

To obtain the asymptotic solution for the $(1,2)$-dromion solution, again we focus on, for example, the (x,t)-dependent plane wave and the 1st (y,t)-dependent plane wave. Hence η_2 and ρ_1 are fixed, and the corresponding dromion is the $(2,1)$th dromion. Then, from (3.31), one can write the asymptotic solution for the $(1,2)$-dromion solution

$$\begin{aligned} P_-^{(1,2)} &= |P_1^{(1,2)}| + |P_2^{(1,2)}|e^{\eta_2} + |P_3^{(1,2)}|e^{\rho_1} + |P_4^{(1,2)}|e^{\eta_2+\rho_1} \\ P_+^{(1,2)} &= |P_5^{(1,2)}| + |P_6^{(1,2)}|e^{\eta_2} + |P_7^{(1,2)}|e^{\rho_1} + |P_8^{(1,2)}|e^{\eta_2+\rho_1}, \end{aligned} \tag{3.50}$$

where the pfaffians $P_i^{(1,2)}$ $(i = 1 \ldots 8)$ are defined in terms of minors of (3.47), (3.48), and (3.49). To apply the notations used in the last section, we rearrange the corresponding skew-symmetric determinants $S_i^{(1,2)}$ of the pfaffians $|P_i^{(1,2)}|$ in the same way as it is done for the $(2,2)$-dromion solution.

The $S_i^{(1,2)} = \left(P_i^{(1,2)}\right)^2$ are as follows:

$$S_1^{(1,2)} = \left|\begin{array}{c|ccc} 0 & -1 & -1 & -1 \\ \hline 1 & 0 & c_{23} & c_{24} \\ 1 & -c_{23} & 0 & c_{34} \\ 1 & -c_{24} & -c_{34} & 0 \end{array}\right| = \left|\begin{array}{cc|cccc} 0 & -1 & -1 & -1 & -1 & -1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ \hline 1 & -1 & 0 & c_{23} & c_{24} & 1 \\ 1 & -1 & -c_{23} & 0 & c_{34} & 1 \\ 1 & -1 & -c_{24} & -c_{34} & 0 & 1 \\ 1 & -1 & -1 & -1 & -1 & 0 \end{array}\right|$$

$$= \left|\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right| \left|\begin{array}{cccc} 0 & c_{23} & c_{24} & 1 \\ -c_{23} & 0 & c_{34} & 1 \\ -c_{24} & -c_{34} & 0 & 1 \\ -1 & -1 & -1 & 0 \end{array}\right| = B_{<;<}\, C_{\geqslant;\geqslant} = (c_{23} - c_{24} + c_{34})^2$$

$$|P_1^{(1,2)}| = |b_{<;<}\, c_{\geqslant;\geqslant}| = |c_{23} - c_{24} + c_{34}|$$

$$S_2^{(1,2)} = \left|\begin{array}{cc|cc} 0 & -1 & -1 & -1 \\ 1 & 0 & 1 & 1 \\ \hline 1 & -1 & 0 & c_{34} \\ 1 & -1 & -c_{34} & 0 \end{array}\right|$$

$$= \left|\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right| \left|\begin{array}{cc} 0 & c_{34} \\ -c_{34} & 0 \end{array}\right| = B_{\leqslant;<}\, C_{>;\geqslant} = (c_{34})^2$$

$$|P_2^{(1,2)}| = |b_{\leqslant;<}\, c_{>;\geqslant}| = |c_{34}|$$

$$S_3^{(1,2)} = \left|\begin{array}{cccc} 0 & -1 & -1 & -1 \\ 1 & 0 & -1 & c_{24} \\ 1 & 1 & 0 & 1 \\ 1 & -c_{24} & -1 & 0 \end{array}\right| = \left|\begin{array}{cc|cc} 0 & -1 & -1 & -1 \\ 1 & 0 & 1 & 1 \\ \hline 1 & -1 & 0 & c_{24} \\ 1 & -1 & -c_{24} & 0 \end{array}\right|$$

$$= \left|\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right| \left|\begin{array}{cc} 0 & c_{24} \\ -c_{24} & 0 \end{array}\right| = B_{<;\leqslant}\, C_{\geqslant;>} = (c_{24})^2$$

$$|P_3^{(1,2)}| = |b_{<;\leqslant}\, c_{\geqslant;>}| = |c_{24}|$$

$$S_4^{(1,2)} = \left|\begin{array}{ccc|c} 0 & -1 & -1 & -1 \\ 1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ \hline 1 & -1 & -1 & 0 \end{array}\right| = \left|\begin{array}{cccc|cc} 0 & -1 & -1 & -1 & -1 & -1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & -1 & 1 & 1 \\ 1 & -1 & 1 & 0 & -1 & -1 \\ \hline 1 & -1 & -1 & 1 & 0 & 1 \\ 1 & -1 & -1 & 1 & -1 & 0 \end{array}\right|$$

$$= \begin{vmatrix} 0 & -1 & -1 & -1 \\ 1 & 0 & 1 & 1 \\ 1 & -1 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{vmatrix} \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} = B_{\leqslant;\leqslant}\, C_{>;>} = (1)^2$$

$$|P_4^{(1,2)}| = |b_{\leqslant;\leqslant}\, c_{>;>}| = 1$$

$$S_5^{(1,2)} = \left|\begin{array}{ccc|c} 0 & 2 & 2 & 1 \\ -2 & 0 & c_{23} & 1 \\ -2 & -c_{23} & 0 & 1 \\ \hline -1 & -1 & -1 & 0 \end{array}\right| = \left|\begin{array}{cccc|cc} 0 & 2 & 2 & 1 & 1 & -1 \\ -2 & 0 & c_{23} & 1 & 1 & -1 \\ -2 & -c_{23} & 0 & 1 & 1 & -1 \\ -1 & -1 & -1 & 0 & 1 & -1 \\ \hline -1 & -1 & -1 & -1 & 0 & -1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{array}\right|$$

$$= \begin{vmatrix} 0 & 2 & 2 & 1 \\ -2 & 0 & c_{23} & 1 \\ -2 & -c_{23} & 0 & 1 \\ -1 & -1 & -1 & 0 \end{vmatrix} \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} = C_{\leqslant;\leqslant}\, B_{>;>} = (c_{23})^2$$

$$|P_5^{(1,2)}| = |c_{\leqslant;\leqslant}\, b_{>;>}| = |c_{23}|$$

$$S_6^{(1,2)} = \begin{vmatrix} 0 & -1 & 2 & 1 \\ 1 & 0 & 1 & -1 \\ -2 & -1 & 0 & 1 \\ -1 & 1 & -1 & 0 \end{vmatrix} = \left|\begin{array}{cc|cc} 0 & 2 & -1 & 1 \\ -2 & 0 & -1 & 1 \\ \hline 1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{array}\right|$$

$$= \begin{vmatrix} 0 & 2 \\ -2 & 0 \end{vmatrix} \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} = C_{<;\leqslant}\, B_{\geqslant;>} = (2)^2$$

$$|P_6^{(1,2)}| = |c_{<;\leqslant}\, b_{\geqslant;>}| = 2$$

$$
S_7^{(1,2)} = \left|\begin{array}{cc|cc} 0 & 2 & -1 & 1 \\ -2 & 0 & -1 & 1 \\ \hline 1 & 1 & 0 & \frac{l_2-l_1}{l_2+l_1} \\ -1 & -1 & -\frac{l_2-l_1}{l_2+l_1} & 0 \end{array}\right|
$$

$$
= \left|\begin{array}{cc} 0 & 2 \\ -2 & 0 \end{array}\right| \left|\begin{array}{cc} 0 & \frac{l_2-l_1}{l_2+l_1} \\ -\frac{l_2-l_1}{l_2+l_1} & 0 \end{array}\right| = C_{\leqslant;<}\, B_{>;\geqslant} = \left(2\frac{l_2-l_1}{l_2+l_1}\right)^2
$$

$$
|P_7^{(1,2)}| = |c_{\leqslant;<}\, b_{>;\geqslant}| = \left|2\frac{l_2-l_1}{l_2+l_1}\right|
$$

$$
S_8^{(1,2)} = \left|\begin{array}{c|ccc} 0 & -1 & -1 & 1 \\ \hline 1 & 0 & 1 & -1 \\ 1 & -1 & 0 & \frac{l_2-l_1}{l_2+l_1} \\ -1 & 1 & -\frac{l_2-l_1}{l_2+l_1} & 0 \end{array}\right| = \left|\begin{array}{cc|cccc} 0 & 1 & -1 & -1 & 1 & -1 \\ -1 & 0 & -1 & -1 & 1 & -1 \\ \hline 1 & 1 & 0 & 1 & -1 & -1 \\ 1 & 1 & -1 & 0 & \frac{l_2-l_1}{l_2+l_1} & 1 \\ -1 & -1 & 1 & -\frac{l_2-l_1}{l_2+l_1} & 0 & -1 \\ 1 & 1 & 1 & -1 & 1 & 0 \end{array}\right|
$$

$$
= \left|\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right| \left|\begin{array}{cccc} 0 & 1 & -1 & -1 \\ -1 & 0 & \frac{l_2-l_1}{l_2+l_1} & 1 \\ 1 & -\frac{l_2-l_1}{l_2+l_1} & 0 & -1 \\ 1 & -1 & 1 & 0 \end{array}\right| = C_{<;<}\, B_{\geqslant;\geqslant} = \left(\frac{l_2-l_1}{l_2+l_1}\right)^2
$$

$$
|P_8^{(1,2)}| = |c_{<;<}\, b_{\geqslant;\geqslant}| = \left|\frac{l_2-l_1}{l_2+l_1}\right|,
$$

hence, from (3.50), the asymptotic expressions, (when the $(2,1)$th dromion is fixed) for the $(1,2)$-dromion solution are

$$
\begin{aligned}
P_-^{(1,2)} &= |c_{23} - c_{24} + c_{34}| + |c_{34}|e^{\eta_2} + |c_{24}|e^{\rho_1} + e^{\eta_2+\rho_1} \\
P_+^{(1,2)} &= |c_{23}| + 2e^{\eta_2} + \left|2\frac{l_2-l_1}{l_2+l_1}\right| e^{\rho_1} + \left|\frac{l_2-l_1}{l_2+l_1}\right| e^{\eta_2+\rho_1}.
\end{aligned}
$$

3.7 Plots of Dromion Interactions

In the generic case, both before and after the interaction, the $(2,2)$-dromion solution has four dromions (see figure 3.3) situated in the U-plane at the corners of the rectangle formed by the plane-wave solitons in the V-plane (see figure 3.4). Using (3.37) and (3.38) we determine the class of pfaffians c which makes the amplitude of some of these dromions zero either as $t \to -\infty$ or as $t \to \infty$. In order to get a particular dromion interaction, we also need to find out the asymptotics in each case when a particular dromion is fixed. In section 3.5 we found the asymptotic expressions when the $(2,1)$th dromion is fixed. Similarly, if we follow the same procedure, we get the following asymptotic expressions as $t \to -\infty$

(when the $(1,1)$th dromion is fixed)

$$P_{-1}^{(2,2)} = |c_{12}c_{34} - c_{13}c_{24} + c_{14}c_{23}| + |c_{23} - c_{24} + c_{34}|\, e^{\eta_1} + |c_{12} + c_{24} - c_{14}| e^{\rho_1} + |c_{24}|\, e^{\eta_1+\rho_1}$$

(when the $(1,2)$th dromion is fixed)

$$P_{-2}^{(2,2)} = |c_{12} - c_{14} + c_{24}| + |c_{24}| e^{\eta_1} + \left| c_{12}\frac{l_2 - l_1}{l_2 + l_1} \right| e^{\rho_2} + \left| \frac{l_2 - l_1}{l_2 + l_1} \right| e^{\eta_1+\rho_2}$$

(when the $(2,1)$th dromion is fixed)

$$P_{-3}^{(2,2)} = |c_{23} - c_{24} + c_{34}| + \left| c_{34}\frac{k_2 - k_1}{k_2 + k_1} \right| e^{\eta_2} + |c_{24}| e^{\rho_1} + \left| \frac{k_2 - k_1}{k_2 + k_1} \right| e^{\eta_2+\rho_1}$$

(when the $(2,2)$th dromion is fixed)

$$P_{-4}^{(2,2)} = |c_{24}| + \left| \frac{k_2 - k_1}{k_2 + k_1} \right| e^{\eta_2} + \left| \frac{l_2 - l_1}{l_2 + l_1} \right| e^{\rho_2} + \left| \frac{k_2 - k_1}{k_2 + k_1}\frac{l_2 - l_1}{l_2 + l_1} \right| e^{\eta_2+\rho_2}$$

and as $t \to \infty$

(when the $(1,1)$th dromion is fixed)

$$P_{+1}^{(2,2)} = |c_{13}| + \left| \frac{k_2 - k_1}{k_2 + k_1} \right| e^{\eta_1} + \left| \frac{l_2 - l_1}{l_2 + l_1} \right| e^{\rho_1} + \left| \frac{k_2 - k_1}{k_2 + k_1}\frac{l_2 - l_1}{l_2 + l_1} \right| e^{\eta_1+\rho_1}$$

(when the $(1,2)$th dromion is fixed)

$$P_{+2}^{(2,2)} = |c_{13} - c_{14} + c_{34}| + \left| c_{34}\frac{k_2 - k_1}{k_2 + k_1} \right| e^{\eta_1} + |c_{13}| e^{\rho_2} + \left| \frac{k_2 - k_1}{k_2 + k_1} \right| e^{\eta_1+\rho_2}$$

(when the $(2,1)$th dromion is fixed)

$$P_{+3}^{(2,2)} = |c_{12} - c_{13} + c_{23}| + |c_{13}| e^{\eta_2} + \left| c_{12}\frac{l_2 - l_1}{l_2 + l_1} \right| e^{\rho_1} + \left| \frac{l_2 - l_1}{l_2 + l_1} \right| e^{\eta_2+\rho_1}$$

(when the $(2,2)$th dromion is fixed)

$$P_{+4}^{(2,2)} = |c_{12}c_{34} - c_{13}c_{24} + c_{14}c_{23}| + |c_{13} - c_{14} + c_{34}|\, e^{\eta_2} + |c_{12} + c_{23} - c_{13}| e^{\rho_2} + |c_{13}|\, e^{\eta_2+\rho_2}.$$

To avoid singularities in the solution, we restrict the constants such that

$$c_{13} < 0, \quad c_{34} > 0, \quad c_{12} < 0, \quad c_{24} < 0,$$

$$c_{14} - c_{34} - c_{13} < 0, \quad c_{24} - c_{23} - c_{34} < 0, \quad c_{12} + c_{23} - c_{13} < 0,$$

$$c_{12} + c_{24} - c_{14} < 0, \quad c_{12}c_{34} - c_{13}c_{24} + c_{14}c_{23} < 0,$$

so that the coefficients in the asymptotic expressions have the same sign. We determine the dromion amplitudes from (3.37). The amplitudes $U_0^- = 0$ and $U_0^+ = 0$ when the phase shifts $F_\perp^- = 0$ and $F_\perp^+ = 0$, respectively. Thus, as $t \to -\infty$,
the $(1,1)$th dromion does not appear when

$$c_{24}(c_{12}c_{34} - c_{13}c_{24} + c_{14}c_{23}) = (c_{24} - c_{23} - c_{34})(c_{12} + c_{24} - c_{14}),$$

the $(1,2)$th dromion does not appear when $\quad c_{14} - c_{12} - c_{24} = c_{12}c_{24}$,
the $(2,1)$th dromion does not appear when $\quad c_{24} - c_{23} - c_{34} = c_{24}c_{34}$,
the $(2,2)$th dromion does not appear when $\quad c_{24} = -1$,
and as $t \to \infty$
the $(1,1)$th dromion does not appear when $\quad c_{13} = -1$,
the $(1,2)$th dromion does not appear when $\quad c_{14} - c_{34} - c_{13} = c_{13}c_{34}$,
the $(2,1)$th dromion does not appear when $\quad c_{13} - c_{12} - c_{23} = c_{13}c_{12}$,
the $(2,2)$th dromion does not appear when

$$c_{13}(c_{12}c_{34} - c_{13}c_{24} + c_{14}c_{23}) = (c_{14} - c_{13} - c_{34})(c_{12} - c_{13} + c_{23}).$$

In particular, we consider the case in which the $(1,1)$th and the $(2,2)$th dromions do not appear before and the $(1,2)$th, $(2,1)$th dromions do not appear after the interaction to produce a solution describing the 90° scattering of two dromions. This kind of scattering is shown in figure 3.5; two dromions with equal amplitudes are approaching each other and interacting (a head-on collision) at time zero. During the interaction a double dromion occurs as the two dromions collide. After the interaction two dromions appear, one with bigger amplitude and faster than before the interaction, the other dromion has smaller amplitude with different sign. For this plot, we have taken the following values for the parameters $k_1 = l_1 = 1$, $k_2 = l_2 = \frac{3}{2}$, $c_{12} = -\frac{1}{2}$, $c_{13} = -\frac{8}{9}$, $c_{14} = -\frac{5}{6}$, $c_{23} = -\frac{5}{6}$, $c_{24} = -1$, $c_{34} = \frac{1}{2}$.

Next we consider the case in which the $(1,2)$th and the $(2,1)$th dromions do not appear before and the $(1,2)$th, $(2,1)$th dromions do not appear after the interaction to produce a solution describing the 0° scattering of two dromions. This kind of scattering is shown in figure 3.6; two dromions one with positive amplitude and the other with negative amplitude are approaching each other and interacting. At the interaction, all four dromions appear and after the interaction two of them disappear and they almost recover their initial profile. This kind of interaction is similar to soliton interactions, as they do not change their initial profile upon interaction. Details of this interaction are shown in figure 3.7. For this plot, we have taken the following values for the parameters $k_1 = l_1 = 1$, $k_2 = l_2 = \frac{3}{2}$, $c_{12} = -3$, $c_{13} = -\frac{1}{2}$, $c_{14} = 1$, $c_{23} = 1$, $c_{24} = -2$, $c_{34} = 3$.

Next we consider the case in which the $(1,1)$th and the $(2,2)$th dromions do not appear before and after the interaction. Two dromions one with negative amplitude are approaching each other and collide. This collision is not exactly an interaction like in other dromion interactions. After the collision the dromions bounce away and have bigger amplitudes than before the collision. This kind of scattering is shown in figure 3.8. For this plot, we have taken the following values for the parameters $k_1 = l_1 = 1$, $k_2 = l_2 = \frac{3}{2}$, $c_{12} = -3$, $c_{13} = -1$, $c_{14} = -\frac{3}{2}$, $c_{23} = -\frac{1}{2}$, $c_{24} = -1$, $c_{34} = 2$.

Next we consider the 2×1 dromion scattering obtained from the $(1,2)$-dromion solution. The asymptotic expressions for the $(1,2)$-dromion solution are as follows as $t \to -\infty$

(when the $(2,1)$th dromion is fixed)

$$P_{-1}^{(1,2)} = |c_{23} - c_{24} + c_{34}| + |c_{34}|e^{\eta_2} + |c_{24}|e^{\rho_1} + e^{\eta_2+\rho_1}$$

(when the $(2,2)$th dromion is fixed)

$$P_{-2}^{(1,2)} = |c_{23}| + e^{\eta_2} + \left|\frac{l_2 - l_1}{l_2 + l_1}\right| e^{\rho_2} + \left|\frac{l_2 - l_1}{l_2 + l_1}\right| e^{\eta_2+\rho_2}$$

and as $t \to \infty$

(when the $(2,1)$th dromion is fixed)

$$P_{+1}^{(1,2)} = |c_{23}| + 2e^{\eta_2} + \left|2\frac{l_2 - l_1}{l_2 + l_1}\right| e^{\rho_1} + \left|\frac{l_2 - l_1}{l_2 + l_1}\right| e^{\eta_2+\rho_1}$$

(when the $(2,2)$th dromion is fixed)

$$P_{+2}^{(1,2)} = |c_{12} - c_{13} + c_{23}| + |c_{23}|e^{\eta_2} + |c_{13}|e^{\rho_2} + e^{\eta_2+\rho_2}.$$

This kind of scattering is shown in figure 3.9; initially two dromions with positive amplitudes are approaching each other and interacting. During the interaction one dromion absorbs the other dromion, and after the interaction we see only one dromion with a bigger amplitude then the dromions before the interaction. For this plot, we have taken the following values for the parameters $k_1 = 0$, $l_1 = 1$, $k_2 = l_2 = \frac{3}{2}$, $c_{12} = -\frac{1}{2}$, $c_{13} = -1$, $c_{14} = -1$, $c_{23} = -1$, $c_{24} = -\frac{3}{2}$, $c_{34} = \frac{1}{2}$.

We also investigate whether there exists an annihilation property for the $(2,2)$-dromion solution. We look for a scattering that has two dromions before the interaction, but no dromion afterwards. That means that the dromions after the interaction must have zero amplitudes. In order to have all dromion amplitudes zero, we must take the following values of constants $c_{13} = c_{14} = c_{23} = c_{24} = -1$. These values of constants also makes the amplitudes of dromions, before the interaction, zero. Therefore, the annihilation property for the NVN equations does not exist.

Next we also consider the case in which the $(1,1)$th and the $(1,2)$th dromions do not appear before and the $(1,1)$th, $(1,2)$th, $(2,1)$th dromions do not appear after the interaction to produce a solution describing the 2×1 scattering of two dromions. This kind of scattering is shown in figure 3.10; two dromions one with positive amplitude and the other negative amplitude are approaching each other and interacting. During the interaction the dromion with the positive amplitude absorbs the dromion with the negative amplitude, and after the interaction only one dromion emerges, that has smaller positive amplitude than the dromion before the interaction. This shows that the dromion with negative amplitude has negative effect on the other dromion's amplitude. For this plot, we have taken the following values for the parameters $k_1 = l_1 = 1$, $k_2 = l_2 = \frac{3}{2}$, $c_{12} = -1$, $c_{13} = -1$, $c_{14} = -1$, $c_{23} = -1$, $c_{24} = -2$, $c_{34} = 2$.

It turns out that in contrast to one-dimensional solitons these two-dimensional solitons (dromions) do not in general preserve their form upon interaction. Only for a special choice of the parameters do these solutions preserve their form. In general the computer generated plots show that the dromions have different interaction properties as compared to soliton interactions, as their initial profile is changed after the interaction.

Appendix

To factorize the pfaffian P_1 and the skew-symmetric determinant $S_1 = (P_1)^2$, we take its skew-symmetric matrix form and denote with G, hence

$$\det(G) = S_1 = P_1^2.$$

For convenient we write G in the following block structure:

$$G = \begin{pmatrix} B_1 & A_1 & B_2 & A_2 \\ -A_1^T & C_1 & A_3 & C_2 \\ -B_2^T & -A_3^T & B_3 & A_4 \\ -A_2^T & -C_2^T & -A_4^T & C_3 \end{pmatrix},$$

where

$$\begin{aligned} B_1 &= (B)_{\substack{1\leqslant i<m \\ 1\leqslant j<m}}, & A_1 &= (A)_{\substack{1\leqslant i<m \\ m\leqslant j\leqslant M}}, \\ B_2 &= (B)_{\substack{1\leqslant i<m \\ M+1\leqslant j<M+n}}, & A_2 &= (A)_{\substack{1\leqslant i<m \\ M+n\leqslant j\leqslant M+N}}, \\ C_1 &= (C)_{\substack{m\leqslant i\leqslant M \\ m\leqslant <j\leqslant M}}, & A_3 &= (-A^T)_{\substack{m\leqslant i\leqslant M \\ M+1\leqslant j<M+n}}, \\ C_2 &= (C)_{\substack{m\leqslant i\leqslant M \\ M+n\leqslant j\leqslant M+N}}, & B_3 &= (B)_{\substack{M+1\leqslant i<M+n \\ M+1\leqslant j<M+n}}, \\ A_4 &= (A)_{\substack{M+1\leqslant i<M+n \\ M+n\leqslant j\leqslant M+N}}, & C_3 &= (C)_{\substack{M+n\leqslant i\leqslant M+N \\ M+n\leqslant j\leqslant M+N}}. \end{aligned}$$

In order to have simpler structure for factorization, we exploit the matrix G in the following way. We interchange columns and rows in the determinant, so that the determinants S_i can be written in the form of block structure. The numbers of the interchanges of the columns and rows are given by r_i. We interchange the rows in the second block-row with the rows in the third block-row, so the number of interchanges is $\tilde{r} = (M-m+1)\times(n-1)$, and we also interchange the corresponding columns in the second block-column with the columns in the third block-column, so again the number of interchanges is $\tilde{r} = (M-m+1)\times(n-1)$. Hence the number of total interchanges in the matrix G is $r_1 = 2\times(M-m+1)\times(n-1)$.

The interchanges of the columns and rows do not affect the value of the determinant, since whenever two columns are interchanged so the corresponding two rows are also interchanged, nullifying the change in sign. Therefore the numbers r_i are even for the determinants S_i, but may not be even for the pfaffians P_i. Therefore, we take the absolute values of the pfaffians, namely $|P_i|$.

This process is achieved simply by pre-multiplying the permutation matrix P by G and P^T.

$$P = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix},$$

where I and 0 are identity and zero block-matrices respectively.

Hence we get $\tilde{G} = P^T\ G\ P$. Next we take $\tilde{G}$ to continue with the proof, since $\det(\tilde{G}) = \det(G)$. We have

$$\begin{aligned} \tilde{G} &= \begin{pmatrix} B_1 & B_2 & A_1 & A_2 \\ -B_2^T & B_3 & -A_3^T & A_4 \\ -A_1^T & A_3 & C_1 & C_2 \\ -A_2^T & -A_4^T & -C_2^T & C_3 \end{pmatrix} \\ &= \begin{pmatrix} \tilde{B} & \tilde{A} \\ -\tilde{A}^T & \tilde{C} \end{pmatrix}, \end{aligned} \tag{3.51}$$

where $\tilde{B}$ and $\tilde{C}$ are skew-symmetric matrices, and $\tilde{A}$ is a rank-1 matrix given by

$$\tilde{B} = \begin{pmatrix} B_1 & B_2 \\ -B_2^T & B_3 \end{pmatrix} \quad \tilde{A} = \begin{pmatrix} A_1 & A_2 \\ -A_3^T & A_4 \end{pmatrix} \quad \tilde{C} = \begin{pmatrix} C_1 & C_2 \\ -C_2^T & C_3 \end{pmatrix}.$$

Next we factorize the matrix (3.51) in this form

$$\tilde{G} = \begin{pmatrix} \tilde{B} & 0 \\ -\tilde{A}^T & I \end{pmatrix} \begin{pmatrix} I & \tilde{B}^{-1}\tilde{A} \\ 0 & \tilde{C} + \tilde{A}^T\tilde{B}^{-1}\tilde{A} \end{pmatrix} \tag{3.52}$$

where, again I and 0 are identity and zero block-matrices respectively. Taking the determinants of the factorized matrices in (3.52), we get

$$\det(\tilde{G}) = \det(\tilde{B})\ \det(\tilde{C} + \tilde{A}^T\tilde{B}^{-1}\tilde{A}).$$

The expression $\tilde{A}^T\tilde{B}^{-1}\tilde{A}$ is a zero matrix, as long as the $\tilde{A}$ is a rank-1 matrix and the $\tilde{B}$ is a skew-symmetric matrix. If $\tilde{B}$ is skew-symmetric then so is $\tilde{B}^{-1}$ a skew-symmetric matrix. (If $\tilde{B}$ is odd-dimensional, and hence not invertible, then see page 46 for explanation.)

Thus we get

$$\begin{aligned}
\det(\tilde{G}) &= \det(\tilde{B})\ \det(\tilde{C}) \\
S_1 &= B_{<;<}\ C_{\geqslant;\geqslant} \\
P_1^2 &= b_{<;<}^2\ c_{\geqslant;\geqslant}^2 \\
|P_1| &= |b_{<;<}\ c_{\geqslant;\geqslant}|.
\end{aligned}$$

Further we can also factorize the matrix $\tilde{B}$ in terms of skew-symmetric matrices B_1 and B_3 defined by the parameters k_i and l_j respectively.

$$\begin{aligned}
\tilde{B} &= \begin{pmatrix} B_1 & B_2 \\ -B_2^T & B_3 \end{pmatrix} \\
&= \begin{pmatrix} B_1 & 0 \\ -B_2^T & I \end{pmatrix} \begin{pmatrix} I & B_1^{-1}B_2 \\ 0 & B_3 + B_2^T B_1^{-1} B_2 \end{pmatrix} \\
\det(\tilde{B}) &= \det(B_1)\det(B_3 + B_2^T B_1^{-1} B_2) \\
&= \det(B_1)\det(B_3) \\
B_{<;<} &= K_<\ L_< \\
b_{<;<}^2 &= k_<^2\ l_<^2 \\
|b_{<;<}| &= |k_<\ l_<|,
\end{aligned}$$

where I and 0 are identity and zero matrices respectively, B_2 is a rank-1 matrix, B_1 and B_3 are skew-symmetric matrices and the entries for the matrices B_1 and B_3 are given in (3.24) and (3.25) respectively. Hence

$$\begin{aligned}
S_1 &= B_{<;<}\ C_{\geqslant;\geqslant} \\
&= K_<\ L_<\ C_{\geqslant;\geqslant} \\
|P_1| &= |b_{<;<}\ c_{\geqslant;\geqslant}| \\
&= |k_<\ l_<\ c_{\geqslant;\geqslant}|.
\end{aligned}$$

The proof for the other pfaffians $|P_i|$ (i=2 ... 8) can be done in a similar way.

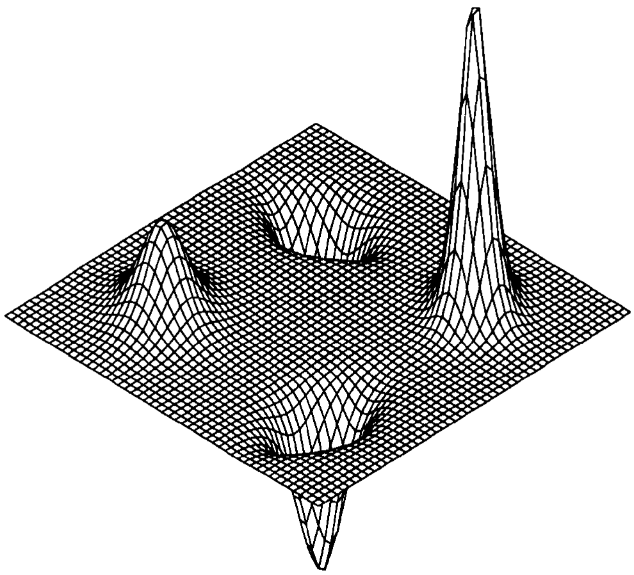


Figure 3.3: The location of the dromions in the U-plane.

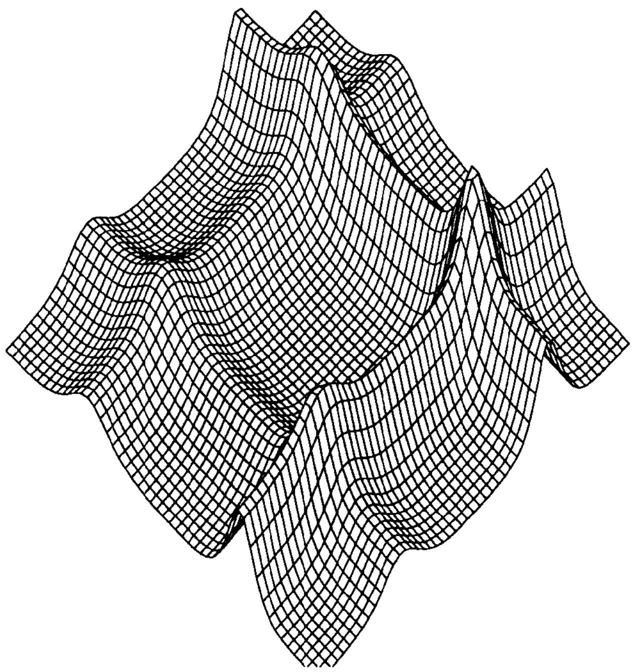


Figure 3.4: The interaction of the plane waves and the location of the dromions in the V-plane.

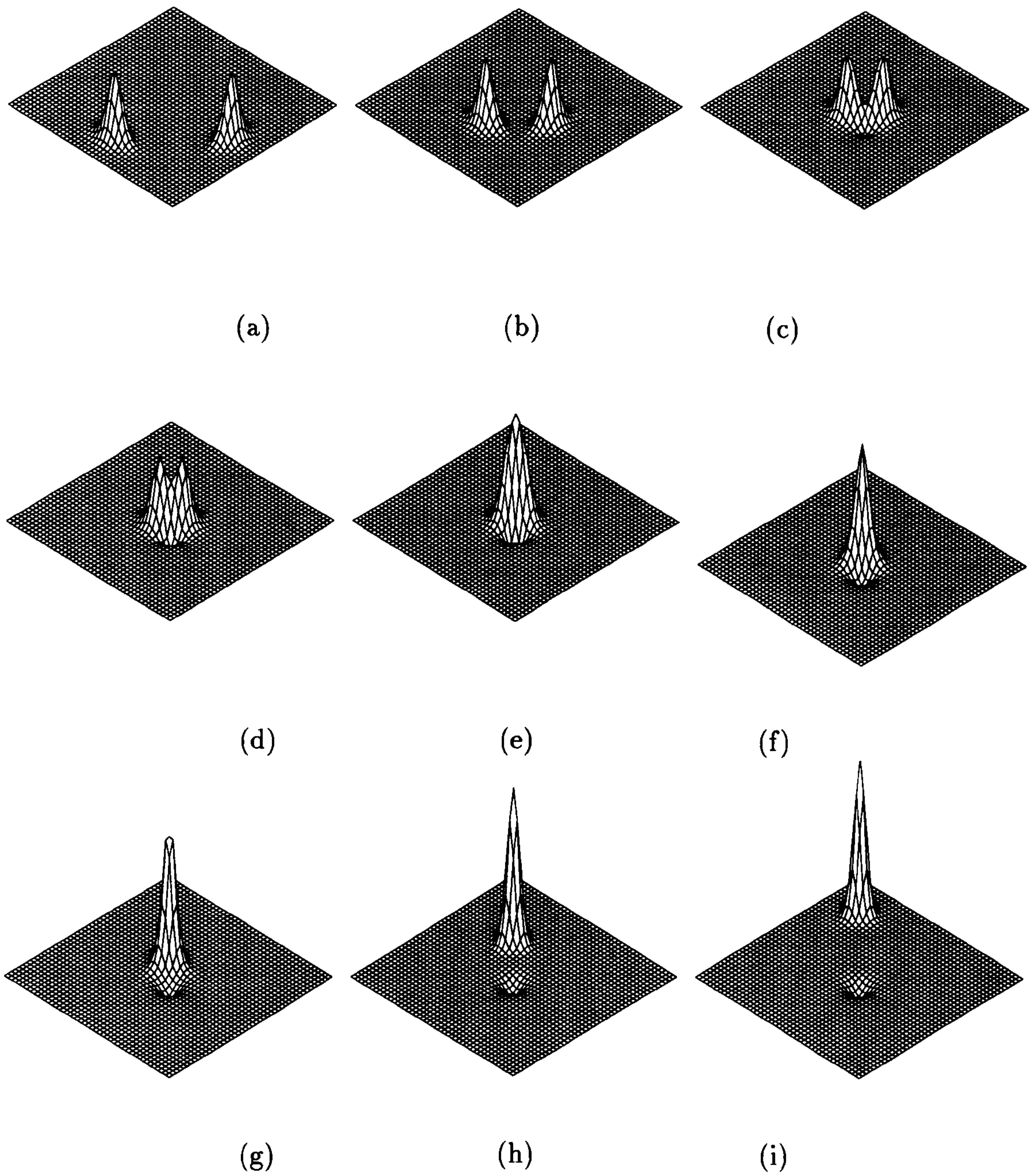
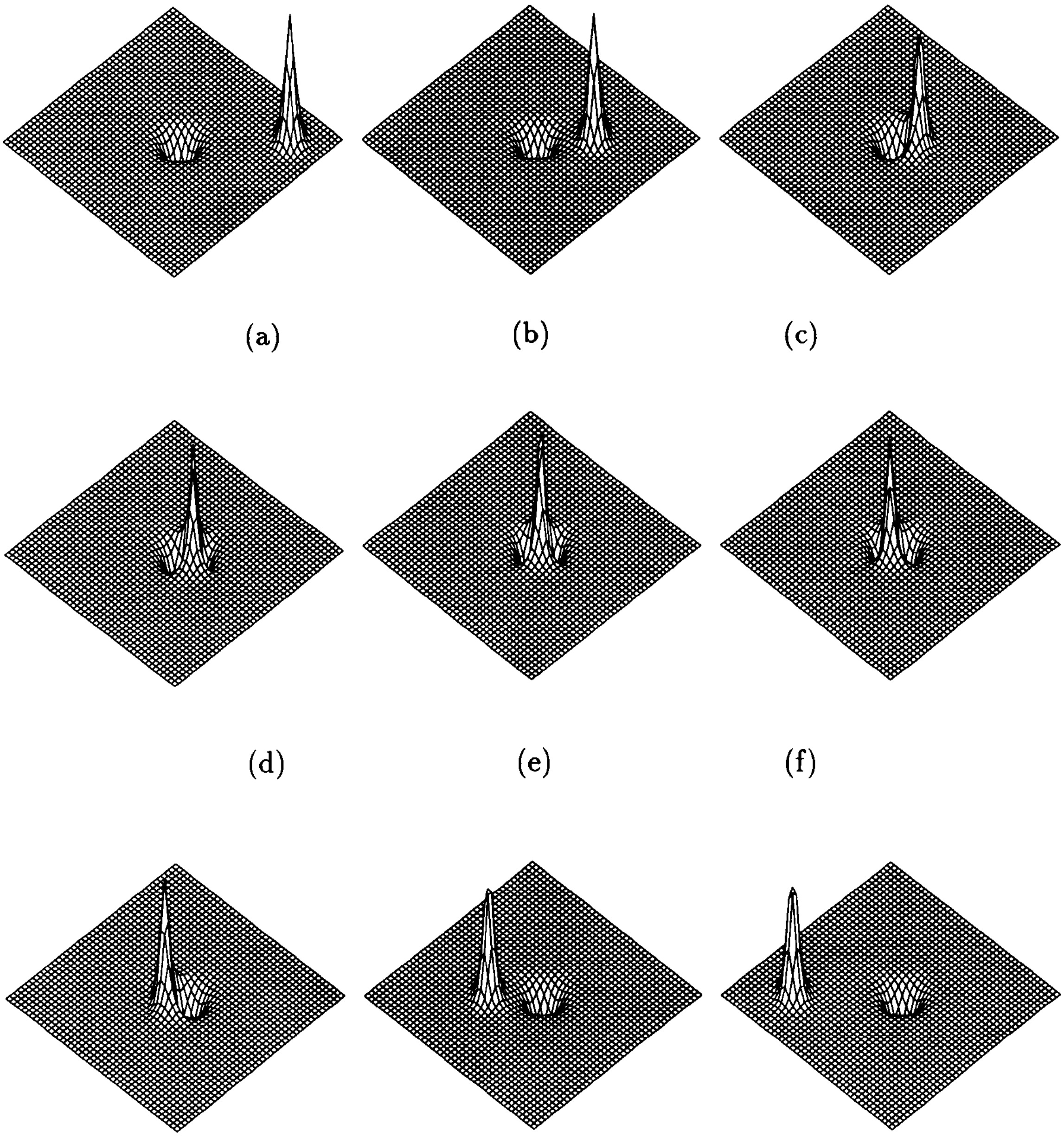


Figure 3.5: 90° Dromion scattering. (a)$t = -10$; (b)$t = -5$; (c)$t = -2$; (d)$t = -1$; (e)$t = 0$; (f)$t = 1$; (g)$t = 2$; (h)$t = 5$; (i)$t = 10$.



(a) (b) (c)

(d) (e) (f)

(g) (h) (i)

Figure 3.6: 0° Dromion scattering. (a)$t = -10$; (b)$t = -5$; (c)$t = -2$; (d)$t = -1$; (e)$t = 0$; (f)$t = 1$; (g)$t = 2$; (h)$t = 5$; (i)$t = 10$.

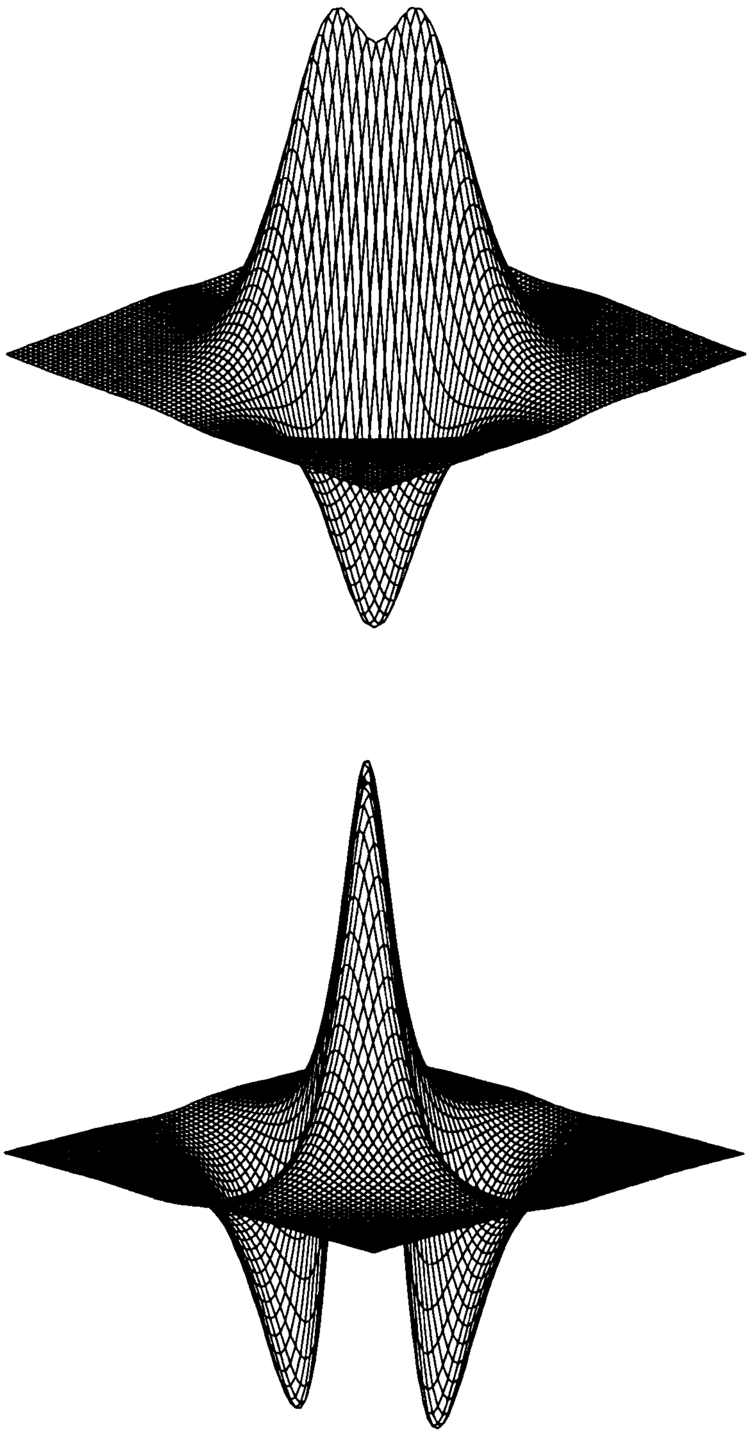


Figure 3.7: Details of the interaction shown in figure 3.6.

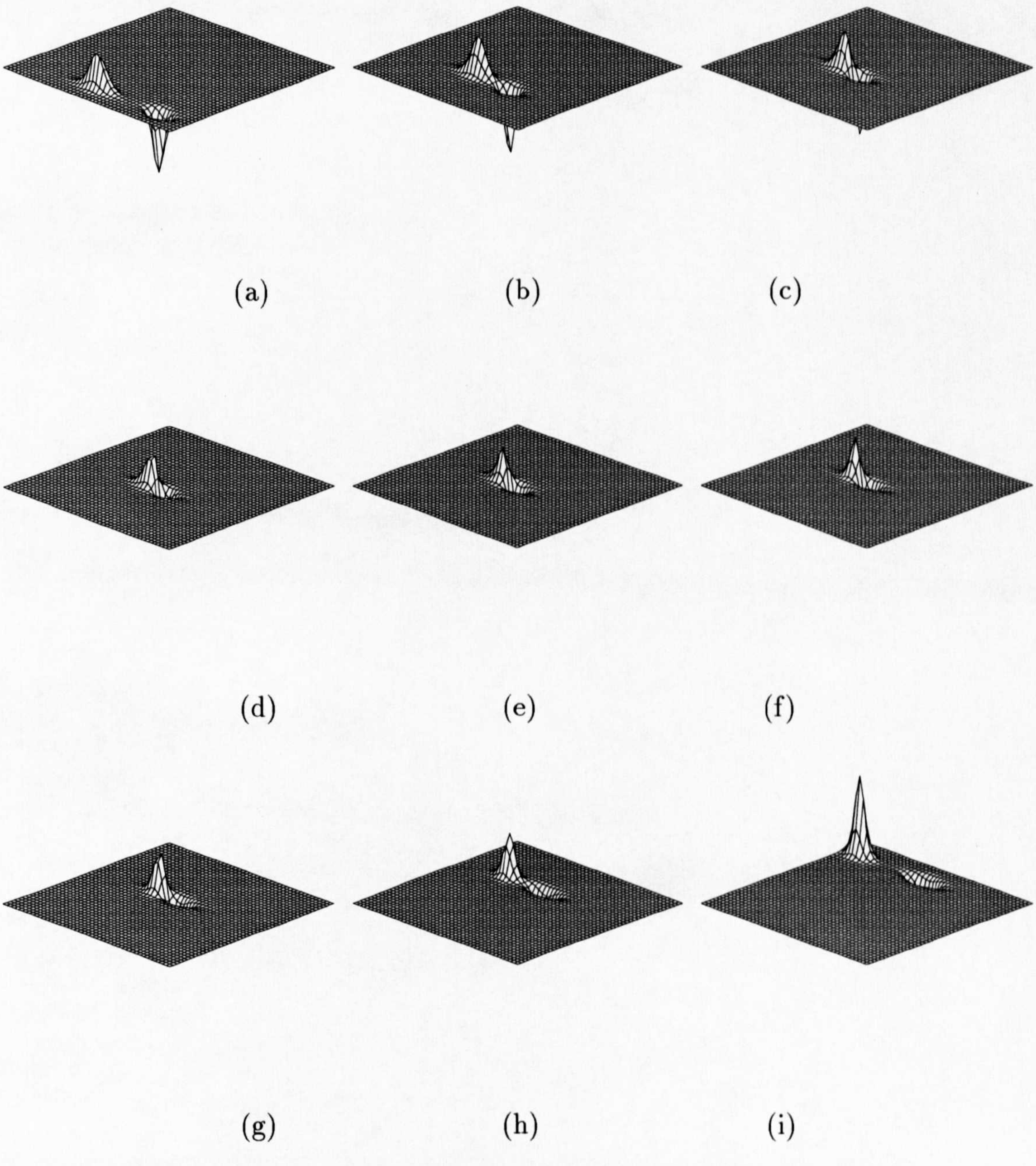


Figure 3.8: Dromion interaction. (a)$t = -10$; (b) $t = -5$; (c) $t = -2$; (d)$t = -1$; (e)$t = 0$; (f)$t = 1$; (g)$t = 2$; (h)$t = 5$; (i)$t = 10$.

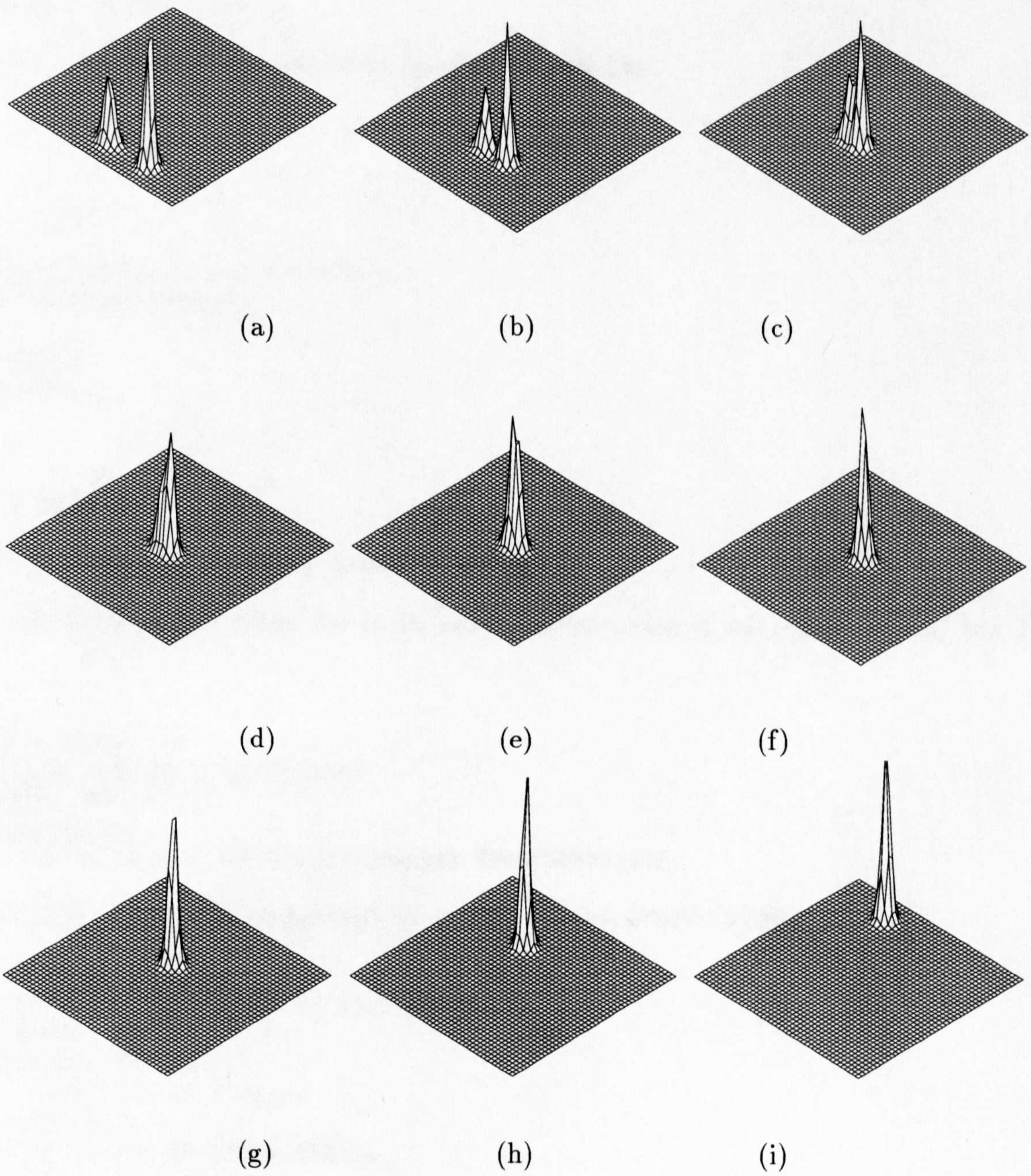


Figure 3.9: 2×1 Dromion scattering. (a)$t = -10$; (b)$t = -5$; (c) $t = -2$; (d)$t = -1$; (e)$t = 0$; (f)$t = 1$; (g)$t = 2$; (h)$t = 5$; (i)$t = 10$.

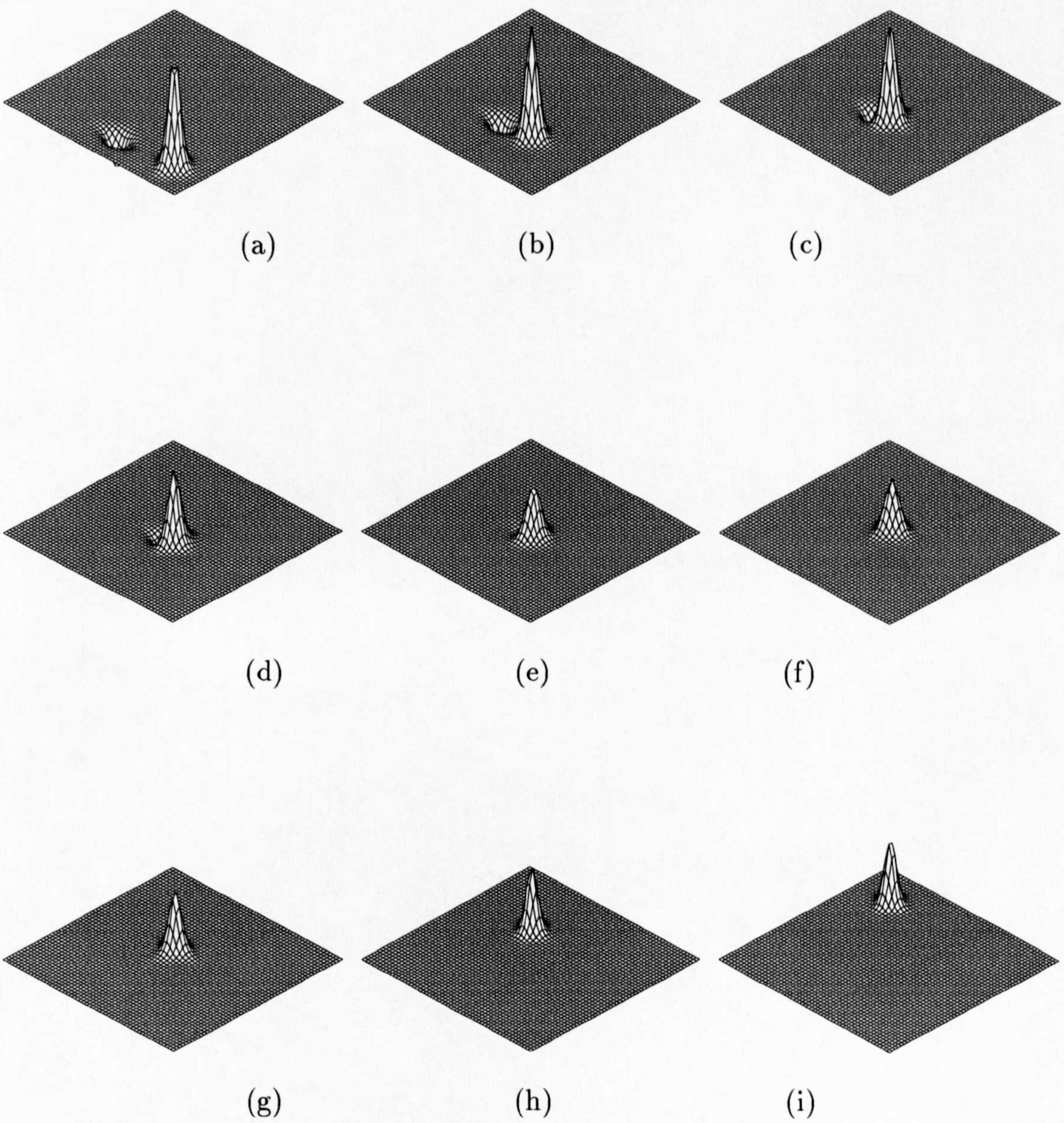


Figure 3.10: 2×1 Dromion interaction. (a)$t = -10$; (b) $t = -5$; (c) $t = -2$; (d)$t = -1$; (e)$t = 0$; (f)$t = 1$; (g)$t = 2$; (h)$t = 5$; (i)$t = 10$.

Chapter 4

Algebraic Solutions of Soliton equations

4.1 Introduction

In recent years solutions to soliton equations in soliton theory have been given in many ways such as by means of Grammians, Wronskians, Pfaffians etc.. The diversity of expressing solutions reflects the richness of algebraic structures which the soliton equations possess in common. It is Sato [14] that unveiled the structures by means of the method of algebraic analysis in the study of the Kadomtsev-Petviashvili (KP) hierarchy. Among the variety of soliton equations, the KP hierarchy is the most widely studied one.

In this chapter, we exploit the algebraic structure of soliton equations and find solutions in terms of fermion particles. These particles can either be charged or neutral, and they can have one component structure or they can have more than one, depending on the structure of the equation. An example of their fermionic structure is shown in the table below for some soliton equations. We write the τ-function for charged and neutral free fermions in terms of determinants and pfaffians respectively, and show that these two concepts are analogous to one another. We write the τ-function for charged fermions in the following form

$$\tau_{\psi} = \det(A)\det(A^{-1} + V),$$

where A is a constant matrix and V is a matrix with the entries of charged fermions. And we also write the corresponding pfaffian analogue of this τ-function in the following form

$$\tau_\phi = \mathrm{Pf}(A)\,\mathrm{Pf}(A' + V),$$

where A, A' are constant triangular matrices, A' is the analogue of the inverse A and V is also a triangular matrix with the entries of neutral fermions. Observe that τ_ψ is in determinant form and τ_ϕ is in pfaffian form. These are explained in more detail in the later sections. In section 5, we derive new general formulae for charged fermions, from which the rational solutions and soliton solutions for the KP hierarchy can be obtained. In section 6, we derive formulae for the rational solutions of the 1-component and 2-component BKP hierarchies. In section 7, we give general formulae for the soliton solutions of the 1-component and 2-component BKP hierarchies. Examples of how to get the soliton and dromion solutions to various soliton equations, from τ-functions are also given.

Fermions	1 component	2 component
charged (ψ_i)	KP	DS
neutral (ϕ_i)	BKP	NVN

4.2 Preliminaries

Here we recall some results from [54]. Let $\mathbf{A}$ be an associative algebra over $\mathbb{C}$ with generators ψ_i, ψ_i^* $(i \in \mathbb{Z})$, satisfying the anti-commutator relations

$$[\psi_i, \psi_j]_+ = 0, \qquad [\psi_i, \psi_j^*]_+ = \delta_{ij}, \qquad [\psi_i^*, \psi_j^*]_+ = 0, \tag{4.1}$$

where $[X, Y]_+ = XY + YX$. The generators ψ_i, ψ_i^* $(i \in \mathbb{Z})$ will be referred to as *free fermions.*

Here $\langle \quad \rangle$ denotes a linear form on $\mathbf{A}$, called the *(vacuum) expectation value,* defined as follows. For $a \in \mathbb{C}$ or quadratic in free fermions

$$\langle a \rangle = a, \qquad \langle \psi_i \rangle = \langle \psi_i^* \rangle = 0 \qquad (i \in \mathbb{Z})$$

$$\langle \psi_i \psi_j \rangle = 0, \qquad \langle \psi_i^* \psi_j^* \rangle = 0, \qquad (i, j \in \mathbb{Z}) \tag{4.2}$$

$$\langle\psi_i\psi_j^*\rangle = \begin{cases} \delta_{i,j} & (i<0) \\ 0 & (\text{otherwise}) \end{cases}, \qquad \langle\psi_j^*\psi_i\rangle = \begin{cases} \delta_{i,j} & (i\geq 0) \\ 0 & (\text{otherwise}) \end{cases}.$$

For a general product $w_1\cdots w_r$ of free fermions w_i, we apply *Wick's theorem* to compute the expectation values

$$\langle w_1\cdots w_r\rangle = \begin{cases} 0 & (\text{r odd}) \\ \sum_\sigma \operatorname{sgn}\sigma\langle w_{\sigma(1)}w_{\sigma(2)}\rangle\cdots\langle w_{\sigma(r-1)}w_{\sigma(r)}\rangle & (\text{r even}) \end{cases}$$

where σ runs over the permutations such that $\sigma(1)<\sigma(2),\cdots,\sigma(r-1)<\sigma(r)$ and $\sigma(1)<\sigma(3),\cdots,\sigma(r-1)$. We see that this theorem gives the expectation value of the general product of free fermions $w_1\cdots w_r$ in terms of a pfaffian. Therefore, Wick's theorem can be expressed in terms of pfaffians in the following way

$$\langle w_1\cdots w_r\rangle = \begin{cases} 0 & (\text{r odd}) \\ \operatorname{Pf}(\langle w_iw_j\rangle) & (\text{r even}) \end{cases}.$$

Lemma 4.2.1 *From Wick's theorem we have*

$$\langle\psi_{i_1}\cdots\psi_{i_r}\psi_{j_1}^*\cdots\psi_{j_s}^*\rangle = \begin{cases} (-1)^{\frac{1}{2}r(r+1)}\det(\langle\psi_{i_p}\psi_{j_q}^*\rangle) & r=s \\ 0 & r\neq s \end{cases}.$$

Proof.

$$(\langle\psi_{i_1}\cdots\psi_{i_r}\psi_{j_1}^*\cdots\psi_{j_s}^*\rangle)^2 = \det\begin{pmatrix} \langle\psi_{i_k}\psi_{i_l}\rangle & \langle\psi_{i_k}\psi_{j_n}^*\rangle \\ -\langle\psi_{i_l}\psi_{j_m}^*\rangle & \langle\psi_{j_m}^*\psi_{j_n}^*\rangle \end{pmatrix}$$

where $k,l=1,\cdots,r$ and $m,n=1,\cdots,s$. Using the definition of the expectation value in (4.2)

$$\begin{aligned}(\langle\psi_{i_1}\cdots\psi_{i_r}\psi_{j_1}^*\cdots\psi_{j_s}^*\rangle)^2 &= \det\begin{pmatrix} 0 & \langle\psi_{i_k}\psi_{j_n}^*\rangle \\ -\langle\psi_{i_l}^*\psi_{j_m}\rangle & 0 \end{pmatrix} \\ &= \begin{cases} \det(\langle\psi_{i_p}\psi_{j_q}^*\rangle)^2 & r=s \\ 0 & r\neq s \end{cases},\end{aligned}$$

hence

$$\langle\psi_{i_1}\cdots\psi_{i_r}\psi_{j_1}^*\cdots\psi_{j_s}^*\rangle = \begin{cases} \pm\det(\langle\psi_{i_p}\psi_{j_q}^*\rangle) & r=s \\ 0 & r\neq s \end{cases}.$$

Looking at the term $\langle\psi_{i_1}\psi_{j_1}^*\rangle\cdots\langle\psi_{i_r}\psi_{j_r}^*\rangle$ on each side we see that the sign is $(-1)^{\frac{1}{2}r(r+1)}$.

□

It is convenient to use the generating functions for free fermions defined as

$$\psi(p) := \sum_{i\in\mathbf{Z}} \psi_i p^i, \qquad \psi^*(q) := \sum_{j\in\mathbf{Z}} \psi_j^* q^{-j}. \tag{4.3}$$

Lemma 4.2.2 *The expectation values are given by*

$$\langle\psi(p_i)\psi^*(q_j)\rangle = \frac{q_j}{p_i - q_j}, \qquad \langle\psi^*(q_j)\psi(p_i)\rangle = -\frac{q_j}{p_i - q_j}$$

and

$$\langle\psi^*(q_j)\psi_0\rangle = \langle\psi_0^*\psi(p_i)\rangle = \langle\psi_0^*\psi_0\rangle = 1.$$

Proof. From the definition (4.3)

$$\begin{aligned}\langle\psi(p_i)\psi^*(q_j)\rangle &= \sum_{m,n\in\mathbf{Z}} \langle\psi_m\psi_n^*\rangle p_i^m q_j^{-n} = \sum_{m,n<0} \delta_{m,n} p_i^m q_j^{-n} \\ &= \sum_{m=1}^{\infty}\left(\frac{q_j}{p_i}\right)^m = \frac{q_j}{p_i - q_j}\end{aligned}$$

and similiarly

$$\begin{aligned}\langle\psi^*(q_j)\psi(p_i)\rangle &= \sum_{m,n\in\mathbf{Z}} \langle\psi_n^*\psi_m\rangle p_i^m q_j^{-n} = \sum_{m,n\geq 0} \delta_{m,n} p_i^m q_j^{-n} \\ &= \sum_{m=0}^{\infty}\left(\frac{p_i}{q_j}\right)^m = -\frac{q_j}{p_i - q_j}.\end{aligned}$$

□

4.2.1 Partitions

Here we introduce some notations from [56] which will be used later in this chapter.

A partition is any sequence

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r, \ldots) \tag{4.4}$$

of non-negative integers in non-increasing order,

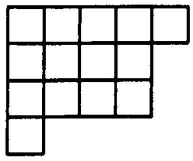
$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq \ldots$$

and containing only finitely many non-zero terms. The non-zero λ_i in (4.4) are called the parts of λ. Sometimes it is convenient to use a notation which indicates the number of times each integer occurs as a part:

$$\lambda = (r^{m_r}, \cdots, 2^{m_2}, 1^{m_1})$$

means that exactly m_i of the parts of λ are equal to i.

The diagram of a partition λ may be formally defined as the set of points $(i,j) \in \mathbb{Z}^2$ such that $1 \geq j \geq \lambda_i$. In drawing such diagrams we shall adopt the convention, as with matrices, that the first coordinate i (the row index) increases as one goes downwards, and the second cordinate j (the column index) increases as one goes from left to right. For example, the diagram of the partition (5441) is



consisting of 5 squares in the top row, 4 in the second row, 4 in the third row, and 1 in the fourth row. We shall usually denote the diagram of a partition λ by the same symbol λ.

Another notation for partitions which is occasionally useful is the following, due to Frobenius. Suppose that the main diagonal of the diagram of λ consists of r squares (i,i) $(1 \leq i \leq r)$. Let $\alpha_i = \lambda_i - i$ be the number of squares in the ith row of λ to the right of (i,i), for $(1 \leq i \leq r)$, and let $\beta_i = \lambda_i' - i$ be the number of squares in the ith column of λ below (i,i), for $(1 \leq i \leq r)$. We have $\alpha_1 > \alpha_2 > \cdots > \alpha_r \geq 0$ and $\beta_1 > \beta_2 > \cdots > \beta_r \geq 0$, and we denote the partition λ by

$$\lambda = (\alpha_1, \ldots, \alpha_r | \beta_1, \ldots, \beta_r) = (\alpha|\beta).$$

For example, if $\lambda = (5441)$, then we have $\alpha = (421)$ and $\beta = (310)$, hence $\lambda = (421|310)$.

Let $a, b \geq 0$, then $(a|b)$ is the Frobenius notation for the partition $(a+1, 1^b)$.

4.3 Charged Free Fermions

We give "time" evolution to the free fermions via a hamiltonian $H(\underline{x})$. For $a \in \mathbf{A}$

$$a(\underline{x}) := e^{H(\underline{x})} a e^{-H(\underline{x})},$$

where $H(\underline{x}) = \sum_{n \geq 1} \left(x_n \sum_{i \in \mathbb{Z}} \psi_i \psi_{i+n}^* \right)$.

Note first of all that

$$\begin{aligned}[H(\underline{x}),\psi(p)] &= \sum_{n\geq 1,k\in\mathbb{Z}} x_n\left[\sum_{i\in\mathbb{Z}}\psi_i\psi^*_{i+n},\psi_k\right]p^k \\ &= \sum_{n\geq 1,k\in\mathbb{Z}} x_n p^k\left(\sum_{i\in\mathbb{Z}}(\psi_i\psi^*_{i+n}\psi_k-\psi_k\psi_i\psi^*_{i+n})\right).\end{aligned}$$

Then using (4.1) we have

$$\begin{aligned}[H(\underline{x}),\psi(p)] &= \sum_{n\geq 1,k\in\mathbb{Z}} x_n p^k\left(\sum_{i\in\mathbb{Z}}\delta_{k,i+n}\psi_i\right) \\ &= \sum_{n\geq 1,k\in\mathbb{Z}} x_n p^k\psi_{k-n} = \sum_{n\geq 1} x_n p^n\psi(p) \\ &= \xi(\underline{x},p)\psi(p),\end{aligned}$$

where $\xi(\underline{x},p) := \sum_{n\geq 1} x_n p^n$. Then

$$\begin{aligned}\psi(p,\underline{x}) &= e^{H(\underline{x})}\psi(p)e^{-H(\underline{x})} \\ &= e^{(\mathrm{ad}\,H(\underline{x}))}\psi(p) = (1+\mathrm{ad}\,H(\underline{x})+\frac{1}{2}(\mathrm{ad}\,H(\underline{x}))^2+\dots)\psi(p),\end{aligned}$$

where $(\mathrm{ad}\,H(\underline{x}))X = [H(\underline{x}),X]$ and so $(\mathrm{ad}\,H(\underline{x}))^j\psi(p) = (\xi(\underline{x},p))^j\psi(p)$, hence

$$\psi(p,\underline{x}) = e^{\xi(\underline{x},p)}\psi(p). \tag{4.5}$$

Similarly, $[H(\underline{x}),\psi^*(q)] = -\xi(\underline{x},q)\psi^*(q)$ and so

$$\psi^*(q,\underline{x}) = e^{H(\underline{x})}\psi^*(q)e^{-H(\underline{x})} = e^{-\xi(\underline{x},q)}\psi^*(q). \tag{4.6}$$

We call a polynomial $\tau(x)$ a *τ-function* if it is representable in the following form for some g:

$$\tau_l(\underline{x},g) = \langle l|g(\underline{x})|l\rangle := \langle\Psi^*_l g(\underline{x})\Psi_l\rangle \tag{4.7}$$

for each $l\in\mathbb{Z}$, where

$$\Psi^*_i = \begin{cases}\psi_{-1}\cdots\psi_i & i<0\\ 1 & i=0\\ \psi^*_0\cdots\psi^*_{i-1} & i>0\end{cases}, \qquad \Psi_i = \begin{cases}\psi^*_i\cdots\psi^*_{-1} & i<0\\ 1 & i=0\\ \psi_{i-1}\cdots\psi_0 & i>0\end{cases}.$$

For example we take

$$g = \psi(p_1)\cdots\psi(p_r)\psi^*(q_1)\cdots\psi^*(q_s) \tag{4.8}$$

for some r, s. Then for $r = s$,

$$\begin{aligned}
\tau_0(\underline{x}, g) &= \langle 0|g(\underline{x})|0\rangle \\
&= e^{\sum_{i=1}^{r}\xi(\underline{x},p_i)-\xi(\underline{x},q_i)}\langle g\rangle \\
&= e^{\sum_{i=1}^{r}\xi(\underline{x},p_i)-\xi(\underline{x},q_i)}\det(\langle\psi(p_i)\psi^*(q_j)\rangle)
\end{aligned}$$

and then, using Lemma 4.2.2, we get

$$\tau_0(\underline{x}, g) = \det\left(e^{\xi(\underline{x},p_i)-\xi(\underline{x},q_j)}\frac{q_j}{p_i - q_j}\right). \tag{4.9}$$

Next we wish to express the τ-function τ_1. From (4.7)

$$\begin{aligned}
\tau_1(\underline{x}, g) &= \langle 1|g(\underline{x})|1\rangle \\
&= \langle \Psi_1^* g(\underline{x})\Psi_1\rangle \\
&= \langle \psi_0^* g(\underline{x})\psi_0\rangle \\
&= e^{\sum_{i=1}^{r}\xi(\underline{x},p_i)-\xi(\underline{x},q_i)}\langle\psi_0^*\psi(p_1)\cdots\psi(p_r)\psi^*(q_1)\cdots\psi^*(q_r)\psi_0\rangle \\
&= e^{\sum_{i=1}^{r}\xi(\underline{x},p_i)-\xi(\underline{x},q_i)}\begin{vmatrix}
-\langle\psi^*(q_1)\psi_0\rangle & \cdots & -\langle\psi^*(q_r)\psi_0\rangle & -\langle\psi_0^*\psi_0\rangle \\
\langle\psi(p_1)\psi^*(q_1)\rangle & \cdots & \langle\psi(p_1)\psi^*(q_r)\rangle & -\langle\psi_0^*\psi(p_1)\rangle \\
\vdots & \ddots & \vdots & \vdots \\
\langle\psi(p_r)\psi^*(q_1)\rangle & \cdots & \langle\psi(p_r)\psi^*(q_r)\rangle & -\langle\psi_0^*\psi(p_r)\rangle
\end{vmatrix}
\end{aligned}$$

using Lemma 4.2.2 we get

$$\tau_1(\underline{x}, g) = e^{\sum_{i=1}^{r}\xi(\underline{x},p_i)-\xi(\underline{x},q_i)}\begin{vmatrix}
-1 & \cdots & -1 & -1 \\
\frac{q_1}{p_1-q_1} & \cdots & \frac{q_r}{p_1-q_r} & -1 \\
\vdots & \ddots & \vdots & \vdots \\
\frac{q_1}{p_r-q_1} & \cdots & \frac{q_r}{p_r-q_r} & -1
\end{vmatrix}. \tag{4.10}$$

Next we do the following operation in the determinant in (4.10): we subtract the first row of the determinant from the rest of the rows. We get

$$
\begin{aligned}
\tau_1(\underline{x},g) &= e^{\sum_{i=1}^r \xi(\underline{x},p_i)-\xi(\underline{x},q_i)} \begin{vmatrix} -1 & \cdots & -1 & -1 \\ \frac{p_1}{p_1-q_1} & \cdots & \frac{p_r}{p_1-q_r} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \frac{p_1}{p_r-q_1} & \cdots & \frac{p_r}{p_r-q_r} & 0 \end{vmatrix} \\
&= (-1)^r e^{\sum_{i=1}^r \xi(\underline{x},p_i)-\xi(\underline{x},q_i)} \begin{vmatrix} \frac{p_1}{p_1-q_1} & \cdots & \frac{p_r}{p_1-q_r} \\ \vdots & \ddots & \vdots \\ \frac{p_1}{p_r-q_1} & \cdots & \frac{p_r}{p_r-q_r} \end{vmatrix} \\
&= (-1)^r \det\left(e^{\xi(\underline{x},p_i)-\xi(\underline{x},q_j)} \frac{p_i}{p_i-q_j} \right). \qquad (4.11)
\end{aligned}
$$

Lemma 4.3.1 *For all $g \in A$,*

$$\tau_l(\underline{x},g) = \tau_{l-m}(\underline{x},\iota_m(g(\underline{x}))), \qquad (4.12)$$

where $\iota_l(\psi_i) = \psi_{i-l}$ and $\iota_l(\psi_i^*) = \psi_{i-l}^*$.

Proof. In order to proof the lemma, we first need to show

$$\tau_1(\underline{x},g) = \tau_0(\underline{x},\iota_1(g)), \qquad (4.13)$$

where g is given by (4.8). Using the definition of fermions from (4.3)

$$\iota_l(\psi(p)) = \sum_{i\in\mathbf{Z}} \iota_l(\psi_i)p^i = \sum_{i\in\mathbf{Z}} \psi_{i-l}p^i = p^l\psi(p)$$

and

$$\iota_l(\psi^*(q)) = \sum_{j\in\mathbf{Z}} \iota_l(\psi_j^*)q^{-j} = \sum_{j\in\mathbf{Z}} \psi_{j-l}^*q^{-j} = q^{-l}\psi^*(q),$$

and hence

$$\iota_1(g) = \frac{\Pi_{i=1}^r p_i}{\Pi_{i=1}^s q_i} g.$$

Using these results we can show the equality in (4.13):

$$
\begin{aligned}
\tau_0(\underline{x},\iota_1(g)) &= \frac{\Pi_{i=1}^r p_i}{\Pi_{i=1}^s q_i}\tau_0(\underline{x},g) \\
&= \frac{\Pi_{i=1}^r p_i}{\Pi_{i=1}^s q_i} \det\left(e^{\xi(\underline{x},p_i)-\xi(\underline{x},q_j)} \frac{q_j}{p_i-q_j} \right) \\
&= \det\left(e^{\xi(\underline{x},p_i)-\xi(\underline{x},q_j)} \frac{p_i}{p_i-q_j} \right) \\
&= \tau_1(\underline{x},g).
\end{aligned}
$$

Then, clearly

$$\begin{aligned}\tau_l(\underline{x}, g) &= \tau_{l-1}(\underline{x}, \iota_1(g)) = \tau_{l-2}(\underline{x}, \iota_1(\iota_1(g))) = \tau_{l-2}(\underline{x}, \iota_2(g)) \cdots \\ &= \tau_{l-l}(\underline{x}, \iota_l(g)) = \tau_0(\underline{x}, \iota_l(g))\end{aligned}$$

and

$$\begin{aligned}\tau_l(\underline{x}, g) &= \tau_{l-m}(\underline{x}, \iota_m(g)) \\ &= \tau_0(\underline{x}, \iota_{l-m}(\iota_m(g))) \\ &= \tau_0(\underline{x}, \iota_l(g)).\end{aligned}$$

Every product of free fermions is the coefficient of some power of $p_1, \cdots, p_r, q_1, \cdots, q_r$ in g defined in (4.8). By expanding both sides of (4.12) with respect to these parameters, the full result follows. Hence the result is proved for g given by (4.8). □

Now we wish to express the τ-function τ_0 in terms of Schur functions [56]. In general a Schur function S_λ, where $\lambda = (\alpha_1, \ldots, \alpha_r | \beta_1, \ldots, \beta_r)$, is defined by

$$S_\lambda = \det(S_{(\alpha_i|\beta_j)}), \tag{4.14}$$

where

$$S_{(\alpha|\beta)} = \sum_{k=0}^{\beta}(-1)^k h_{\alpha+1+k}(x) e_{\beta-k}(x),$$

where $h_i(x)$ and $e_j(x)$ are the complete and elementary symmetric functions, respectively.

The element g as given by (4.8) can be written as

$$g = \sum_{\substack{i_1, \cdots, i_n \in \mathbf{Z} \\ j_1, \cdots, j_n \in \mathbf{Z}}} p_1^{i_1} \cdots p_n^{i_n} q_1^{-j_1} \cdots q_n^{-j_n} g',$$

where

$$g' = \psi_{i_1} \cdots \psi_{i_n} \psi^*_{j_1} \cdots \psi^*_{j_n}.$$

Then (4.9) can be used as a generating function to determine $\tau_0(\underline{x}; g')$ by looking at the coefficients of $p_1^{i_1} \cdots p_n^{i_n} q_1^{-j_1} \cdots q_n^{-j_n}$, where $i_1 > i_2 \cdots > i_n, j_1 > j_2 > \cdots > j_n \in \mathbb{Z}$. Next we expand the entries of the determinant in (4.9) in the following way:

$$e^{\xi(\underline{x}, p_i)} = \sum_{i=0}^{\infty} h_i(x) p^i, \qquad e^{-\xi(\underline{x}, q_j)} = \sum_{j=0}^{\infty} (-1)^j e_j(x) q^j \tag{4.15}$$

and

$$\frac{q_j}{p_i - q_j} = -\sum_{k=0}^{\infty} \left(\frac{p_i}{q_j}\right)^k.$$

Thus the (i,j)th entry in (4.9) can be written as

$$\begin{aligned} e^{\sum_{i=1}^{r}(\xi(\underline{x},p_i)-\xi(\underline{x},q_i))} \frac{q_j}{p_i - q_j} &= \sum_{i,j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{j+1} h_i(x) e_j(x) p^{i+k} q^{j-k} \\ &= \sum_{i,j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{k-j+1} h_{i-k}(x) e_{k-j}(x) p^i q^{-j}, \qquad (4.16) \end{aligned}$$

where $h_n(x) = e_n(x) = 0$ for $n < 0$ and $h_0(x) = e_0(x) = 1$.

Lemma 4.3.2 *The coefficients of $p_1^{i_1} \cdots p_n^{i_n} q_1^{-j_1} \cdots q_n^{-j_n}$ in (4.16) can be expressed in terms of Schur function in the following form:*

$$\begin{aligned} \sum_{k=0}^{\infty} (-1)^{k-j+1} h_{i-k}(x) e_{k-j}(x) &= (-1)^{-j-1} S_{(i|-j-1)} \\ &= (-1)^{-j-1} S_{(i+1,1^{-j-1})}. \end{aligned}$$

Proof.

$$\sum_{k=j}^{i} (-1)^{k-j+1} h_{i-k}(x) e_{k-j}(x) = \sum_{i=0}^{k} (-1)^{i+1} h_{k-i}(x) e_i(x) = 0 \qquad (4.17)$$

since

$$\begin{aligned} 1 &= e^{\sum_{i=0} p^i x_i} e^{-\sum_{k=0} p^k x_k} \\ &= \sum_{i=0} h_i(x) p^i \quad \sum_{k=0} (-1)^k e_k(x) p^k \\ &= \sum_{k=0} \sum_{i=0}^{k} (-1)^i h_{k-i}(x) e_i(x) p^k \\ 0 &= \sum_{k=1} \sum_{i=0}^{k} (-1)^i h_{k-i}(x) e_i(x) p^k \\ &= \sum_{i=0}^{k} (-1)^i h_{k-i}(x) e_i(x). \end{aligned}$$

From (4.17)

$$\begin{aligned}
\sum_{k=j}^{-1}(-1)^{k-j}h_{i-k}(x)e_{k-j}(x) &+ \sum_{k=0}^{i}(-1)^{k-j}h_{i-k}(x)e_{k-j}(x) = 0 \\
\sum_{k=0}^{i}(-1)^{k-j+1}h_{i-k}(x)e_{k-j}(x) &= \sum_{k=j}^{-1}(-1)^{k-j}h_{i-k}(x)e_{k-j}(x) \\
&= \sum_{k=0}^{-j-1}(-1)^{-k-j-1}h_{i+k+1}(x)e_{-k-j-1}(x) \\
&= (-1)^{-j-1}S_{(i|-j-1)}
\end{aligned}$$

□

Hence (4.9) can be written as

$$\tau_0(\underline{x}, g) = \det\left(\sum_{i,j=0}^{\infty}(-1)^{-j-1}S_{(i|-j-1)}p^i q^{-j}\right) \tag{4.18}$$

and

$$\begin{aligned}
\tau_0(\underline{x}, g') &= (-1)^{-j_1-\cdots-j_n-n}\det\left(S_{(i|-j-1)}\right) \\
&= (-1)^{-j_1-\cdots-j_n-n}\det\left(S_{(i+1,1^{-j-1})}\right).
\end{aligned}$$

Similiarly $\tau_1(\underline{x}, g)$ can be written from (4.18) by using Lemma 4.3.1 as

$$\tau_1(\underline{x}, g) = \det\left(\sum_{i,j=0}^{\infty}(-1)^{-j}S_{(i-1|-j)}p^i q^{-j}\right)$$

and

$$\begin{aligned}
\tau_1(\underline{x}, g') &= (-1)^{-j_1-\cdots-j_n}\det\left(S_{(i-1|-j)}\right) \\
&= (-1)^{-j_1-\cdots-j_n}\det\left(S_{(i,1^{-j})}\right).
\end{aligned}$$

In general

$$\tau_l(\underline{x}, g) = \det\left(\sum_{i,j=0}^{\infty}(-1)^{-j+l-1}S_{(i-l|-j+l-1)}p^i q^{-j}\right)$$

and

$$\begin{aligned}
\tau_l(\underline{x}, g') &= (-1)^{-j_1-\cdots-j_n+nl-n}\det\left(S_{(i-l|-j+l-1)}\right) \\
&= (-1)^{-j_1-\cdots-j_n+nl-n}\det\left(S_{(i-l+1,1^{-j+l-1})}\right).
\end{aligned}$$

Hence, by (4.14), each τ_l is a Schur function. These give rational solutions of the KP equation, where $u = 2\partial_x^2(\log\tau)$.

4.4 Charged Free Fermions with 2 Components

In this section we consider free fermions with 2 components. Consider free fermions $\psi_n^{(j)}$, $\psi_n^{(j)*}$ indexed by $n \in \mathbb{Z}$ and $j = 1, 2$, satisfying the anti-commutator relations

$$[\psi_m^{(j)}, \psi_n^{(k)}]_+ = 0, \qquad [\psi_m^{(j)*}, \psi_n^{(k)*}]_+ = 0, \qquad [\psi_m^{(j)}, \psi_n^{(k)*}]_+ = \delta_{jk}\delta_{mn}, \tag{4.19}$$

where $[X, Y]_+ = XY + YX$. Such fermions are obtainable by renumbering the fermions of a single component. For example, the simplest choice is

$$\begin{aligned} \psi_n^{(1)} &= \psi_{2n}, \quad \psi_n^{(2)} = \psi_{2n+1} \\ \psi_n^{(1)*} &= \psi_{2n}^*, \quad \psi_n^{(2)*} = \psi_{2n+1}^*. \end{aligned} \tag{4.20}$$

Fixing the renumbering (4.20), we identify the vacuum expectation values for the 2 component fermions with the single component fermions. The time evolution for the 2 component fermions are induced by the following Hamiltonian

$$H(\underline{x}^{(1)}, \underline{x}^{(2)}) = \sum_{\substack{i \geq 1 \\ n \in \mathbb{Z} \\ j=1,2}} x_i^{(j)} \psi_n^{(j)} \psi_{i+n}^{(j)*},$$

where the time variables are $\underline{x}^{(j)} = (\underline{x}_1^{(j)}, \underline{x}_2^{(j)}, \dots)(j = 1, 2)$.

Lemma 4.4.1 *The expectation values of 2 component free fermions are given by*

$$\langle \psi^{(h)}(p_i^{(h)}) \psi^{(h)*}(q_j^{(h)}) \rangle = \frac{q_j^{(h)}}{p_i^{(h)} - q_j^{(h)}}, \qquad \langle \psi^{(h)*}(q_j^{(h)}) \psi^{(h)}(p_i^{(h)}) \rangle = -\frac{q_j^{(h)}}{p_i^{(h)} - q_j^{(h)}}$$

and

$$\langle \psi^{(h)}(p_i^{(h)}) \psi^{(k)*}(q_j^{(k)}) \rangle = \langle \psi^{(h)*}(q_j^{(h)}) \psi^{(k)}(p_i^{(k)}) \rangle = 0,$$

$$\langle \psi^{(h)}(p_i^{(h)}) \psi^{(h)}(p_j^{(h)}) \rangle = \langle \psi^{(h)*}(q_i^{(h)}) \psi^{(h)*}(q_j^{(h)}) \rangle = 0,$$

where $h, k = 1, 2$.

Proof. From the definition (4.3) and (4.20)

$$\begin{aligned}
\langle\psi^{(h)}(p_i^{(h)})\psi^{(h)*}(q_j^{(h)})\rangle &= \sum_{m,n\in\mathbf{Z}}\langle\psi_m^{(h)}\psi_n^{(h)*}\rangle(p_i^{(h)})^m(q_j^{(h)})^{-n} \\
&= \sum_{m,n\in\mathbf{Z}}\langle\psi_{2m}\psi_{2n}^*\rangle(p_i^{(h)})^m(q_j^{(h)})^{-n} \\
&= \sum_{m,n<0}\delta_{2m,2n}(p_i^{(h)})^m(q_j^{(h)})^{-n} \\
&= \sum_{m=1}^{\infty}\left(\frac{q_j^{(h)}}{p_i^{(h)}}\right)^m = \frac{q_j^{(h)}}{p_i^{(h)}-q_j^{(h)}}
\end{aligned}$$

and similiarly

$$\begin{aligned}
\langle\psi^{(h)*}(q_j^{(h)})\psi^{(h)}(p_i^{(h)})\rangle &= \sum_{m,n\in\mathbf{Z}}\langle\psi_n^{(h)*}\psi_m^{(h)}\rangle(p_i^{(h)})^m(q_j^{(h)})^{-n} \\
&= \sum_{m,n\in\mathbf{Z}}\langle\psi_{2n}^*\psi_{2m}\rangle(p_i^{(h)})^m(q_j^{(h)})^{-n} \\
&= \sum_{m,n\geq 0}\delta_{2m,2n}(p_i^{(h)})^m(q_j^{(h)})^{-n} \\
&= \sum_{m=0}^{\infty}\left(\frac{p_i^{(h)}}{q_j^{(h)}}\right)^m = -\frac{q_j^{(h)}}{p_i^{(h)}-q_j^{(h)}}.
\end{aligned}$$

□

Note first of all that

$$\begin{aligned}
[H(\underline{x}^{(1)},\underline{x}^{(2)}),\psi^{(j)}(p^{(j)})] &= \sum_{i\geq 1,k\in\mathbf{Z}} x_i^{(j)}\left[\sum_{n\in\mathbf{Z}}\psi_n^{(j)}\psi_{i+n}^{(j)*},\psi_k^{(j)}\right]p^{(j)k} \\
&= \sum_{i\geq 1,k\in\mathbf{Z}} x_i^{(j)}p^{(j)k}\left(\sum_{n\in\mathbf{Z}}(\psi_n^{(j)}\psi_{i+n}^{(j)*}\psi_k^{(j)}-\psi_k^{(j)}\psi_n^{(j)}\psi_{i+n}^{(j)*})\right)
\end{aligned}$$

then using (4.19), we have

$$\begin{aligned}
[H(\underline{x}^{(1)},\underline{x}^{(2)}),\psi^{(j)}(p^{(j)})] &= \sum_{i\geq 1,k\in\mathbf{Z}} x_i^{(j)}p^{(j)k}\left(\sum_{n\in\mathbf{Z}}\delta_{k,i+n}\psi_n^{(j)}\right) \\
&= \sum_{i\geq 1,k\in\mathbf{Z}} x_i^{(j)}p^{(j)k}\psi_{k-i}^{(j)} = \sum_{i\geq 1} x_i^{(j)}p^{(j)i}\psi^{(j)}(p^{(j)}) \\
&= \xi(\underline{x}^{(j)},p^{(j)})\psi^{(j)}(p^{(j)}),
\end{aligned}$$

where $\xi(\underline{x}^{(j)},p^{(j)}) := \sum_{i\geq 1} x_i^{(j)}p^{(j)i}$. Then the time evolution

$$\begin{aligned}
\psi^{(h)}(p^{(h)},\underline{x}^{(1)},\underline{x}^{(2)}) &= e^{H(\underline{x}^{(1)},\underline{x}^{(2)})}\psi^{(j)}(p^{(j)})e^{-H(\underline{x}^{(1)},\underline{x}^{(2)})} \\
&= e^{(\mathrm{ad}\,H(\underline{x}^{(1)},\underline{x}^{(2)}))}\psi^{(j)}(p^{(j)}) \\
&= (1+\mathrm{ad}\,H(\underline{x}^{(1)},\underline{x}^{(2)})+\frac{1}{2}(\mathrm{ad}\,H(\underline{x}^{(1)},\underline{x}^{(2)}))^2+\dots)\psi^{(j)}(p^{(j)}),
\end{aligned}$$

where $(\mathrm{ad}\, H(\underline{x}^{(1)}, \underline{x}^{(2)}))X = [H(\underline{x}^{(1)}, \underline{x}^{(2)}), X]$ and so
$(\mathrm{ad}\, H(\underline{x}^{(1)}, \underline{x}^{(2)}))^m \psi^{(j)}(p^{(j)}) = (\xi(\underline{x}^{(j)}, p^{(j)}))^m \psi^{(j)}(p^{(j)})$. Hence

$$\psi^{(j)}(p^{(j)}, \underline{x}^{(1)}, \underline{x}^{(2)}) = e^{\xi(\underline{x}^{(j)}, p^{(j)})} \psi^{(j)}(p^{(j)}). \tag{4.21}$$

Similarly, $[H(\underline{x}^{(1)}, \underline{x}^{(2)}), \psi^{(j)*}(q^{(j)})] = -\xi(\underline{x}^{(j)}, q^{(j)})\psi^{(j)*}(q^{(j)})$ and the time evolution for

$$\begin{aligned} \psi^{(j)*}(q^{(j)}, \underline{x}^{(1)}, \underline{x}^{(2)}) &= e^{H(\underline{x}^{(1)}, \underline{x}^{(2)})} \psi^{(j)*}(q^{(j)}) e^{-H(\underline{x}^{(1)}, \underline{x}^{(2)})} \\ &= e^{-\xi(\underline{x}^{(j)}, q^{(j)})} \psi^{(j)*}(q^{(j)}). \end{aligned}$$

The τ-functions for the 2 component free fermions with the total charge $l_1 + l_2$, we define in the following form for some $g = g^{(1)} g^{(2)}$:

$$\begin{aligned} \tau_{l_1, l_2; l_3}(\underline{x}^{(1)}, \underline{x}^{(2)}) &= \langle l_1, l_2 | e^{H(\underline{x}^{(1)}, \underline{x}^{(2)})} g | l_2 - l_3, l_1 + l_3 \rangle \\ &= \langle \Psi_{l_1}^{(1)*} \Psi_{l_2}^{(2)*} e^{H(\underline{x}^{(1)}, \underline{x}^{(2)})} g \Psi_{l_2 - l_3}^{(2)} \Psi_{l_1 + l_3}^{(1)} \rangle, \end{aligned} \tag{4.22}$$

where

$$\Psi_l^{(i)*} = \begin{cases} \psi_{-1}^{(i)} \cdots \psi_l^{(i)} & l < 0 \\ 1 & l = 0 \\ \psi_0^{(i)*} \cdots \psi_{l-1}^{(i)*} & l > 0 \end{cases}, \qquad \Psi_l^{(i)} = \begin{cases} \psi_l^{(i)*} \cdots \psi_{-1}^{(i)*} & l < 0 \\ 1 & l = 0 \\ \psi_{l-1}^{(i)} \cdots \psi_0^{(i)} & l > 0 \end{cases}.$$

For example we take

$$g^{(j)} = \psi^{(j)}(p_1^{(j)}) \cdots \psi^{(j)}(p_r^{(j)}) \psi^{(j)*}(q_1^{(j)}) \cdots \psi^{(j)*}(q_s^{(j)}) \tag{4.23}$$

for some r, s. Then for $r = s$,

$$\begin{aligned} \tau_{0,0;0}(\underline{x}^{(1)}, \underline{x}^{(2)}) &= \langle 0, 0 | g(\underline{x}^{(1)}, \underline{x}^{(2)}) | 0, 0 \rangle \\ &= e^{\sum_{i=1, j=1,2}^{r} \xi(\underline{x}^{(j)}, p_i^{(j)}) - \xi(\underline{x}^{(j)}, q_i^{(j)})} \langle g \rangle \end{aligned}$$

and, using Lemma 4.4.1, we get

$$\begin{aligned} &\tau_{0,0;0}(\underline{x}^{(1)}, \underline{x}^{(2)}) \\ &= e^{\sum_{i=1, j=1,2}^{r} \xi(\underline{x}^{(j)}, p_i^{(j)}) - \xi(\underline{x}^{(j)}, q_i^{(j)})} \det(\langle \psi^{(1)}(p_i^{(1)}) \psi^{(1)*}(q_j^{(1)}) \rangle) \det(\langle \psi^{(2)}(p_i^{(2)}) \psi^{(2)*}(q_j^{(2)}) \rangle) \\ &= (-1)^{r(r-1)} \det\left(e^{\xi(\underline{x}^{(1)}, p_i^{(1)}) - \xi(\underline{x}^{(1)}, q_j^{(1)})} \frac{q_j^{(1)}}{p_i^{(1)} - q_j^{(1)}} \right) \det\left(e^{\xi(\underline{x}^{(2)}, p_i^{(2)}) - \xi(\underline{x}^{(2)}, q_j^{(2)})} \frac{q_j^{(2)}}{p_i^{(2)} - q_j^{(2)}} \right). \end{aligned}$$

4.5 Charged Free Fermions In General

In this section, we wish to express g more generally in the following form

$$g = e^{\sum_{i,j=1}^{N} a_{ij}\psi^i\psi^{j*}}, \tag{4.24}$$

where the ψ^i, ψ^{j*} $(i,j = 1,\dots,N)$ can be either one-component or two-component fermions. For example, in this chapter we will take $\psi^i = \psi(p_i)$ for the one-component case and $\psi^i = \psi^{(1)}(p^{(1)})$ or $\psi^i = \psi^{(2)}(p^{(2)})$ for the two-component case. Then the τ-function τ_0 is

$$\begin{aligned}
\tau_0 &= \langle g(\underline{x})\rangle \\
&= 1 + \sum_{i_1,j_1=1}^{N} a_{i_1j_1}\langle\psi^{i_1}\psi^{j_1*}\rangle + \sum_{i_1,j_1,i_2,j_2=1}^{N} a_{i_1j_1}a_{i_2j_2}\langle\psi^{i_1}\psi^{j_1*}\psi^{i_2}\psi^{j_2*}\rangle + \cdots \\
&\quad + \sum_{i_1,j_1,\cdots,i_N,j_N=1}^{N} a_{i_1j_1}\cdots a_{i_Nj_N}\langle\psi^{i_1}\psi^{j_1*}\cdots\psi^{i_N}\psi^{j_N*}\rangle.
\end{aligned} \tag{4.25}$$

If we give the following expectation values

$$\langle\psi^i\psi^j\rangle := \langle\psi^{i*}\psi^{j*}\rangle := 0 \tag{4.26}$$

and using Wick's theorem, the τ-function in (4.25) can be written in the $(N \times N)$ determinantal structure in the following form:

$$\tau_0 = \det(I + AV), \tag{4.27}$$

where I is the identity matrix, A is a constant matrix with the entries $A = [a_{ji}]$, and V is a matrix with the entries of expectation values of quadratic free fermions $V = [\langle\psi^i\psi^{j*}\rangle]$. See Appendix for the proof.

Next we give a general M order τ-function, from which the higher order τ-functions can be obtained. This formula can be written in the following form

$$\begin{aligned}
\tau_M &= \langle\psi^{i_1*}\cdots\psi^{i_M*}g\psi^{i_1}\cdots\psi^{i_M}\rangle \\
&= \begin{vmatrix} X & Y \\ G & Z \end{vmatrix},
\end{aligned} \tag{4.28}$$

where

$$X = \begin{pmatrix} \langle\psi^{i_1}\psi^{1*}\rangle & \cdots & \langle\psi^{i_1}\psi^{N*}\rangle \\ \vdots & \ddots & \vdots \\ \langle\psi^{i_M}\psi^{1*}\rangle & \cdots & \langle\psi^{i_M}\psi^{N*}\rangle \end{pmatrix},$$

$$Y = \begin{pmatrix} \langle\psi^{i_1}\psi^{i_1*}\rangle & \cdots & \langle\psi^{i_1}\psi^{i_M*}\rangle \\ \vdots & \ddots & \vdots \\ \langle\psi^{i_M}\psi^{i_1*}\rangle & \cdots & \langle\psi^{i_M}\psi^{i_M*}\rangle \end{pmatrix},$$

$$Z = \begin{pmatrix} \langle\psi^{1}\psi^{i_1*}\rangle & \cdots & \langle\psi^{1}\psi^{i_M*}\rangle \\ \vdots & \ddots & \vdots \\ \langle\psi^{N}\psi^{i_1*}\rangle & \cdots & \langle\psi^{N}\psi^{i_M*}\rangle \end{pmatrix},$$

and while g can be given either as

$$g = \psi^{1*}\cdots\psi^{N*}\psi^{1}\cdots\psi^{N} \tag{4.29}$$

or as in (4.24), then the matrix G can be either $G = \left(e^{\xi(\underline{x},p_i)-\xi(\underline{x},q_j)}\frac{q_j}{p_i-q_j}\right)$ or $G = I+AV$, respectively. Thus, G depends on the choice of g. For example, for $M = 1$, $N = r$ and g is given by (4.29), we have

$$\begin{aligned} \tau_1 &= \langle\psi^{i_1*}g(\underline{x})\psi^{i_1}\rangle \\ &= \begin{vmatrix} \langle\psi^{i_1}\psi^{1*}\rangle & \cdots & \langle\psi^{i_1}\psi^{r*}\rangle & \langle\psi^{i_1}\psi^{i_1*}\rangle \\ & & & \langle\psi^{1}\psi^{i_1*}\rangle \\ & G & & \vdots \\ & & & \langle\psi^{r}\psi^{i_1*}\rangle \end{vmatrix}. \end{aligned}$$

If we choose $\psi^{i_1} = \psi_0$, $\psi^{i_1*} = \psi_0^*$ and $\psi^j = \psi(p_j)$, $\psi^{j*} = \psi^*(q_j)$ $(j = 1,\cdots,r)$, then we recover the τ_1-function in (4.10).

In order to get the soliton solution for the 1-component case, we choose the constants $a_{ij} = 0 \quad (i \neq j)$. From (4.25)

$$\begin{aligned} \tau &= 1 + \sum_{i_1=1}^{N} a_{i_1i_1}\langle\psi^{i_1}\psi^{i_1*}\rangle + \sum_{i_1<i_2=1}^{N} a_{i_1i_1}a_{i_2i_2}\langle\psi^{i_1}\psi^{i_1*}\psi^{i_2}\psi^{i_2*}\rangle + \cdots + \\ & \sum_{i_1<i_2<\cdots<i_N=1}^{N} a_{i_1i_1}\cdots a_{i_Ni_N}\langle\psi^{i_1}\psi^{i_1*}\cdots\psi^{i_N}\psi^{i_N*}\rangle \end{aligned}$$

gives rise to the N-soliton solution.

For example, for $N=2$ from (4.25) we have the following solution

$$\begin{aligned}
\tau_0 &= 1+\sum_{i_1,j_1=1}^{2} a_{i_1j_1}\langle\psi^{i_1}\psi^{j_1*}\rangle+\sum_{i_1,j_1,i_2,j_2=1}^{2} a_{i_1j_1}a_{i_2j_2}\langle\psi^{i_1}\psi^{j_1*}\psi^{i_2}\psi^{j_2*}\rangle\\
&= 1+a_{11}\langle\psi^1\psi^{1*}\rangle+a_{12}\langle\psi^1\psi^{2*}\rangle+a_{21}\langle\psi^2\psi^{1*}\rangle+a_{22}\langle\psi^2\psi^{2*}\rangle\\
&\quad +a_{11}a_{22}\langle\psi^1\psi^{1*}\psi^2\psi^{2*}\rangle+a_{12}a_{21}\langle\psi^1\psi^{2*}\psi^2\psi^{1*}\rangle\\
&= 1+a_{11}\langle\psi^1\psi^{1*}\rangle+a_{12}\langle\psi^1\psi^{2*}\rangle+a_{21}\langle\psi^2\psi^{1*}\rangle+a_{22}\langle\psi^2\psi^{2*}\rangle\\
&\quad +(a_{11}a_{22}-a_{12}a_{21})(\langle\psi^1\psi^{1*}\rangle\langle\psi^2\psi^{2*}\rangle-\langle\psi^1\psi^{2*}\rangle\langle\psi^2\psi^{1*}\rangle)\\
&= \begin{vmatrix} 1+a_{11}\langle\psi^1\psi^{1*}\rangle+a_{21}\langle\psi^2\psi^{1*}\rangle & a_{11}\langle\psi^1\psi^{2*}\rangle+a_{21}\langle\psi^2\psi^{2*}\rangle\\ a_{12}\langle\psi^1\psi^{1*}\rangle+a_{22}\langle\psi^2\psi^{1*}\rangle & 1+a_{12}\langle\psi^1\psi^{2*}\rangle+a_{22}\langle\psi^2\psi^{2*}\rangle \end{vmatrix}
\end{aligned} \tag{4.30}$$

and this can be written as in (4.27), $\tau=\det(I_2+A_2V_2)$ where $I_2=\mathrm{diag}(1,1)$, $A_2=\begin{pmatrix} a_{11} & a_{21}\\ a_{12} & a_{22}\end{pmatrix}$ and $V_2=\begin{pmatrix}\langle\psi^1\psi^{1*}\rangle & \langle\psi^1\psi^{2*}\rangle\\ \langle\psi^2\psi^{1*}\rangle & \langle\psi^2\psi^{2*}\rangle\end{pmatrix}$.

In general the 2-soliton solution can be written from (4.30) by choosing $a_{12}=a_{21}=0$ (the reason for this choice will be explained in a later example)

$$\tau=\begin{vmatrix} 1+a_{11}\langle\psi^1\psi^{1*}\rangle & a_{11}\langle\psi^1\psi^{2*}\rangle\\ a_{22}\langle\psi^2\psi^{1*}\rangle & 1+a_{22}\langle\psi^2\psi^{2*}\rangle\end{vmatrix}. \tag{4.31}$$

Example 4.5.1 *As an example for the one component fermions, we put* $\psi^1=\psi(p_1)$, $\psi^{1*}=\psi^*(q_1)$ *and* $\psi^2=\psi(p_2)$, $\psi^{2*}=\psi^*(q_2)$ *in (4.31). The* τ*-function is*

$$\begin{aligned}
\tau &= 1+a_{11}\langle\psi(p_1)\psi^*(q_1)\rangle+a_{22}\langle\psi(p_2)\psi^*(q_2)\rangle\\
&\quad +a_{11}a_{22}(\langle\psi(p_1)\psi^*(q_1)\rangle\langle\psi(p_2)\psi^*(q_2)\rangle-\langle\psi(p_1)\psi^*(q_2)\rangle\langle\psi(p_2)\psi^*(q_1)\rangle).
\end{aligned}$$

Taking the expectation values and choosing $a_{11}=\dfrac{p_1-q_1}{q_1}$, $a_{22}=\dfrac{p_2-q_2}{q_2}$ *we get*

$$\tau=1+e^{\eta_1}+e^{\eta_2}+A_{12}e^{\eta_1+\eta_2},$$

where

$$\eta_i=\xi(p_i,x)-\xi(q_i,x) \qquad i=1,2$$

and

$$A_{12}=\frac{(p_1-p_2)(q_1-q_2)}{(p_1-q_2)(q_1-p_2)}.$$

Hence $u=2\partial_x^2(\log\tau)$ *gives the 2-soliton solution [12] for the KP equation*

$$(u_x+6uu_x+u_{xxx})_x+3u_{yy}=0.$$

Example 4.5.2 *Here we give an example for the resonant soliton solution for the KP equation. The 4-soliton solution for the KP equation arises from the following choice*

$$g = e^{\sum_{k=1}^{4} a_{kk}\psi(p_k)\psi^*(q_k)}.$$

The τ-function τ is

$$\begin{aligned}
\tau &= \langle g(\underline{x})\rangle \\
&= 1 + a_{11}\langle\psi(p_1)\psi^*(q_1)\rangle + a_{22}\langle\psi(p_2)\psi^*(q_2)\rangle + a_{33}\langle\psi(p_3)\psi^*(q_3)\rangle + a_{44}\langle\psi(p_4)\psi^*(q_4)\rangle \\
&\quad + a_{11}a_{22}\langle\psi(p_1)\psi^*(q_1)\psi(p_2)\psi^*(q_2)\rangle + a_{11}a_{33}\langle\psi(p_1)\psi^*(q_1)\psi(p_3)\psi^*(q_3)\rangle \\
&\quad + a_{11}a_{44}\langle\psi(p_1)\psi^*(q_1)\psi(p_4)\psi^*(q_4)\rangle + a_{22}a_{33}\langle\psi(p_2)\psi^*(q_2)\psi(p_3)\psi^*(q_3)\rangle \\
&\quad + a_{22}a_{44}\langle\psi(p_2)\psi^*(q_2)\psi(p_4)\psi^*(q_4)\rangle + a_{33}a_{44}\langle\psi(p_3)\psi^*(q_3)\psi(p_4)\psi^*(q_4)\rangle \\
&\quad + a_{11}a_{22}a_{33}\langle\psi(p_1)\psi^*(q_1)\psi(p_2)\psi^*(q_2)\psi(p_3)\psi^*(q_3)\rangle \\
&\quad + a_{11}a_{22}a_{44}\langle\psi(p_1)\psi^*(q_1)\psi(p_2)\psi^*(q_2)\psi(p_4)\psi^*(q_4)\rangle \\
&\quad + a_{11}a_{33}a_{44}\langle\psi(p_1)\psi^*(q_1)\psi(p_3)\psi^*(q_3)\psi(p_4)\psi^*(q_4)\rangle \\
&\quad + a_{22}a_{33}a_{44}\langle\psi(p_2)\psi^*(q_2)\psi(p_3)\psi^*(q_3)\psi(p_4)\psi^*(q_4)\rangle \\
&\quad + a_{11}a_{22}a_{33}a_{44}\langle\psi(p_1)\psi^*(q_1)\psi(p_2)\psi^*(q_2)\psi(p_3)\psi^*(q_3)\psi(p_4)\psi^*(q_4)\rangle.
\end{aligned}$$

Now we make the following choice of the parameters, namely we substitute $p_2 = p_1$, then $p_3 = p_2$, $p_4 = p_2$, $q_4 = q_2$, and $q_3 = q_1$. Then the τ-function

$$\begin{aligned}
\tau &= 1 + a_{11}\langle\psi(p_1)\psi^*(q_1)\rangle + a_{22}\langle\psi(p_1)\psi^*(q_2)\rangle + a_{33}\langle\psi(p_2)\psi^*(q_1)\rangle + a_{44}\langle\psi(p_2)\psi^*(q_2)\rangle \\
&\quad + a_{11}a_{44}\langle\psi(p_1)\psi^*(q_1)\psi(p_2)\psi^*(q_2)\rangle + a_{22}a_{33}\langle\psi(p_1)\psi^*(q_2)\psi(p_2)\psi^*(q_1)\rangle.
\end{aligned}$$

Taking the expectation values and choosing $a_{11} = \dfrac{p_1 - q_1}{q_1}$, $a_{22} = \dfrac{p_1 - q_2}{q_2}$, $a_{33} = \dfrac{p_2 - q_1}{q_1}$, $a_{44} = \dfrac{p_2 - q_2}{q_2}$ *we get*

$$\begin{aligned}
\tau &= 1 + e^{\xi(p_1,x)-\xi(q_1,x)} + e^{\xi(p_1,x)-\xi(q_2,x)} + e^{\xi(p_2,x)-\xi(q_1,x)} + e^{\xi(p_2,x)-\xi(q_2,x)} \\
&\quad + A_{12}e^{\xi(p_1,x)-\xi(q_1,x)+\xi(p_2,x)-\xi(q_2,x)},
\end{aligned}$$

where

$$A_{12} = \frac{(p_2 - p_1)(q_1 - q_2)(p_1 - p_2)(q_1 - q_2)}{(p_1 - q_2)(p_2 - q_1)(p_1 - q_1)(p_2 - q_2)}.$$

This gives rise to the resonant 4-soliton solution of the KP equation.

In order to get the τ-function for the 2-soliton solution, as in the previous example, here we choose $a_{22} = a_{33} = 0$. The reason for this choice is that this makes the 2nd *and 3rd terms vanish in the τ-function. Hence the resonant behaviour that gives rise to a solitoff [36] vanishes and we get the 2-soliton solution.*

Example 4.5.3 *For the two component free fermions, we take $a_{11} = a_{22} = 0$ and put $\psi^1 = \psi^{(1)}(p^{(1)})$, $\psi^{1*} = \psi^{(1)*}(q^{(1)})$ and $\psi^2 = \psi^{(2)}(p^{(2)})$, $\psi^{2*} = \psi^{(2)*}(q^{(2)})$ in (4.30). Then the τ-function τ_1 from (4.30) is*

$$\begin{aligned}\tau_1 &= \begin{vmatrix} 1 + a_{21}\langle\psi^2\psi^{1*}\rangle & a_{21}\langle\psi^2\psi^{2*}\rangle \\ a_{12}\langle\psi^1\psi^{1*}\rangle & 1 + a_{12}\langle\psi^1\psi^{2*}\rangle \end{vmatrix} \\ &= 1 + a_{12}\langle\psi^{(1)}(p^{(1)})\psi^{(2)*}(q^{(2)})\rangle + a_{21}\langle\psi^{(2)}(p^{(2)})\psi^{(1)*}(q^{(1)})\rangle \\ &\quad + a_{12}a_{21}(\langle\psi^{(1)}(p^{(1)})\psi^{(2)*}(q^{(2)})\rangle\langle\psi^{(2)}(p^{(2)})\psi^{(1)*}(q^{(1)})\rangle \\ &\quad - \langle\psi^{(1)}(p^{(1)})\psi^{(1)*}(q^{(1)})\rangle\langle\psi^{(2)}(p^{(2)})\psi^{(2)*}(q^{(2)})\rangle).\end{aligned} \tag{4.32}$$

Taking the expectation values and choosing $a_{12} = \dfrac{p^{(1)} - q^{(1)}}{q^{(1)}}$, $a_{21} = \dfrac{q^{(2)} - p^{(2)}}{q^{(2)}}$, we get the τ_1-function for the two component KP-hierachy (DS equations)

$$\tau_1 = 1 + e^{\eta^{(1)} + \eta^{(2)}},$$

where

$$\eta^{(j)} = \xi(p^{(j)}, x) - \xi(q^{(j)}, x) \qquad j = 1, 2. \tag{4.33}$$

Hence $u = \frac{\tau_3}{\tau_1}$ gives the 1-soliton solution [54] to the DS equations, where τ_3 is the bordered determinant of (4.32).

Example 4.5.4 *In order to get the 1-dromion solution for the DS equations we define the following functions [54] (see (4.22))*

$$\begin{aligned}\tau_1(z) &= \langle l_1, l_2 - 1|e^{H(z)}g|l_2 - l - 1, l_1 + l\rangle, \\ \tau_2(z) &= \langle l_1 + 1, l_2 - 2|e^{H(z)}g|l_2 - l - 1, l_1 + l\rangle, \\ \tau_3(z) &= \langle l_1 - 1, l_2|e^{H(z)}g|l_2 - l - 1, l_1 + l\rangle,\end{aligned} \tag{4.34}$$

then the DS equations can be written in the following bilinear equations

$$\begin{aligned}&D_2^{(2)}\tau_2 \cdot \tau_1 + D_1^{(2)^2}\tau_1 \cdot \tau_2 = 0, \\ &D_2^{(1)}\tau_1 \cdot \tau_2 + D_1^{(1)^2}\tau_2 \cdot \tau_1 = 0, \\ &D_2^{(2)}\tau_1 \cdot \tau_3 + D_1^{(2)^2}\tau_3 \cdot \tau_1 = 0, \\ &D_2^{(1)}\tau_3 \cdot \tau_1 + D_1^{(1)^2}\tau_1 \cdot \tau_3 = 0, \\ &D_1^{(1)}D_1^{(2)}\tau_1 \cdot \tau_1 - 2\tau_2 \cdot \tau_3 = 0.\end{aligned} \tag{4.35}$$

The τ-functions in (4.34) provide a method for generating solutions of the two-component KP hierarchy of bilinear equations. In what follows we take the simplest choice $l = l_1 = l_2 - 1 = 0$. Hence we have

$$\tau_1(z) = \langle g(z)\rangle, \quad \tau_2(z) = \langle \psi_0^{(1)*}\psi_{-1}^{(2)} g(z)\rangle, \quad \tau_3(z) = \langle \psi_{-1}^{(1)}\psi_0^{(2)*} g(z)\rangle,$$

where $g(z) = e^{H(z)}ge^{-H(z)}$. The τ_1-function is as in (4.30)

$$\begin{aligned}\tau_1 &= \begin{vmatrix} 1 + a_{11}\langle\psi^1\psi^{1*}\rangle + a_{21}\langle\psi^2\psi^{1*}\rangle & a_{11}\langle\psi^1\psi^{2*}\rangle + a_{21}\langle\psi^2\psi^{2*}\rangle \\ a_{12}\langle\psi^1\psi^{1*}\rangle + a_{22}\langle\psi^2\psi^{1*}\rangle & 1 + a_{12}\langle\psi^1\psi^{2*}\rangle + a_{22}\langle\psi^2\psi^{2*}\rangle \end{vmatrix} \\ &= 1 + a_{11}\langle\psi^1\psi^{1*}\rangle + a_{12}\langle\psi^1\psi^{2*}\rangle + a_{21}\langle\psi^2\psi^{1*}\rangle + a_{22}\langle\psi^2\psi^{2*}\rangle \\ &\quad +(a_{11}a_{22} - a_{12}a_{21})(\langle\psi^1\psi^{1*}\rangle\langle\psi^2\psi^{2*}\rangle - \langle\psi^1\psi^{2*}\rangle\langle\psi^2\psi^{1*}\rangle),\end{aligned}$$

where we choose $\psi^1 = \psi^{(1)}(p^{(1)})$, $\psi^{1*} = \psi^{(1)*}(q^{(1)})$ $\psi^2 = \psi^{(2)}(p^{(2)})$, $\psi^{2*} = \psi^{(2)*}(q^{(2)})$ and $a_{11} = \dfrac{p^{(1)} - q^{(1)}}{q^{(1)}}$, $a_{22} = \dfrac{p^{(2)} - q^{(2)}}{q^{(2)}}$, $a_{12} = \dfrac{p^{(1)} - q^{(2)}}{q^{(2)}}$, $a_{21} = \dfrac{p^{(2)} - q^{(1)}}{q^{(1)}}$. Taking the expectation values, we get

$$\tau_1 = 1 + e^{\eta^{(1)}} + e^{\eta^{(2)}} + A_{12}e^{\eta^{(1)}+\eta^{(2)}},$$

where

$$\eta^{(j)} = \xi(p^{(j)}, z) - \xi(q^{(j)}, z) \qquad j = 1, 2$$

and

$$A_{12} = \frac{(p^{(2)} - p^{(1)})(q^{(2)} - q^{(1)})}{(p^{(1)} - q^{(1)})(p^{(2)} - q^{(2)})}.$$

The τ_2-function can be written from (4.28) in the following form:

$$\tau_2 = \begin{vmatrix} \langle\psi^{i_1}\psi^{1*}\rangle & \langle\psi^{i_1}\psi^{2*}\rangle & \langle\psi^{i_1}\psi^{i_1*}\rangle \\ 1 + a_{11}\langle\psi^1\psi^{1*}\rangle + a_{21}\langle\psi^2\psi^{1*}\rangle & a_{11}\langle\psi^1\psi^{2*}\rangle + a_{21}\langle\psi^2\psi^{2*}\rangle & \langle\psi^1\psi^{i_1*}\rangle \\ a_{12}\langle\psi^1\psi^{1*}\rangle + a_{22}\langle\psi^2\psi^{1*}\rangle & 1 + a_{12}\langle\psi^1\psi^{2*}\rangle + a_{22}\langle\psi^2\psi^{2*}\rangle & \langle\psi^2\psi^{i_1*}\rangle \end{vmatrix},$$

where we choose $\psi^1 = \psi^{(1)}(p^{(1)})$, $\psi^{1*} = \psi^{(1)*}(q^{(1)})$ $\psi^2 = \psi^{(2)}(p^{(2)})$, $\psi^{2*} = \psi^{(2)*}(q^{(2)})$ and $\psi^{i_1*} = \psi_0^{(1)*}$, $\psi^{i_1} = \psi_{-1}^{(2)}$, $a_{12} = \dfrac{p^{(1)} - q^{(2)}}{q^{(2)}}$. Then the τ_2-function is

$$\tau_2 = \frac{q^{(1)}(p^{(1)} - q^{(2)})}{p^{(1)} - q^{(1)}} e^{2\xi(p^{(1)},z) - \xi(q^{(1)},z) - \xi(q^{(2)},z)}.$$

Similarly the τ_3-function can be written from (4.28) in the following form

$$\tau_3 = \begin{vmatrix} \langle\psi^{i_1}\psi^{1*}\rangle & \langle\psi^{i_1}\psi^{2*}\rangle & \langle\psi^{i_1}\psi^{i_1*}\rangle \\ 1 + a_{11}\langle\psi^1\psi^{1*}\rangle + a_{21}\langle\psi^2\psi^{1*}\rangle & a_{11}\langle\psi^1\psi^{2*}\rangle + a_{21}\langle\psi^2\psi^{2*}\rangle & \langle\psi^1\psi^{i_1*}\rangle \\ a_{12}\langle\psi^1\psi^{1*}\rangle + a_{22}\langle\psi^2\psi^{1*}\rangle & 1 + a_{12}\langle\psi^1\psi^{2*}\rangle + a_{22}\langle\psi^2\psi^{2*}\rangle & \langle\psi^2\psi^{i_1*}\rangle \end{vmatrix},$$

where we choose $\psi^1 = \psi^{(1)}(p^{(1)})$, $\psi^{1*} = \psi^{(1)*}(q^{(1)})$ $\psi^2 = \psi^{(2)}(p^{(2)})$, $\psi^{2*} = \psi^{(2)*}(q^{(2)})$ *and* $\psi^{i_1 *} = \psi_0^{(2)*}$, $\psi^{i_1} = \psi_{-1}^{(1)}$, $a_{21} = \dfrac{p^{(2)} - q^{(1)}}{q^{(1)}}$. *Then the* τ_3*-function becomes*

$$\tau_3 = \frac{q^{(2)}(p^{(2)} - q^{(1)})}{p^{(2)} - q^{(2)}} e^{2\xi(p^{(2)},z) - \xi(q^{(2)},z) - \xi(q^{(1)},z)}.$$

The transformations $P = \frac{\tau_2}{\tau_1}$, $Q = \frac{\tau_3}{\tau_1}$, $U = \log \tau_1$ *yield the bilinear equations (4.35) to the following DS equations*

$$\begin{aligned}
&-iP_t + P_{xx} + P_{yy} + 2(U_{xx} + U_{yy})P = 0, \\
&iQ_t + Q_{xx} + Q_{yy} + 2(U_{xx} + U_{yy})Q = 0, \\
&QP = 4U_{xy}.
\end{aligned}$$

Hence $Q = \frac{\tau_3}{\tau_1}$ *is the* 1*-dromion solution [54] for the DS equations.*

4.6 Neutral Free Fermions

In the previous section we constructed charged free fermions ψ_n and ψ_n^* $(n \in \mathbb{Z})$ for the KP-Hierarchy. Here we exploit neutral free fermions ϕ_n $(n \in \mathbb{Z})$ [50], [54] for the BKP-Hierarchy, satisfying the anti-commutation relation

$$[\phi_m, \phi_n]_+ = (-1)^m \delta_{m,-n}.$$

The charged free fermions introduced in the previous section can be split into two sets of neutral free fermions. Namely, if we set

$$\phi_m = \frac{\psi_m + (-1)^m \psi_{-m}^*}{\sqrt{2}}, \qquad \hat{\phi}_m = i\frac{\psi_m - (-1)^m \psi_{-m}^*}{\sqrt{2}}, \qquad (m \in \mathbb{Z}),$$

we have $[\phi_m, \phi_n]_+ = (-1)^m \delta_{m,-n}$, $[\hat{\phi}_m, \hat{\phi}_n]_+ = (-1)^m \delta_{m,-n}$ and $[\phi_m, \hat{\phi}_n]_+ = 0$.

The expectation values of neutral free fermions are defined by

$$\langle \phi_m \phi_n \rangle = \begin{cases} (-1)^m \delta_{m,-n} & n > 0 \\ \frac{1}{2}\delta_{m,0} & n = 0 \\ 0 & n < 0 \end{cases}$$

and the generating function for neutral free fermions is defined as

$$\phi(p) := \sum_{i \in \mathbb{Z}} \phi_i p^i. \tag{4.36}$$

Lemma 4.6.1 *The expectation values are given by*

$$\langle\phi(p_i)\phi(q_j)\rangle = \frac{1}{2}\frac{p_i - q_j}{p_i + q_j}, \qquad \langle\phi(p_i)\phi_0\rangle = \frac{1}{2}.$$

Proof. From the definition (4.36)

$$\begin{aligned}\langle\phi(p_i)\phi(q_j)\rangle &= \sum_{m,n\in\mathbf{Z}}\langle\phi_m\phi_n\rangle p_i^m q_j^n = \frac{1}{2} + \sum_{n>0}(-1)^m\delta_{m,-n}p_i^m q_j^n \\ &= \frac{1}{2} + \sum_{n>0}^{\infty}(-1)^n\left(\frac{q_j}{p_i}\right)^n = \frac{1}{2}\frac{p_i - q_j}{p_i + q_j}\end{aligned}$$

and similiarly

$$\langle\phi(p_i)\phi_0\rangle = \sum_{m,n\in\mathbf{Z}}\langle\phi_m\phi_0\rangle p_i^m = \sum_{m\geq 0}\frac{1}{2}\delta_{m,0}p_i^m = \frac{1}{2}.$$

□

The time evolution for the neutral free fermions

$$\phi(\underline{x}, p_i) = e^{H(\underline{x})}\phi(p_i)e^{-H(\underline{x})} = e^{\xi(\underline{x},p_i)}\phi(p_i), \tag{4.37}$$

where the Hamiltonian $H(\underline{x})$ is defined as

$$H(\underline{x}) = \frac{1}{2}\sum_{\substack{n\in\mathbf{Z}\\ i=1,3,\ldots}}(-1)^{n+1}x_i\phi_n\phi_{-n-l}.$$

The τ-function

$$\tau = \langle g(\underline{x})\rangle,$$

where we choose

$$g = \phi(p_1)\cdots\phi(p_{2r}), \tag{4.38}$$

for some r. Then using Wick's theorem

$$\begin{aligned}\tau_\phi(\underline{x}, g) &= e^{\sum_{i=1}^{2r}\xi(\underline{x},p_i)}\langle g\rangle \\ &= e^{\sum_{i=1}^{2r}\xi(\underline{x},p_i)}\left|\begin{matrix}\langle\phi(p_1)\phi(p_2)\rangle & \cdots & \cdots & \langle\phi(p_1)\phi(p_{2r})\rangle \\ & \langle\phi(p_2)\phi(p_3)\rangle & \cdots & \langle\phi(p_2)\phi(p_{2r})\rangle \\ & & \ddots & \vdots \\ & & & \langle\phi(p_{2r-1})\phi(p_{2r})\rangle\end{matrix}\right|\end{aligned}$$

and using Lemma 4.6.1, we get

$$\tau_\phi = \langle \phi(x,p_1)\cdots\phi(x,p_{2r})\rangle = \frac{1}{2^r}\prod_{i<j}\frac{p_i - p_j}{p_i + p_j}e^{\sum_{i=1}^{2r}\xi(x,p_i)}. \tag{4.39}$$

Now we wish to express the τ-function τ_ϕ in terms of another class of symmetric functions, Schur's Q-functions [56], [57], [58]. In general, for a partition $\lambda = (\lambda_1,\ldots,\lambda_m)$, a Q-function is defined by

$$Q_\lambda(x;t) = \prod_{1\le i<j\le m}(\partial^{(i)} - \partial^{(j)})(\partial^{(i)} - t\partial^{(j)})^{-1}\prod_{i=1}^{m} q_{\lambda_i}(x^{(i)};t)|_{x^{(i)}=x}$$

which reduces to the Schur function $S_\lambda(x)$ when $t=0$, so that $Q_\lambda(x;0) = S_\lambda(x)$. In particular we have the Q-function for the partition $\lambda = (ij)$, namely

$$Q_{(ij)} = q_i q_j + 2\sum_{k=0}^{j-1}(-1)^{k+1}q_{i+k+1}q_{j-k-1}. \tag{4.40}$$

For a general λ, we define a triangular matrix $A = \left(Q_{(\lambda_i\lambda_j)}\right)$. Then, if λ has even number of parts $Q_\lambda = \mathrm{Pf}(A)$, and if λ has odd number of parts then

$$Q_\lambda = \left|\begin{array}{cc} & q_{\lambda_1} \\ A & q_{\lambda_2} \\ & \vdots \\ & q_{\lambda_m}\end{array}\right|.$$

The element g is given by (4.38) can be written as

$$g = \sum_{i_1,\cdots,i_{2r}\in\mathbf{Z}} p_1^{i_1}\cdots p_{2r}^{i_{2r}}g',$$

where

$$g' = \phi_{i_1}\cdots\phi_{i_{2r}}.$$

Then (4.39) can be used as a generating function to determine $\tau_\phi(\underline{x},g')$ by looking at the coefficients of $p_1^{i_1}\cdots p_{2r}^{i_{2r}}$, where $i_1 > i_2 \cdots > i_{2r}\in\mathbf{Z}$. Next we expand the entries of the determinant in (4.39), in the following way:

$$\frac{1}{2}\frac{p_i - p_j}{p_i + p_j} = \frac{1}{2} + \sum_{n=0}^{\infty}\left(-\frac{p_j}{p_i}\right)^{n+1}, \qquad e^{\xi(\underline{x},p_i)} = \sum_{k=0}^{\infty} q_k(x)p_i^k, \tag{4.41}$$

where $q_k(x)$ are the complete symmetric functions. Thus the (i,j)th entry in (4.39) can be written as

$$\begin{aligned}
\langle\phi(\underline{x},p_i)\phi(\underline{x},p_j)\rangle &= \frac{1}{2}\frac{p_i-p_j}{p_i+p_j}e^{\xi(\underline{x},p_i)+\xi(\underline{x},p_j)} \\
&= \left(\frac{1}{2}+\sum_{n=0}^{\infty}\left(-\frac{p_j}{p_i}\right)^{n+1}\right)\sum_{k=0}^{\infty}q_k(x)p_i^k\sum_{l=0}^{\infty}q_l(x)p_j^l \\
&= \frac{1}{2}\sum_{\substack{k=0\\l=0}}^{\infty}q_kq_lp_i^kp_j^l+\sum_{\substack{k=0\\l=0}}^{\infty}\sum_{n=0}^{\infty}(-1)^{n+1}q_kq_lp_i^{k-n-1}p_j^{l+n+1} \\
&= \frac{1}{2}\sum_{\substack{k=-1\\l=0}}^{\infty}q_kq_lp_i^kp_j^l+\sum_{\substack{k=-1\\l=0}}^{\infty}\sum_{n=0}^{\infty}(-1)^{n+1}q_{k+n+1}q_{l-n-1}p_i^kp_j^l \\
&= \frac{1}{2}\sum_{\substack{k=-1\\l=0}}^{\infty}\left(q_kq_l+2\sum_{n=0}^{\infty}(-1)^{n+1}q_{k+n+1}q_{l-n-1}\right)p_i^kp_j^l, \qquad (4.42)
\end{aligned}$$

where $q_i(x)=0$ for $i<0$ and $q_0(x)=1$. The coefficients of $p_i^kp_j^l$ in (4.42) are the Q-functions defined in (4.40). Therefore, we can write the (i,j)th entry of neutral free fermions in terms of Q-functions as follows:

$$\langle\phi(\underline{x},p_i)\phi(\underline{x},p_j)\rangle=\frac{1}{2}\sum_{\substack{k=-1\\l=0}}^{\infty}Q_{(kl)}p_i^kp_j^l.$$

Hence the τ-function in (4.39) can be written in the following form:

$$\begin{aligned}
\tau_\phi(\underline{x},g) &= 2^{-r}\,\mathrm{Pf}\left(\sum_{\substack{k_i=-1\\l_j=0}}^{\infty}Q_{(k_il_j)}p_i^{k_i}p_j^{l_j}\right) \\
&= 2^{-r}\sum_{\substack{k_1,\cdots,k_r=-1\\l_1,\cdots,l_r=0}}^{\infty}\mathrm{Pf}\left(Q_{(k_il_j)}\right)p_1^{k_1}\cdots p_r^{k_r}p_1^{l_1}\cdots p_r^{l_r}
\end{aligned}$$

and

$$\tau_\phi(\underline{x},g')=2^{-r}\,\mathrm{Pf}\left(Q_{(ij)}\right).$$

Hence by (4.40) each τ_ϕ is a Q-function. These give rational solutions [50] of the BKP equation, where $u=2\partial_x(\log\tau_\phi)$.

4.7 Neutral Free Fermions In General

Next we wish to express g more generally for the neutral free fermions in the following form:

$$g=e^{\sum_{i<j=1}^{2N}a_{ij}\phi^i\phi^j},$$

where ϕ^i, ϕ^j $(i < j = 1, \ldots, 2N)$ can be either one-component or two-component fermions. For example we will take $\phi^i = \phi(p_i)$ for the one-component case and $\phi^i = \phi^{(1)}(p^{(1)})$ or $\phi^i = \phi^{(2)}(p^{(2)})$ for the two-component case. Then the τ-function is

$$\begin{aligned}
\tau_\phi &= \langle g(\underline{x})\rangle \\
&= 1 + \sum_{i_1<j_1=1}^{2N} a_{i_1j_1}\langle\phi^{i_1}\phi^{j_1}\rangle + \sum_{i_1<j_1,i_2<j_2=1}^{2N} a_{i_1j_1}a_{i_2j_2}\langle\phi^{i_1}\phi^{j_1}\phi^{i_2}\phi^{j_2}\rangle + \cdots + \\
&\quad \sum_{i_1<j_1,\cdots,i_N<j_N=1}^{2N} a_{i_1j_1}\cdots a_{i_Nj_N}\langle\phi^{i_1}\phi^{j_1}\cdots\phi^{i_N}\phi^{j_N}\rangle. \qquad (4.43)
\end{aligned}$$

Next we give the following expectation value

$$\langle\phi^i\phi^i\rangle := 0. \qquad (4.44)$$

Using the definition of the expectation value in (4.44) and Wick's theorem, the τ_ϕ function in (4.43) can be written in the following pfaffian form:

$$\tau_\phi = \mathrm{Pf}(A)\,\mathrm{Pf}(S), \qquad (4.45)$$

where A and A' are constant tri-angular matrices with the entries $A_{i<j} = [a_{ij}]$, $A'_{i<j} = [a'_{ij}]$ respectively and S is the tri-angular matrix with the entries $S_{i<j} = [a'_{ij} + \langle\phi^i\phi^j\rangle]$. A and A' have the relations $A = \mathrm{Pf}(A)(A')^\dagger$ and $(\mathrm{Pf}(A))^{-1} = \mathrm{Pf}(A')$.

Consider a triangular array $A = [a_{ij}]_{i<j=1,\ldots,n}$ of size n, i.e.

$$A = \begin{bmatrix} a_{12} & a_{13} & \cdots & a_{1n} \\ & a_{23} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ & & & a_{n-1,n} \end{bmatrix}.$$

Then one may define an adjoint array $A^\dagger = [(-1)^{i+j+1}a^\dagger_{ij}]_{i<j=1,\ldots,n}$ whose entries are pfaffians of subarrays of A. To be precise $a^\dagger_{kl}$ is the pfaffian of the array obtained by deleting the k-th and l-th lines in A. Now we define

$$A' = \frac{1}{\mathrm{Pf}(A)}A^\dagger.$$

Note that this array is the analogue of the inverse of a matrix; for a matrix M,

$$M^{-1} = \frac{1}{\det(M)}M^\dagger.$$

Indeed if W and W' are the skew-symmetric matrices whose upper triangles are A and A' respectively, then $W' = W^{-1}$.

For example, for $N = 2$ from (4.43) we have the following solution:

$$
\begin{aligned}
\tau_{\phi_2} &= 1 + \sum_{i_1<j_1=1}^{4} a_{i_1j_1}\langle\phi^{i_1}\phi^{j_1}\rangle + \sum_{i_1<j_1, i_2<j_2=1}^{4} a_{i_1j_1}a_{i_2j_2}\langle\phi^{i_1}\phi^{j_1}\phi^{i_2}\phi^{j_2}\rangle \\
&= 1 + a_{12}\langle\phi^1\phi^2\rangle + a_{13}\langle\phi^1\phi^3\rangle + a_{14}\langle\phi^1\phi^4\rangle + a_{23}\langle\phi^2\phi^3\rangle + a_{24}\langle\phi^2\phi^4\rangle + a_{34}\langle\phi^3\phi^4\rangle \\
&\quad + a_{12}a_{34}\langle\phi^1\phi^2\phi^3\phi^4\rangle + a_{13}a_{24}\langle\phi^1\phi^3\phi^2\phi^4\rangle + a_{14}a_{23}\langle\phi^1\phi^4\phi^2\phi^3\rangle \\
&= 1 + a_{12}\langle\phi^1\phi^2\rangle + a_{13}\langle\phi^1\phi^3\rangle + a_{14}\langle\phi^1\phi^4\rangle + a_{23}\langle\phi^2\phi^3\rangle + a_{24}\langle\phi^2\phi^4\rangle + a_{34}\langle\phi^3\phi^4\rangle \\
&\quad + (a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})(\langle\phi^1\phi^2\rangle\langle\phi^3\phi^4\rangle - \langle\phi^1\phi^3\rangle\langle\phi^2\phi^4\rangle + \langle\phi^1\phi^4\rangle\langle\phi^2\phi^3\rangle) \\
&= \mathrm{Pf}(A_2)\Bigg(\frac{1}{\mathrm{Pf}(A_2)} + \frac{a_{12}}{\mathrm{Pf}(A_2)}\langle\phi^1\phi^2\rangle + \frac{a_{13}}{\mathrm{Pf}(A_2)}\langle\phi^1\phi^3\rangle + \frac{a_{14}}{\mathrm{Pf}(A_2)}\langle\phi^1\phi^4\rangle \\
&\quad + \frac{a_{23}}{\mathrm{Pf}(A_2)}\langle\phi^2\phi^3\rangle + \frac{a_{24}}{\mathrm{Pf}(A_2)}\langle\phi^2\phi^4\rangle + \frac{a_{34}}{\mathrm{Pf}(A_2)}\langle\phi^3\phi^4\rangle + \langle\phi^1\phi^2\rangle\langle\phi^3\phi^4\rangle \\
&\quad - \langle\phi^1\phi^3\rangle\langle\phi^2\phi^4\rangle + \langle\phi^1\phi^4\rangle\langle\phi^2\phi^3\rangle\Bigg) \\
&= \mathrm{Pf}(A_2)\Big(\mathrm{Pf}(A_2') + a_{34}'\langle\phi^1\phi^2\rangle - a_{24}'\langle\phi^1\phi^3\rangle + a_{23}'\langle\phi^1\phi^4\rangle + a_{14}'\langle\phi^2\phi^3\rangle \\
&\quad - a_{13}'\langle\phi^2\phi^4\rangle + a_{12}'\langle\phi^3\phi^4\rangle + \langle\phi^1\phi^2\rangle\langle\phi^3\phi^4\rangle - \langle\phi^1\phi^3\rangle\langle\phi^2\phi^4\rangle \\
&\quad + \langle\phi^1\phi^4\rangle\langle\phi^2\phi^3\rangle\Big),
\end{aligned}
\tag{4.46}
$$

where

$$
A_2 = \begin{bmatrix} a_{12} & a_{13} & a_{14} \\ & a_{23} & a_{24} \\ & & a_{34} \end{bmatrix} = \mathrm{Pf}(A_2) \begin{bmatrix} a_{34}' & -a_{24}' & a_{23}' \\ & a_{14}' & -a_{13}' \\ & & a_{12}' \end{bmatrix}
$$

and

$$
A_2' = \begin{bmatrix} a_{12}' & a_{13}' & a_{14}' \\ & a_{23}' & a_{24}' \\ & & a_{34}' \end{bmatrix}.
$$

Now we can write the expression τ_{ϕ_2} in (4.46) in the form of (4.45)

$$
\tau_{\phi_2} = \mathrm{Pf}(A_2)\,\mathrm{Pf}(S_2), \tag{4.47}
$$

where

$$S_2 = \begin{bmatrix} a'_{12} + \langle\phi^1\phi^2\rangle & a'_{13} + \langle\phi^1\phi^3\rangle & a'_{14} + \langle\phi^1\phi^4\rangle \\ & a'_{23} + \langle\phi^2\phi^3\rangle & a'_{24} + \langle\phi^2\phi^4\rangle \\ & & a'_{34} + \langle\phi^3\phi^4\rangle \end{bmatrix}.$$

Example 4.7.1 *For the soliton solution for the one-component case, we put $\phi^1 = \phi(p_1)$, $\phi^2 = \phi(q_1)$, $\phi^3 = \phi(p_2)$, $\phi^4 = \phi(q_2)$ and choose $a_{12} = 2\frac{p_1+q_1}{p_1-q_1}$, $a_{34} = 2\frac{p_2+q_2}{p_2-q_2}$, $a_{13} = a_{14} = a_{23} = a_{24} = 0$. The τ-function from (4.47) is*

$$\begin{aligned}
\tau &= \left|\begin{matrix} 1 + a_{12}\langle\phi(p_1)\phi(q_1)\rangle & a_{12}\langle\phi(p_1)\phi(p_2)\rangle & a_{12}\langle\phi(p_1)\phi(q_2)\rangle \\ & a_{34}\langle\phi(q_1)\phi(p_2)\rangle & a_{34}\langle\phi(q_1)\phi(q_2)\rangle \\ & & 1 + a_{34}\langle\phi(p_2)\phi(q_2)\rangle \end{matrix}\right| \\
&= 1 + a_{12}\langle\phi(p_1)\phi(q_1)\rangle + a_{34}\langle\phi(p_2)\phi(q_2)\rangle + a_{12}a_{34}(\langle\phi(p_1)\phi(q_1)\rangle\langle\phi(p_2)\phi(q_2)\rangle \\
&\quad -\langle\phi(p_1)\phi(p_2)\rangle\langle\phi(q_1)\phi(q_2)\rangle + \langle\phi(p_1)\phi(q_2)\rangle\langle\phi(q_1)\phi(p_2)\rangle) \\
&= 1 + e^{\eta_1} + e^{\eta_2} + B_{12}e^{\eta_1+\eta_2},
\end{aligned}$$

where $\eta_i = \xi(x,p_i) + \xi(x,q_i) \quad (i = 1,2)$ and

$$B_{12} = \frac{(p_1-p_2)(q_1-q_2)(p_1-q_2)(q_1-p_2)}{(p_1+p_2)(q_1+q_2)(p_1+q_2)(q_1+p_2)}.$$

Hence $u = 2\partial_x(\log\tau)$ gives the 2-soliton solution [58] for the BKP equation

$$(u_t + 15uu_{3x} + 15u_x^3 - 15u_xu_y + u_{5x})_x + 5u_{3x,y} - 5u_{yy} = 0.$$

Example 4.7.2 *For the two-component case, we put $\phi^1 = \phi^{(1)}(p^{(1)})$, $\phi^2 = \phi^{(1)}(q^{(1)})$, $\phi^3 = \phi^{(2)}(p^{(2)})$, $\phi^4 = \phi^{(2)}(q^{(2)})$ in (4.47). Then the τ-function from (4.47) is*

$$\begin{aligned}
\tau &= \left|\begin{matrix} 1 + a_{12}\langle\phi^{(1)}(p^{(1)})\phi^{(1)}(q^{(1)})\rangle & a_{13}\langle\phi^{(1)}(p^{(1)})\phi^{(1)}(q^{(1)})\rangle & a_{14}\langle\phi^{(1)}(p^{(1)})\phi^{(1)}(q^{(1)})\rangle \\ & a_{23}\langle\phi^{(2)}(p^{(2)})\phi^{(2)}(q^{(2)})\rangle & a_{24}\langle\phi^{(2)}(p^{(2)})\phi^{(2)}(q^{(2)})\rangle \\ & & 1 + a_{34}\langle\phi^{(2)}(p^{(2)})\phi^{(2)}(q^{(2)})\rangle \end{matrix}\right| \\
&= 1 + a_{12}\langle\phi^{(1)}(p^{(1)})\phi^{(1)}(q^{(1)})\rangle + a_{34}\langle\phi^{(2)}(p^{(2)})\phi^{(2)}(q^{(2)})\rangle \\
&\quad +(a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})\langle\phi^{(1)}(p^{(1)})\phi^{(1)}(q^{(1)})\rangle\langle\phi^{(2)}(p^{(2)})\phi^{(2)}(q^{(2)})\rangle \\
&= 1 + a_{12}\frac{1}{2}\frac{p^{(1)}-q^{(1)}}{p^{(1)}+q^{(1)}}e^{\eta^{(1)}} + a_{34}\frac{1}{2}\frac{p^{(2)}-q^{(2)}}{p^{(2)}+q^{(2)}}e^{\eta^{(2)}} + B_{12}e^{\eta^{(1)}+\eta^{(2)}},
\end{aligned}$$

where $\eta^{(i)} = \xi(x^{(i)}, p^{(i)}) + \xi(x^{(i)}, q^{(i)}) \quad (i = 1, 2)$ *and*

$$B_{12} = (a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})\frac{1}{4}\frac{p^{(1)} - q^{(1)}}{p^{(1)} + q^{(1)}}\frac{p^{(2)} - q^{(2)}}{p^{(2)} + q^{(2)}}.$$

Hence $u = \partial_{xy}(\log \tau)$ *gives the* 1*-dromion solution [44] for the NVN equations*

$$\begin{aligned} u_t &= u_{xxx} + u_{yyy} + 3(\Phi_{xx}u)_x + 3(\Phi_{yy}u)_y \\ u &= \Phi_{xy}. \end{aligned}$$

We note that the τ-function in (4.45) is the pfaffian analogue of the τ-function in (4.27), namely from (4.45) τ can be written in the following form

$$\tau = \mathrm{Pf}(A)\,\mathrm{Pf}(S) = \mathrm{Pf}(A)\,\mathrm{Pf}(A' + V),$$

where $V_{i<j} = [\langle\phi^i\phi^j\rangle]$ and the corresponding determinantal form can be written in the following form, from (4.27):

$$\tau = \det(I + AV) = \det(A(A^{-1} + V)) = \det(A)\det(A^{-1} + V).$$

Appendix

To show that the expression given in (4.27) is valid, we take $W = AV$ (where W is an $(N \times N)$ matrix with the entries $W = [w_{ij}]$) and expand in the following form [55]:

$$\begin{aligned} \tau_0 &= \det(I + AV) = \det(I + W) \\ &= 1 + \sum_{i=1}^{N} w_{ii} + \sum_{i<j=1}^{N} \begin{vmatrix} w_{ii} & w_{ij} \\ w_{ji} & w_{jj} \end{vmatrix} + \cdots + \begin{vmatrix} W \end{vmatrix}, \end{aligned} \tag{4.48}$$

where $w_{ij} = a_{ki}\langle\psi^k\psi^{j*}\rangle$. Hence the expression in (4.25) is reduced to the expansion in (4.48) after eliminating some terms according to the definition given in (4.26).

Chapter 5

The KR and mNVN Equations: The Pfaffian Technique

5.1 Introduction

In this chapter we study two $(2+1)$-dimensional integrable nonlinear evolution equations; both have dimensional reductions to known integrable equations in $(1+1)$-dimensions. If the two spatial variables appear on an equal footing and hence allow such reductions in either variable one calls the $(2+1)$-dimensional system a strong generalization of the $(1+1)$-dimensional system. For example, the KdV equation has two generalizations to $(2+1)$-dimension, namely the KP equation which is a weak generalization, and the NVN equations which are a strong generalization of the KdV equation. The Konopelchenko-Rogers (KR) equations are a $(2+1)$-dimensional strong generalization of the $(1+1)$-dimensional sine-Gordon(sG) equation analogous to the modified Novikov-Veselov-Nitzhik (mNVN) equations

$$\begin{aligned}4u_t &= u_{xxx} - u_{yyy} + 3u_x v_{xx} - 3u_y v_{yy} - u_x^3 + u_y^3 \\ v_{xy} &= u_x u_y.\end{aligned}$$

These reduce to the potential mKdV equation

$$2u_t = u_{xxx} + 2u_x^3$$

when $y \to -x$. Hence the mNVN equations are a strong generalization of the potential mKdV equation. Both the KR and mNVN equations have pfaffian solutions.

In section 2, we recall some results from [61] and [62]; we apply the gauge transformation to the Lax pair of the KR equations and, after rescaling the Lax pair, we derive the weak Lax pair for the KR equations. The compatibility of this Lax pair leads to the KR equations. In section 3 we carry out the same procedure as in section 2 and obtain a new result for the mNVN equations. In section 4, we show that the KR and mNVN equations in the bilinear form reduce to the identity of pfaffians. These are new results.

5.2 The two-dimensional sine-Gordon equation

The system of Konopelchenko and Rogers [59] arises as the compatibility conditions for the triad of operators

$$
\begin{aligned}
L_1 &= \partial_X + S\partial_Y, \\
L_2 &= \partial_t\partial_Y - V\partial_Y - W_Y, \\
L_3 &= \partial_t\partial_X - V\partial_X - W_X,
\end{aligned}
$$

where $\partial_z = \dfrac{\partial}{\partial z}$ is a derivative with respect to the indicated variable and

$$
S = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}, \tag{5.1}
$$

$$
V = \frac{1}{2}\begin{pmatrix} 0 & -\theta_t \\ \theta_t & 0 \end{pmatrix},
$$

$$
W_Y = -\frac{1}{2\sin\theta}\begin{pmatrix} \phi_X - \cos\theta\phi_Y & \tilde{\phi_Y}\sin\theta \\ -\phi_Y\sin\theta & -(\tilde{\phi_X} + \cos\theta\tilde{\phi_Y}) \end{pmatrix},
$$

$$
W_X = -\frac{1}{2\sin\theta}\begin{pmatrix} \phi_Y - \cos\theta\phi_X & \tilde{\phi_X}\sin\theta \\ -\phi_X\sin\theta & -(\tilde{\phi_Y} + \cos\theta\tilde{\phi_X}) \end{pmatrix},
$$

in which $\theta_t = \phi + \tilde{\phi}$.

Next we will transform the Lax pair of the 2-dimensional sine-Gordon(sG) equation into a new Lax pair which is gauge equivalent to a pair of operators. The Lax pair of the 2-dimensional sine-Gordon or Konopelchenko-Rogers (KR) equations is

$$\begin{aligned} L &:= g^{-1}L_1g = \partial_X - J\partial_Y + Q \\ M &:= g^{-1}(L_2 - JL_3)g = \partial_t\partial_Y + S\partial_t\partial_X + \tfrac{1}{2}(\theta_Y A + \partial_X SA)\partial_t \\ &\quad + \tfrac{1}{2}(\theta_t\theta_Y + \theta_{Yt}A + \theta_{Xt}SA) + \tilde{W}_1 + \tilde{W}_2, \end{aligned}$$

where g is the gauge and

$$J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{5.2}$$

is the reflection matrix,

$$Q = \frac{1}{2}\begin{pmatrix} 0 & \theta_X + \theta_Y \\ \theta_Y - \theta_X & 0 \end{pmatrix},$$

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{5.3}$$

$$\tilde{W}_1 = \frac{\theta_{Xt}}{4\cos\frac{\theta}{2}}\begin{pmatrix} \sin\frac{3\theta}{2} & -\cos\frac{3\theta}{2} \\ -\cos\frac{3\theta}{2} & -\sin\frac{3\theta}{2} \end{pmatrix},$$

$$\tilde{W}_2 = \frac{1}{4}\begin{pmatrix} (\rho_X + \theta_{Yt} - \rho_Y)\tan\frac{\theta}{2} & (\rho_X - \theta_{Yt} + \rho_Y) \\ (-\rho_X + \theta_{Yt} + \rho_Y) & (\rho_X + \theta_{Yt} + \rho_Y)\tan\frac{\theta}{2} \end{pmatrix},$$

in which $\rho = \phi - \tilde{\phi}$. The matrix $S = S_\theta$ can be written in the following form

$$S_\theta = -R_\theta J = -JR_{-\theta},$$

where

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{5.4}$$

is the rotation matrix and J is the reflection matrix given in (5.2). The gauge is chosen to be a 'half-rotation and reflection', thus $g := S_{\theta/2}$, for which $g^2 = I$. If we rotate axes $(X, Y) \to (x, y)$ so that $\partial_x = \partial_X + \partial_Y$, $\partial_y = \partial_X - \partial_Y$ we get the following Lax pair

$$\begin{aligned} L &= \begin{pmatrix} \partial_x & \frac{1}{2}\theta_x \\ -\frac{1}{2}\theta_y & \partial_y \end{pmatrix}, \\ M &= \begin{pmatrix} \partial_y\partial_t + \Theta_y & \frac{1}{2}\theta_y\partial_t \\ -\frac{1}{2}\theta_x\partial_t & \partial_x\partial_t + \Theta_x \end{pmatrix}, \end{aligned}$$

where

$$\Theta_x = -\frac{\rho_y + \theta_{yt}\cos\theta}{2\sin\theta}, \qquad \Theta_y = \frac{\rho_x - \theta_{xt}\cos\theta}{2\sin\theta}.$$

To simplify the notation and to free θ for more conventional usage, we rescale $\theta \to 2u$ and $\Theta \to v_t$. In these variables the Lax pair is

$$L = \begin{pmatrix} \partial_x & u_x \\ -u_y & \partial_y \end{pmatrix} \tag{5.5}$$

$$M = \begin{pmatrix} \partial_y\partial_t + v_{yt} & u_y\partial_t \\ -u_x\partial_t & \partial_x\partial_t + v_{xt} \end{pmatrix}. \tag{5.6}$$

Let

$$\Phi = \begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix} \tag{5.7}$$

be a common solution of $L\Phi = M\Phi = 0$ for L and M given by (5.5) and (5.6) respectively. If we write $L\Phi = 0$ and $M\Phi = 0$ in component form we get the linear equations for the KR equations

$$\phi^1_x + u_x\phi^2 = 0, \tag{5.8}$$

$$\phi^2_y - u_y\phi^1 = 0, \tag{5.9}$$

$$\phi^1_{yt} + v_{yt}\phi^1 + u_y\phi^2_t = 0, \tag{5.10}$$

$$\phi^2_{xt} + v_{xt}\phi^2 - u_x\phi^1_t = 0. \tag{5.11}$$

The commutator for (5.5) and (5.6) is

$$[L, M] = LM - ML = \begin{pmatrix} [L,M]_{1,1} & [L,M]_{1,2} \\ [L,M]_{2,1} & [L,M]_{2,2} \end{pmatrix},$$

where

$$\begin{aligned}
[L,M]_{1,1} &= v_{xyt} + u_y u_{yt} - (u_x^2 - u_y^2)\partial_t, \\
[L,M]_{1,2} &= u_x v_{xt} - u_x v_{yt} - u_{xyt} - u_{xt}\partial_y + (u_x + u_y)(\partial_x - \partial_y)\partial_t, \\
[L,M]_{2,1} &= u_y v_{xt} - u_y v_{yt} + u_{xyt} + u_{yt}\partial_x + (u_x + u_y)(\partial_x - \partial_y)\partial_t, \\
[L,M]_{2,2} &= v_{xyt} + u_x u_{xt} + (u_x^2 - u_y^2)\partial_t.
\end{aligned}$$

To eliminate the operators, we use the linear equations (5.8)-(5.11) in the above commutator, then we get

$$\begin{aligned}
[L, M]_{1,1} &= v_{xyt} - u_y u_{xt} - u_x u_{yt}, \\
[L, M]_{1,2} &= -u_x v_{yt} - u_{xyt} - u_y v_{xt}, \\
[L, M]_{2,1} &= u_y v_{xt} + u_{xyt} + u_x v_{yt}, \\
[L, M]_{2,2} &= v_{xyt} - u_x u_{yt} - u_y u_{xt}.
\end{aligned}$$

Solving these equations for the compatibility condition $[L, M] = 0$, we get the KR equations

$$u_{xyt} + u_x v_{yt} + u_y v_{xt} = 0, \tag{5.12}$$

$$v_{xy} - u_x u_y = 0. \tag{5.13}$$

5.3 Lax pair for the modified Novikov-Veselov-Nithzik equations

It is known that a Lax pair that gives pfaffian solutions of nonlinear equations satisfies the following equation [63]

$$\partial_X L + L^{\dagger}\partial_X = 0, \tag{5.14}$$

where L is one of the Lax pair and $L^{\dagger}$ is the adjoint of L. As shown below this is satisfied by L_1 for the KR equations. We wish to study similar Lax pairs which obey this constraint.

We take the following Lax pair

$$\begin{aligned}
\tilde{L} &= \partial_Y + S\partial_X \\
\tilde{M} &= \partial_t + (E\partial_X^2 + \partial_X^2 F + T\partial_X + \partial_X H + B)\partial_X,
\end{aligned}$$

where S is given in (5.1) and E, F, T, H, B are arbitrary 2×2 matrices, which we will find out later. The adjoint pair is

$$\begin{aligned}
\tilde{L}^{\dagger} &= -\partial_Y - \partial_X S^T, \\
\tilde{M}^{\dagger} &= -\partial_t - \partial_X(\partial_X^2 E^T + F^T\partial_X^2 - \partial_X T^T - H^T\partial_X + B^T).
\end{aligned}$$

Next we substitute these into the left hand side of (5.14). We get

$$\partial_X \tilde{L} + \tilde{L}^\dagger \partial_X = \partial_X(\partial_Y + S\partial_X) + (-\partial_Y - \partial_X S^T)\partial_X = 0$$

and

$$\begin{aligned}
&\partial_X \tilde{M} + \tilde{M}^\dagger \partial_X = \\
&\partial_X(\partial_t + (E\partial_X^2 + \partial_X^2 F + T\partial_X + \partial_X H + B)\partial_X) \\
&+(-\partial_t - \partial_X(\partial_X^2 E^T + F^T\partial_X^2 - \partial_X T^T - H^T\partial_X + B^T))\partial_X = 0
\end{aligned}$$

when $F = E^T$, $H = -T^T$ and $B = B^T$. The compatibility condition for this Lax pair

$$\begin{aligned}
[\tilde{L}, \tilde{M}] &= [\partial_Y, \partial_t] + [\partial_Y, E\partial_X^3] + [\partial_Y, \partial_X^2 E^T\partial_X] + [\partial_Y, T\partial_X^2] + [\partial_Y, -\partial_X T^T\partial_X] \\
&\quad +[\partial_Y, B\partial_X] + [S\partial_X, \partial_t] + [S\partial_X, E\partial_X^3] + [S\partial_X, \partial_X^2 E^T\partial_X] \\
&\quad +[S\partial_X, T\partial_X^2] + [S\partial_X, -\partial_X T^T\partial_X] + [S\partial_X, B\partial_X] = 0
\end{aligned}$$

gives $E = S$ and $T = -T^T$. Hence we can write the Lax pair (strong) [63] for the mNVN equations in the following form:

$$\tilde{L} = \partial_Y + S\partial_X \tag{5.15}$$

$$\tilde{M} = \partial_t + (S\partial_X^2 + \partial_X^2 S + T\partial_X + \partial_X T + B)\partial_X, \tag{5.16}$$

where S is given in (5.1), $T = wA$ is a skew-symmetric matrix in which A is given by (5.3), and B is a symmetric real matrix. The matrices S and A have the following properties

$$S^2 = I, \quad A^2 = -I, \quad SAS = -A, \quad ASA = S, \quad AS = -SA \tag{5.17}$$

and

$$(AS)^2 = (SA)^2 = I, \tag{5.18}$$

where I is the 2×2 identity matrix. To find the entries of the matrices T and B, we make use of the compatibility condition of the Lax pair, $[\tilde{L}, \tilde{M}] = 0$. This commutator is

$$
\begin{aligned}
[\tilde{L}, \tilde{M}] &= [\partial_Y, \partial_t] + [\partial_Y, S\partial_X^3] + [\partial_Y, \partial_X^2 S\partial_X] + [\partial_Y, T\partial_X^2] + [\partial_Y, \partial_X T\partial_X] \\
&\quad + [\partial_Y, B\partial_X] + [S\partial_X, \partial_t] + [S\partial_X, S\partial_X^3] + [S\partial_X, \partial_X^2 S\partial_X] \\
&\quad + [S\partial_X, T\partial_X^2] + [S\partial_X, \partial_X T\partial_X] + [S\partial_X, B\partial_X] \\
&= 2(S_Y - SS_X - S_X S + ST - TS)\partial_X^3 + (2S_{XY} + 2T_Y - 3SS_{XX} \\
&\quad - S_{XX}S - 4S_X^2 + 3ST_X - 4TS_X - T_X S + SB - BS)\partial_X^2 \\
&\quad + (S_{XXY} + T_{XY} + B_Y - S_t - SS_{XXX} - S_{XX}S_X - 2S_X S_{XX} \\
&\quad - 2TS_{XX} + ST_{XX} - T_X S_X + SB_X - BS_X)\partial_X. \qquad (5.19)
\end{aligned}
$$

Before looking at this in more detail we will determine the necessary derivatives of the matrices S and A. Using the properties of the matrices S and A given in (5.17), (5.18), we have

$$S_X = \theta_X SA, \qquad S_Y = \theta_Y SA, \qquad S_t = \theta_t SA, \qquad (5.20)$$

$$S_{XX} = \theta_{XX} SA - \theta_X^2 S, \qquad S_{XY} = \theta_{XY} SA - \theta_X \theta_Y S, \qquad (5.21)$$

$$S_{XXX} = \theta_{XXX} SA - 3\theta_X \theta_{XX} S - \theta_X^3 SA, \qquad (5.22)$$

$$S_{XXY} = \theta_{XXY} SA - \theta_Y \theta_{XX} S - 2\theta_X \theta_{XY} S - \theta_X^2 \theta_Y SA, \qquad (5.23)$$

$$T_X = w_X A, \qquad T_{XX} = w_{XX} A, \qquad T_{XY} = w_{XY} A. \qquad (5.24)$$

In order that the compatibility condition $[\tilde{L}, \tilde{M}] = 0$ is satisfied, the coefficients of the operators ∂_X, ∂_X^2, ∂_X^3 must vanish. Using the derivatives (5.20)-(5.24) and the relations in (5.17) and (5.18), from (5.19), the coefficient of ∂_X^3 vanishes if

$$S_Y - SS_X - S_X S + ST - TS = SA(\theta_Y + 2w) = 0.$$

Hence $w = -\frac{1}{2}\theta_Y$ and $T = -\frac{1}{2}\theta_Y A$. From the coefficient of ∂_X^2, we get

$$(2\theta_{XX} + \theta_{YY})A = SB - BS \qquad (5.25)$$

and, from the coefficient of ∂_X, we get

$$
\begin{aligned}
(\theta_{XXY} - 2\theta_t)SA - 3\theta_X \theta_{XY} S + 2B_Y + 2SB_X - 2\theta_X BSA& \\
= (\theta_{XYY} + 2\theta_{XXX})A.& \qquad (5.26)
\end{aligned}
$$

From (5.25) $B = SBS + (2\theta_{XX} + \theta_{YY})SA$. To write B explicitly, we take B in the following form

$$B = (\theta_{XX} + \frac{1}{2}\theta_{YY})SA + \alpha S + \beta I, \tag{5.27}$$

where α and β are to be found in terms of dependent variables, and then

$$B_X = (\theta_{XXX} + \frac{1}{2}\theta_{XYY})SA - \theta_X(\theta_{XX} + \frac{1}{2}\theta_{YY})S + \alpha_X S + \alpha\theta_X SA + \beta_X, \tag{5.28}$$

$$B_Y = (\theta_{XXY} + \frac{1}{2}\theta_{YYY})SA - \theta_Y(\theta_{XX} + \frac{1}{2}\theta_{YY})S + \alpha_Y S + \alpha\theta_Y SA + \beta_Y. \tag{5.29}$$

Next, we substitute (5.27), (5.28) and (5.29) into (5.26) and, after simplifying, we get the following equations

$$\begin{aligned} 3\theta_{XXY} + \theta_{YYY} - 2\theta_t + 2\alpha\theta_Y - 2\beta\theta_X &= 0 && (5.30)\\ -3\theta_X\theta_{XY} - 2\theta_Y\theta_{XX} - \theta_Y\theta_{YY} + 2\alpha_Y + 2\beta_X &= 0 && (5.31)\\ -\theta_X(2\theta_{XX} + \theta_{YY}) + \alpha_X + \beta_Y &= 0. && (5.32) \end{aligned}$$

From (5.31), we write

$$(\alpha - \frac{1}{4}\theta_X^2 - \frac{1}{4}\theta_Y^2)_Y = (\theta_X\theta_Y - \beta)_X,$$

and define the following relations

$$\psi_X = \alpha - \frac{1}{4}\theta_X^2 - \frac{1}{4}\theta_Y^2, \qquad \psi_Y = \theta_X\theta_Y - \beta. \tag{5.33}$$

From (5.32), we write

$$(\alpha - \theta_X^2 + \frac{1}{2}\theta_Y^2)_X = (\theta_X\theta_Y - \beta)_Y, \tag{5.34}$$

and then we substitute the expressions defined by (5.33) into (5.34). We get the following relation

$$\psi_{XX} - \psi_{YY} = \frac{3}{4}(\theta_X^2 - \theta_Y^2)_X. \tag{5.35}$$

Hence using the definition (5.33), the B in (5.27) can be written in the following form

$$B = (\theta_{XX} + \frac{1}{2}\theta_{YY})SA + (\psi_X + \frac{1}{4}\theta_X^2 + \frac{1}{4}\theta_Y^2)S + (\theta_X\theta_Y - \psi_Y)I. \tag{5.36}$$

In summary we have

$$\begin{aligned}\tilde{L} &= \partial_Y + S\partial_X \\ \tilde{M} &= \partial_t + (S\partial_X^2 + \partial_X^2 S + T\partial_X + \partial_X T + B)\partial_X\end{aligned}$$

where S is given in (5.1), $T = wA$ is a skew-symmetric matrix in which A is given by (5.3), and B is given by (5.36), and the compatibility condition gives (5.30) and (5.35).

5.3.1 Alternative form of the Lax pair

Now we will use a Lax pair which is gauge equivalent to the pair $\tilde{L}$, $\tilde{M}$ given by (5.15), (5.16) respectively. To do this, again we observe that the matrix $S = S_\theta$ may be written in terms of the rotation matrix given in (5.4) and the reflexion matrix given in (5.2), so that

$$S_\theta = -R_\theta J = -JR_{-\theta}.$$

Again, the gauge is chosen to be a 'half-rotation and reflection',

$$g := S_{\theta/2}$$

for which $g^2 = I$, and $g^{-1} = g$. For further usage, we also observe the following equalities of matrices

$$Ag = -gA, \qquad gAg = -A, \qquad (gA)^2 = I, \qquad gSg = gSAgA = -J,$$

$$-JA = gSgA = C,$$

where C is defined as

$$C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and the derivatives for g are

$$g_X = \frac{1}{2}\theta_X gA \qquad g_Y = \frac{1}{2}\theta_Y gA$$

$$g_{XX} = \frac{1}{2}\theta_{XX} gA - \frac{1}{4}\theta_X^2 g \qquad g_{XXX} = \frac{1}{2}\theta_{XXX} gA - \frac{3}{4}\theta_X\theta_{XX} g - \frac{1}{8}\theta_X^3 gA.$$

Then the gauge transformation for the Lax operator L is

$$
\begin{aligned}
L = g^{-1}\tilde{L}g &= g(\partial_Y + S\partial_X)g \\
&= gg_Y + g^2\partial_Y + gSg_X + gSg\partial_X \\
&= \partial_Y - J\partial_X + \frac{1}{2}\theta_Y A - \frac{1}{2}\theta_X JA
\end{aligned}
$$

and hence L can be written in the following form:

$$
L = \partial_Y - J\partial_X + Q, \tag{5.37}
$$

where

$$
Q = \frac{1}{2}\begin{pmatrix} 0 & \theta_X + \theta_Y \\ \theta_X - \theta_Y & 0 \end{pmatrix}, \tag{5.38}
$$

and the gauge transformation for the Lax operator M is

$$
\begin{aligned}
M &= g^{-1}\tilde{M}g \\
&= g(\partial_t + S\partial_X^3 + \partial_X^2 S\partial_X + T\partial_X^2 + \partial_X T\partial_X + B\partial_X)g \\
&= gg_t + g^2\partial_t + gS\partial_X^3 g + g\partial_X^2 S\partial_X g + gT\partial_X^2 g + g\partial_X T\partial_X g + gB\partial_X g \\
&= \partial_t + (\theta_{XX}C + \frac{1}{4}\theta_X^2 J + \frac{1}{2}\theta_{XY}A - \frac{1}{2}\theta_{YY}C - \psi_X J - \frac{1}{4}\theta_Y^2 J - \psi_Y)\partial_X \\
&\quad +(\theta_X C + \theta_Y A)\partial_X^2 - 2J\partial_X^3 + \frac{1}{2}\theta_t A + \frac{1}{4}\theta_X^2\theta_Y A - \frac{1}{2}\theta_X\psi_Y A \\
&\quad +\theta_{XXX}C - \frac{1}{8}\theta_X^3 C + \frac{1}{2}\theta_X\psi_X C + \frac{1}{8}\theta_X\theta_Y^2 C - \frac{1}{2}\theta_X\theta_{XX}J \\
&\quad -\frac{1}{4}\theta_X\theta_{YY}J - \frac{1}{2}\theta_{XX}\theta_Y - \frac{1}{4}\theta_X\theta_{XY}.
\end{aligned}
$$

Hence M can be written in the following form

$$
M = \partial_t + P\partial_X + 2Q\partial_X^2 - 2J\partial_X^3 + R, \tag{5.39}
$$

where

$$
P = \begin{pmatrix} -\frac{1}{4}(\theta_X^2 - \theta_Y^2) + \psi_X - \psi_Y & \theta_{XX} - \frac{1}{2}\theta_{YY} + \frac{1}{2}\theta_{XY} \\ \theta_{XX} - \frac{1}{2}\theta_{YY} - \frac{1}{2}\theta_{XY} & \frac{1}{4}(\theta_X^2 - \theta_Y^2) - \psi_X - \psi_Y \end{pmatrix},
$$

Q is given by (5.38), and

$$
R = \begin{pmatrix} R_1 & R_2 \\ R_3 & R_4 \end{pmatrix}
$$

in which

$$\begin{aligned}
R_1 &= \frac{1}{2}\theta_{XX}(\theta_X - \theta_Y) + \frac{1}{4}\theta_X(\theta_{YY} - \theta_{XY}), \\
R_2 &= \frac{1}{2}\theta_t + \frac{1}{4}\theta_X\theta_Y(\theta_X + \frac{1}{2}\theta_Y) + \frac{1}{2}\theta_X(\psi_X - \psi_Y) + \theta_{XXX} - \frac{1}{8}\theta_X^3, \\
R_3 &= -\frac{1}{2}\theta_t - \frac{1}{4}\theta_X\theta_Y(\theta_X - \frac{1}{2}\theta_Y) + \frac{1}{2}\theta_X(\psi_X + \psi_Y) + \theta_{XXX} - \frac{1}{8}\theta_X^3, \\
R_4 &= -\frac{1}{2}\theta_{XX}(\theta_X + \theta_Y) - \frac{1}{4}\theta_X(\theta_{YY} + \theta_{XY}).
\end{aligned}$$

If we rotate axes $(X,Y) \to (x,y)$ so that $\partial_x = \partial_X + \partial_Y$, $\partial_y = \partial_X - \partial_Y$, then we have

$$\partial_X = \frac{\partial_x + \partial_y}{2}, \qquad \partial_Y = \frac{\partial_x - \partial_y}{2} \tag{5.40}$$

and

$$\theta_X = \frac{1}{2}(\theta_x + \theta_y), \qquad \theta_Y = \frac{1}{2}(\theta_x - \theta_y), \qquad \theta_{XY} = \frac{1}{4}(\theta_{xx} - \theta_{yy}),$$

$$\theta_{XX} = \frac{1}{4}(\theta_{xx} + 2\theta_{xy} + \theta_{yy}), \qquad \theta_{YY} = \frac{1}{4}(\theta_{xx} - 2\theta_{xy} + \theta_{yy}),$$

$$\theta_{XXX} = \frac{1}{8}(\theta_{xxx} + 3\theta_{xxy} + 3\theta_{xyy} + \theta_{yyy}).$$

After substituting these expressions into (5.37) and (5.39), we get the following Lax pair

$$L = \begin{pmatrix} \partial_X + \partial_Y & \frac{1}{2}(\theta_X + \theta_Y) \\ \frac{1}{2}(\theta_X - \theta_Y) & \partial_Y - \partial_X \end{pmatrix} = \begin{pmatrix} \partial_x & \frac{1}{2}\theta_x \\ \frac{1}{2}\theta_y & -\partial_y \end{pmatrix} \tag{5.41}$$

and

$$M = \partial_t + \frac{1}{2}P(\partial_x + \partial_y) + \frac{1}{2}Q(\partial_x + \partial_y)^2 - \frac{1}{4}J(\partial_x + \partial_y)^3 + R, \tag{5.42}$$

where

$$P = \begin{pmatrix} -\frac{1}{4}\theta_x\theta_y + \psi_y & \frac{1}{4}\theta_{xx} + \frac{3}{4}\theta_{xy} \\ \frac{1}{4}\theta_{yy} + \frac{3}{4}\theta_{xy} & \frac{1}{4}\theta_x\theta_y - \psi_x \end{pmatrix}, \qquad Q = \frac{1}{2}\begin{pmatrix} 0 & \theta_x \\ \theta_y & 0 \end{pmatrix},$$

$$R = \begin{pmatrix} r_1 & r_2 \\ r_3 & r_4 \end{pmatrix},$$

in which

$$\begin{aligned}
r_1 &= \frac{1}{16}\theta_y(2\theta_{xx}+3\theta_{xy}+3\theta_{yy})+\frac{1}{16}\theta_x(\theta_{yy}-\theta_{xy}),\\
r_2 &= \frac{1}{2}\theta_t+\frac{1}{32}(\theta_x^3-\theta_x^2\theta_y-3\theta_x\theta_y^2-\theta_y^3)+\frac{1}{4}(\theta_x+\theta_y)\psi_y\\
&\quad +\frac{1}{8}(\theta_{xxx}+3\theta_{xxy}+3\theta_{xyy}+\theta_{yyy}),\\
r_3 &= -\frac{1}{2}\theta_t-\frac{1}{32}(\theta_x^3+\theta_x\theta_y^2+3\theta_y\theta_x^2-\theta_y^3)+\frac{1}{4}(\theta_x+\theta_y)\psi_x\\
&\quad +\frac{1}{8}(\theta_{xxx}+3\theta_{xxy}+3\theta_{xyy}+\theta_{yyy}),\\
r_4 &= -\frac{1}{16}\theta_x(2\theta_{yy}+3\theta_{xy}+3\theta_{xx})-\frac{1}{16}\theta_y(\theta_{xx}-\theta_{xy}).
\end{aligned}$$

If we simplify (5.42) by taking out the multiples and powers of L, we then get the following:

$$M=\begin{pmatrix} M_1 & M_2\\ M_3 & M_4\end{pmatrix},$$

where

$$\begin{aligned}
M_1 &= \partial_t+\frac{1}{4}\partial_y^3+\frac{1}{2}\psi_y\partial_y-\frac{3}{16}\theta_x\theta_y\partial_y+\frac{3}{16}\theta_y\theta_{yy},\\
M_2 &= \frac{1}{8}\theta_x\partial_x^2-\frac{1}{8}\theta_{xx}\partial_x+\frac{1}{8}\theta_{yyy}+\frac{1}{4}\psi_y\theta_y+\frac{1}{2}\theta_t+\frac{1}{32}(\theta_x^3-3\theta_x\theta_y^2-\theta_y^3),\\
M_3 &= \frac{1}{8}\theta_y\partial_y^2-\frac{1}{8}\theta_{yy}\partial_y+\frac{1}{8}\theta_{xxx}+\frac{1}{4}\psi_x\theta_x-\frac{1}{2}\theta_t-\frac{1}{32}(\theta_x^3+3\theta_y\theta_x^2-\theta_y^3),\\
M_4 &= \partial_t-\frac{1}{4}\partial_x^3-\frac{1}{2}\psi_x\partial_x+\frac{3}{16}\theta_x\theta_y\partial_x-\frac{3}{16}\theta_x\theta_{xx}.
\end{aligned}$$

To simplify the notation we rescale the variables

$$\theta\to 2u,\qquad \psi\to\frac{3}{2}(v_x+v_y) \tag{5.43}$$

so that we can get the appropriate form of the mNVN equations. In these variables the Lax pair for the mNVN equations is

$$L=\begin{pmatrix}\partial_x & u_x\\ u_y & -\partial_y\end{pmatrix}\qquad\text{and}\qquad M=\frac{1}{4}\begin{pmatrix}M_1 & M_2\\ M_3 & M_4\end{pmatrix}, \tag{5.44}$$

where

$$\begin{aligned}
M_1 &= 4\partial_t+\partial_y^3+3(v_{xy}+v_{yy}-u_xu_y)\partial_y+3u_yu_{yy}\\
M_2 &= u_x\partial_x^2-u_{xx}\partial_x+3(u_yv_{xy}+u_yv_{yy}-u_xu_y^2)+u_{yyy}+4u_t+u_x^3-u_y^3\\
M_3 &= u_y\partial_y^2-u_{yy}\partial_y+3(u_xv_{xx}+u_xv_{xy}-u_yu_x^2)+u_{xxx}-4u_t-u_x^3+u_y^3\\
M_4 &= 4\partial_t-\partial_x^3-3(v_{xx}+v_{xy}-u_xu_y)\partial_x-3u_xu_{xx}.
\end{aligned}$$

Now, we take the common solution Φ given by (5.7) so that $L\Phi = 0$ and $M\Phi = 0$ and write them in component form. Thus we get the following linear equations for the mNVN equations

$$\phi^1_x + u_x\phi^2 = 0, \tag{5.45}$$

$$\phi^2_y - u_y\phi^1 = 0, \tag{5.46}$$

$$\begin{aligned} 4\phi^1_t + \phi^1_{yyy} + 3(v_{xy} + v_{yy} - u_xu_y)\phi^1_y + 3u_yu_{yy}\phi^1 \\ +u_x\phi^2_{xx} - u_{xx}\phi^2_x + 3(u_yv_{xy} + u_yv_{yy} - u_xu_y^2)\phi^2 \\ + u_{yyy}\phi^2 + 4u_t\phi^2 + (u_x^3 - u_y^3)\phi^2 &= 0, \end{aligned} \tag{5.47}$$

$$\begin{aligned} 4\phi^2_t - \phi^2_{xxx} - 3(v_{xx} + v_{xy} - u_xu_y)\phi^2_x - 3u_xu_{xx}\phi^2 \\ +u_y\phi^1_{yy} - u_{yy}\phi^1_y + 3(u_xv_{xx} + u_xv_{xy} - u_yu_x^2)\phi^1 \\ + u_{xxx}\phi^1 - 4u_t\phi^1 - (u_x^3 - u_y^3)\phi^1 &= 0. \end{aligned} \tag{5.48}$$

Next we write the compatibility for (5.44) and in order to write it in the simplest form we exploit the evolution equations given by (5.45)-(5.48). Hence we get

$$[L, M] = LM - ML = \frac{1}{4}\begin{pmatrix} 3\Lambda\partial_y - \Delta u_x & 3\Lambda u_y + \Delta\partial_x \\ 3\Lambda u_x + \Delta\partial_y & 3\Lambda\partial_x + \Delta u_y \end{pmatrix},$$

where

$$\Lambda = v_{xxy} + v_{xyy} - (u_xu_y)_x - (u_xu_y)_y$$

and

$$\Delta = 4u_t - u_{xxx} + u_{yyy} + u_x^3 - u_y^3 - 3u_x(v_{xx} + v_{xy} + u_y^2) + 3u_y(v_{yy} + v_{xy} + u_x^2).$$

The compatibility condition $[L, M] = 0$ for (5.44) is only satisfied when $\Lambda = 0$ and $\Delta = 0$. Thus, under this condition, we get the mNVN equations

$$4u_t = u_{xxx} - u_{yyy} + 3u_xv_{xx} - 3u_yv_{yy} - u_x^3 + u_y^3 \tag{5.49}$$

$$v_{xy} = u_xu_y. \tag{5.50}$$

Using (5.49) and (5.50), the equations (5.45)-(5.48) can be simplified to

$$\begin{aligned} \phi_x^1 + u_x\phi^2 &= 0, && (5.51) \\ \phi_y^2 - u_y\phi^1 &= 0, && (5.52) \\ 4\phi_t^1 + \phi_{yyy}^1 + 3v_{yy}\phi_y^1 + 3u_yu_{yy}\phi^1 + u_x\phi_{xx}^2 - u_{xx}\phi_x^2 & && \\ +(3u_xv_{xx} + u_{xxx})\phi^2 &= 0, && (5.53) \\ 4\phi_t^2 - \phi_{xxx}^2 - 3v_{xx}\phi_x^2 - 3u_xu_{xx}\phi^2 + u_y\phi_{yy}^1 - u_{yy}\phi_y^1 & && \\ + (3u_yv_{yy} + u_{yyy})\phi^1 &= 0. && (5.54) \end{aligned}$$

The compatibility condition of the strong Lax pair in (5.15), (5.16), as we expect, gives the same equations as in (5.49), (5.50). From (5.30), after substituting values for α and β from (5.33) into (5.30), we rotate the axes as in (5.40) and rescale the variables as given by (5.43), and then we recover the equation $\Delta = 0$. Similiarly from (5.35), we do the same scaling and rotation of axes and then we recover the equation $\Lambda = 0$. The compatibility of the strong Lax pair given in (5.15), (5.16) and the weak Lax pair given in (5.44) lead to the same nonlinear equation given in (5.49), (5.50).

5.4 The KR and mNVN equations: The Pfaffian Technique

In this section we will prove that Pfaffians satisfy the KR and mNVN equations.

The bilinear form of KR equations can be written in the following way

$$u = u_0 + i\ln(G/F), \qquad v = v_0 + \ln(GF), \tag{5.55}$$

where G and F are complex conjugates of one another. Introducing this change into (5.13) we get

$$v_{0xy} - u_{0x}u_{0y} + (FG)^{-1}[D_xD_y - i(u_{0x}D_y + u_{0y}D_x)]G \cdot F = 0. \tag{5.56}$$

We suppose that (u_0, v_0) is itself a solution of (5.12), (5.13) and so from (5.56) we find the bilinear equation

$$[D_xD_y - i(u_{0x}D_y + u_{0y}D_x)]G \cdot F = 0. \tag{5.57}$$

Now considering (5.12) in a similar way we get

$$\begin{aligned} u_{0xyt} &+ u_{0x}v_{0yt} + u_{0y}v_{0xt} \\ &+ (FG)^{-1}[i(D_xD_yD_t + v_{0xt}D_y + v_{0yt}D_x) + u_{0x}D_yD_t + u_{0y}D_xD_t]G\cdot F \\ &+ (FG)^{-2}(-iD_tG\cdot F)[D_xD_y - i(u_{0x}D_y + u_{0y}D_x)]G\cdot F = 0. \end{aligned}$$

Since we suppose that (u_0, v_0) satisfies (5.12), and using (5.57), we get a second bilinear equation

$$[D_xD_yD_t + v_{0xt}D_y + v_{0yt}D_x - i(u_{0x}D_yD_t + u_{0y}D_xD_t)]G\cdot F = 0. \tag{5.58}$$

The pair (5.57), (5.58) are the Hirota form of (5.12), (5.13). Particularly, if we take $u_0 = 0$, $v_{0xt} = \lambda$ and $v_{0yt} = \mu$, then the Hirota form (5.57), (5.58) simplifies to become

$$\begin{aligned} (D_xD_yD_t + \lambda D_y + \mu D_x)G\cdot F &= 0, \\ D_xD_yG\cdot F &= 0. \end{aligned}$$

We introduce pfaffians denoted by $(1, 2, \cdots, 2n)$ (see chapter 2) which represent the functions G and F in the following form

$$G = (1, 2, \cdots, 2n) \tag{5.59}$$

and

$$F = (1, 2, \cdots, 2n)^* \tag{5.60}$$

whose elements are given by the skew-product

$$S[\theta_i, \theta_j] = \int_{(x_0,y_0)}^{(x,y)} W_x[\theta_i, \theta_j]dx - W_y[\theta_i, \theta_j]dy + (\theta_i^*(\theta_j)_t - (\theta_i)_t\theta_j^*)dt, \tag{5.61}$$

where $*$ denotes the complex conjugate, for $X = x$ or y or t, $W_X[a, b] = ab_X - a_Xb$ and

$$\theta_k = \phi_k^1 + i\phi_k^2$$

in which, for $k = 1, \ldots, 2n$, ϕ_k^1 and ϕ_k^2 satisfy the equations (5.8)-(5.11). The integral in (5.61) is written so that it is exact, thus

$$\begin{aligned} &W_{xy}[\theta_i, \theta_j] = -W_{yx}[\theta_i, \theta_j], \quad W_{xt}[\theta_i, \theta_j] = (\theta_i^*(\theta_j)_t - (\theta_i)_t\theta_j^*)_x, \\ &W_{yt}[\theta_i, \theta_j] = -(\theta_i^*(\theta_j)_t - (\theta_i)_t\theta_j^*)_y. \end{aligned}$$

Therefore, we may find the t dependent element in (5.61) by using these equalities with the linear equations given in (5.8)-(5.11). The (i,j)-th element of pfaffians in (5.59), (5.60) is $(i,j) = S[\theta_i, \theta_j]$ and $(I^i, j) = \phi_j^i$, $(I^i, I^j) = 0$. Thus the (i,j)-th element can be written in the following form

$$\begin{aligned}
(i,j) &= \int (W_x[\phi_i^1, \phi_j^1] - W_x[\phi_i^2, \phi_j^2])dx - (W_y[\phi_i^1, \phi_j^1] - W_y[\phi_i^2, \phi_j^2])dy \\
&\quad + (W_t[\phi_i^1, \phi_j^1] + W_t[\phi_i^2, \phi_j^2])dt + i((W_x[\phi_i^1, \phi_j^2] + W_x[\phi_i^2, \phi_j^1])dx \\
&\quad - (W_y[\phi_i^1, \phi_j^2] + W_y[\phi_i^2, \phi_j^1])dy + (\phi_i^1\phi_j^2 - \phi_i^2\phi_j^1)_t dt).
\end{aligned}$$

In order to prove the equations (5.12), (5.13) we exploit the identities of pfaffians which correspond to the Jacobi identity of determinants, and show that the derivatives of the pfaffians are represented by the sum of the pfaffians. From (2.1) and (2.2), for $m = 2$, we have the following pfaffian identities

$$\begin{aligned}
&(a_1, a_2, a_3, a_4, b_1, b_2, \cdots, b_{2m})(b_1, b_2, \cdots, b_{2m}) \\
&= (a_1, a_2, b_1, b_2, \cdots, b_{2m})(a_3, a_4, b_1, b_2, \cdots, b_{2m}) \\
&-(a_1, a_3, b_1, b_2, \cdots, b_{2m})(a_2, a_4, b_1, b_2, \cdots, b_{2m}) \\
&+(a_1, a_4, b_1, b_2, \cdots, b_{2m})(a_2, a_3, b_1, b_2, \cdots, b_{2m})
\end{aligned} \tag{5.62}$$

and

$$\begin{aligned}
&(a_1, a_2, a_3, b_1, b_2, \cdots, b_{2m-1})(b_1, b_2, \cdots, b_{2m}) \\
&= (a_1, b_1, b_2, \cdots, b_{2m-1}))(a_2, a_3, b_1, b_2, \cdots, b_{2m}) \\
&-(a_2, b_1, b_2, \cdots, b_{2m-1}))(a_1, a_3, b_1, b_2, \cdots, b_{2m}) \\
&+(a_3, b_1, b_2, \cdots, b_{2m-1}))(a_1, a_2, b_1, b_2, \cdots, b_{2m}).
\end{aligned} \tag{5.63}$$

With these properties we prove that the KR and mNVN equations reduce to the identity of pfaffians. To write the derivatives of the pfaffians we introduce the symbol ∂_Z^j such that $(\partial_Z^j, i) = \phi_{iZ}^j$ where $Z = x, y, t, xy, xt, yt, xyt$ etc. and $(\partial_Z^i, \partial_Z^j) = 0$. In this notation we have the derivatives of (5.59), (5.60)

$$\begin{aligned}
G_t = (\partial_t^1, I^1, 1, 2, \cdots, 2n) + (\partial_t^2, I^2, 1, 2, \cdots, 2n) \\
+i(\partial_t^2, I^1, 1, 2, \cdots, 2n) + i(I^2, \partial_t^1, 1, 2, \cdots, 2n),
\end{aligned}$$

$$G_x = -u_x(I^2, I^1, 1, 2, \cdots, 2n) + (I^2, \partial_x^2, 1, 2, \cdots, 2n) + i(\partial_x^2, I^1, 1, 2, \cdots, 2n),$$

$$G_y = -u_y(I^2, I^1, 1, 2, \cdots, 2n) + (I^1, \partial_y^1, 1, 2, \cdots, 2n) + i(I^2, \partial_y^1, 1, 2, \cdots, 2n),$$

$$\begin{aligned} G_{xt} &= -u_{xt}(I^2, I^1, 1, 2, \cdots, 2n) - u_x(\partial_t^2, I^1, 1, 2, \cdots, 2n) \\ &\quad +(\partial_t^2, \partial_x^2, 1, 2, \cdots, 2n) + (I^2, \partial_x^2, \partial_t^1, I^1, 1, 2, \cdots, 2n) \\ &\quad -iv_{xt}(I^2, I^1, 1, 2, \cdots, 2n) + iu_x(\partial_t^1, I^1, 1, 2, \cdots, 2n) \\ &\quad +i(\partial_x^2, \partial_t^1, 1, 2, \cdots, 2n) + i(I^2, \partial_x^2, \partial_t^2, I^1, 1, 2, \cdots, 2n), \end{aligned}$$

$$\begin{aligned} G_{yt} &= -u_{yt}(I^2, I^1, 1, 2, \cdots, 2n) + u_y(\partial_t^1, I^2, 1, 2, \cdots, 2n) \\ &\quad +(\partial_t^1, \partial_y^1, 1, 2, \cdots, 2n) + (I^1, \partial_y^1, \partial_t^2, I^2, 1, 2, \cdots, 2n) \\ &\quad -iv_{yt}(I^2, I^1, 1, 2, \cdots, 2n) + iu_y(\partial_t^2, I^2, 1, 2, \cdots, 2n) \\ &\quad +i(\partial_t^2, \partial_y^1, 1, 2, \cdots, 2n) + i(I^1, \partial_y^1, I^2, \partial_t^1, 1, 2, \cdots, 2n), \end{aligned}$$

$$\begin{aligned} G_{xy} &= -u_x(I^2, \partial_y^1, 1, 2, \cdots, 2n) + u_y(I^1, \partial_x^2, 1, 2, \cdots, 2n) \\ &\quad +(I^1, \partial_y^1, I^2, \partial_x^2, 1, 2, \cdots, 2n) - iu_xu_y(I^2, I^1, 1, 2, \cdots, 2n) \\ &\quad +i(\partial_x^2, \partial_y^1, 1, 2, \cdots, 2n), \end{aligned}$$

$$\begin{aligned} G_{xyt} &= u_xv_{yt}(I^2, I^1, 1, 2, \cdots, 2n) + u_yv_{xt}(I^2, I^1, 1, 2, \cdots, 2n) \\ &\quad -u_xu_y(\partial_t^1, I^1, 1, 2, \cdots, 2n) - u_{xt}(I^2, \partial_y^1, 1, 2, \cdots, 2n) \\ &\quad -u_x(\partial_t^2, \partial_y^1, 1, 2, \cdots, 2n) + u_{yt}(I^1, \partial_x^2, 1, 2, \cdots, 2n) \\ &\quad +u_y(\partial_t^1, \partial_x^2, 1, 2, \cdots, 2n) - u_xu_y(\partial_t^2, I^2, 1, 2, \cdots, 2n) \\ &\quad +(I^1, \partial_y^1, \partial_t^2, \partial_x^2, 1, 2, \cdots, 2n) - (\partial_y^1, \partial_t^1, I^2, \partial_x^2, 1, 2, \cdots, 2n) \\ &\quad -iv_{xyt}(I^2, I^1, 1, 2, \cdots, 2n) - iv_{xt}(I^2, \partial_y^1, 1, 2, \cdots, 2n) \\ &\quad -iv_{yt}(\partial_x^2, I^1, 1, 2, \cdots, 2n) - iu_xu_y(\partial_t^2, I^1, 1, 2, \cdots, 2n) \\ &\quad -iu_xu_y(I^2, \partial_t^1, 1, 2, \cdots, 2n) + iu_x(\partial_t^1, \partial_y^1, 1, 2, \cdots, 2n) \\ &\quad -iu_y(\partial_x^2, \partial_t^2, 1, 2, \cdots, 2n) + i(I^1, \partial_y^1, \partial_x^2, \partial_t^1, 1, 2, \cdots, 2n) \\ &\quad +i(I^2, \partial_x^2, \partial_t^2, \partial_y^1, 1, 2, \cdots, 2n) \end{aligned}$$

where we have taken $u_0 = u$ and $v_0 = v$, for simplicity. Substituting these results into the left hand side of (5.57), we obtain

$$
\begin{aligned}
&2((1,2,\cdots,2n)(I^1,\partial_y^1,I^2,\partial_x^2,1,2,\cdots,2n)\\
&+(I^2,I^1,1,2,\cdots,2n)(\partial_x^2,\partial_y^1,1,2,\cdots,2n)\\
&-(I^2,\partial_x^2,1,2,\cdots,2n)(I^1,\partial_y^1,1,2,\cdots,2n)\\
&-(\partial_x^2,I^1,1,2,\cdots,2n)(I^2,\partial_y^1,1,2,\cdots,2n))
\end{aligned}
$$

which vanishes by virtue of the pfaffian identity of the form (5.62). Similarly substituting these derivatives into the left hand side of (5.58), we obtain

$$
\begin{aligned}
&2i((I^1,I^2,1,2,\cdots,2n)(\partial_y^1,\partial_t^1,\partial_x^2,I^2,1,2,\cdots,2n)\\
&-(\partial_t^1,\partial_x^2,I^1,I^2,1,2,\cdots,2n)(\partial_y^1,I^2,1,2,\cdots,2n)\\
&+(\partial_y^1,\partial_x^2,I^1,I^2,1,2,\cdots,2n)(\partial_t^1,I^2,1,2,\cdots,2n)\\
&-(\partial_y^1,\partial_t^1,I^1,I^2,1,2,\cdots,2n)(\partial_x^2,I^2,1,2,\cdots,2n)\\
&+(I^2,I^1,1,2,\cdots,2n)(\partial_y^1,\partial_t^2,\partial_x^2,I^1,1,2,\cdots,2n)\\
&-(\partial_t^2,\partial_x^2,I^2,I^1,1,2,\cdots,2n)(\partial_y^1,I^1,1,2,\cdots,2n)\\
&+(\partial_y^1,\partial_x^2,I^2,I^1,1,2,\cdots,2n)(\partial_t^2,I^1,1,2,\cdots,2n)\\
&-(\partial_y^1,\partial_t^2,I^2,I^1,1,2,\cdots,2n)(\partial_x^2,I^1,1,2,\cdots,2n))
\end{aligned}
$$

which vanishes by virtue of two pfaffian identities of the form (5.63).

Next we prove that the mNVN equations reduce to the pfaffian identities. We have already shown that equation (5.50) reduces to a pfaffian identity. Similarly, the bilinear form of equation (5.49) can be written by substituting the expressions in (5.55) into (5.49). We get

$$
\begin{aligned}
&4u_{0t}-u_{0xxx}+u_{0yyy}-3u_{0x}v_{0xx}+3u_{0y}v_{0yy}+u_{0x}^3-u_{0y}^3\\
&+(GF)^{-1}[i(4D_t-D_x^3+D_y^3-3v_{0xx}D_x+3v_{0yy}D_y+3u_{0x}^2D_x-3u_{0y}^2D_y)\\
&\qquad -3u_{0x}D_x^2+3u_{0y}D_y^2]G\cdot F=0.
\end{aligned}
$$

Since we suppose that (u_0,v_0) satisfies (5.49), then we get the following bilinear form

$$
\begin{aligned}
&[4D_t-D_x^3+D_y^3-3(v_{0xx}-u_{0x}^2)D_x+3(v_{0yy}-u_{0y}^2)D_y\\
&\qquad +3i(u_{0x}D_x^2-u_{0y}D_y^2)]G\cdot F=0. \qquad (5.64)
\end{aligned}
$$

Particularly, again if we take $u_0 = 0$, $v_{0xt} = \lambda$ and $v_{0yt} = \mu$, then the Hirota form simplifies to become

$$(4D_t - D_x^3 + D_y^3)G \cdot F = 0.$$

For the mNVN equations, the elements of G are given by the skew-product

$$\begin{aligned}(i,j) &= \int (W_x[\phi_i^1,\phi_j^1] - W_x[\phi_i^2,\phi_j^2])dx - (W_y[\phi_i^1,\phi_j^1] - W_y[\phi_i^2,\phi_j^2])dy \\ &\quad -(W_t[\phi_i^1,\phi_j^1] + W_t[\phi_i^2,\phi_j^2] - \tfrac{1}{2}W_x[\phi_{i,x}^2,\phi_{j,x}^2] + \tfrac{1}{2}W_y[\phi_{i,y}^1,\phi_{j,y}^1])dt \\ &\quad +i((W_x[\phi_i^1,\phi_j^2] + W_x[\phi_i^2,\phi_j^1])dx - (W_y[\phi_i^1,\phi_j^2] + W_y[\phi_i^2,\phi_j^1])dy \\ &\quad +(\phi_i^1\phi_j^2 - \phi_i^2\phi_j^1)_t dt)\end{aligned}$$

in which, for $k = 1,\ldots,2n$, ϕ_k^1 and ϕ_k^2 satisfy the linear equations (5.51)-(5.54). The x and y dependent elements of this integral are the same as in (5.61), and also exact, and hence we can write the t dependent elements (i.e. by differentiating the x dependent element with respect to t and integrating with respect to x, and exploiting the linear equations given in (5.51)-(5.54)).

In order to reduce the bilinear form (5.64) to pfaffian identities, we need the following derivatives together with the previous derivatives of G, again for simplicity , we take $u_0 = u$ and $v_0 = v$.

$$\begin{aligned}G_t &= (I^1,\partial_t^1,1,2,\cdots,2n) + (I^2,\partial_t^2,1,2,\cdots,2n) \\ &\quad +\frac{1}{2}(\partial_y^1,\partial_{yy}^1,1,2,\cdots,2n) + \frac{1}{2}(\partial_{xx}^2,\partial_x^2,1,2,\cdots,2n) \\ &\quad +i(\partial_t^2,I^1,1,2,\cdots,2n) + i(I^2,\partial_t^1,1,2,\cdots,2n)\end{aligned}$$

$$\begin{aligned}G_{xx} &= -u_{xx}(I^2,I^1,1,2,\cdots,2n) - u_x(\partial_x^2,I^1,1,2,\cdots,2n) \\ &\quad +(I^2,\partial_{xx}^2,1,2,\cdots,2n) + i(\partial_{xx}^2,I^1,1,2,\cdots,2n) \\ &\quad -iu_x(\partial_x^2,I^2,1,2,\cdots,2n)\end{aligned}$$

$$\begin{aligned}G_{yy} &= -u_{yy}(I^2,I^1,1,2,\cdots,2n) - u_y(I^2\partial_y^1,1,2,\cdots,2n) \\ &\quad +(I^1,\partial_{yy}^1,1,2,\cdots,2n) + i(I^2,\partial_{yy}^1,1,2,\cdots,2n) \\ &\quad +iu_y(I^1,\partial_y^1,1,2,\cdots,2n)\end{aligned}$$

$$
\begin{aligned}
G_{xxx} = &-u_{xxx}(I^2, I^1, 1, 2, \cdots, 2n) - 2u_{xx}(\partial_x^2, I^1, 1, 2, \cdots, 2n) \\
&-u_x(\partial_{xx}^2, I^1, 1, 2, \cdots, 2n) + u_x^2(\partial_x^2, I^2, 1, 2, \cdots, 2n) \\
&+(\partial_x^2, \partial_{xx}^2, 1, 2, \cdots, 2n) + (I^2, \partial_{xxx}^2, 1, 2, \cdots, 2n) \\
&+i(\partial_{xxx}^2, I^1, 1, 2, \cdots, 2n) - 2iu_x(\partial_{xx}^2, I^2, 1, 2, \cdots, 2n) \\
&+i(I^2, \partial_x^2, \partial_{xx}^2, I^1, 1, 2, \cdots, 2n) - iu_{xx}(\partial_x^2, I^2, 1, 2, \cdots, 2n)
\end{aligned}
$$

$$
\begin{aligned}
G_{yyy} = &-u_{yyy}(I^2, I^1, 1, 2, \cdots, 2n) - 2u_{yy}(I^2, \partial_y^1, 1, 2, \cdots, 2n) \\
&-u_y^2(I^1, \partial_y^1, 1, 2, \cdots, 2n) - u_y(I^2, \partial_{yy}^1, 1, 2, \cdots, 2n) \\
&+(\partial_y^1, \partial_{yy}^1, 1, 2, \cdots, 2n) + (I^1, \partial_{yyy}^1, 1, 2, \cdots, 2n) \\
&+i(I^2, \partial_{yyy}^1, 1, 2, \cdots, 2n) + 2iu_y(I^1, \partial_{yy}^1, 1, 2, \cdots, 2n) \\
&+i(I^1, \partial_y^1, I^2, \partial_{yy}^1, 1, 2, \cdots, 2n) + iu_{yy}(I^1, \partial_y^1, 1, 2, \cdots, 2n)
\end{aligned}
$$

Substituting these results into the left hand side of (5.64), we obtain

$$
\begin{aligned}
&4i((1, 2, \cdots, 2n)(I^2, \partial_x^2, \partial_{xx}^2, I^1, 1, 2, \cdots, 2n) \\
&-(I^2, I^1, 1, 2, \cdots, 2n)(\partial_x^2, \partial_{xx}^2, 1, 2, \cdots, 2n) \\
&+(I^2, \partial_{xx}^2, 1, 2, \cdots, 2n)(\partial_x^2, I^1, 1, 2, \cdots, 2n) \\
&-(I^2, \partial_x^2, 1, 2, \cdots, 2n)(\partial_{xx}^2, I^1, 1, 2, \cdots, 2n) \\
&+(I^1, \partial_y^1, 1, 2, \cdots, 2n)(I^2, \partial_{yy}^1, 1, 2, \cdots, 2n) \\
&-(I^2, \partial_y^1, 1, 2, \cdots, 2n)(I^1, \partial_{yy}^1, 1, 2, \cdots, 2n) \\
&+(I^2, I^1, 1, 2, \cdots, 2n)(\partial_y^1, \partial_{yy}^1, 1, 2, \cdots, 2n) \\
&+(1, 2, \cdots, 2n)(I^2, \partial_{yy}^1, \partial_y^1, I^1, 1, 2, \cdots, 2n))
\end{aligned}
$$

which vanishes by virtue of the pfaffian identity of the form (5.62).

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